THE MATHEMATICAL
ASSOCIATION OF VICTORIA

2021 VCE REVISION BOOKLET

Specialist Mathematics

Student: ______________________________

CONTENTS

INTRODUCTION

Specialist Mathematical Syllabus Outline

- Further trigonometric functions
- Vectors
- Complex Numbers
- Calculus
- Differential equations
- Mechanics
- Probability and Statistics

Further trigonometric functions

including cosec, sec, cot, identities, graphs, compound and double angle formulae, the restricted inverse circular functions $\sin^{-1}(x) = \arcsin(x)$ $\cos^{-1}(x) = \arccos(x)$ $\tan^{-1}(x) = \arctan(x)$ and simple transformations.

Vectors

Including magnitude of vectors, notation, scalar products, angle between two vectors, scalar and vector resolutes, differentiation and integration of vectors, including position, velocity and acceleration vectors, applications to projectile motion. Sketching the graphs of parametrically defined graphs, and vector functions, application of vectors to geometrical proofs. Differentiation and integration of vectors,

Complex Numbers

Solving quadratics with negative discriminants, polar form of complex numbers, operations on complex numbers in both rectangular and polar form and conversions. De Moivre's theorem for powers and roots of complex numbers, factorisation including the fundamental theorem of algebra and the conjugate root theorem, solving polynomial equations. Subsets of the complex plane, circles, lines and rays.

Calculus

Standard derivatives, chain product and quotient rule. Derivatives of inverse circular functions. Second derivatives and their use in analysis of graphs, including points of inflexion and concavity. Differentiation including implicit differentiation and integration of functions, and techniques of integration, including linear and non-linear substitutions and the evaluation of definite integrals. Integrals involving inverse trigonometric functions and involving algebraic expressions into partial fractions Graphs of functions and their anti-derivatives. Applications of integration to areas and areas between curves and volumes of revolution. Arc length for both functional and parametric representation of curves.

Differential equations

Related rate problems, verifying solutions to a differential equation, setting up and solving first order differential equations, including variables separable, and applications to growth and decay, radioactivity, Newton's Law of Cooling and beam deflections and variable acceleration. Numerical solution of differential equations by Euler's method, direction, slope fields.

Mechanics

Including momentum, kinematics, displacement, velocity and acceleration. Newton's Laws of motion, motion of particles acted upon by a system of forces, including inclined plane problems and connected particles.

Probability and statistics

Linear combinations of random variables, sample means, simulation of repeated random sampling, confidence intervals, hypothesis testing, one and two tailed tests, p-values, and Type I and II errors.

NOTE:

TI Nspire screen shots will be outlined in **green**.

CASIO screen shots will be outlined in **orange**.

Written Examination 2

The examination will consist of two parts, a total of 80 marks

Part A – 20 Multiple-choice questions worth one mark each.
Part B – Extended answer questions, involving multi-stage solutions of increasing complexity worth 60 marks.

Students have 2 hours writing time and 15 minutes reading time.
This examination contributes 44% to the overall total score in Specialist Mathematics, the remaining 34% consists of internally assessed course work SACTS.

The following material are permitted in this examination. Normal stationary: this includes pens, pencils, highlighters, erasers, sharpeners and rulers. (not liquid paper)
One bound reference that may be annotated. The reference may be a textbook.
An approved calculator is allowed in this examination.

General Advice

- Answer questions when required to the number of decimal places specified.
- Answers to questions assume an **exact** answer, decimal approximations are not acceptable.
- When an exact answer is required, appropriate **working** must be shown.
- Marks will not be awarded to questions worth more than one mark if appropriate working is not shown.
- Label graphs carefully, coordinates for intercepts (in exact form) and stationary points and equations for any asymptotes.
- Pay attention to **detail** when sketching graphs.

Strategies in Examination 2 - Section A

- Make sure that you answer every question in the multiple-choice section, there is no penalty for incorrect answers.
- Some questions require you to work through every multiple-choice option when this happens don't panic!
- Eliminate responses that you think are incorrect and focus on the remaining ones.
- Questions generally require only one or two steps, however, you should still expect to do some calculations.

Strategies in Examination 2 - Section B

- If you find you are spending too much time on a question, leave it and move on to the next question.
- When a question says to "show" that a certain result is true, you can use this information to progress through to the next stage of the question.

Specialist Mathematics formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$
curved surface area of a cylinder	$2\pi rh$
volume of a cylinder	$\pi r^2 h$
volume of a cone	$\frac{1}{3}\pi r^2 h$
volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a pyramid	$\frac{1}{3}Ah$
area of triangle	$\frac{1}{2}bc\sin(A)$
sine rule	$\frac{a}{\sin(A)}=\frac{b}{\sin(B)}=\frac{c}{\sin(C)}$
cosine rule	$c^2=a^2+b^2-2ab\cos(C)$

Circular (trigonometric) functions

$\cos^2(x)+\sin^2(x)=1$	
$1+\tan^2(x)=\sec^2(x)$	$\cot^2(x)+1=\operatorname{cosec}^2(x)$
$\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y)$	$\sin(x-y)=\sin(x)\cos(y)-\cos(x)\sin(y)$
$\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y)$	$\cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$
$\tan(x+y)=\frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$	$\tan(x-y)=\frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$
$\cos(2x)=\cos^2(x)-\sin^2(x)$ $=2\cos^2(x)-1$ $=1-2\sin^2(x)$	
$\sin(2x)=2\sin(x)\cos(x)$	$\tan(2x)=\frac{2\tan(x)}{1-\tan^2(x)}$

Circular (trigonometric) functions - continued

Function	$\sin^{-1}$ (arcsin)	$\cos^{-1}$ (arccos)	$\tan^{-1}$ (arctan)
Domain	$[-1, 1]$	$[-1, 1]$	R
Range	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[0, \pi]$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Algebra (complex numbers)

$z = x + yi = r\left(\cos(\theta) + i\sin(\theta)\right) = r\,\text{cis}(\theta)$	
$\lvert z\rvert = \sqrt{x^2 + y^2} = r$	$-\pi < \text{Arg}(z) \le \pi$
$z_1 z_2 = r_1 r_2\,\text{cis}(\theta_1 + \theta_2)$	$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,\text{cis}(\theta_1 - \theta_2)$
$z^n = r^n \text{cis}(n\theta)$ (de Moivre's theorem)	

Vectors in two and three dimensions

$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$
$\lvert\underset{\sim}{r}\rvert = \sqrt{x^2 + y^2 + z^2} = r$
$\dot{\underset{\sim}{r}} = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$
$\underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 = r_1 r_2 \cos(\theta) = x_1 x_2 + y_1 y_2 + z_1 z_2$

Mechanics

momentum	$\underset{\sim}{p} = m\underset{\sim}{v}$
equation of motion	$\underset{\sim}{R} = m\underset{\sim}{a}$

Probability and statistics

for random variables X and Y	$E(aX + b) = aE(X) + b$ $E(aX + bY) = aE(X) + bE(Y)$ $\text{Var}(aX + b) = a^2\,\text{Var}(X)$
for independent random variables X and Y	$\text{Var}(aX + bY) = a^2\,\text{Var}(X) + b^2\,\text{Var}(Y)$
approximate confidence interval for μ	$\left(\bar{x} - z\frac{s}{\sqrt{n}},\ \bar{x} + z\frac{s}{\sqrt{n}}\right)$
distribution of sample mean $\bar{X}$	mean $E(\bar{X}) = \mu$ variance $\text{Var}(\bar{X}) = \frac{\sigma^2}{n}$

Calculus

$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$	$\int x^n dx=\frac{1}{n+1}x^{n+1}+c\ ,\ n\neq -1$
$\frac{d}{dx}\left(e^{ax}\right)=ae^{ax}$	$\int e^{ax}dx=\frac{1}{a}e^{ax}+c$
$\frac{d}{dx}\left(\log_e(x)\right)=\frac{1}{x}$	$\int \frac{1}{x}dx=\log_e\lvert x\rvert+c$
$\frac{d}{dx}\left(\sin(ax)\right)=a\cos(ax)$	$\int \sin(ax)dx=-\frac{1}{a}\cos(ax)+c$
$\frac{d}{dx}\left(\cos(ax)\right)=-a\sin(ax)$	$\int \cos(ax)dx=\frac{1}{a}\sin(ax)+c$
$\frac{d}{dx}\left(\tan(ax)\right)=a\sec^2(ax)$	$\int \sec^2(ax)dx=\frac{1}{a}\tan(ax)+c$
$\frac{d}{dx}\left(\sin^{-1}(x)\right)=\frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{a^2-x^2}}dx=\sin^{-1}\left(\frac{x}{a}\right)+c,\ a>0$
$\frac{d}{dx}\left(\cos^{-1}(x)\right)=\frac{-1}{\sqrt{1-x^2}}$	$\int \frac{-1}{\sqrt{a^2-x^2}}dx=\cos^{-1}\left(\frac{x}{a}\right)+c,\ a>0$
$\frac{d}{dx}\left(\tan^{-1}(x)\right)=\frac{1}{1+x^2}$	$\int \frac{a}{a^2+x^2}dx=\tan^{-1}\left(\frac{x}{a}\right)+c$
	$\int (ax+b)^n dx=\frac{1}{a(n+1)}(ax+b)^{n+1}+c,\ n\neq -1$
	$\int (ax+b)^{-1} dx=\frac{1}{a}\log_e\lvert ax+b\rvert+c$
product rule	$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$
quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$
Euler's method	$\frac{dy}{dx}=f(x),\ x_0=a$ and $y_0=b$, then $x_{n+1}=x_n+h$ and $y_{n+1}=y_n+hf\left(x_n\right)$
acceleration	$a=\frac{d^2x}{dt^2}=\frac{dv}{dt}=v\frac{dv}{dx}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$
arc length	$\int_{x_1}^{x_2}\sqrt{1+\left(f'(x)\right)^2}\,dx$ or $\int_{t_1}^{t_2}\sqrt{\left(x'(t)\right)^2+\left(y'(t)\right)^2}\,dt$

END OF FORMULA SHEET

WORKED SOLUTIONS

Trigonometry and Curve Sketching

Exam 1 Questions (% indicates percentage of students obtaining full marks for the question)

Question 1 (Q5 VCAA SM E1 2018 4 marks 15%)

Sketch the graph of $f(x)=\dfrac{x+1}{x^2-4}$ labelling any asymptotes with their equations and any intercepts with their coordinates.

Question 2 (Q9 VCAA SM E1 2016 3 marks 44%)

Given that $\cos(x-y)=\dfrac{3}{5}$ and $\tan(x)\tan(y)=2$, find $\cos(x+y)$.

Question 3 (Q7 VCAA SM E1 2015 3+2=5 marks 11%)

a. Solve $\sin(2x)=\sin(x)$, $x\in[0,2\pi]$

b. Find $\operatorname{cosec}(2x)<\operatorname{cosec}(x)$, $x\in\left(0,\dfrac{\pi}{2}\right)\cup\left(\dfrac{\pi}{2},\pi\right)$

Question 4 (Q4 VCAA SM E1 2008 3 marks 23%)

Given that $\sin\left(\dfrac{\pi}{10}\right)=\dfrac{\sqrt{5}-1}{4}$, find the value of $\sec\left(\dfrac{\pi}{5}\right)$ in the form $a\sqrt{5}+b$, where $a,b\in R$

Exam 2 - Multiple Choice Questions

Question 1 (VCAA Q1 2011 34%)

The number of straight line asymptotes of the graph $y=\dfrac{2x^3+x^2-1}{x^2-x-2}$ is

A. 0
B. 1
C. 2
D. 3
E. 4

Question 2 (VCAA Q3 2014 61%)

The features of the graph with rule $f(x)=\dfrac{x^2-4x+3}{x^2-x-6}$ include

A. asymptotes at $x=1$ and $x=-2$

B. asymptotes at $x=3$ and $x=-2$

C. an asymptote at $x=1$ and a point of discontinuity at $x=3$

D. an asymptote at $x=-2$ and a point of discontinuity at $x=3$

E. an asymptote at $x=3$ and a point of discontinuity at $x=-2$

Question 3 (VCAA Q2 2017 37%)

The solutions to $\cos(x)>\dfrac{1}{4}\operatorname{cosec}(x)$ for $x\in(0,2\pi)\setminus\pi$ are given by

A. $x\in\left(\dfrac{\pi}{12},\dfrac{5\pi}{12}\right)\cup\left(\dfrac{5\pi}{12},\dfrac{13\pi}{12}\right)\cup\left(\dfrac{17\pi}{12},2\pi\right)$

B. $x\in\left(\dfrac{\pi}{12},\dfrac{5\pi}{12}\right)\cup\left(\dfrac{13\pi}{12},\dfrac{17\pi}{12}\right)$

C. $x\in\left(\dfrac{\pi}{12},\dfrac{5\pi}{12}\right)\cup\left(\pi,\dfrac{13\pi}{12}\right)\cup\left(\dfrac{13\pi}{12},2\pi\right)$

D. $x\in\left(\dfrac{\pi}{12},\dfrac{13\pi}{12}\right)\cup\left(\dfrac{17\pi}{12},2\pi\right)$

E. $x\in\left(\dfrac{\pi}{12},\dfrac{5\pi}{12}\right)\cup\left(\pi,\dfrac{13\pi}{12}\right)\cup\left(\dfrac{17\pi}{12},2\pi\right)$

Question 4 (VCAA Q3 2019 65%)

The implied domain of the function with the rule $f(x)=1-\sec\left(x+\frac{\pi}{4}\right)$ is

A. R

B. $[0,2]$

C. $R\backslash\left\{\frac{(4n-1)\pi}{4}\right\},\ n\in Z$

D. $R\backslash\left\{\frac{(4n+1)\pi}{4}\right\},\ n\in Z$

E. $R\backslash\left\{\frac{(2n-1)\pi}{2}\right\},\ n\in Z$

Question 5 (VCAA Q2 2018 60%)

Consider the function with the rule $f(x)=\frac{1}{\sqrt{\sin^{-1}(cx+d)}}$, where $c,d\in R$ and $c>0$.

The domain is

A. $x>-\frac{d}{c}$

B. $-\frac{d}{c}<x<\frac{1-d}{c}$

C. $\frac{-1-d}{c}<x<\frac{1-d}{c}$

D. $x\in R\backslash\left\{-\frac{d}{c}\right\}$

E. $x\in R$

Exam 2 Questions

Question 1 (Q1 VCAA E2 2017 1+2+2+3+2+1=11 marks 13%)

Let $f : D \to R$, $f(x) = \dfrac{x}{1+x^3}$, where D is the maximal domain of f.

a.i. Find the equations of any asymptotes of the graph of f.

ii. Find $f'(x)$ and state the coordinates of any stationary points of the graph of f, correct to two decimal places.

iii. Find the coordinates of any points of inflection on the graph of f, correct to two decimal places.

b. Sketch the graph of $f(x) = \dfrac{x}{1+x^3}$ from $x = -3$ to $x = 3$ on the axes below, marking all stationary points, points of inflexion and axial intercepts with axes, labelling them with their coordinates. Show any asymptotes and label them with their equations.

c. The region S, bounded by the graph of f, the x-axis and the line $x = 3$, is rotated about the x-axis to form a solid of revolution. The line $x = a$, where $0 < a < 3$, divides the region S into two regions such that, when the regions are rotated about the x-axis, they generate solids of equal volume.

i. Write down an equation involving definite integrals that can be used to determine a.

ii. Hence, find the value of a, correct to two decimal places.

Solutions

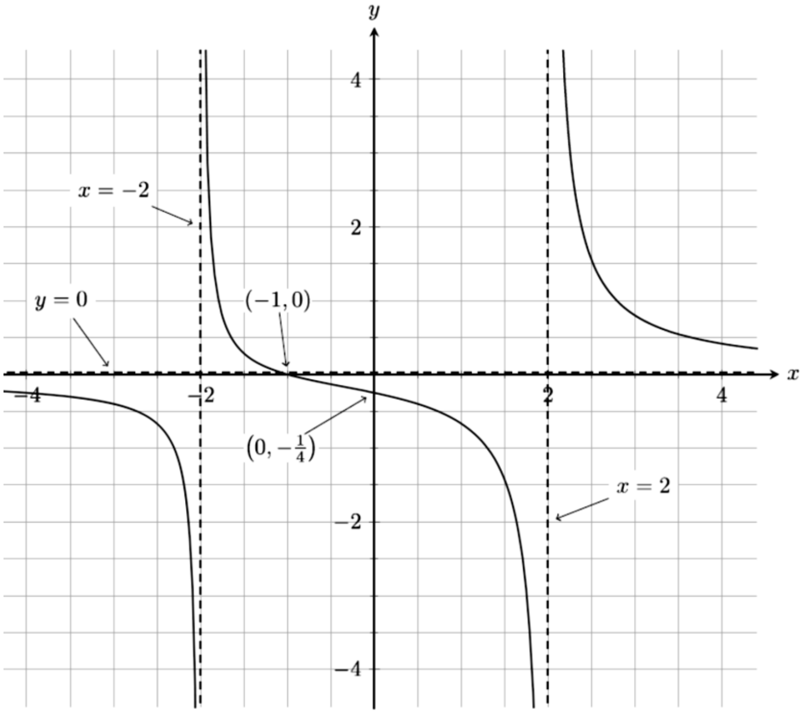
Question 1

$$y = f(x) = \frac{x+1}{x^2-4} = \frac{x+1}{(x+2)(x-2)}$$

vertical asymptotes at $x = \pm 2$

$\lim\limits_{x\to\infty} f(x) = 0$ so there is a horizontal asymptote at $y = 0$ and crosses the x-axis at $x = -1$ $(-1, 0)$ so it crosses the horizontal asymptote, crosses the y-axis when $x = 0$ at $y = -\frac{1}{4}$ $\left(0, -\frac{1}{4}\right)$

$$x > 2, f(x) > 0$$
$$x < -2, f(x) < 0$$



Question 2

(1) $\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y) = \frac{3}{5}$

(2) $\tan(x)\tan(y) = \frac{\sin(x)\sin(y)}{\cos(x)\cos(y)} = 2$

$\Rightarrow \sin(x)\sin(y) = 2\cos(x)\cos(y)$ into (1)

$\cos(x)\cos(y) + 2\cos(x)\cos(y) = 3\cos(x)\cos(y) = \frac{3}{5}$

$\Rightarrow \cos(x)\cos(y) = \frac{1}{5}$

(3) $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

$= \cos(x)\cos(y) - 2\cos(x)\cos(y)$

$= -\cos(x)\cos(y)$

$= -\frac{1}{5}$

Question 3

a. $\sin(2x) = \sin(x) \quad , \quad x \in [0, 2\pi]$

$2\sin(x)\cos(x) - \sin(x) = 0$

$\sin(x)(2\cos(x) - 1) = 0$

$\sin(x) = 0 \qquad \text{and} \qquad \cos(x) = \frac{1}{2}$

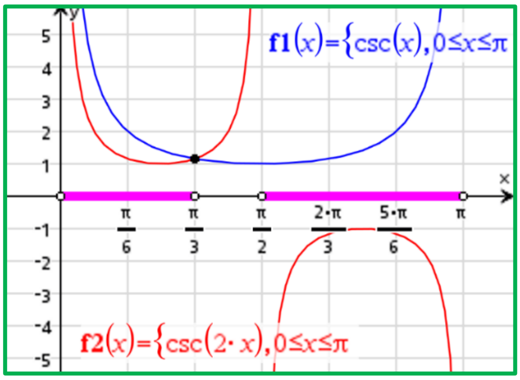
$x = 0, \pi, 2\pi \qquad\qquad x = \frac{\pi}{3}, \; 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

$x = 0, \; \frac{\pi}{3}, \; \pi, \; \frac{5\pi}{3}, \; 2\pi$

b. $\operatorname{cosec}(2x) < \operatorname{cosec}(x)$

for $\quad x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$

$\Rightarrow \quad \left(0, \frac{\pi}{3}\right) \cup \left(\frac{\pi}{2}, \pi\right)$



Question 4

Given $\sin\left(\frac{\pi}{10}\right) = \frac{\sqrt{5}-1}{4}$, let $A = \frac{\pi}{10}$, $2A = \frac{\pi}{5}$

using $\cos(2A) = 1 - 2\sin^2(A)$ gives

$$\cos\left(\frac{\pi}{5}\right) = 1 - 2\left(\frac{\sqrt{5}-1}{4}\right)^2$$

$$\cos\left(\frac{\pi}{5}\right) = 1 - \frac{\left(5 - 2\sqrt{5} + 1\right)}{8} = \frac{8 - \left(6 - 2\sqrt{5}\right)}{8}$$

$$\cos\left(\frac{\pi}{5}\right) = \frac{1+\sqrt{5}}{4}$$

now $\quad \sec\left(\frac{\pi}{5}\right) = \frac{1}{\cos\left(\frac{\pi}{5}\right)} = \frac{1}{\frac{1+\sqrt{5}}{4}} = \frac{4}{1+\sqrt{5}} \times \frac{1-\sqrt{5}}{1-\sqrt{5}} = \frac{4\left(1-\sqrt{5}\right)}{1-5}$

$$\sec\left(\frac{\pi}{5}\right) = -1 + \sqrt{5} \qquad \Rightarrow \quad a = 1 \quad b = -1$$

MC Solutions

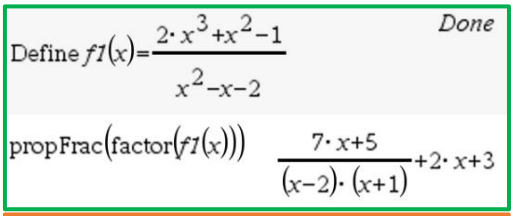
Question 1 **Answer D**

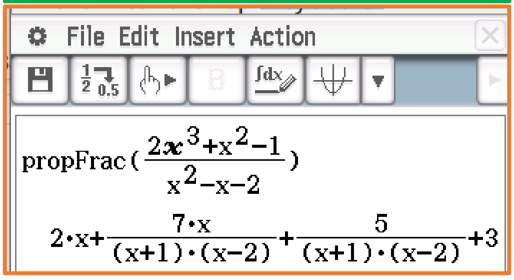
$$y=\frac{2x^3+x^2-1}{x^2-x-2}=2x+3+\frac{7x+5}{(x-2)(x+1)}$$

$x=2$ and $x=-1$ are both vertical asymptotes

$y=2x+3$ is a straight line oblique asymptote

total of 3 straight line asymptotes.





Question 2 **Answer D**

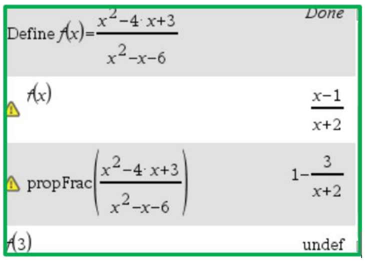
$$y=\frac{x^2-4x+3}{x^2-x-6}=\frac{(x-3)(x-1)}{(x-3)(x+2)}$$

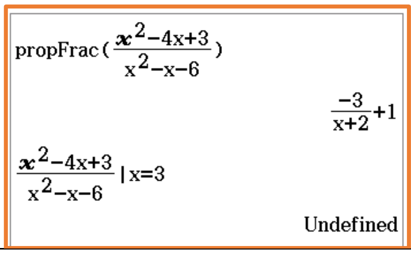
$$=\frac{x-1}{x+2}=\frac{x+2-3}{x+2}=1-\frac{3}{x+2}$$

vertical asymptote $x=-2$

horizontal asymptote $y=1$

point of discontinuity at $x=3$

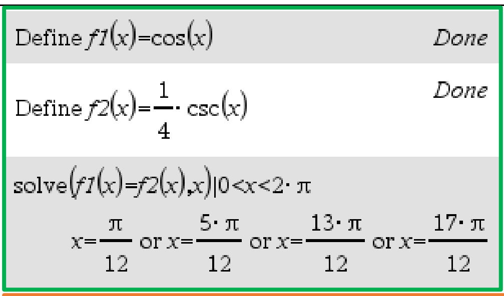


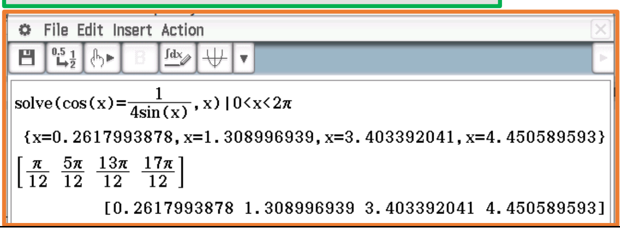


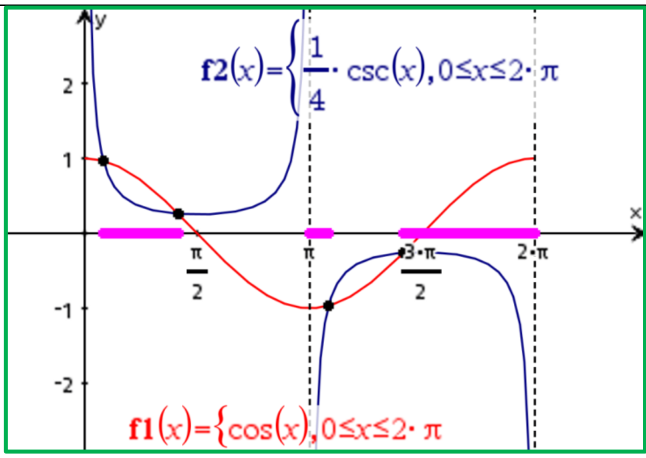
Question 3 **Answer E**

$$\cos(x)=\frac{1}{4}\operatorname{cosec}(x) \quad \Rightarrow \quad x=\frac{\pi}{12},\frac{5\pi}{12},\frac{13\pi}{12},\frac{17\pi}{12}$$

$$\cos(x)>\frac{1}{4}\operatorname{cosec}(x) \quad x\in(0,2\pi)\setminus\{\pi\} \quad \Rightarrow \quad x\in\left(\frac{\pi}{12},\frac{5\pi}{12}\right)\cup\left(\pi,\frac{13\pi}{12}\right)\cup\left(\frac{17\pi}{12},2\pi\right)$$







Question 4 **Answer D**

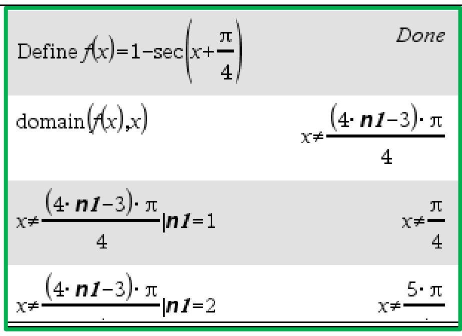
$$f(x)=1-\sec\left(x+\frac{\pi}{4}\right)=1-\frac{1}{\cos\left(x+\frac{\pi}{4}\right)}$$

