Topic 1 — Logic and proof

1.2 Logic

1.2 Exercise

1 $\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

2

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

3

p	$p \vee p$	$p \leftrightarrow (p \vee p)$
T	T	T
F	F	T

4

p	q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Same

5

p	q	$p \leftrightarrow q$	$\neg p$	$\neg q$	$\neg p \leftrightarrow \neg q$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Same

6 Implication $p \rightarrow q$. If it is raining then I take an umbrella.
Contrapositive $\neg q \rightarrow \neg p$. If I do not take an umbrella then it is not raining.
Inverse $\neg p \rightarrow \neg q$. If it is not raining then I will not take an umbrella.
Converse $q \rightarrow p$. If a take an umbrella then it is raining.
Negation $\neg(p \rightarrow q) \leftrightarrow p \wedge \neg q$. It is raining and I do not take an umbrella.

7 Converse of $p \rightarrow q$ is $q \rightarrow p$
Inverse of $p \rightarrow q$ is $\neg p \rightarrow \neg q$

p	q	$q \rightarrow p$	$\neg p$	$\neg q$	$\neg p \rightarrow \neg q$
T	T	T	F	F	T
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	T	T

Same

8 See table at bottom of the page*

9 See table at bottom of the page*

***8**

p	q	r	$q \wedge r$	$p \wedge (q \wedge r)$	$p \wedge q$	$(p \wedge q) \wedge r$
T	T	T	T	T	T	T
T	T	F	F	F	T	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	T	F	F	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

Same

***9**

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Same

10 See table at bottom of the page*

11 **a** All natural numbers satisfy $x > -1$, T.

b All natural numbers are positive, T.

12 **a** There are natural numbers that satisfy $x < -1$, F.

b There is a rational number which satisfies $3x = 2$, T, $x = \frac{2}{3}$.

13 **a** There is an integer which satisfies $3x + 6 = 0$, T, $x = -2$.

b There is an integer whose square is less than 1, T, $x = 0$.

14 **a** The sum of every two natural numbers is positive, T.

b The sum of the squares of every two real numbers is positive, F, $x = 0, y = 0$.

15 **a** For every positive integer, there is a negative integer, such that their sum is zero, T.

b For every integer, there is an integer greater than it, T.

16 **a** The multiplicative reciprocal of every rational number is rational, F, not true when $y = 0$.

b The square of every rational number is rational, T.

c There is a real number which is the reciprocal of the square root of a rational number, T.

17 **a** For every integer, there is an integer which is half of it, F.

b For every integer, there is a rational number which is half of it, T.

18 There is an integer such that its sum with all integers is zero, F.

19 For every non-zero rational number there is a multiplicative reciprocal, T.

20 Every quadratic with real coefficients has rational solutions, F.

1.2 Exam questions

1

p	q	$p \to q$	$\neg(p \to q)$	$\neg q$	$p \wedge \neg q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

Same

Award 1 mark for setting up a truth table with the correct column headers. Award 1 mark for correct truth values and for showing that the 4th and 6th columns are equivalent.

2 See table at bottom of the page*

Award 1 mark for setting up a truth table with the correct column headers. Award 1 mark for correct truth values and award 1 mark for showing that the 7th column is true whenever the 6th column is.

3 Every quadratic with a positive discriminant and real coefficients has a real solution [1 mark]. This is true. [1 mark]

1.3 Direct Proofs

1.3 Exercise

1 $S = \{1, 2, 3, ... 18, 19, 20\}$

a $P = \{2, 3, 5, 7, 11, 13, 17, 19\}$

b $O = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$

c $T = \{3, 6, 9, 12, 15, 18\}$

d $\mathrm{P} \cap \mathrm{O} = \{3, 5, 7, 11, 13, 17, 19\}$

e $\mathrm{P} \cap \mathrm{T} = \{3\}$

***10**

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Same

***2**

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$(p \to q) \wedge (q \to r) \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

All true ↑

2 To prove: the sum of two even integers is even
Proof: Let $a = 2j$ and $b = 2k$
Where $j,\ k \in Z$
So a, b are both even
$a + b = 2j + 2k$
$\quad = 2(j + k)$
$\quad = 2i \quad$ where $i = j + k$, so $i \in Z$
So $a + b$ is even

3 To prove: If x is even then $4x - 7$ is odd
Proof: Let $x = 2j$, where $j \in Z$
So x is even
$4x - 7 = 4 \times (2j) - 7$
$\quad = 8j - 7$
$\quad = 2(4j - 4) + 1$
$\quad = 2i + 1 \quad$ where $i = 4j - 4,\ i \in Z$
So $4x - 7$ is odd

4 To prove: If x is odd then $3x + 11$ is even
Let $x = 2j + 1$, when $j \in Z$
$3x + 11 = 3(2j + 1) + 11$
$\quad = 6j + 3 + 11$
$\quad = 6j + 14$
$\quad = 2(3j + 7)$
$\quad = 2i \quad$ where $i = 3j + 7,\ i \in Z$
So $3x + 11$ is even

5 To prove: the square of an even number is even
Proof: Let $x = 2j$, where $j \in Z$
So x is even
$x^2 = (2j)^2$
$\quad = 4j^2$
$\quad = 2(2j^2)$
$\quad = 2i \quad$ where $i = 2j^2,\ i \in Z$
So x^2 is even

6 To prove: the cube of an odd number is odd
Proof: Let $x = 2j + 1$ when $j \in Z$
So x is odd
$x^3 = (2j + 1)^3$
$\quad = 8j^3 + 12j^2 + 6j + 1$
$\quad = 2(4j^3 + 6j^2 + 3j) + 1$
$\quad = 2i + 1 \quad$ where $i = 4j^3 + 6j^2 + 3j, i \in Z$
So x^3 is odd

7 To prove: If x is even, then $x^2 + 4x + 5$ is odd
Proof: Let $x = 2j, j \in Z$
So x is even
$x^2 + 4x + 5 = (2j)^2 + 4(2j) + 5$
$\quad = 4j^2 + 8j + 5$
$\quad = 2(2j^2 + 4j + 2) + 1$
$\quad = 2i + 1 \quad$ where $i = 2j^2 + 4j + 2,\ i \in Z$
So $x^2 + 4x + 5$ is odd

8 To prove: If x is odd then $x^2 - 5x + 8$ is even
Proof: let $x = 2j + 1$ where $j \in Z$
So x is odd
$x^2 - 5x + 8 = (2j + 1)^2 - 5\,(2j + 1) + 8$
$\quad = 4j^2 + 4j + 1 - 10j - 5 + 8$
$\quad = 4j^2 - 6j + 4$
$\quad = 2\left(2j^2 - 3j + 2\right)$
$\quad = 2i \quad i \in Z$
So $x^2 - 5x + 8$ is even

9 To prove: If x is odd then $x^2 - 7x + 18$ is even
Proof: Let $x = 2j + 1, j \in Z$
So x is odd
$x^2 - 7x + 18 = (2j + 1)^2 - 7(2j + 1) + 18$
$\quad = 4j^2 + 4j + 1 - 14j - 7 + 18$
$\quad = 4j^2 - 10j + 12$
$\quad = 2(2j^2 - 5j + 6)$
$\quad = 2i \quad$ where $i = 2j^2 - 5j + 6, i \in Z$
So $x^2 - 7x + 18$ is even

10 To prove: If x is odd then $x^2 - 1$ is divisible by 4
Proof: Let $x = 2k + 1,\ k \in Z$
So x is odd
$x^2 - 1 = (2k + 1)^2 - 1$
$\quad = 4k^2 + 4k + 1 - 1$
$\quad = 4k^2 + 4k$
$\quad = 4(k^2 + k)$
$\quad = 4i$, where $i = k^2 + k, i \in Z$
So $x^2 - 1$ is divisible by 4

11 To prove: If $10^n - 1$ is prime, then n is odd
Proof:
Let $p\ :\ 10^n - 1$ is prime
$\quad q\ :\ n$ is odd
Contrapositive.
If n is even then $10^n - 1$ is not prime
Since n is even, let $n = 2k,\ k \in Z$
$10^{2k} - 1 = \left(10^k + 1\right)\left(10^k - 1\right)$
This has factors, so it not prime
We have proved $\neg q \to \neg p$.
The contrapositive is true, so the original statement is true.

12 **a** To prove: $x, y \in R\ \ x^2 + y^2 \geq 2xy$
Proof: Consider $(x - y)^2 \geq 0$
Expanding
$x^2 - 2xy + y^2 \geq 0$
$x^2 + y^2 \geq 2xy$

b To prove: $x,\ y,\ z \in R$
$x^2 + y^2 + z^2 \geq xy + yz + xz$
Proof: Consider $(x - y)^2 + (y - z)^2 + (z - x)^2 \geq 0$
Expanding
$x^2 - 2xy + y^2 + y^2 - 2yz + z^2 + z^2 - 2xz + x^2 \geq 0$
$2x^2 + 2y^2 + 2z^2 \geq 2xy + 2yz + 2xz$
$2(x^2 + y^2 + z^2) \geq 2(xy + yz + xz)$
$x^2 + y^2 + z^2 \geq xy + yz + xz$

13 To prove: If $a|b$ then $a^2|b^2$
Proof: Since $a|b\ \exists\, d_1$,
$b = ad_1$
$b^2 = a^2 d_1^2$
$\quad = a^2\, d_3$
Where $d_3 = d_1^2$
So $b^2|a^2$

14 **a** $a,\ b,\ c \in Z$
To prove: If $a|b$ and $a|c$ then $a|\,(b + c)$
Proof: Since $a|b\ \exists\ d_1\ \in Z$
$b = d_1 a$
Since $a|c\ \exists\ d_2\ \in Z$
$\quad c = d_2 a$
$b + c = d_1 a + d_2 a$
$\quad = a\,(d_1 + d_2)$
$\quad = ad_3 \quad$ where $d_3 = d_1 + d_2\ \in Z$
So $a|(b + c)$

b Statement is false
$6|6$ is true so $6|(2 + 4)$

$6|12$ is true so $6|(2+10)$
but $6 \nmid 14$
$6|14$ is false

15 $a, b, c, d \in Z$
To prove: If $a|b$ and $c|d$ then $ac|bd$
Proof: Since $a|b$, $\exists\, d_1 \in Z$
$b = d_1 a$
Since $c|d$, $\exists\, d_2 \in Z$
$d = d_2 c$
$bd = d_1 a \times d_2 c$
$= d_1 d_2\, ac$
$= d_3\, ac \quad$ where $d_3 = d_1 d_2 \in Z$
So $ac|bd$

16 **a** $a, b, c \in Z$
If $a|b$ and $c|b$ then $ac|b$
Note that $2|12$ and $3|12$ and $6|12$ are all true,
but $6|12$ and $4|12$ but $24 \nmid 12$
So it is false.

b If $a, b \in Z$ and $5\left|\left(a^2 + b^2\right)\right.$ then $5|a^2$ and $5|b^2$
This is false.
Example: Suppose $a = 6$ and $b = 8$.
$a^2 + b^2 = 36 + 64 = 100$.
$5|100$ but $5|36$ and $5|64$ are both false.

17 To prove: If $5|n$ then $5|n^2$, $n \in Z$
If $5|n \Rightarrow n = 5k \qquad k \in Z$
$n^2 = 25k^2$
$= 5\left(5k^2\right)$
$= 5j \qquad j \in Z$
So $5|n^2$

18 **a** $\gcd(18, 27) = 9$
b $\gcd(12, 60) = 12$
c $\gcd(7, 11) = 1$
d $\text{lcm}(18, 27) = 54$
e $\text{lcm}(12, 60) = 60$
f $\text{lcm}(7, 11) = 77$

19 To prove:
If $x \in R$ and $\dfrac{x+1}{x^2-4} > 0$
then $x > 2$ or $-2 < x < -1$.
Proof: $\dfrac{x+1}{(x+2)(x-2)} > 0$
Case 1: $x+1 > 0, x+2 > 0, x-2 > 0$
$x > -1, \quad x > -2, \quad x > 2$
So $x > 2$
Case 2: $x+1 > 0, x+2 < 0, x-2 < 0$
$x > -1, \quad x < -2, \quad x < 2$
not possible
Case 3: $x+1 < 0, x+2 < 0, x-2 > 0$
$x < -1, \quad x < -2, \quad x > 2$
not possible
Case 4: $x+1 < 0, x+2 > 0, x-2 < 0$
$x < -1, \quad x > -2, \quad x < 2$
So $-2 < x < -1$
So Case 1 or 4
$x > 2$ or $-2 < x < -1$

20 To prove:
If $x, y \in R$ and $\dfrac{x^2-16}{x+7} > 0$
then $x > 4$ or $-7 < x < -4$
Proof: $\dfrac{(x+4)(x-4)}{x+7} > 0$
Case 1: $x+4 > 0, x-4 > 0, x+7 > 0$
$x > -4, \quad x > 4, \quad x > -7$
So $x > 4$
Case 2: $x+4 > 0, x-4 < 0, x+7 < 0$
$x > -4, \quad x < 4, \quad x < -7$
not possible
Case 3: $x+4 < 0, x-4 < 0, x+7 > 0$
$x < -4, \quad x < 4, \quad x > -7$
So $-7 < x < -4$
Case 4: $x+4 < 0, x-4 > 0, x+7 < 0$
$x < -4, \quad x > 4, \quad x < -7$
not possible
So Case 1 or 3
$x > 4$ or $-7 < x < -4$

21 To prove:
If $x, y \in R$ $(36 - x^2)(x+4) \geq 0$
then $x \leq -6$ or $-4 \leq x \leq 6$
Proof: $(6-x)(6+x)(x+4) \geq 0$
Case 1: $6 - x \geq 0, 6 + x \geq 0, x + 4 \geq 0$
$x \leq 6, \quad x \geq -6, \quad x \geq -4$
So $-4 \leq x \leq 6$
Case 2: $6 - x \geq 0, 6 + x \leq 0, x + 4 \leq 0$
$x \leq 6, \quad x \leq -6, \quad x \leq -4$
So $x \leq -6$
Case 3: $6 - x \leq 0, 6 + x \leq 0, x + 4 \geq 0$
$x \geq 6, \quad x \geq -6, \quad x \geq -4$
Case 4: $6 - x \leq 0, 6 + x \geq 0, x + 4 \leq 0$
$x \geq 6, \quad x \geq -6, \quad x \leq -4$
So Case 1 or 2
$x \leq -6$ or $-4 \leq x \leq 6$

22 To prove: If $x \in Z$, then $x^2 + 5x + 8$ is even
Proof:
Case 1: x is even
$x = 2k, \ k \in Z$
$x^2 + 5x + 8 = (2k)^2 + 5(2k) + 8$
$= 4k^2 + 10k + 8$
$= 2(2k^2 + 5k + 4)$
$= 2i$ where $i = 2k^2 + 5k + 4, i \in Z$
So $x^2 + 5x + 8$ is even
Case 2: x is odd
$x = 2k + 1, \ k \in Z$
$x^2 + 5x + 8 = (2k+1)^2 + 5(2k+1) + 8$
$= 4k^2 + 4k + 1 + 10k + 5 + 8$
$= 4k^2 + 14k + 14$
$= 2(2k^2 + 7k + 7)$
$= 2i$, where $i = 2k^2 + 7k + 7, i \in Z$
So $x^2 + 5x + 8$ is even
By cases, if x is odd or even $x^2 + 5x + 8$ is even.

23 **a** $\gcd(996,\ 524) = 4$

b $\gcd(417,\ 819) = 3$

c $\gcd(1025,\ 450) = 25$

d $\text{lcm}(996,\ 524) = 130\,476$

e $\text{lcm}(417,\ 819) = 113\,841$

f $\text{lcm}(1025,\ 450) = 18\,450$

24

Natural numbers	Number of prime numbers
1–100	25
101–200	21
201–300	16
301–400	16
401–500	17
501–600	14
601–700	16
701–800	14
801–900	15
901–1000	14
1–1000	168
1001–2000	135

No! Peter is wrong.

1.3 Exam questions

1 To prove: Every odd integer is a difference of two squares.

Proof: Let $x = 2k + 1,\ k \in Z$ [1 mark]

So x is odd

$x = 2k + 1$

$= k^2 + 2k + 1 - k^2$

$= (k + 1)^2 - k^2$ [1 mark]

So $3 = 2^2 - 1$

$5 = 3^2 - 2^2$

$7 = 4^2 - 3^2$

$9 = 5^2 - 4^2$

$11 = 6^2 - 5^2$

$13 = 7^2 - 6^2$

2 The statement is false by counter example. [1 mark]

$2|12$ is true or $6|12$ is true.

But $8 \nmid 12$ [1 mark]

3 To prove: If $x \in Z$, then $3x^2 + 7x + 11$ is odd

Proof:

Case 1: x is even

$x = 2k,\ k \in Z$

$3x^2 + 7x + 11 = 3(2k)^2 + 7(2k) + 11$

$= 12k^2 + 14k + 11$

$= 2(6k^2 + 7k + 5) + 1$

$= 2i + 1,$ where $i = 6k^2 + 7k + 5, i \in Z$

So $3x^2 + 7x + 11$ is odd [1 mark]

Case 2: x is odd

$x = 2k + 1,\ k \in Z$

$3x^2 + 7x + 11 = 3(2k + 1)^2 + 7\,(2k + 1) + 11$

$= 3(4k^2 + 4k + 1) + 14k + 7 + 11$

$= 12k^2 + 26k + 21$

$= 2(6k^2 + 13k + 10) + 1$

$= 2i + 1,$ where $i = 6k^2 + 13k + 10, i \in Z$

So $3x^2 + 7x + 11$ is odd [1 mark]

By cases, if x is odd or even $3x^2 + 7x + 11$ is odd. [1 mark]

1.4 Indirect Proofs

1.4 Exercise

1 To prove: If $2^n - 1$ is prime then n is odd

Proof:

Let, $p :\ 2^n - 1$ is prime

$q : n$ is odd

The contrapositive

If n is even then $2^n - 1$ is not prime

Let $n = 2k, \quad k \in Z$

$2^{2k} - 1 = (2^k - 1)(2^k + 1)$

This has factors, so it is not prime

We have proved $\neg q \to \neg p$ is true. Since the contrapositive true, the original statement is true.

2 To prove: If n is an integer and $3n + 7$ is even then n is odd

Proof:

Let, $p :\ n \in Z,\ 3n + 7$ is even

$q : n$ is odd

The contrapositive

If n is even then $3n + 7$ is odd

$\neg p : n \in Z, 3n + 7$ is odd

$\neg q : n$ is even

Let $n = 2k, \quad k \in Z$

$3n + 7 = 3(2k) + 7$

$= 6k + 7$

$= 2(3k + 3) + 1$

$= 2i + 1, \qquad i \in Z$

So $3n + 7$ is odd

We have proved $\neg q \to \neg p$. Since the contrapositive is true, the original statement is true.

3 To prove: If n is an integer and n^2 is even, then n is even.

Proof:

Let, $p : n \in Z, n^2$ is even

$q : n$ is even

The contrapositive

If n is odd then n^2 is odd

$\neg p : n^2$ is odd

$\neg q : n$ is odd

Let, $n = 2k + 1, \quad k \in Z$

$n^2 = (2k + 1)^2$

$= 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$

$= 2i + 1, \qquad i \in Z$

So n^2 is odd

We have proved $\neg q \to \neg p$

Since the contrapositive is true, the original statement is true.

4 To prove: If n an integer and n^2 is not divisible by 4, then n is odd

Proof:

Let, $p : n \in Z, n^2$ not divisible by 4

$q : n$ is odd

The contrapositive

If n is even then n^2 is not divisible by 4

$\neg p : n^2$ is divisible by 4

$\neg q : n$ is even

Let $n = 2k \quad k \in Z$

$n^2 = (2k)^2$
$= 4k^2$
$= 4i, \quad i \in Z$
So n^2 is not divisible by 4
We have proved $\neg q \rightarrow \neg p$
Since the contrapositive is true, the original statement is true.

5 To prove: If n^3 is odd then n is odd.
Proof:
Let, $p : n \in Z, n^3$ is odd
$q : n$ is odd
The contrapositive
If n is even then n^3 is even
$\neg p : n^3$ is even
$\neg q : n$ is even
Let, $n = 2k \quad k \in Z$
$n^3 = (2k)^3$
$= 8k^3$
$= 2(4k^3)$
$= 2i, \quad i \in Z$
So n^3 is even
We have proved $\neg q \rightarrow \neg p$
Since the contrapositive is true, the original statement is true.

6 To prove: If $x \in R, x^3 + 2x > 4x^2 + 9$, then $x > 0$
Proof:
Let, $p : x^3 + 2x > 4x^2 + 9$
$q : x > 0$
The contrapositive
If $x < 0$ then $x^3 + 2x < 4x^2 + 9$
$\neg p : x^3 + 2x < 4x^2 + 9$
$\neg q : x < 0$
If $x < 0$, a negative number to odd powers is negative, so
$x^3 + 2x < 0$
If $x < 0$ a negative number to even powers is positive, so
$4x^2 + 9 > 0$
A positive number is greater than a negative number, so
$4x^2 + 9 > x^3 + 2x$
We have proved $\neg q \rightarrow \neg p$
As the contrapositive is true, the original statement is true.

7 To prove: If $x \in R, x^5 + 2x^3 + x > x^6 + 3x^4 + x^2 + 8$ then $x > 0$.
Proof:
Let: $p : x^5 + 2x^3 + x > x^6 + 3x^4 + x^2 + 8$
$q : x > 0$
The contrapositive
If $x < 0$ then $x^5 + 2x^3 + x < x^6 + 3x^4 + x^2 + 8$
$\neg p : x^5 + 2x^3 + x < x^6 + 3x^4 + x^2 + 8$
$\neg q : x < 0$
If $x < 0$, negative numbers to odd powers are negative, so
$x^5 + 2x^3 + x < 0$
If $x < 0$, negative numbers to even powers are positive, adding positive numbers to positive numbers gives a positive so
$x^6 + 3x^4 + x^2 + 8 > 0$
A positive number is greater then a negative number, so
$x^6 + 3x^4 + x^2 + 8 > x^5 + 2x^3 + x$
We have proved $\neg q \rightarrow \neg p$
As the contrapositive is true, the original statement is true.

8 To prove: If $x, y \in R, y^3 + yx^2 < x^3 + xy^2$ then $y < x$
Proof:
Let, $p : y^3 + yx^2 < x^3 + xy^2$
$q : y < x$
The contrapositive
If $x > y$ then $y^3 + yx^2 > x^3 + xy^2$
$\neg p : y^3 + yx^2 > x^3 + xy^2$
$\neg q : y > x$
If $y > x$
Then $y^3 + yx^2 = y(y^2 + x^2)$ since $x^2 + y^2 \geq 0$
and $y > x$
$y\left(y^2 + x^2\right) > x\left(y^2 + x^2\right) = x^3 + xy^2$
So $\neg p$ is true.
We have proved $\neg q \rightarrow \neg p$
As the contrapositive is true, the original statement is true.

9 To prove: If the product of two integers is odd, then both of the integers are odd
Proof:
Let, $p : a, b \in Z, ab$ is odd
$q : a$ is odd
$r : b$ is odd
$p \rightarrow (q \wedge r)$
The contrapositive is
$\neg (q \wedge r) \rightarrow \neg p$ by Demorgen's law
$(\neg q \vee \neg r) \rightarrow \neg p$
If a or b is even, then ab is even.
So, let:
$a = 2j, \quad j \in Z$
$b = 2k + 1, \quad k \in Z$
$ab = (2j)(2k + 1)$
$= 4jk + 2j$
$= 2(j + 2jk)$
$= 2i \quad i \in Z$
So ab is even
We have proved the contrapositive.
As the contrapositive is true, the original statement is true.

10 To prove: No integers a, b exist for which $4a + 8b = 10$
Proof: Assume the hypothesis and that the conclusion is false.
Assume $\exists\ a, b \in Z$ such that
$4a + 8b = 10$
$4(a + 2b) = 10$
$a + 2b = \frac{5}{2}$
This is a contradiction, as the sum of two integers is an integer, not a rational number. Therefore the original statement must be true, there are no integers a and b for which $4a + 8b = 10$

11 To prove: No integers a, b exist for which $2a - 8b = 21$
Proof: Assume the hypothesis and that the conclusion is false.
Assume $\exists\ a, b$ such that
$2a - 8b = 21$
$2(a - 4b) = 21$
$a - 4b = \frac{21}{2}$
This is a contradiction, as the difference of two integers is an integer. Therefore the original statement must be true, there are no integers a and b for which
$2a - 8b = 21$.

12 To prove: $a, b \in Z$, if $a + b > 19$, then $a \geq 10$ or $b \geq 10$
Proof:
Assume that $a + b > 19$ and that $a < 10$ and $b < 10$
$a + b \leq 9 + 9$
≤ 18
This is a contradiction.
So $a \geq 10$ or $b \geq 10$
OR
$p : a + b \geq 19, a, b \in Z$
$q : a \geq 10$

$r : b \geq 10$

To prove $p \to (q \vee r)$ the contrapositive

$\neg(q \vee r) \to \neg p$

$(\neg q \wedge \neg r) \to \neg p$

If $a < 10$ and $b < 10$ then $a + b < 19$

Let,

$$\begin{aligned} a &= 10 - j, & j > 0, \quad j \in Z^+ \\ b &= 10 - k, & k > 0, \quad k \in Z^+ \\ a + b &= 10 - j + 10 - k; \\ &= 20 - (j + k) \\ a + b &\leq 18 < 19 \end{aligned}$$

We have proved that the contrapositive is true, so the original statement is true.

13 Let $p : 2|n$ and $q : 2|n^2$ where $n \in Z$.

a. To prove $p \to q$

Proof: $2|n$ is true so that

$n = 2k$ where $k \in Z$

$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2) \\ &= 2j \qquad \text{where } j = 2k^2,\ j \in Z \end{aligned}$$

So $2|n^2$ is true

b To prove $q \to p$

Proof: prove the contrapositive $\neg p \to \neg q$

$\neg p$ is $2 \nmid n$

So if n is not divisible by 2, then n is odd, it must have a remainder of 1.

Let, $n = 2k + 1$, where $k \in Z$

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \\ &= 2j + 1 \quad \text{where } j = 2k^2 + 2k, j \in Z \end{aligned}$$

So n^2 is odd, so n^2 is not divisible by 2, it must have a remainder of 1 so $2 \nmid n^2$.

So $\neg p \to \neg q$ has been shown to be true, so $q \to p$ by contrapositive is true.

14 To prove $\sqrt{2}$ is irrational

Proof: Assume $\sqrt{2}$ is rational

$\sqrt{2} = \dfrac{p}{q}$ when $p, q \in z, q \neq 0$ with no common factors.

$2 = \dfrac{p^2}{q^2}$

$2q^2 = p^2$

So p^2 is even

Therefore p is even (from Q3 proof)

Let $p = 2k, k \in Z$

$p^2 = 4k^2$

$2q^2 = 4k^2$

$q^2 = 2k^2$

So q^2 is even

Then q is even

So p and q are both even but we assumed p, q had no common factor, they both have 2 as a factor, this is a contradiction, so $\sqrt{2}$ is not a rational number.

15 To prove $n^2 = 10$ then n is irrational $n = \pm\sqrt{10}$

Proof: Assume $\sqrt{10}$ is rational

$\sqrt{10} = \dfrac{p}{q}$ when $p, q \in Z, q \neq 0$ with no common factors

$10 = \dfrac{p^2}{q^2}$

$$\begin{aligned} p^2 &= 10q^2 \\ &= 2(5q^2) \\ &= 2i \ \ i \in Z \end{aligned}$$

As p^2 is even, p is also even (Q3)

Let $p = 2k, k \in Z$

$p^2 = 4k^2$

$4k^2 = 10q^2$

$5q^2 = 2k^2$

So $5q^2$ must be even since $2k^2$ is even, 5 is odd therefore q^2 is even, so q is also even.

So p and q are both even, but we assumed p, q had no common factors, they both have 2 as a factor, this is a contradiction, so $\sqrt{10}$ is not a rational number.

Simlarly $-\sqrt{10} = \dfrac{p}{q}$, $p > 0$ and $q < 0$ or $p < 0$ and $q > 0$

16 To prove: $\log_3(8)$ is irrational

Let $x = \log_3(8)$

$3^x = 8$

Proof: assume x is rational and $3^x = 8$

$x = \dfrac{p}{q}$ where $p, q \in Z^+$ with no common factors

$3^{\frac{p}{q}} = 8 = 2^3$

$3^p = 2^{3q}$

Now 3^p where $p \in Z^+$ is always odd and 2^{3q} when $q \in Z^+$ is always even.

We have a contradiction, therefore the original statement $\log_3(8)$ is irrational is true.

17 To prove: If n is an integer, $n^3 + 5$ is odd then n is even.

Proof:

Let $p : n \in Z, n^3 + 5$ is odd

$q : n$ is even

The contrapositive if n is odd then $n^3 + 5$ is even.

$\neg p$: $n^3 + 5$ is even

$\neg q$ n is odd

Let $n = 2k + 1, k \in Z$

$$\begin{aligned} n^3 + 5 &= (2k + 1)^3 + 5 \\ &= 8k^3 + 12k^2 + 6k + 1 + 5 \\ &= 8k^3 + 12k^2 + 6k + 6 \\ &= 2(4k^3 + 6k^2 + 3k + 3) \\ &= 2i, \text{where } i = 4k^3 + 6k^2 + 3k + 3, i \in Z \end{aligned}$$

So $n^3 + 5$ is even

We have proved $\neg q \to \neg p$

As the contrapositive is true, the original statement is true.

18 To prove: If $a, b \in Z \backslash \{0\}$ $\quad \dfrac{a}{b} + \dfrac{b}{a} \geq 2$

Proof: Assume the hypothesis $a, b \in Z \backslash \{0\}$ and the conclusion is false.

Assume $\dfrac{a}{b} + \dfrac{b}{a} < 2$

$\dfrac{a^2 + b^2}{ab} < 2$

$a^2 + b^2 < 2ab$

Since $a, b \in Z \backslash \{0\}$ $\qquad ab > 0$

$a^2 - 2ab + b^2 < 0$

$(a - b)^2 < 0$

But the square of any number cannot be negative, so this is a contradiction, therefore the original statement is true.

1.4 Exam questions

1 To prove: When $x,\ y \in R$, if $x + y > 20$ then $x > 10$ or $y > 10$

Proof:

Assume the hypothesis and suppose the conclusion is false.

Then $x \le 10$ and $y \le 10$

$x + y \le 20$

This is a contradiction to

$x + y > 20$

So $x > 10$ or $y > 10$ is true.

OR

$p : x + y > 20, \quad x,\ y \in R$

$q : x > 10$

$r : y > 10$

To prove $p \to (q \vee r)$

$(\neg q \wedge \neg r) \to \neg p$

$\neg q : x \le 10$, let $x = 10 - j,\ j \ge 0,\ j \in R^+$

$\neg r : y \le 10 \qquad y = 10 - k,\ k \ge 0,\ k \in R^+$

$x + y \le 20$

$x + y = 20 - (j + k)$

$x + y \le 20$

We have proved the contrapositive is true, so the original statement is true.

Award 1 mark for selecting an appropriate method of proof.

Award 1 mark for correct reasoning.

Award 1 mark for correct conclusion.

2 To prove: $\log_2(5)$ is irrational.

Let $x = \log_2(5)$

$2^x = 5$

Proof: Assume x rational and $2^x = 5$ [1 mark]

$x = \dfrac{p}{q}$ where $p, q \in Z^+$ with no common factors

$2^{\frac{p}{q}} = 5$

$2^p = 5^q$

Now 2^p when $p \in Z^+$ is always even

5^q when $q \in Z^+$ is always odd. [1 mark]

We have a contradiction, therefore the original statement $\log_2(5)$ is irrational is true. [1 mark]

3 To prove: If n is an integer, $n^3 + 5$ is odd, then n is even.

Proof: Assume the hypothesis and that the conclusion is false.

Assume $n^3 + 5$ is odd and n is odd [1 mark]

$n = 2k + 1,\ k \in Z$

$$\begin{aligned} n^3 + 5 &= (2k+1)^3 + 5 \\ &= 8k^3 + 12k^2 + 6k + 1 + 5 \\ &= 8k^3 + 12k^2 + 6k + 6 \\ &= 2(4k^3 + 6k^2 + 3k + 3) \\ &= 2i, i \in Z \qquad \text{[1 mark]} \end{aligned}$$

So $n^3 + 5$ is even, this is a contradiction since we assumed $n^3 + 5$ was odd, so if $n^3 + 5$ is odd then n is even is a true statement by proof by contradiction. [1 mark]

1.5 Proof by mathematical induction

1.5 Exercise

1 $P(n)$: $1 + 2 + 3 + \ldots + n = \sum\limits_{r=1}^{n} r = \dfrac{1}{2}n(n+1)$

LHS of $P(1) = 1$.

RHS of $P(1) = \dfrac{1}{2} \times 1 \times 2 = 1$ $P(1)$ is true

Assume that $P(k)$ is true.

$\sum\limits_{r=1}^{k} r = \dfrac{1}{2}k(k+1)$

$$\begin{aligned} \text{Consider } P(k+1).\ \text{LHS} &= \sum_{r=1}^{k+1} r \\ &= \sum_{r=1}^{k} r + (k+1) \\ &= \frac{1}{2}k(k+1) + (k+1) \\ &= \frac{1}{2}(k+1)[k+2] \\ &= \frac{1}{2}(k+1)(k+1+1) = \text{RHS of } P(k+1) \end{aligned}$$

$P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n \in N$.

2 $1 \times 2 + 2 \times 3 + \ldots n(n+1) = \sum\limits_{r=1}^{n} r(r+1)$

$$P(n)\text{: } \sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$$

Consider $P(1)$: LHS $= 1 \times 2 = 2$

RHS $= \dfrac{1}{3} \times 1 \times 2 \times 3 = 2$

$P(1)$ is true.

Assume that $P(k)$ is true.

$\sum\limits_{r=1}^{k} r(r+1) = \dfrac{1}{3}k(k+1)(k+2)$

$$\begin{aligned} \text{Consider } P(k+1).\ \text{LHS} &= \sum_{r=1}^{k+1} r(r+1) \\ &= \sum_{r=1}^{k} r(r+1) + (k+1)(k+2) \\ &= \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2) \\ &= \frac{1}{3}(k+1)(k+2)[k+3] \\ &= \frac{1}{3}(k+1)(k+2)(k+3) \\ &= \text{RHS of } P(k+1) \end{aligned}$$

$P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, P(n) is true $\forall\, n \in N$.

3 $\dfrac{1}{2 \times 5} + \dfrac{1}{5 \times 8} + \ldots \dfrac{1}{8 \times 11} + \ldots. + \dfrac{1}{(3n-1)(3n+2)} = \sum\limits_{r=1}^{n} \dfrac{1}{(3r-1)(3r+2)}$

$P(n)$: $\sum\limits_{r=1}^{n} \dfrac{1}{(3r-1)(3r+2)} = \dfrac{n}{6n+4}$

Consider $P(1)$: LHS $= \dfrac{1}{2 \times 5} = \dfrac{1}{10}$

RHS $\dfrac{1}{6+4} = \dfrac{1}{10}$, $P(1)$ is true.

Assume that $P(k)$ is true.

$\sum\limits_{r=1}^{k} \dfrac{1}{(3r-1)(3r+2)} = \dfrac{k}{6k+4}$

$$\begin{aligned}\text{Consider } P(k+1).\text{ LHS} &= \sum_{r=1}^{k+1}\frac{1}{(3r-1)(3r+2)}\\ &= \sum_{r=1}^{k}\frac{1}{(3r-1)(3r+2)}\\ &\quad+\frac{1}{(3(k+1)-1)(3(k+1)+2)}\\ &= \frac{k}{6k+4}+\frac{1}{(3k+2)(3k+5)}\\ &= \frac{k}{2(3k+2)}+\frac{1}{(3k+2)(3k+5)}\\ &= \frac{k(3k+5)+2}{2(3k+2)(3k+5)}\\ &= \frac{3k^2+5k+2}{2(3k+2)(3k+5)}\\ &= \frac{(k+1)(3k+2)}{2(3k+2)(3k+5)}\\ &= \frac{k+1}{2(3k+5)}\\ &= \frac{k+1}{6(k+1)+4}=\text{RHS of } P(k+1)\end{aligned}$$

$P(k+1)$ is true.
We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n\in N$.

4 $1\times 4+2\times 4^2+3\times 4^3+\ldots+n\times 4^n=\sum\limits_{r=1}^{n} r\times 4^r$

$$P(n):\ \sum_{r=1}^{n} r\times 4^r=\frac{1}{9}(4(3n-1)\times 4^n+4)$$

Consider $P(1)$: LHS $=1\times 4=4$
RHS $=\frac{1}{9}\left(4\times 2\times 4^1+4\right)=\frac{1}{9}\times 36=4$
$P(1)$ is true.
Assume that $P(k)$ is true.

$$\sum_{r=1}^{k} r\times 4^r=\frac{1}{9}\left(4(3k-1)\times 4^k+4\right)$$

$$\begin{aligned}\text{Consider } P(k+1).\text{ LHS} &= \sum_{r=1}^{k+1} r\times 4^r\\ &= \sum_{r=1}^{k} r\times 4^r+(k+1)\times 4^{k+1}\\ &= \frac{1}{9}\left(4(3k-1)\times 4^k+4\right)+(k+1)\times 4^{k+1}\\ &= \frac{1}{9}\left[4\times 4^k(3k-1)+4+9(k+1)\times 4^{k+1}\right]\\ &= \frac{1}{9}\left[4^{k+1}(3k-1)+4+9(k+1)\times 4^{k+1}\right]\\ &= \frac{1}{9}\left[(3k-1+9k+9)\times 4^{k+1}+4\right]\\ &= \frac{1}{9}\left[(12k+8)\times 4^{k+1}+4\right]\\ &= \frac{1}{9}\left[4(3(k+1)-1)\times 4^{k+1}+4\right]\\ &= \text{RHS of } P(k+1)\end{aligned}$$

$P(k+1)$ is true.
We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n\in N$.

5 a $P(n)$: 6^n-1 is divisible by 5, $\forall\, n\in N$.
Consider $P(1)$: $6^1-1=5$, which is divisible by 5, so $P(1)$ is true.
Assume that $P(k)$ is true.
6^k-1 is divisible by 5

$$\begin{aligned}\text{Consider } P(k+1) &= 6^{k+1}-1\\ &= 6\times 6^k-1\\ &= (5+1)\times 6^k-1\\ &= 5\times 6^k+6^k-1\end{aligned}$$

Since 6^k-1 is divisible by 5 and 5×6^k is divisible by 5, LHS of $P(k+1)$ is divisible by 5. $P(k+1)$ is true.
We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n\in N$.

b $P(n)$: 6^n+4 is divisible by 5.
Consider $P(1)$: $6^1+4=10$ which is divisible by 5.
Therefore $P(1)$ is true.
Assume that $P(k)$ is true.
6^k+4 is divisible by 5.
Consider $P(k+1)$.

$$\begin{aligned}\text{LHS} &= 6^{k+1}+4\\ &= 6\times 6^k+4\\ &= (5+1)\,6^k+4\\ &= 5\times 6^k+6^k+4\end{aligned}$$

5×6^k is divisible by 5, and 6^k+4 is divisible by 5 by assumption.
Therefore LHS of $P(k+1)=5\times 6^k+6^k+4$ is divisible by 5.
$P(k+1)$ is true.
We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n\in N$.

c $P(n)$: $2^{3n}-3^n$ is divisible by 5.
Consider $P(1)$: $2^3-3^1=8-3=5$ which is divisible by 5.
Therefore $P(1)$ is true.
Assume that $P(k)$ is true.
$2^{3k}-3^k$ is divisible by 5.
Consider $P(k+1)$.

$$\begin{aligned}\text{LHS} &= 2^{3(k+1)}-3^{k+1}\\ &= 2^3\times 2^{3k}-3\times 3^k\\ &= 8\times 2^{3k}-3\times 3^k\\ &= (5+3)\times 2^{3k}-3\times 3^k\\ &= 5\times 2^{3k}+3\left(2^{3k}-3^k\right)\end{aligned}$$

5×2^{3k} is divisible by 5, and $3\left(2^{3k}-3^k\right)$ is divisible by 5 by assumption.
Therefore LHS of $P(k+1)=5\times 2^{3k}+3\left(2^{3k}-3^k\right)$ is divisible by 5.
$P(k+1)$ is true.
We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n\in N$.

6 a $P(n)$: $17n^3+103n$.
Consider $P(1)$: $17+103=120$, which is divisible by 6, $P(1)$ is true.
Assume that $P(k)$ is true.
$17k^3+103k$ is divisible by 6

Consider $P(k+1)$.

$$\begin{aligned}\text{LHS} &= 17(k+1)^3 + 103\,(k+1)\\ &= 17(k^3 + 3k^2 + 3k + 1) + 103k + 103\\ &= 17k^3 + 51k^2 + 51k + 17 + 103k + 103\\ &= 17k^3 + 103k + 120 + 51k^2 + 51k\\ &= 17k^3 + 103k + 120 + 3 \times 17 \times k\,(k+1)\end{aligned}$$

Now $k\,(k+1)$ is always even so it is divisible by 2.

Since $17k^3 + 103k$ is divisible by 6, 120 is divisible by 6, $17 \times 3 \times k\,(k+1)$ is divisible by 6, so LHS of $P(k+1)$ is divisible by 6. $P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n \in N$.

b $P(n)$: $17n^3 + 103n$ is divisible by 6.

Consider $P(1)$: $17 + 103 = 120$ which is divisible by 6.

Therefore $P(1)$ is true.

Assume that $P(k)$ is true.

$17k^3 + 103k$ is divisible by 6.

Consider $P(k+1)$.

$$\begin{aligned}\text{LHS} &= 17(k+1)^3 + 103\,(k+1)\\ &= 17\left(k^3 + 3k^2 + 3k + 1\right) + 103k + 103\\ &= 17k^3 + 51k^2 + 51k + 17 + 103k + 103\\ &= 17k^3 + 103k + 120 + 51k^2 + 51k\\ &= 17k^3 + 103k + 120 + 3 \times 17 \times k\,(k+1)\end{aligned}$$

120 is divisible by 6, $17k^3 + 103k$ is divisible by 6 by assumption and $3 \times 17 \times k\,(k+1)$ is divisible by 6 since $k\,(k+1)$ is even.

Therefore

LHS of $P(k+1) = 17k^3 + 103k + 120 + 3 \times 17 \times k\,(k+1)$ is divisible by 6.

$P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n \in N$.

c $P(n)$: $n\,(n+1)\,(2n+1)$ is divisible by 6.

Consider $P(1)$: $1 \times 2 \times 3 = 6$ which is divisible by 6.

Therefore $P(1)$ is true.

Assume that $P(k)$ is true.

$k\,(k+1)\,(2k+1)$ is divisible by 6.

Therefore $k\,(k+1)\,(2k+1) = 6a,\ a \in Z$.

Consider $P(k+1)$.

$$\begin{aligned}\text{LHS} &= (k+1)\,(k+1+1)\,(2\,(k+1)+1)\\ &= (k+1)\,(k+2)\,(2k+3)\\ &= (k+2)\,(k+1)\,(2k+3)\\ &= k\,(k+1)\,(2k+3) + 2\,(k+1)\,(2k+3)\\ &= k\,(k+1)\,(2k+1+2) + 2\,(k+1)\,(2k+3)\\ &= k\,(k+1)\,(2k+1) + 2k\,(k+1) + 2\,(k+1)\,(2k+3)\\ &= k\,(k+1)\,(2k+1) + (k+1)\,(2k + 2\,(2k+3))\\ &= 6a + (k+1)\,(2k + 4k + 6)\\ &= 6a + (k+1)\,(6k+6)\\ &= 6a + 6(k+1)^2\end{aligned}$$

$6a$ is divisible by 6 by assumption and $6\,(k+1)^2$ is divisible by 6.

Therefore LHS of $P(k+1) = 6a + 6(k+1)^2$ is divisible by 6.

$P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n \in N$.

7 a $P(n)$: $2^{n+2} + 3^{2n+1}$

Consider $P(1)$: $2^3 + 3^3 = 8 + 27 = 35$, which is divisible by 7, $P(1)$ is true.

Assume that $P(k)$ is true.

$2^{k+2} + 3^{2k+1}$ is divisible by 7

Consider $P(k+1)$. $\begin{aligned}[t]\text{LHS} &= 2^{k+1+2} + 3^{2(k+1)+1}\\ &= 2 \times 2^{k+2} + 3^2 \times 3^{2k+1}\\ &= 2 \times 2^{k+2} + 9 \times 3^{2k+1}\\ &= 2(2^{k+2} + 3^{2k+1}) + 7 \times 3^{2k+1}\end{aligned}$

Since $2^{k+2} + 3^{2k+1}$ is divisible by 7 and $7 \times 3^{2k+1}$ is divisible by 7, so LHS of $P(k+1)$ is divisible by 7. $P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n \in N$.

b $P(n)$: $8^{2n+1} + 6^{2n-1}$ is divisible by 7.

Consider $P(1)$: $8^3 + 6^1 = 512 + 6 = 518$ which is divisible by 7.

Therefore $P(1)$ is true.

Assume that $P(k)$ is true.

$8^{2k+1} + 6^{2k-1}$ is divisible by 7.

Consider $P(k+1)$.

$$\begin{aligned}\text{LHS} &= 8^{2(k+1)+1} + 6^{2(k+1)-1}\\ &= 8^2 \times 8^{2k+1} + 6^2 \times 6^{2k-1}\\ &= 64 \times 8^{2k+1} + 36 \times 6^{2k-1}\\ &= 64\left(8^{2k+1} + 6^{2k-1}\right) - 28 \times 6^{2k-1}\\ &= 64\left(8^{2k+1} + 6^{2k-1}\right) - 7\left(4 \times 6^{2k-1}\right)\end{aligned}$$

$7\left(4 \times 6^{2k-1}\right)$ is divisible by 7, and $8^{2k+1} + 6^{2k-1}$ is divisible by 7 by assumption.

Therefore LHS of $P(k+1) = 4\left(8^{2k+1} + 6^{2k-1}\right) - 7\left(4 \times 6^{2k-1}\right)$ is divisible by 7.

$P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n \in N$.

8 $1 + 3 + 5 + ... + (2n-1) = \sum\limits_{r=1}^{n} (2r-1)$

$P(n)$: $\sum\limits_{r=1}^{n}(2r-1) = n^2,\ \forall n \in N$

Consider $P(1)$: LHS = 1, RHS = $1^2 = 1$

$P(1)$ is true.

Assume that $P(k)$ is true.

$$\sum_{r=1}^{k} (2r-1) = k^2$$

Consider $P(k+1)$: $\begin{aligned}[t]\text{LHS} = \sum_{r=1}^{k+1} (2r-1) &= \sum_{r=1}^{k} (2r-1) + 2(k+1) - 1\\ &= k^2 + 2k + 1\\ &= (k+1)^2 = \text{RHS of } P(k+1)\end{aligned}$

$P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n \in N$.

9 $3^{2n-1}+1$ is divisible by 4

Let $P(n)=3^{2n-1}+1$

Consider $P(1)$: $3+1=4$, which is divisible by 4

$P(1)$ is true.

Assume that $P(k)$ is true.

$3^{2k-1}+1$ is divisible by 4

$$\begin{aligned}\text{Consider } P(k+1)\text{: LHS} &= 3^{2(k+1)-1}+1\\ &= 3^{2k+1}+1\\ &= 3^{2k-1+2}+1\\ &= 9\times 3^{2k-1}+1\\ &= (4+5)\times 3^{2k-1}+1\\ &= 4\times 3^{2k-1}+5\times 3^{2k-1}+1\\ &= 4\times 3^{2k-1}+5\left(3^{2k-1}+1\right)-4\\ &= 4\times\left(3^{2k-1}-1\right)+5\left(3^{2k-1}+1\right)\end{aligned}$$

Now $4\times\left(3^{2k-1}-1\right)$ is divisible by 4.

$5\left(3^{2k-1}+1\right)$ is divisible by 4.

$P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n\in N$.

10 $a+a^2+a^3+\ldots+a^n=\displaystyle\sum_{r=1}^{n}a^r=\frac{1}{a-1}\left(a^{n+1}-a\right)$

$P(n)$: $\displaystyle\sum_{r=1}^{n}a^r=\frac{1}{a-1}\left(a^{n+1}-a\right)$

Consider $P(1)$:

LHS $=a$

$$\begin{aligned}\text{RHS} &= \frac{1}{a-1}\left(a^2-a\right)\\ &= \frac{1}{a-1}a(a-1)=a\end{aligned}$$

$P(1)$ is true, $a>1$.

Assume that $P(k)$ is true.

$\displaystyle\sum_{r=1}^{k}a^k=\frac{1}{a-1}\left(a^{k+1}-a\right)$

$$\begin{aligned}\text{Consider } P(k+1)\text{. LHS} &= \sum_{r=1}^{k+1}a^k\\ &= \sum_{r=1}^{k}a^k+a^{k+1}\\ &= \frac{1}{a-1}\left(a^{k+1}-a\right)+a^{k+1}\\ &= \frac{1}{a-1}\left(a^{k+1}-a+(a-1)\,a^{k+1}\right)\\ &= \frac{1}{a-1}\left(a^{k+1}-a+a\times a^{k+1}-a^{k+1}\right)\\ &= \frac{1}{a-1}\left(a\times a^{k+1}-a\right)\\ &= \frac{1}{a-1}\left(a^{k+2}-a\right)\\ &= \frac{1}{a-1}\left(a^{k+1+1}-a\right)=\text{RHS of }P(k+1)\end{aligned}$$

$P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n\in N$.

11 $P(n)$: $1\times 1!+2\times 2!+3\times 3!+\ldots+n\times n!$

$$=\sum_{r=1}^{n}r\times r!=(n+1)!-1$$

Consider $P(1)$. LHS $=1\times 1!=1$

RHS $=2!-1=1$ $P(1)$ is true.

Assume that $P(k)$ is true.

$\displaystyle\sum_{r=1}^{k}r\times r!=(k+1)!-1$

$$\begin{aligned}\text{Consider } P(k+1)\text{. LHS} &= \sum_{r=1}^{k+1}r\times r!\\ &= \sum_{r=1}^{k}r\times r!+(k+1)(k+1)!\\ &= (k+1)!-1+(k+1)(k+1)!\\ &= (k+1)!\,[1+(k+1)]-1\\ &= (k+1)!\,(k+2)-1\\ &= (k+2)!-1=\text{RHS of }P(k+1)\end{aligned}$$

$P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n\in N$.

12 $P(n)$: $(1+x)^n\geq 1+nx \qquad x>-1$

When $n=1$ LHS $=1+x$ $=$ RHS, $P(1)$ is true.

Assume that $P(k)$ is true.

$(1+x)^k\geq 1+kx$

$$\begin{aligned}\text{Consider } P(k+1)\text{. LHS} = (1+x)^{k+1} &= (1+x)^k(1+x)\\ &\geq (1+kx)(1+x)\\ &= 1+(k+1)x+kx^2\\ &\geq 1+(k+1)x\\ &\quad \text{since } kx^2\geq 0,\ k\in N.\end{aligned}$$

$P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n\in N$.

13 $\left(1-\dfrac{2}{3}\right)\left(1-\dfrac{2}{4}\right)\left(1-\dfrac{2}{5}\right)\ldots\left(1-\dfrac{2}{n}\right)=\displaystyle\prod_{r=1}^{n}\left(1-\frac{2}{r}\right)$

$$P(n)\text{: }\prod_{r=1}^{n}\left(1-\frac{2}{r}\right)=\frac{2}{n\,(n-1)}\ :\ n\geq 3$$

Consider $P(3)$: LHS $=1-\dfrac{2}{3}=\dfrac{1}{3}$

RHS $=\dfrac{2}{3\times 2}=\dfrac{1}{3}$ so $P(3)$ is true.

Assume that $P(k)$ is true, $k\geq 3$.

$\displaystyle\prod_{r=1}^{n}\left(1-\frac{2}{1}\right)=\frac{2}{k\,(k-1)}$

$$\begin{aligned}\text{Consider } P(k+1)\text{. LHS} &= \prod_{r=1}^{n}\left(1-\frac{2}{1}\right)\\ &= \left[\prod_{r=1}^{n}\left(1-\frac{2}{1}\right)\right]\left(1-\frac{2}{k+1}\right)\\ &= \left(\frac{2}{k\,(k-1)}\right)\left(1-\frac{2}{k+1}\right)\\ &= \frac{2}{k\,(k-1)}\left(\frac{k+1-2}{k+1}\right)\\ &= \frac{2\times(k-1)}{k\,(k-1)\,(k+1)}\\ &= \frac{2}{k\,(k+1)}\\ &= \frac{2}{(k+1)\,(k+1-1)}=\text{RHS of }P(k+1)\end{aligned}$$

$P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n\in N$.

14 $\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\cdots\left(1-\frac{1}{n^2}\right)=\prod_{r=2}^{n}\left(1-\frac{1}{r^2}\right)$

$P(n)$: $\prod_{r=2}^{n}\left(1-\frac{1}{r^2}\right)=\frac{n+1}{2n}$

LHS of $P(2)=1-\frac{1}{4}=\frac{3}{4}$

RHS of $P(2)=\frac{2+1}{2\times 2}=\frac{3}{4}$

$P(2)$ is true.

Assume that $P(k)$ is true.

$$\prod_{r=2}^{k}\left(1-\frac{1}{r^2}\right)=\frac{k+1}{2k}$$

Consider $P(k+1)$

$$\prod_{r=2}^{k+1}\left(1-\frac{1}{r^2}\right)=\frac{(k+1)+1}{2(k+1)}$$

$$\begin{aligned}\text{LHS of } P(k+1)&=\prod_{r=2}^{k+1}\left(1-\frac{1}{r^2}\right)\\&=\prod_{r=2}^{k}\left(1-\frac{1}{r^2}\right)\left(1-\frac{1}{(k+1)^2}\right)\\&=\frac{k+1}{2k}\left(1-\frac{1}{(k+1)^2}\right)\\&=\frac{k+1}{2k}-\frac{1}{2k(k+1)}\\&=\frac{(k+1)^2}{2k(k+1)}-\frac{1}{2k(k+1)}\\&=\frac{(k+1)^2-1}{2k(k+1)}\\&=\frac{k^2+2k}{2k(k+1)}\\&=\frac{k+2}{2(k+1)}\\&=\frac{(k+1)+1}{2(k+1)}\\&=\text{RHS of } P(k+1)\end{aligned}$$

We have shown that if $P(k)$ is true, $P(k+1)$ is also true. $P(2)$ is true, therefore by mathematical induction

$P(n)$: $\prod_{r=2}^{n}\left(1-\frac{1}{r^2}\right)=\frac{n+1}{2n}$ is true for natural numbers $n\geq 2$.

15 $f_1=f_2=1, f_{n+2}=f_{n+1}+f_n \quad n\in N$

$P(n)$: $f_1+f_3+f_5+\ldots+f_{2n-1}=f_{2n}$

Consider $P(1)$: LHS $=f_1=1$ RHS $=f_2=1$, $P(1)$ is true.

Assume that $P(k)$ is true.

$f_1+f_3+\ldots+f_{2k-1}=f_{2k}$

$$\begin{aligned}\text{Consider } P(k+1)\text{: LHS }&=f_1+f_3+\ldots+f_{2k-1}+f_{2(k+1)-1}\\&=f_{2k}+f_{2k+1}\\&=f_{2k+2}\\&=f_{2(k+1)}=\text{RHS of } P(k+1)\end{aligned}$$

$P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n\in N$.

16 $P(n)$: $f_2+f_4+f_6+\ldots+f_{2n}=f_{2n+1}-1 \quad n\in 1$

Consider $P(1)$: LHS $=f_2=1$ RHS $=f_3-1=2-1=1$

$P(1)$ is true.

Assume that $P(k)$ is true.

$f_2+f_4+f_6+\ldots+f_{2k}=f_{2k+1}-1$

$$\begin{aligned}\text{Consider } P(k+1)\text{: LHS }&=f_2+f_4+f_6+\ldots+f_{2k}+f_{2(k+1)}\\&=f_{2k+1}-1+f_{2k+2}\\&=f_{2k+2}+f_{2k+1}-1\\&=f_{2k+3}-1\\&=f_{2(k+1)+1}-1=\text{RHS of } P(k+1)\end{aligned}$$

$P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n\in N$.

17 Let, $a=\frac{1+\sqrt{5}}{2} \quad b=\frac{1-\sqrt{5}}{2}$

$$\begin{aligned}1+\frac{1}{a}&=1+\frac{2}{1+\sqrt{5}}=1+\frac{2}{1+\sqrt{5}}\times\frac{1-\sqrt{5}}{1-\sqrt{5}}\\&=1+\frac{2\left(1-\sqrt{5}\right)}{1-5}=1+\frac{1}{2}\left(\sqrt{5}-1\right)=\frac{1+\sqrt{5}}{2}\end{aligned}$$

So $1+\frac{1}{a}=a$, similarly

$$\begin{aligned}1+\frac{1}{b}&=1+\frac{2}{1-\sqrt{5}}=1+\frac{2}{1-\sqrt{5}}\times\frac{1+\sqrt{5}}{1+\sqrt{5}}\\&=1+\frac{2\left(1+\sqrt{5}\right)}{1-5}=1-\frac{1}{2}\left(1+\sqrt{5}\right)=\frac{1-\sqrt{5}}{2}\end{aligned}$$

So $1+\frac{1}{b}=b$

$P(n)$: $f_n=\frac{a^n-b^n}{\sqrt{5}}$

Consider $P(1)$: RHS $=\frac{1}{\sqrt{5}}(a-b)=\frac{1}{\sqrt{5}}$

$\left(\frac{1}{2}\left(1+\sqrt{5}\right)-\frac{1}{2}\left(1-\sqrt{5}\right)\right)=1$

LHS $=f_1=1$

$P(1)$ is true.

Assume that $P(k)$ is true.

$f_k=\frac{1}{\sqrt{5}}\left(a^k-b^k\right)$

$$\begin{aligned}\text{Consider } f_{k+1}\text{: LHS }&=f_k+f_{k-1}\\&=\frac{1}{\sqrt{5}}\left(a^k-b^k\right)+\frac{1}{\sqrt{5}}\left(a^{k-1}-b^{k-1}\right)\\&=\frac{1}{\sqrt{5}}\left(a^k+a^{k-1}-b^k-b^{k-1}\right)\\&=\frac{1}{\sqrt{5}}\left(a^k\left(1+\frac{1}{a}\right)-b^k\left(1+\frac{1}{b}\right)\right)\\&=\frac{1}{\sqrt{5}}\left(a^k\times a-b^k\times b\right)\end{aligned}$$

$f_{k+1}=\frac{1}{\sqrt{5}}\left(a^{k+1}-b^{k+1}\right)=$ RHS of $P(k+1)$

$P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n\in N$.

18 $f_1=f_2=1,\ f_{n+2}=f_{n+1}+f_n \quad n\in N.$

$$\sum_{r=1}^{n}f_r=f_{r+2}-1$$

Consider $P(1)$: LHS $=f_1$ RHS $=f_3-1=2-1=1$

$P(1)$ is true.

Assume that $P(k)$ is true.

$$\sum_{r=1}^{k}f_r=f_1+f_2+\ldots\ldots+f_k=f_{k+2}-1$$

$$\begin{aligned} P(k+1)\text{: LHS} &= \sum_{r=1}^{k+1} f_r = f_1 + f_2 + ... + f_k + f_{k+1} \\ &= f_{k+2} - 1 + f_{k+1} \\ &= f_{k+1} + f_{k+2} - 1 \\ &= f_{k+3} - 1 \\ &= f_{(k+1)+2} - 1 = \text{RHS of } P(k+1) \end{aligned}$$

$P(k+1)$ is true.
We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n \in N$.

19 $A = \begin{bmatrix} 4 & 0 \\ -1 & 5 \end{bmatrix}$ show $A^n = \begin{bmatrix} 4^n & 0 \\ 4^n - 5^n & 5^n \end{bmatrix}$

When $n = 1$ $A^1 = \begin{bmatrix} 4^1 & 0 \\ 4^1 - 5^1 & 5^1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ -1 & 5 \end{bmatrix}$

$P(1)$ is true.
Assume that $P(k)$ is true.

$$A^k = \begin{bmatrix} 4^k & 0 \\ 4^k - 5^k & 5^k \end{bmatrix}$$

Consider
$$\begin{aligned} A^{k+1} &= A^k \cdot A \\ &= \begin{bmatrix} 4^k & 0 \\ 4^k - 5^k & 5^k \end{bmatrix} \begin{bmatrix} 4 & 0 \\ -1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 4 \times 4^k & 0 \\ 4\left(4^k - 5^k\right) - 1 \times 5^k & 5 \times 5^k \end{bmatrix} \\ &= \begin{bmatrix} 4^{k+1} & 0 \\ 4 \times 4^k - 5 \times 5^k & 5^{k+1} \end{bmatrix} \\ &= \begin{bmatrix} 4^{k+1} & 0 \\ 4^{k+1} - 5^{k+1} & 5^{k+1} \end{bmatrix} \end{aligned}$$

$P(k+1)$ is true.
We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n \in N$.

20 $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ $A^n = \begin{bmatrix} n+1 & n \\ -n & 1-n \end{bmatrix}$ $\forall\, n \in N$

When $n = 1$ $A^1 = \begin{bmatrix} 1+1 & 1 \\ -1 & 1-1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

$P(1)$ is true.
Assume that $P(k)$ is true.

$$A^k = \begin{bmatrix} k+1 & k \\ -k & 1-k \end{bmatrix}$$

Consider
$$\begin{aligned} A^{k+1} &= A^k \cdot A \\ &= \begin{bmatrix} k+1 & k \\ -k & 1-k \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2(k+1) - k & k+1 \\ -2k - (1-k) & -k \end{bmatrix} \\ &= \begin{bmatrix} (k+1)+1 & k+1 \\ -(k+1) & 1-(k+1) \end{bmatrix} \end{aligned}$$

$P(k+1)$ is true.
We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n \in N$.

21 $B = \begin{bmatrix} 2 & b \\ 0 & 1 \end{bmatrix}$ to show $B^n = \begin{bmatrix} 2^n & b\left(2^n - 1\right) \\ 0 & 1 \end{bmatrix}$

When $n = 1$, $B^1 = \begin{bmatrix} 2^1 & b\left(2^1 - 1\right) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & b \\ 0 & 1 \end{bmatrix}$

So it is true when $n = 1$.
Assume it is true when $n = k$.

$$B^k = \begin{bmatrix} 2^k & b\left(2^k - 1\right) \\ 0 & 1 \end{bmatrix}$$

Consider
$$\begin{aligned} B^{k+1} &= B^k \cdot B \\ &= \begin{bmatrix} 2^k & b\left(2^k - 1\right) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & b \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2^k \times 2 + b\left(2^k - 1\right) \times 0 & 2^k \times b + b\left(2^k - 1\right) \times 1 \\ 0 \times 2 + 1 \times 0 & 0 \times b + 1 \times 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \times 2^k & b \times 2^k + b\left(2^k - 1\right) \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2^{k+1} & b\left(2^k + 2^k - 1\right) \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2^{k+1} & b\left(2^{k+1} - 1\right) \\ 0 & 1 \end{bmatrix} \end{aligned}$$

So it is true when $n = k+1$, B^n is true by mathematical induction $\forall n \in N$.

22 **a** Let the numbers be n, $n+1$, $n+2$, $n \in N$.

$$\begin{aligned} \text{Sum} &= n + (n+1) + (n+2) \\ &= 3n + 3 \\ &= 3(n+1) \end{aligned}$$

Let $P(n)$ be the proposition $3(n+1)$ is divisible by 3.
$P(1)$ is true as $3(1+1) = 6$ is divisible by 3.
Assume that $P(k)$ is true. $3(k+1)$ is divisible by 3.
Consider $P(k+1)$.

$$\begin{aligned} 3(k+1+1) &= 3(k+2) \\ &= 3k + 6 \\ &= (3k+3) + 3 \\ &= 3(k+1) + 3 \end{aligned}$$

By assumption, $3(k+1)$ is divisible by 3. Therefore $3(k+1) + 3$ is divisible by 3.
We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall n \in N$.

b Product
$$\begin{aligned} P(n) &= n(n+1)(n+2) \\ &= n\left(n^2 + 3n + 2\right) \\ &= n^3 + 3n^2 + 2n \end{aligned}$$

Consider $P(1)$: $= 1 + 3 + 2 = 6$ is divisible by 3.
$P(1)$ is true.
Assume that $P(k)$ is true.
$k^3 + 3k^2 + 2k$ is divisible by 3
Consider $P(k+1)$:

$$\begin{aligned} \text{LHS} &= (k+1)^3 + 3(k+1)^2 + 2(k+1) \\ &= k^3 + 3k^2 + 3k + 1 + 3\left(k^2 + 2k + 1\right) + 2k + 2 \\ &= k^3 + 6k^2 + 11k + 6 \\ &= k^3 + 3k^2 + 2k + \left(3k^2 + 9k + 6\right) \\ &= k^3 + 3k^2 + 2k + 3\left(k^2 + 3k + 2\right) \end{aligned}$$

Since $P(k)$ is divisible by 3, and $3(k^2+3k+2)$ is divisible by 3, $P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall n \in N$.

23 $P(n)\colon \cos(n\pi) = (-1)^n \qquad n \in Z^+ \cup \{0\}$

Consider $P(0)$: LHS $= \cos(0) = 1$ RHS $= (-1)^0 = 1$

$P(0)$ is true.

Assume that $P(k)$ is true.

$$\cos(k\pi) = (-1)^k$$

Consider $P(k+1)$:

$$\begin{aligned}\text{LHS} &= \cos((k+1)\pi)\\ &= \cos(k\pi+\pi)\\ &= \cos(k\pi)\cos(\pi) - \sin(k\pi)\sin(\pi)\\ &= -1\times\cos(k\pi) - 0\times\sin(k\pi)\\ &= -1\times(-1)^k\\ &= (-1)^{k+1} = \text{RHS of } P(k+1)\end{aligned}$$

$P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(0)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n \in Z^+ \cup \{0\}$.

24 $P(n)\colon \tan(x+n\pi) = \tan(x) \qquad n \in Z^+ \cup \{0\}$

Consider $P(0)$: LHS $= \tan(x) =$ RHS

$P(0)$ is true.

Assume that $P(k)$ is true.

$\tan(x+k\pi) = \tan(x)$

Consider $P(k+1)$: LHS $= \tan(x+(k+1)\pi)$

$$\begin{aligned}&= \tan(x+k\pi+\pi)\\ &= \tan((x+k\pi)+\pi)\\ &= \frac{\tan(x+k\pi)+\tan(\pi)}{1-\tan(x+k\pi)\tan(\pi)}\end{aligned}$$

Now $\tan(\pi) = 0$

$$\begin{aligned}&= \tan(x+k\pi)\\ &= \tan(x) = \text{RHS of } P(k+1)\end{aligned}$$

$P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(0)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n \in Z^+ \cup \{0\}$.

25 $P(n)\colon \cos(x+2n\pi) = \cos(x) \quad \forall\, n \in Z \cup \{0\}$

Consider $P(0)$. LHS $= \cos(x) =$ RHS

$P(0)$ is true.

Assume that $P(k)$ is true.

$\cos(x+2k\pi) = \cos(x)$

Consider $P(k+1)$: LHS $= \cos(x+2(k+1)\pi)$

$$\begin{aligned}&= \cos(x+2k\pi+2\pi)\\ &= \cos((x+2k\pi)+2\pi)\\ &= \cos(x+2k\pi)\cos(2\pi) - \sin(x+2k\pi)\sin(2\pi)\\ &= 1\times\cos(x+2k\pi) - 0\times\sin(x+2k\pi)\\ &= \cos(x+2k\pi)\\ &= \cos(x) = \text{RHS of } P(k+1)\end{aligned}$$

$P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(0)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n \in Z^+ \cup \{0\}$.

26 $P(n)\colon \sin(x+2n\pi) = \sin(x) \quad n \in Z^+ \cup \{0\}$

Consider $P(0)$.

LHS $= \sin(x) =$ RHS. $P(0)$ is true.

Assume that $P(k)$ is true.

$\sin(x+2k\pi) = \sin(x)$

Consider $P(k+1)$: LHS $= \sin(x+2(k+1)\pi)$

$$\begin{aligned}&= \sin(x+2k\pi+2\pi)\\ &= \sin((x+2k\pi)+2\pi)\\ &= \sin(x+2k\pi)\cos(2\pi) + \cos(x+2k\pi)\sin(2\pi)\\ &= 1\times\sin(x+2k\pi) + 0\times\cos(x+2k\pi)\\ &= \sin(x+2k\pi)\\ &= \sin(x) = \text{RHS of } P(k+1)\end{aligned}$$

$P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(0)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n \in Z^+ \cup \{0\}$.

27 $P(n)\colon \cos(x+n\pi) = (-1)^n\cos(x) \quad n \in Z^+ \cup \{0\}$

Consider $P(0)$.

LHS $= \cos(x)$ RHS $= (-1)^0\cos(x) = \cos(x)$

$P(0)$ is true.

Assume that $P(k)$ is true.

$\cos(x+k\pi) = (-1)^k\cos(x)$

Consider $P(k+1)$: LHS $= \cos(x+(k+1)\pi)$

$$\begin{aligned}&= \cos(x+k\pi+\pi)\\ &= \cos((x+k\pi)+\pi)\\ &= \cos(x+k\pi)\cos(\pi) - \sin(x+k\pi)\sin(\pi)\\ &= \cos(x+k\pi)\times(-1) - \sin(x+k\pi)\times 0\\ &= (-1)\cos(x+k\pi)\\ &= (-1)(-1)^k\cos(x)\\ &= (-1)^{k+1}\cos(x) = \text{RHS of } P(k+1)\end{aligned}$$

$P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(0)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n \in Z^+ \cup \{0\}$.

28 $P(n)\colon \sin(x+n\pi) = (-1)^n\sin(x) \quad n \in Z^+ \cup \{0\}$

Consider $P(0)$.

LHS $= \sin(x)$ RHS $= (-1)^0\sin(x) = \sin(x)$

$P(0)$ is true.

Assume that $P(k)$ is true.

$\sin(x+k\pi) = (-1)^k\sin(x)$

Consider $P(k+1)$: LHS $= \sin(x+(k+1)\pi)$

$$\begin{aligned}&= \sin(x+k\pi+\pi)\\ &= \sin((x+k\pi)+\pi)\\ &= \sin(x+k\pi)\cos\pi + \cos(x+k\pi)\sin\pi\\ &= \sin(x+k\pi)\times(-1) + 0\times\cos(x+k\pi)\\ &= (-1)\times\sin(x+k\pi)\\ &= (-1)\times(-1)^k\sin(x)\\ &= (-1)^{k+1}\sin(x) = \text{RHS of } P(k+1)\end{aligned}$$

$P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(0)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n \in Z^+ \cup \{0\}$.

29 **a**

$$\begin{aligned}\tan\left(x+\frac{\pi}{2}\right) &= \frac{\sin\left(x+\frac{\pi}{2}\right)}{\cos\left(x+\frac{\pi}{2}\right)}\\ &= \frac{\sin(x)\cos\left(\frac{\pi}{2}\right)+\cos(x)\sin\left(\frac{\pi}{2}\right)}{\cos(x)\cos\left(\frac{\pi}{2}\right)-\sin(x)\sin\left(\frac{\pi}{2}\right)}\\ &= \frac{0\times\sin(x)+1\times\cos(x)}{0\times\cos(x)-1\times\sin(x)}\\ &= \frac{\cos(x)}{-\sin(x)} = \frac{-1}{\tan(x)}\end{aligned}$$

b $P(n)\colon \tan\left(\dfrac{(2n+1)\pi}{4}\right) = (-1)^n \; n \in Z^+ \cup \{0\}$

Consider $P(0)$: LHS $= \tan\left(\dfrac{\pi}{4}\right) = 1$

RHS $= (-1)^0 = 1$

$P(0)$ is true.

Assume that $P(k)$ is true.

$$\tan\left(\frac{(2k+1)\pi}{4}\right) = (-1)^k$$

$$\begin{aligned}\text{Consider } P(k+1)\text{: LHS} &= \tan\left(\frac{(2(k+1)+1)\pi}{4}\right)\\ &= \tan\left(\frac{(2k+3)\pi}{4}\right)\\ &= \tan\left(\frac{(2k+1+2)\pi}{4}\right)\\ &= \tan\left(\frac{(2k+1)\pi}{4}+\frac{\pi}{2}\right)\\ &= \frac{-1}{\tan\left(\frac{(2k+1)\pi}{4}\right)}\\ &= \frac{-1}{(-1)^k}\\ &= (-1)\times(-1)^{-k}\\ &= (-1)^{-k},\ \text{since } (-1)^{-k} = (-1)^k\ \ k\in \text{Z}\\ &= (-1)^{(k+1)}\end{aligned}$$

$P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(0)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n \in Z^{+} \cup \{0\}$.

30 [1] $\cos(nx) = 2\cos(x)\cos((n-1)x) - \cos((n-2)x)$

[2] $\sin(nx) = 2\cos(x)\sin((n-1)x) - \sin((n-2)x)$

a Assuming [1] true, performing induction on [2]:

When $n = 2$, LHS $= \sin(2x)$

$$\begin{aligned}\text{RHS} &= 2\cos(x)\sin(x) - \sin(0)\\ &= \sin(2x) = \text{LHS}\end{aligned}$$

So it is true when $n = 2$.

Assume it is true when $n = k$.

[2] $\sin(kx) = 2\cos(x)\sin((k-1)x) - \sin((k-2)x)$

Consider $\sin((k+1)x)$

$$\begin{aligned}&= \sin(kx + x)\\ &= \sin(kx)\cos(x) + \cos(kx)\sin(x)\\ &= \cos(x)\left[2\cos(x)\sin((k-1)x) - \sin((k-2)x)\right]\\ &\quad + \sin(x)\left[2\cos(x)\cos((k-1)x) - \cos((k-2)x)\right]\end{aligned}$$

Using assumption by [1]

$$\begin{aligned}&= 2\cos^2(x)\sin((k-1)x) - \cos(x)\sin((k-2)x)\\ &\quad + 2\sin(x)\cos(x)\cos((k-1)x) - \sin(x)\cos((k-2)x)\\ &\quad \text{Regrouping:}\\ &= 2\cos(x)\left[\cos(x)\sin((k-1)x) + \sin(x)\cos((k-1)x)\right]\\ &\quad - \left[\cos(x)\sin((k-2)x) + \sin(x)\cos((k-2)x)\right]\\ &= 2\cos(x)\sin(x + (k-1)x) - \sin(x + (k-2)x)\\ &= 2\cos(x)\sin(kx) - \sin(kx - x)\\ &= 2\cos(x)\sin(((k+1)-1)x) - \sin((k+1-2)x)\\ &= \text{RHS of [2]}\end{aligned}$$

By mathematical induction, the statement [1] is true $\forall\, n \geq 2$.

b Assuming [2] true, performing induction on [1]:

When $n = 2$ LHS $= \cos(2x)$

$$\begin{aligned}\text{RHS} &= 2\cos(x)\cos(x) - \cos(0)\\ &= 2\cos^2(x) - 1 = \text{LHS}\end{aligned}$$

So it is true when $n = 2$.

Assume it is true when $n = k$.

[1] $\cos(kx) = 2\cos(x)\cos((k-1)x) - \cos((k-2)x))$

Consider $\cos((k+1)x)$

$$\begin{aligned}&= \cos(kx + x)\\ &= \cos(kx)\cos(x) - \sin(x)\sin(kx)\\ &= \cos(x)\left[2\cos(x)\cos((k-1)x) - \cos((k-2)x)\right]\\ &\quad - \sin(x)\left[2\cos(x)\sin((k-1)x) - \sin((k-2)x)\right]\end{aligned}$$

Using assumption by [2]

$$\begin{aligned}&= 2\quad \cos^2(x)\cos((k-1)x) - \cos(x)\cos((k-2)x)\\ &\qquad -2\sin(x)\cos(x)\sin((k-1)x) + \sin(x)\sin((k-2)x)\\ &\qquad \text{Regrouping:}\\ &= 2\cos(x)\left[\cos((k-1)x)\cos(x) - \sin(x)\sin((k-1)x)\right]\\ &\qquad - \left[\cos(x)\cos((k-2)x) - \sin(x)\sin((k-2)x)\right]\\ &= 2\cos(x)\cos((k-1)x + x) - \cos(x + (k-2)x)\\ &= 2\cos(x)\cos(kx) - \cos(kx - x)\\ &= 2\cos(x)\cos(((k+1)-1)x) - \cos((k+1-2)x)\\ &= \text{RHS of [1]}\end{aligned}$$

By mathematical induction, the statement [2] is true $\forall\, n \geq 2$.

1.5 Exam questions

1 $1\times2\times3 + 2\times3\times4 + \ldots + n(n+1)(n+2) = \sum_{r=1}^{n} r(r+1)(r+2)$

$$P(n)\text{: } \sum_{r=1}^{n} r(r+1)(r+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$$

Consider $P(1)$: LHS $= 1\times2\times3 = 6$

RHS $= \frac{1}{4}\times1\times2\times3\times4 = 6$, $P(1)$ is true. [1 mark]

Assume that $P(k)$ is true.

$$\sum_{r=1}^{k} r(r+1)(r+2) = \frac{1}{4}k(k+1)(k+2)(k+3) \qquad \text{[1 mark]}$$

Consider $P(k+1)$:

$$\begin{aligned}\text{LHS} &= \sum_{r=1}^{k+1} r(r+1)(r+2)\\ &= \sum_{r=1}^{k} r(r+1)(r+2) + (k+1)(k+2)(k+3)\\ &= \frac{1}{4}k(k+1)(k+2)(k+3) + (k+1)(k+2)(k+3)\\ &= \frac{1}{4}(k+1)(k+2)(k+3)[k+4]\\ &= \frac{1}{4}(k+1)(k+1+1)(k+1+2)(k+1+3) \quad \text{[1 mark]}\end{aligned}$$

$P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n \in N$. [1 mark]

2 $P(n)$: $5^{2n+1} + 2^{2n+1}$ is divisible by 7.

Consider $P(1)$: LHS $= 5^3 + 2^3 = 125 + 8 = 133 = 19\times7$, which is divisible by 7, so $P(1)$ is true. [1 mark]

Assume that $P(k)$ is true.

$5^{2k+1} + 2^{2k+1}$ is divisible by 7. [1 mark]

Consider

$$\begin{aligned}P(k+1):\ \text{LHS} &= 5^{2(k+1)+1} + 2^{2(k+1)+1}\\ &= 5^{2k+3} + 2^{2k+3}\\ &= 5^{2k+1+2} + 2^{2k+1+2}\\ &= 5^2 \times 5^{2k+1} + 2^2 \times 2^{2k+1}\\ &= 25 \times 5^{2k+1} + 4 \times 2^{2k+1}\\ &= (4+21) \times 5^{2k+1} + 4 \times 2^{2k+1}\\ &= 4\left(5^{2k+1} + 2^{2k+1}\right) + 21 \times 5^{2k+1}\\ &= 4\left(5^{2k+1} + 2^{2k+1}\right) + 7 \times \left(3 \times 5^{2k+1}\right)\end{aligned}$$

Since $5^{2k+1} + 2^{2k+1}$ is divisible by 7 and $7 \times \left(3 \times 5^{2k+1}\right)$ is divisible by 7, RHS of $P(k+1)$ is divisible by 7. [1 mark]

$P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n \in N$. [1 mark]

3 Let $P(n)$ be the proposition $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + ... + \frac{n}{(n+1)!}$

$= 1 - \frac{1}{(n+1)!}, \forall n \in N$

When $n = 1$ $LHS = \frac{1}{2!} = \frac{1}{2}$ $RHS = 1 - \frac{1}{2!} = 1 - \frac{1}{2} = \frac{1}{2}$ so it is true when $n = 1$ [1 mark]

Assume it is true when $n = k$ that is $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + ...$

$+ \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}$ [1 mark]

Now consider:

$$\begin{aligned}&\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + ... + \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!}\\ &= 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!}\\ &= 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)(k+1)!}\\ &= 1 - \frac{(k+2)-(k+1)}{(k+2)(k+1)!}\\ &= 1 - \frac{1}{(k+2)!} = 1 - \frac{1}{(k+1+1)!}\end{aligned}$$

[1 mark]

As $P(k)$ is true and $P(k+1)$ is true and since $P(1)$ is true, by the principle of mathematical

Induction, the statement is true, that is $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + ...$

$+ \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}, \forall n \in N$ [1 mark]

1.6 Review

1.6 Exercise

Technology free: short answer

1 a

p	q	$p \vee q$	$\neg p$	$(p \vee q) \wedge \neg p$	$((p \vee q) \wedge \neg p) \to q$
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

↑
All true

b

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \wedge$ $(q \to p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Same

2 Implication $p \to q$. If I study hard then I will pass the test.

Contrapositive $\neg q \to \neg p$. If I do not pass the test then I did not study hard.

Inverse $\neg p \to \neg q$. If I do not study hard then I will not pass the test.

Converse $q \to p$. If I pass the test then I study hard.

Negation $\neg(p \to q) \leftrightarrow p \wedge \neg q$. I study hard and I do not pass the test.

3 a There is an integer x, which satisfies $3x = 2$, F.

b The square of all integers is positive or zero, T.

c All natural numbers are integers, T.

4 $F(x)$: 'x is my friend', $S(x)$: 'x is sincere'

a $\forall x\,(F(x) \wedge S(x))$

All of my friends are sincere.

b $\exists x\,(F(x) \wedge S(x))$

Some of my friends are sincere.

c $\exists x\,(\neg F(x) \wedge S(x))$

There are some people who are not my friends who are sincere.

d $\exists x\,(F(x) \wedge \neg S(x))$

Some of my friends are not sincere.

e $\exists x\,(\neg F(x) \wedge \neg S(x))$

There are some people who are not my friends and who are not sincere.

f $\exists \neg x\,(F(x) \wedge S(x))$

There isn't any person who is my friend and sincere.

5 a To prove: If a is even and b is odd then $a + b$ is odd

Proof:

$$\begin{aligned}\text{Let } a &= 2j\\ b &= 2k+1,\ j,\ k \in Z\\ a + b &= 2j + 2k + 1\\ &= 2(j+k) + 1\\ &= 2i + 1, \quad i \in Z\end{aligned}$$

Is odd

b To prove: if a is even and b is odd then ab is even

Proof:

$$\begin{aligned}\text{Let } a &= 2j\\ b &= 2k+1,\ j,\ k \in Z\\ ab &= 2j(2k+1)\\ &= 4jk + 2j\\ &= 2(2jk + j)\\ &= 2i, \quad i \in Z\end{aligned}$$

Is even

6 To prove: $2\sqrt{x}\sqrt{y} \le x + y,\ \forall x,\ y \in R^+$

Proof: If $x, y \in R^+$ then $2\sqrt{x}\sqrt{y} \le x + y$

Assume $2\sqrt{xy} > x + y$

Square both sides

$4xy > (x + y)^2$

$4xy > x^2 + 2xy + y^2$

Since $x \ge 0$ and $y \ge 0$

$0 > x^2 - 2xy + y^2$

$0 > (x - y)^2$

This is a contradiction

Since $(x - y)^2 \ge 0$

Therefore $2\sqrt{x}\sqrt{y} \le x + y$

7 $1^3 + 2^3 + \ldots\ldots. + n^3 = \displaystyle\sum_{r=1}^{n} r^3$

$P(n)\colon\ \displaystyle\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2,\ \forall n \in N$

Consider $P(1)$: LHS $= 1$

RHS $= \frac{1}{4} \times 1 \times 2^2 = 1$

$P(1)$ is true.

Assume that $P(k)$ is true.

$$\sum_{r=1}^{k} r^3 = \frac{1}{4}k^2(k+1)^2$$

$$\begin{aligned}\text{Consider } P(k+1)\colon \text{ LHS} = \sum_{r=1}^{k+1} r^3 &= \sum_{r=1}^{k} r^3 + (k+1)^3 \\ &= \frac{1}{4}k^2(k+1)^2 + (k+1)^3 \\ &= \frac{1}{4}(k+1)^2\left[k^2 + 4(k+1)\right] \\ &= \frac{1}{4}(k+1)^2\left[k^2 + 4k + 4\right] \\ &= \frac{1}{4}(k+1)^2(k+2)^2 \\ &= \frac{1}{4}(k+1)^2(k+1+1)^2 \\ &= \text{RHS of } P(k+1)\end{aligned}$$

$P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\ n \in N$.

8 Let $P(n)$ be the proposition $6n$ is divisible by 6.

$P(1)$ is true as $6 \times 1 = 6$ is divisible by 6.

Assume that $P(k)$ is true. $6k$ is divisible by 6.

Consider $P(k+1)$.

$6(k+1) = 6k + 6$

By assumption, $6k$ is divisible by 6. Therefore $6k + 6$ is divisible by 6.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall n \in N$.

Technology active: multiple choice

9

p	$\neg p$	$p \wedge \neg p$	q	$(p \wedge \neg p) \wedge q$
T	F	F	T	F
T	F	F	F	F
F	T	F	T	F
F	T	F	F	F

The propositional statement $(p \wedge \neg p) \wedge q$ is always false.

The correct answer is **C**.

10

p	$\neg p$	q	r	$(p \vee \neg p) \to r$	$p \to r$	$q \to r$	$(p \to r) \vee (q \to r)$
T	F	T	T	T	T	T	T
T	T	T	F	F	F	F	F
T	F	F	T	T	T	T	T
T	F	F	F	F	F	T	T
F	T	T	T	T	T	T	T
F	T	T	F	F	T	F	T
F	T	F	T	T	T	T	T
F	T	F	F	F	T	T	T

$((p \vee \neg p) \to r) \leftrightarrow ((p \to r) \vee (q \to r))$ is sometimes true and sometimes false since the 5th and last columns are not equal.

The correct answer is **B**.

11 The set of rational numbers, Q, is closed under multiplication. This means that the product of any two rational numbers is a rational number. Therefore $\forall x \in Q,\ \exists y \in Q,\ y = x^2$ is true.

The correct answer is **E**.

12

p	q	$\neg p$	$q \wedge \neg p$	$q \to p$	$\neg(q \to p)$	$\neg(q \to p) \leftrightarrow (q \wedge \neg p)$
T	T	F	F	T	F	T
T	F	F	F	T	F	T
F	T	T	T	F	T	T
F	F	T	F	T	F	T

The converse of $p \to q$ is $q \to p$ the negation of this is $\neg(q \to p) \leftrightarrow (q \wedge \neg p)$

The correct answer is **C**.

13 D is false as the sum of an even and an odd number is odd. For example, $1 + 2 = 3$ which is odd.

The correct answer is **D**.

14 A is false as there are an infinite number of primes. B is false since 2 is an even prime. C is false since 3 is prime and 10 is not prime. D is false since $2^{11} - 1 = 2047 = 23 \times 89$. E is true, there is no function capable of determining the nth prime.

The correct answer is **E**.

15 To prove $p \to q$ using a direct proof**,** we assume p is true and show that q is true.

The correct answer is **A**.

16 To prove $p \to q$ using a proof by contraposition, we assume $\neg q$ is true and show that $\neg p$ is true.

The correct answer is **D**.

17 To prove $p \to q$ using a proof by contradiction assume that the negation of $p \to q$ is true. That is, assume $p \wedge \neg q$ is true. Then, show that this leads to a contradiction of the form $r \wedge \neg r$.
The correct answer is **E**.

18 If $n \in N$, to prove a proof by induction about $P(n)$ we show a base case, then show that if $P(n)$ is true, $P(n+1)$ must also be true, that is $P(n) \to P(n+1)$.
The correct answer is **D**.

Technology active: extended response

19 a $1 \times 3 + 2 \times 3^2 + 3 \times 3^3 + ... + n \times 3^n = \sum_{r=1}^{n} r \times 3^r$

$$P(n)\text{: } \sum_{r=1}^{n} r \times 3^r = \frac{3}{4}\left[(2n-1) \times 3^n + 1\right]$$

Consider $P(1)$: LHS $= 1 \times 3 = 3$

RHS $= \frac{3}{4}[(2-1) \times 3 + 1] = 3$

$P(1)$ is true.

Assume that $P(k+1)$ is true.

$$\sum_{r=1}^{k} r \times 3^r = \frac{3}{4}\left[(2k-1) \times 3^k + 1\right]$$

Consider $P(k+1)$:

$$\begin{aligned}\text{LHS} &= \sum_{r=1}^{k+1} r \times 3^r = \sum_{r=1}^{k} r \times 3^r + (k+1) \times 3^{k+1}\\ &= \frac{3}{4}\left[(2k-1) \times 3^k + 1\right] + (k+1) \times 3^{k+1}\\ &= \frac{3}{4}\left[(2k-1) \times 3^k + 1 + 4(k+1) \times 3^k\right]\\ &= \frac{3}{4}\left[3^k(2k - 1 + 4k + 4) + 1\right]\\ &= \frac{3}{4}\left[3^k(6k+3) + 1\right]\\ &= \frac{3}{4}\left[3^k \times 3(2k+1) + 1\right]\\ &= \frac{3}{4}\left[(2k+1) \times 3^{k+1} + 1\right]\\ &= \frac{3}{4}\left[(2(k+1) - 1) \times 3^{k+1} + 1\right]\\ &= \text{RHS of } P(k+1)\end{aligned}$$

$P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n \in N$.

b $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + ... + \frac{1}{n(n+1)} = \sum_{r=1}^{n} \frac{1}{r(r+1)}$

$$P(n)\text{: } \sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$$

Consider $P(1)$:

LHS $= \frac{1}{1 \times 2} = \frac{1}{2}$ RHS $= \frac{1}{2}$

So $P(1)$ is true.

Assume that $P(k)$ is true.

$$\sum_{r=1}^{k} \frac{1}{r(r+1)} = \frac{k}{k+1}$$

Consider $P(k+1)$:

$$\begin{aligned}\text{LHS} &= \sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \sum_{r=1}^{k} \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+2)}\\ &= \frac{k}{k+1} + \frac{1}{(k+2)(k+1)}\\ &= \frac{k(k+2)+1}{(k+1)(k+2)}\\ &= \frac{k^2+2k+1}{(k+1)(k+2)}\\ &= \frac{(k+1)^2}{(k+1)(k+2)}\\ &= \frac{k+1}{k+2}\\ &= \frac{k+1}{(k+1)+1} = \text{RHS of } P(k+1)\end{aligned}$$

$P(k+1)$ is true.

We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n \in N$.

20 a $p(n) = \frac{n^5}{20} - \frac{5n^4}{8} + \frac{17n^3}{6} - \frac{43n^2}{8} + \frac{307n}{60}$

$$\left.\begin{aligned}p(1) &= 2\\ p(2) &= 3\\ p(3) &= 5\\ p(4) &= 7\\ p(5) &= 11\end{aligned}\right\}\text{First 5 prime numbers}$$

But $p(6) = 28$ not prime.

Peter's assertion is false.

b $q(n) = n^2 - n + 41$

$$\left.\begin{aligned}q(1) &= 41\\ q(2) &= 43\\ q(3) &= 47\\ q(4) &= 53\\ q(40) &= 1601\end{aligned}\right\}\text{all primes}$$

But $q(41) = 1681 = 41^2$ not prime

Quentin's assertion is false.

1.6 Exam questions

1 To prove: If x is even then $x^2 + 5x - 11$ is odd

Proof: Let $x = 2j$ where $j \in Z$ [1 mark]

So x is even

$$\begin{aligned}x^2 + 5x - 11 &= (2j)^2 + 5(2j) - 11\\ &= 4j^2 + 10j - 11\\ &= 2\left(2j^2 + 5j - 6\right) + 1\\ &= 2i + 1,\ i \in Z\end{aligned}$$

So $x^2 + 5x - 11$ is odd [1 mark]

2 To prove: $\sqrt{5}$ is irrational.

Proof: assume $\sqrt{5}$ is rational

$\sqrt{5} = \frac{p}{q}$ where $p,\ q \in Z,\ q \neq 0$ with no common factors. [1 mark]

$$5 = \frac{p^2}{q^2}$$

$$p^2 = 5q^2$$

So 5 is a factor of p^2

Since 5 is a prime, if 5 divides into p^2, it must also divide into p.

So 5 is a factor of p
Let $p = 5k, \quad k \in Z$
$p^2 = 25k^2 = 5q^2$
$q^2 = 5k^2$
So 5 is a factor of q^2
So 5 is a factor of q [1 mark]
So both p and q have a common factor of 5. This is a contradiction as we assumed p and q had no common factors.
So $\sqrt{5}$ is not a rational number. [1 mark]

3 $P(n)\colon f_n \geq \left(\frac{3}{2}\right)^{n-2} \qquad n \in N$

Consider $P(1)$: LHS $= f_1 = 1$, RHS $= \left(\frac{3}{2}\right)^{-1} = \frac{2}{3}$

So $P(1)$ is true since $1 \geq \frac{2}{3}$.

Consider $P(2)$: LHS $= f_2 = 1$ RHS $= \left(\frac{3}{2}\right)^{0} = 1$

So $P(2)$ is true since $1 \geq 1$. [1 mark]
Assume that $P(k)$ is true. $k \geq 2$ [1 mark]
Consider $P(k+1)$: LHS $= f_{k+1} = f_k + f_{k-1}$

$$\begin{aligned} &\geq \left(\frac{3}{2}\right)^{k-2} + \left(\frac{3}{2}\right)^{k-3} \\ &= \left(\frac{3}{2}\right)^{k-1}\left(\left(\frac{3}{2}\right)^{-1} + \left(\frac{3}{2}\right)^{-2}\right) \\ &= \left(\frac{3}{2}\right)^{k-1}\left(\frac{2}{3} + \frac{4}{9}\right) \\ &= \left(\frac{3}{2}\right)^{k-1} \times \frac{10}{9} > \left(\frac{3}{2}\right)^{k-1} \quad \text{[1 mark]} \end{aligned}$$

$P(k+1)$ is true.
We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ and $P(2)$ are true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n \in N$. [1 mark]

4 $P(n)\colon 4^{n+1} + 5^{2n-1}$ is divisible by 21.
Consider $P(1)$: LHS $= 4^2 + 5^1 = 21$, which is divisible by 21. [1 mark]
Assume that $P(k)$ is true.
$4^{k+1} + 5^{2k-1}$ is divisible by 21. [1 mark]
Consider $P(k+1)$: LHS $= 4^{k+2} + 5^{2(k+1)-1}$

$$\begin{aligned} &= 4^{k+2} + 5^{2k+2-1} \\ &= 4 \times 4^{k+1} + 25 \times 5^{2k-1} \\ &= 4 \times 4^{k+1} + (21+4)\,5^{2k-1} \\ &= 4\left(4^{k+1} + 5^{2k-1}\right) + 21 \times 5^{2k-1} \quad \text{[1 mark]} \end{aligned}$$

Now $4^{k+1} + 5^{2k-1}$ is divisible by 21
$21 \times 5^{2k-1}$ is divisible by 21
So LHS of $P(k+1)$ is divisible by 21
$P(k+1)$ is true.
We have shown that if $P(k)$ is true, $P(k+1)$ is true. $P(1)$ is true, therefore, by mathematical induction, $P(n)$ is true $\forall\, n \in N$. [1 mark]

5 Two consecutive odd numbers are $2a+1$ and $2a-1$
$a \in Z$ [1 mark]
$(2a+1)^2 - (2a-1)^2 = 4a^2 + 4a + 1 - \left(4a^2 - 4a + 1\right)$
$= 8a$ [1 mark]
Therefore the difference between the squares of two consective odd numbers is divisible by 8. [1 mark]

Topic 2 — Complex Numbers

2.2 Complex numbers in Cartesian form

2.2 Exercise

1 **a** $\text{Re}(8 - 9i) = 8$

b $$\begin{aligned}\text{Im}(12 - i^2 + 2i^5) &= \text{Im}\left(12 - -1 + 2i(i^2)^2\right)\\ &= \text{Im}(13 + 2i)\\ &= 2\end{aligned}$$

c $\text{Im}(17 - i) = -1$

d $\text{Im}(7 - 3i^2 + i^4 - 2i)$
$$\begin{aligned}&= \text{Im}(7 - 3 \times -1 + (i^2)^2 - 2i)\\ &= \text{Im}(11 - 2i)\\ &= -2\end{aligned}$$

2 $u = 3 - i, \quad v = 4 - 3i$

a $$\begin{aligned}u + v &= (3 - i) + (4 - 3i)\\ &= 7 - 4i\end{aligned}$$

b $$\begin{aligned}u - v &= (3 - i) - (4 - 3i)\\ &= -1 + 2i\end{aligned}$$

c $$\begin{aligned}3u - 2v &= 3(3 - i) - 2(4 - 3i)\\ &= 9 - 3i - (8 - 6i)\\ &= 1 + 3i\end{aligned}$$

3 $u = 3 - i, \quad v = 4 - 3i$

a $$\begin{aligned}uv &= (3 - i)(4 - 3i)\\ &= 12 - 4i - 9i + 3i^2\\ &= 9 - 13i\end{aligned}$$

b $$\begin{aligned}v^2 &= (4 - 3i)^2\\ &= 16 - 24i + 9i^2\\ &= 7 - 24i\end{aligned}$$

4 $u = 1 + 3i, \quad v = 3 + 4i$

a $$\begin{aligned}(u - v)^2 &= ((1 + 3i) - (3 + 4i))^2\\ &= (-2 - i)^2\\ &= 4 + 4i + i^2\\ &= 3 + 4i\end{aligned}$$

b $$\begin{aligned}(3u - 2v)^2 &= (3(1 + 3i) - 2(3 + 4i))^2\\ &= (3 + 9i - (6 + 8i))^2\\ &= (-3 + i)^2\\ &= 9 - 6i + i^2\\ &= 8 - 6i\end{aligned}$$

5 $u = 3 - i, \quad v = 3 + i$

a $\text{Re}((3 - i)(3 + i))$
$$\begin{aligned}&= \text{Re}(9 - 3i + 3i - i^2)\\ &= \text{Re}(9 - i^2)\\ &= \text{Re}(10 + 0i)\\ &= 10\end{aligned}$$

b $\text{Im}((3 - i)(3 + i))$
$$\begin{aligned}&= \text{Im}(10 + 0i)\\ &= 0\end{aligned}$$

6 $u = 3 - i, \quad v = 4 - 3i$

a $$\begin{aligned}\frac{1}{u} &= \frac{1}{3 - i} \times \frac{3 + i}{3 + i}\\ &= \frac{3 + i}{9 - i^2}\\ &= \frac{3 + i}{10}\\ &= \frac{3}{10} + \frac{1}{10}i\end{aligned}$$

b $$\begin{aligned}\frac{u}{v} &= \frac{3 - i}{4 - 3i} \times \frac{4 + 3i}{4 + 3i}\\ &= \frac{12 - 4i + 9i - 3i^2}{16 - 9i^2}\\ &= \frac{15 + 5i}{25}\\ &= \frac{3}{5} + \frac{1}{5}i\end{aligned}$$

7 $u = 1 + 3i, \quad v = 3 + 4i$
$$\begin{aligned}\frac{u + v}{u - v} &= \frac{(1 + 3i) + (3 + 4i)}{(1 + 3i) - (3 + 4i)}\\ &= \frac{4 + 7i}{-2 - i} \times \frac{-2 + i}{-2 + i}\\ &= \frac{-8 - 14i + 4i + 7i^2}{4 - i^2}\\ &= \frac{-15 - 10i}{5}\\ &= -3 - 2i\end{aligned}$$

8 $$\begin{aligned}4x - 2iy + 3ix - 4y &= -6 - i\\ 4x - 4y + i(3x - 2y) &= -6 - i\end{aligned}$$
$$\begin{array}{ll}\text{Re}: (1) & 4x - 4y = -6\\ \text{Im}: (2) & 3x - 2y = -1\end{array}$$
$$\begin{array}{rr}(1) & 4x - 4y = -6\\ 2 \times -(2) & -6x + 4y = 2\\ & -2x = -4\\ & x = 2\end{array}$$
$$\begin{aligned}\Rightarrow 2y &= 3x + 1 = 7\\ y &= \frac{7}{2}\end{aligned}$$

9 $$\begin{aligned}(x + iy)(3 - 2i) &= 6 - i\\ 3x + 3iy - 2ix - 2i^2y &= 6 - i\\ (3x + 2y) + (3y - 2x)i &= 6 - i\end{aligned}$$
$$\begin{array}{ll}\text{Re}: (1) & 3x + 2y = 6\\ \text{Im}: (2) & 3y - 2x = -1\\ 2 \times (1) & 6x + 4y = 12\\ 3 \times (2) & 9y - 6x = -3\\ & 13y = 9\\ & y = \dfrac{9}{13}\end{array}$$

$$
\begin{aligned}
\Rightarrow 3x &= 6 - 2y \\
&= 6 - \frac{18}{13} \\
&= \frac{78 - 18}{13} \\
3x &= \frac{60}{13} \\
x &= \frac{20}{13}
\end{aligned}
$$

OR

$$
\begin{aligned}
(x + iy)(3 - 2i) &= 6 - i \\
x + iy &= \frac{6 - i}{3 - 2i} \times \frac{3 + 2i}{3 + 2i} \\
&= \frac{18 - 3i + 12i - 2i^2}{9 - 4i^2} \\
&= \frac{20 + 9i}{13} \\
x + iy &= \frac{20}{13} + \frac{9}{13}i \\
\Rightarrow x &= \frac{20}{13}, \quad y = \frac{9}{13}
\end{aligned}
$$

10
$$
\begin{aligned}
\text{Im}\left(\frac{25}{4 - 3i} + i^{77}\right) &= \text{Im}\left(\frac{25}{4 - 3i} \times \frac{4 + 3i}{4 + 3i} + i^{19 \times 4}i\right) \\
&= \text{Im}\left(\frac{25(4 + 3i)}{16 - 9i^2} + \left(i^4\right)^{19} i\right) \\
&= \text{Im}(4 + 3i + i) \\
&= \text{Im}(4 + 4i) \\
&= 4
\end{aligned}
$$

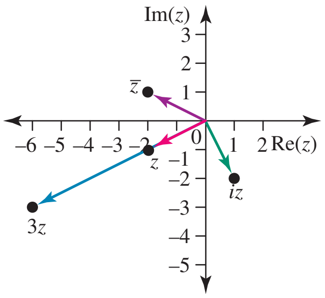
11
$$
\begin{aligned}
\text{Re}\left(\frac{10}{1 + 3i} + i^{96}\right) &= \text{Re}\left(\frac{10}{1 + 3i} \times \frac{1 - 3i}{1 - 3i} + i^{4 \times 24}\right) \\
&= \text{Re}\left(\frac{10(1 - 3i)}{1 - 9i^2} + \left(i^4\right)^{24}\right) \\
&= \text{Re}(1 - 3i + 1) \\
&= \text{Re}(2 - 3i) \\
&= 2
\end{aligned}
$$

12
$$
\begin{aligned}
z &= -2 - i \\
3z &= -6 - 3i \\
\bar{z} &= -2 + i \\
iz &= i(-2 - i) \\
&= -2i - i^2 \\
&= 1 - 2i
\end{aligned}
$$

$3z$ has a length of 3 times z

$\bar{z}$ is the reflection in the real axis

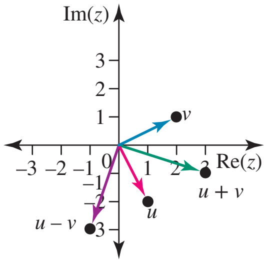
iz is a rotation of 90° anticlockwise from z



13 $u = 1 - 2i, \quad v = 2 + i$

$u + v = 3 - i$

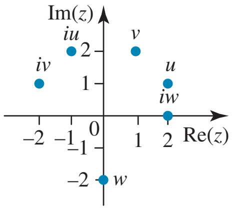
$u - v = -1 - 3i$



14
$$
\begin{aligned}
\frac{z - i}{z + i} &= 2 + i \\
(z - i) &= (2 + i)(z + i) \\
z - i &= 2z + iz + 2i + i^2 \\
z + iz &= -3i - i^2 \\
z(1 + i) &= 1 - 3i \\
z &= \frac{1 - 3i}{1 + i} \times \frac{1 - i}{1 - i} \\
&= \frac{1 - 3i - i + 3i^2}{1 - i^2} \\
&= \frac{-2 - 4i}{2} \\
&= -1 - 2i
\end{aligned}
$$

15
$$
\begin{aligned}
\frac{5(z + 2i)}{z - 2} &= 11 - 2i \\
5(z + 2i) &= (11 - 2i)(z - 2) \\
5z + 10i &= 11z - 2iz - 22 + 4i \\
6z - 2iz &= 6i + 22 \\
z(6 - 2i) &= 6i + 22 \\
z &= \frac{22 + 6i}{6 - 2i} \\
&= \frac{11 + 3i}{3 - i} \times \frac{3 + i}{3 + i} \\
&= \frac{33 + 9i + 11i + 3i^2}{9 - i^2} \\
&= \frac{30 + 20i}{10} \\
z &= 3 + 2i
\end{aligned}
$$

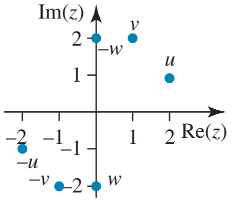
16 a



b As shown in **Q16a**

c Multiplying a complex number z by i rotates z 90° anticlockwise about the origin.

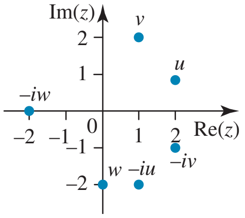
17 a



b As above in **Q17a**

c Multiplying a complex number z by −1 rotates z by 180° about the origin.

18 a



b As shown in **Q18a**

c Multiplying a complex number z by $-i$ rotates z 90° clockwise about the origin.

19 a $\bar{z} = x - iy$

$$\begin{aligned} z + \bar{z} &= x + iy + x - iy \\ &= 2x \\ &= 2\text{Re}(z) \end{aligned}$$

b
$$\begin{aligned} z - \bar{z} &= x + iy - (x - iy) \\ &= 2iy \\ &= 2i\,\text{Im}(z) \end{aligned}$$

20 a
$$\begin{aligned} \text{LHS} &= \overline{z_1} + \overline{z_2} = a - bi + c - di \\ &= a + c - i(b + d) \\ \text{RHS} = \overline{z_1 + z_2} &= \overline{a + bi + c + di} \\ &= \overline{(a + c) + i(b + d)} \\ &= a + c - i(b + d) = \text{LHS} \end{aligned}$$

b
$$\begin{aligned} \text{LHS} = \overline{z_1} \cdot \overline{z_2} &= (a - bi) \cdot (c - di) \\ &= ac + bdi^2 - bci - adi \\ &= ac - bd - i(bc + ad) \\ \text{RHS} = \overline{z_1\, z_2} &= \overline{(a + bi)\,(c + di)} \\ &= \overline{ac + bdi^2 + bci + adi} \\ &= \overline{ac - bd + i(bc + ad)} \\ &= ac - bd - i(bc + ad) = \text{LHS} \end{aligned}$$

2.2 Exam questions

1 $z = a + bi, \quad a = \text{Re}(z) \in R \backslash \{0\}, \quad b = \text{Im}(z) \in R$

$\bar{z} = a - bi, \quad \bar{z} + z = 2a, \quad \bar{z}z = a^2 - b^2 i^2 = a^2 + b^2$

$$\begin{aligned} \frac{4z\bar{z}}{(z + \bar{z})^2} &= \frac{4(a^2 + b^2)}{4a^2} \\ &= 1 + \frac{b^2}{a^2} \\ &= 1 + \left(\frac{\text{Im}(z)}{\text{Re}(z)}\right)^2 \end{aligned}$$

The correct answer is **A**.

2 $i^{1!} = i\,,\ i^{2!} = -1\,,\ i^{3!} = -1\,,\ i^{4!} = i^{5!} = \ldots\ldots = i^{100!} = 1$

Sum $= i - 2 + 97 = 95 + i$

$i^{1!} + i^{2!} + i^{3!} + \ldots\ldots + i^{100!} = 95 + i$

The correct answer is **C**.

3
$$\begin{aligned} \sum_{n=1}^{4} ni^n &= 1 \times i + 2 \times i^2 + 3 \times i^3 + 4 \times i^4 \\ &= i - 2 - 3i + 4 = 2 - 2i \\ \sum_{n=5}^{8} ni^n &= 5 \times i^5 + 6 \times i^6 + 7 \times i^7 + 8 \times i^8 \\ &= 5i - 6 - 7i + 8 = 2 - 2i \\ \sum_{n=1}^{100} ni^n &= 25(2 - i) = 50 - 50i \end{aligned}$$

Alternatively, use a CAS calculator and evaluate:

$$\sum_{n=1}^{100} ni^n = 50 - 50i.$$

The correct answer is **A**.

2.3 Complex numbers in polar form

2.3 Exercise

1 a $z = 1 + \sqrt{3}i$

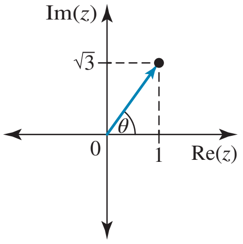
First quadrant

$x = 1,\ y = \sqrt{3}$

$$\begin{aligned} |z| &= \sqrt{x^2 + y^2} \\ &= \sqrt{1 + 3} \\ &= 2 \end{aligned}$$

$$\begin{aligned} \tan(\theta) &= \frac{y}{x} \\ &= \sqrt{3} \\ \theta &= \text{Arg}(z) \\ &= \tan^{-1}\left(\sqrt{3}\right) \\ &= \frac{\pi}{3} \end{aligned}$$

$z = 2\,\text{cis}\left(\frac{\pi}{3}\right)$



b $z = -1 + i$

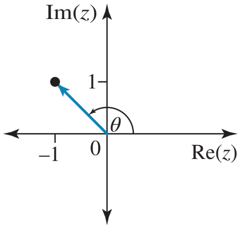
Second quadrant

$$\begin{aligned} |z| &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-1)^2 + (1)^2} \\ &= \sqrt{2} \end{aligned}$$

$x = -1,\ y = 1$

$$\begin{aligned} \theta &= \text{Arg}(z) \\ &= \tan^{-1}(-1) + \pi \\ &= -\frac{\pi}{4} + \pi \\ &= \frac{3\pi}{4} \end{aligned}$$

$z = \sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)$



c $z = -2 - 2\sqrt{3}i$

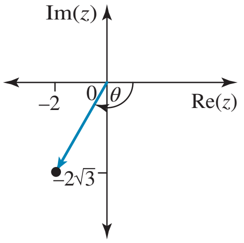
Third quadrant

$x = -2, y = -2\sqrt{3}$

$$
\begin{aligned}
|z| &= \sqrt{x^2+y^2}\\
&= \sqrt{(-2)^2+\left(-2\sqrt{3}\right)^2}\\
&= \sqrt{4+12}\\
&= \sqrt{16}\\
&= 4
\end{aligned}
$$

$$
\begin{aligned}
\theta &= \text{Arg}(z)\\
&= -\pi + \tan^{-1}\left(\sqrt{3}\right)\\
&= -\pi + \frac{\pi}{3}\\
&= -\frac{2\pi}{3}
\end{aligned}
$$

$$z = 4\,\text{cis}\left(-\frac{2\pi}{3}\right)$$



d $z = \sqrt{3} - i$

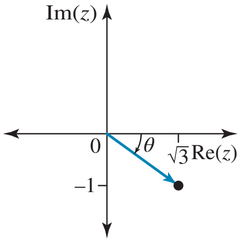
Fourth quadrant

$x = \sqrt{3},\ y = -1$

$$
\begin{aligned}
|z| &= \sqrt{x^2+y^2}\\
&= \sqrt{3+1}\\
&= \sqrt{4}\\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
\theta &= \text{Arg}(z)\\
&= \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)\\
&= -\frac{\pi}{6}
\end{aligned}
$$

$$z = 2\,\text{cis}\left(-\frac{\pi}{6}\right)$$



e $z = 4$

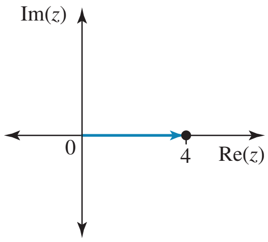
$x = 4$

$y = 0$

$|z| = 4$

$$
\begin{aligned}
\theta &= \text{Arg}(z)\\
&= 0
\end{aligned}
$$

$$z = 4\,\text{cis}(0)$$



f $z = -2i$

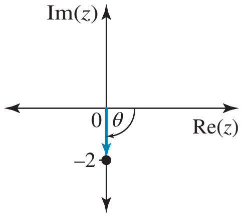
$x = 0$

$y = -2$

$$
\begin{aligned}
|z| &= \sqrt{x^2+y^2}\\
&= \sqrt{0+4}\\
&= 2
\end{aligned}
$$

$$\theta = -\frac{\pi}{2}$$

$$z = 2\,\text{cis}\left(-\frac{\pi}{2}\right)$$



2 **a** $z = \sqrt{3} + i$

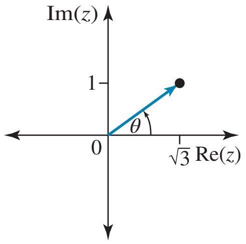
First quadrant

$x = \sqrt{3},\ y = 1$

$$
\begin{aligned}
|z| &= \sqrt{x^2+y^2}\\
&= \sqrt{3+1}\\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
\theta &= \text{Arg}(z)\\
&= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\\
&= \frac{\pi}{6}
\end{aligned}
$$

$$z = 2\,\text{cis}\left(\frac{\pi}{6}\right)$$



b $z = -1 + \sqrt{3}i$

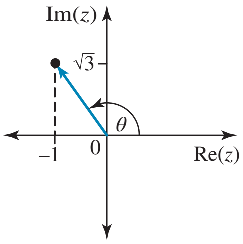
Second quadrant

$x = -1,\ y = \sqrt{3}$

$$
\begin{aligned}
|z| &= \sqrt{x^2+y^2}\\
&= \sqrt{1+3}\\
&= 2
\end{aligned}
$$

$$\begin{aligned}\theta &= \text{Arg}(z)\\ &= \pi - \tan^{-1}\left(\sqrt{3}\right)\\ &= \pi - \frac{\pi}{3}\\ &= \frac{2\pi}{3}\end{aligned}$$

$$z = 2\,\text{cis}\left(\frac{2\pi}{3}\right)$$



c $z = -\sqrt{3} - i$

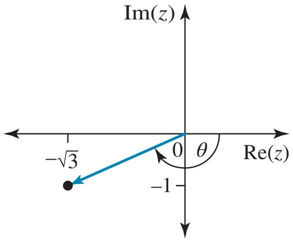
Third quadrant

$x = -\sqrt{3},\ y = -1$

$$\begin{aligned}|z| &= \sqrt{x^2 + y^2}\\ &= \sqrt{3 + 1}\\ &= 2\end{aligned}$$

$$\begin{aligned}\theta &= \text{Arg}(z)\\ &= -\pi + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\\ &= -\pi + \frac{\pi}{6}\\ &= -\frac{5\pi}{6}\end{aligned}$$

$$z = 2\,\text{cis}\left(-\frac{5\pi}{6}\right)$$



d $z = 2 - 2i$

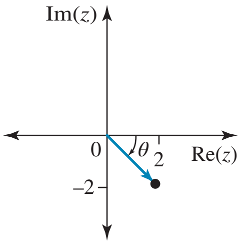
Fourth quadrant

$x = 2,\ y = -2$

$$\begin{aligned}|z| &= \sqrt{x^2 + y^2}\\ &= \sqrt{4 + 4}\\ &= \sqrt{8}\\ &= 2\sqrt{2}\end{aligned}$$

$$\begin{aligned}\theta &= \text{Arg}(z)\\ &= \tan^{-1}\left(-\frac{\pi}{4}\right)\end{aligned}$$

$$z = 2\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$$

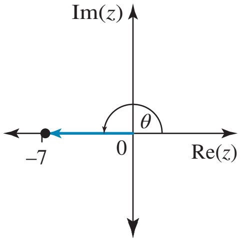


e $z = -7$

$x = -7,\ y = 0$

$|z| = 7,\ \theta = \pi$

$z = 7\,\text{cis}(\pi)$

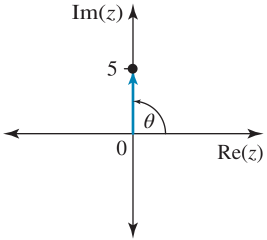


f $z = 5i$

$x = 0,\ y = 5$

$|z| = 5,\ \theta = \dfrac{\pi}{2}$

$$z = 5\,\text{cis}\left(\frac{\pi}{2}\right)$$



3 **a** $$\begin{aligned}4\,\text{cis}\left(-\frac{\pi}{3}\right) &= 4\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)\\ &= 4\cos\left(\frac{\pi}{3}\right) - 4i\sin\left(\frac{\pi}{3}\right)\\ &= 4 \times \frac{1}{2} - 4i \times \frac{\sqrt{3}}{2}\\ &= 2 - 2\sqrt{3}i\end{aligned}$$

b $$\begin{aligned}8\,\text{cis}\left(-\frac{\pi}{2}\right) &= 8\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)\\ &= 8 \times 0 + 8 \times i \times -1\\ &= -8i\end{aligned}$$

4 $$\begin{aligned}6\sqrt{2}\,\text{cis}(-135°) &= 6\sqrt{2}(\cos(-135°) + i\sin(-135°))\\ &= 6\sqrt{2}\cos(135°) - 6\sqrt{2}i\sin(135°)\\ &= 6\sqrt{2} \times -\frac{1}{\sqrt{2}} - 6\sqrt{2} \times \frac{1}{\sqrt{2}}i\\ &= -6 - 6i\end{aligned}$$

5 $u = 6\,\text{cis}\left(-\dfrac{\pi}{3}\right)$

$\overline{u} = 6\,\text{cis}\left(\dfrac{\pi}{3}\right)$

$$\begin{aligned}
\overline{u}^{-1} &= \frac{1}{6\,\text{cis}\left(\frac{\pi}{3}\right)}\\
&= \frac{1}{6}\,\text{cis}\left(-\frac{\pi}{3}\right)\\
&= \frac{1}{6}\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)\\
&= \frac{1}{6}\left(\cos\left(\frac{\pi}{3}\right) - i\sin\left(\frac{\pi}{3}\right)\right)\\
&= \frac{1}{6}\left(\frac{1}{2} - i\times\frac{\sqrt{3}}{2}\right)\\
&= \frac{1}{12} - \frac{\sqrt{3}}{12}i
\end{aligned}$$

6
$$\begin{aligned}
u &= \frac{\sqrt{2}}{4}\,\text{cis}\left(\frac{3\pi}{4}\right)\\
\overline{u} &= \frac{\sqrt{2}}{4}\,\text{cis}\left(-\frac{3\pi}{4}\right)\\
\overline{u}^{-1} &= \frac{4}{\sqrt{2}}\,\text{cis}\left(\frac{3\pi}{4}\right)\\
&= \frac{4}{\sqrt{2}}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)\\
&= \frac{4}{\sqrt{2}}\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)\\
&= -2 + 2i
\end{aligned}$$

7
$$\begin{aligned}
u &= 4\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)\\
v &= \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)
\end{aligned}$$

a
$$\begin{aligned}
uv &= \left(4\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)\left(\sqrt{2}\,\text{cis}\right)\left(\frac{\pi}{4}\right)\right)\\
&= 4\sqrt{2}\times\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4} + \frac{\pi}{4}\right)\\
&= 8\,\text{cis}\left(-\frac{\pi}{2}\right)\\
&= -8i
\end{aligned}$$

b
$$\begin{aligned}
\frac{u}{v} &= \frac{4\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)}{\left(\sqrt{2}\,\text{cis}\right)\left(\frac{\pi}{4}\right)}\\
&= 4\,\text{cis}\left(-\frac{3\pi}{4} - \frac{\pi}{4}\right)\\
&= 4\,\text{cis}(-\pi)\\
&= -4
\end{aligned}$$

8
$$\begin{aligned}
u &= 4\,\text{cis}\left(\frac{\pi}{3}\right)\\
v &= \frac{1}{2}\,\text{cis}\left(-\frac{2\pi}{3}\right)
\end{aligned}$$

a
$$\begin{aligned}
uv &= \left(4\,\text{cis}\left(\frac{\pi}{3}\right)\left(\frac{1}{2}\,\text{cis}\right)\left(-\frac{2\pi}{3}\right)\right)\\
&= 4\times\frac{1}{2}\,\text{cis}\left(\frac{\pi}{3} - \frac{2\pi}{3}\right)\\
&= 2\,\text{cis}\left(-\frac{\pi}{3}\right)\\
&= 2\cos\left(-\frac{\pi}{3}\right) + 2i\sin\left(-\frac{\pi}{3}\right)\\
&= 1 - \sqrt{3}i
\end{aligned}$$

b
$$\begin{aligned}
\frac{u}{v} &= \frac{4\,\text{cis}\left(\frac{\pi}{3}\right)}{\frac{1}{2}\,\text{cis}\left(-\frac{2\pi}{3}\right)}\\
&= 4\times 2\,\text{cis}\left(\frac{\pi}{3} + \frac{2\pi}{3}\right)\\
&= 8\,\text{cis}(\pi)\\
&= -8
\end{aligned}$$

9 **a**
$$\begin{aligned}
u &= -1 - i\\
x &= -1,\ y = -1\\
|z| &= \sqrt{2}\\
\theta &= \text{Arg}(z)\\
&= -\frac{3\pi}{4}\\
u &= \sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)\\
\arg\left(u^{10}\right) &= -\frac{3\pi}{4}\times 10\\
&= -\frac{15\pi}{2}\\
\text{Arg}\left(u^{10}\right) &= -\frac{15\pi}{2} + 8\pi\\
&= \frac{\pi}{2}
\end{aligned}$$

b
$$\begin{aligned}
u^{10} &= \left(\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)\right)^{10}\\
&= \left(\sqrt{2}\right)^{10}\text{cis}\left(\frac{\pi}{2}\right)\\
&= 2^5\,\text{cis}\left(\frac{\pi}{2}\right)\\
&= 32i
\end{aligned}$$

10
$$\begin{aligned}
\frac{(-1+i)^6}{\left(\sqrt{3-i}\right)^4} &= \frac{\left(\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)\right)^6}{\left(2\,\text{cis}\left(-\frac{\pi}{6}\right)\right)^4}\\
&= \frac{\left(\sqrt{2}\right)^6}{2^4}\,\text{cis}\left(-\frac{6\pi}{4} + \frac{4\times\pi}{6}\right)\\
&= \frac{1}{2}\,\text{cis}\left(-\frac{5\pi}{6}\right)\\
&= \frac{1}{2}\left(\cos\left(\frac{5\pi}{6}\right) - i\sin\left(\frac{5\pi}{6}\right)\right)\\
&= \frac{1}{2}\left(\frac{\sqrt{3}}{2} - i\times\frac{1}{2}\right)\\
&= -\frac{\sqrt{3}}{4} - \frac{1}{4}i
\end{aligned}$$

11 **a**
$$\begin{aligned}
\tan\left(\frac{5\pi}{12}\right) &= \tan\left(\frac{\pi}{4} + \frac{\pi}{6}\right)\\
&= \frac{\tan\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{6}\right)}{1 - \tan\left(\frac{\pi}{6}\right)\tan\left(\frac{\pi}{6}\right)}\\
&= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}}\\
&= \frac{1 + \frac{\sqrt{3}}{3}}{1 - \frac{\sqrt{3}}{3}}
\end{aligned}$$

$$= \frac{\frac{3+\sqrt{3}}{3}}{\frac{3-\sqrt{3}}{3}}$$

$$= \frac{3+\sqrt{3}}{3-\sqrt{3}} \times \frac{3+\sqrt{3}}{3+\sqrt{3}}$$

$$= \frac{9+6\sqrt{3}+3}{9-3}$$

$$= \frac{12+6\sqrt{3}}{6}$$

$$= 2+\sqrt{3}$$

b $u = 1 + \left(\sqrt{3+2}\right) i$

$$\text{Arg}(u) = \tan^{-1}\left(2+\sqrt{3}\right)$$

$$= \frac{5\pi}{12}$$

$$iu = \left(1 + \left(\sqrt{3}+2\right) i\right) i$$

$$= i + \left(\sqrt{3+2}\right) i^2$$

$$= -\sqrt{3} - 2 + i$$

$$\text{Arg}(iu) = \frac{5\pi}{12} + \frac{\pi}{2}$$

$$= \frac{11\pi}{12}$$

12 $\tan\left(\frac{11\pi}{12}\right) = \tan\left(\frac{3\pi}{4} + \frac{\pi}{6}\right)$

$$= \frac{\tan\left(\frac{3\pi}{4}\right) + \tan\left(\frac{\pi}{6}\right)}{1 - \tan\left(\frac{3\pi}{4}\right)\tan\left(\frac{\pi}{6}\right)}$$

$$= \frac{-1 + \frac{\sqrt{3}}{3}}{1 + \frac{\sqrt{3}}{3}}$$

$$= \frac{-3+\sqrt{3}}{3+\sqrt{3}} \times \frac{3-\sqrt{3}}{3-\sqrt{3}}$$

$$= \frac{-9+6\sqrt{3}-3}{9-3}$$

$$= \frac{6\sqrt{3}-12}{6}$$

$$= \sqrt{3} - 2$$

$u = 1 + \left(\sqrt{3} - 2\right) i$, in fourth quadrant

$$\text{Arg}(u) = \tan^{-1}\left(\sqrt{3} - 2\right)$$

$$= \frac{11\pi}{12} - \pi$$

$$= -\frac{\pi}{12}$$

13 $0 = \left(1 + \sqrt{3}i\right)^n - \left(1 - \sqrt{3}i\right)^n$

$$0 = \left(2\,\text{cis}\left(\frac{\pi}{3}\right)\right)^n - \left(2\,\text{cis}\left(-\frac{\pi}{3}\right)\right)^n$$

$$0 = 2^n\,\text{cis}\left(\frac{n\pi}{3}\right) - 2^n\,\text{cis}\left(-\frac{n\pi}{3}\right)$$

$$0 = 2^n\left(\frac{n\pi}{3}\right) + i\sin\left(\frac{n\pi}{3}\right) - \left(\cos\left(\frac{n\pi}{3}\right) - i\sin\left(\frac{n\pi}{3}\right)\right)$$

$$0 = 2 \times 2^n i\sin\left(\frac{n\pi}{3}\right)$$

$$0 = \sin\left(\frac{n\pi}{3}\right)$$

$$\frac{n\pi}{3} = k\pi$$

$$n = 3k, k \in Z$$

14 $0 = \left(1 + \sqrt{3}i\right)^n + \left(1 - \sqrt{3}i\right)$

$$0 = \left(2\,\text{cis}\left(\frac{\pi}{6}\right)\right)^n + \left(2\,\text{cis}\left(-\frac{\pi}{6}\right)\right)^n$$

$$0 = 2^n\,\text{cis}\left(\frac{n\pi}{6}\right) + 2^n\,\text{cis}\left(-\frac{n\pi}{6}\right)$$

$$0 = 2^n\left(\cos\left(\frac{n\pi}{6}\right) + i\sin\left(\frac{n\pi}{6}\right) + \left(\cos\left(\frac{n\pi}{6}\right) - i\sin\left(\frac{n\pi}{6}\right)\right)\right)$$

$$0 = 2 \times 2^n\,\cos\left(\frac{n\pi}{6}\right)$$

$$0 = \cos\left(\frac{n\pi}{6}\right)$$

$$\frac{n\pi}{6} = (2k+1)\frac{\pi}{2}$$

$$n = \frac{3}{2}(2k+1)$$

$$= 3k + \frac{3}{2} \qquad k \in Z$$

15 **a** $z = 2 + 2i$

$$= 2\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$$

$$z^8 = \left(2\sqrt{2}\right)^8\,\text{cis}\left(8 \times \frac{\pi}{4}\right)$$

$$= 4096\,\text{cis}(2\pi)$$

$$= 4096\,\text{cis}(0)$$

$$= 4096$$

b $z = -3\sqrt{3} + 3i$

$$x = -3\sqrt{3} \qquad y = 3$$

$$r = \sqrt{\left(-3\sqrt{3}\right)^2 + (3)^2}$$

$$= 6$$

$$\text{Arg}(z) = \pi - \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$

$$= \pi - \frac{\pi}{6}$$

$$= \frac{5\pi}{6}$$

$$z = -3\sqrt{3} + 3i$$

$$= 6\,\text{cis}\left(\frac{5\pi}{6}\right)$$

$$z^6 = 6^6\,\text{cis}\left(6 \times \frac{5\pi}{6}\right)$$

$$= 6^6\,\text{cis}(5\pi)$$

$$= 46\,656\,\text{cis}(5\pi - 4\pi)$$

$$= 46\,656\,\text{cis}(\pi)$$

$$= -46\,656$$

c $z = -\frac{5}{2} - \frac{5\sqrt{3}}{2}i$

$$x = -\frac{5}{2} \qquad y = -\frac{5\sqrt{3}}{2}$$

$$r = \sqrt{\left(-\frac{5}{2}\right)^2 + \left(-\frac{5\sqrt{3}}{2}\right)^2}$$

$$= 5$$

$$\begin{aligned}\operatorname{Arg}(z) &= -\pi + \tan^{-1}\left(\sqrt{3}\right)\\ &= -\pi + \frac{\pi}{3}\\ &= -\frac{2\pi}{3}\end{aligned}$$

$$z = 5\operatorname{cis}\left(-\frac{2\pi}{3}\right)$$

$$\begin{aligned}z^9 &= 5^9\operatorname{cis}\left(9 \times -\frac{2\pi}{3}\right)\\ &= 5^9\operatorname{cis}(-4\pi)\\ &= 1\,953\,125\operatorname{cis}(0)\\ &= 1\,953\,125\end{aligned}$$

d $z = 2\sqrt{3} - 2i$

$x = 2\sqrt{3} \qquad y = -2$

$$\begin{aligned}r &= \sqrt{\left(2\sqrt{3}\right)^2 + (-2)^2}\\ &= 4\end{aligned}$$

$$\begin{aligned}\operatorname{Arg}(z) &= \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\\ &= -\frac{\pi}{6}\end{aligned}$$

$$z = 4\operatorname{cis}\left(-\frac{\pi}{6}\right)$$

$$\begin{aligned}z^7 &= 4^7\operatorname{cis}\left(-\frac{7\pi}{6}\right)\\ &= 4^7\operatorname{cis}\left(-\frac{7\pi}{6} + 2\pi\right)\\ &= 16\,384\operatorname{cis}\left(\frac{5\pi}{6}\right)\\ &= 16\,384\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)\\ &= 16\,384\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)\\ &= -8192\sqrt{3} + 8192i\end{aligned}$$

16 $u = \frac{1}{2}\left(\sqrt{3} - i\right)$

$= \operatorname{cis}\left(-\frac{\pi}{6}\right)$

a
$$\begin{aligned}\overline{u} &= \operatorname{cis}\left(\frac{\pi}{6}\right)\\ &= \frac{1}{2}\left(\sqrt{3} + i\right)\end{aligned}$$

$$\begin{aligned}\left(\frac{1}{u}\right) &= \operatorname{cis}\left(\frac{\pi}{6}\right)\\ &= \frac{1}{2}\left(\sqrt{3} + i\right)\end{aligned}$$

$$\begin{aligned}u^6 &= \operatorname{cis}(-\pi)\\ &= \operatorname{cis}(\pi)\\ &= -1\end{aligned}$$

b $\operatorname{Arg}(\overline{u}) = \frac{\pi}{6}$

$\operatorname{Arg}\left(\frac{1}{u}\right) = \frac{\pi}{6}$

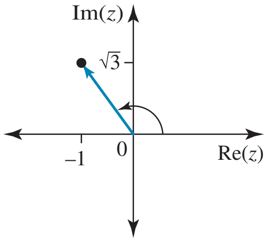
$\operatorname{Arg}(u^6) = \pi$

c $\operatorname{Arg}(\overline{u}) = -\operatorname{Arg}(u)$ Yes

d $\operatorname{Arg}\left(\frac{1}{u}\right) = -\operatorname{Arg}(u)$ Yes

e $\operatorname{Arg}(u^6) = 6\operatorname{Arg}(u)$ No

17 **a**



$u = -1 + \sqrt{3}i$

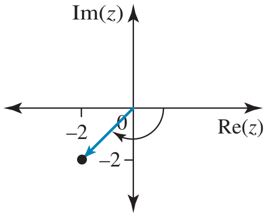
$x = -1 \qquad y = \sqrt{3}$

$$\begin{aligned}r &= \sqrt{1 + 3}\\ &= 2\end{aligned}$$

$$\begin{aligned}\theta &= \operatorname{Arg}(u)\\ &= \pi - \tan^{-1}\left(\sqrt{3}\right)\\ &= \pi - \frac{\pi}{3}\\ &= \frac{2\pi}{3}\end{aligned}$$

$$\begin{aligned}u &= -1 + \sqrt{3}i\\ &= 2\operatorname{cis}\left(\frac{2\pi}{3}\right)\end{aligned}$$

b



$v = -2 - 2i$

$x = -2 \qquad y = -2$

$$\begin{aligned}r &= \sqrt{4 + 4}\\ &= 2\sqrt{2}\end{aligned}$$

$$\begin{aligned}\theta &= \operatorname{Arg}(v)\\ &= -\pi - \tan^{-1}(1)\\ &= -\pi + \frac{\pi}{4}\\ &= -\frac{3\pi}{4}\end{aligned}$$

$$\begin{aligned}v &= -2 - 2i\\ &= 2\sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4}\right)\end{aligned}$$

c
$$\begin{aligned}uv &= 2\operatorname{cis}\left(\frac{2\pi}{3}\right) \times 2\sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4}\right)\\ &= 4\sqrt{2}\operatorname{cis}\left(\frac{2\pi}{3} - \frac{3\pi}{4}\right)\\ &= 4\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{12}\right)\end{aligned}$$

$$\Rightarrow \operatorname{Arg}(uv) = -\frac{\pi}{12}$$

d $$\begin{aligned}\frac{u}{v} &= \frac{2\operatorname{cis}\left(\frac{2\pi}{3}\right)}{2\sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4}\right)}\\ &= \frac{1}{\sqrt{2}}\operatorname{cis}\left(\frac{2\pi}{3}+\frac{3\pi}{4}\right)\\ &= \frac{\sqrt{2}}{2}\operatorname{cis}\left(\frac{17\pi}{12}-2\pi\right)\\ &= \frac{\sqrt{2}}{2}\operatorname{cis}\left(-\frac{7\pi}{12}\right)\end{aligned}$$

$$\operatorname{Arg}\left(\frac{u}{v}\right) = -\frac{7\pi}{12}$$

so $\operatorname{Arg}(u) = \frac{2\pi}{3}$, $\operatorname{Arg}(v) = -\frac{3\pi}{4}$

e $\operatorname{Arg}(uv) = \operatorname{Arg}(u) + \operatorname{Arg}(v)$

In this case, yes, but not in general.

f $\operatorname{Arg}\left(\frac{u}{v}\right) = -\frac{7\pi}{12} \neq \operatorname{Arg}(u) - \operatorname{Arg}(v)$

No

18 $u = \sqrt{2}(1-i)$

$x = \sqrt{2} \qquad y = -\sqrt{2}$

$$\begin{aligned}r &= \sqrt{2+2}\\ &= 2\end{aligned}$$

$$\begin{aligned}\theta &= \operatorname{Arg}(u)\\ &= \tan^{-1}(-1)\\ &= -\frac{\pi}{4}\end{aligned}$$

$$\begin{aligned}u &= \sqrt{2}(1-i)\\ &= 2\operatorname{cis}\left(-\frac{\pi}{4}\right)\end{aligned}$$

$$\begin{aligned}v &= 2\operatorname{cis}\left(\frac{2\pi}{3}\right)\\ &= 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)\\ &= \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\\ &= -1+\sqrt{3}i\end{aligned}$$

a $$\begin{aligned}uv &= \left(2-\sqrt{2}i\right)\left(-1+\sqrt{3}i\right)\\ &= -\sqrt{2}+\sqrt{2}i+\sqrt{6}i-\sqrt{6}i^2\\ &= \left(\sqrt{6}-\sqrt{2}\right)+\left(\sqrt{6}+\sqrt{2}\right)i\end{aligned}$$

b $$\begin{aligned}uv &= 2\operatorname{cis}\left(-\frac{\pi}{4}\right)\times 2\operatorname{cis}\left(\frac{2\pi}{3}\right)\\ &= 4\operatorname{cis}\left(-\frac{\pi}{4}+\frac{2\pi}{3}\right)\\ &= 4\operatorname{cis}\left(\frac{5\pi}{12}\right)\end{aligned}$$

c $uv = \cos\left(\frac{5\pi}{12}\right) + 4i\sin\left(\frac{5\pi}{12}\right)$

Equating imaginary parts

$$4\sin\left(\frac{5\pi}{12}\right) = \sqrt{6}+\sqrt{2}$$

$$\sin\left(\frac{5\pi}{12}\right) = \frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$$

d $$\begin{aligned}\sin\left(\frac{5\pi}{12}\right) &= \sin\left(\frac{2\pi}{3}-\frac{\pi}{4}\right)\\ &= \sin\left(\frac{2\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{2\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)\\ &= \frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2} - \frac{1}{2}\times\frac{\sqrt{2}}{2}\\ &= \frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)\end{aligned}$$

19 $u = -4-4\sqrt{3}i$

$x = -4 \qquad y = -4\sqrt{3}$

$$\begin{aligned}r &= \sqrt{16+48}\\ &= 8\end{aligned}$$

$$\begin{aligned}\theta &= \operatorname{Arg}(u)\\ &= -\pi + \tan^{-1}\left(\sqrt{3}\right)\\ &= -\pi + \frac{\pi}{3}\\ &= -\frac{2\pi}{3}\end{aligned}$$

$$\begin{aligned}u &= -4-4\sqrt{3}i\\ &= 8\operatorname{cis}\left(-\frac{2\pi}{3}\right)\end{aligned}$$

$$\begin{aligned}v &= \sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4}\right)\\ &= -1-i\end{aligned}$$

a $$\begin{aligned}\frac{u}{v} &= \frac{-4-4\sqrt{3}i}{-1-i} = \frac{4\left(1+\sqrt{3}i\right)}{1+i}\times\frac{1-i}{1-i}\\ &= \frac{4\left(1+\sqrt{3}i - i\sqrt{3}i^2\right)}{1-i^2}\\ &= 2\left(\sqrt{3}+1\right) + 2\left(\sqrt{3}-1\right)i\end{aligned}$$

b $$\begin{aligned}\frac{u}{v} &= \frac{8\operatorname{cis}\left(-\frac{2\pi}{3}\right)}{\sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4}\right)}\\ &= \frac{8}{\sqrt{2}}\operatorname{cis}\left(-\frac{2\pi}{3}+\frac{3\pi}{4}\right)\times\frac{\sqrt{2}}{\sqrt{2}}\\ &= 4\sqrt{2}\operatorname{cis}\left(\frac{\pi}{12}\right)\end{aligned}$$

c $\frac{u}{v} = 4\sqrt{2}\cos\left(\frac{\pi}{12}\right) + i4\sqrt{2}\sin\left(\frac{\pi}{12}\right)$

Equating real parts

$$4\sqrt{2}\cos\left(\frac{\pi}{12}\right) = 2\left(\sqrt{3}+1\right)$$

$$\begin{aligned}\cos\left(\frac{\pi}{12}\right) &= \frac{2}{4\sqrt{2}}\left(\sqrt{3}+1\right)\times\frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right) \qquad \text{shown}\end{aligned}$$

d $$\begin{aligned}\cos\left(\frac{\pi}{12}\right) &= \cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\\ &= \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)\\ &= \frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\times\frac{1}{2}\\ &= \frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)\end{aligned}$$

20 $u = -1 - \sqrt{3}i$

$x = -1 \qquad y = -\sqrt{3}$

$r = \sqrt{1+3}$

$= 2$

$\theta = \text{Arg}(u)$

$= -\pi + \tan^{-1}\left(\sqrt{3}\right)$

$= -\pi + \dfrac{\pi}{3}$

$= -\dfrac{2\pi}{3}$

$u = -1 - \sqrt{3}i$

$= 2\,\text{cis}\left(-\dfrac{2\pi}{3}\right)$

$v = \sqrt{2}\,\text{cis}\left(\dfrac{3\pi}{4}\right)$

$= \sqrt{2}\left(\cos\left(\dfrac{3\pi}{4}\right) + i\sin\left(\dfrac{3\pi}{4}\right)\right)$

$= \sqrt{2\left(-\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i\right)}$

$= -1 + i$

$= \sqrt{2}\,\text{cis}\left(\dfrac{3\pi}{4}\right)$

a $\dfrac{u}{v} = \dfrac{-1-\sqrt{3}i}{-1+i} = \dfrac{1+\sqrt{3}i}{1-i} \times \dfrac{1+i}{1+i}$

$= \dfrac{1+\sqrt{3}i + i + \sqrt{3}i^2}{1-i^2}$

$= \dfrac{1}{2}\left(1-\sqrt{3}\right) + \dfrac{1}{2}\left(\sqrt{3}+1\right)i$

b $\dfrac{u}{v} = \dfrac{2\,\text{cis}\left(-\frac{2\pi}{3}\right)}{\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)}$

$= \dfrac{2}{\sqrt{2}}\,\text{cis}\left(-\dfrac{2\pi}{3} - \dfrac{3\pi}{4}\right)$

$= \sqrt{2}\,\text{cis}\left(-\dfrac{17\pi}{12} + 2\pi\right)$

$= \sqrt{2}\,\text{cis}\left(\dfrac{7\pi}{12}\right)$

c $\dfrac{u}{v} = \sqrt{2}\cos\left(\dfrac{7\pi}{12}\right) + i\sqrt{2}\sin\left(\dfrac{7\pi}{12}\right)$

Equating real parts

$\sqrt{2}\cos\left(\dfrac{7\pi}{12}\right) = \dfrac{1}{2}\left(1-\sqrt{3}\right)$

$\cos\left(\dfrac{7\pi}{12}\right) = \dfrac{1}{2}\left(1-\sqrt{3}\right) \times \dfrac{1}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}}$

$= \dfrac{1}{4}\left(\sqrt{2}-\sqrt{6}\right)$ shown

d $\cos\left(\dfrac{7\pi}{12}\right) = \cos\left(\dfrac{3\pi}{4} - \dfrac{\pi}{6}\right)$

$= \cos\left(\dfrac{3\pi}{4}\right)\cos\left(\dfrac{\pi}{6}\right) + \sin\left(\dfrac{3\pi}{4}\right)\sin\left(\dfrac{\pi}{6}\right)$

$= \dfrac{-\sqrt{2}}{2} \times \dfrac{\sqrt{3}}{2} + \dfrac{\sqrt{2}}{2} \times \dfrac{1}{2}$

$= \dfrac{1}{4}\left(\sqrt{2}-\sqrt{6}\right)$

21 a $\tan(2A) = \dfrac{2\tan(A)}{1-\tan^2(A)}$

$A = \dfrac{\pi}{8} \qquad 2A = \dfrac{\pi}{4} \qquad$ let $a = \tan\left(\dfrac{\pi}{8}\right)$

$\dfrac{2a}{1-a^2} = 1$

$1 - a^2 = 2a$

$a^2 + 2a = 1$

$a^2 + 2a + 1 = 2$

$(a+1)^2 = 2$

$a + 1 = \pm\sqrt{2}$ but $a = \tan\left(\dfrac{\pi}{8}\right) > 0,$ take positive

$a = \tan\left(\dfrac{\pi}{8}\right)$

$= \sqrt{2} - 1$ shown

b $u = 1 + \left(\sqrt{2}-1\right)i$

u is in the first quadrant

$\text{Arg}(u) = \tan^{-1}\left(\sqrt{2}-1\right)$

$= \dfrac{\pi}{8}$

c $u = i + \left(\sqrt{2}-1\right)i^2$

$= 1 - \sqrt{2} + i$

iu is a rotation of $\dfrac{\pi}{2}$ anticlockwise from u

so $\text{Arg}(iu) = \text{Arg}\left(1-\sqrt{2}\right) + i$

$= \dfrac{\pi}{2} + \dfrac{\pi}{8}$

$= \dfrac{5\pi}{8}$

22 a $\tan\left(\dfrac{7\pi}{12}\right) = \tan\left(\dfrac{3\pi}{4} - \dfrac{\pi}{6}\right)$

$= \dfrac{\tan\left(\frac{3\pi}{4}\right) - \tan\left(\frac{\pi}{6}\right)}{1 + \tan\left(\frac{3\pi}{4}\right)\tan\left(\frac{\pi}{6}\right)}$

$= \dfrac{-1 - \frac{\sqrt{3}}{3}}{1 - \frac{\sqrt{3}}{3}}$

$= \dfrac{\frac{-3-\sqrt{3}}{3}}{\frac{3-\sqrt{3}}{3}}$

$= \dfrac{-3-\sqrt{3}}{3-\sqrt{3}} \times \dfrac{-3+\sqrt{3}}{3+\sqrt{3}}$

$= \dfrac{-9-6\sqrt{3}+3}{9-3}$

$= -\left(\sqrt{3}+2\right)$

b Let $u = -1 + \left(\sqrt{3}+2\right)i$

u is in the second quadrant

$\text{Arg}(u) = \text{Arg}\left(-1 + \left(\sqrt{3}+2\right)i\right)$

$= \dfrac{7\pi}{12}$

c $iu = -i + \left(\sqrt{3}+2\right)i^2$

$= -\left(\sqrt{3}+2\right) - i$

$i^2u = 1 - \left(\sqrt{3}+2\right)i$ is a rotation of 180° anticlockwise

$$\begin{aligned}\text{so Arg}\left(1-\left(\sqrt{3}+2\right)i\right) &= \frac{7\pi}{12}+\pi-2\pi\\ &= -\frac{5\pi}{12}\end{aligned}$$

d $i^3u = -iu$

$$= \sqrt{3}+2+i$$

Is a rotation of 270° anticlockwise

$$\begin{aligned}\text{so Arg}\left(\sqrt{3}+2+i\right) &= \frac{7\pi}{12}+\frac{3\pi}{2}-2\pi\\ &= \frac{\pi}{12}\end{aligned}$$

23 a $0 = (1+i)^n + (1-i)^n$

$$0 = \left(\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)\right)^n + \left(\sqrt{2}\,\text{cis}\left(\frac{-\pi}{4}\right)\right)^n$$

$$0 = \left(\sqrt{2}\right)^n \text{cis}\left(\frac{n\pi}{4}\right) + \left(\sqrt{2}\right)^4 \text{cis}\left(\frac{-n\pi}{4}\right)$$

$$0 = \left(\sqrt{2}\right)^n \left(\cos\left(\frac{n\pi}{4}\right) + i\sin\left(\frac{n\pi}{4}\right) + \cos\left(\frac{n\pi}{4}\right) + i\,\sin\left(\frac{-n\pi}{4}\right)\right)$$

$$0 = 2\left(\sqrt{2}\right)^n \cos\left(\frac{n\pi}{4}\right)$$

$$\Rightarrow \cos\left(\frac{n\pi}{4}\right) = 0$$

$$\frac{n\pi}{4} = (2k\pi)\,\frac{\pi}{2}$$

$$n = 2\,(2k+1) \qquad k\in Z$$

b $(1+i)^n - (1-i)^n = 0$

$$2\left(\sqrt{2}\right)^n i\sin\left(\frac{n\pi}{4}\right) = 0$$

$$\sin\left(\frac{n\pi}{4}\right) = 0$$

$$\frac{n\pi}{4} = k\pi$$

$$n = 4k \qquad k\in Z$$

24 $z = \text{cis}(\theta)$

$$= \cos(\theta) + i\sin(\theta)$$

a LHS: $z + 1 = \cos(\theta) + 1 + i\sin(\theta)$

$$\begin{aligned}|z+1| &= \sqrt{(1+\cos(\theta))^2 + (\sin(\theta))^2}\\ &= \sqrt{1+2\cos(\theta)+\cos^2(\theta)+\sin^2(\theta)}\\ &= \sqrt{2+2\cos(\theta)}\\ &= \sqrt{2(1+\cos(\theta))}\\ &= \sqrt{2\times 2\,\cos^2\left(\frac{\theta}{2}\right)}\\ &= 2\cos\left(\frac{\theta}{2}\right)\\ &= \text{RHS}\end{aligned}$$

b LHS: $\text{Arg}(1+z) = \tan^{-1}\left(\dfrac{\sin(\theta)}{1+\cos(\theta)}\right)$

$$\begin{aligned}&= \tan^{-1}\left(\frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\theta}{2}\right)}\right)\\ &= \tan^{-1}\left(\tan\left(\frac{\theta}{2}\right)\right)\\ &= \left(\frac{\theta}{2}\right)\\ &= \text{RHS}\end{aligned}$$

25 a The identity $z\bar{z} = |z|^2$ will be used in the following proof.

$|z_1z_2| = |z_1|\,|z_2|$

$$\begin{aligned}\text{LHS} &= |z_1z_2|\\ \text{LHS}^2 &= |z_1z_2|^2 \quad \text{Raise to the power of 2}\\ &= (z_1z_2)\,(\bar{z}_1\bar{z}_2) \ \text{ from identity } z\bar{z} = |z|^2\\ &= (z_1\bar{z}_1)\,(z_2\bar{z}_2)\\ &= |z_1|^2|z_2|^2\\ &= (|z_1|\,|z_2|)^2\\ &= (|z_1|\,|z_2|)\ \text{Raise to the power of } \frac{1}{2}\end{aligned}$$

$\therefore$ LHS = RHS

Hence, $|z_1z_2| = |z_1|\,|z_2|$

b Let, $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Then,

$\arg(z_1) = \theta_1$, $\arg(z_2) = \theta_2$

Since,

$z_1 \times z_2 = r_1r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$

Then,

$\arg(z_1z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$

26 a $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

Let $z_1 = |z_1|\ \text{cis}(\theta_1)$ and $z_2 = |z_2|\ \text{cis}(\theta_2)$

$\therefore\ \arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$

$$\begin{aligned}\text{LHS} &= \left|\frac{z_1}{z_2}\right|\\ &= \left|\frac{|z_1|\ \text{cis}(\theta_1)}{|z_2|\ \text{cis}(\theta_2)}\right|\\ &= \left|\frac{|z_1|}{|z_2|}\ \text{cis}(\theta_1-\theta_2)\right|\\ &= \frac{|z_1|}{|z_2|}\\ &= \text{RHS}\end{aligned}$$

$\therefore$LHS = RHS

b $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Then,

$\arg(z_1) = \theta_1$, $\arg(z_2) = \theta_2$

Since,

$z_1 \div z_2 = r_1 \div r_2\left(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right)$

Then,

$\arg\left(\dfrac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$

27 a $|z^n| = |z|^n$

Let $z = |z|\ \text{cis}(\theta)$

$$\begin{aligned}\text{LHS} &= |z^n|\\ &= |(|z|\ \text{cis}(\theta))^n|\\ &= |(|z|^n\ \text{cis}(n\times\theta))|\\ &= |z|^n\\ &= \text{RHS}\end{aligned}$$

$\therefore$ LHS = RHS

b $\arg(z^n) = n\arg(z)$

$$\begin{aligned}\text{LHS} &= \arg(z^n)\\ &= \arg((r\,\text{cis}(\theta))^n)\\ &= \arg(r^n\,\text{cis}(n\,\theta))\\ &= n\,\theta\end{aligned}$$

$$\begin{aligned}\text{RHS} &= n\arg(z)\\ &= n \times \arg(r\,cis(\theta))\\ &= n\times\theta\\ &= n\theta\\ &= \text{LHS}\end{aligned}$$

$\therefore$ LHS = RHS

28 a $\left|\dfrac{1}{z^n}\right| = \dfrac{1}{|z|^n}$

$$\begin{aligned}z^n &= (r\,\text{cis}(\theta))^n\\ &= r^n\,\text{cis}(n\theta), \text{ by de Moivre's theorem}\end{aligned}$$

$$\begin{aligned}\text{LHS} &= \left|\frac{1}{z^n}\right|\\ &= \left|\frac{1}{(r\,\text{cis}(\theta))^n}\right|\\ &= \left|\frac{1}{r^n\,\text{cis}(n\,\theta)}\right|\\ &= \frac{1}{r^n}\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \frac{1}{|z|^n}\\ &= \frac{1}{|r\,\text{cis}(\theta)|^n}\\ &= \frac{1}{r^n}\\ &= \text{LHS}\end{aligned}$$

b $\arg\left(\dfrac{1}{z^n}\right) = -n\ \arg(z),\ z \neq 0$

$$\begin{aligned}\text{LHS} &= \arg\left(\frac{1}{z^n}\right)\\ &= \arg\left(\frac{1}{(r\,\text{cis}(\theta))^n}\right)\\ &= \arg((r\,\text{cis}(\theta))^{-n})\\ &= \arg(r^{-n}\,\text{cis}(-n\theta))\\ &= -n\,\theta\end{aligned}$$

$$\begin{aligned}\text{RHS} &= -n\ \arg(z)\\ &= -n\times\arg(r\,\text{cis}(\theta))\\ &= -n\times\theta\\ &= -n\,\theta\\ &= \text{LHS}\end{aligned}$$

$\therefore$ LHS = RHS

29

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{r_1(\cos(\theta_1)+i\sin(\theta_1))}{r_2(\cos(\theta_2)+i\sin(\theta_2))}\\ &= \frac{r_1(\cos(\theta_1)+i\sin(\theta_1))}{r_2(\cos(\theta_2)+i\sin(\theta_2))}\times\frac{(\cos(\theta_2)-i\sin(\theta_2))}{(\cos(\theta_2)-i\sin(\theta_2))}\\ &= \frac{r_1(\cos(\theta_1)\times\cos(\theta_2)+\cos(\theta_1)\times -i\sin(\theta_2)+i\sin(\theta_1)\times\cos(\theta_2)+i\sin(\theta_1)\times -i\sin(\theta_2))}{r_2(\cos(\theta_2)\times\cos(\theta_2)+\cos(\theta_2)\times -i\sin(\theta_2)+i\sin(\theta_2)\times\cos(\theta_2)+i\sin(\theta_2)\times -i\sin(\theta_2))}\\ &= \frac{r_1(\cos(\theta_1)\cos(\theta_2)-i\cos(\theta_1)\sin(\theta_2)+i\cos(\theta_2)\sin(\theta_1)+\sin(\theta_1)\sin(\theta_2))}{r_2(\cos^2(\theta_2)-\cancel{i\cos(\theta_2)\sin(\theta_2)}+\cancel{i\cos(\theta_2)\sin(\theta_2)}+\sin^2(\theta_2))}\\ &= \frac{r_1(\cos(\theta_1)\cos(\theta_2)+\sin(\theta_1)\sin(\theta_2)+i(\cos(\theta_2)\sin(\theta_1)-\cos(\theta_1)\sin(\theta_2)))}{r_2(\cos^2(\theta_2)+\sin^2(\theta_2))}\end{aligned}$$

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{r_1(\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2))}{r_2}\\ &= \frac{r_1}{r_2}\text{cis}(\theta_1-\theta_2)\end{aligned}$$

30 Let $P(n)$ be the proposition that $(r\,\text{cis}(\theta))^n = r^n\text{cis}(n\theta)$, $n \in N$, $n \geq 1$.
RTP: Both $P(1)$ is true and $P(k+1)$ is given $P(k)$ is true.

$$\begin{aligned} n = 1;\ \text{LHS} &= (r\,\text{cis}(\theta))^1 \\ &= r\,\text{cis}(\theta) \\ \text{RHS} &= r^1\text{cis}(1 \times \theta) \\ &= r\,\text{cis}(\theta) \end{aligned}$$

LHS = RHS $\Rightarrow P(1)$ is true.
Assume $P(k)$ is true $k \in N, k \geq 1$
$(r\,\text{cis}(\theta))^k = r^k\text{cis}(k\theta)$
$\left[r(\cos(\theta) + i\sin(\theta))\right]^k = r^k(\cos(k\theta) + i\sin(k\theta))$
RTP: $(r\,\text{cis}(\theta))^{k+1} = r^{k+1}\text{cis}((k+1)\theta)$
LHS:

$$\begin{aligned} (r\,\text{cis}(\theta))^{k+1} &= (r\,\text{cis}(\theta))^k \times (r\,\text{cis}(\theta))^1 \\ &= r^k\text{cis}(k\theta) \times r^1\text{cis}(\theta)^1 \\ &= r^{k+1}(\cos(k\theta) + i\sin(k\theta))\,(\cos(\theta) + i\sin(\theta)) \\ &= r^{k+1}\begin{pmatrix} \cos(k\theta)\cos(\theta) + \cos(k\theta)i\sin(\theta) \\ +i\sin(k\theta)\cos(\theta) + i^2\sin(k\theta)\sin(\theta) \end{pmatrix} \\ &= r^{k+1}\begin{pmatrix} (\cos(k\theta)\cos(\theta) - \sin(k\theta)\sin(\theta)) \\ +i\begin{pmatrix} \cos(k\theta)\sin(\theta) \\ +\sin(k\theta)\cos(\theta) \end{pmatrix} \end{pmatrix} \\ &= r^{k+1}(\cos(k\theta + \theta) + i\sin(k\theta + \theta)) \\ &= r^{k+1}(\cos((k+1)\theta) + i\sin((k+1)\theta)) \\ &= r^{k+1}\text{cis}((k+1)\theta) \\ &= \text{RHS of } P(k+1) \end{aligned}$$

That is $P(k+1)$ is true whenever $P(k)$ is true.
By the principle of mathematical induction it is true for all values of n. That is, $(r\,\text{cis}(\theta))^n = r^n\text{cis}(n\theta)$ is true for all values of $n \in N, n \geq 1$.

2.3 Exam questions

1 $z = a + bi$, $b = \text{Im}(z) > 0$, $\bar{z} = a - bi$

$$\frac{\bar{z}z}{z - \bar{z}} = \frac{a^2 - b^2i^2}{2bi} = \frac{a^2 + b^2}{2bi} \times \frac{i}{i} = \frac{-\left(a^2 + b^2\right)}{2b}i$$

$$\text{Re}\left(\frac{\bar{z}z}{z - \bar{z}}\right) = 0, \quad \text{Im}\left(\frac{\bar{z}z}{z - \bar{z}}\right) = \frac{-\left(a^2 + b^2\right)}{2b} < 0$$

$$\text{Arg}\left(\frac{\bar{z}z}{z - \bar{z}}\right) = -\frac{\pi}{2}$$

The correct answer is **A**.

2 $z = r\,\text{cis}(\theta)$, $z^2 = r^2\,\text{cis}(2\theta)$
$\text{Im}(z^2) = 0 \quad \Rightarrow \quad \sin(2\theta) = 0$

$$\arg(z) = \theta = \frac{k\pi}{2}, \quad k \in Z$$

The correct answer is **A**.

3 **a** $z = 3 - \sqrt{3}i$

$$|z| = \sqrt{(3)^2 + \left(-\sqrt{3}\right)^2} = \sqrt{12} = 2\sqrt{3}$$

$$\text{Arg}(z) = \tan^{-1}\left(\frac{-\sqrt{3}}{3}\right) = -\frac{\pi}{6}$$

$$z = 2\sqrt{3}\,\text{cis}\left(-\frac{\pi}{6}\right)$$

Award 1 mark for correctly converting between the two forms.

VCAA Examination Report **note:**

Students were required to show that $3 - \sqrt{3}i = 2\sqrt{3}\,\text{cis}\left(-\frac{\pi}{6}\right)$ and students generally did this quite well. Some particular errors were noted. Some students wrote such things as $\tan\left(\frac{\sqrt{3}}{3}\right)$ or $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6} = -\frac{\pi}{6}$.

b $z^3 = \left(3 - \sqrt{3}i\right)^3 = \left(2\sqrt{3}\,\text{cis}\left(-\frac{\pi}{6}\right)\right)^3$

$z^3 = \left(2\sqrt{3}\right)^3 \text{cis}\left(-\frac{3\pi}{6}\right) = 24\sqrt{3}\,\text{cis}\left(-\frac{\pi}{2}\right)$

$z^3 = 24\sqrt{3}\left(\cos\left(-\frac{\pi}{2}\right) + i\ \sin\left(\frac{\pi}{2}\right)\right) = 0 - 24\sqrt{3}i$

Award 1 mark for using De Moivre's Theorem.
Award 1 mark for the correct answer.

VCAA Examination Report **note:**

The efficient method was to use de Moivre's theorem although some students attempted to expand $\left(3 - \sqrt{3}i\right)^3$. Students who chose the latter approach generally did not score as well.

c $\text{Im}\left(\left(3 - \sqrt{3}i\right)^n\right) = \left(2\sqrt{3}\right)^n \sin\left(-\frac{n\pi}{6}\right) = 0$

$\sin\left(-\frac{n\pi}{6}\right) = 0,\ \frac{n\pi}{6} = k\pi$

$n = 6k,\ k \in Z$ [1 mark]

n is an integer multiple of 6.

VCAA Examination Report **note:**

There were several ways to answer this question. Some students realised that if n was a positive or negative multiple of 6 then $\left(3 - \sqrt{3}i\right)^n$ was real, but were unable to express this mathematically. Some students did not indicate that k was a member of Z, the set of integers.

d $\left(3 - \sqrt{3}i\right)^n = ai,\ a \in R$

$\text{Re}\left(\left(3 - \sqrt{3}i\right)^n\right) = \left(2\sqrt{3}\right)^n \cos\left(-\frac{n\pi}{6}\right) = 0$

$\cos\left(\frac{n\pi}{6}\right) = 0,\ \frac{n\pi}{6} = \frac{\pi}{2} + k\pi$

$n = 6k + 3,\ k \in Z$ [1 mark]

n is an odd integer multiple of 3.

VCAA Examination Report **note:**

This question was answered poorly. There were a number of equivalent correct answers but many students were unable to find a general solution.

2.4 Solving polynomial equations over *C*

2.4 Exercise

1 $0 = z^2 - 14z + 74$

$0 = z^2 - 14z + 49 - 49 + 74$

$0 = (z - 7)^2 + 25$

$0 = (z - 7)^2 - 25i^2$

$0 = (z - 7)^2 - (5i)^2$

$0 = (z - 7 + 5i)(z - 7 - 5i)$

The roots occur as a pair complex conjugates:

$z = 7 + 5i,\ 7 - 5i$

2 $0 = 4z^2 + 4z + 82$

$0 = 4\left(z^2 + z + \frac{41}{2}\right)$

$0 = z^2 + z + \frac{41}{2}$

$0 = z^2 + z + \frac{1}{4} - \frac{1}{4} + \frac{41}{2}$

$0 = \left(z + \frac{1}{2}\right)^2 + \frac{81}{4}$

$0 = \left(z + \frac{1}{2}\right)^2 - \frac{81}{4}i^2$

$0 = \left(z + \frac{1}{2}\right)^2 - \left(\frac{9}{2}i\right)^2$

$0 = \left(z + \frac{1}{2} + \frac{9}{2}i\right)\left(z + \frac{1}{2} - \frac{9}{2}i\right)$

The roots occur as a pair complex conjugates:

$z = -\frac{1}{2} + \frac{9}{2}i,\ -\frac{1}{2} - \frac{9}{2}i$

3 $z(z + 4) = -29$

$z^2 + 4z + 29 = 0$

$a = 1,\ b = 4,\ c = 29$

$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$z = \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times 29}}{2 \times 1}$

$z = \frac{-4 \pm \sqrt{-100}}{2}$

$z = \frac{-4 \pm \sqrt{100i^2}}{2}$

$z = \frac{-4 \pm 10i}{2}$

$z = -2 \pm 5i$

$z = -2 + 5i,\ -2 - 5i$

4 $P(-1 + 13i) = 0$

Let $\alpha = -1 + 13i$

Let $\beta = -1 - 13i$

$P(z) = (z - \alpha)(z - \beta)$

$\quad = (z + 1 - 13i)(z + 1 + 13i)$

$P(z) = (z + 1)^2 - (13i)^2$

$\quad = z^2 + 2z + 1 - 169i^2$

$\quad = z^2 + 2z + 170$

$P(z) = z^2 + 2z + 170$

5 $P(6 - 5i) = 0$

Let $\alpha = 6 - 5i$

Let $\beta = 6 + 5i$

$P(z) = (z - \alpha)(z - \beta)$

$\quad = (z - 6 + 5i)(z - 6 - 5i)$

$P(z) = (z - 6)^2 - (5i)^2$

$\quad = z^2 - 12z + 36 - 25i^2$

$\quad = z^2 - 12z + 61$

$P(z) = z^2 - 12z + 61$

6 $z^3 + 9z^2 + 24z - 34 = 0$

Trial and error for the real root:

$P(1) = 1^3 + 9 \times 1^2 + 24 \times 1 - 34$

$\quad = 0$

Therefore $(z - 1)$ is a factor.

$P(z) = z^3 + 9z^2 + 24z - 34$

$\quad = (z - 1)\left(z^2 + bz + c\right)$

$-34 = -c$ [1]

$c = 34$

$24z = -bz + cz$ [2]

$24z = -bz + 34z$

$b = \frac{24z - 34z}{-z}$

$b = 10$

$$\begin{aligned}
P(z) &= z^3 + 9z^2 + 24z - 34\\
&= (z-1)\left(z^2 + 10z + 34\right)\\
0 &= (z-1)\left(z^2 + 10z + 25 - 25 + 34\right)\\
0 &= (z-1)\left((z+5)^2 + 9\right)\\
0 &= (z-1)\left((z+5)^2 - 9i^2\right)\\
0 &= (z-1)\left((z+5)^2 - (3i)^2\right)\\
0 &= (z-1)(z+5+3i)(z+5-3i)
\end{aligned}$$

The roots are one real and one pair of complex conjugates:
$z = 1, -5-3i, -5+3i$

7 $z^3 + 13z^2 + 97z + 85 = 0$

Trial and error for the real root:

$$\begin{aligned}
P(-1) &= (-1)^3 + 13(-1)^2 + 97(-1) + 85\\
&= 0
\end{aligned}$$

Therefore $(z+1)$ is a factor.

$$\begin{aligned}
P(z) &= z^3 + 13z^2 + 97z + 85\\
&= (z+1)\left(z^2 + bz + c\right)\\
85 &= c \qquad [1]\\
c &= 85\\
97z &= bz + cz \qquad [2]\\
97z &= bz + 85z\\
b &= \frac{97z - 85z}{z}\\
b &= 12\\
P(z) &= z^3 + 13z^2 + 97z + 85\\
&= (z+1)\left(z^2 + 12z + 85\right)\\
0 &= (z+1)\left(z^2 + 12z + 36 - 36 + 85\right)\\
0 &= (z+1)\left((z+6)^2 + 49\right)\\
0 &= (z+1)\left((z+6)^2 - 49i^2\right)\\
0 &= (z+1)\left((z+6)^2 - (7i)^2\right)\\
0 &= (z+1)(z+6+7i)(z+6-7i)
\end{aligned}$$

The roots are one real and one pair of complex conjugates:
$z = -1, -6-7i, -6+7i$

8 $z^3 + 4z^2 - 2z - 20 = 0$

Trial and error for the real root:

$$\begin{aligned}
P(2) &= (2)^3 + 4(2)^2 - 2(2) - 20\\
&= 0
\end{aligned}$$

Therefore $(z-2)$ is a factor.

$$\begin{aligned}
P(z) &= z^3 + 4z^2 - 2z - 20\\
&= (z-2)\left(z^2 + bz + c\right)\\
-20 &= -2c \qquad [1]\\
c &= 10\\
-2z &= -2bz + cz \qquad [2]\\
-2z &= -2bz + 10z\\
b &= \frac{-2z - 10z}{-2z}\\
b &= 6\\
P(z) &= z^3 + 4z^2 - 2z - 20\\
&= (z-2)\left(z^2 + 6z + 10\right)\\
0 &= (z-2)\left(z^2 + 6z + 9 - 9 + 10\right)\\
0 &= (z-2)\left((z+3)^2 + 1\right)\\
0 &= (z-2)\left((z+3)^2 - i^2\right)\\
0 &= (z-2)\left((z+3)^2 - (i)^2\right)\\
0 &= (z-2)(z+3+i)(z+3-i)
\end{aligned}$$

The roots are one real and one pair of complex conjugates:
$z = 2, -3-i, -3+i$

9

$$\begin{aligned}
\alpha &= -2 + 3i\\
\beta &= -2 - 3i\\
\alpha + \beta &= -4\\
\alpha\beta &= 4 - 9i^2\\
&= 13
\end{aligned}$$

so $z^2 + 4z + 13$ is a factor

$$\begin{aligned}
z^3 + bz^2 + cz - 39 &= 0\\
\left(z^2 + 4z + 13\right)(z-3) &= 0
\end{aligned}$$

$z^2\colon b = 4 - 3 = 1$

$z\colon c = 13 - 12 = 1$

All the roots are $-2 \pm 3i, 3$

10

$$\begin{aligned}
\alpha &= 5i\\
\beta &= -5i\\
\alpha + \beta &= 0\\
\alpha\beta &= -25i^2\\
&= 25
\end{aligned}$$

so $z^2 + 25$ is a factor

$$\begin{aligned}
z^3 + bz^2 + cz - 50 &= 0\\
\left(z^2 + 25\right)(z-2) &= 0
\end{aligned}$$

$z^2\colon b = -2$

$z\colon c = 25$

All the roots are $\pm 5i, 2$

11

$$\begin{aligned}
z^3 - 2iz^2 + 4z - 8i &= 0\\
z^2(z-2i) + 4(z-2i) &= 0\\
\left(z^2 + 4\right)(z-2i) &= 0\\
\left(z^2 - 4i^2\right)(z-2i) &= 0\\
(z+2i)(z-2i)^2 &= 0\\
z &= \pm 2i
\end{aligned}$$

12

$$\begin{aligned}
z^3 + 3iz^2 + 7z + 21i &= z^2(z+3i) + 7(z+3i)\\
&= \left(z^2 + 7\right)(z+3i)\\
&= \left(z^2 - 7i^2\right)(z+3i)\\
&= \left(z - \sqrt{7}i\right)\left(z + \sqrt{7}i\right)(z+3i)
\end{aligned}$$

13

$$\begin{aligned}
p(z) &= z^3 + (-2+3i)z^2 + 4z + 12i - 8\\
p(2-3i) &= (2-3i)^3 + (-2+3i)(2-3i)^2\\
&\quad + 4(2-3i) + 12i - 8\\
&= (2-3i)^3 - (2-3i)^3 + 8 - 12i + 12i - 8\\
&= 0 \text{ shown}
\end{aligned}$$

$\Rightarrow z - 2 + 3i$ is a factor

$$\begin{aligned}
z^3 + (-2+3i)z^2 + 4z + 12i - 8 &= 0\\
(z-2+3i)\left(z^2 + 4\right) &= 0\\
(z-2+3i)\left(z^2 - 4i^2\right) &= 0\\
(z-2+3i)(z+2i)(z-2i) &= 0
\end{aligned}$$

$z = \pm 2i, 2 - 3i$

14

$$\begin{aligned}
P(z) &= 2z^3 - (4i+3)z^2 + 10z - 20i - 15\\
P\left(\frac{3}{2} + 2i\right) &= 2\left(\frac{3}{2} + 2i\right)^3 - (4i+3)\left(\frac{3}{2} + 2i\right)^2\\
&\quad + 10\left(\frac{3}{2} + 2i\right) - 20i - 15\\
&= 2\left(\frac{3}{2} + 2i\right)^3 - 2\left(\frac{3}{2} + 2i\right)^3 + 15\\
&\quad + 20i - 20i - 15\\
&= 0 \text{ shown}
\end{aligned}$$

$\Rightarrow z - \frac{3}{2} - 2i$ is a factor

$$\begin{aligned} 2z^3 - (4i+3)z^2 + 10z - 20i - 15 &= 0 \\ (2z - 3 - 4i)\left(z^2 + 5\right) &= 0 \\ (2z - 3 - 4i)\left(z^2 - 5i^2\right) &= 0 \\ (2z - 3 - 4i)\left(z + \sqrt{5}i\right)\left(z - \sqrt{5}i\right) &= 0 \\ z &= \frac{3}{2} + 2i,\ \pm\sqrt{5}i \end{aligned}$$

15 $z^4 - z^2 - 20 = 0$

Let $u = z^2$

$0 = u^2 - u - 20$

$0 = (u-5)(u+4)$

$0 = \left(z^2 - 5\right)\left(z^2 + 4\right)$

$0 = \left(z^2 - 5\right)\left(z^2 - 4i^2\right)$

$0 = \left(z + \sqrt{5}\right)\left(z - \sqrt{5}\right)(z + 2i)(z - 2i)$

$z = \pm 2i,\ \pm\sqrt{5}$

16 $2z^4 - 3z^2 - 9 = 0$

$\left(2z^2 + 3\right)\left(z^2 - 3\right) = 0$

$z^2 = \frac{-3}{2}$ or $z^2 = 3$

$z^2 = \frac{3i^2}{2}$ or $z^2 = 3$

$z = \pm\frac{\sqrt{3}i}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}},\quad z = \pm\sqrt{3}$

$z = \pm\frac{\sqrt{6}i}{2},\ \pm\sqrt{3}$

17 **a** $0 = z^4 + 5z^2 - 36$

$0 = \left(z^2 + 9\right)\left(z^2 - 4\right)$

$0 = (z + 3i)(z - 3i)(z + 2)(z - 2)$

$z = \pm 3i, \pm 2$

b $0 = z^4 + 4z^2 - 21$

$0 = \left(z^2 - 3\right)\left(z^2 + 7\right)$

$0 = \left(z + \sqrt{3}\right)\left(z - \sqrt{3}\right)\left(z + \sqrt{7}i\right)\left(z - \sqrt{7}i\right)$

$z = \pm\sqrt{3}, \pm\sqrt{7}i$

18 **a** $0 = z^4 - 3z^2 - 40$

$0 = \left(z^2 + 5\right)\left(z^2 - 8\right)$

$0 = \left(z + \sqrt{5}i\right)\left(z - \sqrt{5}i\right)\left(z + 2\sqrt{2}\right)\left(z - 2\sqrt{2}\right)$

$z = \pm\sqrt{5}i, \pm 2\sqrt{2}$

b $0 = z^4 + 9z^2 + 18$

$0 = \left(z^2 + 6\right)\left(z^2 + 3\right)$

$0 = \left(z + \sqrt{6}i\right)\left(z - \sqrt{6}i\right)\left(z + \sqrt{3}i\right)\left(z - \sqrt{3}i\right)$

$z = \pm\sqrt{6}i, \pm\sqrt{3}i$

19 $P(z) = z^4 + az^3 + 34z^2 - 54z + 225$

$$\begin{aligned} P(3i) = (3i)^4 + a(3i)^3 - 54 \times 3i + 225 &= 0 \\ 0 &= 81i^4 + 27ai^3 + 306i^2 - 162i + 225 \\ 0 &= 81 - 27ai - 306 - 162i + 225 \\ 0 &= -(27a + 162)i \end{aligned}$$

$\Rightarrow 27a + 162 = 0$

$\Rightarrow a = -6$

or $P(3i) = 0$,

$P(-3i) = 0$

$\Rightarrow z^2 + 9$ is a factor

$z^4 + az^3 + 34z^2 - 54z + 225 = 0$

$\left(z^2 + 9\right)\left(z^2 + bz + 25\right) = 0$

$z^3 : a = b$

$z^2 : 34 = 9 + 25$

$z : -54 = 9b$

$\Rightarrow b = -6$

$\Rightarrow a = -6$

$$\begin{aligned} \left(z^2 + 9\right)\left(z^2 - 6z + 25\right) &= 0 \\ (z + 3i)(z - 3i)\left(z^2 - 6z + 9 + 16\right) &= 0 \\ (z + 3i)(z - 3i)\left((z-3)^2 - 16i^2\right) &= 0 \end{aligned}$$

$z = \pm 3i, 3 \pm 4i$

20 $\alpha = -3 - 4i$

$\beta = -3 + 4i$

$\alpha + \beta = -6$

$\alpha\beta = 9 - 16i^2$

$= 25$

So $z^2 + bz + 25$ is a factor

$z^4 + 6z^3 + 29z^2 + bz + 100 = 0$

$\left(z^2 + 6z + 25\right)\left(z^2 + cz + 4\right) = 0$

$z^3 : 6 = 6 + c = 1 \quad \Rightarrow c = 0$

$z^2 : 29 = 25 + 4 + 6c$

$z : b = 25c + 24 \quad \Rightarrow b = 24$

$\left(z^2 + 6z + 25\right)\left(z^2 + 4\right) = 0$

$z = -3 \pm 4i, \pm 2i$

21 $P(z) = z^4 + 10z^3 + 25z^2 - 90z - 306$

Given, $P(3) = 0$

Let $z_1 = 3$

Let $z_2 = -3$

$$\begin{aligned} P(z) &= (z - z_1)(z - z_2)(z - z_3)(z - z_4) \\ &= (z - 3)(z + 3)(z - z_3)(z - z_4) \\ &= \left(z^2 - 9\right)\left(az^2 + bz + c\right) \end{aligned}$$

$$\begin{array}{r|l} & z^2 + 10z + 34 \\ \cline{2-2} z^2 - 9 & z^4 + 10z^3 + 25z^2 - 90z - 306 \\ & -(z^4 \quad \downarrow \quad -9z^2 \quad \downarrow \quad \downarrow) \\ \cline{2-2} & \qquad 10z^3 + 34z^2 - 90z - 306 \\ & -(\qquad 10z^3 + \downarrow \quad -90z \quad \downarrow) \\ \cline{2-2} & \qquad\qquad 34z^2 \qquad -306 \\ & -(\qquad\quad 34z^2 \qquad -306) \\ \cline{2-2} & \qquad\qquad\qquad\qquad 0 \end{array}$$

$$\begin{aligned} \therefore P(z) &= \left(z^2 - 9\right)\left(z^2 + 10z + 34\right) \\ P(z) &= \left(z^2 - 9\right)\left(z^2 + 10z + 25 - 25 + 34\right) \\ &= \left(z^2 - 9\right)\left((z+5)^2 + 9\right) \\ &= \left(z^2 - 9\right)\left(\left((z+5)^2 - (3i)^2\right)\right) \\ 0 &= (z + 3)(z - 3)(z + 5 + 3i)(z + 5 - 3i) \\ z &= -3,\ 3,\ -5 - 3i,\ -5 + 3i \end{aligned}$$

22 $P(z) = z^3 + 8z^2 + 69z - 468$

$z = 4$ is a zero

Let $z_1 = 4$

$P(z) = (z-4)(z-z_2)(z-z_3)$

$= (z-4)\left(az^2 + bz + c\right)$

$$\begin{array}{r|l} & z^2 + 12z + 117 \\ \hline z-4 & z^3 + 8z^2 + 69z - 468 \\ & -(z^3 - 4z^2 \quad \downarrow \quad \downarrow\ ) \\ \hline & \quad 12z^2 + 69z - 468 \\ & -(\ \ 12z^2 - 48z \quad \downarrow\ ) \\ \hline & \qquad 117z - 468 \\ & -(\qquad 117z - 468) \\ \hline & \qquad\qquad 0 \end{array}$$

$\therefore P(z) = (z-4)\left(z^2 + 12z + 117\right)$

$P(z) = (z-4)\left(z^2 + 12z + 36 - 36 + 117\right)$

$= (z-4)\left((z+6)^2 + 81\right)$

$= (z-4)\left(\left((z+6)^2 - (9i)^2\right)\right)$

$0 = (z-4)(z+6+9i)(z+6-9i)$

$z = 4,\ -6-9i,\ -6+9i$

23 $P(z) = z^4 - 2z^3 - 6z^2 + 32z - 160$

$= (z-1-3i)(z-z_2)(z-z_3)(z-z_4)$

$(z-1-3i)$ is a factor

$\therefore\ (z-1+3i)$ is a factor

$P(z) = (z-1-3i)(z-1+3i)(z-z_3)(z-z_4)$

$= \left((z-1)^2 - (3i)^2\right)(z-z_3)(z-z_4)$

$= \left(z^2 - 2z + 10\right)(z-z_3)(z-z_4)$

$$\begin{array}{r|l} & z^2 - 16 \\ \hline z^2 - 2z + 10 & z^4 - 2z^3 - 6z^2 + 32z - 160 \\ & -(z^4 - 2z^3 + 10z^2 \quad \downarrow \quad \downarrow\ ) \\ \hline & \qquad -16z^2 + 32z - 160 \\ & -(\qquad -16z^2 + 32z - 160) \\ \hline & \qquad\qquad 0 \end{array}$$

$P(z) = \left(z^2 - 2z + 10\right)\left(z^2 - 16\right)$

$= (z-1-3i)(z-1+3i)(z-4)(z+4)$

Let $P(z) = 0$

$0 = (z-1-3i)(z-1+3i)(z-4)(z+4)$

Null Factor Law

$z = 4, -4, 1+3i, 1-3i$

24 $\alpha = ai$

$\beta = -ai$

$\alpha + \beta = 0$

$\alpha\beta = a^2$

So $z^2 + a^2$ is a factor (using CAS)

$\left(z^2 + 16\right)(z+3-4i)(z+3+4i) = 0$

$a = \pm 4i,\ -3 \pm 4i$

25 $P(z) = z^4 + bz^3 + 18z^2 + 32z + 32$

It is known, $P(4i) = 0$

$(z-4i)$ is a factor $\therefore\ (z+4i)$ is a factor

$P(z) = (z-4i)(z+4i)\left(az^2 + bz + c\right)$

$P(z) = \left(z^2 + 16\right)\left(az^2 + bz + c\right)$

$P(z) = az^4 + bz^3 + cz^2 + 16az^2 + 16bz + 16c$

$P(z) = az^4 + bz^3 + (c + 16a)z^2 + 16bz + 16c$

Given,

$P(z) = z^4 + bz^3 + 18z^2 + 32z + 32$

$a = 1,\ b = 2,\ c = 2$

$\therefore P(z) = z^4 + 2z^3 + 18z^2 + 32z + 32$

$P(z) = \left(z^2 + 16\right)\left(z^2 + 2z + 2\right)$

Let $P(z) = 0$

$0 = \left(z^2 + 16\right)\left(z^2 + 2z + 2\right)$

$0 = (z-4i)(z+4i)\left(z^2 + 2z + 1 - 1 + 2\right)$

$0 = (z-4i)(z+4i)\left((z+1)^2 + 1\right)$

$0 = (z-4i)(z+4i)\left((z+1)^2 - (i)^2\right)$

$0 = (z-4i)(z+4i)(z+1+i)(z+1-i)$

Null Factor Law

$b = 2$

$z = \pm 4i,\ z = -1 \pm i$

26 $P(z) = \left(az^2 + bz + c\right)\left(z^2 - 10z + 41\right) + d$

$P(z) = z^4 - 8z^3 + 23z^2 + 62z + 81$

$$\begin{array}{r|l} & z^2 + 2z + 2 \\ \hline z^2 - 10z + 41 & z^4 - 8z^3 + 23z^2 + 62z + 81 \\ & -(z^4 - 10z^3 + 41z^2 \quad \downarrow \quad \downarrow\ ) \\ \hline & \qquad 2z^3 - 18z^2 + 62z + 81 \\ & -(\quad 2z^3 - 20z^2 + 82z \quad \downarrow) \\ \hline & \qquad\qquad 2z^2 - 20z + 81 \\ & -(\qquad\quad 2z^2 - 20z + 82) \\ \hline & \qquad\qquad\qquad -1 \end{array}$$

$P(z) = \left(z^2 + 2z + 2\right)\left(z^2 - 10z + 41\right) - 1$

$d = -1$

2.4 Exam questions

1 a $z^2 + 2z + 2 = 0$

$z^2 + 2z + 1 = -1 = i^2$

$(z+1)^2 = i^2$

$z + 1 = \pm i$

$z = -1 \pm i$ [1 mark]

b $z^2 + 2\bar{z} + 2 = 0$

Let $z = a + bi,\ a, b \in R,\ z \in C$

$(a+bi)^2 + 2(a-bi) + 2 = 0$

$a^2 + 2abi + b^2i^2 + 2a - 2bi + 2 = 0$

$(a^2 + 2a + 2 - b^2) + i(2ab - 2b) = 0$

Re: (1) $a^2 + 2a + 2 - b^2 = 0$

Im: (2) $2ab - 2b = 2b(a-1) = 0$

(2) $\Rightarrow$ $a = 1$ or $b = 0$

If $b = 0$ then $a^2 + 2a + 2 = 0 \Rightarrow a = 1 \pm i$, but $a \in R$

So $a = 1,\ b^2 = 1 + 2 + 2 = 5,\ b = \pm\sqrt{5}$

$z = 1 \pm \sqrt{5}i$

Award 1 mark for setting-up using Cartesian form.

Award 1 mark for equating real and imaginary parts.

Award 1 mark for the final correct values of z.

2 $P(z) = z^3 + az^2 + bz + c$

$P(-2) = 0,\ P(3\mathrm{i}) = 0$

$z = -2,\ z = \pm 3i$

$P(z) = (z+2)(z^2+9)$

$P(z) = z^3 + 2z^2 + 9z + 18$

$a = 2,\ b = 9,\ c = 18$

The correct answer is **C**.

3 Solve the equation to determine the number of distinct roots.

$\left(z^4 - 1\right)\left(z^2 + 3iz - 2\right) = 0$

$(z + i)^2 (z - i)(z + 2i)(z - 1)(z + 1) = 0$

$z = -i,\ i,\ -2i,\ -1,\ 1$

There are 5 distinct roots

To solve this equation using CAS, complete the entry line as: cSolve $\left(\left(z^4 - 1\right)\left(z^2 + 3iz - 2\right) = 0,\ z\right)$, remembering to use the correct symbol for i, rather than the variable i.

The correct answer is **D**.

2.5 Subsets of the complex plane

2.5 Exercise

1 Let $z = x + iy$

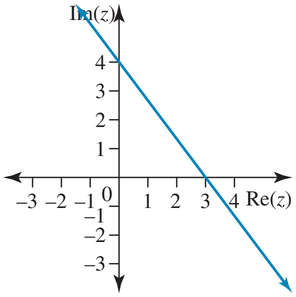
$\text{Re}(z) = x, \text{Im}(z) = y$

$4\text{Re}(z) + 3\,\text{Im}(z) = 12$

$4x + 3y = 12$

Crosses real axis $y = 0,\ \ x = 3 \quad (3, 0)$

Imaginary axis $x = 0,\ \ y = 4 \quad (0, 4)$



2 Line $\quad y = mx + c$

$m = 2 \quad c = 4$

$y = 2x + 4$

$\text{Im}(z) = 2\text{Re}(z) + 4$

$-2\text{Re}(z) + \text{Im}(z) = 4$

$-4\text{Re}(z) + 2\text{Im}(z) = 8$

$\therefore a = -4 \quad b = 2$

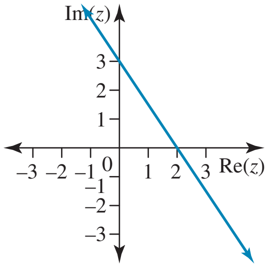
3 **a** $(z : 3\text{Re}(z) + 2\text{Im}(z) = 6)$

Let $z = x + iy \quad \text{Re}(z) = x, \quad \text{Im}(z) = y$

$3x + 2y = 6$

Crosses imaginary axis $x = 0,\ \ y = 3 \quad (0, 3)$

Crosses real axis $y = 0,\ \ x = 2 \quad (2, 0)$



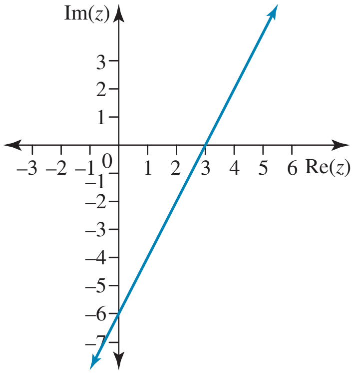
b $(z : 2\text{Re}(z) - \text{Im}(z) = 6)$

Let $z = x + iy \quad \text{Re}(z) = x, \quad \text{Im}(z) = y$

$2x - y = 6$

Crosses imaginary axis $x = 0,\ \ y = -6 \quad (0, -6)$

Crosses real axis $y = 0,\ \ x = 3 \quad (3, 0)$



4 $|z + 3i| = |z - 3|$

Let $z = x + iy$

$|x + (y + 3)i| = |(x - 3) + iy|$

$\sqrt{x^2 + (y + 3)^2} = \sqrt{(x - 3)^2 + y^2}$

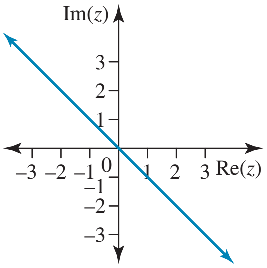
Square both sides and expand

$x^2 + y^2 + 6y + 9 = x^2 - 6x + 9 + y^2$

$6y = -6x$

$y = -x$

Set of points, equidistant from (0, −3) to (3, 0)



5 $(z : |z - 2| = |z - 4|)$

Let $z = x + iy$

$|(x - 2) + iy| = |(x - 4) + iy|$

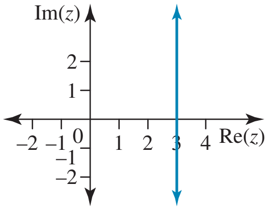
$\sqrt{(x - 2)^2 + y^2} = \sqrt{(x - 4)^2 + y^2}$

$x^2 - 4x + 4 + y^2 = x^2 - 8x + 16 + y^2$

$4x = 12$

$x = 3$

Set of points equidistant from (2, 0) and (4, 0)



b $(z : |z + 4i| = |z - 4|)$

Let $z = x + iy$

$|x + (y + 4)i| = |(x - 4) + iy|$

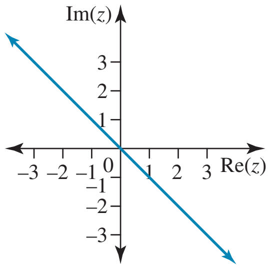
$\sqrt{x^2 + (y + 4)^2} = \sqrt{(x - 4)^2 + y^2}$

$x^2 + y^2 + 8y + 16 = x^2 - 8x + 16 + y^2$

$8x + 8y = 0$

$y = -x$

Set of points equidistant from (0, −4) and (4, 0)



6 **a** $(z: |z+4| = |z-2i|)$

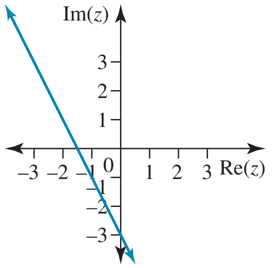
Let $z = x + iy$

$$|(x+4)+iy| = |x+(y-2)i|$$
$$\sqrt{(x+4)^2+y^2} = \sqrt{x^2+(y-2)^2}$$
$$x^2+8x+16+y^2 = x^2+y^2-4y+4$$
$$8x+4y = -12$$
$$2x+y = -3$$

Crosses imaginary axis $x = 0 \quad y = -3 \quad (0, -3)$

Crosses real axis $y = 0 \quad x = -\frac{3}{2} \quad \left(-\frac{3}{2}, 0\right)$

Set of points equidistant from (−4 , 0) and (0, 2)

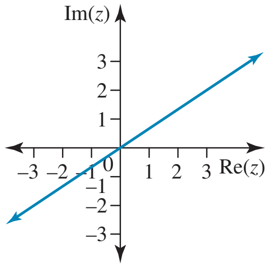


b $\{z: |z+2-3i| = |z-2+3i|\}$

Let $z = x + iy$

$$|(x+2)+(y-3)i| = |(x-2)+(y+3)i|$$
$$\sqrt{(x+2)^2+(y-3)^2} = \sqrt{(x-2)^2+(y+3)^2}$$
$$x^2+4x+4+y^2-6y+9 = x^2-4x+4+y^2+6y+9$$
$$8x-12y = 0$$
$$2x-3y = 0$$

Set of points equidistant from (−2, 3) and (2, −3)

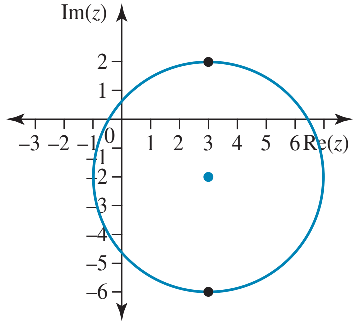


7 $|z-3+2i| = 4$

Let $z = x + iy$

$$|(x-3)+(y+2)i| = 4$$
$$\sqrt{(x-3)^2+(y+2)^2} = 4$$
$$(x-3)^2+(y+2)^2 = 16$$

A circle centre (3, −2) radius 4



8 Centre of the circle is at (3, −3), radius is 3.

$|z-c| = r \qquad c = 3-3i, r = 3$

$|z-(a+bi)| = r$

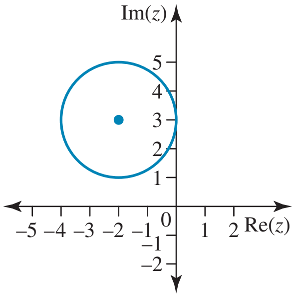
$a = 3, b = -3, r = 3$

9 **a** $(z: |z+2-3i| = 2)$

Let $z = x + iy$

$$|(x+2)+i(y-3)| = 2$$
$$\sqrt{(x+2)^2+(y-3)^2} = 2$$
$$(x+2)^2+(y-3)^2 = 4$$

Circle centre (−2, 3), radius 2

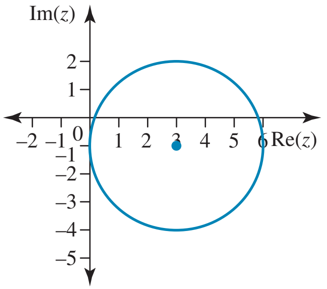


b $(z: |z-3+i| = 3)$

Let $z = x + iy$

$$|(x-3)+(y+1)i| = 3$$
$$\sqrt{(x-3)^2+(y+1)^2} = 3$$
$$(x-3)^2+(y+1)^2 = 9$$

Circle centre origin (3, −1), radius 3



10 Consider $\frac{z-2i}{z-3}$

Let $z = x + iy$

$$= \frac{x+(y-2)i}{(x-3)+yi} \times \frac{(x-3)-iy}{(x-3)-iy}$$
$$= \frac{x(x-3)-y(y-2)i^2+((y-2)(x-3)-xy)i}{(x-3)^2+y^2}$$
$$= \frac{x^2-3x+y^2-2y+(xy-2x-3y-xy+6)i}{(x-3)^2+y^2}$$

a $\text{Im}\left(\dfrac{z-2i}{z-3}\right)=0$

$$\begin{aligned}\Rightarrow -2x-3y+6&=0\\ 3y&=-2x+6\\ y&=-\frac{2x}{3}+2 \quad \text{line}\end{aligned}$$

b $\text{Re}\left(\dfrac{z-2i}{z-3}\right)=0$

$$\begin{aligned}\Rightarrow x^2-3x+y^2-2y&=0\\ \left(x^2-3x+\frac{9}{4}\right)+\left(y^2-2y+1\right)&=\frac{9}{4}+1\\ \left(x-\frac{3}{2}\right)^2+\left(y^2-2y+1\right)&=\frac{9}{4}+1\\ \left(x-\frac{3}{2}\right)^2+(y-1)^2&=\frac{13}{4}\end{aligned}$$

Circle centre $\left(\frac{3}{2},1\right)$ radius $\frac{\sqrt{13}}{2}$

11 a

$$\begin{aligned}|z-3|&=2\,|z+3i|\\ |(x-3)+iy|&=2\,|x+(y+3)i|\\ \sqrt{(x-3)^2+y^2}&=2\sqrt{x^2+(y+3)^2}\\ (x-3)^2+y^2&=4\left(x^2+(y+3)^2\right)\\ x^2-6x+9+y^2&=4x^2+4y^2+24y+36\\ 3x^2+6x+3y^2+24y&=36\\ x^2+2x+y^2+8y&=-9\\ x^2+2x+1+y^2+8y+16&=-9+1+16\\ (x+1)^2+(y+4)^2&=8\end{aligned}$$

Circle centre $(-1,-4)$

Radius $\sqrt{8}=2\sqrt{2}$

b

$$\begin{aligned}|z+3|&=2\,|z+6i|\\ |(x+3)+iy|&=2\,|x+(y+6)i|\\ \sqrt{(x+3)^2+y^2}&=2\sqrt{x^2+(y+6)^2}\\ (x+3)^2+y^2&=4\left(x^2+(y+6)^2\right)\\ x^2+6x+9+y^2&=4x^2+4y^2+48y+144\\ 3x^2-6x+3y^2+48y&=-135\\ x^2-2x+y^2+16y&=-45\\ x^2-2x+1+y^2+16y+64&=-45+1+64\\ (x-1)^2+(y+8)^2&=20\end{aligned}$$

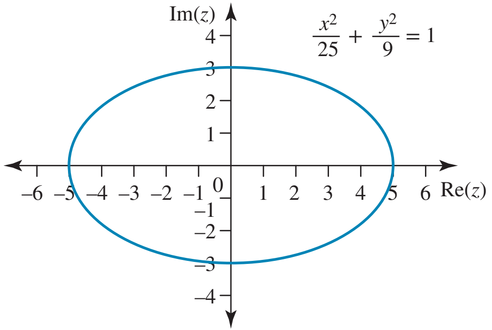
Circle centre $(1,-8)$

Radius $\sqrt{20}=2\sqrt{5}$

12. $|(x-4)+yi|+|(x+4)+yi|=10$

$$\begin{aligned}&\sqrt{(x-4)^2+y^2}+\sqrt{(x+4)^2+y^2}=10\\ &\sqrt{(x-4)^2+y^2}=10-\sqrt{(x+4)^2+y^2}\\ &(x-4)^2+y^2=100+(x+4)^2+y^2-20\sqrt{(x+4)^2+y^2}\\ &20\sqrt{(x+4)^2+y^2}=100+(x+4)^2+y^2-\left((x-4)^2+y^2\right)\\ &20\sqrt{(x+4)^2+y^2}=100+x^2+8x+16+y^2-\left(x^2-8x+16+y^2\right)\\ &20\sqrt{(x+4)^2+y^2}=100+16x=4\,(25+4x)\\ &5\sqrt{(x+4)^2+y^2}=25+4x,\ x\geq-\frac{25}{4}\\ &25\left((x+4)^2+y^2\right)=(25+4x)^2\\ &25\left(x^2+8x+16+y^2\right)=625+200x+16x^2\\ &25x^2+200x+400+25y^2=625+200x+16x^2\\ &9x^2+25y^2=225\\ &\frac{x^2}{25}+\frac{y^2}{9}=1\end{aligned}$$

The graph is an ellipse with semi-major and minor axes 5 and 3 respectively.



13 $\left(z:\text{Arg}(z-2)=\dfrac{\pi}{6}\right)$

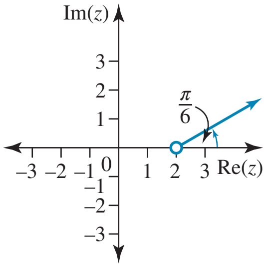
Ray starts from the point $(2,0)$ making an angle of $\frac{\pi}{6}$ with the positive real axis

$$\begin{aligned}\text{Arg}(z-2)&=\frac{\pi}{6}\\ \text{Arg}(x+yi-2)&=\frac{\pi}{6}\\ \text{Arg}((x-2)+yi)&=\frac{\pi}{6}\end{aligned}$$

$\tan^{-1}\left(\dfrac{y}{x-2}\right)=\dfrac{\pi}{6}$, for $x>2$

$\dfrac{y}{x-2}=\tan\left(\dfrac{\pi}{6}\right)=\dfrac{1}{\sqrt{3}}$, for $x>2$

$\sqrt{3}y=(x-2)$, for $x>2$

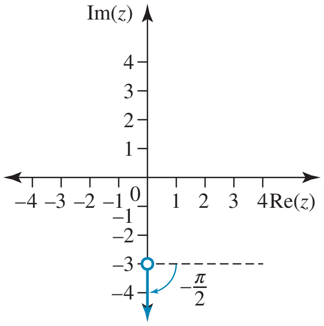


14 $\left(z:\text{Arg}(z+3i)=-\dfrac{\pi}{2}\right)$

Ray starts from the point $(0,-3)$ making an angle of $-\frac{\pi}{2}$ with the positive real axis

Cartesian equation

$x=0$, for $y<-3$



15 a $\left(z: \operatorname{Arg}(z) = \frac{\pi}{6}\right)$

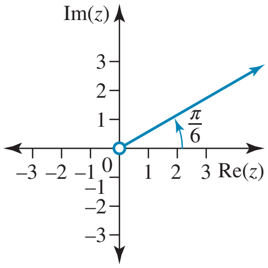
If $z = x + iy$

$$\operatorname{Arg}(z) = \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{6}, \text{ for } x > 0$$

$$\frac{y}{x} = \tan\left(\frac{\pi}{6}\right)$$

$$y = \frac{x}{\sqrt{3}} \quad \text{for } x > 0$$

Ray from origin (not included) making an angle of 30° with the real axis



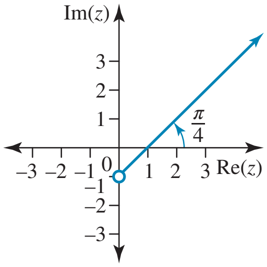
b $\left(z: \operatorname{Arg}(z + i) = \frac{\pi}{4}\right)$

$$\tan^{-1}\left(\frac{y+1}{x}\right) = \frac{\pi}{4}, \text{for } x > 0$$

$$\frac{y+1}{x} = \tan\left(\frac{\pi}{4}\right)$$

$$y = x - 1$$

Ray from origin (0, −1) (not included) making an angle of 45° with the real axis



16 a $S: (z: |z| = 3)$

$T: (z: 3\operatorname{Re}(z) + 4\operatorname{Im}(z) = 12)$

$S: x^2 + y^2 = 9 \quad (1)$

$T: 3x + 4y = 12 \quad (2)$

$(2) \quad 4y = 12 - 3x$

$$y = \frac{12 - 3x}{4} \quad \text{into (1)}$$

$$x^2 + \left(\frac{12 - 3x}{4}\right)^2 = 9$$

$$x^2 + \frac{(12 - 3x)^2}{16} = 9$$

$$16x^2 + 144 - 72x + 9x^2 = 9 \times 16$$

$$25x^2 - 72x = 0$$

$$x(25x - 72) = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad x = \frac{72}{25}$$

$$\text{if } x = 0 \quad \Rightarrow \quad y = 3$$

$$\text{if } x = \frac{72}{25} \quad \Rightarrow \quad y = \frac{1}{4}\left(12 - 3 \times \frac{72}{25}\right) = \frac{21}{25}$$

$$(0, 3) \left(\frac{72}{25}, \frac{21}{25}\right)$$

b $S: (z: |z| = 4)$

$T: (z: 4\operatorname{Re}(z) - 2\operatorname{Im}(z) = k)$

$S: x^2 + y^2 = 16 \quad (1)$

$T: 4x - 2y = k \quad (2)$

$(2) \quad 2y = 4x - k$

$$y = \frac{1}{2}(4x - k) \quad \text{into (1)}$$

$$x^2 + \left(\frac{1}{2}(4x - k)\right)^2 = 16$$

$$x^2 + \frac{1}{4}\left(16x^2 - 8xk + k^2\right) = 16$$

$$4x^2 + 16x^2 - 8xk + k^2 = 64$$

$$20x^2 - 8xk + \left(k^2 - 64\right) = 0$$

For one solution $\Delta = 0$

$a = 20 \quad b = -8k \quad c = k^2 - 64$

$$b^2 - 4ac = 0$$

$$64k^2 - 4 \times 20(k^2 - 64) = 0$$

$$k^2 - 320 = 0$$

$$k = \pm 8\sqrt{5}$$

17 a

$$|z - 6| = 2\,|z - 3i|$$

$$|(x - 6) + iy| = 2\,|x + (y - 3)i|$$

$$\sqrt{(x - 6)^2 + y^2} = 2\sqrt{x^2 + (y - 3)^2}$$

$$(x - 6)^2 + y^2 = 4\left(x^2 + (y - 3)^2\right)$$

$$x^2 - 12x + 36 + y^2 = 4x^2 + 4y^2 - 24y + 36$$

$$3x^2 + 12x + 3y^2 - 24y = 0$$

$$x^2 + 4x + y^2 - 8y = 0$$

$$x^2 + 4x + 4 + y^2 - 8y + 16 = 4 + 16$$

$$(x + 2)^2 + (y - 4)^2 = 20$$

Circle centre (−2, 4) Radius $\sqrt{20} = 2\sqrt{5}$

Circle T: $|z - (a + bi)| = r$

$a = -2, b = 4, r = 2\sqrt{5}$

b

$$S: |z + 3| = 2\,|z - 3i|$$

$$|(x + 3) + iy| = 2\,|x + (y - 3)i|$$

$$\sqrt{(x + 3)^2 + y^2} = 2\sqrt{x^2 + (y - 3)^2}$$

$$(x + 3)^2 + y^2 = 4\left(x^2 + (y - 3)^2\right)$$

$$x^2 + 6x + 9 + y^2 = 4x^2 + 4y^2 - 24y + 36$$

$$3x^2 - 6x + 3y^2 - 24y = -2$$

$$x^2 - 2x + y^2 - 8y = -9$$

$$x^2 - 2x + 1 + y^2 - 8y + 16 = -9 + 1 + 16$$

$$(x - 1)^2 + (y - 4)^2 = 8$$

Circle centre (1, 4) radius $\sqrt{8} = 2\sqrt{2}$

Circle T: $|z - (a + bi)| = r$

$a = 1, \; b = 4, \; r = 2\sqrt{2}$

18 a $S: \left(z: |z| = \sqrt{29}\right)$

$T: (z: 3\operatorname{Re}(z) - \operatorname{Im}(z) = 1)$

$S: x^2 + y^2 = 29 \quad (1)$

$T: 3x - y = 1 \quad (2)$

$y = 3x - 1 \quad \text{into (1)}$

$$x^2 + (3x - 1)^2 = 29$$

$$x^2 + 9x^2 - 6x + 1 = 29$$

$$10x^2 - 6x - 28 = 0$$

$$5x^2 - 3x - 14 = 0$$

$$(5x + 7)(x - 2) = 0$$

$$x = 2 \quad \text{or} \quad -\frac{7}{5}$$

$$\text{If} \quad x = 2 \Rightarrow y = 6 - 1 = 5$$

$$x = -\frac{7}{5} \quad y = 3 \times -\frac{7}{5} - 1 = -\frac{26}{5}$$

$$(2, 5) \quad \left(-\frac{7}{5}, -\frac{26}{5}\right)$$

b $S\colon (z\colon |z| = 5)$

$T\colon (z\colon 2\text{Re}(z) - 3\text{Im}(z) = k)$

$S\colon x^2 + y^2 = 25 \quad (1)$

$T\colon 2x - 3y = k \quad (2)$

$y = \frac{1}{3}(2x - k) \quad \text{into} \quad (1)$

$$x^2 + \left(\frac{1}{3}(2x - k)\right)^2 = 25$$

$$x^2 + \frac{1}{9}\left(4x^2 - 4xk + k^2\right) = 25$$

$$9x^2 + \left(4x^2 - 4xk + k^2\right) = 225$$

$$13x^2 - 4xk + k^2 - 225 = 0$$

For one solution $\Delta = 0$

$a = 13 \quad b = -4k \quad c = k^2 - 225$

$$b^2 - 4ac = 0$$

$$16k^2 - 4 \times 13(k^2 - 225) = 0$$

$$11700 - 36k^2 = 0$$

$$-36(k^2 - 325) = 0$$

$$k^2 = 325$$

$$k = \pm 5\sqrt{13}$$

19 $S = (z\colon |z| = 3)$

$T = \left(z\colon \text{Arg}(z) = -\frac{\pi}{4}\right)$

Cartesian equation of S is $x^2 + y^2 = 9$

Cartesian equation of T is $y = -x$

$S = T$ when

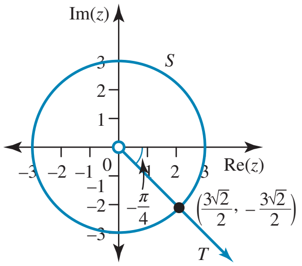
$$x^2 + x^2 = 9$$

$$2x^2 = 9$$

$$x^2 = \frac{9}{2}$$

$$x = \frac{3\sqrt{2}}{2} \quad \text{as } x > 0$$

when $\quad x = \frac{3\sqrt{2}}{2}, y = -\frac{3\sqrt{2}}{2}$



20 a $S = (z\colon |z + 3 + i| = 5)$

Let $z = x + iy$

$|(x + 3) + (y + 1)i| = 5$

$(x + 3)^2 + (y + 1)^2 = 25$

Circle center $(-3, -1)$ radius 5

b $R = \left(z\colon \text{Arg}(z + 3) = -\frac{3\pi}{4}\right)$

Ray starting from the point $(-3, 0)$ making an angle of $-135°$ with the real axis

$$\tan^{-1}\left(\frac{y}{x + 3}\right) = -\frac{3\pi}{4}$$

$$\frac{y}{x + 3} = \tan\left(-\frac{3\pi}{4}\right) = 1$$

$$y = x + 3 \text{ for } x < -3$$

c $\quad S = R$

$$y^2 + (y + 1)^2 = 25$$

$$2y^2 + 2y + 1 = 25$$

$$2y^2 + 2y - 24 = 0$$

$$2(y^2 + y - 12) = 0$$

$$2(y + 4)(y - 3) = 0$$

If $y = -4 \Rightarrow x = -7(-7, -4) \; u = -7 - 4i$

$y = 3 \Rightarrow x = 0$ not included since $x < -3$

21 $S\colon |z| = 3 \qquad z = x + iy$

$$x^2 + y^2 = 9 \quad (1)$$

$T\colon 3\text{Re}(z) + 4\text{Im}(z) = 15$

$$3x + 4y = 15 \quad (2)$$

$$y = \frac{-3x + 15}{4}$$

$$x^2 + \left(\frac{-3x + 15}{4}\right)^2 = 9$$

$$x^2 + \left(\frac{3(5 - x)}{4}\right)^2 = 9$$

$$16x^2 + 9\left(25 - 10x + x^2\right) = 144$$

$$25x^2 - 90x + 81 = 0$$

$$(5x - 9)^2 = 0$$

$$x = \frac{9}{5} \Rightarrow y = \frac{15 - 3 \times \frac{9}{5}}{4} = \frac{12}{5} \quad \left(\frac{9}{5}, \frac{12}{5}\right) \quad (2)$$

22 $S\colon |z| = 6$

$x^2 + y^2 = 36 \quad (1)$

$T\colon 3\text{Re}(z) - 4\text{Im}(z) = k$

$3x - 4y = k \quad (2)$

$y = \frac{3x - k}{4} \quad (2) \text{ into } (1)$

$$x^2 + \left(\frac{3x - k}{4}\right)^2 = 36$$

$$16x^2 + 9x^2 - 6xk + k^2 = 576$$

$$25x^2 - 6xk + \left(k^2 - 576\right) = 0$$

For one solution $\Delta = 0$

$$(6k)^2 - 4 \times 25\left(k^2 - 576\right) = 0$$

$$36k^2 - 100k^2 + 57600 = 0$$

$$-64k^2 + 57600 = 0$$

$$-64\left(k^2 - 900\right) = 0$$

$$r = \pm 30$$

23 $|z-a|^2 - |z-bi|^2 = a^2 + b^2 \quad b \neq 0, a, b \in R$

$z = x + iy$

$$\begin{aligned}
|(x-a)+iy|^2 - |x+(y-b)i|^2 &= a^2+b^2 \\
(x-a)^2 + y^2 - \left(x^2+(y-b)^2\right) &= a^2+b^2 \\
x^2 - 2ax + a^2 + y^2 - \left(x^2 + \left(y^2 - 2by + b^2\right)\right) &= a^2+b^2 \\
-2ax + 2by &= 2b^2 \\
by &= ax + b^2 \\
y &= \frac{ax}{b} + b \text{ line}
\end{aligned}$$

24 $3z\bar{z} + 6z + 6\bar{z} + 2 = 0$

Let $z = x + iy$

$\bar{z} = x - iy$

$z\bar{z} = x^2 + y^2$

$$\begin{aligned}
3z\bar{z} + 6z + 6\bar{z} + 2 &= 0 \\
3(x^2+y^2) + 6(x+iy) + 6(x-iy) + 2 &= 0 \\
3x^2 + 3y^2 + 6x + 6iy + 6x - 6iy + 2 &= 0 \\
3x^2 + 12x + 3y^2 &= -2 \\
3(x^2+4x) + 3y^2 &= -2 \\
3(x^2+4x+4) + 3y^2 &= -2 + 12 \\
3(x+2)^2 + 3y^2 &= 10 \\
(x+2)^2 + y^2 &= \frac{10}{3}
\end{aligned}$$

Circle centre $(-2, 0)$ radius $\sqrt{\frac{10}{3}} = \frac{\sqrt{30}}{3}$

25 a. i. $w = \text{cis}(\theta) \quad z = \frac{1}{2}(9w - \overline{w})$

$\overline{w} = \text{cis}(-\theta)$

$$\begin{aligned}
z &= \frac{1}{2}(9w - \overline{w}) \\
&= \frac{1}{2}(9\,\text{cis}(\theta) - \text{cis}(-\theta)) \\
&= \frac{1}{2}\left[9\cos(\theta) + 9i\sin(\theta) - \left(\cos(-\theta) + i\sin(-\theta)\right)\right] \\
&= \frac{1}{2}\left[8\cos(\theta) + 10i\sin(\theta)\right] \\
&= 4\cos(\theta) + 5i\sin(\theta)
\end{aligned}$$

ii. $z = x + iy$

$x = 4\cos(\theta)$ (1)

$y = 5\sin(\theta)$ (2)

$\cos(\theta) = \frac{x}{4}, \ \sin(\theta) = \frac{y}{5}$

$\cos^2(\theta) + \sin^2(\theta) = 1$

$\Rightarrow \frac{x^2}{16} + \frac{y^2}{25} = 1$ shown

iii.
$$\begin{aligned}
|z-3i|^2 &= |4\cos(\theta) + i(5\sin(\theta) - 3)|^2 \\
&= \left(4\cos(\theta)^2\right) + (5\sin(\theta) - 3)^2 \\
&= 16\cos^2(\theta) + 25\sin^2(\theta) - 30\sin(\theta) + 9 \\
&= 16\left(1 - \sin^2(\theta)\right) + 25\sin^2(\theta) - 30\sin(\theta) + 9 \\
&= 25 - 30\sin(\theta) + 9\sin^2(\theta) \\
&= (5 - 3\sin(\theta))^2
\end{aligned}$$

$$\begin{aligned}
|z+3i|^2 &= |4\cos(\theta) + i(5\sin(\theta) + 3)|^2 \\
&= 16\cos^2(\theta) + 25\sin^2(\theta) + 30\sin(\theta) + 9 \\
&= 25 + 30\sin(\theta) + 9\sin^2(\theta) \\
&= (5 + 3\sin(\theta))^2
\end{aligned}$$

$|z-3i| = 5 - 3\sin(\theta)$

Since $|z-3i| > 0$

$5 - 3\sin(\theta) > 0$

$|z+3i| = 5 + 3\sin(\theta)$

iv. $|z+3i| + |z-3i|$

$= 5 - 3\sin(\theta) + 5 + 3\sin(\theta)$

$= 10$

b. Let $z = x + iy$

$|z-3i| + |z+3i| = 10$

$|x+(y-3)i| + |x+(y+3)i| = 10$

$\sqrt{x^2+(y-3)^2} + \sqrt{x^2+(y+3)^2} = 10$

$\sqrt{x^2+(y-3)^2} = 10 - \sqrt{x^2+(y+3)^2}$

Square both sides

$x^2 + (y-3)^2 = 100 - 20\sqrt{x^2+(y+3)^2} + x^2 + (y+3)^2$

$x^2 + y^2 - 6y + 9 = 100 - 20\sqrt{x^2+(y+3)^2} + x^2 + y^2 + 6y + 9$

$20\sqrt{x^2+(y+3)^2} = 100 + 12y$

$5\sqrt{x^2+(y+3)^2} = 25 + 3y$

So $25 + 3y > 0 \Rightarrow y > -\frac{25}{3}$

Square both sides again

$25\left[x^2 + (y+3)^2\right] = 625 + 150y + 9y^2$

$25x^2 + 25y^2 + 150y + 225 = 625 + 150y + 9y^2$

$25x^2 + 16y^2 = 400$

$\frac{x^2}{16} + \frac{y^2}{25} = 1$

Since $y \in [-5, 5]$

$y > -\frac{25}{3}$ is always satisfied

Whole ellipse.

26. $\{z : |z-c| + |z+c| = 2a\}$ Let $z = x + yi$

$|(x-c)+yi| + |(x+c)+yi| = 2a$

$\sqrt{(x-c)^2+y^2} + \sqrt{(x+c)^2+y^2} = 2a$

$\sqrt{(x-c)^2+y^2} = 2a - \sqrt{(x+c)^2+y^2}$

squaring both sides gives:

$(x-c)^2 + y^2 = 4a^2 + (x+c)^2 + y^2 - 4a\sqrt{(x+c)^2+y^2}$

$x^2 - 2cx + c^2 = x^2 + 2cx + c^2 + 4a^2 - 4a\sqrt{(x+c)^2+y^2}$

$4a\sqrt{(x+c)^2+y^2} = 4a^2 + 4xc$, for $x > -\frac{a^2}{c}$

$a\sqrt{(x+c)^2+y^2} = a^2 + xc$

squaring both sides again gives:

$$\begin{aligned}
a^2\left(x^2 + 2cx + c^2\right) + a^2y^2 &= a^4 + x^2c^2 + 2a^2xc \\
x^2\left(a^2 - c^2\right) + a^2y^2 &= a^4 - a^2c^2 \\
&= a^2\left(a^2 - c^2\right)
\end{aligned}$$

$\frac{x^2}{a^2} + \frac{y^2}{\left(a^2 - c^2\right)} = 1$ if $a^2 > c^2$ an ellipse

27 a $z\bar{z} + (3+2i)z + (3-2i)\bar{z} + 4 = 0$

Let $z = x + iy$

$\bar{z} = x - iy$

$z\bar{z} = x^2 + y^2$

$$\begin{aligned}
x^2 + y^2 + (3+2i)(x+iy) + (3-2i)(x-iy) + 4 &= 0 \\
x^2 + y^2 + 3x + 2ix + 3iy + 2i^2y + 3x - 2ix - 3iy + 2i^2y + 4 &= 0 \\
x^2 + 6x + 9 + y^2 - 4y + 4 &= 9 \\
(x+3)^2 + (y-2)^2 &= 9
\end{aligned}$$

Circle centre $(-3, 2)$ radius 3

b $az\bar{z} + \bar{b}z + b\bar{z} + c = 0 \quad a, c \in R \quad b = \alpha + i\beta$

Let $z = x + iy \quad \bar{z} = x - iy$

$$0 = a\left(x^2 + y^2\right) + (\alpha - i\beta)(x + iy) + (\alpha + i\beta)(x - iy) + c$$

$$0 = ax^2 + ay^2 + \alpha x - \beta ix + \alpha iy - \beta i^2 y + \alpha x + \beta ix$$
$$- \alpha iy - \beta i^2 y + c$$

$$-c = ax^2 + 2\alpha x + ay^2 + 2\beta y$$

$$a\left(x^2 + \frac{2\alpha}{a}x + \frac{\alpha^2}{a^2}\right) + a\left(y^2 + \frac{2\beta}{a}y + \frac{\beta^2}{a^2}\right) = -c + \frac{\alpha^2}{a} + \frac{\beta^2}{a}$$

$$a\left(x + \frac{\alpha}{a}\right)^2 + a\left(y + \frac{\beta}{a}\right)^2 = \frac{\alpha^2 + \beta^2 - ac}{a}$$

$$\left(x + \frac{\alpha}{a}\right)^2 + \left(y + \frac{\beta}{a}\right)^2 = \frac{\alpha^2 + \beta^2 - ac}{a^2}$$

$$= \frac{b\bar{b} - ac}{a^2}$$

$b = \alpha + i\beta \quad \bar{b} = \alpha - i\beta \quad b\bar{b} = \alpha^2 + \beta^2$

Circle centre $\left(-\frac{\alpha}{a}, -\frac{\beta}{a}\right)$ radius $\frac{\sqrt{b\bar{b} - ac}}{a}$

Provided $b\bar{b} > ac$,

$a \neq 0$

28 Consider $\frac{z - ai}{z - b}, \quad ab \neq 0$

Let $z = x + iy$

$$= \frac{x + (y - a)i}{(x - b) + yi} \times \frac{(x - b) - iy}{(x - b) - iy}$$

$$= \frac{x(x - b) - y(y - a)i^2 + ((y - a)(x - b) - xy)i}{(x - b)^2 - i^2y^2}$$

$$= \frac{x(x - b) - y(y - a) + ((y - a)(x - b) - xy)i}{(x - b)^2 + y^2}$$

a $\text{Im}\left(\frac{z - ai}{z - b}\right) = 0$

$$\Rightarrow (y - a)(x - b) - xy = 0$$

$$xy - ax - by + ab - xy = 0$$

$$ab = ax + by$$

$$by = -ax + ab$$

$$y = -\frac{a}{b}x + a$$

$ab \neq 0 \quad$ line

b $\text{Re}\left(\frac{z - ai}{z - b}\right) = 0$

$$\Rightarrow x(x - b) + y(y - a) = 0$$

$$x^2 - bx + \frac{b^2}{4} + y^2 - ay + \frac{a^2}{4} = \frac{a^2 + b^2}{4}$$

$$\left(x - \frac{b}{2}\right)^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2 + b^2}{4}$$

Circle centre $\left(\frac{b}{2}, \frac{a}{2}\right)$ radius $\frac{\sqrt{a^2 + b^2}}{2}$

29 Let $z = x + iy \qquad \bar{z} = x - iy \qquad z, c \in C$

$c = a + bi \qquad \bar{c} = a - bi \qquad a, b \in R$

$z\bar{z} = x^2 + y^2, \qquad c\bar{c} = a^2 + b^2$

$$r^2 = z\bar{z} - c\bar{z} - c\bar{z} + c\bar{c}$$

$$r^2 = x^2 + y^2 - (a + bi)(x - iy) - (a - bi)(x + iy) + a^2 + b^2$$

$$r^2 = x^2 + y^2 - \left(ax + bix - aiy - bi^2y\right) - \left(ax - bix + aiy - byi^2\right)$$
$$+ a^2 + b^2$$

$$r^2 = x^2 - 2ax + a^2 + y^2 - 2by + b^2$$

$$r^2 = (x - a)^2 + (y - b)^2$$

Circle centre (a, b) radius r

30 $\quad |z - c| = 2\,|z - \bar{c}|$

$$|(x - a) + (y - b)i| = 2\,|(x - a) + (y + b)i|$$

$$\sqrt{(x - a)^2 + (y - b)^2} = 2\sqrt{(x - a)^2 + (y + b)^2}$$

$$(x - a)^2 + (y - b)^2 = 4\left((x - a)^2 + (y + b)^2\right)$$

$$3(x - a)^2 + 4(y + b)^2 - (y - b)^2 = 0$$

$$3(x - a)^2 + 4y^2 + 8by + 4b^2 - \left(y^2 - 2by + b^2\right) = 0$$

$$3(x - a)^2 + 3y^2 + 10by + 3b^2 = 0$$

$$(x - a)^2 + y^2 + \frac{10}{3}by + b^2 = 0$$

$$(x - a)^2 + \left(y^2 + \frac{10}{3}by\right) = -b^2$$

$$(x - a)^2 + \left(y^2 + \frac{10}{3}by + \frac{25b^2}{9}\right) = -b^2 + \frac{25b^2}{9}$$

$$(x - a)^2 + \left(y + \frac{5b}{3}\right)^2 = \frac{16b^2}{9}$$

Circle centre $\left(a, \frac{-5b}{3}\right)$ radius $\frac{4b}{3}$

2.5 Exam questions

1 The circle is centred around the point $z = 2 + \sqrt{3}i$.

$|z|$ represents the distance of a complex number from the origin. The maximum distance from the origin will be the distance of the centre of the circle from the origin, plus the radius of the circle.

$$|z|_{\max} = \left|2 + \sqrt{3}i\right| + 1$$
$$= \sqrt{2^2 + \left(\sqrt{3}\right)^2} + 1$$
$$= \sqrt{4 + 3} + 1$$
$$= \sqrt{7} + 1$$

The correct answer is **D**.

2 **a** $u = -2 - i, v = -4 - 3i$

$|z - u| = |z - v|, z = x + yi$

$$|(x + 2) + i(y + 1)| = |(x + 4) + i(y + 3)|$$

$$(x + 2)^2 + (y + 1)^2 = (x + 4)^2 + (y + 3)^2$$

$$x^2 + 4x + 4 + y^2 + 2y + 1 = x^2 + 8x + 16 + y^2 + 6y + 9$$

$$4x + 4y + 20 = 0$$

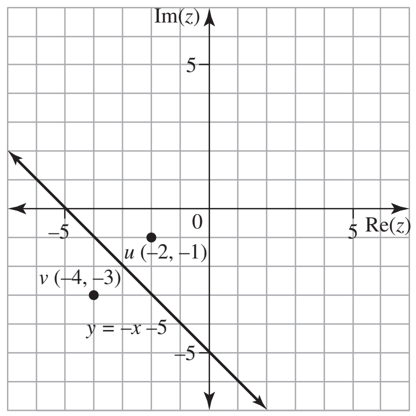
$$y = -x - 5, m = -1, c = -5$$

Award 1 mark for expressing in modulus form.

Award 1 mark for simplifying.

Award 1 mark for the correct line.

b



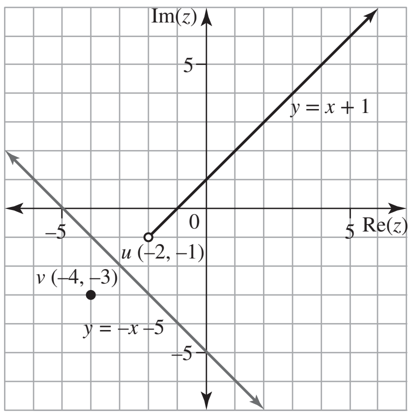
Award 1 mark for the two points.

Award 1 mark for the correctly graphed line.

c The line is the perpendicular bisector of the line joining u and v, or the set points equidistant from both u and v.

Award 1 mark for the correct geometrical interpretation.

d i



Award 1 mark correctly sketching the ray on the diagram in part **b**.

Note this diagram must show an open circle at u.

ii $\text{Arg}(z-u)=\dfrac{\pi}{4}$

$\text{Arg}((x+2)+i(y+1))=\dfrac{\pi}{4}$

$\tan^{-1}\left(\dfrac{y+1}{x+2}\right)=\dfrac{\pi}{4}$ for $x>-2$

$\dfrac{y+1}{x+2}=1$

$y=x+1$ for $x>-2$

A ray with open circle at $u(-2,-1)$

Award 1 mark for the correct line.

e $z_c=a+bi$

$|z-z_c|=r,(x-a)^2+(y-b)^2=r^2$

$(-2,-1)\quad(1)\quad(-2-a)^2+(-1-b)^2=r^2$

$(-4,-3)\quad(2)\quad(-4-a)^2+(-3-b)^2=r^2$

$(0,-5)\quad(3)\quad a^2+(-5-b)^2=r^2$

Solving using CAS:

$a=-\dfrac{5}{3},b=-\dfrac{10}{3},r=\dfrac{5\sqrt{2}}{3}$

$z_c=-\dfrac{5}{3}-\dfrac{10}{3}i=-\dfrac{5}{3}(1+2i)$

Award 1 mark for correctly setting up three equations.

Award 1 mark for the correct value of a and b.

Award 1 mark for the correct value of r.

3 $\text{Arg}(z-2)=\dfrac{\pi}{4}$

$\tan^{-1}\left(\dfrac{y}{x-2}\right)=\dfrac{\pi}{4},x>2$

$\dfrac{y}{x-2}=\tan\left(\dfrac{\pi}{4}\right)=1$

$(1)\ \ y=x-2,x>2$

$\text{Arg}(z-(5+i))=\dfrac{5\pi}{6}$

$\tan^{-1}\left(\dfrac{y-1}{x-5}\right)=\dfrac{5\pi}{6},x<5$

$\dfrac{y-1}{x-5}=\tan\left(\dfrac{5\pi}{6}\right)$

$(2)\ \ y=\dfrac{\sqrt{3}(5-x)}{3}+1,x<5$

$(1)\ \ x=y+2,\ \text{into}\ (2)\ \ y=\dfrac{\sqrt{3}(5-x)}{3}+1$

$y=\dfrac{\sqrt{3}(5-y-2)}{3}+1=\dfrac{(3-y)}{\sqrt{3}}+1$

$\sqrt{3}(y-1)=3-y$

$y\left(\sqrt{3}+1\right)=3+\sqrt{3}$

$y=\dfrac{3+\sqrt{3}}{\sqrt{3}+1}\times\dfrac{\sqrt{3}-1}{\sqrt{3}-1}=\dfrac{3-3+2\sqrt{3}}{3-1}=\sqrt{3}$

The correct answer is **D**.

2.6 Roots of complex numbers

2.6 Exercise

1 $z^2=5\,\text{cis}\left(\dfrac{\pi}{6}\right)$

$z=\sqrt[2]{5}\,\text{cis}\left(\dfrac{\frac{\pi}{6}}{2}+\dfrac{2k\pi}{2}\right)$

$z=\sqrt{5}\,\text{cis}\left(\dfrac{\pi}{12}+k\pi\right)$

Let $k=0$ $z_1=\sqrt{5}\,\text{cis}\left(\dfrac{\pi}{12}+0\times\pi\right)$

$=\sqrt{5}\,\text{cis}\left(\dfrac{\pi}{12}\right)$

Let $k=1$ $z_2=\sqrt{5}\,\text{cis}\left(\dfrac{\pi}{12}+1\times\pi\right)$

$=\sqrt{5}\,\text{cis}\left(\dfrac{13\pi}{12}\right)$

$=\sqrt{5}\,\text{cis}\left(-\dfrac{11\pi}{12}\right)$

$z=\sqrt{5}\,\text{cis}\left(\dfrac{\pi}{12}\right),\sqrt{5}\,\text{cis}\left(-\dfrac{11\pi}{12}\right)$

2 $z^3=27\,\text{cis}\left(\dfrac{2\pi}{3}\right)$

$z=\sqrt[3]{27}\,\text{cis}\left(\dfrac{\frac{2\pi}{3}}{3}+\dfrac{2k\pi}{3}\right)$

$z=3\,\text{cis}\left(\dfrac{2\pi}{9}+\dfrac{2k\pi}{3}\right)$

Let $k=0$ $z_1=3\,\text{cis}\left(\dfrac{2\pi}{9}+\dfrac{2\times0\times\pi}{3}\right)$

$=3\,\text{cis}\left(\dfrac{2\pi}{9}\right)$

$$\begin{aligned}\text{Let } k = 1 \; z_2 &= 3\operatorname{cis}\left(\frac{2\pi}{9} + \frac{2\times 1\times\pi}{3}\right)\\ &= 3\operatorname{cis}\left(\frac{8\pi}{9}\right)\end{aligned}$$

$$\begin{aligned}\text{Let } k = 2 \; z_3 &= 3\operatorname{cis}\left(\frac{2\pi}{9} + \frac{2\times 2\times\pi}{3}\right)\\ &= 3\operatorname{cis}\left(\frac{14\pi}{9}\right)\\ &= 3\operatorname{cis}\left(-\frac{4\pi}{9}\right)\end{aligned}$$

$$z = 3\operatorname{cis}\left(\frac{2\pi}{9}\right), 3\operatorname{cis}\left(\frac{8\pi}{9}\right), 3\operatorname{cis}\left(-\frac{4\pi}{9}\right)$$

3 $z^4 = 16\operatorname{cis}\left(\frac{-3\pi}{4}\right)$

$$\begin{aligned}z &= \sqrt[4]{16}\operatorname{cis}\left(\frac{\frac{-3\pi}{4}}{4} + \frac{2k\pi}{4}\right)\\ &= 2\operatorname{cis}\left(\frac{-3\pi}{16} + \frac{k\pi}{2}\right)\end{aligned}$$

$$\begin{aligned}\text{Let } k = 0 \; z_1 &= 2\operatorname{cis}\left(-\frac{3\pi}{16} + \frac{0\times\pi}{2}\right)\\ &= 2\operatorname{cis}\left(-\frac{3\pi}{16}\right)\end{aligned}$$

$$\begin{aligned}\text{Let } k = 1 \; z_2 &= 2\operatorname{cis}\left(-\frac{3\pi}{16} + \frac{1\times\pi}{2}\right)\\ &= 2\operatorname{cis}\left(\frac{5\pi}{16}\right)\end{aligned}$$

$$\begin{aligned}\text{Let } k = 2 \; z_3 &= 2\operatorname{cis}\left(-\frac{3\pi}{16} + \frac{2\times\pi}{2}\right)\\ &= 2\operatorname{cis}\left(\frac{13\pi}{16}\right)\end{aligned}$$

$$\begin{aligned}\text{Let } k = 3 \; z_4 &= 2\operatorname{cis}\left(-\frac{3\pi}{16} + \frac{3\times\pi}{2}\right)\\ &= 2\operatorname{cis}\left(-\frac{11\pi}{16}\right)\end{aligned}$$

$$z = 2\operatorname{cis}\left(\frac{-3\pi}{16}\right), 2\operatorname{cis}\left(\frac{5\pi}{16}\right), 2\operatorname{cis}\left(\frac{13\pi}{16}\right), 2\operatorname{cis}\left(\frac{-11\pi}{16}\right)$$

4 **a** $z^3 = -1 - \sqrt{3}i$

$$\begin{aligned}|z| &= \sqrt{(-1)^2 + \left(-\sqrt{3}\right)^2}, & \theta &= \tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right)\\ &= 2 & &= \frac{-2\pi}{3}\end{aligned}$$

$$\therefore z = -1 - \sqrt{3}i = 2\operatorname{cis}\left(\frac{-2\pi}{3}\right)$$

$$z^3 = 2\operatorname{cis}\left(-\frac{2\pi}{3}\right)$$

$$z = \sqrt[3]{2}\operatorname{cis}\left(\frac{\frac{-2\pi}{3}}{3} + \frac{2k\pi}{3}\right)$$

$$z = \sqrt[3]{2}\operatorname{cis}\left(\frac{-2\pi}{9} + \frac{2k\pi}{3}\right)$$

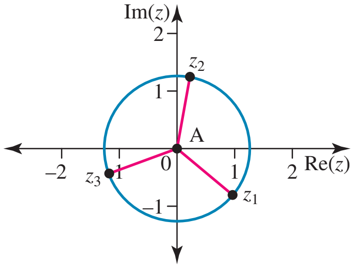
$$\begin{aligned}\text{Let } k = 0 \; z_1 &= \sqrt[3]{2}\operatorname{cis}\left(-\frac{2\pi}{9} + \frac{2\times 0\times\pi}{3}\right)\\ &= \sqrt[3]{2}\operatorname{cis}\left(-\frac{2\pi}{9}\right)\end{aligned}$$

$$\begin{aligned}\text{Let } k = 1 \; z_2 &= \sqrt[3]{2}\operatorname{cis}\left(-\frac{2\pi}{9} + \frac{2\times 1\times\pi}{3}\right)\\ &= \sqrt[3]{2}\operatorname{cis}\left(\frac{4\pi}{9}\right)\end{aligned}$$

$$\begin{aligned}\text{Let } k = 2 \; z_3 &= \sqrt[3]{2}\operatorname{cis}\left(-\frac{2\pi}{9} + \frac{2\times 2\times\pi}{3}\right)\\ &= \sqrt[3]{2}\operatorname{cis}\left(\frac{10\pi}{9}\right)\\ &= \sqrt[3]{2}\operatorname{cis}\left(-\frac{8\pi}{9}\right)\end{aligned}$$

$$z = \sqrt[3]{2}\operatorname{cis}\left(-\frac{2\pi}{9}\right), \sqrt[3]{2}\operatorname{cis}\left(\frac{4\pi}{9}\right), \sqrt[3]{2}\operatorname{cis}\left(-\frac{8\pi}{9}\right)$$

b



The three roots are equally spaced around a circle of radius $\sqrt[3]{2}$. The angle between each solution is $\frac{2\pi}{3}$.

5 **a** $z^4 = -5 - 5i$

$$\begin{aligned}|z| &= \sqrt{(-5)^2 + (-5)^2}, & \theta &= \tan^{-1}\left(\frac{-5}{-5}\right)\\ &= 5\sqrt{2} & \theta &= \frac{-3\pi}{4}\end{aligned}$$

$$\therefore z = -5 - 5i = 5\sqrt{2}\operatorname{cis}\left(\frac{-3\pi}{4}\right)$$

$$z^4 = 5\sqrt{2}\operatorname{cis}\left(\frac{-3\pi}{4}\right)$$

$$z = \sqrt[4]{5\sqrt{2}}\operatorname{cis}\left(\frac{\frac{-3\pi}{4}}{4} + \frac{2k\pi}{4}\right)$$

$$z = \sqrt[4]{5\sqrt{2}}\operatorname{cis}\left(\frac{-3\pi}{16} + \frac{k\pi}{2}\right)$$

$$\begin{aligned}\text{Let } k = 0,\; z_1 &= \sqrt[4]{5\sqrt{2}}\operatorname{cis}\left(\frac{-3\pi}{16} + \frac{0\times\pi}{2}\right)\\ &= \sqrt[4]{5\sqrt{2}}\operatorname{cis}\left(-\frac{3\pi}{16}\right)\end{aligned}$$

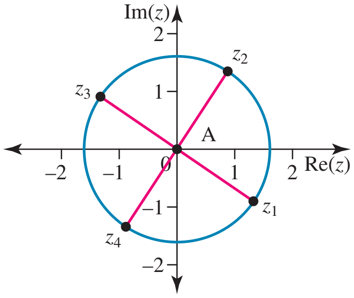
$$\begin{aligned}\text{Let } k = 1,\; z_2 &= \sqrt[4]{5\sqrt{2}}\operatorname{cis}\left(\frac{-3\pi}{16} + \frac{1\times\pi}{2}\right)\\ &= \sqrt[4]{5\sqrt{2}}\operatorname{cis}\left(\frac{5\pi}{16}\right)\end{aligned}$$

$$\begin{aligned}\text{Let } k = 2,\; z_3 &= \sqrt[4]{5\sqrt{2}}\operatorname{cis}\left(\frac{-3\pi}{16} + \frac{2\times\pi}{2}\right)\\ &= \sqrt[4]{5\sqrt{2}}\operatorname{cis}\left(\frac{13\pi}{16}\right)\end{aligned}$$

$$\begin{aligned}\text{Let } k = 3,\; z_4 &= \sqrt[4]{5\sqrt{2}}\operatorname{cis}\left(\frac{-3\pi}{16} + \frac{3\times\pi}{2}\right)\\ &= \sqrt[4]{5\sqrt{2}}\operatorname{cis}\left(\frac{21\pi}{16}\right)\\ &= \sqrt[4]{5\sqrt{2}}\operatorname{cis}\left(-\frac{11\pi}{16}\right)\end{aligned}$$

$$z=\sqrt[4]{5\sqrt{2}}\,\text{cis}\left(-\frac{3\pi}{16}\right), \sqrt[4]{5\sqrt{2}}\,\text{cis}\left(\frac{5\pi}{16}\right),$$
$$\sqrt[4]{5\sqrt{2}}\,\text{cis}\left(\frac{13\pi}{16}\right), \sqrt[4]{5\sqrt{2}}\,\text{cis}\left(-\frac{11\pi}{16}\right)$$

b



The three roots are equally spaced around a circle of radius $\sqrt[4]{5\sqrt{2}} \approx 1.631$. The angle between each solution is $\frac{\pi}{2}$.

6 a $z^6 = 3+\sqrt{3}i$

$$|z| = \sqrt{(3)^2+\left(\sqrt{3}\right)^2},\ \theta = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$$
$$=\sqrt{12} = 2\sqrt{3}\quad \theta = \frac{\pi}{6}$$
$$\therefore z = 3+\sqrt{3}i = 2\sqrt{3}\,\text{cis}\left(\frac{\pi}{6}\right)$$
$$z^6 = 2\sqrt{3}\,\text{cis}\left(\frac{\pi}{6}\right)$$
$$z=\sqrt[6]{2\sqrt{3}}\,\text{cis}\left(\frac{\frac{\pi}{6}}{6}+\frac{2k\pi}{6}\right)$$
$$z=\sqrt[6]{2\sqrt{3}}\,\text{cis}\left(\frac{\pi}{36}+\frac{k\pi}{3}\right)$$

Let $k=0,\ z_1=\sqrt[6]{2\sqrt{3}}\,\text{cis}\left(\frac{\pi}{36}+\frac{0\times\pi}{3}\right)$
$$=\sqrt[6]{2\sqrt{3}}\,\text{cis}\left(\frac{\pi}{36}\right)$$

Let $k=1,\ z_2=\sqrt[6]{2\sqrt{3}}\,\text{cis}\left(\frac{\pi}{36}+\frac{1\times\pi}{3}\right)$
$$=\sqrt[6]{2\sqrt{3}}\,\text{cis}\left(\frac{13\pi}{36}\right)$$

Let $k=2,\ z_3=\sqrt[6]{2\sqrt{3}}\,\text{cis}\left(\frac{\pi}{36}+\frac{2\times\pi}{3}\right)$
$$=\sqrt[6]{2\sqrt{3}}\,\text{cis}\left(\frac{25\pi}{36}\right)$$

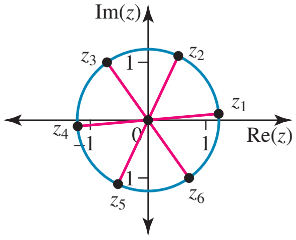
Let $k=3,\ z_4=\sqrt[6]{2\sqrt{3}}\,\text{cis}\left(\frac{\pi}{36}+\frac{3\times\pi}{3}\right)$
$$=\sqrt[6]{2\sqrt{3}}\,\text{cis}\left(\frac{37\pi}{36}\right)$$
$$=\sqrt[6]{2\sqrt{3}}\,\text{cis}\left(-\frac{35\pi}{36}\right)$$

Let $k=4,\ z_5=\sqrt[6]{2\sqrt{3}}\,\text{cis}\left(\frac{\pi}{36}+\frac{4\times\pi}{3}\right)$
$$=\sqrt[6]{2\sqrt{3}}\,\text{cis}\left(\frac{49\pi}{36}\right)$$
$$=\sqrt[6]{2\sqrt{3}}\,\text{cis}\left(-\frac{23\pi}{36}\right)$$

Let $k=5,\ z_6=\sqrt[6]{2\sqrt{3}}\,\text{cis}\left(\frac{\pi}{36}+\frac{5\times\pi}{3}\right)$
$$=\sqrt[6]{2\sqrt{3}}\,\text{cis}\left(\frac{61\pi}{36}\right)$$
$$=\sqrt[6]{2\sqrt{3}}\,\text{cis}\left(-\frac{11\pi}{36}\right)$$

$$z=\sqrt[6]{2\sqrt{3}}\,\text{cis}\left(\frac{\pi}{36}\right), \sqrt[6]{2\sqrt{3}}\,\text{cis}\left(\frac{13\pi}{36}\right),$$
$$\sqrt[6]{2\sqrt{3}}\,\text{cis}\left(\frac{25\pi}{36}\right), \sqrt[6]{2\sqrt{3}}\,\text{cis}\left(-\frac{35\pi}{36}\right),$$
$$\sqrt[6]{2\sqrt{3}}\,\text{cis}\left(-\frac{23\pi}{36}\right), \sqrt[6]{2\sqrt{3}}\,\text{cis}\left(-\frac{11\pi}{36}\right)$$

b



The three roots are equally spaced around a circle of radius $\sqrt[6]{2\sqrt{3}} \approx 1.230$. The angle between each solution is $\frac{\pi}{3}$.

7 a $z^3 = i$

$$i = 1\,\text{cis}\left(\frac{\pi}{2}\right)$$
$$z=\sqrt[3]{1}\,\text{cis}\left(\frac{\frac{\pi}{2}}{3}+\frac{2k\pi}{3}\right)$$
$$z=1\,\text{cis}\left(\frac{\pi}{6}+\frac{2k\pi}{3}\right)$$

Let $k=0,\ z_1=1\,\text{cis}\left(\frac{\pi}{6}+\frac{2\times0\times\pi}{3}\right)$
$$=\text{cis}\left(\frac{\pi}{6}\right)$$

Let $k=1,\ z_2=1\,\text{cis}\left(\frac{\pi}{6}+\frac{2\times1\times\pi}{3}\right)$
$$=\text{cis}\left(\frac{5\pi}{6}\right)$$

Let $k=2,\ z_3=1\,\text{cis}\left(\frac{\pi}{6}+\frac{2\times2\times\pi}{3}\right)$
$$=\text{cis}\left(\frac{9\pi}{6}\right)$$
$$=\text{cis}\left(-\frac{3\pi}{6}\right)$$
$$=\text{cis}\left(-\frac{\pi}{2}\right)$$

$$z=\text{cis}\left(\frac{\pi}{6}\right), \text{cis}\left(\frac{5\pi}{6}\right), \text{cis}\left(-\frac{\pi}{2}\right)$$

b $z^4 = 64i$

$$64i = 64\,\text{cis}\left(\frac{\pi}{2}\right)$$
$$z^4 = 64\,\text{cis}\left(\frac{\pi}{2}\right)$$
$$z=\sqrt[4]{64}\,\text{cis}\left(\frac{\frac{\pi}{2}}{4}+\frac{2k\pi}{4}\right)$$
$$z=2\sqrt{2}\,\text{cis}\left(\frac{\pi}{8}+\frac{k\pi}{2}\right)$$

$$\text{Let } k=0,\ z_1 = 2\sqrt{2}\,\text{cis}\left(\frac{\pi}{8}+\frac{0\times\pi}{2}\right)$$
$$= 2\sqrt{2}\,\text{cis}\left(\frac{\pi}{8}\right)$$
$$\text{Let } k=1,\ z_2 = 2\sqrt{2}\,\text{cis}\left(\frac{\pi}{8}+\frac{1\times\pi}{2}\right)$$
$$= 2\sqrt{2}\,\text{cis}\left(\frac{5\pi}{8}\right)$$
$$\text{Let } k=2,\ z_3 = 2\sqrt{2}\,\text{cis}\left(\frac{\pi}{8}+\frac{2\times\pi}{2}\right)$$
$$= 2\sqrt{2}\,\text{cis}\left(\frac{9\pi}{8}\right)$$
$$= 2\sqrt{2}\,\text{cis}\left(\frac{-7\pi}{8}\right)$$
$$\text{Let } k=3,\ z_4 = 2\sqrt{2}\,\text{cis}\left(\frac{\pi}{8}+\frac{3\times\pi}{2}\right)$$
$$= 2\sqrt{2}\,\text{cis}\left(\frac{13\pi}{8}\right)$$
$$= 2\sqrt{2}\,\text{cis}\left(\frac{-3\pi}{8}\right)$$
$$z = 2\sqrt{2}\,\text{cis}\left(\frac{\pi}{8}\right),\ 2\sqrt{2}\,\text{cis}\left(\frac{5\pi}{8}\right),\ 2\sqrt{2}\,\text{cis}\left(\frac{-7\pi}{8}\right),$$
$$2\sqrt{2}\,\text{cis}\left(\frac{-3\pi}{8}\right)$$

8 $z^2 = 2+2\sqrt{3}i$

Let $z = x+yi$ where $x, y \in R$

$$(x+yi)^2 = 2+2\sqrt{3}i$$
$$x^2+2xyi+y^2i^2 = 2+2\sqrt{3}i$$
$$x^2-y^2+2xyi = 2+2\sqrt{3}i$$

Equate the real and imaginary parts.

$x^2-y^2=2 \qquad [1]$

$2xy = 2\sqrt{3} \qquad [2]$

From [2]: $y = \dfrac{\sqrt{3}}{x}$

Substitute $y = \dfrac{\sqrt{3}}{x}$ this into [1]

$$x^2-\left(\frac{\sqrt{3}}{x}\right)^2 = 2$$
$$x^2-\frac{3}{x^2} = 2$$
$$x^4-3 = 2x^2$$
$$x^4-2x^2-3 = 0$$
$$\left(x^2\right)^2-2(x^2)-3 = 0$$

Let $\alpha = x^2$

$$\alpha^2-2\alpha-3=0$$
$$(\alpha-3)(\alpha+1)=0$$
$$\therefore \alpha = 3,\ \alpha=-1$$
$$\Rightarrow\ x^2=3 \text{ and } x^2=-1$$

We can ignore the solutions to $x^2=-1$ since $x \in R$

$\therefore x^2 = 3 \ \Rightarrow\ x = \pm\sqrt{3}$

When $x = +\sqrt{3},\ y = \dfrac{\sqrt{3}}{\sqrt{3}} = 1$

When $x = -\sqrt{3},\ y = \dfrac{\sqrt{3}}{-\sqrt{3}} = -1$

The solutions to the equation $z^2 = 2+2\sqrt{3}i$ are $z = \sqrt{3}+i$ and $z = -\sqrt{3}-i$

9 RTP:

$z_1 \times z_2 \times z_3 = 1$

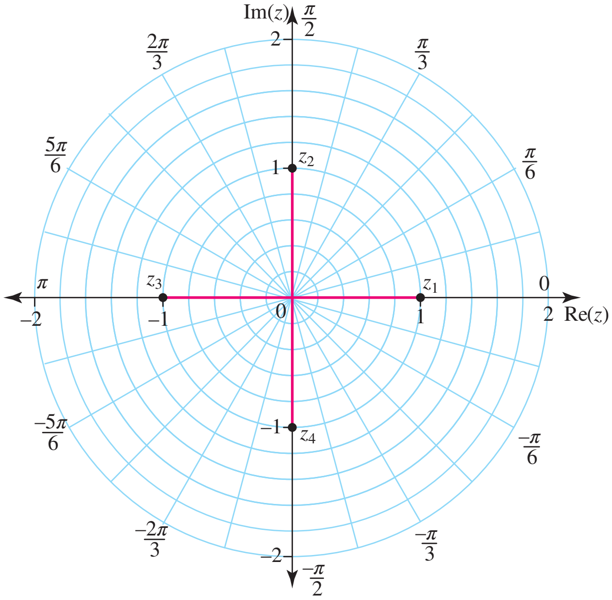
LHS

$$= 1\times\text{cis}\left(\frac{2\pi}{3}\right)\times\text{cis}\left(-\frac{2\pi}{3}\right)$$
$$= 1\times\text{cis}\left(\frac{2\pi}{3}+\left(-\frac{2\pi}{3}\right)\right)$$
$$= 1\times\text{cis}(0)$$
$$= \cos(0)+i\sin(0)$$
$$= 1$$
$$= \text{RHS}$$

10 a $1 = \text{cis}(0)$

$$z^4 = \text{cis}(0)$$
$$z = \sqrt[4]{1}\,\text{cis}(\frac{0}{4}+\frac{2k\pi}{4})$$
$$z = 1\,\text{cis}\left(\frac{2k\pi}{4}\right)$$
$$\text{Let } k=0,\ z_1 = \text{cis}\left(\frac{2\times0\times\pi}{4}\right)$$
$$= \text{cis}(0)$$
$$= 1$$
$$\text{Let } k=1,\ z_2 = \text{cis}\left(\frac{2\times1\times\pi}{4}\right)$$
$$= \text{cis}\left(\frac{\pi}{2}\right)$$
$$\text{Let } k=2,\ z_3 = \text{cis}\left(\frac{2\times2\times\pi}{4}\right)$$
$$= \text{cis}(\pi)$$
$$\text{Let } k=3,\ z_4 = \text{cis}\left(\frac{2\times3\times\pi}{4}\right)$$
$$= \text{cis}\left(\frac{3\pi}{2}\right)$$
$$= \text{cis}\left(-\frac{\pi}{2}\right)$$
$$z = 1,\ \text{cis}\left(\frac{\pi}{2}\right),\ \text{cis}(\pi),\ \text{cis}\left(\frac{-\pi}{2}\right)$$

b



11 RTP:

$z_1 + z_2 + z_3 + z_4 = 0$

LHS

$= 1 + \text{cis}\left(\frac{\pi}{2}\right) + \text{cis}(\pi) + \text{cis}\left(\frac{-\pi}{2}\right)$

$= 1 + (i) + (-1) + (-i)$

$= 0$

$=$ RHS

12 **a** $1 = \text{cis}(0)$

$z^5 = \text{cis}(0)$

$$z = \sqrt[5]{1}\,\text{cis}\left(\frac{0}{5} + \frac{2k\pi}{5}\right)$$

$$z = 1\,\text{cis}\left(\frac{2k\pi}{5}\right)$$

$$\text{Let } k = 0,\ z_1 = \text{cis}\left(\frac{2\times 0\times \pi}{5}\right)$$

$$= \text{cis}(0)$$

$$= 1$$

$$\text{Let } k = 1,\ z_2 = \text{cis}\left(\frac{2\times 1\times \pi}{5}\right)$$

$$= \text{cis}\left(\frac{2\pi}{5}\right)$$

$$\text{Let } k = 2,\ z_3 = \text{cis}\left(\frac{2\times 2\times \pi}{5}\right)$$

$$= \text{cis}\left(\frac{4\pi}{5}\right)$$

$$\text{Let } k = 3,\ z_4 = \text{cis}\left(\frac{2\times 3\times \pi}{5}\right)$$

$$= \text{cis}\left(\frac{6\pi}{5}\right)$$

$$= \text{cis}\left(-\frac{4\pi}{5}\right)$$

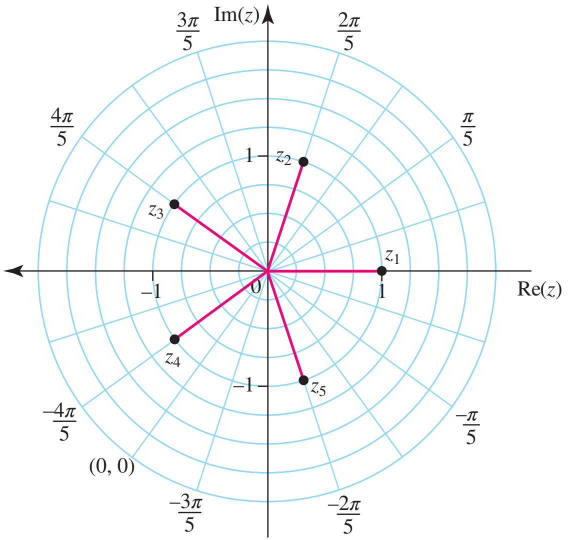
$$\text{Let } k = 4,\ z_4 = \text{cis}\left(\frac{2\times 4\times \pi}{5}\right)$$

$$= \text{cis}\left(\frac{8\pi}{5}\right)$$

$$= \text{cis}\left(-\frac{2\pi}{5}\right)$$

$$z = 1,\ \text{cis}\left(\frac{2\pi}{5}\right),\ \text{cis}\left(\frac{4\pi}{5}\right),\ \text{cis}\left(\frac{-4\pi}{5}\right),\ \text{cis}\left(\frac{-2\pi}{5}\right)$$

b



13 The solutions would be equally spaced by $\frac{2\pi}{5}$ radians and lie on a circle of radius $10^{\frac{1}{5}}$. The modulus of each solution would change from $|z| = 1$ to $|z| = 10^{\frac{1}{5}}$.

14 **a** $z^3 = 8$

$8 = 8\,\text{cis}(0)$

$z^3 = 8\,\text{cis}(0)$

$$z = \sqrt[3]{8}\,\text{cis}\left(\frac{0}{3} + \frac{2k\pi}{3}\right)$$

$$z = 2\,\text{cis}\left(\frac{2k\pi}{3}\right)$$

$$\text{Let } k = 0,\ z_1 = 2\,\text{cis}\left(\frac{2k\pi}{3}\right)$$

$$= 2\,\text{cis}\left(\frac{2\times 0\times \pi}{3}\right)$$

$$= 2$$

$$\text{Let } k = 1,\ z_2 = 2\,\text{cis}\left(\frac{2k\pi}{3}\right)$$

$$= 2\,\text{cis}\left(\frac{2\times 1\times \pi}{3}\right)$$

$$= 2\,\text{cis}\left(\frac{2\pi}{3}\right)$$

$$\text{Let } k = 2,\ z_3 = 2\,\text{cis}\left(\frac{2k\pi}{3}\right)$$

$$= 2\,\text{cis}\left(\frac{2\times 2\times \pi}{3}\right)$$

$$= 2\,\text{cis}\left(\frac{4\pi}{3}\right)$$

$$= 2\,\text{cis}\left(\frac{-2\pi}{3}\right)$$

Solutions in polar form are:

$$z = 2,\ 2\,\text{cis}\left(\frac{2\pi}{3}\right),\ 2\,\text{cis}\left(\frac{-2\pi}{3}\right)$$

In Cartesian form:

$$z = 2$$

$$z = 2\,\text{cis}\left(\frac{2\pi}{3}\right) = 2\cos\left(\frac{2\pi}{3}\right) + 2i\sin\left(\frac{2\pi}{3}\right)$$

$$= -1 + \sqrt{3}i$$

$$z = 2\,\text{cis}\left(\frac{-2\pi}{3}\right) = 2\cos\left(\frac{-2\pi}{3}\right) + 2i\sin\left(\frac{-2\pi}{3}\right)$$

$$= -1 - \sqrt{3}i$$

b $z^4 = 256$

$$256 = 256\,\text{cis}(0)$$

$$z^4 = 256\,\text{cis}(0)$$

$$z = \sqrt[4]{256}\,\text{cis}\left(\frac{0}{4} + \frac{2k\pi}{4}\right)$$

$$z = 4\,\text{cis}\left(\frac{k\pi}{2}\right)$$

$$\text{Let } k = 0,\ z_1 = 4\,\text{cis}\left(\frac{0 \times \pi}{2}\right)$$

$$= 4$$

$$\text{Let } k = 1,\ z_2 = 4\,\text{cis}\left(\frac{1 \times \pi}{2}\right)$$

$$= 4\,\text{cis}\left(\frac{\pi}{2}\right)$$

$$\text{Let } k = 2,\ z_3 = 4\,\text{cis}\left(\frac{2 \times \pi}{2}\right)$$

$$= 4\,\text{cis}(\pi)$$

$$\text{Let } k = 3,\ z_4 = 4\,\text{cis}\left(\frac{3 \times \pi}{2}\right)$$

$$= 4\,\text{cis}\left(\frac{3\pi}{2}\right)$$

$$= 4\,\text{cis}\left(\frac{-\pi}{2}\right)$$

Solutions in polar form are:

$$z = 4,\ 4\,\text{cis}\left(\frac{\pi}{2}\right),\ 4\,\text{cis}(\pi),\ 4\,\text{cis}\left(\frac{-\pi}{2}\right)$$

In Cartesian form:

$$z = 4$$

$$z = 4\,\text{cis}\left(\frac{\pi}{2}\right) = 4\cos\left(\frac{\pi}{2}\right) + 4i\sin\left(\frac{\pi}{2}\right) = 4i$$

$$z = 4\,\text{cis}(\pi) = 4\cos(\pi) + 4i\sin(\pi) = -4$$

$$z = 4\,\text{cis}\left(\frac{-\pi}{2}\right) = 4\cos\left(\frac{-\pi}{2}\right) + 4i\sin\left(\frac{-\pi}{2}\right) = -4i$$

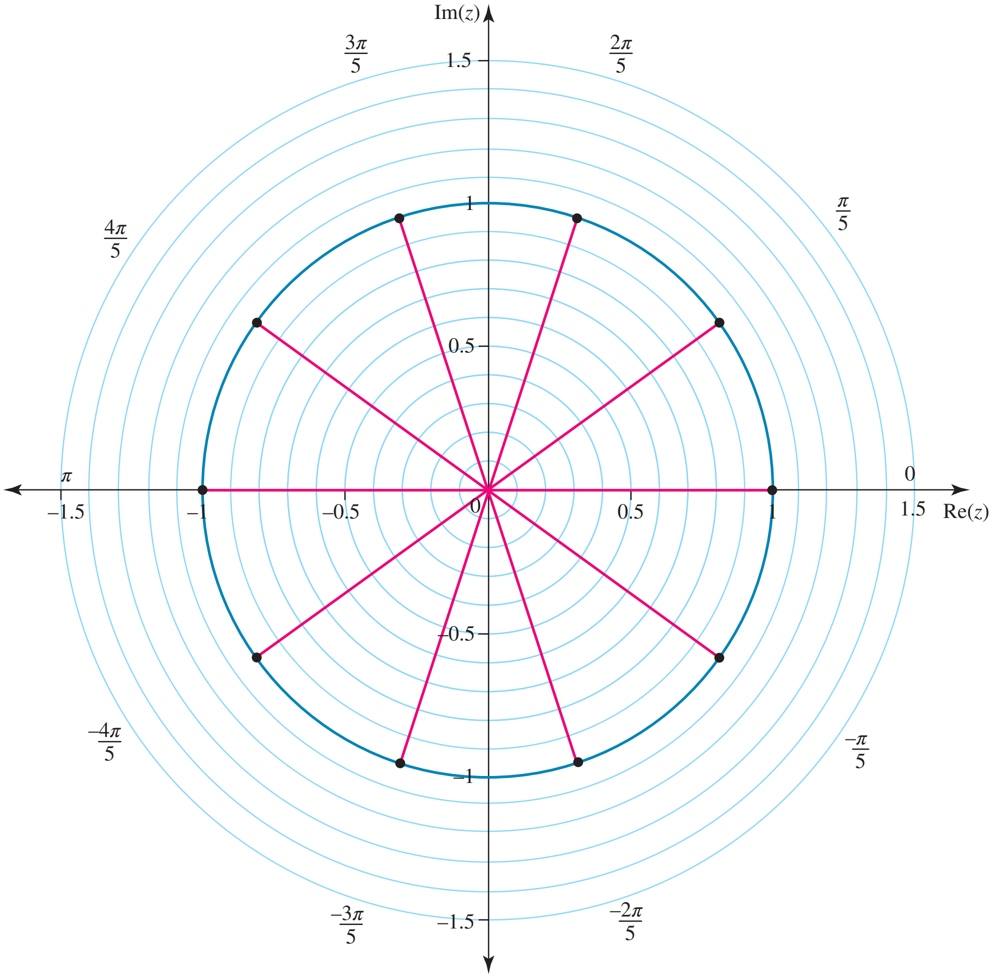
15 See image at the bottom*

16 Let $z^3 = 1$

$$z^3 = 1\ \text{cis}(0 + 2k\pi)$$

$$z = 1\ \text{cis}\left(\frac{2k\pi}{3}\right)$$

***15**



$k = 0 \quad z = 1$

$k = 1 \quad z = 1 \text{ cis}\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i = u$

$k = -1 \; z = 1 \text{ cis}\left(-\frac{2\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i = v$

a So $v = \bar{u}$

b
$$\begin{aligned} u^2 &= \left(1 \text{ cis}\left(\frac{2\pi}{3}\right)\right)^2 \\ &= 1 \text{ cis}\left(\frac{4\pi}{3}\right) \\ &= \text{cis}\left(\frac{4\pi}{3} - 2\pi\right) \\ &= \text{cis}\left(-\frac{2\pi}{3}\right) = v \end{aligned}$$

So $u^2 = v$ shown

c
$$\begin{aligned} 1 + u + v &= 1 + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \\ &= 0 \quad \text{shown} \end{aligned}$$

17
$$\begin{aligned} u &= 2 - 2i \\ &= 2\sqrt{2} \text{ cis}\left(-\frac{\pi}{4}\right) \end{aligned}$$

a
$$\begin{aligned} u^4 &= \left(2\sqrt{2}\right)^4 \text{ cis}\left(4 \times -\frac{\pi}{4}\right) \\ &= 64 \text{ cis}(-\pi) \\ &= 64 \text{ cis}(\pi) \\ &= -64 \end{aligned}$$

So $\text{Arg}(u^4) = \pi, \quad u^4 = -64$

b
$$\begin{aligned} z^4 + 64 &= 0 \\ z^4 &= -64 \\ &= 64 \text{ cis}(\pi + 2k\pi) \\ z &= \sqrt[4]{64} \text{ cis}\left(\frac{\pi}{4} + \frac{k\pi}{2}\right) \\ &= 2\sqrt{2} \text{ cis}\left(\frac{\pi}{4} + \frac{k\pi}{2}\right) \end{aligned}$$

$k = 0 \quad z = 2\sqrt{2} \text{ cis}\left(\frac{\pi}{4}\right) = 2\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)\right)$
$\quad = 2 + 2i$

$k = 1 \quad z = 2\sqrt{2} \text{ cis}\left(\frac{3\pi}{4}\right) = 2\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right)\right)$
$\quad = -2 + 2i$

$k = -1 \quad z = 2\sqrt{2} \text{ cis}\left(-\frac{\pi}{4}\right) = 2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right)\right)$
$\quad = 2 - 2i$

$k = -2 \quad z = 2\sqrt{2} \text{ cis}\left(-\frac{3\pi}{4}\right) = 2\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i \sin\left(-\frac{3\pi}{4}\right)\right)$
$\quad = -2 - 2i$

c Four answers: $\pm(2 + 2i)$, $\pm(2 - 2i)$, all the roots are on a circle of radius $2\sqrt{2}$, and are equally spaced around the circle by 90°, four roots, consist of two pairs of complex conjugates.
See image at the bottom*

18 a
$$\begin{aligned} z^6 - 64 &= 0 \\ z^6 &= 64 \\ &= 64 \text{ cis}(0 + 2k\pi) \\ z &= \sqrt[6]{64} \text{ cis}\left(\frac{k\pi}{3}\right) \\ &= 2 \text{ cis}\left(\frac{k\pi}{3}\right) \end{aligned}$$

$k = 0 \quad z = 2 \text{ cis}(0) = 2$

$k = 1 \quad z = 2 \text{ cis}\left(\frac{\pi}{3}\right) = 1 + \sqrt{3}i$

$k = 2 \quad z = 2 \text{ cis}\left(\frac{2\pi}{3}\right) = -1 + \sqrt{3}i$

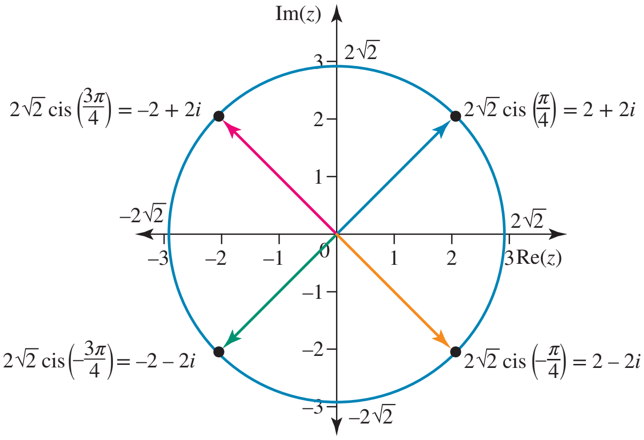
$k = 3 \quad z = 2 \text{ cis}(\pi) = -2$

$k = -1 \quad z = 2 \text{ cis}\left(-\frac{\pi}{3}\right) = 1 - \sqrt{3}i$

$k = -2 \quad z = 2 \text{ cis}\left(\frac{-2\pi}{3}\right) = -1 - \sqrt{3}i$

6 answers: $1 \pm \sqrt{3}i, -1 \pm \sqrt{3}i, \pm 2,$
$2 \text{ cis}\left(\pm\frac{\pi}{3}\right), 2 \text{ cis}\left(\pm\frac{2\pi}{3}\right), 2 \text{ cis}(\pi), 2 \text{ cis}(0)$

***17c**



b $z^6 + 64 = 0$

$$\begin{aligned} z^6 &= -64 \\ &= 64\,\text{cis}(\pi + 2k\pi) \\ z &= \sqrt[6]{64}\,\text{cis}\left(\frac{\pi}{6} + \frac{2k\pi}{6}\right) \\ &= 2\,\text{cis}\left(\frac{\pi}{6} + \frac{k\pi}{3}\right) \end{aligned}$$

$k = 0 \quad z = 2\text{ cis}\left(\frac{\pi}{6}\right) = \sqrt{3} + i$

$k = 1 \quad z = 2\text{ cis}\left(\frac{\pi}{2}\right) = 2i$

$k = 2 \quad z = 2\text{ cis}\left(\frac{5\pi}{6}\right) = -\sqrt{3} + i$

$k = -1 \quad z = 2\text{ cis}\left(-\frac{\pi}{6}\right) = \sqrt{3} - i$

$k = -2 \quad z = 2\text{ cis}\left(-\frac{\pi}{2}\right) = -2i$

$k = -3 \quad z = 2\text{ cis}\left(-\frac{5\pi}{6}\right) = -\sqrt{3} - i$

6 answers: $-\sqrt{3} \pm i, \sqrt{3} \pm i, \pm 2i,$
$2\text{ cis}\left(\pm\frac{5\pi}{6}\right), 2\text{ cis}\left(\pm\frac{\pi}{6}\right), 2\text{ cis}\left(\pm\frac{\pi}{2}\right)$

19 $z^8 - 16 = 0$

$$\begin{aligned} z^8 &= 16 \\ &= 16\text{ cis}(0 + 2k\pi) \\ z &= \sqrt[8]{16}\text{ cis}\left(\frac{2k\pi}{8}\right) \\ &= \sqrt{2}\text{ cis}\left(\frac{k\pi}{4}\right) \end{aligned}$$

$k = 0 \quad z = \sqrt{2}\text{ cis}(0) = \sqrt{2}$

$k = 1 \quad z = \sqrt{2}\text{ cis}\left(\frac{\pi}{4}\right) = 1 + i$

$k = 2 \quad z = \sqrt{2}\text{ cis}\left(\frac{\pi}{2}\right) = \sqrt{2}i$

$k = 3 \quad z = \sqrt{2}\text{ cis}\left(\frac{3\pi}{4}\right) = -1 + i$

$k = 4 \quad z = \sqrt{2}\text{ cis}(\pi) = -\sqrt{2}$

$k = -1 \quad z = \sqrt{2}\text{ cis}\left(-\frac{\pi}{4}\right) = -1 - i$

$k = -2 \quad z = \sqrt{2}\text{ cis}\left(-\frac{\pi}{2}\right) = -\sqrt{2}i$

$k = -3 \quad z = \sqrt{2}\text{ cis}\left(-\frac{3\pi}{4}\right) = -1 - i$

8 answers: $1 \pm i, -1 \pm i, \pm\sqrt{2}i,\ \pm\sqrt{2}, \sqrt{2}\text{cis}\left(\pm\frac{\pi}{4}\right),$
$\sqrt{2}\text{ cis}\left(\pm\frac{\pi}{2}\right),\ 2\text{ cis}\left(\pm\frac{3\pi}{4}\right),\ \sqrt{2}\text{ cis}(0), \sqrt{2}\text{ cis}(\pi)$

All the 8 roots are on a circle of radius $\sqrt{2}$, there are 3 pairs of complex conjugate roots (2 real).
All the roots are equally spaced around the circle by 45° $\left(\frac{\pi}{4}\right)$.

20 $z^{12} = 4096$

$$\begin{aligned} z^{12} &= 4096\text{ cis}(0 + 2k\pi) \\ z &= \sqrt[12]{4096}\text{ cis}\left(\frac{k\pi}{6}\right) \\ &= 2\text{ cis}\left(\frac{k\pi}{6}\right) \end{aligned}$$

$k = 0 \quad z = 2\text{ cis}(0) = 2$

$k = 1 \quad z = 2\text{ cis}\left(\frac{\pi}{6}\right) = \sqrt{3} + i$

$k = 2 \quad z = 2\text{ cis}\left(\frac{\pi}{3}\right) = 1 + \sqrt{3}i$

$k = 3 \quad z = 2\text{ cis}\left(\frac{\pi}{2}\right) = 2i$

$k = 4 \quad z = 2\text{ cis}\left(\frac{2\pi}{3}\right) = -1 + \sqrt{3}i$

$k = 5 \quad z = 2\text{ cis}\left(\frac{5\pi}{6}\right) = -\sqrt{3} + i$

$k = 6 \quad z = 2\text{ cis}(\pi) = -2$

$k = -1 \quad z = 2\text{ cis}\left(-\frac{\pi}{6}\right) = \sqrt{3} - i$

$k = -2 \quad z = 2\text{ cis}\left(-\frac{\pi}{3}\right) = 1 - \sqrt{3}i$

$k = -3 \quad z = 2\text{ cis}\left(-\frac{\pi}{2}\right) = -2i$

$k = -4 \quad z = 2\text{ cis}\left(-\frac{2\pi}{3}\right) = -1 - \sqrt{3}i$

$k = -5 \quad z = 2\text{ cis}\left(-\frac{5\pi}{6}\right) = -\sqrt{3} - i$

12 answers: $-\sqrt{3} \pm i,\ \sqrt{3} \pm i,\ -1 \pm \sqrt{3}i, 1 \pm \sqrt{3}i,\ \pm 2,\ \pm 2i$ or
$2\text{cis}\left(\pm\frac{\pi}{6}\right), 2\text{ cis}\left(\pm\frac{\pi}{3}\right), 2\text{ cis}\left(\pm\frac{\pi}{2}\right), 2\text{ cis}\left(\pm\frac{5\pi}{6}\right),$
$2\text{ cis}\left(\pm\frac{2\pi}{3}\right), 2\text{ cis}(0), 2\text{ cis}(\pi).$

All the 12 roots are on a circle of radius 2, and there are 5 pairs of complex conjugate roots (2 real).
All the roots are equally spaced around the circle by 30° $\left(\frac{\pi}{6}\right)$.

2.6 Exam questions

1 $z^3 = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i = \text{cis}\left(-\frac{\pi}{4} + 2k\pi\right) = \text{cis}\left(\frac{\pi(8k-1)}{4}\right)$

$z = \text{cis}\left(\frac{\pi(8k-1)}{12}\right)$

$k = 0, \quad z = \text{cis}\left(-\frac{\pi}{12}\right)$

$k = 1, \quad z = \text{cis}\left(\frac{7\pi}{12}\right)$

$k = -1, \quad z = \text{cis}\left(\frac{-9\pi}{12}\right) = \text{cis}\left(-\frac{3\pi}{4}\right)$

Award 1 mark for using *k*s with DeMoivre's Theorem.
Award 2 marks for the correct three values.

2 $z^n = 1 + i,\ n \in Z^+$

$z^n = \sqrt{2}\,\text{cis}\left(\frac{\pi}{4} + 2k\pi\right)$

$= 2^{\frac{1}{2}}\text{cis}\left(\frac{\pi}{4} + 2k\pi\right)$

$z = 2^{\frac{1}{2n}}\text{cis}\left(\frac{\pi}{4n} + \frac{2k\pi}{n}\right),\ k \in Z$

The correct answer is **E**.

3 **a** $z^3 = 8i, z \in C$

$z^3 = 8\,\text{cis}\left(\frac{\pi}{2} + 2k\pi\right)$

$z = 8^{\frac{1}{3}}\text{cis}\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right)$

$k = 0, z = 2\,\text{cis}\left(\frac{\pi}{6}\right) = \sqrt{3} + i$ [1 mark]

$k = 1, z = 2\,\text{cis}\left(\frac{5\pi}{6}\right) = -\sqrt{3} + i$ [1 mark]

$k = -1, z = 2\,\text{cis}\left(-\frac{\pi}{2}\right) = -2i$ [1 mark]

b $(z - 2i)^3 = 8i, z \in C$

$z - 2i = \sqrt{3} + i, -\sqrt{3} + i, -2i$

$z = \sqrt{3} + 3i, -\sqrt{3} + 3i, 0$ [1 mark]

2.7 Review

2.7 Exercise

Technology free: short answer

1 **a** $= \text{Re}\left(\frac{5+2i}{5-2i}\right) + \text{Im}\left(\frac{5-2i}{5+2i}\right)$

$= \text{Re}\left(\frac{5+2i}{5-2i} \times \frac{5+2i}{5+2i}\right) + \text{Im}\left(\frac{5-2i}{5+2i} \times \frac{5-2i}{5-2i}\right)$

$= \text{Re}\left(\frac{25+20i+4i^2}{25-4i^2}\right) + \text{Im}\left(\frac{25-20i+4i^2}{25-4i^2}\right)$

$= \text{Re}\left(\frac{21+20i}{29}\right) + \text{Im}\left(\frac{21-20i}{29}\right)$

$= \frac{21}{29} + \frac{-20}{29}$

$= \frac{1}{29}$

b $\text{Im}\left(\frac{d-2i}{3-7i}\right) = 0$

$\text{Im}\left(\frac{d-2i}{3-7i} \times \frac{3+7i}{3+7i}\right) = 0$

$\text{Im}\left(\frac{3d-6i+7di-14i^2}{9-49i^2}\right) = 0$

$\text{Im}\left(\frac{3d+14+i(7d-6)}{58}\right) = 0$

$\Rightarrow 7d - 6 = 0$

$d = \frac{6}{7}$

2 **a** $\frac{(-1+i)^4}{\left(-\sqrt{3}+i\right)^6}$

$= \frac{\left[\sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right)\right]^4}{\left[2\text{cis}\left(\frac{5\pi}{6}\right)\right]^6}$

$= \frac{\left(\sqrt{2}\right)^4\text{cis}\left(4 \times \frac{3\pi}{4}\right)}{(2)^6\text{cis}\left(6 \times \frac{5\pi}{6}\right)}$

$= \frac{2^2}{2^6}\frac{\text{cis}(3\pi)}{\text{cis}(5\pi)}$

$= \frac{1}{16}\frac{\text{cis}(-\pi)}{\text{cis}(\pi)}$

$= \frac{1}{16}\text{cis}(-2\pi)$

$= \frac{1}{16}$

b $(-1-i)^7\left(\sqrt{3}-i\right)^6$

$= \left[\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)\right]^7\left[2\,\text{cis}\left(-\frac{\pi}{6}\right)\right]^6$

$= \left(\sqrt{2}\right)^7\text{cis}\left(-7 \times \frac{3\pi}{4}\right)2^6\text{cis}\left(6 \times \frac{-\pi}{6}\right)$

$= \left(\sqrt{2}\right)^7 \times 2^6\text{cis}\left(\frac{-21\pi}{4} - \pi\right)$

$= \left(\sqrt{2}\right)^{19}\text{cis}\left(\frac{-25\pi}{4}\right)$

$= 512\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

$= 512\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$

$= 512 - 512i$

3 **a** $z^3 - 2iz^2 + 10z - 20i = 0$

$z^2(z - 2i) + 10(z - 2i) = 0$

$\left(z^2 + 10\right)(z - 2i) = 0$

$\left(z^2 - 10i^2\right)(z - 2i) = 0$

$\left(z + \sqrt{10}i\right)\left(z - \sqrt{10}i\right)(z - 2i) = 0$

$z = \pm\sqrt{10}i,\ 2i$

b $2z^4 - z^2 - 45 = 0$

$\left(2z^2 + 9\right)\left(z^2 - 5\right) = 0$

$2\left(z^2 + \frac{9}{2}\right)\left(z^2 - 5\right) = 0$

$2\left(z^2 - \frac{9}{2}i^2\right)\left(z^2 - 5\right) = 0$

$2\left(z + \frac{3i}{\sqrt{2}}\right)\left(z - \frac{3i}{\sqrt{2}}\right)\left(z + \sqrt{5}\right)\left(z - \sqrt{5}\right) = 0$

$z = \pm\frac{3\sqrt{2}}{2}i,\ \pm\sqrt{5}$

4 **a** $(5 - 12i)^n - (5 + 12i)^n = 0$

$[13\,\text{cis}(-\alpha)]^n - [13\,\text{cis}(\alpha)]^n = 0$

Where $\alpha = \tan^{-1}\left(\frac{12}{5}\right)$

$13^n\text{cis}(-n\alpha) - 13^n\text{cis}(n\alpha) = 0$

$13^n(\text{cis}(-n\alpha) - \text{cis}(n\alpha)) = 0$

$13^n[\cos(-n\alpha) + i\sin(-n\alpha) - (\cos(-n\alpha) + i\sin(n\alpha))] = 0$

$-2\times 13^n i\sin(n\alpha)=0$

$\sin(n\alpha)=0$

$n\alpha=k\pi$

$n=\dfrac{k\pi}{\alpha}$

$=\dfrac{k\pi}{\tan^{-1}\left(\frac{12}{5}\right)},\quad k\in Z$

b $(3+4i)^n+(3-4i)^n=0$

$[5\operatorname{cis}(\alpha)]^n+[5\operatorname{cis}(-\alpha)]^n=0$

Where $\alpha=\tan^{-1}\left(\dfrac{4}{3}\right)$

$5^n\operatorname{cis}(n\alpha)+5^n\operatorname{cis}(-n\alpha)=0$

$5^n[\operatorname{cis}(n\alpha)+\operatorname{cis}(-n\alpha)]=0$

$5^n[\cos(n\alpha)+i\sin(n\alpha)+\cos(-n\alpha)+i\sin(-n\alpha)]=0$

$2\times 5^n\cos(n\alpha)=0$

$\cos(n\alpha)=0$

$n\alpha=(2k+1)\dfrac{\pi}{2}$

$n=\dfrac{(2k+1)\pi}{2\tan^{-1}\left(\frac{4}{3}\right)},\quad k\in Z$

5 a $|z-4|=2|z-1|$

Let $z=x+iy$

$|(x-4)+iy|=2|(x-1)+iy|$

$\sqrt{(x-4)^2+y^2}=2\sqrt{(x-1)^2+y^2}$

$x^2-8x+16+y^2=4\left[\left(x^2-2x+1\right)+y^2\right]$

$=4x^2-8x+4+4y^2$

$3x^2+3y^2=12$

$x^2+y^2=4$

Circle centre at origin radius = 2

b $|z-4|=|z-1|$

Let $z=x+iy$

$|(x-4)+iy|=|(x-1)+iy|$

$\sqrt{(x-4)^2+y^2}=\sqrt{(x-1)^2+y^2}$

$x^2-8x+16+y^2=x^2-2x+1+y^2$

$15=6x$

$x=\dfrac{5}{2}$ line

c $|z-4i|=\operatorname{Im}(z)-2$

Let $z=x+iy\quad \operatorname{Im}(z)=y$

$|x+(y-4)i|=y-2$

$\sqrt{x^2+(y-4)^2}=(y-2)^2$

$x^2+y^2-8y+16=y^2-4y+4$

$x^2+12=4y$

$y=\dfrac{x^2}{4}+3$ parabola

d $|z-4|=2(\operatorname{Re}(z)-1)\qquad \operatorname{Re}(z)\geq 1$

Let $z=x+iy\quad \operatorname{Re}(z)=x$

$|(x-4)+iy|=2(x-1)$

$\sqrt{(x-4)^2+y^2}=2(x-1)$

$(x-4)^2+y^2=4(x-1)^2$

$x^2-8x+16+y^2=4x^2-8x+4$

$12=3x^2-y^2$

$\dfrac{x^2}{4}-\dfrac{y^2}{12}=1$ hyperbola

$x>1$ right hand branch only

6 a $z^2+i=0$

$z^2=-i$

$=1\operatorname{cis}\left(-\dfrac{\pi}{2}+2k\pi\right)$

$z=\sqrt{1}\operatorname{cis}\left(-\dfrac{\pi}{4}+k\pi\right)$

$k=0\quad z=1\operatorname{cis}\left(-\dfrac{\pi}{4}\right)=\dfrac{\sqrt{2}}{2}-i\dfrac{\sqrt{2}}{2}$

$k=1\quad z=1\operatorname{cis}\left(\dfrac{3\pi}{4}\right)=\dfrac{-\sqrt{2}}{2}+i\dfrac{\sqrt{2}}{2}$

2 answers: $z=\dfrac{\sqrt{2}}{2}(1-i),\ \dfrac{\sqrt{2}}{2}(-1+i)$

b $z^3+27=0$

$z^3=-27$

$=27\operatorname{cis}(\pi+2k\pi)$

$z=\sqrt[3]{27}\operatorname{cis}\left(\dfrac{\pi+2k\pi}{3}\right)$

$z=3\operatorname{cis}\left(\dfrac{\pi}{3}+\dfrac{2k\pi}{3}\right)$

$k=0\quad z=3\operatorname{cis}\left(\dfrac{\pi}{3}\right)=3\cos\left(\dfrac{\pi}{3}\right)$

$+3i\sin\left(\dfrac{\pi}{3}\right)=\dfrac{3}{2}+\dfrac{3}{2}\sqrt{3}i$

$k=-1\quad z=3\operatorname{cis}\left(-\dfrac{\pi}{3}\right)=3\cos\left(-\dfrac{\pi}{3}\right)$

$+3i\sin\left(-\dfrac{\pi}{3}\right)=\dfrac{3}{2}-\dfrac{3}{2}\sqrt{3}i$

$k=1\quad z=3\operatorname{cis}(\pi)=-3$

3 answers: $z=-3,\ \dfrac{3}{2}\left(1\pm\sqrt{3}i\right)$

c $z^3+27i=0$

$z^3=-27i$

$=27\operatorname{cis}\left(-\dfrac{\pi}{2}+2k\pi\right)$

$z=\sqrt[3]{27}\operatorname{cis}\left(-\dfrac{\pi}{6}+\dfrac{2k\pi}{3}\right)$

$=3\operatorname{cis}\left(-\dfrac{\pi}{6}+\dfrac{2k\pi}{3}\right)$

$k=0\quad z=3\operatorname{cis}\left(-\dfrac{\pi}{6}\right)=3\cos\left(-\dfrac{\pi}{6}\right)$

$+3i\sin\left(-\dfrac{\pi}{6}\right)=\dfrac{3\sqrt{3}}{2}-\dfrac{3}{2}i$

$k=1\quad z=3\operatorname{cis}\left(\dfrac{\pi}{2}\right)=3i$

$k=-1\quad z=3\operatorname{cis}\left(-\dfrac{5\pi}{6}\right)=3\cos\left(-\dfrac{5\pi}{6}\right)$

$+3i\sin\left(-\dfrac{5\pi}{6}\right)=-\dfrac{3\sqrt{3}}{2}-\dfrac{3}{2}i$

3 answers: $z=3i,\ \dfrac{3}{2}\left(\sqrt{3}-i\right),\ -\dfrac{3}{2}\left(\sqrt{3}+i\right)$

All on a circle of radius 3, equally spaced by 120°, they do not occur in conjugate pairs.

d $z^4+1024=0$

$z^4=-1024$

$=1024\operatorname{cis}(\pi+2k\pi)$

$z=\sqrt[4]{1024}\operatorname{cis}\left(\dfrac{\pi}{4}+\dfrac{k\pi}{2}\right)$

$=4\sqrt{2}\operatorname{cis}\left(\dfrac{\pi}{4}+\dfrac{k\pi}{2}\right)$

$k = 0 \quad z = 4\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right) = 4 + 4i$

$k = 1 \quad z = 4\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right) = -4 + 4i$

$k = -1 \quad z = 4\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right) = 4 - 4i$

$k = -2 \quad z = 4\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right) = -4 - 4i$

4 answers; $z = 4(1 \pm i),\ 4(-1 \pm i)$

All on a circle of radius $4\sqrt{2}$, equally spaced by $\frac{\pi}{2}$, 2 pairs of complex conjugate roots.

7 **a** $S = \{z\colon |z + 1 - 2i| = 5\}$ let $z = x + y$

$T\colon \{z : \text{Re}(z) + 2\text{Im}(z) = 8\}$

S (1) $(x + 1)^2 + (y - 2)^2 = 25$

T (2) $x + 2y = 8$

From (2) $x = 8 - 2y$ into (1)

$(9 - 2y)^2 + (y - 2)^2 = 25$

$81 - 36y + 4y^2 + y^2 - 4y + 4 = 25$

$5y^2 - 40y + 60 = 0$

$5(y^2 - 8y + 12) = 0$

$5(y - 6)(y - 2) = 0$

$y = 2 \quad \Rightarrow \quad x = 8 - 4 = 4$

$y = 6 \quad \Rightarrow \quad x = 8 - 12 = -4$

Two points of intersection $(4, 2)$, $(-4, 6)$

b $S = \{z : |z| = 2\}$

$T = \{z : a\,\text{Re}(z) - 2\text{Im}(z) = 5\}$

S (1) $x^2 + y^2 = 4$

T (2) $ax - 2y = 5$

From (2) $y = \frac{1}{2}(ax - 5)$ into (1)

$x^2 + \left(\frac{1}{2}(ax - 5)\right)^2 = 4$

$x^2 + \frac{1}{4}\left(a^2x^2 - 10ax + 25\right) = 4$

$4x^2 + a^2x^2 - 10ax + 25 = 16$

$\left(a^2 + 4\right)x^2 - 10ax + 9 = 0$

For T to be a tangent, there is only one solution.

$\Delta = (10a)^2 - 4 \times 9 \times \left(a^2 + 4\right) = 0$

$100a^2 - 36\left(a^2 + 4\right) = 0$

$100a^2 - 36a^2 = 36 \times 4$

$64a^2 = 36 \times 4$

$a^2 = \frac{36 \times 4}{64} = \frac{9}{4}$

$a = \pm\frac{3}{2}$

Technology active: multiple choice

8 $\text{cis}(\pi) = \cos(\pi) + i\sin(\pi) = -1$

The correct answer is **C**.

9 $z = \frac{1}{a} + bi$

$z^{-1} = \frac{1}{\frac{1}{a} + bi} \times \left(\frac{\frac{1}{a} - bi}{\frac{1}{a} - bi}\right)$

$z^{-1} = \frac{\frac{1}{a} - bi}{\frac{1}{a^2} - b^2i^2}$

$z^{-1} = \frac{\frac{1-abi}{a}}{\frac{1+a^2b^2}{a^2}}$

$z^{-1} = \frac{a(1 - abi)}{1 + a^2b^2}$

$z^{-1} = \frac{a}{1 + a^2b^2} - \frac{a^2bi}{1 + a^2b^2}$

The correct answer is **D**.

10 If $z = a + bi \quad \text{Arg}(z) = \tan^{-1}\left(\frac{b}{a}\right)$ is only true when $a > 0$ in the first and fourth quadrants

The correct answer is **B**.

11 $z = 7\text{cis}\left(\frac{\pi}{11}\right)$

$\bar{z} = 7\text{cis}\left(-\frac{\pi}{11}\right)$

$(\bar{z})^{-1} = \frac{1}{7}\text{cis}\left(\frac{\pi}{11}\right)$

The correct answer is **A**.

12 $z = -1 - i$

$z = \sqrt{2}\text{cis}\left(-\frac{3\pi}{4}\right)$

$\arg(z) = -\frac{3\pi}{4}, \quad \arg(z^7) = -\frac{21\pi}{4}$

$\text{Arg}(z^7) = -\frac{21\pi}{7} + 6\pi = \frac{3\pi}{4}$

The correct answer is **E**.

13 One root is $z = 4i$

$z^3 = 64i^3 = -64i$

$z^3 + 64i = 0$

The correct answer is **A**.

14 A polynomial with real coefficients must follow the conjugate root theorem, which states that any complex roots occur in complex conjugate pairs. Therefore it is impossible to have an odd number of complex roots. Therefore Betty and Colin are incorrect. Andrew, Daisy and Edward are all correct.

The correct answer is **E**.

15 $|z + 8i| = 2\,|z + 2i|$ let $z = x + yi$

$|x + (y + 8)i| = 2\,|x + (y + 2)i|$

$\sqrt{x^2 + (y + 8)^2} = 2\sqrt{x^2 + (y + 2)^2}$

$x^2 + y^2 + 16y + 64 = 4\left(x^2 + y^2 + 4y + 4\right)$

$3x^2 + 3y^2 = 48$

$x^2 + y^2 = 16$

Circle

The correct answer is **B**.

16 $z = -a - ai = -a(1 + i) = \sqrt{2}a\,\text{cis}\left(-\frac{3\pi}{4}\right)$

$z^n = \left(\sqrt{2}a\text{cis}\left(-\frac{3\pi}{4}\right)\right)^n = 2^{\frac{n}{2}}a^n\text{cis}\left(-\frac{3n\pi}{4}\right)$

For z^n to be real, then $\text{Im}(z^n) = 0 \Rightarrow \sin\left(-\frac{3n\pi}{4}\right) = 0$

$\frac{3n\pi}{4} = k\pi$

$n = \frac{4k}{3}$ where $k \in Z$.

The correct answer is **A**.

17 $z^2 = 5 - 12i$ let $z = a + bi, \quad a, b \in R$

$z^2 = a^2 + 2abi + b^2i^2 = \left(a^2 - b^2\right) + 2abi$

Re(1) $a^2 - b^2 = 5$

Im(2) $2ab = -12, \quad a = -\frac{6}{b}$

(1) $\left(-\frac{6}{b}\right)^2 - b^2 = 5$

$$\frac{36}{b^2} - b^2 = 5$$
$$b^4 + 5b^2 - 36 = 0$$
$$\left(b^2 - 4\right)\left(b^2 + 9\right) = 0$$
$$b = \pm 2, \quad \pm 3i, \quad b \in R$$
$$a = \mp 3$$
$$z = 3 - 2i, \quad -3 + 2i$$

The correct answer is **D**.

Technology active: extended response

18 **a** $\cos\left(\frac{\pi}{10}\right) = \frac{1}{4}\sqrt{2\left(5 + \sqrt{5}\right)}$

$$\sin^2\left(\frac{\pi}{10}\right) + \cos^2\left(\frac{\pi}{10}\right) = 1$$
$$\sin^2\left(\frac{\pi}{10}\right) = 1 - \cos^2\left(\frac{\pi}{10}\right)$$
$$= 1 - \left[\frac{1}{4}\sqrt{2\left(5 + \sqrt{5}\right)}\right]^2$$
$$= 1 - \frac{1}{16}\left(2\left(5 + \sqrt{5}\right)\right)$$
$$= \frac{1}{16}\left[16 - 10 - 2\sqrt{5}\right]$$
$$= \frac{1}{16}\left(6 - 2\sqrt{5}\right)$$
$$\sin\left(\frac{\pi}{10}\right) = \frac{1}{4}\sqrt{6 - 2\sqrt{5}}$$

Now $\left(\sqrt{5} - 1\right)^2 = 5 + 1 - 2\sqrt{5} = 6 - 2\sqrt{5}$

So $\sin\left(\frac{\pi}{10}\right) = \frac{1}{4}\left(\sqrt{5} - 1\right)$ shown

b $u = \frac{1}{4}\sqrt{2\left(5 + \sqrt{5}\right)} + \frac{1}{4}\left(\sqrt{5} - 1\right)i$

$$= \cos\left(\frac{\pi}{10}\right) + i\sin\left(\frac{\pi}{10}\right)$$
$$= \text{cis}\left(\frac{\pi}{10}\right)$$

c $u^{20} = \text{cis}\left(20 \times \frac{\pi}{10}\right)$

$$= \text{cis}(2\pi)$$
$$= \text{cis}(0)$$
$$= 1$$
$$u^{30} = \text{cis}\left(\frac{30\pi}{10}\right)$$
$$= \text{cis}(3\pi)$$
$$= \text{cis}(-\pi)$$
$$= -1$$
$$u^{45} = \text{cis}\left(\frac{45\pi}{10}\right)$$
$$= \text{cis}\left(\frac{9\pi}{2}\right)$$
$$= \text{cis}\left(\frac{9\pi}{2} - 4\pi\right)$$
$$= \text{cis}\left(\frac{\pi}{2}\right)^2$$
$$= i$$

d $\text{Re}(u^n) = 0$

$$\text{Re}\left(\text{cis}\left(\frac{\pi}{10}\right)\right)^n = 0$$
$$\text{Re}\left(\text{cis}\left(\frac{\text{n}\pi}{10}\right)\right)n = 0$$
$$\text{Re}\left(\cos\left(\frac{n\pi}{10}\right) + i\sin\left(\frac{n\pi}{10}\right)\right) = 0$$
$$\cos\left(\frac{n\pi}{10}\right) = 0$$
$$\frac{n\pi}{10} = (2k+1)\frac{\pi}{2}$$
$$n = 5(2k+1), \quad k \in Z$$

19 **a** $|z - 8| = 4\sqrt{3}$ let $z = x + yi$

$$|(x - 8) + yi| = 4\sqrt{3}$$
$$\sqrt{(x-8)^2 + y^2} = 4\sqrt{3}$$
$$(x-8)^2 + y^2 = 48$$

A circle centre $(8, 0)$ radius $4\sqrt{3}$.

b $\text{Im}\left(\frac{z + i\sqrt{3}}{z - 1}\right) + \frac{\sqrt{3}}{|z - 1|^2} = 0$ let $z = x + yi$

$$\text{Im}\left(\frac{x + \left(y + \sqrt{3}\right)i}{x - 1 + yi}\right) + \frac{\sqrt{3}}{|x - 1 + yi|^2} = 0$$
$$\text{Im}\left(\frac{x + \left(y + \sqrt{3}\right)i}{x - 1 + yi} \times \frac{x - 1 - yi}{x - 1 - yi}\right) + \frac{\sqrt{3}}{\left|\sqrt{(x-1)^2 + y^2}\right|^2} = 0$$
$$\text{Im}\left(\frac{x(x-1) + y\left(y + \sqrt{3}\right) + i\left(\left(y + \sqrt{3}\right)(x - 1) - xy\right)}{(x-1)^2 + y^2}\right)$$
$$+ \frac{\sqrt{3}}{(x-1)^2 + y^2} = 0$$
$$\frac{\left(y + \sqrt{3}\right)(x - 1) - xy}{(x-1)^2 + y^2} + \frac{\sqrt{3}}{(x-1)^2 + y^2} = 0$$
$$xy + \sqrt{3}x - \sqrt{3} - y - xy + \sqrt{3} = 0$$
$$y = \sqrt{3}x$$

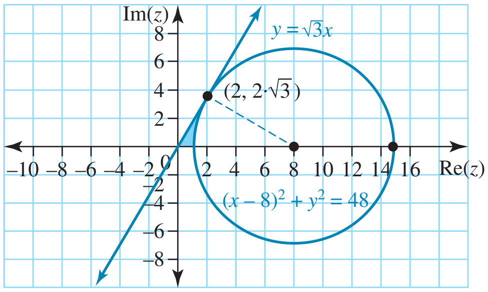
c $\alpha = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$

d Substitute $y = \sqrt{3}x$ into $(x - 8)^2 + y^2 = 48$

$$(x-8)^2 + 3x^2 = 48$$
$$x^2 - 16x + 64 + 3x^2 - 48 = 0$$
$$4x^2 - 16x + 16 = 0$$
$$4\left(x^2 - 4x + 4\right) = 0$$
$$4(x-2)^2 = 0$$
$$x = 2 \;\Rightarrow\; y = 2\sqrt{3} \quad \left(2, 2\sqrt{3}\right)$$

Since the line touches the circle, it is a tangent to the circle at the point $\left(2, 2\sqrt{3}\right)$.

e



f Area of the triangle minus the area of the sector

$$A = \frac{1}{2}bh - \frac{1}{2}r^2\theta,\ b = 8,\ h = 4\sqrt{3}\,\sin\left(30^0\right) = 2\sqrt{3},$$

$$r = 4\sqrt{3},\ \theta = \frac{\pi}{6}$$

$$A = \frac{1}{2}\times 8\times 2\sqrt{3} - \frac{1}{2}\times 48\times\frac{\pi}{6}$$

$$A = 8\sqrt{3} - 4\pi$$

g $|z - k| = |z - i|$ let $k = a + bi = r\,\mathrm{cis}(\theta)$ and $z = x + yi$

$$|(x - a) + (y - b)i| = |x + (y - 1)i|$$

$$\sqrt{(x - a)^2 + (y - b)^2} = \sqrt{x^2 + (y - 1)^2}$$

$$x^2 - 2xa + a^2 + y^2 - 2yb + b^2 = x^2 + y^2 - 2y + 1$$

$$2y - 2yb = 2xa + 1 - a^2 - b^2$$

$$2y\,(1 - b) = 2xa + 1 - a^2 - b^2$$

$$y = \frac{xa}{1 - b} + \frac{1 - a^2 - b^2}{2\,(1 - b)} = \sqrt{3}\,x$$

$$(1)\quad \frac{a}{1 - b} = \sqrt{3}\quad\text{and}\quad b \neq 1\qquad (2)\quad 1 - a^2 - b^2 = 0$$

$$(1)\quad a = \sqrt{3}\,(1 - b)\quad\text{into}\quad(2)\quad 1 - 3\,(1 - b)^2 - b^2 = 0$$

$$4b^2 - 6b + 2 = 0$$

$$2\left(2b^2 - 3b + 1\right) = 0$$

$$2\,(b - 1)\,(2b - 1) = 0$$

$$\text{So}\quad b = \frac{1}{2}\quad\Rightarrow\quad a = \frac{\sqrt{3}}{2}$$

$$k = \frac{\sqrt{3}}{2} + \frac{1}{2}\,i = \mathrm{cis}\left(\frac{\pi}{6}\right),\quad |k| = r = 1,\quad \theta = \frac{\pi}{6}$$

h $\beta \in \left(\frac{\pi}{3},\ \pi\right] \cup \left(-\pi,\ -\frac{\pi}{3}\right)$

20 **a** $R_1 = \left\{z\text{: Arg}(z) = \frac{\pi}{12}\right\},\quad \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{12}$

$$\tan\left(\frac{\pi}{12}\right) = 2 - \sqrt{3} = \frac{y}{x}$$

$$R_1\text{:}\quad y = \left(2 - \sqrt{3}\right)x,\quad x > 0$$

$$R_2 = \left\{z\text{: Arg}(z) = \frac{5\pi}{12}\right\},\ \tan^{-1}\left(\frac{y}{x}\right) = \frac{5\pi}{12}$$

$$\tan\left(\frac{5\pi}{12}\right) = 2 + \sqrt{3} = \frac{y}{x}$$

$$R_2\text{: } y = \left(2 + \sqrt{3}\right)x, x > 0$$

$$|z - 2 - 2i| = \sqrt{2},\quad z = x + yi$$

$$|(x - 2) + (y - 2)i| = \sqrt{2}$$

$(x - 2)^2 + (y - 2)^2 = 2$, circle centre (2, 2) radius 2

b Solving R_1, $y = \left(2 - \sqrt{3}\right)x$ with the circle

$$(x - 2)^2 + (y - 2)^2 = 2,$$

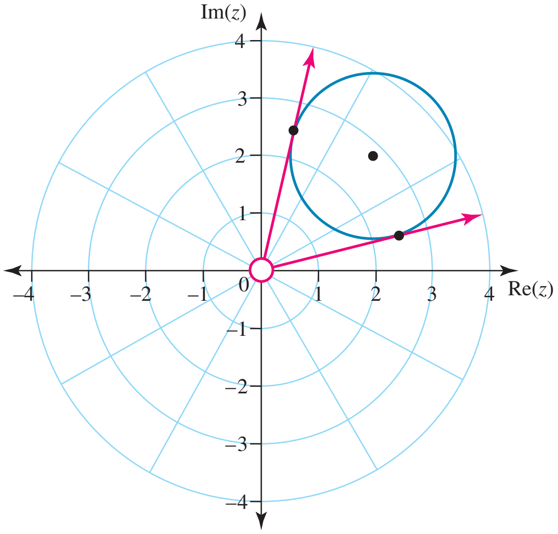
Gives $A\left(\frac{3 + \sqrt{3}}{2},\ \frac{3 - \sqrt{3}}{2}\right)$.

Solving R_2, $y = \left(2 + \sqrt{3}\right)x$ with the circle

$$(x - 2)^2 + (y - 2)^2 = 2,$$

Gives $B\left(\frac{3 - \sqrt{3}}{2},\ \frac{3 + \sqrt{3}}{2}\right)$.

c Open circles for both rays at the origin



d Since $\angle BPA = \frac{2\pi}{3},\quad \angle OPA = \angle OPB = \frac{\pi}{3},$

$$\angle OAP = \angle OBP = \frac{\pi}{2}$$

$$|OP| = \sqrt{8},\quad |PA| = |PB| = \sqrt{2},\quad |OA| = |OB| = \sqrt{6}$$

The area is 2 [area of triangle OAP – area of sector]

$$= 2\left[\frac{1}{2}\times\sqrt{6}\times\sqrt{2} - \frac{1}{2}\left(\sqrt{2}\right)^2\frac{\pi}{3}\right] = 2\left[\frac{\sqrt{12}}{2} - \frac{\pi}{3}\right]$$

$$= 2\sqrt{3} - \frac{2\pi}{3}$$

2.7 Exam questions

1 **a.** **i** $p(z) = z^3 + \alpha z^2 + \beta z + \gamma,\quad z \in C, \alpha, \beta, \gamma \in R,$

By the conjugate root theorem, and since the coefficient are real, z_2 and z_3 are conjugate pairs. [1 mark]

ii Let $z_2 = a + bi,\ z_3 = a - bi$

$$z_2 + z_3 = 2a,\quad |z_2 + z_3| = 0,$$

$$\Rightarrow\ a = 0$$

$$z_2 - z_3 = 2bi, |z_2 - z_3| = |2bi| = 2\,|bi| = 2\,|b| = 6$$

$$\Rightarrow\ b = \pm 3$$

So $(z - 3i)\,(z + 3i) = \left(z^2 + 9\right)$ is a factor

$$p(z) = z^3 + \alpha z^2 + \beta z + \gamma = (z + c)\left(z^2 + 9\right)$$

$$= z^3 + cz^2 + 9z + 9c$$

$$\alpha = c, \beta = 9, \gamma = 9c$$

$$p(2) = -13$$

$$-13 = 8 + 4\alpha + 2\beta + \gamma$$

$$-13 = 8 + 4c + 18 + 9c = 26 + 13c$$

$$13c = -39,\quad c = -3$$

$$\alpha = -3,\ \beta = 9, \gamma = -27$$

$$p(z) = z^3 - 3z^2 + 9z - 27 = (z - 3)\left(z^2 + 9\right)$$

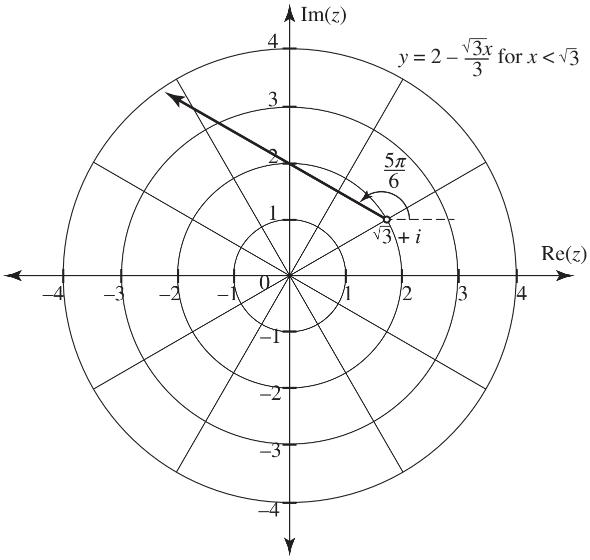
Award 1 mark for setting up cubic coefficents.

Award 1 mark for the correct values of a and b.

Award 1 mark for the correct values of α, β and γ.

b. $z_4 = \sqrt{3} + i$, the ray $\text{Arg}(z - z_4) = \frac{5\pi}{6}$ is the line

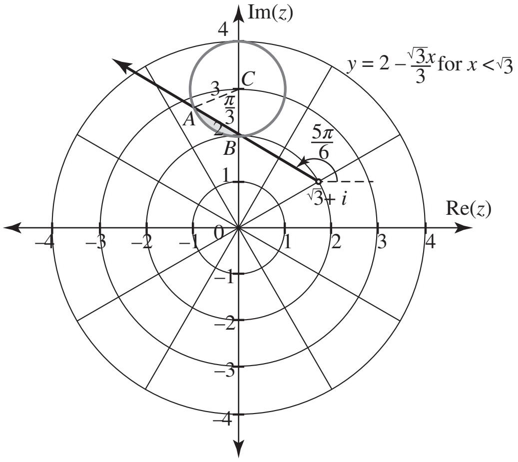
$y = 2 - \frac{\sqrt{3}x}{3}$ for $x < \sqrt{3}$.



Award 1 mark for the correct ray.

Award 1 mark for including open circle at z_4.

c. i The circle $|z - 3i| = 1$ is $x^2 + (y - 3)^2 = 1$ with centre at $C(0, 3)$ and radius $r = 1$. [1 mark]



ii The ray and the circle intersect at the points $A\left(-\frac{\sqrt{3}}{2}, \frac{5}{2}\right)$, $B(0, 2)$, and triangle ABC is equilateral, so $\angle BCA = \frac{\pi}{3}$, the area of the minor segment is

$$\frac{1}{2}r^2(\theta - \sin(\theta)) = \frac{1}{2}\left(\frac{\pi}{3} - \sin\left(\frac{\pi}{3}\right)\right)$$
$$= \frac{1}{2}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$$
$$= \frac{1}{12}\left(2\pi - 3\sqrt{3}\right)$$

Award 1 mark for correctly identifying the segment.

Award 1 mark for correctly calculating the area of the segment.

2 a $|z - (1 + 2i)| = 2, z = x + yi$

$(x - 1)^2 + (y - 2)^2 = 4$

Circle center (1, 2) radius 2 [1 mark]

***VCAA Examination Report* notes:**

Some students gave only one of the two required parts of the answer. An incorrect radius of $\sqrt{2}$ was occasionally given. Students were not asked to find the expression of the circle at this point but a number did so.

b $|z + 1| = \sqrt{2}\,|z - i|$

$$|(x + 1) + iy| = \sqrt{2}\,|x + (y - 1)i|$$
$$\sqrt{(x + 1)^2 + y^2} = \sqrt{2}\sqrt{x^2 + (y - 1)^2}$$
$$x^2 + 2x + 1 + y^2 = 2\left[x^2 + y^2 - 2y + 1\right]$$
$$x^2 - 2x + y^2 - 4y + 1 = 0$$
$$x^2 - 2x + 1 + y^2 - 4y + 4 = 4$$
$$(x - 1)^2 + (y - 2)^2 = 4$$

Circle centre (1, 2) radius 2

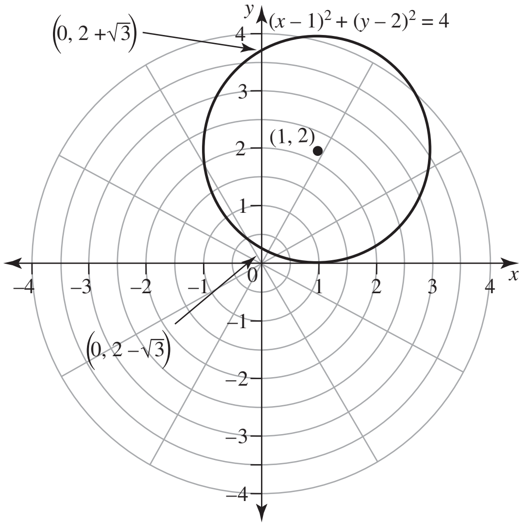
Award 1 mark for correctly setting up an equation involving x and y.

Award 1 mark for correctly manipulating this equation to give the same cartesian equation found in part **a**.

***VCAA Examination Report* notes:**

Most students were able to correctly find an expression that did not involve i. In a 'show that' question such as this, students are expected to explicitly show that the given relation leads to the required conclusion. The working shown above is an example of a suitable response.

c



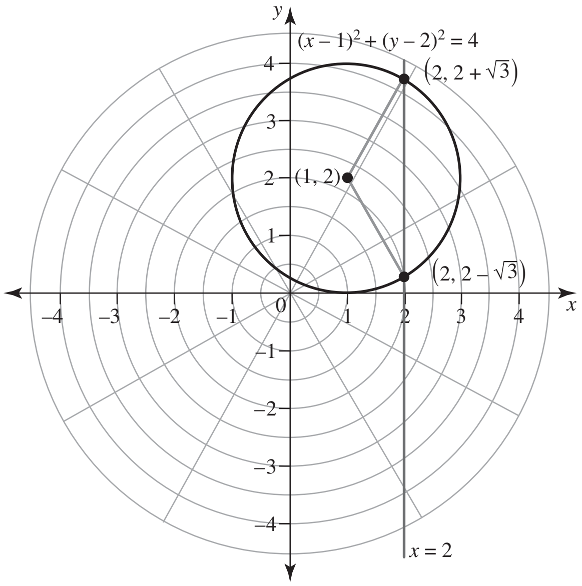
Award 1 mark for the correct circle.

Award 1 mark for correct intercepts.

***VCAA Examination Report* notes:**

The circle was generally drawn correctly. Students did not always supply the coordinates of the y-intercepts as required by the question. Some coordinates were incorrectly given imaginary numbers.

d $|z - 1| = |z - 3|$ is the line $x = 2$, solving $x = 2$ and $(x - 1)^2 + (y - 2)^2 = 4$ gives $y = 2 \pm \sqrt{3}$, $\left(2, 2 + \sqrt{3}\right)$, $\left(2, 2 - \sqrt{3}\right)$



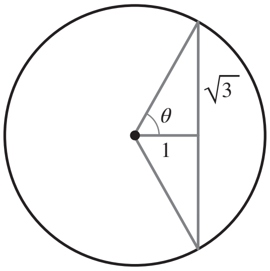
Award 1 mark for drawing a correct graph.

Award 1 mark for correctly labelling the points of intersection.

***VCAA Examination Report* notes:**

While the vertical line was usually sketched correctly, coordinates of the points of intersection with the circle were not always shown. Coordinates were sometimes presented as decimal approximations.

e



$$\tan(\theta) = \frac{\sqrt{3}}{1}$$

$$\theta = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$$

The angle subtended is $2\theta = \dfrac{2\pi}{3}$.

$$A = \frac{1}{2}r^2(\theta - \sin(\theta))$$

$$A = \frac{1}{2} \times 4\left(\frac{2\pi}{3} - \sin\left(\frac{2\pi}{3}\right)\right) = 2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

$$A = \frac{1}{3}\left(4\pi - 3\sqrt{3}\right)$$

Award 1 mark for identifying the required area.

Award 1 mark for correctly calculating the area.

***VCAA Examination Report* notes:**

Students who used standard formulas to find the segment area were generally more successful than those who took a definite integral approach. Many students did not start the problem with a correct sector angle.

3 a Let $z = x + yi$

$$|(x-1)+iy| = |(x+2)+(y-3)i|$$

$$\sqrt{(x-1)^2+y^2} = \sqrt{(x+2)^2+(y-3)^2}$$

$$x^2-2x+1+y^2 = x^2+4x+4+y^2-6y+9$$

$$6x-6y+12=0$$

$$y = x+2$$

Award 1 mark for find the linear equation.

Award 1 mark for correct linear equation.

***VCAA Assessment Report* note:**

The majority of correct answers resulted from substituting $z = x + yi$ into the expression provided. Very few students used a perpendicular bisector approach at this stage. The most common error was a negative gradient.

b The circle $|z-1| = 3$ has centre $(1,\ 0)$ and radius 3.

$$|(x-1)+iy| = 3$$

$$\sqrt{(x-1)^2+y^2} = 3$$

$$(x-1)^2+y^2 = 9$$

For points of intersection, solve

(1) $y = x + 2$ and (2) $(x-1)^2+y^2 = 9$

$$(x-1)^2+(x+2)^2 = 9$$

$$x^2-2x+1+x^2+4x+4 = 9$$

$$x^2+x-2=0$$

$$(x+2)(x-1)=0 \Rightarrow x = 1,\ -2$$

$$(1,\ 3),\ (-2,\ 0)$$

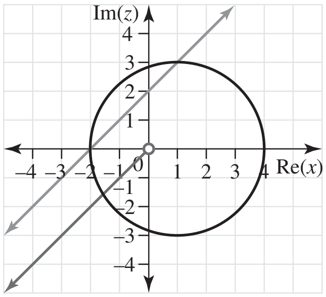
Award 1 mark for solving the line and circle.

Award 1 mark for final correct coordinates of the two points.

***VCAA Assessment Report* note:**

Most students were able to find the cartesian expression for the circle and the line and then apply substitution or use technology to find the required points.

c



Award 1 mark for the correct circle and centre axial intercepts.

Award 1 mark for correct line and intersection points.

***VCAA Assessment Report* note:**

This question was generally well answered. The circle was sketched correctly in cases. However, the line was not always placed with sufficient accuracy. Some students who were unable to find the correct equation in Question 2a. were able to use a perpendicular bisector approach to draw a correct line.

d The area of the major segment is the area of the right-angled isosceles triangle with two side of 3 units, plus the area of the sector, $\frac{1}{2}r^2\theta$, where $r = 3$ and $\theta = \frac{3\pi}{2}$.

$$A = \frac{1}{2} \times 3 \times 3 + \frac{1}{2} \times 3^2 \times \frac{3\pi}{2}$$
$$= \frac{18 + 27\pi}{4}$$

Award 1 mark for the section area.

Award 1 mark for the correct area.

***VCAA Assessment Report* note:**

Students found this question more difficult than previous parts of Question 2. A correct answer was most easily found by adding a right-angled triangle to three-questions of a circle. Subtracting the minor segment area from the circle area was a common approach. Some students correctly used a segment area formula and a larger number set up elaborate definite integrals to find the area, occasionally successfully, but this was not an efficient approach. Sign and factorisation errors meant that some students moved from a correct approach and evaluation to an incorrect final answer.

e The ray $\text{Arg}(z) = -\frac{3\pi}{4}$ is the line $y = x$ for $x < 0$ and has an open circle at the origin.

Since the origin is not included and the ray makes an angle of 135° measured clockwise from the positive real axis.

Award 1 mark for the correct ray on the diagram in part **c**.

***VCAA Assessment Report* note:**

Many students were not able to sketch the required ray. Some students sketched a line but did not restrict their ray appropriately, either including or extending past the origin.

f Since $-\pi < \text{Arg}(z) \leq \pi$, the ray $\text{Arg}(z) = \alpha\pi$ is parallel to the line $|z - 1| = |z + 2 - 3i|$ when $\text{Arg}(z) = -\frac{3\pi}{4}$ or $\text{Arg}(z) = \frac{\pi}{4}$ and therefore does not intersect the line,

$$\alpha \in \left(-1, -\frac{3}{4}\right) \cup \left(-\frac{1}{4}, 1\right].$$

Award 1 mark for each correct value.

***VCAA Assessment Report* note:**

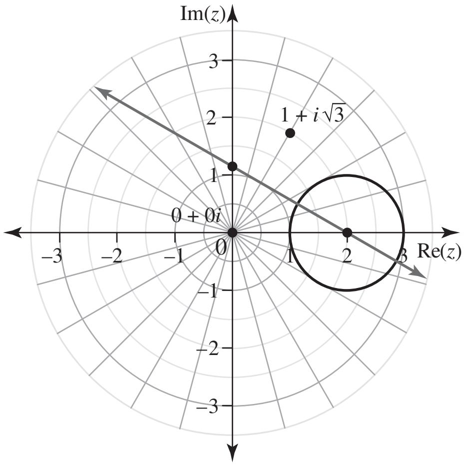
Students found this question demanding, with few students giving a fully correct answer. Many students did not respond to this question. Common incorrect answers contained multiples of π or included the endpoint $\alpha = -1$. Some students did not note that the principal value of the argument was used in the question. Of the correct answers, a variety of correct notations were presented.

4
$$z = \frac{1 + i\sqrt{3}}{1 + i}$$
$$= \frac{2\,\text{cis}\left(\frac{\pi}{3}\right)}{\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)}$$
$$= \frac{2}{\sqrt{2}}\,\text{cis}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$
$$= \sqrt{2}\,\text{cis}\left(\frac{\pi}{12}\right)$$

$$z^5 = \left(\sqrt{2}\,\text{cis}\left(\frac{\pi}{12}\right)\right)^5$$
$$= \left(\sqrt{2}\right)^5 \text{cis}\left(\frac{5\pi}{12}\right)$$
$$= 4\sqrt{2}\,\text{cis}\left(\frac{5\pi}{12}\right)$$

The correct answer is **B**.

5 a i



Award 1 mark for each correctly positioned point.

***VCAA Assessment Report* note:**

This question was answered quite well, but many students could not accurately position $1 + i\sqrt{3}$, not realising it lay on the circle of radius 2. A number of students did not fully label both points.

ii $\left|z - \left(1 + i\sqrt{3}\right)\right| = |z|$ is the line equidistant or the perpendicular bisector from the points $0 + 0i$ and $1 + i\sqrt{3}$.

$|z - 2| = 1$ is the circle, center at $2 + 0i$ and radius 1.

Award 1 mark for the line and 1 mark for the circle on the diagram in **a. i.**

***VCAA Assessment report* note:**

Most students graphed the circle correctly, although some circles were poorly drawn.

A common error was to draw a straight line with a positive gradient. A number of students terminated their line at (2, 0). Few students seemed to realise that the required line was the perpendicular bisector of the line interval joining (0, 0) and $\left(1, \sqrt{3}\right)$.

iii $\left|z - \left(1 + i\sqrt{3}\right)\right| = |z|$

Let $z = x + yi$

$$\left|(x - 1) + \left(y - \sqrt{3}\right)i\right| = |x + yi|$$
$$\sqrt{(x - 1)^2 + \left(y - \sqrt{3}\right)^2} = \sqrt{x^2 + y^2}$$
$$x^2 - 2x + 1 + y^2 - 2\sqrt{3}y + 3 = x^2 + y^2$$
$$x + \sqrt{3}y = 2$$

[1 mark]

***VCAA Assessment Report* note:**

This question was generally well answered. The most common error was the gradient given as positive.

iv (1) $\left|z - \left(1 + i\sqrt{3}\right)\right| = |z| \Rightarrow x + \sqrt{3}y = 2$

(2) $|z - 2| = 1 \Rightarrow (x - 2)^2 + y^2 = 1$

Substitute (1) $x = 2 - \sqrt{3}\,y$ into (2)

$\left(-\sqrt{3}y\right)^2 + y^2 = 4y^2 = 1$

$y^2 = \frac{1}{4} \Rightarrow y = \pm\frac{1}{2}$

$y = \frac{1}{2} \Rightarrow x = 2 - \frac{\sqrt{3}}{2}$

$y = -\frac{1}{2} \Rightarrow x = 2 + \frac{\sqrt{3}}{2}$

$\frac{1}{2}\left(4-\sqrt{3}\right) + \frac{1}{2}i$ and $\frac{1}{2}\left(4+\sqrt{3}\right) - \frac{1}{2}i$

Award 1 mark for substitution.

Award 1 mark for each correct point of intersection.

VCAA Assessment Report **note:**

The question was reasonably well answered. Common errors were answers given in the wrong form and sign errors. Most students attempted to solve the equations of the line and circle simultaneously.

b i $z^2 - 4\cos(\alpha)z + 4 = 0,\ \alpha \in R,\ 0 < \alpha < \frac{\pi}{2}$

$$\begin{aligned}\Delta &= (4\cos(\alpha))^2 - 4\times 4 = 16\cos^2(\alpha) - 16\\ &= 16\left(\cos^2(\alpha) - 1\right) = -16\sin^2(\alpha)\\ &= 16i^2\sin^2(\alpha)\end{aligned}$$

$$z = \frac{4\cos(\alpha) \pm \sqrt{16i^2\sin^2(\alpha)}}{2}$$

$z = 2\cos(\alpha) \pm 2i|\sin(\alpha)|,$ since $0 < \alpha < \frac{\pi}{2}$

$z_1 = 2(\cos(\alpha) + i\sin(\alpha)) = 2\operatorname{cis}(\alpha)$

$z_2 = 2(\cos(\alpha) - i\sin(\alpha)) = 2\operatorname{cis}(-\alpha)$

Award 1 mark for using the quadratic formula.

Award 1 mark for simplifications.

Award 1 mark for both correct roots.

VCAA Assessment Report **note:**

Most students attempted to apply the quadratic formula or complete the square, but few managed to find the values of z in polar form. Dealing with the discriminant proved to be a problem for many. A number of students left answers in cartesian form, and some erroneously converted the correct cartesian form answer to $2\sqrt{2}\operatorname{cis}(\alpha)$ and $2\sqrt{2}\operatorname{cis}(-\alpha)$.

ii $\frac{z_1}{z_2} = \frac{2\ \operatorname{cis}(\alpha)}{2\ \operatorname{cis}(-\alpha)} = \operatorname{cis}(2\alpha)$

$\operatorname{Arg}\left(\frac{z_1}{z_2}\right) = 2\alpha = \frac{5\pi}{6}$

$\alpha = \frac{5\pi}{12}$ [1 mark]

VCAA Assessment Report **note:**

Many students did not attempt this question. Common errors were unsimplified expressions involving $z,\ -\frac{5\pi}{12}$ and $\frac{5\pi}{6}$.

Topic 3 — Vectors

3.2 Vectors in two dimensions

3.2 Exercise

1 $$\begin{aligned} 4\overrightarrow{CB} - \overrightarrow{AB} + 4\overrightarrow{AC} &= 4\left(\overrightarrow{AC} + \overrightarrow{CB}\right) - \overrightarrow{AB} \\ &= 4\overrightarrow{AB} - \overrightarrow{AB} \\ &= 3\overrightarrow{AB} \end{aligned}$$

2 $$\begin{aligned} 2\overrightarrow{BO} - 5\overrightarrow{AO} - 2\overrightarrow{BA} &= 2\overrightarrow{BO} + 5\overrightarrow{OA} - 2\left(\overrightarrow{OA} - \overrightarrow{OB}\right) \\ &= 2\overrightarrow{BO} + 5\overrightarrow{OA} - 2\overrightarrow{OA} + 2\overrightarrow{OB} \\ &= -2\overrightarrow{BO} + 2\overrightarrow{BO} + 5\overrightarrow{OA} - 2\overrightarrow{OA} \\ &= 3\overrightarrow{OA} \end{aligned}$$

3 **a** $$\begin{aligned} 2\overrightarrow{AC} - \overrightarrow{CB} + \overrightarrow{AB} &= 2\left(\overrightarrow{OC} - \overrightarrow{OA}\right) - \left(\overrightarrow{OB} - \overrightarrow{OC}\right) \\ &\quad +\overrightarrow{OB} - \overrightarrow{OA} \\ &= 3\overrightarrow{OC} - 3\overrightarrow{OA} \\ &= 3\left(\overrightarrow{AO} + \overrightarrow{OC}\right) \\ &= 3\overrightarrow{AC} \end{aligned}$$

b $$\begin{aligned} & 5\overrightarrow{CA} + \overrightarrow{BC} + 4\overrightarrow{OC} - \overrightarrow{BO} \\ &= 5\left(\overrightarrow{OA} - \overrightarrow{OC}\right) + \left(\overrightarrow{OC} - \overrightarrow{OB}\right) + 4\overrightarrow{OC} + \overrightarrow{OB} \\ &= 5\overrightarrow{OA} \end{aligned}$$

4 **a** $$\begin{aligned} & \overrightarrow{OC} + 6\overrightarrow{AB} + \overrightarrow{CA} + 5\overrightarrow{OA} \\ &= \overrightarrow{OC} + 6\left(\overrightarrow{OB} - \overrightarrow{OA}\right) + \left(\overrightarrow{OA} - \overrightarrow{OC}\right) + 5\overrightarrow{OA} \\ &= 6\overrightarrow{OB} \end{aligned}$$

b $$\begin{aligned} & 3\overrightarrow{OB} + \overrightarrow{AB} - 3\overrightarrow{AC} + 4\overrightarrow{BC} - \overrightarrow{OC} \\ &= 3\overrightarrow{OB} + \left(\overrightarrow{OB} - \overrightarrow{OA}\right) - 3\left(\overrightarrow{OC} - \overrightarrow{OA}\right) \\ &\quad + 4\left(\overrightarrow{OC} - \overrightarrow{OB}\right) - \overrightarrow{OC} \\ &= \overrightarrow{OA}\,(3 - 1) + \overrightarrow{OB}\,(3 + 1 - 4) + \overrightarrow{OC}\,(4 - 3 - 1) \\ &= 2\overrightarrow{OA} \end{aligned}$$

5 $$\begin{aligned} \overrightarrow{AO} + \overrightarrow{OB} - 2\overrightarrow{BO} - 2\overrightarrow{OC} &= \underset{\sim}{0} \\ \overrightarrow{AO} + \overrightarrow{OB} &= 2\overrightarrow{BO} + 2\overrightarrow{OC} \\ \overrightarrow{AO} + \overrightarrow{OB} &= 2\left(\overrightarrow{BO} + \overrightarrow{OC}\right) \\ \overrightarrow{AB} &= 2\overrightarrow{BC} \end{aligned}$$

Point B is common

So A, B, C are collinear

6 $$\begin{aligned} \overrightarrow{PO} - 4\overrightarrow{RO} + 3\overrightarrow{QO} &= \underset{\sim}{0} \\ \overrightarrow{PO} - \overrightarrow{RO} - 3\overrightarrow{RO} + 3\overrightarrow{QO} &= \underset{\sim}{0} \\ \overrightarrow{PO} - \overrightarrow{RO} &= 3\overrightarrow{RO} - 3\overrightarrow{QO} \\ \overrightarrow{PO} + \overrightarrow{OR} &= 3\left(\overrightarrow{RO} + \overrightarrow{OQ}\right) \\ \overrightarrow{PR} &= 3\overrightarrow{RQ} \end{aligned}$$

Point R is common

So $P,\ Q,\ R$ are collinear

7 **a** $$\begin{aligned} \overrightarrow{AO} + \overrightarrow{OB} &= \overrightarrow{BO} + \overrightarrow{OC} \\ \overrightarrow{AB} &= \overrightarrow{BC} \end{aligned}$$

so A, B, C collinear

b $$\begin{aligned} 3\overrightarrow{OA} - 2\overrightarrow{OB} &= \overrightarrow{OC} \\ 2\overrightarrow{OA} + \overrightarrow{OA} - 2\overrightarrow{OB} - \overrightarrow{OC} &= 0 \\ 2\overrightarrow{BO} + 2\overrightarrow{OA} &= \overrightarrow{OC} - \overrightarrow{OA} \\ 2\left(\overrightarrow{BO} + \overrightarrow{OA}\right) &= \overrightarrow{AO} + \overrightarrow{OC} \\ 2\overrightarrow{BA} &= \overrightarrow{AC}, \qquad \text{point } A \text{ in common} \end{aligned}$$

So A, B, C collinear

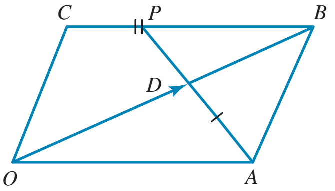
8 **a** $$\begin{aligned} \overrightarrow{BO} + 4\overrightarrow{AO} - 5\overrightarrow{CO} &= \underset{\sim}{0} \\ \overrightarrow{BO} + 4\overrightarrow{AO} - 5\overrightarrow{CO} &= \underset{\sim}{0} \\ \overrightarrow{BO} + 4\overrightarrow{AO} - 4\overrightarrow{CO} + \overrightarrow{OC} &= \underset{\sim}{0} \\ 4\left(\overrightarrow{AO} + \overrightarrow{OC}\right) &= -\overrightarrow{OC} - \overrightarrow{BO} \\ 4\overrightarrow{AC} &= \overrightarrow{CO} + \overrightarrow{OB} \\ 4\overrightarrow{AC} &= \overrightarrow{CB}, \qquad \text{point } C \text{ in common} \end{aligned}$$

So A, B, C collinear

b $$\begin{aligned} 3\overrightarrow{BO} - 5\overrightarrow{CO} + 2\overrightarrow{AO} &= \underset{\sim}{0} \\ 3\overrightarrow{BO} - 3\overrightarrow{CO} - 2\overrightarrow{CO} + 2\overrightarrow{AO} &= \underset{\sim}{0} \\ 3\overrightarrow{BO} + 3\overrightarrow{OC} &= 2\overrightarrow{CO} - 2\overrightarrow{AO} \\ 3\left(\overrightarrow{BO} + \overrightarrow{OC}\right) &= 2\left(\overrightarrow{CO} + \overrightarrow{OA}\right) \\ 3\overrightarrow{BC} &= 2\overrightarrow{CA}, \qquad \text{point } C \text{ in common} \end{aligned}$$

So A, B, C collinear

9 Given



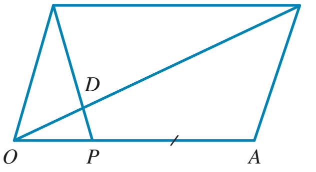
$$\overrightarrow{AD} = \frac{2}{3}\overrightarrow{AP}$$

$$\begin{aligned} \overrightarrow{OD} &= \overrightarrow{OC} + \overrightarrow{CP} + \overrightarrow{PD} \\ &= \overrightarrow{OC} + \frac{1}{2}\overrightarrow{CB} + \frac{1}{3}\overrightarrow{PA} \\ &= \overrightarrow{OC} + \frac{1}{2}\overrightarrow{OA} + \frac{1}{3}\left(\overrightarrow{PB} + \overrightarrow{BA}\right) \\ &= \overrightarrow{AB} + \frac{1}{2}\overrightarrow{OA} + \frac{1}{6}\overrightarrow{OA} + \frac{1}{3}\overrightarrow{BA} \\ &= \overrightarrow{AB}\left(1 - \frac{1}{3}\right) + \overrightarrow{OA}\left(\frac{1}{2} + \frac{1}{6}\right) \\ &= \frac{2}{3}\overrightarrow{AB} + \frac{2}{3}\overrightarrow{OA} \\ &= \frac{2}{3}\left(\overrightarrow{OA} + \overrightarrow{AB}\right) \end{aligned}$$

$$\overrightarrow{OD} = \frac{2}{3}\overrightarrow{OB}$$

So $O,\ D,\ B$ are collinear

10



Given

$\overrightarrow{OP} = \frac{1}{3}\overrightarrow{OA}$

$\overrightarrow{PD} = \frac{1}{4}\overrightarrow{PC}$

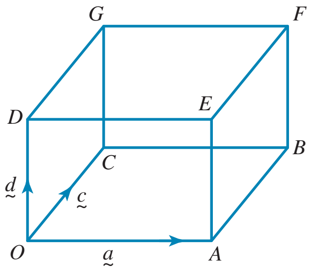
$OABC$ is a parallelogram

$\overrightarrow{OC} = \overrightarrow{AB}$

$$\begin{aligned}
\overrightarrow{OD} &= \overrightarrow{OP} + \overrightarrow{PD} \\
&= \frac{1}{3}\overrightarrow{OA} + \frac{1}{4}\overrightarrow{PC} \\
&= \frac{1}{3}\overrightarrow{OA} + \frac{1}{4}\left(\overrightarrow{PO} + \overrightarrow{OC}\right) \\
&= \frac{1}{3}\overrightarrow{OA} + \frac{1}{4}\left(\overrightarrow{OC} - \overrightarrow{OP}\right) \\
&= \frac{1}{3}\overrightarrow{OA} + \frac{1}{4}\left(\overrightarrow{OC} - \frac{1}{3}\overrightarrow{OA}\right) \\
&= \left(\frac{1}{3} - \frac{1}{12}\right)\overrightarrow{OA} + \frac{1}{4}\overrightarrow{AB} \\
&= \frac{1}{4}\left(\overrightarrow{OA} + \overrightarrow{AB}\right) \\
&= \frac{1}{4}\overrightarrow{OB}
\end{aligned}$$

So O, D, B are collinear

11



$\underset{\sim}{a} = \overrightarrow{OA}$

$\underset{\sim}{d} = \overrightarrow{OD}$

a $\underset{\sim}{a} = \overrightarrow{OA} = \overrightarrow{CB} = \overrightarrow{DE} = \overrightarrow{GF}$

b $\underset{\sim}{c} = \overrightarrow{OC} = \overrightarrow{AB} = \overrightarrow{EF} = \overrightarrow{DG}$

c $\underset{\sim}{d} = \overrightarrow{OD} = \overrightarrow{AE} = \overrightarrow{CG} = \overrightarrow{BF}$

d **i** $\overrightarrow{DF} = \overrightarrow{DG} + \overrightarrow{GF}$

$= \underset{\sim}{a} + \underset{\sim}{c}$

ii $\overrightarrow{EB} = \overrightarrow{EA} + \overrightarrow{AB}$

$= \underset{\sim}{c} - \underset{\sim}{d}$

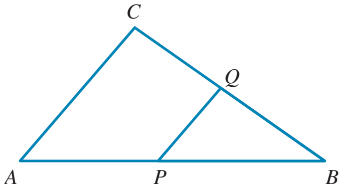
iii $\overrightarrow{FO} = \overrightarrow{FE} + \overrightarrow{ED} + \overrightarrow{DO}$

$= -\left(\underset{\sim}{a} + \underset{\sim}{d} + \underset{\sim}{c}\right)$

iv $\overrightarrow{DB} = \overrightarrow{DO} + \overrightarrow{OA} + \overrightarrow{AB}$

$= \underset{\sim}{a} + \underset{\sim}{c} - \underset{\sim}{d}$

12



P is the midpoint of $\overrightarrow{AB}$

$\overrightarrow{AP} = \overrightarrow{PB} = \frac{1}{2}\overrightarrow{AB}$

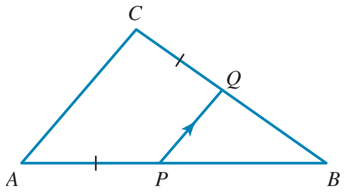
Q is the midpoint of $\overrightarrow{BC}$

$\overrightarrow{BQ} = \overrightarrow{QC} = \frac{1}{2}\overrightarrow{BC}$

$$\begin{aligned}
\overrightarrow{PQ} &= \overrightarrow{PB} + \overrightarrow{BQ} \\
&= \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} \\
&= \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{BC}\right) \\
&= \frac{1}{2}\overrightarrow{AC}
\end{aligned}$$

So $\overrightarrow{PQ}$ is parallel to $\overrightarrow{AC}$ and half the length

13



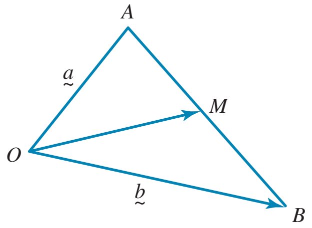
$\overrightarrow{PB} = \frac{1}{3}\overrightarrow{AB}$

$\overrightarrow{BQ} = \frac{1}{3}\overrightarrow{BC}$

$$\begin{aligned}
\overrightarrow{PQ} &= \overrightarrow{PB} + \overrightarrow{BQ} \\
&= \frac{1}{3}\overrightarrow{AB} + \frac{1}{3}\overrightarrow{BC} \\
&= \frac{1}{3}\left(\overrightarrow{AB} + \overrightarrow{BC}\right) \\
&= \frac{1}{3}\overrightarrow{AC}
\end{aligned}$$

So $\overrightarrow{PQ}$ is parallel to $\overrightarrow{AC}$ and one-third the length

14

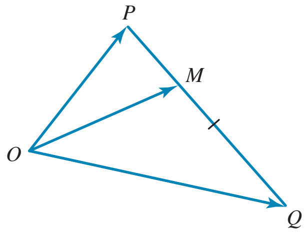


$\underset{\sim}{a} = \overrightarrow{OA}, \quad \underset{\sim}{b} = \overrightarrow{OB}$

M is the midpoint of $\overrightarrow{AB}$

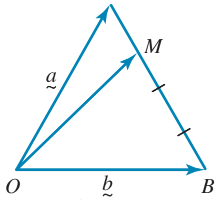
$$\begin{aligned}
\overrightarrow{OM} &= \overrightarrow{OA} + \overrightarrow{AM} \\
&= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB} \\
&= \overrightarrow{OA} + \frac{1}{2}\left(\overrightarrow{OB} - \overrightarrow{OA}\right) \\
&= \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB}\right) \\
&= \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{b}\right)
\end{aligned}$$

15



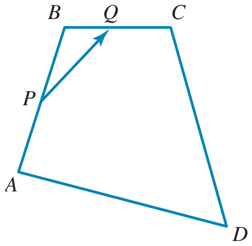
$$\begin{aligned}\overrightarrow{PM} &= \frac{1}{3}\overrightarrow{PQ}\\ \overrightarrow{OM} &= \overrightarrow{OP} + \overrightarrow{PM}\\ &= \overrightarrow{OP} + \frac{1}{3}\overrightarrow{PQ}\\ &= \overrightarrow{OP} + \frac{1}{3}\left(\overrightarrow{OQ} - \overrightarrow{OP}\right)\\ &= \frac{1}{3}\left(2\overrightarrow{OP} + \overrightarrow{OQ}\right)\end{aligned}$$

16



$$\begin{aligned}\overrightarrow{AM} &= \frac{1}{4}\overrightarrow{AB}\\ \overrightarrow{OM} &= \overrightarrow{OA} + \overrightarrow{AM}\\ &= \overrightarrow{OA} + \frac{1}{4}\overrightarrow{AB}\\ &= \overrightarrow{OA} + \frac{1}{4}\left(\overrightarrow{OB} - \overrightarrow{OA}\right)\\ &= \frac{1}{4}\left(3\overrightarrow{OA} + \overrightarrow{OB}\right)\\ &= \frac{1}{4}\left(3\underset{\sim}{a} + \underset{\sim}{b}\right)\end{aligned}$$

17



ABCD is a quadrilateral

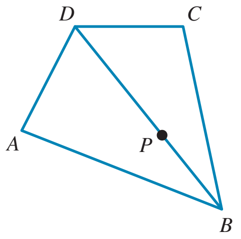
P is the midpoint of $\overrightarrow{AB}$

$\overrightarrow{AP} = \overrightarrow{PB} = \frac{1}{2}\overrightarrow{AB}$

Q is the midpoint of $\overrightarrow{BC}$

$$\begin{aligned}\overrightarrow{BQ} &= \overrightarrow{QC}\\ &= \frac{1}{2}\overrightarrow{BC}\\ \overrightarrow{PQ} &= \overrightarrow{PB} + \overrightarrow{BQ}\\ &= \overrightarrow{AP} + \overrightarrow{QC}\end{aligned}$$

18

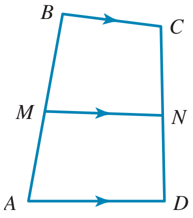


ABCD is a quadrilateral *P* is any point $\overrightarrow{BD}$ given

$$\begin{aligned}\overrightarrow{AP} + \overrightarrow{PB} + \overrightarrow{PD} &= \overrightarrow{PC}\\ \overrightarrow{AP} + \overrightarrow{PB} &= \overrightarrow{PC} - \overrightarrow{PD}\\ \overrightarrow{AP} + \overrightarrow{PB} &= \overrightarrow{DP} + \overrightarrow{PC}\\ \overrightarrow{AB} &= \overrightarrow{DC}\end{aligned}$$

⇒ *ABCD* is a parallelogram

19



Since *M* is the midpoint of $\overrightarrow{AB}$,

$\overrightarrow{AM} = \overrightarrow{MB} = \frac{1}{2}\overrightarrow{AB}$

Since *N* is the midpoint of $\overrightarrow{DC}$

$\overrightarrow{DN} = \overrightarrow{NC} = \frac{1}{2}\overrightarrow{DC}$

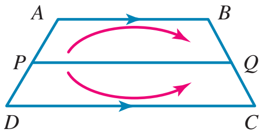
$$\begin{aligned}\overrightarrow{MN} &= \overrightarrow{MB} + \overrightarrow{BC} + \overrightarrow{CN}\\ &= \frac{1}{2}\overrightarrow{AB} + \overrightarrow{BC} + \frac{1}{2}\overrightarrow{CD}\\ \overrightarrow{MN} &= \overrightarrow{MA} + \overrightarrow{AD} + \overrightarrow{DN}\\ &= \frac{1}{2}\overrightarrow{BA} + \overrightarrow{AD} + \frac{1}{2}\overrightarrow{DC}\end{aligned}$$

$$2\overrightarrow{MN} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BA} + \overrightarrow{BC} + \overrightarrow{AD} + \frac{1}{2}\overrightarrow{CD} + \frac{1}{2}\overrightarrow{DC}$$

$$2\overrightarrow{MN} = \overrightarrow{BC} + \overrightarrow{AD}$$

Shown

20



ABCD trapezium

Let $\overrightarrow{AB} = \underset{\sim}{a}$, since $\overrightarrow{DC}$ is parallel to $\overrightarrow{AB}$

$$\begin{aligned}\overrightarrow{DC} &= \lambda\underset{\sim}{a} \quad \lambda \in R\\ \overrightarrow{PQ} &= \overrightarrow{PA} + \overrightarrow{AB} + \overrightarrow{BQ}\\ \overrightarrow{PQ} &= \overrightarrow{PD} + \overrightarrow{DC} + \overrightarrow{CQ}\\ 2\overrightarrow{PQ} &= \overrightarrow{PA} + \overrightarrow{PD} + \overrightarrow{AB} + \overrightarrow{DC} + \overrightarrow{BQ} + \overrightarrow{CQ}\end{aligned}$$

Since *P* is the midpoint of $\overrightarrow{AD}$

$\overrightarrow{AP} = \overrightarrow{PD} = \frac{1}{2}\overrightarrow{AD}$

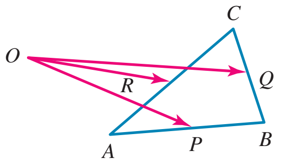
Since *Q* is the midpoint of $\overrightarrow{BC}$

$\overrightarrow{BQ} = \overrightarrow{QC} = \frac{1}{2}\overrightarrow{BC}$

$$\begin{aligned}\overrightarrow{PQ} &= \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{DC}\right)\\ &= \frac{1}{2}\left(\underset{\sim}{a} + \lambda\underset{\sim}{a}\right)\\ &= \frac{1}{2}(\lambda + 1)\underset{\sim}{a}\end{aligned}$$

So $\overrightarrow{PQ}$ is parallel to $\underset{\sim}{a}$ and half the sum of the lengths of $\overrightarrow{AB}$ and $\overrightarrow{DC}$.

21



Since P is the midpoint of $\overrightarrow{AB}$

$$\overrightarrow{AP} = \overrightarrow{PB} = \frac{1}{2}\overrightarrow{AB}$$

Since Q is the midpoint of $\overrightarrow{BC}$

$$\overrightarrow{BQ} = \overrightarrow{QC} = \frac{1}{2}\overrightarrow{BC}$$

Since R is the midpoint of $\overrightarrow{CA}$

$$\overrightarrow{CR} = \overrightarrow{RA} = \frac{1}{2}\overrightarrow{CA}$$

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OA} + \overrightarrow{AP}\\ &= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}\\ &= \overrightarrow{OA} + \frac{1}{2}\left(\overrightarrow{OB} - \overrightarrow{OA}\right)\\ &= \frac{1}{2}\left(\overrightarrow{OB} + \overrightarrow{OA}\right)\end{aligned}$$

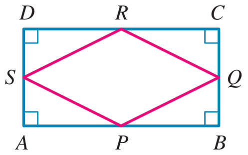
$$\begin{aligned}\overrightarrow{OQ} &= \overrightarrow{OB} + \overrightarrow{BQ}\\ &= \overrightarrow{OB} + \frac{1}{2}\overrightarrow{BC}\\ &= \overrightarrow{OB} + \frac{1}{2}\left(\overrightarrow{OC} - \overrightarrow{OB}\right)\\ &= \frac{1}{2}\left(\overrightarrow{OB} + \overrightarrow{OC}\right)\end{aligned}$$

$$\begin{aligned}\overrightarrow{OR} &= \overrightarrow{OC} + \overrightarrow{CR}\\ &= \overrightarrow{OC} + \frac{1}{2}\overrightarrow{CA}\\ &= \overrightarrow{OC} + \frac{1}{2}\left(\overrightarrow{OA} - \overrightarrow{OC}\right)\\ &= \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OC}\right)\end{aligned}$$

So

$$\begin{aligned}\overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OR} &= \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB}\right) + \frac{1}{2}\left(\overrightarrow{OB} + \overrightarrow{OC}\right)\\ &\quad + \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OC}\right)\\ &= \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}\end{aligned}$$

22



Let $ABCD$ be a rectangle

P be the midpoint of $\overrightarrow{AB}$, $\quad \overrightarrow{AP} = \overrightarrow{PB} = \frac{1}{2}\overrightarrow{AB}$

Q be the midpoint of $\overrightarrow{BC}$, $\quad \overrightarrow{BQ} = \overrightarrow{QC} = \frac{1}{2}\overrightarrow{BC}$

R be the midpoint of $\overrightarrow{CD}$, $\quad \overrightarrow{CR} = \overrightarrow{RD} = \frac{1}{2}\overrightarrow{CD}$

S be the midpoint of $\overrightarrow{DA}$, $\quad \overrightarrow{DS} = \overrightarrow{SA} = \frac{1}{2}\overrightarrow{DA}$

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{PB} + \overrightarrow{BQ}\\ &= \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}\\ &= \frac{1}{2}\overrightarrow{AC}\end{aligned}$$

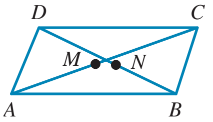
$$\begin{aligned}\overrightarrow{SR} &= \overrightarrow{SD} + \overrightarrow{DR}\\ &= \frac{1}{2}\overrightarrow{AD} + \frac{1}{2}\overrightarrow{DC}\\ &= \frac{1}{2}AC\end{aligned}$$

Since $\overrightarrow{PQ} = \overrightarrow{SR}$

$\overrightarrow{PQ}$ and $\overrightarrow{SR}$ are parallel and qual in length,

so $PQRS$ is a parallelogram.

23



$ABCD$ is a parallelogram, let M be the midpoint of $\overrightarrow{AC}$ and N be the midpoint of $\overrightarrow{BD}$

Show M, N are coincident.