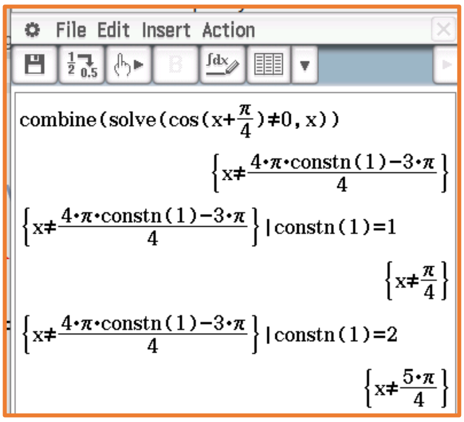
$$\cos\left(x+\frac{\pi}{4}\right)\neq 0$$

$$x+\frac{\pi}{4}\neq\frac{\pi(2k+1)}{2}$$

$$x\neq\frac{(4k+2)\pi}{4}-\frac{\pi}{4}$$

$$x\neq\frac{(4k+1)\pi}{4}$$





Question 5 **Answer B**

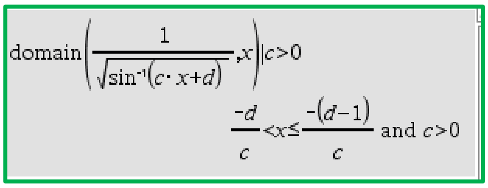
$$f(x)=\frac{1}{\sqrt{\sin^{-1}(cx+d)}}\quad c,d\in R\ ,\ c>0$$

for maximal domain require $\sin^{-1}(cx+d)>0$

$$0<cx+d\leq 1$$

$$-d<cx\leq 1-d$$

$$-\frac{d}{c}<x\leq\frac{1-d}{c}$$



Exam 2 Solutions

Question 1

a.i. $f(x)=\dfrac{x}{1+x^3}$, $1+x^3=0 \Rightarrow x^3=-1$, $x=-1$ is a vertical asymptote

and $y=0$ is the horizontal asymptote

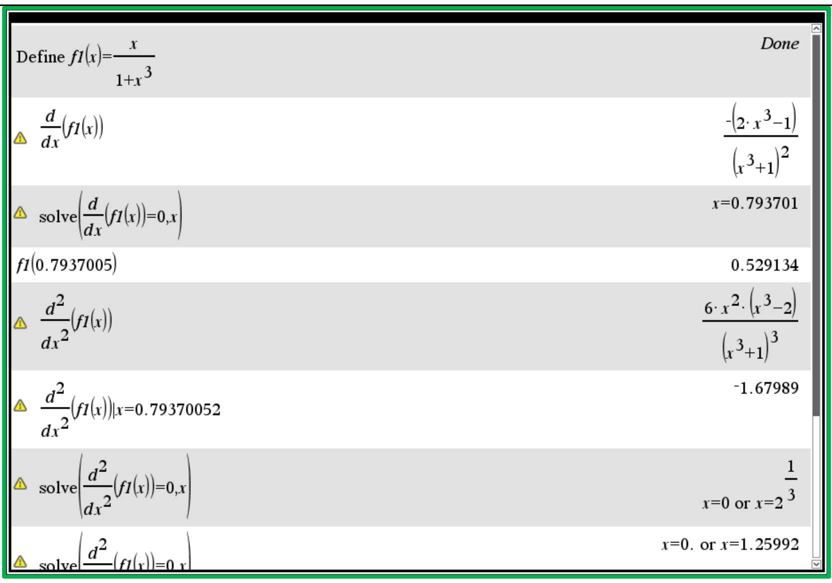
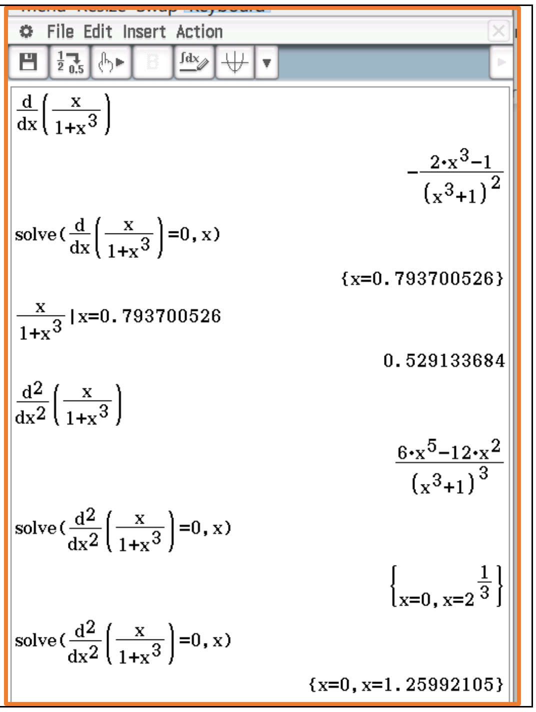
ii. $f'(x)=\dfrac{1-2x^3}{\left(1+x^3\right)^2}$, stationary points when $f'(x)=0$

solving $f'(x)=0$ $\quad x=\dfrac{1}{\sqrt[3]{2}}\approx 0.7937$, $f(0.7937)=0.529$

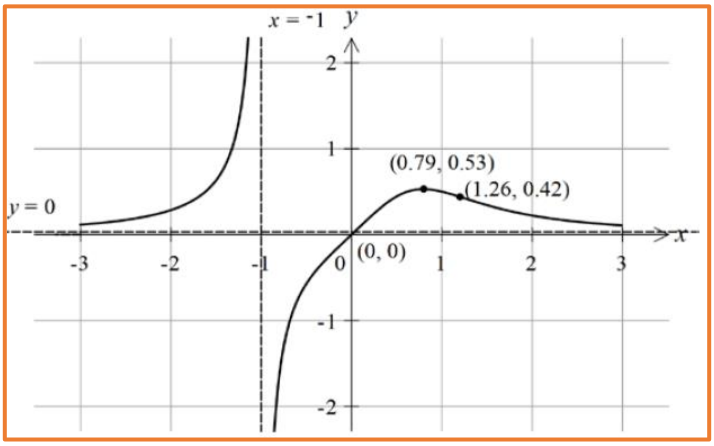
$(0.79,0.53)$ is a local max.

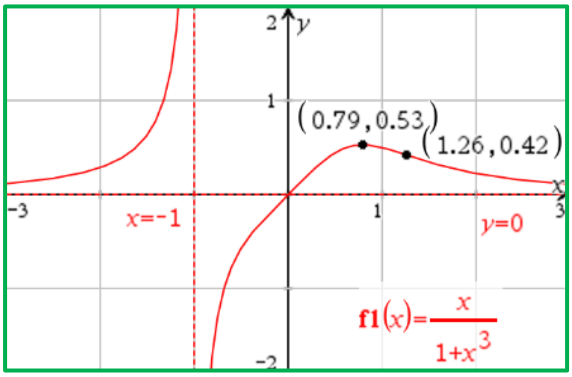
iii. $f''(x)=\dfrac{6x^2\left(x^3-2\right)}{\left(1+x^3\right)^2}$ $\quad f''(0.79)<0$, inflexion points $f''(x)=0$

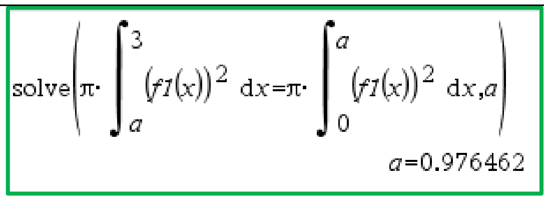
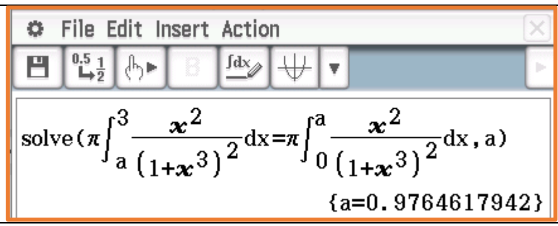
solving $f''(x)=0$ $\quad x=\sqrt[3]{2}\approx 1.2599$, $f(1.26)=0.42$, $(1.26,0.42)$

b. The graph crosses the horizontal asymptote at $x=0$, the origin but the point at the origin $(0,0)$ is not an inflexion point.





c.i. $V=\pi\int_0^a \frac{x^2}{\left(1+x^3\right)^2}dx=\pi\int_a^3 \frac{x^2}{\left(1+x^3\right)^2}dx$ **ii.** solving $a=0.98$	
	

Vectors and Vector Calculus

Exam 1 Questions (% indicates percentage of students obtaining full marks for the question)

Question 1 (Q5 VCAA SM E1 2017 2+2=4 marks 11%)

Relative to a fixed origin, the points *B*, *C* and *D* are defined respectively by the position vectors $\underset{\sim}{b} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{c} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{d} = a\underset{\sim}{i} - 2\underset{\sim}{j}$, where *a* is a real constant.

Given that the magnitude of the angle *BCD* is $\dfrac{\pi}{3}$, find *a*.

Question 2 (Q5 VCAA SM E1 2016 2+2=4 marks 54%)

Consider the vectors $\underset{\sim}{a} = 3\underset{\sim}{i} + 5\underset{\sim}{j} - 2\underset{\sim}{k}$, $\underset{\sim}{b} = \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$, $\underset{\sim}{c} = \underset{\sim}{i} + d\,\underset{\sim}{k}$, where *d* is a real constant.

a. Find the vector resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$.

b. Find the value of *d* if the vectors are **linearly dependent**.

Question 3 (Q1 VCAA SM E1 2014 2+2+2=6 marks 42%)

Consider the vector $\underset{\sim}{a} = \sqrt{3}\underset{\sim}{i} - \underset{\sim}{j} - \sqrt{2}\underset{\sim}{k}$, where $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors in the positive directions, of the *x*, *y* and *z* axes respectively.

a. Find a unit vector in the direction of $\underset{\sim}{a}$.

b. Find the acute angle that $\underset{\sim}{a}$ makes with the positive direction of the *x*-axis.

c. The vector $\underset{\sim}{b} = 2\sqrt{3}\underset{\sim}{i} + m\,\underset{\sim}{j} - 5\underset{\sim}{k}$.

Given that $\underset{\sim}{b}$ is perpendicular to $\underset{\sim}{a}$, find the value of *m*.

Question 4 (Q2 VCAA SM E1 2014 1+2+2=5 marks 42%)

The position vector of a particle at a time *t*, $t \geq 0$ is given by

$\underset{\sim}{r}(t) = (t-2)\underset{\sim}{i} + (t^2 - 4t + 1)\underset{\sim}{j}$

a. Find the cartesian equation of the path followed by the particle.

b. Sketch the path followed by the particle, on the axes below,

labelling all important features.

c. Find the speed of the particle when $t = 1$.

Exam 2 Multiple Choice Questions

Question 1 (VCAA 2014 Q15 69%)

If θ is the angle between $\underset{\sim}{a} = \sqrt{3}\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} - 4\underset{\sim}{j} + \sqrt{3}\underset{\sim}{k}$, then $\cos(2\theta)$ is

A. $-\frac{4}{5}$

B. $\frac{7}{25}$

C. $-\frac{7}{25}$

D. $\frac{14}{25}$

E. $-\frac{24}{25}$

Question 2 (VCAA 2014 Q16 77%)

Two vectors are given by $\underset{\sim}{a} = 4\underset{\sim}{i} + m\underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{b} = -2\underset{\sim}{i} + n\underset{\sim}{j} - \underset{\sim}{k}$, where $m, n \in R^+$.

If $|\underset{\sim}{a}| = 10$ and $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$ then m and n respectively are

A. $5\sqrt{3}$, $\frac{\sqrt{3}}{3}$

B. $5\sqrt{3}$, $\sqrt{3}$

C. $-5\sqrt{3}$, $\sqrt{3}$

D. $\sqrt{93}$, $\frac{5\sqrt{93}}{93}$

E. 5 , 1

Question 3 (VCAA 2014 Q17 62%)

The acceleration vector of a particle that starts from rest is given by

$\underset{\sim}{a}(t) = -4\sin(2t)\underset{\sim}{i} + 20\cos(2t)\underset{\sim}{j} - 20e^{-2t}\underset{\sim}{k}$ where $t \geq 0$.

The velocity vector of the particle $\underset{\sim}{v}(t)$, is given by

A. $-8\cos(2t)\underset{\sim}{i} - 40\sin(2t)\underset{\sim}{j} + 40e^{-2t}\underset{\sim}{k}$

B. $2\cos(2t)\underset{\sim}{i} + 10\sin(2t)\underset{\sim}{j} + 10e^{-2t}\underset{\sim}{k}$

C. $(8 - 8\cos(2t))\underset{\sim}{i} - 40\sin(2t)\underset{\sim}{j} + (40e^{-2t} - 40)\underset{\sim}{k}$

D. $(2\cos(2t) - 2)\underset{\sim}{i} + 10\sin(2t)\underset{\sim}{j} + (10e^{-2t} - 10)\underset{\sim}{k}$

E. $(4\cos(2t) - 4)\underset{\sim}{i} + 20\sin(2t)\underset{\sim}{j} + (20 - 20e^{-2t})\underset{\sim}{k}$

Question 4 (VCAA 2019 Q11 66%)

Let point M have coordinates $(a, 1, -2)$ and let N have coordinates $(-3, b, -1)$.

If the coordinates of the midpoint of $\overrightarrow{MN}$ are $\left(-5, \frac{3}{2}, c\right)$ and a, b and c are real constants, then the values of a, b and c respectively are

A. -13 , 2 and $-\frac{1}{2}$

B. -2 , $\frac{1}{2}$ and -3

C. -7 , -2 and $-\frac{3}{2}$

D. -2 , $-\frac{1}{2}$ and -3

E. -7 , 2 and $-\frac{3}{2}$

Question 5 (VCAA 2011 Q10 58%)

The diagram below shows a rhombus, spanned by two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

It follows that

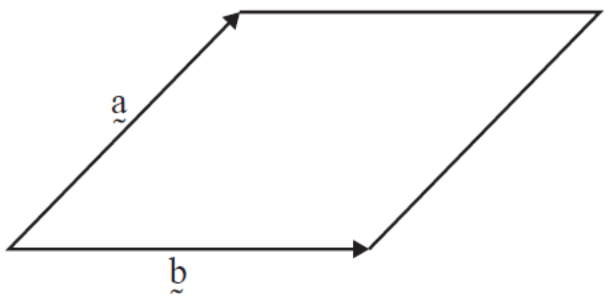
A. $\underset{\sim}{a}.\underset{\sim}{b} = 0$

B. $\underset{\sim}{a} = \underset{\sim}{b}$

C. $(\underset{\sim}{a} + \underset{\sim}{b}).(\underset{\sim}{a} - \underset{\sim}{b}) = 0$

D. $|\underset{\sim}{a} + \underset{\sim}{b}| = |\underset{\sim}{a} - \underset{\sim}{b}|$

E. $2\underset{\sim}{a} + 2\underset{\sim}{b} = \underset{\sim}{0}$



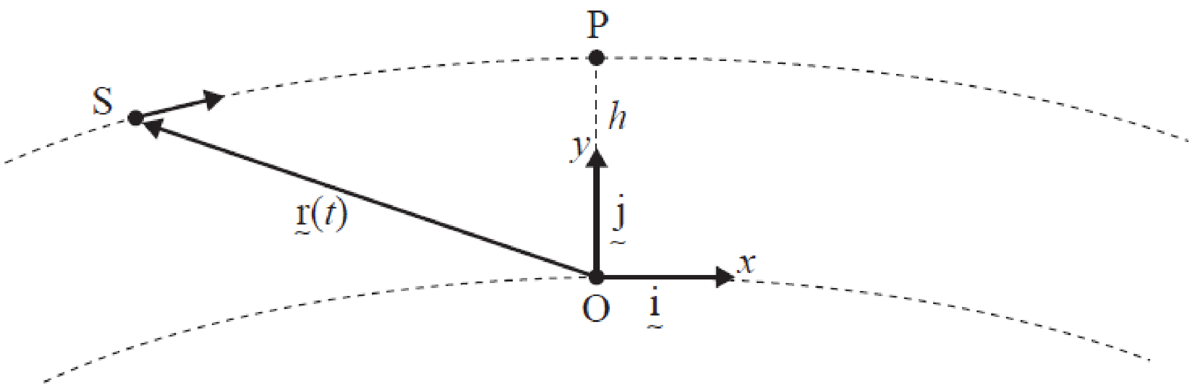
Exam 2 Section B Questions

Question 1 (Q4 VCAA E2 2012 1+3+2+2+3=11 marks 29%)

The position of the *International Space Station* (S), when visible above the horizon from a radar tracking station location (O), is modelled by

$$\underset{\sim}{r}(t) = 6800\sin(\pi(1.3t-0.1))\underset{\sim}{i} + (6800\cos(\pi(1.3t-0.1))-6400)\underset{\sim}{j},\ t \in [0, 0.154]$$

where $\underset{\sim}{i}$ is a unit vector relative to O as shown, and $\underset{\sim}{j}$ is a unit vector vertically up from point O. Time t is measured in hours and displacement components are measured in kilometres.



a. Find the height, h km, of the space station above the surface of Earth when it is at point P, directly above point O.

b. Find the acceleration of the space station and show that its acceleration is perpendicular to its velocity.

c. Find the speed of the space station in km/hr.
Give your answer to the nearest integer.

d. Find the equation of the path followed by the space station in cartesian form.

e. Find the times when the space station is at a distance of 1000 km from the radar tracking location O. Give your answers in hours, correct to two decimal places.

Solutions

Question 1

$\underset{\sim}{b}=\underset{\sim}{i}-\underset{\sim}{j}+2\underset{\sim}{k}$, $\underset{\sim}{c}=2\underset{\sim}{i}-\underset{\sim}{j}+\underset{\sim}{k}$, $\underset{\sim}{d}=a\underset{\sim}{i}-2\underset{\sim}{j}$

$\overrightarrow{CB}=\overrightarrow{OB}-\overrightarrow{OC}=-\underset{\sim}{i}+\underset{\sim}{k}$, $\left|\overrightarrow{CB}\right|=\sqrt{2}$

$\overrightarrow{CD}=\overrightarrow{OD}-\overrightarrow{OC}=(a-2)\underset{\sim}{i}-\underset{\sim}{j}-\underset{\sim}{k}$, $\left|\overrightarrow{CD}\right|=\sqrt{(a-2)^2+2}=\sqrt{a^2-4a+6}$

$\overrightarrow{CB}.\overrightarrow{CD}=-(a-2)-1=1-a$

$$\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}=\frac{\overrightarrow{CB}.\overrightarrow{CD}}{\left|\overrightarrow{CB}\right|\left|\overrightarrow{CD}\right|}=\frac{1-a}{\sqrt{2}\sqrt{a^2-4a+6}}$$

$\sqrt{2}\sqrt{a^2-4a+6}=2(1-a)$, but $1-a>0$

$2(a^2-4a+6)=4-8a+4a^2$

$2a^2-8=0$

$a^2=4 \;\Rightarrow\; a=\pm 2$, but $1-a>0$

$a=-2$ only

Question 2

$\underset{\sim}{a}=3\underset{\sim}{i}+5\underset{\sim}{j}-2\underset{\sim}{k}$, $\underset{\sim}{b}=\underset{\sim}{i}-2\underset{\sim}{j}+3\underset{\sim}{k}$, $\underset{\sim}{c}=\underset{\sim}{i}+d\,\underset{\sim}{k}$

a. $\underset{\sim}{a}.\underset{\sim}{b}=3-10-6=-13$, $\left|\underset{\sim}{b}\right|=\sqrt{1+4+9}=\sqrt{14}$, $\hat{\underset{\sim}{b}}=\frac{1}{\sqrt{14}}\left(\underset{\sim}{i}-2\underset{\sim}{j}+3\underset{\sim}{k}\right)$

vector resolute $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is $\left(\underset{\sim}{a}.\hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}=-\frac{13}{14}\left(\underset{\sim}{i}-2\underset{\sim}{j}+3\underset{\sim}{k}\right)$

b. for linearly dependent $\underset{\sim}{c}=m\underset{\sim}{a}+n\underset{\sim}{b}$

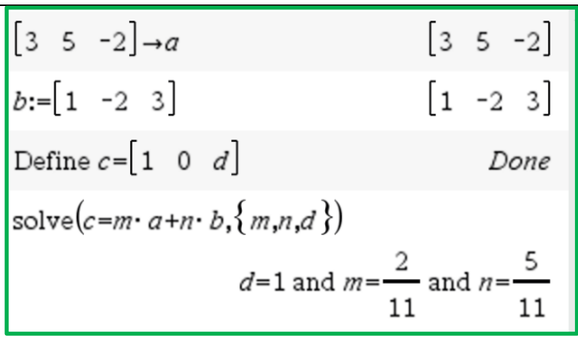
$\underset{\sim}{c}=\underset{\sim}{i}+d\,\underset{\sim}{k}=m\left(3\underset{\sim}{i}+5\underset{\sim}{j}-2\underset{\sim}{k}\right)+n\left(\underset{\sim}{i}-2\underset{\sim}{j}+3\underset{\sim}{k}\right)$

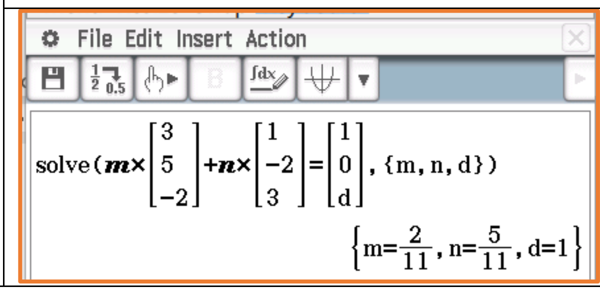
$\underset{\sim}{i}$ (1) $1=3m+n$

$\underset{\sim}{j}$ (2) $0=5m-2n$

$\underset{\sim}{k}$ (3) $d=-2m+3n$

$(3)+(2)-(1) \;\Rightarrow\; d=1$





Question 3

a $\underset{\sim}{a} = \sqrt{3}\underset{\sim}{i} - \underset{\sim}{j} - \sqrt{2}\underset{\sim}{k}$

$$|\underset{\sim}{a}| = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2 + \left(-\sqrt{2}\right)^2} = \sqrt{3+1+2} = \sqrt{6}$$

$$\hat{\underset{\sim}{a}} = \frac{1}{\sqrt{6}}\left(\sqrt{3}\underset{\sim}{i} - \underset{\sim}{j} - \sqrt{2}\underset{\sim}{k}\right)$$

b. $\cos(\alpha) = \dfrac{\underset{\sim}{a}.\underset{\sim}{i}}{|\underset{\sim}{a}|} = \dfrac{\sqrt{3}}{\sqrt{6}} = \dfrac{1}{\sqrt{2}}$

$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$\alpha = \frac{\pi}{4} = 45^0$$

c. $\underset{\sim}{a} = \sqrt{3}\underset{\sim}{i} - \underset{\sim}{j} - \sqrt{2}\underset{\sim}{k}$ is perpendicular to $\underset{\sim}{b} = 2\sqrt{3}\underset{\sim}{i} + m\underset{\sim}{j} - 5\underset{\sim}{k}$

$$\underset{\sim}{a}.\underset{\sim}{b} = 2\times 3 - m + 5\sqrt{2} = 0$$

$$m = 6 + 5\sqrt{2}$$

Question 4

a. $\underset{\sim}{r}(t) = (t-2)\underset{\sim}{i} + \left(t^2 - 4t + 1\right)\underset{\sim}{j}$

$x = t - 2 \quad \Rightarrow t = x + 2$

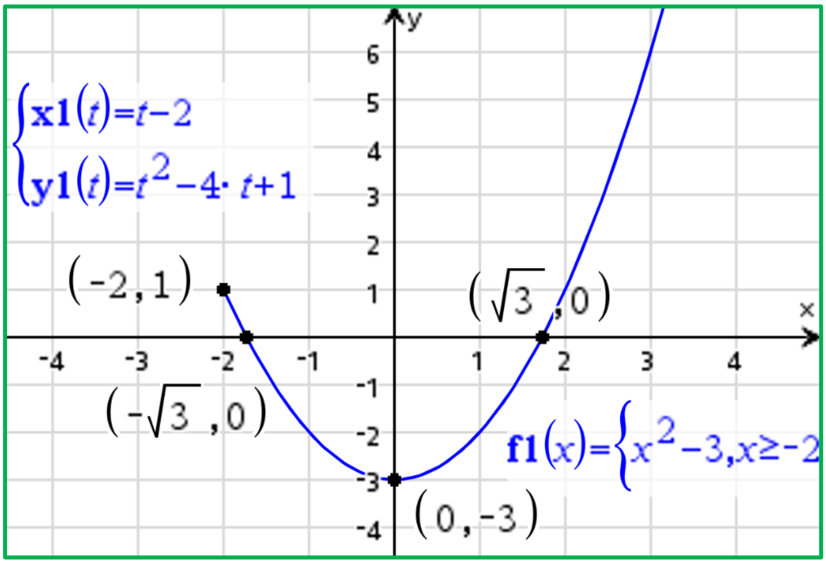
$y = t^2 - 4t + 1$

$y = (x+2)^2 - 4(x+2) + 1$

$y = x^2 + 4x + 4 - 4x - 8 + 1$

$y = x^2 - 3 \quad t \geq 0 \quad \Rightarrow x \geq -2$

b.



c. $\dot{\underset{\sim}{r}}(t) = \underset{\sim}{i} + (2t-4)\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(1) = \underset{\sim}{i} - 2\underset{\sim}{j}$

$$|\dot{\underset{\sim}{r}}(1)| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$$

MC Solutions

Question 1 **Answer B**

$\underset{\sim}{a}=\sqrt{3}\underset{\sim}{i}+4\underset{\sim}{j}-\underset{\sim}{k}\quad,\quad \underset{\sim}{b}=\underset{\sim}{i}-4\underset{\sim}{j}+\sqrt{3}\underset{\sim}{k}$

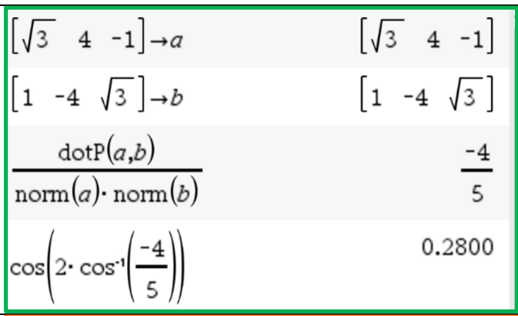
$|\underset{\sim}{a}|=\sqrt{3+16+1}=\sqrt{20}$