Since M is the midpoint of $\overrightarrow{AC}$

$$\overrightarrow{AM} = \overrightarrow{MC} = \frac{1}{2}\overrightarrow{AC}$$

since N is the midpoint of $\overrightarrow{BD}$

$$\overrightarrow{BN} = \overrightarrow{ND} = \frac{1}{2}\overrightarrow{BC}$$

$$\begin{aligned}\overrightarrow{MN} &= \overrightarrow{MA} + \overrightarrow{AB} + \overrightarrow{BN}\\ &= \frac{1}{2}\overrightarrow{CA} + \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BD}\\ &= \frac{1}{2}\left(\overrightarrow{CD} + \overrightarrow{DA}\right) + \overrightarrow{AB} + \frac{1}{2}\left(\overrightarrow{BA} + \overrightarrow{AD}\right)\\ &= \frac{1}{2}\overrightarrow{CD} + \frac{1}{2}\overrightarrow{DA} + \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BA} + \frac{1}{2}\overrightarrow{AD}\end{aligned}$$

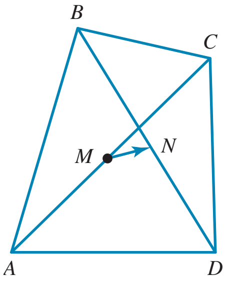
Since $ABCD$ is a parallelogram

$\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$

So $\overrightarrow{MN} = \underset{\sim}{0}$

So M and N are the same point·

24



Since M is the midpoint of $\overrightarrow{AC}$

$$\overrightarrow{AM} = \overrightarrow{MC} = \frac{1}{2}\overrightarrow{AC}$$

Since N is the midpoint of $\overrightarrow{BD}$

$$\overrightarrow{BN} = \overrightarrow{ND} = \frac{1}{2}\overrightarrow{BD}$$

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AM} + \overrightarrow{MN} + \overrightarrow{NB}\\ &= \frac{1}{2}\overrightarrow{AC} + \overrightarrow{MN} + \frac{1}{2}\overrightarrow{DB}\end{aligned}$$

$$\begin{aligned}\overrightarrow{CB} &= \overrightarrow{CM}+\overrightarrow{MN}+\overrightarrow{NB}\\ &= \frac{1}{2}\overrightarrow{CA}+\overrightarrow{MN}+\frac{1}{2}\overrightarrow{DB}\end{aligned}$$

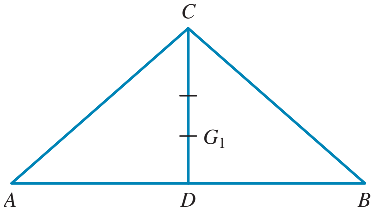
$$\begin{aligned}\overrightarrow{CD} &= \overrightarrow{CM}+\overrightarrow{MN}+\overrightarrow{ND}\\ &= \frac{1}{2}\overrightarrow{CA}+\overrightarrow{MN}+\frac{1}{2}\overrightarrow{BD}\end{aligned}$$

$$\begin{aligned}\overrightarrow{AD} &= \overrightarrow{AM}+\overrightarrow{MN}+\overrightarrow{ND}\\ &= \frac{1}{2}\overrightarrow{AC}+\overrightarrow{MN}+\frac{1}{2}\overrightarrow{BD}\end{aligned}$$

So $\overrightarrow{AB}+\overrightarrow{AD}+\overrightarrow{CB}+\overrightarrow{CD}=\overrightarrow{AC}+4\overrightarrow{MN}+\overrightarrow{DB}+\overrightarrow{CA}+\overrightarrow{BD}$

$= 4\overrightarrow{MN}$ shown

25 Let $\underset{\sim}{a}=\overrightarrow{OA}$ $\quad\underset{\sim}{b}=\overrightarrow{OB}$ $\quad\underset{\sim}{c}=\overrightarrow{OC}$



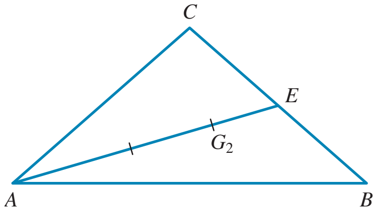
a Since D is the midpoint of $\overrightarrow{AB}$

$$\overrightarrow{AD}=\overrightarrow{DB}=\frac{1}{2}\overrightarrow{AB}$$

$$\begin{aligned}\overrightarrow{CD} &= \overrightarrow{CA}+\overrightarrow{AD}\\ &= \overrightarrow{OA}-\overrightarrow{OC}+\frac{1}{2}\left(\overrightarrow{OB}-\overrightarrow{OA}\right)\\ &= \frac{1}{2}\left(\underset{\sim}{a}+\underset{\sim}{b}-2\underset{\sim}{c}\right)\end{aligned}$$

$$\begin{aligned}\overrightarrow{OG_1} &= \overrightarrow{OC}+\overrightarrow{CG_1}\\ &= \overrightarrow{OC}+\frac{2}{3}\overrightarrow{CD}\\ &= \underset{\sim}{c}+\frac{1}{3}\left(\underset{\sim}{a}+\underset{\sim}{b}-2\underset{\sim}{c}\right)\\ &= \frac{1}{3}\left(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}\right)\end{aligned}$$

b



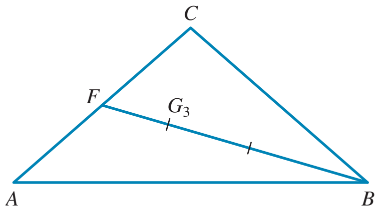
Since E is the midpoint of $\overrightarrow{CB}$

$$\overrightarrow{CE}=\overrightarrow{EB}=\frac{1}{2}\overrightarrow{CB}$$

$$\begin{aligned}\overrightarrow{AE} &= \overrightarrow{AC}+\overrightarrow{CE}\\ &= \overrightarrow{AC}+\frac{1}{2}\overrightarrow{CB}\\ &= \overrightarrow{OC}-\overrightarrow{OA}+\frac{1}{2}\left(\overrightarrow{OB}-\overrightarrow{OC}\right)\\ &= \frac{1}{2}\left(\underset{\sim}{b}+\underset{\sim}{c}-2\underset{\sim}{a}\right)\end{aligned}$$

$$\begin{aligned}\overrightarrow{OG_2} &= \overrightarrow{OA}+\overrightarrow{AG_2}\\ &= \overrightarrow{OA}+\frac{2}{3}\overrightarrow{AE}\\ &= \underset{\sim}{a}+\frac{1}{3}\left(\underset{\sim}{b}+\underset{\sim}{c}-2\underset{\sim}{a}\right)\\ &= \frac{1}{3}\left(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}\right)\end{aligned}$$

c



Since F is the midpoint of $\overrightarrow{AC}$

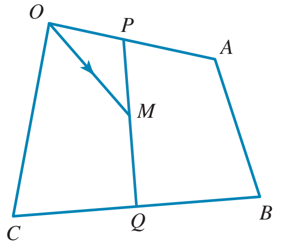
$$\overrightarrow{AF}=\overrightarrow{FC}=\frac{1}{2}\overrightarrow{AC}$$

$$\begin{aligned}\overrightarrow{BF} &= \overrightarrow{BA}+\overrightarrow{AF}\\ &= \overrightarrow{BA}+\frac{1}{2}\overrightarrow{AC}\\ &= \overrightarrow{OA}-\overrightarrow{OB}+\frac{1}{2}\left(\overrightarrow{OC}-\overrightarrow{OA}\right)\\ &= \frac{1}{2}\left(\underset{\sim}{a}+\underset{\sim}{c}-2\underset{\sim}{b}\right)\end{aligned}$$

$$\begin{aligned}\overrightarrow{OG_3} &= \overrightarrow{OB}+\overrightarrow{BG_3}\\ &= \overrightarrow{OB}+\frac{2}{3}\overrightarrow{BF}\\ &= \underset{\sim}{b}+\frac{1}{3}\left(\underset{\sim}{a}+\underset{\sim}{c}-2\underset{\sim}{b}\right)\\ &= \frac{1}{3}\left(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}\right)\end{aligned}$$

d G is the centroid of the triangle, points G_1, G_2, G_3 are all co-incident.

26 a



$\underset{\sim}{a}=\overrightarrow{OA}$, $\quad\underset{\sim}{b}=\overrightarrow{OB}$, $\tilde{c}=\overrightarrow{OC}$

P is the midpoint of $\overrightarrow{OA}$

$$\overrightarrow{OP}=\overrightarrow{PA}=\frac{1}{2}\overrightarrow{OA}$$

Q is the midpoint of $\overrightarrow{BC}$

$$\overrightarrow{CQ}=\overrightarrow{QB}=\frac{1}{2}\overrightarrow{CB}$$

M is the midpoint of $\overrightarrow{PQ}$

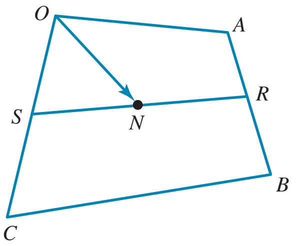
$$\overrightarrow{PM}=\overrightarrow{MQ}=\frac{1}{2}\overrightarrow{PQ}$$

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OP}+\overrightarrow{PM}\\ &= \frac{1}{2}\overrightarrow{OA}+\frac{1}{2}\overrightarrow{PQ}\\ &= \frac{1}{2}\overrightarrow{OA}+\frac{1}{2}\left(\overrightarrow{PA}+\overrightarrow{AB}+\overrightarrow{BQ}\right)\\ &= \frac{1}{2}\overrightarrow{OA}+\frac{1}{2}\left(\frac{1}{2}\overrightarrow{OA}+\overrightarrow{AB}+\frac{1}{2}\overrightarrow{BC}\right)\\ &= \left(\frac{1}{2}+\frac{1}{4}\right)\overrightarrow{OA}+\frac{1}{2}\overrightarrow{AB}+\frac{1}{4}\overrightarrow{BC}\\ &= \frac{3}{4}\overrightarrow{OA}+\frac{1}{2}\left(\overrightarrow{OB}-\overrightarrow{OA}\right)+\frac{1}{4}\left(\overrightarrow{OC}-\overrightarrow{OB}\right)\end{aligned}$$

$$= \left(\frac{3}{4}-\frac{1}{2}\right)\overrightarrow{OA}+\left(\frac{1}{2}-\frac{1}{4}\right)\overrightarrow{OB}+\frac{1}{4}\overrightarrow{OC}$$

$$= \frac{1}{4}\left(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}\right)$$

b



R is the midpoint of $\overrightarrow{AB}$

$\overrightarrow{AR}=\overrightarrow{RB}=\frac{1}{2}\overrightarrow{AB}$

S is the midpoint of $\overrightarrow{OC}$

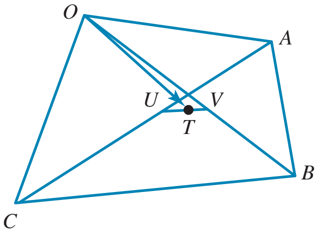
$\overrightarrow{OS}=\overrightarrow{SC}=\frac{1}{2}\overrightarrow{OC}$

N is the midpoint of $\overrightarrow{SR}$

$\overrightarrow{SN}=\overrightarrow{NR}=\frac{1}{2}\overrightarrow{SR}$

$$\begin{aligned}
\overrightarrow{ON}&=\overrightarrow{OS}+\overrightarrow{SN}\\
&=\frac{1}{2}\overrightarrow{OC}+\frac{1}{2}\overrightarrow{SR}\\
&=\frac{1}{2}\overrightarrow{OC}+\frac{1}{2}\left(\overrightarrow{SC}+\overrightarrow{CB}+\overrightarrow{BR}\right)\\
&=\frac{1}{2}\overrightarrow{OC}+\frac{1}{2}\left(\frac{1}{2}\overrightarrow{OC}+\overrightarrow{CB}+\frac{1}{2}\overrightarrow{BA}\right)\\
&=\left(\frac{1}{2}+\frac{1}{4}\right)\overrightarrow{OC}+\frac{1}{2}\overrightarrow{CB}+\frac{1}{4}\overrightarrow{BA}\\
&=\frac{3}{4}\overrightarrow{OC}+\frac{1}{2}\left(\overrightarrow{OB}-\overrightarrow{OC}\right)+\frac{1}{4}\left(\overrightarrow{OA}-\overrightarrow{OB}\right)\\
&=\frac{1}{4}\overrightarrow{OA}+\left(\frac{1}{2}-\frac{1}{4}\right)\overrightarrow{OB}+\left(\frac{3}{4}-\frac{1}{2}\right)\overrightarrow{OC}\\
&=\frac{1}{4}\left(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}\right)
\end{aligned}$$

c



U is the midpoint of $\overrightarrow{AC}$

$\overrightarrow{AU}=\overrightarrow{UC}=\frac{1}{2}\overrightarrow{AC}$

V is the midpoint of $\overrightarrow{OB}$

$\overrightarrow{OV}=\overrightarrow{VB}=\frac{1}{2}\overrightarrow{OB}$

T is the midpoint of $\overrightarrow{UV}$

$\overrightarrow{UT}=\overrightarrow{TV}=\frac{1}{2}\overrightarrow{UV}$

$$\begin{aligned}
\overrightarrow{OT}&=\overrightarrow{OU}+\overrightarrow{UT}\\
&=\overrightarrow{OA}+\overrightarrow{AU}+\frac{1}{2}\overrightarrow{UV}\\
&=\overrightarrow{OA}+\frac{1}{2}\overrightarrow{AC}+\frac{1}{2}\left(\overrightarrow{UA}+\overrightarrow{AB}+\overrightarrow{BV}\right)\\
&=\overrightarrow{OA}+\frac{1}{2}\overrightarrow{AC}+\frac{1}{2}\left(\frac{1}{2}\overrightarrow{CA}+\overrightarrow{AB}+\frac{1}{2}\overrightarrow{BO}\right)\\
&=\overrightarrow{OA}+\left(\frac{1}{2}-\frac{1}{4}\right)\overrightarrow{AC}+\frac{1}{2}\overrightarrow{AB}+\frac{1}{4}\overrightarrow{BO}\\
&=\overrightarrow{OA}+\frac{1}{4}\overrightarrow{AC}+\frac{1}{2}\overrightarrow{AB}+\frac{1}{4}\overrightarrow{BO}\\
&=\overrightarrow{OA}+\frac{1}{4}\left(\overrightarrow{OC}-\overrightarrow{OA}\right)+\frac{1}{2}\left(\overrightarrow{OB}-\overrightarrow{OA}\right)+\frac{1}{4}\overrightarrow{BO}\\
&=\left(1-\frac{1}{4}-\frac{1}{2}\right)\overrightarrow{OA}+\left(\frac{1}{2}-\frac{1}{4}\right)\overrightarrow{OB}+\frac{1}{4}\overrightarrow{OC}\\
&=\frac{1}{4}\left(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}\right)
\end{aligned}$$

d M, N and T are co-incident.

3.2 Exam questions

1 **i** $\overrightarrow{AC}=\underset{\sim}{u}\quad \overrightarrow{CB}=\underset{\sim}{v}$

$\overrightarrow{AN}=\overrightarrow{AC}+\overrightarrow{CN}=\overrightarrow{AC}+\frac{1}{2}\overrightarrow{CB}$

$=\underset{\sim}{u}+\frac{1}{2}\underset{\sim}{v}$ [1 mark]

ii $\overrightarrow{CM}=\overrightarrow{CA}+\overrightarrow{AM}=\overrightarrow{CA}+\frac{1}{2}\overrightarrow{AB}$

$=-\overrightarrow{AC}+\frac{1}{2}\left(\overrightarrow{AC}+\overrightarrow{CB}\right)=\frac{1}{2}\left(\overrightarrow{CB}-\overrightarrow{AC}\right)$

$=\frac{1}{2}\left(\underset{\sim}{v}-\underset{\sim}{u}\right)$ [1 mark]

$\overrightarrow{BP}=\overrightarrow{BC}+\overrightarrow{CP}=-\overrightarrow{CB}-\frac{1}{2}\overrightarrow{AC}$

$=-\underset{\sim}{v}-\frac{1}{2}\underset{\sim}{u}$ [1 mark]

iii $\overrightarrow{AN}+\overrightarrow{CM}+\overrightarrow{BP}$

$=\underset{\sim}{u}+\frac{1}{2}\underset{\sim}{v}+\frac{1}{2}\left(\underset{\sim}{v}-\underset{\sim}{u}\right)+\underset{\sim}{v}-\frac{1}{2}\underset{\sim}{u}$

$=\underset{\sim}{0}$ [1 mark]

2 $\overrightarrow{OQ}=\underset{\sim}{q},\quad \overrightarrow{OS}=\underset{\sim}{s}$

$\overrightarrow{QP}=\frac{1}{2}\left(\overrightarrow{QS}\right)$

$=\frac{1}{2}\left(\overrightarrow{OS}-\overrightarrow{OQ}\right)=\frac{1}{2}\left(\underset{\sim}{s}-\underset{\sim}{q}\right)$ [1 mark]

$\overrightarrow{MN}=\overrightarrow{MR}+\overrightarrow{RN}$

$=\frac{1}{2}\overrightarrow{QR}+\frac{1}{2}\overrightarrow{RS}$

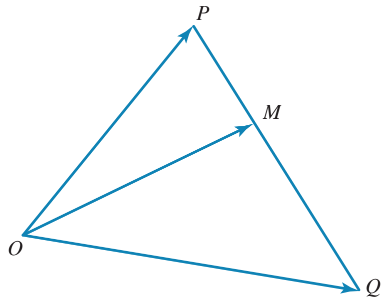
$=\frac{1}{2}\left(\overrightarrow{QR}+\overrightarrow{RS}\right)$ [1 mark]

$=\frac{1}{2}\left(\overrightarrow{OR}-\overrightarrow{OQ}+\overrightarrow{OS}-\overrightarrow{OR}\right)$

$=\frac{1}{2}\left(\overrightarrow{OS}-\overrightarrow{OQ}\right)=\frac{1}{2}\left(\underset{\sim}{s}-\underset{\sim}{q}\right)$

Since $\overrightarrow{QP}=\overrightarrow{MN}$ $QMNP$ is a parallelogram. [1 mark]

3



$$\overrightarrow{PM} = \frac{3}{7}\overrightarrow{PQ}$$
$$\overrightarrow{OP} = \underset{\sim}{p} \quad \text{and} \quad \overrightarrow{OQ} = \underset{\sim}{q}$$
$$\overrightarrow{OM} = \overrightarrow{OP} + \overrightarrow{PM}$$
$$\overrightarrow{OM} = \overrightarrow{OP} + \frac{3}{7}\overrightarrow{PQ}$$
$$\overrightarrow{OM} = \overrightarrow{OP} + \frac{3}{7}\left(\overrightarrow{PO} + \overrightarrow{OQ}\right)$$
$$\overrightarrow{OM} = \overrightarrow{OP} + \frac{3}{7}\left(\overrightarrow{OQ} - \overrightarrow{OP}\right)$$
$$\overrightarrow{OM} = \frac{1}{7}\left(4\underset{\sim}{p} + 3\underset{\sim}{q}\right)$$

The correct answer is **D**.

3.3 Vectors in three dimensions

3.3 Exercise

1 $P(2, -2, -1)$

a $\overrightarrow{OP} = 2\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$

b
$$\begin{aligned} \left|\overrightarrow{OP}\right| &= \sqrt{2^2 + (-2)^2 + (-1)^2} \\ &= \sqrt{4 + 4 + 1} \\ &= \sqrt{9} \\ &= 3 \end{aligned}$$
$$\hat{\overrightarrow{OP}} = \frac{1}{3}\left(2\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}\right)$$

2 $\underset{\sim}{a} = 4\underset{\sim}{i} - 8\underset{\sim}{j} - 2\underset{\sim}{k}$
$$\begin{aligned} |\underset{\sim}{a}| &= \sqrt{4^2 + (-8)^2 + (-2)^2} \\ &= \sqrt{16 + 64 + 4} \\ &= \sqrt{84} \\ &= \sqrt{4 \times 21} \\ &= 2\sqrt{21} \end{aligned}$$
$$\hat{\underset{\sim}{a}} = \frac{1}{\sqrt{21}}\left(2\underset{\sim}{i} - 4\underset{\sim}{j} - \underset{\sim}{k}\right)$$

3 $P(2, 1, -3) \qquad Q(4, -1, 2)$
$$\begin{aligned} \overrightarrow{OP} &= 2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k} \\ \overrightarrow{OQ} &= 4\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k} \\ \overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP} \\ &= 2\underset{\sim}{i} - 2\underset{\sim}{j} + 5\underset{\sim}{k} \\ \left|\overrightarrow{PQ}\right| &= \sqrt{2^2 + (-2)^2 + 5^2} \\ &= \sqrt{4 + 4 + 25} \\ &= \sqrt{33} \end{aligned}$$
$$\hat{\overrightarrow{PQ}} = \frac{1}{\sqrt{33}}\left(2\underset{\sim}{i} - 2\underset{\sim}{j} + 5\underset{\sim}{k}\right)$$

4 $A(-1, 2, -4) \qquad B(2, 6, 8)$
$$\begin{aligned} \overrightarrow{OA} &= -\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k} \\ \overrightarrow{OB} &= 2\underset{\sim}{i} + 6\underset{\sim}{j} + 8\underset{\sim}{k} \\ \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= 3\underset{\sim}{i} + 4\underset{\sim}{j} + 12\underset{\sim}{k} \\ \left|\overrightarrow{AB}\right| &= \sqrt{3^2 + 4^2 + 12^2} \\ &= \sqrt{9 + 16 + 144} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

5
$$\begin{aligned} \underset{\sim}{c} &= \underset{\sim}{a} - 2\underset{\sim}{b} \\ &= -7\underset{\sim}{i} + 12\underset{\sim}{j} + (4 - z)\underset{\sim}{k} \end{aligned}$$
$$4 - z = 0$$
$$z = 4$$

6 $C(x, -2, 4) \qquad D(2, y, -3)$
$$\begin{aligned} \overrightarrow{OC} &= x\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k} \\ \overrightarrow{OD} &= 2\underset{\sim}{i} + y\underset{\sim}{j} - 3\underset{\sim}{k} \\ \overrightarrow{CD} &= \overrightarrow{OD} - \overrightarrow{OC} \\ &= \left(2\underset{\sim}{i} + y\underset{\sim}{j} - 3\underset{\sim}{k}\right) - \left(x\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}\right) \\ &= (2 - x)\underset{\sim}{i} + (y + 2)\underset{\sim}{j} - 7\underset{\sim}{k} \\ &= 3\underset{\sim}{i} + 4\underset{\sim}{j} + z\underset{\sim}{k} \end{aligned}$$
$\underset{\sim}{i}$: $2 - x = 3 \Rightarrow x = -1$

$\underset{\sim}{j}$: $y + 2 = 4 \Rightarrow y = 2$

$\underset{\sim}{k}$: $z = -7$

7 **a** $\underset{\sim}{r} = 2\underset{\sim}{i} - \underset{\sim}{j} + z\underset{\sim}{k} \qquad \underset{\sim}{s} = -4\underset{\sim}{i} + 2\underset{\sim}{j} + 5\underset{\sim}{k}$
$$\begin{aligned} |\underset{\sim}{r}| &= 5 \\ |\underset{\sim}{r}| &= \sqrt{2^2 + (-1)^2 + (z)^2} \\ &= \sqrt{4 + 1 + z^2} \end{aligned}$$
$$\begin{aligned} \sqrt{5 + z^2} &= 5 \\ 5 + z^2 &= 25 \\ z^2 &= 20 \\ z &= \pm\sqrt{20} \\ z &= \pm 2\sqrt{5} \end{aligned}$$

b $\underset{\sim}{r} = \lambda\underset{\sim}{s} \qquad \lambda = -\frac{1}{2}$
$$\underset{\sim}{k}: z = \frac{-1}{2} \times 5$$
$$z = \frac{-5}{2}$$

8 **a** $\underset{\sim}{a} = 3\underset{\sim}{i} + y\underset{\sim}{j} - 4\underset{\sim}{k} \qquad \underset{\sim}{b} = -6\underset{\sim}{i} + 3\underset{\sim}{j} + 8\underset{\sim}{k}$
$$\begin{aligned} |\underset{\sim}{a}| &= \sqrt{3^2 + y^2 + (-4)^2} \\ &= \sqrt{9 + y^2 + 16} \\ |\underset{\sim}{b}| &= \sqrt{(-6)^2 + 3^2 + 8^2} \\ &= \sqrt{36 + 9 + 64} \\ &= \sqrt{109} \end{aligned}$$
$$|\underset{\sim}{a}| = |\underset{\sim}{b}|$$
$$\Rightarrow \sqrt{109} = \sqrt{25 + y^2}$$
$$\begin{aligned} 109 &= 25 + y^2 \\ y^2 &= 84 \\ y &= \pm\sqrt{84} \\ y &= \pm 2\sqrt{21} \end{aligned}$$

b $-2\underset{\sim}{a} = \underset{\sim}{b}$

$\underset{\sim}{j}: \ -2y = 3$

$y = \dfrac{-3}{2}$

9 $\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k} \quad \underset{\sim}{b} = -2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k} \quad \underset{\sim}{c} = 2\underset{\sim}{i} + \underset{\sim}{j} + z\underset{\sim}{k}$

Let $\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$

$2\underset{\sim}{i} + \underset{\sim}{j} + z\underset{\sim}{k} = m\left(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}\right) + n\left(-2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right)$

$\underset{\sim}{i}$ (1) $2 = 2m - 2n$

$\underset{\sim}{j}$ (2) $1 = -m + 2n$

$\underset{\sim}{k}$ (3) $z = 3m - n$

(1) + (2) $m = 3 \Rightarrow 2n = 4 \quad n = 2$

into (3) $z = 9 - 2 = 7$

$m = 3, \ n = 2, \ z = 7$

10 $\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k} \quad \underset{\sim}{b} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k} \quad \underset{\sim}{c} = x\underset{\sim}{i} + 2\underset{\sim}{j}$

$\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$

$x\underset{\sim}{i} + 2\underset{\sim}{j} = m\left(3\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}\right) + n\left(2\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}\right)$

$\underset{\sim}{i}$ (1) $x = 3m + 2n$

$\underset{\sim}{j}$ (2) $2 = -2m - 3n$

$\underset{\sim}{k}$ (3) $0 = 4m + 5n$

$2 \times$ (2) $4 = -4m - 6n$

(3) $0 = 4m + 5n$

$2 \times$ (2) + (3) $4 = -n$

$n = -4$

into (2) $2m = -3n - 2$

$= 12 - 2$

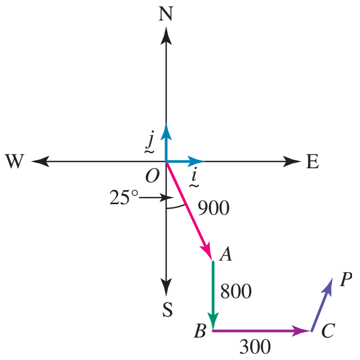
$= 10$

$m = 5$

into (1) $x = 15 - 8 = 7$

$x = 7$

11 a



$\overrightarrow{OA} = 900 \sin(25°)\,\underset{\sim}{i} - 900 \cos(25°)\,\underset{\sim}{j}$

$\overrightarrow{AB} = -800\underset{\sim}{j}$

$\overrightarrow{BC} = 300\underset{\sim}{i}$

$\overrightarrow{CP} = 150\underset{\sim}{k}$

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CP}$

$= 900 \sin(25°)\,\underset{\sim}{i} - 900 \cos(25°)\,\underset{\sim}{j} - 800\underset{\sim}{j} + 300\underset{\sim}{i} + 150\underset{\sim}{k}$

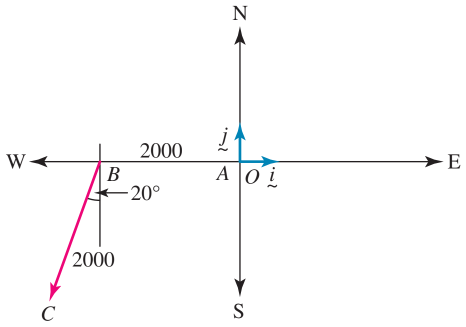
$= (300 + 900 \sin(25°))\,\underset{\sim}{i} - (800 + 900 \cos(25°))\,\underset{\sim}{j} + 150\underset{\sim}{k}$

$= 680.36\underset{\sim}{i} - 1615.68\underset{\sim}{j} + 150\underset{\sim}{k}$

b $\left|\overrightarrow{OP}\right| = \sqrt{680.36^2 + (-1615.68)^2 + 150^2}$

$= 1759.5$ metres

12 a



$\overrightarrow{OA} = 800\underset{\sim}{k}$ (in metres)

$\overrightarrow{AB} = -2000\underset{\sim}{i}$

$\overrightarrow{BC} = -2000 \sin(20°)\,\underset{\sim}{i} - 2000 \cos(20°)\,\underset{\sim}{j}$

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC}$

$= 800\underset{\sim}{k} - 2000\underset{\sim}{i} - 2000 \sin(20°)\,\underset{\sim}{i} - 2000 \cos(20°)\,\underset{\sim}{j}$

$= -(2000 + 2000 \sin(20°))\,\underset{\sim}{i} - 2000 \cos(20°)\,\underset{\sim}{j} + 800\underset{\sim}{k}$

$= -2684\underset{\sim}{i} - 1879.4\underset{\sim}{j} + 800\underset{\sim}{k}$

b $\left|\overrightarrow{OP}\right| = \sqrt{(-2684)^2 + (-1879.4)^2 + 800^2}$

$= 3373$ metres

13 $A = \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix} \ B = \begin{bmatrix} 3 \\ -6 \\ 4 \end{bmatrix} \ C = \begin{bmatrix} 1 \\ -7 \\ 8 \end{bmatrix}$

$\underset{\sim}{a} = 2\underset{\sim}{i} - 5\underset{\sim}{j} + 4\underset{\sim}{k}, \quad \underset{\sim}{b} = 3\underset{\sim}{i} - 6\underset{\sim}{j} + 4\underset{\sim}{k}, \quad \underset{\sim}{c} = \underset{\sim}{i} - 7\underset{\sim}{j} + 8\underset{\sim}{k}$

$C = \alpha A + \beta B$

$$\begin{bmatrix} 1 \\ -7 \\ 8 \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ -6 \\ 4 \end{bmatrix}$$

(1) $1 = 2\alpha + 3\beta$

(2) $-7 = -5\alpha - 6\beta$

(3) $8 = 4\alpha + 4\beta$

$2 \times$ (1) $2 = 4\alpha + 6\beta$

(2) $-7 = -5\alpha - 6\beta$

add $-5 = -\alpha$

$\alpha = 5$ into (1) $3\beta = 1 - 2\alpha = 1 - 10 = -9$

$\beta = -3$

check in (3)

RHS $= 4\alpha + 4\beta = 20 - 12 = 8 =$ LHS

$C = 5A - 3B$

$\underset{\sim}{c} = 5\underset{\sim}{a} - 3\underset{\sim}{b}$

So $A, \ B, \ C$ are linearly dependent

14 $A = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix} \quad B = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \quad C = \begin{bmatrix} 7 \\ 1 \\ -1 \end{bmatrix}$

$\underset{\sim}{a} = 2\underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k}, \underset{\sim}{b} = -\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}, \underset{\sim}{c} = 7\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$

$C = \alpha A + \beta B$

(1) $7 = 2\alpha - \beta$

(2) $1 = -4\alpha - 3\beta$

(3) $-1 = 3\alpha + 2\beta$

$2 \times$ (1) $14 = 4\alpha - 2\beta$

(2) $1 = -4\alpha - 3\beta$

add $15 = -5\beta$

$\beta = -3$ into (1) $2\alpha = 7 + \beta = 4$

$\alpha = 2$

check in (3)

RHS $3\alpha + 2\beta = 6 - 6 = 0 \neq -1$

$$C = 2A - 3B$$

$$\underset{\sim}{c} = 2\underset{\sim}{a} - 3\underset{\sim}{b}$$

Contradiction so they are linearly independent

15 a Let $\underset{\sim}{a} = \underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}$

$$\begin{aligned}|\underset{\sim}{a}| &= \sqrt{1^2 + (-2)^2 + (2)^2}\\ &= \sqrt{9}\\ &= 3\end{aligned}$$

$$\text{So } \hat{\underset{\sim}{a}} = \frac{1}{3}\left(\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}\right)$$

$$\begin{aligned}6\hat{\underset{\sim}{a}} &= 2\left(\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}\right)\\ &= 2\underset{\sim}{i} - 4\underset{\sim}{j} + 4\underset{\sim}{k}\end{aligned}$$

b Let $\underset{\sim}{b} = -3\underset{\sim}{i} + 4\underset{\sim}{j} + 12\underset{\sim}{k}$

$$\begin{aligned}|\underset{\sim}{b}| &= \sqrt{(-3)^2 + 4^2 + 12^2}\\ &= \sqrt{9 + 16 + 144}\\ &= 13\end{aligned}$$

$$\text{So } \hat{\underset{\sim}{b}} = \frac{1}{13}\left(-3\underset{\sim}{i} + 4\underset{\sim}{j} + 12\underset{\sim}{k}\right)$$

$$\begin{aligned}-26\hat{\underset{\sim}{b}} &= \frac{-26}{13}\left(-3\underset{\sim}{i} + 4\underset{\sim}{j} + 12\underset{\sim}{k}\right)\\ &= 6\underset{\sim}{i} - 8\underset{\sim}{j} - 24\underset{\sim}{k}\end{aligned}$$

c Let $\underset{\sim}{c} = -5\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}$

$$\begin{aligned}|\underset{\sim}{c}| &= \sqrt{(-5)^2 + (-3)^2 + 4^2}\\ &= \sqrt{50}\\ &= 5\sqrt{2}\end{aligned}$$

$$\text{So } \hat{\underset{\sim}{c}} = \frac{1}{5\sqrt{2}}\left(-5\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}\right)$$

$$\begin{aligned}-10\sqrt{2}\hat{\underset{\sim}{c}} &= \frac{-10\sqrt{2}}{5\sqrt{2}}\left(-5\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}\right)\\ &= 10\underset{\sim}{i} + 6\underset{\sim}{j} - 8\underset{\sim}{k}\end{aligned}$$

16 a $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k} \qquad \underset{\sim}{b} = -3\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$

$$\begin{aligned}2\underset{\sim}{a} + \underset{\sim}{b} &= 2\left(2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}\right) + \left(-3\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}\right)\\ &= 4\underset{\sim}{i} + 6\underset{\sim}{j} - 2\underset{\sim}{k} - 3\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}\\ &= \underset{\sim}{i} + 7\underset{\sim}{j}\end{aligned}$$

No $\underset{\sim}{k}$ so in the xy plane

b $\underset{\sim}{p} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - 5\underset{\sim}{k} \qquad \underset{\sim}{q} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$

$$\begin{aligned}2\underset{\sim}{p} - 3\underset{\sim}{q} &= 2\left(3\underset{\sim}{i} + 2\underset{\sim}{j} - 5\underset{\sim}{k}\right) - 3\left(2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\right)\\ &= 6\underset{\sim}{i} + 4\underset{\sim}{j} - 10\underset{\sim}{k} - 6\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}\\ &= \underset{\sim}{j} - 4\underset{\sim}{k}\end{aligned}$$

No $\underset{\sim}{i}$ so in the yz plane

c $\underset{\sim}{r} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k} \qquad \underset{\sim}{s} = \underset{\sim}{i} + y\underset{\sim}{j} - 2\underset{\sim}{k}$

$$\begin{aligned}4\underset{\sim}{r} + 3\underset{\sim}{s} &= 4\left(2\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}\right) + 3\left(\underset{\sim}{i} + y\underset{\sim}{j} - 2\underset{\sim}{k}\right)\\ &= 8\underset{\sim}{i} - 12\underset{\sim}{j} + 20\underset{\sim}{k} + 3\underset{\sim}{i} + 3y\underset{\sim}{j} - 6\underset{\sim}{k}\\ &= 11\underset{\sim}{i} + (3y - 12)\underset{\sim}{j} + 14\underset{\sim}{k}\end{aligned}$$

If in the xz plane, $\underset{\sim}{j}$ component is zero

$$\begin{aligned}3y - 12 &= 0\\ 3y &= 12\\ y &= 4\end{aligned}$$

17 a $A\,(2, -1, 3) \quad B\,(8, -7, 15) \quad C\,(4, -3, 7)$

$$\begin{aligned}\overrightarrow{OA} &= 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}\\ \overrightarrow{OB} &= 8\underset{\sim}{i} - 7\underset{\sim}{j} + 15\underset{\sim}{k}\\ \overrightarrow{OC} &= 4\underset{\sim}{i} - 3\underset{\sim}{j} + 7\underset{\sim}{k}\\ \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA}\\ &= 6\underset{\sim}{i} - 6\underset{\sim}{j} + 12\underset{\sim}{k}\\ \overrightarrow{BC} &= \overrightarrow{OC} - \overrightarrow{OB}\\ &= -4\underset{\sim}{i} + 4\underset{\sim}{j} - 8\underset{\sim}{k}\end{aligned}$$

$$\text{So } -3\overrightarrow{BC} = 2\overrightarrow{AB}$$

$$\overrightarrow{AB} = \frac{-3}{2}\overrightarrow{BC}$$

So A, B, C are collinear

b $P\,(2, 1, 4) \quad Q\,(1, -2, 3) \quad R\,(-1, -8, 1)$

$$\begin{aligned}\overrightarrow{OP} &= 2\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}\\ \overrightarrow{OQ} &= \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}\\ \overrightarrow{OR} &= -\underset{\sim}{i} - 8\underset{\sim}{j} + \underset{\sim}{k}\\ \overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP}\\ &= -\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k}\\ \overrightarrow{QR} &= \overrightarrow{OR} - \overrightarrow{OQ}\\ &= -2\underset{\sim}{i} - 6\underset{\sim}{j} - 2\underset{\sim}{k}\\ 2\overrightarrow{PQ} &= \overrightarrow{QR}\\ \overrightarrow{PQ} &= \frac{1}{2}\overrightarrow{QR}\end{aligned}$$

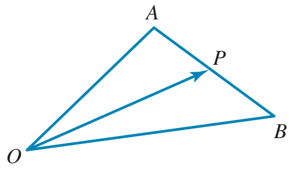
So $P,\ Q,\ R$ are collinear

c $A\,(x, 1, 2) \quad B\,(2, y, -1) \quad C\,(3, -4, 5)$

$$\begin{aligned}\overrightarrow{OA} &= x\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}\\ \overrightarrow{OB} &= 2\underset{\sim}{i} + y\underset{\sim}{j} - \underset{\sim}{k}\\ \overrightarrow{OC} &= 3\underset{\sim}{i} - 4\underset{\sim}{j} + 5\underset{\sim}{k}\\ \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA}\\ &= (2 - x)\underset{\sim}{i} + (y - 1)\underset{\sim}{j} - 3\underset{\sim}{k}\\ \overrightarrow{BC} &= \overrightarrow{OC} - \overrightarrow{OB}\\ &= \underset{\sim}{i} + (-4 - y)\underset{\sim}{j} + 6\underset{\sim}{k}\\ -2\overrightarrow{AB} &= \overrightarrow{BC}\end{aligned}$$

$$\begin{aligned}-2\,(2 - x) &= 1 & -2\,(y - 1) &= -4 - y\\ 2 - x &= \frac{-1}{2} & y - 1 &= 2 + \frac{y}{2}\\ x &= \frac{5}{2} & y &= 6\end{aligned}$$

18 a



$A\,(3, 1, -2) \qquad B\,(5, 3, 4)$

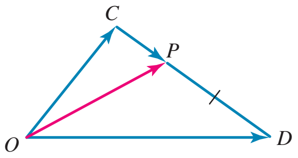
$$\begin{aligned}\overrightarrow{OA} &= 3\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\\ \overrightarrow{OB} &= 5\underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k}\end{aligned}$$

P is the midpoint of $\overrightarrow{AB}$

$$\begin{aligned}\overrightarrow{AP} &= \overrightarrow{PB}\\ &= \frac{1}{2}\overrightarrow{AB}\\ \overrightarrow{OP} &= \overrightarrow{OA} + \overrightarrow{AP}\\ &= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}\end{aligned}$$

$$\begin{aligned}
&= \overrightarrow{OA} + \frac{1}{2}\left(\overrightarrow{OB} - \overrightarrow{OA}\right)\\
&= \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB}\right)\\
&= \frac{1}{2}\left(\left(3\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\right) + \left(5\underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k}\right)\right)\\
&= \frac{1}{2}\left(8\underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k}\right)\\
&= 4\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}
\end{aligned}$$

b



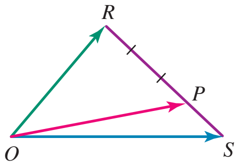
$C(-2, 4, 1) \qquad D(-5, 1, 4)$

$$\begin{aligned}
\overrightarrow{OC} &= -2\underset{\sim}{i} + 4\underset{\sim}{j} + \underset{\sim}{k}\\
\overrightarrow{OD} &= -5\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}
\end{aligned}$$

P is $\frac{1}{3}\overrightarrow{CD}$

$$\begin{aligned}
\overrightarrow{CP} &= \frac{1}{3}\overrightarrow{CD}\\
\overrightarrow{OP} &= \overrightarrow{OC} + \frac{1}{3}\overrightarrow{CD}\\
&= \overrightarrow{OC} + \frac{1}{3}\left(\overrightarrow{OD} - \overrightarrow{OC}\right)\\
&= \frac{1}{3}\left(2\overrightarrow{OC} + \overrightarrow{OD}\right)\\
&= \frac{1}{3}\left(2\left(-2\underset{\sim}{i} + 4\underset{\sim}{j} + \underset{\sim}{k}\right) + \left(-5\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}\right)\right)\\
&= \frac{1}{3}\left(-9\underset{\sim}{i} + 9\underset{\sim}{j} + 6\underset{\sim}{k}\right)\\
&= -3\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}
\end{aligned}$$

c



$R(1, -2, -3) \qquad S(-3, 2, 5)$

$$\begin{aligned}
\overrightarrow{OR} &= \underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k}\\
\overrightarrow{OS} &= -3\underset{\sim}{i} + 2\underset{\sim}{j} + 5\underset{\sim}{k}
\end{aligned}$$

P is $\frac{3}{4}\overrightarrow{RS}$

$$\begin{aligned}
\overrightarrow{RP} &= \frac{3}{4}\overrightarrow{RS}\\
\overrightarrow{OP} &= \overrightarrow{OR} + \frac{3}{4}\overrightarrow{RS}\\
&= \overrightarrow{OR} + \frac{3}{4}\left(\overrightarrow{OS} - \overrightarrow{OR}\right)\\
&= \frac{1}{4}\left(\overrightarrow{OR} + 3\overrightarrow{OS}\right)\\
&= \frac{1}{4}\left(\left(\underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k}\right) + 3\left(-3\underset{\sim}{i} + 2\underset{\sim}{j} + 5\underset{\sim}{k}\right)\right)\\
&= \frac{1}{4}\left(-8\underset{\sim}{i} + 4\underset{\sim}{j} + 12\underset{\sim}{k}\right)\\
&= -2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}
\end{aligned}$$

19 a

$$\begin{aligned}
\underset{\sim}{a} &= 2\underset{\sim}{i} - 4\underset{\sim}{j} - 6\underset{\sim}{k}\\
\underset{\sim}{b} &= 3\underset{\sim}{i} + 6\underset{\sim}{j} - 9\underset{\sim}{k}\\
\underset{\sim}{c} &= 3\underset{\sim}{i} + 10\underset{\sim}{j} - 9\underset{\sim}{k}\\
\underset{\sim}{c} &= \alpha\underset{\sim}{a} + \beta\underset{\sim}{b}
\end{aligned}$$

$$3\underset{\sim}{i} + 10\underset{\sim}{j} - 9\underset{\sim}{k} = \alpha\left(2\underset{\sim}{i} - 4\underset{\sim}{j} - 6\underset{\sim}{k}\right) + \beta\left(3\underset{\sim}{i} + 6\underset{\sim}{j} - 9\underset{\sim}{k}\right)$$

$$\begin{aligned}
\underset{\sim}{i}\,(1) &\quad 3 = 2\alpha + 3\beta\\
\underset{\sim}{j}\,(2) &\quad 10 = -4\alpha + 6\beta\\
\underset{\sim}{k}\,(3) &\quad -9 = -6\alpha - 9\beta\\
2\times(1) &\quad 6 = 4\alpha + 6\beta\\
(2) &\quad 10 = -4\alpha + 6\beta\\
2\times(1) + (2) &\quad 16 = 12\beta
\end{aligned}$$

$\beta = \frac{4}{3}$ and $\alpha = \frac{1}{2}(3 - 4) = \frac{-1}{2}$

So $\underset{\sim}{c} = \frac{4}{3}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}$

So $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ are linearly dependent

b

$$\begin{aligned}
\underset{\sim}{p} &= 2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}\\
\underset{\sim}{q} &= 2\underset{\sim}{i} - \underset{\sim}{j} + 5\underset{\sim}{k}\\
\underset{\sim}{r} &= 2\underset{\sim}{i} + y\underset{\sim}{j} + 8\underset{\sim}{k}\\
\underset{\sim}{r} &= \alpha\underset{\sim}{p} + \beta\underset{\sim}{q}
\end{aligned}$$

$$2\underset{\sim}{i} + y\underset{\sim}{j} + 8\underset{\sim}{k} = \alpha\left(2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}\right) + \beta\left(2\underset{\sim}{i} - \underset{\sim}{j} + 5\underset{\sim}{k}\right)$$

$$\begin{aligned}
\underset{\sim}{i}\,(1) &\quad 2 = 2\alpha + 2\beta\\
\underset{\sim}{j}\,(2) &\quad y = -3\alpha - \beta\\
\underset{\sim}{k}\,(3) &\quad 8 = 4\alpha + 5\beta\\
2\times(1) &\quad 4 = 4\alpha + 4\beta\\
-(3) &\quad -8 = -4\alpha - 5\beta\\
2\times(1) - (3) &\quad -4 = -\beta\\
&\quad \beta = 4
\end{aligned}$$

$$\begin{aligned}
2 &= 2\alpha + 8\\
2\alpha &= -6\\
\alpha &= -3
\end{aligned}$$

$$\begin{aligned}
\text{into (2)}\quad y &= -3\alpha - \beta\\
&= 9 - 4\\
y &= 5
\end{aligned}$$

c

$$\begin{aligned}
\underset{\sim}{a} &= 2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}\\
\underset{\sim}{b} &= 3\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}\\
\underset{\sim}{c} &= -7\underset{\sim}{i} + 8\underset{\sim}{j} + z\underset{\sim}{k}\\
\underset{\sim}{c} &= \alpha\underset{\sim}{a} + \beta\underset{\sim}{b}
\end{aligned}$$

$$-7\underset{\sim}{i} + 8\underset{\sim}{j} + z\underset{\sim}{k} = \alpha\left(2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}\right) + \beta\left(3\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}\right)$$

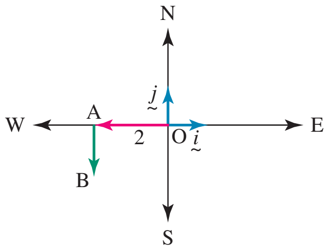
$$\begin{aligned}
\underset{\sim}{i}\,(1) &\quad -7 = 2\alpha + 3\beta\\
\underset{\sim}{j}\,(2) &\quad 8 = -3\alpha - 4\beta\\
\underset{\sim}{k}\,(3) &\quad z = 4\alpha + 2\beta\\
3\times(1) &\quad -21 = 6\alpha + 9\beta\\
2\times(2) &\quad 16 = -6\alpha - 8\beta\\
3\times(1) + 2\times(2) &\quad -5 = \beta
\end{aligned}$$

$\beta = -5$ and $-7 = 2\alpha - 15$

$\alpha = 4$

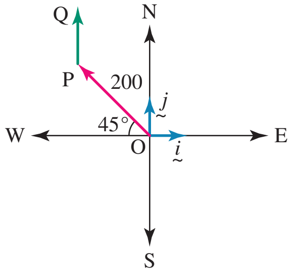
$$\begin{aligned}
\text{into (3)}\quad z &= 4\alpha + 2\beta\\
&= 16 - 10\\
z &= 6
\end{aligned}$$

20



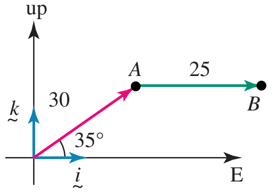
$$\begin{aligned}
\overrightarrow{OA} &= -2\underset{\sim}{i}\\
\overrightarrow{AB} &= -\underset{\sim}{j}\\
\overrightarrow{BC} &= 0.5\underset{\sim}{k}\\
\overrightarrow{OC} &= \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC}\\
&= -2\underset{\sim}{i} - \underset{\sim}{j} + 0.5\underset{\sim}{k}\\
\left|\overrightarrow{OC}\right| &= \sqrt{(-2)^2 + (-1)^2 + (0.5)^2}\\
&= 2.29 \text{ km}
\end{aligned}$$

21



$$\begin{aligned}
\overrightarrow{OP} &= -200\sin(45°)\,\underset{\sim}{i} + 200\cos(45°)\,\underset{\sim}{j}\\
\overrightarrow{PQ} &= 50\underset{\sim}{j}\\
\overrightarrow{QR} &= -5\cos(50°)\,\underset{\sim}{j} + 5\sin(50°)\,\underset{\sim}{k}\\
\overrightarrow{OR} &= \overrightarrow{OP} + \overrightarrow{PQ} + \overrightarrow{QR}\\
&= -200\sin(45°)\,\underset{\sim}{i} + (200\cos(45°) + 50 - 5\cos(50°))\underset{\sim}{j}\\
&\quad + 5\sin(50°)\,\underset{\sim}{k}\\
&= -141.42\underset{\sim}{i} + 188.21\underset{\sim}{j} + 3.83\underset{\sim}{k}\\
\left|\overrightarrow{OR}\right| &= \sqrt{(-141.42)^2 + (188.21)^2 + (3.83)^2}\\
&= 235.45 \text{ m}
\end{aligned}$$

22



$$\overrightarrow{OA} = 30\cos(35°)\,\underset{\sim}{i} + 30\sin(35°)\,\underset{\sim}{k}$$

300 km/hr for 5 minutes

$$300 \times \frac{5}{60} = 25 \text{ km}$$

$$\begin{aligned}
\overrightarrow{AB} &= 25\underset{\sim}{i}\\
\overrightarrow{OB} &= \overrightarrow{OA} + \overrightarrow{AB}\\
&= (25 + 30\cos(35°))\,\underset{\sim}{i} + 30\sin(35°)\,\underset{\sim}{k}\\
&= 49.575\underset{\sim}{i} + 17.207\underset{\sim}{k}\\
\left|\overrightarrow{OB}\right| &= \sqrt{(49.575)^2 + (17.207)^2}\\
&= 52.48 \text{ km}
\end{aligned}$$

23 $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$

$$\begin{aligned}
|\underset{\sim}{a}| &= \sqrt{1^2 + 2^2 + (-3)^2}\\
&= \sqrt{1 + 4 + 9}\\
\cos(\gamma) &= \frac{-3}{\sqrt{14}}\\
\gamma &= \cos^{-1}\left(\frac{-3}{\sqrt{14}}\right)\\
&= 143.3°
\end{aligned}$$

24 $\underset{\sim}{a} = \underset{\sim}{i} + z\underset{\sim}{k}$

$$|\underset{\sim}{a}| = \sqrt{1^2 + z^2}$$

$$\cos(\gamma) = \frac{z}{|\underset{\sim}{a}|}, \quad \gamma = 150°$$

$$\begin{aligned}
\cos(150°) &= -\frac{\sqrt{3}}{2}\\
&= \frac{z}{\sqrt{1 + z^2}} \quad \text{So } z < 0\\
-\sqrt{3}\sqrt{1 + z^2} &= 2z\\
3\left(1 + z^2\right) &= 4z^2\\
3 + 3z^2 &= 4z^2\\
z^2 &= 3\\
z &= \pm\sqrt{3}
\end{aligned}$$

but $z < 0$

$z = -\sqrt{3}$ only

25 **a** $A(-2, 4, 1)$

$$\begin{aligned}
\overrightarrow{OA} &= -2\underset{\sim}{i} + 4\underset{\sim}{j} + \underset{\sim}{k}\\
\left|\overrightarrow{OA}\right| &= \sqrt{(-2)^2 + 4^2 + 1^2}\\
&= \sqrt{4 + 16 + 1}\\
&= \sqrt{21}\\
\hat{\underset{\sim}{a}} &= \frac{1}{\sqrt{21}}\left(-2\underset{\sim}{i} + 4\underset{\sim}{j} + \underset{\sim}{k}\right)
\end{aligned}$$

b
$$\begin{aligned}
\cos(\alpha) &= \frac{-2}{\sqrt{21}}\\
\alpha &= \cos^{-1}\left(\frac{-2}{\sqrt{21}}\right)\\
&= 115.88°
\end{aligned}$$

26 **a** $B(3, 5, -2)$

$$\begin{aligned}
\overrightarrow{OB} &= 3\underset{\sim}{i} + 5\underset{\sim}{j} - 2\underset{\sim}{k}\\
\left|\overrightarrow{OB}\right| &= \sqrt{3^2 + 5^2 + (-2)^2}\\
&= \sqrt{9 + 25 + 4}\\
&= \sqrt{38}\\
\hat{\underset{\sim}{b}} &= \frac{1}{\sqrt{38}}\left(3\underset{\sim}{i} + 5\underset{\sim}{j} - 2\underset{\sim}{k}\right)
\end{aligned}$$

b
$$\begin{aligned}
\cos(\beta) &= \frac{5}{\sqrt{38}}\\
\beta &= \cos^{-1}\left(\frac{5}{\sqrt{38}}\right)\\
&= 35.80°
\end{aligned}$$

27 **a** $C(4, 6, -8)$

$$\begin{aligned}
\overrightarrow{OC} &= 4\underset{\sim}{i} + 6\underset{\sim}{j} - 8\underset{\sim}{k}\\
\left|\overrightarrow{OC}\right| &= \sqrt{4^2 + 6^2 + (-8)^2}\\
&= \sqrt{16 + 36 + 64}\\
&= \sqrt{116}\\
&= \sqrt{4 \times 29}\\
&= 2\sqrt{29}
\end{aligned}$$

$$\begin{aligned}\hat{\underset{\sim}{c}} &= \frac{1}{2\sqrt{29}}\left(4\underset{\sim}{i} + 6\underset{\sim}{j} - 8\underset{\sim}{k}\right)\\ &= \frac{1}{\sqrt{29}}\left(2\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k}\right)\end{aligned}$$

b $\cos(\gamma) = \dfrac{-4}{\sqrt{29}}$

$$\begin{aligned}\gamma &= \cos^{-1}\left(\frac{-4}{\sqrt{29}}\right)\\ &= 137.97^\circ\end{aligned}$$

28 a $A(3, 5, -2) \qquad B(2, -1, 3)$

$$\begin{aligned}\overrightarrow{OA} &= 3\underset{\sim}{i} + 5\underset{\sim}{j} - 2\underset{\sim}{k}\\ \overrightarrow{OB} &= 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}\\ \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA}\\ &= -\underset{\sim}{i} - 6\underset{\sim}{j} + 5\underset{\sim}{k}\\ \left|\overrightarrow{AB}\right| &= \sqrt{(-1)^2 + (-6)^2 + 5^2}\\ &= \sqrt{1 + 36 + 25}\\ &= \sqrt{62}\end{aligned}$$

b $P(-2, 4, 1) \qquad Q(3, -5, 2)$

$$\begin{aligned}\overrightarrow{OP} &= -2\underset{\sim}{i} + 4\underset{\sim}{j} + \underset{\sim}{k}\\ \overrightarrow{OQ} &= 3\underset{\sim}{i} - 5\underset{\sim}{j} + 2\underset{\sim}{k}\\ \overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP}\\ &= 5\underset{\sim}{i} - 9\underset{\sim}{j} + \underset{\sim}{k}\\ \left|\overrightarrow{PQ}\right| &= \sqrt{5^2 + (-9)^2 + 1^2}\\ &= \sqrt{107}\\ \cos(\beta) &= \frac{-9}{\sqrt{107}}\\ &= 150.47^\circ\end{aligned}$$

c $R(4, 3, -1) \qquad S(6, 1, -7)$

$$\begin{aligned}\overrightarrow{OR} &= 4\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}\\ \overrightarrow{OS} &= 6\underset{\sim}{i} + \underset{\sim}{j} - 7\underset{\sim}{k}\\ \overrightarrow{SR} &= \overrightarrow{OR} - \overrightarrow{OS}\\ &= -2\underset{\sim}{i} + 2\underset{\sim}{j} + 6\underset{\sim}{k}\\ \left|\overrightarrow{SR}\right| &= \sqrt{(2)^2 + 2^2 + 6^2}\\ &= \sqrt{4 + 4 + 36}\\ &= \sqrt{44}\\ &= 2\sqrt{11}\\ \hat{\overrightarrow{SR}} &= \frac{1}{2\sqrt{11}}\left(-2\underset{\sim}{i} + 2\underset{\sim}{j} + 6\underset{\sim}{k}\right)\\ &= \frac{1}{\sqrt{11}}\left(-\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}\right)\end{aligned}$$

Only says parallel (can be opposite direction) to $\overrightarrow{SR}$

So $\hat{\overrightarrow{SR}} = \pm\dfrac{1}{\sqrt{11}}\left(-\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}\right)$

29 a $\underset{\sim}{a} = \underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}$

$\underset{\sim}{b} = x\underset{\sim}{i} + 6\underset{\sim}{j} - 12\underset{\sim}{k}$

$\underset{\sim}{a}$ is // to $\underset{\sim}{b}$

$\underset{\sim}{b} = -3\underset{\sim}{a}$

$\underset{\sim}{i}: \ x = -3$

b

$$\begin{aligned}|\underset{\sim}{b}| &= \sqrt{x^2 + 6^2 + (-12)^2}\\ &= \sqrt{x^2 + 36 + 144}\\ &= \sqrt{x^2 + 180}\end{aligned}$$

$$\begin{aligned}|\underset{\sim}{b}| &= 10\sqrt{2}\\ &= \sqrt{x^2 + 180}\\ 200 &= x^2 + 180\\ x^2 &= 20\\ &= 4 \times 5\\ x &= \pm 2\sqrt{5}\end{aligned}$$

30 a $\underset{\sim}{r} = 3\underset{\sim}{i} + y\underset{\sim}{j} + \underset{\sim}{k}$

$$\begin{aligned}|\underset{\sim}{r}| &= \sqrt{3^2 + y^2 + 1^2}\\ &= \sqrt{10 + y^2}\\ |\underset{\sim}{r}| &= 10\\ &= \sqrt{10 + y^2}\\ 100 &= 10 + y^2\\ y^2 &= 90\\ y &= \pm\sqrt{90}\\ &= \pm 3\sqrt{10}\end{aligned}$$

b $\cos(\beta) = \dfrac{y}{\sqrt{10 + y^2}}$

$$\begin{aligned}\beta &= \cos^{-1}\left(\frac{-1}{3}\right)\\ \frac{-1}{3} &= \frac{y}{\sqrt{10 + y^2}}\\ -\sqrt{10 + y^2} &= 3y\\ 10 + y^2 &= 9y^2\\ 8y^2 &= 10\\ y^2 &= \frac{5}{4}\\ y &= \pm\frac{\sqrt{5}}{2}\\ y &= \frac{-\sqrt{5}}{2} \text{ since } y < 0 \text{ only}\end{aligned}$$

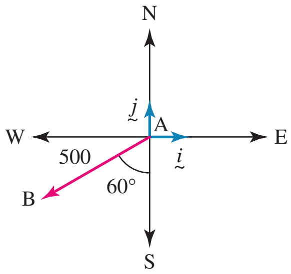
31 a $\underset{\sim}{u} = 2\sqrt{2}\underset{\sim}{i} + 2\underset{\sim}{j} + z\underset{\sim}{k}$

$$\begin{aligned}|\underset{\sim}{u}| &= \sqrt{\left(2\sqrt{2}\right)^2 + 2^2 + z^2}\\ &= \sqrt{8 + 4 + z^2}\\ &= \sqrt{12 + z^2}\\ |\underset{\sim}{u}| &= 6\\ \sqrt{12 + z^2} &= 6\\ 12 + z^2 &= 36\\ z^2 &= 24\\ z &= \pm\sqrt{24}\\ &= \pm\sqrt{4 \times 6}\\ &= \pm 2\sqrt{6}\end{aligned}$$

b $\cos(120^\circ) = \dfrac{-1}{2}$

$$\begin{aligned}&= \frac{z}{\sqrt{12 + z^2}}\\ -2z &= \sqrt{12 + z^2}\\ 4z^2 &= 12 + z^2\\ 3z^2 &= 12\\ z^2 &= 4\\ z &= \pm 2 \ \text{ but } z < 0\\ z &= -2 \ \text{ only}\end{aligned}$$

32



$$\overrightarrow{OA} = 600\underset{\sim}{k}$$
$$\overrightarrow{AB} = -500\cos(60^\circ)\underset{\sim}{j} - 500\sin(60^\circ)\underset{\sim}{i}$$

120 km/hr for 1 minute

$$120 \times \frac{1}{60} = 2\,\text{km} = 2000\,\text{m}$$
$$\begin{aligned}\overrightarrow{BC} &= 2000\cos(50^\circ)\left(-\cos(45^\circ)\underset{\sim}{i} - \sin(45^\circ)\underset{\sim}{j}\right)\\ &\quad + 2000\sin(50^\circ)\underset{\sim}{k}\end{aligned}$$
$$\begin{aligned}\overrightarrow{OC} &= \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC}\\ &= (-500\sin(60^\circ) - 2000\cos(50^\circ)\cos(45^\circ))\underset{\sim}{i}\\ &\quad (-500\cos(60^\circ) - 2000\cos(50^\circ)\sin(45^\circ))\underset{\sim}{j}\\ &\quad + 600 + 2000\sin(50^\circ)\\ &= -1342.05\underset{\sim}{i} - 1159.04\underset{\sim}{j} + 2132.09\underset{\sim}{k}\end{aligned}$$
$$\begin{aligned}\left|\overrightarrow{OC}\right| &= \sqrt{(-1342.05)^2 + (1159.04)^2 + (2132.09)^2}\\ &= 2773.14\,\text{m}\end{aligned}$$

33 a

$$\underset{\sim}{a} = (m-1)\underset{\sim}{i} + (m+1)\underset{\sim}{j} + m\underset{\sim}{k}$$
$$\begin{aligned}|\underset{\sim}{a}| &= \sqrt{(m-1)^2 + (m+1)^2 + m^2}\\ &= \sqrt{m^2 - 2m + 1 + m^2 + 2m + 1 + m^2}\\ &= \sqrt{17}\end{aligned}$$
$$\begin{aligned}\sqrt{3m^2 + 2} &= \sqrt{17}\\ 3m^2 + 2 &= 17\\ 3m^2 &= 15\\ m^2 &= 5\\ m &= \pm\sqrt{5}\end{aligned}$$

b

$$\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$$
$$|\underset{\sim}{a}| = 1$$
$$\Rightarrow x^2 + y^2 + z^2 = 1$$
$$x = \cos(45^\circ) = \frac{\sqrt{2}}{2}$$
$$y = \cos(120^\circ) = \frac{-1}{2}$$
$$\begin{aligned}\text{so } z^2 &= 1 - \left(x^2 + y^2\right)\\ &= 1 - \left(\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{-1}{2}\right)^2\right)\\ &= 1 - \left(\frac{2}{4} + \frac{1}{4}\right)\\ &= \frac{1}{4}\end{aligned}$$
$$z = \pm\frac{1}{2}$$
$$\cos(\alpha) = \frac{1}{2}$$
$$\alpha = 60^\circ$$

Since $0 < \alpha < 90^\circ$

acute

c

$$\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$$
$$|\underset{\sim}{a}| = 1$$
$$\Rightarrow x^2 + y^2 + z^2 = 1$$
$$x = \cos(60^\circ) = \frac{1}{2}$$
$$z = \cos(120^\circ) = \frac{-1}{2}$$
$$\begin{aligned}\text{so } y^2 &= 1 - x^2 - z^2\\ &= 1 - \left(x^2 + z^2\right)\\ &= 1 - \left(\frac{1}{4} + \frac{1}{4}\right)\\ &= \frac{1}{2}\end{aligned}$$
$$\begin{aligned}y &= \pm\frac{1}{\sqrt{2}}\\ &= \pm\frac{\sqrt{2}}{2}\end{aligned}$$

but $y < 0$ $y = \cos(\beta)$

$$\cos(\beta) = -\frac{\sqrt{2}}{2}$$
$$\beta = 135^\circ$$

34 a

$$\underset{\sim}{a} = 4\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}$$
$$\underset{\sim}{b} = -\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$$
$$\underset{\sim}{c} = 4\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$$
$$\underset{\sim}{d} = 7\underset{\sim}{i} + \underset{\sim}{j} + 11\underset{\sim}{k}$$
$$\underset{\sim}{d} = p\underset{\sim}{a} + q\underset{\sim}{b} + r\underset{\sim}{c}$$

$\underset{\sim}{i}$: (1) $7 = 4p - q + 4r$

$\underset{\sim}{j}$: (2) $1 = -3p + 2q - r$

$\underset{\sim}{k}$: (3) $11 = 2p - 3q + 2r$

$$\text{Let } A = \begin{bmatrix}4 & -1 & 4\\ -3 & 2 & -1\\ 2 & -3 & 2\end{bmatrix} \quad X = \begin{bmatrix}p\\ q\\ r\end{bmatrix} \quad K = \begin{bmatrix}7\\ 1\\ 11\end{bmatrix}$$
$$AX = K \quad \Rightarrow X = A^{-1}K$$
$$\begin{aligned}X = A^{-1}K &= \frac{1}{20}\begin{bmatrix}1 & -10 & -7\\ 4 & 0 & -8\\ 5 & 10 & 5\end{bmatrix}\begin{bmatrix}7\\ 1\\ 11\end{bmatrix}\\ &= \frac{1}{20}\begin{bmatrix}-80\\ -60\\ 100\end{bmatrix} = \begin{bmatrix}-4\\ -3\\ 5\end{bmatrix}\end{aligned}$$
$$\Rightarrow p = -4,\ q = -3,\ r = 5$$

b

$$\underset{\sim}{a} = x\underset{\sim}{i} + 2y\underset{\sim}{j} + 3\underset{\sim}{k}$$
$$\underset{\sim}{b} = 2x\underset{\sim}{i} + y\underset{\sim}{j} + 4\underset{\sim}{k}$$
$$|\underset{\sim}{a}| = |\underset{\sim}{b}| = 5$$
$$\sqrt{x^2 + 4y^2 + 9} = 5 \quad \text{from } |\underset{\sim}{a}|$$
$$x^2 + 4y^2 + 9 = 25$$
$$(1) \qquad x^2 + 4y^2 = 16$$
$$\sqrt{4x^2 + y^2 + 16} = 5 \quad \text{from } |\underset{\sim}{b}|$$
$$4x^2 + y^2 + 16 = 25$$
$$(2) \qquad 4x^2 + y^2 = 9$$

solving (1) (2) use CAS

$$x = \pm\frac{2\sqrt{3}}{3} \text{ and } y = \pm\frac{\sqrt{33}}{3}$$

3.3 Exam questions

1 $\underset{\sim}{a} = -\underset{\sim}{i} + 6\underset{\sim}{j} - 3\underset{\sim}{k},\ \underset{\sim}{b} = 2\underset{\sim}{i} - 8\underset{\sim}{j} + 5\underset{\sim}{k},\ \underset{\sim}{c} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + \left|1 - p^2\right|\underset{\sim}{k}$

For linearly dependent $\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$

$3\underset{\sim}{i} + 2\underset{\sim}{j} + \left|1 - p^2\right|\underset{\sim}{k} = m\left(-\underset{\sim}{i} + 6\underset{\sim}{j} - 3\underset{\sim}{k}\right) + n\left(2\underset{\sim}{i} - 8\underset{\sim}{j} + 5\underset{\sim}{k}\right)$

$\underset{\sim}{i}$: (1) $3 = -m + 2n$

$\underset{\sim}{j}$: (2) $2 = 6m - 8n$

$\underset{\sim}{k}$: (3) $\left|1 - p^2\right| = -3m + 5n$

(1) $\times 4 +$ (2) $\Rightarrow 14 = 2m,\ m = 7,\ 2n = m + 3 = 10,\ n = 5$

(3) $\left|1 - p^2\right| = 25 - 21 = 4$

$1 - p^2 = \pm 4$

$p^2 = 5, -3$

$p = \pm\sqrt{5}$

For linearly independent $p \in R\backslash\{\pm\sqrt{5}\}$

Award 1 mark for 3 equations.

Award 1 mark for solving for *m* and *n*.

Award 1 mark for solving for values of *p*.

Award 1 mark for the correct values of *p* for linearly independent.

2 $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}, \underset{\sim}{b} = \lambda\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k},\ \underset{\sim}{c} = \underset{\sim}{i} + \underset{\sim}{k}$

$\underset{\sim}{b} = m\underset{\sim}{a} + n\underset{\sim}{c}$

$\lambda\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k} = m\left(\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right) + n\left(\underset{\sim}{i} + \underset{\sim}{k}\right)$

$\underset{\sim}{i}$: (1) $\lambda = m + n$

$\underset{\sim}{j}$: (2) $3 = 2m$

$\underset{\sim}{k}$: (3) $2 = -m + n$

Solving:

$\lambda = 5, m = \frac{3}{2}, n = \frac{7}{2}$

The correct answer is **E**.

3 $\underset{\sim}{a} = 3\underset{\sim}{i} + 5\underset{\sim}{j} - 2\underset{\sim}{k},\ \underset{\sim}{b} = \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k},\ \underset{\sim}{c} = \underset{\sim}{i} + d\,\underset{\sim}{k}$

For linear dependence, $\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$

$\underset{\sim}{c} = \underset{\sim}{i} + d\,\underset{\sim}{k} = m\left(3\underset{\sim}{i} + 5\underset{\sim}{j} - 2\underset{\sim}{k}\right) + n\left(\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}\right)$

$\underset{\sim}{i}$ (1) $1 = 3m + n$

$\underset{\sim}{j}$ (2) $0 = 5m - 2n$

$\underset{\sim}{k}$ (3) $d = -2m + 3n$

(3) + (2) − (1) $\Rightarrow d = 1$

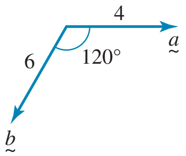
Award 1 mark for expressing three equations using linear dependence.

Award 1 mark for the correct value of *d*.

3.4 Scalar product and applications

3.4 Exercise

1



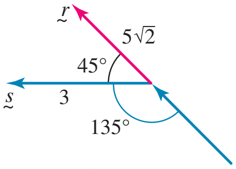
$|\underset{\sim}{a}| = 4,\ |\underset{\sim}{b}| = 6,\ \theta = 120°$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \cos(120°)$

$= 4 \times 6 \times \frac{-1}{2}$

$= -12$

2



$|\underset{\sim}{r}| = 5\sqrt{2},\ |\underset{\sim}{s}| = 3,\ \theta = 45°$

$\underset{\sim}{r} \cdot \underset{\sim}{s} = |\underset{\sim}{r}|\,|\underset{\sim}{s}| \cos(\theta)$

$= 5\sqrt{2} \times 3 \times \cos(45°)$

$= 5\sqrt{2} \times 3 \times \frac{1}{\sqrt{2}}$

$= 15$

3 a $|\underset{\sim}{a}| = 8,\ |\underset{\sim}{b}| = 3,\ \theta = 60°$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \cos(60°)$

$= 8 \times 3 \times \frac{1}{2}$

$= 12$

b $|\underset{\sim}{a}| = 6\sqrt{2},\ |\underset{\sim}{b}| = 5,\ \theta = 135°$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \cos(135°)$

$= 6\sqrt{2} \times 5 \times \left(\frac{-1}{\sqrt{2}}\right)$

$= -30$

c $|\underset{\sim}{a}| = 7,\ |\underset{\sim}{b}| = 3,\ \theta = 180°$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \cos(180°)$

$= 7 \times 3 \times (-1)$

$= -21$

4 $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k},\ \underset{\sim}{b} = -\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = \left(2\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}\right) \cdot \left(-\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}\right)$

$= 2 \times -1 + -3 \times 3 + 5 \times -2$

$= -2 - 9 - 10$

$= -21$

5 $\underset{\sim}{r} = \underset{\sim}{i} + y\underset{\sim}{j} + 4\underset{\sim}{k},\ \underset{\sim}{s} = -2\underset{\sim}{i} + 6\underset{\sim}{j} - 7\underset{\sim}{k}$

$\underset{\sim}{r} \cdot \underset{\sim}{s} = \left(\underset{\sim}{i} + y\underset{\sim}{j} + 4\underset{\sim}{k}\right) \cdot \left(-2\underset{\sim}{i} + 6\underset{\sim}{j} - 7\underset{\sim}{k}\right) = 12$

$12 = 1 \times -2 + 6y + 4 \times -7$

$12 = -2 + 6y - 28$

$12 = 6y - 30$

$y = 7$

6 a $\underset{\sim}{u} = 2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}$

$\underset{\sim}{v} = -\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$

$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left(2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}\right) \cdot \left(-\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}\right)$

$= 2 \times -1 + -1 \times 2 + 4 \times -3$

$= -2 - 2 - 12$

$= -16$

b $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$

$\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = \left(\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}\right) \cdot \left(2\underset{\sim}{i} + \underset{\sim}{j}\right)$

$= 1 \times 2 + 1 \times 1 + -3 \times 0$

$= 3$

c $\underset{\sim}{r} = 2\underset{\sim}{i} - 3\underset{\sim}{j}$

$\underset{\sim}{s} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}$

$$\begin{aligned}\underset{\sim}{r}\cdot\underset{\sim}{s} &= \left(2\underset{\sim}{i}-3\underset{\sim}{j}\right)\cdot\left(3\underset{\sim}{i}+2\underset{\sim}{j}+4\underset{\sim}{k}\right)\\ &= 2\times 3+-3\times 2+0\\ &= 0\end{aligned}$$

7 $\underset{\sim}{a}=2\underset{\sim}{i}+3\underset{\sim}{j}-\underset{\sim}{k},\ \underset{\sim}{b}=4\underset{\sim}{i}-3\underset{\sim}{j}+z\underset{\sim}{k}$

$$\begin{aligned}\underset{\sim}{a}\cdot\underset{\sim}{b}=0 &= \left(2\underset{\sim}{i}+3\underset{\sim}{j}-\underset{\sim}{k}\right)\cdot\left(4\underset{\sim}{i}-3\underset{\sim}{j}+z\underset{\sim}{k}\right)\\ &= 2\times 4+3\times -3+-1\times z\\ &= 8-9-z\\ z&=-1\end{aligned}$$

8 $\underset{\sim}{a}=x\underset{\sim}{i}-\underset{\sim}{j}+x\underset{\sim}{k},\ \underset{\sim}{b}=x\underset{\sim}{i}+5\underset{\sim}{j}-4\underset{\sim}{k}$

$$\begin{aligned}\underset{\sim}{a}\cdot\underset{\sim}{b} &= \left(x\underset{\sim}{i}-\underset{\sim}{j}+x\underset{\sim}{k}\right)\cdot\left(x\underset{\sim}{i}+5\underset{\sim}{j}-4\underset{\sim}{k}\right)\\ &= x^2-5-4x=0\\ &= x^2-4x-5=0\\ 0&=(x-5)(x+1)\\ x&=-1,\ 5\end{aligned}$$

9 a $\underset{\sim}{p}=2\underset{\sim}{i}-\underset{\sim}{j}+z\underset{\sim}{k}$

$\underset{\sim}{q}=-6\underset{\sim}{i}+3\underset{\sim}{j}+5\underset{\sim}{k}$

$\underset{\sim}{p}$ is parallel to $\underset{\sim}{q}$

$$\begin{aligned}\underset{\sim}{q}&=-3\underset{\sim}{p}\\ \underset{\sim}{k}:\ 5&=-3z\\ z&=\frac{-5}{3}\end{aligned}$$

b $\underset{\sim}{p}$ is ⊥ to q

$$\begin{aligned}\underset{\sim}{p}\cdot\underset{\sim}{q}&=-12-3+5z\\ &=0\\ 5z&=15\\ z&=3\end{aligned}$$

c
$$\begin{aligned}|\underset{\sim}{p}|&=\sqrt{2^2+(-1)^2+z^2}\\ &=\sqrt{5+z^2}\\ &=5\\ z^2+5&=25\\ z^2&=20\\ &=4\times 5\\ z&=\pm 2\sqrt{5}\end{aligned}$$

10 a $\underset{\sim}{r}=6\underset{\sim}{i}+y\underset{\sim}{j}-9\underset{\sim}{k}$

$\underset{\sim}{s}=-4\underset{\sim}{i}+2\underset{\sim}{j}+6\underset{\sim}{k}$

$\underset{\sim}{r}$ is parallel to $\underset{\sim}{s}$

$$\begin{aligned}-4\underset{\sim}{r}&=6\underset{\sim}{s}\\ \underset{\sim}{j}:\ -4y&=12\\ y&=-3\end{aligned}$$

b $\underset{\sim}{r}$ is ⊥ to $\underset{\sim}{s}$

$$\begin{aligned}\underset{\sim}{r}\cdot\underset{\sim}{s}&=-24+2y-54\\ &=0\\ 2y&=78\\ y&=39\end{aligned}$$

c
$$\begin{aligned}|\underset{\sim}{r}|&=\sqrt{6^2+y^2+(-9)^2}\\ &=\sqrt{117+y^2}\\ |\underset{\sim}{s}|&=\sqrt{(-4)^2+2^2+6^2}\\ &=\sqrt{56}\\ |\underset{\sim}{r}|&=2\,|\underset{\sim}{s}|\Rightarrow\sqrt{117+y^2}=2\sqrt{56}\\ 117+y^2&=4\times 56\\ y&=\pm\sqrt{107}\end{aligned}$$

11 a $\underset{\sim}{a}=x\underset{\sim}{i}+2\underset{\sim}{j}-3\underset{\sim}{k}$

$\underset{\sim}{b}=\underset{\sim}{i}-2\underset{\sim}{j}-\underset{\sim}{k}$

$\underset{\sim}{c}=2\underset{\sim}{i}+4\underset{\sim}{j}-5\underset{\sim}{k}$

$$\begin{aligned}2\underset{\sim}{a}-3\underset{\sim}{b}&=2\left(x\underset{\sim}{i}+2\underset{\sim}{j}-3\underset{\sim}{k}\right)-3\left(\underset{\sim}{i}-2\underset{\sim}{j}-\underset{\sim}{k}\right)\\ &=(2x-3)\,\underset{\sim}{i}+10\underset{\sim}{j}-3\underset{\sim}{k}\end{aligned}$$

$2\underset{\sim}{a}-3\underset{\sim}{b}$ // to yz plane $\quad\left(2\underset{\sim}{a}-3\underset{\sim}{b}\right)\cdot\underset{\sim}{i}=0$

$$\begin{aligned}2x-3&=0\\ x&=\frac{3}{2}\end{aligned}$$

b
$$\begin{aligned}&\left(2\underset{\sim}{a}-3\underset{\sim}{b}\right)\cdot\underset{\sim}{c}=0\\ &\Rightarrow 2\,(2x-3)+40+15=0\\ 2\,(2x-3)&=-55\\ 2x-3&=\frac{-55}{2}\\ x&=\frac{-49}{4}\end{aligned}$$

c
$$\begin{aligned}11&=|2\underset{\sim}{a}-3\underset{\sim}{b}|\\ 11&=\sqrt{(2x-3)^2+10^2+(-3)^2}\\ 11&=\sqrt{(2x-3)^2+100+9}\\ 121&=(2x-3)^2+109\\ 12&=(2x-3)^2\\ \pm 2\sqrt{3}&=2x-3\\ 2x&=3\pm 2\sqrt{3}\\ x&=\frac{3\pm 2\sqrt{3}}{2}\end{aligned}$$

12
$$\begin{aligned}\underset{\sim}{a}&=4\underset{\sim}{i}-5\underset{\sim}{j}-3\underset{\sim}{k}\\ |\underset{\sim}{a}|&=\sqrt{4^2+(5)^2+(-3)^2}\\ &=\sqrt{16+25+9}\\ &=\sqrt{50}\\ &=5\sqrt{2}\\ \underset{\sim}{b}&=2\underset{\sim}{i}-\underset{\sim}{j}+\underset{\sim}{k}\\ |\underset{\sim}{b}|&=\sqrt{2^2+(-1)^2+1^2}\\ &=\sqrt{6}\\ \underset{\sim}{a}\cdot\underset{\sim}{b}&=8+5-3\\ &=10\\ \cos(\theta)&=\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}|\,|\underset{\sim}{b}|}\\ &=\frac{10}{5\sqrt{2}\times\sqrt{6}}\\ &=\frac{2}{\sqrt{12}}=\frac{\sqrt{3}}{3}\\ \theta&=\cos^{-1}\left(\frac{\sqrt{3}}{3}\right)\\ &=54.74^\circ\end{aligned}$$

13
$$\begin{aligned}\underset{\sim}{u}&=x\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k}\\ |\underset{\sim}{u}|&=\sqrt{x^2+1+1}\\ &=\sqrt{x^2+2}\end{aligned}$$

$$\begin{aligned}
\underset{\sim}{v} &= \underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}\\
|\underset{\sim}{v}| &= \sqrt{1^2 + (-1)^2 + 1^2}\\
&= \sqrt{3}
\end{aligned}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = x - 1 + 1 = x, \qquad \theta = 120^\circ$$

$$\begin{aligned}
\cos(120^\circ) &= \frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{u}|\,|\underset{\sim}{v}|}\\
&= \frac{-1}{2}\\
&= \frac{x}{\sqrt{3}\sqrt{x^2+2}}
\end{aligned}$$

$$\begin{aligned}
\sqrt{3}\sqrt{x^2+2} &= -2x && \text{but } x < 0\\
3\left(x^2+2\right) &= 4x^2\\
3x^2 + 6 &= 4x^2\\
x^2 &= 6\\
x &= \pm\sqrt{6} && \text{but } x < 0\\
x &= -\sqrt{6} \text{ only}
\end{aligned}$$

14 $|\underset{\sim}{a}| = 7, |\underset{\sim}{b}| = 5, \underset{\sim}{a} \cdot \underset{\sim}{b} = 26$

$$\begin{aligned}
|\underset{\sim}{a} - \underset{\sim}{b}|^2 &= \left(\underset{\sim}{a} - \underset{\sim}{b}\right) \cdot \left(\underset{\sim}{a} - \underset{\sim}{b}\right)\\
&= \underset{\sim}{a} \cdot \underset{\sim}{a} - \underset{\sim}{b} \cdot \underset{\sim}{a} - \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{b}\\
&= |\underset{\sim}{a}|^2 - 2\underset{\sim}{a} \cdot \underset{\sim}{b} + |\underset{\sim}{b}|^2\\
&= 7^2 - 2 \times 26 + 5^2\\
&= 49 - 52 + 25\\
&= 22\\
|\underset{\sim}{a} - \underset{\sim}{b}| &= \sqrt{22}
\end{aligned}$$

15 $|\underset{\sim}{r}| = 2, |\underset{\sim}{s}| = 4, \underset{\sim}{r} \cdot \underset{\sim}{s} = 8$

$$\begin{aligned}
|2\underset{\sim}{s} - \underset{\sim}{r}|^2 &= \left(2\underset{\sim}{s} - \underset{\sim}{r}\right) \cdot \left(2\underset{\sim}{s} - \underset{\sim}{r}\right)\\
&= 4\underset{\sim}{s} \cdot \underset{\sim}{s} - 2\underset{\sim}{r} \cdot \underset{\sim}{s} - 2\underset{\sim}{s} \cdot \underset{\sim}{r} + \underset{\sim}{r} \cdot \underset{\sim}{r}\\
&= 4|\underset{\sim}{s}|^2 - 4\underset{\sim}{r} \cdot \underset{\sim}{s} + |\underset{\sim}{r}|^2\\
&= 4 \times 4^2 - 4 \times 8 + 2^2\\
&= 64 - 32 + 4\\
&= 36\\
|2\underset{\sim}{s} - \underset{\sim}{r}| &= \sqrt{36}\\
&= 6
\end{aligned}$$

16 **a**
$$\begin{aligned}
\underset{\sim}{a} &= 3\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}\\
\underset{\sim}{b} &= \underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}\\
\underset{\sim}{c} &= 4\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}
\end{aligned}$$

$$\begin{aligned}
\underset{\sim}{a} \cdot \underset{\sim}{b} &= 3 \times 1 + -2 \times 1 + 4 \times -3\\
&= 3 - 2 - 12\\
&= -11
\end{aligned}$$

$$\begin{aligned}
\underset{\sim}{a} \cdot \underset{\sim}{c} &= 3 \times 4 + -2 \times -3 + 4 \times 5\\
&= 12 + 6 + 20\\
&= 38
\end{aligned}$$

$$\begin{aligned}
\underset{\sim}{b} + \underset{\sim}{c} &= 5\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}\\
\underset{\sim}{a} \cdot \left(\underset{\sim}{b} + \underset{\sim}{c}\right) &= 3 \times 5 + -2 \times -2 + 2 \times 4\\
&= 15 + 4 + 8\\
&= 27
\end{aligned}$$

So $\underset{\sim}{a} \cdot \left(\underset{\sim}{b} + \underset{\sim}{c}\right) = \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}$

$27 = -11 + 38$ shown

b
$$\begin{aligned}
\underset{\sim}{b} - \underset{\sim}{c} &= -3\underset{\sim}{i} + 4\underset{\sim}{j} - 8\underset{\sim}{k}\\
\underset{\sim}{a} \cdot \left(\underset{\sim}{b} - \underset{\sim}{c}\right) &= 3 \times -3 + -2 \times 4 + 4 \times -8\\
&= -9 - 8 - 32\\
&= -49
\end{aligned}$$

So $\underset{\sim}{a} \cdot \left(\underset{\sim}{b} - \underset{\sim}{c}\right) = \underset{\sim}{a} \cdot \underset{\sim}{b} - \underset{\sim}{a} \cdot \underset{\sim}{c}$

$-49 = -11 - 38$ shown

c The dot product is distributive over addition and subtraction of a vector.

d $\underset{\sim}{a} \cdot \underset{\sim}{b} \cdot \underset{\sim}{c}$ is meaningless.

Cannot find the dot product of a vector and a scalar

17 $\underset{\sim}{u} = 5\underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k} \qquad \underset{\sim}{v} = \underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k}$

$$\begin{aligned}
|\underset{\sim}{v}| &= \sqrt{1^2 + (-2)^2 + (-2)^2}\\
&= \sqrt{1 + 4 + 4}\\
&= \sqrt{9}\\
&= 3
\end{aligned}$$

a $\underset{\sim}{u} \cdot \underset{\sim}{v} = 5 + 6 + 4 = 15$

Scalar resolute

$$\underset{\sim}{u} \cdot \hat{\underset{\sim}{v}} = \frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{v}|} = \frac{15}{3} = 5$$

b Vector resolute $\underset{\sim}{u}$ parallel to $\underset{\sim}{v}$

$$\left(\underset{\sim}{u} \cdot \hat{\underset{\sim}{v}}\right)\hat{\underset{\sim}{v}} = \frac{5}{3}\left(\underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k}\right)$$

c Vector resolute $\underset{\sim}{u}$ perpendicular to $\underset{\sim}{v}$

$$\begin{aligned}
\underset{\sim}{u} - \left(\underset{\sim}{u} \cdot \hat{\underset{\sim}{v}}\right)\hat{\underset{\sim}{v}} &= \left(5\underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k}\right) - \frac{5}{3}\left(\underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k}\right)\\
&= \frac{1}{3}\left(\left(15\underset{\sim}{i} - 9\underset{\sim}{j} - 6\underset{\sim}{k}\right) - \left(5\underset{\sim}{i} - 10\underset{\sim}{j} - 10\underset{\sim}{k}\right)\right)\\
&= \frac{1}{3}\left(10\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}\right)
\end{aligned}$$

18 $\underset{\sim}{r} = 2\underset{\sim}{i} + 4\underset{\sim}{k} \qquad \underset{\sim}{s} = \underset{\sim}{i} - 4\underset{\sim}{j} - 2\underset{\sim}{k}$

a
$$\begin{aligned}
|\underset{\sim}{s}| &= \sqrt{1^2 + (-4)^2 + (-2)^2}\\
&= \sqrt{1 + 16 + 4}\\
&= \sqrt{21}
\end{aligned}$$

$$\begin{aligned}
\underset{\sim}{r} \cdot \underset{\sim}{s} &= 2 + 0 - 8\\
&= -6
\end{aligned}$$

$$\begin{aligned}
\left(\underset{\sim}{r} \cdot \hat{\underset{\sim}{s}}\right)\hat{\underset{\sim}{s}} &= \frac{-6}{21}\left(\underset{\sim}{i} - 4\underset{\sim}{j} - 2\underset{\sim}{k}\right)\\
&= \frac{-2}{7}\left(\underset{\sim}{i} - 4\underset{\sim}{j} - 2\underset{\sim}{k}\right)\\
&= \frac{2}{7}\left(-\underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k}\right)
\end{aligned}$$

b
$$\begin{aligned}
\underset{\sim}{r} - \left(\underset{\sim}{r} \cdot \hat{\underset{\sim}{s}}\right)\hat{\underset{\sim}{s}} &= \left(2\underset{\sim}{i} + 4\underset{\sim}{k}\right) + \frac{2}{7}\left(\underset{\sim}{i} - 4\underset{\sim}{j} - 2\underset{\sim}{k}\right)\\
&= \frac{1}{7}\left(7\left(2\underset{\sim}{i} + 4\underset{\sim}{k}\right) + 2\left(\underset{\sim}{i} - 4\underset{\sim}{j} - 2\underset{\sim}{k}\right)\right)\\
&= \frac{1}{7}\left(\left(14\underset{\sim}{i} + 28\underset{\sim}{k}\right) + 2\left(2\underset{\sim}{i} - 8\underset{\sim}{j} - 4\underset{\sim}{k}\right)\right)\\
&= \frac{1}{7}\left(16\underset{\sim}{i} - 8\underset{\sim}{j} + 24\underset{\sim}{k}\right)\\
&= \frac{8}{7}\left(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}\right)
\end{aligned}$$

19
$$\begin{aligned}
&A(2, 3, 2) && \overrightarrow{OA} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}\\
&B(4, p, 0) && \overrightarrow{OB} = 4\underset{\sim}{i} + p\underset{\sim}{j}\\
&C(-1, -1, 0) && \overrightarrow{OC} = -\underset{\sim}{i} - \underset{\sim}{j}\\
&D(-2, 2, 1) && \overrightarrow{OD} = -2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}
\end{aligned}$$

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA}\\ &= 2\underset{\sim}{i} + (p-3)\underset{\sim}{j} - 2\underset{\sim}{k}\\ \overrightarrow{DC} &= \overrightarrow{OC} - \overrightarrow{OD}\\ &= \underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k}\end{aligned}$$

a $\overrightarrow{AB}$ // to $\overrightarrow{DC}$

$$\begin{aligned}\overrightarrow{AB} &= 2\overrightarrow{DC}\\ \underset{\sim}{j} : p - 3 &= -6\\ p &= -3\end{aligned}$$

b $\overrightarrow{AB} \perp$ to $\overrightarrow{DC}$

$$\begin{aligned}\overrightarrow{AB} \cdot \overrightarrow{DC} &= 0\\ 2 - 3(p-3) + 2 &= 0\\ 2 - 3p + 9 + 2 &= 0\\ 3p &= 13\\ p &= \frac{13}{3}\end{aligned}$$

c

$$\begin{aligned}\left|\overrightarrow{AB}\right| &= \left|\overrightarrow{DC}\right|\\ \sqrt{2^2 + (p-3)^2 + (-2)^2} &= \sqrt{1^2 + (-3)^2 + (-1)^2}\\ \sqrt{8 + (p-3)^2} &= \sqrt{11}\\ 8 + (p-3)^2 &= 11\\ (p-3)^2 &= 3\\ p - 3 &= \pm\sqrt{3}\\ p &= 3 \pm \sqrt{3}\end{aligned}$$

d Scalar resolute of $\overrightarrow{AB}$ parallel to $\overrightarrow{DC}$

$$\begin{aligned}\frac{\overrightarrow{AB} \cdot \overrightarrow{DC}}{|\overrightarrow{DC}|} &= \frac{4}{\sqrt{11}}\\ \frac{2 - 3(p-3) + 2}{\sqrt{11}} &= \frac{4}{\sqrt{11}}\\ \frac{13 - 3p}{\sqrt{11}} &= \frac{4}{\sqrt{11}}\\ 3p &= 9\\ p &= 3\end{aligned}$$

20 $P(4, p, -3)$ $\quad \overrightarrow{OP} = 4\underset{\sim}{i} + p\underset{\sim}{j} - 3\underset{\sim}{k}$

$Q(-1, -4, -6)$ $\quad \overrightarrow{OQ} = -\underset{\sim}{i} - 4\underset{\sim}{j} - 6\underset{\sim}{k}$

$R(1, 6, -1)$ $\quad \overrightarrow{OR} = \underset{\sim}{i} + 6\underset{\sim}{j} - \underset{\sim}{k}$

$$\begin{aligned}\overrightarrow{QP} &= \overrightarrow{OP} - \overrightarrow{OQ}\\ &= 5\underset{\sim}{i} + (p+4)\underset{\sim}{j} + 3\underset{\sim}{k}\\ \overrightarrow{RP} &= \overrightarrow{OP} - \overrightarrow{OR}\\ &= 3\underset{\sim}{i} + (p-6)\underset{\sim}{j} - 2\underset{\sim}{k}\end{aligned}$$

$\overrightarrow{QP} \cdot \overrightarrow{RP} = 0$ since $\overrightarrow{QP}$ is ⊥ to $\overrightarrow{RP}$

$$\begin{aligned}15 + (p+4)(p-6) - 6 &= 0\\ 15 + p^2 - 2p - 24 - 6 &= 0\\ p^2 - 2p - 15 &= 0\\ (p+3)(p-5) &= 0\\ p &= 5, -3\end{aligned}$$

21 a

$$\begin{aligned}\underset{\sim}{a} &= x\underset{\sim}{i} - 3\underset{\sim}{j} - 4\underset{\sim}{k}\\ \underset{\sim}{b} &= -\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}\\ |\underset{\sim}{a}| &= \sqrt{x^2 + (-3)^2 + (-4)^2}\\ &= \sqrt{x^2 + 25}\\ |\underset{\sim}{b}| &= \sqrt{(-1)^2 + 2^2 + 1^2}\\ &= \sqrt{6}\end{aligned}$$

$$\begin{aligned}\underset{\sim}{a} \cdot \underset{\sim}{b} &= -x - 6 - 4\\ &= -x - 10\\ \cos(\theta) &= \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}|\,|\underset{\sim}{b}|},\ \theta = 150^\circ\\ \cos(150^\circ) &= \frac{-\sqrt{3}}{2}\\ &= \frac{-(x+10)}{\sqrt{6}\ \sqrt{x^2 + 25}} \quad \text{and } x > -10\\ \sqrt{3} \times \sqrt{6}\sqrt{x^2 + 25} &= 2(x+10)\\ 18\left(x^2 + 25\right) &= 4(x+10)^2\\ 9\left(x^2 + 25\right) &= 2\left(x^2 + 20x + 100\right)\\ 9x^2 + 225 &= 2x^2 + 40x + 200\\ 7x^2 - 40x + 25 &= 0\\ (x-5)(7x-5) &= 0\\ x &= 5, \frac{5}{7} \text{ both answers check}\end{aligned}$$

b

$$\begin{aligned}\underset{\sim}{p} &= 6\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}\\ \underset{\sim}{q} &= \underset{\sim}{i} + y\underset{\sim}{j} - 2\underset{\sim}{k}\\ |\underset{\sim}{p}| &= \sqrt{6^2 + 2^2 + 3^2}\\ &= \sqrt{49} = 7\\ |\underset{\sim}{q}| &= \sqrt{1^2 + y^2 + (-2)^2}\\ &= \sqrt{5 + y^2}\\ \underset{\sim}{p} \cdot \underset{\sim}{q} &= 6 + 2y - 6\\ &= 2y\\ \theta &= \cos^{-1}\left(\frac{4}{21}\right)\\ \cos(\theta) &= \frac{4}{21}\\ &= \frac{\underset{\sim}{p} \cdot \underset{\sim}{q}}{|\underset{\sim}{p}|\,|\underset{\sim}{q}|}\\ \frac{4}{21} &= \frac{2y}{7\sqrt{5 + y^2}}\\ 3y &= 2\sqrt{5 + y^2}\\ 9y^2 &= 4\left(5 + y^2\right)\\ &= 20 + 4y^2\\ 5y^2 &= 20\\ y^2 &= 4\\ y &= \pm 2, \quad \text{but } y > 0\\ y &= 2 \quad \text{only}\end{aligned}$$

c

$$\begin{aligned}\underset{\sim}{u} &= 2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\\ \underset{\sim}{v} &= 4\underset{\sim}{j} + z\underset{\sim}{k}\\ |\underset{\sim}{u}| &= \sqrt{2^2 + (-2)^2 + 1^2}\\ &= 3\\ |\underset{\sim}{v}| &= \sqrt{16 + z^2}\\ \underset{\sim}{u} \cdot \underset{\sim}{v} &= -8 + z \qquad \theta = \cos^{-1}\left(\frac{-1}{3}\right) \text{ and } z < 8\\ \cos(\theta) &= \frac{-1}{3}\\ &= \frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{u}|\,|\underset{\sim}{v}|}\\ \frac{-1}{3} &= \frac{z - 8}{3\sqrt{z^2 + 16}}\end{aligned}$$

$$\begin{aligned} z-8 &= -\sqrt{z^2+16} \\ (z-8)^2 &= z^2+16 \\ z^2-16z+64 &= z^2+16 \\ 16z &= 48 \\ z &= 3 \end{aligned}$$

22 a $|\underset{\sim}{u}| = 3$

$|\underset{\sim}{v}| = 4$

$\underset{\sim}{u} \cdot \underset{\sim}{v} = 6$

$$\begin{aligned} |\underset{\sim}{u}+\underset{\sim}{v}|^2 &= (\underset{\sim}{u}+\underset{\sim}{v}) \cdot (\underset{\sim}{u}+\underset{\sim}{v}) \\ &= \underset{\sim}{u} \cdot \underset{\sim}{u} + \underset{\sim}{v} \cdot \underset{\sim}{u} + \underset{\sim}{u} \cdot \underset{\sim}{v} + \underset{\sim}{v} \cdot \underset{\sim}{v} \\ |\underset{\sim}{u}|^2 + 2\underset{\sim}{u} \cdot \underset{\sim}{v} + |\underset{\sim}{v}|^2 &= 3^2 + 2 \times 6 + 4^2 \\ &= 9 + 12 + 16 \\ &= 37 \end{aligned}$$

So $|\underset{\sim}{u}+\underset{\sim}{v}| = \sqrt{37}$

b $|\underset{\sim}{u}| = 3$

$|\underset{\sim}{v}| = 4$

$\underset{\sim}{u} \cdot \underset{\sim}{v} = 6$

$$\begin{aligned} |\underset{\sim}{u}-\underset{\sim}{v}|^2 &= (\underset{\sim}{u}-\underset{\sim}{v}) \cdot (\underset{\sim}{u}-\underset{\sim}{v}) \\ &= \underset{\sim}{u} \cdot \underset{\sim}{u} - \underset{\sim}{v} \cdot \underset{\sim}{u} - \underset{\sim}{u} \cdot \underset{\sim}{v} + \underset{\sim}{v} \cdot \underset{\sim}{v} \\ &= |\underset{\sim}{u}|^2 - 2\underset{\sim}{u} \cdot \underset{\sim}{v} + |\underset{\sim}{v}|^2 \\ &= 3^2 - 2 \times 6 + 4^2 \\ &= 9 - 12 + 16 \\ &= 13 \end{aligned}$$

So $|\underset{\sim}{u}-\underset{\sim}{v}| = \sqrt{13}$

c $|\underset{\sim}{u}| = 3$

$|\underset{\sim}{v}| = 4$

$\underset{\sim}{u} \cdot \underset{\sim}{v} = 6$

$$\begin{aligned} |3\underset{\sim}{u}-2\underset{\sim}{v}|^2 &= (3\underset{\sim}{u}-2\underset{\sim}{v}) \cdot (3\underset{\sim}{u}-2\underset{\sim}{v}) \\ &= 9\underset{\sim}{u} \cdot \underset{\sim}{u} - 6\underset{\sim}{v} \cdot \underset{\sim}{u} - 6\underset{\sim}{u} \cdot \underset{\sim}{v} + 4\underset{\sim}{v} \cdot \underset{\sim}{v} \\ &= 9\,|\underset{\sim}{u}|^2 - 12\underset{\sim}{u} \cdot \underset{\sim}{v} + 4\,|\underset{\sim}{v}|^2 \\ &= 9 \times 3^2 - 12 \times 6 + 4 \times 4^2 \\ &= 81 - 72 + 64 \\ &= 73 \end{aligned}$$

So $|3\underset{\sim}{u}-2\underset{\sim}{v}| = \sqrt{73}$

23 a $|\underset{\sim}{r}| = 4\sqrt{2}$

$|\underset{\sim}{s}| = 5\sqrt{3}$

$\underset{\sim}{r} \cdot \underset{\sim}{s} = -6$

$$\begin{aligned} |\underset{\sim}{r}+\underset{\sim}{s}|^2 &= (\underset{\sim}{r}+\underset{\sim}{s}) \cdot (\underset{\sim}{r}+\underset{\sim}{s}) \\ &= \underset{\sim}{r} \cdot \underset{\sim}{r} + \underset{\sim}{s} \cdot \underset{\sim}{r} + \underset{\sim}{r} \cdot \underset{\sim}{s} + \underset{\sim}{s} \cdot \underset{\sim}{s} \\ &= |\underset{\sim}{r}|^2 + 2\underset{\sim}{s} \cdot \underset{\sim}{r} + |\underset{\sim}{s}|^2 \\ &= \left(4\sqrt{2}\right)^2 + 2 \times -6 + \left(5\sqrt{3}\right)^2 \\ &= 32 - 12 + 75 \\ &= 95 \end{aligned}$$

So $|\underset{\sim}{r}+\underset{\sim}{s}| = \sqrt{95}$

b $|\underset{\sim}{r}| = 4\sqrt{2}$

$|\underset{\sim}{s}| = 5\sqrt{3}$

$\underset{\sim}{r} \cdot \underset{\sim}{s} = -6$

$$\begin{aligned} |\underset{\sim}{r}-\underset{\sim}{s}|^2 &= (\underset{\sim}{r}-\underset{\sim}{s}) \cdot (\underset{\sim}{r}-\underset{\sim}{s}) \\ &= \underset{\sim}{r} \cdot \underset{\sim}{r} - \underset{\sim}{s} \cdot \underset{\sim}{r} - \underset{\sim}{r} \cdot \underset{\sim}{s} + \underset{\sim}{s} \cdot \underset{\sim}{s} \\ &= |\underset{\sim}{r}|^2 - 2\underset{\sim}{s} \cdot \underset{\sim}{r} + |\underset{\sim}{s}|^2 \\ &= \left(4\sqrt{2}\right)^2 - 2 \times -6 + \left(5\sqrt{3}\right)^2 \\ &= 119 \end{aligned}$$

So $|\underset{\sim}{r}-\underset{\sim}{s}| = \sqrt{119}$

c $|\underset{\sim}{r}| = 4\sqrt{2}$

$|\underset{\sim}{s}| = 5\sqrt{3}$

$\underset{\sim}{r} \cdot \underset{\sim}{s} = -6$

$$\begin{aligned} |4\underset{\sim}{r}+3\underset{\sim}{s}|^2 &= (4\underset{\sim}{r}+3\underset{\sim}{s}) \cdot (4\underset{\sim}{r}+3\underset{\sim}{s}) \\ &= 16\underset{\sim}{r} \cdot \underset{\sim}{r} + 12\underset{\sim}{s} \cdot \underset{\sim}{r} + 12\underset{\sim}{r} \cdot \underset{\sim}{s} + 9\underset{\sim}{s} \cdot \underset{\sim}{s} \\ &= 16\,|\underset{\sim}{r}|^2 + 24\underset{\sim}{r} \cdot \underset{\sim}{s} + 9\,|\underset{\sim}{s}|^2 \\ &= 16 \times \left(4\sqrt{2}\right)^2 + 24 \times -6 + 9 \times \left(5\sqrt{3}\right)^2 \\ &= 1043 \end{aligned}$$

So $|4\underset{\sim}{r}-3\underset{\sim}{s}| = \sqrt{1043}$

24 a $\underset{\sim}{a} = 2\underset{\sim}{i} - 4\underset{\sim}{j} + \underset{\sim}{k}$

$\underset{\sim}{b} = 3\underset{\sim}{i} - \underset{\sim}{j} - 4\underset{\sim}{k}$

$$\begin{aligned} |\underset{\sim}{b}| &= \sqrt{3^2 + (-1)^2 + (-4)^2} \\ &= \sqrt{26} \\ \hat{\underset{\sim}{b}} &= \frac{1}{\sqrt{26}}\left(3\underset{\sim}{i} - \underset{\sim}{j} - 4\underset{\sim}{k}\right) \end{aligned}$$

b $$\begin{aligned} \underset{\sim}{a} \cdot \underset{\sim}{b} &= 6 + 4 - 4 \\ &= 6 \\ \underset{\sim}{a} \cdot \hat{\underset{\sim}{b}} &= \frac{6}{\sqrt{26}} \end{aligned}$$

c $$\begin{aligned} \left(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}} &= \frac{6}{\sqrt{26}} \times \frac{1}{\sqrt{26}}\left(3\underset{\sim}{i} - \underset{\sim}{j} - 4\underset{\sim}{k}\right) \\ &= \frac{3}{13}\left(3\underset{\sim}{i} - \underset{\sim}{j} - 4\underset{\sim}{k}\right) \end{aligned}$$

d $$\begin{aligned} \underset{\sim}{a} - \left(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}} &= \left(2\underset{\sim}{i} - 4\underset{\sim}{j} + \underset{\sim}{k}\right) - \frac{3}{13}\left(3\underset{\sim}{i} - \underset{\sim}{j} - 4\underset{\sim}{k}\right) \\ &= \frac{1}{13}\left(13\left(2\underset{\sim}{i} - 4\underset{\sim}{j} + \underset{\sim}{k}\right) - 3\left(3\underset{\sim}{i} - \underset{\sim}{j} - 4\underset{\sim}{k}\right)\right) \\ &= \frac{1}{13}\left(17\underset{\sim}{i} - 49\underset{\sim}{j} + 25\underset{\sim}{k}\right) \end{aligned}$$

e $$\begin{aligned} |\underset{\sim}{a}| &= \sqrt{2^2 + (-4)^2 + 1^2} \\ &= \sqrt{21} \\ \cos(\theta) &= \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}|\,|\underset{\sim}{b}|} \\ \theta &= \cos^{-1}\left(\frac{6}{\sqrt{21}\sqrt{26}}\right) \\ \theta &= 75.12^\circ \end{aligned}$$

25 a $\underset{\sim}{p} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - 5\underset{\sim}{k}$

$\underset{\sim}{q} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$

$$\begin{aligned} |\underset{\sim}{q}| &= \sqrt{2^2 + 1^2 + (-2)^2} \\ &= 3 \\ \hat{\underset{\sim}{q}} &= \frac{1}{3}\left(2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\right) \end{aligned}$$

b $\underset{\sim}{p}\cdot\underset{\sim}{q} = 6 + 2 + 10$

$= 18$

$\underset{\sim}{p}\cdot\hat{\underset{\sim}{q}} = \dfrac{18}{3}$

$= 6$

c $\left(\underset{\sim}{p}\cdot\hat{\underset{\sim}{q}}\right)\hat{\underset{\sim}{q}} = 2\left(2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\right)$

d $\underset{\sim}{p} - \left(\underset{\sim}{p}\cdot\hat{\underset{\sim}{q}}\right)\hat{\underset{\sim}{q}} = \left(3\underset{\sim}{i} + 2\underset{\sim}{j} - 5\underset{\sim}{k}\right) - 2\left(2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\right)$

$= -\underset{\sim}{i} - \underset{\sim}{k}$

e $|\underset{\sim}{p}| = \sqrt{3^2 + 2^2 + (-5)^2}$

$= \sqrt{38}$

$\cos(\theta) = \dfrac{\underset{\sim}{p}\cdot\underset{\sim}{q}}{|\underset{\sim}{p}|\,|\underset{\sim}{q}|}$

$\theta = \cos^{-1}\left(\dfrac{18}{3\sqrt{38}}\right)$

$\theta = 13.26°$

26 $\underset{\sim}{r} = 3\underset{\sim}{i} - 4\underset{\sim}{j} + \underset{\sim}{k}$

$\underset{\sim}{s} = \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$

$|\underset{\sim}{s}| = \sqrt{1^2 + (-2)^2 + 3^2}$

$= \sqrt{14}$

$\hat{\underset{\sim}{s}} = \dfrac{1}{\sqrt{14}}\left(\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}\right)$

$\underset{\sim}{r}\cdot\underset{\sim}{s} = 3 + 8 + 3$

$= 14$

$\underset{\sim}{r}\cdot\hat{\underset{\sim}{s}} = \dfrac{14}{\sqrt{14}}$

$= \sqrt{14}$

a $\left(\underset{\sim}{r}\cdot\hat{\underset{\sim}{s}}\right)\hat{\underset{\sim}{s}} = \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$

$\underset{\sim}{r} - \left(\underset{\sim}{r}\cdot\hat{\underset{\sim}{s}}\right)\hat{\underset{\sim}{s}} = 3\underset{\sim}{i} - 4\underset{\sim}{j} + \underset{\sim}{k} - \left(\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}\right)$

$= 2\underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k}$

b $|\underset{\sim}{r}| = \sqrt{3^2 + (-4)^2 + 1^2}$

$= \sqrt{26}$

$\cos(\theta) = \dfrac{\underset{\sim}{r}\cdot\underset{\sim}{s}}{|\underset{\sim}{r}|\,|\underset{\sim}{s}|}$

$= \dfrac{14}{\sqrt{26}\sqrt{14}}$

$\theta = \cos^{-1}\left(\dfrac{14}{\sqrt{26}\sqrt{14}}\right)$

$= 42.79°$

27 **a** $A(3, -2, 5)$ $\quad \overrightarrow{OA} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + 5\underset{\sim}{k}$

$B(-1, 0, 4)$ $\quad \overrightarrow{OB} = -\underset{\sim}{i} + 4\underset{\sim}{k}$

$C(2, -1, 3)$ $\quad \overrightarrow{OC} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= -4\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= 3\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$

$\left|\overrightarrow{BC}\right| = \sqrt{3^2 + (-1)^2 + (-1)^2}$

$= \sqrt{11}$

$\hat{\overrightarrow{BC}} = \dfrac{1}{\sqrt{11}}\left(3\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}\right)$

b $\overrightarrow{AB}\cdot\overrightarrow{BC} = -12 - 2 + 1$

$= -13$

$\left(\overrightarrow{AB}\cdot\hat{\overrightarrow{BC}}\right)\hat{\overrightarrow{BC}} = \dfrac{-13}{11}\left(3\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}\right)$

c $\overrightarrow{AB} - \left(\overrightarrow{AB}\cdot\hat{\overrightarrow{BC}}\right)\hat{\overrightarrow{BC}} = -4\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k} + \dfrac{13}{11}\left(3\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}\right)$

$= \dfrac{1}{11}\left(11\left(-4\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right)\right.$

$\left.+13\left(3\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}\right)\right)$

$= \dfrac{1}{11}\left(-5\underset{\sim}{i} + 9\underset{\sim}{j} - 24\underset{\sim}{k}\right)$

d $\left|\overrightarrow{AB}\right| = \sqrt{(-4)^2 + (2)^2 + (-1)^2}$

$= \sqrt{21}$

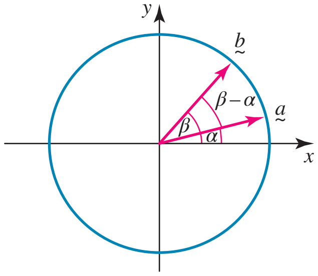
$\cos(\theta) = \dfrac{\overrightarrow{AB}\cdot\overrightarrow{BC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{BC}\right|}$

$= \dfrac{-13}{\sqrt{21}\times\sqrt{11}}$

$\theta = \cos^{-1}\left(\dfrac{-13}{\sqrt{21}\sqrt{11}}\right)$

$= 148.80°$

28



$\underset{\sim}{a} = \cos(\alpha)\,\underset{\sim}{i} + \sin(\alpha)\,\underset{\sim}{j}$

$\underset{\sim}{b} = \cos(\beta)\,\underset{\sim}{i} + \sin(\beta)\,\underset{\sim}{j}$

$\underset{\sim}{a}\cdot\underset{\sim}{b} = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

$|\underset{\sim}{a}| = \sqrt{\cos^2(\alpha) + \sin^2(\alpha)}$

$= 1$

$|\underset{\sim}{b}| = \sqrt{\cos^2(\beta) + \sin^2(\beta)}$

$= 1$

Angle between $\underset{\sim}{b}$ and $\underset{\sim}{a}$ is $\beta - \alpha$

$\cos(\theta) = \dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}|\,|\underset{\sim}{b}|}$

$\Rightarrow \cos(\beta - \alpha) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

29 $\underset{\sim}{F} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}$

$A(1, -2, 2)\, B(2, 1, -4)$

$\underset{\sim}{S} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \underset{\sim}{i} + 3\underset{\sim}{j} + 6\underset{\sim}{k}$

$W = \underset{\sim}{F}\cdot\underset{\sim}{S}$

$= \left(3\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}\right)\cdot\left(\underset{\sim}{i} + 3\underset{\sim}{j} + 6\underset{\sim}{k}\right)$

$= 3 + 6 + 24$

$= 33$

30 **a** $|\underset{\sim}{a}| = 3$

$|\underset{\sim}{b}| = 4$

$|\underset{\sim}{c}| = 5$

$\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}=\underset{\sim}{0}$

So $\underset{\sim}{a}+\underset{\sim}{b}=-\underset{\sim}{c}$

$\left(\underset{\sim}{a}+\underset{\sim}{b}\right)^2=\left(-\underset{\sim}{c}\right)^2$

Using dot products

$|\underset{\sim}{a}|^2+2\underset{\sim}{a}\cdot\underset{\sim}{b}+|\underset{\sim}{b}|^2=|\underset{\sim}{c}|^2$

$9+2\underset{\sim}{a}\cdot\underset{\sim}{b}+16=25$

$2\underset{\sim}{a}\cdot\underset{\sim}{b}=0$

$\underset{\sim}{a}\cdot\underset{\sim}{b}=0$

b $|\underset{\sim}{p}|=5$

$|\underset{\sim}{q}|=12$

$|\underset{\sim}{r}|=13$

$5^2+12^2=13^2$

$\underset{\sim}{p}\cdot\underset{\sim}{q}=\underset{\sim}{0}$

Right-angled triangle

Then, $\underset{\sim}{p}+\underset{\sim}{q}+\underset{\sim}{r}=\underset{\sim}{0}$

c If $\underset{\sim}{a}\cdot\underset{\sim}{b}=\underset{\sim}{c}\cdot\underset{\sim}{b}$

$\underset{\sim}{a}\cdot\underset{\sim}{b}-\underset{\sim}{c}\cdot\underset{\sim}{b}=0$

$\underset{\sim}{b}\cdot\underset{\sim}{a}-\underset{\sim}{b}\cdot\underset{\sim}{c}=0$

$\underset{\sim}{b}\cdot\left(\underset{\sim}{a}-\underset{\sim}{c}\right)=0$

It is possible that $\underset{\sim}{a}=\underset{\sim}{c}$

It is possible that $\underset{\sim}{b}$ is perpendicular to $\underset{\sim}{a}-\underset{\sim}{c}$

31 a $|\underset{\sim}{u}+\underset{\sim}{v}|^2=\left(\underset{\sim}{u}+\underset{\sim}{v}\right)\cdot\left(\underset{\sim}{u}+\underset{\sim}{v}\right)$

$=\underset{\sim}{u}\cdot\underset{\sim}{u}+2\underset{\sim}{u}\cdot\underset{\sim}{v}+\underset{\sim}{v}\cdot\underset{\sim}{v}$

$=|\underset{\sim}{u}|^2+2\underset{\sim}{u}\cdot\underset{\sim}{v}+|\underset{\sim}{v}|^2 \qquad (1)$

$|\underset{\sim}{u}-\underset{\sim}{v}|^2=\left(\underset{\sim}{u}-\underset{\sim}{v}\right)\cdot\left(\underset{\sim}{u}-\underset{\sim}{v}\right)$

$=\underset{\sim}{u}\cdot\underset{\sim}{u}-2\underset{\sim}{u}\cdot\underset{\sim}{v}+\underset{\sim}{v}\cdot\underset{\sim}{v}$

$=|\underset{\sim}{u}|^2-2\underset{\sim}{u}\cdot\underset{\sim}{v}+|\underset{\sim}{v}|^2 \qquad (2)$

subtract $(1)-(2)\Rightarrow|\underset{\sim}{u}+\underset{\sim}{v}|^2-|\underset{\sim}{u}-\underset{\sim}{v}|^2=4\underset{\sim}{u}\cdot\underset{\sim}{v}$

so $\underset{\sim}{u}\cdot\underset{\sim}{v}=\frac{1}{4}\left(|\underset{\sim}{u}+\underset{\sim}{v}|^2-|\underset{\sim}{u}-\underset{\sim}{v}|^2\right) \qquad (3)$

add $(1)+(2)\Rightarrow|\underset{\sim}{u}+\underset{\sim}{v}|^2+|\underset{\sim}{u}-\underset{\sim}{v}|^2$

$=2\left(|\underset{\sim}{u}|^2+|\underset{\sim}{v}|^2\right)$ (4) shown

b $|\underset{\sim}{u}+\underset{\sim}{v}|^2=17, \qquad |\underset{\sim}{u}-\underset{\sim}{v}|^2=13$

So (3) $\frac{1}{4}(17-13)=1=\underset{\sim}{u}\cdot\underset{\sim}{v}$

(4) $\frac{1}{2}(17+13)=|\underset{\sim}{u}|^2+|\underset{\sim}{v}|^2=15$

But $\underset{\sim}{u}\cdot\underset{\sim}{v}=|\underset{\sim}{u}|\,|\underset{\sim}{v}|\cos(\theta)$ and $\cos(\theta)=\frac{1}{\sqrt{50}}$

$\underset{\sim}{u}\cdot\underset{\sim}{v}=1=|\underset{\sim}{u}|\,|\underset{\sim}{v}|\times\frac{1}{\sqrt{50}}$

$|\underset{\sim}{u}||\underset{\sim}{v}|=\sqrt{50}$

$|\underset{\sim}{v}|=\frac{\sqrt{50}}{|\underset{\sim}{u}|}$

$|\underset{\sim}{u}|^2+|\underset{\sim}{v}|^2=15$

$|\underset{\sim}{u}|^2+\frac{\left(\sqrt{50}\right)^2}{|\underset{\sim}{u}|^2}=15$

$|\underset{\sim}{u}|^4-15|\underset{\sim}{u}|^2+50=0$

$\left(|\underset{\sim}{u}|^2-5\right)\left(|\underset{\sim}{u}|^2-10\right)=0$

$|\underset{\sim}{u}|=\sqrt{5}$ or $\sqrt{10}$

$|\underset{\sim}{v}|=\sqrt{10}$ or $\sqrt{5}$

32 a $A(4,-1,-3) \qquad B(3,-4,1)$

$\overrightarrow{OA}=4\underset{\sim}{i}-\underset{\sim}{j}-3\underset{\sim}{k}$

$\overrightarrow{OB}=3\underset{\sim}{i}-4\underset{\sim}{j}+\underset{\sim}{k}$

$\left|\overrightarrow{OA}\right|=\sqrt{4^2+(-1)^2+(-3)^2}$

$=\sqrt{26}$

$\left|\overrightarrow{OB}\right|=\sqrt{26}$

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$

$=-\underset{\sim}{i}-3\underset{\sim}{j}+4\underset{\sim}{k}$

$\left|\overrightarrow{AB}\right|=\sqrt{26}$

b $\overrightarrow{OM}=\overrightarrow{OA}+\overrightarrow{AM}$

$=\overrightarrow{OA}+\frac{1}{2}\overrightarrow{AB}$

$=4\underset{\sim}{i}-\underset{\sim}{j}-3\underset{\sim}{k}+\frac{1}{2}\left(-\underset{\sim}{i}-3\underset{\sim}{j}+4\underset{\sim}{k}\right)$

$=\frac{1}{2}\left(7\underset{\sim}{i}-5\underset{\sim}{j}-2\underset{\sim}{k}\right)$

$\overrightarrow{OG}=\frac{2}{3}\overrightarrow{OM}$

$=\frac{2}{3}\left(\frac{1}{2}\left(7\underset{\sim}{i}-5\underset{\sim}{j}-2\underset{\sim}{k}\right)\right)$

$=\frac{1}{3}\left(7\underset{\sim}{i}-5\underset{\sim}{j}-2\underset{\sim}{k}\right)$

c $P(x,y,z)$

$\overrightarrow{OP}=x\underset{\sim}{i}+y\underset{\sim}{j}+z\underset{\sim}{k}$

$\overrightarrow{GP}=\overrightarrow{OP}-\overrightarrow{OG}$

$=\left(x-\frac{7}{3}\right)\underset{\sim}{i}+\left(y+\frac{5}{3}\right)\underset{\sim}{j}+\left(z+\frac{2}{3}\right)\underset{\sim}{k}$

$\overrightarrow{GP}\cdot\overrightarrow{OG}=\frac{7}{3}\left(x-\frac{7}{3}\right)-\frac{5}{3}\left(y+\frac{5}{3}\right)-\frac{2}{3}\left(z+\frac{2}{3}\right)=0$

$7\left(x-\frac{7}{3}\right)-5\left(y+\frac{5}{3}\right)-2\left(z+\frac{2}{3}\right)=0$

$7x-5y-2z=\frac{49}{3}+\frac{25}{3}+\frac{4}{3}$

$7x-5y-2z=26 \qquad (1)$

d $\left|\overrightarrow{OP}\right|=\sqrt{x^2+y^2+z^2}=5\sqrt{14}$

$x^2+y^2+z^2=350 \qquad (2)$

$\overrightarrow{AP}=\overrightarrow{OP}-\overrightarrow{OA}$

$=(x-4)\underset{\sim}{i}+(y+1)\underset{\sim}{j}+(z+3)\underset{\sim}{k}$

$\left|\overrightarrow{AP}\right|=5\sqrt{14}$

$\Rightarrow\sqrt{(x-4)^2+(y+1)^2+(z+3)^2}$

$=5\sqrt{14}$

$350=x^2-8x+16+y^2+2y+1+z^2+6z+9$

$-26=-8x+2y+6z \qquad (3)$

$\overrightarrow{BP}=\overrightarrow{OP}-\overrightarrow{OB}$

$=(x-3)\underset{\sim}{i}+(y+4)\underset{\sim}{j}+(z-1)\underset{\sim}{k}$

$\left|\overrightarrow{BP}\right|=5\sqrt{14}$

$\Rightarrow\sqrt{(x-3)^2+(y+4)^2+(z-1)^2}$

$=5\sqrt{14}$

$350=x^2-6x+9+y^2+8y+16+z^2-2z+1$

$-26=-6x+8y-2z \qquad (4)$

e Solving (1) (2) (3) (4)

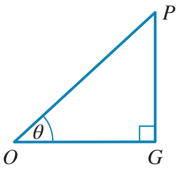
$\Rightarrow x=13, y=9, z=10 \qquad P(13,9,10)$

or

$x = \frac{-25}{3}, y = \frac{-37}{3}, z = \frac{-34}{3}$

$P\left(\frac{-25}{3}, \frac{-37}{3}, \frac{-34}{3}\right)$

f



$|\overrightarrow{OG}| = \sqrt{\left(\frac{7}{3}\right)^2 + \left(\frac{-5}{3}\right)^2 + \left(\frac{-2}{3}\right)^2} = \frac{\sqrt{78}}{3}$

$|\overrightarrow{OP}| = 5\sqrt{14}$

$h = |\overrightarrow{GP}|$

$= \sqrt{\left(5\sqrt{14}\right)^2 - \left(\frac{\sqrt{78}}{3}\right)^2}$

$= \frac{32\sqrt{3}}{3}$

g $\cos(\theta) = \frac{|\overrightarrow{OG}|}{|\overrightarrow{OP}|}$

$= \frac{\frac{\sqrt{78}}{3}}{5\sqrt{14}}$

$\theta = \cos^{-1}\left(\frac{\sqrt{273}}{105}\right)$

$= 80.95°$

33 a $A(-2, -1, 2) \qquad C(1, 2, 2)$

$\overrightarrow{OA} = -2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$

$\overrightarrow{OC} = \underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k} \qquad D(x, y, z)$

$\overrightarrow{OA} \cdot \overrightarrow{OC} = -2 - 2 + 4 = 0$

so $\overrightarrow{OA}$ is $\perp$ to $\overrightarrow{OC}$

$|\overrightarrow{OA}| = \sqrt{4 + 1 + 4}$

$= 3$

$|\overrightarrow{OC}| = \sqrt{1 + 4 + 4}$

$= 3$

Since $|\overrightarrow{OA}| = |\overrightarrow{OC}|$ and $\overrightarrow{OA}$ is $\perp$ to $\overrightarrow{OC}$

then $OABC$ is square

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OA} + \overrightarrow{OC} = -\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}$

$B(-1, 1, 4)$

b $\overrightarrow{OE} = \frac{1}{2}\overrightarrow{OB}$

$= \frac{1}{2}\left(-\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}\right)$

$E\left(\frac{-1}{2}, \frac{1}{2}, 2\right)$

c $\overrightarrow{ED} = \overrightarrow{OD} - \overrightarrow{OE}$

$= \left(x + \frac{1}{2}\right)\underset{\sim}{i} + \left(y - \frac{1}{2}\right)\underset{\sim}{j} + (z - 2)\underset{\sim}{k}$

$\overrightarrow{ED} \cdot \overrightarrow{OE} = 0$

$\Rightarrow \frac{-1}{2}\left(x + \frac{1}{2}\right) + \frac{1}{2}\left(y - \frac{1}{2}\right) + 2(z - 2) = 0$

$-\left(x + \frac{1}{2}\right) + \left(y - \frac{1}{2}\right) + 4(z - 2) = 0$

$-x + y + 4z = \frac{1}{2} + \frac{1}{2} + 8$

$-x + y + 4z = 9 \qquad (1)$

d $\overrightarrow{OD} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

$|\overrightarrow{OD}| = \sqrt{x^2 + y^2 + z^2}$

$= \sqrt{\frac{891}{2}}$

So $x^2 + y^2 + z^2 = \frac{891}{2}$

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$

$= (x + 2)\underset{\sim}{i} + (y + 1)\underset{\sim}{j} + (z - 2)\underset{\sim}{k}$

$|\overrightarrow{AD}| = \sqrt{(x + 2)^2 + (y + 1)^2 + (z - 2)^2}$

$= \sqrt{x^2 + 4x + 4 + y^2 + 2y + 1 + z^2 - 4z + 4}$

$= \sqrt{x^2 + y^2 + z^2 + 4x + 2y - 4z + 9}$

$|\overrightarrow{AD}|^2 = |\overrightarrow{OD}|^2$

$x^2 + y^2 + z^2 + 4x + 2y - 4z + 9 = x^2 + y^2 + z^2$

$4x + 2y - 4z = -9 \qquad (2)$

$\overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB}$

$= (x + 1)\underset{\sim}{i} + (y - 1)\underset{\sim}{j} + (z - 4)\underset{\sim}{k}$

$|\overrightarrow{BD}| = \sqrt{(x + 1)^2 + (y - 1)^2 + (z - 4)^2}$

$|\overrightarrow{BD}|^2 = |\overrightarrow{OD}|^2$

$x^2 + 2x + 1 + y^2 - 2y + 1 + z^2 - 8z + 16 = x^2 + y^2 + z^2$

$2x - 2y - 8z = -18 \qquad (3)$

$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$

$= (x - 1)\underset{\sim}{i} + (y - 2)\underset{\sim}{j} + (z - 2)\underset{\sim}{k}$

$|\overrightarrow{CD}| = \sqrt{(x - 1)^2 + (y - 2)^2 + (z - 2)^2}$

$|\overrightarrow{CD}|^2 = |\overrightarrow{OD}|^2$

$x^2 - 2x + 1 + y^2 - 4y + 4 + z^2 - 4z + 4 = x^2 + y^2 + z^2$

$-2x - 4y - 4z = -9$

$2x + 4y + 4z = 9 \qquad (4)$

e Solving with $x^2 + y^2 + z^2 = \frac{891}{2}$

$\Rightarrow x = \frac{-29}{2}, y = \frac{29}{2}, z = -5 \quad D\left(\frac{-29}{2}, \frac{29}{2}, -5\right)$

or $x = \frac{27}{2}, y = \frac{-27}{2}, z = 9 \quad D\left(\frac{27}{2}, \frac{-27}{2}, 9\right)$

$\overrightarrow{ED} = \left(x + \frac{1}{2}\right)\underset{\sim}{i} + \left(x - \frac{1}{2}\right)\underset{\sim}{j} + (z - 2)\underset{\sim}{k}$

$= -14\underset{\sim}{i} + 14\underset{\sim}{j} - 7\underset{\sim}{k}$

$|\overrightarrow{ED}| = \sqrt{(-14)^2 + (14)^2 + (-7)^2}$

$= 21$

∴ The height of the pyramid is 21 units.

f $|\overrightarrow{OD}| = \frac{9\sqrt{22}}{2}$

$|\overrightarrow{OE}| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 2^2} = \frac{3\sqrt{2}}{2}$

$$\begin{aligned}\cos(\theta) &= \frac{|\overrightarrow{OE}|}{|\overrightarrow{OD}|}\\ &= \frac{\frac{3\sqrt{2}}{2}}{\frac{9\sqrt{22}}{2}}\\ &= \frac{3\sqrt{2}}{9\sqrt{22}}\\ &= \frac{1}{3\sqrt{11}}\\ \theta &= \cos^{-1}\left(\frac{1}{3\sqrt{11}}\right)\\ &= 84.23°\end{aligned}$$

3.4 Exam questions

1 Scalar resolute $\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}} = -4$

$\underset{\sim}{b} = -\sqrt{3}\underset{\sim}{i} \quad |\underset{\sim}{b}| = \sqrt{3}, \ \hat{\underset{\sim}{b}} = \underset{\sim}{i}$

Vector resolute $\left(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}} = 4\underset{\sim}{i}$

2 $\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{k}, \ \underset{\sim}{b} = -\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = -3 - 6 = -9, \ |\underset{\sim}{b}| = \sqrt{1+4+9} = \sqrt{14}$

$\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}} = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|} \quad = \dfrac{-9}{\sqrt{14}} \quad = -\dfrac{9\sqrt{14}}{14}$

3 a $\underset{\sim}{a} = \sqrt{3}\underset{\sim}{i} - \underset{\sim}{j} - \sqrt{2}\underset{\sim}{k}$

$$\begin{aligned}|\underset{\sim}{a}| &= \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2 + \left(-\sqrt{2}\right)^2}\\ &= \sqrt{3+1+2} = \sqrt{6}\\ \hat{\underset{\sim}{a}} &= \frac{1}{\sqrt{6}}\left(\sqrt{3}\underset{\sim}{i} - \underset{\sim}{j} - \sqrt{2}\underset{\sim}{k}\right)\end{aligned}$$

Award 1 mark for the correct unit vector.

b
$$\begin{aligned}\cos(\alpha) &= \frac{\underset{\sim}{a} \cdot \underset{\sim}{i}}{|\underset{\sim}{a}|}\\ &= \frac{\sqrt{3}}{\sqrt{6}}\\ &= \frac{1}{\sqrt{2}}\\ \alpha &= \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)\\ \alpha &= \frac{\pi}{4}\\ &= 45°\end{aligned}$$

Award 1 mark for method (using direction cosines).

Award 1 mark for the correct answer.

c $\underset{\sim}{a} = \sqrt{3}\underset{\sim}{i} - \underset{\sim}{j} - \sqrt{2}\underset{\sim}{k}$ is perpendicular to $\underset{\sim}{b} = 2\sqrt{3}\underset{\sim}{i} + m\underset{\sim}{j} - 5\underset{\sim}{k}$.

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 2 \times 3 - m + 5\sqrt{2} = 0$

$m = 6 + 5\sqrt{2}$

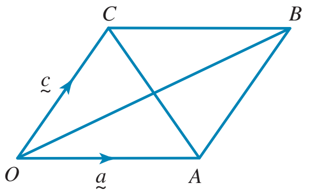
Award 1 mark for method (dot product equated to zero).

Award 1 mark for the correct answer.

3.5 Vector proofs using the scalar product

3.5 Exercise

1



Let *OABC* be a rhombus

Let $\underset{\sim}{a} = \overrightarrow{OA} = \overrightarrow{CB}$

$\quad \underset{\sim}{c} = \overrightarrow{OC} = \overrightarrow{AB}$

Since it is a rhombus $|\underset{\sim}{a}| = |\underset{\sim}{c}|$

$$\begin{aligned}\overrightarrow{OB} &= \overrightarrow{OA} + \overrightarrow{AB} & \overrightarrow{AC} &= \overrightarrow{AO} + \overrightarrow{OC}\\ &= \overrightarrow{OA} + \overrightarrow{OC} & &= \overrightarrow{OC} - \overrightarrow{OA}\\ &= \underset{\sim}{a} + \underset{\sim}{c} & &= \underset{\sim}{c} - \underset{\sim}{a}\end{aligned}$$

Consider $\overrightarrow{OB} \cdot \overrightarrow{AC} = 0$

$= \underset{\sim}{a} \cdot \underset{\sim}{c} + \underset{\sim}{c} \cdot \underset{\sim}{c} - \underset{\sim}{c} \cdot \underset{\sim}{a} - \underset{\sim}{a} \cdot \underset{\sim}{a}$

$= |\underset{\sim}{c}|^2 - |\underset{\sim}{a}|^2$

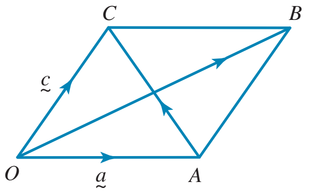
$= 0 \quad$ since $\ |\underset{\sim}{a}| = |\underset{\sim}{c}|$

Since $\overrightarrow{OB} \cdot \overrightarrow{AO} = 0$

$\Rightarrow \overrightarrow{OB}$ is $\perp$ to $\overrightarrow{AC}$

So the diagonals are perpendicular

2



Let *OABC* be a parallelogram

Let $\underset{\sim}{a} = \overrightarrow{OA} = \overrightarrow{CB}$

$\quad \underset{\sim}{c} = \overrightarrow{OC} = \overrightarrow{AB}$

Diagonals:

$$\begin{aligned}\overrightarrow{OB} &= \overrightarrow{OA} + \overrightarrow{AB} & \overrightarrow{AC} &= \overrightarrow{OC} - \overrightarrow{OA}\\ &= \overrightarrow{OA} + \overrightarrow{OC} & &= \underset{\sim}{c} - \underset{\sim}{a}\\ &= \underset{\sim}{a} + \underset{\sim}{c} & &\end{aligned}$$

If the diagonals are in equal in length

$|\overrightarrow{OB}| = |\overrightarrow{AC}|$

$|\overrightarrow{OB}|^2 = |\overrightarrow{AC}|^2$

$$\left(\underset{\sim}{a} + \underset{\sim}{c}\right) \cdot \left(\underset{\sim}{a} + \underset{\sim}{c}\right) = \left(\underset{\sim}{c} - \underset{\sim}{a}\right) \cdot \left(\underset{\sim}{c} - \underset{\sim}{a}\right)$$

$$\underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{c} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{c} + \underset{\sim}{c} \cdot \underset{\sim}{c} = \underset{\sim}{c} \cdot \underset{\sim}{c} - \underset{\sim}{a} \cdot \underset{\sim}{c} - \underset{\sim}{c} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{a}$$

$$|\underset{\sim}{a}|^2 + 2\underset{\sim}{a} \cdot \underset{\sim}{c} + |\underset{\sim}{c}|^2 = |\underset{\sim}{c}|^2 - 2\underset{\sim}{a} \cdot \underset{\sim}{c} + |\underset{\sim}{a}|^2$$

$\Rightarrow 4\underset{\sim}{a} \cdot \underset{\sim}{c} = 0$

$\Rightarrow \underset{\sim}{a} \cdot \underset{\sim}{c} = 0$

So $\overrightarrow{OA}$ is $\perp$ to $\overrightarrow{OC}$, $\quad$ so *OABC* is a rectangle

3 $A(8,3,-1) \quad B(4,5,-2) \quad C(7,9,-6)$

$\overrightarrow{OA} = 8\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$

$\overrightarrow{OB} = 4\underset{\sim}{i} + 5\underset{\sim}{j} - 2\underset{\sim}{k}$

$\overrightarrow{OC} = 7\underset{\sim}{i} + 9\underset{\sim}{j} - 6\underset{\sim}{k}$

$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$

$= 4\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= 3\underset{\sim}{i} + 4\underset{\sim}{j} - 4\underset{\sim}{k}$

$\overrightarrow{BA} \cdot \overrightarrow{BC} = 12 - 8 - 4$

$= 0$ so $\overrightarrow{BA}$ is $\perp$ to $\overrightarrow{BC}$

$|\overrightarrow{BA}| = \sqrt{4^2 + (-2)^2 + 1^2}$

$= \sqrt{21}$

$|\overrightarrow{BC}| = \sqrt{3^2 + 4^2 + (-4)^2}$

$= \sqrt{41}$

So area of the triangle $= \frac{1}{2}\sqrt{21} \times \sqrt{41}$

$= \frac{1}{2}\sqrt{861}$

4 $A(-3,5,4) \quad B(2,3,5) \quad C(4,6,1)$

$\overrightarrow{OA} = -3\underset{\sim}{i} + 5\underset{\sim}{j} + 4\underset{\sim}{k}$

$\overrightarrow{OB} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + 5\underset{\sim}{k}$

$\overrightarrow{OC} = 4\underset{\sim}{i} + 6\underset{\sim}{j} + \underset{\sim}{k}$

$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$

$= -5\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= 2\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k}$

$\overrightarrow{BA} \cdot \overrightarrow{BC} = -10 + 6 + 4$

$= 0$ so $\overrightarrow{BA}$ is $\perp$ to $\overrightarrow{BC}$

$|\overrightarrow{BA}| = \sqrt{(-5)^2 + 2^2 + (-1)^2}$

$= \sqrt{30}$

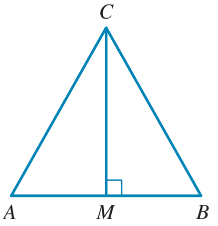
$|\overrightarrow{BC}| = \sqrt{2^2 + 3^2 + (-4)^2}$

$= \sqrt{29}$

So area of the triangle $= \frac{1}{2}\sqrt{30} \times \sqrt{29}$

$= \frac{1}{2}\sqrt{870}$

5



$A(4,7,3) \quad B(8,7,1) \quad C(6,5,2)$

$\overrightarrow{OA} = 4\underset{\sim}{i} + 7\underset{\sim}{j} + 3\underset{\sim}{k}$

$\overrightarrow{OB} = 8\underset{\sim}{i} + 7\underset{\sim}{j} + \underset{\sim}{k}$

$\overrightarrow{OC} = 6\underset{\sim}{i} + 5\underset{\sim}{j} + 2\underset{\sim}{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= 2\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= -2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$

$|\overrightarrow{AC}| = \sqrt{2^2 + (-2)^2 + (-1)^2}$

$= \sqrt{9}$

$= 3$

$|\overrightarrow{BC}| = \sqrt{(-2)^2 + (-2)^2 + 1^2}$

$= \sqrt{9}$

$= 3$

So $|\overrightarrow{AC}| = |\overrightarrow{BC}|$

ABC is an isosceles triangle

$\overrightarrow{AB} = 4\underset{\sim}{i} - 2\underset{\sim}{k}$

$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$

$= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$

$= \overrightarrow{OA} + \frac{1}{2}\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$

$= \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB}\right)$

$= 6\underset{\sim}{i} + 7\underset{\sim}{j} + 2\underset{\sim}{k}$

$\overrightarrow{MC} = \overrightarrow{OC} - \overrightarrow{OM}$

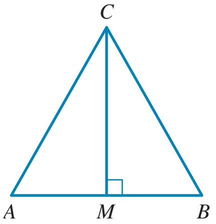
$= -2\underset{\sim}{j}$

$\overrightarrow{MC} \cdot \overrightarrow{AB} = -2\underset{\sim}{j} \cdot \left(4\underset{\sim}{i} - 2\underset{\sim}{k}\right)$

$= 0$

So $\overrightarrow{MC}$ is $\perp$ to $\overrightarrow{AB}$

6



$A(3,-3,4) \quad B(5,3,6) \quad C(3,1,3)$

$\overrightarrow{OA} = 3\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}$

$\overrightarrow{OB} = 5\underset{\sim}{i} + 3\underset{\sim}{j} + 6\underset{\sim}{k}$

$\overrightarrow{OC} = 3\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= 4\underset{\sim}{j} - \underset{\sim}{k}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= -2\underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k}$

$|\overrightarrow{AC}| = \sqrt{4^2 + (-1)^2}$

$= \sqrt{17}$

$|\overrightarrow{BC}| = \sqrt{(-2)^2 + (-2)^2 + (-3)^2}$

$= \sqrt{17}$

Since $|\overrightarrow{AC}| = |\overrightarrow{BC}|$

ABC is an isosceles triangle

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= 2\underset{\sim}{i} + 6\underset{\sim}{j} + 2\underset{\sim}{k}$

$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$

Since M is the midpoint of AB

$= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$

$= \overrightarrow{OA} + \frac{1}{2}\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$

$= \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB}\right)$

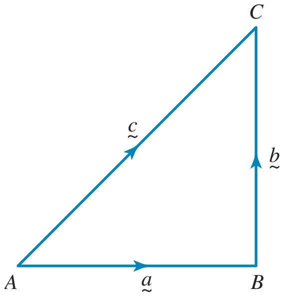
$= 4\underset{\sim}{i} + 5\underset{\sim}{k} \qquad M(4,0,5)$

$$\overrightarrow{MC} = \overrightarrow{OC} - \overrightarrow{OM}$$
$$= -\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$$

Now $\overrightarrow{AB} \cdot \overrightarrow{MC} = -2 + 6 - 4 = 0$

So $\overrightarrow{AB}$ is $\perp$ to $\overrightarrow{MC}$ shown

7



Let $\overrightarrow{AB} = \underset{\sim}{a}$
$\overrightarrow{BC} = \underset{\sim}{b}$
$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$
$\underset{\sim}{c} = \underset{\sim}{a} + \underset{\sim}{b}$

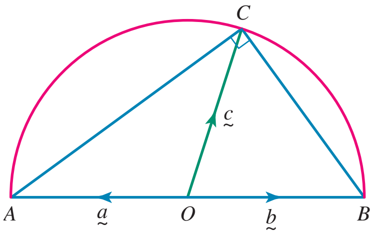
Since there is right angle at B

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

Consider $\underset{\sim}{c} \cdot \underset{\sim}{c} = \left(\underset{\sim}{a} + \underset{\sim}{b}\right) \cdot \left(\underset{\sim}{a} + \underset{\sim}{b}\right)$
$= \underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{b} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{b}$

$|\underset{\sim}{c}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2$
$\left|\overrightarrow{AC}\right|^2 = \left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{BC}\right|^2$
This is Pythagoras' Theorem

8



Let ACB be the semicircle with centre O

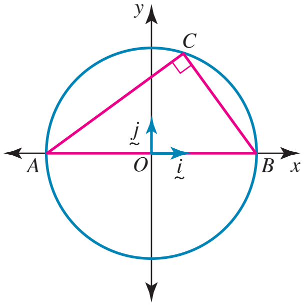
$\overrightarrow{OA} = \underset{\sim}{a}$
$\overrightarrow{OB} = \underset{\sim}{b}$
$\overrightarrow{OC} = \underset{\sim}{c}$

Now $|\underset{\sim}{a}| = |\underset{\sim}{b}| = |\underset{\sim}{c}|$ since they are all radii

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$
$= \underset{\sim}{c} - \underset{\sim}{a}$
$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$
$= \underset{\sim}{c} - \underset{\sim}{b}$

Consider $\overrightarrow{AC} \cdot \overrightarrow{BC} = \left(\underset{\sim}{c} - \underset{\sim}{a}\right) \cdot \left(\underset{\sim}{c} - \underset{\sim}{b}\right)$
$= \underset{\sim}{c} \cdot \underset{\sim}{c} - \underset{\sim}{a} \cdot \underset{\sim}{c} - \underset{\sim}{b} \cdot \underset{\sim}{c} + \underset{\sim}{a} \cdot \underset{\sim}{b}$
but $\underset{\sim}{b} = -\underset{\sim}{a}$ so
$\overrightarrow{AC} \cdot \overrightarrow{CB} = |\underset{\sim}{c}|^2 - \underset{\sim}{c}\left(\underset{\sim}{a} + \underset{\sim}{b}\right) + \underset{\sim}{a}\left(-\underset{\sim}{a}\right)$
$= |\underset{\sim}{c}|^2 - 0 - |\underset{\sim}{a}|^2$
$= 0$ since $|\underset{\sim}{a}| = |\underset{\sim}{c}|$
$\Rightarrow \overrightarrow{AC}$ is $\perp$ to $\overrightarrow{CB}$
$\angle ACB$ is 90°

9



Circle radius r, centre $(0, 0)$
$A(-r, 0)$ $B(r, 0)$
$C(a, b)$
$\overrightarrow{OA} = -r, \underset{\sim}{i}$
$\overrightarrow{OB} = r, \underset{\sim}{i}$
$\overrightarrow{OC} = a\underset{\sim}{i} + b\underset{\sim}{j}$

But C lies on the circle

So $a^2 + b^2 = r^2$

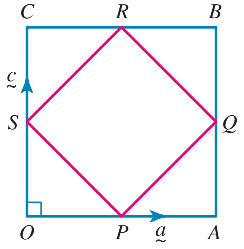
$\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC}$
$= (r - a)\underset{\sim}{i} - b\underset{\sim}{j}$
$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$
$= -(r + a)\underset{\sim}{i} - b\underset{\sim}{j}$

Consider $\overrightarrow{CB} \cdot \overrightarrow{CA} = -(r - a)(r + a) + b^2$
$= -\left(r^2 - ar + ar - a^2\right) + b^2$
$= a^2 + b^2 - r^2$
$= 0$

Since $a^2 + b^2 = r^2$

So $\overrightarrow{CB}$ is $\perp$ to $\overrightarrow{CA}$ shown

10



Let $\underset{\sim}{a} = \overrightarrow{OA} = \overrightarrow{CB}$
$\underset{\sim}{c} = \overrightarrow{OC} = \overrightarrow{AB}$
$|\underset{\sim}{c}| = |\underset{\sim}{a}|$ it is square

$\overrightarrow{PS} = \overrightarrow{PO} + \overrightarrow{OS}$
$= \frac{1}{2}\overrightarrow{AO} + \frac{1}{2}\overrightarrow{OC}$
$= \frac{1}{2}\left(\underset{\sim}{c} - \underset{\sim}{a}\right)$

$\overrightarrow{PQ} = \overrightarrow{PA} + \overrightarrow{AQ}$
$= \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$
$= \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{c}\right)$

$\overrightarrow{SR} = \overrightarrow{SC} + \overrightarrow{CR}$
$= \frac{1}{2}\overrightarrow{OC} + \frac{1}{2}\overrightarrow{CB}$
$= \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{c}\right)$

$$\begin{aligned}\overrightarrow{QR} &= \overrightarrow{QB} + \overrightarrow{BR}\\ &= \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}\\ &= \frac{1}{2}\left(\underset{\sim}{c} - \underset{\sim}{a}\right)\end{aligned}$$

$$\begin{aligned}\left|\overrightarrow{PS}\right|^2 &= \frac{1}{2}\left(\underset{\sim}{c} - \underset{\sim}{a}\right)\cdot\frac{1}{2}\left(\underset{\sim}{c} - \underset{\sim}{a}\right) = \left|\overrightarrow{QR}\right|^2\\ &= \frac{1}{4}\left(\underset{\sim}{c}\cdot\underset{\sim}{c} - 2\underset{\sim}{a}\cdot\underset{\sim}{c} + \underset{\sim}{a}\cdot\underset{\sim}{a}\right)\\ &= \frac{1}{4}\left(\left|\underset{\sim}{c}\right|^2 + \left|\underset{\sim}{a}\right|^2\right)\\ &= \left|\overrightarrow{QR}\right|^2\end{aligned}$$

$$\begin{aligned}\left|\overrightarrow{PQ}\right|^2 &= \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{c}\right)\cdot\frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{c}\right)\\ &= \frac{1}{4}\left(\underset{\sim}{a}\cdot\underset{\sim}{a} + 2\underset{\sim}{a}\cdot\underset{\sim}{c} + \underset{\sim}{c}\cdot\underset{\sim}{c}\right)\\ &= \frac{1}{4}\left(\left|\underset{\sim}{a}\right|^2 + \left|\underset{\sim}{c}\right|^2\right)\\ &= \left|\overrightarrow{SR}\right|^2\end{aligned}$$

$$\begin{aligned}\overrightarrow{PQ}\cdot\overrightarrow{QR} &= \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{c}\right)\cdot\frac{1}{2}\left(\underset{\sim}{c} - \underset{\sim}{a}\right)\\ &= \frac{1}{4}\left(\underset{\sim}{a}\cdot\underset{\sim}{c} + \underset{\sim}{c}\cdot\underset{\sim}{c} - \underset{\sim}{a}\cdot\underset{\sim}{a} - \underset{\sim}{a}\cdot\underset{\sim}{c}\right)\\ &= \frac{1}{4}\left(\left|\underset{\sim}{c}\right|^2 - \left|\underset{\sim}{a}\right|^2\right)\\ &= 0\end{aligned}$$

Also $\overrightarrow{SR}\cdot\overrightarrow{PS} = 0$

⇒ *PQRS* is a square

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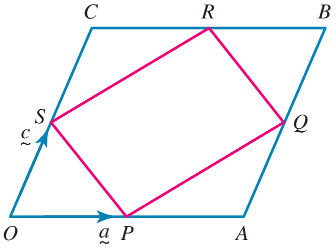

$$\begin{aligned}\overrightarrow{OA} &= \underset{\sim}{a} = \overrightarrow{CB}\\ \overrightarrow{OC} &= \underset{\sim}{c} = \overrightarrow{AB}\end{aligned}$$

$\left|\underset{\sim}{a}\right| = \left|\underset{\sim}{c}\right|$ it is a rhombus

$$\begin{aligned}\overrightarrow{PS} &= \overrightarrow{PO} + \overrightarrow{OS}\\ &= \frac{1}{2}\overrightarrow{AO} + \frac{1}{2}\overrightarrow{OC}\\ &= \frac{1}{2}\left(\underset{\sim}{c} - \underset{\sim}{a}\right)\end{aligned}$$

$$\begin{aligned}\overrightarrow{QR} &= \overrightarrow{QB} + \overrightarrow{BR}\\ &= \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}\\ &= \frac{1}{2}\left(\underset{\sim}{c} - \underset{\sim}{a}\right)\end{aligned}$$

So $\overrightarrow{PS} = \overrightarrow{QR}$

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{PA} + \overrightarrow{AQ}\\ &= \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}\\ &= \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{c}\right)\end{aligned}$$

$$\begin{aligned}\overrightarrow{SR} &= \overrightarrow{SC} + \overrightarrow{CR}\\ &= \frac{1}{2}\overrightarrow{OC} + \frac{1}{2}\overrightarrow{CB}\\ &= \frac{1}{2}\left(\underset{\sim}{c} + \underset{\sim}{a}\right)\end{aligned}$$

So $\overrightarrow{PQ} = \overrightarrow{SR}$

$$\begin{aligned}\overrightarrow{PQ}\cdot\overrightarrow{QR} &= \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{c}\right)\cdot\frac{1}{2}\left(\underset{\sim}{c} - \underset{\sim}{a}\right)\\ &= \frac{1}{4}\left(\underset{\sim}{a}\cdot\underset{\sim}{c} + \underset{\sim}{c}\cdot\underset{\sim}{c} - \underset{\sim}{a}\cdot\underset{\sim}{c} - \underset{\sim}{a}\cdot\underset{\sim}{a}\right)\\ &= \frac{1}{4}\left(\left|\underset{\sim}{c}\right|^2 - \left|\underset{\sim}{a}\right|^2\right)\\ &= 0\end{aligned}$$

So $\overrightarrow{PQ}$ is ⊥ to $\overrightarrow{QR}$

and $\overrightarrow{SR}$ is ⊥ to $\overrightarrow{PS}$

⇒ *PQRS* is a rectangle

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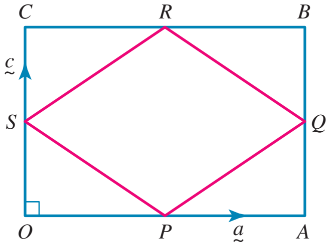


Let *OABC* be a rectangle

$$\begin{aligned}\overrightarrow{OA} &= \underset{\sim}{a} = \overrightarrow{CB}\\ \overrightarrow{OC} &= \underset{\sim}{c} = \overrightarrow{AB}\end{aligned}$$

$\underset{\sim}{a}\cdot\underset{\sim}{c} = 0$

P midpoint of $\overrightarrow{OA}$

$\overrightarrow{OP} = \overrightarrow{PA} = \frac{1}{2}\overrightarrow{OA}$

Q midpoint of $\overrightarrow{AB}$

$\overrightarrow{AQ} = \overrightarrow{QB} = \frac{1}{2}\overrightarrow{AB}$

R midpoint of $\overrightarrow{CB}$

$\overrightarrow{CR} = \overrightarrow{RB} = \frac{1}{2}\overrightarrow{CB}$

S midpoint of $\overrightarrow{OC}$

$\overrightarrow{OS} = \overrightarrow{SC} = \frac{1}{2}\overrightarrow{OC}$

To show *PQRS* is a rhombus

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{PA} + \overrightarrow{AQ}\\ &= \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OC}\\ &= \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{c}\right)\end{aligned}$$

$$\begin{aligned}\overrightarrow{SR} &= \overrightarrow{SC} + \overrightarrow{CR}\\ &= \frac{1}{2}\overrightarrow{OC} + \frac{1}{2}\overrightarrow{CB}\\ &= \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{c}\right)\end{aligned}$$

So $\overrightarrow{PQ} = \overrightarrow{SR}$

$\overrightarrow{PS} = \overrightarrow{PO} + \overrightarrow{OS}$

$= \frac{-1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OC}$

$= \frac{1}{2}\left(\underset{\sim}{c} - \underset{\sim}{a}\right)$

$\overrightarrow{QR} = \overrightarrow{QB} + \overrightarrow{BR}$

$= \frac{1}{2}\overrightarrow{OC} - \frac{1}{2}\overrightarrow{OA}$

$= \frac{1}{2}\left(\underset{\sim}{c} - \underset{\sim}{a}\right)$

So $\overrightarrow{PS} = \overrightarrow{QR}$

To show $\left|\overrightarrow{PS}\right| = \left|\overrightarrow{PQ}\right|$

$\left|\overrightarrow{PS}\right|^2 = \frac{1}{2}\left(\underset{\sim}{c} - \underset{\sim}{a}\right) \cdot \frac{1}{2}\left(\underset{\sim}{c} - \underset{\sim}{a}\right)$

$= \frac{1}{4}\left(\underset{\sim}{c} \cdot \underset{\sim}{c} - \underset{\sim}{a} \cdot \underset{\sim}{c} - \underset{\sim}{a} \cdot \underset{\sim}{c} + \underset{\sim}{a} \cdot \underset{\sim}{a}\right)$

$= \frac{1}{4}\left(|\underset{\sim}{c}|^2 + |\underset{\sim}{a}|^2\right)$

$\left|\overrightarrow{PQ}\right|^2 = \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{c}\right) \cdot \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{c}\right)$

$= \frac{1}{4}\left(\underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{c} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{c} + \underset{\sim}{c} \cdot \underset{\sim}{c}\right)$

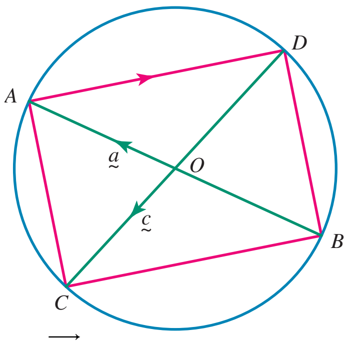
$= \frac{1}{4}\left(|\underset{\sim}{c}|^2 + |\underset{\sim}{a}|^2\right)$

So $\left|\overrightarrow{PS}\right| = \left|\overrightarrow{PQ}\right|$

and $\overrightarrow{SR}$ is // to $\overrightarrow{PQ}$·

⇒ *PQRS* is a rhombus·

13



$\underset{\sim}{a} = \overrightarrow{OA}$

$\underset{\sim}{c} = \overrightarrow{OC}$

Since *O* is the centre it is the midpoint of $\overrightarrow{CD}$

⇒ $\overrightarrow{OB} = -\underset{\sim}{a}$ and $\overrightarrow{OD} = -\underset{\sim}{c}$

$\overrightarrow{AD} = \overrightarrow{AO} + \overrightarrow{OD}$

$= -\underset{\sim}{a} - \underset{\sim}{c}$

So $\overrightarrow{DA} = \underset{\sim}{a} + \underset{\sim}{c}$

$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$

$= \underset{\sim}{c} - \underset{\sim}{a}$

$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC}$

$= \overrightarrow{OA} + \overrightarrow{OC}$

$= \underset{\sim}{a} + \underset{\sim}{c}$

$\overrightarrow{DB} = \overrightarrow{DO} + \overrightarrow{OB}$

$= \underset{\sim}{c} - \underset{\sim}{a}$

$\left|\overrightarrow{AD}\right| = \left|\overrightarrow{CB}\right| \qquad \overrightarrow{AD} = \overrightarrow{CB}$

$\left|\overrightarrow{AC}\right| = \left|\overrightarrow{DB}\right| \qquad \overrightarrow{AC} = \overrightarrow{DB}$

$\overrightarrow{DA} \cdot \overrightarrow{AC} = \left(\underset{\sim}{a} + \underset{\sim}{c}\right) \cdot \left(\underset{\sim}{c} - \underset{\sim}{a}\right)$

$= \underset{\sim}{a} \cdot \underset{\sim}{c} + \underset{\sim}{c} \cdot \underset{\sim}{c} - \underset{\sim}{a} \cdot \underset{\sim}{a} - \underset{\sim}{c} \cdot \underset{\sim}{a}$

$= \left(|\underset{\sim}{c}|^2 - |\underset{\sim}{a}|^2\right)$

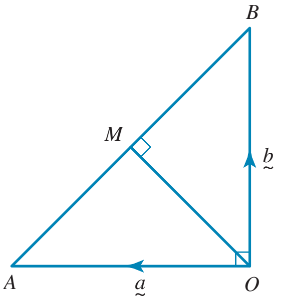
$= 0$

Since both are radii $|\underset{\sim}{c}| = |\underset{\sim}{a}|$

So $\overrightarrow{DA}$ is ⊥ to $\overrightarrow{AC}$

So *ACBD* is a rectangle

14 a



OAB is an isosceles triangle

$\underset{\sim}{a} = \overrightarrow{OA} \quad \underset{\sim}{b} = \overrightarrow{OB}, \quad$ So $\quad |\underset{\sim}{a}| = |\underset{\sim}{b}|$

There is a right angle at *B* $\Rightarrow \underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

M is the midpoint of $\overrightarrow{AB}$

$\overrightarrow{AM} = \overrightarrow{MB} = \frac{1}{2}\overrightarrow{AB}$

$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$

$= \underset{\sim}{a} + \frac{1}{2}\overrightarrow{AB}$

$= \underset{\sim}{a} + \frac{1}{2}\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$

$= \underset{\sim}{a} + \frac{1}{2}\left(\underset{\sim}{b} - \underset{\sim}{a}\right)$

$= \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{b}\right)$

b To show $\overrightarrow{OM}$ is ⊥ to $\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$

$\overrightarrow{OM} \cdot \overrightarrow{AB} = \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{b}\right) \cdot \left(\underset{\sim}{b} - \underset{\sim}{a}\right)$

$= \frac{1}{2}\left(\underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{b} - \underset{\sim}{a} \cdot \underset{\sim}{a} - \underset{\sim}{a} \cdot \underset{\sim}{b}\right)$

$= \frac{1}{2}\left(|\underset{\sim}{b}|^2 - |\underset{\sim}{a}|^2\right)$

$= 0 \quad$ since $\quad |\underset{\sim}{a}| = |\underset{\sim}{b}|$

So $\overrightarrow{OM}$ is ⊥ to $\overrightarrow{AB}$

c $\left|\overrightarrow{OM}\right|^2 = \overrightarrow{OM} \cdot \overrightarrow{OM}$

$= \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{b}\right) \cdot \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{b}\right)$

$= \frac{1}{4}\left(\underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{b} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{b}\right)$

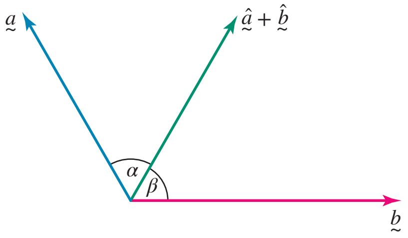
$= \frac{1}{4}\left(|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2\right)$

$\left|\overrightarrow{AB}\right|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2$

So $\left|\overrightarrow{OM}\right| = \frac{1}{2}\left|\overrightarrow{AB}\right|$

Since $\left|\overrightarrow{OM}\right| = \frac{1}{2}\left|\overrightarrow{AB}\right| = \frac{1}{2}\sqrt{|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2}$

15



To show $\alpha=\beta$

Let α be the angle between $\underset{\sim}{a}$ and $\underset{\sim}{\hat{a}}+\underset{\sim}{\hat{b}}$

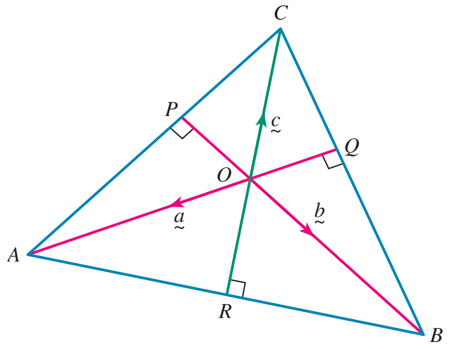
Let β be the angle between $\underset{\sim}{b}$ and $\underset{\sim}{\hat{a}}+\underset{\sim}{\hat{b}}$

$$\begin{aligned}\cos(\alpha) &= \frac{\underset{\sim}{\hat{a}}\cdot\left(\underset{\sim}{\hat{a}}+\underset{\sim}{\hat{b}}\right)}{\left|\underset{\sim}{\hat{a}}\right|\left|\underset{\sim}{\hat{a}}+\underset{\sim}{\hat{b}}\right|}\\ &= \frac{\underset{\sim}{\hat{a}}\cdot\underset{\sim}{\hat{a}}+\underset{\sim}{\hat{a}}\cdot\underset{\sim}{\hat{b}}}{\left|\underset{\sim}{\hat{a}}+\underset{\sim}{\hat{b}}\right|}\\ &= \frac{1+\underset{\sim}{\hat{a}}\cdot\underset{\sim}{\hat{b}}}{\left|\underset{\sim}{\hat{a}}+\underset{\sim}{\hat{b}}\right|}\end{aligned}$$

$$\begin{aligned}\cos(\beta) &= \frac{\underset{\sim}{\hat{b}}\cdot\left(\underset{\sim}{\hat{a}}+\underset{\sim}{\hat{b}}\right)}{\left|\underset{\sim}{\hat{b}}\right|\left|\underset{\sim}{\hat{a}}+\underset{\sim}{\hat{b}}\right|}\\ &= \frac{\underset{\sim}{\hat{b}}\cdot\underset{\sim}{\hat{a}}+\underset{\sim}{\hat{b}}\cdot\underset{\sim}{\hat{b}}}{\left|\underset{\sim}{\hat{a}}+\underset{\sim}{\hat{b}}\right|}\\ &= \frac{1+\underset{\sim}{\hat{a}}\cdot\underset{\sim}{\hat{b}}}{\left|\underset{\sim}{\hat{a}}+\underset{\sim}{\hat{b}}\right|}\end{aligned}$$

$\therefore \cos(\alpha)=\cos(\beta)$

16



a P is the midpoint of $\overrightarrow{AC}$,

$$\text{So}\quad \overrightarrow{AP}=\overrightarrow{PC}=\frac{1}{2}\overrightarrow{AC}$$

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OA}+\overrightarrow{AP}\\ &= \overrightarrow{OA}+\frac{1}{2}\overrightarrow{AC}\\ &= \overrightarrow{OA}+\frac{1}{2}\left(\overrightarrow{OC}-\overrightarrow{OA}\right)\\ &= \frac{1}{2}\left(\overrightarrow{OA}+\overrightarrow{OC}\right)\\ &= \frac{1}{2}\left(\underset{\sim}{a}+\underset{\sim}{c}\right)\end{aligned}$$

Q is the midpoint of $\overrightarrow{CB}$,

$$\text{So } \overrightarrow{CQ}=\overrightarrow{QB}=\frac{1}{2}\overrightarrow{CB}$$

$$\begin{aligned}\overrightarrow{OQ} &= \overrightarrow{OB}+\overrightarrow{BQ}\\ &= \overrightarrow{OB}+\frac{1}{2}\overrightarrow{\mathrm{BC}}\\ &= \overrightarrow{OB}+\frac{1}{2}\left(\overrightarrow{OC}-\overrightarrow{OB}\right)\\ &= \frac{1}{2}\left(\underset{\sim}{b}+\underset{\sim}{c}\right)\end{aligned}$$

b $\overrightarrow{OP}$ is $\perp$ to $\overrightarrow{AC}$, $\overrightarrow{AC}=\underset{\sim}{c}-\underset{\sim}{a}$

$$\begin{aligned}\text{So } \overrightarrow{OP}\cdot\overrightarrow{AC} &= 0\\ \frac{1}{2}\left(\underset{\sim}{a}+\underset{\sim}{c}\right)\cdot\left(\underset{\sim}{c}-\underset{\sim}{a}\right) &= 0\\ \frac{1}{2}\left(\underset{\sim}{a}\cdot\underset{\sim}{c}+\underset{\sim}{c}\cdot\underset{\sim}{c}-\underset{\sim}{a}\cdot\underset{\sim}{c}-\underset{\sim}{a}\cdot\underset{\sim}{a}\right) &= 0\\ \frac{1}{2}\left(\left|\underset{\sim}{c}\right|^2-\left|\underset{\sim}{a}\right|^2\right) &= 0\end{aligned}$$

$\left|\underset{\sim}{c}\right|=\left|\underset{\sim}{a}\right|$

$\overrightarrow{OQ}$ is $\perp$ to $\overrightarrow{BC}$, $\overrightarrow{BC}=\underset{\sim}{c}-\underset{\sim}{b}$

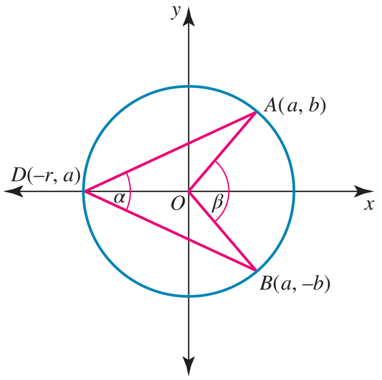
$$\begin{aligned}\text{So } \overrightarrow{OQ}\cdot\overrightarrow{BC} &= 0\\ \frac{1}{2}\left(\underset{\sim}{b}+\underset{\sim}{c}\right)\cdot\left(\underset{\sim}{c}-\underset{\sim}{b}\right) &= 0\\ \frac{1}{2}\left(\underset{\sim}{b}\cdot\underset{\sim}{c}+\underset{\sim}{c}\cdot\underset{\sim}{c}-\underset{\sim}{c}\cdot\underset{\sim}{b}-\underset{\sim}{b}\cdot\underset{\sim}{b}\right) &= 0\\ \frac{1}{2}\left(\left|\underset{\sim}{c}\right|^2-\left|\underset{\sim}{b}\right|^2\right) &= 0\\ \left|\underset{\sim}{c}\right| &= \left|\underset{\sim}{b}\right|\end{aligned}$$

So $\left|\underset{\sim}{a}\right|=\left|\underset{\sim}{b}\right|=\left|\underset{\sim}{c}\right|$ shown

c

$$\begin{aligned}\overrightarrow{OR} &= \overrightarrow{OA}+\overrightarrow{AR}\\ &= \overrightarrow{OA}+\frac{1}{2}\overrightarrow{AB}\\ &= \overrightarrow{OA}+\frac{1}{2}\left(\overrightarrow{OB}-\overrightarrow{OA}\right)\\ &= \frac{1}{2}\left(\underset{\sim}{a}+\underset{\sim}{b}\right)\\ \overrightarrow{AB} &= \overrightarrow{AO}+\overrightarrow{OB}\\ &= \underset{\sim}{b}-\underset{\sim}{a}\\ \overrightarrow{OR}\cdot\overrightarrow{AB} &= \frac{1}{2}\left(\underset{\sim}{a}+\underset{\sim}{b}\right)\cdot\left(\underset{\sim}{b}-\underset{\sim}{a}\right)\\ &= \frac{1}{2}\left(\underset{\sim}{a}\cdot\underset{\sim}{b}-\underset{\sim}{a}\cdot\underset{\sim}{a}+\underset{\sim}{b}\cdot\underset{\sim}{b}-\underset{\sim}{b}\cdot\underset{\sim}{a}\right)\\ &= \frac{1}{2}\left(\left|\underset{\sim}{b}\right|^2-\left|\underset{\sim}{a}\right|^2\right)\\ \text{But } \left|\underset{\sim}{a}\right| &= \left|\underset{\sim}{b}\right|\\ \therefore \overrightarrow{OR}\cdot\overrightarrow{AB} &= 0\\ \overrightarrow{OR} &\perp \overrightarrow{AB}\end{aligned}$$

17



a

$$\begin{aligned}\overrightarrow{OA} &= a\underset{\sim}{i}+b\underset{\sim}{j}\\ \overrightarrow{OB} &= a\underset{\sim}{i}-b\underset{\sim}{j}\\ \left|\overrightarrow{OA}\right| &= \left|\overrightarrow{OB}\right|\\ &= \sqrt{a^2+b^2}\\ &= \sqrt{a^2+(-b)^2}\\ &= r\end{aligned}$$

Both are radii of the circle

So $a^2 + b^2 = r^2$

$$\begin{aligned}\cos(\beta) &= \frac{\overrightarrow{OA} \cdot \overrightarrow{OB}}{\left|\overrightarrow{OA}\right| \cdot \left|\overrightarrow{OB}\right|}\\ &= \frac{a^2 - b^2}{r^2}\\ &= \frac{a^2 - b^2}{a^2 + b}\end{aligned}$$

b $\overrightarrow{OD} = -r\underset{\sim}{i}$

$\left|\overrightarrow{OD}\right| = r$

$$\begin{aligned}\overrightarrow{DA} &= \overrightarrow{OA} - \overrightarrow{OD}\\ &= (a + r)\,\underset{\sim}{i} + b\underset{\sim}{j}\\ \overrightarrow{DB} &= \overrightarrow{OB} - \overrightarrow{OD}\\ &= (a + r)\,\underset{\sim}{i} - b\underset{\sim}{j}\end{aligned}$$

$$\begin{aligned}\left|\overrightarrow{DA}\right| &= \left|\overrightarrow{DB}\right|\\ &= \sqrt{(a + r)^2 + b^2}\end{aligned}$$

$$\begin{aligned}\cos(\alpha) &= \frac{\overrightarrow{DA} \cdot \overrightarrow{DB}}{\left|\overrightarrow{DA}\right| \cdot \left|\overrightarrow{DB}\right|}\\ &= \frac{(a + r)^2 - b^2}{(a + r)^2 + b^2}\\ &= \frac{a^2 + 2ar + a^2 + b^2 - b^2}{a^2 + 2ar + a^2 + 2b^2}\\ &= \frac{2a\,(a + r)}{2\left(a^2 + b^2\right) + 2ar}\\ &= \frac{a\,(a + r)}{r^2 + ar}\\ &= \frac{a\,(a + r)}{r\,(a + r)}\\ &= \frac{a}{r}\\ &= \frac{a}{\sqrt{a^2 + b^2}}\end{aligned}$$

c To show $\beta = 2\alpha$

$$\begin{aligned}\cos(2\alpha) &= 2\cos^2(\alpha) - 1\\ &= 2\left(\frac{a}{\sqrt{a^2 + b^2}}\right)^2 - 1\\ &= \frac{2a^2}{a^2 + b^2} - 1\\ &= \frac{2a^2 - \left(a^2 + b^2\right)}{a^2 + b^2}\\ &= \frac{a^2 - b^2}{a^2 + b^2}\\ &= \cos(\beta)\end{aligned}$$

Since $\cos(2\alpha) = \cos(\beta)$

Then $2\alpha = \beta$ shown

3.5 Exam questions

1 $\overrightarrow{OM} = a\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$, $\overrightarrow{ON} = -3\underset{\sim}{i} + b\underset{\sim}{j} - \underset{\sim}{k}$

Let D be the midpoint of $\overline{MN}$.

$$\overrightarrow{OD} = \overrightarrow{OM} + \overrightarrow{MD}$$

$$\overrightarrow{OD} = \overrightarrow{OM} + \frac{1}{2}\left(\overrightarrow{ON} - \overrightarrow{OM}\right) = \frac{1}{2}\left(\overrightarrow{ON} + \overrightarrow{OM}\right)$$

$$\overrightarrow{OD} = \left(\frac{a - 3}{2}\right)\underset{\sim}{i} + \frac{(b + 1)}{2}\underset{\sim}{j} - \frac{3}{2}\underset{\sim}{k} = -5\underset{\sim}{i} + \frac{3}{2}\underset{\sim}{j} + c\underset{\sim}{k}$$

$\underset{\sim}{i}$: $\dfrac{a - 3}{2} = -5$, $a = -7$

$\underset{\sim}{j}$: $\dfrac{b + 1}{2} = \dfrac{3}{2}$, $b = 2$

$\underset{\sim}{k}$: $c = -\dfrac{3}{2}$

The correct answer is **E**.

2 $\left|\underset{\sim}{a} + \underset{\sim}{b}\right| = \left|\underset{\sim}{a}\right| + \left|\underset{\sim}{b}\right|$ by the triangle rule implies that $\underset{\sim}{a}$ is parallel to $\underset{\sim}{b}$.

The correct answer is **A**.

3 **a** $\overrightarrow{OA} = a\underset{\sim}{i}$, $\overrightarrow{OC} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$

$\left|\overrightarrow{OA}\right| = a = \left|\overrightarrow{OC}\right| = \sqrt{1 + 1 + 1}$

$a = \sqrt{3}$

Award 1 mark for finding the value of a.

b
$$\begin{aligned}\overrightarrow{OB} &= \overrightarrow{OA} + \overrightarrow{AB}\\ &= \overrightarrow{OA} + \overrightarrow{OC}\\ &= \left(1 + \sqrt{3}\right)\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\\ \overrightarrow{AC} &= \overrightarrow{OC} - \overrightarrow{OA}\\ &= \left(1 - \sqrt{3}\right)\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\\ \overrightarrow{OB} \cdot \overrightarrow{AC} &= \left(1 + \sqrt{3}\right)\left(1 - \sqrt{3}\right) + 1 + 1\\ &= 1 - 3 + 2\\ &= 0\end{aligned}$$

Since $\overrightarrow{OB} \cdot \overrightarrow{AC} = 0$, it follows that $\overrightarrow{OB}$ is perpendicular to $\overrightarrow{AC}$, so the diagonals of the rhombus are perpendicular.

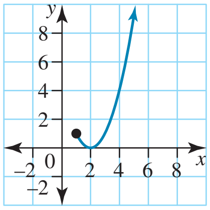
Award 1 mark for the finding both diagonals.

Award 1 mark for the correct dot product and statement.

3.6 Parametric equations

3.6 Exercise

1



$\underset{\sim}{r}(t) = (t + 1)\,\underset{\sim}{i} + (t - 1)^2\underset{\sim}{j}, \quad t \geq 0$

(1) $x(t) = t + 1$

(2) $y(t) = (t - 1)^2$

(1) $\Rightarrow t = x - 1$

$y = (x - 2)^2$ but $t \geq 0$

Part of parabola

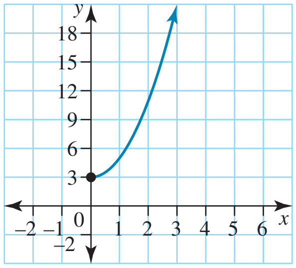
So $x \geq 1$

$y \geq 0$

Domain $[1, \infty)$

Range $[0, \infty)$

2



$\underset{\sim}{r}(t) = \sqrt{t}\underset{\sim}{i} + (2t+3)\underset{\sim}{j}, \quad t \geq 0$

(1) $x(t) = \sqrt{t}$

(2) $y(t) = 2t + 3$

(1) $\Rightarrow t = x^2$

$y = 2x^2 + 3$ but $t \geq 0$

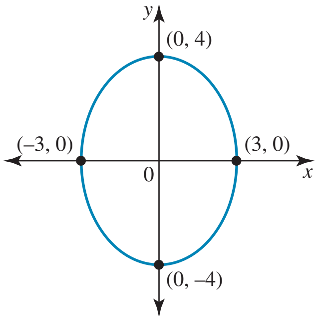
So $x \geq 0$

$y \geq 3$

Domain $[0, \infty)$

Range $[3, \infty)$

3



$\underset{\sim}{r}(t) = 3\cos(t)\underset{\sim}{i} + 4\sin(t)\underset{\sim}{j}, \quad t \geq 0$

(1) $x = 3\cos(t)$

(2) $y = 4\sin(t)$

$\frac{x}{3} = \cos(t)$

$\frac{y}{4} = \sin(t)$

$\cos^2(t) + \sin^2(t) = 1$

$\frac{x^2}{9} + \frac{y^2}{16} = 1$

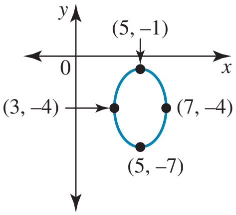
Ellipse centre $(0, 0)$

Semi-minor, major 3, 4

Domain $[-3, 3]$

Range $[-4, 4]$

4



$\underset{\sim}{r}(t) = (5 - 2\cos(t))\underset{\sim}{i} + (3\sin(t) - 4)\underset{\sim}{j}, \quad t \geq 0$

(1) $x = 5 - 2\cos(t)$

(2) $y = 3\sin(t) - 4$

(1) $\cos(t) = \frac{x-5}{-2}$

(2) $\sin(t) = \frac{y+4}{3}$

$\cos^2(t) + \sin^2(t) = 1$

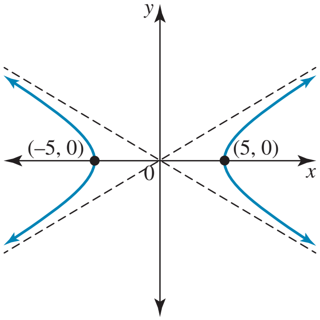
$\frac{(x-5)^2}{4} + \frac{(y+4)^2}{9} = 1$

Ellipse centre $(5, -4)$

Domain $[3, 7]$

Range $[-7, -1]$

5



$\underset{\sim}{r}(t) = 5\sec(2t)\underset{\sim}{i} + 3\tan(2t)\underset{\sim}{j}, \quad t \geq 0$

(1) $x = 5\sec(2t)$

(2) $y = 3\tan(2t)$

(1) $\frac{x}{5} = \sec(2t)$

(2) $\frac{y}{3} = \tan(2t)$

$\sec^2(2t) - \tan^2(2t) = 1$

$\frac{x^2}{25} - \frac{y^2}{9} = 1$

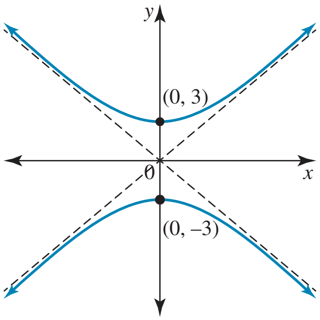
Hyperbola centre $(0, 0)$

Asymptote $y = \pm\frac{3x}{5}$

Domain $(-\infty, -5] \cup [5, \infty)$

Range R

6



$\underset{\sim}{r}(t) = 4\cot\left(\frac{t}{2}\right)\underset{\sim}{i} + 3\operatorname{cosec}\left(\frac{t}{2}\right)\underset{\sim}{j}$

(1) $x = 4\cot\left(\frac{t}{2}\right)$

(2) $y = 3\operatorname{cosec}\left(\frac{t}{2}\right)$

(1) $\cot\left(\frac{t}{2}\right) = \frac{x}{4}$

(2) $\operatorname{cosec}\left(\frac{t}{2}\right) = \frac{y}{3}$

$\operatorname{cosec}^2\left(\frac{t}{2}\right) - \cot^2\left(\frac{t}{2}\right) = 1$

$\frac{y^2}{9} - \frac{x^2}{16} = 1$

Hyperbola centre $(0, 0)$

Asymptote $y = \pm\frac{3x}{4}$

Domain R

Range $(-\infty, -3] \cup [3, \infty)$

7 (1) $x = \frac{5}{2}\left(t + \frac{1}{t}\right)$

(2) $y = \frac{3}{2}\left(t - \frac{1}{t}\right)$

$$\frac{2x}{5} = t + \frac{1}{t}$$

$$\frac{2y}{3} = t - \frac{1}{t} \quad \text{squaring}$$

$$\frac{4x^2}{25} = t^2 + 2 + \frac{1}{t^2}$$

$$\frac{4y^2}{9} = t^2 - 2 + \frac{1}{t^2} \quad \text{subtract}$$

$$4 = \frac{4x^2}{25} - \frac{4y^2}{9}$$

$$1 = \frac{x^2}{25} - \frac{y^2}{9} \quad \text{hyperbola}$$

8 $x = \frac{6t}{1 + t^2}, \quad y = \frac{3\left(1 - t^2\right)}{1 + t^2}$

$$x^2 = \frac{36t^2}{\left(1 + t^2\right)^2}, \quad y^2 = \frac{9\left(1 - t^2\right)^2}{\left(1 + t^2\right)^2}$$

$$\begin{aligned} x^2 + y^2 &= \frac{36t^2 + 9 - 18t^2 + 9t^4}{\left(1 + t^2\right)^2} \\ &= \frac{9\left(1 + 2t^2 + t^4\right)}{\left(1 + t^2\right)^2} \\ &= \frac{9\left(1 + t^2\right)^2}{\left(1 + t^2\right)^2} \end{aligned}$$

$$x^2 + y^2 = 9 \quad \text{shown}$$

9 a $\underset{\sim}{r}(t) = 2t\underset{\sim}{i} + 4t^2\underset{\sim}{j} \quad t \ge 0$

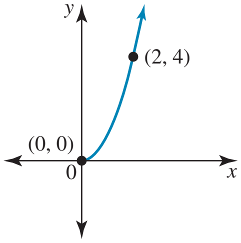
(1) $x = 2t \Rightarrow t = \frac{x}{2}$

(2) $y = 4t^2 \Rightarrow y = 4\left(\frac{x}{2}\right)^2 = x^2$

Part of parabola

Domain $[0, \infty)$

Range $[0, \infty)$



b $\underset{\sim}{r}(t) = (t - 1)\underset{\sim}{i} + 3t\underset{\sim}{j} \quad t \ge 0$

(1) $x = t - 1 \Rightarrow t = x + 1$

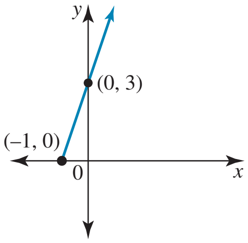
(2) $y = 3t \qquad y = 3(x + 1)$

$\qquad\qquad\qquad = 3x + 3$

Part of a line

Domain $[-1, \infty)$

Range $[0, \infty)$



c $\underset{\sim}{r}(t) = 2t\underset{\sim}{i} + 8t^3\underset{\sim}{j} \quad t \ge 0$

(1) $x = 2t \Rightarrow t = \frac{x}{2}$

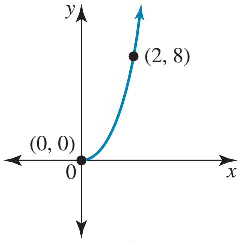
(2) $y = 8t^3 \quad y = 8\left(\frac{x}{2}\right)^3$

$\qquad\qquad y = x^3$

Part of a cubic

Domain $[0, \infty)$

Range $[0, \infty)$



10 a $\underset{\sim}{r}(t) = 2t\underset{\sim}{i} + \frac{1}{t}\underset{\sim}{j} \quad t > 0$

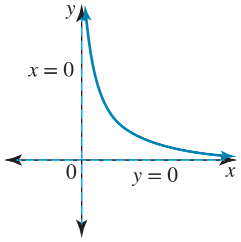
(1) $x = 2t \quad \Rightarrow t = \frac{x}{2}$

(2) $y = \frac{1}{t} \quad y = \frac{2}{x}$

Part of a hyperbola

Domain $(0, \infty)$

Range $(0, \infty)$



b $\underset{\sim}{r}(t) = 2t\underset{\sim}{i} + \left(t^2 - 4t\right)\underset{\sim}{j} \quad t \ge 0$

(1) $x = 2t \qquad \Rightarrow t = \frac{x}{2}$

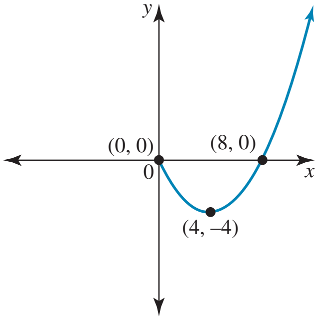
(2) $y = t^2 - 4t$

$$\begin{aligned} y &= \frac{x^2}{4} - 2x \\ &= \frac{1}{4}\left(x^2 - 8x\right) \\ &= \frac{x}{4}(x - 8) \end{aligned}$$

Part of a parabola

Domain $[0, \infty)$

Range $[-4, \infty)$



c $\underset{\sim}{r}(t) = \left(t + \frac{1}{t}\right)\underset{\sim}{i} + \left(t - \frac{1}{t}\right)\underset{\sim}{j}$

(1) $x = t + \frac{1}{t} \qquad x^2 = t^2 + 2 + \frac{1}{t^2}$

(2) $y = t - \frac{1}{t} \qquad y^2 = t^2 - 2 + \frac{1}{t^2}$

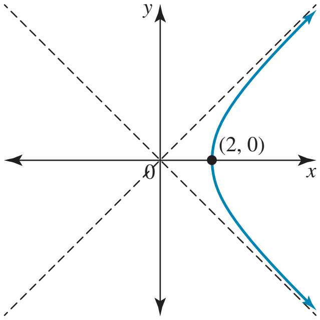
$y^2 = x^2 - 4 \qquad x^2 - y^2 = 4$

$y = \sqrt{x^2 - 4}$

Positive branch only, part of hyperbola asymptotes $y = \pm x$

Domain $[2, \infty)$

Range R



11 a $\underset{\sim}{r}(t) = e^{-2t}\underset{\sim}{i} + e^{2t}\underset{\sim}{j}$

(1) $x = e^{-2t}$

(2) $y = e^{2t}$

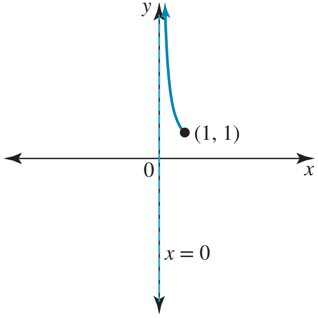
$xy = 1$

$y = \frac{1}{x}$

Part of Hyperbola

Domain $(0, 1]$

Range $[1, \infty)$



b $\underset{\sim}{r}(t) = e^{-t}\underset{\sim}{i} + \left(2 + e^{2t}\right)\underset{\sim}{j} \quad t \geq 0$

(1) $x = e^{-t} \qquad e^t = \frac{1}{x}$

(2) $y = 2 + e^{2t}$

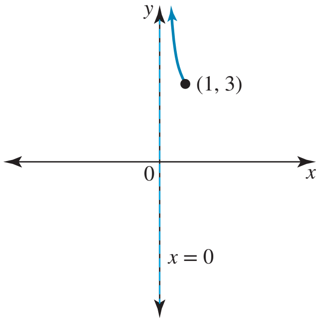
$= 2 + \left(e^t\right)^2$

$= 2 + \frac{1}{x^2}$

Part of truncus

Domain $(0, 1]$

Range $[3, \infty)$



c $\underset{\sim}{r}(t) = e^t\underset{\sim}{i} + \left(2 + e^{2t}\right)\underset{\sim}{j}$

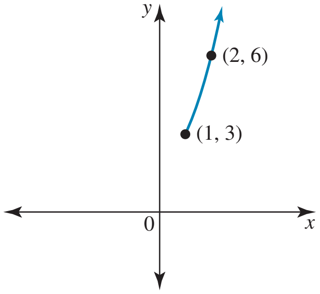
(1) $x = e^t$

(2) $y = \left(2 + e^{2t}\right) \qquad y = 2 + x^2$

Part of parabola

Domain $[1, \infty)$

Range $[3, \infty)$



12 a $\underset{\sim}{r}(t) = 3\cos(t)\underset{\sim}{i} + 3\sin(t)\underset{\sim}{j} \quad t \geq 0$

(1) $x = 3\cos(t) \qquad \cos(t) = \frac{x}{3}$

(2) $y = 3\sin(t) \qquad \sin(t) = \frac{y}{3}$

$\sin^2(t) + \cos^2(t) = 1$

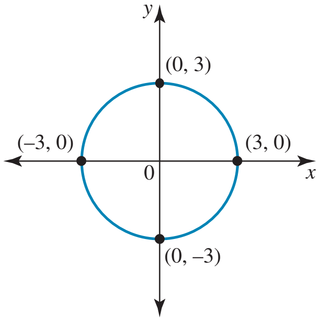
$\frac{x^2}{9} + \frac{y^2}{9} = 1$

$x^2 + y^2 = 9$

Circle centre $(0, 0)$ radius 3

Domain $[-3, 3]$

Range $[-3, 3]$



b $\underset{\sim}{r}(t) = 4\cos(t)\underset{\sim}{i} + 3\sin(t)\underset{\sim}{j} \quad t \geq 0$

(1) $x = 4\cos(t) \qquad \cos(t) = \frac{x}{4}$

(2) $y = 3\sin(t) \qquad \sin(t) = \frac{y}{3}$

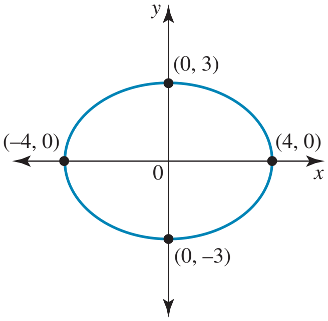
$\sin^2(t) + \cos^2(t) = 1$

$\frac{x^2}{16} + \frac{y^2}{9} = 1$

Ellipse centre at origin

Domain $[-4, 4]$

Range $[-3, 3]$



c $\underset{\sim}{r}(t) = 4\sec(t)\underset{\sim}{i} + 3\tan(t)\underset{\sim}{j} \quad t \geq 0$

(1) $x = 4\sec(t) \qquad \sec(t) = \frac{x}{4}$

(2) $y = 3\tan(t) \qquad \tan(t) = \frac{y}{3}$

$1 + \tan^2(t) = \sec^2(t)$

$1 + \frac{y^2}{9} = \frac{x^2}{16}$

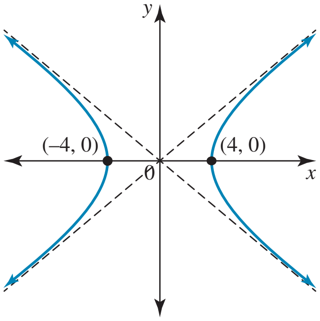
$\frac{x^2}{16} - \frac{y^2}{9} = 1$

Hyperbola, centre at origin

Asymptote $y = \pm\frac{3x}{4}$

Domain $(-\infty, -4] \cup [4, \infty)$

Range R



13 a $\underset{\sim}{r}(t) = (1 + 3\cos(t))\underset{\sim}{i} + (3\sin(t) - 2)\underset{\sim}{j} \quad t \geq 0$

(1) $x = 1 + 3\cos(t) \qquad \cos(t) = \frac{x-1}{3}$

(2) $y = 3\sin(t) - 2 \qquad \sin(t) = \frac{y+2}{3}$

$\sin^2(t) + \cos^2(t) = 1$

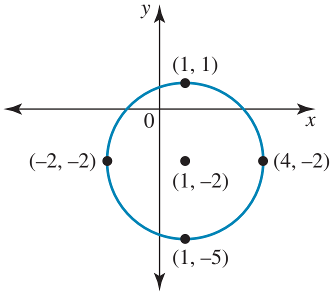
$\frac{(x-1)^2}{9} + \frac{(y+2)^2}{9} = 1$

$(x-1)^2 + (y+2)^2 = 9$

Circle, centre $(1, -2)$ radius 3

Domain $[-2, 4]$

Range $[-5, 1]$



b $\underset{\sim}{r}(t) = (4 + 3\cos(t))\underset{\sim}{i} + (2\sin(t) - 3)\underset{\sim}{j} \quad t \geq 0$

(1) $x = 4 + 3\cos(t) \qquad \cos(t) = \frac{x-4}{3}$

(2) $y = 2\sin(t) - 3 \qquad \sin(t) = \frac{y+3}{2}$

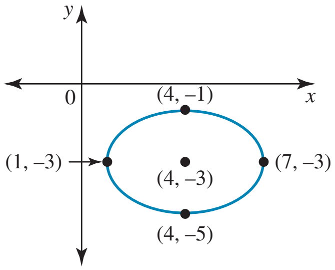
$\cos^2(t) + \sin^2(t) = 1$

$\frac{(x-4)^2}{9} + \frac{(y+3)^2}{4} = 1$

Ellipse, centre $(4, -3)$ semi-major, minor $3, 2$

Domain $[1, 7]$

Range $[-5, -1]$



c $\underset{\sim}{r}(t) = (2 - 3\sec(t))\underset{\sim}{i} + (5\tan(t) - 4)\underset{\sim}{j} \quad t \geq 0$

(1) $x = 2 - 3\sec(t) \qquad \sec(t) = \frac{x-2}{-3}$

(2) $y = 5\tan(t) - 4 \qquad \tan(t) = \frac{y+4}{5}$

$1 + \tan^2(t) = \sec^2(t)$

$1 + \frac{(y+4)^2}{25} = \frac{(x-2)^2}{9}$

$\frac{(x-2)^2}{9} - \frac{(y+4)^2}{25} = 1$

Hyperbola, centre $(2, -4)$

Domain $(-\infty, -1] \cup [5, \infty)$

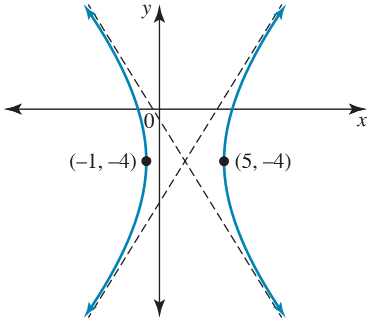
Range R

Asymptote : $\frac{y+4}{5} = \pm\frac{x-2}{3}$

$$y = \pm\frac{5}{3}(x-2) - 4$$

$$y = \frac{5x}{3} - \frac{22}{3}$$

$$y = \frac{-5x}{3} - \frac{2}{3}$$



14 a $\underset{\sim}{r}(t) = \cos^2(t)\underset{\sim}{i} + \sin^2(t)\underset{\sim}{j} \quad t \geq 0$

(1) $x = \cos^2(t) \quad 0 \leq x \leq 1$

(2) $y = \sin^2(t) \quad 0 \leq y \leq 1$

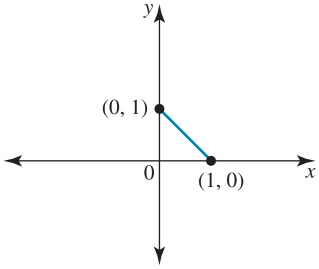
$x + y = 1$

$y = 1 - x,$

Part of a straight line

Domain [0, 1]

Range [0, 1]



b $\underset{\sim}{r}(t) = \cos^3(t)\underset{\sim}{i} + \sin^3(t)\underset{\sim}{j} \qquad t \geq 0$

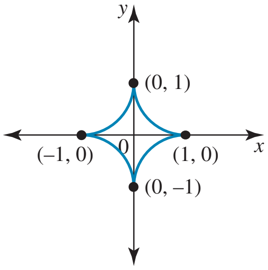
(1) $x = \cos^3(t) \qquad \cos^2(t) = x^{\frac{2}{3}}$

(2) $y = \sin^3(t) \qquad \sin^2(t) = y^{\frac{2}{3}}$

$\Rightarrow x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$

Domain [−1, 1]

Range [−1, 1]



c $\underset{\sim}{r}(t) = \cos^4(t)\underset{\sim}{i} + \sin^4(t)\underset{\sim}{j}$

(1) $x = \cos^4(t) \qquad \sqrt{x} = \cos^2(t)$

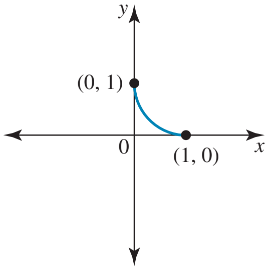
(2) $y = \sin^4(t) \qquad \sqrt{y} = \sin^2(t)$

$\Rightarrow \sin^2(t) + \cos^2(t) = 1$

$\Rightarrow \sqrt{x} + \sqrt{y} = 1$

Domain [0, 1]

Range [0, 1]



15 a $\underset{\sim}{r}(t) = \cos^2(t)\underset{\sim}{i} + \cos(2t)\underset{\sim}{j}, \quad t \geq 0$

(1) $x = \cos^2(t)$

(2) $y = \cos(2t)$

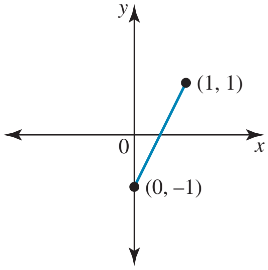
$y = 2\cos^2(t) - 1$

$y = 2x - 1$

Part of a straight line

But $0 \leq x \leq 1$ Domain [0, 1]

$-1 \leq y \leq 1$ Range [−1, 1]



b $\underset{\sim}{r}(t) = \cos(t)\underset{\sim}{i} + \cos(2t)\underset{\sim}{j}, \quad t \geq 0$

(1) $x = \cos(t)$

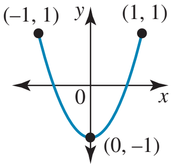
(2) $y = \cos(2t) = 2\cos^2(t) - 1$

$y = 2x^2 - 1$

Part of a parabola

But $-1 \leq x \leq 1$ Domain [−1, 1]

$-1 \leq y \leq 1$ Range [−1, 1]



c $\underset{\sim}{r}(t) = \sin(t)\underset{\sim}{i} + \sin(2t)\underset{\sim}{j}, \quad t \geq 0$

(1) $x = \sin(t)$

(2) $y = \sin(2t) = 2\sin(t)\cos(t)$

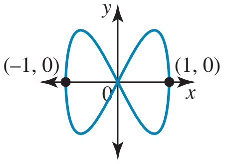
$y^2 = 4\sin^2(t)\cos^2(t)$

$= 4\sin^2(t)\left(1 - \sin^2(t)\right)$

$y = \pm 2x\sqrt{1 - x^2}$

Domain [−1, 1]

Range [−1, 1]



16 a $\underset{\sim}{r}(t) = a\cos(t)\underset{\sim}{i} + a\sin(2t)\underset{\sim}{j}$

(1) $x = a\cos(t) \quad \cos(t) = \dfrac{x}{a}$

(2) $y = a\sin(t) \quad \sin(t) = \dfrac{y}{a}$

$\sin^2(t) + \cos^2(t) = 1$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$$

$$x^2 + y^2 = a^2$$

Circle center at origin, radius a

$$\underset{\sim}{r}(t) = \left(\frac{2at}{1+t^2}\right)\underset{\sim}{i} + \left(\frac{a\left(1-t^2\right)}{1+t^2}\right)\underset{\sim}{j}$$

(1) $x = \dfrac{2at}{1+t^2}$

(2) $y = \dfrac{a\left(1-t^2\right)}{1+t^2}$

$$\begin{aligned} x^2 + y^2 &= \frac{4a^2t^2}{\left(1+t^2\right)^2} + \frac{a^2\left(1-2t^2+t^4\right)}{\left(1+t^2\right)^2} \\ &= \frac{a^2\left(1+2t^2+t^4\right)}{\left(1+t^2\right)^2} \\ &= \frac{a^2\left(1+t^2\right)^2}{\left(1+t^2\right)^2} \\ &= a^2 \end{aligned}$$

So $x^2 + y^2 = a^2$

b $\underset{\sim}{r}(t) = a\cos(t)\underset{\sim}{i} + b\sin(t)\underset{\sim}{j}$

(1) $x = a\cos(t) \qquad \cos(t) = \dfrac{x}{a}$

(2) $y = b\sin(t) \qquad \sin(t) = \dfrac{y}{b}$

$\sin^2(t) + \cos^2(t) = 1$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Ellipse, centre at origin, semi-minor, major axes a, b respectively

$$\underset{\sim}{r}(t) = \left(\frac{2at}{1+t^2}\right)\underset{\sim}{i} + \left(\frac{b\left(1-t^2\right)}{1+t^2}\right)\underset{\sim}{j}$$

(1) $x = \dfrac{2at}{1+t^2}$

(2) $y = \dfrac{b\left(1-t^2\right)}{1+t^2}$

$$\frac{x}{a} = \frac{2t}{1+t^2},\ \frac{y}{b} = \frac{1-t^2}{1+t^2}$$

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} &= \frac{4t^2}{\left(1+t^2\right)^2} + \frac{\left(1-2t^2+t^4\right)}{\left(1+t^2\right)^2} \\ &= \frac{1+2t^2+t^4}{\left(1+t^2\right)^2} \\ &= \frac{\left(1+t^2\right)^2}{\left(1+t^2\right)^2} \\ &= 1 \end{aligned}$$

So $\dfrac{x^2}{a} + \dfrac{y^2}{b} = 1$

c $\underset{\sim}{r}(t) = \dfrac{a}{2}\left(e^{2t} + e^{-2t}\right)\underset{\sim}{i} + \dfrac{b}{2}\left(e^{2t} - e^{-2t}\right)\underset{\sim}{j}$

(1) $x = \dfrac{a}{2}\left(e^{2t} + e^{-2t}\right)$

(2) $y = \dfrac{b}{2}\left(e^{2t} - e^{-2t}\right)$

$$\frac{2x}{a} = e^{2t} + e^{-2t}$$

$$\frac{2y}{b} = e^{2t} - e^{-2t} \qquad \text{square both}$$

$$\frac{4x^2}{a^2} = e^{4t} + 2 + e^{-4t}$$

$$\frac{4y^2}{b^2} = e^{4t} - 2 + e^{-4t} \quad \text{subtract}$$

$$\frac{4x^2}{a^2} - \frac{4y^2}{b^2} = 4$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{hyperbola}$$

$\underset{\sim}{r}(t) = a\sec(t)\underset{\sim}{i} + b\tan(t)\underset{\sim}{j}$

(1) $x = a\sec(t) \qquad \sec(t) = \dfrac{x}{a}$

(2) $y = b\tan(t) \qquad \tan(t) = \dfrac{y}{b}$

$1 + \tan^2(t) = \sec^2(t)$

$$1 + \frac{y^2}{b^2} = \frac{x^2}{a^2}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Hyperbola, centre at origin, asymptote $y = \pm\dfrac{bx}{a}$

$$\underset{\sim}{r}(t) = \frac{a}{2}\left(t + \frac{1}{t}\right)\underset{\sim}{i} + \frac{b}{2}\left(t - \frac{1}{t}\right)\underset{\sim}{j}$$

(1) $x = \dfrac{a}{2}\left(t + \dfrac{1}{t}\right)$

(2) $y = \dfrac{b}{2}\left(t - \dfrac{1}{t}\right)$

$$\frac{2x}{a} = t + \frac{1}{t},\ \frac{2y}{b} = t - \frac{1}{t} \quad \text{square both}$$

$$\frac{4x^2}{a^2} = t^2 + 2 + \frac{1}{t^2}$$

$$\frac{4y^2}{b^2} = t^2 - 2 + \frac{1}{t^2} \qquad \text{subtract}$$

$$\frac{4x^2}{a^2} - \frac{4y^2}{b^2} = 4$$

So $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

17 $\underset{\sim}{r}(t) = 2\sin(t)\underset{\sim}{i} + 2\sin(t)\tan(t)\underset{\sim}{j}$

(1) $x = 2\sin(t)$

(2) $y = 2\sin(t)\tan(t)$

$$\begin{aligned} \text{RHS} &= \frac{x^2}{\sqrt{4-x^2}} \\ &= \frac{4\sin^2(t)}{\sqrt{4-4\sin^2(t)}} \\ &= \frac{4\sin^2(t)}{\sqrt{4\left(1-\sin^2(t)\right)}} \end{aligned}$$

$$\begin{aligned}
&= \frac{4\sin^2(t)}{2\cos(t)}\\
&= 2\sin(t) \times \frac{\sin(t)}{\cos(t)}\\
&= 2\sin(t)\tan(t)\\
&= y\\
&= \text{LHS}
\end{aligned}$$

18 $x = at,\ y = \dfrac{a}{1+t^2}$

$$\begin{aligned}
\text{RHS} &= \frac{a^3}{a^2+x^2}\\
&= \frac{a^3}{a^2+a^2t^2}\\
&= \frac{a^3}{a^2\left(1+t^2\right)}\\
&= \frac{a}{1+t^2}\\
&= y\\
&= \text{LHS}
\end{aligned}$$

19 a
$$\begin{aligned}
\text{LHS} &= \cos(3A)\\
&= \cos(2A + A)\\
&= \cos(2A)\cos(A) - \sin(2A)\sin(A)\\
&= \left(2\cos^2(A) - 1\right)\cos(A) - 2\sin^2(A)\cos(A)\\
&= \left(2\cos^2(A) - 1\right)\cos(A) - 2\cos(A)\left(1 - \cos^2(A)\right)\\
&= 2\cos^3(A) - \cos(A) - 2\cos(A) + 2\cos^3(A)\\
&= 4\cos^3(A) - 3\cos(A)\\
&= \text{RHS}
\end{aligned}$$

b $x = 2\cos(t) \Rightarrow \cos(t) = \dfrac{x}{2}$

$$\begin{aligned}
y &= 2\cos(3t)\\
y &= 2\left(4\cos^3(t) - 3\cos(t)\right)\\
&= 2\left(4\times\left(\frac{x}{2}\right)^3 - 3\times\left(\frac{x}{2}\right)\right)\\
&= 2\left(\frac{4x^3}{8} - \frac{3x}{2}\right)\\
y &= x^3 - 3x
\end{aligned}$$

20 a
$$\begin{aligned}
\text{LHS} &= \cos\ (4A)\\
&= \cos\ (2\times 2A)\\
&= 2\cos^2(2A) - 1\\
&= 2\left(2\cos^2(A) - 1\right)^2 - 1\\
&= 2\left(4\cos^4(A) - 4\cos^2(A) + 1\right) - 1\\
&= 8\cos^4(A) - 8\cos^2(A) + 1\\
&= \text{RHS}
\end{aligned}$$

b $x = 2\cos^2(t)\ \Rightarrow\ \cos^2(t) = \dfrac{x}{2}$

$$\begin{aligned}
y &= \cos(4t)\\
&= 8\ \cos^4(t) - 8\ \cos^2(t) + 1\\
&= 8\left(\frac{x}{2}\right)^2 - 8\times\left(\frac{x}{2}\right) + 1\\
&= 2x^2 - 4x + 1
\end{aligned}$$

21 $\underset{\sim}{r}(t) = 2\tan(t)\underset{\sim}{i} + 2\operatorname{cosec}(2t)\underset{\sim}{j}$

$$\begin{aligned}
x &= 2\tan(t)\\
y &= 2\operatorname{cosec}(2t)
\end{aligned}$$

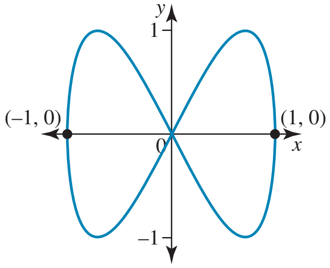
$$\begin{aligned}
\text{RHS} &= \frac{x^2+4}{2x}\\
&= \frac{4\tan^2(t) + 4}{4\tan(t)}\\
&= \frac{4\left(1 + \tan^2(t)\right)}{4\tan(t)}\\
&= \frac{\sec^2(t)}{\tan(t)}\\
&= \frac{1}{\cos^2(t)} \times \frac{\cos(t)}{\sin(t)}\\
&= \frac{1}{\sin(t)\cos(t)}\\
&= \frac{2}{2\sin(t)\cos(t)}\\
&= \frac{2}{\sin(2t)}\\
&= 2\operatorname{cosec}(2t)\\
&= \text{LHS}
\end{aligned}$$

22
$$\begin{aligned}
x &= \cos(t)\left(\sec(t) + \mathrm{a}\ \cos(t)\right)\\
y &= \sin(t)\left(\sec(t) + \mathrm{a}\ \cos(t)\right)\\
x^2 &= \cos^2(t)\left(\sec(t) + a\ \cos(t)\right)^2\\
y^2 &= \sin^2(t)\left(\sec(t) + a\ \cos(t)\right)^2\\
x^2 + y^2 &= \left(\sec(t) + a\cos(t)\right)^2\left(\cos^2(t) + \sin^2(t)\right)\\
&= \left(\sec(t) + a\cos(t)\right)^2\\
\text{LHS} &= (x-1)\left(x^2+y^2\right) - ax^2\\
&= \left(\cos(t)\left(\sec(t) + a\cos(t)\right) - 1\right)\left(\sec(t) + a\cos(t)\right)^2\\
&\quad - a\cos^2(t)\left(\sec(t) + a\cos(t)\right)^2\\
&= \left(\sec(t) + a\cos(t)\right)^2\left(\cos(t)\left(\sec(t) + a\cos(t)\right)\right.\\
&\quad \left.- 1 - a^2\cos^2(t)\right)\\
&= \left(\sec(t) + a\cos(t)\right)^2\left(1 + a^2\cos^2(t) - 1 - a^2\cos^2(t)\right)\\
&= 0\\
&= \text{RHS}
\end{aligned}$$

23
$$\begin{aligned}
x &= 4\ \cos(t)\\
y &= \frac{4\sin^2(t)}{2+\sin(t)}\\
\text{LHS} &= y^2\left(16 - x^2\right)\\
&= \frac{16\sin^4(t)}{(2+\sin(t))^2}\left(16 - 16\cos^2(t)\right)\\
&= \frac{16\sin^4(t)}{(2+\sin(t))^2} \times 16\left(1 - \cos^2(t)\right)\\
&= \frac{256\sin^6(t)}{(2+\sin(t))^2}\\
\text{RHS} &= \left(x^2 + 8y - 16\right)^2\\
&= \left(16\left(\cos^2(t) - 1\right) + \frac{32\sin^2(t)}{2+\sin(t)}\right)^2\\
&= \left(\frac{32\sin^2(t)}{2+\sin(t)} - 16\sin^2(t)\right)^2\\
&= \left(\frac{32\sin^2(t) - 16\sin^2(t)\left(2+\sin(t)\right)}{2+\sin(t)}\right)^2\\
&= \left(\frac{32\sin^2(t) - 32\sin^2(t) - 16\sin^3(t)}{2+\sin(t)}\right)^2\\
&= \frac{256\sin^6(t)}{(2+\sin(t))^2}\\
&= \text{LHS}
\end{aligned}$$

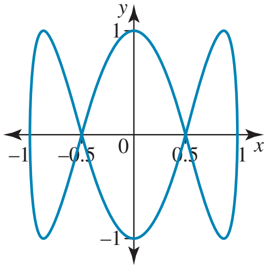
24 **a** $x = \cos(2t)$

$y = \sin(4t)$



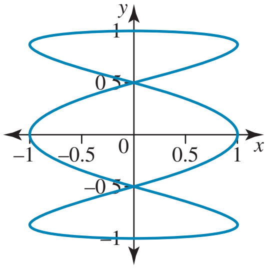
b $x = \cos(2t)$

$y = \sin(6t)$



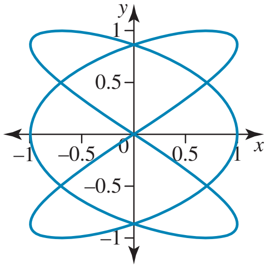
25 **a** $x = \cos(3t)$

$y = \sin(t)$



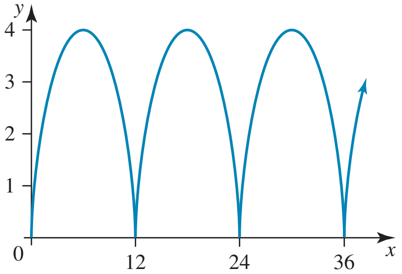
b $x = \cos(3t)$

$y = \sin(2t)$



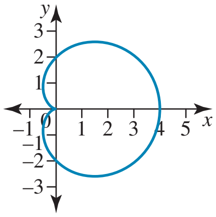
26 **a** Cycloid $x = 2(1 - \sin(t))$

$y = 2(1 - \cos(t)) \quad t \geq 0$



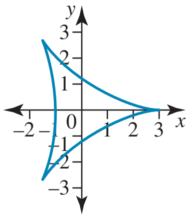
b Cardioid $x = 2\cos(t)(1 + \cos(t)) \quad t \geq 0$

$y = 2\sin(t)(1 + \cos(t))$



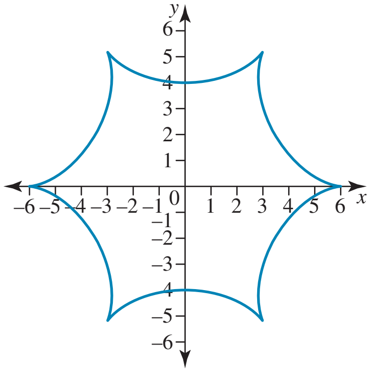
27 Deltoid $x = 2\cos(t) + \cos(2t)$

$y = 2\sin(t) - \sin(2t) \quad t \geq 0$



28 Hypercycloid $x = 5\cos(t) + \cos(5t)$

$y = 5\sin(t) - \sin(5t) \quad t \geq 0$



3.6 Exam questions

1 $\underset{\sim}{r}(t) = \left(1 - \sqrt{a}\sin(t)\right)\underset{\sim}{i} + \left(1 - \frac{1}{b}\cos(t)\right)\underset{\sim}{j}$

$\Rightarrow x = 1 - \sqrt{a}\sin(t),\ y = 1 - \frac{1}{b}\cos(t),\ a, b \in R^+$

$\sin(t) = \frac{1 - x}{\sqrt{a}},\ \cos(t) = b(1 - y), \sin^2(t) + \cos^2(t) = 1,$

To be a circle $(x - h)^2 + (y - k)^2 = r^2$

$$\left(\frac{1 - x}{\sqrt{a}}\right)^2 + (b(1 - y))^2 = 1$$

$$\frac{(1 - x)^2}{a} + b^2(1 - y)^2 = 1$$

$\Rightarrow b^2 = \frac{1}{a},\ ab^2 = 1$

The correct answer is **A**.

2 $\frac{(x - 2)^2}{9} + \frac{(y - 3)^2}{4} = 1$

$\cos^2(t) + \sin^2(t) = 1$

$\cos(t) = \frac{x - 2}{3},\ \sin(t) = \frac{y - 3}{2}$

$x = 2 + 3\cos(t)$ and $y = 3 + 2\sin(t)$

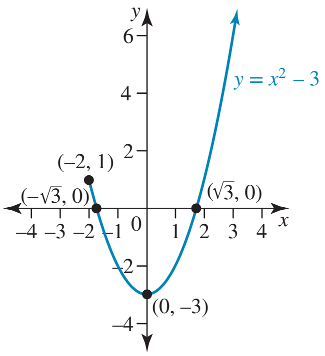
The correct answer is **E**.

3 **a** $\underset{\sim}{r}(t) = (t - 2)\underset{\sim}{i} + (t^2 - 4t + 1)\underset{\sim}{j}$

$$x = t - 2 \quad \Rightarrow t = x + 2$$
$$y = t^2 - 4t + 1$$
$$y = (x+2)^2 - 4(x+2) + 1$$
$$y = x^2 + 4x + 4 - 4x - 8 + 1$$
$$y = x^2 - 3 \quad t \geq 0 \quad \Rightarrow x \geq -2$$

Award 1 mark for the correct proof.

b



Award 1 mark for a correct graph showing the restricted domain and endpoint.

Award 1 mark for the correct axial-intercepts:

$(0, -3), \left(\sqrt{3}, 0\right), \left(-\sqrt{3}, 0\right)$.

3.7 Review

3.7 Exercise

Technology free: short answer

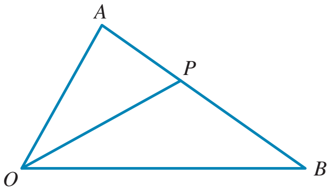
1 $A(4, -2, 3) \qquad B(10, 12, -3)$

$\overrightarrow{OA} = 4\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k} \qquad \overrightarrow{OB} = 10\underset{\sim}{i} + 12\underset{\sim}{j} - 3\underset{\sim}{k}$

a $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (10\underset{\sim}{i} + 12\underset{\sim}{j} - 3\underset{\sim}{k}) - (4\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k})$

$$= 6\underset{\sim}{i} + 14\underset{\sim}{j} - 6\underset{\sim}{k}$$
$$\left|\overrightarrow{AB}\right| = \sqrt{6^2 + 14^2 + (-6)^2} = \sqrt{268}$$
$$= \sqrt{4 \times 67}$$
$$= 2\sqrt{67}$$

b



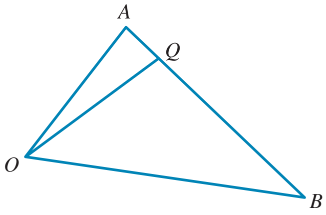
$$\cos(\theta) = \frac{-6}{2\sqrt{67}}$$
$$\theta = \cos^{-1}\left(\frac{-6}{2\sqrt{67}}\right)$$
$$= 111.5°$$

c $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$$= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$$
$$= \overrightarrow{OA} + \frac{1}{2}\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$$
$$= \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB}\right)$$
$$= \frac{1}{2}\left[\left(4\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}\right) + \left(10\underset{\sim}{i} + 12\underset{\sim}{j} - 3\underset{\sim}{k}\right)\right]$$
$$= 7\underset{\sim}{i} + 5\underset{\sim}{j}$$

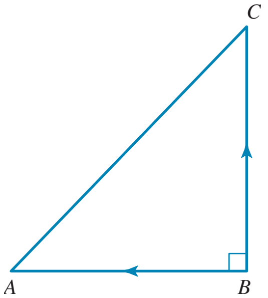
So $P(7, 5, 0)$

d



$$\overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{AQ}$$
$$= \overrightarrow{OA} + \frac{1}{3}\overrightarrow{AB}$$
$$= \overrightarrow{OA} + \frac{1}{3}\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$$
$$= \frac{1}{3}\left(2\overrightarrow{OA} + \overrightarrow{OB}\right)$$
$$= \frac{1}{3}\left[2\left(4\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}\right) + \left(10\underset{\sim}{i} + 12\underset{\sim}{j} - 3\underset{\sim}{k}\right)\right]$$
$$= 6\underset{\sim}{i} + \frac{8}{3}\underset{\sim}{j} + \underset{\sim}{k}$$
$$Q\left(6, \frac{8}{3}, 1\right)$$

2



$A\,(8, -3, 5) \qquad B\,(2, -1, 3) \qquad C(5, 4, -1)$

$$\overrightarrow{OA} = 8\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}$$
$$\overrightarrow{OB} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$$
$$\overrightarrow{OC} = 5\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}$$
$$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$$
$$= 6\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}$$
$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$
$$= 3\underset{\sim}{i} + 5\underset{\sim}{j} - 4\underset{\sim}{k}$$
$$\overrightarrow{BA} \cdot \overrightarrow{BC} = 18 - 10 - 8 = 0$$

So $\overrightarrow{BA}$ is $\perp$ to $\overrightarrow{BC}$

$$\left|\overrightarrow{BA}\right| = \sqrt{6^2 + (-2)^2 + 2^2} = \sqrt{44}$$
$$= 2\sqrt{11}$$
$$\left|\overrightarrow{BC}\right| = \sqrt{3^2 + 5^2 + (-4)^2} = \sqrt{50}$$
$$= 5\sqrt{2}$$

$$\text{Area of the triangle} = \frac{1}{2}\left|\overrightarrow{AB}\right| \left|\overrightarrow{BC}\right|$$
$$= \frac{1}{2} \times 2\sqrt{11} \times 5\sqrt{2}$$
$$= 5\sqrt{22}$$

3 $\underset{\sim}{a} = 4\underset{\sim}{i} - 3\underset{\sim}{j} - 8\underset{\sim}{k},\ \underset{\sim}{b} = -2\underset{\sim}{i} + y\underset{\sim}{j} + 4\underset{\sim}{k}$

a $|\underset{\sim}{b}| = \sqrt{(-2)^2 + y^2 + 4^2} = 5$

$$\sqrt{4+y^2+16}=5$$
$$\sqrt{20+y^2}=5$$
$$20+y^2=25$$
$$y^2=5$$
$$y=\pm\sqrt{5}$$

b $3\underset{\sim}{a}+2\underset{\sim}{b}=3\left(4\underset{\sim}{i}-3\underset{\sim}{j}-8\underset{\sim}{k}\right)+2\left(-2\underset{\sim}{i}+y\underset{\sim}{j}+4\underset{\sim}{k}\right)$

$=8\underset{\sim}{i}+(2y-9)\underset{\sim}{j}-16\underset{\sim}{k}$

Parallel to xz plane

$\left(3\underset{\sim}{a}+2\underset{\sim}{b}\right)\cdot\underset{\sim}{j}=0\Rightarrow 2y-9=0$

$2y=9$

$y=\dfrac{9}{2}$

c $\underset{\sim}{a}=\lambda\underset{\sim}{b}$

$\underset{\sim}{a}=-2\underset{\sim}{b}$ from $\underset{\sim}{i}$ and $\underset{\sim}{k}$

$\underset{\sim}{j}\Rightarrow -3=-2y$

$y=\dfrac{3}{2}$

d $\underset{\sim}{a}\cdot\underset{\sim}{b}=-8-3y-32=0$

$3y=-40$

$y=\dfrac{-40}{3}$

4 **a** $B(1, 4, 3)$ $\quad A(-2, 2, 1)$

$\overrightarrow{OB}=\underset{\sim}{i}+4\underset{\sim}{j}+3\underset{\sim}{k}$ $\quad\overrightarrow{OA}=-2\underset{\sim}{i}+2\underset{\sim}{j}+\underset{\sim}{k}$

$\left|\overrightarrow{OA}\right|=\sqrt{(-2)^2+2^2+1}=\sqrt{9}=3$

$\hat{\underset{\sim}{a}}=\dfrac{1}{3}\left(-2\underset{\sim}{i}+2\underset{\sim}{j}+\underset{\sim}{k}\right)$

$\underset{\sim}{b}\cdot\underset{\sim}{a}=-2+8+3=9$

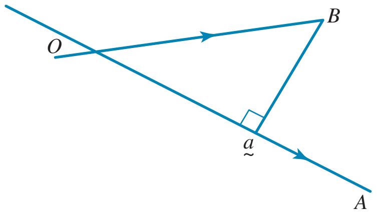
$\left(\underset{\sim}{b}\cdot\hat{\underset{\sim}{a}}\right)\hat{\underset{\sim}{a}}=\dfrac{9}{3}\times\dfrac{1}{3}\left(-2\underset{\sim}{i}+2\underset{\sim}{j}+\underset{\sim}{k}\right)$

$=-2\underset{\sim}{i}+2\underset{\sim}{j}+\underset{\sim}{k}$

$\underset{\sim}{b}-\left(\underset{\sim}{b}\cdot\hat{\underset{\sim}{a}}\right)\hat{\underset{\sim}{a}}=\left(\underset{\sim}{i}+4\underset{\sim}{j}+3\underset{\sim}{k}\right)-\left(-2\underset{\sim}{i}+2\underset{\sim}{j}+\underset{\sim}{k}\right)$

$=3\underset{\sim}{i}+2\underset{\sim}{j}+2\underset{\sim}{k}$

b



$\left|\underset{\sim}{b}-\left(\underset{\sim}{b}\cdot\hat{\underset{\sim}{a}}\right)\hat{\underset{\sim}{a}}\right|=\sqrt{3^2+2^2+2^2}$

$=\sqrt{17}$

5 $P(x, 3, -1)$ $\quad Q(2, -1, 5)$ $\quad R(-1, 1, 2)$

$\overrightarrow{OP}=x\underset{\sim}{i}+3\underset{\sim}{j}-\underset{\sim}{k}$

$\overrightarrow{OQ}=2\underset{\sim}{i}-\underset{\sim}{j}+5\underset{\sim}{k}$

$\overrightarrow{OR}=-\underset{\sim}{i}+\underset{\sim}{j}+2\underset{\sim}{k}$

$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}$

$=(2-x)\underset{\sim}{i}-4\underset{\sim}{j}+6\underset{\sim}{k}$

a $\overrightarrow{PQ}$ // to yz plane $\overrightarrow{PQ}\cdot\underset{\sim}{i}=0$

$2-x=0$

$x=2$

b $\overrightarrow{QR}=\overrightarrow{OR}-\overrightarrow{OQ}$

$=-3\underset{\sim}{i}+2\underset{\sim}{j}-3\underset{\sim}{k}$

$\overrightarrow{PQ}=-2\ \overrightarrow{QR}$

$\underset{\sim}{i}$: $\quad 2-x=6$

$x=-4$

c $\overrightarrow{PQ}$ is $\perp$ to $\overrightarrow{QR}$

$\overrightarrow{PQ}\cdot\overrightarrow{QR}=0$

$-3(2-x)-8-18=0$

$-3(2-x)=26$

$3(2-x)=\dfrac{-26}{3}$

$x=2+\dfrac{26}{9}$

$=\dfrac{44}{9}$

d $\left|\overrightarrow{PQ}\right|=9$

$\left|\overrightarrow{PQ}\right|=\sqrt{(2-x)^2+(-4)^2+6^2}$

$=\sqrt{(2-x)^2+52}=9$

$(2-x)^2+52=81$

$(2-x)^2=29$

$x-2=\pm\sqrt{29}$

$x=2\pm\sqrt{29}$

e $\cos(\alpha)=\dfrac{x}{\left|\overrightarrow{PQ}\right|}$ $\qquad\alpha=\cos^{-1}\left(\dfrac{-1}{\sqrt{13}}\right)$

$\dfrac{-1}{\sqrt{13}}=\dfrac{x}{\sqrt{(2-x)^2+52}}$

$-\sqrt{13}x=\sqrt{(2-x)^2+52}$

$13x^2=(2-x)^2+52$

$13x^2=4-4x+x^2+52$

$12x^2+4x-56=0$

$4\left(3x^2+x-14\right)=0$

$4(3x+7)(x-2)=0$

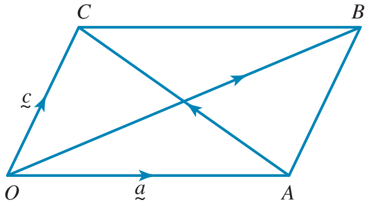
$x=2$ or $\dfrac{-7}{3}$ but $x<0$

$x=\dfrac{-7}{3}$

6 Let $OABC$ be a parallelogram

Let $\underset{\sim}{a}=\overrightarrow{OA}$

$\underset{\sim}{c}=\overrightarrow{OC}$



To show $\left|\overrightarrow{OB}\right|^2+\left|\overrightarrow{AC}\right|^2=\left|\overrightarrow{OA}\right|^2+\left|\overrightarrow{AB}\right|^2+\left|\overrightarrow{BC}\right|^2+\left|\overrightarrow{CO}\right|^2$

Now $\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OA}+\overrightarrow{OC}$

$=\underset{\sim}{a}+\underset{\sim}{c}$

$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}$

$=\underset{\sim}{c}-\underset{\sim}{a}$

LHS $=\left|\overrightarrow{OB}\right|^2+\left|\overrightarrow{AC}\right|^2$

$=\overrightarrow{OB}\cdot\overrightarrow{OB}+\overrightarrow{AC}\cdot\overrightarrow{AC}$

$=\left(\underset{\sim}{a}+\underset{\sim}{c}\right)\cdot\left(\underset{\sim}{a}+\underset{\sim}{c}\right)+\left(\underset{\sim}{c}-\underset{\sim}{a}\right)\cdot\left(\underset{\sim}{c}-\underset{\sim}{a}\right)$

$=\underset{\sim}{a}\cdot\underset{\sim}{a}+\underset{\sim}{c}\cdot\underset{\sim}{a}+\underset{\sim}{a}\cdot\underset{\sim}{c}+\underset{\sim}{c}\cdot\underset{\sim}{c}+\underset{\sim}{c}\cdot\underset{\sim}{c}-\underset{\sim}{a}\cdot\underset{\sim}{c}-c\cdot a+\underset{\sim}{a}\cdot\underset{\sim}{a}$

Since $\underset{\sim}{a}\cdot\underset{\sim}{c}=\underset{\sim}{c}\cdot\underset{\sim}{a}$ and $\underset{\sim}{a}\cdot\underset{\sim}{a}=\left|\underset{\sim}{a}\right|^2$

$= 2\,|\underset{\sim}{a}|^2 + 2\,|\underset{\sim}{c}|^2$

RHS since it is parallelogram

$|\overrightarrow{OC}| = |\overrightarrow{AB}|$ and $|\overrightarrow{OA}| = |\overrightarrow{CB}|$

RHS $|\overrightarrow{OA}|^2 + |\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2 + |\overrightarrow{OC}|^2$

$= 2|\overrightarrow{OA}|^2 + 2|\overrightarrow{OC}|$

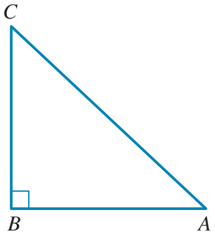
$= 2|\underset{\sim}{a}|^2 + 2|\underset{\sim}{c}|^2$

So LHS = RHS proved.

7 **a** $\underset{\sim}{a} = \overrightarrow{OA}$

$\underset{\sim}{b} = \overrightarrow{OB}$

$\underset{\sim}{c} = \overrightarrow{OC}$



$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \underset{\sim}{b} - \underset{\sim}{a}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= \underset{\sim}{c} - \underset{\sim}{b}$

But $\overrightarrow{AB}$ is ⊥ to $\overrightarrow{BC}$

So $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$

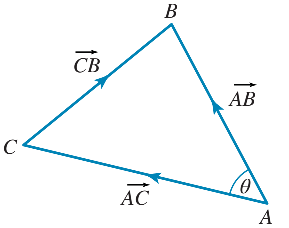
$(b - a) \cdot (c - b) = 0$

$\underset{\sim}{b} \cdot \underset{\sim}{c} - \underset{\sim}{a} \cdot \underset{\sim}{c} - \underset{\sim}{b} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

$\underset{\sim}{b} \cdot \underset{\sim}{b} = \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{c} - \underset{\sim}{a} \cdot c$

$|\underset{\sim}{b}|^2 = \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{c} - \underset{\sim}{c} \cdot \underset{\sim}{a}$ shown

b



$\overrightarrow{CB} = \overrightarrow{AB} - \overrightarrow{AC}$

$\overrightarrow{CB} \cdot \overrightarrow{CB} = \left(\overrightarrow{AB} - \overrightarrow{AC}\right) \cdot \left(\overrightarrow{AB} - \overrightarrow{AC}\right)$

$= \overrightarrow{AB} \cdot \overrightarrow{AB} - \overrightarrow{AC} \cdot \overrightarrow{AB} - \overrightarrow{AB} \cdot \overrightarrow{AC} + \overrightarrow{AC} \cdot \overrightarrow{AC}$

$|\overrightarrow{CB}|^2 = |\overrightarrow{AB}|^2 + |\overrightarrow{AC}|^2 - 2\overrightarrow{AB} \cdot \overrightarrow{AC}$

$= |\overrightarrow{AB}|^2 + |\overrightarrow{AC}|^2 - 2\,|\overrightarrow{AB}| \cdot |\overrightarrow{AC}| \cos(\theta)$

8 $r(t) = (2 + 3\cos(t))\,\underset{\sim}{i} + (4 + 2\sin(t))\,\underset{\sim}{j}$

(1) $x = 2 + 3\cos(t)$

(2) $y = 4 + 2\sin(t)$

(1) $\Rightarrow \cos(t) = \dfrac{x-2}{3}$

(2) $\Rightarrow \sin(t) = \dfrac{y-4}{2}$

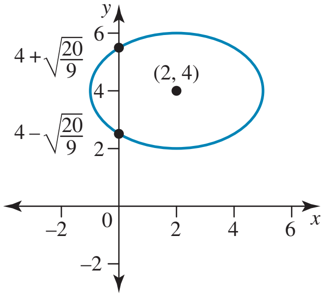
$\cos^2(t) + \sin^2(t) = 1$

$\dfrac{(x-2)^2}{9} + \dfrac{(y-4)^2}{4} = 1$

Ellipse, centre $(2, 4)$

Domain $[-1, 5]$

Range $[2, 6]$



Technology active: multiple choice

9 $\underset{\sim}{a} = \frac{1}{3}\left(2\underset{\sim}{i} + \sqrt{3}\underset{\sim}{j} + 2\underset{\sim}{k}\right), \quad |\underset{\sim}{a}| = \frac{1}{3}(4 + 3 + 4) = \frac{10}{3} \neq 1$

All of the other are unit vectors as they have a magnitude of 1.

The correct answer is **D**.

10 $\underset{\sim}{a} = -2\underset{\sim}{i} + \underset{\sim}{j} - 2\,\underset{\sim}{k}, \quad |\underset{\sim}{a}| = \sqrt{4 + 1 + 4} = 3$

$\hat{\underset{\sim}{a}} = \frac{1}{3}\left(-2\underset{\sim}{i} + \underset{\sim}{j} - 2\,\underset{\sim}{k}\right)$

$-9\hat{\underset{\sim}{a}} = 3\left(2\underset{\sim}{i} - \underset{\sim}{j} + 2\,\underset{\sim}{k}\right)$

The correct answer is **B**.

11 $\underset{\sim}{a} = \underset{\sim}{i} - m\underset{\sim}{j} - 2\,\underset{\sim}{k}\ , \quad \underset{\sim}{b} = -2\underset{\sim}{i} + \underset{\sim}{j} - n\,\underset{\sim}{k}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = -2 - m + 2n = 0$

$2n - m = 2$

Only satisfied by $m = 2$ and $n = 2$.

The correct answer is **A**.

12 $\underset{\sim}{a} = \underset{\sim}{i} - \underset{\sim}{j} + t\underset{\sim}{k}$

$|\underset{\sim}{a}| = \sqrt{1 + 1 + t^2} = \sqrt{2 + t^2} = 4$

$2 + t^2 = 16$

$t^2 = 14$

$t = \pm\sqrt{14}$

The correct answer is **D**.

13 $\underset{\sim}{a} = n\underset{\sim}{i} + \sqrt{m}\underset{\sim}{j} - n\underset{\sim}{k} \quad \underset{\sim}{b} = 4\underset{\sim}{i} - 2\sqrt{m}\,\underset{\sim}{j} - \sqrt{m}\underset{\sim}{k}$

$\underset{\sim}{b} = -2\underset{\sim}{a}$

$\underset{\sim}{i}: \quad 4 = -2n \Rightarrow n = -2$

$\underset{\sim}{k}: \quad -\sqrt{m} = 2n = -4 \Rightarrow m = 16$

The correct answer is **C**.

14 $\underset{\sim}{a} = \frac{1}{2}\left(-\underset{\sim}{i} + \underset{\sim}{j} + z\underset{\sim}{k}\right)$

$\cos\left(135^\circ\right) = \dfrac{\frac{z}{2}}{\frac{1}{2}\sqrt{2 + z^2}} = -\dfrac{\sqrt{2}}{2}, \quad z < 0$

$2z = -\sqrt{2}\sqrt{2 + z^2}$

$4z^2 = 2\left(2 + z^2\right) = 4 + 2z^2$

$2z^2 = 4$

$z^2 = 2$

$z = -\sqrt{2}$ since $z < 0$

The correct answer is **A**.

15 $\underset{\sim}{c} - \underset{\sim}{b} = t\left(\underset{\sim}{a} - \underset{\sim}{b}\right)$

$\underset{\sim}{c} = t\underset{\sim}{a} + (1 - t)\,\underset{\sim}{b}$

$\underset{\sim}{c} - \underset{\sim}{b} = t\left(\underset{\sim}{a} - \underset{\sim}{b}\right)$

$\overrightarrow{OC} - \overrightarrow{OB} = t\left(\overrightarrow{OA} - \overrightarrow{OB}\right)$

$\overrightarrow{BC} = t\overrightarrow{BA}$

So A, B and C are collinear and $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are linearly dependent.

The correct answer is **C**.

16 $\frac{1}{2}\left(\underset{\sim}{a}+\underset{\sim}{b}\right)\cdot\left(\underset{\sim}{b}-\underset{\sim}{a}\right)=0$

$\frac{1}{2}\left(\underset{\sim}{a}\cdot\underset{\sim}{b}+\underset{\sim}{b}\cdot\underset{\sim}{b}-\underset{\sim}{a}\cdot\underset{\sim}{a}-\underset{\sim}{a}\cdot\underset{\sim}{b}\right)=0$

$\frac{1}{2}\left(|\underset{\sim}{b}|^2-|\underset{\sim}{a}|^2\right)=0$

$|\underset{\sim}{b}|=|\underset{\sim}{a}|$

The correct answer is **E**.

17 $\underset{\sim}{a}\cdot\underset{\sim}{b}=0,\ \ \underset{\sim}{a}\cdot\underset{\sim}{a}=1$ and $\underset{\sim}{b}\cdot\underset{\sim}{b}=4$

$\underset{\sim}{a}$ is perpendicular to the vector $\underset{\sim}{b}$ and $|\underset{\sim}{a}+\underset{\sim}{b}|=\sqrt{5}$.

The correct answer is **E**.

18 $\underset{\sim}{r}(t)=\left(t+\frac{1}{t}\right)\underset{\sim}{i}+\left(t^2+\frac{1}{t^2}\right)\underset{\sim}{j}$

$x=t+\frac{1}{t}\ ,\ \ y=t^2+\frac{1}{t^2}$

$x^2=t^2+2+\frac{1}{t^2}$

$x^2=y+2$

$y=x^2-2$

Parabolic path

The correct answer is **B**.

Technology active: extended response

19 $\underset{\sim}{r}(t)=\frac{3}{2}\left(e^{2t}+e^{-2t}\right)\underset{\sim}{j}+\frac{5}{2}\left(e^{2t}-e^{-2t}\right)\underset{\sim}{j}$

(1) $x=\frac{3}{2}\left(e^{2t}+e^{-2t}\right)$ (2) $y=\frac{5}{2}\left(e^{2t}-e^{-2t}\right)$

Square both equations

$x^2=\frac{9}{4}\left(e^{4t}+2+e^{-4t}\right)$

$y^2=\frac{25}{4}\left(e^{4t}-2+e^{-4t}\right)$

$\frac{4x^2}{9}=e^{4t}+2+e^{-4t}$

$\frac{4y^2}{25}=e^{4t}-2+e^{-4t}$

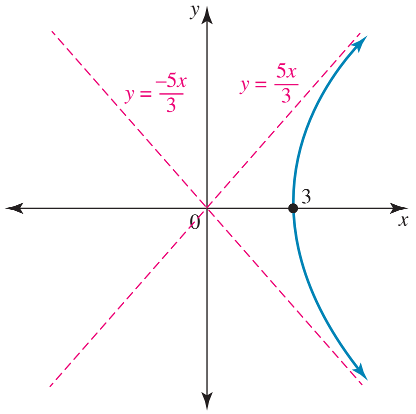
Subtract

$\frac{4x^2}{9}-\frac{4y^2}{25}=4$

So $\frac{x^2}{9}-\frac{y^2}{25}=1$ asymptotes $y=\pm\frac{5x}{3}$

but $t\geq 0$ So $x\geq 0$

It is only right branch of the hyperbola

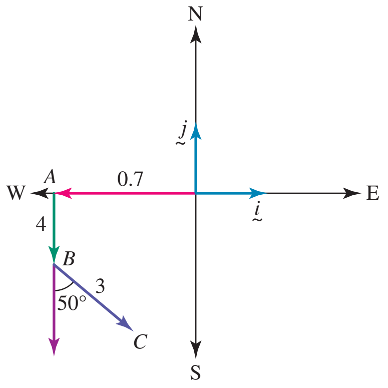


Domain $[3,\infty)$, range R

20 Units in km

$\overrightarrow{OA}=-0.7\underset{\sim}{i}$

$\overrightarrow{AB}=-4\underset{\sim}{j}$



$\overrightarrow{BC}=3\sin(50°)\underset{\sim}{i}-3\cos(50°)\underset{\sim}{j}$

$\overrightarrow{CM}=0.05\underset{\sim}{k}$

$\overrightarrow{OM}=\overrightarrow{OA}+\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CM}$

$=(3\sin(50°)-0.7)\underset{\sim}{i}+(4+3\cos(50°))\underset{\sim}{j}+0.05\underset{\sim}{k}$

$=1.598\underset{\sim}{i}-5.928\underset{\sim}{j}+0.05\underset{\sim}{k}$

$\left|\overrightarrow{OM}\right|=\sqrt{1.598^2+(-5.928)^2+0.05^2}$

$=6.1402$ km

So 6140 metres

3.7 Exam questions

1 **a** $\underset{\sim}{a}=2\underset{\sim}{i}-3\underset{\sim}{j}+\underset{\sim}{k},\ \underset{\sim}{b}=\underset{\sim}{i}+m\underset{\sim}{j}-\underset{\sim}{k}$

$|\underset{\sim}{b}|=\sqrt{1+m^2+1}=\sqrt{2+m^2},\ \underset{\sim}{a}\cdot\underset{\sim}{b}=2-3m-1=1-3m$

$\left(\underset{\sim}{a}\cdot\hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}=\left(\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\hat{\underset{\sim}{b}}|}\right)\hat{\underset{\sim}{b}}$

$=\frac{1-3m}{2+m^2}\left(\underset{\sim}{i}+m\underset{\sim}{j}-\underset{\sim}{k}\right)$

$=-\frac{11}{18}\left(\underset{\sim}{i}+m\underset{\sim}{j}-\underset{\sim}{k}\right)$

$\frac{1-3m}{2+m^2}=-\frac{11}{18}$

$18(1-3m)=-11\left(2+m^2\right)$

$18-54m=-22-11m^2$

$11m^2-54m+40=0$

$(11m-10)(m-4)=0$

$m=4,\ m=\frac{10}{11},\ m\in Z$

$m=4$ only

Award 1 mark for the correct scalar product and magnitude.

Award 1 mark for solving for m.

Award 1 mark for the final correct value of m.

b $\underset{\sim}{a}-\left(\underset{\sim}{a}\cdot\hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}=\left(2\underset{\sim}{i}-3\underset{\sim}{j}+\underset{\sim}{k}\right)+\frac{11}{18}\left(\underset{\sim}{i}+4\underset{\sim}{j}-\underset{\sim}{k}\right)$

$\underset{\sim}{a}-\left(\underset{\sim}{a}\cdot\hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}=\frac{1}{18}\left[\left(36\underset{\sim}{i}-54\underset{\sim}{j}+18\underset{\sim}{k}\right)+\left(11\underset{\sim}{i}+44\underset{\sim}{j}-11\underset{\sim}{k}\right)\right]$

$=\frac{1}{18}\left(47\underset{\sim}{i}-10\underset{\sim}{j}+7\underset{\sim}{k}\right)$

Award 1 mark for the correct vector.

2 $\underset{\sim}{a}=\underset{\sim}{i}+2\underset{\sim}{j}+2\underset{\sim}{k},\ \underset{\sim}{b}=2\underset{\sim}{i}-4\underset{\sim}{j}+4\underset{\sim}{k}$

$|\underset{\sim}{a}|=\sqrt{1+4+4}=3,\ |\underset{\sim}{b}|=\sqrt{4+16+16}=6$

$\underset{\sim}{a}\cdot\underset{\sim}{b}=2-8+8=2$

$$\begin{aligned}\cos(\theta) &= \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}|\,|\underset{\sim}{b}|}\\ &= \frac{2}{3\times 6}\\ &= \frac{1}{9}\end{aligned}$$

$$\Rightarrow \sin(\theta) = \frac{\sqrt{80}}{9}$$

$$\begin{aligned}\sin(2\theta) &= 2\sin(\theta)\cos(\theta)\\ &= 2\times\frac{\sqrt{80}}{9}\times\frac{1}{9}\\ &= \frac{8\sqrt{5}}{81}\end{aligned}$$

The correct answer is **D**.

3 **a** $A(2, -1, 3)$, $B(4, -2, 1)$, $C(a, b, c)$, $D(4, 3, -1)$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$

$\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD} = (a-4)\underset{\sim}{i} + (b-3)\underset{\sim}{j} + (c+1)\underset{\sim}{k}$

$\overrightarrow{AB} = \overrightarrow{DC}$

$a - 4 = 2,\ b - 3 = -1,\ c + 1 = -2$

$a = 6,\ b = 2,\ c = -3$

Award 1 mark for the correct vectors.

Award 1 mark for all correct values.

b $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$

$\quad = 2\underset{\sim}{i} + 4\underset{\sim}{j} - 4\underset{\sim}{k}$

$\left|\overrightarrow{AB}\right| = \sqrt{4+1+4} = 3, \left|\overrightarrow{AD}\right| = \sqrt{4+16+16} = 6$

$\overrightarrow{AB}\cdot\overrightarrow{AD} = 4 - 4 + 8 = 8$

$$\cos(\theta) = \frac{\overrightarrow{AB}\cdot\overrightarrow{AD}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AD}\right|} = \frac{8}{6\times 3} = \frac{4}{9}$$

Award 1 mark for the correct method.

Award 1 mark for the correct result.

c $A = 2\times\frac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AD}\right|\sin(\theta) = 3\times 6\times\frac{\sqrt{65}}{9} = 2\sqrt{65}$

Award 1 mark for the correct method.

Award 1 mark for the correct result.

d $\underset{\sim}{h} = 6\underset{\sim}{i} + 2\underset{\sim}{j} + 5\underset{\sim}{k}$

$\underset{\sim}{h}\cdot\overrightarrow{AB} = 12 - 2 - 10 = 0$

$\underset{\sim}{h}\cdot\overrightarrow{AD} = 12 + 8 - 20 = 0$

So that $\underset{\sim}{h}$ is perpendicular both $\overrightarrow{AB}$ and $\overrightarrow{AD}$

$|\underset{\sim}{h}| = \sqrt{36+4+25} = \sqrt{65}$

$$\hat{\underset{\sim}{h}} = \frac{1}{\sqrt{65}}\left(6\underset{\sim}{i} + 2\underset{\sim}{j} + 5\underset{\sim}{k}\right)$$

Award 1 mark for showing $\underset{\sim}{h}$ is perpendicular to $\overrightarrow{AB}$.

Award 1 mark for showing $\underset{\sim}{h}$ is perpendicular to $\overrightarrow{AD}$.

Award 1 mark for the correct unit vector.

e The height of the pyramid H, is the scalar resolute of $\overrightarrow{AP}$ onto $\underset{\sim}{h}$

$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}$

$$\overrightarrow{AP}\cdot\underset{\sim}{h} = 12 - 6 + 30 = 36,\ H = \frac{\overrightarrow{AP}\cdot\underset{\sim}{h}}{|\underset{\sim}{h}|} = \frac{36}{\sqrt{65}}$$

$$V = \frac{1}{3}AH = \frac{1}{3}\times 2\sqrt{65}\times\frac{36}{\sqrt{65}} = 24$$

Award 1 mark for the correct scalar resolute.

Award 1 mark for the correct volume.

4 $\underset{\sim}{r}(t) = \sec(t)\underset{\sim}{i} + \frac{\sqrt{2}}{2}\tan(t)\underset{\sim}{j}, t\in R$

$$x = \sec(t),\ y = \frac{\sqrt{2}}{2}\tan(t)$$

$$\begin{aligned}x^2 - 2y^2 &= \sec^2(t) - 2\times\frac{2}{4}\tan^2(t)\\ &= \sec^2(t) - \tan^2(t)\\ &= 1\end{aligned}$$

Award 1 mark for stating the correct equations for x and y.

Award 1 mark for correctly substituting x and y into the expression to show the required result.

5 $\underset{\sim}{b} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k},\ \underset{\sim}{c} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k},\ \underset{\sim}{d} = a\underset{\sim}{i} - 2\underset{\sim}{j},$

$\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC} = -\underset{\sim}{i} + \underset{\sim}{k}, \left|\overrightarrow{CB}\right| = \sqrt{2}$

$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = (a-2)\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k},$

$\left|\overrightarrow{CD}\right| = \sqrt{(a-2)^2 + 2} = \sqrt{a^2 - 4a + 6}$

$\overrightarrow{CB}\cdot\overrightarrow{CD} = -(a-2) - 1 = 1 - a$

$$\begin{aligned}\cos\left(\frac{\pi}{3}\right) &= \frac{1}{2}\\ \frac{\overrightarrow{CB}\cdot\overrightarrow{CD}}{\left|\overrightarrow{CB}\right|\left|\overrightarrow{CD}\right|} &= \frac{1}{2}\\ \frac{1-a}{\sqrt{2}\sqrt{a^2-4a+6}} &= \frac{1}{2}\end{aligned}$$

$\sqrt{2}\sqrt{a^2-4a+6} = 2(1-a)$, but $1 - a > 0$

$2\left(a^2 - 4a + 6\right) = 4 - 8a + 4a^2$

$2a^2 - 8 = 0$

$a^2 = 4 \Rightarrow a = \pm 2$, but $1 - a > 0$

$a = -2$

Award 1 mark for the correct vectors.

Award 1 mark for the scalar product.

Award 1 mark for solving.

Award 1 mark for the correct solution for a.

Topic 4 — Vector equations of lines and planes

4.2 Vector cross products

4.2 Exercise

1 a $\begin{vmatrix}-3 & 4\\ 2 & -1\end{vmatrix}$

$$\begin{aligned}&= -3\times -1 - 2\times 4\\ &= 3 - 8\\ &= -5\end{aligned}$$

b $\begin{vmatrix}4 & -1\\ -3 & 2\end{vmatrix}$

$$\begin{aligned}&= 4\times 2\ -\ -3\times -1\\ &= 8 - 3\\ &= 5\end{aligned}$$

2 a $\begin{vmatrix}5 & 2 & 1\\ 3 & 4 & -2\\ 1 & 2 & 3\end{vmatrix}$

$$\begin{aligned}&= 5\begin{vmatrix}4 & -2\\ 2 & 8\end{vmatrix} - 2\begin{vmatrix}3 & -2\\ 1 & 3\end{vmatrix} + 1\begin{vmatrix}3 & 4\\ 1 & 2\end{vmatrix}\\ &= 5\,(12+4) - 2\,(9+2) + 1\,(6-4)\\ &= 80 - 22 + 2\\ &= 60\end{aligned}$$

b $\begin{vmatrix}1 & -2 & 3\\ 2 & 4 & 2\\ 5 & 3 & 1\end{vmatrix}$

$$\begin{aligned}&= 1\begin{vmatrix}4 & 2\\ 3 & 1\end{vmatrix} + 2\begin{vmatrix}2 & 2\\ 5 & 1\end{vmatrix} + 3\begin{vmatrix}2 & 4\\ 5 & 3\end{vmatrix}\\ &= (4-6) + 2\,(2-10) + 3\,(6-20)\\ &= -2 - 16 - 42\\ &= -60\end{aligned}$$

3 a $\underset{\sim}{u} = 2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k},\ \underset{\sim}{v} = -\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$

$$\begin{aligned}\underset{\sim}{u}\times\underset{\sim}{v} &= \begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ 2 & -1 & 4\\ -1 & 2 & -3\end{vmatrix}\\ &= \underset{\sim}{i}\begin{vmatrix}-1 & 4\\ 2 & -3\end{vmatrix} - \underset{\sim}{j}\begin{vmatrix}2 & 4\\ -1 & -3\end{vmatrix} + \underset{\sim}{k}\begin{vmatrix}2 & -1\\ -1 & 2\end{vmatrix}\\ &= -5\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}\end{aligned}$$

b $\left(\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}\right)\times\left(2\underset{\sim}{i} + \underset{\sim}{j}\right)$

$$\begin{aligned}&= \begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ 1 & 1 & -3\\ 2 & 1 & 0\end{vmatrix}\\ &= \underset{\sim}{i}\begin{vmatrix}1 & -3\\ 1 & 0\end{vmatrix} - \underset{\sim}{j}\begin{vmatrix}1 & -3\\ 2 & 0\end{vmatrix} + \underset{\sim}{k}\begin{vmatrix}1 & 1\\ 2 & 1\end{vmatrix}\\ &= 3\underset{\sim}{i} - 6\underset{\sim}{j} - \underset{\sim}{k}\end{aligned}$$

c $\underset{\sim}{r} = 2\underset{\sim}{i} - 3\underset{\sim}{j}\quad \underset{\sim}{s} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}$

$$\begin{aligned}\underset{\sim}{r}\times\underset{\sim}{s} &= \begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ 2 & -3 & 0\\ 3 & 2 & 4\end{vmatrix}\\ &= \underset{\sim}{i}\begin{vmatrix}-3 & 0\\ 2 & 4\end{vmatrix} - \underset{\sim}{j}\begin{vmatrix}2 & 0\\ 3 & 4\end{vmatrix} + \underset{\sim}{k}\begin{vmatrix}2 & -3\\ 3 & 2\end{vmatrix}\\ &= -12\underset{\sim}{i} - 8\underset{\sim}{j} + 13\underset{\sim}{k}\end{aligned}$$

4 a Let $\underset{\sim}{a} = 3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k},\ \underset{\sim}{b} = 4\underset{\sim}{i} - 2\underset{\sim}{j} + 5\underset{\sim}{k}$

$$\begin{aligned}\underset{\sim}{a}\times\underset{\sim}{b} &= \begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ 3 & -1 & 2\\ 4 & -2 & 5\end{vmatrix}\\ &= \underset{\sim}{i}\begin{vmatrix}-1 & 2\\ -2 & 5\end{vmatrix} - \underset{\sim}{j}\begin{vmatrix}3 & 2\\ 4 & 5\end{vmatrix} + \underset{\sim}{k}\begin{vmatrix}3 & -1\\ 4 & -2\end{vmatrix}\\ &= -\underset{\sim}{i} - 7\underset{\sim}{j} - 2k\\ |\underset{\sim}{a}\times\underset{\sim}{b}| &= \sqrt{(-1)^2 + (-7)^2 + (-2)^2}\\ &= \sqrt{54} = 3\sqrt{6}\\ \underset{\sim}{\hat{n}} &= \pm\frac{\underset{\sim}{a}\times\underset{\sim}{b}}{|\underset{\sim}{a}\times\underset{\sim}{b}|} = \pm\frac{1}{3\sqrt{6}}\left(\underset{\sim}{i} + 7\underset{\sim}{j} + 2\underset{\sim}{k}\right)\end{aligned}$$

b Let $\underset{\sim}{a} = 5\underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k}\quad b = -2\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$

$$\begin{aligned}\underset{\sim}{a}\times\underset{\sim}{b} &= \begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ 5 & -2 & -3\\ -2 & 3 & 1\end{vmatrix}\\ &= \underset{\sim}{i}\begin{vmatrix}-2 & -3\\ 3 & 1\end{vmatrix} - \underset{\sim}{j}\begin{vmatrix}5 & -3\\ -2 & 1\end{vmatrix} + \underset{\sim}{k}\begin{vmatrix}5 & -2\\ -2 & 3\end{vmatrix}\\ &= 7\underset{\sim}{i} + \underset{\sim}{j} + 11\underset{\sim}{k}\\ |\underset{\sim}{a}\times\underset{\sim}{b}| &= \sqrt{7^2 + 1^2 + 11^2}\\ &= \sqrt{171} = \sqrt{9\times 19}\\ &= 3\sqrt{19}\\ \underset{\sim}{\hat{n}} &= \pm\frac{\underset{\sim}{a}\times\underset{\sim}{b}}{|\underset{\sim}{a}\times\underset{\sim}{b}|} = \pm\frac{1}{3\sqrt{19}}\left(7\underset{\sim}{i} + \underset{\sim}{j} + 11\underset{\sim}{k}\right)\end{aligned}$$

c Let $\underset{\sim}{a} = -5\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}\quad \underset{\sim}{b} = 2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$

$$\begin{aligned}\underset{\sim}{a}\times\underset{\sim}{b} &= \begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ -5 & 2 & 3\\ 2 & -1 & -2\end{vmatrix}\\ &= \underset{\sim}{i}\begin{vmatrix}2 & 3\\ -1 & -2\end{vmatrix} - \underset{\sim}{j}\begin{vmatrix}-5 & 3\\ 2 & -2\end{vmatrix} + \underset{\sim}{k}\begin{vmatrix}-5 & 2\\ 2 & -1\end{vmatrix}\\ &= -\underset{\sim}{i} - 4\underset{\sim}{j} + \underset{\sim}{k}\\ |\underset{\sim}{a}\times\underset{\sim}{b}| &= \sqrt{(-1)^2 + (-4)^2 + 1^2}\\ &= \sqrt{18}\\ &= 3\sqrt{2}\\ \underset{\sim}{\hat{n}} &= \pm\frac{\underset{\sim}{a}\times\underset{\sim}{b}}{|\underset{\sim}{a}\times\underset{\sim}{b}|} = \pm\frac{1}{3\sqrt{2}}\left(\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}\right)\end{aligned}$$

5 a $\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}, \quad \underset{\sim}{b} = \underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}, \quad \underset{\sim}{c} = 4\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}$

$\underset{\sim}{b} + \underset{\sim}{c} = 5\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}$

$$\begin{aligned}
\underset{\sim}{a} \times \left(\underset{\sim}{b} + \underset{\sim}{c}\right) &= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 3 & -2 & 4 \\ 5 & -2 & 2 \end{vmatrix} \\
&= \underset{\sim}{i}\begin{vmatrix} -2 & 4 \\ -2 & 2 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 3 & 4 \\ 5 & 2 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 3 & -2 \\ 5 & -2 \end{vmatrix} \\
&= 4\underset{\sim}{i} + 14\underset{\sim}{j} + 4\underset{\sim}{k}
\end{aligned}$$

$$\begin{aligned}
\underset{\sim}{a} \times \underset{\sim}{b} &= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 3 & -2 & 4 \\ 1 & 1 & -3 \end{vmatrix} \\
&= \underset{\sim}{i}\begin{vmatrix} -2 & 4 \\ 1 & -3 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 3 & 4 \\ 1 & -3 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} \\
&= 2\underset{\sim}{i} + 13\underset{\sim}{j} + 5\underset{\sim}{k}
\end{aligned}$$

$$\begin{aligned}
\underset{\sim}{a} \times \underset{\sim}{c} &= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 3 & -2 & 4 \\ 4 & -3 & 5 \end{vmatrix} \\
&= \underset{\sim}{i}\begin{vmatrix} -2 & 4 \\ -3 & 5 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 3 & -2 \\ 4 & -3 \end{vmatrix} \\
&= 2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}
\end{aligned}$$

So $\underset{\sim}{a} \times \underset{\sim}{b} + \underset{\sim}{a} \times \underset{\sim}{c} = 4\underset{\sim}{i} + 14\underset{\sim}{j} + 4\underset{\sim}{k}$

$= \underset{\sim}{a} \times \left(\underset{\sim}{b} + \underset{\sim}{c}\right)$

b $\underset{\sim}{b} - \underset{\sim}{c} = -3\underset{\sim}{i} + 4\underset{\sim}{j} - 8\underset{\sim}{k}$

$$\begin{aligned}
\underset{\sim}{a} \times \left(\underset{\sim}{b} - \underset{\sim}{c}\right) &= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 3 & -2 & 2 \\ -3 & 4 & -8 \end{vmatrix} \\
&= 12\underset{\sim}{j} + 6\underset{\sim}{k}
\end{aligned}$$

$$\begin{aligned}
\underset{\sim}{a} \times \underset{\sim}{b} - \underset{\sim}{a} \times \underset{\sim}{c} &= \left(2\underset{\sim}{i} + 13\underset{\sim}{j} + 5\underset{\sim}{k}\right) - \left(2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}\right) \\
&= 12\underset{\sim}{j} + 6\underset{\sim}{k}
\end{aligned}$$

So $\underset{\sim}{a} \times \left(\underset{\sim}{b} - \underset{\sim}{c}\right) = \underset{\sim}{a} \times \underset{\sim}{b} - \underset{\sim}{a} \times \underset{\sim}{c}$

c The cross product is distributive over addition and subtraction.

d $\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}, \quad \underset{\sim}{b} = \underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$

$\underset{\sim}{a} + \underset{\sim}{b} = 4\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}, \quad \underset{\sim}{a} - \underset{\sim}{b} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 7\underset{\sim}{k}$

$$\begin{aligned}
\left(\underset{\sim}{a} + \underset{\sim}{b}\right) \times \left(\underset{\sim}{a} - \underset{\sim}{b}\right) &= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 4 & -1 & 1 \\ 2 & -3 & 7 \end{vmatrix} \\
&= \underset{\sim}{i}\begin{vmatrix} -1 & 1 \\ -3 & 7 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 4 & 1 \\ 2 & 7 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 4 & -1 \\ 2 & -3 \end{vmatrix} \\
&= -4\underset{\sim}{i} - 26\underset{\sim}{j} - 10\underset{\sim}{k}
\end{aligned}$$

$\underset{\sim}{a} \times \underset{\sim}{b} = 2\underset{\sim}{i} + 13\underset{\sim}{j} + 5\underset{\sim}{k}$ so $\underset{\sim}{b} \times \underset{\sim}{a} = -\underset{\sim}{a} \times \underset{\sim}{b}$

$= -2\underset{\sim}{i} - 13\underset{\sim}{j} - 5\underset{\sim}{k}$

$\underset{\sim}{b} \times \underset{\sim}{a} - \underset{\sim}{a} \times \underset{\sim}{b}$

$= 2(\underset{\sim}{b} \times \underset{\sim}{a})$

$= -2(\underset{\sim}{a} \times \underset{\sim}{b})$

$= -4\underset{\sim}{i} - 26\underset{\sim}{j} - 10\underset{\sim}{k}$

Now $\left(\underset{\sim}{a} + \underset{\sim}{b}\right) \times \left(\underset{\sim}{a} - \underset{\sim}{b}\right)$

$= \underset{\sim}{a} \times \left(\underset{\sim}{a} - \underset{\sim}{b}\right) + \underset{\sim}{b} \times \left(\underset{\sim}{a} - \underset{\sim}{b}\right)$

$= \underset{\sim}{a} \times \underset{\sim}{a} - \underset{\sim}{a} \times \underset{\sim}{b} + \underset{\sim}{b} \times \underset{\sim}{a} - \underset{\sim}{b} \times \underset{\sim}{b}$

$= -\underset{\sim}{a} \times \underset{\sim}{b} + \underset{\sim}{b} \times \underset{\sim}{a}$

$= -\underset{\sim}{a} \times \underset{\sim}{b} - \underset{\sim}{a} \times \underset{\sim}{b}$

$= -2(\underset{\sim}{a} \times \underset{\sim}{b})$

$= -4\underset{\sim}{i} - 26\underset{\sim}{j} - 10\underset{\sim}{k}$

e
$$\begin{aligned}
\underset{\sim}{a} \times \underset{\sim}{c} &= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 3 & -2 & 4 \\ 4 & -3 & 5 \end{vmatrix} \\
&= \underset{\sim}{i}\begin{vmatrix} -2 & 4 \\ -3 & 5 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 3 & -2 \\ 4 & -3 \end{vmatrix} \\
&= 2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}
\end{aligned}$$

So $\left(\underset{\sim}{a} + \underset{\sim}{c}\right) \times \left(\underset{\sim}{a} - \underset{\sim}{c}\right)$

$= \underset{\sim}{a} \times \left(\underset{\sim}{a} - \underset{\sim}{c}\right) + \underset{\sim}{c} \times \left(\underset{\sim}{a} - \underset{\sim}{c}\right)$

$= \underset{\sim}{a} \times \underset{\sim}{a} - \underset{\sim}{a} \times \underset{\sim}{c} + \underset{\sim}{c} \times \underset{\sim}{a} - \underset{\sim}{c} \times \underset{\sim}{c}$

$= -\underset{\sim}{a} \times \underset{\sim}{c} + \underset{\sim}{c} \times \underset{\sim}{a}$

$= -\underset{\sim}{a} \times \underset{\sim}{c} - \underset{\sim}{a} \times \underset{\sim}{c}$

$= -2(\underset{\sim}{a} \times \underset{\sim}{c})$

$= -4\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}$

6 a $A(2, -1, 3)\ B(1, -2, 4)\ C(4, 3, -1)$

$\overrightarrow{OA} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$

$\overrightarrow{OB} = \underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}$

$\overrightarrow{OC} = 4\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} \qquad \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$\quad = -\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k} \qquad\qquad = 3\underset{\sim}{i} + 5\underset{\sim}{j} - 5\underset{\sim}{k}$

$$\begin{aligned}
\overrightarrow{AB} \times \overrightarrow{BC} &= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -1 & -1 & 1 \\ 3 & 5 & -5 \end{vmatrix} \\
&= \underset{\sim}{i}\begin{vmatrix} -1 & 1 \\ 5 & -5 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} -1 & 1 \\ 3 & -5 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} -1 & -1 \\ 3 & 5 \end{vmatrix} \\
&= -2\underset{\sim}{j} - 2\underset{\sim}{k}
\end{aligned}$$

$$\begin{aligned}
\left|\overrightarrow{AB} \times \overrightarrow{BC}\right| &= \sqrt{(-2)^2 + (-2)^2} = \sqrt{8} \\
&= 2\sqrt{2}
\end{aligned}$$

$$\begin{aligned}
\text{Area of triangle} &= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| \\
&= \sqrt{2}
\end{aligned}$$

b $P(3, 1, -1)\ Q(1, 0, 2)\ R(2, 2, -3)$

$\overrightarrow{OP} = 3\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$

$\overrightarrow{OQ} = \underset{\sim}{i} + 2\underset{\sim}{k}$

$\overrightarrow{OR} = 2\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} \qquad \overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ}$

$\quad = -2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k} \qquad = \underset{\sim}{i} + 2\underset{\sim}{j} - 5\underset{\sim}{k}$

$$\begin{aligned}
\overrightarrow{PQ} \times \overrightarrow{QR} &= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -2 & -1 & 3 \\ 1 & 2 & -5 \end{vmatrix} \\
&= \underset{\sim}{i}\begin{vmatrix} -1 & 3 \\ 2 & -5 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} -2 & 3 \\ 1 & 5 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} -2 & -1 \\ 1 & 2 \end{vmatrix} \\
&= -\underset{\sim}{i} - 7\underset{\sim}{j} - 3\underset{\sim}{k}
\end{aligned}$$

$$\left|\overrightarrow{PQ} \times \overrightarrow{QR}\right| = \sqrt{(-1)^2 + (-7)^2 + (-3)^2} = \sqrt{59}$$

$$\begin{aligned}
\text{Area of triangle} &= \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{QR}\right| \\
&= \frac{1}{2}\sqrt{59}
\end{aligned}$$

c $A(-4, 2, -1)\ B(-2, 1, -3)\ C(-3, 2, 1)$

$\overrightarrow{OA} = -4\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$

$\overrightarrow{OB} = -2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$

$\overrightarrow{OC} = -3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} \qquad \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$\quad = 2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k} \qquad = -\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}$

$$\overrightarrow{AB}\times\overrightarrow{BC}=\begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ 2 & -1 & -2\\ -1 & 1 & 4\end{vmatrix}$$

$$=\underset{\sim}{i}\begin{vmatrix}-1 & -2\\ 1 & 4\end{vmatrix}-\underset{\sim}{j}\begin{vmatrix}2 & -2\\ -1 & 4\end{vmatrix}+\underset{\sim}{k}\begin{vmatrix}2 & -1\\ -1 & 1\end{vmatrix}$$

$$=-2\underset{\sim}{i}-6\underset{\sim}{j}+\underset{\sim}{k}$$

$$\left|\overrightarrow{AB}\times\overrightarrow{BC}\right|=\sqrt{(-2)^2+(-6)^2+1^2}=\sqrt{41}$$

$$\text{Area of triangle }=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$$

$$=\frac{1}{2}\sqrt{41}$$

7 $\underset{\sim}{a}=5\underset{\sim}{i}-\underset{\sim}{j}+7\underset{\sim}{k}\quad \underset{\sim}{b}=5\underset{\sim}{i}+3\underset{\sim}{j}+4\underset{\sim}{k}$

a $|\underset{\sim}{a}|=\sqrt{5^2+(-1)^2+7^2}=\sqrt{75}$

$=5\sqrt{3}$

b $|\underset{\sim}{b}|=\sqrt{5^2+3^2+4^2}=\sqrt{50}$

$=5\sqrt{2}$

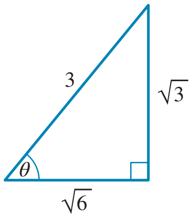
c $\underset{\sim}{a}\cdot\underset{\sim}{b}=25-3+28$

$=50$

d $\cos(\theta)=\dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}|\,|\underset{\sim}{b}|}=\dfrac{50}{5\sqrt{3}\times5\sqrt{2}}=\dfrac{2}{\sqrt{6}}\times\dfrac{\sqrt{6}}{\sqrt{6}}$

$=\dfrac{\sqrt{6}}{3}$

$\theta=\cos^{-1}\left(\dfrac{\sqrt{6}}{3}\right)$



e $\underset{\sim}{a}\times\underset{\sim}{b}=\begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ 5 & -1 & 7\\ 5 & 3 & 4\end{vmatrix}$

$=\underset{\sim}{i}\begin{vmatrix}-1 & 7\\ 3 & 4\end{vmatrix}-\underset{\sim}{j}\begin{vmatrix}5 & 7\\ 5 & 4\end{vmatrix}+\underset{\sim}{k}\begin{vmatrix}5 & -1\\ 5 & 3\end{vmatrix}$

$=-25\underset{\sim}{i}+15\underset{\sim}{j}+20\underset{\sim}{k}$

f $|\underset{\sim}{a}\times\underset{\sim}{b}|=\sqrt{(-25)^2+(15)^2+20^2}=25\sqrt{2}$

$\underset{\sim}{\hat{n}}=\dfrac{\underset{\sim}{a}\times\underset{\sim}{b}}{|\underset{\sim}{a}\times\underset{\sim}{b}|}=\dfrac{1}{\sqrt{2}}\left(-\underset{\sim}{i}+\dfrac{3}{5}\underset{\sim}{j}+\dfrac{4}{5}\underset{\sim}{k}\right)$

$=\dfrac{1}{5\sqrt{2}}\left(-5\underset{\sim}{i}+3\underset{\sim}{j}+4\underset{\sim}{k}\right)$

g $\text{RHS}=|\underset{\sim}{a}|\,|\underset{\sim}{b}|\sin(\theta)\cdot\underset{\sim}{\hat{n}}$

$=5\sqrt{3}\times5\sqrt{2}\times\dfrac{\sqrt{3}}{3}\times\dfrac{1}{5\sqrt{2}}\left(-5\underset{\sim}{i}+3\underset{\sim}{j}+4\underset{\sim}{k}\right)$

$=5\left(-5\underset{\sim}{i}+3\underset{\sim}{j}+4\underset{\sim}{k}\right)$

$=-25\underset{\sim}{i}+15\underset{\sim}{j}+20\underset{\sim}{k}=\text{LHS shown}$

h $\text{LHS}=|\underset{\sim}{a}\times\underset{\sim}{b}|^2=\left(25\sqrt{2}\right)^2$

$=1250$

$\text{RHS}=|\underset{\sim}{a}|^2|\underset{\sim}{b}|^2-\left(\underset{\sim}{a}\cdot\underset{\sim}{b}\right)^2$

$=\left(5\sqrt{3}\right)^2\left(5\sqrt{2}\right)^2-50^2$

$=75\times50-50^2$

$=1250$

$=\text{LHS shown}$

8 $\underset{\sim}{a}=5\underset{\sim}{i}-2\underset{\sim}{j}+4\underset{\sim}{k}\quad \underset{\sim}{b}=2\underset{\sim}{i}+3\underset{\sim}{j}-\underset{\sim}{k}$

a $\underset{\sim}{c}=\underset{\sim}{a}\times\underset{\sim}{b}=\begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ 5 & -2 & 4\\ 2 & 3 & -1\end{vmatrix}$

$=\underset{\sim}{i}\begin{vmatrix}-2 & 4\\ 3 & -1\end{vmatrix}-\underset{\sim}{j}\begin{vmatrix}5 & 4\\ 2 & -1\end{vmatrix}+\underset{\sim}{k}\begin{vmatrix}5 & -2\\ 2 & -1\end{vmatrix}$

$\underset{\sim}{c}=-10\underset{\sim}{i}+13\underset{\sim}{j}+19\underset{\sim}{k}$

b $\underset{\sim}{a}\cdot\underset{\sim}{b}=10-6-4=0$

c $\underset{\sim}{a}\cdot\underset{\sim}{c}=-50-26+76$

$=0$

d $\underset{\sim}{b}\cdot\underset{\sim}{c}=-20+39-19$

$=0$

e $\underset{\sim}{c}$ is perpendicular to both $\underset{\sim}{a}$ and $\underset{\sim}{b}$

f $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are all mutually perpendicular, so they are linearly independent. This can also be seen with the following working out:

$\alpha\underset{\sim}{a}+\beta\underset{\sim}{b}+\gamma\underset{\sim}{c}=0$

(1) $5\alpha+2\beta-10\gamma=0$

(2) $-2\alpha+3\beta+13\gamma=0$

(3) $4\alpha-\beta+19\gamma=0$

Only solution is $\alpha=\beta=\gamma=0$

9 **a** $\underset{\sim}{p}=x\underset{\sim}{i}+y\underset{\sim}{j}-\underset{\sim}{k},\quad \underset{\sim}{q}=4\underset{\sim}{i}-2\underset{\sim}{j}+2\underset{\sim}{k}$

$\underset{\sim}{p}\times\underset{\sim}{q}=\underset{\sim}{0}\Rightarrow\underset{\sim}{p}$ is perpendicular to $\underset{\sim}{q}$

$\underset{\sim}{j}:\ -2x=4\quad\text{so}\quad x=-2$

$\underset{\sim}{i}:\ -2y=-2\quad\text{so}\quad y=1$

OR

$\underset{\sim}{p}\times\underset{\sim}{q}=\begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ x & y & -1\\ 4 & -2 & 2\end{vmatrix}$

$=\underset{\sim}{i}\begin{vmatrix}y & -1\\ -2 & 2\end{vmatrix}-\underset{\sim}{j}\begin{vmatrix}x & -1\\ 4 & 2\end{vmatrix}+\underset{\sim}{k}\begin{vmatrix}x & y\\ 4 & -2\end{vmatrix}$

$=(2y-2)\,\underset{\sim}{i}-(2x+4)\,\underset{\sim}{j}-(2x+4y)\,\underset{\sim}{k}=\underset{\sim}{0}$

$\Rightarrow 2y-2=0\quad\Rightarrow y=1$

$2x+4=0\quad\Rightarrow x=-2$

$-2x-4y=0\quad\text{check}$ ✓

b $\underset{\sim}{p}\cdot\underset{\sim}{q}=4x-2y-2=0$

$2x-y=1\qquad(1)$

and $|\underset{\sim}{p}|=\sqrt{x^2+y^2+1}=\sqrt{14}$

$x^2+y^2+1=14$

$x^2+y^2=13\qquad(2)$

$y=(2x-1)$

$x^2+(2x-1)^2=13$

$x^2+4x^2-4x+1=13$

$5x^2-4x-12=0$

$(5x+6)(x-2)=0$

$x=\dfrac{-6}{5}\quad\text{or}\quad x=2$

$\Rightarrow y=\dfrac{-17}{5}\quad\text{or}\quad y=3$

So $x=2, y=3$ or $x=\dfrac{-6}{5}, y=\dfrac{-17}{5}$

10 a $\underset{\sim}{p} = 3\underset{\sim}{i} + y\underset{\sim}{j} - \underset{\sim}{k} \quad \underset{\sim}{q} = 4\underset{\sim}{i} + y\underset{\sim}{j} - 2\underset{\sim}{k}$

$$\begin{aligned}\underset{\sim}{p} \times \underset{\sim}{q} &= \begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ 3 & y & -1\\ 4 & y & -2\end{vmatrix}\\ &= \underset{\sim}{i}\begin{vmatrix}y & -1\\ y & -2\end{vmatrix} - \underset{\sim}{j}\begin{vmatrix}3 & -1\\ 4 & -2\end{vmatrix} + \underset{\sim}{k}\begin{vmatrix}3 & y\\ 4 & y\end{vmatrix}\\ &= -y\underset{\sim}{i} + 2\underset{\sim}{j} - y\underset{\sim}{k} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}\\ &\Rightarrow y = -3\end{aligned}$$

b $\underset{\sim}{r} = x\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k} \quad \underset{\sim}{s} = 4\underset{\sim}{i} - 4\underset{\sim}{j} + z\underset{\sim}{k}$

$$\underset{\sim}{r} \times \underset{\sim}{s} = \begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ x & 2 & -1\\ 4 & -4 & z\end{vmatrix}$$

$$= (2z - 4)\,\underset{\sim}{i} - (xz + 4)\,\underset{\sim}{j} - (4x + 8)\,\underset{\sim}{k} = \underset{\sim}{0}$$

$\underset{\sim}{i} \Rightarrow 2z - 4 = 0 \Rightarrow z = 2$

$\underset{\sim}{k} \Rightarrow 4x + 8 = 0 \Rightarrow x = -2$

Check $xz + 4 = 0$

Alternatively $\underset{\sim}{r} \times \underset{\sim}{s} = 0$

$\Rightarrow \underset{\sim}{r}$ is parallel to $\underset{\sim}{s}$

$-2\underset{\sim}{r} = \underset{\sim}{s}$

$\underset{\sim}{i}$: $-2x = 4 \Rightarrow x = -2$

$\underset{\sim}{k}$: $2 = z \Rightarrow z = 2$

c $\underset{\sim}{a} = t\underset{\sim}{i} - \underset{\sim}{j} + 2t\underset{\sim}{k} \qquad \underset{\sim}{b} = 2t\underset{\sim}{i} + 3\underset{\sim}{j} - t\underset{\sim}{k}$

$$\begin{aligned}\underset{\sim}{a} \times \underset{\sim}{b} &= \begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ t & -1 & 2t\\ 2t & 3 & -t\end{vmatrix}\\ &= \underset{\sim}{i}\begin{vmatrix}-1 & 2t\\ 3 & -t\end{vmatrix} - \underset{\sim}{j}\begin{vmatrix}t & 2t\\ 2t & -t\end{vmatrix} + \underset{\sim}{k}\begin{vmatrix}t & -1\\ 2t & 3\end{vmatrix}\\ &= -5t\underset{\sim}{i} + 5t^2\underset{\sim}{j} + 5t\underset{\sim}{k}\end{aligned}$$

$$\begin{aligned}|\underset{\sim}{a} \times \underset{\sim}{b}| &= \sqrt{(-5t)^2 + \left(5t^2\right)^2 + (5t)^2}\\ &= \sqrt{25\left(2t^2 + t^4\right)} = 10\sqrt{6}\\ &= 5\sqrt{t^2\left(2 + t^2\right)} = 10\sqrt{6}\end{aligned}$$

$$\begin{aligned}\sqrt{t^2\left(2 + t^2\right)} &= 2\sqrt{6}\\ t^2\left(2 + t^2\right) &= 24\\ t^4 + 2t^2 - 24 &= 0\\ \left(t^2 - 4\right)\left(t^2 + 6\right) &= 0\\ t^2 &= 4\\ t &= \pm 2\end{aligned}$$

11 a $\underset{\sim}{a} = \cos(\alpha)\underset{\sim}{i} + \sin(\alpha)\underset{\sim}{j}$

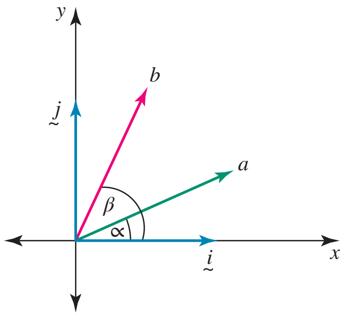
$|\underset{\sim}{a}| = \sqrt{\cos^2(\alpha) + \sin^2(\alpha)} = 1$

$\underset{\sim}{b} = \cos(\beta)\underset{\sim}{i} + \sin(\beta)\underset{\sim}{j}$

$|\underset{\sim}{b}| = \sqrt{\cos^2(\beta) + \sin^2(\beta)} = 1$

$$\begin{aligned}\underset{\sim}{a} \times \underset{\sim}{b} &= \begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ \cos(\alpha) & \sin(\alpha) & 0\\ \cos(\beta) & \sin(\beta) & 0\end{vmatrix}\\ &= \begin{vmatrix}\cos(\alpha) & \sin(\alpha)\\ \cos(\beta) & \sin(\beta)\end{vmatrix}\underset{\sim}{k}\\ &= \big(\cos(\alpha)\sin(\beta) - \cos(\beta)\sin(\alpha)\big)\underset{\sim}{k}\end{aligned}$$

Now $\underset{\sim}{a} \times \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}|\sin(\theta)\,\underset{\sim}{\hat{n}}$



$|\underset{\sim}{a}| = 1$, $|\underset{\sim}{b}| = 1\ \underset{\sim}{\hat{n}} = \underset{\sim}{k}$

Angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is $\beta - \alpha$

$\sin(\beta - \alpha) = \sin(\beta)\cos(\alpha) - \cos(\beta)\sin(\alpha)$ show

b $\underset{\sim}{F} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k} \quad A(1, -1, 2)\, B(2, -1, 4)$

$\underset{\sim}{s} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \underset{\sim}{i} + 2\underset{\sim}{k}$

$$\begin{aligned}\underset{\sim}{T} = \underset{\sim}{F} \times \underset{\sim}{s} &= \begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ 3 & 2 & -1\\ 1 & 0 & 2\end{vmatrix}\\ &= \underset{\sim}{i}\begin{vmatrix}2 & -1\\ 0 & 2\end{vmatrix} - \underset{\sim}{j}\begin{vmatrix}3 & -1\\ 1 & 2\end{vmatrix} + \underset{\sim}{k}\begin{vmatrix}3 & 2\\ 1 & 0\end{vmatrix}\\ &= 4\underset{\sim}{i} - 7\underset{\sim}{j} - 2\underset{\sim}{k}\end{aligned}$$

$|\underset{\sim}{T}| = \sqrt{4^2 + (-7)^2 + (-2)^2} = \sqrt{69}$

12 a $\underset{\sim}{a} \times \underset{\sim}{b} \cdot \underset{\sim}{c} = \left(\underset{\sim}{a} \times \underset{\sim}{b}\right) \cdot \underset{\sim}{c}$

Brackets not needed since $\underset{\sim}{a} \times \underset{\sim}{b}$ is a vector

$\underset{\sim}{a} \times \underset{\sim}{b} \cdot \underset{\sim}{c}$ is a scalar

b $\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j} - 4\underset{\sim}{k}$, $\underset{\sim}{b} = 3\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{c} = \underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$

$$\begin{aligned}\underset{\sim}{a} \times \underset{\sim}{b} &= \begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ 2 & 1 & -4\\ 3 & -2 & -1\end{vmatrix}\\ &= \underset{\sim}{i}\begin{vmatrix}1 & -4\\ -2 & -1\end{vmatrix} - \underset{\sim}{j}\begin{vmatrix}2 & -4\\ 3 & -1\end{vmatrix} + \underset{\sim}{k}\begin{vmatrix}2 & 1\\ 3 & -2\end{vmatrix}\\ &= -9\underset{\sim}{i} - 10\underset{\sim}{j} - 7\underset{\sim}{k}\end{aligned}$$

$$\begin{aligned}\left(\underset{\sim}{a} \times \underset{\sim}{b}\right) \cdot \underset{\sim}{c} &= \left(-9\underset{\sim}{i} - 10\underset{\sim}{j} - 7\underset{\sim}{k}\right) \cdot \left(\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}\right)\\ &= -9 - 20 + 21\\ &= -8\end{aligned}$$

c

$$\begin{aligned}\underset{\sim}{b} \times \underset{\sim}{c} &= \begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ 3 & -2 & -1\\ 1 & 2 & -3\end{vmatrix}\\ &= \underset{\sim}{i}\begin{vmatrix}-2 & -1\\ 2 & -3\end{vmatrix} - \underset{\sim}{j}\begin{vmatrix}3 & -1\\ 1 & -3\end{vmatrix} + \underset{\sim}{k}\begin{vmatrix}3 & -2\\ 1 & 2\end{vmatrix}\\ &= 8\underset{\sim}{i} + 8\underset{\sim}{j} + 8\underset{\sim}{k}\end{aligned}$$

$$\begin{aligned}a \cdot \left(\underset{\sim}{b} \times \underset{\sim}{c}\right) &= \left(2\underset{\sim}{i} + \underset{\sim}{j} - 4\underset{\sim}{k}\right) \cdot \left(8\underset{\sim}{i} + 8\underset{\sim}{j} + 8\underset{\sim}{k}\right)\\ &= 16 + 8 - 32\\ &= -8\end{aligned}$$

d

$$\begin{aligned}\underset{\sim}{c} \times \underset{\sim}{a} &= \begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ 1 & 2 & -3\\ 2 & 1 & -4\end{vmatrix}\\ &= \underset{\sim}{i}\begin{vmatrix}2 & -3\\ 1 & -4\end{vmatrix} - \underset{\sim}{j}\begin{vmatrix}1 & -3\\ 2 & -4\end{vmatrix} + \underset{\sim}{k}\begin{vmatrix}1 & 2\\ 2 & 1\end{vmatrix}\\ &= -5\underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k}\end{aligned}$$

$$\begin{aligned}\left(\underset{\sim}{c} \times \underset{\sim}{a}\right) \cdot b &= \left(-5\underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k}\right) \cdot \left(3\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}\right)\\ &= -15 + 4 + 3\\ &= -8\end{aligned}$$

e $\begin{vmatrix} 2 & 1 & -4 \\ 3 & -2 & -1 \\ 1 & 2 & -3 \end{vmatrix}$

$$= 2\begin{vmatrix} -2 & -1 \\ 2 & -3 \end{vmatrix} - 1\begin{vmatrix} 3 & -1 \\ 1 & -3 \end{vmatrix} - 4\begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix}$$

$$= 2 \times 8 - 1 \times -8 - 4 \times 8$$

$$= 16 + 8 - 32$$

$$= -8$$

13 a $\underset{\sim}{a} \times \underset{\sim}{b} \times \underset{\sim}{c}$ is ambiguous, could be

$\left(\underset{\sim}{a} \times \underset{\sim}{b}\right) \times \underset{\sim}{c}$ or $\underset{\sim}{a} \times \left(\underset{\sim}{b} \times \underset{\sim}{c}\right)$

These are not equivalent.

b i $\underset{\sim}{a} = \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k},\ \underset{\sim}{b} = 2\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k},\ \underset{\sim}{c} = 3\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k}$

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & -2 & 3 \\ 2 & -1 & -1 \end{vmatrix}$$

$$= \underset{\sim}{i}\begin{vmatrix} -2 & 3 \\ -1 & -1 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix}$$

$$= 5\underset{\sim}{i} + 7\underset{\sim}{j} + 3\underset{\sim}{k}$$

$$\left(\underset{\sim}{a} \times \underset{\sim}{b}\right) \times \underset{\sim}{c} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 5 & 7 & 3 \\ 3 & 4 & -2 \end{vmatrix}$$

$$= \underset{\sim}{i}\begin{vmatrix} 7 & 3 \\ 4 & -2 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 5 & 3 \\ 3 & -2 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 5 & 7 \\ 3 & 4 \end{vmatrix}$$

$$= -26\underset{\sim}{i} + 19\underset{\sim}{j} - \underset{\sim}{k}$$

ii

$$\underset{\sim}{b} \times \underset{\sim}{c} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & -1 & -1 \\ 3 & 4 & -2 \end{vmatrix}$$

$$= \underset{\sim}{i}\begin{vmatrix} -1 & -1 \\ 4 & -2 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix}$$

$$= 6\underset{\sim}{i} + \underset{\sim}{j} + 11\underset{\sim}{k}$$

$$\underset{\sim}{a} \times \left(\underset{\sim}{b} \times \underset{\sim}{c}\right) = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & -2 & 3 \\ 6 & 1 & 11 \end{vmatrix}$$

$$= \underset{\sim}{i}\begin{vmatrix} -2 & 3 \\ 1 & 11 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 1 & 3 \\ 6 & 11 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 1 & -2 \\ 6 & 1 \end{vmatrix}$$

$$= -25\underset{\sim}{i} + 7\underset{\sim}{j} + 13\underset{\sim}{k}$$

iii So $\left(\underset{\sim}{a} \times \underset{\sim}{b}\right) \times \underset{\sim}{c} \neq \underset{\sim}{a} \times \left(\underset{\sim}{b} \times \underset{\sim}{c}\right)$. No, not equal.

iv $\underset{\sim}{a} \cdot \underset{\sim}{c} = \left(\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}\right) \cdot \left(3\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k}\right)$

$$= 3 - 8 - 6$$

$$= -11$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \left(\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}\right) \cdot \left(2\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}\right)$$

$$= 2 + 2 - 3$$

$$= 1$$

$$\underset{\sim}{b} \cdot \underset{\sim}{c} = \left(2\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}\right) \cdot \left(3\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k}\right)$$

$$= 6 - 4 + 2$$

$$= 4$$

RHS

$$= \left(\underset{\sim}{a} \cdot \underset{\sim}{c}\right)\underset{\sim}{b} - \left(\underset{\sim}{a} \cdot \underset{\sim}{b}\right)\underset{\sim}{c}$$

$$= -11\left(2\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}\right) - \underset{\sim}{i}\left(3\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k}\right)$$

$$= -25\underset{\sim}{i} + 7\underset{\sim}{j} + 13\underset{\sim}{k} = \underset{\sim}{a} \times \left(\underset{\sim}{b} \times \underset{\sim}{c}\right) = \text{LHS shown}$$

v RHS

$$= \left(\underset{\sim}{a} \cdot \underset{\sim}{c}\right)\underset{\sim}{b} - \left(\underset{\sim}{b} \cdot \underset{\sim}{c}\right)\underset{\sim}{a}$$

$$= -11\left(2\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}\right) - 4\left(\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}\right)$$

$$= -26\underset{\sim}{i} + 19\underset{\sim}{j} - \underset{\sim}{k} = \left(\underset{\sim}{a} \times \underset{\sim}{b}\right) \times \underset{\sim}{c} \quad = \text{LHS shown}$$

14 a $a = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k} \quad \underset{\sim}{b} = 3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ x & y & z \\ 3 & -1 & 2 \end{vmatrix}$$

$$= (2y + z)\underset{\sim}{i} - (2x - 3z)\underset{\sim}{j} - (x + 3y)\underset{\sim}{k}$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = 3x - y + 2z$$

Now $\underset{\sim}{a} \times \underset{\sim}{b} = -2\underset{\sim}{i} + 8\underset{\sim}{k} + 7\underset{\sim}{k}$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = 17$$

$$\left.\begin{array}{ll} (1) & 3x - y + 2z = 17 \\ (2) & 2y + z = -2 \\ (3) & -2x + 3z = 8 \\ (4) & -x - 3y = 7 \end{array}\right\} \text{solving } x = 2, y = -3, z = 4$$

b $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k} \quad \underset{\sim}{b} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}$

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ x & y & z \\ 2 & -3 & 2 \end{vmatrix}$$

$$= (2y + 3z)\underset{\sim}{i} - (2x - 2z)\underset{\sim}{j} - (3x + 2y)\underset{\sim}{k}$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = 0 = 2x - 3y + 2z$$

$$\underset{\sim}{a} \times \underset{\sim}{b} = 5\underset{\sim}{i} + 18\underset{\sim}{j} + 22\underset{\sim}{k}$$

$$\left.\begin{array}{ll} (1) & 2x - 3y + 2z = 0 \\ (2) & 2y + 3z = 5 \\ (3) & -2x + 2z = 18 \\ (4) & -3x - 2y = 22 \end{array}\right\} \text{solving } x = -6,\ y = -2,\ z = 3$$

c $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k} \quad \underset{\sim}{b} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - 5\underset{\sim}{k}$

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ x & y & z \\ 2 & 3 & -5 \end{vmatrix}$$

$$= (-5y - 3z)\underset{\sim}{i} + (5x + 2z)\underset{\sim}{j} + (3x - 2y)\underset{\sim}{k}$$

$$= 7\underset{\sim}{i} + 12\underset{\sim}{j} + 10\underset{\sim}{k}$$

(1) $-5y - 3z = 7$

(2) $5x + 2z = 12$

(3) $3x - 2y = 10$

(4) $x^2 + y^2 + z^2 = 9$ since $|\underset{\sim}{a}| = 3$

Solving

$$x = 2,\ y = -2,\ z = 1 \text{ or } x = \frac{52}{19},\ y = \frac{-17}{19},\ z = \frac{-16}{19}$$

d $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j} + 2\underset{\sim}{k} \quad \underset{\sim}{b} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}$

$|\underset{\sim}{a}| = 3\sqrt{5} \ \ |\underset{\sim}{a}|^2 = 45$

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ x & y & z \\ 3 & -2 & 4 \end{vmatrix}$$

$$= (4y + 2z)\underset{\sim}{i} + (3z - 4x)\underset{\sim}{j} - (2x + 3y)\underset{\sim}{k}$$

$$= 26\underset{\sim}{i} + 23\underset{\sim}{j} - 8\underset{\sim}{k}$$

(1) $4y + 2z = 26$

(2) $3z - 4x = 23$

(3) $2x + 3y = 8$

(4) $x^2 + y^2 + z^2 = 45$

Solving $x = -2, y = 4,\ z = 5$ or $x = \frac{-94}{29},\ y = \frac{140}{29},\ z = \frac{97}{29}$

4.2 Exam questions

1 $\underset{\sim}{a}\times\underset{\sim}{b}=|\underset{\sim}{a}|\,|\underset{\sim}{b}|\sin(\theta)=0$

$\sin(\theta)=0$

$\therefore\theta=0$

$|\underset{\sim}{a}|=0$

$|\underset{\sim}{b}|=0$

When the vector product is zero, the angle between the vectors is 0, or at least one of the vectors is a zero vector.

The correct answer is **B**.

2 $P(3,0,0),\ Q(0,-2,0),\ R(0,0,1)$

$\overrightarrow{OP}=3\underset{\sim}{i}$

$\overrightarrow{OQ}=-2\underset{\sim}{j}$

$\overrightarrow{OR}=\underset{\sim}{k}$ [1 mark]

$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}\quad \overrightarrow{QR}=\overrightarrow{OR}-\overrightarrow{OQ}$

$=-3\underset{\sim}{i}-2\underset{\sim}{j}\qquad =2\underset{\sim}{j}+\underset{\sim}{k}$

$$\overrightarrow{PQ}\times\overrightarrow{QR}=\begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ -3 & -2 & 0\\ 0 & 2 & 1\end{vmatrix}$$

$$=\underset{\sim}{i}\begin{vmatrix}-2 & 0\\ 2 & 1\end{vmatrix}-\underset{\sim}{j}\begin{vmatrix}-3 & 0\\ 0 & 1\end{vmatrix}+\underset{\sim}{k}\begin{vmatrix}-3 & -2\\ 0 & 2\end{vmatrix}$$

$-2\underset{\sim}{i}+3\underset{\sim}{j}-6\underset{\sim}{k}$ [1 mark]

$\left|\overrightarrow{PQ}\times\overrightarrow{QR}\right|=\sqrt{(-2)^2+3^2+(-6)^2}$

$=\sqrt{49}$

$=7$

Area of triangle $=\dfrac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{QR}\right|$

$=\dfrac{7}{2}$ [1 mark]

3 $|\underset{\sim}{a}\times\underset{\sim}{b}|^2=\left||\underset{\sim}{a}|\,|\underset{\sim}{b}|\sin(\theta)\cdot\hat{\underset{\sim}{n}}\right|^2$ since $|\underset{\sim}{n}|=1$

$=|\underset{\sim}{a}|^2|\underset{\sim}{b}|^2\sin^2(\theta)$ [1 mark]

RHS $=|\underset{\sim}{a}|^2|\underset{\sim}{b}|^2-\left(\underset{\sim}{a}\cdot\underset{\sim}{b}\right)^2$

$=|\underset{\sim}{a}|^2|\underset{\sim}{b}|^2-\left(|\underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta)\right)^2$

$=|\underset{\sim}{a}|^2|\underset{\sim}{b}|^2-|\underset{\sim}{a}|^2|\underset{\sim}{b}|^2\cos^2(\theta)$

$=|\underset{\sim}{a}|^2|\underset{\sim}{b}|^2\left(1-\cos^2(\theta)\right)$

$=|\underset{\sim}{a}|^2|\underset{\sim}{b}|^2\sin^2(\theta)$

= LHS shown [1 mark]

4.3 Lines in three dimensions

4.3 Exercise

1 a Does the point $(8,5,4)$ lie on the line?

$t=\dfrac{x-5}{-3}=\dfrac{y+1}{-6}=\dfrac{z+2}{2}$

$x=5-3t$, if $x=8\Rightarrow 8=5-3t,\qquad t=-1$

$y=-1-6t$, if $y=5\Rightarrow 5=-1-6t\qquad t=-1$

$z=-2+2t$, if $z=4\Rightarrow 4=-2+2t$

$2t=6$

$t=3$

No, the point does not lie on the line.

b When $t=6$,

$\dfrac{x-5}{-3}=6,\quad \dfrac{y+1}{-6}=6,\quad \dfrac{z+2}{2}=6$

$x-5=-18\quad y+1=-36\quad z+2=12$

$x=-13\quad y=-37\quad z=10$

The point corresponding to $t=6$ is $(-13,\ -37,\ 10)$.

2 a Line $\dfrac{x-4}{-1}=\dfrac{y+4}{2}=\dfrac{z-5}{-3}=t$

Does the point $(1,2,-4)$ lie on the line?

$x=4-t,\ y=-4+2t,\ z=5-3t$

$x=1\Rightarrow 1=4-t\Rightarrow t=3$

$y=2\Rightarrow 2=-4+2t\Rightarrow 2t=6\qquad t=3$

$z=-4\Rightarrow -4=5-3t\Rightarrow 3t=9\qquad t=3$

Yes the point $(1,2,-4)$ does lie on the line.

b $\dfrac{x-6}{4}=\dfrac{y+3}{2}=\dfrac{z-5}{-3}=t$

$x=6+4t,\ y=-3+2t,\ z=5-3t$

If $t=10,\ x=46,\ y=17,\ z=-25\ (46,17,-25)$

If $t=5\ x=26,\ y=7,\ z=-10\ (26,7,-10)$

If $t=0\ x=6,\ y=-3,\ z=5\ (6,-3,5)$

3 a $P_0(2,-3),\ \underset{\sim}{v}=\underset{\sim}{i}+2\underset{\sim}{j}$

$\overrightarrow{OP_0}=2\underset{\sim}{i}-3\underset{\sim}{j}$

$\underset{\sim}{r}(t)=\overrightarrow{OP_0}+t\underset{\sim}{v}$

$=2\underset{\sim}{i}-3\underset{\sim}{j}+t\left(\underset{\sim}{i}+2\underset{\sim}{j}\right)$

$x=2+t,\ y=-3+2t$

$t=\dfrac{x-2}{1}=\dfrac{y+3}{2}$

b $P_0(2,-3,4),\ \underset{\sim}{v}=\underset{\sim}{i}+2\underset{\sim}{j}-3\underset{\sim}{k}$

$\overrightarrow{OP_0}=2\underset{\sim}{i}-3\underset{\sim}{j}+4\underset{\sim}{k}$

$\underset{\sim}{r}(t)=\overrightarrow{OP_0}+t\underset{\sim}{v}$

$=2\underset{\sim}{i}-3\underset{\sim}{j}+4\underset{\sim}{k}+t\left(\underset{\sim}{i}+2\underset{\sim}{j}-3\underset{\sim}{k}\right)$

$x=2+t,\ y=-3+2t,\ z=4-3t$

$t=\dfrac{x-2}{1}=\dfrac{y+3}{2}=\dfrac{z-4}{-3}$

c $P_0(3,5,1),\ v=2\underset{\sim}{i}+\underset{\sim}{k}$

$\overrightarrow{OP_0}=3\underset{\sim}{i}+5\underset{\sim}{j}+\underset{\sim}{k}$

$\underset{\sim}{r}(t)=\overrightarrow{OP_0}+t\underset{\sim}{v}$

$=3\underset{\sim}{i}+5\underset{\sim}{j}+\underset{\sim}{k}+t\left(2\underset{\sim}{i}+\underset{\sim}{k}\right)$

$x=3+t,\ y=5,\ z=1+t$

$t=\dfrac{x-3}{2}=z-1;\ y=5$

4 a $P_0(2,-3,4),\ P_1(5,2,-2)$

$\overrightarrow{OP_0}=2\underset{\sim}{i}-3\underset{\sim}{j}+4\underset{\sim}{k},\ \overrightarrow{OP_1}=5\underset{\sim}{i}+2\underset{\sim}{j}-2\underset{\sim}{k}$

$\underset{\sim}{r}=\overrightarrow{P_0P_1}=\overrightarrow{OP_1}-\overrightarrow{OP_0}=3\underset{\sim}{i}+5\underset{\sim}{j}-6\underset{\sim}{k}$

$r(t)=\overrightarrow{OP_0}+t\underset{\sim}{r}=2\underset{\sim}{i}-3\underset{\sim}{j}+4\underset{\sim}{k}+t\left(3\underset{\sim}{i}+5\underset{\sim}{j}-6\underset{\sim}{k}\right)$

$x=2+3t,\ y=-3+5t,\ z=4-6t$

$t=\dfrac{x-2}{3}=\dfrac{y+3}{5}=\dfrac{z-4}{-6}$

b $P_0(1,0,-3),\ P_1(-3,2,5)$

$\overrightarrow{OP_0}=\underset{\sim}{i}-3\underset{\sim}{k}\ \overrightarrow{OP_1}=-3\underset{\sim}{i}+2\underset{\sim}{j}+5\underset{\sim}{k}$

$\underset{\sim}{r}=\overrightarrow{P_0P_1}=\overrightarrow{OP_1}-\overrightarrow{OP_0}=-4\underset{\sim}{i}+2\underset{\sim}{j}+8\underset{\sim}{k}$

$r(t)=\overrightarrow{OP_0}+t\underset{\sim}{r}=\underset{\sim}{i}-3\underset{\sim}{k}+t\left(-4\underset{\sim}{i}+2\underset{\sim}{j}+8\underset{\sim}{k}\right)$

$x=1-4t,\ y=2t,\ z=-3+8t$

$t=\dfrac{x-1}{-4}=\dfrac{y}{2}=\dfrac{z+3}{8}$

c $P_0(3,4,-2),\ P_1(0,6,-2)$

$\overrightarrow{OP_0}=3\underset{\sim}{i}+4\underset{\sim}{j}-2\underset{\sim}{k}\qquad \overrightarrow{OP_1}=6\underset{\sim}{j}-2\underset{\sim}{k}$

$\underset{\sim}{r}=\overrightarrow{P_0P_1}=\overrightarrow{OP_1}-\overrightarrow{OP_0}=-3\underset{\sim}{i}+2\underset{\sim}{j}$

$\underset{\sim}{r}(t)=\overrightarrow{OP_0}+t\underset{\sim}{r}=3\underset{\sim}{i}+4\underset{\sim}{j}-2\underset{\sim}{k}+t\left(-3\underset{\sim}{i}+2\underset{\sim}{j}\right)$

$x = 3 - 3t, y = 4 + 2t, z = -2$

$t = \frac{x-3}{-3} = \frac{y-4}{2}; z = -2$

5 a $\frac{x-14}{-3} = \frac{y+5}{1} = \frac{z-11}{-2}$

Direction $\underset{\sim}{v}_1 = -3\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$

$x = -6 - 4t, y = -7 - 3t, z = 7 + 2t$

Direction $\underset{\sim}{v}_2 = -4\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}$

$|\underset{\sim}{v}_1| = \sqrt{(-3)^2 + (1)^2 + (-2)^2} = \sqrt{14}$

$|\underset{\sim}{v}_2| = \sqrt{(-4)^2 + (-3)^2 + (2)^2} = \sqrt{29}$

$\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2 = 12 - 3 - 4 = 5$

$$\begin{aligned}\cos(\theta) &= \frac{|\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2|}{|\underset{\sim}{v}_1||\underset{\sim}{v}_2|}\\ &= \frac{5}{\sqrt{14}\sqrt{29}}\end{aligned}$$

$$\theta = \cos^{-1}\left(\frac{5}{\sqrt{14}\sqrt{29}}\right) = 75.6°$$

b $\frac{x-13}{3} = \frac{y-4}{2} = \frac{z+7}{-2}$

Direction $\underset{\sim}{v}_1 = 3\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$

$x = -1 + t, y = 8 - 2t, z = 14 - 3t$

Direction $\underset{\sim}{v}_2 = \underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k}$

$|\underset{\sim}{v}_1| = \sqrt{9 + 4 + 4} = \sqrt{17}$

$|\underset{\sim}{v}_2| = \sqrt{1 + 4 + 9} = \sqrt{14}$

$\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2 = 3 - 4 + 6 = 5$

$$\begin{aligned}\cos(\theta) &= \frac{\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2}{|\underset{\sim}{v}_1||\underset{\sim}{v}_2|}\\ &= \frac{5}{\sqrt{17}\sqrt{14}}\end{aligned}$$

$$\theta = \cos^{-1}\left(\frac{5}{\sqrt{17}\ \sqrt{14}}\right) = 71.1°$$

c $\frac{x-7}{-3} = \frac{y+10}{4} = \frac{z+1}{2}$

Direction $\underset{\sim}{v}_1 = -3\underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k}$

$x = -2 + 2t, y = -4 - 2t, z = 2 - t$

Direction $\underset{\sim}{v}_2 = 2\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$

$|\underset{\sim}{v}_1| = \sqrt{9 + 16 + 4} = \sqrt{29}$

$|\underset{\sim}{v}_2| = \sqrt{4 + 4 + 1} = 3$

$\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2 = -6 - 8 - 2 = -16$

$$\begin{aligned}\cos(\theta) &= \frac{|\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2|}{|\underset{\sim}{v}_1||\underset{\sim}{v}_2|}\\ &= \frac{16}{3\sqrt{29}}\end{aligned}$$

$$\theta = \cos^{-1}\left(\frac{16}{3\sqrt{29}}\right) = 8.0°$$

6 a Let $s = \frac{x-14}{-3} = \frac{y+5}{1} = \frac{z-11}{-2}$

$x = 14 - 3s, y = -5 + s, z = 11 - 2s$

$x = -6 - 4t, y = -7 - 3t, z = 7 + 2t$

(1) $14 - 3s = -6 - 4t$

(2) $-5 + s = -7 - 3t$

(3) $11 - 2s = 7 + 2t$

from (2) $s = -2 - 3t$ into (1)

$$\begin{aligned}14 - 3(-2 - 3t) &= -6 - 4t\\ 14 + 6 + 9t &= -6 - 4t\\ 13t &= -26\end{aligned}$$

$t = -2 \Rightarrow s = -2 + 6 = 4$

Check in (3)

LHS $11 - 2s = 11 - 8 = 3$

RHS $7 + 2t = 7 - 4 = 3$

$t = -2 \Rightarrow x = 2, y = -1, z = 3$

$s = 4 \Rightarrow x = 2, y = -1, z = 3$

Point of intersection $(2, -1, 3)$

b Let $s = \frac{x-13}{3} = \frac{y-4}{2} = \frac{z+7}{-2}$

$x = 13 + 3s, y = 4 + 2s, z = -7 - 2s$

$x = -1 + t, y = 8 - 2t, z = 14 - 3t$

(1) $13 + 3s = -1 + t$

(2) $4 + 2s = 8 - 2t$

(3) $-7 - 2s = 14 - 3t$

(1) $\Rightarrow t = 14 + 3s$ into (2)

$$\begin{aligned}4 + 2s &= 8 - 2(14 + 3s)\\ 4 + 2s &= 8 - 28 - 6s\\ 8s &= -24\\ s &= -3 \Rightarrow t = 14 - 9 = 5\\ t &= 5\end{aligned}$$

Check in (3)

LHS $= -7 - 2s = -7 + 6 = -1$

RHS $= 14 - 3t = 14 - 15 = -1$

If $s = -3 \Rightarrow x = 4, y = -2, z = -1$

$t = 5 \Rightarrow x = 4, y = -2, z = -1$

Point of intersection $(4, -2, -1)$

c $s = \frac{x-7}{-3} = \frac{y+10}{4} = \frac{z+1}{2}$

$x = 7 - 3s, y = -10 + 4s, z = -1 + 2s$

$x = -2 + 2t, y = -4 - 2t, z = 2 - t$

(1) $7 - 3s = -2 + 2t$

(2) $-10 + 4s = -4 - 2t$

(3) $-1 + 2s = 2 - t$

From (1) $2t = 9 - 3s$ into (2)

$-10 + 4s = -4 - (9 - 3s)$

$-10 + 4s = -4 - 9 + 3s$

$s = -3 \Rightarrow 2t = 9 + 9 = 18$

$t = 9$

Check in (3)

LHS $= -1 - 6 = -7$

RHS $= 2 - 9 = -7$

RHS = LHS

$s = -3 \Rightarrow x = 7 + 9 = 16, y = -10 - 12 = -22,$

$z = -1 - 6 = -7$

$t = 9 \Rightarrow x = -2 + 18 = 16, y = -4 - 18 = -22,$

$z = 2 - 9 = -7$

Point of intersection $(16, -22, -7)$

7 a Let $s = \frac{x-2}{-5} = \frac{y-1}{3} = \frac{z+3}{4}$

Line 1: $x = 2 - 5s, y = 1 + 3s, z = -3 + 4s$

Line 2: $x = -4 - t, y = 6 + 2t, z = 3 + t$

(1) $2 - 5s = -4 - t$

(2) $1 + 3s = 6 + 2t$

(3) $-3 + 4s = 3 + t$

From (1) $t = -6 + 5s$ substitute into (2)

$1 + 3s = 6 + 2(-6 + 5s)$

$1 + 3s = 6 - 12 + 10s$

$7s = 7$

$s = 1 \Rightarrow t = -6 + 5 = -1$

Check in (3)

$\text{LHS} = -3 + 4s = -3 + 4 = 1$

$\text{RHS} = 3 + t = 3 - 1 = 2$

$1 = 2$

Contradiction

⇒ No solution

b Let $s = \dfrac{x+7}{2} = \dfrac{y+1}{3} = z - 3$

Line 1: $x = -7 + 2s, y = -1 + 3s, z = 3 + s$

Line 2: $x = -3 + t, y = 4 + 2t, z = -2 + t$

(1) $-7 + 2s = -3 + t$

(2) $-1 + 3s = 4 + 2t$

(3) $3 + s = -2 + t$

From (1) $t = -4 + 2s$ substitute into (2)

$-1 + 3s = 4 + 2(-4 + 2s)$

$-1 + 3s = 4 - 8 + 4s$

$s = 3 \Rightarrow t = -4 + 6 = 2$

Check in (3)

$\text{LHS} = 3 + s = 6$

$\text{RHS} = -2 + t = 0$

$6 = 0$

Contradiction

⇒ No solution

c Let $s = \dfrac{x-2}{3} = \dfrac{y+3}{-4} = \dfrac{z-5}{2}$

Line 1: $x = 2 + 3s, y = -3 - 4s, z = 5 + 2s$

Line 2: $x = -8 - 4t, y = 1 + 3t, z = 7 + 2t$

(1) $2 + 3s = -8 - 4t$

(2) $-3 - 4s = 1 + 3t$

(3) $5 + 2s = 7 + 2t$

(1) $3s + 4t = -10$

(2) $4s + 3t = -4$

(1) × 4: $12s + 16t = -40$

(2) × 3: $12s + 9t = -12$

$-7t = 28$

$t = -4 \Rightarrow 3s = -10 - 4t = -10 + 16 = 6$

$s = 2$

Check in (3)

$\text{LHS} = 5 + 2s = 9$

$\text{RHS} = 7 + 2t = 7 - 8 = -1$

$9 = -1$

Contradiction

⇒ No solution

8 a Line $\dfrac{x-2}{3} = \dfrac{y+4}{-2} = \dfrac{z-5}{1}$ 0 (0, 0, 0)

Direction of line $\underset{\sim}{v} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$

Line $x = 2 + 3t, y = -4 - 2t, z = 5 + t$

When $t = 0$ $P_0(2, -4, 5)$

$\overrightarrow{OP_0} = 2\underset{\sim}{i} - 4\underset{\sim}{j} + 5\underset{\sim}{k}$

$$\overrightarrow{OP_0} \times \underset{\sim}{v} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & -4 & 5 \\ 3 & -2 & 1 \end{vmatrix}$$

$$= \underset{\sim}{i}\begin{vmatrix} -4 & 5 \\ -2 & 1 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 2 & -4 \\ 3 & -2 \end{vmatrix}$$

$$= 6\underset{\sim}{i} + 13\underset{\sim}{j} + 8\underset{\sim}{k}$$

$|\underset{\sim}{v}| = \sqrt{(3)^2 + (-2)^2 + (1)^2} = \sqrt{14}$

$|\overrightarrow{OP_0} \times \underset{\sim}{v}| = \sqrt{6^2 + 13^2 + 8^2} = \sqrt{269}$

$$d = \frac{|\overrightarrow{OP_0} \times \underset{\sim}{v}|}{|\underset{\sim}{v}|} = \frac{\sqrt{269}}{\sqrt{14}} \times \frac{\sqrt{14}}{\sqrt{14}}$$

$$= \frac{\sqrt{3766}}{14}$$

b Line $\dfrac{x+2}{5} = \dfrac{y+3}{2} = \dfrac{z-4}{-2}$ 0 (0, 0, 0)

Direction of line $\underset{\sim}{v} = 5\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$

Line $x = -2 + 5t, y = -3 + 2t, z = 4 - 2t$

When $t = 0$ $P_0(-2, -3, 4)$

$\overrightarrow{OP_0} = -2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}$

$$\overrightarrow{OP_0} \times \underset{\sim}{v} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -2 & -3 & 4 \\ 5 & 2 & -2 \end{vmatrix}$$

$$= \underset{\sim}{i}\begin{vmatrix} -3 & 4 \\ 2 & -2 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} -2 & 4 \\ 5 & -2 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} -2 & -3 \\ 5 & 2 \end{vmatrix}$$

$$= -2\underset{\sim}{i} + 16\underset{\sim}{j} + 11\underset{\sim}{k}$$

$|\underset{\sim}{v}| = \sqrt{5^2 + 2^2 + (-2)^2} = \sqrt{33}$

$|\overrightarrow{OP_0} \times \underset{\sim}{v}| = \sqrt{(-2)^2 + 16^2 + 11^2} = \sqrt{381}$

$$d = \frac{|\overrightarrow{OP_0} \times \underset{\sim}{v}|}{|\underset{\sim}{v}|} = \frac{\sqrt{381}}{\sqrt{33}} \times \frac{\sqrt{33}}{\sqrt{33}}$$

$$= \frac{\sqrt{1397}}{11}$$

c Line $\dfrac{x+2}{-3} = \dfrac{y-2}{4}; z = 1$ 0 (0, 0, 0)

Direction of line $\underset{\sim}{v} = -3\underset{\sim}{i} + 4\underset{\sim}{j}$

Line $x = -2 - 3t, y = 2 + 4t, z = 1$

When $t = 0$ $P_0(-2, 2, 1)$

$\overrightarrow{OP_0} = -2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$

$$|\overrightarrow{OP_0} \times \underset{\sim}{v}| = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -2 & 2 & 1 \\ -3 & 4 & 0 \end{vmatrix}$$

$$= \underset{\sim}{i}\begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} -2 & 1 \\ -3 & 0 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} -2 & 2 \\ -3 & 4 \end{vmatrix}$$

$$= -4\underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k}$$

$|\overrightarrow{OP_0} \times \underset{\sim}{v}| = \sqrt{(-4)^2 + (-3)^2 + (-2)^2} = \sqrt{29}$

$|\underset{\sim}{v}| = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5$

$$d = \frac{|\overrightarrow{OP_0} \times \underset{\sim}{v}|}{|\underset{\sim}{v}|} = \frac{\sqrt{29}}{5}$$

9 a $A(-1, -2, 6)$ $x = 3 + 2t, y = 2 - 2t, z = 2$

Line $\underset{\sim}{v} = 2\underset{\sim}{i} - 2\underset{\sim}{j}$ $t = 0$ $P_0(3, 2, 2)$

$$\overrightarrow{AP_0} = \overrightarrow{OP_0} - \overrightarrow{OA}$$

$$= 4\underset{\sim}{i} + 4\underset{\sim}{j} - 4\underset{\sim}{k}$$

$$\overrightarrow{AP_0} \times \underset{\sim}{v} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 4 & 4 & -4 \\ 2 & -2 & 0 \end{vmatrix}$$

$$= \underset{\sim}{i}\begin{vmatrix} 4 & -4 \\ -2 & 0 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 4 & -4 \\ 2 & 0 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 4 & 4 \\ 2 & -2 \end{vmatrix}$$

$$= -8\underset{\sim}{i} - 8\underset{\sim}{j} - 16\underset{\sim}{k}$$

$$|\overrightarrow{AP_0} \times \underset{\sim}{v}| = \sqrt{(-8)^2 + (-8)^2 + (-16)^2} = \sqrt{384} = \sqrt{64 \times 6}$$

$$= 8\sqrt{6}$$

$$|\underset{\sim}{v}| = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$$

$$d = \frac{|\overrightarrow{AP_0} \times \underset{\sim}{v}|}{|\underset{\sim}{v}|} = \frac{8\sqrt{6}}{2\sqrt{2}}$$

$$= 4\sqrt{3}$$

b $A(1, 0, -3)$ line $x = 2;\ t = \dfrac{y-2}{-3} = \dfrac{z-1}{4}$

Line $x = 2;\ y = 2 - 3t;\ z = 1 + 4t$

$t = 0\ \ P_0(2, 2, 1)\ \ \underset{\sim}{v} = -3\underset{\sim}{j} + 4\underset{\sim}{k}$

$$\overrightarrow{AP_0} = \overrightarrow{OP_0} - \overrightarrow{OA}$$

$$= \underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}$$

$$\overrightarrow{AP_0} \times \underset{\sim}{v} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 2 & 4 \\ 0 & -3 & 4 \end{vmatrix}$$

$$= \underset{\sim}{i}\begin{vmatrix} 2 & 4 \\ -3 & 4 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 1 & 4 \\ 0 & 4 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix}$$

$$= 20\underset{\sim}{i} - 4\underset{\sim}{j} - 3\underset{\sim}{k}$$

$$|\overrightarrow{AP_0} \times \underset{\sim}{v}| = \sqrt{20^2 + (-4)^2 + (-3)^2} = \sqrt{425} = 5\sqrt{17}$$

$$|\underset{\sim}{v}| = \sqrt{(-3)^2 + 4^2} = 5$$

$$d = \frac{|\overrightarrow{AP_0} \times \underset{\sim}{v}|}{|\underset{\sim}{v}|} = \frac{5\sqrt{17}}{5} = \sqrt{17}$$

c $A(2, -3, 4)\qquad t = \dfrac{x+3}{-2} = \dfrac{y-4}{5} = \dfrac{z+2}{3}$

Line $x = -3 - 2t,\ y = 4 + 5t;\ z = -2 + 3t$

Direction of line $\underset{\sim}{v} = -2\underset{\sim}{i} + 5\underset{\sim}{j} + 3\underset{\sim}{k}$

$t = 0,\ \ P_0(-3, 4, -2)$

$$\overrightarrow{AP_0} = \overrightarrow{OP_0} - \overrightarrow{OA}$$

$$= -5\underset{\sim}{i} + 7\underset{\sim}{j} - 6\underset{\sim}{k}$$

$$\overrightarrow{AP_0} \times \underset{\sim}{v} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -5 & 7 & -6 \\ -2 & 5 & 3 \end{vmatrix}$$

$$= \underset{\sim}{i}\begin{vmatrix} 7 & -6 \\ 5 & 3 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} -5 & -6 \\ -2 & 3 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} -5 & 7 \\ -2 & 5 \end{vmatrix}$$

$$= 51\underset{\sim}{i} + 27\underset{\sim}{j} - 11\underset{\sim}{k}$$

$$|\overrightarrow{AP_0} \times \underset{\sim}{v}| = \sqrt{51^2 + 27^2 + (-11)^2} = \sqrt{3451}$$

$$|\underset{\sim}{v}| = \sqrt{(-2)^2 + 5^2 + 3^2} = \sqrt{38}$$

$$d = \frac{|\overrightarrow{AP_0} \times \underset{\sim}{v}|}{|\underset{\sim}{v}|} = \frac{\sqrt{3451}}{\sqrt{38}} \times \frac{\sqrt{38}}{\sqrt{38}}$$

$$= \frac{\sqrt{131\,138}}{38}$$

10 a $A(1, -2, 3)\ \ B(3, -4, 2)$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\underset{\sim}{v} = 2\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$

Line $t = \dfrac{x-1}{2} = \dfrac{y+2}{-2} = \dfrac{z-3}{-1}$

$x = 1 + 2t,\ y = -2\ -2t,\ z = 3 - t$

Line $x = 3 - 2s,\ y = -1 + s,\ z = -1 + 2s$

(1) $1 + 2t = 3 - 2s\quad \Rightarrow 2s + 2t = 2$

(2) $-2 - 2t = -1 + s\quad \Rightarrow s + 2t = -1$ subtract

(3) $3 - t = -1 + 2s$

(1) – (2): $s = 3\ \ \Rightarrow t = -2$

Check in (3)

LHS $= 3 - t = 5$

RHS $= -1 + 2s = 5$

Let $s = 3$, line (2) $x = -3,\ y = 2,\ z = 5$

$t = -2$, line (1) $x = -3,\ y = 2,\ z = 5$

Point of intersection $(-3, 2, 5)$

Line 1: $\underset{\sim}{v}_1 = 2\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}\quad |\underset{\sim}{v}_1| = \sqrt{4 + 4 + 1} = 3$

Line 2: $\underset{\sim}{v}_2 = -2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}\ \ |\underset{\sim}{v}_2| = \sqrt{4 + 1 + 4} = 3$

$\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2 = -4 - 2 - 2 = -8$

$$\cos(\theta) = \frac{|\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2|}{|\underset{\sim}{v}_1||\underset{\sim}{v}_2|} = \frac{8}{3 \times 3} = \frac{8}{9}$$

$$\theta = \cos^{-1}\left(\frac{8}{9}\right) = 27.3°$$

b $A(4, 5, -2)\ \ B(8, 11, -4)$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= 4\underset{\sim}{i} + 6\underset{\sim}{j} - 2\underset{\sim}{k}$$

Line $t = \dfrac{x-4}{4} = \dfrac{y-5}{6} = \dfrac{z+2}{-2}$

$x = 4 + 4t,\ y = 5 + 6t,\ z = -2 - 2t$

Line $x = -6 + s,\ y = 4 - 2s,\ z = -3 + s$

(1) $4 + 4t = -6 + s \Rightarrow s - 4t = 10\quad$ (1)

(2) $5 + 6t = 4 - 2s\quad 2s + 6t = -1\quad$ (2)

(3) $-2 - 2t = -3 + s\quad s + 2t = 1\quad$ (3)

(3) – (1): $6t = -9$

$$t = -\frac{3}{2}\ \Rightarrow s = 4$$

Check in (2)

LHS $= 5 + 6t = -4$

RHS $= 4 - 2s = -4$

If $s = 4\ \Rightarrow x = -2,\ y = -4,\ z = 1$

$t = -\dfrac{3}{2}\ \Rightarrow x = -2,\ y = -4,\ z = 1$

Point of intersection $(-2, -4, 1)$

Line 1:

$\underset{\sim}{v}_1 = 4\underset{\sim}{i} + 6\underset{\sim}{j} - 2\underset{\sim}{k}\qquad |\underset{\sim}{v}_1| = \sqrt{16 + 36 + 4} = \sqrt{56} = 2\sqrt{14}$

Line 2: $\underset{\sim}{v}_2 = \underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\qquad |\underset{\sim}{v}_2| = \sqrt{1 + 4 + 1} = \sqrt{6}$

$\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2 = 4 - 12 - 2 = -10$

$$\cos(\theta) = \frac{|\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2|}{|\underset{\sim}{v}_1||\underset{\sim}{v}_2|} = \frac{10}{2\sqrt{14}\sqrt{6}} = \frac{5}{\sqrt{84}} = \frac{5}{2\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{5}{2\sqrt{21}}\right) = 56.9°$$

11 a $A(2, -5, 4)\ \ B(x_0, y_0, 8)$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= (x_0 - 2)\,\underset{\sim}{i} + (y_0 + 5)\,\underset{\sim}{j} + 4\underset{\sim}{k}$$

Line $\dfrac{x-1}{3} = \dfrac{y+2}{2} = \dfrac{z-3}{-2}$

Direction $\underset{\sim}{v} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$

$\overrightarrow{AB}$ is // to $\underset{\sim}{v}$

$\overrightarrow{AB} = \lambda\underset{\sim}{v}$

$$(x_0 - 2)\,\underset{\sim}{i} + (y_0 + 5)\,\underset{\sim}{j} + 4\underset{\sim}{k} = \lambda\left(3\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}\right)$$

From $\underset{\sim}{k}$: $4 = -2\lambda \Rightarrow \lambda = -2$

$\Rightarrow \underset{\sim}{i}$: $x_0 - 2 = -6 \Rightarrow x_0 = -4$

$\underset{\sim}{j}$: $y_0 + 5 = -4 \Rightarrow y_0 = -9$

b $P(x_0, y_0, 0) \quad \dfrac{x - x_0}{a} = \dfrac{y - y_0}{b}$; $z_0 = 0$

$y - y_0 = \dfrac{b}{a}(x - x_0), \underset{\sim}{v} = a\underset{\sim}{i} + b\underset{\sim}{j}$

$y = mx + c, \qquad m = \dfrac{b}{a}$

$$c = y_0 - \frac{bx_0}{a}$$

12 a $P_0(x_0, y_0, z_0) \quad P_1(x_1, y_1, z_1)$

$\overrightarrow{OP}_0 = x_0\underset{\sim}{i} + y_0\underset{\sim}{j} + z_0\underset{\sim}{k}$

$\overrightarrow{OP}_1 = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$

$\underset{\sim}{v}\overrightarrow{P_0P_1} = \overrightarrow{OP}_1 - \overrightarrow{OP}_0$

$\underset{\sim}{v} = (x_1 - x_0)\,\underset{\sim}{i} + (y_1 - y_0)\,\underset{\sim}{j} + (z_1 - z_0)\,\underset{\sim}{k}$

Direction of the line

Use P_0

$x = x_0 + (x_1 - x_0)\,t$ parametric equation

$y = y_0 + (y_1 - y_0)\,t$

$z = z_0 + (z_1 - z_0)\,t$

$\Rightarrow t = \dfrac{x - x_0}{x_1 - x_0} = \dfrac{y - y_0}{y_1 - y_0} = \dfrac{z - z_0}{z_0 - z_1}$ shown

b $s = \dfrac{x + 2}{a} = \dfrac{y - y_0}{-3} = \dfrac{z - 2}{1}$

First line direction $\underset{\sim}{v}_1 = a\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}$

Second line $x = -4 + 4t,\ y = -6 + 2t,\ z = 9 - 2t$

Has direction $\underset{\sim}{v}_2 = 4\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$

The lines intersect at 90° $\Rightarrow \underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2 = 0$

$\Rightarrow 4a - 6 - 2 = 0$

$4a = 8$

$a = 2$

Line 1: $x = -2 + 9s = -2 + 2s$

$y = y_0 - 3s$

$z = 2 + s$

$$\left.\begin{aligned} &\Rightarrow (1)\ -2 + 2s = -4 + 4t \\ &\quad (2)\ \ y_0 - 3s = -6 + 2t \\ &\quad (3)\ \quad 2 + s = 9 - 2t \end{aligned}\right\}\ \text{given they intersect}$$

(3) $s = 7 - 2t$ into (1)

$-2 + 2\,(7 - 2t) = -4 + 4t$

$-2 + 14 - 4t = -4 + 4t$

$16 = 8t$

$t = 2 \Rightarrow s = 7 - 4 = 3$

$t = 2 \quad s = 3$

$\Rightarrow x = -2 + 6 = -4 + 8 = 4$

$z = 2 + 3 = 9 - 4 = 5$

$y = -6 + 4 = -2 = y_0 - 9$

$\Rightarrow y_0 = 7 \quad a = 2$

13 Line 1: $x = x_0 - 2t,\ y = -1 + bt,\ z = -6 + t$

Line 2: $s = \dfrac{x + 5}{a} = \dfrac{y - 11}{4} = \dfrac{z - 8}{c}$

Intersect at $(1, y_0, -2)$ at 30°

Line 1:

(1) $1 = x_0 - 2t$

(2) $y_0 = -1 + bt$

(3) $-2 = -6 + t \Rightarrow t = 4$

(1) $x_0 = 1 + 2t \Rightarrow x_0 = 9$

(2) $y_0 = -1 + 4b$

Line 2:

$x = -5 + as,\ y = 11 + 4s,\ z = 8 + \text{cs}$

(4) $1 = -5 + as \quad \Rightarrow as = 6$

(5) $y_0 = 11 + 4s \ \Rightarrow 11 + 4s = -1 + 4b$

(6) $-2 = 8 + cs \quad \Rightarrow cs = -10$

(5) $12 = 4b - 4s \ \Rightarrow b - s = 3$

(4) $s = \dfrac{6}{a} = \dfrac{-10}{c} \quad \Rightarrow 5a = -3c$

$\underset{\sim}{v}_1 = -2\underset{\sim}{i} + b\underset{\sim}{j} + \underset{\sim}{k}, \qquad v_2 = a\underset{\sim}{i} + 4\underset{\sim}{j} + c\underset{\sim}{k}$

$|\underset{\sim}{v}_1| = \sqrt{5 + b^2} \qquad |\underset{\sim}{v}_2| = \sqrt{16 + a^2 + c^2}$

$\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2 = -2a + 4b + c$

(5) $b + \dfrac{10}{c} = 3$, (4) $5a + 3c = 0$

(7) $\cos(30°) = \dfrac{\sqrt{3}}{2} = \dfrac{-2a + 4b + c}{\sqrt{5 + b^2}\,\sqrt{16 + a^2 + c^2}}$

Solving (4) (5) (7) $a, b, c \in z$ using CAS $a, b, c \in z$

$\Rightarrow a = -3,\ b = 1,\ c = 5,\ x_0 = 9,\ y_0 = 3$

4.3 Exam questions

1 $\underset{\sim}{a} = -3\underset{\sim}{i} + 5\underset{\sim}{k}$

$\underset{\sim}{b} = 2\underset{\sim}{i} + 5\underset{\sim}{j} - 7\underset{\sim}{k}$

$\underset{\sim}{v} = \underset{\sim}{b} - \underset{\sim}{a}$

$= 5\underset{\sim}{i} + 5\underset{\sim}{j} - 12\underset{\sim}{k}$

$\underset{\sim}{r} = \underset{\sim}{a} + t\underset{\sim}{v}$

$= \left(-3\underset{\sim}{i} + 5\underset{\sim}{k}\right) + t\left(5\underset{\sim}{i} + 5\underset{\sim}{j} - 12\underset{\sim}{k}\right)$

The correct answer is **A**.

2 $\dfrac{x - 2}{a} = \dfrac{y + 1}{-2} = \dfrac{z - 7}{-4} = t$

$x = 2 + at,\ y = -2t - 1,\ z = 7 - 4t$ [1 mark]

$y = -5 = -2t - 1$

$2t = 4$

$t = 2$

$x = 8 = 2 + 2a$ [1 mark]

$2a = 6$

$a = 3$

$\underset{\sim}{r} = \left(2\underset{\sim}{i} - \underset{\sim}{j} + 7\underset{\sim}{k}\right) + t\left(3\underset{\sim}{i} - 2\underset{\sim}{j} - 4\underset{\sim}{k}\right)$ [1 mark]

3 Line 1: $A(5, 4, -1)\ \ B(9, 14, -7)$

$\underset{\sim}{v}_1 = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= 4\underset{\sim}{i} + 10\underset{\sim}{j} - 6\underset{\sim}{k}$

$t = \dfrac{x - 5}{4} = \dfrac{y - 4}{10} = \dfrac{z + 1}{-6}$

$x = 5 + 4t,\ y = 4 + 10t,\ z = -1 - 6t$ [1 mark]

Line 2: $P(5, 3, 10)\ \ Q(6, 5, 14)$

$\underset{\sim}{v}_2 = \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= \underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}$

$s = \dfrac{x - 5}{1} = \dfrac{y - 3}{2} = \dfrac{z - 10}{4}$

$x = 5 + s,\ y = 3 + 2s,\ z = 10 + 4s$ [1 mark]

(1) $5 + 4t = 5 + s \qquad \Rightarrow s = 4t$ into (2)

(2) $4 + 10t = 3 + 2s \qquad 4 + 10t = 3 + 8t$

(3) $-1 - 6t = 10 + 4s \qquad 2t = -1$

$t = -\dfrac{1}{2} \Rightarrow s = -2$

Check in (3) LHS $= -1 - 6t = 2$

RHS $= 10 + 4s = 2$ [1 mark]

$t = -\dfrac{1}{2} \Rightarrow x = 3,\ y = -1,\ z = 2$

$s = -2 \Rightarrow x = 3,\ y = -1,\ z = 2$

Point of intersection $(3, -1, 2)$

$\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2 = 4 + 20 - 24 = 0$ [1 mark]

Lines intersect at 90° [1 mark]

4.4 Planes

4.4 Exercise

1 Given the point $(5, -2, 3)$, $x = 5$, $y = -2$, $z = 3$

LHS $= 2x + 3y + z = 2 \times 5 + 3 \times (-2) - 3 = 1 =$ RHS

Yes, the point does lie on the plane.

2 a Given the point $(2, -1, -3)$, $x = 2$, $y = -1$, $z = -3$,

LHS $= x - 2y - 3z = 2 + 2 + 9 = d = 13$

b $(x_0, 2, -4)$, $x = x_0,$ $y = 2$, $z = -4$, plane $2x + y - 3z = 10$

LHS $= 2x_0 + 2 + 12 = 10$, $2x_0 = -4$, $x_0 = -2$

3 a $P_0(2, -3, 5)$ $\quad \underset{\sim}{n} = \underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}$

$1(x - 2) + 3(y + 3) - 2(z - 5) = 0$

$x - 2 + 3y + 9 - 2z + 10 = 0$

$-x - 3y + 2z = 17$

b $P_0(1, 0, 4)$ $\quad \underset{\sim}{n} = -\underset{\sim}{j} + 3\underset{\sim}{k}$

$0(x - 1) - 2(y - 0) + 3(z - 4) = 0$

$-2y + 3z - 12 = 0$

$-2y + 3z = 12$

c $P_0(3, -2, -1)$ $\quad \underset{\sim}{n} = -\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}$

$-(x - 3) + 1(y + 2) + 4(z + 1) = 0$

$-x + 3 + y + 2 + 4z + 4 = 0$

$-x + y + 4z = -9$

$x - y - 4z = 9$

4 a $A(1, -2, 3)$ $\quad B(-2, 0, 2)$ $\quad C(3, -4, 2)$

$\overrightarrow{OA} = \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$

$\overrightarrow{OB} = -2\underset{\sim}{i} + 2\underset{\sim}{k}$

$\overrightarrow{OC} = 3\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ $\qquad \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 2\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -3 & 2 & -1 \\ 2 & -2 & -1 \end{vmatrix}$$

$$= \underset{\sim}{i}\begin{vmatrix} 2 & -1 \\ -2 & -1 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} -3 & -1 \\ 2 & -1 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} -3 & 2 \\ 2 & -2 \end{vmatrix}$$

$= -4\underset{\sim}{i} - 5\underset{\sim}{j} + 2\underset{\sim}{k}$ $\quad$ use $B(-2, 0, 2)$

Plane $\quad -4(x + 2) - 5(y - 0) + 2(z - 2) = 0$

$-4x - 8 - 5y + 2z - 4 = 0$

$-4x - 5y + 2z = 12$

b $A(-2, 3, 1)$ $\quad B(2, -1, 3)$ $\quad C(-1, -1, 0)$

$\overrightarrow{OA} = -2\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$

$\overrightarrow{OB} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$

$\overrightarrow{OC} = -\underset{\sim}{i} - \underset{\sim}{j}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 4\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}$ $\qquad \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \underset{\sim}{i} - 4j - \underset{\sim}{k}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 4 & -4 & 2 \\ 1 & -4 & -1 \end{vmatrix}$$

$$= \underset{\sim}{i}\begin{vmatrix} -4 & 2 \\ -4 & -1 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 4 & 2 \\ 1 & -1 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 4 & -4 \\ 1 & -4 \end{vmatrix}$$

$= 12\underset{\sim}{i} + 6\underset{\sim}{j} - 12\underset{\sim}{k}$

Use $\underset{\sim}{n} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$ $\quad C(-1, -1, 0)$

$2(x + 1) + 1(y + 1) - 2(z - 0) = 0$

$2x + 2 + y + 1 - 2z = 0$

$2x + y - 2z = -3$

$-2x - y + 2z = 3$

c $A(3, -2, 5)$ $\quad B(1, 2, 3)$ $\quad C(2, -1, 3)$

$\overrightarrow{OA} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + 5\underset{\sim}{k}$

$\overrightarrow{OB} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$

$\overrightarrow{OC} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -2\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k}$ $\qquad \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -\underset{\sim}{i} + j - 2\underset{\sim}{k}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -2 & 4 & -2 \\ -1 & 1 & -2 \end{vmatrix}$$

$$= \underset{\sim}{i}\begin{vmatrix} 4 & -2 \\ 1 & -2 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} -2 & -2 \\ -1 & -2 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} -2 & 4 \\ -1 & 1 \end{vmatrix}$$

$= -6\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}$

Or

$\underset{\sim}{n} = 3\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$ $\quad$ use $A(3, -2, 5)$

Plane $\quad 3(x - 3) + 1(y + 2) - 1(z - 5) = 0$

$3x - 9 + y + 2 - z + 5 = 0$

$3x + y - z = 2$

5 a $2x - y + 2z = 6$

$\underset{\sim}{n} = 2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ $\quad |\underset{\sim}{n}| = \sqrt{2^2 + (-1)^2 + 2^2} = 3$

$$\text{Distance} = \frac{d}{|\underset{\sim}{n}|} = \frac{6}{3} = 2$$

b $3x + 5y - 4z = 10$

$\underset{\sim}{n} = 3\underset{\sim}{i} + 5\underset{\sim}{j} - 4\underset{\sim}{k}$ $\quad |\underset{\sim}{n}| = \sqrt{3^2 + 5^2 + (-4)^2} = \sqrt{50} = 5\sqrt{2}$

$$\text{Distance} = \frac{d}{|\underset{\sim}{n}|} = \frac{10}{5\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \sqrt{2}$$

c $-4x - 5y + 2z = 15$

$\underset{\sim}{n} = -4\underset{\sim}{i} - 5\underset{\sim}{j} + 2\underset{\sim}{k}$

$|\underset{\sim}{n}| = \sqrt{(4)^2 + (-5)^2 + 2^2} = \sqrt{45} = 3\sqrt{5}$

$$\text{Distance} = \frac{d}{|\underset{\sim}{n}|} = \frac{15}{3\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \sqrt{5}$$

6 a $A(1, -2, 4)$ $\quad 2x - 2y + 3z = 6$

$\overrightarrow{OA} = \underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}$ $\quad \underset{\sim}{n} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$

$|\underset{\sim}{n}| = \sqrt{2^2 + (-2)^2 + 3^2} = \sqrt{17}$

Let $x = 3$, $y = 0$, $z = 0$ $\quad P_0(3, 0, 0)$ lies on the plane

$\overrightarrow{OP_0} = 3\underset{\sim}{i}$

$\overrightarrow{AP_0} = \overrightarrow{OP_0} - \overrightarrow{OA}$

$= 2\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k}$

$\underset{\sim}{n} \cdot \overrightarrow{AP_0} = 4 - 4 - 12 = -12$

$$D = \frac{|\underset{\sim}{n} \cdot \overrightarrow{AP_0}|}{|\underset{\sim}{n}|} = \frac{12}{\sqrt{17}} = \frac{12\sqrt{17}}{17}$$

b $A(3, 2, -1)$ $\quad -3x + 6y + 2z = 2$

$\overrightarrow{OA} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ $\quad \underset{\sim}{n} = -3\underset{\sim}{i} + 6\underset{\sim}{j} + 2\underset{\sim}{k}$

$|\underset{\sim}{n}| = \sqrt{(-3)^2 + 6^2 + 2^2} = 7$

Let $z = 1 \Rightarrow x = 0$, $y = 0$ $\quad P_0(0, 0, 1)$ lies on the plane

$\overrightarrow{OP}_0 = \underset{\sim}{k}$

$\overrightarrow{AP}_0 = \overrightarrow{OP}_0 - \overrightarrow{OA}$

$= -3\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}$

$\underset{\sim}{n} \cdot \overrightarrow{AP}_0 = 9 - 12 + 4 = 1$

$$D = \frac{\left|\underset{\sim}{n} \cdot \overrightarrow{AP}_o\right|}{\left|\underset{\sim}{n}\right|} = \frac{1}{7}$$

c $A(-2, 1, -3) \qquad -2x + y - 2z = 5$

$\overrightarrow{OA} = -2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k} \qquad \underset{\sim}{n} = -2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$

$\left|\underset{\sim}{n}\right| = \sqrt{(-2)^2 + 1^2 + (-2)^2} = 3$

Let $y = 5 \quad x = 0,\ z = 0 \quad P_0(0, 5, 0)$ lines on the plane

$\overrightarrow{OP}_0 = 5\underset{\sim}{j}$

$\overrightarrow{AP}_0 = \overrightarrow{OP}_0 - \overrightarrow{OA}$

$= 2\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k}$

$\underset{\sim}{n} \cdot \overrightarrow{AP}_0 = -4 + 4 - 6 = -6$

$$D = \frac{\left|\underset{\sim}{n} \cdot \overrightarrow{AP}_0\right|}{\left|\underset{\sim}{n}\right|} = \frac{6}{3} = 2$$

7 a $-x + y + 3z = 1$

$\underset{\sim}{n}_1 = -\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k} \qquad \left|\underset{\sim}{n}_1\right| = \sqrt{(-1)^2 + 1^2 + 3^2} = \sqrt{11}$

$2x - 2y - 6z = 3$

$\underset{\sim}{n}_2 = 2\underset{\sim}{i} - 2\underset{\sim}{j} - 6\underset{\sim}{k} \qquad \left|\underset{\sim}{n}_2\right| = \sqrt{2^2 + (-2)^2 + (-6)^2} = \sqrt{44}$

$\underset{\sim}{n}_2 = -2\underset{\sim}{n}_1$

Planes on opposite sides of the origin

$D = D_1 + D_2$

$$= \frac{1}{\sqrt{11}} + \frac{3}{\sqrt{44}} \qquad = \frac{1}{\sqrt{11}} + \frac{3}{2\sqrt{11}}$$

$$= \frac{\sqrt{11}}{11} + \frac{3\sqrt{11}}{22}$$

$$= \frac{5\sqrt{11}}{22}$$

b $3x - 4y + 12z = 17$

$6x - 8y + 24z = 8$

$\underset{\sim}{n}_1 = 3\underset{\sim}{i} - 4\underset{\sim}{j} + 12\underset{\sim}{k} \qquad \left|\underset{\sim}{n}_1\right| = \sqrt{3^2 + (-4)^2 + 12^2} = 13$

$\underset{\sim}{n}_2 = 6\underset{\sim}{i} - 8\underset{\sim}{j} + 24\underset{\sim}{k} \qquad \left|\underset{\sim}{n}_2\right| = \sqrt{6^2 + (-8)^2 + 24^2} = 26$

$\underset{\sim}{n}_2 = 2\underset{\sim}{n}_1$

Planes on same side of the origin

$$D = \frac{17}{13} - \frac{8}{26}$$

$D = 1$

c $3x - 6y - 2z = 2$

$\underset{\sim}{n}_1 = 3\underset{\sim}{i} - 6\underset{\sim}{j} - 2\underset{\sim}{k} \qquad \left|\underset{\sim}{n}_1\right| = \sqrt{3^2 + (-6)^2 + (-2)^2} = 7$

$-6x + 12y + 4z = 9$

$\underset{\sim}{n}_2 = -6\underset{\sim}{i} + 12\underset{\sim}{j} + 4\underset{\sim}{k} \qquad \left|\underset{\sim}{n}_2\right| = \sqrt{(-6)^2 + 12^2 + 4^2} = 14$

$\underset{\sim}{n}_2 = -2\underset{\sim}{n}_1$

Planes on opposite sides of the origin

$D = D_1 + D_2$

$$= \frac{2}{7} + \frac{9}{14}$$

$$= \frac{13}{14}$$

8 a Line $\dfrac{x + 16}{3} = \dfrac{y - 13}{-2} = \dfrac{z + 13}{2}$

$\underset{\sim}{v} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k} \qquad \left|\underset{\sim}{v}\right| = \sqrt{3^2 + (-2)^2 + 2^2} = \sqrt{17}$

Plane $4x + 2y + 5z = 13$

$\underset{\sim}{n} = 4\underset{\sim}{i} + 2\underset{\sim}{j} + 5\underset{\sim}{k} \qquad \left|\underset{\sim}{n}\right| = \sqrt{4^2 + 2^2 + 5^2} = \sqrt{45}$

$\underset{\sim}{v} \cdot \underset{\sim}{n} = 12 - 4 + 10 = 18$

$$\sin(\theta) = \frac{\left|\underset{\sim}{v} \cdot \underset{\sim}{n}\right|}{\left|\underset{\sim}{v}\right|\left|\underset{\sim}{n}\right|} = \frac{18}{\sqrt{17}\sqrt{45}}$$

$$\theta = \sin^{-1}\left(\frac{18}{\sqrt{17}\sqrt{45}}\right) = 40.6°$$

b Line $\dfrac{x - 13}{3} = \dfrac{y - 4}{2} = \dfrac{z + 7}{-2}$

$\underset{\sim}{v} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k} \qquad \left|\underset{\sim}{v}\right| = \sqrt{3^2 + 2^2 + (-2)^2} = \sqrt{17}$

Plane $2x + 3y - 2z = 4$

$\underset{\sim}{n} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k} \qquad \left|\underset{\sim}{n}\right| = \sqrt{2^2 + 3^2 + (-2)^2} = \sqrt{17}$

$\underset{\sim}{v} \cdot \underset{\sim}{n} = 6 + 6 + 4 = 16$

$$\sin(\theta) = \frac{\left|\underset{\sim}{v} \cdot \underset{\sim}{n}\right|}{\left|\underset{\sim}{v}\right|\left|\underset{\sim}{n}\right|} = \frac{16}{\sqrt{17}\sqrt{17}}$$

$$\theta = \sin^{-1}\left(\frac{16}{17}\right) = 70.3°$$

c Line $\dfrac{x - 9}{-2} = \dfrac{y + 5}{3} = \dfrac{z - 10}{-4}$

$\underset{\sim}{v} = -2\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k} \qquad \left|\underset{\sim}{v}\right| = \sqrt{(-2)^2 + 3^2 + (-4)^2} = \sqrt{29}$

Plane $4x - y + 3z = 2$

$\underset{\sim}{n} = 4\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k} \qquad \left|\underset{\sim}{n}\right| = \sqrt{4^2 + (-1)^2 + 3^2} = \sqrt{26}$

$\underset{\sim}{v} \cdot \underset{\sim}{n} = -8 - 3 - 12 = -23$

$$\sin(\theta) = \frac{\left|\underset{\sim}{v} \cdot \underset{\sim}{n}\right|}{\left|\underset{\sim}{v}\right|\left|\underset{\sim}{n}\right|}$$

$$\theta = \sin^{-1}\left(\frac{23}{\sqrt{29}\sqrt{26}}\right) = 56.9°$$

9 a Line $t = \dfrac{x + 16}{3} = \dfrac{y - 13}{-2} = \dfrac{z + 13}{2}$

$x = -16 + 3t,\ y = 13 - 2t,\ z = -13 + 2t$

Plane $4x + 2y + 5z = 5$

$4(-16 + 3t) + 2(13 - 2t) + 5(-13 + 2t) = 5$

$-64 + 12t + 26 - 4t - 65 + 10t = 5$

$18t = 5 + 64 + 65 - 26 = 108$

$t = 6$

So $x = -16 + 18 = 2,\ y = 13 - 12 = 1,\ z = -13 + 12 = -1$

Point of intersection $(2, 1, -1)$

b Line $t = \dfrac{x - 13}{3} = \dfrac{y - 4}{2} = \dfrac{z + 7}{-2}$

$x = 13 + 3t,\ y = 4 + 2t,\ z = -7 - 2t$

Plane $2x + 3y - 2z = 4$

$2(13 + 3t) + 3(4 + 2t) - 2(-7 - 2t) = 4$

$26 + 6t + 12 + 6t + 14 + 4t = 4$

$16t = 4 - 14 - 12 - 26 = -48$

$t = -3$

So $x = 13 - 9 = 4,\ y = 4 - 6 = -2,\ z = -7 + 6 = -1$

Point of intersection $(4, -2, -1)$

c Line $t = \dfrac{x - 9}{-2} = \dfrac{y + 5}{3} = \dfrac{z - 10}{-4}$

$x = 9 - 2t,\ y = -5 + 3t,\ z = 10 - 4t$

Plane $4x - y + 3z = 2$

$4(9 - 2t) - (-5 + 3t) + 3(10 - 4t) = 2$

$36 - 8t + 5 - 3t + 30 - 12t = 2$

$36 + 5 + 30 - 2 = 23t$

$69 = 23t$

$t = 3$

So $x = 9 - 6 = 3$, $y = -5 + 9 = 4$, $z = 10 - 12 = -2$

Point of intersection $(3, 4, -2)$

10 a Line $t = \frac{x+3}{3} = \frac{y-2}{2} = \frac{z+1}{4}$

$x = -3 + 3t, y = 2 + 2t, z = -1 + 4t$

Plane $2x + 3y - 3z = 5$

$2(-3 + 3t) + 3(2 + 2t) - 3(-11 + 4t) = 5$

$-6 + 6t + 6 + 6t + 3 - 12t = 5$

$3 = 5$

Contradiction, no solution. The line and plane do not intersect.

b Line $t = \frac{x-13}{3} = \frac{y-4}{2} = \frac{z+7}{-2}$

$x = 13 + 3t, y = 4 + 2t, z = -7 - 2t$

Plane $2x - 4y - z = 17$

$2(13 + 3t) - 4(4 + 2t) - (-7 - 2t) = 17$

$26 + 6t - 16 - 8t + 7 + 2t = 17$

$0 = 0$

Consistent, true line lies in the plane, intersection is the line.

$\frac{x-13}{3} = \frac{y-4}{2} = \frac{z+7}{-2}$

c Line $t = \frac{x-5}{3} = \frac{y-3}{-2} = \frac{z-1}{2}$

$x = 5 + 3t, y = 3 - 2t, z = 1 + 2t$

Plane $2x - 2y - 5z = 2$

$2(5 + 3t) - 2(3 - 2t) - 5(1 + 2t) = 2$

$10 + 6t - 6 + 4t - 5 - 10t = 2$

$-1 = 2$

Contradiction, no solution. The line and plane do not intersect.

11 a $x + 2y + 3z = 1$ (1)

$2x - y + 4z = 3$ (2) let $z = t$

$x + 2y = 1 - 3t$ (1)

$2x - y = 3 - 4t$ (2)

$x + 2y = 1 - 3t$ (1)

$4x - 2y = 6 - 8t$ (2) × 2

Add $5x = 7 - 11t$

$x = \frac{1}{5}(7 - 11t)$

$2x + 4y = 2 - 6t$ (1) × 2

$2x - y = 3 - 4t$ (2)

Subtract $5y = -1 - 2t$

$y = \frac{1}{5}(-1 - 2t)$

Line $t = \frac{5x-7}{-11} = \frac{5y+1}{-2} = z$

Line $x = \frac{1}{5}(7 - 11t), \quad y = \frac{1}{5}(-1 - 2t), \quad z = t$

$x = \frac{1}{5}(7 - 11t), \quad y = -\frac{1}{5}(1 + 2t), \quad z = t$

b $-x + y + 3z = 1$ (1)

$2x - 2y - 6z = 3$ (2) let $z = t$

$-x + y = 1 - 3t$ (1)

$2x - 2y = 3 + 6t$ (2)

$-2x + 2y = 2 - 6t$ (1) × 2

$2x - 2y = 3 + 6t$ (2)

$0 = 5$ contradiction, planes are parallel, no solution.

c $2x - 3y + z = 11$ (1)

$3x + y + 2z = 7$ (2) let $z = t$

$2x - 3y = 11 - t$ (1)

$3x + y = 7 - 2t$ (2)

$2x - 3y = 11 - t$ (1)

$9x + 3y = 21 - 6t$ (2) × 3

Add $11x = 32 - 7t$

$x = \frac{1}{11}(32 - 7t)$

$-6x + 9y = -33 + 3t$ (1) × −3

$6x + 2y = 14 - 4t$ (2) × 2

Add $11y = -19 - t$

$y = \frac{1}{11}(-19 - t)$

Line $t = \frac{11y+19}{-1} = \frac{11x-32}{-7}$

$x = \frac{1}{11}(32 - 7t), \quad y = \frac{1}{11}(-19 - t), \quad z = t$

$x = \frac{1}{11}(32 - 7t), \quad y = \frac{-1}{11}(19 + t), \quad z = t$

12 a $x + 2y + 3z = 1$

Plane 1: $\underset{\sim}{n}_1 = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$

$|\underset{\sim}{n}_1| = \sqrt{1 + 4 + 9} = \sqrt{14}$

$2x - y + 4z = 3$

Plane 2: $\underset{\sim}{n}_2 = 2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}$

$|\underset{\sim}{n}_2| = \sqrt{2^2 + (-1)^2 + 4^2} = \sqrt{21}$

$\underset{\sim}{n}_1 \cdot \underset{\sim}{n}_2 = 2 - 2 + 12 = 12$

$\cos(\theta) = \frac{\underset{\sim}{n}_1 \cdot \underset{\sim}{n}_2}{|\underset{\sim}{n}_1||\underset{\sim}{n}_2|} = \frac{12}{\sqrt{14}\sqrt{21}}$

$\theta = \cos^{-1}\left(\frac{12}{\sqrt{14 \times 21}}\right) = 45.6°$

b $-x + y + 3z = 1$

Plane 1: $\underset{\sim}{n}_1 = -\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$

$\underset{\sim}{n}_2 = -2\underset{\sim}{n}_1$

$2x - 2y - 6z = 3$

Plane 2: $\underset{\sim}{n}_2 = 2\underset{\sim}{i} - 2\underset{\sim}{j} - 6\underset{\sim}{k}$

Normal parallel, angle is 0° or 18.0°

c $2x - 3y + z = 11$

Plane 1: $\underset{\sim}{n}_1 = 2\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}$

$|\underset{\sim}{n}_1| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14}$

$3x + y + 2z = 7$

Plane 2: $\underset{\sim}{n}_2 = 3\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$

$|\underset{\sim}{n}_2| = \sqrt{9 + 1 + 4} = \sqrt{14}$

$\underset{\sim}{n}_1 \cdot \underset{\sim}{n}_2 = 6 - 3 + 2 = 5$

$\cos(\theta) = \frac{\underset{\sim}{n}_1 \cdot \underset{\sim}{n}_2}{|\underset{\sim}{n}_1||\underset{\sim}{n}_2|} = \frac{5}{14}$

$\theta = \cos^{-1}\left(\frac{5}{14}\right) = 69.1°$

13 a Line through $A(1, -2, -3)$, $B(2, 1, 4)$
$\overrightarrow{OA} = \underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k}$ $\qquad \overrightarrow{OB} = 2\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}$
$\underset{\sim}{v} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \underset{\sim}{i} + 3\underset{\sim}{j} + 7\underset{\sim}{k}$
Line $t = \dfrac{x-1}{1} = \dfrac{y+2}{3} = \dfrac{z+7}{7}$ through A
$x = 1 + t, y = -2 + 3t, z = -3 + 7t$
Plane $-3x - y + 2z = 1$
$-3(1 + t) - (-2 + 3t) + 2(-3 + 7t) = 1$
$-3 - 3t + 2 - 3t - 6 + 14t = 1$
$8t = 8$
$t = 1$
Point $x = 2, y = -2 + 3 = 1, z = -3 + 7 = 4$
Point of intersection $(2, 1, 4)$
Direction of the line
$\underset{\sim}{v} = \underset{\sim}{i} + 3\underset{\sim}{j} + 7\underset{\sim}{k} \qquad |\underset{\sim}{v}| = \sqrt{1 + 9 + 49} = \sqrt{59}$
Normal to the plane
$\underset{\sim}{n} = -3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k} \qquad |\underset{\sim}{n}| = \sqrt{(-3)^2 + (-1)^2 + 2^2} = \sqrt{14}$
$\underset{\sim}{v} \cdot \underset{\sim}{n} = -3 - 3 + 14 = 8$
$\sin(\theta) = \dfrac{\underset{\sim}{v} \cdot \underset{\sim}{n}}{|\underset{\sim}{v}|\,|\underset{\sim}{n}|} = \dfrac{8}{\sqrt{59}\sqrt{14}}$
$\theta = \sin^{-1}\left(\dfrac{8}{\sqrt{59}\sqrt{14}}\right) = 16.2°$

b Plane through $A(2, 1, 5)$, $B(3, 2, 6)$, $C(1, -1, 1)$
$\overrightarrow{OA} = 2\underset{\sim}{i} + \underset{\sim}{j} + 5\underset{\sim}{k}$
$\overrightarrow{OB} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + 6\underset{\sim}{k}$
$\overrightarrow{OC} = \underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$
$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} \qquad \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$
$= \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k} \qquad = -\underset{\sim}{i} - 2\underset{\sim}{j} - 4\underset{\sim}{k}$
$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 1 & 1 \\ -1 & -2 & -4 \end{vmatrix}$
$= \underset{\sim}{i}\begin{vmatrix} 1 & 1 \\ -3 & -4 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 1 & 1 \\ -1 & -4 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix}$
$= -2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$
Plane through A
$-2(x - 2) + 3(y - 1) - 1(z - 5) = 0$
$-2x + 4 + 3y - 3 - z + 5 = 0$
$-2x + 3y - z = -6$
$2x - 3y + z = 6$
Line $t = \dfrac{x+2}{2} = \dfrac{y-1}{-1} = \dfrac{z-1}{-3}$
$x = -2 + 2t, y = 1 - t, z = 1 - 3t$
$2(-2 + 2t) - 3(1 - t) + (1 - 3t) = 6$
$-4 + 4t - 3 + 3t + 1 - 3t = 6$
$4t = 12$
$t = 3$
Point $x = -2 + 6 = 4, y = 1 - 3 = -2, z = 1 - 9 = -8$
Point of intersection $(4, -2, -8)$
Direction of line
$\underset{\sim}{v} = 2\underset{\sim}{i} - \underset{\sim}{j} - 3\underset{\sim}{k} \qquad |\underset{\sim}{v}| = \sqrt{2^2 + (-1)^2 + (-3)^2} = \sqrt{14}$
Normal to plane
$\underset{\sim}{n} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k} \qquad |\underset{\sim}{n}| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14}$
$\underset{\sim}{v} \cdot \underset{\sim}{n} = 4 + 3 - 3 = 4$
$\sin(\theta) = \dfrac{\underset{\sim}{v} \cdot \underset{\sim}{n}}{|\underset{\sim}{v}|\,|\underset{\sim}{n}|} = \dfrac{4}{\sqrt{14}\sqrt{14}}$
$\theta = \sin^{-1}\left(\dfrac{2}{7}\right)$
$= 16.6°$

c Plane $A(1, 0, -4)$ $B(2, 8, 1)$ $C(3, 2, -1)$
$\overrightarrow{OA} = \underset{\sim}{i} - 4\underset{\sim}{k}$
$\overrightarrow{OB} = 2\underset{\sim}{i} + 8\underset{\sim}{j} + \underset{\sim}{k}$
$\overrightarrow{OC} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$
$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \underset{\sim}{i} + 8\underset{\sim}{j} + 5\underset{\sim}{k}$
$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 2\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$
$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 8 & 5 \\ 2 & 2 & 3 \end{vmatrix}$
$= \underset{\sim}{i}\begin{vmatrix} 8 & 5 \\ 2 & 3 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 1 & 8 \\ 2 & 2 \end{vmatrix}$
$= 14\underset{\sim}{i} + 7\underset{\sim}{j} - 14\underset{\sim}{k}$
Take $\underset{\sim}{n} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$
Plane through A:
$2(x - 1) + 1(y - 0) - 2(z + 4) = 0$
$2x - 2 + y - 2z - 8 = 0$
$2x + y - 2z = 10$
Line through $P(-1, 6, 8)$ $Q(-3, 9, 13)$
$\overrightarrow{OP} = -\underset{\sim}{i} + 6\underset{\sim}{j} + 8\underset{\sim}{k}$
$\overrightarrow{OQ} = -3\underset{\sim}{i} + 9\underset{\sim}{j} + 13\underset{\sim}{k}$
$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$
$= -2\underset{\sim}{i} + 3\underset{\sim}{j} + 5\underset{\sim}{k}$
Line through P: $t = \dfrac{x+1}{-2} = \dfrac{y-6}{3} = \dfrac{z-8}{5}$
$x = -1 - 2t, y = 6 + 3t, z = 8 + 5t$
Intersection of line and plane
$2(-1 - 2t) + (6 + 3t) - 2(8 + 5t) = 10$
$-2 - 4t + 6 + 3t - 16 - 10t = 10$
$-11t = 10 + 16 + 2 - 6 = 22$
$t = -2$
So $x = -1 + 4 = 3, y = 6 - 6 = 0, z = 8 - 10 = -2$
Point of intersection $(3, 0, -2)$
Direction of line
$\underset{\sim}{v} = -2\underset{\sim}{i} + 3\underset{\sim}{j} + 5\underset{\sim}{k}$
$|\underset{\sim}{v}| = \sqrt{(-2)^2 + 3^2 + 5^2} = \sqrt{38}$
Normal to plane
$\underset{\sim}{n} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$
$|\underset{\sim}{n}| = \sqrt{2^2 + 1^2 + (-2)^2} = 3$
$\underset{\sim}{v} \cdot \underset{\sim}{n} = -4 + 3 - 10 = -11$
Angle $\sin(\theta) = \dfrac{|\underset{\sim}{v} \cdot \underset{\sim}{n}|}{|\underset{\sim}{v}|\,|\underset{\sim}{n}|} = \dfrac{11}{3\sqrt{38}}$
$\theta = \sin^{-1}\left(\dfrac{11}{3\sqrt{38}}\right)$
$= 36.5°$

14 Line $t = \dfrac{x-4}{2} = \dfrac{y-3}{-1} = \dfrac{z-1}{c}$
$x = 4 + 2t, y = 3 - t, z = 1 + tc$
Plane $3x - 2y - 2z = d$

$3(4+2t)-2(3-t)-2(1+tc)=\mathrm{d}$

$12+6t-6+2t-2-2tc=d$

$(8-2c)t=d-4$

a Unique solution $c\neq 4, \quad d\in R$

b Infinite $c=4, d=4 \quad 0=0$

c No solution $c=4, d\neq 4 \quad 0=x, x\neq 0$

15 Line $t=\dfrac{x-1}{2}=\dfrac{y-3}{b}=\dfrac{z-2}{-4}$

$x=1+2t, y=3+bt, z=2-4t$

Plane $4x-2y+3z=d$

$4(1+2t)-2(3+bt)+3(2-4t)=d$

$4+8t-6-2bt+6-12t=d$

$4-6+6-d=12t-8t+2bt$

$-d+4=(4+2b)t$

a Unique $b\neq -2, \quad d\in R$

b Infinite $b=-2, d=4 \quad 0=0$

c No solution $b=-2, d\neq 4 \quad 0=x, x\neq 0$

16 **a** $P_0(x_0,y_0,z_0) \qquad P_1(x_1,y_1,z_1) \qquad P_2(x_2,y_2,z_2)$

$\overrightarrow{P_1P_2}=(x_2-x_1)\underset{\sim}{i}+(y_2-y_1)\underset{\sim}{j}+(z_2-z_1)\underset{\sim}{k}=\overrightarrow{OP_2}-\overrightarrow{OP_1}$

$\overrightarrow{P_0P_1}=(x_1-x_0)\underset{\sim}{i}+(y_1-y_0)\underset{\sim}{j}+(z_1-z_0)\underset{\sim}{k}=\overrightarrow{OP_1}-\overrightarrow{OP_0}$

Normal to the plane $\underset{\sim}{n}=\overrightarrow{P_1P_2}\times\overrightarrow{P_0P_1}$

$$\underset{\sim}{n}=\begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ x_2-x_1 & y_2-y_1 & z_2-z_1\\ x_1-x_0 & y_1-y_0 & z_1-z_0\end{vmatrix}$$

$$=A\underset{\sim}{i}+B\underset{\sim}{j}+C\underset{\sim}{k}$$

Where $A=\begin{vmatrix}y_2-y_1 & z_2-z_1\\ y_1-y_0 & z_1-z_0\end{vmatrix}$

$B=-\begin{vmatrix}x_2-x_1 & z_2-z_1\\ x_1-x_0 & z_1-z_0\end{vmatrix}$

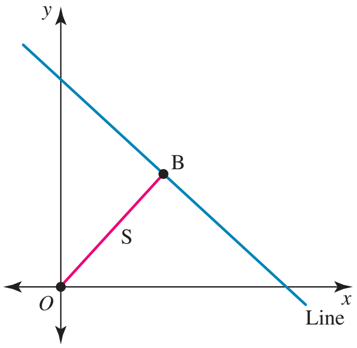
$C=\begin{vmatrix}x_2-x_1 & y_2-y_1\\ x_1-x_0 & y_1-y_0\end{vmatrix}$

Plane $A(x-x_0)+B(y-y_0)+C(z-z_0)=0$ this is equal to

$$\begin{vmatrix}x-x_0 & y-y_0 & z-z_0\\ x_2-x_1 & y_2-y_1 & z_2-z_1\\ x_1-x_0 & y_1-y_0 & z_1-z_0\end{vmatrix}=0 \text{ shown}$$

b Line $ax+by=d$

Method I: using calculus



$B(x,y)$

$s=d(OB)$

$=\sqrt{x^2+y^2}$

But $y=\dfrac{1}{b}(d-ax)$

$$s=\sqrt{x^2+\frac{1}{b^2}(d-ax)^2}$$

$$s^2=x^2+\frac{1}{b^2}(d-ax)^2$$

$$\frac{d(s^2)}{dx}=2x-\frac{2a}{b^2}(d-ax)=0$$

$$x=\frac{a}{b^2}(d-ax)$$

$$b^2x=ad-a^2x$$

$$(a^2+b^2)x=ad$$

$$x=\frac{ad}{a^2+b^2}$$

$$s_{mn}^2=\left(\frac{ad}{a^2+b^2}\right)^2+\frac{1}{b^2}\left(d-\frac{a^2d}{a^2+b^2}\right)^2$$

$$=\frac{a^2d^2}{(a^2+b^2)^2}+\frac{1}{b^2}\frac{(d(a^2+b^2)-a^2d)^2}{(a^2+b^2)^2}$$

$$=\frac{a^2d^2}{(a^2+b^2)^2}+\frac{d^2b^4}{b^2(a^2+b^2)^2}$$

$$s_{mn}^2=\frac{d^2(a^2+b^2)}{(a^2+b^2)^2}=\frac{d^2}{(a^2+b^2)}$$

So $s_{mn}=\dfrac{d}{\sqrt{a^2+b^2}}$ shown

Method II using geometry

Line $ax+by=d$

$by=d-ax$

$y=\dfrac{d}{b}-\dfrac{a}{b}x \qquad m_1=-\dfrac{a}{b}$

Perpendicular line $m_2=\dfrac{b}{a}$ passes through the origin

$y=\dfrac{bx}{a}$ this line intersect $ax+by=d$

at B:

$$ax+\frac{b^2x}{a}=d$$

$$\frac{(a^2+b^2)x}{a}=d$$

$$X_B=\frac{ad}{a^2+b^2} \qquad Y_B=\frac{bd}{a^2+b^2}$$

$$B\left(\frac{ad}{a^2+b^2},\frac{bd}{a^2+b^2}\right)$$

$$d(OB)=\sqrt{\frac{a^2d^2}{(a^2+b^2)^2}+\frac{b^2d^2}{(a^2+b^2)^2}}$$

$$=\sqrt{\frac{(a^2+b^2)d^2}{(a^2+b^2)^2}}$$

$$=\frac{d}{\sqrt{a^2+b^2}} \quad \text{shown}$$

17 **a** Plane 1 $\quad 2x-y+2z=6$

$\underset{\sim}{n}_1=2\underset{\sim}{i}-\underset{\sim}{j}+2\underset{\sim}{k} \qquad |\underset{\sim}{n}_1|=\sqrt{4+1+4}=3$

Distance from origin is $\dfrac{d_1}{|\underset{\sim}{n}_1|}=\dfrac{6}{3}=2$

Plane 2: $\quad ax+by+cz=45$

$\underset{\sim}{n}_2=a\underset{\sim}{i}+b\underset{\sim}{j}+c\underset{\sim}{k}$, since the planes are parallel

$\underset{\sim}{n}_2=\lambda\underset{\sim}{n}_1 \Rightarrow a=2\lambda, b=-\lambda, c=2\lambda$

$\lambda>0 \quad |\underset{\sim}{n}_2|=\sqrt{a^2+b^2+c^2}=3\lambda$

Distance from origin $\dfrac{45}{3\lambda}=D_2$

$$D = D_2 - D_1 = \frac{45}{3\lambda} - 2 = 3$$

$$\frac{45}{3\lambda} = 5$$

$$\lambda = 3$$

Plane $6x - 3y + 6z = 45$

$a = 6, b = -3, c = 6$

b Line let $t = \dfrac{x-4}{a} = \dfrac{y+7}{b} = \dfrac{z+7}{c}$

Point $(1, -2, -3)$ lies on the line

$$x = 1 \Rightarrow (1)\ -\frac{3}{a} = t \qquad a = -\frac{3}{t}$$

$$y = -2 \Rightarrow (2)\ \frac{5}{b} = t \qquad b = \frac{5}{t}$$

$$z = -3 \Rightarrow (3)\ \frac{4}{c} = t \qquad c = \frac{4}{t}$$

Plane $2x - y - z = d$

$(1, -2, -3) \Rightarrow d = 2 + 2 + 3 = 7 \qquad d = 7$

Normal to plane $\underset{\sim}{n} = 2\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k} \qquad |\underset{\sim}{n}| = \sqrt{6}$

Direction of line $\underset{\sim}{v} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$

$\underset{\sim}{v} \cdot \underset{\sim}{n} = 2a - b - c$

$|\underset{\sim}{v}| = \sqrt{a^2 + b^2 + c^2} = \sqrt{50} = 5\sqrt{2}$

Angel between line and plane is 60°

$$\sin(60°) = \frac{\underset{\sim}{v} \cdot \underset{\sim}{n}}{|\underset{\sim}{v}|\,|\underset{\sim}{n}|}$$

$$\sin(60°) = \frac{\sqrt{3}}{2} = \frac{2a - b - c}{5\sqrt{2} \times \sqrt{6}}$$

(4) $2a - b - c = 15$ from (1) (2) (3)

$$\frac{-6}{t} - \frac{5}{t} - \frac{4}{t} = 15$$

$$-\frac{15}{t} = 15 \Rightarrow t = -1$$

$\Rightarrow a = 3, b = -5, c = -4, d = 7$

4.4 Exam questions

1 $A(1, 3, -4)$, $B(2, 0, -1)$ and $C(0, -3, 4)$

$\overrightarrow{AB} = \underset{\sim}{i} - 3\underset{\sim}{j} + 3\underset{\sim}{k}$, $\overrightarrow{AC} = -\underset{\sim}{i} - 6\underset{\sim}{j} + 8\underset{\sim}{k}$ [1 mark]

$$\underset{\sim}{n} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & -3 & 3 \\ -1 & -6 & 8 \end{vmatrix}$$

$\underset{\sim}{n} = -6\underset{\sim}{i} - 11\underset{\sim}{j} - 9\underset{\sim}{k}$ [1 mark]

$0 = -6(x - 1) - 11(y - 3) - 9(z + 4)$

$0 = -6x + 6 - 11y + 33 - 9z - 36$

$6x + 11y + 9z = 3$ [1 mark]

2 $\underset{\sim}{n} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k} \qquad P_0 = (1, 3, -1)$

$2(x - 1) + 3(y - 3) - (z + 1) = 0$ [1 mark]

$2x - 2 + 3y - 9 - z - 1 = 0$

$2x + 3y - z = 12$ [1 mark]

3 $x = t + 3$, $y = 5 - t$ and $z = 2t + 1$

$7 = 3x - 2y + z$

$7 = 3(t + 3) - 2(5 - t) + 1(2t + 1)$

$7 = 7t$

$t = 1$ [1 mark]

$x = 1 + 3 = 4$

$y = 5 - 1 = 4$

$z = 2 \times 1 + 1 = 3$

Point $(4, 4, 3)$ [1 mark]

4.5 Review

4.5 Exercise

Technology free: short answer

1 a i $\underset{\sim}{a} = 3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k} \qquad \underset{\sim}{b} = 4\underset{\sim}{i} - 2\underset{\sim}{j} + 5\underset{\sim}{k}$

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 3 & -1 & 2 \\ 4 & -2 & 5 \end{vmatrix}$$

$$= \underset{\sim}{i}\begin{vmatrix} -1 & 2 \\ -2 & 5 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 3 & -1 \\ 4 & -2 \end{vmatrix}$$

$$= -\underset{\sim}{i} - 7\underset{\sim}{j} - 2\underset{\sim}{k}$$

$$|\underset{\sim}{a} \times \underset{\sim}{b}| = \sqrt{1 + 49 + 4} = \sqrt{54} = \sqrt{9 \times 6} = 3\sqrt{6}$$

$$\hat{\underset{\sim}{n}} = \pm\frac{1}{3\sqrt{6}}(\underset{\sim}{i} + 7\underset{\sim}{j} + 2\underset{\sim}{k})$$

ii $\underset{\sim}{r} = 5\underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k} \qquad \underset{\sim}{s} = -2\underset{\sim}{i} + 3\underset{\sim}{j} + k$

$$\underset{\sim}{r} \times \underset{\sim}{s} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 5 & -2 & -3 \\ -2 & 3 & 1 \end{vmatrix}$$

$$= \underset{\sim}{i}\begin{vmatrix} -2 & -3 \\ 3 & 1 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 5 & -3 \\ -2 & 1 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 5 & -2 \\ -2 & 3 \end{vmatrix}$$

$$= 7\underset{\sim}{i} + \underset{\sim}{j} + 11\underset{\sim}{k}$$

$$|\underset{\sim}{r} \times \underset{\sim}{s}| = \sqrt{7^2 + 1^2 + 11^2} = \sqrt{171} = \sqrt{9 \times 19} = 3\sqrt{19}$$

$$\hat{\underset{\sim}{n}} = \pm\frac{1}{3\sqrt{19}}\left(7\underset{\sim}{i} + \underset{\sim}{j} + 11\underset{\sim}{k}\right)$$

b i $A(1, -2, -3), B(2, 1, 4), C(3, -1, -4)$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \underset{\sim}{i} + 3\underset{\sim}{j} + 7\underset{\sim}{k}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \underset{\sim}{i} - 2\underset{\sim}{j} - 8\underset{\sim}{k}$

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 3 & 7 \\ 1 & -2 & -8 \end{vmatrix}$$

$$= \underset{\sim}{i}\begin{vmatrix} 3 & 7 \\ -2 & -8 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 1 & 7 \\ 1 & -8 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 1 & 3 \\ 1 & -2 \end{vmatrix}$$

$$= -10\underset{\sim}{i} + 15\underset{\sim}{j} - 5\underset{\sim}{k}$$

$$\text{Area} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$= \frac{1}{2}\sqrt{(-10)^2 + 15^2 + (-5)^2}$$

$$= \frac{1}{2}\sqrt{350} = \frac{1}{2}\sqrt{25 \times 14}$$

$$= \frac{5\sqrt{14}}{2}$$

ii $\underset{\sim}{n} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}$

Plane $2(x-1) - 3(y+2) + (z+3) = 0$

$2x - 2 - 3y - 6 + z + 3 = 0$

$2x - 3y + z = 5$

c **i** Line through $P(-1, 3, 5)$, $Q(5, -1, 2)$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = 6\underset{\sim}{i} - 4\underset{\sim}{j} - 3\underset{\sim}{k}$ direction of the line through P

$$t = \frac{x+1}{6} = \frac{y-3}{-4} = \frac{z-5}{-3}$$

$x = -1 + 6t, y = 3 - 4t, z = 5 - 3t$

ii Plane through $P(-1, 3, 5)$, $Q(5, -1, 2)$, $R(2, 1, -1)$

$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = 3\underset{\sim}{i} - 2\underset{\sim}{j} - 6\underset{\sim}{k}$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 6 & -4 & -3 \\ 3 & -2 & -6 \end{vmatrix}$$

$$= \underset{\sim}{i}\begin{vmatrix} -4 & -3 \\ -2 & -6 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 6 & -3 \\ 3 & -6 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 6 & -4 \\ 3 & -2 \end{vmatrix}$$

$= 18\underset{\sim}{i} + 27\underset{\sim}{j}$

Let $\underset{\sim}{n} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$

Plane $2(x+1) + 3(y-3) = 0$

$2x + 2 + 3y - 9 = 0$

$2x + 3y = 7$

$x = -1 + 6t, y = 3 - 4t$

$2(-1 + 6t) + 3(3 - 4t) = 7$

$-2 + 12t + 9 - 12t = 7$

$0 = 0$ true

Line lies in the plane

iii Area of triangle $= \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|$

$$= \frac{1}{2}\sqrt{18^2 + 27^2}$$

$$= \frac{1}{2}\sqrt{1053} = \frac{1}{2}\sqrt{81 \times 13}$$

$$= \frac{9\sqrt{13}}{2}$$

2 **a** Line $x = 2 - t, y = 3 + 2t, z = -4 - 2t$

Let $s = \frac{x-1}{2} = y = \frac{z+5}{3}$

$x = 1 + 2s, y = s, z = -5 + 3s$

(1) $2 - t = 1 + 2s$

(2) $3 + 2t = s \Rightarrow$ (2) $s = 3 + 2t$

(3) $-4 - 2t = -5 + 3s$

(2) into (1)

$2 - t = 1 + 2(3 + 2t)$

$2 - t = 1 + 6 + 4t$

$5t = -5$

$t = -1 \Rightarrow s = 3 - 2 = 1$

Check in (3)

LHS $= -4 - 2t = -4 + 2 = -2$

RHS $= -5 + 3s = -5 + 3 = -2$

$t = -1 \quad x = 3, \quad y = 1, \quad z = -2$

$s = 1 \quad x = 3, \quad y = 1, \quad z = -2$

Point of intersection $(3, 1, -2)$

Line 1: $\underset{\sim}{v}_1 = \underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k} \quad \left|\underset{\sim}{v}_1\right| = \sqrt{(-1)^2 + 2^2 + (-2)^2} = 3$

Line 2: $\underset{\sim}{v}_2 = 2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k} \quad \left|\underset{\sim}{v}_2\right| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$

$\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2 = -2 + 2 - 6 = -6$

$$\text{Angle between: } \cos(\theta) = \frac{\left|\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2\right|}{\left|\underset{\sim}{v}_1\right|\left|\underset{\sim}{v}_2\right|} = \frac{6}{3\sqrt{14}}$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right) = 57.7°$$

b Line $t = \frac{x+5}{4} = \frac{y+6}{2} = \frac{z-7}{-3}$

$x = -5 + 4t, y = -6 + 2t, z = 7 - 3t$

Plane $2x + y - z = 3$

$2(-5 + 4t) + (-6 + 2t) - (7 - 3t) = 3$

$-10 + 8t - 6 + 2t - 7 + 3t = 3$

$13t = 26$

so $t = 2$

$x = -5 + 8 = 3, y = -6 + 4 = -2, z = 7 - 6 = 1$

Point of intersection $(3, -2, 1)$

Direction of line $\underset{\sim}{v} = 4\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$

$|\underset{\sim}{v}| = \sqrt{4^2 + 2^2 + (-3)^2} = \sqrt{29}$

Normal to plane $\underset{\sim}{n} = 2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$

$|\underset{\sim}{n}| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6}$

$\underset{\sim}{v} \cdot \underset{\sim}{n} = 8 + 2 + 3 = 13$

$$\text{Angle: } \sin(\theta) = \frac{\underset{\sim}{v} \cdot \underset{\sim}{n}}{|\underset{\sim}{v}||\underset{\sim}{n}|} = \frac{13}{\sqrt{29}\sqrt{6}}$$

$$\theta = \sin^{-1}\left(\frac{13}{\sqrt{29 \times 6}}\right)$$

$= 80.2°$

c $2x + y - z = 3$ (1)

$3x - 2y + 2z = 1$ (2)

Let $z = t$

(1) $2x + y = 3 + t$

(2) $3x - 2y = 1 - 2t$

(1) × 2 $4x + 2y = 6 + 2t$

add $3x - 2y = 1 - 2t$

$7x = 7$

$x = 1$

(1) × 3 $6x + 3y = 9 + 3t$

(2) × 2 $6x - 4y = 2 - 4t$

Subtract $7y = 7 + 7t$

$y = 1 + t$

Solution line $x = 1, y = 1 + t, z = t$

Plane 1 direction

$\underset{\sim}{n}_1 = 2\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k} \quad \left|\underset{\sim}{n}_1\right| = \sqrt{4 + 1 + 1} = \sqrt{6}$

Plane 2 direction

$\underset{\sim}{n}_2 = 3\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k} \quad \left|\underset{\sim}{n}_2\right| = \sqrt{9 + 4 + 4} = \sqrt{17}$

$\underset{\sim}{n}_1 \cdot \underset{\sim}{n}_2 = 6 - 2 - 2 = 2$

$$\cos(\theta) = \frac{n_1 \cdot n_2}{\left|\underset{\sim}{n}_1\right|\left|\underset{\sim}{n}_2\right|}$$

$$\cos(\theta) = \frac{2}{\sqrt{6}\sqrt{17}}$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{6\times 17}}\right)$$
$$= 78.6°$$

3 a Line $A(1, -2, 3),\ B(2, 1, 4)$

$\underset{\sim}{v} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \underset{\sim}{i} + 3\underset{\sim}{j} + 7k$

Line 1, use A: $\quad t = \dfrac{x-1}{1} = \dfrac{y+2}{3} = \dfrac{z+3}{7}$

$x = 1 + t,\ y = -2 + 3t,\ z = -3 + 7t$

Let $s = \dfrac{x+7}{3} = \dfrac{y-13}{-4} = \dfrac{z+2}{2}$

Line 2: $\quad x = -7 + 3s,\ y = 13 - 4s,\ z = -2 + 2s$

(1) $\quad 1 + t = -7 + 3s \Rightarrow t = 3s - 8$

(2) $\quad -2 + 3t = 13 - 4s$

(3) $\quad -3 + 7t = -2 + 2s$

Into (2) $\quad -2 + 3(3s - 8) = 13 - 4s$

$-2 + 9s - 24 = 13 - 4s$

$13s = 13 + 24 + 2 = 39$

$s = 3, \quad \Rightarrow \quad t = 9 - 8 = 1$

Check in (3) $\quad$ LHS $= -3 + 7 = 4$

RHS $= -2 + 6 = 4$ checks ok

$s = 3 \quad \Rightarrow \quad x = 2, \quad y = 1, \quad z = 4$

$t = 1 \quad \Rightarrow \quad x = 2, \quad y = 1, \quad z = 4$

Point of intersection $(2, 1, 4)$

Angle $\underset{\sim}{v}_1 = \underset{\sim}{i} + 3\underset{\sim}{j} + 7\underset{\sim}{k} \qquad |\underset{\sim}{v}_1| = \sqrt{1 + 9 + 49} = \sqrt{59}$

$\underset{\sim}{v}_2 = 3\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k} \qquad |\underset{\sim}{v}_2| = \sqrt{9 + 16 + 4} = \sqrt{29}$

$\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2 = 3 - 12 + 14 = 5$

$$\cos(\theta) = \frac{\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2}{|\underset{\sim}{v}_1||\underset{\sim}{v}_2|}$$

$$\cos(\theta) = \frac{5}{\sqrt{59}\sqrt{29}}$$

$$\theta = \cos^{-1}\left(\frac{5}{\sqrt{59\times 29}}\right)$$
$$= 83.1^{\circ}$$

b $-3x - y + 2z = 1$

$-3(1 + t) - (-2 + 3t) + 2(-3 + 7t) = 1$

$-3 - 3t + 2 - 3t - 6 + 14t = 1$

$8t = 8$

$t = 1$

$\Rightarrow \quad$ Point $\quad (2, 1, 4)$

$\underset{\sim}{n} = -3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k} \qquad |\underset{\sim}{n}| = \sqrt{9 + 1 + 4} = \sqrt{14}$

$\underset{\sim}{v} = \underset{\sim}{i} + 3\underset{\sim}{j} + 7\underset{\sim}{k} \qquad |\underset{\sim}{v}| = \sqrt{59}$

$\underset{\sim}{v} \cdot \underset{\sim}{n} = -3 - 3 + 14 = 8$

$$\sin\theta = \frac{\underset{\sim}{v} \cdot \underset{\sim}{n}}{|\underset{\sim}{v}||\underset{\sim}{n}|}$$
$$= \frac{8}{\sqrt{59}\sqrt{14}}$$

$$\theta = \sin^{-1}\left(\frac{8}{\sqrt{59\times 14}}\right)$$
$$= 16.2°$$

4 a i Line $t = \dfrac{x+1}{-2} = \dfrac{y-3}{4} = \dfrac{z+2}{5} \qquad 0(0, 0, 0)$

$x = -i - 2t,\ y = 3 + 4t,\ z = -2 + 5t$

$\underset{\sim}{v} = -2\underset{\sim}{i} + 4\underset{\sim}{j} + 5\underset{\sim}{k} \qquad |\underset{\sim}{v}| = \sqrt{4 + 16 + 25} = 3\sqrt{5}$

$P_0\ (-1, 3, -2)$

$$\overrightarrow{OP_0} \times \underset{\sim}{v} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -1 & 3 & -2 \\ -2 & 4 & 5 \end{vmatrix}$$

$$= \underset{\sim}{i}\begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} -1 & -2 \\ -2 & 5 \end{vmatrix} + k\begin{vmatrix} -1 & 3 \\ -2 & 4 \end{vmatrix}$$

$$= 23\underset{\sim}{i} + 9\underset{\sim}{j} + 2\underset{\sim}{k}$$

$|\overrightarrow{OP_0} \times \underset{\sim}{v}| = \sqrt{23^2 + 9^2 + 2^2} = \sqrt{614}$

Distance $\quad \dfrac{\sqrt{614}}{3\sqrt{5}} \times \dfrac{\sqrt{5}}{\sqrt{5}} = \dfrac{\sqrt{3070}}{3}$

ii Plane $-4 + 3y + 12z = 13 \qquad 0(0, 0, 0)$

$\underset{\sim}{n} = -4\underset{\sim}{i} + 3\underset{\sim}{j} + 12\underset{\sim}{k} \qquad |\underset{\sim}{n}| = \sqrt{16 + 9 + 144} = 13$

Distance $\quad \dfrac{d}{|\underset{\sim}{n}|} = 1$

b i Line $t = \dfrac{x-3}{2} = \dfrac{y+2}{-1} = \dfrac{z+4}{-3}\ A(2, -1, -3)$

$x = 3 + 2t,\ y = -2 - t,\ z = -4 - 3t$

$\underset{\sim}{v} = 2\underset{\sim}{i} - \underset{\sim}{j} - 3\underset{\sim}{k} \qquad |\underset{\sim}{v}| = \sqrt{4 + 1 + 9} = \sqrt{14}$

$t = 0 \quad P_0(3, -2, -4) \quad \overrightarrow{OA} = 2\underset{\sim}{i} - \underset{\sim}{j} - 3\underset{\sim}{k}$

$\overrightarrow{AP_0} = \overrightarrow{OP_0} - \overrightarrow{OA}$

$= \underset{\sim}{i} - j - \underset{\sim}{k}$

$$\overrightarrow{AP_0} \times \underset{\sim}{v} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & -1 & -1 \\ 2 & -1 & -3 \end{vmatrix}$$

$$= \underset{\sim}{i}\begin{vmatrix} -1 & -1 \\ -1 & -3 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix}$$

$$= 2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$$

$|\overrightarrow{AP_0} \times \underset{\sim}{v}| = \sqrt{4 + 1 + 1} = \sqrt{6}$

Distance $|\overrightarrow{AP_0} \times \underset{\sim}{v}| = \dfrac{\sqrt{6}}{\sqrt{14}} \times \dfrac{\sqrt{14}}{\sqrt{14}} = \dfrac{\sqrt{21}}{7}$

ii Plane $3\alpha - 4z = 8 \qquad A(2, -1, -3)$

$P_0(0, 0, -2)\, z = -2, x = 0, y = 0$

$\underset{\sim}{n} = 3\underset{\sim}{i} - 4\underset{\sim}{k} \qquad |\underset{\sim}{n}| = \sqrt{9 + 16} = 5$

$\overrightarrow{OP_0} = -2\underset{\sim}{k} \qquad \overrightarrow{OA} = 2\underset{\sim}{i} - \underset{\sim}{j} - 3\underset{\sim}{k}$

$\overrightarrow{AP_0} = \overrightarrow{OP_0} - \overrightarrow{OA} = -2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$

$\underset{\sim}{n} \cdot \overrightarrow{AP_0} = -6 - 4 = -10$

Distance $\dfrac{|\underset{\sim}{n} \cdot \overrightarrow{AP_0}|}{|\underset{\sim}{n}|} = \dfrac{10}{5} = 2$

5 a $3x - 2y - 2z = 6 \quad (1)$

$2x + 3y = 7 \quad (2)$

Let $x = t$

$3x - 2y = 6 + 2t \quad (1)$

$2x + 3y = 7 \qquad (2)$

$6x - 4y = 12 + 4t \quad (1) \times 2$

$6x + 9y = 21 \qquad (2) \times 3$

$13y = 9 - 4t$

$y = \dfrac{1}{13}(9 - t)$

$9x - 6y = 18 + 6t \quad (1) \times 3$
$4x + 6y = 14 \qquad (2) \times 2$
$13x = 32 + 6t$
$x = \frac{1}{13}(32 + 6t)$
$x = \frac{1}{13}(32 + 6t),\ y = \frac{1}{13}(9 - 4t),\ z = t$
$\underset{\sim}{n}_1 = 3\underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k} \quad |\underset{\sim}{n}_1| = \sqrt{9 + 4 + 4} = \sqrt{17}$
$\underset{\sim}{n}_2 = 2\underset{\sim}{i} + 3\underset{\sim}{j} \qquad |\underset{\sim}{n}_2| = \sqrt{13}$
$\underset{\sim}{n}_1 \cdot \underset{\sim}{n}_2 = 6 - 6 = 0$
$\Rightarrow$ Planes are at right angles
90°

b $3x - 2y - 2z = 6 \quad (1)$
$-6x + 4y + 4z = 10 \quad (2)$
Plane 1: $\underset{\sim}{n}_1 = 3\underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k} \quad |\underset{\sim}{n}_1| = \sqrt{9 + 4 + 4} = \sqrt{17}$
Distance to the origin $\frac{6}{\sqrt{17}}$
Plane 2: $\underset{\sim}{n}_2 = -6\underset{\sim}{i} + 4\underset{\sim}{j} + 4\underset{\sim}{k} = -2\underset{\sim}{n}_1$
$|\underset{\sim}{n}_2| = \sqrt{36 + 16 + 16} = \sqrt{68} = 2\sqrt{17}$
Distance to origin $\frac{10}{2\sqrt{17}}$
Planes on opposite sides of the origin, distance between planes:
$\frac{5}{\sqrt{17}} \times \frac{\sqrt{17}}{\sqrt{17}} + \frac{6}{\sqrt{17}} \times \frac{\sqrt{17}}{\sqrt{17}}$
$= \frac{11\sqrt{17}}{17}$

c Plane $3x - 2y - 2z = 6$ Point $A(5, -2, 4)$
$\underset{\sim}{n} = 3\underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k} \quad |\underset{\sim}{n}_1| = \sqrt{17}$
Point on the plane $x = 2 \quad y = 0 \quad z = 0 \quad P_0(2, 0, 0)$
$\overrightarrow{AP_0} = \overrightarrow{OP_0} - \overrightarrow{OA} = -3\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k}$
$\underset{\sim}{n} \cdot \overrightarrow{AP_0} = -9 - 4 + 8 = -5$
Distance $\frac{|\underset{\sim}{n} \cdot AP_0|}{|\underset{\sim}{n}|} = \frac{5}{\sqrt{17}} \times \frac{\sqrt{17}}{\sqrt{17}}$
$= \frac{5\sqrt{17}}{17}$

6 Plane $5x - 2y + 4z = 12$

a Line $t = \frac{x+4}{2} = \frac{y-2}{-1} = \frac{z-9}{-3}$
$x = -4 + 2t,\ y = 2 - t,\ z = 9 - 3t$
$5x - 2y + 4z = 12$
$5(-4 + 2t) - 2(2 - t) + 4(9 - 3t) = 12$
$-20 + 10t - 4 + 2t + 36 - 12t = 12$
$0 = 0$ true, infinite number of solutions.
Intersection is the line $\frac{x+4}{2} = \frac{y-2}{-1} = \frac{z-9}{-3}$

b Line $t = \frac{x-8}{2} = \frac{y-2}{-1} = \frac{z-11}{-3}$
$x = -8 + 2t,\ y = 2 - t,\ z = 11 - 3t$
$5x - 2y + 4z = 12$
$5(-8 + 2t) - 2(2 - t) + 4(11 - 3t) = 12$
$-40 + 10t - 4 + 2t + 44 - 12t = 12$
$0 = 12$ contradiction, no solution,
the line and plane do not intersect.

c Line $t = \frac{x-6}{-2} = \frac{y-1}{2} = \frac{z-9}{-3}$
$x = 6 - 2t,\ y = 1 + 2t,\ z = 9 - 3t$
$5x - 2y + 4z = 12$
$5(6 - 2t) - 2(1 + 2t) + 4(9 - 3t) = 12$
$30 - 10t - 2 - 4t + 36 - 12t = 12$
$52 = 26t$
$t = 2$
When, $t = 2$
$x = 6 - 4 = 2,\ y = 1 + 4 = 5,\ z = 9 - 6 = 3$
Point of intersection $(2, 5, 3)$

7 Line $t = \frac{x-4}{5} = \frac{y+2}{1} = \frac{z-3}{3}$
$x = 4 + 5t, y = -2 + t, z = 3 + 3t$

a Plane $2x - y - 3z = 1$
$2(4 + 5t) - 2(-2 + t) - 3(3 + 3t) = 1$
$8 + 10t + 2 - t - 9 - 9t = 1$
$0 = 0$ consistent
Infinite number of solutions, intersection in the line
$\frac{x-4}{5} = \frac{y+2}{1} = \frac{z-3}{3}$

b Plane $x - 2y - z = 4$
$4 + 5t - 2(-2 + t) - (3 + 3t) = 4$
$4 + 5t + 4 - 2t - 3 - 3t = 4$
$1 = 0$ contradiction no solution, the line and plane do not intersect.

c Plane $x - 2y - z = 3$
$4 + 5t - 3(-2 + t) - (3 + 3t) = 3$
$4 + 5t + 6 - 3t + 3 + 3t = 3$
$5t = -10$
$t = -2$
$x = 4 - 10 = -6, y = -2 - 2 = -4, z = 3 - 6 = -3$
Point of intersection $(-6, -4, -3)$

Technology active: multiple choice

8 The line $x = 2$; $y = 3$; $z = 2 + 3t$ is parallel to the z-axis.
The correct answer is **D**.

9 $A(0, 2, -3)$, $B(0, -6, 9)$ and the line $x = 0$; $\frac{y}{-2} = \frac{z}{3}$, is satisfied by both points A and B which lie on the line.
The correct answer is **A**.

10 Line 1: $\frac{x-1}{2} = \frac{y-3}{-4} = \frac{z+2}{1} = t, \quad x = 1 + 2t,$
$y = 3 - 4t, \quad z = -2 + t$
Line 2: $\frac{x-3}{-2} = \frac{y+1}{4} = \frac{z+1}{-1} = s, \quad x = 3 - 2s,$
$y = -1 + 4s, \quad z = -1 - s$ now let $s = -1 - t$ gives the same line, so the two lines are in fact the same line.
The correct answer is **A**.

11 Line 1: $\frac{x-3}{3} = \frac{y+2}{2} = \frac{z-2}{2} = s$ has direction
$\underset{\sim}{v}_1 = 3\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$

Line 2: $x = 1 + 2t,\ y = -2t,\ z = 3 - t$ has direction $\underset{\sim}{v}_2 = 2\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$

Now $\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2 = 6 - 4 - 2 = 0$, so the two lines are perpendicular. Solving:

$$(1)\quad x = 3 + 3s = 1 + 2t$$
$$(2)\quad y = -2 + 2s = -2t$$
$$(3)\quad z = 2 + 2s = 3 - t$$
$$(3)\ \Rightarrow\ t = 1 - 2s \text{ into } (2)$$
$$-2 + 2s = -2(1 - 2s) = -2 + 4s,$$
$$s = 0,\ t = 1$$
$$x = 3,\ y = -2,\ z = 2$$

So, the two lines intersect in a unique point.

The correct answer is **D**.

12 A point on the line $\frac{x}{3} = \frac{y-1}{4};\ z = 0$, is $(0, 1, 0)$, so $\overrightarrow{OP_0} = \underset{\sim}{j}$

The direction of the line is $\underset{\sim}{v} = 3\underset{\sim}{i} + 4\underset{\sim}{j},\ |\underset{\sim}{v}| = \sqrt{3^2 + 4^2} = 5$

$$\overrightarrow{OP_0} \times \underset{\sim}{v} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 0 & 1 & 0 \\ 3 & 4 & 0 \end{vmatrix} = -3\underset{\sim}{k}$$

The distance of the line from the origin is

$$d = \frac{|\overrightarrow{OP_0} \times \underset{\sim}{v}|}{|\underset{\sim}{v}|} = \frac{3}{5}.$$

The correct answer is **B**.

13 The plane $-x + 2y + 2z = 3$ has a normal $\underset{\sim}{n} = -\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$.

The distance of the plane from the origin is

$$\frac{d}{|\underset{\sim}{n}|} = \frac{3}{\sqrt{(-1)^2 + 2^2 + 2^2}} = \frac{3}{\sqrt{9}} = 1$$

The correct answer is **B**.

14 The first plane $2x + y - z = 0$ passes through the origin and has normal $\underset{\sim}{n}_1 = 2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$

The second plane $-4x - 2y + 2z = 2$ has normal $\underset{\sim}{n}_2 = -4\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k} = -2\underset{\sim}{n}_1$

So the two planes are parallel.

The correct answer is **C**.

15 The point $(1, -2, 3)$ lies on the line $\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-3}{3}$ and the point $(1, -2, 3)$ lies on the plane $x - y - z = 0$.

The line $\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-3}{3} = t,\quad x = 1 + t,$ $y = -2 - 2t,\ z = 3 + 3t$ and the plane substituting the line into the plane gives $1 + t - (-2 - 2t) - (3 + 3t) = 0 = 0$ a consistent equation, so the line lies in the plane.

The correct answer is **E**.

16 $x = 3 + t,\ y = 2 + 6t,\ z = 2 - t$

$$2x - y - 4z = 5$$
$$2(3 + t) - (2 + 6t) - 4(2 - t) = 5$$
$$6 + 2t - 2 - 6t - 8 + 4t = 5$$
$$-4 = 5$$

Contradiction, the line and the plane do not intersect.

The correct answer is **E**.

17 When $t = 2$, P is on the line, when $t = 3$, Q is on the line,

A. is true

When $t = -3$, P is on the line, when $t = -2$, Q is on the line,

B. is true

D. and E are satisfied and true

C. is false

The correct answer is **C**.

Technology active: extended response

18 a The line $x = 5 - t,\ y = 4t - 8,\ z = 6 - 2t$

In the xy plane,

$z = 0,\ 6 = 2t,\ t = 3,\ x = 2, y = 4,$ $A(2, 4, 0)$

In the yz plane,

$x = 0,\ t = 5,\ y = 12,\ z = -4,\ B(0, 12, -4)$

In the xz plane,

$y = 0,\ 4t = 8,\ t = 2,\ x = 3, z = 2,\ C(3, 0, 2)$

b $P(0, 2, -3),\ Q(1, 0, -2)\ R(4, -1, 0)$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k},\quad \overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = 3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$

$$\overrightarrow{PQ} \times \overrightarrow{QR} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & -2 & 1 \\ 3 & -1 & 2 \end{vmatrix}$$

$$\overrightarrow{PQ} \times \overrightarrow{QR} = \underset{\sim}{i}\begin{vmatrix} -2 & 1 \\ -1 & 2 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix}$$

$$\overrightarrow{PQ} \times \overrightarrow{QR} = -3\underset{\sim}{i} + \underset{\sim}{j} + 5\underset{\sim}{k}$$

Plane through $P(0, 2, -3)$

$$-3(x - 0) + 1(y - 2) + 5(z + 3) = 0$$
$$-3x + y - 2 + 5z + 15 = 0$$
$$3x - y - 5z = 13$$

Area of triangle PQR is

$$\frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{QR}| = \frac{1}{2}\sqrt{9 + 1 + 25} = \frac{1}{2}\sqrt{34}$$

19 a Line 1: $x = t + 2,\ y = 2t - 3,\ z = 3 - t$

Line 2: $x = 2s - 1,\ y = s - 3,\ z = 3s - 4$

$$(1)\quad x = t + 2 = 2s - 1$$
$$(2)\quad y = 2t - 3 = s - 3$$
$$(3)\quad z = -t + 3 = 2s - 3$$
$$(1)\ \Rightarrow\ t = 2s - 3 \text{ into } (2)$$
$$2(2s - 3) - 3 = s - 3$$
$$4s - 6 - 3 = s - 3$$
$$3s = 6$$
$$s = 2,\ \Rightarrow\ t = 1$$

Check in (3) yes, point $P(3, -1, 2)$

Line 1 direction $\underset{\sim}{v}_1 = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, line 2 direction $\underset{\sim}{v}_2 = 2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$

$$\underset{\sim}{v}_1 \times \underset{\sim}{v}_2 = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{vmatrix}$$

$$\underset{\sim}{v}_1 \times \underset{\sim}{v}_2 = \underset{\sim}{i}\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

$$\underset{\sim}{n} = \underset{\sim}{v}_1 \times \underset{\sim}{v}_2 = 7\underset{\sim}{i} - 5\underset{\sim}{j} - 3\underset{\sim}{k}$$

Plane through $P(3, -1, 2)$

$$7(x - 3) - 5(y + 1) - 3(z - 2) = 0$$
$$7x - 21 - 5y - 5 - 3z + 6 = 0$$
$$7x - 5y - 3z = 20$$

b Both lines $x = 2t + 1,\ y = t + 3,\ z = 4 - t$ and $x = 2t - 1,\ y = t - 2,\ z = 6 - t$ have direction $\underset{\sim}{v} = 2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$

Consider the point $t = 0$ on line 1, $A(1, 3, 4)$, $t = 0$ on line 2, $B(-1, -2, 6)$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -2\underset{\sim}{i} - 5\underset{\sim}{j} + 2\underset{\sim}{k}$ the vector $\overrightarrow{AB}$ lies in the plane, so $\underset{\sim}{n} = \underset{\sim}{v} \times \overrightarrow{AB}$ is a normal to the plane.

$$\underset{\sim}{n} = \underset{\sim}{v} \times \overrightarrow{AB} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & 1 & -1 \\ -2 & -5 & 2 \end{vmatrix}$$

$$\underset{\sim}{n} = \underset{\sim}{v} \times \overrightarrow{AB} = \underset{\sim}{i}\begin{vmatrix} 1 & -1 \\ -5 & 2 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 2 & -1 \\ -2 & 2 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 2 & 1 \\ -2 & -5 \end{vmatrix}$$

$\underset{\sim}{n} = -3\underset{\sim}{i} - 2\underset{\sim}{j} - 8\underset{\sim}{k}$

Or let $\underset{\sim}{n} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + 8\underset{\sim}{k}$

Plane through $A(1, 3, 4)$

$3(x-1) + 2(y-3) + 8(z-4) = 0$

$3x - 3 + 2y - 6 + 8z - 32 = 0$

$3x + 2y + 8z = 41$

20 **a** Substitute the line $x = 2t + 6,\ y = t + 1,\ z = -2t - 1$ into the plane.

$4x - y - z = 6$

$4(2t + 6) - (t + 1) - (-2t - 1) = 6$

$8t + 24 - t - 1 + 2t + 1 = 6$

$9t = -18$

$t = -2,\ x = 2,\ y = -1,\ z = 3,\ \ A(2, -1, 3)$

b Normal to the plane $\underset{\sim}{n} = 4\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$ direction of the line 1, $\underset{\sim}{v}_1 = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$

$|\underset{\sim}{n}| = \sqrt{16 + 1 + 1} = \sqrt{18} = 3\sqrt{2},$

$|\underset{\sim}{v}_1| = \sqrt{4 + 1 + 4} = \sqrt{9} = 3$

$\underset{\sim}{n} \cdot \underset{\sim}{v}_1 = 8 - 1 + 2 = 9$

$$\sin(\theta) = \frac{\underset{\sim}{n} \cdot \underset{\sim}{v}_1}{|\underset{\sim}{n}|\,|\underset{\sim}{v}_1|} = \frac{9}{3 \times 3\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$\theta = 45°$

c Direction of the line 2, $\underset{\sim}{v}_2 = a\underset{\sim}{i} + b\underset{\sim}{j} + \underset{\sim}{k}$ the two lines are perpendicular, so

$\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2 = 2a + b - 2 = 0$

(1) $2a + b = 2$

The point $A(2, -1, 3)$ also lies on the line

$$\frac{x - x_0}{a} = \frac{y - 3}{b} = z - 1 = s$$

$z = 3 = 1 + s \Rightarrow s = 2$

(2) $x = 2 = x_0 + 2a$

(3) $y = -1 = 3 + 2b \Rightarrow b = -2$, (1) $a = \dfrac{1}{2}(2 - b) = 2$

$x_0 = 2 - 2a = -2$

4.5 Exam questions

1 Direction of the line: $\underset{\sim}{v}_1 = a_1\underset{\sim}{i} + b_1\underset{\sim}{j} + c_1\underset{\sim}{k}$

Normal to the plane: $\underset{\sim}{n} = a_2\underset{\sim}{i} + b_2\underset{\sim}{j} + c_2\underset{\sim}{k}$

The acute angle between these two vectors is:

$$\frac{\pi}{2} - \cos^{-1}\left(\frac{|\underset{\sim}{v}_1 \cdot \underset{\sim}{n}|}{|\underset{\sim}{v}_1|\,|\underset{\sim}{n}|}\right) = \sin^{-1}\left(\frac{|\underset{\sim}{v}_1 \cdot \underset{\sim}{n}|}{|\underset{\sim}{v}_1|\,|\underset{\sim}{n}|}\right)$$

$$= \sin^{-1}\left(\frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right)$$

The correct answer is **C**.

2 Direction of the first line: $\underset{\sim}{v}_1 = a_1\underset{\sim}{i} + b_1\underset{\sim}{j} + c_1\underset{\sim}{k}$, direction of the second line: $\underset{\sim}{v}_2 = a_2\underset{\sim}{i} + b_2\underset{\sim}{j} + c_2\underset{\sim}{k}$.

The acute angle between these two vectors is:

$$\cos^{-1}\left(\frac{|\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2|}{|\underset{\sim}{v}_1|\,|\underset{\sim}{v}_2|}\right) = \cos^{-1}\left(\frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right).$$

The correct answer is **D**.

3 In two dimensions it is a line in the xy plane, in three dimensions it is plane parallel to the z-axis.

The correct answer is **D**.

4 Line: $\dfrac{x}{2} = -y = z,\ \ y = -\dfrac{x}{2}.$

Substitute into the plane $x + y - z = 0$. $x - \dfrac{x}{2} - \dfrac{x}{2} = 0 = 0.$

Consistent. There is an infinite number of solutions, as the line lies in the plane.

The correct answer is **E**.

5

$\overrightarrow{AB} = \left(-2\underset{\sim}{i} + 7\underset{\sim}{j} + 9\underset{\sim}{k}\right) - \left(-3\underset{\sim}{i} + 5\underset{\sim}{j} + 6\underset{\sim}{k}\right)$

$= \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$

$|\overrightarrow{AB}| = \sqrt{14}$

$\overrightarrow{BC} = \left(2\underset{\sim}{i} + \underset{\sim}{j} + 7\underset{\sim}{k}\right) - \left(-2\underset{\sim}{i} + 7\underset{\sim}{j} + 9\underset{\sim}{k}\right)$

$= 4\underset{\sim}{i} - 6\underset{\sim}{j} - 2\underset{\sim}{k}$

$|\overrightarrow{BC}| = \sqrt{56}$

$\overrightarrow{CA} = \left(-3\underset{\sim}{i} + 5\underset{\sim}{j} + 6\underset{\sim}{k}\right) - \left(-2\underset{\sim}{i} + \underset{\sim}{j} + 7\underset{\sim}{k}\right)$

$= -5\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}$

$|\overrightarrow{CA}| = \sqrt{42}$ [1 mark]

$\overrightarrow{AB} \cdot \overrightarrow{BC} = \left(\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}\right) \cdot \left(4\underset{\sim}{i} - 6\underset{\sim}{j} - 2\underset{\sim}{k}\right)$

$= 4 + -24 + -6$

$= -14$

$\overrightarrow{BC} \cdot \overrightarrow{CA} = \left(4\underset{\sim}{i} - 6\underset{\sim}{j} + 2\underset{\sim}{k}\right) \cdot \left(-5\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}\right)$

$= -20 + -24 + 2$

$= -42$

$\overrightarrow{AB} \cdot \overrightarrow{CA} = \left(\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}\right) \cdot \left(-5\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}\right)$

$= -5 + 8 - 3$

$= 0$ [1 mark]

$\therefore \overrightarrow{AB} \cdot \overrightarrow{CA}$ are $\perp$

Triangle ABC is a right-angled scalene triangle. [1 mark]

Topic 5 — Differential calculus

5.2 Review of differentiation techniques

5.2 Exercise

1 a $f(x)=\dfrac{2}{3x^2+5}$

Let $u=3x^2+5$

$$y=\frac{2}{u}=2u^{-1}$$

$$\frac{dy}{du}=-2u^{-2}=-\frac{2}{u^2}$$

$$\frac{du}{dx}=6x$$

$$\begin{aligned}f'(x)&=\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}\\&=-\frac{2}{u^2}\times 6x\\&=-\frac{12x}{\left(3x^2+5\right)^2}\end{aligned}$$

$$\begin{aligned}f'(-1)&=\frac{12}{8^2}\\&=\frac{3}{16}\end{aligned}$$

b $y=5\sin^3(2x)$

Let $u=\sin(2x)$

$$y=5u^3$$

$$\frac{dy}{du}=15u^2$$

$$\frac{du}{dx}=2\cos(2x)$$

$$\begin{aligned}\frac{dy}{dx}&=\frac{dy}{du}\cdot\frac{du}{dx}=30u^2\cos(2x)\\&=30\sin^2(2x)\ \cos(2x)\end{aligned}$$

2 a $\dfrac{d}{dx}\left[\dfrac{1}{2x+5}\right]=\dfrac{d}{dx}\left[u^{-1}\right]$

Where $u=2x+5$

$$\begin{aligned}&=\frac{d}{du}\left[u^{-1}\right]\frac{du}{dx}\\&=-u^{-2}\times 2\\&=-\frac{2}{u^2}\\&=-\frac{2}{(2x+5)^2}\end{aligned}$$

b $f(x)=6\sqrt{\cos(4x)}$

$$y=6u^{\frac{1}{2}}$$

Where $u=\cos(4x)$

$$\frac{dy}{du}=6\times\frac{1}{2}u^{-\frac{1}{2}}$$

$$\frac{du}{dx}=-4\sin(4x)$$

$$\begin{aligned}f'(x)&=\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}\\&=-\frac{12\sin(4x)}{\sqrt{\cos(4x)}}\end{aligned}$$

$$\begin{aligned}f'\left(\frac{\pi}{12}\right)&=-\frac{12\sin\left(\frac{\pi}{3}\right)}{\sqrt{\cos\left(\frac{\pi}{3}\right)}}\\&=-\frac{12\times\frac{\sqrt{3}}{2}}{\sqrt{\frac{1}{2}}}\\&=-12\times\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{1}\\&=-6\sqrt{6}\end{aligned}$$

3 a $y=2\sin^4(3x)$

Let $u=\sin(3x)$

$$\frac{du}{dx}=3\cos(3x)$$

Let $y=2u^4$

$$\frac{dy}{du}=8u^3$$

$$\frac{dy}{dx}=24u^3\cos(3x)$$

$$\frac{dy}{dx}=24\sin^3(3x)\cos(3x)$$

b $y=5\cos^3(4x)$

Let $u=\cos(4x)$

$$\frac{du}{dx}=-4\sin(4x)$$

Let $y=5u^3$

$$\frac{dy}{du}=15u^2$$

$$\frac{dy}{dx}=-60u^2\sin(4x)$$

$$\frac{dy}{dx}=-60\cos^2(4x)\sin(4x)$$

4 a $y=x\sqrt{2x^2+9}$

Let $v=\sqrt{2x^2+9}$

$$v=\left(2x^2+9\right)^{\frac{1}{2}}$$

$$\frac{dv}{dx}=4x\times\frac{1}{2}\times\left(2x^2+9\right)^{-\frac{1}{2}}$$

$$\frac{dv}{dx}=\frac{2x}{\sqrt{2x^2+9}}$$

Let $u=x$

$$\frac{du}{dx}=1$$

$$\frac{dy}{dx}=\frac{2x}{\sqrt{2x^2+9}}+\sqrt{2x^2+9}$$

$$\frac{dy}{dx}=\frac{2x^2+\left(2x^2+9\right)}{\sqrt{2x^2+9}}$$

$$\frac{dy}{dx}=\frac{4x^2+9}{\sqrt{2x^2+9}}$$

b $y=\dfrac{x}{\sqrt{3x^2+5}}$

Let $v=\sqrt{3x^2+5}$

$$v=\left(3x^2+5\right)^{\frac{1}{2}}$$

$$\frac{dv}{dx} = 6x \times \frac{1}{2} \times (3x^2+5)^{-\frac{1}{2}}$$

$$\frac{dv}{dx} = \frac{3x}{\sqrt{3x^2+5}}$$

Let $u = x$

$$\frac{du}{dx} = 1$$

$$y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{\sqrt{3x^2+5} - \frac{3x^2}{\sqrt{3x^2+5}}}{3x^2+5}$$

$$\frac{dy}{dx} = \frac{1}{3x^2+5}\left[\frac{3x^2+5-3x^2}{\sqrt{3x^2+5}}\right]$$

$$\frac{dy}{dx} = \frac{5}{\left(3x^2+5\right)^{\frac{3}{2}}}$$

$$\frac{dy}{dx} = \frac{5}{\sqrt{\left(3x^2+5\right)^3}}$$

5 **a** $f(x) = y$

$$y = e^{\frac{-1}{2}x^2}$$

Let $u = -\frac{1}{2}x^2$

$$\frac{du}{dx} = -x$$

$$y = e^u$$

$$\frac{dy}{du} = e^u$$

$$f'(x) = -xe^u$$

$$f'(x) = -xe^{\frac{-1}{2}x^2}$$

b $f(x) = e^{\cos(2x)}$

Let $u = \cos(2x)$

$$\frac{du}{dx} = -2\sin(2x)$$

$$y = e^u$$

$$\frac{dy}{du} = e^u$$

$$f'(x) = -2\sin(2x)e^u$$

$$f'(x) = -2\sin(2x)e^{\cos(2x)}$$

6 **a** $g(x) = \cos\left(e^{2x}\right)$

Let $u = e^{2x}$

$$\frac{du}{dx} = 2e^{2x}$$

$$y = \cos(u)$$

$$\frac{dy}{du} = -\sin(u)$$

$$g'(x) = -2e^{2x}\sin(u)$$

$$g'(x) = -2e^{2x}\sin(e^{2x})$$

b $g(x) = e^{\sqrt{x}}$

Let $u = \sqrt{x} = x^{\frac{1}{2}}$

$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

$$y = e^u$$

$$\frac{dy}{du} = e^u$$

$$g'(x) = \frac{1}{2\sqrt{x}}e^u$$

$$g'(x) = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

7 **a** $f(x) = 6\cos\left(\frac{3}{x}\right)$

$$y = 6\cos(u)$$

Where $u = \frac{3}{x} = 3x^{-1}$

$$\frac{dy}{du} = -6\sin(u)$$

$$\frac{du}{dx} = -3x^{-2} = -\frac{3}{x^2}$$

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{18}{x^2}\sin(u)$$

$$= \frac{18}{x^2}\sin\left(\frac{3}{x}\right)$$

$$f'\left(\frac{18}{\pi}\right) = 18 \times \left(\frac{\pi}{18}\right)^2 \sin\left(3 \times \frac{\pi}{18}\right)$$

$$= \frac{\pi^2}{18}\sin\left(\frac{\pi}{6}\right)$$

$$= \frac{\pi^2}{18} \times \frac{1}{2}$$

$$= \frac{\pi^2}{36}$$

b $y = e^{\sin(2x)}$

$$y = e^u$$

Where $u = \sin(2x)$

$$\frac{dy}{du} = e^u$$

$$\frac{du}{dx} = 2\cos(2x)$$

$$f(x) = \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= 2e^u\cos(2x)$$

$$= 2\cos(2x)e^{\sin(2x)}$$

8 **a** $y = \sin\left(\sqrt{x}\right)$

$$y = \sin(u)$$

Where $u = \sqrt{x} = x^{\frac{1}{2}}$

$$\frac{dy}{du} = \cos(u)$$

$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{1}{2\sqrt{x}}\cos(u)$$

$$= \frac{\cos(u)}{2\sqrt{x}} = \frac{\cos\left(\sqrt{x}\right)}{2\sqrt{x}}$$

b $h(x) = e^{\cos(2x)}$

$y = e^u$

Where $u = \cos(2x)$

$$\frac{dy}{du} = e^u$$

$$\frac{du}{dx} = -2\sin(2x)$$

$$\begin{aligned} h'(x) &= \frac{dh}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \\ &= -2\sin(2x)e^u \\ &= -2\sin(2x)e^{\cos(2x)} \end{aligned}$$

$$\begin{aligned} h'\left(\frac{\pi}{6}\right) &= -2\sin\left(\frac{\pi}{3}\right)e^{\cos\left(\frac{\pi}{3}\right)} \\ &= -2 \times \frac{\sqrt{3}}{2}e^{\frac{1}{2}} \\ &= -\sqrt{3e} \end{aligned}$$

9 a $y = \log_e\left(\sin\left(\frac{x}{2}\right)\right)$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{d}{dx}\left(\sin\left(\frac{x}{2}\right)\right)}{\sin\left(\frac{x}{2}\right)} \\ &= \frac{\frac{1}{2}\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)} \\ &= \frac{1}{2}\cot\left(\frac{x}{2}\right) \end{aligned}$$

b $y = \log_e\left(\frac{4x^2+9}{4x^2-9}\right)$

$= \log_e\left(4x^2+9\right) - \log_e\left(4x^2-9\right)$

$$\begin{aligned} \frac{dy}{dx} &= \frac{8x}{\left(4x^2+9\right)} - \frac{8x}{\left(4x^2-9\right)} \\ &= \frac{8x\left(4x^2-9\right) - 8x\left(4x^2+9\right)}{\left(4x^2+9\right)\left(4x^2-9\right)} \\ &= \frac{32x^3 - 72x - 32x^3 - 72x}{16x^4 - 81} \\ &= -\frac{144x}{16x^4 - 81} \end{aligned}$$

10 a $f(x) = x^3\cos(4x) = u \cdot v$

Where $u = x^3$

$v = \cos(4x)$

$$\frac{du}{dx} = 3x^2$$

$$\frac{dv}{dx} = -4\sin(4x)$$

$$\begin{aligned} f'(x) &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= -4x^3\sin(4x) + 3x^2\cos(4x) \\ &= x^2(3\cos(4x) - 4x\sin(4x)) \end{aligned}$$

b $y = x^4e^{-3x} = u \cdot v$

$u = x^4$

$v = e^{-3x}$

$$\frac{du}{dx} = 4x^3$$

$$\frac{dv}{dx} = -3e^{-3x}$$

$$\begin{aligned} \frac{dv}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= -3x^4e^{-3x} + 4x^3e^{-3x} \\ &= x^3e^{-3x}(4 - 3x) \end{aligned}$$

11 a $y = e^{-3x}\cos(2x) = u \cdot v$

$u = e^{-3x}$

$v = \cos(2x)$

$$\frac{du}{dx} = -3e^{-3x}$$

$$\frac{dv}{dx} = -2\sin(2x)$$

$$\begin{aligned} \frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= -2e^{-3x}\sin(2x) + -3e^{-3x}\cos(2x) \\ &= -e^{-3x}(2\sin(2x) + 3\cos(2x)) \end{aligned}$$

b $f(x) = y = x^2e^{-x^2} = u \cdot v$

$u = x^2$

$v = e^{-x^2}$

$$\frac{du}{dx} = 2x$$

$$\frac{dv}{dx} = -2xe^{-x^2}$$

$$\begin{aligned} f'(x) &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= -2x^3e^{-x^2} + 2xe^{-x^2} \\ &= 2xe^{-x^2}\left(1 - x^2\right) \end{aligned}$$

$$\begin{aligned} f'(2) &= 4e^{-4}(1 - 4) \\ &= -12e^{-4} \end{aligned}$$

12 a $y = x^3\sin(5x)$

Let $v = \sin(5x)$

$$\frac{dv}{dx} = 5\cos(5x)$$

Let $u = x^3$

$$\frac{du}{dx} = 3x^2$$

$$\frac{dy}{dx} = 5x^3\cos(5x) + 3x^2\sin(5x)$$

$$\frac{dy}{dx} = x^2(5x\cos(5x) + 3\sin(5x))$$

b $y = x^4\cos(4x)$

Let $v = \cos(4x)$

$$\frac{dv}{dx} = -4\sin(4x)$$

Let $u = x^4$

$$\frac{du}{dx} = 4x^3$$

$$\frac{dy}{dx} = -4x^4\sin(4x) + 4x^3\cos(4x)$$

$$\frac{dy}{dx} = 4x^3(\cos(4x) - x\sin(4x))$$

13 a

$$\begin{aligned} \frac{d}{dx}\left[x^3e^{-4x}\right] &= x^3\frac{d}{dx}\left(e^{-4x}\right) + e^{-4x}\frac{d}{dx}\left(x^3\right) \\ &= x^3 \times \left(-4e^{-4x}\right) + e^{-4x} \times 3x^2 \\ &= x^2e^{-4x}(3 - 4x) \end{aligned}$$

b

$$\begin{aligned} \frac{d}{dx}\left[e^{-3x}\sin(2x)\right] &= e^{-3x}\frac{d}{dx}(\sin(2x)) + \sin(2x)\frac{d}{dx}\left(e^{-3x}\right) \\ &= e^{-3x} \times 2\cos(2x) + \sin(2x) \times \left(-3e^{-3x}\right) \\ &= e^{-3x}(2\cos(2x) - 3\sin(2x)) \end{aligned}$$

14 a $y = \dfrac{3\cos(3x)}{2x^3} = \dfrac{u}{v}$

$$u = 3\cos(3x)$$
$$v = 2x^3$$
$$\frac{du}{dx} = -9\sin(3x)$$
$$\frac{dv}{dx} = 6x^2$$
$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$= \frac{-18x^3\sin(3x) - 18x^2\cos(3x)}{\left(2x^3\right)^2}$$
$$= \frac{-18x^2\,(x\sin(3x) + \cos(3x))}{4x^6}$$
$$= -\frac{9}{2x^4}\,(x\sin(3x) + \cos(3x))$$

b $f(x) = y = \dfrac{x}{\sqrt{4x+9}} = \dfrac{u}{v}$

$$u = x$$
$$v = \sqrt{4x+9}$$
$$\frac{du}{dx} = 1$$
$$\frac{dv}{dx} = \frac{2}{\sqrt{4x+9}}$$
$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$= \frac{\sqrt{4x+9} - \frac{2x}{\sqrt{4x+9}}}{4x+9}$$
$$= \frac{1}{4x+9}\left[\frac{4x+9-2x}{\sqrt{4x+9}}\right]$$
$$= \frac{2x+9}{\sqrt{(4x+9)^3}}$$

15 a $f(x) = \dfrac{1}{3xe^{2x}} = \dfrac{e^{-2x}}{3x} = \dfrac{u}{v}$

$$u = e^{-2x}$$
$$v = 3x$$
$$\frac{du}{dx} = -2e^{-2x}$$
$$\frac{dv}{dx} = 3$$
$$f'(x) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$= \frac{-6xe^{-3x} - 3e^{-2x}}{9x^2}$$
$$= -\frac{3e^{-2x}\,(1+2x)}{9x^2}$$
$$= -\frac{(1+2x)\,e^{-2x}}{3x^2}$$

b Let $y = \left[\dfrac{3x^2+5}{3x^2-5}\right]^2 = w^2$

$$w = \frac{3x^2+5}{3x^2-5}$$
$$\frac{dy}{dw} = 2w$$
$$\frac{dw}{dx} = \frac{6x\left(3x^2-5\right) - 6x\left(3x^2+5\right)}{\left(3x^2-5\right)^2}$$
$$= \frac{18x^3 - 30x - 18x^3 - 30x}{\left(3x^2-5\right)^2}$$
$$= -\frac{60x}{\left(3x^2-5\right)^2}$$
$$\frac{dy}{dx} = \frac{dy}{dw}\cdot\frac{dw}{dx}$$
$$= -\frac{120x}{\left(3x^2-5\right)^2} \times \left(\frac{3x^2+5}{3x^2-5}\right)$$
$$= -\frac{120x\left(3x^2+5\right)}{\left(3x^2-5\right)^3}$$

16 a $y = \dfrac{3\sin(3x)}{2x^3}$

Let $v = 2x^3$
$$\frac{dv}{dx} = 6x^2$$
Let $u = 3\sin(3x)$
$$\frac{du}{dx} = 9\cos(3x)$$
$$y = \frac{u}{v}$$
$$\frac{dy}{dx} = \frac{18x^3\cos(3x) - 18x^2\sin(3x)}{4x^6}$$
$$\frac{dy}{dx} = \frac{18x^2\,(x\cos(3x) - \sin(3x))}{4x^6}$$
$$\frac{dy}{dx} = \frac{9}{2x^4}\,(x\cos(3x) - \sin(3x))$$

b $y = \dfrac{4\cos(2x)}{3x^4}$

Let $v = 3x^4$
$$\frac{dv}{dx} = 12x^3$$
Let $u = 4\cos(2x)$
$$\frac{du}{dx} = -8\sin(2x)$$
$$y = \frac{u}{v}$$
$$\frac{dy}{dx} = \frac{-24x^4\sin(2x) - 48x^3\cos(2x)}{9x^8}$$
$$\frac{dy}{dx} = \frac{-24x^3\,(x\sin(2x) + 2\cos(2x))}{9x^8}$$
$$\frac{dy}{dx} = \frac{-8}{3x^5}\,(2\cos(2x) + x\sin(2x))$$

17 a $\dfrac{d}{dx}\left[\dfrac{e^{3x}}{x^2}\right] = \dfrac{x^2\frac{d}{dx}\left(e^{3x}\right) - e^{3x}\frac{d}{dx}\left(x^2\right)}{x^4}$

$$= \frac{3x^2e^{3x} - 2xe^{3x}}{x^4}$$
$$= \frac{xe^{3x}\,(3x-2)}{x^4}$$
$$= \frac{e^{3x}\,(3x-2)}{x^3}$$

b $$\begin{aligned}\frac{d}{dx}\left[\frac{1}{x^3e^{2x}}\right] &= \frac{d}{dx}\left[\frac{e^{-2x}}{x^3}\right]\\ &= \frac{x^3\frac{d}{dx}\left(e^{-2x}\right)-e^{-2x}\frac{d}{dx}\left(x^3\right)}{x^6}\\ &= \frac{-2x^3e^{-2x}-3x^2e^{-2x}}{x^6}\\ &= -\frac{x^2e^{-2x}\left(2x+3\right)}{x^6}\\ &= -\frac{(2x+3)}{x^4e^{2x}}\end{aligned}$$

18 a $$\begin{aligned}y &= 6\tan^4\left(\frac{x}{3}\right)\\ y &= 6u^4\\ u &= \tan\left(\frac{x}{3}\right)\\ \frac{dy}{du} &= 24u^3\\ \frac{du}{dx} &= \frac{1}{3}\sec^2\left(\frac{x}{3}\right)\\ \frac{dy}{dx} &= 24u^3\times\frac{1}{3}\sec^2\left(\frac{x}{3}\right)\\ &= 8\tan^3\left(\frac{x}{3}\right)\sec^2\left(\frac{x}{3}\right)\end{aligned}$$

b $$\begin{aligned}f(x) &= x^4\tan\left(\frac{x}{4}\right) = u\cdot v\\ u &= x^4\\ v &= \tan\left(\frac{x}{4}\right)\\ \frac{du}{dx} &= 4x^3\\ \frac{dv}{dx} &= \frac{1}{4}\sec\left(\frac{x}{4}\right)\\ f'(x) &= \frac{1}{4}x^4\sec^2\left(\frac{x}{4}\right)+4x^3\tan\left(\frac{x}{4}\right)\\ &= x^3\left(\frac{x}{4}\sec^2\left(\frac{x}{4}\right)+4\tan\left(\frac{x}{4}\right)\right)\\ f'(\pi) &= \pi^3\left(\frac{\pi}{4}\sec^2\left(\frac{\pi}{4}\right)+4\tan\left(\frac{\pi}{4}\right)\right)\\ &= \pi^3\left(\frac{\pi}{4}\times 2+4\times 1\right)\\ &= \frac{\pi^3\left(\pi+8\right)}{2}\end{aligned}$$

19 a $$\begin{aligned}g(x) &= \frac{\tan(3x)}{x} = \frac{u}{v}\\ u &= \tan(3x)\\ \frac{du}{dx} &= 3\sec^2(3x)\\ \frac{dv}{dx} &= 1\\ g'(x) &= \frac{3x\sec^2(3x)-\tan(3x)}{x^2}\\ g'\left(\frac{\pi}{9}\right) &= \frac{3\times\frac{\pi}{9}\sec^2\left(\frac{\pi}{3}\right)-\tan\left(\frac{\pi}{3}\right)}{\left(\frac{\pi}{9}\right)^2}\\ &= \frac{81}{\pi^2}\left(\frac{\pi}{3}\times 4-\sqrt{3}\right)\\ &= \frac{81}{\pi^2}\left(\frac{4\pi-3\sqrt{3}}{3}\right)\\ &= \frac{27\left(4\pi-3\sqrt{3}\right)}{\pi^2}\end{aligned}$$

b $$\begin{aligned}y &= 5\tan\left(\frac{2}{x}\right)\\ y &= 5\tan(u)\\ u &= \frac{2}{x} = 2x^{-1}\\ \frac{dy}{du} &= 5\sec^2(u)\\ \frac{du}{dx} &= -2x^{-2}\\ \frac{dy}{dx} &= -\frac{10}{x^2}\sec^2\left(\frac{2}{x}\right)\\ \left.\frac{dy}{dx}\right|_{x=\frac{12}{\pi}} &= -10\times\left(\frac{\pi}{12}\right)^2\sec^2\left(2\times\frac{\pi}{12}\right)\\ &= -\frac{10\pi^2}{144}\times\left(\frac{2\sqrt{3}}{3}\right)^2\\ &= -\frac{5\pi^2}{54}\end{aligned}$$

20 a $$\begin{aligned}f(x) &= \log_e\left(\sqrt{4x^2+9}\right)\\ &= \frac{1}{2}\log_e\left(4x^2+9\right)\\ f'(x) &= \frac{1}{2}\times\frac{8x}{4x^2+9}\\ f'(-1) &= \frac{-4}{4+9} = -\frac{4}{13}\end{aligned}$$

b $$\begin{aligned}y &= \cos\left(\log_e\left(\frac{x}{2}\right)\right)\\ u &= \log_e\left(\frac{x}{2}\right)\\ y &= \cos(u)\\ \frac{dy}{du} &= -\sin(u)\\ \frac{du}{dx} &= \frac{1}{x}\\ \frac{dy}{dx} &= -\frac{1}{x}\sin\left(\log_e\left(\frac{x}{2}\right)\right)\end{aligned}$$

21 $y = x^2\log_e(5x+4)$

Let $v = \log_e(5x+4)$

$$\frac{dv}{dx} = \frac{5}{5x+4}$$

Let $u = x^2$

$$\frac{du}{dx} = 2x$$

$$\frac{dy}{dx} = 2x\log_e(5x+4)+\frac{5x^2}{5x+4}$$

22 a $$\begin{aligned}y &= \log_e\left(\frac{4x-9}{4x+9}\right)\\ &= \log_e(4x-9)-\log_e(4x+9)\\ \frac{dy}{dx} &= \frac{4}{4x-9}-\frac{4}{4x+9}\\ &= \frac{4(4x+9)-4(4x-9)}{(4x-9)(4x+9)}\\ &= \frac{72}{16x^2-81}\end{aligned}$$

b $y = \log_e\left(\dfrac{3x^2+5}{3x^2-5}\right)$

$$= \log_e(3x^2+5) - \log_e(3x^2-5)$$

$$\frac{dy}{dx} = \frac{6x}{3x^2+5} - \frac{6x}{3x^2-5}$$

$$= \frac{6x(3x^2-5) - 6x(3x^2+5)}{(3x^2+5)(3x^2-5)}$$

$$= \frac{60x}{25-9x^4}$$

23 a $y = g(x)$

$g(x) = \log_e(\sin(3x))$

Let $u = \sin(3x)$

$$\frac{du}{dx} = 3\cos(3x)$$

$$y = \log_e(u)$$

$$\frac{dy}{du} = \frac{1}{u}$$

$$g'(x) = \frac{3\cos(3x)}{\sin(3x)}$$

$$= 3\cot(3x)$$

$$= \frac{3}{\tan(3x)}$$

$$g'\left(\frac{\pi}{12}\right) = \frac{3}{\tan\left(\frac{\pi}{4}\right)}$$

$$= 3$$

b $y = h(x)$

$h(x) = \log_e(\tan(2x))$

Let $u = \tan(2x)$

$$\frac{du}{dx} = 2\sec^2(2x)$$

$$h(x) = \log_e(u)$$

$$\frac{dy}{du} = \frac{1}{u}$$

$$h'(x) = \frac{2\sec^2(2x)}{\tan(2x)}$$

$$= \frac{2}{\cos^2(2x) \times \frac{\sin(2x)}{\cos(2x)}}$$

$$= \frac{2}{\sin(2x)\cos(2x)}$$

$$= \frac{4}{\sin(4x)}$$

$$h'\left(\frac{\pi}{12}\right) = \frac{4}{\sin\left(\frac{\pi}{3}\right)}$$

$$= \frac{4}{\frac{\sqrt{3}}{2}}$$

$$= \frac{8}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{8\sqrt{3}}{3}$$

24 a $f(x) = 4\cos\left(\dfrac{2}{x}\right)$

Let $u = \dfrac{2}{x} = 2x^{-1}$

$$\frac{du}{dx} = -2x^{-2}$$

$$= -\frac{2}{x^2}$$

$$y = 4\cos(u)$$

$$\frac{dy}{du} = -4\sin(u)$$

$$f'(x) = \frac{8}{x^2}\sin\left(\frac{2}{x}\right)$$

$$f'\left(\frac{3}{2\pi}\right) = 8 \times \left(\frac{2\pi}{3}\right)^2 \sin\left(\frac{4\pi}{3}\right)$$

$$= 8 \times \frac{4\pi^2}{9} \times \frac{-\sqrt{3}}{2}$$

$$= -\frac{16\pi^2\sqrt{3}}{9}$$

b $f(x) = 2\tan\left(\dfrac{3}{x}\right)$

Let $u = \dfrac{3}{x} = 3x^{-1}$

$$\frac{du}{dx} = -3x^{-2}$$

$$= -\frac{3}{x^2}$$

$$y = 2\tan(u)$$

$$\frac{dy}{du} = 2\sec^2(u)$$

$$f'(x) = -\frac{6}{x^2}\sec^2\left(\frac{3}{x}\right)$$

$$f'\left(\frac{18}{\pi}\right) = -6 \times \left(\frac{\pi}{18}\right)^2 \sec^2\left(\frac{\pi}{6}\right)$$

$$= -6 \times \frac{\pi^2}{18^2} \times \frac{1}{\cos^2\left(\frac{\pi}{6}\right)}$$

$$= -6 \times \frac{\pi^2}{324} \times \frac{4}{3}$$

$$= -\frac{2\pi^2}{81}$$

25 a $y = \sec(kx)$

$$= \frac{1}{\cos(kx)}$$

Let $u = \cos(kx)$

$$\frac{du}{dx} = -k\sin(kx)$$

$$y = \frac{1}{u}$$

$$\frac{dy}{du} = -\frac{1}{u^2}$$

$$\frac{dy}{dx} = \frac{k\sin(kx)}{\cos^2(kx)}$$

$$= \frac{k\sin(kx)}{\cos(kx)} \times \frac{1}{\cos(kx)}$$

$$\frac{d}{dx}(\sec(kx)) = k\sec(kx)\tan(kx)$$

b $y = \operatorname{cosec}(kx)$

$$= \frac{1}{\sin(kx)}$$

Let $u = \sin(kx)$

$$\frac{du}{dx} = k\cos(kx)$$

$$y = \frac{1}{u}$$

$$\frac{dy}{du} = -\frac{1}{u^2}$$

$$\frac{dy}{dx} = -\frac{k\cos(kx)}{\sin^2(kx)}$$

$$= -\frac{k\cos(kx)}{\sin(kx)} \times \frac{1}{\sin(kx)}$$

$$\frac{d}{dx}(\operatorname{cosec}(kx)) = -k\operatorname{cosec}(kx)\cot(kx)$$

c $y = \cot(kx)$

$$= \frac{\cos(kx)}{\sin(kx)}$$

$$\frac{dy}{dx} = \frac{-k\sin^2(kx) - k\cos^2(kx)}{\sin^2(kx)}$$

$$= -\frac{k\left(\sin^2(kx) + \cos^2(kx)\right)}{\sin^2(kx)}$$

$$-k\operatorname{cosec}^2(kx)$$

$$\frac{d}{dx}(\cot(kx)) = -k\operatorname{cosec}^2(kx)$$

26 $y = \log_e\left(\cot\left(\frac{x}{2}\right) + \operatorname{cosec}\left(\frac{x}{2}\right)\right)$

$$\frac{dy}{dx} = \frac{\frac{d}{dx}\left(\cot\left(\frac{x}{2}\right) + \operatorname{cosec}\left(\frac{x}{2}\right)\right)}{\cot\left(\frac{x}{2}\right) + \operatorname{cosec}\left(\frac{x}{2}\right)}$$

$$= \frac{-\frac{1}{2}\operatorname{cosec}^2\left(\frac{x}{2}\right) - \frac{1}{2}\operatorname{cosec}\left(\frac{x}{2}\right)\cot\left(\frac{x}{2}\right)}{\cot\left(\frac{x}{2}\right) + \operatorname{cosec}\left(\frac{x}{2}\right)}$$

$$= -\frac{\frac{1}{2}\operatorname{cosec}\left(\frac{x}{2}\right)\left(\operatorname{cosec}\left(\frac{x}{2}\right) + \cot\left(\frac{x}{2}\right)\right)}{\cot\left(\frac{x}{2}\right) + \operatorname{cosec}\left(\frac{x}{2}\right)}$$

$$= -\frac{1}{2}\operatorname{cosec}\left(\frac{x}{2}\right)$$

27 **a** $y = \sin(nx + \alpha)$

$$= \sin(nx)\cos(\alpha) + \cos(nx)\sin(\alpha)$$

$$\frac{dy}{dx} = n\cos(nx)\cos(\alpha) - n\sin(nx)\sin(\alpha)$$

$$= n\cos(nx + \alpha)$$

b $y = \cos(nx + \alpha)$

$$= \cos(nx)\cos(\alpha) - \sin(nx)\sin(\alpha)$$

$$\frac{dy}{dx} = -n\sin(nx)\cos(\alpha) - n\cos(nx)\sin(\alpha)$$

$$= -n\sin(nx + \alpha)$$

28 **a** $y = \sin^n(kx)$

Let $u = \sin(kx)$

$$\frac{du}{dx} = k\cos(kx)$$

$$y = u^n$$

$$\frac{dy}{du} = nu^{n-1}$$

$$\frac{dy}{dx} = nku^{n-1}\cos(kx)$$

$$= nk\sin^{n-1}(kx)\cos(kx)$$

b $y = \cos^n(kx)$

Let $u = \cos(kx)$

$$\frac{du}{dx} = -k\sin(kx)$$

$$y = u^n$$

$$\frac{dy}{du} = nu^{n-1}$$

$$\frac{dy}{dx} = -nku^{n-1}\sin(kx)$$

$$= -nk\cos^{n-1}(kx)\sin(kx)$$

29 **a** $y = e^{kx}\sin(bx)$

Let $v = \sin(bx)$

$$\frac{dv}{dx} = b\cos(bx)$$

Let $u = e^{kx}$

$$\frac{du}{dx} = ke^{kx}$$

$$y = uv$$

$$\frac{dy}{dx} = be^{kx}\cos(bx) + ke^{kx}\sin(bx)$$

$$= e^{kx}(b\cos(bx) + k\sin(bx))$$

b $y = e^{kx}\cos(bx)$

Let $v = \cos(bx)$

$$\frac{dv}{dx} = -b\sin(bx)$$

Let $u = e^{kx}$

$$\frac{du}{dx} = ke^{kx}$$

$$y = uv$$

$$\frac{dy}{dx} = -be^{kx}\sin(bx) + ke^{kx}\cos(bx)$$

$$= e^{kx}(k\cos(bx) - b\sin(bx))$$

30 $y = x^n\sin(kx)$

Let $v = \sin(kx)$

$$\frac{dv}{dx} = k\cos(kx)$$

Let $u = x^n$

$$\frac{du}{dx} = nx^{n-1}$$

$$y = uv$$

$$\frac{dy}{dx} = kx^n\cos(kx) + nx^{n-1}\sin(kx)$$

$$= x^{n-1}(n\sin(kx) + kx\cos(kx))$$

31 **a** $y = \dfrac{\sin(kx)}{x^n}$

Let $v = x^n$

$$\frac{dv}{dx} = nx^{n-1}$$

Let $u = \sin(kx)$

$$\frac{du}{dx} = k\cos(kx)$$

$$y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{kx^n\cos(kx) - nx^{n-1}\sin(kx)}{x^{2n}}$$

$$= \frac{x^{n-1}(kx\cos(kx) - n\sin(kx))}{x^{2n}}$$

$$= \frac{1}{x^{n+1}}(kx\cos(kx) - n\sin(kx))$$

b $y = \dfrac{\cos(kx)}{x^n}$

Let $v = x^n$

$$\frac{dv}{dx} = nx^{n-1}$$

Let $u = \cos(kx)$

$$\frac{du}{dx} = -k\sin(kx)$$

$$y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{-kx^n\sin(kx) - nx^{n-1}\cos(kx)}{x^{2n}}$$

$$= \frac{-1}{x^{n+1}}(kx\sin(kx) + n\cos(kx))$$

32 **a** $y = x^ne^{kx}$

Let $v = e^{kx}$

$$\frac{dv}{dx} = ke^{kx}$$

Let $u = x^n$

$$\begin{aligned}
\frac{du}{dx} &= nx^{n-1}\\
y &= uv\\
\frac{dy}{dx} &= kx^n e^{kx} + nx^{n-1}e^{kx}\\
&= x^{n-1}e^{kx}(n + kx)
\end{aligned}$$

b $y = \dfrac{e^{kx}}{x^n}$

Let $v = x^n$

$$\frac{dv}{dx} = nx^{n-1}$$

Let $u = e^{kx}$

$$\begin{aligned}
\frac{du}{dx} &= ke^{kx}\\
y &= \frac{u}{v}\\
\frac{dy}{dx} &= \frac{kx^n e^{kx} - nx^{n-1}e^{kx}}{x^{2n}}\\
&= \frac{x^{n-1}e^{kx}(kx - n)}{x^{2n}}\\
&= \frac{e^{kx}(kx - n)}{x^{n+1}}
\end{aligned}$$

33 a

$$\begin{aligned}
y &= \log_e\left(\sqrt{ax^2 + b}\right)\\
&= \log_e(ax^2 + b)^{\frac{1}{2}}\\
&= \frac{1}{2}\log_e\left(ax^2 + b\right)\\
\frac{dy}{dx} &= \frac{1}{2} \times \frac{2ax}{9x^2 + b}\\
&= \frac{ax}{ax^2 + b}
\end{aligned}$$

b

$$\begin{aligned}
y &= \log_e\left(\frac{ax + b}{cx + d}\right)\\
&= \log_e(ax + b) - \log_e(cx + d)\\
\frac{dy}{dx} &= \frac{a}{ax + b} - \frac{c}{cx + d}\\
&= \frac{a(cx + d) - c(ax + b)}{(ax + b)(cx + d)}\\
&= \frac{ad - bc}{(ax + b)(cx + d)}
\end{aligned}$$

34 a

$$\begin{aligned}
y &= \log_e\left(\frac{ax^2 + b}{cx^2 + d}\right)\\
&= \log_e\left(ax^2 + b\right) - \log_e\left(cx^2 + d\right)\\
\frac{dy}{dx} &= \frac{2ax}{ax^2 + b} - \frac{2cx}{cx^2 + d}\\
&= \frac{2ax\left(cx^2 + d\right) - 2cx\left(ax^2 + b\right)}{\left(ax^2 + b\right)\left(cx^2 + d\right)}\\
&= \frac{2x(ad - bc)}{\left(ax^2 + b\right)\left(cx^2 + d\right)}
\end{aligned}$$

b

$$\begin{aligned}
y &= \log_e(\sin^n(bx))\\
&= n\log_e(\sin(bx))\\
\frac{dy}{dx} &= \frac{nb\cos(bx)}{\sin(bx)}\\
&= \frac{nb}{\tan(bx)}
\end{aligned}$$

35 a

$$\begin{aligned}
y &= \log_e(\cos^n(bx))\\
&= n\log_e(\cos(bx))\\
\frac{dy}{dx} &= -\frac{nb\sin(bx)}{\cos(bx)}\\
&= -nb\tan(bx)
\end{aligned}$$

b

$$\begin{aligned}
y &= \log_e(\tan^n(bx))\\
&= n\log_e(\tan(bx))\\
\frac{dy}{dx} &= \frac{nb\sec^2(bx)}{\tan(bx)}\\
&= \frac{nb}{\cos^2(bx)\frac{\sin(bx)}{\cos(bx)}}\\
&= \frac{nb}{\cos(bx)\sin(bx)}\\
&= \frac{2nb}{2\sin(bx)\cos(bx)}\\
&= \frac{2nb}{\sin(2bx)}
\end{aligned}$$

36 a

$$\begin{aligned}
y &= u \cdot v \cdot w\\
&= (u \cdot v)w\\
\frac{dy}{dx} &= w\frac{d}{dx}(uv) + uv\frac{d}{dx}(w)\\
&= w\left[u\frac{dv}{dx} + v\frac{du}{dx}\right] + uv\frac{dw}{dx}\\
&= vw\frac{du}{dx} + uw\frac{dv}{dx} + uv\frac{dw}{dx}
\end{aligned}$$

b $y = x^3e^{-4x}\cos(2x)$

Let $y = u \cdot v \cdot w$

$$\begin{aligned}
u &= x^3\\
v &= e^{-4x}\\
w &= \cos(2x)\\
\frac{dy}{dx} &= 3x^2e^{-4x}\cos(2x) - 4e^{-4x}x^3\cos(2x) - 2x^3e^{-4x}\sin(2x)\\
&= e^{-4x}\left[(3x^2 - 4x^3)\cos(2x) - 2x^3\sin(2x)\right]
\end{aligned}$$

37 a

$$\begin{aligned}
f(x) &= \sin(kx)\\
f(x + h) &= \sin(kx + kh)\\
f'(x) &= \lim_{h\to 0}\frac{f(x + h) - f(x)}{h}\\
f'(x) &= \lim_{h\to 0}\frac{\sin(kx + kh) - \sin(kx)}{h}\\
f'(x) &= \lim_{h\to 0}\frac{2\cos\left(kx + \frac{kh}{2}\right)\sin\left(\frac{kh}{2}\right)}{h}\\
f'(x) &= k\lim_{h\to 0}\frac{\sin\left(\frac{kh}{2}\right)}{\frac{kh}{2}}\lim_{h\to 0}\cos\left(kx + \frac{kh}{2}\right)\\
f'(x) &= k\cos(kx)
\end{aligned}$$

b

$$\begin{aligned}
f(x) &= \cos(kx)\\
f(x + h) &= \cos(kx + kh)\\
f'(x) &= \lim_{h\to 0}\frac{f(x + h) - f(x)}{h}\\
f'(x) &= \lim_{h\to 0}\frac{\cos(kx + kh) - \cos(kx)}{h}\\
f'(x) &= \lim_{h\to 0}\frac{-2\sin\left(kx + \frac{kh}{2}\right)\sin\left(\frac{kh}{2}\right)}{h}\\
f'(x) &= k\lim_{h\to 0}\frac{\sin\left(\frac{kh}{2}\right)}{\frac{kh}{2}}\quad\lim_{h\to 0} -\sin\left(kx + \frac{kh}{2}\right)\\
f'(x) &= -k\sin(kx)
\end{aligned}$$

c $f(x) = \tan(kx)$

$f(x+h) = \tan(kx+kh)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\tan(kx+kh) - \tan(kx)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\frac{\sin(kx+kh)}{\cos(kx+kh)} - \frac{\sin(kx)}{\cos(kx)}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\frac{\sin(kx+kh)\cos(kx) - \sin(kx)\cos(kx+kh)}{\cos(kx+kh)\cos(kx)}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\frac{\sin(kh)}{\cos(kx+kh)\cos(kx)}}{h}$$

$$f'(x) = k \lim_{h\to 0} \frac{\sin(kh)}{kh} \quad \lim_{h\to 0} \frac{1}{\cos(kx+kh)\cos(kx)}$$

$$f'(x) = \frac{k}{\cos^2(kx)} = k\sec^2(kx)$$

38 a $f(x) = \sec(kx)$

$f(x+h) = \sec(kx+kh)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\sec(kx+kh) - \sec(kx)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{\cos(kx+kh)} - \frac{1}{\cos(kx)}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\frac{\cos(kx) - \cos(kx+kh)}{\cos(kx+kh)\cos(kx)}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{-2\,\sin\left(kx + \frac{kh}{2}\right)\sin\left(-\frac{kh}{2}\right)}{h\,\cos(kx+kh)\cos(kx)}$$

$$f'(x) = k \lim_{h\to 0} \frac{\sin\left(\frac{kh}{2}\right)}{\frac{kh}{2}} \quad \lim_{h\to 0} \frac{\sin\left(kx + \frac{kh}{2}\right)}{\cos(kx+kh)\cos(kx)}$$

$$f'(x) = \frac{k\,\sin(kx)}{\cos^2(kx)} = k\,\tan(kx)\sec(kx)$$

b $f(x) = \text{cosec}(kx)$

$f(x+h) = \text{cosec}(kx+kh)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\text{cosec}(kx+kh) - \text{cosec}(kx)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{\sin(kx+kh)} - \frac{1}{\sin(kx)}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\frac{\sin(kx) - \sin(kx+kh)}{\sin(kx+kh)\sin(kx)}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{2\,\cos(kx + \frac{kh}{2})\sin\left(-\frac{kh}{2}\right)}{h\sin(kx+kh)\sin(kx)}$$

$$f'(x) = k \lim_{h\to 0} \frac{\sin\left(\frac{kh}{2}\right)}{\frac{kh}{2}} \quad \lim_{h\to 0} \frac{-\cos\left(kx + \frac{kh}{2}\right)}{\sin(kx+kh)\sin(kx)}$$

$$f'(x) = \frac{-k\,\cos(kx)}{\sin^2(kx)} = -k\cot(kx)\text{cosec}(kx)$$

c $f(x) = \cot(kx)$

$f(x+h) = \cot(kx+kh)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\cot(kx+kh) - \cot(kx)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\frac{\cos(kx+kh)}{\sin(kx+kh)} - \frac{\cos(kx)}{\sin(kx)}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\frac{\cos(kx+kh)\sin(kx) - \cos(kx)\sin(kx+kh)}{\sin(kx+kh)\sin(kx)}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\frac{\sin(-kh)}{\sin(kx+kh)\sin(kx)}}{h}$$

$$f'(x) = k \lim_{h\to 0} \frac{\sin(kh)}{kh} \quad \lim_{h\to 0} \frac{-1}{\sin(kx+kh)\sin(kx)}$$

$$f'(x) = \frac{-k}{\sin^2(kx)} = -k\,\text{cosec}^2(kx)$$

5.2 Exam questions

1 $y = \log_e\left(\sqrt{2x^2+9}\right)$

$= \log_e\left(2x^2+9\right)^{\frac{1}{2}}$

$= \frac{1}{2}\log_e\left(2x^2+9\right)$ [1 mark]

$\frac{dy}{dx} = \frac{1}{2} \times \frac{4x}{2x^2+9}$

$= \frac{2x}{2x^2+9}$ [1 mark]

2 $y = \log_e(\tan(3x) + \sec(3x))$

$\frac{dy}{dx} = \frac{\frac{d}{dx}(\tan(3x) + \sec(3x))}{\tan(3x) + \sec(3x)}$

$= \frac{3\sec^2(3x) + 3\sec(3x)\tan(3x)}{\tan(3x) + \sec(3x)}$ [1 mark]

$= \frac{3\sec(3x)(\sec(3x) + \tan(3x))}{\tan(3x) + \sec(3x)}$

$= 3\sec(3x)$ [1 mark]

3 $y = x^n\cos(kx)$

Let $v = \cos(kx)$

$\frac{dv}{dx} = -k\sin(kx)$

Let $u = x^n$

$\frac{du}{dx} = nx^{n-1}$

$y = uv$

$\frac{dy}{dx} = -kx^n\sin(kx) + nx^{n-1}\cos(kx)$

$= x^{n-1}(n\cos(nx) - kx\sin(nx))$

Award 1 mark for correctly identifying u and v and their derivatives.

Award 1 mark for correctly applying the product rule to obtain the result.

5.3 Applications of differentiation

5.3 Exercise

1 $y = x^2 - 2x - 8$

When $x = 3$

$y = 9 - 6 - 8 = -5$

P(3, −5)

$\frac{dy}{dx} = 2x - 2$

When $x = 3$

$\frac{dy}{dx} = 6 - 2 = 4 = m_T$

$T\colon\ y + 5 = 4(x - 3)$

$y + 5 = 4x - 12$

$y = 4x - 17$

2 $y = \sqrt{4x + 5}$ at $x = 1$

$y = \sqrt{9} = 3$

P(1, 3)

$\frac{dy}{dx} = 4 \times \frac{1}{2} \times (4x + 5)^{\frac{-1}{2}}$

$= \frac{2}{\sqrt{4x + 5}}$

When $x = 1$

$\frac{dy}{dx} = \frac{2}{\sqrt{9}} = \frac{2}{3} = m_T$

$T\colon\ y - 3 = \frac{2}{3}(x - 1)$

$y = \frac{2}{3}x - \frac{2}{3} + 3$

$= \frac{2x}{3} + \frac{7}{3}$

Or

$3y = 2x + 7$

$3y - 2x - 7 = 0$

3 **a** $y = 9 - x^2$at $x = 2$

When $x = 2$

$y = 9 - 4$

$= 5$

P (2, 5)

$\frac{dy}{dx} = -2x$

When $x = 2$

$\frac{dy}{dx} = -4 = m_T$

$T\colon\ y - 5 = -4\,(x - 2)$

$y = -4x + 13$

b $y = 5\ \sin(2x)$ at $x = \frac{\pi}{3}$

When $x = \frac{\pi}{3}$

$y = 5\ \sin\left(\frac{2\pi}{3}\right)$

$= \frac{5\sqrt{3}}{2}$

$P\left(\frac{\pi}{3}, \frac{5\sqrt{3}}{2}\right)$

$\frac{dy}{dx} = 10\cos(2x)$

When $x = \frac{\pi}{3}$

$\frac{dy}{dx} = 10\cos\left(\frac{2\pi}{3}\right)$

$= -5$

$T\colon\ y - \frac{5\sqrt{3}}{2} = -5\left(x - \frac{\pi}{3}\right)$

$y = -5x + \frac{5\pi}{3} + \frac{5\sqrt{3}}{2}$

4 **a** $y = \sqrt{2x + 1}$ at $x = 4$

When $x = 4$

$y = \sqrt{8 + 1}$

$= 3$

P (4, 3)

$y = (2x + 1)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2} \times 2 \times (2x + 1)^{-\frac{1}{2}}$

$= \frac{1}{\sqrt{2x + 1}}$

At $x = 4, m_T = \frac{1}{3}$

$T\colon\ y - 3 = \frac{1}{3}(x - 4)$

$3y - 9 = x - 4$

$3y - x - 5 = 0$

b $y = 4\cos(3x)$ at $x = \frac{\pi}{12}$

When $x = \frac{\pi}{12}$

$y = 4\cos\left(\frac{\pi}{4}\right)$

$= 2\sqrt{2}$

$P\left(\frac{\pi}{12}, 2\sqrt{2}\right)$

$\frac{dy}{dx} = -12\sin(3x)$

At $x = \frac{\pi}{12}$

$m_T = -12\ \sin\left(\frac{\pi}{4}\right)$

$= -6\sqrt{2}$

$T\colon\ y - 2\sqrt{2} = -6\sqrt{2}\left(x - \frac{\pi}{12}\right)$

$y = -6\sqrt{2}x + \frac{\sqrt{2}\pi}{2} + 2\sqrt{2}$

5 $y = -x^2 + 2x + 15$

When $x = 2$

$y = -4 + 4 + 15$

P(2, 15)

When $x = 2$

$\frac{dy}{dx} = -4 + 2 = -2 = m_T$

$m_N = \frac{1}{2}$

$N\colon\ y - 15 = \frac{1}{2}(x - 2)$

$2y - 30 = x - 2$

$2y - x - 28 = 0$

6 $y = \frac{4}{3x - 2}$

When $x = 1$

$y = \frac{4}{3 - 2} = 4$

P(1, 4)

$y = 4\,(3x - 2)^{-1}$

$\frac{dy}{dx} = -12\,(3x - 2)^{-2} = -\frac{12}{(3x - 2)^2}$

When $x = 1$

$\frac{dy}{dx} = -\frac{12}{1} = -12 = m_T$

$m_N = \frac{1}{12}$

$$N\colon\ y-4=\frac{1}{12}(x-1)$$
$$12y-48=x-1$$
$$12y-x-47=0$$

7 $y=x^2-6x+5$

$$\begin{aligned}
&y=2x+c\\
&\frac{dy}{dx}=2x-6\\
&m_T=2\\
&2x-6=2\\
&2x=8\\
&x=4\\
&y=16-24+5=-3\\
&P(4,\ -3)\ -3=8+c\\
&c=-11
\end{aligned}$$

8 $y=x^2+4x+12$

$$\begin{aligned}
&\frac{dy}{dx}=2x+4\\
&2y-x+c=0\\
&2y=x-c\\
&y=\frac{x}{2}-\frac{c}{2}\\
&m_N=\frac{1}{2}\\
&m_T=-2\\
&2x+4=-2\\
&2x=-6
\end{aligned}$$

When $x=-3$

$$\begin{aligned}
&y=9-12+12=9\\
&P(-3,\ 9)\\
&18+3+c=0\\
&c=-21
\end{aligned}$$

9 **a** $y=x^2-8x-9$

$$\begin{aligned}
&\frac{dy}{dx}=2x-8\\
&y=-12x+c\\
&m_T=-12
\end{aligned}$$

So $2x-8=-12$

$$\begin{aligned}
&2x=-4\\
&x=-2
\end{aligned}$$

At $x=-2$

$$\begin{aligned}
&y=4+16-9=11\\
&P(-2,\ 11)
\end{aligned}$$

So $11=24+c$

$$c=-13$$

b $y=\dfrac{2}{(2x-3)^2}$

$$\begin{aligned}
&y=2(2x-3)^{-2}\\
&y=8x+c\\
&m_T=8\\
&\frac{dy}{dx}=-8(2x-3)^{-3}\\
&\quad=-\frac{8}{(2x-3)^3}
\end{aligned}$$

So $-\dfrac{8}{(2x-3)^3}=8$

$$\begin{aligned}
&(2x-3)^3=-1\\
&2x-3=-1\\
&2x=2\\
&x=1\\
&y(1)=2\\
&P\ (1,\ 2)
\end{aligned}$$

So $2=8+c$

$$c=-6$$

10 **a** $3y-4x+c=0$

$$\begin{aligned}
&3y=4x-c\\
&y=\frac{4}{3}x-\frac{c}{3}\\
&m_N=\frac{4}{3}\Rightarrow m_T=-\frac{3}{4}\\
&y=\frac{3}{x^2}=3x^{-2}\\
&\frac{dy}{dx}=-6x^{-3}\\
&\quad=-\frac{6}{x^3}
\end{aligned}$$

So $-\dfrac{6}{x^3}=-\dfrac{3}{4}$

$$\begin{aligned}
&x^3=8\\
&x=2\\
&y(2)=\frac{3}{4}\\
&P\left(2,\ \frac{3}{4}\right)
\end{aligned}$$

So $c=-3y+4x$

$$\begin{aligned}
&c=-\frac{9}{4}+8\\
&\quad=\frac{23}{4}
\end{aligned}$$

b $2y+x+c=0$

$$\begin{aligned}
&2y=-x-c\\
&y=-\frac{x}{2}-\frac{c}{2}\\
&m_N=-\frac{1}{2}\Rightarrow m_T=2\\
&y=\sqrt{4x-3}=(4x-3)^{\frac{1}{2}}\\
&\frac{dy}{dx}=\frac{1}{2}\times 4\times(4x-3)^{-\frac{1}{2}}\\
&\quad=\frac{2}{\sqrt{4x-3}}\\
&\frac{2}{\sqrt{4x-3}}=2
\end{aligned}$$

So $\sqrt{4x-3}=1$

$$\begin{aligned}
&4x-3=1\\
&x=1\\
&y(1)=1\\
&P(1,\ 1)\\
&c=-2y-x
\end{aligned}$$

So $c=-2-1$

$$c=-3$$

11 $y = 4\sin\left(\frac{x}{2}\right)$

$x = \frac{\pi}{3}$

$y = 4\sin\left(\frac{\pi}{6}\right) = 2$

$\frac{dy}{dx} = 2\cos\left(\frac{x}{2}\right)$

$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{3}} = 2\cos\left(\frac{\pi}{6}\right) = \sqrt{3}$

$m_T = \sqrt{3}$

$\mathrm{P}\left(\frac{\pi}{3}, 2\right)$

$T\colon\ y - 2 = \sqrt{3}\left(x - \frac{\pi}{3}\right)$

$y = \sqrt{3}x - \frac{\sqrt{3}\pi}{3} + 2$

$N\colon\ m_N = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

$y - 2 = -\frac{\sqrt{3}}{3}\left(x - \frac{\pi}{3}\right)$

$y = -\frac{\sqrt{3}}{3}x + \frac{\sqrt{3}\pi}{9} + 2$

12 $y = -3e^{-2x} + 4$

at $x = 0, y = 1$

$\frac{dy}{dx} = 6e^{-2x}$

$\left.\frac{dy}{dx}\right|_{x=0} = 6 = m_T$

$\mathrm{P}(0, 1)$

$T\colon\ y - 1 = 6(x - 0)$

$y = 6x + 1$

$N\colon\ m_N = -\frac{1}{6}$

$y - 1 = -\frac{1}{6}(x - 0)$

$6y - 6 = -x$

$6y + x - 6 = 0$

13 **a** $y = 16 - x^2$ at $x = 3$

When $x = 3$

$y = 16 - 9 = 7$

$\mathrm{P}(3,\ 7)$

$\frac{dy}{dx} = -2x$

At $x = 3$

$m_T = -6 \Rightarrow m_n = \frac{1}{6}$

$N\colon\ y - 7 = \frac{1}{6}(x - 3)$

$6y - 42 = x - 3$

$6y - x - 39 = 0$

b $y = \tan(3x)$ at $x = \frac{\pi}{9}$

When $x = \frac{\pi}{9}$

$y = \tan\left(\frac{\pi}{3}\right)$

$= \sqrt{3}$

$P\left(\frac{\pi}{9}, \sqrt{3}\right)$

$\frac{dy}{dx} = 3\sec^2(3x)$

$= \frac{3}{\cos^2(3x)}$

At $x = \frac{\pi}{9}$

$m_T = \frac{3}{\cos^2\left(\frac{\pi}{3}\right)} = 12 \Rightarrow m_n = -\frac{1}{12}$

$N\colon\ y - \sqrt{3} = -\frac{1}{12}\left(x - \frac{\pi}{9}\right)$

$12y - 12\sqrt{3} = -x + \frac{\pi}{9}$

$12y + x - 12\sqrt{3} - \frac{\pi}{9} = 0$

14 **a** $y = \log_e(4x - 3)$ at $x = 1$

When $x = 1$

$y = \log_e(1) = 0$

$\mathrm{P}(1, 0)$

$\frac{dy}{dx} = \frac{4}{4x - 3}$

When $x = 1$

$m_T = \frac{4}{1} = 4 \Rightarrow m_n = -\frac{1}{4}$

$N\colon\ y - 0 = -\frac{1}{4}(x - 1)$

$4y = -x + 1$

$4y + x = 1$

b $y = \frac{3}{(2x - 3)^2}$ at $x = 3$

When $x = 3$

$y = \frac{3}{9} = \frac{1}{3}$

$\mathrm{P}\left(3, \frac{1}{3}\right)$

$y = 3(2x - 3)^{-2}$

$\frac{dy}{dx} = 3 \times 2 \times -2 \times (2x - 3)^{-3}$

$= -\frac{12}{(2x - 3)^3}$

When $x = 3$

$m_T = -\frac{12}{3^3} = -\frac{4}{9} \Rightarrow m_n = \frac{9}{4}$

$N\colon\ y - \frac{1}{3} = \frac{9}{4}(x - 3)$

$4y - \frac{4}{3} = 9x - 27$

$12y - 4 = 27x - 81$

$12y - 27x + 77 = 0$

15 $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

16 $V = \frac{\pi}{3}r^2h$

a $\frac{dV}{dr} = \frac{2}{3}\pi rh$

b $\frac{dV}{dh} = \frac{\pi}{3}r^2$

17 $f(x) = 5 - 3e^{-2x}$

a $f'(x) = 6e^{-2x} > 0$

So $f(x)$ is always increasing.

b Crosses x-axis at $y = 0$

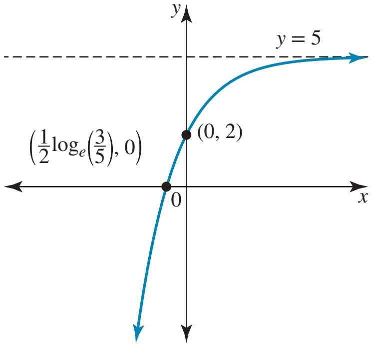
$$5 - 3e^{-2x} = 0$$
$$5 = 3e^{-2x}$$
$$e^{-2x} = \frac{5}{3}$$
$$x = -\frac{1}{2}\log_e\left(\frac{5}{3}\right)$$
$$= \frac{1}{2}\log_e\left(\frac{3}{5}\right)$$
$$= -0.26$$
$$\text{P}\left(\frac{1}{2}\log_e\left(\frac{3}{5}\right),0\right)$$

Crosses y-axis at $x = 0$

$f(0) = 2$

P$(0, 2)$

$y = 5$ is horizontal asymptote



c $f\left(\frac{1}{2}\right) = 5 - 3e^{-1}$

$$f'\left(\frac{1}{2}\right) = 6e^{-1} = m_T$$
$$y - \left(5 - 3e^{-1}\right) = 6e^{-1}\left(x - \frac{1}{2}\right)$$
$$y - 5 + \frac{3}{e} = \frac{6x}{e} - \frac{3}{e}$$
$$y = \frac{6x}{e} + 5 - \frac{6}{e}$$

18 f: $R \to$ R

$f(x) = 4 - 3e^{2x}$

a $f'(x) = -6e^{2x} < 0$

So $f(x)$ is always decreasing

b Crosses x-axis at $y = 0$

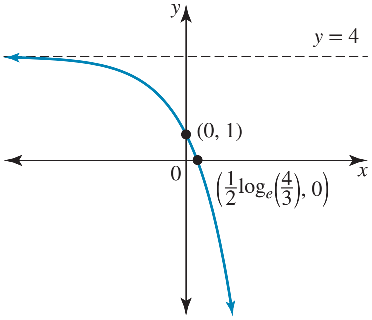
$$4 - 3e^{2x} = 0$$
$$e^{2x} = \frac{4}{3}$$
$$x = \frac{1}{2}\log_e\left(\frac{4}{3}\right)$$
$$= 0.14$$
$$\text{P}\left(\frac{1}{2}\log_e\left(\frac{4}{3}\right), 0\right)$$

Crosses y-axis at $x = 0$

$f(0) = 1$

P$(0, 1)$

$y = 4$ is horizontal asymptote



c $f\left(\frac{1}{2}\right) = 4 - 3e$

$$f'\left(\frac{1}{2}\right) = -6e = m_T$$
$$m_N = \frac{1}{6e}$$
$$N\text{: } y - (4 - 3e) = \frac{1}{6e}\left(x - \frac{1}{2}\right)$$
$$y = \frac{x}{6e} - \frac{1}{12e} - 3e + 4$$

19 $y = 3\log_e(4x - 5)$

a $4x - 5 > 0$

$$x > \frac{5}{4}$$

Domain: $\left(\frac{5}{4}, \infty\right)$

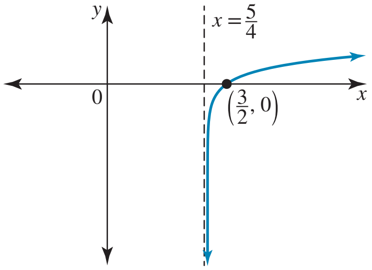
$x = \frac{5}{4}$ is a vertical asymptote

Range: R

Does not cross y-axis

Crosses x-axis at $y = 0$

$$\log_e(4x - 5) = 0$$
$$4x - 5 = 1$$
$$x = \frac{3}{2}$$
$$\text{P}\left(\frac{3}{2}, 0\right)$$



b $\frac{dy}{dx} = \frac{12}{4x - 5}$

Since $x > \frac{5}{4}$

$\frac{dy}{dx} > 0.$

No stationary points

c At $x = 2$

$y = 3\log_e(3)$

$$m_T = \left.\frac{dy}{dx}\right|_{x=2} = 4$$

T: $y - 3\log_e(3) = 4(x - 2)$

$y = 4x - 8 + \log_e(27)$

20 $f(x) = 2\log_e(5 - 2x)$

a $5 - 2x > 0$

$x < \frac{5}{2}$

Domain: $\left(-\infty, \frac{5}{2}\right)$

$x = \frac{5}{2}$ is a vertical asymptote

Range: R

Crosses x-axis at $y = 0$

$5 - 2x = 1$

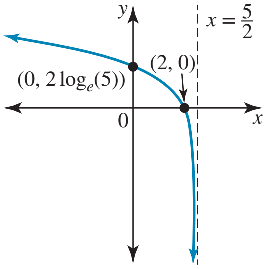
$x = 2$

P $(2, 0)$

Crosses y-axis at $x = 0$

$y = 2\log_e(5)$

P $(0, 2\log_e(5))$



b $f'(x) = -\frac{4}{5 - 2x}$

Since $x < \frac{5}{2}$

$f'(x) < 0$

So there are no stationary points, so the function is a one-one function.

c At $x = 1$

$f(1) = 2\log_e(3)$

P $(1, 2\log_e(3))$

$f'(1) = -\frac{4}{3} = m_T$

$m_N = \frac{3}{4}$

$N: y - 2\log_e(3) = \frac{3}{4}(x - 1)$

$y = \frac{3x}{4} - \frac{3}{4} + \log_e(9)$

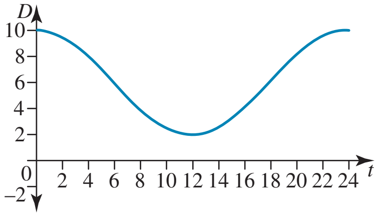
21 $D(t) = 6 + 4\cos\left(\frac{\pi t}{12}\right)$

a $D(6) = 6 + 4\cos\left(\frac{\pi}{2}\right) = 6$ m

b Period: $T = \frac{2\pi}{\frac{\pi}{12}} = 24$

Amplitude: 4

Range: $[2, 10]$



c $D = 8 = 6 + 4\cos\left(\frac{\pi t}{12}\right)$

$\cos\left(\frac{\pi t}{12}\right) = \frac{1}{2}$

$\frac{\pi t}{12} = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}$

$= \frac{\pi}{3}, \frac{5\pi}{3}$

$t = 4, 20$ between 4 am and 8 pm.

d $\frac{dD}{dt} = -4 \times \frac{\pi}{12}\sin\left(\frac{\pi t}{12}\right)$

$\left.\frac{dD}{dt}\right|_{t=3} = -\frac{\pi}{3}\sin\left(\frac{\pi}{4}\right) = -\frac{\pi\sqrt{2}}{6}$ m/s

e $D(6) = 6$

$D(0) = 10$

$\frac{D(6) - D(0)}{6 - 0} = \frac{6 - 10}{6} = -\frac{2}{3}$ m/s

22 $N = N(t) = N_0 e^{kt}$

a $t = 0$ 2010 $N(0) = N_0 = 500\,000$

$t = 10$ 2020 $N(10) = 750\,000$

$750\,000 = 500\,000e^{10k}$

$e^{10k} = \frac{3}{2}$

$10k = \log_e\left(\frac{3}{2}\right)$

$k = \frac{1}{10}\log_e\left(\frac{3}{2}\right) \approx 0.0405$

b $N(t) = 500\,000e^{0\cdot0405t}$

$N(15) = 500\,000e^{0\cdot0405\times15}$

$= 918\,559$

c $\frac{dN}{dt} = 500\,000 \times ke^{kt}$

$\left.\frac{dN}{dt}\right|_{t=15} = 37\,244.3$

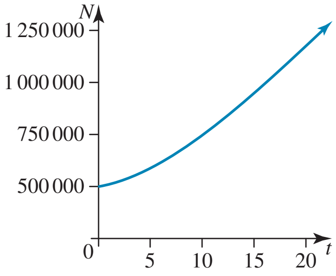
d $N(0) = 500\,000$

$N(30) = 1\,687\,500$

$\frac{N(30) - N(0)}{30 - 0} = \frac{1\,687\,500 - 500\,000}{30}$

$= 39\,583.3$

e



23 **a** $y = k - x^2, k > 0$

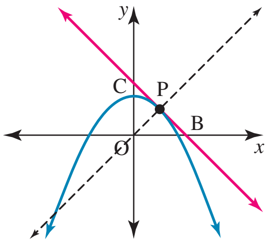
At P, $x = a, a > 0$

$y = k - a^2$

P $(a, k - a^2)$

$\frac{dy}{dx} = -2x$

$m_T = -2a$



$T\text{: } y - \left(k - a^2\right) = -2a\,(x - a)$

Crosses x-axis at $y = 0$

$-k + a^2 = -2a\,(x - a)$

$$x - a = -\frac{a^2 - k}{2a}$$

$$= \frac{k - a^2}{2a}$$

$$x = \frac{k - a^2}{2a} + a$$

$$= \frac{k - a^2 + 2a^2}{2a}$$

$$= \frac{a^2 + k}{2a}$$

$$\mathrm{B}\left(\frac{a^2 + k}{2a}, 0\right)$$

Crosses y-axis at $x = 0$

$y - k + a^2 = 2a^2$

$y = a^2 + k$

$\mathrm{C}\left(0,\ a^2 + k\right)$

$$\text{Area OBC} = \frac{1}{2}\left(a^2 + k\right) \times \left(\frac{a^2 + k}{2a}\right)$$

$$= \frac{\left(a^2 + k\right)^2}{4a}$$

b Normal at P $m_N = \dfrac{1}{2a}$

$y - \left(k - a^2\right) = \dfrac{1}{2a}\,(x - a)$ passes through origin

$$a^2 - k = -\frac{1}{2}$$

$$k = a^2 + \frac{1}{2}$$

24 **a** $i\,(t) = 120e^{-3t}\cos(10t)$

$$\frac{di}{dt} = -3 \times 120e^{-3t}\cos(10t) - 10 \times 120e^{-3t}\sin(10t)$$

$$= -120e^{-3t}\,(3\ \cos(10t) + 10\ \sin(10t))$$

$$\left.\frac{di}{dt}\right|_{t=0.01} = -120e^{-0.03}\left(10\ \cos\left(\frac{1}{10}\right) + 10\ \sin\left(\frac{1}{10}\right)\right)$$

$$= -463.875\ \text{mA/s}$$

b $i\,(0) = -120$

$$i\,(0.02) = 120e^{-0.06}\cos\left(\frac{1}{5}\right)$$

$$= 110.759$$

$$\frac{i\,(0.02) - i\,(0)}{0.02 - 0} = -462.048\ \text{mA/s}$$

25 $D\,(t) = 30te^{-\frac{t}{3}}$

a $D\,(1) = 30e^{-\frac{1}{3}}$

$D\,(2) = 60e^{-\frac{2}{3}}$

$$\text{Average } \frac{D\,(2) - D\,(1)}{2 - 1} = \frac{60e^{-\frac{2}{3}} - 30e^{-\frac{1}{3}}}{2 - 1}$$

$$= 9.31\ \text{mg}$$

b $\dfrac{dD}{dt} = 30e^{-\frac{t}{3}} + 30 \times t \times -\dfrac{1}{3}e^{-\frac{t}{3}}$

$$= e^{-\frac{t}{3}}\,(30 - 10t)$$

$$\left.\frac{dD}{dt}\right|_{t=1.5} = e^{-\frac{1}{2}}\left(30 - 10 \times \frac{3}{2}\right)$$

$$= 9.1\ \text{mg/h}$$

c $\dfrac{dD}{dt} = 0 \rightarrow t = 3$ hours

$D\,(3) = 90e^{-1}$

$= 33.11$ mg

d $D\,(t) = 30te^{-\frac{t}{3}} = 10$

$t = 0.378$ or 10.289 above 10 mg for 9.91 hours.

5.3 Exam questions

1 **a** $y = \dfrac{1}{x}$ when $x = a,\ a > 0$: $\mathrm{P}\left(a, \dfrac{1}{a}\right)$ [1 mark]

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

$$m_T = -\frac{1}{a^2}$$

$$T\text{: } y - \frac{1}{a} = -\frac{1}{a^2}\,(x - a)$$

Crosses x-axis when $y = 0$ at point B.

$-\dfrac{1}{a} = -\dfrac{1}{a^2}\,(x - a)$ [1 mark]

$x - a = a$

$x = 2a$

$\mathrm{B}\,(2a, 0)$

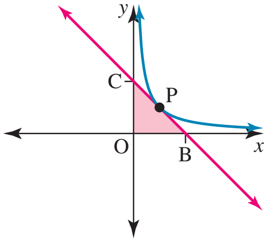
Crosses y-axis when $x = 0$ at point C.

$y - \dfrac{1}{a} = -\dfrac{1}{a^2}\,(-a)$ [1 mark]

$$y - \frac{1}{a} = \frac{1}{a}$$

$$y = \frac{2}{a}$$

$$\mathrm{C}\left(0, \frac{2}{a}\right)$$



$$\text{Area of } \Delta\text{OBC} = \frac{1}{2} \times 2a \times \frac{2}{a}$$

$= 2$ [1 mark]

2 **a** $f(x) = x^3e^{-2x}$

$f'(x) = 3x^2e^{-2x} - 2x^3e^{-2x}$

$= x^2e^{-2x}\,(3 - 2x)$

At $x = 1$

$$f(1) = e^{-2}$$
$$f'(1) = e^{-2}$$
$$T\text{: } y - e^{-2} = e^{-2}(x-1)$$
$$y = xe^{-2}$$ [1 mark]

At $x = 0$
$$f(0) = 0$$
$$f(0) = 0$$
$$T: y = 0$$

b At $x = a$ [1 mark]
$$P\left(a, a^3e^{-2a}\right)$$
$$f'(a) = e^{-2a}\left(3a^2 - 2a^3\right)$$
Tangent at $x = a$
$$y - a^3e^{-2a} = e^{-2a}\left(3a^2 - 2a^3\right)(x - a)$$ [1 mark]

If the tangent passes through the origin $x = 0, y = 0$
$$-a^3e^{-2a} = e^{-2a}\left(3a^2 - 2a^3\right) \times -a$$
$$-a^3 = -a\left(3a^2 - 2a^3\right)$$
$$a^2 = 3a^2 - 2a^3$$
$$2a^3 - 2a^2 = 0$$
$$2a^2(a-1) = 0$$
So $a = 0$ or $a = 1$

Are the only tangents which pass through the origin. [1 mark]

3 $P(t) = \dfrac{520}{0.3 + e^{-0.15t}}$

a $P(0) = \dfrac{520}{1.3} = 400$ [1 mark]

b $P(t) = 520\left(0.3 + e^{-0.15t}\right)^{-1}$
$$\frac{dP}{dt} = 0.15e^{-0.15t} \times 520\left(0.3 + e^{-0.15t}\right)^{-2}$$
$$= \frac{78e^{-0.15t}}{\left(0.3 + e^{-0.15t}\right)^2}$$
$$\left.\frac{dP}{dt}\right|_{t=10} = \frac{78e^{-0.15\times10}}{\left(0.3 + e^{-0.15\times10t}\right)^2}$$ [1 mark]
$$= 63.6$$

c $P(10) = 994.016$
$$\frac{P(10) - P(0)}{10 - 0} = \frac{994 - 400}{10}$$
$$= 59.4$$ [1 mark]

5.4 Implicit and parametric differentiation

5.4 Exercise

1 $x^3 + y^3 = 27$
$$\frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(y^3\right) = \frac{d}{dx}(27)$$
$$3x^2 + 3y^2\frac{dy}{dx} = 0$$
$$3x^2 = -3y^2\frac{dy}{dx}$$
$$\frac{dy}{dx} = -\frac{x^2}{y^2}$$

2
$$\sqrt{x} + \sqrt{y} = 4$$
$$\frac{d}{dx}\left(\sqrt{x}\right) + \frac{d}{dx}\left(\sqrt{y}\right) = \frac{d}{dx}(4)$$
$$\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} = 0$$
$$\frac{1}{2\sqrt{x}} = -\frac{1}{2\sqrt{y}}\frac{dy}{dx}$$
$$\frac{dy}{dx} = -\sqrt{\frac{y}{x}}$$

3 **a**
$$y^2 - 2x = 3$$
$$\frac{d}{dx}\left(y^2\right) - \frac{d}{dx}(2x) = \frac{d}{dx}(3)$$
$$2y\frac{dy}{dx} - 2 = 0$$
$$2y\frac{dy}{dx} = 2$$
$$\frac{dy}{dx} = \frac{1}{y}$$

b
$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$
$$\frac{1}{4}\frac{d}{dx}\left(x^2\right) + \frac{1}{9}\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(1)$$
$$\frac{x}{2} + \frac{2y}{9}\frac{dy}{dx} = 0$$
$$\frac{2y}{9}\frac{dy}{dx} = -\frac{x}{2}$$
$$\frac{dy}{dx} = -\frac{9x}{4y}$$

c
$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$
$$\frac{1}{16}\frac{d}{dx}\left(x^2\right) - \frac{1}{9}\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(1)$$
$$\frac{x}{8} - \frac{2y}{9}\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{9x}{16y}$$

d $4x - 2y - 3x^2 + y^2 = 10$
$$\frac{d}{dx}(4x) - \frac{d}{dx}(2y) - \frac{d}{dx}\left(3x^2\right) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(10)$$
$$4 - 2\frac{dy}{dx} - 6x + 2y\frac{dy}{dx} = 0$$
$$(2y - 2)\frac{dy}{dx} = 6x - 4$$
$$\frac{dy}{dx} = \frac{6x - 4}{2y - 2}$$
$$= \frac{3x - 2}{y - 1}$$

4 **a**
$$\frac{d}{dx}\left(4x^2y^2\right) = 4x^2\frac{d}{dx}\left(y^2\right) + y^2\frac{d}{dx}\left(4x^2\right)$$
$$= 8x^2y\frac{dy}{dx} + 8xy^2$$

b $9x^3 + 4x^2y^2 - 3y^3 + 2x - 5y + 12 = 0$

$$\frac{d}{dx}\left(9x^3\right) + \frac{d}{dx}\left(4x^2y^2\right) - \frac{d}{dx}\left(3y^3\right) + \frac{d}{dx}(2x)$$

$$-\frac{d}{dx}(5y) + \frac{d}{dx}(12) = 0$$

$$27x^2 + 8x^2y\frac{dy}{dx} + 8xy^2 - 9y^2\frac{dy}{dx} + 2 - 5\frac{dy}{dx} = 0$$

$$27x^2 + 8xy^2 + 2 = \left(5 + 9y^2 - 8x^2y\right)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{27x^2 + 8xy^2 + 2}{5 + 9y^2 - 8x^2y}$$

5 $x^2 - 4xy + 2y^2 - 3x + 5y - 7 = 0$

When $x = 2$

$$4 - 8y + 2y^2 - 6 + 5y - 7 = 0$$

$$2y^2 - 3y - 9 = 0$$

$$(2y + 3)(y - 3) = 0$$

$$y = -\frac{3}{2}, 3$$

So $y = 3$

P$(2, 3)$

$$\frac{d}{dx}\left(x^2\right) - \frac{d}{dx}(4xy) + \frac{d}{dx}\left(2y^2\right) - \frac{d}{dx}(3x)$$

$$+\frac{d}{dx}(5y) - \frac{d}{dx}(7) = 0$$

$$2x - 4y - 4x\frac{dy}{dx} + 4y\frac{dy}{dx} - 3 + 5\frac{dy}{dx} = 0$$

$$2x - 4y - 3 = (4x - 4y - 5)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x - 4y - 3}{4x - 4y - 5}$$

$$\left.\frac{dy}{dx}\right|_{(2,3)} = \frac{4 - 12 - 3}{8 - 12 - 5} = \frac{-11}{-9}$$

$$m_T = \frac{11}{9}$$

$$m_N = -\frac{9}{11}$$

6 **a** $\frac{d}{dx}\left(2x^2\right) + \frac{d}{dx}(3xy) - \frac{d}{dx}\left(3y^2\right) + \frac{d}{dx}(8) = 0$

$$4x + 3x\frac{dy}{dx} + 3y - 6y\frac{dy}{dx} = 0$$

$$4x + 3y = (6y - 3x)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{4x + 3y}{6y - 3x}$$

b $x^2 + x^2y^2 - 6x + 5 = 0$

$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(x^2y^2\right) - \frac{d}{dx}(6x) + \frac{d}{dx}(5) = 0$$

$$2x + 2xy^2 + 2x^2y\frac{dy}{dx} - 6 = 0$$

$$2x^2y\frac{dy}{dx} = 6 - 2x - 2xy^2$$

$$\frac{dy}{dx} = \frac{3 - x - xy^2}{x^2y}$$

c $x^3 - 3x^2y + 3xy^2 - y^3 - 27 = 0$

$$\frac{d}{dx}\left(x^3\right) - \frac{d}{dx}\left(3x^2y\right) + \frac{d}{dx}\left(3xy^2\right) - \frac{d}{dx}\left(y^3\right) - \frac{d}{dx}(27) = 0$$

$$3x^2 - \left(3x^2\frac{dy}{dx} + 6xy\right) + \left(3y^2 + 6xy\frac{dy}{dx}\right) - 3y^2\frac{dy}{dx} = 0$$

$$3x^2 - 6xy + 3y^2 = 3x^2\frac{dy}{dx} - 6xy\frac{dy}{dx} + 3y^2\frac{dy}{dx}$$

$$= \left(3x^2 - 6xy + 3y^2\right)\frac{dy}{dx}$$

$$\frac{dy}{dx} = 1$$

Alternatively

$$x^3 - 3x^2y + 3xy^2 - y^3 = 27$$

$$(x - y)^3 = 27$$

$$\frac{d}{dx}(x - y)^3 = \frac{d}{dx}(27)$$

$$3(x - y)^2\left(1 - \frac{dy}{dx}\right) = 0$$

If $x \neq y$ then $\frac{dy}{dx} = 1$

d $y^3 - y^2 - 3x - x^2 + 9 = 0$

$$\frac{d}{dx}\left(y^3\right) - \frac{d}{dx}\left(y^2\right) - \frac{d}{dx}(3x) - \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(9) = 0$$

$$3y^2\frac{dy}{dx} - 2y\frac{dy}{dx} - 3 - 2x = 0$$

$$\left(3y^2 - 2y\right)\frac{dy}{dx} = 2x + 3$$

$$\frac{dy}{dx} = \frac{2x + 3}{3y^2 - 2y}$$

$$= \frac{2x + 3}{y(3y - 2)}$$

7 $2xy + e^{-\left(x^2+y^2\right)} = c$

$$2xy + e^{-x^2} \cdot e^{-y^2} = c$$

$$\frac{d}{dx}(2xy) + \frac{d}{dx}\left(e^{-x^2} \cdot e^{-y^2}\right) = \frac{d}{dx}(c)$$

$$2x\frac{dy}{dx} + 2y + e^{-x^2}\frac{d}{dy}\left(e^{-y^2}\right) \cdot \frac{dy}{dx} + e^{-y^2}\frac{d}{dx}\left(e^{-x^2}\right) = 0$$

$$2x\frac{dy}{dx} + 2y - 2ye^{-\left(x^2+y^2\right)}\frac{dy}{dx} - 2xe^{-\left(x^2+y^2\right)} = 0$$

$$2y - 2xe^{-\left(x^2+y^2\right)} = 2ye^{-\left(x^2+y^2\right)}\frac{dy}{dx} - 2x\frac{dy}{dx}$$

$$y - xe^{-\left(x^2+y^2\right)} = \left(ye^{-\left(x^2+y^2\right)} - x\right)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y - xe^{-\left(x^2+y^2\right)}}{ye^{-\left(x^2+y^2\right)} - x}$$

8 $\sin(3x + 2y) + x^2 + y^2 = c$

$$\frac{d}{dx}(\sin(3x + 2y)) + \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(c)$$

Consider $\frac{d}{dx}(\sin(3x + 2y))$

Let $v = 3x + 2y$

$$\frac{d}{dv}(\sin(v)) \cdot \frac{dv}{dx} = \cos(v)\left(3 + 2\frac{dy}{dx}\right)$$

So $\cos(3x + 2y)\left(3 + 2\frac{dy}{dx}\right) + 2x + 2y\frac{dy}{dx} = 0$

$$3\cos(3x + 2y) + 2\cos(3x + 2y)\frac{dy}{dx} + 2x + 2y\frac{dy}{dx} = 0$$

$$3\cos(3x + 2y) + 2x = -2y\frac{dy}{dx} - 2\cos(3x + 2y)\frac{dy}{dx}$$

$$= -2(y + \cos(3x + 2y))\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{3\cos(3x + 2y) + 2x}{2(y + \cos(3x + 2y))}$$

9 a $\frac{y^2-2x}{3x^2+4y}=6x$

$$y^2-2x=6x\left(3x^2+4y\right)$$
$$=18x^3+24xy$$
$$2y\frac{dy}{dx}-2=54x^2+24y+24x\frac{dy}{dx}$$
$$2y\frac{dy}{dx}-24x\frac{dy}{dx}=54x^3+24y+2$$
$$(2y-24x)\frac{dy}{dx}=54x^2+24y+2$$
$$\frac{dy}{dx}=\frac{27x^2+12y+1}{y-12x}$$

b $\frac{x^3+8y^3}{x^2-2xy+4y^2}=x^2$

$$x^3+8y^3=x^2\left(x^2-2xy+4y^2\right)$$
$$x^3+8y^3=x^4-2x^3y+4x^2y^2$$
$$3x^2+24y^2\frac{dy}{dx}=4x^3-\left(6x^2y+2x^3\frac{dy}{dx}\right)+\left(8xy^2+8x^2y\frac{dy}{dx}\right)$$
$$\left(24y^2+2x^3-8x^2y\right)\frac{dy}{dx}=4x^3-6x^2y-3x^2+8xy^2$$
$$\frac{dy}{dx}=\frac{4x^3-6x^2y-3x^2+8xy^2}{24y^2+2x^3-8x^2y}$$

c

$$e^{x+y}+\cos(y)-y^2=0$$
$$e^x\cdot e^y+\cos(y)-y^2=0$$
$$e^xe^y+e^xe^y\frac{dy}{dx}-\sin(y)\frac{dy}{dx}-2y\frac{dy}{dx}=0$$
$$e^{x+y}=\frac{dy}{dx}\left(2y+\sin(y)-e^{x+y}\right)$$
$$\frac{dy}{dx}=\frac{e^{x+y}}{2y+\sin(y)-e^{x+y}}$$

d $e^{xy}+\cos(xy)+x^2=0$

Let $u=xy$

$$\frac{du}{dx}=x\frac{dy}{dx}+y$$
$$e^u+\cos(u)+x^2=0$$
$$\frac{d}{dx}\left(e^u\right)+\frac{d}{dx}(\cos(u))+\frac{d}{dx}\left(x^2\right)=0$$
$$e^u\frac{du}{dx}-\sin(u)\frac{du}{dx}+2x=0$$
$$\left(e^{xy}-\sin(xy)\right)\left(x\frac{dy}{dx}+y\right)+2x=0$$
$$xe^{xy}\frac{dy}{dx}-x\sin(xy)\frac{dy}{dx}+ye^{xy}-y\sin(xy)+2x=0$$
$$\left(xe^{xy}-x\sin(xy)\right)\frac{dy}{dx}=y\sin(xy)-ye^{xy}-2x$$
$$\frac{dy}{dx}=\frac{y\sin(xy)-ye^{xy}-2x}{xe^{xy}-x\sin(xy)}$$

10 a $\log_e(2x+3y)+4x-5y=10$

Let $u=2x+3y$

$$\frac{du}{dx}=2+3\frac{dy}{dx}$$
$$\frac{d}{dx}(\log_e(u))=\frac{d}{du}(\log_e(u))\frac{du}{dx}$$

So $\frac{1}{u}\left(2+3\frac{dy}{dx}\right)+4-5\frac{dy}{dx}=0$

$$\left(2+3\frac{dy}{dx}\right)=\left(5\frac{dy}{dx}-4\right)u$$
$$2+3\frac{dy}{dx}=\left(5\frac{dy}{dx}-4\right)(2x+3y)$$
$$2+3\frac{dy}{dx}=10x\frac{dy}{dx}-8x+15y\frac{dy}{dx}-12y$$
$$2+8x+12y=(15y+10x-3)\frac{dy}{dx}$$
$$\frac{dy}{dx}=\frac{2+8x+12y}{15y+10x-3}$$

b

$$\log_e(3xy)+x^2+y^2-9=0$$
$$\log_e(3x)+\log_e(y)+x^2+y^2-9=0$$
$$\frac{1}{x}+\frac{1}{y}\frac{dy}{dx}+2x+2y\frac{dy}{dx}=0$$
$$\frac{1}{x}+2x=-\left(2y+\frac{1}{y}\right)\frac{dy}{dx}$$
$$\frac{1+2x^2}{x}=-\left(\frac{2y^2+1}{y}\right)\frac{dy}{dx}$$
$$\frac{dy}{dx}=-\frac{y\left(2x^2+1\right)}{x\left(2y^2+1\right)}$$

c

$$\frac{1}{x}+2xy+\frac{1}{y}-6=0$$
$$x^{-1}+2xy+y^{-1}-6=0$$
$$-x^{-2}+\left(2y+2x\frac{dy}{dx}\right)-y^{-2}\frac{dy}{dx}=0$$
$$-\frac{1}{x^2}+2y=\frac{1}{y^2}\frac{dy}{dx}-2x\frac{dy}{dx}$$
$$\frac{2x^2y-1}{x^2}=\frac{1-2xy^2}{y^2}\frac{dy}{dx}$$
$$\frac{dy}{dx}=\frac{y^2\left(2x^2y-1\right)}{x^2\left(1-2xy^2\right)}$$

d $2y^3-\frac{\sin(3x)}{\sec(2y)}+x^2=0$

$$2y^3-\sin(3x)\cos(2y)+x^2=0$$
$$6y^2\frac{dy}{dx}+2\sin(3x)\sin(2y)\frac{dy}{dx}-3\cos(3x)\cos(2y)+2x=0$$
$$\left(6y^2+2\sin(3x)\sin(2y)\right)\frac{dy}{dx}=3\cos(3x)\cos(2y)-2x$$
$$\frac{dy}{dx}=\frac{3\cos(3x)\cos(2y)-2x}{6y^2+2\sin(3x)\sin(2y)}$$

11 a $y^2=x^3(2-x)$ at $(1,1)$

$$y^2=2x^3-x^4$$
$$2y\frac{dy}{dx}=6x^2-4x^3$$
$$\frac{dy}{dx}=\frac{3x^2-2x^3}{y}\text{ at }x=1,\ y=1$$
$$\left.\frac{dy}{dx}\right|_{(1,1)}=\frac{3-2}{1}=1$$

b

$$x^3+3xy+y^3+13=0$$
$$3x^2+3y+3x\frac{dy}{dx}+3y^2\frac{dy}{dx}=0$$
$$3x^2+3y=-\left(3y^2+3x\right)\frac{dy}{dx}$$
$$\frac{dy}{dx}=-\frac{x^2+y}{y^2+x}\text{ at }(1,-2)$$

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{(1,-2)} &= -\frac{1-2}{4+1}\\ &= \frac{1}{5}\\ &= m_T\end{aligned}$$

P$(1,-2)$

$$\begin{aligned}T\colon y+2 &= \frac{1}{5}(x-1)\\ 5(y+2) &= x-1\\ 5y+10 &= x-1\\ 5y-x+11 &= 0\\ x-5y-11 &= 0\end{aligned}$$

c $x^2+4x+y^2-3y+6xy-4=0$ at $(2,-1)$

$$2x+4+2y\frac{dy}{dx}-3\frac{dy}{dx}+6y+6x\frac{dy}{dx}=0$$

$$2x+4+6y=(3-2y-6x)\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{2x+4+6y}{3-2y-6x} \text{ at } x=2, \quad y=-1$$

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{(2,-1)} &= \frac{4+4-6}{3+2-12}\\ &= -\frac{2}{7}=m_T\end{aligned}$$

d $3x^2+2xy+4y^2+5x-10y-8=0$

$$6x+2y+2x\frac{dy}{dx}+8y\frac{dy}{dx}+5-10\frac{dy}{dx}=0$$

$$\begin{aligned}6x+2y+5 &= (10-8y-2x)\frac{dy}{dx}\\ \frac{dy}{dx} &= \frac{6+4+5}{10-16-2}\\ &= -\frac{15}{8}=m_T\\ \text{So } m_N &= \frac{8}{15}\end{aligned}$$

12 a $x^2+y^2=25$ at $(3,-4)$

$$\begin{aligned}2x+2y\frac{dy}{dx} &= 0\\ \frac{dy}{dx} &= -\frac{x}{y}\\ \left.\frac{dy}{dx}\right|_{(3,-4)} &= \frac{-3}{-4}\\ &= \frac{3}{4}=m_T\end{aligned}$$

P$(3,-4)$

$$\begin{aligned}T\colon y+4 &= \frac{3}{4}(x-3)\\ 4(y+4) &= 3(x-3)\\ 4y+16 &= 3x-9\\ 4y-3x+25 &= 0\\ 3x-4y-25 &= 0\end{aligned}$$

b $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$ at $x=1$

$$\begin{aligned}\frac{1}{4}+\frac{y^2}{9} &= 1\\ \frac{y^2}{9} &= 1-\frac{1}{4}\\ &= \frac{3}{4}\\ y^2 &= \frac{27}{4}\end{aligned}$$

So $y=\dfrac{3\sqrt{3}}{2}$ since $y>0$

$$\frac{2x}{4}+\frac{2y}{9}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{9x}{4y} \text{ at } \left(1,\frac{3\sqrt{3}}{2}\right)$$

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{\left(1,\frac{3\sqrt{3}}{2}\right)} &= -\frac{9}{4\times\frac{3\sqrt{3}}{2}}\times\frac{\sqrt{3}}{\sqrt{3}}\\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

c $\dfrac{x^2}{16}-\dfrac{y^2}{9}=1$ at $x=5$

$$\begin{aligned}\frac{25}{16}-\frac{y^2}{9} &= 1\\ \frac{y^2}{9} &= \frac{25}{16}-1\\ &= \frac{9}{16}\end{aligned}$$

But $y<0$

$$y=-\frac{9}{4}$$

$$\frac{2x}{16}-\frac{2y}{9}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{9x}{16y} \text{ at } \left(5,-\frac{9}{4}\right)$$

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{\left(5,-\frac{9}{4}\right)} &= \frac{9\times5}{16\times-\frac{9}{4}}\\ &= -\frac{5}{4}=m_T\\ m_N &= \frac{4}{5}\end{aligned}$$

d $2\left(x^2+y^2\right)^2=25\left(x^2-y^2\right)$ at $(3,1)$

$$2x^4+4x^2y^2+2y^4=25x^2-25y^2$$

$$8x^3+8xy^2+8x^2y\frac{dy}{dx}+8y^3\frac{dy}{dx}=50x-50y\frac{dy}{dx}$$

$$8x^3+8xy^2-50x=-\left(8x^2y+8y^3+50y\right)\frac{dy}{dx}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{50x-8xy^2-8x^3}{8x^2y+8y^3+50y}\\ &= \frac{25x-4xy^2-4x^3}{4x^2y+4y^3+25y}\\ \left.\frac{dy}{dx}\right|_{(3,1)} &= \frac{75-12-108}{36+4+25}\\ &= -\frac{9}{13}\\ m_N &= \frac{13}{9}\end{aligned}$$

13 a

$$\begin{aligned}
x &= 4\cos(t)\\
y &= 4\sin(t)\\
\frac{dx}{dt} &= \dot{x} = -4\sin(t)\\
\frac{dy}{dt} &= \dot{y} = 4\cos(t)\\
\frac{dy}{dx} &= \frac{dt}{dx}\cdot\frac{dy}{dt} = \frac{\dot{y}}{\dot{x}}\\
&= -\frac{4\cos(t)}{4\sin(t)}\\
&= -\cot(t)
\end{aligned}$$

b

$$\begin{aligned}
\cos(t) &= \frac{x}{4}\\
\sin(t) &= \frac{y}{4}\\
\cos^2(t) + \sin^2(t) &= 1\\
\frac{x^2}{16} + \frac{y^2}{16} &= 1\\
x^2 + y^2 &= 16\\
2x + 2y\frac{dy}{dx} &= 0\\
\frac{dy}{dx} = -\frac{x}{y} &= -\frac{4\cos(t)}{4\sin(t)}\\
&= -\cot(t)
\end{aligned}$$

14 a

$$\begin{aligned}
x &= t^2\\
y &= 2t - t^4\\
\frac{dx}{dt} &= \dot{x} = 2t\\
\frac{dy}{dt} &= \dot{y} = 2 - 4t^3\\
\frac{dy}{dx} &= \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{\dot{y}}{\dot{x}}\\
&= \frac{2-4t^3}{2t}\\
&= \frac{1-2t^3}{t}
\end{aligned}$$

b

$$\begin{aligned}
t &= \sqrt{x}\\
y &= 2t - t^4\\
&= 2\sqrt{x} - x^2\\
\frac{dy}{dx} &= \frac{1}{\sqrt{x}} - 2x\\
&= \frac{1}{t} - 2t^2\\
&= \frac{1-2t^3}{t}
\end{aligned}$$

15 a

$$\begin{aligned}
x &= 2t^2\\
y &= 4t\\
\frac{dx}{dt} &= \dot{x} = 4t\\
\frac{dy}{dt} &= \dot{y} = 4\\
\frac{dy}{dx} &= \frac{\dot{y}}{\dot{x}} = \frac{4}{4t} = \frac{1}{t}
\end{aligned}$$

b

$$\begin{aligned}
x &= 3\sin(2t)\\
y &= 4\cos(2t)\\
\frac{dx}{dt} &= \dot{x} = 6\cos(2t)\\
\frac{dy}{dt} &= \dot{y} = -8\sin(2t)\\
\frac{dy}{dx} &= \frac{\dot{y}}{\dot{x}}\\
&= -\frac{8\sin(2t)}{6\cos(2t)}\\
&= -\frac{4}{3}\tan(2t)
\end{aligned}$$

c

$$\begin{aligned}
x &= 5t\\
y &= \frac{5}{t} = 5t^{-1}\\
\frac{dx}{dt} &= \dot{x} = 5\\
\frac{dy}{dt} &= \dot{y} = -5t^{-2} = -\frac{5}{t^2}\\
\frac{dy}{dx} &= \frac{\dot{y}}{\dot{x}} = \frac{-\frac{5}{t^2}}{5} = -\frac{5}{t^2}\times\frac{1}{5}\\
&= -\frac{1}{t^2}
\end{aligned}$$

d

$$\begin{aligned}
x &= 3\sec(t)\\
y &= 4\tan(t)\\
\frac{dx}{dt} &= \dot{x} = 3\sec(t)\tan(t)\\
\frac{dy}{dt} &= \dot{y} = 4\sec^2(t)\\
\frac{dy}{dx} &= \frac{\dot{y}}{\dot{x}} = \frac{4\sec^2(t)}{3\sec(t)\tan(t)}\\
&= \frac{4}{3}\frac{1}{\cos(t)}\times\frac{\cos(t)}{\sin(t)}\\
&= \frac{4}{3}\operatorname{cosec}(t)
\end{aligned}$$

16 a

$$\begin{aligned}
x &= at^2\\
y &= 2at\\
\frac{dx}{dt} &= \dot{x} = 2at\\
\frac{dy}{dt} &= \dot{y} = 2a\\
\frac{dy}{dx} &= \frac{\dot{y}}{\dot{x}} = \frac{2a}{2at} = \frac{1}{t}
\end{aligned}$$

b

$$\begin{aligned}
x &= a\cos(t)\\
y &= b\sin(t)\\
\frac{dx}{dt} &= \dot{x} = -a\sin(t)\\
\frac{dy}{dt} &= \dot{y} = b\cos(t)\\
\frac{dy}{dx} &= \frac{\dot{y}}{\dot{x}} = -\frac{b\cos(t)}{a\sin(t)} = -\frac{b}{a}\cot(t)
\end{aligned}$$

c

$$\begin{aligned}
x &= at\\
y &= \frac{a}{t} = at^{-1}\\
\frac{dx}{dt} &= \dot{x} = a\\
\frac{dy}{dt} &= \dot{y} = -at^{-2} = -\frac{a}{t^2}\\
\frac{dy}{dx} &= \frac{\dot{y}}{\dot{x}} = \frac{-\frac{a}{t^2}}{a} = -\frac{a}{t^2}\times\frac{1}{a} = -\frac{1}{t^2}
\end{aligned}$$

d $x = a\sec(t)$

$y = b\tan(t)$

$$\frac{dx}{dt} = \dot{x} = a\sec(t)\tan(t)$$

$$\frac{dy}{dt} = \dot{y} = b\sec^2(t)$$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{b\sec^2(t)}{a\sec(t)\tan(t)} = \frac{b}{a}\text{cosec}(t)$$

17 a [1] $x = at^2$

[2] $y = 2\,at \rightarrow t = \dfrac{y}{2a}$ into [1]

$$x = a\left(\frac{y}{2a}\right)^2 = \frac{y^2}{4a}$$

$$y^2 = 4ax$$

$$2y\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}$$

$$= \frac{2a}{2at}$$

$$= \frac{1}{t}$$

b [1] $x = a\cos(t) \rightarrow \cos(t) = \dfrac{x}{a}$

[2] $y = b\sin(t) \rightarrow \sin(t) = \dfrac{y}{b}$

$$\cos^2(t) + \sin^2(t) = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y} = -\frac{b^2a\cos(t)}{a^2b\sin(t)}$$

$$= -\frac{b}{a}\cot(t)$$

c $x = at \rightarrow t = \dfrac{x}{a}$

$y = \dfrac{a}{t} \rightarrow t = \dfrac{a}{y}$

$$\frac{x}{a} = \frac{a}{y}$$

$$xy = a^2$$

$$y + x\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{y}{x} = \frac{-\frac{a}{t}}{at} = -\frac{1}{t^2}$$

d $x = a\sec(t) \rightarrow \sec(t) = \dfrac{x}{a}$

$y = b\tan(t) \rightarrow \tan(t) = \dfrac{y}{b}$

$$\tan^2(t) + 1 = \sec^2(t)$$

$$\frac{y^2}{b^2} + 1 = \frac{x^2}{a^2}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{b^2x}{a^2y} = \frac{b^2 \times a\ \sec(t)}{a^2 \times b\tan(t)}$$

$$= \frac{b}{a} \times \frac{1}{\cos(t)} \times \frac{\cos(t)}{\sin(t)}$$

$$= \frac{b}{a}\text{cosec}(t)$$

18 a

$$x^3 - 3axy + y^3 = 0$$

$$3x^2 - 3ay - 3ax\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

$$3x^2 - 3ay = \left(3ax - 3y^2\right)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{x^2 - ay}{ax - y^2}$$

b $x = \dfrac{3at}{1 + t^3}$

$y = \dfrac{3at^2}{1 + t^3}$

LHS: $x^3 - 3axy + y^3$

$$= \left(\frac{3at}{1+t^3}\right)^3 - 3a\left(\frac{3at}{1+t^3}\right)\left(\frac{3at^2}{1+t^3}\right) + \left(\frac{3at^2}{1+t^3}\right)^3$$

$$= \frac{27a^3t^3}{\left(1+t^3\right)^3} - \frac{27a^3t^3}{\left(1+t^3\right)^2} + \frac{27a^3t^6}{\left(1+t^3\right)^3}$$

$$= \frac{27a^3t^3 - 27a^3t^3\left(1+t^3\right) + 27a^3t^6}{\left(1+t^3\right)^3}$$

$= 0$

= RHS shown

$x = \dfrac{3at}{1 + t^3}$

Quotient rule:

$$\frac{dx}{dt} = \dot{x} = \frac{3a\left(1+t^3\right) - 3t^2 \times 3at}{\left(1+t^3\right)^2}$$

$$= \frac{3a - 6at^3}{\left(1+t^3\right)^2}$$

$$= \frac{3a\left(1-2t^3\right)}{\left(1+t^3\right)^2}$$

$y = \dfrac{3at^2}{1 + t^3}$

$$\frac{dy}{dt} = \dot{y} = \frac{6at\left(1+t^3\right) - 3t^2 \times \left(3at^2\right)}{\left(1+t^3\right)^2}$$

$$= \frac{6at - 3at^4}{\left(1+t^3\right)^2}$$

$$= \frac{3at\left(2-t^3\right)}{\left(1+t^3\right)^2}$$

So $\dfrac{dy}{dx} = \dfrac{\dot{y}}{\dot{x}}$

$$= \frac{3at\left(2-t^3\right)}{\left(1+t^3\right)^2} \times \frac{\left(1+t^3\right)^2}{3a\left(1-2t^3\right)}$$

$$= \frac{t\left(t^3-2\right)}{2t^3-1}$$

19 a $\left(x^2+y^2\right)^2 = 2\left(x^2-y^2\right)$

$$x^4+2x^2y^2+y^4-2x^2+2y^2=0$$

$$4x^3+4xy^2+4x^2y\frac{dy}{dx}+4y^3\frac{dy}{dx}-4x+4y\frac{dy}{dx}=0$$

$$\left(4x^2y+4y^3+4y\right)\frac{dy}{dx}=4x-4x^3-4xy^2$$

$$\frac{dy}{dx}=\frac{x-x^3-xy^2}{x^2y+y^3+y}$$

$$=\frac{x\left(1-x^2-y^2\right)}{y\left(x^2+y^2+1\right)}$$

$$=-\frac{x\left(x^2+y^2-1\right)}{y\left(x^2+y^2+1\right)}$$

b $x=\dfrac{\sqrt{2}\cos(t)}{1+\sin^2(t)}$

$$x^2=\frac{2\cos^2(t)}{\left[1+\sin^2(t)\right]^2}$$

$$y=\frac{\sqrt{2}\sin(t)\cos(t)}{1+\sin^2(t)}$$

$$y^2=\frac{2\sin^2(t)\cos^2(t)}{\left[1+\sin^2(t)\right]^2}$$

LHS: $\left(x^2+y^2\right)^2-2\left(x^2-y^2\right)$

$$=\left[\frac{2\cos^2(t)\left(1+\sin^2(t)\right)}{\left[1+\sin^2(t)\right]^2}\right]^2-2\left[\frac{2\cos^2(t)-2\sin^2(t)\cos^2(t)}{\left[1+\sin^2(t)\right]^2}\right]$$

$$=\frac{4\cos^4(t)}{\left[1+\sin^2(t)\right]^2}-2\left[\frac{2\cos^2(t)\left[1-\sin^2(t)\right]}{\left[1+\sin^2(t)\right]^2}\right]$$

$$=\frac{4\cos^4(t)-4\cos^4(t)}{\left[1+\sin^2(t)\right]^2}$$

$$=0$$

= LHS shown

$$x=\frac{\sqrt{2}\cos(t)}{1+\sin^2(t)}$$

$$\frac{dx}{dt}=\dot{x}=-\frac{\sqrt{2}\sin(t)\left(\cos^2(t)+2\right)}{\left[1+\sin^2(t)\right]^2}$$

$$y=\frac{\sqrt{2}\sin(t)\cos(t)}{1+\sin^2(t)}$$

$$\frac{dy}{dt}=\dot{y}=\frac{\sqrt{2}\left(2\cos^2(t)-\sin^2(t)-1\right)}{\left[1+\sin^2(t)\right]^2}$$

$$\frac{dy}{dx}=\frac{\dot{y}}{\dot{x}}=\frac{2-3\cos^2(t)}{\sin(t)\left(2+\cos^2(t)\right)}$$

20 a

$$y^2=\frac{x^3}{2a-x}$$

$$y^2\left(2a-x\right)=x^3$$

$$2ay^2-xy^2-x^3=0$$

$$4ay\frac{dy}{dx}-y^2-2xy\frac{dy}{dx}-3x^2=0$$

$$\left(4ay-2xy\right)\frac{dy}{dx}=y^2+3x^2$$

$$\frac{dy}{dx}=\frac{y^2+3x^2}{2y\left(2a-x\right)}$$

b $x=\dfrac{2at^2}{1+t^2}$

$$y=\frac{2at^3}{1+t^2}$$

Consider $y^2\left(2a-x\right)=x^3$

LHS: $y^2\left(2a-x\right)=\dfrac{4a^2t^6}{\left[1+t^2\right]^2}\left[2a-\dfrac{2at^2}{1+t^2}\right]$

$$=\frac{4a^2t^6}{\left[1+t^2\right]^2}\left[\frac{2a\left(1+t^2\right)-2at^2}{1+t^2}\right]$$

$$=\frac{4a^2t^6\left(2a+2at^2-2at^2\right)}{\left[1+t^2\right]^3}$$

$$=\frac{8a^3t^6}{\left[1+t^2\right]^3}$$

$$=\left[\frac{2at^2}{1+t^2}\right]^3$$

$$=x^3$$

= RHS shown

$$\frac{dx}{dt}=\dot{x}=\frac{4at\left(1+t^2\right)-2t\left(2at^2\right)}{\left[1+t^2\right]^2}=\frac{4at}{\left[1+t^2\right]^2}$$

$$\frac{dy}{dt}=\dot{y}=\frac{6at^2\left(1+t^2\right)-2t\left(2at^3\right)}{\left[1+t^2\right]^2}=\frac{2at^2\left(t^2+3\right)}{\left[1+t^2\right]^2}$$

$$\frac{dy}{dx}=\frac{\dot{y}}{\dot{x}}=\frac{t\left(t^2+3\right)}{2}$$

c $x=2a\sin^2(t)$

$$y=\frac{2a\sin^3(t))}{\cos(t)}$$

$$y^2\left(2a-x\right)=x^3$$

LHS: $=\dfrac{4a^2\sin^6(t)}{\cos^2(t)}\left(2a-2a\sin^2(t)\right)$

$$=\frac{4a^2\sin^6(t)}{\cos^2(t)}\left(2a\left(1-\sin^2(t)\right)\right)$$

$$=8a^3\sin^6(t)$$

$$=\left(2a\sin^2(t)\right)^3$$

$$=x^3$$

= RHS as shown

$$x=2a\sin^2(t)$$

$$\frac{dx}{dt}=\dot{x}=4a\sin(t)\cos(t)$$

$$y=\frac{2a\sin^3(t)}{\cos(t)}$$

$$\frac{dy}{dt}=\dot{y}=2a\tan^2(t)\left(2\cos^2(t)+1\right)$$

$$\frac{dy}{dx}=\frac{\dot{y}}{\dot{x}}=\frac{\cancel{2}\cancel{a}\tan^2(t)\left(2\cos^2(t)+1\right)}{{}_2\cancel{4}\cancel{a}\sin(t)\cos(t)}$$

$$=\frac{\sin(t)\left(2\cos^2(t)+1\right)}{2\cos^3(t)}$$

5.4 Exam questions

1 $\sin\left(x^2\right)+\cos\left(y^2\right)=\frac{3\sqrt{2}}{\pi}xy$ using implicit differentiation

$$\frac{d}{dx}\left(\sin\left(x^2\right)\right)+\frac{d}{dx}\left(\cos\left(y^2\right)\right)=\frac{3\sqrt{2}}{\pi}\frac{d}{dx}(xy)$$

$$2x\cos\left(x^2\right)-2y\sin\left(y^2\right)\frac{dy}{dx}=\frac{3\sqrt{2}}{\pi}\left(y+x\frac{dy}{dx}\right)$$

$$2x\cos\left(x^2\right)-\frac{3\sqrt{2}y}{\pi}=\frac{dy}{dx}\left(\frac{3\sqrt{2}x}{\pi}+2y\sin\left(y^2\right)\right)$$

Now at the point $\left(\frac{\sqrt{\pi}}{\sqrt{6}},\frac{\sqrt{\pi}}{\sqrt{3}},\right)$ $x=\frac{\sqrt{\pi}}{\sqrt{6}},y=\frac{\sqrt{\pi}}{\sqrt{3}}$

$$2\frac{\sqrt{\pi}}{\sqrt{6}}\cos\left(\frac{\pi}{6}\right)-\frac{3\sqrt{2}}{\pi}\times\frac{\sqrt{\pi}}{\sqrt{3}}$$
$$=\frac{dy}{dx}\left(\frac{3\sqrt{2}}{\pi}\times\frac{\sqrt{\pi}}{\sqrt{6}}+2\frac{\sqrt{\pi}}{\sqrt{3}}\sin\left(\frac{\pi}{3}\right)\right)$$

$$2\frac{\sqrt{\pi}}{\sqrt{6}}\times\frac{\sqrt{3}}{2}-\frac{3\sqrt{2}}{\pi}\times\frac{\sqrt{\pi}}{\sqrt{3}}$$
$$=\frac{dy}{dx}\left(\frac{3\sqrt{2}}{\pi}\times\frac{\sqrt{\pi}}{\sqrt{6}}+2\frac{\sqrt{\pi}}{\sqrt{3}}\times\frac{\sqrt{3}}{2}\right)$$

$$\frac{\sqrt{\pi}}{\sqrt{2}}-\frac{\sqrt{6}}{\sqrt{\pi}}=\frac{dy}{dx}\left[\sqrt{\pi}+\frac{\sqrt{3}}{\sqrt{\pi}}\right]$$

$$\frac{\pi-\sqrt{12}}{\sqrt{2\pi}}=\frac{dy}{dx}\left(\frac{\pi+\sqrt{3}}{\sqrt{\pi}}\right)$$

$$\frac{dy}{dx}=\frac{\pi-2\sqrt{3}}{\sqrt{2}\left(\pi+\sqrt{3}\right)}\quad a=2,\ b=3$$

Award 1 mark for using implicit differentiation and product rule.
Award 1 mark for correctly rearranging.
Award 1 mark for correctly substituting the x and y values.
Award 1 mark for transforming.
Award 1 mark for the final correct values of a and b.

2 $3xy^2+2y=x$ using implicit differentiation, with product rule in the first term

$$3y^2+6xy\frac{dy}{dx}+2\frac{dy}{dx}=1$$

$$\frac{dy}{dx}(6xy+2)=1-3y^2$$

$$\frac{dy}{dx}=\frac{1-3y^2}{6xy+2}\text{ at the point }(1,-1)$$

$$\left.\frac{dy}{dx}\right|_{(1,-1)}=\frac{1-3}{-6+2}$$
$$=\frac{1}{2}$$

$$T\colon y+1=\frac{1}{2}(x-1)$$
$$2y+2=x-1$$
$$2y-x+3=0$$

Award 1 mark for using implicit differentiation with the product rule.
Award 1 mark for the correct gradient.
Award 1 mark for the correct tangent line.

3 $x=\sin(t)-\cos(t)\qquad y=\frac{1}{2}\sin(2t)$

$$\frac{dx}{dt}=\dot{x}=\cos(t)+\sin(t)\quad\frac{dy}{dt}=\dot{y}=\cos(2t)$$

$$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=\frac{\dot{y}}{\dot{x}}=\frac{\cos(2)}{\cos(t)+\sin(t)}$$
$$=\frac{\cos(2t)}{\cos(t)+\sin(t)}\times\frac{\cos(t)-\sin(t)}{\cos(t)-\sin(t)}$$
$$=\frac{\cos(2t)\left(\cos(t)-\sin(t)\right)}{\cos^2(t)-\sin^2(t)}$$
$$=\frac{\cos(2t)\left(\cos(t)-\sin(t)\right)}{\cos(2t)}$$
$$=\cos(t)-\sin(t)$$

The correct answer is **A**.

5.5 Second derivatives

5.5 Exercise

1 $f(x)=\frac{8\sqrt{x^3}}{3x},\ x\neq 0$

$$=\frac{8}{3}\cdot\frac{x^{\frac{3}{2}}}{x}$$
$$=\frac{8}{3}x^{\frac{1}{2}}$$
$$f'(x)=\frac{4}{3}x^{-\frac{1}{2}}$$
$$f''(x)=-\frac{2}{3}x^{-\frac{3}{2}}$$
$$=-\frac{2}{3\sqrt{x^3}}$$
$$f''(4)=-\frac{2}{3\sqrt{4^3}}$$
$$=-\frac{2}{3\times 2^3}$$
$$=-\frac{1}{12}$$

2 $f(x)=8\cos\left(\frac{x}{2}\right)$

$$f'(x)=-4\sin\left(\frac{x}{2}\right)$$
$$f''(x)=-2\cos\left(\frac{x}{2}\right)$$
$$f''\left(\frac{\pi}{3}\right)=-2\cos\left(\frac{\pi}{6}\right)$$
$$=-2\times\frac{\sqrt{3}}{2}$$
$$=-\sqrt{3}$$

3 $f(x)=\frac{4x^2}{3\sqrt{x}}$

$$x>0$$
$$f(x)=\frac{4}{3}x^{2-\frac{1}{2}}$$
$$f(x)=\frac{4}{3}x^{\frac{3}{2}}$$
$$f'(x)=\frac{4}{3}\times\frac{3}{2}x^{\frac{1}{2}}$$
$$=2x^{\frac{1}{2}}$$

$$\begin{aligned} f''(x) &= 2\times\frac{1}{2}\times x^{-\frac{1}{2}} \\ &= \frac{1}{\sqrt{x}} \\ f''(4) &= \frac{1}{\sqrt{4}} = \frac{1}{2} \end{aligned}$$

4
$$\begin{aligned} f(x) &= \frac{2}{3x-5} \\ &= 2(3x-5)^{-1} \\ f'(x) &= -6(3x-5)^{-2} \\ f''(x) &= 12\times 3\times(3x-5)^{-3} \\ &= \frac{36}{(3x-5)^3} \\ f''(1) &= \frac{36}{(-2)^3} \\ &= -\frac{9}{2} \end{aligned}$$

5
$$\begin{aligned} y &= x^3\log_e\left(2x^2+5\right) \\ \frac{dy}{dx} &= x^3\frac{d}{dx}\left(\log_e\left(2x^2+5\right)\right)+\log_e\left(2x^2+5\right)\frac{d}{dx}\left(x^3\right) \\ &= \frac{x^3\times 4x}{2x^2+5}+3x^2\log_e\left(2x^2+5\right) \\ &= \frac{4x^4}{2x^2+5}+3x^2\log_e\left(2x^2+5\right) \\ \frac{d^2y}{dx^2} &= \frac{\frac{d}{dx}\left(4x^4\right)\cdot\left(2x^2+5\right)-\frac{d}{dx}\left(2x^2+5\right)\cdot\left(4x^4\right)}{\left(2x^2+5\right)^2}+ \\ &\quad \frac{3x^2\times 4x}{2x^2+5}+6x\log_e\left(2x^2+5\right) \\ &= \frac{16x^3\left(2x^2+5\right)-4x\times 4x^4}{\left(2x^2+5\right)^2}+\frac{12x^3}{2x^2+5}+ \\ &\quad 6x\log_e\left(2x^2+5\right) \\ &= \frac{32x^5+80x^3-16x^5}{\left(2x^2+5\right)^2}+\frac{12x^3}{2x^2+5}+6x\log_e\left(2x^2+5\right) \\ &= \frac{16x^5+80x^3}{\left(2x^2+5\right)^2}+\frac{12x^3}{2x^2+5}+6x\log_e\left(2x^2+5\right) \\ &= \frac{16x^5+80x^3+12x^3\left(2x^2+5\right)}{\left(2x^2+5\right)^2}+6x\log_e\left(2x^2+5\right) \\ &= \frac{40x^5+140x^3}{\left(2x^2+5\right)^2}+6x\ \log_e\left(2x^2+5\right) \\ &= \frac{20x^3\left(2x^2+7\right)}{\left(2x^2+5\right)^2}+6x\log_e\left(2x^2+5\right) \end{aligned}$$

6
$$\begin{aligned} y &= \frac{x^4}{e^{3x}} = x^4e^{-3x} \\ \frac{dy}{dx} &= x^4\frac{d}{dx}\left(e^{-3x}\right)+e^{-3x}\frac{d}{dx}\left(x^4\right) \\ &= -3e^{-3x}x^4+4x^3e^{-3x} \\ &= x^3e^{-3x}(4-3x) \\ &= 4x^3e^{-3x}-3e^{-3x}x^4 \\ \frac{d^2y}{dx^2} &= 4x^3\frac{d}{dx}\left(e^{-3x}\right)+e^{-3x}\frac{d}{dx}\left(4x^3\right) \\ &\quad -\left[3x^4\frac{d}{dx}\left(e^{-3x}\right)+e^{-3x}\frac{d}{dx}\left(3x^4\right)\right] \\ &= -12x^3e^{-3x}+12x^2e^{-3x}+9x^4e^{-3x}-12x^3e^{-3x} \\ &= \left(12x^2+9x^4-24x^3\right)e^{-3x} \\ &= 3x^2\left(4+3x^2-8x\right)e^{-3x} \end{aligned}$$

7
$$\begin{aligned} f(x) &= 4\log_e(2x-3) \\ f'(x) &= \frac{8}{(2x-3)} = 8(2x-3)^{-1} \\ f''(x) &= -16(2x-3)^{-2} \\ &= -\frac{16}{(2x-3)^2} \\ f''(3) &= -\frac{16}{3^2} \\ &= -\frac{16}{9} \end{aligned}$$

8
$$\begin{aligned} f(x) &= e^{x^2} \\ f'(x) &= 2xe^{x^2} \\ f''(x) &= 2e^{x^2}+4x^2e^{x^2} \\ f''(1) &= 2e^1+4e^1 \\ &= 6e \end{aligned}$$

9 a
$$\begin{aligned} y &= \log_e\left(x^2+4x+13\right) \\ \frac{dy}{dx} &= \frac{2x+4}{x^2+4x+13} \\ \frac{d^2y}{dx^2} &= \frac{2\left(x^2+4x+13\right)-(2x+4)^2}{\left(x^2+4x+13\right)^2} \\ &= \frac{2x^2+8x+26-\left(4x^2+16x+16\right)}{\left(x^2+4x+13\right)^2} \\ &= \frac{-2x^2-8x+10}{\left(x^2+4x+13\right)^2} \\ &= \frac{-2\left(x^2+4x-5\right)}{\left(x^2+4x+13\right)^2} \end{aligned}$$

b
$$\begin{aligned} y &= e^{3x}\cos(4x) \\ \frac{dy}{dx} &= 3e^{3x}\cos(4x)-4e^{3x}\sin(4x) \\ &= e^{3x}(3\cos(4x)-4\sin(4x)) \\ \frac{d^2y}{dx^2} &= 3e^{3x}(3\cos(4x)-4\sin(4x)) \\ &\quad +e^{3x}(-12\sin(4x)-16\cos(4x)) \\ &= e^{3x}(-7\cos(4x)-24\sin(4x)) \\ &= -e^{3x}(7\cos(4x)+24\sin(4x)) \end{aligned}$$

c
$$\begin{aligned} y &= x^3e^{-2x} \\ \frac{dy}{dx} &= 3x^2e^{-2x}-2x^3e^{-2x} \\ &= e^{-2x}\left(3x^2-2x^3\right) \\ \frac{d^2y}{dx^2} &= -2e^{-2x}\left(3x^2-2x^3\right)+e^{-2x}\left(6x-6x^2\right) \\ &= e^{-3x}\left(-12x^2+4x^3+6x\right) \\ &= 2xe^{-2x}\left(2x^2-6x+3\right) \end{aligned}$$

10 a
$$\begin{aligned} y &= x^2\cos(3x) \\ \frac{dy}{dx} &= 2x\ \cos(3x)-3x^2\sin(3x) \\ \frac{d^2y}{dx^2} &= 2\ \cos(3x)-6x\ \sin(3x)-6x\ \sin(3x)-9x^2\cos(3x) \\ &= \left(2-9x^2\right)\cos(3x)-12x\ \sin(3x) \end{aligned}$$

b
$$\begin{aligned} y &= x\log_e(6x+7) \\ \frac{dy}{dx} &= \log_e(6x+7)+\frac{6x}{(6x+7)} \\ \frac{d^2y}{dx^2} &= \frac{6}{(6x+7)}+\frac{6\,(6x+7)-6x\times 6}{(6x+7)^2} \end{aligned}$$

$$
\begin{aligned}
&= \frac{6}{(6x+7)} + \frac{42}{(6x+7)^2} \\
&= \frac{6\,(6x+7)+42}{(6x+7)^2} \\
&= \frac{36x+84}{(6x+7)^2} \\
&= \frac{12\,(3x+7)}{(6x+7)^2}
\end{aligned}
$$

c

$$
\begin{aligned}
y &= \log_e\left(x+\sqrt{x^2+16}\right) \\
u &= x+\sqrt{x^2+16} \\
y &= \log_e(u) \\
\frac{dy}{du} &= \frac{1}{u} \\
\frac{du}{dx} &= 1+\frac{x}{\sqrt{x^2+16}} \\
&= \frac{\sqrt{x^2+16}+x}{\sqrt{x^2+16}} \\
&= \frac{u}{\sqrt{x^2+16}} \\
\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} \\
&= \frac{1}{u}\times\frac{u}{\sqrt{x^2+16}} \\
&= \frac{1}{\sqrt{x^2+16}} \\
&= \left(x^2+16\right)^{-\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d^2y}{dx^2} &= 2x\times-\frac{1}{2}\times\left(x^2+16\right)^{-\frac{3}{2}} \\
&= -\frac{x}{\sqrt{\left(x^2+16\right)^3}}
\end{aligned}
$$

11

$$
\begin{aligned}
y^2 &= a+bx^2 \\
2y\frac{dy}{dx} &= 2bx \\
2y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2 &= 2b \\
y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2 &= b
\end{aligned}
$$

12

$$
\begin{aligned}
y^3 &= x\left(a+bx^3\right) \\
y^3 &= ax+bx^4 \\
3y^2\frac{dy}{dx} &= a+4bx^3 \\
3y^2\frac{d^2y}{dx^2}+6y\left(\frac{dy}{dx}\right)^2 &= 12bx^2 \\
y^2\frac{d^2y}{dx^2}+2y\left(\frac{dy}{dx}\right)^2 &= 4bx^2
\end{aligned}
$$

13 a

$$
\begin{aligned}
y^2 &= 4ax \\
2y\frac{dy}{dx} &= 4a \\
\frac{dy}{dx} &= \frac{2a}{y} \\
y\frac{dy}{dx} &= 2a \text{ implicit again}
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2} &= 0 \\
y\frac{d^2y}{dx^2} &= -\left(\frac{dy}{dx}\right)^2 \\
y\frac{d^2y}{dx^2} &= -\frac{4a^2}{y^2} \\
\frac{d^2y}{dx^2} &= -\frac{4a^2}{y^3}
\end{aligned}
$$

b

$$
\begin{aligned}
&\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1 \\
&\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx} = 0 \\
&\frac{dy}{dx} = \frac{b^2x}{a^2y} \text{ implicit again} \\
&\frac{x}{a^2}-\frac{y}{b^2}\frac{dy}{dx} = 0 \\
&\frac{1}{a^2}-\left(\frac{1}{b^2}\left(\frac{dy}{dx}\right)^2+\frac{y}{b^2}\frac{d^2y}{dx^2}\right) = 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{y}{b^2}\frac{d^2y}{dx^2} &= \frac{1}{a^2}-\frac{1}{b^2}\left(\frac{dy}{dx}\right)^2 \\
&= \frac{1}{a^2}-\frac{1}{b^2}\left(\frac{b^2x}{a^2y}\right)^2 \\
&= \frac{1}{a^2}-\frac{b^2x^2}{a^4y^2} \\
&= \frac{a^2y^2-b^2x^2}{a^4y^2} \\
&= \frac{-a^2b^2}{a^4y^2}, \text{ since } b^2x^2-a^2y^2=a^2b^2
\end{aligned}
$$

So $\dfrac{d^2y}{dx^2} = -\dfrac{b^4}{a^2y^3}$

c

$$
\begin{aligned}
&xy = a^2 \\
&x\frac{dy}{dx}+y = 0 \\
&\frac{dy}{dx} = -\frac{y}{x} \text{ implicit again} \\
&\frac{dy}{dx}+x\frac{d^2y}{dx^2}+\frac{dy}{dx} = 0
\end{aligned}
$$

$$
\begin{aligned}
x\frac{d^2y}{dx^2} &= -2\frac{dy}{dx} \\
&= \frac{2y}{x}
\end{aligned}
$$

$$
\frac{d^2y}{dx^2} = \frac{2y}{x^2} = \frac{2a^2}{x^3}, \text{ since } y = \frac{a^2}{x}
$$

Or

$$
\begin{aligned}
y &= \frac{a^2}{x} = a^2x^{-1} \\
\frac{dy}{dx} &= -a^2x^{-2} \\
\frac{d^2y}{dx^2} &= 2a^2x^{-3} = \frac{2a^2}{x^3}
\end{aligned}
$$

14

$$
\begin{aligned}
x &= 4\cos(t) \\
y &= 4\sin(t) \\
\dot{x} &= \frac{dx}{dt} = -4\sin(t) \\
\dot{y} &= \frac{dy}{dt} = 4\cos(t)
\end{aligned}
$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt}\cdot\frac{dt}{dx}\\ &= -\frac{4\cos(t)}{4\sin(t)}\\ &= -\cot(t)\\ \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(-\cot(t)\right)\\ &= \frac{d}{dt}\left(-\frac{\cos(t)}{\sin(t)}\right)\frac{dt}{dx}\\ &= \frac{\sin^2(t)+\cos^2(t)}{\sin^2(t)}\times\left(-\frac{1}{4\sin(t)}\right)\\ &= -\frac{1}{4\sin^3(t)}\end{aligned}$$

15 $x = t^2$

$$\begin{aligned}y &= 2t - t^4\\ \dot{x} &= \frac{dx}{dt} = 2t\\ \dot{y} &= \frac{dy}{dt} = 2 - 4t^3\\ \frac{dy}{dx} &= \frac{dy}{dt}\cdot\frac{dt}{dx}\\ &= \frac{2-4t^3}{2t}\\ &= \frac{1}{t} - 2t^2\\ \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{1}{t}-2t^2\right)\frac{dt}{dx}\\ &= \left(-t^{-2}-4t\right)\times\frac{1}{2t}\\ &= -\left(\frac{1}{t^2}+4t\right)\times\frac{1}{2t} = \frac{-\left(1+4t^3\right)}{2t^3}\end{aligned}$$

When $t = 2$

$$\frac{d^2y}{dx^2} = \frac{-33}{16}$$

16 **a** $x = at^2$

$$\begin{aligned}y &= 2at\\ \frac{dx}{dt} &= \dot{x} = 2at\\ \frac{dy}{dt} &= \dot{y} = 2a\\ \frac{dy}{dx} &= \frac{\dot{y}}{\dot{x}} = \frac{2a}{2at} = \frac{1}{t} = t^{-1}\\ \frac{d^2y}{dx^2} &= \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}\\ &= \frac{d}{dt}\left(t^{-1}\right)\frac{1}{2at}\\ &= -t^{-2}\times\frac{1}{2at}\\ &= -\frac{1}{2at^3}\end{aligned}$$

b $x = at$

$$\begin{aligned}y &= \frac{a}{t} = at^{-1}\\ \frac{dx}{dt} &= \dot{x} = a\\ \frac{dy}{dt} &= \dot{y} = -at^{-2} = -\frac{a}{t^2}\\ \frac{dy}{dx} &= \frac{\dot{y}}{\dot{x}} = -\frac{a}{t^2}\times\frac{1}{a} = -t^{-2}\\ \frac{d^2y}{dx^2} &= \frac{d}{dt}\left(-t^{-2}\right)\times\frac{dt}{dx}\\ &= 2t^{-3}\times\frac{1}{a}\\ &= \frac{2}{at^3}\end{aligned}$$

c $x = a\cos(t)$

$$\begin{aligned}y &= b\sin(t)\\ \frac{dx}{dt} &= \dot{x} = -a\sin(t)\\ \frac{dy}{dt} &= \dot{y} = b\cos(t)\\ \frac{dy}{dx} &= \frac{\dot{y}}{\dot{x}} = -\frac{b\cos(t)}{a\sin(t)} = -\frac{b}{a}\cot(t)\\ \frac{d^2y}{dx^2} &= \frac{d}{dt}\left(-\frac{b}{a}\cot(t)\right)\frac{dt}{dx}\\ &= \frac{b}{a}\text{cosec}^2(t)\times -\frac{1}{a\sin(t)}\\ &= \frac{-b}{a^2\sin^3(t)}\end{aligned}$$

17 **a** $y^2 = a + bx$

$$\begin{aligned}2y\frac{dy}{dx} &= b\\ 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 &= 0\\ y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 &= 0\end{aligned}$$

b $y = x\sqrt{a+bx}$

$$\begin{aligned}y^2 &= x^2(a+bx)\\ &= ax^2 + bx^3\\ 2y\frac{dy}{dx} &= 2ax + 3bx^2\\ 2\left(y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right) &= 2a + 6bx\\ y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 &= a + 3bx\end{aligned}$$

18 **a** Let $y = (ax+b)^n$

$$\begin{aligned}\frac{dy}{dx} &= na(ax+b)^{n-1}\\ \frac{d^2y}{dx^2} &= na\times(a)(n-1)(ax+b)^{n-2}\end{aligned}$$

So $\dfrac{d^2}{dx^2}\left[(ax+b)^n\right] = a^2n(n-1)(ax+b)^{n-2}$

b Let $y = \log_e\left((ax+b)^n\right)$

$$\begin{aligned}&= n\log_e(ax+b)\\ \frac{dy}{dx} &= \frac{na}{ax+b} = na(ax+b)^{-1}\\ \frac{d^2y}{dx^2} &= -na^2(ax+b)^{-2}\end{aligned}$$

So

$$\frac{d^2}{dx^2}\left[\log_e(ax+b)^n\right] = -\frac{a^2n}{(ax+b)^2}$$

c

$$
\begin{aligned}
y &= \log_e\left(\left(ax^2+b\right)^n\right)\\
&= n\log_e\left(ax^2+b\right)\\
\frac{dy}{dx} &= \frac{n\times 2ax}{ax^2+b} = \frac{2anx}{ax^2+b}\\
\frac{d^2y}{dx^2} &= \frac{2an\left(ax^2+b\right)-2ax\times(2anx)}{\left(ax^2+b\right)^2}\\
&= \frac{2a^2nx^2+2anb-4a^2nx^2}{\left(ax^2+b\right)^2}
\end{aligned}
$$

So $\dfrac{d^2}{dx^2}\left[\log_e\left(ax^2+b\right)^n\right] = -\dfrac{2an\left(ax^2-b\right)}{\left(ax^2+b\right)^2}$

19 a $x^2+y^2=a^2$

$$
\begin{aligned}
2x+2y\frac{dy}{dx} &= 0\\
\frac{dy}{dx} &= -\frac{x}{y}
\end{aligned}
$$

Implicit again

$$
\begin{aligned}
2+2y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2 &= 0\\
y\frac{d^2y}{dx^2} &= -1-\left(\frac{dy}{dx}\right)^2\\
&= -1-\left(-\frac{x}{y}\right)^2\\
&= -1-\frac{x^2}{y^2}\\
y\frac{d^2y}{dx^2} &= -\frac{\left(x^2+y^2\right)}{y^2}\\
\frac{d^2y}{dx^2} &= -\frac{a^2}{y^3}
\end{aligned}
$$

b

$$
\begin{aligned}
\rho &= \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}\\
&= \frac{\left(1+\frac{x^2}{y^2}\right)^{\frac{3}{2}}}{-\frac{a^2}{y^3}}\\
&= \frac{\left(\frac{y^2}{y^2}+\frac{x^2}{y^2}\right)^{\frac{3}{2}}}{-\frac{a^2}{y^3}}\\
&= \left(\frac{x^2+y^2}{y^2}\right)^{\frac{3}{2}}\times-\frac{y^3}{a^2}\\
&= \left(\frac{a^2}{y^2}\right)^{\frac{3}{2}}\times-\frac{y^3}{a^2}\\
&= \frac{a^3}{y^3}\times-\frac{y^3}{a^2}\\
&= -a
\end{aligned}
$$

So $|\rho| = a$

20

$$
\begin{aligned}
x &= \cos\theta+\theta\sin(\theta)\\
\frac{dx}{d\theta} &= -\sin(\theta)+\sin(\theta)+\theta\cos(\theta)\\
&= \theta\cos(\theta)\\
y &= \sin(\theta)-\theta\cos(\theta)\\
\frac{dy}{d\theta} &= \cos(\theta)-(\cos(\theta)-\theta\sin(\theta))\\
&= \theta\sin(\theta)\\
\frac{dy}{dx} &= \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{\theta\sin(\theta)}{\theta\cos(\theta)}\\
&= \tan(\theta)\\
\frac{d^2y}{dx^2} &= \frac{d}{dx}(\tan(\theta))\\
&= \frac{d}{d\theta}(\tan(\theta))\frac{d\theta}{dx}\\
&= \frac{\sec^2(\theta)}{\theta\cos(\theta)}
\end{aligned}
$$

$$
\begin{aligned}
\rho &= \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}\\
&= \frac{\left(1+\tan^2(\theta)\right)^{\frac{3}{2}}}{\frac{\sec^2(\theta)}{\theta\cos(\theta)}}\\
&= \left(\sec^2(\theta)\right)^{\frac{3}{2}}\times\frac{\theta\cos(\theta)}{\sec^2(\theta)}\\
&= \frac{\sec^3(\theta)\times\theta\times\cos(\theta)}{\sec^2(\theta)}\\
&= \sec(\theta)\times\theta\times\cos(\theta)\\
&= \theta
\end{aligned}
$$

21 a

$$
\begin{aligned}
x &= a(\theta-\sin(\theta))\\
\frac{dx}{d\theta} &= a(1-\cos(\theta))\\
&= a\left(1-\left(1-2\sin^2\left(\frac{\theta}{2}\right)\right)\right)\\
&= 2a\sin^2\left(\frac{\theta}{2}\right)\\
y &= a(1-\cos(\theta))\\
&= a-a\cos(\theta)\\
\frac{dy}{d\theta} &= a\sin(\theta) = 2a\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\\
\frac{dy}{dx} &= \frac{dy}{d\theta}\frac{d\theta}{dx}\\
&= \frac{2a\ \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2a\sin^2\left(\frac{\theta}{2}\right)}\\
&= \cot\left(\frac{\theta}{2}\right)
\end{aligned}
$$

b

$$
\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\cot\left(\frac{\theta}{2}\right)\right)\\
&= \frac{d}{d\theta}\left(\cot\left(\frac{\theta}{2}\right)\right)\frac{d\theta}{dx}\\
&= -\frac{1}{2}\text{cosec}^2\left(\frac{\theta}{2}\right)\times\frac{1}{2a\sin^2\left(\frac{\theta}{2}\right)}\\
&= -\frac{1}{4a\sin^4\left(\frac{\theta}{2}\right)}
\end{aligned}
$$

c $\rho = \dfrac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$

$= \dfrac{\left(1+\cot^2\left(\frac{\theta}{2}\right)\right)^{\frac{3}{2}}}{-\frac{1}{4a\sin^4\left(\frac{\theta}{2}\right)}}$

$= \left(\text{cosec}^2\left(\dfrac{\theta}{2}\right)\right)^{\frac{3}{2}} \times -4a\sin^4\left(\dfrac{\theta}{2}\right)$

$= \text{cosec}^3\left(\dfrac{\theta}{2}\right) \times -4a\sin^4\left(\dfrac{\theta}{2}\right)$

$= \dfrac{1}{\sin^3\left(\frac{\theta}{2}\right)} \times -4a\sin^4\left(\dfrac{\theta}{2}\right)$

$= -4a\sin\left(\dfrac{\theta}{2}\right)$

22 **a** $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

$\dfrac{2}{3}x^{-\frac{1}{3}} + \dfrac{2}{3}y^{-\frac{1}{3}}\dfrac{dy}{dx} = 0$

$\dfrac{2}{3\sqrt[3]{x}} = -\dfrac{2}{3\sqrt[3]{y}}\dfrac{dy}{dx}$

$\dfrac{dy}{dx} = -\sqrt[3]{\dfrac{y}{x}}$

b (1) $x = a\cos^3(\theta)$

$\cos(\theta) = \sqrt[3]{\dfrac{x}{a}}$

(2) $y = a\sin^3(\theta)$

$\sin(\theta) = \sqrt[3]{\dfrac{y}{a}}$

$\cos^2(\theta) + \sin^2(\theta) = 1$

$\left(\dfrac{x}{a}\right)^{\frac{2}{3}} + \left(\dfrac{x}{a}\right)^{\frac{2}{3}} = 1$

$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

$x = a\cos^3(\theta)$

$\dfrac{dx}{d\theta} = -3a\cos^2(\theta)\sin(\theta)$

$y = a\sin^3(\theta)$

$\dfrac{dy}{d\theta} = 3a\sin^2(\theta)\cos(\theta)$

$\dfrac{dy}{dx} = \dfrac{dy}{d\theta}\dfrac{d\theta}{dx}$

$= \dfrac{3a\sin^2(\theta)\cos(\theta)}{-3a\cos^2(\theta)\sin(\theta)}$

$= -\tan(\theta)$

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$

$= \dfrac{d}{d\theta}\left(-\tan(\theta)\right)\dfrac{d\theta}{dx}$

$= \dfrac{-\sec^2(\theta)}{-3a\cos^2(\theta)\sin(\theta)}$

$= \dfrac{1}{3a\cos^4(\theta)\sin(\theta)}$

23 **a** $y = e^{-2x}\sin(3x)$

$\dfrac{dy}{dx} = e^{-2x}\dfrac{d}{dx}(\sin(3x)) + \sin(3x)\dfrac{d}{dx}\left(e^{-2x}\right)$

$= 3e^{-2x}\cos(3x) - 2e^{-2x}\sin(3x)$

$= e^{-2x}(3\cos(3x) - 2\sin(3x))$

$\dfrac{d^2y}{dx^2} = e^{-2x}\dfrac{d}{dx}(3\cos(3x) - 2\sin(3x))$

$+ (3\cos(3x) - 2\sin(3x))\dfrac{d}{dx}\left(e^{-2x}\right)$

$= e^{-2x}(-9\sin(3x) - 6\cos(3x))$

$+ (3\cos(3x) - 2\sin(3x))(-2e^{-2x})$

$= -e^{-2x}(5\sin(3x) + 12\cos(3x))$

b $y = u \cdot V$

$\dfrac{dy}{dx} = u\dfrac{dV}{dx} + V\dfrac{du}{dx}$

$\dfrac{d^2y}{dx^2} = u\dfrac{d^2V}{dx^2} + \dfrac{du}{dx}\cdot\dfrac{dV}{dx} + \dfrac{dV}{dx}\cdot\dfrac{du}{dx} + V\dfrac{d^2u}{dx^2}$

$= u\dfrac{d^2V}{dx^2} + 2\dfrac{du}{dx}\cdot\dfrac{dV}{dx} + V\dfrac{d^2u}{dx^2}$

$y = e^{-2x}\sin(3x)$

$u = e^{-2x}$

$V = \sin(3x)$

$\dfrac{du}{dx} = -2e^{-2x}$

$\dfrac{dV}{dx} = 3\cos(3x)$

$\dfrac{d^2u}{dx^2} = 4e^{-2x}$

$\dfrac{d^2V}{dx^2} = -9\sin(3x)$

$\dfrac{d^2y}{dx^2} = -9e^{-2x}\sin(3x) + 12e^{-2x}\cos(3x) + 4e^{-2x}\sin(3x)$

$= -e^{-2x}(5\sin(3x) + 12\cos(3x))$

24 **a** $x = f(t)$

$y = h(t)$

$\dot{x} = h'(t) = \dfrac{dx}{dt}$

$\dot{y} = h'(t) = \dfrac{dy}{dt}$

$\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \dfrac{\dot{y}}{\dot{x}}$

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{\dot{y}}{\dot{x}}\right)$

$= \dfrac{d}{dt}\left(\dfrac{\dot{y}}{\dot{x}}\right)\dfrac{dt}{dx}$

$= \dfrac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2} \times \dfrac{1}{\dot{x}}$

$= \dfrac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}$

b Given $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$

Let $u = \dfrac{dx}{dy}$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right)\\ &= \frac{d}{dx}\left(u^{-1}\right)\\ &= \frac{d}{du}\left(u^{-1}\right)\frac{du}{dx}\\ &= -u^{-2}\frac{d}{dx}\left(\frac{dx}{dy}\right)\\ &= -u^{-2}\frac{d}{dy}\left(\frac{dx}{dy}\right)\cdot\frac{dy}{dx}\\ &= -u^{-2}\frac{d^2x}{dy^2}\times u^{-1}\\ &= -u^{-3}\frac{d^2x}{dy^2}\end{aligned}$$

So $\frac{d^2y}{dx^2} = -\left(\frac{dy}{dx}\right)^3\frac{d^2x}{dy^2}$

25 **a** $y = f(x) = x^3$

$\frac{dy}{dx} = f'(x) = 3x^2$

$\frac{d^2y}{dx^2} = f''(x) = 6x$

$\frac{d^3y}{dx^3} = f'''(x) = 6 = 3 \times 2 \times 1 = 3!$

b $y = f(x) = x^4$

$\frac{dy}{dx} = f'(x) = 4x^3$

$\frac{d^2y}{dx^2} = f''(x) = 12x^2$

$\frac{d^3y}{dx^3} = f'''(x) = 24x$

$\frac{d^4y}{dx^4} = f^{(4)}(x) = 24 = 4 \times 3 \times 2 \times 1 = 4!$

26 To show $\frac{d^n}{dx^n}(x^n) = n!$

The first base step is to prove it is true for the base case, that is when $n = 1$.

$\frac{d}{dx}(x) = 1 = 1!$ so it is true when $n = 1$.

We now assume it is true when $n = k$, that is assume $\frac{d^k}{dx^k}\left(x^k\right) = k!$ for the kth derivative is true, we now consider the $(k+1)$th derivative, assuming also that $\frac{d}{dx}(x^n) = nx^{n-1}$

$\frac{d^{k+1}}{dx^{k+1}}\left(x^{k+1}\right) = \frac{d^k}{dx^k}\left(\frac{d}{dx}\left(x^{k+1}\right)\right)$ the $(k+1)$th derivative is the kth derivative of the first derivative

$\frac{d^{k+1}}{dx^{k+1}}\left(x^{k+1}\right) = \frac{d^k}{dx^k}\left((k+1)x^k\right)$ take the constant factor $(k+1)$ outside the derivative

$\frac{d^{k+1}}{dx^{k+1}}\left(x^{k+1}\right) = (k+1)\frac{d^k}{dx^k}\left(\left(x^k\right)\right) = (k+1)k!$ since we assumed that $\frac{d^k}{dx^k}\left(x^k\right) = k!$

$\frac{d^{k+1}}{dx^{k+1}}\left(x^{k+1}\right) = (k+1)!$

Which is what we want to show, this completes the proof by induction, so it is true for all $n \in N$.

27 To show $\frac{d^n}{dx^n}(xe^x) = (x+n)e^x$, the first base step is to prove it is true for the base case, that is when $n = 1$, using the product rule $\frac{d}{dx}(xe^x) = x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x) = xe^x + e^x = (x+1)e^x$ so it is true when $n = 1$.

We now assume it is true when $n = k$, that is assume $\frac{d^k}{dx^k}(xe^x) = (x+k)e^x$ for the kth derivative is true, we now consider the $(k+1)$th derivative, that is

$\frac{d^{k+1}}{dx^{k+1}}(xe^x) = \frac{d}{dx}\left(\frac{d^k}{dx^k}\left(xe^k\right)\right)$ using the assumption

$\frac{d^{k+1}}{dx^{k+1}}(xe^x) = \frac{d}{dx}((x+k)e^x)$ using the product rule

$\frac{d^{k+1}}{dx^{k+1}}(xe^x) = (x+k)\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x+k)$

$\frac{d^{k+1}}{dx^{k+1}}(xe^x) = (x+k)e^x + e^x$

$\frac{d^{k+1}}{dx^{k+1}}(xe^x) = (x+k+1)e^x$

Which is what we want to show, this completes the proof by induction, so it is true for all $n \in N$.

28 To show $\frac{d^n}{dx^n}(e^{mx}) = m^n e^{mx}$, the first base step is to prove it is true for the base case, that is when $n = 1$, $\frac{d}{dx}(e^{mx}) = me^{mx}$ so it is true when $n = 1$.

We now assume it is true when $n = k$, that is assume $\frac{d^k}{dx^k}(e^{mx}) = m^k e^{mx}$ for the kth derivative is true, we now consider the $(k+1)$th derivative, that is

$\frac{d^{k+1}}{dx^{k+1}}(e^{mx}) = \frac{d}{dx}\left(\frac{d^k}{dx^k}(e^{mx})\right)$ using the assumption

$\frac{d^{k+1}}{dx^{k+1}}(e^{mx}) = \frac{d}{dx}\left(m^k e^{mx}\right) = m^k\frac{d}{dx}(e^{mx})$ taking the constant outside the derivative

$\frac{d^{k+1}}{dx^{k+1}}(e^{mx}) = m^k(me^{mx})$

$\frac{d^{k+1}}{dx^{k+1}}(e^{mx}) = m^{k+1}e^{mx}$

Which is what we want to show, this completes the proof by induction, so it is true for all $n \in N$.

29 To show $\frac{d^n}{dx^n}\left(\log_e(x)\right) = \frac{(-1)^{n-1}(n-1)!}{x^n}$, the first base step is to prove it is true for the base case, that is when $n = 1$,

LHS $= \frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$

RHS $= \frac{(-1)^{1-1}(1-1)!}{x^1} = \frac{1}{x}$ since $0! = 1$ so it is true when $n = 1$.

We now assume it is true when $n = k$, that is assume $\frac{d^k}{dx^k}\left(\log_e(x)\right) = \frac{(-1)^{k-1}(k-1)!}{x^k}$ for the kth derivative is true, we now consider the $(k+1)$th derivative, that is

$\frac{d^{k+1}}{dx^{k+1}}\left(\log_e(x)\right) = \frac{d}{dx}\left(\frac{d^k}{dx^k}\left(\log_e(x)\right)\right)$ using the assumption

$\frac{d^{k+1}}{dx^{k+1}}\left(\log_e(x)\right) = \frac{d}{dx}\left(\frac{(-1)^{k-1}(k-1)!}{x^k}\right)$

$\frac{d^{k+1}}{dx^{k+1}}\left(\log_e(x)\right) = (-1)^{k-1}(k-1)!\frac{d}{dx}\left(x^{-k}\right)$ taking the constant factor outside the derivative

$\frac{d^{k+1}}{dx^{k+1}}\left(\log_e(x)\right) = (-1)^{k-1}(k-1)!\left(-kx^{-k-1}\right)$ simplifying

$\frac{d^{k+1}}{dx^{k+1}}\left(\log_e(x)\right) = (-1) \times (-1)^{k-1}k(k-1)!\left(\frac{1}{x^{k+1}}\right)$

$\frac{d^k}{dx^k}\left(\log_e(x)\right) = \frac{(-1)^k k!}{x^{k+1}}$

Which is what we want to show, this completes the proof by induction, so it is true for all $n \in N$.

30 **a** To show $\frac{d^n}{dx^n}(\sin(mx)) = m^n \sin\left(mx + \frac{n\pi}{2}\right)$, the first base step is to prove it is true for the base case, that is when $n = 1$, LHS $= \frac{d}{dx}(\sin(mx)) = m\cos(mx)$

RHS $= m\sin\left(mx + \frac{\pi}{2}\right) =$

$m\left(\sin(mx)\cos\left(\frac{\pi}{2}\right) + \cos(x)\sin\left(\frac{\pi}{2}\right)\right) = m\cos(x)$ since $\sin\left(\frac{\pi}{2}\right) = 1$ and $\cos\left(\frac{\pi}{2}\right) = 0$ so it is true when $n = 1$.

We now assume it is true when $n = k$, that is assume

$\frac{d^k}{dx^k}(\sin(mx)) = m^k \sin\left(mx + \frac{k\pi}{2}\right)$

for the kth derivative is true, we now consider the $(k+1)$th derivative, that is

$\frac{d^{k+1}}{dx^{k+1}}(\sin(mx)) = \frac{d}{dx}\left(\frac{d^k}{dx^k}(\sin(mx))\right)$ using the assumption

$\frac{d^{k+1}}{dx^{k+1}}(\sin(mx)) = \frac{d}{dx}\left(m^k \sin\left(mx + \frac{k\pi}{2}\right)\right)$

$\frac{d^{k+1}}{dx^{k+1}}(\sin(mx)) = m^k \frac{d}{dx}\left(\sin\left(mx + \frac{k\pi}{2}\right)\right)$ taking the constant factor outside the derivative

$\frac{d^{k+1}}{dx^{k+1}}(\sin(mx)) = m^k\left(m\cos\left(mx + \frac{k\pi}{2}\right)\right) =$

$m^{k+1}\cos\left(mx + \frac{k\pi}{2}\right)$ since $\cos(\theta) = \sin\left(\theta + \frac{\pi}{2}\right)$

$\frac{d^{k+1}}{dx^{k+1}}(\sin(mx)) = m^{k+1}\sin\left(mx + \frac{k\pi}{2} + \frac{\pi}{2}\right)$

$\frac{d^{k+1}}{dx^{k+1}}(\sin(mx)) = m^{k+1}\sin\left(mx + \frac{(k+1)\pi}{2}\right)$

Which is what we want to show, this completes the proof by induction, so it is true for all $n \in N$.

b To show $\frac{d^n}{dx^n}(\cos(mx)) = m^n \cos\left(mx + \frac{n\pi}{2}\right)$, the first base step is to prove it is true for the base case, that is when $n = 1$, LHS $= \frac{d}{dx}(\cos(mx)) = -m\sin(mx)$

RHS $= m\cos\left(mx + \frac{\pi}{2}\right) =$

$m\left(\cos(mx)\cos\left(\frac{\pi}{2}\right) - \sin(x)\sin\left(\frac{\pi}{2}\right)\right) = -m\sin(x)$

since $\sin\left(\frac{\pi}{2}\right) = 1$ and $\cos\left(\frac{\pi}{2}\right) = 0$ so it is true when $n = 1$.

We now assume it is true when $n = k$, that is assume

$\frac{d^k}{dx^k}(\cos(mx)) = m^k \cos\left(mx + \frac{k\pi}{2}\right)$

for the kth derivative is true, we now consider the $(k+1)$th derivative, that is

$\frac{d^{k+1}}{dx^{k+1}}(\cos(mx)) = \frac{d}{dx}\left(\frac{d^k}{dx^k}(\cos(mx))\right)$ using the assumption

$\frac{d^{k+1}}{dx^{k+1}}(\cos(mx)) = \frac{d}{dx}\left(m^k \cos\left(mx + \frac{k\pi}{2}\right)\right)$

$\frac{d^{k+1}}{dx^{k+1}}(\cos(mx)) = m^k \frac{d}{dx}\left(\cos\left(mx + \frac{k\pi}{2}\right)\right)$ taking the constant factor outside the derivative

$\frac{d^{k+1}}{dx^{k+1}}(\cos(mx)) = m^k\left(-m\sin\left(mx + \frac{k\pi}{2}\right)\right) =$

$-m^{k+1}\sin\left(mx + \frac{k\pi}{2}\right)$

Now since $\sin(\theta) = -\cos\left(\theta + \frac{\pi}{2}\right)$

$\frac{d^{k+1}}{dx^{k+1}}(\cos(mx)) = m^{k+1}\cos\left(mx + \frac{k\pi}{2} + \frac{\pi}{2}\right)$

$\frac{d^{k+1}}{dx^{k+1}}(\cos(mx)) = m^{k+1}\cos\left(mx + \frac{(k+1)\pi}{2}\right)$

Which is what we want to show, this completes the proof by induction, so it is true for all $n \in N$.

5.5 Exam questions

1 $\frac{dy}{dx} = e^x \tan^{-1}(y)$ at $(0, 1)$ When $x = 0$, $y = 1$,

$\frac{dy}{dx} = \tan^{-1}(1) = \frac{\pi}{4}$, using implicit differentiation and the product rule:

$\frac{d^2y}{dx^2} = e^x \tan^{-1}(y) + \frac{e^x}{1+y^2}\frac{dy}{dx}$

When $x = 0$, $y = 1$,

$$\begin{aligned}\frac{d^2y}{dx^2} &= \tan^{-1}(1) + \frac{1}{2}\frac{dy}{dx}\\ &= \frac{\pi}{4} + \frac{\pi}{8}\\ &= \frac{3\pi}{8}\end{aligned}$$

The correct answer is **B**.

2 To show $\frac{d}{dx}(x^n) = nx^{n-1}$,

the first base step is to prove it is true for the base case, that is when $n = 1$,

$\frac{d}{dx}\left(x^1\right) = 1 = 1x^0 = 1$ so it is true when $n = 1$.

We now assume it is true when $n = k$, that is assume $\frac{d}{dx}\left(x^k\right) = kx^{k-1}$ the kth derivative is true, we now consider the derivative of x^{k+1}

$\frac{d}{dx}\left(x^{k+1}\right) = \frac{d}{dx}\left(x^k \times x\right)$ using the product rule

$\frac{d}{dx}\left(x^{k+1}\right) = x\frac{d}{dx}\left(x^k\right) + x^k\frac{d}{dx}(x)$

$\frac{d}{dx}\left(x^{k+1}\right) = x \times kx^{k-1} + x^k \times 1 = kx^k + x^k$

$\frac{d}{dx}\left(x^{k+1}\right) = (k+1)x^k$

Which is what we want to show, this completes the proof by induction, so it is true for all $n \in N$.

Award 1 mark for base case when $n = 1$.

Award 1 mark for assumption case.

Award 1 mark for simplifying using the product rule.

Award 1 mark for the final conclusion.

3 Let

$$\begin{aligned}y &= \left(ax^2+b\right)^n\\ \frac{dy}{dx} &= n \times 2ax\left(ax^2+b\right)^{n-1}\\ &= 2an\left[x \times \left(ax^2+b\right)^{n-1}\right] && \text{[1 mark]}\\ \frac{d^2y}{dx^2} &= 2an\left[\left(ax^2+b\right)^{n-1} + x\frac{d}{dx}\left(ax^2+b\right)^{n-1}\right] && \text{[1 mark]}\\ &= 2an\left[\left(ax^2+b\right)^{n-1} + 2ax^2(n-1)\left(ax^2+b\right)^{n-2}\right]\\ &= 2an\left(ax^2+b\right)^{n-2}\left[\left(ax^2+b\right) + 2ax^2(n-1)\right] && \text{[1 mark]}\\ &= 2an\left(ax^2+b\right)^{n-2}\left(ax^2(1+2(n-1)) + b\right)\\ &= 2an\left(ax^2+b\right)^{n-2}\left(a(2n-1)x^2 + b\right)\end{aligned}$$

$$\frac{d^2}{dx^2}\left[\left(ax^2+b\right)^n\right] = 2an\left(ax^2+b\right)^{n-2}\left(a(2n-1)x^2+b\right)$$ [1 mark]

5.6 Derivatives of inverse trigonometric functions

5.6 Exercise

1 a $y = \sin^{-1}\left(\frac{x}{5}\right)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{25-x^2}}$$

$x \in (-5, 5)$

b

$$\begin{aligned} y &= \sin^{-1}(5x) \\ u &= 5x \\ y &= \sin^{-1}(u) \\ \frac{dy}{du} &= \frac{1}{\sqrt{1-u^2}} \\ \frac{du}{dx} &= 5 \\ \frac{dy}{dx} &= \frac{5}{\sqrt{1-25x^2}} \end{aligned}$$

$x \in \left(-\frac{1}{5}, \frac{1}{5}\right)$

2 $f(x) = \sin^{-1}(4x)$

$$\begin{aligned} u &= 4x \\ y &= \sin^{-1}(u) \\ \frac{dy}{du} &= \frac{1}{\sqrt{1-u^2}} \\ \frac{du}{dx} &= 4 \\ \frac{dy}{dx} &= \frac{4}{\sqrt{1-16x^2}} \end{aligned}$$

$|x| < \frac{1}{4}$

$$\begin{aligned} f'\left(\frac{1}{8}\right) &= \frac{4}{\sqrt{1-16\times\frac{1}{64}}} \\ &= \frac{4}{\sqrt{1-\frac{1}{4}}} \\ &= \frac{4}{\sqrt{\frac{3}{4}}} \\ &= \frac{8}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\ &= \frac{8\sqrt{3}}{3} \end{aligned}$$

3 a $y = \sin^{-1}\left(\frac{x}{3}\right)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{9-x^2}}$$

$|x| < 3$

b

$$\begin{aligned} y &= \sin^{-1}(3x) \\ u &= 3x \\ y &= \sin^{-1}(u) \\ \frac{dy}{du} &= \frac{1}{\sqrt{9-u^2}} \\ \frac{du}{dx} &= 3 \\ \frac{dy}{dx} &= \frac{3}{\sqrt{1-9x^2}} \end{aligned}$$

$|x| < \frac{1}{3}$

4 a

$$\begin{aligned} y &= \sin^{-1}\left(\frac{4x}{3}\right) \\ u &= 4x \\ y &= \sin^{-1}\left(\frac{u}{3}\right) \\ \frac{dy}{du} &= \frac{1}{\sqrt{9-u^2}} \\ \frac{du}{dx} &= 4 \\ \frac{dy}{dx} &= \frac{4}{\sqrt{9-16x^2}} \end{aligned}$$

$|x| < \frac{3}{4}$

b $y = \sin^{-1}\left(\frac{4x+3}{5}\right)$

$$\begin{aligned} u &= 4x+3 \\ y &= \sin^{-1}\left(\frac{u}{5}\right) \\ \frac{dy}{du} &= \frac{1}{\sqrt{25-u^2}} \\ \frac{du}{dx} &= 4 \\ \frac{dy}{dx} &= \frac{4}{\sqrt{25-(4x+3)^2}} \\ &= \frac{4}{\sqrt{(5-(4x+3))(5+(4x+3))}} \\ &= \frac{4}{\sqrt{(8+4x)(2-4x)}} \\ &= \frac{\sqrt{2}}{\sqrt{(1-2x)(x+2)}} \end{aligned}$$

$$\begin{aligned} \left|\frac{4x+3}{5}\right| &< 1 \\ -5 < 4x+3 &< 5 \\ -8 < 4x &< 2 \\ -2 < x &< \frac{1}{2} \end{aligned}$$

5 a

$$\begin{aligned} y &= \cos^{-1}\left(\frac{3x}{4}\right) \\ u &= 3x \\ y &= \cos^{-1}\left(\frac{u}{4}\right) \\ \frac{dy}{du} &= -\frac{1}{\sqrt{16-u^2}} \qquad \left|\frac{3x}{4}\right| < 1 \\ \frac{du}{dx} &= 3 \qquad |x| < \frac{4}{3} \\ \frac{dy}{dx} &= -\frac{3}{\sqrt{16-u^2}} \\ &= -\frac{3}{\sqrt{16-9x^2}} \qquad \text{for } x \in \left(-\frac{4}{3}, \frac{4}{3}\right) \end{aligned}$$

b $y = \cos^{-1}\left(\dfrac{2x-3}{5}\right)$

$u = 2x - 3$

$y = \cos^{-1}\left(\dfrac{u}{5}\right)$

$$\frac{dy}{du} = -\frac{1}{\sqrt{25-u^2}}$$

$$\frac{du}{dx} = 2$$

$$\frac{dy}{dx} = -\frac{2}{\sqrt{25-(2x-3)^2}}$$

$$= \frac{2}{\sqrt{(5+(2x-3))(5-(2x-3)}}$$

$$= -\frac{2}{\sqrt{(2x+2)(8-2x)}}$$

$$= -\frac{1}{\sqrt{(x+1)(4-x)}} \text{ for } x \in (-1, 4)$$

$$\left|\frac{2x-3}{5}\right| < 1$$

$-5 < 2x - 3 < 5$

$-2 < 2x < 8$

$-1 < x < 4$

6 $y = \cos^{-1}\left(\dfrac{3x-4}{5}\right)$

$u = 3x - 4$

$y = \cos^{-1}\left(\dfrac{u}{5}\right)$

$$\frac{dy}{du} = -\frac{1}{\sqrt{25-u^2}}$$

$$\frac{du}{dx} = 3$$

$$\frac{dy}{dx} = -\frac{3}{\sqrt{25-(3x-4)^2}}$$

When $x = \dfrac{4}{3}$

$$\frac{dy}{dx} = -\frac{3}{5}$$

7 a $y = \cos^{-1}\left(\dfrac{x}{4}\right)$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{16-x^2}} \quad |x| < 4$$

b $y = \cos^{-1}(4x)$

$u = 4x$

$y = \cos^{-1}(u)$

$$\frac{dy}{du} = -\frac{1}{\sqrt{1-u^2}} \quad |x| < \frac{1}{4}$$

$$\frac{du}{dx} = 4$$

$$\frac{dy}{dx} = -\frac{4}{\sqrt{1-16x^2}}$$

8 a $y = \cos^{-1}\left(\dfrac{3x}{4}\right)$

$u = 3x$

$y = \cos^{-1}\left(\dfrac{u}{4}\right)$

$$\frac{dy}{du} = -\frac{1}{\sqrt{16-u^2}}$$

$$\frac{du}{dx} = 3$$

$$\frac{dy}{dx} = -\frac{3}{\sqrt{16-9x^2}} \qquad |x| < \frac{4}{3}$$

b $y = \cos^{-1}\left(\dfrac{3x+5}{7}\right)$

$u = 3x + 5$

$y = \cos^{-1}\left(\dfrac{u}{7}\right)$

$$\frac{dy}{du} = -\frac{1}{\sqrt{49-u^2}}$$

$$\frac{du}{dx} = 3$$

$$\frac{dy}{dx} = -\frac{3}{\sqrt{49-(3x+5)^2}}$$

$$= -\frac{3}{\sqrt{(7-(3x+5)(7+(3x+5))}}$$

$$= -\frac{3}{\sqrt{(2-3x)(12+3x)}}$$

$$= -\frac{\sqrt{3}}{\sqrt{(2-3x)(x+4)}}$$

For $\left|\dfrac{3x+5}{7}\right| < 1$

$-7 < 3x + 5 < 7$

$-12 < 3x < 2$

$-4 < x < \dfrac{2}{3}$

9 a $y = \tan^{-1}(4x)$

$u = 4x$

$y = \tan^{-1}(u)$

$$\frac{dy}{du} = \frac{1}{1+u^2}$$

$$\frac{du}{dx} = 4$$

$$\frac{dy}{dx} = \frac{4}{1+16x^2}$$

b $y = \tan^{-1}\left(\dfrac{2x-3}{5}\right)$

$u = 2x - 3$

$y = \tan^{-1}\left(\dfrac{u}{5}\right)$

$$\frac{dy}{du} = \frac{5}{25+u^2}$$

$$\frac{du}{dx} = 2$$

$$\frac{dy}{dx} = \frac{10}{25+u^2}$$

$$= \frac{10}{25+(2x-3)^2}$$

$$= \frac{10}{25+4x^2-12x+9}$$

$$= \frac{10}{4x^2-12x+34}$$

$$= \frac{5}{2x^2-6x+17}$$

10 $y = \tan^{-1}\left(\dfrac{3x-5}{7}\right)$

$u = 3x - 5$

$y = \tan^{-1}\left(\dfrac{u}{7}\right)$

$$\frac{dy}{du} = \frac{7}{49+u^2}$$

$$\frac{du}{dx} = 3$$

$$\frac{dy}{dx} = \frac{21}{49+u^2}$$
$$= \frac{21}{49+(3x-5)^2}$$
$$\left.\frac{dy}{dx}\right|_{x=4} = \frac{21}{49+49} = \frac{3}{14}$$

11 **a** $y = \tan^{-1}\left(\frac{x}{6}\right)$

$$\frac{dy}{dx} = \frac{6}{x^2+36}$$

b $y = \tan^{-1}(6x)$

$u = 6x$

$y = \tan^{-1}(u)$

$$\frac{dy}{du} = \frac{1}{1+u^2}$$
$$\frac{du}{dx} = 6$$
$$\frac{dy}{dx} = \frac{6}{1+36x^2}$$

12 **a** $y = \tan^{-1}\left(\frac{5x}{6}\right)$

$u = 5x$

$y = \tan^{-1}\left(\frac{u}{6}\right)$

$$\frac{dy}{du} = \frac{6}{36+u^2}$$
$$\frac{du}{dx} = 5$$
$$\frac{dy}{dx} = \frac{30}{36+25x^2}$$

b $y = \tan^{-1}\left(\frac{6x+5}{4}\right)$

$u = 6x+5$

$y = \tan^{-1}\left(\frac{u}{4}\right)$

$$\frac{dy}{du} = \frac{4}{16+u^2}$$
$$\frac{du}{dx} = 6$$
$$\frac{dy}{dx} = \frac{24}{16+(6x+5)^2}$$
$$= \frac{24}{36x^2+60x+41}$$

13 $y = \cos^{-1}\left(\frac{2x}{3}\right)$

$u = 2x$

$y = \cos^{-1}\left(\frac{u}{3}\right)$

$$\frac{dy}{du} = -\frac{1}{\sqrt{9-u^2}}$$
$$\frac{du}{dx} = 2$$
$$\frac{dy}{dx} = -\frac{2}{\sqrt{9-4x^2}} = -2\left(9-4x^2\right)^{-\frac{1}{2}}$$
$$\frac{d^2y}{dx^2} = -2 \times \left(\frac{-1}{2}\right) \times (-8x) \times \left(9-4x^2\right)^{-\frac{3}{2}}$$
$$= -\frac{8x}{\sqrt{\left(9-4x^2\right)^3}} \qquad |x| < \frac{3}{2}$$

14 $y = \tan^{-1}\left(\frac{4x}{3}\right)$

$u = 4x$

$y = \tan^{-1}\left(\frac{u}{3}\right)$

$$\frac{dy}{du} = \frac{3}{9+u^2}$$
$$\frac{du}{dx} = 4$$
$$\frac{dy}{dx} = \frac{12}{9+16x^2}$$
$$= 12\left(9+16x^2\right)^{-1}$$
$$\frac{d^2y}{dx^2} = -12 \times 32x \times \left(9+16x^2\right)^{-2}$$
$$= -\frac{384x}{\left(9+16x^2\right)^2}$$

15 $y = \sin^{-1}\left(\frac{2}{x}\right)$

$u = \frac{2}{x} = 2x^{-1}$

$y = \sin^{-1}(u)$

$$\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$$
$$\frac{du}{dx} = -2x^{-2} = -\frac{2}{x^2}$$
$$\frac{dy}{dx} = -\frac{2}{x^2} \times \frac{1}{\sqrt{1-\frac{4}{x^2}}}$$
$$= -\frac{2}{x^2} \times \frac{1}{\sqrt{\frac{x^2-4}{x^2}}}$$
$$= -\frac{2}{x^2} \times \frac{|x|}{\sqrt{x^2-4}}$$
$$-\frac{2}{|x|\sqrt{x^2-4}}$$

For $x^2 > 4$

$|x| > 2$

16 $y = \tan^{-1}\left(\frac{4}{\sqrt{x}}\right)$

$u = \frac{4}{\sqrt{x}} = 4x^{-\frac{1}{2}}$

$y = \tan^{-1}(u)$

$$\frac{dy}{du} = \frac{1}{1+u^2}$$
$$\frac{du}{dx} = -2x^{-\frac{3}{2}} = -\frac{2}{x^{\frac{3}{2}}}$$

$$\frac{dy}{dx} = -\frac{2}{x^{\frac{3}{2}}} \times \frac{1}{1+\frac{16}{x}}$$

$$= -\frac{2}{x^{\frac{3}{2}}}\left(\frac{1}{\frac{16+x}{x}}\right)$$

$$= -\frac{2x}{x^{\frac{3}{2}}} \times \frac{1}{16+x}$$

$$= -\frac{2}{\sqrt{x}\,(16+x)}$$

For $x > 0$

17 a $y = \sin^{-1}\left(\frac{\sqrt{x}}{3}\right)$

$$u = \sqrt{x} = x^{\frac{1}{2}}$$

$$y = \sin^{-1}\left(\frac{u}{3}\right)$$

$$\frac{dy}{du} = \frac{1}{\sqrt{9-u^2}}$$

$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \qquad 9x - x^2 > 0$$

$$\text{for } x(9-x) > 0$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \times \frac{1}{\sqrt{9-x}} \qquad 0 < x < 9$$

$$= \frac{1}{2\sqrt{x(9-x)}}$$

b $y = \sin^{-1}\left(\frac{3}{4x}\right)$

$$u = \frac{3}{4x} = \frac{3}{4}x^{-1}$$

$$y = \sin^{-1}(\text{u})$$

$$\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$$

$$\frac{du}{dx} = -\frac{3}{4}x^{-2} = -\frac{3}{4x^2}$$

$$\frac{dy}{dx} = -\frac{3}{4x^2} \times \frac{1}{\sqrt{1-\frac{9}{16x^2}}}$$

$$= -\frac{3}{4x^2}\,\frac{4\,|x|}{\sqrt{16x^2-9}}$$

$$= -\frac{3}{|x|\sqrt{(4x+3)(4x-3)}} \qquad 16x^2 > 9, \quad |x| > \frac{3}{4}$$

c $y = \cos^{-1}\left(\frac{e^{2x}}{4}\right)$

$$u = e^{2x}$$

$$y = \cos^{-1}\left(\frac{u}{4}\right)$$

$$\frac{dy}{du} = -\frac{1}{\sqrt{16-u^2}} \qquad e^{4x} < 16$$

$$e^{2x} < 4$$

$$\frac{du}{dx} = 2e^{2x} \qquad 2x < \log_e(4)$$

$$x < \log_e(2)$$

$$\frac{dy}{dx} = \frac{2e^{2x}}{\sqrt{16-e^{4x}}}$$

18 a $y = \cos^{-1}\left(\frac{4}{3x}\right)$

$$u = \frac{4}{3x} = \frac{4}{3}x^{-1}$$

$$y = \cos^{-1}(u)$$

$$\frac{dy}{du} = -\frac{1}{\sqrt{1-u^2}}$$

$$\frac{du}{dx} = -\frac{4}{3}x^{-2} = -\frac{4}{3x^2}$$

$$\frac{dy}{dx} = \frac{4}{3x^2} \times \frac{1}{\sqrt{1-\left(\frac{16}{9x^2}\right)}}$$

$$= \frac{4}{3x^2}\,\frac{1}{\sqrt{\frac{9x^2-16}{9x^2}}}$$

$$= \frac{4}{3x^2}\,\frac{3\,|x|}{\sqrt{9x^2-16}}$$

$$= \frac{4}{|x|\sqrt{9x^2-16}} \qquad 9x^2 > 16, \quad |x| > \frac{4}{3}$$

b $y = \tan^{-1}\left(\frac{x^2}{3}\right)$

$$u = x^2$$

$$y = \tan^{-1}\left(\frac{u}{3}\right)$$

$$\frac{dy}{du} = \frac{3}{9+u^2} \quad x \in R$$

$$\frac{du}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{6x}{x^4+9}$$

c $y = \tan^{-1}\left(\frac{6}{5x}\right)$

$$u = \frac{6}{5x} = \frac{6}{5}x^{-1}$$

$$y = \tan^{-1}(u)$$

$$\frac{dy}{du} = \frac{1}{1+u^2}$$

$$\frac{du}{dx} = -\frac{6}{5}x^{-2} = -\frac{6}{5x^2}$$

$$\frac{dy}{dx} = -\frac{6}{5x^2} \times \frac{1}{\frac{25x^2+36}{25x^2}}$$

$$= -\frac{6}{5x^2} \times \frac{25x^2}{25x^2+36}$$

$$= -\frac{30}{25x^2+36} \qquad x \in R\backslash\{0\}$$

19 a $y = \sin^{-1}\left(\frac{5x}{4}\right)$

$$u = 5x$$

$$y = \sin^{-1}\left(\frac{u}{4}\right)$$

$$\frac{dy}{du} = \frac{1}{\sqrt{16-u^2}}$$

$$\frac{du}{dx} = 5$$

$$\frac{dy}{dx} = \frac{5}{\sqrt{16-25x^2}} = 5\left(16-25x^2\right)^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = 5 \times -\frac{1}{2} \times -50x \times \left(16-25x^2\right)^{-\frac{3}{2}}$$

$$= \frac{125x}{\sqrt{\left(16-25x^2\right)^3}} \qquad |x| < \frac{4}{5}$$

b $y = \cos^{-1}\left(\dfrac{6x}{5}\right)$

$u = 6x$

$y = \cos^{-1}\left(\dfrac{u}{5}\right)$

$$\frac{dy}{du} = -\frac{1}{\sqrt{25 - u^2}}$$

$$\frac{du}{dx} = -\frac{6}{\sqrt{25 - 36x^2}} = -6\left(25 - 36x^2\right)^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = -6 \times -\frac{1}{2} \times -72x \times \left(25 - 36x^2\right)^{-\frac{3}{2}}$$

$$= -\frac{216x}{\sqrt{\left(25 - 36x^2\right)^3}}$$

c $y = \tan^{-1}\left(\dfrac{7x}{6}\right)$

$u = 7x$

$y = \tan^{-1}\left(\dfrac{u}{6}\right)$

$$\frac{dy}{du} = \frac{6}{36 + u^2}$$

$$\frac{du}{dx} = 7$$

$$\frac{dy}{dx} = \frac{42}{36 + 49x^2} = 42\left(36 + 49x^2\right)^{-1}$$

$$\frac{d^2y}{dx^2} = 42 \times 98x \times -1 \times \left(36 + 49x^2\right)^{-2}$$

$$= -\frac{4116x}{\left(36 + 49x^2\right)^2}$$

5.6 Exam questions

1 $f(x) = \arctan(3x - 6) + \pi$

$f(x) = y = \tan^{-1}(u) + \pi, u = 3x - 6$

$$\frac{dy}{du} = \frac{1}{1 + u^2}, \frac{du}{dx} = 3$$

$$f'(x) = \frac{3}{1 + (3x - 6)^2}$$

$$f'(x) = \frac{3}{9x^2 - 36x + 37}$$

Award 1 mark for the correct use of the chain rule.

2 Using the product rule

$$\frac{d}{dx}\left(x \arccos\left(\frac{x}{a}\right)\right), a > 0$$

$$= \frac{d}{dx}(x) \arccos\left(\frac{x}{a}\right) + x\frac{d}{dx}\left(\arccos\left(\frac{x}{a}\right)\right)$$

$$= \arccos\left(\frac{x}{a}\right) - \frac{x}{\sqrt{a^2 - x^2}}$$

Award 1 mark for the correct use of the product rule.

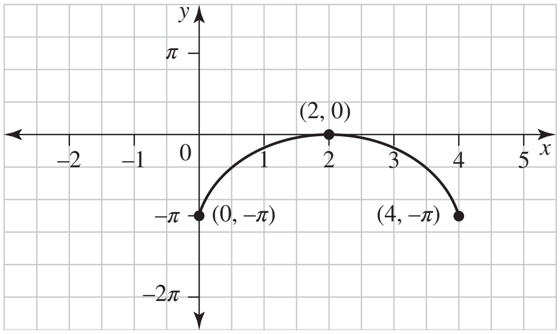
3 $y = (2 - x)\sin^{-1}\left(\dfrac{x}{2} - 1\right)$

Domain $\left|\dfrac{x}{2} - 1\right| \le 1 \Rightarrow -1 \le \dfrac{x}{2} - 1 \le 1$

$0 \le \dfrac{x}{2} \le 2 \Rightarrow x \in [0, 4]$

$f(0) = -\pi, f(4) = -\pi, f(2) = 0$

Range $[-\pi, 0]$



The correct answer is **A**.

5.7 Related rates

5.7 Exercise

1 Radius r (m), area $A\left(\text{m}^2\right)$, time t (s)

$$\frac{dr}{dt} = 0.5 \text{ m/s}$$

Find $\dfrac{dA}{dt} = ?$ when $r = 20$

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt} = 2\pi r \times 0.5$$

$$\left.\frac{dA}{dt}\right|_{r=20} = 2\pi \times 20 \times \frac{1}{2} = 20\pi \text{ m}^2/\text{s}$$

2 Radius r (m), area $A\left(\text{m}^2\right)$, time t (s)

$$\frac{dA}{dt} = 40\pi \text{ cm}^2/\text{s}$$

Find $\dfrac{dr}{dt} = ?$ when $r = 10$

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

$$\frac{dr}{dt} = \frac{dr}{dA} \cdot \frac{dA}{dt} = \frac{1}{2\pi r} \times 40\pi$$

$$\left.\frac{dr}{dt}\right|_{r=10} = \frac{40\pi}{2\pi \times 10} = 2 \text{ cm/s}$$

3 **a** Square length L (cm), area $A\left(\text{cm}^2\right)$, time t (s)

$$A = L^2$$

$$\frac{dL}{dt} = 2 \text{ cm/s}$$

$$\frac{dA}{dt} = ?$$

$$\frac{dA}{dL} = 2L$$

$$\frac{dA}{dt} = \frac{dA}{dL} \cdot \frac{dL}{dt}$$

$$= 2L \times 2$$

$$\left.\frac{dA}{dt}\right|_{L=4} = 16 \text{ cm}^2/\text{s}$$

b Circle area $A\left(\text{m}^2\right)$, radius r (m), time t (s)

$$\frac{dA}{dt} = 2 \text{ m}^2/\text{s}$$

$$\frac{dr}{dt} = ?$$

$$A = \pi r^2$$
$$\frac{dA}{dr} = 2\pi r$$
$$\frac{dr}{dt} = \frac{dr}{dA} \cdot \frac{dA}{dt}$$
$$= \frac{1}{2\pi r} \times 2$$
$$\left.\frac{dr}{dt}\right|_{r=4} = \frac{1}{4\pi} \text{ m/s}$$

4 Radius r (mm), volume $V\left(\text{mm}^3\right)$, surface area $S\left(\text{mm}^2\right)$, time $t\,(s)$

$$\frac{dr}{dt} = 2 \text{ mm/s}$$
$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dr} = S = 4\pi r^2$$

a
$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$
$$= 4\pi r^2 \times 2$$
$$\left.\frac{dV}{dr}\right|_{r=10} = 800\pi \text{ mm}^3\text{/s}$$

b
$$S = 4\pi r^2$$
$$\frac{dS}{dr} = 8\pi r$$
$$\frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dt}$$
$$= 8\pi r \times 2$$
$$\left.\frac{dS}{dt}\right|_{r=10} = 160\pi \text{ mm}^2\text{/s}$$

5 Volume $V\left(\text{m}^3\right)$, radius r (cm), time t (s)

$$\frac{dV}{dt} = -6 \text{ cm}^3\text{/s}$$

Find $\frac{dr}{dt} = ?$ when $r = 6$

$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dr} = 4\pi r^2$$
$$\frac{dr}{dt} = \frac{dr}{dV} \cdot \frac{dV}{dr} = \frac{1}{4\pi r^2} \times -6$$
$$\left.\frac{dr}{dt}\right|_{r=6} = -\frac{6}{4\pi \times 6^2} = -\frac{1}{24\pi}\text{cm/s}$$

Radius decreasing by $\frac{1}{24\pi}$ cm/s

6 Surface area $S\left(\text{cm}^2\right)$

$$\frac{dr}{dt} = -3 \text{ cm/s}$$

Find $\frac{dS}{dt} = ?$ when $r = 2$

$$S = \frac{dv}{dr} = 4\pi r^2$$
$$\frac{dS}{dr} = 8\pi r$$
$$\frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dt} = 8\pi r \times -3 = -48\pi \text{ cm}^2\text{/s}$$

Surface area is decreasing by 48π cm^2/s.

7 Radius r (mm), volume $V\left(\text{mm}^3\right)$, surface area $S\left(\text{mm}^2\right)$, time t (weeks)

$$\frac{dr}{dt} = -0.2 \text{ mm/week}$$
$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dr} = S = 4\pi r^2$$

a
$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$
$$= -0.2S$$

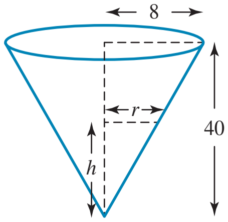
So $\frac{dV}{dt} \propto S$

b $r = 30 - 0.2t$

When $r = 0$

$$t = \frac{30}{0.2} = 150 \text{ weeks}$$

8 Volume $V\left(\text{cm}^3\right)$, radius r (cm), height h (cm), time t (s)



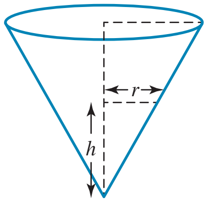
$$\frac{r}{h} = \frac{8}{40}$$
$$r = \frac{h}{5}$$
$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{5}\right)^2 h$$
$$= \frac{\pi h^3}{75}$$
$$\frac{dV}{dh} = \frac{\pi h^2}{25}$$

Given $\frac{dV}{dt} = -6$ cm^3/s find $\frac{dh}{dt} = ?$ when $h = 16$

$$\frac{dh}{dt} = \frac{dh}{dv} \cdot \frac{dV}{dt}$$
$$= \frac{25}{\pi h^2} \times -6 = \frac{-75}{128\pi} \text{ cm/s}$$

Water level is decreasing by $\frac{75}{128\pi}$ cm/s.

9



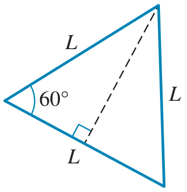
$$r = \frac{h}{2}$$
$$h = 2r$$
$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2(2r) = \frac{2\pi r^3}{3}$$
$$\frac{dV}{dr} = 2\pi r^2 \text{ and } \frac{dr}{dt} = -2$$
$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 2\pi r^2 \times -2$$
$$\left.\frac{dV}{dt}\right|_{r=4} = 2\pi \times 16 \times -2 = -64\pi \text{ cm}^3\text{/s}\cdot$$

Volume is decreasing by 64π cm^3/s.

10 Length L (cm), area $A\left(\text{cm}^2\right)$, altitude h (cm), time t (s)

a $A = \frac{1}{2}L^2\sin(60°)$

$= \frac{\sqrt{3}L^2}{4}$



Given $\frac{dL}{dt} = 2$ cm/s

$$\frac{dA}{dL} = \frac{\sqrt{3}}{2}L$$

$$\frac{dA}{dt} = \frac{dA}{dL}\cdot\frac{dL}{dt}$$

$$= \sqrt{3}L$$

$$\left.\frac{dA}{dt}\right|_{L=2\sqrt{3}} = 6\,\text{cm}^2/\text{s}$$

b $\sin(60°) = \frac{h}{L} = \frac{\sqrt{3}}{2}$

$$h = \frac{\sqrt{3}}{2}L$$

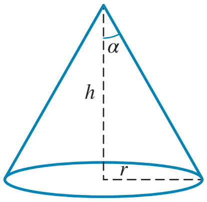
$$\frac{dh}{dL} = \frac{\sqrt{3}}{2}$$

$$\frac{dh}{dt} = \frac{dh}{dL}\cdot\frac{dL}{dt}$$

$$= \frac{\sqrt{3}}{2}\times 2$$

$$= \sqrt{3}\text{ cm/s}$$

11 a



$$V = \frac{1}{3}\pi r^2 h$$

$$\tan(\alpha) = \frac{r}{h}$$

$$r = h\tan(\alpha)$$

$$V = \frac{1}{3}\pi(h\tan(\alpha))^2 h$$

$$= \frac{1}{3}\pi h^3\tan^2(\alpha)$$

b $\alpha = 71.57°$

$\tan(71.57°) = 3 = \frac{r}{h}$

$\Rightarrow r = 3h$

$$\frac{dh}{dt} = 2\text{ cm/s}$$

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi(3h)^2 h$$

$$= 3\pi h^3$$

$$\frac{dV}{dh} = 9\pi h^2$$

$$\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt}$$

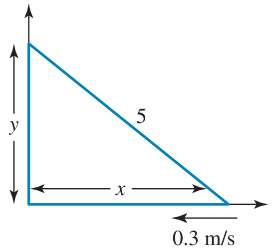
$$= 9\pi h^2\times 2$$

$$= 18\pi h^2$$

$$\left.\frac{dV}{dt}\right|_{h=5} = 18\pi\times 5^2$$

$$= 450\pi\text{ cm}^3/\text{s}$$

12



Given $\frac{dx}{dt} = -0.3$ m/s

$$x^2 + y^2 = 25$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

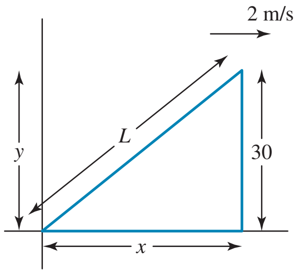
$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

When $x = 3,\ y = 4$

$$\frac{dy}{dt} = \frac{3}{4}\times 0.3 = 0.225\text{ m/s}$$

Moves up

13



Given $\frac{dx}{dt} = 2$

$$x^2 + 30^2 = L^2$$

$$2x\frac{dx}{dt} = 2L\frac{dL}{dt}$$

$$\frac{dL}{dt} = \frac{x}{L}\frac{dx}{dt}$$

When $L = 50,\ x = 40$

$$\frac{dL}{dt} = \frac{40}{50}\times 2 = \frac{8}{5}\text{ m/s}$$

String length is increasing

14 **a** $V=\frac{1}{3}\pi h^2(3r-h),\ r=10$

$=\frac{1}{3}\pi h^2(30-h)$

$=\frac{\pi}{3}\left(30h^2-h^3\right)$

$\frac{dV}{dh}=\frac{\pi}{3}\left(60h-3h^2\right)$

$=\pi\left(20h-h^2\right)$

Given $\frac{dV}{dt}=3\text{ cm}^3/\text{s}$

$\frac{dh}{dt}=\frac{dh}{dV}\cdot\frac{dV}{dt}$

$=\frac{3}{\pi\left(20h-h^2\right)}$

$\left.\frac{dh}{dt}\right|_{h=5}=\frac{3}{\pi(100-25)}$

$=\frac{1}{25\pi}$ cm/s

b $V=\frac{\pi}{432}\left(h^3+108h^2+388h\right)$

$\frac{dV}{dh}=\frac{\pi}{432}\left(3h^2+216h+388\right)$

Given $\frac{dV}{dt}=-7\text{ cm}^3/\text{s}$

$\frac{dh}{dt}=\frac{dh}{dV}\cdot\frac{dV}{dt}$

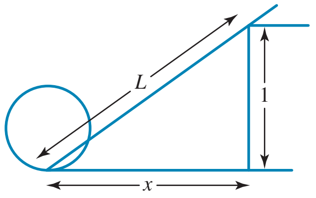
$=\frac{432}{\pi\left(3h^2+216h+388\right)}\times(-7)$

$\left.\frac{dh}{dt}\right|_{h=6}=\frac{-7\times 432}{\pi(3\times 36+216\times 6+388)}$

$=-\frac{27}{16\pi}$ cm/s

Falling by $\frac{27}{16\pi}$ cm/s.

15 **a**



$\frac{dL}{dt}=26$ m/ min

$x^2+1=L^2$

$2x\frac{dx}{dt}=2L\frac{dL}{dt}$

$\frac{dx}{dt}=\frac{L}{x}\frac{dL}{dt}$

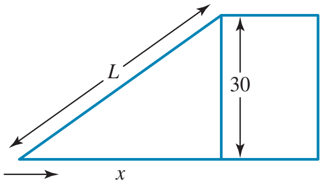
When $x=10$

$L=\sqrt{101}$

$\frac{dx}{dt}=\frac{\sqrt{101}}{10}\times 26$

≈ 26.13 m/ min

b



$\frac{dx}{dt}=-54$ km/hr

$=\frac{-54\times 1000}{60\times 60}$

$=-15$ m/s

$x^2+30^2=L^2$

$2x\frac{dx}{dt}=2L\frac{dL}{dt}$

$\frac{dL}{dt}=\frac{x}{L}\frac{dx}{dt}$

When $x=40$

$L=50$

$\frac{dL}{dt}=\frac{40}{50}\times -15$

$=-12$ m/s

16 $q=\frac{10p}{p-10}$

a $\frac{dq}{dp}=\frac{10(p-10)-1\times 10p}{(p-10)^2}$

$=-\frac{100}{(p-10)^2}$

b $\frac{dp}{dt}=0.2$

$\frac{dq}{dt}=?$

$\frac{dq}{dt}=\frac{dq}{dp}\cdot\frac{dp}{dt}$

$=-\frac{100}{(p-10)^2}\times 0.2$

$\left.\frac{dq}{dt}\right|_{p=12}=-\frac{100}{4}\times 0.2$

$=-5$ cm/s

Decreases by 5 cm/s

17 **a** $PV^{1.4}=C$

$P=1.01\times 10^5$

$V=22.4\times 10^3$

$C=\left(1.01\times 10^5\right)\left(22.4\times 10^{-3}\right)^{1.4}=495.072$

$\frac{dV}{dt}=0.005\text{ m}^3/\text{s}$

$P=CV^{-1.4}$

$\frac{dP}{dV}=-1.4CV^{-2.4}$

$\frac{dP}{dt}=\frac{dP}{dV}\cdot\frac{dV}{dt}$

$=-1.4CV^{-2.4}\times 0.005$

$=-1.4\times 495.072\times\left(22.4\times 10^{-3}\right)^{-2.4}\times 0.005$

$=-31\,562.5$ pa/s

Decreases by 31 562.5 pa/s

b

$$PV^n=C$$

$$\frac{d}{dt}\left(PV^n\right)=\frac{d}{dt}(C)$$

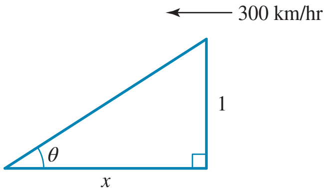
$$P\frac{d}{dV}\left(V^n\right)\frac{dV}{dt}+V^n\frac{d}{dP}(P)\frac{dP}{dt}=0$$

$$PnV^{n-1}\frac{dV}{dt}+V^n\frac{dP}{dt}=0$$

$$PnV^{n-1}\frac{dV}{dt}=-V^n\frac{dP}{dt}$$

$$\frac{dV}{dt}=-\frac{V}{nP}\frac{dP}{dt}$$

18



$$\frac{dx}{dt}=-300\text{ km/hr}=\frac{-300}{60\times 60}=\frac{-1}{12}\text{ km/s}$$

$$\tan(\theta)=\frac{1}{x}$$

$$x=\cot(\theta)$$

$$\frac{dx}{d\theta}=-\text{cosec}^2(\theta)$$

$$\frac{d\theta}{dt}=\frac{d\theta}{dx}\cdot\frac{dx}{dt}$$

$$=\frac{1}{12\text{cosec}^2\theta}$$

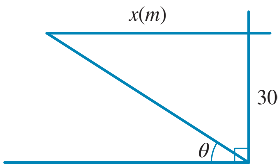
$$=\frac{1}{12}\sin^2\theta,\text{ when }x=30,\ \sin(\theta)=\frac{1}{\sqrt{901}}$$

$$=\frac{1}{12}\times\left(\frac{1}{\sqrt{901}}\right)^2\text{ rad/s}$$

$$=\frac{1}{10812}\times\frac{180}{\pi}$$

$$=0.005°/\text{s}$$

19



$$\frac{dx}{dt}=108\text{ km/hr}$$

$$=\frac{108\times 1000}{60\times 60}$$

$$=30\text{ m/s}$$

$$\tan(\theta)=\frac{300}{x}$$

$$x=300\cot(\theta)$$

$$\frac{dx}{d\theta}=-300\text{ cosec}^2(\theta)$$

$$\frac{d\theta}{dx}=\frac{d\theta}{dx}\cdot\frac{dx}{dt}$$

$$=\frac{1}{-300\text{ cosec}^2(\theta)}\times 30\text{ m/s}$$

When $x=0.4$ km

$$\sin(\theta)=\frac{3}{5}=400\text{ m}$$

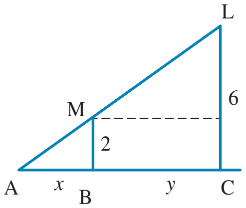
$$\frac{d\theta}{dt}=-\frac{1}{10}\sin^2(\theta)$$

$$=-\frac{9}{250}\text{m/s}$$

$$=-\frac{9}{250}\times\frac{180}{\pi}$$

$$=-2.1°/\text{s}$$

20



Light is at L

Man is at M

$x=AB$

$y=BC$

Shadow is AB

Given $\frac{dy}{dt}=1.5$ m/s

a $\frac{x}{2}=\frac{x+y}{6}$ by similar triangles

$$3x=x+y$$

$$2x=y$$

$$\frac{dy}{dx}=2$$

$$\frac{dx}{dt}=\frac{dx}{dy}\cdot\frac{dy}{dt}$$

$$=\frac{1}{2}\times\frac{3}{2}$$

$$=\frac{3}{4}=0.75\text{ m/s}$$

b Since $AC=x+y$

$$\frac{d}{dt}(x+y)=\frac{dx}{dt}+\frac{dy}{dt}$$

$$=\frac{3}{4}+\frac{3}{2}=2.25\text{ m/s}$$

c Let $z=ML$

$$z^2=y^2+16$$

$$2z\frac{dz}{dt}=2y\frac{dy}{dt}$$

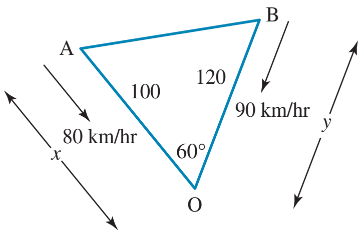
$$\frac{dz}{dt}=\frac{y}{z}\cdot\frac{dy}{dt}$$

When $y=8,\quad z=4\sqrt{5}$

$$\frac{dz}{dt}=\frac{8}{4\sqrt{5}}\times 1.5$$

$$=1.3\text{ m/s}$$

21



a OA = 100 – 80t

OB = 120 – 90t

By cosine rule:

$$\begin{aligned}S^2 = \mathrm{AB}^2 &= \mathrm{OA}^2 + \mathrm{OB}^2 - 2\mathrm{OA}\cdot\mathrm{OB}\times\cos(60°)\\ &= (100-80t)^2 + (120-90t)^2\\ &\quad - 2\,(100-80t)\,(120-90t)\times\frac{1}{2}\end{aligned}$$

$$S = \sqrt{7300t^2 - 19000t + 12400}$$

Since $7300t^2 - 19000t + 12400 \neq 0$

Trains do not crash

b Method 1

$$\frac{dS}{dt} = \frac{10(73t-95)}{\sqrt{73t^2 - 190t + 124}}$$

$$\left.\frac{dS}{dt}\right|_{t=1} = -\frac{220\sqrt{7}}{7}\text{ km/hr}$$

Distance between them is decreasing, so the trains are approaching at a rate of $\frac{220\sqrt{7}}{7}$ km/hr.

Method 2

$$\frac{dx}{dt} = -80$$

$$\frac{dy}{dt} = -90$$

After 1 hour, $t = 1, x = 20, y = 30$

$$S^2 = x^2 + y^2 - xy$$

$$S = 10\sqrt{7}$$

$$2S\frac{dS}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} - x\frac{dy}{dt} - y\frac{dx}{dt}$$

$$20\sqrt{7}\frac{dS}{dt} = 2\times 20\times -80 + 2\times 30\times -90 - 20\times -90 - 30\times -80$$

$$\frac{dS}{dt} = \frac{-220\sqrt{7}}{7}\text{km/hr}$$

Distance between them is decreasing, so the trains are approaching at a rate of $\frac{220\sqrt{7}}{7}$ km/hr.

5.7 Exam questions

1 Given $\frac{dV}{dt} = 1.5\text{ m}^3\text{/min}$

$$\tan(60°) = \frac{r}{h} = \sqrt{3},\quad r = \sqrt{3}h$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\sqrt{3}h\right)^2 h = \pi h^3$$

$$\frac{dV}{dh} = 3\pi h^2$$

$$\frac{dh}{dt} = \frac{dh}{dV}\cdot\frac{dV}{dt} = \frac{1.5}{3\pi h^2},\qquad \left.\frac{dh}{dt}\right|_{h=0.5} = \frac{1.5}{3\pi(0.5)^2} = 0.64\text{ m/min}$$

The correct answer is **C**.

2 $x = \tan^{-1}(t)$, $A(x) = 6x^2$

$$A(t) = 6\left(\tan^{-1}(t)\right)^2$$

$$\frac{dA}{dt} = \frac{6\times 2\tan^{-1}(t)}{1+t^2}$$

When $t = 1$:

$$\begin{aligned}\frac{dA}{dt} &= \frac{12\tan^{-1}(1)}{2}\\ &= 6\times\frac{\pi}{4}\\ &= \frac{3\pi}{2}\text{ mm}^2\text{/ day}\end{aligned}$$

Award 1 mark for correctly expressing A in terms of t.
Award 1 mark for the correct rate.
Award 1 mark for evaluating the rate.
Award 1 mark for the final correct result.

3 a

$$\begin{aligned}\frac{r}{h} &= \frac{0.5}{1}\\ &= \frac{1}{2}\\ r &= \frac{h}{2}\\ V &= \frac{1}{3}\pi r^2 h\\ &= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h\\ &= \frac{\pi}{12}h^3 \qquad \text{[1 mark]}\end{aligned}$$

b

$$\begin{aligned}\frac{dV}{dt} &= 0.02\pi - 0.01\pi\sqrt{h}\\ &= \frac{\pi}{100}\left(2-\sqrt{h}\right) \qquad \text{[1 mark]}\\ \frac{dV}{dh} &= \frac{\pi h^2}{4} \qquad \text{[1 mark]}\\ \frac{dh}{dt} &= \frac{dh}{dV}\cdot\frac{dV}{dt}\\ &= \frac{4}{\pi h^2}\times\frac{\pi}{100}\left(2-\sqrt{h}\right)\\ &= \frac{2-\sqrt{h}}{25h^2} \qquad \text{[1 mark]}\end{aligned}$$

$$\begin{aligned}\left.\frac{dh}{dt}\right|_{h=0.25} &= \frac{2-\sqrt{0.25}}{25(0.25)^2}\\ &= \frac{1.5}{\left(\frac{25}{16}\right)}\\ &= \frac{24}{25}\\ &= 0.96\text{ m/ min} \qquad \text{[1 mark]}\end{aligned}$$

5.8 Review

5.8 Exercise

Technology free: short answer

1 a

$$\begin{aligned}y &= x^5e^{-3x}\\ \frac{dy}{dx} &= x^5\frac{d}{dx}\left(e^{-3x}\right) + e^{-3x}\frac{d}{dx}\left(x^5\right)\\ &= -3x^5e^{-3x} + 5x^4e^{-3x}\\ &= x^4e^{-3x}(5-3x)\end{aligned}$$

b

$$\begin{aligned}y &= \frac{\cos(2x)}{x^3}\\ \frac{dy}{dx} &= \frac{-2x^3\sin(2x) - 3x^2\cos(2x)}{x^6}\\ &= -\frac{x^2\left(2x\sin(2x) + 3\cos(2x)\right)}{x^6}\\ &= -\frac{1}{x^4}\left(2x\sin(2x) + 3\cos(2x)\right)\end{aligned}$$

c $y = \frac{x}{\sqrt{3x^2+5}}$

$u = x \qquad V = \sqrt{3x^2+5} = \left(3x^2+5\right)^{\frac{1}{2}}$

$\frac{du}{dx} = 1 \qquad \frac{dV}{dx} = \frac{1}{2} \times 6x\left(3x^2+5\right)^{-\frac{1}{2}}$

$= \frac{3x}{\sqrt{3x^2+5}}$

$\frac{dy}{dx} = \frac{\sqrt{3x^2+5} - \frac{3x^2}{\sqrt{3x^2+5}}}{3x^2+5}$

$= \frac{1}{3x^2+5}\left[\frac{\left(3x^2+5\right) - 3x^2}{\sqrt{3x^2+5}}\right]$

$= \frac{5}{\sqrt{\left(3x^2+5\right)^3}}$

2 a $y = \sin^{-1}\left(\frac{4x}{5}\right) = \sin^{-1}\left(\frac{u}{5}\right) \qquad u = 4x$

$\frac{dy}{du} = \frac{1}{\sqrt{25-u^2}} \qquad \frac{du}{dx} = 4$

$\frac{dy}{dx} = \frac{4}{\sqrt{25-16x^2}} = 4\left(25-16x^2\right)^{\frac{-1}{2}}$

$\frac{d^2y}{dx^2} = 4 \times \left(\frac{-1}{2}\right) \times (-32x) \times \left(25-16x^2\right)^{\frac{-3}{2}}$

$= \frac{64x}{(25-16x^2)^{\frac{3}{2}}}$

b $y = e^{-3x}\sin(2x)$

$\frac{dy}{dx} = -3e^{-3x}\sin(2x) + 2\cos(2x)e^{-3x}$

$= e^{-3x}(2\cos(2x) - 3\sin(2x))$

$\frac{d^2y}{dx^2} = -3e^{-3x}(2\cos(2x) - 3\sin(2x))$

$+ e^{-3x}(-4\sin(2x) - 6\cos(2x))$

$= e^{-3x}(5\sin(2x) - 12\cos(2x))$

c $y = x^3\cos(2x)$

$\frac{dy}{dx} = 3x^2\cos(2x) - 2x^3\sin(2x)$

$\frac{d^2y}{dx^2} = 6x\cos(2x) - 6x^2\sin(2x) - 6x^2\sin(2x) - 4x^3\cos(2x)$

$= \left(6x - 4x^3\right)\cos(2x) - 12x^2\sin(2x)$

3 a $y = 6\sin\left(\frac{x}{3}\right) \qquad y\left(\frac{\pi}{2}\right) = 6\sin\left(\frac{\pi}{6}\right) = 3$

$\frac{dy}{dx} = 2\cos\left(\frac{x}{3}\right) \qquad \mathrm{P}\left(\frac{\pi}{2}, 3\right)$

$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{2}} = 2\cos\left(\frac{\pi}{6}\right) = \sqrt{3} = m_T \qquad m_N = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

$N: \quad y - 3 = -\frac{\sqrt{3}}{3}\left(x - \frac{\pi}{2}\right)$

$y = -\frac{3\sqrt{3}x}{3} + \frac{\sqrt{3}\pi + 18}{6}$

b $y = \tan\left(\frac{2x}{3}\right) \qquad y\left(\frac{\pi}{2}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

$\frac{dy}{dx} = \frac{2}{3}\sec^2\left(\frac{2x}{3}\right) \qquad \mathrm{P}\left(\frac{\pi}{2}, \sqrt{3}\right)$

$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{2}} = \frac{2}{3\cos^2\left(\frac{\pi}{3}\right)} = \frac{2}{3\left(\frac{1}{2}\right)^2} = \frac{8}{3} = m_T \Rightarrow \quad m_N = \frac{-3}{8}$

$N: \quad y - \sqrt{3} = \frac{-3}{8}\left(x - \frac{\pi}{2}\right)$

$y = \frac{-3x}{8} + \frac{3\pi}{16} + \sqrt{3}$

c $y = \sin^{-1}\left(\frac{\sqrt{x}}{2}\right) \qquad y(1) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$y = \sin^{-1}\left(\frac{u}{2}\right) \qquad u = \sqrt{x} \qquad \mathrm{P}\left(1, \frac{\pi}{6}\right)$

$\frac{dy}{du} = \frac{1}{\sqrt{4-u^2}} \qquad \frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}\sqrt{4-x}}$

$\left.\frac{dy}{dx}\right|_{x=1} = \frac{1}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{6} = m_T \Rightarrow m_N = -2\sqrt{3}$

$N: \quad y - \frac{\pi}{6} = -2\sqrt{3}(x-1)$

$y = -2\sqrt{3}x + 2\sqrt{3} + \frac{\pi}{6}$

4 a $2x^3 - 6x^2y^2 + 3y^3 - 2 = 0$

$\frac{d}{dx}\left(2x^3\right) - \frac{d}{dx}\left(6x^2y^2\right) + \frac{d}{dx}\left(3y^3\right) - \frac{d}{dx}(2) = 0$

$6x^2 - 12xy^2 - 12x^2y\frac{dy}{dx} + 9y^2\frac{dy}{dx} = 0$

$6x^2 - 12xy^2 = \left(12x^2y - 9y^2\right)\frac{dy}{dx}$

$6x\left(x - 2y^2\right) = 3y\left(4x^2 - 3y\right)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{2x\left(x-2y^2\right)}{y\left(4x^2-3y\right)}$

b $4x^4 - 7xy - 3y^4 - 8 = 0$

$\frac{d}{dx}\left(4x^4\right) - \frac{d}{dx}(7xy) - \frac{d}{dx}\left(3y^4\right) - \frac{d}{dx}(8) = 0$

$16x^3 - 7y - 7x\frac{dy}{dx} - 12y^3\frac{dy}{dx} = 0$

$16x^3 - 7y = \left(7x + 12y^3\right)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{16x^3 - 7y}{7x + 12y^3}$

c $x^3 - 4xy + y^3 - 15 = 0 \quad \mathrm{P}\ (2, -1)$

$\frac{d}{dx}\left(x^3\right) - \frac{d}{dx}(4xy) + \frac{d}{dx}\left(y^3\right) = 0$

$3x^2 - 4x\frac{dy}{dx} - 4y + 3y^2\frac{dy}{dx} = 0$

$3x^2 - 4y = \left(4x - 3y^2\right)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{3x^2 - 4y}{4x - 3y^2}$

$\left.\frac{dy}{dx}\right|_{(2,-1)} = \frac{12+4}{8-3} = \frac{16}{5} = m_T \Rightarrow m_N = \frac{-5}{16}$

$N: \quad y + 1 = \frac{-5}{16}(x-2)$

$16(y+1) = -5x + 10$

$16y + 5x + 6 = 0$

d $4x^2 + 4xy + 9y^2 - 6y - 2x - 23 = 0$

When $x = 2$

$16 + 8y + 9y^2 - 6y - 4 - 23 = 0$

$9y^2 + 2y - 11 = 0$

$(9y + 11)(y - 1) = 0$

$y = 1, \dfrac{-11}{9}$ in 1st quadrant $y > 0$

P(2, 1)

$\dfrac{d}{dx}\left(4x^2\right) + \dfrac{d}{dx}(4xy) + \dfrac{d}{dx}\left(9y^2\right) - \dfrac{d}{dx}(6y) - \dfrac{d}{dx}(2x) = 0$

$8x + 4y + 4x\dfrac{dy}{dx} + 18y\dfrac{dy}{dx} - 6\dfrac{dy}{dx} - 2 = 0$

$8x + 4y - 2 = (6 - 4x - 18y)\dfrac{dy}{dx}$

$\dfrac{dy}{dx} = \dfrac{4x + 2y - 1}{3 - 2x - 9y}$

$\left.\dfrac{dy}{dx}\right|_{(2,1)} = \dfrac{8 + 2 - 1}{3 - 4 - 9} = \dfrac{-9}{10}$

$T: \quad y - 1 = \dfrac{-9}{10}(x - 2)$

$y = \dfrac{-9x}{10} + \dfrac{18}{10} + 1$

$y = \dfrac{-9x}{10} + \dfrac{28}{10}$

$10y = -9x + 28$

$10y + 9x - 28 = 0$

5 $\dfrac{dA}{dt} = 50\pi\ \text{m}^2/\text{hr}, \ \dfrac{dr}{dt} = ?$

$A = \pi r^2$

$\dfrac{dA}{dr} = 2\pi r$

$\dfrac{dr}{dt} = \dfrac{dr}{dA} \cdot \dfrac{dA}{dt}$

$= \dfrac{50\cancel{\pi}}{2\cancel{\pi}r}$

$\dfrac{dr}{dt} = \dfrac{50}{2 \times 5} = 5\ \text{m/hr}$

6 **a** $\dfrac{dr}{dt} = 2\ \text{cm/s}$

$V = \dfrac{4}{3}\pi r^3$

$\dfrac{dV}{dt} = 4\pi r^2$

$\dfrac{dV}{dt} = \dfrac{dV}{dr} \cdot \dfrac{dr}{dt}$

$= 4\pi r^2 \times 2$

$\left.\dfrac{dV}{dt}\right|_{r=6} = 4\pi \times 36 \times 2 = 288\pi\ \text{cm}^3/\text{s}$

b $S = \dfrac{dV}{dr} = 4\pi \text{r}^2$

$\dfrac{dS}{dr} = 8\pi r$

$\dfrac{dS}{dt} = \dfrac{dS}{dr} \cdot \dfrac{dr}{dt}$

$= 8\pi r \times 2$

$\left.\dfrac{dS}{dr}\right|_{r=6} = 8\pi \times 6 \times 2 = 96\pi\ \text{cm}^2/\text{s}$

Technology active: multiple choice

7 $\dfrac{d}{dx}\left[\sqrt{9x^2 + 16}\right]$

$= \dfrac{d}{dx}\left[\left(9x^2 + 16\right)^{\frac{1}{2}}\right]$

$= \dfrac{1}{2} \times 18x \times \left(9x^2 + 16\right)^{-\frac{1}{2}}$

$= \dfrac{9x}{\sqrt{9x^2 + 16}}$

The correct answer is **B**.

8 $y = \cos\left(\dfrac{x}{2}\right)$ when $x = \dfrac{\pi}{3}, \ y = \cos\left(\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2}$

$\dfrac{dy}{dx} = -\dfrac{1}{2}\sin\left(\dfrac{x}{2}\right)$

$m_T = \left.\dfrac{dy}{dx}\right|_{x=\frac{\pi}{3}} = -\dfrac{1}{2}\sin\left(\dfrac{\pi}{6}\right) = -\dfrac{1}{4}$

$m_N = 4, \quad P\left(\dfrac{\pi}{3}, \dfrac{\sqrt{3}}{2}\right)$

$N: \ y - \dfrac{\sqrt{3}}{2} = 4\left(x - \dfrac{\pi}{3}\right)$

$y = 4x - \dfrac{4\pi}{3} + \dfrac{\sqrt{3}}{2}$

The correct answer is **D**.

9 $y = \sin^{-1}\left(\dfrac{2x}{3}\right)$

$\dfrac{dy}{dx} = \dfrac{2}{\sqrt{9 - 4x^2}}$

$\left.\dfrac{dy}{dx}\right|_{x=\frac{3}{4}} = \dfrac{2}{\sqrt{9 - 4 \times \frac{9}{16}}} = \dfrac{2}{\frac{3}{2}\sqrt{3}} = \dfrac{4\sqrt{3}}{9}$

The correct answer is **B**.

10 $y = x^2\cos(2x)$

$\dfrac{dy}{dx} = 2x\cos(2x) - 2x^2\sin(2x)$

$\left.\dfrac{dy}{dx}\right|_{x=\frac{\pi}{6}} = 2 \times \dfrac{\pi}{6}\cos\left(\dfrac{\pi}{3}\right) - 2\left(\dfrac{\pi}{6}\right)^2\sin\left(\dfrac{\pi}{3}\right)$

$= \dfrac{\pi}{6} - \dfrac{\sqrt{3}\pi^2}{36} = \dfrac{\pi\left(6 - \sqrt{3}\,\pi\right)}{36}$

The correct answer is **E**.

11 $2x^2 + 3y^2 - 4xy - 9 = 0$

$4x + 6y\dfrac{dy}{dx} - 4y - 4x\dfrac{dy}{dx} = 0$

$4x - 4y = (4x - 6y)\dfrac{dy}{dx}$

$\dfrac{dy}{dx} = \dfrac{2x - 2y}{2x - 3y}$

$\left.\dfrac{dy}{dx}\right|_{(-3,-1)} = \dfrac{-6 + 2}{-6 + 3} = \dfrac{4}{3}$

The correct answer is **C**.

12 $f(x) = \sqrt{2x + 1}$

$f(4) = \sqrt{9} = 3, \quad f(0) = 1$

$\dfrac{f(4) - f(0)}{4 - 0} = \dfrac{3 - 1}{4} = \dfrac{1}{2}$

The correct answer is **A**.

13 $y = \cos^{-1}(1 - 2x)$

$\dfrac{dy}{dx} = -\dfrac{-2}{\sqrt{1 - (1 - 2x)^2}}$

$\dfrac{dy}{dx} = \dfrac{2}{\sqrt{1 - \left(1 - 4x + 4x^2\right)}}$

$$\frac{dy}{dx} = \frac{2}{\sqrt{4x(1-x)}}$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{x(1-x)}}$$

The correct answer is **C**.

14 $x = e^{-t}, \quad y = e^{2t}$

$$\frac{dx}{dt} = \dot{x} = -e^{-t}, \quad \frac{dy}{dt} = \dot{y} = 2e^{2t}$$
$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{2e^{2t}}{-e^{-t}} = -2e^{3t}$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(-2e^{3t}\right)\frac{dt}{dx}$$
$$= -6e^{3t} \times \frac{1}{-e^{-t}} = 6e^{4t}$$

The correct answer is **A**.

15 $x^2 + y^2 = 169$ when $x = -5,\ y < 0 \ \Rightarrow \ y = -12$

$$2x + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{x}{y}$$
$$\left.\frac{dy}{dx}\right|_{(-5,\,-12)} = -\left(\frac{-5}{-12}\right) = -\frac{5}{12} = m_T$$
$$m_N = \frac{12}{5}$$

The correct answer is **B**.

16 $\dfrac{dV}{dt} = -2, \quad V = L^3, \quad \dfrac{dV}{dL} = 3L^2$

$$\frac{dL}{dt} = \frac{dL}{dV}\frac{dV}{dt} = -\frac{2}{3L^2}$$
$$\left.\frac{dL}{dt}\right|_{L=2} = \frac{-2}{12} = \frac{-1}{6}$$

Decrease by $\dfrac{1}{6}$

The correct answer is **E**.

Technology active: extended response

17 $x = ct \qquad \dfrac{dx}{dt} = \dot{x} = c$

$$y = \frac{c}{t} = ct^{-1} \qquad \frac{dy}{dt} = \dot{y} = -ct^{-2} = \frac{-c}{t^2}$$
$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{-c}{t^2} \times \frac{1}{c} = \frac{-1}{t^2} = m_T$$

a $T: \quad y - \dfrac{c}{t} = \dfrac{-1}{t^2}(x - ct)$

$$y - \frac{c}{t} = -\frac{x}{t^2} + \frac{c}{t}$$
$$y = -\frac{x}{t^2} + \frac{2c}{t}$$
$$x + t^2y = 2ct$$

b $N: \quad m_N = t^2$

$$y - \frac{c}{t} = t^2(x - ct)$$
$$y - \frac{c}{t} = xt^2 - ct^3$$
$$y = xt^2 - ct^3 + \frac{c}{t}$$

c $x = ct \quad \Rightarrow \quad t = \dfrac{x}{c}$

$$y = \frac{c}{t} \qquad t = \frac{c}{y}$$
$$\frac{x}{c} = \frac{c}{y}$$
$$xy = c^2$$

18 a $x = a\cos(\theta) \qquad \cos(\theta) = \dfrac{x}{a}$

$$y = b\sin(\theta) \qquad \sin(\theta) = \frac{y}{b}$$

$$\cos^2(\theta) + \sin^2(\theta) = 1$$
$$\Rightarrow \ \frac{x^2}{y^2} + \frac{y^2}{b^2} = 1$$

$$x = a\cos(\theta) \qquad \frac{dx}{d\theta} = -a\sin(\theta)$$
$$y = b\sin(\theta) \qquad \frac{dy}{d\theta} = b\cos(\theta)$$
$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{-b\cos(\theta)}{a\sin(\theta)} = m_T$$
$$T: \ y - b\sin(\theta) = \frac{-b\cos(\theta)}{a\sin(\theta)}(x - a\cos(\theta))$$
$$(y - b\sin(\theta))\ a\sin(\theta) = -b\cos(\theta)(x - a\cos(\theta))$$
$$a\,y\sin(\theta) - b\,a\sin^2(\theta) = -b\,x\cos(\theta) + ab\cos^2(\theta)$$
$$a\,y\ \sin(\theta) + b\,x\cos(\theta) = a\,b\ \left(\cos^2(\theta) + \sin^2(\theta)\right)$$
$$a\,y\ \sin(\theta) + b\,x\cos(\theta) = ab$$
$$\frac{x\ \cos(\theta)}{a} + \frac{y\ \sin(\theta)}{b} = 1$$

b i $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \qquad$ at $\quad \mathrm{P}(x_1, y_1)$

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{b^2x}{a^2y}$$
$$\left.\frac{dy}{dx}\right|_{\mathrm{P}} = \frac{b^2x_1}{a^2y_1}$$
$$T: \ y - y_1 = \frac{b^2x_1^2}{a^2y_1}(x - x_1)$$
$$a^2y_1(y - y_1) = b^2x_1(x - x_1)$$
$$xb^2x_1 - ya^2y_1 = b^2x_1^2 - a^2y_1^2$$
$$\frac{x\,x_1}{a^2} - \frac{y\,y_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$$
$$\frac{x\,x_1}{a^2} - \frac{y\,y_1}{b^2} = 1$$

ii $x = a\sec(\theta) \quad y = b\tan(\theta)$

$$\frac{x^2}{a^2} = \sec^2(\theta) \qquad \frac{y^2}{b^2} = \tan^2(\theta)$$

$$\sec^2(\theta) - \tan^2(\theta) = 1$$
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

19 a Let $y = x\log_e(ax+b)$

$$\frac{dy}{dx} = \log_e(ax+b) + \frac{ax}{ax+b}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{a}{ax+b} + \frac{a(ax+b) - a^2x}{(ax+b)^2}\\ &= \frac{a}{ax+b} + \frac{ab}{(ax+b)^2}\\ &= \frac{a(ax+b)+ab}{(ax+b)^2}\\ &= \frac{a^2x+2ab}{(ax+b)^2}\\ &= \frac{a(ax+2b)}{(ax+b)^2}\end{aligned}$$

So $\frac{d^2}{dx^2}\left[x\log_e(ax+b)\right] = \frac{a(ax+2b)}{(ax+b)^2}$ shown

b Let $y = x\log_e\left(ax^2+b\right)$

$$\frac{dy}{dx} = \log_e\left(ax^2+b\right) + \frac{2ax^2}{ax^2+b}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{2ax}{ax^2+b} + \frac{4ax\left(ax^2+b\right) - 2ax\left(2ax^2\right)}{\left(ax^2+b\right)^2}\\ &= \frac{2ax}{ax^2+b} + \frac{4axb}{\left(ax^2+b\right)^2}\\ &= \frac{2ax\left(ax^2+b\right)+4axb}{\left(ax^2+b\right)^2}\\ &= \frac{2a^2x^3+6axb}{\left(ax^2+b\right)^2}\\ &= \frac{2ax\left(ax^2+3b\right)}{\left(ax^2+b\right)^2}\end{aligned}$$

So $\frac{d^2}{dx^2}\left[x\log_e\left(ax^2+b\right)\right] = \frac{2ax\left(ax^2+3b\right)}{\left(ax^2+b\right)^2}$ shown

c Let $y = x^2\log_e(ax+b)$

$$\frac{dy}{dx} = 2x\log_e(ax+b) + \frac{ax^2}{ax+b}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= 2\log_e(ax+b) + \frac{2ax}{ax+b} + \frac{2ax(ax+b) - a\left(ax^2\right)}{(ax+b)^2}\\ &= 2\log_e(ax+b) + \frac{2ax}{ax+b} + \frac{a^2x^2+2abx}{(ax+b)^2}\\ &= 2\log_e(ax+b) + \frac{2ax(ax+b)+a^2x^2+2abx}{(ax+b)^2}\\ &= 2\log_e(ax+b) + \frac{3a^2x^2+4abx}{(ax+b)^2}\\ &= 2\log_e(ax+b) + \frac{ax(3ax+4b)}{(ax+b)^2}\end{aligned}$$

So $\frac{d^2}{dx^2}\left[x^2\log_e(ax+b)\right] =$

$2\log_e(ax+b) + \frac{ax(3ax+4b)}{(ax+b)^2}$ shown

20 a $y = \frac{8a^3}{x^2+4a^2} = 8a^3\left(x^2+4a^2\right)^{-1}$

$$\begin{aligned}\frac{dy}{dx} &= -8a^3 \times 2x\left(x^2+4a^2\right)^{-2}\\ &= \frac{-16a^3x}{\left(x^2+4a^2\right)^2}\end{aligned}$$

b $x = 2a\cot(t)$

$y = a(1-\cos(2t))$

$x^2 = 4a^2\cot^2(t)$

$$\begin{aligned}x^2+4a^2 &= 4a^2\cot^2(t)+4a^2\\ &= 4a^2\left(\cot^2(t)+1\right)\\ &= 4a^2\operatorname{cosec}^2(t)\end{aligned}$$

$$\begin{aligned}\text{RHS} = \frac{8a^3}{x^2+4a^2} &= \frac{8a^3}{4a^2\operatorname{cosec}^2(t)} = 2a\sin^2(t)\\ &= a(1-\cos(2t))\\ &= y = \text{LHS}\end{aligned}$$

c $x = 2a\cot(t)$ $\quad \frac{dx}{dt} = \dot{x} = -2a\operatorname{cosec}^2(t)$

$y = a(1-\cos(2t))$ $\quad \frac{dy}{dt} = \dot{y} = 2a\ \sin(2t)$

$$\begin{aligned}\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} &= \frac{2a\sin(2t)}{-2a\operatorname{cosec}^2(t)}\\ &= \frac{4a\sin(t)\cos(t)}{-2a}\times\sin^2(t)\\ &= -2\sin^3(t)\cos(t)\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{-16a^3x}{\left(x^2+4a^2\right)^2} = \frac{-16a^3\times 2a\cot(t)}{16a^4\operatorname{cosec}^4(t)}\\ &= \frac{-2\ \cos(t)}{\sin(t)}\times\sin^4(t)\\ &= -2\sin^3(t)\cos(t) \quad \text{shown}\end{aligned}$$

21 a $x = 2\sin(t)$ $\quad \frac{dx}{dt} = \dot{x} = 2\cos(t)$

$$\begin{aligned}y &= 2\tan(t)\ \sin(t)\\ &= \frac{2\sin^2(t)}{\cos(t)}\end{aligned}$$

$$\begin{aligned}\frac{dy}{dt} &= \frac{4\sin(t)\cos^2(t)+2\sin^3(t)}{\cos^2(t)}\\ &= \frac{2\sin(t)\left[2\cos^2(t)+\sin^2(t)\right]}{\cos(t)}\\ &= \frac{2\sin(t)\left(\cos^2(t)+1\right)}{\cos(t)}\end{aligned}$$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\sin(t)\left(\cos^2(t)+1\right)}{\cos^2(t)}$$

b $x^2 = 4\sin^2(t),\ x^4 = 16\sin^4(t)$

$$\begin{aligned}&x^4 + x^2y^2 - 4y^2\\ &= 16\sin^4(t) + 4\sin^2(t)\times 4\tan^2(t)\sin^2(t)\times 16\tan^2(t)\sin^2(t)\\ &= 4\sin^2(t)\left[4\sin^2(t)+4\tan^2(t)\sin^2(t)-4\tan^2(t)\right]\\ &= 4\sin^2(t)\left[4\sin^2(t)+\tan^2(t)\left(4\sin^2(t)-4\right)\right]\\ &= 4\sin^2(t)\left[4\sin^2(t)+\frac{\sin^2(t)}{\cos^2(t)}\left(4\sin^2(t)-4\right)\right]\\ &= 4\sin^2(t)\left[\frac{4\sin^2(t)\cos^2(t)+4\sin^4(t)-4\sin^2(t)}{\cos^2(t)}\right]\\ &= 4\tan^2(t)\left[4\sin^2(t)\left(\cos^2(t)+\sin^2(t)\right)-4\sin^2(t)\right]\\ &= 0 = \text{RHS} \quad \text{shown}\end{aligned}$$

$$x^4 + x^2y^2 - 4y^2 = 0$$

$$4x^3 + 2xy^2 + 2x^2y\frac{dy}{dx} - 8y\frac{dy}{dx} = 0$$

$$4x^3 + 2xy^2 = \left(8y - 2x^2y\right)\ \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{x\left(2x^2+y^2\right)}{y\left(4-x^2\right)}$$

c $x^4 = 4y^2 - x^2y^2$

$$x^4 = y^2\left(4 - x^2\right)$$

$$y^2 = \frac{x^4}{4 - x^2}$$

$$y = \pm\frac{x^2}{\sqrt{4 - x^2}}$$

Consider positive

$$y = \frac{x^2}{\sqrt{4 - x^2}}$$

$$\frac{dy}{dx} = \frac{2x \times \sqrt{4 - x^2} - \frac{1}{2} \times (-2x)\left(4 - x^2\right)^{\frac{-1}{2}} \times x^2}{\left(4 - x^2\right)}$$

$$= \frac{2x\sqrt{4 - x^2} + \frac{x^3}{\sqrt{4 - x^2}}}{\left(4 - x^2\right)}$$

$$= \frac{1}{\left(4 - x^2\right)}\left[\frac{2x\left(4 - x^2\right) + x^3}{\sqrt{4 - x^2}}\right]$$

$$= \frac{8x - x^3}{\sqrt{\left(4 - x^2\right)^3}}$$

22 a $x = t\cos(t)$ $\dfrac{dx}{dt} = \dot{x} = \cos(t) - t\sin(t)$

$y = t\sin(t)$ $\dfrac{dy}{dt} = \dot{y} = \sin(t) + t\cos(t)$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\sin(t) + t\cos(t)}{\cos(t) - t\sin(t)}$$

b $x^2 = t^2\cos^2(t)$

$y^2 = t^2\sin^2(t)$

$\cos^2(t) + \sin^2(t) = 1$

$x^2 + y^2 = t^2$

$$\frac{t\ \sin(t)}{t\ \cos(t)} = \tan(t) = \frac{y}{x}$$

$$t = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\text{So } x^2 + y^2 = \left(\tan^{-1}\left(\frac{y}{x}\right)\right)^2$$

c Use implicit first consider

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{y}{x}\right)\right)^2 \quad \text{let } u = \frac{y}{x} = y\,x^{-1}$$

$$= \frac{d}{du}\left(\tan^{-1}(u)\right)^2\frac{du}{dx} \qquad \frac{du}{dx} = -y\,x^{-2} + x^{-1}\frac{dy}{dx}$$

$$= 2\tan^{-1}(u) \cdot \frac{1}{1 + u^2} \cdot \frac{du}{dx} = \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2}$$

$$\text{So } \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(\tan^{-1}\left(\frac{y}{x}\right)\right)^2$$

$$2x + 2y\frac{dy}{dx} = 2\tan^{-1}\left(\frac{y}{x}\right)\left(\frac{1}{1 + \frac{y^2}{x^2}}\right) . \left(\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2}\right)$$

$$x + y\frac{dy}{dx} = \tan^{-1}\left(\frac{y}{x}\right)\left(\frac{x^2}{x^2 + y^2}\right)\left(\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2}\right)$$

$$\left(x + y\left(\frac{dy}{dx}\right)\right)\left(x^2 + y^2\right) = \left(x\frac{dy}{dx} - y\right)\tan^{-1}\left(\frac{y}{x}\right)$$

$$x\left(x^2 + y^2\right) + y\left(x^2 + y^2\right)\frac{dy}{dx} = x\tan^{-1}\left(\frac{y}{x}\right)\frac{dy}{dx} - y\tan^{-1}\left(\frac{y}{x}\right)$$

$$x\left(x^2 + y^2\right) + y\tan^{-1}\left(\frac{y}{x}\right) = \left[x\tan^{-1}\left(\frac{y}{x}\right) - y\left(x^2 + y^2\right)\right]\frac{dy}{dx}$$

$$\text{So } \frac{dy}{dx} = \frac{x^3 + xy^2 + y\tan^{-1}\left(\frac{y}{x}\right)}{x\tan^{-1}\left(\frac{y}{x}\right) - yx^2 - y^3}$$

5.8 Exam questions

1 $e^xe^{2y} + e^{4y^2} = 2e^4$

$\dfrac{d}{dx}\left(e^xe^{2y}\right) + \dfrac{d}{dx}\left(e^{4y^2}\right) = \dfrac{d}{dx}\left(2e^4\right)$ using the product rule on the first term

$$e^{2y}\frac{d}{dx}\left(e^x\right) + e^x\frac{d}{dx}\left(e^{2y}\right) + \frac{d}{dy}\left(e^{4y^2}\right)\frac{dy}{dx} = 0$$

$$e^xe^{2y} + 2e^xe^{2y}\frac{d}{dx} + 8ye^{4y^2}\frac{dy}{dx} = 0$$

Substitute $x = 1,\ y = 2$

$$e^2e^2 + 2e^2e^2\frac{dy}{dx} + 8e^4\frac{dy}{dx} = 0$$

$$e^4 = -10e^4\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{10}$$

Award 1 mark for applying the product rule.

Award 1 mark for correctly using implicit differentiation.

Award 1 mark for the final correct gradient.

2 $\cos(y) + y\ \sin(x) = x^2, \left(0, -\dfrac{\pi}{2}\right)$

Using implicit differentiation with the product rule on second term:

$$-\sin(y)\frac{dy}{dx} + \sin(x)\frac{dy}{dx} + y\ \ \cos(x) = 2x$$

$$\frac{dy}{dx}\left[\sin(x) - \sin(y)\right] = 2x - y\ \cos(x)$$

$$\frac{dy}{dx} = \frac{2x - y\ \cos(x)}{\sin(x) - \sin(y)}$$

At $x = 0, y = -\dfrac{\pi}{2}$:

$$m_T = \frac{0 + \frac{\pi}{2}\cos(0)}{\sin(0) - \sin(-\frac{\pi}{2})}$$

$$= \frac{\pi}{2}$$

$$\Rightarrow\ m_N = -\frac{2}{\pi}$$

Find the equation of *N*:

$$y + \frac{\pi}{2} = -\frac{2}{\pi}(x - 0)$$

$$y = -\frac{2x}{\pi} - \frac{\pi}{2}$$

Award 1 mark for using implicit differentiation and applying the product rule.

Award 1 mark for finding the correct gradient at the indicated point.

Award 1 mark for gradient of the normal.

Award 1 mark for the equation of the normal.

3 a $x^2 - xy + \dfrac{3}{2}y^2 = 9$ implicit differentiation

$$2x - y - x\frac{dy}{dx} + 3y\frac{dy}{dx} = 0$$

$$2x - y = (x - 3y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x - y}{x - 3y}$$

Award 1 mark for using implicit differentiation.

Award 1 mark for the correct gradient.

b $A\colon (3,0)\, m_A = \left.\dfrac{dy}{dx}\right|_{(3,0)} = 2 \quad T_A\colon y - 0 = 2(x-3)$

$T_A\colon y = 2x - 6$

$B\colon \left(0, \sqrt{6}\right) m_B = \left.\dfrac{dy}{dx}\right|_{\left(0,\sqrt{6}\right)} = \dfrac{1}{3} \quad T_B\colon y - \sqrt{6} = \dfrac{1}{3}(x-0)$

$T_B\colon y = \dfrac{x}{3} + \sqrt{6}$

Award 1 mark for the correct tangent at $(3,0)$.

Award 1 mark for the correct tangent at $\left(0,\sqrt{6}\right)$.

c Let $m_B = \tan(\theta_1) = \dfrac{1}{3} \quad \Rightarrow \theta_1 = \tan^{-1}\left(\dfrac{1}{3}\right)$ and

$m_A = \tan(\theta_2) = 2 \ \Rightarrow\ \theta_2 = \tan^{-1}(2)$

$\theta = \theta_2 - \theta_1$

$\tan(\theta) = \tan\left(\theta_2 - \theta_1\right)$

$$= \frac{\tan(\theta_2) - \tan(\theta_1)}{1 + \tan(\theta_2)\tan(\theta_1)} = \frac{2 - \frac{1}{3}}{1 + 2 \times \frac{1}{3}} = \frac{\frac{5}{3}}{\frac{5}{3}} = 1$$

$\theta = \tan^{-1}(1)$

$\theta = \dfrac{\pi}{4}$

Award 1 mark for using the tan angle formula.

Award 1 mark for the final correct angle.

4 $y^2 + \dfrac{3e^{x-1}}{x-2} = c$

Using implicit differentiation and the quotient rule.

$$2y\frac{dy}{dx} + \frac{3e^{\left(x-1\right)}(x-2) - 3e^{\left(x-1\right)}}{(x-2)^2} = 0$$

$$2y\frac{dy}{dx} + \frac{3e^{\left(x-1\right)}(x-3)}{(x-2)^2} = 0$$

When $x = 1, \dfrac{dy}{dx} = 2$

$4y + \dfrac{3e^0 \times -2}{(-1)^2} = 0 \ \Rightarrow\ 4y - 6 = 0 \ \Rightarrow\ y = \dfrac{3}{2}$

When $x = 1$ and $y = \dfrac{3}{2}, c = \dfrac{9}{4} + \dfrac{3e^0}{-1}$

$c = \dfrac{9}{4} - 3$

$c = -\dfrac{3}{4}$

Award 1 mark for using implicit differentiation.

Award 1 mark for using the quotient rule.

Award 1 mark for substituting.

Award 1 mark for using the final correct value c.

5 To show $\dfrac{d^n}{dx^n}\left(\dfrac{1}{ax+b}\right) = \dfrac{(-1)^n\, n!\, a^n}{(ax+b)^{n+1}}$

The first base step is to prove it is true for the base case, that is when $n = 1$,

$\text{LHS} = \dfrac{d}{dx}\left(\dfrac{1}{ax+b}\right) = \dfrac{d}{dx}\left((ax+b)^{-1}\right)$

$= -a(ax+b)^{-2} = \dfrac{-a}{(ax+b)^2}$

$\text{RHS} = \dfrac{(-1)^1 1!\, a^1}{(ax+b)^2} = \dfrac{-a}{(ax+b)^2}$ so is true when $n = 1$.

We now assume it is true when $n = k$, that is assume

$\dfrac{d^k}{dx^k}\left(\dfrac{1}{ax+b}\right) = \dfrac{(-1)^k\, k!\, a^k}{(ax+b)^{k+1}}$ for the kth derivative is true,

we now consider the $(k+1)$th derivative, that is

$\dfrac{d^{k+1}}{dx^{k+1}}\left(\dfrac{1}{ax+b}\right) = \dfrac{d}{dx}\left(\dfrac{d^k}{dx^k}\left(\dfrac{1}{ax+b}\right)\right)$ using the assumption

$$\frac{d^{k+1}}{dx^{k+1}}\left(\frac{1}{ax+b}\right) = \frac{d}{dx}\left(\frac{(-1)^k\, k!\, a^k}{(ax+b)^{k+1}}\right)$$

Taking the constant factor outside the derivative simplifying

$$\frac{d^{k+1}}{dx^{k+1}}\left(\frac{1}{ax+b}\right) = (-1)^k\, k!\, a^k \frac{d}{dx}\left((ax+b)^{-k-1}\right)$$

$$\frac{d^{k+1}}{dx^{k+1}}\left(\frac{1}{ax+b}\right) = (-1)^k\, k!\, a^k\,(-k-1) \times a\left((ax+b)^{-k-2}\right)$$

$$\frac{d^{k+1}}{dx^{k+1}}\left(\frac{1}{ax+b}\right) = -(-1)^k\,(k+1)\,k!\,a^{k+1}\left(\frac{1}{(ax+b)^{k+2}}\right)$$

$$\frac{d^{k+1}}{dx^{k+1}}\left(\frac{1}{ax+b}\right) = \frac{(-1)^{k+1}\,(k+1)!\,a^{k+1}}{(ax+b)^{k+2}}$$

Which is what we want to show, this completes the proof by induction, so it is true for all $n \in N$.

Award 1 mark for base case when $n = 1$.

Award 1 mark for assumption case.

Award 2 marks for simplifying.

Award 1 mark for the final conclusion.

Topic 6 — Functions and graphs

6.2 Sketching graphs of cubics and quartics

6.2 Exercise

1 $y = x^3 - 4x^2 + 4x$

$= x\left(x^2 - 4x + 4\right)$

$= x(x-2)^2$

Crosses x-axis when $y = 0$, $x = 0$ or $x = 2$

$\Rightarrow (0, 0), (2, 0)$

$\frac{dy}{dx} = 3x^2 - 8x + 4$

$= (3x - 2)(x - 2) = 0$

$\frac{dy}{dx} = 0$ when $x = 2$ or $x = \frac{2}{3}$

When $x = \frac{2}{3}$

$$y = \frac{2}{3}\left(\frac{2}{3} - 2\right)^2 = \frac{32}{27}$$

Stationary points are $(2, 0)$ and $\left(\frac{2}{3}, \frac{23}{27}\right)$

$$\frac{d^2y}{dx^2} = 6x - 8 = 0$$

When $x = \frac{4}{3}$

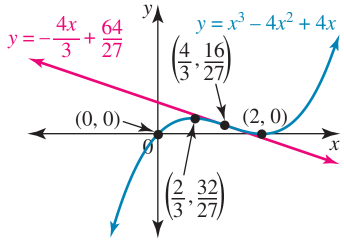
$$y = \frac{4}{3}\left(\frac{4}{3} - 2\right)^2 = \frac{16}{27}$$

When $x = \frac{2}{3}$	$\frac{d^2y}{dx^2} = -4 < 0$	$\left(\frac{2}{3}, \frac{23}{27}\right)$	local max
$x = 2$	$\frac{d^2y}{dx^2} = 4 > 0$	$(2, 0)$	local min
$x = \frac{4}{3}$	$\frac{dy}{dx} = 2\left(\frac{4}{3} - 2\right) = -\frac{4}{3}$	$\left(\frac{4}{3}, \frac{16}{27}\right)$	inflection

Tangent: $y - \frac{16}{27} = -\frac{4}{3}\left(x - \frac{4}{3}\right)$

$y = -\frac{4x}{3} + \frac{16}{9} + \frac{16}{27}$

$y = -\frac{4x}{3} + \frac{64}{27}$



2 **a** $y = x^3 - 27x$

$= x\left(x^2 - 27\right)$

$= x\left(x + 3\sqrt{3}\right)\left(x - 3\sqrt{3}\right)$

Crosses x-axis at $y = 0$

$\Rightarrow x = 0, \pm 3\sqrt{3}$

$(0, 0)$, $\left(\pm 3\sqrt{3}, 0\right)$

$\frac{dy}{dx} = 3x^2 - 27 = 0$

$= 3\left(x^2 - 9\right) = 0$

$= 3(x + 3)(x - 3) = 0$

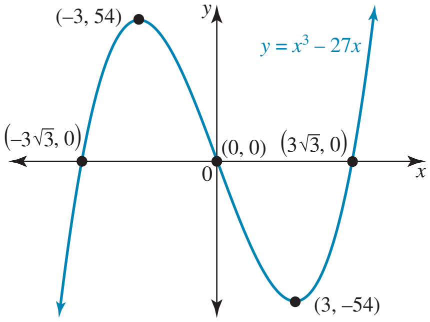
$\Rightarrow x = \pm 3$

When $x = -3$, $y = (-3)^3 - 27 \times -3 = 54$

$x = 3$, $y = (3)^3 - 27 \times 3 = -54$

$\frac{d^2y}{dx^2} = 6x$

When $x = 3$	$y'' > 0$	$(3, -54)$	local min
$x = -3$	$y'' < 0$	$(-3, 54)$	local max
$\frac{d^2y}{dx^2} = 0 \Rightarrow x = 0$		$(0, 0)$	inflection



b $y = 9x - x^3$

$= x\left(9 - x^2\right)$

$= x(3 + x)(3 - x)$

Crosses x-axis at $y = 0$

$\Rightarrow x = 0, \pm 3$

$(0, 0)$, $(\pm 3, 0)$

$\frac{dy}{dx} = 9 - 3x^2 = 0$

$= 3\left(3 - x^2\right) = 0$

$= 3\left(x + \sqrt{3}\right)\left(x - \sqrt{3}\right) = 0$

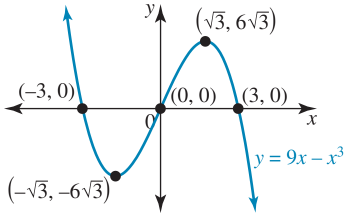
$\Rightarrow x = \pm\sqrt{3}$

When $x = \sqrt{3}$, $y = 9\sqrt{3} - \left(\sqrt{3}\right)^3 = 6\sqrt{3}$

$x = -\sqrt{3}$, $y = -9\sqrt{3} - \left(-\sqrt{3}\right)^3 = -6\sqrt{3}$

$\frac{d^2y}{dx^2} = -6x$

When $x = \sqrt{3}$	$y'' < 0$	$\left(\sqrt{3}, 6\sqrt{3}\right)$	local max
$x = -\sqrt{3}$	$y'' > 0$	$\left(-\sqrt{3}, -6\sqrt{3}\right)$	local min
$\frac{d^2y}{dx^2} = 0 \Rightarrow x = 0$		$(0, 0)$	inflection



3 a $y = x^3 + 12x^2 + 36x$

$= x\left(x^2 + 12x + 36\right)$

$= x(x+6)^2$

Crosses x-axis at $y = 0$

$\Rightarrow x = 0, -6$

$(0, 0), (-6, 0)$

$$\frac{dy}{dx} = 3x^2 + 24x + 36 = 0$$

$$= 3\left(x^2 + 8x + 12\right) = 0$$

$$= 3(x+2)(x+6) = 0$$

$$\Rightarrow x = -2, -6$$

When $x = -2,\ y = -2(-2+6)^2 = -32$

$x = -6,\ y = 0$

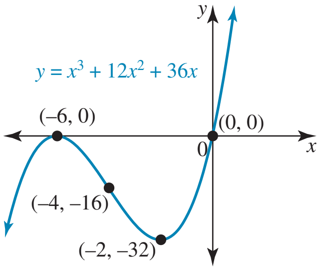
$$\frac{d^2y}{dx^2} = 6x + 24$$

$$= 6(x+4)$$

When $x = 2$ $\quad y'' > 0$ $\quad (-2, -32)$ $\quad$ local min

$x = -6$ $\quad y'' < 0$ $\quad (-6, 0)$ $\quad$ local max

$\frac{d^2y}{dx^2} = 0 \Rightarrow x = -4$ $\quad (-4, -16)$ $\quad$ inflection



b $y = -x^3 + 10x^2 - 25x$

$= -x\left(x^2 - 10x + 25\right)$

$= -x(x-5)^2$

Crosses x-axis at $y = 0$

$\Rightarrow x = 0, 5$

$(0, 0), (5, 0)$

$$\frac{dy}{dx} = -3x^2 + 20x - 25 = 0$$

$$= (5 - 3x)(x - 5) = 0$$

$$\Rightarrow x = 5, \frac{5}{3}$$

When $x = \frac{5}{3}\ y = -\frac{5}{3}\left(\frac{5}{3} - 5\right)^2 = -\frac{500}{27} = -18\frac{14}{27}$

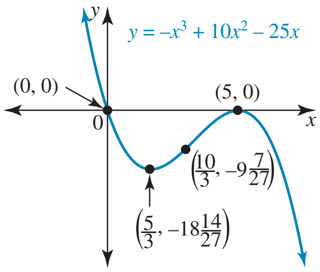
$x = 5\ \ y = 0$

$$\frac{d^2y}{dx^2} = -6x + 20$$

When $x = 5$ $\quad y'' < 0$ $\quad (5, 0)$ $\quad$ local max

$x = \frac{5}{3}$ $\quad y'' > 0$ $\quad \left(\frac{5}{3}, -18\frac{14}{27}\right)$ $\quad$ local min

$\frac{d^2y}{dx^2} = 0 \Rightarrow x = \frac{10}{3}$ $\quad \left(\frac{10}{3}, -9\frac{7}{27}\right)$ $\quad$ inflection



4 a $y = x^3 - 3x^2 - 9x - 5$

$f(-1) = -1 - 3 + 9 - 5 = 0 \quad \Rightarrow (x+1)$ is a factor

$y = (x+1)\left(x^2 - 4x - 5\right)$

$= (x+1)^2(x-5)$

y-intercept $x = 0 \Rightarrow y = 5 \Rightarrow (0, -5)$

Crosses x-axis $x = -1, 5 \Rightarrow (-1, 0)\ (5, 0)$

$$\frac{dy}{dx} = 3x^2 - 6x - 9 = 0$$

$$= 3\left(x^2 - 2x - 3\right) = 0$$

$$= 3(x+1)(x-3) = 0$$

$$\Rightarrow x = -1, 3$$

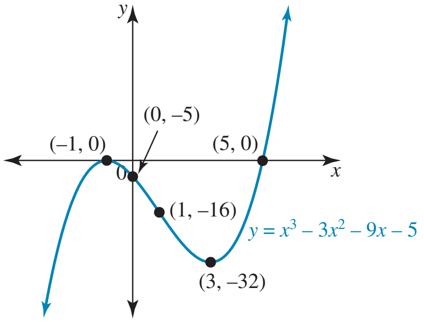
When $x = 3\ y = 16 \times -2 = -32$

$$\frac{d^2y}{dx^2} = 6x - 6$$

When $x = -1$ $\quad y'' < 0$ $\quad (-1, 0)$ $\quad$ local max

$x = 3$ $\quad y'' > 0$ $\quad (3, -32)$ $\quad$ local min

$\frac{d^2y}{dx^2} = 0 \Rightarrow x = 1$ $\quad (1, -16)$ $\quad$ inflection



b $y = -x^3 + 9x^2 - 15x - 25$

$f(-1) = 1 + 9 + 15 - 25 = 0 \quad \Rightarrow (x+1)$ is a factor

$y = -(x+1)(x^2 - 10x + 25)$

$= -(x+1)(x-5)^2$

y-intercept $x = 0 \Rightarrow y = -25 \Rightarrow (0, -25)$

Crosses x-axis $x = -1, 5 \quad \Rightarrow (-1, 0)\ (5, 0)$

$$\frac{dy}{dx} = -3x^2 + 18x - 15 = 0$$

$$= -3\left(x^2 - 6x + 5\right) = 0$$

$$= -3(x-5)(x-1) = 0$$

$$\Rightarrow x = 1, 5$$

When $x = 1 \quad y = -2 \times (-4)^2 = -32$

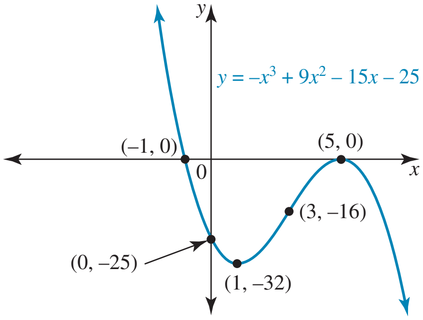
$$\frac{d^2y}{dx^2} = -6x + 18$$

$$= 6(3 - x)$$

When $x = 1 \quad y'' > 0 \quad (1, -32)$ local min

$x = 5 \quad y'' < 0 \quad (5, 0)$ local max

$\frac{d^2y}{dx^2} = 0 \Rightarrow x = 3 \quad (3, -16)$ inflection



5 a $y = x^3 - x^2 - 16x + 16$

$f(1) = 1 - 1 - 16 + 16 = 0 \Rightarrow (x - 1)$ is a factor

$$y = (x - 1)\left(x^2 - 16\right)$$
$$= (x - 1)(x + 4)(x - 4)$$

y-intercept $x = 0 \Rightarrow y = 16 \Rightarrow (0, 16)$

Crosses x-axis $x = 1, \pm 4 \quad \Rightarrow (1, 0)\,(-4, 0)\,(4, 0)$

$$\frac{dy}{dx} = 3x^2 - 2x - 16 = 0$$
$$= (x + 2)(3x - 8) = 0$$
$$\Rightarrow x = \frac{8}{3}, -2$$

When $x = -2 \quad y = (-2)^3 - (-2)^2 + 32 + 16 = 36$

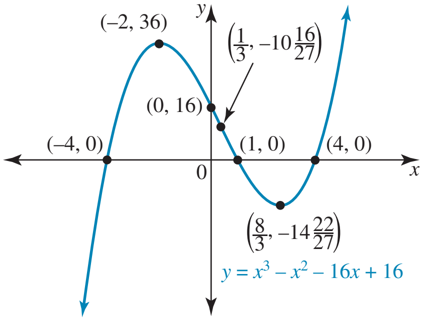
$x = \frac{8}{3} \quad y = \left(\frac{8}{3}\right)^3 - \left(\frac{8}{3}\right)^2 - 16 \times \frac{8}{3} + 16 = -14\frac{22}{27}$

$\frac{d^2y}{dx^2} = 6x - 2 = 0 \quad \Rightarrow x = \frac{1}{3}$

When $x = -2 \quad y'' < 0 \quad (-2, 36)$ local max

$x = \frac{8}{3} \quad y'' > 0 \quad \left(\frac{8}{3}, -14\frac{22}{27}\right)$ local min

$\frac{d^2y}{dx^2} = 0 \Rightarrow x = \frac{1}{3} \quad \left(\frac{1}{3}, 10\frac{16}{27}\right)$ inflection



b $y = -x^3 - 5x^2 + 8x + 12$

$f(-1) = 1 - 5 - 8 + 12 = 0 \quad \Rightarrow (x + 1)$ is a factor

$$y = -(x + 1)\left(x^2 + 4x - 12\right)$$
$$= -(x + 1)(x - 2)(x + 6)$$

y-intercept $x = 0 \Rightarrow y = 12 \quad \Rightarrow (0, 12)$

Crosses x-axis $x = -6, -1, 2 \quad \Rightarrow (-1, 0)\,(2, 0)\,(-6, 0)$

$$\frac{dy}{dx} = -3x^2 - 10x + 8 = 0$$
$$= -\left(3x^2 + 10x - 8\right) = 0$$
$$= -(3x - 2)(x + 4) = 0$$
$$\Rightarrow x = -4, \frac{2}{3}$$

When $x = -4 \quad y = -36$

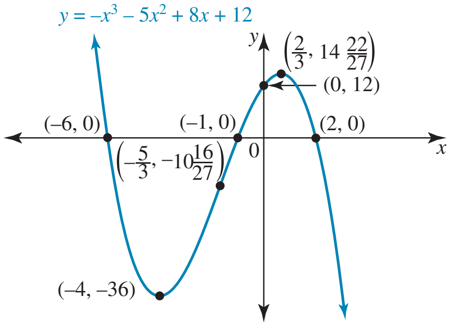
$x = \frac{2}{3} \quad y = 14\frac{22}{27}$

$\frac{d^2y}{dx^2} = -6x - 10 = 0 \quad \Rightarrow x = \frac{-5}{3} \quad \Rightarrow y = -10\frac{16}{27}$

When $x = -4 \quad y'' > 0 \quad (-4, -36)$ local min

$x = \frac{2}{3} \quad y'' < 0 \quad \left(\frac{2}{3}, 14\frac{22}{27}\right)$ local max

$\frac{d^2y}{dx^2} = 0 \Rightarrow x = \frac{-5}{3} \quad \left(-\frac{5}{3}, -10\frac{16}{27}\right)$ inflection



6 $y = x^4 - 24x^2 + 80$

$$= \left(x^2 - 4\right)\left(x^2 - 20\right)$$
$$= (x + 2)(x - 2)\left(x + \sqrt{20}\right)\left(x - \sqrt{20}\right)$$

Crosses y-axis $x = 0 \quad (0, 80)$

Crosses x-axis $y = 0 \quad x = \pm 2, \ \pm\sqrt{20} = \pm 2\sqrt{5}$

$(\pm 2, 0)\left(\pm 2\sqrt{5}, 0\right)$

$$\frac{dy}{dx} = 4x^3 - 48x$$
$$= 4x\left(x^2 - 12\right)$$
$$= 4x\left(x + 2\sqrt{3}\right)\left(x - 2\sqrt{3}\right)$$

$\frac{dy}{dx} = 0 \quad \Rightarrow x = 0, \ x = \pm 2\sqrt{3}$

$$\frac{d^2y}{dx^2} = 12x^2 - 48$$
$$= 12\left(x^2 - 4\right)$$

When $x = 0, \ y = 80, \ y'' < 0, \ (0, 80)$ local max

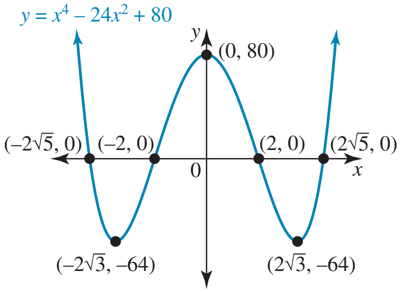
When $x = \pm 2\sqrt{3}, \ y = 144 - 24 \times 12 + 80 = -64$

$y'' > 0 \left(\pm 2\sqrt{3}, -64\right)$ both min

When $\frac{d^2y}{dx^x} = 0 \ x^2 = 4 \ x = \pm 2, \ (\pm 2, 0)$ inflection

$f(x) = x^4 - 24x^2 + 80$

$f(-x) = f(x)$ symmetrical about the y-axis



7 a $y = f(x) = x^4 - 4x^3$

$$= x^3(x - 4)$$

Crosses x-axis $x = 0, 4 \quad \Rightarrow (0, 0)\,(4, 0)$

$$\frac{dy}{dx} = f'(x) = 4x^3 - 12x^2 = 0$$
$$= 4x^2(x - 3) = 0$$

$\Rightarrow x = 0, 3$

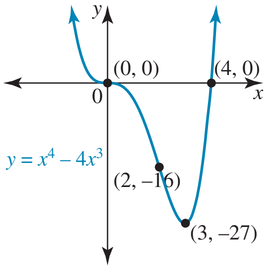
When $x = 3, f(3) = 81 - 108 = -27$

$$\frac{d^2y}{dx^2} = 12x^2 - 24x = 0$$
$$= 12x(x-2) = 0$$

$\Rightarrow x = 0, 2$

When $x = 2, f(2) = 16 - 32 = -16$

$(0, 0)$	$f'(0) = 0$	$f''(0) = 0$	S.P.I
$(2, -16)$	$f'(2) \neq 0$	$f''(2) = 0$	inflection
$(3, -27)$	$f'(3) = 0$	$f''(3) = 36 > 0$	min



b $y = f(x) = 4x^2 - x^4$
$$= x^2\left(4 - x^2\right)$$
$$= x^2(2+x)(2-x)$$

Crosses x-axis $x = 0, \pm 2 \Rightarrow (0, 0)\,(2, 0)\,(-2, 0)$

$$\frac{dy}{dx} = f'(x) = 8x - 4x^3 = 0$$
$$= 4x\left(2 - x^2\right) = 0$$

$\Rightarrow x = 0, \pm\sqrt{2}$

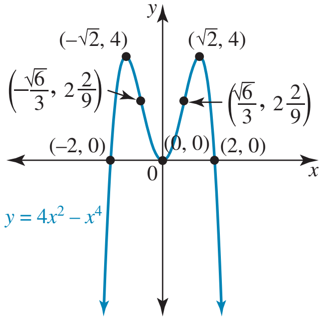
When $x = \pm\sqrt{2}, f\left(\pm\sqrt{2}\right) = 8 - 4 = 4$

$$\frac{d^2y}{dx^2} = f''(x) = 8 - 12x^2 = 0$$
$$x^2 = \frac{8}{12}$$
$$\Rightarrow x = \pm\sqrt{\frac{2}{3}} = \pm\frac{\sqrt{6}}{3}$$

When $x = \pm\frac{\sqrt{6}}{3}, f\left(\pm\frac{\sqrt{6}}{3}\right) = 2\frac{2}{9}$

$\left(\pm\sqrt{2}, 4\right)$	$f''\left(\pm\sqrt{2}\right) = -16 < 0$	max TP
$\left(\pm\frac{\sqrt{6}}{3}, 2\frac{2}{9}\right)$		inflection
$(0, 0)$	$f''(0) = 8 > 0$	local min

$f(x) = f(-x)$ even function, symmetrical about the y-axis



8 a $y = f(x) = x^4 + 4x^3 - 16x - 16$

$f(1) = 1 + 4 - 16 - 16 = -27$

$f(2) = 16 + 32 - 32 - 16 = 0$ $(x-2)$ is a factor

$f(-2) = 16 - 32 + 32 - 16 = 0$ $(x+2)$ is a factor

$f(x) = (x-2)(x+2)^3$

y-intercept $(0, -16)$

Crosses x-axis $x = 2, -2 \Rightarrow (2, 0)\,(-2, 0)$

$$\frac{dy}{dx} = f'(x) = 4x^3 + 12x^2 - 16$$

$f'(1) = 4 + 12 - 16 = 0$

$f'(-2) = -32 + 48 - 16 = 0$

$f'(x) = 4(x-1)(x+2)^2 = 0$

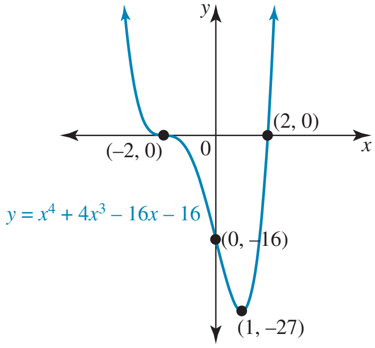
$\Rightarrow x = 1, -2$

When $x = 1, f(1) = -27$

$$\frac{d^2y}{dx^2} = f''(x) = 12x^2 + 24x = 0$$
$$= 12x(x+2) = 0$$

$\Rightarrow x = 0, -2$

$(0, -16)$	$f'(0) \neq 0$	$f''(0) = 0$	inflection
$(1, -27)$	$f'(1) = 0$	$f''(1) = 36 > 0$	min
$(-2, 0)$	$f'(-2) = 0$	$f''(-2) = 0$	S.P.I



b $y = f(x) = x^4 - 6x^2 + 8x - 3$

$f(1) = 1 - 6 + 8 - 3 = 0$ $(x-1)$ is a factor

$f(2) = 16 - 24 + 16 - 3 \neq 0$

$f(3) = 81 - 54 + 24 - 3 \neq 0$

$f(-3) = 81 - 54 - 24 - 3 = 0$ $(x+3)$ is a factor

$f(x) = (x-1)^3(x+3)$

y-intercept $(0, -3)$

Crosses x-axis $x = 1, -3 \Rightarrow (1, 0)\,(-3, 0)$

$$\frac{dy}{dx} = f'(x) = 4x^3 - 12x + 8 = 0$$
$$= 4(x^3 - 3x + 2) = 0$$
$$= 4(x-1)^2(x+2) = 0$$

$\Rightarrow x = 1, -2$

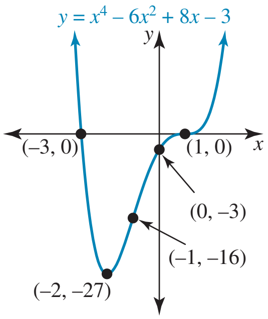
When $x = -2, f(-2) = 16 - 24 - 16 - 3 = -27$

$$\frac{d^2y}{dx^2} = f''(x) = 12x^2 - 12 = 0$$
$$= 12(x+1)(x-1) = 0$$

$\Rightarrow x = \pm 1$

When $x = -1, f(-1) = 1 - 6 - 8 - 3 = -16$

$(1, 0)$	$f'(1) = 0$	$f''(1) = 0$	S.P.I
$(-1, -16)$	$f'(-1) \neq 0$	$f''(-1) = 0$	inflection
$(-2, -27)$	$f'(-2) = 0$	$f''(-2) > 0$	min



9 $y = x^3 + bx^2 + cx + d$

Crosses y-axis at $x = 0 \quad y = d = 5$

$\frac{dy}{dx} = 3x^2 + 2bx + c$

$\frac{d^2y}{dx^2} = 6x + 2b$

When $x = 1, \frac{d^2y}{dx^2} = 0$

$6 + 2b = 0$

$b = -3$

$y = x^3 - 3x^2 + cx + 5$

When $x = 1, \quad y = -21$

$1 - 3 + c + 5 = -21$

$c = -24$

$\frac{dy}{dx} = 3x^2 - 6x - 24$

$\left.\frac{dy}{dx}\right|_{x=1} = 3 - 6 - 24 = -27$

Tangent: $y + 21 = -27(x - 1)$

$y + 21 = -27x + 27$

$y = 6 - 27x$

10 SPI $(1, -2)$

$y = (x - 1)^3 - 2$

$= (x^3 - 3x^2 + 3x - 1) - 2$

$= x^3 - 3x^2 + 3x - 3$

$b = -3 \quad c = 3 \quad d = -3$

11 $y = x^3 + bx^2 + cx + d$

$\frac{dy}{dx} = 3x^2 + 2bx + c$

$\frac{d^2y}{dx^2} = 6x + 2b = 0$

$$x = -\frac{2b}{6} = -\frac{b}{3} = 2$$

So $b = -6$

Crosses x-axis at $x = 3$

$f(x) = x^3 - 6x^2 + cx + d$

$f(3) = 27 - 54 + 3c + d$

(1) $3c + d = 27$

$(2, -4)$ point of inflection

$f(2) = -4$

$-4 = 8 - 24 + 2c + d$

(2) $2c + d = 12$

Subtract $c = 15 \quad \Rightarrow d = -18$

$b = -6 \quad c = 15 \quad d = -18$

12 $f(x) = ax^3 + bx^2 + cx$

$f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b$

At $x = -2$

Tangent: $y = 21x + 8$

$f''(-2) = 0 \quad \Rightarrow 2b - 12a = 0 \quad \Rightarrow b = 6a \qquad (1)$

$f'(-2) = 21 \quad \Rightarrow 21 = 12a - 4b + c \qquad (2)$

When $x = -2 \quad y = -42 + 8 = -34$

$f(-2) = -34 \quad \Rightarrow -34 = -8a + 4b - 2c \qquad (3)$

(1) $b = 6a$ into (2)

$21 = 12a - 24a + c$

$21 = -12a + c \qquad (4)$

(1) $b = 6a$ into (3)

$-34 = -8a + 24a - 2c$

$-34 = 16a - 2c \qquad (5)$

$(4) \times 2 \qquad 42 = -24a + 2c$

$(5) \qquad -34 = 16a - 2c$

Add $\quad 8 = -8a$

$a = -1 \qquad b = -6 \qquad c = 21 + 12a = 9$

13 $y = ax^4 + bx^2 + c$

TP at $(0, -8)$ inflections at $\left(\pm\sqrt{2}, 12\right)$

So when $x = 0, \ y = -8 = c$

$\frac{dy}{dx} = 4ax^3 + 2bx$

$= 2x\left(2ax^2 + b\right)$

Turning points when $x = 0$

$\frac{d^2y}{dx^2} = 12ax^2 + 2b$

Now $\frac{d^2y}{dx^2} = 0$ when $x = \sqrt{2}$

$0 = 24a + 2b \quad (1)$

Also when $x = \sqrt{2}, \quad y = 12$

$12 = 4a + 2b - 8$

$4a + 2b = 20 \quad (2)$

Subtract

$20a = -20$

$a = -1$

$b = -12a$

$b = 12$

$y = -x^4 + 12x^2 - 8$

$\frac{dy}{dx} = 2x\left(12 - 2x^2\right)$

$= 4x\left(6 - x^2\right)$

Turning points at:

$x^2 = 6 \Rightarrow x = \pm\sqrt{6}$

$y = -36 + 12 \times 6 - 8$

$= 28$

When $x = \pm\sqrt{6}, \frac{d^2y}{dx^2} = -48 < 0$

$\left(\sqrt{6}, 28\right)$ and $\left(-\sqrt{6}, 28\right)$ both maximum turning points

14 $y = x^3 - 2ax^2 + a^2x$

$= x\left(x^2 - 2ax + a^2\right)$

$= x(x - a)^2$

When $y = 0, \quad x = 0$ or $x = a$

So it crosses the x-axis at $(0, 0)$ and $(a, 0)$

$\frac{dy}{dx} = 3x^2 - 4ax + a^2$

$= (3x - a)(x - a) = 0$

When $x = a, \ y = \frac{a}{3}\left(\frac{a}{3} - a\right)^2 = \frac{4a^3}{27}$

There is a turning points at $\left(\frac{a}{3}, \frac{4a^3}{27}\right)$

$$\frac{d^2y}{dx^2} = 6x - 4a = 0$$

$$x = \frac{2a}{3} \text{ for inflection point}$$

When $x = \frac{2a}{3}$

$$y = \frac{2a}{3}\left(\frac{2a}{3} - a\right)^2 = \frac{2a^3}{27}$$

Inflection point at $\left(\frac{2a}{3}, \frac{2a^3}{37}\right)$

When $x = \frac{2a}{3}$

$$\frac{dy}{dx} = a\left(\frac{2a}{3} - a\right) = -\frac{a^2}{3}$$

Tangent: $y - \frac{2a^3}{27} = -\frac{a^2}{3}\left(x - \frac{2a}{3}\right)$

$$y = -\frac{a^2x}{3} + \frac{2a^3}{9} + \frac{2a^3}{27}$$

$$y = \frac{8a^3}{27} - \frac{a^2x}{3}$$

15 a $y = x^3 - a^2x$

$= x\left(x^2 - a^2\right)$

Crosses x-axis $x = 0, x = \pm a$

$$\frac{dy}{dx} = 3x^2 - a^2 = 0$$

$$x^2 = \frac{a^2}{3}$$

$$x = \pm\frac{a}{\sqrt{3}} = \pm\frac{\sqrt{3}a}{3}$$

When $x = \frac{\sqrt{3}a}{3}$ $y = \frac{\sqrt{3}a}{3}\left(\frac{a^2}{3} - a^2\right) = -\frac{2a^3\sqrt{3}}{9}$

$x = -\frac{\sqrt{3}a}{3}$ $y = -\frac{\sqrt{3}a}{3}\left(\frac{a^2}{3} - a^2\right) = \frac{2a^3\sqrt{3}}{9}$

$$\frac{d^2y}{dx^2} = 6x = 0$$

$\Rightarrow (0, 0)$ inflection

$\left(\frac{\sqrt{3}a}{3}, -\frac{2a^3\sqrt{3}}{9}\right), \left(-\frac{\sqrt{3}a}{3}, \frac{2a^3\sqrt{3}}{9}\right)$ TP

b $y = (x - a)^2(x - b)$ $a \neq b$

Crosses x-axis $x = a, b$

$\Rightarrow (a, 0)\ (b, 0)$

$$\frac{dy}{dx} = f'(x) = 2(x - a)(x - b) + 1 \times (x - a)^2$$

$$= (x - a)[2(x - b) + (x - a)]$$

$$= (x - a)(3x - a - 2b)$$

SP $\frac{dy}{dx} = 0$ $\Rightarrow x = a, x = \frac{a + 2b}{3}$

$$f\left(\frac{a + 2b}{3}\right) = \left(\frac{a + 2b}{3} - a\right)^2\left(\frac{a + 2b}{3} - b\right)$$

$$= \left(\frac{a + 2b - 3a}{3}\right)^2\left(\frac{a + 2b - 3b}{3}\right)$$

$$= \frac{4}{27}(a - b)^3$$

$$\frac{d^2y}{dx^2} = f''(x) = 6x - 4a - 2b = 0$$

$$x = \frac{2a + b}{3}$$

$$f\left(\frac{2a + b}{3}\right) = \left(\frac{2a + b}{3} - a\right)^2\left(\frac{2a + b}{3} - b\right)$$

$$= \left(\frac{2a + b - 3a}{3}\right)^2\left(\frac{2a + b - 3b}{3}\right)$$

$$= \frac{2}{27}(a - b)^3$$

$$f''\left(\frac{a + 2b}{3}\right) = 2(b - a)$$

$$f''(a) = 2(a - b)$$

So $(a, 0)$ $\left(\frac{a + 2b}{3}, \frac{4}{27}(a - b)^3\right)$ TP

$\left(\frac{2a + b}{3}, \frac{2}{27}(a - b)^3\right)$ inflection

16 $y = (x - a)^3(x - b)$ $a \neq b$

Crosses x-axis $x = a, b$

$\Rightarrow (a, 0)\ (b, 0)$

$$\frac{dy}{dx} = f'(x) = 3(x - a)^2(x - b) + (x - a)^3 \times 1$$

$$= (x - a)^2[3(x - b) + (x - a)]$$

$$= (x - a)^2(4x - 3b - a)$$

SP $\frac{dy}{dx} = 0$ $\Rightarrow x = a, x = \frac{a + 3b}{4}$

$$f\left(\frac{a + 3b}{4}\right) = \left(\frac{a + 3b}{4} - a\right)^3\left(\frac{a + 3b}{4} - b\right)$$

$$= \left(\frac{a + 3b - 4a}{4}\right)^3\left(\frac{a + 3b - 4b}{4}\right)$$

$$= -\frac{27}{256}(a - b)^4$$

$$\frac{d^2y}{dx^2} = f''(x) = 2(x - a)(4x - 3b - a) + (x - a)^2 \times 4 = 0$$

$$= 6(x - a)(2x - a - b) = 0$$

$$x = a, \frac{a + b}{2}$$

$$f\left(\frac{a + b}{2}\right) = \left(\frac{a + b}{2} - a\right)^3\left(\frac{a + b}{2} - b\right)$$

$$= -\frac{(a - b)^4}{16}$$

So $\left(\frac{a + 3b}{4}, -\frac{27}{256}(a - b)^4\right)$ TP

$\left(\frac{a + b}{2}, -\frac{1}{16}(a - b)^4\right)$ inflection

$(a, 0)$ SPI

6.2 Exam questions

1 $f'(x) = 2(x-3)^3 + 5$

$f''(x) = 6(x-3)^2 \geq 0$

$f(x) = \frac{1}{2}(x-3)^4 + 5x$

$f(x)$ is a quartic, and has a minimum point, has no inflection points, the second derivative does not change sign.

(All cubics have an inflection point)

The correct answer is **B**.

2 $f(a) = 1, \;\; f(-a) = -1 \;\; f(b) = -1, \;\; f(-b) = 1$

$f''(x) = \frac{(x+a)^2(x-b)}{g(x)}, \;\; g(x) < 0, \; f''(x) = 0$

$\Rightarrow x = -a, \; x = b, f(b) = -1 \;\; |f(b) = 1|$

$x = -a$ is a turning point, $(b, 1)$ is an inflection point.

The correct answer is **E**.

3 $f(x) = x^3 - mx^2 + 4$

$f'(x) = 3x^2 - 2mx$

$f''(x) = 6x - 2m$

$f''(x) \geq 0 \quad \Rightarrow \quad x \geq \frac{m}{3}$

The correct answer is **B**.

6.3 Sketching graphs of rational functions

6.3 Exercise

1 $y = \frac{16 - x^3}{4x}$

$= \frac{4}{x} - \frac{x^2}{4}$

$= 4x^{-1} - \frac{x^2}{4}$

Crosses x-axis $y = 0$

$16 - x^3 = 0$

$x^3 = 16$

$x = \sqrt[3]{16} \approx 2.52 \quad \Rightarrow (2.52, 0)$

Does not cross y-axis

Quadratic asymptote at $y = -\frac{x^2}{4}$

Vertical asymptote at $x = 0$

$\frac{dy}{dx} = -4x^{-2} - \frac{x}{2} = 0$

$x^3 = -8 \quad \text{so } x = -2$

$y = \frac{16+8}{-8} = -3$

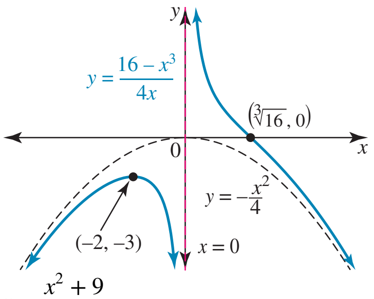
$\frac{d^2y}{dx^2} = 8x^{-3} - \frac{1}{2}$

$= \frac{8}{x^3} - \frac{1}{2}$

When $x = -2$

$\frac{d^2y}{dx^2} = -\frac{3}{2} < 0$

So $(-2, -3)$ is a max TP



2 $y = \frac{x^2 + 9}{2x}$

$= \frac{x}{2} + \frac{9}{2x}$

$= \frac{x}{2} + \frac{9}{2}x^{-1}$

Crosses x-axis $x^2 + 9 = 0 \Rightarrow x^2 = -9$ no solution

Does not cross x or y-axis

Oblique asymptote at $y = \frac{x}{2}$

Vertical asymptote at $x = 0$

$\frac{dy}{dx} = \frac{1}{2} - \frac{9}{2}x^{-2} = 0$

$\Rightarrow x^2 = 9$

$x = \pm 3$

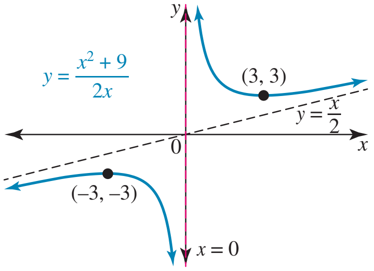
$y(3) = \frac{9+9}{6} = 3$

$y(-3) = \frac{9+9}{-6} = -3$

$\frac{d^2y}{dx^2} = 9x^{-3}$

When $x = 3, \; y'' > 0 \quad (3, 3) \quad$ minimum TP

When $x = -3, \; y'' < 0 \quad (-3, -3) \quad$ maximum TP



3 a $y = \frac{x^2 + 4}{2x}$

$= \frac{x}{2} + \frac{2}{x}$

$= \frac{x}{2} + 2x^{-1}$

x-intercepts $x^2 + 4 = 0$

Does not cross x-axis

Does not cross y-axis

Oblique asymptote $y = \frac{x}{2}$

Vertical asymptote $x = 0$

$\frac{dy}{dx} = \frac{1}{2} - 2x^{-2} = 0$

$x^2 = 4$

$x = \pm 2$

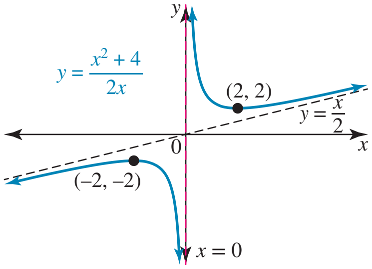
When $x = 2, y = 2$

When $x = -2, y = -2$

$$\frac{d^2y}{dx^2} = 4x^{-3} = \frac{4}{x^3}$$

$$\text{At } (2, 2) \quad y'' > 0 \text{ min}$$

$$\text{At } (-2, -2) \quad y'' < 0 \text{ local max}$$



b $y = \frac{x^3 + 16}{2x}$

$$= \frac{x^2}{2} + \frac{8}{x}$$

$$= \frac{x^2}{2} + 8x^{-1}$$

x-intercepts

$x^3 + 16 = 0$

$$x = \sqrt[3]{-16} \approx -2.52$$

$\Rightarrow (-2.52, 0)$

Does not cross y-axis

Quadratic asymptote $y = \frac{x^2}{2}$

Vertical asymptote $x = 0$

$$\frac{dy}{dx} = \frac{2x}{2} - 8x^{-2} = 0$$

$$x - \frac{8}{x^2} = 0$$

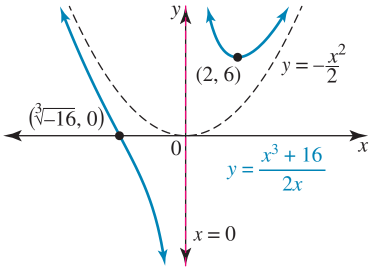
$$x^3 = 8$$

$\Rightarrow x = 2$

When $x = 2, y = 6$

$$\frac{d^2y}{dx^2} = 1 + 16x^{-3} > 0$$

$(2, 6)$ local min



4 $y = \frac{16}{16 - x^2} = 16\left(16 - x^2\right)^{-1}$

Vertical asymptotes

$16 - x^2 = 0$

$x^2 = 16$

$x = \pm 4$

Horizontal asymptote at $y = 0$

$$\frac{dy}{dx} = 2x \times 16\left(16 - x^2\right)^{-2}$$

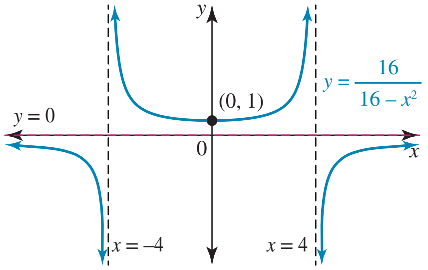
$$= \frac{32x}{\left(16 - x^2\right)^2} = 0$$

When $x = 0, y = 1$ $(0, 1)$ local minimum

y-intercept $x = 0, y = 1$ $(0, 1)$

Domain $x \in R \backslash \{\pm 4\}$

Range $(-\infty, 0) \cup [1, \infty)$



5 $y = \frac{12}{x^2 - 4x - 12}$

$$= 12\left(x^2 - 4x - 12\right)^{-1}$$

$$= \frac{12}{(x - 6)(x + 2)}$$

Vertical asymptotes $x = -2, x = 6$

Horizontal asymptote $y = 0$

$$\frac{dy}{dx} = -12(2x - 4)\left(x^2 - 4x - 12\right)^{-2}$$

$$= \frac{-12(2x - 4)}{\left(x^2 - 4x - 12\right)^2} = 0$$

$2x - 4 = 0$

$x = 2$

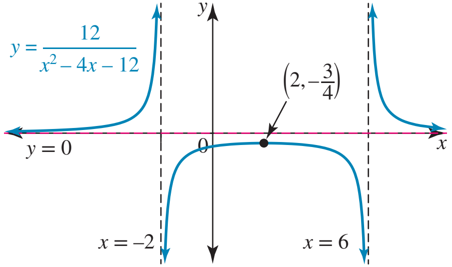
$$y(2) = \frac{12}{4 - 8 - 12} \Rightarrow \text{local max } \left(2, -\frac{3}{4}\right)$$

$$= -\frac{3}{4}$$

Since $x^2 - 4x - 12$ has a local min at $x = 2$

Domain $x \in R \backslash \{-2, 6\}$

Range $\left(-\infty, -\frac{3}{4}\right] \cup (0, \infty)$



6 **a** $y = \frac{18}{x^2 - 9} = 18\left(x^2 - 9\right)^{-1}$

Vertical asymptotes

$x^2 - 9 = 0$

$x^2 = 9$

$x = \pm 3$

Horizontal asymptote at $y = 0$

$$\frac{dy}{dx} = -36x\left(x^2 - 9\right)^{-2}$$

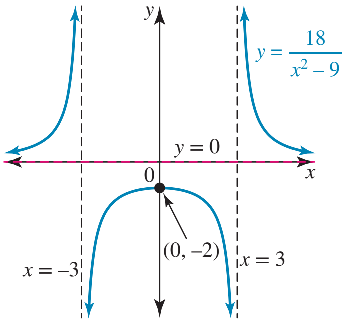
$$= \frac{-36x}{\left(x^2 - 9\right)^2} = 0$$

When $x = 0, y = -2$ $(0, -2)$ local maximum

Domain $x \in R \backslash \{\pm 3\}$

Range $(-\infty, -2] \cup (0, \infty)$

$f(-x) = f(x)$ so symmetrical about the y-axis



b $y = \frac{18}{8+2x-x^2} = 18\left(8+2x-x^2\right)^{-1}$

$= \frac{18}{-\left(x^2-2x-8\right)}$

$= \frac{-18}{(x-4)(x+2)}$

Crosses y-axis $x = 0 \Rightarrow y = \frac{9}{4} \Rightarrow \left(0, \frac{9}{4}\right)$

Vertical asymptotes

$8 + 2x - x^2 = 0$

$x = 4, -2$

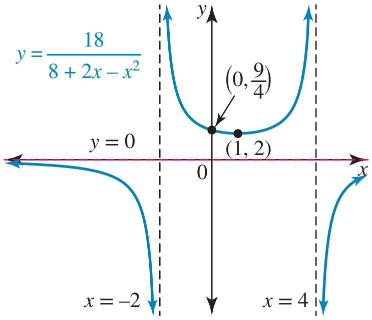
Horizontal asymptote at $y = 0$

$\frac{dy}{dx} = -18(2-2x)\left(8+2x-x^2\right)^{-1} = 0$

When $x = 1, y = 2\,(1, 2)$ local minimum

Domain $x \in R \backslash \{-2, 4\}$

Range $(-\infty, 0) \cup [2, \infty]$



7 a $y = \frac{x^3-32}{2x^2}$

$= \frac{x}{2} - 16x^{-2}$

x-intercepts

$x^3 - 32 = 0$

$x^3 = 32$

$x = \sqrt[3]{32} \approx 3.17$

$\Rightarrow (3.17, 0)$

Does not cross y-axis

Oblique asymptote $y = \frac{x}{2}$

Vertical asymptote $x = 0$

$\frac{dy}{dx} = \frac{1}{2} + 32x^{-3} = 0$

$x^3 = -64$

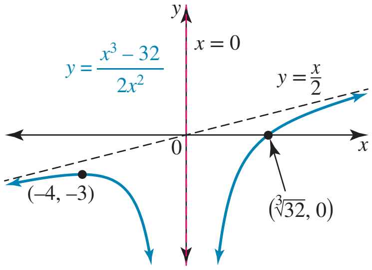
$x = \sqrt[3]{-64}$

$\Rightarrow x = -4$

When $x = -4, y = -3$

$\frac{d^2y}{dx^2} = -96x^{-4} < 0$

$(-4, -3)$ local max



b $y = \frac{-\left(x^3+4\right)}{x^2}$

$= -x - \frac{4}{x^2}$

$= -x - 4x^{-2}$

x-intercepts

$x^3 + 4 = 0$

$x^3 = -4$

$x = \sqrt[3]{-4} \approx -1.59$

$\Rightarrow (-1.59, 0)$

Does not cross y-axis

Oblique asymptote $y = -x$

Vertical asymptote $x = 0$

$\frac{dy}{dx} = -1 + 8x^{-3} = 0$

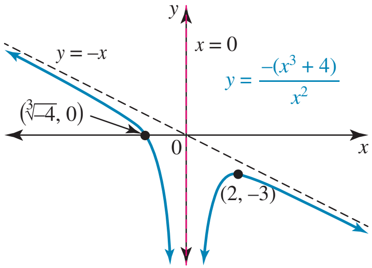
$x^3 = 8$

$\Rightarrow x = 2$

When $x = 2,\ y = -3$

$\frac{d^2y}{dx^2} = -24x^{-4} < 0$

$(2, -3)$ local max



8 a $y = \frac{x^4-81}{2x^2}$

$= \frac{x^2}{2} - \frac{81}{2x^2}$

$= \frac{x^2}{2} - \frac{81}{2}x^{-2}$

x-intercepts

$x^4 - 81 = 0$

$x^4 = 81$

$x = \pm 3$

$\Rightarrow (\pm 3, 0)$

Does not cross y-axis

Quadratic asymptote $y = \frac{x^2}{2}$

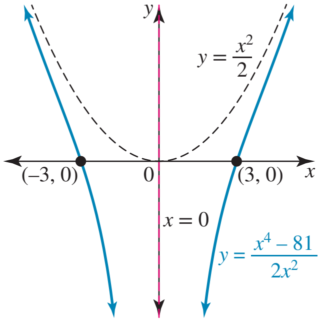
Vertical asymptote $x = 0$

$\frac{dy}{dx} = x + 81x^{-3} = 0$

$x^4 = -81$

No turning points

$f(-x) = f(x)$, symmetrical about the y-axis



b $y = \frac{-\left(x^4 + 16\right)}{2x^2}$

$= \frac{-x^2}{2} - \frac{8}{x^2}$

$= -\frac{x^2}{2} - 8x^{-2}$

x-intercepts

$-\left(x^4 + 16\right) = 0$

$x^4 = -16$

No solution

Does not cross x-axis

Does not cross y-axis

Quadratic asymptote $y = -\frac{x^2}{2}$

Vertical asymptote $x = 0$

$\frac{dy}{dx} = -x + 16x^{-3} = 0$

$x^4 = 16$

$x^2 = \pm 4$

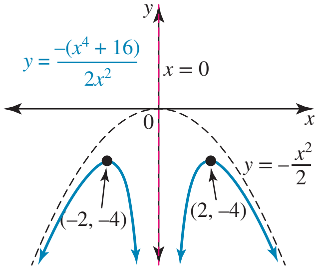
$x = \pm 2$

When $x = \pm 2,\ y = -4$

$\frac{d^2y}{dx^2} = -1 - 48x^{-4} < 0$ when $x = \pm 2$

$(2, -4)$ and $(-2, -4)$ both max

$f(-x) = f(x)$, symmetrical about the y-axis



9 $y = \frac{A}{x^2 + bx + 7}$ $\left(-4, -\frac{4}{3}\right)$ local max

$\frac{dy}{dx} = 0 \quad \Rightarrow 2x + b = 0$

$x = -\frac{b}{2} = -4$

$\Rightarrow b = 8$

$f(-4) = \frac{A}{16 - 32 + 7} = \frac{A}{-9} = -\frac{4}{3}$

$\Rightarrow A = 12$

$f(x) = \frac{12}{x^2 + 8x + 7}$

$= \frac{12}{(x + 1)(x + 7)}$

Vertical asymptotes $x = -1, x = -7$

Horizontal asymptotes $y = 0$

Domain $x \in R \setminus \{-7, -1\}$

Range $\left(-\infty, -\frac{4}{3}\right] \cup (0, \infty)$

10 $y = \frac{A}{bx + c - x^2}$ $(3, 2)$ local min

$\frac{dy}{dx} = 0 \quad \Rightarrow b - 2x = 0$

$x = \frac{b}{2} = 3$

$\Rightarrow b = 6$

$-x^2 + 6x + c = 0$ has $x = 8$ as zero

$-\left(x^2 - 6x - c\right) = 0$

$-(x - 8)(x + 2) = 0$

$\Rightarrow c = 16$

$f(3) = \frac{A}{18 + 16 - 9} = \frac{A}{25} = 2$

$\Rightarrow A = 50$

Vertical asymptotes $x = -2, x = 8$

Horizontal asymptotes $y = 0$

Domain $x \in R \setminus \{-2, 8\}$

Range $(-\infty, 0) \cup [2, \infty)$

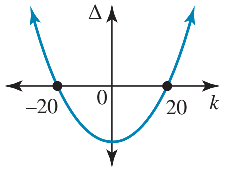
11 $y = \frac{1}{kx - 4x^2 - 25}$

$g(x) = -4x^2 + kx - 25$

$\Delta = k^2 - 4 \times -4 \times -25$

$= k^2 - 400$

$= (k + 20)(k - 20)$



a 2 vertical asymptotes $\Delta > 0$

$k > 20$ or $k < -20$ or $|k| > 20$

$(-\infty, -20) \cup (20, \infty)$

b One vertical asymptote $\Delta = 0$

$k = \pm 20$

c No vertical asymptotes $\Delta < 0$

$-20 < k < 20$

$= (-20, 20)$ or $|k| < 20$

12 $f(x) = \frac{18}{x^2 + 9}$

Crosses y-axis $x = 0,\ y = 2 \quad \Rightarrow (0, 2)$

$x^2 + 9 \neq 0$

No vertical asymptotes

Horizontal asymptote is the x-axis

$f'(x) = \frac{-36x}{\left(x^2 + 9\right)^2}$

$f'(x) = 0 \Rightarrow x = 0$

$\Rightarrow (0, 2)$

$f''(x) = \frac{-36\left(x^2 + 9\right) - 4x \times -36x}{\left(x^2 + 9\right)^3}$

$= \frac{108\left(x^2 - 3\right)}{\left(x^2 + 9\right)^3}$

$f''(0) < 0$

So $(0, 2)$ max

$$f''(x) = 0 \quad \Rightarrow x = \pm\sqrt{3}$$

$$f\left(\pm\sqrt{3}\right) = \frac{3}{2}$$

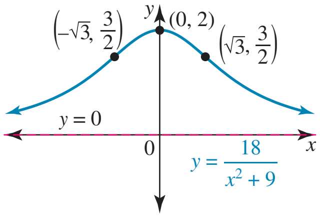
$$\Rightarrow \left(\pm\sqrt{3}, \frac{3}{2}\right) \quad \text{inflection points}$$

$$f'\left(\sqrt{3}\right) = \frac{-36\sqrt{3}}{12^2} = -\frac{\sqrt{3}}{4}$$

$$\text{Tangent: } y - \frac{3}{2} = -\frac{\sqrt{3}}{4}\left(x - \sqrt{3}\right)$$

$$y = -\frac{\sqrt{3}x}{4} + \frac{3}{4} + \frac{3}{2}$$

$$y = -\frac{\sqrt{3}x}{4} + \frac{9}{4}$$



13 $y = f(x) = \dfrac{ax^3 + b}{x^2}$

$$= ax + \frac{b}{x^2}$$

Oblique asymptote $y = ax$
Vertical asymptote $x = 0$
Crosses x-axis
$ax^3 + b = 0$

$$x^3 = -\frac{b}{a}$$

$$x = \sqrt[3]{-\frac{b}{a}}$$

$$\frac{dy}{dx} = f'(x) = a - \frac{2b}{x^3} = 0$$

$$x^3 = \frac{2b}{a}$$

$$x = \sqrt[3]{\frac{2b}{a}}$$

$$f\left(\sqrt[3]{\frac{2b}{a}}\right) = \frac{a \times \frac{2b}{a} + b}{\sqrt[3]{\left(\frac{2b}{a}\right)^2}} = \frac{3}{2}\sqrt[3]{2a^2b}$$

So crosses x-axis at $\sqrt[3]{-\frac{b}{a}}$

Turning point at $\left(\sqrt[3]{\frac{2b}{a}}, \frac{3}{2}\sqrt[3]{2a^2b}\right)$

14 $y = f(x) = \dfrac{ax^3 + b}{x}$

$$= ax^2 + \frac{b}{x}$$

Quadratic asymptote $y = ax^2$
Vertical asymptote $x = 0$
Crosses x-axis
$ax^3 + b = 0$

$$x^3 = -\frac{b}{a}$$

$$x = \sqrt[3]{-\frac{b}{a}}$$

$$\frac{dy}{dx} = f'(x) = 2ax - \frac{b}{x^2} = 0$$

$$x^3 = \frac{b}{2a}$$

$$x = \sqrt[3]{\frac{b}{2a}}$$

$$f\left(\sqrt[3]{\frac{b}{2a}}\right) = \frac{\left(\frac{b}{2} + b\right)}{\sqrt[3]{\frac{b}{2a}}} = \frac{3b}{2} \times 2^{\frac{1}{3}}a^{\frac{1}{3}}b^{-\frac{1}{3}}$$

$$= \frac{3}{2}\sqrt[3]{2ab^2}$$

So crosses x-axis at $\sqrt[3]{-\frac{b}{a}}$

Turning point at $\left(\sqrt[3]{\frac{b}{2a}}, \frac{3}{2}\sqrt[3]{2ab^2}\right)$

15 $y = f(x) = \dfrac{ax^4 + b}{x^2}$

$$= ax^2 + \frac{b}{x^2}$$

Quadratic asymptote $y = ax^2$
Vertical asymptote $x = 0$
Crosses x-axis
$ax^4 + b = 0$

$$x^4 = -\frac{b}{a} \quad \text{if } ab > 0$$

Does not cross the x-axis

$$\frac{dy}{dx} = f'(x) = 2ax - \frac{2b}{x^3} = 0$$

$$x^4 = \frac{b}{a}$$

$$x = \pm\sqrt[4]{\frac{b}{a}}$$

If $ab > 0$ turning point $\left(\pm\sqrt[4]{\frac{b}{a}}, 2\sqrt{ab}\right)$ and no x-intercepts

If $ab < 0$ no turning points, crosses x-axis at $x = \pm\sqrt[4]{-\frac{b}{a}}$

16 $y = f(x) = \dfrac{ax^2 + b}{x}$

$$= ax + \frac{b}{x}$$

Oblique asymptote $y = ax$
Vertical asymptote $x = 0$

$$\frac{dy}{dx} = f'(x) = a - \frac{b}{x^2} = 0$$

$$x^2 = \frac{b}{a}$$

$$x = \pm\sqrt{\frac{b}{a}}$$

If $ab > 0 \Rightarrow$ T.P

$$f\left(\sqrt{\frac{b}{a}}\right) = \frac{2b}{\sqrt{\frac{b}{a}}} = 2\sqrt{ab}$$

Crosses x-axis

$ax^2 + b = 0$

$x^2 = -\frac{b}{a}$ if $ab < 0$

$x = \pm\sqrt{-\frac{b}{a}}$

So if $ab > 0$ two turning points at $\left(\pm\sqrt{\frac{b}{a}}, \pm 2\sqrt{ab}\right)$ and no x-intercepts

If $ab < 0$ no turning points but two x-intercepts at

$x = \pm\sqrt{\frac{-b}{a}}$

17 **a** $y = \frac{x^2 + 5x + 4}{x} = x + 5 + \frac{4}{x}$

$(-1, 0)$ $(-4, 0)$

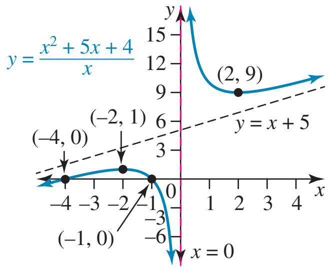
Does not cross the y-axis

$y = x + 5$ oblique asymptote

$x = 0$ vertical asymptote

$(2, 9)$ local min

$(-2, 1)$ local max



b $y = \frac{2x^2 + x - 6}{x} = 2x + 1 - \frac{6}{x}$

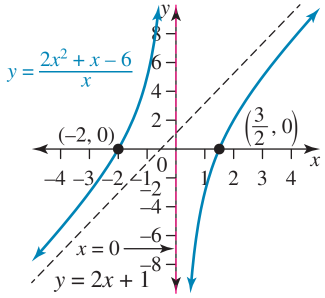
$(-2, 0)$ $\left(\frac{3}{2}, 0\right)$

Does not cross the y-axis

$y = 2x + 1$ oblique asymptote

$x = 0$ vertical asymptote

No turning points



c $y = \frac{5x - 6 - x^2}{x} = 5 - x - \frac{6}{x}$

Crosses the x-axis $(2, 0)$ $(3, 0)$

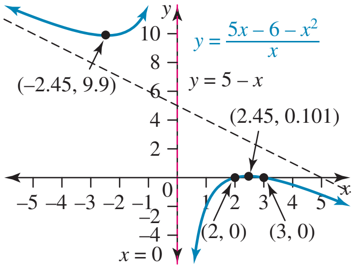
Does not cross the y-axis

$y = 5 - x$ oblique asymptote

$x = 0$ vertical asymptote

$(2.45, 0.101)$ local max

$(-2.45, 9.9)$ local min



18 $f(x) = \frac{ax^2 + bx + c}{x} = ax + b + \frac{c}{x} = ax + b + cx^{-1}$

$x = 0$ is a vertical asymptote

$y = ax + b$ is an oblique asymptote

$f'(x) = a - \frac{c}{x^2} = 0, \quad a = \frac{c}{x^2}, \quad x^2 = \frac{c}{a}$

So if $ac > 0$ then there are turning points at $x = \pm\sqrt{\frac{c}{a}}$

$f\left(\sqrt{\frac{c}{a}}\right) = a\sqrt{\frac{c}{a}} + b + c\sqrt{\frac{a}{c}} = b + 2\sqrt{ac}$

$f\left(-\sqrt{\frac{c}{a}}\right) = -a\sqrt{\frac{c}{a}} + b - c\sqrt{\frac{a}{c}} = b - 2\sqrt{ac}$

Turning points at $\left(\sqrt{\frac{c}{a}}, b + 2\sqrt{ac}\right)$ and

$\left(-\sqrt{\frac{c}{a}}, b - 2\sqrt{ac}\right)$

So if $ac < 0$ then there are no turning points

$f''(x) = \frac{2c}{x^3} \neq 0$ if $c \neq 0$ so there are no inflection points

19 **a** $y = \frac{x + 2}{x^2 - 16} = \frac{x + 2}{(x - 4)(x + 4)}$

Crosses the x-axis $(-2, 0)$

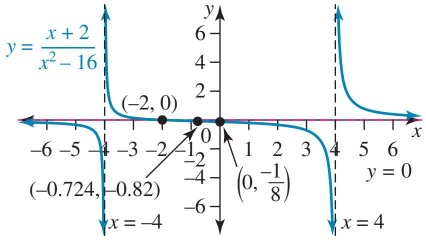
Cross the y-axis $\left(0, -\frac{1}{8}\right)$

$y = 0$ horizontal asymptote

$x = \pm 4$ vertical asymptote

No turning points

$(-0.724, -0.082)$ point of inflection



b $y = \frac{x^2 - 16}{x + 2} = x - 2 - \frac{12}{x + 2}$

Crosses the x-axis $(\pm 4, 0)$

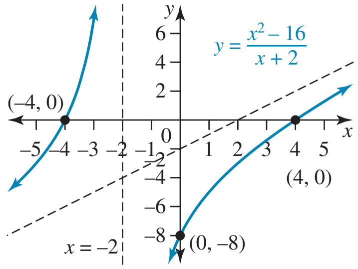
Crosses the y-axis $(0, -8)$

$y = x - 2$ oblique asymptote

$x = -2$ vertical asymptote

No turning points

No points of inflection



20 $y = \dfrac{x+4}{x^2-16} = \dfrac{1}{x-4},\ x \neq -4$

Does not cross the x-axis

Crosses the y-axis $\left(0,\ -\dfrac{1}{4}\right)$

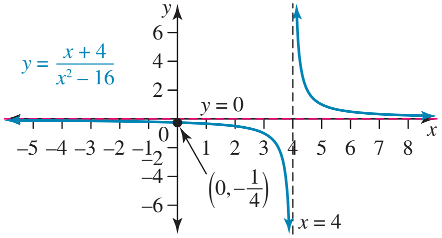
Point of discontinuity at $\left(-4,\ -\dfrac{1}{8}\right)$

$y = 0$ horizontal asymptote

$x = -4$ vertical asymptote

No turning points

No points of inflection



21 $f(x) = \dfrac{x^2-2x-8}{x^2-x-6} = \dfrac{(x-4)(x+2)}{(x-3)(x+2)}$

$f(x) = \dfrac{x-4}{x-3} = \dfrac{x-3-1}{x-3} = 1 - \dfrac{1}{x-3},\quad x \neq -2$

$\left(-2,\ \dfrac{6}{5}\right)$ point of discontinuity

$x = 3$ and $y = 1$ are asymptotes

22 $f(x) = \dfrac{x^3-6x^2+9x}{x^3-9x}$

$f(x) = \dfrac{x\left(x^2-6x+9\right)}{x\left(x^2-9\right)} = \dfrac{x(x-3)^2}{x(x+3)(x-3)}$

$f(x) = \dfrac{x-3}{x+3},\ \ x \neq 0, 3$

$(0, -1),\ \ (3, 0)$ are both points of discontinuity

$x = -3$ and $y = 1$ are asymptotes

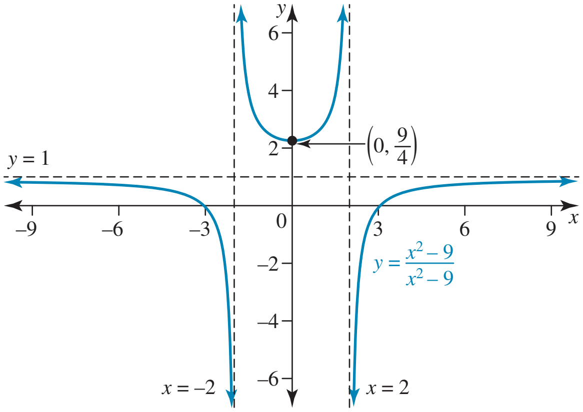
23 a $y = \dfrac{x^2-9}{x^2-4} = \dfrac{(x-3)(x+3)}{(x-2)(x+2)} = 1 - \dfrac{5}{x^2-4}$

$\dfrac{dy}{dx} = \dfrac{10x}{\left(x^2-4\right)^2},\quad \dfrac{d^2y}{dx^2} = \dfrac{-10\left(3x^2+4\right)}{\left(x^2-4\right)^3}$

Crosses x-axis $(\pm 3, 0)$ crosses y-axis $\left(0,\ \dfrac{9}{4}\right)$ local min

Asymptotes $x = \pm 2$, $y = 1$

No inflection points



b $y = \dfrac{x^2+9}{x^2-4} = \dfrac{x^2+9}{(x-2)(x+2)} = 1 + \dfrac{13}{x^2-4}$

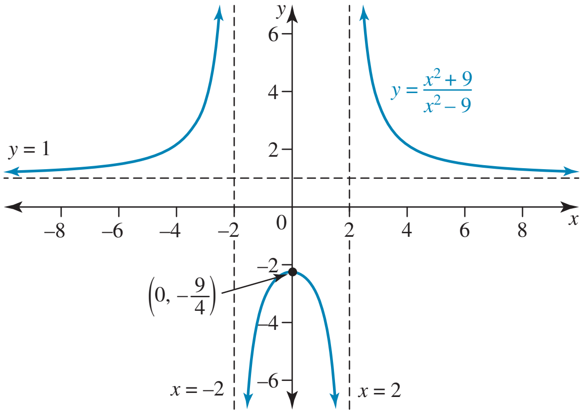
$\dfrac{dy}{dx} = \dfrac{-26x}{\left(x^2-4\right)^2},\quad \dfrac{d^2y}{dx^2} = \dfrac{26\left(3x^2+4\right)}{\left(x^2-4\right)^3}$

Doesn't cross x-axis

Crosses y-axis $\left(0,\ -\dfrac{9}{4}\right)$ local max

Asymptotes $x = \pm 2$, $y = 1$

No inflection points



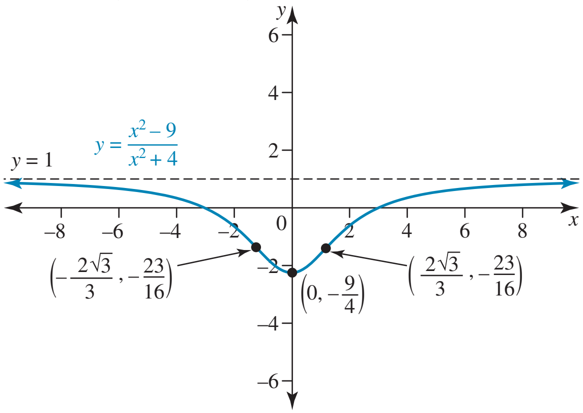
c $y = \dfrac{x^2-9}{x^2+4} = \dfrac{(x-3)(x+3)}{x^2+4} = 1 - \dfrac{13}{x^2+4}$

$\dfrac{dy}{dx} = \dfrac{26x}{\left(x^2+4\right)^2},\quad \dfrac{d^2y}{dx^2} = \dfrac{-26\left(3x^2-4\right)}{\left(x^2+4\right)^3}$

Crosses x-axis $(\pm 3, 0)$ crosses y-axis $\left(0,\ -\dfrac{9}{4}\right)$ local min

Asymptote $y = 1$

Inflection $\left(\pm\dfrac{2\sqrt{3}}{3}, -\dfrac{23}{16}\right)$



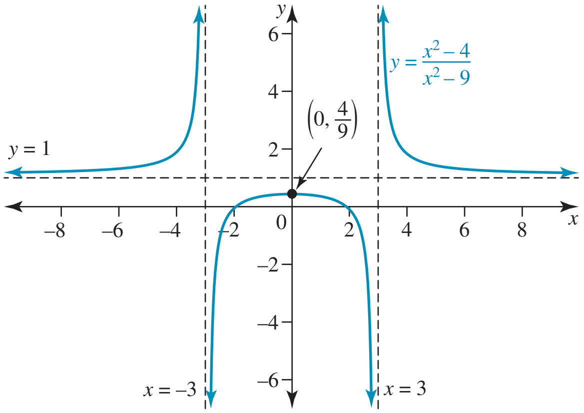
24 a $y = \dfrac{x^2-4}{x^2-9} = \dfrac{(x-2)(x+2)}{(x-3)(x+3)} = 1 + \dfrac{5}{x^2-9}$

$\dfrac{dy}{dx} = \dfrac{-10x}{\left(x^2-9\right)^2},\quad \dfrac{d^2y}{dx^2} = \dfrac{30\left(x^2+3\right)}{\left(x^2-9\right)^3}$

Crosses x-axis $(\pm 2, 0)$, crosses y-axis $\left(0,\ \dfrac{4}{9}\right)$ local max

Asymptotes $x = \pm 3$, $y = 1$

No inflection points



b $y = \dfrac{x^2+4}{x^2-9} = \dfrac{x^2+4}{(x-3)(x+3)} = 1 + \dfrac{13}{x^2-9}$

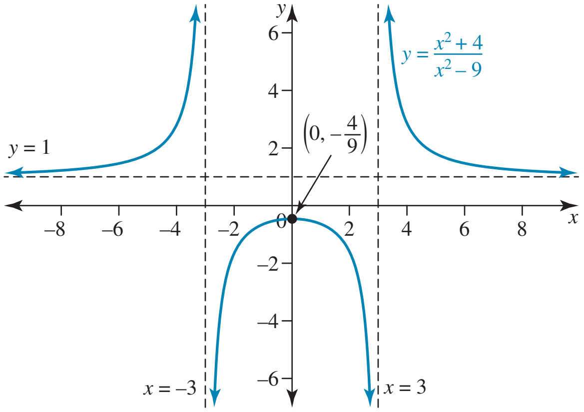
$\dfrac{dy}{dx} = \dfrac{-26x}{\left(x^2-9\right)^2}, \quad \dfrac{d^2y}{dx^2} = \dfrac{78\left(x^2+3\right)}{\left(x^2-9\right)^3}$

Doesn't cross x-axis

Crosses y-axis $\left(0, -\dfrac{4}{9}\right)$ local max

Asymptotes $x = \pm 3$, $y = 1$

No inflection points



c $y = \dfrac{x^2+4}{x^2+9} = 1 - \dfrac{5}{x^2+9}$

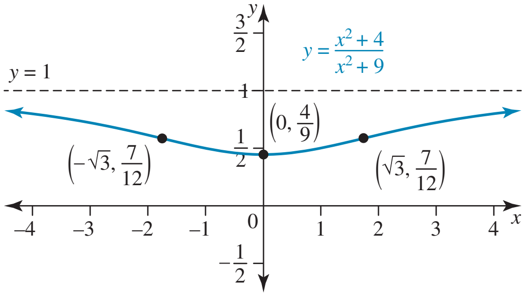
$\dfrac{dy}{dx} = \dfrac{10x}{\left(x^2+9\right)^2}, \quad \dfrac{d^2y}{dx^2} = \dfrac{-30\left(x^2-3\right)}{\left(x^2+9\right)^3}$

Doesn't cross x-axis

Crosses y-axis $\left(0, \dfrac{4}{9}\right)$ local min

Asymptote $y = 1$

Inflection $\left(\pm\sqrt{3}, \dfrac{7}{12}\right)$



25 a $y = \dfrac{x^2+4}{x^2-5x+4} = 1 + \dfrac{5x}{(x-4)(x-1)}$

$\dfrac{dy}{dx} = \dfrac{-5(x+2)(x-2)}{\left(x^2-5x+4\right)^2}, \quad \dfrac{d^2y}{dx^2} = \dfrac{10\left(x^3-12x+20\right)}{\left(x^2-5x+4\right)^3}$

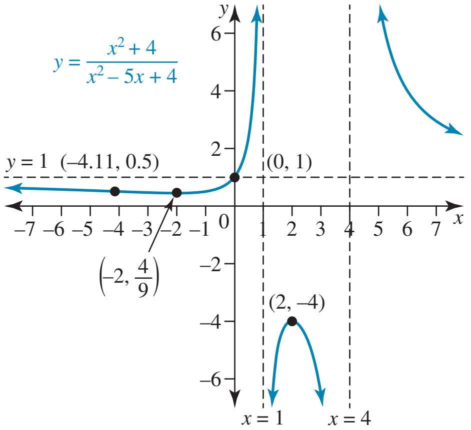
Doesn't cross x-axis

Crosses y-axis and asymptote at $(0, 1)$

$\left(-2, \dfrac{4}{9}\right)$ local min, $(2, -4)$ local max

Asymptotes $x = 1$, $x = 4$, $y = 1$

Inflection $(-4.11, 0.5)$



b $y = \dfrac{2x^2+2x+3}{2x^2-2x+5} = 1 + \dfrac{2(2x-1)}{2x^2-2x+5}$

$\dfrac{dy}{dx} = \dfrac{-8(x+1)(x-2)}{\left(2x^2-2x+5\right)^2},$

$\dfrac{d^2y}{dx^2} = \dfrac{8(2x-1)\left(2x^2-2x-13\right)}{\left(2x^2-2x+5\right)^3}$

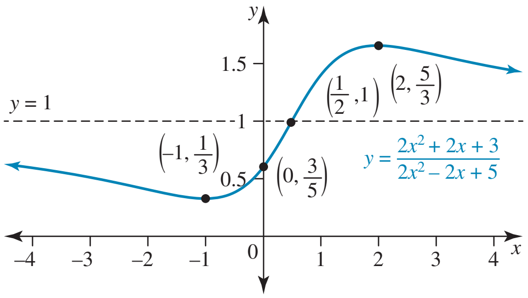
Doesn't cross x-axis

Cross y-axis $\left(0, \dfrac{3}{5}\right)$

$\left(-1, \dfrac{1}{3}\right)$ min, $\left(2, \dfrac{5}{3}\right)$ max

Asymptote $y = 1$ crosses at $x = \dfrac{1}{2}$

Inflection $(-2.1, 0.43)$, $\left(\dfrac{1}{2}, 1\right)$, $(3.1, 1.58)$



26 a Numerator has two solutions $b^2 > 4c$

b Numerator has no solutions $b^2 < 4c$

c Denominator has two solutions $p^2 > 4q$

d Denominator has no solutions $p^2 < 4q$

e Inflection points when no vertical asymptotes, denominator has no solutions $p^2 < 4q$

f $y = \dfrac{x^2 + bx + c}{x^2 + px + q} = 1 + \dfrac{(b - p)x + c - q}{x^2 + px + q}$ crosses horizontal asymptote at $y = 1$ when $(b - p)x + c - q = 0$ at $x = \dfrac{q - c}{b - p}$

6.3 Exam questions

1 $f(x) = \dfrac{x + 1}{x^2 - 4}$

Vertical asymptotes at $x = \pm 2$

Horizontal asymptote at $y = 0$

Crosses the x-axis (horizontal asymptote) at $x = -1, (-1, 0)$

Crosses the y-axis at $y = -\dfrac{1}{4}\left(0, -\dfrac{1}{4}\right)$

$x > 2, f(x) > 0$

$x < -2, f(x) < 0$

See graph at the bottom of the page*

Award 1 mark for correct vertical asymptotes.