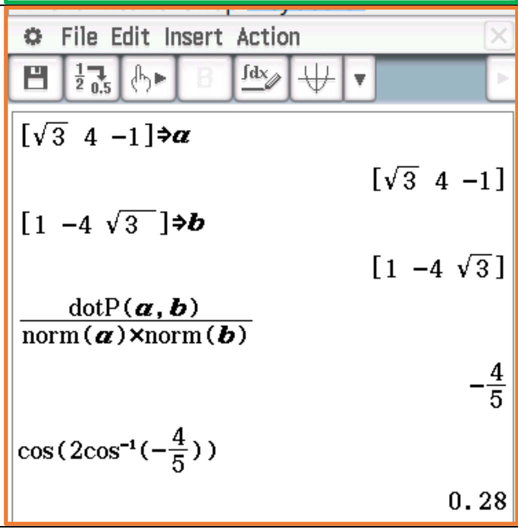
$|\underset{\sim}{b}|=\sqrt{1+16+3}=\sqrt{20}$

$\underset{\sim}{a}.\underset{\sim}{b}=\sqrt{3}-16-\sqrt{3}=-16$

$$\cos(\theta)=\frac{\underset{\sim}{a}.\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}=\frac{-16}{20}=-\frac{4}{5}$$

$$\cos(2\theta)=2\cos^2(\theta)-1=2\left(-\frac{4}{5}\right)^2-1=\frac{7}{25}$$





Question 2 **Answer A**

$\underset{\sim}{a}=4\underset{\sim}{i}+m\underset{\sim}{j}-3\underset{\sim}{k}\quad,\quad \underset{\sim}{b}=-2\underset{\sim}{i}+n\,\underset{\sim}{j}-\underset{\sim}{k}$

$|\underset{\sim}{a}|=\sqrt{16+m^2+9}=\sqrt{25+m^2}=10$

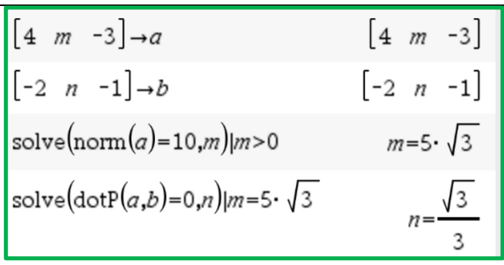
$m^2+25=100$

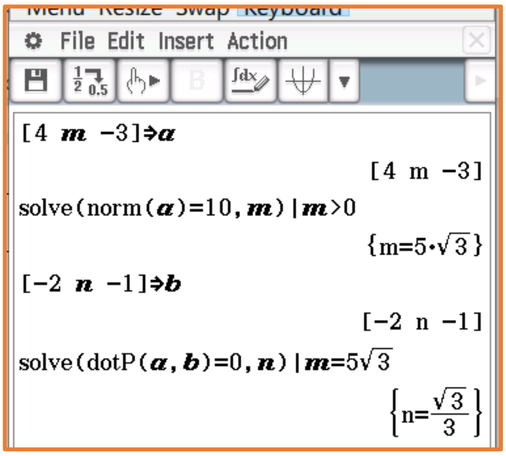
$m^2=75=25\times 3$

$m=5\sqrt{3}$ since $m>0$

$\underset{\sim}{a}.\underset{\sim}{b}=-8+mn+3=0\quad\Rightarrow mn=5$

$m=5\sqrt{3}$ and $n=\dfrac{\sqrt{3}}{3}$





Question 3 **Answer D**

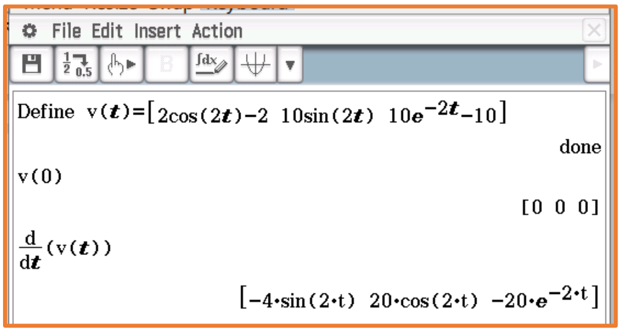
$$\underset{\sim}{a}(t) = -4\sin(2t)\underset{\sim}{i} + 20\cos(2t)\underset{\sim}{j} - 20e^{-2t}\underset{\sim}{k} \quad t \geq 0$$

$$\underset{\sim}{v}(t) = -\int 4\sin(2t)\,dt\,\underset{\sim}{i} + \int 20\cos(2t)\,dt\,\underset{\sim}{j} - \int 20e^{-2t}\,dt\,\underset{\sim}{k}$$

$$\underset{\sim}{v}(t) = 2\cos(2t)\underset{\sim}{i} + 10\sin(2t)\underset{\sim}{j} + 10e^{-2t}\underset{\sim}{k} + \underset{\sim}{c}$$

$$\underset{\sim}{v}(0) = \underset{\sim}{0} = 2\underset{\sim}{i} + 10\underset{\sim}{k} + \underset{\sim}{c} \;\Rightarrow\; \underset{\sim}{c} = -2\underset{\sim}{i} - 10\underset{\sim}{k}$$

$$\underset{\sim}{v}(t) = \left(2\cos(2t) - 2\right)\underset{\sim}{i} + 10\sin(2t)\underset{\sim}{j} + \left(10e^{-2t} - 10\right)\underset{\sim}{k}$$



Question 4 **Answer E**

$$\overrightarrow{OM} = a\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k} \;,\; \overrightarrow{ON} = -3\underset{\sim}{i} + b\underset{\sim}{j} - \underset{\sim}{k}$$

$$\overrightarrow{OD} = \overrightarrow{OM} + \overrightarrow{MD} = \overrightarrow{OM} + \frac{1}{2}\overrightarrow{MN}$$

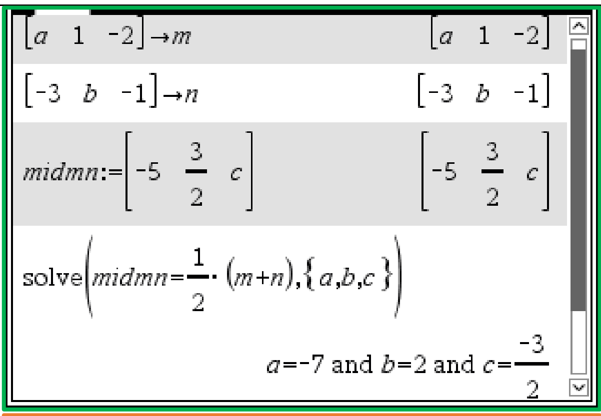
$$\overrightarrow{OD} = \overrightarrow{OM} + \frac{1}{2}\left(\overrightarrow{ON} - \overrightarrow{OM}\right) = \frac{1}{2}\left(\overrightarrow{ON} + \overrightarrow{OM}\right)$$

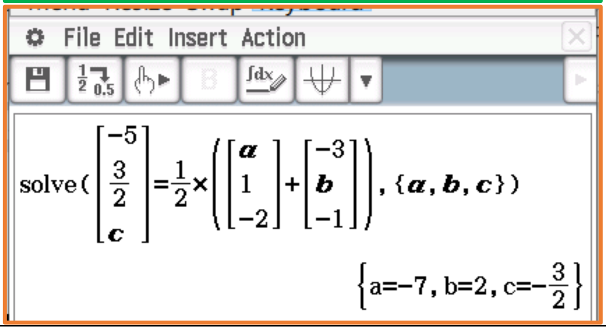
$$\overrightarrow{OD} = \left(\frac{a-3}{2}\right)\underset{\sim}{i} + \frac{(b+1)}{2}\underset{\sim}{j} - \frac{3}{2}\underset{\sim}{k} = -5\underset{\sim}{i} + \frac{3}{2}\underset{\sim}{j} + c\underset{\sim}{k}$$

$$\underset{\sim}{i}: \quad \frac{a-3}{2} = -5 \;,\; a = -7$$

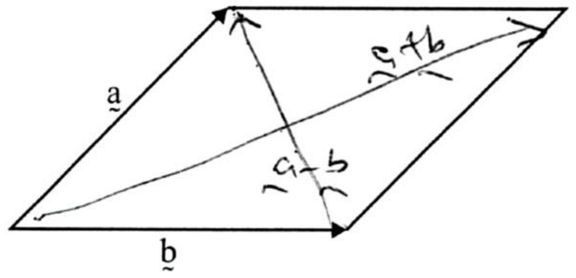
$$\underset{\sim}{j}: \quad \frac{b+1}{2} = \frac{3}{2} \;,\; b = 2$$

$$\underset{\sim}{k}: \quad c = -\frac{3}{2}$$





Question 5 **Answer C**



Since the diagonals of a rhombus are perpendicular,

$(\underset{\sim}{a}+\underset{\sim}{b}).(\underset{\sim}{a}-\underset{\sim}{b})=\underset{\sim}{a}.\underset{\sim}{a}+\underset{\sim}{a}.\underset{\sim}{b}-\underset{\sim}{a}.\underset{\sim}{b}-\underset{\sim}{b}.\underset{\sim}{b}=0$

$|\underset{\sim}{a}|^2-|\underset{\sim}{b}|^2=0 \Rightarrow |\underset{\sim}{a}|=|\underset{\sim}{b}|$ so the lengths are equal it is a rhombus.

Exam 2 Solutions

Question 1

$\underset{\sim}{r}(t)=6800\sin\left(\pi(1.3t-0.1)\right)\underset{\sim}{i}+\left(6800\cos\left(\pi(1.3t-0.1)\right)-6400\right)\underset{\sim}{j}\ ,\ t\in[0,0.154]$

a. $x(t)=6800\sin\left(\pi(1.3t-0.1)\right)$ and $y(t)=6800\cos\left(\pi(1.3t-0.1)\right)-6400$

$x(t)=0 \quad \sin(0)=0 \ , \ 1.3t-0.1=0$

$\Rightarrow \ t=\dfrac{0.1}{1.3} \qquad h=y\left(\dfrac{0.1}{1.3}\right)=400 \text{ km}$

b. Let $\theta=\pi(1.3t-0.1) \ \Rightarrow \ \dfrac{d\theta}{dt}=1.3\pi$

velocity $\underset{\sim}{v}(t)=1.3\times 6800\pi\cos(\theta)\underset{\sim}{i}-1.3\times 6800\pi\sin(\theta)\underset{\sim}{j}$

acceleration $\underset{\sim}{a}(t)=-1.3^2\times 6800\pi^2\sin(\theta)\underset{\sim}{i}-1.3^2\times 6800\pi^2\cos(\theta)\underset{\sim}{j}$

$\underset{\sim}{v}(t).\underset{\sim}{a}(t)=-1.3^3\times 6800^2\pi^3\sin(\theta)\cos(\theta)+1.3^3\times 6800^2\pi^3\sin(\theta)\cos(\theta)=0$

so $\underset{\sim}{v}(t)$ is perpendicular to $\underset{\sim}{a}(t)$

c. speed $=\left|\underset{\sim}{v}(t)\right|=\sqrt{\left(1.3\times 6800\pi\cos(\theta)\right)^2+\left(1.3\times 6800\pi\sin(\theta)\right)^2}$

speed $=\left|\underset{\sim}{v}(t)\right|=\sqrt{\left(1.3^2\times 6800^2\pi^2\left(\sin^2(\theta)+\cos^2(\theta)\right)\right)}=1.3\times 6800\pi=27771.7$

speed $=\left|\underset{\sim}{v}(t)\right|=27772$ km/hr

d. $x=6800\sin(\theta) \quad y=6800\cos(\theta)-6400$

$\sin(\theta)=\dfrac{x}{6800} \quad \cos(\theta)=\dfrac{y+6400}{6800}$

$\sin^2(\theta)+\cos^2(\theta)=1$

$x^2+(y+6400)^2=6800^2$

e. solving $\sqrt{x(t)^2+y(t)^2}=1000$ with $t\in[0,0.154]$

$t=0.04$ and 0.11 hours

Define $x1(t)=6800\cdot\sin(\pi\cdot(1.3\cdot t-0.1))$	*Done*
Define $y1(t)=6800\cdot\cos(\pi\cdot(1.3\cdot t-0.1))-6400$	*Done*
solve$(x1(t)=0,t)\vert 0<t<0.154$	$t=0.0769$
$y1(0.076923076923078)$	400.0000
norm$\left(\left[\dfrac{d}{dt}(x1(t)) \quad \dfrac{d}{dt}(y1(t))\right]\right)$	27771.6791
solve$\left(\sqrt{(x1(t))^2+(y1(t))^2}=1000,t\right)\vert 0<t<0.154$	$t=0.0429$ or $t=0.1110$

```
File Edit Insert Action
Define x(t)=6800sin(π(1.3t-0.1))
                                        done
Define y(t)=6800cos(π(1.3t-0.1))-6400
                                        done
solve(x(t)=0,t)|0<t<0.15
                           {t=0.07692307692}
y(0.07692307692)
                                         400
norm(d/dt(x(t)),d/dt(y(t)))
simplify(norm([d/dt(x(t))  d/dt(y(t))]))
                                 27771.67906
solve(norm([x(t)  y(t)])=1000,t)|0<t<0.154
              {t=0.04287819235,t=0.1109679615}
```

Complex Numbers

Exam 1 Questions (% indicates percentage of students obtaining full marks for the question)

Question 1 (Q2 VCAA SM E1 2018 1+2=3 marks 37%)

a. Show that $1+i=\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$.

b. Evaluate $\frac{\left(\sqrt{3}-i\right)^{10}}{(1+i)^{12}}$, giving your answer in the form $a+bi$ where $a,\ b\in R$.

Question 2 (Q3 VCAA SM E1 2014 2+3=5 marks 48%)

Let f be a function of a complex variable, defined by the rule, $f(z)=z^4-4z^3+7z^2-4z+6$.

a. Given that $z=i$ is a solution of $f(z)=0$, write down a quadratic factor of $f(z)$.

b. Given that the other quadratic factor of $f(z)$ has the form az^2+bz+c, find all solutions of $z^4-4z^3+7z^2-4z+6=0$ in cartesian form.

Question 3 (Q7 VCAA SM E1 2019 1+2+1+1=5 marks 26%)

a. Show that $3-\sqrt{3}i=2\sqrt{3}\,\text{cis}\left(-\frac{\pi}{6}\right)$

b. Find $\left(3-\sqrt{3}i\right)^3$, expressing your answer in the form $x+yi$, where $x,y\in R$.

c. Find the integer values of n for which $\left(3-\sqrt{3}i\right)^n$ is real.

d. Find the integer values of n for which $\left(3-\sqrt{3}i\right)^n=ai$, where a is a real number.

Question 4 (Q4 VCAA SM E1 2015 3+1=4 marks 40%)

a. Find all solutions of $z^3=8i$, $z\in C$ in cartesian form.

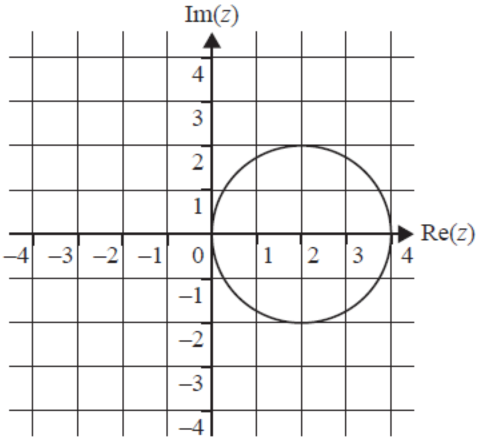
b. Find all solutions of $(z-2i)^3=8i$, $z\in C$ in cartesian form.

Exam 2 Multiple Choice Questions

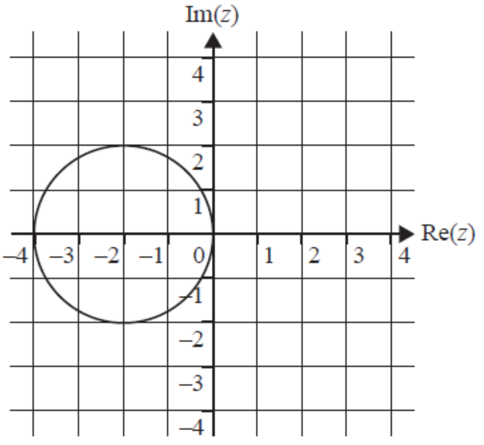
Question 1 (VCAA NHT 2018 Q5)

Which of the following graphs shows the set of points in the complex plane specified by the relation $\{ z : (z+2)(\overline{z}+2) = 4 \},\ z \in C$.

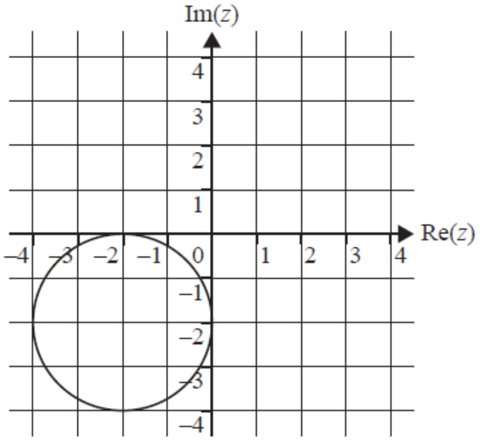
A.



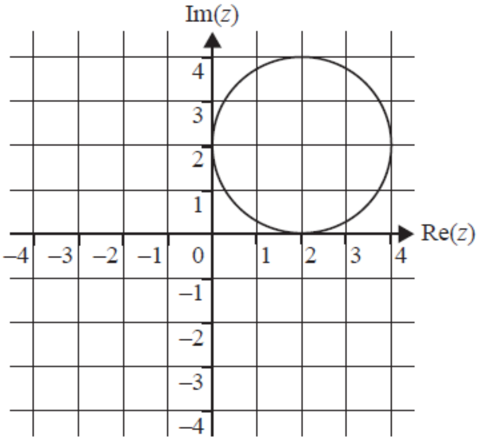
B.



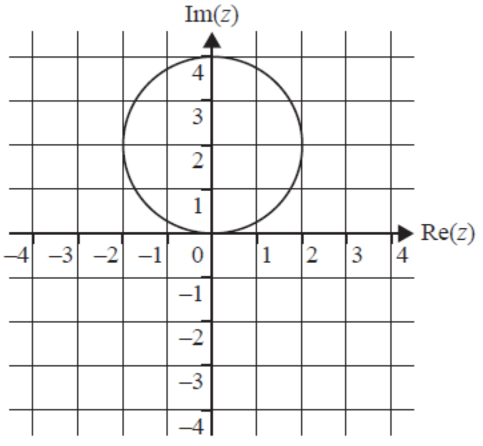
C.



D.



E.



Question 2 (VCAA NHT 2018 Q6)

Given that $(z-3i)$ is a factor of $P(z)=z^3+2z^2+9z+18$, which one of the following is **false?**

A. $P(3i)=0$

B. $P(-3i)=0$

C. $P(z)$ has three linear factor over C.

D. $P(z)$ has no real roots.

E. $P(z)$ has two complex conjugate roots

Question 3 (VCAA Q5 2012 74%)

If $z=\sqrt{2}\,\text{cis}\left(-\frac{4\pi}{5}\right)$ and $w=z^9$, then $w=$

A. $16\sqrt{2}\,\text{cis}\left(\frac{36\pi}{5}\right)$

B. $16\sqrt{2}\,\text{cis}\left(-\frac{\pi}{5}\right)$

C. $16\sqrt{2}\,\text{cis}\left(\frac{4\pi}{5}\right)$

D. $9\sqrt{2}\,\text{cis}\left(-\frac{\pi}{5}\right)$

E. $9\sqrt{2}\,\text{cis}\left(\frac{4\pi}{5}\right)$

Question 4 (VCAA Q6 2012 37%)

For any complex number, z, the location on an Argand diagram of the complex number $u=i^3\overline{z}$ can be found by

A. rotating z through $\frac{3\pi}{2}$ in an anticlockwise direction about the origin.

B. reflecting z about the x-axis and then reflecting about the y-axis.

C. reflecting z about the y-axis and then rotating anticlockwise through $\frac{\pi}{2}$ about the origin.

D. reflecting z about the x-axis and then rotating anticlockwise through $\frac{\pi}{2}$ about the origin.

E. rotating z through $\frac{3\pi}{2}$ in a clockwise direction about the origin.

Exam 2 Section B Questions

Question (Q2 VCAA NHT E2 2018 2+2+2+1+2+2=11 marks)

In the complex plane, L is the line given by $|z+1| = \left|z + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right|$.

a. Show that the cartesian equation of the line L is given by $y = -\frac{1}{\sqrt{3}}x$.

b. Find the point(s) of intersection of L and the graph of $z\,\overline{z} = 4$ in cartesian form.

c. Sketch L and the graph of the relation $z\,\overline{z} = 4$ on the Argand diagram below.

The part of the line L in the fourth quadrant can be expressed in the form $\text{Arg}(z) = \alpha$

d. State the value of α .

e. Find the area enclosed by L and the graphs of the relations $z\,\overline{z} = 4$, $\text{Arg}(z) = \frac{\pi}{3}$ and $\text{Re}(z) = \sqrt{3}$.

f. The straight line L can be written in the form $z = k\,\overline{z}$, where $k \in C$.

Find k in the form $r\,\text{cis}(\theta)$, where θ is the principal argument of k.

Solutions

Question 1

a. $u = 1 + i$

$|u| = \sqrt{1+1} = \sqrt{2} \quad \operatorname{Arg}(u) = \tan^{-1}(1) = \frac{\pi}{4} \;,\; u = 1 + i = \sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$

b. $v = \sqrt{3} - i$

$|v| = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = 2 \quad \operatorname{Arg}(v) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6} \;,\; v = \sqrt{3} - i = 2\operatorname{cis}\left(-\frac{\pi}{6}\right)$

$$\frac{\left(\sqrt{3}-i\right)^{10}}{(1+i)^{12}} = \frac{\left(2\operatorname{cis}\left(-\frac{\pi}{6}\right)\right)^{10}}{\left(\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)\right)^{12}} = \frac{2^{10}\operatorname{cis}\left(-\frac{5\pi}{3}\right)}{2^6\operatorname{cis}(3\pi)}$$

$$= 2^4\operatorname{cis}\left(-\frac{5\pi}{3} - 3\pi\right)$$

$$= 2^4\operatorname{cis}\left(-\frac{5\pi}{3} - 3\pi + 4\pi\right) = 16\operatorname{cis}\left(-\frac{2\pi}{3}\right)$$

$$= 16\cos\left(\frac{2\pi}{3}\right) - 16i\sin\left(\frac{2\pi}{3}\right)$$

$$= 16 \times -\frac{1}{2} - 16 \times \frac{\sqrt{3}}{2}$$

$$= -8 - 8\sqrt{3}\,i$$

Question 2

a. $f(z) = z^4 - 4z^3 + 7z^2 - 4z + 6$

by the conjugate root theorem

$f(i) = f(-i) = 0$

so $(z-i)(z+i) = z^2 - i^2 = z^2 + 1$ is a quadratic factor

b. $z^4 - 4z^3 + 7z^2 - 4z + 6 = 0$

$$\left(z^2+1\right)\left(z^2-4z+6\right) = 0$$

$$\left(z^2+1\right)\left(z^2-4z+4+2\right) = 0$$

$$\left(z^2+1\right)\left((z-2)^2+2\right) = 0$$

$$\left(z^2-i^2\right)\left((z-2)^2-2i^2\right) = 0$$

$$(z+i)(z-i)\left(z-2+\sqrt{2}i\right)\left(z-2-\sqrt{2}i\right) = 0$$

$$z = \pm i \;,\; 2 \pm \sqrt{2}i$$

Question 3

a. $z = 3 - \sqrt{3}i$

$$|z| = \sqrt{(3)^2 + \left(-\sqrt{3}\right)^2} = \sqrt{12} = 2\sqrt{3}$$

$$\text{Arg}(z) = \tan^{-1}\left(\frac{-\sqrt{3}}{3}\right) = -\frac{\pi}{6}$$

$$z = 2\sqrt{3}\,\text{cis}\left(-\frac{\pi}{6}\right)$$

b. $z^3 = \left(3 - \sqrt{3}i\right)^3 = \left(2\sqrt{3}\,\text{cis}\left(-\frac{\pi}{6}\right)\right)^3$

$$z^3 = \left(2\sqrt{3}\right)^3 \text{cis}\left(-\frac{3\pi}{6}\right) = 24\sqrt{3}\,\text{cis}\left(-\frac{\pi}{2}\right)$$

$$z^3 = 24\sqrt{3}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right) = 0 - 24\sqrt{3}i$$

c. $z^n = \left(3 - \sqrt{3}i\right)^n = \left(2\sqrt{3}\,\text{cis}\left(-\frac{\pi}{6}\right)\right)^n = \left(2\sqrt{3}\right)^n \text{cis}\left(-\frac{n\pi}{6}\right)$

$$z^n = \left(2\sqrt{3}\right)^n \left(\cos\left(-\frac{n\pi}{6}\right) + i\sin\left(-\frac{n\pi}{6}\right)\right) = \left(2\sqrt{3}\right)^n \left(\cos\left(\frac{n\pi}{6}\right) - i\sin\left(\frac{n\pi}{6}\right)\right)$$

$$\text{Im}\left(\left(3 - \sqrt{3}i\right)^n\right) = \left(2\sqrt{3}\right)^n \sin\left(\frac{n\pi}{6}\right) = 0$$

$$\sin\left(\frac{n\pi}{6}\right) = 0\ ,\quad \frac{n\pi}{6} = k\pi$$

$$n = 6k\ ,\ k \in Z$$

d. $\left(3 - \sqrt{3}i\right)^n = ai\ ,\ a \in R$

$$\text{Re}\left(\left(3 - \sqrt{3}i\right)^n\right) = \left(2\sqrt{3}\right)^n \cos\left(\frac{n\pi}{6}\right) = 0$$

$$\cos\left(\frac{n\pi}{6}\right) = 0\ ,\quad \frac{n\pi}{6} = \frac{\pi}{2} + k\pi$$

$$n = 3 + 6k\ ,\ k \in Z$$

Question 4

a. $z^3 = 8i,\ z \in C$

$z^3 = 8\,\mathrm{cis}\left(\frac{\pi}{2} + 2k\pi\right)$

$z = 8^{\frac{1}{3}}\,\mathrm{cis}\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right)$

$k = 0,\ z = 2\,\mathrm{cis}\left(\frac{\pi}{6}\right) = \sqrt{3} + i$

$k = 1,\ z = 2\,\mathrm{cis}\left(\frac{5\pi}{6}\right) = -\sqrt{3} + i$

$k = -1,\ z = 2\,\mathrm{cis}\left(-\frac{\pi}{2}\right) = -2i$

b. $(z - 2i)^3 = 8i\ ,\ z \in C$

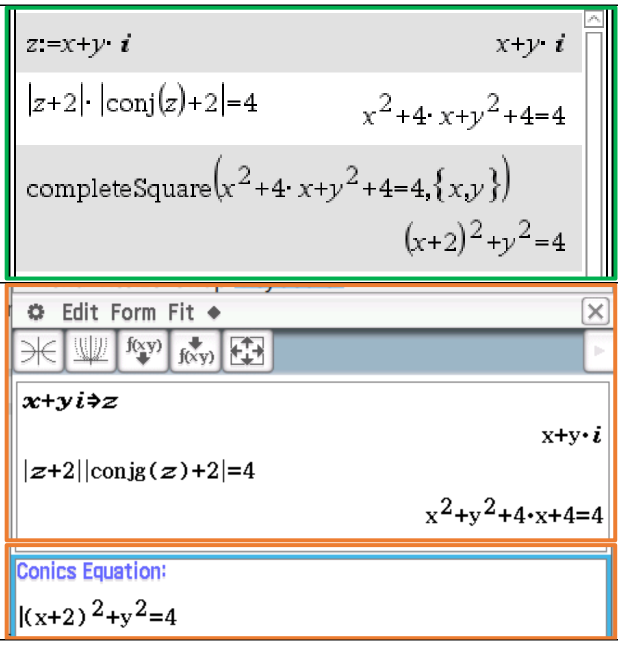
$z - 2i = \sqrt{3} + i\ ,\ -\sqrt{3} + i\ ,\ -2i$

$z = \sqrt{3} + 3i\ ,\ -\sqrt{3} + 3i\ ,\ 0$

MC Solutions

Question 1 **Answer B**

$$\left(z+2\right)\left(\overline{z}+2\right)=4$$
$$z\,\overline{z}+2z+2\overline{z}+4=4$$
$$\left(x+yi\right)\left(x-yi\right)+2\left(x+yi\right)+2\left(x-yi\right)=0$$
$$x^2-y^2i^2+4x=0$$
$$x^2+4x+4+y^2=4$$
$\left(x+2\right)^2+y^2=4$ a circle centre at $\left(-2,0\right)$ radius 2.

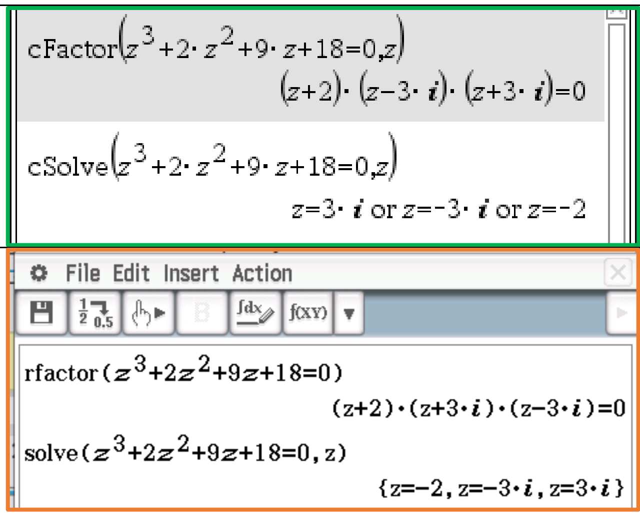


Question 2 **Answer D**

By the conjugate root theorem, the roots occur in conjugate pair, since the coefficients are all real given $z-3i$ is a factor so is $z+3i$,

$$P\left(z\right)=z^3+2z^2+9z+18$$
$$=\left(z+3i\right)\left(z-3i\right)\left(z+2\right)$$
$$=\left(z^2+9\right)\left(z+2\right)=0$$

so $P\left(3i\right)=P\left(-3i\right)=0$, $P\left(z\right)$ has three linear factors and two complex conjugate roots, it has one real root, **D** is false.



Question 3 **Answer C**

$$z=\sqrt{2}\,\text{cis}\left(-\frac{4\pi}{5}\right)$$
$$w=z^9=\left(\sqrt{2}\,\text{cis}\left(-\frac{4\pi}{5}\right)\right)^9=\left(\sqrt{2}\right)^9\left(-\frac{9\times 4\pi}{5}\right)$$
$$w=16\sqrt{2}\,\text{cis}\left(-\frac{36\pi}{5}\right)=16\sqrt{2}\,\text{cis}\left(-\frac{36\pi}{5}+8\pi\right)$$
$$w=16\sqrt{2}\,\text{cis}\left(\frac{4\pi}{5}\right)$$

Question 4 **Answer C**

$u = i^3\overline{z}$, $\overline{z}$ is a reflection in x-axis, i^3 is a rotation by $\frac{3\pi}{2}$ anticlockwise.

But none of the options have these, equivalently reflecting in the y-axis,

and then rotating $\frac{\pi}{2}$ anticlockwise about the origin.

$z = r\text{cis}(\theta)$, $\overline{z} = r\text{cis}(-\theta)$

$$i^3\overline{z} = -ir\text{cis}(-\theta) = -ir\left(\cos(-\theta) + i\sin(-\theta)\right) = -ir\left(\cos(\theta) - i\sin(\theta)\right) = r\left(-i\cos(\theta) + i^2\sin(\theta)\right)$$

$$i^3\overline{z} = -r\left(\sin(\theta) + i\cos(\theta)\right) = -r\left(\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right)$$

Exam 2 Solutions

Solution

a. $|z+1| = \left|z + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right|$, $z = x + yi$

$$|(x+1) + yi| = \left|\left(x + \frac{1}{2}\right) + \left(y - \frac{\sqrt{3}}{2}\right)i\right|$$

$$\sqrt{(x+1)^2 + y^2} = \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(y - \frac{\sqrt{3}}{2}\right)^2}$$

$$x^2 + 2x + 1 + y^2 = x^2 + x + \frac{1}{4} + y^2 - \sqrt{3}y + \frac{3}{4}$$

$$x = -\sqrt{3}\,y$$

$$y = -\frac{1}{\sqrt{3}}x$$

b. $z\overline{z} = 4$, $z = x + yi$, $\overline{z} = x - yi$

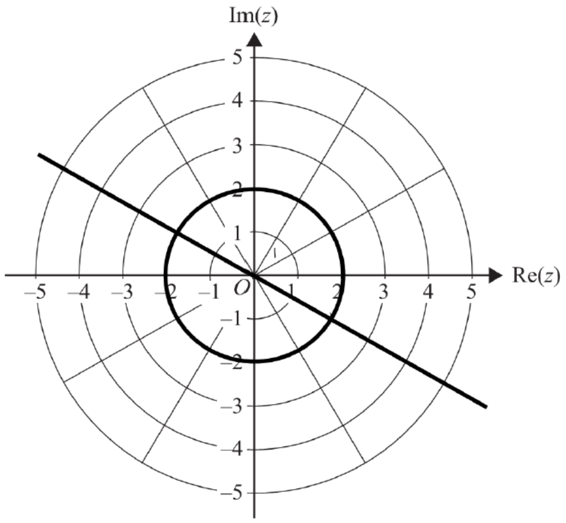
$z\overline{z} = x^2 - y^2i^2 = x^2 + y^2 = 4$ substitute $y = -\frac{1}{\sqrt{3}}x$

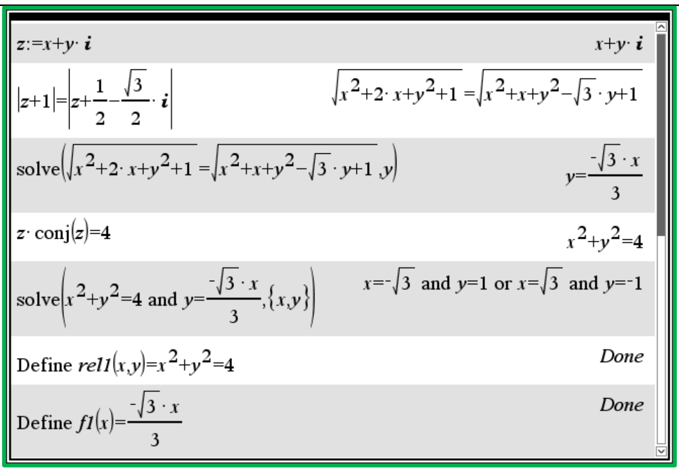
$$x^2 + \frac{x^2}{3} = \frac{4x^2}{3} = 4$$

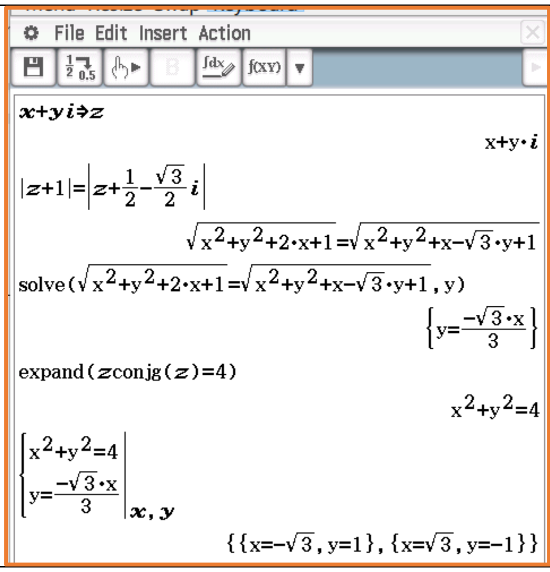
$$x = \pm\sqrt{3} \quad y = \mp 1$$

$$\left(\sqrt{3}, -1\right) , \left(-\sqrt{3}, 1\right)$$

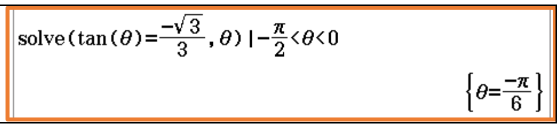
c.







d. $\alpha = -\dfrac{\pi}{6}$



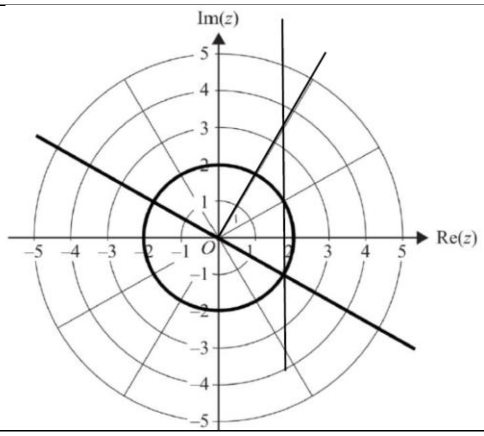
e. required area is the area of a triangle plus the area of a sector

$$A=\frac{1}{2}bc\sin(A)+\frac{1}{2}r^2\theta$$

$$b=c=2\ ,\quad A=60^0\ ,\quad r=2\ ,\quad \theta=\frac{\pi}{6}$$

$$A=\frac{1}{2}\times 2^2\sin\left(60^0\right)+\frac{1}{2}\times 2^2\times\frac{\pi}{6}$$

$$A=\sqrt{3}+\frac{\pi}{3}$$



f. $z = k\bar{z}$, $z = x + yi$

$$x + yi = k(x - yi) = kx - kiy$$

$$iy(k+1) = x(k-1)$$

$$y = \frac{x(k-1)}{i(k+1)} \times \frac{i}{i}$$

$$y = -\frac{xi(k-1)}{k+1} = -\frac{x}{\sqrt{3}}$$

$$\frac{i(k-1)}{k+1} = \frac{1}{\sqrt{3}}$$

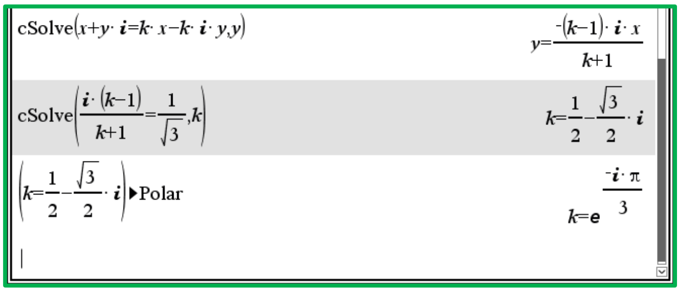
$$\sqrt{3}\,i(k-1) = k+1$$

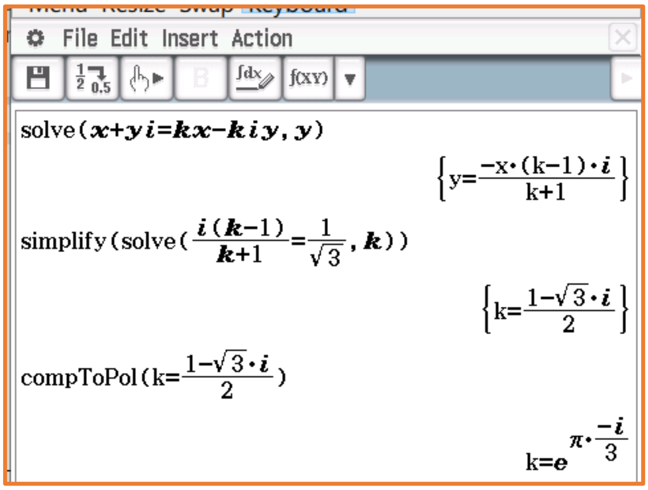
$$k\left(-1+\sqrt{3}\,i\right) = 1+\sqrt{3}\,i$$

$$k = \frac{1+\sqrt{3}\,i}{-1+\sqrt{3}\,i}$$

$$k = \frac{1+\sqrt{3}\,i}{-1+\sqrt{3}\,i} \times \frac{-1-\sqrt{3}\,i}{-1-\sqrt{3}\,i} = \frac{-1-3i^2-2\sqrt{3}\,i}{1-3i^2} = \frac{2-2\sqrt{3}\,i}{4}$$

$$k = \frac{1}{2} - \frac{\sqrt{3}}{2}i = 1\operatorname{cis}\left(-\frac{\pi}{3}\right) \;,\quad r = 1 \;,\; \theta = -\frac{\pi}{3}$$





Calculus

Exam 1 Questions (% indicates percentage of students obtaining full marks for the question)

Question 1 (Q7 VCAA SM E1 NHT 2019 5 marks)

Given that $3x^2 + 2xy + y^2 = 6$, find $\frac{d^2y}{dx^2}$ at the point $(1,1)$.

Question 2 (Q7 VCAA SM E1 2017 4 marks 30%)

The position vector of a particle moving along a curve at time t is given by $\underset{\sim}{r}(t) = \cos^3(t)\underset{\sim}{i} + \sin^3(t)\underset{\sim}{j}$,

$0 \le t \le \frac{\pi}{4}$.

Find the length of path that the particle travels along the curve from $t = 0$ to $t = \frac{\pi}{4}$.

Question 3 (Q2 VCAA SM E1 2013 4 marks 42%)

Evaluate $\int_0^1 \frac{x-5}{x^2 - 5x + 6}\,dx$

Question 4 (Q8 VCAA SM E1 NHT 2019 4 marks)

Find the length of the arc of the curve defined by $y = \frac{x^4}{4} + \frac{1}{8x^2} + 3$ from

$x = 1$ to $x = 2$. Give your answer in the form $\frac{a}{b}$ where a and b are positive integers.

Question 5 (Q1 VCAA SM E1 2012 2 marks 46%)

Find an antiderivative of $\frac{6+x}{x^2+4}$.

Exam 2 Multiple Choice Questions

Question 1 (VCAA 2018 Q3 46%)

Which of the following, where *A*, *B*, *C* and *D* are non-zero real numbers, is the partial fractions form for the expression $\dfrac{2x^2+3x+1}{(2x+1)^3(x^2-1)}$

A. $\dfrac{A}{2x+1}+\dfrac{B}{x-1}+\dfrac{C}{x+1}$

B. $\dfrac{A}{2x+1}+\dfrac{B}{(2x+1)^2}+\dfrac{C}{(2x+1)^3}+\dfrac{Dx}{x^2-1}$

C. $\dfrac{A}{2x+1}+\dfrac{Bx+C}{x^2-1}$

D. $\dfrac{A}{2x+1}+\dfrac{B}{(2x+1)^2}+\dfrac{C}{x-1}$

E. $\dfrac{A}{2x+1}+\dfrac{Bx+C}{(2x+1)^2}+\dfrac{D}{x-1}$

Question 2 (VCAA 2017 Q8 29%)

Let $f(x)=x^3-mx^2+4$, where $m,\ x\in R$.

The **gradient** of *f* will always be strictly increasing for

A. $x\geq 0$

B. $x\geq \dfrac{m}{3}$

C. $x\leq \dfrac{m}{3}$

D. $x\geq \dfrac{2m}{3}$

E. $x\leq \dfrac{2m}{3}$

Question 3 (VCAA 2017 Q7 60%)

With a suitable substitution $\int_1^2 x^2\sqrt{2-x}\,dx$ can be expressed as

A. $-\int_1^2 \left(4u^{\frac{1}{2}} - 4u^{\frac{3}{2}} + u^{\frac{5}{2}}\right) du$

B. $\int_1^2 \left(4u^{\frac{1}{2}} - 4u^{\frac{3}{2}} + u^{\frac{5}{2}}\right) du$

C. $\int_0^1 \left(-4u^{\frac{1}{2}} + 4u^{\frac{3}{2}} - u^{\frac{5}{2}}\right) du$

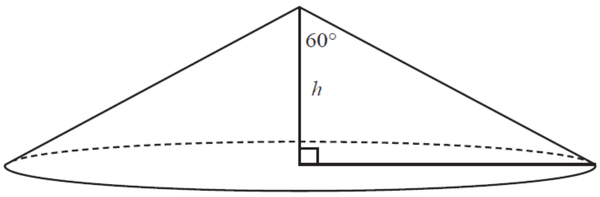
D. $-\int_1^0 \left(4u^{\frac{1}{2}} - 4u^{\frac{3}{2}} + u^{\frac{5}{2}}\right) du$

E. $\int_1^0 \left(4u^{\frac{1}{2}} - 4u^{\frac{3}{2}} + u^{\frac{5}{2}}\right) du$

Question 4 (VCAA 2019 Q10 58%)

Sand falls from a chute to form a pile in the shape of a right circular cone with semi vertex- angle 60^0. Sand is added to the pile at a rate of $1.5\,\text{m}^3$ per minute.

The rate at which the height *h* metres of the pile in increasing, in metres per minutes, when the height of the pile is 0.5m, correct to two decimal places, is

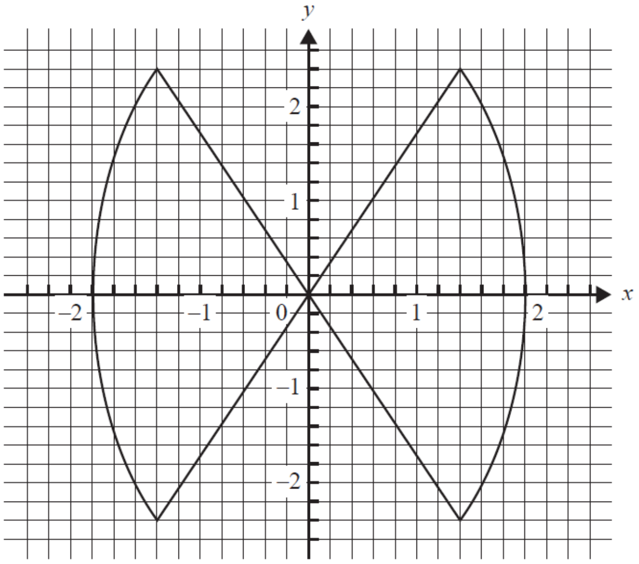


A. 0.21
B. 0.31
C. 0.64
D. 3.82
E. 3.53

Exam 2 Section B Questions

Question 1   (Q3 VCAA E2 2017 1+1+3+3+2=10 marks 16%)

A brooch is designed using inverse circular functions to make the shape shown in the diagram below.



The edges of the brooch in the first quadrant are described by the piecewise functions

$$f(x)=\begin{cases}3\arcsin\left(\dfrac{x}{2}\right), & 0\le x\le\sqrt{2}\\ 3\arccos\left(\dfrac{x}{2}\right), & \sqrt{2}\le x\le 2\end{cases}$$

a. Write down the coordinates of the corner point of the brooch in the first quadrant.

b. Specify the piecewise function that describes the edges of the brooch in the third quadrant.

c. Given that each unit in the diagram represents one centimetre, find the area of the brooch. Give your answer in square centimetres correct to one decimal place.

d. Find the acute angle between the edges of the brooch at the origin. Give your answer in degrees, correct to one decimal place.

e. The perimeter of the brooch has a border of gold. Show that the total length of the gold border needed is given by a definite integral of the form,

$$\int_0^2 \sqrt{a+\frac{b}{4-x^2}}\,dx \text{ where } a, b\in R.$$ Find the values of a and b.

Solutions

Question 1

$3x^2+2xy+y^2=6$ using implicit differentiation, with the product rule on the second term

$\frac{d}{dx}\left(3x^2\right)+\frac{d}{dx}\left(2xy\right)+\frac{d}{dx}\left(y^2\right)=\frac{d}{dx}(6) \Rightarrow \quad 6x+2y+2x\frac{dy}{dx}+2y\frac{dy}{dx}=0$ taking out a factor of 2

$(1) \quad 3x+y+x\frac{dy}{dx}+y\frac{dy}{dx}=0$

$3x+y=-(x+y)\frac{dy}{dx}$ at the point $(1,1)$ substitute $x=1$, $y=1$

$3+1=4=-2\frac{dy}{dx} \quad , \quad \frac{dy}{dx}=-2$

using implicit differentiation again on (1)

$\frac{d}{dx}(3x)+\frac{d}{dx}(y)+\frac{d}{dx}\left(x\frac{dy}{dx}\right)+\frac{d}{dx}\left(y\frac{dy}{dx}\right)=0$

using the product rule on the last two terms

$3+\frac{dy}{dx}+\frac{dy}{dx}+x\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}=0$

$3+2\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^2=-(x+y)\frac{d^2y}{dx^2}$

at the point $(1,1)$ substitute $x=1$, $y=1$ and $\frac{dy}{dx}=-2$

$3-4+4=-2\frac{d^2y}{dx^2}$

$\frac{d^2y}{dx^2}=-\frac{3}{2}$

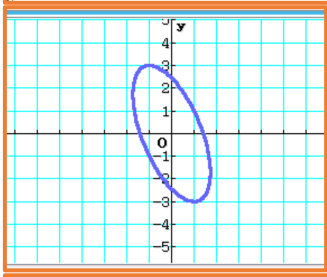
$\operatorname{impDif}\left(3\cdot x^2+2\cdot x\cdot y+y^2=6,x,y,2\right)$

$\frac{-2\cdot\left(3\cdot x^2+2\cdot x\cdot y+y^2\right)}{(x+y)^3}$

$\frac{-2\cdot\left(3\cdot x^2+2\cdot x\cdot y+y^2\right)}{(x+y)^3}|x=1 \text{ and } y=1 \qquad \frac{-3}{2}$

solve$(3x^2+2xy+y^2=6, y)$

$\{y=-x-\sqrt{-2\cdot(x^2-3)}, y=-x+\sqrt{-2\cdot(x^2-3)}\}$



$\frac{d^2}{dx^2}\left(-x+\sqrt{-2\cdot(x^2-3)}\right)|x=1|y=1$

$-\frac{3}{2}$

Question 2

$\underset{\sim}{r}(t)=\cos^3(t)\underset{\sim}{i}+\sin^3(t)\underset{\sim}{j}, \ 0\le t\le\frac{\pi}{4}, \quad s=\int_a^b\sqrt{\dot{x}^2+\dot{y}^2}\,dt$

$x=x(t)=\cos^3(t) \quad , \quad y=y(t)=\sin^3(t)$

$\dot{x}=\frac{dx}{dt}=-3\cos^2(t)\sin(t) \quad , \quad \dot{y}=\frac{dy}{dt}=3\sin^2(t)\cos(t)$

$\sqrt{\dot{x}^2+\dot{y}^2}=\sqrt{9\cos^4(t)\sin^2(t)+9\sin^4(t)\cos^2(t)}=\sqrt{9\cos^2(t)\sin^2(t)\left(\cos^2(t)+\sin^2(t)\right)}$

$\sqrt{\dot{x}^2+\dot{y}^2}=|3\sin(t)\cos(t)|$ but $0\le t\le\frac{\pi}{4}$ so $\sqrt{\dot{x}^2+\dot{y}^2}=\frac{3}{2}\sin(2t)$

$s=\int_0^{\frac{\pi}{4}}\frac{3}{2}\sin(2t)\,dt=\left[-\frac{3}{4}\cos(2t)\right]_0^{\frac{\pi}{4}}=-\frac{3}{4}\cos\left(\frac{\pi}{2}\right)+\frac{3}{4}\cos(0)$

$s=\frac{3}{4}$ units

Question 3

by partial fractions

$$\frac{x-5}{x^2-5x+6}=\frac{A}{x-3}+\frac{B}{x-2}=\frac{A(x-2)+B(x-3)}{(x-3)(x-2)}=\frac{x(A+B)-2A-3B}{x^2-5x+6}$$

$x-5=A(x-2)+B(x-3)$ alternatively

let $x=2 \quad -3=-B \quad \Rightarrow B=3$ $\quad (1)\ A+B=1 \quad (2)\ -2A-3B=-5$

let $x=3 \quad -2=A \quad \Rightarrow A=-2$ $\quad 2\times(1)+(2) \Rightarrow B=3\ ,\ A=-2$

$$\int_0^1 \frac{x-5}{x^2-5x+6}\,dx=\int_0^1\left(\frac{3}{x-2}-\frac{2}{x-3}\right)dx$$

$$=\left[3\log_e|x-2|-2\log_e|x-3|\right]_0^1$$

$$=\left(3\log_e|-1|-2\log_e|-2|\right)-\left(3\log_e|-2|-2\log_e|-3|\right)$$

$$=-2\log_e(2)-3\log_e(2)+2\log_e(3)$$

$$=\log_e\left(\frac{9}{4\times 8}\right)$$

$$=\log_e\left(\frac{9}{32}\right)$$

Question 4

$$s=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$y=\frac{x^4}{4}+\frac{1}{8x^2}+3=\frac{x^4}{4}+\frac{1}{8}x^{-2}+3 \quad , \quad \frac{dy}{dx}=x^3-\frac{1}{4}x^{-3}=x^3-\frac{1}{4x^3}$$

$$1+\left(\frac{dy}{dx}\right)^2=1+\left(x^3-\frac{1}{4x^3}\right)^2=1+x^6-\frac{1}{2}+\frac{1}{16x^6}=x^6+\frac{1}{2}+\frac{1}{16x^6}=\left(x^3+\frac{1}{4x^3}\right)^2$$