Award 1 mark for correct horizontal asymptotes.

Award 1 mark for correct axial intercepts.

Award 1 mark for correct shape and horizontal asymptote intercept.

***VCAA Examination Report* note:**

Most students realised that $x = -2$ and $x = 2$ were vertical asymptotes, although the horizontal asymptote $y = 0$ was often not stated. Students who found the axis intercepts were not always able to position them correctly on the axes. Some students showed a stationary point of inflection on their graph or were missing the outer branches.

2 $f(x) = \dfrac{x^3 - ax}{x^2} = x - \dfrac{a}{x}$

$x = 0$ is a vertical asymptote

$y = x$ is an oblique asymptote

The correct answer is **C**.

3 a $f(x) = \dfrac{4 + x^2 + x^3}{x}, \ x \in R \backslash \{0\}$

$f'(x) = \dfrac{2x^3 + x^2 - 4}{x^2}$

For stationary points, solving $f'(x) = 0$ gives

$x = 1.1134, \ f(1.1134) = 5.946$

$(1.11, 5.95)$

Award 1 mark for the correct coordinates of a stationary point.

***VCAA Examination Report* note:**

This question was answered very well. A small number of students gave the coordinates for the point of inflection rather than the stationary point.

b $f''(x) = \dfrac{2(x^3 + 4)}{x^3}$

For inflection points, $f''(x) = 0$:

$x = \sqrt[3]{-4} = -1.587, \ f(-1.587) = -1.587$

$(-1.59, -1.59)$

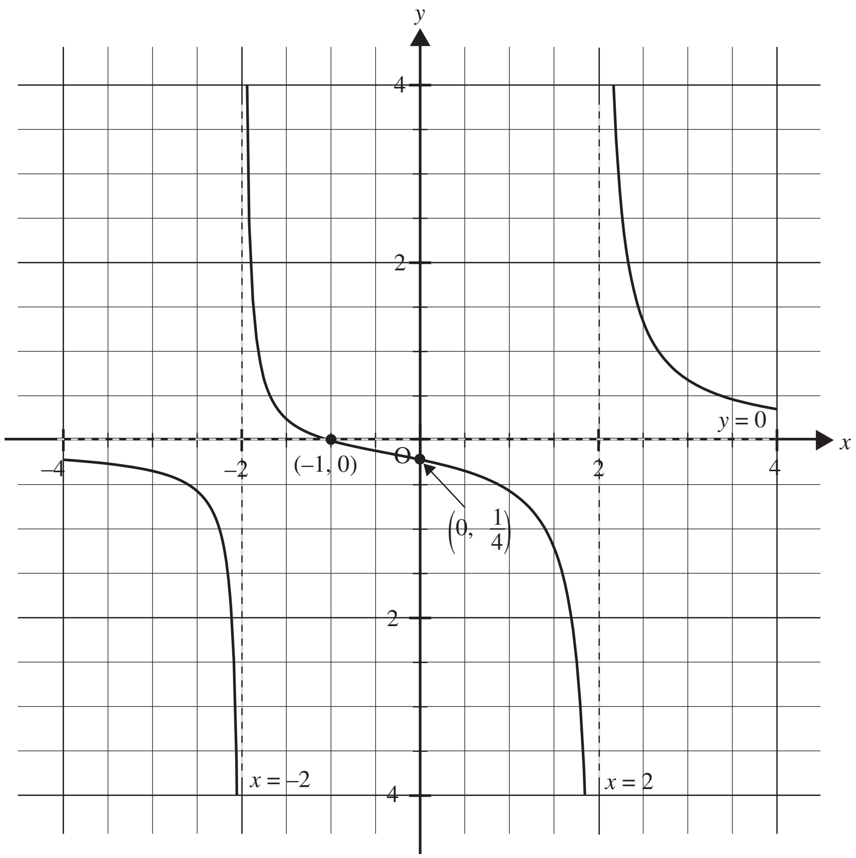
Award 1 mark for setting the second derivative to zero.

Award 1 mark for the correct coordinates of the inflection point.

***VCAA Assessment Report* note:**

This question was generally answered well. The most common error resulted from substituting a rounded x value, yielding an incorrect y value of -1.58. Some students left off the negative sign of the y value.

***1**



c See graph at the bottom of the page*
Award 1 mark for the correct graph on the domain.
Award 1 mark for the stationary point and inflection point.
Award 1 mark for the endpoints and vertical asymptote.

6.4 Sketching graphs of product and quotient functions

6.4 Exercise

1 $f(x) = \dfrac{x^4}{e^{2x}} = x^4e^{-2x}$

$f'(x) = x^4\dfrac{d}{dx}\left(e^{-2x}\right) + e^{-2x}\dfrac{d}{dx}\left(x^4\right)$

$= -2x^4e^{-2x} + 4x^3e^{-2x}$

$= 2x^3e^{-2x}(2 - x)$

$f''(x) = -2x^4\dfrac{d}{dx}(e^{-2x}) - e^{-2x}\dfrac{d}{dx}(2x^4)$

$+e^{-2x}\dfrac{d}{dx}(4x^3) + 4x^3\dfrac{d}{dx}(e^{-2x})$

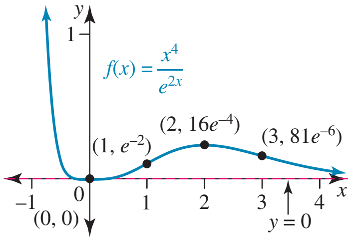
$= 4x^4e^{-2x} - 8x^3e^{-2x} + 12x^2e^{-2x} - 8x^3e^{-2x}$

$= (4x^4 - 16x^3 + 12x^2)e^{-2x}$

$= 4x^2(x^2 - 4x + 3)e^{-2x}$

$= 4x^2(x - 3)(x - 1)e^{-2x}$

For inflection $f''(x) = 0$ so $x = 0, 1, 3$
For stationary points $f'(x) = 0 \quad x = 0, 2$
So $(0, 0)$ is an absolute minimum
Local maximum $(2, 16e^{-4})$
Inflection points $(1, e^{-2})(3, 81e^{-6})$
As $x \to \infty \quad y \to 0$
$y = 0$ is a horizontal asymptote



2 $f(x) = \dfrac{3}{\pi}\sin^{-1}\left(\dfrac{3x - 2}{4}\right)$

Domain: $\left|\dfrac{3x - 2}{4}\right| \le 1$

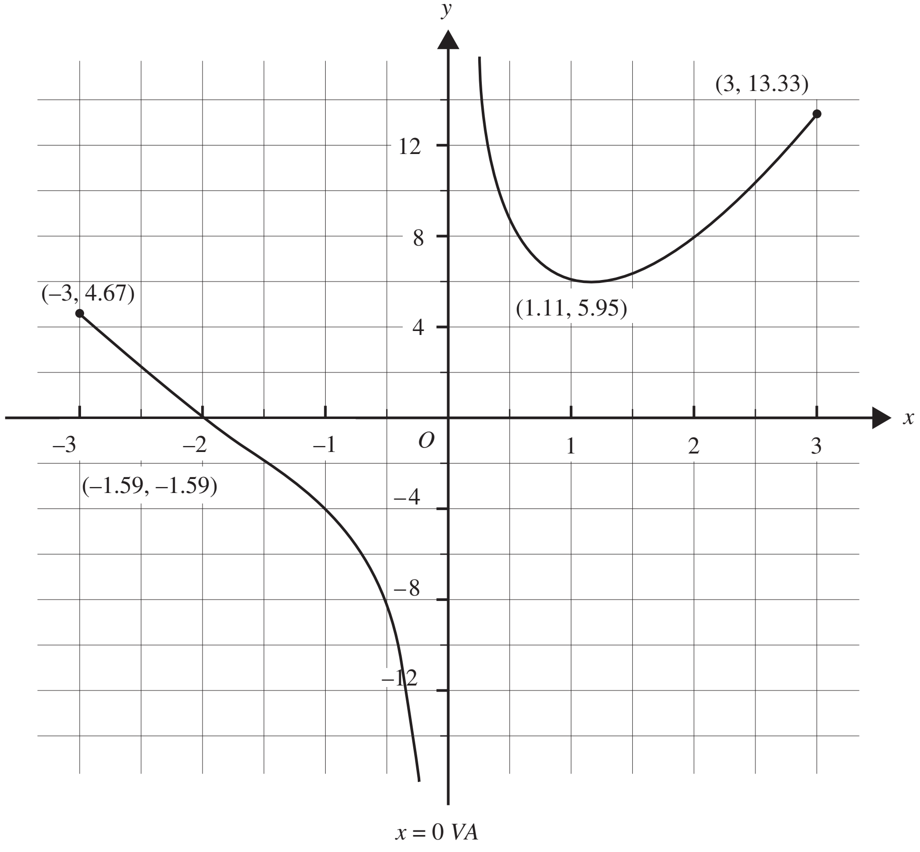
$-4 \le 3x - 2 \le 4$

$-2 \le 3x \le 6$

$-\dfrac{2}{3} \le x \le 2$, domain $\left[-\dfrac{2}{3}, 2\right]$

Endpoints: $f(2) = \dfrac{3}{\pi}\sin^{-1}(1) = \dfrac{3}{\pi} \times \dfrac{\pi}{2} = \dfrac{3}{2}$

***3c**



$f\left(-\frac{2}{3}\right) = \frac{3}{\pi}\sin^{-1}(-1) = \frac{3}{\pi} \times -\frac{\pi}{2} = -\frac{3}{2}$

$\left(2, \frac{3}{2}\right) \quad \left(-\frac{2}{3}, -\frac{3}{2}\right)$

$y = f(x) = \frac{3}{\pi}\sin^{-1}\left(\frac{u}{4}\right), \; u = 3x - 2$

$\frac{dy}{du} = \frac{3}{\pi\sqrt{16 - u^2}} \qquad \frac{du}{dx} = 3$

$$\begin{aligned} f'(x) &= \frac{9}{\pi\sqrt{16 - (3x-2)^2}} \\ &= \frac{9}{\pi\sqrt{16 - (9x^2 - 12x + 4)}} \\ &= \frac{9}{\pi\sqrt{-3(3x^2 - 4x - 4)}} \\ &= \frac{3\sqrt{3}}{\pi\sqrt{4 + 4x - 3x^2}} \\ &= \frac{3\sqrt{3}}{\pi\sqrt{(2-x)(3x+2)}}, \quad -\frac{2}{3} < x < 2 \end{aligned}$$

$f'(x) \neq 0$ so no stationary points

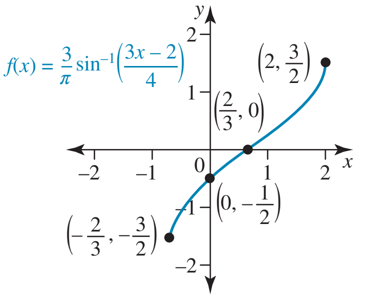
$f'(x) = \frac{3\sqrt{3}}{\pi}(4 + 4x - 3x^2)^{\frac{-1}{2}}$

$f'(x) = \frac{-3\sqrt{3}(4 - 6x)}{2\pi\sqrt{(4 - 3x - x^2)^3}} = 0$

$4 - 6x = 0$

$x = \frac{2}{3} \quad f\left(\frac{2}{3}\right) = 0$

So $\left(\frac{2}{3}, 0\right)$ is an inflection point



3 $f(x) = \frac{2}{\pi}\cos^{-1}\left(\frac{5 - 2x}{4}\right)$

Domain: $\left|\frac{5 - 2x}{4}\right| \leq 1$

$-4 \leq 5 - 2x \leq 4$

$-9 \leq -2x \leq -1$

$1 \leq 2x \leq 9$

$\frac{1}{2} \leq x \leq \frac{9}{2}, \quad \text{domain}: \left[\frac{1}{2}, \frac{9}{2}\right]$

Endpoints: $f\left(\frac{9}{2}\right) = \frac{2}{\pi}\cos^{-1}(-1) = 2$

$f\left(\frac{1}{2}\right) = \frac{2}{\pi}\cos^{-1}(1) = 0$

$\left(\frac{9}{2}, 2\right) \quad \left(\frac{1}{2}, 0\right)$

$y = f(x) = \frac{2}{\pi}\cos^{-1}\left(\frac{u}{4}\right) \quad u = 5 - 2x$

$\frac{dy}{du} = \frac{-2}{\pi\sqrt{16 - u^2}} \qquad \frac{du}{dx} = -2$

$$\begin{aligned} f'(x) &= \frac{4}{\pi\sqrt{16 - (5 - 2x)^2}} \\ &= \frac{4}{\pi\sqrt{16 - (25 - 20x + 4x^2)}} \\ &= \frac{4}{\pi\sqrt{-4x^2 + 20x - 9}} \\ &= \frac{4}{\pi\sqrt{(1 - 2x)(2x - 9)}} \quad \frac{1}{2} < x < \frac{9}{2} \end{aligned}$$

$f'(x) \neq 0$ so no stationary point

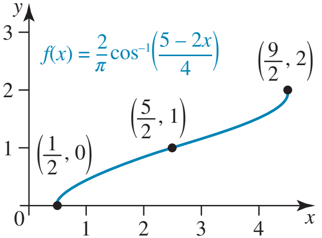
$f'(x) = \frac{4}{\pi}(-4x^2 + 20x - 9)^{\frac{-1}{2}}$

$f''(x) = \frac{-2}{\pi}\frac{(-8x + 20)}{\sqrt{(-4x^2 + 20x - 9)^3}} = 0$

$-8x + 20 = 0$

$x = \frac{5}{2} \quad f\left(\frac{5}{2}\right) = \frac{2}{\pi}\cos^{-1}(0) = \frac{2}{\pi} \times \frac{\pi}{2} = 1$

$\left(\frac{5}{2}, 1\right)$ is an inflection point



4 $f(x) = \frac{4}{\pi}\tan^{-1}(x - 1)$

Domain R

Range $(-2, 2)$

Crosses x-axis $y = 0, x - 1 = 0, \; x = 1, \; (1, 0)$

Crosses y-axis $x = 0, y = \frac{4}{\pi}\tan^{-1}(-1) = \frac{4}{\pi} \times \frac{-\pi}{4} = -1$

$(0, -1)$

$y = f(x) = \frac{4}{\pi}\tan^{-1}(u) \quad u = x - 1$

$\frac{dy}{du} = \frac{4}{\pi(1 + u^2)} \qquad \frac{du}{dx} = 1$

$$\begin{aligned} f'(x) &= \frac{4}{\pi(1 + (x - 1)^2} \\ &= \frac{4}{\pi(x^2 - 2x + 2)} \end{aligned}$$

$f'(x) \neq 0$ no stationary points

$f'(x) = \frac{4}{\pi}(x^2 - 2x + 2)^{-1}$

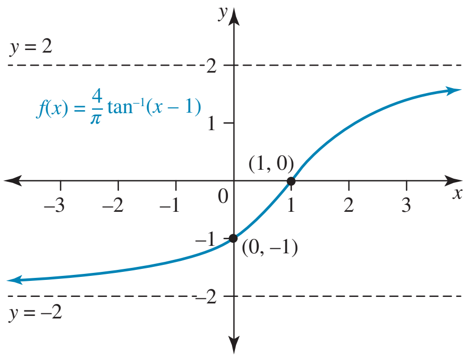
$f''(x) = \frac{-4(2x - 2)}{\pi(x^2 - 2x + 2)^2} = 0$

$2x - 2 = 0$

$x = 1, f(1) = \frac{4}{\pi}\tan^{-1}(0) = 0$

$(1, 0)$ is an inflection point

$y = \pm 2$ horizontal asymptotes



5 $f(x) = x^2 \log_e(x)$

The maximal domain is $x > 0$

The curve $y = x^2$ is a quadratic asymptote

Crosses x-axis $x = 1$ $(1, 0)$

$$\frac{dy}{dx} = f'(x) = 2x\log_e(x) + x = 0 \text{ for SP}$$

$$x = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}},\ f\left(\frac{1}{\sqrt{e}}\right) = -\frac{1}{2e}$$

So $\left(\frac{1}{\sqrt{e}}, -\frac{1}{2e}\right)$ or $(0.61, -0.18)$ minimum TP

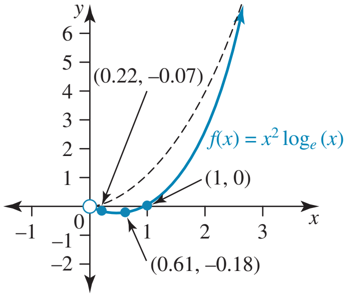
Range $\left[-\frac{1}{2e}, \infty\right)$

$$\frac{d^2y}{dx^2} = f''(x) = 2\log_e(x) + 3 = 0 \text{ for inflection}$$

$$x = e^{-\frac{3}{2}},\ f\left(e^{-\frac{3}{2}}\right) = \frac{-3e^{-3}}{2}$$

$\left(\frac{1}{\sqrt{e^3}}, \frac{-3}{2e^3}\right)$ or $(0.22, -0.07)$

is an inflection point.



6 $f(x) = \frac{e^{-2x}}{x}$

The maximal domain is $R\backslash\{0\}$

The graph does not cross x or y-axis

$x = 0$ is a verified asymptote

$\lim_{x\to\infty} f(x) = 0$ so $y = 0$ is a horizontal asymptote

$$f'(x) = -\frac{(2x+1)e^{-2x}}{x^2} = 0 \text{ for SP}$$

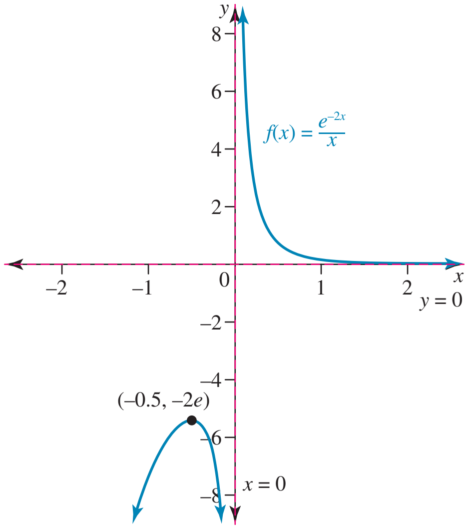
$$x = -\frac{1}{2}\quad f\left(-\frac{1}{2}\right) = -2e$$

$(-0.5, -5.44)$ is a local max TP

$$f''(x) = \frac{2(2x^2 + 2x + 1)e^{-2x}}{x^3} \neq 0$$

No inflection points

Range $(-\infty, -2e] \cup (0, \infty)$



7 a $f(x) = \frac{\sin^{-1}(x)}{x}$

Domain: $[-1, 0) \cup (0, 1]$

Endpoints: $\left(-1, \frac{\pi}{2}\right)$ $\left(1, \frac{\pi}{2}\right)$

$x = 0$ is a point of discontinuity

No asymptotes

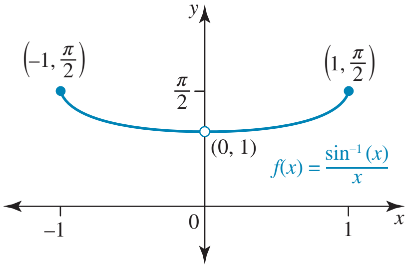
$$f'(x) = \frac{1}{x\sqrt{1-x^2}} - \frac{\sin^{-1}(x)}{x} \neq 0$$

$$f''(x) = \frac{2\sin^{-1}(x)}{x^3} - \frac{2}{x^2\sqrt{1-x^2}} + \frac{1}{\sqrt{(1-x^2)^3}} \neq 0$$

No turning points

No inflection points

Range $\left(1, \frac{\pi}{2}\right]$



b $f(x) = \frac{1}{\sin^{-1}\left(\frac{x}{2}\right)}$

For $\sin^{-1}\left(\frac{x}{2}\right)$, the domain is $[-2, 2]$

For $\frac{1}{\sin^{-1}\left(\frac{x}{2}\right)}$ the domain is $[-2, 0) \cup (0, 2]$

$\sin^{-1}\left(\frac{x}{2}\right) = 0$ when $x = 0$

So $x = 0$ is a vertical asymptote

$$f(2) = \frac{2}{\pi},\quad f(-2) = -\frac{2}{\pi}$$

Endpoints: $\left(2, \frac{2}{\pi}\right)$ $\left(-2, -\frac{2}{\pi}\right)$

Range $\left(-\infty, -\frac{2}{\pi}\right] \cup \left[\frac{2}{\pi}, \infty\right)$

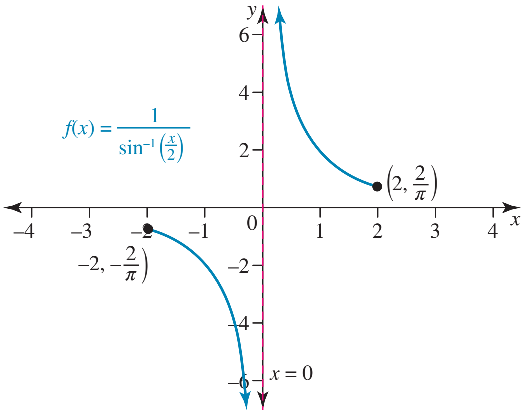
Does not cross x or y-axis

$$f'(x) = \frac{-1}{\sqrt{(x+2)(2-x)}\left(\sin^{-1}\left(\frac{x}{2}\right)\right)^2}$$

$f''(x) \neq 0$

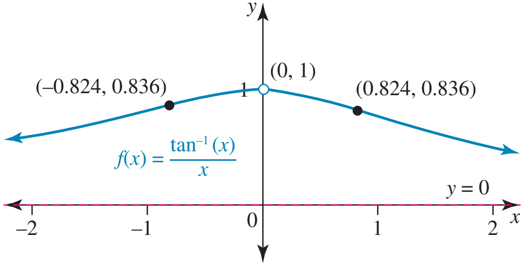
No stationary points

No inflection points



8 $f(x)=\dfrac{\tan^{-1}(x)}{x}$

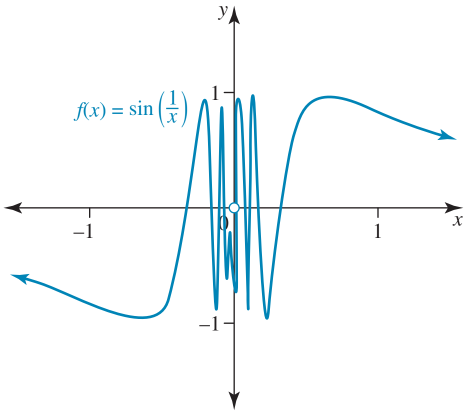
Now $\tan^{-1}(x)=0$ when $x=0$
but $f(0)$ does not exist
$\lim\limits_{x\to0}\left(\dfrac{\tan^{-1}(x)}{x}\right)=1$
So the point (0, 1) is a point of discontinuity
$f'(x)=\dfrac{1}{x(x^2+1)}-\dfrac{\tan^{-1}(x)}{x^2}\neq0$
$f''(x)=\dfrac{2\tan^{-1}(x)}{x^3}-\dfrac{2(2x^2+1)}{x^2(x^2+1)^2}=0$
When $x=\pm0.824$
$f(\pm0.824)=0.836$
$(\pm0.824, 0.836)$ inflection points
Domain $R\backslash\{0\}$
Range $(0, 1)$



9 a $f(x)=\sin\left(\dfrac{1}{x}\right)$

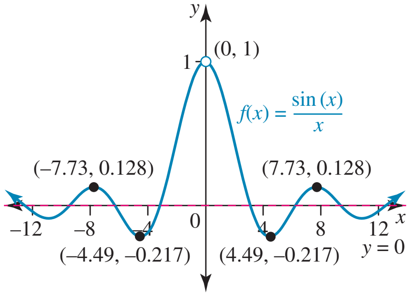
Domain $R\backslash\{0\}$
Range $[-1, 1]$
The point at the origin $x=0$ is a point of discontinuity. The graph oscillates widely near the origin, and has infinitely many turning points and many inflection points and no asymptotes.

$f'(x)=\dfrac{-\cos\left(\frac{1}{x}\right)}{x^2}\qquad f''(x)=\dfrac{2\cos\left(\frac{1}{x}\right)}{x^3}-\dfrac{\sin\left(\frac{1}{x}\right)}{x^4}$



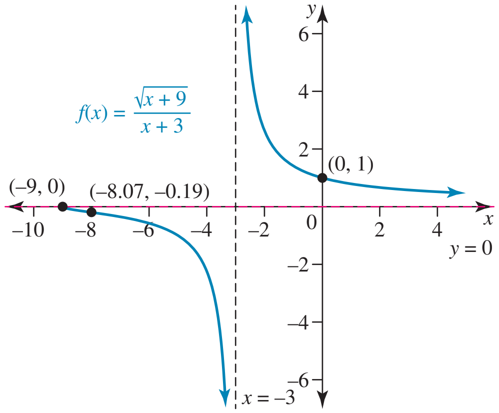
b $f(x)=\dfrac{\sin(x)}{x}$

Domain $R\backslash\{0\}$
The point at the origin $x=0$ is a point of discontinuity
$f'(x)=\dfrac{x\cos(x)-\sin(x)}{x^2}=0$
$x=\pm4.49,\quad\pm7.73.\dots$
$(\pm4.49, -0.217)$ absolute minimums.
$(\pm7.73, 0.128)$ local maximums.
As $x\to\infty\quad y\to0$
$y=0$ is a horizontal asymptote
Range $[-0.217, 1)$
$f''(x)=\left(\dfrac{2}{x^3}-\dfrac{1}{x}\right)\sin(x)-\dfrac{2\cos(x)}{x^2}$
infinitely many inflection points



10 $f(x)=\dfrac{\sqrt{x+9}}{x+3}$

Domain $x+9\geq0,\quad x\geq-9$ but $x\neq-3$
So domain $[-9, -3)\cup(-3, \infty)$
Range R
$x=-3$ is a vertical asymptote
$\lim\limits_{x\to\infty}\left(\dfrac{\sqrt{x+9}}{x+3}\right)\to0$
So $y=0$ is a horizontal asymptote
Crosses x-axis $y=0, x=-9\ (-9, 0)$
Crosses y-axis $x=0, f(0)=\dfrac{\sqrt{9}}{3}=1\quad(0, 1)$
Crosses the horizontal asymptote
$f'(x)=\dfrac{-(x+15)}{2(x+3)^2\sqrt{x+9}}=0$ when $x=-15$
But not in the domain, so no stationary points
$f''(x)=\dfrac{3(x^2+30x+177)}{4(x+3)^3(x+9)^{\frac{3}{2}}}=0$
Solving $x^2+30x+177=0$ with $x\geq-9$
gives $x=-8.07,\ f(-8.07)=-0.19$
So $(-8.07, -0.19)$ is an inflection point.



11 $f(x) = \dfrac{\sqrt{x+4}}{x-1}$

Domain $x + 4 \geq 0$, $x \geq -4$ but $x \neq 1$

So domain $[-4, 1) \cup (1, \infty)$

Range R

$x = 1$ is a vertical asymptote

$$\lim_{x \to \infty} \left(\frac{\sqrt{x+4}}{x-1} \right) \to 0$$

So $y = 0$ is a horizontal asymptote

Crosses x-axis $y = 0$, $x = -4$ $(-4, 0)$

Crosses y-axis $x = 0$, $f(0) = \dfrac{\sqrt{4}}{-1} = -2$ $(0, -2)$

Crosses the horizontal asymptote

$$f'(x) = \frac{-(x+9)}{2(x-1)^2\sqrt{x+4}} = 0 \text{ when } x = -9$$

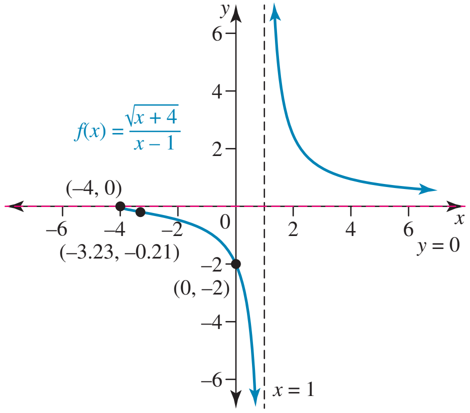
But not in domain, so no stationary points

$$f''(x) = \frac{3x^2 + 54x + 143}{4(x-1)^3(x+4)^{\frac{3}{2}}} = 0$$

Solving $3x^2 + 54x + 143 = 0$ with $x \geq -4$

gives $x = -3.23$, $f(-3.23) = -0.21$

So $(-3.23, -0.21)$ is an inflection point.



12 $f(x) = \dfrac{\sqrt{x^2+4}}{x^2-4} = \dfrac{\sqrt{x^2+4}}{(x+2)(x-2)}$

Does not cross the x-axis

Crosses y-axis $x = 0$, $f(0) = \dfrac{\sqrt{4}}{-4} = -\dfrac{1}{2}$ $\left(0, -\dfrac{1}{2}\right)$

$x = \pm 2$ vertical asymptotes

$$\lim_{x \to \infty} \left(\frac{\sqrt{x^2+4}}{x^2-4} \right) \to 0$$

So $y = 0$ horizontal asymptote

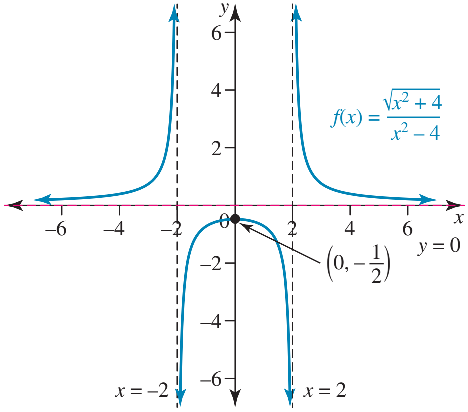
$$f'(x) = \frac{-x(x^2+12)}{(x-2)^2(x+2)^2\sqrt{x^2+4}} = 0 \text{ When } x = 0$$

So $\left(0, -\dfrac{1}{2}\right)$ is a local maximum

$$f''(x) = \frac{2(x^6 + 30x^4 + 96x^2 + 96)}{(x-2)^3(x+2)^3(x^2+4)^{\frac{3}{2}}} \neq 0$$

So there are no inflection points

Range $\left(-\infty, -\dfrac{1}{2}\right] \cup (0, \infty)$



13 a $f(x) = \dfrac{\sqrt{x^2+4x+4}}{(x+2)^2}$

$$= \frac{\sqrt{(x+2)^2}}{(x+2)^2} = \frac{|x+2|}{(x+2)^2}$$

$$= \frac{1}{|x+2|}$$

$x = -2$ is a vertical asymptote

Does not cross x-axis

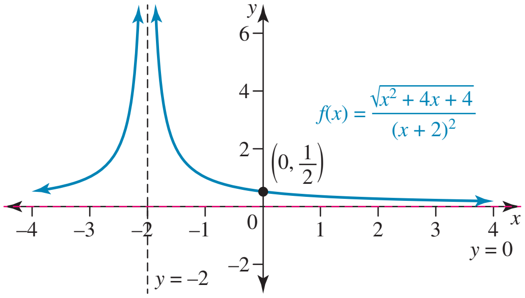
Crosses y-axis $x = 0$, $y = \dfrac{1}{2}$ $\left(0, \dfrac{1}{2}\right)$

$y = 0$ is a horizontal asymptote

Domain $R\backslash\{-2\}$

Range $(0, \infty)$

No turning points, no inflection points



b $\dfrac{1}{f(x)} = \dfrac{(x+2)^2}{\sqrt{x^2+4x+4}}$

$$= \frac{(x+2)^2}{\sqrt{(x+2)^2}}$$

$$= \frac{(x+2)^2}{|x+2|} = |x+2|, \quad x \neq -2$$

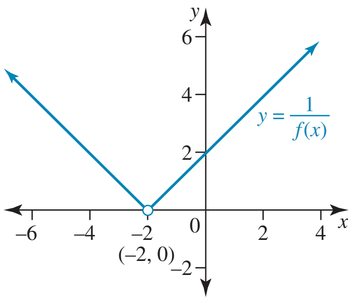
$x = -2$ is a point of discontinuity

Does not cross x-axis

Crosses y-axis $x = 0$ $y = 2$ $(0, 2)$

Domain $R\backslash\{0\}$, range $(0, \infty)$

No turning points, no inflection points.



14 $f(x) = \frac{(x+2)^2}{\sqrt{x^2-4}} = \frac{(x+2)^2}{\sqrt{(x+2)(x-2)}}$

$= \frac{(x+2)^{\frac{3}{2}}}{\sqrt{x-2}}$

So $x = -2$ is a point of discontinuity

$\lim_{x \to -2}\left(\frac{(x+2)^2}{\sqrt{x^2-4}}\right) = 0$

$x = 2$ is a vertical asymptote

Does not cross x or y-axis

$f'(x) = \frac{(x+2)(x^2-2x-8)}{(x^2-4)^{\frac{3}{2}}} = \frac{(x-4)(x+2)^2}{(x^2-4)^{\frac{3}{2}}}$

$f'(x) = 0$ When $x = -4$, $f(4) = 6\sqrt{3}$

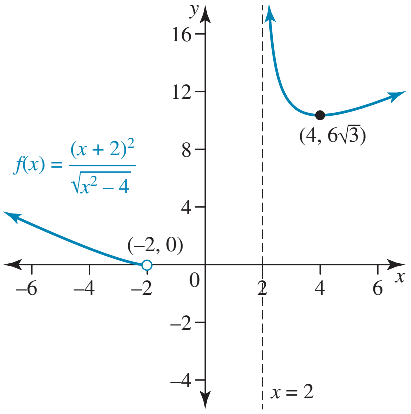
$(4, 6\sqrt{3})$ is a local minimum

Domain $(-\infty, -2) \cup (2, \infty)$

Range $(0, \infty)$

$f''(x) = \frac{12(x+2)^2}{(x^2-4)^{\frac{5}{2}}} = 0$

No inflection points



15 $x = -1$ vertical asymptote

Domain $(-1, \infty)$

$y = x^n$ oblique or non-linear asymptote

Crosses $y = x^n$ at $x = e - 1$

n even $(0, 0)$ is an inflection point.

n odd $(0, 0)$ is an absolute minimum turning point.

16 $y = 0$ horizontal asymptote

n odd $(0, 0)$ is an inflection point,
four other inflection points, one absolute maximum and one absolute minimum turning points.

n even $(0, 0)$ is an absolute minimum turning point.
four other inflection points, two absolute maximum turning points.

17 **a** n odd $(0, 0)$ is an absolute minimum turning point,
two other inflection points, two absolute maximum turning points.

n even $(0, 0)$ is an inflection point, two other inflection points
one absolute maximum and minimum turning points

b n odd $(0, 0)$ is an inflection point, two inflection points,
one local maximum and one level minimum turning points,
n even $(0, 0)$ is a minimum turning point, two other inflection points, two absolute maximum turning points.

18 **a** $f(x) = \frac{40}{\pi}\sin^{-1}\left(\frac{x^2-16}{16}\right)$

Domain $\left|\frac{x^2-16}{16}\right| \leq 1$

$-16 \leq x^2 - 16 \leq 16$

$0 \leq x^2 \leq 32$

$|x| \leq \sqrt{32}$

$-4\sqrt{2} \leq x \leq 4\sqrt{2}$ domain $\left[-4\sqrt{2}, 4\sqrt{2}\right]$

Endpoints: $f\left(4\sqrt{2}\right) = 20 \quad f\left(-4\sqrt{2}\right) = 20$

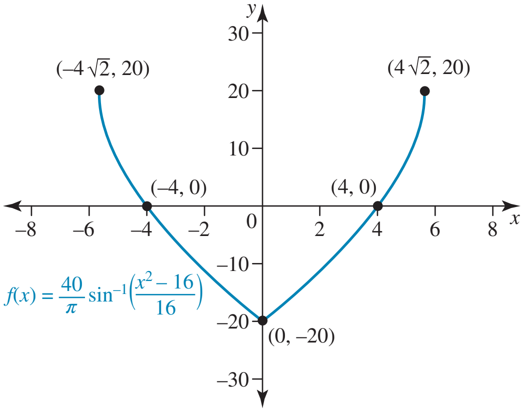
$\left(4\sqrt{2}, 20\right) \quad \left(-4\sqrt{2}, 20\right)$

Crosses x-axis $y = 0$, $x^2 - 16 = 0$, $x = \pm 4$

$(4, 0) \quad (-4, 0)$

Crosses y-axis $x = 0, f(0) = \frac{40}{\pi}\sin^{-1}(-1) = \frac{40}{\pi} \times \frac{-\pi}{2}$

$= -20$ $(0, -20)$



b $f'(x) = \begin{cases} \frac{80}{\pi\sqrt{32-x^2}}, & 0 < x < 4\sqrt{2} \\ \frac{-80}{\pi\sqrt{32-x^2}}, & -4\sqrt{2} < x < 0 \end{cases}$

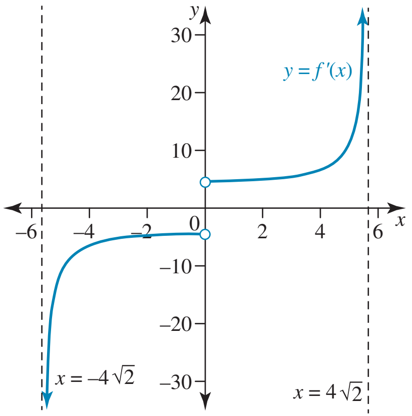
$f'(x) \neq 0$ no turning points

$f''(x) = 0 \Rightarrow x = 0$ but not in domain

No inflection points

graph of $f'(x)$ has points of discontinuity at $\left(0, \pm\frac{10\sqrt{2}}{\pi}\right)$

and vertical asymptote at $x = \pm 4\sqrt{2}$



19 a $f(x) = \frac{30}{\pi}\cos^{-1}\left(\frac{x^2-49}{49}\right)$

Domain: $\left|\frac{x^2-49}{49}\right| \le 1$

$-49 \le x^2 - 49 \le 49$

$0 \le x^2 \le 98$

$|x| \le \sqrt{98}$

$-7\sqrt{2} \le x \le 7\sqrt{2}$ domain $\left[-7\sqrt{2}, 7\sqrt{2}\right]$

Endpoints: $f\left(7\sqrt{2}\right) = 0,\ f\left(-7\sqrt{2}\right) = 0$

$\left(7\sqrt{2}, 0\right)$ $\left(-7\sqrt{2}, 0\right)$ also x-intercepts

Crosses y-axis $x = 0$, $f(0) = \frac{30}{\pi}\cos^{-1}(-1) = \frac{30}{\pi} \times \pi = 30$

$(0, 30)$

See graph at the bottom of the page*

b $f'(x) = \begin{cases} \dfrac{-60}{\pi\sqrt{98-x^2}} & 0 < x < 7\sqrt{2} \\ \dfrac{60}{\pi\sqrt{98-x^2}} & -7\sqrt{2} < x < 0 \end{cases}$

$f'(x) \ne 0$ no turning points

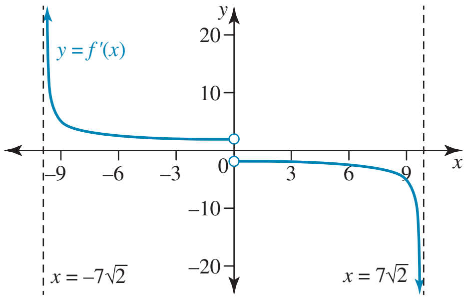
$f''(x) = 0 \Rightarrow x = 0$ but not in domain

No inflection points

Graph of $f'(x)$ has point of

discontinuity at $\left(0, \pm\frac{30\sqrt{2}}{7\pi}\right)$

and vertical asymptote at $x = \pm 7\sqrt{2}$



20 a i $A(t) = 30te^{-\frac{t}{3}}$

Product rule:

$\frac{dA}{dt} = 30e^{-\frac{t}{3}} + 30t \times -\frac{1}{3}e^{-\frac{t}{3}}$

$= 30e^{-\frac{t}{3}}\left(1 - \frac{t}{3}\right)$

$= (30 - 10t)\,e^{-\frac{t}{3}}$

When $\frac{dA}{dt} = 0 \Rightarrow t = 3$

$A_{\max} = A(3) = 90e^{-1} \approx 33.11$ mg

ii $\frac{d^2A}{dt^2} = -10e^{-\frac{t}{3}} + -\frac{1}{3}e^{-\frac{t}{3}}(30 - 10t)$

$= \left(\frac{10t}{3} - 20\right)e^{-\frac{t}{3}}$

Inflection $\frac{d^2A}{dt^2} = 0 \Rightarrow t = 6$

$A(6) = 180e^{-2} \approx 24.36$ mg

b i $B(t) = 15t^2e^{-\frac{t}{2}}$

$\frac{dB}{dt} = 30te^{-\frac{t}{2}} + 15t^2 \times -\frac{1}{2}e^{-\frac{t}{2}}$

$= 15te^{-\frac{t}{2}}\left(2 - \frac{t}{2}\right)$

$= \left(30t - \frac{15t^2}{2}\right)e^{-\frac{t}{2}}$

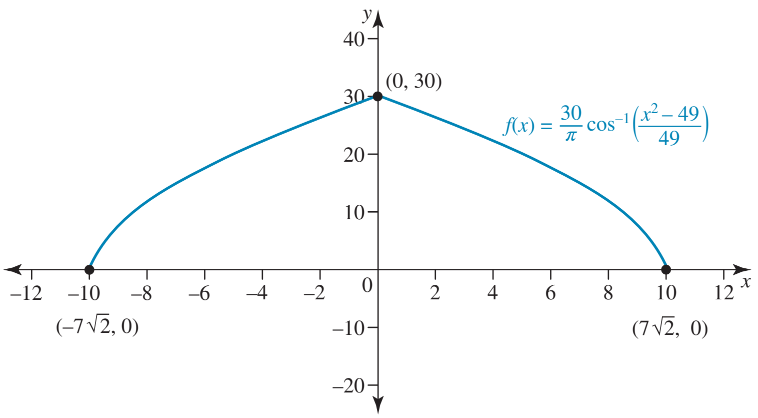
When $\frac{dB}{dt} = 0 \quad \Rightarrow t = 4$

$B_{\max} = B(4) = 240e^{-2} \approx 32.48$ mg

ii $\frac{d^2B}{dt^2} = (30 - 15t)\,e^{-\frac{t}{2}} + -\frac{1}{2}\left(30t - \frac{15t^2}{2}\right)e^{-\frac{t}{2}}$

$= \left(30 - 30t + \frac{15t^2}{4}\right)e^{-\frac{t}{2}}$

***19a**



Inflection $\frac{d^2B}{dt^2} = 0 \quad \Rightarrow \frac{15t^2}{4} - 30t + 30 = 0$

$$t = 2\left(2 \pm \sqrt{2}\right) \approx 1.17, 6.83$$

$$B\left(2\left(2+\sqrt{2}\right)\right) \approx 23.01$$

$$B\left(2\left(2-\sqrt{2}\right)\right) \approx 11.46$$

c CAS : $A(t) = B(t)$

$30te^{-\frac{t}{3}} = 15t^2e^{-\frac{t}{2}}$

$\Rightarrow t = 3.714, 9.073$

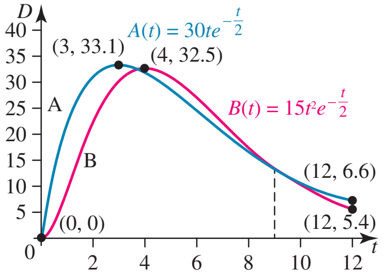
The difference $9.073 - 3.714 = 5.358$

Over 12 hours: $\frac{5.358}{12} \times 100 = 44.7\%$

End points:

$A(12) = 360e^{-4} \approx 6.59$

$B(12) = 2160e^{-6} \approx 5.35$



6.4 Exam questions

1 For local maximum $y = \frac{1}{(\cos(ax)+1)^2+3}, \quad a \in R\backslash\{0\}$

$(\cos(ax)+1)^2 + 3$ has local minimum of 3

When $\cos(ax) = -1$

$ax = \cos^{-1}(-1) + 2k\pi$

$ax = \pi + 2k\pi = \pi(1+2k)$

Local max = $\left(\frac{\pi(1+2k)}{a}, \frac{1}{3}\right), \quad k \in Z$

The correct answer is **E**.

2 a $f(x) = x^2e^{-x}$

$f'(x) = (2x - x^2)e^{-x}$

Solving $f'(x) = x(2-x)e^{-x} = 0$ for stationary points:

$x = 0, 2$ since $e^{-x} \neq 0$

Absolute minimum $(0, 0)$

Local maximum $(2, 4e^{-2}) \approx (2, 0.54)$

Award 1 mark for the correct derivative.

Award 1 mark for the correct derivative.

b $y = 0$ is a horizontal asymptote.

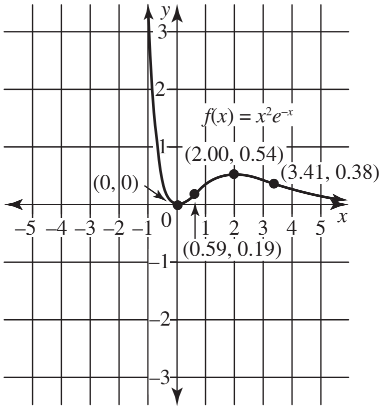
Award 1 mark for the correct equation.

c $f''(x) = (x^2 - 4x + 2)e^{-x}$

Solving $f''(x) = (x^2 - 4x + 2)e^{-x} = 0$ for inflection points:

$x = 2 \pm \sqrt{2} \approx 0.59, 3.41$ since $e^{-x} \neq 0$

Inflection points $(0.59, 0.19), (3.41, 0.38)$



Award 1 mark for the correct second derivative and inflection points.

Award 2 marks for the correct graph and labelled diagram.

d $g(x) = x^ne^{-x}$

$g''(x) = x^{n-2}\left(x^2 - 2nx + n(n-1)\right)e^{-x}$

Award 1 mark for the correct expression.

e i $g''(x) = 0 \quad \Rightarrow \quad x = n \pm \sqrt{n}$

Award 1 mark for the correct values.

ii

Number of points of inflection	**Value(s) of n (where $n \in Z$)**
0	$n \leq 0$
1	1
2	$2, 4, 6 \ldots n$ even $n = 2k, k \in Z^+$
3	$3, 5, 7 \ldots n$ odd $n = 2k+1, k \in Z^+$

When n is odd, the point at the origin is a horizontal point of inflection, but when n is even, the point at the origin is an absolute minimum turning point (not an inflection point).

Award 1 mark for the correct table.

Award 1 mark for the correct explanation.

3 $f(x) = \frac{e^x}{x-1}$

$x = 1$ is a vertical asymptote.

$y = 0$ is a horizontal asymptote.

The graph has a minimum turning point at $x = 2$.

The graph does cross the y-axis, $f(0) = -1$.

The graph does not have a point of inflection, $f''(x) \neq 0$

The correct answer is **E**.

6.5 Review

6.5 Exercise

Technology free: short answer

1 a $y = x^3 - 12x$

$$= x\left(x^2 - 12\right)$$

$$= x\left(x + 2\sqrt{3}\right)\left(x - 2\sqrt{3}\right)$$

Crosses x-axis where $y = 0$ $x = 0, \ \pm 2\sqrt{3}$

$(0, 0)\left(2\sqrt{3}, 0\right)\left(-2\sqrt{3}, 0\right)$

$$\frac{dy}{dx} = 3x^2 - 12$$
$$= 3\left(x^2 - 4\right)$$
$$= 3(x+2)(x-2)$$

$\frac{dy}{dx} = 0$ When $x = \pm 2$

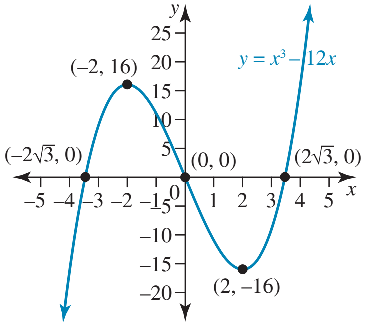
When $x = 2$, $y = 8 - 24 = -16$

$x = -2$, $y = -8 + 24 = 16$

Stationary points are $(2, -16)$ and $(2, 16)$

$$\frac{d^2y}{dx^2} = 6x = 0$$

When $x = 0, y = 0$ $(0, 0)$ inflection



b $y = x^4 - 12x^3 + 48x^2 - 64x$
$$= x\left(x^3 - 12x^2 + 48x - 64\right)$$
$$= x(x-4)^3$$

Crosses x-axis $x = 0, 4$

$(0, 0)$ $(4, 0)$

$$\frac{dy}{dx} = 4x^3 - 36x^2 + 96x - 64$$
$$= 4\left(x^3 - 9x^2 + 24x - 16\right)$$
$$= 4(x-1)\left(x^2 - 8x + 16\right)$$
$$= 4(x-1)(x-4)^2$$

$\frac{dy}{dx} = 0$ $x = 1, 4$

When $x = 1$, $y = 16(1-4)^3 = -27$

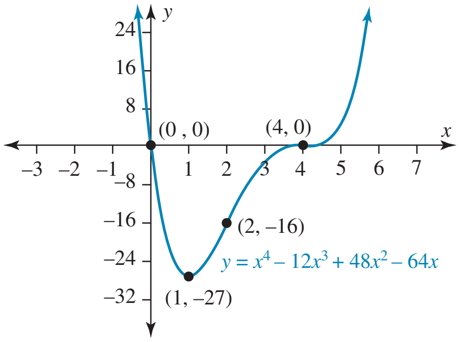
$(1, -27)$ absolute min

$$\frac{d^2y}{dx^2} = 12x^2 - 72x + 96$$
$$= 12\left(x^2 - 6x + 8\right)$$
$$= 12(x-4)(x-2)$$

$\frac{d^2y}{dx^2} = 0$ $x = 2, 4$

When $x = 2$, $y = 2(2-4)^3 = -16$

$(2, -16)$ $(4, 0)$ inflection points



2 a $y = x^4 - 8x^3$
$$\frac{dy}{dx} = 4x^3 - 24x^2$$
$$\frac{d^2y}{dx^2} = 12x^2 - 48x$$
$$= 12x(x-4)$$

For concave up $\frac{d^2y}{dx^2} > 0$

$x > 4$ and $x < 0$

$(-\infty, 0) \cup (4, \infty)$

b $y = x^4 + 2b^3 + 3cx^2$
$$\frac{dy}{dx} = 4x^3 + 6bx^2 + 6cx$$
$$\frac{d^2y}{dx^2} = 12x^2 + 12bx + 6c$$
$$\Delta = (12b)^2 - 4 \times 12 \times 6c$$
$$= 144b^2 - 288c$$
$$= 144\left(b^2 - 2c\right)$$

No inflection points $\Delta < 0$

$b^2 - 2c < 0$ $c > 0$

$b^2 < 2c$

$-\sqrt{2c} < b < \sqrt{2c}$

$b \in \left(-\sqrt{2c}, \sqrt{2c}\right)$

3 $y = f(x) = x^3 - 6x^2 + px + q$

$f(1) = 9 = 1 - 6 + p + q$

(1) $p + q = 14$

$f'(x) = 3x^2 - 12x + p$

The function has a stationary point at $(1, 9)$, $f'(1) = 0$

$f'(1) = 3 - 12 + p = 0$

$p = 9$, $q = 5$

$f(x) = x^3 - 6x^2 + 9x + 5$

$f'(x) = 3x^2 - 12x + 9 = 3\left(x^2 - 4x + 3\right)$

$f'(x) = 3(x-1)(x-3) = 0$, $x = 3$

$f(3) = 27 - 54 + 27 + 5 = 5$, $(3, 5)$ is the other turning point

$f''(x) = 6x - 12 = 0$, $x = 2$

$f(2) = 8 - 24 + 18 + 5 = 7$

$(2, 7)$ is the inflection point

4 a $y = \frac{12}{x^2 + 4x} = 12\left(x^2 + 4x\right)^{-1}$

Vertical asymptotes

$x^2 + 4x = 0$

$x(x+4) = 0$

$x = 0, -4$

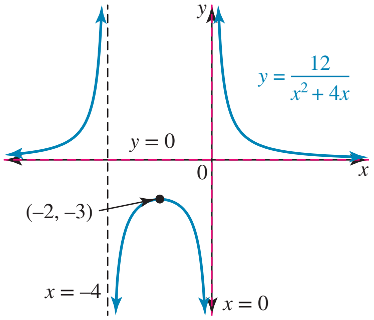
Horizontal asymptote at $y = 0$

$$\frac{dy}{dx} = -(2x+4) \times 12\left(x^2 + 4x\right)^{-2} = 0$$
$$2x + 4 = 0$$
$$x = -2$$

When $x = -2, y = -3, (-2, -3)$ local maximum

Domain $x \in R \backslash \{0, -4\}$

Range $(-\infty, -3] \cup (0, \infty)$



b $y = \frac{8}{7 - 6x - x^2} = 8\left(7 - 6x - x^2\right)^{-1}$

$= \frac{8}{-\left(x^2 + 6x - 7\right)}$

$= \frac{-8}{(x+7)(x-1)}$

Crosses y-axis $x = 0 \Rightarrow y = \frac{8}{7} \Rightarrow \left(0, \frac{8}{7}\right)$

Vertical asymptotes

$x = -7$

$x = 1$

Horizontal asymptote at $y = 0$

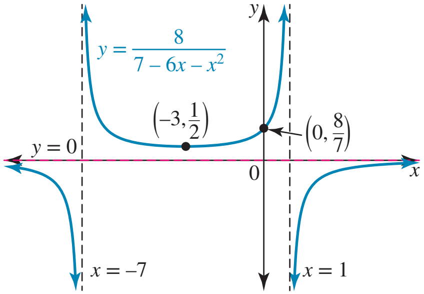
$\frac{dy}{dx} = 8 \times -1 \times (-6 - 2x)\left(7 - 6x - x^2\right)^{-2}$

$= \frac{8(2x+6)}{\left(-x^2 - 6x + 7\right)^2} = 0, \text{ when } x = -3$

When $x = -3, y = \frac{1}{2}$ $\left(-3, \frac{1}{2}\right)$ local minimum

Domain $x \in R \backslash \{-7, 1\}$

Range $(-\infty, 0) \cup \left[\frac{1}{2}, \infty\right)$



5 $y = \frac{A}{x^2 + bx + 12}$

For a local maximum $2x + b = 0$

At $x = 4$, $8 + b = 0$

$b = -8$

So $y = \frac{A}{x^2 - 8x + 12}$

$y = \frac{A}{(x-6)(x-2)}$

When $x = 4$ $y = -2$

$-2 = \frac{A}{-4}$

$A = 8$

$y = 0$ is a horizontal asymptote

$x = 2$ and $x = 6$ are vertical asymptotes

Domain $R \backslash \{2, 6\}$

Range $(-\infty, -2] \cup (0, \infty)$

6 $y = \frac{x-5}{x^2 - 25}$

$= \frac{x-5}{(x+5)(x-5)}$

$= \frac{1}{x+5} \quad x \neq 5$

$\lim_{x \to 5}\left(\frac{x-5}{x^2 - 25}\right) = \lim_{x \to 5}\left(\frac{1}{x+5}\right) = \frac{1}{10}$

The point $\left(5, \frac{1}{10}\right)$ is a point of discontinuity

Crosses y-axis $x = 0, y = \frac{1}{5}$ $\left(0, \frac{1}{5}\right)$

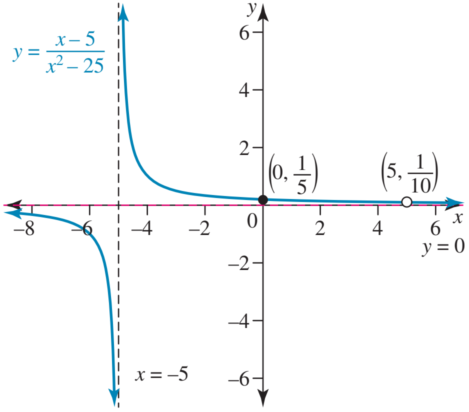
$x = -5$ is a vertical asymptote

$y = 0$ is a horizontal asymptote

No turning points, no inflection points

Domain $R \backslash \{\pm 5\}$

Range $R \backslash \left\{0, \frac{1}{10}\right\}$



7 **a** $f(x) = \frac{\sqrt{x^2 - 6x + 9}}{(x-3)^2}$

$= \frac{\sqrt{(x-3)^2}}{(x-3)^2}$

$= \frac{|x-3|}{(x-3)^2}$

$= \frac{1}{|x-3|}, \quad x \neq 3$

Does not cross x-axis

Cross y-axis $x = 0, y = \frac{1}{3}$ $\left(0, \frac{1}{3}\right)$

$f'(x) \neq 0 \quad f''(x) \neq 0$

No turning points

No inflection points

$x = 3$ is a vertical asymptote

$y = 0$ is a horizontal asymptote

See graph at the bottom of the page*

b $f(x) = \dfrac{(x-3)^2}{\sqrt{x^2-6x+9}}$

$= \dfrac{(x-3)^2}{\sqrt{(x-3)^2}}$

$= \dfrac{(x-3)^2}{|x-3|}$

$= |x-3|\,;\ x \neq 3$

Does not cross x-axis

Crosses y-axis $x = 0$, $y = |-3| = 3$ $(0, 3)$

$$\lim_{x\to 3}\left(\frac{(x-3)^2}{\sqrt{x^2-6x+9}}\right) = \lim_{x\to 3}(|x-3|) = 0$$

$(3, 0)$ is a point of discontinuity

No turning points

No inflection points

See graph at the bottom of the page*

8 $y = \dfrac{x^2-a^2}{x-b} \qquad a \neq b$

$= \dfrac{x^2-b^2+b^2-a^2}{x-b}$

$= \dfrac{(x-b)(x+b)+b^2-a^2}{(x-b)}$

$y = x+b+\dfrac{b^2-a^2}{x-b}$

$= x+b+\left(b^2-a^2\right)(x-b)^{-1}$

$\dfrac{dy}{dx} = 1-(x-b)^{-2}\left(b^2-a^2\right) = 0$

$1 = \dfrac{b^2-a^2}{(x-b)^2}$

$(x-b)^2 = b^2-a^2$

$(x-b) = \pm\sqrt{b^2-a^2}$

$x = b \pm \sqrt{b^2-a^2} \qquad |b| > |a|$

$\dfrac{d^2y}{dx^2} = 2(x-b)^{-3}\left(b^2-a^2\right) \neq 0$

So there is no inflection points

Technology active: multiple choice

9 $f(x) = x^3 + 3kx^2$

$f'(x) = 3x^2 + 6kx$

$f''(x) = 6x + 6k$

For the gradient to be decreasing $f''(x) \leq 0$

$6x + 6k \leq 0$

$x \leq -k$ or $\left(-\infty, -k\right]$

The correct answer is **B**.

10 The graph of the function

$y = \dfrac{x^2+4x-12}{x-2} = \dfrac{(x-2)(x+6)}{x-2} = x+6,\ x \neq 2.$

Has a point of discontinuity at $x = 2$.

The correct answer is **A**.

11 $y = \dfrac{x-2}{x^2-2x-8} = \dfrac{x-2}{(x+2)(x-4)}$

The graph crosses the x-axis at $x = 2$ and the y-axis at $y = \dfrac{1}{4}$.

The graph has two vertical asymptotes at $x = -2$ and $x = 4$ and a horizontal asymptote at $y = 0$ and crosses the asymptote at the point $(2, 0)$.

There is not a point of discontinuity at $x = 2$.

The correct answer is **A**.

12 The graph of the $y = \dfrac{\sqrt{x}}{x+2}$ has a maximal domain of $[0, \infty)$

as $\displaystyle\lim_{x\to\infty}\left(\frac{\sqrt{x}}{x+2}\right) = 0$ so $y = 0$ is a horizontal asymptote.

The graph does not have a vertical asymptote.

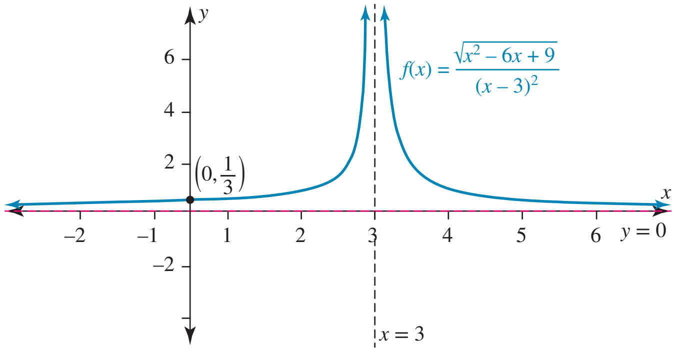
There is an absolute maximum turning point at $(2, 0.35)$

There is an axial intercept at the origin $(0, 0)$

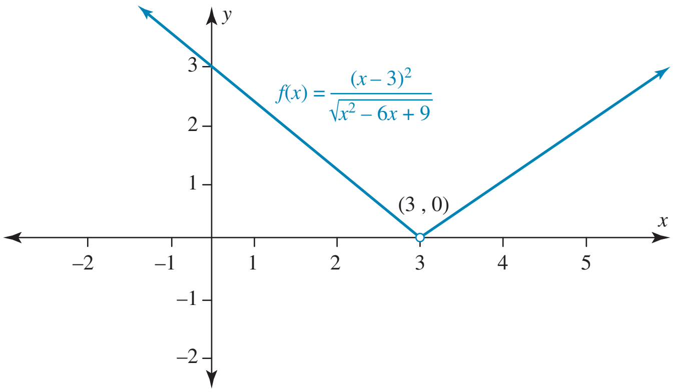
There is a point of inflection at $(4.31, 0.33)$

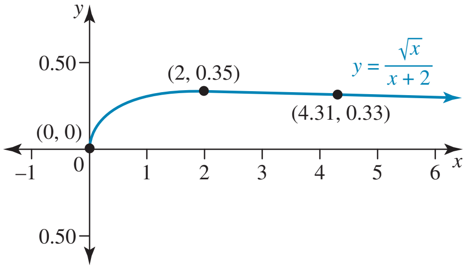
The correct answer is **B**.

***7a**



***7b**

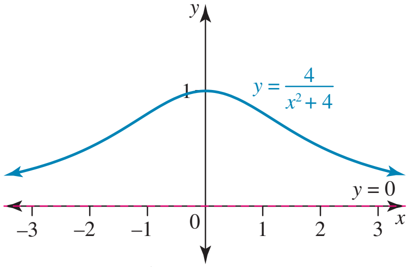




13 $y = \dfrac{4}{x^2 + 4}$

There is only one horizontal asymptote at $y = 0$.

The correct answer is **C**.



14 $y = \dfrac{1}{ax^2 + b(1-a)x - b^2}$ the graph has vertical asymptotes when the denominator is zero

$ax^2 + b(1-a)x - b^2 = 0$

$(x-b)(ax+b) = 0$

$x = b, -\dfrac{b}{a}$

Also as $x \to \infty$, $y \to 0$ so that the x-axis or $y = 0$ is also a horizontal asymptote.

The asymptotes are $x = b$, $x = -\dfrac{b}{a}$ and $y = 0$.

The correct answer is **B**.

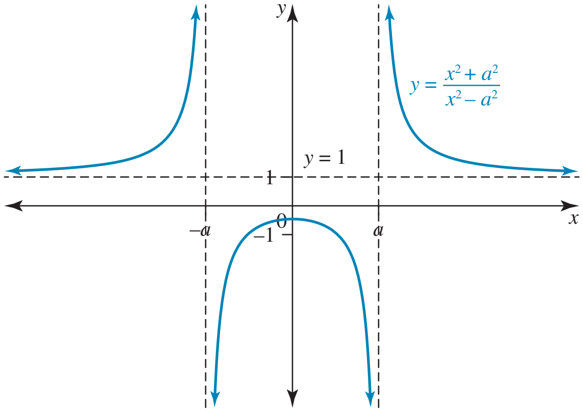
15 $y = \dfrac{x^2 + a^2}{x^2 - a^2} = 1 + \dfrac{2a^2}{x^2 - a^2}$

The maximal domain is $R \backslash \{\pm a\}$

The range is $(-\infty, -1] \cup (1, \infty)$.

The graph has vertical asymptotes at $x = \pm a$ and a horizontal asymptote at $y = 1$. The graph crosses the y-axis at $y = -1$ and this point $(0, -1)$ is a maximum turning point.

The correct answer is **E**.



16 The graph is that of a reciprocal inverted parabola, with asymptotes at $x = a$ and $x = b$, note that $a < 0$ while $b > 0$.

Its rule is

$$f(x) = \frac{-1}{(x-a)(x-b)} = \frac{-1}{x^2 - (a+b)x + ab}$$

$$= \frac{1}{(a+b)x - ab - x^2}$$

The correct answer is **E**.

17 The x-axis is a horizontal asymptote, option **A** is true.

When $x = 0$, $y = \dfrac{1}{c}$ as the y-intercept, option **B** is true.

When $-4 - 2x = 0 \Rightarrow x = -2$ is a minimum turning point. option **C** is true.

The quadratic in the denominator $c - 4x - x^2$ has a discriminant of $\Delta = (-4)^2 - 4 \times {}^-1 \times c = 16 + 4c = 4(4 + c)$ so

If $\Delta > 0$, $c > -4$ the quadratic has two real solutions, and hence $f(x)$ has two vertical asymptote, so option **D** is false.

If $\Delta = 0$, $c = -4$ the quadratic has one (repeated) real solution, and hence $f(x)$ has one vertical asymptote **E** is true.

The correct answer is **D**.

18 The quadratic in the denominator $x^2 + 2bx + 9$ has a discriminant of $\Delta = (2b)^2 - 4 \times 1 \times 9 = 4b^2 - 36 = 4(b^2 - 9)$ so if $\Delta < 0$, $|b| < 3$ the quadratic has no real solutions, and hence $f(x)$ has no vertical asymptotes, option **A** is true.

If $\Delta > 0$, $|b| > 3$ the quadratic has two real solutions, and hence $f(x)$ has two vertical asymptotes, option **B** is true.

The x-axis is a horizontal asymptote, option **C** is true.

However option **D** is false, when $2x + 2b = 0$, $x = -b$, the point $\left(-b, \dfrac{1}{9 - b^2}\right)$ is a maximum turning point.

When $x = 0$, $y = \dfrac{1}{9}$ as the y-intercept, option **E** is true.

The correct answer is **D**.

Technology active: extended response

19 a $f(x) = \dfrac{\sqrt{x+9}}{x-3}$

Cross y-axis $x = 0$	$y = -1$	$(0, -1)$
Cross x-axis $y = 0$	$x = -9$	$(-9, 0)$

$x = 3$ is a vertical asymptote

$y = 0$ horizontal asymptote

Domain $[-9, 3) \cup (3, \infty)$

Range R

$f'(x) = \dfrac{-(x+21)}{2(x-3)^2\sqrt{x+9}}$

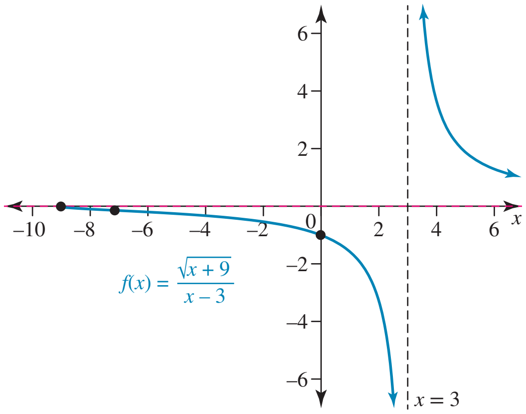
$f'(x) = 0 \Rightarrow x = -21$ but not in domain

$f''(x) = \dfrac{3(x^2 + 42x + 249)}{4(x-3)^3\sqrt{(x+9)^3}}$

$f''(x) = 0 \Rightarrow x^2 + 42x + 249 = 0$

$\Rightarrow x = -7.144 \;\; y = -0.134$

Inflection point $(-7.14, -0.13)$



b Domain $x \geq -a \quad R\backslash\{-b\}$

i $a = b$

ii $a < b$

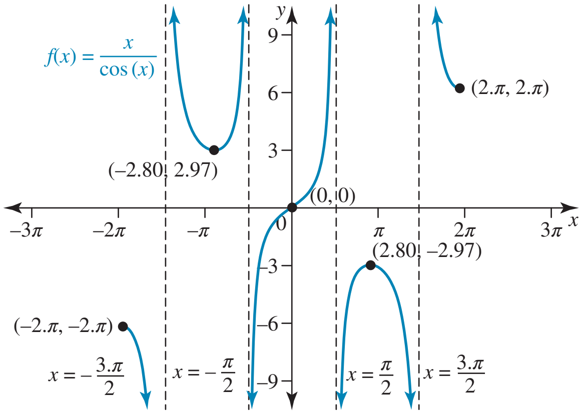
20 Endpoints $(2\pi, 2\pi)$, $(-2\pi, -2\pi)$

$x = \pm\dfrac{\pi}{2}$, $x = \pm\dfrac{3\pi}{2}$ vertical asymptotes

Local main $(-2.80, 2.97)$

Local max $(2.80, -2.97)$

Inflection point $(0, 0)$



6.5 Exam questions

1 a $f(x) = \dfrac{(2x-3)(x+5)}{(x-1)(x+2)}$

$f(x) = 2 + \dfrac{5x-11}{(x-1)(x+2)}$ [1 mark]

b Vertical asymptotes $x = 1$, $x = -2$ [1 mark]

Horizontal asymptote $y = 2$ [1 mark]

c The graph crosses x-axis $\left(\dfrac{3}{2}, 0\right)$, $(-5, 0)$

The graph crosses y-axis $\left(0, \dfrac{15}{2}\right)$

The graph crosses horizontal asymptote at $\left(\dfrac{11}{5}, 2\right)$

The turning points are $(-0.04, 7.49)$, $(4.44, 2.51)$

The inflection point is $(6.79, 2.45)$

See graph at the bottom of the page*

Award 1 mark for the correct graph shapes.

Award 1 mark for the correct axial intercepts.

Award 1 mark for the correct turning points and inflection point.

d i $g_k(x) = \dfrac{(2x-3)(x+5)}{(x-k)(x+2)}$ $k \in R$

There are two asymptotes: one horizontal and one vertical.

$k = -5$, $g_k(x) = \dfrac{2x-3}{x+2}$ $k \in R$

$k = \dfrac{3}{2}$ $g_k(x) = \dfrac{2(x+5)}{x+2}$

$k = -2$ $g_k(x) = \dfrac{(2x-3)(x+5)}{(x+2)^2}$

Award 2 marks for all 3 correct values of k.

ii For more than two asymptotes but no stationary points, solving the discriminant for k for values less than zero when $\dfrac{dg_k(x)}{dx} = 0$

gives $k < -5$ or $k > \dfrac{3}{2}$

Award 1 mark for each correct value of k.

2 a $f(x) = \arctan(3x - 6) + \pi$

$f(x) = y = \tan^{-1}(u) + \pi, u = 3x - 6$

$\dfrac{dy}{du} = \dfrac{1}{1+u^2}, \dfrac{du}{dx} = 3$

$f'(x) = \dfrac{3}{1+(3x-6)^2}$

$f'(x) = \dfrac{3}{9x^2 - 36x + 37}$

Award 1 mark for the correct derivative.

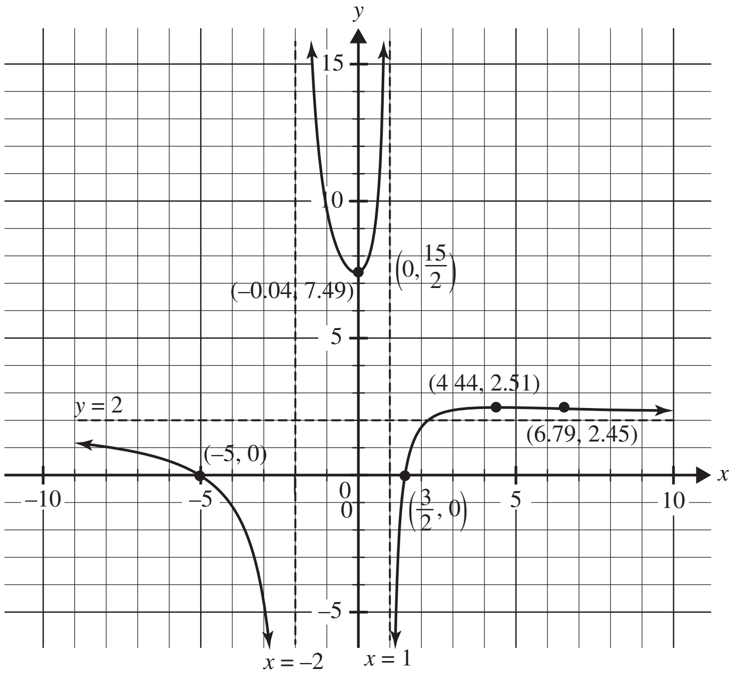
b $f'(x) = 3(9x^2 - 36x + 37)^{-1}$

$f''(x) = -3(18x - 36)(9x^2 - 36x + 37)^{-2}$

$f''(x) = \dfrac{-3 \times 18(x-2)}{(9x^2 - 36x + 37)^2} = 0$ for point of inflection

---

***1 c**



$f''(1) > 0, f''(3) < 0$: change of sign, so $x = 2$ is a point of inflection.

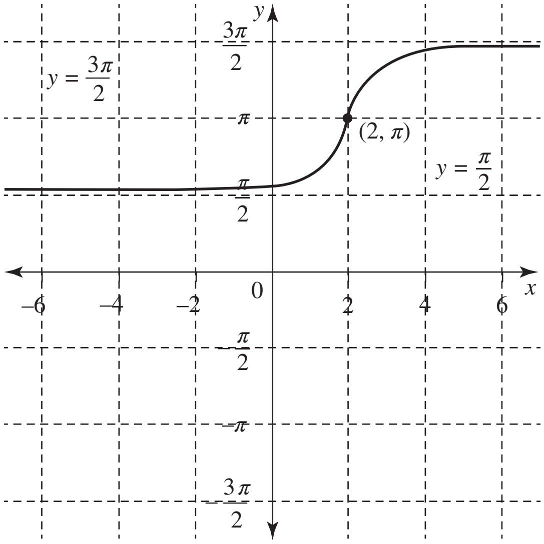
Award 1 mark for the correct second derivative.

Award 1 mark for the correct value of the point of inflection.

c $f(2) = \tan^{-1}(0) + \pi = \pi \to (2, \pi)$

The range of $y = \tan^{-1}(x)$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

So $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) + \pi, y = \frac{3\pi}{2}$ and $y = \frac{\pi}{2}$ are horizontal asymptotes.



Award 1 mark for the correct graph shape.

Award 1 mark for the correct asymptotes.

3 $f(x) = \frac{x^2 + 1}{2x - 8} = \frac{17}{2(x - 4)} + \frac{x}{2} + 2$

$x = 4$ is a vertical asymptote.

$y = \frac{x}{2} + 2$ is an oblique asymptote.

The correct answer is **C**.

4 a i $f(x) = \frac{x}{1 + x^3}, x = -1$ is a vertical asymptote and $y = 0$ is the horizontal asymptote

Award 1 mark for both correct asymptotes.

***VCAA Examination Report* note:**

The majority of students stated the vertical asymptote but significantly fewer stated the horizontal asymptote. Various incorrect attempts at partial fraction forms were made.

ii $f'(x) = \frac{1 - 2x^3}{(1 + x^3)^2}$

Stationary points at $f'(x) = 0$

$x = \frac{1}{\sqrt[3]{2}} \approx 0.7937, \; f(0.7937) = 0.529$

$(0.79, 0.53)$ is a local maximum.

Award 1 mark for the correct derivative.

Award 1 mark for the correct coordinate of the stationary point.

***VCAA Examination Report* note:**

This question was generally answered well. Some students did not give the coordinates of the stationary point in the required form.

iii $f''(x) = \frac{6x^2(x^3 - 2)}{(1 + x^3)^2} f''(0.79) < 0$

Inflection points at $f''(x) = 0$

$x = \sqrt[3]{2} \approx 1.2599, \; f(1.26) = 0.42$

$(1.26, 0.42)$ is a point of inflection

Award 1 mark for setting the second derivative to zero.

Award 1 mark for the correct coordinate of inflection point.

***VCAA Examination Report* note:**

The majority of students provided the correct inflection point. A common error was to erroneously include the point $(0, 0)$, which is another point where $f''(x) = 0$, but it is not a point of inflection as there is no change of concavity; $f''(x)$ does not change sign.

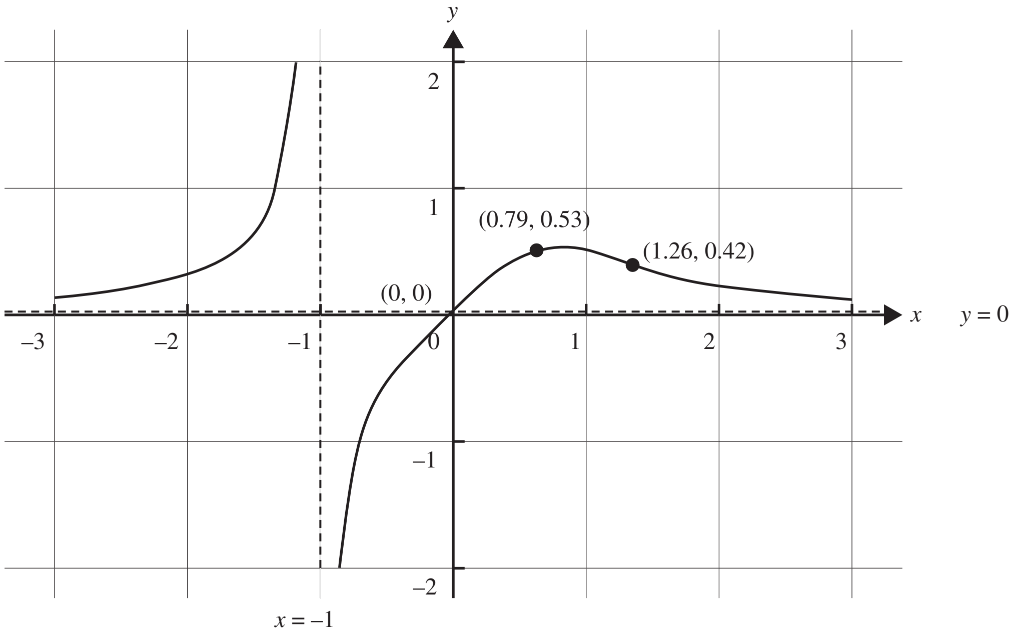
b See graph at the bottom of the page*

Award 1 mark for the correct graph on the domain with the correct intercept.

Award 1 mark for the stationary point and the inflection point.

Award 1 mark for the correct asymptote and its equation.

***4b**



***VCAA Examination Report* note:**

Graphing was generally completed to a reasonable standard. In some cases the shape of the graph was poor and the required points were not marked clearly or were not placed in the correct position.

5 $y = \dfrac{x^2 - 4x + 3}{x^2 - x - 6}$

$$= \frac{(x-3)(x-1)}{(x-3)(x+2)},\ x \neq 3$$

$$= \frac{x-1}{x+2}$$

$$= \frac{x+2-3}{x+2}$$

$$= 1 - \frac{3}{x+2}$$

The graph has a vertical asymptote at $x = -2$, a horizontal asymptote at $y = 1$ and a point of discountinutity at $x = 3$.

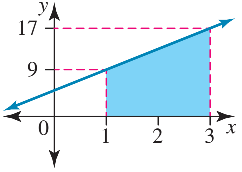
The correct answer is **D**.

Topic 7 — Integral calculus

7.2 Areas under and between curves

7.2 Exercise

1 $y = 4x + 5$



$$A = \int_1^3 (4x + 5)dx$$

$$= \left[2x^2 + 5x\right]_1^3$$

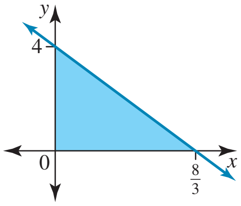
$$= 26$$

Or

$$A = \frac{2}{2}(9 + 17)$$

$$= 26$$

2 $y = 4 - \dfrac{3x}{2}$



$$A = \int_0^{\frac{8}{3}} \left(4 - \frac{3x}{2}\right) dx$$

$$= \left[4x - \frac{3x^2}{4}\right]_0^{\frac{8}{3}}$$

$$= \frac{16}{3}$$

Or

$$A = \frac{1}{2} \times 4 \times \frac{8}{3}$$

$$= \frac{16}{3}$$

3 **a** $y = 6 - 2x$

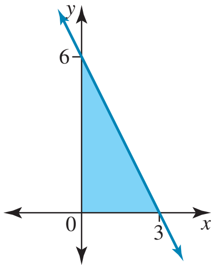
$$A = \int_0^3 (6 - 2x)\, dx$$

$$= \left[6x - x^2\right]_0^3$$

$$= (18 - 9) - 0$$

$$= 9$$

Area of triangle $= \dfrac{1}{2} \times 3 \times 6 = 9$



b $y = 3x + 5$

$$A = \int_1^4 (3x + 5)\, dx$$

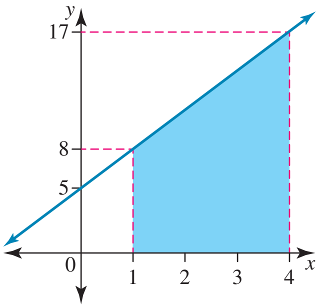
$$= \left[\frac{3x^2}{2} + 5x\right]_1^4$$

$$= \left(\frac{3}{2} \times 16 + 20\right) - \left(\frac{3}{2} + 5\right)$$

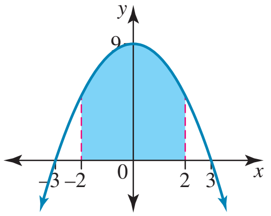
$$= 37.5$$

$$\text{Area} = \frac{1}{2} \times 3 \times (8 + 17)$$

$$= 37.5$$



4 $y = 9 - x^2$



$$A = \int_{-2}^{2} 9 - x^2 dx$$

$$= 2\int_0^2 9 - x^2 dx$$

$$= 2\left[9x - \frac{1}{3}x^3\right]_0^2$$

$$= \frac{92}{3}$$

5 $y = b - 3x^2$

$$A = \int_{-1}^{1} b - 3x^2 dx = 16$$

$$= 2\int_{0}^{1} b - 3x^2 dx = 16$$

$$= \int_{0}^{1} b - 3x^2 dx = 8$$

$$= \left[bx - x^3\right]_0^1$$

$$= b - 1 = 8$$

$$b = 9$$

6 $y = 12 - 3x^2$

$= 3(4 - x^2)$

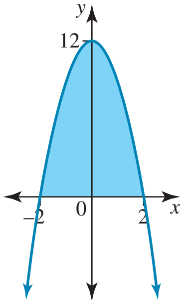
$= 3(2 + x)(2 - x)$

a $\int_{-2}^{2} \left(12 - 3x^2\right) dx = 2\int_{0}^{2} \left(12 - 3x^2\right) dx$

$= 2\left[12x - x^3\right]_0^2$

$= 2[24 - 8 - 0]$

$= 32$

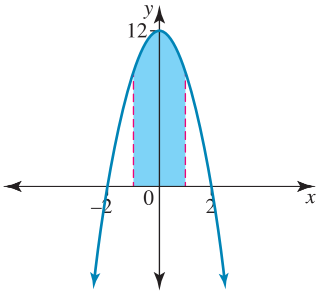


b $\int_{-1}^{1} \left(12 - 3x^2\right) dx = 2\int_{0}^{1} \left(12 - 3x^2\right) dx$

$= 2\left[12x - x^3\right]_0^1$

$= 2[12 - 1 - 0]$

$= 22$



7 $y = x^2 - 25$

$= (x + 5)(x - 5)$

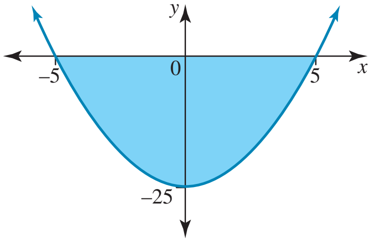
a $A = \left|\int_{-5}^{5} \left(x^2 - 25\right) dx\right|$

$$= 2\int_{0}^{5} \left(25 - x^2\right) dx$$

$$= 2\left[25x - \frac{1}{3}x^3\right]_0^5$$

$$= 2\left[25 \times 5 - \frac{1}{3} \times 125 - 0\right]$$

$$= 166\frac{2}{3}$$



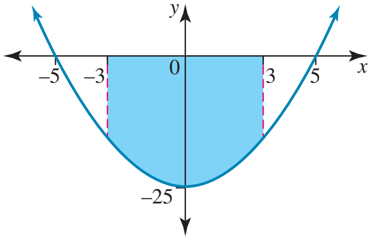
b $A = \left|\int_{-3}^{3} \left(x^2 - 25\right) dx\right|$

$$= 2\int_{0}^{3} \left(25 - x^2\right) dx$$

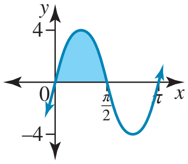
$$= 2\left[25x - \frac{1}{3}x^3\right]_0^3$$

$$= 2\left[75 - \frac{1}{3} \times 3 \times 9 - 0\right]$$

$$= 132$$



8 $y = 4\sin(2x)$



Amplitude = 4

Period $= \frac{2\pi}{2} = \pi$

$$A = \int_{0}^{\frac{\pi}{2}} 4\sin(2x)dx$$

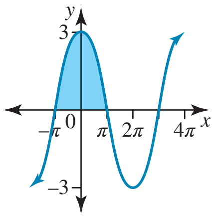
$$= \frac{4}{2}\left[-\cos(2x)\right]_0^{\frac{\pi}{2}}$$

$$= 2\left[-\cos(\pi) + \cos(0)\right]$$

$$= 2[1 + 1]$$

$$= 4$$

9 $y = 3\cos\left(\frac{x}{2}\right)$



Amplitude = 3

$$\text{Period} = \frac{2\pi}{\frac{1}{2}} = 4\pi$$

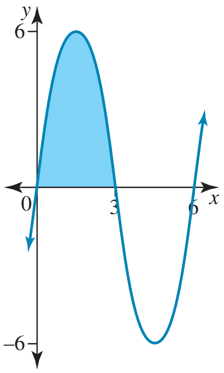
$$\begin{aligned} A &= \int_{-\pi}^{\pi} 3\cos\left(\frac{x}{2}\right) dx \\ &= 2\int_{0}^{\pi} 3\cos\left(\frac{x}{2}\right) dx \\ &= 6\left[2\sin\left(\frac{x}{2}\right)\right]_0^{\pi} \\ &= 12\left[\sin\left(\frac{\pi}{2}\right) - \sin(0)\right] \\ &= 12\,[1 - 0] \\ &= 12 \end{aligned}$$

10 a $y = 6\sin\left(\frac{\pi x}{3}\right)$

Amplitude 6

$$\text{Period } T = \frac{2\pi}{\frac{\pi}{3}} = 6$$

$$\begin{aligned} \int_0^3 6\sin\left(\frac{\pi x}{3}\right) dx &= 6\left[-\frac{3}{\pi}\cos\left(\frac{\pi x}{3}\right)\right]_0^3 \\ &= -\frac{18}{\pi}\,[\cos(\pi) - \cos(0)] \\ &= \frac{36}{\pi} \end{aligned}$$

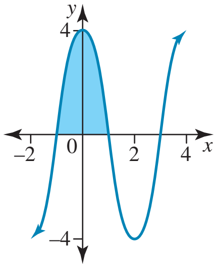


b $y = 4\cos\left(\frac{\pi x}{2}\right)$

Amplitude 4

$$\text{Period } T = \frac{2\pi}{\frac{\pi}{2}} = 4$$

$$\begin{aligned} \int_{-1}^{1} 4\cos\left(\frac{\pi x}{2}\right) dx &= 2\int_0^1 4\cos\left(\frac{\pi x}{2}\right) dx \\ &= 8 \times \frac{2}{\pi}\left[\sin\left(\frac{\pi x}{2}\right)\right]_0^1 \\ &= \frac{16}{\pi}\left[\sin\left(\frac{\pi}{2}\right) - \sin(0)\right] \\ &= \frac{16}{\pi} \end{aligned}$$



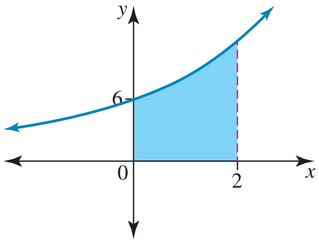
c $y = a\sin(nx)$

Amplitude a

$$\text{Period } T = \frac{2\pi}{n}$$

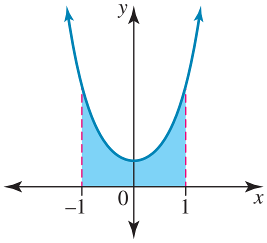
$$\begin{aligned} \text{Area} &= \int_0^{\frac{\pi}{n}} a\sin(nx)dx \\ &= -\frac{a}{n}\,[\cos(nx)]_0^{\frac{\pi}{n}} \\ &= -\frac{a}{n}\,[\cos(\pi) - \cos(0)] \\ &= \frac{2a}{n} \end{aligned}$$

11. $y = 6e^{3x}$



$$\begin{aligned} A &= \int_0^2 6e^{3x}dx \\ &= 6\left[\frac{1}{3}e^{3x}\right]_0^2 \\ &= 2\left[e^6 - e^0\right] \\ &= 2\left[e^6 - 1\right] \end{aligned}$$

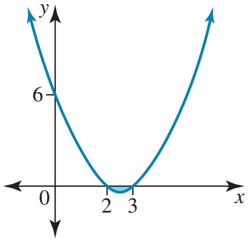
12 $y = 6\left(e^{-2x} + e^{2x}\right)$



$$\begin{aligned} A &= \int_{-1}^{1} 6\left(e^{-2x} + e^{2x}\right) dx \\ &= 12\int_0^1 \left(e^{-2x} + e^{2x}\right) dx \\ &= 12\left[-\frac{1}{2}e^{-2x} + \frac{1}{2}e^{2x}\right]_0^1 \\ &= 12\left[-\frac{1}{2}e^{-2} + \frac{1}{2}e^2 + \frac{1}{2} - \frac{1}{2}\right] \\ &= 6\left[e^2 - \frac{1}{e^2}\right] \end{aligned}$$

13 $y = x^2 - 5x + 6$

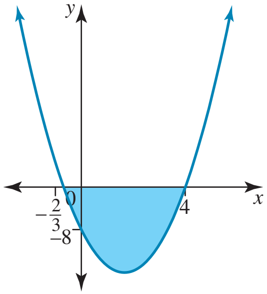
$= (x - 3)(x - 2)$



$$
\begin{aligned}
A &= \left|\int_2^3 x^2 - 5x + 6dx\right| \\
&= -\left[\frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x\right]_2^3 \\
&= -\left[\left(\frac{1}{3}\times 27 - \frac{5}{2}\times 9 + 18\right) - \left(\frac{1}{3}\times 8 - \frac{5}{2}\times 4 + 12\right)\right] \\
&= \frac{1}{6}
\end{aligned}
$$

14 $y = 3x^2 - 10x - 8$

$= (3x + 2)(x - 4)$

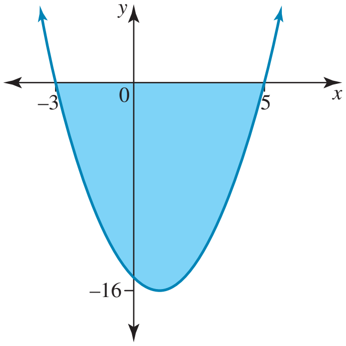


$$
\begin{aligned}
A &= \left|\int_{-\frac{2}{3}}^{4} 3x^2 - 10x - 8dx\right| \\
&= -\left[x^3 - 5x^2 - 8x\right]_{-\frac{2}{3}}^{4} \\
&= -\left[(4^3 - 5\times 4^2 - 8\times 4) - \left(\left(-\frac{2}{3}\right)^3\right.\right. \\
&\qquad \left.\left. -5\times\left(-\frac{2}{3}\right)^2 + 8\times\frac{2}{3}\right)\right] \\
&= \frac{1372}{27}
\end{aligned}
$$

15 $y = x^2 - 2x - 15$

$= (x - 5)(x + 3)$

$$
\begin{aligned}
A &= \left|\int_{-3}^{5} \left(x^2 - 2x - 15\right) dx\right| \\
&= -\int_{-3}^{5} \left(x^2 - 2x - 15\right) dx \\
&= -\left[\frac{1}{3}x^3 - x^2 - 15x\right]_{-3}^{5} \\
&= -\left[\left(\frac{1}{3}\times 125 - 25 - 15\times 5\right) - \left(\frac{1}{3}\times -27 - 9 + 45\right)\right] \\
&= 85\frac{1}{3}
\end{aligned}
$$

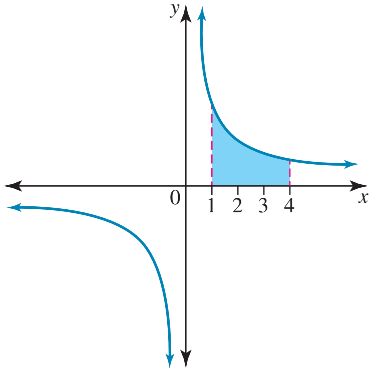


16 $y = \frac{1}{x}$

a $\int_1^4 \frac{1}{x}\,dx = \left[\log_e(x)\right]_1^4$

$= \log_e(4) - \log_e(1)$

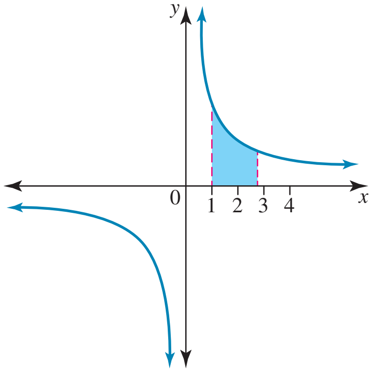
$= \log_e(4)$



b $\int_1^e \frac{1}{x}\,dx = \left[\log_e|x|\right]_1^e$

$= \log_e(e) - \log_e(1)$

$= 1$



c $\int_1^a \frac{1}{x}\,dx = \left[\log_e|x|\right]_1^a$

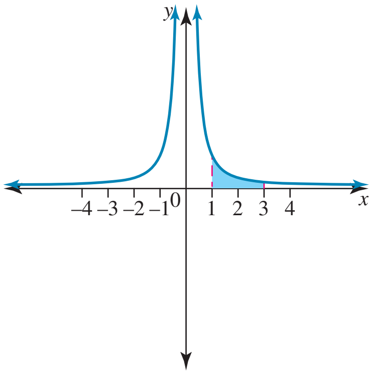
$= \log_e(a)$

17 $y = \frac{1}{x^2}$

a $A = \int_1^3 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^3$

$= -\frac{1}{3} + 1$

$= \frac{2}{3}$



b $A = \int_1^a \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^a$

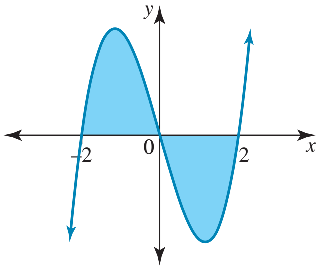
$= -\frac{1}{a} + 1$

$= 1 - \frac{1}{a}$

18 $y = x^3 - 4x$

$= x(x^2 - 4)$

$= x(x + 2)(x - 2)$



$$A = 2\int_{-2}^{0} x^3 - 4xdx$$

$$= 2\left[\frac{1}{4}x^4 - 2x^2\right]_{-2}^{0}$$

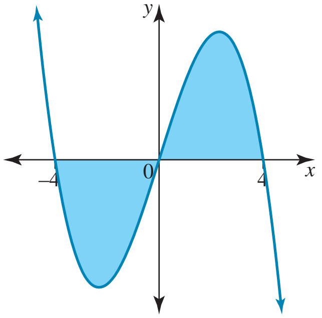
$$= 2\left[0 - \frac{1}{4}(-2)^4 + 2(-2)^2\right]$$

$$= 8$$

19 $y = 16x - x^3$

$= x(16 - x^2)$

$= x(4 + x)\,(4 - x)$



$$A = 2\int_0^4 16x - x^3dx$$

$$= 2\left[8x^2 - \frac{1}{4}x^4\right]_0^4$$

$$= 2\left[8 \times 16 - \frac{1}{4} \times 16^2\right]$$

$$= 128$$

20 $y = x^2 + 3x - 18$

$= (x + 6)\,(x - 3)$

$$A_1 = \int_{-3}^{3} \left(x^2 + 3x - 18\right) dx$$

$$= \left[\frac{1}{3}x^3 + \frac{3}{2}x^2 - 18x\right]_{-3}^{3}$$

$$= \left[\left(\frac{1}{3} \times 9 \times 3 + \frac{3}{2} \times 9 - 18 \times 3\right) - \left(\frac{1}{3} \times -3 \times 9 + \frac{3}{2} \times 9 + 18 \times 3\right)\right]$$

$$= -90$$

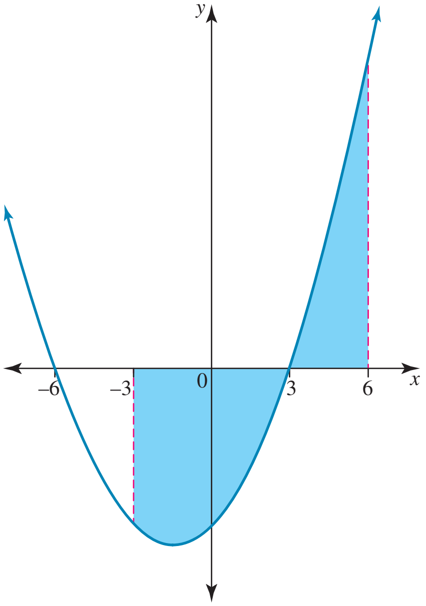
$$A_2 = \int_3^6 \left(x^2 + 3x - 18\right) dx$$

$$= \left[\frac{1}{3}x^3 + \frac{3}{2}x^2 - 18x\right]_3^6$$

$$= \left[\left(\frac{1}{3} \times 6 \times 36 + \frac{3}{2} \times 36 - 18 \times 6\right) - \left(\frac{1}{3} \times 9 \times 3 + \frac{3}{2} \times 9 - 18 \times 3\right)\right]$$

$$= 49.5$$

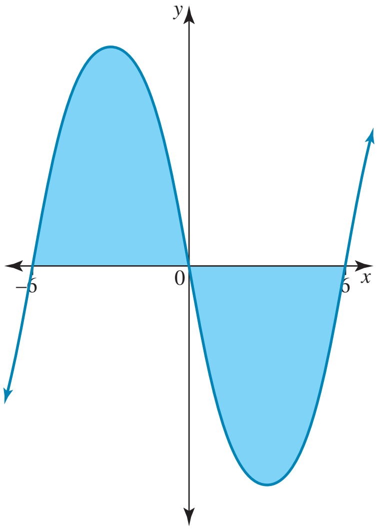
Area $= |-90| + 49.5 = 139.5$



21 $y = x^3 - 36x$

$= x(x^2 - 36)$

$= x(x + 6)(x - 6)$



a $A_1 = \int_0^6 \left(x^3 - 36x\right) dx$

$$= \left[\frac{1}{4}x^4 - 18x^2\right]_0^6$$

$$= \left[\frac{1}{4} \times 36 \times 36 - 18 \times 36 - 0\right]$$

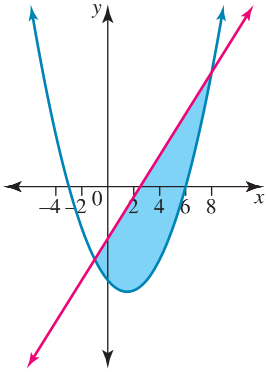
$$= -324$$

Required area = 648

b $$\int_{-3}^{0}\left(x^3-36x\right)dx=\left[\frac{1}{4}x^4-18x^2\right]_{-3}^{0}$$
$$=(0)-\left(\left(\frac{1}{4}\times(-3)^4\right)-\left(18\times(-3)^2\right)\right)$$
$$=141\frac{3}{4}$$

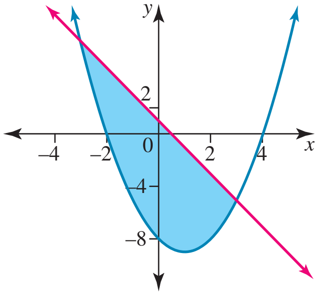
So required area
$$\int_{-3}^{0}\left(x^3-36x\right)\ dx+\left|\int_{0}^{6}\left(x^3-36x\right)\ dx\right|$$
$$=141\frac{3}{4}+324$$
$$=465\frac{3}{4}$$

22
$$\begin{aligned}y_1&=x^2-3x-18\\&=(x-6)(x+3)\\y_2&=4x-10\\y_1&=y_2\\x^2-3x-18&=4x-10\\x^2-7x-8&=0\\(x-8)(x+1)&=0\\x&=8,-1\end{aligned}$$



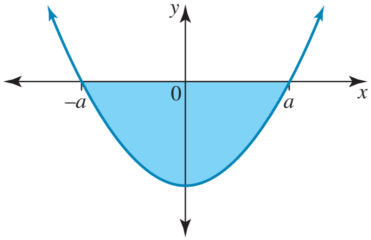
$$A=\int_a^b y_2-y_1dx$$
$$=\int_{-1}^{8}-x^2+7x+8dx$$
$$=\left[-\frac{1}{3}x^3+\frac{7}{2}x^2+8x\right]_{-1}^{8}$$
$$=\frac{243}{2}$$
$$=121\frac{1}{2}\text{units}^2$$

23
$$\begin{aligned}y_1&=x^2-2x-8\\&=(x-4)(x+2)\\y_2&=1-2x\\y_1&=y_2\\x^2-2x-8&=1-2x\\x^2-9&=0\\(x+3)(x-3)&=0\\x&=\pm3\end{aligned}$$



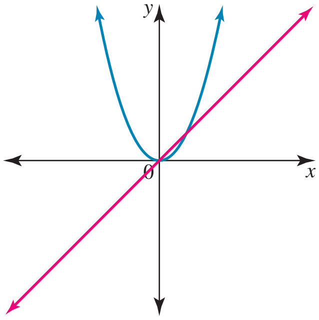
$$A=\int_{-3}^{3}9-x^2dx$$
$$=\left[9x-\frac{1}{3}x^3\right]_{-3}^{3}$$
$$=36$$

24 $y=x^2-a^2$
$=(x+a)(x-a)$



$$A=-\int_{-a}^{a}\left(x^2-a^2\right)\ dx$$
$$=-2\int_{0}^{a}\left(x^2-a^2\right)\ dx$$
$$=2\left[a^2x-\frac{1}{3}x^3\right]_{0}^{a}$$
$$=2\left[\left(a^3-\frac{1}{3}a^3\right)-0\right]$$
$$=\frac{4a^3}{3}$$

25 **a** $y_1=x^2$
$y_2=x$
$y_1=y_2$
$x^2=x$
$x^2-x=0$
$x(x-1)=0$
$x=0,1$



$$\begin{aligned}A &= \int_0^1 \left(x - x^2\right)\,dx\\ &= \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1\\ &= \frac{1}{2} - \frac{1}{3}\\ &= \frac{1}{6}\end{aligned}$$

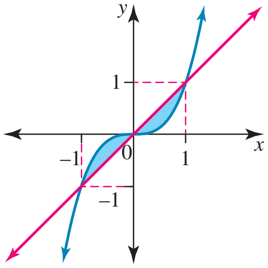
b $y_1 = x^3$

$y_2 = x$

$y_1 = y_2$

$x^3 = x$

$$\begin{aligned}x^3 - x &= 0\\ x\left(x^2 - x\right) &= 0\\ x(x+1)(x-1) &= 0\\ x = 0,\ x = -1,\ &x = 1\end{aligned}$$



$$\begin{aligned}A &= \int_0^1 \left(x - x^3\right)\,dx\\ &= \left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1\\ &= \left(\frac{1}{2} - \frac{1}{4}\right) - 0\\ &= \frac{1}{4}\end{aligned}$$

So area $= \frac{1}{2}$

c $y_1 = x^4$

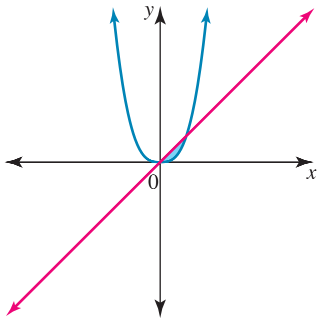
$y_2 = x$

$y_1 = y_2$

$x^4 = x$

$$\begin{aligned}x^4 - x &= 0\\ x(x^3 - 1) &= 0\\ x &= 0, 1\end{aligned}$$

$$\begin{aligned}A &= \int_0^1 \left(x - x^4\right)\,dx\\ &= \left[\frac{1}{2}x^2 - \frac{1}{5}x^5\right]_0^1\\ &= \left(\frac{1}{2} - \frac{1}{5}\right) - 0\\ &= \frac{3}{10}\end{aligned}$$



d $y_1 = x^5$

$y_2 = x$

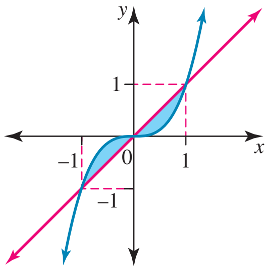
$y_1 = y_2$

$x^5 = x$

$$\begin{aligned}x^5 - x &= 0\\ x(x^4 - x) &= 0\\ x(x^2 + 1)\left(x^2 - 1\right) &= 0\\ x(x^2 + 1)(x+1)(x-1) &= 0\\ x = 0,\ \pm 1\end{aligned}$$

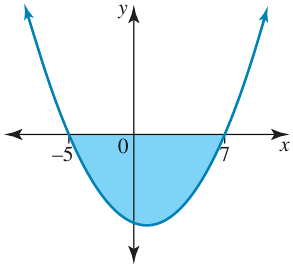
$$\begin{aligned}A &= \int_0^1 \left(x - x^5\right)\,dx\\ &= \left[\frac{1}{2}x^2 - \frac{1}{6}x^6\right]_0^1\\ &= \left(\frac{1}{2} - \frac{1}{6}\right) - 0\\ &= \frac{1}{3}\end{aligned}$$

Area is $= \frac{2}{3}$



26 a $y = x^2 - 2x - 35$

$= (x - 7)(x + 5)$



$$\begin{aligned} A &= \left|\int_{-5}^{7} \left(x^2 - 2x - 35\right) dx\right| \\ &= -\left[\frac{1}{3}x^3 - x^2 - 35x\right]_{-5}^{7} \\ &= -\left[\left(\frac{1}{3} \times 7 \times 49 - 49 - 35 \times 7\right)\right. \\ &\qquad \left. - \left(\frac{1}{3} \times -5 \times 25 - 25 + 35 \times 5\right)\right] \\ &= 288 \end{aligned}$$

b $y_1 = x^2 - 2x - 35$

$y_2 = 4x - 8$

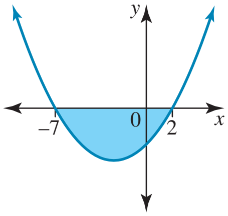
$y_1 = y_2$

$$\begin{aligned} x^2 - 2x - 35 &= 4x - 8 \\ x^2 - 6x - 27 &= 0 \\ (x - 9)(x + 3) &= 0 \\ x &= -3, 9 \end{aligned}$$

$$\begin{aligned} A &= \int_a^b (y_2 - y_1)\, dx = \int_{-3}^{9} \left(-x^2 + 6x + 27\right) dx \\ &= \left[-\frac{1}{3}x^3 + 3x^2 + 27x\right]_{-3}^{9} \\ &= \left(-\frac{1}{3} \times 9 \times 81 + 3 \times 81 + 27 \times 9\right) \\ &\qquad - \left(-\frac{1}{3} \times -3 \times 9 + 3 \times 9 - 27 \times 3\right) \\ &= 288 \end{aligned}$$

27 a $y = x^2 + 5x - 14$

$$\begin{aligned} &= (x + 7)(x - 2) \\ A &= \left|\int_{-7}^{2} \left(x^2 + 5x - 14\right) dx\right| \\ &= -\left[\frac{1}{3}x^3 + \frac{5}{2}x^2 - 14x\right]_{-7}^{2} \\ &= -\left[\left(\frac{1}{3} \times 8 + \frac{5}{2} \times 4 - 14 \times 2\right)\right. \\ &\qquad \left. - \left(\frac{1}{3} \times -7 \times 49 + \frac{5}{2} \times 49 + 14 \times 7\right)\right] \\ &= 121\frac{1}{2} \end{aligned}$$



b $y_1 = x^2 + 5x - 14$

$y_2 = 2x + 4$

$y_1 = y_2$

$$\begin{aligned} x^2 + 5x - 14 &= 2x + 4 \\ x^2 + 3x - 18 &= 0 \\ (x + 6)(x - 3) &= 0 \\ x &= 3, -6 \end{aligned}$$

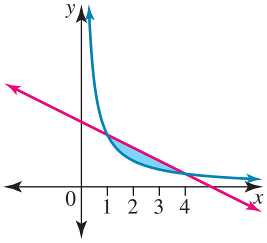
$$\begin{aligned} A &= \int_a^b (y_2 - y_1)\, dx \\ &= \int_{-6}^{3} \left(-x^2 - 3x + 18\right) dx \\ &= \left[-\frac{1}{3}x^3 - \frac{3}{2}x^2 + 18x\right]_{-6}^{3} \\ &= \left(-\frac{1}{3} \times 27 - \frac{3}{2} \times 9 + 18 \times 3\right) \\ &\qquad - \left(-\frac{1}{3} \times -6 \times 36 - \frac{3}{2} \times 36 - 18 \times 6\right) \\ &= 121\frac{1}{2} \end{aligned}$$

28 a $2y + x - 5 = 0$

$$\begin{aligned} 2y &= 5 - x \\ y_1 &= \frac{5 - x}{2} \\ y_2 &= \frac{2}{x} \end{aligned}$$

Points of intersection $y_1 = y_2$

$$\begin{aligned} \frac{5 - x}{2} &= \frac{2}{x} \\ 4 &= x(5 - x) \\ 4 &= 5x - x^2 \\ x^2 - 5x + 4 &= 0 \\ (x - 1)(x - 4) &= 0 \\ x &= 1, 4 \end{aligned}$$



$$\begin{aligned} A &= \int_1^4 \left(\frac{5 - x}{2} - \frac{2}{x}\right) dx \\ &= \int_1^4 \left(\frac{5}{2} - \frac{x}{2} - \frac{2}{x}\right) dx \\ &= \left[\frac{5x}{2} - \frac{x^2}{4} - 2\log_e |x|\right]_1^4 \\ &= \left(10 - 4 - 2\log_e(4)\right) - \left(\frac{5}{2} - \frac{1}{4} - 2\log_e(1)\right) \\ &= \frac{15}{4} - 2\log_e(4) \\ &\approx 0.9774 \end{aligned}$$

b $9y + 3x - 10 = 0$

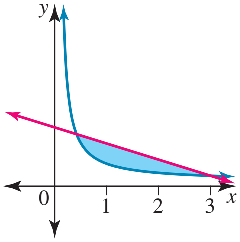
$$\begin{aligned} 9y &= 10 - 3x \\ y_1 &= \frac{10 - 3x}{9} \\ y_2 &= \frac{1}{3x} \end{aligned}$$

Points of intersection $y_1 = y_2$

$$\begin{aligned}\frac{10-3x}{9}&=\frac{1}{3x}\\ 10-3x&=\frac{9}{3x}=\frac{3}{x}\\ x(10-3x)&=3\\ 3x^2-10x+3&=0\\ (3x-1)(x-3)&=0\end{aligned}$$

$$x=\frac{1}{3},3$$

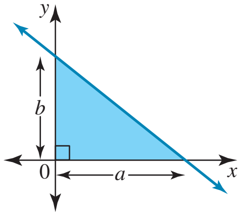
$$\begin{aligned}A&=\int_{\frac{1}{3}}^{3}\left(\frac{10-3x}{9}-\frac{1}{3x}\right)dx\\ &=\int_{\frac{1}{3}}^{3}\left(\frac{10}{9}-\frac{x}{3}-\frac{1}{3x}\right)dx\\ &=\left[\frac{10x}{9}-\frac{x^2}{6}-\frac{1}{3}\log_e|x|\right]_{\frac{1}{3}}^{3}\\ &=\left(\frac{10}{3}-\frac{3}{2}-\frac{1}{3}\log_e(3)\right)-\left(\frac{10}{27}-\frac{1}{54}-\frac{1}{3}\log_e\left(\frac{1}{3}\right)\right)\\ &=\frac{40}{27}+\frac{1}{3}\log_e\left(\frac{1}{9}\right)\\ &\approx 0.7491\end{aligned}$$



29 **a** $y=mx+c$

$$m=-\frac{b}{a}$$

$$c=b$$



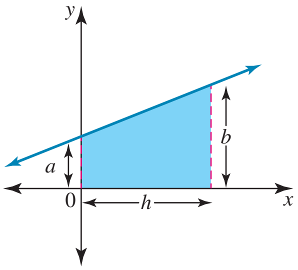
$$\begin{aligned}A&=\int\limits_0^a\left(b-\frac{b}{a}x\right)dx\\ &=\left[bx-\frac{bx^2}{2a}\right]_0^a\\ &=\left(ab-\frac{ba^2}{2a}\right)-(0)\\ &=\frac{1}{2}ab\end{aligned}$$

b $y=mx+c$

$$m=\frac{b-a}{h}$$

$$c=a$$

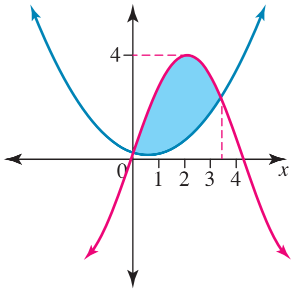
$$y=\left(\frac{b-a}{h}\right)x+a$$



$$\begin{aligned}A&=\int\limits_0^h\left(\left(\frac{b-a}{h}\right)x+a\right)dx\\ &=\left[\left(\frac{b-a}{2h}\right)x^2+ax\right]_0^h\\ &=\left(\left(\frac{b-a}{2h}\right)h^2+ah\right)-(0)\\ &=\frac{1}{2}(b-a)h+ah\\ &=h\left[\frac{1}{2}(b-a)+a\right]\\ &=\frac{1}{2}h[b-a+2a]\\ &=\frac{1}{2}h(b+a)\end{aligned}$$

30 **a** $y_1=\frac{x^2}{3}$

$$y_2=4\sin\left(\frac{x}{2}\right)$$



Using CAS:

$y_1=y_2$

$x=3.4443$

$$\begin{aligned}y&=\frac{(3.4443)^2}{3}\\ &=4\sin\left(\frac{3.4443}{2}\right)\\ &=3.9543\end{aligned}$$

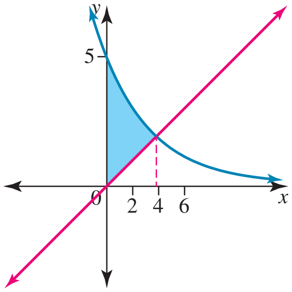
Coordinates point of intersection = (3.4443, 3.9543)

b $A=\int\limits_a^b(y_2-y_1)\,dx$

$$\begin{aligned}&=\int\limits_0^{3.4443}\left(4\sin\left(\frac{x}{2}\right)-\frac{x^2}{3}\right)dx\\ &=4.6662\end{aligned}$$

31 **a** $y_1 = \frac{x}{2}$

$y_2 = 5e^{-\frac{x}{4}}$



Using CAS

$y_1 = y_2$

$x = 3.8343$

$y = \frac{3.8343}{3} = 5e^{-\frac{3.8343}{4}} = 1.9172$

Coordinates of point of intersection = (3.8343, 1.9172)

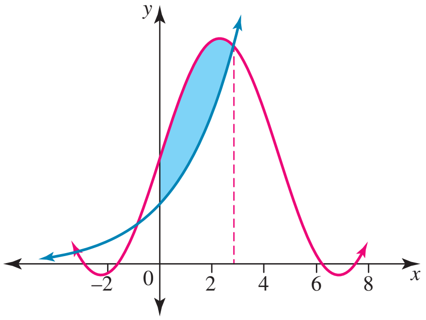
b $A = \int_a^b (y_2 - y_1)\, dx$

$= \int_0^{3.8343} \left(5e^{-\frac{x}{4}} - \frac{x}{2}\right) dx$

$= 8.6558$

32 **a** $y_1 = 23e^{\frac{x}{2}}$

$y_2 = 45 \sin\left(\frac{2x}{3}\right) + 42$



$y_1 = y_2$

$x = 2.6420$

$y = 23e^{\frac{2.6420}{2}} = 45 \sin\left(\frac{2}{3} \times 2.6420\right) + 42$

$= 86.1855$

Coordinates of point of intersection = (2.6420, 86.1855)

b $A = \int_0^{2.6420} \left(45 \sin\left(\frac{2x}{3}\right) + 42 - 23e^{\frac{x}{2}}\right) dx$

$= 64.8779$

7.2 Exam questions

1 $y = x^2 - 2x - 24$

$= (x - 6)(x + 4)$

$A_1 = \int_2^6 \left(x^2 - 2x - 24\right) dx$

$= \left[\frac{1}{3}x^3 - x^2 - 24x\right]_2^6$

$= \left[\left(\frac{1}{3} \times 6 \times 36 - 36 - 24 \times 6\right) - \left(\frac{1}{3} \times 8 - 4 - 48\right)\right]$

$= -58\frac{2}{3}$

$A_2 = \int_6^8 \left(x^2 - 2x - 24\right) dx$

$= \left[\frac{1}{3}x^3 - x^2 - 24x\right]_6^8$

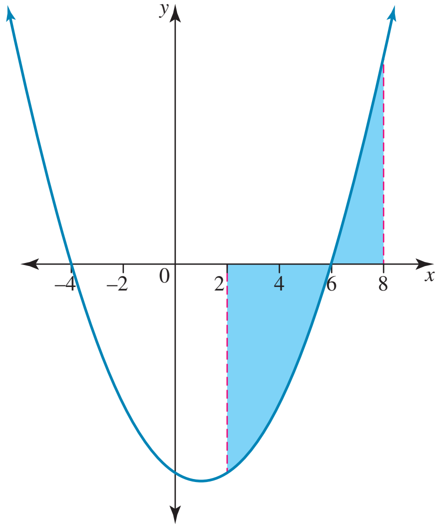
$= \left[\left(\frac{1}{3} \times 8^3 - 8^2 - 24 \times 8\right) - \left(\frac{1}{3} \times 6^3 - 6^2 - 24 \times 6\right)\right]$

$= 22\frac{2}{3}$

Area $= |A_1| + A_2$

$= \left|-58\frac{2}{3}\right| + 22\frac{2}{3}$

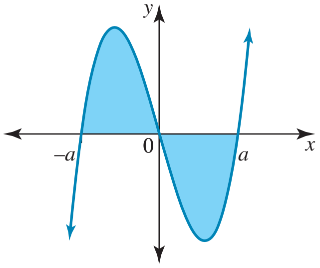
$= 81\frac{1}{3}$ [1 mark]



2 $y = x^3 - a^2x$

$= x(x^2 - a^2)$

$= x(x + a)(x - a)$



$$A_1 = \int_{-a}^{0} \left(x^3 - a^2x\right)\, dx$$
$$= \left[\frac{1}{4}x^4 - \frac{1}{2}a^2x^2\right]_{-a}^{0}$$
$$= (0) - \left[\frac{1}{4}a^4 - \frac{1}{2}a^2a^2\right]$$
$$= \frac{1}{4}a^4 \qquad \text{[1 mark]}$$

So required area $= \frac{1}{2}a^4$ [1 mark]

3 **a** $y_1 = \dfrac{190}{x^2} - 5$

$y_2 = -32\cos\left(\dfrac{x}{5}\right)$

$y_1 = y_2$

$x = 7.5882,\ 24.2955$

Points of intersection (7.5882, −1.7003)
(24.2955, −4.6781) [1 mark]

b $$A = \int_{7.5882}^{24.2955} \left(-32 \quad \cos\left(\frac{x}{5}\right) - \frac{190}{x^2} + 5\right) dx$$
$= 384.3732$ [1 mark]

7.3 Linear substitutions

7.3 Exercise

1 $\int (5x-9)^6\, dx$

Let
$u = 5x - 9$
$\dfrac{du}{dx} = 5$
$\dfrac{dx}{du} = \dfrac{1}{5}$
$dx = \dfrac{1}{5}du$

$$= \int u^6 \times \frac{1}{5}du$$
$$= \frac{1}{5}\int u^6 du$$
$$= \frac{1}{5}\left[\frac{1}{7}u^7\right] + c$$
$$= \frac{1}{35}(5x-9)^7 + c$$

2 $\int (3x+4)^7\, dx$

Let
$u = 3x + 4$
$\dfrac{du}{dx} = 3$
$\dfrac{dx}{du} = \dfrac{1}{3}$
$dx = \dfrac{1}{3}du$

$$= \int u^7 \times \frac{1}{3}du$$
$$= \frac{1}{3}\int u^7 du$$
$$= \frac{1}{3}\left[\frac{1}{8}u^8\right] + c$$
$$= \frac{1}{24}(3x+4)^8 + c$$

3 Let
$u = 3x + 5$
$\dfrac{du}{dx} = 3$
$\dfrac{dx}{du} = \dfrac{1}{3}$
$dx = \dfrac{1}{3}du$

a $$\int (3x+5)^6\, dx = \int u^6 \cdot \frac{1}{3}du$$
$$= \frac{1}{3}\int u^6 du$$
$$= \frac{1}{3}\left(\frac{1}{7}u^7\right) + c$$
$$= \frac{1}{21}(3x+5)^7 + c$$

b $$\int \frac{1}{(3x+5)^2}\, dx = \int \frac{1}{u^2} \cdot \frac{1}{3}du$$
$$= \frac{1}{3}\int u^{-2} du$$
$$= \frac{1}{3}\left(-u^{-1}\right) + c$$
$$= -\frac{1}{3(3x+5)} + c$$

c $$\int \frac{1}{(3x+5)^3}\, dx = \int \frac{1}{u^3} \cdot \frac{1}{3}du$$
$$= \frac{1}{3}\int u^{-3} du$$
$$= \frac{1}{3}\left(-\frac{1}{2}u^{-2}\right) + c$$
$$= -\frac{1}{6u^2} + c$$
$$= -\frac{1}{6(3x+5)^2} + c$$

d $$\int \frac{1}{\sqrt[3]{3x+5}}\, dx = \int u^{-\frac{1}{3}} \cdot \frac{1}{3}du$$
$$= \frac{1}{3}\int u^{-\frac{1}{3}} du$$
$$= \frac{1}{3} \times \frac{3}{2}u^{\frac{2}{3}} + c$$
$$= \frac{1}{2}\sqrt[3]{u^2} + c$$
$$= \frac{1}{2}\sqrt[3]{(3x+5)^2} + c$$

4 $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{16x+25}}$

$$y = \int \frac{1}{\sqrt{16x+25}}dx$$

Let

$$u = 16x + 25$$

$$\frac{du}{dx} = 16$$

$$\frac{dx}{du} = \frac{1}{16}$$

$$dx = \frac{1}{16}du$$

$$y = \int u^{-\frac{1}{2}} \times \frac{1}{16}du$$

$$= \frac{2}{16}u^{\frac{1}{2}} + c$$

$$y = \frac{1}{8}\sqrt{16x + 25} + c$$

When $x = 0, y = 0$

$$0 = \frac{1}{8}\sqrt{25} + c$$

$$c = -\frac{5}{8}$$

$$y = \frac{1}{8}\left(\sqrt{16x + 25} - 5\right)$$

5 $f'(x) = \dfrac{1}{(3x - 7)^2}$

$$f(x) = \int \frac{1}{(3x - 7)^2}\,dx$$

$$u = 3x - 7$$

$$\frac{du}{dx} = 3$$

$$\frac{dx}{du} = \frac{1}{3}$$

$$dx = \frac{1}{3}du$$

$$f(x) = \int u^{-2} \times \frac{1}{3}du$$

$$= -\frac{1}{3}u^{-1} + c$$

$$= -\frac{1}{3\,(3x - 7)} + c$$

$f(2) = 3 \rightarrow y = 3$ when $x = 2$

$$3 = \frac{1}{3} + c$$

$$c = \frac{8}{3}$$

$$f(x) = -\frac{1}{3\,(3x - 7)} + \frac{8}{3}$$

$$f(1) = \frac{11}{4}$$

6 $\displaystyle\int \frac{1}{3x - 5}dx = \frac{1}{3}\log_e|3x - 5| + c$

7 $\displaystyle\int \frac{1}{7 - 2x}dx = -\frac{1}{2}\log_e|7 - 2x| + c$

8 Let

$$u = 6x + 7$$

$$\frac{du}{dx} = 6$$

$$\frac{dx}{du} = \frac{1}{6}$$

$$dx = \frac{1}{6}du$$

a $\displaystyle\int (6x + 7)^8\,dx = \int u^8 \cdot \frac{1}{6}du$

$$= \frac{1}{6}\int u^8 du$$

$$= \frac{1}{3} \times \frac{1}{9}u^9 + c$$

$$= \frac{1}{54}\,(6x + 7)^9 + c$$

b $\displaystyle\int \frac{1}{\sqrt{6x + 7}}\,dx = \int u^{-\frac{1}{2}} \cdot \frac{1}{6}du$

$$= \frac{1}{6}\int u^{-\frac{1}{2}} du$$

$$= \frac{1}{6} \times 2u^{\frac{1}{2}} + c$$

$$= \frac{1}{3}\sqrt{6x + 7} + c$$

c $\displaystyle\int \frac{1}{(6x + 7)}\,dx = \int \frac{1}{u} \cdot \frac{1}{6}du$

$$= \frac{1}{6}\int \frac{1}{u}\,du$$

$$= \frac{1}{6}\log_e|u| + c$$

$$= \frac{1}{6}\log_e|(6x + 7)| + c$$

d $\displaystyle\int \frac{1}{(6x + 7)^2}\,dx = \int \frac{1}{u^2} \cdot \frac{1}{6}du$

$$= \frac{1}{6}\int u^{-2}\,du$$

$$= \frac{1}{6}\left(-u^{-1}\right) + c$$

$$= -\frac{1}{6\,(6x + 7)} + c$$

9 $\displaystyle\int_{-1}^{0} \frac{9}{(2x + 3)^2}dx$

Let

$$u = 2x + 3$$

$$\frac{du}{dx} = 2$$

$$\frac{dx}{du} = \frac{1}{2}$$

$$dx = \frac{1}{2}du$$

Terminals

$x = 0, u = 3$

$x = -1, u = 1$

$$\int_1^3 9u^{-3} \times \frac{1}{2}du = \frac{9}{2}\int_1^3 u^{-3}du$$

$$= \frac{9}{2}\left[-\frac{1}{2}u^{-2}\right]_1^3$$

$$= -\frac{9}{4}\left[\frac{1}{u^2}\right]_1^3$$

$$= 2$$

10 $A = \int_0^4 \frac{6}{\sqrt{3x+4}}dx$

Let

$$u = 3x + 4$$
$$\frac{du}{dx} = 3$$
$$\frac{dx}{du} = \frac{1}{3}$$
$$dx = \frac{1}{3}du$$

Terminals

$x = 4, u = 16$

$x = 0, u = 4$

$$\int_4^{16} 6u^{-\frac{1}{2}} \times \frac{1}{3}du = 2\int_4^{16} u^{-\frac{1}{2}}du$$
$$= 2 \times 2\left[u^{\frac{1}{2}}\right]_4^{16}$$
$$= 4\left[\sqrt{16} - \sqrt{4}\right]$$
$$= 8$$

11 a $\int x(5x-9)^5\,dx$

Let

$$u = 5x - 9$$
$$\frac{du}{dx} = 5$$
$$\frac{dx}{du} = \frac{1}{5}$$
$$dx = \frac{1}{5}du$$
$$= \int \frac{1}{5}(u+9)u^5 \times \frac{1}{5}du$$
$$= \frac{1}{25}\int (u+9)u^5 du$$
$$= \frac{1}{25}\int u^6 + 9u^5 du$$
$$= \frac{1}{25}\left[\frac{1}{7}u^7 + \frac{3}{2}u^6\right] + c$$
$$= \frac{1}{175}(5x-9)^7 + \frac{3}{50}(5x-9)^6 + c$$

b $\int \frac{2x-1}{9x^2-24x+16}dx = \int \frac{2x-1}{(3x-4)^2}dx$

Let

$$u = 3x - 4$$
$$\frac{du}{dx} = 3$$
$$\frac{dx}{du} = \frac{1}{3}$$
$$dx = \frac{1}{3}du$$
$$3x = u + 4$$
$$x = \frac{1}{3}(u+4)$$
$$2x = \frac{2}{3}(u+4)$$
$$= \frac{2u}{3} + \frac{8}{3}$$
$$2x - 1 = \frac{2u}{3} + \frac{8}{3} - 1$$
$$\frac{2u}{3} + \frac{5}{3}$$
$$= \frac{1}{3}(2u + 5)$$
$$= \int \frac{\frac{1}{3}(2u+5)}{u^2} \times \frac{1}{3}du$$
$$= \frac{1}{9}\int \frac{(2u+5)}{u^2}du$$
$$= \frac{1}{9}\int \frac{2}{u} + 5u^{-2}du$$
$$= \frac{1}{9}\left[2\log_e|u| - 5u^{-1}\right] + c$$
$$= \frac{2}{9}\log_e|3x-4| - \frac{5}{9(3x-4)} + c$$

12 a $\int \frac{x}{(2x+7)^3}dx$

Let

$$u = 2x + 7$$
$$\frac{du}{dx} = 2$$
$$\frac{dx}{du} = \frac{1}{2}$$
$$dx = \frac{1}{2}du$$
$$2x = u - 7$$
$$x = \frac{1}{2}(u-7)$$
$$= \int \frac{\frac{1}{2}(u-7)}{u^3} \times \frac{1}{2}du$$
$$= \frac{1}{4}\int \frac{(u-7)}{u^3}du$$
$$= \frac{1}{4}\int u^{-2} - 7u^{-3}du$$
$$= \frac{1}{4}\left[-u^{-1} + \frac{7}{2}u^{-2}\right] + c$$
$$= \frac{1}{4}\left[\frac{7-2u}{2u^2}\right] + c$$
$$= \frac{1}{8u^2}[7 - 2(2x+7)] + c$$
$$= \frac{1}{8u^2}[7 - 4x - 14] + c$$
$$= \frac{1}{8(2x+7)^2}(-4x-7) + c$$
$$= \frac{-(4x+7)}{8(2x+7)^2} + c$$

b $\displaystyle\int \frac{x}{\sqrt{6x+5}}dx$

Let

$$u = 6x + 5$$

$$\frac{du}{dx} = 6$$

$$\frac{dx}{du} = \frac{1}{6}$$

$$dx = \frac{1}{6}du$$

$$6x = u - 5$$

$$x = \frac{1}{6}(u-5)$$

$$= \int \frac{\frac{1}{6}(u-5)}{u^{\frac{1}{2}}} \times \frac{1}{6}du$$

$$= \frac{1}{36}\int (u-5)\,u^{-\frac{1}{2}}du$$

$$= \frac{1}{36}\int u^{\frac{1}{2}} - 5u^{-\frac{1}{2}}du$$

$$= \frac{1}{36}\left[\frac{2}{3}u^{\frac{3}{2}} - 10u^{\frac{1}{2}}\right] + c$$

$$= \frac{1}{36}\left[2u^{\frac{1}{2}}\left(\frac{u}{3} - 5\right)\right] + c$$

$$= \frac{1}{18}\sqrt{u}\left[\frac{u-15}{3}\right] + c$$

$$= \frac{1}{54}\sqrt{u}\,[6x + 5 - 15] + c$$

$$= \frac{1}{54}\sqrt{u}\,[6x - 10] + c$$

$$= \frac{1}{27}(3x-5)\sqrt{6x+5} + c$$

13 Let

$$u = 3x + 5$$

$$\frac{du}{dx} = 3$$

$$\frac{dx}{du} = \frac{1}{3}$$

$$dx = \frac{1}{3}du$$

$$3x = u - 5$$

$$x = \frac{1}{3}(u-5)$$

a $\displaystyle\int x(3x+5)^6\,dx = \int \frac{1}{3}(u-5)\,u^6 \cdot \frac{1}{3}\,du$

$$= \frac{1}{9}\int (u-5)\,u^6 du$$

$$= \frac{1}{9}\int \left(u^7 - 5u^6\right)\,du$$

$$= \frac{1}{9}\left[\frac{1}{8}u^8 - \frac{5}{7}u^7\right] + c$$

$$= \frac{1}{72}(3x+5)^8 - \frac{5}{63}(3x+5)^7 + c$$

Or

$$= \frac{1}{9}\left[\frac{u^7}{56}(7u-40)\right] + c$$

$$= \frac{1}{504}(7(3x+5) - 40)(3x+5)^7 + c$$

$$= \frac{1}{504}(21x-5)(3x+5)^7 + c$$

b $\displaystyle\int \frac{1}{(3x+5)^2}\,dx = \int \frac{\frac{1}{3}(u-5)}{u^2} \cdot \frac{1}{3}\,du$

$$= \frac{1}{9}\int \frac{(u-5)}{u^2}\,du$$

$$= \frac{1}{9}\int \left(\frac{1}{u} - \frac{5}{u^2}\right)\,du$$

$$= \frac{1}{9}\int \left(\frac{1}{u} - 5u^{-2}\right)\,du$$

$$= \frac{1}{9}\left[\log_e|u| + 5u^{-1}\right] + c$$

$$= \frac{1}{9}\log_e|3x+5| + \frac{5}{9(3x+5)} + c$$

c $\displaystyle\int \frac{x}{(3x+5)^3}\,dx = \int \frac{\frac{1}{3}(u-5)}{u^3} \cdot \frac{1}{3}\,du$

$$= \frac{1}{9}\int (u-5)\,u^{-3}du$$

$$= \frac{1}{9}\int u^{-2} - 5u^{-3}du$$

$$= \frac{1}{9}\left[-u^{-1} + \frac{5}{2}u^{-2}\right] + c$$

$$= \frac{1}{9}\left[-\frac{1}{u} + \frac{5}{2u^2}\right] + c$$

$$= \frac{1}{9}\left[\frac{-2u+5}{2u^2}\right] + c$$

$$= \frac{1}{18}\left(\frac{-(3x+5)+5}{(3x+5)^2}\right) + c$$

$$= \frac{-(6x+5)}{18(3x+5)^2} + c$$

d $\displaystyle\int \frac{x}{\sqrt[3]{3x+5}}\,dx = \int \frac{1}{3}(u-5)\,u^{-\frac{1}{3}} \cdot \frac{1}{3}\,du$

$$= \frac{1}{9}\int u^{\frac{2}{3}} - 5u^{-\frac{1}{3}}du$$

$$= \frac{1}{9}\left[\frac{3}{5}u^{\frac{5}{3}} - \frac{15}{2}u^{\frac{2}{3}}\right] + c$$

$$= \frac{1}{9}\left[\frac{u^{\frac{2}{3}}(6u-75)}{10}\right] + c$$

$$= \frac{1}{90}[6(3x+5) - 75]\,u^{\frac{2}{3}} + c$$

$$= \frac{1}{90}(18x-45)\,u^{\frac{2}{3}} + c$$

$$= \frac{1}{90} \times 9(2x-5)\,u^{\frac{2}{3}} + c$$

$$= \frac{1}{10}(2x-5)(3x+5)^{\frac{2}{3}} + c$$

14 Let

$u = 6x + 7$

$\frac{du}{dx} = 6$

$\frac{dx}{du} = \frac{1}{6}$

$dx = \frac{1}{6}du$

$6x = u - 7$

$x = \frac{1}{6}(u - 7)$

a
$$\begin{aligned}\int x(6x+7)^8\,dx &= \int \frac{1}{6}(u-7)\,u^8 \cdot \frac{1}{6}du\\ &= \frac{1}{36}\int u^9 - 7u^8 du\\ &= \frac{1}{36}\left[\frac{1}{10}u^{10} - \frac{7}{9}u^9\right] + c\\ &= \frac{1}{360}(6x+7)^{10} - \frac{7}{324}(6x+7)^9 + c\end{aligned}$$

Or

$$\begin{aligned}\frac{1}{36}\left[\frac{u^9}{90}(9u-70)\right] + c &= \frac{1}{3240}(6x+7)^9(9(6x+7)-70) + c\\ &= \frac{1}{3240}(54x-7)(6x+7)^9 + c\end{aligned}$$

b
$$\begin{aligned}\int \frac{x}{\sqrt{6x+7}}dx &= \int \frac{\frac{1}{6}(u-7)}{\sqrt{u}} \cdot \frac{1}{6}du\\ &= \frac{1}{36}\int (u-7)\;u^{-\frac{1}{2}}du\\ &= \frac{1}{36}\int u^{\frac{1}{2}} - 7u^{-\frac{1}{2}}du\\ &= \frac{1}{36}\left[\frac{2}{3}u^{\frac{3}{2}} - 14u^{\frac{1}{2}}\right] + c\\ &= \frac{1}{36} \times 2u^{\frac{1}{2}}\left[\frac{u}{3} - 7\right] + c\\ &= \frac{\sqrt{u}}{18}\left(\frac{u-21}{3}\right) + c\\ &= \frac{\sqrt{u}}{54}(6x+7-21) + c\\ &= \frac{\sqrt{u}}{54} \times 2(3x-7) + c\\ &= \frac{1}{27}(3x-7)\sqrt{6x+7} + c\end{aligned}$$

c
$$\begin{aligned}\int \frac{x}{(6x+7)}\,dx &= \int \frac{\frac{1}{6}(u-7)}{u} \cdot \frac{1}{6}du\\ &= \frac{1}{36}\int \frac{(u-7)}{u}du\\ &= \frac{1}{36}\int 1 - \frac{7}{u}du\\ &= \frac{1}{36}\left[u - 7\log_e|(u)|\right] + c\\ &= \frac{6x+7}{36} - \frac{7}{36}\log_e|(6x+7)| + c\\ &= \frac{x}{6} - \frac{7}{36}\log_e|(6x+7)| + d\end{aligned}$$

d
$$\begin{aligned}\int \frac{x}{(6x+7)^2}\,dx &= \int \frac{\frac{1}{6}(u-7)}{u^2} \cdot \frac{1}{6}du\\ &= \frac{1}{36}\int \frac{(u-7)}{u^2}\,du\\ &= \frac{1}{36}\int u^{-1} - 7u^{-2}du\\ &= \frac{1}{36}\left[\log_e|u| + 7u^{-1}\right] + c\\ &= \frac{7}{36(6x+7)} + \frac{1}{36}\log_e|(6x+7)| + c\end{aligned}$$

15 $\int \frac{6x}{3x+4}dx$

Let

$u = 3x + 4$

$\frac{du}{dx} = 3$

$\frac{dx}{du} = \frac{1}{3}$

$dx = \frac{1}{3}du$

$3x = u - 4$

$x = \frac{1}{3}(u - 4)$

$$\begin{aligned}6x &= \frac{6}{3}(u-4)\\ &= 2x - 8\\ &= \int \frac{2u-8}{u} \times \frac{1}{3}du\\ &= \frac{1}{3}\int 2 - \frac{8}{u}du\\ &= \frac{1}{3}\left[2u - 8\log_e|u|\right] + c\\ &= \frac{2}{3}(3x+4) - \frac{8}{3}\log_e|3x+4| + c\\ &= 2x - \frac{8}{3}\log_e|3x+4| + d\end{aligned}$$

16 $\int_0^1 \frac{4x}{2x-5}dx$

Let

$u = 2x - 5$

$\frac{du}{dx} = 2$

$\frac{dx}{du} = \frac{1}{2}$

$dx = \frac{1}{2}du$

$2x = u + 5$

$x = \frac{1}{2}(u + 5)$

$4x = 2u + 10$

Terminals

$x = 1, u = -3$

$x = 0, u = -5$

$$= \frac{1}{2}\int_{-5}^{-3} \frac{2u+10}{u}\,du$$
$$= \frac{1}{2}\int_{-5}^{-3} 2+\frac{10}{u}\,du$$
$$= \frac{1}{2}\left[2u+10\log_e|u|\right]_{-5}^{-3}$$
$$= \frac{1}{2}\left[-6+10\log_e(3)+10-10\log_e(5)\right]$$
$$= 2+5\log_e\left(\frac{3}{5}\right)$$

17 $\int\limits_0^5 \frac{x}{\sqrt{3x+1}}dx$

Let

$u = 3x+1$

$\frac{du}{dx} = 3$

$\frac{dx}{du} = \frac{1}{3}$

$dx = \frac{1}{3}du$

$3x = u-1$

$x = \frac{1}{3}(u-1)$

Terminals

$x = 0, u = 1$

$x = 5, u = 16$

$$= \int_1^{16} \frac{\frac{1}{3}(u-1)}{\sqrt{u}} \times \frac{1}{3}du$$
$$= \frac{1}{9}\int_1^{16} (u-1)\,u^{-\frac{1}{2}}du$$
$$= \frac{1}{9}\int_1^{16} u^{\frac{1}{2}} - u^{-\frac{1}{2}}du$$
$$= \frac{1}{9}\left[\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right]_1^{16}$$
$$= \frac{1}{9}\left[\frac{2}{3}(16)^{\frac{3}{2}} - 2\sqrt{16} - \frac{2}{3}(1)^{\frac{3}{2}} + 2\sqrt{1}\right]$$
$$= 4$$

18 $\int\limits_0^1 \frac{15x}{(3x+2)^2}dx$

Let

$u = 3x+2$

$\frac{du}{dx} = 3$

$\frac{dx}{du} = \frac{1}{3}$

$dx = \frac{1}{3}du$

$3x = u-2$

$x = \frac{1}{3}(u-2)$

$15x = 5(u-2)$

Terminals

$x = 1, u = 5$

$x = 0, u = 2$

$$= \int_2^5 \frac{5(u-2)}{u^2} \times \frac{1}{3}du$$
$$= \frac{5}{3}\int_2^5 \frac{(u-2)}{u^2}\,du$$
$$= \frac{5}{3}\int_2^5 u^{-1} - 2u^{-2}du$$
$$= \frac{5}{3}\left[\log_e|u| + 2u^{-1}\right]_2^5$$
$$= \frac{5}{3}\left[\log_e|u| + \frac{2}{u}\right]_2^5$$
$$= \frac{5}{3}\left[\log_e(5) + \frac{2}{5} - \log_e(2) - \frac{2}{2}\right]$$
$$= \frac{5}{3}\left[\log_e\left(\frac{5}{2}\right) - \frac{3}{5}\right]$$
$$= \frac{5}{3}\log_e\left(\frac{5}{2}\right) - 1$$

19 a $\frac{dx}{dt} = \frac{1}{(2-5t)^2}$

$x(0) = 0$

$x = \int \frac{1}{(2-5t)^2}dt$

Let

$u = 2-5t$

$\frac{du}{dt} = -5$

$\frac{dt}{du} = -\frac{1}{5}$

$dt = -\frac{1}{5}du$

$x = \int u^{-2} \cdot -\frac{1}{5}du$

$= \frac{1}{5}u^{-1} + c$

$x = \frac{1}{5(2-5t)} + c$

When $t = 0,\ x = 0$

$0 = \frac{1}{10} + c \rightarrow c = -\frac{1}{10}$

$x = \frac{1}{5(2-5t)} - \frac{1}{10}$

$= \frac{2-(2-5t)}{10(2-5t)}$

$x(t) = \frac{t}{2(2-5t)}$

b $\frac{dy}{dx} = \frac{1}{\sqrt{3-2x}}; x < \frac{3}{2}$

$y = \int (3-2x)^{-\frac{1}{2}} \cdot dx$

Let

$u = 3-2x$

$\frac{du}{dx} = -2$

$dx = -\frac{1}{2}du$

$y = \int u^{-\frac{1}{2}} \cdot -\frac{1}{2} \cdot dx = -u^{\frac{1}{2}} + c$

$y = -\sqrt{3-2x} + c$

When $x = -\frac{1}{2}, y = -2$

$$-2 = -\sqrt{4} + c \to c = 0$$

$$y = -\sqrt{3 - 2x}$$

20 $f'(x) = \frac{3}{3 - 2x}$

$$f(x) = \int \frac{3}{3 - 2x}\, dx = -\frac{3}{2} \log_e |(3 - 2x)| + c$$

When $x = 0, y = 0$

$$0 = -\frac{3}{2} \log_e |3| + c \to c = \frac{3}{2} \log_e |3|$$

$$f(x) = \frac{3}{2} \log_e |3| - \frac{3}{2} \log_e |3 - 2x|$$

$$= \frac{3}{2} \log_e \left(\frac{3}{|3 - 2x|} \right)$$

$$f(1) = \frac{3}{2} \log_e (3)$$

21 $\frac{dy}{dx} = \frac{x}{\sqrt{2x + 9}}$

$$y = \int \frac{x}{\sqrt{2x + 9}} dx$$

Let

$$u = 2x + 9$$

$$\frac{du}{dx} = 2$$

$$dx = \frac{1}{2} du$$

$$x = \frac{1}{2} (u - 9)$$

$$y = \int \frac{\frac{1}{2}(u - 9)}{\sqrt{u}} \cdot \frac{1}{2}\, du = \frac{1}{4} \int (u - 9)\, u^{-\frac{1}{2}} du$$

$$= \frac{1}{4} \int u^{\frac{1}{2}} - 9u^{-\frac{1}{2}} du$$

$$= \frac{1}{4} \left[\frac{2}{3} u^{\frac{3}{2}} - 18u^{\frac{1}{2}} \right] + c$$

$$= \frac{1}{4} u^{\frac{1}{2}} \left(\frac{2u}{3} - 18 \right) + c$$

$$= \frac{1}{4} \sqrt{u} \left(\frac{2\,(u - 27)}{3} \right) + c$$

$$= \frac{1}{2} \sqrt{u} \times \frac{1}{3} (2x + 9 - 27) + c$$

$$y = \frac{1}{3} (x - 9) \sqrt{2x + 9} + c$$

When $x = 0, y = 0$

$$0 = \frac{1}{3} \times -9\sqrt{9} + c \to c = 9$$

$$y = \frac{1}{3} (x - 9) \sqrt{2x + 9} + 9$$

22 **a** $\int_1^2 (3x - 4)^5\, dx$

Let

$$u = 3x - 4$$

$$\frac{du}{dx} = 3$$

$$dx = \frac{1}{3} du$$

Terminals

$x = 2, u = 2$

$x = 1, u = -1$

$$= \int_{-1}^{2} u^5 \cdot \frac{1}{3} du$$

$$= \frac{1}{3} \int_{-1}^{2} u^5 du$$

$$= \frac{1}{3} \left[\frac{1}{6} u^6 \right]_{-1}^{2}$$

$$= \frac{1}{18} \left(2^6 - (-1)^6 \right)$$

$$= \frac{7}{2}$$

b $\int_1^2 x\,(3x - 4)^5\, dx$

Let

$u = 3x - 4$

$$x = \frac{1}{3} (u + 4)$$

$$= \int_{-1}^{2} \frac{1}{3} (u + 4)\, u^5 \cdot \frac{1}{3} du$$

$$= \frac{1}{9} \int_{-1}^{2} (u + 4)\, u^5 du$$

$$= \frac{1}{9} \int_{-1}^{2} u^6 + 4u^5 du$$

$$= \frac{1}{9} \left[\frac{1}{7} u^7 + \frac{4}{6} u^6 \right]_{-1}^{2}$$

$$= \frac{1}{9} \left[\frac{1}{7} (2)^7 + \frac{2}{3} (2)^6 - \frac{1}{7} (-1)^7 - \frac{2}{3} (-1)^6 \right]$$

$$= \frac{1}{9} \left[\frac{1}{7} (128 + 1) + \frac{2}{3} (64 - 1) \right]$$

$$= \frac{47}{7}$$

c $\int_0^{13} \frac{1}{\sqrt[3]{2t + 1}}\, dt$

Let

$$u = 2t + 1$$

$$\frac{du}{dt} = 2$$

$$dt = \frac{1}{2} du$$

Terminals

$t = 13, \; u = 27$

$t = 0, \; u = 1$

$$\int_1^{27} u^{-\frac{1}{3}} \cdot \frac{1}{2} du = \frac{1}{2}\left[\frac{3}{2}u^{\frac{2}{3}}\right]_1^{27}$$

$$= \frac{3}{4}\left[27^{\frac{2}{3}} - 1^{\frac{2}{3}}\right]$$

$$= \frac{3}{4}[9 - 1]$$

$$= 6$$

d $\displaystyle\int_0^{13} \frac{t}{\sqrt[3]{2t+1}} dt = \int_1^{27} \frac{\frac{1}{2}(u-1)}{u^{\frac{1}{3}}} \cdot \frac{1}{2} du$

$$= \frac{1}{4}\int_1^{27} (u-1)\, u^{-\frac{1}{3}} du$$

$$= \frac{1}{4}\int_1^{27} u^{\frac{2}{3}} - u^{-\frac{1}{3}} du$$

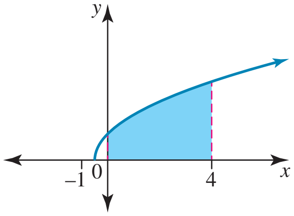
$$= \frac{1}{4}\left[\frac{3}{5}u^{\frac{5}{3}} - \frac{3}{2}u^{\frac{2}{3}}\right]_1^{27}$$

$$= \frac{1}{4}\left[\frac{3}{5}\left(27^{\frac{5}{3}} - 1^{\frac{2}{3}}\right) - \frac{3}{2}\left(27^{\frac{2}{3}} - 1^{\frac{2}{3}}\right)\right]$$

$$= \frac{1}{4}\left[\frac{3}{5}\left(3^5 - 1\right) - \frac{3}{2}\left(3^2 - 1\right)\right]$$

$$= \frac{333}{10}$$

23 a $y = \sqrt{2x+1}$



$$A = \int_0^4 \sqrt{2x+1}\, dx$$

Let

$u = 2x + 1$

$\dfrac{du}{dx} = 2$

Terminals

$x = 4, \ u = 9$

$x = 0, \ u = 1$

$$A = \int_1^9 u^{\frac{1}{2}} \cdot \frac{1}{2} du$$

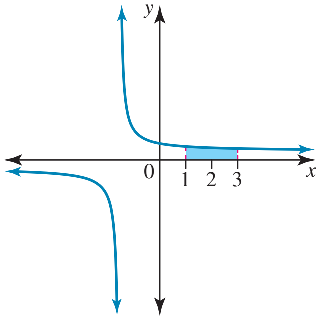
$$= \frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^9$$

$$= \frac{1}{3}\left[9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right]$$

$$= \frac{1}{3}[27 - 1]$$

$$= \frac{26}{3}$$

b $y = \dfrac{1}{3x+5}$



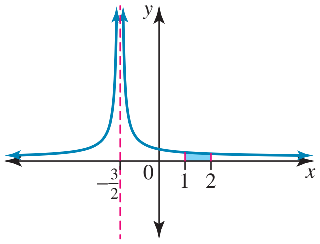
$$A = \int_0^3 \frac{1}{3x+5}\, dx$$

$$= \frac{1}{3}\left[\log_e(3x+5)\right]_0^3$$

$$= \frac{1}{3}\left[\log_e(14) - \log_e(5)\right]$$

$$= \frac{1}{3}\log_e\left(\frac{14}{5}\right)$$

24 a $y = \dfrac{1}{(2x+3)^2}$



$$A = \int_1^2 \frac{1}{(2x+3)^2}\, dx$$

Let

$u = 2x + 3$

$\dfrac{du}{dx} = 2$

$dx = \dfrac{1}{2}du$

Terminals

$x = 2, \ u = 7$

$x = 1, \ u = 5$

$$A = \int_5^7 u^{-2} \cdot \frac{1}{2} du$$

$$= \frac{1}{2}\int_5^7 u^{-2} du$$

$$= \frac{1}{2}\left[-\frac{1}{u}\right]_5^7$$

$$= \frac{1}{2}\left[-\frac{1}{7} + \frac{1}{5}\right]$$

$$= \frac{1}{35}$$

b $\int_0^5 \frac{x}{\sqrt{16-3x}}\,dx$

Let

$u = 16 - 3x$

$\frac{du}{dx} = -3$

$dx = -\frac{1}{3}du$

$3x = 16 - u$

$x = \frac{1}{3}(16 - u)$

Terminals

$x = 5, u = 1$

$x = 0, u = 16$

$$A = \int_{16}^{1} \frac{\frac{1}{3}(16-u)}{\sqrt{u}} \cdot -\frac{1}{3}du$$

$$= -\frac{1}{9}\int_{16}^{1} (16-u)\,u^{-\frac{1}{2}}du$$

$$= -\frac{1}{9}\int_{16}^{1} 16u^{-\frac{1}{2}} - u^{\frac{1}{2}}du$$

$$= \frac{1}{9}\left[32u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_1^{16}$$

$$= \frac{1}{9}\left[32\sqrt{16} - \frac{2}{3}16^{\frac{3}{2}} - 32\sqrt{1} + \frac{2}{3} \times 1^{\frac{3}{2}}\right]$$

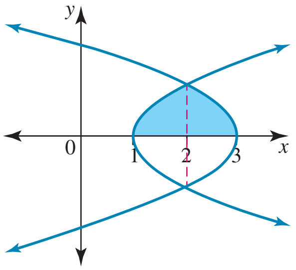
$$= 6$$

25 **a** $y_1^2 = 16(x-1)$

$y_1 = \pm 4\sqrt{x-1}$

$y_0^2 = 16(3-x)$

$y_0 = \pm 4\sqrt{3-x}$



To find intersection points, $y_1^2 = y_0^2$

$16(x-1) = 16(3-x)$

$x - 1 = 3 - x$

$2x = 4$

$x = 2$

$$A = \int_1^2 4\sqrt{x-1}\cdot dx + \int\limits_2^3 4\sqrt{3-x}\cdot dx$$

Let

$u = x - 1$

$\frac{du}{dx} = 1$

Terminals

$x = 2, u = 1$

$x = 1, u = 0$

Let

$v = 3 - x$

$\frac{dv}{dx} = -1$

Terminals

$x = 3, v = 0$

$x = 2, v = 1$

$$A = 4\int_0^1 u^{\frac{1}{2}}\cdot du + 4\int_1^0 -v^{\frac{1}{2}}\cdot du$$

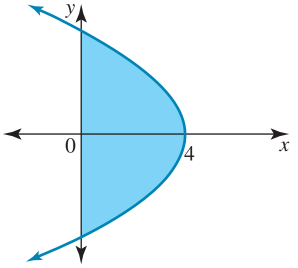
$$= \frac{4\times 2}{3}\left[u^{\frac{3}{2}}\right]_0^1 + \frac{4\times 2}{3}\left[v^{\frac{3}{2}}\right]_0^1$$

$$= \frac{8}{3}\left[1^{\frac{3}{2}} - 0\right] + \frac{8}{3}\left[1^{\frac{3}{2}} - 0\right]$$

$$= \frac{16}{3}$$

b $y^2 = 4 - x$

$y = \pm\sqrt{4-x}$



$$A = 2\int_0^4 \sqrt{4-x}dx$$

Let

$u = 4 - x$

$\frac{du}{dx} = -1$

Terminals

$x = 4, u = 0$

$x = 0, u = 4$

$$A = 2\int_4^0 -u^{\frac{1}{2}}du$$

$$= 2\left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^4$$

$$= \frac{4}{3}\left[4^{\frac{3}{2}} - 0\right]$$

$$= \frac{4}{3}[8 - 0]$$

$$= \frac{32}{3}$$

26 **a** $\int \sqrt{ax+b}.\,dx$

Let

$u = ax + b$

$\frac{du}{dx} = a$

$\frac{dx}{du} = \frac{1}{a}$

$dx = \frac{1}{a}du$

$$= \int u^{\frac{1}{2}}\frac{1}{a}du$$

$$= \frac{1}{a}\int u^{\frac{1}{2}}du$$

$$= \frac{1}{a}\frac{2}{3}u^{\frac{3}{2}} + c$$

$$= \frac{2}{3a}(ax+b)^{\frac{3}{2}} + c$$

b $\int x\sqrt{ax+b}\cdot dx$

$$u = ax + b$$

$$ax = u - b$$

$$x = \frac{1}{a}(u-b)$$

$$= \int \frac{1}{a}(u-b)\,u^{\frac{1}{2}}\frac{1}{a}du$$

$$= \frac{1}{a^2}\int u^{\frac{3}{2}} - bu^{\frac{1}{2}}du$$

$$= \frac{1}{a^2}\left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}bu^{\frac{3}{2}}\right] + c$$

$$= \frac{1}{15a^2}\left[6u^{\frac{5}{2}} - 10bu^{\frac{3}{2}}\right] + c$$

$$= \frac{2u^{\frac{3}{2}}}{15a^2}(3u - 5b) + c$$

$$= \frac{2u^{\frac{3}{2}}}{15a^2}(3(ax+b) - 5b) + c$$

$$= \frac{2}{15a^2}(3ax - 2b)(ax+b)^{\frac{3}{2}} + c$$

c $\int \frac{1}{ax+b}dx = \frac{1}{a}\log_e|ax+b| + c$

d $\int \frac{x}{ax+b}dx$

$$= \frac{1}{a^2}\int \frac{u-b}{u}du$$

$$= \frac{1}{a^2}\int \left(1 - \frac{b}{u}\right)du$$

$$= \frac{1}{a^2}\left[u - b\log_e|u|\right] + c$$

$$= \frac{u}{a^2} - \frac{b}{a^2}\log_e|u| + c$$

$$= \frac{ax+b}{a^2} - \frac{b}{a^2}\log_e|u| + c$$

$$= \frac{ax}{a^2} + \frac{b}{a^2} - \frac{b}{a^2}\log_e|u| + c$$

$$= \frac{x}{a} - \frac{b}{a^2}\log_e|ax+b| + d$$

27 a $\int \frac{1}{\sqrt{ax+b}}dx = \int u^{-\frac{1}{2}}\cdot\frac{1}{a}\cdot du$

$$= \frac{2}{a}\times u^{\frac{1}{2}} + c$$

$$= \frac{2}{a}\sqrt{ax+b} + c$$

b $\int \frac{x}{\sqrt{ax+b}}dx = \int \frac{\frac{1}{a}(u-b)}{u^{\frac{1}{2}}}\cdot\frac{1}{a}\cdot du$

$$= \frac{1}{a^2}\int u^{-\frac{1}{2}}(u-b)\cdot du$$

$$= \frac{1}{a^2}\int \left(u^{\frac{1}{2}} - bu^{-\frac{1}{2}}\right)du$$

$$= \frac{1}{a^2}\left[\frac{2}{3}u^{\frac{3}{2}} - 2bu^{\frac{1}{2}}\right] + c$$

$$= \frac{2u^{\frac{1}{2}}}{a^2}\left(\frac{u-3b}{3}\right) + c$$

$$= \frac{2\sqrt{u}}{3a^2}(ax + b - 3b) + c$$

$$= \frac{2}{3a^2}(ax - 2b)\sqrt{(ax+b)} + c$$

c $\int \frac{dx}{(ax+b)^2}$

$$= \int u^{-2}\frac{1}{a}du$$

$$= \frac{1}{a}\times -u^{-1} + c$$

$$= -\frac{1}{au} + c$$

$$= -\frac{1}{a(ax+b)} + c$$

d $\int \frac{x}{(ax+b)^2}dx$

$$= \frac{1}{a^2}\int \frac{u-b}{u^2}du$$

$$= \frac{1}{a^2}\int \left(\frac{1}{u} - \frac{b}{u^2}\right)du$$

$$= \frac{1}{a^2}\left[\log_e|u| + \frac{b}{u}\right] + c$$

$$= \frac{b}{a^2(ax+b)} + \frac{1}{a^2}\log_e|ax+b| + c$$

28 a $\int \frac{cx+d}{ax+b}dx$

Let

$$u = ax + b$$

$$dx = \frac{1}{a}du$$

$$x = \frac{1}{a}(u-b)$$

$$cx + d = \frac{c}{a}(u-b) + d$$

$$= \frac{1}{a}[cu - bc + ad]$$

$$= \frac{1}{a^2}\int \frac{ad - bc + cu}{u}du$$

$$= \frac{1}{a^2}\int \frac{ad-bc}{u} + c\cdot du$$

$$= \frac{1}{a^2}\left[(ad-bc)\log_e|u| + cu\right] + k$$

$$= \frac{ad-bc}{a^2}\log_e|u| + \frac{c(ax+b)}{a^2} + k$$

$$= \frac{ad-bc}{a^2}\log_e|ax+b| + \frac{cx}{a} + l$$

b $$\int \frac{cx+d}{(ax+b)^2}dx$$

$$= \frac{1}{a^2}\int \frac{ad-bc+cu}{u^2}du$$

$$= \frac{1}{a^2}\int \frac{ad-bc}{u^2}+\frac{c}{u}du$$

$$= \frac{1}{a^2}\left[-\frac{ad-bc}{u}+c\log_e|u|\right]+k$$

$$= \frac{c}{a^2}\log_e|ax+b|-\frac{ad-bc}{a^2\,(ax+b)}+k$$

c $$\int \frac{cx^2+d}{(ax+b)^2}dx$$

$$= \frac{1}{a^3}\int \frac{c(u-b)^2+a^2d}{u^2}\cdot du$$

$$= \frac{1}{a^3}\int \frac{c\left(u^2-2bu+b^2\right)+a^2d}{u^2}\cdot du$$

$$\frac{1}{a^3}\int c-\frac{2bc}{u}+\frac{a^2d+cb^2}{u^2}\cdot du$$

$$= \frac{1}{a^3}\left[cu-2bc\log_e|u|-\frac{1}{u}\left(a^2d+cb^2\right)\right]+k$$

$$= \frac{1}{a^3}\left[c\,(ax+b)-2bc\log_e|u|-\frac{1}{u}\left(a^2d+cb^2\right)\right]+k$$

$$= \frac{cx}{a^2}-\frac{2bc}{a^3}\log_e|ax+b|-\frac{a^2d+cb^2}{a^3\,(ax+b)}+l$$

d $$\int \frac{cx^2+d}{ax+b}dx$$

Let

$$u=ax+b$$

$$dx=\frac{1}{a^2}du$$

$$x^2=\frac{1}{a^2}(u-b)^2$$

$$cx^2+d=\frac{c}{a^2}\,(u-b)^2+d$$

$$=\frac{1}{a^2}\left[c\,(u-b)^2+a^2d\right]$$

$$= \frac{1}{a^3}\int \frac{c(u-b)^2+a^2d}{u}du$$

$$= \frac{1}{a^3}\int \frac{c\left(u^2-2bu+b^2\right)+a^2d}{u}\cdot du$$

$$= \frac{1}{a^3}\int \left(cu-2bc+\frac{cb^2+a^2d}{u}\right)\cdot du$$

$$= \frac{1}{a^3}\left[\frac{1}{2}cu^2-2bcu+\left(a^2d+cb^2\right)\log_e|u|\right]+k$$

$$= \frac{1}{a^3}\left[\frac{cu}{2}\,(u-4b)+\left(a^2d+cb^2\right)\log_e|u|\right]+k$$

$$= \frac{1}{a^3}\left[\frac{c\,(ax-3b)\,(ax+b)}{2}+\left(a^2d+cb^2\right)\log_e|u|\right]+k$$

$$= \frac{a^2d+cb^2}{a^3}\log_e|ax+b|+\frac{cx^2}{2a}-\frac{bcx}{a^2}+l$$

29 a $$\int \frac{x^2}{ax+b}=\frac{1}{a^3}\int \frac{(u-b)^2}{u}\cdot du$$

$$= \frac{1}{a^3}\int \frac{u^2-2bu+b^2}{u}\cdot du$$

$$= \frac{1}{a^3}\int u-2b+\frac{b^2}{u}.du$$

$$= \frac{1}{a^3}\left[\frac{1}{2}u^2-2bu+b^2\log_e|u|\right]+c$$

$$= \frac{1}{a^3}\left[\frac{u}{2}\,(u-4b)+b^2\log_e|u|\right]+c$$

$$= \frac{1}{2a^3}\,(ax-3b)\,(ax+b)+\frac{b^2}{a^3}\log_e|ax+b|+c$$

b

$$\int \frac{x^2}{(ax+b)^2}=\frac{1}{a^3}\int \frac{(u-b)^2}{u^2}\cdot du$$

$$= \frac{1}{a^3}\int \frac{u^2-2bu+b^2}{u^2}\cdot du$$

$$= \frac{1}{a^3}\int 1-\frac{2b}{u}+\frac{b^2}{u^2}\cdot du$$

$$= \frac{1}{a^3}\left[u-2b\quad\log_e|u|-\frac{b^2}{u}\right]+c$$

$$= \frac{1}{a^3}\left[u-\frac{b^2}{u}-2b\quad\log_e|u|\right]+c$$

$$= \frac{1}{a^3}\left[\frac{u^2-b^2}{u}-2b\quad\log_e|u|\right]+c$$

$$= \frac{1}{a^3}\left[\frac{(ax+b)^2-b^2}{u}-2b\quad\log_e|u|\right]+c$$

$$= \frac{1}{a^3}\left[\frac{a^2x^2+2axb+b^2-b^2}{u}-2b\quad\log_e|u|\right]+c$$

$$= \frac{1}{a^3}\left[\frac{ax\,(xa+2b)}{u}-2b\quad\log_e|u|\right]+c$$

$$= \frac{x\,(xa+2b)}{a^2\,(ax+b)}-\frac{2b}{a^3}\log_e|ax+b|+c$$

c $$\int \frac{x^2}{(ax+b)^3}$$

Let

$$u=ax+b$$

$$dx=\frac{1}{a}du$$

$$x=\frac{1}{a}\,(u-b)$$

$$= \frac{1}{a^3}\int \frac{(u-b)^2}{u^3}\cdot du$$

$$= \frac{1}{a^3}\int \frac{u^2-2bu+b^2}{u^3}\cdot du$$

$$= \frac{1}{a^3}\int \frac{1}{u}-\frac{2b}{u^2}+\frac{b^2}{u^3}\cdot du$$

$$= \frac{1}{a^3}\left[\log_e|u|+\frac{2b}{u}-\frac{b^2}{2u^2}\right]+c$$

$$= \frac{1}{a^3}\left[\log_e|u|+\frac{4bu-b^2}{2u^2}\right]+c$$

$$= \frac{1}{a^3}\left[\log_e|u| + \frac{b(4u-b)}{2u^2}\right] + c$$

$$= \frac{1}{a^3}\left[\log_e|u| + \frac{b}{2u^2}(4(ax+b)-b)\right] + c$$

$$= \frac{1}{a^3}\left[\log_e|ax+b| + \frac{b(4ax+3b)}{2(ax+b)^2}\right] + c$$

CAS gives

$$\frac{1}{a^3}\log_e|ax+b| - \frac{x(3ax+2b)}{2a^2(ax+b)^2}$$

Differ by $\frac{3}{2a^3}$

d $$\int \frac{x^2}{\sqrt{ax+b}}dx = \frac{1}{a^3}\int (u-b)^2 u^{-\frac{1}{2}} \cdot du$$

$$= \frac{1}{a^3}\int \left(u^2 - 2bu + b^2\right) u^{-\frac{1}{2}} \cdot du$$

$$= \frac{1}{a^3}\int u^{\frac{3}{2}} - 2bu^{\frac{1}{2}} + b^2u^{-\frac{1}{2}} \cdot du$$

$$= \frac{1}{a^3}\left[\frac{2}{5}u^{\frac{5}{2}} - \frac{4bu^{\frac{3}{2}}}{3} + 2b^2u^{\frac{1}{2}}\right] + c$$

$$= \frac{1}{15a^3}\left[6u^{\frac{5}{2}} - 20bu^{\frac{3}{2}} + 30b^2u^{\frac{1}{2}}\right] + c$$

$$= \frac{u^{\frac{1}{2}}}{15a^3}\left[6u^2 - 20bu + 30b^2\right] + c$$

$$= \frac{u^{\frac{1}{2}}}{15a^3}\left[6(ax+b)^2 - 20b(ax+b) + 30b^2\right] + c$$

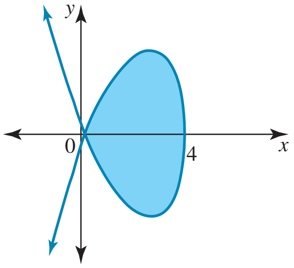
$$= \frac{u^{\frac{1}{2}}}{15a^3}\left[6\left(a^2x^2 + 2abx + b^2\right) - 20b(ax+b) + 30b^2\right] + c$$

$$= \frac{u^{\frac{1}{2}}}{15a^3}\left[6a^2x^2 - 8abx + 16b^2\right] + c$$

$$= \frac{2}{15a^3}\left[3a^2x^2 - 4abx + 8b^2\right]\sqrt{ax+b} + c$$

30 **a** $y^2 = x^2(4-x)$

$y = x\sqrt{4-x}$



$$A = 2\int_0^4 x\sqrt{4-x}\,dx$$

Let

$u = 4 - x$

$x = 4 - u$

$$A = 2\int_0^4 (4-u)\,u^{\frac{1}{2}}du$$

$$= 2\int_0^4 4u^{\frac{1}{2}} - u^{\frac{3}{2}}du$$

$$= 2\left[\frac{8}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]_0^4$$

$$= 2\left[\frac{8}{3}(4)^{\frac{3}{2}} - \frac{2}{5}(4)^{\frac{5}{2}} - 0\right]$$

$$= \frac{256}{15}$$

b $y^2 = a - x$

$$A = 2\int_0^a \sqrt{a-x}dx$$

Let

$u = a - x$

$\frac{du}{dx} = -1$

Terminals

$x = a, u = 0$

$x = 0, u = a$

$$A = 2\int_0^a u^{\frac{1}{2}} \cdot du$$

$$= 2\left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^a$$

$$= 2\left[\frac{2}{3}a^{\frac{3}{2}} - 0\right]$$

$$= \frac{4a^{\frac{3}{2}}}{3} = \frac{4}{3}\sqrt{a^3}$$

c $$A = 2\int_0^a x\sqrt{a-x}dx$$

$$= 2\int_a^0 (a-u)\,u^{\frac{1}{2}} \times -du$$

$$= 2\int_0^a au^{\frac{1}{2}} - u^{\frac{3}{2}}du$$

$$= 2\left[\frac{2a}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]_0^a$$

$$= 2\left[\frac{2a}{3}a^{\frac{3}{2}} - \frac{2}{5}a^{\frac{5}{2}}\right]$$

$$= 2\left[\frac{2}{3}a^{\frac{5}{2}} - \frac{2}{5}a^{\frac{5}{2}}\right]$$

$$= \frac{8a^{\frac{5}{2}}}{15} = \frac{8}{15}\sqrt{a^5}$$

7.3 Exam questions

1 Let $u = 1 - x,\ x = 1 - u,\ x + 1 = 2 - u,\ \dfrac{du}{dx} = -1.$
Terminals $x = 0,\ u = 1,\ x = -1,\ u = 2$

$$I = \int_{-1}^{0} \frac{1 + x}{\sqrt{1 - x}} dx$$

$$= -\int_{2}^{1} \frac{2 - u}{\sqrt{u}} du$$

$$= \int_{1}^{2} (2 - u)u^{-\frac{1}{2}} du$$

$$= \int_{1}^{2} \left(2u^{-\frac{1}{2}} - u^{\frac{1}{2}}\right) du$$

$$= \left[4u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_{1}^{2}$$

$$= \left(4\sqrt{2} - \frac{2}{3} \times 2\sqrt{2}\right) - \left(4 - \frac{2}{3}\right)$$

$$= \frac{8\sqrt{2}}{3} - \frac{10}{3}$$

Award 1 mark for the substitution.
Award 1 mark for changing terminals.
Award 1 mark for the correct definite integral.
Award 1 mark for the final correct answer.

2 $\displaystyle\int_{1}^{5} (2x - 1)\sqrt{2x + 1} dx$

Let $u = 2x + 1,\quad 2x = u - 1,\quad 2x - 1 = u - 2,\ \dfrac{du}{dx} = 2$

Terminals, when $x = 5,\ u = 11$ and when $x = 1,\ u = 3$

$$\int_{1}^{5} (2x - 1)\sqrt{2x - 1}\, dx = \frac{1}{2} \int_{3}^{11} (u - 2)\sqrt{u}\, du$$

$$= \frac{1}{2} \int_{3}^{11} \left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right) du$$

The correct answer is **E**.

3 $I = \displaystyle\int_{1}^{2} x^2\sqrt{2 - x}\, dx$

$u = 2 - x\ \dfrac{du}{dx} = -1$

$x = 2 - u,\ x^2 = (2 - u)^2 = 4 - 4u + u^2$

$x = 2 \Rightarrow u = 0,\ x = 1 \Rightarrow u = 1$

$$I = -\int_{1}^{0} \left(4 - 4u + u^2\right) u^{\frac{1}{2}} du$$

$$I = -\int_{1}^{0} \left(4u^{\frac{1}{2}} - 4u^{\frac{3}{2}} + u^{\frac{5}{2}}\right) du$$

The correct answer is **D**.

7.4 Non-linear substitutions

7.4 Exercise

1 $\displaystyle\int \frac{8x}{\left(x^2 + 16\right)^3} dx = \int 8x\left(x^2 + 16\right)^{-3} \frac{dx}{du} \cdot du$

Let

$u = x^2 + 16$

$\dfrac{du}{dx} = 2x$

$\dfrac{dx}{du} = \dfrac{1}{2x}$

$dx = \dfrac{1}{2x} du$

$$= \int 8x \cdot u^{-3} \frac{1}{2x} du$$

$$= 4\int u^{-3} du$$

$$= -2u^{-2} + c$$

$$= -\frac{2}{u^2} + c$$

$$= \frac{-2}{\left(x^2 + 16\right)^2} + c$$

2 $\displaystyle\int \frac{5x}{\sqrt{2x^2 + 3}} dx = \int 5x\left(2x^2 + 3\right)^{-\frac{1}{2}} \frac{dx}{du} \cdot du$

Let

$u = 2x^2 + 3$

$\dfrac{du}{dx} = 4x$

$\dfrac{dx}{du} = \dfrac{1}{4x}$

$dx = \dfrac{1}{4x} du$

$$= \int 5xu^{-\frac{1}{2}} \cdot \frac{1}{4x} du$$

$$= \frac{5}{4}\int u^{-\frac{1}{2}} du$$

$$= \frac{5}{4} \times 2u^{\frac{1}{2}} + c$$

$$= \frac{5}{2}\sqrt{2x^2 + 3} + c$$

3 **a** $\displaystyle\int x\left(x^2 + 4\right)^5 dx$

Let

$u = x^2 + 4$

$\dfrac{du}{dx} = 2x$

$\dfrac{dx}{du} = \dfrac{1}{2x}$

$$\int x \cdot u^5 \cdot \frac{dx}{du} \cdot du = \int x \cdot u^5 \cdot \frac{1}{2x} \cdot du$$

$$= \frac{1}{2}\int u^5 \cdot du$$

$$= \frac{1}{2}\left[\frac{1}{6}u^6\right] + c$$
$$= \frac{1}{12}\left(x^2+4\right)^6 + c$$

b $\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\int \frac{2x}{x^2+4}\,dx$

$$= \frac{1}{2}\log_e\left(x^2+4\right) + c$$

c $\int \frac{x}{\left(x^2+9\right)^2}\,dx$

Let

$u = x^2+9$

$\frac{du}{dx} = 2x$

$\frac{dx}{du} = \frac{1}{2x}$

$$= \int x \cdot u^{-2} \cdot \frac{dx}{du} \cdot du$$
$$= \int x \cdot u^{-2} \cdot \frac{1}{2x} \cdot du$$
$$= \frac{1}{2}\int u^{-2} \cdot du$$
$$= -\frac{1}{2}u^{-1} + c$$
$$= -\frac{1}{2\left(x^2+9\right)} + c$$

d $\int \frac{x}{\sqrt{x^2+9}}\,dx = \int x\left(x^2+9\right)^{-\frac{1}{2}} \cdot \frac{dx}{du} \cdot du$

$$= \int x \cdot u^{-\frac{1}{2}} \cdot \frac{1}{2x} \cdot du$$
$$= \frac{1}{2}\int u^{-\frac{1}{2}} \cdot du$$
$$= \frac{1}{2} \times 2u^{\frac{1}{2}} + c$$
$$= \sqrt{x^2+9} + c$$

4 $\int \frac{x+2}{x^2+4x+29}dx = \frac{1}{2}\int \frac{2(x+2)}{x^2+4x+29}\,dx$

$$= \frac{1}{2}\int \frac{2x+4}{x^2+4x+29}\,dx$$
$$= \frac{1}{2}\log_e\left(x^2+4x+29\right) + c$$

5 $\int \frac{x^2}{x^3+9}dx = \frac{1}{3}\int \frac{3x^2}{x^3+9}dx$

$$= \frac{1}{3}\log_e\left(\left|x^3+9\right|\right) + c$$

6 **a** $\int \frac{x^2}{\left(x^3+27\right)^3}$

Let

$u = x^3+27$

$\frac{du}{dx} = 3x^2$

$\frac{dx}{du} = \frac{1}{3x^2}$

$$= \int x^2 \cdot u^{-3} \cdot \frac{dx}{du} \cdot du$$
$$= \int x^2 \cdot u^{-3} \cdot \frac{1}{3x^2} \cdot du$$
$$= \frac{1}{3}\int u^{-3} \cdot du$$
$$= -\frac{1}{6}u^{-2} + c$$
$$= -\frac{1}{6\left(x^3+27\right)^2} + c$$

b $\int \frac{x^2}{\sqrt{x^3+27}}\,dx = \int x^2\left(x^3+27\right)^{-\frac{1}{2}}\frac{dx}{du}du$

$$= \int x^2 \cdot u^{-\frac{1}{2}} \cdot \frac{1}{3x^2} \cdot du$$
$$= \frac{1}{3}\int u^{-\frac{1}{2}} \cdot du$$
$$= \frac{2}{3}u^{\frac{1}{2}} + c$$
$$= \frac{2}{3}\sqrt{x^3+27} + c$$

c $\int \frac{x^2}{x^3+8} \cdot dx = \frac{1}{3}\int \frac{3x^2}{x^3+8} \cdot dx$

$$= \frac{1}{3}\log\left|x^3+8\right| + c$$

d $\int x^2\left(x^3+8\right)^3 dx$

Let

$u = x^3+8$

$\frac{du}{dx} = 3x^2$

$\frac{dx}{du} = \frac{1}{3x^2}$

$$\int x^2 \cdot u^3 \cdot \frac{1}{3x^2} \cdot du = \frac{1}{3}\int u^3 \cdot du$$
$$= \frac{1}{3}\left[\frac{1}{4}u^4\right] + c$$
$$= \frac{1}{12}\left(x^3+8\right)^4 + c$$

7 **a** $\int (x-2)\left(x^2-4x+13\right)^3 dx$

Let

$u = x^2-4x+13$

$\frac{du}{dx} = 2x-4 = 2(x-2)$

$$\int (x-2)u^3 \cdot \frac{dx}{du} \cdot du = \int (x-2)u^3 \cdot \frac{1}{2(x-2)} \cdot du$$
$$= \frac{1}{2}\int u^3 \cdot du$$
$$= \frac{1}{2}\left[\frac{1}{4}u^4\right] + c$$
$$= \frac{1}{8}\left(x^2-4x+13\right)^4 + c$$

b

$$\begin{aligned}\int \frac{(x-2)}{\left(x^2-4x+13\right)^2}\cdot dx &= \int (x-2)\left(x^2-4x+13\right)^{-2}\frac{dx}{du}\cdot du\\ &= \int (x-2)\,u^{-2}\frac{1}{2\,(x-2)}.du\\ &= \frac{1}{2}\int u^{-2}\cdot du\\ &= -\frac{1}{2}u^{-1}+c\\ &= -\frac{1}{2\left(x^2-4x+13\right)}+c\end{aligned}$$

c $\int \frac{4-x}{\sqrt{x^2-8x+25}}\cdot dx$

Let

$u = x^2-8x+25$

$\frac{du}{dx} = 2x-8 = 2\,(x-4)$

$$\begin{aligned}&\int (4-x)\cdot u^{-\frac{1}{2}}\frac{dx}{du}\cdot du\\ &= \int (4-x)\cdot u^{-\frac{1}{2}}\frac{1}{2\,(x-4)}\cdot du\\ &= -\frac{1}{2}\int u^{-\frac{1}{2}}\cdot du\\ &= -u^{\frac{1}{2}}+c\\ &= -\sqrt{x^2-8x+25}+c\end{aligned}$$

d

$$\begin{aligned}\int \frac{4-x}{x^2-8x+25}\cdot dx &= -\frac{1}{2}\int \frac{2x-8}{x^2-8x+25}\;dx\\ &= -\frac{1}{2}\log_e\left(x^2-8x+25\right)+c\end{aligned}$$

8 $\int \frac{1}{x^2}\sin\left(\frac{1}{x}\right)dx$

Let

$u = \frac{1}{x} = x^{-1}$

$\frac{du}{dx} = -x^{-2} = -\frac{1}{x^2}$

$\frac{dx}{du} = -x^2$

$dx = -x^2du$

$$\begin{aligned}&= \int \frac{1}{x^2}\sin(u)\times -x^{-2}du\\ &= -\int \sin(u)du\\ &= \cos(u)+c\\ &= \cos(\frac{1}{x})+c\end{aligned}$$

9 $\int x\cos(x^2)\,dx$

Let

$u = x^2$

$\frac{du}{dx} = 2x$

$\frac{dx}{du} = \frac{1}{2x}$

$dx = \frac{1}{2x}du$

$$\begin{aligned}&= \int x\cos(u)\cdot\frac{1}{2x}du\\ &= \frac{1}{2}\sin(u)+c\\ &= \frac{1}{2}\sin(x^2)+c\end{aligned}$$

10 a $\int \cos(3x)e^{\sin(3x)}dx$

Let

$u = \sin(3x)$

$\frac{du}{dx} = 3\cos(3x)$

$\frac{dx}{du} = \frac{1}{3\cos(3x)}$

$dx = \frac{1}{3\cos(3x)}\,du$

$$\begin{aligned}&= \int \cos(3x)e^u\frac{1}{3\cos(3x)}\,du\\ &= \frac{1}{3}\int e^u du\\ &= \frac{1}{3}e^u+c\\ &= \frac{1}{3}e^{\sin(3x)}+c\end{aligned}$$

b $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx = \int \frac{1}{\sqrt{x}}\cos(\sqrt{x})\;dx$

Let

$u = \sqrt{x} = x^{\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$\frac{dx}{du} = 2\sqrt{x}$

$dx = 2\sqrt{x}du$

$$\begin{aligned}&= \int \frac{1}{\sqrt{x}}\cos(u)\times 2\sqrt{x}\,du\\ &= 2\int \cos(u)du\\ &= 2\sin(u)+c\\ &= 2\sin(\sqrt{x})+c\end{aligned}$$

11 a $\int \sec^2(2x)e^{\tan(2x)}dx$

Let

$u = \tan(2x)$

$\frac{du}{dx} = 2\;\sec^2(2x)$

$\frac{dx}{du} = \frac{1}{2\;\sec^2(2x)}$

$dx = \frac{1}{2\;\sec^2(2x)}du$

$$= \int \sec^2(2x) e^u \frac{1}{2\sec^2(2x)}\ du$$

$$= \frac{1}{2}\int e^u du$$

$$= \frac{1}{2}e^u + c$$

$$= \frac{1}{2}e^{\tan(2x)} + c$$

b $\int \frac{\log_e(3x)}{4x} dx$

Let

$u = \log_e(3x)$

$\frac{du}{dx} = \frac{1}{x}$

$\frac{dx}{du} = x$

$dx = x\, du$

$$= \int \frac{1}{4x}\log_e(3x) dx$$

$$= \int \frac{1}{4x} u \cdot x\ du$$

$$= \frac{1}{4}\int u\ du$$

$$= \frac{1}{4}\left(\frac{1}{2}u^2\right) + c$$

$$= \frac{1}{8}\left(\log_e(3x)\right)^2 + c$$

12 a $\int \frac{e^{2x}}{4e^{2x}+5} dx$

Let

$u = 4e^{2x} + 5$

$\frac{du}{dx} = 8e^{2x}$

$\frac{dx}{du} = \frac{1}{8e^{2x}}$

$$\int e^{2x} \cdot \frac{1}{u} \cdot \frac{1}{8e^{2x}} \cdot du = \frac{1}{8}\int \frac{1}{u} \cdot du$$

$$= \frac{1}{8}\log_e |u| + c$$

$$= \frac{1}{8}\log_e(4e^{2x} + 5\,) + c$$

b $\int \frac{e^{-3x}}{\left(2e^{-3x}-5\right)^2} dx = \int e^{-3x}\left(2e^{-3x}-5\right)^{-2}\frac{dx}{du} \cdot du$

Let

$u = 2e^{-3x} - 5$

$\frac{du}{dx} = -6e^{-3x}$

$\frac{dx}{du} = -\frac{1}{6e^{-3x}}$

$$\int e^{-3x} \cdot u^{-2} \cdot -\frac{1}{6e^{-3x}} \cdot du = -\frac{1}{6}\int u^{-2} \cdot du$$

$$= \frac{1}{6}u^{-1} + c$$

$$= \frac{1}{6\left(2e^{-3x}-5\right)} + c$$

c $\int \frac{e^{-2x}}{\left(3e^{-2x}+4\right)^3} \cdot dx = \int e^{-2x}\left(3e^{-2x}+4\right)^{-3} \cdot \frac{dx}{du} \cdot du$

Let

$u = 3e^{-2x} + 4$

$\frac{du}{dx} = -\,6e^{-2x}$

$\frac{dx}{du} = -\frac{1}{6e^{-2x}}$

$$\int e^{-2x} \cdot u^{-3} \cdot -\frac{1}{6e^{-2x}} \cdot du = -\frac{1}{6}\int u^{-3} \cdot du$$

$$= \frac{1}{12}u^{-2} + c$$

$$= \frac{1}{12\left(3e^{-2x}+4\right)^2} + c$$

d $\int \frac{2e^{2x}+1}{\left(e^{2x}+x\right)^2} \cdot dx$

Let

$u = e^{2x} + x$

$\frac{du}{dx} = 2e^{2x} + 1$

$\frac{dx}{du} = \frac{1}{2e^{2x}+1}$

$$= \int u^{-2} \cdot du$$

$$= -\frac{1}{u} + c$$

$$= -\frac{1}{e^{2x}+x} + c$$

13 $\int_0^{2\sqrt{2}} \frac{s}{\sqrt{2s^2+9}} ds$

Let

$u = 2s^2 + 9$

$\frac{du}{ds} = 4s$

$\frac{ds}{du} = \frac{1}{4s}$

$ds = \frac{1}{4s}du$

Terminals

$s = 2\sqrt{2},\ u = 25$

$s = 0,\ u = 9$

$$= \int_9^{25} su^{-\frac{1}{2}} \cdot \frac{1}{4s}du$$

$$= \frac{1}{4}\int_9^{25} u^{-\frac{1}{2}}du$$

$$= \frac{1}{4} \times 2\left[u^{\frac{1}{2}}\right]_9^{25}$$

$$= \frac{1}{2}\left[\sqrt{25} - \sqrt{9}\right]$$

$$= 1$$

14 $\int_0^1 \frac{p}{\left(3p^2+5\right)^2}\ dp$

$u = 3p^2 + 5$

$\frac{du}{dp} = 6p$

$\frac{dp}{du} = \frac{1}{6p}$

$dp = \frac{1}{6p}du$

Terminals

$p = 1, u = 8$

$p = 0, u = 5$

$= \int_5^8 pu^{-2}\frac{1}{6p}du$

$= \frac{1}{6}\int_5^8 u^{-2}du$

$= \frac{1}{6}\left[-\frac{1}{u}\right]_5^8$

$= \frac{1}{6}\left[-\frac{1}{8}+\frac{1}{5}\right]$

$= \frac{1}{80}$

15 a $\int\limits_3^5 \frac{3t}{\sqrt{t^2-9}} \cdot dt$

Let

$u = t^2 - 9$

$\frac{du}{dt} = 2t$

$\frac{dt}{du} = \frac{1}{2t}$

Terminals when

$t = 3, u = 0$

$t = 5, u = 16$

$= \int_0^{16} 3t \cdot u^{-\frac{1}{2}} \cdot \frac{1}{2t} \cdot du$

$= \frac{3}{2}\int_0^{16} u^{-\frac{1}{2}} \cdot du$

$= \frac{3}{2} \times 2\left[u^{\frac{1}{2}}\right]_0^{16}$

$= 3\left[\sqrt{16}-\sqrt{0}\right]$

$= 12$

b $\int\limits_0^2 \frac{x}{4x^2+9} \cdot dx$

$= \frac{1}{8}\int\limits_0^2 \frac{8x}{4x^2+9} \cdot dx$

$= \frac{1}{8}\left[\log_e(4x^2+9)\right]_0^2$

$= \frac{1}{8}\left[\log_e(25) - \log_e(9)\right]$

$= \frac{1}{8}\log_e\left(\frac{25}{9}\right)$

$= \frac{1}{8}\log_e\left(\frac{5}{3}\right)^2$

$= \frac{1}{4}\log_e\left(\frac{5}{3}\right)$

c $\int_1^2 \frac{x}{\left(2x^2+1\right)^2} \cdot dx$

Let

$u = 2x^2 + 1$

$\frac{du}{dx} = 4x$

Terminals

$x = 1, u = 3$

$x = 2, u = 9$

$= \frac{1}{4}\int_3^9 u^{-2} \cdot du$

$= -\frac{1}{4}\left[u^{-1}\right]_3^9$

$= -\frac{1}{4}\left[\frac{1}{9}-\frac{1}{3}\right]$

$= \frac{1}{18}$

d $\int_{-1}^1 \frac{s}{\sqrt{2s^2+3}} .dx$

Let

$u = 2s^2 + 3$

$\frac{du}{ds} = 4s$

Terminals

$s = -1,\ u = 5$

$s = 1,\ u = 5$

$= \frac{1}{4}\int_5^5 u^{-\frac{1}{2}} \cdot du$

$= 0$

16 a $\int_4^5 \frac{3-x}{x^2-6x+34} .dx$

Let

$u = x^2 - 6x + 34$

$= \left(x^2 - 6x + 9\right) + 25$

$= (x-3)^2 + 25$

$\frac{du}{dx} = 2(x-3)$

Terminals

$x = 5, u = 29$

$x = 4, u = 26$

$= -\frac{1}{2}\int_{26}^{29} \frac{1}{u}.du$

$= -\frac{1}{2}\left[\log_e |u|\right]_{26}^{29}$

$= -\frac{1}{2}\left[\log_e(29) - \log_e(26)\right]$

$= -\frac{1}{2}\log_e\left(\frac{29}{26}\right)$

$= \frac{1}{2}\log_e\left(\frac{26}{29}\right)$

b $\int_2^3 \frac{2-x}{\left(x^2-4x+5\right)^2}.dx$

Let

$u = x^2 - 4x + 5$

$= x^2 - 4x + 4 + 1$

$= (x-2)^2 + 1$

$\frac{du}{dx} = 2(x-2)$

Terminals

$x = 3, u = 2$

$x = 2, u = 1$

$= -\frac{1}{2}\int_1^2 u^{-2} \cdot du$

$= -\frac{1}{2}\left[-u^{-1}\right]_1^2$

$= \frac{1}{2}\left(\frac{1}{2} - 1\right)$

$= -\frac{1}{4}$

c $\int_1^4 \frac{e\sqrt{p}}{\sqrt{p}} dp$

Let

$u = \sqrt{p} = p^{\frac{1}{2}}$

$\frac{du}{dp} = \frac{1}{2}p^{-\frac{1}{2}} = \frac{1}{2\sqrt{p}}$

Terminals

$p = 4, u = 2$

$p = 1, u = 1$

$= 2\int_1^2 e^u \cdot du$

$= 2\left[e^u\right]_1^2$

$= 2\left(e^2 - e^1\right)$

$= 2e(e-1)$

d $\int_0^{\frac{\pi}{8}} \sec^2(2\theta)e^{\tan(2\theta)} \cdot d\theta$

Let

$u = \tan(2\theta)$

$\frac{du}{d\theta} = 2\sec^2(2\theta)$

Terminals

$x = \frac{\pi}{8},\ u = \tan\left(\frac{\pi}{4}\right) = 1$

$x = 0,\ u = \tan(0) = 0$

$= \frac{1}{2}\int_0^1 e^u \cdot du$

$= \frac{1}{2}\left[e^u\right]_0^1$

$= \frac{1}{2}\left(e^1 - e^0\right)$

$= \frac{1}{2}(e-1)$

17 a $\int_0^{\frac{\pi}{4}} \sin(2\theta)e^{\cos(2\theta)} \cdot d\theta$

Let

$u = \cos(2\theta)$

$\frac{du}{d\theta} = -2\sin(2\theta)$

$d\theta = -\frac{1}{2\sin(2\theta)} \cdot du$

Terminals

$\theta = \frac{\pi}{4}, u = \cos\left(\frac{\pi}{2}\right) = 0$

$\theta = 0, u = \cos(0) = 1$

$= -\frac{1}{2}\int_1^0 e^u \cdot du$

$= \frac{1}{2}\left[e^u\right]_0^1$

$= \frac{1}{2}\left(e^1 - e^0\right)$

$= \frac{1}{2}(e-1)$

b $\int_{\frac{1}{2}}^{\frac{e}{2}} \frac{\log_e(2t)}{3t} \cdot dt = \int_{\frac{1}{2}}^{\frac{e}{2}} \frac{1}{3t}\log_e(2t) \cdot \frac{dt}{du} \cdot du$

Let

$u = \log_e(2t)$

$\frac{du}{dt} = \frac{1}{t}$

$\frac{dt}{du} = t$

Terminals

$t = \frac{e}{2}, u = \log_e e = 1$

$t = \frac{1}{2}, u = \log_e(1) = 0$

$= \int_0^1 \frac{1}{3t}u \cdot t \cdot du$

$= \frac{1}{3}\int_0^1 u \cdot du$

$= \frac{1}{3}\left[\frac{1}{2}u^2\right]_0^1$

$= \frac{1}{6}(1-0)$

$= \frac{1}{6}$

c $\int_{\frac{6}{\pi}}^{\frac{3}{\pi}} \frac{1}{x^2}\cos\left(\frac{1}{x}\right) dx$

Let

$u = \frac{1}{x} = x^{-1}$

$\frac{du}{dx} = \frac{-1}{x^2}$

Terminals

$x = \frac{3}{\pi}, u = \frac{\pi}{3}$

$x = \frac{6}{\pi}, u = \frac{\pi}{6}$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{x^2}\cos(u)\cdot -x^2\cdot du$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} -\cos(u)du$$

$$= [-\sin(u)]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{-\sqrt{3}}{2} + \frac{1}{2}$$

$$= \frac{1}{2}\left(1-\sqrt{3}\right)$$

d $\displaystyle\int_1^4 \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

Let

$$u = \sqrt{x} = x^{\frac{1}{2}}$$

$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Terminals

$x = 4,\ u = 2$

$x = 1,\ u = 1$

$$= \int_1^2 \frac{1}{\sqrt{x}}\sin(u)\cdot 2\sqrt{x}\cdot du$$

$$= 2\int_1^2 \sin(u)\cdot du$$

$$= -2\,[\cos(u)]_1^2$$

$$= -2\,[\cos(2)-\cos(1)]$$

$$= 2\cos(1) - 2\cos(2)$$

18 $\displaystyle\int_0^{\frac{1}{3}} \frac{\sin^{-1}(3x)}{\sqrt{1-9x^2}}\,dx$

Let

$$u = \sin^{-1}(3x)$$

$$\frac{du}{dx} = \frac{3}{\sqrt{1-9x^2}}$$

$$\frac{dx}{du} = \frac{\sqrt{1-9x^2}}{3}$$

$$dx = \frac{\sqrt{1-9x^2}}{3}du$$

Terminals

$x = \frac{1}{3}, u = \sin^{-1}(1) = \frac{\pi}{2}$

$x = 0, u = \sin^{-1}(0) = 0$

$$= \int_0^{\frac{\pi}{2}} u\cdot\frac{1}{3}du$$

$$= \frac{1}{3}\left[\frac{1}{2}u^2\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{6}\left[\frac{\pi^2}{4} - 0\right]$$

$$= \frac{\pi^2}{24}$$

19 $\displaystyle\int_0^4 \frac{\tan^{-1}\left(\frac{x}{4}\right)}{16+x^2}\,dx$

Let

$$u = \tan^{-1}\left(\frac{x}{4}\right)$$

$$\frac{du}{dx} = \frac{4}{16+x^2}$$

$$\frac{dx}{du} = \frac{16+x^2}{4}$$

$$dx = \frac{16+x^2}{4}du$$

Terminals

$x = 4, u = \tan^{-1}(1) = \frac{\pi}{4}$

$x = 0, u = \tan^{-1}(0) = 0$

$$= \int_0^4 \tan^{-1}\left(\frac{x}{4}\right)\frac{1}{16+x^2}du$$

$$= \int_0^{\frac{\pi}{4}} u\cdot\frac{1}{16+x^2}\times\frac{16+x^2}{4}du$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} u\,du$$

$$= \frac{1}{4}\left[\frac{1}{2}u^2\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{8}\left[\left(\frac{\pi}{4}\right)^2 - 0\right]$$

$$= \frac{\pi^2}{128}$$

20 **a** $\displaystyle\int \frac{1}{x}\sin(\log_e(4x))dx$

Let

$$u = \log_e(4x)$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$\frac{dx}{du} = x$$

$$= \int \frac{1}{x}\sin(u)\cdot x\cdot du$$

$$= \int \sin(u)\cdot du$$

$$= -\ \cos(u) + c$$

$$= -\ \cos(\log_e(4x)) + c$$

b $\displaystyle\int \frac{1}{x}\cos(\log_e(3x))dx$

Let

$$u = \log_e(3x)$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$\frac{dx}{du} = x$$

$$= \int \frac{1}{x}\cos(u)\cdot x\cdot du$$

$$= \int \cos(u).du$$

$$= \sin(u) + c$$

$$= \sin(\log_e(3x)) + c$$

c $\displaystyle\int \frac{\sin^{-1}\left(\frac{x}{2}\right)}{\sqrt{4-x^2}}\,dx = \int \sin^{-1}\left(\frac{x}{2}\right)\frac{1}{\sqrt{4-x^2}}\cdot\frac{dx}{du}\cdot du$

Let

$$u = \sin^{-1}\left(\frac{x}{2}\right)$$

$$\frac{du}{dx} = \frac{1}{\sqrt{4-x^2}}$$

$$\frac{dx}{du} = \sqrt{4-x^2}$$

$$= \int u\cdot du$$

$$= \frac{1}{2}u^2 + c$$

$$= \frac{1}{2}\left(\sin^{-1}\left(\frac{x}{2}\right)\right)^2 + c$$

d $\displaystyle\int \frac{\tan^{-1}(2x)}{1+4x^2}\,dx$

Let

$$u = \tan^{-1}(2x)$$

$$\frac{du}{dx} = \frac{2}{1+4x^2}$$

$$\frac{dx}{du} = \frac{1+4x^2}{2}$$

$$= \int \frac{1}{2}u\cdot du$$

$$= \frac{1}{4}u^2 + c$$

$$= \frac{1}{4}\left(\tan^{-1}(2x)\right)^2 + c$$

21 a $\displaystyle\frac{dy}{dx} = x\sin(x^2)$

$$y = \int x\sin(x^2)\cdot dx$$

Let

$$u = x^2$$

$$\frac{du}{dx} = 2x$$

$$\frac{dx}{du} = \frac{1}{2x}$$

$$y = \int x\ \sin(u)\cdot\frac{1}{2x}\cdot du$$

$$= \frac{1}{2}\int \sin(u)\cdot du$$

$$= -\frac{1}{2}\cos(u) + c$$

$$y = -\frac{1}{2}\cos(x^2) + c$$

When $x = 0, y = 0$

$$0 = -\frac{1}{2}\cos(0) + c$$

$$c = \frac{1}{2}$$

$$y = \frac{1}{2}\left(1-\cos(x^2)\right)$$

b $\displaystyle\frac{dy}{dx} = \frac{1}{x^2}\sec^2\left(\frac{1}{x}\right)$

$$y = \int \frac{1}{x^2}\sec^2\left(\frac{1}{x}\right)\,dx$$

Let

$$u = \frac{1}{x} = x^{-1}$$

$$\frac{du}{dx} = -x^{-2} = -\frac{1}{x^2}$$

$$\frac{dx}{du} = -x^2$$

$$y = \int \frac{1}{x^2}\sec^2(u)\cdot -x^2du$$

$$= -\int \sec^2(u)\cdot du$$

$$= -\tan(u) + c$$

$$y = -\tan\left(\frac{1}{x}\right) + c$$

When $x = \dfrac{4}{\pi}, y = 0$

$$0 = -\tan\left(\frac{\pi}{4}\right) + c$$

$$c = 1$$

$$y = 1 - \tan\left(\frac{1}{x}\right)$$

When $x = \dfrac{3}{\pi}$

$$y = 1 - \tan\left(\frac{\pi}{3}\right)$$

$$= 1 - \sqrt{3}$$

c $\displaystyle f'(x) = \frac{5-x}{x^2-10x+29}$

$$f(x) = -\frac{1}{2}\int \frac{2x-10}{x^2-10x+29}\cdot dx$$

$$= -\frac{1}{2}\log_e\left(x^2-10x+29\right) + c$$

$f(0) = 0,\ x = 0,\ y = 0$

$$0 = -\frac{1}{2}\log_e(29) + c$$

$$c = \frac{1}{2}\log_e(29)$$

$$f(x) = \frac{1}{2}\log_e\left(\frac{29}{x^2-10x+29}\right)$$

$$f(1) = \frac{1}{2}\log_e\left(\frac{29}{20}\right)$$

d $\displaystyle\frac{dy}{dx} = \sin(2x)e^{\cos(2x)}$

$$y = \int \sin(2x)e^{\cos(2x)}\cdot dx$$

Let

$$u = \cos(2x)$$

$$\frac{du}{dx} = -2\sin(2x)$$

$$\frac{dx}{du} = -\frac{1}{2\sin(2x)}$$

$$y = \int \sin(2x)\,e^u\cdot\frac{dx}{du}\cdot du$$

$$= \int \sin(2x)\,e^u\cdot -\frac{1}{2\sin(2x)}\cdot du$$

$$= -\frac{1}{2}\int e^u\cdot du$$

$$= -\frac{1}{2}e^{\cos(2x)} + c$$

When $x = \frac{\pi}{4}, y = 0$

$0 = -\frac{1}{2}e^{\cos\left(\frac{\pi}{2}\right)} + c$

$c = \frac{1}{2}$

$y = \frac{1}{2}(1 - e^{\cos(2x)})$

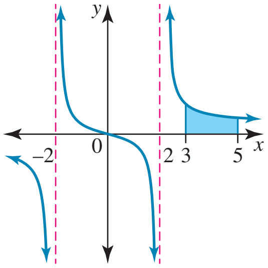
When $x = 0$

$y = \frac{1}{2}\left(1 - e^{\cos(0)}\right)$

$= \frac{1}{2}(1 - e)$

22 **a** Vertical asymptote $x = \pm 2$

$x^2 = c = 4$



$A = \int_3^5 \frac{x}{x^2 - 4} \cdot dx$

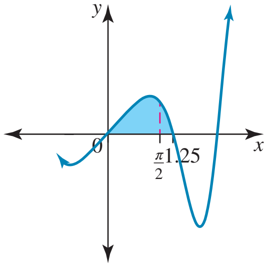
$= \frac{1}{2}\left[\log_e |x^2 - 4|\right]_3^5$

$= \frac{1}{2}\left[\log_e(21) - \log_e(5)\right]$

$= \frac{1}{2}\log_e\left(\frac{21}{5}\right)$

b $y = x\cos(x^2)$

$A = \int_0^{\frac{\sqrt{\pi}}{2}} x\cos(x^2) \cdot dx$



Let

$u = x^2$

$\frac{du}{dx} = 2x$

Terminals

$x = \frac{\sqrt{\pi}}{2}, u = \frac{\pi}{4}$

$x = 0, u = 0$

$A = \int_0^{\frac{\pi}{4}} x\cos(u) \cdot \frac{1}{2x} \cdot du$

$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos(u) \cdot du$

$= \frac{1}{2}\left[\sin(u)\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{2}\left[\sin\left(\frac{\pi}{4}\right) - \sin(0)\right]$

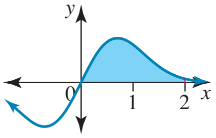
$= \frac{1}{2} \times \frac{\sqrt{2}}{2}$

$= \frac{\sqrt{2}}{4}$

23 **a** $y = xe^{-x^2}$

$x = 0, 2$

$A = \int_0^2 xe^{-x^2} \cdot dx$



Let

$u = x^2$

$\frac{du}{dx} = 2x$

Terminals

$x = 2, u = 4$

$x = 0, u = 0$

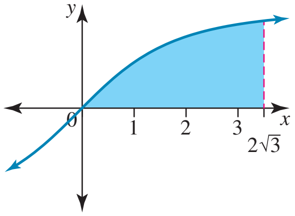
$A = \int_0^4 \frac{1}{2}e^{-u} \cdot du$

$= -\frac{1}{2}\left[e^{-u}\right]_0^4$

$= -\frac{1}{2}\left[e^{-4} - e^0\right]$

$= \frac{1}{2}\left[1 - e^{-4}\right]$

b $A = \int_0^{2\sqrt{3}} \frac{x}{\sqrt{x^2 + 4}} \cdot dx$



Let

$u = x^2 + 4$

$\frac{du}{dx} = 2x$

Terminals

$x = 2\sqrt{3}, u = 16$

$x = 0, u = 4$

$A = \int_0^{2\sqrt{3}} x(x^2 + 4)^{-\frac{1}{2}} \frac{dx}{du} \cdot du$

$= \frac{1}{2}\int_4^{16} u^{-\frac{1}{2}} \cdot du$

$= \left[u^{\frac{1}{2}}\right]_4^{16}$

$= \sqrt{16} - \sqrt{4}$

$= 4 - 2$

$= 2$

24 **a** $\int \frac{x}{\sqrt{ax^2 + b}} dx$

Let

$u = ax^2 + b$

$\frac{du}{dx} = 2ax$

$\frac{dx}{du} = \frac{1}{2ax}$

$= \int x \cdot u^{-\frac{1}{2}} \frac{dx}{du} \cdot du$

$= \int x \cdot u^{-\frac{1}{2}} \frac{1}{2ax} \cdot du$

$= \frac{1}{2a} \int u^{-\frac{1}{2}} \cdot du$

$= \frac{1}{a} \cdot u^{\frac{1}{2}} + c$

$= \frac{1}{a}\sqrt{ax^2 + b} + c$

b $\int \frac{x}{\left(ax^2 + b\right)^2} \cdot dx$

$= \int x.u^{-2} \frac{dx}{du} \cdot du$

$= \int x.u^{-2} \frac{1}{2ax} \cdot du$

$= \frac{1}{2a} \int u^{-2} \cdot du$

$= -\frac{1}{2a} \cdot u^{-1} + c$

$= -\frac{1}{2a\left(ax^2 + b\right)} + c$

c $\int x(ax^2 + b)^n \cdot dx$

$= \frac{1}{2a} \int u^n \cdot du$

$= \frac{1}{2a} . \frac{1}{n + 1} \cdot u^{n+1} + c$

$= \frac{1}{2a\,(n + 1)} \left(ax^2 + b\right)^{n+1} + c$

d $\int \frac{x}{ax^2 + b} \cdot dx$

$= \frac{1}{2a} \int \frac{2ax}{ax^2 + b} \cdot dx$

$= \frac{1}{2a} \log_e \left|ax^2 + b\right| + c$

25 **a** $\int \frac{f'(x)}{f(x)} \cdot dx = \log_e(|f(x)|) + c$

b $\int \frac{f'(x)}{f(x)^2} \cdot dx$

Let

$u = f(x)$

$\frac{du}{dx} = f'(x)$

$= \int f'(x) \cdot u^{-2} \cdot \frac{dx}{du} \cdot du$

$= \int f'(x) \cdot u^{-2} \cdot \frac{1}{f'(x)} \cdot du$

$= \int u^{-2} \cdot du$

$= -\frac{1}{u} + c$

$= -\frac{1}{f(x)} + c$

c $\int \frac{f'(x)}{\sqrt{f(x)}} \cdot dx$

$= \int f'(x) \cdot u^{-\frac{1}{2}} \cdot \frac{dx}{du} \cdot du$

$= \int f'(x) \cdot u^{-\frac{1}{2}} \cdot \frac{1}{f'(x)} \cdot du$

$= \int u^{-\frac{1}{2}} \cdot du$

$= 2u^{\frac{1}{2}} + c$

$= 2\sqrt{f(x)} + c$

d $\int f'(x) \cdot e^{f(x)} \cdot dx$

$= \int f'(x) \cdot e^u \cdot \frac{1}{f'(x)} \cdot du$

$= \int e^u \cdot du$

$= e^u + c$

$= e^{f(x)} + c$

7.4 Exam questions

1 $\int_0^{\frac{\pi}{6}} \tan^2(x) \sec^2(x) dx$

Let $u = \tan(x)$.

$\frac{du}{dx} = \sec^2(x)$

Terminals: when $x = \frac{\pi}{6}$, $u = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$

and when $x = 0$, $u = \tan(0) = 0$.

$$\int_0^{\frac{\pi}{6}} \tan^2(x) \sec^2(x) dx = \int_0^{\frac{1}{\sqrt{3}}} u^2 du$$

The correct answer is **E**.

2 $I = \int_a^b \left(x^3 e^{2x^4}\right) dx$

Let $u = x^4$: $\frac{du}{dx} = 4x^3 \Rightarrow \frac{dx}{du} = \frac{1}{4x^3}$.

Terminals: $x = a \Rightarrow u = a^4$, $x = b \Rightarrow u = b^4$

$I = \int_{a^4}^{b^4} x^3 e^{2u} \frac{1}{4x^3} du = \frac{1}{4} \int_{a^4}^{b^4} \left(e^{2u}\right) du$

The correct answer is **D**.

3 $\int_{e^3}^{e^4} \frac{1}{x\log_e(x)}dx$ Let $u=\log_e(x)$, $\frac{du}{dx}=\frac{1}{x}$

Terminals: $x=e^4$, $u=\log_e\left(e^4\right)=4$ and $x=e^3$, $u=\log_e\left(e^3\right)=3$

$\int_{e^3}^{e^4} \frac{1}{x\log_e(x)}dx=\int_3^4 \frac{1}{u}du=\int_a^b \frac{1}{u}du$, so $a=3$ and $b=4$.

The correct answer is **B**.

7.5 Integrals of powers of trigonometric functions

7.5 Exercise

1 **a**
$$\begin{aligned}\int \sin^2\left(\frac{x}{4}\right)dx &= \frac{1}{2}\int\ 1-\cos\left(\frac{x}{2}\right)dx\\ &= \frac{1}{2}\left[x-2\sin\left(\frac{x}{2}\right)\right]+c\\ &= \frac{x}{2}-\sin\left(\frac{x}{2}\right)+c\end{aligned}$$

b
$$\begin{aligned}\frac{1}{2}\int 2\sin\left(\frac{x}{4}\right)\cos\left(\frac{x}{4}\right)dx &= \frac{1}{2}\int\ \sin\left(\frac{x}{2}\right)dx\\ &= -\frac{1}{2}\times 2\cos\left(\frac{x}{2}\right)+c\\ &= -\cos\left(\frac{x}{2}\right)+c\end{aligned}$$

2 **a**
$$\begin{aligned}\int_0^{\frac{\pi}{6}} 4\quad \sin^2(2x)\ dx &= 2\int_0^{\frac{\pi}{6}} 1-\cos(4x)dx\\ &= 2\left[x-\frac{1}{4}\sin(4x)\right]_0^{\frac{\pi}{6}}\\ &= 2\left[\frac{\pi}{6}-\frac{1}{4}\sin\left(\frac{2\pi}{3}\right)-0\right]\\ &= \frac{\pi}{3}-\frac{1}{2}\times\frac{\sqrt{3}}{2}\\ &= \frac{4\pi-3\sqrt{3}}{12}\end{aligned}$$

b
$$\begin{aligned}\frac{1}{4}\int_0^{\frac{\pi}{3}} 4\quad \sin^2(2x)\cos^2(2x)\,dx &= \frac{1}{4}\int_0^{\frac{\pi}{3}} (\sin(4x))^2dx\\ &= \frac{1}{8}\int_0^{\frac{\pi}{3}} 1-\cos(8x)dx\\ &= \frac{1}{8}\left[x-\frac{1}{8}\sin(8x)\right]_0^{\frac{\pi}{3}}\end{aligned}$$

$$= \frac{1}{8}\left[\frac{\pi}{3}-\frac{1}{8}\sin\left(\frac{8\pi}{3}\right)-0\right]$$

$$= \frac{\pi}{24}-\frac{\sqrt{3}}{128}$$

3 **a** $\int \cos(4x)\ \sin^5(4x)dx$

Let

$u=\sin(4x)$

$\frac{du}{dx}=4\cos(4x)$

$$\begin{aligned}&= \frac{1}{4}\int u^5du\\ &= \frac{1}{4}\left[\frac{1}{6}u^6\right]+c\\ &= \frac{1}{24}\sin^6(4x)+c\end{aligned}$$

b $\int_0^{\frac{\pi}{4}} \sin(2x)\cos^3(2x)dx$

Let

$u=\cos(2x)$

$\frac{du}{dx}=-2\sin(2x)$

Terminals

$x=\frac{\pi}{4}, u=\cos\left(\frac{\pi}{2}\right)=0$

$x=0, u\ =\cos(0)\quad =1$

$$\begin{aligned}&= \int_1^0 \sin(2x)\quad u^3\times-\frac{1}{2\quad \sin(2x)}du\\ &= -\frac{1}{2}\int_1^0 u^3du\\ &= \frac{1}{2}\int_0^1 u^3du\\ &= \frac{1}{2}\left[\frac{1}{4}u^4\right]_0^1\\ &= \frac{1}{8}\,[1-0]\\ &= \frac{1}{8}\end{aligned}$$

4 **a**
$$\begin{aligned}&\int \cos(2x)\sin(2x)\cdot dx\\ &= \frac{1}{2}\int \sin(4x)\cdot dx\\ &= -\frac{1}{8}\cos(4x)+c\end{aligned}$$

b
$$\begin{aligned}&\int \cos^2(2x)+\sin^2(2x)\cdot dx\\ &= \int 1\cdot dx\\ &= x+c\end{aligned}$$

5 **a** $\int \cos^3(2x)\sin(2x)\cdot dx$

Let

$u=\cos(2x)$

$\frac{du}{dx}=-2\sin(2x)$

$$\begin{aligned}&= \int u^3\times-\frac{1}{2}\cdot du\\ &= -\frac{1}{2}\int u^3\cdot du\\ &= -\frac{1}{8}u^4+c\\ &= -\frac{1}{8}\cos^4(2x)+c\end{aligned}$$

b $\int \cos(2x)\sin^3(2x)\cdot dx$

Let

$u = \sin(2x)$

$\frac{du}{dx} = 2\cos(2x)$

$= \int u^3 \times \frac{1}{2}\cdot du$

$= \frac{1}{2}\int u^3 \cdot du$

$= \frac{1}{8}u^4 + c$

$= \frac{1}{8}\sin^4(2x) + c$

6 a $\int \cos^5(4x)\sin^2(4x)dx = \int \cos(4x)\cos^4(4x)\sin^2(4x)dx$

$= \int \cos(4x)\left(\cos^2(4x)\right)^2\sin^2(4x)dx$

$= \int \cos(4x)\left(1 - \sin^2(4x)\right)^2\sin^2(4x)dx$

Let

$u = \sin(4x)$

$\frac{du}{dx} = 4\cos(4x)$

$= \int \cos(4x)\left(1 - u^2\right)^2 u^2 \frac{1}{4\cos(4x)}du$

$= \frac{1}{4}\int \left(1 - 2u^2 + u^4\right) u^2 du$

$= \frac{1}{4}\int \left(u^2 - 2u^4 + u^6\right) du$

$= \frac{1}{4}\left[\frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7\right] + c$

$= \frac{1}{4}\left[\frac{1}{3}\sin^3(4x) - \frac{2}{5}\sin^5(4x) + \frac{1}{7}\sin^7(4x)\right]$

$= \frac{1}{12}\sin^3(4x) - \frac{1}{10}\sin^5(4x) + \frac{1}{28}\sin^7(4x)$

b $\int_0^{\frac{\pi}{12}} \sin^3(3x)dx = \int_0^{\frac{\pi}{12}} \sin(3x)\sin^2(3x)dx$

$= \int_0^{\frac{\pi}{12}} \sin(3x)\left(1 - \cos^2(3x)\right)dx$

Let

$u = \cos(3x)$

$\frac{du}{dx} = -3\sin(3x)$

Terminals

$x = \frac{\pi}{12},\ u = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$x = 0,\ u = \cos(0) = 1$

$= -\frac{1}{3}\int_1^{\frac{\sqrt{2}}{2}} 1 - u^2 du$

$= -\frac{1}{3}\left[u - \frac{1}{3}u^3\right]_1^{\frac{\sqrt{2}}{2}}$

$= -\frac{1}{3}\left[\frac{\sqrt{2}}{2} - \frac{1}{3}\left(\frac{\sqrt{2}}{2}\right)^3 - 1 + \frac{1}{3}\right]$

$= \frac{1}{3}\left[\frac{2}{3} - \frac{5}{6}\frac{\sqrt{2}}{2}\right]$

$= \frac{8 - 5\sqrt{2}}{36}$

7 a $\int_0^{\frac{\pi}{4}} \cos(2x)\sin^4(2x)\cdot dx$

Let

$u = \sin(2x)$

$\frac{du}{dx} = 2\cos(2x)$

Terminals

$x = 0,\ u = 0$

$x = \frac{\pi}{4},\ u = \sin\left(\frac{\pi}{2}\right) = 1$

$= \int u^4 \cdot \frac{1}{2}\cdot du$

$= \frac{1}{2} \times \frac{1}{5}\left[u^5\right]_0^1$

$= \frac{1}{10}$

b $\int_0^{\frac{\pi}{4}} \cos^2(2x)\sin^3(2x)\cdot dx$

$= \int_0^{\frac{\pi}{4}} \cos^2(2x)\sin(2x)\sin^2(2x)\cdot dx$

$= \int_0^{\frac{\pi}{4}} \cos^2(2x)(1 - \cos^2(2x))\sin(2x)\cdot dx$

Let

$u = \cos(2x)$

$\frac{du}{dx} = -2\sin(2x)$

Terminals

$x = 0,\ u = \cos(0) = 1$

$x = \frac{\pi}{4},\ u = \cos\left(\frac{\pi}{2}\right) = 0$

$\int_1^0 u^2\left(1 - u^2\right) \times -\frac{1}{2}\cdot du$

$= -\frac{1}{2}\int_1^0 \left(u^2 - u^4\right)\cdot du$

$= -\frac{1}{2}\left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right]_1^0$

$= -\frac{1}{2}\left[0 - \frac{1}{3} + \frac{1}{5}\right]$

$= \frac{1}{15}$

8 a $\int \sin^4(2x)dx = \int \left(\sin^2(2x)\right)^2 dx$

$= \int \left(\frac{1}{2}(1 - \cos(4x))\right)^2 dx$

$= \frac{1}{4}\int 1 - 2\cos(4x) + \cos^2(4x)dx$

$= \frac{1}{4}\int 1 - 2\cos(4x) + \frac{1}{2}(1 + \cos(8x))\,dx$

$= \frac{1}{4}\int \frac{3}{2} - 2\cos(4x) + \frac{1}{2}\cos(8x)dx$

$$= \frac{1}{4}\left[\frac{3}{2}x - \frac{1}{2}\sin(4x) + \frac{1}{16}\sin(8x)\right] + c$$

$$= \frac{3}{8}x - \frac{1}{8}\sin(4x) + \frac{1}{64}\sin(8x) + c$$

b $\int_0^{\frac{\pi}{12}} \cos^4(3x)dx = \int_0^{\frac{\pi}{12}} \left(\cos^2(3x)\right)^2 dx$

$$= \int_0^{\frac{\pi}{12}} \left(\frac{1}{2}\left(1 + \cos(6x)\right)\right)^2 dx$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{12}} \left(1 + 2\cos(6x) + \cos^2(6x)\right)\ dx$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{12}} \left(1 + 2\cos(6x)\right) + \frac{1}{2}\left(1 + \cos(12x)\right)\ dx$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{12}} \frac{3}{2} + 2\cos(6x) + \frac{1}{2}\cos(12x)\,dx$$

$$= \frac{1}{4}\left[\frac{3}{2}x + \frac{1}{3}\sin(6x) + \frac{1}{24}\sin(12x)\right]_0^{\frac{\pi}{12}}$$

$$= \frac{1}{4}\left[\frac{3}{2} \times \frac{\pi}{12} + \frac{1}{3}\sin(\frac{\pi}{2}) + \frac{1}{24}\sin(\pi) - 0\right]$$

$$= \frac{\pi}{32} + \frac{1}{12}$$

9 a $\int_0^{\frac{\pi}{4}} \cos^2(2x)\sin^2(2x) \cdot dx$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{4}\sin^2(4x)\right) \cdot dx$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} \frac{1}{2}\left(1 - \cos(8x)\right) \cdot dx$$

$$= \frac{1}{8}\left[x - \sin(8x)\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{8}\left[\frac{\pi}{4} - \sin(2\pi) - 0\right]$$

$$= \frac{\pi}{32}$$

b $\int_0^{\frac{\pi}{4}} \cos^3(2x)\sin^2(2x) \cdot dx$

$$= \int_0^{\frac{\pi}{4}} \cos(2x)\cos^2(2x)\sin^2(2x) \cdot dx$$

$$= \int_0^{\frac{\pi}{4}} \cos(2x)(1 - \sin^2(2x))\sin^2(2x) \cdot dx$$

Let

$u = \sin(2x)$

$\frac{du}{dx} = 2\cos(2x)$

Terminals

$x = 0,\ u = \sin(0) = 0$

$x = \frac{\pi}{4},\ u = \sin(\frac{\pi}{2}) = 1$

$$\int_0^1 u^2\left(1 - u^2\right) \times \frac{1}{2} \cdot du$$

$$= \frac{1}{2}\int_0^1 \left(u^2 - u^4\right) \cdot du$$

$$= \frac{1}{2}\left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{3} - \frac{1}{5}\right]$$

$$= \frac{1}{15}$$

10 a $\int \cos^2(4x)\sin^2(4x) \cdot dx$

$$= \int \frac{1}{4}\sin^2(8x) \cdot dx$$

$$= \frac{1}{8}\int \left(1 - \cos(16x)\right) \cdot dx$$

$$= \frac{x}{8} - \frac{1}{128}\sin(16x) + c$$

b $\int \cos^2(4x)\sin^3(4x) \cdot dx$

$$= \int \cos^2(4x)\sin(4x)\sin^2(4x) \cdot dx$$

$$= \int \cos^2(4x)\sin(4x)\left(1 - \cos^2(4x)\right) \cdot dx$$

Let

$u = \cos(4x)$

$\frac{du}{dx} = -4\sin(4x)$

$$= -\frac{1}{4}\int u^2\left(1 - u^2\right) \cdot du$$

$$= -\frac{1}{4}\int \left(u^2 - u^4\right) \cdot du$$

$$= -\frac{1}{4}\left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right] + c$$

$$= \frac{1}{20}\cos^5(4x) - \frac{1}{12}\cos^3(4x) + c$$

11 a $\int \cos^3(4x)\sin^2(4x) \cdot dx$

$$= \int \cos(4x)\cos^2(4x)\sin^2(4x) \cdot dx$$

$$= \int \cos(4x)\left(1 - \sin^2(4x)\right)\sin^2(4x) \cdot dx$$

Let

$u = \sin(4x)$

$\frac{du}{dx} = 4\ \cos(4x)$

$$= \frac{1}{4}\int u^2\left(1 - u^2\right) \cdot du$$

$$= \frac{1}{4}\int \left(u^2 - u^4\right) \cdot du$$

$$= \frac{1}{4}\left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right] + c$$

$$= \frac{1}{12}\sin^3(4x) - \frac{1}{20}\sin^5(4x) + c$$

b $\int \cos^3(4x)\sin^4(4x) \cdot dx$

$$= \int \cos(4x)\cos^2(4x)\sin^4(4x) \cdot dx$$

$$= \int \cos(4x)\left(1 - \sin^2(4x)\right)\sin^4(4x) \cdot dx$$

Let

$$u = \sin(4x)$$

$$\frac{du}{dx} = 4\cos(4x)$$

$$= \frac{1}{4}\int u^4\left(1 - u^2\right)\cdot du$$

$$= \frac{1}{4}\int \left(u^4 - u^6\right)\cdot du$$

$$= \frac{1}{4}\left[\frac{1}{5}u^5 - \frac{1}{7}u^7\right] + c$$

$$= \frac{1}{20}\sin^5(4x) - \frac{1}{28}\sin^7(4x) + c$$

12 a $\displaystyle\int_0^{\frac{\pi}{6}} \sin^2(3x)\cdot dx$

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} (1 - \cos(6x))\cdot dx$$

$$= \frac{1}{2}\left[x - \frac{1}{6}\sin(6x)\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{2}\left[\frac{\pi}{6} - \frac{1}{6}\sin(\pi) - 0\right]$$

$$= \frac{\pi}{12}$$

b $\displaystyle\int_0^{\frac{\pi}{6}} \cos^3(3x)\cdot dx$

$$= \int_0^{\frac{\pi}{6}} \cos(3x)\cos^2(3x)\cdot dx$$

$$= \int_0^{\frac{\pi}{6}} \cos(3x)\left(1 - \sin^2(3x)\right)\cdot dx$$

Let

$$u = \sin(3x)$$

$$\frac{du}{dx} = 3\cos(3x)$$

Terminals

$$x = \frac{\pi}{6},\ u = \sin\left(\frac{\pi}{2}\right) = 1$$

$$x = 0,\ u = \sin(0) = 0$$

$$= \int_0^1 \frac{1}{3}\left(1 - u^2\right)\cdot du$$

$$= \frac{1}{3}\left[u - \frac{1}{3}u^3\right]_0^1$$

$$= \frac{1}{3}\left[1 - \frac{1}{3} - 0\right]$$

$$= \frac{2}{9}$$

13 a $\displaystyle\int_0^{\frac{\pi}{6}} \sin^4(3x)\cdot dx$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{4}(1 - \cos(6x))^2\cdot dx$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{6}} (1 - 2\cos(6x)) + \cos^4(6x)\cdot dx$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{6}} (1 - 2\cos(6x)) + \frac{1}{2}(1 + \cos(12x))\cdot dx$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{6}} \frac{3}{2} - 2\cos(6x) + \frac{1}{2}\cos(12x)\cdot dx$$

$$= \left[\frac{3x}{8} - \frac{1}{12}\sin(6x) + \frac{1}{96}\sin(12x)\right]_0^{\frac{\pi}{6}}$$

$$= \frac{3}{8}\frac{\pi}{6} - \frac{1}{12}\sin(\pi) + \frac{1}{96}\sin(2\pi) - 0$$

$$= \frac{\pi}{16}$$

b $\displaystyle\int_0^{\frac{\pi}{6}} \cos^5(3x)\cdot dx$

$$= \int_0^{\frac{\pi}{6}} \cos(3x)\cos^4(3x)\cdot dx$$

$$= \int_0^{\frac{\pi}{6}} \cos(3x)\left(1 - \sin^2(3x)\right)^2\cdot dx$$

Let

$$u = \sin(3x)$$

$$\frac{du}{dx} = 3\cos(3x)$$

Terminals

$$x = \frac{\pi}{6},\ u = \sin\left(\frac{\pi}{2}\right) = 1$$

$$x = 0,\ u = \sin(0) = 0$$

$$= \frac{1}{3}\int_0^1 \left(1 - u^2\right)^2\cdot du$$

$$= \frac{1}{3}\int_0^1 1 - 2u^2 + u^4\cdot du$$

$$= \frac{1}{3}\left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_0^1$$

$$= \frac{1}{3}\left[1 - \frac{2}{3} + \frac{1}{5} - 0\right]$$

$$= \frac{8}{45}$$

14 a $\displaystyle\int (\cos(2x) + \sin(2x))^2\cdot dx$

$$= \int \cos^2(2x) + 2\sin(2x)\cos(2x) + \sin^2(2x)\cdot dx$$

$$= \int (1 + \sin(4x))\cdot dx$$

$$= x - \frac{1}{4}\cos(4x) + c$$

b $\displaystyle\int \cos^3(2x) + \sin^3(2x)\cdot dx$

$$= \int \cos(2x)\cos^2(2x) + \sin(2x)\sin^2(2x)\cdot dx$$

$$= \int \cos(2x)\left(1 - \sin^2(2x)\right)\cdot dx + \int \sin(2x)\left(1 - \cos^2(2x)\right)\cdot dx$$

Let

$$u = \sin(2x)$$

$$\frac{du}{dx} = 2\cos(2x)$$

$$v = \cos(2x)$$

$$\frac{dv}{dx} = -2\sin(2x)$$

$$= \frac{1}{2}\int \left(1-u^2\right)\cdot du + -\frac{1}{2}\int \left(1-v^2\right)\cdot dv$$
$$= \frac{1}{2}\left(u-\frac{1}{3}u^3\right)-\frac{1}{2}\left(v-\frac{1}{3}v^3\right)+c$$
$$= \frac{1}{2}(u-v)+\frac{1}{6}\left(v^3-u^3\right)+c$$
$$= \frac{1}{2}(\sin(2x)-\cos(2x))+\frac{1}{6}\left(\cos^3(2x)-\sin^3(2x)\right)+c$$

15 $\int \sin^3(2x)\cos^3(2x)\cdot dx$

a Using $\sin(A)\cos(A)=\frac{1}{2}\sin(2A)$

$$= \int\left(\frac{1}{2}\sin(4x)\right)^3\cdot dx$$
$$= \frac{1}{8}\int \sin^3(4x)\cdot dx$$
$$= \frac{1}{8}\int \sin(4x)\sin^2(4x)\cdot dx$$
$$= \frac{1}{8}\int \sin(4x)\left(1-\cos^2(4x)\right)\cdot dx$$

Let

$u=\cos(4x)$

$\frac{du}{dx}=-4\sin(4x)$

$$= -\frac{1}{32}\int\left(1-u^2\right)\cdot du$$
$$= -\frac{1}{32}\left(u-\frac{1}{3}u^3\right)+c$$
$$= \frac{1}{96}\cos^3(4x)-\frac{1}{32}\cos(4x)+c_1$$

b $\int \sin^3(2x)\cos^3(2x)\cdot dx$

$$= \int \sin(2x)\sin^2(2x)\cos^3(2x)\cdot dx$$
$$= \int \sin(2x)\left(1-\cos^2(2x)\right)\cos^3(2x)\cdot dx$$

Let

$u=\cos(2x)$

$\frac{du}{dx}=-2\sin(2x)$

$$= -\frac{1}{2}\int\left(1-u^2\right)u^3\cdot du$$
$$= -\frac{1}{2}\left(u^3-u^5\right)\cdot du$$
$$= -\frac{1}{2}\left[\frac{1}{4}u^4-\frac{1}{6}u^6\right]+c$$
$$= \frac{1}{12}\cos^6(2x)-\frac{1}{8}\cos^4(2x)+c_2$$

c $\int \sin^3(2x)\cos^3(2x)\cdot dx$

$$= \int \sin^3(2x)\cos^2(2x)\cos(2x)\cdot dx$$
$$= \int \sin^3(2x)\cos(2x)\left(1-\sin^2(2x)\right)\cdot dx$$

Let

$u=\sin(2x)$

$\frac{du}{dx}=2\cos(2x)$

$$= \frac{1}{2}\int\left(1-u^2\right)u^3\cdot du$$
$$= \frac{1}{2}\left(u^3-u^5\right)\cdot du$$
$$= \frac{1}{2}\left[\frac{1}{4}u^4-\frac{1}{6}u^6\right]+c$$
$$= \frac{1}{8}\sin^4(2x)-\frac{1}{12}\sin^6(2x)+c_3$$

16 a $\int \tan\left(\frac{x}{2}\right)dx=\int\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}dx$

Let

$u=\cos\left(\frac{x}{2}\right)$

$\frac{du}{dx}=-\frac{1}{2}\sin\left(\frac{x}{2}\right)$

$\frac{dx}{du}=-\frac{2}{\sin\left(\frac{x}{2}\right)}$

$$= \int\frac{1}{u}\cdot -2du$$
$$= -2\log_e|u|+c$$
$$= -2\log_e\left|\cos\left(\frac{x}{2}\right)\right|+c$$
$$= 2\log_e\left|\sec\left(\frac{x}{2}\right)\right|+c$$

b $\int_0^{\frac{\pi}{12}}\tan(4x)\,dx=\int_0^{\frac{\pi}{12}}\frac{\sin(4x)}{\cos(4x)}dx$

Let

$u=\cos(4x)$

$\frac{du}{dx}=-4\sin(4x)$

$\frac{dx}{du}=-\frac{1}{4\sin(4x)}$

Terminals

$x=\frac{\pi}{12},\ u=\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$

$x=0,\ u=\cos(0)=1$

$$= \int_1^{\frac{1}{2}}\frac{1}{u}\times-\frac{1}{4}du$$
$$= -\frac{1}{4}\left[\log_e|u|\right]_1^{\frac{1}{2}}$$
$$= -\frac{1}{4}\left[\log_e\left(\frac{1}{2}\right)-\log_e(1)\right]$$
$$= \frac{1}{4}\log_e(2)$$

17 a $\int \tan^2\left(\frac{x}{3}\right)dx=\int\sec^2\left(\frac{x}{3}\right)-1dx$

$$= 3\tan\left(\frac{x}{3}\right)-x+c$$

b $\int_0^{\frac{\pi}{16}}\tan^2(4x)dx=\int_0^{\frac{\pi}{16}}\sec^2(4x)-1dx$

$$\left[\frac{1}{4}\tan(4x)-x\right]_0^{\frac{\pi}{16}}$$
$$= \frac{1}{4}\tan\left(\frac{\pi}{4}\right)-\frac{\pi}{16}-0$$
$$= \frac{4-\pi}{16}$$

18 a $\int \tan(3x)\cdot dx = \int \frac{\sin(3x)}{\cos(3x)}\cdot dx$

Let

$u = \cos(3x)$

$\frac{du}{dx} = -3\sin(3x)$

$= -\frac{1}{3}\int \frac{1}{u}\cdot du$

$= -\frac{1}{3}\log_e|u| + c$

$= \frac{1}{3}\log_e\left|\frac{1}{u}\right| + c$

$= \frac{1}{3}\log_e(|\sec(3x)|) + c$

b $\int \cot(3x)\cdot dx = \int \frac{\cos(3x)}{\sin(3x)}\cdot dx$

Let

$u = \sin(3x)$

$\frac{du}{dx} = 3\cos(3x)$

$= \frac{1}{3}\int \frac{1}{u}\cdot du$

$= \frac{1}{3}\log_e|u| + c$

$= \frac{1}{3}\log_e(|\sin(3x)|) + c$

19 a $\int \tan(3x)\sec^2(3x).dx$

Let

$u = \tan(3x)$

$\frac{du}{dx} = 3\ \sec^2(3x)$

$= \frac{1}{3}\int u\cdot du$

$= \frac{1}{6}u^2 + c$

$= \frac{1}{6}\tan^2(3x) + c$

b $\int \tan^2(3x)\sec^2(3x)\cdot dx$

$= \frac{1}{3}\int u^2\cdot du$

$= \frac{1}{9}u^3 + c$

$= \frac{1}{9}\tan^3(3x) + c$

20 a $\int_0^{\frac{\pi}{20}} \tan(5x)\cdot dx = \int_0^{\frac{\pi}{20}} \frac{\sin(5x)}{\cos(5x)}\cdot dx$

Let

$u = \cos(5x)$

$\frac{du}{dx} = -5\sin(5x)$

Terminals

$x = \frac{\pi}{20},\ u = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$x = 0,\ u = \cos(0) = 1$

$= -\frac{1}{5}\int_1^{\frac{\sqrt{2}}{2}} \frac{1}{u}\cdot du$

$= -\frac{1}{5}\left[\log_e(u)\right]_1^{\frac{\sqrt{2}}{2}}$

$= -\frac{1}{5}\left[\log_e\left(\frac{\sqrt{2}}{2}\right) - \log_e(1)\right]$

$= -\frac{1}{5}\log_e\left(\frac{\sqrt{2}}{2}\right)$

$= -\frac{1}{5}\log_e\left(\frac{1}{\sqrt{2}}\right)$

$= \frac{1}{10}\log_e(2)$

b $\int_0^{\frac{\pi}{20}} \tan^2(5x)\cdot dx$

$= \int_0^{\frac{\pi}{20}} \sec^2(5x) - 1\,dx$

$= \left[\frac{1}{5}\tan(5x) - x\right]_0^{\frac{\pi}{20}}$

$= \frac{1}{5}\tan(\frac{\pi}{4}) - \frac{\pi}{20} - 0$

$= \frac{4-\pi}{20}$

21 a $\int_0^{\frac{\pi}{20}} \tan^3(5x)\sec^2(5x)\cdot dx$

Let

$u = \tan(5x)$

$\frac{du}{dx} = 5\ \sec^2(5x)$

Terminals

$x = \frac{\pi}{20},\ u = \tan\left(\frac{\pi}{4}\right) = 1$

$x = 0,\ u = \tan(0) = 0$

$= \frac{1}{5}\int_0^1 u^3\cdot du$

$= \frac{1}{20}\left[u^4\right]_0^1$

$= \frac{1}{20}$

b $\int_0^{\frac{\pi}{20}} \tan^2(5x)\sec^4(5x)\cdot dx$

$= \int_0^{\frac{\pi}{20}} \tan^2(5x)\sec^2(5x)\sec^2(5x)\cdot dx$

$= \int_0^{\frac{\pi}{20}} \tan^2(5x)\left(1 + \tan^2(5x)\right)\sec^2(5x)\cdot dx$

Let

$u = \tan(5x)$

$\frac{du}{dx} = 5\sec^2(5x)$

$= \frac{1}{5}\int_0^1 u^2\left(1 + u^2\right)\cdot du$

$= \frac{1}{5}\int_0^1 u^2 + u^4\cdot du$

$$= \frac{1}{5}\left[\frac{1}{3}u^3 + \frac{1}{5}u^5\right]_0^1$$

$$= \frac{1}{5}\left(\frac{1}{3} + \frac{1}{5}\right)$$

$$= \frac{8}{75}$$

22 a $\int \operatorname{cosec}^2(2x)\cos(2x) \cdot dx$

$$= \int \frac{\cos(2x)}{\sin^2(2x)} \cdot dx$$

Let

$u = \sin(2x)$

$\frac{du}{dx} = 2\cos(2x)$

$$= \frac{1}{2}\int u^{-2} \cdot du$$

$$= -\frac{1}{2}u^{-1} + c$$

$$= -\frac{1}{2u} + c$$

$$= -\frac{1}{2\sin(2x)} + c$$

$$= -\frac{1}{2}\operatorname{cosec}(2x) + c$$

b $\int \sec^2(2x)\sin(2x) \cdot dx$

$$= \int \frac{\sin(2x)}{\cos^2(2x)} \cdot dx$$

Let

$u = \cos(2x)$

$\frac{du}{dx} = -2\sin(2x)$

$$= -\frac{1}{2}\int u^{-2} \cdot du$$

$$= \frac{1}{2}u^{-1} + c$$

$$= \frac{1}{2u} + c$$

$$= \frac{1}{2\cos(2x)} + c$$

$$= \frac{1}{2}\sec(2x) + c$$

c $\int \frac{\sin(2x)}{\cos^3(2x)} \cdot dx$

Let

$u = \cos(2x)$

$\frac{du}{dx} = -2\sin(2x)$

$$= -\frac{1}{2}\int u^{-3} \cdot du$$

$$= \frac{1}{4}u^{-2} + c$$

$$= \frac{1}{4u^2} + c$$

$$= \frac{1}{4}\sec^2(2x) + c$$

d $\int \frac{\cos(2x)}{\sin^3(2x)} \cdot dx$

Let

$u = \sin(2x)$

$\frac{du}{dx} = 2\cos(2x)$

$$= \frac{1}{2}\int u^{-3} \cdot du$$

$$= -\frac{1}{4}u^{-2} + c$$

$$= -\frac{1}{4u^2} + c$$

$$= -\frac{1}{4}\operatorname{cosec}^2(2x) + c$$

23 a $\int \sin(ax)\cos^n(ax) \cdot dx$

Let

$u = \cos(ax)$

$\frac{du}{dx} = -a\sin(ax)$

$$= -\frac{1}{a}\int u^n \cdot du$$

$$= -\frac{1}{a(n+1)}u^{n+1} + c$$

$$= -\frac{1}{a(n+1)}\cos^{n+1}(ax) + c$$

b $\int \cos(ax)\sin^n(ax) \cdot dx$

Let

$u = \sin(ax)$

$\frac{du}{dx} = a\cos(ax)$

$$= \frac{1}{a}\int u^n \cdot du$$

$$= \frac{1}{a(n+1)}u^{n+1} + c$$

$$= \frac{1}{a(n+1)}\sin^{n+1}(ax) + c$$

c $\int \tan^n(ax)\sec^2(ax) \cdot dx$

Let

$u = \tan(ax)$

$\frac{du}{dx} = a\sec^2(ax)$

$$= \frac{1}{a}\int u^n \cdot du$$

$$= \frac{1}{a(n+1)}u^{n+1} + c$$

$$= \frac{1}{a(n+1)}\tan^{n+1}(ax) + c$$

24 a $\int \tan(ax) \cdot dx = \int \frac{\sin(ax)}{\cos(ax)} \cdot dx$

Let

$u = \cos(ax)$

$\frac{du}{dx} = -a\sin(ax)$

$$= -\frac{1}{a}\int \frac{1}{u} \cdot du$$

$$= -\frac{1}{a}\log_e |u| + c$$

$$= -\frac{1}{a}\log_e |\cos(ax)| + c$$

b $\int \tan^2(ax) \cdot dx$

$$= \int \sec^2(ax) - 1 \cdot dx$$

$$= \frac{1}{a}\tan(ax) - x + c$$

c $\int \tan^3(ax) \cdot dx$

$$= \int \tan(ax)\tan^2(ax) \cdot dx$$

$$= \int \tan(ax)\left(\sec^2(ax) - 1\right) \cdot dx$$

$$= \int \tan(ax)\sec^2(ax)dx - \int \tan(ax)dx$$

Let

$u = \tan(ax)$ and using the result from Q24a:

$$= \frac{1}{a}\int u \cdot du - \frac{1}{a}\log_e |\cos(ax)| + c$$

$$= \frac{1}{2a}u^2 - \frac{1}{a}\log_e |\cos(ax)| + c$$

$$= \frac{1}{2a}\tan^2(ax) + \frac{1}{a}\log_e |\cos(ax)| + c$$

d $\int \tan^4(ax) \cdot dx$

$$= \int \tan^2(ax)\tan^2(ax) \cdot dx$$

$$= \int \tan^2(ax)\left(\sec^2(ax) - 1\right) \cdot dx$$

$$= \int \tan^2(ax)\sec^2(ax)dx - \int \tan^2(ax)dx$$

Let

$u = \tan(ax)$ and using the result from Q24b:

$$= \frac{1}{a}\int u^2 \cdot du - \frac{1}{a}\tan(ax) - x + c$$

$$= \frac{1}{3a}u^3 - \frac{1}{a}\tan(ax) - x + c$$

$$= \frac{1}{3a}\tan^3(ax) - \frac{1}{a}\tan(ax) + x + c$$

25 a $\int \sin^2(ax) \cdot dx$

$$= \frac{1}{2}\int 1 - \cos(2ax) \cdot dx$$

$$= \frac{1}{2}\left[x - \frac{1}{2a}\sin(2ax)\right] + c$$

$$= \frac{x}{2} - \frac{1}{4a}\sin(2ax) + c$$

b $\int \sin^3(ax) \cdot dx$

$$= \int \sin(ax)\sin^2(ax)$$

$$= \int \sin(ax)\left(1 - \cos^2(2ax)\right) \cdot dx$$

Let

$u = \cos(ax)$

$\frac{du}{dx} = -a\sin(ax)$

$$= -\frac{1}{a}\int 1 - u^2 \cdot du$$

$$= -\frac{1}{a}\left[u - \frac{1}{3}u^3\right] + c$$

$$= \frac{1}{3a}\cos^3(ax) - \frac{1}{a}\cos(ax) + c$$

c $\int \sin^4(ax) \cdot dx$

$$= \int \frac{1}{4}(1 - \cos(2ax))^2 \cdot dx$$

$$= \frac{1}{4}\int (1 - 2\cos(2ax)) + \cos^2(2ax) \cdot dx$$

$$= \frac{1}{4}\int (1 - 2\cos(2ax)) + \frac{1}{2} + \frac{1}{2}\cos(4ax) \cdot dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos(2ax)\right) + \frac{1}{2}\cos(4ax) \cdot dx$$

$$= \frac{1}{4}\left[\frac{3x}{2} - \frac{1}{a}\sin(2ax) + \frac{1}{8a}\sin(4ax)\right] + c$$

$$= \frac{3x}{8} - \frac{1}{4a}\sin(2ax) + \frac{1}{32a}\sin(4ax) + c$$

d $\int \sin^5(ax) \cdot dx$

$$= \int \sin(ax)\sin^4(ax) \cdot dx$$

$$= \int \sin(ax)\left(\sin^2(ax)\right)^2 \cdot dx$$

$$= \int \sin(ax)\left(1 - \cos^2(2ax)\right)^2 \cdot dx$$

Let

$u = \cos(ax)$

$\frac{du}{dx} = -a\sin(ax)$

$$= -\frac{1}{a}\int \left(1 - u^2\right)^2 \cdot du$$

$$= -\frac{1}{a}\int 1 - 2u^2 + u^4 \cdot du$$

$$= -\frac{1}{a}\left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right] + c$$

$$= -\frac{1}{a}\cos(ax) + \frac{2}{3a}\cos^3(ax) - \frac{1}{5a}\cos^5(ax) + c$$

26 a $\int \cos^2(ax) \cdot dx$

$$= \frac{1}{2}\int 1 + \cos(2ax) \cdot dx$$

$$= \frac{1}{2}\left[x + \frac{1}{2a}\sin(2ax)\right] + c$$

$$= \frac{x}{2} + \frac{1}{4a}\sin(2ax) + c$$

b $\int \cos^3(ax) \cdot dx$

$= \int \cos(ax)\cos^2(ax)$

$= \int \cos(ax)\left(1 - \sin^2(ax)\right) \cdot dx$

Let

$u = \sin(ax)$

$\frac{du}{dx} = a\cos(ax)$

$= \frac{1}{a}\int 1 - u^2 \cdot du$

$= \frac{1}{a}\left[u - \frac{1}{3}u^3\right] + c$

$= \frac{1}{a}\sin(ax) - \frac{1}{3a}\sin^3(ax) + c$

c $\int \cos^4(ax) \cdot dx$

$= \int \left(\cos^2(ax)\right)^2 \cdot dx$

$= \int \frac{1}{4}(1 + \cos(2ax))^2 \cdot dx$

$= \frac{1}{4}\int (1 + 2\cos(2ax)) + \cos^2(2ax) \cdot dx$

$= \frac{1}{4}\int (1 + 2\cos(2ax)) + \frac{1}{2} + \frac{1}{2}\cos(4ax) \cdot dx$

$= \frac{1}{4}\int \left(\frac{3}{2} + 2\cos(2ax)\right) + \frac{1}{2}\cos(4ax) \cdot dx$

$= \frac{1}{4}\left[\frac{3x}{2} + \frac{1}{a}\sin(2ax) + \frac{1}{8a}\sin(4ax)\right] + c$

$= \frac{3x}{8} + \frac{1}{4a}\sin(2ax) + \frac{1}{32a}\sin(4ax) + c$

d $\int \cos^5(ax) \cdot dx$

$= \int \cos(ax)\sin^4(ax) \cdot dx$

$= \int \cos(ax)\left(\cos^2(ax)\right)^2 \cdot dx$

$= \int \cos(ax)\left(1 - \sin^2(2ax)\right)^2 \cdot dx$

Let

$u = \sin(ax)$

$\frac{du}{dx} = a\cos(ax)$

$= \frac{1}{a}\int \left(1 - u^2\right)^2 \cdot du$

$= \frac{1}{a}\int 1 - 2u^2 + u^4 \cdot du$

$= \frac{1}{a}\left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right] + c$

$= \frac{1}{a}\sin(ax) - \frac{2}{3a}\sin^3(ax) + \frac{1}{5a}\sin^5(ax) + c$

7.5 Exam questions

1 $\int_0^{\frac{\pi}{3}} \cos^2\left(\frac{3x}{2}\right) dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{3}} (1 + \cos(3x))dx$

$= \frac{1}{2}\left[x + \frac{1}{3}\sin(3x)\right]_0^{\frac{\pi}{3}} = \frac{1}{2}\left(\frac{\pi}{3} + \frac{1}{3}\sin(\pi) - \left(0 + \frac{1}{3}\sin(0)\right)\right)$

$= \frac{\pi}{6}$

The correct answer is **B**.

2 Let $u = \tan(3x)$, $\frac{du}{dx} = 3\sec^2(3x)$

$\int \sec^2(3x)\tan^2(3x)\, dx$

$= \frac{1}{3}\int u^2 du = \frac{1}{3}u^3 + c$

$= \frac{1}{9}\tan^3(3x) + c$

The correct answer is **D**.

3 $\int_0^p \left(\sin^2(5x) - \cos^2(5x)\right) dx = -\int_0^p \cos(10x)\, dx$ [1 mark]

$= \left[-\frac{1}{10}\sin(10x)\right]_0^p$

$= -\frac{1}{10}\sin(10p)$ [1 mark]

7.6 Integrals involving inverse trigonometric functions

7.6 Exercise

1 a $\int \frac{1}{\sqrt{100 - x^2}} dx = \sin^{-1}\left(\frac{x}{10}\right) + c$

b $\int \frac{12}{\sqrt{64 - 9x^2}} dx$

Let

$u = 3x$

$\frac{du}{dx} = 3$

$= 12\int \frac{1}{\sqrt{64 - u^2}} \cdot \frac{1}{3}du$

$= 4 \quad \sin^{-1}\left(\frac{u}{8}\right) + c$

$= 4 \quad \sin^{-1}\left(\frac{3x}{8}\right) + c$

2 a $\displaystyle\int \frac{1}{\sqrt{36-25x^2}}\,dx$

Let

$$u = 5x$$
$$\frac{du}{dx} = 5$$
$$= \frac{1}{5}\int \frac{1}{\sqrt{36-u^2}}\,du$$
$$= \frac{1}{5}\sin^{-1}\left(\frac{u}{6}\right) + c$$
$$= \frac{1}{5}\sin^{-1}\left(\frac{5x}{6}\right) + c$$

b $\displaystyle\int \frac{x}{\sqrt{36-25x^2}}\,dx$

Let

$$u = 36-25x^2$$
$$\frac{du}{dx} = -50x$$
$$= \int xu^{-\frac{1}{2}}\frac{dx}{du}du$$
$$= \int xu^{-\frac{1}{2}}\left(-\frac{1}{50x}\right)du$$
$$= -\frac{1}{50}\int u^{-\frac{1}{2}}du$$
$$= -\frac{1}{25}u^{\frac{1}{2}} + c$$
$$= -\frac{1}{25}\sqrt{36-25x^2} + c$$

3 a $\displaystyle\int \frac{1}{\sqrt{16-x^2}}\cdot dx = \sin^{-1}\left(\frac{x}{4}\right) + c$

b $\displaystyle\int \frac{1}{\sqrt{1-16x^2}}\cdot dx$

Let

$$u = 4x$$
$$\frac{du}{dx} = 4$$
$$= \frac{1}{4}\int \frac{1}{\sqrt{1-u^2}}\cdot dx$$
$$= \frac{1}{4}\sin^{-1}(u) + c$$
$$= \frac{1}{4}\sin^{-1}(4x) + c$$

4 a $\displaystyle\int \frac{10}{\sqrt{49-25x^2}}\cdot dx$

Let

$$u = 5x$$
$$\frac{du}{dx} = 5$$
$$= \frac{10}{5}\int \frac{1}{\sqrt{49-u^2}}\cdot dx$$
$$= 2\sin^{-1}\left(\frac{u}{7}\right) + c$$
$$= 2\sin^{-1}\left(\frac{5x}{7}\right) + c$$

b $\displaystyle\int \frac{10x}{\sqrt{49-25x^2}}\cdot dx = \int 10x\left(49-25x^2\right)^{-\frac{1}{2}}\cdot dx$

Let

$$u = 49-25x^2$$
$$\frac{du}{dx} = -50x$$
$$dx = -\frac{1}{50x}\cdot dx$$
$$= \int 10x \times u^{-\frac{1}{2}} \times -\frac{1}{50x}\cdot du$$
$$= -\frac{1}{5}\int u^{-\frac{1}{2}}\cdot du$$
$$= -\frac{2}{5}u^{\frac{1}{2}} + c$$
$$= -\frac{2}{5}\sqrt{49-25x^2} + c$$

5 a $\displaystyle\frac{dy}{dx} = -\frac{2}{\sqrt{25-16x^2}}$

$$y = 2\int -\frac{1}{\sqrt{25-16x^2}}\,dx$$

Let

$$u = 4x$$
$$\frac{du}{dx} = 4$$
$$y = 2\int \frac{-1}{\sqrt{25-u^2}} \times \frac{1}{4}\,du$$
$$= -\frac{1}{2}\sin^{-1}\left(\frac{u}{5}\right) + c$$
$$= -\frac{1}{2}\sin^{-1}\left(\frac{4x}{5}\right) + c$$

When $x = 0, y = 0$

$$0 = -\frac{1}{2}\sin^{-1}(0) + c$$
$$c = 0$$
$$y = -\frac{1}{2}\sin^{-1}\left(\frac{4x}{5}\right)$$

Or

$$\frac{1}{2}\cos^{-1}\left(\frac{4x}{5}\right) - \frac{\pi}{4} = \frac{1}{4}\left(2\cos^{-1}\left(\frac{4x}{5}\right) - \pi\right)$$

b $\displaystyle\frac{dy}{dx} = -\frac{2}{\sqrt{4-x^2}}$

$$y = 2\int -\frac{1}{\sqrt{4-x^2}}\,dx$$
$$y = 2\cos^{-1}\left(\frac{x}{2}\right) + c$$

When $x = 2, y = 3$

$$3 = 2\cos^{-1}(1) + c$$
$$c = 3$$
$$y = 2\cos^{-1}\left(\frac{x}{2}\right) + 3$$

6 a $\displaystyle\int \frac{-1}{\sqrt{4-x^2}}\cdot dx = \cos^{-1}\left(\frac{x}{2}\right) + c$

b $\displaystyle\int -\frac{2}{\sqrt{1-4x^2}}\cdot dx$

Let

$$u = 2x$$
$$\frac{du}{dx} = 2$$

$$= \int \frac{-1}{\sqrt{1-u^2}} \cdot dx$$
$$= \cos^{-1}(u) + c$$
$$= \cos^{-1}(2x) + c$$

7 a $\displaystyle\int -\frac{3x}{\sqrt{36-49x^2}} \cdot dx$

Let

$$u = 36 - 49x^2$$
$$\frac{du}{dx} = -98x$$
$$dx = -\frac{1}{98x} \cdot du$$
$$= \int -3x \times u^{-\frac{1}{2}} \times -\frac{1}{98x} \cdot du$$
$$= \frac{3}{98} \int u^{-\frac{1}{2}} \cdot du$$
$$= \frac{3}{49} u^{\frac{1}{2}} + c$$
$$= \frac{3}{49}\sqrt{36-49x^2} + c$$

b $\displaystyle\int -\frac{3}{\sqrt{36-49x^2}} \cdot dx$

Let

$$u = 7x$$
$$\frac{du}{dx} = 7$$
$$= -\frac{3}{7} \int \frac{1}{\sqrt{36-u^2}} \cdot dx$$
$$= \frac{3}{7} \cos^{-1}\left(\frac{u}{6}\right) + c$$
$$= \frac{3}{7} \cos^{-1}\left(\frac{7x}{6}\right) + c$$

8 a $\displaystyle\int \frac{1}{100+x^2}\, dx = \frac{1}{10} \tan^{-1}\left(\frac{x}{10}\right) + c$

b $\displaystyle\int \frac{12}{64+9x^2}\, dx$

Let

$$u = 3x$$
$$\frac{du}{dx} = 3$$
$$= 12 \int \frac{1}{64+u^2} \times \frac{1}{3}\, dx$$
$$= 4 \times \frac{1}{8} \tan^{-1}\left(\frac{u}{8}\right) + c$$
$$= \frac{1}{2} \tan^{-1}\left(\frac{3x}{8}\right) + c$$

9 a $\displaystyle\int \frac{1}{36+25x^2}\, dx$

Let

$$u = 5x$$
$$\frac{du}{dx} = 5$$
$$= \int \frac{1}{36+u^2} \times \frac{1}{5} du$$
$$= \frac{1}{5} \times \frac{1}{6} \tan^{-1}\left(\frac{u}{6}\right) + c$$
$$= \frac{1}{30} \tan^{-1}\left(\frac{5x}{6}\right) + c$$

b $\displaystyle\int \frac{5x}{64+25x^2} dx$

Let

$$u = 64 + 25x^2$$
$$\frac{du}{dx} = 50x$$
$$= \int \frac{5x}{u} \times \frac{1}{50x} du$$
$$= \frac{1}{10} \int \frac{1}{u} du$$
$$= \frac{1}{10} \log_e |u| + c$$
$$= \frac{1}{10} \log_e(64 + 25x^2) + c$$

10 a $\displaystyle\int_0^{\frac{9}{8}} \frac{1}{\sqrt{81-16x^2}}\, dx$

Let

$$u = 4x$$
$$\frac{du}{dx} = 4$$

Terminals

$$x = \frac{9}{8},\ u = \frac{9}{2}$$
$$x = 0,\ u = 0$$
$$= \int_0^{\frac{9}{2}} \frac{1}{\sqrt{81-u^2}} \times \frac{1}{4}\, du$$
$$= \frac{1}{4}\left[\sin^{-1}\left(\frac{u}{9}\right)\right]_0^{\frac{9}{2}}$$
$$= \frac{1}{4}\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)\right]$$
$$= \frac{1}{4} \times \frac{\pi}{6}$$
$$= \frac{\pi}{24}$$

b $\displaystyle\int_0^{\frac{9}{4}} \frac{1}{81+16x^2}\, dx$

Let

$$u = 4x$$
$$\frac{du}{dx} = 4$$

Terminals

$$x = \frac{9}{4},\ u = 9$$
$$x = 0,\ u = 0$$
$$= \int_0^{9} \frac{1}{81+u^2} \times \frac{1}{4}\, du$$
$$= \frac{1}{4}\left[\tan^{-1}\left(\frac{u}{9}\right)\right]_0^{9}$$
$$= \frac{1}{36}\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$
$$= \frac{1}{36} \times \frac{\pi}{4}$$
$$= \frac{\pi}{144}$$

11 a $\int \frac{21}{49+36x^2}\cdot dx$

Let

$u = 6x$

$\frac{du}{dx} = 6$

$= \frac{21}{6}\int \frac{1}{49+u^2}\cdot dx$

$= \frac{21}{6}\times\frac{1}{7}\tan^{-1}\left(\frac{u}{7}\right)+c$

$= \frac{1}{2}\tan^{-1}\left(\frac{6x}{7}\right)+c$

b $\int \frac{8}{1+16x^2}\cdot dx$

Let

$u = 4x$

$\frac{du}{dx} = 4$

$= \frac{8}{4}\int \frac{1}{1+u^2}\cdot dx$

$= 2\ \tan^{-1}(u)+c$

$= 2\ \tan^{-1}(4x)+c$

12 a $\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{3+2x^2}dx$

Let

$u = \sqrt{2}x$

$\frac{du}{dx} = \sqrt{2}$

$dx = \frac{1}{\sqrt{2}}du$

Terminals

$x = \frac{1}{\sqrt{2}},\ u = 1$

$x = 0,\ u = 0$

$= \frac{1}{\sqrt{2}}\int_0^1 \frac{1}{3+u^2}du$

$= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)\right]_0^1$

$= \frac{1}{\sqrt{6}}\left[\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)-\tan^{-1}(0)\right]$

$= \frac{1}{\sqrt{6}}\times\frac{\pi}{6}\times\frac{\sqrt{6}}{\sqrt{6}}$

$= \frac{\pi\sqrt{6}}{36}$

b $\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{2-3x^2}}dx$

Let

$u = \sqrt{3}x$

$\frac{du}{dx} = \sqrt{3}$

$dx = \frac{1}{\sqrt{3}}du$

Terminals

$x = \frac{1}{\sqrt{2}}, u = \sqrt{\frac{3}{2}}$

$x = 0, u = 0$

$= \frac{1}{\sqrt{3}}\int_0^{\sqrt{\frac{3}{2}}} \frac{1}{\sqrt{2-u^2}}du$

$= \frac{1}{\sqrt{3}}\left[\sin^{-1}\left(\frac{u}{\sqrt{2}}\right)\right]_0^{\sqrt{\frac{3}{2}}}$

$= \frac{1}{\sqrt{3}}\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)-\sin^{-1}(0)\right]$

$= \frac{1}{\sqrt{3}}\times\frac{\pi}{3}\times\frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{\pi\sqrt{3}}{9}$

13 a $\int_0^3 \frac{x}{\sqrt{9-x^2}}$

Let

$u = 9-x^2$

$\frac{du}{dx} = -2x$

$dx = -\frac{1}{2x}\cdot du$

Terminals

$x = 3, u = 0$

$x = 0, u = 9$

$= -\frac{1}{2}\int_9^0 u^{-\frac{1}{2}}\cdot du$

$= \frac{1}{2}\int_0^9 u^{-\frac{1}{2}}\cdot du$

$= \left[u^{\frac{1}{2}}\right]_0^9$

$= \sqrt{9}$

$= 3$

b $\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx = \left[\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3$

$= \sin^{-1}(1)-\sin^{-1}(0)$

$= \frac{\pi}{2}$

14 a $\int_0^3 \frac{1}{9+x^2}\cdot dx = \frac{1}{3}\left[\tan^{-1}\left(\frac{x}{3}\right)\right]_0^3$

$= \frac{1}{3}\left[\tan^{-1}(1)-\tan^{-1}(0)\right]$

$= \frac{1}{3}\times\frac{\pi}{4}$

$= \frac{\pi}{12}$

b $\int_0^3 \frac{x}{9+x^2}\cdot dx = \frac{1}{2}\left[\log_e(9+x^2)\right]_0^3$

$= \frac{1}{2}\left[\log_e(18)-\log_e(9)\right]$

$= \frac{1}{2}\log_e(2)$

15 a $\displaystyle\int_0^{\frac{5}{6}} \frac{1}{\sqrt{25-9x^2}} \cdot dx$

Let

$u = 3x$

$\dfrac{du}{dx} = 3$

Terminals

$x = \dfrac{5}{6},\ u = \dfrac{5}{2}$

$x = 0,\ u = 0$

$$\frac{1}{3}\int_0^{\frac{5}{2}} \frac{1}{\sqrt{25-u^2}} \cdot du$$

$$= \frac{1}{3}\left[\sin^{-1}\left(\frac{u}{5}\right)\right]_0^{\frac{5}{2}}$$

$$= \frac{1}{3}\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)\right]$$

$$= \frac{1}{3}\left[\frac{\pi}{6} - 0\right]$$

$$= \frac{\pi}{18}$$

b $\displaystyle\int_0^{\frac{5}{3}} \frac{x}{25+9x^2} \cdot dx$

Let

$u = 25 + 9x^2$

$\dfrac{du}{dx} = 18x$

Terminals

$x = \dfrac{5}{3},\ u = 50$

$x = 0,\ u = 25$

$$= \frac{1}{18}\int_{35}^{50} \frac{1}{u} \cdot du$$

$$= \frac{1}{18}\left[\log_e |u|\right]_{25}^{50}$$

$$= \frac{1}{18}\left[\log_e(50) - \log_e(25)\right]$$

$$= \frac{1}{18}\log_e(2)$$

16 a $\displaystyle\int_0^{\frac{5}{3}} \frac{x}{\sqrt{25-9x^2}} \cdot dx$

Let

$u = 25 - 9x^2$

$\dfrac{du}{dx} = -18x$

Terminals

$x = \dfrac{5}{3}, u = 0$

$x = 0, u = 25$

$$= -\frac{1}{18}\int_{25}^{0} u^{-\frac{1}{2}} \cdot dx$$

$$= \frac{1}{18}\int_{0}^{25} u^{-\frac{1}{2}} \cdot dx$$

$$= \frac{1}{9}\left[u^{\frac{1}{2}}\right]_0^{25}$$

$$= \frac{1}{9}\left[\sqrt{25} - \sqrt{0}\right]$$

$$= \frac{5}{9}$$

b $\displaystyle\int_0^{\frac{5}{3}} \frac{1}{25+9x^2} \cdot dx$

Let

$u = 3x$

$\dfrac{du}{dx} = 3$

Terminals

$x = \dfrac{5}{3},\ u = 5$

$x = 0,\ u = 0$

$$= \frac{1}{3}\int_0^{5} \frac{1}{25+u^2} \cdot du$$

$$= \frac{1}{3}\left[\frac{1}{5}\tan^{-1}\left(\frac{u}{5}\right)\right]_0^{5}$$

$$= \frac{1}{15}\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$

$$= \frac{1}{15} \times \frac{\pi}{4}$$

$$= \frac{\pi}{60}$$

17 a $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{4-x^2}}$

$$y = \int \frac{1}{\sqrt{4-x^2}} \cdot dx$$

$$= \sin^{-1}\left(\frac{x}{2}\right) + c$$

When

$x = \sqrt{3}, y = \pi$

$\pi = \sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right) + c$

$\pi = \dfrac{\pi}{3} + c$

$c = \dfrac{2\pi}{3}$

$y = \sin^{-1}\left(\dfrac{x}{2}\right) + \dfrac{2\pi}{3}$

b $\dfrac{dy}{dx} = \dfrac{1}{1+4x^2}$

$$y = \int \frac{1}{1+4x^2} \cdot dx$$

Let

$u = 2x$

$\dfrac{du}{dx} = 2$

$$y = \frac{1}{2}\int \frac{1}{1+u^2} \cdot du$$

$$= \frac{1}{2}\tan^{-1}(u) + c$$

$$y = \frac{1}{2}\tan^{-1}(2x) + c$$

When

$x = \frac{1}{2}, y = \pi$

$\pi = \frac{1}{2}\tan^{-1}(1) + c$

$= \frac{\pi}{8} + c$

$c = \frac{7\pi}{8}$

$y = \frac{1}{2}\tan^{-1}(2x) + \frac{7\pi}{8}$

18 a $\frac{dy}{dx} + \frac{1}{3+x^2} = 0$

$\frac{dy}{dx} = -\frac{1}{3+x^2}$

$y = \int -\frac{1}{3+x^2} \cdot dx$

$= -\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + c$

When

$x = 1, y = 0$

$0 = -\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + c$

$c = \frac{1}{\sqrt{3}} \times \frac{\pi}{6} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{\pi\sqrt{3}}{18}$

$y = -\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + \frac{\pi\sqrt{3}}{18}$

When

$x = 0, y = \frac{\pi\sqrt{3}}{18}$

b $\frac{dy}{dx} + \frac{1}{\sqrt{6-x^2}} = 0$

$\frac{dy}{dx} = -\frac{1}{\sqrt{6-x^2}}$

$y = \int -\frac{1}{\sqrt{6-x^2}} \cdot dx$

$y = \cos^{-1}\left(\frac{x}{\sqrt{6}}\right) + c$

When

$x = \sqrt{3}, y = 0$

$0 = \cos^{-1}\left(\frac{\sqrt{3}}{\sqrt{6}}\right) + c$

$c = -\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$= -\frac{\pi}{4}$

$y = \cos^{-1}\left(\frac{x}{\sqrt{6}}\right) - \frac{\pi}{4}$

When

$x = 0$

$y = \cos^{-1}(0) - \frac{\pi}{4}$

$y = \frac{\pi}{2} - \frac{\pi}{4}$

$y = \frac{\pi}{4}$

19 a $\int_0^1 \frac{1}{\sqrt{4-3x^2}}dx$

Let

$u = \sqrt{3}x$

$\frac{du}{dx} = \sqrt{3}$

Terminals

$x = 1, u = \sqrt{3}$

$x = 0, u = 0$

$= \frac{1}{\sqrt{3}}\int_0^{\sqrt{3}} \frac{1}{\sqrt{4-u^2}} \cdot du$

$= \frac{1}{\sqrt{3}}\left[\sin^{-1}\left(\frac{u}{2}\right)\right]_0^{\sqrt{3}}$

$= \frac{1}{\sqrt{3}}\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}(0)\right]$

$= \frac{1}{\sqrt{3}} \times \frac{\pi}{3} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{\pi\sqrt{3}}{9}$

b $\int_0^1 \frac{1}{1+3x^2}dx$

Let

$u = \sqrt{3}x$

$\frac{du}{dx} = \sqrt{3}$

Terminals

$x = 1, u = \sqrt{3}$

$x = 0, u = 0$

$= \frac{1}{\sqrt{3}}\int_0^{\sqrt{3}} \frac{1}{1+u^2} \cdot du$

$= \frac{1}{\sqrt{3}}\left[\tan^{-1}(u)\right]_0^{\sqrt{3}}$

$= \frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\sqrt{3}\right) - \tan^{-1}(0)\right]$

$= \frac{1}{\sqrt{3}} \times \frac{\pi}{3} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{\pi\sqrt{3}}{9}$

20 a $\int_0^1 \frac{x}{1+3x^2} \cdot dx = \frac{1}{6}\left[\log_e\left(1+3x^2\right)\right]_0^1$

$= \frac{1}{6}\left[\log_e(4) - \log_e(1)\right]$

$= \frac{1}{6}\left[\log_e(2^2)\right]$

$= \frac{1}{3}\log_e(2)$

b $\int_0^1 \frac{x}{\sqrt{4-3x^2}} \cdot dx$

Let

$u = 4 - 3x^2$

$\frac{du}{dx} = -6x$

Terminals

$x = 1, u = 1$

$x = 0, u = 4$

$= -\frac{1}{6}\int_4^1 u^{-\frac{1}{2}} \cdot du$

$= \frac{1}{6}\int_1^4 u^{-\frac{1}{2}} \cdot du$

$= \frac{1}{3}\left[u^{\frac{1}{2}}\right]_1^4$

$= \frac{1}{3}\left[\sqrt{4} - \sqrt{1}\right]$

$= \frac{1}{3}[2 - 1]$

$= \frac{1}{3}$

21 a $4x^2 - 12x + 25 = \left(4x^2 - 12x + 9\right) + 16$

$= (2x - 3)^2 + 16$

$\int \frac{1}{4x^2 - 12x + 25}\,dx = \int \frac{1}{(2x-3)^2 + 16}\,dx$

Let

$u = 2x - 3$

$\frac{du}{dx} = 2$

$= \frac{1}{2}\int \frac{1}{u^2 + 16}\,du$

$= \frac{1}{2} \times \frac{1}{4}\tan^{-1}\left(\frac{u}{4}\right) + c$

$= \frac{1}{8}\tan^{-1}\left(\frac{2x-3}{4}\right) + c$

b $7 + 12x - 4x^2 = -\left(4x^2 - 12x + 9\right) + 16$

$= 16 - (2x - 3)^2$

$\int \frac{1}{\sqrt{7x - 12x - 4x^2}}\,dx = \int \frac{1}{\sqrt{16 - (2x-3)^2}}\,dx$

Let

$u = 2x - 3$

$\frac{du}{dx} = 2$

$= \frac{1}{2}\int \frac{1}{\sqrt{16 - u^2}}\,du$

$= \frac{1}{2}\sin^{-1}\left(\frac{u}{4}\right) + c$

$= \frac{1}{2}\sin^{-1}\left(\frac{2x-3}{4}\right) + c$

22 a $25x^2 - 20x + 13 = \left(25x^2 - 20x + 4\right) + 9$

$= (5x - 2)^2 + 9$

$\int \frac{1}{25x^2 - 20x + 13}\,dx = \int \frac{1}{(5x-2)^2 + 9}\,dx$

Let

$u = 5x - 2$

$\frac{du}{dx} = 5$

$= \frac{1}{5}\int \frac{1}{u^2 + 9}\,du$

$= \frac{1}{5} \times \frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right) + c$

$= \frac{1}{15}\tan^{-1}\left(\frac{5x-2}{3}\right) + c$

b $12 + 20x - 25x^2 = -\left(25x^2 - 20x + 4\right) + 16$

$= 16 - (5x - 2)^2$

$\int \frac{1}{\sqrt{12 + 20x - 25x^2}}\,dx = \int \frac{1}{\sqrt{16 - (5x-2)^2}}\,dx$

Let

$u = 5x - 2$

$\frac{du}{dx} = 5$

$= \frac{1}{5}\int \frac{1}{\sqrt{16 - u^2}}\,du$

$= \frac{1}{5}\sin^{-1}\left(\frac{u}{4}\right) + c$

$= \frac{1}{5}\sin^{-1}\left(\frac{5x-2}{4}\right) + c$

23 a Consider

$5 - 4x - x^2 = -\left(x^2 + 4x + 4\right) + 9$

$= 9 - (x + 2)^2$

$\int \frac{2}{5 - 4x - x^2} \cdot dx = \int \frac{2}{9 - (x+2)^2} \cdot dx$

Let

$u = x + 2$

$\frac{du}{dx} = 1$

$\int \frac{2}{9 - u^2} \cdot du = 2\sin^{-1}\left(\frac{u}{3}\right) + c$

$= 2\sin^{-1}\left(\frac{x+2}{3}\right) + c$

b Consider

$x^2 + 4x + 13 = \left(x^2 + 4x + 4\right) + 9$

$= (x + 2)^2 + 9$

$\int \frac{2}{x^2 + 4x + 13} \cdot dx = \int \frac{2}{(x+2)^2 + 9} \cdot dx$

Let

$u = x + 2$

$\frac{du}{dx} = 1$

$= \int \frac{2}{u^2 + 9} \cdot dx$

$= \frac{2}{3}\tan^{-1}\left(\frac{u}{3}\right) + c$

$= \frac{2}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + c$

24 a Consider

$24 - 30x - 9x^2 = -\left(9x^2 + 30x\right) + 24$

$= -\left(9x^2 + 30x + 25\right) + 24 + 25$

$= -(3x + 5)^2 + 49$

$\int \frac{6}{\sqrt{24 - 30x - 9x^2}} \cdot dx = \int \frac{6}{\sqrt{49 - (3x+5)^2}} \cdot dx$

Let

$$u = 3x + 5$$

$$\frac{du}{dx} = 3$$

$$= 6 \times \frac{1}{3} \int \frac{1}{\sqrt{49 - u^2}} \cdot du$$

$$= 2\sin^{-1}\left(\frac{u}{7}\right) + c$$

$$= 2\sin^{-1}\left(\frac{3x+5}{7}\right) + c$$

b $74 + 30x + 9x^2 = \left(9x^2 + 30x + 25\right) + 74 - 25$

$$= \left(9x^2 + 30x + 25\right) + 49$$

$$= (3x + 5)^2 + 49$$

$$\int \frac{6}{74 + 30x + 9x^2} \cdot dx = \int \frac{6}{(3x+5)^2 + 49} \cdot dx$$

Let

$$u = 3x + 5$$

$$\frac{du}{dx} = 3$$

$$= 6 \times \frac{1}{3} \int \frac{1}{u^2 + 49} \cdot du$$

$$= 2 \times \frac{1}{7} \tan^{-1}\left(\frac{u}{7}\right) + c$$

$$= \frac{2}{7} \tan^{-1}\left(\frac{3x+5}{7}\right) + c$$

25 a $\displaystyle\int \frac{5 - 3x}{\sqrt{25 - 9x^2}}\, dx = \int \frac{5}{\sqrt{25 - 9x^2}}\, dx - \int \frac{3x}{\sqrt{25 - 9x^2}}\, dx$

Let

$$u = 3x$$

$$\frac{du}{dx} = 3$$

$$v = 25 - 9x^2$$

$$\frac{dv}{dx} = -18x$$

$$= \frac{5}{3} \int \frac{1}{\sqrt{25 - u^2}}\, du + \frac{3}{18} \int v^{-\frac{1}{2}} dv$$

$$= \frac{5}{3} \sin^{-1}\left(\frac{u}{5}\right) + \frac{1}{6} \times 2v^{\frac{1}{2}} + c$$

$$= \frac{5}{3} \sin^{-1}\left(\frac{3x}{5}\right) + \frac{1}{3}\sqrt{25 - 9x^2} + c$$

b $\displaystyle\int \frac{3x + 5}{9x^2 + 25}\, dx = \int \frac{3x}{9x^2 + 25}\, dx + \int \frac{5}{9x^2 + 25}\, dx$

Let

$$u = 9x^2 + 25$$

$$\frac{du}{dx} = 18x$$

$$v = 3x$$

$$\frac{dv}{dx} = 3$$

$$= \frac{1}{18} \int \frac{3}{u}\, du + \frac{5}{3} \int \frac{1}{v^2 + 25}\, dv$$

$$= \frac{1}{6} \log_e |u| + \frac{5}{3} \times \frac{1}{5} \tan^{-1}\left(\frac{v}{5}\right) + c$$

$$= \frac{1}{6} \log_e (9x^2 + 25 + \frac{1}{3} \tan^{-1}\left(\frac{3x}{5}\right) + c$$

26 a $\displaystyle\int \frac{3x - 4}{\sqrt{9 - 16x^2}} \cdot dx = \int \frac{3x}{\sqrt{9 - 16x^2}} \cdot dx - \int \frac{4}{\sqrt{9 - 16x^2}} \cdot dx$

Let

$$u = 9 - 16x^2$$

$$\frac{du}{dx} = -32x$$

$$v = 4x$$

$$\frac{dv}{dx} = 4$$

$$= -\frac{3}{32} \int u^{-\frac{1}{2}} \cdot du - \frac{4}{4} \int \frac{1}{\sqrt{9 - v^2}} \cdot dv$$

$$= -\frac{3}{32} \times \frac{2}{1} \times u^{\frac{1}{2}} + \cos^{-1}\left(\frac{v}{3}\right) + c$$

$$= \cos^{-1}\left(\frac{4x}{3}\right) - \frac{3}{16}\sqrt{9 - 16x^2} + c$$

b $\displaystyle\int \frac{3 + 4x}{9 + 16x^2} \cdot dx = \int \frac{3}{9 + 16x^2} \cdot dx + \int \frac{4x}{9 + 16x^2} \cdot dx$

Let

$$u = 4x$$

$$\frac{du}{dx} = 4$$

$$v = 9 + 16x^2$$

$$\frac{dv}{dx} = 32x$$

$$= \frac{3}{4} \int \frac{1}{9 + u^2} \cdot du + \frac{1}{8} \int \frac{1}{v} \cdot dv$$

$$= \frac{3}{4} \times \frac{1}{3} \tan^{-1}\left(\frac{u}{3}\right) + \frac{1}{8} \log_e |v| + c$$

$$= \frac{1}{4} \tan^{-1}\left(\frac{4x}{3}\right) + \frac{1}{8} \log_e \left(9 + 16x^2\right) + c$$

27 a $\displaystyle\int \frac{5 - 2x}{\sqrt{5 - 2x^2}} dx = \int \frac{5}{\sqrt{5 - 2x^2}} dx - \int \frac{2x}{\sqrt{5 - 2x^2}} dx$

Let

$$u = \sqrt{2}x$$

$$\frac{du}{dx} = \sqrt{2}$$

$$v = 5 - 2x^2$$

$$\frac{dv}{dx} = -4x$$

$$= \frac{5}{\sqrt{2}} \int \frac{1}{\sqrt{5 - u^2}} du + \frac{2}{4} \int v^{-\frac{1}{2}} dv$$

$$= \frac{5}{\sqrt{2}} \sin^{-1}\left(\frac{u}{\sqrt{5}}\right) + \frac{1}{2} \times 2v^{\frac{1}{2}} + c$$

$$= \frac{5\sqrt{2}}{2} \sin^{-1}\left(\frac{\sqrt{2}x}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}\right) + \sqrt{v} + c$$

$$= \sqrt{5 - 2x^2} + \frac{5\sqrt{2}}{2} \sin^{-1}\left(\frac{\sqrt{10}x}{5}\right) + c$$

b $\displaystyle\int \frac{5 - 2x}{25 + 2x^2} dx = \int \frac{5}{25 + 2x^2} dx - \int \frac{2x}{25 + 2x^2} dx$

Let

$$u = \sqrt{2}x$$

$$\frac{du}{dx} = \sqrt{2}$$

$$v = 5 + 2x^2$$

$$\frac{dv}{dx} = 4x$$

$$\begin{aligned}
&= \frac{5}{\sqrt{2}}\int \frac{1}{25+u^2}du - \frac{1}{2}\int \frac{1}{v}dv \\
&= \frac{5}{\sqrt{2}} \times \frac{1}{5}\tan^{-1}\left(\frac{u}{5}\right) - \frac{1}{2}\log_e(v) + c \\
&= \frac{\sqrt{2}}{\sqrt{2}} \times \frac{5}{\sqrt{2}} \times \frac{1}{5}\tan^{-1}\left(\frac{\sqrt{2}x}{5}\right) - \frac{1}{2}\log_e\left(25+2x^2\right) + c \\
&= \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}x}{5}\right) - \frac{1}{2}\log_e\left(25+2x^2\right) + c
\end{aligned}$$

28 **a** $\displaystyle\int \frac{1}{\sqrt{p^2-q^2x^2}}\cdot dx$

Let

$$\begin{aligned}
u &= qx \\
\frac{du}{dx} &= q \\
&= \frac{1}{q}\int \frac{1}{\sqrt{p^2-u^2}}\cdot du \\
&= \frac{1}{q}\sin^{-1}\left(\frac{u}{p}\right) + c \\
&= \frac{1}{q}\sin^{-1}\left(\frac{qx}{p}\right) + c
\end{aligned}$$

b $\displaystyle\int \frac{1}{p^2+q^2x^2}\cdot dx$

Let

$$\begin{aligned}
u &= qx \\
\frac{du}{dx} &= q \\
&= \frac{1}{q}\int \frac{1}{p^2+u^2}\cdot du \\
&= \frac{1}{q}\left(\frac{1}{p}\tan^{-1}\left(\frac{u}{p}\right)\right) + c \\
&= \frac{1}{pq}\tan^{-1}\left(\frac{qx}{p}\right) + c
\end{aligned}$$

c $\displaystyle\int \frac{ax+b}{\sqrt{p^2-q^2x^2}}\cdot dx = \int \frac{ax}{\sqrt{p^2-q^2x^2}}\cdot dx + \int \frac{b}{\sqrt{p^2-q^2x^2}}\cdot dx$

Let

$$\begin{aligned}
u &= p^2-q^2x^2 \\
\frac{du}{dx} &= -2q^2x \\
&= -\frac{a}{2q^2}\int u^{-\frac{1}{2}}\cdot du + \int \frac{b}{\sqrt{p^2-q^2x^2}}dx \\
&= -\frac{a}{2q^2}\times 2u^{\frac{1}{2}} + \int \frac{b}{\sqrt{p^2-q^2x^2}}dx \\
&= -\frac{a}{q^2}\sqrt{p^2-q^2x^2} + \frac{b}{q}\sin^{-1}\left(\frac{qx}{p}\right) + c
\end{aligned}$$

d $\displaystyle\int \frac{ax+b}{p^2+q^2x^2}\cdot dx = \int \frac{ax}{p^2+q^2x^2}\cdot dx + \int \frac{b}{p^2+q^2x^2}\cdot dx$

Let

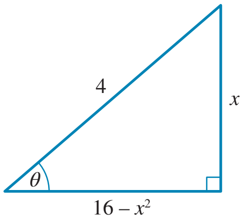
$$\begin{aligned}
u &= p^2+q^2x^2 \\
\frac{du}{dx} &= 2q^2x \\
&= \frac{a}{2q^2}\int \frac{1}{u}\cdot du + \int \frac{b}{p^2+q^2x^2}dx \\
&= \frac{a}{2q^2}\log_e(u) + \int \frac{b}{p^2+q^2x^2}dx \\
&= \frac{a}{2q^2}\log_e\left(p^2+q^2x^2\right) + \frac{b}{pq}\tan^{-1}\left(\frac{qx}{p}\right) + c
\end{aligned}$$

29 **a** **i** $\displaystyle\int \sqrt{16-x^2}\cdot dx$

Let

$$\begin{aligned}
x &= 4\sin(\theta) \\
\frac{dx}{d\theta} &= 4\cos(\theta) \\
dx &= 4\cos(\theta)d\theta \\
x^2 &= 16\ \sin^2(\theta) \\
16-x^2 &= 16-16\ \sin^2(\theta) \\
&= 16\left(1-\sin^2(\theta)\right) \\
&= 16\ \cos^2(\theta) \\
\sqrt{16-x^2} &= 4\cos(\theta)
\end{aligned}$$

$$\begin{aligned}
&\int \sqrt{16-x^2}\cdot dx \\
&= \int 4\cos(\theta)\cdot 4\ \cos(\theta)\cdot d\theta \\
&= 16\int \cos^2(\theta)\cdot d\theta \\
&= 8\int 1+\cos(2\theta)\cdot d\theta \\
&= 8\left[\theta+\frac{1}{2}\sin(2\theta)\right] + c \\
&= 8\theta + 4\times 2\sin(\theta)\cos(\theta) + c
\end{aligned}$$



$$\begin{aligned}
\sin(\theta) &= \frac{x}{4} \\
\theta &= \sin^{-1}\left(\frac{x}{4}\right) \\
\cos(\theta) &= \frac{\sqrt{16-x^2}}{4}
\end{aligned}$$

$$\begin{aligned}
&\int \sqrt{16-x^2}dx \\
&= 8\sin^{-1}\left(\frac{x}{4}\right) + \frac{x}{2}\sqrt{16-x^2} + c
\end{aligned}$$

ii Terminals

$x = 4, \theta = \dfrac{\pi}{2}$

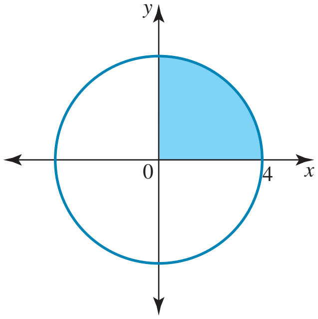
$x = 0, \theta = 0$

$$\begin{aligned}
&\int_0^4 \sqrt{16-x^2}\cdot dx = 16\int_0^{\frac{\pi}{2}} \cos^2(\theta)\cdot d\theta \\
&= 8\int_0^{\frac{\pi}{2}} 1+\cos(2\theta)\cdot d\theta \\
&= 8\left[\theta+\frac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{2}}
\end{aligned}$$

$$= 8\left[\frac{\pi}{2} + \frac{1}{2}\sin(\pi) - 0\right]$$
$$= 4\pi$$

If

$$y = \sqrt{16 - x^2}$$
$$y^2 = 16 - x^2$$
$$x^2 + y^2 = 16$$



$$A = \int_a^b y \cdot dx$$

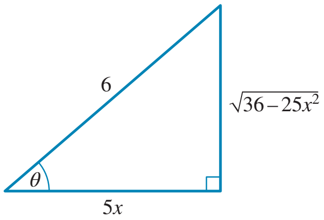
Area is $\frac{1}{4}$ of the area of the circle of radius 4.

$$A = \frac{1}{4}\pi r^2$$
$$= \frac{1}{4}\pi \times 16$$
$$= 4\pi$$

b i $\int \sqrt{36 - 25x^2} \cdot dx$

Let

$$x = \frac{6}{5}\cos(\theta)$$
$$\frac{dx}{d\theta} = -\frac{6}{5}\sin(\theta)$$
$$dx = -\frac{6}{5}\sin(\theta)d\theta$$
$$x^2 = \frac{36}{25}\cos^2(\theta)$$
$$25x^2 = 36\cos^2(\theta)$$
$$36 - 25x^2 = 36 - 36\cos^2(\theta)$$
$$= 36\left(1 - \cos^2(\theta)\right)$$
$$= 36\sin^2(\theta)$$
$$\sqrt{36 - 25x^2} = 6\sin(\theta)$$
$$\int \sqrt{36 - 25x^2} \cdot dx$$
$$= \int 6\sin(\theta) \times -\frac{6}{5}\sin(\theta) \cdot d\theta$$
$$= -\frac{36}{5}\int \sin^2(\theta) \cdot d\theta$$
$$= -\frac{18}{5}\int 1 - \cos(2\theta) \cdot d\theta$$
$$= -\frac{18}{5}\left[\theta - \frac{1}{2}\sin(2\theta)\right] + c$$
$$= -\frac{18\theta}{5} + \frac{9}{5}\sin(2\theta) + c$$
$$= -\frac{18}{5}\theta + \frac{18}{5}\sin(\theta)\cos(\theta) + c$$



$$\cos(\theta) = \frac{5x}{6}$$
$$\theta = \cos^{-1}\left(\frac{5x}{6}\right)$$
$$= -\frac{18}{5}\cos^{-1}\left(\frac{5x}{6}\right) + \frac{18}{5} \times \frac{\sqrt{36 - 25x^2}}{6} \times \frac{5x}{6} + c$$
$$= \frac{x}{2}\sqrt{36 - 25x^2} - \frac{18}{5}\cos^{-1}\left(\frac{5x}{6}\right) + c$$

ii $\int_0^{\frac{6}{5}} \sqrt{36 - 25x^2} \cdot dx$

Let

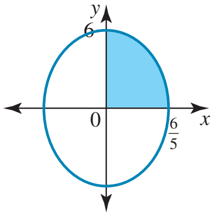
$$u = \frac{6}{5}\cos(\theta)$$

Terminals

$$x = \frac{6}{5}, \cos(\theta) = 1 \to \theta = 0$$
$$x = 0, \cos(\theta) = 0 \to \theta = \frac{\pi}{2}$$
$$\int_{\frac{\pi}{2}}^{0} -\frac{36}{5}\sin^2(\theta) \cdot d\theta$$
$$= \frac{18}{5}\int_0^{\frac{\pi}{2}} 1 - \cos(2\theta) \cdot d\theta$$
$$= \frac{18}{5}\left[\theta - \frac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{2}}$$
$$= \frac{18}{5}\left[\frac{\pi}{2} - \frac{1}{2}\sin(\pi) - 0\right]$$
$$= \frac{9\pi}{5}$$

If

$$y = \sqrt{36 - 25x^2}$$
$$y^2 = 36 - 25x^2$$
$$25x^2 + y^2 = 36$$
$$\frac{25x^2}{36} + \frac{y^2}{36} = 1$$
$$\frac{x^2}{\left(\frac{6}{5}\right)^2} + \frac{y^2}{6^2} = 1$$



Area is one-quarter of the ellipse.

$$a = \frac{6}{5}, b = 6$$

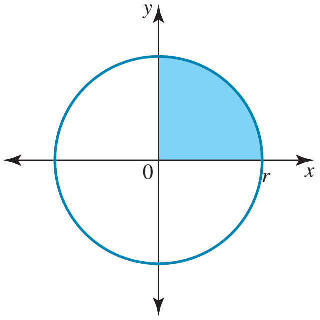
$$\frac{1}{4}\pi ab = \frac{1}{4}\pi \times \frac{6}{5} \times 6$$

$$= \frac{9\pi}{5}$$

30 a $x^2 + y^2 = r^2$

$$y^2 = r^2 - x^2$$

$$y = \pm\sqrt{r^2 - x^2}$$



Symmetry A $= 4\int_0^r \sqrt{r^2 - x^2} \cdot dx$

Let

$$x = r\sin(\theta)$$

$$\frac{dx}{d\theta} = r\cos(\theta)$$

$$dx = r\cos(\theta) \cdot d\theta$$

$$x^2 = r^2\sin^2(\theta)$$

$$r^2 - x^2 = r^2 - r^2\sin^2(\theta)$$

$$= r^2\left(1 - \sin^2(\theta)\right)$$

$$= r^2\cos^2(\theta)$$

$$\sqrt{r^2 - x^2} = r\cos(\theta)$$

Terminals

$x = r, \sin(\theta) = 1 \to \theta = \frac{\pi}{2}$

$x = 0, \sin(\theta) = 0 \to \theta = 0$

$$A = 4\int_0^{\frac{\pi}{2}} r^2\cos^2(\theta) \cdot d\theta$$

$$= 2r^2\int_0^{\frac{\pi}{2}} 1 + \cos(2\theta) \cdot d\theta$$

$$= 2r^2\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{2}}$$

$$= 2r^2\left[\frac{\pi}{2} + \frac{1}{2}\sin(\theta) - 0\right]$$

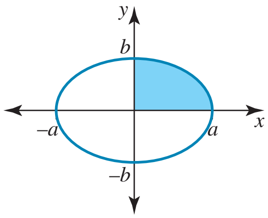
$$= \pi r^2$$

b $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2}\left(a^2 - x^2\right)$$

$$y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$



Symmetry

$$A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2} \cdot dx$$

$$= \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2} \cdot dx$$

$$= \frac{4b}{a}\left(\frac{\pi a^2}{4}\right)$$

$$= \pi ab$$

31 a $\int_3^5 \frac{x - 4}{x^2 - 6x + 13} \cdot dx$

$$= \frac{1}{2}\int_3^5 \frac{2(x - 4)}{x^2 - 6x + 13} \cdot dx$$

$$= \frac{1}{2}\int_3^5 \frac{2x - 6 - 2}{x^2 - 6x + 13} \cdot dx$$

$$= \frac{1}{2}\int_3^5 \frac{2x - 6}{x^2 - 6x + 13} \cdot dx - \int_3^5 \frac{1}{x^2 - 6x + 13} \cdot dx$$

$$= \frac{1}{2}\left[\log_e\left(x^2 - 6x + 13\right)\right]_3^5 - \left[\frac{1}{2}\tan^{-1}\left(\frac{x - 3}{2}\right)\right]_3^5$$

$$= \frac{1}{2}\left[\log_e(8) - \log_e(4)\right] - \frac{1}{2}\tan^{-1}(1) + \frac{1}{2}\tan^{-1}(0)$$

$$= \frac{1}{2}\log_e(2) - \frac{\pi}{8}$$

b $\int_{\frac{3}{2}}^{\frac{7}{2}} \frac{2x + 1}{4x^2 - 12x + 25} \cdot dx$

$$= \frac{1}{4}\int_{\frac{3}{2}}^{\frac{7}{2}} \frac{4(2x + 1)}{4x^2 - 12x + 25} \cdot dx$$

$$= \frac{1}{4}\int_{\frac{3}{2}}^{\frac{7}{2}} \frac{8x - 12 + 16}{4x^2 - 12x + 25} \cdot dx$$

$$= \frac{1}{4}\int_{\frac{3}{2}}^{\frac{7}{2}} \frac{8x - 12}{4x^2 - 12x + 25} \cdot dx + 4\int_{\frac{3}{2}}^{\frac{7}{2}} \frac{1}{4x^2 - 12x + 25} \cdot dx$$

$$= \frac{1}{4}\left[\log_e(4x^2 - 12x + 25)\right]_{\frac{3}{2}}^{\frac{7}{2}} + 4 \times \frac{1}{2} \times \left[\frac{1}{4}\tan^{-1}\left(\frac{2x - 3}{4}\right)\right]_{\frac{3}{2}}^{\frac{7}{2}}$$

$$= \frac{1}{4}[\log_e(32) - \log_e(16)] + \frac{1}{2}[\tan^{-1}(1) - \tan^{-1}(0)]$$

$$= \frac{1}{4}\log_e(2) + \frac{\pi}{8}$$

32 a $\int_1^4 \frac{3x+2}{\sqrt{8+2x-x^2}} \cdot dx$

$$= -\frac{1}{2}\int_1^4 \frac{-2(3x+2)}{\sqrt{8+2x-x^2}} \cdot dx$$

$$= -\frac{1}{2}\int_1^4 \frac{-6x+6-10}{\sqrt{8+2x-x^2}} \cdot dx$$

$$= -\frac{3}{2}\int_1^4 \frac{-2x+2}{\sqrt{8+2x-x^2}} \cdot dx + 5\int_1^4 \frac{1}{\sqrt{8+2x-x^2}} \cdot dx$$

$$= -\frac{3}{2}\int_1^4 \frac{2-2x}{\left(8+2x-x^2\right)^{\frac{1}{2}}} \cdot dx + 5\int_1^4 \frac{1}{\sqrt{9-(x-1)^2}} \cdot dx$$

$$= -\frac{3}{2}\int_1^4 \frac{2-2x}{\left(8+2x-x^2\right)^{\frac{1}{2}}} \cdot dx + 5\left[\sin^{-1}\left(\frac{x-1}{3}\right)\right]_1^4$$

$$= -\frac{3}{2}\int_1^4 \frac{2-2x}{\left(8+2x-x^2\right)^{\frac{1}{2}}} \cdot dx + 5\left[\sin^{-1}(1)-\sin^{-1}(0)\right]$$

$$= -\frac{3}{2}\int_1^4 \frac{2-2x}{\left(8+2x-x^2\right)^{\frac{1}{2}}} \cdot dx + \frac{5\pi}{2}$$

Let

$u = 8 + 2x - x^2$

$\frac{du}{dx} = 2 - 2x$

Terminals

$x = 4, u = 0$

$x = 1, u = 9$

$$= -\frac{3}{2}\int_9^0 u^{-\frac{1}{2}} \cdot dx + \frac{5\pi}{2}$$

$$= \frac{3}{2}\int_0^9 u^{-\frac{1}{2}} \cdot dx + \frac{5\pi}{2}$$

$$= 3\left[u^{\frac{1}{2}}\right]_0^9 + \frac{5\pi}{2}$$

$$= 3\left[\sqrt{9}-\sqrt{0}\right] + \frac{5\pi}{2}$$

$$= 9 + \frac{5\pi}{2}$$

b $\int_{-2}^1 \frac{2x-3}{\sqrt{12-8x-4x^2}} \cdot dx$

If

$u = 12 - 8x - 4x^2$

$\frac{du}{dx} = -8 - 8x$

Terminals

$x = 1, u = 0$

$x = -2, u = 12$

Now

$$\begin{aligned}-4x^2 - 8x + 12 &= \left(4x^2 + 8x\right) + 12\\ &= \left(4x^2 + 8x + 4\right) + 16\\ &= -(2x+2)^2 + 16\end{aligned}$$

$$= -\frac{1}{4}\int_{-2}^1 \frac{-4(2x-3)}{\sqrt{12-8x-4x^2}} \cdot dx$$

$$= -\frac{1}{4}\int_{-2}^1 \frac{-8x-8+20}{\sqrt{12-8x-4x^2}} \cdot dx$$

$$= -\frac{1}{4}\int_{-2}^1 \frac{-8x-8}{\sqrt{12-8x-4x^2}} \cdot dx - 5\int_{-2}^1 \frac{1}{\sqrt{12-8x-4x^2}} \cdot dx$$

$$= -\frac{1}{4}\int_{12}^0 u^{-\frac{1}{2}} \cdot du - 5\int_{-2}^1 \frac{1}{\sqrt{16-(2x+2)^2}} \cdot dx$$

$$= -\frac{1}{4} \times 2\left[u^{\frac{1}{2}}\right]_{12}^0 - 5\left[\frac{1}{2}\sin^{-1}\left(\frac{2x+2}{4}\right)\right]_{-2}^1$$

$$= -\frac{1}{2}\left[\sqrt{0}-\sqrt{12}\right] - \frac{5}{2}\left[\sin^{-1}\left(\frac{x+1}{2}\right)\right]_{-2}^1$$

$$= \frac{1}{2}\sqrt{4}\times\sqrt{3} - \frac{5}{2}\left[\sin^{-1}(1) - \sin^{-1}\left(-\frac{1}{2}\right)\right]$$

$$= \sqrt{3} - \frac{5}{2}\left[\frac{\pi}{2} + \frac{\pi}{6}\right]$$

$$= \sqrt{3} - \frac{5\pi}{3}$$

7.6 Exam questions

1 $\int_0^1 \frac{2x+1}{x^2+1}dx = \int_0^1 \frac{2x}{x^2+1}dx + \int_0^1 \frac{1}{x^2+1}dx$

$$\begin{aligned}&= \left[\log_e(x^2+1) + \tan^{-1}(x)\right]_0^1\\ &= (\log_e(2) + \tan^{-1}(1)) - (\log_e(1) + \tan^{-1}(0))\\ &= \log_e(2) + \frac{\pi}{4}\end{aligned}$$

Award 1 mark for two integrals.
Award 1 mark for either correct integrals and evaluating.
Award 1 mark for final correct answer.

2 Let $u = \sqrt{5}x$, $\frac{du}{dx} = \sqrt{5}$

$$\int \frac{dx}{2+5x^2}$$

$$= \frac{1}{\sqrt{5}}\int \frac{du}{2+u^2} = \frac{1}{\sqrt{5}}\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + c$$

$$= \frac{1}{\sqrt{10}}\tan^{-1}\left(\frac{\sqrt{5}x}{\sqrt{2}}\right) + c$$

$$= \frac{\sqrt{10}}{10}\tan^{-1}\left(\frac{\sqrt{10}x}{2}\right) + c$$

The correct answer is **D**.

3 $\int \frac{\frac{a}{b}}{\sqrt{\left(b^2 - a^2x^2\right)}}dx = \int \frac{\frac{a}{b}}{\sqrt{a^2\left(\left(\frac{b^2}{a^2}\right) - x^2\right)}}dx$ [1 mark]

$$= \int \frac{\frac{1}{b}}{\sqrt{\left(\frac{b^2}{a^2}\right) - x^2}}dx$$

$= \frac{1}{b}\sin^{-1}\left(\frac{ax}{b}\right)$ [1 mark]

7.7 Integrals involving partial fractions

7.7 Exercise

1 a $\int \frac{1}{100-x^2}\,dx$

$$\begin{aligned}\frac{1}{100-x^2} &= \frac{A}{10-x} + \frac{B}{10+x}\\ &= \frac{A(10+x) + B(10-x)}{(10-x)(10+x)}\\ &= \frac{x(A-B) + 10(A+B)}{100-x^2}\end{aligned}$$

$(1)\, A - B = 0 \rightarrow A = B = \frac{1}{20}$

$(2)\, 10(A + B) = 1$

$$\int \frac{1}{100 - x^2}\, dx = \frac{1}{20} \int \frac{1}{10 - x} + \frac{1}{10 - x}\, dx$$

$$= \frac{1}{20}\left[-\log_e |10 - x| + \log_e |10 + x|\right] + c$$

$$= \frac{1}{20} \log_e \left|\frac{x + 10}{x - 10}\right| + c$$

b $\displaystyle\int \frac{12}{64 - 9x^2}\, dx$

$$\frac{12}{64 - 9x^2} = \frac{A}{8 - 3x} + \frac{B}{8 + 3x}$$

$$= \frac{A(8 - 3x) + B(8 + 3x)}{(8 - 3x)(8 + 3x)}$$

$$= \frac{-3x(A - B) + 8(A + B)}{64 - 9x^2}$$

$(1)\, A - B = 0 \rightarrow A = B$

$(2)\, 8(A + B) = 12 \rightarrow A = B = \frac{3}{4}$

$$\int \frac{1}{64 - 9x^2}\, dx = \frac{3}{4} \int \frac{1}{8 - 3x} + \frac{1}{8 - 3x} dx$$

$$= \frac{3}{4}\left[-\frac{1}{3} \log_e |8 - 3x| + \frac{1}{3} \log_e |8 + 3x|\right] + c$$

$$= \frac{1}{4} \log_e \left|\frac{3x + 8}{3x - 8}\right| + c$$

2 a $\displaystyle\int \frac{1}{36 - 25x^2}\, dx$

$$\frac{1}{36 - 25x^2} = \frac{A}{6 - 5x} + \frac{B}{6 + 5x}$$

$$= \frac{A(6 + 5x) + B(6 - 5x)}{(6 - 5x)(6 + 5x)}$$

$$= \frac{5x(A - B) + 6(A + B)}{36 - 25x^2}$$

$(1)\, A - B = 0 \rightarrow A = B = \frac{1}{12}$

$(2)\, 6(A + B) = 1$

$$\int \frac{1}{36 - 25x^2}\, dx = \frac{1}{12} \int \frac{1}{6 - 5x} + \frac{1}{6 + 5x}\, dx$$

$$= \frac{1}{12}\left[-\frac{1}{5} \log_e |6 - 5x| + \frac{1}{5} \log_e |6 + 5x|\right] + c$$

$$= \frac{1}{60} \log_e \left|\frac{5x + 6}{5x - 6}\right| + c$$

b $\displaystyle\int \frac{x}{36 - 25x^2}\, dx = -\frac{1}{50} \int \frac{50x}{36 - 25x^2}\, dx$

$$= -\frac{1}{50} \log_e |36 - 25x^2| + c$$

3 a $\displaystyle\frac{1}{x^2 - 4} = \frac{1}{(x + 2)(x - 2)}$

$$= \frac{A}{(x + 2)} + \frac{B}{(x - 2)}$$

$$= \frac{A(x - 2) + B(x + 2)}{(x + 2)(x - 2)}$$

So $1 = A(x - 2) + B(x + 2)$

Let

$x = 2, 1 = 4B \rightarrow B = \frac{1}{4}$

$x = -2, 1 = -4A \rightarrow A = -\frac{1}{4}$

$$= \int \frac{1}{x^2 - 4} dx = \frac{1}{4} \int -\frac{1}{(x + 2)} + \frac{1}{(x - 2)}\, dx$$

$$= \frac{1}{4}\left[\log_e |x - 2| - \log_e |x + 2|\right] + c$$

$$= \frac{1}{4} \log_e \left(\left|\frac{x - 2}{x + 2}\right|\right) + c$$

b $\displaystyle\int \frac{1}{16 + x^2}\, dx$

$$= 2 \times \frac{1}{4} \tan^{-1}\left(\frac{x}{4}\right) + c$$

$$= \frac{1}{2} \tan^{-1}\left(\frac{x}{4}\right) + c$$

4 a $\displaystyle\int \frac{x}{x^2 - 25}\, dx$

$$= \frac{1}{2} \int \frac{2x}{x^2 - 25}\, dx$$

$$= \frac{1}{2} \log_e |x^2 - 25| + c$$

b $\displaystyle\frac{2x - 3}{x^2 - 36} = \frac{1}{(x + 6)(x - 6)}$

$$= \frac{A}{(x + 6)} + \frac{B}{(x - 6)}$$

$$= \frac{A(x - 6) + B(x + 6)}{(x + 6)(x - 6)}$$

$x = 6, 9 = 12B \rightarrow B = \frac{3}{4}$

$x = -6, -15 = -12A \rightarrow A = \frac{5}{4}$

$$\int \frac{2x - 3}{x^2 - 36} dx$$

$$= \frac{1}{4} \int \left(\frac{5}{x + 6} + \frac{3}{x - 6}\right) dx$$

$$= \frac{1}{4}\left[5 \log_e |x + 6| + 3 \log_e |x - 6|\right] + c$$

$$= \frac{1}{4} \log_e(|x + 6|^5 |x - 6|^3) + c$$

5 a $\displaystyle\frac{x + 13}{x^2 + 2x - 15} = \frac{x + 13}{(x + 5)(x - 3)}$

$$\frac{A}{(x + 5)} + \frac{B}{(x - 3)}$$

$$= \frac{A(x - 3) + B(x + 5)}{(x + 5)(x - 3)}$$

$x + 13 = A(x - 3) + B(x + 5)$

$x = 3, 16 = 8B \rightarrow B = 2$

$x = -5, 8 = -8A \rightarrow A = -1$

$$\int \frac{x + 13}{x^2 + 2x - 15}\, dx = \int \frac{-1}{(x + 5)} + \frac{2}{(x - 3)}\, dx$$

$$= -\log_e |x + 5| + 2 \log_e |x - 3| + c$$

$$= \log_e \left(\left|\frac{(x - 3)^2}{x + 5}\right|\right) + c$$

b $$\begin{aligned}\frac{x-11}{x^2+3x-4} &= \frac{x-11}{(x+4)(x-1)}\\ &\frac{A}{(x+4)}+\frac{B}{(x-1)}\\ &= \frac{A(x-1)+B(x+4)}{(x+4)(x-1)}\end{aligned}$$

$x-11 = A(x-1)+B(x+4)$

$x = 1, -10 = 5B \rightarrow B = -2$

$x = -4, -15 = -5A \rightarrow A = 3$

$$\begin{aligned}\int \frac{x-11}{x^2+3x-4}\,dx &= \int \frac{3}{(x+4)} - \frac{2}{(x-1)}\,dx\\ &= 3\log_e|x+4| - 2\log_e|x-1| + c\\ &= \log_e\left(\left|\frac{(x+4)^3}{(x-1)^2}\right|\right) + c\end{aligned}$$

6 a $$\begin{aligned}\frac{x+11}{x^2+x-12} &= \frac{x+11}{(x+4)(x-3)}\\ &= \frac{A}{(x+4)}+\frac{B}{(x-3)}\\ &= \frac{A(x-3)+B(x+4)}{(x+4)(x-3)}\end{aligned}$$

So $x+11 = A(x-3)+B(x+4)$

Let

$x = 3, 14 = 7B \rightarrow B = 2$

$x = -4, 7 = -7A \rightarrow A = -1$

So

$$\begin{aligned}&\int \frac{x+11}{x^2+x-12}\,dx\\ &= \int \left(\frac{2}{x-3}+\frac{-1}{x+4}\right)dx\\ &= 2\log_e|x-3| - \log_e|x+4| + c\\ &= \log_e\left(\frac{(x-3)^2}{|x+4|}\right) + c\end{aligned}$$

b $$\begin{aligned}\frac{5x-9}{x^2-2x-15} &= \frac{5x-9}{(x-5)(x+3)}\\ &= \frac{A}{(x-5)}+\frac{B}{(x+3)}\\ &= \frac{A(x+3)+B(x-5)}{(x-5)(x-3)}\end{aligned}$$

So $5x-9 = A(x+3)+B(x-5)$

Let

$x = 5, 16 = 8A \rightarrow A = 2$

$x = -3, -24 = -8B \rightarrow B = 3$

So

$$\begin{aligned}\int \frac{5x-9}{x^2-2x-15}\,dx &= \int \left(\frac{2}{x-5}+\frac{3}{x+3}\right)dx\\ &= 2\log_e|x-5| + 3\log_e|x+3| + c\\ &= \log_e\left((x-5)^2|x+3|^3\right) + c\end{aligned}$$

7 a $$\begin{aligned}\frac{2x-19}{x^2+x-6} &= \frac{2x-19}{(x+3)(x-2)}\\ &= \frac{A}{(x+3)}+\frac{B}{(x-2)}\\ &= \frac{A(x-2)+B(x+3)}{(x+3)(x-2)}\end{aligned}$$

So $2x-19 = A(x-2)+B(x+3)$

Let

$x = 2, -15 = 5B \rightarrow B = -3$

$x = -3, -25 = -5A \rightarrow A = 5$

$$\begin{aligned}\int \frac{2x-19}{x^2+x-6}\,dx &= \int \frac{5}{x+3}dx - \int \frac{3}{x-2}\,dx\\ &= 5\log_e|x+3| - 3\log_e|x-2| + c\\ &= \log_e\left(\frac{|x+3|^5}{|x-2|^3}\right) + c\end{aligned}$$

b $$\begin{aligned}\frac{11}{x^2-3x-28} &= \frac{11}{(x-7)(x+4)}\\ &= \frac{A}{(x-7)}+\frac{B}{(x+4)}\\ &= \frac{A(x+4)+B(x-7)}{(x-7)(x+4)}\end{aligned}$$

So $11 = A(x+4)+B(x-7)$

Let

$x = 7, 11 = 11A \rightarrow A = 1$

$x = -4, 11 = -11B \rightarrow B = -1$

$$\begin{aligned}\int \frac{11}{x^2-3x-28}\,dx &= \int \left(\frac{1}{x-7}-\frac{1}{x+4}\right)dx\\ &= \log_e|x-7| - \log_e|x+4| + c\\ &= \log_e\left(\frac{|x-7|}{|x+4|}\right) + c\end{aligned}$$

8 a $$\begin{aligned}\frac{2x+1}{x^2+6x+9} &= \frac{2x+1}{(x+3)^2}\\ &= \frac{A}{(x+3)}+\frac{B}{(x+3)^2}\\ &= \frac{A(x+3)+B}{(x+3)^2}\\ &= \frac{Ax+3A+B}{(x+3)^2}\end{aligned}$$

$A = 2$

$3A + B = 1 \rightarrow B = -5$

$$\begin{aligned}\int \frac{2x+1}{x^2+6x+9}\,dx &= \int \frac{2}{(x+3)} - \frac{5}{(x+3)^2}\,dx\\ &= \log_e(x+3)^2 + \frac{5}{x+3} + c\end{aligned}$$

b $$\begin{aligned}\frac{2x-1}{9x^2-24x+16} &= \frac{2x-1}{(3x-4)^2}\\ &= \frac{A}{(3x-4)}+\frac{B}{(3x-4)^2}\\ &= \frac{A(3x-4)+B}{(3x-4)^2}\\ &= \frac{3Ax+\mathrm{B}-4A}{(3x-4)^2}\end{aligned}$$

$3A = 2 \rightarrow A = \frac{2}{3}$

$B - 4A = -1 \rightarrow B = \frac{5}{3}$

$$\begin{aligned}\int \frac{2x-1}{9x^2-24x+16}\,dx &= \int \frac{2}{3(3x-4)} + \frac{5}{3(3x-4)^2}\,dx\\ &= \frac{2}{9}\log_e(|3x-4|) - \frac{5}{9(3x-4)} + c\end{aligned}$$

9 a $$\begin{aligned}\frac{2x+3}{x^2-6x+9} &= \frac{2x+3}{(x-3)^2}\\ &= \frac{A}{(x-3)}+\frac{B}{(x-3)^2}\\ &= \frac{A(x-3)+B}{x^2-6x+9}\\ &= \frac{Ax+B-3A}{x^2-6x+9}\end{aligned}$$

So

$$A = 2$$

$$B - 3A = 3$$

$$\rightarrow B = 3 + 3A = 9$$

$$\int \frac{2x+3}{x^2-6x+9}\,dx$$

$$= \int \left(\frac{2}{x-3} + \frac{9}{(x-3)^2}\right) dx$$

$$= 2\log_e |x-3| - \frac{9}{x-3} + c$$

b $$\frac{2x-5}{x^2+4x+4} = \frac{2x-5}{(x+2)^2}$$

$$= \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$$

$$= \frac{A(x+2)+B}{(x+2)^2}$$

$$= \frac{Ax+2A+B}{x^2+4x+4}$$

So

$$A = 2$$

$$2A + B = -5$$

$$\rightarrow B = -5 - 2A = -9$$

$$\int \frac{2x-5}{x^2+4x+14}\,dx$$

$$= \int \left(\frac{2}{x+2} - \frac{9}{(x+2)^2}\right) dx$$

$$= 2\log_e |x+2| + \frac{9}{x+2} + c$$

$$= \log_e |x+2|^2 + \frac{9}{x+2} + c$$

10 a $$\frac{4x}{4x^2+12x+9} = \frac{4x}{(2x+3)^2}$$

$$= \frac{A}{(2x+3)} + \frac{B}{(2x+3)^2}$$

$$= \frac{A(2x+3)+B}{(2x+3)^2}$$

$$= \frac{2Ax+3A+B}{4x^2+12x+9}$$

So

$$2A = 4 \rightarrow A = 2$$

$$3A + B = 0 \rightarrow B = -3A = -6$$

$$\int \frac{4x}{4x^2+12x+9}\,dx$$

$$= \int \left(\frac{2}{2x+3} - \frac{6}{(2x+3)^2}\right) dx$$

$$= \log_e |2x+3| + \frac{3}{2x+3} + c$$

b $$\frac{6x-19}{9x^2-30x+15} = \frac{6x-19}{(3x-5)^2}$$

$$= \frac{A}{(3x-5)} + \frac{B}{(3x-5)^2}$$

$$= \frac{A(3x-5)+B}{(3x-5)^2}$$

$$= \frac{3Ax\ +\ B-5A}{9x^2-30x+15}$$

So

$$3A = 6 \rightarrow A = 2$$

$$B - 5A = -19 \rightarrow B = -9$$

$$\int \frac{6x-19}{9x^2-30x+15}\,dx$$

$$= \int \left(\frac{2}{3x-5} - \frac{9}{(3x-5)^2}\right) dx$$

$$= \frac{2}{3}\log_e |3x-5| + \frac{3}{3x-5} + c$$

11 $$\int \frac{1}{x^2+kx+25}dx$$

a $k = 0$

$$\int \frac{1}{x^2+25}dx = \frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) + c$$

b $k = 26$

$$\frac{1}{x^2+26x+25} = \frac{A}{x+25} + \frac{B}{x+1}$$

$$= \frac{A(x+1)+B(x+25)}{(x+25)(x+1)}$$

So $1\ =\ A(x+1) + B(x+25)$

Let

$$x = -1, 1 = 24B \rightarrow B = \frac{1}{24}$$

$$x = -25, 1 = -24A \rightarrow A = -\frac{1}{24}$$

$$\int \frac{1}{x^2+26x+25}\,dx = \frac{1}{24}\int\left(\frac{1}{x+1} - \frac{1}{x+25}\right) dx$$

$$= \frac{1}{24}\left[\log_e |x+1| - \log_e |x+25|\right] + c$$

$$= \frac{1}{24}\log_e\left(\left|\frac{x+1}{x+25}\right|\right) + c$$

c $k = -10$

$$\int \frac{1}{x^2-10x+25}\,dx = \int \frac{1}{(x-5)^2}\,dx$$

$$= -\frac{1}{x-5} + c$$

12 a $$\frac{3x^2+10x-4}{x^2+3x-10} = \frac{3\left(x^2+3x-10\right)+x+26}{x^2+3x-10}$$

$$= 3 + \frac{x+26}{x^2+3x-10}$$

$$= 3 + \frac{A}{(x+5)} + \frac{B}{(x-2)}$$

$$= 3 + \frac{A(x-2)+B(x+5)}{(x+5)(x-2)}$$

$$x + 26 = A(x-2) + B(x+5)$$

$$x = 2, 28 = 7B \rightarrow \mathrm{B} = 4$$

$$x = -5, 21 = -7A \rightarrow A = -3$$

$$\int \frac{3x^2+10x-4}{x^2+3x-10}dx = \int 3 - \frac{3}{x+5} + \frac{4}{x-2}dx$$

$$= 3x - 3\log_e |x+5| + 4\log_e |x-2\,| + c$$

$$= 3x + \log_e\left(\frac{(x-2)^4}{|x+5|^3}\right) + c$$

b
$$\begin{aligned}\frac{-2x^2-x+20}{x^2+x-6} &= \frac{-2x^2-2x+12+x-8}{x^2+x-6}\\ &= \frac{-2\left(x^2+x-6\right)+x+8}{x^2+x-6}\\ &= -2+\frac{x+8}{x^2+x-6}\\ &= -2+\frac{A}{(x+3)}+\frac{B}{(x-2)}\\ &= -2+\frac{A(x-2)+B(x+3)}{(x+3)(x-2)}\end{aligned}$$

$x+8=A(x-2)+B(x+3)$

$x=2, 10=5B \rightarrow B=2$

$x=-3, 5=-5A \rightarrow A=-1$

$$\begin{aligned}\int \frac{-2x^2-x+20}{x^2+x-6}dx &= \int -2-\frac{1}{x+3}+\frac{2}{x-2}dx\\ &= -2x-\log_e|x+3|+2\log_e|x-2|+c\\ &= \log_e\left(\frac{(x-2)^2}{|x+3|}\right)-2x+c\end{aligned}$$

13 a

$$\begin{array}{r} 1 \\ x^2+x-12\overline{\smash{)}x^2-4x-11} \\ -\left(x^2+x-12\right) \\ -5x+1 \end{array}$$

$$\frac{x^2-4x-11}{x^2+x-12}$$

$$= 1+\frac{1-5x}{x^2+x-12}$$

$$\begin{aligned}\frac{1-5x}{x^2+x-12} &= \frac{1-5x}{(x+4)(x-3)}\\ &= \frac{A}{(x+4)}+\frac{B}{(x-3)}\\ &= \frac{A(x-3)+B(x+4)}{(x+4)(x-3)}\end{aligned}$$

So $1-5x=A(x-3)+B(x+4)$

Let

$x=3, -14=7B \rightarrow B \quad =-2$

$x=-4, 21=-7A \rightarrow A=-3$

So

$$\begin{aligned}\int \frac{x^2-4x-11}{x^2+x-12}\,dx &= \int\left(1-\frac{3}{x+4}-\frac{2}{x-3}\right)dx\\ &= x-3\log_e|x+4|-2\log_e|x-3|+c\\ &= x-\log_e\left(\left|(x-3)^2(x+4)^3\right|\right)+c\end{aligned}$$

b

$$\begin{array}{r} -3 \\ x^2+2x-3\overline{\smash{)}-3x^2-4x-5} \\ -\left(-3x^2-6x+9\right) \\ 2x-14 \end{array}$$

$$\frac{-3x^2-4x-5}{x^2+2x-3} = -3+\frac{2x-14}{x^2+2x-3}$$

$$\begin{aligned}\frac{2x-14}{x^2+2x-3} &= \frac{2x-14}{(x+3)(x-1)}\\ &= \frac{A}{(x+3)}+\frac{B}{(x-1)}\\ &= \frac{A(x-1)+B(x+3)}{(x+3)(x-1)}\end{aligned}$$

So $2x-14=A(x-1)+B(x+3)$

Let

$x=1, -12=4B \rightarrow B=-3$

$x=-3, -20=-4A \rightarrow A=5$

$$\begin{aligned}\int \frac{-3x^2-4x-5}{x^2+2x-3}\,dx &= \int\left(-3+\frac{5}{x+3}-\frac{3}{x-1}\right)dx\\ &= -3x+5\log_e|x+3|-5\log_e|x-1|+c\\ &= -3x+\log_e\left|\frac{(x+3)^5}{(x-1)^3}\right|+c\end{aligned}$$

14 a

$$\begin{array}{r} -3 \\ 16-9x^2\overline{\smash{)}27x^2} \\ -(27x^2-48) \\ 48 \end{array}$$

$$\begin{aligned}\frac{-x+22}{x^2-4x-12} &= \frac{-x+22}{(x-6)(x+2)}\\ &= \frac{A}{x-6}+\frac{B}{x+2}\\ &= \frac{A(x+2)+B(x-6)}{(x-6)(x+2)}\end{aligned}$$

So $-x+22=A(x+2)+B(x-6)$

Let

$x=6, 16=8A \rightarrow A=2$

$x=-2, 24=-8B \rightarrow B=-3$

$$\begin{aligned}\int \frac{4x^2-17x-26}{x^2-4x-12}\,dx &= \int\left(4+\frac{2}{x-6}-\frac{3}{x+2}\right)dx\\ &= 4x+2\ \log_e|x-6|-3\log_e|x+2|+c\\ &= 4x+\log_e\left|\frac{(x-6)^2}{(x+2)^3}\right|+c\end{aligned}$$

b

$$\begin{array}{r} -2x \\ x^2-6x+8\overline{\smash{)}-2x^3+12x^2-17x} \\ -2x^3+12x^2-16x \\ -x \end{array}$$

$$\begin{aligned}\frac{-x}{x^2-6x+8} &= \frac{-x}{(x-4)(x-2)}\\ &= \frac{A}{x-4}+\frac{B}{x-2}\\ &= \frac{A(x-2)+B(x-4)}{(x-4)(x-2)}\end{aligned}$$

So $-x=A(x-2)+B(x-4)$

Let

$x=2, -2=-2B \rightarrow B=1$

$x=4, -4=2A \rightarrow A=-2$

$$\begin{aligned}\int \frac{-2x^3+12x^2-17x}{x^2-6x-8}\,dx &= \int\left(-2x-\frac{2}{x-4}+\frac{1}{x-2}\right)dx\\ &= -x^2-2\log_e|x-4|+\log_e|x-2|+c\\ &= -x^2+\log_e\left(\frac{|x-2|}{(x-4)^2}\right)+c\end{aligned}$$

15 a $\displaystyle\int_1^2 \frac{1}{x^2+4x}\,dx$

$$\begin{aligned}\frac{1}{x^2+4x} &= \frac{1}{x(x+4)}\\ &= \frac{A}{x}+\frac{B}{(x+4)}\\ &= \frac{A(x+4)+Bx}{x(x+4)}\end{aligned}$$

$$1 = A(x+4)+Bx$$
$$x = 0, 1 = 4A \rightarrow A = \frac{1}{4}$$
$$x = -4, 1 = -4B \rightarrow B = -\frac{1}{4}$$
$$\begin{aligned}\int_1^2 \frac{1}{x^2+4x}\,dx &= \frac{1}{4}\int_1^2 \frac{1}{x} - \frac{1}{x+4}\,dx\\ &= \frac{1}{4}\left[\log_e|x| - \log_e|x+4|\right]_1^2\\ &= \frac{1}{4}\left[\log_e\left|\frac{2}{6}\right| - \log_e\left|\frac{1}{5}\right|\right]\\ &= \frac{1}{4}\log_e\left(\frac{5}{3}\right)\end{aligned}$$

b $\displaystyle\int_5^6 \frac{1}{x^2-16}\,dx$

$$\begin{aligned}\frac{1}{x^2-16} &= \frac{1}{(x+4)(x-4)}\\ &= \frac{A}{x+4} + \frac{B}{x-4}\\ &= \frac{A(x-4)+B(x+4)}{(x+4)(x-4)}\end{aligned}$$
$$1 = A(x-4)+B(x+4)$$
$$x = 4, 1 = 8B \rightarrow B = \frac{1}{8}$$
$$x = -4, 1 = -8A \rightarrow A = -\frac{1}{8}$$
$$\begin{aligned}\int_5^6 \frac{1}{x^2-16}\,dx &= \frac{1}{8}\int_5^6 \frac{1}{x-4} - \frac{1}{x+4}\,dx\\ &= \frac{1}{8}\left[\log_e|x-4| - \log_e|x+4|\right]_5^6\\ &= \frac{1}{8}\left[\log_e\left(\left|\frac{2}{10}\right|\right) - \log_e\left(\left|\frac{1}{9}\right|\right)\right]\\ &= \frac{1}{8}\log_e\left(\frac{9}{5}\right)\end{aligned}$$

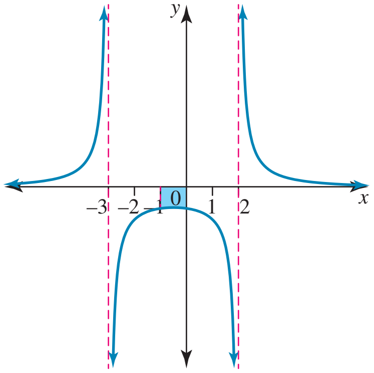
16 a $\displaystyle\int_{-1}^1 \frac{3x+8}{x^2+6x+8}\,dx$

$$\begin{aligned}\frac{3x+8}{x^2+6x+8} &= \frac{3x+8}{(x+4)(x+2)}\\ &= \frac{A}{x+4} + \frac{B}{x+2}\\ &= \frac{A(x+2)+B(x+4)}{(x+4)(x+2)}\end{aligned}$$
$$3x + 8 = A(x+2)+B(x+4)$$
$$x = -2, 2 = 2B \rightarrow B = 1$$
$$x = -4, -4 = -2A \rightarrow A = 2$$
$$\begin{aligned}\int_{-1}^1 \frac{3x+8}{x^2+6x+8}\,dx &= \int_{-1}^1 \frac{2}{x+4} + \frac{1}{x+2}\,dx\\ &= \left[2\log_e|x+4| + \log_e|x+2|\right]_{-1}^1\\ &= \log_e(25\times 3) - \log_e(9\times 1)\\ &= \log_e\left(\frac{25}{3}\right)\end{aligned}$$

b $\displaystyle\int_3^4 \frac{x-6}{x^2-4x+4}\,dx$

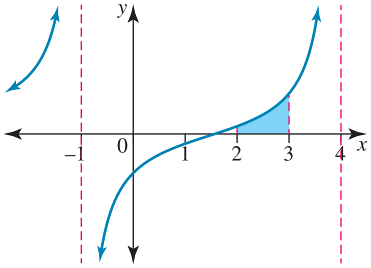
$$\begin{aligned}\frac{x-6}{x^2-4x+4} &= \frac{x-6}{(x-2)^2}\\ &= \frac{A}{x-2} + \frac{B}{(x-2)^2}\\ &= \frac{A(x-2)+B}{(x-2)^2}\\ &= \frac{Ax+B-2A}{(x-2)^2}\end{aligned}$$
$$A = 1$$
$$B - 2A = -6 \rightarrow B = -4$$
$$\begin{aligned}\int_3^4 \frac{x-6}{x^2-4x+4}\,dx &= \int_3^4 \frac{1}{x-2} - \frac{4}{(x-2)^2}\,dx\\ &= \left[\log_e|x-2| + \frac{4}{x-2}\right]_3^4\\ &= \log_e(2) + \frac{4}{2} - \log_e(1) - \frac{4}{1}\\ &= \log_e(2) - 2\end{aligned}$$

17 a $\displaystyle y = \frac{5}{x^2+x-6} = \frac{5}{(x+3)(x-2)}$



$$\text{Area} = \left|\int_{-1}^0 \frac{5}{x^2+x-6}\right|dx$$
$$\begin{aligned}\frac{5}{x^2+x-6} &= \frac{A}{x+3} + \frac{B}{x-2}\\ &= \frac{A(x-2)+B(x+3)}{(x+3)(x-2)}\end{aligned}$$
$$5 = A(x-2)+B(x+3)$$
$$x = 2, 5 = 5B \rightarrow B = 1$$
$$x = -3, 5 = -5A \rightarrow A = -1$$
$$\begin{aligned}&\text{Area} \int_0^{-1} \frac{-1}{(x+3)} + \frac{1}{(x-2)}\,dx\\ &= \left[-\log_e|x+3| + \log_e|x-2|\right]_0^{-1}\\ &= \left[\log_e\left|\frac{x-2}{x+3}\right|\right]_0^{-1}\\ &= \log_e\left(\left|\frac{-3}{2}\right|\right) - \log_e\left(\left|\frac{-2}{3}\right|\right)\\ &= \log_e\left(\left|\frac{9}{4}\right|\right)\\ &= \log_e\left(\frac{9}{4}\right)\end{aligned}$$

b $y = \dfrac{2x-3}{4+3x-x^2}$



Area $\displaystyle\int_3^2 \frac{2x-3}{4+3x-x^2}\,dx$

$= \left[\log_e|4+3x-x^2|\right]_3^2$

$= \log_e(6) - \log_e(4)$

$= \log_e\left(\dfrac{3}{2}\right)$

18 a $y = \dfrac{21}{40-11x-2x^2} = \dfrac{21}{(5-2x)(x+8)}$

$$\frac{21}{(5-2x)(x+8)} = \frac{A}{(5-2x)} + \frac{B}{(x+8)}$$
$$= \frac{A(x+8)+B(5-2x)}{(5-2x)(x+8)}$$

$21 = A(x+8) + B(5-2x)$

$x = -8, 21 = 21B \rightarrow B = 1$

$x = \dfrac{5}{2}, 21 = 10\dfrac{1}{2}A \rightarrow A = 2$

$$\int_{-5}^{0} \frac{21}{40-11x-2x^2}\,dx = \int_{-5}^{0} \frac{2}{(5-2x)} + \frac{1}{(x+8)}\,dx$$
$$= \left[-\log_e|5-2x| + \log_e|x+8|\right]_{-5}^{0}$$
$$= \left[\log_e\left|\frac{x+8}{5-2x}\right|\right]_{-5}^{0}$$
$$= \log_e\left(\frac{8}{5}\right) - \log_e\left(\frac{3}{15}\right)$$
$$= \log_e(8)$$

b

$$\begin{array}{r} x \\ x^2-9\overline{\smash{)}x^3-9x+9} \\ -\left(x^3-9x\right) \end{array}$$

$$\frac{x^3-9x+9}{x^2-9} = x + \frac{9}{x^2-9}$$

$$\frac{9}{x^2-9} = \frac{A}{x+3} + \frac{B}{x-3}$$
$$= \frac{A(x-3)+B(x+3)}{x^2-9}$$

$9 = A(x-3) + B(x+3)$

$x = 3, 9 = 6B \rightarrow B = \dfrac{3}{2}$

$x = -3, 9 = -6A \rightarrow A = -\dfrac{3}{2}$

Area $\displaystyle\int_4^6 \frac{x^3-9x+9}{x^2-9}\,dx$

$= \displaystyle\int_4^6 x + \frac{3}{2}\left(\frac{1}{x-3} - \frac{1}{x+3}\right)dx$

$= \left[\dfrac{1}{2}x^2 + \dfrac{3}{2}\log_e\left|\dfrac{x-3}{x+3}\right|\right]_4^6$

$= \dfrac{1}{2}\times 36 + \dfrac{3}{2}\log_e\left(\dfrac{3}{9}\right) - \dfrac{1}{2}\times 16 - \dfrac{3}{2}\log_e\left(\dfrac{1}{7}\right)$

$= \dfrac{1}{2}(36-16) + \dfrac{3}{2}\log_e\left(\dfrac{3}{9}\times 7\right)$

$= 10 + \dfrac{3}{2}\log_e\left(\dfrac{7}{3}\right)$

19 a $\dfrac{2}{4x-x^2} = \dfrac{2}{x(4-x)}$

$$= \frac{A}{x} + \frac{B}{4-x}$$
$$= \frac{A(4-x)+Bx}{x(4-x)}$$

$2 = A(4-x) + Bx$

Let

$x = 0, 2 = 4A \rightarrow A = \dfrac{1}{2}$

$x = 4, 2 = 4B \rightarrow B = \dfrac{1}{2}$

$$\int_1^2 \frac{2}{4x-x^2}\,dx = \frac{1}{2}\int_1^2 \frac{1}{x} + \frac{1}{(4-x)}\,dx$$
$$= \frac{1}{2}\left[\log_e\left|\frac{x}{4-x}\right|\right]_1^2$$
$$= \frac{1}{2}\left(\log_e\left(\frac{2}{2}\right) - \log_e\left(\frac{1}{3}\right)\right)$$
$$= \log_e\left(\sqrt{3}\right)$$
$$a = \sqrt{3}$$

b $\displaystyle\int_1^2 \frac{2}{\sqrt{4x-x^2}}\,dx$

$4x - x^2 = -\left(x^2-4x\right)$

$= -\left(x^2-4x+4\right)+4$

$= 4-(x-2)^2$

Let

$u = x-2$

$\dfrac{du}{dx} = 1$

$du = dx$

Terminals

$x = 1, u = -1$

$x = 2, u = 0$

$$\int_{-1}^{0} \frac{2}{\sqrt{4-u^2}}\,du = 2\left[\sin^{-1}\left(\frac{u}{2}\right)\right]_{-1}^{0}$$
$$= 2\sin^{-1}(0) - 2\sin^{-1}\left(-\frac{1}{2}\right)$$
$$= \frac{\pi}{3}$$
$$= \pi b$$
$$b = \frac{1}{3}$$

c $\dfrac{3x}{x^2-x-2} = \dfrac{3x}{(x-2)(x+1)}$

$$= \frac{A}{(x-2)} + \frac{B}{(x+1)}$$
$$= \frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$$

$3x = A(x+1) + B(x-2)$

$$x = 2, 6 = 3A \rightarrow A = 2$$
$$x = -1, -3 = -3B \rightarrow B = 1$$
$$\begin{aligned}\int_3^4 \frac{3}{x^2 - x - 2} dx &= \int_3^4 \frac{2}{(x-2)} + \frac{1}{(x+1)} dx \\ &= \left[2\log_e |x-2| + \log_e |x+1|\right]_3^4 \\ &= \left[\log_e (x-2)^2 (x+1)\right]_3^4 \\ &= \log_e(5) \\ &= \log_e(c) \\ c &= 5\end{aligned}$$

d
$$\begin{aligned}x^2 - x + 1 &= \left(x^2 - x + \frac{1}{4}\right) + 1 - \frac{1}{4} \\ &= \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}\end{aligned}$$
$$\int_{\frac{1}{2}}^2 \frac{3}{x^2 - x + 1}\, dx = \int_{\frac{1}{2}}^2 \frac{3}{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}}\, dx$$

Let
$$u = x - \frac{1}{2}$$
$$\frac{du}{dx} = 1$$
$$du = dx$$

Terminals
$$x = \frac{1}{2}, u = 0$$
$$x = 2, u = \frac{3}{2}$$
$$\begin{aligned}\int_0^{\frac{3}{2}} \frac{3}{u^2 + \frac{3}{4}} du &= \left[\frac{3}{\sqrt{\frac{3}{4}}} \tan^{-1}\left(\frac{u}{\frac{\sqrt{3}}{2}}\right)\right]_0^{\frac{3}{2}} \\ &= 2\sqrt{3}\left[\tan^{-1}\left(\sqrt{3}\right) - \tan^{-1}(0)\right] \\ &= \frac{2\pi\sqrt{3}}{3} \\ &= \pi d \\ d &= \frac{2\sqrt{3}}{3}\end{aligned}$$

20 a $y = \dfrac{16x^2 + 25}{16x^2 - 25}$

$$\begin{array}{r} 1 \\ 16x^2 - 25 \big) \overline{16x^2 + 25} \\ -\left(16x^2 - 25\right) \end{array}$$

$$\frac{16x^2 + 25}{16x^2 - 25} = 1 + \frac{50}{16x^2 - 25}$$
$$\begin{aligned}\frac{50}{16x^2 - 25} &= \frac{50}{(4x-5)(4x+5)} \backslash \\ &= \frac{A}{(4x-5)} + \frac{B}{(4x+5)} \\ &= \frac{A(4x+5) + B(4x-5)}{(4x-5)(4x+5)} \\ &= \frac{4x(A+B) + 5(A-B)}{16x^2 - 25}\end{aligned}$$
$$x\colon\ 4(A+B) = 0$$
$$x^0\colon\ 5(A-B) = 50$$
$$A = -B$$
$$-5B = 50$$
$$B = -5$$
$$A = 5$$
$$\begin{aligned}\int \frac{16x^2 + 25}{16x^2 - 25}\, dx &= \int 1 + \frac{5}{4x-5} - \frac{5}{4x+5}\, dx \\ &= x + \frac{5}{4}\log_e|4x-5| - \frac{5}{4}\log_e|4x+5| + c \\ &= x + \frac{5}{4}\log_e\left(\left|\frac{4x-5}{4x+5}\right|\right) + c\end{aligned}$$

b
$$\begin{aligned}\frac{1}{y} &= \frac{16x^2 - 25}{16x^2 + 25} \\ &= \frac{16x^2 + 25 - 50}{16x^2 + 25} \\ &= 1 - \frac{50}{16x^2 + 25}\end{aligned}$$
$$\int \frac{1}{y} dy = \int 1 - \frac{50}{16x^2 + 25} dx$$

Let
$$u = 4x$$
$$\frac{du}{dx} = 4$$
$$\begin{aligned}&= x - \frac{50}{4}\int \frac{1}{u^2 + 25} du \\ &= x - \frac{50}{4} \times \frac{1}{5}\tan^{-1}\left(\frac{u}{5}\right) + c \\ &= x - \frac{5}{2}\tan^{-1}\left(\frac{4x}{5}\right) + c\end{aligned}$$

21 a

$$\begin{array}{r} -3 \\ 16 - 9x^2 \big) \overline{27x^2} \\ -(27x^2 - 48) \\ 48 \end{array}$$

$$\frac{27x^2}{16 - 9x^2} = -3 + \frac{48}{16 - 9x^2}$$
$$\begin{aligned}\frac{48x^2}{16 - 9x^2} &= \frac{A}{4 + 3x} + \frac{B}{4 - 3x} \\ &= \frac{A(4-3x) + B(4+3x)}{(4+3x)(4-3x)} \\ &= \frac{3x(B-A) + 4(A+B)}{16 - 9x^2}\end{aligned}$$
$$x\colon B - A = 0 \rightarrow B = A$$
$$x^0\colon\ 4(A+B) = 48 \rightarrow B = A = 6$$
$$\begin{aligned}\int_0^1 \frac{27x^2}{16 - 9x^2}\, dx &= \int_0^1 -3 + \frac{6}{4+3x} + \frac{6}{4-3x}\, dx \\ &= \left[-3x + 2\log_e|4+3x| - 2\log_e|4-3x|\right]_0^1 \\ &= \left[-3x + 2\log_e\left|\frac{4+3x}{4-3x}\right|\right]_0^1 \\ &= -3 + 2\log_e\left(\frac{7}{1}\right) - 2\log_e\left(\frac{4}{4}\right) \\ &= 2\log_e(7) - 3\end{aligned}$$

b

$$\begin{array}{r} 5 \\ 4x^2 + 4x + 1 \big) \overline{20x^2} \\ -(20x^2 + 20x + 5) \\ -20x - 5 \end{array}$$

$$\begin{aligned}
\frac{20x^2}{4x^2+4x+1} &= 5-\frac{20x+5}{(2x+1)^2}\\
\frac{20x+5}{(2x+1)^2} &= \frac{A}{(2x+1)}+\frac{B}{(2x+1)^2}\\
&= \frac{A(2x+1)+B}{(2x+1)^2}\\
&= \frac{2Ax+B+A}{(2x+1)^2}
\end{aligned}$$

$x{:}2A = 20 \rightarrow A = 10$

$x^0{:}A + B = 5 \rightarrow B = -5$

$$\begin{aligned}
&\int_0^2 \frac{20x^2}{4x^2+4x+1}dx = \int_0^2 5-\frac{10}{2x+1}+\frac{5}{(2x+1)^2}dx\\
&= \left[5x-5\log_e|2x+1|-\frac{5}{2(2x+1)}\right]_0^2\\
&= 10-5\log_e(5)-\frac{5}{2\times5}-0+5\log_e(1)+\frac{5}{2}\\
&= 12-5\log_e(5)
\end{aligned}$$

22 a Let

$f(x) = x^3 - 2x^2 + 9x - 18$

$f(1) = 1 - 2 + 9 - 18 \neq 0$

$f(2) = 8 - 8 + 18 - 18 = 0$

$\rightarrow x = 2$ is a factor.

$$\begin{aligned}
x^3-2x^2+9x-18 &= (x-2)\left(x^2+9\right)\\
\frac{19-3x}{x^3-2x^2+9x-18} &= \frac{19-3x}{(x-2)\left(x^2+9\right)}\\
&= \frac{A}{(x-2)}+\frac{Bx+\mathrm{C}}{\left(x^2+9\right)}\\
&= \frac{A(x^2+9)+(Bx+C)(x-2)}{(x-2)\left(x^2+9\right)}\\
&= \frac{x^2(A+B)+x(C-2B)+9A-2C}{x^3-2x^2+9x-18}
\end{aligned}$$

$(1) : A + B = 0$

$(2) : C - 2B = -3$

$(3) : 9A - 2C = 19$

$\therefore A = 1, B = -1, C = -5$

$$\begin{aligned}
2\times(1) : 2A+2B &= 0\\
C-2B &= -3\\
\text{add}\quad 2A+C &= -3\times(2)\\
4A+2C &= -6\\
(3)\quad 9A-2C &= 19
\end{aligned}$$

$13A = 13 \rightarrow A = 1$

$$\begin{aligned}
\int\frac{19-3x}{x^3-2x^2+9x-18}dx &= \int\frac{1}{x-2}-\frac{x+5}{\left(x^2+9\right)}dx\\
&= \int\frac{1}{x-2}-\frac{x}{\left(x^2+9\right)}-\frac{5}{\left(x^2+9\right)}dx\\
&= \log_e|x-2|-\frac{1}{2}\log_e\left|x^2+9\right|-\frac{5}{3}\tan^{-1}\left(\frac{x}{3}\right)+c\\
&= \log_e\left(\frac{|x-2|}{\sqrt{x^2+9}}\right)-\frac{5}{3}\tan^{-1}\left(\frac{x}{3}\right)+c
\end{aligned}$$

b Let

$f(x) = x^3 + 3x^2 + 16x + 48$

$f(1) = 1 + 3 + 16 + 48 \neq 0$

$f(2) = 8 + 12 + 36 + 48 \neq 0$

$f(-3) = -27 + 27 - 48 + 48 = 0$

$\rightarrow x + 3$ is a factor.

$$\begin{aligned}
x^3+3x^2+16x+48 &= (x+3)\left(x^2+16\right)\\
\frac{25}{x^3+3x^2+16x+48} &= \frac{25}{(x+3)\left(x^2+16\right)}\\
&= \frac{A}{(x+3)}+\frac{Bx+C}{\left(x^2+16\right)}\\
&= \frac{A(x^2+16)+(Bx+C)(x+3)}{(x+3)\left(x^2+16\right)}\\
&= \frac{x^2(A+B)+x(3B+C)+16A+3C}{x^3+3x^2+16x+48}
\end{aligned}$$

$(1) : A + B = 0$

$(2) : 3B + C = 0$

$(3) : 16A + 3C = 25$

$$\begin{aligned}
3\times(1)-(2){:}\ 3A-C &= 0\\
9A-3C &= 0
\end{aligned}$$

add $25\,A = 25$

$\therefore A = 1, B = -1, C = 3$

$$\begin{aligned}
\int\frac{25}{x^3+3x^2+16x+48} &= \int\frac{1}{x+3}+\frac{3-x}{\left(x^2+16\right)}dx\\
&= \int\frac{1}{x+3}-\frac{x}{\left(x^2+16\right)}+\frac{3}{\left(x^2+16\right)}dx\\
&= \log_e|x+3|-\frac{1}{2}\log_e\left|x^2+16\right|\\
&\quad+\frac{3}{4}\tan^{-1}\left(\frac{x}{4}\right)+c\\
&= \log_e\left(\frac{|x+3|}{\sqrt{x^2+16}}\right)+\frac{3}{4}\tan^{-1}\left(\frac{x}{4}\right)+c
\end{aligned}$$

23 a

$$\begin{aligned}
\frac{x^2-2x+9}{x^3+9x} &= \frac{x^2-2x+9}{x(x^2+9)}\\
&= \frac{A}{x}+\frac{Bx+C}{\left(x^2+9\right)}\\
&= \frac{A(x^2+9)+(Bx+C)x}{x(x^2+9)}\\
&= \frac{x^2(A+B)+Cx+9A}{x(x^2+9)}
\end{aligned}$$

$x^2\!: A + B = 1$

$x^1\!: C = -2$

$x^0\!: 9A = 9 \rightarrow A = 1, B = 0$

$$\begin{aligned}
\int_{\sqrt{3}}^3\frac{x^2-2x+9}{x^3+9x}dx &= \int_{\sqrt{3}}^3\frac{1}{x}-\frac{2}{\left(x^2+9\right)}dx\\
&= \left[\log_e|x|-\frac{2}{3}\tan^{-1}\left(\tfrac{x}{3}\right)\right]_{\sqrt{3}}^3\\
&= \log_e(3)-\frac{2}{3}\tan^{-1}(1)-\log_e\left(\sqrt{3}\right)\\
&\quad+\frac{2}{3}\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)\\
&= \log_e\left(\frac{3}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}\right)-\frac{2}{3}\times\frac{\pi}{4}+\frac{2}{3}\times\frac{\pi}{6}\\
&= \log_e\left(\sqrt{3}\right)-\frac{\pi}{18}\\
&= \frac{1}{2}\log_e(3)-\frac{\pi}{18}
\end{aligned}$$

b $$\begin{aligned}\frac{4x^2-16x+19}{(2x-3)^3}&=\frac{A}{(2x-3)}+\frac{B}{(2x-3)^2}+\frac{C}{(2x-3)^3}\\&=\frac{A(2x-3)^2+B(2x-3)+C}{(2x-3)^3}\\&=\frac{A(4x^2-12x+9)+B(2x-3)+C}{(2x-3)^3}\\&=\frac{4Ax^2+x(2B-12A)+9A+C-3B}{(2x-3)^3}\end{aligned}$$

$x^2{:}4A=4\to A=1$

$x^1{:}2B-12A=-16\to B=-2$

$x^0{:}9A+C-3B=19\to C=4$

$$\begin{aligned}&\int_2^3\frac{4x^2-16x+19}{(2x-3)^3}dx\\&=\int_2^3\frac{1}{(2x-3)}-\frac{2}{(2x-3)^2}+\frac{4}{(2x-3)^3}\,dx\\&=\left[\frac{1}{2}\log_e|2x-3|+\frac{1}{(2x-3)}-\frac{1}{(2x-3)^2}\right]_2^3\\&=\frac{1}{2}\log_e(3)+\frac{1}{3}-\frac{1}{3^2}-\left(\frac{1}{2}\log_e(1)+\frac{1}{1}-\frac{1}{1}\right)\\&=\frac{1}{2}\log_e(3)+\frac{2}{9}\end{aligned}$$

24 a $$\int\frac{1}{b^2x^2-a^2}\,dx$$

$$\begin{aligned}\frac{1}{b^2x^2-a^2}&=\frac{1}{(bx+a)\,(bx-a)}\\&=\frac{A}{(bx+a)}+\frac{B}{(bx-a)}\\&=\frac{A(bx-a)+B(bx+a)}{(bx+a)\,(bx-a)}\\&=\frac{x(Ab+Bb)+Ba-Aa}{(bx+a)\,(bx-a)}\end{aligned}$$

(1): $b(A+B)=0$

(2): $a(B-A)=1$

(1): $A=-B$

Add $B=\dfrac{1}{2a}$

$$\begin{aligned}&\frac{1}{2a}\int\frac{-1}{(bx+a)}+\frac{1}{(bx-a)}\,dx\\&=\frac{1}{2a}\left[\frac{1}{b}\log_e|bx-a|\ -\frac{1}{b}\log_e|bx+a|\right]+c\\&=\frac{1}{2ab}\log_e\left(\frac{|bx-a|}{|bx+a|}\right)+c\end{aligned}$$

b $$\begin{aligned}\int\frac{x}{b^2x^2-a^2}\,dx&=\frac{1}{2b^2}\int\frac{2b^2x}{b^2x^2-a^2}\,dx\\&=\frac{1}{2b^2}\log_e(|b^2x^2-a^2|)+c\end{aligned}$$

25 a $$\int\frac{x}{(ax-b)^2}\,dx$$

$$\begin{aligned}\frac{x}{(ax-b)^2}&=\frac{A}{(ax-b)}+\frac{B}{(ax-b)^2}\\&=\frac{A(ax-b)+B}{(ax-b)^2}\end{aligned}$$

$(1){:}Aa=1\to A=\dfrac{1}{a}$

$(2){:}B-Ab=0\to B=\dfrac{b}{a}$

$$\begin{aligned}\int\frac{x}{(ax-b)^2}\,dx&=\frac{1}{a}\int\frac{1}{(ax-b)}+\frac{b}{(ax-b)^2}\,dx\\&=\frac{1}{a}\left[\frac{1}{a}\log_e|ax-b|-\frac{b}{a\,(ax-b)}\right]+c\\&=\frac{1}{a^2}\log_e|ax-b|-\frac{b}{a^2(ax-b)}+c\end{aligned}$$

b $$\int\frac{1}{(px+a)\,(qx+b)}\,dx$$

$$\begin{aligned}\frac{1}{(px+a)\,(qx+b)}&=\frac{A}{(px+a)}+\frac{B}{(qx+b)}\\&=\frac{A(qx+b)+B(px+a)}{(px+a)\,(qx+b)}\\&=\frac{x(Aq+Bp)+Ba+Ab}{(px+a)\,(qx+b)}\end{aligned}$$

$x{:}(1){:}Aq+Bp=0$

$x^0{:}(2){:}Ba+Ab=1$

$(1)\times a{:}aAq+Bap=0$

$(2)\times p{:}Bpa+Apb=p$

Subtract $(pb-aq)\,A=p$

$A=\dfrac{p}{pb-aq}=-\dfrac{p}{aq-bp}$

$(1)\times b{:}bAq+Bbp=0$

$(2)\times q{:}Bqa+Aqb=q$

Subtract $(aq-bp)\,B=q$

$B=\dfrac{q}{aq-bp}$

$$\begin{aligned}\int\frac{1}{(px+a)\,(qx+b)}dx&=\frac{1}{aq-bp}\int\frac{-p}{(px+a)}+\frac{q}{(qx+b)}dx\\&=\frac{1}{aq-bp}\left[\log_e|qx+b|-\log_e|px+a|\right]+c\\&=\frac{1}{aq-bp}\log_e\left(\left|\frac{qx+b}{px+a}\right|\right)+c\end{aligned}$$

26 Expanding

$$\begin{array}{r}x^6-4x^5+5x^4-4x^2+4\\1+x^2\overline{\big)x^8-4x^7+6x^6-4x^5+x^4}\\\underline{-(x^8+0x^7+\ \ x^6)\qquad\qquad\quad}\\-4x^7+5x^6-4x^5\qquad\\\underline{-(-4x^7+0x^6-4x^5)\qquad}\\5x^6+0x^5+\ \ x^4\\\underline{-(5x^6+0x^5+5x^4)}\\-4x^4\\\underline{-(-4x^4-4x^2)}\\4x^2\\\underline{-(4x^2+4)}\\-4\end{array}$$

So

$$\begin{aligned}\int_0^1\frac{x^4(1-x)^4}{1+x^2}\,dx&=\int_1^1x^6-4x^5+5x^4-4x^2+4-\frac{4}{x^2+1}\,dx\\&=\left[\frac{1}{7}x^7-\frac{2}{3}x^6+x^5-\frac{4}{3}x^3+4x-4\tan^{-1}(x)\right]_0^1\\&=\frac{22}{7}-\frac{4\pi}{4}\\&=\frac{22}{7}-\pi\end{aligned}$$

7.7 Exam questions

1 $$\frac{2x^2+3x+1}{(2x+1)^3\left(x^2-1\right)} = \frac{(2x+1)(x+1)}{(2x+1)^3(x+1)(x-1)}$$
$$= \frac{1}{(2x+1)^2(x-1)} \quad x \neq \pm 1, -\frac{1}{2}$$
$$= \frac{A}{2x+1} + \frac{B}{(2x+1)^2} + \frac{C}{x-1}$$

The correct answer is **D**.