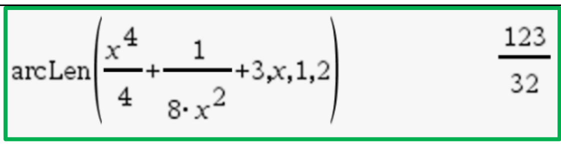
$$\sqrt{\left(1+\left(\frac{dy}{dx}\right)^2\right)}=x^3+\frac{1}{4x^3}$$

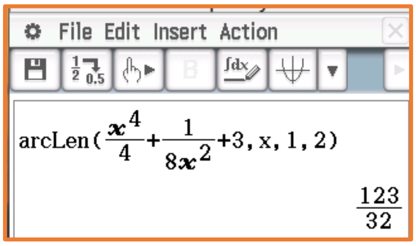
$$s=\int_1^2\left(x^3+\frac{1}{4x^3}\right)dx=\int_1^2\left(x^3+\frac{1}{4}x^{-3}\right)dx$$

$$s=\left[\frac{x^4}{4}-\frac{1}{8}x^{-2}\right]_1^2=\left[\frac{x^4}{4}-\frac{1}{8x^2}\right]_1^2$$

$$s=\left(4-\frac{1}{32}\right)-\left(\frac{1}{4}-\frac{1}{8}\right)=4-\left(\frac{1+8-4}{32}\right)=4-\frac{5}{32}=\frac{128-5}{32}$$

$$s=\frac{123}{32}$$





Question 5

$$\int \frac{6+x}{x^2+4}\,dx = \int \frac{6}{x^2+4}\,dx + \int \frac{x}{x^2+4}\,dx$$

$$= 3\tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{2}\log_e\left(x^2+4\right)$$

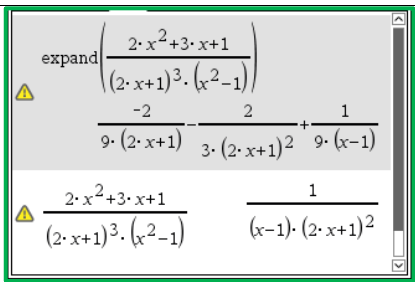
MC Solutions

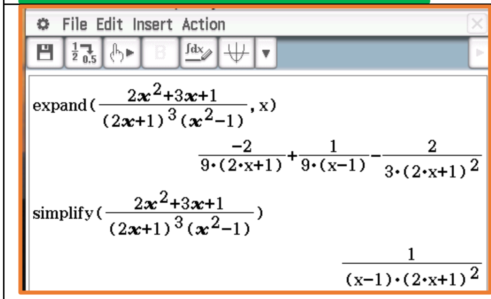
Question 1 Answer D

$$\frac{2x^2+3x+1}{(2x+1)^3\left(x^2-1\right)} = \frac{(2x+1)(x+1)}{(2x+1)^3(x+1)(x-1)}$$

$$= \frac{1}{(2x+1)^2(x-1)} \qquad x \neq \pm 1, -\frac{1}{2}$$

$$= \frac{A}{2x+1} + \frac{B}{(2x+1)^2} + \frac{C}{x-1}$$





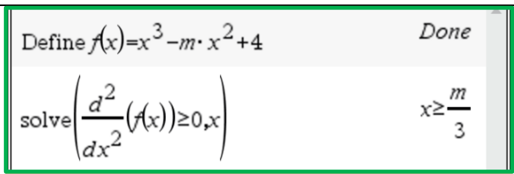
Question 2 Answer B

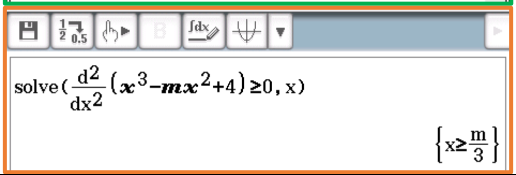
$$f(x) = x^3 - mx^2 + 4$$

$$f'(x) = 3x^2 - 2mx$$

$$f''(x) = 6x - 2m$$

$$f''(x) \geq 0 \quad \Rightarrow \quad x \geq \frac{m}{3}$$





Question 3 Answer D

$$I = \int_1^2 x^2\sqrt{2-x}\,dx$$

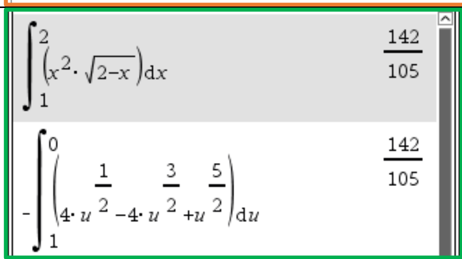
$$u = 2-x \quad \frac{du}{dx} = -1$$

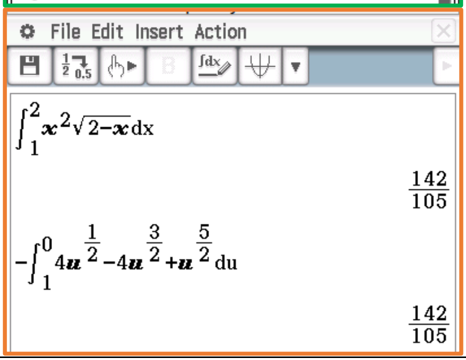
$$x = 2-u \ , \ x^2 = (2-u)^2 = 4-4u+u^2$$

$$x = 2 \Rightarrow u = 0 \ , \ x = 1 \Rightarrow u = 1$$

$$I = -\int_1^0 \left(4-4u+u^2\right)u^{\frac{1}{2}}\,du$$

$$I = -\int_1^0 \left(4u^{\frac{1}{2}} - 4u^{\frac{3}{2}} + u^{\frac{5}{2}}\right)du$$





Question 4 **Answer C**

given $\dfrac{dV}{dt} = 1.5 \text{ m}^3/\text{min}$

$\tan(60^0) = \dfrac{r}{h} = \sqrt{3}$, $r = \sqrt{3}\,h$ $V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi\left(\sqrt{3}h\right)^2 h = \pi h^3$

$\dfrac{dV}{dh} = 3\pi h^2$

$\dfrac{dh}{dt} = \dfrac{dh}{dV}\dfrac{dV}{dt} = \dfrac{1.5}{3\pi h^2}$, $\left.\dfrac{dh}{dt}\right|_{h=0.5} = \dfrac{1.5}{3\pi(0.5)^2} = 0.64\,\text{m/min}$

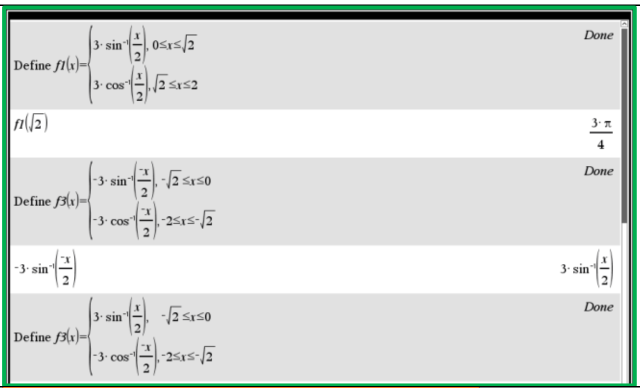
Exam 2 Solutions

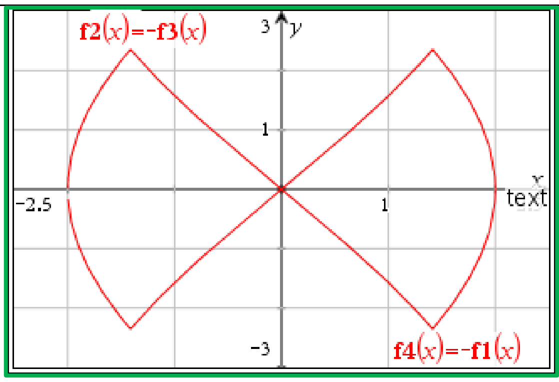
Question 1

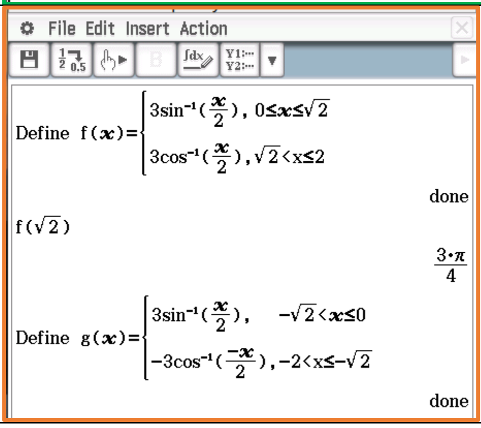
a. $f(x) = \begin{cases} 3\sin^{-1}\left(\dfrac{x}{2}\right), & 0 \le x \le \sqrt{2} \\ 3\cos^{-1}\left(\dfrac{x}{2}\right), & \sqrt{2} \le x \le 2 \end{cases}$

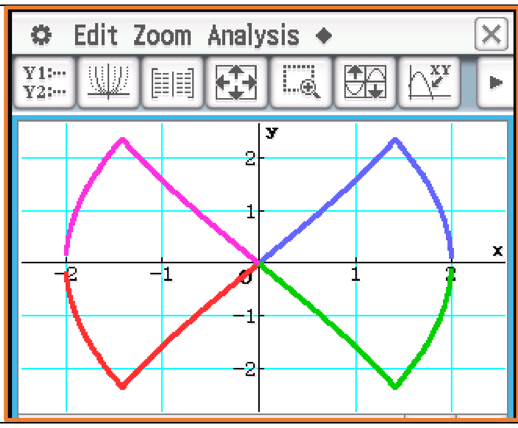
$f\left(\sqrt{2}\right) = 3\sin^{-1}\left(\dfrac{\sqrt{2}}{2}\right) = \dfrac{3\pi}{4}$

$\left(\sqrt{2}, \dfrac{3\pi}{4}\right)$









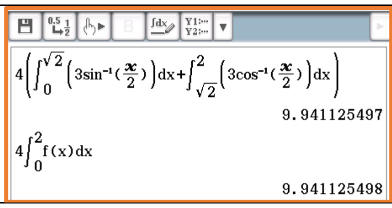
b. reflect in the x-axis and reflect in the y-axis.

$$f(x)=\begin{cases} 3\sin^{-1}\left(\dfrac{x}{2}\right), & -\sqrt{2}\le x\le 0 \\ -3\cos^{-1}\left(-\dfrac{x}{2}\right), & -2\le x\le \sqrt{2} \end{cases}$$

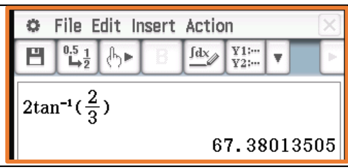
c. The total area of the brooch using symmetry is

$$A=4\left[\int_0^{\sqrt{2}} 3\sin^{-1}\left(\frac{x}{2}\right)dx+\int_{\sqrt{2}}^{2} 3\cos^{-1}\left(\frac{x}{2}\right)dx\right]$$

$$A=9.9\,\text{cm}^2$$



d. $y=3\sin^{-1}\left(\dfrac{x}{2}\right)$, $\dfrac{dy}{dx}=\dfrac{3}{\sqrt{4-x^2}}$ when $x=0$, $\left.\dfrac{dy}{dx}\right|_{x=0}=\dfrac{3}{2}=\tan(\theta)$ this is the angle with the x-axis. The acute angle between the brooch at the origin is $2\left(90^0-\tan^{-1}\left(\dfrac{3}{2}\right)\right)=2\tan^{-1}\left(\dfrac{2}{3}\right)=67.4^0$



e. $y_1=3\sin^{-1}\left(\dfrac{x}{2}\right)$ $\quad y_2=3\cos^{-1}\left(\dfrac{x}{2}\right)$

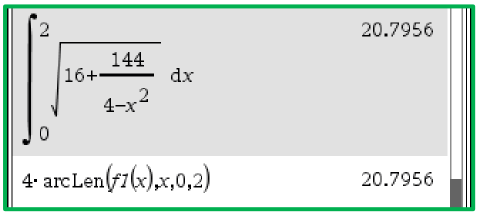
$$\frac{dy_1}{dx}=\frac{3}{\sqrt{4-x^2}}\qquad \frac{dy_2}{dx}=\frac{-3}{\sqrt{4-x^2}}$$

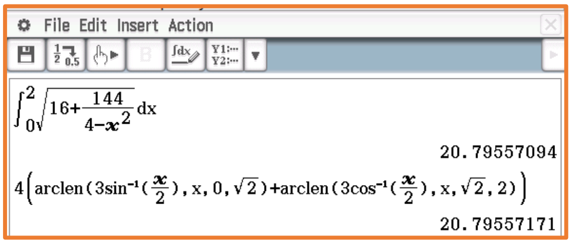
$$s=4\left(\int_0^{\sqrt{2}}\sqrt{1+\left(\frac{dy_1}{dx}\right)^2}\,dx+\int_{\sqrt{2}}^{2}\sqrt{1+\left(\frac{dy_2}{dx}\right)^2}\,dx\right)$$

by properties of definite integrals

$$s=4\int_0^2\sqrt{1+\left(\frac{3}{\sqrt{4-x^2}}\right)^2}\,dx=\sqrt{16}\int_0^2\sqrt{1+\left(\frac{9}{4-x^2}\right)}dx$$

$$s=\int_0^2\sqrt{16+\frac{144}{4-x^2}}\,dx\ ,\ a=16\ ,\ b=144$$





Differential Equations

Exam 1 Questions (% indicates percentage of students obtaining full marks for the question)

Question 1 (Q2 VCAA SM E1 2011 3 marks 49%)

Find the value of the real constant k, given that kxe^{2x} is a solution of the differential equation

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+5y=e^{2x}(15x+6)$$

Question 2 (Q10 VCAA SM E1 2016 5 marks 14%)

Solve the differential equation $\sqrt{2-x^2}\frac{dy}{dx}=\frac{1}{2-y}$, given that $y(1)=0$.

Express y as a function of x.

Question 3 (Q5 VCAA SM E1 2013 2+3=5 marks 41%)

A container of water is heated to boiling point, (100 °C) and then placed in a room that has a constant temperature of 20°C. After five minutes the temperature of the water is 80°C.

a. Use Newton's law of cooling, $\frac{dT}{dt}=-k(T-20)$, where T°C is the temperature of the water at a time t minutes, after the water is placed in the room, to show that $e^{-5k}=\frac{3}{4}$.

b. Find the temperature of the water 10 minutes after it is placed in the room.

Exam 2 Multiple Choice Questions

Question 1 (VCAA Q16 2011 25%)

The gradient of the normal to a curve at any point $P(x, y)$ is twice the gradient of the line joining P and the point $Q(1,1)$. The coordinates on the curve satisfy the differential equation

A. $\frac{dy}{dx}+\frac{x-1}{2(y-1)}=0$

B. $\frac{dy}{dx}-\frac{x-1}{2(y-1)}=0$

C. $\frac{dy}{dx}+\frac{2(y-1)}{x-1}=0$

D. $\frac{dy}{dx}+\frac{2(x-1)}{y-1}=0$

E. $\frac{dy}{dx}-\frac{2(y-1)}{x-1}=0$

Question 2 (VCAA Q17 2011 52%)

The differential equation which best represents the above direction field is

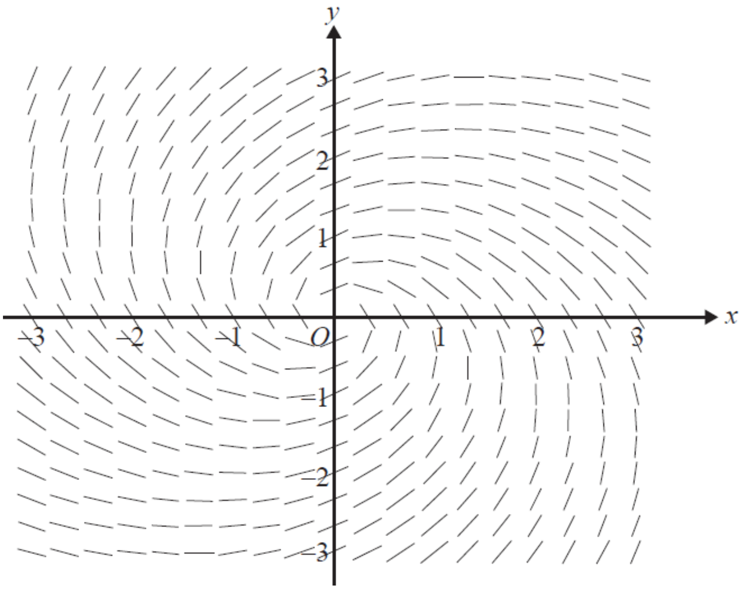
A. $\frac{dy}{dx}=\frac{y-2x}{2y+x}$

B. $\frac{dy}{dx}=\frac{2x-y}{y-2x}$

C. $\frac{dy}{dx}=\frac{2y-x}{y+2x}$

D. $\frac{dy}{dx}=\frac{y-2x}{2y-x}$

E. $\frac{dy}{dx}=\frac{x-2y}{2y+x}$



Question 3 (VCAA Q13 2013 57%)

Water container 2 grams of salt per litre flows at a rate of 10 litres per minute into a tank that initially contained 50 litres of pure water. The concentration of the salt is kept uniform by stirring and the mixture flows out of the tank at the rate of 6 litres per minute.

If Q grams is the amount of salt in the tank t minutes after the water begins to flow, the differential equation relating Q and t is

A. $\frac{dQ}{dt} = 20 - \frac{3Q}{25+2t}$

B. $\frac{dQ}{dt} = 10 - \frac{3Q}{25+2t}$

C. $\frac{dQ}{dt} = 20 - \frac{3Q}{25-2t}$

D. $\frac{dQ}{dt} = 10 - \frac{3Q}{25-2t}$

E. $\frac{dQ}{dt} = 20 - \frac{3Q}{25}$

Question 4 (VCAA Q9 2012 59%)

Euler's formula is used to find y_2 where $\frac{dy}{dx} = \cos(x)$, $x_0 = 0,\ y_0 = 1$ and $h = 0.1$.

The value of y_2 correct to four decimal places is

A. 1.1000 and this is an underestimate of $y(0.2)$

B. 1.1995 and this is an overestimate of $y(0.2)$

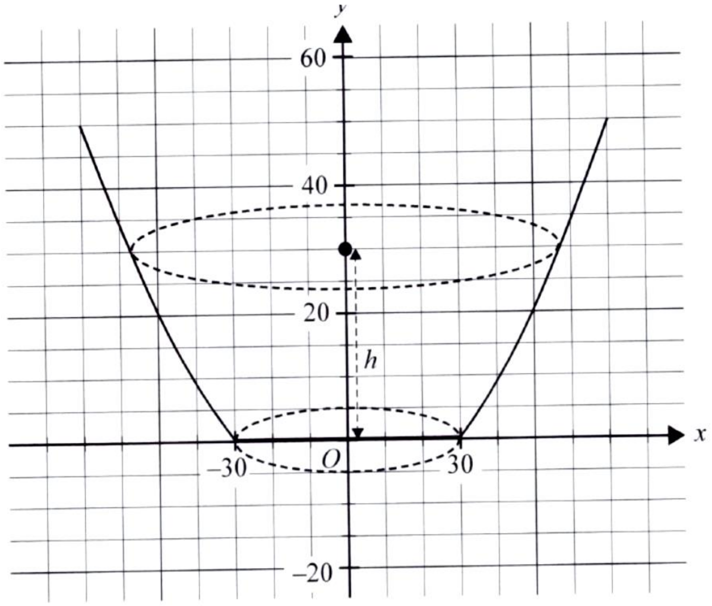
C. 1.1995 and this is an underestimate of $y(0.2)$

D. 1.2975 and this is an underestimate of $y(0.2)$

E. 1.2975 and this is an overestimate of $y(0.2)$

Exam 2 Questions

Question 1 (Q3 VCAA NHT E2 2019 1+1+2+3+2=9 marks)



The vertical cross-section of a barrel is shown above. The radius of the circular base (along the x-axis) is 30 cm and the radius of the circular top is 70 cm. The curved sides of the cross-sectional shown are parts of the parabola with rule $y=\frac{x^2}{80}-\frac{45}{4}$. The height of the barrel is 50 cm.

a.i. Show that the volume of the barrel is given by $V=\pi\int_0^{50}(900+80y)\,dy$

ii. Find the volume of the barrel in cubic centimetres.

The barrel is initially full of water. Water begins to leak from the bottom of the barrel such that $\frac{dV}{dt}=\frac{-8000\pi\sqrt{h}}{A}$ cubic centimetres per second, where after t seconds the depth of the water in the barrel is h centimetres, the volume of water remaining in the barrel is V cubic centimetres and the uppermost surface area of the water is A square centimetres.

b. Show that $\frac{dV}{dt}=\frac{-400\sqrt{h}}{4h+45}$.

c. Find $\frac{dh}{dt}$ in terms of h. Express your answer in the form $\frac{dh}{dt}=\frac{-a\sqrt{h}}{\pi(b+ch)^2}$ where, a, b and c are positive integers.

d. Using a definite integral in terms of h, find the **time, in hours**, correct to one decimal place for the barrel to empty.

Solutions

Question 1

$y = kxe^{2x}$ using the product rule

$$\frac{dy}{dx} = ke^{2x}\frac{d}{dx}(x) + kx\frac{d}{dx}\left(e^{2x}\right)$$

$$\frac{dy}{dx} = k\left(e^{2x} + 2xe^{2x}\right) = ke^{2x}(1+2x) \text{ using the product rule again}$$

$$\frac{d^2y}{dx^2} = k(1+2x)\frac{d}{dx}\left(e^{2x}\right) + ke^{2x}\frac{d}{dx}(1+2x)$$

$$\frac{d^2y}{dx^2} = 2ke^{2x}(1+2x) + 2ke^{2x} = ke^{2x}(4x+4)$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = ke^{2x}\left[(4x+4) - 2(1+2x) + 5x\right]$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = ke^{2x}(5x+2) = e^{2x}(15x+6)$$

$$\Rightarrow k = 3$$

Question 2

$$\sqrt{2-x^2}\frac{dy}{dx} = \frac{1}{2-y} \quad \text{using variables separable}$$

$$\int (2-y)\,dy = \int \frac{1}{\sqrt{2-x^2}}\,dx$$

$$2y - \frac{y^2}{2} = \sin^{-1}\left(\frac{x}{\sqrt{2}}\right) + c$$

when $y = 0$, $x = 1$ $\Rightarrow 0 = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + c \Rightarrow c = -\frac{\pi}{4}$

$$4y - y^2 = 2\sin^{-1}\left(\frac{x}{\sqrt{2}}\right) - \frac{\pi}{2}$$

$$y^2 - 4y + 4 = (y-2)^2 = 4 + \frac{\pi}{2} - 2\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)$$

$$y - 2 = \pm\sqrt{4 + \frac{\pi}{2} - 2\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)} \quad \text{since } y = 0 \text{ when } x = 1 \text{ take the negative}$$

$$y = 2 - \sqrt{4 + \frac{\pi}{2} - 2\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)}$$

Question 3

a. $\dfrac{dT}{dt} = -k(T-20)$

let $\theta = T - 20$ $\quad \dfrac{d\theta}{dt} = -k\theta \quad \Rightarrow \quad \theta = \theta_0 e^{-kt}$

when $t = 0 \quad T = 100 \quad \theta_0 = 80$

$t = 5 \quad T = 80 \quad \theta(5) = 60$

$60 = 80e^{-5k}$

$\Rightarrow e^{-5k} = \dfrac{60}{80} = \dfrac{3}{4}$ shown

b. when $t = 10 \quad \theta(10) = 80e^{-10k} = 80\left(e^{-5k}\right)^2$

$$= 80 \times \left(\frac{3}{4}\right)^2 = \frac{80 \times 9}{16} = 45$$

so the temperature of the water is $65^0 C$

MC Solutions

Question 1 Answer A

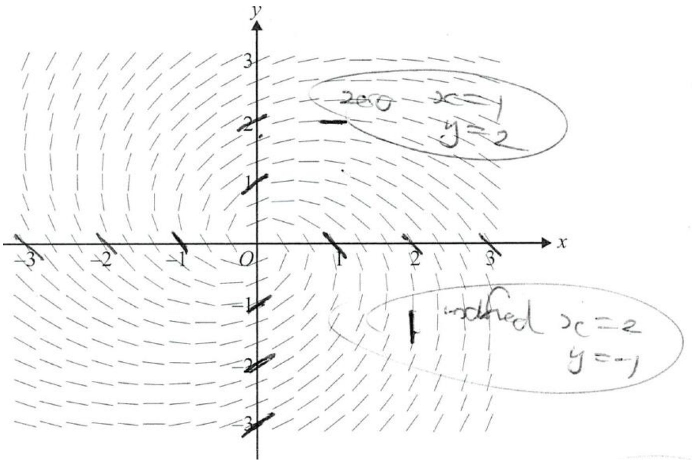
$$m_T=\frac{dy}{dx} \quad \text{normal is } m_N=-\frac{dx}{dy}=2\left(\frac{y-1}{x-1}\right)$$

$$\Rightarrow \frac{dy}{dx}=-\frac{x-1}{2(y-1)} \quad \text{or} \quad \frac{dy}{dx}+\frac{x-1}{2(y-1)}=0$$

Question 2 Answer A

at $x=0$ $y=-3,-2,-1,1,2,3$ slope is positive
at $y=0$ $x=-3,-2,-1,1,2,3$ slope is negative
at $x=1$ and $y=2$ slope is zero and at $x=2$ and $y=-1$ slope is undefined,

only option **A.** slope $=\dfrac{dy}{dx}=\dfrac{y-2x}{2y+x}$



Question 3 Answer A

$$\frac{dQ}{dt}=\text{inflow}-\text{outflow}$$

$$\frac{dQ}{dt}=10\times 2-\frac{6Q}{V(t)} \quad \text{where } V(t)=50+(10-6)t$$

$$\frac{dQ}{dt}=20-\frac{6Q}{50+4t}=20-\frac{3Q}{25+2t}$$

Question 4 Answer B

$$\frac{dy}{dx}=f(x)=\cos(x)$$

$h=0.1 \quad y_0=1 \quad x_0=0 \quad x_1=0.1$

$y_1=y_0+hf(x_0)=1+0.1\cos(0)=1.1$

$y_2=y_1+hf(x_1)=1.1+0.1\cos(0.1)=1.1995$

$\dfrac{dy}{dx}=\cos(x)$ exact

$y=\int\cos(x)\,dx=\sin(x)+c$

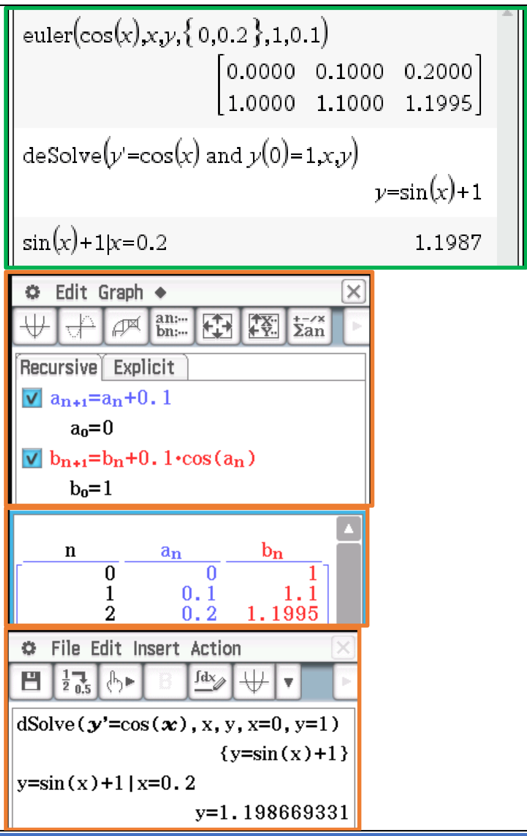
Now $y=1$ when $x=0 \Rightarrow c=1$

$y(x)=1+\sin(x)$

$y(0.2)=1+\sin(0.2)=1.1987$

Euler's method overestimates,

as the graph of $\sin(x)$ is concave down



Exam 2 Solutions

Question 1

a.i $y=\dfrac{x^2}{80}-\dfrac{45}{4}$ $\quad V=\pi\int_a^b x^2\,dy$

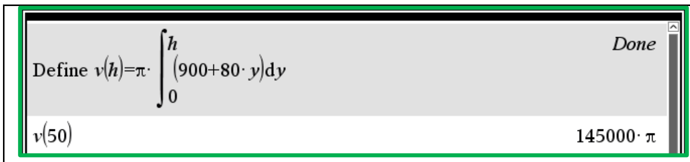
$$y+\frac{45}{4}=\frac{x^2}{80}\quad\Rightarrow\quad x^2=80y+900$$

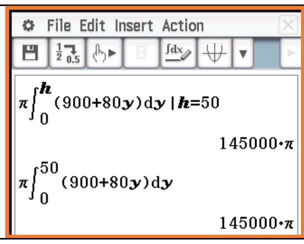
$$V=\pi\int_0^{50}\left(900+80y\right)dy$$

ii. $V=\pi\int_0^{50}\left(900+80y\right)dy$

$$V=\pi\left[900y+40y^2\right]_0^{50}=\pi\left(900\times50+40\times50^2\right)$$

$$V=145\,000\pi$$

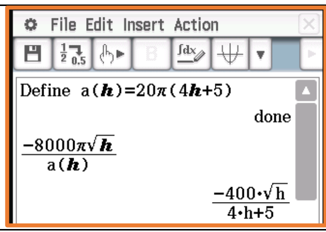




b. $A=\pi x^2=\pi\left(80h+900\right)=20\pi\left(4h+45\right)$

$$\frac{dV}{dt}=\frac{-8000\pi\sqrt{h}}{A(x)}=\frac{-8000\pi\sqrt{h}}{20\pi\left(4h+45\right)}$$

$$\frac{dV}{dt}=\frac{-400\sqrt{h}}{4h+45}$$



c. $V(h)=\pi\int_0^h\left(900+80y\right)dy \quad\Rightarrow\quad \dfrac{dV}{dh}=\pi\left(900+80h\right)$ using related rates

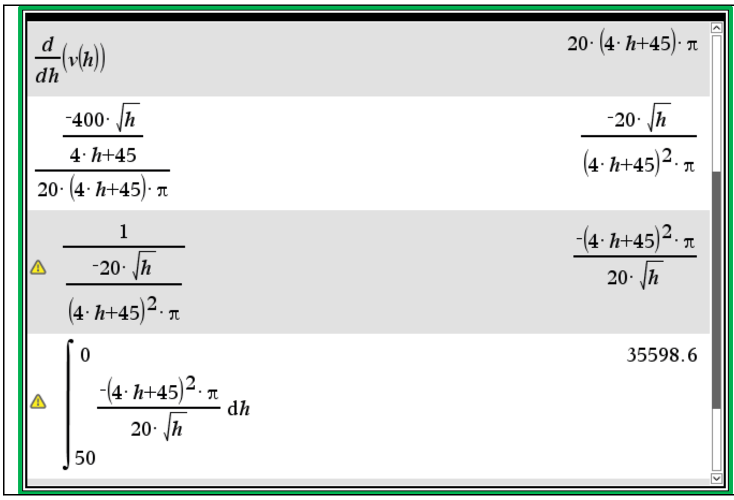
$$\frac{dh}{dt}=\frac{dh}{dV}\cdot\frac{dV}{dt}=\frac{-400\sqrt{h}}{4h+45}\times\frac{1}{\pi\left(900+80h\right)}$$

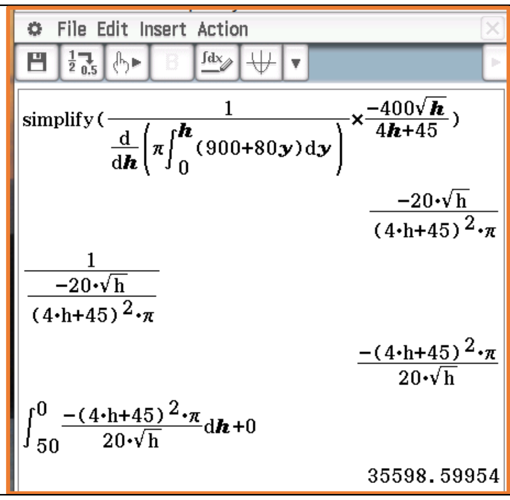
$$\frac{dh}{dt}=\frac{-20\sqrt{h}}{\pi\left(4h+45\right)^2}\quad,\ a=20\,,\ b=45\,,\ c=4$$

d. $\dfrac{dt}{dh} = \dfrac{-\pi(4h+45)^2}{20\sqrt{h}}$ inverting

$$T = \int_{50}^{0} \frac{-\pi(4h+45)^2}{20\sqrt{h}}\, dh = 35598.6 \text{ sec}$$

$$T = \frac{35598.6}{60 \times 60} = 9.9 \text{ hours}$$





Mechanics

Exam 1 Questions (% indicates percentage of students obtaining full marks for the question)

Question 1 (Q2 VCAA SM E1 2009 2+2=4 marks 52%)

A 50 kg student stands in a lift which accelerates downwards at a rate of 2ms^{-2}.

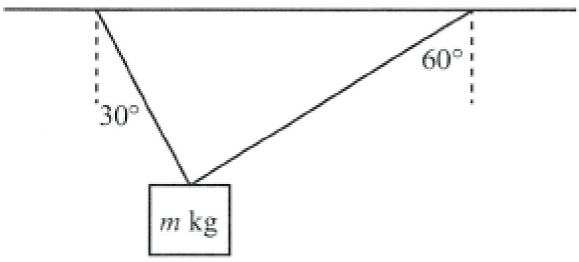
a. Find the reaction of the lift floor on the student.

b. A few minutes later the lift accelerates upwards at a rate of 2ms^{-2}.

Find the reaction of the lift floor on the student, during this stage of the motion.

Question 2 (Q7 VCAA SM E1 2011 1+3=4 marks 41%)

A flowerpot of mass m kg is held in equilibrium by two ropes, both of which are connected to a ceiling. The first rope makes an angle of 30^0 to the ceiling and has a tension T_1 newtons. The second makes an angle of 60^0 to the vertical and has a tension of T_2 newtons.



a. Show that $T_2 = \frac{T_1}{\sqrt{3}}$

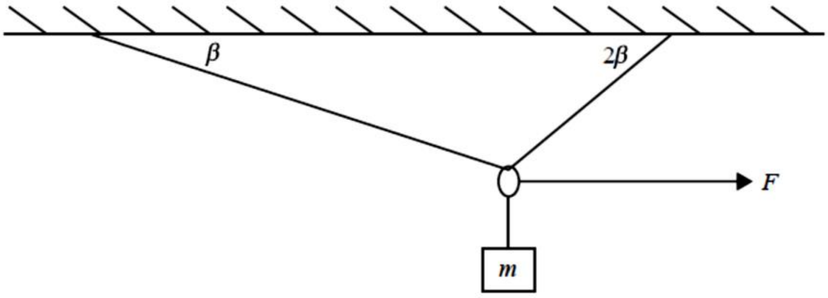
b. The first rope is strong, but the second rope will break if the tension in it exceeds 98 newtons. Find the maximum value of m for which the flowerpot remains in equilibrium.

Question 3 (Q6 VCAA SM E1 2015 4 marks 59%)

The acceleration $a\text{ ms}^{-2}$ of a body moving in a straight line in terms of the velocity $v\text{ ms}^{-1}$ is given by $a = 4v^2$. Given that $v = e$ when $x = 1$, where x is the displacement of the body in metres, find the velocity of the body when $x = 2$.

Question 4 (Q9b VCAA SM E1 2019 3 marks 10%)

A different light inextensible string is connected at each end to a horizontal ceiling. A mass of m kilograms hangs from a smooth ring on the string. A horizontal force of F newtons is applied to the ring. The tension in the string has a constant magnitude and the system is in equilibrium. At one end the string makes an angle of β with the ceiling and at the other end the string makes an angle of 2β with the ceiling, as shown in the diagram below.



Show that $F = mg\left(\dfrac{1-\cos(\beta)}{\sin(\beta)}\right)$.

Exam 2 Multiple Choice Questions

Question 1 (VCAA Q14 2019 36%)

A 4 kg mass is held at rest on a smooth surface. It is connected by a light inextensible string that passes over a smooth pulley to 2 kg mass which is in turn connected by the same type of string to a 1 kg mass. This is shown in the diagram below. When the 4 kg is released, the tension is the string connecting the 1 and 2 kg masses is T newtons. The value of T is

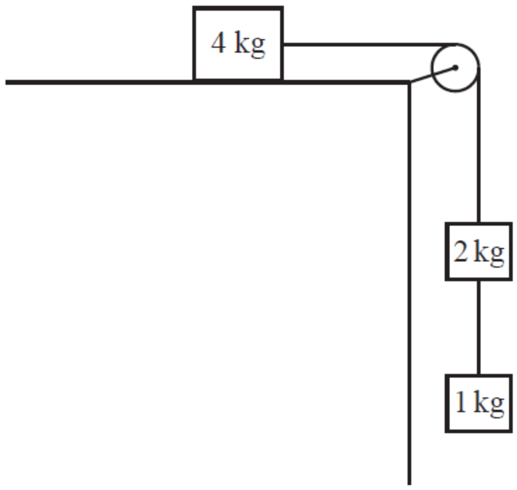
A. $\dfrac{4g}{7}$

B. $\dfrac{3g}{7}$

C. $\dfrac{g}{7}$

D. $\dfrac{6g}{7}$

E. g



Question 2 (VCAA Q21 2014 52%)

The acceleration in ms^{-2}, of a particle moving in a straight line is given by $-4x$, where x is the displacement from a fixed origin O. If the particle is at rest $x = 5$, then the speed of the particle, in ms^{-1}, where $x = 3$ is

A. 8

B. $8\sqrt{2}$

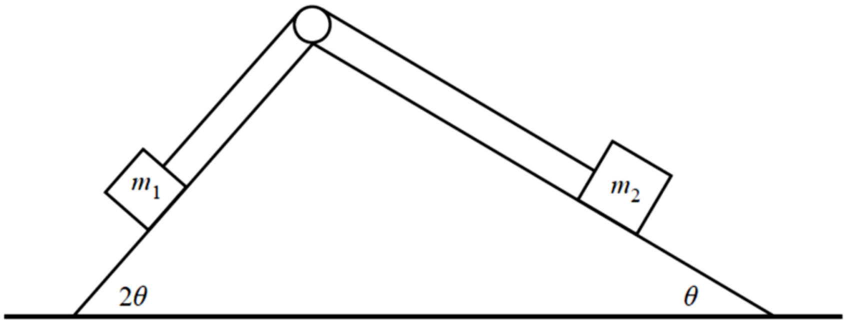
C. 12

D. $4\sqrt{2}$

E. $2\sqrt{34}$

Question 3 (VCAA Q14 2017 51%)

Two particles with masses m_1 kilograms and m_2 kilograms are connected by a taut light string that passes over a smooth pulley. The particles sit on smooth inclined planes, as shown in the diagram below.



If the system is in equilibrium, then $\frac{m_1}{m_2}$ is equal to

A. $\frac{\sec(\theta)}{2}$

B. $2\sec(\theta)$

C. $2\cos(\theta)$

D. $\frac{1}{2}$

E. 1

Question 4 (VCAA Q16 2017 58%)
An object of mass 20 kg, initially at rest, is pulled along a rough horizontal surface by a force of 80 N acting at an angle of 40^0 upwards from the horizontal. A frictional force of 20 N opposes the motion. After the pulling force has acted for 5 s, the magnitude of the momentum, in $\text{kg}\,\text{m}\,\text{s}^{-1}$, of the object is closest to

A. 10
B. 40
C. 160
D. 210
E. 4100

Question 5 (VCAA Q15 2016 58%)
A variable force of F newtons acts on a 3 kg mass so that it moves in a straight line.
At a time t seconds, $t \geq 0$, its velocity v metres per second and position x metres from the origin are given by $v = 3 - x^2$. It follows that

A. $F = -2x$
B. $F = -6x$
C. $F = 2x^3 - 6x$
D. $F = 6x^3 - 18x$
E. $F = 9x - 3x^3$

Question 6 (VCAA Q16 2016 46%)
A cricket ball is hit from the ground at an angle of 30^0 to the horizontal with a velocity of $20\ \text{m}\,\text{s}^{-1}$. The ball is subject to gravity and air resistance is negligible. Given that the field is level, the horizontal distance travelled by the ball, in metres, to the point of impact is

A. $\dfrac{10\sqrt{3}}{g}$
B. $\dfrac{20}{g}$
C. $\dfrac{100\sqrt{3}}{g}$
D. $\dfrac{200\sqrt{3}}{g}$
E. $\dfrac{400}{g}$

Question 7 (VCAA Q17 2016 63%)

A body of mass 3 kg is moving to the left in a straight line at $2\ \mathrm{m\,s^{-1}}$. It experiences a force for a period of time, after which it is then moving to the right at $2\ \mathrm{m\,s^{-1}}$. The change in the momentum of the particle in $\mathrm{kg\,m\,s^{-1}}$, in the direction of the final motion is

A. -6

B. 0

C. 4

D. 6

E. 12

Question 8 (VCAA Q22 2015 42%)

A ball is thrown vertically up with an initial velocity of $7\sqrt{6}\ \mathrm{m\,s^{-1}}$, and is subject to gravity and air resistance. The acceleration of the ball is given by $\ddot{x} = -\left(g + 0.1v^2\right)$, where x is its vertical displacement, and $v\ \mathrm{m\,s^{-1}}$ is its velocity at a time t seconds. The time taken for the ball to reach maximum height is

A. $\dfrac{\pi}{3}$

B. $\dfrac{5\pi}{21\sqrt{2}}$

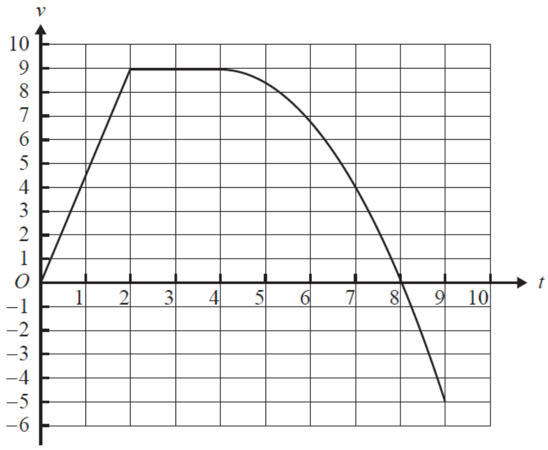
C. $\log_e(4)$

D. $\dfrac{10\pi}{21\sqrt{2}}$

E. $10\log_e(4)$

Question 9 (VCAA Q22 2014 49%)

The velocity-time graph shows the motion of a body travelling in a straight line, where $v\ \text{ms}^{-1}$ is its velocity after a time *t* seconds.



The velocity of the body over the time interval $t \in [4,9]$ is given by

$v(t) = -\frac{9}{16}(t-4)^2 + 9$. The total distance in metres, travelled by the body over the first nine seconds is closest to

A. 45.6

B. 47.5

C. 48.6

D. 51.0

E. 53.4

Question 10 (VCAA Q21 2010 63%)

A light inextensible string passes over a smooth, light pulley. A mass of 5 kg is attached to one end of the string and a mass of M kg is attached to the other end as shown.

The M kg mass accelerates downwards at $\frac{7}{5}\ \text{ms}^{-2}$. The value of M is

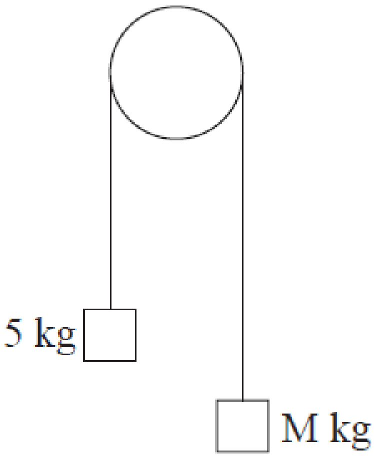
A. $5\frac{2}{7}$

B. $6\frac{2}{5}$

C. $6\frac{2}{3}$

D. $8\frac{2}{5}$

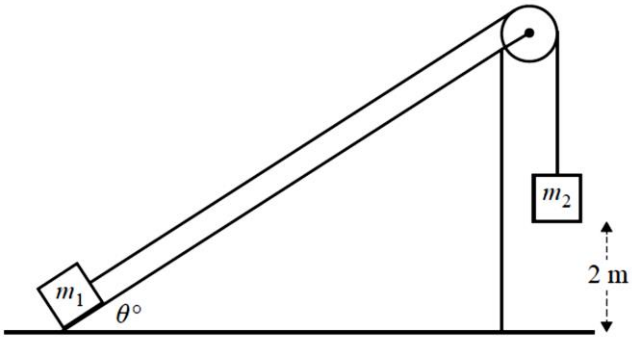
E. $9\frac{4}{5}$



Exam 2 Questions

Question 1 (Q5 VCAA E2 2019 1+1+1+2+5=10 marks 9%)

A mass of m_1 kilograms is initially held at rest near the bottom of a smooth plane inclined at an angle of θ degrees to the horizontal. It is connected to a mass of m_2 kilograms by a light inextensible string parallel to the plane, which passes over a smooth pulley at the end of the plane. The mass m_2 is 2 m above the horizontal floor. The situation is shown in the diagram below.



a. After the mass m_1 is released, the following forces, measured in newtons, act on the system

- weight forces W_1 and W_2
- the normal reaction N
- the tension in the string T

On the diagram, clearly label the forces acting on each of the masses.

b. If the system remains in equilibrium after the mass m_1 is released, show that $\sin(\theta) = \frac{m_2}{m_1}$.

c. After the mass m_1 is released, the mass m_2 falls to the floor.

i. For what values of θ will this occur? Express your answers as an inequality in terms of m_1 and m_2.

ii. Find the magnitude of the acceleration, in m s^{-2}, of the system after the mass m_1 is released and before the mass m_2 hits the floor. Express your answer in terms of m_1, m_2 and θ.

d. After the mass m_1 is released, it moves up the plane. Find the maximum distance, in metres, that the mass m_1 will move up the plane if $m_1 = 2m_2$ and $\sin(\theta) = \frac{1}{4}$.

Solutions

Question 1

a. Since the lift is moving down
the equation of motion is

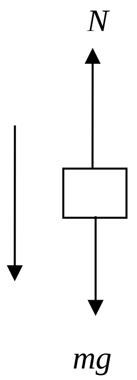
$ma = mg - N$

Now $m = 50$ kg and $a = 2\,\text{ms}^{-2}$

$N = mg - ma$

$N = 50 \text{ x } 9.8 - 50 \text{ x } 2 = 490 - 100$

$N = 390$ newtons



b. Since the lift is moving up
the equation of motion is

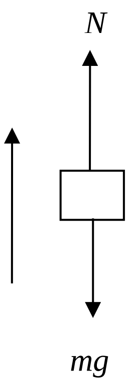
$ma = N - mg$

$N = ma + mg$

Now $m = 50$ kg and $a = 2\,\text{ms}^{-2}$

$N = 50 \text{ x } 2 + 50 \text{ x } 9.8 = 100 + 490$

$N = 590$ newtons



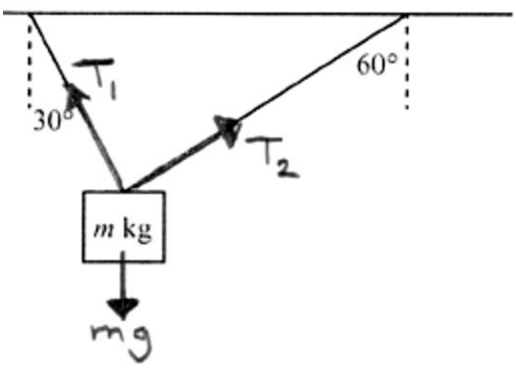
Question 2

a. resolving horizontally

$T_2 \sin\left(60^0\right) - T_1 \sin\left(30^0\right) = 0$

$T_2 \times \dfrac{\sqrt{3}}{2} = T_1 \times \dfrac{1}{2}$

$\Rightarrow\ T_2 = \dfrac{T_1}{\sqrt{3}}$



b. Now $T_2 = 98$ and $T_1 = \sqrt{3}\,T_2$

resolving vertically

$T_2 \cos\left(60^0\right) + T_1 \cos\left(30^0\right) - mg = 0$

$T_2 \cos\left(60^0\right) + \sqrt{3}T_2 \cos\left(30^0\right) = mg$

$T_2\left(\dfrac{1}{2} + \sqrt{3} \times \dfrac{\sqrt{3}}{2}\right) = 98 \times 2 = 20 \times 9.8 = mg$

$\Rightarrow\ m = 20$ kg

Question 3

$$a = v\frac{dv}{dx} = 4v^2$$

$$\Rightarrow \frac{dv}{dx} = 4v$$

$$\frac{dx}{dv} = \frac{1}{4v}$$

$$\Rightarrow x = \frac{1}{4}\int \frac{1}{v}\,dv = \frac{1}{4}\log_e(|v|) + c$$

when $x = 1 \quad v = e \qquad 1 = \frac{1}{4}\log_e(e) + c \quad \Rightarrow c = 1 - \frac{1}{4} = \frac{3}{4}$

$$x = \frac{1}{4}\log_e(|v|) + \frac{3}{4} \quad \Rightarrow \quad 4x - 3 = \log_e(|v|)$$

Now when $x = 2 \qquad \log_e(|v|) = 5$

$v = e^5 \ \text{ms}^{-1}$

Question 4

b. the tensions in the strings are equal

resolving horizontally

$(1) \quad T\cos(2\beta) - T\cos(\beta) + F = 0$

resolving vertically

$(2) \quad T\sin(2\beta) + T\sin(\beta) - mg = 0$

$(1) \quad F = T(\cos(\beta) - \cos(2\beta))$

$(2) \quad mg = T(\sin(2\beta) + \sin(\beta))$

$$\frac{F}{mg} = \frac{\cos(\beta) - \cos(2\beta)}{\sin(2\beta) + \sin(\beta)}$$

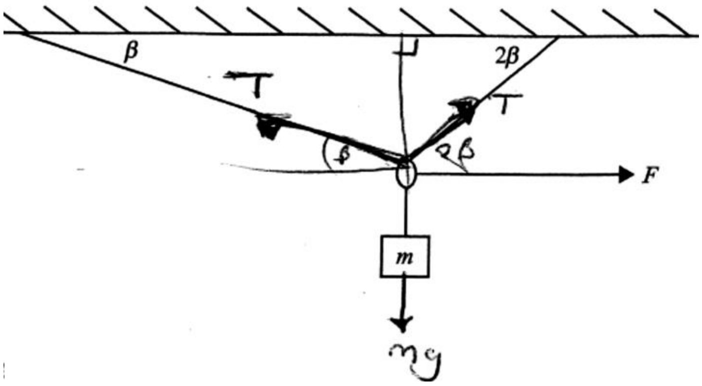
$$\frac{F}{mg} = \frac{\cos(\beta) - (2\cos^2(\beta) - 1)}{2\sin(\beta)\cos(\beta) + \sin(\beta)}$$

$$\frac{F}{mg} = \frac{-(2\cos^2(\beta) - \cos(\beta) - 1)}{\sin(\beta)(2\cos(\beta) + 1)}$$

$$\frac{F}{mg} = \frac{-(2\cos(\beta) + 1)(\cos(\beta) - 1)}{\sin(\beta)(2\cos(\beta) + 1)}$$