2 Using partial fractions

$$\frac{1}{x\left(1+x^2\right)} = \frac{A}{x} + \frac{Bx+C}{1+x^2}$$
$$= \frac{A(1+x^2)+x(Bx+C)}{x(1+x^2)}$$
$$= \frac{x^2(A+B)+Cx+A}{x(1+x^2)}$$

$A+B=0,\ A=1,\ C=0,\ B=-1$

$$\int_1^{\sqrt{3}} \frac{1}{x\left(1+x^2\right)}dx = \int_1^{\sqrt{3}} \left(\frac{1}{x} - \frac{x}{1+x^2}\right)dx$$
$$= \left[\log_e |x| - \frac{1}{2}\log_e(1+x^2)\right]_1^{\sqrt{3}}$$
$$= \left[\frac{1}{2}\log_e\left(\frac{x^2}{1+x^2}\right)\right]_1^{\sqrt{3}}$$
$$= \left(\frac{1}{2}\log_e\left(\frac{3}{4}\right) - \frac{1}{2}\log_e\left(\frac{1}{2}\right)\right)$$
$$= \frac{1}{2}\log_e\left(\frac{3}{2}\right)$$
$$= \log_e\left(\sqrt{\frac{3}{2}}\right)$$

Award 1 mark for the correct partial fractions decomposition.
Award 1 mark for the solving for the coefficients.
Award 1 mark for the correct integration.
Award 1 mark for the correct final values of *a* and *b*.

3 By partial fractions,

$$\frac{a}{x(x-a)} = \frac{A}{x} + \frac{B}{x-a} = \frac{A(x-a)+Bx}{x(x-a)} = \frac{x(A+B)-Aa}{x(x-a)}$$

(1) $A+B=0$ (2) $-Aa=a \Rightarrow A=-1 \quad B=1$

$$\int \frac{a}{x(x-a)}dx = \int \left(\frac{1}{x-a} - \frac{1}{x}\right)dx$$
$$= \log_e(x-a) - \log_e(x)$$
$$= \log_e\left(\frac{x-a}{x}\right) + c \quad \text{since } x > a > 0$$

The correct answer is **B**.

7.8 Review

7.8 Exercise

Technology free: short answer

1 a $$\int \frac{1}{\sqrt{5-2x}}dx$$

Let

$$u = 5-2x$$
$$\frac{du}{dx} = -2$$
$$\frac{dx}{du} = -\frac{1}{2}$$
$$dx = -\frac{1}{2}du$$
$$\int u^{-\frac{1}{2}} \times -\frac{1}{2}du = -\frac{1}{2}\int u^{-\frac{1}{2}}du$$
$$= -\frac{1}{2}\left[2u^{\frac{1}{2}}\right] + c$$
$$= -\sqrt{u} + c$$
$$= -\sqrt{5-2x} + c$$

b $$\int \frac{1}{\sqrt{25-4x^2}}dx$$

Let

$$u = 2x$$
$$\frac{du}{dx} = 2$$
$$\frac{dx}{du} = \frac{1}{2}$$
$$dx = \frac{1}{2}du$$
$$\int \frac{1}{\sqrt{25-u^2}} \times \frac{1}{2}du = \frac{1}{2}\int \frac{1}{\sqrt{25-u^2}}du$$
$$= \frac{1}{2}\sin^{-1}\left(\frac{u}{5}\right) + c$$
$$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{5}\right) + c$$

c $$\int \frac{x}{\sqrt{25-4x^2}}dx$$

Let

$$u = 25-4x^2$$
$$\frac{du}{dx} = -8x$$
$$\frac{dx}{du} = -\frac{1}{8x}$$
$$dx = -\frac{1}{8x}du$$
$$\int x.u^{-\frac{1}{2}} \times -\frac{1}{8x}du = -\frac{1}{8}\int u^{-\frac{1}{2}}du$$
$$= -\frac{1}{8}\left[2u^{\frac{1}{2}}\right] + c$$
$$= -\frac{1}{4}\sqrt{u} + c$$
$$= -\frac{1}{4}\sqrt{25-4x^2} + c$$

d $$\int \frac{x}{25-4x^2}dx = -\frac{1}{8}\int \frac{-8x}{25-4x^2}dx$$
$$= -\frac{1}{8}\log_e(|25-4x^2|) + c$$

2 a
$$\begin{aligned}\frac{1}{25-4x^2} &= \frac{1}{(5+2x)(5-2x)}\\ &= \frac{A}{(5+2x)}+\frac{B}{(5-2x)}\\ &= \frac{A(5-2x)+B(5+2x)}{(5+2x)(5-2x)}\\ &= \frac{2x(B-A)+5(A+B)}{(5+2x)(5-2x)}\end{aligned}$$

$$B-A=0 \rightarrow A=B$$

$$5(A+B)=1 \rightarrow A=B=\frac{1}{10}$$

$$\begin{aligned}\int \frac{1}{25-4x^2}dx &= \frac{1}{10}\int \frac{1}{(5+2x)}+\frac{1}{(5-2x)}dx\\ &= \frac{1}{10}\left[\frac{1}{2}\log_e|5+2x|-\frac{1}{2}\log_e|5-2x|\right]+c\\ &= \frac{1}{20}\log_e\left|\frac{5+2x}{5-2x}\right|+c\end{aligned}$$

b $\displaystyle\int \frac{1}{25+4x^2}dx$

Let

$$u=2x$$
$$\frac{du}{dx}=2$$
$$dx=\frac{1}{2}du$$

$$\begin{aligned}\int \frac{1}{25-u^2}\cdot\frac{1}{2}du &= \frac{1}{2}\int \frac{1}{25-u^2}du\\ &= \frac{1}{2}\left[\frac{1}{5}\tan\frac{1}{5}^{-1}\left(\frac{u}{5}\right)\right]+c\\ &= \frac{1}{10}\tan^{-1}\left(\frac{2x}{5}\right)+c\end{aligned}$$

c $\displaystyle\int \frac{1}{5-2x}\,dx$

Let

$$u=5-2x$$
$$\frac{du}{dx}=-2$$
$$dx=-\frac{1}{2}du$$
$$=-\frac{1}{2}\int \frac{1}{u}du$$
$$=-\frac{1}{2}\log_e|u|+c$$
$$=-\frac{1}{2}\log_e|5-2x|+c$$

d Using division

$$\frac{x^2}{25-4x^2}=-\frac{1}{4}+\frac{25}{4}\times\frac{1}{25-4x^2}$$

$$\begin{aligned}\int \frac{x^2}{25-4x^2}\,dx &= \int -\frac{1}{4}+\frac{25}{4}\times\frac{1}{25-4x^2}dx\\ &= \int -\frac{1}{4}dx+\frac{25}{4}\int \frac{1}{25-4x^2}dx\\ &= -\frac{x}{4}+\frac{25}{4}\times\frac{1}{20}\log_e\left|\frac{5+2x}{5-2x}\right|+c\\ &= -\frac{x}{4}+\frac{5}{16}\log_e\left|\frac{5+2x}{5-2x}\right|+c\end{aligned}$$

3 a $\displaystyle\int \frac{x}{\left(25-4x^2\right)^2}dx$

Let

$$u=25-4x^2$$
$$\frac{du}{dx}=-8x$$
$$=\int xu^{-2}\times-\frac{1}{8x}du=-\frac{1}{8}\int u^{-2}du$$
$$=\frac{1}{8}u^{-1}+c$$
$$=\frac{1}{8(25-4x^2)}+c$$

b $\displaystyle\int \frac{x}{5-2x}dx$

Let

$$u=5-2x$$
$$\frac{du}{dx}=-2$$
$$dx=-\frac{1}{2}du$$
$$2x=5-u$$
$$x=\frac{1}{2}(5-u)$$
$$=\int \frac{\frac{1}{2}(5-u)}{u}\times-\frac{1}{2}\,du=-\frac{1}{4}\int \frac{5-u}{u}du$$
$$=-\frac{1}{4}\int \frac{5}{u}-1du$$
$$=-\frac{1}{4}\left[5\log_e|u|-u\right]+c$$
$$=\frac{u}{4}-\frac{5}{4}\log_e|u|+c$$
$$=\frac{5-2x}{4}-\frac{5}{4}\log_e|u|+c$$
$$=-\frac{x}{2}-\frac{5}{4}\log_e|5-2x|+d$$

c
$$\begin{aligned}\int \frac{x}{(5-2x)^2}dx &= \int \frac{\frac{1}{2}(5-u)}{u^2}\times-\frac{1}{2}du\\ &= -\frac{1}{4}\int \frac{(5-u)}{u^2}du\\ &= -\frac{1}{4}\int 5u^{-2}-\frac{1}{u}du\\ &= -\frac{1}{4}\left[-5u^{-1}-\log_e|u|\right]+c\\ &= \frac{1}{4}\log_e|5-2x|+\frac{5}{4(5-2x)}+c\end{aligned}$$

d $$\int \frac{x}{\sqrt{5-2x}}dx = \int \frac{\frac{1}{2}(5-u)}{\sqrt{u}} \times -\frac{1}{2}du$$
$$= -\frac{1}{4}\int (5-u)\, u^{-\frac{1}{2}}du$$
$$= -\frac{1}{4}\int 5u^{-\frac{1}{2}} - u^{\frac{1}{2}}du$$
$$= -\frac{1}{4}\left[10u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right] + c$$
$$= -\frac{1}{4} \times 2u^{\frac{1}{2}}\left(5 - \frac{u}{3}\right) + c$$
$$= -\frac{\sqrt{u}}{2}\left(\frac{15-u}{3}\right) + c$$
$$= -\frac{\sqrt{u}}{6}(15 - (5-2x)) + c$$
$$= -\frac{\sqrt{u}}{6}(10+2x) + c$$
$$= -\frac{1}{3}(x+5)\sqrt{5-2x} + c$$

4 **a** $$\int_0^1 \frac{1}{4-3x}dx$$

Let

$u = 4 - 3x$

$\frac{du}{dx} = -3$

$dx = -\frac{1}{3}du$

Terminals

$x = 1, u = 1$

$x = 0, u = 4$

$$\int_4^1 \frac{1}{u} \times -\frac{1}{3}du = -\frac{1}{3}\left[\log_e |u|\right]_4^1 + c$$
$$= -\frac{1}{3}\left[\log_e(1) - \log_e(4)\right]$$
$$= \frac{1}{3}\log_e(4)$$
$$= \frac{1}{3}\log_e(2^2)$$
$$= \frac{2}{3}\log_e(2)$$

b $$\int_0^1 \frac{1}{\sqrt{4-3x}}dx = -\frac{1}{3}\int_4^1 u^{-\frac{1}{2}}du$$
$$= -\frac{1}{3}\left[2u^{\frac{1}{2}}\right]_4^1$$
$$= -\frac{1}{3}2\left(\sqrt{1} - \sqrt{4}\right)$$
$$= \frac{2}{3}$$

c $$\int_0^1 \frac{1}{(4-3x)^2}dx = -\frac{1}{3}\int_4^1 u^{-2}du$$
$$= \frac{1}{3}\left[\frac{1}{u}\right]_4^1$$
$$= \frac{1}{3}\left(1 - \frac{1}{4}\right)$$
$$= \frac{1}{4}$$

d $$\int_0^1 \frac{x}{(4-3x)^2}dx$$

Let

$u = 4 - 3x$

$\frac{du}{dx} = -3$

$x = \frac{1}{3}(4-u)$

$$= -\frac{1}{9}\int_4^1 \frac{4-u}{u^2}du$$
$$= \frac{1}{9}\int_1^4 4u^{-2} - \frac{1}{u}du$$
$$= \frac{1}{9}\left[-4u^{-1} - \log_e |u|\right]_1^4$$
$$= -\frac{1}{9}\left[\frac{4}{4} + \log_e(4) - 4 - \log_e(1)\right]$$
$$= \frac{1}{9}\left(3 - \log_e(4)\right)$$

5 **a** $$\int_0^1 \frac{x}{4-3x}dx = \int_4^1 \frac{\frac{1}{3}(4-u)}{u} \times -\frac{1}{3}du$$
$$= -\frac{1}{9}\int_4^1 \frac{(4-u)}{u}du$$
$$= \frac{1}{9}\int_1^4 \frac{4}{u} - 1du$$
$$= \frac{1}{9}\left[4\log_e |u| - u\right]_1^4$$
$$= \frac{1}{9}\left[4\log_e(4) - 4 - 4\log_e(1) + 1\right]$$
$$= \frac{1}{9}\left[8\log_e(2) - 3\right]$$

b $$\int_0^1 \frac{x}{\sqrt{4-3x}}dx = -\frac{1}{9}\int_4^1 (4-u)\,u^{-\frac{1}{2}}du$$
$$= \frac{1}{9}\int_1^4 4u^{-\frac{1}{2}} - u^{\frac{1}{2}}du$$
$$= \frac{1}{9}\left[8u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_1^4$$
$$= \frac{1}{9}\left[8\sqrt{4} - \frac{2}{3} \times 4^{\frac{3}{2}} - 8\sqrt{1} + \frac{2}{3} \times 1^{\frac{3}{2}}\right]$$
$$= \frac{10}{27}$$

c $\displaystyle\int_0^{\frac{2}{3}} \frac{1}{\sqrt{16-9x^2}}dx$

Let

$u = 3x$

$\dfrac{du}{dx} = 3$

$dx = \dfrac{1}{3}du$

Terminals

$x = \dfrac{2}{3},\ u = 2$

$x = 0,\ u = 0$

$$= \int_0^2 \frac{1}{\sqrt{16-u^2}} \times \frac{1}{3}du$$

$$= \frac{1}{3}\int_0^2 \frac{1}{\sqrt{16-u^2}}du$$

$$= \frac{1}{3}\left[\sin^{-1}\left(\frac{u}{4}\right)\right]_0^2$$

$$= \frac{1}{3}\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)\right]$$

$$= \frac{\pi}{18}$$

d $\displaystyle\int_0^{\frac{4}{3}} \frac{x}{\sqrt{16-9x^2}}dx$

Let

$u = 16 - 9x^2$

$\dfrac{du}{dx} = -18x$

Terminals

$x = \dfrac{4}{3},\ u = 0$

$x = 0,\ u = 16$

$$= \int_{16}^0 x.u^{-\frac{1}{2}} \times -\frac{1}{18x}du$$

$$= -\frac{1}{18}\int_{16}^0 u^{-\frac{1}{2}}du$$

$$= \frac{1}{18}\int_0^{16} u^{-\frac{1}{2}}du$$

$$= \frac{1}{18}\left[2u^{\frac{1}{2}}\right]_0^{16}$$

$$= \frac{1}{9}\left[\sqrt{16} - \sqrt{0}\right]$$

$$= \frac{4}{9}$$

6 a $\displaystyle\int_0^{\frac{4}{3}} \frac{1}{16+9x^2}dx$

Let

$u = 3x$

$\dfrac{du}{dx} = 3$

$dx = \dfrac{1}{3}du$

Terminals

$x = \dfrac{4}{3}, u = 4$

$x = 0, u = 0$

$$= \int_0^4 \frac{1}{16+u^2} \times \frac{1}{3}du$$

$$= \frac{1}{3}\int_0^4 \frac{1}{16+u^2}du$$

$$= \frac{1}{3}\left[\frac{1}{4}\tan^{-1}\left(\frac{u}{4}\right)\right]_0^4$$

$$= \frac{1}{12}\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$

$$= \frac{1}{12}\left[\frac{\pi}{4} - 0\right]$$

$$= \frac{\pi}{48}$$

b $\displaystyle\int_0^{\frac{4}{3}} \frac{x}{16+9x^2}dx = \frac{1}{18}\int_0^{\frac{4}{3}} \frac{18x}{16+9x^2}dx$

$$= \frac{1}{18}\left[\log_e(16+9x^2)\right]_0^{\frac{4}{3}}$$

$$= \frac{1}{18}\left[\log_e(32) - \log_e(16)\right]$$

$$= \frac{1}{18}\log_e(2)$$

c $\dfrac{1}{16-9x^2} = \dfrac{1}{(4+3x)(4-3x)}$

$$= \frac{A}{(4+3x)} + \frac{B}{(4-3x)}$$

$$= \frac{A(4-3x) + B(4+3x)}{(4+3x)(4-3x)}$$

$$= \frac{3x(B-A) + 4(A+B)}{16-9x^2}$$

$x: B - A = 0 \rightarrow A = B = \dfrac{1}{8}$

$x^0: 4(A+B) = 1$

$$\int_0^{\frac{2}{3}} \frac{1}{16-9x^2}dx = \frac{1}{8}\int_0^{\frac{2}{3}} \frac{1}{4+3x} + \frac{1}{4-3x}dx$$

$$= \frac{1}{8}\left[\frac{1}{3}\log_e|4+3x| - \frac{1}{3}\log_e|4-3x|\right]_0^{\frac{2}{3}}$$

$$= \frac{1}{24}\left[\log_e\left|\frac{4+3x}{4-3x}\right|\right]_0^{\frac{2}{3}}$$

$$= \frac{1}{24}\left[\log_e\left|\frac{6}{2}\right| - \log_e|1|\right]$$

$$= \frac{1}{24}\log_e(3)$$

d $$\int_0^{\frac{2}{3}} \frac{x}{16-9x^2}\,dx = -\frac{1}{18}\left[\log_e(16-9x^2)\right]_0^{\frac{2}{3}}$$
$$= -\frac{1}{18}\left[\log_e(12) - \log_e(16)\right]$$
$$= -\frac{1}{18}\log_e\left(\frac{12}{16}\right)$$
$$= \frac{1}{18}\log_e\left(\frac{4}{3}\right)$$

Technology active: multiple choice

7 Let $u = 5 - 2x$

$u - 5 = -2x$

$x = \frac{1}{2}(5 - u)$

$\frac{du}{dx} = -2$

when $x = 3, u = -1$

when $x = 4, u = -3$

$$\int_4^3 \frac{2}{5-2x}\,dx = \int_{-3}^{-1} \frac{2}{u}\,\frac{du}{-2} = \int_{-1}^{-3} \frac{1}{u}\,du$$

The correct answer is **C**.

8 $x^2 - 4x = 2x$

$x^2 - 6x = 0$

$x(x-6) = 0$

$x = 0, 6$

$$A = \int_0^6 (2x - x^2 + 4x)dx$$
$$= \int_0^6 (6x - x^2)dx = 36$$

The correct answer is **D**.

9 Let $u = 2x - 1$

$x = \frac{1}{2}(u + 1)$

$\frac{du}{dx} = 2$

when $x = 1, u = 1$

when $x = 2, u = 3$

$$\int_1^3 \frac{2}{(u+1)\sqrt{u}}\,\frac{du}{2} = \int_1^3 \frac{1}{(u+1)\sqrt{u}}\,du$$

The correct answer is **A**.

10 Let $u = 1 + 2x^2$

$x^2 = \frac{u-1}{2}$

$x = \sqrt{\frac{u-1}{2}}$

$\frac{du}{dx} = 4x$

When $x = 0, u = 1$

When $x = 2, u = 9$

$$\int_1^9 \frac{\frac{u-1}{2} \times x}{\sqrt{u}}\,\frac{du}{4x} = \frac{1}{8}\int_1^9 \frac{u-1}{\sqrt{u}}\,du$$

The correct answer is **C**.

11 Let $u = \sqrt{x}$

$u^2 = x$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$dx = 2\sqrt{x}du$

When $x = 1, u = 1$

When $x = 4, u = 2$

$$\int_1^2 \frac{e^u}{\sqrt{x}}\,2\sqrt{x}du = 2\int_1^2 e^u\,du$$

The correct answer is **E**.

12 Let $u = \sin(x)$

$\frac{du}{dx} = \cos(x)$

$dx = \frac{du}{\cos(x)}$

when $x = 0, u = 0$

when $x = \frac{\pi}{6}, u = \frac{1}{2}$

$$\int_0^{\frac{1}{2}} \cos(x)\left(1 - u^2\right)u^4\,\frac{du}{\cos(x)} = \int_0^{\frac{1}{2}} \left(u^4 - u^6\right)du$$

The correct answer is **B**.

13 Using CAS

$$\int \sin^2(2x)\,dx = \frac{x}{2} - \frac{\sin(2x)\cos(2x)}{4}$$
$$= \frac{x}{2} - \frac{2\sin(2x)\cos(2x)}{8} = \frac{x}{2} - \frac{\sin(4x)}{8}$$

The correct answer is **B**.

14 $$\int \frac{a}{a^2+x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right)$$
$$\int \frac{1}{25+16x^2}\,dx = \frac{4}{5}\int \frac{\frac{5}{4}}{16\left(\frac{25}{16}+x^2\right)}\,dx$$
$$= \frac{4}{5\times 16}\int \frac{\frac{5}{4}}{\left(\frac{25}{16}+x^2\right)}\,dx$$
$$= \frac{1}{20}\int \frac{\frac{5}{4}}{\left(\frac{25}{16}+x^2\right)}\,dx$$
$$= \frac{1}{20}\tan^{-1}\left(\frac{4x}{5}\right) + c$$

The correct answer is **E**.

15 $$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)$$
$$\int \frac{1}{\sqrt{25-16x^2}}\,dx = \int \frac{1}{\sqrt{16\left(\frac{25}{16}-x^2\right)}}\,dx$$
$$= \int \frac{1}{4\sqrt{\left(\frac{25}{16}-x^2\right)}}\,dx$$
$$= \frac{1}{4}\sin^{-1}\left(\frac{4x}{5}\right) + c$$

The correct answer is **A**.

16 A repeated linear term and a quadratic term requires the form of D.

The correct answer is **D**.

Technology active: extended response

17 **a** $\int \sin^2(4x)dx = \frac{1}{2}\int (1-\cos(8x))\,dx$

$$= \frac{1}{2}\left[x - \frac{1}{8}\sin(8x)\right] + c$$

$$= \frac{x}{2} - \frac{1}{16}\sin(8x) + c$$

b $\int \cos(4x)\sin^2(4x)dx$

Let

$u = \sin(4x)$

$\frac{du}{dx} = 4\cos(4x)$

$$= \int \cos(4x)u^2.\frac{1}{4\cos(4x)}du$$

$$= \frac{1}{4}\int u^2 du$$

$$= \frac{1}{4}\left[\frac{1}{3}u^3\right] + c$$

$$= \frac{1}{12}\sin^3(4x) + c$$

c $\int \cos^2(4x)\sin^2(4x)dx = \frac{1}{4}\int \left(2\cos(4x)\sin(4x)\right)^2 dx$

$$= \frac{1}{4}\int \sin^2(8x)dx$$

$$= \frac{1}{8}\int 1 - \cos(16x)\,dx$$

$$= \frac{1}{8}\left[x - \frac{1}{16}\sin(16x)\right] + c$$

$$= \frac{x}{8} - \frac{1}{128}\sin(16x) + c$$

d $\int \cos^3(4x)\sin^2(4x)dx = \int \cos(4x)\cos^2(4x)\sin^2(4x)dx$

$$= \int \cos(4x)\left(1-\sin^2(4x)\right)\sin^2(4x)dx$$

Let

$u = \sin(4x)$

$\frac{du}{dx} = 4\cos(4x)$

$$= \int \cos(4x)\left(1-u^2\right)u^2 \times \frac{1}{4\cos(4x)}du$$

$$= \frac{1}{4}\int u^2 - u^4 du$$

$$= \frac{1}{4}\left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right] + c$$

$$= \frac{1}{12}\sin^3(4x) - \frac{1}{20}\sin^5(4x) + c$$

18 **a** $\int_0^{\frac{4}{3}} \frac{x^3}{\sqrt{16-9x^2}}dx$

Let

$u = 16 - 9x^2$

$\frac{du}{dx} = -18x$

$9x^2 = 16 - u$

$x^2 = \frac{1}{9}(16-u)$

Terminals

$x = \frac{4}{3}, u = 0$

$x = 0, u = 16$

$$= \int_{16}^{0} \frac{x \times \frac{1}{9}(16-u)}{\sqrt{u}} \times -\frac{1}{18x}du$$

$$= \frac{1}{9} \times -\frac{1}{18}\int_{16}^{0} \frac{(16-u)}{\sqrt{u}}du$$

$$= \frac{1}{162}\int_0^{16} 16u^{-\frac{1}{2}} - u^{\frac{1}{2}}du$$

$$= \frac{1}{162}\left[32u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_0^{16}$$

$$= \frac{1}{162}\left[32\sqrt{16} - \frac{2}{3} \times 16^{\frac{3}{2}} - 32\sqrt{0} + \frac{2}{3} \times 0\right]$$

$$= \frac{128}{243}$$

b $\int_0^{\frac{2}{3}} \frac{x^2}{16-9x^2}dx = -\frac{1}{9}\int_0^{\frac{2}{3}} \frac{-9x^2}{16-9x^2}dx$

$$= -\frac{1}{9}\int_0^{\frac{2}{3}} \frac{16-9x^2-16}{16-9x^2}dx$$

$$= -\frac{1}{9}\int_0^{\frac{2}{3}} \left(\frac{16-9x^2}{16-9x^2}\right) - \left(\frac{16}{16-9x^2}\right)dx$$

$$= -\frac{1}{9}\int_0^{\frac{2}{3}} 1\,dx - \frac{1}{9}\int_0^{\frac{2}{3}} \frac{-16}{16-9x^2}dx$$

$$= -\frac{1}{9}\int_0^{\frac{2}{3}} 1\,dx + \frac{16}{9}\int_0^{\frac{2}{3}} \frac{1}{16-9x^2}dx$$

$$= -\frac{1}{9}\int_0^{\frac{2}{3}} 1\,dx + \frac{16}{9} \times \frac{1}{8}\int_0^{\frac{2}{3}} \frac{1}{4+3x} + \frac{1}{4-3x}dx$$

$$= -\frac{1}{9}\left[x\right]_0^{\frac{2}{3}} + \frac{2}{9}\left[\frac{1}{3}\log_e|4+3x| - \frac{1}{3}\log_e|4-3x|\right]_0^{\frac{2}{3}}$$

$$= -\frac{1}{9}\left[x\right]_0^{\frac{2}{3}} + \frac{2}{9} \times \frac{1}{3}\left[\log_e\frac{|4+3x|}{|4-3x|}\right]_0^{\frac{2}{3}}$$

$$= -\frac{1}{9}\left[\frac{2}{3} - 0\right] + \frac{2}{27}\left[\log_e\left|\frac{6}{2}\right|\right]$$

$$= -\frac{2}{27} + \frac{2}{27}\left[\log_e(3)\right]$$

$$= \frac{2}{27}\left(\log_e(3) - 1\right)$$

c $\int_0^{\frac{4}{3}} \sqrt{16-9x^2}dx$

Let

$x = \frac{4}{3}\sin(\theta)$

$\frac{dx}{d\theta} = \frac{4}{3}\cos(\theta)$

$dx = \frac{4}{3}\cos(\theta)d\theta$

$$\begin{aligned}
x^2 &= \frac{16}{9}\sin^2(\theta)\\
9x^2 &= 16\sin^2(\theta)\\
16 - 9x^2 &= 16 - 16\sin^2(\theta)\\
&= 16\left(1 - \sin^2(\theta)\right)\\
&= 16\cos^2(\theta)\\
\sqrt{16 - 9x^2} &= 4\cos(\theta)
\end{aligned}$$

Terminals

$x = \frac{4}{3}, \sin(\theta) = 1 \rightarrow \theta = \frac{\pi}{2}$

$x = 0, \sin(\theta) = 0 \rightarrow \theta = 0$

$$\begin{aligned}
&= \int_0^{\frac{\pi}{2}} 4\cos(\theta).\frac{4}{3}\cos(\theta)d\theta\\
&= \frac{16}{3}\int_0^{\frac{\pi}{2}} \cos^2(\theta)\,d\theta\\
&= \frac{8}{3}\int_0^{\frac{\pi}{2}} 1 + \cos(2\theta)\,d\theta\\
&= \frac{8}{3}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{2}}\\
&= \frac{8}{3}\left[\frac{\pi}{2} + \frac{1}{2}\sin(\pi) - 0 - \frac{1}{2}\sin(0)\right]\\
&= \frac{4\pi}{3}
\end{aligned}$$

d $$\begin{aligned}
\int_0^{\frac{4}{3}} \frac{x^2}{\sqrt{16 - 9x^2}}dx &= \int_0^{\frac{\pi}{2}} \frac{\frac{16}{9}\sin^2(\theta)}{4\cos(\theta)} \times \frac{4}{3}\cos(\theta)d\theta\\
&= \frac{16}{27}\int_0^{\frac{\pi}{2}} \sin^2(\theta)d\theta\\
&= \frac{8}{27}\int_0^{\frac{\pi}{2}} 1 - \cos(2\theta)d\theta\\
&= \frac{8}{27}\left[\theta - \frac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{2}}\\
&= \frac{8}{27}\left[\frac{\pi}{2} - \frac{1}{2}\sin(\pi) - 0 + \frac{1}{2}\sin(0)\right]\\
&= \frac{4\pi}{27}
\end{aligned}$$

19 a $$\int_0^{\frac{\pi}{12}} \cos(4x)\sin^3(4x)dx$$

Let

$u = \sin(4x)$

$\frac{du}{dx} = 4\cos(4x)$

Terminals

$x = \frac{\pi}{12}, u = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$x = 0, u = \sin(0) = 0$

$$\begin{aligned}
&= \int_0^{\frac{\sqrt{3}}{2}} \cos(4x).u^3 \times \frac{1}{4\cos(4x)}du\\
&= \frac{1}{4}\int_0^{\frac{\sqrt{3}}{2}} u^3du\\
&= \frac{1}{4}\left[\frac{1}{4}u^4\right]_0^{\frac{\sqrt{3}}{2}}\\
&= \frac{1}{16}\left[\left(\frac{\sqrt{3}}{2}\right)^4 - 0\right]\\
&= \frac{9}{256}
\end{aligned}$$

b $$\begin{aligned}
\int_0^{\frac{\pi}{12}} \cos^2(4x)\sin^3(4x)dx &= \int_0^{\frac{\pi}{12}} \cos^2(4x)\sin(4x)\sin^2(4x)dx\\
&= \int_0^{\frac{\pi}{12}} \left(1 - \cos^2(4x)\right)\sin(4x)\cos^2(4x)dx
\end{aligned}$$

Let

$u = \cos(4x)$

$\frac{du}{dx} = -4\sin(4x)$

Terminals

$x = \frac{\pi}{12}, u = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$x = 0, u = \cos(0) = 1$

$$\begin{aligned}
&= \int_1^{\frac{1}{2}} \left(1 - u^2\right)u^2 \times -\frac{1}{4}du\\
&= -\frac{1}{4}\int_1^{\frac{1}{2}} u^2 - u^4du\\
&= \frac{1}{4}\left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right]_{\frac{1}{2}}^1\\
&= \frac{47}{1920}
\end{aligned}$$

c $$\begin{aligned}
\int_0^{\frac{\pi}{12}} \cos^3(4x)\sin^3(4x)dx &= \int_0^{\frac{\pi}{12}} \cos(4x)\cos^2(4x)\sin^3(4x)dx\\
&= \int_0^{\frac{\pi}{12}} \cos(4x)\left(1 - \sin^2(4x)\right)\sin^3(4x)dx
\end{aligned}$$

Let

$u = \sin(4x)$

$\frac{du}{dx} = 4\cos(4x)$

Terminals

$x = \frac{\pi}{12}, u = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$x = 0, u = \sin(0) = 0$

$$= \frac{1}{4}\int_0^{\frac{\sqrt{3}}{2}} \left(1-u^2\right)u^3 du$$

$$= \frac{1}{4}\int_0^{\frac{\sqrt{3}}{2}} u^3 - u^5 du$$

$$= \frac{1}{4}\left[\frac{1}{4}u^4 - \frac{1}{6}u^6\right]_0^{\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{4}\left[\frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)^4 - \frac{1}{6}\left(\frac{\sqrt{3}}{2}\right)^6 - 0\right] = \frac{9}{512}$$

d $\displaystyle\int_0^{\frac{\pi}{12}} \cos^4(4x)\sin^3(4x)dx = \int_0^{\frac{\pi}{12}} \cos^4(4x)\sin(4x)\sin^2(4x)dx$

$$= \int_0^{\frac{\pi}{12}} \cos^4(4x)\sin(4x)\left(1-\cos^2(4x)\right)dx$$

Let

$u = \cos(4x)$

$\dfrac{du}{dx} = -4\sin(4x)$

Terminals

$x = \dfrac{\pi}{12}, u = \cos\left(\dfrac{\pi}{3}\right) = \dfrac{1}{2}$

$x = 0, u = \cos(0) = 1$

$$= \int_1^{\frac{1}{2}} u^4\sin(4x)\left(1-u^2\right)\times -\frac{1}{4\sin(4x)}du$$

$$= -\frac{1}{4}\int_1^{\frac{1}{2}} u^4\left(1-u^2\right)du$$

$$= \frac{1}{4}\int_{\frac{1}{2}}^{1} \left(u^4 - u^6\right)du$$

$$= \frac{1}{4}\left[\frac{1}{5}u^5 - \frac{1}{7}u^7\right]_{\frac{1}{2}}^{1}$$

$$= \frac{233}{17920}$$

20 a $\displaystyle\int \frac{x^3}{\sqrt{25-4x^2}}dx = \int \frac{x\times x^2}{\sqrt{25-4x^2}}$

Let

$u = 25 - 4x^2$

$\dfrac{du}{dx} = -8x$

$4x^2 = 25 - u$

$x^2 = \dfrac{1}{4}(25-u)$

$$= \int \frac{\frac{1}{4}x\,(25-u)}{\sqrt{u}} \times -\frac{1}{8x}du$$

$$= -\frac{1}{32}\int (25-u)\,u^{-\frac{1}{2}}du$$

$$= -\frac{1}{32}\left[50u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right] + c$$

$$= -\frac{1}{32}\times 2u^{\frac{1}{2}}\left(25 - \frac{u}{3}\right) + c$$

$$= -\frac{1}{16}\sqrt{u}\left(\frac{75-\left(25-4x^2\right)}{3}\right) + c$$

$$= -\frac{1}{48}\sqrt{u}\left(50+4x^2\right) + c$$

$$= -\frac{1}{24}(2x^2+25)\sqrt{25-4x^2} + c$$

b $\displaystyle\int x^3\sqrt{25-4x^2}dx = \int x\times x^2\sqrt{25-4x^2}dx$

Let

$u = 25 - 4x^2$

$\dfrac{du}{dx} = -8x$

$4x^2 = 25 - u$

$x^2 = \dfrac{1}{4}(25-u)$

$$= \int x\times\frac{1}{4}(25-u)\sqrt{u}\times -\frac{1}{8x}du$$

$$= -\frac{1}{32}\int \left(25u^{\frac{1}{2}} - u^{\frac{3}{2}}\right)du$$

$$= -\frac{1}{32}\left[\frac{50}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right] + c$$

$$= -\frac{1}{32}\times 2u^{\frac{3}{2}}\left(\frac{25}{3} - \frac{u}{5}\right) + c$$

$$= -\frac{u^{\frac{3}{2}}}{16}\left(\frac{(125-3u)}{15}\right) + c$$

$$= -\frac{u^{\frac{3}{2}}}{16}\times\frac{1}{15}\left(125 - 3\left(25-4x^2\right)\right) + c$$

$$= -\frac{u^{\frac{3}{2}}}{16}\times\frac{1}{15}\left(12x^2+50\right) + c$$

$$= -\frac{u^{\frac{3}{2}}}{16}\times\frac{1}{15}\times 2\left(6x^2+25\right) + c$$

$$= -\frac{1}{120}\left(6x^2+25\right)\left(25-4x^2\right)^{\frac{3}{2}} + c$$

c $\displaystyle\int \sqrt{25-4x^2}dx$

Let

$x = \dfrac{5}{2}\sin(\theta)$

$dx = \dfrac{5}{2}\cos(\theta)d\theta$

$$x^2 = \frac{25}{4}\sin^2(\theta)$$

$$4x^2 = 25\sin^2(\theta)$$

$$\begin{aligned} 25 - 4x^2 &= 25 - 25\sin^2(\theta) \\ &= 25\left(1-\sin^2(\theta)\right) \\ &= 25\cos^2(\theta) \end{aligned}$$

$$\sqrt{25-4x^2} = 5\cos(\theta)$$

$$= \int 5\cos(\theta)\times\frac{5}{2}\cos(\theta)\,d\theta$$

$$= \frac{25}{2}\int \cos^2(\theta)\,d\theta$$

$$
\begin{aligned}
&= \frac{25}{4}\int 1+\cos(2\theta)\,d\theta\\
&= \frac{25}{4}\left[\theta+\frac{1}{2}\sin(2\theta)\right]+c\\
&= \frac{25}{4}\theta+\frac{25}{4}\sin(\theta)\cos(\theta)+c
\end{aligned}
$$

$$\sin(\theta)=\frac{2x}{5}$$

Using right angled triangles,

$$\cos(\theta)=\frac{\sqrt{25-4x^2}}{5}$$

$$
\begin{aligned}
&= \frac{25}{4}\sin^{-1}\left(\frac{2x}{5}\right)+\frac{25}{4}\times\frac{2x}{5}\times\frac{\sqrt{25-4x^2}}{5}+c\\
&= \frac{25}{4}\sin^{-1}\left(\frac{2x}{5}\right)+\frac{x}{2}\sqrt{25-4x^2}+c
\end{aligned}
$$

d $\int x^2\sqrt{25-4x^2}dx$

Let

$$x=\frac{5}{2}\sin(\theta)$$

$$dx=\frac{5}{2}\cos(\theta)d\theta$$

$$
\begin{aligned}
x^2 &= \frac{25}{4}\sin^2(\theta)\\
4x^2 &= 25\sin^2(\theta)\\
25-4x^2 &= 25-25\sin^2(\theta)\\
&= 25\left(1-\sin^2(\theta)\right)\\
&= 25\cos^2(\theta)\\
\sqrt{25-4x^2} &= 5\cos(\theta)
\end{aligned}
$$

$$
\begin{aligned}
&= \int \frac{25}{4}\sin^2(\theta).5\cos(\theta)\times\frac{5}{2}\cos(\theta)\,d\theta\\
&= \frac{625}{8}\int \sin^2(\theta)\cos^2(\theta)\,d\theta\\
&= \frac{625}{8}\int \frac{1}{4}(2\sin(\theta)\cos(\theta))^2 d\theta\\
&= \frac{625}{32}\int \sin^2(2\theta)\,d\theta\\
&= \frac{625}{64}\int 1-\cos(4\theta)d\theta\\
&= \frac{625}{64}\left[\theta-\frac{1}{4}\sin(4\theta)\right]+c\\
&= \frac{625}{64}\left[\theta-\frac{1}{4}\times 2\sin(2\theta)\cos(2\theta)\right]+c\\
&= \frac{625}{64}\left[\theta-\frac{1}{4}\times 2\times 2\sin(\theta)\cos(\theta)\left(\cos^2(\theta)-\sin^2(\theta)\right)\right]+c\\
&= \frac{625}{64}\theta-\frac{625}{64}\sin(\theta)\cos(\theta)\left(\cos^2(\theta)-\sin^2(\theta)\right)+c
\end{aligned}
$$

But

$$\sin(\theta)=\frac{2x}{5}$$

$$\cos(\theta)=\frac{\sqrt{25-4x^2}}{5}$$

$$
\begin{aligned}
&= \frac{625}{64}\theta-\frac{625}{64}\times\frac{2x}{5}\times\frac{\sqrt{25-4x^2}}{5}\left(\frac{25-4x^2}{25}-\frac{4x^2}{25}\right)+c\\
&= \frac{625}{64}\sin^{-1}\left(\frac{2x}{5}\right)+\frac{x}{32}\left(8x^2-25\right)\sqrt{25-4x^2}+c
\end{aligned}
$$

7.8 Exam questions

1 $I=\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sec^2(x)}{\sec^2(x)-3\tan(x)+1}dx$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sec^2(x)}{1+\tan^2(x)-3\tan(x)+1}dx$$

using $1+\tan^2(x)=\sec^2(x)$

Let $u=\tan(x)$.

$$\frac{du}{dx}=\sec^2(x)$$

Terminals

$$x=\frac{\pi}{3},\ u=\tan\left(\frac{\pi}{3}\right)=\sqrt{3},\ x=\frac{\pi}{4},\ u=\tan\left(\frac{\pi}{4}\right)=1$$

$$
\begin{aligned}
I &= \int_1^{\sqrt{3}}\frac{1}{1+u^2-3u+1}du\\
&= \int_1^{\sqrt{3}}\frac{1}{u^2-3u+2}du
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{u^2-3u+2} &= \frac{1}{(u-1)(u-2)}\\
&= \frac{A}{u-1}+\frac{B}{u-2}\\
&= \frac{A(u-2)+B(u-1)}{(u-1)(u-2)}\\
&= \frac{u(A+B)-2A-B}{u^2-3u+2}
\end{aligned}
$$

$$A+B=0,\ 2A+B=-1\Rightarrow A=-1,\ B=1$$

$$I=\int_1^{\sqrt{3}}\left(\frac{1}{u-2}-\frac{1}{u-1}\right)du$$

The correct answer is **C**.

2 $\displaystyle\int\tan(2x)dx=\int\frac{\sin(2x)}{\cos(2x)}dx$

Let $u=\cos(2x)$, $\dfrac{du}{dx}=-2\sin(2x)$

$$
\begin{aligned}
\int\tan(2x)dx &= -\frac{1}{2}\int\frac{1}{u}du=-\frac{1}{2}\log_e(|u|)+c\\
&= -\frac{1}{2}\log_e(|\cos(2x)|)+c\\
&= \frac{1}{2}\log_e\left(\frac{1}{|\cos(2x)|}\right)+c\\
&= \frac{1}{2}\log_e(|\sec(2x)|)+c
\end{aligned}
$$

Award 1 mark for the correct substitution.

Award 1 mark for the correct result.

3 a $f(x)=3x\tan^{-1}(2x)$, range $f=[0,\infty)$

Award 1 mark for the correct range.

b $f'(x)=3\tan^{-1}(2x)+3x\times\dfrac{2}{1+4x^2}$, using the product rule

$$=3\tan^{-1}(2x)+\frac{6x}{1+4x^2}\qquad\text{[1 mark]}$$

c $\frac{d}{dx}\left[3x\tan^{-1}(2x)\right] = 3\tan^{-1}(2x) + \frac{6x}{1+4x^2}$

$$\int\left(3\tan^{-1}(2x) + \frac{6x}{1+4x^2}\right)dx = 3x\tan^{-1}(2x)$$

$$\int 3\tan^{-1}(2x)dx + \int\frac{6x}{1+4x^2}dx = 3x\tan^{-1}(2x)$$

$$\int 3\tan^{-1}(2x)dx = 3x\tan^{-1}(2x) - \int\frac{6x}{1+4x^2}dx$$

$$3\int\tan^{-1}(2x)dx = 3x\tan^{-1}(2x) - \frac{3}{4}\log_e\left(1+4x^2\right)$$

$$\int\tan^{-1}(2x)dx = x\tan^{-1}(2x) - \frac{1}{4}\log_e\left(1+4x^2\right)$$

$$A = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\tan^{-1}(2x)dx$$

$$= \left[x\tan^{-1}(2x) - \frac{1}{4}\log_e\left(1+4x^2\right)\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$$

$$= \left(\frac{\sqrt{3}}{2}\tan^{-1}\left(\sqrt{3}\right) - \frac{1}{4}\log_e(4)\right) - \left(\frac{1}{2}\tan^{-1}(1) - \frac{1}{4}\log_e(2)\right)$$

$$= \left(\frac{\sqrt{3}}{2}\times\frac{\pi}{3} - \frac{1}{4}\log_e(4)\right) - \left(\frac{1}{2}\times\frac{\pi}{4} - \frac{1}{4}\log_e(2)\right)$$

$$= \pi\left(\frac{\sqrt{3}}{6} - \frac{1}{8}\right) + \frac{1}{4}\log_e\left(\frac{1}{2}\right)$$

$$= \frac{\left(4\sqrt{3}-3\right)\pi}{24} - \frac{1}{4}\log_e(2)$$

Award 1 mark for the correct definite integral for the area.

Award 1 mark for deducing the correct antiderivative.

Award 1 mark for the correct final area.

4 $\int\frac{a^2x^2+b^2}{a^2x^2-b^2}dx = \int\frac{a^2x^2-b^2+2b^2}{a^2x^2-b^2}dx =$

$$\int\left(1+\frac{2b^2}{a^2x^2-b^2}\right)dx$$

By partial fractions,

$$\frac{2b^2}{a^2x^2-b^2} = \frac{A}{ax+b} + \frac{B}{ax-b}$$

$$= \frac{A(ax-b)+B(ax+b)}{(ax+b)(ax-b)} = \frac{ax(A+B)+b(B-A)}{(ax+b)(ax-b)}$$

(1) $B - A = 2b$ (2) $A + B = 0$ add $\Rightarrow B = b$ $A = -b$

$$\int\left(1+\frac{2b^2}{a^2x^2-b^2}\right)dx = \int\left(1+\frac{b}{ax-b} - \frac{b}{ax+b}\right)dx$$

$$= x + \frac{b}{a}\log_e(|ax-b|) - \frac{b}{a}\log_e(|ax+b|)$$

$$= x + \frac{b}{a}\log_e\left(\left|\frac{ax-b}{ax+b}\right|\right)$$

The correct answer is **C**.

5 $\int_a^b\left(\text{cosec}^2(3x)e^{3\cot(3x)}\right)dx = \left[-\frac{1}{9}e^{3\cot(3x)}\right]_a^b$

$$= -\frac{1}{9}\left(e^{3\cot(3b)} - e^{3\cot(3a)}\right)$$ [1 mark]

Topic 8 — Differential equations

8.2 Verifying solutions to differential equations

8.2 Exercise

1

$$\begin{aligned}
y &= x^2\\
\frac{dy}{dx} &= 2x\\
\frac{d^2y}{dx^2} &= 2\\
\text{LHS} &= x^2\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 + 2y\\
&= x^2 \times 2 - (2x)^2 + 2(x^2)\\
&= 2x^2 - 4x^2 + 2x^2\\
&= 0\\
&= \text{RHS}
\end{aligned}$$

2

$$\begin{aligned}
y &= x^4\\
\frac{dy}{dx} &= 4x^3\\
\frac{d^2y}{dx^2} &= 12x^2\\
\text{LHS} &= x^4\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 + 4x^2y\\
&= x^4 \times 12x^2 - \left(4x^3\right)^2 + 4x^2(x^4)\\
&= 12x^6 - 16x^6 + 4x^6\\
&= 0\\
&= \text{RHS}
\end{aligned}$$

3

$$\begin{aligned}
y &= 2x^2 - 3x + 5\\
\frac{dy}{dx} &= 4x - 3\\
\text{LHS} &= \left(\frac{dy}{dx}\right)^2 - 8y + 31\\
&= (4x-3)^2 - 8\left(2x^2 - 3x + 5\right) + 31\\
&= 16x^2 - 24x + 9 - 16x^2 + 24x - 40 + 31\\
&= 0\\
&= \text{RHS}
\end{aligned}$$

4 a

$$\begin{aligned}
y &= x^3 - 3x^2 - \frac{3x}{2} + 1\\
\frac{dy}{dx} &= 3x^2 - 6x - \frac{3}{2}\\
\frac{d^2y}{dx^2} &= 6x - 6\\
\text{LHS} &= \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 6y + 6x^2\\
&= 6x - 6 - 2x\left(3x^2 - 6x - \frac{3}{2}\right) +\\
&\quad 6\left(x^3 - 3x^2 - \frac{3x}{2} + 1\right) + 6x^2\\
&= 6x - 6 - 6x^3 + 12x^2 + 3x + 6x^3 - 18x^2 - 9x +\\
&\quad 6 + 6x^2\\
&= 0\\
&= \text{RHS}
\end{aligned}$$

b

$$\begin{aligned}
y &= ax^3 + bx^2\\
\frac{dy}{dx} &= 3\,ax^2 + 2\,bx\\
\frac{d^2y}{dx^2} &= 6\,ax + 2\,b\\
\text{LHS} &= x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y\\
&= x^2(6\,ax + 2\,b) - 4x\left(3\,ax^2 + 2\,bx\right) + 6\left(ax^3 + bx^2\right)\\
&= 6\,ax^3 + 2\,bx^2 - 12\,ax^3 - 8\,bx^2 + 6\,ax^3 + 6\,bx^2\\
&= 0\\
&= \text{RHS}
\end{aligned}$$

5

$$\begin{aligned}
y &= e^{kx}\\
\frac{dy}{dx} &= k\,e^{kx}\\
\frac{d^2y}{dx^2} &= k^2\,e^{kx}\\
\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 10y &= 0\\
k^2e^{kx} + 3ke^{kx} - ke^{kx} &= 0\\
e^{kx}\left(k^2 + 3k - 10\right) &= 0\\
e^{kx} &\neq 0\\
k^2 + 3k - 10 &= 0\\
(k+5)(k-2) &= 0
\end{aligned}$$

$$k = -5, 2$$

6

$$\begin{aligned}
y &= \cos(kx)\\
\frac{dy}{dx} &= -k\sin(kx)\\
\frac{d^2y}{dx^2} &= -k^2\cos(kx)
\end{aligned}$$

$$\begin{aligned}
\frac{d^2y}{dx^2} + 9y &= 0\\
-k^2\cos(kx) + 9\cos(kx) &= 0\\
\cos(kx)\left(9 - k^2\right) &= 0\\
(3+k)(3-k) &= 0
\end{aligned}$$

$$k = \pm 3$$

7 a

$$\begin{aligned}
y &= a + bx + cx^2\\
\frac{dy}{dx} &= b + 2cx\\
\frac{d^2y}{dx^2} &= 2c\\
\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y &= 4x^2\\
4x^2 &= 2c + 2(b + 2cx) + 4\left(a + bx + cx^2\right)\\
4x^2 &= 2c + 2b + 4a + x(4c + 4b) + 4cx^2
\end{aligned}$$

$$\begin{aligned}
&x^2: 4c = 4 \Rightarrow c = 1\\
&x^1: 4c + 4b = 0 \Rightarrow b = -c = -1\\
&x^0: 2c + 2b + 4a = 0
\end{aligned}$$

$$\begin{aligned}
4a &= -2c - 2b\\
&= 0
\end{aligned}$$

$$\begin{aligned}
a &= 0\\
b &= -1\\
c &= 1
\end{aligned}$$

b $y = ax^3 + bx^2 + cx + d$

$$\frac{dy}{dx} = 3ax^2 + 2bx + c$$

$$\frac{d^2y}{dx^2} = 6ax + 2b$$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^3$$

$$\begin{aligned} x^3 &= 6ax + 2b + 2\left(3ax^2 + 2bx + c\right) + \\ &\quad \left(ax^3 + bx^2 + cx + d\right) \\ x^3 &= 2b + 2c + d + x(6a + 4b + c) + \\ &\quad x^2\,(6a + b) + ax^3 \end{aligned}$$

$x^3: a = 1$

$x^2: 6a + b = 0 \Rightarrow b = -6a = -6$

$x^1: 6a + 4b + c = 0 \Rightarrow c = -4b - 6a = 18$

$x^0: 2b + 2c + d = 0 \Rightarrow d = -2b - 2c = -24$

$a = 1$

$b = -6$

$c = 18$

$d = -24$

8 a $y = x^n$

$$\frac{dy}{dx} = nx^{n-1}$$

$$\frac{d^2y}{dx^2} = n(n-1)x^{n-2}$$

$$\begin{aligned} \text{LHS} &= x^2y\frac{d^2y}{dx^2} - x^2\left(\frac{dy}{dx}\right)^2 + ny^2 \\ &= x^2 \times x^n \times n(n-1)x^{n-2} - x^2\left(nx^{n-1}\right)^2 + n(x^n)^2 \\ &= n(n-1)x^{2n} - n^2x^{2n} + nx^{2n} \\ &= x^{2n}(n^2 - n - n^2 + n) \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

b $y = x^n$

$$\frac{dy}{dx} = nx^{n-1}$$

$$\frac{d^2y}{dx^2} = n(n-1)x^{n-2}$$

$$\begin{aligned} 0 &= x^2\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 - 10y \\ 0 &= x^2 \times n(n-1)x^{n-2} - 2x \times nx^{n-1} - 10x^n \\ 0 &= n(n-1)x^n - 2nx^n - 10x^n \\ 0 &= (n(n-1) - 2n - 10)\,x^n \\ 0 &= \left(n^2 - 3n - 10\right)x^{10} \\ 0 &= (n-5)\,(n+2) \end{aligned}$$

$n = -2, 5$

9 a $x = e^{3t} + e^{-4t}$

$$\frac{dx}{dt} = 3e^{3t} - 4e^{-4t}$$

$$\frac{d^2x}{dt^2} = 9e^{3t} + 16e^{-4t}$$

$$\begin{aligned} \text{LHS} &= \frac{d^2x}{dt^2} + \frac{dx}{dt} - 12x \\ &= 9e^{3t} + 16e^{-4t} + 3e^{3t} - 4e^{-4t} - 12\left(e^{3t} + e^{-4t}\right) \\ &= 12e^{3t} + 12e^{-4t} - 12e^{3t} - 12e^{-4t} \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

b $y = Ae^{3x} + Be^{-3x}$

$$\frac{dy}{dx} = 3Ae^{3x} - 3Be^{-3x}$$

$$\frac{d^2y}{dt^2} = 9Ae^{3x} + 9Be^{-3x}$$

$$\begin{aligned} \text{LHS} &= \frac{d^2y}{dx^2} - 9y \\ &= 9Ae^{3x} + 9Be^{-3x} - 9\left(Ae^{3x} + Be^{-3x}\right) \\ &= 9Ae^{3x} + 9Be^{-3x} - 9e^{3x} - 9Be^{-3x} \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

10 $y = e^{kx}$

$$\frac{dy}{dx} = ke^{kx}$$

$$\frac{d^2y}{dx^2} = k^2e^{kx}$$

$$\begin{aligned} \frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y &= 0 \\ k^2e^{kx} + 5ke^{kx} - 6e^{kx} &= 0 \\ e^{kx}\left(k^2 + 5k - 6\right) &= 0 \\ e^{kx} &\neq 0 \\ k^2 + 5k - 6 &= 0 \\ (k+6)\,(k-1) &= 0 \end{aligned}$$

$k = -6,\ 1$

11 a $y = 3\sin(2x) + 4\cos(2x)$

$$\frac{dy}{dx} = 6\cos(2x) - 8\sin(2x)$$

$$\frac{d^2y}{dx^2} = -12\sin(2x) - 16\cos(2x)$$

$$\begin{aligned} \text{LHS} &= \frac{d^2y}{dx^2} + 4y \\ &= -12\sin(2x) - 16\cos(2x) + 4\,(3\sin(2x) + 4\cos(2x)) \\ &= -12\sin(2x) - 16\cos(2x) + 12\sin(2x) + 16\cos(2x) \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

b $y = A\sin(3x) + B\cos(3x)$

$$\frac{dy}{dx} = 3A\cos(3x) - 3B\sin(3x)$$

$$\frac{d^2y}{dx^2} = -9A\sin(3x) - 9B\cos(3x)$$

$$\begin{aligned} \text{LHS} &= \frac{d^2y}{dx^2} + 9y \\ &= -9A\sin(3x) - 9B\cos(3x) \\ &\quad + 9(A\sin(3x) + B\cos(3x)) \\ &= -9A\sin(3x) - 9B\cos(3x) \\ &\quad + 9A\sin(3x) + 9B\cos(3x) \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

12 a $y = a\sin(nx) + b\cos(nx)$

$$\frac{dy}{dx} = n\,a\cos(nx) - b\,n\sin(nx)$$

$$\frac{d^2y}{dx^2} = -n^2a\sin(nx) - b\,n^2\cos(nx)$$

$$\begin{aligned}
\text{LHS} &= \frac{d^2y}{dx^2} + n^2 y \\
&= -n^2 a\sin(nx) - b\,n^2\cos(nx) \\
&\quad + n^2\,(a\sin(nx) + b\cos(nx)) \\
&= -n^2 a\sin(nx) - b\,n^2\cos(nx) \\
&\quad + n^2 a\sin(nx) + n^2 b\cos(nx) \\
&= 0 \\
&= \text{RHS}
\end{aligned}$$

b

$$\begin{aligned}
x &= a\sin(pt) \\
\frac{dx}{dt} &= a\,p\cos(pt) \\
\frac{d^2x}{dt^2} &= -a\,p^2\sin(pt) \\
\frac{d^2x}{dt^2} + 9x &= 0 \\
0 &= -a\,p^2\sin(pt) + 9\,a\sin(pt) \\
0 &= a\sin(pt)\left(9 - p^2\right) \\
a\ \sin(pt) &\neq 0 \\
9 - p^2 &= 0 \\
p &= \pm 3
\end{aligned}$$

13

$$\begin{aligned}
y &= xe^{3x} \\
\frac{dy}{dx} &= x\frac{d}{dx}\left(e^{3x}\right) + e^{3x}\frac{d}{dx}(x) \\
&= 3\,x\,e^{3x} + e^{3x} \\
&= e^{3x}(3x + 1) \\
\frac{d^2y}{dx^2} &= \frac{d}{dx}(3x+1)\,e^{3x} + (3x+1)\,\frac{d}{dx}\left(e^{3x}\right) \\
&= 3e^{3x} + 3\,(3x+1)\,e^{3x} \\
&= e^{3x}(9x + 6) \\
\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y &= e^{3x}(9x+6) - 6e^{3x}(3x+1) + 9xe^{3x} \\
&= e^{3x}(9x + 6 - 18x - 6 + 9x) \\
&= 0 \\
&= \text{RHS}
\end{aligned}$$

14

$$\begin{aligned}
y &= Ax\cos(2x) \\
\frac{dy}{dx} &= A\frac{d}{dx}(x)\cos(2x) + Ax\frac{d}{dx}\big(\cos(2x)\big) \\
&= A\cos(2x) - 2Ax\sin(2x) \\
\frac{d^2y}{dx^2} &= -2A\sin(2x) - 2A\left(\frac{d}{dx}(x)\sin(2x) + x\frac{d}{dx}(\sin(2x))\right) \\
&= -2A\sin(2x) - 2A\sin(2x) - 4Ax\cos(2x) \\
&= -4A\sin(2x) - 4Ax\cos(2x) \\
\frac{d^2y}{dx^2} + 4y &= 8\sin(2x) \\
8\sin(2x) &= -4A\sin(2x) - 4Ax\cos(2x) + 4Ax\cos(2x) \\
8\sin(2x) &= -4A\sin(2x) \\
-4A &= 8 \\
A &= -2
\end{aligned}$$

15 a

$$\begin{aligned}
y &= e^{x^2} \\
&= e^u \text{ where } u = x^2 \\
\frac{dy}{du} &= e^u \qquad \frac{du}{dx} = 2x \\
\frac{dy}{dx} &= 2xe^{x^2} \text{ Product rule} \\
\frac{d^2y}{dx^2} &= 2x\frac{d}{dx}\left(e^{x^2}\right) + e^{x^2}\frac{d}{dx}(2x) \\
&= 2x \times \left(2xe^{x^2}\right) + e^{x^2} \times 2 \\
&= e^{x^2}\left(4x^2 + 2\right)
\end{aligned}$$

$$\begin{aligned}
\text{LHS} &= \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 2y \\
&= e^{x^2}\left(4x^2 + 2\right) - 2x\left(2xe^{x^2}\right) - 2e^{x^2} \\
&= e^{x^2}\left(4x^2 + 2 - 4x^2 - 2\right) \\
&= 0 \\
&= \text{RHS}
\end{aligned}$$

b

$$\begin{aligned}
y &= \cos\left(x^2\right) \\
&= \cos(u) \text{ where } u = x^2 \\
\frac{dy}{du} &= -\sin(u) \qquad \frac{du}{dx} = 2x \\
\frac{dy}{dx} &= -2x\ \sin\left(x^2\right) \\
\frac{d^2y}{dx^2} &= -2x\frac{d}{dx}\left(\sin\left(x^2\right)\right) - \sin\left(x^2\right)\frac{d}{dx}(2x) \\
&= -4x^2\cos\left(x^2\right) - 2\sin\left(x^2\right) \\
\text{LHS} &= x\frac{d^2y}{dx^2} - \frac{dy}{dx} + 4x^3 y \\
&= x\left(-4x^2\cos\left(x^2\right) - 2\sin\left(x^2\right)\right) - \left(-2\,x\sin\left(x^2\right)\right) \\
&\quad + 4x^3\cos\left(x^2\right) \\
&= -4x^3\cos\left(x^2\right) - 2\,x\sin\left(x^2\right) + 2\,x\sin\left(x^2\right) \\
&\quad + 4x^3\cos\left(x^2\right) \\
&= 0 \\
&= \text{RHS}
\end{aligned}$$

16 a

$$\begin{aligned}
y &= ax + b\sqrt{x^2+1} \\
\frac{dy}{dx} &= a + \frac{bx}{\sqrt{x^2+1}} \\
\frac{d^2y}{dx^2} &= \frac{b\sqrt{x^2+1} - bx \times \frac{x}{\sqrt{x^2+1}}}{\left(\sqrt{x^2+1}\right)^2} \\
&= \frac{\frac{b\left(x^2+1\right) - bx^2}{\sqrt{x^2+1}}}{x^2+1} \\
&= \frac{b}{\left(x^2+1\right)\sqrt{x^2+1}} \\
&= b\left(x^2+1\right)^{-\frac{3}{2}} \\
\text{LHS} &= \left(x^2+1\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y \\
&= \left(x^2+1\right) \times b\left(x^2+1\right)^{\frac{-3}{2}} + x\left(a + \frac{bx}{\sqrt{x^2+1}}\right) \\
&\quad - \left(ax + b\sqrt{x^2+1}\right) \\
&= \frac{b}{\sqrt{x^2+1}} + ax + \frac{bx^2}{\sqrt{x^2+1}} - ax - b\sqrt{x^2+1} \\
&= \frac{b + bx^2 - b\left(x^2+1\right)}{\sqrt{x^2+1}} \\
&= 0 \\
&= \text{RHS}
\end{aligned}$$

b

$$\begin{aligned}
y &= \log_e\left(x + \sqrt{x^2-9}\right) \\
\frac{dy}{dx} &= \frac{1}{\left(x + \sqrt{x^2-9}\right)}\left(1 + \frac{x}{\sqrt{x^2-9}}\right) \\
&= \frac{1}{\left(x + \sqrt{x^2-9}\right)}\left(\frac{\sqrt{x^2-9} + x}{\sqrt{x^2-9}}\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{x^2-9}} = \left(x^2-9\right)^{-\frac{1}{2}}\\
\frac{d^2y}{dx^2} &= -\frac{1}{2}\times 2x\times\left(x^2-9\right)^{-\frac{3}{2}}\\
&= \frac{-x}{\left(x^2-9\right)^{\frac{3}{2}}}
\end{aligned}$$

$$\begin{aligned}
\text{LHS} &= \left(x^2-9\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx}\\
&= \left(x^2-9\right)\times\frac{-x}{\left(x^2-9\right)^{\frac{3}{2}}} + x\times\left(x^2-9\right)^{-\frac{1}{2}}\\
&= \frac{-x}{\sqrt{x^2-9}} + \frac{x}{\sqrt{x^2-9}}\\
&= 0\\
&= \text{RHS}
\end{aligned}$$

17 a

$$\begin{aligned}
y &= \tan(ax)\\
\frac{dy}{dx} &= a\sec^2(ax)\\
&= a\left(\tan^2(ax)+1\right)\\
\frac{d^2y}{dx^2} &= a\left(2a\tan(ax)\sec^2(ax)\right)\\
&= 2a^2\tan(ax)\left(\tan^2(ax)+1\right)\\
&= 2a^2y\left(1+y^2\right)\\
&= \text{RHS}
\end{aligned}$$

b

$$\begin{aligned}
y &= \tan^2(ax)\\
\frac{dy}{dx} &= 2a\tan(ax)\sec^2(ax)\\
&= 2a\tan(ax)\left(\tan^2(ax)+1\right)\\
&= 2a\tan^3(ax) + 2a\tan(ax)\\
\frac{d^2y}{dx^2} &= 6a^2\tan^2(ax)\sec^2(ax) + 2a^2\sec^2(ax)\\
&= 6a^2\tan^2(ax)\left(1+\tan^2(ax)\right) + 2a^2\left(1+\tan^2(ax)\right)\\
&= 2a^2\left(1+\tan^2(ax)\right)\left(1+3\tan^2(ax)\right)\\
&= 2a^2(1+y)(1+3y)\\
&= 2a^2\left(1+4y+3y^2\right)\\
&= \text{RHS}
\end{aligned}$$

18

$$\begin{aligned}
y &= \log_e(ax+b)\\
\frac{dy}{dx} &= \frac{a}{ax+b}\\
&= a(ax+b)^{-1}\\
\frac{d^2y}{dx^2} &= -a^2(ax+b)^{-2}\\
&= \frac{-a^2}{(ax+b)^2}
\end{aligned}$$

Also $e^y = ax+b,\quad e^{-y} = \frac{1}{ax+b},\quad e^{-2y} = \frac{1}{(ax+b)^2}$

$$\begin{aligned}
\frac{d^2y}{dx^2} + a^2e^{-2y} &= \frac{-a^2}{(ax+b)^2} + \frac{a^2}{(ax+b)^2}\\
&= 0\\
&= \text{RHS}
\end{aligned}$$

19 a

$$\begin{aligned}
y &= \sin^{-1}(3x)\\
\frac{dy}{dx} &= \frac{3}{\sqrt{1-9x^2}}\\
&= 3\left(1-9x^2\right)^{-\frac{1}{2}}\\
\frac{d^2y}{dx^2} &= -\frac{1}{2}\times 3\times -18x\left(1-9x^2\right)^{-\frac{3}{2}}\\
&= \frac{27x}{\left(1-9x^2\right)^{\frac{3}{2}}}
\end{aligned}$$

$$\begin{aligned}
\text{LHS} &= \left(1-9x^2\right)\frac{d^2y}{dx^2} - 9x\frac{dy}{dx}\\
&= \left(1-9x^2\right)\times\frac{27x}{\left(1-9x^2\right)^{\frac{3}{2}}} - \frac{9x\times 3}{\sqrt{1-9x^2}}\\
&= \frac{27x}{\sqrt{1-9x^2}} - \frac{27x}{\sqrt{1-9x^2}}\\
&= 0\\
&= \text{RHS}
\end{aligned}$$

b

$$\begin{aligned}
y &= \cos^{-1}\left(\frac{x}{4}\right)\\
\frac{dy}{dx} &= \frac{-1}{\sqrt{16-x^2}}\\
&= -\left(16-x^2\right)^{-\frac{1}{2}}\\
\frac{d^2y}{dx^2} &= -\frac{1}{2}\times 2x\times\left(16-x^2\right)^{-\frac{3}{2}}\\
&= \frac{-x}{\left(16-x^2\right)^{\frac{3}{2}}}
\end{aligned}$$

$$\begin{aligned}
\text{LHS} &= \left(16-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx}\\
&= \left(16-x^2\right)\times\frac{-x}{\left(16-x^2\right)^{\frac{3}{2}}} - \frac{x\times -1}{\sqrt{16-x^2}}\\
&= \frac{-x}{\sqrt{16-x^2}} + \frac{x}{\sqrt{16-x^2}}\\
&= 0\\
&= \text{RHS}
\end{aligned}$$

20 a

$$\begin{aligned}
v &= \frac{mg}{k}\left(1-e^{-kt}\right)\\
&= \frac{mg}{k} - \frac{mg}{k}e^{-kt}\\
\frac{dv}{dt} &= 0 - \frac{mg}{k}\times -ke^{-kt}\\
&= mge^{-kt}
\end{aligned}$$

$$\begin{aligned}
\text{LHS} &= \frac{dv}{dt} + kv\\
&= mge^{-kt} + k\left(\frac{mg}{k}\left(1-e^{-kt}\right)\right)\\
&= mge^{-kt} + mg\left(1-e^{-kt}\right)\\
&= mge^{-kt} + mg - mge^{-kt}\\
&= mg\\
&= \text{RHS}
\end{aligned}$$

b

$$\begin{aligned}
i &= 3e^{-2t}\sin(3t)\\
\frac{di}{dt} &= 3\left(-2e^{-2t}\sin(3t) + 3e^{-2t}\cos(3t)\right)\\
&= 3e^{-2t}(3\cos(3t) - 2\sin(3t))\\
\frac{d^2i}{dt^2} &= -6e^{-2t}(3\cos(3t) - 2\sin(3t))\\
&\quad + 3e^{-2t}(-9\sin(3t) - 6\cos(3t))\\
&= 3e^{-2t}(-12\cos(3t) - 5\sin(3t))
\end{aligned}$$

$$\begin{aligned}\text{LHS} &= \frac{d^2i}{dt^2}+4\frac{di}{dt}+13i\\ &= 3e^{-2t}\left(-12\cos(3t)-5\sin(3t)+12\cos(3t)\right.\\ &\quad \left.-8\sin(3t)+13\sin(3t)\right)\\ &= 0\\ &= \text{RHS}\end{aligned}$$

21 a $y = e^{3x}\cos(2x)$

$$\begin{aligned}\frac{dy}{dx} &= 3e^{3x}\cos(2x)-2e^{3x}\sin(2x)\\ &= e^{3x}\left(3\cos(2x)-2\sin(2x)\right)\\ \frac{d^2y}{dx^2} &= 3e^{3x}\left(3\cos(2x)-2\sin(2x)\right)\\ &\quad +e^{3x}\left(-6\sin(2x)-4\cos(2x)\right)\\ &= e^{3x}\left(5\cos(2x)-12\sin(2x)\right)\end{aligned}$$

$$\begin{aligned}\text{LHS} &= \frac{d^2y}{dx^2}-6\frac{dy}{dx}+13y\\ &= e^{3x}\left(5\cos(2x)-12\sin(2x)-18\cos(2x)\right.\\ &\quad \left.+12\sin(2x)+13\cos(2x)\right)\\ &= 0\\ &= \text{RHS}\end{aligned}$$

b $x = t\left(a\cos(3t)+b\sin(3t)\right)$

$$\begin{aligned}\frac{dx}{dt} &= a\cos(3t)+b\sin(3t)\\ &\quad +t\left(-3a\sin(3t)+3b\cos(3t)\right)\\ \frac{d^2x}{dt^2} &= -3a\sin(3t)+3b\cos(3t)\\ &\quad -3a\sin(3t)+3b\cos(3t)\\ &\quad +t\left(-9a\cos(3t)-9b\sin(3t)\right)\end{aligned}$$

$$\begin{aligned}\text{so } \frac{d^2x}{dt^2}+9x &= -6a\sin(3t)+6b\cos(3t)\\ &= 6\cos(3t)\end{aligned}$$

$a = 0$

$b = 1$

22 a $y = xe^{-3x}$

$$\begin{aligned}\frac{dy}{dx} &= e^{-3x}-3xe^{-3x}\\ &= (1-3x)e^{-3x}\\ \frac{d^2y}{dx^2} &= -3e^{-3x}-3(1-3x)e^{-3x}\\ &= (9x-6)e^{-3x}\end{aligned}$$

$$\begin{aligned}\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by &= 0\\ (9x-6)e^{-3x}+a(1-3x)e^{-3x}+bxe^{-3x} &= 0\\ e^{-3x}\left((b+9-3a)x+(a-6)\right) &= 0\end{aligned}$$

$\Rightarrow a-6=0 \Rightarrow a=6$

$b+9-3a=0$

$\Rightarrow b=3a-9=9$

b $y = e^{kx}(Ax+B)$

$$\begin{aligned}\frac{dy}{dx} &= ke^{kx}(Ax+B)+Ae^{kx}\\ &= e^{kx}(A+kB+kAx)\\ \frac{d^2y}{dx^2} &= ke^{kx}(A+kB+kAx)+e^{kx}(kA)\\ &= e^{kx}\left(2kA+k^2B+k^2Ax\right)\end{aligned}$$

$$\begin{aligned}\text{LHS} &= \frac{d^2y}{dx^2}-2k\frac{dy}{dx}+k^2y\\ &= e^{kx}\left(2kA+k^2B+k^2Ax-2k(A+kB+kAx)\right.\\ &\quad \left.+k^2(Ax+B)\right)\\ &= e^{kx}\left(2kA+k^2B+k^2Ax-2kA-2k^2B-2k^2Ax\right.\\ &\quad \left.+\ k^2Ax+k^2B\right)\\ &= 0\\ &= \text{RHS}\end{aligned}$$

23 a $y = Ax^2e^{-3x}$

$$\begin{aligned}\frac{dy}{dx} &= 2Axe^{-3x}-3Ax^2e^{-3x}\\ &= A\left(2x-3x^2\right)e^{-3x}\\ \frac{d^2y}{dx^2} &= A(2-6x)e^{-3x}-3A\left(2x-3x^2\right)e^{-3x}\\ &= A\left(2-12x+9x^2\right)e^{-3x}\end{aligned}$$

$$\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y = 10e^{-3x}$$

$$\begin{aligned}10e^{-3x} &= Ae^{-3x}\left(2-12x+9x^2+6\left(2x-3x^2\right)+9x^2\right)\\ 10e^{-3x} &= Ae^{-3x}\left(2-12x+9x^2+12x-18x^2+9x^2\right)\\ 2Ae^{-3x} &= 10e^{-3x}\\ 2A &= 10\\ A &= 5\end{aligned}$$

b $y = Ax^2e^{-kx}$

$$\begin{aligned}\frac{dy}{dx} &= 2Axe^{-kx}-kAx^2e^{-kx}\\ &= Ae^{-kx}\left(2x-kx^2\right)\\ \frac{d^2y}{dx^2} &= -kAe^{-kx}\left(2x-kx^2\right)+Ae^{-kx}(2-2kx)\\ &= Ae^{-kx}\left(2-4kx+k^2x^2\right)\end{aligned}$$

$$\begin{aligned}\frac{d^2y}{dx^2}+2k\frac{dy}{dx}+k^2y &= Ae^{-kx}\left(2-4kx+k^2x^2+4kx-\ 2k^2x^2+k^2x^2\right)\\ &= 2Ae^{-kx}\\ &= Be^{-kx}\end{aligned}$$

$\Rightarrow B = 2A$ shown

24 $y = \sqrt{\dfrac{\pi}{x}}\sin(x)$

$$\begin{aligned} &= \sqrt{\pi}x^{-\frac{1}{2}}\sin(x)\\ \frac{dy}{dx} &= \sqrt{\pi}\left(x^{-\frac{1}{2}}\cos(x)-\frac{1}{2}x^{-\frac{3}{2}}\sin(x)\right)\\ \frac{d^2y}{dx^2} &= \sqrt{\pi}\left(-x^{-\frac{1}{2}}\sin(x)-\frac{1}{2}x^{-\frac{3}{2}}\cos(x)+\frac{3}{4}x^{-\frac{5}{2}}\sin(x)\right.\\ &\quad \left.-\frac{1}{2}x^{-\frac{3}{2}}\cos(x)\right)\\ &= \sqrt{\pi}\left(\left(\frac{3}{4}x^{-\frac{5}{2}}-x^{-\frac{1}{2}}\right)\ \sin(x)\right)-\sqrt{\pi}x^{-\frac{3}{2}}\cos(x)\end{aligned}$$

$$\begin{aligned}\text{LHS} &= 4x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+\left(4x^2-1\right)y\\ &= 4x^2\sqrt{\pi}\left(\frac{3}{4}x^{-\frac{5}{2}}-x^{-\frac{1}{2}}\right)\sin(x)-4x^2\sqrt{\pi}x^{-\frac{3}{2}}\cos(x)\\ &\quad +4x\sqrt{\pi}\left(x^{-\frac{1}{2}}\cos(x)-\frac{1}{2}x^{-\frac{3}{2}}\sin(x)\right)\\ &\quad +\ \left(4x^2-1\right)\sqrt{\pi}x^{-\frac{1}{2}}\sin(x)\end{aligned}$$

$$= \left(3\sqrt{\pi}x^{-\frac{1}{2}} - 4x^{\frac{3}{2}}\sqrt{\pi}\right)\sin(x) - 4x^{\frac{1}{2}}\sqrt{\pi}\cos(x)$$
$$+4\sqrt{\pi}x^{\frac{1}{2}}\cos(x) - 2\sqrt{\pi}x^{-\frac{1}{2}}\sin(x) + 4\sqrt{\pi}x^{\frac{3}{2}}\sin(x)$$
$$-\sqrt{\pi}x^{-\frac{1}{2}}\sin(x)$$
$$= 0$$
$$= \text{RHS}$$

25 a $n = 3, y = P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$

$$\frac{dy}{dx} = \frac{15}{2}x^2 - \frac{3}{2}$$
$$\frac{d^2y}{dx^2} = 15x$$
$$\text{LHS} = \left(1 - x^2\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 12y$$
$$= \left(1 - x^2\right) \times 15x - 2x\left(\frac{15}{2}x^2 - \frac{3}{2}\right) + 12 \times \frac{1}{2}\left(5x^3 - 3x\right)$$
$$= 15x - 15x^3 - 15x^3 + 3x + 30x^3 - 18x$$
$$= 0$$
$$= \text{RHS}$$

$n = 4 \quad y = P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$

$$\frac{dy}{dx} = \frac{1}{8}\left(140x^3 - 60x\right)$$
$$= \frac{35x^3}{2} - \frac{15x}{2}$$
$$\frac{d^2y}{dx^2} = \frac{1}{8}\left(420x^2 - 60\right)$$
$$= \frac{105}{2}x^2 - \frac{15}{2}$$
$$\text{LHS} = \left(1 - x^2\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 20y$$
$$= \left(1 - x^2\right) \times \frac{1}{2}\left(105x^2 - 15\right) - 2x \times \frac{1}{2}\left(35x^3 - 15x\right)$$
$$+ 20 \times \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$$
$$= \left(-\frac{105}{2}x^4 + 60x^2 - \frac{15}{2}\right) - \left(35x^4 - 15x^2\right)$$
$$+ \left(\frac{175}{2}x^4 - 75x^2 + \frac{15}{2}\right)$$
$$= 0$$
$$= \text{RHS}$$

b $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(\left(x^2 - 1\right)^n\right)$

$n = 2$

$$P_2(x) = \frac{1}{2^2 2!}\frac{d^2}{dx^2}\left(\left(x^2 - 1\right)^2\right)$$
$$= \frac{1}{8}\frac{d^2}{dx^2}\left(x^4 - 2x^2 + 1\right)$$
$$= \frac{1}{8}\frac{d}{dx}\left(4x^3 - 4x\right)$$
$$= \frac{1}{8}\left(12x^2 - 4\right)$$
$$= \frac{1}{2}\left(3x^2 - 1\right)$$

$n = 3$

$$P_3(x) = \frac{1}{2^3 3!}\frac{d^3}{dx^3}\left(x^2 - 1\right)^3$$
$$= \frac{1}{48}\frac{d^3}{dx^3}\left(x^6 - 3x^4 + 3x^2 - 1\right)$$
$$= \frac{1}{48}\frac{d^2}{dx^2}\left(6x^5 - 12x^3 + 6x\right)$$
$$= \frac{1}{48}\frac{d}{dx}\left(30x^4 - 36x^2 + 6\right)$$
$$= \frac{1}{48}\left(120x^3 - 72x\right)$$
$$= \frac{1}{2}\left(5x^3 - 3x\right)$$

c $P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$

$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$

$n = 2$

$$\int_{-1}^{1}\left(P_2(x)\right)^2 dx = \int_{-1}^{1}\left(\frac{1}{2}\left(3x^2 - 1\right)\right)^2 dx$$
$$= \int_{-1}^{1}\frac{1}{4}\left(9x^4 - 6x^2 + 1\right)dx$$
$$= \frac{1}{4}\left[\frac{9x^5}{5} - \frac{6x^3}{3} + x\right]_{-1}^{1}$$
$$= \frac{1}{4}\left(\left(\frac{9}{5} - 2 + 1\right) - \left(-\frac{9}{5} + 2 - 1\right)\right)$$
$$= \frac{2}{5}$$

$n = 2 \quad \frac{2}{2n + 1} = \frac{2}{5}$ shown

$n = 3$

$$\int_{-1}^{1}\left(P_3(x)\right)^2 dx = \int_{-1}^{1}\left(\frac{1}{2}\left(5x^3 - 3x\right)\right)^2 dx$$
$$= \int_{-1}^{1}\frac{1}{4}\left(25x^6 - 30x^4 + 9x^2\right)dx$$
$$= \frac{1}{4}\left[\frac{25x^7}{7} - \frac{30x^5}{5} + 3x^3\right]_{-1}^{1}$$
$$= \frac{1}{4}\left(\left(\frac{25}{7} - 6 + 3\right) - \left(-\frac{25}{7} + 6 - 3\right)\right)$$
$$= \frac{2}{7}$$

$n = 3 \quad \frac{2}{2n + 1} = \frac{2}{7}$ shown

$$\int_{-1}^{1}P_2(x)P_3(x)\,dx = \int_{-1}^{1}\frac{1}{2}\left(3x^2 - 1\right) \times \frac{1}{2}\left(5x^3 - 3x\right)dx$$
$$= \frac{1}{4}\int_{-1}^{1}\left(15x^5 - 14x^3 + 3x\right)dx$$
$$= \frac{1}{4}\left[\frac{15x^6}{6} - \frac{7}{2}x^4 + \frac{3}{2}x^2\right]_{-1}^{1}$$
$$= \frac{1}{4}\left(\left(\frac{15}{6} - \frac{7}{2} + \frac{3}{2}\right) - \left(\frac{15}{6} - \frac{7}{2} + \frac{3}{2}\right)\right)$$
$$= 0 \quad \text{shown}$$

8.2 Exam questions

1
$$y = x\sin(x)$$
$$\frac{dy}{dx} = x\cos(x) + \sin(x)$$
$$\frac{d^2y}{dx^2} = \cos(x) - x\sin(x) + \cos(x)$$
$$\frac{d^2y}{dx^2} = 2\cos(x) - x\sin(x)$$
$$\frac{d^2y}{dx^2} + x\sin(x) = 2\cos(x)$$
$$\frac{d^2y}{dx^2} + y = 2\cos(x)$$

The correct answer is **E**.

2
$$y = e^{mx}$$
$$\frac{dy}{dx} = me^{mx}$$
$$\frac{d^2y}{dx^2} = m^2e^{mx} \qquad \text{[1 mark]}$$
$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$$
$$m^2e^{mx} + 4me^{mx} + 13e^{mx} = 0$$
$$e^{mx}\left(m^2 + 4m + 13\right) = 0$$
$$e^{mx} \neq 0$$
$$m^2 + 4m + 13 = 0 \qquad \text{[1 mark]}$$
$$m^2 + 4m + 4 = -13 + 4$$
$$(m+2)^2 = -9$$
$$(m+2)^2 = 9i^2$$
$$m + 2 = \pm 3i$$
$$m = -2 \pm 3i \qquad \text{[1 mark]}$$

3
$$y = \tan^{-1}(2x)$$
$$\frac{dy}{dx} = \frac{2}{1+4x^2} \qquad \text{[1 mark]}$$
$$= 2\left(1+4x^2\right)^{-1}$$
$$\frac{d^2y}{dx^2} = -1 \times 2 \times 8x\left(1+4x^2\right)^{-2}$$
$$= \frac{-16x}{\left(1+4x^2\right)^2} \qquad \text{[1 mark]}$$

$$\text{LHS} = \left(1+4x^2\right)\frac{d^2y}{dx^2} + 8x\frac{dy}{dx}$$
$$= \left(1+4x^2\right)\frac{-16x}{\left(1+4x^2\right)^2} + 8x \times \frac{2}{1+4x^2} \qquad \text{[1 mark]}$$
$$= \frac{-16x}{1+4x^2} + \frac{16x}{1+4x^2}$$
$$= 0$$
$$= \text{RHS} \qquad \text{[1 mark]}$$

8.3 Solving Type 1 differential equations, $\frac{dy}{dx} = f(x)$

8.3 Exercise

1 a
$$\frac{dy}{dx} + 12x^3 = 0$$
$$\frac{dy}{dx} = -12x^3$$
$$y = \int -12x^3\,dx$$
$$y = c - 3x^4$$

b
$$\frac{dy}{dx} + 6x = 0, \quad y(2) = 1$$
$$\frac{dy}{dx} = -6x$$
$$y = \int -6x\,dx$$
$$y = -3x^2 + c$$
$$x = 2 \qquad y = 1$$
$$1 = -12 + c$$
$$c = 13$$
$$y = 13 - 3x^2$$

2 a
$$\frac{dy}{dx} + 12\cos(2x) = 0,$$
$$\frac{dy}{dx} = -12\cos(2x)$$
$$y = \int -12\cos(2x)\,dx$$
$$y = c - 6\sin(2x)$$

b
$$\frac{dy}{dx} + 6\sin(3x) = 0, \quad y(0) = 0$$
$$\frac{dy}{dx} = -6\sin(3x)$$
$$y = \int -6\sin(3x)\,dx$$
$$y = 2\cos(3x) + c$$
$$x = 0 \qquad y = 0$$
$$0 = 2\cos(0) + c$$
$$c = -2$$
$$y = 2\cos(3x) - 2$$
$$= 2\left(\cos(3x) - 1\right)$$

3 a
$$\frac{dy}{dx} - 4x = 3$$
$$\frac{dy}{dx} = 4x + 3$$
$$y = \int (4x+3)\,dx$$
$$y = 2x^2 + 3x + c$$

b
$$\frac{dy}{dx} - (3x-5)(x+4) = 0$$
$$\frac{dy}{dx} = (3x-5)(x+4)$$
$$= 3x^2 + 7x - 20$$
$$y = \int \left(3x^2 + 7x - 20\right)dx$$
$$= x^3 + \frac{7}{2}x^2 - 20x + c$$

4
$$(5-4x)^2\frac{dy}{dx} + 1 = 0, \qquad y(1) = 2$$
$$(5-4x)^2\frac{dy}{dx} = -1,$$
$$\frac{dy}{dx} = \frac{-1}{(5-4x)^2}$$
$$y = \frac{-1}{4(5-4x)} + c$$

$x = 1 \quad y = 2$

$2 = \dfrac{-1}{4} + c$

$c = \dfrac{9}{4}$

$$\begin{aligned} y &= \frac{-1}{4\,(5-4x)} + \frac{9}{4} \\ &= \frac{-1 + 9\,(5-4x)}{4\,(5-4x)} \\ &= \frac{44 - 36x}{4\,(5-4x)} \\ &= \frac{4\,(11 - 9x)}{4\,(5-4x)} \\ &= \frac{9x - 11}{4x - 5}, \quad x \in R\backslash\left\{\frac{5}{4}\right\} \end{aligned}$$

5 $(7-4x)\,\dfrac{dy}{dx} + 2 = 0, \quad y(2) = 3$

$$\begin{aligned} (7-4x)\,\frac{dy}{dx} &= -2, \\ \frac{dy}{dx} &= \frac{-2}{(7-4x)} \end{aligned}$$

$$\begin{aligned} y &= \int \frac{-2}{7-4x}dx \\ &= \frac{1}{2}\log_e(|7-4x|) + c \end{aligned}$$

$x = 2 \qquad y = 3$

$3 = \dfrac{1}{2}\log_e(|-1|) + c$

$c = 3$

$y = 3 + \dfrac{1}{2}\log_e(|7-4x|), \quad x \in R\backslash\left\{\dfrac{7}{4}\right\}$

6 $\sqrt{2x-5}\dfrac{dy}{dx} + 1 = 0, \quad y(3) = 0$

$$\begin{aligned} \sqrt{2x-5}\frac{dy}{dx} &= -1, \\ \frac{dy}{dx} &= \frac{-1}{\sqrt{2x-5}} \end{aligned}$$

$$\begin{aligned} y &= \int \frac{-1}{\sqrt{2x-5}}dx \\ &= -\sqrt{2x-5} + c \end{aligned}$$

$x = 3, \qquad y = 0$

$0 = -\sqrt{1} + c$

$c = 1$

$y = 1 - \sqrt{2x-5}, \quad x > \dfrac{5}{2}$

7 $\sqrt{x}\dfrac{dy}{dx} + 2 = 0, \qquad y(4) = 3$

$$\begin{aligned} \sqrt{x}\frac{dy}{dx} &= -2, \\ \frac{dy}{dx} &= \frac{-2}{\sqrt{x}} \end{aligned}$$

$$\begin{aligned} y &= -2\int x^{-\frac{1}{2}}dx \\ &= -4x^{\frac{1}{2}} + c \\ &= -4\sqrt{x} + c \end{aligned}$$

$x = 4 \qquad y = 3$

$3 = -4\sqrt{4} + c$

$c = 11$

$y = 11 - 4\sqrt{x}, \quad x > 0$

8 $\sqrt{64 - x^2}\dfrac{dy}{dx} - 6 = 0, \quad y(4) = 0$

$$\begin{aligned} \sqrt{64 - x^2}\frac{dy}{dx} &= 6, \\ \frac{dy}{dx} &= \frac{6}{\sqrt{64 - x^2}} \end{aligned}$$

$$\begin{aligned} y &= \int \left(\frac{6}{\sqrt{64 - x^2}}\right)dx \\ &= 6\sin^{-1}\left(\frac{x}{8}\right) + c \end{aligned}$$

$x = 4 \quad y = 0$

$0 = 6\sin^{-1}\left(\dfrac{1}{2}\right) + c$

$c = -6 \times \dfrac{\pi}{6} = -\pi$

$y = 6\ \sin^{-1}\left(\dfrac{x}{8}\right) - \pi, \ x \in (-8,\ 8)$

9 $\left(16 + x^2\right)\dfrac{dy}{dx} + 4 = 0, \quad y(4) = \dfrac{\pi}{4}$

$$\begin{aligned} (16 + x^2)\,\frac{dy}{dx} &= -4, \\ \frac{dy}{dx} &= \frac{-4}{16 + x^2} \end{aligned}$$

$$\begin{aligned} y &= \int \frac{-4}{16 + x^2}dx \\ &= -\tan^{-1}\left(\frac{x}{4}\right) + c \end{aligned}$$

$x = 4 \qquad y = \dfrac{\pi}{4}$

$$\begin{aligned} \frac{\pi}{4} &= -\tan^{-1}(1) + c \\ &= \frac{-\pi}{4} + c \end{aligned}$$

$c = \dfrac{\pi}{2}$

$y = \dfrac{\pi}{2} - \tan^{-1}\left(\dfrac{x}{4}\right), \quad x \in R$

10 **a** $3x\dfrac{dy}{dx} - 2x^2 = 5, \quad y\,(1) = 3$

$$\begin{aligned} 3x\frac{dy}{dx} &= 5 + 2x^2 \\ \frac{dy}{dx} &= \frac{5 + 2x^2}{3x} \end{aligned}$$

$y = \dfrac{1}{3}\displaystyle\int \left(\frac{5}{x} + 2x\right)\,dx$

$y = \dfrac{1}{3}\left(5\log_e(|x|) + x^2\right) + c$

When $x = 1,\ y = 3$

$3 = \dfrac{1}{3}\left(5\log_e(|1|) + 1\right) + c$

$c = 3 - \dfrac{1}{3} = \dfrac{8}{3}$

$y = \dfrac{1}{3}\left[\left(5\log_e(|x|) + x^2\right) + 8\right], \quad x \in R\backslash\{0\}$

b $\dfrac{dy}{dx} = 6\left(e^{-3x} + e^{3x}\right), \quad y(0) = 0$

$y = \displaystyle\int \left(6e^{-3x} + 6e^{3x}\right)\, dx$

$y = -2e^{-3x} + 2e^{3x} + c$

When $x = 0, \quad y = 0$

$0 = -2e^{0} + 2e^{0} + c$

$c = 0$

$y = 2\left(e^{3x} - e^{-3x}\right), \ x \in R$

11 a $\dfrac{dy}{dx} - 4\sin(2x) = 0, \quad y(0) = 2$

$\dfrac{dy}{dx} = 4\sin(2x)$

$y = \displaystyle\int 4\sin(2x)\, dx$

$y = -2\cos(2x) + c$

When $x = 0, \quad y = 2$

$2 = -2\cos(0) + c$

$c = 4$

$y = 4 - 2\cos(2x), \quad x \in R$

b $\dfrac{dy}{dx} + 6\cos(3x) = 0, \quad y\left(\dfrac{\pi}{2}\right) = 5$

$\dfrac{dy}{dx} = -6\cos(3x)$

$y = \displaystyle\int -6\cos(3x)\, dx$

$y = -2\sin(3x) + c$

When $x = \dfrac{\pi}{2}, \quad y = 5$

$5 = -2\sin\left(\dfrac{3\pi}{2}\right) + c$

$5 = 2 + c$

$c = 3$

$y = 3 - 2\sin(3x), \quad x \in R$

12 a $\dfrac{dy}{dx} - 8\sin^2(2x) = 0, \quad y(0) = 0$

$\dfrac{dy}{dx} = 8\sin^2(2x)$

$y = \displaystyle\int 8\sin^2(2x)\, dx$

$y = 4\displaystyle\int (1 - \cos(4x))\, dx$

$y = 4x - \sin(4x) + c$

When $x = 0, \quad y = 0$

$0 = 0 - \sin(0) + c$

$c = 0$

$y = 4x - \sin(4x), \ x \in R$

b $\dfrac{dy}{dx} - 12\cos^2(3x) = 0, \quad y(0) = 0$

$\dfrac{dy}{dx} = 12\cos^2(3x)$

$y = \displaystyle\int 12\cos^2(3x)\, dx$

$y = 6\displaystyle\int (1 + \cos(6x))\, dx$

$y = 6x + \sin(6x) + c$

When $x = 0, \quad y = 0$

$0 = 0 + \sin(0) + c$

$c = 0$

$y = 6x + \sin(6x), \ x \in R$

13 a $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{4x + 9}}, \quad y(0) = 0$

$y = \displaystyle\int \left(\dfrac{1}{\sqrt{4x + 9}}\right) dx$

Let $u = 4x + 9 \quad \dfrac{du}{dx} = 4$

$y = \dfrac{1}{4}\displaystyle\int u^{-\frac{1}{2}} du$

$y = \dfrac{1}{2}u^{\frac{1}{2}} + c$

$y = \dfrac{1}{2}\sqrt{4x + 9} + c$

When $x = 0, \quad y = 0$

$0 = \dfrac{1}{2}\sqrt{9} + c$

$c = -\dfrac{3}{2}$

$y = \dfrac{1}{2}\left(\sqrt{4x + 9} - 3\right)$ for $x > -\dfrac{9}{4}$

b $\dfrac{dy}{dx} + \dfrac{1}{3 - 2x} = 0, \quad y(2) = 1$

$\dfrac{dy}{dx} = \dfrac{-1}{3 - 2x}$

$y = \displaystyle\int \dfrac{1}{2x - 3} dx$

$y = \dfrac{1}{2}\log_e(|2x - 3|) + c$

When $x = 2, \ y = 1$

$1 = \dfrac{1}{2}\log_e(1) + c$

$c = 1$

$y = 1 + \dfrac{1}{2}\log_e(|2x - 3|), \quad x \in R\backslash\left\{\dfrac{3}{2}\right\}$

14 a $\dfrac{dy}{dx} = \dfrac{1}{(3x - 5)^2}, \quad y(2) = 3$

$y = \displaystyle\int \left(\dfrac{1}{(3x - 5)^2}\right) dx$

$y = \dfrac{-1}{3(3x - 5)} + c$

When $x = 2, \ y = 3$

$3 = \dfrac{-1}{3} + c$

$c = \dfrac{10}{3}$

$y = \dfrac{1}{3}\left(10 - \dfrac{1}{3x - 5}\right)$

$= \dfrac{1}{3}\left(\dfrac{10(3x - 5) - 1}{3x - 5}\right)$

$= \dfrac{1}{3}\left(\dfrac{30x - 51}{3x - 5}\right)$

$y = \dfrac{10x - 17}{3x - 5}, \quad x \in R\backslash\left\{\dfrac{5}{3}\right\}$

b $\dfrac{dy}{dx} = \dfrac{8}{7 - 4x}, \quad y(2) = 5$

$y = \displaystyle\int \left(\dfrac{8}{7 - 4x}\right) dx$

$y = -2\log_e(|7 - 4x|) + c$

When $x = 2,\ y = 5$

$5 = -2\log_e(1) + c$

$c = 5$

$y = 5 - 2\log_e(|7 - 4x|)\, x \in R\backslash\left\{\frac{7}{4}\right\}$

15 a $\left(x^2 + 9\right)\frac{dy}{dx} - 3x = 0, \quad y(0) = 0$

$\left(x^2 + 9\right)\frac{dy}{dx} = 3x$

$\frac{dy}{dx} = \frac{3x}{x^2 + 9}$

$y = \frac{3}{2}\int\frac{2x}{x^2 + 9}dx$

$y = \frac{3}{2}\log_e\left(x^2 + 9\right) + c$

When $x = 0,\ y = 0$

$0 = \frac{3}{2}\log_e(9) + c$

$c = -\frac{3}{2}\log_e(9)$

$y = \frac{3}{2}\log_e\left(\frac{x^2 + 9}{9}\right), \quad x \in R$

b $\sqrt{x^2 + 4}\frac{dy}{dx} + x = 0, \quad y(0) = 0$

$\sqrt{x^2 + 4}\frac{dy}{dx} = -x$

$\frac{dy}{dx} = \frac{-x}{\sqrt{x^2 + 4}}$

$y = \int\frac{-x}{\sqrt{x^2 + 4}}dx$

Let $u = x^2 + 4 \qquad \frac{du}{dx} = 2x$

$y = -\frac{1}{2}\int\left(u^{-\frac{1}{2}}\right)du$

$y = -u^{\frac{1}{2}} + c$

$y = -\sqrt{x^2 + 4} + c$

When $x = 0,\ y = 0$

$0 = -\sqrt{4} + c$

$c = 2$

$y = 2 - \sqrt{x^2 + 4}, \quad x \in R$

16 a $\left(x^2 + 6x + 13\right)\frac{dy}{dx} - x = 3, \quad y(0) = 0$

$\left(x^2 + 6x + 13\right)\frac{dy}{dx} = 3 + x$

$\frac{dy}{dx} = \frac{x + 3}{x^2 + 6x + 13}$

$y = \frac{1}{2}\int\frac{2x + 6}{x^2 + 6x + 13}dx$

$y = \frac{1}{2}\log_e\left(x^2 + 6x + 13\right) + c$

When $x = 0,\ y = 0$

$0 = \frac{1}{2}\log_e(13) + c$

$c = -\frac{1}{2}\log_e(13)$

$y = \frac{1}{2}\log_e\left(\frac{x^2 + 6x + 13}{13}\right), \quad x \in R$

b $\left(x^2 - 4x + 9\right)\frac{dy}{dx} + x = 2, \quad y(0) = 0$

$\left(x^2 - 4x + 9\right)\frac{dy}{dx} = 2 - x$

$\frac{dy}{dx} = \frac{2 - x}{x^2 - 4x + 9}$

$y = -\frac{1}{2}\int\frac{2x - 4}{x^2 - 4x + 9}dx$

$y = -\frac{1}{2}\log_e\left(x^2 - 4x + 9\right) + c$

When $x = 0,\ y = 0$

$0 = -\frac{1}{2}\log_e(9) + c$

$c = \frac{1}{2}\log_e(9)$

$y = -\frac{1}{2}\log_e\left(x^2 - 4x + 9\right) + \frac{1}{2}\log_e(9)$

$y = \frac{1}{2}\log_e\left(\frac{9}{x^2 - 4x + 9}\right)$

$y = \log_e\left(\frac{3}{\sqrt{x^2 - 4x + 9}}\right), x \in R$

17 $\sec(2x)\frac{dy}{dx} + \sin^3(2x) = 0, \quad y(0) = 0$

$\sec(2x)\frac{dy}{dx} = -\sin^3(2x)$

$\frac{dy}{dx} = -\cos(2x)\sin^3(2x)$

$y = \int -\cos(2x)\sin^3(2x)\,dx$

Let $u = \sin(2x) \quad \frac{du}{dx} = 2\cos(2x)$

$y = -\frac{1}{2}\int u^3 du$

$= -\frac{1}{8}u^4 + c$

$y = -\frac{1}{8}\sin^4(2x) + c$

When $x = 0, \quad y = 0$

$0 = -\frac{1}{8}\sin^4(0) + c$

$c = 0$

$y = -\frac{1}{8}\sin^4(2x), \ x \in R$

18 a $\frac{dy}{dx} + \log_e(2x) = 4, \quad y\left(\frac{1}{2}\right) = 1$

$\frac{dy}{dx} = 4 - \log_e(2x)$

$y = \int\left(4 - \log_e(2x)\right)\,dx$

$= \int 4dx - \int\log_e(2x)\,dx$

Let $u = \log_e(2x) \quad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{x}, \quad v = x$

$$y = 4x - \left(x\log_e(2x) - \int 1 \cdot dx\right)$$

$$= 4x - x\log_e(2x) + x + c$$

When $x = \frac{1}{2}, \quad y = 1$

$$1 = 2 - \frac{1}{2}\log_e(1) + \frac{1}{2} + c$$

$$c = -\frac{3}{2}$$

$$y = 5x - x\log_e(2x) - \frac{3}{2}, \quad x > 0$$

b $e^x\frac{dy}{dx} + x = 5, \quad y(0) = 0$

$$e^x\frac{dy}{dx} = 5 - x$$

$$\frac{dy}{dx} = 5e^{-x} - xe^{-x}$$

$$y = \int 5e^{-x}dx - \int xe^{-x}dx$$

Let $u = x \quad \frac{dv}{dx} = e^{-x}$

$$\frac{du}{dx} = 1, \qquad v = -e^{-x}$$

$$y = -5e^{-x} - \left(-xe^{-x} + \int e^{-x}dx\right)$$

$$y = -5e^{-x} + xe^{-x} + e^{-x} + c$$

$$y = -4e^{-x} + xe^{-x} + c$$

When $x = 0, \ y = 0$

$$0 = -4 + c$$

$$c = 4$$

$$y = (x - 4)e^{-x} + 4, \ x \in R$$

19 a $\frac{dy}{dx} = e^{\frac{1}{x}} \, y(1) = 3$

$$y = \int_0^x e^{\frac{1}{t}}dt + c$$

when $x = 1$ $y = 3$

$$3 = \int_0^1 e^{\frac{1}{t}}dt + c$$

$$c = 3 - \int_0^1 e^{\frac{1}{t}}dt$$

$$y = \int_0^x e^{\frac{1}{t}}dt + 3 - \int_0^1 e^{\frac{1}{t}}dt$$

$$= \int_0^x e^{\frac{1}{t}}dt + \int_1^0 e^{\frac{1}{t}}dt + 3$$

$$= \int_1^x e^{\frac{1}{t}}dt + 3$$

b When $x = 2$ $y = \int_1^2 e^{\frac{1}{t}}dt + 3 = 5.020$

20 a $\frac{dy}{dx} = \sin^{-1}\left(x^2\right) \quad y(0 \cdot 1) = 1$

$y = \int_0^x \sin^{-1}\left(u^2\right)du + c \quad$ when $x = 0.1$ $y = 1$

$$1 = \int_0^{0.1} \sin^{-1}\left(u^2\right)du + c$$

$$c = 1 - \int_0^{0.1} \sin^{-1}\left(u^2\right)du$$

$$y = \int_0^x \sin^{-1}\left(u^2\right)du + 1 - \int_0^{0.1} \sin^{-1}\left(u^2\right)du$$

$$= \int_0^x \sin^{-1}\left(u^2\right)du + \int_{0.1}^0 \sin^{-1}\left(u^2\right)du + 1$$

$$y = \int_{0.1}^x \sin^{-1}\left(u^2\right)du + 1$$

b When $x = 0.5$ $y = \int_{0.1}^{0.5} \sin^{-1}\left(u^2\right)du + 1 = 1.042$

21 a $\left(4x^2 + 9\right)\frac{dy}{dx} + 2x = 3, \quad y(0) = 0$

$$(4x^2 + 9)\frac{dy}{dx} = 3 - 2x$$

$$\frac{dy}{dx} = \frac{3 - 2x}{4x^2 + 9}$$

$$y = \int \frac{3}{4x^2 + 9}dx - \int \frac{2x}{4x^2 + 9}dx$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{2x}{3}\right) - \frac{1}{4}\log_e\left(4x^2 + 9\right) + c$$

When $x = 0, \ y = 0$

$$0 = \frac{1}{2}\tan^{-1}(0) - \frac{1}{4}\log_e(9) + c$$

$$c = \frac{1}{4}\log_e(9)$$

$$y = \frac{1}{2}\tan^{-1}\left(\frac{2x}{3}\right) - \frac{1}{4}\log_e\left(4x^2 + 9\right) + \frac{1}{4}\log_e(9)$$

$$y = \frac{1}{2}\tan^{-1}\left(\frac{2x}{3}\right) + \frac{1}{4}\log_e\left(\frac{9}{4x^2 + 9}\right) \quad x \in R$$

b $\sqrt{9 - 4x^2}\frac{dy}{dx} + 2x = 3, \quad y(0) = 0$

$$\sqrt{9 - 4x^2}\frac{dy}{dx} = 3 - 2x$$

$$\frac{dy}{dx} = \frac{3 - 2x}{\sqrt{9 - 4x^2}}$$

$$y = \int \frac{3}{\sqrt{9 - 4x^2}}dx - \frac{1}{4}\int \frac{8x}{\sqrt{9 - 4x^2}}dx$$

$$= \frac{3}{2}\sin^{-1}\left(\frac{2x}{3}\right) + \frac{1}{2}\sqrt{9 - 4x^2} + c$$

When $x = 0, \ y = 0$

$$0 = \frac{3}{2}\sin^{-1}(0) + \frac{1}{2}\sqrt{9} + c$$

$$c = -\frac{3}{2}$$

$$y = \frac{3}{2}\sin^{-1}\left(\frac{2x}{3}\right) + \frac{1}{2}\sqrt{9 - 4x^2} - \frac{3}{2}$$

$$y = \frac{3}{2}\left(\sin^{-1}\left(\frac{2x}{3}\right) - 1\right) + \frac{\sqrt{9 - 4x^2}}{2} \quad \text{for } |x| < \frac{3}{2}$$

22 a i $\sqrt{a^2 - x^2}\frac{dy}{dx} + b = 0, \quad y(0) = 0$

$$\sqrt{a^2 - x^2}\frac{dy}{dx} = -b$$

$$\frac{dy}{dx} = \frac{-b}{\sqrt{a^2 - x^2}}$$

$$y = -\int \frac{b}{\sqrt{a^2 - x^2}}dx$$

$$y = -b\sin^{-1}\left(\frac{x}{a}\right) + c$$

When $x = 0,\ y = 0$

$$0 = -b\sin^{-1}(0) + c$$

$$c = 0$$

$$y = -b\sin^{-1}\left(\frac{x}{a}\right) \qquad |x| < a$$

ii $\left(a^2 - x^2\right)\dfrac{dy}{dx} + b = 0, \quad y(0) = 0$

$$\left(a^2 - x^2\right)\frac{dy}{dx} = -b$$

$$\frac{dy}{dx} = \frac{-b}{a^2 - x^2}$$

$$y = \int \frac{-b}{a^2 - x^2}dx$$

Partial fractions

$$\frac{-b}{a^2 - x^2} = \frac{A}{a - x} + \frac{B}{a + x} = \frac{A(a + x) + B(a - x)}{a^2 - x^2}$$

$$= \frac{x(A - B) + aA + Ba}{a^2 - x^2}$$

$$\Rightarrow A - B = 0$$

$$\Rightarrow A = B$$

$$aA + Ba = a(A + B) = -b$$

$$\Rightarrow A = \frac{-b}{2a}$$

$$y = \frac{-b}{2a}\int\left(\frac{1}{a - x} + \frac{1}{a + x}\right)dx$$

$$= \frac{-b}{2a}\left(-\log_e(|a - x|) + \log_e(|a + x|)\right) + c$$

When $x = 0,\ y = 0$

$$0 = \frac{-b}{2a}\left(-\log_e(|a|) + \log_e(|a|)\right) + c$$

$$c = 0$$

$$y = \frac{b}{2a}\log_e\left(\left|\frac{a - x}{a + x}\right|\right), \qquad |x| < a$$

iii $(a + bx)^2\dfrac{dy}{dx} + 1 = 0, \quad y(0) = 0$

$$(a + bx)^2\frac{dy}{dx} = -1$$

$$\frac{dy}{dx} = \frac{-1}{(a + bx)^2}$$

$$y = \int\frac{-1}{(a + bx)^2}dx$$

$$= \frac{1}{b(a + bx)} + c$$

When $x = 0,\ y = 0$

$$0 = \frac{1}{ab} + c$$

$$c = -\frac{1}{ab}$$

$$y = \frac{1}{b(a + bx)} - \frac{1}{ab}$$

$$= \frac{a - (a + bx)}{ab(a + bx)}$$

$$= \frac{-x}{a(a + bx)}, \qquad \text{for } x \neq -\frac{a}{b}$$

b $e^{2x}\dfrac{dy}{dx} + \cos(3x) = 0, \quad y(0) = 0$

$$e^{2x}\frac{dy}{dx} = -\cos(3x)$$

$$y = \int -e^{-2x}\cos(3x)\,dx$$

$$= \frac{1}{13}e^{-2x}(2\cos(3x) - 3\sin(3x)) + c$$

When $x = 0,\ y = 0$

$$0 = \frac{1}{13} \times 1(2 - 0) + c$$

$$c = -\frac{2}{13}$$

$$y = \frac{e^{-2x}}{13}(2\cos(3x) - 3\sin(3x)) - \frac{2}{13}$$

8.3 Exam questions

1 $e^{2x}\dfrac{dy}{dx} + 6 = 2e^{4x}$

$$e^{2x}\frac{dy}{dx} = 2e^{4x} - 6$$

$$\frac{dy}{dx} = \frac{2e^{4x} - 6}{e^{2x}} \qquad \text{[1 mark]}$$

$$y = \int\left(2e^{2x} - 6e^{-2x}\right)dx$$

$$y = e^{2x} + 3e^{-2x} + c \qquad \text{[1 mark]}$$

2 $\sqrt{x^2 + 9}\dfrac{dy}{dx} - x = 0$

$$\sqrt{x^2 + 9}\frac{dy}{dx} = x$$

$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 9}} \qquad \text{[1 mark]}$$

$$y = \int\frac{x}{\sqrt{x^2 + 9}}$$

$$y = \sqrt{x^2 + 9} + c \qquad \text{[1 mark]}$$

3 $\operatorname{cosec}(3x)\dfrac{dy}{dx} + 9\cos^2(3x) = 0, \qquad y(0) = 0$

$$\operatorname{cosec}(3x)\frac{dy}{dx} = -9\cos^2(3x)$$

$$\frac{dy}{dx} = -9\sin(3x)\cos^2(3x)$$

$$y = \int -9\sin(3x)\cos^2(3x)\,dx \qquad \text{[1 mark]}$$

Let $u = \cos(3x) \qquad \dfrac{du}{dx} = -3\sin(3x)$

$$y = 3\int u^2du$$

$$= u^3 + c$$

$$y = \cos^3(3x) + c \qquad \text{[1 mark]}$$

When $x = 0, \quad y = 0$

$$0 = \cos^3(0) + c$$

$$c = -1$$

$$y = \cos^3(3x) - 1, \quad x \in R\backslash\frac{n\pi}{3}, n \in Z \qquad \text{[1 mark]}$$

8.4 Solving Type 2 differential equations, $\frac{dy}{dx}=f(y)$

8.4 Exercise

1 $\sqrt{y}\frac{dy}{dx}+4=0$

$$\sqrt{y}\frac{dy}{dx}=-4$$
$$\frac{dy}{dx}=\frac{-4}{\sqrt{y}}$$
$$\frac{dx}{dy}=\frac{-\sqrt{y}}{4}$$
$$x=\frac{-1}{4}\int\sqrt{y}dy$$
$$x=\frac{-1}{4}\int y^{\frac{1}{2}}dy$$
$$x=\frac{-1}{4}\left(\frac{2}{3}y^{\frac{3}{2}}+c\right)$$
$$-4x=\frac{2}{3}y^{\frac{3}{2}}+c$$
$$\frac{2}{3}y^{\frac{3}{2}}=-4x-c$$
$$=A-4x \quad \text{Let}\, A=-c$$
$$y^{\frac{3}{2}}=\frac{3A}{2}-6x$$
$$=B-6x \quad \text{Let}\, B=\frac{3A}{2}$$
$$y=\sqrt[3]{(B-6x)^2}$$

2 $\frac{dy}{dx}-\tan(2y)=0$

$$\frac{dy}{dx}=\tan(2y)$$
$$\frac{dx}{dy}=\frac{1}{\tan(2y)}$$
$$=\frac{\cos(2y)}{\sin(2y)}$$
$$x=\int\frac{\cos(2y)}{\sin(2y)}dy$$
$$=\frac{1}{2}\log_e(|\sin(2y)|)+c$$
$$2x-A=\log_e|\sin(2y)|$$
$$\sin(2y)=\pm e^{2x-A}=e^{2x}\cdot B$$
$$B=e^{-A}$$
$$2y=\pm\sin^{-1}\left(Be^{2x}\right)$$
$$y=\frac{1}{2}\sin^{-1}\left(Be^{2x}\right)$$

3 **a** $\frac{dy}{dx}=\frac{y^2}{4}$

$$\frac{dx}{dy}=\frac{4}{y^2}$$
$$=4y^{-2}$$
$$x=\int 4y^{-2}dy$$
$$=-4y^{-1}+c$$
$$=\frac{-4}{y}+c$$
$$\frac{4}{y}=c-x$$
$$y=\frac{4}{c-x}$$

b $\frac{dy}{dx}=y+4$

$$\frac{dx}{dy}=\frac{1}{y+4}$$
$$x=\int\frac{1}{y+4}dy$$
$$x=\log_e|y+4|+c$$
$$\log_e|y+4|=x-c$$
$$y+4=\pm e^{x-c}=\pm e^x\cdot e^{-c}$$
$$y+4=Ae^x,\ \text{Let}\, A=\pm e^{-c}$$
$$y=Ae^x-4$$

4 **a** $\frac{dy}{dx}=\frac{y}{4}$

$$\frac{dx}{dy}=\frac{4}{y}$$
$$x=4\int\frac{1}{y}dy$$
$$x=4\log_e|y|+c$$
$$4\log_e|y|=x-c$$
$$\log_e|y|=\frac{x}{4}-\frac{c}{4}$$
$$y=\pm e^{\frac{x}{4}-\frac{c}{4}}=\pm e^{\frac{x}{4}}\cdot e^{\frac{c}{4}},\ \text{Let}\, A=\pm e^{-\frac{c}{4}}$$
$$y=Ae^{\frac{x}{4}}$$

b $\frac{dy}{dx}=\frac{4}{y^2}$

$$\frac{dx}{dy}=\frac{y^2}{4}$$
$$x=\frac{1}{4}\int y^2dy$$
$$x=\frac{1}{12}y^3+c$$
$$y^3=12x-12c$$
$$=12x+A, \quad \text{Let}\, A=-12c$$
$$y=\sqrt[3]{12x+A}$$

5 $\frac{dy}{dx}+(5-4y)^2=0, \quad y(1)=2$

$$\frac{dy}{dx}=-(5-4y)^2$$
$$\frac{dx}{dy}=\frac{-1}{(5-4y)^2}$$
$$x=\int\frac{-1}{(5-4y)^2}dy$$
$$x=\frac{-1}{4(5-4y)}+c$$
$$x=1 \quad y=2$$
$$1=\frac{1}{12}+c$$
$$c=\frac{11}{12}$$

$$x = \frac{-1}{4\,(5-4y)} + \frac{11}{12}$$

$$\frac{1}{4\,(5-4y)} = \frac{11}{12} - x$$

$$\frac{1}{4\,(5-4y)} = \frac{11-12x}{12}$$

$$5 - 4y = \frac{3}{11-12x}$$

$$4y = 5 - \frac{3}{11-12x}$$

$$4y = \frac{5\,(11-12x) - 3}{11-12x}$$

$$4y = \frac{52x - 60x}{11-12x}$$

$$4y = \frac{4\,(13-15x)}{11-12x}$$

$$y = \frac{15x-13}{12x-11}, \qquad x \neq \frac{11}{12}$$

6 $\dfrac{dy}{dx} + 4y - 7 = 0, \quad y\,(0) = 3$

$$\frac{dy}{dx} = 7 - 4y$$

$$\frac{dx}{dy} = \frac{1}{7-4y}$$

$$x = \int \frac{1}{7-4y}dy$$

$$x = \frac{-1}{4}\log_e(|7-4y|) + c$$

$x = 0, \quad y = 3$

$$0 = \frac{-1}{4}\log_e(|-5|) + c$$

$$c = \frac{1}{4}\log_e(5)$$

$$x = \frac{1}{4}\log_e(5) - \frac{1}{4}\log_e(|7-4y\,|)$$

$$x = \frac{1}{4}\log_e\left(\frac{5}{|7-4y\,|}\right)$$

$$e^{4x} = \frac{5}{|7-4y\,|}$$

$$5e^{-4x} = |7-4y\,|$$

Since $y = 3$, $x = 0$

$$7 - 4y = -5e^{-4x}$$

$$4y = 7 + 5e^{-4x}$$

$$y = \frac{1}{4}\left(7 + 5e^{-4x}\right)$$

7 $\dfrac{dy}{dx} + 3y = 0 \qquad y\,(0) = 5$

$$\frac{dy}{dx} = -3y$$

$$\frac{dx}{dy} = \frac{-1}{3y}$$

$$x = \frac{-1}{3}\int \frac{1}{y}dy$$

$$x = \frac{-1}{3}\log_e|y| + c$$

$x = 0, \qquad y = 5$

$$0 = \frac{-1}{3}\log_e(5) + c$$

$$c = \frac{1}{3}\log_e(5)$$

$$x = \frac{1}{3}\log_e(5) - \frac{1}{3}\log_e(y)$$

$$x = \frac{1}{3}\log_e\left(\left|\frac{5}{y}\right|\right)$$

$$3x = \log_e\left(\left|\frac{5}{y}\right|\right)$$

$$\left|\frac{5}{y}\right| = e^{3x}$$

$$\left|\frac{y}{5}\right| = e^{-3x}$$

$$y = 5e^{-3x}$$

Note: The absolute value can be removed since LHS = RHS when the given condition is substituted in.

8 $\dfrac{dy}{dx} - 5y = 0, \qquad y\,(0) = 3$

$$\frac{dy}{dx} = 5y$$

$$\frac{dx}{dy} = \frac{1}{5y}$$

$$x = \frac{1}{5}\int \frac{1}{y}dy$$

$$x = \frac{1}{5}\log_e(|y|) + c$$

$x = 0, \qquad y = 3$

$$0 = \frac{1}{5}\log_e(3) + c$$

$$c = \frac{-1}{5}\log_e(3)$$

$$x = \frac{1}{5}\log_e(|y|) - \frac{1}{3}\log_e(3)$$

$$x = \frac{1}{5}\log_e\left(\left|\frac{y}{3}\right|\right)$$

$$5x = \log_e\left(\left|\frac{y}{3}\right|\right)$$

$$\left|\frac{y}{3}\right| = e^{5x}$$

$$y = 3e^{5x}$$

Note: The absolute value can be removed since LHS = RHS when the given condition is substituted in.

9 **a** $\dfrac{dy}{dx} + 5y = 0, \qquad y\,(0) = 4$

$$\frac{dy}{dx} = -5y$$

$$\frac{dx}{dy} = \frac{-1}{5y}$$

$$x = \frac{-1}{5}\int \frac{1}{y}dy$$

$$x = \frac{-1}{5}\log_e|y| + c$$

$y\,(0) = 4$

$$0 = \frac{-1}{5}\log_e(4) + c$$

$$c = \frac{1}{5}\log_e(4)$$

$$x = \frac{-1}{5}\log_e(|y|) + \frac{1}{5}\log_e(4)$$

$$x = \frac{1}{5}\left(\log_e(4) - \log_e(|y|)\right)$$

$$5x = \log_e\left(\left|\frac{4}{y}\right|\right)$$

$$\left|\frac{4}{|y|}\right| = e^{5x}$$

$$\left|\frac{y}{4}\right| = e^{-5x}$$

$$y = 4e^{-5x}$$

Note: The absolute value can be removed since LHS = RHS when the given condition is substituted in.

b $\frac{dy}{dx} - 3y = 0, \qquad y(1) = 2$

$$\frac{dy}{dx} = 3y$$

$$\frac{dx}{dy} = \frac{1}{3y}$$

$$x = \frac{1}{3}\int \frac{1}{y}dy$$

$$x = \frac{1}{3}\log_e(|y|) + c$$

$$y(1) = 2$$

$$1 = \frac{1}{3}\log_e(2) + c$$

$$c = 1 - \frac{1}{3}\log_e(2)$$

$$x = \frac{1}{3}\log_e(|y|) + 1 - \frac{1}{3}\log_e(2)$$

$$x - 1 = \frac{1}{3}\left(\log_e(|y|) - \log_e(2)\right)$$

$$3x - 3 = \log_e\left(\left|\frac{y}{2}\right|\right)$$

$$\left|\frac{y}{2}\right| = e^{3x-3}$$

$$y = 2e^{3x-3}$$

Note: The absolute value can be removed since LHS = RHS when the given condition is substituted in.

10 a $\frac{dy}{dx} + 2y = 5, \qquad y(0) = 3$

$$\frac{dy}{dx} = 5 - 2y$$

$$\frac{dx}{dy} = \frac{1}{5 - 2y}$$

$$x = \int \frac{1}{5 - 2y}dy$$

$$x = -\frac{1}{2}\log_e|5 - 2y| + c$$

$$y(0) = 3$$

$$0 = -\frac{1}{2}\log_e|5 - 6| + c$$

$$0 = -\frac{1}{2}\log_e|1| + c$$

$$c = 0$$

$$x = -\frac{1}{2}\log_e|5 - 2y|$$

$$-2x = \log_e|5 - 2y|$$

$$e^{-2x} = |5 - 2y| \quad \text{since } x = 0 \quad \text{when } y = 3$$

$$e^{-2x} = 2y - 5$$

$$2y = 5 + e^{-2x}$$

$$y = \frac{1}{2}\left(5 + e^{-2x}\right)$$

b $\frac{dy}{dx} - 3y + 4 = 0, \qquad y(0) = 2$

$$\frac{dy}{dx} = 3y - 4$$

$$\frac{dx}{dy} = \frac{1}{3y - 4}$$

$$x = \int \frac{1}{3y - 4}dy$$

$$x = \frac{1}{3}\log_e|3y - 4| + c$$

$$y(0) = 2$$

$$0 = \frac{1}{3}\log_e 2 + c$$

$$c = -\frac{1}{3}\log_e(2)$$

$$x = \frac{1}{3}\log_e|3y - 4| - \frac{1}{3}\log_e(2)$$

$$= \frac{1}{3}\left(\log_e(|3y - 4|) - \log_e(2)\right)$$

$$3x = \log_e\left(\frac{|3y - 4|}{2}\right)$$

$$e^{3x} = \frac{|3y - 4|}{2}$$

$$2e^{3x} = 3y - 4$$

Note: The absolute value can be removed since LHS = RHS when the given condition is substituted in.

$$3y = 4 + 2e^{3x}$$

$$3y = 2\left(2 + e^{3x}\right)$$

$$y = \frac{2}{3}\left(2 + e^{3x}\right)$$

11 $\sqrt{64 - y^2} - 6\frac{dy}{dx} = 0 \qquad y(0) = 8$

$$\sqrt{64 - y^2} = 6\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{\sqrt{64 - y^2}}{6}$$

$$\frac{dx}{dy} = \frac{6}{\sqrt{64 - y^2}}$$

$$x = 6\int \frac{1}{\sqrt{64 - y^2}}dy$$

$$x = 6\sin^{-1}\left(\frac{y}{8}\right) + c$$

$$x = 0, \quad y = 8$$

$$0 = 6\sin^{-1}(1) + c$$

$$\mathrm{c} = -6 \times \frac{\pi}{2}$$

$$c = -3\pi$$

$$x = 6\sin^{-1}\left(\frac{y}{8}\right) - 3\pi$$

$$x + 3\pi = 6\sin^{-1}\left(\frac{y}{8}\right)$$

$$\sin^{-1}\left(\frac{y}{8}\right) = \frac{x}{6} + \frac{\pi}{2}$$

$$\frac{y}{8} = \sin\left(\frac{x}{6} + \frac{\pi}{2}\right)$$

$$= \sin\left(\frac{x}{6}\right)\cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{x}{6}\right)\sin\left(\frac{\pi}{2}\right)$$

$$\frac{y}{8} = \cos\left(\frac{x}{6}\right)$$

$$y = 8\cos\left(\frac{x}{6}\right)$$

$$\frac{-\pi}{2} < \frac{x}{6} + \frac{\pi}{2} < \frac{\pi}{2}$$

$$-\pi < \frac{x}{6} < 0$$

$$-6\pi < x < 0$$

12 $16 + y^2 - 4\dfrac{dy}{dx} = 0, \qquad y(0) = 0$

$$16 + y^2 = 4\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{16 + y^2}{4}$$

$$\frac{dx}{dy} = \frac{4}{16 + y^2}$$

$$x = \int \frac{4}{16 + y^2}dy$$

$$= \tan^{-1}\left(\frac{y}{4}\right) + c$$

$$x = 0, \quad y = 0$$

$$0 = \tan^{-1}(0) + c$$

$$c = 0$$

$$x = \tan^{-1}\left(\frac{y}{4}\right)$$

$$\frac{y}{4} = \tan(x)$$

$$y = 4\tan(x) \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

13 a $\dfrac{dy}{dx} = \sqrt{y}, \qquad y(1) = 4$

$$\frac{dy}{dx} = y^{\frac{1}{2}}$$

$$\frac{dx}{dy} = \frac{1}{y^{\frac{1}{2}}}$$

$$= y^{-\frac{1}{2}}$$

$$x = \int y^{-\frac{1}{2}}dy$$

$$x = 2y^{\frac{1}{2}} + c$$

$$y(1) = 4$$

$$1 = 2\sqrt{4} + c$$

$$c = -3$$

$$x = 2\sqrt{y} - 3$$

$$2\sqrt{y} = x + 3$$

$$\sqrt{y} = \frac{1}{2}(x + 3)$$

$$y = \frac{1}{4}(x + 3)^2, \qquad x \in R$$

b $\dfrac{dy}{dx} = y^2, \qquad y(1) = 3$

$$\frac{dx}{dy} = \frac{1}{y^2}$$

$$x = \int y^{-2}dy$$

$$x = -y^{-1} + c$$

$$x = \frac{-1}{y} + c$$

$$y(1) = 3$$

$$1 = \frac{-1}{3} + c$$

$$c = \frac{4}{3}$$

$$x = \frac{-1}{y} + \frac{4}{3}$$

$$\frac{1}{y} = \frac{4}{3} - x$$

$$\frac{1}{y} = \frac{4 - 3x}{3} \qquad \text{Require} \quad 4 - 3x \neq 0$$

$$y = \frac{3}{4 - 3x}, \qquad x \neq \frac{4}{3}$$

14 a $\dfrac{dy}{dx} = 4e^{2y}, \qquad y(2) = 0$

$$\frac{dx}{dy} = \frac{1}{4e^{2y}}$$

$$x = \frac{1}{4}\int e^{-2y}dy$$

$$x = \frac{-1}{8}e^{-2y} + c$$

$$y(2) = 0$$

$$2 = \frac{-1}{8}e^0 + c$$

$$c = \frac{17}{8}$$

$$x = \frac{-1}{8}e^{-2y} + \frac{17}{8}$$

$$8x = -e^{-2y} + 17$$

$$e^{-2y} = 17 - 8x$$

$$-2y = \log_e(17 - 8x)$$

$$y = -\frac{1}{2}\log_e(17 - 8x)$$

Require $17 - 8x > 0$

$$x < \frac{17}{8}$$

b $\dfrac{dy}{dx} + 6e^{3y} = 0, \qquad y(1) = 0$

$$\frac{dy}{dx} = -6e^{3y}$$

$$\frac{dx}{dy} = \frac{-1}{6e^{3y}}$$

$$x = \frac{-1}{6}\int e^{-3y}dy$$

$$x = \frac{1}{18}e^{-3y} + c$$

$$y(1) = 0$$

$$1 = \frac{1}{18}e^0 + c$$

$$c = \frac{17}{18}$$
$$x = \frac{1}{18}e^{-3y} + \frac{17}{18}$$
$$18x = e^{-3y} + 17$$
$$e^{-3y} = 18x - 17$$
$$-3y = \log_e(18x - 17)$$
$$y = -\frac{1}{3}\log_e(18x - 17),$$

Require $18x - 17 > 0$
$$x > \frac{17}{18}$$

15 a $\frac{dy}{dx} = (5 - 2y)^2, \qquad y(1) = 3$
$$\frac{dx}{dy} = \frac{1}{(5 - 2y)^2}$$
$$x = \int \frac{1}{(5 - 2y)^2}\,dy$$
$$x = \frac{1}{2(5 - 2y)} + c$$
$$y(1) = 3$$
$$1 = \frac{-1}{2} + c$$
$$c = \frac{3}{2}$$
$$x = \frac{1}{2(5 - 2y)} + \frac{3}{2}$$
$$x - \frac{3}{2} = \frac{1}{2(5 - 2y)}$$
$$\frac{1}{2(5 - 2y)} = \frac{2x - 3}{2}$$
$$5 - 2y = \frac{1}{2x - 3}$$
$$2y = 5 - \frac{1}{2x - 3}$$
$$2y = \frac{5(2x - 3) - 1}{2x - 3}$$
$$2y = \frac{10x - 16}{2x - 3}$$
$$2y = \frac{2(5x - 8)}{2x - 3}$$
$$y = \frac{5x - 8}{2x - 3}$$

Require $2x - 3 \neq 0$
$$x \neq \frac{3}{2}$$

b $\frac{dy}{dx} + (7 - 3y)^2 = 0, \qquad y(3) = 2$
$$\frac{dy}{dx} = -(7 - 3y)^2$$
$$\frac{dx}{dy} = \frac{-1}{(7 - 3y)^2}$$
$$x = \int \frac{-1}{(7 - 3y)^2}\,dy$$
$$x = \frac{-1}{3(7 - 3y)} + c$$
$$y(3) = 2$$
$$3 = \frac{-1}{3} + c$$
$$c = \frac{10}{3}$$
$$x = \frac{-1}{3(7 - 3y)} + \frac{10}{3}$$
$$\frac{10}{3} - x = \frac{1}{3(7 - 3y)}$$
$$\frac{10 - 3x}{3} = \frac{1}{3(7 - 3y)}$$
$$7 - 3y = \frac{1}{10 - 3x}$$
$$3y = 7 - \frac{1}{10 - 3x}$$
$$3y = \frac{7(10 - 3x) - 1}{10 - 3x}$$
$$= \frac{69 - 21x}{10 - 3x}$$
$$3y = \frac{-3(7x - 23)}{10 - 3x}$$
$$y = \frac{7x - 23}{3x - 10}$$

Require $3x - 10 \neq 0$
$$x \neq \frac{10}{3}$$

16 a $\frac{dy}{dx} = \sqrt{4y + 9}, \qquad y(0) = 0$
$$\frac{dx}{dy} = \frac{1}{\sqrt{4y + 9}}$$
$$x = \int (4y + 9)^{-\frac{1}{2}}dy$$

Let $u = 4y + 9 \qquad \frac{du}{dy} = 4$
$$x = \int \frac{1}{4}u^{-\frac{1}{2}}dy$$
$$= \frac{1}{4} \times 2u^{\frac{1}{2}} + c$$
$$x = \frac{1}{2}\sqrt{4y + 9} + c$$

When $x = 0, \qquad y = 0$
$$0 = \frac{1}{2}\sqrt{9} + c$$
$$c = \frac{-3}{2}$$
$$x = \frac{1}{2}\left(\sqrt{4y + 9} - 3\right)$$
$$2x = \sqrt{4y + 9} - 3$$
$$\sqrt{4y + 9} = 2x + 3$$
$$4y + 9 = (2x + 3)^2$$
$$= 4x^2 + 12x + 9$$
$$4y = 4x^2 + 12x$$
$$= 4\left(x^2 + 3x\right)$$
$$y = x^2 + 3x$$

b $\dfrac{dy}{dx} = 4y^2 + 9, \qquad y(0) = 0$

$$\frac{dx}{dy} = \frac{1}{4y^2+9}$$

$$x = \int \frac{1}{4y^2+9}dy$$

Let $u = 2y, \quad \dfrac{du}{dy} = 2$

$$x = \frac{1}{2}\int \frac{1}{u^2+9}du$$

$$x = \frac{1}{2} \times \frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right) + c$$

$$x = \frac{1}{6}\tan^{-1}\left(\frac{2y}{3}\right) + c$$

When $x = 0, \quad y = 0$

$$0 = \frac{1}{6}\tan^{-1}(0) + c$$

$$c = 0$$

$$6x = \tan^{-1}\left(\frac{2y}{3}\right)$$

$$\frac{2y}{3} = \tan(6x)$$

since dom $\tan^{-1}$ is $\left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$

$$-\frac{\pi}{2} < 6x < \frac{\pi}{2}$$

$$-\frac{\pi}{12} < x < \frac{\pi}{12}$$

$$y = \frac{3}{2}\tan(6x)$$

17 a $\dfrac{dy}{dx} + 4y = y^2, \qquad y(0) = 3$

$$\frac{dy}{dx} = y^2 - 4y$$

$$\frac{dx}{dy} = \frac{1}{y^2 - 4y}$$

$$x = \int \frac{1}{y^2-4y}dy$$

Partial fractions

$$\begin{aligned}\frac{1}{y^2-4y} &= \frac{1}{y(y-4)}\\ &= \frac{A}{y} + \frac{B}{y-4}\\ &= \frac{A(y-4)+By}{y(y-4)}\\ 1 &= A(y-4) + By\end{aligned}$$

Let $y = 4 \quad \Rightarrow B = \dfrac{1}{4}$

$y = 0 \quad \Rightarrow A = \dfrac{-1}{4}$

$$x = \frac{1}{4}\int\left(\frac{1}{y-4} - \frac{1}{y}\right)dy$$

$$4x = \log_e\left|\frac{y-4}{y}\right| + c$$

When $x = 0, \quad y = 3$

$$0 = \log_e\left|\frac{-1}{3}\right| + c$$

$$c = -\log_e\left(\frac{1}{3}\right)$$

$$\begin{aligned}4x &= \log_e\left|\frac{3(y-4)}{y}\right|\\ &= \log_e\left|\frac{3(4-y)}{y}\right|\end{aligned}$$

$$e^{4x} = \frac{3(4-y)}{y}$$

Note: The absolute value can be removed since LHS = RHS when the given condition is substituted in.

$$\begin{aligned}ye^{4x} &= 3(4-y)\\ ye^{4x} &= 12 - 3y\\ 12 &= ye^{4x} + 3y\\ &= y\left(3 + e^{4x}\right)\\ y &= \frac{12}{3+e^{4x}}\end{aligned}$$

b $\dfrac{dy}{dx} - 3y = y^2, \qquad y(0) = 6$

$$\frac{dy}{dx} = 3y + y^2$$

$$\frac{dx}{dy} = \frac{1}{y(y+3)}$$

$$x = \int \frac{1}{y(y+3)}dy$$

Partial fractions

$$\begin{aligned}\frac{1}{y(y+3)} &= \frac{A}{y} + \frac{B}{y+3}\\ &= \frac{A(y+3)+By}{y(y+3)}\\ 1 &= A(y+3) + By\end{aligned}$$

Let $y = 0 \quad \Rightarrow A = \dfrac{1}{3}$

$y = -3 \quad \Rightarrow B = \dfrac{-1}{3}$

$$x = \frac{1}{3}\int\left(\frac{1}{y} - \frac{1}{y+3}\right)dy$$

$$3x = \left(\log_e\left|\frac{y}{y+3}\right|\right) + c$$

When $y = 6, \quad x = 0$

$$0 = \log_e\left(\frac{6}{9}\right) + c$$

$$c = -\log_e\left(\frac{2}{3}\right)$$

$$3x = \log_e\left|\frac{3y}{2(y+3)}\right|$$

$$e^{3x} = \frac{3y}{2(y+3)}$$

Note: The absolute value can be removed since LHS = RHS when the given condition is substituted in.

$$\begin{aligned}3y &= 2(y+3)e^{3x}\\ 3y &= 2ye^{3x} + 6e^{3x}\\ 6e^{3x} &= 3y - 2ye^{3x}\\ &= y\left(3 - 2e^{3x}\right)\\ y &= \frac{6e^{3x}}{3-2e^{3x}}\end{aligned}$$

18 a $\dfrac{dy}{dx} + 7y = y^2 + 12, \qquad y(0) = 0$

$$\frac{dy}{dx} = y^2 - 7y + 12$$

$$\frac{dx}{dy} = \frac{1}{y^2 - 7y + 12}$$

$$x = \int \frac{1}{y^2 - 7y + 12} dy$$

Partial fractions

$$\frac{1}{y^2 - 7y + 12} = \frac{1}{(y-3)(y-4)}$$

$$= \frac{A}{y-3} + \frac{B}{y-4}$$

$$= \frac{A(y-4) + B(y-3)}{(y-3)(y-4)}$$

$$1 = A(y-4) + B(y-3)$$

Let $y = 4 \quad \Rightarrow B = 1$

$y = 3 \quad \Rightarrow A = -1$

$$x = \int \left(\frac{1}{y-4} - \frac{1}{y-3}\right) dy$$

$$x = \log_e\left(\left|\frac{y-4}{y-3}\right|\right) + c$$

When $x = 0, \quad y = 0$

$$0 = \log_e\left|\frac{4}{3}\right| + c$$

$$c = -\log_e\left|\frac{4}{3}\right|$$

$$x = \log_e\left|\frac{3(y-4)}{4(y-3)}\right|$$

$$e^x = \frac{3(y-4)}{4(y-3)}$$

Note: The absolute value can be removed since LHS = RHS when the given condition is substituted in.

$$3(y-4) = 4(y-3)e^x$$

$$3y - 12 = 4ye^x - 12e^x$$

$$-12 + 12e^x = 4ye^x - 3y$$

$$12(e^x - 1) = y(4e^x - 3)$$

$$y = \frac{12(e^x - 1)}{4e^x - 3}$$

b $\dfrac{dy}{dx} - 6y = y^2 + 8, \qquad y(0) = 0$

$$\frac{dy}{dx} = y^2 + 6y + 8$$

$$\frac{dx}{dy} = \frac{1}{y^2 + 6y + 8}$$

$$x = \int \frac{1}{y^2 + 6y + 8} dy$$

Partial fractions

$$\frac{1}{y^2 + 6y + 8} = \frac{1}{(y+2)(y+4)}$$

$$= \frac{A}{y+2} + \frac{B}{y+4}$$

$$= \frac{A(y+4) + B(y+2)}{(y+2)(y+4)}$$

$$1 = A(y+4) + B(y+2)$$

Let $y = -2 \quad \Rightarrow A = \dfrac{1}{2}$

$y = -4 \quad \Rightarrow B = \dfrac{-1}{2}$

$$x = \frac{1}{2}\int \left(\frac{1}{y+2} - \frac{1}{y+4}\right) dy$$

$$2x = \log_e\left(\left|\frac{y+2}{y+4}\right|\right) + c$$

When $x = 0, \quad y = 0$

$$0 = \log_e\left(\frac{1}{2}\right) + c$$

$$c = -\log_e\left(\frac{1}{2}\right)$$

$$2x = \log_e\left(\left|\frac{2(y+2)}{y+4}\right|\right)$$

$$e^{2x} = \frac{2(y+2)}{y+4}$$

Note: The absolute value can be removed since LHS = RHS when the given condition is substituted in.

$$2(y+2) = (y+4)e^{2x}$$

$$2y + 4 = ye^{2x} + 4e^{2x}$$

$$2y - ye^{2x} = 4e^{2x} - 4$$

$$y\left(2 - e^{2x}\right) = 4\left(e^{2x} - 1\right)$$

$$y = \frac{4\left(e^{2x} - 1\right)}{2 - e^{2x}}$$

$$= \frac{4\left(1 - e^{2x}\right)}{e^{2x} - 2}$$

$$e^{2x} \neq 2$$

$$2x \neq \log_e(2)$$

$$x \neq \frac{1}{2}\log_e(2)$$

$$x \neq \log_e\left(\sqrt{2}\right)$$

19 $\dfrac{dy}{dx} + ky = 0, \qquad y(0) = y_0$

$$\frac{dy}{dx} = -ky$$

$$\frac{dx}{dy} = \frac{-1}{ky}$$

$$x = \frac{-1}{k}\int \frac{1}{y} dy$$

$$x = \frac{-1}{k}\log_e|y| + c$$

When $x = 0, \quad y = y_0$

$$0 = \frac{-1}{k}\log_e|y_0| + c$$

$$c = \frac{1}{k}\log_e|y_0|$$

$$x = \frac{1}{k}\log_e|y_0| - \frac{1}{k}\log_e|y|$$

$$x = \frac{1}{k}\left(\log_e\left|\frac{y_0}{y}\right|\right)$$

$$kx = \log_e\left|\frac{y_0}{y}\right|$$

$$\frac{y_0}{y} = e^{kx}$$

Note: The absolute value can be removed since LHS = RHS when the given condition is substituted in.

$$\frac{y}{y_0} = e^{-kx}$$
$$y = y_0 e^{-kx}$$

20 $\dfrac{dy}{dx} + ay = b, \qquad y(0) = c$

$$\frac{dy}{dx} = b - ay$$
$$\frac{dx}{dy} = \frac{1}{b - ay}$$
$$x = \int \frac{1}{b - ay} dy$$
$$x = \frac{-1}{a} \log_e |b - ay| + k$$

When $x = 0, \quad y = c$

$$0 = \frac{-1}{a} \log_e |b - ac| + k$$
$$k = \frac{1}{a} \log_e |b - ac|$$
$$x = \frac{1}{a} \left(\log_e |b - ac| - \log_e |b - ay|\right)$$
$$ax = \log_e \left|\frac{b - ac}{b - ay}\right|$$
$$e^{ax} = \frac{b - ac}{b - ay}$$

Note: The absolute value can be removed since LHS = RHS when the given condition is substituted in.

$$e^{-ax} = \frac{b - ay}{b - ac}$$
$$b - ay = (b - ac)\, e^{-ax}$$
$$ay = b - (b - ac)\, e^{-ax}$$
$$= b - be^{-ax} + ace^{-ax}$$
$$y = \frac{b}{a} - \frac{b}{a} e^{-ax} + ce^{-ax}$$
$$y = \left(c - \frac{b}{a}\right) e^{-ax} + \frac{b}{a}$$

21 a $\dfrac{dy}{dx} = (ay + b)^2, \qquad y(0) = 0$

$$\frac{dx}{dy} = \frac{1}{(ay + b)^2}$$
$$x = \int \frac{1}{(ay + b)^2} dy$$
$$x = \frac{-1}{a(ay + b)} + K$$

$y(0) = 0$

$$0 = \frac{-1}{ab} + K$$
$$K = \frac{1}{ab}$$
$$x = \frac{1}{ab} - \frac{1}{a(ay + b)}$$
$$\frac{1}{a(ay + b)} = \frac{1}{ab} - x$$
$$\frac{1}{a(ay + b)} = \frac{1 - abx}{ab}$$
$$ay + b = \frac{b}{1 - abx}$$
$$ay = \frac{b}{1 - abx} - b$$
$$= \frac{b - (1 - abx)\, b}{1 - abx}$$
$$ay = \frac{b - b + ab^2x}{1 - abx}$$
$$y = \frac{b^2x}{1 - abx} \qquad x \neq \frac{1}{ab}$$

b $\dfrac{dy}{dx} = b^2y^2 + a^2, \qquad y(0) = 0$

$$\frac{dx}{dy} = \frac{1}{b^2y^2 + a^2}$$
$$x = \int \frac{1}{b^2y^2 + a^2} dy$$
$$x = \frac{1}{ab} \tan^{-1}\left(\frac{by}{a}\right) + c$$

$x = 0 \quad y = 0$

$\Rightarrow c = 0$

$$\tan^{-1}\left(\frac{by}{a}\right) = abx$$
$$\frac{by}{a} = \tan(abx)$$
$$y = \frac{a}{b} \tan(abx)$$

22 a $\dfrac{dy}{dx} = (y + a)(y + b), \qquad y(0) = 0$

$$\frac{dx}{dy} = \frac{1}{(y + a)(y + b)}$$
$$x = \int \frac{1}{(y + a)(y + b)} dy$$

Partial fractions

$$\frac{A}{y + a} + \frac{B}{y + b} = \frac{A(y + b) + B(y + a)}{(y + a)(y + b)}$$
$$= \frac{y(A + B) + bA + Ba}{(y + a)(y + b)}$$

$A + B = 0 \qquad \Rightarrow B = -A$

$bA + Ba = 1 \qquad bA - aA = 1$

$$A(b - a) = 1$$
$$A = \frac{1}{b - a}$$
$$= \frac{-1}{a - b}$$
$$x = \frac{-1}{a - b} \int \left(\frac{1}{y + a} - \frac{1}{y + b}\right) dy$$
$$-(a - b)\, x = \log_e \left|\frac{y + a}{y + b}\right| + K$$

$x = 0$ When $y = 0$

$$0 = \log_e \left|\frac{a}{b}\right| + K$$
$$K = -\log_e \left(\frac{a}{b}\right)$$
$$-(a - b)\, x = \log_e \left|\frac{y + a}{y + b}\right| - \log_e \left(\frac{a}{b}\right)$$
$$= \log_e \left|\frac{b(y + a)}{a(y + b)}\right|$$
$$e^{-(a-b)x} = \frac{b(y + a)}{a(y + b)}$$

Note: The absolute value can be removed since LHS = RHS when the given condition is substituted in.

$$\text{let} \quad T = e^{-(a-b)x}$$

$$\frac{b(y+a)}{a(y+b)} = T$$

$$b(y+a) = Ta(y+b)$$

$$by + ab = Tay + Tab$$

$$by - Tay = Tab - ab$$

$$y(b - aT) = ab(T - 1)$$

$$y = \frac{ab(T-1)}{b - aT} = \frac{ab\left(e^{-(a-b)x} - 1\right)}{b - ae^{-(a-b)x}}$$

$$= \frac{ab\left(1 - e^{-(a-b)x}\right)}{ae^{-(a-b)x} - b}$$

b As $x \to \infty$

Since $a - b > 0$

$a > b$

$ae^{-(a-b)x} \to 0$

$y \to -a$

8.4 Exam questions

1 $\dfrac{dy}{dx} = 1 - \dfrac{y}{3}, \quad y(2) = 4$

$$\frac{dy}{dx} = \frac{3-y}{3} \Rightarrow \frac{dy}{dx} = \frac{3}{3-y}$$

$$x = 3\int \frac{1}{3-y}dy = -3\log_e(|3-y|) + \text{c}$$

$$2 = -3\log_e(|1|) + \text{c}, \qquad \text{c} = 2$$

$$x = -3\log_e(|3-y|) + 2$$

$$\frac{2-x}{3} = \log_e(|3-y|) \quad \text{since } x = 2 \text{ when } y = 4$$

$$\text{y} - 3 = \text{e}^{\frac{2-x}{3}}$$

$$\text{y} = 3 + \text{e}^{\frac{2-x}{3}} = 3 + \text{e}^{\frac{-(x-2)}{3}}$$

Or use CAS to solve.

The correct answer is **B**.

2 $\dfrac{dy}{dx} + 6\,\text{cosec}\left(\dfrac{y}{2}\right) = 0, \qquad y\left(\dfrac{1}{3}\right) = 0$

$$\frac{dy}{dx} = -6 \quad \text{cosec}\left(\frac{y}{2}\right)$$

$$\frac{dy}{dx} = \frac{-1}{-6 \quad \text{cosec}\left(\frac{y}{2}\right)}$$

$$x = \frac{-1}{6}\int \sin\left(\frac{y}{2}\right) \quad dy$$

$$x = \frac{1}{3}\cos\left(\frac{y}{2}\right) + c \qquad \text{[1 mark]}$$

When $y = 0, \quad x = \dfrac{1}{3}$

$$\frac{1}{3} = \frac{1}{3}\cos(0) + c$$

$$c = 0 \qquad \text{[1 mark]}$$

$$x = \frac{1}{3}\cos\left(\frac{y}{2}\right)$$

$$3x = \cos\left(\frac{y}{2}\right)$$

$$\frac{y}{2} = \cos^{-1}(3x)$$

$$y = 2\cos^{-1}(3x) \qquad \text{[1 mark]}$$

Require $|3x| \leq 1$

$$|x| \leq \frac{1}{3} \qquad \text{[1 mark]}$$

3 $\dfrac{dy}{dx} + ay = by^2, \qquad y(0) = c$

$$\frac{dy}{dx} = by^2 - ay$$

$$\frac{dy}{dx} = \frac{1}{by^2 - ay}$$

$$= \frac{1}{y(by-a)}$$

$$x = \int \frac{1}{y(by-a)}dy \qquad \text{[1 mark]}$$

Partial fractions

$$\frac{1}{y(by-a)} = \frac{A}{y} + \frac{B}{by-a}$$

$$= \frac{A(by-a) + By}{y(by-a)}$$

$$= \frac{y(\text{A}b + B) - aA}{y(by-a)}$$

$$aA = -1 \qquad \Rightarrow A = \frac{-1}{a}$$

$$Ab + B = 0 \qquad \Rightarrow B = -Ab = \frac{b}{a}$$

$$x = \frac{1}{a}\int\left(\frac{b}{by-a} - \frac{1}{y}\right)dy \qquad \text{[1 mark]}$$

$$ax = \log_e\left|\frac{by-a}{y}\right| + K \qquad \text{[1 mark]}$$

When $x = 0, \quad y = c$

$$0 = \log_e\left|\frac{bc-a}{c}\right| + K$$

$$K = -\log_e\left|\frac{bc-a}{c}\right|$$

$$ax = \log_e\left|\frac{c(by-a)}{(bc-a)y}\right|$$

$$e^{ax} = \frac{c(by-a)}{y(bc-a)} \qquad \text{[1 mark]}$$

Note: The absolute value can be removed since LHS = RHS when the given condition is substituted in.

$$\text{y}(bc-a)e^{ax} = c(by-a) = bcy - ac$$

$$-ac = y(bc-a)e^{ax} - bcy$$

$$ac = y(bc + (a - bc)e^{ax})$$

$$y = \frac{ac}{bc + (a-bc)e^{ax}} \qquad \text{[1 mark]}$$

8.5 Solving Type 3 differential equations, $\dfrac{dy}{dx} = f(x)g(y)$

8.5 Exercise

1 $\dfrac{dy}{dx} = \dfrac{x+2}{y^3+8}$

$$\int\left(y^3 + 8\right)dy = \int(x+2)\,dx$$

$$\frac{1}{4}y^4 + 8y = \frac{1}{2}x^2 + 2x + c$$

$$\frac{y^4}{4} + 8y - \frac{x^2}{2} - 2x = c$$

2 $\dfrac{dy}{dx} = \dfrac{y^2 + 4}{x^2y^2}$

$$\int \frac{y^2}{y^2 + 4} dy = \int \frac{1}{x^2} dx$$

$$\int \frac{y^2 + 4 - 4}{y^2 + 4} dy = \int x^{-2} dx$$

$$\int 1 - \frac{4}{y^2 + 4} dy = \int x^{-2} dx$$

$$y - 2\tan^{-1}\left(\frac{y}{2}\right) = \frac{-1}{x} + c$$

$$y - 2\tan^{-1}\left(\frac{y}{2}\right) + \frac{1}{x} = c$$

3 a $\dfrac{dy}{dx} = \dfrac{x^2 + 4}{y^2 + 4}$

$$\int \left(y^2 + 4\right) dy = \int \left(x^2 + 4\right) dx$$

$$\frac{1}{3}y^3 + 4y = \frac{1}{3}x^3 + 4x + c$$

$$\frac{1}{3}y^3 + 4y - \frac{1}{3}x^3 - 4x = c$$

b $\dfrac{dy}{dx} = \dfrac{xy}{y^2 + 4}$

$$\int \frac{y^2 + 4}{y} dy = \int x\, dx$$

$$\int \left(y + \frac{4}{y}\right) dy = \int x\, dx$$

$$\frac{1}{2}y^2 + 4\log_e |y| = \frac{1}{2}x^2 + c$$

$$\frac{1}{2}y^2 + 4\log_e |y| - \frac{1}{2}x^2 = c$$

4 a $\dfrac{dy}{dx} = \dfrac{x^2y^2}{y^2 + 4}$

$$\int \frac{y^2 + 4}{y^2} dy = \int x^2 dx$$

$$\int \left(1 + 4y^{-2}\right) dy = \int x^2 dx$$

$$y - 4y^{-1} = \frac{1}{3}x^3 + c$$

$$y - \frac{4}{y} - \frac{1}{3}x^3 = c$$

b $\dfrac{dy}{dx} = \dfrac{xy^2e^{x^2}}{y^3 + 8}$

$$\int \frac{y^3 + 8}{y^2}\, dy = \int xe^{x^2} dx$$

let $u = x^2 \quad \dfrac{du}{dx} = 2x$

$$\int \left(y + 8y^{-2}\right) dy = \frac{1}{2}\int e^u du$$

$$\frac{1}{2}y^2 - 8y^{-1} = \frac{1}{2}e^{x^2} + c$$

$$\frac{1}{2}y^2 - \frac{8}{y} - \frac{1}{2}e^{x^2} = c$$

5 $\dfrac{dy}{dx} - y = 3x^2y, \qquad y(0) = 1$

$$\frac{dy}{dx} = y + 3x^2y$$

$$\frac{dy}{dx} = y\left(1 + 3x^2\right)$$

$$\int \frac{1}{y} dy = \int \left(1 + 3x^2\right) dx$$

$$\log_e |y| = x + x^3 + c$$

$$y = 1, \qquad x = 0$$

$$\log_e(1) = 0 + c$$

$$c = 0$$

$$\log_e |y| = x + x^3$$

$$y = e^{x^3 + x}$$

Note: The absolute value can be removed since LHS = RHS when the given condition is substituted in.

6 $\dfrac{dy}{dx} + y^2 = 2xy^2, \quad y(2) = 1$

$$\frac{dy}{dx} = 2xy^2 - y^2$$

$$= y^2(2x - 1)$$

$$\int \frac{1}{y^2} dy = \int (2x - 1)\, dx$$

$$\frac{-1}{y} = x^2 - x + c$$

$$y = 1, \quad x = 2$$

$$-1 = 4 - 2 + c$$

$$c = -3$$

$$\frac{-1}{y} = x^2 - x - 3$$

$$\frac{1}{y} = 3 + x - x^2$$

$$y = \frac{1}{3 + x - x^2},$$

$$3 + x - x^2 = 0$$

$$\Delta = 1 - 4 \times -1 \times 3$$

$$= 13$$

$$x \neq \frac{1 \pm \sqrt{13}}{2}$$

7 $\dfrac{dy}{dx} - 2x\sqrt{64 - y^2} = 0, \qquad y(0) = 0$

$$\frac{dy}{dx} = 2x\sqrt{64 - y^2}$$

$$\int \frac{1}{\sqrt{64 - y^2}}\, dy = \int 2x dx$$

$$\sin^{-1}\left(\frac{y}{8}\right) = x^2 + c$$

$$y = 0, \quad x = 0$$

$$\sin^{-1}(0) = 0 + c$$

$$c = 0$$

$$\sin^{-1}\left(\frac{y}{8}\right) = x^2 \qquad |x^2| < \frac{\pi}{2}$$

$$\frac{y}{8} = \sin\left(x^2\right) \qquad |x| < \frac{\sqrt{\pi}}{\sqrt{2}}$$

$$y = 8\sin\left(x^2\right) \qquad -\frac{\sqrt{2\pi}}{2} < x < \frac{\sqrt{2\pi}}{2}$$

8 $2\dfrac{dy}{dx} - x\left(16 + y^2\right) = 0, \qquad y(0) = 0$

$$2\frac{dy}{dx} = x\left(16 + y^2\right)$$

$$\int \frac{1}{16 + y^2}\,dy = \int \frac{x}{2}\,dx$$

$$\frac{1}{4}\tan^{-1}\left(\frac{y}{4}\right) = \frac{x^2}{4} + c$$

$$y = 0 \qquad x = 0$$

$$\frac{1}{4}\tan^{-1}(0) = 0 + c$$

$$c = 0$$

$$\frac{1}{4}\tan^{-1}\left(\frac{y}{4}\right) = \frac{x^2}{4} \qquad |x^2| < \frac{\pi}{2}$$

$$\tan^{-1}\left(\frac{y}{4}\right) = x^2 \qquad |x| < \frac{\sqrt{\pi}}{\sqrt{2}}$$

$$\frac{y}{4} = \tan\left(x^2\right)$$

$$y = 4\tan\left(x^2\right) \qquad -\frac{\sqrt{2\pi}}{2} < x < \frac{\sqrt{2\pi}}{2}$$

9 a $\dfrac{dy}{dx} - \dfrac{y^2}{x} = 0 \qquad y(1) = 1$

$$\frac{dy}{dx} = \frac{y^2}{x}$$

$$\int \frac{1}{y^2}\,dy = \int \frac{1}{x}\,dx$$

$$\frac{-1}{y} = \log_e(|x|) + c$$

$$x = 1, \qquad y = 1$$

$$-1 = \log_e(1) + c$$

$$c = -1$$

$$\frac{-1}{y} = \log_e(|x|) - 1$$

$$\frac{1}{y} = 1 - \log_e(|x|)$$

$$y = \frac{1}{1 - \log_e(|x|)}, \qquad x \neq 0$$

b $\dfrac{dy}{dx} + 12y^2\sin(4x) = 0, \quad y(\pi) = 1$

$$\frac{dy}{dx} = -12y^2\sin(4x)$$

$$\int \frac{1}{y^2}\,dy = \int -12\sin(4x)dx$$

$$\frac{-1}{y} = 3\cos(4x) + c$$

$$x = \pi, \qquad y = 1$$

$$-1 = 3\cos(4\pi) + c$$

$$c = -4$$

$$\frac{-1}{y} = 3\cos(4x) - 4$$

$$\frac{1}{y} = 4 - 3\cos(4x)$$

$$y = \frac{1}{4 - 3\cos(4x)}$$

10 a $\dfrac{dy}{dx} + \dfrac{x}{y} = 0, \qquad y(1) = 2$

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$\int y\,dy = -\int x\,dx$$

$$\frac{1}{2}y^2 = \frac{-1}{2}x^2 + c$$

$$x = 1, \qquad y = 2$$

$$2 = \frac{-1}{2} + c$$

$$c = \frac{5}{2}$$

$$\frac{1}{2}y^2 = \frac{-1}{2}x^2 + \frac{5}{2}$$

$$y^2 = 5 - x^2$$

$$y = \pm\sqrt{5 - x^2}$$

But $x = 1, y = 2$

$$\therefore y = \sqrt{5 - x^2}, \qquad |x| \leq \sqrt{5}$$

b $\dfrac{dy}{dx} + 6y^2x^2 = 0, \qquad y(1) = 3$

$$\frac{dy}{dx} = -6y^2x^2$$

$$\int \frac{-1}{y^2}\,dy = \int 6x^2\,dx$$

$$\frac{1}{y} = 2x^3 + c$$

$$x = 1, \qquad y = 3$$

$$\frac{1}{3} = 2 + c$$

$$c = \frac{-5}{3}$$

$$\frac{1}{y} = 2x^3 - \frac{5}{3}$$

$$\frac{1}{y} = \frac{6x^3 - 5}{3}$$

$$y = \frac{3}{6x^3 - 5}, \qquad x \neq \sqrt[3]{\frac{5}{6}}$$

11 a $\dfrac{dy}{dx} + 18x^3y^2 = 0, \qquad y(-1) = 2$

$$\frac{dy}{dx} = -18x^3y^2$$

$$\int \frac{1}{y^2}\,dy = \int -18x^3\,dx$$

$$\frac{-1}{y} = \frac{-18}{4}x^4 + c$$

$$= \frac{-9}{2}x^4 + c$$

$$x = -1, \qquad y = 2$$

$$\frac{-1}{2} = \frac{-9}{2} + c$$

$$c = 4$$

$$\frac{-1}{y} = \frac{-9}{2}x^4 + 4$$

$$\frac{-1}{y} = \frac{8 - 9x^4}{2}$$

$$y = \frac{2}{9x^4 - 8}, \quad x \neq \pm\sqrt[4]{\frac{8}{9}}$$

b $\frac{dy}{dx} - \frac{y^2}{x^2} = 0 \quad y(1) = 2$

$$\frac{dy}{dx} = \frac{y^2}{x^2}$$
$$\int \frac{1}{y^2} dy = \int \frac{1}{x^2} dx$$
$$\frac{-1}{y} = \frac{-1}{x} + c$$
$$x = 1, \quad y = 2$$
$$\frac{-1}{2} = -1 + c$$
$$c = \frac{1}{2}$$
$$\frac{-1}{y} = \frac{-1}{x} + \frac{1}{2}$$
$$\frac{-1}{y} = \frac{x-2}{2x}$$
$$y = \frac{2x}{2-x}, \quad x \neq 2$$

12 a $\frac{dy}{dx} = y^2 e^{2x}, \quad y(0) = 1$

$$\int \frac{1}{y^2} dy = \int e^{2x} dx$$
$$\frac{-1}{y} = \frac{1}{2} e^{2x} + c$$
$$x = 0, \quad y = 1$$
$$-1 = \frac{1}{2} + c$$
$$c = \frac{-3}{2}$$
$$\frac{-1}{y} = \frac{1}{2} e^{2x} - \frac{3}{2}$$
$$\frac{-1}{y} = \frac{1}{2}(e^{2x} - 3)$$
$$y = \frac{2}{3 - e^{2x}}, \quad x \neq \log_e\left(\sqrt{3}\right)$$

b $\frac{dy}{dx} + 12x^5y^2 = 0, \quad y(1) = 2$

$$\frac{dy}{dx} = -12x^5y^2$$
$$\int \frac{1}{y^2} dy = \int -12x^5 dx$$
$$\frac{-1}{y} = -2x^6 + c$$
$$x = 1, \quad y = 2$$
$$\frac{-1}{2} = -2 + c$$
$$c = \frac{3}{2}$$
$$\frac{-1}{y} = -2x^6 + \frac{3}{2}$$
$$\frac{1}{y} = 2x^6 - \frac{3}{2}$$
$$\frac{1}{y} = \frac{4x^6 - 3}{2}$$
$$y = \frac{2}{4x^6 - 3}$$

13 a $\frac{dy}{dx} + y = 3x^2y, \quad y(0) = 1$

$$\frac{dy}{dx} = 3x^2y - y$$
$$= y\left(3x^2 - 1\right)$$
$$\int \frac{1}{y} dy = \int \left(3x^2 - 1\right) dx$$
$$\log_e |y| = x^3 - x + c$$
$$y = 1, \quad x = 0$$
$$\log_e (1) = 0 + c$$
$$c = 0$$
$$y = e^{x^3 - x}$$

Note: The absolute value can be removed since
LHS = RHS when the given condition is substituted in.

b $\frac{dy}{dx} + 6x^2y^2 = y^2 \quad y(-1) = 2$

$$\frac{dy}{dx} = y^2 - 6x^2y^2$$
$$= y^2\left(1 - 6x^2\right)$$
$$\int \frac{1}{y^2} dy = \int \left(1 - 6x^2\right) dx$$
$$\frac{-1}{y} = x - 2x^3 + c$$
$$y = 2, \quad x = -1$$
$$\frac{-1}{2} = -1 + 2 + c$$
$$c = \frac{-3}{2}$$
$$\frac{-1}{y} = x - 2x^3 - \frac{3}{2}$$
$$\frac{-1}{y} = \frac{2x - 4x^3 - 3}{2}$$
$$y = \frac{2}{4x^3 - 2x + 3}$$

14 a $\frac{dy}{dx} + 2xy^2 = y^2 \quad y(1) = 2$

$$\frac{dy}{dx} = y^2 - 2xy^2$$
$$= y^2(1 - 2x)$$
$$\int \frac{1}{y^2} dy = \int (1 - 2x) dx$$
$$\frac{-1}{y} = x - x^2 + c$$
$$y = 2, \quad x = 1$$
$$\frac{-1}{2} = 1 - 1 + c$$
$$c = \frac{-1}{2}$$
$$\frac{-1}{y} = x - x^2 - \frac{1}{2}$$
$$\frac{-1}{y} = \frac{2x - 2x^2 - 1}{2}$$
$$y = \frac{2}{2x^2 - 2x + 1}$$

b $\frac{dy}{dx} + 8x^3y^4 = y^4 \quad y(0) = 1$

$$\frac{dy}{dx} = y^4 - 8x^3y^4$$
$$= y^4\left(1 - 8x^3\right)$$

$$\int \frac{1}{y^4}dy = \int \left(1 - 8x^3\right) dx$$
$$\int y^{-4}dy = \int \left(1 - 8x^3\right) dx$$
$$\frac{-1}{3}y^{-3} = x - 2x^4 + c$$
$$y = 1, \qquad x = 0$$
$$\frac{-1}{3} = c$$
$$\frac{-1}{3y^3} = x - 2x^4 - \frac{1}{3}$$
$$\frac{-1}{3y^3} = \frac{3x - 6x^4 - 1}{3}$$
$$y^3 = \frac{1}{6x^4 - 3x + 1}$$
$$y = \frac{1}{\sqrt[3]{6x^4 - 3x + 1}}$$

15 a $x\frac{dy}{dx} + 2y = y^2 \qquad y(1) = 1$

$$x\frac{dy}{dx} = y^2 - 2y$$
$$\int \frac{1}{y^2 - 2y}dy = \int \frac{1}{x}dx$$

Partial fractions

$$\frac{1}{y^2 - 2y} = \frac{1}{y(y-2)}$$
$$= \frac{A}{y} + \frac{B}{y-2}$$
$$= \frac{A(y-2) + By}{y(y-2)}$$
$$= \frac{y(A+B) - 2A}{y^2 - 2y}$$
$$A + B = 0$$
$$\Rightarrow -2A = 1$$
$$\Rightarrow A = \frac{-1}{2}$$
$$\Rightarrow B = \frac{1}{2}$$
$$\frac{1}{2}\int \left(\frac{1}{y-2} - \frac{1}{y}\right) dy = \int \frac{1}{x}dx$$
$$\frac{1}{2}\left(\log_e |y-2| - \log_e |y|\right) = \log_e |x| + c$$
$$\frac{1}{2}\log_e \left|\frac{y-2}{y}\right| = \log_e |x| + c$$
$$y = 1, \quad x = 1$$
$$c = 0$$
$$\log_e \left|\frac{y-2}{y}\right| = \left(2\log_e |x|\right)$$
$$\frac{2-y}{y} = x^2 \quad \text{since } x = 1 \text{ when } y = 1$$
$$2 - y = yx^2$$
$$y + yx^2 = 2$$
$$y\left(1 + x^2\right) = 2$$
$$y = \frac{2}{1 + x^2}$$

b $x\frac{dy}{dx} - 4y = y^2, \qquad y(1) = 1$

$$x\frac{dy}{dx} = y^2 + 4y$$
$$\int \frac{1}{y^2 + 4y}dy = \int \frac{1}{x}dx$$

Partial fractions

$$\frac{1}{y^2 + 4y} = \frac{1}{y(y+4)}$$
$$= \frac{A}{y} + \frac{B}{y+4}$$
$$= \frac{A(y+4) + By}{y(y+4)}$$
$$= \frac{y(A+B) + 4A}{y^2 + 4y}$$
$$A + B = 0$$
$$\Rightarrow 4A = 1$$
$$A = \frac{1}{4}$$
$$\Rightarrow B = \frac{-1}{4}$$
$$\frac{1}{4}\int \frac{1}{y} - \frac{1}{y+4}dy = \int \frac{1}{x}dx$$
$$\frac{1}{4}\left(\log_e |y| - \log_e |y+4|\right) = \log_e |x| + c$$
$$\frac{1}{4}\log_e \left|\frac{y}{y+4}\right| = \log_e |(x)| + c$$
$$y = 1, \qquad x = 1$$
$$\frac{1}{4}\log_e \left(\frac{1}{5}\right) = \log_e(1) + c$$
$$c = \frac{1}{4}\log_e \left(\frac{1}{5}\right)$$
$$\frac{1}{4}\log_e \left(\left|\frac{y}{y+4}\right|\right) = \log_e (|x|) + \frac{1}{4}\log_e \left(\frac{1}{5}\right)$$
$$\log_e \left(\left|\frac{y}{y+4}\right|\right) = 4\log_e (|x|) + \log_e \left(\frac{1}{5}\right)$$
$$= \log_e \left(\frac{x^4}{5}\right)$$
$$\frac{y}{y+4} = \frac{x^4}{5}$$

Note: The absolute value can be removed since LHS = RHS when the given condition is substituted in.

$$5y = (y+4)x^4$$
$$5y = yx^4 + 4x^4$$
$$5y - yx^4 = 4x^4$$
$$y\left(5 - x^4\right) = 4x^4$$
$$y = \frac{4x^4}{5 - x^4}, \ x \neq \pm\sqrt[4]{5}$$

16 a $\left(4 + \boldsymbol{x}^2\right)\frac{dy}{dx} - 2xy = 0 \quad y(0) = 1$

$$\left(4 + \boldsymbol{x}^2\right)\frac{dy}{dx} = 2xy$$
$$\int \frac{1}{y}dy = \int \frac{2x}{4 + x^2}dx$$
$$\log_e (|y|) = \log_e \left(4 + x^2\right) + c$$

$y = 1 \qquad x = 0$

$\log_e(1) = \log_e(4) + c$

$c = -\log_e(4)$

$\log_e(|y|) = \log_e\left(4 + x^2\right) - \log_e(4)$

$$\log_e(|y|) = \log_e\left(\frac{4 + x^2}{4}\right)$$

$$y = \frac{1}{4}\left(x^2 + 4\right)$$

Note: The absolute value can be removed since LHS = RHS when the given condition is substituted in.

b $\dfrac{y^2 + 4}{x^2 + 9} - \dfrac{y}{x}\dfrac{dy}{dx} = 0 \qquad y(0) = 2$

$$\frac{y^2 + 4}{x^2 + 9} = \frac{y}{x}\frac{dy}{dx}$$

$$\int \frac{y}{y^2 + 4}dy = \int \frac{x}{x^2 + 9}dx$$

$$\frac{1}{2}\log_e\left(y^2 + 4\right) = \frac{1}{2}\log_e\left(x^2 + 9\right) + c$$

$y = 2, \quad x = 0$

$$\frac{1}{2}\log_e(8) = \frac{1}{2}\log_e(9) + c$$

$$c = \frac{1}{2}\log_e\left(\frac{8}{9}\right)$$

$$\frac{1}{2}\log_e\left(y^2 + 4\right) = \frac{1}{2}\log_e\left(x^2 + 9\right) + \frac{1}{2}\log_e\left(\frac{8}{9}\right)$$

$$\log_e\left(y^2 + 4\right) = \log_e\left(\frac{8\left(x^2 + 9\right)}{9}\right)$$

$$\begin{aligned} y^2 + 4 &= \frac{8\left(x^2 + 9\right)}{9} \\ &= \frac{8x^2}{9} + 8 \\ y^2 &= \frac{8x^2}{9} + 4 \\ &= \frac{1}{9}\left(8x^2 + 36\right) \\ &= \frac{4}{9}\left(2x^2 + 9\right) \\ y &= \frac{2}{3}\sqrt{2x^2 + 9} \quad \text{since } y > 0 \end{aligned}$$

17 a $\dfrac{dy}{dx} - x\left(25 + y^2\right) = 0 \qquad y(0) = 0$

$$\frac{dy}{dx} = x\left(25 + y^2\right)$$

$$\int \frac{1}{25 + y^2}dy = \int x\,dx$$

$$\frac{1}{5}\tan^{-1}\left(\frac{y}{5}\right) = \frac{1}{2}x^2 + c$$

$y = 0 \quad x = 0$

$\Rightarrow c = 0$

$$\begin{aligned} \tan^{-1}\left(\frac{y}{5}\right) &= \frac{5x^2}{2} \\ \frac{y}{5} &= \tan\left(\frac{5x^2}{2}\right) \\ y &= 5\tan\left(\frac{5x^2}{2}\right) \end{aligned}$$

$$\begin{aligned} \left|\frac{5x^2}{2}\right| &< \frac{\pi}{2} \\ |x^2| &< \frac{\pi}{5} \\ |x| &< \sqrt{\frac{\pi}{5}} \\ |x| &< \frac{\sqrt{5\pi}}{5} \end{aligned}$$

b $\dfrac{dy}{dx} + 4x\sqrt{25 - y^2} = 0 \quad y(0) = 5$

$$\frac{dy}{dx} = -4x\sqrt{25 - y^2}$$

$$\int \frac{-1}{\sqrt{25 - y^2}}dy = \int 4x dx$$

$$\cos^{-1}\left(\frac{y}{5}\right) = 2x^2 + c$$

$y = 5 \quad x = 0$

$\cos^{-1}(1) = 0 + c$

$c = 0$

$$\begin{aligned} \cos^{-1}\left(\frac{y}{5}\right) &= 2x^2 \\ \frac{y}{5} &= \cos\left(2x^2\right) \\ y &= 5\cos\left(2x^2\right) \end{aligned}$$

$$\begin{aligned} |2x^2| &\le \pi \\ |x^2| &\le \frac{\pi}{2} \\ |x| &\le \sqrt{\frac{\pi}{2}} \\ |x| &\le \frac{\sqrt{2\pi}}{2} \end{aligned}$$

18 $v = \dfrac{y}{x}$

$y = x \cdot v$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$x\dfrac{dy}{dx} + 3y = 4x \qquad y(2) = 1$

$$\begin{aligned} x\frac{dy}{dx} &= 4x - 3y \\ \frac{dy}{dx} &= \frac{4x - 3y}{x} \\ &= 4 - \frac{3y}{x} \\ v + x\frac{dv}{dx} &= 4 - 3v \\ x\frac{dv}{dx} &= 4 - 4v \\ &= 4(1 - v) \end{aligned}$$

$$\int \frac{1}{1 - v}dv = \int \frac{4}{x}dx$$

$$-\log_e(|1 - v|) = 4\log_e(|x|) + c$$

$y = 1 \qquad x = 2 \ \Rightarrow v = \dfrac{1}{2}$

$$-\log_e\left(\frac{1}{2}\right) = 4\log_e(2) + c$$

$$\begin{aligned} c &= -\log_e\left(\frac{1}{2}\right) - 4\log_e(2) \\ &= \log_e(2) - 4\log_e(2) \\ &= -3\log_e(2) \end{aligned}$$

$$-\log_e|1-v| = 4\log_e(|x|) - \log_e(8)$$

$$-\log_e|1-v| = \log_e\left(\frac{x^4}{8}\right)$$

$$\log_e|1-v| = \log_e\left(\frac{8}{x^4}\right)$$

$$1 - v = 1 - \frac{y}{x} = \frac{8}{x^4}$$

Note: The absolute value can be removed since LHS = RHS when the given condition is substituted in.

$$\frac{y}{x} = 1 - \frac{8}{x^4}$$

$$= \frac{x^4-8}{x^4},$$

$$y = \frac{x^4-8}{x^3}, \qquad x \neq 0$$

19 $x\dfrac{dy}{dx} - y = 4x, \qquad y(1) = 2$

$$x\frac{dy}{dx} = 4x + y$$

$$\frac{dy}{dx} = \frac{4x+y}{x}$$

$$= 4 + \frac{y}{x}$$

$$v + x\frac{dv}{dx} = 4 + v$$

$$x\frac{dv}{dx} = 4$$

$$\int dv = 4\int \frac{1}{x}dx$$

$$v = 4\log_e(|x|) + c$$

$y = 2 \quad x = 1 \quad v = 2$

$2 = 4\log_e(1) + c$

$c = 2$

$$v = 4\log_e(|x|) + 2$$

$$\frac{y}{x} = 2\left(1 + 2\log_e(|x|)\right)$$

$$y = 2x\left(1 + 2\log_e(|x|)\right), \qquad x \neq 0$$

20 **a** $a = -1$

$$x\frac{dy}{dx} - y = bx$$

$$x\frac{dy}{dx} = bx + y$$

$$\frac{dy}{dx} = b + \frac{y}{v}$$

$$v + x\frac{dv}{dx} = b + v$$

$$x\frac{dv}{dx} = b$$

$$\int dv = b\int\left(\frac{1}{x}\right)dx$$

$$v = \frac{y}{x} = b\log_e(|x|) + c$$

$$y = bx\log_e(|x|) + cx$$

$$y = x\left(c + b\log_e(|x|)\right), \ x \neq 0$$

b $x\dfrac{dy}{dx} + ay = bx$

$$x\frac{dy}{dx} = bx - ay$$

$$\frac{dy}{dx} = \frac{bx - ay}{x}$$

$$= b - \frac{ay}{x}$$

$$= b - av$$

$$v + x\frac{dv}{dx} = b - av$$

$$x\frac{dv}{dx} = b - (a+1)v$$

$$\int \frac{1}{b-(a+1)v}dv = \int \frac{1}{x}dx \qquad a \neq -1$$

$$\frac{-1}{a+1}\log_e|b-(a+1)v| = \log_e(|x|) + \log_e A$$

$$= \log_e(|Ax|)$$

$$\log_e|b-(a+1)v| = -(a+1)\log_e(|Ax|)$$

$$\log_e|b-(a+1)v| = \log_e\left(\frac{1}{|Ax|}\right)^{a+1}$$

$$b - (a+1)v = \frac{\pm 1}{(Ax)^{a+1}}$$

$$b - (a+1)\frac{y}{x} = \frac{B}{x^{a+1}} \qquad B = \frac{\pm 1}{A^{a+1}}$$

$$bx - (a+1)y = \frac{B}{x^a}$$

$$(a+1)y = bx + \frac{D}{x^a} \qquad D = -B$$

$$y = \frac{bx}{a+1} + \frac{c}{x^a}, \ c = D/(\mathrm{a}+1)$$

8.5 Exam questions

1 $\dfrac{dy}{dx} = \dfrac{2ye^{2x}}{1+e^{2x}}$ Separating the variables:

$\log_e(|y|) = \log_e\left(1+e^{2x}\right) + c$ To find c use $x = 0, \ y = \pi$

$\log_e(\pi) = \log_e(2) + c, \quad c = \log_e\left(\dfrac{\pi}{2}\right)$

$\log_e(y) = \log_e\left(1+e^{2x}\right) + \log_e\left(\dfrac{\pi}{2}\right)$

Since $y > 0$

modulus is not needed

$\log_e(y) = \log_e\left(\dfrac{\pi}{2}\left(1+e^{2x}\right)\right)$

$$y = \frac{\pi}{2}\left(1+e^{2x}\right)$$

Award 1 mark for separating variables.

Award 1 mark for correct integration.

Award 1 mark for finding constant of integration.

Award 1 mark for correct rule as $y = \ldots$

2 $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$

$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$

$\sin(x+y) - \sin(x-y) = 2\cos(x)\sin(y)$

$$\frac{dy}{dx} = \frac{2}{\sin(x+y) - \sin(x-y)}$$

$$\frac{dy}{dx} = \frac{2}{2\cos(x)\sin(y)}$$

$$\int \sin(y)dy = \int \frac{1}{\cos(x)}dx = \int \sec(x)dx$$

The correct answer is **D**.

3 $\frac{dy}{dx} = \frac{-x}{1+y^2}, y(-1) = 1$

$$\int \left(1+y^2\right) dy = \int -x\, dx$$

$$y + \frac{1}{3}y^3 = -\frac{1}{2}x^2 + c$$

When $x = -1,\ y = 1$

$$1 + \frac{1}{3} = -\frac{1}{2} + c \Rightarrow c = \frac{11}{6}$$

$$y + \frac{1}{3}y^3 = -\frac{1}{2}x^2 + \frac{11}{6}$$

$2y^3 + 6y + 3x^2 - 11 = 0, \quad a = 2,\ b = 6,\ c = 3,\ d = -11$

Award 1 mark for the correct separation of the variables.

Award 1 mark for the correct values.

8.6 Solving Type 4 differential equations, $\frac{d^2y}{dx^2} = f(x)$

8.6 Exercise

1 $\frac{d^2y}{dx^2} + 30x^4 = 0,$

$$\frac{d^2y}{dx^2} = -30x^4$$

$$\frac{dy}{dx} = \int -30x^4 dx$$

$$= -6x^5 + c_1$$

$$y = \int \left(c_1 - 6x^5\right) dx$$

$$= c_2 + c_1 x - x^6$$

2 $\frac{d^2y}{dx^2} + 36\ \sin(3x) = 0,$

$$\frac{d^2y}{dx^2} = -36\ \sin(3x)$$

$$\frac{dy}{dx} = \int -36 \sin(3x) dx$$

$$= 12\cos(3x) + c_1$$

$$y = \int (c_1 + 12\cos(3x))\, dx$$

$$= c_2 + c_1 x + 4\sin(3x)$$

3 **a** $x^3\frac{d^2y}{dx^2} + 4 = 0$

$$x^3\frac{d^2y}{dx^2} = -4$$

$$\frac{d^2y}{dx^2} = \frac{-4}{x^3}$$

$$\frac{dy}{dx} = \int -4x^{-3} dx$$

$$\frac{dy}{dx} = 2x^{-2} + c_1$$

$$y = \int \left(2x^{-2} + c_1\right) dx$$

$$y = \frac{-2}{x} + c_1 x + c_2, \qquad x \neq 0$$

b $\frac{d^2y}{dx^2} + (x+4)(2x-5) = 0$

$$\frac{d^2y}{dx^2} = -(x+4)(2x-5)$$

$$\frac{dy}{dx} = \int -\left(2x^2 + 3x - 20\right) dx$$

$$\frac{dy}{dx} = \frac{-2x^3}{3} - \frac{3x^2}{2} + 20x + c_1$$

$$y = \int \left(\frac{-2x^3}{3} - \frac{3x^2}{2} + 20x + c_1\right) dx$$

$$y = \frac{-x^4}{6} - \frac{x^3}{2} + 10x^2 + c_1 x + c_2$$

4 **a** $x^3\frac{d^2y}{dx^2} + 2x - 5 = 0$

$$x^3\frac{d^2y}{dx^2} = 5 - 2x$$

$$\frac{d^2y}{dx^2} = \frac{5-2x}{x^3}$$

$$\frac{dy}{dx} = \int \left(5x^{-3} - 2x^{-2}\right) dx$$

$$\frac{dy}{dx} = -\frac{5}{2}x^{-2} + 2x^{-1} + c_1$$

$$y = \int \left(-\frac{5}{2}x^{-2} + 2x^{-1} + c_1\right) dx$$

$$y = \frac{5}{2}x^{-1} + 2\log_e(|x|) + c_1 x + c_2,$$

$$y = c_2 + c_1 x + 2\log_e(|x|) + \frac{5}{2x}, \qquad x \neq 0$$

b $e^{3x}\frac{d^2y}{dx^2} + 5 = 2e^{2x}$

$$e^{3x}\frac{d^2y}{dx^2} = 2e^{2x} - 5$$

$$\frac{d^2y}{dx^2} = \frac{2e^{2x} - 5}{e^{3x}}$$

$$\frac{dy}{dx} = \int \left(2e^{-x} - 5e^{-3x}\right) dx$$

$$\frac{dy}{dx} = -2e^{-x} + \frac{5}{3}e^{-3x} + c_1$$

$$y = \int \left(-2e^{-x} + \frac{5}{3}e^{-3x} + c_1\right) dx$$

$$y = 2e^{-x} - \frac{5}{9}e^{-3x} + c_1 x + c_2$$

5 $\frac{d^2y}{dx^2} + 24x^2 = 0, \qquad y'(-1) = 3, \quad y(-1) = 2$

$$\frac{d^2y}{dx^2} = -24x^2$$

$$\frac{dy}{dx} = \int -24x^2 dx$$

$$= -8x^3 + c_1$$

$$\frac{dy}{dx} = 3, \quad x = -1$$

$$3 = 8 + c_1$$

$$c_1 = -5$$

$$\frac{dy}{dx} = -8x^3 - 5$$

$$y = \int \left(-8x^3 - 5\right) dx$$

$$= -2x^4 - 5x + c_2$$

$$y = 2, \qquad x = -1$$

$$2 = -2 + 5 + c_2$$

$$c_2 = -1$$

$$y = -2x^4 - 5x - 1$$

6 $$\frac{d^2y}{dx^2}+12\sin(2x)=0, \qquad y'\left(\frac{\pi}{4}\right)=6, \quad y\left(\frac{\pi}{4}\right)=4$$
$$\frac{d^2y}{dx^2}=-12\sin(2x)$$
$$\frac{dy}{dx}=\int -12\sin(2x)dx$$
$$=6\cos(2x)+c_1$$
$$\frac{dy}{dx}=6 \qquad x=\frac{\pi}{4}$$
$$6=6\cos\left(\frac{\pi}{2}\right)+c_1$$
$$c_1=6$$
$$\frac{dy}{dx}=6\cos(2x)+6$$
$$y=\int(6\cos(2x)+6)\,dx$$
$$=3\sin(2x)+6x+c_2$$
$$x=\frac{\pi}{4}, \qquad y=4$$
$$4=3\sin\left(\frac{\pi}{2}\right)+\frac{3\pi}{2}+c_2$$
$$c_2=1-\frac{3\pi}{2}$$
$$y=3\sin(2x)+6x+1-\frac{3\pi}{2}$$

7 $$\frac{d^2y}{dx^2}+\frac{12}{(3x+16)^3}=0, \qquad y'(0)=0, \quad y(0)=0$$
$$\frac{d^2y}{dx^2}=\frac{-12}{(3x+16)^3}$$
$$\frac{dy}{dx}=\int\frac{-12}{(3x+16)^3}dx$$
$$=\frac{2}{(3x+16)^2}+c_1$$
$$\frac{dy}{dx}=0 \quad x=0$$
$$0=\frac{2}{256}+c_1$$
$$c_1=\frac{-1}{128}$$
$$\frac{dy}{dx}=\frac{2}{(3x+16)^2}-\frac{1}{128}$$
$$y=\int\left(\frac{2}{(3x+16)^2}-\frac{1}{128}\right)dx$$
$$=\frac{-2}{3\,(3x+16)}-\frac{x}{128}+c_2$$
$$x=0, \qquad y=0$$
$$0=-\frac{2}{48}+c_2$$
$$c_2=\frac{1}{24}$$
$$y=\frac{-2}{3\,(3x+16)}-\frac{x}{128}+\frac{1}{24}$$
$$=\frac{-128\times 2-x\times 3\,(3x+16)+16\,(3x+16)}{384\,(3x+16)}$$
$$=\frac{-256-9x^2-48x+48x+256}{384\,(3x+16)}$$
$$=\frac{-9x^2}{384\,(3x+16)}$$
$$=\frac{-3x^2}{128\,(3x+16)}, \qquad x\neq-\frac{16}{3}$$

8 $$\frac{d^2y}{dx^2}+\frac{12}{\sqrt{(2x+9)^3}}=0, \qquad y(0)=0 \qquad y'(0)=1,$$
$$\frac{d^2y}{dx^2}=\frac{-12}{\sqrt{(2x+9)^3}}$$
$$\frac{dy}{dx}=-12\int(2x+9)^{\frac{-3}{2}}dx$$
$$=12(2x+9)^{\frac{-1}{2}}+c_1$$
$$\frac{dy}{dx}=1 \qquad x=0$$
$$1=\frac{12}{\sqrt{9}}+c_1$$
$$c_1=-3$$
$$\frac{dy}{dx}=\frac{12}{\sqrt{2x+9}}-3$$
$$y=\int\left(12(2x+9)^{\frac{-1}{2}}-3\right)dx$$
$$=12(2x+9)^{\frac{1}{2}}-3x+c_2$$
$$y=12\sqrt{2x+9}-3x+c_2$$
$$x=0, \qquad y=0$$
$$0=12\sqrt{9}-0+c_2$$
$$c_2=-36$$
$$y=12\sqrt{2x+9}-3x-36, \qquad x>-\frac{9}{2}$$

9 $$\frac{d^2y}{dx^2}+6x=0, \qquad y(1)=2, \quad y(2)=3$$
$$\frac{d^2y}{dx^2}=-6x$$
$$\frac{dy}{dx}=\int -6x dx$$
$$\frac{dy}{dx}=-3x^2+c_1$$
$$y=\int\left(-3x^2+c_1\right)dx$$
$$y=-x^3+c_1x+c_2,$$
When $y=3, \quad x=2$

(1) $3=-8+2c_1+c_2 \Rightarrow 2c_1+c_2=11$

When $y=2, \quad x=1$

(2) $2=-1+c_1+c_2 \Rightarrow c_1+c_2=3$

$$c_1=8, \quad c_2=-5$$
$$y=-x^3+8x-5$$

10 $$\frac{d^2y}{dx^2}+8\left(e^{2x}+e^{-2x}\right)=0, \qquad x=0, \qquad \frac{dy}{dx}=0, y=0$$
$$\frac{d^2y}{dx^2}=-8e^{2x}-8e^{-2x}$$
$$\frac{dy}{dx}=\int\left(-8e^{2x}-8e^{-2x}\right)dx$$
$$\frac{dy}{dx}=-4e^{2x}+4e^{-2x}+c_1$$
$$\frac{dy}{dx}=0, \qquad x=0$$
$$0=-4+4+c_1$$
$$c_1=0$$
$$\frac{dy}{dx}=-4e^{2x}+4e^{-2x}$$

$$y = \int \left(-4e^{2x} + 4e^{-2x}\right) dx$$
$$= -2e^{2x} - 2e^{-2x} + c_2$$
When $y = 0, \quad x = 0$
$$0 = -2 - 2 + c_2$$
$$c_2 = 4$$
$$y = 4 - 2e^{2x} - 2e^{-2x}$$

11 a $\dfrac{d^2y}{dx^2} + 64\sin(4x) = 0, \quad y(0) = 4, \quad y'(0) = 8$
$$\frac{d^2y}{dx^2} = -64\sin(4x)$$
$$\frac{dy}{dx} = \int -64\sin(4x)dx$$
$$= 16\cos(4x) + c_1$$
$$\frac{dy}{dx} = 8, \quad x = 0$$
$$8 = 16\cos(0) + c_1$$
$$c_1 = -8$$
$$\frac{dy}{dx} = 16\cos(4x) - 8$$
$$y = \int (16\cos(4x) - 8)\, dx$$
$$= 4\sin(4x) - 8x + c_2$$
$$y = 4, \quad x = 0$$
$$4 = 4\sin(0) - 0 + c_2$$
$$c_2 = 4$$
$$y = 4\sin(4x) - 8x + 4$$

b $\dfrac{d^2y}{dx^2} + 27\cos(3x) = 0, \quad y\left(\dfrac{\pi}{6}\right) = 3, \quad y'\left(\dfrac{\pi}{6}\right) = 9$
$$\frac{d^2y}{dx^2} = -27\cos(3x)$$
$$\frac{dy}{dx} = \int -27\cos(3x)dx$$
$$= -9\sin(3x) + c_1$$
$$\frac{dy}{dx} = 9, \quad x = \frac{\pi}{6}$$
$$9 = -9\sin\left(\frac{\pi}{2}\right) + c_1$$
$$c_1 = 18$$
$$\frac{dy}{dx} = -9\sin(3x) + 18$$
$$y = \int (18 - 9\sin(3x))\, dx$$
$$= 18x + 3\cos(3x) + c_2$$
$$x = \frac{\pi}{6}, \quad y = 3$$
$$3 = 18\left(\frac{\pi}{6}\right) + 3\cos\left(\frac{\pi}{2}\right) + c_2$$
$$c_2 = 3 - 3\pi$$
$$y = 18x + 3\cos(3x) + 3 - 3\pi$$

12 a $\dfrac{d^2y}{dx^2} + 32\sin^2(2x) = 0, \quad y(0) = 0, \quad y'(0) = 0$
$$\frac{d^2y}{dx^2} = -32\sin^2(2x)$$
$$\frac{dy}{dx} = \int -32\sin^2(2x)dx$$
$$= -16\int (1 - \cos(4x))\, dx$$
$$= -16\left(x - \frac{1}{4}\sin(4x)\right) + c_1$$
$$\frac{dy}{dx} = 0, \quad x = 0$$
$$0 = 0 + c_1$$
$$c_1 = 0$$
$$\frac{dy}{dx} = 16\left(\frac{1}{4}\sin(4x) - x\right)$$
$$= 4\sin(4x) - 16x$$
$$y = \int (4\sin(4x) - 16x)\, dx$$
$$= -\cos(4x) - 8x^2 + c_2$$
$$y = 0, \quad x = 0$$
$$0 = -\cos(0) - 0 + c_2$$
$$c_2 = 1$$
$$y = 1 - \cos(4x) - 8x^2$$

b $\dfrac{d^2y}{dx^2} + 16\cos^2(4x) = 0, \quad y(0) = 0, \quad y'(0) = 0$
$$\frac{d^2y}{dx^2} = -16\cos^2(4x)$$
$$\frac{dy}{dx} = \int -16\cos^2(4x)dx$$
$$= -8\int (1 + \cos(8x))\, dx$$
$$= -8\left(x + \frac{1}{8}\sin(8x)\right) + c_1$$
$$\frac{dy}{dx} = 0, \quad x = 0$$
$$0 = -8\,(0) + c_1$$
$$c_1 = 0$$
$$\frac{dy}{dx} = -8x - \sin(8x)$$
$$y = \int (-8x - \sin(8x))\, dx$$
$$= -4x^2 + \frac{1}{8}\cos(8x) + c_2$$
$$x = 0, \quad y = 0$$
$$0 = 0 + \frac{1}{8}\cos(0) + c_2$$
$$c_2 = \frac{-1}{8}$$
$$y = \frac{1}{8}\cos(8x) - 4x^2 - \frac{1}{8}$$

13 a $\dfrac{d^2y}{dx^2} = \dfrac{1}{(3x+2)^3}, \quad y(0) = 0, \quad y'(0) = 0$
$$\frac{dy}{dx} = \int \frac{1}{(3x+2)^3}dx$$
$$= \frac{-1}{6(3x+2)^2} + c_1$$
$$\frac{dy}{dx} = 0, \quad x = 0$$
$$0 = \frac{-1}{24} + c_1$$
$$c_1 = \frac{1}{24}$$
$$\frac{dy}{dx} = \frac{-1}{6\,(3x+2)^2} + \frac{1}{24}$$

$$y = \int \left(\frac{1}{24} - \frac{1}{6(3x+2)^2}\right) dx$$
$$y = \frac{x}{24} + \frac{1}{18\,(3x+2)} + c_2$$
$$x = 0, \quad y = 0$$
$$0 = 0 + \frac{1}{36} + c_2$$
$$c_2 = \frac{-1}{36}$$
$$y = \frac{x}{24} + \frac{1}{18\,(3x+2)} - \frac{1}{36}$$
$$= \frac{9x\,(3x+2) + 12 - 6\,(3x+2)}{216\,(3x+2)}$$
$$= \frac{x^2}{8\,(3x+2)}, \qquad x \neq -\frac{2}{3}$$

b $$\frac{d^2y}{dx^2} + \frac{1}{\sqrt{(2x+9)^3}} = 0, \quad y\,(0) = 0, \quad y'\,(0) = 0$$
$$\frac{d^2y}{dx^2} = -\frac{1}{\sqrt{(2x+9)^3}}$$
$$\frac{dy}{dx} = \int -(2x+9)^{\frac{-3}{2}}\, dx$$
$$= (2x+9)^{\frac{-1}{2}} + c_1$$
$$\frac{dy}{dx} = 0, \quad x = 0$$
$$0 = \frac{1}{\sqrt{9}} + c_1$$
$$c_1 = \frac{-1}{3}$$
$$\frac{dy}{dx} = (2x+9)^{\frac{-1}{2}} - \frac{1}{3}$$
$$y = \int \left((2x+9)^{\frac{-1}{2}} - \frac{1}{3}\right) dx$$
$$y = (2x+9)^{\frac{1}{2}} - \frac{x}{3} + c_2$$
$$y = 0, \quad x = 0$$
$$0 = \sqrt{9} + c_2$$
$$c_2 = -3$$
$$y = \sqrt{2x+9} - \frac{x}{3} - 3, \quad x > -\frac{9}{2}$$

14 a $$\frac{d^2y}{dx^2} = k,$$
$$\frac{dy}{dx} = \int k\, dx$$
$$= kx + c_1$$
$$y = \int (kx + c_1)\, dx$$
$$y = \frac{kx^2}{2} + c_1x + c_2 \qquad \text{parabolas}$$

b $$\frac{d^2y}{dx^2} = -12, \qquad \text{TP}\,(-2, 4),$$
$$\frac{dy}{dx} = \int -12dx$$
$$= -12x + c_1$$
When $x = -2 \quad \dfrac{dy}{dx} = 0$
$$0 = 24 + c_1$$
$$c_1 = -24$$
$$\frac{dy}{dx} = -12x - 24$$
$$y = \int ((-24 - 12x))\, dx$$
$$y = -24x - 6x^2 + c_2$$
$$y = 4, \quad x = -2$$
$$4 = 48 - 24 + c_2$$
$$c_2 = -20$$
$$y = -6x^2 - 24x - 20$$

15 a $$\frac{d^2y}{dx^2} = kx,$$
$$\frac{dy}{dx} = \int kx\, dx$$
$$= \frac{kx^2}{2} + c_1$$
$$y = \int \left(\frac{kx^2}{2} + c_1\right) dx$$
$$= \frac{kx^3}{6} + c_1x + c_2 \qquad \text{cubics}$$

b $$\frac{d^2y}{dx^2} = 18x, \qquad \text{TP}\,(-2, 0),$$
$$\frac{dy}{dx} = \int 18x\, dx$$
$$= 9x^2 + c_1$$
$$\frac{dy}{dx} = 0 \quad x = -2$$
$$0 = 36 + c_1$$
$$c_1 = -36$$
$$\frac{dy}{dx} = 9x^2 - 36$$
$$y = \int \left(\left(9x^2 - 36\right)\right) dx$$
$$y = 3x^3 - 36x + c_2$$
$$y = 0, \quad x = -2$$
$$0 = -24 + 72 + c_2$$
$$c_2 = -48$$
$$y = 3x^3 - 36x - 48$$

16 a $$\frac{d^2y}{dx^2} + \frac{20}{\sqrt{4x+9}} = 0, \quad y\,(0) = 0, \quad y'\,(0) = 0$$
$$\frac{d^2y}{dx^2} = \frac{-20}{\sqrt{4x+9}}$$
$$\frac{dy}{dx} = -20\int (4x+9)^{\frac{-1}{2}}\, dx$$
$$= -10\,(4x+9)^{\frac{1}{2}} + c_1$$
$$\frac{dy}{dx} = 0, \quad x = 0$$
$$0 = -10\sqrt{9} + c_1$$
$$c_1 = 30$$
$$\frac{dy}{dx} = -10\sqrt{4x+9} + 30$$
$$y = \int \left(30 - 10\,(4x+9)^{\frac{1}{2}}\right) dx$$
$$y = 30x - \frac{5}{3}\,(4x+9)^{\frac{3}{2}} + c_2$$

$x = 0, \quad y = 0$

$0 = 0 - \frac{5}{3}(9)^{\frac{3}{2}} + c_2$

$c_2 = 45$

$y = 30x + 45 - \frac{5}{3}\sqrt{(4x+9)^3}, \quad x > -\frac{9}{4}$

b $\frac{d^2y}{dx^2} + \frac{16}{(4x+9)^2} = 0, \quad y(0) = 0, \quad y'(0) = 0$

$\frac{d^2y}{dx^2} = \frac{-16}{(4x+9)^2}$

$\frac{dy}{dx} = \int \frac{-16}{(4x+9)^2}dx$

$= \frac{4}{4x+9} + c_1$

$\frac{dy}{dx} = 0, \quad x = 0$

$0 = \frac{4}{9} + c_1$

$c_1 = \frac{-4}{9}$

$\frac{dy}{dx} = \frac{4}{4x+9} - \frac{4}{9}$

$y = \int \left(\frac{4}{4x+9} - \frac{4}{9}\right) dx$

$= \log_e(|4x+9|) - \frac{4x}{9} + c_2$

$x = 0, \quad y = 0$

$0 = \log_e(|9|) + c_2$

$c_2 = -\log_e(9)$

$y = \log_e(|4x+9|) - \frac{4x}{9} - \log_e(9)$

$y = \log_e\left(\frac{|4x+9|}{9}\right) - \frac{4x}{9}, \quad x \neq -\frac{9}{4}$

17 a $\frac{d^2y}{dx^2} = k\left(x^2 - Lx\right), 0 \le x \le L$

$\frac{dy}{dx} = k\int \left(x^2 - Lx\right) dx$

$\frac{dy}{dx} = k\left(\frac{1}{3}x^3 - \frac{Lx^2}{2} + c_1\right)$

$y = k\int \left(\frac{1}{3}x^3 - \frac{Lx^2}{2} + c_1\right) dx$

$= k\left(\frac{x^4}{12} - \frac{Lx^3}{6} + c_1x + c_2\right)$

When $x = 0$, $y = 0$ and therefore $c_2 = 0$

When $x = L$, $y = 0$

$0 = k\left(\frac{L^4}{12} - \frac{L^4}{6} + c_1L\right)$

$c_1 = \frac{1}{12}L^3$

$y = k\left(\frac{x^4}{12} - \frac{Lx^3}{6} + \frac{L^3}{12}x\right)$

$= \frac{k}{12}\left(x^4 - 2Lx^3 + L^3x\right)$

$\frac{dy}{dx} = \frac{k}{12}\left(4x^3 - 6Lx^2 + L^3\right)$

$x = \frac{L}{2}$ in the middle of the beam.

$\frac{dy}{dx} = \frac{k}{12}\left(4\left(\frac{L}{2}\right)^3 - 6L\left(\frac{L}{2}\right)^2 + L^3\right)$

$= \frac{k}{12}\left(\frac{4L^3}{8} - \frac{6L^3}{4} + L^3\right)$

$= 0$

b $y_{\max} = y\left(\frac{L}{2}\right) = \frac{k}{12}\left(\left(\frac{L}{2}\right)^4 - 2L\left(\frac{L}{2}\right)^3 + L^3\left(\frac{L}{2}\right)\right)$

$= \frac{k}{12}\left(\frac{L^4}{16} - \frac{L^4}{4} + \frac{L^4}{2}\right)$

$= \frac{5kL^4}{192}$

18 $\frac{d^2y}{dx^2} = k(L-x), 0 \le x \le L$

$\frac{dy}{dx} = k\int (L-x)\,dx$

$\frac{dy}{dx} = k\left(Lx - \frac{x^2}{2} + c_1\right)$

When $x = 0$, $\frac{dy}{dx} = 0 \Rightarrow c_1 = 0$

$\frac{dy}{dx} = k\left(Lx - \frac{1}{2}x^2\right)$

$y = k\int \left(Lx - \frac{1}{2}x^2\right) dx$

$= k\left(\frac{Lx^2}{2} - \frac{1}{6}x^3 + c_2\right)$

When $x = 0$, $y = 0 \Rightarrow c_2 = 0$

$y = k\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right)$

$= \frac{k}{6}\left(3Lx^2 - x^3\right) \quad 0 \le x \le L$

$y_{\max}$ occurs at the end point when $x = L$

$y_{\max} = y(L) = \frac{k}{6}\left(3L^3 - L^3\right) = \frac{kL^3}{3}$

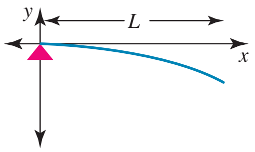
Check when $\frac{dy}{dx} = 0$

$Lx - \frac{1}{2}x^2 = 0$

$x\left(L - \frac{x}{2}\right) = 0$

$\Rightarrow x = 0$ or $x = 2L$, this is outside the domain.

19 a



$\frac{d^2y}{dx^2} = \frac{W}{2EI}(L-x)^2 \quad 0 \le x \le L$

$\frac{dy}{dx} = \frac{W}{2EI}\int \left(L^2 - 2Lx + x^2\right) dx$

$= \frac{W}{2EI}\left[L^2x - Lx^2 + \frac{1}{3}x^3 + c_1\right]$

When $x = 0$, $\frac{dy}{dx} = 0 \Rightarrow c_1 = 0$

$$y = \frac{W}{2EI}\int\left(L^2x - Lx^2 + \frac{1}{3}x^3\right)dx$$

$$= \frac{W}{2EI}\left[\frac{L^2x^2}{2} - \frac{Lx^3}{3} + \frac{1}{12}x^4 + c_2\right]$$

When $x = 0, y = 0 \Rightarrow c_2 = 0$

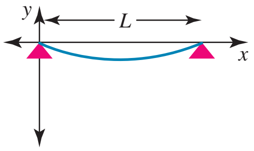
$$y = \frac{W}{24EI}\left(6L^2x^2 - 4Lx^3 + x^4\right)$$

When $x = L$ the maximum occurs at the end point.

$$y_{\text{max}} = \frac{W}{24EI}\left(6L^4 - 4L^4 + L^4\right)$$

$$= \frac{WL^4}{8EI}$$

b



$$EI\frac{d^2y}{dx^2} = \frac{W}{2}\left(x^2 - Lx + \frac{L^2}{6}\right)$$

$$\frac{dy}{dx} = \frac{W}{2EI}\int\left(x^2 - Lx + \frac{L^2}{6}\right)dx$$

$$= \frac{W}{2EI}\left(\frac{1}{3}x^3 - \frac{Lx^2}{2} + \frac{L^2x}{6} + c_1\right)$$

$$y = \frac{W}{2EI}\int\left(\frac{1}{3}x^3 - \frac{Lx^2}{2} + \frac{L^2x}{6} + c_1\right)dx$$

$$= \frac{W}{2EI}\left(\frac{1}{12}x^4 - \frac{Lx^3}{6} + \frac{L^2x^2}{12} + c_1x + c_2\right)$$

When $x = 0,\ y = 0\ \Rightarrow c_2 = 0$

When $x = L, y = 0$

$$0 = \frac{W}{2EI}\left(\frac{L^4}{12} - \frac{L^4}{6} + \frac{L^4}{12} + c_1L\right)$$

$\Rightarrow c_1 = 0$

$$y = \frac{W}{24EI}\left(x^4 - 2Lx^3 + L^2x^2\right)$$

$$= \frac{Wx^2}{24EI}\left(x^2 - 2Lx + L^2\right)$$

$$= \frac{Wx^2}{24EI}(x - L)^2$$

Now

$$\frac{dy}{dx} = \frac{W}{24EI}\left(4x^3 - 6Lx^2 + 2L^2x\right) = 0$$

When $x = \frac{L}{2}$

$$\frac{W}{24EI}\left(\frac{L^3}{2} - \frac{3L^3}{2} + L^3\right) = 0$$

$$y_{\text{max}} = y\left(\frac{L}{2}\right) = \frac{W}{24EI}\left(\frac{L}{2}\right)^2\left(\frac{L}{2} - L\right)^2$$

$$= \frac{WL^4}{384EI}$$

20 $\frac{d^2y}{dx^2} + \frac{1}{(ax+b)^3} = 0, \quad y(0) = 0, \quad y'(0) = 0$

$$\frac{d^2y}{dx^2} = \frac{-1}{(ax+b)^3}$$

$$\frac{dy}{dx} = \int\frac{-1}{(ax+b)^3}dx$$

$$= \frac{1}{2a(ax+b)^2} + c_1$$

$$\frac{dy}{dx} = 0 \quad x = 0$$

$$0 = \frac{1}{2ab^2} + c_1$$

$$c_1 = \frac{-1}{2ab^2}$$

$$\frac{dy}{dx} = \frac{1}{2a(ax+b)^2} - \frac{1}{2ab^2}$$

$$y = \int\left(\frac{1}{2a(ax+b)^2} - \frac{1}{2ab^2}\right)dx$$

$$= \frac{-1}{2a^2(ax+b)} - \frac{x}{2ab^2} + c_2$$

$x = 0, \quad y = 0$

$$0 = \frac{-1}{2a^2b} + c_2$$

$$c_2 = \frac{1}{2a^2b}$$

$$y = \frac{-1}{2a^2(ax+b)} - \frac{x}{2ab^2} + \frac{1}{2a^2b}$$

$$= \frac{-b^2 - ax(ax+b) + b(ax+b)}{2a^2b^2(ax+b)}$$

$$= \frac{-b^2 - a^2x^2 - abx + abx + b^2}{2a^2b^2(ax+b)}$$

$$y = \frac{-x^2}{2b^2(ax+b)}, \qquad x \neq -\frac{b}{a}$$

21 $\frac{d^2y}{dx^2} + \frac{1}{(ax+b)^2} = 0, \quad y(0) = 0, \quad y'(0) = 0$

$$\frac{d^2y}{dx^2} = \frac{-1}{(ax+b)^2}$$

$$\frac{dy}{dx} = \int\frac{-1}{(ax+b)^2}dx$$

$$= \frac{1}{a(ax+b)} + c_1$$

$$\frac{dy}{dx} = 0 \quad x = 0$$

$$0 = \frac{1}{ab} + c_1$$

$$c_1 = \frac{-1}{ab}$$

$$\frac{dy}{dx} = \frac{1}{a(ax+b)} - \frac{1}{ab}$$

$$y = \int\left(\frac{1}{a(ax+b)} - \frac{1}{ab}\right)dx$$

$$= \frac{1}{a^2}\log_e(|ax+b|) - \frac{x}{ab} + c_2$$

$x = 0, y = 0$

$$0 = \frac{1}{a^2}\log_e(b) + c_2$$

$$c_2 = \frac{-1}{a^2}\log_e(b)$$

$$y = \frac{1}{a^2}\log_e(|ax+b|) - \frac{x}{ab} - \frac{1}{a^2}\log_e(b)$$

$$= \frac{1}{a^2}\log_e\left(\frac{|ax+b|}{b}\right) - \frac{x}{ab}, \qquad x \neq -\frac{b}{a}$$

22 a $\frac{d}{dx}\left(\frac{x}{\sqrt{9+4x^2}}\right)$

Using quotient rule,

$$\frac{1\sqrt{9+4x^2}-\frac{1}{2}\times 8x\times\left(9+4x^2\right)^{\frac{-1}{2}}\times x}{9+4x^2}$$

$$=\frac{\sqrt{9+4x^2}-\frac{4x^2}{\sqrt{9+4x^2}}}{9+4x^2}$$

$$=\frac{1}{9+4x^2}\left(\frac{9+4x^2-4x^2}{\sqrt{9+4x^2}}\right)$$

$$=\frac{9}{\sqrt{\left(9+4x^2\right)^3}}\quad\text{shown}$$

b $\frac{d^2y}{dx^2}+\frac{9}{\sqrt{\left(9+4x^2\right)^3}}=0,$

$$\frac{d^2y}{dx^2}=\frac{-9}{\sqrt{\left(9+4x^2\right)^3}}$$

$$\frac{dy}{dx}=-\int\frac{9}{\sqrt{\left(9+4x^2\right)^3}}dx$$

$$=\frac{-x}{\sqrt{9+4x^2}}+c_1$$

$$y=\int\left(c_1-\frac{x}{\sqrt{9+4x^2}}\right)dx$$

$$y=c_1x+c_2-\frac{1}{4}\sqrt{9+4x^2}$$

8.6 Exam questions

1 $\frac{d^2y}{dx^2}+24x^2=0,\qquad y(1)=2,\;\; y(2)=3$

$$\frac{d^2y}{dx^2}=-24x^2$$

$$\frac{dy}{dx}=\int -24x^2dx$$

$$\frac{dy}{dx}=-8x^3+c_1$$

$$y=\int\left(-8x^3+c_1\right)dx$$

$$y=-2x^4+c_1x+c_2$$

When $y=2,\;\; x=1$

(1) $2=-2+c_1+c_2\Rightarrow c_1+c_2=4$

When $y=3,\;\; x=2$

(2) $3=-32+2c_1+c_2\Rightarrow 2c_1+c_2=35$

$c_1=31,\; c_2=-27$

$y=-2x^4+31x-27$ [1 mark]

2 $e^x\frac{d^2y}{dx^2}+4e^{-2x}=5,\qquad x=0,\;\frac{dy}{dx}=0,\; y=0$

$$e^x\frac{d^2y}{dx^2}=5-4e^{-2x}$$

$$\frac{dy}{dx}=\int\left(5e^{-x}-4e^{-3x}\right)dx$$

$$=5e^{-x}+\frac{4}{3}e^{-3x}+c1$$

$\frac{dy}{dx}=0,\qquad x=0$

$$0=-5+\frac{4}{3}+c_1$$

$$c_1=\frac{11}{3}$$

$\frac{dy}{dx}=-5e^{-x}+\frac{4}{3}e^{-3x}+\frac{11}{3}$ [1 mark]

$$y=\int\left(-5e^{-x}+\frac{4}{3}e^{-3x}+\frac{11}{3}\right)dx$$

$$y=5e^{-x}-\frac{4}{9}e^{-3x}+\frac{11x}{3}+c_2$$

$y=0,\qquad x=0$

$$0=5-\frac{4}{9}+c_2$$

$$c_2=\frac{-41}{9}$$

$y=5e^{-x}-\frac{4}{9}e^{-3x}+\frac{11x}{3}-\frac{41}{9}$ [1 mark]

3 a $\frac{d}{dx}\left(\frac{x}{\sqrt{a+bx^2}}\right)$

Using the quotient rule,

$$=\frac{1\sqrt{a+bx^2}-2bx\times\frac{1}{2}\left(a+bx^2\right)^{-\frac{1}{2}}\times x}{a+bx^2}$$

$$=\frac{1}{a+bx^2}\left(\sqrt{a+bx^2}-\frac{bx^2}{\sqrt{a+bx^2}}\right)$$

$$=\frac{1}{a+bx^2}\left(\frac{a+bx^2-bx^2}{\sqrt{a+bx^2}}\right)$$

$=\frac{a}{\left(\sqrt{a+bx^2}\right)^3}$ [1 mark]

b $\frac{d^2y}{dx^2}+\frac{1}{\sqrt{\left(a+bx^2\right)^3}}=0$

$$\frac{d^2y}{dx^2}=-\frac{1}{\sqrt{\left(a+bx^2\right)^3}}$$

$$\frac{dy}{dx}=-\int\frac{1}{\sqrt{\left(a+bx^2\right)^3}}dx$$

$$=\frac{-x}{a\sqrt{a+bx^2}}+c_1$$

$$y=\frac{-1}{a}\int\left(\frac{x}{\sqrt{a+bx^2}}+c_1\right)dx$$

$$=\frac{-1}{a}\times\frac{1}{2b}\times 2\sqrt{a+bx^2}+c_1x+c_2$$

$=c_2+c_1x-\frac{1}{ab}\sqrt{a+bx^2}$ [1 mark]

8.7 Review

8.7 Exercise

Technology free: short answer

1 a $y = \sin^2(3x)$

$$\begin{aligned}\frac{dy}{dx} &= 2 \times 3\sin(3x)\cos(3x)\\ &= 3\sin(6x)\end{aligned}$$

$$\frac{d^2y}{dx^2} = 18\cos(6x)$$

$$\begin{aligned}\frac{d^2y}{dx^2} + 36y - 18 &= 18\cos(6x) + 36\sin^2(3x) - 18\\ &= 18\cos(6x) + 36\left(1 - \cos^2(3x)\right) - 18\\ &= 18\cos(6x) + 36\left(1 - \frac{1}{2}(1 + \cos(6x))\right)\\ &\quad - 18\\ &= 18\cos(6x) + 36 - 18 - 18\cos(6x) - 18\\ &= 0\\ &= \text{RHS}\end{aligned}$$

b $y = \sin^{-1}\left(\frac{x}{3}\right)$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\sqrt{9 - x^2}}\\ &= \left(9 - x^2\right)^{\frac{-1}{2}}\end{aligned}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= -2x \times \frac{-1}{2}\left(9 - x^2\right)^{\frac{-3}{2}}\\ &= \frac{x}{\left(9 - x^2\right)^{\frac{3}{2}}}\end{aligned}$$

$$\begin{aligned}\left(9 - x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} &= \left(9 - x^2\right) \times \frac{x}{\left(9 - x^2\right)^{\frac{3}{2}}} - x \times \left(9 - x^2\right)^{\frac{-1}{2}}\\ &= \frac{x}{\sqrt{9 - x^2}} - \frac{x}{\sqrt{9 - x^2}}\\ &= 0\\ &= \text{RHS}\end{aligned}$$

2 a $y = \log_e\left(x + \sqrt{x^2 + 9}\right)$

$$\frac{dy}{dx} = \left(1 + \frac{x}{\sqrt{x^2 + 9}}\right) \times \frac{1}{x + \sqrt{x^2 + 9}}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{\sqrt{x^2 + 9} + x}{\sqrt{x^2 + 9}} \times \frac{1}{x + \sqrt{x^2 + 9}}\\ &= \frac{1}{\sqrt{x^2 + 9}}\end{aligned}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 + 9}} = \left(x^2 + 9\right)^{-\frac{1}{2}}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= -\frac{1}{2} \times 2x \times \left(x^2 + 9\right)^{-\frac{3}{2}}\\ &= \frac{-x}{\left(x^2 + 9\right)^{\frac{3}{2}}}\end{aligned}$$

$$\left(x^2 + 9\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$$

$$\left(x^2 + 9\right)\frac{-x}{\left(x^2 + 9\right)^{\frac{3}{2}}} + x\left(x^2 + 9\right)^{-\frac{1}{2}} = 0$$

$$-x\left(x^2 + 9\right)^{-\frac{1}{2}} + x\left(x^2 + 9\right)^{-\frac{1}{2}} = 0$$

b $y = Ax\sin(3x)$

$$\frac{dy}{dx} = A\sin(3x) + 3Ax\cos(3x)$$

$$\frac{d^2y}{dx^2} = 3A\cos(3x) + 3A\cos(3x) - 9Ax\sin(3x)$$

$$\frac{d^2y}{dx^2} + 9y = 6A\cos(3x) = 18\cos(3x)$$

$$6A = 18$$

$$A = 3$$

3 a $\frac{dy}{dx} - x^2 = 4, \quad y(1) = 2$

$$\frac{dy}{dx} = 4 + x^2$$

$$\begin{aligned}y &= \int\left(4 + x^2\right)dx\\ &= 4x + \frac{1}{3}x^3 + c\end{aligned}$$

When $y = 2, \quad x = 1$

$$2 = 4 + \frac{1}{3} + c$$

$$c = -\frac{7}{3}$$

$$y = 4x + \frac{1}{3}\left(x^3 - 7\right)$$

b $x^2\frac{dy}{dx} + 4 = 2x, \quad y(1) = 2$

$$x^2\frac{dy}{dx} = 2x - 4$$

$$\frac{dy}{dx} = \frac{2x - 4}{x^2}$$

$$\begin{aligned}y &= \int\left(\frac{2}{x} - 4x^{-2}\right)dx\\ &= 2\log_e(|x|) + 4x^{-1} + c\end{aligned}$$

When $x = 1 \quad y = 2$

$$2 = 2\log_e(1) + 4 + c$$

$$c = -2$$

$$y = 2\log_e(|x|) + \frac{4}{x} - 2, \quad x \neq 0$$

4 a $\left(x^2 + 9\right)\frac{dy}{dx} + 3 = 0$

$$\frac{dy}{dx} = -\frac{3}{x^2 + 9}$$

$$y = -\int\frac{3}{x^2 + 9}dx$$

$$y = -\tan^{-1}\left(\frac{x}{3}\right) + c$$

Since $y(0) = 2$,

$2 = -\tan^{-1}(0) + c$

$c = 2$

$y = -\tan^{-1}\left(\frac{x}{3}\right) + 2$

b $\sqrt{9+x^2}\dfrac{dy}{dx}+3x=0, \quad y(0)=2$

$$\sqrt{9+x^2}\frac{dy}{dx}=-3x$$

$$\frac{dy}{dx}=\frac{-3x}{\sqrt{9+x^2}}$$

$$y=\int\left(\frac{-3x}{\sqrt{9+x^2}}\right)dx$$

Let $u=9+x^2, \dfrac{du}{dx}=2x, dx=\dfrac{du}{2x}$

$$y=-\frac{3}{2}\int\left(\frac{x}{\sqrt{u}}\times\frac{du}{x}\right)$$

$$y=-\frac{3}{2}\int\left(\frac{1}{\sqrt{u}}du\right)$$

$$y=-\frac{3}{2}\times 2\sqrt{u}+c$$

$$y=-3\sqrt{9+x^2}+c, \; y(0)=2$$

$$2=-3\times 3+c$$

$$c=11$$

$$y=11-3\sqrt{9+x^2}$$

c $\left(9-x^2\right)\dfrac{dy}{dx}+3=0, \quad y(0)=2$

$$\left(9-x^2\right)\frac{dy}{dx}=-3$$

$$\frac{dy}{dx}=\frac{-3}{9-x^2}$$

$$=\frac{3}{x^2-9}$$

$$\frac{3}{x^2-9}=\frac{A}{x+3}+\frac{B}{x-3}$$

$$=\frac{A(x-3)+B(x+3)}{(x+3)(x-3)}$$

$$=\frac{x(A+B)+3B-3A}{x^2-9}$$

$$A+B=0 \quad A\Rightarrow -B$$

$$3(B-A)=3$$

$$B=\frac{1}{2}\;A=-\frac{1}{2}$$

$$y=\frac{1}{2}\int\left(\frac{1}{x-3}-\frac{1}{x+3}\right)dx$$

$$=\frac{1}{2}\log_e\left(\left|\frac{x-3}{x+3}\right|\right)+c$$

When $x=0 \;\; y=2$

$$2=\frac{1}{2}\log_e(1)+c$$

$$c=2$$

$$y=2+\frac{1}{2}\log_e\left(\left|\frac{x-3}{x+3}\right|\right), \quad x\neq\pm 3$$

5 a $\dfrac{dy}{dx}+9y^2=0, \; y(0)=1$

$$\frac{dy}{dx}=-9y^2$$

$$\frac{dx}{dy}=\frac{-1}{9y^2}$$

$$x=\frac{-1}{9}\int\left(y^{-2}\right)\,dy$$

$$x=\frac{1}{9}y^{-1}+c$$

$$=\frac{1}{9y}+c$$

When $y=1 \;\; x=0$

$$0=\frac{1}{9}+c$$

$$c=-\frac{1}{9}$$

$$x=\frac{1}{9y}-\frac{1}{9}$$

$$\frac{1}{9y}=x+\frac{1}{9}$$

$$\frac{1}{9y}=\frac{9x+1}{9}$$

$$y=\frac{1}{9x+1}, \quad x\neq-\frac{1}{9}$$

b $\dfrac{dy}{dx}+\dfrac{9}{y}=0, \;\; y(0)=1$

$$\frac{dy}{dx}=\frac{-9}{y}$$

$$\frac{dx}{dy}=\frac{-y}{9}$$

$$x=\frac{-1}{9}\int y\,dy$$

$$x=\frac{-1}{9\times 2}y^2+c$$

$$x=\frac{-1}{18}y^2+c$$

When $x=0 \;\; y=1$

$$0=\frac{-1}{18}+c$$

$$c=\frac{1}{18}$$

$$x=\frac{1}{18}-\frac{1}{18}y^2$$

$$=\frac{1}{18}\left(1-y^2\right)$$

$$18x=1-y^2$$

$$y^2=1-18x$$

$$y=\sqrt{1-18x}, \qquad x<\frac{1}{18}$$

c $\dfrac{dy}{dx}+9y^3=0, \;\; y(0)=1$

$$\frac{dy}{dx}=-9y^3$$

$$\frac{dx}{dy}=\frac{-1}{9y^3}$$

$$x=\frac{-1}{9}\int y^{-3}dy$$

$$x=\frac{1}{18}y^{-2}+c$$

When $x=0, \;\; y=1$

$$0 = \frac{1}{18} + c$$
$$c = -\frac{1}{18}$$
$$x = \frac{1}{18y^2} - \frac{1}{18}$$
$$x = \frac{1}{18}\left(\frac{1}{y^2} - 1\right)$$
$$18x = \frac{1}{y^2} - 1$$
$$18x + 1 = \frac{1}{y^2}$$
$$y^2 = \frac{1}{18x + 1}$$
$$y = \frac{1}{\sqrt{18x + 1}}, \quad x > -\frac{1}{18}$$

6 a $\frac{dy}{dx} + 4y = 7, \quad y(0) = 2$

$$\frac{dy}{dx} = 7 - 4y$$
$$\frac{dx}{dy} = \frac{1}{7 - 4y}$$
$$x = \int \frac{1}{7 - 4y}dy$$
$$x = \frac{-1}{4}\log_e(|7 - 4y|) + c$$

When $x = 0, \qquad y = 2$

$$0 = -\frac{1}{4}\log_e|-1|$$
$$c = 0$$
$$x = \frac{-1}{4}\log_e(|7 - 4y|)$$
$$-4x = \log_e(|7 - 4y|)$$
$$e^{-4x} = |7 - 4y|$$
$$e^{-4x} = 4y - 7 \qquad \text{since } x = 0 \text{ when } y = 2$$
$$4y = 7 + e^{-4x}$$
$$y = \frac{1}{4}\left(7 + e^{-4x}\right)$$

b $\frac{dy}{dx} + (5 - 2y)^2 = 0, \quad y(1) = 2$

$$\frac{dy}{dx} = -(5 - 2y)^2$$
$$\frac{dx}{dy} = \frac{-1}{(5 - 2y)^2}$$
$$x = \int \frac{-1}{(5 - 2y)^2}dy$$
$$x = \frac{-1}{2(5 - 2y)} + c$$

When $\quad x = 1, \qquad y = 2$

$$1 = \frac{-1}{2} + c$$
$$c = \frac{3}{2}$$
$$x = \frac{-1}{2(5 - 2y)} + \frac{3}{2}$$
$$\frac{1}{2(5 - 2y)} = \frac{3}{2} - x$$
$$\frac{1}{2(5 - 2y)} = \frac{3 - 2x}{2}$$
$$5 - 2y = \frac{1}{3 - 2x}$$
$$2y = 5 - \frac{1}{3 - 2x}$$
$$= \frac{5(3 - 2x) - 1}{3 - 2x}$$
$$= \frac{14 - 10x}{3 - 2x}$$
$$y = \frac{5x - 7}{2x - 3}, \qquad x \neq \frac{3}{2}$$

c $\frac{dy}{dx} + \sqrt{9 - y^2} = 0, \quad y(0) = 3$

$$\frac{dy}{dx} = -\sqrt{9 - y^2}$$
$$\frac{dx}{dy} = -\frac{1}{\sqrt{9 - y^2}}$$
$$x = \int -\frac{1}{\sqrt{9 - y^2}}dy$$
$$x = \cos^{-1}\left(\frac{y}{3}\right) + c$$

When $x = 0, \quad y = 3$

$$0 = \cos^{-1}(1) + c$$
$$c = 0$$
$$x = \cos^{-1}\left(\frac{y}{3}\right)$$
$$\frac{y}{3} = \cos(x)$$
$$y = 3\cos(x), \quad 0 \le x \le \pi$$

Technology active: multiple choice

7 $y = x^n$

$$\frac{dy}{dx} = nx^{n-1}$$
$$\frac{d^2y}{dx^2} = n(n - 1)x^{n-2}$$
$$x^2y'' + xy' - y = 0$$
$$0 = x^n(n(n - 1) + n - 1)$$
$$0 = x^n\left(n^2 - n + n - 1\right)$$
$$0 = n^2 - 1$$
$$n = \pm 1$$

The correct answer is **D**.

8 $y' = -ke^{-kx}$

$$y'' = k^2e^{-kx}$$
$$0 = k^2 - k - 6$$
$$0 = (k - 3)(k + 2)$$
$$k = 3, -2$$

The correct answer is **B**.

9 $y' = \cos(3x) - 3x\sin(3x)$

$$y'' = -3\sin(3x) - 3\sin(3x) - 9x\cos(3x)$$
$$y'' + 9x\cos(3x) + 6\sin(3x) = 0$$

The correct answer is **C**.

10 $y' = 2 \times \frac{1}{2}\cos\left(\frac{x}{2}\right)$

$y'' = -2 \times \frac{1}{4}\sin\left(\frac{x}{2}\right)$

$= -\frac{1}{2}\sin\left(\frac{x}{2}\right)$

$4y'' + y = 0$

The correct answer is **D**.

11 $y' = \frac{dy}{dx} = -36x$

$y = \int -36x\,dx$

$= -18x^2 + c$

The correct answer is **B**.

12 $y = \int \frac{1}{(7x-6)^2}\,dx$

$= \frac{-1}{7\,(7x-6)} + c$

$0 = \frac{1}{42} + c$

$y = \frac{-1}{42} - \frac{1}{7\,(7x-6)}$

$= \frac{-7x+6-6}{42\,(7x-6)}$

$= \frac{-x}{6\,(7x-6)}$

$= \frac{x}{6\,(6-7x)}$

The correct answer is **C**.

13 $y = Ae^{\frac{-x}{3}}$

The correct answer is **A**.

14 $\frac{dy}{dx} = -3x^2y^2$

$\int \frac{1}{y^2}dy = \int -3x^2dx$

$\frac{-1}{y} = -x^3 + c$

$\frac{1}{y} = x^3 + A$

$y = \frac{1}{x^3+c}$

The correct answer is **E**.

15 $\int \frac{1}{y^2+4}\,dy = \int 1.\,dx$

$\frac{1}{2}\tan^{-1}\frac{y}{2} = x + c$

$y(0) = 0,\ c = 0$

$\tan^{-1}\left(\frac{y}{2}\right) = 2x$

$\frac{y}{2} = \tan(2x)$

$y = 2\tan(2x)$

The correct answer is **E**.

16 $y'' = -36x$

$\frac{dy}{dx} = \int -36x\,dx$

$= -18x^2 + c_1$

$y = \int \left(c_1 - 18x^2\right)\,dx$

$= c_1x + c_2 - 6x^3$

The correct answer is **B**.

Technology active: extended response

17 a $\frac{dy}{dx} - y^2 = 4, \quad y(0) = 0$

$\frac{dy}{dx} = 4 + y^2$

$\frac{dx}{dy} = \frac{1}{4+y^2}$

$x = \int \frac{1}{4+y^2}\,dy$

$x = \frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right) + c$

When $x = 0, \quad y = 0$

$0 = \frac{1}{2}\tan^{-1}(0) + c$

$c = 0$

$x = \frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right)$

$2x = \tan^{-1}\left(\frac{y}{2}\right)$

$\frac{y}{2} = \tan(2x), \quad \frac{-\pi}{2} < 2x < \frac{\pi}{2}$

$y = 2\tan(2x), \quad \frac{-\pi}{4} < x < \frac{\pi}{4}$

b $\frac{dy}{dx} - y^2 = 4y, \quad y(0) = 2$

$\frac{dy}{dx} = y^2 + 4y$

$\frac{dx}{dy} = \frac{1}{y^2+4y}$

$x = \int \frac{1}{y\,(y+4)}\,dy$

$\frac{1}{y^2+4y} = \frac{A}{y} + \frac{B}{y+4} = \frac{A\,(y+4) + By}{y\,(y+4)}$

$= \frac{y\,(A+B) + 4A}{y^2+4y}$

$4A = 1 \quad A = \frac{1}{4}$

$A + B = 0 \quad B = -A$

$x = \frac{1}{4}\int \left(\frac{1}{y} - \frac{1}{y+4}\right)\,dy$

$= \frac{1}{4}\left(\log_e|y| - \log_e|y+4|\right) + c$

When $x = 0, \quad y = 2$

$0 = \frac{1}{4}\left(\log_e(2) - \log_e(6)\right) + c$

$c = \frac{1}{4}\log_e(3)$

$4x = \log_e\left|\frac{3y}{y+4}\right|$

$e^{4x} = \frac{3y}{y+4}$

Note: The absolute value can be removed since LHS = RHS when the given condition is substituted in.

$$3y = (y+4)\,e^{4x}$$
$$3y = ye^{4x} + 4e^{4x}$$
$$4e^{4x} = 3y - ye^{4x}$$
$$4e^{4x} = y\left(3 - e^{4x}\right) \qquad e^{4x} \neq 3$$
$$y = \frac{4e^{4x}}{3 - e^{4x}} \qquad x \neq \frac{1}{4}\log_e(3)$$

c $16\dfrac{dy}{dx} + e^{4y} = 0, \ \ y(0) = 0$

$$16\frac{dy}{dx} = -e^{4y}$$
$$\frac{dy}{dx} = \frac{-e^{4y}}{16}$$
$$\frac{dx}{dy} = -16e^{-4y}$$
$$x = \int -16e^{-4y}dy$$
$$x = 4e^{-4y} + c$$

When $x = 0, \ \ y = 0$

$$0 = 4 + c$$
$$c = -4$$
$$x = 4e^{-4y} - 4$$
$$4e^{-4y} = x + 4$$
$$e^{-4y} = \frac{x+4}{4}$$
$$e^{4y} = \frac{4}{x+4}$$
$$4y = \log_e\left|\frac{4}{x+4}\right|$$
$$y = \frac{1}{4}\log_e\left(\frac{4}{x+4}\right), \qquad x > -4$$

18 a $\dfrac{dy}{dx} - 6x^2y^2 = 0, \quad y(1) = 1$

$$\frac{dy}{dx} = 6x^2y^2$$
$$\frac{1}{y^2}\frac{dy}{dx} = 6x^2$$
$$\int \frac{1}{y^2}dy = \int 6x^2dx$$
$$-\frac{1}{y} = 2x^3 + c$$

When $y = 1, \quad x = 1$

$$-1 - 2 = c \Rightarrow c = -3$$
$$-\frac{1}{y} = 2x^3 - 3$$
$$y = \frac{1}{3 - 2x^3}$$

b $\dfrac{dy}{dx} = 4x^3e^{-y}, \ \ y(0) = 0$

$$e^y\frac{dy}{dx} = 4x^3$$
$$\int e^ydy = \int 4x^3dx$$
$$e^y = x^4 + c$$

When $y = 0, \ \ x = 0$

$$c = 1$$
$$e^y = x^4 + 1$$
$$y = \log_e\left(x^4 + 1\right)$$

c $\dfrac{dy}{dx} - y^2 = 4xy^2, \quad y(1) = 1$

$$\frac{dy}{dx} = 4xy^2 + y^2$$
$$= y^2(4x+1)$$
$$\frac{1}{y^2}\frac{dy}{dx} = (4x+1)$$
$$\int \frac{1}{y^2}dy = \int (4x+1)\ dx$$
$$-\frac{1}{y} = 2x^2 + x + c$$

When $x = 1, \quad y = 1$

$$-1 = 2 + 1 + c$$
$$c = -4$$
$$-\frac{1}{y} = 2x^2 + x - 4$$
$$y = \frac{1}{4 - x - 2x^2}$$

19 a $x^2\dfrac{d^2y}{dx^2} - 4 = 2\sqrt{x}, \quad y(1) = 3, \ \ y'(1) = 2$

$$x^2\frac{d^2y}{dx^2} = 4 + 2\sqrt{x}$$
$$\frac{d^2y}{dx^2} = \frac{4 + 2\sqrt{x}}{x^2}$$
$$= 4x^{-2} + 2x^{\frac{-3}{2}}$$
$$\frac{dy}{dx} = \int\left(4x^{-2} + 2x^{\frac{-3}{2}}\right)dx$$
$$= -4x^{-1} - 4x^{\frac{-1}{2}} + c_1$$
$$\frac{dy}{dx} = 2 \text{ when } x = 1$$
$$2 = -4 - 4 + c_1$$
$$c_1 = 10$$
$$\frac{dy}{dx} = -4x^{-1} - 4x^{\frac{-1}{2}} + 10$$
$$y = \int\left(-4x^{-1} - 4x^{\frac{-1}{2}} + 10\right)dx$$
$$y = -4\log_e(|x|) - 8x^{\frac{1}{2}} + 10x + c_2$$

When $y = 3 \quad x = 1$

$$3 = -4\log_e(1) - 8 + 10 + c_2$$
$$c_2 = 1$$
$$y = 10x + 1 - 8\sqrt{x} - 4\log_e(x), \quad x > 0$$

b $\dfrac{d^2y}{dx^2} + \dfrac{49}{(4x+7)^3} = 0, \quad y(0) = 0, \ \ y'(0) = 0$

$$\frac{d^2y}{dx^2} = \frac{-49}{(4x+7)^3}$$
$$\frac{dy}{dx} = \int\left(\frac{-49}{(4x+7)^3}\right)dx$$
$$= \frac{49}{8(4x+7)^2} + c_1$$

$$\frac{dy}{dx}=0 \text{ when } x=0$$

$$0=\frac{1}{8}+c_1$$

$$c_1=\frac{-1}{8}$$

$$\frac{dy}{dx}=\frac{49}{8(4x+7)^2}-\frac{1}{8}$$

$$y=\int\left(\frac{49}{8(4x+7)^2}-\frac{1}{8}\right)dx$$

$$y=\frac{-49}{32\,(4x+7)}-\frac{x}{8}+c_2$$

When $x=0 \quad y=0$

$$0=\frac{-49}{32\times 7}+c_2$$

$$c_2=\frac{7}{32}$$

$$\begin{aligned}y&=\frac{-49}{32\,(4x+7)}-\frac{x}{8}+\frac{7}{32}\\&=\frac{-49-4x\,(4x+7)+7\,(4x+7)}{32\,(4x+7)}\\&=\frac{-49-16x^2-28x+28x+49}{32\,(4x+7)}\\&=\frac{-x^2}{2\,(4x+7)}\qquad x\neq\frac{-7}{4}\end{aligned}$$

c $\dfrac{d^2y}{dx^2}+\dfrac{42}{\sqrt{7x+4}}=0,\quad y\,(0)=0,\ y'\,(0)=0$

$$\frac{d^2y}{dx^2}=\frac{-42}{\sqrt{7x+4}}$$

$$\begin{aligned}\frac{dy}{dx}&=\int\left(\frac{-42}{\sqrt{7x+4}}\right)dx\\&=\int -42(7x+4)^{\frac{-1}{2}}\,dx\\&=\frac{-42}{7\times\frac{1}{2}}(7x+4)^{\frac{1}{2}}+c_1\\&=-12\sqrt{7x+4}+c_1\end{aligned}$$

$$\frac{dy}{dx}=0 \text{ when } x=0$$

$$0=-12\sqrt{4}+c_1$$

$$c_1=24$$

$$\frac{dy}{dx}=-12\sqrt{7x+4}+24$$

$$y=\int\left(-12(7x+4)^{\frac{1}{2}}+24\right)dx$$

$$\begin{aligned}y&=\frac{-12}{3}\times\frac{2}{7}(7x+4)^{\frac{3}{2}}+24x+c_2\\&=\frac{-8}{7}(7x+4)^{\frac{3}{2}}+24x+c_2\end{aligned}$$

When $x=0 \quad y=0$

$$0=\frac{-8}{7}(4)^{\frac{3}{2}}+c_2$$

$$c_2=\frac{8}{7}\times 8$$

$$\begin{aligned}y&=\frac{-8}{7}(7x+4)^{\frac{3}{2}}+24x+\frac{64}{7}\\&=\frac{8}{7}\left(21x+8-(7x+4)^{\frac{3}{2}}\right),\qquad x>\frac{-4}{7}\end{aligned}$$

20 a $y=\cos^{-1}\left(\dfrac{2x}{3}\right)$

$$\frac{dy}{dx}=\frac{-2}{\sqrt{9-4x^2}}$$

Let $u=9-4x^2,\ \dfrac{du}{dx}=-8x$

$$\begin{aligned}\frac{d^2y}{dx^2}&=-2\times-\frac{1}{2}\left(9-4x^2\right)^{\frac{-3}{2}}\times -8x\\&=\frac{-8x}{\left(9-4x^2\right)^{\frac{3}{2}}}\end{aligned}$$

$$\frac{d^2y}{dx^2}-ax\left(\frac{dy}{dx}\right)^3=0$$

Substituting:

$$\frac{-8x}{\left(9-4x^2\right)^{\frac{3}{2}}}-ax\times\left(\frac{-2}{\left(9-4x^2\right)^{\frac{1}{2}}}\right)^3=0$$

$$\frac{-8x}{\left(9-4x^2\right)^{\frac{3}{2}}}-\frac{-8ax}{\left(9-4x^2\right)^{\frac{3}{2}}}=0$$

$\Rightarrow a=1$

b $y=b\,x^2e^{-3x}$

$$\begin{aligned}\frac{dy}{dx}&=2b\,xe^{-3x}-3b\,x^2e^{-3x}\\&=e^{-3x}\left(2bx-3bx^2\right)\end{aligned}$$

$$\begin{aligned}\frac{d^2y}{dx^2}&=e^{-3x}(2b-6bx)-3e^{-3x}\left(2bx-3bx^2\right)\\&=be^{-3x}\left(9x^2-12x+2\right)\end{aligned}$$

$$\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=8e^{-3x}$$

Substituting,

$$\begin{aligned}be^{-3x}\left(9x^2-12x+2\right)+6e^{-3x}\left(2bx-3bx^2\right)+9bx^2e^{-3x}&=8e^{-3x}\\e^{-3x}\left(9bx^2-12bx+2b+12bx-18bx^2+9bx^2\right)&=8e^{-3x}\\e^{-3x}\,(2b)&=8e^{-3x}\\b&=4\end{aligned}$$

8.7 Exam questions

1 $y=\sin(x^2)$

$\dfrac{dy}{dx}=2x\cos(x^2)$ [1 mark]

$\dfrac{d^2y}{dx^2}=2\ \cos(x^2)-4x^2\ \sin(x^2)$ [1 mark]

$$\begin{aligned}x\frac{d^2y}{dx^2}-\frac{dy}{dx}+4x^3y&=x(2\ \cos(x^2)-4x^2\sin(x^2))\\&\quad-2\,x\ \cos(x^2)+4\,x^3\sin(x^2)\\&=2\,x\ \cos(x^2)-4x^3\sin(x^2)\\&\quad-2\,x\ \cos(x^2)+4\,x^3\sin(x^2)\\&=0\\&=\text{RHS}\end{aligned}$$

[1 mark]

2 $y = 4e^{-2x}\sin(3x)$

$$\frac{dy}{dx} = -8e^{-2x}\sin(3x) + 12e^{-2x}\cos(3x) \qquad \text{[1 mark]}$$

$$= 4e^{-2x}(3\cos(3x) - 2\sin(3x))$$

$$\frac{d^2y}{dx^2} = -8e^{-2x}(3\cos(3x) - 2\sin(3x))$$

$$+ 4e^{-2x}(-6\sin(3x) - 6\cos(3x))$$

$$= -4e^{-2x}(5\sin(3x) + 12\cos(3x)) \qquad \text{[1 mark]}$$

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

$$0 = 4e^{-2x}(-5\sin(3x) - 12\cos(3x)$$

$$+ a(3\cos(3x) - 2\sin(3x)) + b\sin(3x)) \qquad \text{[1 mark]}$$

$$0 = 4e^{-2x}((-5 - 2a + b)\sin(3x) + (-12 + 3a)\cos(3x))$$

$$-12 + 3a = 0$$

$$\Rightarrow a = 4 \qquad \text{[1 mark]}$$

$$-5 - 2a + b = 0$$

$$\Rightarrow b = 2a + 5$$

$$b = 13 \qquad \text{[1 mark]}$$

3 $(5x+3)^2\dfrac{dy}{dx} + 4 = 0, \quad y(-1) = 2$

$$(5x+3)^2\frac{dy}{dx} = -4$$

$$\frac{dy}{dx} = \frac{-4}{(5x+3)^2}$$

$$y = \int \frac{-4}{(5x+3)^2}dx$$

$$= \frac{4}{5(5x+3)} + c \qquad \text{[1 mark]}$$

When $x = -1,\ y = 2$

$$2 = \frac{2}{-5} + c$$

$$c = \frac{12}{5} \qquad \text{[1 mark]}$$

$$y = \frac{4}{5(5x+3)} + \frac{12}{5}$$

$$y = \frac{4 + 12(5x+3)}{5(5x+3)}$$

$$y = \frac{60x + 40}{5(5x+3)}$$

$$y = \frac{12x+8}{5x+3}, x \neq \frac{-3}{5} \qquad \text{[1 mark]}$$

4 $y = \tan^{-1}\left(\dfrac{x}{3}\right)$

$$\frac{dy}{dx} = \frac{3}{x^2+9}$$

$$\left(x^2+9\right)\frac{dy}{dx} = 3$$

Therefore, $y = \tan^{-1}\left(\dfrac{x}{3}\right)$ is the solution to the differential equation

$$\left(x^2+9\right)\frac{dy}{dx} - 3 = 0.$$

The correct answer is **A**.

5 $y = c\,t\,\sin(2t)$

$$\frac{dy}{dx} = c\,[\sin(2t) + 2t\,\cos(2t)] \qquad \text{[1 mark]}$$

$$\frac{d^2y}{dx^2} = c\,[2\cos(2t) + 2\cos(2t) - 4t\,\sin(2t)] \qquad \text{[1 mark]}$$

$$= c\,[4\,\cos(2t) - 4t\,\sin(2t)]$$

$$\frac{d^2y}{dx^2} + 4y = 4\,c\,\cos(2t) - 4\,ct\,\sin(2t) + 4ct\,\sin(2t)$$

$$= 4c\,\cos(2t)$$

$$4c\,\cos(2t) = 8\cos(2t)$$

$$c = 2 \qquad \text{[1 mark]}$$

Topic 9 — Further integration techniques and applications

9.2 Integration by parts

9.2 Exercise

1 a $\int x e^{-2x}\, dx$

$$u = x \qquad \frac{dv}{dx} = e^{-2x}$$

$$\frac{du}{dx} = 1 \qquad v = -\frac{1}{2}e^{-2x}$$

$$\int x e^{-2x} dx = -\frac{1}{2}xe^{-2x} + \int \frac{1}{2}e^{-2x} dx$$

$$= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + c$$

$$= -\frac{1}{4}e^{-2x}(1 + 2x) + c$$

b $\int x e^{\frac{x}{3}}\, dx$

$$u = x \quad \frac{dv}{dx} = e^{\frac{x}{3}}$$

$$\frac{du}{dx} = 1 \quad v = 3e^{\frac{x}{3}}$$

$$\int xe^{\frac{x}{3}} dx = 3xe^{\frac{x}{3}} - \int 3e^{\frac{x}{3}}\, dx$$

$$= 3xe^{\frac{x}{3}} - 9\, e^{\frac{x}{3}} + c$$

$$= 3e^{\frac{x}{3}}(x - 3) + c$$

2 a $\int x \sin(3x)\, dx$

$$u = x \qquad \frac{dv}{dx} = \sin(3x)$$

$$\frac{du}{dx} = 1 \qquad v = -\frac{1}{3}\cos(3x)$$

$$\int x\ \sin(3x)\, dx = -\frac{x}{3}\cos(3x) + \frac{1}{3}\int \cos(3x)\, dx$$

$$= -\frac{x}{3}\cos(3x) + \frac{1}{9}\sin(3x) + c$$

$$= \frac{1}{9}\left(\sin(3x) - 3x\cos(3x)\right) + c$$

b $\int x\cos\left(\frac{x}{2}\right)\, dx$

$$u = x \qquad \frac{dv}{dx} = \cos\left(\frac{x}{2}\right)$$

$$\frac{du}{dx} = 1 \qquad v = 2\sin\left(\frac{x}{2}\right)$$

$$\int x\cos\left(\frac{x}{2}\right)\, dx = 2x\sin\left(\frac{x}{2}\right) - \int 2\ \sin\left(\frac{x}{2}\right) dx$$

$$= 2x\sin\left(\frac{x}{2}\right) + 4\cos\left(\frac{x}{2}\right) + c$$

$$= 2\left(x\sin\left(\frac{x}{2}\right) + 2\cos\left(\frac{x}{2}\right)\right) + c$$

3 a $\int \log_e(3x) dx$

$$= \int 1 \cdot \log_e(3x) dx$$

$$u = \log_e(3x) \qquad \frac{dv}{dx} = 1$$

$$\frac{du}{dx} = \frac{1}{x} \qquad v = x$$

$$\int \log_e(3x)\, dx = x\log_e(3x) - \int x \times \frac{1}{x}\, dx$$

$$= x\log_e(3x) - \int 1\, dx$$

$$= x\log_e(3x) - x + c$$

$$= x\left(\log_e(3x) - 1\right) + c$$

b $\int x\log_e\left(\frac{x}{2}\right)\, dx$

$$u = \log_e\left(\frac{x}{2}\right) \qquad \frac{dv}{dx} = x$$

$$\frac{du}{dx} = \frac{1}{x} \qquad v = \frac{1}{2}x^2$$

$$\int x\log_e\left(\frac{x}{2}\right)\, dx = \frac{1}{2}x^2\log_e\left(\frac{x}{2}\right) - \int \frac{1}{x} \times \frac{1}{2}x^2\, dx$$

$$= \frac{1}{2}x^2\log_e\left(\frac{x}{2}\right) - \frac{1}{2}\int x\, dx$$

$$= \frac{1}{2}x^2\log_e\left(\frac{x}{2}\right) - \frac{x^2}{4} + c$$

$$= \frac{1}{2}x^2\left(\log_e\left(\frac{x}{2}\right) - \frac{1}{2}\right) + c$$

4 a $\int \sin^{-1}\left(\frac{x}{3}\right)\, dx$

$$= \int 1 \cdot \sin^{-1}\left(\frac{x}{3}\right)\, dx$$

$$u = \sin^{-1}\left(\frac{x}{3}\right) \qquad \frac{dv}{dx} = 1$$

$$\frac{du}{dx} = \frac{1}{\sqrt{9 - x^2}} \qquad v = x$$

$$\int \sin^{-1}\left(\frac{x}{3}\right)\, dx = x\sin^{-1}\left(\frac{x}{3}\right) - \int \frac{x}{\sqrt{9 - x^2}}\, dx$$

$$\text{let } t = 9 - x^2$$

$$\frac{dt}{dx} = -2x$$

$$= x\sin^{-1}\left(\frac{x}{3}\right) + \frac{1}{2}\int t^{-\frac{1}{2}}\, dt$$

$$= x\sin^{-1}\left(\frac{x}{3}\right) + t^{\frac{1}{2}} + c$$

$$= x\sin^{-1}\left(\frac{x}{3}\right) + \sqrt{9 - x^2} + c$$

b $\int \cos^{-1}(4x)\,dx$

$= \int 1 \cdot \cos^{-1}(4x)\,dx$

$u = \cos^{-1}(4x) \qquad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{-4}{\sqrt{1-16x^2}} \qquad v = x$

$\int \cos^{-1}(4x)\,dx = x\cos^{-1}(4x) + \int \frac{4x}{\sqrt{1-16x^2}}\,dx$

let $t = 1 - 16x^2$

$\frac{dt}{dx} = -32x$

$= x\cos^{-1}(4x) - \frac{1}{8}\int t^{-\frac{1}{2}}\,dt$

$= x\cos^{-1}(4x) - \frac{1}{4}t^{\frac{1}{2}} + c$

$= x\cos^{-1}(4x) - \frac{1}{4}\sqrt{1-16x^2} + c$

5 a $\int \tan^{-1}\left(\frac{x}{5}\right)\,dx$

$= \int 1 \cdot \tan^{-1}\left(\frac{x}{5}\right)\,dx$

$u = \tan^{-1}\left(\frac{x}{5}\right) \qquad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{5}{x^2+25} \qquad v = x$

$\int \tan^{-1}\left(\frac{x}{5}\right)\,dx = x\tan^{-1}\left(\frac{x}{5}\right) - \int \frac{5x}{x^2+25}\,dx$

$= x\tan^{-1}\left(\frac{x}{5}\right) - \frac{5}{2}\log_e\left(x^2+25\right) + c$

b $\int x\tan^{-1}(2x)\,dx$

$u = \tan^{-1}(2x) \qquad \frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{2}{1+4x^2} \qquad v = \frac{1}{2}x^2$

$\int x\tan^{-1}(2x)\,dx = \frac{1}{2}x^2\tan^{-1}(2x) - \int \frac{x^2}{1+4x^2}\,dx$

$= \frac{1}{2}x^2\tan^{-1}(2x) - \frac{1}{4}\int \frac{4x^2+1-1}{1+4x^2}\,dx$

$= \frac{1}{2}x^2\tan^{-1}(2x) - \frac{1}{4}\int\left(1 - \frac{1}{1+4x^2}\right)\,dx$

$= \frac{1}{2}x^2\tan^{-1}(2x) - \frac{1}{4}\left[x - \frac{1}{2}\tan^{-1}(2x)\right] + c$

$= \frac{1}{8}\left(1+4x^2\right)\tan^{-1}(2x) - \frac{x}{4} + c$

6 a $\int_{\frac{1}{3}}^{1} \frac{\log_e(3x)}{x^3}\,dx$

$u = \log_e(3x) \qquad \frac{dv}{dx} = \frac{1}{x^3} = x^{-3}$

$\frac{du}{dx} = \frac{1}{x} \qquad v = -\frac{1}{2}x^{-2} = -\frac{1}{2x^2}$

$\int_{\frac{1}{3}}^{1} \frac{\log_e(3x)}{x^3}\,dx = \left[\frac{-1}{2x^2}\log_e(3x)\right]_{\frac{1}{3}}^{1} + \frac{1}{2}\int_{\frac{1}{3}}^{1}\frac{1}{x^3}\,dx$

$= \left[\frac{-1}{2x^2}\log_e(3x) - \frac{1}{4x^2}\right]_{\frac{1}{3}}^{1}$

$= \left[\frac{-1}{4x^2}\left(1 + 2\log_e(3x)\right)\right]_{\frac{1}{3}}^{1}$

$= \left[\frac{-1}{4}\left(1 + 2\log_e(3)\right)\right] - \left[\frac{-9}{4}\left(1 + 2\log_e(1)\right)\right]$

$= 2 - \frac{1}{2}\log_e(3)$

b $\int_{0}^{\frac{\pi}{8}} x\sin(2x)\cos(2x)\,dx$

$= \frac{1}{2}\int_{0}^{\frac{\pi}{8}} x\sin(4x)\,dx$

$u = x \qquad \frac{dv}{dx} = \sin(4x)$

$\frac{du}{dx} = 1 \qquad v = -\frac{1}{4}\cos(4x)$

$\frac{1}{2}\int_{0}^{\frac{\pi}{8}} x\sin(4x)\,dx = \frac{1}{2}\left[\left[\frac{-x}{4}\cos(4x)\right]_{0}^{\frac{\pi}{8}} + \frac{1}{4}\int_{0}^{\frac{\pi}{8}}\cos(4x)\,dx\right]$

$= \frac{1}{2}\left[\frac{-x}{4}\cos(4x) + \frac{1}{16}\sin(4x)\right]_{0}^{\frac{\pi}{8}}$

$= \frac{1}{32}\left[\sin(4x) - 4x\cos(4x)\right]_{0}^{\frac{\pi}{8}}$

$= \frac{1}{32}\left[\sin\left(\frac{\pi}{2}\right) - \frac{\pi}{2}\cos\left(\frac{\pi}{2}\right)\right] - \frac{1}{32}\left[\sin(0) - 0\right]$

$= \frac{1}{32}$

7 a $\int_{0}^{\frac{\pi}{8}} x\cos^2(2x)\,dx$

$= \frac{1}{2}\int_{0}^{\frac{\pi}{8}} x\,(1 + \cos(4x))\,dx$

$= \frac{1}{2}\int_{0}^{\frac{\pi}{8}} x\,dx + \frac{1}{2}\int_{0}^{\frac{\pi}{8}} x\cos(4x)\,dx$

$= \frac{1}{2}\left[\frac{1}{2}x^2\right]_{0}^{\frac{\pi}{8}} \qquad$ let $u = x \qquad \frac{dv}{dx} = \cos(4x)$

$= \frac{1}{4}\left(\frac{\pi^2}{64} - 0\right) \qquad \frac{du}{dx} = 1 \qquad v = \frac{1}{4}\sin(4x)$

$= \frac{\pi^2}{256} + \frac{1}{2}\left[\frac{x}{4}\sin(4x)\right]_{0}^{\frac{\pi}{8}} - \frac{1}{8}\int_{0}^{\frac{\pi}{8}}\sin(4x)\,dx$

$= \frac{\pi^2}{256} + \left[\frac{x}{8}\sin(4x) + \frac{1}{32}\cos(4x)\right]_{0}^{\frac{\pi}{8}}$

$$= \frac{\pi^2}{256} + \left(\frac{\pi}{64}\sin\left(\frac{\pi}{2}\right) + \frac{1}{32}\cos\left(\frac{\pi}{2}\right) - 0 - \frac{1}{32}\cos(0)\right)$$

$$= \frac{\pi^2}{256} + \frac{\pi}{64} - \frac{1}{32}$$

$$= \frac{1}{256}\left(\pi^2 + 4\pi - 8\right)$$

b $\displaystyle\int_0^{\frac{\pi}{9}} x\sin^2(3x)\,dx$

$$= \frac{1}{2}\int_0^{\frac{\pi}{9}} x\left(1 - \cos(6x)\right)\,dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{9}} x\,dx - \frac{1}{2}\int_0^{\frac{\pi}{9}} x\cos(6x)\,dx$$

$$= \frac{1}{4}[x^2]_0^{\frac{\pi}{9}} \quad u = x \quad \frac{dv}{dx} = \cos(6x)$$

$$\frac{du}{dx} = 1 \qquad v = \frac{1}{6}\sin(6x)$$

$$\frac{1}{4}\left(\frac{\pi^2}{81} - 0\right) - \frac{1}{2}\left[\frac{x}{6}\sin(6x)\right]_0^{\frac{\pi}{9}} - \frac{1}{2}\int_0^{\frac{\pi}{9}} \frac{1}{6}\sin(6x)\,dx$$

$$= \frac{\pi^2}{324} - \left[\frac{x}{12}\sin(6x) + \frac{1}{72}\cos(6x)\right]_0^{\frac{\pi}{9}}$$

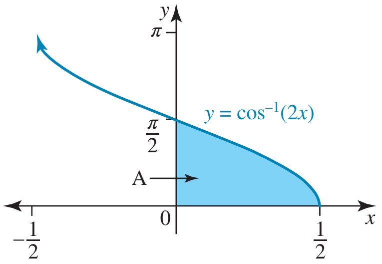
$$= \frac{\pi^2}{324} - \left[\frac{\pi}{108}\sin\left(\frac{2\pi}{3}\right) + \frac{1}{72}\cos\left(\frac{2\pi}{3}\right) - 0 - \frac{1}{72}\cos(0)\right]$$

$$= \frac{\pi^2}{324} - \frac{\pi}{108} \times \frac{\sqrt{3}}{2} - \frac{1}{72} \times \frac{-1}{2} + \frac{1}{72}$$

$$= \frac{\pi^2}{324} - \frac{\sqrt{3}\pi}{216} + \frac{1}{48}$$

$$= \frac{1}{1296}\left(4\pi^2 - 6\pi\sqrt{3} + 27\right)$$

8 $f : \left\{x : |x| \le \frac{1}{2}\right\} \to R,\ f(x) = \cos^{-1}(2x)$



$$A = \int_0^{\frac{1}{2}} \cos^{-1}(2x)\,dx = \int_0^{\frac{1}{2}} 1 \cdot \cos^{-1}(2x)\,dx$$

Let $u = \cos^{-1}(2x)$ $\quad \dfrac{dv}{dx} = 1$

$$\frac{du}{dx} = \frac{-2}{\sqrt{1 - 4x^2}} \qquad v = x$$

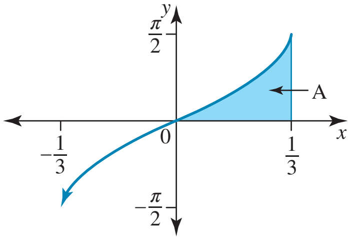
$$A = \left[x\cos^{-1}(2x)\right]_0^{\frac{1}{2}} + \int_0^{\frac{1}{2}} \frac{2x}{\sqrt{1 - 4x^2}}\,dx$$

$$= \left[x\cos^{-1}(2x)\right]_0^{\frac{1}{2}} - \left[\frac{1}{2}\sqrt{1 - 4x^2}\right]_0^{\frac{1}{2}}$$

$$= \frac{1}{2}\cos^{-1}(1) - 0 - 0 + \frac{1}{2}$$

$$= \frac{1}{2}\text{ units}^2$$

9



$$A = \int_0^{\frac{1}{3}} \sin^{-1}(3x)\,dx = \int_0^{\frac{1}{3}} 1 \cdot \sin^{-1}(3x)\,dx$$

$$u = \sin^{-1}(3x) \qquad \frac{dv}{dx} = 1$$

$$\frac{du}{dx} = \frac{3}{\sqrt{1 - 9x^2}} \qquad v = x$$

$$A = \left[x\sin^{-1}(3x)\right]_0^{\frac{1}{3}} - \int_0^{\frac{1}{3}} \frac{3x}{\sqrt{1 - 9x^2}}\,dx$$

$$= \left[x\sin^{-1}(3x)\right]_0^{\frac{1}{3}} + \left[\frac{1}{3}\sqrt{1 - 9x^2}\right]_0^{\frac{1}{3}}$$

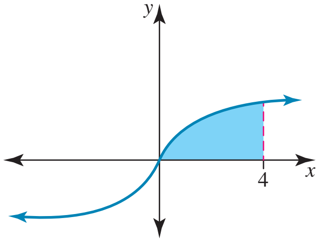
$$= \frac{1}{3}\sin^{-1}(1) - 0 + \frac{1}{3}\sqrt{0} - \frac{1}{3}$$

$$= \frac{1}{3} \times \frac{\pi}{2} - \frac{1}{3}$$

$$= \frac{\pi}{6} - \frac{1}{3}$$

$$= \frac{1}{6}(\pi - 2)\text{ units}^2$$

10



$$A = \int_0^4 \tan^{-1}\left(\frac{x}{4}\right)\,dx = \int_0^4 1 \cdot \tan^{-1}\left(\frac{x}{4}\right)\,dx$$

$$u = \tan^{-1}\left(\frac{x}{4}\right) \qquad \frac{dv}{dx} = 1$$

$$\frac{du}{dx} = \frac{4}{16 + x^2} \qquad v = x$$

$$A = \left[x\tan^{-1}\left(\frac{x}{4}\right)\right]_0^4 - \int_0^4 \frac{4x}{16+x^2}\,dx$$

$$= \left[x\tan^{-1}\left(\frac{x}{4}\right)\right]_0^4 - \left[2\log_e(16+x^2)\right]_0^4$$

$$= 4\tan^{-1}(1) - 0 - 2\log_e 32 + 2\log_2 16$$

$$= 4 \times \frac{\pi}{4} + 2\log_2\left(\frac{16}{32}\right)$$

$$= \pi + 2\log_e\left(\frac{1}{2}\right)$$

$$= \pi - \log_e(4)\,\text{units}^2$$

11 a $\int xe^{ax}\,dx$

$u = x \qquad \frac{dv}{dx} = e^{ax}$

$\frac{du}{dx} = 1 \qquad v = \frac{1}{a}e^{ax}$

$$\int xe^{ax}dx = \frac{x}{a}e^{ax} - \frac{1}{a}\int e^{ax}dx$$

$$= \frac{x}{a}e^{ax} - \frac{1}{a^2}e^{ax} + c$$

$$= \frac{1}{a^2}e^{ax}(ax-1) + c$$

b $\int x\cos(ax)\,dx$

$u = x \qquad \frac{dv}{dx} = \cos(ax)$

$\frac{du}{dx} = 1 \qquad v = \frac{1}{a}\sin(ax)$

$$\int x\cos(ax)dx = \frac{x}{a}\sin(ax) - \frac{1}{a}\int \sin(ax)\,dx$$

$$= \frac{x}{a}\sin(ax) + \frac{1}{a^2}\cos(ax) + c$$

$$= \frac{1}{a^2}\left(ax\sin(ax) + \cos(ax)\right) + c$$

12 a $\int x\sin(ax)\,dx$

$u = x \qquad \frac{dv}{dx} = \sin(ax)$

$\frac{du}{dx} = 1 \qquad v = -\frac{1}{a}\cos(ax)$

$$\int x\sin(ax) = -\frac{x}{a}\cos(ax) + \frac{1}{a}\int \cos(ax)\,dx$$

$$= -\frac{x}{a}\cos(ax) + \frac{1}{a^2}\sin(ax) + c$$

$$= \frac{1}{a^2}\left(\sin(ax) - ax\cos(ax)\right) + c$$

b $\int x^n\log_e(ax)\,dx \quad \text{if } n \neq -1$

$u = \log_e(ax) \qquad \frac{dv}{dx} = x^n$

$\frac{du}{dx} = \frac{1}{x} \qquad v = \frac{x^{n+1}}{n+1}$

$$\int x^n\log_e(ax)\,dx = \frac{1}{n+1}x^{n+1}\log_e(ax) - \frac{1}{n+1}\int x^{n+1} \times \frac{1}{x}\,dx$$

$$= \frac{1}{n+1}x^{n+1}\log_e(ax) - \frac{1}{n+1}\int x^n dx$$

$$= \frac{1}{n+1}x^{n+1}\log_e(ax) - \frac{1}{(n+1)^2}x^{n+1} + c$$

$$= \frac{x^{n+1}}{(n+1)^2}\left[(n+1)\log_e(ax) - 1\right] + c$$

$n \neq -1$

If $n = -1$

$$\int \frac{\log_e(ax)}{x}dx$$

let $u = \log_e(ax) \qquad \frac{du}{dx} = \frac{1}{x}$

$$= \int u\,du$$

$$= \frac{1}{2}u^2 + c$$

$$= \frac{1}{2}\left(\log_e(ax)\right)^2 + c$$

13 a $\int \sin^{-1}\left(\frac{x}{a}\right)dx = \int 1\cdot\sin^{-1}\left(\frac{x}{a}\right)dx$

$u = \sin^{-1}\left(\frac{x}{a}\right) \qquad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{\sqrt{a^2-x^2}} \qquad v = x$

$$\int \sin^{-1}\left(\frac{x}{a}\right)dx = x\sin^{-1}\left(\frac{x}{a}\right) - \int \frac{x}{\sqrt{a^2-x^2}}\,dx$$

$$= x\sin^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2-x^2} + c$$

b $\int \cos^{-1}\left(\frac{x}{a}\right)dx = \int 1\cdot\cos^{-1}\left(\frac{x}{a}\right)dx$

$u = \cos^{-1}\left(\frac{x}{a}\right) \qquad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{-1}{\sqrt{a^2-x^2}} \qquad v = x$

$$\int \cos^{-1}\left(\frac{x}{a}\right)dx = x\cos^{-1}\left(\frac{x}{a}\right) + \int \frac{x}{\sqrt{a^2-x^2}}\,dx$$

$$= x\cos^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2-x^2} + c$$

14 a $\int \tan^{-1}\left(\frac{x}{a}\right)dx = \int 1\cdot\tan^{-1}\left(\frac{x}{a}\right)dx$

$u = \tan^{-1}\left(\frac{x}{a}\right) \qquad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{a}{a^2+x^2} \qquad v = x$

$$\int \tan^{-1}\left(\frac{x}{a}\right)dx = x\tan^{-1}\left(\frac{x}{a}\right) - \int \frac{ax}{a^2+x^2}\,dx$$

$$= x\tan^{-1}\left(\frac{x}{a}\right) - \frac{a}{2}\log_e(a^2+x^2) + c$$

b $\int x\tan^{-1}\left(\frac{x}{a}\right)dx$

$u = \tan^{-1}\left(\frac{x}{a}\right) \qquad \frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{a}{a^2+x^2} \qquad v = \frac{1}{2}x^2$

$$\begin{aligned}\int x\tan^{-1}\left(\frac{x}{a}\right)dx &= \frac{1}{2}x^2\tan^{-1}\left(\frac{x}{a}\right)-\int\frac{ax^2}{2\left(a^2+x^2\right)}dx\\ &= \frac{1}{2}x^2\tan^{-1}\left(\frac{x}{a}\right)-\frac{a}{2}\int\left(\frac{x^2+a^2-a^2}{x^2+a^2}\right)dx\\ &= \frac{1}{2}x^2\tan^{-1}\left(\frac{x}{a}\right)-\frac{a}{2}\int\left(1-\frac{a^2}{x^2+a^2}\right)dx\\ &= \frac{1}{2}x^2\tan^{-1}\left(\frac{x}{a}\right)-\frac{a}{2}\left[x-a\tan^{-1}\left(\frac{x}{a}\right)\right]+c\\ &= \frac{1}{2}\left(x^2+a^2\right)\tan^{-1}\left(\frac{x}{a}\right)-\frac{ax}{2}+c\end{aligned}$$

15 a $\int x^m\sin(ax)dx$

let $u = x^m$ $\quad \dfrac{dv}{dx} = \sin(ax)$

$\dfrac{du}{dx} = mx^{m-1}$ $\quad v = -\dfrac{1}{a}\cos(ax)$

$$\int x^m\sin(ax)\,dx = -\frac{1}{a}x^m\cos(ax)+\frac{m}{a}\int x^{m-1}\cos(ax)\,dx$$

consider $\int x^{m-1}\cos(ax)\,dx$

let $u = x^{m-1}$ $\quad \dfrac{dv}{dx} = \cos(ax)$

$\dfrac{du}{dx} = (m-1)\,x^{m-2}$ $\quad v = \dfrac{1}{a}\sin(ax)$

$$\int x^{m-1}\cos(ax) = \frac{1}{a}x^{m-1}\sin(ax)-\frac{m-1}{a}\int x^{m-2}\sin(ax)\,dx$$

$$\begin{aligned}\text{So}\int x^m\sin(ax) &= -\frac{1}{a}x^m\cos(ax)\\ &+\frac{m}{a}\left[\frac{1}{a}x^{m-1}\sin(ax)-\frac{m-1}{a}\int x^{m-2}\sin(ax)\,dx\right]\\ &= -\frac{1}{a}x^m\cos(ax)+\frac{m}{a^2}x^{m-1}\sin(ax)\\ &-\frac{m(m-1)}{a^2}\int x^{m-2}\sin(ax)\,dx\end{aligned}$$

b Let $m = 4,\ a = 2$

$$\begin{aligned}\int x^4\sin(2x)\,dx &= -\frac{1}{2}x^4\cos(2x)+x^3\sin(2x)\\ &-3\int x^2\sin(2x)\,dx\end{aligned}$$

Let $m = 2,\ a = 2$

$$\begin{aligned}\int x^2\sin(2x)\,dx &= -\frac{1}{2}x^2\cos(2x)+\frac{1}{2}x\sin(2x)\\ &-\frac{1}{2}\int\sin(2x)\,dx\\ &= -\frac{1}{2}x^2\cos(2x)+\frac{1}{2}x\sin(2x)\\ &+\frac{1}{4}\cos(2x)\end{aligned}$$

$$\begin{aligned}\text{So}\int x^4\sin(2x)\,dx &= -\frac{1}{2}x^4\cos(2x)+x^3\sin(2x)\\ &+\frac{3}{2}x^2\cos(2x)-\frac{3}{2}x\sin(2x)\\ &-\frac{3}{4}\cos(2x)+c\end{aligned}$$

16 a $I_n(x) = \int\sin^n(x)\,dx$

$$= \int\sin(x)\sin^{n-1}(x)\,dx$$

$u = \sin^{n-1}(x)$ $\quad \dfrac{dv}{dx} = \sin(x)$

$\dfrac{du}{dx} = (n-1)\sin^{n-2}(x)\cos(x)$ $\quad v = -\cos(x)$

$$\begin{aligned}I_n &= -\cos(x)\sin^{n-1}(x)+(n-1)\int\sin^{n-2}(x)\cos^2(x)\,dx\\ &= -\cos(x)\sin^{n-1}(x)+(n-1)\int\sin^{n-2}(x)\left(1-\sin^2(x)\right)dx\\ &= -\cos(x)\sin^{n-1}(x)+(n-1)\int\sin^{n-2}(x)\,dx\\ &\qquad -(n-1)\int\sin^n(x)dx\\ &= -\cos(x)\sin^{n-1}(x)+(n-1)I_{n-2}-(n-1)I_n\end{aligned}$$

$$\begin{aligned}I_n &= -\cos(x)\sin^{n-1}(x)+(n-1)I_{n-2}-nI_n+I_n\\ nI_n &= -\cos(x)\sin^{n-1}(x)+(n-1)I_{n-2}\\ I_n &= -\frac{1}{n}\sin^{n-1}(x)\cos(x)+\frac{n-1}{n}I_{n-2}\ \text{shown}\end{aligned}$$

b Now $I_1 = \int\sin(x)\,dx = -\cos(x)$

$$\begin{aligned}I_3 &= -\frac{1}{3}\cos(x)\sin^2(x)+\frac{2}{3}I_1\\ &= -\frac{1}{3}\cos(x)\sin^2(x)-\frac{2}{3}\cos(x)\\ I_5 &= -\frac{1}{5}\cos(x)\sin^4(x)+\frac{4}{5}I_3\\ &= -\frac{1}{5}\cos(x)\sin^4(x)+\frac{4}{5}\left[-\frac{1}{3}\cos(x)\sin^2(x)-\frac{2}{3}\cos(x)\right]\\ I_5 &= -\frac{1}{5}\cos(x)\sin^4(x)-\frac{4}{15}\cos(x)\sin^2(x)-\frac{8}{15}\cos(x)\end{aligned}$$

$$\begin{aligned}&\int_0^{\frac{\pi}{2}}\sin^5(x)\,dx\\ &= -\frac{1}{5}\cos\left(\frac{\pi}{2}\right)\sin^4\left(\frac{\pi}{2}\right)-\frac{4}{15}\cos\left(\frac{\pi}{2}\right)\sin^2\left(\frac{\pi}{2}\right)\\ &\quad -\frac{8}{15}\cos\left(\frac{\pi}{2}\right)+\frac{1}{5}\cos(0)\sin^4(0)+\frac{4}{15}\cos(0)\sin^2(0)\\ &\quad +\frac{8}{15}\cos(0)\\ &= \frac{8}{15}\end{aligned}$$

17 a $C_n = \int\cos^n(x)\,dx$

$$= \int\cos(x)\cos^{n-1}(x)\,dx$$

$u = \cos^{n-1}(x)$ $\quad \dfrac{dv}{dx} = \cos(x)$

$\dfrac{du}{dx} = -(n-1)\cos^{n-2}(x)\sin(x)$ $\quad v = \sin(x)$

$$\begin{aligned}C_n &= \sin(x)\cos^{n-1}(x)+(n-1)\int\cos^{n-2}(x)\sin^2(x)\,dx\\ &= \sin(x)\cos^{n-1}(x)+(n-1)\int\cos^{n-2}(x)\left(1-\cos^2(x)\right)dx\end{aligned}$$

$$= \sin(x)\cos^{n-1}(x) + (n-1)\int \cos^{n-2}(x)\,dx - (n-1)\int \cos^{n}(x)\,dx$$

$$C_n = \sin(x)\cos^{n-1}(x) + (n-1)\int \cos^{n-2}(x)\,dx - n\int \cos^{n}(x)\,dx + \int \cos^{n}(x)\,dx$$

$$n\int \cos^{n}(x)\,dx = \sin(x)\cos^{n-1}(x) + (n-1)\,C_{n-2}$$

$$C_n = \frac{1}{n}\sin(x)\cos^{n-1}(x) + \frac{n-1}{n}C_{n-2}$$

b $C_1 = \int \cos(x)\,dx = \sin(x)$

$$C_3 = \int \cos^3(x)\,dx$$

$$= \frac{1}{3}\sin(x)\cos^2(x) + \frac{2}{3}C_1$$

$$= \frac{1}{3}\sin(x)\cos^2(x) + \frac{2}{3}\sin(x)$$

$$C_5 = \frac{1}{5}\sin(x)\cos^4(x) + \frac{4}{5}C_3$$

$$= \frac{1}{5}\sin(x)\cos^4(x) + \frac{4}{5}\left(\frac{1}{3}\sin(x)\cos^2(x) + \frac{2}{3}\sin(x)\right)$$

$$= \frac{1}{5}\sin(x)\cos^4(x) + \frac{4}{15}\sin(x)\cos^2(x) + \frac{8}{15}\sin(x)$$

$$\int_0^{\frac{\pi}{2}} \cos^5(x)\,dx = \frac{1}{5}\sin\left(\frac{\pi}{2}\right)\cos^4\left(\frac{\pi}{2}\right) + \frac{4}{15}\sin\left(\frac{\pi}{2}\right)\cos^2\left(\frac{\pi}{2}\right)$$

$$+ \frac{8}{15}\sin\left(\frac{\pi}{2}\right) - \frac{1}{5}\sin(0)\cos^4(0)$$

$$- \frac{4}{15}\sin(0)\cos^2(0) - \frac{8}{15}\sin(0)$$

$$= \frac{8}{15}$$

18 a $SL(x) = \int \sin\left(\log_e(x)\right)\,dx = \int 1 \cdot \sin\left(\log_e(x)\right)\,dx$

$$\frac{du}{dx} = 1 \qquad v = \sin\left(\log_e(x)\right)$$

$$u = x \qquad \frac{dv}{dx} = \frac{1}{x} \times \cos\left(\log_e(x)\right)$$

$$SL(x) = x\sin\left(\log_e(x)\right) - \int x \times \frac{1}{x}\cos\left(\log_e(x)\right)\,dx$$

$$= x\sin\left(\log_e(x)\right) - \int \cos\left(\log_e(x)\right)\,dx$$

$$SL = x\sin\left(\log_e(x)\right) - CL(x) \qquad (1)$$

b $CL(x) = \int \cos\left(\log_e(x)\right)\,dx = \int 1 \cdot \cos\left(\log_e(x)\right)\,dx$

$$\frac{du}{dx} = 1 \qquad v = \cos\left(\log_e(x)\right)$$

$$u = x \qquad \frac{dv}{dx} = -\frac{1}{x}\sin\left(\log_e(x)\right)$$

$$CL(x) = x\cos\left(\log_e(x)\right) + \int x \times \frac{1}{x}\sin\left(\log_e(x)\right)\,dx$$

$$= x\cos\left(\log_e(x)\right) + \int \sin\left(\log_e(x)\right)\,dx$$

$$CL = x\cos\left(\log_e(x)\right) + SL(x) \qquad (2)$$

c Substitute (2) into (1)

$$SL(x) = x\sin\left(\log_e(x)\right) - \left[x\cos\left(\log_e(x)\right) + SL(x)\right]$$

$$2SL(x) = x\sin\left(\log_e(x)\right) - x\cos\left(\log_e(x)\right)$$

$$SL(x) = \frac{x}{2}\left[\sin\left(\log_e(x)\right) - \cos\left(\log_e(x)\right)\right] \text{ shown}$$

Substitute (1) into (2)

$$CL(x) = x\cos\left(\log_e(x)\right) + \left[x\sin\left(\log_e(x)\right) - CL(x)\right]$$

$$2\,CL(x) = x\left[\cos\left(\log_e(x)\right) + \sin\left(\log_e(x)\right)\right]$$

$$CL(x) = \frac{x}{2}\left[\cos\left(\log_e(x)\right) + \sin\left(\log_e(x)\right)\right] \text{ shown}$$

19 a $S = \int e^{ax}\sin(bx)\,dx$

$$u = \sin(bx) \qquad \frac{dv}{dx} = e^{ax}$$

$$\frac{du}{dx} = b\cos(bx) \qquad v = \frac{1}{a}e^{ax}$$

$$S = \frac{1}{a}e^{ax}\sin(bx) - \frac{b}{a}\int e^{ax}\cos(bx)\,dx$$

$$\text{so } S = \frac{1}{a}e^{ax}\sin(bx) - \frac{b}{a}C \quad (1)$$

b $C = \int e^{ax}\cos(bx)\,dx$

$$u = \cos(bx) \qquad \frac{dv}{dx} = e^{ax}$$

$$\frac{du}{dx} = -b\sin(bx) \qquad v = \frac{1}{a}e^{ax}$$

$$C = \frac{1}{a}e^{ax}\cos(bx) + \frac{b}{a}\int e^{ax}\sin(bx)\,dx$$

$$C = \frac{1}{a}e^{ax}\cos(bx) + \frac{b}{a}S \quad (2)$$

c Substitute (2) into (1)

$$S = \frac{1}{a}e^{ax}\sin(bx) - \frac{b}{a}\left[\frac{1}{a}e^{ax}\cos(bx) + \frac{b}{a}S\right]$$

$$= \frac{1}{a}e^{ax}\sin(bx) - \frac{b}{a^2}e^{ax}\cos(bx) - \frac{b^2}{a^2}S$$

$$S\left(1 + \frac{b^2}{a^2}\right) = \frac{1}{a}e^{ax}\sin(bx) - \frac{b}{a^2}e^{ax}\cos(bx)$$

$$S\left(\frac{a^2 + b^2}{a^2}\right) = \frac{1}{a^2}e^{ax}\left(a\sin(bx) - b\cos(bx)\right)$$

$$\text{so } S = \frac{e^{ax}}{a^2 + b^2}\left(a\sin(bx) - b\cos(bx)\right) \text{ shown}$$

Substitute (1) into (2)

$$C = \frac{1}{a}e^{ax}\cos(bx) + \frac{b}{a}\left[\frac{1}{a}e^{ax}\sin(bx) - \frac{b}{a}C\right]$$

$$= \frac{1}{a}e^{ax}\cos(bx) + \frac{b}{a^2}e^{ax}\sin(bx) - \frac{b^2}{a^2}C$$

$$C\left(1 + \frac{b^2}{a^2}\right) = \frac{1}{a^2}e^{ax}\left(a\cos(bx) + b\sin(bx)\right)$$

$$C\left(\frac{a^2 + b^2}{a^2}\right) = \frac{1}{a^2}e^{ax}\left(a\cos(bx) + b\sin(bx)\right)$$

$$\text{so } C = \frac{e^{ax}}{a^2 + b^2}\left(a\cos(bx) + b\sin(bx)\right) \text{ shown}$$

d $y = e^{-2x}\sin(3x)$

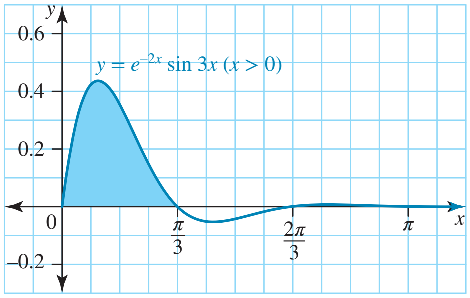
Crosses x − axis:

$y = 0$

$\sin(3x) = 0$

$3x = 0, \pi$

$x = 0, \dfrac{\pi}{3}$



$$A = \int_0^{\frac{\pi}{3}} e^{-2x} \sin(3x)\, dx$$

$a = -2 \qquad b = 3$

$$A = \left[-\frac{e^{-2x}}{13}\left[2\sin(3x) + 3\cos(3x)\right]\right]_0^{\frac{\pi}{3}}$$

$$= -\frac{e^{-\frac{2\pi}{3}}}{13}(2\sin(\pi) + 3\cos(\pi)) + \frac{1}{13}(2\sin(0) + 3\cos(0))$$

$$= \frac{3e^{-\frac{2\pi}{3}}}{13} + \frac{3}{13}$$

$$= \frac{3}{13}\left(1 + e^{-\frac{2\pi}{3}}\right) \text{ units}^2$$

20 a $\frac{d}{dx}(x^n) = nx^{n-1}$

when $n = 1$ $\frac{d}{dx}(x) = 1$ true

Assume its true when $n = k$

$\frac{d}{dx}\left(x^k\right) = kx^{k-1}$

Consider $\frac{d}{dx}\left(x^{k+1}\right)$

$= \frac{d}{dx}\left(x \times x^k\right)$ product rule

$= x\frac{d}{dx}\left(x^k\right) + x^k\frac{d}{dx}(x)$

$= x \times k \times x^{k-1} + 1 \times x^k$

$= kx^k + x^k$

$= (k+1)x^k$

So by induction true $n \in Z$

b $\int x^n dx = \frac{x^{n+1}}{n+1}$

When $n = 1$ $\quad \int x\, dx = \frac{1}{2}x^2$ true

Assume true when $n = k$ $\quad \int x^k dx = \frac{x^{k+1}}{k+1}$

Consider $\int x^{k+1}\, dx$

$= \int x \times x^k\, dx$

$u = x \qquad \frac{dv}{dx} = x^k$

$\frac{du}{dx} = 1 \qquad v = \frac{1}{k+1}x^{k+1}$

$$\int x^{k+1}dx = \frac{1}{k+1}x \times x^{k+1} - \int \frac{1}{k+1}x^{k+1}dx$$

$$= \frac{1}{k+1}x^{k+2} - \frac{1}{k+1}\int x^{k+1}\, dx$$

$$\left(1 + \frac{1}{k+1}\right)\int x^{k+1}dx = \frac{1}{k+1}x^{k+2}$$

$$\frac{k+2}{k+1}\int x^{k+1}dx = \frac{1}{k+1}x^{k+2}$$

$$\int x^{k+1}dx = \frac{1}{k+2}x^{k+2}$$

Shown for $n \in z$

21 Let $I_n(x) = \int_0^x t^n e^{-t}dt$ for $n \in N \quad n \geq 0$

$$I_n(x) = n!\left[1 - e^{-x}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots + \frac{x^n}{n!}\right)\right]$$

When $n = 0$

LHS $I_0(x) = \int_0^x t^0 e^{-t}dt = \int_0^x e^{-1}dt = \left[-e^{-t}\right]_0^x = \left(-e^{-x} + 1\right) =$ $1 - e^{-x}$

RHS $0!\left[1 - e^{-x}(1)\right] = 1 - e^{-x}$

Assume it true when $n = k$

$$I_k(x) = k!\left[1 - e^{-x}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots + \frac{x^k}{k!}\right)\right]$$

Now consider $I_{k+1}(x) = \int_0^x t^{k+1}e^{-t}dt$ use integration by parts

$u = t^{k+1} \qquad \frac{dv}{dt} = e^{-t} \qquad \frac{du}{dt} = (k+1)t^k \qquad v = -e^{-t}$

$$I_{k+1}(x) = \left[-t^{k+1}e^{-t}\right]_0^x + (k+1)\int_0^x t^k e^{-t}dt$$

$I_{k+1}(x) = -x^{k+1}e^{-x} + (k+1)I_k(x)$ from inductive assumption

$$I_{k+1}(x) = -x^{k+1}e^{-x} + (k+1)\left[k!\left[1 - e^{-x}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots + \frac{x^k}{k!}\right)\right]\right]$$

$$I_{k+1}(x) = -x^{k+1}e^{-x} + (k+1)k!\left[1 - e^{-x}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots + \frac{x^k}{k!}\right)\right]$$

$$I_{k+1}(x) = -x^{k+1}e^{-x} + (k+1)!\left[1 - e^{-x}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots + \frac{x^k}{k!}\right)\right]$$

$$I_{k+1}(x) = (k+1)!\left[1 - e^{-x}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots + \frac{x^k}{k!} + \frac{x^{k+1}}{(k+1)!}\right)\right]$$

Which is true for $n = k + 1$, hence by induction it is true for all $n \geq 0$.

22 a $I_0 = \int_0^{-4}\sqrt{x+4}\, dx = \int_0^{-4}(x+4)^{\frac{1}{2}}\, dx$

$$I_0 = \left[\frac{2}{3}(x+4)^{\frac{3}{2}}\right]_0^{-4}$$

$$= \frac{2}{3}\left[(0) - 4^{\frac{3}{2}}\right] = \frac{16}{3}$$

$$I_1 = \int_0^{-4} x\sqrt{x+4}\, dx$$

Let $u = x + 4, \quad x = u - 4, \quad \frac{du}{dx} = 1$

$x = -4, \quad u = 0, \quad x = 0, \quad u = 4$

$$I_1 = \int_4^0 (u-4)\sqrt{u}\, du = \int_4^0 (u-4)\, u^{\frac{1}{2}}\, du$$

$$I_1 = \int_4^0 \left(u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\right)\, du$$

$$I_1 = \left[\frac{2}{5}u^{\frac{5}{2}} - \frac{8}{3}u^{\frac{3}{2}}\right]_4^0$$

$$I_1 = \left[0 - \left(\frac{2}{5}(4)^{\frac{5}{2}} - \frac{4}{3}(4)^{\frac{3}{2}}\right)\right] = -\frac{64}{5} + \frac{64}{3}$$

$$I_1 = 64\left(\frac{5-3}{15}\right) = \frac{128}{15}$$

b $I_{k+1} = \int_0^{-4} x^{k+1}\sqrt{x+4}\,dx$ parts

$u = x^{k+1}$ $\quad \dfrac{dv}{dx} = \sqrt{x+4} = (x+4)^{\frac{1}{2}}$

$\dfrac{du}{dx} = (k+1)\,x^k$ $\quad v = \dfrac{2}{3}(x+4)^{\frac{3}{2}}$

$$I_{k+1} = \left[\frac{2x^{k+1}}{3}(x+4)^{\frac{3}{2}}\right]_0^{-4} - \frac{2(k+1)}{3}\int_0^{-4} x^k(x+4)^{\frac{3}{2}}\,dx$$

$$I_{k+1} = 0 - \frac{2(k+1)}{3}\int_0^{-4}\left(x^k\right)(x+4)\sqrt{x+4}\,dx$$

$$I_{k+1} = -\frac{2(k+1)}{3}\int_0^{-4}\left(x^{k+1} + 4x^k\right)\sqrt{x+2}\,dx$$

$$I_{k+1} = -\frac{2(k+1)}{3}\left[\int_0^{-4} x^{k+1}\sqrt{x+2}\,dx + 4\int_0^{-4} x^k\sqrt{x+2}\,dx\right]$$

$$I_{k+1} = -\frac{2(k+1)}{3}\left[I_{k+1} + 4I_k\right]$$

$$I_{k+1}\left(1 + \frac{2(k+1)}{3}\right) = -\frac{8(k+1)}{3}I_k$$

$$\left(\frac{2k+5}{3}\right)I_{k+1} = -\frac{8(k+1)}{3}I_k$$

$$I_{k+1} = \frac{-8(k+1)}{2k+5}I_k$$

c To show $I_n = \int_0^{-4} x^n\sqrt{x+4}\,dx$ then $I_n = \dfrac{-8n}{2n+3}\,I_{n-1}$

When $n = 1$ LHS $= I_1 = \dfrac{128}{15}$

RHS $= \dfrac{-8}{3+2}\,I_0 = -\dfrac{8}{5} \times \dfrac{-16}{3} = \dfrac{128}{15}$

So it is true when $n = 1$

Assume it is true when $n = k$ that is $I_k = \int_0^{-4} x^k\sqrt{x+4}\,dx$

Consider $I_{k+1} = \int_0^{-4} x^{k+1}\sqrt{x+4}\,dx$

We have shown that

$I_{k+1} = -\dfrac{8(k+1)}{2k+5}\,I_k = -\dfrac{8(k+1)}{2(k+1)+3}\,I_k$

So it is true for $n = k + 1$ by the principle of induction.

It is true for $n = 1, 2, 3....$

23 a $I_n = \int_1^{e^2}\left(\log_e(x)\right)^n\,dx$

$$I_0 = \int_1^{e^2} 1\,dx = [x]_1^{e^2} = e^2 - 1$$

$$I_1 = \int_1^{e^2}\left(\log_e(x)\right)\,dx = \int_1^{e^2} 1 \times \left(\log_e(x)\right)\,dx \text{ parts}$$

$u = \log_e(x)$ $\quad \dfrac{dv}{dx} = 1$

$\dfrac{du}{dx} = \dfrac{1}{x}$ $\quad v = x$

$$I_1 = \left[x\log_e(x)\right]_1^{e^2} - \int_1^{e^2} x \times \frac{1}{x}\,dx$$

$$I_1 = \left[e^2\log_e(e^2) - \log_e(1)\right] - \int_1^{e^2} 1\,dx$$

$$I_1 = 2e^2 - [x]_1^{e^2} = 2e^2 - \left(e^2 - 1\right)$$

$$I_1 = e^2 + 1$$

b $I_{k+1} = \int_1^{e^2}\left(\log_e(x)\right)^{k+1}\,dx$

$u = \left(\log_e(x)\right)^{k+1}$ $\quad \dfrac{dv}{dx} = 1$

$\dfrac{du}{dx} = (k+1)\left(\log_e(x)\right)^k \times \dfrac{1}{x}$ $\quad v = x$

$$I_{k+1} = \left[x\left(\log_e(x)\right)^{k+1}\right]_1^{e^2} - \int_1^{e^2} x\,(k+1)\left(\log_e(x)\right)^k \times \frac{1}{x}\,dx$$

$$I_{k+1} = \left[e^2\left(\log_e(e^2)\right)^{k+1} - \log_e(1)\right] - (k+1)\int_1^{e^2}\left(\log_e(x)\right)^k\,dx$$

$$I_{k+1} = 2^{k+1}e^2 - (k+1)\,I_k$$

Using the recurrence relation

$I_3 = 2^3e^2 - 3I_2 = 8e^2 - 3\left(2e^2 - 2\right)$

$I_3 = 2e^2 + 6$

$I_4 = 2^4e^2 - 4I_3 = 16e^2 - 4\left(2e^2 + 6\right)$

$I_4 = 8e^2 - 24$

c Induction, consider $I_n = 2^n e^2 - nI_{n-1}$

When $n = 1$ LHS $= I_1 = 2^1e^2 - I_0 = 2e^2 - \left(e^2 - 1\right) = e^2 + 1 =$ RHS from (a)

So it is true when $n = 1$

Consider $I_{k+1} = \int_1^{e^2}\left(\log_e(x)\right)^{k+1}\,dx$, we have shown

$I_{k+1} = 2^{k+1}e^2 - (k+1)\,I_k$ from (b)

So it is true for $n = k + 1$ by the principle of induction.

It is true for $n = 1, 2, 3....$

24 a $I_0 = \int_0^1 xe^{x^2}\,dx = \dfrac{1}{2}\left[e^{x^2}\right]_0^1 = \dfrac{1}{2}\,(e - 1)$

b $I_n = \int_0^1 x^{2n+1}e^{x^2}\,dx = \int_0^1 x^{2n} \times xe^{x^2}\,dx$ parts

$u = x^{2n}$ $\quad \dfrac{dv}{dx} = xe^{x^2}$

$\dfrac{du}{dx} = 2nx^{2n-1}$ $\quad v = \dfrac{1}{2}e^{x^2}$

$$I_n = \left[\frac{1}{2}x^{2n}e^{x^2}\right]_0^1 - \int_0^1 2nx^{2n-1}\,\frac{e^{x^2}}{2}\,dx$$

$$I_n = \left(\frac{e}{2} - 0\right) - n\int_0^1 x^{2n-1}e^{x^2}\,dx$$

$$I_n = \frac{e}{2} - n\int_0^1 x^{2(n-1)+1}e^{x^2}\,dx$$

$$I_n = \frac{e}{2} - nI_{n-1}$$

$$I_0 = \frac{1}{2}(e-1)$$

$$I_1 = \frac{e}{2} - I_o = \frac{e}{2} - \frac{1}{2}(e-1) = \frac{1}{2}$$

$$I_2 = \frac{e}{2} - 2I_1 = \frac{e}{2} - 1$$

$$I_3 = \frac{e}{2} - 3I_2 = \frac{e}{2} - 3\left(\frac{e}{2} - 1\right) = 3 - e$$

$$I_4 = \frac{e}{2} - 4I_3 = \frac{e}{2} - 4(3-e) = \frac{9e}{2} - 12$$

9.2 Exam questions

1 $\int x^3\log_e(2x)\,dx$

$$u = \log_e(2x) \qquad \frac{dv}{dx} = x^3$$

$$\frac{du}{dx} = \frac{1}{x} \qquad v = \frac{1}{4}x^4$$

$$\int x^3\log_e(2x)\,dx = \frac{1}{4}x^4\log_e(2x) - \int \frac{1}{4}x^4\cdot\frac{1}{x}\,dx \quad \text{[1 mark]}$$

$$= \frac{1}{4}x^4\log_e(2x) - \frac{1}{4}\int x^3\,dx$$

$$= \frac{1}{4}x^4\log_e(2x) - \frac{1}{16}x^4 + c$$

$$= \frac{1}{16}x^4\left(4\log_e(2x) - 1\right) + c \quad \text{[1 mark]}$$

2 $n \in Z$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx$$

$$u = x \qquad \frac{dv}{dx} = \cos(nx)$$

$$\frac{du}{dx} = 1 \qquad v = \frac{1}{n}\sin(nx)$$

$$a_n = \frac{1}{\pi}\left[\left[\frac{x}{n}\sin(nx)\right]_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi}\sin(nx)\,dx\right] \quad \text{[1 mark]}$$

$$= \frac{1}{\pi}\left[\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\sin(n\pi) + \frac{1}{n^2}\cos(n\pi) + \frac{\pi}{n}\sin(-n\pi) - \frac{1}{n^2}\cos(-n\pi)\right]$$

Now $\sin(n\pi) = 0 \quad \sin(-n\pi) = 0$

$\cos(n\pi) = \cos(-n\pi) \quad = (-1)^n$

So $a_n = 0$ [1 mark]

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx$$

$$u = x \qquad \frac{dv}{dx} = \sin(nx)$$

$$\frac{du}{dx} = 1 \quad v = \frac{-1}{n}\cos(nx)$$

$$b_n = \frac{1}{\pi}\left[\left[-\frac{x}{n}\cos(nx)\right]_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\cos(nx)\,dx\right] \quad \text{[1 mark]}$$

$$= \frac{1}{\pi}\left[-\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\cos(n\pi) + \frac{1}{n^2}\sin(n\pi) - \frac{\pi}{n}\cos(-n\pi) - \frac{1}{n^2}\sin(-n\pi)\right]$$

$$= -\frac{2}{n}\cos(n\pi)$$

$$= -\frac{2}{n}(-1)^n \quad \text{[1 mark]}$$

3 **a** $J_n = \int x^n e^{kx}\,dx$

$$u = x^n \qquad \frac{dv}{dx} = e^{kx}$$

$$\frac{du}{dx} = nx^{n-1} \quad v = \frac{1}{k}e^{kx}$$

$$J_n = \frac{1}{k}x^n e^{kx} - \frac{n}{k}\int x^{n-1}e^{kx}\,dx$$

$$J_n = \frac{1}{k}x^n e^{kx} - \frac{n}{k}J_{n-1} \quad \text{[1 mark]}$$

b $J_0 = \int e^{kx}\,dx = \frac{1}{k}e^{kx}$

$$J_1 = \int xe^{kx}dx = \frac{1}{k}xe^{kx} - \frac{1}{k}J_0$$

$$= \frac{1}{k}xe^{kx} - \frac{1}{k^2}e^{kx}$$

$$J_2 = \int x^2e^{kx}dx = \frac{1}{k}x^2e^{kx} - \frac{2}{k}J_1$$

$$= \frac{1}{k}x^2e^{kx} - \frac{2}{k}\left(\frac{1}{k}xe^{kx} - \frac{1}{k^2}e^{kx}\right)$$

$$= \frac{1}{k}e^{kx}\left(x^2 - \frac{2x}{k} + \frac{2}{k^2}\right)$$

$$J_3 = \int x^3e^{kx}\,dx = \frac{1}{k}x^3e^{kx} - \frac{3}{k}J_2$$

$$= \frac{1}{k}x^3e^{kx} - \frac{3}{k}\left(\frac{1}{k}e^{kx}\left(x^2 - \frac{2x}{k} + \frac{2}{k^2}\right)\right)$$

$$= \frac{1}{k}e^{kx}\left(x^3 - \frac{3x^2}{k} + \frac{6x}{k^2} - \frac{6}{k^3}\right) \quad \text{[1 mark]}$$

So $\int_0^1 x^3e^{2x}\,dx \quad k = 2 \qquad n = 3$

$$= \left[\frac{1}{2}e^{2x}\left(x^3 - \frac{3x^2}{2} + \frac{6x}{4} - \frac{6}{8}\right)\right]_0^1$$

$$= \frac{1}{2}e^2\left(1 - \frac{3}{2} + \frac{6}{4} - \frac{3}{4}\right) - \frac{1}{2}e^0\left(0 - \frac{3}{4}\right)$$

$$= \frac{1}{8}\left(e^2 + 3\right) \quad \text{[1 mark]}$$

9.3 Integration by recognition and graphs of anti-derivatives

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9.3 Exercise

1 $y = \sin^{-1}\left(\frac{\sqrt{x}}{2}\right)$

$$= \sin^{-1}\left(\frac{u}{2}\right) \quad u = \sqrt{x} = x^{\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{1}{\sqrt{4-u^2}} \quad \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \times \frac{1}{\sqrt{4-x}}$$

$$\frac{d}{dx}\left(\sin^{-1}\left(\frac{\sqrt{x}}{2}\right)\right) = \frac{1}{2\sqrt{4x-x^2}} \text{ for } \qquad 0 < x < 4$$

$$\begin{aligned}\int_1^4 \frac{1}{\sqrt{4x-x^2}}\,dx &= \left[2\sin^{-1}\left(\frac{\sqrt{x}}{2}\right)\right]_1^4\\ &= 2\left[\sin^{-1}(1) - \sin^{-1}\left(\frac{1}{2}\right)\right]\\ &= 2\left(\frac{\pi}{2} - \frac{\pi}{6}\right)\\ &= 2\left(\frac{\pi(3-1)}{6}\right)\\ &= \frac{2\pi}{3}\end{aligned}$$

2 $y = \arcsin\left(\frac{2}{x}\right) = \sin^{-1}\left(\frac{2}{x}\right)$

$$= \sin^{-1}(u) \qquad u = \frac{2}{x} = 2x^{-1}$$

$$\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}} \qquad \frac{du}{dx} = -2x^{-2} = \frac{-2}{x^2}$$

$$\begin{aligned}\frac{dy}{dx} &= -\frac{2}{x^2}\,\frac{1}{\sqrt{1-\left(\frac{4}{x^2}\right)}}\\ &= -\frac{2}{x^2}\,\frac{1}{\sqrt{\frac{x^2-4}{x^2}}}\\ &= -\frac{2\,|x|}{x^2\sqrt{x^2-4}}\\ &= -\frac{2}{|x|\sqrt{x^2-4}},\ |x| > 2\end{aligned}$$

$$\frac{d}{dx}\left(\arcsin\left(\frac{2}{x}\right)\right) = \frac{-2}{|x|\sqrt{x^2-4}}$$

$$\begin{aligned}\int_{\frac{4\sqrt{3}}{3}}^{4} \frac{1}{x\sqrt{x^2-4}}\,dx &= -\frac{1}{2}\left[\arcsin\left(\frac{2}{x}\right)\right]_{\frac{4\sqrt{3}}{3}}^{4}\\ &= -\frac{1}{2}\left(\arcsin\left(\frac{1}{2}\right) - \arcsin\left(\frac{\sqrt{3}}{2}\right)\right)\\ &= -\frac{1}{2}\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\\ &= -\frac{1}{2}\left(\frac{\pi(1-2)}{6}\right)\\ &= \frac{\pi}{12}\end{aligned}$$

3 $y = \sin^{-1}\left(\frac{x^3}{8}\right)$

$$= \sin^{-1}\left(\frac{u}{8}\right) \qquad u = x^3$$

$$\frac{dy}{du} = \frac{1}{\sqrt{64-u^2}} \qquad \frac{du}{dx} = 3x^2$$

$$\frac{d}{dx}\left(\sin^{-1}\left(\frac{x^3}{8}\right)\right) = \frac{3x^2}{\sqrt{64-x^6}} \quad |x| \quad < 2$$

$$\begin{aligned}\int_2^{\sqrt[3]{4}} \frac{x^2}{\sqrt{64-x^6}}\,dx &= \frac{1}{3}\left[\sin^{-1}\left(\frac{x^3}{8}\right)\right]_2^{\sqrt[3]{4}}\\ &= \frac{1}{3}\left(\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(1)\right)\\ &= \frac{1}{3}\left(\frac{\pi}{6} - \frac{\pi}{2}\right)\\ &= -\frac{\pi}{9}\end{aligned}$$

4 $y = \cos^{-1}\left(\frac{4}{\sqrt{x}}\right)$

$$= \cos^{-1}(u) \qquad u = \frac{4}{\sqrt{x}} = 4x^{-\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{-1}{\sqrt{1-u^2}} \qquad \frac{du}{dx} = -2x^{\frac{-3}{2}} = \frac{-2}{\sqrt{x^3}}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{2}{\sqrt{x^3}} \times \frac{1}{\sqrt{1-\frac{16}{x}}}\\ &= \frac{2}{\sqrt{x^3}}\,\frac{1}{\sqrt{\frac{x-16}{x}}}\\ &= \frac{2}{\sqrt{x^3}} \times \frac{\sqrt{x}}{\sqrt{x-16}}\end{aligned}$$

$$\frac{d}{dx}\left(\cos^{-1}\left(\frac{4}{\sqrt{x}}\right)\right) = \frac{2}{x\sqrt{x-16}},\ x > 16$$

$$\begin{aligned}\int_{\frac{64}{3}}^{64} \frac{1}{x\sqrt{x-16}} &= \frac{1}{2}\left[\cos^{-1}\left(\frac{4}{\sqrt{x}}\right)\right]_{\frac{64}{3}}^{64}\\ &= \frac{1}{2}\left(\cos^{-1}\left(\frac{1}{2}\right) - \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\\ &= \frac{1}{2}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)\\ &= \frac{\pi}{12}\end{aligned}$$

5 $y = \arccos\left(\frac{6}{x}\right) = \cos^{-1}\left(\frac{6}{x}\right)$

$$= \cos^{-1}(u) \qquad u = \frac{6}{x} = 6x^{-1}$$

$$\frac{dy}{du} = \frac{-1}{\sqrt{1-u^2}} \quad \frac{du}{dx} = -6x^{-2} = -\frac{6}{x^2}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{6}{x^2} \times \frac{1}{\sqrt{1-\frac{36}{x^2}}}\\ &= \frac{6}{x^2}\,\frac{1}{\sqrt{\frac{x^2-36}{x^2}}}\\ &= \frac{6}{x^2}\,\frac{|x|}{\sqrt{x^2-36}}\end{aligned}$$

$$\frac{d}{dx}\left(\cos^{-1}\left(\frac{6}{x}\right)\right) = \frac{6}{|x|\sqrt{x^2-36}},\ |x| > 6$$

$$\begin{aligned}\int_{4\sqrt{3}}^{12} \frac{1}{x\sqrt{x^2-36}}\,dx &= \frac{1}{6}\left[\cos^{-1}\left(\frac{6}{x}\right)\right]_{4\sqrt{3}}^{12}\\ &= \frac{1}{6}\left(\cos^{-1}\left(\frac{1}{2}\right) - \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\\ &= \frac{1}{6}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)\\ &= \frac{\pi}{36}\end{aligned}$$

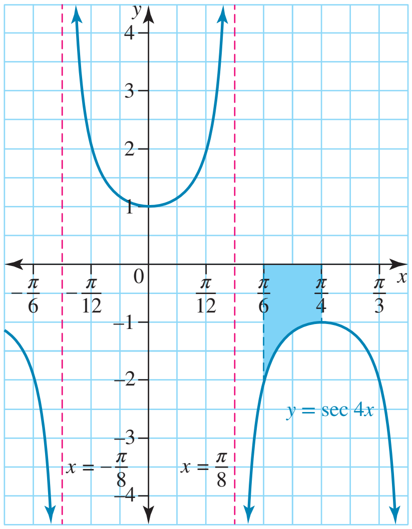
6 $$y = \tan^{-1}\left(\frac{4}{\sqrt{x}}\right)$$
$$= \tan^{-1}(u) \quad u = \frac{4}{\sqrt{x}} = 4x^{-\frac{1}{2}}$$
$$\frac{dy}{du} = \frac{1}{1+u^2} \quad \frac{du}{dx} = -2x^{-\frac{3}{2}} = \frac{-2}{\sqrt{x^3}}$$
$$\frac{dy}{dx} = \frac{-2}{\sqrt{x^3}} \times \frac{1}{1+\frac{16}{x}}$$
$$= \frac{-2}{\sqrt{x^3}} \times \frac{1}{\frac{x+16}{x}}$$
$$= \frac{-2x}{\sqrt{x^3}\,(x+16)}$$
$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{4}{\sqrt{x}}\right)\right) = \frac{-2}{\sqrt{x}\,(x+16)} \quad x > 0$$
$$\int_{\frac{16}{3}}^{16} \frac{1}{\sqrt{x}\,(x+16)}\,dx = -\frac{1}{2}\left[\tan^{-1}\left(\frac{4}{\sqrt{x}}\right)\right]_{\frac{16}{3}}^{16}$$
$$= -\frac{1}{2}\left(\tan^{-1}(1) - \tan^{-1}\left(\sqrt{3}\right)\right)$$
$$= -\frac{1}{2}\left(\frac{\pi}{4} - \frac{\pi}{3}\right)$$
$$= \frac{\pi}{24}$$

7 $$y = \tan^{-1}\left(\frac{\sqrt{x}}{3}\right)$$
$$= \tan^{-1}\left(\frac{u}{3}\right) \quad u = \sqrt{x} = x^{\frac{1}{2}}$$
$$\frac{dy}{du} = \frac{3}{9+u^2} \quad \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$
$$\frac{dy}{dx} = \frac{3}{2\sqrt{x}} \times \frac{1}{9+x}$$
$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{\sqrt{x}}{3}\right)\right) = \frac{3}{2\sqrt{x}\,(9+x)}, \quad x > 0$$
$$\int_{9}^{27} \frac{1}{\sqrt{x}\,(9+x)}\,dx = \frac{2}{3}\left[\tan^{-1}\left(\frac{\sqrt{x}}{3}\right)\right]_{9}^{27}$$
$$= \frac{2}{3}\left(\tan^{-1}\left(\sqrt{3}\right) - \tan^{-1}(1)\right)$$
$$= \frac{2}{3}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$
$$= \frac{2}{3}\left(\frac{\pi(4-3)}{12}\right)$$
$$= \frac{\pi}{18}$$

8 $$y = \tan^{-1}\left(\frac{x^2}{4}\right)$$
$$= \tan^{-1}\left(\frac{u}{4}\right) \quad u = x^2$$
$$\frac{dy}{du} = \frac{4}{16+u^2} \quad \frac{du}{dx} = 2x$$
$$\frac{dy}{dx} = \frac{8x}{16+x^4}$$
$$\int_{0}^{2} \frac{x}{x^4+16}\,dx = \frac{1}{8}\left[\tan^{-1}\left(\frac{x^2}{4}\right)\right]_{0}^{2}$$
$$= \frac{1}{8}\left(\tan^{-1}(1) - \tan^{-1}(0)\right)$$
$$= \frac{1}{8}\left(\frac{\pi}{4} - 0\right)$$
$$= \frac{\pi}{32}$$

9 $$y = \log_e(\tan(x))$$
$$= \log_e(\sin(x)) - \log_e(\cos(x))$$
$$\frac{dy}{dx} = \frac{\cos(x)}{\sin(x)} + \frac{\sin(x)}{\cos(x)}$$
$$= \frac{\cos^2(x) + \sin^2(x)}{\sin(x)\cos(x)}$$
$$= \frac{2}{2\sin(x)\cos(x)}$$
$$= \frac{2}{\sin(2x)}$$
$$\frac{d}{dx}\left(\log_e(\tan(x))\right) = 2\operatorname{cosec}(2x)$$
$$A = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \operatorname{cosec}(2x)\,dx$$
$$= \frac{1}{2}\left[\log_e(\tan(x))\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$
$$= \frac{1}{2}\left[\log_e\left(\tan\left(\frac{\pi}{3}\right)\right) - \log_e\left(\tan\left(\frac{\pi}{6}\right)\right)\right]$$
$$= \frac{1}{2}\left[\log_e\left(\sqrt{3}\right) - \log_e\left(\frac{\sqrt{3}}{3}\right)\right]$$
$$= \frac{1}{2}\log_e(3)$$
$$= \log_e\left(\sqrt{3}\right)$$

10 Let $y = \log_e\sqrt{\dfrac{\sin(2x) + \cos(2x)}{\sin(2x) - \cos(2x)}}$
$$= \frac{1}{2}\log_e\left(\frac{\sin(2x) + \cos(2x)}{\sin(2x) - \cos(2x)}\right)$$
$$y = \frac{1}{2}\log_e(\sin(2x) + \cos(2x)) - \frac{1}{2}\log_e(\sin(2x) - \cos(2x))$$
$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{2\cos(2x) - 2\sin(2x)}{\sin(2x) + \cos(2x)}\right) - \frac{1}{2}\left(\frac{2\cos(2x) + 2\sin(2x)}{\sin(2x) - \cos(2x)}\right)$$
$$= \frac{\cos(2x) - \sin(2x)}{\sin(2x) + \cos(2x)} - \frac{\cos(2x) + \sin(2x)}{\sin(2x) - \cos(2x)}$$
$$= \frac{-(\cos(2x) - \sin(2x))^2 - (\cos(2x) + \sin(2x))^2}{\sin^2(2x) - \cos^2(2x)}$$
$$= \frac{(\cos(2x) - \sin(2x))^2 + (\cos(2x) + \sin(2x))^2}{\cos(4x)}$$
$$= \frac{\begin{gathered}\cos^2(2x) + \sin^2(2x) - 2\sin(2x)\cos(2x) + \cos^2(2x)\\ + \sin^2(2x) + 2\sin(2x)\cos(2x)\end{gathered}}{\cos(4x)}$$
$$= 2\sec(4x)$$



$$A = \left|\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec(4x)\,dx\right|$$

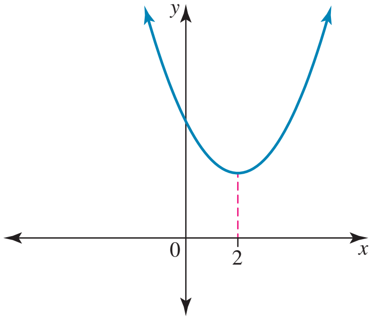
$$= \left|\frac{1}{4}\left[\log_e\left(\frac{\sin(2x) + \cos(2x)}{\sin(2x) - \cos(2x)}\right)\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}\right|$$

$$= \left|0 - \frac{1}{4}\log_e\left(\sqrt{3}+2\right)\right|$$

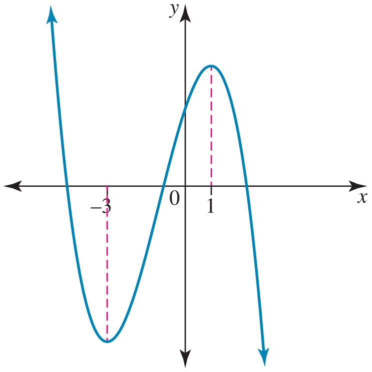
$$= \frac{1}{4}\log_e\left(\sqrt{3}+2\right) \qquad \text{area is below } x\text{-axis}$$

It is an area $\frac{1}{4}\log_e\left(\sqrt{3} + 2\right)$ units^2

11 a At $x = 2$ gradient goes from negative to positive.
$x = 2$ is a minimum turning point.

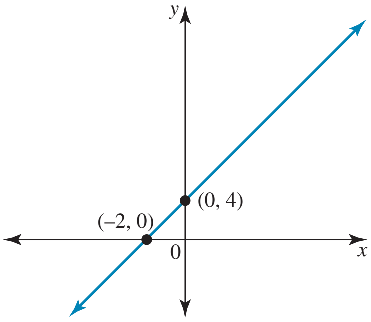


b At $x = -3$ minimum turning point,
$x = 1$ maximum turning point,
$x = -1$ point of inflexion.

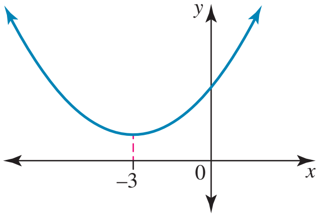


12 At $x = \pm 4$ minimum turning points,
$x = 0$ maximum turning point.

13 a A straight line with gradient of 2.



b $x = -3$ is a minimum turning point.



14 $\frac{dy}{dx} = \tan\left(\frac{1}{x}\right) \quad y\left(\frac{\pi}{8}\right) = \frac{1}{2}$

$$y = \int_0^x \tan\left(\frac{1}{t}\right)\,dt + c$$

when $x = \frac{\pi}{8}$ $y = \frac{1}{2}$

$$\frac{1}{2} = \int_0^{\frac{\pi}{8}} \tan\left(\frac{1}{t}\right)\,dt + c$$

$$c = \frac{1}{2} - \int_0^{\frac{\pi}{8}} \tan\left(\frac{1}{t}\right)\,dt$$

$$y = \int_0^x \tan\left(\frac{1}{t}\right)\,dt + \frac{1}{2} - \int_0^{\frac{\pi}{8}} \tan\left(\frac{1}{t}\right)\,dt$$

$$= \int_0^x \tan\left(\frac{1}{t}\right)\,dt + \int_{\frac{\pi}{8}}^0 \tan\left(\frac{1}{t}\right)\,dt + \frac{1}{2}$$

$$= \int_{\frac{\pi}{8}}^x \tan\left(\frac{1}{t}\right)\,dt + \frac{1}{2}$$

9.3 Exam questions

1 $y = \cos^{-1}\left(\frac{4}{x^2}\right)$

$$= \cos^{-1}(u) \qquad u = \frac{4}{x^2} = 4x^{-2}$$

$$\frac{dy}{du} = \frac{-1}{\sqrt{1-u^2}} \qquad \frac{du}{dx} = -8x^{-3} = -\frac{8}{x^3}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{8}{x^3}\sqrt{\frac{1}{1-\frac{16}{x^4}}}\\ &= \frac{8}{x^3}\sqrt{\frac{x^4}{x^4-16}}\\ &= \frac{8}{x^3}\times\frac{x^2}{\sqrt{x^4-16}} && \text{[1 mark]}\end{aligned}$$

$$\frac{d}{dx}\left(\cos^{-1}\left(\frac{4}{x^2}\right)\right) = \frac{8}{x\sqrt{x^4-16}}, \quad |x|>2 \qquad \text{[1 mark]}$$

$$\begin{aligned}\int_2^{2\sqrt{2}}\frac{1}{x\sqrt{x^4-16}}\,dx &= \frac{1}{8}\left[\cos^{-1}\left(\frac{4}{x^2}\right)\right]_2^{2\sqrt{2}} && \text{[1 mark]}\\ &= \frac{1}{8}\left(\cos^{-1}\left(\frac{1}{2}\right)-\cos^{-1}(1)\right)\\ &= \frac{1}{8}\left(\frac{\pi}{3}-0\right)\\ &= \frac{\pi}{24} && \text{[1 mark]}\end{aligned}$$

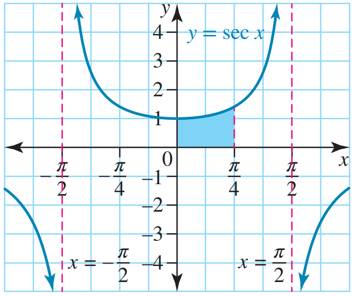
2 $\frac{d}{dx}(\sec(x)) = \tan(x)\sec(x),\ \frac{d}{dx}(\tan(x))$

$$= \sec^2(x) \qquad \text{[1 mark]}$$

If $y = \log_e(\tan(x)+\sec(x))$

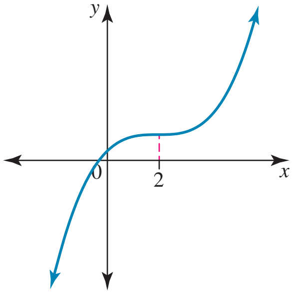
$$\begin{aligned}\frac{dy}{dx} &= \frac{\sec^2(x)+\tan(x)\sec(x)}{\tan(x)+\sec(x)}\\ &= \frac{\sec(x)(\tan(x)+\sec(x))}{\tan(x)+\sec(x)}\\ &= \sec(x) && \text{[1 mark]}\end{aligned}$$

So $\int \sec(x)\,dx = \log_e(\tan(x)+\sec(x))+c$



$$\begin{aligned}A &= \int_0^{\frac{\pi}{4}}\sec(x)\,dx\\ &= \left[\log_e(\tan(x)+\sec(x))\right]_0^{\frac{\pi}{4}} && \text{[1 mark]}\\ &= \log_e\left(\tan\left(\frac{\pi}{4}\right)+\sec\left(\frac{\pi}{4}\right)\right)-\log_e(\tan(0)+\sec(0))\\ &= \log_e\left(1+\sqrt{2}\right) && \text{[1 mark]}\end{aligned}$$

3 A stationary point of inflection at $x = 2$.

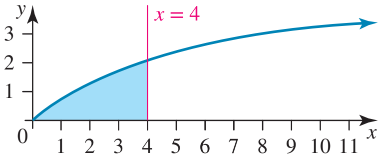


Award 1 mark for correct shape of graph and 1 mark for the stationary point of inflection at $x = 2$.

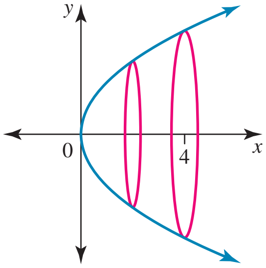
9.4 Solids of revolution

9.4 Exercise

1 a Region bounded by $y = \sqrt{x}$, the x-axis and the line $x = 4$.



b



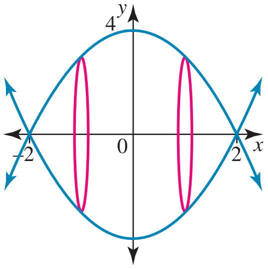
$$\begin{aligned}y &= \sqrt{x}\\ x &= 0, 4\\ V &= \pi\int_a^b [f(x)]^2dx\\ &= \pi\int_0^4 xdx\\ &= \pi\left[\frac{1}{2}x^2\right]_0^4\\ &= 8\pi\text{ units}^3\end{aligned}$$

2 $y = 4 - x^2$

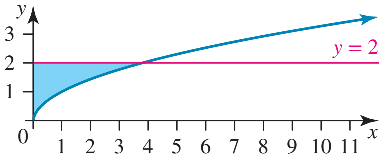
Solve $0 = 4 - x^2$ to determine the x-intercepts

$\therefore x = \pm 2$

$$\begin{aligned}V &= \pi\int_a^b [f(x)]^2dx\\ &= \pi\int_{-2}^{2}\left(4-x^2\right)^2dx\\ &= 2\pi\int_0^2\left(x^4-8x^2+16\right)dx\\ &= 2\pi\left[\frac{1}{5}x^5-\frac{8}{3}x^3+16x\right]_0^2\\ &= 2\pi\left[\frac{1}{5}(2)^5-\frac{8}{3}(2)^3+16(2)-0\right]\\ &= \frac{512\pi}{15}\text{units}^3\end{aligned}$$



3 **a** Region bounded by the curve $y = \sqrt{x}$, the y-axis, the origin and the line $y = 2$.



b $y = \sqrt{x} \rightarrow y^2 = x$

$y = 0, 2$

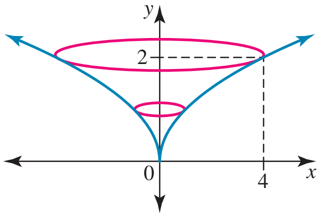
$x^2 = y^4$

$$V = \pi \int_a^b [f(y)]^2 dy$$

$$= \pi \int_0^2 y^4 dy$$

$$= \pi \left[\frac{1}{5}y^5\right]_0^2$$

$$= \frac{32\pi}{5}\text{units}^3$$



4 $y = 4 - x^2 \rightarrow x^2 = 4 - y$

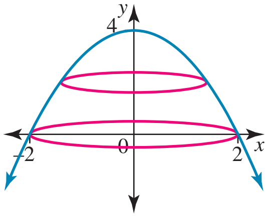
$$V = \pi \int_a^b [f(y)]^2 dy$$

$$= \pi \int_0^4 (4 - y) dy$$

$$= \pi \left[4y - \frac{1}{2}y^2\right]_0^4$$

$$= \pi \left[4(4) - \frac{1}{2}(4)^2 - 0\right]$$

$$= 8\pi\,\text{units}^3$$



5 **a** $V = \pi \int_a^b [f(x)]^2 dx$

$$= \pi \int_0^5 9x^2 dx$$

$$= \frac{9\pi}{3}\left[x^3\right]_0^5$$

$$= 3\pi\left(5^3 - 0\right)$$

$$= 375\pi\ \text{units}^3$$

b $y = 3x$

$x = \frac{y}{3}$

$$V = \pi \int_a^b [f(y)]^2 dy$$

$$= \pi \int_0^5 \frac{y^2}{9} dx$$

$$= \frac{\pi}{9}\left[\frac{1}{3}y^3\right]_0^5$$

$$= \frac{\pi}{27}\left(5^3 - 0\right)$$

$$= \frac{125\pi}{27}\text{units}^3$$

6 **a** $2x + 3y = 6$

$3y = 6 - 2x$

$y = \frac{1}{3}(6 - 2x)$

$$V = \pi \int_a^b [f(x)]^2 dx$$

$$= \pi \int_0^3 \left(\frac{1}{3}(6 - 2x)\right)^2 dx$$

$$= \frac{\pi}{9} \int_0^3 \left(36 - 24x + 4x^2\right) dx$$

$$= \frac{\pi}{9}\left[36x - 12x^2 + \frac{4}{3}x^3\right]_0^3$$

$$= \frac{\pi}{9}\left(36 \times 3 - 12 \times 9 + \frac{4}{3} \times 27 - 0\right)$$

$$= 4\pi\,\text{units}^3$$

b $2x = 6 - 3y$

$x = \frac{1}{2}(6 - 3y)$

$$V = \pi \int_a^b [f(y)]^2 dy$$

$$= \pi \int_0^2 \left(\frac{1}{2}(6 - 3y)\right)^2 dy$$

$$= \frac{\pi}{4} \int_0^2 \left(36 - 36y + 9y^2\right) dy$$

$$= \frac{\pi}{4} \int_0^2 \left[36y - 18y^2 + 3y^3\right]_0^2$$

$$= \frac{\pi}{4}(72 - 18 \times 4 + 3 \times 8 - 0)$$

$$= 6\pi\,\text{units}^3$$

7 $y = \sec(2x)$

$x = 0, \frac{\pi}{8}$

$$V = \pi \int_a^b [f(x)]^2 dx$$

$$= \frac{\pi}{2}\left[\tan(2x)\right]_0^{\frac{\pi}{8}}$$

$$= \frac{\pi}{2}\left(\tan\left(\frac{\pi}{4}\right) - \tan(0)\right)$$

$$= \frac{\pi}{2}\text{units}^3$$

8 $y = 2e^{\frac{x}{2}}$

$x = 0, 2$

$$V = \pi\int_a^b [f(x)]^2 dx$$

$$= 4\pi\left[e^x\right]_0^2$$

$$= 4\pi\left(e^2 - e^0\right)$$

$$= 4\pi\left(e^2 - 1\right)\text{ units}^3$$

9 Bucket

$y = mx + c$

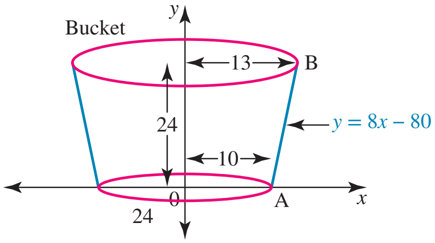
A $(10, 0)$

B $(13, 24)$

$$m = \frac{24}{13 - 10} = 8$$

$[1]: y - 0 = 8(x - 10) \rightarrow y = 8x - 80$

$[2]: 8x = y + 80 \rightarrow x = \frac{1}{8}(y + 80)$



$$V = \pi\int_a^b [f(y)]^2 dy$$

$$= \pi\int_0^{24}\left[\frac{1}{8}(y + 80)\right]^2 dy$$

$$= \frac{\pi}{64}\int_0^{24}\left(y^2 + 160y + 6400\right) dy$$

$$= \frac{\pi}{64}\left[\frac{1}{3}y^3 + 80y^2 + 6400y\right]_0^{24}$$

$$= \frac{\pi}{64}\left[\frac{1}{3}(24)^3 + 80(24)^2 + 6400(24) - 0\right]$$

$$= 3192\pi\text{ cm}^3$$

$$= 3192\pi\text{ mL}$$

$$= \frac{3192\pi}{1000}\text{ L}$$

$$\approx 10\text{ L}$$

10 $y = ax^2 + c$

A $(6, 0) \rightarrow 0 = 36a + c$ [1]

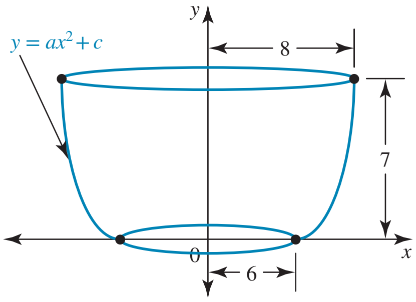
B $(8, 7) \rightarrow 7 = 64a + c$ [2]

$[2] - [1] \rightarrow a = \frac{1}{4} \rightarrow c = -9$

Rotate about y-axis

$$y = \frac{1}{4}x^2 - 9$$

$$x^2 = 4y + 36$$



$$V = \pi\int_a^b [f(y)]^2 dy$$

$$= \pi\int_0^7 (4y + 36)\, dy$$

$$= \pi\left[2y^2 + 36y\right]_0^7$$

$$= \pi\left[2(7)^2 + 36(7) - 0\right]$$

$$= 350\pi\text{ cm}^3$$

$350\pi \approx 1099.56$

The capacity of the soup bowl to the nearest mL is 1100 mL.

11 **a** $y = 3\sin(2x)$

Crosses x-axis at $y = 0$

$3\sin(2x) = 0$

$2x = 0, \pi$

$x = 0, \frac{\pi}{2}$

$$V = \pi\int_a^{b^2} [f(x)]^2 dx = \pi\int_0^{\frac{\pi}{2}} 9\sin^2(2x) dx$$

$$= \frac{9\pi}{2}\int_0^{\frac{\pi}{2}} (1 - \cos(4x))\, dx$$

$$= \frac{9\pi}{2}\left[x - \frac{1}{4}\sin(4x)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{9\pi}{2}\left(\frac{\pi}{2} - \frac{1}{4}\sin(2\pi) - (0)\right)$$

$$= \frac{9\pi^2}{4}\text{units}^3$$

b $y = 4\cos(3x)$

Crosses x-axis at $y = 0$

$4\cos(3x) = 0$

$3x = \frac{\pi}{2}$

$x = \frac{\pi}{6}$

$$V = \pi\int_a^b [f(x)]^2\, dx = \pi\int_0^{\frac{\pi}{6}} 16\cos^2(3x) dx$$

$$= 8\pi\int_0^{\frac{\pi}{6}} (1 + \cos(6x))\, dx$$

$$= 8\pi\left[x + \frac{1}{6}\sin(6x)\right]_0^{\frac{\pi}{6}}$$

$$= 8\pi\left(\frac{\pi}{6} + \frac{1}{6}\sin(\pi) - 0\right)$$

$$= \frac{4\pi^2}{3}\text{units}^3$$

12 a $y = 3x^2 + 4$

Limits are between $x = 0$ and $x = 2$

$$V = \pi \int_a^b \left[f(x)\right]^2 dx$$

$$= \pi \int_0^2 \left(3x^2 + 4\right)^2 dx$$

$$= \pi \int_0^2 \left(9x^4 + 24x^2 + 16\right) dx$$

$$= \pi \left[\frac{9}{5}x^5 + 8x^3 + 16x\right]_0^2$$

$$= \pi \left(\frac{9}{5} \times 32 + 8 \times 8 + 16 \times 2 - (0)\right)$$

$$= \frac{768\pi}{5} \text{units}^3$$

b $3x^2 = y - 4$

$$x^2 = \frac{1}{3}(y - 4)$$

$$y = 4, 10$$

$$V = \pi \int_a^b \left[f(y)\right]^2 dy$$

$$V = \pi \int_4^{10} \frac{1}{3}(y - 4)\, dy$$

$$= \frac{\pi}{3}\left[\frac{1}{2}y^2 - 4y\right]_4^{10}$$

$$= \frac{\pi}{3}\left[\left(\frac{1}{2} \times 100 - 40\right) - \left(\frac{1}{2} \times 16 - 16\right)\right]$$

$$= 6\pi \,\text{units}^3$$

13 a $V = \pi \int_a^b \left[f(x)\right]^2 dx$

$$= 2\pi \int_0^3 \left(x^2 - 9\right)^2 dx$$

$$= 2\pi \int_0^3 \left(x^4 - 18x^2 + 81\right) dx$$

$$= 2\pi \left[\frac{1}{5}x^5 - 6x^3 + 81x\right]_0^3$$

$$= 2\pi \left[\left(\frac{1}{5} \times 243 - 6 \times 27 + 81 \times 3 - (0)\right)\right]$$

$$= \frac{1296\pi}{5} \text{units}^3$$

b $V = \pi \int_a^b \left[f(y)\right]^2 dy$

$$V = \pi \int x^2 dy$$

$$= \pi \int_{-9}^{0} (y + 9)\, dy$$

$$= \pi \left[\frac{1}{2}y^2 + 9y\right]_{-9}^{0}$$

$$= \pi \left[0 - \left(\frac{81}{2} - 81\right)\right]$$

$$= \frac{81\pi}{2} \text{units}^3$$

14 a $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$

$$\frac{y^2}{16} = 1 - \frac{x^2}{25}$$

$$= \frac{25 - x^2}{25}$$

$$y^2 = \frac{16}{25}\left(25 - x^2\right)$$

$$V = \pi \int_a^b \left[f(x)\right]^2 dx$$

$$= \pi \int_{-5}^{5} \frac{16}{25}\left(25 - x^2\right) dx$$

$$= \frac{32\pi}{25} \int_0^5 \left(25 - x^2\right) dx$$

$$= \frac{32\pi}{25}\left[25x - \frac{1}{3}x^3\right]_0^5$$

$$= \frac{32\pi}{25}\left[125 - \frac{1}{3}5^3 - 0\right]$$

$$= \frac{320\pi}{3} \text{ units}^3$$

b $\dfrac{x^2}{25} = 1 - \dfrac{y^2}{16}$

$$x^2 = \frac{25}{16}\left(16 - y^2\right)$$

$$V = \pi \int_a^b \left[f(y)\right]^2 dy$$

$$= \pi \int_{-4}^{4} \frac{25}{16}\left(16 - y^2\right) dy$$

$$= \frac{25\pi}{8} \int_0^4 \left(16 - y^2\right) dy$$

$$= \frac{25\pi}{8}\left[16y - \frac{1}{3}y^3\right]_0^4$$

$$= \frac{25\pi}{8}\left[16 \times 4 - \frac{1}{3}4^3 - 0\right]$$

$$= \frac{400\pi}{3} \text{ units}^3$$

c $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\frac{y^2}{b^2} = \frac{a^2 - x^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2}\left(a^2 - x^2\right)$$

$$V = \pi \int_{x_1}^{x_2} \left[f(x)\right]^2 dx$$

$$= \pi \int_{-a}^{a} \frac{b^2}{a^2}\left(a^2 - x^2\right) dx$$

$$= \frac{2\pi b^2}{a^2} \int_{0}^{a} \left(a^2 - x^2\right) dx$$

$$= \frac{2\pi b^2}{a^2} \left[a^2 x - \frac{1}{3}x^3\right]_0^a$$

$$= \frac{2\pi b^2}{a^2} \left[a^3 - \frac{1}{3}a^3 - 0\right]$$

$$= \frac{4}{3}\pi a b^2$$

d $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{x^2}{a^2} = 1 - \frac{y^2}{b^2}$$

$$\frac{x^2}{a^2} = \frac{b^2 - y^2}{b^2}$$

$$x^2 = \frac{a^2}{b^2}\left(b^2 - y^2\right)$$

$$V = \pi \int_{y_1}^{y_2} [f(x)]^2 dy$$

$$= \pi \int_{-b}^{b} \frac{a^2}{b^2}\left(b^2 - y^2\right) dy$$

$$= \frac{2\pi a^2}{b^2} \int_{o}^{b} \left(b^2 - y^2\right) dy$$

$$= \frac{2\pi a^2}{b^2} \left[b^2 y - \frac{1}{3}y^3\right]_0^b$$

$$= \frac{2\pi a^2}{b^2} \left[b^3 - \frac{1}{3}b^3 - 0\right]$$

$$= \frac{4}{3}\pi a^2 b$$

e Egg $b = \frac{1}{2} \times 57, a = \frac{1}{2} \times 44$

If rotating about the x-axis: $\frac{x^2}{\left(\frac{57}{2}\right)^2} + \frac{y^2}{22^2} = 1$

$$\frac{4x^2}{57^2} + \frac{y^2}{22^2} = 1$$

If rotating about the y-axis: $\frac{x^2}{22^2} + \frac{y^2}{\left(\frac{57}{2}\right)^2} = 1$

$$\frac{x^2}{22^2} + \frac{4y^2}{57^2} = 1$$

f If rotating about the x-axis: $\frac{x^2}{\left(\frac{57}{2}\right)^2} + \frac{y^2}{22^2} = 1$

$$\frac{4x^2}{57^2} + \frac{y^2}{22^2} = 1$$

$$\text{V} = \frac{4}{3}\pi a b^2$$

$$= \frac{4}{3}\pi\left(\frac{57}{2}\right)(22)^2$$

$$= 18392\pi$$

If rotating about the y-axis: $\frac{x^2}{22^2} + \frac{y^2}{\left(\frac{57}{2}\right)^2} = 1$

$$\frac{x^2}{22^2} + \frac{4y^2}{57^2} = 1$$

$$\text{V} = \frac{4}{3}\pi a^2 b$$

$$= \frac{4}{3}\pi (22)^2\left(\frac{57}{2}\right)$$

$$= 18392\pi$$

The volume of the egg is $18392\pi \approx 57780\,\text{mm}^3$

15 **a** A (0, 30)

B (30, 20)

$y = ax^2 + c$ by symmetry

A (0, 30) → C = 30

B (30, 20) → 20 = 900a + 30

$-10 = 900a$

$$a = -\frac{1}{90}$$

$y = 30 - \frac{x^2}{90}$ for $- 30 \le x \le 30$

b $V = \pi \int_{a}^{b} [f(x)]^2 dx$

$$= \pi \int_{-30}^{30} \left(30 - \frac{x^2}{90}\right)^2 dx$$

$$= 2\pi \int_{0}^{30} \left(900 - 2 \times 30 \times \frac{x^2}{90} + \frac{x^4}{8100}\right) dx$$

$$= 2\pi \left[900x - \frac{2}{9}x^3 + \frac{x^5}{5 \times 8100}\right]_0^{30}$$

$$= 2\pi \left[900 \times 30 - \frac{2}{9} \times 30^3 + \frac{1}{5 \times 8100} \times 30^5\right]$$

$$= 43200\pi\,\text{cm}^3$$

$$= \frac{43200\pi}{1000}\ \text{L}$$

$$\approx 136\text{L}$$

16 **a** A (6, 0)

B (8, 20)

AB line

$$y = mx + c$$

$$m = \frac{20 - 0}{8 - 6}$$

$$= 10$$

$$y - 0 = 10(x - 6)$$

$$y = 10x - 60$$

$$10x = y + 60$$

$$x = \frac{1}{10}(y + 60)$$

$$V = \pi \int_{a}^{b} [f(y)]^2 dy$$

$$= \pi \int_{0}^{20} \left(\frac{1}{10}(y + 60)\right)^2 dy$$

$$= \frac{\pi}{100}\left[\frac{1}{3}(y + 60)^3\right]_0^{20}$$

$$= \frac{\pi}{300}\left[80^3 - 60^3\right]$$

$$= \frac{2960\pi}{3}$$

$$= 986\frac{2}{3}\pi$$

$$= 3099.70\,\text{cm}^3$$

b A (6, 0)

B (8, 20)

Parabola $y = ax^2 + c$

$A(6, 0) \rightarrow 0 = 36a + c$

$B(8, 20) \rightarrow 20 = 64a + c$

$20 = 28a$

$a = \frac{5}{7} \rightarrow c = -36a = \frac{-36 \times 5}{7} = \frac{-180}{7}$

So

$$y = \frac{5x^2}{7} - \frac{180}{7}$$

$$7y = 5x^2 - 180$$

$$5x^2 = 7y + 180$$

$$x^2 = \frac{1}{5}(7y + 180)$$

$$\begin{aligned} V &= \pi \int_a^b [f(y)]^2 dy \\ &= \frac{\pi}{5} \int_0^{20} (7y + 180)\, dy \\ &= \frac{\pi}{5} \left[\frac{1}{14}(7y + 180)^2\right]_0^{20} \\ &= \frac{\pi}{70} \left[320^2 - 180^2\right] \\ &= 1000\pi \\ &= 3141.59\,\text{cm}^3 \end{aligned}$$

c A (6, 0)

B (8, 20)

Cubic $y = ax^3 + b$

$A(6, 0) \rightarrow 0 = 216a + b$

$B(8, 20) \rightarrow 20 = 512a + b$

$a = \frac{20}{512 - 216} = \frac{5}{74} \rightarrow b = -216a$

$b = -216 \times \frac{5}{74} = \frac{-540}{37}$

$$y = \frac{5}{74}x^3 - \frac{540}{37}$$

$$\frac{5x^3}{74} = y + \frac{540}{37} = \frac{37y + 540}{37}$$

$$x^3 = \frac{2}{5}(37y + 540)$$

$$\begin{aligned} V &= \pi \int_a^b [f(y)]^2 dy \\ &= \pi \int_0^{20} \left[\frac{2}{5}(37y + 540)\right]^{\frac{2}{3}} dy \\ &= \pi \times \frac{2^{\frac{2}{3}}}{5^{\frac{2}{3}}} \left[\frac{(37y + 540)^{\frac{5}{3}}}{\frac{5}{3} \times 37}\right]_0^{20} \\ &= \pi \times \frac{2^{\frac{2}{3}}}{5^{\frac{2}{3}}} \times \frac{3}{185} \left[1280^{\frac{5}{3}} - 540^{\frac{5}{3}}\right] \\ &= \frac{37488\pi}{37} \\ &= 3183.03\,\text{cm}^3 \end{aligned}$$

d A (6, 0)

B (8, 20)

$y = ax^4 + b$

$A(6, 0) \rightarrow 0 = 1296a + b$

$B(8, 20) \rightarrow 20 = 4096a + b$

$$\begin{aligned} a &= \frac{20}{4096 - 1296} = \frac{1}{140} \rightarrow b = -1296\text{a} \\ &= -1296 \times \frac{1}{140} \\ &= -\frac{324}{35} \end{aligned}$$

$$\therefore y = \frac{1}{140}x^4 - \frac{324}{35}$$

$$\frac{1}{140}x^4 = y + \frac{324}{35} = \frac{35y + 324}{35}$$

$$x^4 = 4(35y + 324)$$

$$x^2 = 2\sqrt{35y + 324}$$

$$\begin{aligned} \text{V} &= \pi \int_a^b [f(y)]^2 dy \\ &= 2\pi \int_0^{20} \sqrt{35y + 324} dy \\ &= 2\pi \left[\frac{2}{3} \times \frac{1}{35}(35y + 324)^{\frac{3}{2}}\right]_0^{20} \\ &= \frac{4\pi}{105} \left[1024^{\frac{3}{2}} - 324^{\frac{3}{2}}\right] \\ &= \frac{15392\pi}{15} \\ &\approx 3223.69\,\text{cm}^3 \end{aligned}$$

17 a $\frac{1}{x} + \frac{1}{y} = \frac{1}{4}$

$$\begin{aligned} \frac{1}{y} &= \frac{1}{4} - \frac{1}{x} \\ &= \frac{x - 4}{4x} \\ y &= \frac{4x}{x - 4} \\ &= 4 + \frac{16}{x - 4} \end{aligned}$$

$$\begin{aligned} V &= \pi \int_a^b [f(x)]^2 dx \\ &= \pi \int_0^2 \left(4 + \frac{16}{x - 4}\right)^2 dx \\ &= \pi \int_0^2 \left(16 + \frac{128}{x - 4} + \frac{256}{(x - 4)^2}\right) dx \\ &= \pi \left[16x + 128 \log_e |x - 4| - \frac{256}{(x - 4)}\right]_0^2 \\ &= \pi \left[\left(32 + 128 \log_e |-2| - \frac{256}{(-2)}\right)\right. \\ &\quad \left. - \left(0 + 128 \log_e |-4| - \frac{256}{(-4)}\right)\right] \\ &= 32\pi \left[3 - 4 \log_e(2)\right] \text{ units}^3 \end{aligned}$$

b $\sqrt{x}+\sqrt{y}=2$

$$\begin{aligned}\sqrt{y}&=2-\sqrt{x}\\ y&=\left(2-\sqrt{x}\right)^2\\ &=4-4\sqrt{x}+x\end{aligned}$$

$$\begin{aligned}y^2&=\left(2-\sqrt{x}\right)^4\\ &=2^4-4\times 2^3\times\sqrt{x}+6\times 2^2\times\left(\sqrt{x}\right)^2\\ &\quad-4\times 2\times\left(\sqrt{x}\right)^3+\left(\sqrt{x}\right)^4\\ &=16-32\sqrt{x}+24x-8x^{\frac{3}{2}}+x^2\end{aligned}$$

$$\begin{aligned}V&=\pi\int_a^b\left[f(x)\right]^2dx\\ &=\pi\int_0^4\left(16-32\sqrt{x}+24x-8x^{\frac{3}{2}}+x^2\right)dx\\ &=\pi\left[16x-\frac{2\times 32\times x^{\frac{3}{2}}}{3}+12x^2-\frac{16x^{\frac{5}{2}}}{5}+\frac{x^3}{3}\right]_0^4\\ &=\pi\left[64-\frac{512}{3}+192-\frac{512}{5}+\frac{64}{3}\right]\\ &=\frac{64\pi}{15}\text{units}^3\end{aligned}$$

18 A sphere of radius 20 cm can be formed by rotating a circle with radius 20 cm around the y-axis. The equation of the circle will be $x^2+y^2=20^2$.

The base of the bowl has a radius of 12 cm. Solve for y if $x=12$.

$$\begin{aligned}12^2+y^2&=20^2\\ y^2&=256\\ y&=\pm 16\end{aligned}$$

Because we are looking for a value for the base of the bowl, the solution will be $y=-16$.

The height of the water in the bowl is 28 cm, therefore the other boundary is $-16+28=12$

$$V=\pi\int_{-16}^{12}\left[f(y)\right]^2dy$$

$$\begin{aligned}\text{V}&=\pi\int_{-16}^{12}400-y^2dy\\ &=\pi\left[400y-\frac{1}{3}y^3\right]_{-16}^{12}\\ &=\pi\left[400\times 12-\frac{1}{3}12^3+400\times 16-\frac{1}{3}16^3\right]\\ &=\frac{27\,776\pi}{3}\text{ cm}^3\end{aligned}$$

9.4 Exam questions

1 $y=2\sqrt{\dfrac{x^2+x+1}{(x+1)\left(x^2+1\right)}}$

$$V=\pi\int_a^b y^2\,dx=4\pi\int_0^{\sqrt{3}}\frac{x^2+x+1}{(x+1)\left(x^2+1\right)}dx$$

By partial fractions,

$$\begin{aligned}\frac{A}{x+1}+\frac{Bx+C}{x^2+1}&=\frac{A\left(x^2+1\right)+(x+1)(Bx+C)}{(x+1)\left(x^2+1\right)}\\ &=\frac{x^2(A+B)+x(B+C)+\text{A}+\text{C}}{(x+1)\left(x^2+1\right)}\\ &=\frac{x^2+x+1}{(x+1)\left(x^2+1\right)}\end{aligned}$$

Equating coefficients:

$x^2: A+B=1,\ x^1: B+C=1,\ x^0: A+C=1$

$\Rightarrow A=B=C=\dfrac{1}{2}$

$$V=2\pi\int_0^{\sqrt{3}}\left(\frac{1}{x+1}+\frac{x+1}{x^2+1}\right)dx$$

$$V=2\pi\int_0^{\sqrt{3}}\left(\frac{1}{x+1}+\frac{x}{x^2+1}+\frac{1}{x^2+1}\right)dx$$

$$V=2\pi\left[\log_e(x+1)+\frac{1}{2}\log_e\left(x^2+1\right)+\tan^{-1}(x)\right]_0^{\sqrt{3}}$$

$$V=2\pi\left[\log_e\left(1+\sqrt{3}\right)+\frac{1}{2}\log_e(4)+\tan^{-1}\left(\sqrt{3}\right)-0\right]$$

$$V=2\pi\left(\log_e\left(2+2\sqrt{3}\right)+\frac{\pi}{3}\right)\text{ units}^3,\ a=2+2\sqrt{3},\ b=\frac{\pi}{3}$$

Award 1 mark for the correct define integral for the volume.
Award 1 mark for using partial fractions.
Award 2 mark for correct integration.
Award 1 mark for the final correct volume.

2 $y=\sqrt{\dfrac{1+2x}{1+x^2}},\ V=\pi\displaystyle\int_0^b y^2\,dx$

$$V=\pi\int_0^1\left(\frac{1+2x}{1+x^2}\right)dx$$

$$V=\pi\left[\int_0^1\frac{1}{1+x^2}dx+\int_0^1\frac{2x}{1+x^2}dx\right]$$

$$V=\pi\left[\tan^{-1}(x)+\log_e\left(1+x^2\right)\right]_0^1$$

$$V=\pi\left[\tan^{-1}(1)+\log_e(2)-\tan^{-1}(0)-\log_e(1)\right]$$

$$V=\pi\left(\frac{\pi}{4}+\log_e(2)\right)\text{ units}^3$$

Award 1 mark for the correct definite integral.
Award 1 mark each for the correct integration of each term.
Award 2 mark for the correct final volume.

VCAA Examination Report **note:**

Most students were able to write down the correct integral to find the volume of solid of revolution. Some students did not recognise the way in which the integrand split naturally and had difficulty proceeding further with the question. Some attempted solutions using partial fractions were seen. Many students who were able to successfully split the integrand used a substitution method to integrate $\displaystyle\int_0^1\frac{2x}{1+x^2}dx$. This was unnecessary and resulted in a loss of marks if not done correctly.

3 $y=2x^2-3$

$x^2=\dfrac{1}{2}(y+3)$

$$V_y = \pi \int_a^b x^2 \, dy$$

$$= \pi \int_{-3}^{5} \frac{1}{2}(y+3)\, dy \qquad [1 \text{ mark}]$$

$$= \frac{\pi}{2}\left[\frac{1}{2}y^2 + 3y\right]_{-3}^{5} \qquad [1 \text{ mark}]$$

$$= \frac{\pi}{2}\left[\left(\frac{25}{2}+15\right) - \left(\frac{9}{2}-9\right)\right]$$

$$= 16\pi \text{ units}^3 \qquad [1 \text{ mark}]$$

9.5 Volumes

9.5 Exercise

1 a $y = 9 - x^2,\ y = 5$

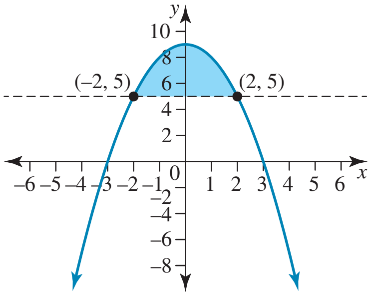
$5 = 9 - x^2$

$x^2 = 4$

$x = \pm 2$

$r_2 = y$

$r_1 = 5$



$$\text{V} = \pi \int_a^b r_2^2 - r_1^2 dx$$

$$= \pi \int_{-2}^{2} \left(y^2 - 25\right) dx$$

$$= 2\pi \int_0^2 \left(\left(9 - x^2\right)^1 - 25\right) dx$$

$$= 2\pi \int_0^2 \left(x^4 - 18x^2 + 56\right) dx$$

$$= 2\pi \left[\frac{1}{5}x^5 - 6x^3 + 56x\right]_0^2$$

$$= 2\pi \left[\frac{1}{5}(2)^5 - 6(2)^3 + 56(2) - 0\right]$$

$$= \frac{704\pi}{5} \text{ units}^3$$

b $$\text{V} = \pi \int_a^b \left[f(y)\right]^2 dy$$

$$= \pi \int_5^9 (9 - y)\, dy$$

$$= \pi \left[9y - \frac{1}{2}y^2\right]_5^9$$

$$= \pi \left[9(9) - \frac{1}{2}(9)^2 - 9(5) + \frac{1}{2}(5)^2\right]$$

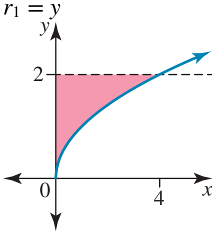
$$= 8\pi \text{ units}^3$$

2 a $y = \sqrt{x},\ y = 2$

$x = 4,\ y = \sqrt{4} = 2$

$r_2 = 2$

$r_1 = y$



$$\text{V} = \pi \int_a^b r_2^2 - r_1^2 dx$$

$$= \pi \int_0^4 4 - y^2 dx$$

$$= \pi \int_0^4 4 - x dx$$

$$= \pi \left[4x - \frac{1}{2}x^2\right]_0^4$$

$$= \pi \left[4(4) - \frac{1}{2}(4)^2 - 0\right]$$

$$= 8\pi \text{ units}^3$$

b $$\text{V} = \pi \int_a^b \left[f(y)\right]^2 dy$$

$$= \pi \int_0^2 y^4 dy$$

$$= \pi \left[\frac{1}{5}y^5\right]_0^2$$

$$= \frac{\pi}{5}[32 - 0]$$

$$= \frac{32\pi}{5} \text{ units}^3$$

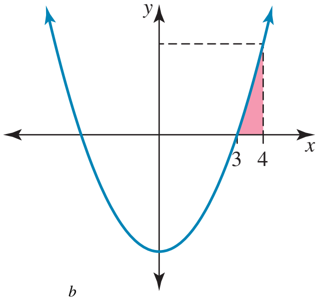
3 a $y = x^2 - 9$

$x = 4,\ y = 7$

$r_2 = 4$

$r_1 = x$

Rotate about the y-axis



$$V = \pi \int_a^b r_2^2 - r_1^2 dy$$

$$= \pi \int_0^7 \left(16 - x^2\right) dy$$

$$= \pi \int_{0}^{7} (16 - (y + 9))\,dy$$

$$= \pi \int_{0}^{7} (7 - y)\,dy$$

$$= \pi \left[7y - \frac{1}{2}y^2\right]_{0}^{7}$$

$$= \pi \left[7(7) - \frac{1}{2}(7)^2 - 0\right]$$

$$= \frac{49\pi}{2}\text{ units}^3$$

b Rotate about the x-axis

$$V = \pi \int_{a}^{b} y^2 dx$$

$$= \pi \int_{3}^{4} \left(x^2 - 9\right)^2 dx$$

$$= \pi \int_{3}^{4} \left(x^4 - 18x^2 + 81\right) dx$$

$$= \pi \left[\frac{1}{5}x^5 - 6x^3 + 81x\right]_{3}^{4}$$

$$= \pi \left[\frac{1}{5}(4)^5 - 6(4)^3 + 81\,(4) - \frac{1}{5}(3)^5 + 6(3)^3 - 81(3)\right]$$

$$= \frac{76\pi}{5}\text{ units}^3$$

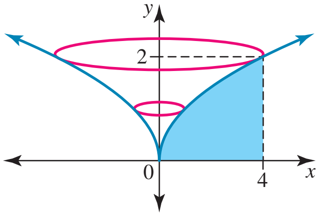
4 a $y = \sqrt{x}$,

$x = 4,\ y = 2$

$r_2 = 4$

$r_1 = x$

Rotate about the y-axis



$$V = \pi \int_{a}^{b} r_2^2 - r_1^2 dy$$

$$V = \pi \int_{0}^{2} \left(16 - x^2\right) dy$$

$$= \pi \int_{0}^{2} \left(16 - y^4\right) dy$$

$$= \pi \left[16y - \frac{1}{5}y^5\right]_{0}^{2}$$

$$= \pi \left[16\,(2) - \frac{1}{5}(2)^5 - 0\right]$$

$$= \frac{128\pi}{5}\text{ units}^3$$

b $$V = \pi \int_{a}^{b} y^2 dx$$

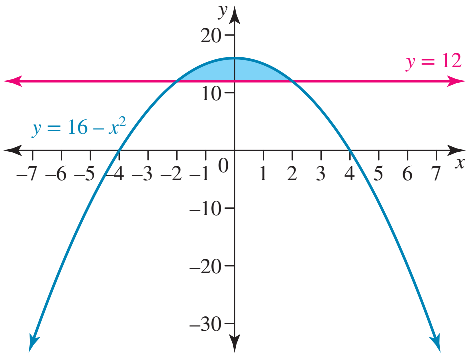
$$= \pi \int_{0}^{4} x\,dx$$

$$= \pi \left[\frac{1}{2}x^2\right]_{0}^{4}$$

$$= \pi\,[8 - 0]$$

$$= 8\pi\text{ units}^3$$

5



a $y = 16 - x^2,\ y = 12$

$12 = 16 - x^2$

$x^2 = 4$

$x = \pm 2$

$r_2 = y,\ r_1 = 12$

$$V = \pi \int_{a}^{b} r_2^2 - r_1^2 dx$$

$$= \pi \int_{-2}^{2} \left(y^2 - 144\right) dx$$

$$= 2\pi \int_{0}^{2} \left(\left(16 - x^2\right)^2 - 144\right) dx$$

$$= 2\pi \int_{0}^{2} \left(112 - 32x^2 + x^4\right) dx$$

$$= 2\pi \left[112x - \frac{32x^3}{3} + \frac{x^5}{5}\right]_{0}^{2}$$

$$= 2\pi \left[224 - \frac{32}{3} \times 8 + \frac{32}{5} - 0\right]$$

$$= \frac{4352\pi}{15}\text{ units}^3$$

b $$V = \pi \int_{a}^{b} [f(y)]^2 dy$$

$$= \pi \int_{12}^{16} 16 - y\,dy$$

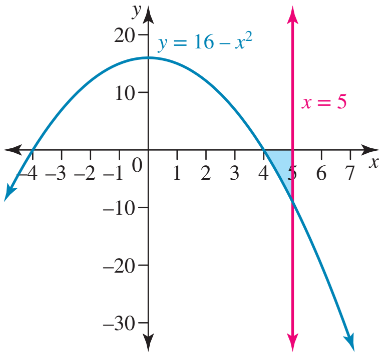
$$= \pi \left[16y - \frac{1}{2}y^2\right]_{12}^{16}$$

$$= \pi \left[\left(16 \times 16 - \frac{1}{2} \times 16^2\right) - \left(16 \times 12 - \frac{1}{2} \times 12^2\right)\right]$$

$$= \pi\,[128 - 120]$$

$$= 8\pi\text{ units}^3$$

6



a $y = x^2 - 16$, $x = 5$

When

$x = 5$, $y = 9$

$r_2 = 5$, $r_1 = x$

$$V = \pi \int_a^b r_2^2 - r_1^2 dy$$

$$= \pi \int_0^9 \left(25 - x^2\right) dy$$

$$= \pi \int_0^9 (25 - (y + 16))\, dy$$

$$= \pi \int_0^9 (9 - y) dy$$

$$= \pi \left[9y - \frac{1}{2}y^2\right]_0^9$$

$$= \pi \left[81 - \frac{1}{2} \times 81 - 0\right]$$

$$= \frac{81\pi}{2} \text{units}^3$$

b $V = \pi \int_a^b \left[f(x)\right]^2 dx$

$$= \pi \int_4^5 \left(x^2 - 16\right)^2 dx$$

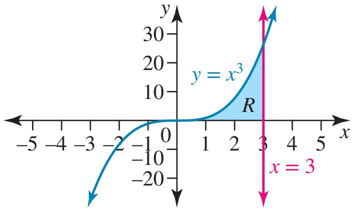
$$= \pi \int_4^5 \left(x^4 - 32x^2 + 256\right) dx$$

$$= \pi \left[\frac{1}{5}x^5 - \frac{32}{3}x^3 + 256x\right]_4^5$$

$$= \pi \left[\left(\frac{1}{5} \times 5^5 - \frac{32}{3} \times 5^3 + 256 \times 5\right) - \left(\frac{1}{5} \times 4^5 - \frac{32}{3} \times 4^3 + 256 \times 4\right)\right]$$

$$= \frac{383\pi}{15} \text{ units}^3$$

7



a $y = x^3$

$x = 3$

$x = y^{\frac{1}{3}}$

$x^2 = y^{\frac{2}{3}}$

About y-axis

$r_2 = 3$, $r_1 = x$

$$V = \pi \int_0^{27} r_2^2 - r_1^2 dy$$

$$= \pi \int_0^{27} 9 - y^{\frac{2}{3}} dy$$

$$= \pi \left[9y - \frac{3}{5}y^{\frac{5}{3}}\right]_0^{27}$$

$$= \frac{486\pi}{5} \text{ units}^3$$

b About x-axis

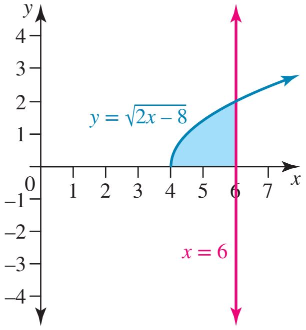
$$V = \pi \int_a^b \left[f(x)\right]^2 dx$$

$$= \pi \int_0^3 x^6 dx$$

$$= \pi \left[\frac{1}{7}x^7\right]_0^3$$

$$= \frac{2187\pi}{7} \text{ units}^3$$

8



a $y = \sqrt{2x - 8}$, $x = 6$

$r_2 = 6$, $r_1 = x$

$x = 6$, $y = 2$

$y^2 = 2x - 8$

$x = \frac{1}{2}\left(y^2 + 8\right)$

$$V = \pi \int_0^2 r_2^2 - r_1^2 dy$$

$$= \pi \int_0^2 \left(36 - \frac{1}{4}\left(y^2 + 8\right)^2\right) dy$$

$$= \pi \int_0^2 \left(20 - 4y^2 - \frac{1}{4}y^4\right) dy$$

$$= \pi \left[20y - \frac{4}{3}y^3 - \frac{1}{20}y^5\right]_0^2$$

$$= \pi\left[20(2) - \frac{4}{3}(2)^3 - \frac{1}{20}(2)^5 - 0\right]$$

$$= \frac{416\pi}{15} \text{ units}^3$$

b About the x-axis

$$V = \pi\int_a^b [f(x)]^2\, dx$$

$$= \pi\int_4^6 \left(\sqrt{2x-8}\right)^2 dx$$

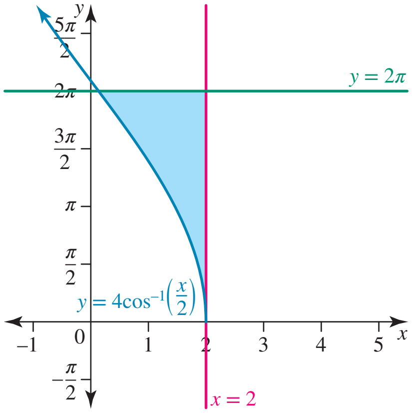
$$= \pi\int_4^6 (2x-8)\, dx$$

$$= \pi\left[x^2 - 8x\right]_4^6$$

$$= \pi[(36-48)-(16-32)]$$

$$= 4\pi \text{ units}^3$$

9 a



$$y = 4\cos^{-1}\left(\frac{x}{2}\right)$$

$y = 2\pi, \quad x = 2$

Rotate about y-axis

$r_2 = 2, \quad r_1 = x$

$$\frac{y}{4} = \cos^{-1}\left(\frac{x}{2}\right)$$

$$x = 2\cos\left(\frac{y}{4}\right)$$

$$V = \pi\int_0^{2\pi}\left(4 - 4\cos^2\left(\frac{y}{4}\right)\right) dy$$

$$= \pi\int_0^{2\pi} 4\sin^2\left(\frac{y}{4}\right) dy$$

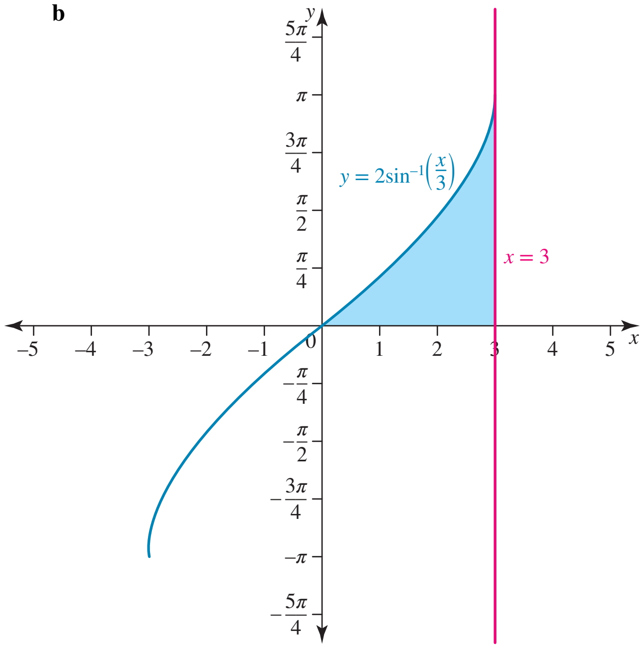
$$= 2\pi\int_0^{2\pi}\left(1 - \cos\left(\frac{y}{2}\right)\right) dy$$

$$= 2\pi\left[y - 2\sin\left(\frac{y}{2}\right)\right]_0^{2\pi}$$

$$= 2\pi[2\pi - 2\sin(\pi) - 0 + 2\sin(0)]$$

$$= 4\pi^2 \text{ units}^3$$

b



$$y = 2\sin^{-1}\left(\frac{x}{3}\right)$$

$y = 2\sin^{-1}(1) = \pi, \; x = 3$

Rotate about x-axis

$r_2 = 3, \quad r_1 = x$

$$\frac{y}{2} = \sin^{-1}\left(\frac{x}{3}\right)$$

$$\frac{x}{3} = \sin\left(\frac{y}{2}\right)$$

$$x = 3\sin\left(\frac{y}{2}\right)$$

$$V = \pi\int_0^{\pi} r_2^2 - r_1^2 dy$$

$$= \pi\int_0^{\pi} 9 - 9\sin^2\left(\frac{y}{2}\right) dy$$

$$= \pi\int_0^{\pi} 9\cos^2\left(\frac{y}{2}\right) dy$$

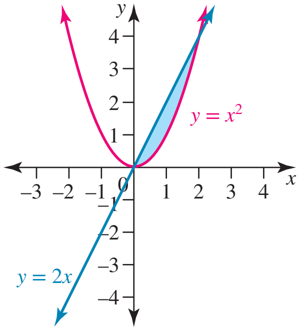
$$= \frac{9\pi}{2}\int_0^{\pi} 1 + \cos y\, dy$$

$$= \frac{9\pi}{2}\left[y + \sin(y)\right]_0^{\pi}$$

$$= \frac{9\pi}{2}[\pi + \sin(\pi) - 0 - \sin(0)]$$

$$= \frac{9\pi^2}{2} \text{ units}^3$$

10



$$y_1 = x^2,\ y_2 = 2x$$
$$y_1 = y_2$$
$$2x = x^2$$
$$x(x-2) = 0$$
$$x = 0,\ 2$$

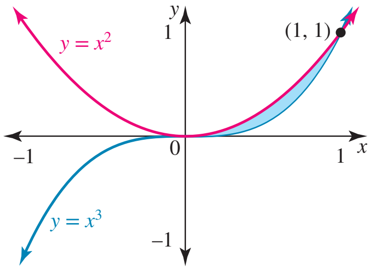
a About x-axis

$$V = \pi \int_0^2 r_2^2 - r_1^2 dx$$
$$= \pi \int_0^2 4x^2 - x^4 dx$$
$$= \pi \left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^2$$
$$= \pi \left[\frac{4}{3}(2)^3 - \frac{1}{5}(2)^5 - 0\right]$$
$$= \frac{64\pi}{15}\text{units}^3$$

b About y-axis

$$r_1 = x_1,\ r_2 = x_2$$
$$r_1^2 = \frac{y^2}{4},\ r_2^2 = y$$
$$V = \pi \int_0^4 \left(y - \frac{y^2}{4}\right) dy$$
$$= \pi \left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4$$
$$= \pi \left[8 - \frac{16}{3}\right]$$
$$= \pi \times \frac{8}{3}$$
$$= \frac{8\pi}{3}\text{ units}^3$$

11



$$y_1 = x^2, y_2 = x^3$$
$$y_1 = y_2$$
$$x^3 = x^2$$
$$x^2(x-1) = 0$$
$$x = 0, 1$$

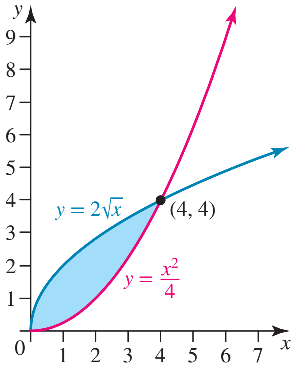
a About x-axis

$$V = \pi \int_0^1 \left(r_2^2 - r_1^2\right) dx$$
$$= \pi \int_0^1 \left(x^4 - x^6\right) dx$$
$$= \pi \left[\frac{1}{5}x^5 - \frac{1}{7}x^7\right]_0^1$$
$$= \pi \left[\frac{1}{5}(1)^5 - \frac{1}{7}(1)^7 - 0\right]$$
$$= \frac{2\pi}{35}\text{units}^3$$

b About y-axis

$$r_1 = x_1,\ r_2 = x_2$$
$$r_1^2 = x^2 = y$$
$$r_2^2 = x_2^2 = y^{\frac{2}{3}}$$
$$V = \pi \int_0^1 \left(y^{\frac{2}{3}} - y\right) dy$$
$$= \pi \left[\frac{3}{5}y^{\frac{5}{3}} - \frac{1}{2}y^2\right]_0^1$$
$$= \pi \left[\frac{3}{5} - \frac{1}{2} - 0\right]$$
$$= \frac{\pi}{10}\text{ units}^3$$

12 a



$$y_1 = \frac{x^2}{4},\ y_2 = 2\sqrt{x}$$
$$y_1 = y_2$$
$$\frac{x^2}{4} = 2\sqrt{x}$$
$$\frac{x^4}{16} = 4x$$
$$x^4 = 64 \times x$$
$$x\left(x^3 - 64\right) = 0$$
$$x = 0,\ 4$$
$$V = \pi \int_a^b \left(r_2^2 - r_1^2\right) dx$$
$$= \pi \int_0^4 \left(4x - \frac{x^4}{16}\right) dx$$
$$= \pi \left[2x^2 - \frac{x^5}{80}\right]_0^4$$

$$= \pi\left[2(4)^2 - \frac{(4)^5}{80} - 0\right]$$

$$= \frac{96\pi}{5}\text{units}^3$$

b $y_1 = \frac{x^2}{a},\ y_2 = \sqrt{ax}$

$$y_1 = y_2$$

$$\frac{x^2}{a} = \sqrt{ax}$$

$$\frac{x^4}{a^2} = ax$$

$$x^4 = a^3x$$

$$x\left(x^3 - a^3\right) = 0$$

$$x = 0,\ a$$

$$r_2 = y_2$$

$$r_1 = y_1$$

$$V = \pi\int_a^b \left(r_2^2 - r_1^2\right)dx$$

$$= \pi\int_0^a \left(ax - \frac{x^4}{a^2}\right)dx$$

$$= \pi\left[\frac{ax^2}{2} - \frac{x^5}{5a^2}\right]_0^a$$

$$= \pi\left[\frac{a(a)^2}{2} - \frac{(a)^5}{5a^2} - 0\right]$$

$$= \frac{3\pi a^3}{10}\text{units}^3$$

13 a

$$y_1 = x^2$$

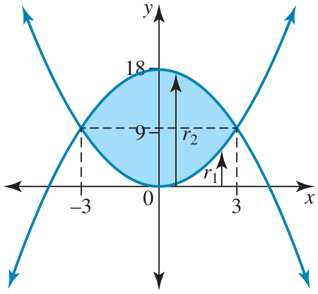
$$y_2 = 18 - x^2$$

$$y_1 = y_2$$

$$18 - x^2 = x^2$$

$$2x^2 = 18$$

$$x = \pm 3$$



$$V = \pi\int_a^b \left(r_2^2 - r_1^2\right)dx$$

$$r_2 = y_2 = 18 - x^2$$

$$r_1 = y_1 = x^2$$

$$V = \pi\int_{-3}^{3} \left(\left(18 - x^2\right)^2 - x^4\right)dx$$

$$= 2\pi\int_0^3 \left(324 - 36x^2\right)dx$$

$$= 2\pi\left[324x - 12x^3\right]_0^3$$

$$= 2\pi\left[324(3) - 12(3)^3 - 0\right]$$

$$= 1296\pi\,\text{units}^3$$

b $V = \pi\int_0^9 ydy + \pi\int_9^{18} 18 - ydy$

$$= \pi\left[\frac{1}{2}y^2\right]_0^9 + \pi\left[18y - \frac{1}{2}y^2\right]_9^{18}$$

$$= \pi\left[\frac{1}{2}(9)^2\right] + \pi\left[18(18) - \frac{1}{2}(18)^2 - 18(9) + \frac{1}{2}(9)^2\right]$$

$$= 81\pi\ \text{units}^3$$

14

$$y_1 = (x-2)^2 = x^2 - 4x + 4$$

$$y_2 = 4 + 4x - x^2 = 8 - (x-2)^2$$

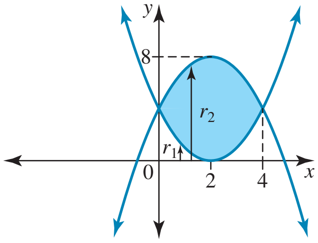
$$y_1 = y_2$$

$$x^2 - 4x + 4 = 8 - (x-2)^2$$

$$2x^2 - 8x = 0$$

$$2x(x-4) = 0$$

$$x = 0,\ 4$$



$$V = \pi\int_a^b \left(r_2^2 - r_1^2\right)dx$$

$$r_2 = y_2 = 8 - (x-2)^2$$

$$r_1 = y_1 = (x-2)^2$$

$$V = \pi\int_0^4 \left(\left(8 - (x-2)^2\right)^2 - \left((x-2)^2\right)^2\right)dx$$

$$= \pi\int_0^4 \left(64 - 16\left(x^2 - 4x + 4\right)\right)dx$$

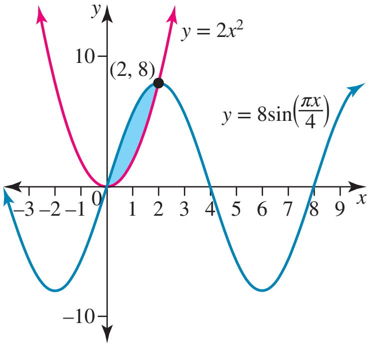
$$= \pi\int_0^4 \left(64x - 16x^2\right)dx$$

$$= \pi\left[32x^2 - \frac{16}{3}x^3\right]_0^4$$

$$= \pi\left[32(4)^2 - \frac{16}{3}(4)^3 - 0\right]$$

$$= \frac{512}{3}\pi\,\text{units}^3$$

15



Use technology to locate intersections at (0, 0) and (2, 8).

$y_2 = 8\ \sin\left(\frac{\pi x}{4}\right)$

$y_1 = 2x^2$

Rotate about x-axis

$r_2 = y_2$

$r_2 = y_1$

$$V = \pi \int_a^b \left(r_2^2 - r_1^2\right) dx$$

$$= \pi \int_0^2 \left(64 \sin^2\left(\frac{\pi x}{4}\right) - 4x^4\right) dx$$

$$= \pi \int_0^2 \left(64\left(\frac{1}{2}\left(1 - \cos\left(\frac{\pi x}{2}\right)\right)\right) - 4x^4\right) dx$$

$$= \pi \int_0^2 \left(32 - 32\cos\left(\frac{\pi x}{2}\right) - 4x^4\right) dx$$

$$= \pi \left[32x - \frac{64}{\pi}\sin\left(\frac{\pi x}{2}\right) - \frac{4x^5}{5}\right]_0^2$$

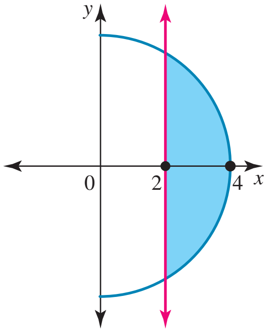
$$= \pi \left[32(2) - \frac{64}{\pi}\sin(\pi) - \frac{4(2)^5}{5} - 0\right]$$

$$= \frac{192\pi}{5} \text{ units}^3$$

16 **a** The sphere of radius 4 cm can be modelled by rotating a circle of radius 4. The equation of this circle is $x^2 + y^2 = 4^2$.

A cylinder of radius 2 cm can be found by rotating the line $x = 2$ about the y-axis.

This means that the volume of remaining cheese can be found by rotating the region shown about the y-axis.



Area shown is between $x = \sqrt{16 - y^2}$ and $x = 2$

Identifying point of intersection: $2 = \sqrt{16 - y^2}$

$4 = 16 - y^2$

$y^2 = 12$

$y = \pm 2\sqrt{3}$

Rotate about y – axis

$$\text{V} = \pi \int_{-2\sqrt{3}}^{2\sqrt{3}} \left(x^2 - 4\right) dy$$

$$= 2\pi \int_0^{2\sqrt{3}} \left(16 - y^2 - 4\right) dy$$

$$= 2\pi \int_0^{2\sqrt{3}} \left(12 - y^2\right) dy$$

$$= 2\pi \left[12y - \frac{y^3}{3}\right]_0^{2\sqrt{3}}$$

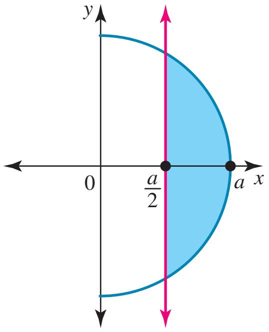
$$= 2\pi \left[12\left(2\sqrt{3}\right) - \frac{\left(2\sqrt{3}\right)^3}{3} - 0\right]$$

$$= 32\sqrt{3}\pi \text{ units}^3$$

b The sphere of radius a can be modelled by rotating a circle of radius a. The equation of this circle is $x^2 + y^2 = a^2$.

A cylinder of radius $\frac{a}{2}$ can be found by rotating the line $x = \frac{a}{2}$ about the y-axis.

This means that the volume remaining can be found by rotating the region shown about the y-axis.



Area shown is between and $x = \sqrt{a^2 - y^2}$ and $x = \frac{a}{2}$

Identifying point of intersection: $\frac{a}{2} = \sqrt{a^2 - y^2}$

$\frac{a^2}{4} = a^2 - y^2$

$y^2 = \frac{3a^2}{4}$

$y = \pm\frac{a\sqrt{3}}{2}$

$$\text{V} = \pi \int_{-\frac{\sqrt{3}a}{2}}^{\frac{\sqrt{3}a}{2}} \left(x^2 - \frac{a^2}{4}\right) dy$$

$$= 2\pi \int_0^{\frac{\sqrt{3}a}{2}} \left(a^2 - y^2 - \frac{a^2}{4}\right) dy$$

$$= 2\pi \int_0^{\frac{\sqrt{3}a}{2}} \left(\frac{3a^2}{4} - y^2\right) dy$$

$$= 2\pi \left[\frac{3a^2 y}{4} - \frac{y^3}{3}\right]_0^{\frac{a\sqrt{3}}{2}}$$

$$= 2\pi \left[\frac{3a^2}{4}\left(\frac{a\sqrt{3}}{2}\right) - \frac{1}{3}\left(\frac{a\sqrt{3}}{2}\right)^3 - 0\right]$$

$$= 2\pi \left(\frac{3\sqrt{3}a^3}{8} - \frac{3\sqrt{3}a^3}{24}\right)$$

$$= 2\pi \left(\frac{9\sqrt{3}a^3}{24} - \frac{3\sqrt{3}a^3}{24}\right)$$

$$= 2\pi\left(\frac{6\sqrt{3}a^3}{24}\right)$$

$$= \frac{a^3\pi\sqrt{3}}{2}\text{ units}^3$$

17 a $x^2 + (y-8)^2 = 16$

$$(y-8)^2 = 16 - x^2$$

$$y = 8 \pm \sqrt{16 - x^2}$$

$$r_2 = y_2 = 8 + \sqrt{16 - x^2}$$

$$r_1 = y_1 = 8 - \sqrt{16 - x^2}$$

Volume rotated about x-axis

$$\text{V} = \pi\int_a^b \left(r_2^2 - r_1^2\right) dx$$

$$= \pi\int_{-4}^{4}\left(\left(8+\sqrt{16-x^2}\right)^2 - \left(8-\sqrt{16-x^2}\right)^2\right) dx$$

$$= 2\pi\int_0^4 32\sqrt{16-x^2}dx$$

$$= 64\pi\left[\frac{1}{4}\pi \times 4^2\right]_0^4$$

$$= 256\pi\text{ units}^3$$

b $(x-a)^2 + y^2 = r^2$

$$(x-a)^2 = r^2 - y^2$$

$$x = a \pm \sqrt{r^2 - y^2}$$

$$r_2 = a + \sqrt{r^2 - y^2}$$

$$r_1 = a - \sqrt{r^2 - y^2}$$

$$\text{V} = \pi\int_{-r}^{r}\left(r_2^2 - r_1^2\right) dy$$

$$= 2\pi\int_0^r\left(\left(a+\sqrt{r^2-y^2}\right)^2 - \left(a-\sqrt{r^2-y^2}\right)^2\right) dy$$

$$= 2\pi\int_0^r 4a\sqrt{r^2-y^2}dy$$

$$= 8\pi a\int_0^r \sqrt{r^2-y^2}dy$$

$$= 8\pi a\left[\frac{1}{4}\pi r^2\right]$$

$$= 2\pi^2 ar^2\text{ units}^3$$

18 a $y = a^2 - x^2$

$$y = \frac{a^2}{4} = a^2 - x^2$$

$$x^2 = \frac{3a^2}{4}$$

$$x = \pm\frac{\sqrt{3a^2}}{2} = \pm\frac{\sqrt{3}a}{2}$$

$$r_1 = \frac{a^2}{4}$$

$$r_2 = y$$

$$V = \pi\int_{-r}^{r}\left(r_2^2 - r_1^2\right) dx$$

$$= \pi\int_{-\frac{\sqrt{3}a}{2}}^{\frac{\sqrt{3}a}{2}}\left(y^2 - \frac{a^4}{16}\right) dx$$

$$= 2\pi\int_0^{\frac{\sqrt{3}a}{2}}\left(\left(a^2-x^2\right)^2 - \frac{a^4}{16}\right) dx$$

$$= 2\pi\int_0^{\frac{\sqrt{3}a}{2}}\left(a^4 - 2a^2x^2 + x^4 - \frac{a^4}{16}\right) dx$$

$$= 2\pi\int_0^{\frac{\sqrt{3}a}{2}}\left(\frac{15a^4}{16} - 2a^2x^2 + x^4\right) dx$$

$$= 2\pi\left[\frac{15a^4}{16}x - \frac{2}{3}a^2x^3 + \frac{1}{5}x^5\right]_0^{\frac{\sqrt{3}a}{2}}$$

$$= 2\pi\left[\frac{15a^4}{16}\left(\frac{\sqrt{3}a}{2}\right) - \frac{2}{3}a^2\left(\frac{\sqrt{3}a}{2}\right)^3 + \frac{1}{5}\left(\frac{\sqrt{3}a}{2}\right)^5 - 0\right]$$

$$= \frac{11\sqrt{3}\pi a^5}{20}\text{ units}^3$$

b

$$\text{V} = \pi\int_{\frac{a^2}{4}}^{a^2} x^2 dy$$

$$= \pi\int_{\frac{a^2}{4}}^{a^2}\left(a^2 - y\right) dy$$

$$= \pi\left[a^2y - \frac{1}{2}y^2\right]_{\frac{a^2}{4}}^{a^2}$$

$$= \pi\left[\left(a^2\left(a^2\right) - \frac{1}{2}\left(a^2\right)^2\right) - \left(a^2\left(\frac{a^2}{4}\right) - \frac{1}{2}\left(\frac{a^2}{4}\right)^2\right)\right]$$

$$= \frac{9\pi a^4}{32}\text{ units}^3$$

9.5 Exam questions

1 a $x^2 - 2(x-1)^2 = 1$

$$x^2 - 2\left(x^2 - 2x + 1\right) = 1$$

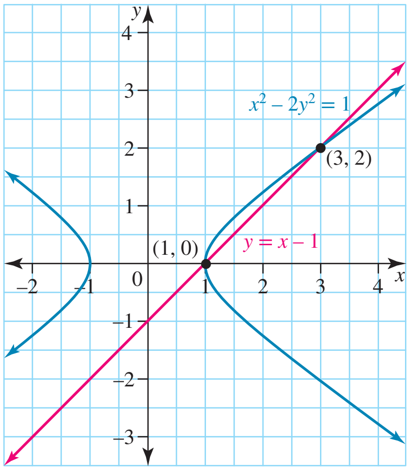
$$x^2 - 4x + 3 = 0$$

$x = 1, x = 3$ [1 mark]

***VCAA Examination Report* note:**

Students needed to substitute $y = x - 1$ into the equation $x^2 - 2y^2 = 1$ and solve the resulting quadratic equation for x. A number of students gave the coordinates of the points of intersection and in some cases did not do this correctly.

b



$a = 1,\ b = 3,\ y_2^2 = \dfrac{x^2 - 1}{2},\ y_1 = x - 1$

$$V = \pi \int_1^3 \left(\frac{x^2 - 1}{2} - (x - 1)^2 \right) dx \qquad [1 \text{ mark}]$$

$$V = \frac{\pi}{2} \int_1^3 \left(-x^2 + 4x - 3 \right) dx$$

$$V = \frac{\pi}{2} \left[-\frac{x^3}{3} + 2x^2 - 3x \right]_1^3$$

$$V = \frac{\pi}{2} \left[(-9 + 18 - 9) - \left(-\frac{1}{3} + 2 - 3 \right) \right]$$

$$V = \frac{2\pi}{3} \text{ units}^3 \qquad [1 \text{ mark}]$$

VCAA Examination Report **note:**

Students found this question challenging. Some students did not apply the formula for the volume of a solid of revolution correctly. Many students made algebraic or arithmetic errors. A small number of students realised that the volume required could be found by finding the volume obtained by rotating the region bounded by the hyperbola, the x-axis and the lines $x = 1$ and $x = 3$ about the x-axis and then subtracting the volume of an appropriate cone.

2 Volume around y-axis: $V_y = \pi \displaystyle\int_a^{b^2} dy$

$y = x^{\frac{4}{3}} \Rightarrow y^3 = x^4 \Rightarrow x^2 = y^{\frac{3}{2}},\ a = 0,\ b = 3$

$$V = \pi \int_0^3 y^{\frac{3}{2}}\, dy$$

$$= \pi \left[\frac{2}{5} y^{\frac{5}{2}} \right]_0^3$$

$$= \frac{2\pi}{5} \left(3^{\frac{5}{2}} - 0 \right)$$

$$= \frac{2\pi}{5} \times 3^2 \times 3^{\frac{1}{2}}$$

$$= \frac{18\pi 3^{\frac{1}{2}}}{5}$$

The correct answer is **D**.

3 $V = \pi \displaystyle\int_a^b \left(y_2^2 - y_1^2 \right) dx$

$a = 0,\ b = \dfrac{\pi}{3},\ y_1 = \sin(x),\ y_2 = 3x$

$$V = \pi \int_0^{\frac{\pi}{3}} \left(9x^2 - \sin^2(x) \right) dx$$

$$V = \pi \int_0^{\frac{\pi}{3}} \left(9x^2 - \frac{1}{2}(1 - \cos(2x)) \right) dx$$

$$V = \pi \int_0^{\frac{\pi}{3}} \left(9x^2 - \frac{1}{2} + \frac{1}{2}\cos(2x) \right) dx$$

$$V = \pi \left[3x^3 - \frac{x}{2} + \frac{1}{4}\sin(2x) \right]_0^{\frac{\pi}{3}}$$

$$V = \pi \left[3 \times \frac{\pi^3}{27} - \frac{\pi}{6} + \frac{1}{4} \sin\left(\frac{2\pi}{3} \right) \right]$$

$$V = \frac{\pi^4}{9} - \frac{\pi^2}{6} + \frac{\sqrt{3}\pi}{8}$$

Award 1 mark for the correct volume.
Award 1 mark for using the double angle formula.
Award 1 mark for the correct integration.
Award 1 mark for the correct final volume.

VCAA Assessment Report **note:**

This question was well done by some students and poorly by others. Common errors included π being omitted from the outset, an incorrect expression for the volume - using $(3x - \sin x)^2$ as the integrand was common - the double angle formula not being used or being used incorrectly (for example, a sign error), occasional mistakes with the integration step (for example, a sign error), algebraic simplification errors at the end and finding the area rather than the volume. Several students did not distribute a negative through the brackets correctly. Too many students were unable to evaluate $\sin\left(\dfrac{2\pi}{3}\right)$ and many errors were made when expanding brackets, especially sign errors. Also, $(3x)^2 = 6x^2$ was often seen. A few students treated the shape to be rotated as a triangle. Some students arrived at the correct answer but then made errors when attempting to use a common denominator (which was unnecessary).

9.6 Arc length and surface area

9.6 Exercise

1 $y = \dfrac{2}{3} x^{\frac{3}{2}} \quad \dfrac{dy}{dx} = x^{\frac{1}{2}} \quad [0, 3]$

$$s = \int_0^3 \sqrt{1 + x}\, dx$$

$$= \int_0^3 (1 + x)^{\frac{1}{2}}\, dx$$

$$= \left[\frac{2}{3} (1 + x)^{\frac{3}{2}} \right]_0^3$$

$$= \frac{2}{3} \left[4^{\frac{3}{2}} - 1 \right]$$

$$s = \frac{14}{3}$$

2 $y = \sqrt{x^3} = x^{\frac{3}{2}} \quad \frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} = \frac{3\sqrt{x}}{2} \quad [0, 5]$

$$s = \int_0^5 \sqrt{1 + \frac{9x}{4}}\, dx = \int_0^5 \sqrt{\frac{9x+4}{4}}\, dx$$
$$= \frac{1}{2}\int_0^5 (9x+4)^{\frac{1}{2}}\, dx$$
$$= \frac{1}{2} \times \frac{2}{3} \times \frac{1}{9}\left[(9x+4)^{\frac{3}{2}}\right]_0^5$$
$$= \frac{1}{27}\left[49^{\frac{3}{2}} - 4^{\frac{3}{2}}\right]$$
$$s = \frac{335}{27}$$

3 $f(x) = \sqrt{4x+5}$ over $[0, 4]$

$$f'(x) = 4 \times \frac{1}{2} \times (4x+5)^{-\frac{1}{2}} = \frac{2}{\sqrt{4x+5}}$$
$$S = 2\pi\int_0^4 \sqrt{4x+5}\sqrt{1 + \frac{4}{4x+5}}\, dx$$
$$= 2\pi\int_0^4 \sqrt{4x+5}\sqrt{\frac{4x+5+4}{4x+5}}\, dx$$
$$= 2\pi\int_0^4 \sqrt{4x+9}\, dx$$
$$= 2\pi\left[\frac{2}{3} \times \frac{1}{4}\,(4x+9)^{\frac{3}{2}}\right]_0^4$$
$$= \frac{\pi}{3}\left[25^{\frac{3}{2}} - 9^{\frac{3}{2}}\right]$$
$$= \frac{\pi}{3}\,(125 - 27)$$
$$S = \frac{98\pi}{3}$$

4 **a** $y = 3x + 5$ from $x = 1$ to $x = 6$

$$\frac{dy}{dx} = 3$$
$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$
$$= \int_1^6 \sqrt{1+9}\, dx$$
$$= \sqrt{10}\int_1^6 1\, dx$$
$$= \sqrt{10}\,[x]_1^6$$
$$= \sqrt{10}\,(6-1)$$
$$s = 5\sqrt{10}$$

b $S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$$= 2\pi\int_1^6 (3x+5)\,\sqrt{10}\, dx$$
$$= 2\sqrt{10}\,\pi\left[\frac{3x^2}{2} + 5x\right]_1^6$$
$$= 2\sqrt{10}\,\pi\left(\frac{3 \times 36}{2} + 30 - \frac{3}{2} - 5\right)$$
$$= \sqrt{10}\,\pi\,(108 + 60 - 3 - 10)$$
$$= 155\,\pi\sqrt{10}$$

5 $f(x) = x^3 \quad f'(x) = 3x^2$

$$S = 2\pi\int_0^1 x^3\sqrt{1+9x^4}\, dx$$

Let $u = 9x^4 \quad \frac{dy}{dx} = 36x^3$

$x = 1 \quad u = 9 \quad x = 0 \quad u = 0$

$$S = 2\pi\int_0^9 \frac{1}{36}\sqrt{1+u}\, du$$
$$= \frac{\pi}{18}\left[\frac{2}{3}\,(1+u)^{\frac{3}{2}}\right]_0^9$$
$$= \frac{\pi}{27}\left[10^{\frac{3}{2}} - 1\right]$$
$$S = \frac{\pi}{27}\left(10\sqrt{10} - 1\right)$$

6 $f(x) = \frac{x^3}{6} + \frac{1}{2x} = \frac{x^3}{6} + \frac{1}{2}x^{-1} \quad [1, 2]$

$$f'(x) = \frac{x^2}{2} - \frac{1}{2}x^{-2} = \frac{x^2}{2} - \frac{1}{2x^2}$$
$$1 + (f'(x))^2 = \frac{x^4}{4} - 2 \times \frac{x^2}{2} \times \frac{1}{2x^2} + \frac{1}{4x^4} + 1$$
$$= \frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4} = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2$$

a $s = \int_1^2 \left(\frac{x^2}{2} + \frac{1}{2}x^{-2}\right) dx$

$$= \left[\frac{x^3}{6} - \frac{1}{2x}\right]_1^2 = \frac{8}{6} - \frac{1}{4} - \frac{1}{6} + \frac{1}{2} = \frac{16 - 3 - 2 + 6}{12}$$
$$= \frac{17}{12}$$

b $S = 2\pi\int_1^2 \left(\frac{x^3}{6} + \frac{1}{2x}\right)\left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx$

$$= 2\pi\int_1^2 \left(\frac{x^5}{12} + \frac{x}{4} + \frac{x}{12} + \frac{1}{4x^3}\right) dx$$
$$= 2\pi\int_1^2 \left(\frac{x^5}{12} + \frac{x}{3} + \frac{1}{4}x^{-3}\right) dx$$
$$= 2\pi\left[\frac{x^6}{72} + \frac{x^2}{6} - \frac{1}{8x^2}\right]_1^2$$
$$= 2\pi\left[\frac{64}{72} + \frac{4}{6} - \frac{1}{32} - \frac{1}{72} - \frac{1}{6} + \frac{1}{8}\right]$$
$$= 2\pi\left[\frac{8}{9} + \frac{2}{3} - \frac{1}{32} - \frac{1}{72} - \frac{1}{6} + \frac{1}{8}\right]$$
$$S = \frac{47\pi}{16}$$

7 $y = \frac{x^6 + 2}{8x^2}$

$$y = \frac{x^4}{8} + \frac{1}{4x^2} = \frac{x^4}{8} + \frac{1}{4}x^{-2} \quad [1, 2]$$
$$\frac{dy}{dx} = \frac{x^3}{2} - \frac{1}{2}x^{-3} = \frac{x^3}{2} - \frac{1}{2x^3}$$

$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{x^6}{4}-2\times\frac{x^3}{2}\times\frac{1}{2x^3}+\frac{1}{4x^6}$$

$$=\frac{x^6}{4}+\frac{1}{2}+\frac{1}{4x^6}=\left(\frac{x^3}{2}+\frac{1}{2x^3}\right)^2$$

a $s=\int_1^2\left(\frac{x^3}{2}+\frac{1}{2}x^{-3}\right)dx$

$$=\left[\frac{x^4}{8}-\frac{1}{4x^2}\right]_1^2$$

$$=2-\frac{1}{16}-\frac{1}{8}+\frac{1}{4}$$

$$s=\frac{33}{16}$$

b $S=2\pi\int_1^2\left(\frac{x^4}{8}+\frac{1}{4x^2}\right)\left(\frac{x^3}{2}+\frac{1}{2x^3}\right)dx$

$$=2\pi\int_1^2\left(\frac{x^7}{16}+\frac{x}{16}+\frac{x}{8}+\frac{1}{8x^5}\right)dx$$

$$=2\pi\int_1^2\left(\frac{x^7}{16}+\frac{3x}{16}+\frac{1}{8}x^{-5}\right)dx$$

$$=2\pi\left[\frac{x^8}{128}+\frac{3x^2}{32}-\frac{1}{32x^4}\right]_1^2$$

$$=2\pi\left[\frac{256}{128}+\frac{3\times4}{32}-\frac{1}{512}-\frac{1}{128}-\frac{3}{32}+\frac{1}{32}\right]$$

$$=2\pi\left[2+\frac{3}{8}-\frac{1}{512}-\frac{1}{128}-\frac{1}{16}\right]$$

$$S=\frac{1179\pi}{256}$$

8 $y=\frac{1}{2}\left(e^x+e^{-x}\right)$ $[0,1]$

$$\frac{dy}{dx}=\frac{1}{2}\left(e^x-e^{-x}\right)$$

$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{1}{4}\left(e^{2x}-2+e^{-2x}\right)$$

$$=\frac{1}{4}\left(e^{2x}+2+e^{-2x}\right)=\left[\frac{1}{2}\left(e^x+e^{-x}\right)\right]^2$$

a $s=\int_0^1\frac{1}{2}\left(e^x+e^{-x}\right)dx$

$$=\frac{1}{2}\left[e^x-e^{-x}\right]_0^1$$

$$=\frac{1}{2}\left(e-e^{-1}-1+1\right)$$

$$s=\frac{1}{2}\left(e-\frac{1}{e}\right)$$

b $S=2\pi\int_0^1\frac{1}{2}\left(e^x+e^{-x}\right)\times\frac{1}{2}\left(e^x+e^{-x}\right)dx$

$$=\frac{\pi}{2}\int_0^1\left(e^{2x}+2+e^{-2x}\right)dx$$

$$=\frac{\pi}{2}\left[\frac{1}{2}e^{2x}+2x-\frac{1}{2}e^{-2x}\right]_0^1$$

$$=\frac{\pi}{2}\left[\frac{1}{2}e^2+2-\frac{1}{2}e^{-2}-\frac{1}{2}+\frac{1}{2}\right]$$

$$S=\frac{\pi}{4}\left(e^2+4-e^{-2}\right)$$

9 $y=\frac{4\sqrt{2}}{3}x^{\frac{3}{2}}-1$ from $x=0$ to $x=1$

$$\frac{dy}{dx}=2\sqrt{2}x^{\frac{1}{2}}$$

$$1+\left(\frac{dy}{dx}\right)^2=1+8x$$

$$s=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=\int_0^1\sqrt{1+8x}\,dx$$

Let $u=1+8x$ $\frac{du}{dx}=8$

Terminals $x=1$ $u=9$, $x=0$ $u=1$

$$s=\int_1^9u^{\frac{1}{2}}\frac{1}{8}\,du$$

$$=\frac{1}{8}\times\frac{2}{3}\left[u^{\frac{3}{2}}\right]_1^9$$

$$=\frac{1}{12}\left[9^{\frac{3}{2}}-1^{\frac{3}{2}}\right]$$

$$=\frac{1}{12}(27-1)$$

$$s=\frac{13}{6}$$

10 $27y^2=4(x-2)^3$ from $x=3$ to $x=8$

$$y^2=\frac{4}{27}(x-2)^3$$

$$y=\frac{2}{3\sqrt{3}}(x-2)^{\frac{3}{2}}$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{3}}(x-2)^{\frac{1}{2}}$$

$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{1}{3}(x-2)=\frac{x+1}{3}$$

$$s=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=\int_3^8\frac{1}{\sqrt{3}}\sqrt{x+1}\,dx$$

Let $u=x+1$ $\frac{du}{dx}=1$

Terminals $x=3$ $u=4$, $x=8$ $u=9$

$$=\frac{1}{\sqrt{3}}\int_4^{9^{\frac{1}{2}}}du$$

$$=\frac{2}{3\sqrt{3}}\left[u^{\frac{3}{2}}\right]_4^9$$

$$=\frac{2}{3\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}\left[9^{\frac{3}{2}}-4^{\frac{3}{2}}\right]$$

$$=\frac{2\sqrt{3}}{9}(27-8)$$

$$s=\frac{38\sqrt{3}}{9}$$

11 $y=\frac{2}{3}\sqrt{(x-1)^3}$ from $x=1$ to $x=9$

$=\frac{2}{3}(x-1)^{\frac{3}{2}}$

$\frac{dy}{dx}=(x-1)^{\frac{1}{2}}$

$1+\left(\frac{dy}{dx}\right)^2=1+(x-1)=x$

$s=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$=\int_1^9\sqrt{x}\,dx=\int_1^9 x^{\frac{1}{2}}\,dx$

$=\left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^9$

$=\frac{2}{3}\left(9^{\frac{3}{2}}-1\right)$

$=\frac{2}{3}(27-1)$

$s=\frac{52}{3}$

12 $y=\frac{2}{3}\sqrt{(2x-3)^3}$ from $x=\frac{5}{2}$ to $x=\frac{9}{2}$

$=\frac{2}{3}(2x-3)^{\frac{3}{2}}$

$\frac{dy}{dx}=2\sqrt{2x-3}$

$1+\left(\frac{dy}{dx}\right)^2=1+4(2x-3)=8x-11$

$s=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$=\int_{\frac{5}{2}}^{\frac{9}{2}}\sqrt{8x-11}\,dx$

Let $u=8x-11$ $\frac{du}{dx}=8$

Terminals $x=\frac{9}{2}$ $u=25$, $x=\frac{5}{2}$ $u=9$

$=\frac{1}{8}\int_9^{25}u^{\frac{1}{2}}\,du$

$\frac{2}{3}\times\frac{1}{8}\left[u^{\frac{3}{2}}\right]_9^{25}$

$=\frac{1}{12}\left[25^{\frac{3}{2}}-9^{\frac{3}{2}}\right]$

$=\frac{1}{12}[125-27]$

$=\frac{49}{6}$

13 $f(x)=\frac{\sqrt{x}(4x-3)}{6}=\frac{1}{6}\left(4x^{\frac{3}{2}}-3x^{\frac{1}{2}}\right)$ over $[1,\ 4]$

$f'(x)=\frac{1}{6}\left(4\times\frac{3}{2}\times x^{\frac{1}{2}}-\frac{3}{2}\times x^{-\frac{1}{2}}\right)=\frac{1}{6}\left(6\sqrt{x}-\frac{3}{2\sqrt{x}}\right)$

$1+(f'(x))^2=1+\frac{1}{36}\left(36x-2\times6\sqrt{x}\times\frac{3}{2\sqrt{x}}+\frac{9}{4x}\right)$

$=1+\frac{1}{36}\left(36x-18+\frac{9}{4x}\right)$

$=\frac{1}{36}\left(36x+18+\frac{9}{4x}\right)=\frac{1}{6}\left(6\sqrt{x}+\frac{3}{2\sqrt{x}}\right)^2$

a $s=\int_1^4\frac{1}{6}\left(6x^{\frac{1}{2}}+\frac{3}{2}x^{-\frac{1}{2}}\right)dx$

$=\frac{1}{6}\left[\frac{12}{3}x^{\frac{3}{2}}+3\times x^{\frac{1}{2}}\right]_1^4$

$=\frac{1}{6}\left[4x^{\frac{3}{2}}+3\sqrt{x}\right]_1^4$

$=\frac{1}{6}\left[\left(4\times4^{\frac{3}{2}}+3\sqrt{4}\right)-(4+3)\right]$

$=\frac{1}{6}[4\times8+6-7]$

$s=\frac{31}{6}$

b $S=2\pi\int_1^4\frac{1}{6}\left(4x^{\frac{3}{2}}-3x^{\frac{1}{2}}\right)\times\frac{1}{6}\left(6x^{\frac{1}{2}}+\frac{3}{2}x^{-\frac{1}{2}}\right)dx$

$=\frac{2\pi}{36}\int_1^4\left(24x^2-18x+6x-\frac{9}{2}\right)dx$

$=\frac{2\pi}{36}\int_1^4\left(24x^2-12x-\frac{9}{2}\right)dx$

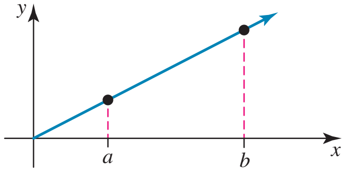
$=\frac{\pi}{18}\left[8x^3-6x^2-\frac{9x}{2}\right]_1^4$

$=\frac{\pi}{36}\left[x\left(16x^2-12x-9\right)\right]_1^4$

$=\frac{\pi}{36}(4(256-48-9)-(16-12-9))$

$S=\frac{89\pi}{4}$

14



$y=mx+c$

$\frac{dy}{dx}=m$

$s=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$=\int_a^b\sqrt{1+m^2}\,dx$

$=\left[\sqrt{1+m^2}\,x\right]_a^b$

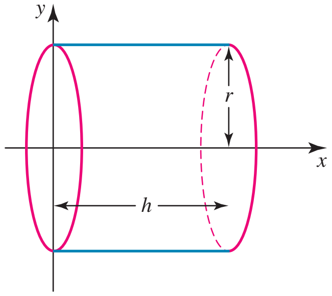
$=(b-a)\sqrt{1+m^2}$

When $x=a$ $y=ma+c$ $A(a,\ ma+c)$

$x=b$ $y=mb+c$ $B(b,\ mb+c)$

$$\begin{aligned} s = d(AB) &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(b-a)^2 + [(mb + c) - (ma + c)]^2} \\ &= \sqrt{(b-a)^2 + (m(b-a))^2} \\ &= \sqrt{(b-a)^2 (1 + m^2)} \\ &= (b-a)\sqrt{1+m^2} \qquad \text{shown} \end{aligned}$$

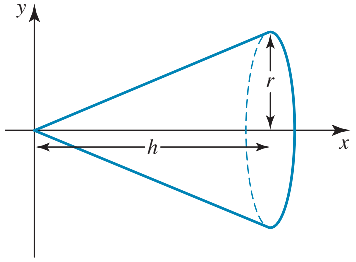
15 **a**



$$y = r \quad \frac{dy}{dx} = 0$$

$$\begin{aligned} S &= 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \\ &= 2\pi \int_0^h r\sqrt{1+0}\,dx \\ &= 2\pi r \int_0^h 1\,dx \\ &= 2\pi r\,[x]_0^h \\ &= 2\pi r\,(h - 0) \\ S &= 2\pi rh \end{aligned}$$

b



$$y = \frac{rx}{h} \quad \frac{dy}{dx} = \frac{r}{h}$$

$$\begin{aligned} S &= 2\pi \int_0^h \frac{rx}{h}\sqrt{1 + \frac{r^2}{h^2}}\,dx \\ &= \frac{2\pi r}{h} \int_0^h x\sqrt{\frac{r^2 + h^2}{h^2}}\,dx \\ &= \frac{2\pi r\sqrt{r^2 + h^2}}{h^2} \int_0^h x\,dx \\ &= \frac{2\pi r\sqrt{r^2 + h^2}}{h^2} \left[\frac{1}{2}x^2\right]_0^h \\ &= \frac{2\pi r\sqrt{r^2 + h^2}}{h^2} \left(\frac{1}{2}h^2 - 0\right) \\ S &= \pi r\sqrt{r^2 + h^2} \end{aligned}$$

16 $y = \sqrt{9 - x^2} \qquad [0, 3]$

a $$\frac{dy}{dx} = \frac{-x}{\sqrt{9 - x^2}}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{9 - x^2} = \frac{9 - x^2 + x^2}{9 - x^2} = \frac{9}{9 - x^2}$$

$$\begin{aligned} s &= \int_0^3 \frac{3}{\sqrt{9 - x^2}}\,dx \\ &= 3\left[\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3 \\ &= 3\left(\sin^{-1}(1) - \sin^{-1}(0)\right) \\ s &= \frac{3\pi}{2} \end{aligned}$$

This is $\frac{1}{4}$ of the circumference of a circle of radius 3.

b $$\begin{aligned} S &= 2\pi \int_0^3 \sqrt{9 - x^2} \times \frac{3}{\sqrt{9 - x^2}}\,dx \\ &= 2\pi \int_0^3 3\,dx \\ &= 6\pi\,[x]_0^3 \\ &= 6\pi\,(3 - 0) \\ S &= 18\pi \end{aligned}$$

This is $\frac{1}{2}$ the surface area of a sphere of radius 3.

17 **a** $y = \log_e\left(x + \sqrt{x^2 + 1}\right) + x\sqrt{x^2 + 1}$

Let $u = x + \sqrt{x^2 + 1} = x + \left(x^2 + 1\right)^{\frac{1}{2}}$

$$\begin{aligned} \frac{du}{dx} &= 1 + 2x \times \frac{1}{2} \times \left(x^2 + 1\right)^{-\frac{1}{2}} \\ &= 1 + \frac{x}{\sqrt{x^2 + 1}} = \frac{x + \sqrt{x^2 + 1}}{\sqrt{x^2 + 1}} \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\cancel{x + 1\sqrt{x^2 + 1}}} \times \frac{\cancel{x + \sqrt{x^2 + 1}}}{\sqrt{x^2 + 1}} + \sqrt{x^2 + 1} + \frac{x \times x}{\sqrt{x^2 + 1}} \\ &= \frac{1}{\sqrt{x^2 + 1}} + \sqrt{x^2 + 1} + \frac{x^2}{\sqrt{x^2 + 1}} \\ &= \frac{1 + x^2 + 1 + x^2}{\sqrt{x^2 + 1}} \\ &= \frac{2\left(x^2 + 1\right)}{\sqrt{x^2 + 1}} = 2\sqrt{x^2 + 1} \end{aligned}$$

So $\int \sqrt{x^2 + 1}\,dx = \frac{1}{2}\left[\log_e\left(x + \sqrt{x^2 + 1}\right) + x\sqrt{x^2 + 1}\right]$

b $f(x) = x^2 \qquad f'(x) = 2x$

Arc length $s = \int_0^1 \sqrt{1 + 4x^2}\,dx$

Let $u = 2x \qquad \frac{du}{dx} = 2$

Terminals $x = 0 \quad u = 0$

$x = 1 \quad u = 2$

$$\begin{aligned} s &= \frac{1}{2}\int_0^1 \sqrt{1 + u^2}\,du = \frac{1}{4}\left[\log_e\left(u + \sqrt{u^2 + 1}\right) + u\sqrt{u^2 + 1}\right]_0^2 \\ &= \frac{1}{4}\left[\log_e\left(2 + \sqrt{5}\right) + 2\sqrt{5} - \log_e(1) - 0\right] \\ s &= \frac{1}{4}\log_e\left(2 + \sqrt{5}\right) + \frac{\sqrt{5}}{2} \end{aligned}$$

18 $f(x) = \sin(x) \quad f'(x) = \cos(x)$

$$S = 2\pi \int_0^{\frac{\pi}{2}} \sin(x)\sqrt{1+\cos^2(x)}\,dx$$

Let $u = \cos(x) \quad \dfrac{du}{dx} = -\sin(x)$

$x = \dfrac{\pi}{2} \quad u = \cos\left(\dfrac{\pi}{2}\right) = 0$

$x = 0 \quad u = \cos(0) = 1$

$$S = 2\pi \int_1^0 -\sqrt{1+u^2}\,du$$

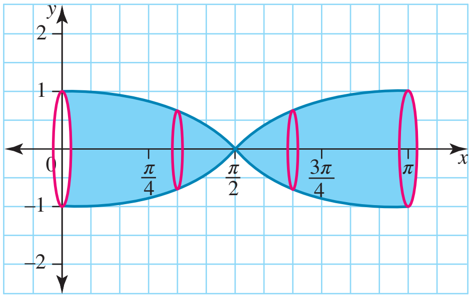
$$= 2\pi \int_0^1 \sqrt{1+u^2}\,du$$

$$= \pi\left[\log_e\left(u+\sqrt{u^2+1}\right)+u\sqrt{u^2+1}\right]_0^1$$

$$= \pi\left[\left(\log_e\left(1+\sqrt{2}\right)+\sqrt{2}\right)-0\right]$$

$$S = \pi\left(\log_e\left(1+\sqrt{2}\right)+\sqrt{2}\right)$$

19



$f(x) = \cos(x) \quad f'(x) = -\sin(x)$

$$S = 2\times 2\pi \int_0^{\frac{\pi}{2}} \cos(x)\sqrt{1+\sin^2(x)}\,dx$$

$u = \sin(x) \quad \dfrac{du}{dx} = \cos(x)$

$x = \dfrac{\pi}{2} \quad u = \sin\left(\dfrac{\pi}{2}\right) = 1$

$x = 0 \quad u = \sin(0) = 0$

$$S = 4\pi \int_0^1 \sqrt{1+u^2}\,du$$

$$S = 2\pi\left(\log_e\left(1+\sqrt{2}\right)+\sqrt{2}\right)$$

20 **a** Let $y = \sec(x) = \dfrac{1}{\cos(x)} = \dfrac{1}{u} = u^{-1} \quad u = \cos(x)$

$\dfrac{dy}{du} = -u^{-2} = \dfrac{-1}{u^2} \quad \dfrac{du}{dx} = -\sin(x)$

$$\frac{d}{dx}(\sec(x)) = \frac{\sin(x)}{\cos^2(x)} = \frac{\sin(x)}{\cos(x)} \times \frac{1}{\cos(x)}$$

$$= \tan(x)\sec(x)$$

$$\frac{d}{dx}\left[\log_e(\sec(x)+\tan(x))\right] = \frac{\frac{d}{dx}[\sec(x)+\tan(x)]}{\sec(x)+\tan(x)}$$

$$= \frac{\tan(x)\sec(x)+\sec^2(x)}{\sec(x)+\tan(x)}$$

$$= \frac{\sec(x)(\tan(x)+\sec(x))}{\sec(x)+\tan(x)} = \sec(x)$$

b $$s = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$y = \log_e(\sec(x)) \quad \frac{dy}{dx} = \frac{\frac{d}{dx}(\sec(x))}{\sec(x)} = \frac{\tan(x)\sec(x)}{\sec(x)}$$

$$= \tan(x)$$

$$s = \int_0^{\frac{\pi}{3}} \sqrt{1+\tan^2(x)}\,dx = \int_0^{\frac{\pi}{3}} \sqrt{\sec^2(x)}\,dx$$

$$= \int_0^{\frac{\pi}{3}} |\sec(x)|\,dx$$

$$= \log_e(\sec(x)+\tan(x))]_0^{\frac{\pi}{3}}$$

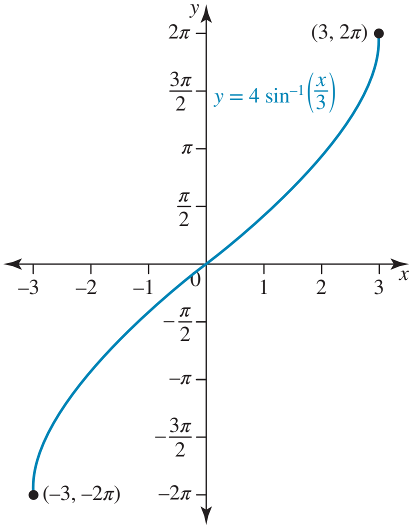
$$= \log_e\left(\sec\left(\frac{\pi}{3}\right)+\tan\left(\frac{\pi}{3}\right)\right) - \log_e(\sec(0)+\tan(0))$$

$$s = \log_e\left(2+\sqrt{3}\right)$$

21 $y = 4\sin^{-1}\left(\dfrac{x}{3}\right) \quad \dfrac{dy}{dx} = \dfrac{4}{\sqrt{9-x^2}}$

a Domain $[-3, 3]$

Range $[-2\pi, 2\pi]$



b $$A = 2\int_0^3 4\sin^{-1}\left(\frac{x}{3}\right)dx$$

$$= 12(\pi - 2)$$

c $$s = 2\int_0^3 \sqrt{1+\frac{16}{9-x^2}}\,dx$$

$$= 2\int_0^3 \sqrt{\frac{25-x^2}{9-x^2}}\,dx$$

$$= 14.1808$$

d $$V_x = 2\pi \int_0^3 16\left(\sin^{-1}\left(\frac{x}{3}\right)\right)^2 dx$$

$$= 24\pi\left(\pi^2-8\right)$$

e $$S = 4\pi \int_0^3 4\sin^{-1}\left(\frac{x}{3}\right)\sqrt{\frac{25-x^2}{9-x^2}}\,dx$$

$$= 267.3693$$

f $y = 4\sin^{-1}\left(\frac{x}{3}\right)$

$\frac{y}{4} = \sin^{-1}\left(\frac{x}{3}\right)$

$\frac{x}{3} = \sin\left(\frac{y}{4}\right)$

$x = 3\sin\left(\frac{y}{4}\right) \qquad \frac{dx}{dy} = \frac{3}{4}\cos\left(\frac{y}{4}\right)$

$$V_y = 2\pi\int_0^{2\pi} 9\sin^2\left(\frac{y}{4}\right)dy$$
$$= 18\pi^2$$

g $S_y = 2\pi\int x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

$$= 4\pi\int_0^{2\pi} 3\sin\left(\frac{y}{4}\right)\sqrt{1+\frac{9}{16}\cos^2\left(\frac{y}{4}\right)}\,dy$$
$$= 2\pi\left(15+16\log_e(2)\right)$$

22 a $y = 3e^{-2x} \quad [0,\ 1]$

$\frac{dy}{dx} = -6e^{-2x}$

$$s = \int_0^1 \sqrt{1+36e^{-4x}}\,dx$$
$X = 2.8323$

b $f(x) = 3e^{-2x} \qquad [0,\ 1]$

$f'(x) = -6e^{-2x}$

$$S = 2\pi\int_0^1 3e^{-2x}\sqrt{1+36e^{-4x}}\,dx$$
$$= 29.2171$$

23 a $y = \log_e(2x+1) \quad [0, 3]$

$\frac{dy}{dx} = \frac{2}{2x+1}$

$$s = \int_0^3 \sqrt{1+\frac{4}{(2x+1)^2}}\,dx$$
$$= 3.6837$$

b $f(x) = \log_e(2x+1) \qquad [0, 3]$

$f'(x) = \frac{2}{2x+1}$

$$S = 2\pi\int_0^3 \log_e(2x+1)\sqrt{1+\frac{4}{(2x+1)^2}}\,dx$$
$$= 27.1426$$

24 a $y = 3\cos^{-1}\left(\frac{x}{2}\right) \qquad [-2, 2]$

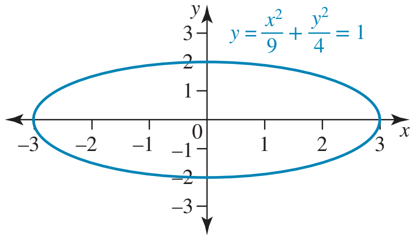
$\frac{dy}{dx} = \frac{-3}{\sqrt{4-x^2}}$

$$s = \int_{-2}^{2}\sqrt{1+\frac{9}{4-x^2}}\,dx$$
$$= 10.3978$$

b $f(x) = 3\cos^{-1}\left(\frac{x}{2}\right) \qquad [-2,\ 2]$

$$S = 2\pi\int_{-2}^{2} 3\cos^{-1}\left(\frac{x}{2}\right)\sqrt{1+\frac{9}{4-x^2}}\,dx$$
$$= 307.8661$$

25 a



$\frac{x^2}{9} + \frac{y^2}{4} = 1$

$\frac{y^2}{4} = 1 - \frac{x^2}{9}$

$= \frac{9-x^2}{9}$

$y = \frac{2}{3}\sqrt{9-x^2}$

$\frac{dy}{dx} = \frac{2}{3}\times(-2x)\times\frac{1}{2}\left(9-x^2\right)^{-\frac{1}{2}} \quad = \frac{-2x}{3\sqrt{9-x^2}}$

$$s = 4\int_0^3\sqrt{1+\frac{4x^2}{9\left(9-x^2\right)}}\,dx$$
$$= 15.8654$$

b $S_x = 2\pi\int_0^3 2\times\frac{2}{3}\sqrt{9-x^2}\sqrt{1+\frac{4x^2}{9\left(9-x^2\right)}}\,dx$

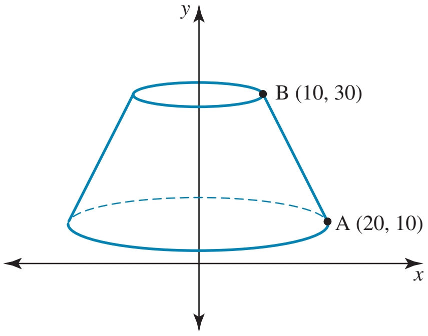
$= 21.5409\,\pi = 67.6729$

c $\frac{x^2}{9} = 1 - \frac{y^2}{4} = \frac{4-y^2}{4}$

$x = \frac{3}{2}\sqrt{4-y^2} \quad \frac{dx}{dy} = \frac{-3y}{2\sqrt{4-y^2}}$

$$S_y = 2\pi\int_0^2 2\times\frac{3}{2}\sqrt{4-y^2}\sqrt{1+\frac{9y^2}{4\left(4-y^2\right)}}\,dy$$
$$= 89.0007$$

26 a



$y = mx + c$

$(20,\ 10) \quad \Rightarrow \quad 10 = 20m + c \ \ (1)$

$(10,\ 30) \quad \Rightarrow \quad 30 = 10m + c \ \ (2)$

$(2) - (1) \qquad 20 = -10m$

$m = -2 \Rightarrow \ c = 50$

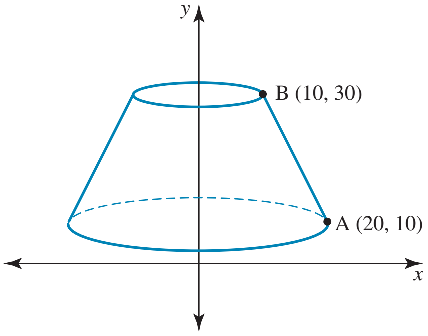
$y = -2x + 50$

$2x = 50 - y$

$x = 25 - \frac{y}{2} \qquad \frac{dx}{dy} = -\frac{1}{2}$

$$\begin{aligned}
S &= 2\pi\int_a^b x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy\\
&= 2\pi\int_{10}^{30}\left(25-\frac{y}{2}\right)\sqrt{1+\frac{1}{4}}\,dy\\
&= 2\pi\int_{10}^{30}\frac{50-y}{2}\sqrt{\frac{5}{4}}\,dy\\
&= \frac{\pi\sqrt{5}}{2}\int_{10}^{30}(50-y)\,dy\\
&= \frac{\pi\sqrt{5}}{2}\left[50y-\frac{1}{2}y^2\right]_{10}^{30}\\
&= \frac{\pi\sqrt{5}}{2}\left[1500-\frac{900}{2}-500+50\right]\\
&= 300\,\pi\sqrt{5}\\
&\approx 2107.44\,\text{cm}^3
\end{aligned}$$

b



$$y = \frac{a}{x}+k$$

$$(20, 10) \Rightarrow 10 = \frac{a}{20}+k \quad (1)$$

$$(10, 30) \Rightarrow 30 = \frac{a}{10}+k \quad (2)$$

$$(2)-(1)\ 20 = \frac{a}{10}-\frac{a}{20} = \frac{a}{20}$$

$$a = 400 \Rightarrow k = -10$$

$$y = \frac{400}{x}-10$$

$$y+10 = \frac{400}{x}$$

$$x = \frac{400}{y+10} \qquad \frac{dx}{dy} = \frac{-400}{(y+10)^2}$$

$$\begin{aligned}
S &= 2\pi\int_a^b x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy\\
&= 2\pi\int_{10}^{30}\frac{400}{y+10}\sqrt{1+\frac{160000}{(y+10)^4}}\,dy\\
&= 2006.57\,\text{cm}^3
\end{aligned}$$

9.6 Exam questions

1 $s = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$y = \frac{1}{3}\left(x^2+2\right)^{\frac{3}{2}}$

$$\frac{dy}{dx} = \frac{1}{3}\times 2x\times\frac{3}{2}\left(x^2+2\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = x\sqrt{x^2+2}$$

$$\begin{aligned}
1+\left(\frac{dy}{dx}\right)^2 &= 1+x^2\left(x^2+2\right)\\
&= 1+2x^2+x^4\\
&= \left(x^2+1\right)^2
\end{aligned}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = x^2+1$$

$$\begin{aligned}
s &= \int_0^2\left(x^2+1\right)\,dx\\
&= \left[\frac{x^3}{3}+x\right]_0^2 = \left(\frac{8}{3}+2\right)-0\\
s &= \frac{14}{3}
\end{aligned}$$

Award 1 mark for the correct gradient.
Award 1 mark for substituting into the arc length formula.
Award 1 mark for the correct definite integral giving the arc length.
Award 1 mark for the final correct arc length.

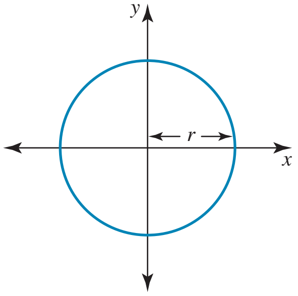
VCAA Assessment Report **note:**

Some students responded very well to this question but others had some difficulty. A number made an error in the formula, despite it being on the formula sheet. Most students found the derivative correctly, but some made errors leading to an impossible integral. A common error was to give the derivative as $\frac{x}{\sqrt{x^2+2}}$. A large proportion of those who found the correct derivative and substituted correctly into the formula were then unable to recognise the perfect square inside the square root. Some of the functions that were used in attempts to substitute could not lead to a correct answer. A small number of students took the square root of individual terms. Some students did not use *dx*.

2 $f(x) = \sqrt{x} \quad f'(x) = \frac{1}{2\sqrt{x}}$

$$\begin{aligned}
S &= 2\pi\int_0^1\sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx && \text{[1 mark]}\\
&= 2\pi\int_0^1\sqrt{x}\sqrt{\frac{4x+1}{4x}}\,dx\\
&= \pi\int_0^1\sqrt{4x+1}\,dx\\
&= \pi\left[\frac{1}{4}\times\frac{2}{3}(4x+1)^{\frac{3}{2}}\right]_0^1 && \text{[1 mark]}\\
&= \frac{\pi}{6}\left[5^{\frac{3}{2}}-1\right]\\
&= \frac{\pi}{6}\left(5\sqrt{5}-1\right) && \text{[1 mark]}
\end{aligned}$$

3 a



$$x^2 + y^2 = r^2$$

$$y = \sqrt{r^2 - x^2} \qquad \frac{dy}{dx} = \frac{-x}{\sqrt{r^2 - x^2}}$$ [1 mark]

$$S = 4\int_0^r \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{r^2 - x^2} = \frac{r^2 - x^2 + x^2}{r^2 - x^2} = \frac{r^2}{r^2 - x^2}$$ [1 mark]

$$s = 4\int_0^r \frac{r}{\sqrt{r^2 - x^2}}\, dx$$

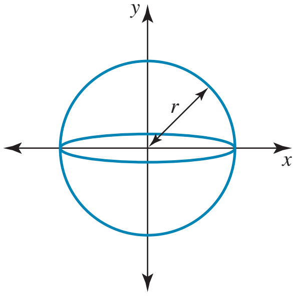
$$= 4r\left[\sin^{-1}\left(\frac{x}{r}\right)\right]_0^r$$ [1 mark]

$$= 4r\left[\sin^{-1}(1) - \sin^{-1}(0)\right]$$

$$= 4r \times \frac{\pi}{2}$$

$$C = 2\pi r$$ [1 mark]

b Sphere



$$y = \sqrt{r^2 - x^2}$$

$$x^2 + y^2 = r^2$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$ [1 mark]

$$S = 2 \times 2\pi \int_0^r y\sqrt{1 + \frac{x^2}{y^2}}\, dx$$

$$= 4\pi\int_0^r y\sqrt{\frac{x^2 + y^2}{y^2}}\, dx$$ [1 mark]

$$= 4\pi\int_0^r r\, dx$$

$$= 4\pi r\int_0^r 1\, dx$$

$$= 4\pi r\,[x]_0^r$$ [1 mark]

$$= 4\pi r\,(r - 0)$$

$$S = 4\pi r^2$$ [1 mark]

9.7 Water flow

9.7 Exercise

1 $y = ax + b$

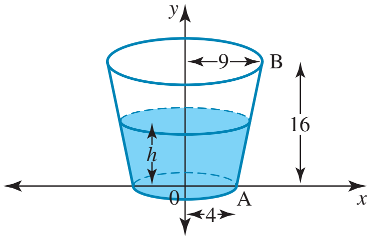
A (4, 0) → $0 = 4a + b$

B (9, 16) → $16 = 9a + b$

$$a = \frac{16}{5}$$

$$b = -\frac{64}{5}$$

$$y = \frac{16}{5}(x - 4)$$



$$V = \pi\int_0^h x^2\, dy$$

$$= \frac{\pi}{256}\int_0^h (5y + 64)^2\, dy$$

$$\frac{dV}{dh} = \frac{\pi}{256}(5h + 64)^2$$

$$\frac{dV}{dt} = -2\sqrt{h}$$

$$\frac{dt}{dh} = \frac{dt}{dV}\cdot\frac{dV}{dh} = \frac{-\pi(5h + 64)^2}{512\sqrt{h}}$$

$$t = \frac{-\pi}{512}\int_{16}^{0} \frac{(5h + 64)^2}{\sqrt{h}}\, dh$$

$x = 431.45$ minutes

2 $y = ax^2 + b$

A (4, 0) → $0 = 16a + b$

B (9, 16) → $16 = 81a + b$

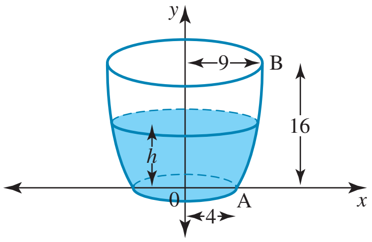
$$a = \frac{16}{65}$$

$$b = -\frac{256}{65}$$

$$y = \frac{16}{65}\left(x^2 - 16\right)$$

$$\frac{65y}{16} = x^2 - 16$$

$$x^2 = \frac{65y + 256}{16}$$



$$V = \pi \int_0^h x^2\,dy$$

$$= \pi \int_0^h \frac{65y + 256}{16}\,dy$$

$$\frac{dV}{dh} = \frac{\pi}{16}(65h + 256)$$

$$\frac{dV}{dt} = -2\sqrt{h}$$

$$\frac{dt}{dh} = \frac{dt}{dV} \cdot \frac{dV}{dh} = \frac{\pi(65h + 256)}{-2\sqrt{h} \times 16}$$

$$t = \frac{-\pi}{32} \int_{16}^{0} \frac{(65h + 256)}{\sqrt{h}}\,dh$$

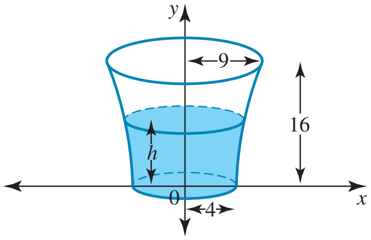
$$= 473.3 \text{ minutes}$$

3 $\dfrac{x^2}{16} - \dfrac{65y^2}{4096} = 1$

$$\frac{x^2}{16} = 1 + \frac{65y^2}{4096}$$

$$\frac{x^2}{16} = \frac{4096 + 65y^2}{4096}$$

$$x^2 = \frac{1}{256}\left(65y^2 + 4096\right)$$



$$V = \pi \int_0^h x^2\,dy$$

$$= \frac{\pi}{256} \int_0^h 65y^2 + 4096\,dy$$

$$\frac{dV}{dh} = \frac{\pi}{256}(65h^2 + 4096)$$

$$\frac{dV}{dt} = -k\sqrt{h}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt} = \frac{-k\sqrt{h}}{\frac{\pi}{256}\left(65h^2 + 4096\right)}$$

$$\frac{dt}{dh} = \frac{65h^2 + 4096}{A\sqrt{h}}$$

$$At + B = \int \frac{65h^2 + 4096}{\sqrt{h}}\,dh$$

$$= \int 65h^{\frac{3}{2}} + 4096h^{-\frac{1}{2}}\,dh$$

$$= 26h^{\frac{5}{2}} + 8192h^{\frac{1}{2}}$$

When

$h = 16,\ t = 0$

$B = 26(16)^{\frac{5}{2}} + 8192(16)^{\frac{1}{2}} = 59\,392$

$h = 9,\ t = 10$

$10A + B = 26(9)^{\frac{5}{2}} + 8192(9)^{\frac{1}{2}} = 30\,894$

$$10A = -28\,498$$

$$A = \frac{-14\,249}{5},\ B = 59\,392$$

When empty, $h = 0$

$$t = -\frac{B}{A} = \frac{59\,392}{\frac{14\,249}{5}}$$

$$= 20.8 \text{ min}$$

So it takes an extra 10.8 minutes to empty

4 $y = ax^2 + b$

A $(4, 0) \rightarrow 0 = 16a + b$

B $(9, 16) \rightarrow 16 = 81a + b$

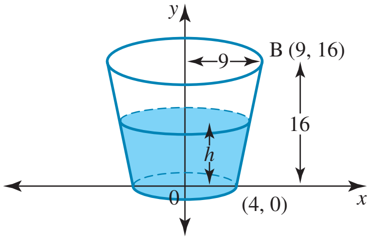
$$a = \frac{16}{65}$$

$$b = -\frac{256}{65}$$

$$y = \frac{16}{65}\left(x^2 - 16\right)$$

$$\frac{65y}{16} = x^2 - 16$$

$$x^2 = \frac{65y + 256}{16}$$



$$V = \pi \int_0^h x^2\,dy$$

$$= \frac{\pi}{16} \int_0^h 65y + 256\,dy$$

$$\frac{dV}{dh} = \frac{\pi}{16}(65h + 256)$$

$$\frac{dV}{dt} = -k\sqrt{h}$$

$$\frac{dt}{dh} = \frac{dt}{dV} \cdot \frac{dV}{dh} = \frac{\frac{\pi}{16}(65h + 256)}{-k\sqrt{h}}$$

$$t = -\frac{\pi}{16k} \int \frac{65h + 256}{\sqrt{h}}$$

$$At + B = \int \left(65h^{\frac{1}{2}} + 256h^{-\frac{1}{2}}\right)$$

$$= \frac{130}{3}h^{\frac{3}{2}} + 512h^{\frac{1}{2}}$$

$h = 16,\ t = 0$

$B = \dfrac{130}{3}(16)^{\frac{3}{2}} + 512(16)^{\frac{1}{2}} = \dfrac{14\,464}{3}$

$h = 9,\ t = 10$

$10A + B = \dfrac{130}{3}(9)^{\frac{3}{2}} + 512(9)^{\frac{1}{2}} = 2706$

$$10A = -\frac{6346}{3}$$

$$A = -\frac{3173}{15},\ B = \frac{14\,464}{3}$$

When empty, $h = 0$

$$t = -\frac{B}{A} = \frac{\frac{14\,464}{3}}{\frac{3173}{15}}$$

$= 22.8$ min

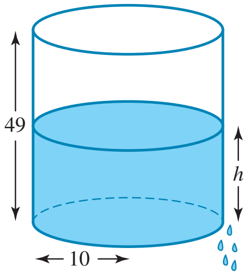
Takes an extra 12.8 minutes.

5 **a** Coffee pot, units cm

$r = 10$

$V = \pi r^2 h$

$V = 100\pi h$



$$\frac{dV}{dh} = 100\pi \qquad \frac{dV}{dt} = -2\sqrt{h}$$

$$\frac{dh}{dt} = \frac{dh}{dV}\cdot\frac{dV}{dt} = \frac{-2\sqrt{h}}{100\pi}$$

$$\frac{dt}{dh} = \frac{-50\pi}{\sqrt{h}}$$

$$t = -50\,\pi \int h^{-\frac{1}{2}}\,dh$$

$$t = -100\,\pi\, h^{\frac{1}{2}} + c$$

When $h = 49 \quad t = 0$

$0 = -100\,\pi\sqrt{49} + c$

$c = 700\,\pi$

$t = -100\,\pi\sqrt{h} + 700\,\pi$

$= 100\,\pi\left(7 - \sqrt{h}\right)$

When $h = 0 \quad t = 700\,\pi$ sec

$$= \frac{700\,\pi}{60}$$

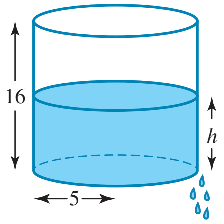
$= 36.65$ minutes.

b Teapot, units cm

$r = 5,\ h = 16$

$V = \pi r^2 h$

$= 25\,\pi h$



$$\frac{dV}{dh} = 25\pi \qquad \frac{dV}{dt} = -k\sqrt{h}$$

$$\frac{dh}{dt} = \frac{dh}{dV}\cdot\frac{dV}{dt} = \frac{-k\sqrt{h}}{25\,\pi} = -A\sqrt{h}$$

$$\frac{dt}{dh} = \frac{-1}{A\sqrt{h}} = -\frac{1}{A}h^{-\frac{1}{2}}$$

$$-At = \int h^{-\frac{1}{2}}\,dh = 2h^{\frac{1}{2}} + c$$

$-At + B = 2\sqrt{h}$

When $\quad h = 16 \quad t = 0 \quad B = 2\sqrt{16} = 8$

$\qquad h = 9 \quad t = 10$

$10A + 4 = \sqrt{9} = 3 \times -10A + 8 = 2\sqrt{9} = 6$

$$A = -\frac{1}{10} \times A = \frac{1}{5}$$

$$\sqrt{h} = 3 - \frac{t}{10} \times 2\sqrt{h} = 6 - \frac{t}{5}$$

When $h = 0 \quad t = 30$ minutes

So it takes 20 more minutes to empty.

6 **a** Bathtub L = 1.5 W = 0.6 m

$V = \text{LWh}$

$= 1.5 \times 0.6 \times h$

$= 0.9h$

$$\frac{dV}{dh} = 0.9 \qquad \frac{dV}{dt} = -2\sqrt{h}\ \text{m}^3/\text{min}$$

$$\frac{dh}{dt} = \frac{dh}{dV}\cdot\frac{dV}{dt} = -\frac{2\sqrt{h}}{0.9}$$

$$\frac{dt}{dh} = -\frac{0.9}{2\sqrt{h}}$$

$$t = -\frac{0.9}{2}\int h^{-\frac{1}{2}}\,dh$$

$$t = -\frac{0.9}{2} \times 2h^{\frac{1}{2}} + c$$

$t = -0.9\sqrt{h} + c$

When $h = 1 \quad t = 0$

$0 = -0.9 + \text{c}$

$c = 0.9$

$t = -0.9\sqrt{h} + 0.9$

$= 0.9\left(1 - \sqrt{h}\right)$

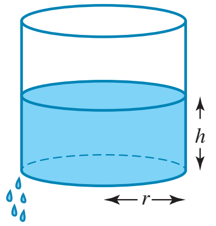
When $h = 0$

$t = 0.9$ min

$= 54$ seconds

b $V = 160$ litres 90 minutes empty

$= \pi r^2 h$



$$\frac{dV}{dh} = \pi r^2 \quad \frac{dV}{dt} = -k\sqrt{h}\ \ \text{cm}^3/\text{min}$$

$$\frac{dh}{dt} = \frac{dh}{dV}\cdot\frac{dV}{dt} = \frac{-k\sqrt{h}}{\pi r^2}$$

$$\frac{dt}{dh} = \frac{-\pi r^2}{k\sqrt{h}} = \frac{-\pi r^2}{k}h^{-\frac{1}{2}}$$

$$t = \frac{-\pi r^2}{k}\int_{169}^{0} h^{-\frac{1}{2}}\,dh$$

$$= \frac{-\pi r^2}{k}\left[2\sqrt{h}\right]_{169}^{0}$$

$$= \frac{-\pi r^2}{k}\left(-2\sqrt{169} + 0\right) = \frac{\pi r^2}{k} \times 26$$

Now $V = \pi r^2 \times 169 = 160\,000$ $\qquad \pi r^2 = \dfrac{160\,000}{169}$

$r = \sqrt{\dfrac{160\,000}{\pi \times 169}} = 17.36$

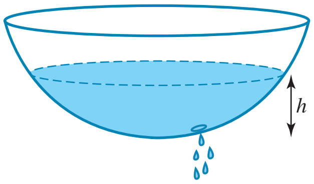
Time to empty $\dfrac{160\,000 \times 26}{k \times 169} = 90$

$k = \dfrac{160\,000 \times 26}{169 \times 90}$

$k = 273.5$

7 **a** $V = \dfrac{\pi h^2}{3}(30 - h)$

$V = \dfrac{\pi}{3}(30h^2 - h^3)$



$\dfrac{dV}{dh} = \dfrac{\pi}{3}\left(60h - 3h^2\right) = \pi\left(20h - h^2\right) \qquad \dfrac{dV}{dt} = -k\sqrt{h}$

$\dfrac{dt}{dh} = \dfrac{dt}{dV} \cdot \dfrac{dV}{dh} = \dfrac{\pi\left(20h - h^2\right)}{-k\sqrt{h}}$

$$\begin{aligned} t &= \frac{-\pi}{k}\int_9^0 \left(\frac{20h - h^2}{\sqrt{h}}\right) dh \\ &= \frac{-\pi}{k}\int_9^0 \left(20h^{\frac{1}{2}} - h^{\frac{3}{2}}\right) dh \\ &= \frac{-\pi}{k}\left[\frac{40}{3}h^{\frac{3}{2}} - \frac{2}{5}h^{\frac{5}{2}}\right]_9^0 \\ &= \frac{-\pi}{k}\left[0 - \left(\frac{40}{3} \times 9^{\frac{3}{2}} - \frac{2}{5} \times 9^{\frac{5}{2}}\right)\right] \\ T &= \frac{\pi}{k} \times \frac{1314}{5} = 1314 \\ \Rightarrow \quad & k = \frac{\pi}{5} \end{aligned}$$

b Drinking trough

$\alpha = 45^0$

$\tan(\alpha) = 1$

$A = 40h + hd$

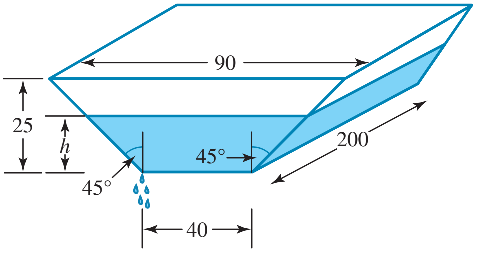
$= 40h + h^2 \tan(\alpha)$

$= 40h + h^2$

$= h(40 + h)$

$V = 200h(40 + h)$

$= 200(40h + h^2)$



$\dfrac{dV}{dh} = 200(40 + 2h) \qquad \dfrac{dV}{dt} = -k\sqrt{h}$

$= 400(20 + h)$

$\dfrac{dh}{dt} = \dfrac{dh}{dV} \cdot \dfrac{dV}{dt} = \dfrac{-k\sqrt{h}}{400(20 + h)}$

$\dfrac{dt}{dh} = \dfrac{400(20 + h)}{-k\sqrt{h}} \qquad t = \dfrac{-400}{k}\displaystyle\int \dfrac{20 + h}{\sqrt{h}}\, dh$

$At + B = \displaystyle\int 20h^{-\frac{1}{2}} + h^{\frac{1}{2}}\, dh$

$= 40h^{\frac{1}{2}} + \dfrac{2}{3}h^{\frac{3}{2}}$

When $h = 25 \quad t = 0$

$B = 40\sqrt{25} + \dfrac{2}{3} \times 25^{\frac{3}{2}} = \dfrac{850}{3}$

When $h = 16 \quad t = 20$

$20A + B = 40\sqrt{16} + \dfrac{2}{3} \times 16^{\frac{3}{2}} = \dfrac{608}{3}$

$20A = \dfrac{-242}{3} \qquad A = \dfrac{-121}{30}$

Empty when $h = 0$

$t = -\dfrac{B}{A} = \dfrac{\frac{850}{3}}{\frac{121}{30}}$

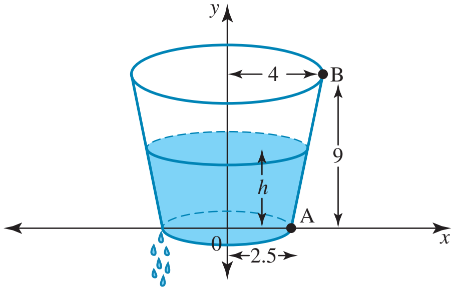
$= 70.25$ minutes

So it takes an extra 50.25 minutes to empty.

8 **a** Coffee cup, units cm

$A(2.5, 0) \quad B(4, 9)$

$m(AB) = \dfrac{9 - 0}{4 - 2.5} = 6$



$y - 0 = 6\left(x - \dfrac{5}{2}\right)$

$y = 6x - 15$

$6x = y + 15$

$x = \dfrac{1}{6}(y + 15)$

$V = \pi\displaystyle\int_0^h x^2\, dy = \int_0^h \dfrac{1}{36}(y + 15)^2\, dy$

$\dfrac{dV}{dh} = \dfrac{\pi}{36}(h + 15)^2 \qquad \dfrac{dV}{dt} = -k\sqrt{h}$

$\dfrac{dh}{dt} = \dfrac{dh}{dV} \cdot \dfrac{dV}{dt} = \dfrac{-k\sqrt{h}}{\frac{\pi}{36}(h + 15)^2}$

$\dfrac{dt}{dh} = \dfrac{-\pi(h + 15)^2}{36k\sqrt{h}}$

$t = \dfrac{-\pi}{36k}\displaystyle\int_9^0 \dfrac{(h + 15)^2}{\sqrt{h}}\, dh = 3$ minutes

$= \dfrac{\pi}{36k} \times \dfrac{9936}{5} = 3$

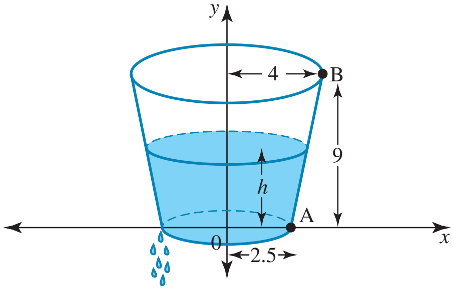
$k = \dfrac{9936\pi}{36 \times 5 \times 3} = \dfrac{92\pi}{5}$

b Plastic bucket, units cm

$A(10, 0)\quad B(13, 24)$

$$m(AB) = \frac{24}{3} = 8$$

$$y - 0 = 8(x - 10)$$



$$y = 8x - 80$$

$$8x = y + 80$$

$$x = \frac{y + 80}{8}$$

$$x^2 = \frac{(y + 80)^2}{64}$$

$$V = \pi \int_0^h \frac{(y + 80)^2}{64}\, dy$$

$$\frac{dV}{dh} = \frac{\pi(h + 80)^2}{64}, \quad \frac{dV}{dt} = -k\sqrt{h}, \quad k > 0$$

$$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt} = \frac{-64k\sqrt{h}}{\pi(h + 80)^2}$$

When $h = 16 \quad \frac{dh}{dt} = \frac{-k}{36\pi} = -\frac{1}{10}$

$$\Rightarrow k = \frac{18\pi}{5}$$

$$\frac{dh}{dt} = \frac{-64 \times 18\pi}{5} \times \frac{\sqrt{h}}{\pi(h + 80)^2}$$

$$= \frac{-1152\sqrt{h}}{5(h + 80)^2}$$

$$\frac{dt}{dh} = \frac{-5(h + 80)^2}{1152\sqrt{h}}$$

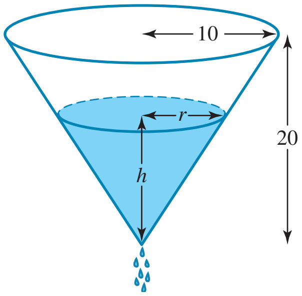
$$t = \int_{16}^{0} \frac{-5(h + 80)^2}{1152\sqrt{h}}\, dh = \frac{6848}{27}$$

$= 253.63$ minutes

9 a $V = \frac{1}{3}\pi r^2 h$, units cm

$$\frac{r}{h} = \frac{10}{20} = \frac{1}{2}$$

$$r = \frac{h}{2}$$



$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}$$

$$\frac{dV}{dh} = \frac{\pi h^2}{4}, \quad \frac{dV}{dt} = -k\sqrt{h}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt} = \frac{-k\sqrt{h}}{\frac{\pi}{4}h^2} = \frac{-4k}{\pi h^{\frac{3}{2}}}$$

$$\frac{dt}{dh} = \frac{-\pi h^{\frac{3}{2}}}{4k}$$

$$t = -A\int h^{\frac{3}{2}}\, dh$$

$$= -A \times \frac{2}{5}h^{\frac{5}{2}} + c$$

So $t = -Bh^{\frac{5}{2}} + c$

When $\quad h = 16 \quad t = 0 \Rightarrow 0 = -B \times 16^{\frac{5}{2}} + c$

$$c = 1024B$$

When $h = 9 \quad t = 10 \Rightarrow 10 = -B \times 9^{\frac{5}{2}} + c$

$$= -243B + 1024B$$

$$10 = 781B$$

$$B = \frac{10}{781}$$

Empty when $h = 0$

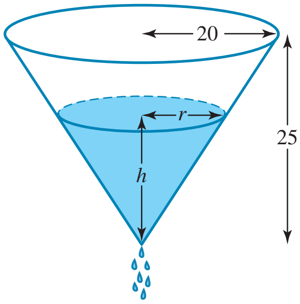
$$t = c = \frac{1024 \times 10}{781} = 13.11$$

An extra 3.11 minutes to empty

b Oil leaking from funnel, units cm

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{r}{h} = \frac{20}{25} \Rightarrow r = \frac{4h}{5}$$



$$V = \frac{1}{3}\pi\left(\frac{4h}{5}\right)^2 h = \frac{16\pi h^3}{75}$$

$$\frac{dV}{dh} = \frac{16\pi h^2}{25} \quad \frac{dV}{dt} = -k\sqrt{h}\,\text{cm}^3/\text{s}$$

$h = 25 \quad t = 0$

$$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt} = \frac{-k\sqrt{h}}{\frac{16\pi h^2}{25}} = \frac{-25k}{16\pi h^{\frac{3}{2}}}$$

$$\frac{dt}{dh} = \frac{-16\pi h^{\frac{3}{2}}}{25k}$$

$$t = \frac{-16\pi}{25k}\int_{25}^{0} h^{\frac{3}{2}}\, dh$$

$$= \frac{-16\pi}{25k}\left[\frac{2}{5}h^{\frac{5}{2}}\right]_{25}^{0}$$

$$= \frac{-16\pi}{25k}\left[0 - \frac{2}{5} \times 25^{\frac{5}{2}}\right]$$

$$= \frac{16\pi}{25k} \times \frac{2}{5} \times 3125$$

$$= \frac{800\pi}{k} = 40 \text{ seconds}$$

$k = 20\pi$

10 $V = 500\left(h^2 - \frac{h^4}{4}\right) \quad 0 \le h \le 1$ units m

a $\frac{dV}{dh} = 500\left(2h - \frac{4h^3}{4}\right)$

$$= 500h\left(2 - h^2\right)$$

$$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt} = \frac{300 - 2\sqrt{h}}{500h\left(2 - h^2\right)}$$

$$\left.\frac{dh}{dt}\right|_{h=0.5} = \frac{300 - 2\sqrt{0.5}}{500 \times 0.5\ \left(2 - 0.5^2\right)}$$

$$= 0.68 \text{ m/hr}$$

b $\frac{dt}{dh} = \frac{500h\ \left(2 - h^2\right)}{300 - 2\sqrt{h}}$

$$t = \int_0^1 \frac{500\,h\left(2 - h^2\right)}{300\ - 2\sqrt{h}}\,dh = 1.26 \text{ hours}$$

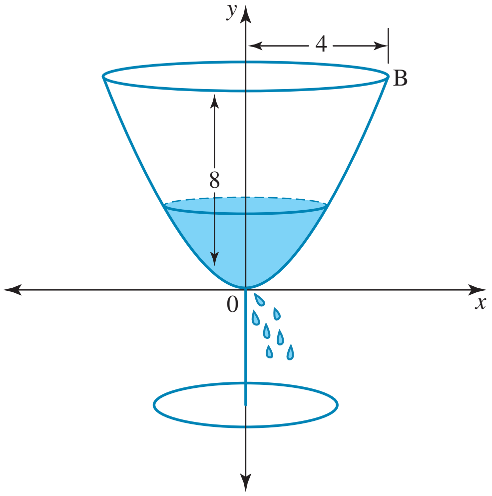
c $T = \int_1^0 \frac{500h\left(2 - h^2\right)}{-2\sqrt{h}}\,dh = 261.9 \text{ hours}$

11 $y = x^{\frac{3}{2}}$, units cm

a $y^2 = x^3$

$x = y^{\frac{2}{3}}$

$x^2 = y^{\frac{4}{3}}$



$$V = \pi\int_0^h x^2\,dy = \pi\int_0^h y^{\frac{4}{3}}\,dh$$

$$\frac{dV}{dh} = \pi h^{\frac{4}{3}}, \quad \frac{dV}{dt} = -k\sqrt{h}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt} = \frac{-k\sqrt{h}}{\pi h^{\frac{4}{3}}} = -Ah^{-\frac{5}{6}}$$

$$\frac{dt}{dh} = -\frac{1}{A}h^{\frac{5}{6}}$$

$$-At + B = \frac{6}{11}h^{\frac{11}{6}}$$

When $x = 4 \ \ y = h = 8\ ,\ t = 0$

$$B = \frac{6}{11} \times 8^{\frac{11}{6}} = \frac{192\sqrt{2}}{11}$$

When $x = 1 \ \ y = 1 \ \ t = 3\ ,\ h = 1$

$$-3A + B = \frac{6}{11}$$

$$A = \frac{2\left(32\sqrt{2} - 1\right)}{11}$$

When $h = 0 \quad t = \frac{B}{A} = \frac{192\sqrt{2}}{11} \times \frac{11}{2\left(32\sqrt{2} - 1\right)}$

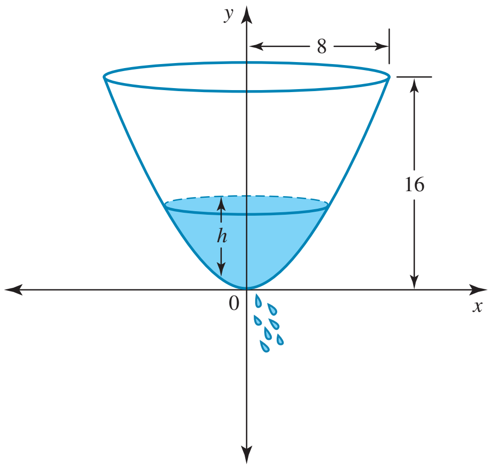
$$= 3.07 \text{ minutes}$$

Takes an extra 0.07 minutes.

b $y = x^{\frac{4}{3}}$ units cm

$y^3 = x^4$

$x^2 = y^{\frac{3}{2}}$



$$V = \pi\int_a^b x^2\,dy$$

$$= \pi\int_0^h y^{\frac{3}{2}}\,dy$$

$$= \pi\left[\frac{2}{5}y^{\frac{5}{2}}\right]_0^h$$

$$V = \frac{2\pi}{5}h^{\frac{5}{2}} \quad \frac{dV}{dh} = \pi h^{\frac{3}{2}}$$

$$\frac{dV}{dt} = -k\sqrt{h}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt} = \frac{-k\sqrt{h}}{\pi h^{\frac{3}{2}}} = -\frac{A}{h}$$

$$\frac{dt}{dh} = \frac{-h}{A}$$

$$t = \frac{-1}{A}\int h\,dh$$

$$\frac{1}{2}h^2 = Bt + c$$

When $t = 0 \quad h = 16$

$$\Rightarrow c = \frac{1}{2} \times 16^2 = 128$$

When $t = 3 \qquad h = 12$

$$\frac{1}{2} \times 12^2 = 72 = 3B + 128$$

$$B = \frac{-56}{3}$$

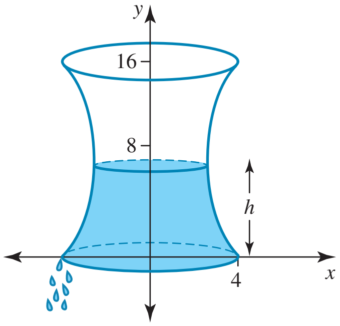
When $h = 0 \;\; t = \dfrac{-c}{B} = 6.86$ minutes

So it takes an extra 3.86 minutes to empty.

12 a $\dfrac{25x^2}{144} - \dfrac{(y-8)^2}{36} = 1$, units cm

$$\frac{25x^2}{144} = 1 + \frac{(y-8)^2}{36}$$

$$\frac{25x^2}{144} = \frac{36 + (y-8)^2}{36}$$



$$x^2 = \frac{4}{25}\left(36 + (y-8)^2\right)$$

$$V = \pi \int_0^h x^2\,dy = \frac{4\pi}{25}\int_0^h \left(36 + (y-8)^2\right) dy$$

$$\frac{dV}{dh} = \frac{4\pi}{25}\left(36 + (h-8)^2\right) \qquad \frac{dV}{dt} = -2\sqrt{h}$$

$$\frac{dt}{dh} = \frac{dt}{dV} \cdot \frac{dV}{dh} = \frac{-4\pi\left(36 + (h-8)^2\right)}{25 \times 2\sqrt{h}}$$

$$t = \frac{-4\pi}{25 \times 2}\int_{16}^{0} \frac{36 + (h-8)^2}{\sqrt{h}}\,dh$$

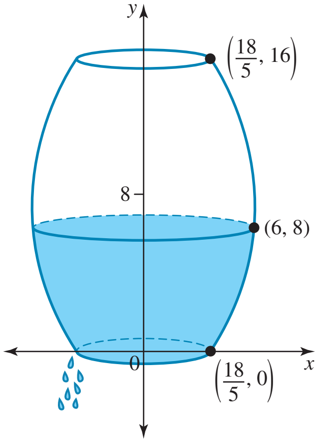
$$= 132.4 \text{ sec}$$

$$V_{max} = \frac{4\pi}{25}\int_0^{16}\left(36 + (y-8)^2\right) dy$$

$$= 461.1 \text{ cm}^3$$

b $\dfrac{x^2}{36} + \dfrac{(y-8)^2}{100} = 1$, units cm

When $x = \dfrac{18}{5} \;\; y = 0$



$y = 16 \;\; x = \dfrac{18}{5}$

$$\frac{x^2}{36} = 1 - \frac{(y-8)^2}{100} = \frac{100 - (y-8)^2}{100}$$

$$x^2 = \frac{36}{100}\left(100 - (y-8)^2\right)$$

$$V = \pi\int_0^h x^2 dy = \frac{36\pi}{100}\int_0^h \left(100 - (y-8)^2\right) dy$$

$$\frac{dV}{dh} = \frac{36\pi}{100}\left(100 - (h-8)^2\right) \qquad \frac{dV}{dt} = -2\sqrt{h}$$

$$\frac{dt}{dh} = \frac{dt}{dV} \cdot \frac{dV}{dh} = \frac{36\pi\left(100 - (h-8)^2\right)}{100 \times -2\sqrt{h}}$$

$$t = \frac{-36\pi}{200}\int_{16}^{0} \frac{100 - (h-8)^2}{\sqrt{h}}\,dh = \frac{12624\pi}{125}$$

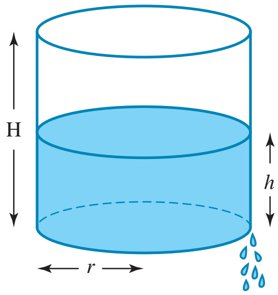
$$= 317.3 \text{ sec}$$

$$V_{\max} = \frac{36\pi}{100}\int_0^{16}\left(100 - (y-8)^2\right) dy = \frac{11328\,\pi}{25}$$

$$= 1423.5 \text{ cm}^3$$

13 a $V = \pi r^2 h$

$$\frac{dv}{dh} = \pi r^2$$



$$\frac{dV}{dt} = -k\sqrt{h}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt} = \frac{-k\sqrt{h}}{\pi r^2}$$

$$\frac{dh}{dt} = -A\sqrt{h}$$

$$\frac{dt}{dh} = -\frac{1}{A}\frac{1}{\sqrt{h}} = -\frac{1}{A}h^{-\frac{1}{2}}$$

$$t = -\frac{1}{A}\int h^{-\frac{1}{2}}\,dh$$

$$t = -\frac{2}{A}h^{\frac{1}{2}} + c_1$$

$$2\sqrt{h} = Bt + c$$

When $t = 0 \;\; h = H$

$\Rightarrow c = 2\sqrt{H}$

When $t = T \quad h = \dfrac{1}{2}H$

$$2\sqrt{\frac{H}{2}} = BT + c$$

$$\frac{2\sqrt{H}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = BT + c$$

$$\sqrt{2}\sqrt{H} = BT + 2\sqrt{H}$$

$$BT = \sqrt{2}\sqrt{H} - 2\sqrt{H}$$

$$B = \frac{\sqrt{H}\left(\sqrt{2} - 2\right)}{T}$$

When $h = 0 \Rightarrow t = \frac{-c}{B} = \frac{-2\sqrt{H}}{\sqrt{H}\frac{(\sqrt{2}-2)}{T}}$

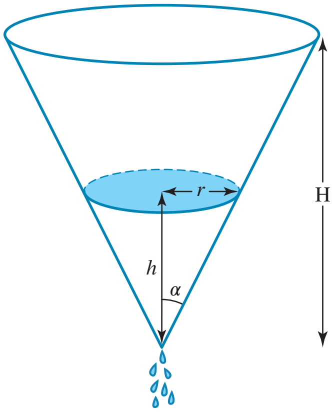
$$= \frac{2T}{2 - \sqrt{2}} \text{ shown}$$

b $V = \frac{1}{3}\pi r^2 h$

$\tan(\alpha) = \frac{r}{h}$

$r = h\tan(\alpha)$

$V = \frac{1}{3}\pi \tan^2(\alpha) . h^3$



$$\frac{dV}{dh} = \pi\tan^2(\alpha)\, h^2$$

$$\frac{dV}{dt} = -k\sqrt{h}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt} = \frac{-k\sqrt{h}}{\pi\tan^2(\alpha)\, h^2} = -Ah^{-\frac{3}{2}}$$

$$\frac{dt}{dh} = -\frac{1}{A}\, h^{\frac{3}{2}}$$

$$t = -\frac{1}{A}\int h^{\frac{3}{2}}\, dh$$

$$= -\frac{2}{5A}\, h^{\frac{5}{2}} + c_1$$

$$\frac{2}{5}h^{\frac{5}{2}} = Bt + c$$

$$t = 0 \;\; h = H \;\; \Rightarrow \;\; c = \frac{2}{5}H^{\frac{5}{2}}$$

$$t = T \; h = \frac{H}{2} \;\; \Rightarrow \;\; \frac{2}{5}\left(\frac{H}{2}\right)^{\frac{5}{2}} = BT + c$$

$$BT = \frac{2}{5}\left(\frac{H}{2}\right)^{\frac{5}{2}} - \frac{2}{5}(H)^{\frac{5}{2}} = \frac{2}{5}H^{\frac{5}{2}}\left(\left(\frac{1}{2}\right)^{\frac{5}{2}} - 1\right)$$

When $h = 0$ $\qquad t = -\frac{c}{B}$

$$= \frac{-\frac{2}{5}H^{\frac{5}{2}}}{\frac{2}{5T}H^{\frac{5}{2}}\left(\left(\frac{1}{2}\right)^{\frac{5}{2}} - 1\right)}$$

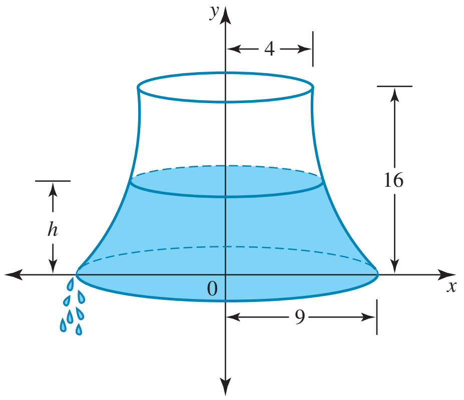
$t = \frac{T}{1 - \sqrt{2^{-5}}}$ shown

14 **a** $A\,(9,0)$ B (4, 16), units cm

$y = \frac{a}{x} + b$

(1) $0 = \frac{a}{9} + b$

(2) $16 = \frac{a}{4} + b$



$$a = \frac{576}{5} \qquad b = -\frac{64}{5}$$

$$y = \frac{576}{5x} - \frac{64}{5}$$

$$y + \frac{64}{5} = \frac{576}{5x} = \frac{5y + 64}{5}$$

$$x = \frac{576}{5y + 64}$$

$$V = \pi\int_0^h x^2\, dy = \pi\int_0^h \frac{576^2}{(5y + 64)^2}\, dy$$

$$\frac{dV}{dh} = \frac{331776\,\pi}{(5h + 64)^2} \qquad \frac{dV}{dt} = -2\sqrt{h}$$

$$\frac{dt}{dh} = \frac{dt}{dV} \cdot \frac{dV}{dh} = \frac{-331776\pi}{2\sqrt{h}\,(5h + 64)^2}$$

$$t = -165888\,\pi\int_{16}^{0} \frac{1}{\sqrt{h}\,(5h + 64)^2}\, dh$$

$$= 609.1 \text{ sec}$$

$$V_{max} = 576^2\pi\int_0^{16} \frac{1}{(5y + 64)^2}\, dy = 576\pi$$

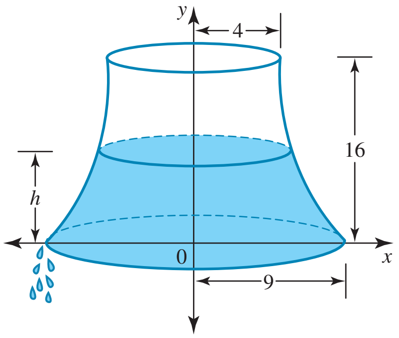
$$= 1809.6 \text{ cm}^3$$

b $A\,(9,0)$ B (4, 16), units cm

$y = \frac{a}{x^2} + b$

(1) $0 = \frac{a}{81} + b$

(2) $16 = \frac{a}{16} + b$



$$a = \frac{20736}{65} \qquad b = -\frac{256}{65}$$

$$y = \frac{20736}{65x^2} - \frac{256}{65}$$

$$y + \frac{256}{65} = \frac{20736}{65x^2} = \frac{65y + 256}{65}$$

$$x^2 = \frac{20736}{65y + 256}$$

$$V = \pi\int_0^h x^2\,dy = \pi\int_0^h \frac{20736}{65y + 256}\,dy$$

$$\frac{dV}{dh} = \frac{20736\,\pi}{65h + 256} \qquad \frac{dV}{dt} = -2\sqrt{h}$$

$$\frac{dt}{dh} = \frac{dt}{dV}\cdot\frac{dV}{dh} = \frac{-20736\,\pi}{2\sqrt{h}\,(65h + 256)}$$

$$t = -10368\,\pi\int_{16}^{0} \frac{1}{\sqrt{h}\,(65h + 256)}\,dh$$

$$= 560.7 \text{ sec}$$

$$V_{\max} = \pi\int_0^{16} \frac{20736}{65y + 256}\,dy = \frac{82944\pi}{65}\log_e\left(\frac{3}{2}\right)$$

$$= 1625.5 \text{ cm}^3$$

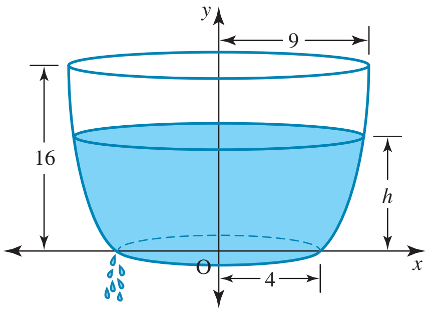
15 a $A\,(4, 0)$ B $(9, 16)$, units cm

$y = ax^3 + b$

(1) $0 = 64a + b$

(2) $16 = 729a + b$

$665a = 16$



$$a = \frac{16}{665} \qquad b = -\frac{1024}{665}$$

$$y = \frac{16}{665}x^3 - \frac{1024}{665} = \frac{16}{665}\left(x^3 - 64\right)$$

$$\frac{665y}{16} = x^3 - 64$$

$$x^3 = \frac{665y}{16} + 64 = \frac{1}{16}(665y + 1024)$$

$$x^2 = \left[\frac{1}{16}(665y + 1024)\right]^{\frac{2}{3}}$$

$$V = \pi\int_0^h x^2\,dy = \pi\int_0^h \left(\frac{1}{16}(665y + 1024)\right)^{\frac{2}{3}} dy$$

$$\frac{dV}{dh} = \pi\left(\frac{665h + 1024}{16}\right)^{\frac{2}{3}} \qquad \frac{dV}{dt} = -2\sqrt{h}$$

$$\frac{dt}{dh} = \frac{dt}{dV}\cdot\frac{dV}{dh} = \frac{-\pi}{2\sqrt{h}}\left(\frac{665h + 1024}{16}\right)^{\frac{2}{3}}$$

$$t = -\frac{\pi}{2}\int_{16}^{0}\left(\frac{665h + 1024}{16}\right)^{\frac{2}{3}} \times \frac{1}{\sqrt{h}}\,dh$$

$$= 515.3 \text{ sec}$$

$$V_{\max} = \pi\int_0^{16}\left(\frac{665y + 1024}{16}\right)^{\frac{2}{3}} dy$$

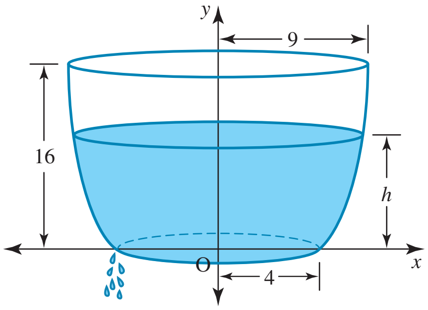
$$= 2631.6 \text{ cm}^3$$

b $A\,(4, 0)$ B $(9, 16)$, units cm

$y = ax^4 + b$

(1) $0 = 256a + b$

(2) $16 = 6561\,a + b$



$$a = \frac{16}{6305} \qquad b = -\frac{4096}{6305}$$

$$y = \frac{16}{6305}\left(x^4 - 256\right)$$

$$\frac{6305y}{16} = x^4 - 256$$

$$x^4 = \frac{6305y}{16} + 256 = \frac{6305y + 4096}{16}$$

$$x^2 = \frac{\sqrt{6305y + 4096}}{4}$$

$$V = \pi\int_0^h x^2\,dy = \frac{\pi}{4}\int_0^h \sqrt{6305y + 4096}\,dy$$

$$\frac{dV}{dh} = \frac{\pi}{4}\sqrt{6305h + 4096} \qquad \frac{dV}{dt} = -2\sqrt{h}$$

$$\frac{dt}{dh} = \frac{dt}{dV}\cdot\frac{dV}{dh} = \frac{-\pi}{8}\sqrt{\frac{6305h + 4096}{h}}$$

$$t = -\frac{\pi}{8}\int_{16}^{0}\sqrt{\frac{6305h + 4096}{h}}\,dh$$

$$= 555.6 \text{ sec}$$

$$V_{\max} = \frac{\pi}{4}\int_0^{16}\sqrt{6305y + 4096}\,dy$$

$$= 2802.8 \text{ cm}^3$$

16 Hot water tank units litres, time hours

Inflow $f(t) = 12\sqrt{t}\sin^2\left(\frac{\pi t}{4}\right) \quad 0 \le t \le T$

Outflow $g(t) = 5\sqrt{t} \quad 0 \le t \le 4$

Initial volume 100 L

a When $t = 3$ $f(t) - g(t) > 0$

Increasing

b When $t = 4$ $f(t) = g(t) < 0$

Decreasing

c Solving $f(t) = g(t)$ for $0 \le t \le 4$

$\Rightarrow t = 0.89, 3.11$ hours

d $100 + \int_0^4 (f(t) - g(t))\,dt = 106.645\text{L}$

e $160 - 106.645 = 53.35$

Solving $\int_4^T f(t)\,dt = 53.35$ with $4 < T < 8$

$\Rightarrow T = 7.04$ hours

9.7 Exam questions

1 $y = x^3 - 8, \quad x^3 = y + 8$

a i $V = \pi \int_0^H (y+8)^{\frac{2}{3}}\,dy$ [1 mark]

ii $V(H) = \frac{3\pi}{5}\left[(H+8)^{\frac{5}{3}} - 32\right]$ [1 mark]

b i $\frac{dV}{dh} = \pi(h+8)^{\frac{2}{3}}, \quad \frac{dV}{dt} = -4\sqrt{h}$ [1 mark]

$\frac{dh}{dt} = \frac{dh}{dV}\frac{dV}{dt} = \frac{-4\sqrt{h}}{\pi(h+8)^{\frac{2}{3}}}$ [1 mark]

ii $\frac{d^2h}{dt^2} = 0 \quad \Rightarrow \quad h = 24, \left.\frac{dh}{dt}\right|_{h=24} = -0.62$ [1 mark]

Decreases at 0.62 cm/min [1 mark]

iii Maximum rate when $h = 50$

$\frac{dV}{dt} = 4\sqrt{50} = 20\sqrt{2}$ cm³/min [1 mark]

c $\frac{dh}{dt} = \frac{40\sqrt{2} - 4\sqrt{h}}{\pi(h+8)^{\frac{2}{3}}}$ [1 mark]

$\frac{dt}{dh} = \frac{\pi(h+8)^{\frac{2}{3}}}{4\left(10\sqrt{2} - \sqrt{h}\right)}$ [1 mark]

$t = \int_{25}^{50} \frac{\pi(h+8)^{\frac{2}{3}}}{4\left(10\sqrt{2} - \sqrt{h}\right)}\,dh = 31.4$ [1 mark]

2 a $y = \frac{1}{2}\sqrt{4x^2 - 1}$

$V = \pi \int_a^{b^2} dy$

$4y^2 = 4x^2 - 1 \Rightarrow x^2 = \frac{1}{4}\left(1 + 4y^2\right)$ [1 mark]

$V = \frac{\pi}{4}\int_0^h \left(1 + 4y^2\right)dy$

$V = \frac{\pi}{4}\left[y + \frac{4y^3}{3}\right]_0^h = \frac{\pi}{4}\left(h + \frac{4h^3}{3}\right)$ [1 mark]

***VCAA Examination Report* note:**

Approximately half of the students were able to either set up an appropriate definite integral or find an anti–derivative and attempt to evaluate the constant of integration. Of these, many did not explicitly show that the first part of their response yielded the required volume.

b When the fountain is full, $h = \frac{\sqrt{3}}{2}$

so $V\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi\sqrt{3}}{4}$. [1 mark]

When the fountain is half-full, $V = \frac{\pi\sqrt{3}}{8}$.

Solving $V = \frac{\pi\sqrt{3}}{8} = \frac{\pi}{4}\left(h + \frac{4h^3}{3}\right)$

gives $h = 0.59$ m [1 mark]

***VCAA Examination Report* note:**

While the approach above was the most common, other correct approaches were used. A common error was to fail to halve the volume.

c i Inflow $0.04\text{m}^3/\text{s}$, outflow $0.05\sqrt{h}\,\text{m}^3/\text{s}$

$\frac{dV}{dt} = \text{inflow} - \text{outflow} = 0.04 - 0.05\sqrt{h}$

$= \frac{1}{100}\left(4 - 5\sqrt{h}\right)$ [1 mark]

Since $V = \frac{\pi}{4}\left(h + \frac{4h^3}{3}\right)$,

$\frac{dV}{dh} = \frac{\pi}{4}\left(1 + 4h^2\right)$

$\frac{dh}{dt} = \frac{dh}{dV}\cdot\frac{dV}{dt} = \frac{\frac{1}{100}\left(4 - 5\sqrt{h}\right)}{\frac{\pi}{4}\left(4h^2 + 1\right)}$

$= \frac{4 - 5\sqrt{h}}{25\pi\left(4h^2 + 1\right)}$ [1 mark]

***VCAA Examination Report* note:**

Most students were able to correctly state $\frac{dV}{dt}$ and find $\frac{dV}{dh}$ and then use this to find $\frac{dh}{dV}$ before proceeding.

A few students did not understand the importance of brackets when multiplying the derivative expressions. Many students moved directly from the product of the derivatives to the required expression, without explicitly showing that their product led to the final (given) answer.

ii $\left.\frac{dh}{dt}\right|_{h=0.25} = \frac{4 - 5\sqrt{0.25}}{25\pi\left(4(0.25)^2 + 1\right)}$

$= 0.0153$ m / s [1 mark]

***VCAA Examination Report* note:**

The majority of students were able to use the supplied derivative to find the required rate for the given depth. Some students did not give the answer in the required decimal form.

d Express the time taken for the depth to reach 0.25 m as a definite integral and evaluate this integral correct to the nearest tenth of a second.

$\frac{dt}{dh} = \frac{25\pi\left(4h^2 + 1\right)}{4 - 5\sqrt{h}}$

$$t = \int_0^{0.25} \left(\frac{25\pi\left(4h^2 + 1\right)}{4 - 5\sqrt{h}} \right) dh \qquad \text{[1 mark]}$$

$$= 9.8 \text{ seconds} \qquad \text{[1 mark]}$$

***VCAA Examination Report* note:**

Most students were able to set up a correct definite integral. Transcription errors were occasionally present in the integrand.

e The water level stabilises when $4 - 5\sqrt{h} = 0$

$$\sqrt{h} = \frac{4}{5}$$

$$h = \frac{16}{25} \qquad \text{[1 mark]}$$

Therefore, the height from the top is

$$\frac{\sqrt{3}}{2} - \frac{16}{25} = 0.23 \text{ m}. \qquad \text{[1 mark]}$$

***VCAA Examination Report* note:**

Most students who attempted this question understood that they need to solve $\frac{dh}{dt} = 0$ to find the limiting water level. Many students who correctly found h did not subtract their value from the height of the top of the fountain.

3 a $\frac{r}{h} = \frac{0.5}{1} = \frac{1}{2} \Rightarrow r = \frac{h}{2}$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h$$

$$V = \frac{\pi}{12}h^3$$

Award 1 mark for the correct expression.

b $\frac{dV}{dt} = 0.02\pi - 0.01\pi\sqrt{h} = \frac{\pi}{100}\left(2 - \sqrt{h}\right)$

$$\frac{dV}{dh} = \frac{\pi h^2}{4}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$$

$$= \frac{4}{\pi h^2} \times \frac{\pi}{100}\left(2 - \sqrt{h}\right)$$

$$= \frac{2 - \sqrt{h}}{25h^2}$$

$$\left.\frac{dh}{dt}\right|_{h=0.25} = \frac{2 - \sqrt{0.25}}{25(0.25)^2}$$

$$= \frac{24}{25}$$

$$= 0.96$$

Award 1 mark for the correct rates.

Award 1 mark for the correct chain rule.

Award 1 mark for the correct rate for h.

Award 1 mark for the correct final rate.

c $\frac{dh}{dt} = \frac{2 - \sqrt{h}}{25h^2}$

Invert: $\frac{dt}{dh} = \frac{25^2}{2 - \sqrt{h}}$

$$t_{\text{full}} = \int_0^1 \frac{25h^2}{2 - \sqrt{h}} dh$$

$$= 7.3688$$

$$= 7.4 \text{ minutes}$$

Award 1 mark for inverting and setting up the correct definite integral.

Award 1 mark for solving using CAS for the correct time.

d $\frac{dV}{dt} = 0.05\pi = \frac{\pi}{20} \text{m}^3/\text{min}$

$$V = \frac{\pi}{48}\left(x^3 + 6x^2 + 12x\right)$$

$$\frac{dV}{dx} = \frac{\pi}{48}\left(3x^2 + 12x + 12\right)$$

$$= \frac{\pi}{16}(x + 2)^2$$

$$\frac{dx}{dt} = \frac{dx}{dV} \cdot \frac{dV}{dt}$$

$$\frac{dx}{dt} = \frac{16}{\pi(x + 2)^2} \times \frac{\pi}{20} = \frac{4}{5(x + 2)^2}$$

$$\frac{dt}{dx} = \frac{5(x + 2)^2}{4}$$

$$t = \frac{5}{4}\int (x + 2)^2 dx$$

$$t = \frac{5}{12}(x + 2)^3 + c$$

$$t = 0,\ x = 0 \quad \Rightarrow\ 0 = \frac{5}{12} \times 8 + c$$

$$c = -\frac{5}{12} \times 8$$

$$t = \frac{5}{12}(x + 2)^3 - \frac{5}{12} \times 8$$

$$t = \frac{5}{12}\left[(x + 2)^3 - 8\right] \quad \text{when } x = 1,\ t = \frac{95}{12}$$

$$\frac{12t}{5} = (x + 2)^3 - 8$$

$$(x + 2)^3 = \frac{12t}{5} + 8$$

$$(x + 2)^3 = \frac{4(3t + 10)}{5}$$

$$x(t) = \sqrt[3]{\frac{4(3t + 10)}{5}} - 2$$

$$= 2\sqrt[3]{(0.3t + 1)} - 2 \quad \text{for } 0 \le t \le \frac{95}{12}$$

Award 1 mark for finding the correct rate.

Award 1 mark for the correct chain rule.

Award 1 mark for inverting and solving the differential equation.

Award 1 mark for the constant of integration.

Award 1 mark for obtaining the correct expression for x

9.8 Review

9.8 Exercise

Technology free: short answer

1 a $\int x\sin(4x)\, dx$

$$u = x \qquad \frac{dv}{dx} = \sin(4x)$$

$$\frac{du}{dx} = 1 \qquad v = -\frac{1}{4}\cos(4x)$$

$$\int x\sin(4x)\,dx = \frac{-x}{4}\cos(4x) + \int \frac{1}{4}\cos(4x)\,dx$$
$$= \frac{-x}{4}\cos(4x) + \frac{1}{16}\sin(4x) + c$$
$$= \frac{1}{16}(\sin(4x) - 4x\cos(4x)) + \text{c}$$

b $\int \cos^{-1}(5x)\,dx$

$$= \int 1\cdot\cos^{-1}(5x)\,dx$$

$$u = \cos^{-1}(5x) \qquad \frac{dv}{dx} = 1$$
$$\frac{du}{dx} = \frac{-5}{\sqrt{1-25x^2}} \qquad v = x$$

$$\int \cos^{-1}(5x)\,dx = x\cos^{-1}(5x)$$

$$\text{let } t = 1-25x^2$$
$$\frac{dt}{dx} = -50x$$
$$5\int \frac{x}{\sqrt{1-25x^2}}\,dx = -\frac{1}{10}\int t^{-\frac{1}{2}}\,dt$$
$$= -\frac{1}{5}t^{\frac{1}{2}} + c$$

$$\int \cos^{-1}(5x)\,dx = x\cos^{-1}(5x) - \frac{1}{5}\sqrt{1-25x^2} + c$$

c $\int \sin^{-1}\left(\frac{2x}{3}\right)\,dx$

$$= \int 1\cdot\sin^{-1}\left(\frac{2x}{3}\right)\,dx$$

$$u = \sin^{-1}\left(\frac{2x}{3}\right) \qquad \frac{dv}{dx} = 1$$
$$\frac{du}{dx} = \frac{2}{\sqrt{9-4x^2}} \qquad v = x$$

$$\int \sin^{-1}\left(\frac{2x}{3}\right)\,dx = x\sin^{-1}\left(\frac{2x}{3}\right)$$

$$\text{Let } t = 9-4x^2$$
$$\frac{dt}{dx} = -8x$$
$$-\int \frac{2x}{\sqrt{9-4x^2}}\,dx = +\frac{1}{4}\int t^{-\frac{1}{2}}\,dt$$
$$= +\frac{1}{2}t^{\frac{1}{2}} + c$$

$$\int \sin^{-1}\left(\frac{2x}{3}\right)\,dx = x\sin^{-1}\left(\frac{2x}{3}\right) + \frac{1}{2}\sqrt{9-4x^2} + c$$

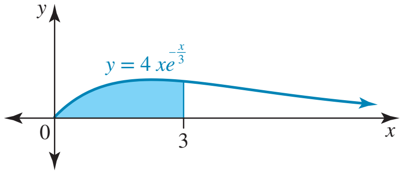
d $\int x^2\log_e(3x)\,dx$

$$u = \log_e(3x) \qquad \frac{dv}{dx} = x^2$$
$$\frac{du}{dx} = \frac{1}{x} \qquad v = \frac{1}{3}x^3$$

$$\int x^2\log_e(3x)\,dx = \frac{1}{3}x^3\log_e(3x)$$
$$-\int \frac{1}{3}x^3\cdot\frac{1}{x}\,dx = -\frac{1}{3}\int x^2\,dx$$
$$= -\frac{1}{9}x^3 + c$$

$$= \frac{1}{9}x^3\left(3\log_e(3x) - 1\right) + c$$

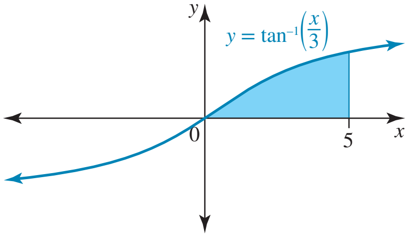
2 a $A = \int_0^3 xe^{-\frac{x}{3}}\,dx$



$$u = x \qquad \frac{dv}{dx} = e^{-\frac{x}{3}}$$
$$\frac{du}{dx} = 1 \qquad v = -3e^{-\frac{x}{3}}$$

$$A = \left[-3xe^{-\frac{x}{3}}\right]_0^3 + 3\int_0^3 e^{-\frac{x}{3}}\,dx$$
$$= \left[-3xe^{-\frac{x}{3}}\right]_0^3 - \left[9e^{-\frac{x}{3}}\right]_0^3$$
$$= -\left[(3x+9)e^{-\frac{x}{3}}\right]_0^3$$
$$= -18e^{-1} + 9e^0$$
$$= 9 - \frac{18}{e}\text{ units}^2$$

b $A = \int_0^5 1\tan^{-1}\left(\frac{x}{5}\right)\,dx$



$$u = \tan^{-1}\left(\frac{x}{5}\right) \qquad \frac{dv}{dx} = 1$$
$$\frac{du}{dx} = \frac{5}{25+x} \qquad v = x$$

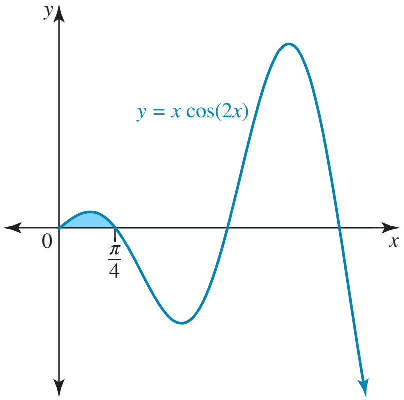
$$A = \left[x\tan^{-1}\left(\frac{x}{5}\right)\right]_0^5 - \int_0^5 \frac{5x}{25+x^2}\,dx$$
$$= \left[x\tan^{-1}\left(\frac{x}{5}\right)\right]_0^5 - \left[\frac{5}{2}\log_e\left(25+x^2\right)\right]_0^5$$
$$= \left[x\tan^{-1}\left(\frac{x}{5}\right) - \frac{5}{2}\log_e\left(25+x^2\right)\right]_0^5$$
$$= 5\tan^{-1}(1) - \frac{5}{2}\log_e(50) - \left(0 - \frac{5}{2}\log_e(25)\right)$$
$$= 5\times\frac{\pi}{4} + \frac{5}{2}\log_e\left(\frac{1}{2}\right)$$
$$= \frac{5\pi}{4} - \frac{5}{2}\log_e(2)$$
$$= \frac{5}{4}\left(\pi - 2\log_e(2)\right)\text{ units}^2$$

c $y = x\cos(2x)$

Crosses x-axis $y = 0$

$x\cos(2x) = 0$

$\cos(2x) = 0$



$$2x = \frac{\pi}{2}, \ \frac{3\pi}{2} \ \ldots$$

$$x = \frac{\pi}{4}, \ \frac{3\pi}{4} \ \ldots$$

$$A = \int_0^{\frac{\pi}{4}} x\cos(2x)\,dx$$

$$u = x \qquad \frac{dv}{dx} = \cos(2x)$$

$$\frac{du}{dx} = 1 \qquad v = \frac{1}{2}\sin(2x)$$

$$A = \left[\frac{1}{2}x\sin(2x)\right]_0^{\frac{\pi}{4}} - \frac{1}{2}\int_0^{\frac{\pi}{4}} \sin(2x)\,dx$$

$$= \left[\frac{1}{2}x\sin(2x)\right]_0^{\frac{\pi}{4}} + \left[\frac{1}{4}\cos(2x)\right]_0^{\frac{\pi}{4}}$$

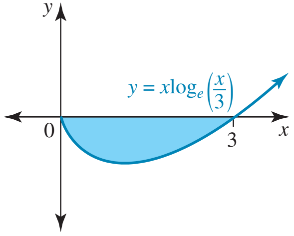
$$= \left[\frac{1}{4}\left(\cos(2x) + 2x\sin(2x)\right)\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{4}\cos\left(\frac{\pi}{2}\right) + \frac{\pi}{8}\sin\left(\frac{\pi}{2}\right) - \frac{1}{4}\cos(0) - 0$$

$$= \frac{\pi}{8} - \frac{1}{4}$$

$$= \frac{1}{8}(\pi - 2)\text{ units}^2$$

d $y = x\log_e\left(\frac{x}{3}\right)$



$$A = \int_0^3 x\log_e\left(\frac{x}{3}\right)\,dx$$

$$u = \log_e\left(\frac{x}{3}\right) \qquad \frac{dv}{dx} = x$$

$$\frac{du}{dx} = \frac{1}{x} \qquad v = \frac{1}{2}x^2$$

$$A = \left[\frac{1}{2}x^2\log_e\left(\frac{x}{3}\right)\right]_0^3 - \int_0^3 \frac{1}{2}x^2 \cdot \frac{1}{x}\,dx$$

$$= \left[\frac{1}{2}x^2\log_e\left(\frac{x}{3}\right)\right]_0^3 - \int_0^3 \frac{1}{2}x\,dx$$

$$= \left[\frac{1}{2}x^2\log_e\left(\frac{x}{3}\right)\right]_0^3 - \left[\frac{1}{4}x^2\right]_0^3$$

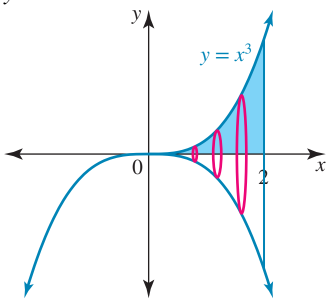
$$= \left[-\frac{1}{4}x^2\left(1 - 2\log_e\left(\frac{x}{3}\right)\right)\right]_0^3$$

$$= -\frac{1}{4} \times 9\left(1 - 2\log_e(1)\right) - 0$$

$$= -\frac{9}{4} \quad \text{but} \quad A < 0$$

$$\text{So } A = \frac{9}{4}\text{ units}^2$$

3 a $y = x^3$



$$V = \pi\int_0^2 x^6\,dx$$

$$= \pi\left[\frac{1}{7}x^7\right]_0^2$$

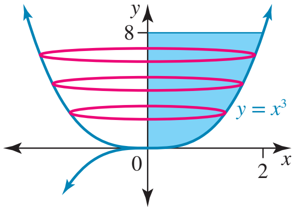
$$= \pi\left(\frac{1}{7}2^7 - 0\right)$$

$$= \frac{128\pi}{7}\text{ units}^3$$

b $y = x^3$

$y^{\frac{1}{3}} = x$

$x^2 = y^{\frac{2}{3}}$



$$V = \pi\int_0^8 y^{\frac{2}{3}}\,dy$$

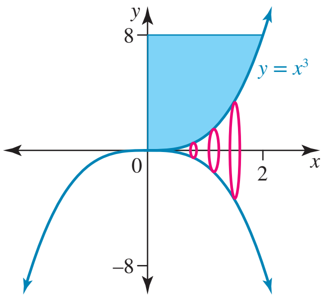
$$= \pi\left[\frac{3}{5}y^{\frac{5}{3}}\right]_0^8$$

$$= \frac{3\pi}{5}\left(8^{\frac{5}{3}} - 0\right)$$

$$= \frac{96\pi}{5}\text{ units}^3$$

c $y = x^3$

$y = 8$



$r_2 = 8 \quad r_1 = y$

$$V = \pi \int_a^b \left(r_2{}^2 - r_1{}^2\right) dx$$

$$= \pi \int_0^2 \left(8^2 - x^6\right) dx$$

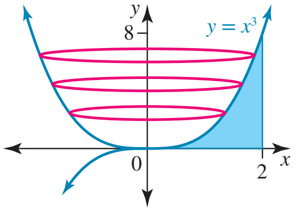
$$= \pi \left[64x - \frac{1}{7}x^7\right]_0^2$$

$$= \pi \left(128 - \frac{128}{7} - 0\right)$$

$$= \pi \left(128\left(1 - \frac{1}{7}\right)\right)$$

$$= \frac{768\,\pi}{7} \text{ units}^3$$

d $r_2 = 2 \quad r_1 = x$



$$V = \pi \int_a^b \left(r_2^2 - r_1^2\right) dy$$

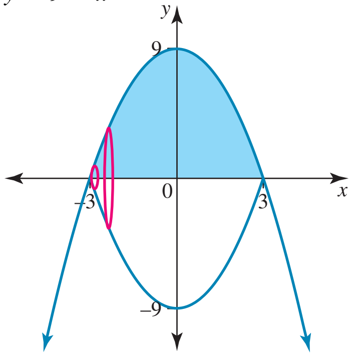
$$= \pi \int_0^8 \left(4 - y^{\frac{2}{3}}\right) dy$$

$$= \pi \left[4y - \frac{3}{5}y^{\frac{5}{3}}\right]_0^8$$

$$= \pi \left[4 \times 8 - \frac{3}{5} \times 8^{\frac{5}{3}} - 0\right]$$

$$= \frac{64\,\pi}{5} \text{ units}^3$$

4 a $y = 9 - x^2$



$$V = \pi \int_a^b y^2 dx$$

$$V = \pi \int_{-3}^{3} \left(9 - x^2\right)^2 dx$$

$$= 2\pi \int_0^3 \left(81 - 18x^2 + x^4\right) dx$$

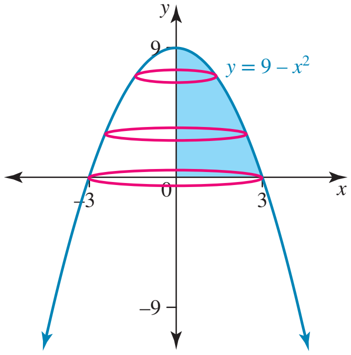
$$= 2\pi \left[81x - 6x^3 + \frac{1}{5}x^5\right]_0^3$$

$$= 2\pi \left[81 \times 3 - 6 \times 27 + \frac{1}{5} \times 243 - 0\right]$$

$$= \frac{1296\,\pi}{5}$$

b $y = 9 - x^2$

$x^2 = 9 - y$



$$V = \pi \int_a^b x^2 dy$$

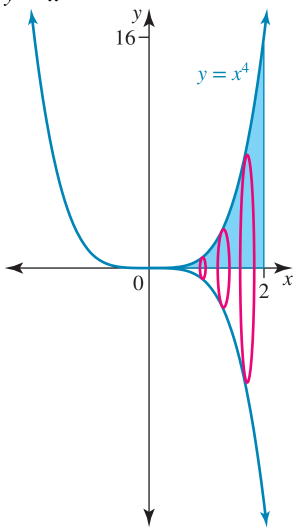
$$= \pi \int_0^9 (9 - y)\, dy$$

$$= \pi \left[9y - \frac{1}{2}y^2\right]_0^9$$

$$= \pi \left(81 - \frac{81}{2} - 0\right)$$

$$= \frac{81\,\pi}{2} \text{ units}^3$$

5 $y = x^4$



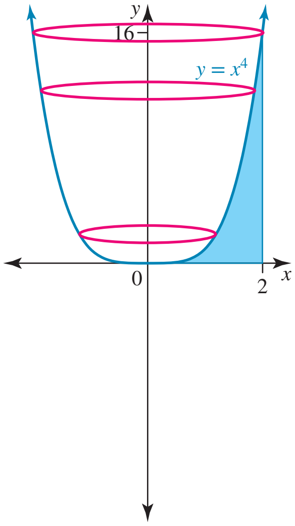
a $V = \pi \int y^2\,dx$

$= \pi \int_0^2 x^8\,dx$

$= \frac{\pi}{9}\left[x^9\right]_0^2$

$= \frac{512\pi}{9}$ units^3

b



$r_2 = 2 \quad r_1 = x$

$V = \pi \int \left(r_2^2 - r_1^2\right)\,dy$

$= \pi \int_0^{16} \left(4 - x^2\right)\,dy$

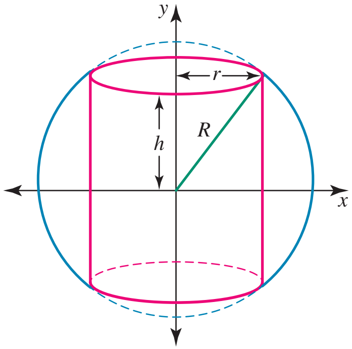
$= \pi \int_0^{16} \left(4 - \sqrt{y}\right)\,dy$

$= \pi\left[4y - \frac{2}{3}y^{\frac{3}{2}}\right]_0^{16}$

$= \pi\left[4 \times 16 - \frac{2}{3} \times 64 - 0\right] = 64\pi\left(1 - \frac{2}{3}\right)$

$= \frac{64\pi}{3}$ units^3

6



$x^2 + y^2 = R^2$
About y-axis
$h^2 + r^2 = R^2$
$h^2 = R^2 - r^2$
$h = \sqrt{R^2 - r^2}$

$V = \pi \int_{-h}^{h} \left(x^2 - r^2\right)\,dy$

$= 2\pi \int_0^h \left(R^2 - y^2 - r^2\right)\,dy$

$= 2\pi \int_0^h \left(\left(R^2 - r^2\right) - y^2\right)\,dy$

$= 2\pi\left[\left(R^2 - r^2\right)y - \frac{1}{3}y^3\right]_0^{\sqrt{R^2 - r^2}}$

$= 2\pi\left[\left(R^2 - r^2\right)\sqrt{R^2 - r^2} - \frac{1}{3}\left(\sqrt{R^2 - r^2}\right)^3\right]$

$= 2\pi\left[\left(1 - \frac{1}{3}\right)\left(R^2 - r^2\right)^{\frac{3}{2}}\right]$

$= \frac{4\pi}{3}\left[R^2\left(1 - \frac{r^2}{R^2}\right)\right]^{\frac{3}{2}}$

$= \frac{4\pi R^3}{3}\left(1 - \frac{r^2}{R^2}\right)^{\frac{3}{2}}$

7 **a** $y = \frac{x^3}{3} + \frac{1}{4x} = \frac{x^3}{3} + \frac{1}{4}x^{-1}$

$\frac{dy}{dx} = x^2 - \frac{1}{4}x^{-2} = x^2 - \frac{1}{4x^2}$

$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

$= \int_1^3 \sqrt{1 + \left(x^2 - \frac{1}{4x^2}\right)^2}\,dx$

$= \int_1^3 \sqrt{1 + \left(x^4 - 2 \times x^2 \times \frac{1}{4x^2} + \frac{1}{16x^4}\right)}\,dx$

$= \int_1^3 \sqrt{1 + x^4 - \frac{1}{2} + \frac{1}{16x^4}}\,dx$

$= \int_1^3 \sqrt{x^4 + \frac{1}{2} + \frac{1}{16x^4}}\,dx$

$= \int_1^3 \sqrt{\left(x^2 + \frac{1}{4x^2}\right)^2}\,dx$

$= \int_1^3 \left(x^2 + \frac{1}{4}x^{-2}\right)\,dx$

$= \left[\frac{1}{3}x^3 - \frac{1}{4}x^{-1}\right]_1^3$

$= \left(\frac{1}{3} \times 9 \times 3 - \frac{1}{12}\right) - \left(\frac{1}{3} - \frac{1}{4}\right)$

$= \frac{53}{6}$

b $y = \frac{2x^4 + 6}{12x} = \frac{x^3}{6} + \frac{1}{2x} = \frac{x^3}{6} + \frac{1}{2}x^{-1}$

$\frac{dy}{dx} = \frac{x^2}{2} - \frac{1}{2}x^{-2} = \frac{x^2}{2} - \frac{1}{2x^2}$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\left(\frac{x^2}{2}-\frac{1}{2x^2}\right)^2$$

$$= 1+\frac{x^4}{4}-2\times\frac{x^2}{2}\times\frac{1}{2x^2}+\frac{1}{4x^4}$$

$$= 1+\frac{x^4}{4}-\frac{1}{2}+\frac{1}{4x^4}$$

$$= \frac{x^4}{4}+\frac{1}{2}+\frac{1}{4x^4}$$

$$= \left(\frac{x^2}{2}+\frac{1}{2x^2}\right)^2$$

$$S=\int_1^5\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

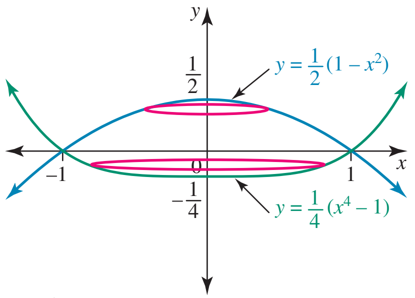
$$=\int_1^5\left(\frac{x^2}{2}+\frac{1}{2}x^{-2}\right)dx$$

$$=\left[\frac{x^3}{6}-\frac{1}{2}x^{-1}\right]_1^5$$

$$=\left(\frac{5^3}{6}-\frac{1}{10}\right)-\left(\frac{1}{6}-\frac{1}{2}\right)$$

$$=\frac{316}{15}$$

8



$$y=\frac{1}{2}\left(1-x^2\right)$$

$$y=\frac{1}{4}\left(x^4-1\right)$$

Top: $V=\pi\int x^2 dy \quad 2y=1-x^2$

$x^2=1-2y$

$$V=\pi\int_0^{\frac{1}{2}}(1-2y)\,dy$$

$$=\pi\left[y-y^2\right]_0^{\frac{1}{2}}$$

$$=\pi\left[\left(\frac{1}{2}-\frac{1}{4}\right)-(0)\right]$$

$$=\frac{\pi}{4}$$

Bottom: $V=\pi\int_{-\frac{1}{4}}^{0} x^2 dy$ $\qquad 4y=x^4-1$

$x^4=1+4y$

$x^2=\sqrt{4y+1}$

$$V=\pi\int_{-\frac{1}{4}}^{0}\sqrt{4y+1}\,dy$$

$$=\pi\left[\frac{2}{3}\frac{1}{4}(4y+1)^{\frac{3}{2}}\right]_{-\frac{1}{4}}^{0}$$

$$=\pi\left[\frac{1}{6}\times 1^{\frac{3}{2}}-\frac{1}{6}(0)\right]$$

$$=\frac{\pi}{6}$$

Total volume $\frac{\pi}{4}+\frac{\pi}{6}=\frac{5\pi}{12}$ units^3

Technology active: multiple choice

9 $\int x^2\cos\left(\frac{x}{2}\right)dx$

$u=x^2 \quad \frac{dv}{dx}=\cos\left(\frac{x}{2}\right)$

$\frac{du}{dx}=2x \quad v=\int\cos\left(\frac{x}{2}\right)dx=2\sin\left(\frac{x}{2}\right)$

$\int u\frac{dv}{dx}\,dx=uv-\int v\frac{du}{dx}\,dx$

$\int x^2\cos\left(\frac{x}{2}\right)dx=2x^2\sin\left(\frac{x}{2}\right)-4\int x\sin\left(\frac{x}{2}\right)dx$

The correct answer is **A**.

10 $r=5, \qquad V=\pi r^2h=25\pi h$

$\frac{dV}{dt}=-5\sqrt{h}, \quad \frac{dV}{dh}=25\pi$

$\frac{dt}{dh}=\frac{dt}{dV}\frac{dV}{dh}=\frac{25\pi}{-5\sqrt{h}}=-\frac{5\pi}{\sqrt{h}}$

$t=-5\pi\int_9^0\frac{1}{\sqrt{h}}\,dh=5\pi\int_0^9\frac{1}{\sqrt{h}}\,dh$

The correct answer is **D**.

11 $y=\sqrt{x-2}$

$V=\pi\int_2^6(x-2)\,dx$

$V=\pi\left[\frac{1}{2}x^2-2x\right]_2^6$

$V=\pi\left[(18-12)-(2-4)\right]$

$V=8\pi$

The correct answer is **B**.

12 $y=\log_e(x+1)$

$x+1=e^y$

$x=(e^y-1)$

$V=\pi\int_0^{\log_e(3)}\left(e^y-1\right)^2dy\approx 3.45$

The correct answer is **C**.

13 $y = 2\sin(2x)$

$$V = \pi\int_0^{\frac{\pi}{4}} \left(4 - 4\sin^2(2x)\right) dx$$

$$V = 4\pi\int_0^{\frac{\pi}{4}} \left(1 - \sin^2(2x)\right) dx$$

$$V = 4\pi\int_0^{\frac{\pi}{4}} \cos^2(2x)\, dx$$

The correct answer is **A**.

14 $y = \log_e(3x)$

$$\frac{dy}{dx} = \frac{1}{x}$$

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$s = \int_1^2 \sqrt{1 + \frac{1}{x^2}}\, dx = \int_1^2 \sqrt{\frac{x^2+1}{x^2}}\, dx$$

$$s = \int_1^2 \frac{\sqrt{x^2+1}}{x}\, dx$$

The correct answer is **B**.

15 $y = \tan(2x)$

$$\frac{dy}{dx} = 2\sec^2(2x)$$

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$s = \int_0^{\frac{\pi}{8}} \sqrt{1 + 4\sec^4(2x)}\, dx \approx 1.08$$

The correct answer is **C**.

16 $y = \tan(2x)$

$y = \tan(2x)$

$$\frac{dy}{dx} = 2\sec^2(2x)$$

$$S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{symmetry}$$

$$S = 4\pi\int_0^{\frac{\pi}{8}} \tan(2x)\sqrt{1 + 4\sec^4(2x)}\, dx$$

The correct answer is **B**.

17 $y = \tan^{-1}(2x)$

$$y = \frac{2}{1+4x^2}$$

$$S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$S = 2\pi\int_0^{\pi} \tan^{-1}(2x)\sqrt{1 + \frac{4}{\left(1+4x^2\right)^2}}\, dx \approx 23.89$$

The correct answer is **E**.

18 $\frac{dy}{dx} = \sqrt{2x^6 + 1}$

$$y = \int_0^x \sqrt{2t^2 + 1}\, dt + c$$

$$y(1) = 5, \quad 5 = \int_0^1 \sqrt{2t^2 + 1}\, dt + c$$

$$c = 5 - \int_0^1 \sqrt{2t^2 + 1}\, dt$$

$$y = \int_0^x \sqrt{2t^2 + 1}\, dt + 5 - \int_0^1 \sqrt{2t^2 + 1}\, dt$$

$$y = \int_0^x \sqrt{2t^2 + 1}\, dt + \int_1^0 \sqrt{2t^2 + 1}\, dt + 5$$

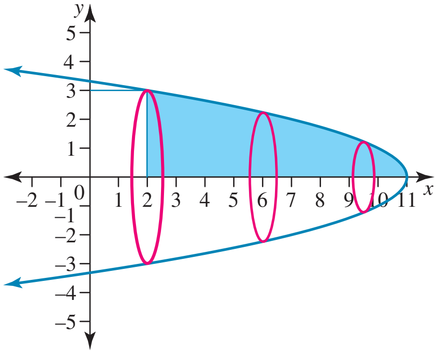
$$y(x) = \int_1^x \sqrt{2t^2 + 1}\, dt + 5$$

$$y(4) = \int_1^4 \sqrt{2t^6 + 1}\, dt + 5$$

The correct answer is **E**.

Technology active: extended response

19



$y = \sqrt{11 - x}$

a $$A = \int_2^{11} \sqrt{11 - x}\, dx$$

$$= \int_2^{11} (11 - x)^{\frac{1}{2}}\, dx$$

$$= \left[-\frac{2}{3}(11 - x)^{\frac{3}{2}}\right]_2^{11}$$

$$= -\frac{2}{3}\left[0 - 9^{\frac{3}{2}}\right]$$

$$= 18$$

b $$V = \pi\int_2^{11^2} dx$$

$$= \pi\int_2^{11} (11 - x)\, dx$$

$$= \pi\left[11x - \frac{1}{2}x^2\right]_2^{11}$$

$$= \pi\left[121 - \frac{1}{2} \times 121 - 22 + 2\right]$$

$$= \frac{81\pi}{2}$$

c $y = (11-x)^{\frac{1}{2}}, \ \frac{dy}{dx} = \frac{-1}{2}(11-x)^{-\frac{1}{2}} = \frac{-1}{2\sqrt{11-x}}$

$$S = \int_2^{11} \sqrt{1 + \frac{1}{4(11-x)}}\, dx$$

$$= \int_2^{11} \sqrt{\frac{4(11-x)+1}{4(11-x)}}\, dx$$

$$S = \frac{1}{2}\int_2^{11} \sqrt{\frac{45-4x}{11-x}}\, dx$$

d $S = 2\pi \int_2^{11} \sqrt{11-x}\sqrt{\frac{45-4x}{11-x}}\, dx$

$$= 2\pi \int_2^{11} \sqrt{45-4x}\, dx$$

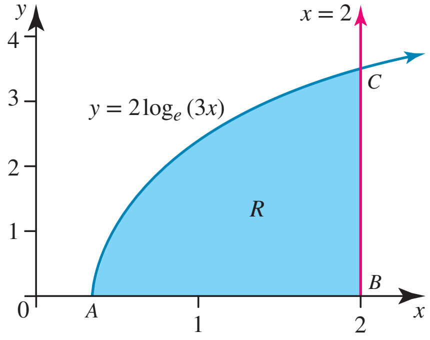
$$= 2\pi \left[\frac{2}{3\times(-4)}(45-4x)^{\frac{3}{2}}\right]_2^{11}$$

$$= \frac{-\pi}{3}\left[1^{\frac{3}{2}} - 37^{\frac{3}{2}}\right]$$

$$= \frac{\pi}{3}\left(37^{\frac{3}{2}} - 1\right)$$

$$= \frac{\pi}{3}\left(37\sqrt{37} - 1\right)$$

20



a $y = 2\log_e(3x)$

Crosses x-axis at A, when $y = 0$

$\log_e(3x) = 0 \quad 3x = 1 \quad x = \frac{1}{3}$

$A\left(\frac{1}{3}, 0\right) \ B(2, 0)$

When $x = 2$ $y = 2\log_e(6)$ $C\left(2, 2\log_e(6)\right)$

$\frac{dy}{dx} = \frac{2}{x}$

$S_{AC} = \int_{\frac{1}{3}}^{2} \sqrt{1 + \frac{4}{x^2}}\, dx = 4.022$

$d(AB) = 1\frac{2}{3} \quad d(BC) = 2\log_e(6)$

Total perimeter $1\frac{2}{3} + 4.027 + \log_e(36)$

$= 9.2718$ units

b $A = \int_{\frac{1}{3}}^{2} 2\log_e(3x)\, dx$

$$= \frac{2}{3}\left(6\log_e(6) - 5\right)$$

$$= 3.8337 \text{ units}^2$$

c $V = \pi \int_{\frac{1}{3}}^{2} \left(2\log_e(3x)\right)^2 dx$

$$= 32.5105 \text{ units}^3$$

d $S = 2\pi \int_{\frac{1}{3}}^{2} 2\log_e(3x)\sqrt{1 + \frac{4}{x^2}}\, dx$

$$= 47.6287 \text{ units}^2$$

e $\int_{\frac{1}{3}}^{k} 2\log_e(3x)\, dx = \frac{1}{2}\int_{\frac{1}{3}}^{2} 2\log_e(3x)\, dx$

$$= \frac{1}{2} \times 3.8337$$

$k = 1.4111$

f $\pi \int_{\frac{1}{3}}^{k} \left(2\log_e 3x\right)^2 dx = \frac{1}{2} \times 32.5105$

$k = 1.5344$

9.8 Exam questions

1 $y^2 = x^2 - 2x, \ V = \pi \int_a^b x^2\, dy$

$y^2 + 1 = x^2 - 2x + 1 = (x-1)^2$

$x = 1 + \sqrt{y^2 + 1}$

$$V = \pi \int_0^{2\sqrt{2}} \left(1 + \sqrt{y^2+1}\right)^2 dy$$

$y = \tan(t), \ y^2 + 1 = \tan^2(t) + 1 = \sec^2(t), \ \frac{dy}{dt} = \sec^2(t)$

$y = 2\sqrt{2}, \ t = \tan^{-1}\left(2\sqrt{2}\right), \ y = 0, \ t = 0$

$$V = \pi \int_0^{\tan^{-1}\left(2\sqrt{2}\right)} (1 + \sec(t))^2 \sec^2(t)dt$$

Award 1 mark for the correct method.

Award 1 mark for the correct definite integral.

VCAA Examination Report **note:**

Very few students answered this question correctly. The most common incorrect answer was an integral in terms of x. Of those that attempted to give an integral in terms of t, most simply replaced dx with dt.

2 **a** $f(x) = \frac{9}{(x+2)(x-4)}$

$$f(x) = \frac{9}{\left(x^2 - 2x - 8\right)}$$

$$= 9\left(x^2 - 2x - 8\right)^{-1}$$

$$f'(x) = \frac{-9(2x-2)}{\left(x^2 - 2x - 8\right)^2} = 0$$

when $x = 1$;

$$f(1) = \frac{9}{3 \times -3}$$

$$= -1$$

Stationary point, local maximum at $(1, -1)$

Award 1 mark for solving the gradient equal to zero.

Award 1 mark for the correct x-value.

Award 1 mark for the correct coordinates.

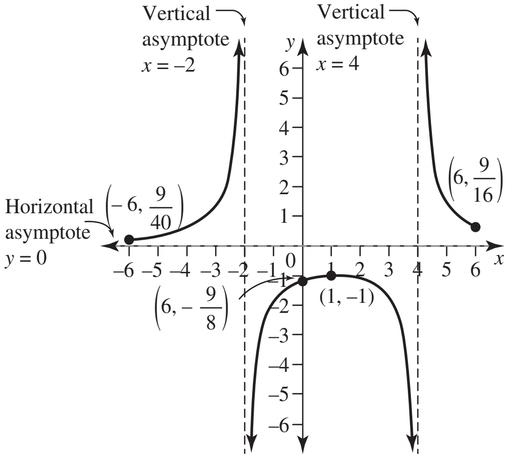
b Vertical asymptotes $x = 4,\ x = -2$

Horizontal asymptote $y = 0$

Award 1 mark for both correct vertical asymptotes.

Award 1 mark for the horizontal asymptote.

c



Award 1 mark for the correct shape over the correct domain.

Award 1 mark for the correct y-intercept.

Award 1 mark for correctly approaching asymptotes.

d **i** $V = \pi \displaystyle\int_0^3 \frac{81}{(x+2)^2(x-4)^2}dx$

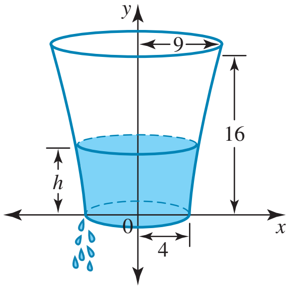
Award 1 mark for the correct terminals.

Award 1 mark for the correct definite integral representing the volume.

ii $V = 12.85$

Award 1 mark for correctly using CAS to obtain the volume.

3



$$\frac{x^2}{16} - \frac{65y^2}{4096} = 1$$

$$\frac{x^2}{16} = 1 + \frac{65y^2}{4096}$$

$$\frac{x^2}{16} = \frac{4096 + 65y^2}{4096}$$

$$x^2 = \frac{1}{256}\left(65y^2 + 4096\right)$$

$$V = \pi\int_0^h x^2\,dy = \int_0^h \frac{\pi}{256}\left(65y^2 + 4096\right)\,dy \quad \text{[1 mark]}$$

$$\frac{dV}{dh} = \frac{\pi}{256}(65h^2 + 4096),\ \frac{dV}{dt} = -k\sqrt{h}$$

$$\frac{dh}{dt} = \frac{dh}{dV}\cdot\frac{dV}{dt} = \frac{-k\sqrt{h}}{\frac{\pi}{256}\left(65h^2 + 4096\right)}$$

$$\frac{dt}{dh} = \frac{65h^2 + 4096}{A\sqrt{h}}$$

$$At + B = \int \frac{65h^2 + 4096}{\sqrt{h}}\,dh$$

$$= \int\left(65h^{\frac{3}{2}} + 4096h^{-\frac{1}{2}}\right)dh$$

$$= 26h^{\frac{5}{2}} + 8192h^{\frac{1}{2}} \qquad \text{[1 mark]}$$

When $h = 16 \quad t = 0$

$$B = 26 \times 16^{\frac{5}{2}} + 8192 \times 16^{\frac{1}{2}} = 59392$$

When $h = 9 \quad t = 10$

$$10 \quad A + B = 26 \times 9^{\frac{5}{2}} + 8192 \times 9^{\frac{1}{2}} = 30894$$

$$10A = -28498$$

$$A = \frac{-14249}{5} \qquad B = 59392$$

When empty $h = 0$

$$t = -\frac{B}{A} = \frac{59392}{\frac{14249}{5}}$$

$= 20.8$ minutes

So it takes an extra 10.8 minutes [1 mark]

4 **a** $\displaystyle\int \cos^n(ax)\,dx = C_n$

$$= \int \cos(ax)\cos^{n-1}(ax)\,dx$$

Let $u = \cos^{n-1}(ax) \quad \dfrac{dv}{dx} = \cos(ax)$ [1 mark]

$$\frac{du}{dx} = -(n-1)\,a\cos^{n-2}(ax)\sin(ax) \qquad v = \int\cos(ax)\,dx$$

$$= \frac{1}{a}\sin(ax)$$

$$C_n = \frac{1}{a}\sin(ax)\cos^{n-1}(ax) + (n-1)\int \sin^2(ax)\cos^{n-2}(ax)\,dx$$

$$= \frac{1}{a}\sin(ax)\cos^{n-1}(ax) + (n-1)\int\left(1 - \cos^2(ax)\right)\cos^{n-2}(ax)\,dx$$

$$C_n = \frac{1}{a}\sin(ax)\cos^{n-1}(ax) + (n-1)\int\cos^{n-2}(ax)\,dx - (n-1)\int\cos^n(ax)\,dx \quad \text{[1 mark]}$$

$$\int\cos^n(ax)\,dx = \frac{1}{a}\sin(ax)\quad\cos^{n-1}(ax) + (n-1)\int\cos^{n-2}(ax)\,dx$$
$$-n\int\cos^n(ax)\,dx + \int\cos^n(ax)\,dx$$

$$n\int\cos^n(ax)\,dx = \frac{1}{a}\sin(ax)\quad\cos^{n-1}(ax) + (n-1)\int\cos^{n-2}(ax)\,dx$$

$$\int\cos^n(ax)\,dx = \frac{1}{an}\sin(ax)\quad\cos^{n-1}(ax)$$
$$+\frac{n-1}{n}\int\cos^{n-2}(ax)\,dx \qquad \text{[1 mark]}$$

b Let $n = 5$, $a = 3$

$$\int \cos^5(3x)\,dx = \frac{1}{15}\sin(3x)\cos^4(3x) + \frac{4}{5}\int \cos^3(3x)\,dx$$ [1 mark]

$$\int \cos^3(3x)\,dx = \frac{1}{9}\sin(3x)\cos^2(3x) + \frac{2}{3}\int \cos(3x)\,dx$$

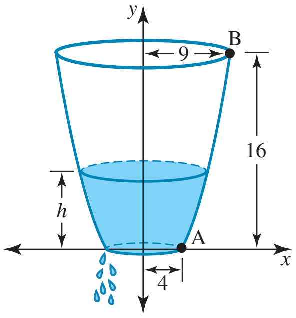
$$= \frac{1}{9}\sin(3x)\cos^2(3x) + \frac{2}{9}\sin(3x)$$ [1 mark]

$$\int \cos^5(3x)\,dx = \frac{1}{15}\sin(3x)\cos^4(3x)$$

$$+\frac{4}{5}\left[\frac{1}{9}\sin(3x)\cos^2(3x) + \frac{2}{9}\sin(3x)\right] + c$$

$$= \frac{1}{45}\sin(3x)\left[3\cos^4(3x) + 4\cos^2(3x) + 8\right] + c$$ [1 mark]

5



$A(4, 0)$ $B(9, 16)$

$y = ax^2 + b$

(1) $0 = 16a + b$

(2) $16 = 81a + b$

$65a = 16$

$a = \frac{16}{65}$ $b = -\frac{256}{65}$

$y = \frac{16}{65}\left(x^2 - 16\right)$ [1 mark]

$\frac{65y}{16} = x^2 - 16$

$x^2 = \frac{65y}{16} + 16 = \frac{65y + 256}{16}$

$V = \pi\int_0^h x^2\,dy = \pi\int_0^h \frac{65y + 256}{16}dy$ [1 mark]

$\frac{dV}{dh} = \frac{\pi}{16}(65h + 256)$, $\frac{dV}{dt} = -2\sqrt{h}$

$\frac{dt}{dh} = \frac{dt}{dV}\cdot\frac{dV}{dh} = \frac{\pi(65h + 256)}{-2\sqrt{h} \times 16}$

$t = \frac{-\pi}{32}\int_{16}^{0} \frac{65h + 256}{\sqrt{h}}dh$ [1 mark]

$= \frac{-\pi}{32} \times -\frac{14464}{3}$

$= 473.3$ minutes [1 mark]

Topic 10 — Applications of first-order differential equations

10.2 Growth and decay

10.2 Exercise

1 a $N = N(t)$ is the population of the city n years after 2015

$$\frac{dN}{dt} = kN \rightarrow N = N_0e^{kt}$$

In 2015 $t = 0$, $N(0) = 1.79 \times 10^6 = N_0$

In 2019 $t = 4$, $N(4) = 2 \times 10^6$

$$2 \times 10^6 = 1.79 \times 10^6 e^{4k}$$

$$\frac{2}{1.79} = e^{4k}$$

$$4k = \log_e\left(\frac{2}{1.79}\right)$$

$$k = \frac{1}{4}\log_e\left(\frac{2}{1.79}\right) \approx 0.0277$$

$$N = 1.79 \times 10^6 e^{0.0277t}$$

b In 2030 $t = 15$

$$N(15) = 1.79 \times 10^6 e^{0.0277\times 15}$$

$$= 2.71 \times 10^6$$

$$= 2.71 \text{ million}$$

c $N = 5 \times 10^6$, $t = ?$

$$5 \times 10^6 = 1.79 \times 10^6 e^{0.0277t}$$

$$\frac{5}{1.79} = e^{0.0277t}$$

$$0.0277t = \log_e\left(\frac{5}{1.79}\right)$$

$$t = \frac{1}{0.0277}\log_e\left(\frac{5}{1.79}\right) \approx 37.04$$

In the year 2015 + 37 = 2052

2 $\frac{dw}{dt} = kw \rightarrow N = N_0e^{kt}$

$N(15) = 297 \rightarrow 297 = N_0e^{15k}$ [1]

$N(26) = 523 \rightarrow 523 = N_0e^{26k}$ [2]

$$\frac{[2]}{[1]} \rightarrow \frac{523}{297} = \frac{N_0e^{26k}}{N_0e^{15k}}$$

$$11k = \log_e\left(\frac{523}{297}\right)$$

$$k = \frac{1}{11}\log_e\left(\frac{523}{297}\right) \approx 0.0514$$

So (1) $N_0 = \frac{297}{e^{15k}} = 137.29$

Initial number of frogs is 137

3 a N number of insects

$$\frac{dN}{dt} = kN$$

$t = 0$, $N(0) = N_0 = 600$

$t = 2$, $N(2) = 1300$

$$N = N_0e^{kt}$$

$$1300 = 600e^{2k}$$

$$\frac{13}{6} = e^{2k}$$

$$k = \frac{1}{2}\log_e\left(\frac{13}{6}\right) \approx 0.3866$$

$$N = 600e^{0.3866t}$$

$$N(5) = 600e^{0.3866\times 5}$$

$$= 4146 \text{ insects}$$

b Initially 10 000 fish

$$N = N(t)$$

$$\frac{dN}{dt} = kN$$

$$k = 0.04$$

Growing continuously at a rate of 4% per year means $k = 0.04$.

$$N_0 = 10\,000$$

$$N(t) = N = 10\,000e^{0.04t}$$

$$N(3) = 10\,000e^{0.04\times 3}$$

$$= 11\,275 \text{ fish}$$

4 N number of possums

$$\frac{dN}{dt} = kN$$

$$N = N_0e^{kt}$$

$t = 0$, $N(0) = N_0 = 521$

$t = 4$, $N(4) = 678$

$$\frac{678}{521} = e^{4k}$$

$$k = \frac{1}{4}\log_e\left(\frac{678}{521}\right) \approx 0.0658$$

$$N(t) = N = 521e^{0.0658t}$$

Further 5 months

$N(9) = N = 521e^{0.0658\times 9} = 942$ possums

5 Australian population N million, t years after 2013

$$N = N_0e^{kt}$$

$t = 0$, $2013 \rightarrow N(0) = 23.131$: $23.131 = N_0e^0$ $\therefore N_0 = 23.131$

$t = 5$, $2018 \rightarrow N(5) = 24.992$: $24.992 = N_0e^{5k}$

$$24.992 = 23.131e^{5k}$$

$$e^{5k} = \frac{24.992}{23.131}$$

$$k = \frac{1}{5}\log_e\left(\frac{24.992}{23.131}\right)$$

$$N = 23.131e^{\frac{t}{5}\log_e\left(\frac{24.992}{23.131}\right)}$$

a In 2025, $t = 12$.

$$N = 23.131e^{\frac{12}{5}\log_e\left(\frac{24.992}{23.131}\right)}$$

$$= 27.851607 \text{ (million)}$$

$$= 27\,851\,607$$

b $N = 30$, $t = ?$

$$30 = 23.131e^{\frac{t}{5}\log_e\left(\frac{24.992}{23.131}\right)}$$

$$\frac{30}{23.131} = e^{\frac{t}{5}\log_e\left(\frac{24.992}{23.131}\right)}$$

$$\frac{t}{5}\log_e\left(\frac{24.992}{23.131}\right) = \log_e\left(\frac{30}{23.131}\right)$$

$$t = \frac{5\log_e\left(\frac{30}{23.131}\right)}{\log_e\left(\frac{24.992}{23.131}\right)} = 16.80$$

The population will first exceed 30 million is the year 2029.

6 World population N billion, t years after 2011 can be modelled by the equation $N = N_0e^{kt}$.

$t = 0,\ 2011 \rightarrow N(0) = 7\ :\ 7 = N_0e^0 \therefore N_0 = 7$

$t = 7,\ 2018 \rightarrow N(7) = 7.632\,819325$

$7.632\,819325 = N_0e^{7k}$

$7.632\,819325 = 7e^{7k}$

$$e^{7k} = \frac{7.632\,819325}{7}$$

$$k = \frac{1}{7}\log_e\left(\frac{7.632\,819325}{7}\right)$$

$$N = 7e^{\frac{t}{7}\log_e\left(\frac{7.632819325}{7}\right)}$$

a In 2024, $t = 13$.

$$N = 7e^{\frac{13}{7}\log_e\left(\frac{7.632819325}{7}\right)}$$

$= 8.220578135$ (billion)

$= 8\,220\,578\,135$

b $N = 10,\ t = ?$

$$10 = 7e^{\frac{t}{7}\log_e\left(\frac{7.632819325}{7}\right)}$$

$$\frac{10}{7} = e^{\frac{t}{7}\log_e\left(\frac{7.632819325}{7}\right)}$$

$$\frac{t}{7}\log_e\left(\frac{7.632819325}{7}\right) = \log_e\left(\frac{10}{7}\right)$$

$$t = \frac{7\log_e\left(\frac{10}{7}\right)}{\log_e\left(\frac{7.632819325}{7}\right)} = 28.85$$

The world population will first exceed 10 billion is the year 2039.

7 $\dfrac{dm}{dt} = -km,\ m = m_0e^{-kt},\ k > 0$

$m(0) = m_0 = 80\,\text{mg}$

$m(1) = 0.9 \times 80 = 72\,\text{mg}$

$72 = 80e^{-k}$

$$e^k = \frac{80}{72}$$

$$k = \log_e\left(\frac{80}{72}\right) = 0.105\,361$$

$m(t) = 80e^{-0.1054t}$

$m(3) = 80e^{-0.1054\times3} = 58.31$ mg

8 $\dfrac{dm}{dt} = -km,\ m = m_0e^{-kt}$

$m(2) = 64 = m_0e^{-2k}$ [1]

$m(4) = 36 = m_0e^{-4k}$ [2]

$$\frac{[1]}{[2]} \rightarrow \frac{64}{36} = \frac{m_0e^{-2k}}{m_0e^{-4k}} = e^{2k}$$

$$e^k = \frac{4}{3}$$

$$k = \log_e\left(\frac{4}{3}\right)$$

$$m_0 = 64e^{2k} = 64e^{2\log_e\left(\frac{4}{3}\right)} = 113.78\,\text{mg}$$

9 $\dfrac{dm}{dt} = -km,\ m = m_0e^{-kt},\ k > 0$

15% disintegrates

$m(3) = 0.85\,m_0$

$0.85m_0 = m_0e^{-3k}$

$0.85 = e^{-3k}$

$-3k = \log_e(0.85)$

$$k = -\frac{1}{3}\log_e(0.85) \approx 0.05\,417$$

$$\text{T} = \frac{1}{k}\log_e(2) = \frac{1}{0.05\,417}\log_e(2)$$

$= 12.8$ years

10 $T = 1601 = \dfrac{1}{k}\log_e(2)$

$$k = \frac{\log_e(2)}{1601} \approx 0.000\,433$$

$m = m_0e^{-kt}$

$t = 1000$

$m = m_0e^{-0.000\,433 \times 1000}$

$= 0.649m_0$

So 65% remains.

11 **a** $\dfrac{dm}{dt} = -km$

$m = m_0e^{-kt}$

m mass in grams, t years

$t = 0,\ m(0) = 1$

$t = 22\,652,\ m = 0.5$

$0.5 = 1e^{-22\,652k}$

$$-22\,652k = \log_e\left(\frac{1}{2}\right)$$

$$k = \frac{1}{-22\,652}\log_e\left(\frac{1}{2}\right) \approx 0.000\,031$$

$m(t) = e^{-0.000\,031t}$

$t = 40\,000$

$m(40\,000) = e^{-0.000\,031\times40\,000} = 0.29$ g

b $t = 6,\ m = 0.85\,m_0$

15% disintegrates

$t = 6,\ m = 0.85\,m_0$

$0.85\,m_0 = m_0e^{-6k}$

$$k = -\frac{1}{6}\log_e(0.85) \approx 0.027\,086$$

$$T = \frac{1}{k}\log_e(2)$$

$$= \frac{1}{0.027\,086}\log_e(2)$$

$= 25.59$ years

12 Iodine–131

$$\frac{dm}{dt} = -km$$

$$T = \frac{1}{k}\log_e(2) = 8$$

$$k = \frac{\log_e(2)}{8} = 0.08\,664$$

$m(0) = 2\,\text{g} = m_0$

$t = ?$

$$m = 120\,\text{mg} = 120 \times 10^{-3} = 0.12$$
$$0.12 = 2e^{-0.08\,664t}$$
$$t = -\frac{1}{0.08\,664}\log_e\left(\frac{0.12}{2}\right) = 32.47\text{ days}$$

13 Strontium-90

$$T = 28.9 = \frac{1}{k}\log_e(2)$$
$$k = \frac{\log_e(2)}{28.9} = 0.024$$

For 60% to decay, 40% remains

$$m = m_0e^{-kt}$$
$$0.4m_0 = m_0e^{-0.024t}$$
$$0.4 = e^{-0.024t}$$
$$t = -\frac{1}{0.024}\log_e(0.4) = 38.2\text{ years}$$

14 Curium-243

$T = 29.1$ years

$$k = \frac{\log_e(2)}{29.1} = 0.02\,382$$
$$m = m_0e^{-kt}$$

After 10 years

$$m = m_0e^{-0.02\,382 \times 10} = 0.788\,m_0$$
$$0.788 \times 100 = 78.8\%\text{ remains}$$

15 Uranium-238

$T = 4.468$ billion years

Big bang 13.8 billion years ago

$$T = 4.468 = \frac{1}{k}\log_e(2)$$
$$k = \frac{\log_e(2)}{4.468} \approx 0.1551$$
$$m = m_0e^{-kt}$$
$$t = 13.8$$
$$m = m_0e^{-0.1551 \times 13.8} = 0.1176\,m_0$$

11.76% remains

16 Type A

$$N_A(0) = 900$$
$$N_A(10) = 3000$$
$$N_A = N_{A0}e^{kt}$$
$$3000 = 900e^{10k_A}$$
$$k_A = \frac{1}{10}\log_e\left(\frac{30}{9}\right) \approx 0.1204$$
$$N_A = 900e^{0.1204t}$$

Type B

$$N_B(8) = 2500$$
$$N_B(4) = 1250$$
$$N_B = N_{B0}e^{kt}$$
$$1250 = N_{B0}e^{4k_B}$$
$$2500 = N_{B0}e^{8k_B}$$
$$\frac{2500}{1250} = 2 = e^{4k_B}$$
$$e^{4k_B} = \log_e(2)$$
$$k_B = \frac{1}{4}\log_e(2) \approx 0.1733$$
$$N_{B0} = 1250e^{-4k_B} = 625$$
$$N_B = 625e^{-0.1733t}$$

For equal numbers

$$N_A = N_B$$
$$900e^{0.1204t} = 625e^{0.1733t}$$
$$\frac{900}{625} = \frac{e^{0.1733t}}{e^{0.1204t}} = e^{(0.1733-0.1240)t}$$
$$\frac{900}{625} = e^{0.0529t}$$
$$0.0529t = \log_e\left(\frac{900}{625}\right)$$
$$t = \frac{1}{0.0529}\log_e\left(\frac{900}{625}\right)$$
$$= 6.89\text{ hours}$$

17 Let $M(t)$ represent the population of Melbourne in millions after 2013.

Let $S(t)$ represent the population of Sydney in millions after 2013.

Let k_m represent the growth rate of Melbourne's population.

Let k_s represent the growth rate of Sydney's population.

a $M(t) = M_0e^{k_mt}$

$$M(0) = M_0 = 4.217$$
$$M(8) = 5.061$$
$$5.061 = 4.217e^{8k_m}$$
$$e^{8k_m} = \frac{5.061}{4.217}$$
$$k_m = \frac{1}{8}\log_e\left(\frac{5.061}{4.217}\right) \approx 0.0228$$
$$S(t) = S_0e^{k_bt}$$
$$S(0) = S_0 = 4.386$$
$$S(8) = 4.992$$
$$4.992 = 4.386e^{8k_s}$$
$$e^{8k_s} = \frac{4.992}{4.386}$$
$$k_s = \frac{1}{8}\log_e\left(\frac{4.992}{4.386}\right) \approx 0.0162$$

Since $k_m > k_s$, Melbourne's growth rate is greater than Sydney's.

b $M(t) = 8,\ t = ?$

$$8 = 4.217e^{k_mt}$$
$$e^{k_mt} = \frac{8}{4.217}$$
$$k_mt = \log_e\left(\frac{8}{4.217}\right)$$
$$t = \frac{1}{k_m}\log_e\left(\frac{8}{4.217}\right)$$
$$t = \frac{8}{\log_e\left(\frac{5.061}{4.217}\right)}\log_e\left(\frac{8}{4.217}\right) \approx 28.08$$

Early in the year 2041.

c $S(t) = 8,\ t = ?$

$$8 = 4.386\,e^{k_st}$$
$$e^{k_st} = \frac{8}{4.386}$$
$$k_st = \log_e\left(\frac{8}{4.386}\right)$$
$$t = \frac{1}{k_s}\log_e\left(\frac{8}{4.386}\right)$$
$$t = \frac{8}{\log_e\left(\frac{4.992}{4.386}\right)}\log_e\left(\frac{8}{4.386}\right) \approx 37.152$$

Early in the year 2050.

18 a $\dfrac{dQ}{dt} = -kQ$

$$\frac{dt}{dQ} = \frac{1}{kQ}$$

$$t = -\frac{1}{k}\int \frac{1}{Q} dQ$$

$$= -\frac{1}{k}\log_e(Q) + C$$

$$kt = -\log_e(Q) + B \text{ where } B = kC$$

$$t = t_0$$

$$Q = Q_0$$

$$kt_0 = -\log_e(Q_0) + B$$

$$B = \log_e(Q_0) + kt_0$$

$$kt = -\log_e(Q) + \log_e(Q_0) + kt_0$$

$$kt - kt_0 = -\log_e(Q) + \log_e(Q_0)$$

$$k(t - t_0) = \log_e\left(\frac{Q_0}{Q}\right)$$

$$\frac{Q_0}{Q} = e^{k(t-t_0)}$$

$$\frac{Q}{Q_0} = e^{-k(t-t_0)}$$

$$Q = Q(t) = Q_0 e^{-k(t-t_0)}$$

b $t - t_0 = 5730$

$$\frac{Q}{Q_0} = \frac{1}{2} = e^{-k\times 5730}$$

$$5730k = \log_e(2)$$

$$k = 0.000121$$

c $\dfrac{Q}{Q_0} = \dfrac{2}{7} = e^{-k(t-t_0)}$

$$(t - t_0) = \frac{\log_e\left(\frac{2}{7}\right)}{-k} = 10\,356 \text{ years}$$

10.2 Exam questions

1 a i $\dfrac{dP}{dt} = kP$

$$\int \frac{1}{P}\, dP = \int k\, dt$$

$\log_e(|P|) = kt + c$ but $P > 0$, so the modulus is not needed

$P = P(t) = e^{kt+c} = P_0\, e^{k\,t}$ where $P_0 = e^c$

(1) $P(a) = P_0\, e^{k\,a} = r$, (2) $P(b) = P_0\, e^{k\,b} = s$

$$\frac{(1)}{(2)} \quad \frac{r}{s} = \frac{e^{k\,a}}{e^{k\,b}} = e^{k(a-b)}, \quad k(a-b) = \log_e\left(\frac{r}{s}\right)$$

$$k = \frac{1}{a-b}\log_e\left(\frac{r}{s}\right)$$

Award 1 mark for correctly integrating and solving.

Award 1 mark for the correct result.

ii $k = \dfrac{1}{a-b}\log_e\left(\dfrac{r}{s}\right) > 0$, since

$P > 0 \Rightarrow \quad r > 0, \ s > 0$

Case (1) $k > 0$, $\dfrac{1}{a-b} > 0$, $\log_e\left(\dfrac{r}{s}\right) > 0 \quad \Rightarrow a > b$ and $r > s > 0$ or

Case (2) $k > 0$, $\dfrac{1}{a-b} < 0$, $\log_e\left(\dfrac{r}{s}\right) < 0 \quad \Rightarrow a < b$ and $s > r > 0$

Award 1 mark each for the correct cases.

b i $\dfrac{dQ}{dt} = e^{t-Q} = \dfrac{e^t}{e^Q}$

$$\int e^Q\, dQ = \int e^t\, dt \qquad \text{[1 mark]}$$

ii $e^Q = e^t + c$ to find c use $Q = 1$, $t = 0$

$$e = 1 + c, \qquad c = e - 1$$

$$e^Q = e^t + e - 1$$

$$Q = \log_e(e^t + e - 1)$$

Award 1 mark for correctly integrating.

Award 1 mark for the correct proof.

iii $\dfrac{d^2Q}{dt^2} = \dfrac{(e-1)e^t}{(e^t + e - 1)^2} \neq 0$ since $e > 1$ and $e^t > 0$ so there are no inflection points.

Award 1 mark for the correct second derivative.

Award 1 mark for the correct argument

2 $m = m_0 e^{-kt}$, $m_0 = 3$

When $t = 38213$, $m = 1$

$$1 = 3e^{-38213k}$$

$$e^{-38213k} = \frac{1}{3}$$

$$-38213k = \log_e\left(\frac{1}{3}\right)$$

$$k = \frac{-\log_e\left(\frac{1}{3}\right)}{38213}$$

$$\approx 0.00002875$$

$m = 3e^{-0.00002875t}$ [1 mark]

The half-life is the time taken for the mass to reduce to half its initial mass.

$$1.5 = 3e^{-0.00002875t}$$

$$e^{-0.00002875t} = \frac{1}{2}$$

$$\log_e\left(\frac{1}{2}\right) = -0.00002875t$$

$$t = \frac{\log_e\left(\frac{1}{2}\right)}{-0.00002875}$$

$$= 24\,109 \text{ years}$$

$$\approx 24\,110 \text{ years, correct to the nearest year [1 mark]}$$

3 $\dfrac{dN}{dt} = kN$

$$N = N_0 e^{kt}$$

$$N = 80.827e^{kt}$$

When $t = 11$, $N = 83.900$

$$83.9 = 80.827e^{11k}$$

$$e^{11k} = \frac{83.9}{80.827}$$

$$11k = \log_e\left(\frac{83.9}{80.827}\right)$$

$$11k = 0.0373$$

$$k = 0.00339$$

$$N = 80.827e^{0.00339t}$$

$$100 = 80.827e^{0.00339t}$$

$$e^{0.00339t} = \frac{100}{80.827}$$

$$0.00339t = \log_e\left(\frac{100}{80.827}\right)$$

$$0.00339t = 0.21286$$

$$t = 62.8$$

62.8 years after 2010 is late in 2072.

The correct answer is **D**.

10.3 Other applications of first-order differential equations

10.3 Exercise

1 $\frac{dP}{dt} = P \times r$

$P = P_0 e^{rt}$

$P_0 = ?$

$t = 2$

$P = \$10\,000$

$r = 5\% = 0.05$

$P_0 = Pe^{-rt}$

$= 10\,000 \times e^{-0.05\times 2}$

$= \$9048.37$

2 Let $\$P$ be the price of the house after 2002

$\frac{dP}{dt} \propto P$

$\frac{dP}{dt} = kP$

$kt = \int \frac{1}{P} dP = \log_e(|P|) + c,\ P > 0$

$P = P_0,\ t = 0$

$0 = \log_e(P_0) + c$

$c = -\log_e(P_0)$

$kt = \log_e(P) - \log_e(P_0)$

$= \log_e\left(\frac{P}{P_0}\right)$

$\frac{P}{P_0} = e^{kt}$

$P = P(t) = P_0 e^{kt}$

$2002:\ t = 0,\ P(0) = P_0 = \$315\,000$

$2018:\ t = 16,\ P(16) = \$1\,260\,000$

$1\,260\,000 = 315\,000 e^{16k}$

$4 = e^{16k}$

$k = \frac{1}{16}\log_e(4)$

$P(8) = 315\,000 e^{8\times\frac{1}{16}\log_e(4)} = \$630\,000$

Doubles every 8 years

3 **a** $P = P(h)$ pressure at height h

$\frac{dP}{dh} = -kP$

$P(0) = P_0$

$\frac{dh}{dP} = \frac{1}{-kP}$

$kh = -\int \frac{1}{P} dP$

$kh = -\log_e(P) + c$

Modulus not needed as $P > 0$

$kh = -\log_e(P) + c$

$h = 0,\ P = P_0$

$0 = -\log_e(P_0) + c$

$c = \log_e(P_0)$

$kh = \log_e(P_0) - \log_e(P)$

$kh = \log_e\left(\frac{P_0}{P}\right)$

$e^{kh} = \frac{P_0}{P}$

$P = P(h) = P_0 e^{kh},\ k > 0$

b $h = 0,\ P(0) = 76 = P_0$

$h = 1,\ P(1) = 62.2235$

$h = ?,\ P(2) = ?$

$62.2235 = 76e^{-k}$

$e^{-k} = \frac{62.2235}{76}$

$k = -\log_e\left(\frac{62.2235}{76}\right) = 0.2$

$P = P(h) = 76e^{-0.2h}$

$h = 2$

$P(2) = 76e^{-0.2\times 2} = 50.94\text{ cm Hg}$

4 **a** Drug D amount at time t

$\frac{dD}{dt} = -kD,\ D(0) = D_0$

$\frac{dt}{dD} = -\frac{1}{kD}$

$kt = -\int \frac{1}{D}\, dD$

$kt = -\log_e(D) + c$

$0 = -\log_e(D_0) + c$

$c = \log_e(D_0)$

$kt = -\log_e(D) + \log_e(D_0)$

$= \log_e\left(\frac{D_0}{D}\right)$

$\frac{D_0}{D} = e^{kt}$

$D(t) = D_0 e^{-kt}$

b $T = \frac{1}{k} = 3\text{ hours}$

$k = \frac{1}{3}$

Initially 0.10% to 0.05%

$0.05 = 0.1e^{-\frac{t}{3}}$

$e^{-\frac{t}{3}} = \frac{0.05}{0.1}$

$-\frac{t}{3} = \log_e\left(\frac{1}{2}\right)$

$t = -3\log_e\left(\frac{1}{2}\right) \approx 2.08\text{ hours}$

5 **a** $\frac{dI}{dx} = -kI$

$\frac{dx}{dI} = -\frac{1}{kI}$

$x = -\frac{1}{k}\int \frac{1}{I}\, dD$

$kx = -\log_e(I) + c$

$x = 0,\ I = I_0$

$$
\begin{aligned}
0 &= -\log_e(I_0) + c\\
c &= \log_e(I_0)\\
kx &= -\log_e(I) + \log_e(I_0)\\
&= \log_e\left(\frac{I_0}{I}\right)\\
\frac{I_0}{I} &= e^{kx}\\
I &= I(x) = I_0e^{-kx},\ k > 0
\end{aligned}
$$

b

$$
\begin{aligned}
I &= 0.95I_0\\
x &= 0.25\\
0.95I_0 &= I_0e^{-0.25k}\\
0.95 &= e^{-0.25k}\\
-0.25k &= \log_e(0.95)\\
k &= -\frac{1}{0.25}\log_e(0.95) = 0.2052\\
I &= I_0e^{-0.2052x}\\
x &= 1\\
I &= I_0e^{-0.2052} = 0.8145I_0
\end{aligned}
$$

So 18.5% is lost

6 a $\frac{dT}{d\theta} = \mu T,\ T(0) = T_0$

$$
\begin{aligned}
\frac{d\theta}{dT} &= \frac{1}{\mu T}\\
\mu\theta &= \int \frac{1}{T}\,dT\\
\mu\theta &= \log_e(T) + c\\
\theta &= 0,\ T = T_0\\
0 &= \log_e(T_0) + c\\
c &= -\log_e(T_0)\\
\mu\theta &= \log_e(T) - \log_e(T_0)\\
&= \log_e\left(\frac{T}{T_0}\right)\\
\frac{T}{T_0} &= e^{\mu\theta}\\
T &= T_0e^{\mu\theta}
\end{aligned}
$$

b $T_0 = 80$ newtons

$$
\begin{aligned}
\theta &= \pi\\
\mu &= 0.3\\
T &= ?\\
T &= 80e^{0.3\pi} = 205.31\text{ newtons}
\end{aligned}
$$

7 $\frac{dN}{dt} = k\sqrt{N}$

N population in millions, t years

2013 : $t = 0, N(0) = 4$ million

2018 : $t = 5, N(4) = 9$ million

$$
\begin{aligned}
\frac{dt}{dN} &= \frac{1}{k\sqrt{N}}\\
kt &= \int N^{-\frac{1}{2}}\,dN\\
kt &= 2N^{\frac{1}{2}} + c\\
t &= 0,\ N = 4\\
0 &= 2\sqrt{4} + c\\
c &= -4\\
kt &= 2\sqrt{N} - 4\\
t &= 5,\ N = 9\\
5k &= 2\sqrt{9} - 4 = 2\\
k &= \frac{2}{5}\\
\frac{2t}{5} &= 2\sqrt{N} - 4\\
2\sqrt{N} &= \frac{2t}{5} + 4\\
\sqrt{N} &= \frac{t}{5} + 2\\
N &= N(t) = \left(\frac{t}{5} + 2\right)^2
\end{aligned}
$$

2028 : $t = 15$

$$N(15) = \left(\frac{15}{5} + 2\right)^2 = 25\text{ million}$$

8 $\frac{dN}{dt} = \frac{k}{N}$

$$
\begin{aligned}
\frac{dt}{dN} &= \frac{N}{k}\\
kt &= \int N\,dN = \frac{1}{2}N^2 + c\\
t &= 0,\ N(0) = 20\\
0 &= \frac{1}{2}(20)^2 + c\\
c &= -200\\
kt &= \frac{1}{2}N^2 - 200\\
t &= 5,\ N = 80\\
5k &= \frac{1}{2}(80)^2 - 200 = 3000\\
k &= 600\\
600t &= \frac{1}{2}N^2 - 200\\
\frac{1}{2}N^2 &= 600t + 200\\
&= 200(3t + 1)\\
N^2 &= 400(3t + 1)\\
N &= N(t) = 20\sqrt{3t + 1}
\end{aligned}
$$

Further 11 months

$$
\begin{aligned}
t &= 16\\
N(16) &= 20\sqrt{3(16) + 1} = 140
\end{aligned}
$$

9 a $\frac{dR}{dT} = \alpha R,\ R(0) = R_0$

$$
\begin{aligned}
\frac{dT}{dR} &= \frac{1}{\alpha R}\\
\alpha T &= \int \frac{1}{R}\,dR\\
\alpha T &= \log_e(R) + c\\
T &= 0,\ R = R_0\\
0 &= \log_e(R_0) + c\\
c &= -\log_e(R_0)\\
\alpha T &= \log_e(R) - \log_e(R_0)\\
&= \log_e\left(\frac{R}{R_0}\right)
\end{aligned}
$$

$$\frac{R}{R_0} = e^{\alpha T}$$
$$R = R_0 e^{\alpha T}$$

b $\alpha = 0.004$
$$R(60) = 40$$
$$R(30) = ?$$
$$40 = R_0 e^{0.004\times 60}$$
$$R_0 = 40e^{-0.004\times 60} = 31.465$$
$$R(T) = 31.465e^{0.004T}$$
$$R(30) = 31.465e^{0.004\times 30} = 35.48 \text{ ohms}$$

10 $L\frac{di}{dt} + Ri = 0,\ i(0) = i_0$
$$L\frac{di}{dt} = -Ri$$
$$\frac{1}{L}\frac{dt}{di} = -\frac{1}{Ri}$$
$$\int \frac{R}{L}t = \int -\frac{1}{i}\,di$$
$$\frac{Rt}{L} = -\log_e(i) + c$$
$$t = 0, \quad i = i_0$$
$$0 = -\log_e(i_0) + c$$
$$c = \log_e(i_0)$$
$$\frac{Rt}{L} = -\log_e(i) + \log_e(i_0)$$
$$= \log_e\left(\frac{i_0}{i}\right)$$
$$e^{\frac{Rt}{L}} = \frac{i_0}{i}$$
$$i = \frac{i_0}{e^{\frac{Rt}{L}}}$$
$$i = i(t) = i_0 e^{-\frac{Rt}{L}}$$

11 a Rabbits growth 25%
$$N(0) = 320$$
$$\frac{dN}{dt} = 0.25N - 40$$
$$\frac{dN}{dt} = \frac{N}{4} - 40 = \frac{N-160}{4},\ N(0) = 320$$

b $\frac{dt}{dN} = \frac{4}{N-160}$
$$t = 4\int \frac{1}{N-160}\,dN = 4\log_e(N-160) + c$$
$$t = 0,\ N = 320$$
$$0 = 4\log_e(320-160) + c$$
$$c = -4\log_e(160)$$
$$t = 4\log_e(N-160) - 4\log_e(160)$$
$$= 4[\log_e(N-160) - \log_e(160)]$$
$$\frac{t}{4} = \log_e\left(\frac{N-160}{160}\right)$$
$$e^{\frac{t}{4}} = \frac{N}{160} - 1$$
$$\frac{N}{160} = 1 + e^{\frac{t}{4}}$$
$$N = N(t) = 160\left(1 + e^{\frac{t}{4}}\right)$$

c After 8 years

$N(8) = 160(1 + e^2) = 1342.25$

1342 rabbits

12 Rabbits growth 20%
$$\frac{dN}{dt} = 0.20N - K$$
$$\frac{dN}{dt} = \frac{N}{5} - K = \frac{N-5K}{5}$$
$$\frac{dt}{dN} = \frac{5}{N-5K}$$
$$t = 5\int \frac{1}{N-5K}\,dN = 5\log_e(N-5K) + c$$
$$t = 0, \qquad N = N_0$$
$$0 = 5\log_e(N_0 - 5K) + c$$
$$c = -5\log_e(N_0 - 5K)$$
$$t = 5\log_e(N-5K) - 5\log_e(N_0 - 5K)$$
$$= 5[\log_e(N-5K) - \log_e(N_0 - 5K)]$$
$$\frac{t}{5} = \log_e\left(\frac{N-5K}{N_0-5K}\right)$$
$$t = 10, \quad N(10) = 6K$$
$$2 = \log_e\left(\frac{K}{N_0-5K}\right)$$
$$\frac{K}{N_0-5K} = e^2$$
$$N_0 - 5K = \frac{K}{e^2}$$
$$N_0 = \frac{K}{e^2} + 5K$$
$$N_0 = K\left(5 + \frac{1}{e^2}\right)$$

13 a $N = N(t)$ be number of cows
$$\frac{dN}{dt} = \frac{N}{20} - 5$$
$$\frac{dN}{dt} = \frac{N-100}{20},\ N(0) = 200$$

b $\frac{dt}{dN} = \frac{20}{N-100}$
$$t = \int \frac{20}{N-100}\,dN$$
$$t = 20\log_e(N-100) + c$$
$$t = 0, N = 200$$
$$0 = 20\log_e(100) + c$$
$$c = -20\log_e(100)$$
$$t = 20\log_e(N-100) - 20\log_e(100)$$
$$t = 20[\log_e(N-100) - \log_e(100)]$$
$$\frac{t}{20} = \log_e\left(\frac{N-100}{100}\right)$$
$$\frac{N-100}{100} = e^{\frac{t}{20}}$$
$$N - 100 = 100e^{\frac{t}{20}}$$
$$N = 100e^{\frac{t}{20}} + 100$$
$$N = N(t) = 100(e^{\frac{t}{20}} + 1)$$

c $N(3) = 100\left(e^{\frac{3}{20}} + 1\right) = 216$ cows

14 $N = N(t)$ be population of the country in 2019

$$\frac{dN}{dt} = 0.04N + 10\,000,\ N(0) = 500\,000$$
$$\frac{dN}{dt} = \frac{4N}{100} + 10\,000 = \frac{1}{25}(N + 250\,000)$$
$$\frac{dt}{dN} = \frac{25}{N + 250\,000}$$
$$t = 25\int \frac{1}{N + 250\,000}\,dN$$
$$t = 25\log_e(N + 250\,000) + c$$
$$t = 0, \quad N = 500\,000$$
$$0 = 25\log_e(750\,000) + c$$
$$c = -25\log_e(750\,000)$$
$$t = 25\log_e(N + 250\,000) - 25\log_e(750\,000)$$
$$t = 25\log_e\left(\frac{N + 250\,000}{750\,000}\right)$$
$$\frac{N + 250\,000}{750\,000} = e^{\frac{t}{25}}$$
$$N = N(t) = 250\,000(3e^{\frac{t}{25}} - 1)$$
$$2024:\ t = 5$$
$$N(5) = 250\ 000(3e^{\frac{1}{5}} - 1) = 666\ 052 \text{ people}$$

15 a $N = N(t)$ koalas on plantation

$$\frac{dN}{dt} = 0.1N - 50,\ N(0) = 400$$
$$\frac{dN}{dt} = \frac{N}{10} - 50 = \frac{N - 500}{10}$$
$$\frac{dt}{dN} = \frac{10}{N - 500}$$
$$t = 10\int \frac{1}{N - 500}\,dN$$
$$t = 10\log_e(N - 500) + c$$
$$10\log_e(N - 500) = t - c$$
$$\log_e(N - 500) = \frac{t}{10} + B\ \left(B = \frac{-c}{10}\right)$$
$$N - 500 = e^{\frac{t}{10} + B} = Ae^{\frac{t}{10}} \text{ where } A = e^B$$
$$t = 0,\ N = 400$$
$$400 - 500 = -100 = A$$
$$N = 500 - 100e^{\frac{t}{10}}$$
$$N = N(t) = 100\left(5 - e^{\frac{t}{10}}\right)$$
$$N = 200,\ t = ?$$
$$200 = 100\left(5 - e^{\frac{t}{10}}\right)$$
$$2 = 5 - e^{\frac{t}{10}}$$
$$e^{\frac{t}{10}} = 3$$
$$\frac{t}{10} = \log_e(3)$$
$$t = 10\log_e(3) = 10.99 \text{ years}$$

Take 11 years.

b $\dfrac{dN}{dt} = 0.1N - s,\ N(0) = 200$

$$\frac{dN}{dt} = \frac{N}{10} - s = \frac{N - 10s}{10}$$
$$\frac{dt}{dN} = \frac{10}{N - 10s}$$
$$t = 10\int \frac{1}{N - 10s}\,dN$$
$$t = 10\log_e(N - 10s) + c$$
$$t = 0, N = 200$$
$$c = -10\log_e(200 - 10s)$$
$$t = 10\log_e(N - 10s) - 10\log_e(200 - 10s)$$
$$\frac{t}{10} = \log_e\left(\frac{N - 10s}{200 - 10s}\right)$$
$$\frac{N - 10s}{200 - 10s} = e^{\frac{t}{10}}$$
$$N - 10s = (200 - 10s)e^{\frac{t}{10}}$$
$$N = 10s + (200 - 10s)e^{\frac{t}{10}}$$
$$N > 200$$
$$200 - 10s > 0$$
$$10s < 200$$
$$s < 20$$

Alternatively

$$\frac{dN}{dt} > 0 \rightarrow N - 10s > 0$$

Initially

$$N = N_0 = 200 \rightarrow 200 - 10s > 0$$
$$s < 20$$

Therefore the maximum number of koalas that can be sold is 19.

16 a $\dfrac{dN}{dt} = k\sqrt{N},\ N(0) = N_0$

$$\frac{dN}{dt} = kN^{\frac{1}{2}}$$
$$\frac{dt}{dN} = \frac{1}{kN^{\frac{1}{2}}}$$
$$kt = \int N^{-\frac{1}{2}}\,dN$$
$$kt = 2N^{\frac{1}{2}} + c$$
$$kt = 2\sqrt{N} + c$$
$$t = 0,\ N = N_0$$
$$0 = 2\sqrt{N_0} + c$$
$$c = -2\sqrt{N_0}$$
$$kt = 2\sqrt{N} - 2\sqrt{N_0}$$
$$2\sqrt{N} = kt - 2\sqrt{N_0}$$
$$\sqrt{N} = \frac{1}{2}kt - \sqrt{N_0}$$
$$N = N(t) = \left(\frac{1}{2}kt - \sqrt{N_0}\right)^2$$

b $\frac{dN}{dt} = kN^{-1} = \frac{k}{N}$

$$\frac{dt}{dN} = \frac{N}{k}$$
$$kt = \int N \quad dN$$
$$= \frac{1}{2}N^2 + c$$
$$t = 0,\ N = N_0$$
$$0 = \frac{1}{2}N_0^2 + c$$
$$c = -\frac{1}{2}N_0^2$$
$$kt = \frac{1}{2}N^2 - \frac{1}{2}N_0^2$$
$$\frac{1}{2}N^2 = kt + \frac{1}{2}N_0^2$$
$$N^2 = 2kt + N_0^2$$
$$N = N(t) = \sqrt{2kt + N_0^2}$$

c $\frac{dN}{dt} = kN^{\frac{3}{2}},\ N(0) = N_0$

$$\frac{dt}{dN} = \frac{1}{kN^{\frac{3}{2}}}$$
$$kt = \int N^{-\frac{3}{2}}\, dN$$
$$= -2N^{-\frac{1}{2}} + c$$
$$kt = -\frac{2}{\sqrt{N}} + c$$
$$t = 0,\ N = N_0$$
$$0 = -\frac{2}{\sqrt{N_0}} + c$$
$$c = \frac{2}{\sqrt{N_0}}$$
$$kt = -\frac{2}{\sqrt{N}} + \frac{2}{\sqrt{N_0}}$$
$$\frac{2}{\sqrt{N}} = \frac{2}{\sqrt{N_0}} - kt$$
$$\frac{2}{\sqrt{N}} = \frac{2 - \sqrt{N_0}kt}{\sqrt{N_0}}$$
$$\sqrt{N} = \frac{2\sqrt{N_0}}{2 - \sqrt{N_0}kt}$$
$$N = N(t) = \frac{4N_0}{\left(2 - kt\sqrt{N_0}\right)^2}$$

d $\frac{dN}{dt} = kN^2,\ N(0) = N_0$

$$\frac{dt}{dN} = \frac{1}{kN^2}$$
$$kt = \int N^{-2}\, dN$$
$$kt = -\frac{1}{N} + c$$
$$t = 0,\ N = N_0$$
$$0 = -\frac{1}{N_0} + c$$
$$c = \frac{1}{N_0}$$
$$kt = -\frac{1}{N} + \frac{1}{N_0}$$
$$\frac{1}{N} = \frac{1}{N_0} - kt$$
$$\frac{1}{N} = \frac{1 - N_0kt}{N_0}$$
$$N = N(t) = \frac{N_0}{1 - ktN_0}$$

17 a $\frac{dN}{dt} = kN - c,\ k > 0,\ c > 0,\ N(0) = N_0$

$$\frac{dt}{dN} = \frac{1}{kN - c}$$
$$t = \int \frac{1}{kN - c}\, dN$$
$$= \frac{1}{k}\log_e(kN - c) + A$$
$$t = 0,\ N = N_0$$
$$0 = \frac{1}{k}\log_e(kN_0 - c) + A$$
$$A = -\frac{1}{k}\log_e(kN_0 - c)$$
$$t = \frac{1}{k}\log_e(kN - c) - \frac{1}{k}\log_e(kN_0 - c)$$
$$kt = \log_e\left(\frac{kN - c}{kN_0 - c}\right)$$
$$e^{kt} = \frac{kN - c}{kN_0 - c}$$
$$kN - c = (kN_0 - c)e^{kt}$$
$$kN = (kN_0 - c)e^{kt} + c$$
$$N = N(t) = \left(N_0 - \frac{c}{k}\right)e^{kt} + \frac{c}{k}$$

b **i** If $N_0 > \frac{c}{k}$ then $N(t) = +e^{kt} + \frac{c}{k}$ (increasing exponentially)

ii If $N_0 < \frac{c}{k}$ then $N(t) = -e^{kt} + \frac{c}{k}$ (decreasing exponentially)

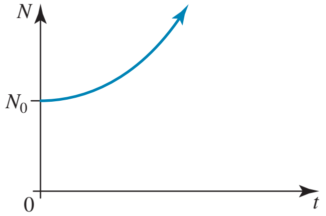
iii If $N_0 = \frac{c}{k}$ then $N(t) = \frac{c}{k}$ (remains constant/stable)

18 $\frac{dN}{dt} = kN$

$$t = 0,\ N = N(0) = N_0$$
$$t = \log_e(8),\ N = 2N_0$$
$$2N_0 = N_0e^{k\log_e(8)}$$
$$2 = e^{k\log_e(8)} = 8^k$$
$$k = \frac{1}{3}$$
$$\frac{dN}{dt} = kN - Q$$
$$\frac{dt}{dN} = \frac{1}{kN - Q}$$
$$t = \int \frac{1}{kN - Q}\, dN$$
$$t = \frac{1}{k}\log_e(kN - Q) + c$$
$$t = 0,\ N = N_0$$
$$0 = \frac{1}{k}\log_e(kN_0 - Q) + c$$
$$c = -\frac{1}{k}\log_e(kN_0 - Q)$$

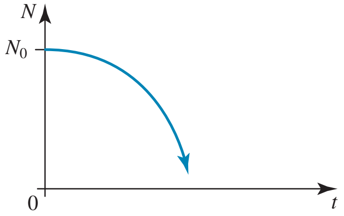
$$t = \frac{1}{k}\log_e(kN - Q) - \frac{1}{k}\log_e(kN_0 - Q)$$
$$kt = \log_e\left(\frac{kN - Q}{kN_0 - Q}\right)$$
$$\frac{kN - Q}{kN_0 - Q} = e^{kt}$$
$$kN - Q = (kN_0 - Q)e^{kt}$$
$$kN = (kN_0 - Q)e^{kt} + Q$$
$$N = N(t) = \left(N_0 - \frac{Q}{k}\right)e^{kt} + \frac{Q}{k}$$
$$k = \frac{1}{3}$$
$$N = N(t) = (N_0 - 3Q)e^{\frac{t}{3}} + 3Q$$

a If $N_0 > 3Q$ population increases



b If $N_0 = 3Q,\ N = N(t) = 3Q$ population remains stable

c If $N_0 < 3Q$ population decreases



10.3 Exam questions

1 a $$\frac{dQ}{dt} = -kQ,\ k > 0,\ Q(0) = Q_0$$
$$Q = Q_0e^{-kt}$$
$$Q(0.5) = 500$$
$$Q(1.0) = 250$$
$$[1] : 500 = Q_0e^{-\frac{k}{2}}$$
$$[2] : 250 = Q_0e^{-k}$$
$$2 = \frac{e^{-\frac{k}{2}}}{e^{-k}} = e^{\frac{k}{2}}$$
$$\frac{k}{2} = \log_e(2)$$
$$k = 2\ \log_e(2)$$
$$k = \log_e(4)$$
$$e^k = 4$$
$$Q_0 = 250e^k = 1000\text{ units}$$ [1 mark]

b $$Q = 1000e^{-kt}$$
$$Q = 125 = 1000e^{-kt}$$
$$\frac{125}{1000} = \frac{1}{8} = e^{-kt}$$
$$-kt = \log_e\left(\frac{1}{8}\right)$$
$$t = -\frac{1}{k}\log_e\left(\frac{1}{8}\right) = \frac{-\log_e\left(\frac{1}{8}\right)}{\log_e(4)}$$
$$= \frac{\log_e(8)}{\log_e(4)} = 1.5\text{ s}$$ [1 mark]

2 $$\frac{dQ}{dt} + \frac{Q}{RC} = 0,\ Q(0) = Q_0$$
$$\frac{dQ}{dt} = -\frac{Q}{RC}$$
$$\frac{dt}{dQ} = -\frac{RC}{Q}$$
$$t = -RC\int\frac{1}{Q}\,dQ$$ [1 mark]
$$= -RC\log_e(Q) + c$$
$$t = 0,\qquad Q = Q_0$$
$$0 = -RC\log_e(Q_0) + c$$
$$c = RC\log_e(Q_0)$$
$$t = -RC\ln(Q) + RC\ln(Q_0)$$ [1 mark]
$$\frac{t}{RC} = \ln\left(\frac{Q_0}{Q}\right)$$
$$\frac{Q_0}{Q} = e^{\frac{t}{RC}}$$
$$\frac{Q}{Q_0} = e^{-\frac{t}{RC}}$$
$$Q = Q(t) = Q_0e^{-\frac{t}{RC}}$$ [1 mark]

3 $$\frac{dI}{dx} = -kI$$
$$\frac{dx}{dI} = -\frac{1}{kI}$$
$$x = -\frac{1}{k}\int\frac{1}{I}dI$$
$$-kx = \log_e(I) + c$$ [1 mark]

When $x = 0,\ I = I_0$
$$c = -\log_e(I_0)$$
$$-kx = \log_e(I) - \log_e(I_0)$$
$$-kx = \log_e\left(\frac{I}{I_0}\right)$$
$$I = I_0e^{-kx}$$ [1 mark]

When $I = 0.95I_0,\ x = 0.40$
$$0.95I_0 = I_0e^{-0.4k}$$
$$0.95 = e^{-0.4k}$$
$$-0.4k = \log_e(0.95)$$
$$k = -\frac{1}{0.4}\log_e(0.95) = 0.1282$$
$$I = I_0e^{-0.1282x}$$
$$x = 1$$
$$I = I_0e^{-0.1282} = 0.880I_0$$

Then the percentage lost is 12.0% [1 mark]

10.4 Bounded growth and Newton's law of cooling

10.4 Exercise

1 $m(0) = 30,\ m(10) = 100,\ P = 200$
$$\frac{dm}{dt} = k(200 - m)$$
$$\frac{dt}{dm} = \frac{1}{k(200 - m)}$$

$$kt = \int \frac{1}{(200 - m)}\, dm$$
$$= -\log_e(200 - m) + c$$
$$t = 0,\ m(0) = 30$$
$$0 = -\log_e(200 - 30) + c$$
$$c = \log_e(170)$$
$$kt = -\log_e(200 - m) + \log_e(170)$$
$$kt = \log_e\left(\frac{170}{200 - m}\right)$$
$$t = 10,\ m = 100$$
$$10k = \log_e\left(\frac{170}{100}\right)$$
$$k = \frac{1}{10}\log_e\left(\frac{17}{10}\right) = 0.0531$$
$$e^{kt} = \frac{170}{200 - m}$$
$$200 - m = 170e^{-kt}$$
$$m = 200 - 170e^{-kt}$$
$$m = m(t) = 10(20 - 17e^{-kt})$$

After 30 weeks

$$m(30) = 10(20 - 17e^{-30k}) = 165.4 \text{ g}$$

2 $N(t)$ number of birds, t months

$$N(0) = N_0 = 200,\ N(5) = 800,\ P = 3000$$
$$\frac{dN}{dt} = k(P - N) = k(3000 - N)$$
$$\frac{dt}{dN} = \frac{1}{k(3000 - N)}$$
$$kt = \int \frac{1}{(3000 - N)}\, dN = -\log_e(3000 - N) + c$$
$$t = 0,\quad N = 200$$
$$0 = -\log_e(3000 - 200) + c$$
$$c = \log_e(2800)$$
$$kt = \log_e\left(\frac{2800}{3000 - N}\right)$$
$$N = 800,\ t = 5$$
$$5k = \log_e\left(\frac{2800}{3000 - 800}\right)$$
$$k = \frac{1}{5}\log_e\left(\frac{14}{11}\right) = 0.0482$$
$$e^{kt} = \frac{2800}{3000 - N}$$
$$3000 - N = 2800e^{-kt}$$
$$N = N(t) = 3000 - 2800e^{-kt}$$
$$N = 1500$$
$$1500 = 3000 - 2800e^{-kt}$$
$$2800e^{-kt} = 1500$$
$$e^{-kt} = \frac{15}{28}$$
$$-kt = \log_e\left(\frac{15}{28}\right)$$
$$t = \frac{5\log_e\left(\frac{28}{15}\right)}{\log_e\left(\frac{14}{11}\right)}$$
$$= 13 \text{ months}$$

3 a Cat mass m = kg, t = weeks

$$m(0) = 0.1,\ m(30) = 4,\ P = 5$$
$$\frac{dm}{dt} = k(p - m) = k(5 - m)$$
$$\frac{dt}{dm} = \frac{1}{k(5 - m)}$$
$$kt = \int \frac{1}{(5 - m)}\, dm$$
$$kt = -\log_e(5 - m) + c$$
$$t = 0,\ m(0) = 0.1$$
$$0 = -\log_e(4.9) + c$$
$$c = \log_e(4.9)$$
$$kt = -\log_e(5 - m) + \log_e(4.9)$$
$$kt = \log_e\left(\frac{4.9}{5 - m}\right)$$
$$t = 30,\ m = 4$$
$$30k = \log_e\left(\frac{4.9}{1}\right)$$
$$k = \frac{1}{30}\log_e(4.9) = 0.052\,075$$
$$e^{kt} = \frac{4.9}{5 - m}$$
$$5 - m = 4.9e^{-kt}$$
$$m = m(t) = 5 - 4.9e^{-kt}$$
$$m(40) = 5 - 4.9e^{-0.052\,075\times 40} = 4.4 \text{ kg}$$

b Toy poodle, m mass = kg, t = weeks

$$m(0) = 0.2,\ m(14) = 1.3,\ P = 4.5$$
$$\frac{dm}{dt} = k(4.5 - m) = k\left(\frac{9}{2} - m\right) = k\left(\frac{9 - 2m}{2}\right)$$
$$\frac{dt}{dm} = \frac{2}{k(9 - 2m)}$$
$$\frac{kt}{2} = \int \frac{1}{(9 - 2m)}\, dm$$
$$\frac{kt}{2} = -\frac{1}{2}\log_e(9 - 2m) + c$$
$$t = 0, m(0) = 0.2$$
$$0 = -\frac{1}{2}\log_e(9 - 2\times 0.2) + c$$
$$c = \frac{1}{2}\log_e\left(\frac{43}{5}\right)$$
$$\frac{kt}{2} = -\frac{1}{2}\log_e(9 - 2m) + \frac{1}{2}\log_e\left(\frac{43}{5}\right)$$
$$kt = \log_e\left(\frac{43}{5(9 - 2m)}\right)$$
$$t = 14, m = 1.3$$
$$14k = \log_e\left(\frac{43}{5(9 - 2\times 1.3)}\right)$$
$$k = \frac{1}{14}\log_e\left(\frac{43}{32}\right) = 0.0211$$
$$e^{kt} = \frac{43}{5(9 - 2m)}$$
$$5(9 - 2m) = 43e^{-kt}$$
$$45 - 10m = 43e^{-kt}$$

$$m = m(t) = \frac{1}{10}\left(45 - 43e^{-kt}\right)$$
$$t = 52$$
$$m(52) = \frac{1}{10}\left(45 - 43e^{-0.0211 \times 52}\right) = 3.06 \text{ kg}$$

4 **a** $N(0) = N_0 = 50,\ N(10) = 500,\ P = 1000$
$$\frac{dN}{dt} = k(P - N) = k(1000 - N)$$
$$\frac{dt}{dN} = \frac{1}{k(1000 - N)}$$
$$kt = \int \frac{1}{(1000 - N)}\, dN = -\log_e(1000 - N) + c$$
$$t = 0,\ N = 50$$
$$0 = -\log_e(1000 - 50) + c$$
$$c = \log_e(950)$$
$$kt = \log_e\left(\frac{950}{1000 - N}\right)$$
$$N = 500,\ t = 10$$
$$10k = \log_e\left(\frac{950}{500}\right)$$
$$k = \frac{1}{10}\log_e\left(\frac{19}{10}\right) = 0.06\,418$$
$$e^{kt} = \frac{950}{1000 - N}$$
$$1000 - N = 950e^{-kt}$$
$$N = N(t) = 1000 - 950e^{-kt}$$
$$= 50(20 - 19e^{-kt})$$
$$t = 20$$
$$N(20) = 50(20 - 19e^{-0.06\,481 \times 20}) = 736 \text{ fish}$$

b $N = 900,\ t = ?$
$$t = \frac{1}{k}\log_e\left(\frac{95}{10}\right)$$
$t = 35.1$ months

5 On a diet $m(t)$ mass in kg, t in weeks
$$m(0) = 84,\ m(10) = 77,\ P = 70$$
$$\frac{dm}{dt} = k(P - m) = k(70 - m)$$
$$\frac{dt}{dm} = \frac{1}{k(70 - m)}$$
$$kt = \int \frac{1}{(70 - \mathrm{m})}\, dm$$
$$kt = -\log_e(m - 70) + c$$
$$t = 0,\ m(0) = 84$$
$$0 = -\log_e(84 - 70) + c$$
$$c = \log_e(14)$$
$$kt = -\log_e(m - 70) + \log_e(14)$$
$$kt = \log_e\left(\frac{14}{m - 70}\right)$$
$$t = 10,\ m = 77$$
$$10k = \log_e\left(\frac{14}{7}\right)$$
$$k = \frac{1}{10}\log_e(2) = 0.069\,315$$
$$e^{kt} = \frac{14}{m - 70}$$
$$m - 70 = 14e^{-14t}$$
$$m = m(t) = 5(14 + e^{-kt})$$
$$t = 20$$
$$m(20) = 5(14 + e^{-0.069\,315 \times 20}) = 71.25\, kg$$
Lost 12.75 kg

6 $T_m = 2°\text{C},\ \theta = T - T_m$
$$t = 0,\ T_0 = 25 \rightarrow \theta_0 = 25 - 2 = 23$$
$$t = 5,\ T_0 = 22 \rightarrow \theta(2) = 22 - 2 = 20$$
$$\frac{dT}{dt} = k(T - T_m)$$
$$\frac{d\theta}{dt} = k\theta \rightarrow \theta = \theta_0 e^{kt}$$
$$20 = 23e^{5k}$$
$$e^{5k} = \frac{20}{23}$$
$$5k = \log_e\left(\frac{20}{23}\right)$$
$$k = \frac{1}{5}\log_e\left(\frac{20}{23}\right) = -0.028$$
$$\theta(t) = 23e^{-0.028t}$$
$$t = ?,\ T = 13,\ \theta = 13 - 2 = 11$$
$$11 = 23e^{-0.028t}$$
$$\frac{11}{23} = e^{-0.028t}$$
$$-0.028t = \log_e\left(\frac{11}{23}\right)$$
$$t = -\frac{1}{0.028}\log_e\left(\frac{11}{23}\right) = 26.34 \text{ min}$$

7 $T_m = 20°\text{C},\ \theta = T - T_m$
$$\frac{dT}{dt} = k(T - T_m)$$
$$\frac{d\theta}{dt} = k\theta \rightarrow \theta = \theta_0 e^{kt}$$
$$t = 0,\ T_0 = 250°\text{C} \rightarrow \theta_0 = 250 - 20 = 230$$
$$t = 6,\ T_0 = 210°\text{C} \rightarrow \theta(2) = 210 - 20 = 190$$
$$190 = 230e^{6k}$$
$$e^{6k} = \frac{19}{23}$$
$$6k = \log_e\left(\frac{19}{23}\right)$$
$$k = \frac{1}{6}\log_e\left(\frac{19}{23}\right) = -0.03184$$
$$\theta(t) = 230e^{-0.03184t}$$
$$t = -\frac{1}{0.03184}\log_e\left(\frac{\theta}{230}\right)$$
$$T = 195,\ \theta = 175$$
$$t_1 = -\frac{1}{0.03184}\log_e\left(\frac{175}{230}\right) = 8.5833$$
$$T = 205,\ \theta = 185$$
$$t_2 = -\frac{1}{0.03184}\log_e\left(\frac{185}{230}\right) = 6.838$$
$$t_1 - t_2 = 8.5833 - 6.838 = 1.75 \text{ min}$$

8 a Let $T = T(t)$ be the temperature of the can.

$$\frac{dT}{dt} = k(T - T_m),\ T_m = 30$$

$$\theta = T - T_m$$

$$\frac{d\theta}{dt} = k\theta \rightarrow \theta = \theta_0 e^{kt}$$

$t = 0,\ T_0 = 3°\text{C} \rightarrow \theta_0 = 3 - 30 = -27$

$t = 2,\ T = 4°\text{C} \rightarrow \theta(2) = 4 - 30 = -26$

$-26 = -27e^{2k}$

$$e^{2k} = \log_e\left(\frac{26}{27}\right)$$

$$k = \frac{1}{2}\log_e\left(\frac{26}{27}\right) = -0.0189$$

$\theta(t) = -27e^{-0.0189t}$

$t = 5$

$\theta(5) = -27e^{-0.0189\times 5} = -24.57$

$T = 30 - 24.57 = 5.43°\text{C}$

b $T(t)$ is temperature of the body

$$\frac{dT}{dt} = k(T - T_m),\ T_m = 18$$

$$\theta = T - T_m$$

$$\frac{d\theta}{dt} = k\theta \rightarrow \theta = \theta_0 e^{kt}$$

$T_0 = ?,\ \theta_0 = ?$

$t = 10,\ T = 22°\text{C} \rightarrow \theta(10) = 22 - 18 = 4$

$t = 20, T = 4°\text{C} \rightarrow \theta(20) = 20 - 18 = 2$

$4 = \theta_0 e^{10k}$

$2 = \theta_0 e^{20k}$

$2 = e^{-10k}$

$-10k = \log_e(2)$

$$k = -\frac{1}{10}\log_e(2) = -0.06\,931$$

$\theta_0 = 4e^{-10k} = 4e^{10\times 0.06\,931} = 8$

$T_0 = \theta_0 + T_m = 8 + 18 = 26°\text{C}$

9 $T(t)$ is temperature of baby's bath

$$\frac{dT}{dt} = k(T - T_m),\ T_m = 20,\ T_0 = 42,\ t = 0$$

$$\theta = T - T_m$$

$$\frac{d\theta}{dt} = k\theta \rightarrow \theta = \theta_0 e^{kt}$$

$t = 0,\ \rightarrow \theta_0 = 42 - 20 = 22$

$t = 2,\ T(2) = 40°\text{C} \rightarrow \theta(2) = 40 - 20 = 20$

$20 = 22e^{2k}$

$$k = \frac{1}{2}\log_e\left(\frac{10}{11}\right) = -0.0477$$

$\theta = 22e^{-0.0477t}$

$T = 38 \rightarrow \theta = 38 - 20 = 18°\text{C}$

$T = 34 \rightarrow \theta = 34 - 20 = 14°\text{C}$

$$\frac{\theta}{22} = e^{-0.0477t}$$

$$-0.0477t = \log_e\left(\frac{\theta}{22}\right)$$

$$t = -\frac{1}{0.0477}\log_e\left(\frac{\theta}{22}\right)$$

$$\theta = 18,\ t_1 = -\frac{1}{0.0477}\log_e\left(\frac{18}{22}\right) = 4.21$$

$$\theta = 14,\ t_2 = -\frac{1}{0.0477}\log_e\left(\frac{14}{22}\right) = 9.48$$

Time in bath: $t_2 - t_1 = 9.48 - 4.21 = 5.27$ min

10 Temperature $T_m = ?$

$$\frac{dT}{dt} = k(T - T_m),\ t = 0 = 8 \text{ am}$$

$$\theta = T - T_m$$

$$\frac{d\theta}{dt} = k_1\theta \rightarrow \theta = \theta_0 e^{k_1 t}$$

8 am : $t = 0,\ \rightarrow \theta = \theta_0 = T_0 - T_m = -18 - T_m$

11 am : $t = 3,\ T = 0 \rightarrow \theta = -T_m$

2 pm : $t = 6,\ T = 10 \rightarrow \theta = 10 - T_m$

[1] : $-T_m = (-18 - T_m)e^{3k_1}$

$$e^{3k_1} = \frac{T_m}{T_m + 18}$$

[2] : $10 - T_m = (-18 - T_m)e^{6k_1}$

$$e^{6k_1} = \frac{T_m - 10}{T_m + 18} = (e^{3k_1})^2$$

$$\frac{T_m - 10}{T_m + 18} = \left(\frac{T_m}{T_m + 18}\right)^2$$

$(T_m + 18)(T_m - 10) = T_m^2$

$T_m^2 + 8T_m - 180 = T_m^2$

$8T_m = 180$

$$T_m = \frac{180}{8} = 22.5°\text{C}$$

$$e^{3k_1} = \frac{T_m}{T_m + 18} = \frac{22.5}{22.5 + 18}$$

$$e^{3k_1} = \frac{5}{9}$$

$$3k_1 = \log_e\left(\frac{5}{9}\right)$$

$$k_1 = \frac{1}{3}\log_e\left(\frac{5}{9}\right)$$

$$T = T_m + \theta_0 e^{kt} = 22.5 - (18 + 22.5)e^{\frac{1}{3}\log_e\left(\frac{5}{9}\right)t}$$

5 pm : $t = 9 \rightarrow T = 22.5 - 40.5e^{\frac{1}{3}\log_e\left(\frac{5}{9}\right)\times 9} = 15.6°\text{C}$

Now placed in the oven $T_m = 200$

$$\frac{dT}{dt} = k_2(T - 200)$$

5 pm : $t = 0,\ \rightarrow T_0 = 15.6°\text{C} \rightarrow \theta_0 = T_0 - T_m = -184.4$

6 pm : $t = 1,\ T = 70 \rightarrow \theta = -130$

$\theta = \theta_0 e^{k_2 t}$

$-130 = -184.4e^{k_2}$

$$e^{k_2} = \frac{130}{184.4}$$

$$k_2 = \log_e\left(\frac{130}{184.4}\right) = -0.3498$$

7 pm : $t = 2 \rightarrow \theta = \theta_0 e^{2k_2} = -184.4e^{2k_2}$

$$= -184.4\left(\frac{130}{184.4}\right)^2 = -91.63$$

So chicken is at $-91.63 + 200 = 108.37°\text{C}$

11 $RC\dfrac{dv}{dt} + v = E,\ v(0) = 0$

$$RC\frac{dv}{dt} = E - v$$

$$\frac{1}{RC}\frac{dt}{dv} = \frac{1}{E - v}$$

$$\frac{t}{RC} = \int \frac{1}{E - v}\,dv$$

$$\frac{t}{RC} = -\log_e(E - v) + c$$

$$t = 0, \quad v = 0$$

$$0 = -\log_e(E) + c \rightarrow c = \log_e(E)$$

$$\frac{t}{RC} = -\log_e(E - v) + \log_e(E)$$

$$\frac{t}{RC} = \log_e\left(\frac{E}{E - v}\right)$$

$$e^{\frac{t}{RC}} = \frac{E}{E - v}$$

$$E - v = Ee^{-\frac{t}{RC}}$$

$$v = v(t) = E - Ee^{-\frac{t}{RC}} = E\left(1 - e^{-\frac{t}{RC}}\right)$$

12 $N = N(t)$ population at time t

Capacity P

$N(0) = N_0 > 0,\ P > 0$

$$\frac{dN}{dt} = k(P - N)$$

$$\frac{dt}{dN} = \frac{1}{k(P - N)}$$

$$kt = \int \frac{1}{(P - N)}\,dN = -\log_e|P - N| + c$$

Case 1:

$N > P$

$$kt = -\log_e(N - P) + c$$

$N(0) = N_0,\ t = 0, N = N_0$

$$0 = -\log_e(N_0 - P) + c$$

$$c = \log_e(N_0 - P)$$

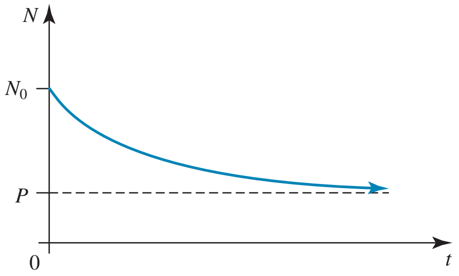
$$kt = -\log_e(N - P) + \log_e(N_0 - P)$$

$$= \log_e\left(\frac{N_0 - P}{N - P}\right)$$

$$e^{kt} = \frac{N_0 - P}{N - P}$$

$$N - P = (N_0 - P)e^{kt}$$

$$N = N(t) = P + (N_0 - P)e^{-kt}$$



Case 2:

$N < P$

$$kt = -\log_e(P - N) + c$$

$N(0) = N_0,\ t = 0, N = N_0$

$$0 = -\log_e(P - N_0) + c$$

$$c = \log_e(P - N_0)$$

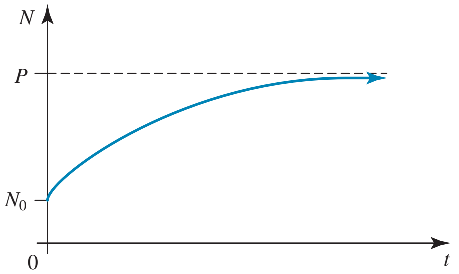
$$kt = -\log_e(P - N) + \log_e(P - N_0)$$

$$= \log_e\left(\frac{P - N_0}{P - N}\right)$$

$$e^{kt} = \frac{P - N_0}{P - N}$$

$$P - N = (P - N_0)e^{kt}$$

$$N = N(t) = P - (P - N_0)e^{-kt} = P + (N_0 - P)e^{-kt}$$



13 **a** $\dfrac{dT}{dt} = k(T - T_m),\ T(0) = T_0$

$$\frac{dt}{dT} = \frac{1}{k(T - T_m)}$$

$$t = \frac{1}{k}\int \frac{1}{(T - T_m)}\,dT$$

$$t = \frac{1}{k}\log_e(T - T_m) + c$$

let $b = kc$

$$kt - b = \log_e(T - T_m)$$

$$e^{kt-b} = e^{kt}\cdot e^{-b} = ae^{kt} = T - T_m \text{ where } a = e^{-b}$$

$$T = T(t) = T_m + ae^{kt}$$

$$T(0) = T_0$$

$$T_0 = T_m + a$$

$$a = T_0 - T_m$$

$$T = T(t) = T_m + (T_0 - T_m)e^{kt}$$

b $T = T(t) = T_m + (T_0 - T_m)\,e^{kt},\ T_m = 17°\text{C}$

$t = 0,\ T = T_0 = 37°\text{C}$

$t = t_1,\ T = 30.4°\text{C}$

$t = t_1 + 1,\ T = 29.9°\text{C}$

$[1]:\ 30.4 = 17 + 20e^{kt_1} \quad \rightarrow 30.4 - 17 = 20e^{kt_1}$

$[2]:\ 29.9 = 17 + 20e^{k(t_1+1)} \rightarrow 29.9 - 17 = 20e^{k(t_1+1)}$

$$= 20e^{kt_1}\cdot e^k$$

$$12.9 = 13.4e^k$$

$$e^k = \frac{12.9}{13.4}$$

$$k = \log_e\left(\frac{12.9}{13.4}\right) = -0.0380$$

$$e^{kt_1} = \frac{13.4}{20}$$

$$kt_1 = \log_e\left(\frac{13.4}{20}\right)$$

$$t_1 = -\frac{1}{0.0380}\log_e\left(\frac{13.4}{20}\right) = 10.53 \text{ hours}$$

8 am − 10.53 hours = 9:28 pm

14 Heating

$$\frac{dH}{dt} = k_1(H - 95),\ 0 \le t \le \frac{3}{2},\ T_0 = 3$$

a $T_h = 95,\ t_1 = \dfrac{3}{2}$

b Cooling

$$\frac{dc}{dt} = k_2(C - 3),\ 0 \le t \le \frac{3}{4}$$

$$T_c = 3,\ t_2 = \frac{3}{4}$$

c $H(t) = 95 - 92e^{k_1 t}$

$$H\left(\frac{3}{2}\right) = 45$$

$$45 = 95 - 92e^{\frac{3}{2}k_1}$$

$$92e^{\frac{3}{2}k_1} = 50$$

$$e^{\frac{3}{2}k_1} = \frac{50}{92}$$

$$\frac{3}{2}k_1 = \log_e\left(\frac{50}{92}\right)$$

$$k_1 = \frac{2}{3}\log_e\left(\frac{50}{92}\right),\ k_1 < 0$$

$$C\,(t) = 3 + 42e^{k_2 t}$$

$$C\left(\frac{3}{4}\right) = 35$$

$$35 = 3 + 42e^{\frac{3}{4}k_2}$$

$$32 = 42e^{\frac{3}{4}k_2}$$

$$e^{\frac{3}{4}k_2} = \frac{32}{42}$$

$$\frac{3}{4}k_2 = \log_e\left(\frac{32}{42}\right)$$

$$k_2 = \frac{4}{3}\log_e\left(\frac{32}{42}\right),\quad k_2 < 0$$

$$\frac{k_1}{k_2} = \frac{\frac{2}{3}\log_e\left(\frac{50}{92}\right)}{\frac{4}{3}\log_e\left(\frac{32}{42}\right)} = 1.12$$

15 a $\frac{dP}{dt} = rP - m,\ P(0) = P_0,\ P(T) = 0$

$\$P_0$ initial amount borrowed

r% rate per annum

m payments made annually

T loan paid of in T years

$$\frac{dt}{dP} = \frac{1}{rP - m}$$

$$t = \int \frac{1}{rP - m}\,dP$$

$$t = \frac{1}{r}\log_e(|rP - m|)\ +\ c$$

$$t = 0,\ P = P_0$$

$$0 = \frac{1}{r}\log_e(|rP_0 - m|) + c$$

$$c = -\frac{1}{r}\log_e(|rP_0 - m|)$$

$$t = \frac{1}{r}\log_e(|rP - m|) - \frac{1}{r}\log_e(|rP_0 - m|)$$

$$t = \frac{1}{r}\left[\log_e\left(\left|\frac{rP - m}{rP_0 - m}\right|\right)\right]$$

$$e^{rt} = \left(\left|\frac{rP - m}{rP_0 - m}\right|\right)$$

$$t = T,\ P = 0$$

$$\frac{m}{m - rP_0} = e^{rT}$$

b To find the payments

$$\frac{m - rP_0}{m} = e^{-rT}$$

$$m - rP_0 = me^{-rT}$$

$$m - me^{-rT} = rP_0$$

$$m(1 - e^{-rT}) = rP_0$$

$$m = \frac{rP_0}{(1 - e^{-rT})}$$

House loan

$P_0 = \$300\,000$

$r = 6\% = \frac{6}{100}$

$T = 20$ years

$$m = \frac{\frac{6}{100} \times 300\,000}{\left(1 - e^{-\frac{6}{100}\times 20}\right)} = \$25\,758.23 \text{ per year}$$

$$\frac{25\,758.23}{12} = \$2146.52 \text{ per month}$$

Total amount paid = $20 \times 25\,758.23 = \$515\,164.6$

Total amount interest paid = $\$515\,164.6 - \$300\,000$

$= \$215\,164.60$

c Repay \$350 per month

8% over 5 years

Loan $P_0 = ?$

$$P_0 = \frac{m}{r}\left(1 - e^{-rT}\right)$$

$m = \$350$

$r = \frac{8}{100} \times \frac{1}{12}$

$T = 5 \times 12$

$$P_0 = \frac{350}{\frac{8}{100} \times \frac{1}{12}}\left(1 - e^{-\frac{8}{100}\times\frac{1}{12}\times 5\times 12}\right) = \$17\,308.20$$

d To find time

$T = ?$

$$rT = \log e\left(\frac{m}{m - rP_0}\right)$$

$P_0 = \$120\,000$

$m = \$1200$

$r = 6.3\% = \frac{6.3}{100} \times \frac{1}{12}$

$$T = \frac{1}{\frac{6.3}{100} \times \frac{1}{12}}\log_e\left(\frac{1200}{1200 - \left(\frac{6.3}{100} \times \frac{1}{12} \times 120\,000\right)}\right)$$

$= 141.8$ months $= 11.8$ years

e $P_0 = \$8000$

$m = \$150$

$T = 6$ years $= 72$ months

$$\frac{m - rP_0}{m} = e^{-rT}$$

$$\frac{150 - \frac{8000 \times r}{12}}{150} = e^{-\frac{r}{12}\times 72}$$

$r = 10.56\%$ per annum

10.4 Exam questions

1 a Let $\theta = T - 20$

$\theta = \theta_0 e^{-kt}$

When $t = 0$, $T = 100$, therefore $\theta_0 = 80$ [1 mark]

When $t = 5$, $T = 80$, therefore $\theta(5) = 60$

$60 = 80e^{-5k}$

$e^{-5k} = \dfrac{60}{80}$

$e^{-5k} = \dfrac{3}{4}$ [1 mark]

b When $t = 10$, $\theta(10) = 80e^{-10k} = 80(e^{-5k})^2 = 80 \times \left(\dfrac{3}{4}\right)^2$

$= \dfrac{80 \times 9}{16} = 45$

So the temperature of the water is 65°C.

Award 1 mark for substituting.

Award 1 mark for the correct indices and finding the temperature difference.

Award 1 mark for the correct value of the water temperature.

VCAA Assessments Report **note:**

Surprisingly, the result of Question 5a.was rarely used directly, the majority of students expressed k in logarithmic form and often struggled with the algebra. In many cases, their solution to the differential equation had not been expressed with T as the subject but was left with t as the subject. Only a small number of students who did have $T = 20 + 80e^{-kt}$ were able to write $e^{-10k} = (e^{-5k})^2$ in order to solve this part efficiently. Some had difficulties with the exponentials and logarithms. There was some poor arithmetic at the end; for example, $T - 20 = 45$ so $T = 25$ was quit common. Several students multiplied first rather than cancelling and made their working more complicated. A small number of students left their answer in the from $T = \dfrac{1040}{16}$, $T = \dfrac{720}{16} + 20$ or $T = \dfrac{130}{2}$ (unsimplified).

2 a $L\dfrac{di}{dt} + Ri = E,\ i(0) = 0$

$L\dfrac{di}{dt} = E - Ri$

$\dfrac{1}{L}\dfrac{dt}{di} = \dfrac{1}{E - Ri}$

$\dfrac{1}{L}\displaystyle\int dt = \int \dfrac{1}{E - Ri}\,di$ [1 mark]

$\dfrac{t}{L} = -\dfrac{1}{R}\log_e(E - Ri) + c$

$t = 0, i = 0$

$0 = -\dfrac{1}{R}\log_e(E) + c$

$\to c = \dfrac{1}{R}\log_e(E)$ [1 mark]

$\dfrac{t}{L} = -\dfrac{1}{R}\log_e(E - Ri) + \dfrac{1}{R}\log_e(E)$ [1 mark]

$\dfrac{t}{L} = \dfrac{1}{R}[\log_e(E) - \log_e(E - Ri)]$

$\dfrac{Rt}{L} = \log_e\left(\dfrac{E}{E - Ri}\right)$

$e^{\frac{Rt}{L}} = \dfrac{E}{E - Ri}$

$(E - Ri)e^{\frac{Rt}{L}} = E$ [1 mark]

$E - Ri = Ee^{-\frac{Rt}{L}}$

$Ri = E - Ee^{-\frac{Rt}{L}} = E\left(1 - e^{-\frac{Rt}{L}}\right)$

$i = i(t) = \dfrac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$ [1 mark]

b Max $i(t) = \dfrac{E}{R},\ t \to \infty,\ e^{-\frac{Rt}{L}} \to 0$

$i(t) = \dfrac{E}{2R},$

$\dfrac{E}{2R} = \dfrac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$ [1 mark]

$e^{-\frac{Rt}{L}} = \dfrac{1}{2}$

$-\dfrac{Rt}{L} = \log_e\left(\dfrac{1}{2}\right)$

$t = -\dfrac{L}{R}\log_e\left(\dfrac{1}{2}\right) = \dfrac{L}{R}\log_e(2)$ [1 mark]

3 Thermometer $T_m = ?$

$t = 0,\ T = 25 \to \theta_0 = 25 - T_m$

$t = 5,\ T = 15 \to \theta(5) = 15 - T_m$

$t = 10,\ T = 10 \to \theta(10) = 10 - T_m$

$\theta = \theta_0 e^{kt}$

$[1] : \quad \theta_0 = 25 - T_m$

$[2] : \quad 15 - T_m = \theta_0 e^{5k}$

$[3] : \quad 10 - T_m = \theta_0 e^{10k}$

$e^{5k} = \dfrac{15 - T_m}{25 - T_m}$

$e^{10k} = \dfrac{10 - T_m}{25 - T_m}$

$e^{10k} = \left(e^{5k}\right)^2$ [1 mark]

$e^{5k} = \dfrac{15 - T_m}{25 - T_m}$

$e^{10k} = \dfrac{10 - T_m}{25 - T_m}$

$e^{10k} = \left(e^{5k}\right)^2$ [1 mark]

$\left(\dfrac{15 - T_m}{25 - T_m}\right)^2 = \dfrac{10 - T_m}{25 - T_m}$

$\dfrac{(15 - T_m)^2}{(25 - T_m)^2} = \dfrac{10 - T_m}{25 - T_m}$

$(15 - T_m)^2 = (25 - T_m)^2 \times \dfrac{10 - T_m}{25 - T_m}$

$(15 - T_m)^2 = (25 - T_m)(10 - T_m)$

$22 - 30T_m + T_m^2 = 250 - 35T_m + T_m^2$

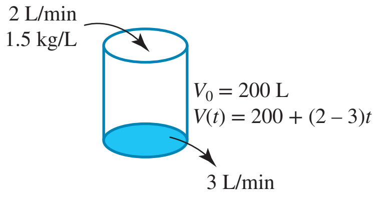
$5T_m = 25$

$T_m = 5°\text{C}$ [1 mark]

10.5 Chemical reactions and dilution problems

10.5 Exercise

1 **a**



$$\frac{dQ}{dt} = 2 \times 1.5 - \frac{3Q}{200 + (2 - 3)t} = 3 - \frac{3Q}{200 - t}$$

$$Q(0) = 0.1 \times 200 = 20$$

b $Q = \frac{3}{2}(200 - t) + C(200 - t)^3$

$$\text{LHS}: \ \frac{dQ}{dt} = -\frac{3}{2} - 3C(200 - t)^2$$

$$\begin{aligned}\text{RHS}: \ 3 - \frac{3Q}{200 - t} &= 3 - \frac{3}{200 - t}\left[\frac{3}{2}(200 - t) + C(200 - t)^3\right]\\ &= 3 - \frac{9}{2} - 3C(200 - t)^2\\ &= -\frac{3}{2} - 3C(200 - t)^2 = \text{LHS}\end{aligned}$$

c

$$\begin{aligned}Q(0) &= 20\\ 20 &= \frac{3}{2} \times 200 + 200^3 \times C\\ 20 &= 300 + 200^3 \times C\\ 200^3 \times C &= -280\\ C &= \frac{-7}{200\,000}\end{aligned}$$

d $Q(t) = \frac{3}{2}(200 - t) - \frac{7}{200\,000}(200 - t)^3$

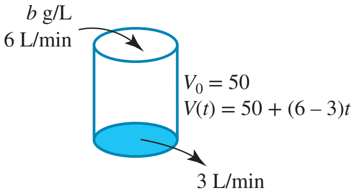
$$t = 100$$

$$Q(100) = \frac{3}{2} \times 100 - \frac{7}{200\,000}(100)^3 = 115$$

$$t = 100, v = 100$$

$$c = \frac{Q}{v} = \frac{115}{100} = 1.15 \text{ kg/L}$$

2 **a**



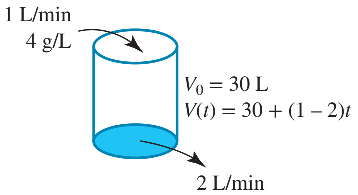
$$\frac{dQ}{dt} = 6b - \frac{3Q}{50 + 3t}$$

b $Q = 2(50 + 3t) + \frac{C}{50 + 3t}$

$$\text{LHS}: \ 6 - \frac{3C}{(50 + 3t)^2}$$

$$\begin{aligned}\text{RHS}: \ 6b - \frac{3}{50 + 3t}\left[2(50 + 3t) + \frac{C}{50 + 3t}\right] &= 6b - 6 - \frac{3C}{(50 + 3t)^2}\\ 6b - 6 &= 6\\ b &= 2\end{aligned}$$

3 **a**



$$\frac{dQ}{dt} = 4 - \frac{2Q}{30 - t}, Q(0) = 0$$

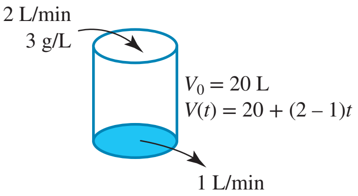
b $Q(t) = \frac{2t(30 - t)}{15}, Q(0) = 0$

$$= \frac{2}{15}(30t - t^2)$$

$$\text{LHS}: \ \frac{dQ}{dt} = \frac{2}{15}(30 - 2t) = \frac{4}{15}(15 - t)$$

$$\begin{aligned}\text{RHS}: \ 4 - \frac{2Q}{30 - t} &= 4 - \frac{2}{30 - t}\left[\frac{2t}{15}(30 - t)\right]\\ &= 4 - \frac{4t}{15}\\ &= \frac{1}{15}(60 - 4t)\\ &= \frac{4}{15}(15 - t) = \text{LHS}\end{aligned}$$

4 **a**



$$\frac{dQ}{dt} = 6 - \frac{Q}{20 + t}, Q(0) = 0$$

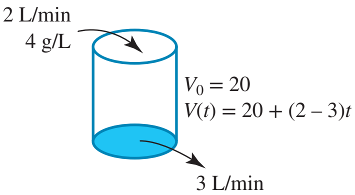
b $Q = \frac{3t(t + 40)}{20 + t} = \frac{3t^2 + 120t}{20 + t}$

$$\begin{aligned}\text{LHS}: \ \frac{dQ}{dt} &= \frac{(6t + 120)(20 + t) - (3t^2 + 120t)}{(20 + t)^2}\\ &= \frac{6t^2 + 240t + 2400 - (3t^2 + 120t)}{(20 + t)^2}\\ &= \frac{3t^2 + 120t + 2400}{(20 + t)^2}\\ &= \frac{3(t^2 + 40t + 800)}{(20 + t)^2}\end{aligned}$$

$$\text{RHS}: \ 6 - \frac{1}{20 + t}\left[\frac{3t(t + 40)}{20 + t}\right]$$

$$\begin{aligned}&= 6 - \frac{3t(t + 40)}{(20 + t)^2}\\ &= \frac{6(20 + t)^2 - 3t(t + 40)}{(20 + t)^2}\\ &= \frac{6(400 + 40t + t^2) - (3t^2 + 120t)}{(20 + t)^2}\\ &= \frac{3t^2 + 120t + 2400}{(20 + t)^2}\\ &= \frac{3(t^2 + 40t + 800)}{(20 + t)^2} = \text{LHS}\end{aligned}$$

5 a



$$\frac{dQ}{dt} = 8 - \frac{3Q}{20 - t},\ Q(0) = 0$$

b LHS : $Q = \frac{1}{100}t(t - 40)\,(t - 20)$

$$= \frac{1}{100}t(t^2 - 60t + 800)$$

$$= \frac{1}{100}(t^3 - 60t^2 + 800t)$$

$$\frac{dQ}{dt} = \frac{1}{100}(3t^2 - 120t + 800)$$

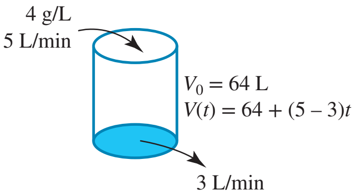
RHS : $8 - \frac{3}{20 - t}\left[\frac{1}{100}t(t - 40)(t - 20)\right]$

$$= 8 + \frac{3}{100}t(t - 40)$$

$$= \frac{1}{100}[800 + 3t(t - 40)]$$

$$= \frac{1}{100}(3t^2 - 120t + 800) = \text{LHS}$$

6 a



$$V(t) = 64 + (5 - 3)t$$

$$V_0 = 64L$$

$$\frac{dQ}{dt} = 4 \times 5 - \frac{3Q}{64 + (5 - 3)t},\ Q(0) = 2 \times 64 = 128$$

$$\frac{dQ}{dt} = 20 - \frac{3Q}{64 + 2t},\ Q(0) = 128$$

b LHS : $Q = 4(64 + 2t) + C(64 + 2t)^{-\frac{3}{2}}$

$$\frac{dQ}{dt} = 8 - \frac{3}{2} \times 2 \times C(64 + 2t)^{-\frac{5}{2}}$$

$$= 8 - 3C(64 + 2t)^{-\frac{5}{2}}$$

RHS:

$$20 - \frac{3Q}{64 + 2t} = 20 - \frac{3}{64 + 2t}\left[4\,(64 + 2t) + (64 + 2t)^{-\frac{3}{2}}\right]$$

$$= 20 - 12 - 3C(64 + 2t)^{-\frac{5}{2}}$$

$$= 8 - 3C(64 + 2t)^{-\frac{5}{2}} = \text{LHS}$$

c $Q(0) = 2 \times 64 = 128$

$$128 = 4 \times 64 + C \times 64^{-\frac{3}{2}}$$

$$= 256 + \frac{C}{512}$$

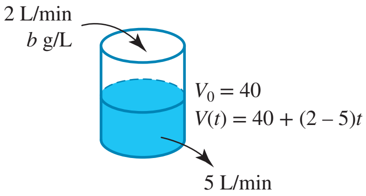
$$C = -128 \times 512 = -2^{16}$$

d $V = 100 = 64 + 2t \rightarrow t = 18, 0 \leq t \leq 18$

$$Q(18) = 4 \times 100 - 2^{16} \times 100^{-\frac{3}{2}} = 334.464 \text{ g}$$

$$c = \frac{Q}{V} = 3.34 \text{ g/L}$$

7



$$V(t) = 40 + (2 - 5)t$$

$$V_0 = 40L$$

$$\frac{dQ}{dt} = 2b - \frac{5Q}{40 - 3t}, Q(0) = 0$$

$$Q = 3\,(40 - 3t) + C(40 - 3t)^{\frac{5}{3}}$$

LHS : $\frac{dQ}{dt} = -9 - 3C \times \frac{5}{3}\,(40 - 3t)^{\frac{2}{3}}$

$$= -9 - 5C(40 - 3t)^{\frac{2}{3}}$$

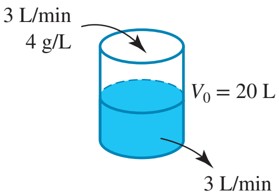
RHS : $2b - \frac{5Q}{40 - 3t} = 2b - \frac{5}{40 - 3t}\left[3(40 - 3t) + C(40 - 3t)^{\frac{5}{3}}\right]$

$$= 2b - 15 - 5C(40 - 3t)^{\frac{2}{3}}$$

$$2b - 15 = -9$$

$$b = 3$$

8 a



$$\frac{dQ}{dt} = 4 \times 3 - \frac{3Q}{20} = \frac{3(80 - Q)}{20}$$

$$Q(0) = 0.25 \times 20 = 5$$

b $\frac{dt}{dQ} = \frac{20}{3\,(80 - Q)}$

$$\frac{3t}{20} = \int \frac{1}{80 - Q}\,dQ = -\log_e(80 - Q)$$

$$Q(0) = 5$$

$$0 = -\log_e(75) + c$$

$$c = \log_e(75)$$

$$\frac{3t}{20} = \log_e(75) - \log_e(80 - Q)$$

$$\frac{3t}{20} = \log_e\left(\frac{75}{80 - Q}\right)$$

$$e^{\frac{3t}{20}} = \frac{75}{80 - Q}$$

$$80 - Q = 75e^{-\frac{3t}{20}}$$

$$Q = 80 - 75e^{-\frac{3t}{20}}$$

$$Q(t) = 5\left(16 - 15e^{-\frac{3t}{20}}\right)$$

c $t = \frac{8}{5}$

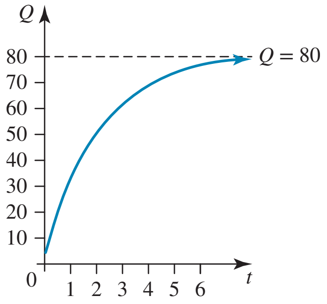
$$Q\left(\frac{8}{5}\right) = 5\left(16 - 15e^{\frac{-3\times 8}{5\times 20}}\right)$$

$$= 21.00$$

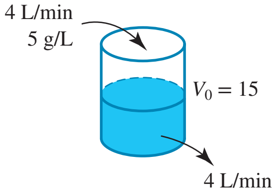
$$c = \frac{Q}{v} = \frac{21.00}{20} = 1.05 \text{ g/L}$$

d $t \to \infty, Q \to 80$

e



9 a



$$\frac{dQ}{dt} = 5\times 4 - \frac{4Q}{15} = \frac{4(75-Q)}{15}$$

$$Q(0) = 0$$

b $\frac{dt}{dQ} = \frac{15}{4\,(75-Q)}$

$$\frac{4t}{15} = \int \frac{1}{(75-Q)}\,dQ = -\log_e(75-Q) + c$$

$$Q(0) = 0$$

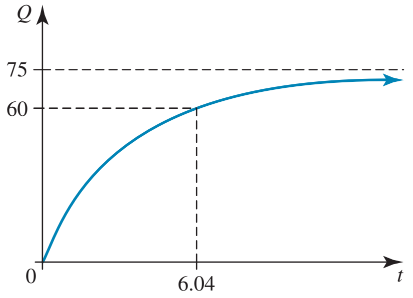
$$0 = -\log_e(75) + c \to c = \log_e(75)$$

$$\frac{4t}{15} = \log_e(75) - \log_e(75-Q) = \log_e\left(\frac{75}{75-Q}\right)$$

$$75 - Q = 75e^{-\frac{4t}{15}}$$

$$Q(t) = 75\left(1 - e^{-\frac{4t}{15}}\right), t \to \infty, Q \to 75$$

c



$t = ?$

$$c = 4 = \frac{Q}{15} \text{ g/L} \to Q = 60$$

$$60 = 75\left(1 - e^{-\frac{4t}{15}}\right)$$

$$\frac{60}{75} = 1 - e^{-\frac{4t}{15}}$$

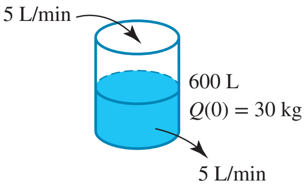
$$e^{-\frac{4t}{15}} = 1 - \frac{60}{75} = \frac{1}{5}$$

$$e^{\frac{4t}{15}} = 5$$

$$\frac{4t}{15} = \log_e(5)$$

$$t = \frac{15}{4}\log_e(5) = 6.04 \text{ min}$$

10 a



$$\frac{dQ}{dt} = 0 - \frac{5Q}{600},\ Q(0) = 30$$

$$\frac{dQ}{dt} = -\frac{Q}{120}$$

$$Q = 30e^{-\frac{t}{120}}$$

b 2 hours

$$t = 120$$

$$Q = 30e^{-1} = 11.04 \text{ kg}$$

c $Q = 15, t = ?$

$$15 = 30e^{-\frac{t}{120}}$$

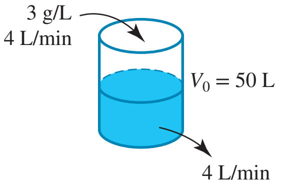
$$e^{-\frac{t}{120}} = \frac{1}{2}$$

$$e^{\frac{t}{120}} = 2$$

$$\frac{t}{120} = \log_e(2)$$

$$t = 120\log_e(2) = 83.18 \text{ min}$$

11



$$\frac{dQ}{dt} = 3\times 4 - \frac{4Q}{50}, Q(0) = 50$$

$$\frac{dQ}{dt} = 12 - \frac{2Q}{25}$$

$$\frac{dQ}{dt} = \frac{2(150-Q)}{25}$$

$$\frac{dt}{dQ} = \frac{25}{2(150-Q)}$$

$$t = \frac{25}{2}\int \frac{1}{(150-Q)}\,dQ$$

$$\frac{2t}{25} = -\log_e(|150-Q|) + c$$

$$0 = -\log_e(100) + c$$

$$c = \log_e(100)$$

$$\frac{2t}{25} = -\log_e(|150-Q|) + \log_e(100) = \log_e\left(\frac{100}{150-Q}\right)$$

Note: The absolute value can be removed since LHS = RHS when the given condition is substituted in.

$$e^{\frac{2t}{25}} = \frac{100}{150-Q}$$

$$150 - Q = 100e^{-\frac{2t}{25}}$$

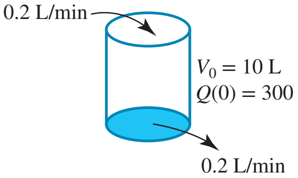
$$Q = 150 - 100e^{-\frac{2t}{25}}$$

$$Q(t) = 50\left(3 - 2e^{-\frac{2t}{25}}\right)$$

$$t = 5, Q(5) = 50\left(3 - 2e^{-\frac{2}{5}}\right) = 82.97 \text{ g}$$

$$c = \frac{Q(5)}{50} = \frac{82.97}{50} = 1.659 \text{ g/L}$$

12 **a**



$$\frac{dQ}{dt} = 0 - \frac{0.2Q}{10}, Q(0) = 300$$

$$\frac{dQ}{dt} = -\frac{Q}{50}$$

$$\frac{dt}{dQ} = \frac{-50}{Q}$$

$$-\frac{t}{50} = \int \frac{1}{Q}\,dQ$$

$$-\frac{t}{50} = \log_e(Q) + c$$

$$0 = \log_e(300) + c$$

$$c = -\log_e(300)$$

$$-\frac{t}{50} = \log_e(Q) - \log_e(300) = \log_e\left(\frac{Q}{300}\right)$$

$$e^{-\frac{t}{50}} = \frac{Q}{300}$$

$$Q(t) = 300e^{-\frac{t}{50}}$$

i $c = 25$ g/L, $Q = 150$

$$250 = 300e^{-\frac{t}{50}}$$

$$\frac{25}{30} = e^{-\frac{t}{50}}$$

$$\frac{6}{5} = e^{\frac{t}{50}}$$

$$\frac{t}{50} = \log_e\left(\frac{6}{5}\right)$$

$$t = 50\log_e\left(\frac{6}{5}\right) = 9.12 \text{ min}$$

ii $c = 15$ g/L, $Q = 150$

$$\frac{Q}{V} = c = 30e^{-\frac{t}{50}} = 15$$

$$e^{-\frac{t}{50}} = \frac{1}{2}$$

$$e^{\frac{t}{50}} = 2$$

$$\frac{t}{50} = \log_e(2)$$

$$t = 50\log_e(2) = 34.66 \text{ min}$$

b Temperature of the coffee cup $T(t)$

$$\frac{dT}{dt} = k(T - T_m),\ T_m = 17$$

$$\theta = T - T_m$$

$$\frac{d\theta}{dt} = k\theta$$

$$\theta = \theta_0 e^{kt}$$

$$t = 0, T_0 = 93, \theta_0 = T - T_m = 93 - 17 = 76$$

$$t = 1, T = 88, \theta(1) = 88 - 17 = 71$$

$$71 = 76e^{k}$$

$$e^{k} = \frac{71}{76}$$

$$k = \log_e\left(\frac{71}{76}\right) = -0.06\ 805$$

$$\theta = 76e^{-0.06\ 805t}$$

$$e^{-0.06\ 805t} = \frac{\theta}{76}$$

$$t = -\frac{1}{0.06\ 805}\log_e\left(\frac{\theta}{76}\right)$$

$$T = 50, \theta = 50 - 17 = 33$$

$$t_1 = -\frac{1}{0.06\ 805}\log_e\left(\frac{33}{76}\right) = 12.259$$

$$T = 65, \theta = 65 - 17 = 48$$

$$t_2 = -\frac{1}{0.06\ 805}\log_e\left(\frac{48}{76}\right) = 6.7528$$

Time to drink

$$t_1 - t_2 = 12.259 - 6.7528 = 5.5 \text{ min}$$

13 **a** $A + B \to X$

$$\frac{dx}{dt} \propto \left(1 - \frac{x}{2}\right)\left(3 - \frac{x}{2}\right)$$

$$\frac{dx}{dt} = k(2 - x)(6 - x),\ x(0) = 0$$

b

$$\frac{dt}{dx} = \frac{1}{k(2 - x)(6 - x)}$$

$$kt = \int \frac{1}{(2 - x)(6 - x)}\,dx$$

$$\frac{1}{(2 - x)(6 - x)} = \frac{A}{2 - x} + \frac{B}{6 - x}$$

$$= \frac{A(6 - x) + B(2 - x)}{(2 - x)(6 - x)}$$

$$= \frac{6A + 2B - x(A + B)}{(2 - x)(6 - x)}$$

$$\to A + B = 0 \to B = -A$$

$$6A + 2B = 1 \to 4A = 1$$

$$A = \frac{1}{4},\ B = -\frac{1}{4}$$

$$kt = \frac{1}{4}\int \frac{1}{2 - x} - \frac{1}{6 - x}\,dx$$

$$= \frac{1}{4}\left[-\log_e(|2 - x|) + \log_e(|6 - x|)\right] + c$$

$$= \frac{1}{4}\log_e\left(\left|\frac{6 - x}{2 - x}\right|\right) + c$$

$$x(0) = 0$$

$$0 = \frac{1}{4}\log_e(3) + c$$

$$c = -\frac{1}{4}\log_e(3)$$

$$kt = \frac{1}{4}\log_e\left(\frac{6-x}{2-x}\right) - \frac{1}{4}\log_e(3)$$

Note: The absolute value can be removed since LHS = RHS when the given condition is substituted in.

$$kt = \frac{1}{4}\log_e\left(\frac{6-x}{3(2-x)}\right)$$

$$k = ? \rightarrow x(3) = 1$$

$$3k = \frac{1}{4}\log_e\left(\frac{5}{3}\right)$$

$$k = \frac{1}{12}\log_e\left(\frac{5}{3}\right)$$

$$4kt = \log_e\left(\frac{6-x}{3(2-x)}\right)$$

$$\frac{6-x}{3(2-x)} = e^{4kt}$$

$$\frac{6-3x}{6-x} = e^{-4kt}$$

$$6-3x = (6-x)\,e^{-4kt}$$

$$= 6e^{-4kt} - xe^{-4kt}$$

$$6 - 6e^{-4kt} = 3x - xe^{-4kt}$$

$$6(1-e^{-4kt}) = x\left(3-e^{-4kt}\right)$$

$$x(t) = \frac{6(1-e^{-4kt})}{3-e^{-4kt}} \text{ where } k = \frac{1}{12}\log_e\left(\frac{5}{3}\right)$$

c $t = 6$

$$x\,(6) = \frac{6\left(1-e^{-24k}\right)}{3-e^{-24k}}$$

$$12k = \log_e\left(\frac{5}{3}\right)$$

$$e^{12k} = \frac{5}{3}$$

$$e^{-24k} = \left(\frac{3}{5}\right)^2$$

$$x(6) = \frac{6\left(1-\frac{9}{25}\right)}{3-\frac{9}{25}}$$

$$= \frac{16}{11}\text{ g}$$

14 a $A + B \rightarrow X$

$$\frac{dx}{dt} \propto \left(4-\frac{x}{2}\right)\left(4-\frac{x}{2}\right)$$

$$\frac{dx}{dt} = k(8-x)^2, x(0) = 0$$

b

$$\frac{dt}{dx} = \frac{1}{k(8-x)^2}$$

$$kt = \int \frac{1}{(8-x)^2}\,dx$$

$$kt = \frac{1}{(8-x)} + c$$

$$x(0) = 0$$

$$0 = \frac{1}{8} + c \rightarrow c = -\frac{1}{8}$$

$$kt = \frac{1}{(8-x)} - \frac{1}{8}$$

$$k = ?,\ x(2) = 3$$

$$2k = \frac{1}{5} - \frac{1}{8} = \frac{8-5}{40} \rightarrow k = \frac{3}{80}$$

$$\frac{3t}{80} = \frac{1}{8-x} - \frac{1}{8}$$

$$\frac{1}{8-x} = \frac{3t}{80} + \frac{1}{8}$$

$$\frac{1}{8-x} = \frac{3t+10}{80}$$

$$8-x = \frac{80}{3t-10}$$

$$x = 8 - \frac{80}{3t-10}$$

$$x = \frac{8\,(3t+10)-80}{3t+10}$$

$$x(t) = \frac{24t}{3t+10}$$

c

$$x = 6,\ t = ?$$

$$6 = \frac{24t}{3t+10}$$

$$6\,(3t+10) = 24t$$

$$18t + 60 = 24t$$

$$60 = 6t$$

$$t = 10$$

Takes another 8 minutes

d

$$x(t) = \frac{24t}{3t+10} = \frac{24}{3+\frac{10}{t}}$$

$$\lim_{t\to\infty} x(t) = 8\text{ g}$$

15 a $A + B \rightarrow X$

$$\frac{dx}{dt} \propto \left(2-\frac{x}{2}\right)\left(4-\frac{x}{2}\right)$$

$$\frac{dx}{dt} = k(4-x)(8-x),\ x(0) = 0$$

b

$$\frac{dt}{dx} = \frac{1}{k(4-x)\,(8-x)}$$

$$kt = \int \frac{1}{(4-x)\,(8-x)}\,dx$$

$$\frac{1}{(4-x)\,(8-x)} = \frac{A}{(4-x)} + \frac{B}{(8-x)}$$

$$= \frac{A(8-x)+B(4-x)}{(4-x)\,(8-x)}$$

$$= \frac{8A+4B-x(A+B)}{(4-x)\,(8-x)}$$

$$\rightarrow A + B = 0 \rightarrow B = -A$$

$$8A + 4B = 1 \rightarrow 4A = 1$$

$$A = \frac{1}{4},\ B = -\frac{1}{4}$$

$$kt = \frac{1}{4}\int \frac{1}{4-x} - \frac{1}{8-x}\,dx$$

$$= \frac{1}{4}\left[-\log_e(|4-x|) + \log_e(|8-x|)\right] + c$$

$$= \frac{1}{4}\log_e\left(\left|\frac{8-x}{4-x}\right|\right) + c$$

$$x(0) = 0$$

$$0 = \frac{1}{4}\log_e(2) + c$$

$$c = -\frac{1}{4}\log_e(2)$$

$$kt = \frac{1}{4}\log_e\left(\frac{8-x}{4-x}\right) - \frac{1}{4}\log_e(2)$$

Note: The absolute value can be removed since LHS = RHS when the given condition is substituted in.

$$kt = \frac{1}{4}\log_e\left(\frac{8-x}{2\,(4-x)}\right)$$

$$\begin{aligned}
k &= ? \rightarrow x(2) = 1\\
2k &= \frac{1}{4}\log_e\left(\frac{7}{6}\right)\\
k &= \frac{1}{8}\log_e\left(\frac{7}{6}\right)\\
4kt &= \log_e\left(\frac{8-x}{2(4-x)}\right)\\
\frac{8-x}{2(4-x)} &= e^{4kt}\\
\frac{2(4-x)}{8-x} &= e^{-4kt}\\
(8-x)e^{-4kt} &= 2(4-x)\\
8e^{-4kt} - xe^{-4kt} &= 8 - 2x\\
2x - xe^{-4kt} &= 8 - 8e^{-4kt}\\
x(2-e^{-4kt}) &= 8\left(1-e^{-4kt}\right)\\
x(t) &= \frac{8\left(1-e^{-4kt}\right)}{2-e^{-4kt}} \text{ where } k = \frac{1}{8}\log_e\left(\frac{7}{6}\right)
\end{aligned}$$

c $t = 4, x = ?$

$$\begin{aligned}
e^{8k} &= \frac{7}{6}\\
e^{-8k} &= \frac{6}{7}\\
x(4) &= \frac{8(1-e^{-16k})}{2-e^{-16k}}\\
&= \frac{8\left(1-\left(\frac{6}{7}\right)^2\right)}{2-\left(\frac{6}{7}\right)^2}\\
&= \frac{52}{31}\text{ g}
\end{aligned}$$

d $\lim\limits_{t\to\infty} x(t) = \dfrac{8}{2} = 4\text{ g},\ k > 0$

16 $A + B \rightarrow X$

$$\frac{dx}{dt} \propto \left(5-\frac{x}{2}\right)\left(2-\frac{x}{2}\right)$$

$x(0) = 0$

$$\begin{aligned}
\frac{dx}{dt} &= k(10-x)(4-x)\\
\frac{dt}{dx} &= \frac{1}{k(10-x)(4-x)}\\
kt &= \int \frac{1}{(10-x)(4-x)}\,dx
\end{aligned}$$

$$\begin{aligned}
\frac{1}{(10-x)(4-x)} &= \frac{A}{10-x} + \frac{B}{4-x}\\
&= \frac{A(4-x)+B(10-x)}{(10-x)(4-x)}\\
&= \frac{4A+10B-x(A+B)}{(10-x)(4-x)}\\
&\rightarrow A+B=0 \rightarrow B=-A\\
4A+10B &= 1 \rightarrow 6B = 1\\
A &= -\frac{1}{6}, B = \frac{1}{6}
\end{aligned}$$

$$\begin{aligned}
kt &= \frac{1}{6}\int \frac{1}{4-x} - \frac{1}{10-x}\,dx\\
&= \frac{1}{6}[-\log_e(|4-x|) + \log_e(|10-x|)] + c\\
&= \frac{1}{6}\log_e\left(\left|\frac{10-x}{4-x}\right|\right) + c
\end{aligned}$$

$$\begin{aligned}
x(0) &= 0\\
0 &= \frac{1}{6}\log_e\left(\frac{10}{4}\right) + c\\
c &= -\frac{1}{6}\log_e\left(\frac{5}{2}\right)\\
kt &= \frac{1}{6}\log_e\left(\frac{10-x}{4-x}\right) - \frac{1}{6}\log_e\left(\frac{5}{2}\right)
\end{aligned}$$

Note: The absolute value can be removed since LHS = RHS when the given condition is substituted in.

$$\begin{aligned}
kt &= \frac{1}{6}\log_e\left(\frac{2(10-x)}{5(4-x)}\right)\\
k &= ? \rightarrow x(3) = 2\\
k &= \frac{1}{18}\log_e\left(\frac{8}{5}\right) = 0.0261\\
6kt &= \log_e\left(\frac{2(10-x)}{5(4-x)}\right)
\end{aligned}$$

a $x = 3$

$$\begin{aligned}
6kt &= \log_e\left(\frac{2\times 7}{5\times 1}\right)\\
t &= \frac{1}{6k}\log_e\left(\frac{14}{5}\right) = 6.57\text{ min}
\end{aligned}$$

b

$$\begin{aligned}
6kt &= \log_e\left(\frac{2\,(10-x)}{5\,(4-x)}\right)\\
e^{6kt} &= \frac{2\,(10-x)}{5\,(4-x)}\\
\frac{5\,(4-x)}{2\,(10-x)} &= e^{-6kt}\\
5\,(4-x) &= 2\,(10-x)e^{-6kt}\\
20-5x &= 20e^{-6kt} - 2xe^{-6kt}\\
20\,(1-e^{-6kt}) &= x(5-2e^{-6kt})\\
x(t) &= \frac{20\,(1-e^{-6kt})}{(5-2e^{-6kt})}\\
t &= 6\\
e^{18k} &= \frac{8}{5}\\
e^{-18k} &= \frac{5}{8}\\
e^{-36k} &= \left(\frac{5}{8}\right)^2\\
x(6) &= \frac{20(1-e^{-36k})}{(5-2e^{-36k})}\\
&= \frac{20\left(1-\left(\frac{5}{8}\right)^2\right)}{\left(5-2\left(\frac{5}{8}\right)^2\right)}\\
&= \frac{26}{9}\text{ g}
\end{aligned}$$

c $t \rightarrow \infty$

$$\begin{aligned}
x &\rightarrow \frac{20}{5}\\
x &\rightarrow 4\text{ g}
\end{aligned}$$

17 a $\dfrac{dx}{dt} = k(a-x),\ x(0) = 0,\ a > 0$

$$\begin{aligned}
\frac{dt}{dx} &= \frac{1}{k(a-x)}\\
kt &= \int \frac{1}{(a-x)}\,dx = -\log_e(|a-x|) + c
\end{aligned}$$

$$0 = -\log_e(|a - 0|) + c$$
$$c = \log_e(a)$$
$$kt = -\log_e(|a - x|) + \log_e(a) = \log_e\left(\frac{a}{a - x}\right)$$

Note: The absolute value can be removed since LHS = RHS when the given condition is substituted in.

$$\frac{a}{a - x} = e^{kt}$$
$$a - x = ae^{-kt}$$
$$x = a - ae^{-kt}$$
$$x(t) = a(1 - e^{-kt})$$

b $a = 5,$

$$x = 2,\ t = 4$$
$$x = ?,\ t = 10$$
$$2 = 5(1 - e^{-4k})$$
$$\frac{2}{5} = 1 - e^{-4k}$$
$$e^{-4k} = 1 - \frac{2}{5} = \frac{3}{5}$$
$$e^{4k} = \frac{5}{3}$$
$$4k = \log_e\left(\frac{5}{3}\right)$$
$$k = \frac{1}{4}\log_e\left(\frac{5}{3}\right) = 0.1277$$
$$x(t) = 5(1 - e^{-0.1277t})$$
$$t = 10$$
$$x(10) = 5(1 - e^{-0.1277\times 10})$$
$$= 3.61 \text{ g}$$

18 a $A + B \to X,\ x(0) = 0$

$$\frac{dx}{dt} = k\left(b - \frac{x}{2}\right)^2$$
$$= k\left(\frac{2b - x}{2}\right)^2$$
$$= \frac{k}{4}(2b - x)^2$$
$$\frac{dt}{dx} = \frac{4}{k}\frac{1}{(2b - x)^2}$$
$$\frac{kt}{4} = \int \frac{1}{(2b - x)^2}\,dx = \frac{1}{2b - x} + c$$
$$0 = \frac{1}{2b} + c \to c = -\frac{1}{2b}$$
$$\frac{kt}{4} = \frac{1}{2b - x} - \frac{1}{2b}$$
$$\frac{1}{2b - x} = \frac{kt}{4} + \frac{1}{2b} = \frac{bkt + 2}{4b}$$
$$2b - x = \frac{4b}{bkt + 2}$$
$$x = 2b - \frac{4b}{bkt + 2} = \frac{2b(2 + bkt) - 4b}{2 + bkt}$$
$$x(t) = \frac{2b^2kt}{2 + bkt}$$

b $b = 5,\ x = 2,\ t = 4$

$$x = ?,\ t = 10$$
$$x(t) = \frac{2b^2kt}{2 + bkt}$$
$$2 = \frac{200k}{2 + 20k}$$
$$2(2 + 20k) = 200k$$
$$k = \frac{1}{40}$$
$$x = \frac{50 \times \frac{1}{40} \times t}{2 + 5 \times \frac{1}{40} \times t}$$
$$x(t) = \frac{10t}{t + 16}$$
$$x(10) = \frac{50}{13} = 3.85 \text{ g}$$

c $\lim\limits_{t\to\infty} \dfrac{10t}{t + 16} = \lim\limits_{t\to\infty} \dfrac{10}{1 + \frac{16}{t}} = 10 \text{ g}$

19 a $\dfrac{dQ}{dt} = bf - \dfrac{fQ}{V_0},\ Q(0) = 0$

$$= \frac{bfV_0 - fQ}{V_0}$$
$$= \frac{f}{V_0}(bV_0 - Q),\ Q(0) = 0$$

b $\dfrac{dt}{dQ} = \dfrac{V_0}{f(bV_0 - Q)}$

$$\frac{ft}{V_0} = \int \frac{1}{(bV_0 - Q)}\,dQ = -\log_e(bV_0 - Q) + c$$
$$0 = -\log_e(bV_0) + c \to c = \log_e(bV_0)$$
$$\frac{ft}{V_0} = -\log_e(bV_0 - Q) + \log_e(bV_0) = \log_e\left(\frac{bV_0}{bV_0 - Q}\right)$$
$$e^{\frac{ft}{V_0}} = \frac{bV_0}{bV_0 - Q}$$
$$bV_0 - Q = bV_0e^{\frac{-ft}{V_0}}$$
$$Q = bV_0 - bV_0e^{\frac{-ft}{V_0}}$$
$$Q(t) = bV_0\left(1 - e^{\frac{-ft}{V_0}}\right)$$
$$t \to \infty,\ f > 0,\ V_0 > 0$$
$$Q \to bV_0$$

20 a $\dfrac{dQ}{dt} = \dfrac{f}{V_0}(bV_0 - Q), Q\,(0) = q_0$

b

$$\frac{dt}{dQ} = \frac{V_0}{f(bV_0 - Q)}$$
$$\frac{ft}{V_0} = \int \frac{1}{(bV_0 - Q)}\,dQ = -\log_e(bV_0 - Q) + c$$
$$0 = -\log_e(bV_0 - q_0) + c \to c = \log_e(b - q_0)$$
$$\frac{ft}{V_0} = -\log_e(bV_0 - Q) + \log_e(bV_0 - q_0) = \log_e\left(\frac{bV_0 - q_0}{bV_0 - Q}\right)$$
$$e^{\frac{ft}{V_0}} = \frac{bV_0 - q_0}{bV_0 - Q}$$
$$bV_0 - Q = (bV_0 - q_0)e^{\frac{-ft}{V_0}}$$
$$Q = bV_0 - (bV_0 - q_0)e^{\frac{-ft}{V_0}}$$
$$Q(t) = bV_0 + (q_0 - bV_0)e^{-\frac{ft}{V_0}}$$

21 $\frac{dx}{dt} = k\left(a - \frac{x}{3}\right)^3,\ x(0) = 0$

$$\begin{aligned}
&= k\left(\frac{3a - x}{3}\right)^3\\
&= \frac{k}{27}(3a - x)^3\\
\frac{dt}{dx} &= \frac{27}{k}\frac{1}{(3a - x)^3}\\
\frac{kt}{27} &= \int \frac{1}{(3a - x)^3}\,dx = \frac{1}{2(3a - x)^2} + c\\
0 &= \frac{1}{18a^2} + c \to c = -\frac{1}{18a^2}\\
\frac{kt}{27} &= \frac{1}{2(3a - x)^2} - \frac{1}{18a^2}\\
&= \frac{9a^2 - (3a - x)^2}{18a^2(3a - x)^2}\\
\frac{kt}{27} &= \frac{9a^2 - (9a^2 - 6ax + x^2)}{18a^2(3a - x)^2}\\
&= \frac{6ax - x^2}{18a^2(3a - x)^2}\\
kt &= \frac{27x(6a - x)}{18a^2(3a - x)^2}\\
t &= \frac{3x(6a - x)}{2a^2k(3a - x)^2}
\end{aligned}$$

22 $x(0) = 0,\ k > 0,\ a > b > 0$

$$\begin{aligned}
\frac{dx}{dt} &= k\left(a - \frac{x}{2}\right)\left(b - \frac{x}{2}\right) = k\left(\frac{2a - x}{2}\right)\left(\frac{2b - x}{2}\right)\\
&= \frac{k}{4}(2a - x)(2b - x)\\
\frac{dt}{dx} &= \frac{4}{k(2a - x)(2b - x)}\\
\frac{kt}{4} &= \int \frac{1}{(2a - x)\,(2b - x)}\,dx
\end{aligned}$$

$$\begin{aligned}
\frac{1}{(2a - x)(2b - x)} &= \frac{A}{(2a - x)} + \frac{B}{(2b - x)}\\
&= \frac{A(2b - x) + B(2a - x)}{(2a - x)(2b - x)}\\
&= \frac{2aB + 2bA - x(A + B)}{(2a - x)(2b - x)}
\end{aligned}$$

$$\begin{aligned}
&\to A + B = 0 \to B = -A\\
&2aB + 2bA = 1\\
&-2aA + 2bA = 1\\
&2A(b - a) = 1\\
&A = -\frac{1}{2(a - b)},\ B = \frac{1}{2(a - b)}\\
&\frac{kt}{4} = \frac{1}{2(a - b)}\int \frac{1}{(2b - x)} - \frac{1}{(2a - x)}\,dx\\
&\frac{(a - b)kt}{2} = -\log_e(2b - x) + \log_e(2a - x) + c\\
&\qquad = \log_e\left(\frac{2a - x}{2b - x}\right) + c\\
&x(0) = 0\\
&0 = \log_e\left(\frac{a}{b}\right) + c \to c = -\log_e\left(\frac{a}{b}\right)\\
&\frac{(a - b)\,kt}{2} = \log_e\left(\frac{2a - x}{2b - x}\right) - \log_e\left(\frac{a}{b}\right) = \log_e\left(\frac{b\,(2a - x)}{a\,(2b - x)}\right)\\
&e^{\frac{(a-b)kt}{2}} = \frac{b(2a - x)}{a(2b - x)}
\end{aligned}$$

$$\begin{aligned}
a(2b - x) &= b(2a - x)e^{-\frac{(a-b)kt}{2}}\\
2ab - ax &= 2abe^u - bxe^u\\
u &= -\frac{(a - b)\,kt}{2}\\
2ab - 2abe^u &= ax - bxe^u\\
x(t) &= \frac{2ab\left(1 - e^{-\frac{(a-b)kt}{2}}\right)}{a - be^{-\frac{(a-b)kt}{2}}}
\end{aligned}$$

$a > b,\ k > 0,\ e^u > 0$

$\lim\limits_{t\to\infty} x(t) = \frac{2ab}{a} = 2b$

23 **a** $\frac{dQ}{dt} = 2\left(2 + \sin\left(\frac{t}{6}\right)\right) - \frac{2Q}{200}$

$Q(0) = 200 \times 0.01 = 2$

b Solve on CAS:

$Q(t) = -\frac{30000}{2509}\cos\left(\frac{t}{6}\right) + \frac{1800}{2509}\sin\left(\frac{t}{6}\right) + ce^{-\frac{t}{100}} + 400$

$Q(0) = 2$:

$2 = -\frac{30000}{2509}\cos(0) + \frac{1800}{2509}\sin(0) + ce^0 + 400$

$c = -\frac{968582}{2509}$

$\therefore Q(t) = -\frac{30000}{2509}\cos\left(\frac{t}{6}\right) + \frac{1800}{2509}\sin\left(\frac{t}{6}\right)$

$-\frac{968582}{2509}e^{-\frac{t}{100}} + 400$

c $Q(100) = -\frac{30000}{2509}\cos\left(\frac{100}{6}\right) + \frac{1800}{2509}\sin\left(\frac{100}{6}\right) -$

$\frac{968582}{2509}e^{-\frac{100}{100}} + 400 = 264.266$ grams

$c = \frac{Q}{V} = \frac{264.266}{200} = 1.32$ grams/litre

24 **a** $\frac{dQ}{dt} = 10 \times 3e^{-\frac{t}{2}} - \frac{5Q}{25 + (10 - 5)\,t}$

$\frac{dQ}{dt} = 30e^{-\frac{t}{2}} - \frac{5Q}{25 + 5t}$

$Q(0) = 0$

b Solve on CAS:

$Q(t) = \frac{c}{t + 5} - \frac{60\,(t + 7)\,e^{-\frac{t}{2}}}{t + 5}$

$Q\,(0) = 0$:

$0 = \frac{c}{5} - \frac{60 \times 7 \times e^0}{5}$

$c = 420$

$\therefore Q(t) = \frac{420}{t + 5} - \frac{60\,(t + 7)\,e^{-\frac{t}{2}}}{t + 5}$

c Max Q occurs when $\frac{dQ}{dt} = 0$

$\frac{dQ}{dt} = \frac{30\left(t^2 + 12t + 39\right)e^{-\frac{t}{2}}}{(t + 5)^2} - \frac{420}{(t + 5)^2}$

Solve $\frac{dQ}{dt} = 0$:

$t_{\max Q} = 3.9858$ minutes

$c = \frac{Q}{V} = \frac{Q}{25 + 5t} = \frac{1}{25 + 5t}\left(\frac{420}{t + 5} - \frac{60\,(t + 7)\,e^{-\frac{t}{2}}}{t + 5}\right)$

Max c occurs when $\frac{dc}{dt} = 0$

$\frac{dc}{dt} = \frac{6(t^2 + 14t + 53)e^{-\frac{t}{2}}}{(t + 5)^3} - \frac{168}{(t + 5)^3}$

Solve $\frac{dc}{dt} = 0$:

$t_{\max c} = 2.3780$ minutes

$\max c = c(2.3780) = 0.9136$ grams/litre

d $V = 325$

$25 + 5t = 325$

$t = 60$ minutes

$c(60) = 0.0199$ grams/litre

e Since $Q(0) = 0$,

Total Q flowed out

$= \text{Total } Q \text{ flowed in } -Q(60)$

$= \int_0^{60} 10 \times 3e^{-\frac{t}{2}} dt - Q(60)$

$= \left[-60e^{-\frac{t}{2}}\right]_0^{60} - Q(60)$

$= -60e^{-30} + 60 - 6.4515$

$= 53.54$ grams

10.5 Exam questions

1 a $\frac{dQ}{dt} = \text{inflow} - \text{outflow}$

$V(t) = 16 + (5 - 3)\text{t}$

$\frac{dQ}{dt} = 5 \times 0 - \frac{3Q}{V(t)} = 5 \times 0 - \frac{3Q}{16 + (5 - 3)t}$ [1 mark]

***VCAA Examination Report* note:**

This problem required students to recognise a difference of rates. The most common error was a failure to explicitly note that the rate in was zero.

b $\log_e(Q) = -\frac{3}{2}\log_e(16 + 2t) + c$ [1 mark]

$Q = \frac{1}{2},\ t = 0 \Rightarrow \log_e\left(\frac{1}{2}\right) + \frac{3}{2}\log_e(16) = c$

$\log_e(Q) =$

$-\frac{3}{2}\log_e(16 + 2t) + \log_e\left(\frac{1}{2}\right) + \frac{3}{2}\log_e(16)$ [1 mark]

$\log_e(Q) - \log_e\left(\frac{1}{2}\right) = -\frac{3}{2}\log_e(16 + 2t) + \frac{3}{2}\log_e(16)$

$\log_e(2Q) = \log_e\left(\frac{16}{16 + 2t}\right)^{\frac{3}{2}}$

$2Q = \left(\frac{16}{16 + 2t}\right)^{\frac{3}{2}} = \frac{\left(\sqrt{16}\right)^3}{(16 + 2t)^{\frac{3}{2}}} = \frac{64}{(16 + 2t)^{\frac{3}{2}}}$

$Q = \frac{32}{(16 + 2t)^{\frac{3}{2}}}$ $a = 32,\ b = 3,\ c = 2$ [1 mark]

***VCAA Examination Report* note:**

The majority of students realised that this was a separable differential equation, but many made errors in the subsequent integration with the arbitrary constant of integration frequently missing. Some students made transcription errors that fundamentally changed the problem. Others encountered arithmetic or algebraic issues. Many students took the common factor of 2 from the $16 + 2t$ expression and evaluated $\frac{1}{2}\int \frac{1}{8 + t} dt$. This unnecessary manipulation made subsequent calculations more difficult for these students.

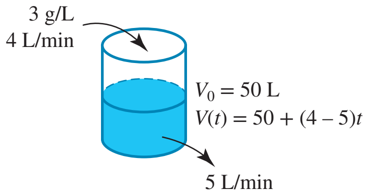
2 $\frac{dQ}{dt} = \text{inflow} - \text{outflow}$

$\frac{dQ}{dt} = 10 \times 2 - \frac{6Q}{V(t)}$ where $V(t) = 50 + (10 - 6)t$

$\frac{dQ}{dt} = 20 - \frac{6Q}{50 + 4t} = 20 - \frac{3Q}{25 + 2t}$

The correct answer is **A**.

3 a



$\frac{dQ}{dt} = 3 \times 4 - \frac{5Q}{50 + (4 - 5)t}$

$\frac{dQ}{dt} = 12 - \frac{5Q}{50 - t}$ [1 mark]

b $Q = 3\,(50 - t) + C(50 - t)^5$

LHS : $\frac{dQ}{dt} = -3 - 5C(50 - t)^4$ [1 mark]

RHS : $12 - \frac{5Q}{50 - t} = 12 - \frac{55}{50 - t}\left[3\,(50 - t) + C(50 - t)^5\right]$

$= 12 - 15 - 5C(50 - t)^4$

$= -3 - 5C(50 - t)^4$ [1 mark]

10.6 The logistic equation

10.6 Exercise

1 a $N(t)$ Number who had heard the rumour, t weeks

$N(0) = 4,\ P = 100$

$\frac{dN}{dt} = kN(100 - N)$

$\frac{dt}{dN} = \frac{1}{kN(100 - N)}$

$kt = \int \frac{1}{N(100 - N)} dN$

$\frac{1}{N(100 - N)} = \frac{A}{N} + \frac{B}{(100 - N)}$

$= \frac{A(100 - N) + BN}{N(100 - N)}$

$= \frac{N(B - A) + 100A}{N(100 - N)}$

$B - A = 0$

$100A = 1$

$A = B = \frac{1}{100}$

$kt = \frac{1}{100}\int \frac{1}{N} + \frac{1}{(100 - N)} dN$

$100kt = \log_e(N) - \log_e(100 - N) + c$

$0 = \log_e(4) - \log_e(96) + c \rightarrow c$

$c = \log_e(96) - \log_e(4) = \log_e\left(\frac{96}{4}\right)$

$100kt = \log_e(N) - \log_e(100 - N) + \log_e(24)$

$= \log_e\left(\frac{24N}{100 - N}\right)$

$k = ?,\ N(2) = 20$

$$\begin{aligned}
200k &= \log_e\left(\frac{24 \times 20}{100-20}\right) = \log_e(6)\\
k &= \frac{1}{200}\log_e(6) = 0.008\,959\\
100kt = 0.8959t &= \log_e\left(\frac{24N}{100-N}\right)\\
e^{100kt} &= \frac{24N}{100-N}\\
100 - N &= 24Ne^{-100kt}\\
100 &= N + 24Ne^{-100kt}\\
&= N\left(1 + 24e^{-100kt}\right)\\
N = N(t) &= \frac{100}{1 + 24e^{-0.8959t}}
\end{aligned}$$

b $N = N(5) = \dfrac{100}{1 + 24e^{-0.8959 \times 5}} = 78.61$

78 have heard it.

c

$$\begin{aligned}
N &= 50,\ t = ?\\
50 &= \frac{100}{1 + 24e^{-0.8959t}}\\
1 + 24e^{-0.8959t} &= 2\\
24e^{-0.8959t} &= 1\\
e^{-0.8959t} &= \frac{1}{24}\\
-0.8959t &= \log_e\left(\frac{1}{24}\right) = -\log_e(24)\\
t &= \frac{1}{0.8959}\log_e(24) = 3.55 \text{ weeks}
\end{aligned}$$

2 a $N(t)$ Number of infected

$N(0) = 2,\ P = 80$

$$\begin{aligned}
\frac{dN}{dt} &= kN(80-N)\\
\frac{dt}{dN} &= \frac{1}{kN(80-N)}\\
kt &= \int \frac{1}{N(80-N)}\,dN\\
\frac{1}{N(80-N)} &= \frac{A}{N} + \frac{B}{(80-N)}\\
&= \frac{A(80-N) + BN}{N(80-N)}\\
&= \frac{N(B-A) + 80A}{N(80-N)}
\end{aligned}$$

$$\begin{aligned}
B - A &= 0\\
80A &= 1\\
A = B &= \frac{1}{80}\\
kt &= \frac{1}{80}\int \frac{1}{N} + \frac{1}{(80-N)}\,dN\\
80kt &= \log_e(N) - \log_e(80-N) + c\\
0 &= \log_e(2) - \log_e(78) + c \rightarrow c\\
&= \log_e(78) - \log_e(2) = \log_e\left(\frac{78}{2}\right)\\
80kt &= \log_e(N) - \log_e(80-N) + \log_e(39)\\
&= \log_e\left(\frac{39N}{80-N}\right)
\end{aligned}$$

$k = ?,\ N(3) = 20$

$$\begin{aligned}
240k &= \log_e\left(\frac{39 \times 20}{80-20}\right) = \log_e(13)\\
k &= \frac{1}{240}\log_e(13) = 0.010\,687\\
80 \times 0.010\,687t &= \log_e\left(\frac{39N}{80-N}\right)\\
e^{0.855t} &= \frac{39N}{80-N}\\
80 - N &= 39Ne^{-0.855t}\\
80 &= N + 39Ne^{-0.855t}\\
N(t) &= \frac{80}{1 + 39e^{-0.855t}}
\end{aligned}$$

b $N(7) = \dfrac{80}{1 + 39e^{-0.855 \times 7}} = 72.85$

72 have the infection

c $60\% \times 80 = 48 = N,\ t = ?$

$$\begin{aligned}
48 &= \frac{80}{1 + 39e^{-0.855t}}\\
1 + 39e^{-0.855t} &= \frac{80}{48}\\
39e^{-0.855t} &= \frac{80}{48} - 1 = \frac{2}{3}\\
e^{-0.855t} &= \frac{2}{3} \times \frac{1}{39} = \frac{2}{117}\\
-0.855t &= \log_e\left(\frac{2}{117}\right)\\
t &= \frac{1}{0.855}\log_e\left(\frac{117}{2}\right) = 4.76 \text{ days}
\end{aligned}$$

3 a $N(t)$ Population of Midgar, t years after 2011.

$$N(t) = \frac{PN_0}{N_0 + (P - N_0)e^{-kt}}$$

$2011 : t = 0,\ N(0) = 3.948 = N_0$

$2015 : t = 4,\ N(4) = 4.256$

$2019 : t = 8,\ N(8) = 4.504$

$$\begin{aligned}
4.256 &= \frac{3.948P}{3.948 + (P - 3.948)e^{-4k}}\\
4.504 &= \frac{3.948P}{3.948 + (P - 3.948)e^{-8k}}\\
k &= 0.087107\\
P &= 5.23594
\end{aligned}$$

Maximum population of Midgar is 5.236 million.

b

$$\begin{aligned}
5 &= \frac{5.2359}{1 + 0.3262e^{-0.0871t}}\\
1 + 0.3262e^{-0.0871t} &= \frac{5.2359}{5}\\
0.3262e^{-0.0871t} &= 0.04719\\
e^{-0.0871t} &= \frac{0.04719}{0.3262}\\
-0.0871t &= \log_e(0.14465)\\
t &= -\frac{1}{0.0871}\log_e(0.14465) = 22.2
\end{aligned}$$

In the year 2033.