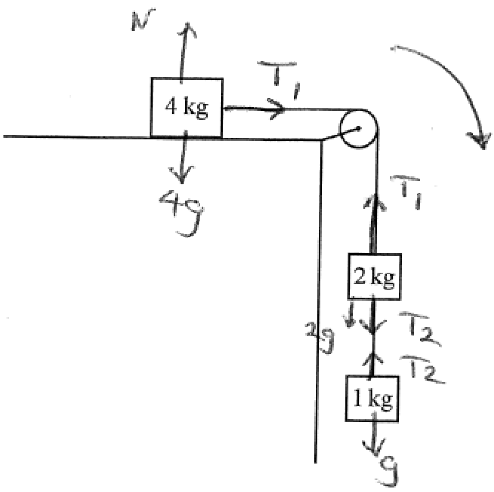
$$\frac{F}{mg} = \frac{1 - \cos(\beta)}{\sin(\beta)} \quad \text{since } \cos(\beta) \neq -\frac{1}{2}$$

$$F = mg\left(\frac{1 - \cos(\beta)}{\sin(\beta)}\right)$$



MC Solutions

Question 1 **Answer A**



resolving around 4 kg (1) $T_1 = 4a$

resolving around 2 kg (2) $2g + T_2 - T_1 = 2a$

resolving around 1 kg (3) $g - T_2 = a$

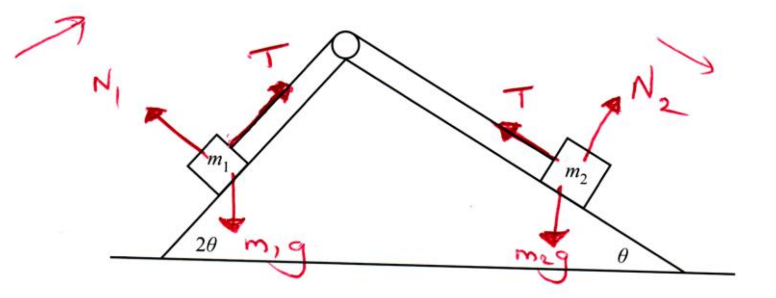
$(1)\ T_1 = 4a\ ,\ (3)\ a = g - T_2$ into (2)

$$2g + T_2 - 4(g - T_2) = 2(g - T_2)$$
$$-2g + 5T_2 = 2g - 2T_2$$
$$7T_2 = 4g$$
$$T_2 = \frac{4g}{7}$$

Question 2 **Answer A**

$$a = \frac{d\left(\frac{1}{2}v^2\right)}{dx} = -4x$$
$$\frac{1}{2}v^2 = \int -4x\,dx = -2x^2 + c$$
$$v = 0\ \ x = 5 \Rightarrow\ \ 0 = -50 + c\quad c = 50$$
$$v^2 = 100 - 4x^2\quad \text{when}\quad x = 3$$
$$v^2 = 100 - 4\times 9 = 64$$
$$|v| = 8$$

Question 3 **Answer A**



resolving around mass m_1

$$T - m_1 g\sin(2\theta) = 0$$

resolving around mass m_2

$$m_2 g\sin(\theta) - T = 0$$
$$T = m_1 g\sin(2\theta) = m_2 g\sin(\theta)$$
$$\frac{m_1}{m_2} = \frac{\sin(\theta)}{\sin(2\theta)} = \frac{\sin(\theta)}{2\sin(\theta)\cos(\theta)} = \frac{1}{2\cos(\theta)}$$
$$\frac{m_1}{m_2} = \frac{\sec(\theta)}{2}$$

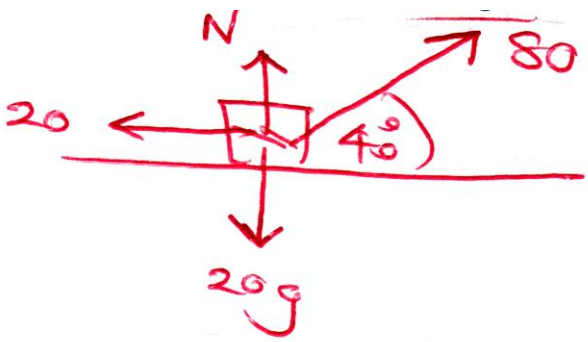
Question 4 **Answer D**

$80\cos\left(40^0\right)-20=20a$

$a=4\cos\left(40^0\right)-1\approx 2.065\ ,\ u=0\ ,\ t=5$

$v=u+at=5\times 2.065\approx 10.32$

$p=mv=20\times 10.32\approx 207$



Question 5 **Answer C**

$v=3-x^2\quad ,\quad m=3$

$\frac{dv}{dx}=-2x$

$F=ma=mv\frac{dv}{dx}$

$F=3\times -2x\left(3-x^2\right)$

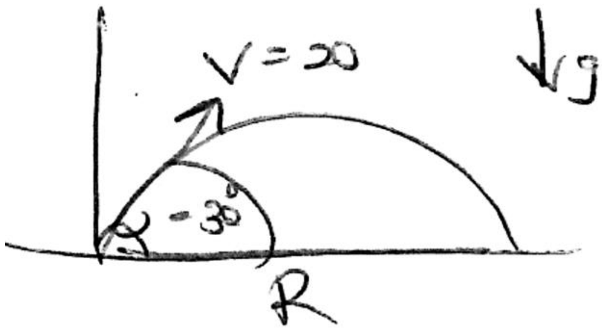
$F=-6x\left(3-x^2\right)$

$F=6x^3-18x$

Question 6 **Answer D**

$V=20\ \text{ms}^{-1}\ ,\ \alpha=30^0\ ,\ R=\frac{V^2\sin\left(2\alpha\right)}{g}$

$R=\frac{20^2\sin\left(60^0\right)}{g}=\frac{400\times\frac{\sqrt{3}}{2}}{g}=\frac{200\sqrt{3}}{g}$



Question 7 **Answer E**

$\left|\underset{\sim}{p}\right|=mv_f-mv_i=m\left(v_f-v_i\right)$

$=3\left(2-{}^{-}2\right)=12$

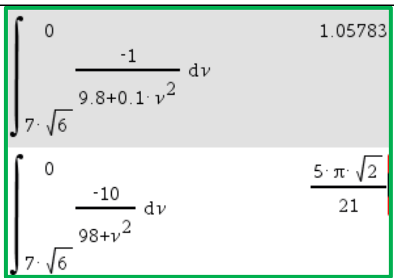
Question 8 **Answer D**

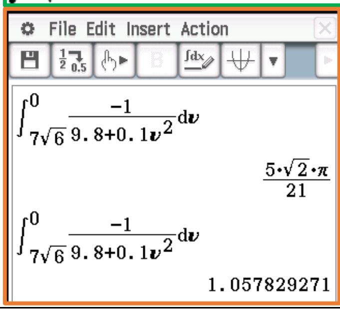
$\ddot{x}=\frac{dv}{dt}=-\left(9.8+0.1v^2\right)$

$\frac{dt}{dv}=\frac{-1}{9.8+0.1v^2}$

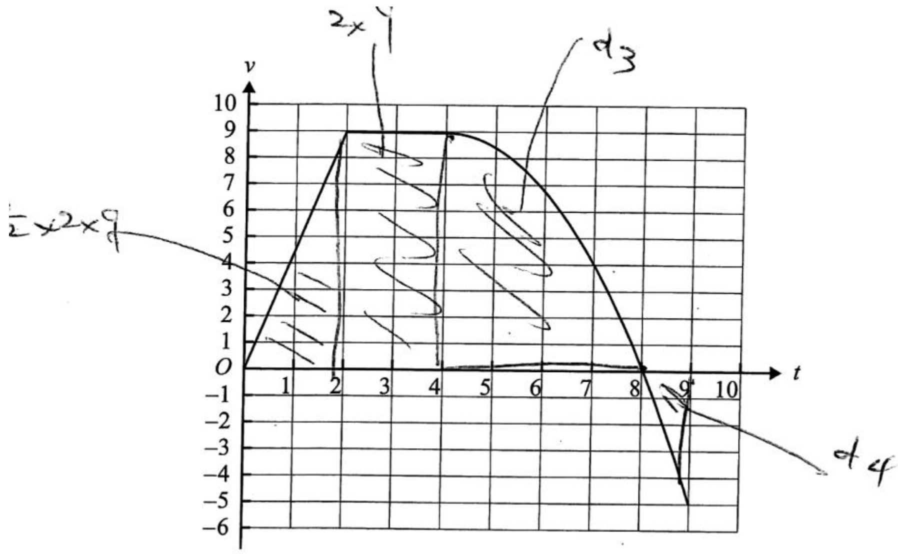
$t=\int_{7\sqrt{6}}^{0}\left(\frac{-10}{98+v^2}\right)dv$

$t=\frac{5\pi\sqrt{2}}{21}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{10\pi}{21\sqrt{2}}$



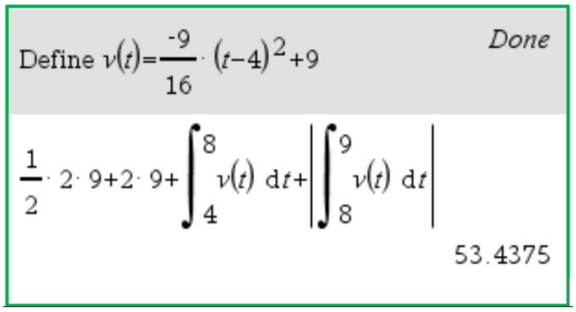


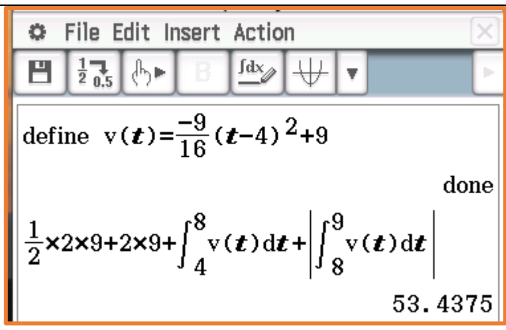
Question 9 **Answer E**



$$d=\frac{1}{2}\times 2\times 9+2\times 9+\int_4^8 v(t)\,dt+\left|\int_8^9 v(t)\,dt\right|$$

$$d=53.4375$$





Question 10 **Answer C**

(1) $Mg-T=Ma=\frac{7M}{5}$

(2) $T-5g=5a=5\times\left(\frac{7}{5}\right)=7$

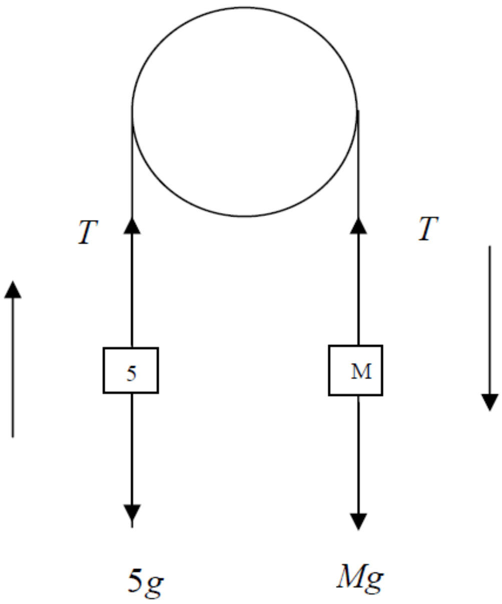
(2) $\Rightarrow$ $T=5g+7$ into (1)

$$Mg-(5g+7)=\frac{7M}{5}$$

$$M\left(g-\frac{7}{5}\right)=5g+7$$

$$M\left(\frac{5g-7}{5}\right)=5g+7$$

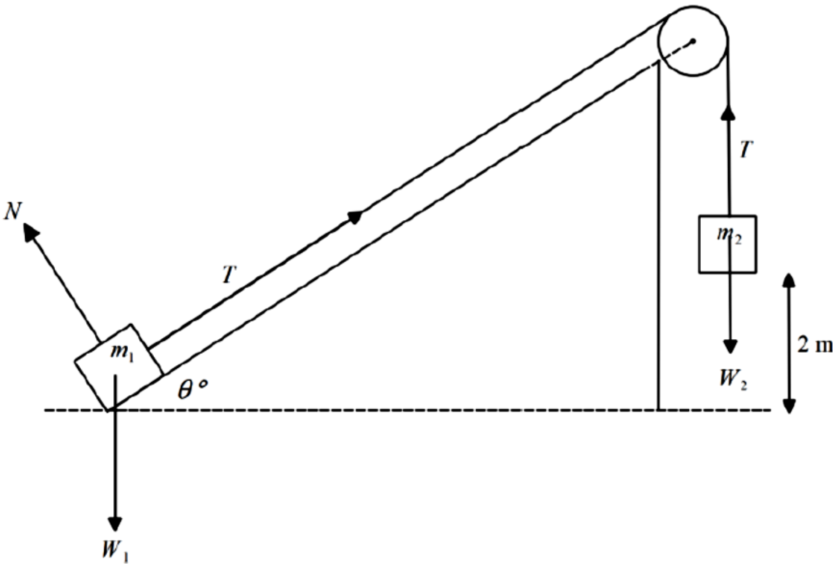
$$M=5\left(\frac{5g+7}{5g-7}\right)=\frac{5\times 56}{42}=\frac{5\times 4}{3}=6\frac{2}{3}\ \text{kg}$$



Exam 2 Solutions

Question 1

a.



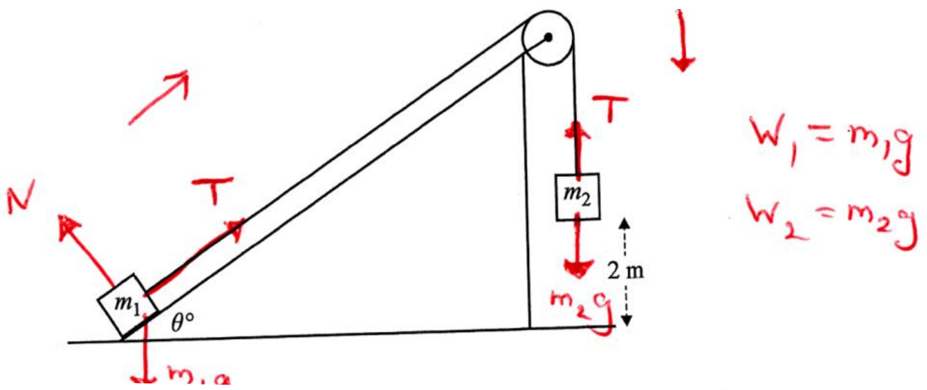
b. $W_1 = m_1 g \quad , \quad W_2 = m_2 g$

in equilibrium, resolving around mass m_2 $\quad (1) \quad m_2 g - T = 0$

resolving around mass m_1 $\quad (2) \quad T - m_1 g \sin(\theta) = 0$

$(1)+(2) \quad m_1 g \sin(\theta) = m_2 g \quad \Rightarrow \quad \sin(\theta) = \dfrac{m_2}{m_1}$

c.



motion with m_2 moving down

resolving around mass m_2 $\quad (1) \quad m_2 g - T = m_2 a$

resolving around mass m_1 $\quad (2) \quad T - m_1 g \sin(\theta) = m_1 a$

$(1)+(2) \quad m_2 g - m_1 g \sin(\theta) = (m_2 + m_2) a$

ii. $a = \dfrac{g\left(m_2 - m_1 \sin(\theta)\right)}{m_1 + m_2}$ $\quad a > 0 \quad , \quad m_2 - m_1 \sin(\theta) > 0$

i. $0 < \sin(\theta) < \dfrac{m_2}{m_1} \quad , \quad 0 < \theta < \sin^{-1}\left(\dfrac{m_2}{m_1}\right)$

d. the mass m_2 falls a distance of 2 metres, so the mass m_1 moves a distance up the plane while the particles are connected, at this instant, the string is slack and there is no more tension in the string.

given that $m_1 = 2m_2 \;,\; \sin(\theta) = \dfrac{1}{4}$ then

$$a = \frac{g\left(m_2 - m_1 \sin(\theta)\right)}{m_1 + m_2} = \frac{g\left(m_2 - 2m_2 \times \frac{1}{4}\right)}{2m_2 + m_2}$$

so $a = \dfrac{g}{6} \;,\; u = 0 \;,\; s = 2 \;,\; v^2 = u^2 + 2as \;,\; v = \sqrt{\dfrac{2g}{3}}$ is its speed when the string becomes slack, it now continues to move up, its acceleration now is $ma = -mg\sin(\theta) \;,\; a = -\dfrac{g}{4}$,

it moves up until it comes to rest, using $a = -\dfrac{g}{4} \;,\; u = \sqrt{\dfrac{2g}{3}} \;,\; v = 0 \;,\; s = ?$,

$v^2 = u^2 + 2as \;,\; 0 = \dfrac{2g}{3} - \dfrac{gs}{2} \;,\; s = \dfrac{4}{3}$,

overall, the mass m_1 will move a total distance of $3\dfrac{1}{3}$ metres up the plane.

Probability and Statistics

Exam 1 Questions (% indicates percentage of students obtaining full marks for the question)

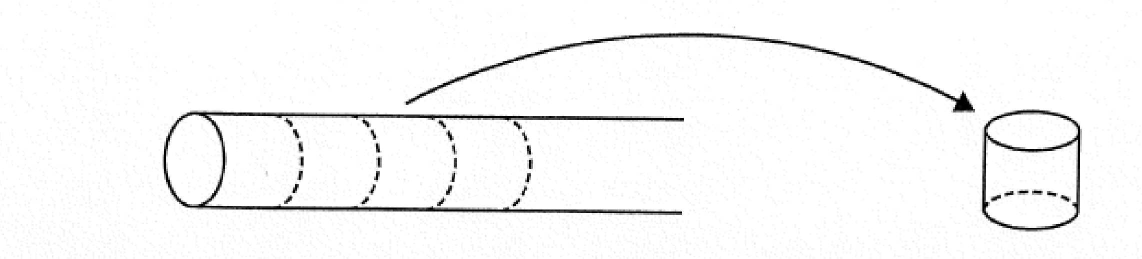
Question 1 (Q4 VCAA SM E1 2017 3 marks 37%)

The volume of soft drink dispensed by a machine into bottles varies normally with a mean of 298 mL and a standard deviation of 3 mL. The soft drink is sold in packs of four bottles.

Find the approximate probability that the mean volume of soft drink per bottle on a randomly selected four-bottle pack is less than 295 mL. Give your answer correct to three decimal places.

Question 2 (Q3 VCAA SM E1 2019 1+1+1=3 marks 30%)

A machine produces chocolate in the form of a continuous cylinder of radius 0.5 cm. Smaller cylindrical pieces that are cut parallel to its end, as shown in the diagram below.



The lengths of the pieces vary with a mean of 3 cm and a standard deviation of 0.1 cm.

a. Find the expected volume of a piece of chocolate in cm^3.

b. Find the variance of the volume of a piece of chocolate in cm^6.

c. Find the expected surface area of a piece of chocolate in cm^2.

Question 3 (Q4 VCAA SM E1 2018 4 marks 43%)

X and Y are independent variables. The mean and the variance of X are both 2, while the mean and variance of Y are 2 and 4 respectively.
Given that a and b are integers, find the values of a and b, if the mean and variance of $aX + bY$ are 10 and 44 respectively.

Exam 2 Multiple Choice Questions

Question 1 (VCAA 2017 Q18 42%)

U and V are independent normally distributed random variables, where U has a mean of 5 and a variance of 1, and V has a mean of 8 and a variance of 1. The random variable W is defined by $W = 4U - 3V$. In terms of the standard normal variable Z, $\Pr(W > 5)$ is equivalent to

A. $\Pr\left(Z > \dfrac{9\sqrt{7}}{7}\right)$

B. $\Pr(Z < 1.8)$

C. $\Pr\left(Z < \dfrac{9\sqrt{7}}{7}\right)$

D. $\Pr(Z > 0.2)$

E. $\Pr(Z > 1.8)$

Question 2 (VCAA 2018 Q18 62%)

A 95% confidence interval for the mean height μ , in centimetres, of a random sample of 36 Irish setter dogs, is $58.42 < \mu < 67.31$

The standard deviation of the height of the population of Irish setter dogs, in centimetres, correct to two decimal places, is

A. 2.26

B. 2.27

C. 13.60

D. 13.61

E. 62.87

Question 3 (VCAA 2016 Q18 61%)

Oranges grown on a citrus farm have a mean mass of 204 grams with a standard deviation of 9 grams.
Lemons grown on the same farm have a mean mass of 76 grams with a standard deviation of 3 grams.
The masses of lemons are independent of the masses of the oranges.
The mean mass and standard deviation, grams, respectively of a set of three of these oranges and two of these lemons are

A. 764, $3\sqrt{29}$

B. 636, 12

C. 764, $\sqrt{33}$

D. 636, $3\sqrt{10}$

E. 636, 33

Question 4 (VCAA 2016 Q20 68%)

The lifetime of a certain brand of batteries is normally distributed with a mean of 20 hours and a standard deviation of two hours. A random sample of 25 batteries is selected.
The probability that the mean lifetime of this sample of 25 batteries exceeds 19.3 hours is

A. 0.0401
B. 0.1368
C. 0.6103
D. 0.8632
E. 0.9599

Exam 2 Questions

Question 1 (Q6 VCAA 2018 1+1+2+1+1+1+1=8 marks 11%)

The heights of mature water buffaloes in northern Australia are known to be normally distributed with a standard deviation of 15 cm. It is claimed that the mean height of the water buffaloes is 150 cm.
To decide whether the claim about the mean height is true, rangers selected a random sample of 50 mature water buffaloes. The mean height of this sample was found to be 145 cm.
A one-tailed statistical test is to be carried out to see if the sample mean height of 145 cm differs significantly from the claimed population mean of 150 cm.
Let $\overline{X}$ denote the mean height of a random sample of 50 mature water buffaloes.

a. State suitable hypotheses H_0 and H_1 for the statistical test.

b. Find the standard deviation of $\overline{X}$.

c. Write down an expression for the *p* value of the statistical test and evaluate your answers correct to four decimal places.

d. State with reason whether H_0 should be rejected at the 5% level of significance.

e. What is the smallest value of the sample mean height that could be observed for H_0 to be **not** rejected? Give your answer in centimetres, correct to two decimal places.

f. If the true height of all mature water buffaloes in northern Australia is in fact 145 cm, what is the probability that H_0 will be accepted at the 5% level of significance?
Give you answer correct to two decimal places.

g. Using the observed sample mean of 145 cm, find a **99% confidence interval** for the mean height of all mature water buffaloes in northern Australia. Express the values in your confidence interval in centimetres, correct to one decimal place.

Solutions

Question 1

$$D \overset{d}{=} N\left(298, 3^2\right), \quad n = 4 \quad , \quad \bar{D} \overset{d}{=} N\left(298, \frac{3^2}{4}\right)$$

$$\Pr\left(\bar{D} < 295\right) = \Pr\left(Z < \frac{295-298}{\frac{3}{2}}\right) = \Pr\left(Z < -2\right) = 0.025$$

Question 2

radius $r = 0.5 = \frac{1}{2}$ constant, height $h \overset{d}{=} D\left(3.0, 0.1^2\right)$

a. $E(h) = 3$, $\text{Var}(h) = 0.1^2 = \left(\frac{1}{10}\right)^2 = 0.01$

$V = \pi r^2 h$, $E(V) = \pi r^2 E(h) = \pi\left(\frac{1}{2}\right)^2 \times 3 = \frac{3\pi}{4}$ cm^3

b. $\text{Var}(V) = \left(\pi r^2\right)^2 \text{Var}(h) = \left(\pi^2 \frac{1}{2}\right)^4 \times \left(\frac{1}{10}\right)^2 = \frac{\pi^2}{1600}$ cm^6.

c. $S = 2\pi r^2 + 2\pi r h$

$E(S) = 2\pi r^2 + 2\pi r\, E(h)$

$E(S) = 2\pi\left(\frac{1}{2}\right)^2 + 2\pi \times \left(\frac{1}{2}\right) \times 3 = \frac{7\pi}{2}$ cm^2

Question 3

$X \overset{d}{=} D(2,2)$ $\quad Y \overset{d}{=} D(2,4)$

$E(X) = 2$, $E(Y) = 2$, $\text{var}(X) = 2$, $\text{var}(Y) = 4$

$E(aX + bY) = aE(X) + bE(Y) = 2a + 2b = 10$

(1) $a + b = 5 \quad \Rightarrow \quad b = 5 - a$

$\text{Var}(aX + bY) = a^2\text{Var}(X) + b^2\text{Var}(Y) = 2a^2 + 4b^2 = 44$

(2) $a^2 + 2b^2 = 22$ from (1) substitute for b

$a^2 + 2(5-a)^2 = a^2 + 2\left(25 - 10a + a^2\right) = 22$

$3a^2 - 20a + 28 = (3a - 14)(a - 2) = 0$

$a = \frac{14}{3}$, 2 but $a \in Z$

$a = 2$, $b = 3$

MC Solutions

Question 1 Answer E

$U \overset{d}{=} N(5,1^2)$, $V \overset{d}{=} N(8,1^2)$.

$$W = 4U - 3V$$

$$E(W) = 4E(U) - 3E(V)$$
$$= 4\times5 - 3\times8 = -4$$

$$\text{Var}(W) = 16\text{Var}(U) + 9\text{Var}(V)$$
$$= 16\times1 + 9\times1 = 25$$

$$W \overset{d}{=} N(-4,5^2)$$

$$\Pr(W > 5) = \Pr\left(Z > \frac{5+4}{5}\right) = \Pr(Z > 1.8)$$

Question 2 Answer D

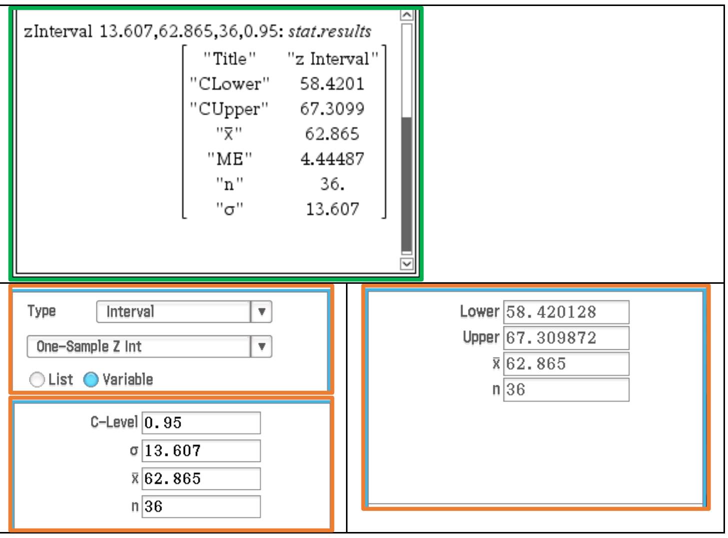
$$CI: \left(\mu - z\times\frac{\sigma}{\sqrt{n}}, \mu + z\times\frac{\sigma}{\sqrt{n}}\right) = (58.42, 67.31)$$