4 $N(t)$ Population of Hillwood, t years after 2011

$$N(t) = \frac{PN_0}{N_0 + (P - N_0)e^{-kt}}$$

$2011 : t = 0,\ N(0) = 1.609 = N_0$

$2013 : t = 2,\ N(2) = 1.735$

$2019 : t = 8,\ N(8) = 2.004$

$$1.735 = \frac{1.609P}{1.609 + (P - 1.609)e^{-2k}}$$
$$2.004 = \frac{1.609P}{1.609 + (P - 1.609)e^{-8k}}$$
$$k = 0.147089$$
$$P = 2.25021$$
$$N(t) = \frac{2.25021}{1 + 0.3985e^{-0.147089t}}$$

a
$$2 = \frac{2.25021}{1 + 0.3985e^{-0.147089t}}$$
$$1 + 0.3985e^{-0.147089t} = \frac{2.25021}{2}$$
$$0.3985e^{-0.147089t} = 0.1251$$
$$e^{-0.147089t} = \frac{0.1251}{0.3985}$$
$$-0.147089t = \log_e\left(\frac{0.1251}{0.3985}\right)$$
$$t = -\frac{1}{0.147089}\log_e\left(\frac{0.1251}{0.3985}\right) = 7.86$$

In the year 2018

b 2030 : $t = 19$
$$N(19) = \frac{2.25021}{1 + 0.3985e^{-0.147089 \times 19}} = 2.197 \text{ million}$$

5 **a** $y = \dfrac{500}{1 + 9e^{-3x}}$ [1]

Manipulating this equation gives:

$\dfrac{y}{500} = \dfrac{1}{1 + 9e^{-3x}}$ [2]

Rearranging [1] gives:
$$1 + 9e^{-3x} = \frac{500}{y}$$
$$9e^{-3x} = \frac{500}{y} - 1$$
$$= \frac{500 - y}{y}$$

$\therefore e^{-3x} = \dfrac{500 - y}{9y}$ [3]

Before we determine $\dfrac{dy}{dx}$ write [1] as
$$y = 500\left(1 + 9e^{-3x}\right)^{-1}$$
$$\frac{dy}{dx} = 500 \times -1 \times -27e^{-3x}\left(1 + 9e^{-3x}\right)^{-2}$$
$$= 500 \times 27e^{-3x} \times \frac{1}{\left(1 + 9e^{-3x}\right)^2}$$
$$= 500 \times 27e^{-3x} \times \frac{1}{1 + 9e^{-3x}} \times \frac{1}{1 + 9e^{-3x}}$$

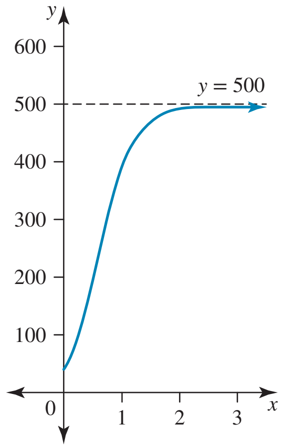
Substituting [2] and [3] into this gives:
$$\frac{dy}{dx} = 500 \times 27\left(\frac{500 - y}{9y}\right) \times \frac{y}{500} \times \frac{y}{500}$$
$$= \left(\frac{27y^2}{500}\right)\left(\frac{500 - y}{9y}\right)$$
$$= \left(\frac{3y}{500}\right)(500 - y)$$
$$\therefore \frac{dy}{dx} = \frac{3y\,(500 - y)}{500}$$

b
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dy}\left(\frac{dy}{dx}\right) \cdot \frac{dy}{dx}$$
$$= \frac{d}{dy}\left(\frac{3}{500}y(500 - y)\right) \cdot \frac{dy}{dx}$$
$$= \frac{d}{dy}\frac{3}{500}\left(500y - y^2\right) \cdot \frac{dy}{dx}$$
$$= \frac{3}{500}(500 - 2y) \times \left(\frac{3}{500}y(500 - y)\right)$$
$$= \frac{9}{250\,000}y(500 - 2y)(500 - y)$$
$$= \frac{9y(250 - y)(500 - y)}{125\,000}$$

c
$$\frac{d^2y}{dx^2} = 0$$
$$\rightarrow (500 - 2y) = 0$$
$$y = 250$$
$$y = 250 = \frac{500}{1 + 9e^{-3x}}$$
$$1 + 9e^{-3x} = 2$$
$$9e^{-3x} = 1$$
$$e^{-3x} = \frac{1}{9}$$
$$e^{3x} = 9$$
$$3x = \log_e(9)$$
$$x = \frac{1}{3}\log_e(9)$$

Inflection point $\left(\dfrac{1}{3}\log_e(9), 250\right) \approx (0.73, 250)$

d $y(0) = 50,\ \lim\limits_{x \to \infty} y(x) = 500$



6 **a**
$$y = \frac{400}{1 + 199e^{-2x}} = 400\left(1 + 199e^{-2x}\right)^{-1}$$
$$\frac{dy}{dx} = 400 \times (-1) \times (-2) \times 199e^{-2x}\left(1 + 199e^{-2x}\right)^{-2}$$
$$= \frac{400 \times 2 \times 199e^{-2x}}{\left(1 + 199e^{-2x}\right)^2}$$
$$= \frac{400}{\left(1 + 199e^{-2x}\right)} \times \frac{2 \times 199e^{-2x}}{\left(1 + 199e^{-2x}\right)}$$

$$1+199e^{-2x}=\frac{400}{y}$$

$$\frac{1}{1+199e^{-2x}}=\frac{y}{400}$$

$$199e^{-2x}=\frac{400-y}{y}$$

$$\frac{dy}{dx}=y\times 2\left(\frac{400-y}{y}\right)\times\frac{y}{400}=\frac{y(400-y)}{200}$$

b $$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dy}\left(\frac{dy}{dx}\right)\cdot\frac{dy}{dx}$$

$$=\frac{d}{dy}\left(\frac{1}{200}\left(400y-y^2\right)\right)\cdot\frac{dy}{dx}$$

$$=\frac{1}{200}(400-2y)\times\left(\frac{1}{200}y(400-y)\right)$$

$$=\frac{y(200-y)(400-y)}{20\,000}$$

c $$\frac{d^2y}{dx^2}=0$$

$$\rightarrow (200-y)=0$$

$$y=200$$

$$y=200=\frac{400}{1+199e^{-2x}}$$

$$1+199e^{-2x}=2$$

$$199e^{-2x}=1$$

$$e^{-2x}=\frac{1}{199}$$

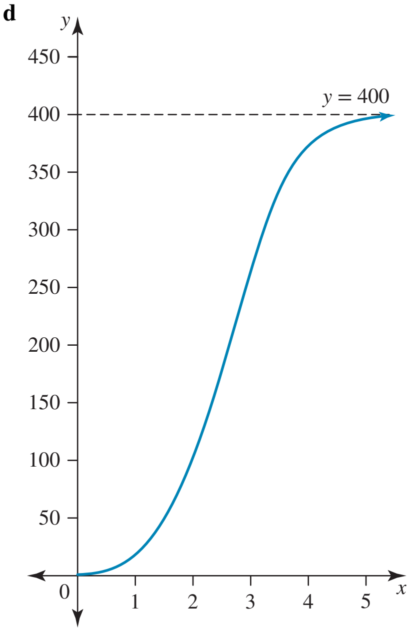
$$e^{2x}=199$$

$$2x=\log_e(199)$$

$$x=\frac{1}{2}\log_e(199)$$

Inflection point $\left(\frac{1}{2}\log_e(199),\ 200\right)\approx(2.65,\ 200)$

d



7 **a** $$y=\frac{600}{1+99e^{-\frac{x}{3}}}=600\left(1+99e^{-\frac{x}{3}}\right)^{-1}$$

$$\frac{dy}{dx}=(600)\times(-1)\times\left(-\frac{1}{3}\right)\times 99e^{-\frac{x}{3}}\left(1+99e^{-\frac{x}{3}}\right)^{-2}$$

$$=\frac{200\times 99e^{-\frac{x}{3}}}{\left(1+99e^{-\frac{x}{3}}\right)^2}=\frac{600\times 99e^{-\frac{x}{3}}}{3\left(1+99e^{-\frac{x}{3}}\right)^2}$$

$$1+99e^{-\frac{x}{3}}=\frac{600}{y}$$

$$99e^{-\frac{x}{3}}=\frac{600-y}{y}$$

$$\frac{dy}{dx}=\frac{y}{3}\left(\frac{600-y}{y}\right)\times\frac{y}{600}=\frac{y(600-y)}{1800}$$

b $$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dy}\left(\frac{dy}{dx}\right)\cdot\frac{dy}{dx}$$

$$=\frac{d}{dy}\left(\frac{1}{1800}\left(600y-y^2\right)\right)\left(\frac{1}{1800}y(600-y)\right)$$

$$=\frac{1}{1800^2}(600-2y)\,y(600-y)$$

$$=\frac{y(300-y)(600-y)}{1620\,000}$$

c $$\frac{d^2y}{dx^2}=0$$

$$y=300=\frac{600}{1+99e^{-\frac{x}{3}}}$$

$$1+99e^{-\frac{x}{3}}=2$$

$$99e^{-\frac{x}{3}}=1$$

$$e^{-\frac{x}{3}}=\frac{1}{99}$$

$$e^{\frac{x}{3}}=99$$

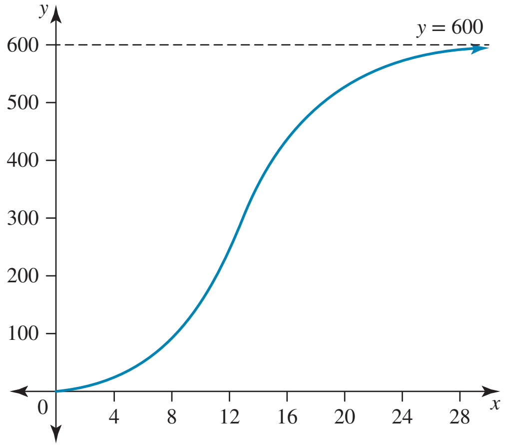
$$\frac{x}{3}=\log_e(99)$$

$$x=3\log_e(99)$$

Inflection point $(3\log_e(99),\ 300)\approx(13.79, 300)$

$y(0)=6,\ \lim\limits_{x\to\infty}y(x)=600$

d



8 $N(t)$ number of fish

a $N(0)=N_0=500,\ P=4000,\ k=8\%$

$$\frac{dN}{dt}=kN\left(1-\frac{N}{P}\right)$$

$$\frac{dN}{dt}=\frac{8}{100}N\left(1-\frac{N}{4000}\right)$$

$$= \frac{2N}{25}\left(\frac{4000-N}{4000}\right)$$

$$= \frac{N(4000-N)}{50\,000},\ N(0) = 500$$

b
$$\frac{dt}{dN} = \frac{50\,000}{N(4000-N)}$$

$$\frac{t}{50\,000} = \int \frac{1}{N(4000-N)}\,dN$$

$$\frac{1}{N(4000-N)} = \frac{A}{N} + \frac{B}{(4000-N)}$$

$$= \frac{A(4000-N)+BN}{N(4000-N)}$$

$$= \frac{N(B-A)+4000A}{N(4000-N)}$$

$$B - A = 0$$

$$4000A = 1$$

$$A = B = \frac{1}{4000}$$

$$\frac{t}{50\,000} = \frac{1}{4000}\int \frac{1}{N} + \frac{1}{(4000-N)}\,dN$$

$$\frac{2t}{25} = \log_e(N) - \log_e(4000-N) + c$$

$$0 = \log_e(500) + \log_e(4000-500) + c \to c$$

$$c = \log_e(3500) - \log_e(500) = \log_e(7)$$

$$\frac{2t}{5} = \log_e(N) - \log_e(4000-N) + \log_e(7)$$

$$= \log_e\left(\frac{7N}{4000-N}\right)$$

$$e^{\frac{2t}{5}} = \frac{7N}{4000-N}$$

$$4000 - N = 7Ne^{-\frac{2t}{5}}$$

$$4000 = N + 7Ne^{-\frac{2t}{5}}$$

$$4000 = N\left(1 + 7e^{-\frac{2t}{5}}\right)$$

$$N = N(t) = \frac{4000}{1 + 7e^{-\frac{2t}{5}}}$$

c $N = N(5) = \dfrac{4000}{1+7e^{-2}} = 2054$ fish

d $N = 3000,\ t = ?$

$$3000 = \frac{4000}{1 + 7e^{-\frac{2t}{5}}}$$

$$1 + 7e^{-\frac{2t}{5}} = \frac{4}{3}$$

$$7e^{-\frac{2t}{5}} = \frac{1}{3}$$

$$e^{-\frac{2t}{5}} = \frac{1}{21}$$

$$e^{\frac{2t}{5}} = 21$$

$$\frac{2t}{5} = \log_e(21)$$

$$t = \frac{5}{2}\log_e(21) = 7.61 \text{ years}$$

9 a $N(t)$ number of zombies, t days

$N(0) = 1,\ P = 360,\ k = N(3) = 60$

$$\frac{dN}{dt} = kN(360-N),\ N(0) = 1$$

b
$$\frac{dt}{dN} = \frac{1}{kN(360-N)}$$

$$kt = \int \frac{1}{N(360-N)}\,dN$$

$$\frac{1}{N(360-N)} = \frac{A}{N} + \frac{B}{(360-N)}$$

$$= \frac{A(360-N)+BN}{N(360-N)}$$

$$= \frac{N(B-A)+360A}{N(360-N)}$$

$$B - A = 0$$

$$360A = 1$$

$$A = B = \frac{1}{360}$$

$$kt = \frac{1}{360}\int \frac{1}{N} + \frac{1}{(360-N)}\,dN$$

$$360kt = \log_e(N) - \log_e(360-N) + c$$

$$0 = \log_e(1) - \log_e(359) + c \to c = \log_e(359)$$

$$360kt = \log_e(N) - \log_e(360-N) + \log_e(359)$$

$$= \log_e\left(\frac{359N}{360-N}\right)$$

$$k = ?,\ N = 60,\ t = 3$$

$$3 \times 360k = \log_e\left(\frac{359 \times 60}{360-60}\right) = \log_e\left(\frac{359}{5}\right)$$

$$360k = \frac{1}{3}\log_e\left(\frac{359}{5}\right) = 1.42\,463$$

$$360kt = \log_e\left(\frac{359N}{360-N}\right)$$

$$\frac{359N}{360-N} = e^{360kt}$$

$$360 - N = 359Ne^{-360kt}$$

$$360 = N(1 + 359e^{-360kt})$$

$$N(t) = \frac{360}{1 + 359e^{-360kt}}$$

$$75\% \times 360 = 270$$

$$N = 270,\ t = ?$$

$$270 = \frac{360}{1 + 359e^{-360kt}}$$

$$1 + 359e^{-360kt} = \frac{360}{270}$$

$$359e^{-360kt} = \frac{1}{3}$$

$$e^{-360kt} = \frac{1}{3 \times 359}$$

$$-360kt = \log_e\left(\frac{1}{3 \times 359}\right)$$

$$t = \frac{1}{360k}\log_e(3 \times 359)$$

$$= \frac{1}{1.42\,463}\log_e(3 \times 359) = 4.9 \text{ days}$$

10 $N(t)$ number of students, t days

a $N(0) = 2,\ P = 2000,\ N(3) = 200$

$$\frac{dN}{dt} = kN(2000-N), N(0) = 2$$

b

$$\frac{dt}{dN} = \frac{1}{kN(2000 - N)}$$

$$kt = \int \frac{1}{N(2000 - N)}\, dN$$

$$\begin{aligned}
\frac{1}{N(2000 - N)} &= \frac{A}{N} + \frac{B}{(2000 - N)} \\
&= \frac{A(2000 - N) + BN}{N(2000 - N)} \\
&= \frac{N(B - A) + 2000A}{N(2000 - N)}
\end{aligned}$$

$$B - A = 0$$

$$2000A = 1$$

$$A = B = \frac{1}{2000}$$

$$kt = \frac{1}{2000}\int \frac{1}{N} + \frac{1}{(2000 - N)}\, dN$$

$$2000kt = \log_e(N) - \log_e(2000 - N) + c$$

$$0 = \log_e(2) - \log_e(2000 - 2) + c \rightarrow c = \log_e(999)$$

$$\begin{aligned}
2000kt &= \log_e(N) - \log_e(2000 - N) + \log_e(999) \\
&= \log_e\left(\frac{999N}{(2000 - N)}\right)
\end{aligned}$$

$$k = ?,\ N = 200,\ t = 3$$

$$6000k = \log_e\left(\frac{999 \times 200}{2000 - 200}\right) = \log_e(111)$$

$$2000k = \frac{1}{3}\log_e(111) = 1.56\,984$$

$$2000kt = \log_e\left(\frac{999N}{(2000 - N)}\right)$$

$$\frac{999N}{(2000 - N)} = e^{2000kt}$$

$$2000 - N = 999Ne^{-2000kt}$$

$$2000 = N(1 + 999e^{-2000kt})$$

$$N(t) = \frac{2000}{1 + 999e^{-1.56984t}}$$

$$N = 1000$$

$$1000 = \frac{2000}{1 + 999e^{-2000kt}}$$

$$1 + 999e^{-2000kt} = 2$$

$$999e^{-2000kt} = 1$$

$$e^{-2000kt} = \frac{1}{999}$$

$$2000kt = \log_e(999)$$

$$t = \frac{\log_e(999)}{\frac{1}{3}\log_e(111)} = 4.4 \text{ days}$$

c The rumour is spreading most rapidly when $N = \dfrac{P}{2}$

That is after 4.4 days

11 $N(t)$ number who have the epidemic

$N(0) = 4,\ P = 10\,000,\ N(4) = 50$

$$\frac{dN}{dt} = kN(10\,000 - N)$$

$$\frac{dt}{dN} = \frac{1}{kN(10\,000 - N)}$$

$$kt = \int \frac{1}{N(10\,000 - N)}\, dN$$

$$\begin{aligned}
\frac{1}{N(10\,000 - N)} &= \frac{A}{N} + \frac{B}{(10\,000 - N)} \\
&= \frac{A(10\,000 - N) + BN}{N(10\,000 - N)} \\
&= \frac{N(B - A) + 10\,000A}{N(10\,000 - N)}
\end{aligned}$$

$$B - A = 0$$

$$10\,000A = 1$$

$$A = B = \frac{1}{10\,000}$$

$$kt = \frac{1}{10\,000}\int \frac{1}{N} + \frac{1}{(10\,000 - N)}\, dN$$

$$10\,000kt = \log_e(N) - \log_e(10\,000 - N) + c$$

$$0 = \log_e(4) - \log_e(10\,000 - 4) + c$$

$$c = \log_e\left(\frac{9996}{4}\right) = \log_e(2499)$$

$$\begin{aligned}
10\,000kt &= \log_e(N) - \log_e(10\,000 - N) + \log_e(2499) \\
&= \log_e\left(\frac{2499N}{(10\,000 - N)}\right)
\end{aligned}$$

$$k = ?,\ N = 50,\ t = 4$$

$$40\,000k = \log_e\left(\frac{2499 \times 50}{10\,000 - 50}\right) = \log_e\left(\frac{2499}{199}\right)$$

$$10\,000k = \frac{1}{4}\log_e\left(\frac{2499}{199}\right) = 0.6326$$

$$10\,000kt = \log_e\left(\frac{2499N}{(10\,000 - N)}\right)$$

$$\frac{2499N}{(10\,000 - N)} = e^{10000kt}$$

$$10\,000 - N = 2499Ne^{-10000kt}$$

$$10\,000 = N(1 + 2499e^{-10000kt})$$

$$N(t) = \frac{10\,000}{1 + 2499e^{-0.6326t}}$$

$$t = 14$$

$$N(14) = \frac{10\,000}{1 + 2499e^{-0.6326 \times 14}} = 7374$$

12 $N(t)$ number of children who have a cold

a $N(0) = 1,\ N(2) = 5,\ N(4) = 15,\ P = ?$

$$\begin{aligned}
N(t) &= \frac{N_0P}{N_0 + (P - N_0)e^{-kt}},\ N_0 = 1 \\
&= \frac{P}{1 + (P - 1)e^{-kt}}
\end{aligned}$$

$$N(2) = 5 \rightarrow 5 = \frac{P}{1 + (P - 1)e^{-2k}}$$

$$N(4) = 15 \rightarrow 15 = \frac{P}{1 + (P - 1)e^{-4k}}$$

$$k = \frac{1}{2}\log_e(6),\ P = 25$$

b $N(t) = \dfrac{25}{1 + 24e^{-0.896t}}$

$$N = \frac{P}{2} = 12.5 = \frac{25}{1 + 24e^{-kt}}$$

$$1 + 24e^{-kt} = 2$$

$$24e^{-kt} = 1$$

$$e^{-kt} = \frac{1}{24}$$

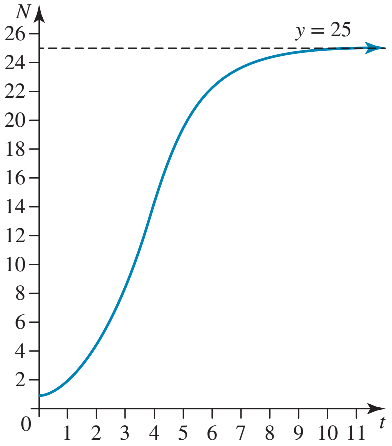
$$e^{kt} = 24$$
$$kt = \log_e(24)$$
$$t = \frac{\log_e(24)}{\frac{1}{2}\log_e(6)} = 3.55 \text{ days}$$

c $t = 6$

$$N(6) = \frac{25}{1 + 24e^{-0.896 \times 6}} = 22.5$$

22 children

d



10.6 Exam questions

1 $N(t) = \dfrac{200}{1 + 99e^{-kt}}$

$$\frac{dN}{dt} = \frac{0 - 200 \times (-k) \times 99e^{-kt}}{\left(1 + 99e^{-kt}\right)} \qquad \text{[1 mark]}$$
$$\frac{dN}{dt} = \frac{200 \times k \times 99e^{-kt}}{\left(1 + 99e^{-kt}\right)^2}$$
$$= \frac{200k}{1 + 99e^{-kt}} \times \frac{200\left(1 + 99e^{-kt}\right) - 200}{200\left(1 + 99e^{-kt}\right)}$$
$$\frac{dN}{dt} = \frac{200k}{1 + 99e^{-kt}} \times \left(\frac{200\left(1 + 99e^{-kt}\right)}{200\left(1 + 99e^{-kt}\right)} - \frac{200}{200\left(1 + 99e^{-kt}\right)}\right)$$
$$\frac{dN}{dt} = kN\left(1 - \frac{N}{200}\right), N(0) = 2 \qquad \text{[1 mark]}$$

2 $N = \dfrac{P}{2}$

$P = 500$

$N = 250$ [1 mark]

3 a $N(0) = N_0,\ P > N_0 > 0$

$$\frac{dN}{dt} = kN\left(1 - \frac{N}{P}\right) = kN\left(\frac{P - N}{P}\right) = \frac{k}{P}N(P - N)$$
$$\frac{dt}{dN} = \frac{P}{kN(P - N)}$$
$$kt = P\int \frac{1}{N(P - N)}dN$$
$$\frac{1}{N(P - N)} = \frac{A}{N} + \frac{B}{(P - N)}$$
$$= \frac{A(P - N) + BN}{N(P - N)}$$
$$= \frac{N(B - A) + PA}{N(P - N)}$$
$$B - A = 0$$
$$PA = 1$$
$$A = B = \frac{1}{P}$$
$$kt = \int \frac{1}{N} + \frac{1}{P - N}dN$$
$$kt = \log_e(N) - \log_e(P - N) + c \qquad \text{[1 mark]}$$
$$0 = \log_e(N_0) - \log_e(P - N_0) + c \rightarrow c = \log_e\left(\frac{P - N_0}{N_0}\right)$$
$$kt = \log_e(N) - \log_e(P - N) + \left(\frac{P - N_0}{N_0}\right)$$
$$= \log_e\left(\frac{N(P - N_0)}{N_0\,(P - N)}\right)$$
$$e^{kt} = \frac{N(P - N_0)}{N_0(P - N)}$$
$$e^{-kt} = \frac{N_0(P - N)}{N(P - N_0)}$$
$$N(P - N_0)e^{-kt} = N_0(P - N) = N_0P - NN_0$$
$$NN_0 + N(P - N_0)e^{-kt} = N_0P$$
$$N(t) = \frac{N_0P}{N_0 + (P - N_0)\,e^{-kt}} \qquad \text{[1 mark]}$$

b $k > 0,\ \lim\limits_{t \to \infty} N(t) = \dfrac{N_0P}{N_0 + 0} = P$ [1 mark]

c $\dfrac{dN}{dt} = kN\left(1 - \dfrac{N}{P}\right) = k\left(N - \dfrac{N^2}{P}\right)$

$$\frac{d^2N}{dt^2} = \frac{d}{dt}\left(\frac{dN}{dt}\right) = \frac{d}{dN}\left(\frac{dN}{dt}\right) \cdot \frac{dN}{dt}$$
$$= k\left(1 - \frac{2N}{P}\right)kN\left(1 - \frac{N}{P}\right)$$
$$= k^2N\left(1 - \frac{2N}{P}\right)\left(1 - \frac{N}{P}\right) \qquad \text{[1 mark]}$$
$$\frac{d^2N}{dt^2} = 0$$
$$1 - \frac{2N}{P} = 0$$
$$N = \frac{P}{2}$$
$$t = ?$$
$$e^{kt} = \frac{N(P - N_0)}{N_0(P - N)}$$
$$= \frac{\frac{P}{2}(P - N_0)}{N_0(P - \frac{P}{2})}$$
$$= \frac{P - N_0}{N_0}$$
$$kt = \log_e\left(\frac{P - N_0}{N_0}\right)$$
$$t = \frac{1}{k}\log_e\left(\frac{P}{N_0} - 1\right)$$

∴ Inflection point occurs at

$$\left(\frac{1}{k}\log_e\left(\frac{P}{N_0} - 1\right), \frac{P}{2}\right) \qquad \text{[1 mark]}$$

d $N(t) = \dfrac{N_0P}{N_0 + (P - N_0)e^{-kt}}$

$$N = N_0P\left(N_0 + (P - N_0)e^{-kt}\right)^{-1}$$

$$\frac{dN}{dt} = -PN_0(P - N_0) \times -ke^{-kt}\left(N_0 + (P - N_0)e^{-kt}\right)^{-2}$$

$$= \frac{kPN_0(P - N_0)e^{-kt}}{\left(N_0 + (P - N_0)e^{-kt}\right)^2}$$

$$= \frac{PN_0}{\left(N_0 + (P - N_0)e^{-kt}\right)} \times k(P - N_0)e^{-kt} \times \frac{1}{N_0 + (P - N_0)e^{-kt}}$$

$$\frac{1}{N_0 + (P - N_0)e^{-kt}} = \frac{N}{PN_0}$$ [1 mark]

$$N_0 + (P - N_0)e^{-kt} = \frac{PN_0}{N}$$

$$(P - N_0)e^{-kt} = \frac{PN_0}{N} - N_0 = \frac{PN_0 - NN_0}{N}$$

$$e^{-kt} = \frac{N_0(P - N)}{N(P - N_0)}$$

$$\frac{dN}{dt} = N \times \frac{k(P - N_0) \times N_0(P - N)}{N(P - N_0)} \times \frac{N}{PN_0}$$

$$= \frac{kN(P - N)}{P}$$

$$= kN\left(1 - \frac{N}{P}\right)$$ [1 mark]

10.7 Euler's method

10.7 Exercise

1 a

$$\frac{dy}{dx} = 3\sqrt{x}, \quad y(4) = 1, \quad h = \frac{1}{4}$$

$$f(x) = 3\sqrt{x}, \quad x_0 = 4, \quad y_0 = 1$$

$$y_1 = y_0 + hf(x_0)$$

$$= 1 + \frac{1}{4}\left(3\sqrt{4}\right)$$

$$= 2.5$$

$$y_2 = y_1 + hf(x_1)$$

$$= 2.5 + \frac{1}{4}\left(3\sqrt{\frac{17}{4}}\right)$$

$$= 4.04\,616$$

$$y_3 = y_2 + hf(x_2), \ x_2 = \frac{9}{2}$$

$$= 4.04\,616 + \frac{1}{4}\left(3\sqrt{\frac{9}{2}}\right)$$

$$= 5.63\,715$$

$$y_4 = y_3 + hf(x_3), \ x_3 = \frac{19}{4}$$

$$= 5.63\,715 + \frac{1}{4}\left(3\sqrt{\frac{19}{4}}\right)$$

$$= 7.2717$$

b x : 4; 4.25; 4.5; 4.75; 5

y : 1; 2.5; 4.0462; 5.6372; 7.2717

$$\frac{dy}{dx} = 3\sqrt{x} = 3x^{\frac{1}{2}}$$

$$y = \int 3x^{\frac{1}{2}}dx$$

$$= 2x^{\frac{3}{2}} + c$$

$$x = 4, \quad y = 1$$

$$1 = 2(4)^{\frac{3}{2}} + c \rightarrow c = -15$$

$$y = 2\sqrt{x^3} - 15$$

$$x = 5$$

$$y = 2\sqrt{5^3} - 15 = 7.3607$$

Euler's method underestimates

$$\frac{7.2717 - 7.3607}{7.3607} = -1.2\%$$

2 $\frac{dy}{dx} = 2\cos\left(\frac{x}{2}\right)$, $y(0) = 2$, $h = \frac{1}{2}$, $x_0 = 0$, $y_0 = 2$

$$f(x) = 2\cos\left(\frac{x}{2}\right)$$

$$y_1 = y_0 + hf(x_0)$$

$$= 2 + \frac{1}{2}(2\cos(0)) = 3$$

$$y_2 = y_1 + hf(x_0), \ x_1 = \frac{1}{2}$$

$$= 3 + \frac{1}{2}\left(2\cos\left(\frac{1}{4}\right)\right) = 3.9689$$

$$\frac{dy}{dx} = 2\cos\left(\frac{x}{2}\right)$$

$$y = \rightarrow \int 2\cos\left(\frac{x}{2}\right)\,dx$$

$$= 4\sin\left(\frac{x}{2}\right) + c \rightarrow c = 2$$

$$y = 4\sin\left(\frac{x}{2}\right) + 2$$

$$x = 1$$

$$y(1) = 4\sin\left(\frac{1}{2}\right) + 2 = 3.9177$$

3 $\frac{dy}{dx} = \sin(3x)$, $y(0) = 3$, $h = \frac{1}{3}$, $x_0 = 0$, $y_0 = 3$

$$f(x) = \sin(3x)$$

$$y_1 = y_0 + hf(x_0)$$

$$= 3 + \frac{1}{3}\sin(0) = 3$$

$$y_2 = y_1 + hf(x_1), \quad x_1 = \frac{1}{3}$$

$$= 3 + \frac{1}{3}\sin(1) = 3.2805$$

$$y_3 = y_2 + hf(x_2), \quad x_2 = \frac{2}{3}$$

$$= 3.2805 + \frac{1}{3}\sin(2) = 3.5836$$

$$\frac{dy}{dx} = \sin(3x)$$

$$y = \int \sin(3x)\,dx$$

$$= -\frac{1}{3}\cos(3x) + c \rightarrow c = \frac{10}{3}$$

$$y = \frac{1}{3}(10 - \cos(3x))$$

$$x = 1$$

$$y(1) = \frac{1}{3}(10 - \cos(3)) = 3.6633$$

4 $\frac{dy}{dx} = 4e^{-2x}, \quad y(0) = 2, \quad h = \frac{1}{4}, \quad x_0 = 0, \quad y_0 = 2$

$$\begin{aligned}
f(x) &= 4e^{-2x} \\
y_1 &= y_0 + hf(x_0) \\
&= 2 + \frac{1}{4}4e^{0} = 3 \\
y_2 &= y_1 + hf(x_1), \quad x_1 = \frac{1}{4} \\
&= 3 + \frac{1}{4}4e^{-\frac{1}{2}} = 3.60\,653 \\
y_3 &= y_2 + hf(x_2), x_2 = \frac{1}{2} \\
&= 3.60\,653 + \frac{1}{4}4e^{-1} = 3.97\,441 \\
y_4 &= y_3 + hf(x_3), \quad x_3 = \frac{3}{4} \\
&= 3.97\,441 + \frac{1}{4}4e^{-\frac{3}{2}} = 4.1975
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dx} &= 4e^{-2x} \\
y &= \int 4e^{-2x}dx \\
&= -2e^{-2x} + c \to c = 4 \\
y &= 4 - 2e^{-2x} \\
x &= 1 \\
y(1) &= 4 - 2e^{-2} = 3.7293
\end{aligned}$$

5 a $\frac{dy}{dx} = -6x, \quad y(1) = 2, \quad h = \frac{1}{2}, \quad x_0 = 1, \quad y_0 = 2$

$$\begin{aligned}
f(x) &= -6x \\
y_1 &= y_0 + hf(x_0) \\
&= 2 + \frac{1}{2}(-6 \times 1) = -1 \\
y_2 &= y_1 + hf(x_1) \\
&= -1 + \frac{1}{2}\left(-6 \times \frac{3}{2}\right) = -5.5
\end{aligned}$$

b $y(1) = 2, h = \frac{1}{3}, x_0 = 1, y_0 = 2,$

$x_1 = \frac{4}{3}, x_2 = \frac{5}{3}, x_3 = 2$

$$\begin{aligned}
y_1 &= y_0 + hf(x_0) \\
&= 2 + \frac{1}{3}(-6 \times 1) = 0 \\
y_2 &= y_1 + hf(x_1) \\
&= 0 + \frac{1}{3}\left(-6 \times \frac{4}{3}\right) = -\frac{8}{3} \\
y_3 &= y_2 + hf(x_2) \\
&= -\frac{8}{3} + \frac{1}{3}\left(-6 \times \frac{5}{3}\right) = -6
\end{aligned}$$

$x\colon 1, \frac{4}{3}, \frac{5}{3}, 2$

$y\colon 2, 0, -\frac{8}{3}, -6$

c $y(1) = 2, \quad h = \frac{1}{4}, \quad x_0 = 1, \quad y_0 = 2,$

$x_1 = \frac{5}{4}, \quad x_2 = \frac{3}{2}, \quad x_3 = \frac{7}{4}, \quad x_4 = 2$

$$\begin{aligned}
f(x) &= -6x \\
y_1 &= y_0 + hf(x_0) \\
&= 2 + \frac{1}{4}(-6 \times 1) = \frac{1}{2} \\
y_2 &= y_1 + hf(x_1) \\
&= \frac{1}{2} + \frac{1}{4}\left(-6 \times \frac{5}{4}\right) = -\frac{11}{8} \\
y_3 &= y_2 + hf(x_2) \\
&= -\frac{11}{8} + \frac{1}{4}\left(-6 \times \frac{3}{2}\right) = -\frac{29}{8} \\
y_4 &= y_3 + hf(x_3) \\
&= -\frac{29}{8} + \frac{1}{4}\left(-6 \times \frac{7}{4}\right) = -\frac{25}{4}
\end{aligned}$$

$x : 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2$

$y : 2, \frac{1}{2}, -\frac{11}{8}, -\frac{29}{8}, -\frac{25}{4}$

d $y(1) = 2, \quad h = \frac{1}{5}, \quad x_0 = 1, \quad y_0 = 2, \quad x_1 = \frac{6}{5},$

$x_2 = \frac{7}{5}, \quad x_3 = \frac{8}{5}, \quad x_4 = \frac{9}{5}, \quad x_5 = 2$

$$\begin{aligned}
f(x) &= -6x \\
y_1 &= y_0 + hf(x_0) \\
&= 2 + \frac{1}{5}(-6 \times 1) = \frac{4}{5} \\
y_2 &= y_1 + hf(x_1) \\
&= \frac{4}{5} + \frac{1}{5}\left(-6 \times \frac{6}{5}\right) = -\frac{16}{25} \\
y_3 &= y_2 + hf(x_2) \\
&= -\frac{16}{25} + \frac{1}{5}\left(-6 \times \frac{7}{5}\right) = -\frac{58}{25} \\
y_4 &= y_3 + hf(x_3) \\
&= -\frac{58}{25} + \frac{1}{5}\left(-6 \times \frac{8}{5}\right) = -\frac{106}{25} \\
y_5 &= y_4 + hf(x_4) \\
&= -\frac{106}{25} + \frac{1}{5}\left(-6 \times \frac{9}{5}\right) = -\frac{32}{5}
\end{aligned}$$

$x\colon 1, \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5}, 2$

$y\colon 2, \frac{4}{5}, -\frac{16}{25}, -\frac{58}{25}, -\frac{106}{25}, -\frac{32}{5}$

6 $\frac{dy}{dx} = 6x^2, \; y(1) = k, \; y_3 = 12, \; h = \frac{1}{3}, \; x_0 = 1, \; y_0 = k$

$$\begin{aligned}
f(x) &= 6x^2 \\
y_1 &= y_0 + hf(x_0) \\
&= k + \frac{1}{3}(6 \times 1^2) = k + 2 \\
y_2 &= y_1 + hf(x_1) \\
&= k + 2 + \frac{1}{3}\left(6 \times \left(\frac{4}{3}\right)^2\right) = k + \frac{50}{9} \\
y_3 &= y_2 + hf(x_2) \\
&= k + \frac{50}{9} + \frac{1}{3}\left(6 \times \left(\frac{5}{3}\right)^2\right) \\
&= k + \frac{50}{9} + \frac{50}{9} \\
&= k + \frac{100}{9} = 12 \\
k &= 12 - \frac{100}{9} \\
k &= \frac{8}{9}
\end{aligned}$$

7 $\dfrac{dy}{dx} + \dfrac{1}{x^2} = 0, \quad y(1) = k, \quad y_3 = \dfrac{431}{1200}, \quad h = \dfrac{1}{3}$

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{1}{x^2}, \quad x_0 = 1, \quad y_0 = k\\
f(x) &= -\frac{1}{x^2}\\
y_1 &= y_0 + hf(x_0)\\
&= k + \frac{1}{3}(-1) = k - \frac{1}{3}\\
y_2 &= y_1 + hf(x_1)\\
&= k - \frac{1}{3} + \frac{1}{3}\left(-\frac{1}{\left(\frac{4}{3}\right)^2}\right)\\
&= k - \frac{1}{3} - \frac{3}{16}\\
&= k - \frac{25}{48}\\
y_3 &= y_2 + hf(x_2)\\
&= k - \frac{25}{48} + \frac{1}{3}\left(-\frac{1}{\left(\frac{5}{3}\right)^2}\right)\\
y_3 &= k - \frac{769}{1200} = \frac{431}{1200}\\
k &= 1
\end{aligned}$$

8 $\dfrac{dy}{dx} = \dfrac{2}{x}, \quad y(1) = k, \quad h = \dfrac{1}{4}, \quad x_0 = 1, \quad y_0 = k, \quad y_4 = 2$

$$\begin{aligned}
f(x) &= \frac{2}{x}\\
y_1 &= y_0 + hf(x_0)\\
&= k + \frac{1}{4}\left(\frac{2}{1}\right) = k + \frac{1}{2}\\
y_2 &= y_1 + hf(x_1)\\
&= k + \frac{1}{2} + \frac{1}{4}\left(\frac{2}{\frac{5}{4}}\right) = k + \frac{9}{10}\\
y_3 &= y_2 + hf(x_2)\\
&= k + \frac{9}{10} + \frac{1}{4}\left(\frac{2}{\frac{3}{2}}\right) = k + \frac{37}{30}\\
y_4 &= y_3 + hf(x_3)\\
&= k + \frac{37}{30} + \frac{1}{4}\left(\frac{2}{\frac{7}{4}}\right) = k + \frac{319}{210} = 2\\
k &= 2 - \frac{319}{210} = \frac{101}{210}
\end{aligned}$$

9 $\dfrac{dy}{dx} = 4x^3, \ y(0) = k, \ h = \dfrac{1}{4}, \ y_4 = 1$

$$\begin{aligned}
f(x) &= 4x^3, \ x_0 = 0, \ y_0 = k\\
y_1 &= y_0 + hf(x_0)\\
&= k + \frac{1}{4}(4 \times 0)\\
&= k\\
y_2 &= y_1 + hf(x_1), \ x_1 = \frac{1}{4}\\
&= k + \frac{1}{4} \times 4\left(\frac{1}{4}\right)^3\\
&= k + \frac{1}{64}\\
y_3 &= y_2 + hf(x_2), \ x_2 = \frac{1}{2}\\
&= k + \frac{1}{64} + \frac{1}{4} \times 4\left(\frac{1}{2}\right)^3\\
&= k + \frac{9}{64}\\
y_4 &= y_3 + hf(x_3), \ x_3 = \frac{3}{4}\\
&= k + \frac{9}{64} + \frac{1}{4} \times 4\left(\frac{3}{4}\right)^3\\
&= k + \frac{9}{16} = 1\\
k &= 1 - \frac{9}{16} = \frac{7}{16}
\end{aligned}$$

10 $\dfrac{dy}{dx} = \log_e(3x + 2), \quad y(1) = 2, \quad x_0 = 1, \quad y_0 = 2, \quad h = \dfrac{1}{3}$

$$\begin{aligned}
f(x) &= \log_e(3x + 2)\\
y_1 &= y_0 + hf(x_0)\\
&= 2 + \frac{1}{3}\log_e(5)\\
y_2 &= y_1 + hf(x_1)\\
&= 2 + \frac{1}{3}\log_e(5) + \frac{1}{3}\log_e(6) = 2 + \frac{1}{3}\log_e(30)\\
y_3 &= y_2 + hf(x_2)\\
&= 2 + \frac{1}{3}\log_e(30) + \frac{1}{3}\log_e(7) = 2 + \frac{1}{3}\log_e(210)
\end{aligned}$$

11 $\dfrac{dy}{dx} = \log_e(4x + 1), \quad y(3) = 5, \quad x_0 = 3, \quad y_0 = 5, \quad h = \dfrac{1}{4}$

$$\begin{aligned}
f(x) &= \log_e(4x + 1)\\
y_1 &= y_0 + hf(x_0)\\
&= 5 + \frac{1}{4}\log_e(13)\\
y_2 &= y_1 + hf(x_1), \quad x_1 = \frac{13}{4}\\
&= 5 + \frac{1}{4}\log_e(13) + \frac{1}{4}\log_e(14) = 5 + \frac{1}{4}\log_e(182)\\
y_3 &= y_2 + hf(x_2), \quad x_2 = \frac{7}{2}\\
&= 25 + \frac{1}{4}\log_e(182) + \frac{1}{4}\log_e(15) = 5 + \frac{1}{4}\log_e(2730)\\
y_4 &= y_3 + hf(x_3), \quad x_3 = \frac{15}{4}\\
&= 5 + \frac{1}{4}\log_e(2730) + \frac{1}{4}\log_e(16) = +\frac{1}{4}\log_e(43\,680)
\end{aligned}$$

12 $\dfrac{dy}{dx} = \log_e(2x + 5), \ y(2) = 4$

a $x_0 = 2, \quad y_0 = 4, \quad h = \dfrac{1}{2}, \quad f(x) = \log_e(2x + 5)$

$$\begin{aligned}
y_1 &= y_0 + hf(x_0)\\
&= 4 + \frac{1}{2}\log_e(9)\\
y_2 &= y_1 + hf(x_1), \quad x_1 = \frac{5}{2}\\
&= 4 + \frac{1}{2}\log_e(9) + \frac{1}{2}\log_e(10) = 4 + \frac{1}{2}\log_e(90)
\end{aligned}$$

b $\dfrac{dy}{dx} = \log_e(2x + 5)$

$$\begin{aligned}
y &= \int \log_e(2x + 5)\,dx\\
u &= \log_e(2x + 5), \quad v = x\\
\frac{du}{dx} &= \frac{2}{2x + 5}, \quad \frac{dv}{dx} = 1\\
y &= x\log_e(2x + 5) - \int \frac{2x}{2x + 5}\,dx
\end{aligned}$$

$$y = x\log_e(2x+5) - \int \frac{2x+5-5}{2x+5}\,dx$$
$$= x\log_e(2x+5) - \int 1 - \frac{5}{2x+5}\,dx$$
$$= x\log_e(2x+5) - x + \frac{5}{2}\log_e(2x+5) + c$$
$$y = \frac{(2x+5)}{2}\log_e(2x+5) - x + c$$
$$x = 2, y = 4$$
$$4 = \frac{9}{2}\log_e(9) - 2 + c \to c = 6 - \frac{9}{2}\log_e(9)$$
$$y = \frac{(2x+5)}{2}\log_e(2x+5) - x - \frac{9}{2}\log_e(9) + 6$$
$$y(3) = \frac{11}{2}\log_e(11) - 3 - \frac{9}{2}\log_e(9) + 6$$
$$= \frac{11}{2}\log_e(11) - \frac{9}{2}\log_e(9) + 3$$

13 a $\frac{dy}{dx} = \frac{2}{y},\ y(2) = 3,\ h = \frac{1}{4}$

$f(y) = \frac{2}{y},\ x_0 = 2,\ y_0 = 3$

$$y_1 = y_0 + hf(y_0)$$
$$= 3 + \frac{1}{4}\left(\frac{2}{3}\right)$$
$$= \frac{19}{6} = 3.1667$$
$$y_2 = y_1 + hf(y_1),\ x_1 = \frac{1}{4}$$
$$= \frac{19}{6} + \frac{1}{4} \times 2\left(\frac{6}{19}\right)$$
$$= 3.3246$$
$$y_3 = y_2 + hf(y_2)$$
$$= 3.3246 + \frac{1}{4} \times \frac{2}{3.3246}$$
$$= 3.4750$$
$$y_4 = y_3 + hf(y_3)$$
$$= 3.4750 + \frac{1}{4} \times \frac{2}{3.4750}$$
$$= 3.6188$$

x: 2; 2.25; 2.5; 2.75; 3

y: 3; 3.1667; 3.3246; 3.4750; 3.6188

b $\frac{dy}{dx} = \frac{2}{y},\quad y(2) = 3$

$$\frac{dx}{dy} = \frac{y}{2}$$
$$2x = \int y\,dx$$
$$2x = \frac{1}{2}y^2 + c$$
$$y^2 = 4x + A \text{ where } A = -2c$$
$$x = 2,\quad y = 3$$
$$9 = 8 + A$$
$$A = 1$$
$$y^2 = 4x + 1$$
$$y = \sqrt{4x+1}$$
$$x = 3$$
$$y = \sqrt{13} = 3.6056$$

$$\%\text{ error} = \frac{3.6188 - 3.6056}{3.6056} = 0.4\%$$

14 $\frac{dy}{dx} = \tan(y),\quad y(0.2) = 0.4,\quad h = \frac{1}{10} = 0.1$

$f(y) = \tan(y),\quad y_0 = 0.4$

$$y_1 = y_0 + hf(y_0)$$
$$= 0.4 + 0.1 \times \tan(0.4) = 0.4423$$
$$y_2 = y_1 + hf(y_1),\quad x_1 = \frac{1}{4}$$
$$= 0.4423 + 0.1 \times \tan(0.4423) = 0.4896$$
$$y_3 = y_2 + hf(y_2)$$
$$= 0.4896 + 0.1 \times \tan(0.4896) = 0.5429$$

x : 0.2; 0.3; 0.4; 0.5

y : 0.4; 0.4423; 0.4896; 0.5429

15 a $\frac{dy}{dx} = \frac{y}{3},\quad y(0) = 4,\quad h = \frac{1}{2},\quad x_0 = 0,\quad y_0 = 4$

$f(y) = \frac{y}{3}$

$$y_1 = y_0 + hf(y_0)$$
$$= 4 + \frac{1}{2}\left(\frac{4}{3}\right) = \frac{14}{3} = 14.66\,666$$
$$y_2 = y_1 + hf(y_1)$$
$$= \frac{14}{3} + \frac{1}{2}\left(\frac{14}{9}\right) = \frac{49}{9} = 5.44\,444$$

x: 0; 0.5; 1

y: 4; 14.66 666; 5.44 444

b $h = \frac{1}{3}$

$$y_1 = y_0 + hf(y_0)$$
$$= 4 + \frac{1}{3}\left(\frac{4}{3}\right) = \frac{40}{9} = 4.4444$$
$$y_2 = y_1 + hf(y_1)$$
$$= \frac{40}{9} + \frac{1}{3}\left(\frac{40}{9} \times \frac{1}{3}\right) = \frac{400}{81} = 4.93\,287$$
$$y_3 = y_2 + hf(y_2)$$
$$= \frac{400}{81} + \frac{1}{3}\left(\frac{400}{81} \times \frac{1}{3}\right) = \frac{4000}{729} = 5.48\,697$$

x: 0; 0.33 333; 0.66 666; 1

y: 4; 4.4444; 4.93 287; 5.48 697

16 a $\frac{dy}{dx} = \frac{y}{2}(5-y),\ y(0) = 1,\ h = \frac{1}{2},\ y_0 = 1$

$f(y) = \frac{y}{2}(5-y)$

$$y_1 = y_0 + hf(y_0)$$
$$= 1 + \frac{1}{2}\left(\frac{1}{2}(5-1)\right) = 2$$
$$y_2 = y_1 + hf(y_1)$$
$$= 2 + \frac{1}{2}\left(\frac{2}{2}(5-2)\right) = \frac{7}{2} = 3.5$$

b $\frac{dy}{dx} = \frac{y}{2}(5-y),\quad y(0) = 1,\quad h = \frac{1}{3},\quad y_0 = 1$

$f(y) = \frac{y}{2}(5-y)$

$$y_1 = y_0 + hf(y_0)$$
$$= 1 + \frac{1}{3}\left(\frac{1}{2}(5-1)\right) = \frac{5}{3} = 1.6666$$
$$y_2 = y_1 + hf(y_1)$$

$$= \frac{5}{3} + \frac{1}{3}\left(\frac{1}{2} \times \frac{5}{3}\left(5 - \frac{5}{3}\right)\right) = \frac{70}{27} = 2.5926$$

$$y_3 = y_2 + hf(y_2)$$

$$= \frac{70}{27} + \frac{1}{3}\left(\frac{1}{2} \times \frac{70}{27}\left(5 - \frac{70}{27}\right)\right) = \frac{7945}{2187} = 3.6328$$

17 a $\frac{dy}{dx} + 2x^3y^2, \quad y(0) = 2, \quad h = \frac{1}{4}$

$$f(x, y) = -2x^3y^2, \quad x_0 = 0$$

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 2 + \frac{1}{4}(0) = 2$$

$$y_2 = y_1 + hf(x_1, y_1), \quad x_1 = \frac{1}{4}$$

$$= 2 + \frac{1}{4}\left(-2 \times \left(\frac{1}{4}\right)^3 \times 2^2\right) = 1.9688$$

$$y_3 = y_2 + hf(x_2, y_2)$$

$$= 1.9688 + \frac{1}{4}\left(-2 \times \left(\frac{1}{2}\right)^3 \times 1.9688^2\right)$$

$$= 1.7265$$

$$y_4 = y_3 + hf(x_3, y_3)$$

$$= 1.7265 + \frac{1}{4}\left(-2 \times \left(\frac{3}{4}\right)^3 \times 1.7265^2\right)$$

$$= 1.0977$$

x: 0, 0.25, 0.5; 0.75; 1

y: 2; 2; 1.9688; 1.7265; 1.0977

b $\frac{dy}{dx} + 2x^3y^2 = 0, \ y(0) = 2$

$$\frac{dy}{dx} = -2x^3y^2$$

$$\frac{1}{y^2}dy = \int -2x^3 dx$$

$$-\frac{1}{y} = -\frac{x^4}{2} + c$$

$$y = 2, \quad x = 0$$

$$-\frac{1}{2} = 0 + c \rightarrow c = -\frac{1}{2}$$

$$\frac{1}{y} = \frac{x^4 + 1}{2}$$

$$y = \frac{2}{x^4 + 1}$$

$$x = 1, \quad y = 1$$

Over estimate by $\frac{1.0977 - 1}{1} = 9.8\%$

18 a $\frac{dy}{dx} = 2y\cos(x), \quad y(0) = 1, \quad h = 0.1$

$$f(x, y) = 2y\cos(x), \quad y_0 = 1$$

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 1 + 0.1(2 \times 1 \times \cos(0)) = 1.2$$

$$y_2 = y_1 + hf(x_1, y_1), \quad x_1 = 0.1$$

$$= 1.2 + 0.1(2 \times 1.2 \times \cos(0.1)) = 1.4388$$

$$y_3 = y_2 + hf(x_2, y_2)$$

$$= 1.4388 + 0.1(2 \times 1.4388 \times \cos(0.2)) = 1.7208$$

b $\frac{dy}{dx} = 2y\cos(x)$

$$\int \frac{1}{y}dy = \int 2\cos(x)\,dx$$

$$\log_e(y) = 2\sin(x) + c$$

$$y = e^{2\sin(x)}$$

$$x = 0.3$$

$$y = e^{2\sin(03)} = 1.8059$$

% error $\frac{1.7208 - 1.8059}{1.8059} = -0.047$

Underestimates by 4.7%

19 $\frac{dy}{dx} + xy^2 = 0, \ y(1) = 2$

$$\frac{dy}{dx} = -xy^2, \ x_0 = 1, \ y_0 = 2$$

$$f(x, y) = -xy^2$$

$$h = \frac{1}{4}$$

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 2 + \frac{1}{4}\left(-1 \times 2^2\right) = 1$$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$= 1 + \frac{1}{4}\left(-\frac{5}{4} \times (1)^2\right) = \frac{11}{16} = 0.6875$$

$$y_3 = y_2 + hf(x_2, y_2)$$

$$= \frac{11}{16} + \frac{1}{4}\left(-\frac{3}{2} \times \left(\frac{11}{16}\right)^2\right) = \frac{1045}{2048} = 0.510\,254$$

$$y_4 = y_3 + hf(x_3, y_3)$$

$$= \frac{1045}{2048} + \frac{1}{4}\left(-\frac{7}{4} \times \left(\frac{1045}{2048}\right)^2\right) = 0.3963$$

20 a $\frac{dy}{dx} = y\sqrt{x^2 + 5} = f(x, y), \quad y(2) = 5, \quad x_0 = 2, \quad y_0 = 5$

$$h = \frac{1}{3}$$

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 5 + \frac{1}{3}\left(5\sqrt{4 + 5}\right) = 10$$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$= 10 + \frac{1}{3}\left(10\sqrt{\frac{49}{9} + 5}\right) = 20.7726$$

$$y_3 = y_2 + hf(x_2, y_2)$$

$$= 20.7726 + \frac{1}{3}\left(20.7726\sqrt{\frac{64}{9} + 5}\right) = 44.8696$$

x: 2.000; 2.3333; 2.6667; 3.0000

y: 5.000; 10.000; 20.7726; 44.8696

b $h = \frac{1}{4}$

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 5 + \frac{1}{4}\left(5\sqrt{4 + 5}\right) = 8.75$$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$= 8.75 + \frac{1}{4}\left(8.75\sqrt{\frac{81}{16} + 5}\right) = 15.6891$$

$$y_3 = y_2 + hf(x_2,\ y_2)$$
$$= 15.6891 + \frac{1}{4}\left(15.6891\sqrt{\frac{25}{4}+5}\right) = 28.8447$$
$$y_4 = y_3 + hf(x_3,\ y_3)$$
$$= 28.8447 + \frac{1}{4}\left(28.8447\sqrt{\frac{121}{16}+5}\right) = 54.4038$$

x: 2.000; 2.2500; 2.5000; 2.7500; 3.0000

y: 5.000; 8.75; 15.6891; 28.8447; 54.4038

21 $\frac{dy}{dx} + \frac{e^{-2x}}{y} = 0,\ y(1) = 5,\ x_0 = 1,\ y_0 = 5$

$$\frac{dy}{dx} = -\frac{e^{-2x}}{y} = f(x,\ y)$$

a $h = \frac{1}{3}$

$$y_1 = y_0 + hf(x_0,\ y_0)$$
$$= 5 + \frac{1}{3}\left(-\frac{1}{5}\times e^{-2}\right) = 4.9910$$
$$y_2 = y_1 + hf(x_1,\ y_1)$$
$$= 4.9910 + \frac{1}{3}\left(-\frac{1}{4.9910}\times e^{-\frac{8}{3}}\right) = 4.9863$$
$$y_3 = y_2 + hf(x_2,\ y_2)$$
$$= 4.9863 + \frac{1}{3}\left(-\frac{1}{4.9863}\times e^{-\frac{10}{3}}\right) = 4.9840$$

x: 1.000; 1.3333; 1.6667; 1.0000

y: 5.000; 4.9910; 4.9863; 4.9840

b $h = \frac{1}{4}$

$$y_1 = y_0 + hf(x_0,\ y_0)$$
$$= 5 + \frac{1}{4}\left(-\frac{1}{5}\times e^{-2}\right) = 4.9932$$
$$y_2 = y_1 + hf(x_1,\ y_1)$$
$$= 4.9932 + \frac{1}{4}\left(-\frac{1}{4.9932}\times e^{-\frac{5}{2}}\right) = 4.9891$$
$$y_3 = y_2 + hf(x_2,\ y_2)$$
$$= 4.9891 + \frac{1}{4}\left(-\frac{1}{4.9891}\times e^{-3}\right) = 4.9866$$
$$y_4 = y_3 + hf(x_3,\ y_3)$$
$$= 4.9866 + \frac{1}{4}\left(-\frac{1}{4.9866}\times e^{-\frac{7}{4}}\right) = 4.9851$$

x: 1.000; 1.2500; 1.5000; 1.7500; 2.0000

y: 5.000; 4.9932; 4.9891; 4.9866; 4.9851

22 a $\frac{dy}{dx} = 4x - y + 2,\quad y(0) = 2,\quad h = \frac{1}{4},\quad x_0 = 0,$

$x_1 = \frac{1}{4},\quad x_2 = \frac{1}{2},\quad x_3 = \frac{3}{4},\quad x_4 = 1$

$$\frac{dy}{dx} = 4x - y + 2 = f(x, y)$$
$$y_1 = y_0 + hf(x_0,\ y_0)$$
$$= 2 + \frac{1}{4}(4\times 0 - 2 + 2) = 2$$
$$y_2 = y_1 + hf(x_1,\ y_1)$$
$$= 2 + \frac{1}{4}\left(4\times\frac{1}{4} - 2 + 2\right) = \frac{9}{4} = 2.25$$
$$y_3 = y_2 + hf(x_2,\ y_2)$$
$$= \frac{9}{4} + \frac{1}{4}\left(4\times\frac{1}{2} - \frac{9}{4} + 2\right) = \frac{43}{16} = 2.6875$$
$$y_4 = y_3 + hf(x_3,\ y_3)$$
$$= \frac{43}{16} + \frac{1}{4}\left(4\times\frac{3}{4} - \frac{43}{16} + 2\right) = \frac{209}{64} = 3.2656$$

x: 0; 0.25; 0.5; 0.75; 1

y: 2; 2; 2.25; 2.6875; 3.2656

b $\frac{dy}{dx} = 4x - y + 2,\ y(0) = 2$

$$y = ax + b + ce^{kx}$$
$$2 = b + c$$
$$\frac{dy}{dx} = a + cke^{kx} = 4x - \left(ax + b + ce^{kx}\right) + 2$$
$$= 2 - b + x(4 - a) - ce^{kx}$$
$$k = -1$$
$$x\ :\ 4 - a = 0 \rightarrow a = 4$$
$$x^0\ :\ 2 - b = a \rightarrow b = -2$$
$$c = 2 - b \rightarrow c = 4$$
$$y = 4e^{-x} + 4x - 2$$
$$y(1) = 4e^{-1} + 4 - 2 = 3.4715$$

23 a $\frac{dy}{dx} = 5x + 2y + 1,\quad y(0) = 1,\quad h = \frac{1}{4},\quad x_0 = 0,$

$x_1 = \frac{1}{4},\quad x_2 = \frac{1}{2},\quad x_3 = \frac{3}{4},\quad x_4 = 1$

$$\frac{dy}{dx} = 5x + 2y + 1 = f(x,\ y)$$
$$y_1 = y_0 + hf(x_0,\ y_0)$$
$$= 1 + \frac{1}{4}(5\times 0 + 2\times 1 + 1) = \frac{7}{4} = 1.75$$
$$y_2 = y_1 + hf(x_1,\ y_1)$$
$$= \frac{7}{4} + \frac{1}{4}\left(5\times\frac{1}{4} + 2\times\frac{7}{4} + 1\right) = \frac{51}{16} = 3.1875$$
$$y_3 = y_2 + hf(x_2,\ y_2)$$
$$= \frac{51}{16} + \frac{1}{4}\left(5\times\frac{1}{2} + 2\times\frac{51}{16} + 1\right) = \frac{181}{32} = 5.6563$$
$$y_4 = y_3 + hf(x_3,\ y_3)$$
$$= \frac{181}{32} + \frac{1}{4}\left(5\times\frac{3}{4} + 2\times\frac{181}{32} + 1\right) = \frac{619}{64} = 9.6719$$

x: 0; 0.25; 0.5; 0.75; 1

y: 1; 1.75; 3.1875; 5.6563; 9.6719

b $y = ax + b + ce^{kx}$

$$\frac{dy}{dx} = 5x + 2y + 1,\ y(0) = 1$$
$$\frac{dy}{dx} = a + cke^{kx} = 5x + 2\left(ax + b + ce^{kx}\right) + 1$$
$$= 2b + 1 + x(2a + 5) + 2ce^{kx}$$
$$k = 2$$
$$x\ :\ 2a + 5 = 0 \rightarrow a = -\frac{5}{2}$$
$$x^0\ :\ 2b + 1 = a \rightarrow b = -\frac{7}{4}$$
$$c = 1 - b \rightarrow c = \frac{11}{4}$$
$$y = \frac{1}{4}\left(11e^{2x} - 10x - 7\right)$$
$$y(1) = \frac{1}{4}\left(11e^2 - 10 - 7\right) = 16.0699$$

24 $\frac{dy}{dx} = x\sin\left(\frac{x}{2}\right), y(0) = 2,\ x_0 = 0,\ y_0 = 2$

$f(x) = x\sin\left(\frac{x}{2}\right)$

a $h = \frac{1}{3}$

$$y_1 = y_0 + hf(x_0)$$
$$= 2 + \frac{1}{3}(0 \times \sin(0)) = 2$$
$$y_2 = y_1 + hf(x_1)$$
$$= 2 + \frac{1}{3}\left(\frac{1}{3}\sin\left(\frac{1}{6}\right)\right)$$
$$y_3 = y_2 + hf(x_2)$$
$$= 2 + \left(\frac{1}{9}\sin\left(\frac{1}{6}\right)\right) + \frac{1}{3}\left(\frac{2}{3}\sin\left(\frac{1}{3}\right)\right)$$
$$= 2 + \frac{1}{9}\left[\sin\left(\frac{1}{6}\right) + 2\sin\left(\frac{1}{3}\right)\right]$$

b $h = \frac{1}{4}$

$$y_1 = y_0 + hf(x_0)$$
$$= 2 + \frac{1}{4}(0 \times \sin(0)) = 2$$
$$y_2 = y_1 + hf(x_1)$$
$$= 2 + \frac{1}{4}\left(\frac{1}{4}\sin\left(\frac{1}{8}\right)\right) = 2 + \frac{1}{16}\sin\left(\frac{1}{8}\right)$$
$$y_3 = y_2 + hf(x_2)$$
$$= 2 + \frac{1}{16}\sin\left(\frac{1}{8}\right) + \frac{1}{4}\left(\frac{1}{2}\sin\left(\frac{1}{4}\right)\right)$$
$$= 2 + \frac{1}{16}\sin\left(\frac{1}{8}\right) + \frac{1}{8}\sin\left(\frac{1}{4}\right)$$
$$y_4 = y_3 + hf(x_3)$$
$$= 2 + \frac{1}{16}\left(\sin\left(\frac{1}{8}\right) + 2\sin\left(\frac{1}{4}\right)\right) + \frac{1}{4}\left(\frac{3}{4}\sin\left(\frac{3}{8}\right)\right)$$
$$2 + \frac{1}{16}\left[\sin\left(\frac{1}{8}\right) + 2\sin\left(\frac{1}{4}\right) + 3\sin\left(\frac{3}{8}\right)\right]$$

c $y = \int x\sin\left(\frac{x}{2}\right)dx$

$$u = x, \quad \frac{dv}{dx} = \sin\left(\frac{x}{2}\right)$$
$$\frac{du}{dx} = 1, \quad v = -2\cos\left(\frac{x}{2}\right)$$
$$y = -2x\cos\left(\frac{x}{2}\right) + \int 2\cos\left(\frac{x}{2}\right)dx$$
$$= -2x\cos\left(\frac{x}{2}\right) + 4\sin\left(\frac{x}{2}\right) + c$$
$$y(0) = 2$$
$$2 = 0 + c \rightarrow c = 2$$
$$= 2 - 2x\cos\left(\frac{x}{2}\right) + 4\sin\left(\frac{x}{2}\right)$$
$$x = 2$$
$$y = 2 - 4\cos(1) + 4\sin(1)$$

25 $\frac{dy}{dx} = xe^{-3x},\ y(0) = 4,\ x_0 = 0,\ y_0 = 4$

$f(x) = xe^{-3x}$

a $h = \frac{1}{3}$

$$y_1 = y_0 + hf(x_0)$$
$$= 4 + \frac{1}{3}\left(0 \times e^0\right) = 4$$
$$y_2 = y_1 + hf(x_1)$$
$$= 4 + \frac{1}{3}\left(\frac{1}{3} \times e^{-1}\right) = 4 + \frac{1}{9}e^{-1}$$
$$y_3 = y_2 + hf(x_2)$$
$$= 4 + \frac{1}{9}e^{-1} + \frac{1}{3}\left(\frac{2}{3} \times e^{-2}\right) = 4 + \frac{1}{9}(e^{-1} + 2e^{-2})$$

b $h = \frac{1}{4}$

$$y_1 = y_0 + hf(x_0)$$
$$= 4 + \frac{1}{4}\left(0 \times e^0\right) = 4$$
$$y_2 = y_1 + hf(x_1)$$
$$= 4 + \frac{1}{4}\left(\frac{1}{4} \times e^{-\frac{3}{4}}\right) = 4 + \frac{1}{16}e^{-\frac{3}{4}}$$
$$y_3 = y_2 + hf(x_2)$$
$$= 4 + \frac{1}{16}e^{-\frac{3}{4}} + \frac{1}{4}\left(\frac{1}{2} \times e^{-\frac{3}{2}}\right) = 4 + \frac{1}{16}\left(e^{-\frac{3}{4}} + 2e^{-\frac{3}{2}}\right)$$
$$y_4 = y_3 + hf(x_3)$$
$$= 4 + \frac{1}{16}\left(e^{-\frac{3}{4}} + 2e^{-\frac{3}{2}}\right) + \frac{1}{4}\left(\frac{3}{4} \times e^{-\frac{9}{4}}\right)$$
$$= 4 + \frac{1}{16}\left(e^{-\frac{3}{4}} + 2e^{-\frac{3}{2}} + 3e^{-\frac{9}{4}}\right)$$

c $y = \int xe^{-3x}dx$

$$u = x,\ \frac{dv}{dx} = e^{-3x}$$
$$\frac{du}{dx} = 1,\ v = -\frac{1}{3}e^{-3x}$$
$$y = -\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}dx$$
$$y = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + c$$
$$y(0) = 4$$
$$4 = 0 - \frac{1}{9} + c \rightarrow c = \frac{37}{9}$$
$$y = \frac{1}{9}\left(-(3x + 1)e^{-3x} + 37\right)$$
$$y(1) = \frac{1}{9}\left(-4e^{-3} + 37\right)$$

26 $\frac{dy}{dx} = f(x),\ y(x_0) = y_0$

$$x_1 = x_0 + h$$
$$y_1 = y_0 + hf(x_0)$$
$$y_2 = y_1 + hf(x_1)$$
$$= y_0 + hf(x_0) + hf(x_0 + h) = y_0 + h(f(x_0) + f(x_0 + h))$$
$$y_3 = y_2 + hf(x_2) \rightarrow x_2 = x_0 + 2h$$
$$= y_0 + hf(x_0) + hf(x_0 + h) + hf(x_0 + 2h)$$
$$= y_0 + h(f(x_0) + f(x_0 + h) + f(x_0 + 2h))$$
$$y_4 = y_3 + hf(x_3) \rightarrow x_3 = x_0 + 3h$$
$$= y_0 + h(f(x_0) + f(x_0 + h) + f(x_0 + 2h)) + hf(x_0 + 3h)$$
$$= y_0 + h(f(x_0) + f(x_0 + h) + f(x_0 + 2h) + f(x_0 + 3h))$$
$$y_n = y_0 + h\sum_{k=0}^{n-1} f(x_0 + kh)$$

10.7 Exam questions

1 $\frac{dy}{dx} = y\sin(x) = f(x, y) \quad x_0 = 1, \quad y_0 = 2, \quad h = 0.1$

$y_1 = y_0 + hf(x_0, y_0)$

$y_1 = 2 + 0.1 \times 2\sin(1)$

$y_1 = 2.168$

$y_2 = y_1 + hf(x_1, y_1)$

$y_2 = 2.168 + 0.1 \times 2.168\sin(1.1)$

$y_2 = 2.3615$

The correct answer is **C**.

2 $\frac{dy}{dx} = f(x) = e^{\cos(x)}, \ y_0 = e, \ x_0 = 0, h = 0.1$

$y_1 = y_0 + hf(x_0)$

$y_1 = e + 0.1e^{\cos(0)}$

$y_1 = e + 0.1e$

$y_2 = y_1 + hf(x_1)$

$y_2 = e + 0.1e^{\cos(0)} + 0.1e^{\cos(0.1)}$

$y_2 = e + 0.1\left(e + e^{\cos(0.1)}\right)$

$y_3 = y_2 + hf(x_2)$

$y_3 = e + 0.1\left(e + e^{\cos(0.1)}\right) + 0.1e^{\cos(0.2)}$

$y_3 = e + 0.1\left(e + e^{\cos(0.1)} + e^{\cos(0.2)}\right)$

The correct answer is **C**.

3 Using Euler's method, with

$x_0 = 1\,, \ y_0 = 2\,, \ h = -0.1, \ \frac{dy}{dx} = f(x) = 2x^2 + x + 1$

So that $x_1 = 0.9$ and $x_2 = 0.8$

$y_1 = y_0 + hf(x_0) = 2 - 0.1(4) = 1.6$

$y_2 = y_1 + hf(x_1) = 1.6 - 0.1(2 \times 0.9^2 + 0.9 + 1)$

$= 1.248$

The correct answer is **B**.

10.8 Slope fields

10.8 Exercise

1 $\frac{dy}{dx} = \frac{2}{x}$

$x = 0$: slope undefined

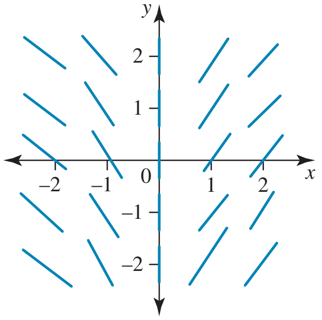
$x = 1$: slope $\frac{dy}{dx} = 2$

$x = 2$: slope $\frac{dy}{dx} = 1$

$x = -1$: slope $\frac{dy}{dx} = -2$

$x = -2$: slope $\frac{dy}{dx} = -1$

$y = \int \frac{2}{x}\, dx = 2\log_e(|x|) + c$



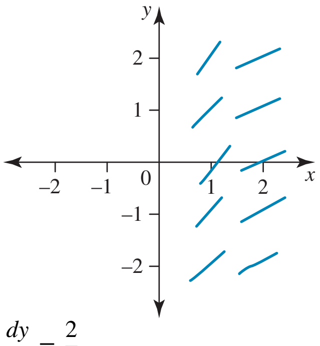
2 $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}, \quad x \neq 0, \ x > 0$

$x = 0$: slope undefined

$x = 1$: slope $\frac{1}{2} = 0.5$

$x = 2$: slope $\frac{1}{2\sqrt{2}} = 0.35$

$y = \int \frac{1}{2\sqrt{x}}\, dx = x^{\frac{1}{2}} + c = \sqrt{x} + c$



3 $\frac{dy}{dx} = \frac{2}{y}$

$y = 0$: slope undefined

$y = 1$: slope $\frac{dy}{dx} = 2$

$y = 2$: slope $\frac{dy}{dx} = 1$

$y = -1$: slope $\frac{dy}{dx} = -2$

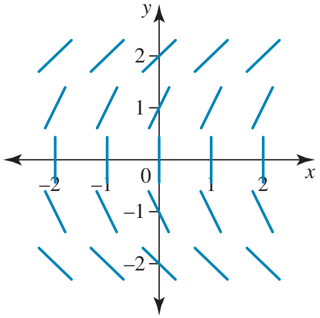
$y = -2$: slope $\frac{dy}{dx} = -1$

$\int y\, dx = \int 2\, dx$

$\frac{1}{2}y^2 = 2x + c$

$y^2 = 4x + B$

$y = \pm\sqrt{4x + B}$



4 $\frac{dy}{dx} = 2\sqrt{y}, \ y \neq 0$

$y = 0$: slope 0

$y = 1$: slope $2\sqrt{1} = 2$

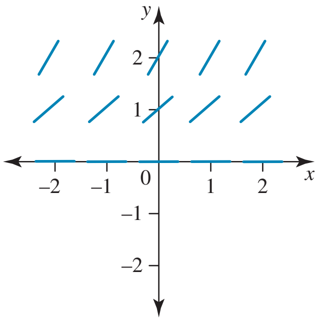
$y = 2$: slope $2\sqrt{2} = 2.8$

$x = \frac{1}{2}\int y^{-\frac{1}{2}} dy = y^{\frac{1}{2}} + c$

$\sqrt{y} = x - c$

$y = (x - c)^2$

$x \geq c, \ y \geq 0$



5 $y\frac{dy}{dx} - x = 0$

$$y\frac{dy}{dx} = x$$

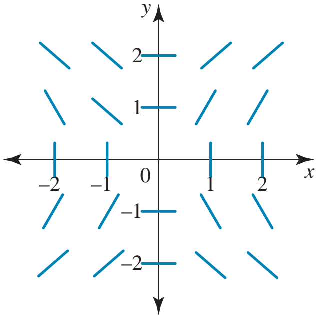
$$\frac{dy}{dx} = \frac{x}{y}$$

$$\int y\,dy = \int x\,dx$$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + c$$

$$y^2 = x^2 + A \text{ where } A = 2c$$

$$y^2 - x^2 = A$$



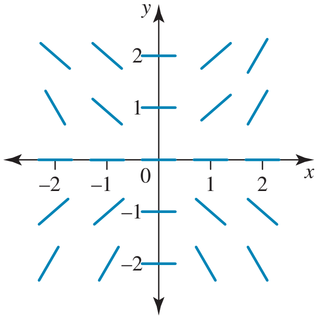
6 $\frac{dy}{dx} = xy$

$$\int \frac{1}{y}dy = \int x\,dx$$

$$\log_e(y) = \frac{1}{2}x^2 + c$$

$$y = e^{\frac{1}{2}x^2 + c}$$

$$y = Ae^{\frac{1}{2}x^2} \text{ where } A = e^c$$



7 Slopes only defined for $x > 0$, so B and E are incorrect.
At $x = 1$ and $y = -1$, slope is -1. A and D are incorrect.
Only C is the only possible solution.
The correct answer is **C**.

8 From the graph, slope = 0 at $x = 1$ and $y = -1$.

A : $\frac{dy}{dx} = \cos(1 - 1) = \cos(0) = 1$

B : $\frac{dy}{dx} = \cos(1 - (-1)) = \cos(2) < 0$

C : $\frac{dy}{dx} = \sin(1 - (-1)) = \sin(2) > 0$

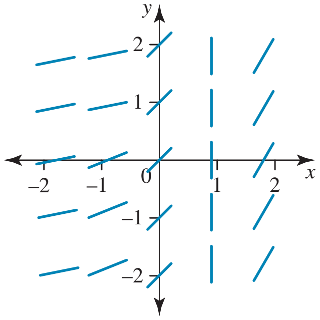
D : $\frac{dy}{dx} = \sin(1 - 1) = \sin(0) = 0$

E : $\frac{dy}{dx} = \tan(1 - (-1)) = \tan(2) > 0$

The correct answer is **D**.

9 a $\frac{dy}{dx} = \frac{2}{(x-1)^2}$

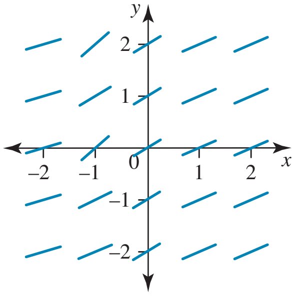
$$y = \int \frac{2}{(x-1)^2} = -\frac{1}{x-1} + c$$



b i $\frac{dy}{dx} = \frac{1}{1+x^2}$

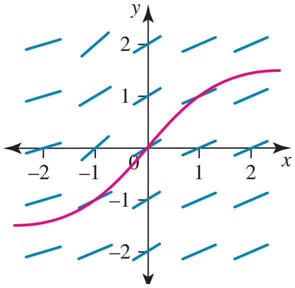
$$y = \int \frac{1}{1+x^2}\,dx$$

$$y = \tan^{-1}(x) + c$$

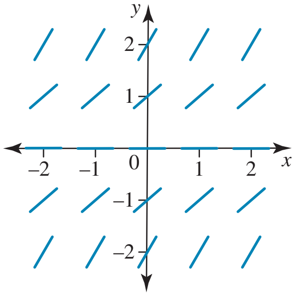


ii $x = 0,\ y = 0,\ c = 0$

$y = \tan^{-1}(x)$



10 a $\frac{dy}{dx} = y^2$



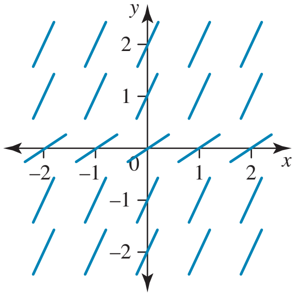
b $$\frac{dy}{dx} = 1 + y^2,\ y(0) = 0$$

$$\int \frac{1}{1 + y^2}\, dy = \text{false}$$

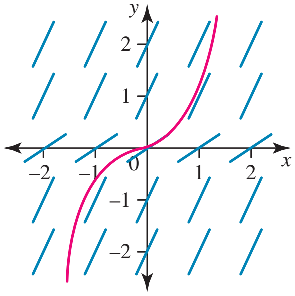
$$\tan^{-1}(y) = x,\ y(0) = 0$$

$$y = \tan(x)$$

i

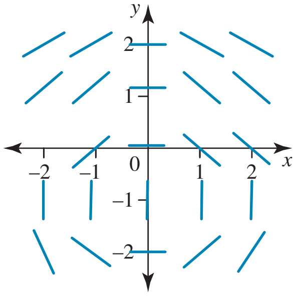


ii

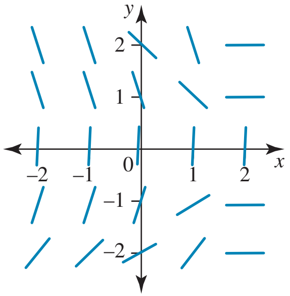


$y = \tan(x)$

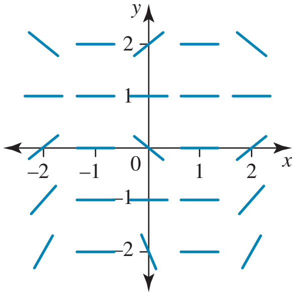
11 a $\frac{dy}{dx} = -\frac{x}{y + 1}$



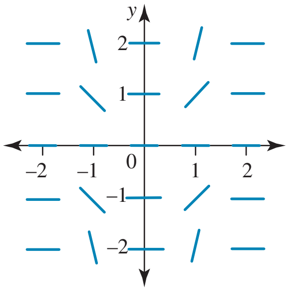
b $\frac{dy}{dx} = \frac{x - 2}{y}$



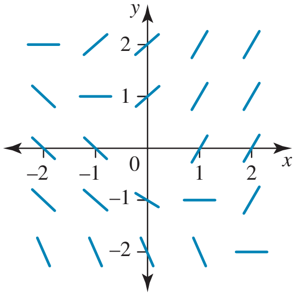
12 a $\frac{dy}{dx} = (y - 1)\cos\left(\frac{\pi x}{2}\right)$



b $\frac{dy}{dx} = y^2 \sin\left(\frac{\pi x}{2}\right)$



13 a $\frac{dy}{dx} = x + y$



b $$y = Ae^{kx} + Bx + C$$

$$\frac{dy}{dx} = kAe^{kx} + B = x + y$$

$$= x + Ae^{kx} + Bx + C$$

$$= Ae^{kx} + x(B + 1) + C$$

$$k = 1$$

$$B + 1 = 0 \rightarrow B = -1$$

$$C = B \rightarrow C = -1$$

$$y(0) = A + C = 0$$

$$A = -C \rightarrow A = 1$$

c $y = e^x - x - 1$

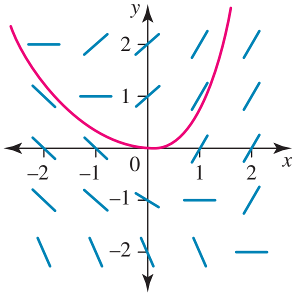
$$\frac{dy}{dx} = e^x - 1$$

$e^x = 1$

$x = 0,\ y = 0$

Turning point at $(0, 0)$

d



14 **a** Slope 4
  b Slope 2
  c Slope 3
  d Slope 1

15 **a** Slope 3
  b Slope 1
  c Slope 4
  d Slope 2

16 **a** Slope 1
  b Slope 3
  c Slope 2
  d Slope 4

17 **a** Slope 2
  b Slope 3
  c Slope 1
  d Slope 4

18 **a** Slope 1
  b Slope 4
  c Slope 2
  d Slope 3

10.8 Exam questions

1 At the point $x = 1,\ y = 1$, the slope is $m = \tan(135°) = -1$, only $\frac{dy}{dx} = y - 2x$ satisfies this.

The correct answer is **D**.

2 $\left.\frac{dy}{dx}\right|_{x=a} = m,\ P(a, b)$

$T: y - b = m(x - a)$

$$y = 0 \Rightarrow x = a - \frac{b}{m} = b$$

$$\frac{b}{m} = a - b,\ m = \frac{b}{a-b}$$

$\frac{dy}{dx} = m = \frac{y}{x-y}$ is underfined when $x = y$ and $\frac{dy}{dx} = 0$ when $y = 0$.

The correct answer is **B**.

3 When the gradient $m = 1$,there are no vertical slopes

$m = \frac{dy}{dx} = \cos(y - x)$ is the only possible differential equation.

The correct answer is **B**.

10.9 Review

10.9 Exercise

Technology free: short answer

1 N number at time t year

$$\frac{dN}{dt} = kN \Rightarrow N = N_0 e^{kt}$$

$$N(0) = N_0 = 250$$

$$N(3) = 450$$

$$450 = 250e^{3k}$$

$$45 = 25e^{3k}$$

$$e^{3k} = \frac{45}{25} = \frac{9}{5}$$

$$3k = \log_e\left(\frac{9}{5}\right)$$

$$k = \frac{1}{3}\log_e\left(\frac{9}{5}\right) \approx 0.196$$

$$N = N(t) = 250e^{0.196t}$$

when $t = 8$:

$$N(8) = 250e^{0.196 \times 8}$$

$$= 1198.58$$

so 1198 kangaroos

2 **a** $\frac{dv}{dt} + \frac{v}{kR} = 0, \quad v(0) = v_0$

$$\frac{dv}{dt} = -\frac{v}{kR}$$

$$\frac{dt}{dv} = -\frac{kR}{v}$$

$$\frac{1}{kR}\int dt = \int -\frac{1}{v}dv$$

$$\frac{t}{kR} = -\log_e(v) + C$$

$t = 0 \quad v = v_0$

$$0 = -\log_e(v_0) + c$$

$$C = \log_e(v_0)$$

$$\frac{t}{RC} = -\log_e(v) + \log_e(v_0)$$

$$= \log_e\left(\frac{v_0}{v}\right)$$

$$\frac{v_0}{v} = e^{\frac{t}{RC}}$$

$$v(t) = v = v_0 e^{-\frac{t}{RC}}$$

b $t = ? \quad v = \frac{1}{2}v_0$

$$\frac{1}{2}v_0 = v_0 e^{-\frac{t}{RC}}$$

$$\frac{1}{2} = e^{-\frac{t}{RC}}$$

$$-\frac{t}{RC} = \log_e\left(\frac{1}{2}\right)$$

$$t = -RC\log_e\left(\frac{1}{2}\right)$$

$$t = RC\log_e(2)$$

3 Let $N = N(t)$ be the % of the population to buy *a* new iPhone, at time *t* weeks.

$p = 70$

$$\frac{dN}{dt} = k(70 - N); \quad N(0) = 0$$

$$\frac{dt}{dN} = \frac{1}{k(70 - N)}$$

$$kt = \int \frac{1}{70 - N} dN$$

$$= -\log_e(|70 - N|) + C$$

Since $2 \leq N < 70$, models are not needed.

To find C, use $N = 0$, $t = 0$ (no one has it yet)

$$0 = -\log_e(70 - 0) + C$$

$$C = \log_e(70)$$

$$kt = -\log_e(70 - N) + \log_e(70)$$

$$= \log_e\left(\frac{70}{70 - N}\right)$$

To find k, use $N(4) = 20$, $t = 4$, $N = 20$

$$4k = \log_e\left(\frac{70}{70 - 20}\right) = \log_e\left(\frac{7}{5}\right)$$

$$k = \frac{1}{4}\log_e\left(\frac{7}{5}\right) \approx 0.084$$

$$e^{kt} = \frac{70}{70 - N}$$

$$70 - N = 70e^{-kt}$$

$$N = 70 - 70e^{-kt}$$

$$N = N(t) = 70(1 - e^{-0.084t})$$

when $N = 50$, $t = ?$

$$50 = 70\left(1 - e^{-0.084t}\right)$$

$$1 - e^{-0.084t} = \frac{50}{70}$$

$$e^{-0.084t} = 1 - \frac{5}{7} = \frac{2}{7}$$

$$-0.084t = \log_e\left(\frac{2}{7}\right)$$

$$t = -\frac{1}{0.084}\log_e\left(\frac{2}{7}\right) \approx 14.8 \text{ weeks}$$

4 half-life 186.1 days

$$T = \frac{1}{k}\log_e(2) = 186.1$$

$$k = \frac{\log_e(2)}{186.1} = 0.003725$$

$$m = m_0e^{-kt}$$

when $t = 90$

$$m = m_0e^{-0.003725 \times 90}$$

$$= 0.715m_0$$

So $(1 - 0.715) = 0.285$

28.5% is left.

5 $\dfrac{dI}{dx} = -kI,\ I(0) = I_0$

$$\frac{dx}{dI} = -\frac{1}{kI}$$

$$x = -\frac{1}{k}\int \frac{1}{I} dI$$

$$x = -\frac{1}{k}\log_e(I) + c$$

$x = 0$, $I = I_0$

$$0 = -\frac{1}{k}\log_e(I_0) + c \Rightarrow c = \frac{1}{k}\log_e(I_0)$$

$$x = \frac{1}{k}\log_e(I_0) - \frac{1}{k}\log_e(I)$$

$$= \frac{1}{k}\log_e\left(\frac{I_0}{I}\right)$$

$$kx = \log_e\left(\frac{I_0}{I}\right)$$

$$\frac{I_0}{I} = e^{kx}$$

$$I = I_0e^{-kx}$$

6 **a** $\dfrac{dT}{dt} = 25 + k(T - 15)$

When $T = 75$, $\dfrac{dT}{dt} = 45$°C/ min

$$45 = 25 + k(75 - 15)$$

$$20 = 60k$$

$$k = \frac{1}{3}$$

$$\frac{dT}{dt} = 25 + \frac{1}{3}(T - 15)$$

$$= 25 + \frac{T}{3} - 5$$

$$\frac{dT}{dt} = 20 + \frac{T}{3}$$

$$\frac{dT}{dt} = \frac{60 + T}{3}, \ T(0) = 15$$

b $\dfrac{dt}{dT} = \dfrac{30}{60 + T}$

$$t = 3\int \frac{1}{60 + T} dT$$

$$t = 3\log_e(60 + T) + c$$

$T = 15$, $t = 0$

$$0 = 3\log_e(75) + c \Rightarrow c = -3\log_e(75)$$

$$t = 3(60 + T) - 3\log_e(75)$$

$$t = 3\ \log_e(\frac{60 + T}{75})$$

$$e^{\frac{t}{3}} = \frac{60 + T}{75}$$

$$60 + T = 75e^{\frac{t}{3}}$$

$$T = 75e^{\frac{t}{3}} - 60$$

$$T(t) = 15\left(5e^{\frac{t}{3}} - 4\right)$$

c When $T = 185$°C

$$185 = 75e^{\frac{t}{3}} - 60$$

$$245 = 75e^{\frac{t}{3}}$$

$$e^{\frac{t}{3}} = \frac{245}{75} = \frac{49}{15}$$

$$t = 3\ \log_e\left(\frac{49}{15}\right)$$

($\approx$ 3.55 min)

Technology active: multiple choice

7 $\dfrac{dN}{dt} = kN,\ N(0) = N_0$

$$\frac{dt}{dN} = \frac{1}{kN}$$

$$t = \frac{1}{k}\int \frac{1}{N}dN$$
$$kt = \log_e(N) + c$$
When $t = 0, N = N_0$
$$0 = \log_e(N_0) + c \Rightarrow c = -\log_e(N)$$
$$kt = \log_e(N) - \log_e(N_0)$$
$$= \log_e\left(\frac{N}{N_0}\right)$$
$$N = N_0e^{kt}$$
When $t = 4,\ N = 2N_0$
$$N = N_0e^{kt}$$
$$2 = e^{4k}$$
$$k = \frac{1}{4}\log_e(2)$$
$$N = 3N_0,\quad t = ?$$
$$3N_0 = N_0e^{kt}$$
$$3 = e^{kt}$$
$$t = \frac{1}{k}\log_e(3)$$
$$t = \frac{1}{\frac{1}{4}\log_e(2)}\log_e(3)$$
$$t = \frac{4\log_e(3)}{\log_e(2)}$$
$$t = \frac{\log_e(3)^4}{\log_e(2)}$$
$$= \frac{\log_e(81)}{\log_e(2)}$$
The correct answer is **A**.

8 $m = m_0e^{-kt}$

When $t = 5,\ m = \frac{1}{2}m_0$
$$\frac{1}{2} = e^{-5k}$$
$$-5k = \log_e\left(\frac{1}{2}\right)$$
$$-5k = -\log_e(2)$$
$$k = \frac{1}{5}\log_e(2)$$
$$m = m_0e^{-\frac{t}{5}\log_e(2)}$$
When $t = 3,\ m = ?$
$$m = m_0e^{-\frac{3}{5}\log_e(2)}$$
$$m = 0.66\,m_0$$
Therefore 34% has disintegrated.

The correct answer is **E**.

9 $\frac{dI}{dx} = -kI$
$$\frac{dx}{dI} = -\frac{1}{kI}$$
$$x = -\frac{1}{k}\int \frac{1}{I}dI$$
$$-kx = \log_e(I) + c$$
When $x = 0,\quad I = I_0$
$$c = -\log_e(I_0)$$
$$-kx = \log_e(I) - \log_e(I_0)$$
$$-kx = \log_e\left(\frac{I}{I_0}\right)$$
$$I = I_0e^{-kx}$$
When $I = 0.95I_0, x = 0.40$
$$0.95I_0 = I_0e^{-0.4k}$$
$$0.95 = e^{-0.4k}$$
$$-0.4k = \log_e(0.95)$$
$$k = -\frac{1}{0.4}\log_e(0.95) = 0.1282$$
$$I = I_0e^{-0.1282x}$$
$$x = 1$$
$$I = I_0e^{-0.1282} = 0.88I_0$$
Then the percentage lost is 12%

The correct answer is **A**.

10 $V = 10\,\text{L}$

Concentarion 5 kg/L

Input 6 L/ min

Output 4 L/ min

Input − Output = $6t - 4t = 2t$ L/ min
$$V(t) = 10 + 2t$$
Inflow rate = $5 \times 6 = 30$ kg/ min

Outflow rate = $\frac{4Q}{V(t)}$
$$\frac{dQ}{dt} = \text{Inflow rate} - \text{Outflow rate}$$
$$\frac{dQ}{dt} = 30 - \frac{4Q}{10 + 2t}$$
The correct answer is **B**.

11 $\frac{dT}{dt} = k(T - T_m)$
$$\frac{dT}{dt} = k(T - 18),\ T(0) = 180$$
The correct answer is **C**.

12 $\frac{dW}{dt} = kW(200 - W) - 20,\ W(0) = 200$

The correct answer is **B**.

13 $N(t) = \frac{200}{1 + 99e^{-kt}}$
$$\frac{dN}{dt} = \frac{0 - 200 \times (-k) \times 99e^{-kt}}{\left(1 + 99e^{-kt}\right)^2}$$
$$\frac{dN}{dt} = \frac{200 \times k \times 99e^{-kt}}{\left(1 + 99e^{-kt}\right)^2} = \frac{200k}{1 + 99e^{-kt}} \times \frac{200 + 200 \times 99e^{-kt} - 200}{200\left(1 + 99e^{-kt}\right)}$$
$$\frac{dN}{dt} = \frac{200k}{1 + 99e^{-kt}} \times \frac{200\left(1 + 99e^{-kt}\right) - 200}{200\left(1 + 99e^{-kt}\right)}$$
$$\frac{dN}{dt} = \frac{200k}{1 + 99e^{-kt}} \times \left(\frac{200\left(1 + 99e^{-kt}\right)}{200\left(1 + 99e^{-kt}\right)} - \frac{200}{200\left(1 + 99e^{-kt}\right)}\right)$$
$$\frac{dN}{dt} = kN\left(1 - \frac{N}{200}\right)$$
The correct answer is **C**.

14 $N = \frac{P}{2}$
$$P = 500$$
$$N = 250$$
The correct answer is **D**.

15 $f(x) = -\frac{1}{9^x},\ y_0 = 1,\ x_0 = 0,\quad h = 0.5$
$$y_1 = y_0 + hf(x_0)$$
$$y_1 = 1 + 0.5 \times -1$$
$$y_1 = 0.5$$
$$y_2 = y_1 + hf(x_1)$$
$$y_2 = 0.5 + 0.5 \times (-\frac{1}{9^{0.5}}) = 0.3333$$
The correct answer is **E**.

16 $\frac{dy}{dx} = \frac{x + 2}{y - 2}$

The correct answer is **D**.

Technology active: extended response

17 **a** $\dfrac{dQ}{dt} = 2 \times 6 - \dfrac{4Q}{20 + (6 - 4)t}$

$\dfrac{dQ}{dt} = 12 - \dfrac{2Q}{10 + t},\ Q(0) = 0$

b $Q = \dfrac{4t(t^2 + 30t + 300)}{(t + 10)^2}$

Satisfies the initial condition, $t = 0,\ Q = 0$

LHS

$$Q = \frac{4t^3 + 120t^2 + 1200t}{(t + 10)^2}$$

$$\frac{dQ}{dt} = \frac{\left(12t^2 + 240t + 1200\right)(t + 10)^2 - 2(t + 10)\left(4t^3 + 120t^2 + 1200t\right)}{(t + 10)^4}$$

$$= \frac{(t + 10)\left[(t + 10)\left(12t^2 + 240t + 1200\right) - 2\left(4t^3 + 120t^2 + 1200t\right)\right]}{(t + 10)^4}$$

$$= \frac{1}{(t + 10)^3}\left[12t^3 + 240t^2 + 1200t + 120t^2 + 2400t + 12\,000 - 8t^3 - 240t^2 - 2400t\right]$$

$$= \frac{4t^3 + 120t^2 + 1200t + 12\,000}{(t + 10)^3}$$

$$= \frac{4\left(t^3 + 30t^2 + 300t + 3\,000\right)}{(t + 10)^3}$$

RHS

$$12 - \frac{2Q}{10 + t}$$

$$12 - \frac{2}{10 + t}\left[\frac{4t\left(t^2 + 30t + 300\right)}{(t + 10)^2}\right]$$

$$= 12 - \frac{2}{(10 + t)^3}\left[4t^3 + 120t^2 + 1200t\right]$$

$$= \frac{1}{(10 + t)^3}\left[12\left(1000 + 300t + 30t^2 + t^3\right) - 2\left(4t^3 + 120t^2 + 1200t\right)\right]$$

$$= \frac{4t^3 + 120t^2 + 1200t + 12\,000}{(10 + t)^3}$$

$$= \frac{4\left(t^3 + 30t^2 + 300t + 3\,000\right)}{(10 + t)^3}$$

$= \text{RHS}$

18 metal bar $T_m = 18^\circ\text{C}$

$T_0 = 400^\circ\text{C}, \quad T(10) = 250, \quad t = ?, \quad T = 120^\circ\text{C}$

$\theta = T - T_m$

$\theta = \theta_0 e^{-kt}$

$\theta_0 = T_0 - T_m = 400 - 18 = 382$

$\theta(10) = 250 - 18 = 232$

$\Rightarrow 232 = 382e^{-10k}$

$e^{-10k} = \dfrac{232}{382}$

$10k = \log_e\left(\dfrac{382}{232}\right)$

$k = \dfrac{1}{10}\log_e\left(\dfrac{191}{116}\right) \approx 0.0499$

So $\theta = 382e^{-0.05t}$

when $T = 120 \Rightarrow \theta = 120 - 18 = 102$

$$102 = 382e^{-0.05t}$$
$$e^{t-0.052} = \frac{102}{382}$$
$$-0.05t = \log_e\left(\frac{102}{382}\right)$$
$$t = -\frac{1}{0.05}\log_e\left(\frac{51}{191}\right)$$
$$= 26.48 \text{ minutes}$$

It takes 16.5 minutes to cool from 250°C to 120°C

19 a $\frac{dT}{dt} = -k(T - T_m) \quad T_m = ?$

$$\theta = T - T_m \quad \theta_0 = T_0 - T_m$$
$$T_0 = -20°\text{C} \quad T(1) = -16 \quad T(2) = -12\frac{1}{3}$$
$$\theta(0) = \theta_0 = -20 - T_m$$
$$\theta = \theta_0 e^{-kt}$$
$$\theta(1) \Rightarrow (-16 - T_m) = (-20 - T_m)\,e^{-k}$$
$$e^{-k} = \frac{16 + T_m}{20 + T_m}$$
$$\theta(2) \Rightarrow \left(-12\frac{1}{3} - T_m\right) = (-20 - T_m)\,e^{-2k}$$
$$e^{-2k} = \frac{12\frac{1}{3} + T_m}{20 + T_m} = \left(e^{-k}\right)^2$$
$$\frac{12\frac{1}{3} + T_m}{20 + T_m} = \left(\frac{16 + T_m}{20 + T_m}\right)^2$$
$$\left(12\frac{1}{3} + T_m\right)(20 + T_m) = (16 + T_m)^2$$
$$\frac{740}{3} + \frac{97}{3}T_m + \cancel{T_m^2} = 256 + 32T_m + \cancel{T_m^2}$$
$$256 - \frac{740}{3} = T_m\left(\frac{97}{3} - 32\right)$$
$$\frac{28}{3} = \frac{Tm}{3}$$
$$T_m = 28°\text{C}$$
$$\text{So } e^{-k} = \frac{16 + 28}{20 + 28} = \frac{44}{48} = \frac{11}{12}$$
$$k = -\log_e\left(\frac{11}{12}\right) = 0.087011$$

b further 3 minutes

when $t = 5 \quad \theta(3) = (-20 - 28)e^{-0.087011 \times 5} = -31.07$

$$T = \theta + T_m \quad = -31.07 + 28$$
$$= -3.07°\text{C}$$

20 a $\frac{dy}{dx} = \frac{y}{2}(4 - y), \quad x = -3, 3 \quad y = 0, 4$

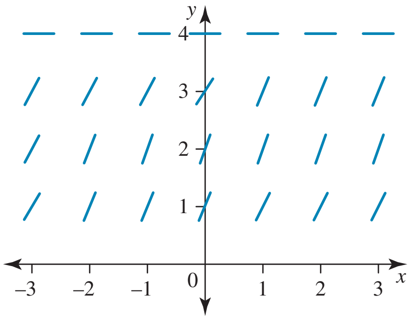
When $y = 0 \quad \frac{dy}{dx} = \text{slope} = 0$

$y = 1 \quad \frac{dy}{dx} = \text{slope} = \frac{3}{2} = 1.5$

$y = 2 \quad \frac{dy}{dx} = \text{slope} = 2$

$y = 3 \quad \frac{dy}{dx} = \text{slope} = \frac{3}{2} = 1.5$

$y = 4 \quad \frac{dy}{dx} = \text{slope} = 0$



b $\frac{dy}{dx} = \frac{y}{2}(4 - y), \quad y(0) = 1, \quad h = \frac{1}{3}$

$f(y) = \frac{y}{2}(4 - y) \quad x_0 = 0, \quad y_0 = 1$

$$y_1 = y_0 + hf(y_0)$$
$$= 1 + \frac{1}{3}\left(\frac{1}{2}(4 - 1)\right)$$
$$= 1.5 = \frac{3}{2}$$
$$y_2 = y_1 + hf(y_1)$$
$$= \frac{3}{2} + \frac{1}{3}\left(\frac{1}{2} \times \frac{3}{2}\left(4 - \frac{3}{2}\right)\right)$$
$$= \frac{17}{8} = 2.125$$
$$y_3 = y_2 + hf(y_2)$$
$$= \frac{17}{8} + \frac{1}{3}\left(\frac{1}{2} \times \frac{17}{8}\left(4 - \frac{17}{8}\right)\right)$$
$$= \frac{357}{128} = 2.78906$$

x	0	$\frac{1}{3}$	$\frac{2}{3}$	1
y	1	1.5	2.125	2.7891

c $\frac{dy}{dx} = \frac{y(4 - y)}{2}; \quad y(0) = 1$

$$\frac{dx}{dy} = \frac{2}{y(4 - y)}$$
$$x = \int \frac{2}{y(4 - y)}dy$$

Partial fractions $\frac{2}{y(4 - y)} = \frac{A}{y} + \frac{B}{4 - y}$

$$= \frac{A(4 - y) + By}{y(4 - y)}$$
$$= \frac{y(B - A) + 4A}{y(4 - y)}$$

y: $B - A = 0$

y': $4A = 2 \quad A = B = \frac{1}{2}$

$$x = \frac{1}{2}\int\left(\frac{1}{y} + \frac{1}{4 - y}\right)dy$$

$2x = \log_e(|y|) - \log_e(|4 - y|) + c$

but $0 < y < 4$

modulus not needed, $y = 1, x = 0$

$$0 = \log_e(1) - \log_e(3) + c$$
$$c = \log_e(3)$$
$$2x = \log_e(y) - \log_e(4-y) + \log_e(3)$$
$$2x = \log_e\left(\frac{3y}{4-y}\right)$$
$$\frac{3y}{4-y} = e^{2x}$$
$$4 - y = 3ye^{-2x}$$
$$4 = y + 3ye^{-2x}$$
$$= y(1+3e^{-2x})$$
$$y = y(x) = \frac{4}{1+3e^{-2x}}$$

d $y = \dfrac{4}{1+3e^{-2x}} \quad y(0) = \dfrac{4}{4} = 1$

$$y = 4\left(1+3e^{-2x}\right)^{-1}$$
$$\frac{dy}{dx} = 4 \times -1 \times -6e^{-2x}\left(1+3e^{-2x}\right)^{-2}$$
$$= \frac{4 \times 6e^{-2x}}{\left(1+3e^{-2x}\right)^2} = \frac{4}{\left(1+3e^{-2x}\right)} \times \frac{6e^{-2x}}{\left(1+3e^{-2x}\right)}$$

Now $\dfrac{1}{1+3e^{-2x}} = \dfrac{y}{4}$

$$1 + 3e^{-2x} = \frac{4}{y}$$
$$3e^{-2x} = \frac{4}{y} - 1 = \frac{4-y}{y}$$
$$6e^{-2x} = \frac{2(4-y)}{y}$$

So $\dfrac{dy}{dx} = y \times \dfrac{2(4-y)}{4} \times \dfrac{y}{4}$

$\dfrac{dy}{dx} = \dfrac{y(4-y)}{2}$ shown

e inflection point:

$$\frac{dy}{dx} = \frac{1}{2}\left(4y - y^2\right)$$
$$\frac{d^2y}{dx^2} = \frac{d}{dy}\left(\frac{1}{2}\left(4y-y^2\right)\right)\frac{dy}{dx}$$
$$= \frac{1}{2}(4-2y)\frac{dy}{dx}$$
$$\frac{d^2y}{dx^2} = \frac{1}{2}(2-y)y(4-y)$$

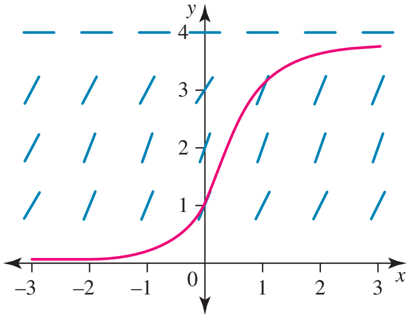
Since $0 < y < 4$

inflection point when $y = 2$

$$2 = \frac{4}{1+3e^{-2x}}$$
$$e^{-2x} = 1 \quad e^{-2x} = \frac{1}{3}$$
$$-2x = \log_e\left(\frac{1}{3}\right)$$
$$x = \frac{1}{2}\log_e(3)$$

Point $\left(\dfrac{1}{2}\log_e(3), 2\right)$

f



10.9 Exam questions

1 When $x = 0$, the gradient $m = 1$; when $y = 0$ the gradient $m = -1$. These conditions are only satified by $m = \dfrac{dy}{dx} = \dfrac{2x+y}{y-2x}$.

The correct answer is **A**.

2 $f(x) = \dfrac{dy}{dx} = 2x^2 - x,\ y_0 = 0 = y(2),\ h = 0.1$

$x_0 = 2,\ x_1 = 2.1,\ x_2 = 2.2$

$f(2) = 6,\ f(2.1) = 6.72,\ f(2.2) = 7.48$

$$y_1 = y_0 + hf(x_0)$$
$$= 0 + 0.1f(2)$$
$$= 0.1 \times 6$$
$$= 0.6$$
$$y_2 = y_1 + hf(x_1)$$
$$= 0.6 + 0.1f(2.1)$$
$$= 0.6 + 0.1 \times 6.72$$
$$= 1.272$$
$$y_3 = y_2 + hf(x_2)$$
$$= 1.272 + 0.1f(2.2)$$

The correct answer is **C**.

3 a Using implicit differentiation $\dfrac{1}{N}\dfrac{dN}{dt} = 3 \times 0.4e^{-0.4t}$

LHS $= \dfrac{1}{N}\dfrac{dN}{dt} + 0.4\log_e(N) - 2.4$ [1 mark]

$= 3 \times 0.4e^{-0.4t} + 0.4\left(6 - 3e^{-0.4t}\right) - 2.4$

$= 3 \times 0.4e^{-0.4t} + 2.4 - 3 \times 0.4e^{-0.4t}$

$- 2.4 = 0$ shown [1 mark]

b When $t = 0$ $\log_e(N) = 6 - 3e^0 = 3 \quad \Rightarrow N = e^3 \approx 20$

$\Rightarrow N = e^3 \approx 20$ [1 mark]

c As $t \to \infty$ $\log_e(N) \to 6 - 3e^{-\infty} = 6$ [1 mark]

$N = e^6 \approx 403$ [1 mark]

d i $\dfrac{d^2N}{dt^2} = \dfrac{d}{dt}\left(\dfrac{dN}{dt}\right) = \dfrac{d}{dt}\left[0.4N(6 - \log_e(N))\right]$

using product rule

$$\frac{d^2N}{dt^2} = 0.4\frac{dN}{dt}(6 - \log_e(N)) + 0.4N \times \frac{-1}{N}\frac{dN}{dt}$$

$= \dfrac{dN}{dt}\left[0.4\left(6 - \log_e(N)\right) - 0.4\right]$ [1 mark]

$$= \frac{dN}{dt}\left(2 - 0.4\log_e(N)\right)$$

substitute for

$$\frac{dN}{dt} = 0.4N\left(6-\log_e(N)\right)$$

$$= 0.4N\left(6-\log_e(N)\right)\left(2-0.4\log_e(N)\right)$$

$$= \frac{4N}{25}\left(6-\log_e(N)\right)\left(5-\log_e(N)\right)$$ [1 mark]

ii For inflection points

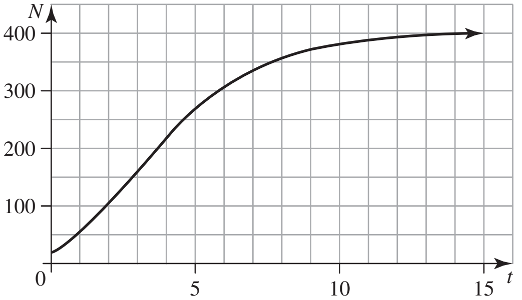
$$\frac{d^2N}{dt^2} = 0 \text{ since } \log_e(N) \neq 6 \Rightarrow \log_e(N) = 5$$

$$\Rightarrow N = e^5 \approx 148$$ [1 mark]

$$5 = 6 - 3e^{-0.4\,t} \Rightarrow e^{-0.4\,t} = \frac{1}{3}$$

$$t = \frac{1}{0.4}\log_e(3) \simeq 2.7 \text{ years}$$ [1 mark]

e



[2 marks]

4 a $\frac{dn}{dt} = k(50-n)$ [1 mark]

$n(0) = 4$

$n(4) = 16$

$$\frac{dt}{dn} = \frac{1}{k(50-n)}$$

$$kt = \int \frac{1}{(50-n)}\,dn$$

$kt = -\log_e(|50-n|) + c$ but $4 \leq n \leq 50$

When $t = 0$, $n = 4$

$0 = -\log_e(46) + c \Rightarrow c = \log_e(46)$

$kt = \log_e(46) - \log_e(50 - \text{n})$

$$kt = \log_e\left(\frac{46}{50-n}\right)$$

When $t = 4$, $n = 16$

$$4k = \log_e\left(\frac{46}{50-16}\right) = \log_e\left(\frac{23}{17}\right)$$

$$k = \frac{1}{4}\log_e\left(\frac{23}{17}\right)$$

$$e^{kt} = \frac{46}{50-n}$$ [1 mark]

$$50 - n = 46e^{-kt}$$

$$n = n(t) = 50 - 46e^{-kt}$$

$$n(t) = 2(25 - 23e^{-kt})$$ [1 mark]

b $t = 8$, $n(8) = 2\left(25 - 23e^{-2\log_e\left(\frac{23}{17}\right)}\right)$

$$= 2\left(25 - 23 \times \left(\frac{17}{23}\right)^2\right)$$

$$= \frac{572}{23}$$

≈ 25 ducks [1 mark]

5 a $N(t) = \frac{PN_0}{N_0 + (P - N_0)e^{-kt}}$

In 2010, $t = 0$ and $N(0) = N_0 = 34.148$

In 2015, $t = 5$ and $N(5) = 36.027$

$$36.027 = \frac{34.148P}{34.148 + (P - 34.148)e^{-5k}} \quad (1)$$

In 2020, $t = 10$ and $N(10) = 37.742$

$$37.742 = \frac{34.148P}{34.148 + (P - 34.148)\,e^{-10k}} \quad (2)$$ [1 mark]

Solving equations (1) and (2) on a CAS calculator:

$P = 48.742$, $k = 0.0383$

The maximum population of Canada will be 48.742 million. [1 mark]

b To determine the population in 2000, let $t = -10$.

$$N(-10) = \frac{48.742 \times 34.148}{34.148 + (48.742 - 34.148)e^{-0.0383 \times -10}}$$

$$= 29.962$$

The population of Canada in 2000 was 29.962 million. [1 mark]

Topic 11 — Kinematics: rectilinear motion

11.2 Differentiating position and velocity

11.2 Exercise

1 $x = t^2 - 4t - 12$

a $x(0) = -12$ m

b $v = \dfrac{dx}{dt} = 2t - 4$

$v(0) = -4$ m/s

c At rest $v = 0$

$2t - 4 = 0$

$t = 2$

$x(2) = 4 - 8 - 12$

$= -16$ m

d Orgin $x = 0 = t^2 - 4t - 12 = 0$

$(t - 6)(t + 2) = 0$

$t = 6$, since $t \geq 0$

$v(6) = 12 - 4$

$= 8$ m/s

2 $x(t) = \dfrac{1}{18}\left(2t^3 - 3t^2 - 12t + 8\right)$

$v = \dfrac{dx}{dt} = \dfrac{1}{18}\left(6t^2 - 6t - 12\right)$

At rest $v = 0$ $\quad 6t^2 - 6t - 12 = 0$

$6\left(t^2 - t - 2\right) = 0$

$6(t - 2)(t + 1) = 0$

$t = 2$, since $t \geq 0$

$a = \dfrac{dv}{dt} = \dfrac{1}{18}(12t - 6)$

$a(t) = \dfrac{1}{3}(2t - 1)$

$a(2) = \dfrac{1}{3} \times (4 - 1)$

$= 1$ m/s^2

3 $x(t) = t^3 - 9t^2 + 15t + 3$

a $x(0) = 3$ m

b $v = \dfrac{dx}{dt} = 3t^2 - 18t + 15$

$v(0) = 15$ m/s

c At rest $v = 0 = 3\left(t^2 - 6t + 5\right) = 0$

$(t - 5)(t - 1) = 0$

$t = 1, \ 5$

$x(1) = 1 - 9 + 15 + 3 = 10$ m

$x(5) = 125 - 225 + 75 + 3 = -22$ m

d $v_{AV} = \dfrac{x(5) - x(0)}{5 - 0} = \dfrac{-22 - 3}{5}$

$= -5$ m/s

e $t = 0 \ \ x = 3, \quad t = 1 \ \ x = 10 \quad t = 5 \ \ x = -2$

total distance $= 7 + 32 = 39$ m

average speed $= \dfrac{\text{total distance}}{\text{total time}} = \dfrac{39}{5}$

$= 7.8$ m/s

4 $x(t) = t^3 - 9t^2 + 24t + 5$

a $v = \dfrac{dx}{dt} = 3t^2 - 18t + 24$

At rest $v = 0 = 3\left(t^2 - 6t + 8\right) = 0$

$(t - 4)(t - 2) = 0$

$t = 2, \ 4$

$x(2) = 8 - 36 + 48 + 5 = 25$ m

$x(4) = 64 - 144 + 96 + 5 = 21$ m

b $v_{AV} = \dfrac{x(4) - x(0)}{4 - 0} = \dfrac{21 - 5}{4 - 0} = \dfrac{16}{4}$

$= 4$ m/s

c $t = 0, x = 5 \quad t = 2, x = 25 \quad t = 4, x = 21$

total disatance $= 20 + 4 = 24$

average speed $= \dfrac{\text{total distance}}{\text{total time}}$

$= \dfrac{24}{4}$

$= 6$ m/s

d $a = \dfrac{dv}{dt} = 6t - 18$

$a(2) = 12 - 18$

$= -6$ m/s^2

5 $h(t) = 40t - 5t^2$

a $v(t) = \dfrac{dh}{dt} = 40 - 10t = 0$

At rest $v = 0 \quad t = 4$

b $h(4) = 160 - 5 \times 16 = 80$ m

c $t = ? \quad h = 0 = t(40 - 5t) = 0$

$t = 0, \ 8$

Leaves Earth at $t = 0$

Returns to Earth at $t = 8$

d $v = \pm 20 \quad \pm 20 = 40 - 10t$

$10t = 40 \pm 20 = 60, \ 20$

$t = 2, \ 6$

e Up/down

$80\,\text{m} + 80\,\text{m} = 160\,\text{m}$

f Average speed $= \dfrac{160}{8} = 20$ m/s

6 $h(t) = 10 + 49t - 4.9t^2$

a $h(0) = 10$ m

b $v = \dfrac{dh}{dt} = 49 - 9.8t$

$v(0) = 49$ m/s

c $v = 0 \Rightarrow 49 = 9.8t$

$t = 5$ s

d $h_{\max} = h(5) = 10 + 49 \times 5 - 4.9 \times 25$

$= 132.5$

e $h = 0 \Rightarrow 4.9t^2 - 49t - 10 = 0$

$t = \dfrac{49 \pm \sqrt{(-49)^2 - 4 \times -10 \times 4.9}}{9.8}$

$= \dfrac{49 \pm \sqrt{2597}}{9.8}$

$= 10.2, \ t \geq 0$

f $v(10.2) = 49 - 9.8 \times 10.2$

$= -50.96\,\text{m/s}$

speed of 50.96 m/s

g $a = \dfrac{dv}{dt} = -9.8$ constant

7 $x(t) = 20t - \dfrac{5}{3}t^2$

a $v = \dfrac{dx}{dt} = 20 - \dfrac{10}{3}t$

$v(0) = 20\text{ m/s} = \dfrac{20 \times 60 \times 60}{1000}$

$= 72\text{ km/hr}$

b $v = 0 \quad 20 - \dfrac{10}{3}t = 0$

$t = 6\,\text{s}$

c $x(6) = 20 \times 6 - \dfrac{5}{3} \times 36$

$= 60\,\text{m}$

8 $x(t) = \dfrac{t^2}{6} - \dfrac{t^3}{1620}$

a $v = \dfrac{dx}{dt} = \dfrac{1}{3}t - \dfrac{1}{540}t^2$

$= \dfrac{t}{3}\left(1 - \dfrac{t}{180}\right)$

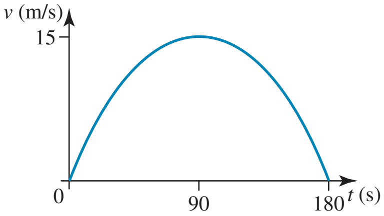
$a(t) = \dfrac{1}{3} - \dfrac{1}{270}t$

$a = 0, \quad t = 90$

$v_{\max} = v(90) = \dfrac{90}{3}\left(1 - \dfrac{90}{180}\right)$

$= 15\text{ m/s}$

$= 54\text{ km/h}$



b $v = 0 \quad t = 180$

3 minutes

c $x(180) = \dfrac{1}{6} \times 180^2 - \dfrac{1}{1620} \times 180^3$

$= 1800\,\text{m}$

$= 1.8\,\text{km}$

d $v_{AV} = \dfrac{1800}{180}$

$= 10\text{ m/s}$

$= 36\text{ km/h}$

9 $x(t) = 6e^{-\frac{t}{2}}$

a $x(0) = 6\,\text{m}$

b $v = \dfrac{dx}{dt} = -3e^{-\frac{t}{2}}$

$v(0) = -3\text{ m/s}$

c $a = \dfrac{dv}{dt} = \dfrac{3}{2}e^{-\frac{t}{2}}$

$a(0) = 1.5\text{ m/s}^2$

d $x = 6e^{-\frac{t}{2}}$

$\dfrac{x}{6} = e^{-\frac{t}{2}}$

$a = \dfrac{3}{2}e^{-\frac{t}{2}}$

$= \dfrac{3}{2} \times \dfrac{x}{6}$

$a = \dfrac{1}{4}x$

10 $x(t) = 9e^{-\frac{t}{3}} - 6e^{-t}$

a $x(0) = 9 - 6 = 3\,\text{m}$

b $v = \dfrac{dx}{dt} = -3e^{-\frac{t}{3}} + 6e^{-t}$

$v(0) = -3 + 6 = 3\,\text{m/s}$

c $v = 0 \quad 3e^{-\frac{t}{3}} = 6e^{-t}$

$e^{-\frac{t}{3}} \cdot e^{t} = 2 = e^{\frac{2t}{3}}$

$\dfrac{2t}{3} = \log_e(2)$

$t = \dfrac{3}{2}\log_e(2)$

d $a(t) = e^{-\frac{t}{3}} - 6e^{-t}$

If $a + bv + cx = 0$

$e^{-\frac{t}{3}} - 6e^{-t} + b\left(-3e^{-\frac{t}{3}} + 6e^{-t}\right) + c\left(9e^{-\frac{t}{3}} - 6e^{-t}\right) = 0$

$e^{-\frac{t}{3}}(1 - 3b + 9c) + e^{-t}(-6 + 6b - 6c) = 0$

(1) $\quad 3b - 9c = 1$

(2) $\quad 6b - 6c = 6$

(1) × 2 $\quad 6b - 18c = 2$

(2) − (1) × 2 $\quad 12c = 4$

$c = \dfrac{1}{3} \qquad b = \dfrac{4}{3}$

11 $x(t) = 4\cos\left(\dfrac{t}{2}\right)$

a $x(0) = 4\,\text{m}$

b $v(t) = -2\sin\left(\dfrac{t}{2}\right)$

$v(0) = 0\text{ m/s}$

c $x(t) = 4\cos\left(\dfrac{t}{2}\right)$

$a(t) = -\cos\left(\dfrac{t}{2}\right)$

$a = -\dfrac{x}{4}$

$\Rightarrow x = -4a$

d $x = 2 = 4\cos\left(\dfrac{t}{2}\right)$

$\cos\left(\dfrac{t}{2}\right) = \dfrac{1}{2}$

$\dfrac{t}{2} = \cos^{-1}\left(\dfrac{1}{2}\right) + 2k\pi$

$$\frac{t}{2} = \pm\frac{\pi}{3} + 2k\pi$$

$$t = \pm\frac{2\pi}{3} + 4k\pi = \frac{2\pi}{3}(6k \pm 1) \quad k \in Z^+$$

e $v = 0 \quad \sin\left(\frac{t}{2}\right) = 0$

$$\frac{t}{2} = k\pi$$

$$t = 2k\pi \qquad k \in Z^+$$

12 $x(t) = 9\cos\left(\frac{t}{3}\right) + 18\sin\left(\frac{t}{3}\right)$

a $x(0) = 9\,\text{m}$

b $v = \frac{dx}{dt} = -3\sin\left(\frac{t}{3}\right) + 6\cos\left(\frac{t}{3}\right)$

$v(0) = 6$ m/s

c $a = \frac{dv}{dt} = -\cos\left(\frac{t}{3}\right) - 2\sin\left(\frac{t}{3}\right)$

$a(0) = -1$ m/s^2

d $x = 9\left(\cos\left(\frac{t}{3}\right) + 2\sin\left(\frac{t}{3}\right)\right)$

$$x^2 = 81\left(\cos\left(\frac{t}{3}\right) + 2\sin\left(\frac{t}{3}\right)\right)^2$$

$$\frac{x^2}{9} = 9\left(\cos^2\left(\frac{t}{3}\right) + 4\sin^2\left(\frac{t}{3}\right) + 4\cos\left(\frac{t}{3}\right)\sin\left(\frac{t}{3}\right)\right)$$

$$= 9\left(1 - \sin^2\left(\frac{t}{3}\right) + 4\sin^2\left(\frac{t}{3}\right) + 2\sin\left(\frac{2t}{3}\right)\right)$$

$$\frac{x^2}{9} = 9\left(1 + 3\sin^2\left(\frac{t}{3}\right) + 2\sin\left(\frac{2t}{3}\right)\right)$$

$$45 - \frac{x^2}{9} = 36 - 27\sin^2\left(\frac{t}{3}\right) - 18\sin\left(\frac{2t}{3}\right)$$

$$v^2 = \left[3\left(-\sin\left(\frac{t}{3}\right) + 2\cos\left(\frac{t}{3}\right)\right)\right]^2$$

$$= 9\left(\sin^2\left(\frac{t}{3}\right) - 4\sin\left(\frac{t}{3}\right)\cos\left(\frac{t}{3}\right) + 4\cos^2\left(\frac{t}{3}\right)\right)$$

$$= 9\left(\sin^2\left(\frac{t}{3}\right) - 2\sin\left(\frac{2t}{3}\right) + 4\left(1 - \sin^2\left(\frac{t}{3}\right)\right)\right)$$

$$= 9\left(4 - 2\sin\left(\frac{2t}{3}\right) - 3\sin^2\left(\frac{t}{3}\right)\right)$$

$$= 36 - 27\sin^2\left(\frac{t}{3}\right) - 18\sin\left(\frac{2t}{3}\right)$$

e $a = -\cos\left(\frac{t}{3}\right) - 2\sin\left(\frac{t}{3}\right)$

$x = 9\cos\left(\frac{t}{3}\right) + 18\sin\left(\frac{t}{3}\right)$

$a = -\frac{x}{9}$ shown

11.2 Exam questions

1 $x = 9\sin\left(\frac{t}{3}\right)$

$v = \frac{dx}{dt} = 3\cos\left(\frac{t}{3}\right)$

$a = \frac{dv}{dt} = -\sin\left(\frac{t}{3}\right)$

$9a + x = 0$

The correct answer is **A**.

2 $x(t) = \frac{t^2}{4} - \frac{t^3}{720}$

$$v(t) = \frac{dx}{dt} = \frac{t}{2} - \frac{1}{240}t^2$$

$$= \frac{t}{2}\left(1 - \frac{t}{120}\right)$$

$$a(t) = \frac{dv}{dt} = \frac{1}{2} - \frac{1}{120}t$$

$$= \frac{1}{2}\left(1 - \frac{t}{60}\right)$$

a $a > 0 \Rightarrow t \in (0, 60)$

$a < 0 \Rightarrow t \in (60, 120)$ [1 mark]

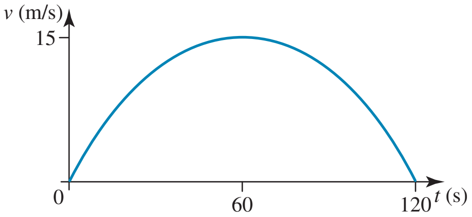
b $v_{\max} = v(60) = \frac{60}{2}\left(1 - \frac{60}{120}\right)$

$= 15$ m/s

$= \frac{15 \times 60 \times 60}{1000} = 54$ km/h [1 mark]

c $v = 0 \quad t = 0,\quad 120\,\text{s}$

2 minutes [1 mark]



d $x(120) = \frac{1}{4} \times 120^2 - \frac{1}{720} \times 120^3$

$= 1200\,\text{m}$

$= 1.2\,\text{km}$ [1 mark]

e $v_{AV} = \frac{1200}{120} = 10$ m/s

$= \frac{10 \times 60 \times 60}{1000}$

$= 36$ km/h [1 mark]

3 $x(t) = 490e^{-\frac{t}{10}} + 49t - 390$

a $x(0) = 490 - 390$

$= 100$ [1 mark]

b $v(t) = \frac{dx}{dt} = -49e^{-\frac{t}{10}} + 49$

$v(0) = 0$ m/s [1 mark]

c $a(t) = \frac{dv}{dt} = 4.9e^{-\frac{t}{10}}$ [1 mark]

d $49 - v(t)$

$$= \cancel{49} - \left[-49e^{-\frac{t}{10}} + \cancel{49}\right]$$

$$= 49e^{-\frac{t}{10}}$$

$$= 10a$$

So $a = \frac{1}{10}(49 - v)$ [1 mark]

11.3 Constant acceleration

11.3 Exercise

1 $u = 3$ m/s, $v = 9$ m/s, $t = 6$

a $a = ?$

$$v = u + at$$
$$9 = 3 + 6a$$
$$6 = 6a$$
$$a = 1 \text{ m/s}^2$$

b $s = ?$

$$s = ut + \frac{1}{2}at^2$$
$$s = 3 \times 6 + \frac{1}{2} \times 1 \times 36$$
$$= 36 \text{ m}$$

c $t = ?$, $v = 12$, $a = 1$, $u = 3$

$$v = u + at$$
$$12 = 3 + 1 \times t$$
$$t = 9 \text{ s}$$

2 a $u = 72$ km/h

$$= \frac{72 \times 1000}{60 \times 60} = 20 \text{ m/s}, \quad v = 0, \quad s = 20, \quad t = ?$$
$$s = \frac{1}{2}(u + v)t$$
$$20 = \frac{1}{2}(20 + 0) \times t$$
$$= 10t$$
$$t = 2 \text{ s}$$

b $u = 20$ m/s, $v = 0$, $t = 20$, $s = ?$

$$s = \frac{1}{2}(u + v)t$$
$$= \frac{1}{2}(0 + 20) \times 20$$
$$= 200 \text{ m}$$

3 a $s = 600$ m, $u = 0$, $t = 30$, $a = ?$

$$s = ut + \frac{1}{2}at^2$$
$$600 = 0 + \frac{1}{2} \times a \times 30^2$$
$$a = \frac{2 \times 600}{30 \times 30} = \frac{4}{3} = 1.\dot{3} \text{ m/s}^2$$

$v = ?$, $t = 15$, $a = 1.\dot{3}$, $u = 0$

$$v = u + at$$
$$= 0 + \frac{4}{3} \times 15$$
$$= 20 \text{ m/s}$$

b $s = D$, $u = 0$, $t = T$, $a = ?$

$$s = ut + \frac{1}{2}at^2$$
$$D = 0 + \frac{1}{2}a \times T^2$$
$$a = \frac{2D}{T^2}$$

$v = ?$, $t = \frac{T}{2}$, $u = 0$

$$v = u + at$$
$$v = 0 + \frac{2D}{T^2} \times \frac{T}{2}$$
$$v = \frac{D}{T}$$

4 a $S = 1$ km, $s = 1000$ m, $u = 0$, $t = 40$, $a = ?$

$$s = ut + \frac{1}{2}at^2$$
$$1000 = 0 + \frac{1}{2} \times a \times 40^2$$
$$a = \frac{2 \times 1000}{40 \times 40} = \frac{5}{4} = 1.25 \text{ m/s}^2$$

$s = 500$, $u = 0$, $a = \frac{5}{4}$, $v = ?$

$$v^2 = u^2 + 2as$$
$$v^2 = 0 + 2 \times \frac{5}{4} \times 500$$
$$= \frac{2500}{2}$$
$$v = \frac{50}{\sqrt{2}} = 25\sqrt{2} \text{ m/s}$$

b $s = D$, $u = 0$, $t = T$, $a = ?$

$$s = ut + \frac{1}{2}at^2$$
$$D = 0 + \frac{1}{2}a \times T^2$$
$$a = \frac{2D}{T^2}$$

$s = \frac{D}{2}$, $u = 0$, $a = \frac{2D}{T^2}, v = ?$

$$v^2 = u^2 + 2as$$
$$= 0 + 2 \times \frac{2D}{T^2} \times \frac{D}{2}$$
$$v^2 = \frac{2D^2}{T^2}$$
$$v = \sqrt{2}\frac{D}{T}$$

5 $u = 54$ km/h $= \frac{54 \times 1000}{60 \times 60} = 15$ m/s $v = 18$ km/h $= \frac{18 \times 1000}{60 \times 60} = 5$ m/s

$t = 6$ s

a

$$v = u + at$$
$$5 = 15 + 6a$$
$$a = -\frac{5}{3} \text{m/s}^2$$

b $s = ?$

$$s = \frac{1}{2}(u + v)t$$
$$= \frac{1}{2}(5 + 15) \times 6$$
$$= 60 \text{ m}$$

c $u = 15$ m/s, $v = 0$, $a = -\frac{5}{3}$

$$v = u + at$$
$$0 = 15 - \frac{5t}{3}$$
$$t = 9 \text{ s}$$

d $s = ?,\ \ u = 15,\ \ v = 0,\ \ a = -\frac{5}{3}$

$$v^2 = u^2 + 2as$$
$$0 = 15^2 - 2 \times \frac{5}{3} \times s$$
$$s = \frac{15 \times 15 \times 3}{2 \times 5}$$
$$= 67.5\text{ m}$$

6 a i $u = 18$ km/h $\quad v = 36$ km/h

$= 5$ m/s $\quad v = 10$ m/s

$a = ? \quad s = D$

$$v^2 = u^2 + 2as$$
$$100 = 25 + 2aD$$
$$2aD = 75$$
$$a = \frac{75}{2D}$$

$v = ? \qquad s = \frac{D}{2}$

$$v^2 = u^2 + 2as$$
$$= 25 + 2 \times \frac{75}{2D} \times \frac{D}{2}$$
$$= 25 + \frac{75}{2} = \frac{125}{2}$$
$$v = \frac{5\sqrt{10}}{2}$$

ii $t = ?\ \ v = u + at$

$$10 = 5 + \frac{75}{2D} \times t$$
$$5 = \frac{75t}{2D}$$
$$t = \frac{5 \times 2D}{75} = \frac{2D}{15}$$
$$v_{AV} = \frac{\text{total distance}}{\text{total time}}$$
$$= \frac{\cancel{D}}{\frac{2\cancel{D}}{15}} = \frac{15}{2} = 7.5\text{ m/s}$$

b i $u = U,\ \ v = V,\ \ s = D,\ \ a = ?$

$$v^2 = u^2 + 2as$$
$$V^2 = U^2 + 2aD$$
$$a = \frac{V^2 - U^2}{2D},\ \ s = \frac{D}{2},\ \ v = ?$$
$$v^2 = u^2 + 2as$$
$$v^2 = U^2 + 2 \times \left(\frac{V^2 - U^2}{2\cancel{D}}\right) \times \frac{\cancel{D}}{2}$$
$$= U^2 + \frac{1}{2}\left(V^2 - U^2\right)$$
$$= \frac{1}{2}\left(U^2 + V^2\right)$$
$$v = \sqrt{\frac{U^2 + V^2}{2}}$$

ii $v = u + at,\ \ t = ?$

$$V = U + \left(\frac{V^2 - U^2}{2D}\right) \times t$$
$$t = \frac{2D(V - U)}{V^2 - U^2} = \frac{2D}{V + U}$$
$$v_{AV} = \frac{\text{total distance}}{\text{total time}}$$
$$= \frac{D}{\frac{2D}{V+U}}$$
$$= \frac{U + V}{2}$$

7 a $t = 4, \quad v = 10$ m/s

$t = 6 \quad v = 14$ m/s

$a = ?,\ u = ?,\ v = u + at$

(1) $\quad 10 = u + 4a$

(2) $\quad 14 = u + 6a$

$$2a = 4$$
$$a = 2\text{ m/s}^2,\ \ u = 2\text{ m/s}$$

b $t = T_1,\ \ v = V_1$

$t = T_2,\ \ v = V_2 \qquad a = ?,\ \ u = ? \quad v = u + at$

(1) $\quad V_1 = u + aT_1$

(2) $\quad V_2 = u + aT_2$

$$V_2 - V_1 = a(T_2 - T_1)$$
$$a = \frac{V_2 - V_1}{T_2 - T_1}$$

$(1) \times T_2 \qquad V_1T_2 = uT_2 + aT_1T_2$

$(2) \times T_1 \qquad V_2T_1 = uT_1 + aT_1T_2$

$$V_2T_1 - V_1T_2 = u(T_1 - T_2)$$
$$u = \frac{V_1T_2 - V_2T_1}{T_2 - T_1}$$

8 a $s = 28, \quad t = 4$

$s = 45, \quad t = 6, \quad u = ?,\ a = ?$

$$s = ut + \frac{1}{2}at^2$$

(1) $\quad 28 = 4u + 8a$

(2) $\quad 45 = 6u + 18a$

$\frac{1}{2}(1) \quad 14 = 2u + 4a$

$\frac{3}{2}(1) \quad 42 = 6u + 12a$

$(2) - \frac{3}{2}(1) \quad 3 = 6a$

$$a = \frac{1}{2} = 0.5\text{ m/s}^2 \quad u = 6\text{ m/s}$$

b $s = d_1,\ \ t = T_1$

$s = d_2,\ \ t = T_2 \quad u = ?,\ a = ?$

$$s = uT + \frac{1}{2}aT^2$$

(1) $\quad d_1 = uT_1 + \frac{1}{2}aT_1^2$

(2) $\quad d_2 = uT_2 + \frac{1}{2}aT_2^2$

$(1) \times T_2 \quad d_1T_2 = uT_1T_2 + \frac{1}{2}aT_1^2T_2$

$(2) \times T_1 \quad d_2T_1 = uT_1T_2 + \frac{1}{2}aT_2^2T_1$

$$(2) \times T_1 - (1) \times T_2 \quad d_2T_1 - d_1T_2 = \frac{1}{2}aT_1T_2(T_2 - T_1)$$

$$a = \frac{2(d_2T_1 - d_1T_2)}{T_1T_2(T_2 - T_1)}$$

9 **a** $v = 5\sqrt{2}, \quad s = 25$

$v = 5\sqrt{5}, \quad s = 100, \quad u = ?, \ a = ?$

$v^2 = u^2 + 2as$

$(1) \quad 50 = u^2 + 50a$

$(2) \quad 125 = u^2 + 200a$

$(2) - (1) \quad 75 = 150a$

$a = 0.5 \text{ m/s}^2, \ u^2 = 25 \text{ m/s}, \quad u = 5 \text{ m/s}$

b $v = V_1, \quad s = d_1$

$v = V_2, \quad s = d_2 \quad u = ?, \ a = ?$

$v^2 = u^2 + 2as$

$(1) \quad V_1^2 = u^2 + 2ad_1$

$(2) \quad V_2^2 = u^2 + 2ad_2$

$(2) - (1) \quad V_2^2 - V_1^2 = 2ad_2 - 2ad_1$

$= 2a(d_2 - d_1)$

$$a = \frac{{V_2}^2 - {V_1}^2}{2(d_2 - d_1)}$$

$(1) \times d_2 \quad V_1^2d_2 = u^2d_2 + 2ad_1d_2$

$(2) \times d_1 \quad V_2^2d_1 = u^2d_1 + 2ad_1d_2$

$(1) \times d_2 - (2) \times d_1 V_1^2d_2V_1^2d_2 - V_2^2d_1 = u^2(d_2 - d_1)$

$$u = \sqrt{\frac{V_1^2d_2 - V_2^2d_1}{d_2 - d_1}}$$

10 **a** $u = 0, \quad t = 2, \quad a = -g = -9.8$

$v = u + at$

$= 0 - 9.8 \times 2$

$= -19.6 \text{ m/s}$

Hits the ground speed 19.6 m/s

$s = ut + \frac{1}{2}at^2$

$= 0 - \frac{1}{2} \times 9.8 \times 4$

$= -19.6$ below cliff

Cliff is 19.6 m high

b $u = 0, \quad t = T, \quad a = -g$

$v = u + at$

$= 0 - gT$

speed $= gT$ m/s

$s = ut + \frac{1}{2}at^2$

$= 0 - \frac{1}{2}gT^2$

Cliff is $\frac{1}{2}gT^2$ m

11 **a** $u = 0, s = -253, \quad a = -g$

$v^2 = u^2 + 2as$

$v^2 = 0 + 2 \times 9.8 \times 253$

$v = 70.419 \text{ m/s}$

$= \frac{70.419 \times 60 \times 60}{1000} \text{ km/h}$

$= 253.5 \text{ km/h}$

$s = ut + \frac{1}{2}at^2$

$-253 = 0 - 4.9t^2$

$t = \sqrt{\frac{253}{4.9}}$

$= 7.19 \text{ s}$

b $s = 37, \ u = ? \ \ v = 0$

$v^2 = u^2 + 2as$

$0 = u^2 - 2 \times 9.8 \times 37$

$u = \sqrt{2 \times 9.8 \times 37}$

$= 26.93 \text{ m/s}$

$= \frac{26.93 \times 60 \times 60}{1000}$

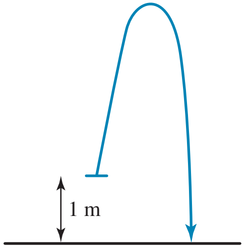
$= 96.95 \text{ km/hr}$

$v = u + at$

$0 = 26.93 - 9.8t$

$t = \frac{26.93}{9.8} = 2.75 \text{ s}$

12 $u = 49 \text{ m/s}, a = -9.8 \text{ m/s}^2$



a $v = u + at$

$0 = 49 - 9.8t$

$t = \frac{49}{9.8} = 5 \text{ s}$

b $s = ut + \frac{1}{2}at^2$

$= 49 \times 5 - \frac{1}{2} \times 9.8 \times 25$

$= 122.5$ m above the firing point so 123.5m above ground.

c Hits the ground $s = -1$

$-1 = 49t - 4.9t^2$

$t = -0.02, 10.02$

Takes 10.02 s to hit the ground, since $t > 0$

d Speed $v = \pm 20$

$\pm 20 = 49 - 9.8t$

$9.8t = 49 \pm 20$

$= 69, \quad 29$

$t = 7.04, \quad 2.96 \text{ s}$

13 **a** $a = -9.8 \text{ m/s}^2 \quad u = 5 - 3 = 2$

$s = -112.5, \ t = ?$

$s = ut + \frac{1}{2}at^2$

$-112.5 = 2t - 4.9t^2$

$4.9t^2 - 2t - 112.5 = 0$

$\Rightarrow t = 5 \text{ s}$

b $u = U,\ s = -H,\ a = -g$

$$s = ut + \frac{1}{2}at^2$$

$$-H = Ut - \frac{1}{2}gt^2$$

$$\frac{1}{2}gt^2 - Ut - H = 0$$

$$t = \frac{U \pm \sqrt{U^2 + 2gH}}{2 \times \frac{1}{2}g} \qquad t \geq 0$$

$$t = \frac{U + \sqrt{U^2 + 2gH}}{g}$$

14 a i Simply dropped

$u = 0\ s = -29.4 \quad a = -9.8,\ t = ?$

$$s = ut + \frac{1}{2}at^2$$

$$-29.4 = 0 - 4.9t^2$$

$$t = \sqrt{\frac{29.4}{4.9}}$$

$$t = \sqrt{6}\text{ s}$$

ii $t = 3 \qquad t = 2$

See graph at the foot of the page.*

$$s = ut + \frac{1}{2}at^2$$

(1) $s = -H,\ u = U,\ t = 3$

$-H = 3U - 4.9 \times 9$

(1) $H + 3U = 44.1$

(2) $s = -H,\ u = -U,\ t = 2$

$-H = -2U - 4.9 \times 4$

(2) $H - 2U = 19.6$

(1) − (2) $5U = 24.5$

$U = 4.9$ m/s

so $H = 2 \times 4.9 + 19.6$

$= 29.4$ m

b i $s = ut + \frac{1}{2}at^2$

See graph at the foot of the page.*

(1) $-H = UT_1 - \frac{1}{2}gT_1^2$

(2) $-H = -UT_2 - \frac{1}{2}gT_2^2$

$$UT_1 - \frac{1}{2}gT_1^2 = -UT_2 - \frac{1}{2}gT_2^2$$

$$U(T_1 + T_2) = \frac{1}{2}g\left(T_1^2 - T_2^2\right)$$

$$U\cancel{(T_1 + T_2)} = \frac{1}{2}g\cancel{(T_1 + T_2)}(T_1 - T_2)$$

$$U = \frac{g}{2}(T_1 - T_2)$$

$$H = UT_2 + \frac{1}{2}gT_2{}^2$$

$$= \frac{g}{2}(T_1 - T_2)T_2 + \frac{1}{2}gT_2{}^2$$

$$= \frac{1}{2}gT_1T_2 - \cancel{\frac{1}{2}gT_2{}^2} + \cancel{\frac{1}{2}gT_2{}^2}$$

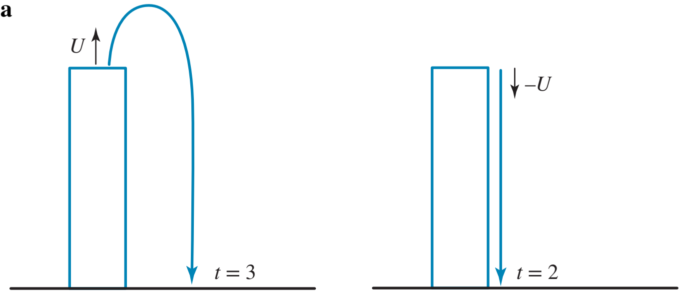
$$H = \frac{g}{2}T_1T_2$$

ii Dropped $u = 0\ \ s = -H$

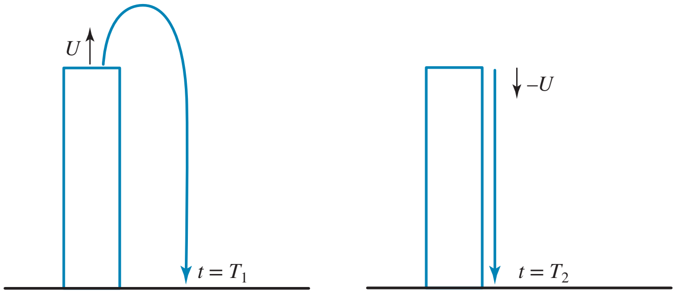
$$\frac{-g}{2}T_1T_2 = 0 - \frac{g}{2}T^2$$

$$T = \sqrt{T_1T_2}$$

***14a**



***14b**



15 Time to fall

$u = 0,\ v = -40,\ t = ?$

$$v = u + at$$
$$-40 = -9.8t$$
$$t = \frac{40}{9.8} = 4.08163$$

Depth of well $S = \left(\frac{u+v}{2}\right)t$

$$= \frac{-40}{2} \times -4.08163$$
$$= 81.6327 \text{ below}$$

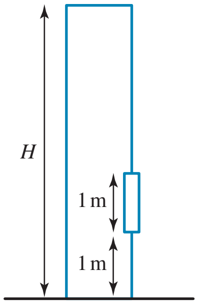
Sound heard $4.3196 - 4.08163$

$$= 0.2380$$

Speed of sound $V = \frac{s}{t} = \frac{81.6327}{0.2380}$

$$= 343 \text{ m/s}$$

16 Let building have height H



$u = 0, \quad a = -g$

$s = -H + 2, \quad t = T$

$s = -H + 1, \quad t = T + 0.03$

$$s = ut + \frac{1}{2}at^2$$

(1) $\quad -H + 2 = -4.9T^2$

(2) $\quad -H + 1 = -4.9(T + 0.03)^2$

$$H = 2 + 4.9T^2 = 1 + 4.9(T + 0.03)^2$$
$$= 1 + 4.9\left(T^2 + 0.06T + 0.03^2\right)$$
$$2 + \cancel{4.9T^2} = 1 + \cancel{4.9T^2} + 4.9 \times 0.06T + 4.9 \times 0.03^2$$
$$1 - 4.9 \times 0.03^2 = 4.9 \times 0.06T$$
$$T = \frac{1 - 4.9 \times 0.03^2}{4.9 \times 0.06}$$
$$= 3.3864 \text{ s}$$

So $\quad H = 2 + 4.9 \times 3.3864^2$

$$= 58.19 \text{ m}$$

17 a $\quad s = ut + \frac{1}{2}at^2$

(1) $\quad s = -10, \ u = 5, \ t = ?$

$$-10 = 5t - 4.9t^2$$
$$t = 2.0271$$

(2) $\quad s = -10, \ u = 0, \ t = ?$

$$-10 = -4.9t^2$$
$$t = \sqrt{\frac{10}{4.9}} = 1.4286$$

difference $= 2.0271 - 1.4256$

$$= 0.5986 \text{ s}$$

b $\quad s = -H, \ u = U, \ t = ?$

$$s = ut + \frac{1}{2}at^2$$
$$-H = Ut - \frac{1}{2}gt^2$$
$$\frac{1}{2}gt^2 - Ut - H = 0$$
$$t = \frac{U \pm \sqrt{U^2 + 2gH}}{2 \times \frac{1}{2}g} \qquad \text{since } t > 0$$
$$T_1 = \frac{U + \sqrt{U^2 + 2gH}}{g}$$

$S = -H, \ u = 0, \ t = ?$

$$-H = -\frac{1}{2}gt^2$$
$$T_2 = t = \sqrt{\frac{2H}{g}} = \frac{\sqrt{2gH}}{g}$$
$$T = T_1 - T_2 = \frac{U + \sqrt{U^2 + 2gH} - \sqrt{2gH}}{g}$$

18 a $\quad s = ut + \frac{1}{2}at^2$

(1) $\quad u = 0, \ s = -5$

$$-5 = 0 - \frac{1}{2}gt^2$$
$$t = \sqrt{\frac{10}{9.8}} = 1.01 \text{ s}$$

(2) $\quad t = ?, \ u = 0, \ s = -10$

$$-10 = \frac{1}{2}gt^2$$
$$t = \sqrt{\frac{20}{9.8}} = 1.4286$$

Particle (2) travels 10 m, $\quad u = ?$ in time $1.4286 - 1.01 = 0.4184$

$$s = ut + \frac{1}{2}at^2$$
$$-10 = -0.4184u - 4.9 \times 0.4184^2$$
$$u = \frac{10}{0.4184} - 4.9 \times 0.4184$$
$$= 21.85 \text{ m/s}$$

b Time to fall to half of cliff height

$$s = -\frac{H}{2} \qquad u = 0$$
$$-\frac{H}{2} = 0 - \frac{1}{2}gt^2 \qquad t = \sqrt{\frac{H}{g}}$$

Time to fall from top of cliff

$$-H = 0 - \frac{1}{2}gt^2 \qquad t = \sqrt{\frac{2H}{g}}$$

Difference $\quad \sqrt{\frac{2H}{g}} - \sqrt{\frac{H}{g}} = \left(\sqrt{2} - 1\right)\sqrt{\frac{H}{g}}$

Now distance of H, $\quad u = ?$

$$-H = -UT - \frac{1}{2}gT^2$$
$$U = \frac{H}{T} - \frac{1}{2}gT$$

$$= \frac{H}{\left(\sqrt{2}-1\right)}\sqrt{\frac{g}{H}} - \frac{1}{2}g\left(\sqrt{2}-1\right)\sqrt{\frac{H}{g}}$$

$$= \sqrt{gH}\left(\frac{1}{\sqrt{2}-1} - \frac{\sqrt{2}-1}{2}\right)$$

$$= \sqrt{gH}\left(\frac{1}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1} - \frac{1}{2}\left(\sqrt{2}-1\right)\right)$$

$$= \sqrt{gH}\left(\frac{\sqrt{2}+1}{2-1} - \frac{1}{2}\left(\sqrt{2}-1\right)\right)$$

$$= \sqrt{gH}\left(\sqrt{2}+1-\frac{1}{2}\sqrt{2}+\frac{1}{2}\right)$$

$$= \sqrt{gH}\left(\frac{\sqrt{2}}{2}+\frac{3}{2}\right)$$

$$= \frac{\sqrt{gH}}{2}\left(\sqrt{2}+3\right)$$

Check $H = 10$

$$U = \frac{\sqrt{9.8 \times 10}}{2}\left(\sqrt{2}+3\right)$$
$$= 21.85$$

11.3 Exam questions

1 $a = -g,\ u = 2,\ s = -50,\ t = ?$

$s = ut + \frac{1}{2}at^2$

$-50 = 2t - 4.9t^2$

Solving for $t \geq 0$ gives $t = 3.4\,s$.

The correct answer is **E**.

2 $u = 5,\ v = -11,\ t = 16,\ s = ?$

Using the constant acceleration formula:

$$s = \left(\frac{u+v}{2}\right)t$$
$$= \left(\frac{5-11}{2}\right) \times 16$$
$$= -48$$

The distance is 48 m.

The correct answer is **B**.

3 $s = -100,\ u = 2,\ a = -9.8,\ t = ?$

$s = ut + \frac{1}{2}at^2$

$-100 = 2t - 4.9t^2$

Solving using CAS

$\Rightarrow t = 4.7\text{ s}$

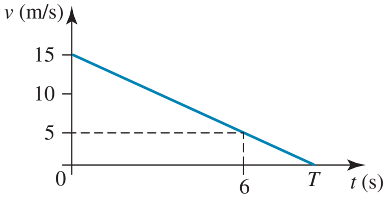
The correct answer is **C**.

11.4 Velocity-time graphs

11.4 Exercise

1 $u = 54\text{ km/h} = \frac{54 \times 1000}{60 \times 60} = 15\text{ m/s}$

$v = 18\text{ km/h} = \frac{18 \times 1000}{60 \times 60} = 5\text{ m/s}$



$\frac{T-6}{5} = \frac{T}{15}$

$15(T-6) = 5T$

$3(T-6) = T$

$3T - 18 = T$

$2T = 18$

$T = 9\text{ s}$

So takes extra 3 s.

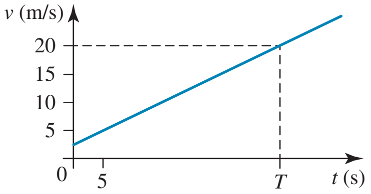
Area $= \frac{1}{2} \times 9 \times 15$

$= 67.5\text{ m}$

2 $u = 9\text{ km/h} = \frac{9 \times 1000}{60 \times 60} = 2.5\text{ m/s}$

$v = 18\text{ km/h} = \frac{18 \times 1000}{60 \times 60} = 5\text{ m/s}$

$v_f = 72\text{ km/h} = \frac{72 \times 1000}{60 \times 60} = 20\text{ m/s}$



$a = \frac{5-2.5}{5} = 0.5\text{ m/s}^2$

$\frac{20-5}{T-5} = \frac{1}{2}$

$2 \times 15 = T - 5$

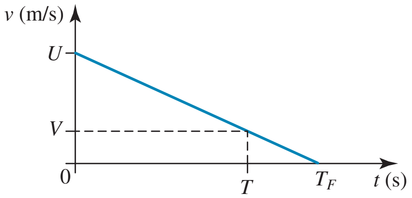
$T = 35\text{ s}$

So takes extra 30 s.

Area $\frac{1}{2} \times (2.5 + 20) \times 35$

$d = 393.75\text{ m}$

3



a Area of trapezium

$d = \frac{1}{2}(U + V)T$

b $\frac{T_f - T}{V} = \frac{T_f}{U}$

$U(T_f - T) = VT_f$

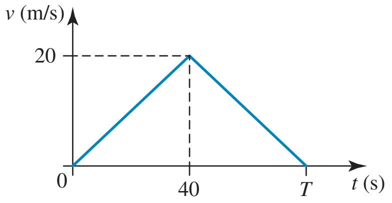
$(U - V)T_f = UT$

$T_f = \frac{UT}{U - V}$

$$\text{Distance} = \frac{1}{2} \times U \times T_f$$

$$= \frac{U^2T}{2(U - V)}$$

4



$$\text{Distance} = \frac{1}{2} \times 20 \times 40 + \frac{1}{2} \times 20 \times (T - 40) = 1000$$

$$400 + 10(T - 40) = 1000$$

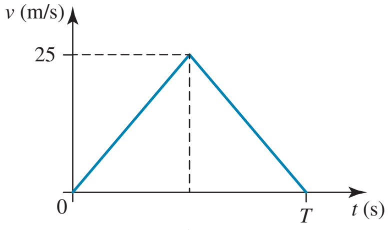
$$10(T - 40) = 600$$

$$T - 40 = 60$$

$$T = 100$$

So decelerates for 60 s

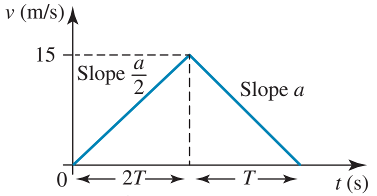
5 $96 \text{ km/h} = \dfrac{90 \times 1000}{60 \times 60} = 25 \text{ m/s}$



$$\text{Distance} = 900 = \frac{1}{2} \times T \times 25$$

$$T = 72 \text{ s}$$

6



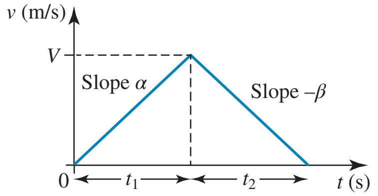
$$\text{Distance} = \frac{1}{2} \times 3T \times 15 = 1500$$

$$3T = 200,\ T = \frac{200}{3}$$

Total time 200 s

Time accelerating $\dfrac{400}{3} = 133\dfrac{1}{3}$ s

7



$$\alpha = \frac{V}{t_1},\ -\beta = -\frac{V}{t_2}$$

$$T = t_1 + t_2 = \frac{V}{\alpha} + \frac{V}{\beta}$$

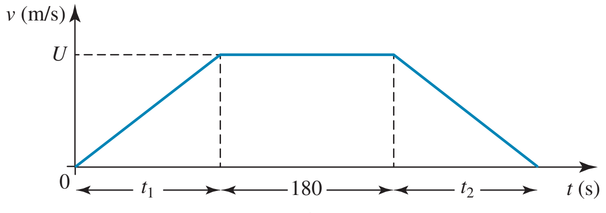
$$T = \frac{V(\alpha + \beta)}{\alpha\beta}$$

$$S = \frac{1}{2} \times Vt_1 + \frac{1}{2}V \times t_2$$

$$= \frac{1}{2}(t_1 + t_2)V$$

$$S = \frac{V^2(\alpha + \beta)}{2\alpha\beta}$$

8 Area of trapezium



$$\frac{V}{2} \times (360 + 180) = 4860$$

$$V \times 540 = 2 \times 4860$$

$$V = \frac{2 \times 4860}{540}$$

$$V = 18 \text{ m/s}$$

9

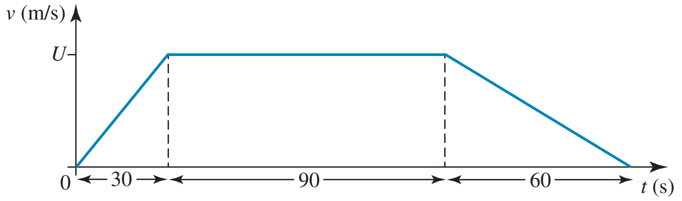
$$v(t) = \begin{cases} \dfrac{t}{4} & 0 \le t \le 60 \\ 15 & 60 < t \le 300 \\ \dfrac{1}{2}(330 - t) & 300 < t \le 330 \end{cases}$$

$$D = \frac{1}{2} \times 15 \times 60 + 240 \times 15 + \frac{1}{2} \times 15 \times 30$$

$$= 450 + 3600 + 225$$

$$= 4275 \text{ m}$$

10



$$v(t) = \begin{cases} \dfrac{t}{b} & 0 \le t \le 30 \\ U & 30 < t < 120 \\ \dfrac{1}{12}(180 - t) & 120 \le t \le T \end{cases}$$

$$\frac{1}{12}(180 - T) = 0,\ T = 180$$

$$\frac{180 - 120}{12} = \frac{60}{12} = 5 = U$$

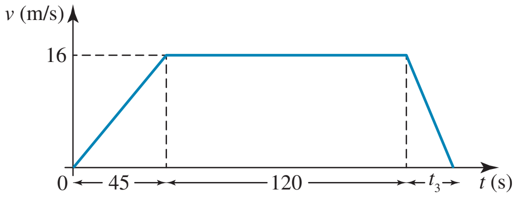
$$\frac{30}{b} = 5,\ b = 6$$

$$D = \frac{1}{2} \times 30 \times 5 + 90 \times 5 + \frac{1}{2} \times 5 \times 60$$

$$= 75 + 450 + 150$$

$$= 675 \text{ m}$$

11



$$\frac{1}{2} \times 45 \times 16 + 16 \times 120 + \frac{1}{2} \times 16 \times t_3 = 2500$$

$$360 + 1920 + 8t_3 = 2500$$

$$8t_3 = 2500 - 360 - 1920$$

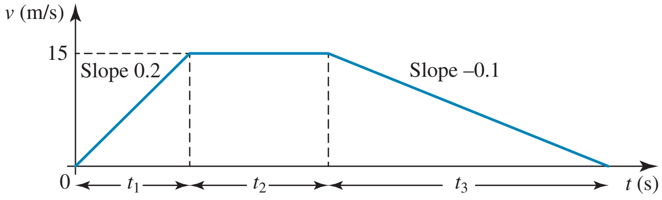
$$= 220$$

$$t_3 = \frac{220}{8} = \frac{55}{2} = 27.5$$

Total time = 45 + 120 + 27.5

= 192.5 s

12



$$0.2 = \frac{15}{t_1} \qquad 0.1 = \frac{15}{t_3}$$

$$t_1 = 75 \qquad t_3 = 150$$

$$\frac{1}{2} \times t_1 \times 15 + t_2 \times 15 + \frac{1}{2} \times t_3 \times 15 = 6000$$

$$\frac{15}{2}(t_1 + 2t_2 + t_3) = 6000$$

$$t_1 + 2t_2 + t_3 = \frac{2 \times 6000}{15} = 800$$

$$75 + 2t_2 + 150 = 800$$

$$2t_2 = 575$$

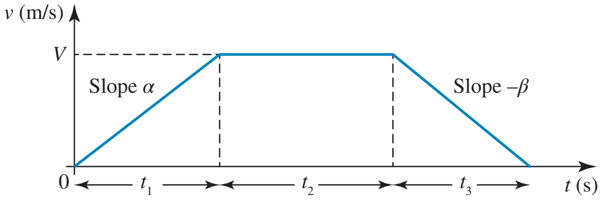
$$t_2 = \frac{575}{2} = 287.5$$

Total time $T = t_1 + t_2 + t_3$

= 800 – 287.5

= 512.5 s

13



$$\alpha = \frac{V}{t_1}, -\beta = -\frac{V}{t_3}$$

$T = t_1 + t_2 + t_3$, $t_2 > 0$

$$S = \frac{1}{2}Vt_1 + Vt_2 + \frac{1}{2}Vt_3$$

$$= \frac{V^2}{2\alpha} + Vt_2 + \frac{V^2}{2\beta}$$

$$\text{So } Vt_2 = S - \left(\frac{V^2}{2\alpha} + \frac{V^2}{2\beta}\right)$$

$$= S - \frac{V^2(\alpha + \beta)}{2\alpha\beta}$$

$$t_2 = \frac{S}{V} - \frac{V(\alpha + \beta)}{2\alpha\beta} > 0$$

$$\text{So } S > \frac{V^2(\alpha + \beta)}{2\alpha\beta}$$

$$\text{Now T} = \frac{V}{\alpha} + \left(S - \frac{V^2(\alpha + \beta)}{2\alpha\beta}\right) \times \frac{1}{V} + \frac{V}{\beta}$$

$$= \frac{V}{\alpha} + \frac{V}{\beta} + \frac{S}{V} - \frac{V(\alpha + \beta)}{2\alpha\beta}$$

$$= \frac{V}{\alpha} + \frac{V}{\beta} + \frac{S}{V} - \frac{V}{2\alpha} - \frac{V}{2\beta}$$

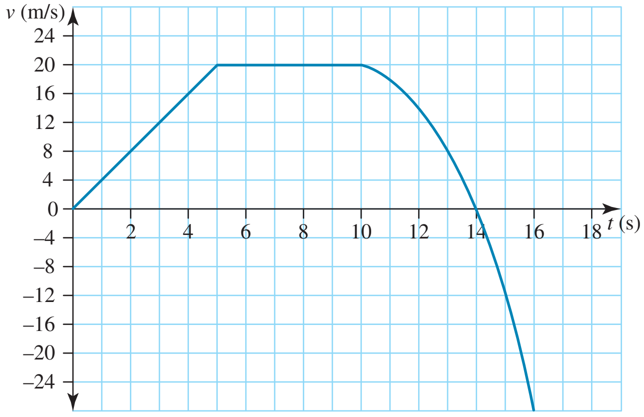
$$= \frac{S}{V} + \frac{V}{2\alpha} + \frac{V}{2\beta}$$

$$= \frac{S}{V} + \frac{V}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$$

14 $$v(t) = \begin{cases} 4t & 0 \le t \le 5 \\ 20 & 5 < t \le 10 \\ 20 - \frac{5}{4}(t - 10)^2 & 10 < t \le 16 \end{cases}$$

a See the graph footer of the page.*

***14a**



b $$\begin{aligned} d_1 &= \int_0^5 4t\,dt \\ &= \left[2t^2\right]_0^5 \\ &= 50 \end{aligned} \qquad \begin{aligned} d_2 &= \int_5^{10} 20\,dt \\ &= [20\,t]_5^{10} \\ &= 200 - 100 \\ &= 100 \end{aligned}$$

$$\begin{aligned} d_3 &= \int_{10}^{16} \left(20 - \frac{5}{4}(t-10)^2\right) dt \\ &= \left[20t - \frac{5}{12}(t-10)^3\right]_{10}^{16} \\ &= 20 \times 16 - \frac{5}{12} \times 6^3 - 20 \times 10 + \frac{5}{12} \times 0 \\ &= 30 \end{aligned}$$

$$\begin{aligned} d_{31} &= \int_{10}^{14} \left(20 - \frac{5}{4}(t-10)^2\right) dt \\ &= \left[20t - \frac{5}{12}(t-10)^3\right]_{10}^{14} \\ &= 20 \times 14 - \frac{5}{12} \times 4^3 - 20 \times 10 + 0 \\ &= \frac{160}{3} = 53\frac{1}{3} \end{aligned}$$

$$\begin{aligned} d_{32} &= \int_{14}^{16} \left(20 - \frac{5}{4}(t-10)^2\right) dt \\ &= \left[20t - \frac{5}{12}(t-10)^3\right]_{14}^{16} \\ &= 20 \times 16 - \frac{5}{12} \times 6^3 - 20 \times 14 + \frac{5}{12} \times 4^3 \\ &= \frac{-70}{3} = -23\frac{1}{3} \end{aligned}$$

Distance traveled $= 50 + 100 + 53\frac{1}{3} + 23\frac{1}{3}$

$= \frac{680}{3} = 226.67\text{ m}$

$= 226\frac{2}{3}\text{ m}$

c Displacement $= 50 + 100 + 53\frac{1}{3} - 23\frac{1}{3}$

$= 180\text{ m}$

15 a $v = 8,\ t = 4$ $\qquad 4a = 8$

$a = 2$

So $b = 8$

$c = 8$

When $t = 12 \quad v = 0$

$$\begin{aligned} 8 - d(4)^2 &= 0 \\ 16d &= 8 \\ d &= \frac{1}{2} \end{aligned}$$

$$v(t) = \begin{cases} 2t & 0 \le t \le 4 \\ 8 & 4 < t \le 8 \\ 8 - \frac{1}{2}(t-8)^2 & 8 < t \le 14 \end{cases}$$

b $$\begin{aligned} d_1 &= \int_0^4 2t\,dt \\ &= \left[t^2\right]_0^4 \\ &= 16 \end{aligned}$$

$$\begin{aligned} d_2 &= \int_4^8 8\,dt \\ &= [8\,t]_4^8 \\ &= 64 - 32 \\ &= 32 \end{aligned}$$

$$\begin{aligned} d_3 &= \int_8^{14} \left(8 - \frac{1}{2}(t-8)^2\right) dt \\ &= \left[8t - \frac{1}{6}(t-8)^3\right]_8^{14} \\ &= \left[8 \times 14 - \frac{1}{6} \times 6^3 - 64 + 0\right] \\ &= 12 \end{aligned}$$

$$\begin{aligned} d_{31} &= \int_8^{12} \left(8 - \frac{1}{2}(t-8)^2\right) dt \\ &= \left[8t - \frac{1}{6}(t-8)^3\right]_8^{12} \\ &= \left[8 \times 12 - \frac{1}{6} \cdot 4^3 - 64 + 0\right] \\ &= \frac{64}{3} \end{aligned}$$

$$\begin{aligned} d_{32} &= \int_{12}^{14} \left(8 - \frac{1}{2}(t-8)^2\right) dt \\ &= \left[8t - \frac{1}{6}(t-8)^3\right]_{12}^{14} \\ &= \left[8 \times 14 - \frac{1}{6} \times 6^3 - 8 \times 12 + \frac{1}{6} \times 4^3\right] \\ &= -\frac{28}{3} \end{aligned}$$

$$\begin{aligned} \text{Distance travelled} &= d_1 + d_2 + d_{31} - d_{32} \\ &= 16 + 32 + \frac{64}{3} + \frac{28}{3} \\ &= \frac{236}{3}\text{m} \end{aligned}$$

c $$\begin{aligned} \text{Displacement} &= d_1 + d_2 + d_3 \\ &= 16 + 32 + 12 \\ &= 60\text{ m} \end{aligned}$$

16 $$\begin{aligned} 126\text{ km/h} &= \frac{126 \times 1000}{60 \times 60} = 35\text{ m/s} \\ 90\text{ km/h} &= \frac{90 \times 1000}{60 \times 60} = 25\text{ m/s} \\ 108\text{ km/h} &= \frac{108 \times 1000}{60 \times 60} = 30\text{ m/s} \end{aligned}$$

Car: $35 \times 4 + \frac{1}{2}(25 + 35) \times 4 + 25(T - 8)$

$= 140 + 120 + 25T - 200$

$= 25T + 60$

Policeman: $\frac{1}{2} \times 16 \times 30 + 30(T - 18)$

$= 240 + 30T - 540$

$= 30T - 300$

$25T + 60 = 30T - 300$

$5T = 360$

$T = 72$ s

So police been in motion for 72 s

17 **a** See graph at the foot of the page.*

b $d_1 = \displaystyle\int_0^{60} \frac{30}{\pi} \sin^{-1}\left(\frac{t}{60}\right) dt$

$= 327.04$ m

c $d_3 = \displaystyle\int_{180}^{220} 15 \cos\left(\frac{\pi}{80}(t - 180)\right) dt$

$= 381.97$ m

d $D = d_1 + 15 \times 120 + d_3$

$= 327.04 + 1800 + 381.97$

$= 2509.01$ m

e $50 \text{ km/hr} = \dfrac{50 \times 1000}{60 \times 60} = \dfrac{125}{9} = 13.\dot{8}$ m/s

Solving $\dfrac{30}{\pi}\sin^{-1}\left(\dfrac{t}{60}\right) = 13.\dot{8}$ gives $t = 59.594$

$15 \cos\left(\dfrac{\pi}{180}(t - 180)\right) = 13.\dot{8}$ gives $t = 189.863$

Time $\dfrac{189.863 - 51.514}{220} \times 100 \approx 59.2\%$

f $a = \dfrac{d}{dt}\left(\dfrac{30}{\pi}\sin^{-1}\left(\dfrac{t}{60}\right)\right)$ when $t = 30$

$= 0.184$ m/s^2

18 **a** $v(t) = \begin{cases} 6\sqrt{t} & 0 \le t \le 25 \\ \dfrac{120}{\pi} \tan^{-1}\left(\dfrac{50 - t}{25}\right) & 25 < t \le 50 \end{cases}$

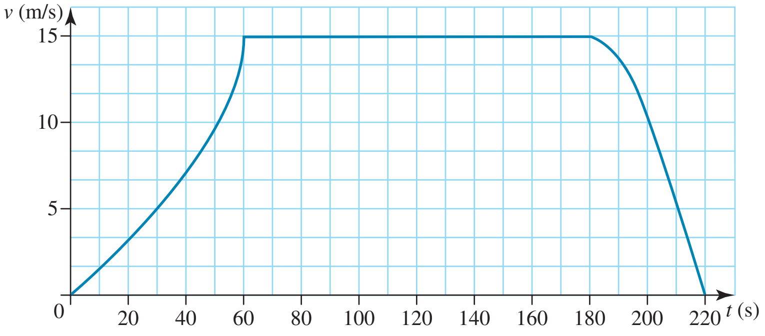
See graph at the foot of the page.*

b $D = \displaystyle\int_0^{25} 6\sqrt{t}\, dt + \int_{25}^{50} \frac{120}{\pi} \tan^{-1}\left(\frac{50 - t}{25}\right) dt$

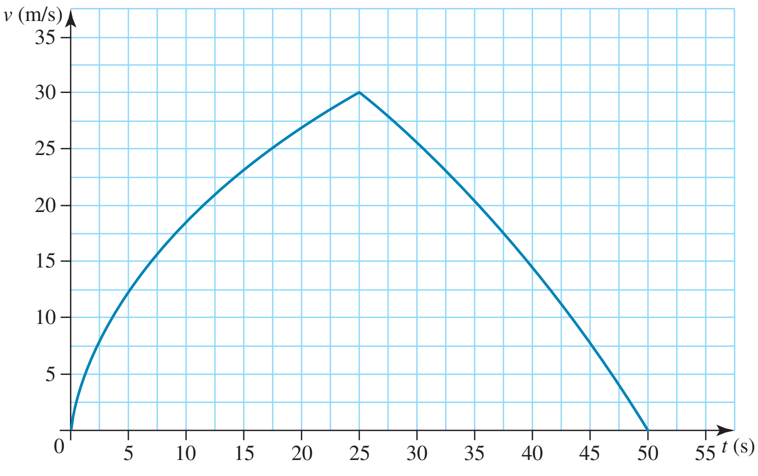
$= 500 + 419.05$

$= 919.05$ m

***17a**



***18a**



c $a = \dfrac{d}{dt}\left(6\sqrt{t}\right) = 3t^{\frac{-1}{2}} = \dfrac{3}{\sqrt{t}}$

When $t = 16$

$$a = \frac{3}{4} = 0.75\,\text{m/s}^2$$

19 **a** $v(t) = \begin{cases} \dfrac{80}{\pi}\tan^{-1}\left(\dfrac{t}{16}\right) & 0 \le t \le 16 \\ 20 - \dfrac{5}{49}(t-16)^2 & 16 < t \le 36 \end{cases}$

See graph at the foot of the page.*

b $D = \displaystyle\int_0^{36} |v(t)|\,dt = 424.24\,\text{m}$

c $D = \displaystyle\int_0^{36} v(t)\,dt = 306.68\,\text{m}$

d $a = \dfrac{-5 \times 2}{49}(t-16)$

$a(23) = \dfrac{-10}{49} \times 7$

$= \dfrac{-10}{7}\,\text{m/s}^2$

20 $v(t) = \begin{cases} \dfrac{80}{\pi}\tan^{-1}\left(\dfrac{t}{90}\right) & 0 \le t \le 90 \\ a + bt & 90 < t \le 270 \\ \dfrac{30}{\pi}\cos^{-1}\left(\dfrac{t-270}{90}\right) & 270 < t \le 360 \end{cases}$

a When $t = 90$ $\quad \dfrac{80}{\pi}\tan^{-1}(1) = \dfrac{80}{\pi} \times \dfrac{\pi}{4} = 20$

(1) $\quad a + 90b = 20$

When $t = 270$ $\quad \dfrac{30}{\pi}\cos^{-1}(0) = \dfrac{30}{\pi} \times \dfrac{\pi}{2} = 15$

(2) $\quad a + 270b = 15$

(2) − (1) $\quad (270 - 90)\,b = -5$

$b = \dfrac{-5}{180} = \dfrac{-1}{36}$

$\Rightarrow a = 20 + \dfrac{90}{36} = \dfrac{45}{2}$

b See the graph footer of the page.*

c $a = \left.\dfrac{dv}{dt}\right|_{t=45} = 0.226\,\text{m/s}^2$

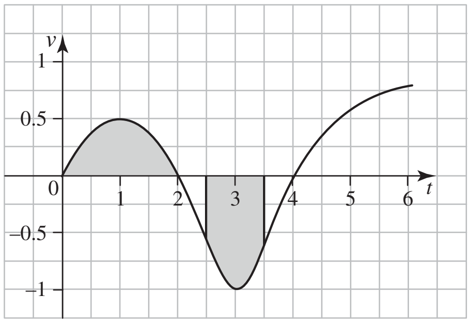
d $D = \displaystyle\int_0^{360} v(t)\,dt$

$= 5015.15\,\text{m}$

11.4 Exam questions

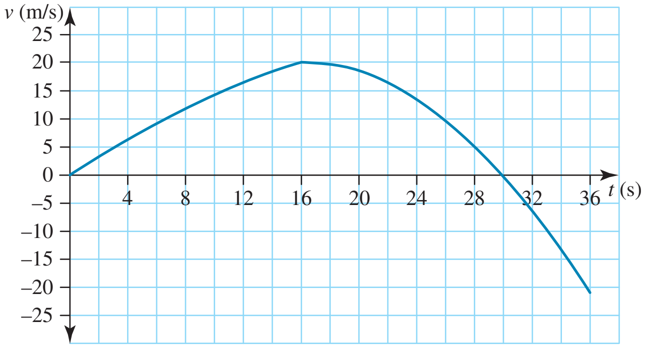
1 The return to its initial position, the areas above and below the x-axis need to be equal in magnitude.

$$\int_0^2 v(t)dt \approx \left|\int_{2.5}^{3.5} v(t)\,dt\right|$$

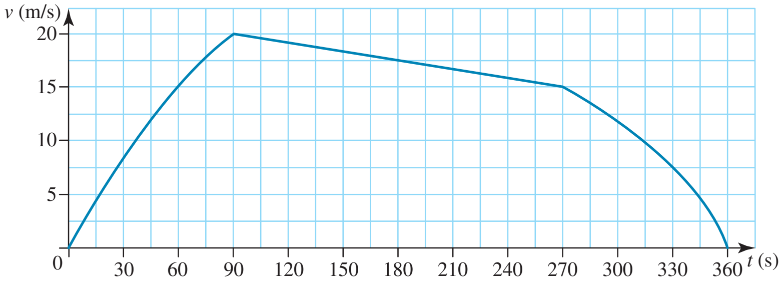


The correct answer is **D**.

***19a**



***20b**

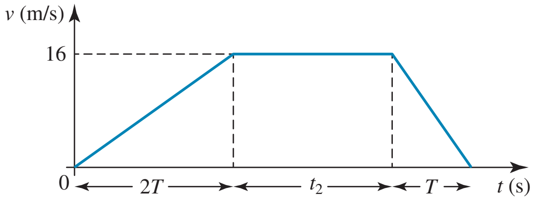


2 $d = \frac{1}{2} \times 2 \times 9 + 2 \times 9 + \int_4^8 v(t)\,dt + \left| \int_8^9 v(t)\,dt \right|$

$d = 53.4375$

The correct answer is **E**.

3



Total time $= 2T + t_2 + T = 3T + t_2 = 240$ [1 mark] (1)

Total distance $= \frac{1}{2} \times 2T \times 16$

$+ 16t_2 + \frac{1}{2} \times 16 \times T = 2760$ [1 mark]

$16T + 16t_2 + 8T = 2760$

$24T + 16t_2 = 2760$

$\frac{3T}{2} + t_2 = \frac{2760}{16} = \frac{345}{2}$ (2)

$(1) - (2)\ \left(3 - \frac{3}{2}\right)T = \frac{-345}{2} + 240$

$\frac{3T}{2} = \frac{135}{2}$

$T = 45$ [1 mark]

$t_2 = 240 - 3 \times 45 = 105$

Distance at top speed $= 105 \times 16 = 1680$ m [1 mark]

11.5 Acceleration that depends on time

11.5 Exercise

1 $a = \frac{dv}{dt} = 3 - 6t$

$v = \int (3 - 6t)\,dt$

$= 3t - 3t^2 + c_1$

$v(0) = 0 \Rightarrow c_1 = 0$

$v = \frac{dx}{dt} = 3t - 3t^2$

$x = \int \left(3t - 3t^2\right)\,dt$

$= \frac{3t^2}{2} - t^3 + c_2$

$x(0) = 0 \Rightarrow c_2 = 0$

$x(t) = \frac{3t^2}{2} - t^3$

$x(1) = \frac{3}{2} - 1 = \frac{1}{2}$ m

2 $a = \frac{dv}{dt} = (24t - 6)$

$v = \int (24t - 6)\,dt$

$= 12t^2 - 6t + c_1$

$v(0) = 1 \Rightarrow c_1 = 1$

$v = \frac{dx}{dt} = 12t^2 - 6t + 1$

$x = \int (12t^2 - 6t + 1)\,dt$

$= 4t^3 - 3t^2 + t + c_2$

$x(1) = 4$

$\Rightarrow 4 = 4 - 3 + 1 + c_2$

$c_2 = 2$

$x(t) = 4t^3 - 3t^2 + t + 2$

3 $\ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = -2\sin\left(\frac{t}{3}\right)$

$v = \int -2\sin\left(\frac{t}{3}\right)\,dt$

$= 6\cos\left(\frac{t}{3}\right) + c_1$

When $t = 0 \quad v = 6$

$6 = 6\cos(0) + c_1 \Rightarrow c_1 = 0$

$v = \frac{dx}{dt} = 6\cos\left(\frac{t}{3}\right)$

$x = \int 6\cos\left(\frac{t}{3}\right)\,dt$

$= 18\sin\left(\frac{t}{3}\right) + c_2$

When $t = 0 \quad x = 0$

$0 = 18\sin(0) + c_2 \Rightarrow c_2 = 0$

$x(t) = 18\sin\left(\frac{t}{3}\right)$

4 $\ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = -2\cos\left(\frac{t}{2}\right)$

$v = \int -2\cos\left(\frac{t}{2}\right)\,dt$

$= -4\sin\left(\frac{t}{2}\right) + c_1$

When $v = 0 \quad t = 0$

$0 = -4\sin(0) + c_1 \Rightarrow c_1 = 0$

$v = \frac{dx}{dt} - 4\sin\left(\frac{t}{2}\right)$

$x = \int -4\sin\left(\frac{t}{2}\right)\,dt$

$= 8\cos\left(\frac{t}{2}\right) + c_2$

When $t = 0 \quad x = 4$

$4 = 8\cos(0) + c_2 \Rightarrow c_2 = -4$

$x(t) = 8\cos\left(\frac{t}{2}\right) - 4$

When $t = \frac{\pi}{2}$

$x\left(\frac{\pi}{2}\right) = 8\cos\left(\frac{\pi}{4}\right) - 4$

$= 8 \times \frac{\sqrt{2}}{2} - 4$

$= 4\left(\sqrt{2} - 1\right)$ meters

5 $a = \frac{dv}{dt} = -3\sin\left(\frac{t}{2}\right)$

$v = \int -3\sin\left(\frac{t}{2}\right)\,dt$

$v(t) = 6\cos\left(\frac{t}{2}\right) + c_1$

$v(0) = 6 = 6 + c_1$

$c_1 = 0$

$v = \frac{dx}{dt} = 6\cos\left(\frac{t}{2}\right)$

$x = \int 6\cos\left(\frac{t}{2}\right)\,dt$

$x(t) = 12\sin\left(\frac{t}{2}\right) + c_2$

$x(0) = 4 = 0 + c_2$

$c_2 = 0$

$x(t) = 12\sin\left(\frac{t}{2}\right)$

$x_{\max} = 12\,\text{m}$

6 **a** $a = \frac{dv}{dt} = 50 - 120\,t$

$v = \int (50 - 120t)\,dt$

$= 50t - 60t^2 + c_1$

$t = 0 \qquad v = 72\,\text{km/hr} = \frac{72 \times 1000}{60 \times 60} = 20\,\text{m/s}$

$20 = 0 + c_1 \;\Rightarrow\; c_1 = 20$

$v(t) = 50t - 60t^2 + 20$

When $v = 10$ m/s (36 km/hr)

$10 = 50t - 60t^2 + 20$

$60t^2 - 50t - 10 = 0$

$6t^2 - 5t - 1 = 0$

$(6t + 1)\,(t - 1) = 0$

So $t = \frac{-1}{6},\quad 1 \quad$ since $\; t > 0$

After one second speed is 36 km/hr

b $D = \int_0^1 \left(50t - 60t^2 + 20\right)\,dt$

$= \left[25t^2 - 20t^3 + 20t\right]_0^1$

$= 25 - 20 + 20$

$= 25$ meters

7 $a = \frac{dv}{dt} = \frac{1}{3} - \frac{t}{270}$

$v = \int \left(\frac{1}{3} - \frac{t}{270}\right)\,dt$

$v = \frac{t}{3} - \frac{t^2}{540} + c_1$

$t = 0 \;\; v = 0 \;\Rightarrow\; c_1 = 0$

$v = \frac{dx}{dt} = \frac{t}{3} - \frac{t^2}{540}$

$x = \int \left(\frac{t}{3} - \frac{t^2}{540}\right)\,dt$

$= \frac{t^2}{6} - \frac{t^3}{1620} + c_2$

$x = 0 \;\; t = 0 \;\Rightarrow\; c_2 = 0$

$x(t) = \frac{t^2}{6} - \frac{t^3}{1620}$

When $v = 0 \;\Rightarrow\; \frac{t}{3}\left(1 - \frac{t}{180}\right) = 0 \to t = 0,\;$ 180 seconds

$x(180) = \frac{180^2}{6} - \frac{180^3}{1620} = 1800\,\text{m}$

Time between stops 180 seconds

Distance between stops 1800 m

8 $a = \frac{dv}{dt} = \frac{-100\,t}{3}$

$v = \int \frac{-100}{3} t\,dt$

$= \frac{-50t^2}{3} + c_1$

$v(0) = \frac{50}{3} = c_1$

$v(t) = \frac{50}{3} - \frac{50t^2}{3}$

Comes to rest $v = 0 \;\Rightarrow\; t = 1$ sec

$v = \frac{dx}{dt} = \frac{50}{3} - \frac{50t^2}{3}$

$x = \int \left(\frac{50}{3} - \frac{50}{3}t^2\right)\,dt$

$= \frac{50t}{3} - \frac{50}{9}t^3 + c_2$

$x(0) = 0 \;\Rightarrow\; c_2 = 0$

$x(t) = \frac{50t}{3} - \frac{50t^3}{9}$

$= \frac{50t}{9}\left(3 - t^2\right)$

$x(1) = \frac{50}{9} \times 2 = 11.\dot{1}$

$= 11\frac{1}{9}\,\text{m}$

9 **a** $\ddot{x} = \frac{-200}{(t+2)^3}$

$\frac{dv}{dt} = -200\,(t+2)^{-3}$

$v = \int -200\,(t+2)^{-3}\;dt$

$= 100\,(t+2)^{-2} + c_1$

$v = \frac{100}{(t+2)^2} + c_1$

When $v = 25 \quad t = 0$

$25 = \frac{100}{4} + c_1 \Rightarrow c_1 = 0$

$v = \frac{dx}{dt} = \frac{100}{(t+2)^2}$

When $v = 16 = \frac{100}{(t+2)^2}$

$\Rightarrow (t+2)^2 = \frac{100}{16}$

$t + 2 = \frac{10}{4}$

$t = \frac{5}{2} - 2 = \frac{1}{2}$ s since $\; t > 0$

b $D = \int_0^{\frac{1}{2}} \frac{100}{(t+2)^2}dt$

$= \left[\frac{-100}{(t+2)}\right]_0^{\frac{1}{2}}$

$= \frac{-100}{\frac{5}{2}} + \frac{100}{2}$

$= 10$ meters

10 $a = \frac{dv}{dt} = \frac{-800}{(t+5)^3}$

$v = \int \frac{-800}{(t+5)^3}\;dt$

$= \frac{400}{(t+5)^2} + c_1$

When $t = 0 \quad v = 16$

$16 = \frac{400}{5^2} + c_1 \;\Rightarrow\; c_1 = 0$

$v = \frac{dx}{dt} = \frac{400}{(t+5)^2}$

$x = \int \frac{400}{(t+5)^2}\;dt$

$= \frac{-400}{(t+5)} + c_2$

$t = 0 \qquad x = 0 \qquad c_2 = 80$

$$x(t) = 80 - \frac{400}{(t+5)}$$

$$= \frac{80t}{t+5}$$

$$x(5) = \frac{80 \times 5}{10}$$

$$= 40\,\text{m}$$

$$\text{OR} \int_0^5 \frac{400}{(t+5)^2}\,dt = 40\,\text{m}$$

11 $a = \dfrac{dv}{dt} = 8e^{2t} - 4$

$$v = \int \left(8e^{2t} - 4\right)\,dt$$

$$v = 4e^{2t} - 4t + c_1$$

When $t = 0$, $v = 0$

$$0 = 4 - 0 + c_1$$

$$c_1 = -4$$

$$v = \frac{dx}{dt} = 4e^{2t} - 4t - 4$$

$$x = \int \left(4e^{2t} - 4t - 4\right)\,dt$$

$$= 2e^{2t} - 2t^2 - 4t + c_2$$

When $x = 0 \qquad t = 0$

$$0 = 2 - 0 + c_2$$

$$c_2 = -2$$

$$x(t) = 2\left(e^{2t} - 1\right) - 2t^2 - 4t$$

12 $a = \dfrac{dv}{dt} = \dfrac{49}{5}e^{\frac{-t}{5}}$

$$v = \frac{49}{5}\int e^{\frac{-t}{5}}\,dt$$

$$= -49e^{\frac{-t}{5}} + c_1$$

$$v(0) = 0 \ = -49 + c_1$$

$$c_1 = 49$$

$$v = \frac{dx}{dt} = 49 - 49e^{\frac{-t}{5}}$$

$$x = \int 49\left(1 - e^{\frac{-t}{5}}\right)\,dt$$

$$= 49\left(t + 5e^{\frac{-t}{5}} + c_2\right)$$

$$x(0) = 20 = 49\,(5 + c_2)$$

$$= 245 + 49\,c_2$$

$$c_2 = \frac{-225}{49}$$

$$x(t) = 49t + 245e^{\frac{-t}{5}} - 225$$

13 $a = \ddot{x} = \dfrac{dv}{dt} = be^{-kt}$

$$v = \int b\,e^{-kt}\,dt$$

$$= -\frac{b}{k}\,e^{-kt} + c_1$$

$$v(0) = U = c_1 - \frac{b}{k}$$

$$c_1 = U + \frac{b}{k}$$

$$v = \frac{dx}{dt} = \frac{-b}{k}e^{-kt} + U + \frac{b}{k}$$

$$x = \int \left(U + \frac{b}{k}\left(1 - e^{-kt}\right)\right)\,dt$$

$$x = Ut + \frac{b}{k}\left(t + \frac{1}{k}e^{-kt} + c_2\right)$$

$$x(0) = 0 = 0 + \frac{b}{k}\left(\frac{1}{k} + c_2\right)$$

$$c_2 = \frac{-1}{k}$$

$$x(t) = Ut + \frac{b}{k}\left(t + \frac{1}{k}\left(e^{-kt} - 1\right)\right)\,\text{m}$$

14 $a = \dfrac{dv}{dt} = 4 - 4e^{-0.1t}$

$$v = \int \left(4 - 4e^{-0.1t}\right)\,dt$$

$$v = 4t + \frac{4}{0.1}e^{-0.1t} + c_1$$

$$12 = 0 + 40 + c_1$$

$$\Rightarrow c_1 = -28$$

$$v = \frac{dx}{dt} = 4t + 40e^{-0.1t} - 28$$

$$S = \int_0^5 \left(4t + 40e^{-0.1t} - 28\right)\,dt$$

$$= \left[2t^2 - 400e^{-0.1t} - 28t\right]_0^5$$

$$= \left(2 \times 25 - 400e^{-0.5} - 28 \times 5\right) - (0 - 400 - 0)$$

$$= 67.388\,\text{m}$$

15 $a = \dfrac{dv}{dt} = \dfrac{-36900\sqrt{2}}{\sqrt{(369t + 128)^3}}$

$$v = -36900\sqrt{2}\int (369t + 128)^{\frac{-3}{2}}\,dt$$

$$= -36900\sqrt{2} \times (-2) \times \frac{1}{369}\ \ (369t + 128)^{\frac{-1}{2}} + c_1$$

$$v = \frac{200\sqrt{2}}{\sqrt{(369t + 128)}} + c_1$$

When $t = 0 \quad v = 25$

$$25 = \frac{200\sqrt{2}}{\sqrt{128}} + c_1 \Rightarrow c_1 = 0$$

$$v(t) = \frac{200\sqrt{2}}{\sqrt{369t + 128}}$$

When $v = 16 = \dfrac{200\sqrt{2}}{\sqrt{369t + 128}}$

$$\Rightarrow \sqrt{369t + 128} = \frac{200\sqrt{2}}{16}$$

$$369t + 128 = \left(\frac{25\sqrt{2}}{2}\right)^2 = \frac{625}{2}$$

$$369t = \frac{625}{2} - 128 = \frac{369}{2}$$

$T = \dfrac{1}{2}$ seconds

$$D = \int_0^{\frac{1}{2}} \frac{200\sqrt{2}}{\sqrt{369t + 128}}\,dt$$

$$= \frac{400}{41}\ \text{m} \ = \ 9\frac{31}{41}\ \text{m}$$

16 $a = \ddot{x} = \dfrac{dv}{dt} = t\cos\left(t^2\right)$

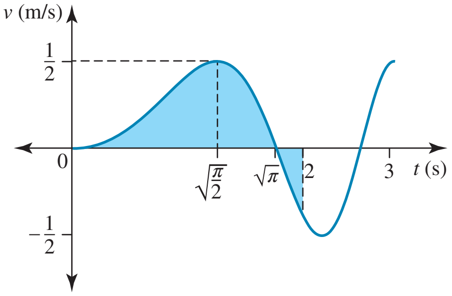
$v = \displaystyle\int t\cos\left(t^2\right)\ dt$

$= \dfrac{1}{2}\sin\left(t^2\right) + c_1$

$v = 0 \quad t = 0$

$\Rightarrow c_1 = 0$

$v = \dfrac{1}{2}\sin\left(t^2\right)$



When $\sin\left(t^2\right) = 0$

$t^2 = \pi$

$t = \sqrt{\pi}$

Distance travelled in first 2 seconds

$D = \displaystyle\int_0^{\sqrt{\pi}} \frac{1}{2}\sin\left(t^2\right) dt + \left|\int_{\sqrt{\pi}}^{2} \frac{1}{2}\sin\left(t^2\right) dt\right|$

$= 0.4474 + |-0.045|$

$= 0.492\text{ m}$

17 $\ddot{x} = \dfrac{dv}{dt} = -e^{-2t}\left(12\cos(3t) + 5\sin(3t)\right)$

$v = \displaystyle\int -e^{-2t}\left(12\cos(3t) + 5\sin(3t)\right) dt$

$= e^{-2t}\left(3\cos(3t) - 2\sin(3t)\right) + c_1$

$v = 3, \quad t = 0$

$3 = 3 + c_1 \Rightarrow c_1 = 0$

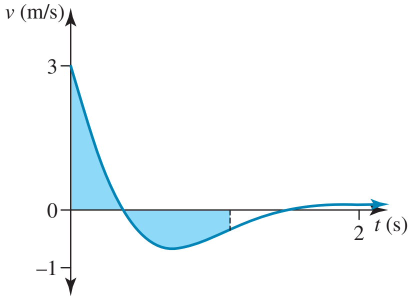
$v = \dfrac{dx}{dt} = e^{-2t}\left(3\cos(3t) - 2\sin(3t)\right)$

$x = \displaystyle\int e^{-2t}\left(3\cos(3t) - 2\sin(3t)\right)\ dt$

$= e^{-2t}\sin(3t) + c_2$

$x = 0 \quad t = 0 \quad c_2 = 0$

$x(t) = e^{-2t}\sin(3t)$



$v(t) = 0 \Rightarrow t = 0.3276$

$D = \displaystyle\int_0^{0.3276} v(t)\,dt + \left|\int_{0.3276}^{1} v(t)\,dt\right|$

$= 0.432117 + |-0.41302|$

$= 0.845\ \text{m}$

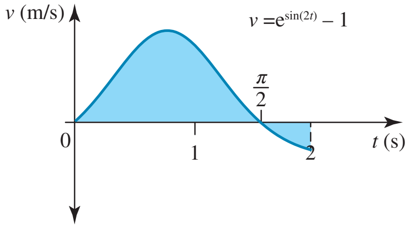
18 $a = \dfrac{dv}{dt} = 2\cos(2t)e^{\sin(2t)}$

$v = \displaystyle\int 2\cos(2t)e^{\sin(2t)}\ dt$

$v = e^{\sin(2t)} + c_1$

$0 = 1 + c_1 \Rightarrow c_1 = -1$

$v(t) = e^{\sin(2t)} - 1$



$v(t) = 0 \quad \Rightarrow e^{\sin(2t)} = 1$

$\sin(2t) = 0$

$2t = \pi$

$t = \dfrac{\pi}{2}$

$D = \displaystyle\int_0^{\frac{\pi}{2}} \left(e^{\sin(2t)} - 1\right) dt + \left|\int_{\frac{\pi}{2}}^{2} \left(e^{\sin(2t)} - 1\right) dt\right|$

$= 1.5336 + |-0.1353|$

$= 1.669\text{ m}$

11.5 Exam questions

1 a $v = \displaystyle\int \left(\frac{76}{5} - 5t\right) dt = \frac{76t}{5} - \frac{5t^2}{2} + c_1$

$v(0) = 0 \Rightarrow c_1 = 0$

$v = v(t) = \dfrac{76t}{5} - \dfrac{5t^2}{2}$

$v(5) = 76 - \dfrac{125}{2} = 13.5\text{ m s}^{-1}$

Award 1 mark for the velocity expression.

Award 1 mark for the correct velocity after 5 seconds.

b $v = \dfrac{dh}{dt} = \dfrac{76t}{5} - \dfrac{5t^2}{2}$

$h = \displaystyle\int \left(\frac{76t}{5} - \frac{5t^2}{2}\right) dt = \frac{38t^2}{5} - \frac{5t^3}{6} + c_2$

$h(0) = 0 \Rightarrow c_2 = 0$

$h = h(t) = \dfrac{38t^2}{5} - \dfrac{5t^3}{6}$

$h(5) = 38 \times 5 - \dfrac{5}{6} \times 125 = 85.83\text{ m}$

Award 1 mark for the height expression.

Award 1 mark for the correct height after 5 seconds.

c $u = 13.5,\ a = -g,\ v = 0,\ s = ?$

Rising under gravity,using $v^2 = u^2 + 2as$:

$0 = 13.5^2 - 2 \times 9.8\,s$

$s = \dfrac{13.5^2}{2 \times 9.8}$

$s = 9.3$

The maximum height reached by the rocket is $9.3 + 85.83 = 95.13\text{ m}$.

Award 1 mark for method, using constant acceleration formulas or integration.

Award 1 mark for the correct maximum height.

d From when the propulsion force stopped, time to rise to maximum height and fall back to the ground:

$s = -85.83,\ u = 13.5,\ a = -g,\ t = ?$

Using $s = ut + \frac{1}{2}at^2$:

$-85.83 = 13.5t - 4.9t^2$

Solving $\Rightarrow t = 5.78$

The total time of flight is $5 + 5.78 = 10.8$ s

Award 1 mark for method, using constant acceleration formulas or intergration.

Award 1 mark for the correct time.

Award 1 mark for the final correct time of flight.

2 $a = \frac{dv}{dt} = 36\cos(3t)$

$$v = \int 36\cos(3\text{t})\,dt$$

$= 12\sin(3t) + c_1$ [1 mark]

$v(0) = 0 \Rightarrow c_1 = 0$

$v = \frac{dx}{dt} = 12\sin(3t)$

$$x = \int 12\sin(3t)\,dt$$

$= -4\cos(3t) + c_2$ [1 mark]

$t = 0,\ x = 2$

$2 = -4\cos(0) + c_2$

$c_2 = 6$

$x(t) = -4\cos(3t) + 6$ [1 mark]

Furthest $\cos(3t) = -1$

$x_{\max} = 10$ m from origin [1 mark]

3 $a = \frac{dv}{dt} = -kt$

$$v = \int -kt\,dt$$

$$= \frac{-kt^2}{2} + c_1$$

$v(0) = U$

$U = c_1$ [1 mark]

$$v = \frac{dx}{dt} = \frac{-kt^2}{2} + U$$

$$x = \int \left(U - \frac{kt^2}{2}\right) dt$$

$$x = Ut - \frac{kt^3}{6} + c_2$$

$x = 0 \quad t = 0 \quad c_2 = 0$ [1 mark]

Comes to rest: $v = 0 \Rightarrow \frac{kt^2}{2} = U$

$t = \sqrt{\frac{2U}{k}}$ [1 mark]

$$D = x\left(\sqrt{\frac{2U}{k}}\right) = U\sqrt{\frac{2U}{k}} - \frac{k}{6}\left(\frac{2U}{k}\right)\sqrt{\frac{2U}{k}}$$

$$= \sqrt{\frac{2U}{k}}\left(U - \frac{U}{3}\right)$$

$= \frac{2U}{3}\sqrt{\frac{2U}{k}}$ [1 mark]

11.6 Acceleration that depends on velocity

11.6 Exercise

1 $\ddot{x} = \frac{dv}{dt} = \frac{1}{2}(8 - v)$

$$\frac{dt}{dv} = \frac{2}{8 - v}$$

$$t = 2\int \frac{1}{8 - v}\,dv$$

$t = -2\log_e(8 - v) + c_1$

Now when $t = 0, \quad v = 0$

$0 = -2\log_e(8) + c_1$

$c_1 = 2\log_e(8)$

$t = -2\log_e(8 - v) + 2\log_e(8)$

$$= -2\left[\log_e\left(\frac{8 - v}{8}\right)\right]$$

$$e^{-\frac{t}{2}} = \frac{8 - v}{8}$$

$8 - v = 8e^{\frac{-t}{2}}$

$v = 8 - 8e^{\frac{-t}{2}}$

$$v = \frac{dx}{dt} = 8 - 8e^{\frac{-t}{2}}$$

$$x = \int \left(8 - 8e^{\frac{-t}{2}}\right) dt$$

$= 8t + 16e^{\frac{-t}{2}} + c_2$

When $t = 0, \quad x = 0$

$0 = 16 + c_2$

$c_2 = -16$

$x = 8t + 16e^{-\frac{t}{2}} - 16$

$$x(t) = 8\left(1 + 2\left(e^{-\frac{t}{2}} - 1\right)\right)$$

2 $\ddot{x} = \frac{dv}{dt} = -\frac{1}{3}(v - 6)$

$$\frac{dt}{dv} = \frac{-3}{v - 6}$$

$$t = -3\int \frac{1}{v - 6}\,dv$$

$= -3\log_e(v - 6) + c_1$

When $t = 0, \quad v = 12$

$0 = -3\log_e 6 + c_1$

$c_1 = 3\log_e(6)$

$t = 3\log_e(6) - 3\log_e(v - 6)$

$$= 3\log_e\left(\frac{6}{v - 6}\right)$$

$$e^{\frac{t}{3}} = \frac{6}{v - 6}$$

$v - 6 = 6e^{-\frac{t}{3}}$

$v = \frac{dx}{dt} = 6 + 6e^{-\frac{t}{3}}$

$$x = \int \left(6 + 6e^{-\frac{t}{3}}\right) dt$$

$x = 6t - 18e^{-\frac{t}{3}} + c_2$

When $x = 0 \quad t = 0$

$0 = -18 + c_2$

$c_2 = 18$

$x = 6t + 18 - 18e^{-\frac{t}{3}}$

$$= 6\left(t + 3\left(1 - e^{-\frac{t}{3}}\right)\right)$$

3 $\ddot{x} = \frac{dv}{dt} = -2v$

$\frac{dt}{dv} = \frac{-1}{2v}$

$t = -\frac{1}{2}\int \frac{1}{v}\,dv$

$= -\frac{1}{2}\log_e(|v|) + c_1$

When $v = 1 \quad t = 0$

$0 = -\frac{1}{2}\log_e(1) + c_1$

$c_1 = 0$

$-2t = \log_e(|v|)$

$v = e^{-2t}$ since $v > 0$

$v = \frac{dx}{dt} = e^{-2t}$

$x = \int e^{-2t}\,dt$

$= -\frac{1}{2}e^{-2t} + c_2$

When $t = 0 \quad x = 0$

$0 = -\frac{1}{2} + c_2$

$c_2 = \frac{1}{2}$

$x = x(t) = \frac{1}{2}\left(1 - e^{-2t}\right)$

4 $\ddot{x} = \frac{dv}{dt} = -3(v+4)$

$\frac{dt}{dv} = \frac{-1}{3\,(v+4)}$

$t = -\frac{1}{3}\int \frac{1}{v+4}\,dv$

$= -\frac{1}{3}\log_e(v+4) + c_1$

When $t = 0, \quad v = 2$

$0 = -\frac{1}{3}\log_e(6) + c_1$

$c_1 = \frac{1}{3}\log_e(6)$

$t = \frac{1}{3}\log_e(6) - \frac{1}{3}\log_e(v+4)$

$t = \frac{1}{3}\log_e\left(\frac{6}{v+4}\right)$

$e^{3t} = \frac{6}{v+4}$

$v + 4 = 6e^{-3t}$

$\frac{dx}{dt} = v = 6e^{-3t} - 4$

$x = \int \left(6\,e^{-3t} - 4\right)\,dt$

$x = -2\,e^{-3t} - 4t + c_2$

When $t = 0 \quad x = 0$

$0 = -2 + c_2$

$c_2 = 2$

$x = 2 - 2e^{-3t} - 4t$

$= 2\left(1 - e^{-3t} - 2t\right)$

5 a $\ddot{x} = -\frac{1}{5}\,v^{\frac{3}{2}}$

Use $\ddot{x} = \frac{dv}{dt}$

$\frac{dv}{dt} = -\frac{1}{5}v^{\frac{3}{2}}$

$\int v^{-\frac{3}{2}}\,dv = \int \frac{-1}{5}\,dt$

$-2v^{\frac{-1}{2}} = \frac{-t}{5} + c_1$

$\frac{-2}{\sqrt{v}} = \frac{-t}{5} + c_1$

When $t = 0 \quad v = 25$

$\frac{-2}{\sqrt{25}} = \frac{-2}{5} = c_1$

$\frac{2}{\sqrt{v}} = \frac{t}{5} + \frac{2}{5}$

When $v = 16$

$\frac{2}{4} = \frac{t}{5} + \frac{2}{5}$

$\frac{t}{5} = \frac{1}{2} - \frac{2}{5} = \frac{1}{10}$

$t = \frac{5}{10}$

$= \frac{1}{2}$ sec

b Use $\ddot{x} = v\frac{dv}{dx}$

$v\frac{dv}{dx} = -\frac{1}{5}v^{\frac{3}{2}}$

$\frac{dv}{dx} = -\frac{1}{5}v^{\frac{1}{2}}$

$\int v^{\frac{-1}{2}}\,dv = \int \frac{-1}{5}\,dx$

$2v^{\frac{1}{2}} = \frac{-x}{5} + c_2$

When $x = 0 \quad v = 25$

$2\sqrt{25} = 10 = c_2$

$2\sqrt{v} = \frac{-x}{5} + 10$

When $v = 16$

$2\sqrt{16} = 8 = \frac{-x}{5} + 10$

$\frac{x}{5} = 10 - 8 = 2$

$x = 10$ meters

6 a $\ddot{x} = \frac{dv}{dt} = -\frac{1}{10}v^{\frac{3}{2}}$

$\frac{dt}{dv} = \frac{-10}{v^{\frac{3}{2}}}$

$t = -10\int v^{-\frac{3}{2}}dv$

$= 20v^{-\frac{1}{2}} + c_1$

When $v = 16, \quad t = 0$

$0 = \frac{20}{\sqrt{16}} + c_1 \Rightarrow c_1 = -5$

$t = \frac{20}{\sqrt{v}} - 5$

When $v = 4$

$t = \frac{20}{\sqrt{4}} - 5$

$T = 5$ seconds

b $\ddot{x} = v\frac{dv}{dx} = \frac{-1}{10}v^{\frac{3}{2}}$

$\frac{dv}{dx} = \frac{-1}{10}v^{\frac{1}{2}}$

$$\frac{dx}{dv}=\frac{-10}{v^{\frac{1}{2}}}$$
$$\frac{-x}{10}=\int v^{-\frac{1}{2}}dv$$
$$\frac{-x}{10}=2v^{\frac{1}{2}}+c_2$$
When $v=16,\quad x=0$
$$0=2\sqrt{16}+c_2\Rightarrow c_2=-8$$
$$\frac{-x}{10}=2\sqrt{v}-8$$
$$x=80-20\sqrt{v}$$
When $v=4$
$$x=80-20\sqrt{4}$$
$$D=40\,\text{m}$$

7 **a** $\ddot{x}=-\frac{4}{5}\sqrt{v}$

Use $\ddot{x}=\frac{dv}{dt}$
$$\frac{dv}{dt}=\frac{-4}{5}v^{\frac{1}{2}}$$
$$\frac{dt}{dv}=\frac{-5}{4v^{\frac{1}{2}}}$$
$$\int\frac{-4}{5}\,dt=\int\frac{1}{v^{\frac{1}{2}}}dv$$
$$\frac{-4t}{5}=2v^{\frac{1}{2}}+c_1$$
When $t=0\quad v=16$
$$0=2\sqrt{16}+c_1\Rightarrow c_1=-8$$
$$\frac{-4t}{5}=2\sqrt{v}-8$$
$$\frac{4t}{5}=8-2\sqrt{v}$$
When $v=4$
$$\frac{4t}{5}=8-2\sqrt{4}=4$$
$t=5$ seconds

b Use $\ddot{x}=v\frac{dv}{dx}$
$$v\frac{dv}{dx}=-\frac{4}{5}v^{\frac{1}{2}}$$
$$\frac{dv}{dx}=-\frac{4}{5}v^{-\frac{1}{2}}$$
$$\int v^{\frac{1}{2}}dv=-\frac{4}{5}\int dx$$
$$-\frac{4x}{5}=\frac{2}{3}v^{\frac{3}{2}}+c_2$$
When $x=0\quad v=16$
$$0=\frac{2}{3}(16)^{\frac{3}{2}}+c_2$$
$$\Rightarrow c_2=-\frac{128}{3}$$
$$-\frac{4x}{5}=\frac{2}{3}v^{\frac{3}{2}}-\frac{128}{3}$$
$$\frac{4x}{5}=\frac{128}{3}-\frac{2}{3}v^{\frac{3}{2}}$$
When $v=4$
$$\frac{4x}{5}=\frac{128}{3}-\frac{2}{3}(4)^{\frac{3}{2}}$$
$$=\frac{128}{3}-\frac{2}{3}\times 8$$
$$\frac{4x}{5}=\frac{112}{3}$$
$$x=\frac{112\times 5}{3\times 4}=\frac{140}{3}$$
$$=46\frac{2}{3}\text{ m}$$

8 $\ddot{x}=\frac{-369}{160000}v^3$

Use $\ddot{x}=\frac{dv}{dt}$
$$\frac{dv}{dt}=\frac{-369}{160000}v^3$$
$$\int v^{-3}dv=\int\frac{-369}{160000}\,dt$$
$$-\frac{1}{2}v^{-2}=\frac{-369}{160000}t+c_1$$
When $v=25\quad t=0$
$$-\frac{1}{2\times(25)^2}=c_1=\frac{-1}{1250}$$
$$\frac{1}{2v^2}=\frac{369\,t}{160000}+\frac{1}{1250}$$
When $v=16$
$$\frac{1}{2\times(16)^2}=\frac{369\,t}{160000}+\frac{1}{1250}$$
$$\frac{369\,t}{160000}=\frac{1}{512}-\frac{1}{1250}$$
$$=\frac{369}{320000}$$
$$T=\frac{1}{2}\text{ sec}$$
Use $\ddot{x}=v\frac{dv}{dx}$
$$v\frac{dv}{dx}=\frac{-369}{160000}v^3$$
$$\frac{dv}{dx}=\frac{-369}{160000}v^2$$
$$\int v^{-2}dv=\int\frac{-369}{160000}dx$$
$$\frac{-1}{v}=\frac{-369}{160000}x+c_2$$
When $v=25\quad x=0$
$$c_2=\frac{-1}{25}$$
$$\frac{1}{v}=\frac{369\,x}{160000}+\frac{1}{25}$$
When $v=16$
$$\frac{1}{16}=\frac{369\,x}{160000}+\frac{1}{25}$$
$$\frac{369\,x}{160000}=\frac{1}{16}-\frac{1}{25}$$
$$=\frac{9}{400}$$
$$x=\frac{160000\times 9}{369\times 400}$$
$$D=\frac{400}{41}\text{ m}$$

9 Use $\ddot{x} = \dfrac{dv}{dt} = -\lambda v^3$

$$\int -\lambda\, dt = \int v^{-3}\, dv$$

$$-\lambda\, t = \frac{-1}{2} v^{-2} + c_1$$

$t = 0 \quad v = U$

$$0 = \frac{-1}{2U^2} + c_1$$

$$c_1 = \frac{1}{2U^2}$$

$$-\lambda t = \frac{-1}{2v^2} + \frac{1}{2U^2}$$

$$\lambda t = \frac{-1}{2U^2} + \frac{1}{2v^2}$$

When $v = \dfrac{1}{2}U,\ t = T$

$$\lambda T = \frac{-1}{2U^2} + \frac{1}{2\left(\frac{U^2}{4}\right)}$$

$$= \frac{-1}{2U^2} + \frac{2}{U^2}$$

$$\lambda = \frac{3}{2U^2T}$$

$$v\frac{dv}{dx} = -\lambda v^3$$

$$\frac{dv}{dx} = -\lambda v^2$$

$$\int -\lambda\, dx = \int v^{-2} dv$$

$$-\lambda x = -v^{-1} + c_2$$

$x = 0 \quad v = U$

$$0 = -\frac{1}{U} + c_2$$

$$c_2 = \frac{1}{U}$$

$$-\lambda x = -\frac{1}{v} + \frac{1}{U}$$

$$\lambda x = \frac{1}{v} - \frac{1}{U}$$

When $x = D \quad v = \dfrac{1}{2}U$

$$\lambda D = \frac{1}{\frac{1}{2}U} - \frac{1}{U}$$

$$\lambda D = \frac{2}{U} - \frac{1}{U} = \frac{1}{U}$$

$$\lambda = \frac{1}{UD}$$

So $\dfrac{3}{2U^2T} = \dfrac{1}{UD}$

$$\frac{D}{T} = \frac{2U}{3}$$

10 $\ddot{x} = -\lambda v^{\frac{3}{2}}$

Use $\ddot{x} = \dfrac{dv}{dt}$

$$\frac{dv}{dt} = -\lambda v^{\frac{3}{2}}$$

$$\int -\lambda\, dt = \int v^{\frac{-3}{2}}\, dv$$

$$-\lambda t = -2v^{\frac{-1}{2}} + c_1$$

When $t = 0 \quad v = U$

$$0 = -\frac{2}{\sqrt{U}} + c_1$$

$$c_1 = \frac{2}{\sqrt{U}}$$

$$-\lambda t = -\frac{2}{\sqrt{v}} + \frac{2}{\sqrt{U}}$$

$$\lambda t = \frac{2}{\sqrt{v}} - \frac{2}{\sqrt{U}}$$

When $t = T \quad v = \dfrac{1}{2}U$

$$\lambda T = \frac{2}{\sqrt{\frac{U}{2}}} - \frac{2}{\sqrt{U}}$$

$$= \frac{2\sqrt{2}}{\sqrt{U}} - \frac{2}{\sqrt{U}}$$

$$\lambda = \frac{2\left(\sqrt{2}-1\right)}{T\sqrt{U}}$$

Use $\ddot{x} = v\dfrac{dv}{dx}$

$$v\frac{dv}{dx} = -\lambda v^{\frac{3}{2}}$$

$$\frac{dv}{dx} = -\lambda v^{\frac{1}{2}}$$

$$\int -\lambda\, dx = \int v^{\frac{-1}{2}} dv$$

$$-\lambda x = 2v^{\frac{1}{2}} + c_2$$

$x = 0 \quad v = U$

$2\sqrt{U} + c_2 = 0$

$c_2 = -2\sqrt{U}$

$-\lambda x = 2\sqrt{v} - 2\sqrt{U}$

$\lambda x = 2\sqrt{U} - 2\sqrt{v}$

When $v = \dfrac{1}{2}U, \quad x = D$

$$\lambda D = 2\sqrt{U} - 2\sqrt{\frac{U}{2}}$$

$$= 2\sqrt{U}\left(1 - \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\right)$$

$$= 2\sqrt{U}\left(1 - \frac{\sqrt{2}}{2}\right)$$

$$\lambda = \frac{\sqrt{U}}{D}\left(2 - \sqrt{2}\right)$$

So $\dfrac{2\left(\sqrt{2}-1\right)}{T\sqrt{U}} = \dfrac{\sqrt{U}\left(2-\sqrt{2}\right)}{D}$

$$\frac{D}{T} = \frac{U\left(2-\sqrt{2}\right)}{2\left(\sqrt{2}-1\right)} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}$$

$$= \frac{U\left(2\sqrt{2} - 2 + 2 - \sqrt{2}\right)}{2(2-1)}$$

$$\frac{D}{T} = \frac{\sqrt{2}U}{2} \qquad \text{shown}$$

11 a

$$\ddot{x} = -kv$$
$$\frac{dv}{dt} = -kv$$
$$\frac{dt}{dv} = \frac{-1}{kv}$$
$$\int -k\,dt = \int \frac{1}{v}\,dv$$
$$-kt = \log_e(v) + c_1$$
$$v = U, \quad t = 0$$
$$0 = \log_e(U) + c_1$$
$$c_1 = -\log_e(U)$$
$$-kt = \log_e(v) - \log_e(U)$$
$$= \log_e\left(\frac{v}{U}\right)$$
$$e^{-kt} = \frac{v}{U}$$
$$v = Ue^{-kt}$$

b

$$v = \frac{dx}{dt} = Ue^{-kt}$$
$$x = \int Ue^{-kt}\,dt$$
$$= -\frac{U}{k}e^{-kt} + c_2$$
$$x = 0 \quad t = 0$$
$$0 = \frac{-U}{k} + c_2$$
$$c_2 = \frac{U}{k}$$
$$x = \frac{U}{k}\left(1 - e^{-kt}\right)$$

c

$$v\frac{dv}{dx} = -kv$$
$$\frac{dv}{dx} = -k$$
$$v = -kx + c_3$$
$$v = U, \quad x = 0$$
$$U = c_3$$
$$v = U - kx$$

12 a

$$\ddot{x} = -kv^2$$
$$v\frac{dv}{dx} = -kv^2$$
$$\frac{dv}{dx} = -kv$$
$$\frac{dx}{dv} = \frac{-1}{kv}$$
$$-kx = \int \frac{1}{v}\,dv$$
$$= \log_e(v) + c_1$$
$$x = 0 \quad v = U$$
$$0 = \log_e(U) + c_1$$
$$c_1 = -\log_e(U)$$
$$-kx = \log_e(v) - \log_e(U)$$
$$= \log_e\left(\frac{v}{U}\right)$$
$$\frac{v}{U} = e^{-kx}$$
$$v = Ue^{-kx}$$

b

$$\frac{dv}{dt} = -kv^2$$
$$\frac{dt}{dv} = -\frac{1}{kv^2}$$
$$-kt = \int \frac{1}{v^2}\,dv$$
$$= -\frac{1}{v} + c_2$$
$$t = 0 \quad v = U$$
$$0 = \frac{-1}{U} + c_2$$
$$c_2 = \frac{1}{U}$$
$$-kt = \frac{-1}{v} + \frac{1}{U}$$
$$\frac{1}{v} = kt + \frac{1}{U}$$
$$\frac{1}{v} = \frac{1 + ktU}{U}$$
$$v = \frac{U}{1 + kUt}$$

13 a

$$\ddot{x} = -(p + qv)$$
$$\frac{dv}{dt} = -(p + qv)$$
$$\frac{dt}{dv} = -\frac{1}{p + qv}$$
$$t = \int \frac{-1}{p + qv}\,dv$$
$$t = -\frac{1}{q}\log_e(p + qv) + c_1$$
$$t = 0 \quad v = U$$
$$0 = -\frac{1}{q}\log_e(p + qU) + c_1$$
$$c_1 = \frac{1}{q}\log_e(p + qU)$$
$$t = \frac{1}{q}\log_e(p + qU) - \frac{1}{q}\log_e(p + qv)$$
$$= \frac{1}{q}\log_e\left(\frac{p + qU}{p + qv}\right)$$

Comes to rest $v = 0$

$$T = \frac{1}{q}\log_e\left(\frac{p + qU}{p}\right)$$
$$= \frac{1}{q}\log_e\left(1 + \frac{qU}{p}\right)$$

b

$$v\frac{dv}{dx} = -(p + qv)$$
$$x = -\int \left(\frac{v}{p + qv}\right)dv$$
$$= -\frac{1}{q}\int \left(\frac{qv + p - p}{p + qv}\right)dv$$
$$= -\frac{1}{q}\int \left(1 - \frac{p}{p + qv}\right)dv$$
$$= -\frac{1}{q}\left(v - \frac{p}{q}\log_e(p + qv) + c_2\right)$$

When $x = 0, \quad v = U$

$$0 = \frac{-1}{q}\left(U - \frac{p}{q}\log_e(p+qU) + c_2\right)$$

$$c_2 = \frac{p}{q}\log_e(p+qU) - U$$

$$x = -\frac{1}{q}\left(v - U + \frac{p}{q}\log_e\left(\frac{p+qU}{p+qv}\right)\right)$$

Comes to rest $x = D, \quad v = 0$

$$D = -\frac{1}{q}\left(-U + \frac{p}{q}\log_e\left(1+\frac{qU}{p}\right)\right)$$

$$= \frac{U}{q} - \frac{p}{q}\left(\frac{1}{q}\log_e\left(1+\frac{qU}{p}\right)\right)$$

$$D = \frac{1}{q}(U - pT)$$

14 $\ddot{x} = -\left(p + qv^2\right)$

$$\frac{dv}{dt} = -\left(p+qv^2\right)$$

$$\frac{dt}{dv} = \frac{-1}{p+qv^2}$$

$$t = -\int \frac{1}{p+qv^2}\,dv$$

$$= \frac{-1}{p}\int \frac{1}{1+b^2v^2}\,dv \qquad b^2 = \frac{q}{p}, \quad b = \sqrt{\frac{q}{p}}$$

$$t = \frac{-1}{pb}\tan^{-1}(vb) + c_1$$

$t = 0, \quad v = U$

$$0 = \frac{-1}{pb}\tan^{-1}(Ub) + c_1$$

$$c_1 = \frac{1}{pb}\tan^{-1}(Ub)$$

$$t = \frac{1}{pb}\left(\tan^{-1}(Ub) - \tan^{-1}(vb)\right)$$

When $v = 0 \quad t = T$

$$T = \frac{1}{pb}\tan^{-1}(Ub)$$

$$= \frac{1}{p}\sqrt{\frac{p}{q}}\tan^{-1}\left(U\sqrt{\frac{q}{p}}\right)$$

$$T = \frac{1}{\sqrt{pq}}\tan^{-1}\left(U\sqrt{\frac{q}{p}}\right)$$

$$v\frac{dv}{dx} = -\left(p+qv^2\right)$$

$$x = -\int \frac{v}{p+qv^2}\,dv$$

$$= -\frac{1}{2q}\log_e\left(p+qv^2\right) + c_2$$

$x = 0 \quad v = U$

$$0 = \frac{-1}{2q}\log_e\left(p+qU^2\right) + c_2$$

$$c_2 = \frac{1}{2q}\log_e\left(p+qU^2\right)$$

$$x = \frac{1}{2q}\left(\log_e\left(p+qU^2\right) - \log_e\left(p+qv^2\right)\right)$$

$$= \frac{1}{2q}\log_e\left(\frac{p+qU^2}{p+qv^2}\right)$$

Comes to rest $v = 0 \quad x = D$

$$D = \frac{1}{2q}\log_e\left(\frac{p+qU^2}{p}\right)$$

$$= \frac{1}{2q}\log_e\left(1+\frac{qU^2}{p}\right)$$

15 **a** Use $\ddot{x} = v\dfrac{dv}{dx} = \dfrac{8820 - v^2}{900}$

$$x = \int \frac{900v}{8820 - v^2}\,dv$$

$$x = -450\ \log_e\left(8820 - v^2\right) + c_1$$

But $x = 0$ when $v = 0$

$$0 = -450\log_e(8820) + c_1$$

$$c_1 = 450\log_e(8820)$$

$$x = 450\log_e\left(\frac{8820}{8820 - v^2}\right)$$

b When $x = 150 \quad v = ?$

$$150 = 450\log_e\left(\frac{8820}{8820 - v^2}\right)$$

$$\log_e\left(\frac{8820}{8820 - v^2}\right) = \frac{1}{3}$$

$$\frac{8820}{8820 - v^2} = e^{\frac{1}{3}}$$

$$8820 - v^2 = 8820e^{-\frac{1}{3}}$$

$$v = \sqrt{8820\left(1 - e^{-\frac{1}{3}}\right)}$$

$$= 50.002\text{ m/s}$$

c $\dfrac{dv}{dt} = \dfrac{8820 - v^2}{900}$

$$\frac{dt}{dv} = \frac{900}{8820 - v^2}$$

$$t = \int \frac{900}{8820 - v^2}\,dv$$

$$\frac{900}{8820 - v^2} = \frac{\text{A}}{42\sqrt{5} - v} + \frac{\text{B}}{42\sqrt{5} + v}$$

$$= \frac{\text{A}\left(42\sqrt{5} + v\right) + \text{B}\left(42\sqrt{5} - v\right)}{\left(42\sqrt{5} - v\right)\left(42\sqrt{5} + v\right)}$$

$$= \frac{42\sqrt{5}(\text{A}+\text{B}) + v(\text{A}-\text{B})}{8820 - v^2}$$

$$42\sqrt{5}(\text{A}+\text{B}) = 900$$

$$\text{A} - \text{B} = 0 \Rightarrow \text{A} = \text{B} = \frac{15\sqrt{5}}{7}$$

$$t = \frac{15\sqrt{5}}{7}\int\left(\frac{1}{42\sqrt{5} - v} + \frac{1}{42\sqrt{5} + v}\right)dv$$

$$= \frac{15\sqrt{5}}{7}\log_e\left(\left|\frac{42\sqrt{5} + v}{42\sqrt{5} - v}\right|\right) + c_2$$

$t = 0, \quad v = 0$

$0 = c_2$

$$t = \frac{15\sqrt{5}}{7}\log_e\left(\frac{42\sqrt{5} + v}{42\sqrt{5} - v}\right)$$

Since $\quad 0 \le v < 42\sqrt{5}$

d $t \Rightarrow \infty \quad \ddot{x} = 0$

$v_{\mathrm{T}} = \sqrt{8820} = 42\sqrt{5}$ m/s

e $$T = \int_{0}^{50.002} \frac{900}{8820 - v^2}\,dv$$

$= 5.69\,s$

OR

$$T = \frac{15\sqrt{5}}{7}\log_e\left(\frac{42\sqrt{5} + 50.002}{42\sqrt{5} - 50.002}\right)$$

$= 5.69$ s

16 $\ddot{x} = g - kv^2$

$\ddot{x} = 0 \;\Rightarrow\; g - kv^2 = 0 \;\Rightarrow\; v_{\mathrm{T}}^2 = \dfrac{g}{k}$

$$v_{\mathrm{T}} = \sqrt{\frac{g}{k}}$$

$$\frac{dv}{dt} = g - kv^2$$

$$\frac{dt}{dv} = \frac{1}{g - kv^2}$$

$$t = \int \frac{1}{g - kv^2}\,dv$$

$$= \frac{1}{k}\int \frac{1}{\frac{g}{k} - v^2}\,dv$$

$$= \frac{1}{k}\int \frac{1}{v_{\mathrm{T}}^2 - v^2}\,dv$$

$$\frac{1}{v_{\mathrm{T}}^2 - v^2} = \frac{\mathrm{A}}{v_{\mathrm{T}} + v} + \frac{\mathrm{B}}{v_{\mathrm{T}} - v}$$

$$= \frac{\mathrm{A}\,(v_{\mathrm{T}} - v) + \mathrm{B}\,(v_{\mathrm{T}} + v)}{v_{\mathrm{T}}^2 - v^2}$$

$$1 = v_{\mathrm{T}}(\mathrm{A} + \mathrm{B}) + v(\mathrm{B} - \mathrm{A})$$

$$\mathrm{B} - \mathrm{A} = 0$$

$$v_{\mathrm{T}}(\mathrm{A} + B) = 1 \;\Rightarrow \mathrm{A} = \mathrm{B} = \frac{1}{2v_{\mathrm{T}}}$$

$$t = \frac{1}{2kv_{\mathrm{T}}}\int \left(\frac{1}{v + v_{\mathrm{T}}} + \frac{1}{v_{\mathrm{T}} - v}\right) dv$$

$$= \frac{1}{2kv_{\mathrm{T}}}\log_e\left(\left|\frac{v + v_{\mathrm{T}}}{v_{\mathrm{T}} - v}\right|\right) + c$$

$v = 0 \quad t = 0 \quad \Rightarrow c = 0$

$$t = \frac{1}{2kv_{\mathrm{T}}}\log_e\left(\frac{v_{\mathrm{T}} + v}{v_{\mathrm{T}} - v}\right)$$

17 a $\ddot{x} = g - kv$

$$\frac{dv}{dt} = g - kv$$

$$\frac{dt}{dv} = \frac{1}{g - kv}$$

$$t = \int \frac{1}{g - kv}\,dv$$

$$= \frac{-1}{k}\log_e(g - kv) + c_1$$

$t = 0 \quad v = 0 \quad x = 0$

$$0 = \frac{-1}{k}\log_e(g) + c_1$$

$$c_1 = \frac{1}{k}\log_e(g)$$

$$t = \frac{1}{k}\left(\log_e(g) - \log(g - kv)\right)$$

$$kt = \log_e\left(\frac{g}{g - kv}\right)$$

$$\frac{g}{g - kv} = e^{kt}$$

$$g - kv = ge^{-kt}$$

$$kv = g - ge^{-kt}$$

$$v = \frac{g}{k}\left(1 - e^{-kt}\right)$$

b $t \to \infty \quad v \to \dfrac{g}{k} \quad$ or $\ddot{x} = 0$

$$v_{\mathrm{T}} = \frac{g}{k}$$

c Method I

When $v = \dfrac{1}{2}\; v_{\mathrm{T}} = \dfrac{g}{2k}$

$$t = \frac{1}{k}\log_e\left(\frac{g}{g - \frac{g}{2}}\right) = \frac{1}{k}\log_e(2)$$

$$v = \frac{dx}{dt} = \frac{g}{k}\left(1 - e^{-kt}\right)$$

$$x = \frac{g}{k}\int \left(1 - e^{-kt}\right) dt$$

$$= \frac{g}{k}\left[t + \frac{1}{k}e^{-kt} + c_2\right]$$

$t = 0 \qquad x = 0$

$$\frac{1}{k} + c_2 = 0 \qquad c_2 = \frac{-1}{k}$$

$$x = \frac{g}{k}\left[t + \frac{1}{k}\left(e^{-kt} - 1\right)\right]$$

When $t = \dfrac{1}{k}\log_e(2) \qquad \mathrm{e}^{-kt} = \dfrac{1}{2}$

so $D = \dfrac{g}{k}\left[\dfrac{1}{k}\log_e(2) + \dfrac{1}{k}\left(\dfrac{1}{2} - 1\right)\right]$

$$D = \frac{g}{k^2}\left(\log_e(2) - \frac{1}{2}\right)$$

Method II

$$v\frac{dv}{dx} = g - kv$$

$$D = \int_{0}^{\frac{g}{2k}} \frac{v}{g - kv}\,dv$$

$$= \frac{g}{k^2}\left(\log_e(2) - \frac{1}{2}\right)$$

18 a Use $\ddot{x} = v\dfrac{dv}{dx} = \dfrac{49 - v^2}{5}$

$$\int \frac{v}{9.8 - 0.2v^2}dv = \int dx$$

$$x = -\frac{5}{2}\log_e\left(\left|v^2 - 49\right|\right) + c_1$$

When $v = 0 \quad x = 0$

$$0=-\frac{5}{2}\log_e(49)+c_1 \quad \Rightarrow c_1=\frac{5}{2}\log_e(49)$$

$$x=\frac{5}{2}\log_e\left(\frac{49}{49-v^2}\right) \quad \text{since } 0\le v<7$$

$$\frac{2x}{5}=\log_e\left(\frac{49}{49-v^2}\right)$$

$$\frac{49}{49-v^2}=e^{\frac{2x}{5}}$$

$$\frac{49-v^2}{49}=e^{-\frac{2x}{5}}$$

$$49-v^2=49e^{-\frac{2x}{5}}$$

$$v^2=49-49e^{-\frac{2x}{5}}$$

$$v^2=49\left(1-e^{-\frac{2x}{5}}\right)$$

$$v=7\sqrt{1-e^{-0.4x}}$$

As $x\to\infty, v\to 7$ m/s

b Use $\dfrac{dv}{dt}=\dfrac{49-v^2}{5}$

$$\frac{dt}{dv}=\frac{5}{49-v^2}$$

$$\int\frac{1}{5}\,dt=\int\frac{1}{49-v^2}\,dv$$

$$\frac{\text{A}}{7-v}+\frac{\text{B}}{7+v}=\frac{1}{49-v^2}$$

$$\frac{\text{A}(7+v)+\text{B}(7-v)}{49-v^2}=\frac{v(\text{A}-\text{B})+7(\text{A}+\text{B})}{49-v^2}$$

$$\Rightarrow \text{A}-\text{B}=0 \quad \Rightarrow \text{A}=\text{B}=\frac{1}{14}$$

$$\Rightarrow (\text{A}+\text{B})=\frac{1}{7}$$

$$\frac{t}{5}=\frac{1}{14}\int\left(\frac{1}{7-v}+\frac{1}{7+v}\right)dv$$

$$\frac{14t}{5}=\log_e(7+v)-\log_e(7-v)+c_2$$

When $t=0 \quad v=0 \quad \Rightarrow c_2=0$

$$\frac{14t}{5}=\log_e\left(\frac{7+v}{7-v}\right)$$

$$\frac{7+v}{7-v}=e^{\frac{14t}{5}}$$

$$\frac{7-v}{7+v}=e^{-\frac{14t}{5}}$$

$$7-v=(7+v)\text{e}^{-\frac{14t}{5}}$$

$$7-v=7e^{-\frac{14t}{5}}+ve^{-\frac{14t}{5}}$$

$$7-7e^{-\frac{14t}{5}}=v+ve^{-\frac{14t}{5}}$$

$$7\left(1-e^{-\frac{14t}{5}}\right)=v\left(1+e^{-\frac{14t}{5}}\right)$$

$$v(t)=\frac{7\left(1-e^{-2.8t}\right)}{1+e^{-2.8t}}$$

as $t\to\infty$

$v\to 7$ m/s

c $v=\dfrac{dx}{dt}=\dfrac{7\left(1-e^{-2.8t}\right)}{1+e^{-2.8t}}\times\dfrac{e^{1.4t}}{e^{1.4t}}$

$$\frac{dx}{dt}=\frac{7\left(e^{1.4t}-e^{-1.4t}\right)}{e^{1.4t}+e^{-1.4t}}$$

$$x=7\int\frac{e^{1.4t}-e^{-1.4t}}{e^{1.4t}+e^{-1.4t}}\,dt$$

$$=\frac{7}{1.4}\log_e\left(e^{1.4t}+e^{-1.4t}\right)+c_3$$

When $t=0 \quad x=0$

$$0=5\log_e(2)+c_3$$

$$c_3=-5\log_e(2)$$

$$x=5\log_e\left(\frac{e^{1.4t}+e^{-1.4t}}{2}\right)$$

19 Going up: $\ddot{x}=-g-kv^2$

$$v\frac{dv}{dx}=-\left(g+kv^2\right)$$

$$x=-\int\frac{v}{g+kv^2}\,dv$$

$$=-\frac{1}{2k}\log_e(g+kv^2)+c_1$$

When $x=0$, $v=U$

$$0=-\frac{1}{2k}\log_e\left(g+kU^2\right)+c_1$$

$$c_1=\frac{1}{2k}\log_e\left(g+kU^2\right)$$

$$x=\frac{1}{2k}\log_e\left(\frac{g+kU^2}{g+kv^2}\right)$$

At maximum height $v=0, \quad x=H$

$$H=\frac{1}{2k}\log_e\left(\frac{g+kU^2}{g}\right)$$

$$=\frac{1}{2k}\log_e\left(1+\frac{kU^2}{g}\right)$$

b $\dfrac{dv}{dt}=-\left(g+kv^2\right)$

$$\frac{dt}{dv}=-\frac{1}{g+kv^2}$$

let $b^2=\dfrac{k}{g}$

$$b=\sqrt{\frac{k}{g}}$$

$$t=-\int\frac{1}{g+kv^2}\,dv$$

$$t=-\frac{1}{g}\int\frac{1}{1+b^2v^2}\,dv$$

$$t=\frac{-1}{gb}\tan^{-1}(bv)+c_2$$

$t=0 \quad v=U$

$$0=\frac{-1}{gb}\tan^{-1}(bU)+c_2$$

$$t=\frac{1}{gb}\left(\tan^{-1}(bU)-\tan^{-1}(bv)\right)$$

Time to rise, at maximum height $v=0$

$$T=\frac{1}{gb}\tan^{-1}(bU)$$

$$=\frac{1}{\sqrt{gk}}\tan^{-1}\left(U\sqrt{\frac{k}{g}}\right)$$

c Coming down: $\ddot{x} = g - kv^2$

$$v\frac{dv}{dx} = g - kv^2$$

$$x = \int \frac{v\,dv}{g - kv^2}$$

$$x = -\frac{1}{2k}\log_e\left(g - kv^2\right) + c_3$$

$x = 0, \quad v = 0$

$$0 = \frac{-1}{2k}\log_e(g) + c_3$$

$$c_3 = \frac{1}{2k}\log_e(g)$$

$$x = \frac{1}{2k}\log_e\left(\frac{g}{g - kv^2}\right)$$

Hits the ground when $x = H \quad v = V$

$$H = \frac{1}{2k}\log_e\left(\frac{g + kU^2}{g}\right) = \frac{1}{2k}\log_e\left(\frac{g}{g - kV^2}\right)$$

$$\frac{g + kU^2}{g} = \frac{g}{g - kV^2}$$

$$\left(g + kU^2\right)\left(g - kV^2\right) = g^2$$

$$\cancel{g^2} + kg\left(U^2 - V^2\right) - k^2U^2V^2 = \cancel{g^2}$$

$$kgU^2 = kgV^2 + k^2U^2V^2$$

$$gU^2 = \left(g + kU^2\right)V^2$$

$$V = U\sqrt{\frac{g}{g + kU^2}}$$

20 $\ddot{x} = -kv^3$

$$v\frac{dv}{dx} = -kv^3$$

$$\frac{dv}{dx} = -kv^2$$

$$\frac{dx}{dv} = \frac{-1}{kv^2}$$

$$x = \frac{-1}{k}\int v^{-2}dv$$

$$= \frac{1}{k}v^{-1} + c_1$$

$v = U \quad x = 0$

$$0 = \frac{1}{kU} + c_1 \quad c_1 = \frac{-1}{kU}$$

$$x = \frac{1}{kv} - \frac{1}{kU}$$

$$\frac{1}{kv} = x + \frac{1}{kU} = \frac{xkU + 1}{kU}$$

$$v = \frac{U}{xkU + 1}$$

$$\frac{dx}{dt} = \frac{U}{xkU + 1}$$

$$\frac{dt}{dx} = \frac{xkU + 1}{U}$$

$$t = \frac{1}{U}\int (xkU + 1)dx$$

$$= \frac{1}{U}\left(\frac{x^2}{2}kU + x + c_2\right)$$

$x = 0, \quad t = 0 \quad c_2 = 0$

$$t = \frac{x}{U} + \frac{kx^2}{2}$$

21 $\ddot{x} = -400\left(v^2 + 10000\right), \quad t = 0 \quad v = 400\text{ m/s}$

a Use $\ddot{x} = \dfrac{dv}{dt} = -400\left(v^2 + 10000\right)$

$$\int \frac{dv}{v^2 + 10000} = -400\int dt$$

$$-400t = \frac{1}{100}\tan^{-1}\left(\frac{v}{100}\right) + c_1$$

When $t = 0 \quad v = 400$

$$0 = \frac{1}{100}\tan^{-1}(4) + c_1$$

$$c_1 = \frac{-1}{100}\tan^{-1}(4)$$

$$-400t = \frac{1}{100}\tan^{-1}\left(\frac{v}{100}\right) - \frac{1}{100}\tan^{-1}(4)$$

$$t = \frac{1}{40000}\left(\tan^{-1}(4) - \tan^{-1}\left(\frac{v}{100}\right)\right)$$

When it comes to rest $v = 0$

$$t = \frac{1}{40000}\tan^{-1}(4)$$

$= 0.000033$ seconds $= 0.033$ ms

b $40000\,t = \tan^{-1}(4) - \tan^{-1}\left(\dfrac{v}{100}\right)$

$$\tan^{-1}\left(\frac{v}{100}\right) = \tan^{-1}(4) - 40000t$$

$$\frac{v}{100} = \tan\left(\tan^{-1}(4) - 40000t\right)$$

$$v = v(t) = 100\tan\left(\tan^{-1}(4) - 40000t\right)$$

$$\mathrm{D} = \int_0^{0.000033} v(t)\mathrm{d}t$$

$= 0.003542$ m

$= 3.5$ mm

Alternatively $\ddot{x} = v\dfrac{dv}{dx} = -400\left(v^2 + 10000\right)$

$$\int \frac{v}{v^2 + 10000}dv = -400\int dx$$

$$\frac{1}{2}\log_e\left(v^2 + 10000\right) = -400x + c_2$$

When $x = 0 \quad v = 400$

$$\frac{1}{2}\log_e\left(400^2 + 10000\right) = c_2 = \frac{1}{2}\log_e(170000)$$

$$400x = \frac{1}{2}\log_e(170000)$$

$$-\frac{1}{2}\log_e\left(v^2 + 10000\right)$$

$$x = \frac{1}{800}\log_e\left(\frac{170000}{v^2 + 10000}\right)$$

When it comes to rest $v = 0$

$$D = \frac{1}{800}\log_e(17)$$

$= 0.003542$ m

$= 3.5$ mm

22 a Proportional to the speed:

$v = U,\quad t = 0;\quad v = V,\quad t = T\quad x = D$

$$\ddot{x} = -\lambda v$$

Use $\ddot{x} = \dfrac{dv}{dt}$

$$\frac{dv}{dt} = -\lambda v$$

$$\int \frac{1}{v} dv = \int -\lambda\, dt$$

$$\log_e(v) = -\lambda t + c_1$$

When $t = 0\quad v = U$

$$\log_e(U) = c_1$$

$$\log_e(v) = -\lambda t + \log_e(U)$$

$$\log_e(v) - \log_e(U) = -\lambda t$$

$$\log_e\left(\frac{v}{U}\right) = -\lambda t$$

When $t = T, v = V$

$$\log_e\left(\frac{V}{U}\right) = -\lambda T$$

$$\lambda = \frac{1}{T}\log_e\left(\frac{U}{V}\right)$$

Use $\ddot{x} = v\dfrac{dv}{dx}$

$$v\frac{dv}{dx} = -\lambda v$$

$$\frac{dv}{dx} = -\lambda$$

$$\int dv = -\lambda \int dx$$

$$v = -\lambda x + c_2$$

When $x = 0\quad v = U$

$$U = c_2$$

$$v = -\lambda x + U$$

When $x = D\quad v = V$

$$V = -\lambda D + U$$

$$\lambda D = U - V$$

$$\lambda = \frac{U - V}{D}$$

$$\frac{1}{T}\log_e\left(\frac{U}{V}\right) = \frac{U - V}{D}$$

$$\frac{D}{T} = \frac{U - V}{\log_e\left(\frac{U}{V}\right)}$$

b Proportional to the square of the speed:

$v = U, t = 0; v = V, t = T, x = D$

$$\ddot{x} = -\lambda v^2$$

Use $\ddot{x} = \dfrac{dv}{dt}$

$$\frac{dv}{dt} = -\lambda v^2$$

$$\int \frac{1}{v^2} dv = -\lambda \int dt$$

$$-\frac{1}{v} = -\lambda t + c_1$$

When $v = U,\quad t = 0$

$$\frac{-1}{U} = c_1$$

$$\frac{1}{v} = \lambda t + \frac{1}{U}$$

When $v = V,\quad t = T$

$$\frac{1}{V} = \lambda T + \frac{1}{U}$$

$$\lambda T = \frac{1}{V} - \frac{1}{U}$$

$$\lambda T = \frac{U - V}{UV}$$

$$\lambda = \frac{U - V}{TUV}$$

Use $\ddot{x} = v\dfrac{dv}{dx}$

$$v\frac{dv}{dx} = -\lambda v^2$$

$$\frac{dv}{dx} = -\lambda v$$

$$\int \frac{1}{v} dv = -\lambda \int dx$$

$$\log_e(v) = -\lambda x + c_2$$

When $v = U, x = 0$

$$\log_e(U) = c_2$$

$$\log_e(v) = -\lambda x + \log_e(U)$$

When $v = V, x = D$

$$\log_e(V) = -\lambda D + \log_e(U)$$

$$\lambda D = \log_e(U) - \log_e(V)$$

$$\lambda D = \log_e\left(\frac{U}{V}\right)$$

$$\lambda = \frac{1}{D}\log_e\left(\frac{U}{V}\right)$$

$$\frac{U - V}{TUV} = \frac{1}{D}\log_e\left(\frac{U}{V}\right)$$

$$\frac{D}{T} = \frac{UV}{U - V}\log_e\left(\frac{U}{V}\right)$$

c Proportional to the cube of the speed:

$v = U,\ t = 0;\ v = V,\ t = T,\ x = D$

$$\ddot{x} = -\lambda v^3$$

Use $\ddot{x} = \dfrac{dv}{dt}$

$$\frac{dv}{dt} = -\lambda v^3$$

$$\int \frac{1}{v^3} dv = -\lambda \int dt$$

$$-\frac{1}{2v^2} = -\lambda t + c_1$$

When $v = U,\quad t = 0$

$$-\frac{1}{2U^2} = c_1$$

$$\frac{1}{2v^2} = \lambda t + \frac{1}{2U^2}$$

When $t = T, v = V$

$$\frac{1}{2V^2} = \lambda T + \frac{1}{2U^2}$$

$$\lambda T = \frac{1}{2V^2} - \frac{1}{2U^2}$$

$$= \frac{U^2 - V^2}{2V^2U^2}$$

$$\lambda = \frac{U^2 - V^2}{2TU^2V^2}$$

Use $\ddot{x} = v\frac{dv}{dx}$

$$v\frac{dv}{dx} = -\lambda v^3$$

$$\frac{dv}{dx} = -\lambda v^2$$

$$\int \frac{1}{v^2}dv = -\lambda \int dx$$

$$-\frac{1}{v} = -\lambda x + c_2$$

When $v = U, \quad x = 0$

$$-\frac{1}{U} = c_2$$

$$-\frac{1}{v} = -\lambda x - \frac{1}{U}$$

When $v = V, \quad x = D$

$$\frac{1}{V} = \lambda D + \frac{1}{U}$$

$$\lambda D = \frac{1}{V} - \frac{1}{U}$$

$$\lambda = \frac{U - V}{UVD}$$

$$\frac{(U - V)(U + V)}{2TU^2V^2} = \frac{U - V}{UVD}$$

$$\frac{D}{T} = \frac{2UV}{U + V}$$

d Proportional to the fourth power of the speed:

$v = U, \quad t = 0; \quad v = V, \quad t = T, \quad x = D$

$$\ddot{x} = -\lambda v^4$$

Use $\ddot{x} = \frac{dv}{dt}$

$$\frac{dv}{dt} = -\lambda v^4$$

$$\int \frac{1}{v^4}dv = -\lambda \int dt$$

$$\int v^{-4}dv = -\lambda t + c_1$$

$$-\frac{1}{3}v^{-3} = -\lambda t + c_1$$

When $v = U \quad t = 0$

$$-\frac{1}{3U^3} = c_1$$

$$-\frac{1}{3v^3} = -\lambda t - \frac{1}{3U^3}$$

When $v = V, \quad t = T$

$$\frac{1}{3V^3} = \lambda T + \frac{1}{3U^3}$$

$$\lambda T = \frac{1}{3V^3} - \frac{1}{3U^3}$$

$$\lambda = \frac{U^3 - V^3}{3U^3V^3T}$$

Use $\ddot{x} = v\frac{dv}{dx}$

$$v\frac{dv}{dx} = -\lambda v^4$$

$$\frac{dv}{dx} = -\lambda v^3$$

$$\int \frac{1}{v^3}dv = -\lambda \int dx$$

$$\int v^{-3} = -\lambda x + c_2$$

$$-\frac{1}{2}v^{-2} = -\lambda x + c_2$$

When $v = U, \quad x = 0$

$$-\frac{1}{2U^2} = c_2$$

$$-\frac{1}{2v^2} = -\lambda x - \frac{1}{2U^2}$$

When $x = D \quad v = V$

$$\frac{1}{2V^2} = \lambda D + \frac{1}{2U^2}$$

$$\lambda D = \frac{1}{2V^2} - \frac{1}{2U^2}$$

$$\lambda = \frac{U^2 - V^2}{2V^2U^2D}$$

$$\frac{U^3 - V^3}{3U^3V^3T} = \frac{U^2 - V^2}{2V^2U^2D}$$

$$\frac{D}{T} = \frac{3\left(U^2 - V^2\right)UV}{2\left(U^3 - V^3\right)}$$

$$= \frac{3(U + V)\cancel{(U - V)}UV}{2\cancel{(U - V)}(U^2 + UV + V^2)}$$

$$\frac{D}{T} = \frac{3UV(U + V)}{2\left(U^2 + UV + V^2\right)}$$

e Proportional to 5^{th} Power:

$\ddot{x} = -\lambda v^5$

use $\ddot{x} = \frac{dv}{dt} = -\lambda v^5$

$$\int v^{-5}dv = \int -\lambda dt$$

$$-\frac{1}{4}v^{-4} = -\lambda t + c_1$$

When $v = U, t = 0$

$$-\frac{1}{4U^4} = c_1$$

$$\frac{1}{4v^4} = \lambda t + \frac{1}{4U^4}$$

When $v = V, t = T$

$$\lambda T = \frac{1}{4V^4} - \frac{1}{4U^4}$$

$$= \frac{U^4 - V^4}{4U^4V^4}$$

$$\lambda = \frac{U^4 - V^4}{4TU^4V^4}$$

Use $\ddot{x} = v\frac{dv}{dx} = -\lambda v^5$

$$\frac{dv}{dx} = -\lambda v^4$$

$$\int v^{-4}dv = \int -\lambda\, dx$$

$$-\frac{1}{3}v^{-3} = -\lambda x + c_2$$

$v = U, x = 0$

$$-\frac{1}{3U^3} = c_2$$

$$\frac{1}{3v^3} = \lambda x + \frac{1}{3U^3}$$

When $x = D, v = V$

$$\lambda D = \frac{1}{3V^3} - \frac{1}{3U^3}$$

$$\lambda D = \frac{U^3 - V^3}{3U^3V^3}$$

$$\lambda = \frac{U^3 - V^3}{3U^3V^3D}$$

$$\frac{U^4 - V^4}{4TU^4V^4} = \frac{U^3 - V^3}{3U^3V^3D}$$

$$\frac{D}{T} = \frac{4UV\left(U^3 - V^3\right)}{3\left(U^4 - V^4\right)}$$

f Proportional to square root of the speed cubed: $\ddot{x} = -\lambda v^{\frac{3}{2}}$

$$\ddot{x} = \frac{dv}{dt} = -\lambda v^{\frac{3}{2}}$$

$$\int v^{-\frac{3}{2}}dv = \int -\lambda dt$$

$$-2v^{-\frac{1}{2}} = -\lambda t + c_1$$

$v = U, t = 0$

$$c_1 = -\frac{2}{\sqrt{U}}$$

$$\frac{2}{\sqrt{v}} = \lambda t + \frac{2}{\sqrt{U}}$$

When $t = T, v = V$

$$\lambda T = \frac{2}{\sqrt{V}} - \frac{2}{\sqrt{U}}$$

$$\lambda = \frac{2\left(\sqrt{U} - \sqrt{V}\right)}{T\sqrt{UV}}$$

use $v\frac{dv}{dx} = -\lambda v^{\frac{3}{2}}$

$$\frac{dv}{dx} = -\lambda v^{\frac{1}{2}}$$

$$\int v^{-\frac{1}{2}}dv = \int -\lambda dx$$

$$2v^{\frac{1}{2}} = -\lambda x + c_2$$

When $v = U, x = 0$

$$c_2 = 2\sqrt{U}$$

$$2\sqrt{v} = -\lambda x + 2\sqrt{U}$$

When $x = D, v = V$

$$\lambda D = 2\sqrt{U} - 2\sqrt{V}$$

$$\lambda = \frac{2\left(\sqrt{U} - \sqrt{V}\right)}{D}$$

so, $\dfrac{2\left(\sqrt{U} - \sqrt{V}\right)}{T\sqrt{UV}} = \dfrac{2\left(\sqrt{U} - \sqrt{V}\right)}{D}$

$$\frac{D}{T} = \sqrt{UV}$$

g $n \in R\backslash\{1, 2\}$

$$\ddot{x} = -\lambda v^n$$

$$\frac{dv}{dt} = -\lambda v^n$$

$$\int v^{-n}dv = \int -\lambda dt$$

$$\frac{v^{-n+1}}{1-n} = -\lambda t + c_1$$

$t = 0, v = U$

$$\frac{U^{1-n}}{1-n} = c_1$$

$$\frac{v^{1-n}}{1-n} = -\lambda t + \frac{U^{1-n}}{1-n}$$

When $v = V, t = T$

$$\lambda T = \frac{U^{1-n} - V^{1-n}}{1-n}$$

$$v\frac{dv}{dx} = -\lambda v^n$$

$$\frac{dv}{dx} = -\lambda v^{n-1}$$

$$\int v^{1-n}dv = \int -\lambda\, dx$$

$$\frac{v^{2-n}}{2-n} = -\lambda x + c_2$$

$x = 0, v = U$

$$\frac{U^{2-n}}{2-n} = c_2$$

$$\frac{v^{2-n}}{2-n} = -\lambda x + \frac{U^{2-n}}{2-n}$$

When $x = D, v = V$

$$\lambda D = \frac{U^{2-n} - V^{2-n}}{2-n}$$

So $\dfrac{D}{T} = \dfrac{\left(U^{2-n} - V^{2-n}\right)(1-n)}{\left(U^{1-n} - V^{1-n}\right)(2-n)} \quad n \neq 1, 2$

Use CAS $\quad n = 3 \quad \dfrac{D}{T} = \dfrac{2UV}{U + V}$

$$n = 4 \quad \frac{D}{T} = \frac{3UV(U+V)}{2\left(U^2 + UV + V^2\right)}$$

$$n = 5 \quad \frac{D}{T} = \frac{4UV\left(U^2 + UV + V^2\right)}{3\left(U^3 + VU^2 + UV^2 + V^3\right)}$$

h Constant deceleration:

$u = U, v = V, s = D, t = T$

$$v^2 = u^2 + 2as$$

$$V^2 = U^2 + 2aD$$

$$a = \frac{V^2 - U^2}{2D}$$

$$v = u + at$$
$$V = U + aT$$
$$a = \frac{V-U}{T}$$
$$\text{So } \frac{V^2-U^2}{2D} = \frac{V-U}{T}$$
$$\frac{D}{T} = \frac{V^2-U^2}{2(V-U)} = \frac{\cancel{(V-U)}(V+U)}{2\cancel{(V-U)}}$$
$$\frac{D}{T} = \frac{U+V}{2}$$

11.6 Exam questions

1 a $u = 0,\ t = 2,\ a = g$

$$s = ut + \frac{1}{2}at^2$$
$$s = 0 + \frac{1}{2} \times 9.8 \times 4$$
$$= 19.6\,\text{m}$$

Award 1 mark for using the constant acceleration formulae or another method.

Award 1 mark for the correct distance.

VCAA Examination Report **note:**

Most students correctly used constant acceleration formulas. A smaller number of students started with an expression for acceleration and integrated correctly, either using definite integrals or evaluating the constant separately.

b $u = 0,\ t = 2,\ a = g$

$$v = u + at$$
$$v = 0 + 9.8 \times 2$$
$$= 19.6\,\text{ms}^{-1}$$

Award 1 mark for the correct proof of speed.

c $a = g - 0.01v^2$

For terminal velocity $a = 0$.

$$v^2 = \frac{g}{0.01}$$
$$= 980$$
$$v = \sqrt{980}$$
$$= 14\sqrt{5}\,\text{ms}^{-1} \quad \left(\text{accept } 31.3\,\text{ms}^{-1}\right)$$

Award 1 mark for the correct terminal velocity.

VCAA Examination Report **note:**

Correct solutions were generally obtained by setting $a = 0$. Some answers were not given in exact form.

d i $a = \dfrac{dv}{dt} = g - 0.01v^2$

$$\frac{dt}{dv} = \frac{1}{g - 0.01v^2}$$
$$t = \int_{19.6}^{30} \frac{1}{g - 0.01\,v^2}\,dv + 2$$

Award 1 mark for the correct form of the acceleration and inverting.

Award 1 mark for the correct definite integral for time.

VCCA Examination Report **note:**

This question was often misinterpreted by students, either by assuming that the model applied form the start of the skydiver's fall (integrating from 0 to 30) or by giving an answer that only gave the time after 2 seconds. Many students did not attempt this question.

ii Solve $t = \displaystyle\int_{19.6}^{30} \frac{1}{g - 0.01v^2}\,dv + 2$ using CAS

$$t = 5.8\,\text{s}$$

Award 1 mark for the correct time.

e $a = v\dfrac{dv}{dx} = g - 0.01v^2$

$$\frac{dx}{dv} = \frac{v}{g - 0.01v^2}$$
$$x = \int_{19.6}^{30} \frac{v}{g - 0.01v^2}\,dv + 19.6 \text{ using CAS}$$
$$x = 120\,\text{m}$$

Award 1 mark for the correct form of the acceleration, simplifying and the inverting.

Award 1 mark for the correct definite integral for the distance.

Award 1 mark for the correct distance.

VCAA Examination Report **note:**

While a variety of solutions was given, the errors apparent in Question 2di., as a result of not taking the first 2 seconds of motion into account, also appeared in responses to this question.

2 $\ddot{x} = \dfrac{dv}{dt} = -\left(9.8 + 0.1v^2\right)$

$$\frac{dt}{dv} = \frac{-1}{9.8 + 0.1v^2}$$
$$t = \int_{7\sqrt{6}}^{0} \left(\frac{-10}{98 + v^2}\right) dv$$
$$= \frac{5\pi\sqrt{2}}{21} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{10\pi}{21\sqrt{2}}$$

Found using CAS.

The correct answer is **D**.

3 $a = \sqrt{v^2 - 1}$, use $a = v\dfrac{dv}{dx}$

$$v\frac{dv}{dx} = \sqrt{v^2-1} \Rightarrow x = \int \frac{v}{\sqrt{v^2-1}}\,dv$$
$$x = \sqrt{v^2-1} + c \text{ when } v = \sqrt{2}, x = 0 \Rightarrow 0 = \sqrt{1} + c,\ c = -1$$
$$x = \sqrt{v^2-1} - 1 \Rightarrow 1 + x = \sqrt{v^2-1}$$
$$v^2 - 1 = (1+x)^2 \Rightarrow v = \sqrt{1 + (1+x)^2}$$

The correct answer is **D**.

11.7 Acceleration that depends on position

11.7 Exercise

1 $\ddot{x} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) \quad = \dfrac{7x-6}{x^3}$

$$\frac{1}{2}v^2 = \int \left(7x^{-2} - 6x^{-3}\right) dx$$
$$= -7x^{-1} + 3x^{-2} + c \quad = \frac{-7}{x} + \frac{3}{x^2} + c$$

When $x = 3,\quad v = 0$

$$0=\frac{3}{9}-\frac{7}{3}+c$$

$$\Rightarrow c=2$$

$$v^2=2\left(\frac{3}{x^2}-\frac{7}{x}+2\right)$$

$$=\frac{2\left(2x^2-7x+3\right)}{x^2}$$

$$v=\frac{\sqrt{2(2x-1)(x-3)}}{x}$$

Comes to rest when $v=0$

$\Rightarrow x=3$ and $x=\frac{1}{2}$m

2 $\ddot{x}=\dfrac{d\left(\frac{1}{2}v^2\right)}{dx}=\dfrac{3-x}{2x^3},\quad x=1,\quad v=0$

$$\frac{1}{2}v^2=\int\frac{1}{2}\left(3x^{-3}-x^{-2}\right)dx$$

$$=\frac{1}{2}\left[-\frac{3}{2x^2}+x^{-1}+c_1\right]$$

When $v=0\quad x=1$

$$0=\frac{1}{2}\left[-\frac{3}{2}+1+c_1\right]\quad\Rightarrow c_1=\frac{1}{2}$$

$$v^2=\frac{1}{2}+\frac{1}{x}-\frac{3}{2x^2}$$

$$=\frac{x^2+2x-3}{2x^2}$$

$$v=\frac{1}{x}\sqrt{\frac{(x+3)(x-1)}{2}}$$

When $v=0\quad x=1$ and $x=-3$

3 $\ddot{x}=\dfrac{d\left(\frac{1}{2}v^2\right)}{dx}=\dfrac{10x-8}{x^3}$

$$\frac{1}{2}v^2=\int\left(10x^{-2}-8x^{-3}\right)dx$$

$$=-10x^{-1}+4x^{-2}+c$$

$$=\frac{-10}{x}+\frac{4}{x^2}+c$$

When $x=2\quad v=0$

$$0=\frac{-10}{2}+\frac{4}{4}+c$$

$$\Rightarrow c=4$$

$$v^2=2\left(\frac{4}{x^2}-\frac{10}{x}+4\right)$$

$$=\frac{2\left(4x^2-10x+4\right)}{x^2}=\frac{4\left(2x^2-5x+2\right)}{x^2}$$

$$v=\frac{\sqrt{4(2x-1)(x-2)}}{x}$$

When $v=0\quad x=2$ and $x=\frac{1}{2}$m

4 $\ddot{x}=\dfrac{d\left(\frac{1}{2}v^2\right)}{dx}=\dfrac{1+bx}{x^3}$

$$\frac{1}{2}v^2=\int\left(x^{-3}+bx^{-2}\right)dx$$

$$\frac{1}{2}v^2=-\frac{1}{2}x^{-2}-bx^{-1}+c_1$$

$$=-\frac{1}{2x^2}-\frac{b}{x}+c_1$$

When $x=1,\quad v=0$

$$0=-\frac{1}{2}-b+c_1=0\quad\Rightarrow c_1=b+\frac{1}{2}$$

$$\frac{1}{2}v^2=-\frac{1}{2x^2}-\frac{b}{x}+\left(b+\frac{1}{2}\right)$$

$$=\frac{-1-2bx+2x^2\left(b+\frac{1}{2}\right)}{2x^2}$$

When $v=0\quad x^2(2b+1)-2bx-1=0$

$$((2b+1)x+1)(x-1)=0$$

It is at rest when:

$x=1$ and $x=\dfrac{-1}{2b+1}=-\dfrac{1}{5}$

$$2b+1=5$$

$$2b=4$$

$$b=2$$

5 $\ddot{x}=\dfrac{-(p+qx)}{x^3}$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right)=-(p+qx)x^{-3}$$

$$-\frac{1}{2}v^2=\int\left(px^{-3}+qx^{-2}\right)dx$$

$$=-\frac{1}{2}px^{-2}-qx^{-1}+c$$

$x=p,\ v=0$

$$0=-\frac{1}{2p}-\frac{q}{p}+c$$

$$c=\frac{1}{2p}+\frac{q}{p}=\frac{2q+1}{2p}$$

$$\frac{1}{2}v^2=\frac{p}{2x^2}+\frac{q}{x}-\frac{2q+1}{2p}$$

Comes to rest $v=0$

$$\frac{p}{2x^2}+\frac{q}{x}-\frac{2q+1}{2p}=0$$

$$\frac{p^2+2pqx-(2q+1)x^2}{2px^2}=0$$

$$(2q+1)x^2-2pqx-p^2=0$$

$$((2q+1)x+p)(x-p)=0$$

At rest $x=p$ or $x=\dfrac{-p}{2q+1}$

6 $\ddot{x}=\dfrac{d\left(\frac{1}{2}v^2\right)}{dx}=4x-6,\quad v=1,\ x=2,\ t=0$

$$\frac{1}{2}v^2=\int(4x-6)dx$$

$$=2x^2-6x+c_1$$

$v=1,\ x=2$

$$\frac{1}{2}=8-12+c_1\ \Rightarrow c_1=\frac{9}{2}$$

$$v^2=4x^2-12x+9$$

$$v^2=(2x-3)^2$$

$v=\dfrac{dx}{dt}=2x-3\quad$ Since $\ v>0\ \ x>\dfrac{3}{2}$

$$\int\frac{dx}{2x-3}=t$$

$$t=\frac{1}{2}\log_e(|2x-3|)+c_2$$

When $x=2\ t=0$

$$0 = \frac{1}{2}\log_e(1) + c_2$$
$$\Rightarrow c_2 = 0$$
$$t = \frac{1}{2}\log_e(2x-3)$$
$$2x - 3 = e^{2t}$$
$$2x = 3 + e^{2t}$$
$$x = \frac{1}{2}\left(3 + e^{2t}\right)$$

7 $\ddot{x} = \dfrac{d\left(\frac{1}{2}v^2\right)}{dx} = 9x - 15,\ v = 1,\ x = 2,\ t = 0$

$$\frac{1}{2}v^2 = \int (9x-15)dx$$
$$= \frac{9x^2}{2} - 15x + c_1$$
$$v = 1, x = 2$$
$$\frac{1}{2} = 18 - 30 + c_1 \Rightarrow c_1 = \frac{25}{2}$$
$$v^2 = 9x^2 - 30x + 25$$
$$v^2 = (3x-5)^2$$
$$v = \frac{dx}{dt} = 3x - 5 \quad \text{Since } v > 0 \quad x \geq 2$$
$$\int \frac{dx}{3x-5} = t$$
$$t = \frac{1}{3}\log_e(|3x-5|) + c_2$$
$$x = 2 \quad t = 0$$
$$0 = \frac{1}{3}\log_e(1) + c_2 \Rightarrow c_2 = 0$$
$$3t = \log_e(3x-5)$$
$$3x - 5 = e^{3t}$$
$$3x = 5 + e^{3t}$$
$$x = \frac{1}{3}\left(5 + e^{3t}\right)$$

8 $\ddot{x} = x - 1$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = x - 1, \quad t = 0,\ x = 0,\ v = 1$$
$$\frac{1}{2}v^2 = \int (x-1)\,dx$$
$$= \frac{1}{2}x^2 - x + c_1, \quad \text{since} \quad x = 0, v = 1$$
$$\frac{1}{2} = 0 + c_1 \Rightarrow c_1 = \frac{1}{2}$$
$$\frac{1}{2}v^2 = \frac{1}{2}x^2 - x + \frac{1}{2}$$
$$v^2 = x^2 - 2x + 1$$
$$= (x-1)^2 \qquad \text{since} \quad v > 0$$
$$v = \frac{dx}{dt} = (x-1)$$
$$\int \frac{dx}{x-1} = t$$
$$t = \log_e|(x-1)| + c \quad \text{since} \quad x = 0$$
$$= \log_e(1-x) + \text{c}$$
When $x = 0, t = 0$
$$0 = \log_e(1) + c \quad \Rightarrow c = 0$$
$$t = \log_e(1-x)$$
$$1 - x = e^t$$
$$x = 1 - e^t$$

9 $\ddot{x} = \dfrac{d\left(\frac{1}{2}v^2\right)}{dx} = -e^{-2x}, \quad t = 0,\ x = 0,\ v = 1$

$$\frac{1}{2}v^2 = \int -e^{-2x}\,dx$$
$$= \frac{1}{2}e^{-2x} + c_1$$
$$v = 1 \ \ x = 0$$
$$\frac{1}{2} = \frac{1}{2}e^0 + c_1 \quad \Rightarrow c_1 = 0$$
$$\frac{1}{2}v^2 = \frac{1}{2}e^{-2x}$$
$$v^2 = e^{-2x}$$
$$v = \frac{dx}{dt} = e^{-x}$$
$$\frac{dt}{dx} = \frac{1}{e^{-x}} = e^x$$
$$t = \int e^x\,dx$$
$$t = e^x + c_2$$
$$t = 0, x = 0$$
$$0 = 1 + c_2 \quad \Rightarrow c_2 = -1$$
$$t = e^x - 1$$
$$e^x = t + 1$$
$$x = \log_e(t+1)$$

10 $\ddot{x} = \dfrac{d\left(\frac{1}{2}v^2\right)}{dx} = 2e^{4x} \quad v = 1,\ x = 0,\ t = 0$

$$\frac{1}{2}v^2 = \int 2e^{4x}\,dx$$
$$= \frac{1}{2}e^{4x} + c_1, \quad x = 0, v = 1$$
$$\frac{1}{2} = \frac{1}{2}e^0 + c_1 \quad \Rightarrow c_1 = 0$$
$$v^2 = e^{4x}$$
$$v = e^{2x} \quad \text{Since} \quad v > 0$$
$$v = \frac{dx}{dt} = e^{2x}$$
$$\frac{dt}{dx} = \frac{1}{e^{2x}} = e^{-2x}$$
$$t = \int e^{-2x}\,dx$$
$$= -\frac{1}{2}e^{-2x} + c_2$$
$$t = 0, x = 0$$
$$0 = -\frac{1}{2} + c_2 \quad \Rightarrow c_2 = \frac{1}{2}$$
$$t = \frac{1}{2} - \frac{1}{2}e^{-2x}$$
$$\frac{1}{2}e^{-2x} = \frac{1}{2} - t$$
$$e^{-2x} = (1-2t)$$
$$-2x = \log_e(1-2t)$$
$$x = -\frac{1}{2}\log_e(1-2t)$$

11 a $v = x^2 \qquad \dfrac{dv}{dx} = 2x$

$$a = v\frac{dv}{dx} = 2x^3$$

b $v = \sqrt{16 - 25x^2}$

$= \left(16 - 25x^2\right)^{\frac{1}{2}}$

$\dfrac{dv}{dx} = \dfrac{1}{2} \times (-50x) \times \left(16 - 25x^2\right)^{-\frac{1}{2}}$

$= \dfrac{-25x}{\sqrt{16 - 25x^2}}$

$a = v\dfrac{dv}{dx} = -25x$

c $v = e^{2x} + e^{-2x}$

$\dfrac{dv}{dx} = 2e^{2x} - 2e^{-2x}$

$v\dfrac{dv}{dx} = \left(e^{2x} + e^{-2x}\right) 2\left(e^{2x} - e^{-2x}\right)$

$= 2\left(e^{4x} + 1 - 1 - e^{-4x}\right)$

$= 2\left(e^{4x} - e^{-4x}\right)$

12 a $v = x^4$

$\dfrac{dv}{dx} = 4x^3$

$a = v\dfrac{dv}{dx} = 4x^7$

b $v = \sqrt{4 + 9x^2}$

$= \left(4 + 9x^2\right)^{\frac{1}{2}}$

$\dfrac{dv}{dx} = \dfrac{1}{2} \times 18x \times \left(4 + 9x^2\right)^{-\frac{1}{2}}$

$= \dfrac{9x}{\sqrt{4 + 9x^2}}$

$a = v\dfrac{dv}{dx} = \dfrac{9x}{\sqrt{4 + 9x^2}} \times \sqrt{4 + 9x^2}$

$a = 9x$

c $v = 3 - 2e^{-2t} \quad \Rightarrow 2e^{-2t} = 3 - v$

$a = \dfrac{dv}{dt} = 4e^{-2t}$

$= 2 \times \left(2e^{-2t}\right)$

$= 2(3 - v)$

13 a $v = x^n$

$\dfrac{dv}{dx} = nx^{n-1}$

$a = v\dfrac{dv}{dx} = nx^n \times x^{n-1}$

$= nx^{2n-1}$

b $v = \sqrt{b - n^2\, x^2}$

$= \left(b - n^2\, x^2\right)^{\frac{1}{2}}$

$\dfrac{dv}{dx} = \dfrac{1}{2} \times \left(-2n^2x\right) \times \left(b - n^2\, x^2\right)^{-\frac{1}{2}}$

$= \dfrac{-n^2x}{\sqrt{b - n^2\, x^2}}$

$a = v\dfrac{dv}{dx} = -n^2x$

14 $v = e^{nx} + e^{-nx}$

$\dfrac{dv}{dx} = ne^{nx} - ne^{-nx}$

$= n\left(e^{nx} - e^{-nx}\right)$

$a = v\dfrac{dv}{dx} = n\left(e^{nx} - e^{-nx}\right)\left(e^{nx} + e^{-nx}\right)$

$= n\left(e^{2nx} - 1 + 1 - e^{-2nx}\right)$

$= n\left(e^{2nx} - e^{-2nx}\right)$

15 $\ddot{x} = \dfrac{d\left(\frac{1}{2}v^2\right)}{dx} = -9x$

$\dfrac{1}{2}v^2 = \displaystyle\int -9x\, dx$

$= -\dfrac{9x^2}{2} + c_1$

$v = 3 \qquad t = 0 \qquad x = 1$

$\dfrac{9}{2} = -\dfrac{9}{2} + c_1 \quad \Rightarrow c_1 = 9$

$v^2 = 18 - 9x^2$

$= 9\left(2 - x^2\right)$

$v = \dfrac{dx}{dt} = 3\sqrt{2 - x^2}$

$t = \dfrac{1}{3}\displaystyle\int \dfrac{1}{\sqrt{2 - x^2}}dx$

$3t = \sin^{-1}\left(\dfrac{x}{\sqrt{2}}\right) + c_2$

$t = 0,\ x = 1$

$0 = \sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right) + c_2 \quad c_2 = -\dfrac{\pi}{4}$

$3t = \sin^{-1}\left(\dfrac{x}{\sqrt{2}}\right) - \dfrac{\pi}{4}$

$3t + \dfrac{\pi}{4} = \sin^{-1}\left(\dfrac{x}{\sqrt{2}}\right)$

$\dfrac{x}{\sqrt{2}} = \sin\left(3t + \dfrac{\pi}{4}\right)$

$= \sin(3t)\cos\left(\dfrac{\pi}{4}\right) + \cos(3t)\sin\left(\dfrac{\pi}{4}\right)$

$x = \sin(3t) + \cos(3t)$

16 $\ddot{x} = \dfrac{d\left(\frac{1}{2}v^2\right)}{dx} = -\dfrac{x}{4}, \quad v = 0, \quad x = 8, \quad t = 0$

$\dfrac{1}{2}v^2 = \displaystyle\int -\dfrac{x}{4}dx$

$= -\dfrac{x^2}{8} + c_1$

When $x = 8 \quad v = 0$

$0 = -8 + c_1 \ \Rightarrow \ c_1 = 8$

$v^2 = 16 - \dfrac{x^2}{4}$

$v^2 = \dfrac{1}{4}(64 - x^2)$

$v = \dfrac{dx}{dt} = \dfrac{1}{2}\sqrt{64 - x^2}$ since $v \geq 0$

$\displaystyle\int \dfrac{dx}{\sqrt{64 - x^2}} = \dfrac{1}{2}\int dt$

$\sin^{-1}\left(\dfrac{x}{8}\right) = \dfrac{t}{2} + c_2$

When $x = 8 \ \ t = 0 \ \ \sin^{-1}(1) = c_2 = \dfrac{\pi}{2}$

$$\sin^{-1}\left(\frac{x}{8}\right)=\frac{t}{2}+\frac{\pi}{2}$$

$$\frac{x}{8}=\sin\left(\frac{t}{2}+\frac{\pi}{2}\right)$$

$$=\sin\left(\frac{t}{2}\right)\cos\left(\frac{\pi}{2}\right)+\cos\left(\frac{t}{2}\right)\sin\left(\frac{\pi}{2}\right)$$

$$=0+1\times\cos\left(\frac{t}{2}\right)$$

$$x=8\cos\left(\frac{t}{2}\right)$$

When $x=4$ $\cos\left(\frac{t}{2}\right)=\frac{1}{2}$

$$\frac{t}{2}=2n\pi\pm\frac{\pi}{3}\quad n\in Z^{+}$$

$$t=\frac{2\pi}{3}(6n\pm1)\quad n\in Z^{+}$$

17 $\ddot{x}=-16x,\quad x=3,\quad v=0,\quad t=0$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx}=-16x$$

$$\frac{1}{2}v^2=\int -16x\,dx$$

$$=-8x^2+c_1,\quad x=3\ \ v=0$$

$$0=-8\times 9+c_1\quad\Rightarrow c_1=72$$

$$\frac{1}{2}v^2=-8x^2+72$$

$$v^2=144-16x^2$$

$$=16(9-x^2)$$

$$v=\frac{dx}{dt}=4\sqrt{9-x^2}$$

$$\int\frac{dx}{\sqrt{9-x^2}}=4\int dt$$

$$4t=\sin^{-1}\left(\frac{x}{3}\right)+c_2$$

$t=0\quad x=3$

$$0=\sin^{-1}(1)+c_2\Rightarrow c_2=-\frac{\pi}{2}$$

$$4t=\sin^{-1}\left(\frac{x}{3}\right)-\frac{\pi}{2}$$

$$\sin^{-1}\left(\frac{x}{3}\right)=4t+\frac{\pi}{2}$$

$$\frac{x}{3}=\sin\left(4t+\frac{\pi}{2}\right)$$

$$=\sin(4t)\cos\left(\frac{\pi}{2}\right)+\cos(4t)\sin\left(\frac{\pi}{2}\right)$$

$$x=3\cos(4t)$$

$x=0\quad\cos(4t)=0$

$$4t=2n\pi\pm\frac{\pi}{2}$$

$$4t=\frac{\pi}{2}(4n\pm1)$$

$$t=\frac{\pi}{8}(2n-1)\quad n\in Z^{+}$$

18 $\ddot{x}=\frac{d\left(\frac{1}{2}v^2\right)}{dx}=-\frac{x}{9},\quad v=6,\quad x=0,\quad t=0$

$$\frac{1}{2}v^2=\int -\frac{x}{9}\,dx$$

$$=-\frac{x^2}{18}+c_1$$

When $v=6,\quad x=0$

$$\frac{1}{2}\times 36=c_1\quad=18$$

$$v^2=36-\frac{x^2}{9}$$

$$=\frac{1}{9}(324-x^2)$$

$$v=\frac{dx}{dt}=\frac{1}{3}\sqrt{324-x^2}\text{ since }v>0$$

$$\int\frac{dx}{\sqrt{324-x^2}}=\frac{1}{3}\int dt$$

$$\sin^{-1}\left(\frac{x}{18}\right)=\frac{t}{3}+c_2$$

When $x=0\quad t=0$

$\Rightarrow c_2=0$

$$\sin^{-1}\left(\frac{x}{18}\right)=\frac{t}{3}$$

$$\frac{x}{18}=\sin\left(\frac{t}{3}\right)$$

$$x=18\sin\left(\frac{t}{3}\right)$$

$$x=9\quad\sin\left(\frac{t}{3}\right)=\frac{1}{2}$$

$$\frac{t}{3}=n\pi-\frac{\pi}{6},\quad\frac{\pi}{6}+2n\pi$$

$$t=\frac{\pi}{2}(12n+5),\quad t=\frac{\pi}{2}(12n+1)\quad n\in Z^{+}\cup$$

19 $\ddot{x}=-n^2x,\quad x=p,\quad v=0,\quad t=0$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right)=-n^2x$$

$$\frac{1}{2}v^2=-n^2\int xdx$$

$$=\frac{-n^2x^2}{2}+c_1$$

When $x=p\quad v=0$

$$0=\frac{-n^2p^2}{2}+c_1\quad\Rightarrow c_1=\frac{n^2p^2}{2}$$

$$v^2=n^2(p^2-x^2)$$

$$v=\frac{dx}{dt}\quad=\pm n\sqrt{p^2-x^2}\text{ take positive since }v\geq 0$$

$$\frac{dt}{dx}=\frac{1}{n\sqrt{p^2-x^2}}$$

$$nt=\int\frac{1}{\sqrt{p^2-x^2}}dx$$

$$=\sin^{-1}\left(\frac{x}{p}\right)+c_2$$

$$x=p,\quad t=0\quad 0=\sin^{-1}(1)+c_2$$

$$c_2=-\frac{\pi}{2}$$

$$nt=\sin^{-1}\left(\frac{x}{p}\right)-\frac{\pi}{2}$$

$$\sin^{-1}\left(\frac{x}{p}\right)=nt+\frac{\pi}{2}$$

$$\frac{x}{p}=\sin\left(nt+\frac{\pi}{2}\right)$$

$$=\sin(nt)\cos\left(\frac{\pi}{2}\right)+\cos(nt)\sin\left(\frac{\pi}{2}\right)$$

$$x=p\cos(nt)$$

20 $\ddot{x} = -\frac{k}{x^3} \qquad v = 0, x = p$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = -kx^{-3}$$

$$\frac{1}{2}v^2 = \int -kx^{-3}dx$$

$$= \frac{k}{2}x^{-2} + c_1$$

$$x = p \quad v = 0$$

$$0 = \frac{k}{2p^2} + c_1 \quad c_1 = -\frac{k}{2p^2}$$

$$\frac{1}{2}v^2 = \frac{k}{2x^2} - \frac{k}{2p^2}$$

$$= \frac{k(p^2 - x^2)}{2x^2p^2}$$

$$v = \frac{dx}{dt} \quad = \frac{\sqrt{k(p^2 - x^2)}}{px}$$

21 $\ddot{x} = \frac{d\left(\frac{1}{2}v^2\right)}{dx} = p(px + q) \qquad v = 2q, x = \frac{q}{p}, t = 0$

$$\frac{1}{2}v^2 = \int (p^2x + qp)\,dx$$

$$= \frac{p^2x^2}{2} + qpx + c_1$$

$$v = 2q, \quad x = \frac{q}{p}$$

$$\frac{1}{2} \times 4q^2 = \frac{p^2}{2} \times \frac{q^2}{p^2} + qp \times \frac{q}{p} + c_1$$

$$2q^2 = \frac{q^2}{2} + q^2 + c_1$$

$$c_1 = \frac{q^2}{2}$$

$$v^2 = p^2x^2 + 2pqx + q^2$$

$$v^2 = (px + q)^2$$

$$v = \pm(px + q)$$

$$v = \frac{dx}{dt} = (px + q)$$

$$t = \int \frac{1}{px + q}\,dx$$

$$= \frac{1}{p}\log_e(|px + q|) + c_2$$

$$t = 0 \quad x = \frac{q}{p}$$

$$0 = \frac{1}{p}\log_e(2q) + c_2$$

$$c_2 = -\frac{1}{p}\log_e(2q)$$

$$pt = \log_e\left(\frac{px + q}{2q}\right)$$

$$x = \frac{q}{p}(2e^{pt} - 1)$$

22 **a** $\ddot{x} = 8x(x^2 - 9), \qquad v = 10, t = 0, x = 2$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = 8x^3 - 72x$$

$$\frac{1}{2}v^2 = \int (8x^3 - 72x)\,dx$$

$$= 2x^4 - 36x^2 + c_1$$

When $t = 0 \quad v = 10, \quad x = 2$

$$50 = 32 - 144 + c_1 \quad \Rightarrow c_1 = 162$$

$$\frac{1}{2}v^2 = 2x^4 - 36x^2 + 162$$

$$\frac{1}{2}v^2 = 2(9 - x^2)^2$$

$$v^2 = 4(9 - x^2)^2$$

$$v = 2(9 - x^2)$$

$$\frac{dx}{dt} = 2(9 - x^2)$$

$$\int \frac{dx}{9 - x^2} = 2\int dt$$

$$\frac{A}{3 - x} + \frac{B}{3 + x} = \frac{A(3 + x) + B(3 - x)}{9 - x^2} \quad = \frac{x(A - B) + 3(A + B)}{9 - x^2}$$

$$3(A + B) = 1$$

$$A - B = 0 \quad \Rightarrow A = B = \frac{1}{6}$$

$$\frac{1}{6}\int \left(\frac{1}{3 - x} + \frac{1}{3 + x}\right)dx = 2\int dt$$

$$\log_e\left|\frac{x + 3}{x - 3}\right| = 12t + c$$

When $x = 2, \quad t = 0 \quad c = \log_e(5)$

$$\log_e\left(\left|\frac{x + 3}{x - 3}\right|\right) = 12t + \log_e(5)$$

$$\log_e\left(\frac{x + 3}{5\,|x - 3|}\right) = 12t$$

$$\log_e\left(\frac{5\,|x - 3|}{x + 3}\right) = -12t$$

Since $x = 2 \quad |x - 3| = 3 - x$

$$e^{-12t} = \frac{5(3 - x)}{x + 3}$$

$$5(3 - x) = (x + 3)e^{-12t}$$

$$15 - 5x = xe^{-12t} + 3e^{-12t}$$

$$15 - 3e^{-12t} = 5x + xe^{-12t}$$

$$3(5 - e^{-12t}) = x(5 + e^{-12t})$$

$$x(t) = \frac{3(5 - e^{-12t})}{5 + e^{-12t}}$$

b As $t \to \infty \quad e^{-12t} \to 0 \quad x \to 3\,\text{m}$

23 $\ddot{x} = \frac{d\left(\frac{1}{2}v^2\right)}{dx} = -\frac{8}{x^2}, \qquad v = 0, x = 4$

$$\frac{1}{2}v^2 = \int -8x^{-2}dx$$

$$= 8x^{-1} + c_1$$

$$v = 0 \quad x = 4$$

$$0 = \frac{8}{4} + c_1 \Rightarrow c_1 = -2$$

$$\frac{1}{2}v^2 = \frac{8}{x} - 2$$

$$v^2 = \frac{16}{x} - 4$$

When $v = 0 \quad \frac{16}{x} - 4 = 0$

$$\frac{16}{x} = 4$$
$$x = 4$$
$$v^2 = \frac{16 - 4x}{x}$$
$$= \frac{4(4 - x)}{x}$$
$$v = -2\sqrt{\frac{4 - x}{x}} = \frac{dx}{dt}, \quad \text{since} \quad v \le 0$$
$$\frac{dt}{dx} = \frac{-1}{2}\sqrt{\frac{x}{4 - x}}$$
$$T = \frac{-1}{2}\int_4^0 \sqrt{\frac{x}{4 - x}}\, dx = 3.14 \text{ s}$$

24 $\ddot{x} = -\frac{k}{x^2}$ when $v = 0, x = p$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = -\frac{k}{x^2}$$
$$\frac{1}{2}v^2 = \int -\frac{k}{x^2}dx$$
$$= \frac{k}{x} + c$$

When $v = 0 \quad x = p$

$$0 = \frac{k}{p} + c \Rightarrow c = -\frac{k}{p}$$
$$\frac{1}{2}v^2 = \frac{k}{x} - \frac{k}{p}$$
$$\frac{1}{2}v^2 = \frac{k(p - x)}{px}$$
$$v^2 = \frac{2k(p - x)}{px}$$
$$v = \sqrt{\frac{2k(p - x)}{px}}$$

11.7 Exam question

1 $a = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$

Given $a = -4x$,

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = -4x$$
$$\frac{1}{2}v^2 = \int -4x\, dx$$
$$\frac{1}{2}v^2 = -2x^2 + c$$

Given $v = 0,\ x = 5$,

$0 = -50 + c$

$c = 50$

$$\frac{1}{2}v^2 = -2x^2 + 50$$
$$v^2 = 100 - 4x^2$$

When $x = 3$,

$v^2 = 100 - 4 \times 9$

$v^2 = 64$

$|v| = 8$

The correct answer is **A**.

2 $\ddot{x} = \frac{d\left(\frac{1}{2}v^2\right)}{dx} = -kx$

$\frac{1}{2}v^2 = -k\int x\, dx$ [1 mark]

$$= -\frac{kx^2}{2} + c_1$$

$v(0) = U \Rightarrow \frac{1}{2}U^2 = c_1$

$\frac{1}{2}v^2 = \frac{-kx^2}{2} + \frac{1}{2}U^2$ [1 mark]

Comes to rest when $v = 0$

$$\frac{1}{2}U^2 = \frac{kx^2}{2}$$

$\Rightarrow x = U\sqrt{\frac{1}{k}}$ [1 mark]

3 $v = b - ne^{-nt} \Rightarrow ne^{-nt} = b - v$ [1 mark]

$$a = \frac{dv}{dt} = n^2e^{-nt}$$
$$= n\left(ne^{-nt}\right)$$
$$= n(b - v)$$

$= -n(v - b)$ [1 mark]

11.8 Review

11.8 Exercise

Technology free: short answer

1 $v(t) = \frac{1}{6}(t^2 - t - 2)$

a $\frac{dx}{dt} = \frac{1}{6}\left(t^2 - t - 2\right)$

$$x = \frac{1}{6}\int \left(t^2 - t - 2\right) dt$$
$$= \frac{1}{6}\left(\frac{t^3}{3} - \frac{t^2}{2} - 2t + c\right)$$

$x = -3, \quad t = 0 \quad c = -18$

$$x(t) = \frac{1}{6}\left(\frac{t^3}{3} - \frac{t^2}{2} - 2t - 18\right)$$
$$= \frac{1}{36}(2t^3 - 3t^2 - 12t - 108) \text{ m}$$

b At rest $v = 0 \quad t^2 - t - 2 = 0$

$(t - 2)(t + 1) = 0$

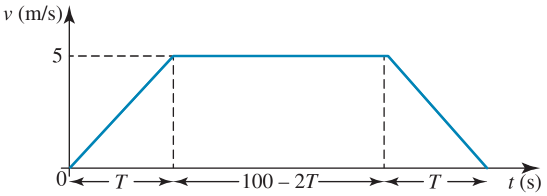
$t = 2$

$$x(2) = \frac{1}{36}(2 \times 8 - 3 \times 4 - 24 - 108)$$
$$= \frac{-32}{9} \text{ m}$$

c $a(t) = \frac{dv}{dt} = \frac{1}{6}(2t - 1)$

$$a(2) = \frac{1}{2} \text{ m/s}^2$$

2



$$\frac{1}{2}\times T\times 5+(100-2T)\times 5+\frac{1}{2}\times 5\times T=400$$

$$\frac{5T}{2}+5(100-2T)+\frac{5T}{2}=400$$

$$5T+500-10T=400$$

$$5T=100$$

$$T=20\,\text{s}$$

$$\text{Acceleration }\ \frac{5}{20}=0.25\text{ m/s}^2$$

3 $\ddot{x}=\dfrac{dv}{dt}=\dfrac{-24}{(3t+1)^{\frac{3}{2}}}$

$$v=-24\int(3t+1)^{-\frac{3}{2}}dt$$

$$=-\frac{24}{3}\times(-2)\times(3t+1)^{-\frac{1}{2}}+c_1$$

$$v=\frac{16}{\sqrt{3t+1}}+c_1$$

When $t=0, v=16\Rightarrow c_1=0$

$$v=\frac{dx}{dt}=\frac{16}{\sqrt{3t+1}}$$

$$x=16\int(3t+1)^{-\frac{1}{2}}dt$$

$$=\frac{16\times 2}{3}(3t+1)^{\frac{1}{2}}+c_2$$

$$=\frac{32}{3}\sqrt{3t+1}+c_2$$

$x=0\quad t=0\quad 0=\dfrac{32}{3}+c_2\Rightarrow c_2=-\dfrac{32}{3}$

$$x(t)=\frac{32}{3}(\sqrt{3t+1}-1)$$

When $t=5$

$$x(5)=\frac{32}{3}(4-1)$$

$$=32\,\text{m}$$

4 $\ddot{x}=\dfrac{dv}{dt}=-(16\cos(2t)+12\sin(2t))$

$$v=\int(-16\cos(2t)-12\sin(2t))\,dt$$

$$=-8\sin(2t)+6\cos(2t)+\mathrm{c}_1$$

$$v(0)=6=0+6+c_1$$

$$\Rightarrow c_1=0$$

$$v=\frac{dx}{dt}=-8\sin(2t)+6\cos(2t)$$

$$x=\int(-8\sin(2t)+6\cos(2t))\,dt$$

$$=4\cos(2t)+3\sin(2t)+c_2$$

$x(0)=4\quad=4+c_2$

$\Rightarrow c_2=0$

$$x(t)=4\cos(2t)+3\sin(2t)$$

$$x\left(\frac{\pi}{8}\right)=4\cos\left(\frac{\pi}{4}\right)+3\sin\left(\frac{\pi}{4}\right)$$

$$=4\frac{\sqrt{2}}{2}+3\frac{\sqrt{2}}{2}$$

$$=\frac{7\sqrt{2}}{2}\text{m}$$

5 $\ddot{x}=\dfrac{dv}{dt}=-\dfrac{112}{15\sqrt{v}},\quad u=16\text{ m/s},\quad v=4\text{ m/s}$

$$\frac{dt}{dv}=\frac{-15\sqrt{v}}{112}$$

$$t=-\frac{15}{112}\int v^{\frac{1}{2}}dv$$

$$=-\frac{15}{112}\times\frac{2}{3}v^{\frac{3}{2}}+c_1$$

$$t=-\frac{5}{56}v^{\frac{3}{2}}+c_1$$

When $t=0\quad v=16$

$$0=-\frac{5}{56}(16)^{\frac{3}{2}}+c_1\Rightarrow c_1=\frac{40}{7}$$

$$t=-\frac{5}{56}v^{\frac{3}{2}}+\frac{40}{7}$$

When $v=4$

$$t=\frac{40}{7}-\frac{5}{56}(4)^{\frac{3}{2}}=5\,\text{sec}$$

$$T=5\,\text{sec}$$

$$v\frac{dv}{dx}=\frac{-112}{15\sqrt{v}}$$

$$\sqrt{v}\,v\,dv=\frac{-112}{15}dx$$

$$\frac{-112}{15}x=\int v^{\frac{3}{2}}dv$$

$$=\frac{2}{5}v^{\frac{5}{2}}+c_2$$

When $v=16\quad x=0$

$$0=\frac{2}{5}(16)^{\frac{5}{2}}+c_2\Rightarrow c_2=-\frac{2048}{5}$$

$$\frac{-112x}{15}=\frac{2}{5}v^{\frac{5}{2}}-\frac{2048}{5}$$

When $v=4$

$$\frac{-112x}{15}=\frac{2}{5}(4)^{\frac{5}{2}}-\frac{2048}{5}$$

$$=\frac{64-2048}{5}=\frac{-1984}{5}$$

$$x=\frac{15\times 1984}{112\times 5}$$

$$=\frac{372}{7}$$

$$D=53\frac{1}{7}\text{ m}$$

6 $\ddot{x}=\dfrac{d\left(\frac{1}{2}v^2\right)}{dx}=\dfrac{17x-12}{x^3}$

$$\frac{1}{2}v^2=\int(17x^{-2}-12x^{-3})dx$$

$$=-17x^{-1}+6x^{-2}+c\ =\frac{-17}{x}+\frac{6}{x^2}+c$$

When $x=3\quad v=0$

$$0=-\frac{17}{3}+\frac{6}{9}+c$$

$$\Rightarrow c=5$$

$$v^2=2\left(\frac{6}{x^2}-\frac{17}{x}+5\right)$$

$$=\frac{2(5x^2-17x+6)}{x^2}$$

$$v=\frac{\sqrt{2(x-3)(5x-2)}}{x}$$

Comes to rest when $v = 0$

$\Rightarrow x = 3$ and $x = \dfrac{2}{5} = 0.4$ m

Technology active: multiple choice

7 $s = 1500\,\text{m}, \quad t = 100, \quad u = 0$

$$s = ut + \frac{1}{2}at^2$$

$$1500 = 0 + \frac{1}{2} \times a \times 100^2$$

$$a = \frac{2 \times 1500}{100^2} = 0.3$$

The correct answer is **D**.

8 $u = 21 \quad v = 6 \quad s = 200 \quad a = ?$

$$v^2 = u^2 + 2as$$

$$6^2 = 21^2 + 2a \times 200$$

$$a = \frac{6^2 - 21^2}{2 \times 200} = -1.0125$$

$u = 21 \quad v = 0 \quad a = -1.0125 \quad s = ?$

$$v^2 = u^2 + 2as$$

$$0 = 21^2 - 2 \times 1.0125 \times s$$

$$s = \frac{21^2}{2 \times 1.0125} = 217.78$$

Extra 17.78

The correct answer is **C**.

9 $u = 5, \quad v = 0 \quad a = -g \quad s = ?$

$$v^2 = u^2 + 2as$$

$$0 = 25 - 2 \times 9.8 \times s$$

$$s = \frac{25}{2 \times 9.8} = 1.276$$

The correct answer is **B**.

10 $a = \dfrac{dv}{dt} = 2t$

$$v = \int 2t\, dt$$

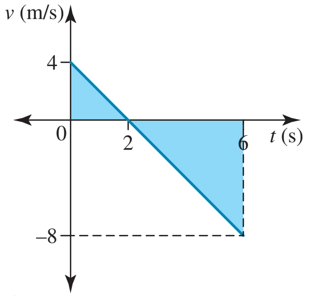
$$v = t^2 + c$$

$$t = 0, v = u \Rightarrow c = u$$

$$v = u + t^2$$

The correct answer is **A**.

11



$$\frac{1}{2} \times 4 \times 8 = 16$$

$$4 + 16 = 20$$

The correct answer is **D**.

12 $\dfrac{dv}{dt} = v$

$$t = \int \frac{1}{v} dv$$

$$t = \log_e(v) + c$$

$t = 0 \qquad v = u \qquad c = -\log_e(u)$

$$t = \log_e\left(\frac{v}{u}\right)$$

$$\frac{v}{u} = e^t \quad v = ue^t$$

The correct answer is **B**.

13 $v = \sin\left(\dfrac{x}{2}\right)$

$$v^2 = \sin^2\left(\frac{x}{2}\right)$$

$$\frac{1}{2}v^2 = \frac{1}{2}\sin^2\left(\frac{x}{2}\right)$$

$$a = \frac{d\left(\frac{1}{2}v^2\right)}{dx} = \frac{d}{dx}\left(\frac{1}{2}\sin^2\left(\frac{x}{2}\right)\right)$$

$$= \frac{1}{\cancel{2}} \times \cancel{2} \times \frac{1}{2}\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$$

$$= \frac{1}{4}\left(2\sin\left(\frac{x}{2}\right)\ \cos\left(\frac{x}{2}\right)\right)$$

$$= \frac{1}{4}\sin(x)$$

The correct answer is **E**.

14 $a = \dfrac{dv}{dt} = a(t)$

$$\int_{v_1}^{v_2} 1\, dv = \int_{t_1}^{t_2} a(t)\, dt$$

$$v_2 - v_1 = \int_{t_1}^{t_2} a(t)\, dt$$

The correct answer is **E**.

15 $v = \dfrac{dv}{dt} = a(v)$

$$\frac{dt}{dv} = \frac{1}{a(v)}$$

$$\int_{t_1}^{t_2} 1\, dt = \int_{v_1}^{v_2} \frac{1}{a(v)}\, dv$$

$$t_2 - t_1 = \int_{v_1}^{v_2} \frac{1}{a(v)}\, dv$$

The correct answer is **C**.

16 $v\dfrac{dv}{dx} = a(x)$

$$\int_{v_1}^{v_2} v\, dv = \int_{x_1}^{x_2} a(x)\, dx$$

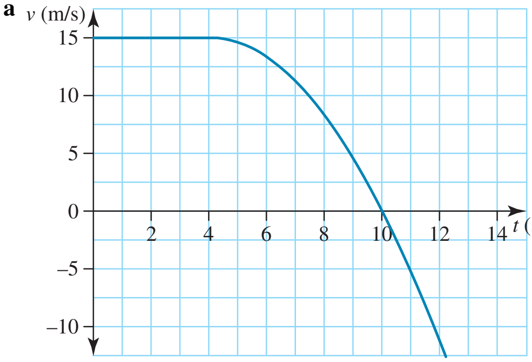
$$\frac{1}{2}v_2^2 - \frac{1}{2}v_1^2 = \int_{x_1}^{x_2} a(x)\, dx$$

The correct answer is **A**.

Technology active: extended response

17 $v(t) = \begin{cases} 15 & 0 \le t \le 4 \\ 15 - \dfrac{5}{12}(t-4)^2 & 4 < t \le 12 \end{cases}$

a



b $d_1 = \int_0^4 15\,dt = 60$

$$d_2 = \int_4^{10} \left(15 - \frac{5}{12}(t-4)^2\right) dt$$
$$= \left[15t - \frac{5}{36}(t-4)^3\right]_4^{10}$$
$$= 150 - \frac{5}{36} \times 6^3 - 60 + 0$$
$$= 60$$

$$d_3 = \int_{10}^{12} \left(15 - \frac{5}{12}(t-4)^2\right) dt$$
$$= \left[15t - \frac{5}{36}(t-4)^3\right]_{10}^{12}$$
$$= 15 \times 12 - \frac{5}{36} \times 8^3 - 150 + \frac{5}{36} \times 6^3$$
$$= \frac{-100}{9} = -11\frac{1}{9}$$

Distance travelled

$$= 60 + 60 + \frac{100}{9} = \frac{1180}{9}$$
$$= 131\frac{1}{9} \text{ m}$$

c Displacement $= 60 + 60 - \dfrac{100}{9} = \dfrac{980}{9}$

$$= 108\frac{8}{9} \text{ m}$$

d $a(t) = \dfrac{dv}{dt} = \begin{cases} 0 & 0 < t < 4 \\ -\dfrac{5}{6}(t-4) & 4 < t < 12 \end{cases}$

$$a(6) = -\frac{5}{6}(6-4)$$
$$= -\frac{5}{3} \text{ m/s}^2$$

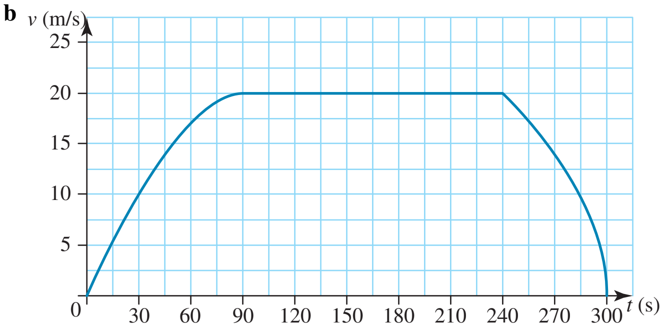
18 **a** $v(t) = \begin{cases} a \sin\left(\dfrac{\pi t}{180}\right) & 0 \le t \le 90 \\ 20 & 90 < t \le 240 \\ b \cos^{-1}\left(\dfrac{t-240}{60}\right) & 240 < t \le 300 \end{cases}$

$t = 90 \quad a \sin\left(\dfrac{\pi}{2}\right) = 20 \quad a = 20$

$t = 240 \quad b \cos^{-1}(0) = b \times \dfrac{\pi}{2} = 20$

$$b = \frac{40}{\pi}$$

b



c $d_1 = \int_0^{90} 20 \sin\left(\dfrac{\pi t}{180}\right) dt$

$$= \left[20 \times -\frac{180}{\pi} \cos\left(\frac{\pi t}{180}\right)\right]_0^{90}$$
$$= -\frac{3600}{\pi}\left(\cos\left(\frac{\pi}{2}\right) - \cos(0)\right)$$
$$= \frac{3600}{\pi}$$

$d_2 = 20 \times 150 = 3000$

Over $0 \le t \le 240$

$D = \dfrac{3600}{\pi} + 3000 \text{ m}$

d $\displaystyle\int_{240}^{300} \frac{40}{\pi} \cos^{-1}\left(\frac{t-240}{60}\right) dt = 763.944$

Total distance $= \dfrac{3600}{\pi} + 3000 + 763.944$

$= 4909.86 \text{ m}$

$= 4910 \text{ m}$

e $a(t) = \begin{cases} \dfrac{\pi}{9} \cos\left(\dfrac{\pi t}{180}\right) & 0 < t < 90 \\ 0 & 90 < t < 240 \\ \dfrac{-40}{\pi\sqrt{480t - t^2 - 54000}} & 240 < t < 300 \end{cases}$

$a(45) = \dfrac{\pi\sqrt{2}}{18} \text{ m/s}^2$

f $a(270) = \dfrac{-4\sqrt{3}}{9\pi} \text{ m/s}^2$

19 Proportional to the squre root of the speed:

$v = U, \quad t = 0; \quad v = V, \quad t = T, \quad x = D$

$\ddot{x} = -\lambda\sqrt{v}$

Use $\ddot{x} = \dfrac{dv}{dt}$

$$\frac{dv}{dt} = -\lambda\sqrt{v}$$

$$\int \frac{1}{\sqrt{v}} dv = -\lambda \int dt$$

$$\int v^{-\frac{1}{2}} dv = -\lambda t + c_1$$

$$2v^{\frac{1}{2}} = -\lambda t + c_1$$

When $t = 0 \quad v = U$

$2\sqrt{U} = c_1$

$2\sqrt{v} = -\lambda t + 2\sqrt{U}$

When $t = T$, $v = V$

$$2\sqrt{V} = -\lambda T + 2\sqrt{U}$$

$$\lambda T = 2\left(\sqrt{U} - \sqrt{V}\right)$$

$$\lambda = 2\frac{\left(\sqrt{U} - \sqrt{V}\right)}{T}$$

Use $\ddot{x} = v\dfrac{dv}{dx}$

$$v\,\frac{dv}{dx} = -\lambda\sqrt{v}$$

$$\int \frac{v\,dv}{\sqrt{v}} = -\lambda \int dx$$

$$\int v^{\frac{1}{2}}dv = -\lambda x + c_2$$

$$\frac{2}{3}v^{\frac{3}{2}} = -\lambda x + c_2$$

When $x = 0$ $v = U$

$$\frac{2}{3}U^{\frac{3}{2}} = c_2$$

$$\frac{2}{3}v^{\frac{3}{2}} = -\lambda x + \frac{2}{3}U^{\frac{3}{2}}$$

When $v = V, x = D$

$$\frac{2}{3}V^{\frac{3}{2}} = -\lambda D + \frac{2}{3}U^{\frac{3}{2}}$$

$$\lambda D = \frac{2}{3}\left(U^{\frac{3}{2}} - V^{\frac{3}{2}}\right)$$

$$\lambda = \frac{2\left(U^{\frac{3}{2}} - V^{\frac{3}{2}}\right)}{3D}$$

$$\frac{2\left(\sqrt{U} - \sqrt{V}\right)}{T} = \frac{2}{3D}\left(U^{\frac{3}{2}} - V^{\frac{3}{2}}\right)$$

$$\frac{D}{T} = \frac{U^{\frac{3}{2}} - V^{\frac{3}{2}}}{3\left(\sqrt{U} - \sqrt{V}\right)} = \frac{\left(\sqrt{U}\right)^3 - \left(\sqrt{V}\right)^3}{3\left(\sqrt{U} - \sqrt{V}\right)}$$

$$= \frac{\cancel{\left(\sqrt{U} - \sqrt{V}\right)}}{3\cancel{\left(\sqrt{U} - \sqrt{V}\right)}}\left(U + \sqrt{UV} + V\right)$$

$$\frac{D}{T} = \frac{U + \sqrt{UV} + V}{3}$$

20 **a** $\ddot{x} = -\lambda v, t = 0, v = U; t = T, v = \dfrac{U}{2}, s = D$

Use $\ddot{x} = \dfrac{dv}{dt} = -\lambda v$

$$\int \frac{1}{v}dv = \int -\lambda\,dt$$

$$\log_e(v) = -\lambda t + c_1$$

$t = 0$, $v = U$

$$\log_e(U) = c_1$$

$$\log_e(v) = -\lambda t + \log_e(U)$$

When $t = T, v = \dfrac{1}{2}U$

$$\log_e\left(\frac{1}{2}U\right) = -\lambda T + \log_e(U)$$

$$\lambda T = \log_e(U) - \log_e\left(\frac{1}{2}U\right)$$

$$\lambda T = \log_e\left(\frac{U}{\frac{1}{2}U}\right)$$

$$= \log_e(2)$$

$$\lambda = \frac{1}{T}\log_e(2)$$

Use $\ddot{x} = v\dfrac{dv}{dx} = -\lambda v$

$$\frac{dv}{dx} = -\lambda$$

$$v = \int -\lambda\,dx$$

$$v = -\lambda x + c_2$$

$x = 0$ $v = U$

$$c_2 = U$$

$$v = -\lambda x + U$$

When $x = D$ $v = \dfrac{1}{2}U$

$$\frac{1}{2}U = -\lambda D + U$$

$$\lambda D = U - \frac{1}{2}U$$

$$\lambda D = \frac{U}{2}$$

$$\lambda = \frac{U}{2D}$$

$$\frac{1}{\mathrm{T}}\log_e(2) = \frac{U}{2D}$$

$$\frac{D}{T} = \frac{U}{2\log_e(2)} \qquad \text{shown}$$

b $\ddot{x} = -\lambda v^2$

Use $\ddot{x} = \dfrac{dv}{dt} = -\lambda v^2$

$$\int v^{-2}dv = -\lambda \int dt$$

$$-\frac{1}{v} = -\lambda t + c_1$$

When $v = U$, $t = 0$

$$-\frac{1}{U} = c_1$$

$$-\frac{1}{v} = -\lambda t - \frac{1}{U}$$

$$\frac{1}{v} = \lambda t + \frac{1}{U}$$

When $v = \dfrac{1}{2}U$, $t = T$

$$\frac{2}{U} = \lambda T + \frac{1}{U}$$

$$\lambda T = \frac{2}{U} - \frac{1}{U}$$

$$= \frac{1}{U}$$

$$\lambda = \frac{1}{UT}$$

Use $\ddot{x} = v\frac{dv}{dx} = -\lambda v^2$

$$\frac{dv}{dx} = -\lambda v$$

$$\int \frac{1}{v}dv = \int -\lambda\, dx$$

$$\log_e(v) = -\lambda x + c_2$$

$$v = U, \quad x = 0$$

$$\log_e(U) = c_2$$

$$\log_e(v) = -\lambda x + \log_e(U)$$

When $v = \frac{1}{2}U, \quad x = D$

$$\log_e\left(\frac{1}{2}U\right) = -\lambda D + \log_e(U)$$

$$\lambda D = \log_e(U) - \log_e\left(\frac{1}{2}U\right)$$

$$= \log_e\left(\frac{U}{\frac{1}{2}U}\right)$$

$$\lambda D = \log_e(2)$$

$$\lambda = \frac{1}{D}\log_e(2)$$

$$\frac{1}{UT} = \frac{1}{D}\log_e(2)$$

$$\frac{D}{T} = U\log_e(2)$$

c $\ddot{x} = -\lambda\sqrt{v}$

Use $\ddot{x} = \frac{dv}{dt} = -\lambda\sqrt{v}$

$$\int \frac{1}{\sqrt{v}}\,dv = \int -\lambda\, dt$$

$$\int v^{-\frac{1}{2}}dv = -\lambda t + c_1$$

$$2v^{\frac{1}{2}} = -\lambda t + c_1$$

$$2\sqrt{v} = -\lambda t + c_1$$

When $t = 0 \quad v = U$

$$2\sqrt{U} = c_1$$

$$2\sqrt{v} = -\lambda t + 2\sqrt{U}$$

When $t = T, \quad v = \frac{1}{2}U$

$$2\sqrt{\frac{1}{2}U} = -\lambda T + 2\sqrt{U}$$

$$\frac{2}{\sqrt{2}}\sqrt{U} = -\lambda T + 2\sqrt{U}$$

$$\sqrt{2U} = -\lambda T + 2\sqrt{U}$$

$$\lambda T = (2-\sqrt{2})\sqrt{U}$$

$$\lambda = \frac{(2-\sqrt{2})\sqrt{U}}{T}$$

use $\ddot{x} = v\frac{dv}{dx} = -\lambda\sqrt{v}$

$$\int \frac{v}{\sqrt{v}}dv = \int -\lambda\, dx$$

$$\int v^{\frac{1}{2}}dv = -\lambda x + c_2$$

$$\frac{2}{3}v^{\frac{3}{2}} = -\lambda x + c_2$$

$$x = 0 \quad v = U$$

$$c_2 = \frac{2}{3}U^{\frac{3}{2}}$$

$$\frac{2}{3}v^{\frac{3}{2}} = -\lambda x + \frac{2}{3}U^{\frac{3}{2}}$$

when $v = \frac{1}{2}U, \quad x = D$

$$\frac{2}{3}\left(\frac{1}{2}U\right)^{\frac{3}{2}} = -\lambda D + \frac{2}{3}U^{\frac{3}{2}}$$

$$\frac{2}{3} \times \frac{1}{2\sqrt{2}}U^{\frac{3}{2}} = -\lambda D + \frac{2}{3}U^{\frac{3}{2}}$$

$$\lambda D = \frac{2}{3}U^{\frac{3}{2}}\left(1 - \frac{1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\right)$$

$$\lambda D = \frac{2}{3}U^{\frac{3}{2}}\left(1 - \frac{\sqrt{2}}{4}\right)$$

$$\lambda = \frac{2}{3}U^{\frac{3}{2}}\left(\frac{4-\sqrt{2}}{4D}\right)$$

$$\frac{\left(2-\sqrt{2}\right)\sqrt{U}}{T} = \frac{U^{\frac{3}{2}}\left(4-\sqrt{2}\right)}{6D}$$

$$\frac{D}{T} = U\left(\frac{4-\sqrt{2}}{6(2-\sqrt{2})}\right) \times \frac{2+\sqrt{2}}{2+\sqrt{2}}$$

$$= \frac{U}{6}\left[\frac{8-2\sqrt{2}+4\sqrt{2}-2}{4-2}\right]$$

$$= \frac{U\left(6+2\sqrt{2}\right)}{12}$$

$$\frac{D}{T} = \frac{U\left(3+\sqrt{2}\right)}{6} \quad \text{shown}$$

d Constant acceleration

$$v = \frac{1}{2}U, \quad u = U, \quad s = D, \quad t = T$$

$$\begin{aligned}
v &= u + at\\
\frac{1}{2}U &= U + aT\\
aT &= \frac{1}{2}U - U\\
&= -\frac{U}{2}\\
a &= -\frac{U}{2T}\\
v^2 &= u^2 + 2as\\
\left(\frac{1}{2}U\right)^2 &= U^2 + 2aD\\
\frac{1}{4}U^2 &= U^2 + 2aD\\
2aD &= \frac{1}{4}U^2 - U^2\\
2aD &= -\frac{3U^2}{4}\\
a &= -\frac{3U^2}{8D}\\
\Rightarrow \frac{U}{2T} &= \frac{3U^2}{8D}\\
\frac{D}{T} &= \frac{3U}{4}
\end{aligned}$$

11.8 Exam questions

1 a $\frac{dx}{dt} = x\sin(t), \quad x(0) = 1$

$\int \frac{1}{x}dx = \int \sin(t)dt$

$\log_e(|x|) = -\cos(t) + c$

$\log_e(1) = -\cos(0) + c, \quad c = 1$

$\log_e(|x|) = 1 - \cos(t)$

$|x| = e^{1-\cos(t)}$

$x = \pm e^{1-\cos(t)}$

since $x = 1$, $t = 0$ take positive

$x = e^{1-\cos(t)}$

Award 1 mark for correctly separating the variables.

Award 1 mark for correct integrals and finding the constant of integration.

Award 1 mark for the final correct expression for the dispalcement.

b $x_{\text{max}} = e^2$ and occurs when

$\cos(t) = -1, \quad t = \pi, 3\pi, 5\pi, \ldots\ldots = (2n-1)\pi, \quad n \in Z^+ = \{1, 2, 3\ldots\}$, since $t \geq 0$

Award 1 mark for the correct maximum displacement.

Award 1 mark for the correct values of the times.

2 $v = e^x\sin(x)$

Using the product rule:

$\frac{dv}{dx} = e^x\sin(x) + e^x\cos(x) = e^x(\sin(x) + \cos(x))$

$a = v\frac{dv}{dx} = e^{2x}\sin(x)(\sin(x) + \cos(x)) = e^{2x}\left(\sin^2(x) + \sin(x)\cos(x)\right)$

$a = e^{2x}\left(\sin^2(x) + \frac{1}{2}\sin(2x)\right)$

The correct answer is **A**.

3 a $\frac{d^2x}{dt^2} = 1.4\left(7 - \frac{dx}{dt}\right)$ using $\frac{d^2x}{dt^2} = v\frac{dv}{dx}$

$$\begin{aligned}
v\frac{dv}{dx} &= 1.4(7 - v)\\
\frac{dv}{dx} &= \frac{1.4(7 - v)}{v}\\
\frac{dx}{dv} &= \frac{v}{1.4(7 - v)}\\
1.4\frac{dx}{dv} &= \frac{v}{7 - v} = \frac{v - 7 + 7}{7 - v}\\
1.4\frac{dx}{dv} &= -1 + \frac{7}{7 - v}
\end{aligned}$$

Award 1 mark for equating using the correct form for acceleration and inverting.

Award 1 mark for the correct verification.

***VCAA Assessment Report* note**

Many students managed to get the correct form for acceleration and inverted, but omitted the step to establish the final form, which could have been division or another valid method.

b $1.4\int dx = \int\left(-1 + \frac{7}{7 - v}\right)dv$

$1.4x = -v - 7\log_e(|7 - v|) + c$ but $0 \leq v < 7$

and $v(0) = 0 \Rightarrow c = 7\log_e(7)$

$1.4x = -v - 7\log_e(7 - v) + 7\log_e(7)$

Award 1 mark for correctly showing the required result.

***VCAA Assessment Report* note**

In this 'show that' question, many students did not include the constant of integration or show its evaluation.

c When $v = 5$, $x = D$

$1.4D = -5 - 7\log_e(7 - 5) + 7\log_e(7)$

$D = \frac{1}{1.4}\left(7\log_e\left(\frac{7}{2}\right) - 5\right)$

$D = 2.7\text{ m}$

Award 1 mark for the correct value.

***VCAA Assessment Report* note:**

A number of students did not attempt this question. A few students incorrectly substituted $v = -5$.

d $a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = 1.4(7 - v)$

$\frac{dt}{dv} = \frac{1}{1.4(7 - v)}$

$$\begin{aligned}
t &= \frac{1}{1.4}\int_0^5 \frac{1}{7 - v}dv\\
&= \frac{1}{1.4}\left[-\log_e(7 - v)\right]_0^5\\
&= \frac{1}{1.4}\left(-\log_e(2) + \log_e(7)\right) = \frac{1}{1.4}\log_e\left(\frac{7}{2}\right) = 0.895
\end{aligned}$$

$t = 0.9\text{ s}$

Award 1 mark for inverting.

Award 1 mark for setting up and solving a definite integral for the time.

Award 1 mark for the correct value of the time.

***VCAA Assessment Report* note:**

Only a small number of students attempted this question. Some attempted to find the time using direct integration instead of a definite integral.

4 $a = v\dfrac{dv}{dx} = 4v^2 \Rightarrow \dfrac{dv}{dx} = 4v$

$\dfrac{dx}{dv} = \dfrac{1}{4v} \Rightarrow x = \dfrac{1}{4}\int \dfrac{1}{v}dv = \dfrac{1}{4}\log_e(|v|) + c$

When $x = 1,\ v = e$

$1 = \dfrac{1}{4}\log_e(e) + c \ \Rightarrow c = 1 - \dfrac{1}{4} = \dfrac{3}{4}$

$x = \dfrac{1}{4}\log_e(|v|) + \dfrac{3}{4} \ \Rightarrow\ 4x - 3 = \log_e(|v|)$

Now when $x = 2$, $\log_e(|v|) = 5$

$v = e^5\ \text{ms}^{-1}$

Award 1 mark for using the correct equation of motion.
Award 1 mark for the correct integration.
Award 1 mark for finding the correct constant of integration.
Award 1 mark for the final correct value of velocity.

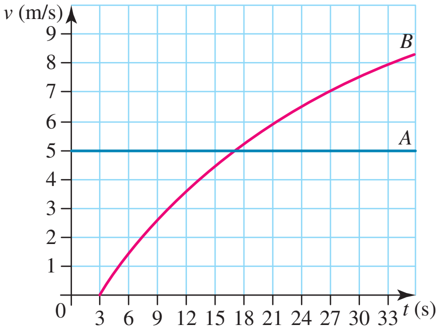
VCAA Assessment Report **note:**

The majority of students understood the need to use the relevant form for the acceleration a. Some used an incorrect form. Quite a few integrated $\dfrac{1}{4v}$ to get $\log_e(4v)$, omitting the coefficient of $\dfrac{1}{4}$. A relatively small number of students forgot the constant of integration, while others made arithmetical slips in finding the constant of integration or a simplification slip in working with it. Some used modulus at the integration step and then gave the final answer as $\pm e^5$, A few were unable to simplify $\sqrt{e^{10}}$. Others integrated with respect to v to obtain a cubic.

5 $v_A(t) = 5 \quad t \geq 0$

$v_B(t) = \dfrac{30}{\pi}\tan^{-1}\left(\dfrac{t-3}{24}\right) \quad t \geq 3$

a



Award 1 mark each for correctly sketching both of the cyclist's velocity-time graphs.

b $\displaystyle\int_3^T \dfrac{30}{\pi}\tan^{-1}\left(\dfrac{t-3}{24}\right) dt = 5T$

$T = 37.175\ \text{s}$

Award 1 mark for the correct value.

Topic 12 — Vector calculus

12.2 Position vectors as functions of time

12.2 Exercise

1 $\underset{\sim}{r}(t) = (t-2)\underset{\sim}{i} + (3t-1)\underset{\sim}{j}$

a $\underset{\sim}{r}(5) = 3\underset{\sim}{i} + 14\underset{\sim}{j}$

$$\begin{aligned}|\underset{\sim}{r}(5)| &= \sqrt{9+196}\\ &= \sqrt{205}\end{aligned}$$

b
$$\begin{aligned}|\underset{\sim}{r}(t)| &= \sqrt{(t-2)^2+(3t-1)^2}\\ &= \sqrt{t^2-4t+4+9t^2-6t+1}\\ &= \sqrt{10t^2-10t+5}\end{aligned}$$

c
$$\frac{d|\underset{\sim}{r}(t)|}{dt} = \frac{1}{2}(20t-10)\left(10t^2-10t+5\right)^{\frac{1}{2}} = 0$$

$$\begin{aligned}\rightarrow 20t-10 &= 0\\ t &= \frac{1}{2}\end{aligned}$$

$$\underset{\sim}{r}\left(\frac{1}{2}\right) = -\frac{3}{2}\underset{\sim}{i} + \frac{1}{2}\underset{\sim}{j}$$

$$\left|\underset{\sim}{r}\left(\frac{1}{2}\right)\right| = \sqrt{\left(-\frac{3}{2}\right)^2+\left(\frac{1}{2}\right)^2} = \frac{\sqrt{10}}{2}$$

2 **a** $\underset{\sim}{r}(t) = \sqrt{t}\underset{\sim}{i} + (2t+3)\underset{\sim}{j}$

$$\begin{aligned}\underset{\sim}{r}(4) &= 2\underset{\sim}{i} + 11\underset{\sim}{j}\\ |\underset{\sim}{r}(4)| &= \sqrt{2^2+11^2}\\ &= 5\sqrt{5}\end{aligned}$$

b
$$\begin{aligned}|\underset{\sim}{r}(t)| &= \sqrt{\left(\sqrt{t}\right)^2+(2t+3)^2} = 15\sqrt{2}\\ t+4t^2+12t+9 &= \left(15\sqrt{2}\right)^2\\ 4t^2+13t+9 &= 450\\ 4t^2+13t-441 &= 0\\ (4t+49)(t-9) &= 0\\ t &= -\frac{49}{4},\ t=9\\ t &\geq 0\\ \therefore t &= 9\end{aligned}$$

3 $\underset{\sim}{r}_A(t) = (2t+6)\underset{\sim}{i} + \left(t^2-6t+45\right)\underset{\sim}{j}$

$\underset{\sim}{r}_B(t) = (t+11)\underset{\sim}{i} + (7t+5)\underset{\sim}{j}$

a
$$\begin{aligned}\underset{\sim}{r}_A(t) &= \underset{\sim}{r}_B(t)\\ &\underset{\sim}{i}\\ 2t+6 &= t+11\\ t-5 &= 0\\ t &= 5\\ &\underset{\sim}{j}\\ t^2-6t+45 &= 7t+5\\ t^2-13t+40 &= 0\\ (t-5)(t-8) &= 0\\ t &= 5,\ 8\end{aligned}$$

Common solution is $t = 5$

$\underset{\sim}{r}_A(5) = (16)\underset{\sim}{i} + (40)\underset{\sim}{j}$

$\underset{\sim}{r}_B(5) = (16)\underset{\sim}{i} + (40)\underset{\sim}{j}$

Collide at $(16,\ 40)$, $t = 5$

b Paths intersect

$$\begin{aligned}\underset{\sim}{r}_A(s) &= (2s+6)\underset{\sim}{i} + \left(s^2-6s+45\right)\underset{\sim}{j}\\ \underset{\sim}{r}_B(t) &= (t+11)\underset{\sim}{i} + (7t+5)\underset{\sim}{j}\\ \underset{\sim}{r}_A(s) &= \underset{\sim}{r}_B(t)\\ &\underset{\sim}{i}\\ 2s+6 &= t+11\\ t &= 2s-5\\ &\underset{\sim}{j}\\ s^2-6s+45 &= 7t+5\\ s^2-6s+45 &= 7(2s-5)+5\\ &= 14s-30\\ s^2-20s+75 &= 0\\ (s-5)(s-15) &= 0\\ s &= 5, 15\\ t &= 5, 25\\ \underset{\sim}{r}_A(15) &= 36\underset{\sim}{i} + 180\underset{\sim}{j}\\ \underset{\sim}{r}_B(25) &= 36\underset{\sim}{i} + 180\underset{\sim}{j}\end{aligned}$$

So paths cross at (36, 180)

c
$$\begin{aligned}\underset{\sim}{r}_A(10) &= 26\underset{\sim}{i} + 85\underset{\sim}{j}\\ \underset{\sim}{r}_B(10) &= 21\underset{\sim}{i} + 75\underset{\sim}{j}\\ \left|\underset{\sim}{r}_A(10) - \underset{\sim}{r}_B(10)\right| &= \left|5\underset{\sim}{i} + 10\underset{\sim}{j}\right| = 5\sqrt{5}\end{aligned}$$

4 $\underset{\sim}{r}_A(t) = \left(-t^2+12t-22\right)\underset{\sim}{i} + (19-3t)\underset{\sim}{j}$

$\underset{\sim}{r}_B(t) = (18-2t)\underset{\sim}{i} + (t+3)\underset{\sim}{j}$

a
$$\begin{aligned}\underset{\sim}{r}_A(t) &= \underset{\sim}{r}_B(t)\\ &\underset{\sim}{i}\\ -t^2+12t-22 &= 18-2t\\ t^2-14t+40 &= 0\\ (t-4)(t-10) &= 0\\ t &= 4, 10\\ &\underset{\sim}{j}\\ 19-3t &= t+3\\ 4t &= 16\\ t &= 4\end{aligned}$$

Common solution is $t = 4$

$\underset{\sim}{r}_A(4) = 10\underset{\sim}{i} + 7\underset{\sim}{j}$

$\underset{\sim}{r}_B(4) = 10\underset{\sim}{i} + 7\underset{\sim}{j}$

Collide at $(10, 7), t = 4$

b Paths intersect

$$\begin{aligned}\underset{\sim}{r}_A(s) &= \underset{\sim}{r}_B(t)\\ &\underset{\sim}{j}\\ 19-3s &= t+3\\ t &= 16-3s\\ &\underset{\sim}{i}\\ -s^2+12s-22 &= 18-2t\\ -s^2+12s-22 &= 18-2(16-3s)\\ -s^2+12s-22 &= 18-32+6s\end{aligned}$$

$$\begin{aligned} s^2 - 6s + 8 &= 0 \\ (s-4)(s-2) &= 0 \\ s &= 2, 4 \\ t &= 4, 10 \end{aligned}$$

$\underset{\sim}{r}_A(2) = -2\underset{\sim}{i} + 13\underset{\sim}{j}$

$\underset{\sim}{r}_B(10) = -2\underset{\sim}{i} + 13\underset{\sim}{j}$

So paths cross at $(-2, 13)$

c $$\begin{aligned} \underset{\sim}{r}_A(5) &= 13\underset{\sim}{i} + 4\underset{\sim}{j} \\ \underset{\sim}{r}_B(5) &= 8\underset{\sim}{i} + 8\underset{\sim}{j} \\ \left|\underset{\sim}{r}_A(5) - \underset{\sim}{r}_B(5)\right| &= \left|5\underset{\sim}{i} - 4\underset{\sim}{j}\right| = \sqrt{41} \end{aligned}$$

5 a $\underset{\sim}{r}(t) = (2t-1)\underset{\sim}{i} + (t-3)\underset{\sim}{j}, t \geq 0$

$$\begin{aligned} \underset{\sim}{r}(4) &= 7\underset{\sim}{i} + \underset{\sim}{j} \\ |\underset{\sim}{r}(4)| &= \sqrt{49+1} \\ &= 5\sqrt{2} \end{aligned}$$

b $$\begin{aligned} |\underset{\sim}{r}(t)|^2 &= (2t-1)^2 + (t-3)^2, t \geq 0 \\ &= 4t^2 - 4t + 1 + t^2 - 6t + 9 \\ |\underset{\sim}{r}(t)| &= \sqrt{5t^2 - 10t + 10} \end{aligned}$$

c $$\begin{aligned} \frac{d|\underset{\sim}{r}(t)|}{dt} &= \frac{10t - 10}{2\sqrt{5t^2 - 10t + 10}} = 0 \\ \rightarrow 10t - 10 &= 0 \\ t &= 1 \\ \underset{\sim}{r}(1) &= \underset{\sim}{i} - 2\underset{\sim}{j} \\ |\underset{\sim}{r}(1)| &= \sqrt{1 + (-2)^2} \\ &= \sqrt{5} \end{aligned}$$

6 a $\underset{\sim}{r}(t) = (4t-3)\underset{\sim}{i} + (3t+4)\underset{\sim}{j}, \; t \geq 0$

$$\begin{aligned} \underset{\sim}{r}(2) &= 5\underset{\sim}{i} + 10\underset{\sim}{j} \\ |\underset{\sim}{r}(2)| &= \sqrt{25 + 100} = \sqrt{125} \\ &= 5\sqrt{5} \end{aligned}$$

b $$\begin{aligned} |\underset{\sim}{r}(t)| &= \sqrt{(4t-3)^2 + (3t+4)^2} \\ &= \sqrt{25t^2 + 25} \\ &= 5\sqrt{t^2 + 1} \end{aligned}$$

c $$\begin{aligned} \frac{d|\underset{\sim}{r}(t)|}{dt} &= \frac{5t}{\sqrt{t^2+1}} = 0 \\ \rightarrow t &= 0 \\ \underset{\sim}{r}(0) &= -3\underset{\sim}{i} + 4\underset{\sim}{j} \\ |\underset{\sim}{r}(0)| &= \sqrt{9 + 16} = 5 \end{aligned}$$

7 $\underset{\sim}{r}(t) = (2t-3)\underset{\sim}{i} + 2\sqrt{t}\underset{\sim}{j}, \; t \geq 0$

a $$\begin{aligned} \underset{\sim}{r}(4) &= 5\underset{\sim}{i} + 4\underset{\sim}{j} \\ |\underset{\sim}{r}(4)| &= \sqrt{25 + 16} = \sqrt{41} \end{aligned}$$

b $$\begin{aligned} |\underset{\sim}{r}(t)|^2 &= (2t-3)^2 + \left(2\sqrt{t}\right)^2 \\ &= 4t^2 - 8t + 9 \\ |\underset{\sim}{r}(t)| &= \sqrt{4t^2 - 8t + 9} \\ \frac{d|\underset{\sim}{r}(t)|}{dt} &= \frac{8t - 8}{2\sqrt{4t^2 - 8t + 9}} = 0 \\ 8t - 8 &= 0 \\ t &= 1 \\ \underset{\sim}{r}(1) &= -\underset{\sim}{i} + 2\underset{\sim}{j} \\ |\underset{\sim}{r}(1)| &= \sqrt{(-1)^2 + 4} \\ &= \sqrt{5} \end{aligned}$$

c $$\begin{aligned} |\underset{\sim}{r}(t)| &= \sqrt{4t^2 - 8t + 9} = 3 \\ 4t^2 - 8t + 9 &= 9 \\ 4t^2 - 8t &= 0 \\ 4t(t-2) &= 0 \\ t &= 0, \; 2 \end{aligned}$$

8 $\underset{\sim}{r}(t) = (2t-7)\underset{\sim}{i} + (2t+2)\underset{\sim}{j}, \; t \geq 0$

a $$\begin{aligned} |\underset{\sim}{r}(t)|^2 &= (2t-7)^2 + (2t+2)^2 \\ &= 8t^2 - 20t + 53 \\ |\underset{\sim}{r}(t)| &= \sqrt{8t^2 - 20t + 53} \\ \frac{d|\underset{\sim}{r}(t)|}{dt} &= \frac{\frac{1}{2}(16t - 20)}{\sqrt{8t^2 - 20t + 53}} = 0 \\ \rightarrow t &= \frac{5}{4} \\ \underset{\sim}{r}\left(\frac{5}{4}\right) &= -\frac{9}{2}\underset{\sim}{i} + \frac{9}{2}\underset{\sim}{j} \\ \left|\underset{\sim}{r}\left(\frac{5}{4}\right)\right| &= \sqrt{\frac{81}{4} + \frac{81}{4}} \\ &= \frac{9\sqrt{2}}{2} \end{aligned}$$

b $$\begin{aligned} |\underset{\sim}{r}(t)| &= \sqrt{8t^2 - 20t + 53} = 9\sqrt{5} \\ 8t^2 - 20t + 53 &= 81 \times 5 = 405 \\ 4(2t+11)(t-8) &= 0 \\ t &= -\frac{11}{2}, \; 8 \\ t &\geq 0 \\ t &= 8 \text{ only} \end{aligned}$$

9 $\underset{\sim}{r}(t) = (at+b)\underset{\sim}{i} + (ct^2 + d)\underset{\sim}{j}, \; t \geq 0$

$\underset{\sim}{r}(2) = 5\underset{\sim}{i} + 7\underset{\sim}{j}$

$\underset{\sim}{r}(2) = (2a+b)\underset{\sim}{i} + (4c+d)\underset{\sim}{j} = 5\underset{\sim}{i} + 7\underset{\sim}{j}$

$\underset{\sim}{i}(1)\colon 2a + b = 5$

$\underset{\sim}{j}(2)\colon 4c + d = 7$

$\underset{\sim}{r}(4) = 13\underset{\sim}{i} + 19\underset{\sim}{j}$

$\underset{\sim}{r}(4) = (4a+b)\underset{\sim}{i} + (16c+d)\underset{\sim}{j} = 13\underset{\sim}{i} + 19\underset{\sim}{j}$

$\underset{\sim}{i}(3)\colon 4a + b = 13$

$\underset{\sim}{j}(4)\colon 16c + d = 19$

$(3) - (1)\colon 2a = 8$

$a = 4 \rightarrow b = -3$

$(4) - (2)\colon 12c = 12$

$c = 1 \rightarrow d = 3$

$\underset{\sim}{r}(t) = (4t-3)\underset{\sim}{i} + (t^2 + 3)\underset{\sim}{j}$

10 $\underset{\sim}{r}(t) = (at+b)\underset{\sim}{i} + (ct^2 + dt)\underset{\sim}{j}, t \geq 0$

$\underset{\sim}{r}(4) = 13\underset{\sim}{i} + 4\underset{\sim}{j}$

$\underset{\sim}{r}(4) = (4a+b)\underset{\sim}{i} + (16c + 4d)\underset{\sim}{j} = 13\underset{\sim}{i} + 4\underset{\sim}{j}$

$\underset{\sim}{i}(1)\colon 4a + b = 13$

$\underset{\sim}{j}(2)\colon 16c + 4d = 4$

$\underset{\sim}{r}(6) = (6a+b)\underset{\sim}{i} + (36c + 6d)\underset{\sim}{j} = 17\underset{\sim}{i} + 18\underset{\sim}{j}$

$\underset{\sim}{i}(3)\colon 6a + b = 17$

$\underset{\sim}{j}(4)\colon 36c + 6d = 18$

$(3) - (1)\colon 2a = 4$

$a = 2 \rightarrow b = 5$

$\frac{1}{4}(2): 4c + d = 1$

$\frac{1}{6}(4): 6c + d = 3$

$2c = 2$

$c = 1 \rightarrow d = -3$

$\underset{\sim}{r}(t) = (2t+5)\underset{\sim}{i} + \left(t^2 - 3t\right)\underset{\sim}{j}$

11 **a** $\underset{\sim}{r}_A(t) = (3t - 43)\underset{\sim}{i} + \left(-t^2 + 26t - 160\right)\underset{\sim}{j}$

$\underset{\sim}{r}_B(t) = (17 - t)\underset{\sim}{i} + (2t - 25)\underset{\sim}{j}$

$t \geq 0$

$\underset{\sim}{r}_A(t) = \underset{\sim}{r}_B(t)$

$\underset{\sim}{i}: 3t - 43 = 17 - t \rightarrow t = 15$

$\underset{\sim}{j}: -t^2 + 26t - 160 = 2t - 25 \rightarrow (t - 15)(t - 9) = 0$

Common solution is $t = 15$

$\underset{\sim}{r}_A(15) = 2i + 5\underset{\sim}{j}$

$\underset{\sim}{r}_B(15) = 2\underset{\sim}{i} + 5\underset{\sim}{j}$

Paths intersect at $(2, 5)$, $t = 15$

b Paths intersect

$\underset{\sim}{r}_A(s) = (3s - 43)\underset{\sim}{i} + \left(-s^2 + 26s - 160\right)\underset{\sim}{j}$

$\underset{\sim}{r}_B(t) = (17 - t)\underset{\sim}{i} + (2t - 25)\underset{\sim}{j}$

$\underset{\sim}{i}: 3s - 43 = 17 - t \rightarrow t = 60 - 3s$

$\underset{\sim}{j}: -s^2 + 26s - 160 = 2t - 25 \rightarrow (s - 15)(s - 17) = 0$

$s = 15, 17$

$t = 15, 9$

$\underset{\sim}{r}_A(17) = 8\underset{\sim}{i} - 7\underset{\sim}{j}$

$\underset{\sim}{r}_B(9) = 8\underset{\sim}{i} - 7\underset{\sim}{j}$

Paths cross at $(8, -7)$

c

$$\begin{aligned}\underset{\sim}{r}_A(12) &= -7\underset{\sim}{i} + 8\underset{\sim}{j}\\ \underset{\sim}{r}_B(12) &= 5\underset{\sim}{i} - \underset{\sim}{j}\\ \left|\underset{\sim}{r}_A(12) - \underset{\sim}{r}_B(12)\right| &= \left|-12\underset{\sim}{i} + 9\underset{\sim}{j}\right|\\ &= \sqrt{144 + 81}\\ &= 15\end{aligned}$$

12 $\underset{\sim}{r}_A(t) = \left(t^2 - 6\right)\underset{\sim}{i} + (2t + 2)\underset{\sim}{j}$

$\underset{\sim}{r}_B(t) = (7t - 16)\underset{\sim}{i} + \frac{1}{5}\left(17t - t^2\right)\underset{\sim}{j}$

$t \geq 0$

a

$$\begin{aligned}\underset{\sim}{r}_A(t) &= \underset{\sim}{r}_B(t)\\ \underset{\sim}{i}: t^2 - 6 &= 7t - 16\\ t^2 - 7t + 10 &= 0\\ (t - 5)(t - 2) &= 0\\ t &= 5, 2\\ \underset{\sim}{j}: 2t + 2 &= \frac{1}{5}\left(17t - t^2\right)\\ t^2 - 7t + 10 &= 0\\ (t - 5)(t - 2) &= 0\\ t &= 5, 2\\ \underset{\sim}{r}_A(2) &= -2\underset{\sim}{i} + 6\underset{\sim}{j}\\ \underset{\sim}{r}_B(2) &= -2\underset{\sim}{i} + 6\underset{\sim}{j}\\ \underset{\sim}{r}_A(5) &= 19\underset{\sim}{i} + 12\underset{\sim}{j}\\ \underset{\sim}{r}_B(5) &= 19\underset{\sim}{i} + 12\underset{\sim}{j}\end{aligned}$$

Collide at

$(-2, 6)$, $t = 2$

$(19, 12)$, $t = 5$

b

$$\begin{aligned}\underset{\sim}{r}_A(10) &= 94\underset{\sim}{i} + 22\underset{\sim}{j}\\ \underset{\sim}{r}_B(10) &= 54\underset{\sim}{i} + 14\underset{\sim}{j}\\ \underset{\sim}{r}_A(10) - \underset{\sim}{r}_B(10) &= 40\underset{\sim}{i} + 8\underset{\sim}{j}\\ \left|\underset{\sim}{r}_A(10) - \underset{\sim}{r}_B(10)\right| &= \sqrt{40^2 + 8^2}\\ &= \sqrt{1600 + 64}\\ &= 8\sqrt{26}\end{aligned}$$

13 $\underset{\sim}{r}_A(t) = \left(-t^2 + 12t + 53\right)\underset{\sim}{i} + (3t + 38)\underset{\sim}{j}$

$\underset{\sim}{r}_B(t) = (2t + 29)\underset{\sim}{i} + (86 - t)\underset{\sim}{j}$

$t \geq 0$

a

$$\begin{aligned}\underset{\sim}{r}_A(t) &= \underset{\sim}{r}_B(t)\\ \underset{\sim}{i}: -t^2 + 12t + 53 &= 2t + 29\\ t^2 - 10t - 24 &= 0\\ (t - 12)(t + 2) &= 0\\ t &= -2,\ 12\\ \underset{\sim}{j}: 3t + 38 &= 86 - t\\ 4t &= 48\\ t &= 12\end{aligned}$$

Common solution $t = 12$

$\underset{\sim}{r}_A(12) = 53\underset{\sim}{i} + 74\underset{\sim}{j}$

$\underset{\sim}{r}_B(2) = 53\underset{\sim}{i} + 74\underset{\sim}{j}$

Collide at $(53, 74)$,$t = 12$

b Paths intersect $\underset{\sim}{r}_A(s) = \underset{\sim}{r}_B(t)$

$$\begin{aligned}\underset{\sim}{i}: -s^2 + 12s + 53 &= 2t + 29\\ \underset{\sim}{j}: 3s + 38 &= 86 - t \rightarrow t = 48 - 3s\\ -s^2 + 12s + 53 &= 2(48 - 3s) + 29\\ &= 96 - 6s + 29\\ s^2 - 18s + 72 &= 0\\ (s - 6)(s - 12) &= 0\\ s &= 6,\ 12 \rightarrow t = 30, 12\\ \underset{\sim}{r}_A(6) &= 89\underset{\sim}{i} + 56\underset{\sim}{j}\\ \underset{\sim}{r}_B(30) &= 89\underset{\sim}{i} + 56\underset{\sim}{j}\end{aligned}$$

Paths cross at $(89,\ 56)$

c

$$\begin{aligned}\underset{\sim}{r}_A(20) &= -107\underset{\sim}{i} + 98\underset{\sim}{j}\\ \underset{\sim}{r}_B(20) &= 69\underset{\sim}{i} + 66\underset{\sim}{j}\\ \left|\underset{\sim}{r}_A(20) - \underset{\sim}{r}_B(20)\right| &= \left|-176\underset{\sim}{i} + 32\underset{\sim}{j}\right|\\ &= \sqrt{32\,000}\\ &= 80\sqrt{5}\end{aligned}$$

14 $\underset{\sim}{r}(t) = (3 - 3\cos(t))\underset{\sim}{i} + (3 + 3\sin(t))\underset{\sim}{j},\ t \geq 0$

a $\underset{\sim}{r}(0) = 3\underset{\sim}{j}$

$\underset{\sim}{r}(\pi) = 6\underset{\sim}{i} + 3\underset{\sim}{j}$

$\underset{\sim}{r}(2\pi) = 3\underset{\sim}{j}$

b $x(t) = 3 - 3\cos(t) \rightarrow \cos(t) = \dfrac{x - 3}{-3}$

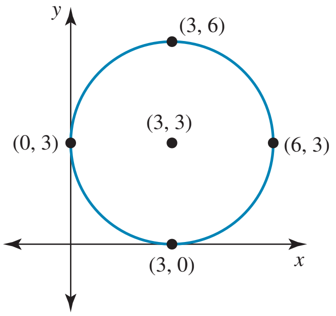
$y(t) = 3 + 3\sin(t) \rightarrow \sin(t) = \dfrac{y - 3}{3}$

$(x - 3)^2 + (y - 3)^2 = 9$

Circle centre $(3, 3)$, radius 3

Domain $[0,\ 6]$

Range $[0,\ 6]$



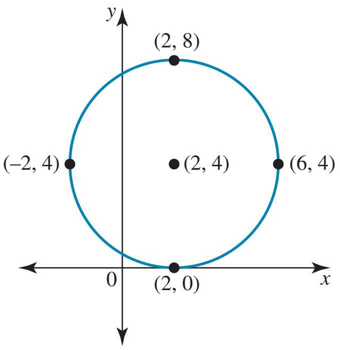
c $|\underset{\sim}{r}(t)| = \sqrt{(3-3\cos(t))^2 + (3+3\sin(t))^2}$
$= \sqrt{27 + 18\sin(t) - 18\cos(t)}$
$= 3\sqrt{(3 + 2\sin(t) - 2\cos(t))}$

d $$\frac{d|\underset{\sim}{r}(t)|}{dt} = \frac{1}{2}(2\cos(t) + 2\sin(t))$$
$$\times 3(3 + 2\sin(t) - 2\cos(t))^{-\frac{1}{2}}$$
$$= 0$$
$$= \frac{3(\cos(t) + \sin(t))}{\sqrt{3 + 2\sin(t) - 2\cos(t)}}$$
$\rightarrow \cos(t) + \sin(t) = 0$
$\cos(t) = -\sin(t)$
$\tan(t) = -1$
$t = \frac{3\pi}{4}, \frac{7\pi}{4}$
$$r\left(\frac{7\pi}{4}\right) = \left(3 - \frac{3\sqrt{2}}{2}\right)\underset{\sim}{i} + \left(3 - \frac{3\sqrt{2}}{2}\right)\underset{\sim}{j}$$
Minimum distance $\sqrt{2}\left(3 - \frac{3\sqrt{2}}{2}\right) = 3\sqrt{2} - 3$

15 $\underset{\sim}{r}(t) = (2 + 4\cos(t))\underset{\sim}{i} + (4 + 4\sin(t))\underset{\sim}{j},\ t \geq 0$

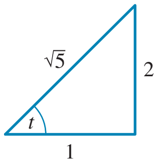
a $\underset{\sim}{r}(0) = 6\underset{\sim}{i} + 4\underset{\sim}{j},\ (6, 4)$
$\underset{\sim}{r}(\pi) = -2\underset{\sim}{i} + 4\underset{\sim}{j},\ (-2, 4)$
$\underset{\sim}{r}(2\pi) = 6\underset{\sim}{i} + 4\underset{\sim}{j},\ (6, 4)$

b $x(t) = 2 + 4\cos(t) \rightarrow \cos(t) = \frac{x-2}{4}$
$y(t) = 4 + 4\sin(t) \rightarrow \sin(t) = \frac{y-4}{4}$
$\cos^2(t) + \sin^2(t) = 1$
$(x-2)^2 + (y-4)^2 = 16$
Circle centre $(2, 4)$, radius 4
Domain $[-2,\ 6]$
Range $[0,\ 8]$



c $|\underset{\sim}{r}(t)|^2 = (2 + 4\cos(t))^2 + (4 + 4\sin(t))^2$
$= 4 + 16\cos(t) + 16\cos^2(t)$
$+ 16 + 32\sin(t) + 16\sin^2(t)$
$= 20 + 16\cos(t) + 32\sin(t)$
$+16\left(\cos^2(t) + \sin^2(t)\right)$
$= 36 + 16\cos(t) + 32\sin(t)$
$= 4(9 + 4\cos(t) + 8\sin(t))$
$|\underset{\sim}{r}(t)| = 2\sqrt{(9 + 4\cos(t) + 8\sin(t))}$

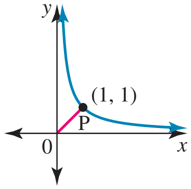
d $$\frac{d|\underset{\sim}{r}(t)|}{dt} = \frac{2 \times \frac{1}{2}(-4\sin(t) + 8\cos(t))}{\sqrt{(9 + 4\cos(t) + 8\sin(t))}} = 0$$
$-4\sin(t) + 8\cos(t) = 0$
$8\cos(t) = 4\sin(t)$
$\tan(t) = 2$
$t = \tan^{-1}(2),\ \pi + \tan^{-1}(2)$
Minimum occurs at $t = \pi + \tan^{-1}(2)$



$$x\left(\pi + \tan^{-1}(2)\right) = 2 + 4\cos\left(\pi + \tan^{-1}(2)\right) = 2 - \frac{4\sqrt{5}}{5}$$
$$y\left(\pi + \tan^{-1}(2)\right) = 4 + 4\sin\left(\pi + \tan^{-1}(2)\right) = 4 - \frac{8\sqrt{5}}{5}$$
$$|\underset{\sim}{r}(t)|_{\text{min}} = \sqrt{\left(2 - \frac{4\sqrt{5}}{5}\right)^2 + \left(4 - \frac{8\sqrt{5}}{5}\right)^2}$$
$$= \sqrt{36 - 16\sqrt{5}}$$
$$= 2\sqrt{5} - 4$$

16 $\underset{\sim}{r}(t) = t\underset{\sim}{i} + \frac{1}{t^2}\underset{\sim}{j},\ t \geq 0$

$x = t,\ y = \frac{1}{t} \rightarrow y = \frac{1}{x}$



$$|\underset{\sim}{r}(t)| = \sqrt{t^2 + \frac{1}{t^2}} = \sqrt{\frac{t^4 + 1}{t^2}},\ t \geq 0$$
$$\frac{d|\underset{\sim}{r}(t)|}{dt} = \frac{\frac{1}{2} \times 4t^3 \times t \times \left(t^4 + 1\right)^{-\frac{1}{2}} - 1 \times \sqrt{1 + t^4}}{t^2}$$
$$= \frac{\frac{2t^4}{\sqrt{t^4+1}} - \sqrt{1 + t^4}}{t^2}$$
$$= \frac{t^4 - 1}{t^2\sqrt{1 + t^4}} = 0$$
$\rightarrow t^4 - 1 = 0$
$\left(t^2 - 1\right)\left(t^2 + 1\right) = 0$
$t = \pm 1,\ t \geq 0$
$t = 1$
$\underset{\sim}{r}(1) = \underset{\sim}{i} + \underset{\sim}{j}$
$|\underset{\sim}{r}(1)|_{\text{min}} = \sqrt{2}$

17 $\underset{\sim}{r}(t) = e^{-t}i + e^{t}\underset{\sim}{j}, t \in R, xy = 1$

$$\begin{aligned}
|\underset{\sim}{r}(t)| &= \sqrt{e^{-2t}+e^{2t}} = \left(e^{-2t}+e^{2t}\right)^{\frac{1}{2}}\\
\frac{d\,|\,\underset{\sim}{r}(t)\,|}{dt} &= \frac{1}{2}\left(-2e^{-2t}+2e^{2t}\right)\left(e^{-2t}+e^{2t}\right)^{-\frac{1}{2}} = 0\\
&= \frac{e^{2t}-e^{-2t}}{\sqrt{e^{-2t}+e^{2t}}} = 0\\
&\to e^{2t}-e^{-2t} = 0\\
e^{2t} &= e^{-2t} = \frac{1}{e^{2t}}\\
e^{4t} &= 1\\
4t &= 0\\
t &= 0\\
\underset{\sim}{r}(0) &= \underset{\sim}{i}+\underset{\sim}{j}\\
|\underset{\sim}{r}(0)|_{\min} &= \sqrt{2}
\end{aligned}$$

18 $\underset{\sim}{r}_1(t) = a\cos(t)\underset{\sim}{i} + a\sin(t)\underset{\sim}{j},\ a \in R^{+}$

$\underset{\sim}{r}_2(t) = a\cos^2(t)\underset{\sim}{i} + a\sin^2(t)\underset{\sim}{j}$

(1) : circle : $x^{a} + y^{a} = a^{2}$

centre : $(0, 0)$

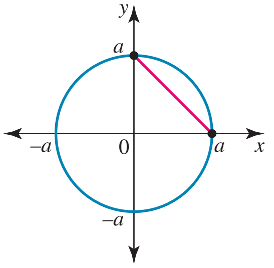
radius : a

domain : $[-a, a]$

range : $[-a, a]$

(2) : $x = a\cos^2(t) \to \frac{x}{a} + \frac{y}{a} = 1$

$y = a\sin^2(t) \to x + y = a$



$$\begin{aligned}
\underset{\sim}{r}_1(0) &= a\underset{\sim}{i}, (a, 0)\\
\underset{\sim}{r}_1\left(\frac{\pi}{2}\right) &= a\underset{\sim}{j}, (0, a)\\
\underset{\sim}{r}_1(\pi) &= -a\underset{\sim}{i}, (-a, 0)\\
\underset{\sim}{r}_1\left(\frac{3\pi}{2}\right) &= -a\underset{\sim}{i}, (0, -a)\\
\underset{\sim}{r}_1(2\pi) &= a\underset{\sim}{i}, (a, 0)\\
\underset{\sim}{r}_2(0) &= a\underset{\sim}{j}, (a, 0)\\
\underset{\sim}{r}_2\left(\frac{\pi}{2}\right) &= a\underset{\sim}{j}, (0, a)\\
\underset{\sim}{r}_2(\pi) &= a\underset{\sim}{i}, (a, 0)\\
\underset{\sim}{r}_2\left(\frac{3\pi}{2}\right) &= a\underset{\sim}{j}, (0, a)\\
\underset{\sim}{r}_2(2\pi) &= a\underset{\sim}{i}, (a, 0)\\
\underset{\sim}{r}_1(t) &= \underset{\sim}{r}_2(t)\\
\underset{\sim}{i} &:\\
a\cos(t) &= a\cos^2(t)\\
a\cos^2(t) - a\cos(t) &= 0\\
a\cos(t)\left(\cos(t) - 1\right) &= 0\\
\cos(t) &= 0, 1\\
t &= \frac{\pi}{2}, \frac{3\pi}{2}, 0, 2\pi\\
\underset{\sim}{j} &:
\end{aligned}$$

$$\begin{aligned}
a\sin(t) &= a\sin^2(t)\\
a\sin^2(t) - a \quad \sin(t) &= 0\\
a\sin(t)\left(\sin(t) - 1\right) &= 0\\
\sin(t) &= 0, 1\\
t &= 0, \quad \frac{\pi}{2}, \quad \pi, \quad 2\pi
\end{aligned}$$

Common solutions $t = 0, \frac{\pi}{2}, 2\pi$

Paths intersect $(a, 0), (0, a)$

19 $\underset{\sim}{r}_1(t) = (a + a\cos(t))\,\underset{\sim}{i} + (a + a\sin(t))\,\underset{\sim}{j},$

$a \in R^{+}$

$\underset{\sim}{r}_2(t) = a\cos^2(t)\underset{\sim}{i} + a\sin^2(t)\underset{\sim}{j}$

$(1): x = a + a\cos(t) \to \cos(t) = \frac{x-a}{a}$

$y = a + a\sin(t) \to \sin(t) = \frac{y-a}{a}$

$(x-a)^2 + (y-a)^2 = a^2$

circle centre: (a, a)

radius: a

domain: $[0, 2a]$

range: $[0, 2a]$

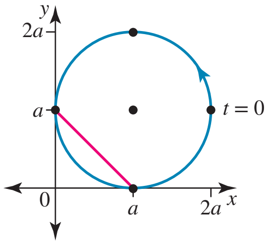
$(2): x = a\cos^2(t)$

$y = a\sin^2(t) \to \frac{x}{a} + \frac{y}{a} = 1$

line $\to x + y = a$

domain: $[0, a]$

range: $[0, a]$



$$\begin{aligned}
\underset{\sim}{r}_1(0) &= 2a\underset{\sim}{i} + a\underset{\sim}{j}, (2a, a)\\
\underset{\sim}{r}_1\left(\frac{\pi}{2}\right) &= a\underset{\sim}{i} + 2a\underset{\sim}{j}, (a, 2a)\\
\underset{\sim}{r}_1(\pi) &= a\underset{\sim}{j}, (0, a)\\
\underset{\sim}{r}_1\left(\frac{3\pi}{2}\right) &= a\underset{\sim}{i}, (a, 0)\\
\underset{\sim}{r}_1(2\pi) &= 2a\underset{\sim}{i} + a\underset{\sim}{j}, (2a, a)\\
\underset{\sim}{r}_2(0) &= a\underset{\sim}{i}, (a, 0)\\
\underset{\sim}{r}_2\left(\frac{\pi}{2}\right) &= a\underset{\sim}{j}, (0, a)\\
\underset{\sim}{r}_2(\pi) &= a\underset{\sim}{i}, (a, 0)\\
\underset{\sim}{r}_2\left(\frac{3\pi}{2}\right) &= a\underset{\sim}{j}, (0, a)\\
\underset{\sim}{r}_2(2\pi) &= a\underset{\sim}{i}, (a, 0)\\
\underset{\sim}{r}_1(t) &= \underset{\sim}{r}_2(t)
\end{aligned}$$

Paths intersect $(a, 0), (0, a)$

20 $\underset{\sim}{r}(t) = (5t-2)\,\underset{\sim}{i} + (12t-2)\,\underset{\sim}{j},\ t \geq 0$

$$\begin{aligned}
|\underset{\sim}{r}(t)| &= \sqrt{(5t-2)^2 + (12t-2)^2}\\
&= \sqrt{169t^2 - 68t + 8}\\
\frac{d|\underset{\sim}{r}(t)|^2}{dt} &= \frac{2\,(169t - 34)}{2\sqrt{169t^2 - 68t + 8}} = 0\\
\to t &= \frac{34}{169}
\end{aligned}$$

$$\begin{aligned}\underset{\sim}{r}\left(\frac{34}{169}\right)&=\frac{1}{169}\left(-168\underset{\sim}{i}+70\underset{\sim}{j}\right)\\ \left|\underset{\sim}{r}(t)\right|_{\min}&=\left|\underset{\sim}{r}\left(\frac{34}{169}\right)\right|=\frac{14}{13}.\end{aligned}$$

21

$$\begin{aligned}\underset{\sim}{r}(t)&=(at+b)\,\underset{\sim}{i}+(ct+d)\underset{\sim}{j},\ t\geq 0\\ \left|\underset{\sim}{r}(t)\right|^2&=(at+b)^2+(ct+d)^2\\ &=\left(a^2+c^2\right)t^2+2(ab+cd)t+b^2+d^2\\ \left|\underset{\sim}{r}(t)\right|&=\sqrt{\left(a^2+c^2\right)t^2+2(ab+cd)t+b^2+d^2}\\ \frac{d\left|\underset{\sim}{r}(t)\right|}{dt}&=\frac{2t\left(a^2+c^2\right)+2(ab+cd)}{2\sqrt{\left(a^2+c^2\right)t^2+2(ab+cd)t+b^2+d^2}}\\ &=0\\ \rightarrow t&=\frac{-(ab+cd)}{a^2+c^2}\\ \underset{\sim}{r}\left(\frac{-(ab+cd)}{a^2+c^2}\right)&=\frac{ad-bc}{\sqrt{a^2+c^2}}\end{aligned}$$

22

$$\begin{aligned}\underset{\sim}{r}(t)&=(4t-3)\,\underset{\sim}{i}+\left(t^2+3\right)\underset{\sim}{j},\ t\geq 0\\ \left|\underset{\sim}{r}(t)\right|^2&=(4t-3)^2+\left(t^2+3\right)^2\\ &=16t^2-24t+9+t^4+6t^2+9\\ &=t^4+22t^2-24t+18\\ \frac{d\left|\underset{\sim}{r}(t)\right|}{dt}&=\frac{2\left(t^3+11t-6\right)}{\sqrt{t^4+22t^2-24t+18}}=0\\ \rightarrow t^3+11t-6&=0\\ t&=0.5318\\ \underset{\sim}{r}(0.5318)&=-0.873\underset{\sim}{i}+3.283\underset{\sim}{j}\\ \left|\underset{\sim}{r}(0.5318)\right|_{\min}&=\sqrt{(-0.873)^2+(3.283)^2}\\ &=3.397\end{aligned}$$

23

$$\begin{aligned}\underset{\sim}{r}(t)&=(2t-1)\,\underset{\sim}{i}+\left(t^2+3t\right)\underset{\sim}{j},\ t\geq 0\\ \left|\underset{\sim}{r}(t)\right|^2&=(2t-1)^2+\left(t^2+3t\right)^2\\ &=4t^2-4t+1+t^4+6t^3+9t^2\\ &=t^4++6t^3+13t^2-4t+1\\ \frac{d\left|\underset{\sim}{r}(t)\right|^2}{dt}&=\frac{2t^3+9t^2+13t-2}{\sqrt{t^4++6t^3+13t^2-4t+1}}=0\\ \rightarrow 2t^3+9t^2+13t-2&=0\\ t&=0.1399\\ \underset{\sim}{r}(0.1399)&=-0.720\underset{\sim}{i}+0.439\underset{\sim}{j}\\ \left|\underset{\sim}{r}(0.5318)\right|_{\min}&=\sqrt{(-0.720)^2+(0.439)^2}=0.844\end{aligned}$$

12.2 Exam questions

1 $\underset{\sim}{r}_A(t)=\left(t^2-1\right)\underset{\sim}{i}+\left(a+\frac{t}{3}\right)\underset{\sim}{j}$ and

$\underset{\sim}{r}_B(t)=\left(t^3-t\right)\underset{\sim}{i}+\left(\arccos\left(\frac{t}{2}\right)\right)\underset{\sim}{j}$

they collide when $\underset{\sim}{r}_A(t)=\underset{\sim}{r}_B(t)$

$\underset{\sim}{i}:\ t^2-1=t^3-t=t\left(t^2-1\right)$

$t=1$ since $0\leq t\leq 2$

$\underset{\sim}{j}:\ a+\frac{1}{3}=\arccos\left(\frac{1}{2}\right)=\frac{\pi}{3}$

$a=\frac{\pi-1}{3}$

Award 1 mark for equating components.

Award 1 mark for solving for t.

Award 1 mark for the correct value of a.

***VCAA Examination Report* note:**

Most students attempted to equate the $\underset{\sim}{i}$ components of $\underset{\sim}{r}_A(t)$ and $\underset{\sim}{r}_B(t)$ in order to determine the value of t when the particles collided. Various algebraic and transcription errors were made by students, which meant they could not be awarded full marks for the question.

2 a $\underset{\sim}{r}_A(t)=(t+1)\underset{\sim}{i}+\left(t^2+2t\right)\underset{\sim}{j}$

$\underset{\sim}{r}_B(t)=t^2\underset{\sim}{i}+\left(t^2+3\right)\underset{\sim}{j},t\geq 0$

Yacht A : $x=t+1,\ y=t^2+2t,\ t=x-1$

$y_A=(x-1)^2+2\,(x-1)$

$y_A=x^2-1,\ x\geq\ 1$ [1 mark]

Yacht B : $x=t^2,\ y=t^2+3$

$y_B=x+3,\ x\ \geq\ 0$ [1 mark]

***VCAA Examination Report* note:**

A variety of less simplified, but correct, forms were given and accepted.

b Solving $x_A\ =x_B$:

$t+1=t^2$

$t=\frac{1+\sqrt{5}}{2}$ since $t\geq 0$.

Solving $y_A\ =y_B$:

$t^2+2t=t^2+3$

$t=\frac{3}{2}$ [1 mark]

They are not in the same place at the same time; therefore, the yachts will not collide. [1 mark]

***VCAA Examination Report* note:**

The approach shown is one of several correct ways to show that the yachts do not collide.

c The yachts' paths cross when $y_A\ =\ y_B$:

$x^2-1=x\ +3$ [1 mark]

Solving gives $x=\ 2.562,\ y\ =\ 5.562$.

The yachts' paths cross at (2.562, 5.562). [1 mark]

***VCAA Examination Report* note:**

Missing the condition that $t\geq 0$, led, incorrectly, to two points being provided. Many otherwise correct responses were not expressed in the required form, as coordinates correct to three decimal places.

3 $\underset{\sim}{r_1}(t)=\left(2+4t^2\right)\underset{\sim}{i}+(3t+2)\underset{\sim}{j}$

$\underset{\sim}{r_2}(t)=(6t)\underset{\sim}{i}+(4+t)\underset{\sim}{j}$

$\underset{\sim}{i}:\ 2+4t^2=6t$

$2t^2-3t+1=0$

$(2t-1)\,(t-1)=0$

$t=\frac{1}{2},\ 1$

$\underset{\sim}{j}:\ 3t+2=4+t$

$2t=2$

$t=1$

This gives a common solution of $t=1$

$\underset{\sim}{r_1}(1)=\underset{\sim}{r_2}(1)=6\underset{\sim}{i}+5\underset{\sim}{j}$

The correct answer is **B**.

12.3 Differentiation of vectors

12.3 Exercise

1 $\underset{\sim}{r}(t)=\left(e^{2t}+e^{-2t}\right)\underset{\sim}{i}+\left(e^{2t}-e^{-2t}\right)\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t)=\left(2e^{2t}-2e^{-2t}\right)\underset{\sim}{i}+\left(2e^{2t}+2e^{-2t}\right)\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(0)=(2-2)\underset{\sim}{i}+(2+2)\underset{\sim}{j}=4\underset{\sim}{j}$

$\left|\dot{\underset{\sim}{r}}(0)\right|=4$

$\hat{\underset{\sim}{r}}(0)=\underset{\sim}{j}$

2
$$\begin{aligned}\underset{\sim}{r}(t) &= \cos(2t)\underset{\sim}{i} + \sin(2t)\underset{\sim}{j}\\ \underset{\sim}{\dot{r}}(t) &= -2\sin(2t)\underset{\sim}{i} + 2\cos(2t)\underset{\sim}{j}\\ \underset{\sim}{\dot{r}}\left(\frac{\pi}{6}\right) &= -2\sin\left(\frac{\pi}{3}\right)\underset{\sim}{i} + 2\cos\left(\frac{\pi}{3}\right)\underset{\sim}{j}\\ &= -\sqrt{3}\underset{\sim}{i} + \underset{\sim}{j}\\ \left|\underset{\sim}{\dot{r}}\left(\frac{\pi}{6}\right)\right| &= \sqrt{\left(-\sqrt{3}\right)^2 + (1)^2} = 2\\ \hat{\underset{\sim}{r}}\left(\frac{\pi}{6}\right) &= \frac{1}{2}\left(-\sqrt{3}\underset{\sim}{i} + \underset{\sim}{j}\right)\end{aligned}$$

3 $\underset{\sim}{r}(t) = te^{-2t}\underset{\sim}{i} + te^{2t}\underset{\sim}{j},\ t \geq 0$

a
$$\begin{aligned}\underset{\sim}{\dot{r}}(t) &= \frac{d}{dt}te^{-2t}\underset{\sim}{i} + \frac{d}{dt}te^{2t}\underset{\sim}{j}\\ &= \left(e^{-2t} - 2te^{-2t}\right)\underset{\sim}{i} + \left(e^{2t} + 2te^{2t}\right)\underset{\sim}{j}\\ &= e^{-2t}(1-2t)\underset{\sim}{i} + e^{2t}(1+2t)\underset{\sim}{j}\end{aligned}$$

b
$$\begin{aligned}|\underset{\sim}{\dot{r}}(t)| &= \sqrt{\left(e^{-2t}(1-2t)\right)^2 + \left(e^{2t}(1+2t)\right)^2}\\ &= \sqrt{e^{-4t}\left(1-4t+4t^2\right) + e^{4t}\left(1+4t+4t^2\right)}\\ \left|\underset{\sim}{\dot{r}}\left(\frac{1}{2}\right)\right| &= \sqrt{e^{-2}(1-2+1) + e^{2}(1+2+1)} = 2e\end{aligned}$$

c
$$\begin{aligned}\underset{\sim}{\ddot{r}}(t) &= \frac{d}{dt}e^{-2t}(1-2t)\underset{\sim}{i} + \frac{d}{dt}e^{2t}(1+2t)\underset{\sim}{j}\\ &= \left(-2e^{-2t}(1-2t) - 2e^{-2t}\right)\underset{\sim}{i} + \left(2e^{2t}(1+2t) + 2e^{2t}\right)\underset{\sim}{j}\\ &= (4t-4)e^{-2t}\underset{\sim}{i} + (4t+4)\,e^{2t}\underset{\sim}{j}\\ &= 4(t-1)e^{-2t}\underset{\sim}{i} + 4(t+1)e^{2t}\underset{\sim}{j}\end{aligned}$$

4
$$\begin{aligned}\underset{\sim}{r}(t) &= \cos^2(t)\underset{\sim}{i} + \sin^2(t)\underset{\sim}{j}, t \geq 0\\ \underset{\sim}{\dot{r}}(t) &= -2\cos(t)\sin(t)\underset{\sim}{i} + 2\sin(t)\cos(t)\underset{\sim}{j}\\ &= -\sin(2t)\underset{\sim}{i} + \sin(2t)\underset{\sim}{j}\\ \underset{\sim}{\dot{r}}\left(\frac{3\pi}{8}\right) &= -\sin\left(\frac{3\pi}{4}\right)\underset{\sim}{i} + \sin\left(\frac{3\pi}{4}\right)\underset{\sim}{j} = -\frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}\underset{\sim}{j}\\ \left|\underset{\sim}{\dot{r}}\left(\frac{3\pi}{8}\right)\right| &= \sqrt{\left(-\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = 1\text{ m/s}\end{aligned}$$

5 $\underset{\sim}{r}(t) = 2t^4\underset{\sim}{i} + 4\cos(2t)\underset{\sim}{j} + 6e^{-2t}\underset{\sim}{k}$

a $\underset{\sim}{\dot{r}}(t) = 8t^3\underset{\sim}{i} - 8\sin(2t)\underset{\sim}{j} - 12e^{-2t}\underset{\sim}{k}$

b $\underset{\sim}{\ddot{r}}(t) = 24t^2\underset{\sim}{i} - 16\cos(2t)\underset{\sim}{j} + 24e^{-2t}\underset{\sim}{k}$

6 $\underset{\sim}{r}(t) = 8\cos\left(\frac{\pi t}{4}\right)\underset{\sim}{i} + 8\sin\left(\frac{\pi t}{4}\right)\underset{\sim}{j} + 4e^{-2t}\underset{\sim}{k}$

$$\begin{aligned}\underset{\sim}{\dot{r}}(t) &= -2\pi\sin\left(\frac{\pi t}{4}\right)\underset{\sim}{i} + 2\pi\cos\left(\frac{\pi t}{4}\right)\underset{\sim}{j} - 8e^{-2t}\underset{\sim}{k}\\ \underset{\sim}{\dot{r}}(t) &= -\frac{\pi^2}{2}\cos\left(\frac{\pi t}{4}\right)\underset{\sim}{i} + \frac{\pi^2}{2}\sin\left(\frac{\pi t}{4}\right)\underset{\sim}{j} + 16e^{-2t}\underset{\sim}{k}\\ \underset{\sim}{\dot{r}}(1) &= -\frac{\pi^2}{2}\cos\left(\frac{\pi}{4}\right)\underset{\sim}{i} + \frac{\pi^2}{2}\sin\left(\frac{\pi}{4}\right)\underset{\sim}{j} + 16e^{-2}\underset{\sim}{k}\\ &= -\frac{\pi^2\sqrt{2}}{4}\underset{\sim}{i} + \frac{\pi^2\sqrt{2}}{4}\underset{\sim}{j} + 16e^{-2}\underset{\sim}{k}\\ |\underset{\sim}{\dot{r}}(1)| &= \sqrt{\left(-\frac{\pi^2\sqrt{2}}{4}\right)^2 + \left(\frac{\pi^2\sqrt{2}}{4}\right)^2 + \left(16e^{-2}\right)^2}\\ &= \sqrt{2 \times \frac{\pi^4 \times 2}{16} + 256e^{-4}}\\ &= \sqrt{\frac{\pi^4}{4} + 256e^{-4}}\\ &= \sqrt{\frac{\pi^4 + 1024e^{-4}}{4}}\\ &= \frac{1}{2}\sqrt{\pi^4 + 1024e^{-4}}\end{aligned}$$

7 $\underset{\sim}{r}(t) = (4 + 3\cos(2t))\,\underset{\sim}{i} + (3 - 2\sin(2t))\,\underset{\sim}{j},\ 0 \leq t \leq \pi$

a
$$\begin{aligned}x &= 4 + 3\cos(2t)\\ \dot{x} &= -3\sin(2t)\\ y &= 3 - 2\sin(2t)\\ \dot{y} &= -4\cos(2t)\\ \frac{dy}{dx} &= \frac{\dot{y}}{\dot{x}} = \frac{-4\cos(2t)}{-3\sin(2t)} = \frac{4}{3}\cot(2t)\\ \frac{dy}{dx} &= 0 \to \cos(2t) = 0,\ \sin(2t) \neq 0,\ 0 \leq t \leq \pi\\ 2t &= \frac{\pi}{2},\ \frac{3\pi}{2}\\ t &= \frac{\pi}{4},\ \frac{3\pi}{4}\\ x\left(\frac{\pi}{4}\right) &= 4 + 3\cos\left(\frac{\pi}{2}\right) = 4\\ y\left(\frac{\pi}{4}\right) &= 3 - 2\sin\left(\frac{\pi}{2}\right) = 1\\ x\left(\frac{3\pi}{4}\right) &= 4 + 3\cos\left(\frac{3\pi}{2}\right) = 4\\ y\left(\frac{3\pi}{4}\right) &= 3 - 2\sin\left(\frac{3\pi}{2}\right) = 5\end{aligned}$$

$(4,\ 1), (4,\ 5)$

b $x = 4 + 3\cos(2t) \to \cos(2t) = \dfrac{x-4}{3}$

$y = 3 - 2\sin(2t) \to \sin(2t) = \dfrac{y-3}{-2}$

$\sin^2(2t) + \cos^2(2t) = 1$

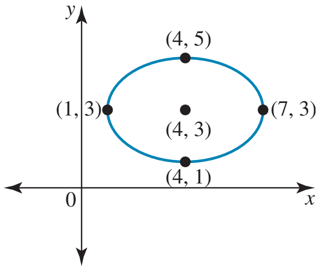
$\dfrac{(x-4)^2}{9} + \dfrac{(y-3)^2}{4} = 1$

Ellipse centre $(4,\ 3)$

Semi major/minor $(3,\ 2)$

Domain $4 \pm 3 = [1,\ 7]$

Range $3 \pm 2 = [1,\ 5]$



c
$$\begin{aligned}|\underset{\sim}{\dot{r}}(t)| &= \sqrt{9\sin^2(2t) + 16\cos^2(2t)}\\ &= \sqrt{16\cos^2(2t) + 9\left(1 - \cos^2(2t)\right)}\\ &= \sqrt{9 + 7\cos^2(2t)}\end{aligned}$$

$\cos(2t) = 0 \to |\underset{\sim}{\dot{r}}(t)|_{min} = \sqrt{9} = 3$

$\cos(2t) = 1 \to |\underset{\sim}{\dot{r}}(t)|_{max} = \sqrt{9+7} = 4$

8 $\underset{\sim}{r}(t) = 3\sec(t)\underset{\sim}{i} + 2\tan(t)\underset{\sim}{j},\ 0 \leq t \leq 2\pi$

a $x = 3\sec(t) \to \sec(t) = \dfrac{x}{3}$

$y = 2\tan(t) \to \tan(t) = \dfrac{y}{2}$

$\sec^2(t) - \tan^2(t) = 1$

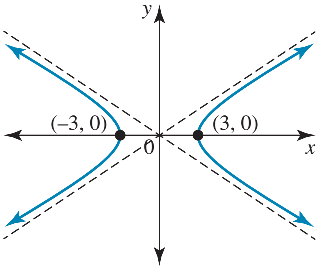
$\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$

Hyperbola crosses x-axis at $y = 0,\ x = \pm 3\,(3,\ 0)\,(-3,\ 0)$.

Domain $|x| \geq 3$

Range R

Asymptotes $y = \pm\dfrac{2x}{3}$



b $$x = 3\sec(t)$$
$$\dot{x} = 3\tan(t)\sec(t)$$
$$y = 2\tan(t)$$
$$\dot{y} = 2\sec^2(t)$$
$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{2\sec^2(t)}{3\tan(t)\sec(t)}$$
$$= \frac{2}{3\cos^2(t)} \times \frac{\cos(t)}{\sin(t)} \times \cos(t)$$
$$= \frac{2}{3\sin(t)} = \frac{4}{3}$$
$$\sin(t) = \frac{1}{2}$$
$$t = \frac{\pi}{6}, \ \frac{5\pi}{6}$$
$$x\left(\frac{\pi}{6}\right) = 3\sec\left(\frac{\pi}{6}\right) = 2\sqrt{3}$$
$$y\left(\frac{\pi}{6}\right) = 2\tan\left(\frac{\pi}{6}\right) = \frac{2\sqrt{3}}{3}$$
$$\left(2\sqrt{3}, \ \frac{2\sqrt{3}}{3}\right)$$
$$x\left(\frac{5\pi}{6}\right) = 3\sec\left(\frac{5\pi}{6}\right) = -2\sqrt{3}$$
$$y\left(\frac{5\pi}{6}\right) = 2\tan\left(\frac{5\pi}{6}\right) = -\frac{2\sqrt{3}}{3}$$
$$\left(-2\sqrt{3}, \ -\frac{2\sqrt{3}}{3}\right)$$

9 $\underset{\sim}{r}(t) = 5t\underset{\sim}{i} + \left(12t - 4.9t^2\right)\underset{\sim}{j}$

a Hits the ground
$$12t - 4.9t^2 = 0$$
$$t\,(12 - 4.9t) = 0$$
$$t = 0, \ t = \frac{12}{4.9} = 2.45 \text{ s.}$$

b $x = 5t$
$$x(2.449) = 5 \times 2.449 = 12.25 \text{ m}$$

c $\dot{\underset{\sim}{r}}(t) = 5\underset{\sim}{i} + (12 - 9.8t)\underset{\sim}{j}$
$$\dot{\underset{\sim}{r}}(0) = 5\underset{\sim}{i} + 12\underset{\sim}{j}$$
$$|\dot{\underset{\sim}{r}}(0)| = \sqrt{25 + 144} = 13 \text{ m/s}$$
$$\alpha = \tan^{-1}\left(\frac{12}{5}\right) = 67.38°$$

d Rises until
$$\dot{y} = 0, \ 12 - 9.8\,t = 0$$
$$t = \frac{12}{9.8} = 1.225 \text{ sec}$$
Max height
$$12(1.225) - 4.9(1.225)^2$$
$$= 7.35 \text{ m}$$

e $x = 5t \rightarrow t = \dfrac{x}{5}$
$$y = 12t - 4.9t^2$$
$$y = \frac{12x}{5} - 4.9\left(\frac{x}{5}\right)^2$$
$$y = \frac{12x}{5} - \frac{49x^2}{250}$$
$$= \frac{x}{250}(600 - 49x)$$

10 **a** $\underset{\sim}{r}(t) = 2t\underset{\sim}{i} + 4t^2\underset{\sim}{j}, \ t \geq 0, \ t = 1$
$$\dot{\underset{\sim}{r}}(t) = 2\underset{\sim}{i} + 8t\underset{\sim}{j}, \ t = 1$$
$$\dot{\underset{\sim}{r}}(1) = 2\underset{\sim}{i} + 8\underset{\sim}{j}$$
$$|\dot{\underset{\sim}{r}}(1)| = \sqrt{2^2 + 8^2} = 2\sqrt{17}$$
$$\dot{\underset{\sim}{s}} = \frac{\dot{\underset{\sim}{r}}(1)}{|\dot{\underset{\sim}{r}}(1)|} = \frac{1}{\sqrt{17}}\left(\underset{\sim}{i} + 4\underset{\sim}{j}\right)$$

b $\underset{\sim}{r}(t) = 2t\underset{\sim}{i} + 8t^3\underset{\sim}{j}, \ t \geq 0, \ t = 1$
$$\dot{\underset{\sim}{r}}(t) = 2\underset{\sim}{i} + 24t^2\underset{\sim}{j}, \ t = 1$$
$$\dot{\underset{\sim}{r}}(1) = 2\underset{\sim}{i} + 24\underset{\sim}{j}$$
$$|\dot{\underset{\sim}{r}}(1)| = \sqrt{2^2 + 24^2} = 2\sqrt{145}$$
$$\dot{\underset{\sim}{s}} = \frac{\dot{\underset{\sim}{r}}(1)}{|\dot{\underset{\sim}{r}}(1)|} = \frac{1}{\sqrt{145}}\left(\underset{\sim}{i} + 12\underset{\sim}{j}\right)$$

c $\underset{\sim}{r}(t) = 3t^2\underset{\sim}{i} + \left(t^2 - 4t\right)\underset{\sim}{j}, \ t \geq 0, \ t = 3$
$$\dot{\underset{\sim}{r}}(t) = 6t\underset{\sim}{i} + (2t - 4)\underset{\sim}{j}, \ t = 3$$
$$\dot{\underset{\sim}{r}}(3) = 18\underset{\sim}{i} + 2\underset{\sim}{j}$$
$$|\dot{\underset{\sim}{r}}(3)| = \sqrt{18^2 + 2^2} = 2\sqrt{82}$$
$$\dot{\underset{\sim}{s}} = \frac{\dot{\underset{\sim}{r}}(3)}{|\dot{\underset{\sim}{r}}(3)|} = \frac{1}{\sqrt{82}}\left(9\underset{\sim}{i} + \underset{\sim}{j}\right)$$

d $\underset{\sim}{r}(t) = \left(t + \dfrac{1}{t}\right)\underset{\sim}{i} + \left(t - \dfrac{1}{t}\right)\underset{\sim}{j}, \ t \geq 0, \ t = 2$
$$\dot{\underset{\sim}{r}}(t) = \left(1 - \frac{1}{t^2}\right)\underset{\sim}{i} + \left(1 + \frac{1}{t^2}\right)\underset{\sim}{j}, \ t = 2$$
$$\dot{\underset{\sim}{r}}(2) = \left(1 - \frac{1}{4}\right)\underset{\sim}{i} + \left(1 + \frac{1}{4}\right)\underset{\sim}{j} = \frac{3}{4}\underset{\sim}{i} + \frac{5}{4}\underset{\sim}{j}$$
$$|\dot{\underset{\sim}{r}}(2)| = \sqrt{\frac{9}{16} + \frac{25}{16}} = \frac{1}{4}\sqrt{34}$$
$$\dot{\underset{\sim}{s}} = \frac{\dot{\underset{\sim}{r}}(3)}{|\dot{\underset{\sim}{r}}(3)|} = \frac{1}{\sqrt{34}}\left(3\underset{\sim}{i} + 5\underset{\sim}{j}\right)$$

11 **a** $\underset{\sim}{r}(t) = e^{-2t}\underset{\sim}{i} + e^{2t}\underset{\sim}{j}, \ t \geq 0, \ t = 0$
$$\dot{\underset{\sim}{r}}(t) = -2e^{-2t}\underset{\sim}{i} + 2e^{2t}\underset{\sim}{j}, \ t = 0$$
$$\dot{\underset{\sim}{r}}(0) = -2\underset{\sim}{i} + 2\underset{\sim}{j}$$
$$|\dot{\underset{\sim}{r}}(0)| = \sqrt{(-2)^2 + 2^2} = 2\sqrt{2}$$
$$\dot{\underset{\sim}{s}} = \frac{\dot{\underset{\sim}{r}}(0)}{|\dot{\underset{\sim}{r}}(0)|} = \frac{1}{\sqrt{2}}\left(-\underset{\sim}{i} + \underset{\sim}{j}\right)$$

b $\underset{\sim}{r}(t) = \cos^2(t)\underset{\sim}{i} + \cos(2t)\underset{\sim}{j}, \ 2 \geq 0 \ <, \ t = \dfrac{\pi}{3}$
$$\dot{\underset{\sim}{r}}(t) = -2\cos(t)\sin(t)\underset{\sim}{i} - 2\sin(2t)\underset{\sim}{j}$$
$$= -\sin(2t)\underset{\sim}{i} - 2\sin(2t)\underset{\sim}{j}$$
$$\dot{\underset{\sim}{r}}\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}\underset{\sim}{i} - \sqrt{3}\underset{\sim}{j}$$

$$\left|\dot{\underset{\sim}{r}}\left(\frac{\pi}{3}\right)\right|=\sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2+\left(\sqrt{3}\right)^2}=\frac{\sqrt{15}}{2}$$

$$\dot{\underset{\sim}{s}}=\frac{\dot{\underset{\sim}{r}}\left(\frac{\pi}{3}\right)}{\left|\dot{\underset{\sim}{r}}\left(\frac{\pi}{3}\right)\right|}=\frac{-\sqrt{3}}{\frac{\sqrt{15}}{2}}\left(\frac{1}{2}\underset{\sim}{i}+\underset{\sim}{j}\right)=-\frac{1}{\sqrt{5}}\left(\underset{\sim}{i}+2\underset{\sim}{j}\right)$$

12 **a** $\underset{\sim}{r}(t)=\left(t^2+9\right)\underset{\sim}{i}+\frac{1}{1+t}j=\left(t^2+9\right)\underset{\sim}{i}+(1+t)^{-1}\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t)=2t\underset{\sim}{i}-(1+t)^{-2}\underset{\sim}{j}=2t\underset{\sim}{i}-\frac{1}{(1+t)^2}\underset{\sim}{j}$

$\ddot{\underset{\sim}{r}}(t)=2\underset{\sim}{i}+2(1+t)^{-3}\underset{\sim}{j}=2\underset{\sim}{i}+\frac{2}{(1+t)^3}\underset{\sim}{j}$

b $\underset{\sim}{r}(t)=\log_e(3t)\underset{\sim}{i}+\left(5t^2+4t\right)\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t)=\frac{1}{t}\underset{\sim}{i}+(10t+4)\underset{\sim}{j}$

$\ddot{\underset{\sim}{r}}(t)=-\frac{1}{t^2}+10\underset{\sim}{j}$

c $\underset{\sim}{r}(t)=3\cos(2t)\underset{\sim}{i}-4\sin(2t)\underset{\sim}{j}+\left(12t-5t^2\right)\underset{\sim}{k}$

$\dot{\underset{\sim}{r}}(t)=-6\sin(2t)\underset{\sim}{i}-8\cos(2t)\underset{\sim}{j}+(12-10t)\underset{\sim}{k}$

$\ddot{\underset{\sim}{r}}(t)=-12\cos(2t)\underset{\sim}{i}+16\sin(2t)\underset{\sim}{j}-10\underset{\sim}{k}$

13 $\underset{\sim}{r}(t)=3\cos(2t)\underset{\sim}{i}+3\sin(2t)\underset{\sim}{j},\ t\geq 0$

a $x=3\cos(2t)\to\cos(2t)=\frac{x}{3}$

$y=3\sin(2t)\to\sin(2t)=\frac{y}{3}$

$\cos^2(2t)+\sin^2(2t)=1$

$\frac{x^2}{9}+\frac{y^2}{9}=1$

$x^2+y^2=9$

Circle centre $(0,\ 0)$

Radius 3

b $\dot{\underset{\sim}{r}}(t)=-6\sin(2t)\underset{\sim}{i}+6\cos(2t)\underset{\sim}{j}$

$|\dot{\underset{\sim}{r}}(t)|=\sqrt{(-6\sin(2t))^2+(6\cos(2t))^2}$

$=6$

c $\ddot{\underset{\sim}{r}}(t)=-12\cos(2t)\underset{\sim}{i}-12\sin(2t)\underset{\sim}{j}$

$=-4\left(3\cos(2t)\underset{\sim}{i}+3\sin(2t)\underset{\sim}{j}\right)$

$=-4\underset{\sim}{r}(t)$

d $\dot{\underset{\sim}{r}}(t)\cdot\ddot{\underset{\sim}{r}}(t)=72\sin(2t)\cos(2t)-72\cos(2t)\sin(2t)=0$

$\dot{\underset{\sim}{r}}(t)$perpendicular $\ddot{\underset{\sim}{r}}(t)$

14 $\underset{\sim}{r}(t)=4\cos(3t)\underset{\sim}{i}+2\sqrt{2}\sin(3t)\left(\underset{\sim}{j}-\underset{\sim}{k}\right),\ t\geq 0$

$=4\cos(3t)\underset{\sim}{i}+2\sqrt{2}\sin(3t)\underset{\sim}{j}-2\sqrt{2}\sin(3t)\underset{\sim}{k}$

a $\dot{\underset{\sim}{r}}(t)=-12\sin(3t)\underset{\sim}{i}+6\sqrt{2}\cos(3t)\underset{\sim}{j}-6\sqrt{2}\cos(3t)\underset{\sim}{k}$

$$|\dot{\underset{\sim}{r}}(t)|=\sqrt{\begin{array}{c}(-12\sin(3t))^2+\left(6\sqrt{2}\cos(3t)\right)^2\\+\left(-6\sqrt{2}\cos(3t)\right)^2\end{array}}$$

$=\sqrt{144\sin^2(3t)+72\cos^2(3t)+72\cos^2(3t)}$

$=\sqrt{144\left(\sin^2(3t)+\cos^2(3t)\right)}$

$=12$

b $\ddot{\underset{\sim}{r}}(t)=-36\cos(3t)\underset{\sim}{i}-18\sqrt{2}\sin(3t)\underset{\sim}{j}+18\sqrt{2}\sin(3t)\underset{\sim}{k}$

$\ddot{\underset{\sim}{r}}(t)\cdot\dot{\underset{\sim}{r}}(t)=12\times 36\sin(3t)\cos(3t)-18\times 6\times 2\sin(3t)$

$\cos(3t)-18\times 6\times 2\sin(3t)\cos(3t)$

$=432\sin(3t)\cos(3t)-432\sin(3t)\cos(3t)$

$=0$

$\ddot{\underset{\sim}{r}}(t)$ perpendicular to $\dot{\underset{\sim}{r}}(t)$

15 **a** $\underset{\sim}{r}(t)=a\cos(nt)\underset{\sim}{i}+a\sin(nt)\underset{\sim}{j},\ t\geq 0$

$x=a\cos(nt)\to\cos(nt)=\frac{x}{a}$

$y=a\sin(nt)\to\sin(nt)=\frac{y}{a}$

$\cos^2(nt)+\sin^2(nt)=1$

$\frac{x^2}{a^2}+\frac{y^2}{a^2}=1$

$x^2+y^2=a^2$

Circle centre $(0,0)$

Radius a

b $\dot{\underset{\sim}{r}}(t)=-na\sin(nt)\underset{\sim}{i}+na\cos(nt)\underset{\sim}{j}$

$|\dot{\underset{\sim}{r}}(t)|=\sqrt{(-na\sin(nt))^2+(na\cos(nt))^2}$

$=\sqrt{n^2a^2\left(\sin^2(nt)+\cos^2(nt)\right)}$

$=na$ constant

c $\ddot{\underset{\sim}{r}}(t)=-n^2a\cos(nt)\underset{\sim}{i}-n^2a\sin(nt)\underset{\sim}{j}$

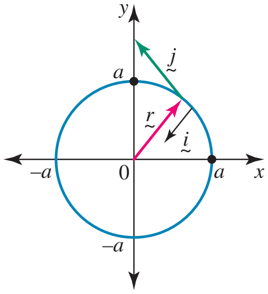
$=-n^2\left[a\cos(nt)\underset{\sim}{i}-a\sin(nt)\underset{\sim}{j}\right]$

$=-n^2\underset{\sim}{r}(t)$

d $\dot{\underset{\sim}{r}}(t)\cdot\ddot{\underset{\sim}{r}}(t)=0$

$\dot{\underset{\sim}{r}}(t)$ perpendicular to $\ddot{\underset{\sim}{r}}(t)$

e Circular motion



16 $\underset{\sim}{r}(t)=\sec(t)\underset{\sim}{i}+3\tan(t)\underset{\sim}{j},0\leq t\leq 2\pi$

a $x=2\sec(t)\to\frac{x}{2}=\sec(t)$

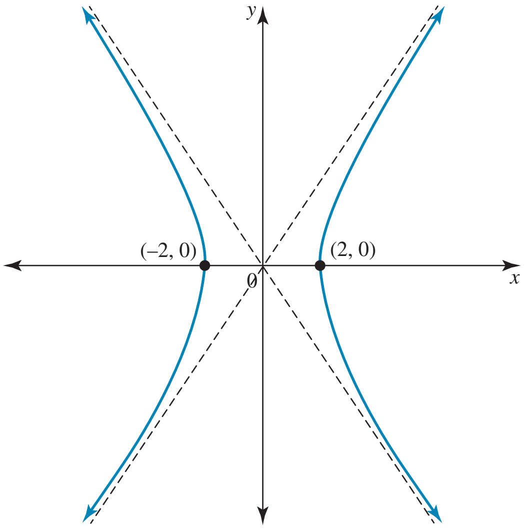
$y=3\tan(t)\to\frac{y}{3}=\tan(t)$

$\sec^2(t)-\tan^2(t)=1$

$\frac{x^2}{4}-\frac{y^2}{9}=1$

Hyperbola crosses x-axis at $x=\pm 2,\ (\pm 2,0)$

Asymptote $y=\frac{\pm 3x}{2}$



Domain $|x| \geq 2$

Range R

b $x = 2\sec(t)$

$$\dot{x} = \frac{dx}{dt} = 2\sec(t)\tan(t)$$

$$y = 3\tan(t)$$

$$\dot{y} = \frac{dy}{dt} = 3\sec^2(t)$$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{3\sec^2(t)}{2\sec(t)\tan(t)} = \frac{3\sec(t)}{2\tan(t)} = \frac{\frac{3}{\cos(t)}}{\frac{2\sin(t)}{\cos(t)}}$$

$$= \frac{3}{2\sin(t)},\ t = \frac{\pi}{4}$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = \frac{3}{2\sin\left(\frac{\pi}{4}\right)} = \frac{3\sqrt{2}}{2}$$

17 $\underset{\sim}{r}(t) = 2\sec^2(t)\underset{\sim}{i} + 3\tan^2(t)\underset{\sim}{j}, 0 \leq t \leq 2\pi$

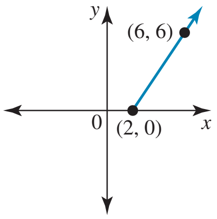
a $x = 2\sec^2(t) \rightarrow \dfrac{x}{2} = \sec^2(t)$

$y = 3\tan^2(t) \rightarrow \dfrac{y}{3} = \tan^2(t)$

$\sec^2(t) - \tan^2(t) = 1$

$$\frac{x}{2} - \frac{y}{3} = 1$$

Straight line when $y = 0,\ x = 2,\ t = 0$



Domain $x \geq 2$, $[2, \infty)$

Range $y \geq 0$, $[0, \infty)$

b $x = 2\sec^2(t) \rightarrow \dot{x} = \dfrac{dx}{dt} = 4\sec^2(t)\tan(t) = \dfrac{4\sin(t)}{\cos^3(t)}$

$y = 3\tan^2(t) \rightarrow \dot{y} = \dfrac{dy}{dt} = 6\tan(t)\sec^2(t) = \dfrac{6\sin(t)}{\cos^3(t)}$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{6\sin(t)}{\cos^3(t)} \times \frac{\cos^3(t)}{4\sin(t)} = \frac{6\sin(t)}{4\sin(t)} = \frac{3}{2}$$

c $|\dot{\underset{\sim}{r}}(t)| = \sqrt{\dfrac{16\sin^2(t) + 36\sin^2(t)}{\cos^6(t)}} = \sqrt{\dfrac{52\sin^2(t)}{\cos^6(t)}}$

$$= \frac{2\sqrt{13}\sin(t)}{\cos^3(t)}$$

$\sin(t) = 0,\ \cos(t) = \pm 1,\ t = n\pi,\ n \in Z$

$|\dot{\underset{\sim}{r}}(t)|_{\min} = 0$

18 $\underset{\sim}{r}(t) = 3\,\text{cosec}(t)\underset{\sim}{i} + 4\cot(t)\underset{\sim}{j},\ 0 \leq t \leq 2\pi$

a $x = 3\,\text{cosec}(t) \rightarrow \dfrac{x}{3} = \text{cosec}(t)$

$y = 4\cot(t) \rightarrow \dfrac{y}{4} = \cot(t)$

$\text{cosec}^2(t) - \cot^2(t) = 1$

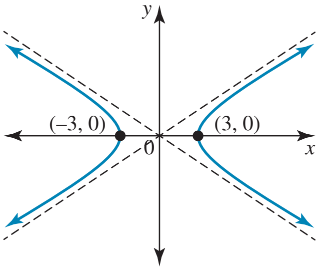
$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

Hyperbola crosses x-axis at $x = \pm 3$, $(\pm 3, 0)$

Asymptote $y = \pm\dfrac{4x}{3}$

Domain $|x| \geq 3$

Range R



b $x = 3\quad \text{cosec}(t) = 3(\sin(t))^{-1} \rightarrow \dfrac{dx}{dt} = -3\dfrac{\cos(t)}{\sin^2(t)}$

$y = 4\cot(t) \rightarrow \dfrac{dy}{dt} = -4\quad \text{cosec}^2(t) = \dfrac{-4}{\sin^2(t)}$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{4}{3\cos(t)}$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{3}} = \frac{4}{3\cos\left(\frac{\pi}{3}\right)} = \frac{8}{3}$$

c $|\dot{\underset{\sim}{r}}(t)| = \sqrt{\dfrac{9\cos^2(t)}{\sin^4(t)} + \dfrac{16}{\sin^4(t)}} = \sqrt{\dfrac{16 + 9\cos^2(t)}{\sin^2(t)}}$

$\cos(t) = 0 \rightarrow \sin(t) = \pm 1$

$|\dot{\underset{\sim}{r}}(t)|_{\min} = 4$

19 $\underset{\sim}{r}(t) = (1 + 2\,\text{cosec}(t))\,\underset{\sim}{i} + (4 - 3\cot(t))\underset{\sim}{j}, 0 \leq t \leq 2\pi$

a $x = 1 + 2\quad \text{cosec}(t) \rightarrow \dfrac{x-1}{2} = \text{cosec}(t)$

$y = 4 - 3\cot(t) \rightarrow \dfrac{y-4}{-3} = \cot(t)$

$\text{cosec}^2(t) - \cot^2(t) = 1$

$$\frac{(x-1)^2}{4} - \frac{(y-4)^2}{9} = 1$$

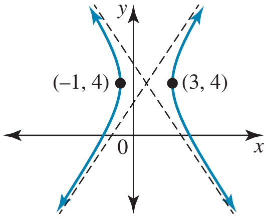
Hyperbola centre at $y = (1, 4)$

Asymptote $y = 4 \pm \dfrac{3}{2}(x - 1)$

$y = \dfrac{-3x}{2} + \dfrac{11}{2},\ y = \dfrac{3x}{2} + \dfrac{5}{2}$

Domain $R\backslash(-1, 3)$

Range R



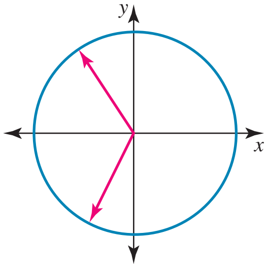
b $x = 1 + 2\,\text{cosec}(t) \to \dot{x} = \dfrac{-2\cos(t)}{\sin^2(t)}$

$y = 4 - 3\cot(t) \to \dot{y} = \dfrac{3}{\sin^2(t)}$

$\dfrac{dy}{dx} = \dfrac{\dot{y}}{\dot{x}} = \dfrac{-3}{2\cos(t)} = 3$

$\cos(t) = -\dfrac{1}{2},\ 0 \le t \le 2\pi$

$t = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$



20 $\underset{\sim}{r}(t) = 10t\underset{\sim}{i} + \left(10t - 4.9t^2\right)\underset{\sim}{j}$

a Hits the ground

$10t - 4.9t^2 = 0$

$t(10 - 4.9t) = 0$

$t = 0,\ t = \dfrac{10}{4.9} = 2.04\text{ sec}$

b $R = x(2.04) = 10 \times 2.04 = 20.4\text{ m}$

c $\underset{\sim}{\dot{r}}(t) = 10\underset{\sim}{i} + (10 - 9.8t)\underset{\sim}{j}$

$\underset{\sim}{\dot{r}}(0) = 10\underset{\sim}{i} + 10\underset{\sim}{j}$

$|\underset{\sim}{\dot{r}}(0)| = 10\sqrt{2} = 14.14\text{ m/s}$

Angle $\tan(\alpha) = 1$

$\alpha = 45°$

d At maximum height

$10 - 9.8t = 0$

$t = \dfrac{10}{9.8} = 1.02\text{ sec}$

$y(1.02) = 10 \times 1.02 - 4.9 \times 1.02^2 = 5.1\text{ m}$

e $x = 10t,\ y = 10t - 4.9t^2$

$t = \dfrac{x}{10},\ y = 10\left(\dfrac{x}{10}\right) - 4.9\left(\dfrac{x}{10}\right)^2 = x - \dfrac{49x^2}{1000}$

$= -\dfrac{x}{1000}(49x - 1000)$

21 $\underset{\sim}{r}(t) = 35t\underset{\sim}{i} + \left(1.8 + 9t - 4.9t^2\right)\underset{\sim}{j},\ t \ge 0$

a Metres above the ground

$\underset{\sim}{r}(0) = 1.8\underset{\sim}{j}$

b $y(t) = 1.8 + 9t - 4.9t^2,\ t \ge 0$

$t = 2.019\ \text{sec}$

c $R = x(2.019) = 35 \times 2.019 = 70.65\text{ m}$

d $\underset{\sim}{\dot{r}}(t) = 35\underset{\sim}{i} + (9 - 9.8t)\underset{\sim}{j} = 0$

$\underset{\sim}{\dot{r}}(0) = 35\underset{\sim}{i} + 9\underset{\sim}{j}$

$|\underset{\sim}{\dot{r}}(0)| = \sqrt{35^2 + 9^2} = 36.14\text{ m/s}$

$\tan(\alpha) = \dfrac{9}{35}$

$\alpha = \tan^{-1}\left(\dfrac{9}{35}\right) = 14.42°$

e $\dot{y} = 9 - 9.8\,t = 0 \to t = 0.918$

$\text{H} = y(0.918) = 1.8 + 9 \times 0.918 - 4.9 \times 0.918^2$

$= 5.93\text{ m}$

f $x = 35t,\ y = 1.8 + 9t - 4.9t^2$

$t = \dfrac{x}{35},\ y = 1.8 + 9\left(\dfrac{x}{35}\right) - 4.9\left(\dfrac{x}{35}\right)^2$

$= 1.8 + \dfrac{9x}{35} - \dfrac{x^2}{250}$

22 $\underset{\sim}{r}(t) = 23t\underset{\sim}{i} + 5t\underset{\sim}{j} + 4\sqrt{2}\sin\left(\dfrac{\pi t}{2}\right)\underset{\sim}{k}$

a Hits the ground

$4\sqrt{2}\sin\left(\dfrac{\pi t}{2}\right) = 0$

$\dfrac{\pi t}{2} = 0,\ \pi$

$t = 2\text{ s}$

b $\underset{\sim}{r}(2) = 46\underset{\sim}{i} + 10\underset{\sim}{j}$

$|\underset{\sim}{r}(2)| = \sqrt{46^2 + 10^2} = 47.1\text{ m}$

c $\underset{\sim}{\dot{r}}(t) = 23\underset{\sim}{i} + 5\underset{\sim}{j} + 2\sqrt{2}\pi\cos\left(\dfrac{\pi t}{2}\right)\underset{\sim}{k}$

$\underset{\sim}{\dot{r}}(0) = 23\underset{\sim}{i} + 5\underset{\sim}{j} + 2\sqrt{2}\pi\underset{\sim}{k}$

$|\underset{\sim}{\dot{r}}(0)| = \sqrt{23^2 + 5^2 + \left(2\sqrt{2}\,\pi\right)^2} = 25.16\text{ m/s}$

d $\sin\left(\dfrac{\pi t}{2}\right) = 1$

$\underset{\sim}{r} \cdot \underset{\sim}{k} = 4\sqrt{2}\text{ m} \approx 5.66\text{ m}$

12.3 Exam questions

1 a $x = 2\sin(2t),\ y = 3\cos(t),\ t \underline{>} 0$

$\underset{\sim}{r}(t) = 2\sin(2t)\underset{\sim}{i} + 3\cos(t)\underset{\sim}{j}$

$\underset{\sim}{r}(t) = 2\sin(2t)\underset{\sim}{i} + 3\cos(t)\underset{\sim}{j}$

$\underset{\sim}{r}\left(\dfrac{\pi}{6}\right) = 2\sin\left(\dfrac{\pi}{3}\right)\underset{\sim}{i} + 3\cos\left(\dfrac{\pi}{6}\right)\underset{\sim}{j}$

$\underset{\sim}{r}\left(\dfrac{\pi}{6}\right) = \sqrt{3}\underset{\sim}{i} + \dfrac{3\sqrt{3}}{2}\underset{\sim}{j}$

$\left|\underset{\sim}{r}\left(\dfrac{\pi}{6}\right)\right| = \sqrt{3 + \dfrac{27}{4}}$

$\left|\underset{\sim}{r}\left(\dfrac{\pi}{6}\right)\right| = \dfrac{\sqrt{39}}{2}$

Award 1 mark for substitution.

Award 1 mark for the correct distance.

b i $\dot{x} = \dfrac{dx}{dt} = 4\cos(2t),\ \dot{y} = \dfrac{dy}{dt} = -3\sin(t)$

$\dfrac{dy}{dx} = \dfrac{\dot{y}}{\dot{x}} = \dfrac{-3\sin(t)}{4\cos(2t)}$

$\left.\dfrac{dy}{dx}\right|_{t=\pi} = \dfrac{-3\sin(\pi)}{4\cos(2\pi)} = 0$

$\underset{\sim}{r}(\pi) = 2\sin(2\pi)\underset{\sim}{i} + 3\cos(\pi)\underset{\sim}{j}$

$\underset{\sim}{r}(\pi) = -3\underset{\sim}{j}$

The tangent is the horizontal line $y = -3$.

Award 1 mark for the correct gradient.

Award 1 mark for the correct velocity.

Award 1 mark for the correct equation.

ii $\dot{\underset{\sim}{r}}(t) = 4\cos(2t)\underset{\sim}{i} - 3\sin(t)\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(\pi) = 4\cos(2\pi)\underset{\sim}{i} - 3\sin(\pi)\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(\pi) = 4\underset{\sim}{i}$

Award 1 mark for the velocity vector.

Award 1 mark for the correct velocity.

iii $\ddot{\underset{\sim}{r}}(t) = -8\sin(2t)\underset{\sim}{i} - 3\cos(t)\underset{\sim}{j}$

$\ddot{\underset{\sim}{r}}(\pi) = -8\sin(2\pi)\underset{\sim}{i} - 3\cos(\pi)\underset{\sim}{j} = 3\underset{\sim}{j}$

$|\ddot{\underset{\sim}{r}}(\pi)| = 3$

Award 1 mark for correct substitution.

Award 1 mark for the correct answer.

c $x = 0 \Rightarrow 2\sin(2t) = 0$

$2t = 0,\ \pi$

$y = 0 \Rightarrow 3\cos(t) = 0$

$t = \dfrac{\pi}{2}$

First time when $t = \dfrac{\pi}{2}$.

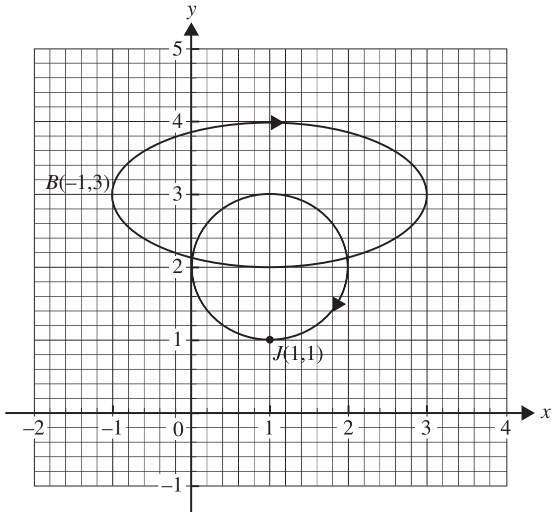
Award 1 mark for the correct time.

2 a $r_{\underset{\sim}{B}}(t) = (1 - 2\cos(t))\,\underset{\sim}{i} + (3 + \sin(t))\,\underset{\sim}{j}$

$r_{\underset{\sim}{B}}(0) = -\underset{\sim}{i} + 3\underset{\sim}{j}$ $B(-1, 3)$

$r_{\underset{\sim}{J}}(t) = (1 - \sin(t))\underset{\sim}{i} + (2 - \cos(t))\underset{\sim}{j}$

$r_{\underset{\sim}{J}}(0) = \underset{\sim}{i} + \underset{\sim}{j}$ $J(1, 1)$



Award 1 mark for the correct directions (arrow clockwise).

Award 1 mark for the correct coordinates of B and J.

VCAA Examination Report **note:**

Students plotted the initial positions correctly but significant numbers of students did not label the direction of motion or clearly identify the jet ski and the boat. Both requirements were explicitly stated in the question.

b i $\dot{r}_{\underset{\sim}{B}}(t) = 2\sin(t)\underset{\sim}{i} + \cos(t)\underset{\sim}{j}$

$\dot{r}_{\underset{\sim}{J}}(t) = -\cos(t)\underset{\sim}{i} + \sin(t)\underset{\sim}{j}$

Speeds

$\left|\dot{r}_{\underset{\sim}{B}}(t)\right| = \sqrt{4\sin^2(t) + \cos^2(t)}$

$\left|\dot{r}_{\underset{\sim}{J}}(t)\right| = \sqrt{\cos^2(t) + \sin^2(t)}$

Equal speeds

$4\sin^2(t) + \cos^2(t) = 1$

$3\sin^2(t) = 0$

$\sin(t) = 0,\ t > 0$

$t = \pi$

Award 1 mark for equating the speeds.

Award 1 mark for the correct value for time.

VCAA Examination Report **note:**

Most students found correct expressions for velocity vectors. The most common error was to equate these velocity vectors rather than equating speeds.

ii $r_{\underset{\sim}{B}}(\pi) = 3\underset{\sim}{i} + 3\underset{\sim}{j}$

$B(3, 3)$

Award 1 mark for the correct coordinate.

VCAA Examination Report **note:**

Some answers were not given in coordinate from.

c i $\dot{r}_{\underset{\sim}{B}}(t) - \dot{r}_{\underset{\sim}{J}}(t) = (\sin(t) - 2\cos(t))\,\underset{\sim}{i}$

$+ (1 + \sin(t) + \cos(t))\,\underset{\sim}{j}$

Distance between $d(t) = \left|\dot{r}_{\underset{\sim}{B}}(t) - \dot{r}_{\underset{\sim}{J}}(t)\right| =$

$\sqrt{(\sin(t) - 2\cos(t))^2 + (1 + \sin(t) + \cos(t))^2}$

Award 1 mark for the final distance.

VCAA Examination Report **note:**

A variety of correct forms was given by students; many of these were likely produced by CAS technology, including expressions involving double angles. Students should take care when transcribing expressions from technology output as errors frequently occur, particularly regarding the number and placement of brackets. Some incorrect answers retained vectors in the expression.

ii Solving $\dfrac{d}{dt}(d(t)) = 0$ using CAS gives the minimum value as $t = 4.28$, minimum distance $d(4.28) = 0.33$ km.

Award 1 mark for the correct minimum distance.

VCAA Examination Report **note:**

Many students found this question difficult. Incorrect answers involving other locally minimum values were frequent.

3 a $\underset{\sim}{r}(t) = (3\sin(2t) - 2)\,\underset{\sim}{i} + (3 - 2\cos(2t))\,\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = 6\cos(2t)\underset{\sim}{i} + 4\sin(2t)$

$|\dot{\underset{\sim}{r}}(t)| = \sqrt{36\cos^2(2t) + 16\sin^2(2t)}$

Award 1 mark for the correct velocity vector.

Award 1 mark for the correct speed.

VCAA Examination Report **note:**

Most students answered this question very well. There were errors seen in the derivative, usually involving sign but sometimes mixing up sin and cos. Occasionally the $\underset{\sim}{\mathbf{i}}$ and $\underset{\sim}{\mathbf{j}}$ were left out at this stage. The most frequent error was not attempting to find the modulus. A small number of students removed the $\underset{\sim}{\mathbf{i}}$ and $\underset{\sim}{\mathbf{j}}$ in an attempt to convert from velocity and speed. A few correctly found an expression for the speed but then made errors in trying to simplify it.

b $\left|\dot{\underset{\sim}{r}}\left(\dfrac{\pi}{12}\right)\right| = \sqrt{36\cos^2\left(\dfrac{\pi}{6}\right) + 16\sin^2\left(\dfrac{\pi}{6}\right)}$

$= \sqrt{36 \times \dfrac{3}{4} + 16 \times \dfrac{1}{4}}$

$\left|\dot{\underset{\sim}{r}}\left(\dfrac{\pi}{12}\right)\right| = \sqrt{31}$ m/s

Award 1 mark for the correct final result.

***VCAA Examination Report* note:**
This question was answered well, with most students who had made a reasonable attempt at part a. answering correctly. Typical errors included leaving the answer as a vector or simplifying incorrectly to obtain $3\sqrt{3}+2$ by taking the square root of individual terms.

12.4 Special parametric curves

12.4 Exercise

1 $\underset{\sim}{r}(t) = 2\sin(t)\underset{\sim}{i} + \cos(2t)\underset{\sim}{j},\ 0 \le t \le 2\pi$

a $x = 2\sin(t)$

$y = \cos(2t)$

$\dot{x} = 2\cos(t)$

$\dot{y} = -2\sin(2t)$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{-2\sin(2t)}{2\cos(t)} = \frac{-\sin(2t)}{\cos(t)}$$

b $\frac{dy}{dx} = 0$

$\sin(2t) = 0$

$\cos(t) \neq 0$

$2t = 0,\ \pi,\ 2\pi,\ 3\pi,\ 4\pi$

$t = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ 2\pi \text{ but } t \neq \frac{\pi}{2},\ \frac{3\pi}{2}$

so $t = 0,\ \pi,\ 2\pi$

$\underset{\sim}{r}(0) = 2\sin(0)\underset{\sim}{i} + \cos(0)\underset{\sim}{j} = \underset{\sim}{j} \rightarrow (0, 1)$

$\underset{\sim}{r}\left(\frac{\pi}{2}\right) = 2\sin\left(\frac{\pi}{2}\right)\underset{\sim}{i} + \cos\left(\frac{\pi}{2}\right)\underset{\sim}{j} = 2\underset{\sim}{i} - \underset{\sim}{j} \rightarrow (2, -1)$

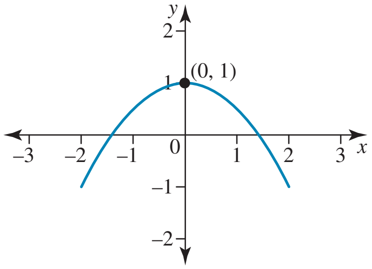
$\underset{\sim}{r}(\pi) = 2\sin(\pi)\underset{\sim}{i} + \cos(\pi)\underset{\sim}{j} = \underset{\sim}{j} \rightarrow (0, 1)$

$\underset{\sim}{r}\left(\frac{3\pi}{2}\right) = 2\sin\left(\frac{3\pi}{2}\right)\underset{\sim}{i} + \cos\left(\frac{3\pi}{2}\right)\underset{\sim}{j} = -2\underset{\sim}{i} - \underset{\sim}{j} \rightarrow (-2, -1)$

$\underset{\sim}{r}(2\pi) = 2\sin(2\pi)\underset{\sim}{i} + \cos(2\pi)\underset{\sim}{j} = \underset{\sim}{j} \rightarrow (0, 1)$

One turning point at (0, 1)

c



d $\dot{\underset{\sim}{r}}(t) = 2\cos(t)\underset{\sim}{i} - 2\sin(2t)\underset{\sim}{j}$

$|\dot{\underset{\sim}{r}}(t)| = \sqrt{4\cos^2(t) + 4\sin^2(2t)}$

$\left|\dot{\underset{\sim}{r}}\left(\frac{\pi}{4}\right)\right| = \sqrt{4\cos^2\left(\frac{\pi}{4}\right) + 4\sin^2\left(\frac{\pi}{2}\right)} = \sqrt{6}$

$|\dot{\underset{\sim}{r}}(\pi)| = \sqrt{4\cos^2(\pi) + 4\sin^2(2\pi)} = 2$

e $t:\ 0,\ \frac{\pi}{4},\ \frac{\pi}{2},\ \frac{3\pi}{4},\ \pi$

$x:\ 0,\ \sqrt{2},\ 2,\ \sqrt{2},\ 0$

$y = 1,\ 0,\ -1,\ 0,\ 1$

Moves from $t = 0 \rightarrow t = \frac{\pi}{4}\ (0, 1)\left(\sqrt{2}, 0\right)$

$$A = 2\int_0^{\frac{\pi}{4}} y \cdot \frac{dx}{dt}dt$$

$$= 2\int_0^{\frac{\pi}{4}} \cos(2t) \cdot 2\cos(t)dt$$

$$= 4\int_0^{\frac{\pi}{4}} \cos(t) \cdot \left(1 - 2\sin^2(t)\right) dt$$

$$= 4\int_0^{\frac{\pi}{4}} \cos(t)dt - 8\int_0^{\frac{\pi}{4}} \cos(t)\sin^2(t)dt$$

$$= 4\,[\sin(t)]_0^{\frac{\pi}{4}} - \frac{8}{3}\left[\sin^3(t)\right]_0^{\frac{\pi}{4}}$$

$$= 2\sqrt{2} - \frac{8}{3}\frac{2\sqrt{2}}{8}$$

$$= \frac{4\sqrt{2}}{3}$$

f $s = 2\int_0^{\frac{\pi}{2}} 2\sqrt{\cos^2(t) + \sin^2(2t)} = 5.9158$

g $x = 2\sin(t) \rightarrow \frac{x}{2} = \sin(t)$

$y = \cos(2t)$

$y = 1 - 2\sin^2(t)$

$y = 1 - 2\left(\frac{x}{2}\right)^2$

$y = 1 - \frac{x^2}{2}$

Crosses x-axis at $y = 0,\ x = \pm\sqrt{2}$

h $A = 2\int_0^{\sqrt{2}} 1 - \frac{x^2}{2}dx$

$$= 2\left[x - \frac{1}{6}x^3\right]_0^{\sqrt{2}}$$

$$= 2\left[\sqrt{2} - \frac{1}{6}\left(\sqrt{2}\right)^3\right]$$

$$= \frac{4\sqrt{2}}{3}$$

2 $\underset{\sim}{r}(t) = a\sin(nt)\underset{\sim}{i} + b\cos(mt)\underset{\sim}{j},\ t \ge 0$

a $x = a\sin(nt)$

$y = b\cos(mt)$

$\dot{x} = na\cos(nt)$

$\dot{y} = -mb\sin(mt)$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{-mb\sin(mt)}{na\cos(nt)}$$

b $|\dot{\underset{\sim}{r}}(t)| = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{a^2n^2\cos^2(nt) + b^2m^2\sin^2(mt)}$

c $a = b,\quad m = n$

$x = b\sin(nt)$

$y = b\cos(nt)$

$$\begin{aligned} x^2 + y^2 &= b^2\sin^2(nt) + b^2\cos^2(nt) \\ &= b^2\left(\sin^2(nt) + \cos^2(nt)\right) \\ &= b^2 \end{aligned}$$

Circle centre at origin with radius $a = b$

d
$$a \neq b, m = n$$
$$x = a\sin(nt) \rightarrow \frac{x}{a} = \sin(nt)$$
$$y = b\cos(nt) \rightarrow \frac{y}{b} = \cos(nt)$$
$$\sin^2(nt) + \cos^2(nt) = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Ellipse

e $m = 2n$
$$x = a\sin(nt) \rightarrow \frac{x}{a} = \sin(nt)$$
$$y = b\cos(2nt) \rightarrow y = b\left(1 - 2\sin^2(nt)\right)$$
$$y = b\left(1 - 2\left(\frac{x}{a}\right)^2\right)$$
$$y = b\left(1 - \frac{2x^2}{a^2}\right)$$
$$y = \frac{b}{a^2}\left(a^2 - 2x^2\right)$$

3 $\underset{\sim}{r}(t) = (2\cos(t) + \cos(2t))\,\underset{\sim}{i} + (2\sin(t) - \sin(2t))\,\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = (-2\sin(t) - 2\sin(2t))\,\underset{\sim}{i} + (2\cos(t) - 2\cos(2t))\,\underset{\sim}{j}$

a
$$x = 2\cos(t) + \cos(2t)$$
$$y = 2\sin(t) - \sin(2t)$$
$$\dot{x} = -2\sin(t) - 2\sin(2t)$$
$$\dot{y} = 2\cos(t) - 2\cos(2t)$$
$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{2(\cos(2t) - \cos(t))}{-2(\sin(t) + \sin(2t))} = \frac{(\cos(2t) - \cos(t))}{(\sin(t) + \sin(2t))}$$

b
$$|\dot{\underset{\sim}{r}}(t)| = \sqrt{4(\sin(t) + \sin(2t))^2 + 4(\cos(t) - \cos(2t))^2}$$
$$= \sqrt{\begin{array}{l} 4\left(\sin^2(t) + 2\sin(t)\sin(2t) + \sin^2(2t)\right) \\ +4\left(\cos^2(t) - 2\cos(t)\cos(2t) + \cos^2(2t)\right) \end{array}}$$
$$= \sqrt{\begin{array}{l} 4\left(\sin^2(t) + \cos^2(t)\right) + 8\left(\sin(t)\sin(2t)\right. \\ \left.- \cos(t)\cos(2t)\right) + 4\left(\sin^2(2t) + \cos^2(2t)\right) \end{array}}$$
$$= \sqrt{4(2 - 2\cos(3t))}$$
$$= \sqrt{8(1 - \cos(3t))}$$
$$= \sqrt{16\sin^2\left(\frac{3t}{2}\right)}$$
$$= 4\sin\left(\frac{3t}{2}\right)$$

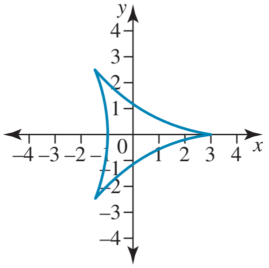
c
$$\underset{\sim}{r}(t) \cdot \dot{\underset{\sim}{r}}(t) = (2\cos(t) + \cos(2t)) \times (-2\sin(t) - 2\sin(2t))$$
$$+ (2\sin(t) - \sin(2t)) \times (2\cos(t) - 2\cos(2t))$$
$$= -4\cos(t)\sin(t) - 4\cos(t)\sin(2t) - 2\sin(t)\cos(2t)$$
$$-2\cos(2t)\sin(2t) + 4\sin(t)\cos(t) - 4\sin(t)\cos(2t)$$
$$-2\cos(t)\sin(2t) + 2\sin(2t)\cos(2t)$$
$$= -4\left(\sin(2t)\cos(t) + \cos(2t)\sin(t)\right) - 2(\sin(2t)\cos(t)$$
$$+ \cos(2t)\sin(t))$$
$$= -6\left(\sin(3t)\right)$$

d $\dot{\underset{\sim}{r}}(t) = -2\left(\cos(t) + 2\cos(2t)\right)\underset{\sim}{i} + 2\left(-\sin(t) + 2\sin(2t)\right)\underset{\sim}{j}$
$$\ddot{x} = -2\left(\cos(t) + 2\cos(2t)\right)$$
$$\ddot{y} = 2\left(-\sin(t) + 2\sin(2t)\right)$$
$$|\ddot{\underset{\sim}{r}}(t)| = \sqrt{\ddot{x}^2 + \ddot{y}^2}$$
$$= \sqrt{4(\cos(t) + 2\cos(2t))^2 + 4(-\sin(t) + 2\sin(2t))^2}$$
$$= \sqrt{4\left(1 + 4 + 4\left(\cos(t)\cos(2t) - \sin(t)\sin(2t)\right)\right)}$$
$$= \sqrt{4\left(5 + 4\cos(3t)\right)}$$
$$= 2\sqrt{4\cos(3t) + 5}$$
$$\cos(3t) = 1 \rightarrow |\ddot{\underset{\sim}{r}}(t)|_{\text{max}} = 6$$
$$\cos(3t) = -1 \rightarrow |\ddot{\underset{\sim}{r}}(t)|_{\text{min}} = 2$$

e $x = 2\cos(t) + \cos(2t)$

$y = 2\sin(t) - \sin(2t)$

Deltoid



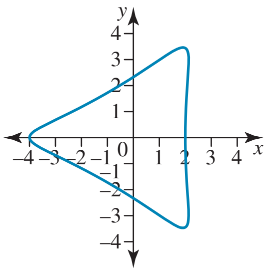
4 $\underset{\sim}{r}(t) = (3\cos(t) - \cos(2t))\,\underset{\sim}{i} + (3\sin(t) + \sin(2t))\,\underset{\sim}{j}$, $0 \leq t \leq 2\pi$

a
$$x = 3\cos(t) - \cos(2t)$$
$$y = 3\sin(t) + \sin(2t)$$
$$\dot{x} = -3\sin(t) + 2\sin(2t)$$
$$\dot{y} = 3\cos(t) + 2\cos(2t)$$
$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{3\cos(t) + 2\cos(2t)}{-3\sin(t) + 2\sin(2t)}$$
$$\left.\frac{dy}{dx}\right|_{t=\frac{3\pi}{4}} = \frac{3\cos\left(\frac{3\pi}{4}\right) + 2\cos\left(\frac{3\pi}{2}\right)}{-3\sin\left(\frac{3\pi}{4}\right) + 2\sin\left(\frac{3\pi}{2}\right)}$$
$$= \frac{-\frac{3\sqrt{2}}{2}}{-\frac{3\sqrt{2}}{2} - 2}$$
$$= \frac{3\sqrt{2}}{3\sqrt{2} + 4} \times \frac{3\sqrt{2} - 4}{3\sqrt{2} - 4}$$
$$= \frac{18 - 12\sqrt{2}}{18 - 16}$$
$$= 9 - 6\sqrt{2}$$

b
$$|\dot{\underset{\sim}{r}}(t)| = \sqrt{\dot{x}^2 + \dot{y}^2}$$
$$\dot{x}^2 = (-3\sin(t) + 2\sin(2t))^2 = 9\sin^2(t)$$
$$-12\sin(t)\sin(2t) + 4\sin^2(2t)$$
$$\dot{y}^2 = (3\cos(t) + 2\cos(2t))^2 = 9\cos^2(t)$$
$$+ 12\cos(t)\cos(2t) + 4\cos^2(2t)$$
$$\dot{x}^2 + \dot{y}^2 = 9\left(\sin^2(t) + \cos^2(t)\right) + 4\left(\sin^2(2t) + \cos^2(2t)\right)$$
$$+ 12\left(\cos(t)\cos(2t) - \sin(t)\sin(2t)\right)$$
$$= 13 + 12\cos(3t)$$
$$|\dot{\underset{\sim}{r}}(t)| = \sqrt{13 + 12\cos(3t)}$$
$$\cos(3t) = -1 \rightarrow |\dot{\underset{\sim}{r}}(t)|_{\text{min}} = 1$$
$$\cos(3t) = 1 \rightarrow |\dot{\underset{\sim}{r}}(t)|_{\text{max}} = 5$$

c $\dot{\underset{\sim}{r}}(t) = (-3\sin(t) + 2\sin(2t))\,\underset{\sim}{i} + (3\cos(t) + 2\cos(2t))\,\underset{\sim}{j}$
$$\ddot{\underset{\sim}{r}}(t) = (-3\cos(t) + 4\cos(2t))\,\underset{\sim}{i} + (-3\sin(t) - 4\sin(2t))\,\underset{\sim}{j}$$
$$|\ddot{\underset{\sim}{r}}(t)| = \sqrt{\ddot{x}^2 + \ddot{y}^2}$$
$$\ddot{x}^2 = (-3\cos(t) + 4\cos(2t))^2 = 9\cos^2(t)$$
$$-24\cos(t)\cos(2t) + 16\cos^2(2t)$$
$$\ddot{y}^2 = (-3\sin(t) - 4\sin(2t))^2 = 9\sin^2(t)$$
$$+ 24\sin(t)\sin(2t) + 16\sin^2(2t)$$
$$\ddot{x}^2 + \ddot{y}^2 = 9\left(\cos^2(t) + \sin^2(t)\right) + 16\left(\cos^2(2t) + \sin^2(2t)\right)$$
$$- 24\left(\cos(t)\cos(2t) - \sin(t)\sin(2t)\right)$$
$$= 25 - 24\cos(3t)$$
$$|\ddot{\underset{\sim}{r}}(t)| = \sqrt{25 - 24\cos(3t)}$$
$$\cos(3t) = 1 \rightarrow |\ddot{\underset{\sim}{r}}(t)|_{\text{min}} = 1$$
$$\cos(3t) = -1 \rightarrow |\ddot{\underset{\sim}{r}}(t)|_{\text{max}} = 7$$

d



$x = 3\cos(t) - \cos(2t)$

$y = 3\sin(t) + \sin(2t)$

Hypocycloid

5 $\underset{\sim}{r}(t) = (t - \sin(t))\,\underset{\sim}{i} + (1 + \cos(t))\,\underset{\sim}{j},\ 0 \le t \le 4\pi$

a $x = t - \sin(t) \to \dot{x} = 1 - \cos(t)$

$y = 1 + \cos(t) \to \dot{y} = -\sin(t)$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{-\sin(t)}{1 - \cos(t)} = -\cot\left(\frac{t}{2}\right)$$

b Turning points $\frac{dy}{dx} = 0, 0 \le t \le 4\pi$

$\sin(t) = 0, \cos(t) \ne 1$

$t = \pi, 3\pi$

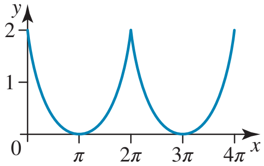
$x(\pi) = \pi, y(\pi) = 1 + \cos(\pi) = 0$

$x(3\pi) = 3\pi, y(3\pi) = 1 + \cos(3\pi) = 0$

$t = \pi, (\pi, 0)$

$t = 3\pi, (3\pi, 0)$ min

c



d $\dot{\underset{\sim}{r}}(t) = (1 - \cos(t))\,\underset{\sim}{i} - \sin(t)\underset{\sim}{j}$

$$\begin{aligned}|\dot{\underset{\sim}{r}}(t)| &= \sqrt{(1 - \cos(t))^2 + (-\sin(t))^2}\\ &= \sqrt{1 - 2\cos(t) + \cos^2(t) + \sin^2(t)}\\ &= \sqrt{2 - 2\cos(t)}\\ &= \sqrt{2\,(1 - \cos(t))}\\ &= \sqrt{2 \times 2\sin^2\left(\frac{t}{2}\right)}\\ &= 2\sin\left(\frac{t}{2}\right)\end{aligned}$$

e
$$\begin{aligned}s &= \int_a^b |\dot{\underset{\sim}{r}}(t)|\ dt\\ &= \int_0^{4\pi} \left|2\sin\left(\frac{t}{2}\right)\right|\ dt\\ &= 4\int_0^{2\pi} \sin\left(\frac{t}{2}\right)\ dt\\ &= 4\left[-2\cos\left(\frac{t}{2}\right)\right]_0^{2\pi}\\ &= 4\,[-2\cos(\pi) + 2\cos(0)]\\ &= 16\ \text{units}\end{aligned}$$

f
$$\begin{aligned}A &= \int_0^{4\pi} y\,dx\\ &= 4\int_0^{\pi} (1 + \cos(t)) \cdot (1 - \cos(t))\ \ dt\\ &= 4\int_0^{\pi} \left(1 - \cos^2(t)\right)\ dt\\ &= 4\int_0^{\pi} \sin^2(t)\,dt\\ &= 2\int_0^{\pi} (1 - \cos(2t))\ dt\\ &= 2\left[t - \frac{1}{2}\sin(2t)\right]_0^{\pi}\\ &= 2\left[\pi - \frac{1}{2}\sin(2\pi) - 0 + \frac{1}{2}\sin(0)\right]\\ &= 2\pi\ \text{units}^2\end{aligned}$$

6 $\underset{\sim}{r}(t) = (\cos(t) + t\sin(t))\,\underset{\sim}{i} + (\sin(t) - t\cos(t))\,\underset{\sim}{j},\ t \ge 0$

a $x = \cos(t) + t\sin(t) \to \dot{x} = t\cos(t)$

$y = \sin(t) - t\cos(t) \to \dot{y} = t\sin(t)$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{t\sin(t)}{t\cos(t)} = \tan(t)$$

b

$\dot{\underset{\sim}{r}}(t) = t\cos(t)\underset{\sim}{i} + t\sin(t)\underset{\sim}{j}$

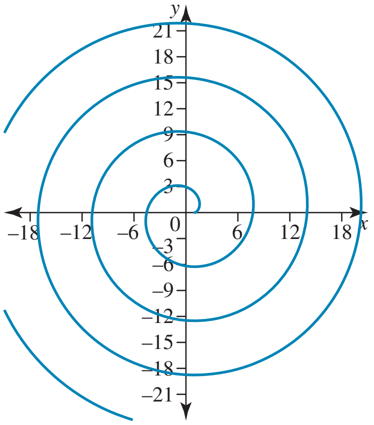
$|\dot{\underset{\sim}{r}}(t)| = \sqrt{t^2\cos^2(t) + t^2\sin^2(t)} = t$

c $\ddot{\underset{\sim}{r}}(t) = (\cos(t) - t\sin(t))\,\underset{\sim}{i} + (\sin(t) + t\cos(t))\,\underset{\sim}{j}$

$$\begin{aligned}|\ddot{\underset{\sim}{r}}(t)| &= \sqrt{(\cos(t) - t\sin(t))^2 + (\sin(t) + t\cos(t))^2}\\ &= \sqrt{1 + t^2}\end{aligned}$$

d
$$\begin{aligned}s &= \int_{t_0}^{t_1} |\dot{\underset{\sim}{r}}(t)|\,dt\\ &= \int_{t_0}^{t_1} tdt\\ &= \left[\frac{1}{2}t^2\right]_{t_0}^{t_1}\\ &= \frac{1}{2}\left(t_1^2 - t_0^2\right)\end{aligned}$$

e



7 a $\underset{\sim}{r}(t) = at\underset{\sim}{i} + \dfrac{a}{1+t^2}\underset{\sim}{j},\ t \in R$

$$x = at \to \dot{x} = a$$

$$y = \frac{a}{1+t^2} \to \dot{y} = -\frac{2at}{\left(1+t^2\right)^2}$$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = -\frac{2t}{\left(1+t^2\right)^2}$$

$$\frac{dy}{dx} = 0$$

Turning point $t = 0 \to \underset{\sim}{r}(0) = 0\underset{\sim}{i} + a\underset{\sim}{j}$

Point is $(0, a)$

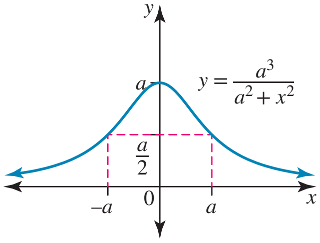
b
$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}\\ &= \frac{d}{dt}\left(\frac{-2t}{\left(1+t^2\right)^2}\right) \times \frac{1}{a}\\ &= \frac{-2\left(1+t^2\right)^2 - 4t\left(1+t^2\right) \times -2t}{\left(1+t^2\right)^4} \times \frac{1}{a}\\ &= \frac{\left(1+t^2\right)\left(-2-2t^2+8t^2\right)}{\left(1+t^2\right)^4} \times \frac{1}{a}\\ &= \frac{2\left(3t^2-1\right)}{a\left(1+t^2\right)^3}\end{aligned}$$

Inflection points when $\dfrac{d^2y}{dx^2} = 0,\ t = \pm\dfrac{1}{\sqrt{3}} = \pm\dfrac{\sqrt{3}}{3}$

$$\underset{\sim}{r}\left(\frac{\sqrt{3}}{3}\right) = \frac{\sqrt{3}a}{3}\underset{\sim}{i} + \frac{3a}{4}\underset{\sim}{j} \quad \left(\frac{\sqrt{3}a}{3}, \frac{3a}{4}\right)$$

$$\underset{\sim}{r}\left(\frac{-\sqrt{3}}{3}\right) = \frac{-\sqrt{3}a}{3}\underset{\sim}{i} + \frac{3a}{4}\underset{\sim}{j} \quad \left(\frac{-\sqrt{3}a}{3}, \frac{3a}{4}\right)$$

c



d $t = 0,\ x = 0,\ y = a,\ (0, a)$

$t = 1,\ x = a,\ y = \dfrac{a}{2},\ \left(a, \dfrac{a}{2}\right)$

$$\begin{aligned}A &= \int_0^1 \frac{a}{1+t^2} \times a\,dt\\ &= a^2\int_0^1 \frac{1}{1+t^2}\,dt\\ &= a^2\left[\tan^{-1}(t)\right]_0^1\\ &= a^2\left[\tan^{-1}(1) - \tan^{-1}(0)\right]\\ &= \frac{\pi a^2}{4}\end{aligned}$$

e $x = at,\ y = \dfrac{a}{1+t^2}$

$$t = \frac{x}{a},\ y = \frac{a}{1+\left(\frac{x}{a}\right)^2} = \frac{a^3}{a^2+x^2}$$

$$\begin{aligned}A &= \int_0^a \frac{a^3}{a^2+x^2}\,dt\\ &= a^3 \times \frac{1}{a}\left[\tan^{-1}\left(\frac{x}{a}\right)\right]_0^a\\ &= a^2\left[\tan^{-1}(1) - \tan^{-1}(0)\right]\\ &= \frac{\pi a^2}{4}\end{aligned}$$

8 $\underset{\sim}{r}(t) = \dfrac{3at}{1+t^3}\underset{\sim}{i} + \dfrac{3at^2}{1+t^3}\underset{\sim}{j},\ t \in R$

a $x^3 + y^3 = 3axy$

$$\begin{aligned}\text{LHS: } x^3 + y^3 &= \left(\frac{3a}{1+t^3}\right)^3 + \left(\frac{3at^2}{1+t^3}\right)^3\\ &= \frac{27a^3t^3 + 27a^3t^6}{\left(1+t^3\right)^3}\\ &= \frac{27a^3t^3}{\left(1+t^3\right)^2}\end{aligned}$$

$$\text{RHS: } 3axy = 3a\left(\frac{3at}{1+t^3}\right)\left(\frac{3at^2}{1+t^3}\right) = \frac{27a^3t^3}{\left(1+t^3\right)^2}$$

$$\therefore x^3 + y^3 = 3axy$$

b $x = \dfrac{3at}{1+t^3}$

$$\begin{aligned}\frac{dx}{dt} &= \frac{3a\left(1+t^3\right) - 3t^2(3at)}{\left(1+t^3\right)^2}\\ &= \frac{3a + 3at^3 - 9at^3}{\left(1+t^3\right)^2}\\ &= \frac{3a\left(1-2at^3\right)}{\left(1+t^3\right)^2}\end{aligned}$$

$$y = \frac{3at^2}{1+t^3}$$

$$\begin{aligned}\frac{dy}{dt} &= \frac{6at\left(1+t^3\right) - 3t^2\left(3at^2\right)}{\left(1+t^3\right)^2}\\ &= \frac{3at\left(2-t^3\right)}{\left(1+t^3\right)^2}\end{aligned}$$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{3at\left(2-t^3\right)}{\left(1+t^3\right)^2} \times \frac{\left(1+t^3\right)^2}{3a\left(1-2at^3\right)} = \frac{t\left(2-t^3\right)}{1-2at^3}$$

Gradient is zero when $t = \sqrt[3]{2},\ 0$

c
$$x^3 + y^3 = 3axy$$

$$3x^2 + 3y^2\left(\frac{dy}{dx}\right) = 3ay + 3ax\left(\frac{dy}{dx}\right)$$

$$\left(3y^2 - 3ax\right)\left(\frac{dy}{dx}\right) = 3ay - 3x^2$$

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

Turning points at $y = \dfrac{x^2}{a}$

$$x^3 + \left(\frac{x^2}{a}\right)^3 = 3ax\left(\frac{x^2}{a}\right)$$

$$x^3 + \frac{x^6}{a^3} = 3x^3$$

$$\frac{x^6}{a^3} - 2x^3 = 0$$

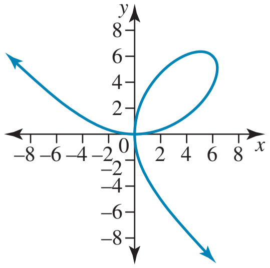
$$x^3\left(\frac{x^3}{a^3} - 2\right) = 0$$

$$x^3 = 2a^3$$

$$x = a\sqrt[3]{2}$$

$$y = \frac{\left(a\sqrt[3]{2}\right)^2}{a} = a * \sqrt[3]{4}$$

d i



ii $\underset{\sim}{r}(t) = \frac{9t}{1+t^3}\underset{\sim}{i} + \frac{9t^2}{1+t^3}\underset{\sim}{j}$

$$\underset{\sim}{r}(2) = \left(\frac{9\times 2}{1+8}\right)\underset{\sim}{i} + \left(\frac{9\times 4}{1+8}\right)\underset{\sim}{j} = 2\underset{\sim}{i} + 4\underset{\sim}{j}$$

iii $\dot{\underset{\sim}{r}}(t) = \frac{9\left(1-2t^3\right)}{\left(1+t^3\right)^2}\underset{\sim}{i} + \frac{9t\left(2-t^3\right)}{\left(1+t^3\right)^2}\underset{\sim}{j}$

$$\dot{\underset{\sim}{r}}(2) = \frac{9\,(1-16)}{(1+8)^2}\underset{\sim}{i} + \frac{9\times 2\,(2-16)}{(1+8)^2}\underset{\sim}{j} = -\frac{5}{3}\underset{\sim}{i} - \frac{4}{3}\underset{\sim}{j}$$

$$|\dot{\underset{\sim}{r}}(2)| = \sqrt{\left(\frac{5}{3}\right)^2 + \left(\frac{4}{3}\right)^2} = \frac{\sqrt{41}}{3}$$

iv $t = 2,\ P(2,4),\ m_T = \frac{4}{5}$

$$y - 4 = \frac{4}{5}(x-2)$$

$$5\,(y-4) = 4\,(x-2)$$

$$5y - 20 = 4x - 8$$

$$5y - 4x - 12 = 0$$

9 a $y = \frac{abx}{x^2+a^2}$

$$\frac{dy}{dx} = \frac{ab(x^2+a^2) - 2x(abx)}{\left(x^2+a^2\right)^2}$$

$$= \frac{ab(a^2-x^2)}{\left(x^2+a^2\right)^2}$$

$$= \frac{ab(a-x)(a+x)}{\left(x^2+a^2\right)^2}$$

For turning points $\frac{dy}{dx} = 0 \to x = \pm a$

$$x = a,\ y = \frac{a^2b}{2a^2} = \frac{b}{2},\ \left(a,\ \frac{b}{2}\right)$$

$$x = -a,\ y = -\frac{a^2b}{2a^2} = -\frac{b}{2},\ \left(-a,\ -\frac{b}{2}\right)$$

b $\frac{d^2y}{dx^2} = \frac{-2abx(x^2+a^2)^2 - ab(a^2-x^2)\times 4x(x^2+a^2)}{\left(x^2+a^2\right)^4}$

$$= \frac{-2abx(x^2+a^2)\left[\left(x^2+a^2\right) + 2\left(a^2-x^2\right)\right]}{\left(x^2+a^2\right)^4}$$

$$= \frac{2abx(x^2-3a^2)}{\left(x^2+a^2\right)^3}$$

For points of inflection $\frac{d^2y}{dx^2} = 0 \to x^2 = 3a^2,\ x = \pm\sqrt{3}a$

$$x = \sqrt{3}a,\ y = \frac{ab\sqrt{3}a}{4a^2} = \frac{\sqrt{3}b}{4},\ \left(\sqrt{3}a,\ \frac{\sqrt{3}b}{4}\right)$$

$$x = -\sqrt{3}a,\ y = -\frac{ab\sqrt{3}a}{4a^2} = -\frac{\sqrt{3}b}{4},\ \left(-\sqrt{3}a,\ -\frac{\sqrt{3}b}{4}\right)$$

c $\underset{\sim}{r}(t) = a\cot(t)\underset{\sim}{i} + \frac{b}{2}\sin(2t)\underset{\sim}{j},\quad t \geq 0$

$$x = a\cot(t)$$

$$y = \frac{b}{2}\sin(2t)$$

$$x^2 + a^2 = \cot^2(t) + a^2$$

$$= a^2\left(\cot^2(t) + 1\right)$$

$$= a^2\operatorname{cosec}^2(t)$$

$$y = \frac{b}{2}\sin(2t)$$

$$= \frac{b}{2}\times 2\cos(t)\sin(t)$$

$$= b\cos(t)\sin(t)$$

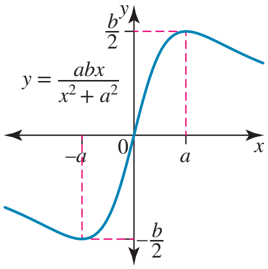
$$= \frac{b\cos(t)}{\sin(t)}\times\sin^2(t)$$

$$= b\cot(t)\times\frac{1}{\operatorname{cosec}^2(t)}$$

$$= b\cdot\frac{x}{a}\times\frac{a^2}{x^2+a^2}$$

$$y = \frac{abx}{x^2+a^2}$$

d



e Method I:

$$A = \int y\,dx$$

$$= \int_0^a \frac{abx}{x^2+a^2}dx$$

$$= \frac{ab}{2}\left[\log_e\left(x^2+a^2\right)\right]_0^a$$

$$= \frac{ab}{2}\left[\log_e\left(2a^2\right) - \log_e\left(a^2\right)\right]$$

$$= \frac{ab}{2}\log_e(2)$$

Method II:

$$t = 0,\ P(\infty,\ 0)$$

$$t = \frac{\pi}{4},\ P\left(a,\ \frac{b}{2}\right)$$

$$t = \frac{\pi}{2},\ P(0,\ 0)$$

$$x = a\cot(t)$$

$$\dot{x} = -a\operatorname{cosec}^2(t)$$

$$A = \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{b}{2}\sin(2t)\times -a\operatorname{cosec}^2(t)\,dt$$

$$= \frac{-ab}{2}\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}\sin(2t)\times\operatorname{cosec}^2(t)\,dt$$

$$= \frac{ab}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\sin(t)\cos(t)\times\frac{1}{\sin^2(t)}dt$$

$$= ab\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\cos(t)}{\sin(t)}dt$$

Let

$$u = \sin(t),\ \ \frac{du}{dt} = \cos(t)$$

$$t = \frac{\pi}{2},\ \ u = \sin\left(\frac{\pi}{2}\right) = 1$$

$$t = \frac{\pi}{4},\ \ u = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$= ab\int_{\frac{1}{\sqrt{2}}}^{1}\frac{1}{u}du$$

$$= ab\left[\log_e(u)\right]_{\frac{1}{\sqrt{2}}}^{1}$$

$$= ab\left[\log_e(1) - \log_e\left(\frac{1}{\sqrt{2}}\right)\right]$$

$$= -ab\log_e(\frac{1}{\sqrt{2}})$$

$$= ab\log_e\left(\sqrt{2}\right)$$

$$= \frac{ab}{2}\log_e(2)$$

10 $\underset{\sim}{r}(t) = 8\cos^3(t)\underset{\sim}{i} + 8\sin^3(t)\underset{\sim}{j},\ 0 \le t \le 2\pi$

a $x = 8\cos^3(t) \to \cos(t) = \left(\frac{x}{8}\right)^{\frac{1}{3}} = \frac{\sqrt[3]{x}}{2}$

$$y = 8\sin^3(t) \to \sin(t) = \left(\frac{y}{8}\right)^{\frac{1}{3}} = \frac{\sqrt[3]{y}}{2}$$

$$\sin^2(t) + \cos^2(t) = 1$$

$$\frac{x^{\frac{2}{3}}}{4} + \frac{y^{\frac{2}{3}}}{4} = 1$$

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$$

b $\dot{x} = -24\cos^2(t)\sin(t)$

$$\dot{y} = 24\sin^2(t)\cos(t)$$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{24\sin^2(t)\cos(t)}{-24\cos^2(t)\sin(t)} = -\tan(t)$$

c $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$

Implicit

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0$$

$$\frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = -\frac{2}{3}x^{-\frac{1}{3}}$$

$$\frac{dy}{dx} = -\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}} = -\sqrt[3]{\frac{y}{x}}$$

Stationary points occur when $\frac{dy}{dx} = 0$:

$$-\sqrt[3]{\frac{y}{x}} = 0$$

$$y = 0$$

Hence,

$$x^{\frac{2}{3}} + 0^{\frac{2}{3}} = 4$$

$$x = \pm 8$$

To test the nature, use the second derivative test:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\left(\frac{y}{x}\right)^{\frac{1}{3}}\right)$$

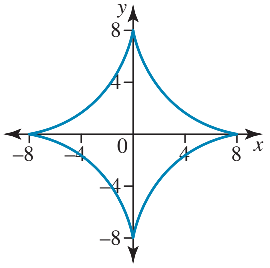
$$= -\frac{1}{3}\left(\frac{y}{x}\right)^{-\frac{2}{3}}\times\frac{d}{dx}\left(\frac{y}{x}\right)$$

$$= -\frac{1}{3}\left(\frac{x}{y}\right)^{\frac{2}{3}}\times\frac{x\frac{dy}{dx} - y}{x^2}$$

$$= -\frac{1}{3}\left(\frac{x}{y}\right)^{\frac{2}{3}}\times\frac{1}{x^2}\times\left(x\times\left(-\left(\frac{y}{x}\right)^{\frac{1}{3}}\right) - y\right)$$

When $y = 0$, $\frac{d^2y}{dx^2}$ is undefined. Hence $(\pm 8, 0)$ are cusps, not stationary points.

d



e $\dot{\underset{\sim}{r}}(t) = -24\cos^2(t)\sin(t)\underset{\sim}{i} + 24\sin^2(t)\cos(t)\underset{\sim}{j}$

$$\left|\dot{\underset{\sim}{r}}(t)\right| = \sqrt{24^2\cos^4(t)\sin^2(t) + 24^2\sin^4(t)\cos^2(t)}$$

$$= 24\cos(t)\sin(t)$$

$$= 12\sin(2t)$$

f $s = \int_{t_0}^{t_1}\left|\dot{\underset{\sim}{r}}(t)\right|\ dt$

$$= \int_0^{2\pi}|12\sin(2t)|\ dt$$

$$= 4\times 12\int_0^{\frac{\pi}{2}}\sin(2t)\ dt$$

$$= -24\left[\cos(2t)\right]_0^{\frac{\pi}{2}}$$

$$= -24\left[\cos(\pi) - \cos(0)\right]$$

$$= 48$$

g $A = \int_a^b y\quad dx$

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$$

$$y^{\frac{2}{3}} = 4 - x^{\frac{2}{3}}$$

$$y = \sqrt{\left(4 - x^{\frac{2}{3}}\right)^3}$$

$$A = 4\int_0^8\sqrt{\left(4 - x^{\frac{2}{3}}\right)^3}dx$$

Let

$$x = 8\cos^3(t)$$

$$x^{\frac{2}{3}} = 4\cos^2(t)$$

$$4 - x^{\frac{2}{3}} = 4 - 4\cos^2(t) = 4\left(1 - \cos^2(t)\right) = 4\sin^2(t)$$

$$\sqrt{\left(4 - x^{\frac{2}{3}}\right)^3} = \sqrt{\left(4\sin^2(t)\right)^3} = 8\sin^3(t)$$

$$dx = -24\cos^2(t)\sin(t)$$

$x = 8, \quad t = 0$

$x = 0, \quad t = \dfrac{\pi}{2}$

$$A = -24 \times 8 \times 4 \int_{\frac{\pi}{2}}^{0} \sin^4(t)\cos^2(t)\,dt$$

$$= 768 \int_0^{\frac{\pi}{2}} \sin^4(t)\cos^2(t)\,dt$$

$$= 24\pi$$

OR

$$A = \int_a^b y dx$$

$$= 4 \int_{\frac{\pi}{2}}^{0} 8\sin^3(t) \times -24\cos^2(t)\sin(t)\,dt$$

$$= 768 \int_0^{\frac{\pi}{2}} \sin^4(t)\cos^2(t)\,dt$$

$$= 24\pi$$

h $x = a\cos^3(t) \to \dot{x} = -3a\cos^2(t)\sin(t)$

$y = a\sin^3(t) \to \dot{y} = 3a\sin^2(t)\cos(t)$

$\dfrac{dy}{dx} = \dfrac{\dot{y}}{\dot{x}} = -\tan(t)$

$P\left(a\cos^3(t), a\sin^3(t)\right)$

$m_T = -\tan(t)$

Tangent $y - y_1 = m(x - x_1)$

$y - a\sin^3(t) = -\tan(t)\left(x - a\cos^3(t)\right)$

$y = a\sin^3(t) - \tan(t)\left(x - a\cos^3(t)\right)$

i Crosses x-axis at Q $y = 0$

$-a\sin^3(t) = -\tan(t)\left(x - a\cos^3(t)\right)$

$-a\sin^3(t) = \dfrac{-\sin(t)}{\cos(t)}\left(x - a\cos^3(t)\right)$

$a\sin^2(t)\cos(t) = x - a\cos^3(t)$

$x = a\cos^3(t) + a\sin^2(t)\cos(t)$

$= a\cos(t)\left[\cos^2(t) + \sin^2(t)\right]$

$x_Q = a\cos(t)$

$Q(a\cos(t), 0)$

Crosses y-axis at R $x = 0$

$y - a\sin^3(t) = -\tan(t) \times -a\cos^3(t)$

$= a\sin(t)\cos^2(t)$

$y = a\sin^3(t) + a\sin(t)\cos^2(t)$

$= a\sin(t)\left[\sin^2(t) + \cos^2(t)\right]$

$y_R = a\sin(t)$

$R(0, \ a\sin(t))$

$$d(RQ) = \sqrt{\left(x_R - x_Q\right)^2 + \left(y_R - y_Q\right)^2}$$

$$= a$$

11 a $y^2\left(x^2 + y^2\right) = a^2x^2$

$y^2x^2 + y^4 = a^2x^2$

$y^4 = a^2x^2 - y^2x^2$

$= x^2\left(a^2 - y^2\right)$

b $x = a\cos(t)\cot(t), \quad y = a\cos(t),$

$0 \le t \le 2\pi$

LHS: $x^2 = a^2\cos^2(t)\cot^2(t)$

RHS: $\dfrac{y^4}{a^2 - y^2} = \dfrac{a^4\cos^4(t)}{a^2 - a^2\cos^2(t)}$

$= \dfrac{a^4\cos^4(t)}{a^2\left(1 - \cos^2(t)\right)}$

$= \dfrac{a^2\cos^4(t)}{\sin^2(t)}$

$= \dfrac{a^2\cos^2(t)\cos^2(t)}{\sin^2(t)}$

$= a^2\cos^2(t)\cot^2(t) = \text{LHS}$

c $x = \dfrac{a\cos^2(t)}{\sin(t)}$

$\dot{x} = \dfrac{-2a\cos(t)\sin^2(t) - a\cos^3(t)}{\sin^2(t)}$

$= \dfrac{-a\cos(t)\left(\sin^2(t) + 1\right)}{\sin^2(t)}$

$y = a\cos(t)$

$\dot{y} = -a\sin(t)$

$\dfrac{dy}{dx} = \dfrac{\dot{y}}{\dot{x}} = \dfrac{\sin^3(t)}{\cos(t)\left(\sin^2(t) + 1\right)}$

Vertical tangent when $\cos(t) = 0, \ t = \dfrac{\pi}{2}, \ \dfrac{3\pi}{2}$

t: $0, \ \dfrac{\pi}{2}, \ \pi, \ \dfrac{3\pi}{2}, \ 2\pi$

x: undef, 0, undef, 0

$y = a, \ 0, \ -a, \ 0, \ a$

So $(0, 0)$ is vertical tangent.

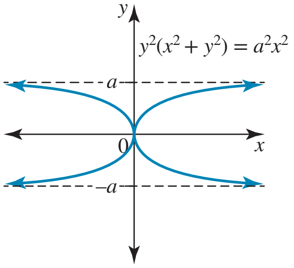
Since $x^2 = \dfrac{y^4}{a^2 - y^2}$

When $a^2 - y^2 = 0, \ y = \pm a$

$\dfrac{dy}{dx} = 0 \to \sin(t) = 0$, but x is undef

No turning points

d



e

$x^2 = \dfrac{y^4}{a^2 - y^2}$

$-y^2 - a^2$

$= a^2 - y^2 \overline{\smash{)}\ y^4}$

$-\left(y^4 - a^2y^2\right)$

a^2y^2

$-\left(a^2y^2 - a^4\right)$

a^4

$= -\left(y^2 + a^2\right) + \dfrac{a^4}{a^2 - y^2}$

$V = \pi \displaystyle\int_0^{\frac{a}{2}} x^2 dy$

Partial fraction

$= \dfrac{a^4}{a^2 - y^2}$

$= \dfrac{a^4}{(a + y)(a - y)}$

$= \dfrac{B}{(a + y)} + \dfrac{C}{(a - y)}$

$$= \frac{B(a-y)+C(a+y)}{a^2-y^2}$$

$$= \frac{y(C-B)+a(C+B)}{a^2-y^2}$$

$$C-B=0 \rightarrow C=B$$

$$a(C+B)=a^4 \rightarrow B=C=\frac{1}{2}a^3$$

$$V=\pi\int_0^{\frac{a}{2}} -y^2-a^2+\frac{1}{2}a^3\left(\frac{1}{a+y}+\frac{1}{a-y}\right)dy$$

$$=\pi\left[-\frac{1}{3}y^3-a^2y+\frac{1}{2}a^3\log_e\left|\frac{a+y}{a-y}\right|\right]_0^{\frac{a}{2}}$$

$$=\pi\left[-\frac{1}{3}\left(\frac{a}{2}\right)^3-a^2\left(\frac{a}{2}\right)+\frac{1}{2}a^3\log_e\left|\frac{a+\frac{a}{2}}{a-\frac{a}{2}}\right|-\frac{1}{2}a^3\log_e(1)\right]$$

$$=\pi a^3\left(\log_e\left(\sqrt{3}\right)-\frac{13}{24}\right)$$

12 a $y=x\sqrt{1-x^2}$

$x=0,\ y=0,\ |x|\le 1$

Domain $[-1,1]$

x-intercepts $(-1,0)\,(0,0)\,(1,0)$

b Product rule

$$u=x$$

$$v=\sqrt{1-x^2}$$

$$\frac{du}{dx}=1$$

$$\frac{dv}{dx}=\frac{-x}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx}=f'(x)=\sqrt{1-x^2}-\frac{x^2}{\sqrt{1-x^2}}$$

$$=\frac{1-2x^2}{\sqrt{1-x^2}}$$

For stationary points $\frac{dy}{dx}=0$

$$1-2x^2=0\rightarrow x=\pm\frac{1}{\sqrt{2}}=\pm\frac{\sqrt{2}}{2}$$

$$x<\frac{\sqrt{2}}{2},\quad \frac{dy}{dx}>0$$

$$y=\frac{\sqrt{2}}{2}\sqrt{1-\frac{1}{2}}=\frac{1}{2}$$

$$x>\frac{\sqrt{2}}{2},\quad \frac{dy}{dx}<0$$

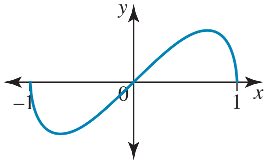
$\left(\frac{\sqrt{2}}{2},\frac{1}{2}\right)$ is max.

$$x<-\frac{\sqrt{2}}{2},\quad \frac{dy}{dx}<0$$

$$x>\frac{-\sqrt{2}}{2},\quad \frac{dy}{dx}>0$$

$$y=-\frac{\sqrt{2}}{2}\sqrt{1-\frac{1}{2}}=\frac{-1}{2}$$

$\left(-\frac{\sqrt{2}}{2},-\frac{1}{2}\right)$ is min.



c $\frac{dy}{dx}=\frac{1-2x^2}{\sqrt{1-x^2}}$

$$u=1-2x^2\rightarrow\frac{du}{dx}=-4x$$

$$v=\sqrt{1-x^2}\rightarrow\frac{dv}{dx}=-\frac{x}{\sqrt{1-x^2}}$$

$$\frac{d^2y}{dx^2}=\frac{-4x\sqrt{1-x^2}+\frac{x\left(1-2x^2\right)}{\sqrt{1-x^2}}}{1-x^2}$$

$$=\frac{2x^3-3x}{\sqrt{\left(1-x^2\right)^3}}$$

For inflection point $\frac{d^2y}{dx^2}=0$

$2x^3-3x=0$

$x=0$

$x^2=\frac{3}{2}\rightarrow x=\pm\sqrt{\frac{3}{2}}$

But this is not in domain $[-1,1]$.

So $(0,0)$ is the only inflection point.

d As $x\rightarrow\ \pm 1$

$$\frac{dy}{dx}\rightarrow\infty$$

There is a vertical tangent at $x=\pm1$.

e $y^2=x^2-x^4$

$$2y\frac{dy}{dx}=2x-4x^3$$

$$\frac{dy}{dx}=\frac{2x-4x^3}{2y}$$

$$=\frac{2x\left(1-2x^2\right)}{2y}$$

$$=\frac{x\left(1-2x^2\right)}{x\sqrt{1-x^2}}$$

$$=\frac{\left(1-2x^2\right)}{\sqrt{1-x^2}}$$

f $\underset{\sim}{r}(t)=\sin(t)\underset{\sim}{i}+\frac{1}{2}\sin(2t)\underset{\sim}{j},\quad t\in[0,2\pi]$

$x=\sin(t)$

$y=\frac{1}{2}\sin(2t)$

LHS: $y^2=\frac{1}{4}\sin^2(2t)$

$=\frac{1}{4}(2\sin(t)\cos(t))^2$

$=\sin^2(t)\cos^2(t)$

$=\sin^2(t)\left(1-\sin^2(t)\right)$

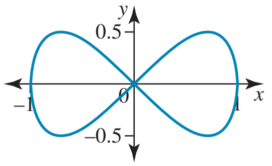
RHS: $x^2-x^4=\sin^2(t)-\sin^4(t)$

$=\sin^2(t)\left(1-\sin^2(t)\right)$

$=$ LHS

So moves on $y^2=x^2-x^4$

g



h $A = \int_a^b y \cdot dx$

$= 4\int_0^1 x\sqrt{1 - x^2}\, dx$

Let

$u = 1 - x^2 \rightarrow \frac{du}{dx} = -2x$

Terminals

$x = 0,\ u = 1$

$x = 1,\ u = 0$

$A = 4\int_1^0 xu^{\frac{1}{2}} \times \frac{1}{-2x} du$

$= -2\int_1^0 u^{\frac{1}{2}} du$

$= 2\int_0^1 u^{\frac{1}{2}} du$

$= 2\left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^1$

$= \frac{4}{3}[1 - 0]$

$= \frac{4}{3}$

i $V = \pi\int_a^b y^2 \cdot dx$

$= 2\pi\int_0^1 x^2 - x^4 dx$

$= 2\pi\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1$

$= \frac{4\pi}{15}$

j $t = 0,\ \underset{\sim}{r}(0) = 0\underset{\sim}{i} + 0\underset{\sim}{j},\ P(0,\ 0)$

$t = \frac{\pi}{2},\ \underset{\sim}{r}\left(\frac{\pi}{2}\right) = \underset{\sim}{i},\ P(1, 0)$

$x = \sin(t) \rightarrow \dot{x} = \cos(t)$

$y = \frac{1}{2}\sin(2t) \rightarrow \dot{y} = \cos(2t)$

$|\underset{\sim}{r}(t)| = \sqrt{\cos^2(2t) + \cos^2(t)}$

$s = 4\int_0^{\frac{\pi}{2}} \sqrt{\cos^2(2t) + \cos^2(t)}\quad dt = 6.0972$ units

13 **a** $f(x) = x^{\frac{3}{2}}\sqrt{4 - x}$

$g(x) = -\sqrt{x^3(4 - x)}$

Domain = $[0, 4]$

$a = 0,\ b = 4$

b g is reflection in x-axis.

c **i** $y^2 = x^3(4 - x) = 4x^3 - x^4$

$2y\frac{dy}{dx} = 12x^2 - 4x^3$

$y\frac{dy}{dx} = 2x^2(3 - x)$

$\frac{dy}{dx} = \frac{2x^2(3 - x)}{y} = \frac{2x^2(3 - x)}{x^{\frac{3}{2}}\sqrt{4 - x}} = \frac{2\sqrt{x}(3 - x)}{\sqrt{4 - x}}$

ii $y\frac{dy}{dx} = 6x^2 - 2x^3$

$\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = 12x - 6x^2$

$y\frac{d^2y}{dx^2} = 12x - 6x^2 - \left(\frac{dy}{dx}\right)^2$

$= 12x - 6x^2 - \frac{4x^4(3 - x)^2}{y^2}$

$= 6x(2 - x) - \frac{4x^4(3 - x)^2}{x^3(4 - x)}$

$= 6x(2 - x) - \frac{4x(3 - x)^2}{(4 - x)}$

$= \frac{2x}{4 - x}\left[3(2 - x)(4 - x) - 2(3 - x)^2\right]$

$y\frac{d^2y}{dx^2} = \frac{2x}{4 - x}\left[24 - 18x + 3x^2 - \left(18 - 12x + 2x^2\right)\right]$

$\frac{d^2y}{dx^2} = \frac{2\left(6 - 6x + x^2\right)}{\sqrt{x(4 - x)^3}}$

iii For turning point $\frac{dy}{dx} = 0$

$4x^2(3 - x)^2 = 0 \rightarrow x = 3,\ 0$

When

$x = 0,\ y = 0$

$\frac{d^2y}{dx^2}$ undef. $(0, 0)$ cusp

$x = 3,\ y = 3\sqrt{3},\ y'' = -2\sqrt{3}$

$x = 3,\ y = -3\sqrt{3},\ y'' = 2\sqrt{3}$

$\left(3,\ 3\sqrt{3}\right)$ max

$\left(3,\ -3\sqrt{3}\right)$ min

iv Inflection

$x^2 - 6x + 6 = 0$

$(x - 3)^2 = 3$

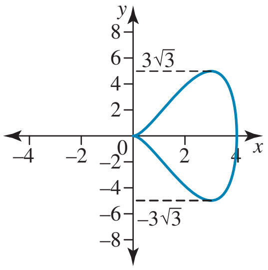
$x = 3 + \sqrt{3}$ is not in $[0,\ 4]$

$x = 3 - \sqrt{3},\ f\left(3 - \sqrt{3}\right) = 2.36$

Inflection $\left(3 - \sqrt{3},\ 2.36\right)$

v As $x \rightarrow 4$ gradient becomes infinite

vi



d **i** $\underset{\sim}{r}(t) = 4\sin^2\left(\frac{t}{2}\right)\underset{\sim}{i} + 16\cos\left(\frac{t}{2}\right)\sin^3\left(\frac{t}{2}\right)\underset{\sim}{j}, t \geq 0$

$x = 4\sin^2\left(\frac{t}{2}\right)$

$y = 16\cos\left(\frac{t}{2}\right)\sin^3\left(\frac{t}{2}\right)$

$$\begin{aligned}\text{RHS}: x^3(4-x) &= 64\sin^6\left(\frac{t}{2}\right)\left(4-4\sin^2\left(\frac{t}{2}\right)\right)\\ &= 256\sin^6\left(\frac{t}{2}\right)\times\left(1-\sin^2\left(\frac{t}{2}\right)\right)\\ &= 256\sin^6\left(\frac{t}{2}\right)\cos^2\left(\frac{t}{2}\right)\\ &= \left[16\cos\left(\frac{t}{2}\right)\sin^3\left(\tfrac{t}{2}\right)\right]^2\\ &= y^2\end{aligned}$$

So particle moves on $y^2 = x^3(4-x)$

ii

$$\begin{aligned}x &= 3\\ 4\sin^2\left(\frac{t}{2}\right) &= 3\\ \sin\left(\frac{t}{2}\right) &= \frac{\sqrt{3}}{2}\\ t &= \frac{2\pi}{3}\end{aligned}$$

iii $x = 4\sin^2\left(\frac{t}{2}\right)$

$$\dot{x} = 4\times 2\times\frac{1}{2}\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right) = 2\sin(t)$$

$$\left.\frac{dx}{dt}\right|_{t=\frac{2\pi}{3}} = \sqrt{3}$$

$t = \frac{2\pi}{3}$ is a maximum

$$\frac{dy}{dx} = 0$$

$$\frac{dy}{dt} = 0$$

$$\dot{\underset{\sim}{r}}\left(\frac{2\pi}{3}\right) = \sqrt{3}\underset{\sim}{i}$$

$$\left|\dot{\underset{\sim}{r}}\left(\frac{2\pi}{3}\right)\right| = \sqrt{3}$$

14 $r(t) = (3\cos(t) - \cos(3t))\,\underset{\sim}{i} + (3\sin(t) - \sin(3t))\,\underset{\sim}{j}$

a

$$\begin{aligned}\dot{\underset{\sim}{r}}(t) &= (-3\sin(t) + 3\sin(3t))\,\underset{\sim}{i}\\ &\quad + (3\cos(t) - 3\cos(3t))\,\underset{\sim}{j}\\ |\dot{\underset{\sim}{r}}(t)| &= \sqrt{\begin{aligned}&(-3\sin(t)+3\sin(3t))^2\\ &+(3\cos(t)-3\cos(3t))^2\end{aligned}}\\ &= \sqrt{\begin{aligned}&18-18\,(\cos(t)\cos(3t)\\ &\quad+\sin(t)\sin(3t))\end{aligned}}\\ &= \sqrt{18\,(1-\cos(2t))}\\ &= \sqrt{18\times 2\sin^2(t)}\\ &= 6\sin(t)\end{aligned}$$

$$|\dot{\underset{\sim}{r}}(t)|_{\max} = 6$$

$$|\dot{\underset{\sim}{r}}(t)|_{\min} = -6$$

b

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{3\cos(t) - 3\cos(3t)}{-3\sin(t) + 3\sin(3t)}$$

$$\frac{dy}{dx} = 0 \rightarrow 3\cos(t) - 3\cos(3t) = 0$$

$$3\cos(t) = 3\left[4\cos^3(t) - 3\cos(t)\right]$$

$$\cos(t) = 0,\ \pm 1$$

$$t = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ 2\pi$$

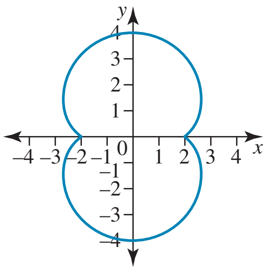
$$\sin(t) \neq 0$$

$$t = \frac{\pi}{2},\ \frac{3\pi}{2}$$

$$\underset{\sim}{r}\left(\frac{\pi}{2}\right) = \left[0\underset{\sim}{i} + 4\underset{\sim}{j}\right],\ p = (0,\ 4)$$

$$\underset{\sim}{r}\left(\frac{3\pi}{2}\right) = \left[0\underset{\sim}{i} - 4\underset{\sim}{j}\right],\ p = (0,\ -4)$$

c



d Show

$$\left(x^2 + y^2 - 4\right)^3 = 108y^2$$

$$\begin{aligned}\text{LHS}: x^2 + y^2 - 4 &= (3\cos(t) - \cos(3t))^2 + (3\sin(t) - \sin(3t))^2 - 4\\ &= -9\cos^2(t) - 6\cos(t)\cos(3t) + \cos^2(t)\\ &\quad +9\sin^2(t) - 6\sin(t)\sin(3t) + \sin^2(t) - 4\\ &= 6 - 6\,(\cos(t)\cos(3t) + \sin(t)\sin(3t))\\ &= 6 - 6\cos(2t)\\ &= 6\,(1 - \cos(2t))\\ &= 12\sin^2(t)\\ \left(x^2 + y^2 - 4\right)^3 &= 1728\sin^6(t)\\ \text{RHS}: 108y^2 &= 108(\sin(t) - \sin(3t))^2\\ 108y^2 &= 108\left(9\sin^2(t) - 6\sin(t)\sin(3t) + \sin^2(3t)\right)\\ \sin(3t) &= 3\sin(t) - 4\sin^3(t)\\ 108y^2 &= 108\left(9\sin^2(t) - 6\sin(t)\left(3\sin(t) - 4\sin^3(t)\right)\right.\\ &\quad \left.+\left(3\sin(t) - 4\sin^3(t)\right)^2\right)\\ &= 1728\sin^6(t) = \text{LHS}\end{aligned}$$

e $t = 0,\ x = 2,\ y = 0 \rightarrow (2, 0)$

$t = \frac{\pi}{2},\ x = 0,\ y = 4 \rightarrow (0, 4)$

$$\begin{aligned}s &= 4\int_0^{\frac{\pi}{2}} |\dot{\underset{\sim}{r}}(t)|\ dt\\ &= 4\int_0^{\frac{\pi}{2}} 6\sin(t)\ dt\\ &= 24\left[-\cos(t)\right]_0^{\frac{\pi}{2}}\\ &= 24\left[-\cos\left(\frac{\pi}{2}\right) + \cos(0)\right]\\ &= 24\text{ units}\end{aligned}$$

f $$A = 4\int_{\frac{\pi}{2}}^{0} (3\sin(t) - \sin(3t))\cdot(-3\sin(t) + 3\sin(3t))\qquad dt = 12\pi$$

15 $\underset{\sim}{r}(t) = 2\cos(t)\,(1 + \cos(t))\,\underset{\sim}{i} + 2\sin(t)\,(1 + \cos(t))\,\underset{\sim}{j}$

$0 \leq t \leq 2\pi$

a

$$\begin{aligned}|\underset{\sim}{r}(t)| &= \sqrt{4\cos^2(t)\,(1 + \cos(t))^2 + 4\sin^2(t)\,(1 + \cos(t))^2}\\ &= \sqrt{4\,(1 + \cos(t))^2\left(\cos^2(t) + \sin^2(t)\right)}\\ &= 2\,(1 + \cos(t))\\ &= 2\left(2\cos^2\left(\frac{t}{2}\right)\right)\\ &= 4\cos^2\left(\frac{t}{2}\right)\end{aligned}$$

b

$$\begin{aligned}\dot{\underset{\sim}{r}}(t) &= \Big(\big(-2\sin(t)\big(1 + \cos(t)\big)\big) - 2\sin(t)\cos(t)\Big)\underset{\sim}{i}\\ &\quad + \Big(2\cos(t)\,\big(1 + \cos(t) - 2\sin^2(t)\big)\ \Big)\underset{\sim}{j}\\ &= -2\sin(t)\,(2\cos(t) + 1)\,\underset{\sim}{i} + \left(4\cos^2(t) + 2\cos(t)\right)\underset{\sim}{j}\end{aligned}$$

$$\begin{aligned}|\underset{\sim}{\dot{r}}(t)| &= \sqrt{4\sin^2(t)(2\cos(t)+1)^2+\left(2\cos(t)(1+\cos(t))-2\sin^2(t)\right)^2}\\ &= 2\sqrt{2(\cos(t)+1)}\\ &= 4\cos\left(\frac{t}{2}\right)\end{aligned}$$

c $\underset{\sim}{r}(t)\cdot\underset{\sim}{\dot{r}}(t) = -4\sin(t)(\cos(t)+1) = 0$

$\sin(t) = 0,$

$\cos(t) = -1$

$t = 0,\ \pi,\ 2\pi$

d
$$\begin{aligned}\underset{\sim}{\dot{r}}(t) &= -2\sin(t)(2\cos(t)+1)\,\underset{\sim}{i}\\ &\quad+\left(4\cos^2(t)+2\cos(t)-2\right)\underset{\sim}{j}\\ \underset{\sim}{\ddot{r}}(t) &= \left(-2\cos(t)(2\cos(t)+1)+4\sin^2(t)\right)\underset{\sim}{i}\\ &\quad+(-8\cos(t)\sin(t)-2\sin(t))\,\underset{\sim}{j}\end{aligned}$$

$|\underset{\sim}{\ddot{r}}(t)| = 2\sqrt{4\cos(t)+5}$

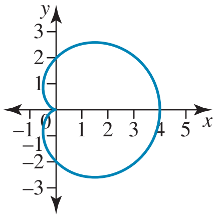
$\cos(t) = 1$

$|\underset{\sim}{\ddot{r}}(t)|_{\max} = 6$

$\cos(t) = -1$

$|\underset{\sim}{\ddot{r}}(t)|_{\min} = 2$

e



f $x = 2\cos(t)(1+\cos(t))$

$y = 2\sin(t)(1+\cos(t))$

$t:\ 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ 2\pi$

$x:\ 4,\ 0,\ 0,\ 0,\ 4$

$y:\ 0,\ 2,\ 0,\ -2,\ 0$

$p:\ (4,0),\ (0,2),\ (0,0),\ (0,-2),\ (4,0)$

$$\begin{aligned}s &= 2\int_0^{\pi}|\underset{\sim}{\dot{r}}(t)|\ dt\\ &= 2\int_0^{\pi}4\cos\left(\frac{t}{2}\right)\ dt\\ &= 16\left[\sin\left(\frac{t}{2}\right)\right]_0^{\pi}\\ &= 16\left[\sin\left(\frac{\pi}{2}\right)-\sin(0)\right]\\ &= 16\text{ units}^2\end{aligned}$$

g
$$\begin{aligned}A &= 2\int_{\pi}^{0}2\sin(t)(1+\cos(t))\times-2\sin(t)(2\cos(t)+1)\,dt\\ &= -8\int_{\pi}^{0}\sin^2(t)(1+\cos(t))(2\cos(t)+1)\,dt\\ &= 6\pi\text{ units}^2\end{aligned}$$

h $x = 2\cos(t)(1+\cos(t))$

$y = 2\sin(t)(1+\cos(t))$

$x^2 = 4\cos^2(t)(1+\cos(t))^2$

$y^2 = 4\sin^2(t)(1+\cos(t))^2$

$$\begin{aligned}x^2+y-2x &= 4\cos^2(t)(1+\cos(t))^2+4\sin^2(t)\\ &\quad(1+\cos(t))^2-4\cos(t)(1+\cos(t))\\ &= 4(1+\cos(t))^2\left(\sin^2(t)+\cos^2(t)\right)\\ &\quad-4\cos(t)(1+\cos(t))\\ &= 4(1+\cos(t))[1+\cos(t)-\cos(t)]\\ &= 4(1+\cos(t))\end{aligned}$$

$$\begin{aligned}\text{LHS}:\ \left(x^2+y-2x\right)^2 &= 16(1+\cos(t))^2\\ \text{RHS}:\ 4\left(x^2+y^2\right) &= 4\left[4\cos^2(t)(1+\cos(t))^2+4\sin^2(t)\right.\\ &\quad\left.(1+\cos(t))^2\right.\\ &= 16(1+\cos(t))^2\left[\cos^2(t)+\sin^2(t)\right]\\ &= 16(1+\cos(t))^2 = \text{LHS}\end{aligned}$$

16 $\underset{\sim}{r}(t) = (n\cos(t)+\cos(nt))\,\underset{\sim}{i}+(n\sin(t)-\sin(nt))\,\underset{\sim}{j},\ 0\le t\le 2\pi$

$0\le t\le 2\pi$

$\underset{\sim}{\dot{r}}(t) = -n(\sin(t)+\sin(nt))\underset{\sim}{i}+n(\cos(t)-\cos(nt))\underset{\sim}{j}$

a $x = n\cos(t)+\cos(nt)$

$\dot{x} = -n(\sin(t)+\sin(nt))$

$y = n\sin(t)-\sin(nt)$

$\dot{y} = n(\cos(t)-\cos(nt))$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{n(\cos(t)-\cos(nt))}{-n(\sin(t)+\sin(nt))} = \frac{\cos(nt)-\cos(t)}{\sin(nt)+\sin(nt)}$$

b
$$\begin{aligned}|\underset{\sim}{\dot{r}}(t)| &= \sqrt{n^2(\sin(t)+\sin(nt))^2+n^2(\cos(t)-\cos(nt))^2}\\ &= \sqrt{n^2(2+2\sin(t)\sin(nt)-2\cos(t)\cos(nt))}\\ &= \sqrt{n^2(2-2\cos(n+1)t)}\\ &= \sqrt{2n^2(1-\cos(n+1)t)}\\ &= \sqrt{2n^2\times 2\cos^2\left(\frac{n+1}{2}\right)t}\\ &= 2n\cos\left(\left(\frac{n+1}{2}\right)t\right)\end{aligned}$$

c
$$\begin{aligned}\underset{\sim}{r}(t)\cdot\underset{\sim}{\dot{r}}(t) &= -n(\sin(t)+\sin(nt))(n\cos(t)+\cos(nt))\\ &\quad+n(\cos(t)-\cos(nt))(n\sin(t)-\sin(nt))\\ &= -n\sin(t)(n\cos(t)+\cos(nt))-n\sin(nt)\\ &\quad(n\cos(t)+\cos(nt))+n\cos(t)(n\sin(t)-\sin(nt))\\ &\quad-n\cos(nt)(n\sin(t)-\sin(nt))\\ &= -n^2(\sin(nt)\cos(t)+\cos(nt)\sin(t))\\ &\quad-n(\sin(t)\cos(nt)+\cos(t)\sin(nt))\\ &= -\left(n^2+n\right)\sin((n+1)t)\end{aligned}$$

d
$$\begin{aligned}\underset{\sim}{\dot{r}}(t) &= -n(\cos(t)+n\cos(nt))\underset{\sim}{i}\\ &\quad+n(-\sin(t)+n\sin(nt))\underset{\sim}{j}\\ |\underset{\sim}{\dot{r}}(t)| &= \sqrt{\begin{matrix}n^2(\cos(t)+n\cos(nt))^2\\ +n^2(-\sin(t)+n\sin(nt))^2\end{matrix}}\\ &= \sqrt{\begin{matrix}n^2\sin^2(t)+\cos^2(t)+2n^3\cos(t)\cos(nt)\\ -2n^3\sin(t)\sin(nt)+n^4\left(\cos^2(nt)+\sin^2(nt)\right)\end{matrix}}\\ &= \sqrt{n^2+n^4+2n^3\cos((n+1)t)}\\ &= n\sqrt{2n\cos((n+1)t)+\left(1+n^2\right)}\end{aligned}$$

$\cos((n+1)t) = 1$

$|\underset{\sim}{\dot{r}}(t)|_{\max} = n\sqrt{n^2+2n+1} = n(n+1)$

$\cos((n+1)t) = -1$

$|\underset{\sim}{\dot{r}}(t)|_{\min} = n\sqrt{n^2-2n+1} = n(n-1)$

12.4 Exam questions

1 $x = 3\sin(t),\ y = 4\cos(t),\ 0 \le t \le \pi$

$\dot{x} = 3\cos(t),\ \dot{y} = -4\sin(t)$

$$s = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2}dt$$

$$s = \int_0^{\pi} \sqrt{9\cos^2(2t) + 16\sin^2(t)}\ dt$$

$$= \int_0^{\pi} \sqrt{9\left(1 - \sin^2(2t)\right) + 16\sin^2(t)}\ dt$$

$$s = \int_0^{\pi} \sqrt{9 + 7\sin^2(t)}\ dt$$

The correct answer is **B**.

2 $\underset{\sim}{r}(t) = \frac{t^3}{3}\underset{\sim}{i}\left(\arcsin(t) + t\sqrt{1-t^2}\right)\underset{\sim}{j},\ 0 \le t \le \frac{3}{4},$

$$x = \frac{t^3}{3}\quad y = \arcsin(t) + t\sqrt{1-t^2}$$

$$\dot{x} = \frac{dx}{dt} = t^2, \dot{y} = \frac{dy}{dt} = \frac{1}{\sqrt{1-t^2}} + \sqrt{1-t^2} - \frac{t^2}{\sqrt{1-t^2}}$$

$$\dot{y} = \frac{dy}{dt} = \frac{1 + 1 - t^2 - t^2}{\sqrt{1-t^2}} = \frac{2\left(1-t^2\right)}{\sqrt{1-t^2}} = 2\sqrt{1-t^2}\quad \text{[1 mark]}$$

$$\sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{t^4 + 4\left(1-t^2\right)} = \sqrt{t^4 - 4t^2 + 4}\quad \text{[1 mark]}$$

$$\sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{\left(t^2 - 2\right)^2} = \left|t^2 - 2\right| = 2 - t^2 > 0$$

$$\text{since } 0 \le t \le \frac{3}{4}\quad \text{[1 mark]}$$

$$d = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2}dt = \int_0^{\frac{3}{4}} \left(at^2 + bt + c\right) dt\quad \text{[1 mark]}$$

$$\sqrt{\dot{x}^2 + \dot{y}^2} = 2 - t^2 = at^2 + bt + c$$

$$a = -1, b = 0, c = 2\quad \text{[1 mark]}$$

VCAA Examination Report **note:**

Only a few students obtained full marks for this question. Most students recognised that the arc length formula needed to be applied, but some had difficultly differentiating $\text{arc}\sin(t) + t\sqrt{1-t^2}$. A number of students applied the product and chain rule correctly to the term and ignored the $\arcsin(t)$ term. Many students had difficulty simplifying $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$ and were unable to proceed further. Of those students who were able to find that $d = \int_0^{\frac{3}{4}} \sqrt{\left(t^2 - 2\right)^2}dt$, only a small number recognised the significance of the domain $0 < t < 1$.

3 $r(t) = \cos^3(t)\underset{\sim}{i} + \sin^3(t)\underset{\sim}{j},\ 0 \le t \le \frac{\pi}{4},\ s = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2}dt$

$$\dot{x} = \frac{dx}{dt} = -3\cos^2(t)\sin(t),\ \dot{y} = \frac{dy}{dt} = 3\sin^2(t)\cos(t)$$

$$\sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cos^2(t)}$$

$$= \sqrt{9\cos^2(t)\sin^2(t)\left(\cos^2(t) + \sin^2(t)\right)}$$

$$\sqrt{\dot{x}^2 + \dot{y}^2} = |3\sin(t)\cos(t)| \text{ but } 0 \le t \le \frac{\pi}{4} \text{ so}$$

$$\sqrt{\dot{x}^2 + \dot{y}^2} = \frac{3}{2}\sin(2t)$$

$$s = \int_0^{\frac{\pi}{4}} \sqrt{\dot{x}^2 + \dot{y}^2}\ dt$$

$$= \int_0^{\frac{\pi}{4}} \frac{3}{2}\sin(2t)dt$$

$$= \left[-\frac{3}{4}\cos(2t)\right]_0^{\frac{\pi}{4}}$$

$$= -\frac{3}{4}\cos\left(\frac{\pi}{2}\right) + \frac{3}{4}\cos(0)$$

$$s = \frac{3}{4}\text{units}$$

Award 1 mark for finding the derivatives of x and y.
Award 1 mark for the correct definite integral setting up the arc length.
Award 1 mark for the correct double angle.
Award 1 mark for the final correct answer.

VCCA Examination Report **note:**

Students had varied success with a question. A number of students were unable to find the necessary derivatives, neglecting to use the chain rule. Of those who did use the chain rule, the question was reasonably well answered, although there were many who did not recognise the appropriate form of the arc length formula. Some incorrect answers involved:

- finding $\left|r\frac{\pi}{4} - r(0)\right|$
- using the formula with $\frac{dy}{dx}$ (sometimes with correct working, expect for using dt rather than dx)
- errors in derivatives
- an inability to correctly simplify the expression under the square root
- taking the square root of individual terms
- correct simplification but an error at the end with terminals or substitution and missing dt in lines of working.

12.5 Integration of vectors

12.5 Exercise

1 $\dot{\underset{\sim}{r}}(t) = (4t - 4)\underset{\sim}{i} - 3\underset{\sim}{j}$

$\underset{\sim}{r}(1) = 3\underset{\sim}{i} + \underset{\sim}{j}$

$$\underset{\sim}{r}(t) = \int (4t - 4)\,dt\underset{\sim}{i} - \int 3dt\underset{\sim}{j} = \left(2t^2 - 4t\right)\underset{\sim}{i} - 3t\underset{\sim}{j} + \underset{\sim}{c}$$

$$\underset{\sim}{r}(1) = (2 - 4)\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{c} = 3\underset{\sim}{i} + \underset{\sim}{j}$$

$$\underset{\sim}{c} = 5\underset{\sim}{i} + 4\underset{\sim}{j}$$

$$\underset{\sim}{r}(t) = \left(2t^2 - 4t + 5\right)\underset{\sim}{i} + (4 - 3t)\underset{\sim}{j}$$

2 $\dot{\underset{\sim}{r}}(t) = 6\sin(3t)\underset{\sim}{i} + 4e^{-2t}\underset{\sim}{j}$

$\underset{\sim}{r}(0) = 3\underset{\sim}{i} + \underset{\sim}{j}$

$$\underset{\sim}{r}(t) = \int 6\sin(3t)dt\underset{\sim}{i} + \int 4e^{-2t}dt\underset{\sim}{j} = -2\cos(3t)\underset{\sim}{i} - 2e^{-2t}\underset{\sim}{j} + \underset{\sim}{c}$$

$$\underset{\sim}{r}(0) = -2\cos(0)\underset{\sim}{i} - 2e^{0}\underset{\sim}{j} + \underset{\sim}{c} = 3\underset{\sim}{i} + \underset{\sim}{j}$$

$$\underset{\sim}{c} = 5\underset{\sim}{i} + 3\underset{\sim}{j}$$

$$\underset{\sim}{r}(t) = (5 - 2\cos(3t))\,\underset{\sim}{i} + \left(3 - 2e^{-2t}\right)\underset{\sim}{j}$$

3 **a** $\dot{\underset{\sim}{r}}(t) = e^{\frac{-t}{3}}\underset{\sim}{i} + 4t^3\underset{\sim}{j},\ \underset{\sim}{r}(0) = \underset{\sim}{0}$

$\underset{\sim}{r}(t) = \int e^{\frac{-t}{3}}dt\underset{\sim}{i} + \int 4t^3 dt\underset{\sim}{j}$

$= -3e^{\frac{-t}{3}}\underset{\sim}{i} + t^4\underset{\sim}{j} + \underset{\sim}{c}_1$

$\underset{\sim}{r}(0) = \underset{\sim}{0} = -3\underset{\sim}{i} + \underset{\sim}{c}_1$

$\underset{\sim}{c}_1 = 3\underset{\sim}{i}$

$\underset{\sim}{r}(t) = 3\left(1 - e^{\frac{-t}{3}}\right)\underset{\sim}{i} + t^4\underset{\sim}{j}$

b $\dot{\underset{\sim}{r}}(t) = 2t\underset{\sim}{i} + 6\sin(2t)\underset{\sim}{j},\ \underset{\sim}{r}(0) = \underset{\sim}{0}$

$\underset{\sim}{r}(t) = \int 2t\,dt\underset{\sim}{i} + \int 6\sin(2t)\,dt\underset{\sim}{j}$

$= t^2\underset{\sim}{i} - 3\cos(2t)\underset{\sim}{j} + \underset{\sim}{c}_1$

$\underset{\sim}{r}(0) = \underset{\sim}{0} = -3\underset{\sim}{j} + \underset{\sim}{c}_1$

$\underset{\sim}{c}_1 = 3\underset{\sim}{j}$

$\underset{\sim}{r}(t) = t^2\underset{\sim}{i} - 3\cos(2t)\underset{\sim}{j} + 3\underset{\sim}{j}$

$= t^2\underset{\sim}{i} + 3(1 - \cos(2t))\underset{\sim}{j}$

4 **a** $\dot{\underset{\sim}{r}}(t) = \frac{1}{\sqrt{16-t^2}}\underset{\sim}{i} - \frac{t}{\sqrt{t^2+9}}\underset{\sim}{j},\ \underset{\sim}{r}(0) = 3\underset{\sim}{i} + 2\underset{\sim}{j}$

$\underset{\sim}{r}(t) = \int \frac{1}{\sqrt{16-t^2}}\,dt\underset{\sim}{i} - \int \frac{t}{\sqrt{t^2+9}}\,dt\underset{\sim}{j}$

$= \sin^{-1}\left(\frac{t}{4}\right)\underset{\sim}{i} - \sqrt{t^2+9}\underset{\sim}{j} + \underset{\sim}{c}$

$\underset{\sim}{r}(0) = \underset{\sim}{0} - 3\underset{\sim}{j} + \underset{\sim}{c}_1 = 3\underset{\sim}{i} + 2\underset{\sim}{j}$

$\underset{\sim}{c}_1 = 3\underset{\sim}{i} + 5\underset{\sim}{j}$

$\underset{\sim}{r}(t) = \sin^{-1}\left(\frac{t}{4}\right)\underset{\sim}{i} - \sqrt{t^2+9}\underset{\sim}{j} + 3\underset{\sim}{i} + 5\underset{\sim}{j}$

$= \left(3 + \sin^{-1}\left(\frac{t}{4}\right)\right)\underset{\sim}{i} + \left(5 - \sqrt{t^2+9}\right)\underset{\sim}{j}$

b $\dot{\underset{\sim}{r}}(t) = \frac{2}{2t+1}\underset{\sim}{i} + \frac{72}{(3t+2)^2}\underset{\sim}{j},\ \underset{\sim}{r}(0) = 5\underset{\sim}{i} + \underset{\sim}{j}$

$\underset{\sim}{r}(t) = \int \frac{2}{2t+1}\,dt\underset{\sim}{i} + \int \frac{72}{(3t+2)^2}\,dt\underset{\sim}{j}$

$= \log_e(2t+1)\underset{\sim}{i} - \frac{24}{(3t+2)}\underset{\sim}{j} + \underset{\sim}{c}$

$\underset{\sim}{r}(0) = \log_e(1)\underset{\sim}{i} - 12\underset{\sim}{j} + \underset{\sim}{c} = 5\underset{\sim}{i} + \underset{\sim}{j}$

$\underset{\sim}{c}_1 = 5\underset{\sim}{i} + 13\underset{\sim}{j}$

$\underset{\sim}{r}(t) = \log_e(2t+1)\underset{\sim}{i} - \frac{12}{(3t+2)^2}\underset{\sim}{j} + 5\underset{\sim}{i} + 13\underset{\sim}{j}$

$= \left(5 + \log_e(2t+1)\right)\underset{\sim}{i} + \left(13 - \frac{24}{(3t+2)}\right)\underset{\sim}{j}$

5 $\ddot{\underset{\sim}{r}}(t) = -12t^2\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(2) = -2\underset{\sim}{i} - 16\underset{\sim}{j}$

$\underset{\sim}{r}(2) = \underset{\sim}{i} + 6\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = \int -12t^2 dt\underset{\sim}{j} = -4t^3\underset{\sim}{j} + \underset{\sim}{c}_1$

$\dot{\underset{\sim}{r}}(2) = -4(2)^3\underset{\sim}{j} + \underset{\sim}{c}_1 = -2\underset{\sim}{i} - 16\underset{\sim}{j} \rightarrow \underset{\sim}{c}_1 = -2\underset{\sim}{i} + 16\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = -4t^3\underset{\sim}{j} + \left(-2\underset{\sim}{i} + 16\underset{\sim}{j}\right)$

$\dot{\underset{\sim}{r}}(t) = -2\underset{\sim}{i} + \left(16 - 4t^3\right)\underset{\sim}{j}$

$\underset{\sim}{r}(t) = \int -2dt\underset{\sim}{i} + \int \left(16 - 4t^3\right)dt\underset{\sim}{j} = -2t\underset{\sim}{i} + \left(16t - t^4\right)\underset{\sim}{j} + \underset{\sim}{c}_2$

$\underset{\sim}{r}(2) = -2(2)\underset{\sim}{i} + \left(16(2) - 2^4\right)\underset{\sim}{j} + \underset{\sim}{c}_2 = \underset{\sim}{i} + 6\underset{\sim}{j} \rightarrow \underset{\sim}{c}_2 = 5\underset{\sim}{i} - 10\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = (5 - 2t)\underset{\sim}{i} + \left(16t - t^4 - 10\right)\underset{\sim}{j}$

6 $\ddot{\underset{\sim}{r}}(t) = 6\underset{\sim}{i} + 2\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(1) = 6\underset{\sim}{i} + 10\underset{\sim}{j}$

$\underset{\sim}{r}(1) = -2\underset{\sim}{i} + 7\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = \int 6dt\underset{\sim}{i} + \int 2dt\underset{\sim}{j} = 6t\underset{\sim}{i} + 2t\underset{\sim}{j} + \underset{\sim}{c}_1$

$\dot{\underset{\sim}{r}}(1) = 6\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{c}_1 = 6\underset{\sim}{i} + 10\underset{\sim}{j} \rightarrow \underset{\sim}{c}_1 = 8\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = 6t\underset{\sim}{i} + (2t + 8)\underset{\sim}{j}$

$\underset{\sim}{r}(t) = \int 6tdt\underset{\sim}{i} + \int (2t + 8)\,dt\underset{\sim}{j} = 3t^2\underset{\sim}{i} + \left(t^2 + 8t\right)\underset{\sim}{j} + \underset{\sim}{c}_2$

$\underset{\sim}{r}(1) = 3\underset{\sim}{i} + (1 + 8(1))\underset{\sim}{j} + \underset{\sim}{c}_2 = -2\underset{\sim}{i} + 7\underset{\sim}{j} \rightarrow \underset{\sim}{c}_2 = -5\underset{\sim}{i} - 2\underset{\sim}{j}$

$\underset{\sim}{r}(t) = \left(3t^2 - 5\right)\underset{\sim}{i} + \left(t^2 + 8t - 2\right)\underset{\sim}{j}$

7 **a** $\ddot{\underset{\sim}{r}}(t) = 8\underset{\sim}{i} + 6\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(0) = \underset{\sim}{0}$

$\underset{\sim}{r}(0) = 3\underset{\sim}{i} - 2\underset{\sim}{j},\ t \geq 0$

$\dot{\underset{\sim}{r}}(t) = \int 8dt\underset{\sim}{i} + \int 6dt\underset{\sim}{j} = 8t\underset{\sim}{i} + 6t\underset{\sim}{j} + \underset{\sim}{c}_1$

$\dot{\underset{\sim}{r}}(0) = \underset{\sim}{c}_1 = \underset{\sim}{0}$

$\dot{\underset{\sim}{r}}(t) = 8t\underset{\sim}{i} + 6t\underset{\sim}{j}$

$\underset{\sim}{r}(t) = \int 8tdt\vec{i} + \int 6tdt\vec{j} = 4t^2\underset{\sim}{i} + 3t^2 + \underset{\sim}{c}_2$

$\underset{\sim}{r}(0) = \underset{\sim}{c}_2 = 3\underset{\sim}{i} - 2\underset{\sim}{j}$

$\underset{\sim}{r}(t) = \left(4t^2 + 3\right)\underset{\sim}{i} + \left(3t^2 - 2\right)\underset{\sim}{j}$

b $\ddot{\underset{\sim}{r}}(t) = 4\underset{\sim}{i} + 2\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(0) = 8\underset{\sim}{j}$

$\underset{\sim}{r}(0) = 3\underset{\sim}{i} + 4\underset{\sim}{j},\ t \geq 0$

$\dot{\underset{\sim}{r}}(t) = \int 4dt\underset{\sim}{i} + \int 2dt\underset{\sim}{j} = 4t\underset{\sim}{i} + 2t\underset{\sim}{j} + \underset{\sim}{c}_1$

$\dot{\underset{\sim}{r}}(0) = \underset{\sim}{c}_1 = 8\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = 4t\underset{\sim}{i} + (2t + 8)\underset{\sim}{j}$

$\underset{\sim}{r}(t) = \int 4t\,dt\underset{\sim}{i} + \int (2t + 8)\ dt\underset{\sim}{j} = 2t^2\underset{\sim}{i} + \left(t^2 + 8t\right)\underset{\sim}{j} + \underset{\sim}{c}_2$

$\underset{\sim}{r}(0) = \underset{\sim}{c}_2 = 3\underset{\sim}{i} + 4\underset{\sim}{j}$

$\underset{\sim}{r}(t) = \left(2t^2 + 3\right)\underset{\sim}{i} + \left(t^2 + 8t + 4\right)\underset{\sim}{j}$

8 **a** $\ddot{\underset{\sim}{r}}(t) = -\frac{9}{(3t+1)^2}\underset{\sim}{i} + \frac{32}{(2t+1)^3}\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(0) = 3\underset{\sim}{i} - 8\underset{\sim}{j},\ t \geq 0$

$\underset{\sim}{r}(0) = 4\underset{\sim}{i} + 3\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = \int -\frac{9}{(3t+1)^2}dt\underset{\sim}{i} + \int \frac{32}{(2t+1)^3}dt\underset{\sim}{j}$

$= \frac{3}{3t+1}\underset{\sim}{i} - \frac{8}{(2t+1)^2}\underset{\sim}{j} + \underset{\sim}{c}_1$

$\dot{\underset{\sim}{r}}(0) = 3\underset{\sim}{i} - 8\underset{\sim}{j} + \underset{\sim}{c}_1 = 3\underset{\sim}{i} - 8\underset{\sim}{j}$

$\underset{\sim}{c}_1 = \underset{\sim}{0}$

$\dot{\underset{\sim}{r}}(t) = \frac{3}{3t+1}\underset{\sim}{i} - \frac{8}{(2t+1)^2}\underset{\sim}{j}$

$\underset{\sim}{r}(t) = \int \frac{3}{3t+1}dt\underset{\sim}{i} + \int \frac{-8}{(2t+1)^2}dt\underset{\sim}{j}$

$= \log_e(3t+1)\underset{\sim}{i} + \frac{4}{2t+1}\underset{\sim}{j} + \underset{\sim}{c}_2$

$\underset{\sim}{r}(0) = 4\underset{\sim}{j} + \underset{\sim}{c}_2 = 4\underset{\sim}{i} + 3\underset{\sim}{j}$

$\underset{\sim}{c}_2 = 4\underset{\sim}{i} - \underset{\sim}{j}$

$\underset{\sim}{r}(t) = \left(\log_e(3t+1) + 4\right)\underset{\sim}{i} + \left(\frac{4}{2t+1} - 1\right)\underset{\sim}{j}$

b
$$\begin{aligned}
&\underset{\sim}{\ddot{r}}(t) = -\frac{9}{(3t+1)^2}\underset{\sim}{i} - \frac{24}{(2t+1)^4}\underset{\sim}{j}\\
&\underset{\sim}{\dot{r}}(0) = 2\underset{\sim}{i} - \underset{\sim}{j}\\
&\underset{\sim}{r}(0) = 6\underset{\sim}{i} + 8\underset{\sim}{j},\ \ t \ge 0\\
&\underset{\sim}{\dot{r}}(t) = \int -\frac{9}{(3t+1)^2}dt\underset{\sim}{i} + \int -\frac{24}{(2t+1)^4}dt\underset{\sim}{j}\\
&\quad = \frac{3}{3t+1}\underset{\sim}{i} + \frac{4}{(2t+1)^3}\underset{\sim}{j} + \underset{\sim}{c}_1\\
&\underset{\sim}{\dot{r}}(0) = 3\underset{\sim}{i} + 4\underset{\sim}{j} + \underset{\sim}{c}_1 = 6\underset{\sim}{i} + 8\underset{\sim}{j}\\
&\underset{\sim}{c}_1 = 3\underset{\sim}{i} + 4\underset{\sim}{j}\\
&\underset{\sim}{\dot{r}}(t) = \left(\frac{3}{3t+1} + 3\right)\underset{\sim}{i} + \left(\frac{4}{(2t+1)^3} + 4\right)\underset{\sim}{j}\\
&\underset{\sim}{r}(t) = \int \frac{3}{3t+1} + 3dt\underset{\sim}{i} + \int \frac{4}{(2t+1)^3} + 4dt\underset{\sim}{j}\\
&\quad = \left(\log_e(3t+1) + 3t\right)\underset{\sim}{i} + \left(4t - \frac{1}{(2t+1)^2}\right)\underset{\sim}{j} + \underset{\sim}{c}_2\\
&\underset{\sim}{r}(0) = -\underset{\sim}{j} + \underset{\sim}{c}_2 = 2\underset{\sim}{i} - \underset{\sim}{j}\\
&\underset{\sim}{c}_2 = 2\underset{\sim}{i}\\
&\underset{\sim}{r}(t) = \left(\log_e(3t+1) + 3t + 2\right)\underset{\sim}{i} + \left(4t - \frac{1}{(2t+1)^2}\right)\underset{\sim}{j}
\end{aligned}$$

9
$$\begin{aligned}
&\underset{\sim}{\ddot{r}}(t) = -45\cos(3t)\underset{\sim}{i} + 45\sin(3t)\underset{\sim}{j},\ \ 0 \le t \le 2\pi\\
&\underset{\sim}{\dot{r}}(0) = -15\underset{\sim}{j}\\
&\underset{\sim}{r}(0) = 3\underset{\sim}{i} + 4\underset{\sim}{j}\\
&\underset{\sim}{\dot{r}}(t) = \int -45\cos(3t)dt\underset{\sim}{i} + \int 45\sin(3t)dt\underset{\sim}{j}\\
&\quad = -15\sin(3t)\underset{\sim}{i} - 15\cos(3t)\underset{\sim}{j} + \underset{\sim}{c}_1\\
&\underset{\sim}{\dot{r}}(0) = -15\sin(0)\underset{\sim}{i} - 15\cos(0)\underset{\sim}{j} + \underset{\sim}{c}_1 = -15\underset{\sim}{j} \rightarrow \underset{\sim}{c}_1 = \underset{\sim}{0}\\
&\underset{\sim}{\dot{r}}(t) = -15\sin(3t)\underset{\sim}{i} - 15\cos(3t)\underset{\sim}{j}\\
&\underset{\sim}{r}(t) = \int -15\sin(3t)dt\underset{\sim}{i} + \int -15\cos(3t)dt\underset{\sim}{j}\\
&\quad = 5\cos(3t)\underset{\sim}{i} - 5\sin(3t)\underset{\sim}{j} + \underset{\sim}{c}_2\\
&\underset{\sim}{r}(0) = 5\cos(0)\underset{\sim}{i} - 5\sin(0)\underset{\sim}{j} + \underset{\sim}{c}_2 = 3\underset{\sim}{i} + 4\underset{\sim}{j} \rightarrow \underset{\sim}{c}_2 = -2\underset{\sim}{i} + 4\underset{\sim}{j}\\
&\underset{\sim}{r}(t) = (-2 + 5\cos(3t))\underset{\sim}{i} + (4 - 5\sin(3t))\underset{\sim}{j}\\
&x = -2 + 5\cos(3t) \rightarrow \cos(3t) = \frac{x+2}{5}\\
&y = 4 - 5\sin(3t) \rightarrow \sin(3t) = \frac{4-y}{5}\\
&\sin^2(3t) + \cos^2(3t) = 1\\
&\frac{(x+2)^2}{25} + \frac{(4-y)^2}{25} = 1\\
&(x+2)^2 + (4-y)^2 = 25
\end{aligned}$$

Circle centre $(-2, 4)$
Radius 5

10
$$\begin{aligned}
&\underset{\sim}{\ddot{r}}(t) = 12\cos(2t)\underset{\sim}{i} - 20\sin(2t)\underset{\sim}{j},\ \ 0 \le t \le \pi\\
&\underset{\sim}{\dot{r}}(0) = 10\underset{\sim}{j}\\
&\underset{\sim}{r}(0) = 2\underset{\sim}{i} - 2\underset{\sim}{j}\\
&\underset{\sim}{\dot{r}}(t) = \int 12\cos(2t)dt\underset{\sim}{i} + \int -20\sin(2t)dt\underset{\sim}{j}\\
&\quad = 6\sin(2t)\underset{\sim}{i} + 10\cos(2t)\underset{\sim}{j} + \underset{\sim}{c}_1\\
&\underset{\sim}{\dot{r}}(0) = 6\sin(0)\underset{\sim}{i} + 10\cos(0)\underset{\sim}{j} + \underset{\sim}{c}_1 = 10\underset{\sim}{j} \rightarrow \underset{\sim}{c}_1 = \underset{\sim}{0}\\
&\underset{\sim}{\dot{r}}(t) = 6\sin(2t)\underset{\sim}{i} + 10\cos(2t)\underset{\sim}{j}\\
&\underset{\sim}{r}(t) = \int 6\sin(2t)dt\underset{\sim}{i} + \int 10\cos(2t)dt\underset{\sim}{j}\\
&\quad = -3\cos(2t)\underset{\sim}{i} + 5\sin(2t)\underset{\sim}{j} + \underset{\sim}{c}_2\\
&\underset{\sim}{r}(0) = -3\cos(0)\underset{\sim}{i} + 5\sin(0)\underset{\sim}{j} + \underset{\sim}{c}_2\\
&\quad = 2\underset{\sim}{i} - 2\underset{\sim}{j} \rightarrow \underset{\sim}{c}_2 = 5\underset{\sim}{i} - 2\underset{\sim}{j}\\
&\underset{\sim}{r}(t) = (5 - 3\cos(2t))\underset{\sim}{i} + (-2 + 5\sin(2t))\underset{\sim}{j}\\
&x = 5 - 3\cos(2t) \rightarrow \cos(2t) = \frac{x-5}{-3}\\
&y = -2 + 5\sin(2t) \rightarrow \sin(2t) = \frac{y+2}{5}\\
&\sin^2(2t) + \cos^2(2t) = 1\\
&\frac{(x-5)^2}{9} + \frac{(y+2)^2}{25} = 1
\end{aligned}$$

Ellipse centre $(5, -2)$
Semi major/minor 3, 5

11 **a**
$$\begin{aligned}
&\underset{\sim}{\ddot{r}}(t) = -10\underset{\sim}{j}\\
&\underset{\sim}{\dot{r}}(0) = 15\underset{\sim}{i} + 20\underset{\sim}{j}\\
&\underset{\sim}{r}(0) = 2\underset{\sim}{j},\ \ t \ge 0\\
&\underset{\sim}{\dot{r}}(t) = \int -10dt\underset{\sim}{j} = -10t\underset{\sim}{j} + \underset{\sim}{c}_1\\
&\underset{\sim}{\dot{r}}(0) = \underset{\sim}{c}_1 = 15\underset{\sim}{i} + 20\underset{\sim}{j}\\
&\underset{\sim}{\dot{r}}(t) = 15\underset{\sim}{i} + (20 - 10t)\underset{\sim}{j}\\
&\underset{\sim}{r}(t) = \int 15dt\underset{\sim}{i} + \int (20 - 10t)\,dt\underset{\sim}{j} = 15t\underset{\sim}{i} + \left(20t - 5t^2\right)\underset{\sim}{j} + \underset{\sim}{c}_2\\
&\underset{\sim}{r}(0) = \underset{\sim}{c}_2 = 2\underset{\sim}{j}\\
&\underset{\sim}{r}(t) = 15t\underset{\sim}{i} + \left(2 + 20t - 5t^2\right)\underset{\sim}{j}\\
&\quad x = 15t \rightarrow y = 2 + 20t - 5t^2\\
&\quad t = \frac{x}{15} \rightarrow y = 2 + 20\left(\frac{x}{15}\right) - 5\left(\frac{x}{15}\right)^2\\
&\quad y = -\frac{x^2}{45} + \frac{4x}{3} + 2,\ \ x \ge 0
\end{aligned}$$

b
$$\begin{aligned}
&\underset{\sim}{\ddot{r}}(t) = -9.8t\underset{\sim}{j}\\
&\underset{\sim}{\dot{r}}(0) = 5\underset{\sim}{i} + 10\underset{\sim}{j}\\
&\underset{\sim}{r}(0) = \underset{\sim}{j},\ \ t \ge 0\\
&\underset{\sim}{\dot{r}}(t) = \int -9.8dt\underset{\sim}{j} = -9.8t\underset{\sim}{j} + \underset{\sim}{c}_1\\
&\underset{\sim}{\dot{r}}(0) = \underset{\sim}{c}_1 = 5\underset{\sim}{i} + 10\underset{\sim}{j}\\
&\underset{\sim}{\dot{r}}(t) = 5\underset{\sim}{i} + (10 - 9.8t)\underset{\sim}{j}\\
&\underset{\sim}{r}(t) = \int 5dt\underset{\sim}{i} + \int (10 - 9.8t)\,dt\underset{\sim}{j} = 5t\underset{\sim}{i} + \left(10t - 4.9t^2\right)\underset{\sim}{j} + \underset{\sim}{c}_2\\
&\underset{\sim}{r}(0) = \underset{\sim}{c}_2 = \underset{\sim}{j}\\
&\underset{\sim}{r}(t) = 5t\underset{\sim}{i} + \left(10t - 4.9t^2 + 1\right)\underset{\sim}{j}\\
&\quad x = 5t \rightarrow y = 10t - 4.9t^2 + 1\\
&\quad t = \frac{x}{5} \rightarrow y = 10\left(\frac{x}{5}\right) - 4.9\left(\frac{x}{5}\right)^2 + 1\\
&\quad y = -\frac{49x^2}{250} + 2x + 1,\ \ x \ge 0
\end{aligned}$$

12 **a**
$$\begin{aligned}
&\underset{\sim}{\ddot{r}}(t) = -e^{-\frac{t}{2}}\underset{\sim}{i} + 2e^{\frac{t}{2}}\underset{\sim}{j}\\
&\underset{\sim}{\dot{r}}(0) = 2\underset{\sim}{i} + 4\underset{\sim}{j}\\
&\underset{\sim}{r}(0) = -2\underset{\sim}{i} + 3\underset{\sim}{j},\ \ t \in R\\
&\underset{\sim}{\dot{r}}(t) = \int -e^{-\frac{t}{2}}dt\underset{\sim}{i} + \int 2e^{\frac{t}{2}}dt\underset{\sim}{j} = 2e^{-\frac{t}{2}}\underset{\sim}{i} + 4e^{\frac{t}{2}}\underset{\sim}{j} + \underset{\sim}{c}_1\\
&\underset{\sim}{\dot{r}}(0) = 2\underset{\sim}{i} + 4\underset{\sim}{j} + \underset{\sim}{c}_1 = 2\underset{\sim}{i} + 4\underset{\sim}{j} \rightarrow \underset{\sim}{c}_1 = \underset{\sim}{0}\\
&\underset{\sim}{\dot{r}}(t) = 2e^{-\frac{t}{2}}\underset{\sim}{i} + 4e^{\frac{t}{2}}\underset{\sim}{j}\\
&\underset{\sim}{r}(t) = \int 2e^{-\frac{t}{2}}dt\underset{\sim}{i} + \int 4e^{\frac{t}{2}}dt\underset{\sim}{j} = -4e^{-\frac{t}{2}}\underset{\sim}{i} + 8e^{\frac{t}{2}}\underset{\sim}{j} + \underset{\sim}{c}_2\\
&\underset{\sim}{r}(0) = -4\underset{\sim}{i} + 8\underset{\sim}{j} + \underset{\sim}{c}_2 = -2\underset{\sim}{i} + 3\underset{\sim}{j} \rightarrow \underset{\sim}{c}_2 = 2\underset{\sim}{i} - 5\underset{\sim}{j}\\
&\underset{\sim}{r}(t) = \left(2 - 4e^{-\frac{t}{2}}\right)\underset{\sim}{i} + \left(8e^{\frac{t}{2}} - 5\right)\underset{\sim}{j}
\end{aligned}$$

$x = 2 - 4e^{-\frac{t}{2}} \rightarrow y = 8e^{\frac{t}{2}} - 5$

$e^{-\frac{t}{2}} = \dfrac{2-x}{4} \rightarrow e^{\frac{t}{2}} = \dfrac{y+5}{8}$

$e^{\frac{t}{2}} = \dfrac{4}{2-x}$

$\dfrac{4}{2-x} = \dfrac{y+5}{8}$

$y = \dfrac{32}{2-x} - 5$

b $\dot{\underset{\sim}{r}}(t) = 8\cos(2t)\underset{\sim}{i} - 8\sin(2t)\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(0) = 4\underset{\sim}{j}$

$\underset{\sim}{r}(0) = \underset{\sim}{i} + 5\underset{\sim}{j},\ 0 \le t \le 2\pi$

$\dot{\underset{\sim}{r}}(t) = \displaystyle\int 8\cos(2t)dt\underset{\sim}{i} + \int -8\sin(2t)dt\underset{\sim}{j}$

$= 4\sin(2t)\underset{\sim}{i} + 4\cos(2t)\underset{\sim}{j} + \underset{\sim}{c}_1$

$\dot{\underset{\sim}{r}}(0) = 4\underset{\sim}{j} + \underset{\sim}{c}_1 = 4\underset{\sim}{j} \rightarrow \underset{\sim}{c}_1 = \underset{\sim}{0}$

$\dot{\underset{\sim}{r}}(t) = 4\sin(2t)\underset{\sim}{i} + 4\cos(2t)\underset{\sim}{j}$

$\underset{\sim}{r}(t) = \displaystyle\int 4\sin(2t)\,dt\underset{\sim}{i} + \int 4\cos(2t)\,dt\underset{\sim}{j}$

$= -2\cos(2t)\underset{\sim}{i} + 2\sin(2t)\underset{\sim}{j} + \underset{\sim}{c}_2$

$\underset{\sim}{r}(0) = -2\underset{\sim}{i} + \underset{\sim}{c}_2 = \underset{\sim}{i} + 5\underset{\sim}{j} \rightarrow \underset{\sim}{c}_2 = 3\underset{\sim}{i} + 5\underset{\sim}{j}$

$\underset{\sim}{r}(t) = (3 - 2\cos(2t))\,\underset{\sim}{i} + (5 + 2\sin(2t))\,\underset{\sim}{j}$

$x = 3 - 2\cos(2t) \rightarrow y = 5 + 2\sin(2t)$

$\cos(2t) = \dfrac{x-3}{-2} \rightarrow \sin(2t) = \dfrac{y-5}{2}$

$\cos^2(2t) + \sin^2(2t) = 1$

$\left(\dfrac{x-3}{-2}\right)^2 + \left(\dfrac{y-5}{2}\right)^2 = 1$

$(x-3)^2 + (y-5)^2 = 4$

Circle centre (3, 5), radius 2.

13 **a** $\ddot{r}(t) = 9\cos(3t)\underset{\sim}{i} + 18\sin(3t)\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(0) = -6\underset{\sim}{j}$

$\underset{\sim}{r}(0) = 3\underset{\sim}{i} + 5\underset{\sim}{j},\ 0 \le t \le 2\pi$

$\dot{\underset{\sim}{r}}(t) = \displaystyle\int 9\cos(3t)dt\underset{\sim}{i} + \int 18\sin(3t)dt\underset{\sim}{j}$

$= 3\sin(3t)\underset{\sim}{i} - 6\cos(3t)\underset{\sim}{j} + \underset{\sim}{c}_1$

$\dot{\underset{\sim}{r}}(0) = -6\underset{\sim}{j} + \underset{\sim}{c}_1 = -6\underset{\sim}{j} \rightarrow \underset{\sim}{c}_1 = \underset{\sim}{0}$

$\dot{\underset{\sim}{r}}(t) = 3\sin(3t)\underset{\sim}{i} - 6\cos(3t)\underset{\sim}{j}$

$\underset{\sim}{r}(t) = \displaystyle\int 3\sin(3t)dt\underset{\sim}{i} + \int -6\cos(3t)\,dt\underset{\sim}{j}$

$= -\cos(3t)\underset{\sim}{i} - 2\sin(3t)\underset{\sim}{j} + \underset{\sim}{c}_2$

$\underset{\sim}{r}(0) = -\underset{\sim}{i} + \underset{\sim}{c}_2 = 3\underset{\sim}{i} + 5\underset{\sim}{j} \rightarrow \underset{\sim}{c}_2 = 4\underset{\sim}{i} + 5\underset{\sim}{j}$

$\underset{\sim}{r}(t) = (4 - \cos(3t))\,\underset{\sim}{i} + (5 - 2\sin(3t))\,\underset{\sim}{j}$

$x = 4 - \cos(3t) \rightarrow y = 5 - 2\sin(3t)$

$\cos(3t) = 4 - x \rightarrow \sin(3t) = \dfrac{5-y}{2}$

$\cos^2(3t) + \sin^2(3t) = 1$

$(x-4)^2 + \dfrac{(y-5)^2}{4} = 1$

Ellipse centre (4, 5), semi-major/minor 1, 2.

b $\ddot{r}(t) = -3\cos\left(\dfrac{t}{2}\right)\underset{\sim}{i} + \sin\left(\dfrac{t}{2}\right)\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(0) = -2\underset{\sim}{j}$

$\underset{\sim}{r}(0) = 5\underset{\sim}{i} + 3\underset{\sim}{j},\ 0 \le t \le 4\pi$

$\dot{\underset{\sim}{r}}(t) = \displaystyle\int -3\cos\left(\frac{t}{2}\right)dt\underset{\sim}{i} + \int \sin\left(\frac{t}{2}\right)dt\underset{\sim}{j}$

$= -6\sin\left(\dfrac{t}{2}\right)\underset{\sim}{i} - 2\cos\left(\dfrac{t}{2}\right)\underset{\sim}{j} + \underset{\sim}{c}_1$

$\dot{\underset{\sim}{r}}(0) = -2\underset{\sim}{j} + \underset{\sim}{c}_1 = -2\underset{\sim}{j} \rightarrow \underset{\sim}{c}_1 = \underset{\sim}{0}$

$\dot{\underset{\sim}{r}}(t) = -6\sin\left(\dfrac{t}{2}\right)\underset{\sim}{i} - 2\cos\left(\dfrac{t}{2}\right)\underset{\sim}{j}$

$\underset{\sim}{r}(t) = \displaystyle\int -6\sin\left(\frac{t}{2}\right)dt\underset{\sim}{i} - \int 2\cos\left(\frac{t}{2}\right)\,dt\underset{\sim}{j}$

$= 12\cos\left(\dfrac{t}{2}\right)\underset{\sim}{i} - 4\sin\left(\dfrac{t}{2}\right)\underset{\sim}{j} + \underset{\sim}{c}_2$

$\underset{\sim}{r}(0) = 12\underset{\sim}{i} + \underset{\sim}{c}_2 = 5\underset{\sim}{i} + 3\underset{\sim}{j} \rightarrow \underset{\sim}{c}_2 = -7\underset{\sim}{i} + 3\underset{\sim}{j}$

$\underset{\sim}{r}(t) = \left(-7 + 12\cos\left(\dfrac{t}{2}\right)\right)\underset{\sim}{i} + \left(3 - 4\sin\left(\dfrac{t}{2}\right)\right)\underset{\sim}{j}$

$x = -7 + 12\cos\left(\dfrac{t}{2}\right) \rightarrow y = 3 - 4\sin\left(\dfrac{t}{2}\right)$

$\cos\left(\dfrac{t}{2}\right) = \dfrac{x+7}{12} \rightarrow \sin\left(\dfrac{t}{2}\right) = \dfrac{3-y}{4}$

$\cos^2\left(\dfrac{t}{2}\right) + \sin^2\left(\dfrac{t}{2}\right) = 1$

$\dfrac{(x+7)^2}{144} + \dfrac{(y-3)^2}{16} = 1$

Ellipse centre (−7, 3), semi-major/minor 12, 4.

14 **a** $\underset{\sim}{r}(t) = -2\cos(t)\underset{\sim}{i} - 8\cos(2t)\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(0) = \underset{\sim}{0}$

$\underset{\sim}{r}(0) = 2\underset{\sim}{i},\quad t \ge 0$

$\dot{\underset{\sim}{r}}(t) = \displaystyle\int -2\cos(t)dt\underset{\sim}{i} + \int -8\cos(2t)dt\underset{\sim}{j}$

$= -2\sin(t)\underset{\sim}{i} - 4\sin(2t)\underset{\sim}{j} + \underset{\sim}{c}_1$

$\dot{\underset{\sim}{r}}(0) = \vec{0} + \underset{\sim}{c}_1 = \vec{0} \rightarrow \underset{\sim}{c}_1 = \vec{0}$

$\dot{\underset{\sim}{r}}(t) = -2\sin(t)\underset{\sim}{i} - 4\sin(2t)\underset{\sim}{j}$

$\underset{\sim}{r}(t) = \displaystyle\int -2\sin(t)dt\underset{\sim}{i} + \int -4\sin(2t)\,dt\underset{\sim}{j}$

$= 2\cos(t)\underset{\sim}{i} + 2\cos(2t)\underset{\sim}{j} + \underset{\sim}{c}_2$

$\underset{\sim}{r}(0) = 2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{c}_2 = 2\underset{\sim}{i} \rightarrow \underset{\sim}{c}_2 = -2\underset{\sim}{j}$

$\underset{\sim}{r}(t) = (2\cos(t))\,\underset{\sim}{i} + (2\cos(2t) - 2)\,\underset{\sim}{j}$

$x = 2\cos(t) \rightarrow y = 2\cos(2t) - 2$

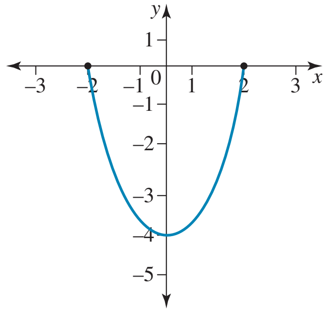
$x^2 = 4\cos^2(t) \rightarrow y = 2\left(2\cos^2(t) - 1\right) - 2$

$= 4\cos^2(t) - 4$

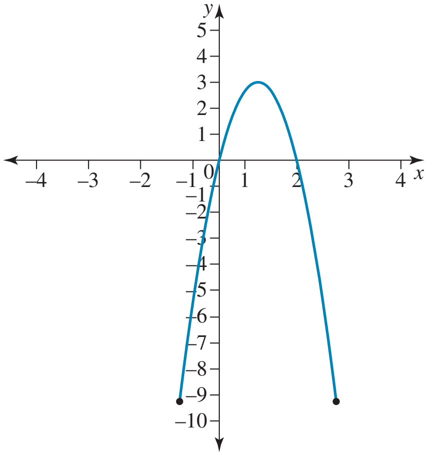
$y = x^2 - 4,\ t \ge 0$

$x \in [-2,\ 2]$

$y \in [-4,\ 0]$



b
$$\begin{aligned}
\ddot{\underset{\sim}{r}}(t) &= -8\sin(2t)\underset{\sim}{i} - 96\cos(4t)\underset{\sim}{j},\ 0 \le t \le \pi \\
\dot{\underset{\sim}{r}}(0) &= 4\underset{\sim}{i} \\
\underset{\sim}{r}(0) &= \underset{\sim}{i} + 3\underset{\sim}{j} \\
\dot{\underset{\sim}{r}}(t) &= \int -8\sin(2t)dt\underset{\sim}{i} + \int -96\cos(4t)dt\underset{\sim}{j} \\
&= 4\cos(2t)\underset{\sim}{i} - 24\sin(4t)\underset{\sim}{j} + \underset{\sim}{c}_1 \\
\dot{\underset{\sim}{r}}(0) &= 4\underset{\sim}{i} + \underset{\sim}{c}_1 = 4\underset{\sim}{i} \rightarrow \underset{\sim}{c}_1 = \underset{\sim}{0} \\
\dot{\underset{\sim}{r}}(t) &= 4\cos(2t)\underset{\sim}{i} - 24\sin(4t)\underset{\sim}{j} \\
\underset{\sim}{r}(t) &= \int 4\cos(2t)dt\underset{\sim}{i} + \int -24\sin(4t)\,dt\underset{\sim}{j} \\
&= 2\sin(2t)\underset{\sim}{i} + 6\cos(4t)\underset{\sim}{j} + \underset{\sim}{c}_2 \\
\underset{\sim}{r}(0) &= 6\underset{\sim}{j} + \underset{\sim}{c}_2 = \underset{\sim}{i} + 3\underset{\sim}{j} \rightarrow \underset{\sim}{c}_2 = \underset{\sim}{i} - 3\underset{\sim}{j} \\
\underset{\sim}{r}(t) &= (2\sin(2t) + 1)\,\underset{\sim}{i} + (6\cos(4t) - 3)\,\underset{\sim}{j} \\
x &= 2\sin(2t) + 1 \rightarrow y = 6\cos(4t) - 3 \\
x - 1 &= 2\sin(2t) \rightarrow y = 6\left(1 - 2\sin^2(2t)\right) - 3 \\
(x-1)^2 &= 4\sin^2(2t) \rightarrow y = 6 - 12\sin^2(2t) - 3 \\
y &= 3 - 12\sin^2(2t) \\
&= 3 - 3 \times 4\sin^2(2t) \\
&= 3 - 3(x-1)^2 \\
&= 3 - 3\left(x^2 - 2x + 1\right) \\
&= 3 - 3x^2 + 6x - 3 \\
y &= -3x(x-2) \\
& \quad 0 \le t \le \pi \\
x &\in [-1,\ 3] \\
y &\in [-9,\ 3]
\end{aligned}$$



15 a Particle A

$\ddot{\underset{\sim}{r}}_A(t) = 2\underset{\sim}{i} + 4\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(0) = -6\underset{\sim}{i} + 5\underset{\sim}{j}$

$\underset{\sim}{r}(0) = 13\underset{\sim}{i} - 17\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = \int 2dt\underset{\sim}{i} + \int 4dt\underset{\sim}{j} = 2t\underset{\sim}{i} + 4t\underset{\sim}{j} + \underset{\sim}{c}_1$

$\dot{\underset{\sim}{r}}(0) = \underset{\sim}{c}_1 = -6\underset{\sim}{i} + 5\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = (2t - 6)\,\underset{\sim}{i} + (4t + 5)\,\underset{\sim}{j}$

$\underset{\sim}{r}(t) = \int (2t - 6)\,dt\underset{\sim}{i} + (4t + 5)\ dt\underset{\sim}{j}$

$\quad = \left(t^2 - 6t\right)\underset{\sim}{i} + \left(2t^2 + 5t\right)\underset{\sim}{j} + \underset{\sim}{c}_2$

$\underset{\sim}{r}(0) = \underset{\sim}{c}_2 = 13\underset{\sim}{i} - 17\underset{\sim}{j}$

$\underset{\sim}{r}_A(t) = \left(t^2 - 6t + 13\right)\underset{\sim}{i} + \left(2t^2 + 5t - 17\right)\underset{\sim}{j}$

Particle B

$\ddot{\underset{\sim}{r}}_B(t) = 6\underset{\sim}{i} + 8\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(0) = -8\underset{\sim}{i} - 20\underset{\sim}{j}$

$\underset{\sim}{r}(0) = \underset{\sim}{i} + 40\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = \int 6dt\underset{\sim}{i} + \int 8dt\underset{\sim}{j} = 6t\underset{\sim}{i} + 8t\underset{\sim}{j} + \underset{\sim}{c}_1$

$\dot{\underset{\sim}{r}}(0) = \underset{\sim}{c}_1 = -8\underset{\sim}{i} - 20\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = (6t - 8)\,\underset{\sim}{i} + (8t - 20)\,\underset{\sim}{j}$

$\underset{\sim}{r}(t) = \int (6t - 8)\,dt\underset{\sim}{i} + \int (8t - 20)\,dt\underset{\sim}{j}$

$\quad = \left(3t^2 - 8t\right)\underset{\sim}{i} + \left(4t^2 - 20t\right)\underset{\sim}{j} + \underset{\sim}{c}_2$

$\underset{\sim}{r}(0) = \underset{\sim}{c}_2 = \underset{\sim}{i} + 40\underset{\sim}{j}$

$\underset{\sim}{r}_B(t) = \left(3t^2 - 8t + 1\right)\underset{\sim}{i} + \left(4t^2 - 20t + 40\right)\underset{\sim}{j}$

Collide when $\underset{\sim}{r}_A(t) = \underset{\sim}{r}_B(t)$

$\underset{\sim}{i}$:

$t^2 - 6t + 13 = 3t^2 - 8t + 1$

$2t^2 - 2t - 12 = 0$

$2\left(t^2 - t - 6\right) = 0$

$2\,(t - 3)\,(t + 2) = 0$

$t = -2,\quad 3 \rightarrow t \ge 0,\quad \therefore t = 3$

$\underset{\sim}{j}$:

$4t^2 - 20t + 40 = 2t^2 + 5t - 17$

$2t^2 - 25t + 57 = 0$

$(2t - 19)\,(t - 3) = 0$

$t = \dfrac{19}{2},\quad t = 3$

Common time is $t = 3$

Collide when

$t = 3$

$\underset{\sim}{r}_A(3) = 4\underset{\sim}{i} + 16\underset{\sim}{j}$

$\underset{\sim}{r}_B(3) = 4\underset{\sim}{i} + 16\underset{\sim}{j}$

$(4,\ 16)$

b Particle A

$\ddot{\underset{\sim}{r}}_A(t) = 2\underset{\sim}{i} - 2\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(1) = -5\underset{\sim}{i} + 6\underset{\sim}{j}$

$\underset{\sim}{r}(1) = -3\underset{\sim}{i} + 2\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = \int 2dt\underset{\sim}{i} - \int 2dt\underset{\sim}{j} = 2t\underset{\sim}{i} - 2t\underset{\sim}{j} + \underset{\sim}{c}_1$

$\dot{\underset{\sim}{r}}(1) = 2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{c}_1 = -5\underset{\sim}{i} + 6\underset{\sim}{j} \rightarrow \underset{\sim}{c}_1$

$\quad = -7\underset{\sim}{i} + 8\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = (2t - 7)\,\underset{\sim}{i} + (-2t + 8)\,\underset{\sim}{j}$

$\underset{\sim}{r}(t) = \int (2t - 7)\,dt\underset{\sim}{i} + (2t + 8)\ dt\underset{\sim}{j}$

$\quad = \left(t^2 - 7t\right)\underset{\sim}{i} + \left(-t^2 + 8t\right)\underset{\sim}{j} + \underset{\sim}{c}_2$

$\underset{\sim}{r}(1) = -6\underset{\sim}{i} + 7\underset{\sim}{j} + \underset{\sim}{c}_2 = -3\underset{\sim}{i} + 2\underset{\sim}{j} \rightarrow \underset{\sim}{c}_2 = 3\underset{\sim}{i} - 5\underset{\sim}{j}$

$\underset{\sim}{r}_A(t) = \left(t^2 - 7t + 3\right)\underset{\sim}{i} + \left(-t^2 + 8t - 5\right)\underset{\sim}{j}$

Particle B

$\ddot{\underset{\sim}{r}}_B(t) = 2\underset{\sim}{i} - 6\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(1) = -2\underset{\sim}{i} + 6\underset{\sim}{j}$

$\underset{\sim}{r}(1) = -15\underset{\sim}{i} + 34\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = \int 2dt\underset{\sim}{i} - \int 6dt\underset{\sim}{j} = 2t\underset{\sim}{i} - 6t\underset{\sim}{j} + \underset{\sim}{c}_1$

$\dot{\underset{\sim}{r}}(1) = 2\underset{\sim}{i} - 6\underset{\sim}{j} + \underset{\sim}{c}_1 = -2\underset{\sim}{i} + 6\underset{\sim}{j} \rightarrow \underset{\sim}{c}_1$

$\quad = -4\underset{\sim}{i} + 12\underset{\sim}{j}$

$\dot{\utilde{r}}(t) = (2t-4)\,\utilde{i} + (-6t+12)\,\utilde{j}$

$\utilde{r}(t) = \int (2t-4)\,dt\utilde{i} + \int (-6t+12)\;dt\utilde{j}$

$\quad = \left(t^2-4t\right)\utilde{i} + \left(-3t^2+12t\right)\utilde{j} + \utilde{c}_2$

$\utilde{r}(1) = -3\utilde{i} + 9\utilde{j} + \utilde{c}_2 = -15\utilde{i} + 34\utilde{j} \rightarrow \utilde{c}_2$

$\quad = -12\utilde{i} + 25\utilde{j}$

$\utilde{r}_{\text{B}}(t) = \left(t^2-4t-12\right)\utilde{i} + \left(-3t^2+12t+25\right)\utilde{j}$

Collide when

$\utilde{r}_{\text{A}}(t) = \utilde{r}_{\text{B}}(t)$

$\utilde{i}$

$t^2-7t+3 = t^2-4t-12$

$3t = 15$

$t = 5$

$\utilde{j}$

$-t^2+8t-5 = -3t^2+12t+25$

$2t^2-4t-30 = 0$

$2\,(t-5)\,(t+3)$

$t = 5,\ -3,\ t \geq 0 \therefore t = 5$

Collide when

$t = 5$

$\utilde{r}_{\text{A}}(5) = -7\utilde{i} + 10\utilde{j}$

$\utilde{r}_{\text{B}}(5) = -7\utilde{i} + 10\utilde{j}$

$(-7,\ 10)$

16 $\ddot{\utilde{r}}(t) = -n^2r\cos(nt)\utilde{i} - n^2r\sin(nt)\utilde{j}$

$\dot{\utilde{r}}(0) = nr\utilde{j}$

$\utilde{r}(0) = (a+r)\,\utilde{i} + b\utilde{j},\ 0 \leq t \leq 2\pi$

$\dot{\utilde{r}}(t) = \int -n^2r\cos(nt)dt\utilde{i} + \int -n^2r\sin(nt)dt\utilde{j}$

$\quad = -nr\sin(nt)\utilde{i} + nr\cos(nt)\utilde{j} + \utilde{c}_1$

$\dot{\utilde{r}}(0) = nr\utilde{j} + \utilde{c}_1 = nr\utilde{j} \rightarrow \utilde{c}_1 = \utilde{0}$

$\dot{\utilde{r}}(t) = -nr\sin(nt)\utilde{i} + nr\cos(nt)\utilde{j}$

$\utilde{r}(t) = r\cos(nt)\utilde{i} + r\sin(nt)\utilde{j} + \utilde{c}_2$

$\utilde{r}(0) = r\utilde{i} + \utilde{c}_2 = (a+r)\,\utilde{i} + b\utilde{j} \rightarrow \utilde{c}_2 = a\utilde{i} + b\utilde{j}$

$\utilde{r}(t) = (a+r\cos(nt))\,\utilde{i} + (b+r\sin(nt))\,\utilde{j}$

$x = a+r\cos(nt) \rightarrow y = b+r\sin(nt)$

$\cos(nt) = \dfrac{x-a}{r} \rightarrow \sin(nt) = \dfrac{y-b}{r}$

$(x-a)^2 + (y-b)^2 = r^2$

Circle centre$(a,\ b)$, radius r

17 $\ddot{\utilde{r}}(t) = -n^2a\cos(nt)\utilde{i} - bn^2\sin(nt)\utilde{j},\quad t \geq 0$

$\dot{\utilde{r}}(0) = bn\utilde{j}$

$\utilde{r}(0) = (h+a)\,\utilde{i} + k\utilde{j}$

$\dot{\utilde{r}}(t) = \int -n^2a\cos(nt)dt\utilde{i} + \int -bn^2\sin(nt)dt\utilde{j}$

$\quad = -na\sin(nt)\utilde{i} + bn\cos(nt)\utilde{j} + \utilde{c}_1$

$\dot{\utilde{r}}(0) = bn\utilde{j} + \utilde{c}_1 = bn\utilde{j} \rightarrow \utilde{c}_1 = \utilde{0}$

$\dot{\utilde{r}}(t) = -na\sin(nt)\utilde{i} + bn\cos(nt)\utilde{j}$

$\utilde{r}(t) = a\cos(nt)\utilde{i} + b\sin(nt)\utilde{j} + \utilde{c}_2$

$\utilde{r}(0) = a\utilde{i} + \utilde{c}_2 = (h+a)\,\utilde{i} + k\utilde{j} \rightarrow \utilde{c}_2 = h\utilde{i} + k\utilde{j}$

$\utilde{r}(t) = (h+a\cos(nt))\,\utilde{i} + (k+b\sin(nt))\,\utilde{j}$

$x = h+a\cos(nt) \rightarrow y = k+b\sin(nt)$

$\cos(nt) = \dfrac{x-h}{a} \rightarrow \sin(nt = \dfrac{y-k}{b}$

$\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

Ellipse centre $(h,\ k)$, semi-major/minor $a,\ b$.

18 $\ddot{\utilde{r}}(t) = (4\cos(2t) - 2\cos(t))\,\utilde{i} + (4\sin(2t) - 2\sin(t))\,\utilde{j}$

$0 \leq t \leq 2\pi$

$\dot{\utilde{r}}(\pi) = -4\utilde{j}$

$\utilde{r}(\pi) = -3\utilde{i}$

$\dot{\utilde{r}}(t) = \int (4\cos(2t) - 2\cos(t))\,dt\utilde{i}$

$\quad + \int (4\sin(2t) - 2\sin(t))\,dt\utilde{j}$

$\quad = (2\sin(2t) - 2\sin(t))\,\utilde{i} +$

$\quad (-2\cos(2t) + 2\cos(t))\,\utilde{j} + \utilde{c}_1$

$\dot{\utilde{r}}(\pi) = (2\sin(2\pi) - 2\sin(\pi))\,\utilde{i} +$

$\quad (-2\cos(2\pi) + 2\cos(\pi))$

$\quad \utilde{j} + \utilde{c}_1 \ = -4\utilde{j} \rightarrow \utilde{c}_1 = \utilde{0}$

$\dot{\utilde{r}}(t) = (2\sin(2t) - 2\sin(t))\,\utilde{i} + (-2\cos(2t) + 2\cos(t))\,\utilde{j}$

$\utilde{r}(t) = \int (2\sin(2t) - 2\sin(t))\,dt\utilde{i} +$

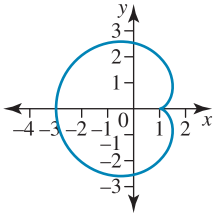
$\quad \int (-2\cos(2t) + 2\cos(t))\,dt\utilde{j}$

$\quad = (-\cos(2t) + 2\cos(t))\,\utilde{i} + (-\sin(2t) + 2\sin(t))\,\utilde{j} + \utilde{c}_2$

$\utilde{r}(\pi) = (-\cos(2\pi) + 2\cos(\pi))\,\utilde{i}$

$\quad + (-\sin(2\pi) + 2\sin(\pi))\,\utilde{j} + \utilde{c}_2 = -3\utilde{i} \rightarrow \utilde{c}_2 = 0$

$\utilde{r}(t) = (2\cos(t) - \cos(2t))\,\utilde{i} + (2\sin(t) - \sin(2t))\,\utilde{j}$



19 $\ddot{\utilde{r}}(t) = -\,(16\cos(4t) + 4\cos(t))\,\utilde{i} + (16\sin(4t) - 4\sin(t))\,\utilde{j},$

$0 \leq t \leq 2\pi$

$\dot{\utilde{r}}\left(\dfrac{\pi}{2}\right) = -4\utilde{i} - 4\utilde{j}$

$\utilde{r}\left(\dfrac{\pi}{2}\right) = \utilde{i} + 4\utilde{j}$

$\dot{\utilde{r}}(t) = \int -\,(16\cos(4t) + 4\cos(t))\,dt\utilde{i}$

$\quad + \int (16\sin(4t) - 4\sin(t))\,dt\utilde{j}$

$\quad = (-4\sin(4t) - 4\sin(t))\,\utilde{i} +$

$\quad (-4\cos(4t) + 4\cos(t))\,\utilde{j} + \utilde{c}_1$

$\dot{\utilde{r}}\left(\dfrac{\pi}{2}\right) = \left(-4\sin(2\pi) - 4\sin\left(\dfrac{\pi}{2}\right)\right)\utilde{i}$

$\quad + \left(-4\cos(2\pi) + 4\cos\left(\dfrac{\pi}{2}\right)\right)\utilde{j} + \utilde{c}_1$

$\quad = -4\utilde{i} - 4\utilde{j} \rightarrow \utilde{c}_1 = \utilde{0}$

$\dot{\utilde{r}}(t) = (-4\sin(4t) - 4\sin(t))\,\utilde{i} + (-4\cos(4t) + 4\cos(t))\,\utilde{j}$

$\utilde{r}(t) = \int (-4\sin(4t) - 4\sin(t))\,dt\utilde{i}$

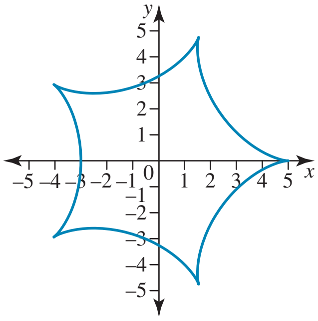
$\quad + \int (-4\cos(4t) + 4\cos(t))\,dt\utilde{j}$

$\quad = (\cos(4t) + 4\cos(t))\,\utilde{i} + (-\sin(4t) + 4\sin(t))\,\utilde{j} + \utilde{c}_2$

$\utilde{r}\left(\dfrac{\pi}{2}\right) = \left(\cos(2\pi) + 4\cos\left(\dfrac{\pi}{2}\right)\right)\utilde{i} + \left(-\sin(2\pi) + 4\sin\left(\dfrac{\pi}{2}\right)\right)$

$\quad \utilde{j} + \utilde{c}_2 = \utilde{i} + 4\utilde{j} \rightarrow \utilde{c}_2 = \utilde{0}$

$\utilde{r}(t) = (4\cos(t) + \cos(4t))\,\utilde{i} + (4\sin(t) - \sin(4t))\,\utilde{j}$



20 $\ddot{\utilde{r}}(t) = -8\cos(2t)\utilde{i} - 108\sin(6t)\utilde{j},\ \ 0 \le t \le 2\pi$

$\dot{\utilde{r}}\left(\frac{\pi}{4}\right) = -4\utilde{i}$

$\utilde{r}\left(\frac{\pi}{4}\right) = -3\utilde{j}$

$$\begin{aligned}\dot{\utilde{r}}(t) &= \int -8\cos(2t)dt\utilde{i} - \int 108\sin(6t)dt\utilde{j}\\ &= -4\sin(2t)\utilde{i} + 18\cos(6t)\utilde{j} + \utilde{c}_1\end{aligned}$$

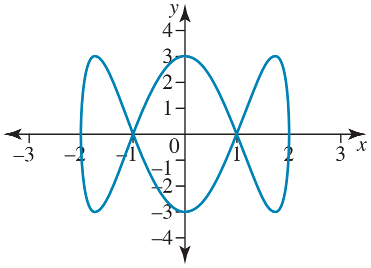
$$\begin{aligned}\dot{\utilde{r}}\left(\frac{\pi}{4}\right) &= -4\sin\left(\frac{\pi}{2}\right)\utilde{i} + 18\cos\left(\frac{3\pi}{2}\right)\utilde{j} + \utilde{c}_1\\ &= -4\utilde{i} \to \utilde{c}_1 = \utilde{0}\end{aligned}$$

$\dot{\utilde{r}}(t) = -4\sin(2t)\utilde{i} + 18\cos(6t)\utilde{j}$

$$\begin{aligned}\utilde{r}(t) &= \int -4\sin(2t)dt\utilde{i} + \int 18\cos(6t)\,dt\utilde{j}\\ &= 2\cos(2t)\utilde{i} + 3\sin(6t)\utilde{j} + \utilde{c}_2\end{aligned}$$

$\utilde{r}\left(\frac{\pi}{4}\right) = 2\cos\left(\frac{\pi}{2}\right)\utilde{i} + 3\sin\left(\frac{3\pi}{2}\right)\utilde{j} + \utilde{c}_2 \to \utilde{c}_2 = \utilde{0}$

$\utilde{r}(t) = 2\cos(2t)\utilde{i} + 3\sin(6t)\utilde{j}$



12.5 Exam questions

1 $\utilde{v}(t) = \dot{\utilde{r}}(t) = (4t-3)\utilde{i} + 2t\utilde{j} - 5\utilde{k}$

$\utilde{r}(t) = \int (4t-3)dt\utilde{i} + \int 2t\,dt\,\utilde{j} - \int 5\,dt\,\utilde{k}$ [1 mark]

$= \left(2t^2 - 3t\right)\utilde{i} + t^2\utilde{j} - 5t\,\utilde{k} + \utilde{c}$ [1 mark]

$\utilde{r}(0) = \utilde{i} - 2\utilde{k} = \utilde{c}$

$\utilde{r}(t) = \left(2t^2 - 3t + 1\right)\utilde{i} + t^2\utilde{j} - (5t+2)\utilde{k}$

$$\begin{aligned}\utilde{r}(2) &= (8-6+1)\utilde{i} + 4\utilde{j} - (10+2)\utilde{k}\\ &= 3\utilde{i} + 4\utilde{j} - 12\utilde{k}\end{aligned}$$

$|\utilde{r}(2)| = \sqrt{9+16+144} = \sqrt{169} = 13\text{ m}$

Award 1 mark for finding the constant vector.

Award 1 mark for the correct final distance.

VCAA Assessment Report **note:**

Most students handled this question very well. Common errors included finding an incorrect anti-derivative vector, forgetting to include a constant (vector) of integration, finding the incorrect constant of integration, or a sign error in consolidating the vector answer. A large proportion of these errors were caused by incorrect use (or lack) of brackets. Some students added $5t$ to 2 and got $7t$. Several students left the displacement vector as the answer, and many made arithmetic errors in calculating the modulus of the vector. Some assumed that the distance from the origin was given by the modulus of $\utilde{r}(2) - \utilde{r}(0)$. Some were unable to recognise that $\sqrt{169} = 13$.

2 $\utilde{a}(t) = -4\sin(2t)\utilde{i} + 20\cos(2t)\utilde{j} - 20e^{-2t}\utilde{k}, \qquad t \ge 0$

$\utilde{v}(t) = -\int 4\sin(2t)dt\,\utilde{i} + \int 20\cos(2t)\,dt\,\utilde{j} - \int 20e^{-2t}\,dt\,\utilde{k}$

$\utilde{v}(t) = 2\cos(2t)\,\utilde{i} + 10\sin(2t)\utilde{j} + 10e^{-2t}\utilde{k} + \utilde{c}$

$\utilde{v}(0) = 0 = 2\utilde{i} + 10\utilde{k} + \utilde{c}$

$\Rightarrow \utilde{c} = -2\utilde{i} - 10\utilde{k}$

$\utilde{v}(t) = (2\cos(2t) - 2)\,\utilde{i} + 10\sin(2t)\utilde{j} + \left(10e^{-2t} - 10\right)\utilde{k}$

The correct answer is **D**.

3

$\ddot{r}(t) = 4e^{-2t}\utilde{i} + 4e^{2t}\utilde{j}$

$\dot{\utilde{r}}(0) = -2\utilde{i} + 2\utilde{j}$

$\utilde{r}(0) = 5\utilde{i} - 2\utilde{j},\ t \ge 0$

$\dot{\utilde{r}}(t) = \int 4e^{-2t}dt\utilde{i} + \int 4e^{2t}dt\utilde{j} = -2e^{-2t}\utilde{i} + 2e^{2t}\utilde{j} + \utilde{c}_1$

$\dot{\utilde{r}}(0) = -2\utilde{i} + 2\utilde{j} + \utilde{c}_1 = -2\utilde{i} + 2\utilde{j} \to \utilde{c}_1 = \utilde{0}$

$\dot{\utilde{r}}(t) = -2e^{-2t}\utilde{i} + 2e^{2t}\utilde{j}$

$\dot{\utilde{r}}(t) = \int -2e^{-2t}dt\utilde{i} + \int 2e^{2t}dt\utilde{j} = -e^{-2t}\utilde{i} + e^{2t}\utilde{j} + \utilde{c}_2$

$\utilde{r}(0) = \utilde{i} + \utilde{j} + \utilde{c}_2 = 5\utilde{i} - 2\utilde{j}$

$\therefore \utilde{c}_2 = 4\utilde{i} - 3\utilde{i}$

$\utilde{r}(t) = \left(4 + e^{-2t}\right)\utilde{i} + \left(e^{2t} - 3\right)\utilde{j}$ [1 mark]

$x = 4 + e^{-2t} \to y = e^{2t} - 3$

$e^{-2t} = x - 4 \to e^{2t} = y - 3$

$e^{-2t} = \frac{1}{e^{2t}} \to y - 3 = \frac{1}{x-4}$

$y = \frac{1}{x-4} + 3$ [1 mark]

12.6 Projectile motion

12.6 Exercise

1 $\alpha = 67.38°$

$V = 13$ m/s

a $T = \frac{2V\sin(\alpha)}{g} = \frac{2 \times 13\sin(67.38°)}{9.8} = 2.45$ sec

b $R = \frac{V^2\sin(2\alpha)}{g} = \frac{13^2\sin(2 \times 67.38°)}{9.8} = 12.24$ m

c $H = \frac{V^2\sin^2(\alpha)}{2g} = \frac{13^2\sin^2(67.38°)}{2 \times 9.8} = 7.35$ m

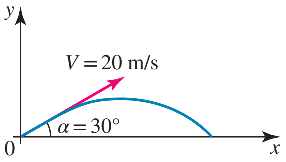
2 $T = \frac{10\sqrt{3}}{g}$

$\frac{R}{H} = \frac{4\sqrt{3}}{3}$

$$\frac{R}{H} = \frac{\frac{V^2\sin(2\alpha)}{g}}{\frac{V^2\sin^2(\alpha)}{2g}}$$

$$= \frac{V^2\sin(2\alpha)}{g} \times \frac{2g}{V^2\sin^2(\alpha)}$$
$$= \frac{2\sin(\alpha)\cos(\alpha)\times 2}{\sin^2(\alpha)}$$
$$= \frac{4\cos(\alpha)}{\sin(\alpha)}$$
$$= \frac{4}{\tan(\alpha)} = \frac{4\sqrt{3}}{3}$$
$$= \tan(\alpha) = \frac{3}{\sqrt{3}} = \sqrt{3}$$
$$\alpha = \tan^{-1}\left(\sqrt{3}\right) = 60°$$
$$T = \frac{2V\sin(\alpha)}{g} = \frac{2V\sin(60°)}{g} = \frac{10\sqrt{3}}{g}$$
$V = 10$ m/s

3



$V = 20$ m/s

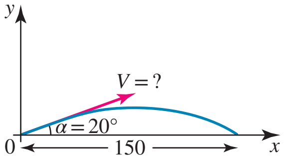
$\alpha = 30°$

a $T = \frac{2V\sin\alpha}{g} = \frac{2\times 20\times\sin(30°)}{9.8} = 2.04$ sec

b $R = \frac{V^2\sin(2\alpha)}{g} = \frac{20^2\sin(60°)}{9.8} = 35.35$ m

c $H = \frac{V^2\sin^2(\alpha)}{2g} = \frac{20^2\sin^2(30°)}{2\times 9.8} = 5.10$ m

4

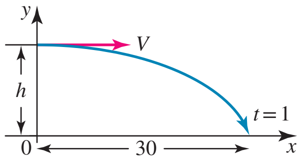


a $R = 150 = \frac{V^2\sin(40°)}{g} \rightarrow V = \sqrt{\frac{150\times 9.8}{\sin(40°)}} = 47.82$ m/s

b $H = \frac{V^2\sin^2(\alpha)}{2g} = \frac{47.82^2\sin^2(20°)}{2\times 9.8} = 13.65$ m

c $T = \frac{2V\sin(\alpha)}{g} = \frac{2\times 47.82\times\sin(20°)}{9.8} = 3.34$ sec

5



a $y = h - \frac{1}{2}gt^2 = 0$

$h = \frac{1}{2}\times 9.8\times t^2 = 4.9$ m

b $x = 30 = Vt = V\times 1 \rightarrow V = 30$ m/s

c $\dot{x} = 30,\quad \dot{y} = -gt = 9.8,\quad t = 1$

Hits the ground

$|\dot{\underset{\sim}{r}}(1)| = \sqrt{30^2 + (-9.8)^2} = 31.56$ m/s

$\tan(\psi) = \frac{\dot{y}}{\dot{x}} = \frac{9.8}{30}$

$\psi = \tan^{-1}\left(\frac{9.8}{30}\right) = 18.1°$

d $x = Vt \rightarrow t = \frac{x}{30}$

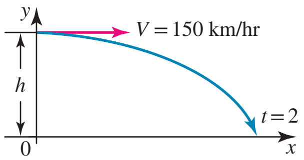
$y = h - \frac{1}{2}gt^2$

$y = 4.9 - 4.9t^2$

$y = 4.9\left(1 - \left(\frac{x}{30}\right)^2\right)$

$y = \frac{49}{9000}\left(900 - x^2\right)$

6



$V = 150 \text{ km/hr} = \frac{150\times 1000}{60\times 60} = 41\frac{2}{3}$ m/s

a $y = h - \frac{1}{2}gt^2$

$y = 0,\quad t = 2$

$h = \frac{1}{2}\times 9.8\times 4 = 19.6$ m

b $x = Vt$

$R = 41\frac{2}{3}\times 2 = 83\frac{1}{3}$ m

c $\dot{x} = 41\frac{2}{3},\quad \dot{y} = -gt = -9.8t,\quad t = 2\dot{y} = -19.6$

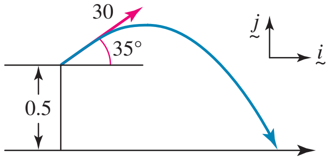
$|\dot{\underset{\sim}{r}}(t)| = \sqrt{\left(41\frac{2}{3}\right)^2 + (-19.6)^2} = 46.05$ m/s

$\tan(\psi) = \frac{\dot{y}}{\dot{x}} = \frac{19.6}{41\frac{2}{3}}$

$\psi = \tan^{-1}\left(\frac{19.6}{41\frac{2}{3}}\right) = 25.2°$

7 $V = 30$ m/s

$\alpha = 35°$



a $\ddot{\underset{\sim}{r}}(t) = -9.8\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(0) = 30\cos(35°)\underset{\sim}{i} + 30\sin(35°)\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = 30\cos(35°)\underset{\sim}{i} + (30\sin(35°) - 9.8t)\underset{\sim}{j}$

$\underset{\sim}{r}(0) = 0.5\underset{\sim}{j}$

$\underset{\sim}{r}(t) = 30t\cos(35°)\underset{\sim}{i} + \left(0.5 + 30t\sin(30°) - 4.9t^2\right)\underset{\sim}{j}$

Hits the ground when

$y = 0.5 + 30t\sin(35°) - 4.9t^2 = 0$

$t = -0.0288...,\ 3.5405...$

$t \geq 0$

Time of flight 3.54 sec

$x(3.5405...) = 30\times 3.5405...\times\cos(35°) = 87$ m

b $\dot{y} = 30\sin(35°) - 9.8t = 0$

$t = 1.76$ sec(1.7558...)

$y(1.7558...) = 0.5 + 30\times 1.7558...$

$\times\sin(35°) - 4.9\times(1.7558...)^2$

$= 15.61$ m

c $\begin{aligned}\dot{\underset{\sim}{r}}(3.5405...) &= 30\cos(35°)\underset{\sim}{i} + (35\sin(35°)\\ &\quad - 9.8 \times 3.5405...)\underset{\sim}{j}\\ &= 24.574...\underset{\sim}{i} - 17.489...\underset{\sim}{j}\end{aligned}$

$|\dot{\underset{\sim}{r}}(3.54)| = \sqrt{24.574...^2 + (-17.489...)^2} = 30.16\text{ m/s}$

$\begin{aligned}\psi &= \tan^{-1}\left(\frac{\dot{y}}{\dot{x}}\right)\\ &= \tan^{-1}\left(\frac{17.489...}{24.574...}\right)\\ &= 35.44°\end{aligned}$

d $x = 30t\cos(35°)$

$t = \dfrac{x}{30\cos(35°)}$

$y = 0.5 + 30t\sin(35°) - 4.9t^2$

$y = 0.5 + 30\left(\dfrac{x}{30\cos(35°)}\right)\sin(35°)$

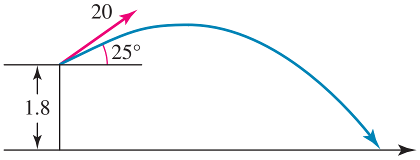
$-4.9\left(\dfrac{x}{30\cos(35°)}\right)^2$

$y = 0.5 + x\tan(35°) - \dfrac{49}{9000}x^2\sec^2(35°)$

Parabolic

8 $V = 20$ m/s

$\alpha = 25°$



a $\ddot{\underset{\sim}{r}}(t) = -9.8\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(0) = 20\cos(25°)\underset{\sim}{i} + 20\sin(25°)\underset{\sim}{j}$

$\underset{\sim}{r}(t) = 20\cos(25°\underset{\sim}{i} + (20\sin(25°) - 9.8t)\underset{\sim}{j}$

$\underset{\sim}{r}(0) = 1.8\underset{\sim}{j}$

$\underset{\sim}{r}(t) = 20t\cos(25°)\underset{\sim}{i} + \left(1.8 + 20t\sin(25°) - 4.9t^2\right)\underset{\sim}{j}$

Hits the ground when

$y = 1.8 + 20t\sin(25°) - 4.9t^2 = 0$

$t = -0.19, \quad 1.92$

$t \geq 0$

Time of flight 1.92 sec

$x(1.92) = 20 \times 1.92 \times \cos(25°) = 34.7$ m

b $\begin{aligned}\dot{y} &= 20\sin(25°) - 9.8t = 0\\ t &= 0.86\text{ sec}\\ y(0.86) &= 1.8 + 20 \times 0.86 \times \sin(25°)\\ &\quad -4.9 \times 0.86^2 = 5.45\text{ m}\end{aligned}$

c $\begin{aligned}\dot{\underset{\sim}{r}}(1.92) &= 20\cos(25°)\underset{\sim}{i} + (20\sin(25°) - 9.8 \times 1.92)\underset{\sim}{j}\\ &= 18.126\underset{\sim}{i} - 10.331\underset{\sim}{j}\end{aligned}$

$|\dot{\underset{\sim}{r}}(1.92)| = \sqrt{18.126^2 + -10.331^2} = 20.86$ m/s

$\begin{aligned}\psi &= \tan^{-1}(\frac{\dot{y}}{\dot{x}})\\ &= \tan^{-1}(\frac{10.331}{18.126})\\ &= 29.76°\end{aligned}$

d $x = 20t\cos(25°)$

$t = \dfrac{x}{20\cos(25°)}$

$y = 1.8 + 20t\sin(25°) - 4.9t^2$

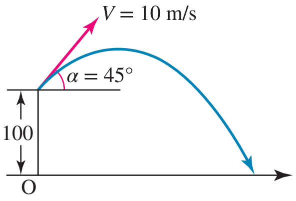
$y = 1.8 + 20(\dfrac{x}{20\cos(25°)})\sin(25°)$

$-4.9\left(\dfrac{x}{20\cos(25°)}\right)^2$

$y = 1.8 + x\tan(25°) - \dfrac{49}{4000}x^2\sec^2(25°)$

Parabolic

9



$V = 10$ m/s

$\alpha = 45°$

$h = 100$

a $y = 100 + 10t\sin(45°) - 4.9t^2$

$x = 10t\cos(45°)$

$y = 0$

$100 + 5\sqrt{2}t - 4.9t^2 = 0$

$50 + 4 \times 100 \times 4.9 = 2010$

$t = \dfrac{-5\sqrt{2} \pm \sqrt{2010}}{-2 \times 4.9} = -3.85,\ 5.296\text{ sec}$

$t \geq 0$

$t = 5.296$ sec

b $R = x(5.296) = 10 \times 5.296 \times \dfrac{\sqrt{2}}{2} = 37.45$ m

c $\dot{y} = 5\sqrt{2} - 9.8t = 0$

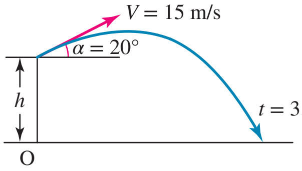
$t = 0.7215$ sec

$\begin{aligned}y(0.7215) &= 100 + 5\sqrt{2} \times 0.7215\\ &\quad - 4.9(0.7215)^2 = 102.6\text{ m}\end{aligned}$

Height above the building is 2.6 m

d $\begin{aligned}\dot{x} &= 5\sqrt{2} = 7.07\text{ m/s}\\ \dot{y}(5.296) &= 5\sqrt{2} - 9.8 \times 5.296 = -44.83\\ |\dot{\underset{\sim}{r}}(t)| &= \sqrt{(7.07)^2 + (-44.83)^2} = 45.39\text{ m/s}\\ \psi &= \tan^{-1}\left(\frac{44.83}{7.07}\right) = 81°\end{aligned}$

10



$V = 15$ m/s

$\alpha = 20°$

a $y = h + Vt\sin(\alpha) - \frac{1}{2}gt^2$

$y = 0, t = 3.053$

$h = \frac{1}{2}gt^2 - Vt\sin(\alpha)$

$= \frac{1}{2} \times 9.8 \times 3.053^2 - 15 \times 3.053 \times \sin(20°)$

$= 30$ m

b $x = Vt\cos(\alpha) = 43.03$ m

c $H = \dfrac{V^2\sin^2(\alpha)}{2g} = 1.34$ m above the edge of the cliff.

d $\dot{x} = V\cos(\alpha) = 15\cos(20°) = 14.095$

$\dot{y} = V\sin(\alpha) - gt = 15\sin(20°)$

$-9.8 \times 3.053 = -24.79$

$|\dot{\underset{\sim}{r}}(t)| = \sqrt{(14.095)^2 + (-24.79)^2} = 28.52$ m/s

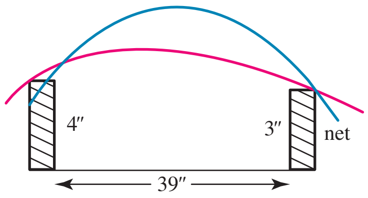
$\psi = \tan^{-1}\left(\dfrac{24.79}{14.095}\right) = 60.4°$

11 $v = 100$ ft/ sec

$g = 32$ ft/ sec

$x = 39$

$y = -1$



$$\frac{gx^2}{2V^2}\tan^2(\alpha) - x\tan(\alpha) + \left(y + \frac{gx^2}{2V^2}\right) = 0$$

$$\frac{32 \times 39^2}{2 \times 100^2}\tan^2(\alpha) - 39\tan(\alpha) + \left(\frac{32 \times 39^2}{2 \times 100^2} - 1\right) = 0$$

$$2.4336\tan^2(\alpha) - 39\tan(\alpha) + 1.4336 = 0$$

$$\Delta = 39^2 - 4 \times 1.4336 \times 2.4336 = 1507.045$$

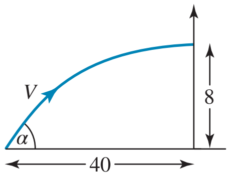
$$\sqrt{\Delta} = 38.8207$$

$$\tan(\alpha) = \frac{39 \pm 38.8207}{2 \times 2.4336} = 15.9883,\quad 0.0368$$

$$\alpha = \tan^{-1}(15.9883),\quad \tan^{-1}(0.0368)$$

$$\alpha = 86.4°,\quad 2.1°$$

12



$V = 30$ m/s

$x = 40$

$y = 8$

$\alpha = ?$

$$\frac{gx^2}{2V^2}\tan^2(\alpha) - x\tan(\alpha) + \left(y + \frac{gx^2}{2V^2}\right) = 0$$

$$\frac{9.8 \times 40^2}{2 \times 30^2}\tan^2(\alpha) - 40\tan(\alpha) + \left(8 + \frac{9.8 \times 40^2}{2 \times 30^2}\right) = 0$$

$$8.711\tan^2(\alpha) - 40\tan(\alpha) + 16.711 = 0$$

$$\tan(\alpha) = \frac{40 \pm 31.902}{2 \times 8.711} = 4.127,\ 0.465$$

$$\alpha = \tan^{-1}(4.127),\ \tan^{-1}(0.465)$$

$$= 76.4°, 25°$$

13 **a** **i** $V = 147$ m/s

$\alpha = 45°$

$x = Vt\cos(\alpha)$

$x = 2000 = 147t\cos(45°)$

$t = 19.24$ sec

$y = Vt\sin(\alpha) - \frac{1}{2}gt^2$

$= 147 \times 19.24\sin(45°) - \frac{1}{2} \times 9.8 \times 19.24^2 = 185.9$ m

ii $R = \dfrac{V^2\sin(2\alpha)}{g} = \dfrac{147^2\sin(90°)}{9.8} = 2205$ m

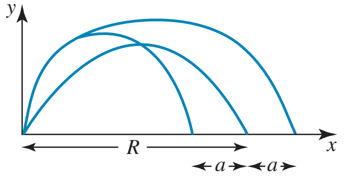
So it passes 205 beyond the target.

iii $V = ?$

$R = 2000 = \dfrac{V^2}{9.8}$

$v = \sqrt{9.8 \times 2000} = 140$ m/s

b



When

α : range $R - a$

θ : range R

β : range $R + a$

$(1)\ R - a = \dfrac{V^2\sin(2\alpha)}{g}$

$(2)\ R = \dfrac{V^2\sin(2\theta)}{g}$

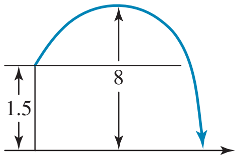
$(3)\ R + a = \dfrac{V^2\sin(2\beta)}{g}$

$(1) + (3)$: $2R = \dfrac{V^2}{g}(\sin(2\alpha) + \sin(2\beta))$

$\dfrac{2V^2\sin(2\theta)}{g} = \dfrac{V^2}{g}(\sin(2\alpha) + \sin(2\beta))$

$\sin(2\theta) = \dfrac{1}{2}(\sin(2\alpha) + \sin(2\beta))$

14



$V = ?$

$H = 8$

$\alpha = 35°$

a $H = 8 - 1.5 = 6.5 = \dfrac{V^2\sin^2(35°)}{2 \times 9.8}$

$V = \sqrt{\dfrac{2 \times 9.8 \times 6.5}{\sin^2(35°)}} = 19.68$ m/s

b $y = 1.5 + 19.68t\sin(35°) - 4.9t^2 = 0$

When it hits the ground

$1.5 + 11.287t - 4.9t^2 = 0$

$$t = \frac{-11.287 \pm \sqrt{11.287^2 + 4 \times 1.5 \times 4.9}}{-2 \times 4.9}$$

$$t = 2.43 \text{ sec}, \quad t \geq 0$$

$$x = 19.68t\cos(35°)$$
$$= 19.68 \times 2.43\cos(35°)$$
$$= 39.163 \text{ m}$$

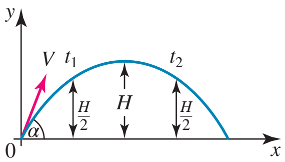
c $\dot{x} = V\cos(\alpha) = 19.68\cos(35°) = 16.12$

$\dot{y} = V\sin(\alpha) - gt = 19.68\sin(35°) - 9.8 \times 2.43 = -12.522$

speed : $\sqrt{16.12^2 + (-12.522)^2} = 20.41$ m/s

angle : $\tan^{-1}\left(\frac{12.522}{16.12}\right) = 37.8°$

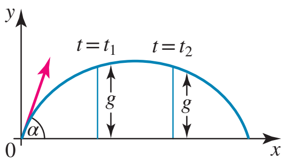
15



$$H = \frac{V^2\sin^2(\alpha)}{2g}$$
$$V^2\sin^2(\alpha) = 2gH$$
$$V^2 = \frac{2gH}{\sin^2(\alpha)}$$

$$y = Vt\sin(\alpha) - \frac{1}{2}gt^2$$
$$y = \frac{H}{2}$$
$$\frac{H}{2} = Vt\sin(\alpha) - \frac{1}{2}gt^2$$
$$\frac{1}{2}gt^2 - Vt\sin(\alpha) + \frac{H}{2} = 0$$
$$t = \frac{V\sin(\alpha) \pm \sqrt{\Delta}}{g}$$
$$\Delta = (V\sin(\alpha))^2 - 4 \times \frac{1}{2}g \times \frac{H}{2}$$
$$= V^2\sin^2(\alpha) - gH$$
$$= 2gH - gH$$
$$= gH$$
$$t_1 = \frac{V\sin(\alpha) - \sqrt{gH}}{g}$$
$$t_2 = \frac{V\sin(\alpha) + \sqrt{gH}}{g}$$
$$t_2 - t_1 = \frac{2\sqrt{gH}}{g}$$
$$= 2\sqrt{\frac{H}{g}}$$

16



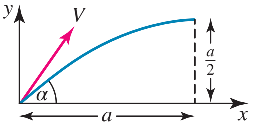
$V = 3g$

$y = g$

$t = ?$

$$y = Vt\sin(\alpha) - \frac{1}{2}gt^2$$
$$g = 3gt\sin(\alpha) - \frac{1}{2}gt^2$$
$$\frac{1}{2}gt^2 - 3gt\sin(\alpha) + g = 0$$
$$t^2 - 6t\sin(\alpha) + 2 = 0$$
$$t = \frac{6\sin(\alpha) \pm \sqrt{36\sin^2(\alpha) - 8}}{2}$$
$$t_1 = 3\sin(\alpha) + \sqrt{9\sin^2(\alpha) - 2}$$
$$t_2 = 3\sin(\alpha) - \sqrt{9\sin^2(\alpha) - 2}$$
$$t_1 - t_2 = 2\sqrt{9\sin^2(\alpha) - 2} = 1$$
$$\sqrt{9\sin^2(\alpha) - 2} = \frac{1}{2}$$
$$9\sin^2(\alpha) - 2 = \frac{1}{4}$$
$$9\sin^2(\alpha) = 2 + \frac{1}{4} = \frac{9}{4}$$
$$\sin^2(\alpha) = \frac{1}{4}$$
$$\sin(\alpha) = \frac{1}{2}$$
$$\alpha = 30°$$
$$\cos(\alpha) = \frac{\sqrt{3}}{2}$$
$$R = \frac{V^2\sin(2\alpha)}{g}$$
$$= \frac{V^2 \times 2\sin(\alpha)\cos(\alpha)}{g}$$
$$= \frac{9g^2 \times 2 \times \frac{1}{2} \times \frac{\sqrt{3}}{2}}{g}$$
$$R = \frac{9g\sqrt{3}}{2}$$

17

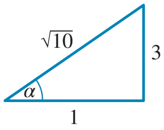


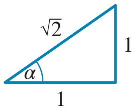
$V = \sqrt{2ga}$

$x = a$

$y = \frac{a}{2}$

$$\frac{gx^2}{2V^2}\tan^2(\alpha) - x\tan(\alpha) + \left(y + \frac{gx^2}{2V^2}\right) = 0$$
$$\frac{ga^2}{2 \times 2ga}\tan^2(\alpha) - a\tan(\alpha) + \left(\frac{a}{2} + \frac{ga^2}{2 \times 2ga}\right) = 0$$
$$\frac{a}{4}\tan^2(\alpha) - a\tan(\alpha) + \frac{3a}{4} = 0$$
$$\tan^2(\alpha) - 4\tan(\alpha) + 3 = 0$$
$$(\tan(\alpha) - 3)(\tan(\alpha) - 1) = 0$$
$$\alpha = \tan^{-1}(3), \tan^{-1}(1)$$





$$x = Vt\cos(\alpha)$$
$$t_1 = \frac{x}{\cos(\alpha_1)}$$
$$t_2 = \frac{x}{\cos(\alpha_2)}$$
$$\frac{t_1}{t_2} = \frac{\sqrt{10}}{\sqrt{2}} = \sqrt{5}$$

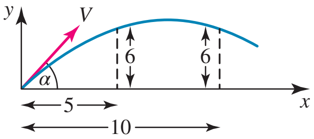
$$\mathrm{H} = \frac{V^2\sin^2(\alpha)}{2g}$$

ratio of heights

$$\frac{\sin^2(\alpha_1)}{\sin^2(\alpha_2)}$$

$$= \frac{\frac{9}{10}}{\frac{1}{2}} = \frac{2}{1} \times \frac{9}{10} = \frac{9}{5}$$

18



a $P(5,6), \quad P(10,6)$

$$y = x\tan(\alpha) - \frac{gx^2}{2V^2}\left(1+\tan^2(\alpha)\right)$$

$$\frac{gx^2}{2V^2}\left(1+\tan^2(\alpha)\right) = x\tan(\alpha) - y$$

$$\frac{g}{2V^2}\left(1+\tan^2(\alpha)\right) = \frac{x\tan(\alpha) - y}{x^2}$$

LHS of this equality is independent of x and y

$$\frac{5\tan(\alpha)-6}{25} = \frac{10\tan(\alpha)-6}{100}$$

$$4\,(5\tan(\alpha)-6) = 10\tan(\alpha)-6$$

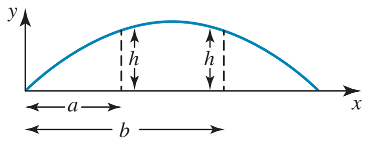
$$20\tan(\alpha) - 24 = 10\tan(\alpha) - 6$$

$$10\tan(\alpha) = 18$$

$$\tan(\alpha) = \frac{9}{5}$$

$$\alpha = \tan^{-1}\left(\frac{9}{5}\right)$$

b



$P(a,h), \quad P(b,h)$

$$\frac{g}{2V^2}\left(1+\tan^2(\alpha)\right) = \frac{x\tan(\alpha) - y}{x^2}$$

$$\frac{a\tan(\alpha)-h}{a^2} = \frac{b\tan(\alpha)-h}{b^2}$$

LHS of this equality is independent of x and y

$$b^2\,(a\tan(\alpha)-h) = a^2\,(b\tan(\alpha)-h)$$

$$ab^2\tan(\alpha) - b^2h = a^2b\tan(\alpha) - a^2h$$

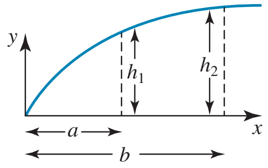
$$\left(ab^2 - a^2b\right)\tan(\alpha) = h\,(b-a)\,(a+b)$$

$$a \neq b$$

$$\tan(\alpha) = \frac{h\,(a+b)}{ab}$$

$$\alpha = \tan^{-1}\left(\frac{h\,(a+b)}{ab}\right)$$

c



$$b > a$$

$$\frac{g}{2V^2}\left(1+\tan^2(\alpha)\right) = \frac{x\tan(\alpha) - y}{x^2}$$

LHS of this equality is independent of x and y

$P_1(a,h_1), P_2(b,h_2)$

$$\frac{a\tan(\alpha)-h_1}{a^2} = \frac{b\tan(\alpha)-h_2}{b^2}$$

$$b^2\,(a\tan(\alpha)-h_1) = a^2\,(b\tan(\alpha)-h_2)$$

$$ab^2\tan(\alpha) - b^2h_1 = a^2b\tan(\alpha) - a^2h_2$$

$$\left(ab^2 - a^2b\right)\tan(\alpha) = b^2h_1 - a^2h_2$$

$$ab\,(b-a)\tan(\alpha) = b^2h_1 - a^2h_2$$

$$\tan(\alpha) = \frac{b^2h_1 - a^2h_2}{ab\,(b-a)}$$

19 $\underset{\sim}{r}(t) = 60\left(1 - e^{-\frac{t}{2}}\right)\underset{\sim}{i} + 2t\underset{\sim}{j} + \left(1 + 12t - 4.9t^2\right)\underset{\sim}{k}$

a Football hits the ground when

$1 + 12t - 4.9t^2 = 0$

$t = -0.08, \quad 2.53$ sec

$t \geq 0$

Football hits the ground after 2.53 sec

$$\underset{\sim}{r}\,(2.53) = 60\left(1 - e^{-\frac{2.53}{2}}\right)\underset{\sim}{i} + 2\,(2.53)\,\underset{\sim}{j} = 43.063\underset{\sim}{i} + 5.059\underset{\sim}{j}$$

$$\left|\underset{\sim}{r}\,(2.53)\right| = \sqrt{43.063^2 + 5.059^2} = 43.36 \text{ m}$$

b $\dot{\underset{\sim}{r}}(t) = 30e^{-\frac{t}{2}}\underset{\sim}{i} + 2\underset{\sim}{j} + (12 - 9.8t)\,\underset{\sim}{k}$

$$\dot{\underset{\sim}{r}}(2.53) = 30e^{-\frac{2.53}{2}}\,\underset{\sim}{i} + 2\underset{\sim}{j} + (12 - 9.8 \times 2.53)\,\underset{\sim}{k}$$

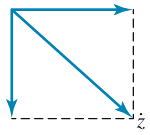
$$= 8.469\underset{\sim}{i} + 2\underset{\sim}{j} - 12.79\underset{\sim}{k}$$

$$\left|\dot{\underset{\sim}{r}}(2.53)\right| = \sqrt{8.469^2 + 2^2 + (-12.79)^2} = 15.47 \text{ m/s}$$

$$\psi = \tan^{-1}\left(\frac{\dot{z}}{\sqrt{\dot{x}+\dot{y}}}\right)$$

$$= \tan^{-1}\left(\frac{12.79}{\sqrt{8.469^2 + 2^2}}\right)$$

$$= 55.77°$$



20 **a** $\underset{\sim}{r}(t) = 20t\underset{\sim}{i} + \left(2\pi t - 3\sin\left(\frac{\pi t}{6}\right)\right)\underset{\sim}{j} + \left(1.8 + 14.4t - 5t^2\right)\underset{\sim}{k}$

Hits the ground when

$1.8 + 14.4t - 5t^2 = 0$

$$t = \frac{-14.4 \pm \sqrt{14.4^2 - 4 \times -5 \times 1.8}}{2}$$

$t = -0.12, \quad 3$

$t \geq 0$

Hits the ground after 3 seconds.

b $\underset{\sim}{r}(3) = 20(3)\underset{\sim}{i} + \left(2\pi(3) - 3\sin\left(\frac{\pi}{2}\right)\right)\underset{\sim}{j}$

$$= 60\underset{\sim}{i} + (6\pi - 3)\,\underset{\sim}{j}$$

$$\left|\underset{\sim}{r}(3)\right| = \sqrt{60^2 + (6\pi - 3)^2} = 62.06 \text{ m}$$

c $\dot{\underset{\sim}{r}}(t) = 20\underset{\sim}{i} + \left(2\pi - \frac{\pi}{2}\cos\left(\frac{\pi t}{6}\right)\right)\underset{\sim}{j} + (14.4 - 10t)\underset{\sim}{k}$

When it hits the ground

$$\dot{\underset{\sim}{r}}(3) = 20\underset{\sim}{i} + \left(2\pi - \frac{\pi}{2}\cos\left(\frac{\pi}{2}\right)\right)\underset{\sim}{j} + (14.4 - 10 \times 3)\underset{\sim}{k}$$

$$= 20\underset{\sim}{i} + 2\pi\underset{\sim}{j} - 15.6\underset{\sim}{k}$$

$$|\dot{\underset{\sim}{r}}(3)| = \sqrt{20^2 + 4\pi^2 + (-15.6)^2} = 26.13 \text{ m/s}$$

$$\tan(\psi) = \frac{\dot{z}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = \frac{15.6}{\sqrt{20^2 + 4\pi^2}}$$

$$\psi = \tan^{-1}\left(\frac{15.6}{\sqrt{20^2 + 4\pi^2}}\right) = 36.65°$$

21 a $\underset{\sim}{r}(t) = 6t\underset{\sim}{i} + 28t\underset{\sim}{j} + \frac{12\sqrt{2}}{5}\sin\left(\frac{\pi t}{2}\right)\underset{\sim}{k}$

$$\dot{\underset{\sim}{r}}(t) = 6\underset{\sim}{i} + 28\underset{\sim}{j} + \frac{6\sqrt{2}}{5}\pi\cos\left(\frac{\pi t}{2}\right)\underset{\sim}{k}$$

$$\dot{\underset{\sim}{r}}(0) = 6\underset{\sim}{i} + 28\underset{\sim}{j} + \frac{6\sqrt{2}}{5}\pi\underset{\sim}{k}$$

$$|\dot{\underset{\sim}{r}}(0)| = \sqrt{(6)^2 + (28)^2 + \left(\frac{6\sqrt{2}}{5}\pi\right)^2} = 29.13 \text{ m/s}$$

b $\tan(\psi) = \dfrac{\frac{6\sqrt{2}}{5}\pi}{\sqrt{6^2 + 28^2}}$

$$\psi = \tan^{-1}(0.186) = 10.5°$$

c Maximum height is $\frac{12\sqrt{2}}{5} = 3.4$ m

When

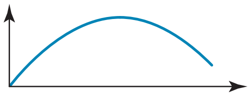
$\sin\left(\frac{\pi t}{2}\right) = 1$

$t = 1$ sec

d i When $\underset{\sim}{k}$ component of $\underset{\sim}{r} = 2.4$:

$$\sin\left(\frac{\pi t}{2}\right) = \frac{1}{\sqrt{2}}$$

$$\frac{\pi t}{2} = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}, \frac{3\pi}{4}$$



Take $\frac{3\pi}{4}$ as it is on the way down

$t = 1.5$ sec

ii $\underset{\sim}{r}\left(\frac{3}{2}\right) = 6\left(\frac{3}{2}\right)\underset{\sim}{i} + 28\left(\frac{3}{2}\right)\underset{\sim}{j} + 2.4\underset{\sim}{k}$

$$= 9\underset{\sim}{i} + 42\underset{\sim}{j} + 2.4\underset{\sim}{k}$$

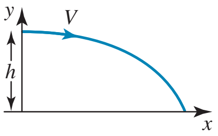
Distance $\sqrt{9^2 + 42^2} = 42.95$ m

iii $\dot{\underset{\sim}{r}}\left(\frac{3}{2}\right) = 6\underset{\sim}{i} + 28\underset{\sim}{j} + \frac{6\sqrt{2}}{5}\pi\cos\left(\frac{3\pi}{4}\right)\underset{\sim}{k}$

$$= 6\underset{\sim}{i} + 28\underset{\sim}{j} - \frac{6\pi}{5}\underset{\sim}{k}$$

$$\left|\dot{\underset{\sim}{r}}\left(\frac{3}{2}\right)\right| = \sqrt{(6)^2 + (28)^2 + \left(-\frac{6\pi}{5}\right)^2} = 28.88 \text{ m/s}$$

22



$x = Vt,$

$y = h - \frac{1}{2}gt^2$

a Hits the ground $y = 0, \quad t = T$

$$h - \frac{1}{2}gT^2 = 0$$

$$T = \sqrt{\frac{2h}{g}}$$

b When

$x = R, \quad t = T$

$R = VT$

c Hits the ground $t = T$

$$\dot{x} = V$$

$$\dot{y} = -gT = -g\sqrt{\frac{2h}{g}} = -\sqrt{2gh}$$

$$|\dot{\underset{\sim}{r}}(t)| = \sqrt{V^2 + 2gh}$$

d $\tan(\psi) = \frac{\dot{y}}{\dot{x}}$

$$\dot{y} = -gT$$

$$\dot{x} = V = \frac{R}{T}$$

$$\psi = \tan^{-1}\left(\frac{\dot{y}}{\dot{x}}\right) = \tan^{-1}\left(\frac{-gT}{V}\right) = \tan^{-1}\left(\frac{-gT}{\frac{R}{T}}\right)$$

$$= \tan^{-1}\left(\frac{-gT^2}{R}\right) = \tan^{-1}\left(\frac{-g}{R} \times \frac{2h}{g}\right)$$

$$= \tan^{-1}\left(\frac{-2h}{R}\right)$$

therefore the angle at which the object strikes the ground is $\tan^{-1}\left(\frac{2h}{R}\right)$.

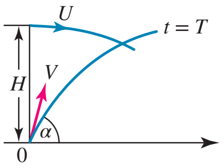
e $t = \frac{x}{V}$

$$y = h - \frac{1}{2}gt^2 = h - \frac{1}{2}g\left(\frac{x}{V}\right)^2 = h - \frac{gx^2}{2V^2}$$

$$= h - \frac{gx^2T^2}{2R^2}$$

$$y = h\left(1 - \frac{x^2}{R^2}\right)$$

23



$(1): x = Ut$

$(2): y = H - \frac{1}{2}gt^2$

$(3): x = Vt\cos(\alpha)$

$(4): y = Vt\sin(\alpha) - \frac{1}{2}gt^2$

At the point of collision $t = T$

$$UT = VT\cos(\alpha)$$

$$\cos(\alpha) = \frac{U}{V}$$

$$\alpha = \cos^{-1}\left(\frac{U}{V}\right)$$

$$VT\sin(\alpha) - \frac{1}{2}gT^2 = H - \frac{1}{2}gT^2$$

$$H = VT\sin(\alpha)$$

$$\sin(\alpha) = \frac{H}{VT}$$

$$\cos(\alpha) = \frac{U}{V}$$
$$\sin^2(\alpha) + \cos^2(\alpha) = 1$$
$$\frac{H^2}{V^2T^2} + \frac{U^2}{V^2} = 1$$
$$V^2T^2 = H^2 + U^2T^2$$
$$V^2 = U^2 + \frac{H^2}{T^2}$$

24 $T = \dfrac{2V\sin(\alpha)}{g}$

$$R = \frac{V^2\sin(2\alpha)}{g}$$
$$H = \frac{V^2\sin^2(\alpha)}{2g}$$

a $T^2 = \dfrac{4V^2\sin^2(\alpha)}{g^2}$

$$\frac{1}{2}gT^2 = \frac{1}{2}g \times \frac{4V^2\sin^2(\alpha)}{g^2} = \frac{2V^2\sin^2(\alpha)}{g}$$
$$= \frac{V^2 \times 2\sin(\alpha)\cos(\alpha)}{g} \times \frac{\sin(\alpha)}{\cos(\alpha)}$$
$$= \frac{V^2\sin(2\alpha)}{g} \times \tan(\alpha)$$
$$\frac{1}{2}gT^2 = R\tan(\alpha)$$

b $\dfrac{4H}{R} = \dfrac{4V^2\sin^2(\alpha)}{2g} \times \dfrac{g}{V^2\sin(2\alpha)} = \dfrac{2\sin^2(\alpha)}{2\sin(\alpha)\cos(\alpha)} = \tan(\alpha)$

c $T^2 = \dfrac{4V^2\sin^2(\alpha)}{g^2} = \dfrac{V^2\sin^2(\alpha)}{2g} \times \dfrac{8}{g}$

$$T^2 = \frac{8H}{g}$$

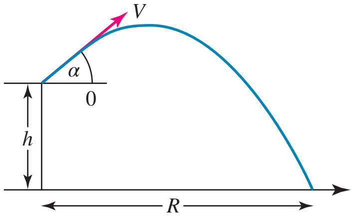
d $y = x\tan(\alpha) - \dfrac{gx^2}{2V^2}\left(1 + \tan^2(\alpha)\right)$

$$x = \frac{R}{2}, y = H, \tan(\alpha) = \frac{4H}{R}$$
$$H = \frac{R}{2} \times \frac{4H}{R} - \frac{g}{2V^2}\left(\frac{R^2}{4}\right)\left(1 + \frac{16H^2}{R^2}\right)$$
$$= 2H - \frac{gR^2}{8V^2} - \frac{2gH^2}{V^2}$$
$$H = \frac{gR^2}{8V^2} + \frac{2gH^2}{V^2}$$
$$gR^2 + 16gH^2 - 8V^2H = 0$$
$$8V^2H = g\left(16H^2 + R^2\right)$$
$$V^2 = \frac{g}{8H}(16)\left(H^2 + \frac{R^2}{16}\right)$$
$$V = \sqrt{2g\left(H + \frac{R^2}{16H}\right)}$$

e $y = x\tan(\alpha) - \dfrac{gx^2}{2V^2}\left(1 + \tan^2(\alpha)\right)$

$$\tan(\alpha) = \frac{4H}{R}, V^2 = \frac{g}{8H}(16)\left(H^2 + \frac{R^2}{16}\right)$$
$$y = \frac{4H}{R}x - \frac{gx^2}{2} \times \frac{8H}{g\left(16H^2 + R^2\right)}\left(1 + \frac{16H^2}{R^2}\right)$$
$$y = \frac{4H}{R}x - \frac{4x^2H}{\left(16H^2 + R^2\right)} \times \left(\frac{16H^2 + R^2}{R^2}\right)$$
$$= \frac{4H}{R}x - \frac{4x^2H}{R^2}$$
$$y = \frac{4Hx}{R^2}(R - x)$$

25



a i $y = x\tan(\alpha) - \dfrac{gx^2}{2V^2}\left(1 + \tan^2(\alpha)\right)$

$$x = R, y = -h$$
$$-h = R\tan(\alpha) - \frac{gR^2}{2V^2}\sec^2(\alpha)$$

Implicit equation $R = R(\alpha)$

For maximum range $\dfrac{dR}{d\alpha} = 0$

$$R\frac{d}{d\alpha}\tan(\alpha) + \tan(\alpha)\frac{dR}{d\alpha} - \frac{g}{2V^2}\sec^2(\alpha) \cdot \frac{dR^2}{d\alpha}$$
$$-\frac{gR^2}{2V^2}\frac{d}{d\alpha}\sec^2(\alpha) = 0$$
$$R\sec^2(\alpha) - \frac{gR^2}{2V^2} \times 2\sec(\alpha)\tan(\alpha)\sec(\alpha) = 0$$
$$R\sec^2(\alpha) = \frac{gR^2}{V^2}\sec^2(\alpha)\tan(\alpha)$$
$$\tan(\alpha) = \frac{V^2}{gR}$$

ii $h = \dfrac{gR^2}{2V^2}\left(1 + \tan^2(\alpha)\right) - R\tan(\alpha)$

$$= \frac{gR^2}{2V^2}\left(1 + \frac{V^4}{g^2R^2}\right) - R\frac{V^2}{gR}$$
$$= \frac{gR^2}{2V^2} + \frac{V^2}{2g} - \frac{V^2}{g}$$
$$= \frac{gR^2}{2V^2} - \frac{V^2}{2g}$$
$$h = \frac{g^2R^2 - V^4}{2V^2g}$$

iii $2V^2gh = g^2R^2 - V^4$

$$g^2R^2 = 2V^2gh + V^4$$
$$R^2 = \frac{V^2}{g^2}\left(V^2 + 2gh\right)$$
$$R = \frac{V}{g}\sqrt{V^2 + 2gh}$$

iv $x = R,\ t = T$

$$x = Vt\cos(\alpha) \rightarrow \cos(\alpha) = \frac{R}{VT}$$
$$y = Vt\sin(\alpha) - \frac{1}{2}gt^2 \rightarrow -h = VT\sin(\alpha) - \frac{1}{2}gT^2$$
$$-h = VT\frac{\sin(\alpha)}{\cos(\alpha)} \times \cos(\alpha) - \frac{1}{2}gT^2$$

$$-h = VT\tan(\alpha)\times\cos(\alpha)-\frac{1}{2}gT^2$$

$$-h = VT\cdot\frac{V^2}{gR}\times\frac{R}{VT}-\frac{1}{2}gT^2$$

$$-h=\frac{V^2}{g}-\frac{1}{2}gT^2$$

$$\frac{1}{2}gT^2=\frac{V^2}{g}+h$$

$$\frac{1}{2}gT^2=\frac{V^2+gh}{g}$$

$$T^2=\frac{V^2+gh}{g^2}\times 2$$

$$T=\frac{\sqrt{2(V^2+gh)}}{g}$$

b $V = 20$ m/s

$h = 2$

$$R=\frac{V}{g}\sqrt{V^2+2gh}$$

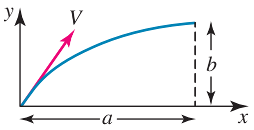
$$=\frac{20}{9.8}\sqrt{20^2+2\times 9.8\times 2}$$

$= 42.77$ m

$$\alpha=\tan^{-1}\left(\frac{V^2}{gR}\right)=43.7^\circ$$

$$T=\frac{\sqrt{2(V^2+gh)}}{g}=2.96\text{ s}$$

3



$V=\sqrt{2ga}$

$x = a$

$y = b$

a $$\frac{gx^2}{2V^2}\tan^2(\alpha)-x\tan(\alpha)+\left(y+\frac{gx^2}{2V^2}\right)=0$$

$$\frac{ga^2}{2\times 2ga}\tan^2(\alpha)-a\tan(\alpha)+\left(b+\frac{ga^2}{2\times 2ga}\right)=0 \quad [1 \text{ mark}]$$

$$\frac{a}{4}\tan^2(\alpha)-a\tan(\alpha)+b+\frac{a}{4}=0$$

$$a\tan^2(\alpha)-4a\tan(\alpha)+(4b+a)=0 \quad [1 \text{ mark}]$$

b $\Delta=(4a)^2-4a(4b+a)$

$=16a^2-16ab-4a^2$

$=12a^2-16ab$

$=4a(3a-4b)$

Two roots for α, $\Delta>0$

No roots for α, $\Delta<0$

$\rightarrow 3a-4b<0$

$4b>3a$ [1 mark]

c If $4b=3a,\ \Delta=0$

$$\rightarrow \tan(\alpha)=\frac{4a}{2a}=2$$

$\alpha=\tan^{-1}(2)$ [1 mark]

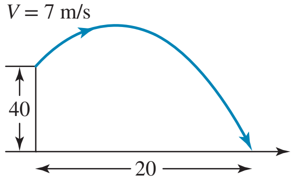
12.6 Exam questions

1 $$V=20\,\text{ms}^{-1},\alpha=30^\circ,R=\frac{V^2\sin(2\alpha)}{g}$$

$$\Rightarrow R=\frac{20^2\sin(60^\circ)}{g}=\frac{400\times\frac{\sqrt{3}}{2}}{g}=\frac{200\sqrt{3}}{g}$$

The correct answer is **D**.

2 $V = 7$ m/s



$V = 7$ m/s

$x = 20$

$y = -40$

$\alpha = ?$

$$\frac{gx^2}{2V^2}\tan^2(\alpha)-x\tan(\alpha)+\left(y+\frac{gx^2}{2V^2}\right)=0$$

$$\frac{9.8\times 20^2}{2\times 7^2}\tan^2(\alpha)-20\tan(\alpha)+\left(8+\frac{9.8\times 20^2}{2\times 7^2}\right)=0 \quad [1 \text{ mark}]$$

$$40\tan^2(\alpha)-20\tan(\alpha)=0$$

$$20\tan(\alpha)(2\tan(\alpha)-1)=0$$

$$\tan(\alpha)=0,\tan(\alpha)=\frac{1}{2}$$

$\alpha=0, 26.6^\circ$ [1 mark]

12.7 Review

12.7 Exercise

Technology free: short answer

1 **a** $\underset{\sim}{r}(t)=6\cos(t)\underset{\sim}{i}+6\sin(t)\underset{\sim}{j}$

$$\underset{\sim}{r}\left(\frac{\pi}{4}\right)=6\cos\left(\frac{\pi}{4}\right)\underset{\sim}{i}+6\sin\left(\frac{\pi}{4}\right)\underset{\sim}{j}$$

$=3\sqrt{2}\underset{\sim}{i}+3\sqrt{2}\underset{\sim}{j}$

$A(3\sqrt{2},3\sqrt{2})$

b $$\underset{\sim}{r}\left(\frac{5\pi}{6}\right)=6\cos\left(\frac{5\pi}{6}\right)\underset{\sim}{i}+6\sin\left(\frac{5\pi}{6}\right)\underset{\sim}{j}$$

$=-3\sqrt{3}\underset{\sim}{i}+3\underset{\sim}{j}$

$B(-3\sqrt{3},3)$

c $x=6\cos(t)$

$y=6\sin(t)$

$x^2+y^2=36$

d $$\underset{\sim}{r}\left(\frac{5\pi}{6}\right)-\underset{\sim}{r}\left(\frac{\pi}{4}\right)=-(3\sqrt{3}+3\sqrt{2})\underset{\sim}{i}+(3-3\sqrt{2})\underset{\sim}{j}$$

$$\left|\underset{\sim}{r}\left(\frac{5\pi}{6}\right)-\underset{\sim}{r}\left(\frac{\pi}{4}\right)\right|=\sqrt{(3\sqrt{3}+3\sqrt{2})^2+(3-3\sqrt{2})^2}$$

$$=\sqrt{27+18+18\sqrt{6}+9-18\sqrt{2}+18}$$

$$=\sqrt{72+18\sqrt{6}-18\sqrt{2}}$$

$$=\sqrt{18(\sqrt{6}-\sqrt{2}+4)}$$

$$=3\sqrt{2(\sqrt{6}-\sqrt{2}+4)}$$

≈ 9.52

e $\left|\underset{\sim}{r}\left(\frac{\pi}{4}\right)\right| = 6$

$\left|\underset{\sim}{r}\left(\frac{5\pi}{6}\right)\right| = 6$

$\underset{\sim}{r}\left(\frac{\pi}{4}\right)\cdot\underset{\sim}{r}\left(\frac{5\pi}{6}\right) = 9\sqrt{2} - 9\sqrt{6}$

$\cos(\theta) = \dfrac{\underset{\sim}{r}\left(\frac{\pi}{4}\right)\cdot\underset{\sim}{r}\left(\frac{5\pi}{6}\right)}{\left|\underset{\sim}{r}\left(\frac{\pi}{4}\right)\right|\left|\underset{\sim}{r}\left(\frac{5\pi}{6}\right)\right|} = \dfrac{9(\sqrt{2}-\sqrt{6})}{36}$

$= \frac{1}{4}(\sqrt{2}-\sqrt{6})$

but $\theta = \dfrac{5\pi}{6} - \dfrac{\pi}{4} = \dfrac{7\pi}{12}$

$\therefore \cos\left(\dfrac{7\pi}{12}\right) = \dfrac{9(\sqrt{2}-\sqrt{6})}{36}$

f $s = \displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

$x = 6\cos(t) \qquad y = 6\sin(t)$

$\dfrac{dx}{dt} = -6\sin(t) \qquad \dfrac{dy}{dt} = 6\cos(t)$

$\sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}$

$= \sqrt{36\sin^2(t) + 36\cos^2(t)}$

$= \sqrt{36\left(\sin^2(t) + \cos^2(t)\right)} = 6$

$s = \displaystyle\int_{\frac{\pi}{4}}^{\frac{5\pi}{6}} 6\,dt = \left[6t\right]_{\frac{\pi}{4}}^{\frac{5\pi}{6}} = 6\left(\frac{5\pi}{6} - \frac{\pi}{4}\right)$

$s = \dfrac{7\pi}{2}$

2 **a** $\underset{\sim}{r}(t) = 8\cos(t)\underset{\sim}{i} + 6\sin(t)\underset{\sim}{j}$

$\underset{\sim}{r}\left(\dfrac{\pi}{4}\right) = 8\cos\left(\dfrac{\pi}{4}\right)\underset{\sim}{i} + 6\sin\left(\dfrac{\pi}{4}\right)\underset{\sim}{j}$

$= 4\sqrt{2}\underset{\sim}{i} + 3\sqrt{2}\underset{\sim}{j}$

$A\left(4\sqrt{2},\, 3\sqrt{2}\right)$

b $\underset{\sim}{r}\left(\dfrac{5\pi}{6}\right) = 8\cos\left(\dfrac{5\pi}{6}\right)\underset{\sim}{i} + 6\sin\left(\dfrac{5\pi}{6}\right)\underset{\sim}{j}$

$= -4\sqrt{3}\underset{\sim}{i} + 3\underset{\sim}{j}$

$B\left(-4\sqrt{3},\, 3\right)$

c $x = 8\cos(t)$

$\cos(t) = \dfrac{x}{8}$

$y = 6\sin(t)$

$\sin(t) = \dfrac{y}{6}$

$\dfrac{x^2}{64} + \dfrac{y^2}{36} = 1$

3 $\underset{\sim}{r}(t) = 8\cos(2t)\underset{\sim}{i} + 6\sin(2t)\underset{\sim}{j}, t \in [0, \pi]$

$\dot{\underset{\sim}{r}}(t) = -16\sin(2t)\underset{\sim}{i} + 12\cos(2t)\underset{\sim}{j}$

a $\underset{\sim}{r}\left(\dfrac{\pi}{6}\right) = 8\cos\left(\dfrac{\pi}{3}\right)\underset{\sim}{i} + 6\sin\left(\dfrac{\pi}{3}\right)\underset{\sim}{j} = 4\underset{\sim}{i} + 3\sqrt{3}\underset{\sim}{j}$

b $\dot{\underset{\sim}{r}}\left(\dfrac{\pi}{6}\right) = -16\sin\left(\dfrac{\pi}{3}\right)\underset{\sim}{i} + 12\cos\left(\dfrac{\pi}{3}\right)\underset{\sim}{j} = -8\sqrt{3}\underset{\sim}{i} + 6\underset{\sim}{j}$

c $\underset{\sim}{r}(t)\cdot\dot{\underset{\sim}{r}}(t) = 8\times(-16\cos(2t)\sin(2t))$

$\qquad + 6\times 12\sin(2t)\cos(2t)$

$= -56\sin(2t)\cos(2t)$

$= -28\sin(4t)$

$\underset{\sim}{r}(t) \perp \dot{\underset{\sim}{r}}(t)$

$\underset{\sim}{r}(t)\cdot\dot{\underset{\sim}{r}}(t) = 0$

$\sin(4t) = 0$

$4t = 0, \pi, 2\pi, 3\pi, 4\pi$

$t = 0, \dfrac{\pi}{4}, \dfrac{\pi}{2}, \dfrac{3\pi}{4}, \pi$

d For parallel

$\underset{\sim}{r}(t) = \lambda\dot{\underset{\sim}{r}}(t)$

$\underset{\sim}{i} \qquad \dfrac{-16\sin(2t)}{8\cos(2t)} = \lambda$

$\underset{\sim}{j} \qquad \dfrac{12\cos(2t)}{6\sin(2t)} = \lambda$

$-2\tan(2t) = \dfrac{2}{\tan(2t)}$

$\tan^2(2t) = -1$

Not possible.

4 **a** $V = 8$ m/s, $\alpha = ?$, $\dfrac{R}{H} = 4$

$\dfrac{R}{H} = \dfrac{V^2\sin(2\alpha)}{g} \times \dfrac{2g}{V^2\sin^2(\alpha)}$

$= \dfrac{4\sin(\alpha)\cos(\alpha)}{\sin^2(\alpha)} = \dfrac{4}{\tan(\alpha)} = 4$

$\Rightarrow \tan(\alpha) = 1, \quad \alpha = 45^{\circ}$

$R = \dfrac{V^2}{g} = \dfrac{8^2}{g} = \dfrac{64}{g}\text{m}$

b $T = \dfrac{2v\sin(\alpha)}{g} = \dfrac{2\times 8\sin(45)}{g} = \dfrac{8\sqrt{2}}{g}s$

5 $\underset{\sim}{p} = 3\cos(2t)\,\underset{\sim}{i} + 4\sin(2t)\underset{\sim}{j}$

$\underset{\sim}{q} = 4\sin(nt)\,\underset{\sim}{i} + 3\cos(nt)\underset{\sim}{j}$

a $\underset{\sim}{p}: \quad x = 3\cos(2t)$

$\qquad y = 4\sin(2t)$

$\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1, \ t \geq 0$

$\underset{\sim}{q}: \ x = 4\sin(nt)$

$\qquad y = 3\cos(nt)$

$\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1, \ t \geq 0$

b Solving

$\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

$\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$

$\left(\dfrac{12}{5}, \dfrac{12}{5}\right) \quad \left(\dfrac{-12}{5}, \dfrac{12}{5}\right)$

$\left(\dfrac{-12}{5}, \dfrac{-12}{5}\right) \quad \left(\dfrac{12}{5}, \dfrac{-12}{5}\right)$

6 $\ddot{\underset{\sim}{r}}(t) = -\cos(t)\underset{\sim}{i} - 9\cos(3t)\underset{\sim}{j}$

$$\dot{\underset{\sim}{r}}(t) = \int -\cos(t)dt\underset{\sim}{i} - \int 9\cos(3t)dt\underset{\sim}{j}$$
$$\dot{\underset{\sim}{r}}(t) = -\sin(t)\underset{\sim}{i} - 3\sin(3t)\underset{\sim}{j} + \underset{\sim}{c}_1$$
$$\dot{\underset{\sim}{r}}(0) = \underset{\sim}{0} = \underset{\sim}{c}_1$$
$$\dot{\underset{\sim}{r}}(t) = -\sin(t)\underset{\sim}{i} - 3\sin(3t)\underset{\sim}{j}$$
$$\underset{\sim}{r}(t) = \int -\sin(t)dt\underset{\sim}{i} - \int 3\sin(3t)dt\underset{\sim}{j}$$
$$\underset{\sim}{r}(t) = \cos(t)\underset{\sim}{i} + \cos(3t)\underset{\sim}{j} + \underset{\sim}{c}_2$$
$$\underset{\sim}{r}(0) = \underset{\sim}{i} + \underset{\sim}{j} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{c}_2$$
$$\underset{\sim}{c}_1 = \underset{\sim}{0}$$
$$\underset{\sim}{r}(t) = \cos(t)\underset{\sim}{i} + \cos(3t)\underset{\sim}{j}$$
$$x = \cos(t)$$
$$y = \cos(3t)$$
$$y = \cos(2t + t)$$
$$= \cos(2t)\cos(t) - \sin(2t)\sin(t)$$
$$= (2\cos^2(t) - 1)\cos(t) - 2\sin^2(t)\cos(t)$$
$$= (2\cos^2(t) - 1)\cos(t) - 2\cos(t)(1 - \cos^2(t))$$
$$= 2\cos^3(t) - \cos(t) - 2\cos(t) + 2\cos^3(t)$$
$$= 4\cos^3(t) - 3\cos(t)$$
$$y = 4x^3 - 3x$$

Technology active: multiple choice

7 $\underset{\sim}{p} = \left(t^2 - 8t + 15\right)\underset{\sim}{i} + \left(t^2 - 4t + 3\right)\underset{\sim}{j}$

$$\underset{\sim}{p} = (t-5)(t-3)\underset{\sim}{i} + (t-3)(t-1)\underset{\sim}{j}$$
$$\underset{\sim}{q} = \left(t^2 - 5t + 6\right)\underset{\sim}{i} + \left(t^2 - 6t + 8\right)\underset{\sim}{j}$$
$$\underset{\sim}{q} = (t-3)(t-2)\underset{\sim}{i} + (t-4)(t-2)\underset{\sim}{j}$$
$$\underset{\sim}{p}(3) = \underset{\sim}{0}$$
$$\underset{\sim}{p}(2) \neq \underset{\sim}{0}$$
$$\underset{\sim}{q}(3) = -\underset{\sim}{j}$$
$$\underset{\sim}{q}(2) = \underset{\sim}{0}$$

The correct answer is **A**.

8 $\underset{\sim}{r}(4) = 8\underset{\sim}{i} + 4\underset{\sim}{j}$

$$|\underset{\sim}{r}(4)| = \sqrt{64 + 16} = \sqrt{80} = 4\sqrt{5}$$

The correct answer is **A**.

9 $\dot{\underset{\sim}{r}}(t) = na\sin(nt)\underset{\sim}{i} + bm\cos(mt)\underset{\sim}{j}$

$$\dot{\underset{\sim}{r}}(0) = bm\underset{\sim}{j}$$
$$\hat{\dot{\underset{\sim}{r}}}(0) = \underset{\sim}{j}$$

The correct answer is **D**.

10 $\dot{\underset{\sim}{r}}(t) = -(\sin(t) + \cos(t))\underset{\sim}{i} + (-\sin(t) + \cos(t))\underset{\sim}{j}$

$$|\dot{\underset{\sim}{r}}(t)| = \sqrt{(\sin(t) + \cos(t))^2 + (-\sin(t) + \cos(t))^2}$$
$$= \sqrt{2}$$

The correct answer is **D**.

11 $\dot{\underset{\sim}{r}}(t) = -6\sin(2t)\underset{\sim}{i} + 8\cos(2t)\underset{\sim}{j}$

$$\frac{dy}{dx} = \frac{8\cos(2t)}{-6\sin(2t)} = \frac{-4}{3\tan(2t)}$$
$$x = 3\cos(2t)$$
$$|\dot{\underset{\sim}{r}}(t)| = \sqrt{36\sin^2(2t) + 64\cos^2(2t)}$$

The correct answer is **C**.

12 $x = 4\cos(2t)$

$$\cos(2t) = \frac{x}{4}$$
$$y = 2\sin(2t)$$
$$\sin(2t) = \frac{y}{2}$$
$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

This is the equation of an ellipse.

The correct answer is **E**.

13 $\dot{\underset{\sim}{r}}(t) = 4e^{\frac{t}{2}}\underset{\sim}{i} - 2\sin\left(\frac{t}{2}\right)\underset{\sim}{j}$

$$\underset{\sim}{r}(0) = 3\underset{\sim}{i} - 3\underset{\sim}{j}$$
$$\underset{\sim}{r}(t) = 8e^{\frac{t}{2}}\underset{\sim}{i} + 4\cos\left(\frac{t}{2}\right)\underset{\sim}{j} + \underset{\sim}{c}$$
$$\underset{\sim}{r}(0) = 8\underset{\sim}{i} + 4\underset{\sim}{j} + \underset{\sim}{c} = 3\underset{\sim}{i} - 3\underset{\sim}{j}$$
$$\Rightarrow \underset{\sim}{c} = -5\underset{\sim}{i} - 7\underset{\sim}{j}$$
$$\underset{\sim}{r}(t) = (8e^{\frac{t}{2}} - 5)\underset{\sim}{i} + (4\cos\left(\frac{t}{2}\right) - 7)\underset{\sim}{j}$$

The correct answer is **C**.

14 The particle moves in a circle.

The correct answer is **B**.

15 $H = \dfrac{V^2\sin^2(\alpha)}{2g}$

$$= \frac{16^2 \times \frac{1}{4}}{2g}$$
$$= \frac{32}{g}$$

The correct answer is **B**.

16 Alicia

$$\dot{\underset{\sim}{r}}(t) = 15\sqrt{2}\,\underset{\sim}{i} + \left(15\sqrt{2} - gt\right)\underset{\sim}{k}$$
$$\dot{\underset{\sim}{r}}(0) = 15\sqrt{2}\,\underset{\sim}{i} + \left(15\sqrt{2}\right)\underset{\sim}{k}$$
$$|\dot{\underset{\sim}{r}}(0)| = \sqrt{2\left(15\sqrt{2}\right)^2}$$
$$= \sqrt{900} = 30$$

Betty

Solving

$$15\sqrt{2}t - \frac{1}{2}gt^2 = 0$$
$$t = 0, 4.33$$

Colin

Maximum

$$y = 15\sqrt{2}t - \frac{1}{2}gt^2$$
$$\frac{dy}{dt} = 15\sqrt{2} - gt = 0$$
$$t = \frac{15\sqrt{2}}{g} \approx 2.16$$
$$H = 15\sqrt{2}(2.16) - \frac{1}{2}g(2.16)^2 = 22.96$$

Edward

Golf ball travels in a parabolic path.

David

$$15\sqrt{2} \times 4.33 = 91.84$$

The correct answer is **E**.

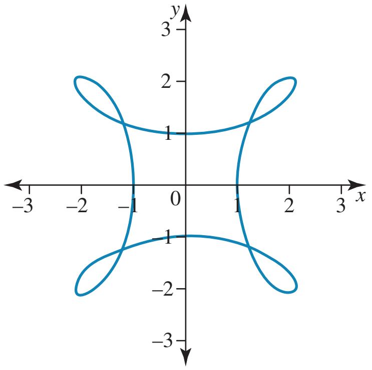
Technology active: extended response

17 $\underset{\sim}{r}(t) = (2\cos(t) - \cos(3t))\,\underset{\sim}{i} + (2\sin(t) + \sin(3t))\,\underset{\sim}{j},\ t \geq 0$

a
$$x = 2\cos(t) - \cos(3t)$$
$$\dot{x} = -2\sin(t) + 3\sin(3t)$$
$$y = 2\sin(t) + \sin(3t)$$
$$\dot{y} = 2\cos(t) + 3\cos(3t)$$
$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{2\cos(t) + 3\cos(3t)}{-2\sin(t) + 3\sin(3t)}$$
$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{6}} = \frac{2\cos\left(\frac{\pi}{6}\right) + 3\cos\left(\frac{\pi}{2}\right)}{-2\sin\left(\frac{\pi}{6}\right) + 3\sin\left(\frac{\pi}{2}\right)} = \frac{\sqrt{3}}{2}$$
$$\dot{\underset{\sim}{r}}(t) = (-2\sin(t) + 3\sin(3t))\,\underset{\sim}{i} + (2\cos(t) + 3\cos(3t))\,\underset{\sim}{j}$$

b
$$|\dot{\underset{\sim}{r}}(t)| = \sqrt{(-2\sin(t) + 3\sin(3t))^2 + (2\cos(t) + 3\cos(3t))^2}$$
$$= \sqrt{4\sin^2(t) - 12\sin(t)\sin(3t) + 9\sin^2(3t) + 4\cos^2(t) + 12\cos(t)\cos(3t) + 9\cos^2(3t)}$$
$$= \sqrt{4 + 12(\cos(t)\cos(3t) - \sin(t)\sin(3t)) + 9}$$
$$= \sqrt{13 + 12\cos(4t)}$$
$$\cos(4t) = 1 \rightarrow |\dot{\underset{\sim}{r}}(t)|_{\max} = 5$$
$$\cos(4t) = -1 \rightarrow |\dot{\underset{\sim}{r}}(t)|_{\min} = 1$$

c



18 a $R = \dfrac{V^2\sin(2\alpha)}{g} = 480$

$V = 70\text{ m/s}$
$$480 = \frac{70^2\sin(2\alpha)}{9.8}$$
$$\sin(2\alpha) = 0.96$$
$$2\alpha = \sin^{-1}(0.96), 180 - \sin^{-1}(0.96)$$
$$\alpha = 36.87°, 53.13°$$

b
$$T = \frac{2V\sin(\alpha)}{g}$$
$$\sin(\alpha_1) = \frac{3}{5}$$
$$\sin(\alpha_2) = \frac{4}{5}$$
$$\frac{T_1}{T_2} = \frac{\sin(\alpha_1)}{\sin(\alpha_2)} = \frac{3}{4}$$
$$T_1 : T_2 = 3 : 4$$

c $H = \dfrac{\sin^2(\alpha_1)}{\sin^2(\alpha_2)} = \dfrac{9}{16}$

$H_1 : H_2 = 9 : 16$

19 a Same range, same initial speed, different angles, times of flight, 1, 2 seconds.
$$(1) : \frac{V^2}{g}\sin(2\alpha_1) = \frac{V^2}{g}\sin(2\alpha_2) = R$$
$$(2) : T = 1 = \frac{2V\sin(\alpha_1)}{g}$$
$$(3) : T = 2 = \frac{2V\sin(\alpha_2)}{g}$$
$$\frac{(2)}{(3)} : \frac{1}{2} = \frac{\sin(\alpha_1)}{\sin(\alpha_2)}$$
$$\sin(2\alpha_2) = \sin(2\alpha_1)$$
$$2\alpha_1 = 180 - 2\alpha_2$$
$$\alpha_1 = 90 - \alpha_2$$
$$0 < \alpha_1 < \alpha_2 < 90^{\text{o}}$$
$$\sin\left(\alpha_1\right) = \sin\left(90^{\text{o}} - \alpha_2\right) = \cos\left(\alpha_2\right)$$
$$\rightarrow \tan\left(\alpha_2\right) = 2$$
$$\alpha_2 = \tan^{-1}(2) = 63.4^{\text{o}}$$
$$\alpha_1 = 26.6^{\text{o}}$$
$$V = \frac{g}{2\sin\left(\alpha_1\right)} = 10.96\text{ m/s}$$
$$R = \frac{V^2\sin\left(2\alpha_1\right)}{g} = 9.8\text{ m}$$

b Angles are α_1, α_2
$$R = \frac{V^2\sin(2\alpha)}{g}$$
$$T = \frac{2V\sin(\alpha)}{g}$$
$$T_1 = \frac{2V\ \sin(\alpha_1)}{g}$$
$$T_2 = \frac{2V\ \sin(\alpha_2)}{g}$$
$$T_1T_2 = \frac{4V^2\sin(\alpha_1)\sin(\alpha_2)}{g^2}$$
$$= \frac{2V^2}{g^2}\left(2\sin(\alpha_1)\sin(\alpha_2)\right)$$
$$= \frac{2V^2}{g^2}\sin(2\alpha_2)$$
$$= \frac{2}{g}\left(\frac{V^2\sin(2\alpha_2)}{g}\right)$$
$$T_1T_2 = \frac{2R}{g}$$
$$R = \frac{1}{2}gT_1T_2$$

20 $\underset{\sim}{r}(t) = \sqrt{t}\ \cos(t)\underset{\sim}{i} + \sqrt{t}\ \sin(t)\underset{\sim}{j},\quad 0 \leq t \leq 9$

a Distance from the origin.
$$|\underset{\sim}{r}(t)| = \sqrt{\left(\sqrt{t}\ \cos(t)\right)^2 + \left(\sqrt{t}\ \sin(t)\right)^2}$$
$$= \sqrt{t(\cos^2(t) + \sin^2(t))}$$
$$= \sqrt{t}$$

b $\dot{\underset{\sim}{r}}(t) = \left(\frac{1}{2\sqrt{t}}\cos(t) - \sqrt{t}\,\sin(t)\right)\underset{\sim}{i} + \left(\frac{1}{2\sqrt{t}}\sin(t) + \sqrt{t}\,\cos(t)\right)\underset{\sim}{j}$

$$|\dot{\underset{\sim}{r}}(t)|^2 = \left(\frac{1}{2\sqrt{t}}\cos(t) - \sqrt{t}\,\sin(t)\right)^2 + \left(\frac{1}{2\sqrt{t}}\sin(t) + \sqrt{t}\,\cos(t)\right)^2$$

$$= \frac{1}{4t}\cos^2(t) - \cos(t)\sin(t) + t\sin^2(t) + \frac{1}{4t}\sin^2(t) + \cos(t)\sin(t) + t\cos^2(t)$$

$$= \frac{1}{4t}(\cos^2(t) + \sin^2(t)) + t(\cos^2(t) + \sin^2(t))$$

$$|\dot{\underset{\sim}{r}}(t)|^2 = \frac{1}{4t} + t = \frac{1+4t^2}{4t}$$

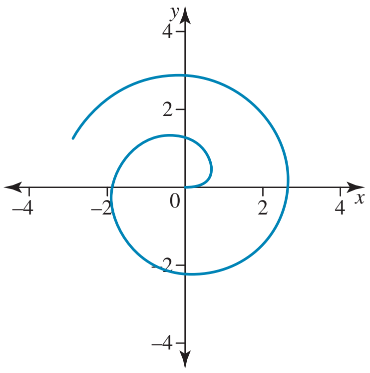
$$|\dot{\underset{\sim}{r}}(t)| = \sqrt{\frac{1+4t^2}{4t}}$$

c $|r(4)| = \sqrt{4} = 2$

d $s = \int_0^4 \sqrt{\frac{1+4t^2}{4t}}\,dt$

$= 6.08$

e



f $\underset{\sim}{r}(t).\dot{\underset{\sim}{r}}(t) = \sqrt{t}\cos(t)\left(\frac{1}{2\sqrt{t}}\cos(t) - \sqrt{t}\,\sin(t)\right) + \sqrt{t}\sin(t)\left(\frac{1}{2\sqrt{t}}\sin(t) + \sqrt{t}\,\cos(t)\right)$

$$= \frac{1}{2}\cos^2(t) - t\cos(t)\sin(t) + \frac{1}{2}\sin^2(t) + t\cos(t)\sin(t)$$

$$= \frac{1}{2}$$

g $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\frac{1}{2\sqrt{t}}\sin(t) + \sqrt{t}\,\cos(t)}{\frac{1}{2\sqrt{t}}\cos(t) - \sqrt{t}\,\sin(t)}$

$$= \frac{\frac{1}{2\sqrt{t}}[\sin(t) + 2t\,\cos(t)]}{\frac{1}{2\sqrt{t}}[\cos(t) - 2t\,\sin(t)]}$$

$$= \frac{\sin(t) + 2t\,\cos(t)}{\cos(t) - 2t\,\sin(t)}$$

$$= \frac{\cos(t)(\tan(t) + 2t)}{\cos(t)(1 - 2t\tan(t))}$$

$$= \frac{\tan(t) + 2t}{1 - 2t\tan(t)}$$

h Gradient = zero

$\frac{dy}{dx} = 0 \Rightarrow \tan(t) + 2t = 0$

Using CAS

$t = 1.84, 4.82, 7.92$

i Gradient = infinite, vertical tangents

$1 - 2t\tan(t) = 0$

Using CAS

$t = 0.65, 3.29.6.36$

12.7 Exam questions

1 a i $A: \underset{\sim}{r}(t) = (-1 + 4\cos(t))\underset{\sim}{i} + \frac{2}{\sqrt{3}}\sin(t)\underset{\sim}{j}$,

$B: \underset{\sim}{s}(t) = (3\sec(t) - 1)\underset{\sim}{i} + \tan(t)\underset{\sim}{j}$

$A: x(t) = -1 + 4\cos(t), y(t) = \frac{2}{\sqrt{3}}\sin(t)$

$\cos(t) = \frac{x+1}{4}, \sin(t) = \frac{\sqrt{3}y}{2}$

$\cos^2(t) + \sin^2(t) = 1$

$\frac{(x+1)^2}{16} + \frac{3y^2}{4} = 1$ [1 mark]

ii $\frac{3y^2}{4} = 1 - \frac{(x+1)^2}{16}$

$\frac{3y^2}{4} = \frac{16 - (x+1)^2}{16}$

$y^2 = \frac{1}{3\times4}\left(16 - \left(x^2 + 2x + 1\right)\right)$

$y^2 = \frac{1}{3\times4}\left(-x^2 - 2x + 15\right)$

$y = \pm\frac{1}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}\sqrt{-x^2 - 2x + 15}$

But in the first quadrant $x > 0,\ y > 0$ take positive

$y = \frac{\sqrt{3}}{6}\sqrt{-x^2 - 2x + 15}$ [1 mark]

b i A and B will collide if $\underset{\sim}{r}(t) = \underset{\sim}{s}(t)$

$\underset{\sim}{i}$: (1) $-1 + 4\cos(t) = 3\sec(t) - 1$ $\underset{\sim}{j}$: (2) $\frac{2}{\sqrt{3}}\sin(t) = \tan(t)$

$4\cos(t) = \frac{3}{\cos(t)}$ $\frac{2}{\sqrt{3}}\sin(t) = \frac{\sin(t)}{\cos(t)}$

$\cos^2(t) = \frac{3}{4}$ $\sin(t) \neq 0,\ \cos(t) = \frac{\sqrt{3}}{2}$

Only common solutions:

$\cos(t) = \frac{\sqrt{3}}{2}, \quad t = \frac{\pi}{6}$ [1 mark]

ii $\underset{\sim}{r}\left(\frac{\pi}{6}\right) = \left(-1 + 4\cos\left(\frac{\pi}{6}\right)\right)\underset{\sim}{i} + \frac{2}{\sqrt{3}}\sin\left(\frac{\pi}{6}\right)\underset{\sim}{j}$

$= \left(2\sqrt{3} - 1\right)\underset{\sim}{i} + \frac{\sqrt{3}}{3}\underset{\sim}{j}$

Collision point is $\left(2\sqrt{3} - 1, \frac{\sqrt{3}}{3}\right)$ [1 mark]

c i $\frac{d}{dx}\left(8\sin^{-1}\left(\frac{x+1}{4}\right)+\frac{(x+1)\sqrt{-x^2-2x+15}}{2}\right)$

Using the product rule

$$=\frac{d}{dx}\left(8\sin^{-1}\left(\frac{x+1}{4}\right)\right)+\frac{1}{2}\sqrt{-x^2-2x+15}\,\frac{d}{dx}(x+1)$$

$$+\frac{(x+1)}{2}\frac{d}{dx}\left(\left(-x^2-2x+15\right)^{\frac{1}{2}}\right)$$

$$=\frac{8}{\sqrt{16-(x+1)^2}}+\frac{1}{2}\sqrt{-x^2-2x+15}$$

$$+\frac{(x+1)}{2}\times\frac{1}{2}\times(-2x-2)\left(-x^2-2x+15\right)^{-\frac{1}{2}}$$

$$=\frac{8}{\sqrt{-x^2-2x+15}}+\frac{\sqrt{-x^2-2x+15}}{2}-\frac{(x+1)^2}{2\sqrt{-x^2-2x+15}}$$

$$=\frac{16+\left(\sqrt{-x^2-2x+15}\right)^2-\left(x^2+2x+1\right)}{2\sqrt{-x^2-2x+15}}$$

$$=\frac{16-x^2-2x+15-\left(x^2+2x+1\right)}{2\sqrt{-x^2-2x+15}}$$

$$=\frac{2\left(-x^2-2x+15\right)}{2\sqrt{-x^2-2x+15}}$$

$$=\frac{\sqrt{-x^2-2x-15}\times\sqrt{-x^2-2x-15}}{\sqrt{-x^2-2x-15}}$$

$$=\sqrt{-x^2-2x+15}$$

Award 1 mark for correct derivatives and using the product rule.

c ii Award 1 mark for the correct proof.

$$A=\frac{\sqrt{3}}{6}\int_{1}^{2\sqrt{3}-1}\sqrt{-x^2-2x+15}\,dx$$

$$A=\frac{\sqrt{3}}{6}\left[8\sin^{-1}\left(\frac{x+1}{4}\right)+\frac{(x+1)\sqrt{-x^2-2x+15}}{2}\right]_{1}^{2\sqrt{3}-1}$$

$$A=\frac{\sqrt{3}}{6}\left[8\sin^{-1}\left(\frac{x+1}{4}\right)+\frac{(x+1)\sqrt{16-(x+1)^2}}{2}\right]_{1}^{2\sqrt{3}-1}$$

$$A=\frac{\sqrt{3}}{6}\left[\left(8\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)+\frac{2\sqrt{3}\sqrt{4}}{2}\right)-\right.$$

$$\left.\left(8\sin^{-1}\left(\frac{1}{2}\right)+\frac{2\sqrt{12}}{2}\right)\right]$$

$$A=\frac{\sqrt{3}}{6}\left(8\left(\frac{\pi}{3}-\frac{\pi}{6}\right)+\sqrt{12}-\sqrt{12}\right)=\frac{8\sqrt{3}}{6}\times\frac{\pi}{6}$$

$$A=\frac{2\sqrt{3}\pi}{9}\ a=2, b=9$$

Award 1 mark for correct anti-derivative and evaluating.

Award 1 mark for the correct answer.

2 a (1) $x=ut\cos(\theta)$, (2) $y=ut\sin(\theta)-\frac{1}{2}gt^2$

(1) $t=\frac{x}{u\cos(\theta)}$ into (2)

$$y=\frac{xu\sin(\theta)}{u\cos(\theta)}-\frac{1}{2}g\left(\frac{x}{u\cos(\theta)}\right)^2$$

$$y=x\tan(\theta)-\frac{4.9x^2}{u^2\cos^2(\theta)}\qquad\text{[1 mark]}$$

b At C, when $x=16$, $y=4$, $\theta=30^0$, $u=?$

Solving $4=16\tan(30^0)-\frac{4.9\times16^2}{u^2\cos^2(30^0)}$

for u gives the minimum speed as 17.87 ms^{-1}

Award 1 mark for setting up the equation to be solved.

Award 1 mark for the correct value of u.

c At C, when $x=16$, $y=4$, $\theta=?$, $u=?$

$$4=16\tan(\theta)-\frac{4.9\times16^2}{u^2\cos^2(\theta)}\qquad(1)$$

$$\frac{dy}{dx}=\tan(\theta)-\frac{9.8x}{u^2\cos^2(\theta)}$$

$$\frac{dy}{dx}=\tan(170°)=\tan(\theta)-\frac{9.8\times16}{u^2\cos^2(\theta)}\qquad(2)$$

Solving (1), (2) for u and θ, gives $u=16.4$, $\theta=34°$

Award 2 marks for identifying the correct equations to be solved.

Award 1 mark for correct values of u and θ.

d $a=v\frac{dv}{ds}=\frac{60}{v}$

$$\int 60ds=\int v^2\,dv$$

$$60s=\frac{1}{3}v^3+c,\ s=0,\ v=0,\ c=0$$

$$v^3=180s,\ v=(180s)^{\frac{1}{3}}$$

Award 1 mark for solving the differential equation.

Award 1 mark for the correct proof.

e AB: $v=20,\ 20=(180s)^{\frac{1}{3}},\ s=\frac{400}{9}=44.444$

$$a=v\frac{dv}{ds}=-9$$

$$\int v\,dv=\frac{1}{2}v^2=-9s+c$$

$$v=0,\quad s=\frac{400}{9},\quad c=400$$

$$v^2=800-18s$$

$$v=\sqrt{800-18s}=(180s)^{\frac{1}{3}},\qquad s=28.082$$

$$WB=\frac{400}{9}-28.082=16.4\,\text{m}$$

Award 1 mark for setting up the differential equation.

Award 1 mark for the correct value of s.

Award 1 mark for the final correct distance.

3 a Aeroplane $\underset{\sim}{r}_A(t)=$

$$\left(450-150\sin\left(\frac{\pi t}{6}\right)\right)\underset{\sim}{i}+\left(400-200\cos\left(\frac{\pi}{6}\right)\right)\underset{\sim}{j}$$

$$\dot{\underset{\sim}{r}}_A(t)=-\frac{150\pi}{6}\cos\left(\frac{\pi t}{6}\right)\underset{\sim}{i}+\frac{200\pi}{6}\sin\left(\frac{\pi t}{6}\right)\underset{\sim}{j}$$

$$=-25\pi\cos\left(\frac{\pi t}{6}\right)\underset{\sim}{j}+\frac{100\pi}{3}\sin\left(\frac{\pi t}{6}\right)\underset{\sim}{j}$$

$$\left|\dot{\underset{\sim}{r}}_A(t)\right|=\sqrt{\left(-25\pi\cos\left(\frac{\pi t}{6}\right)\right)^2+\left(\frac{100\pi}{3}\sin\left(\frac{\pi t}{6}\right)\right)^2}$$

$$=\sqrt{625\pi^2\cos^2\left(\frac{\pi t}{6}\right)+\frac{10000\pi^2}{9}\sin^2\left(\frac{\pi t}{6}\right)}$$

$$=\sqrt{625\pi^2\left(1-\sin^2\left(\frac{\pi t}{6}\right)\right)+\frac{10000\pi^2}{9}\sin^2\left(\frac{\pi t}{6}\right)}$$

$$=\sqrt{625\pi^2+\left(\frac{10000}{9}-625\right)\pi^2\sin^2\left(\frac{\pi t}{6}\right)}$$

When $\sin\left(\frac{\pi}{6}\right)=1$

$$|\dot{\underset{\sim}{r}}_A(t)|_{\max} = \sqrt{\frac{10000\pi^2}{9}}$$
$$= \frac{100\pi}{3}$$

Award 1 mark for the correct velocity vector.
Award 1 mark for finding the speed.
Award 1 mark for the correct maximum speed.

b i $x = 450 - 150\sin\left(\frac{\pi t}{6}\right),\ y = 400 - 200\cos\left(\frac{\pi t}{6}\right)$

$$\sin\left(\frac{\pi t}{6}\right) = \frac{x-450}{-150},\ \cos\left(\frac{\pi t}{6}\right) = \frac{y-400}{-200}$$
$$\sin^2\left(\frac{\pi t}{6}\right) + \cos^2\left(\frac{\pi t}{6}\right) = 1$$
$$\left(\frac{x-450}{-150}\right)^2 + \left(\frac{y-400}{-200}\right) = 1$$
$$\frac{(x-450)^2}{22500} + \frac{(y-400)^2}{40000} = 1$$

Award 1 mark for correctly eliminating the parameter.
Award 1 mark for the correct proof.

ii $\underset{\sim}{r}_A(0) = 450\underset{\sim}{i} + 200\underset{\sim}{j}$ and $\underset{\sim}{r}_A(3) = 300\underset{\sim}{i} + 400\underset{\sim}{j}$
So, the plane moves on an ellipse starting at the point (450, 200) when $t = 0$ and moving clockwise.
See image at the bottom*
Award 2 marks for the correct diagram and direction.
Award 1 mark for the correct initial value.

c Drone: $\underset{\sim}{r}_D(t) = (30t)\underset{\sim}{i} + \left(-t^2 + 40t\right)\underset{\sim}{j}$,

$$x = 30t, y = -t^2 + 40t, t = \frac{x}{30}, 0 \le t \le 40$$

$$y = -\frac{x^2}{900} + \frac{4x}{3}, 0 \le x \le 1200$$

Solving $\frac{(x-450)^2}{22500} + \frac{(y-400)^2}{40000} = 1$ and $y = -\frac{x^2}{900} + \frac{4x}{3}$
gives $x = 316,\ x = 600$.

The point of intersection are (316, 310) and (600, 400)
See Image bottom of the page*
Award 1 mark for solving.
Award 1 mark for the correct points of intersection.
Award 1 mark for the path of the drone on the diagram.

d Solving $x = 450 - 150\sin\left(\frac{\pi t}{6}\right) = 30t$ gives $t = 12.85$
Solving $y = 400 - 200\cos\left(\frac{\pi t}{6}\right) = -t^2 + 40t$ gives
$t = 10, 14.73$,
So, although their paths cross, they are at the intersection point at different times.
Therefore, the drone will **NOT** make contact with the aeroplane.
Award 2 marks for correctly solving x and y.
Award 1 mark for the correct interpretation.

4 a $\underset{\sim}{r}_A(t) = 5(1-t)\underset{\sim}{i} + 3(t+1)\underset{\sim}{j}$
$\underset{\sim}{r}_B(t) = 4(t-2)\underset{\sim}{i} + (5t-2)\underset{\sim}{j}$

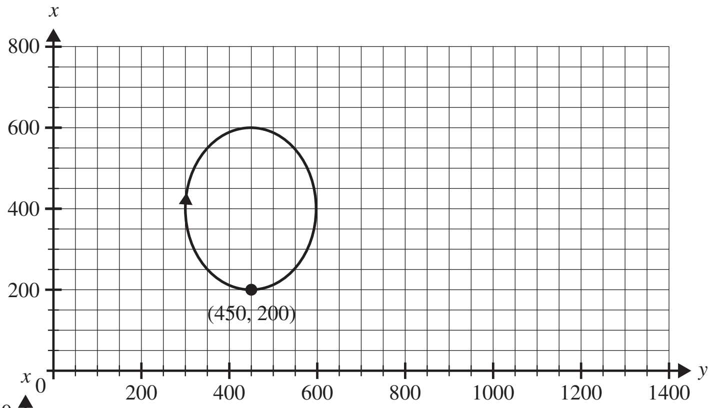
Equating $\underset{\sim}{i}$ components : $5(1-t) = 4(t-2) \Rightarrow t = \frac{13}{9}$

Equating $\underset{\sim}{j}$ components : $3(1+t) = 5t - 2 \Rightarrow t = \frac{5}{2}$

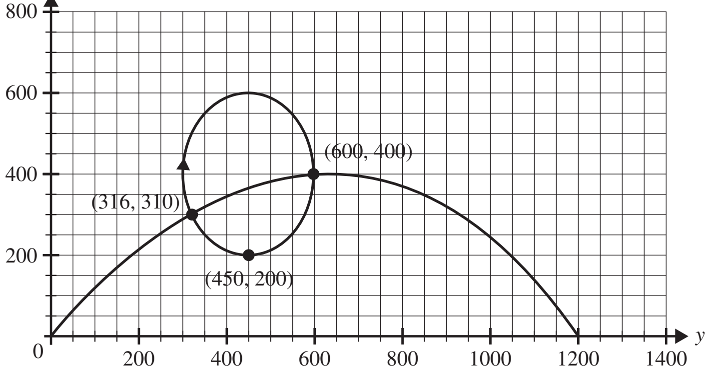
The two ships are not at the same coordinates at the same time.
Therefore, the ships do not collide.
Award 1 mark for solving the equations.
Award 1 mark for the correct conclusion.

b Ship A has parametric equations
$x = 5(1-t), y = 3(1+t)$ or $y = -\frac{3x}{5} + 6$, starting from $\underset{\sim}{r}_A(0) = 5\underset{\sim}{i} + 3\underset{\sim}{j}$, (5, 3).

***3 b ii**



***3 c**



Ship B has parametric equations

$x = 4(t-2), y = 5t - 2$ or $y = \dfrac{5x}{4} + 8$, starting from $\underset{\sim}{r}_B(0) = -8\underset{\sim}{i} - 2\underset{\sim}{j}, (-8, -2)$.

See image at the bottom*

Award 1 mark for the correct line for ship A.

Award 1 mark for the correct line for ship B.

Award 1 mark for the correct intersection and starting points.

***VCAA Assessment Report* note:**

The majority of students found the correct cartesian expressions for the paths. Many students did not take note of when the vector functions applied and consequently plotted the paths over incorrect domains. The instruction to show the direction of motion was usually followed.

c $\underset{\sim}{v}_A(t) = \dot{\underset{\sim}{r}}_A(t) = -5\underset{\sim}{i} + 3\underset{\sim}{j}$

$|\dot{\underset{\sim}{r}}_A(t)| = \sqrt{25+9} = \sqrt{34}$

$\underset{\sim}{v}_B(t) = \dot{\underset{\sim}{r}}_B(t) = 4\underset{\sim}{i} + 5\underset{\sim}{j}$

$|\dot{\underset{\sim}{r}}_B(t)| = \sqrt{16+25} = \sqrt{41}$

$\dot{\underset{\sim}{r}}_A(t) \cdot \dot{\underset{\sim}{r}}_B(t) = -20 + 15 = -5$

$$\cos(\theta) = \frac{-5}{\sqrt{34}\sqrt{41}}$$

$\Rightarrow \theta = 97.7°$

Award 1 mark for the velocity vectors and finding the angle between them.

Award 1 mark for the correct final angle.

***VCAA Assessment Report* note:**

Students who found velocity vectors before finding the angle between them using a scalar product were more successful than those who used the gradients from the cartesian expressions. Some students found the angle between position vectors at chosen times, which indicated that a greater appreciation of the meaning of position and velocity vectors is required. Occasionally students gave the acute angle between the paths.

d i $\underset{\sim}{r}_B(t) - \underset{\sim}{r}_A(t) = (9t-13)\underset{\sim}{i} + (2t-5)\underset{\sim}{j}$

$$\begin{aligned} S(t) &= \left|(9t-13)\underset{\sim}{i} + (2t-5)\underset{\sim}{j}\right| \\ &= \sqrt{(9t-13)^2 + (2t-5)^2} \\ &= \sqrt{85t^2 - 254t + 194} \end{aligned}$$

For closest approach:

$$\frac{ds}{dt} = \frac{85t - 127}{\sqrt{85t^2 - 254t + 194}} = 0$$

$$\Rightarrow t = \frac{127}{85} = 1.494 \text{ hours}$$

Award 1 mark for finding the correct expression for distance and differentiating correctly.

Award 1 mark for the correct time.

***VCAAA Assessment Report* note:**

Students who found a displacement vector frequently went on to find the correct time. Some students used an expression for the difference between position vector magnitudes. Students are reminded that the instruction to give the answer correct to three decimal places must be followed to gain full marks. A rational answer did not suffice here.

ii Minimum distance:

$$s\left(\frac{127}{85}\right) = \frac{19\sqrt{85}}{85} = 2.06 \text{ km}$$

Award 1 mark for the correct minimum distance

***VCAAA Assessment Report* note:**

Many students who did not attempt Question 4di. did not attempt this question.

5 a i $\underset{\sim}{r}(t) = \left(50 + 25\cos\left(\dfrac{\pi t}{30}\right)\right)\underset{\sim}{i} + \left(50 + 25\sin\left(\dfrac{\pi t}{30}\right)\right)\underset{\sim}{j}$

$+\dfrac{2t}{5}\underset{\sim}{k}$

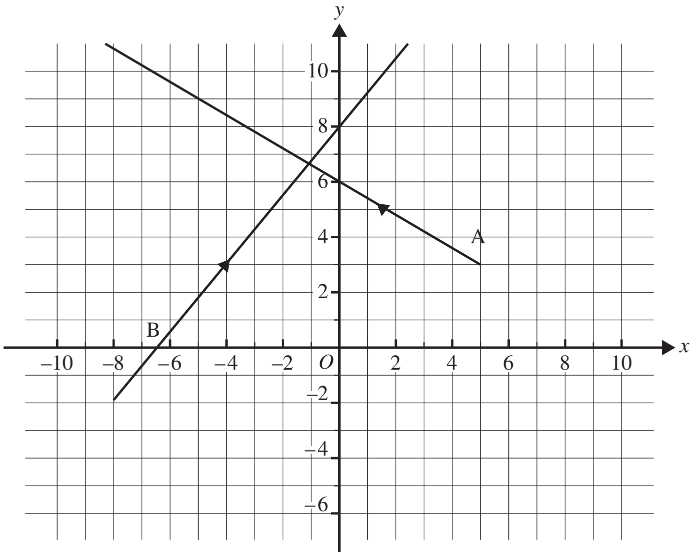
$$\underset{\sim}{r}(t) \cdot \underset{\sim}{k} = \frac{2t}{5} = 60$$

$$t = \frac{60 \times 5}{2} = 150 \text{ s} \qquad [1 \text{ mark}]$$

***VCAA Assessment Report* note:**

Most students answered this question well. A number of students attempted to solve $\underset{\sim}{r}(t) = 60$ failing to realise that only the $\underset{\sim}{k}$ component was 60.

***4 b**



ii $\underset{\sim}{r}(150) = 25\underset{\sim}{i} + 50\underset{\sim}{j} + 60\underset{\sim}{k}$

$$\tan(\theta) = \frac{60}{\sqrt{25^2 + 50^2}}$$

$$\theta = \tan^{-1}\left(\frac{12\sqrt{5}}{25}\right)$$

$$= 47°$$

Award 1 mark for the vector at the time.

Award 1 mark for the correct angle.

VCAA Assessment Report **note:**

A significant number of students seemed not to know what “angle of elevation” meant. A number found the complementary angle — the angle with the vertical. Others found angles made with the $\underset{\sim}{i}$ or $\underset{\sim}{j}$ directions.

A small number of students tried to find the angle of elevation using the velocity vector. A common error was to assume that the helicopter, when at an altitude of 60 m, was directly above its initial location.

b $\underset{\sim}{r}(0) = 75\underset{\sim}{i} + 50\underset{\sim}{j} \Rightarrow \cos\left(\frac{\pi t}{30}\right) = 1$

$\frac{\pi t}{30} = 0, 2\pi \Rightarrow t = 0, 60$ therefore after 60 s [1 mark]

VCAA Assessment Report **note:**

Most students didn’t realise that the period of the horizontal motion was required. Many students gave lengthy answers to this question, which was not necessary.

c $\underset{\sim}{v}(t) = -\frac{5\pi}{6}\sin\left(\frac{\pi t}{30}\right)\underset{\sim}{i} + \frac{5\pi}{6}\cos\left(\frac{\pi t}{30}\right)\underset{\sim}{j} + \frac{2}{5}\underset{\sim}{k}$

$$\underset{\sim}{a}(t) = -\frac{\pi^2}{36}\cos\left(\frac{\pi t}{30}\right)\underset{\sim}{i} - \frac{\pi^2}{36}\sin\left(\frac{\pi t}{30}\right)\underset{\sim}{j}$$

$$\begin{aligned}\underset{\sim}{v}(t)\cdot\underset{\sim}{a}(t) &= -\frac{5\pi}{6}\sin\left(\frac{\pi t}{30}\right)\times\left(-\frac{\pi^2}{36}\cos\left(\frac{\pi t}{30}\right)\right)\\ &\quad+\frac{5\pi}{6}\cos\left(\frac{\pi t}{30}\right)\times-\frac{\pi^2}{36}\sin\left(\frac{\pi t}{30}\right)+0\\ &= 0\end{aligned}$$

So, the velocity vector is perpendicular to the acceleration vector.

Award 1 mark for the velocity vector.

Award 1 mark for the acceleration vector.

Award 1 mark for showing that the dot product is zero.

VCAA Assessment Report **note:**

A significant number of students gave answers with x as the variable instead of t. Some students differentiated using CAS technology in degree mode, and a significant number omitted the $\underset{\sim}{k}$ component from the velocity. Often, $\underset{\sim}{i}, \underset{\sim}{j}$ or $\underset{\sim}{k}$. were just dropped in the midst of working. Some students simply asserted that $\ddot{\underset{\sim}{r}}(t).\dot{\underset{\sim}{r}}(t) = 0$, without setting out the scalar product to show it.

d $\underset{\sim}{v}(t) = -\frac{5\pi}{6}\sin\left(\frac{\pi t}{30}\right)\underset{\sim}{i} + \frac{5\pi}{6}\cos\left(\frac{\pi t}{30}\right)\underset{\sim}{j} + \frac{2}{5}\underset{\sim}{k}$

$$\begin{aligned}\text{Speed } = |\underset{\sim}{v}(t)| &= \sqrt{\frac{25\pi^2}{36}\sin^2\left(\frac{\pi t}{30}\right) + \frac{25\pi^2}{36}\cos^2\left(\frac{\pi t}{30}\right) + \frac{4}{25}}\\ &= \sqrt{\frac{25\pi^2}{36}\left(\sin^2\left(\frac{\pi t}{30}\right) + \cos^2\left(\frac{\pi t}{30}\right)\right) + \frac{4}{25}}\\ &= \sqrt{\frac{25\pi^2}{36} + \frac{4}{25}} = \frac{\sqrt{625\pi^2 + 144}}{30}\\ &= 2.65\,\text{ms}^{-1}\end{aligned}$$

Award 1 mark for finding the speed.

Award 1 mark for the final correct value of the speed.

VAA Assessment Report **note:**

Most students knew that they needed to find $|\dot{\underset{\sim}{r}}(t)|$. Leaving out the $\underset{\sim}{k}$ component was a common error. Some students could not simplify $\left(-\frac{5\pi}{6}\sin\left(\frac{\pi t}{30}\right)\right)^2 + \left(\frac{5\pi}{6}\cos\left(\frac{\pi t}{30}\right)\right)^2$ using the Pythagorean identity.

e $\underset{\sim}{r}_T = 60\underset{\sim}{i} + 40\underset{\sim}{j} + 8\underset{\sim}{k}$

$$\underset{\sim}{r}_H(45) = 50\underset{\sim}{i} + 25\underset{\sim}{j} + 18\underset{\sim}{k}$$

$$\underset{\sim}{r}_H(45) - \underset{\sim}{r}_T = -10\underset{\sim}{i} - 15\underset{\sim}{j} + 10\underset{\sim}{k}$$

$$\begin{aligned}|\underset{\sim}{r}_H(45) - \underset{\sim}{r}_T| &= \sqrt{100 + 225 + 100} = \sqrt{425}\\ &= 20.6\,\text{m}\end{aligned}$$

Award 1 mark for the position vector after 45 seconds.

Award 1 mark for the subtraction of vectors.

Award 1 mark for the correct distance.

VCAA Assessment Report **note:**

This question was moderately well answered. Frequent errors occurred in finding $\underset{\sim}{r}\,(45)$, and many students attempted to find the distance using $|\underset{\sim}{r}\,(45)| - |\underset{\sim}{r}_{\text{tree}}|$ instead of $|\underset{\sim}{r}\,(45) - \underset{\sim}{r}_{\text{tree}}|$. Occasionally an exact value of $5\sqrt{17}$ was given instead of an answer correct to one decimal place.

Topic 13 — Probability and statistics

13.2 Linear combinations of random variables

13.2 Exercise

1 **a** $\mathrm{E}(X-10)=\mathrm{E}(X)-10$

$=15.3-10$

$=5.3$

$V(X-10)=V(X)$

$=1.8$

b $\mathrm{E}(2X)=2\mathrm{E}(X)$

$=2\times 15.3$

$=30.6$

$V(2X)=2^2V(X)$

$=4\times 1.8$

$=7.2$

2 **a** $\mathrm{E}(X+4)=\mathrm{E}(X)+4$

$=13+4$

$=17$

$V(X+4)=V(X)$

$=3.2$

b $\mathrm{E}(3X)=3\mathrm{E}(X)$

$=3\times 13$

$=39$

$V(3X)=3^2V(X)$

$=9\times 3.2$

$=28.8$

3 $\mathrm{E}(2X+3Y)=2\mathrm{E}(X)+3\mathrm{E}(Y)$

$=2\times 3.5+3\times 5.4$

$=23.2$

$V(2X+3Y)=2^2V(X)+3^2V(Y)$

$=4\times 1.1+9\times 2.44$

$=26.36$

4 $\mathrm{E}\left(\frac{1}{2}X+Y\right)=\frac{1}{2}\mathrm{E}(X)+\mathrm{E}(Y)$

$=\frac{1}{2}\times 23.43+12.43$

$=24.145$

$V\left(\frac{1}{2}X+Y\right)=\left(\frac{1}{2}\right)^2V(X)+V(Y)$

$=\frac{1}{4}\times 5.89+9.7$

$=11.1725$

5 $D=X-Y\quad S=X+Y$

$\mathrm{E}(D)=\mathrm{E}(X)-\mathrm{E}(Y),\quad \mathrm{E}(S)=\mathrm{E}(X)+\mathrm{E}(Y)$

$\mathrm{Var}(D)=\mathrm{Var}(S)=\mathrm{Var}(X)+\mathrm{Var}(Y)$

Given $\mathrm{E}(D)=18,\ \ \mathrm{E}(S)=10$

$\mathrm{Var}(X)=\mathrm{Var}(Y)$

$\mathrm{Var}(D)=18$

$18=2\,\mathrm{Var}(X)$

$\mathrm{Var}(X)=\mathrm{Var}(Y)=9$

$2\mathrm{E}(X)=\mathrm{E}(D)+\mathrm{E}(S)=28$

$\mathrm{E}(X)=14$

$2\mathrm{E}(Y)=\mathrm{E}(S)-\mathrm{E}(D)=-8$

$\mathrm{E}(Y)=-4$

6 $X\sim\mathrm{Bi}\left(n=25,p=\frac{1}{5}\right)\quad Y\sim\mathrm{Bi}\left(n=16,p=\frac{1}{4}\right)$

$T=3X-2Y$, full $\mathrm{E}(T)$ $\mathrm{Var}(T)$

$\mathrm{E}(X)=np=25\times\frac{1}{5}=5$

$\mathrm{Var}(X)=npq=25\times\frac{1}{5}\times\frac{4}{5}=4$

$\mathrm{E}(Y)=np=16\times\frac{1}{4}=4$

$\mathrm{Var}(Y)=npq=16\times\frac{1}{4}\times\frac{3}{4}=3$

$T=3X-2Y$

$\mathrm{E}(T)=3\mathrm{E}(X)-2\mathrm{E}(Y)$

$=3\times 5-2\times 4$

$=7$

$\mathrm{Var}(T)=9\,\mathrm{Var}(X)+\ 4\mathrm{Var}(Y)$

$=9\times 4+4\times 3$

$=48$

$\mathrm{sd}(T)=\sqrt{48}=4\sqrt{3}$

7 The distribution $2X$ represents the weight of a puppy multiplied by 2, whereas the distribution X_1+X_2 represents the combined weight of two puppies from the distribution. Both distributions will have the same mean, but the variances will be quite different. The difference in the variances of the distributions can be explain by the fact that the distribution $2X$ will contain many large values and many small values (as any heavy puppies' weight will be multiplied by 2 and any light puppies' weight will be multiplied by 2), whereas it is unlikely that both puppies in the distribution X_1+X_2 will be very heavy, or very light. It is more likely that the combined weights will be closer to two times the mean weight of a puppy.

8 $X\sim N(56,64),\ Y\sim N(13,25)$

Let $T=Y-\frac{X}{4}$.

$\mathrm{E}(T)=\mathrm{E}(Y)-\frac{1}{4}\mathrm{E}(X)$

$=13-\frac{1}{4}\times 56$

$=-1$

$\mathrm{Var}(T)=\mathrm{Var}(Y)+\left(-\frac{1}{4}\right)^2\mathrm{Var}(X)$

$=25+\frac{1}{16}\times 64$

$=29$

$\sigma(T)=\sqrt{29}$

$\Pr\left(Y>\frac{X}{4}\right)=\Pr\left(Y-\frac{X}{4}>0\right)$

$=\Pr(T>0)$

Using the normalcdf tool on a calculator:
$\Pr(T > 0) = 0.426$

9 Let S represent the lengths of Snapper and F represent the lengths of Flathead.
$S \sim N(43, 324)$, $F \sim N(35, 121)$
The distribution representing the total length of fish caught is $T = S_1 + S_2 + S_3 + S_4 + F_1 + F_2$.

$$\begin{aligned}\mathrm{E}(T) &= 4\mathrm{E}(S) + 2\mathrm{E}(F)\\ &= 4 \times 43 + 2 \times 35\\ &= 242\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(T) &= 4\mathrm{Var}(S) + 2\mathrm{Var}(F)\\ &= 4 \times 324 + 2 \times 121\\ &= 1538\end{aligned}$$

Using the normalcdf tool on a calculator:
$\Pr(T > 300) = 0.070$

10 a

x	0	1	2
$\Pr(X = x)$	0.5	0.3	0.2

y	0	1	2
$\Pr(Y = y)$	0.4	0.1	0.5

t	0	1	2	3	4
$\Pr(T = t)$	0.2	0.17	0.36	0.17	0.1

$a = 0.2,\ b = 0.1,\ c = 0.17,\ d = 0.36$

b $\mathrm{E}(X) = 0 \times 0.5 + 1 \times 0.3 + 2 \times 0.2 = 0.7$
$\mathrm{E}(X^2) = 0^2 \times 0.5 + 1 \times 0.3 + 4 \times 0.2 = 1.1$
$\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = 1.1 - 0.7^2 = 0.61$

$\mathrm{E}(Y) = 0 \times 0.4 + 1 \times 0.1 + 2 \times 0.5 = 1.1$
$\mathrm{E}(Y^2) = 0 \times 0.4 + 1 \times 0.1 + 4 \times 0.5 = 2.1$
$\mathrm{Var}(Y) = \mathrm{E}(Y^2) - (\mathrm{E}(Y))^2 = 2.1 - 1.1^2 = 0.89$

$T = X + Y$
$\mathrm{E}(T) = \mathrm{E}(X) + \mathrm{E}(Y) = 0.7 + 1.1 = 1.8$
$\mathrm{Var}(T) = \mathrm{Var}(X) + \mathrm{Var}(Y) = 0.61 + 0.89 = 1.5$

Check.

$$\begin{aligned}\mathrm{E}(T) &= 0 \times 0.2 + 1 \times 0.17 + 2 \times 0.36\\ &\quad +3 \times 0.17 + 4 \times 0.1\\ &= 1.8\end{aligned}$$

$$\begin{aligned}\mathrm{E}(T^2) &= 0 \times 0.2 + 1 \times 0.17 + 4 \times 0.36 + 9\\ &\quad \times 0.17 + 16 \times 0.1\\ &= 4.74\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(T) &= \mathrm{E}(T^2) - (\mathrm{E}(T))^2 = 4.74 - 1.8^2\\ &= 1.5\end{aligned}$$

11 a

x	1	2	3
$\Pr(X = x)$	c	$4c$	$9c$

$\sum \Pr(X = x) = c + 4c + 9c = 14c = 1 \quad c = \frac{1}{14}$

b $\mathrm{E}(X) = c + 8c + 27c \quad = 36c = \frac{36}{14} = \frac{18}{7}$

$\mathrm{E}(X^2) = c + 16c + 81c = 98c = \frac{98}{14} = 7$

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = 7 - \left(\frac{18}{7}\right)^2\\ &= \frac{19}{49}\end{aligned}$$

$T = 7X - 3$

$$\begin{aligned}\mathrm{E}(T) &= 7\mathrm{E}(X) - 3\\ &= 7 \times \frac{18}{7} - 3\\ &= 15\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(T) &= 49\mathrm{Var}(X)\\ &= 49 \times \frac{19}{49}\\ &= 19\end{aligned}$$

12 a $\int_1^3 ky^2 dy = k\left[\frac{y^3}{3}\right]_1^3 = k\left(9 - \frac{1}{3}\right) = 1 \quad k = \frac{3}{26}$

b
$$\begin{aligned}\mathrm{E}(Y) &= \frac{3}{26}\int_1^3 y^3 dy = \frac{3}{26}\left[\frac{1}{4}y^4\right]_1^3 = \frac{3}{104}(81 - 1)\\ &= \frac{30}{13}\end{aligned}$$

$$\begin{aligned}\mathrm{E}(Y^2) &= \frac{3}{26}\int_1^3 y^4 dy = \frac{3}{26 \times 5}\left[y^5\right]_1^3 = \frac{3}{130}(243 - 1)\\ &= \frac{365}{65}\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(Y) &= \mathrm{E}(Y^2) - (\mathrm{E}(Y))^2 = \frac{365}{65} - \left(\frac{30}{13}\right)^2\\ &= \frac{219}{845}\end{aligned}$$

$T = 13Y + 8$

$$\begin{aligned}\mathrm{E}(T) &= 13\mathrm{E}(Y) + 8\\ &= 13 \times \frac{30}{13} + 8\\ &= 38\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(T) &= 13^2\mathrm{Var}(Y)\\ &= 169 \times \frac{219}{845}\\ &= 43.8\end{aligned}$$

13 $X \sim N(10, 4) \qquad Y \sim N(11, 5)$

a $T = Y - X$
$\mathrm{E}(T) = \mathrm{E}(Y) - \mathrm{E}(X) = 11 - 10 = 1$
$\mathrm{Var}(T) = \mathrm{Var}(Y) + \mathrm{Var}(X) = 5 + 4 = 9$
$T \sim N(1, 9)$

$$\begin{aligned}\Pr(Y > X) &= \Pr(Y - X > 0)\\ &= \Pr(T > 0)\\ &= 0.6306\end{aligned}$$

b $D = X - Y$
$\mathrm{E}(D) = \mathrm{E}(X) - \mathrm{E}(Y) = -1$
$D \sim N(-1, 9)$

$$\begin{aligned}\Pr(X > Y) &= \Pr(X - Y > 0)\\ &= \Pr(D > 0)\\ &= 0.3694\end{aligned}$$

c $U = 3X - 2Y$
$\mathrm{E}(U) = 3\mathrm{E}(X) - 2\mathrm{E}(Y) = 3 \times 10 - 2 \times 11 = 8$
$\mathrm{Var}(U) = 9\mathrm{Var}(X) + 4\mathrm{Var}(Y) = 9 \times 4 + 4 \times 5 = 56$
$U \sim N(8, 56)$
$\Pr(3X - 2Y > 5)$
$= \Pr(U > 5)$
$= 0.6558$

14 Holes $H \sim N(0.5, 0.007^2)$
Pegs $P \sim N(0.48, 0.007^2)$

a
$$\begin{aligned} D &= P - H \\ \mathrm{E}(D) &= \mathrm{E}(P) - \mathrm{E}(H) \\ &= 0.48 - 0.50 = -0.02 \\ \mathrm{Var}(D) &= \mathrm{Var}(P) + \mathrm{Var}(H) \\ &= 0.007^2 + 0.007^2 \\ &= 0.00\,0098 \end{aligned}$$

$$\begin{aligned} \Pr(P > H) &= \Pr(P - H > 0) \\ &= \Pr(D > 0) \\ &= 0.0217 \end{aligned}$$

b $\mathrm{E}(P) = ?\ \ D = P - H$
$\Pr(D > 0) = 0.99$

$$\frac{0 - \mathrm{E}(D)}{\sqrt{0.00\,0098}} = -2.236$$

$\mathrm{E}(D) = -0.023 \qquad = \mathrm{E}(P) - 0.5$
$\mathrm{E}(P) = 0.5 - 0.023$
$= 0.477$

15 $X_1 \sim N(15, 5^2)\ \ X_2 \sim N(12, 3^2)\ \ X_3 \sim N(10, 2^2)$

a
$$\begin{aligned} T &= 4X_1 + 3X_2 - 5X_3 \\ \mathrm{E}(T) &= 4\mathrm{E}(X_1) + 3\mathrm{E}(X_2) - 5\mathrm{E}(X_3) \\ &= 4 \times 15 + 3 \times 12 - 5 \times 10 \\ &= 46 \\ \mathrm{Var}(T) &= 16\,\mathrm{Var}(X_1) + 9\,\mathrm{Var}(X_2) + 25\,\mathrm{Var}(X_3) \\ &= 16 \times 5^2 + 9 \times 3^2 + 25 \times 2^2 \\ &= 581 \end{aligned}$$

$T \sim N(46, 581)$
$\Pr(T > 40) = 0.5983$

b
$$\begin{aligned} U &= 4X_1 - 3X_2 - 5X_3 \\ \mathrm{E}(U) &= 4\mathrm{E}(X_1) - 3\mathrm{E}(X_2) - 5\mathrm{E}(X_3) \\ &= 4 \times 15 - 3 \times 12 - 5 \times 10 \\ &= -26 \end{aligned}$$

$U \sim N(-26, 581)$
$\Pr(U > 0) = 0.1404$

16 Fencing posts $F \sim N(12, 0.5^2)$
End pieces $P \sim N(75, 10^2)$
Length of fence
$L = F_1 + F_2 + \ldots + F_{12} + P_1 + P_2$

$$\begin{aligned} \mathrm{E}(L) &= 12\mathrm{E}(F) + 2\mathrm{E}(P) \\ &= 12 \times 12 + 2 \times 75 \\ &= 294 \\ \mathrm{Var}(L) &= 12\mathrm{Var}(F) + 2\mathrm{Var}(P) \\ &= 12 \times 0.5^2 + 2 \times 10^2 \\ &= 203 \end{aligned}$$

$L \sim N(294, 203)$
$\Pr(L > 300) = 0.3368$

17 Let D be the distribution of weights of medium sized dogs, $D \sim N\left(12, 3^2\right)$, and
Let C be the distribution of weights of cats, $C \sim N\left(4, 1.5^2\right)$.
Let T be the total weight of three cats and one dog.
$T = C_1 + C_2 + C_3 + D$

$$\begin{aligned} \mathrm{E}(T) &= 3\mathrm{E}(C) + \mathrm{E}(D) \\ &= 3 \times 4 + 12 \\ &= 24 \\ \mathrm{Var}(T) &= 3\mathrm{Var}(C) + \mathrm{Var}(D) \\ &= 3 \times 1.5^2 + 1 \times 3^2 \\ &= 15.75 \end{aligned}$$

$T \sim N(24, 15.75)$
$\Pr(T > 25) = 0.4005$

18 Let M be the distribution of males, $M \sim N\left(85, 15^2\right)$,
Let F be the distribution of females, $F \sim N\left(65, 20^2\right)$.
Let T be the total weight of 12 males and 8 females.
$T = M_1 + M_2 + \ldots + M_{12} + F_1 + F_2 + \ldots + F_8$

$$\begin{aligned} \mathrm{E}(T) &= 12\mathrm{E}(M) + 8\mathrm{E}(F) \\ &= 12 \times 85 + 8 \times 65 \\ &= 1540 \end{aligned}$$

The weights are a linear combination of 20 independent random variables.

$$\begin{aligned} \mathrm{Var}(T) &= 12\mathrm{Var}(M) + 8\mathrm{Var}(F) \\ &= 12 \times 15^2 + 8 \times 20^2 \\ &= 5900 \end{aligned}$$

$T \sim N(1540, 5900)$
$\Pr(T > 1500) = 0.6987$

13.2 Exam questions

1 $X \sim D(25, 36),\ \ S \sim D(30, 49)$

$$\begin{aligned} S &= mX + n \\ \mathrm{E}(S) &= m\mathrm{E}(X) + n \\ (1)\ 30 &= 25m + n \\ \mathrm{Var}(S) &= m^2\mathrm{Var}(X) \\ (2)\ 49 &= 36m^2,\ \ m > 0 \\ m &= \frac{7}{6},\ \ n = \frac{5}{6} \\ S &= \frac{1}{6}(7X + 5) = \frac{1}{5}(7 \times 32 + 5) \approx 38 \end{aligned}$$

The correct answer is **D**.

2 $\mathrm{E}(X) = \mathrm{E}(Y) = 4,\ \ \mathrm{var}(X) = \mathrm{var}(Y) = 9$

$$\begin{aligned} Z &= aX + bY \\ \mathrm{E}(Z) &= a\mathrm{E}(X) + b\mathrm{E}(Y) \\ (1)\ \ 8 &= 4a + 4b \\ \mathrm{var}(Z) &= a^2\mathrm{var}(X) + b^2\mathrm{var}(Y) \\ (2)\ \ 90 &= 9a^2 + 9b^2 \end{aligned}$$

$(1)\ \ a + b = 2\ \ (2)\ a^2 + b^2 = 10$
$(1)\ \ b = 2 - a$ into (2)
$a^2 + (2 - a)^2 = 10 = a^2 + 4 - 4a + a^2$
$a^2 - 2a - 3 = 0$
$(a - 3)(a + 1) = 0$
$a = 3,\ b = -1$ or $a = -1,\ b = 3$
The correct answer is **C**.

3 Scores in Mathematics $M \sim N(71, 10^2)$
Scores in Statistics: $S \sim N(75, 7^2)$
$T = M - S$
$\Pr(M > S) = \Pr(T > 0)$
$\mathrm{E}(T) = \mathrm{E}(M) - \mathrm{E}(S) = 71 - 75 = -4$
$\mathrm{var}(T) = \mathrm{var}(M) + \mathrm{var}(S) = 100 + 49$
$T \sim N(-4, 149)$

$$\Pr(T > 0) = \Pr\left(Z > \frac{4}{\sqrt{149}}\right) = 0.3716$$

The correct answer is **B**.

13.3 Sample means and simulations

13.3 Exercise

1

Week	Working	$\bar{x}$
1	$\frac{92+43+41+39+35}{5}$	50
2	$\frac{118+81+46+51+38}{5}$	66.8
3	$\frac{62+48+46+41+49}{5}$	49.2
4	$\frac{82+48+42+43+41}{5}$	51.2
5	$\frac{78+51+42+41+38}{5}$	50
6	$\frac{63+62+41+43+44}{5}$	50.6
7	$\frac{55+41+46+41+32}{5}$	43

$$\bar{x} = \frac{50+66.8+49.2+51.2+50+50.6+43}{7}$$

$$\text{The average travel time} = \frac{360.8}{7} = 51.5$$

The mean travel time is 51.5 minutes.

2

Day	Working	$\bar{x}$
Monday	$\frac{115+95+105+95+115+100+90+95}{8}$	101.25
Tuesday	$\frac{95+85+90+90+105+95+75+95}{8}$	91.25
Wednesday	$\frac{110+95+80+110+95+90+105+80}{8}$	95.625
Thursday	$\frac{95+100+90+95+105+100+90+85}{8}$	95
Friday	$\frac{105+95+90+100+105+100+90+105}{8}$	98.75
Saturday	$\frac{90+85+90+110+80+100+90+90}{8}$	91.875
Sunday	$\frac{105+100+100+95+90+90+90+110}{8}$	97.5

$$\bar{x} = \frac{101.25+91.25+95.625+95+98.75+91.875+97.5}{7}$$

$$\text{The average movie length} = \frac{671.25}{7} = 95.9$$

The average movie length is 95.9 minutes.

3 $\mu = 600, \sigma = \sqrt{300}$

a $n = 10$

$$\begin{aligned} \mathrm{E}\left(\bar{X}\right) &= \mu \\ &= 600 \end{aligned}$$

$$\begin{aligned}\text{Var}\left(\overline{X}\right) &= \frac{\sigma^2}{n}\\ &= \frac{300}{10}\\ &= 30\end{aligned}$$

This means that $\sigma_{\overline{x}} = \sqrt{30}$
≈ 5.48

b $n = 20$

$$\begin{aligned}\text{E}\left(\overline{X}\right) &= \mu\\ &= 600\end{aligned}$$

The mean is unchanged.

$$\begin{aligned}\text{Var}\left(\overline{X}\right) &= \frac{\sigma^2}{n}\\ &= \frac{300}{20}\\ &= 15\end{aligned}$$

This means that $\sigma_{\overline{x}} = \sqrt{15}$
≈ 3.87

The standard deviation is reduced.

4 $\mu = 90.5$, $\sigma = 10$

a $n = 15$

$$\begin{aligned}\text{E}\left(\overline{X}\right) &= \mu\\ &= 90.5\end{aligned}$$

$$\begin{aligned}\text{Var}\left(\overline{X}\right) &= \frac{\sigma^2}{n}\\ &= \frac{100}{15}\\ &= 6.\dot{6}\end{aligned}$$

This means that $\sigma_{\overline{x}} = \sqrt{6.\dot{6}}$
≈ 2.58

b $n = 30$

$$\begin{aligned}\text{E}\left(\overline{X}\right) &= \mu\\ &= 90.5\end{aligned}$$

The mean is unchanged.

$$\begin{aligned}\text{Var}\left(\overline{X}\right) &= \frac{\sigma^2}{n}\\ &= \frac{100}{30}\\ &= 3.\dot{3}\end{aligned}$$

This means that $\sigma_{\overline{x}} = \sqrt{3.\dot{3}}$
≈ 1.83

The standard deviation is reduced.

5 **a, b** Answers will vary. The expected average is 50 but it is likely that results will be different to this.

6

	Working	$\bar{x}$
Toss 1	$\dfrac{7+6+10+7+11+2+10+8}{8}$	7.625
Toss 2	$\dfrac{8+9+8+6+6+11+4+10}{8}$	7.75
Toss 3	$\dfrac{2+4+3+10+6+8+8+6}{8}$	5.875
Toss 4	$\dfrac{8+5+6+5+11+7+6+2}{8}$	6.25
Toss 5	$\dfrac{10+12+6+8+10+8+3+4}{8}$	7.625
Toss 6	$\dfrac{4+10+9+5+3+6+5+5}{8}$	5.875
Toss 7	$\dfrac{7+6+5+6+9+10+4+2}{8}$	6.125
Toss 8	$\dfrac{11+8+9+8+9+9+6+10}{8}$	8.75
Toss 9	$\dfrac{4+7+10+10+7+4+12+8}{8}$	7.75
Toss 10	$\dfrac{7+8+3+8+10+4+7+11}{8}$	7.25

Estimate of the mean

$$\bar{x} = \frac{7.625+7.75+5.875+6.25+7.625+5.875+6.125+8.75+7.75+7.25}{10}$$

$$= \frac{70.875}{10}$$

$$\approx 7.09$$

The sample mean is 7.09. This is an estimate of the population mean.

To find the theoretical mean, consider the totals possible from two dice:

+	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

The theoretical mean is

$$\frac{2+2\times3+3\times4+4\times5+5\times6+6\times7+5\times8+4\times9+3\times10+2\times11+12}{36} = 7$$

7 Answers will vary.

8 $\mu = 73, \sigma = 12, n = 20$

$$\mathrm{E}\left(\bar{X}\right) = \mu$$
$$= 73$$
$$\mathrm{Var}\left(\bar{X}\right) = \frac{\sigma^2}{n}$$
$$= \frac{144}{20}$$
$$= 7.2$$

This means that $\sigma_{\bar{x}} = \sqrt{7.2}$

≈ 2.68

9 $n = 30$

$$\begin{aligned}\mathrm{E}\left(\overline{X}\right) &= \mu \\ &= 73 \\ \mathrm{Var}\left(\overline{X}\right) &= \frac{\sigma^2}{n} \\ &= \frac{144}{30} \\ &= 4.8\end{aligned}$$

This means that $\sigma_{\overline{x}} = \sqrt{4.8}$
≈ 2.19

10

$$\begin{aligned}\sigma_{\overline{x}} &= 2 \\ \mathrm{Var}\left(\overline{X}\right) &= 4 \\ \mathrm{Var}\left(\overline{X}\right) &= \frac{\sigma^2}{n} \\ 4 &= \frac{144}{n} \\ n &= \frac{144}{4} \\ n &= 36\end{aligned}$$

A sample size greater than 36 would be needed to reduce the standard deviation to less than 2.

11 $\mu = 123, \sigma = 43, n = 25$

$$\begin{aligned}\mathrm{E}\left(\overline{X}\right) &= \mu \\ &= 123 \\ \mathrm{Var}\left(\overline{X}\right) &= \frac{\sigma^2}{n} \\ &= \frac{43^2}{25} \\ &= 73.96\end{aligned}$$

This means that $\sigma_{\overline{x}} = \sqrt{73.96}$
$= 8.6$

12 $n = 40$

$$\begin{aligned}\mathrm{E}\left(\overline{X}\right) &= \mu \\ &= 123 \\ \mathrm{Var}\left(\overline{X}\right) &= \frac{\sigma^2}{n} \\ &= \frac{43^2}{40} \\ &= 46.225\end{aligned}$$

This means that $\sigma_{\overline{x}} = \sqrt{46.225}$
≈ 6.8

13

$$\begin{aligned}\sigma_{\overline{x}} &= 5 \\ \mathrm{Var}\left(\overline{X}\right) &= 25 \\ \mathrm{Var}\left(\overline{X}\right) &= \frac{\sigma^2}{n} \\ 25 &= \frac{43^2}{n} \\ n &= \frac{1849}{25} \\ n &= 73.96\end{aligned}$$

A sample of at least 74 would be needed to reduce the standard deviation of the distribution of sample means to less than 5.

14 Weights of vegemite jars: $V \sim N(500, 16)$
$n = 4,\ \overline{V} \sim N(500, 4)$
$\Pr\left(\overline{V} < 498\right) = 0.1587$

15 Weights of oranges: $V \sim N\left(500, 17^2\right)$
$n = 3,\ \overline{O} \sim N\left(500, \dfrac{17^2}{3}\right)$
$\Pr\left(\overline{O} > 140\right) = 0.1796$

16 **a** Volume of beer: $B \sim N\left(330, 7^2\right)$
$n = 6,\ \overline{B} \sim N\left(300, \dfrac{49}{6}\right)$
$\Pr\left(\overline{B} < 325\right) = 0.0401$

b Total: $T = 6B$
$E(T) = 6E(B) = 6 \times 330 = 1980$
$\mathrm{Var}(T) = 6\mathrm{Var}(B) = 6 \times 49 = 294$
$T \sim N(1980, 294)$
$\Pr(T > 2000) = 0.1217$

17 **a** Masses of eggs: $G \sim N\left(50, 6^2\right)$
$n = 12,\ \overline{G} \sim N\left(50, \dfrac{36}{12}\right) = N(50, 3)$
$\Pr\left(\overline{G} > 52\right) = 0.1241$

b Total: $T = 12G$
$E(T) = 12E(G) = 12 \times 50 = 600$
$\mathrm{Var}(T) = 12\mathrm{Var}(G) = 12 \times 36 = 432$
$T \sim N(600, 432)$
$\Pr(T < 580) = 0.1680$

18 Weights of baby girls: $G \sim N\left(3.2, 0.4^2\right)$
$n = ?,\ \overline{G} \sim N\left(3.2, \dfrac{0.4^2}{n}\right)$
$\Pr\left(\overline{G} < 3\right) = 0.023$

$$\frac{3 - 3.2}{\frac{0.4}{\sqrt{n}}} = -\frac{\sqrt{n}}{2} = -1.9954$$

$n = 15.93$
16 baby girls

19 Heights of 17 year old boys: $B \sim N\left(175.5, 5^2\right)$
$n = ?,\ \overline{B} \sim N\left(175.5, \dfrac{5^2}{n}\right)$
$\Pr\left(\overline{B} > 176\right) = 0.31$
$\Pr\left(\overline{B} < 176\right) = 0.69$

$$\frac{176 - 175.5}{\frac{5}{\sqrt{n}}} = \frac{\sqrt{n}}{10} = 0.4959$$

$n = 24.59$
25 boys

20 Answers will vary. The initial distribution should be skewed to the right and the distribution of sample means should be symmetrical.

13.3 Exam questions

1 Cats $C \sim N\left(66, \dfrac{16}{9}\right)$, $n = 5, \overline{C} \sim N\left(66, \dfrac{16}{9 \times 5}\right)$

$$\begin{aligned}\Pr\left(\overline{C} > 65\right) &= \Pr\left(Z > \frac{65 - 66}{\frac{4}{3\sqrt{5}}}\right) \\ &= \Pr(Z > -1.6677) \\ &= 0.9532\end{aligned}$$

The correct answer is **E**.

2 **a** $X = N(2005, 6^2)$
$\Pr(X \geq 2000) = 0.798$
79.8%

Award 1 mark for the correct percentage.

VCAA Examination Report **note:**

While responses indicated that this question was understood, many students either did not express their answer as a percentage or did not give the required level of accuracy.

b $T = X_1 + X_2 + \ldots + X_{10}$

$$\begin{aligned} \text{E}(T) &= 10\text{E}(X) \\ &= 10 \times 2005 \\ &= 20050 \end{aligned}$$

$$\text{sd}(T) = n \times \frac{\sigma_X}{\sqrt{n}} = 6\sqrt{10}$$

Award 1 mark for the correct expectation proof.

Award 1 mark for the correct standard deviation proof.

VCAA Examination Report **note:**

The mean was calculated correctly by most students. Errors frequently occurred when calculating the variance, leading to students using unable to show the given standard deviation.

c $\Pr(T > 20000) = 0.996$

99.6%

Award 1 mark for the correct percentage.

VCAA Examination Report **note:**

A significant proportion of otherwise correct answers were not given in the required form.

d $\Pr\left(Z \geq \dfrac{20000 - 20050}{\sqrt{10}\sigma}\right) \geq 0.999$

$$\Pr\left(Z < \frac{-50}{\sqrt{10}\sigma}\right) \leq 0.001$$

$$\begin{aligned} \frac{-50}{\sqrt{10}\sigma} &= -3.09 \\ \sigma &= 5.1 \end{aligned}$$

Award 1 mark for the correct inverse normal.

Award 1 mark for the correct z value equation to be solved.

Award 1 mark for correct standard deviation.

VCAA Examination Report **note:**

While some students were able to find $z = -3.090$, many did not account for the sample size in subsequent calculations.

3 $\overline{X} = N\left(\mu = 20, \sigma^2 = \dfrac{4}{25}\right)$

$\Pr\left(\overline{X} > 19.3\right) = 0.9599$

The correct answer is **E**.

13.4 Confidence intervals

13.4 Exercise

1 The average amount spent on holidays is \$2314.

To calculate a 95% confidence interval, $z = 1.96$

$$\begin{aligned} z\frac{s}{\sqrt{n}} &= 1.96 \times \frac{567}{\sqrt{75}} \\ &\approx 128 \end{aligned}$$

$$\begin{aligned} \bar{x} - z\frac{s}{\sqrt{n}} &= 2314 - 128 \\ &= 2186 \end{aligned}$$

$$\begin{aligned} \bar{x} + z\frac{s}{\sqrt{n}} &= 2314 + 128 \\ &= 2442 \end{aligned}$$

The average amount spent on holidays is $(\$2186,\ \$2442)$.

2 The average distance is 1.2 km.

To calculate a 95% confidence interval, $z = 1.96$

$$\begin{aligned} z\frac{s}{\sqrt{n}} &= 1.96 \times \frac{0.5}{\sqrt{30}} \\ &\approx 0.18 \end{aligned}$$

$$\begin{aligned} \bar{x} - z\frac{s}{\sqrt{n}} &= 1.2 - 0.18 \\ &= 1.02 \end{aligned}$$

$$\begin{aligned} \bar{x} + z\frac{s}{\sqrt{n}} &= 1.2 + 0.18 \\ &= 1.38 \end{aligned}$$

The distance is (1.02 km, 1.38 km).

3 For an 80% confidence interval, $z = 1.28$

$\bar{x} = \$23\,456$

$$\begin{aligned} z\frac{s}{\sqrt{n}} &= 1.28 \times \frac{537}{\sqrt{116}} \\ &\approx 64 \end{aligned}$$

$$\begin{aligned} \bar{x} - z\frac{s}{\sqrt{n}} &= 23\,456 - 64 \\ &= 23\,392 \end{aligned}$$

$$\begin{aligned} \bar{x} + z\frac{s}{\sqrt{n}} &= 23\,456 + 64 \\ &= 23\,520 \end{aligned}$$

The average car value is $(\$23\,392,\ \$23\,520)$.

4 For a 75% confidence interval, $z = 1.150$

$\bar{x} = 25.7$ g

$$\begin{aligned} z\frac{s}{\sqrt{n}} &= 1.150 \times \frac{5.8}{\sqrt{95}} \\ &\approx 0.68 \end{aligned}$$

$$\begin{aligned} \bar{x} - z\frac{s}{\sqrt{n}} &= 25.7 - 0.68 \\ &= 25.02 \end{aligned}$$

$$\begin{aligned} \bar{x} + z\frac{s}{\sqrt{n}} &= 25.7 + 0.68 \\ &= 26.38 \end{aligned}$$

(25.02 g, 26.38 g) of chocolate are consumed per day.

5 At the 95% level of confidence, $z = 1.96$.

$$\begin{aligned} z\frac{s}{\sqrt{n}} &= 100 \\ 1.96 \times \frac{567}{\sqrt{n}} &= 100 \\ \sqrt{n} &= \frac{1.96 \times 567}{100} \\ \sqrt{n} &= 11.1132 \\ n &= 123.5 \end{aligned}$$

A sample of 124 would be needed.

6 At the 95% level of confidence, $z = 1.96$.

$$\begin{aligned} z\frac{s}{\sqrt{n}} &= 0.1 \\ 1.96 \times \frac{0.5}{\sqrt{n}} &= 0.1 \\ \sqrt{n} &= \frac{1.96 \times 0.5}{0.1} \\ \sqrt{n} &= 9.8 \\ n &= 96.04 \end{aligned}$$

A sample of at least 97 would be needed to reduce the interval to ± 0.1 km for a 95% confidence level.

7 a The average amount is \$203.45.

To calculate a 95% confidence interval, $z = 1.96$

$$\begin{aligned} z\frac{s}{\sqrt{n}} &= 1.96 \times \frac{43.32}{\sqrt{50}} \\ &\approx 12.01 \end{aligned}$$

$$\begin{aligned} \bar{x} - z\frac{s}{\sqrt{n}} &= 203.45 - 12.01 \\ &= 191.44 \end{aligned}$$

$$\begin{aligned} \bar{x} + z\frac{s}{\sqrt{n}} &= 203.45 + 12.01 \\ &= 215.46 \end{aligned}$$

The average cash deposit amount is (\$191.44, \$215.46).

b For a 90% confidence interval, $z = 1.645$

$$\begin{aligned} z\frac{s}{\sqrt{n}} &= 1.645 \times \frac{43.32}{\sqrt{50}} \\ &\approx 10.08 \end{aligned}$$

$$\begin{aligned} \bar{x} - z\frac{s}{\sqrt{n}} &= 203.45 - 10.08 \\ &= 193.37 \end{aligned}$$

$$\begin{aligned} \bar{x} + z\frac{s}{\sqrt{n}} &= 203.45 + 10.08 \\ &= 213.53 \end{aligned}$$

The average cash deposit amount is (\$193.37, \$213.53).

c For a 95% confidence level, $z = 1.96$.

$$\begin{aligned} z\frac{s}{\sqrt{n}} &= 2 \\ 1.96 \times \frac{43.32}{\sqrt{n}} &= 2 \\ \sqrt{n} &= \frac{1.96 \times 43.32}{2} \\ \sqrt{n} &= 42.4536 \\ n &= 1802.3 \end{aligned}$$

A sample of at least 1803 would be needed to reduce the interval to ± \$2 for a 95% confidence level.

8 a The average amount is 2314 minutes.

To calculate a 95% confidence interval, $z = 1.96$

$$\begin{aligned} z\frac{s}{\sqrt{n}} &= 1.96 \times \frac{243}{\sqrt{40}} \\ &\approx 75 \end{aligned}$$

$$\begin{aligned} \bar{x} - z\frac{s}{\sqrt{n}} &= 2314 - 75 \\ &= 2239 \end{aligned}$$

$$\begin{aligned} \bar{x} + z\frac{s}{\sqrt{n}} &= 2314 + 75 \\ &= 2389 \end{aligned}$$

The average battery life is (2239, 2389) minutes.

b For a 99% confidence interval, $z = 2.58$

$$\begin{aligned} z\frac{s}{\sqrt{n}} &= 2.58 \times \frac{243}{\sqrt{40}} \\ &\approx 99 \end{aligned}$$

$$\begin{aligned} \bar{x} - z\frac{s}{\sqrt{n}} &= 2314 - 99 \\ &= 2215 \end{aligned}$$

$$\begin{aligned} \bar{x} + z\frac{s}{\sqrt{n}} &= 2314 + 99 \\ &= 2413 \end{aligned}$$

The average battery life is (2215, 2413) minutes.

c For a 95% confidence level, $z = 1.96$.

$$\begin{aligned} z\frac{s}{\sqrt{n}} &= 50 \\ 1.96 \times \frac{243}{\sqrt{n}} &= 50 \\ \sqrt{n} &= \frac{1.96 \times 243}{50} \\ \sqrt{n} &= 9.5256 \\ n &= 90.7 \end{aligned}$$

A sample of at least 91 would be needed to reduce the interval to ± 50 minutes for a 95% confidence level.

9 a The average time is 24.6 minutes.

To calculate a 95% confidence interval, $z = 1.96$

$$\begin{aligned} z\frac{s}{\sqrt{n}} &= 1.96 \times \frac{7.6}{\sqrt{30}} \\ &\approx 2.7 \end{aligned}$$

$$\begin{aligned} \bar{x} - z\frac{s}{\sqrt{n}} &= 24.6 - 2.7 \\ &= 21.9 \end{aligned}$$

$$\begin{aligned} \bar{x} + z\frac{s}{\sqrt{n}} &= 24.6 + 2.7 \\ &= 27.3 \end{aligned}$$

The average time is (21.9, 27.3) minutes.

b For a 90% confidence interval, $z = 1.645$

$$\begin{aligned} z\frac{s}{\sqrt{n}} &= 1.645 \times \frac{7.6}{\sqrt{30}} \\ &\approx 2.3 \end{aligned}$$

$$\begin{aligned} \bar{x} - z\frac{s}{\sqrt{n}} &= 24.6 - 2.3 \\ &= 22.3 \end{aligned}$$

$$\begin{aligned} \bar{x} + z\frac{s}{\sqrt{n}} &= 24.6 + 2.3 \\ &= 26.9 \end{aligned}$$

The average time is (22.3, 26.9) minutes.

c For a 95% confidence level, $z = 1.96$.

$$\begin{aligned} z\frac{s}{\sqrt{n}} &= 2 \\ 1.96 \times \frac{7.6}{\sqrt{n}} &= 2 \\ \sqrt{n} &= \frac{1.96 \times 7.6}{2} \\ \sqrt{n} &= 7.448 \\ n &= 55.5 \end{aligned}$$

A sample of 56 would be needed to reduce the interval to ± 2 minutes for a 95% confidence level.

d For a 90% confidence interval, $z = 1.645$

$$\begin{aligned} z\frac{s}{\sqrt{n}} &= 2 \\ 1.645 \times \frac{7.6}{\sqrt{n}} &= 2 \\ \sqrt{n} &= \frac{1.645 \times 7.6}{2} \\ \sqrt{n} &= 6.251 \\ n &= 39.1 \end{aligned}$$

A sample of 40 would be needed to reduce the interval to ± 2 minutes for a 90% confidence level.

10 $z_{0.005} = 2.58$

$z_{0.025} = 1.96$

$$\bar{x} = \frac{2+3}{2}$$
$$= 2.5$$

Using the 99% confidence interval values:

$$3 = \bar{x} + z_{0.005}\frac{s}{\sqrt{n}}$$
$$3 = 2.5 + z_{0.005}\frac{s}{\sqrt{n}}$$
$$z_{0.005}\frac{s}{\sqrt{n}} = 0.5$$
$$2.58\frac{s}{\sqrt{n}} = 0.5$$
$$\frac{s}{\sqrt{n}} = \frac{1}{5.16}$$

Calculating the 95% confidence interval:

$$x = \bar{x} \pm z_{0.025}\frac{s}{\sqrt{n}}$$
$$= 2.5 \pm 1.96 \times \frac{1}{5.16}$$
$$\approx 2.5 \pm 0.38$$

The 95% confidence interval is (2.12, 2.88) minutes.

11 $z_{0.05} = 1.645$

$z_{0.005} = 2.58$

$$\bar{x} = \frac{85+90}{2}$$
$$= 87.5$$

Using the 90% confidence interval values:

$$90 = \bar{x} + z_{0.05}\frac{s}{\sqrt{n}}$$
$$90 = 87.5 + z_{0.05}\frac{s}{\sqrt{n}}$$
$$z_{0.05}\frac{s}{\sqrt{n}} = 2.5$$
$$1.645\frac{s}{\sqrt{n}} = 2.5$$
$$\frac{s}{\sqrt{n}} = \frac{2.5}{1.645}$$

Calculating a 99% confidence interval:

$$x = \bar{x} \pm z_{0.005}\frac{s}{\sqrt{n}}$$
$$= 87.5 \pm 2.58 \times \frac{2.5}{1.645}$$
$$\approx 87.5 \pm 3.9$$

A 99% confidence interval is (83.6, 91.4) minutes.

12 $z_{0.025} = 1.96$

$z_{0.005} = 2.58$

$$\bar{x} = \frac{215.171 + 238.695}{2}$$
$$= 226.933$$

Using the 95% confidence interval values:

$$238.695 = \bar{x} + z_{0.025}\frac{s}{\sqrt{n}}$$
$$238.695 = 226.933 + z_{0.025}\frac{s}{\sqrt{n}}$$
$$z_{0.05}\frac{s}{\sqrt{n}} = 11.762$$
$$1.96\frac{s}{\sqrt{n}} = 11.762$$
$$\frac{s}{\sqrt{n}} = \frac{11.762}{1.96}$$

Calculating a 99% confidence interval:

$$x = \bar{x} \pm z_{0.005}\frac{s}{\sqrt{n}}$$
$$= 226.933 \pm 2.58 \times \frac{11.762}{1.96}$$
$$\approx 226.933 \pm 15.483$$

A 99% confidence interval is (211.450, 242.416) minutes.

13.4 Exam questions

1 $$\bar{x} = \frac{70.2 + 75.8}{2}$$
$$= 73$$
$$73 + 1.96 \times \frac{\sigma}{\sqrt{100}} = 75.8$$
$$\sigma = 14.2857$$
$$\frac{\bar{x}}{\sigma} = 5.1$$

The correct answer is **C**.

2 $CI: \bar{x} - z \times \frac{\sigma}{\sqrt{n}}, \bar{x} + z \times \frac{\sigma}{\sqrt{n}}$

$n = 36,\ 98\%\ z = 2.32635,\ \ \bar{x} = 65,\ \sigma = 4$

$$65 \pm 2.32635 \times \frac{4}{\sqrt{36}} = (63.4, 66.6)$$

The correct answer is **D**.

3 $$CI: \left(\bar{x} - z \times \frac{\sigma}{\sqrt{n}},\ \bar{x} + z \times \frac{\sigma}{\sqrt{n}}\right) = (58.42, 67.31)$$

$n = 36, 95\%\ z = 1.96,\ \sigma = ?$

$$2z \times \frac{\sigma}{\sqrt{n}} = 67.31 - 58.42 = 8.89$$
$$\sigma = \frac{8.89 \times 6}{2 \times 1.96} = 13.607 \approx 13.61$$

The correct answer is **D**.

13.5 Hypothesis testing

13.5 Exercise

1 H_0: The extra training sessions do not change his race time.
H_1: The extra training sessions improve his race time.

2 H_0: The extra training sessions do not change her race time.
H_1: The extra training sessions improve her race time.

3 **a** A type I error means that the null hypothesis was rejected when it was actually true. The conclusion reached was: The pet food is safe.

b A type II error means that the null hypothesis was accepted when it was actually false. The conclusion reached was: The pet food is contaminated.

4 **a** A type I error means that the null hypothesis was rejected when it was actually true. The conclusion reached was: Aerial spraying is not safe.

b A type II error means that the null hypothesis was accepted when it was actually false. The conclusion reached was: There is insufficient evidence to conclude that aerial spraying is safe.

5 a The null hypothesis is not of the form $H_0: \mu = k,\ k \in R$, therefore it is not valid.

b The null and alternative hypotheses are statements about the population mean, μ, not the sample mean, $\bar{x}$. These hypotheses are therefore not valid.

c The null and alternative hypotheses are statements about the population mean, μ, not the sample mean, $\bar{x}$. These hypotheses are therefore not valid.

d The alternative hypothesis is a statement regarding the sample mean, $\bar{x}$. It is therefore not valid.

e The alternative hypothesis is a statement regarding the sample mean, $\bar{x}$. It is therefore not valid.

f The null hypothesis is a statement regarding the sample mean, $\bar{x}$. It is therefore not valid.

6 a $H_0: \mu = 6$ screws, lengths in mm

$H_1: \mu \neq 6$

b $\mu_0 = 6,\ \bar{x} = 6.06,\ \sigma = 0.4,\ n = 81$

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{6.06 - 6}{\frac{0.4}{\sqrt{81}}} = 1.35$$

$p = 2\Pr(Z > 1.35) = 0.177$

c 5%, $\alpha = 0.05$

$p = 0.177 > 0.05 = \alpha$

Do not reject H_0

7 $H_0 = \mu = 4$ steel rods, lengths in cm

$H_1: \mu \neq 4$

$\mu_0 = 4,\ \bar{x} = 4.12,\ \sigma = 0.3,\ n = 49$

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{4.12 - 4}{\frac{0.3}{\sqrt{49}}} = 2.8$$

$p = 2\Pr(Z > 2.8) = 0.0051$

At 1% $\alpha = 0.01$

$p = 0.0051 < 0.01 = \alpha$

Reject H_0

8 a $H_0: \mu = 40\,000$ distance in km for car tyres

$H_1: \mu > 40\,000$

b $\mu_0 = 40\,000,\ \bar{x} = 43\,250,\ \sigma = 9000,\ n = 32$

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{43\,250 - 40\,000}{\frac{9000}{\sqrt{32}}} = 2.0428$$

$p = \Pr(Z > 2.0428) = 0.0205$

c $p = 0.02 > 0.01 = \alpha$ at 1% $\alpha = 0.01$

Do not reject H_0

9 $H_0: \mu = 2500$ distance pen writes for in m

$H_1\ \mu > 2500$

$\mu = 2500,\ \bar{x} = 2750,\ \sigma = 750,\ n = 35$

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{2750 - 2500}{\frac{750}{\sqrt{35}}} = 1.9720$$

$p = \Pr(Z > 1.9720) = 0.0243$

a 10% $\alpha = 0.1$ $p = 0.0243 < 0.1 < \alpha$

Reject H_0

b 5% $\alpha = 0.05$ $p = 0.0243 < 0.05 = \alpha$

Reject H_0

c 1% $\alpha = 0.01$ $p = 0.0243 > 0.01 = \alpha$

Do not reject H_0

10 a $H_0: \mu = 35\,000$ time in hours for a TV

$H_1: \mu < 35\,000$

b $\bar{x} = 32\,100,\ \sigma = 12\,000\ n = 25$

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{32100 - 35\,000}{\frac{12000}{\sqrt{25}}} = -1.2083$$

$p = \Pr(Z < -1.2083) = 0.1135$

c At 5% level $\alpha = 0.05$

$p = 0.1135 > 0.05 = \alpha$

Do not reject H_0

d. $\Pr\left(\bar{X} < I^* | \mu = 35\,000\right) = 0.05$

$$\frac{I^* - 35\,000}{\frac{12\,000}{\sqrt{25}}} = -1.6449$$

$I^* = 31\,052.35$

If the mean of a sample was any value less than 31 052.35, we would have sufficient evidence to reject the null hypothesis, H_0.

11 $H_0: \mu = 10$ time in mm to complete the task

$H_0: \mu < 10$

$\mu_0 = 10,\ \bar{x} = 9.4,\ s = 2,\ n = 36$

Use s for σ since $n = 36 > 30$

$$z = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} = \frac{9.4 - 10}{\frac{2}{\sqrt{36}}} = -1.8$$

$p = \Pr(Z < -1.8) = 0.0359$

At 5% level $\alpha = 0.05$

$p = 0.0359 < 0.05 = \alpha$

Reject H_0

12 $H_0: \mu = 200$ amount of coffee in cup in mL

$H_0: \mu \neq 200$

$\mu_0 = 200,\ \bar{x} = 208,\ \sigma = 20,\ n = 30$

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{208 - 200}{\frac{20}{\sqrt{30}}} = 2.19$$

$p = 2\Pr(Z > 2.19) = 0.0285$

a 10% $\alpha = 0.1$

$p = 0.0285 < 0.1 = \alpha$

Reject H_0, needs a service

b 5% $\alpha = 0.05$

$p = 0.0285 < 0.05$

Rejected H_0, needs a service

c 1% $\alpha = 0.01$

$p = 0.0285 > 0.01$

Do not reject H_0, no service required

13 a $H_0: \mu = 100$, IQ

$H_0: \mu > 100$

$\mu_0 = 100,\ \bar{x} = 105,\ \sigma = 15,\ n = 25$

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{105 - 100}{\frac{15}{\sqrt{25}}} = 1.667$$

$p = \Pr(Z > 1.667) = 0.0478$

5%, $\alpha = 0.05$

$p = 0.0478 < 0.05 = \alpha$

Reject H_0, above average class

b $H_0: \mu = 100$

$H_1: \mu < 100$

$\mu_0 = 100,\ \bar{x} = 94,\ \sigma = 15,\ n = 36$

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{94 - 100}{\frac{15}{\sqrt{36}}} = -2.4$$

$p = \Pr(Z < -2.4) = 0.0082$

1% $\alpha = 0.01$

$p = 0.0082 < 0.01 = \alpha$

Reject H_0, below average class

c $H_0: \mu = 100$

$H_1: \mu \neq 100$

$\mu_\sigma = 100,\ \bar{x} = 98,\ \sigma = 15,\ n = 40$

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{98 - 100}{\frac{15}{\sqrt{40}}} = -0.8433$$

$p = 2Pr(z < -0.8433) = 0.3991$

5% $\quad p = 0.3991 > 0.05 = \alpha$

Do not reject H_0, students are average

14 **a** $H_0: \mu = 50$ age at a hospital

$H_1: \mu \neq 50$

$\mu_0 = 50, \quad \bar{x} = 54.5, \quad s = 17, \quad n = 40$

$$z = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} = \frac{54.5 - 50}{\frac{17}{\sqrt{40}}} = 1.6741$$

$p = 2\,\text{Pr}(Z > 1.6751) = 0.0941$

5% $\quad \alpha = 0.05$

$p = 0.0941 > 0.05 = \alpha$

Do not reject H_0

b $H_0: \ \mu = 50$

$H_1: \ \mu > 50$

$p = \text{Pr}(Z > 1.6751) = 0.0471$

5% $\quad \alpha = 0.05$

$p = 0.0471 < 0.05 = \alpha$

Reject H_0, appears patients are older

15 **a** $H_0: \ \mu = 10$ days stay in hospital

$H_0: \ \mu \neq 10$

$\mu_0 = 10, \quad \bar{x} = 9, \quad s = 3, \quad n = 40$

$$z = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} = \frac{9 - 10}{\frac{3}{\sqrt{40}}} = -2.1082$$

$p = 2\,\text{Pr}(Z < -2.1082) = 0.035$

5% $\quad \alpha = 0.05$

$p = 0.035 < 0.05$

Reject H_0, patients do not stay 10 days

b $H_0: \mu = 10$

$H_1: \mu < 10$

$p = \text{Pr}(Z < -2.1082) = 0.0175$

$p = 0.0175 < 0.05$

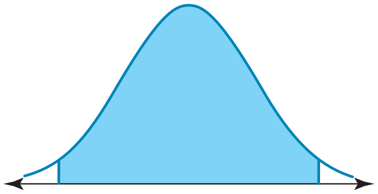
Reject H_0, patients stay less than 10 days

16 $H_0: \mu = 50$

$H_1: \mu \neq 50 \, \bar{x} = ?, \quad \mu_0 = 50, \ \sigma = 17, \ n = 30$

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

a



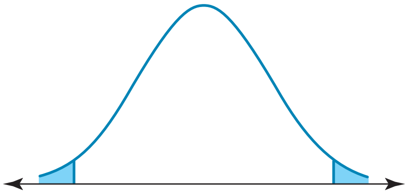
10% $\quad z = \pm 1.645 = \dfrac{\bar{x} - 50}{\frac{17}{\sqrt{30}}}$

$\Rightarrow \bar{x} = 44.89, \ 55.11$

Do not reject H_0

$44.89 \leq \bar{x} \leq 55.11$

b



Reject H_0 at 5% significance: $z = \pm 1.96$

$\Rightarrow \bar{x} = 43.92, \ 56.08$

$\bar{x} < 43.92, \ \bar{x} > 56.08$

c Reject H_0 at 1% significance: $z = \pm 2.576$

$\bar{x} = 42, \ 58$

$\bar{x} < 42.00, \ \bar{x} > 58.00$

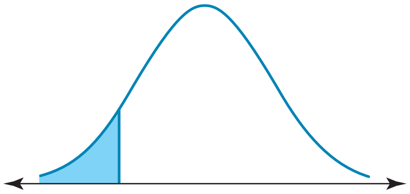
17 $H_0: \mu = 30$

$H_1: \mu > 30$

$\mu_\sigma = 30, \ \sigma = 7, \ n = 36$

a 10% $\quad z = -1.282 = \dfrac{\bar{x} - 30}{\frac{7}{6}}$

$\Rightarrow \bar{x} = 28.50$



reject H_0

$\bar{x} < 28.50$

b 5% $\quad z = -1.645$

$\Rightarrow \bar{x} = 28.08$

reject H_0

$\bar{x} < 28.08$

c 1% $\quad z = -2.326$

$\Rightarrow \bar{x} = 27.29$

Do not reject H_0

$\bar{x} \geq 27.29$

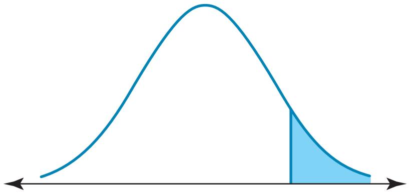
18 $H_0: \mu = 70$

$H_1: \mu > 70$

$n = ? \quad \bar{x} = 75, \quad \sigma = 17, \quad \mu_0 = 70$

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

a 5% $\quad \dfrac{75 - 70}{\frac{17}{\sqrt{n}}} > 1.645$ reject H_0



$n > 31.28$

So $n \geq 32$

b 1% $\quad \dfrac{75 - 70}{\frac{17}{\sqrt{n}}} > 2.326$ reject H_0

$n > 62.54$

$n \geq 63$

13.5 Exam questions

1 **a** $H_0: \ \mu = 200$

$H_1: \ \mu < 200$ [1 mark]

b i $G \sim N(250, 10^2)$, $\overline{G} \sim N\left(250, \dfrac{10^2}{36}\right)$

$$p = \Pr\left(\overline{G} < 195 | \mu = 200\right) = \Pr\left(Z < \frac{195 - 200}{\frac{10}{6}}\right)$$

$$= \Pr(Z < -3)$$

$$p = \frac{1}{2}(1 - \Pr(-3 < Z < 3)) = \frac{1}{2}(1 - 0.9973) = 0.00135$$

Award 1 mark for the correct setting-up of the p value.

Award 1 mark for the correct probability.

ii $p = 0.00135 < 0.01 = a$,

Strong evidence to reject H_0, accept H_1 [1 mark]

c $\bar{x} = 250,\ z = 1.96,\ n = 25,\ \sigma = 10$

$$\bar{x} \pm z\frac{\sigma}{\sqrt{n}} = 250 \pm 1.96 \times \frac{10}{\sqrt{25}}$$

(246.08, 253.92) [1 mark]

2 $\bar{x} = 0.4725,\ n = 100,\ \mu_0 = 0.5,\ \sigma = 0.2887$

$$\overline{X} \overset{d}{=} N\left(0.4725, \frac{0.2887}{\sqrt{100}}\right) = N(0.4725, 0.02887)$$

$$p = \Pr\left(Z < \frac{0.4725 - 0.5}{0.02887}\right) = \Pr(Z < -0.9525)$$

$p = 0.1704$

The correct answer is **B**.

3 a $\mathrm{N} \sim (375, 15^2),\ n = 50,\ \overline{\mathrm{N}} \sim N\left(375, \dfrac{9}{2}\right),\ \mathrm{sd}(\overline{\mathrm{N}}) = \dfrac{3}{\sqrt{2}}$

$\Pr(370 < \overline{\mathrm{N}} < 375) = 0.4908$

$\Pr(\text{at least one}) = 1 - \Pr(\text{none}) = 1 - (1 - 0.4908)^2 = 0.741$

Award 1 mark for the correct sample standard deviation.

Award 1 mark for the correct probability.

VCAA Examination Report **note:**

Of the students who found the standard deviation of the sample mean, about half successfully used a binomial distribution to answer the question.

b $\overline{\mathrm{N}}_1 \sim N\left(375, \dfrac{9}{2}\right), \overline{\mathrm{N}}_2 \sim N\left(375, \dfrac{9}{2}\right)$

$D = \overline{N}_1 - \overline{N}_2$

$\mathrm{E}(D) = \mathrm{E}(\overline{N}_1) - \mathrm{E}(\overline{N}_2) = 0$

$\mathrm{var}(D) = \mathrm{var}(\overline{N}_1) + \mathrm{var}(\overline{N}_2) = 2 \times \dfrac{9}{2} = 9$

$D \sim N(0, 9)$

$\Pr(|D| < 2) = \Pr(-2 < D < 2) = 0.495$

Award 1 mark for the correct differences of means.

Award 1 mark for the correct variance.

Award 1 mark for the correct probability.

VCAA Examination Report **note:**

Many students did not make a reasonable start to this question, or they were unable to correctly find the variance for the combined distributions. Very few students indicated an understanding that the difference between the samples could be negative and found $\Pr(\overline{X}_1 - \overline{X}_2 < 2)$ rather than correctly finding $\Pr(-2 < \overline{X}_1 - \overline{X}_2 < 2)$ or $\Pr\left(|\overline{X}_1 - \overline{X}_2| < 2\right)$.

c $H_0: \mu = 375$

$H_1: \mu \neq 375$ two sided

Award 1 mark for the correct hypothesis.

VCAA Examination Report **note:**

While most students correctly stated the null and alternative hypotheses for a two-tailed test, answers indicating a one-tailed test were relatively frequent.

d $p = 2\Pr(\overline{\mathrm{N}} < 372 | \mu = 375) = 2\Pr\left(Z < \dfrac{372 - 375}{\frac{15}{\sqrt{100}}}\right)$

$p = 2\Pr(Z < -2) = 0.046$

Award 1 mark for the correct p value.

e Since $p < 0.05$ or $-2 < -1.96$, there is evidence to reject H_0 at the 5% level of significance.

The machine is not working properly.

Award 1 mark for the correct conclusion.

f $\Pr(\overline{\mathrm{N}} < m | \mu = 375) = \Pr\left(Z < \dfrac{m - 375}{\frac{15}{\sqrt{100}}}\right)$

$\dfrac{2}{3}(m - 375) = -1.96$

$m = 372.1$

Award 1 mark for the correct value.

VCAA Examination Report **note:**

This question was often not attempted. Most students who did attempt it answered correctly.

The most frequent incorrect response was 372.5, resulting from $\Pr(\overline{X} < x_c) = 0.05$.

13.6 Review

13.6 Exercise

Technology free: short answer

1 a

$$\begin{aligned}\mathrm{E}(Y) &= \mathrm{E}\left(\frac{X}{4} + 10\right)\\ &= \frac{1}{4}\mathrm{E}(X) + 10\\ &= \frac{1}{4} \times 434 + 10\\ &= 118.5\end{aligned}$$

b

$$\begin{aligned}V(Y) &= V\left(\frac{X}{4} + 10\right)\\ &= \left(\frac{1}{4}\right)^2 V(X)\\ &= \frac{1}{16} \times 64\\ &= 4\end{aligned}$$

2 If $T = 2X + 3Y$ and $D = X - Y$

$\mathrm{E}(T) = 41,\ \ \mathrm{E}(D) = 3$

$\mathrm{Var}(T) = 72,\ \ \mathrm{Var}(D) = 13$

Find $\mathrm{E}(X),\ \mathrm{E}(Y),\ \mathrm{Var}(X)$ and $\mathrm{Var}(Y)$.

$$\begin{array}{rrcl} & \mathrm{E}(T) &=& 2\mathrm{E}(X) + 3\mathrm{E}(Y)\\ (1) & 41 &=& 2\mathrm{E}(X) + 3\mathrm{E}(Y)\\ & \mathrm{E}(D) &=& \mathrm{E}(X) - \mathrm{E}(Y)\\ (2) & 3 &=& \mathrm{E}(X) - \mathrm{E}(Y)\\ 3 \times (2) & 9 &=& 3\mathrm{E}(X) - 3\mathrm{E}(Y)\\ & 50 &=& 5\mathrm{E}(X)\\ & \mathrm{E}(X) &=& 10 \Rightarrow \mathrm{E}(Y) = 7\\ & \mathrm{Var}(T) &=& 4\,\mathrm{Var}(X) + 9\,\mathrm{Var}(Y)\end{array}$$

$$\begin{aligned}
(3)\quad && 72 &= 4\,\mathrm{Var}(X)+9\,\mathrm{Var}(Y)\\
&& \mathrm{Var}(D) &= \mathrm{Var}(X)+\mathrm{Var}(Y)\\
(4)\quad && 13 &= \mathrm{Var}(X)+\mathrm{Var}(Y)\\
4\times(4)\quad && 52 &= 4\,\mathrm{Var}(X)+4\,\mathrm{Var}(Y)\\
&& 20 &= 5\,\mathrm{Var}(Y)\\
&& \mathrm{Var}(Y) &= 4 \Rightarrow \mathrm{Var}(X)=9
\end{aligned}$$

$X \sim N(10, 9)$

$Y \sim N(7, 4)$

3 a $\bar{x}+\dfrac{2\sigma}{\sqrt{n}}=36,$ (1) $\bar{x}-\dfrac{2\sigma}{\sqrt{n}}=24$ (2)

(1) + (2) $2\bar{x}=60.$ (2) − (1) $\dfrac{4\sigma}{\sqrt{n}}=12,\quad \sigma=18$

$\bar{x}=30$

$\sqrt{n}=\dfrac{\sigma}{3}=6,\qquad n=36$

b $\dfrac{4\sigma}{\sqrt{n}}=4,\quad \sqrt{n}=\sigma=18$

$n=324$

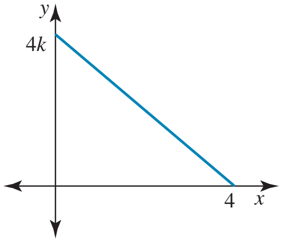
4 a
$$\begin{aligned}
\mathrm{E}(\overline{X}) &= \mu\\
&= 58.9\\
V(\overline{X}) &= \frac{\sigma^2}{n}\\
&= \frac{5.2^2}{20}\\
&= 1.352
\end{aligned}$$

b
$$\begin{aligned}
\frac{\sigma}{\sqrt{n}} &< 1\\
\sigma &< \sqrt{n}\\
\sqrt{n} &> 5.2\\
n &> 27.04
\end{aligned}$$

The sample size needs to be at least 28.

5 $f(x)=\begin{cases} k(4-x) & 0\le x\le 4\\ 0 & \text{otherwise}\end{cases}$

a
$$\begin{aligned}
\int_0^4 k(4-x)dx &= 1\\
k\left[4x-\frac{1}{2}x^2\right]_0^4 &= 1\\
k\,[16-8-0] &= 1\\
k &= \frac{1}{8}
\end{aligned}$$



$\dfrac{1}{2}\times 4k\times 4=1$

$k=\dfrac{1}{8}$

b
$$\begin{aligned}
\mathrm{E}(X) &= \frac{1}{8}\int_0^4 x(4-x)dx\\
&= \frac{1}{8}\int_0^4 (4x-x^2)dx\\
&= \frac{1}{8}\left[2x^2-\frac{x^3}{3}\right]_0^4\\
&= \frac{1}{8}\left[32-\frac{64}{3}-0\right]\\
&= \frac{4}{3}
\end{aligned}$$

c
$$\begin{aligned}
\mathrm{E}(X^2) &= \frac{1}{8}\int_0^4 x^2(4-x)dx.\\
&= \frac{1}{8}\int_0^4 (4x^2-x^3)dx\\
&= \frac{1}{8}\left[\frac{4}{3}x^3-\frac{x^4}{4}\right]_0^4\\
&= \frac{1}{8}\left[\frac{256}{3}-\frac{256}{4}-0\right]\\
&= \frac{8}{3}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}(X^2)-(\mathrm{E}(X))^2\\
&= \frac{8}{3}-\left(\frac{4}{3}\right)^2\\
&= \frac{8}{9}
\end{aligned}$$

d $Y\sim \mathrm{Bi}\left(n=16,\, p=\dfrac{1}{4}\right)$

$\mathrm{E}(Y)=np=16\times\dfrac{1}{4}=4$

$\mathrm{Var}(Y)=npq=16\times\dfrac{1}{4}\times\dfrac{3}{4}=3$

$$\begin{aligned}
Z &= 9X+Y\\
\mathrm{E}(Z) &= 9\mathrm{E}(X)+\mathrm{E}(Y)\\
&= 9\times\frac{4}{3}+4\\
&= 16
\end{aligned}$$

e
$$\begin{aligned}
\mathrm{Var}(Z) &= 81\times\frac{8}{9}+3\\
&= 75\\
\mathrm{sd}(Z) &= \sqrt{75}\\
&= 5\sqrt{3}
\end{aligned}$$

Technology active: multiple choice

6
$$\begin{aligned}
\mathrm{E}(X) &= \mathrm{E}(Y)=12\\
\mathrm{Var}(X) &= \mathrm{Var}(Y)=4\\
\mathrm{E}(T) &= 2\mathrm{E}(X)-\mathrm{E}(Y)\\
&= 2\times 12-12\\
&= 12
\end{aligned}$$

$$\begin{aligned}\text{Var}(T) &= 2^2\text{Var}(X) + \text{Var}(Y)\\ &= 4\times 4+4\\ &= 20\end{aligned}$$

$$T \sim N(12, 20)$$

$$\begin{aligned}\Pr(T > 18) &= \Pr\left(Z > \frac{18-12}{\sqrt{20}}\right)\\ &= \Pr\left(Z > \frac{6}{\sqrt{20}}\right)\end{aligned}$$

The correct answer is **E**.

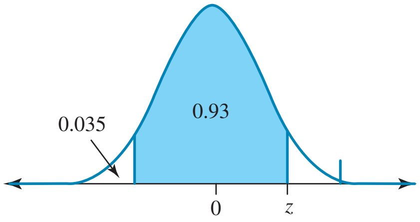
7 $\sigma = 5,\ \bar{x} = 164$

$$\bar{x} + z \times \frac{\sigma}{\sqrt{n}} = 165.18$$

$$164 + \frac{5z}{\sqrt{n}} = 165.18$$

$$\frac{5z}{\sqrt{n}} = 1.18$$

$$\sqrt{n} = \frac{5z}{1.18}$$



The z value can be determined using the inverse normal function on a calculator.

$$\begin{aligned}z &= \text{InvNorm}(0.965,\ 0,\ 1)\\ &= 1.812\end{aligned}$$

$$\begin{aligned}\sqrt{n} &= \frac{5\times 1.812}{1.18}\\ &= 7.68\\ n &= 7.68^2\\ &= 58.95\\ &\approx 59\end{aligned}$$

The correct answer is **D**.

8 A 90% confidence interval means that 90% of the samples will have a mean that falls inside the confidence interval.

The correct answer is **C**.

9 The null hypothesis is that the number of daily customers is unchanged. $H_0: \mu = 342$.

The alternative hypothesis is that the number of daily customers has changed. $H_1: \mu \neq 342$.

Note that we are checking whether the number of customers has increased OR decreased, so the test is 2-sided.

The correct answer is **A**.

10 A Type I error is when the null hypothesis is rejected when it is true.

Type I error: The sports day is cancelled but the weather remains dry.

A Type II error is when the null hypothesis is not rejected when it is false.

Type II error: The sports day is not called off and it rains.

The correct answer is **B**.

11 $\bar{x} = \dfrac{17.75 + 19.95}{2} = 18.85$

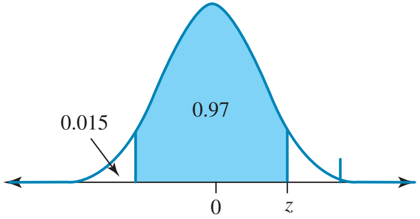
The z value which corresponds with a 99% confidence interval is 2.576.

$$\bar{x} + z\frac{\sigma}{\sqrt{n}} = 19.95$$

$$18.85 + 2.576\frac{\sigma}{\sqrt{n}} = 19.95$$

$$\begin{aligned}\frac{\sigma}{\sqrt{n}} &= \frac{1.1}{2.576}\\ &= 0.427\end{aligned}$$

The z value which corresponds with a 97% confidence interval can be calculated using the inverse normal function on a calculator.



$$\begin{aligned}z &= \text{InvNorm}(0.985, 0, 1)\\ &= 2.1701\end{aligned}$$

97% confidence interval:

$$\begin{aligned}\left(\bar{x} - z\frac{\sigma}{\sqrt{n}}, \bar{x} + z\frac{\sigma}{\sqrt{n}}\right) &= (18.85 - 2.1701\times 0.427,\\ &\quad 18.85 - 2.1701 \times 0.427)\\ &= (\$17.92,\ \$19.78)\end{aligned}$$

The correct answer is **D**.

12 H_0 should be rejected if $p \leq \alpha$.

$\alpha = 0.08$

If $p = 0.1$, H_0 should not be rejected as it is greater than the level of significance.

The correct answer is **D**.

13 The width of a confidence interval is determined by the margin of error, $z\dfrac{\sigma}{\sqrt{n}}$.

Decreasing by 80% reduces the width to $\frac{1}{5}$ of the original width.

$$z\frac{\sigma}{\sqrt{n_1}} = 5 \times z\frac{\sigma}{\sqrt{n_2}}$$

$$\frac{1}{\sqrt{n_1}} = \frac{5}{\sqrt{n_2}}$$

$$\frac{\sqrt{n_2}}{\sqrt{n_1}} = 5$$

$$\frac{n_2}{n_1} = 5^2$$

$$n_2 = 25n_1$$

The correct answer is **B**.

14 $B \sim N\left(376,\ 3^2\right)$

$$\bar{B} \sim N\left(376,\ \frac{3^2}{6}\right) = N(376,\ 1.5)$$

$$\begin{aligned}\Pr\left(\bar{B} < 373\right) &= \text{normcdf}(-\infty, 373, 376, 1.5)\\ &= 0.0072\end{aligned}$$

The correct answer is **E**.

15 $$\begin{aligned} \mathrm{E}(X) &= \int_0^1 x \cdot f(x)\, dx \\ &= \int_0^1 x \cdot 3x^2\, dx \\ &= \int_0^1 3x^3\, dx \\ &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned} \mathrm{E}(X^2) &= \int_0^1 x^2 \cdot f(x)\, dx \\ &= \int_0^1 x^2 \cdot 3x^2\, dx \\ &= \int_0^1 3x^4\, dx \\ &= \frac{3}{5} \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(X) &= \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 \\ &= \frac{3}{5} - \left(\frac{3}{4}\right)^2 \\ &= \frac{3}{80} \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(\overline{X}) &= \frac{\mathrm{Var}(X)}{n} \\ &= \frac{\left(\frac{3}{80}\right)}{75} \end{aligned}$$

$$\begin{aligned} \sigma\left(\overline{X}\right) &= \sqrt{\frac{\left(\frac{3}{80}\right)}{75}} \\ &= \sqrt{\frac{3}{80}} \times \sqrt{\frac{1}{75}} \\ &= \frac{\sqrt{5}}{100} \end{aligned}$$

The correct answer is **A**.

Technology active: extended response

16 **a** $H_0: \mu = 50$

$H_1: \mu > 50, \sigma = 10, \quad n = 25, \ \bar{x} = 54$

b $Z_c = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{54 - 50}{\frac{10}{\sqrt{25}}} = 2 = a$

$p = \Pr(Z > 2) = 0.023$

c $p = 0.023 < 0.05 \quad 5\%$

Reject H_0

d $p = 0.023 > 0.01 \quad 1\%$

Do not reject H_0

e $\bar{x} \pm z\dfrac{\sigma}{\sqrt{n}}$

$\bar{x} = 54, \quad \sigma = 10, \quad n = 25$

$95\% \quad z = 1.96$

$54 \pm 1.96 \times \dfrac{10}{5} = (50.08, 57.92)$

f $99\% \quad z = 2.576$

$54 \pm 2.576 \times \dfrac{10}{5} = (48.85, 59.15)$

17 Brown onions Potatoes

$B \sim N(180, 30^2) \quad P \sim N(200, 50^2)$

a $D = B_1 + B_2 + B_3 + B_4 + B_5 - (P_1 + P_2 + P_3 + P_4)$

$$\begin{aligned} \mathrm{E}(D) &= 5E(B) - 4\mathrm{E}(P) = 5 \times 180 - 4 \times 200 \\ &= 100 \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(D) &= 5\mathrm{Var}(B) + 4\mathrm{Var}(P) = 5 \times 30^2 + 4 \times 50^2 \\ &= 14500 \end{aligned}$$

$\Pr(D > 0) = 0.7969$

b $B_1 + B_2 + \ldots + B_6$ (1 bag brown onions)

$$\begin{aligned} \mathrm{E}(B_1 + B_2 + \ldots + B_6) &= 6\mathrm{E}(B) = 6 \times 180 \\ &= 1080 \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(B_1 + B_2 + \ldots + B_6) &= 6 \times 30^2 \\ &= 5400 \end{aligned}$$

$\Pr(B_1 + B_2 + \ldots + B_6 > 1000) = 0.8618$

c $P_1 + P_2 + \ldots + P_5$ (1 bag potatoes)

$$\begin{aligned} \mathrm{E}(P_1 + P_2 + \ldots + P_5) &= 5\mathrm{E}(P) = 5 \times 200 \\ &= 1000 \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(P_1 + P_2 + \ldots + P_5) &= 5 \times 50^2 \\ &= 12500 \end{aligned}$$

$\Pr(P_1 + P_2 + \ldots + P_5 > 1000) = 0.5$

d $T = B_1 + B_2 + \ldots + B_6 + P_1 + P_2 + \ldots + P_5$

$$\begin{aligned} \mathrm{E}(T) &= 6\mathrm{E}(B) + 5\mathrm{E}(P) = 6 \times 180 + 5 \times 200 \\ &= 2080 \end{aligned}$$

$\mathrm{Var}(T) = 6 \times 30^2 + 5 \times 50^2 = 17900$

$\Pr(T > 2000) = 0.7251$

18 Chocolate $C \sim N(201, 1.0^2)$

a $B = C_1 + C_2 + \ldots + C_5$

$$\begin{aligned} \mathrm{E}(B) &= 5 \times \mathrm{E}(C) = 5 \times 201 \\ &= 1005 \end{aligned}$$

$\mathrm{Var}(B) = 5\,\mathrm{Var}(C) = 5$

$\Pr(B > 1000) = 0.9873$

b $\Pr(C > 200) = 0.95$

$\Pr(C < 200) = 0.05$

$$\begin{aligned} \frac{200 - 201}{\sigma} &= -1.6449 \\ \sigma &= 0.61 \end{aligned}$$

c $H_0 : \mu = 201$

$H_1 \ \ \mu < 201$

$\bar{x} = 200.5, \quad \sigma = 1 \ \ n = 25$

$z_c = \dfrac{200.5 - 201}{\frac{1}{\sqrt{25}}} = -2.5 < -2.326$

$p = \Pr(Z < -2.5) = 0.0062$

$1\% \ \ \alpha = 0.01$

$p = 0.0062 < 0.01 = \alpha$

Reject H_0

Mean weight less than 201 g

d $\dfrac{c^+ - 201}{\frac{1}{\sqrt{25}}} = -2.326$

$\Pr(c^+ > 201 | \mu = 201) = 0.99$

$c^+ = 200.535$

19 **a** $s = 1.23 \quad 95\% \quad (3.20, 3.62)$

$\bar{x} = \dfrac{3.2 + 3.62}{2} = 3.41$

$\bar{x} \pm z \times \dfrac{1.23}{\sqrt{n}}, \quad z = 1.96$

$0.21 = 1.96 \times \frac{1.23}{\sqrt{n}}$

$\sqrt{n} = \frac{1.96 \times 1.23}{0.21} = 11.48$

$n \geq 11.48^2 = 131.79$

$n = 132$

b $n = 25 \quad \bar{x} = 3.41 \quad (2.964, 3.856), \quad s = 1.23$

$2z \times \frac{s}{\sqrt{n}} = 3.856 - 2.964 = 0.892$

$z = \frac{5 \times 0.892}{1.23 \times 2} = 1.8130$

$\Pr(-1.813 \leq z \leq 1.8130) = 0.93$

C is 93%

c $T = B_1 + B_2 + \ldots + B_{10} \quad B \sim N(3.41, 1.23^2)$

$E(T) = 10 \times 3.41 = 34.1$

$\text{Var}(T) = 10 \times 1.23^2, \quad \text{sd}(T) = 3.89$

$\Pr(T > 34) = 0.5103$

20 a $f(x) = \begin{cases} ke^{-4k} & x \geq 0 \\ 0 & x < 0 \end{cases}$

$k\int_0^\infty e^{-4x}dx = 1$

$k\left[-\frac{1}{4}e^{-4x}\right]_0^\infty = 1$

$\lim_{n \to \infty}\left(\frac{-k}{4}e^{-4n} + \frac{k}{4}\right) = 1$

$k = 4$

b $E(x) = 4\int_0^\infty xe^{-4x}dx$

$u = x \quad \frac{dv}{dx} = e^{-4x}$

$\frac{du}{dx} = 1 \quad v = \frac{-1}{4}e^{-4x}$

$E(X) = \left[-\frac{4x}{4}e^{-4x}\right]_0^\infty + \frac{4}{4}\int_0^\infty e^{-4x}dx$

$= \frac{1}{4}$

c $E(X^2) = 4\int_0^\infty x^2e^{-4x}dx$

$u = x^2 \quad \frac{dv}{dx} = e^{-4x}$

$\frac{du}{dx} = 2x \quad v = \frac{-1}{4}e^{-4x}$

$E(X^2) = 4\left[-\frac{x^2}{4}e^{-4x}\right]_0^\infty + 4 \times \frac{2}{4}\int_0^\infty xe^{-4x}dx$

$= 4 \times \frac{2}{4} \times \frac{1}{16}$

$= \frac{1}{8}$

$\text{Var}(X) = E(X^2) - (E(X))^2$

$= \frac{1}{8} - \left(\frac{1}{4}\right)^2$

$= \frac{1}{16}$

d $Y \sim Bi\left(n = 18, \ P = \frac{1}{3}\right)$

$E(Y) = np = 18 \times \frac{1}{3} = 6$

$\text{Var}(Y) = npq = 18 \times \frac{1}{3} \times \frac{2}{3} = 4$

$Z = 4X + 3Y$

$E(Z) = 4E(X) + 3E(Y)$

$= 4 \times \frac{1}{4} + 3 \times 6 = 19$

e $\text{Var}(Z) = 16\,\text{Var}(X) + 9\,\text{Var}(Y)$

$= 16 \times \frac{1}{16} + 9 \times 4$

$= 37$

$\text{sd}(Z) = \sqrt{37}$

13.6 Exam questions

1 a Mass of one employee is $M \sim N(75, 8^2)$

n employees have total mass $T = M_1 + M_2 + \ldots + M_n$

$E(T) = nE(M) = 75n$

$\text{Var}(T) = n\text{Var}(M) = 64n$

$T = M_1 + M_2 + \ldots + M_n \sim N(75n, \ 64n)$

$\Pr(T > 1000) < 0.01$

$2.3263 = \frac{1000 - 75n}{8\sqrt{n}}, n = 12.46$

$n = 12$

Award 1 mark for using a combination of variables.

Award 1 mark for the correct value of n.

b Time for one hot drink $H \sim N\left(2, \left(\frac{1}{2}\right)^2\right)$

Time for four hot drinks: $T = H_1 + H_2 + H_3 + H_4$

$E(T) = 4E(H) = 4 \times 2 = 8$

$\text{Var}(T) = 4\text{Var}(H) = 4 \times \left(\frac{1}{2}\right)^2 = 1$

$T = H_1 + H_2 + H_3 + H_4 \sim N(8, 1)$

$\Pr(T \leq 7.5) = 0.3085$

Award 1 mark for using a combination of variables.

Award 1 mark for the correct probability.

c i $H_0\colon \ \mu = 60\,000$

$H_1\colon \ \mu > 60\,000$ [1 mark]

ii $p = \Pr(\bar{X} > 63500 | \mu = 60\,000)$

$p = 0.0044$ [1 mark]

iii $p = 0.0044 < 0.01$ strong evidence to reject H_0

Accept H_1 advertising is a success [1 mark]

d $p = \Pr(\bar{X} > \bar{x} | \mu = 60\,000) < 0.01$

$\bar{x} \geq 63109$ [1 mark]

e $\Pr(\bar{X} > \bar{x} | \mu = 60000) > 0.05$

$\bar{x} = 62198$ at the 5% level, incorrectly accept null hypothesis

$\Pr(\bar{X} < 62198 | \mu = 63000) = 0.274$

Award 1 mark for the correct value.

Award 1 mark for the correct probability.

2 $T_1 \sim N(30, 5^2),\ \ T_2 \sim d(30, 5^2)$

$D = T_1 - T_2$

$\text{E}(D) = \text{E}(T_1) - \text{E}(T_2) = 0$

$\text{Var}(D) = \text{Var}(T_1) + \text{Var}(T_2) = 2 \times 5^2 = 50$

$D \sim N(0, 50)$

$\Pr(-3 < D < 3) = 0.329$

The correct answer is **C**.

3 a $r = 0.5$, height $h = D(3.0, 0.1^2)$

$$\text{E}(h) = 3,\ \ \text{var}(h) = 0.1^2 = \left(\frac{1}{10}\right)^2 = 0.01$$

$$V = \pi r^2 h,\ \ \text{E}(V) = \pi r^2 \text{E}(h) = \pi\left(\frac{1}{2}\right)^2 \times 3$$

$$= \frac{3\pi}{4}\,\text{cm}^3 \qquad \text{[1 mark]}$$

***VCAA Examination Report* note:**

This question was well done. Occasionally the π was missing from the answer.

b $$\text{Var}(V) = (\pi r^2)^2 \text{Var}(h) = \left(\pi^2 \times \left(\frac{1}{2}\right)^4\right) \times \left(\frac{1}{10}\right)^2$$

$$= \frac{\pi^2}{1600}\,\text{cm}^6. \qquad \text{[1 mark]}$$

***VCAA Examination Report* note:**

Students could use fractions to find $\text{Var}(V) = \text{Var}(\pi r^2 h) = \dfrac{\pi^2}{16} \times \dfrac{1}{100} = \dfrac{\pi^2}{1600}$. Students who used this approach tended to score more highly than those using decimals, who sometimes were not able to evaluate $(\pi \times 0.25)^2 \times (0.1)^2$ correctly. A number of students omitted the π^2 from their answer.

c $S = 2\pi r^2 + 2\pi rh$

$$\text{E}(S) = 2\pi r^2 + 2\pi r\,\text{E}(h)\text{E}(S) = 2\pi\left(\frac{1}{2}\right)^2 + 2\pi \times \left(\frac{1}{2}\right) \times 3$$

$$= \frac{7\pi}{2}\text{cm}^2 \qquad \text{[1 mark]}$$

***VCAA Examination Report* note:**

Some students were unable to evaluate $\dfrac{\pi}{2} + 3\pi$ correctly.

4 $X \sim D(2, 2),\ \ Y \sim D(2, 4)$

$\text{E}(X) = 2,\ \text{E}(Y) = 2,\ \text{Var}(X) = 2,\ \text{Var}(Y) = 4$

$\text{E}(aX + bY) = a\text{E}(X) + b\text{E}(Y) = 2a + 2b = 10$

$(1)\ a + b = 5\ \ \Rightarrow\ \ b = 5 - a$ [1 mark]

$\text{Var}(aX + bY) = a^2\text{Var}(X) + b^2\text{Var}(Y)$

$= 2a^2 + 4b^2 = 44$ [1 mark]

$(2)\ \ a^2 + 2b^2 = 22$

$a^2 + 2(5 - a)^2 = 22$ [1 mark]

$a^2 + 2(25 - 10a + a^2) = 22$

$3a^2 - 20a + 28 = 0$

$(3a - 14)(a - 2) = 0$

$a = \dfrac{14}{3}, 2$ but $a \in Z$

$a = 2,\ \ b = 3$ [1 mark]

***VCAA Examination Report* note:**

From the information given, students needed to write down a pair of simultaneous equations $2a + 2b = 10,\ \ 2a^2 + 4b^2 = 44$ and then solve for a and b. Common problems included failing to reject the non-integer solution and only stating the solution with minimal or no working. Students are reminded that in a question worth more than one mark, appropriate working must be shown.

Algebraic errors were common, with some students having difficulty solving a quadratic equation.

Quite a few students ‘squared’ both sides of the first equation to obtain $4a^2 + 4b^2 = 100$.

5 a $\text{E}(\overline{X}) = \mu = 1.1$

$$\text{Var}(\overline{X}) = \frac{\sigma^2}{n} = \frac{0.16^2}{25}$$

$$\text{sd}(\overline{X}) = \frac{0.16}{5} = 0.032$$

Award 1 mark for the correct expectation.

Award 1 mark for the correct standard deviation.

b $H_0 :\ \mu = 1.1$

$H_1 :\ \mu > 1.1$: one-sided test, to test whether the mean level has increased.

Award 1 mark for the correct null hypothesis.

Award 1 mark for the current alternative hypothesis.

***VCAA Examination Report* note:**

This question was answered very well by the majority of students. Some students failed to use appropriate notation or state the hypotheses clearly. The alternate hypothesis was occasionally written for a two-tail test, that is $H_1 :\ \mu \neq 1.1$.

c i $p = \Pr(\overline{X} > 1.2)$

$$= \Pr\left(Z > \frac{1.2 - 1.1}{0.032}\right)$$

$= \Pr(Z > 3.125)$

$= 0.0009$

Award 1 mark for the correct expression for the p value.

Award 1 mark for the correct p value.

***VCAA Examination Report* note:**

Transcription errors caused some students to miss out on marks, with answers such as 0.009 occurring.

High-scoring answers using the z-distribution were prevalent.

ii Since $p < 0.05$, there is evidence to support the alternative hypothesis, H_1.

That is, there has been an increase in the mean level of pollutant.

Award 1 mark for the correct reason.

***VCAA Examination Report* note:**

A correct response to Question 5ci. was generally followed by a correct answer to this question. Some students did not explicitly test at the 5% level of significance. Two-tail approaches appeared occasionally.

d $\Pr(\overline{X} > \bar{x}_c | \mu = 1.1) = 0.05 \Rightarrow \Pr(\overline{X} < \bar{x}_c | \mu = 1.1) = 0.95$

$$1.64485 = \frac{\bar{x}_c - 1.1}{0.032} \Rightarrow \bar{x}_c = 1.153$$

Award 1 mark for the correct value.

e $\Pr(\overline{X} < 1.163 | \mu = 1.1) = 0.025$

$$\Pr(\overline{X} < 1.163 | \mu = 1.2) = \Pr\left(Z < \frac{1.163 - 1.2}{0.032}\right)$$

$= \Pr(Z < -1.15625)$

$= 0.124$

Award 1 mark for the correct probability.