$n = 36$, 95% $z = 1.96$ $\sigma = ?$

$$\mu = \frac{1}{2}(58.42 + 67.31) = 62.865$$

$$67.31 - 58.42 = 8.89 = 2z\times\frac{\sigma}{\sqrt{n}} = \frac{2\times1.96\sigma}{\sqrt{36}}$$

$$\sigma = \frac{8.89\times6}{2\times1.96} = 13.607 \approx 13.61$$



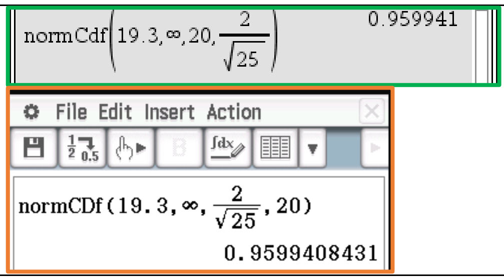
Question 3 Answer A

Let O_i denote 'mass of an orange'
Let L_i denote 'mass of a lemon'
In a set of 3 independent oranges and 2 lemons
mean $E(O_1 + O_2 + O_3 + L_1 + L_2) = 3\times204 + 2\times76 = 764$
variance = $\text{Var}(O_1 + O_2 + O_3 + L_1 + L_2) = 3\times9^2 + 2\times3^2 = 261$
standard deviation = $\sqrt{261} = 3\sqrt{29}$

Question 4 Answer E

$$\bar{X} \overset{d}{=} N\left(\mu = 20, \sigma^2 = \frac{4}{25}\right)$$

$$\Pr(\bar{X} > 19.3) = 0.9599$$



Exam 2 Solutions

Solution

a. $H_o : \mu = 150$

$H_1 : \mu < 150$

b. $sd\left(\bar{X}\right) = \dfrac{15}{\sqrt{50}} = \dfrac{3}{\sqrt{2}} = \dfrac{3\sqrt{2}}{2}$

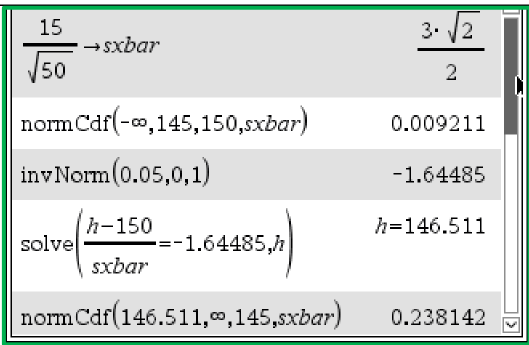
c. $p = \Pr\left(\bar{X} < 145 \mid \mu = 150\right) = 0.0092$

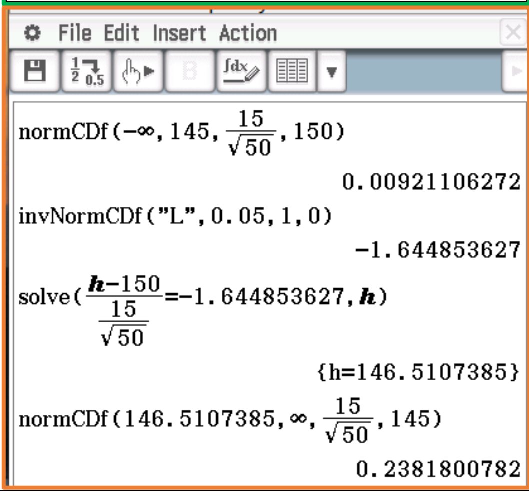
d. as $p < 0.05$ there is evidence to reject H_o accept at the 5% level

e. $\Pr\left(Z < \dfrac{h-150}{\frac{3}{\sqrt{2}}}\right) = 0.05$

$\dfrac{h-150}{\frac{3}{\sqrt{2}}} = -1.645$, $h = 146.51\text{cm}$

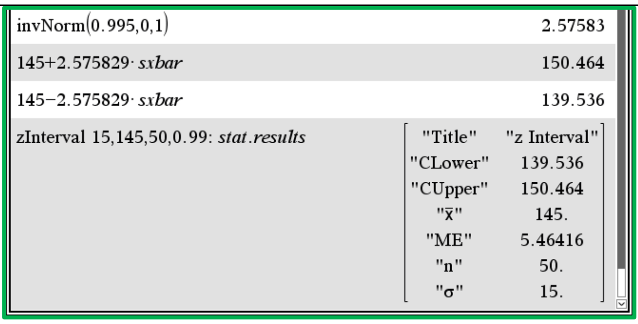
f. $\Pr\left(\bar{X} > 146.511 \mid \mu = 145\right) = 0.24$

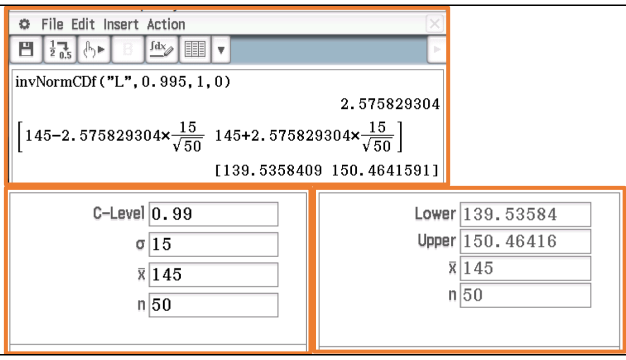




g. $\bar{x} = 145$, $\sigma = \dfrac{3}{\sqrt{2}}$, 99%, $z = 2.5758$, CI $\left(\bar{x} - z \times \dfrac{\sigma}{\sqrt{n}}, \bar{x} + z \times \dfrac{\sigma}{\sqrt{n}}\right)$

$= 145 \pm 2.5758 \times \dfrac{3}{\sqrt{2}} = (139.5, 150.5)$





COMPREHENSIVE NOTES

Graphs of the absolute value (modulus) function
And rational functions of low degree

*Sketch graphs of the absolute value (modulus) function.

- Definition: $f(x)=|x|=\begin{cases} x, & x\geq 0 \\ -x, & x<0 \end{cases}$.

Note: $\sqrt{x^2}=|x|$.

- Transformations of the modulus function: $g(x)=a|x-h|+k$.

'Corner' at $(h,\ k)$.

Note: $|x-h|=\begin{cases} x-h, & x\geq h \\ -(x-h)=h-x, & x<h \end{cases}$.

- Composite functions involving the modulus function:

* $y=|f(x)|=\begin{cases} f(x) & \text{if } f(x)\geq 0 \\ -f(x) \quad (\text{reflection in } x\text{-axis}) & \text{if } f(x)<0 \end{cases}$

* $y=f(|x|)=\begin{cases} f(x) & \text{if } x\geq 0 \\ f(-x) \quad (\text{reflection in } y\text{-axis}) & \text{if } x<0 \end{cases}$

* $y=|f(|x|)|$

*Sketch graphs of rational functions.

A rational function is a function of the form $f(x)=\dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials.

Special case: A rational function of the form $f(x)=\dfrac{p(x)}{x}$ can be expressed in the form

$f(x)=ax^m+bx^{-n}+c$

where $n,m\in Z^+$. It has asymptotes with equations $x=0$ (vertical asymptote) and $y=ax^m+c$ (horizontal, oblique or curvilinear asymptote).

- Sketch graphs using key features: * Asymptotes and asymptotic behaviour.
  - * Axes intercepts.
  - * Stationary points and their nature.
  - * Domain and range.
- Sketch graphs using *addition of ordinates*.

Finding key features of a graph- summary.

FEATURE OF $y = f(x)$	HOW TO FIND THE FEATURE
***y*-intercepts.**	• Substitute $x=0$ into $y=f(x)$ and solve $y=f(0)$ directly for *y*.
***x*-intercepts.**	• Substitute $y=0$ into $y=f(x)$ and solve $0=f(x)$ for *x*. The *procedure* for solving $0=f(x)$ will depend on the type of function.
Stationary Points.	• ***x*-coordinate**: Solve $\frac{dy}{dx}=0$ for *x*. The *procedure* for solving $\frac{dy}{dx}=0$ will depend on the type of function. • ***y*-coordinate**: Substitute *x*-coordinate into $y=f(x)$ and solve directly for *y*.
Nature of Stationary Points.	**Options:** 1. Sign test. 2. Second derivative test.
Vertical Asymptotes: The *vertical line(s)* passing through the value(s) of *x* for which *y* is not defined.	• Determine the values of *x* (*a*, *b*, …) that make $y=f(x)$ undefined: <u>Rational functions</u>: Solve *DENOMINATOR* = 0 for *x*. <u>Logarithmic functions</u>: Solve *ARGUMENT OF LOG* = 0 for *x*. • Equations of vertical asymptotes are $x=a$, $x=b$, …
Horizontal Asymptotes: The *horizontal line* passing through the value that *y* approaches as $\lvert x\rvert \to +\infty$. **Oblique and Curvilinear Asymptotes:** • Oblique (diagonal or slant) asymptote: A linear asymptote that is not parallel or vertical to the axes. • Curvilinear asymptote: An asymptote that is not linear. A rational function $y=f(x)$ has an oblique or curvilinear asymptote $y=h(x)$ if $\lim\limits_{\lvert x\rvert\to+\infty}(f(x)-h(x))=0$	• $f(x)=g(x)+b$ where $\lim\limits_{\lvert x\rvert\to+\infty} g(x)=0$: Equation of horizontal asymptote is $y=b$. • $f(x)=g(x)+h(x)$ where $f(x)$ is a rational function and $\lim\limits_{\lvert x\rvert\to+\infty} g(x)=0$: Equation of oblique or curvilinear asymptote is $y=h(x)$. **Note:** *Polynomial Long Division* is required if $f(x)=\frac{\text{polynomial 1}}{\text{polynomial 2}}$, where the degree of polynomial 1 is **equal to or greater** than the degree of polynomial 2: $\left.\begin{array}{r\|l} & h(x) \\ \hline \text{polynomial 2} & \text{polynomial 1} \\ & \vdots \\ \hline & \text{Remainder} \end{array}\right\} \Rightarrow \frac{\text{polynomial1}}{\text{polynomial 2}}=h(x)+\frac{\text{Remainder}}{\text{polynomial 2}}=h(x)+g(x)$.
Point of Inflection: At a point of inflection there is a change in concavity of $y=f(x)$ and therefore a change in sign of $f''(x)$. Such a change of sign MAY occur at points where $f''(x)=0$.	*Warning*: $f''(x)=0$ is a necessary but NOT sufficient condition for a point of inflection to exist. The nature of the solutions to $f''(x)=0$ must always be tested: a solution to $f''(x)=0$ is only a point of inflection of *f* if it is a turning point of f'. Example: $f(x)=\frac{3}{20}x^5-x^4+2x^3-2$ (see Topic 7: Differential Calculus).

Note: A graph can never touch or cross a vertical asymptote but it is possible for a graph to touch or cross horizontal, oblique or curvilinear asymptotes.

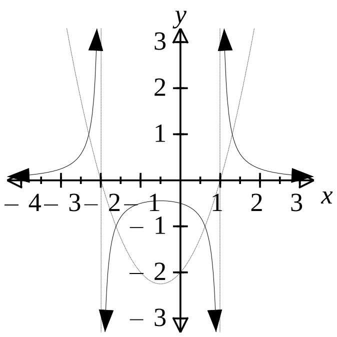
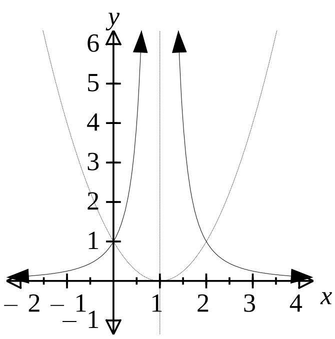
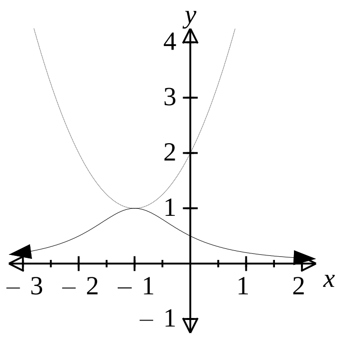
Graphs of quotient (reciprocal) functions

Using the known graph of a function $g(x)$ to sketch the unknown graph of the reciprocal function $y = \frac{1}{g(x)}$:

Feature on graph of $g(x)$	**Feature on graph of** $y = \frac{1}{g(x)}$
x-intercept at $x = a$ (which may or may not also be a stationary point).	Vertical asymptote at $x = a$.
y-intercept at $y = a$.	y-intercept at $y = \frac{1}{a}$.
Stationary point: Turning point (maximum) at (a, b), $b \neq 0$. Turning point (minimum) at (a, b), $b \neq 0$. Stationary point of inflection at (a, b), $b \neq 0$	 Turning point (minimum) at $\left(a, \frac{1}{b}\right)$, $b \neq 0$. Turning point (maximum) at $\left(a, \frac{1}{b}\right)$, $b \neq 0$. Stationary point of inflection at $\left(a, \frac{1}{b}\right)$, $b \neq 0$.
Vertical asymptote with equation $x = a$.	Either a 'hole' on the x-axis ($y = \frac{1}{g(x)}$ is not defined at $x = a$) (see non-routine example 3) or an x-intercept at $x = a$ ($y = \frac{1}{g(x)}$ is defined at $x = a$) (see non-routine example 2). The approach of the graph towards the 'hole' is similar to if there was an x-intercept at $x = a$ (which may or may not also be a stationary point).
Points where $y = 1$. Points where $y = -1$.	Points on the graph of $y = g(x)$ where $y = 1$ or $y = -1$.
Horizontal asymptote with equation $y = a$, $a \neq 0$.	Horizontal asymptote with equation $y = \frac{1}{a}$, $a \neq 0$.

***Graphs of rational functions of the type $f(x)=\dfrac{1}{ax^2+bx+c}$, that is, $f(x)$ is the reciprocal of the quadratic function $g(x)=ax^2+bx+c$.**

- Sketch graphs via consideration of the quadratic function $g(x)=ax^2+bx+c$ and its known graph.
  Classification of basic shapes using the discriminant.
  Domain and range.

Function	$y=\dfrac{1}{x^2+x-2}$, $\Delta>0$	$y=\dfrac{1}{x^2-2x+1}$, $\Delta=0$	$y=\dfrac{1}{x^2+2x+2}$, $\Delta<0$
Graphs			
Domain	$R\setminus\{-2,1\}$	$R\setminus\{1\}$	R
Range	$\left(-\infty,-\dfrac{4}{9}\right]\cup(0,\infty)$	$(0,\infty)$	$(0,1]$

- Sketch graphs via consideration of key features:

  * Asymptotes and asymptotic behaviour:
  Location of *vertical asymptote(s)* will require solution of the equation $0=ax^2+bx+c$.
  Equation of *horizontal asymptote*: $y=0$ (*ie.* x-axis).

  * y-intercept.

  * There is no x-intercept.

  * Turning points and their nature.

  Note: If a turning point exists, its coordinates and nature can be determined either by considering the turning
  point of the quadratic function $g(x)=ax^2+bx+c$ or by using calculus (use either chain rule or quotient rule).

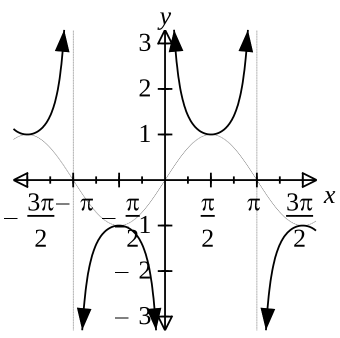
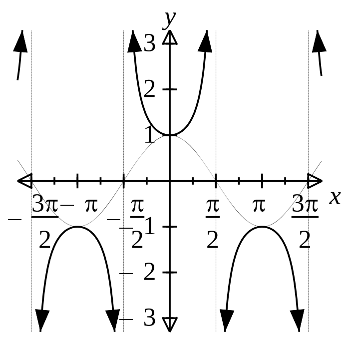
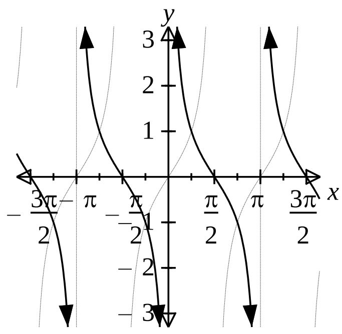
- Vertical translations of $f(x)=\dfrac{1}{ax^2+bx+c}$ such as $y=\dfrac{2x^2+2x-3}{x^2+x-2}=\dfrac{1}{x^2+x-2}+2$

***The reciprocal trigonometric functions $\operatorname{cosec}(x)$, $\sec(x)$ and $\cot(x)$.**

Definitions: $\operatorname{cosec}(x)=\dfrac{1}{\sin(x)}$, $\sec(x)=\dfrac{1}{\cos(x)}$, $\cot(x)=\dfrac{1}{\tan(x)}=\dfrac{\cos(x)}{\sin(x)}$.

Sketch graphs via consideration of the functions $\sin x$, $\cos x$ and $\tan x$ and their known graphs.

Key features include vertical asymptotes, turning points and x-intercepts.

Function	$\text{cosec}\,(x)=\dfrac{1}{\sin(x)}$	$\sec\,(x)=\dfrac{1}{\cos(x)}$	$\cot\,(x)=\dfrac{1}{\tan(x)}=\dfrac{\cos(x)}{\sin(x)}$
Domain	$R\backslash\{m\pi\}$ where $m\in Z$	$R\backslash\left\{\dfrac{(2m+1)\pi}{2}\right\}$ where $m\in Z$	$R\backslash\{m\pi\}$ where $m\in Z$
Range	$(-\infty,-1]\cup[1,\infty)$	$(-\infty,-1]\cup[1,\infty)$	R
Graphs			

***Transformations of the graphs of $f(x)=\text{cosec}\,(x)$, $f(x)=\sec(x)$ and $f(x)=\cot(x)$.**

A sequence of transformations can be used to transform a basic function $y=f(x)$ into a new function $y=g(x)$:

$$f(x)\xrightarrow{\text{Transformations}}g(x)=Af(Bx-C)+D=Af(B[x-\tfrac{C}{B}])+D=Af(B[x-\delta])+D\,.$$

The sequence of translations, dilations and reflections have been applied in the following order (other orders may also be valid for a particular example):

1. $f(x)\xrightarrow[\text{Dilation}]{\text{Horizontal}}f(Bx)$

2. Reflection around y-axis.

3. $f(Bx)\xrightarrow[\text{Translation}]{\text{Horizontal}}f(Bx-C)=f(B[x-\tfrac{C}{B}])=f(B[x-\delta])$

4. $f(B[x-\delta])\xrightarrow[\text{Dilation}]{\text{Vertical}}Af(B[x-\delta])$

5. Reflection around x-axis.

6. $Af(B[x-\delta])\xrightarrow[\text{Translation}]{\text{Vertical}}Af(B[x-\delta])+D=g(x)$

Transformations (functional approach) summary:

TRANSLATIONS	**Translation from *x*-axis:** $f(x) \to g(x) = f(x) + D$. The graph of $y = f(x)$ is shifted in the vertical direction by D units. If D is +ve the shift is up. If D is –ve the shift is down. The point (α, β) on $y = f(x)$ is shifted to the point $(\alpha, \beta + D)$.
	Translation from *y*-axis: $f(x) \to g(x) = f(x - \delta)$ The graph of $y = f(x)$ is shifted in the horizontal direction by δ units. If δ is +ve the shift is to the right. If δ is –ve the shift is to the left. The point (α, β) on $y = f(x)$ is shifted to the point $(\alpha + \delta, \beta)$.
DILATIONS	**Vertical Dilation/Dilation from *x*-axis/Dilation parallel to *y*-axis:** $f(x) \to g(x) = Af(x)$ The graph of $y = f(x)$ is compressed $(A < 1)$ or stretched $(A > 1)$ in the vertical direction (change of scale along the *y*-axis by A). Dilation factor is A. The point (α, β) on $y = f(x)$ is dilated to the point $(\alpha, A\beta)$.
	Horizontal Dilation/Dilation from *y*-axis/Dilation parallel to *x*-axis: $f(x) \to g(x) = f(Bx)$ The graph of $y = f(x)$ is compressed $(B > 1)$ or stretched $(B < 1)$ in the horizontal direction (change of scale along the *x*-axis by $\frac{1}{B}$). Dilation factor is $\frac{1}{B}$. The point (α, β) on $y = f(x)$ is dilated to the point $\left(\frac{\alpha}{B}, \beta\right)$.
REFLECTIONS	**Reflection in *x*-axis:** $f(x) \to g(x) = -f(x)$ The graph of $y = f(x)$ is reflected in the *x*-axis. The point (α, β) on $y = f(x)$ is reflected to the point $(\alpha, -\beta)$.
	Reflection in *y*-axis: $f(x) \to g(x) = f(-x)$ The graph of $y = f(x)$ is reflected in the *y*-axis. The point (α, β) on $y = f(x)$ is reflected to the point $(-\alpha, \beta)$.

Trigonometric functions

***Revision of the basic trigonometric functions (sine, cosine, tan) - Mathematical Methods Unit 2.**

Definitions and the unit circle, symmetry properties, basic graphs, the Pythagorean Identity, solutions of equations (including general solution in terms of a parameter).

***Identities obtained from** $\sin^2(A)+\cos^2(A)=1$

$\sec^2(A)=1+\tan^2(A)$, $\operatorname{cosec}^2(A)=1+\cot^2(A)$

Simplification of expressions.

Identities and their proof.

Solving for trigonometric values.

***Compound and Double Angle formulae.**

$\sin(A\pm B)=\sin(A)\cos(B)\pm\sin(B)\cos(A)$, $\cos(A\pm B)=\cos(A)\cos(B)\mp\sin(A)\sin(B)$,

$$\tan(A\pm B)=\frac{\tan(A)\pm\tan(B)}{1\mp\tan(A)\tan(B)}$$

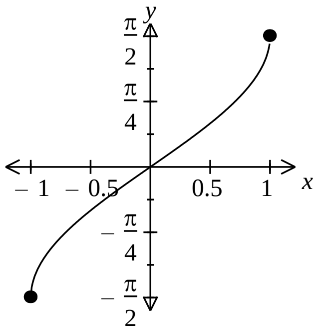
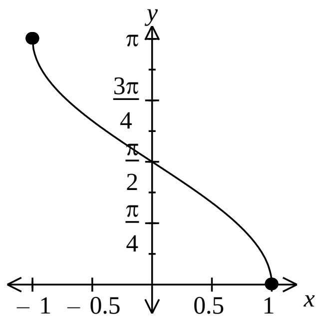
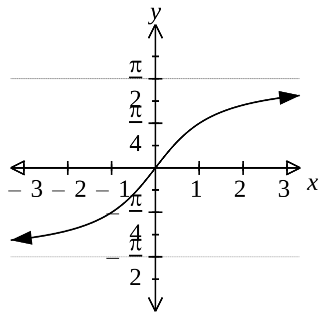
$\sin(2A)=2\sin(A)\cos(A)$, $\cos(2A)=\cos^2(A)-\sin^2(A)\ =1-2\sin^2(A)=2\cos^2(A)-1$,

$$\tan(2A)=\frac{2\tan(A)}{1-\tan^2(A)}$$

***The restricted** $\sin(x)$ **,** $\cos(x)$ **and** $\tan(x)$ **trigonometric functions and their inverses.**

Definitions of the restricted trigonometric functions $\sin(x)$, $\cos(x)$ and $\tan(x)$ and their graphs.

Definitions of the inverse trigonometric functions $\sin^{-1}(x)$ or $\arcsin(x)$, $\cos^{-1}(x)$ or $\arccos(x)$ and $\tan^{-1}(x)$ or $\arctan(x)$ and their graphs.

Function	$\sin^{-1}(x)$	$\cos^{-1}(x)$	$\tan^{-1}(x)$
Domain	$[-1,\ 1]$	$[-1,\ 1]$	R
Range	$\left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]$ (*ie.* 1st and 4th quadrants)	$[0,\ \pi]$ (*ie.* 1st and 2nd quadrants)	$\left(-\frac{\pi}{2},\ \frac{\pi}{2}\right)$ (*ie.* 1st and 4th quadrants)
Meaning	$y=\sin^{-1}x$ means $x=\sin y$ and $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$.	$y=\cos^{-1}x$ means $x=\cos y$ and $0\le y\le\pi$.	$y=\tan^{-1}x$ means $x=\tan y$ and $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$.
Graphs			

Note: The graph of the inverse function $y=f^{-1}(x)$ is obtained from the graph of $y=f(x)$ by reflection in the line $y=x$. The point (α,β) on $y=f(x)$ is reflected to the point (β,α).

*Composite functions.

A composite function is any function that can be written in the form f(w), where w = w(x) is also a function.
For a composite function to exist, the range of w must be a subset of the domain of f.
When this condition fails, the domain of w must be ***restricted*** *so that its range becomes a subset of the domain of f.*
The domain of a composite function is given by the (restricted) domain of w.

Simple transformations of $\sin^{-1}(x)$, $\cos^{-1}(x)$, $\tan^{-1}(x)$ and their graphs.
Maximal domain and range.

Vectors in two and three dimensions

*Review of non-right-angle triangles.

Sine Rule: $\boxed{\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}}$

In other words:
$\boxed{\dfrac{\sin A}{a} = \dfrac{\sin B}{b},\ \dfrac{\sin A}{a} = \dfrac{\sin C}{c},\ \dfrac{\sin B}{b} = \dfrac{\sin C}{c}}$

Cosine Rule: $\boxed{a^2 = b^2 + c^2 - 2bc\cos A}$

*Introduction to scalars and vectors.

A *scalar* is a quantity that has magnitude only.
A *vector* is a quantity that has both magnitude and direction.

Vector notation: $\underset{\sim}{a}$, $\overrightarrow{AB}$ *etc.* Geometric representation of a vector.

Algebraic properties of vectors.

Position vector of a point P relative to a point O. Notation: $\overrightarrow{OP}$. Specification of direction - bearings.
The negative of a vector.

Magnitude of a vector. Notation: $|\underset{\sim}{a}|$, $|\overrightarrow{AB}|$ *etc.* The zero vector $\underset{\sim}{\mathbf{0}}$.

Unit vector parallel to $\underset{\sim}{a}$: $\hat{\underset{\sim}{a}} = \dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$.

Necessary condition for equality of two vectors:

Two vectors are equal if and only if they have the same magnitude and direction.

*Basic operations with vectors - geometric representation.

Multiplication of a vector by a scalar.

Necessary condition for two vectors to be parallel to each other:

$\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel to each other: $\underset{\sim}{a} = \lambda \underset{\sim}{b}$.

Addition and subtraction of vectors - the ‘head to tail rule’.

Note: Pythagoras’ Theorem and/or trigonometry (including *sine* and *cosine rules in non-right-angled triangles*) might need to be used to calculate magnitude and direction of resultant vector.

Linear dependence and independence of a set of vectors - geometric interpretation.

A closed n-sided figure can be constructed from a set of n linearly dependent vectors.

For example: A triangle can be constructed from a set of three linearly dependent vectors.
A quadrilateral can be constructed from a set of four linearly dependent vectors.

***Resolving a vector into Cartesian (rectangular) components.**

Definition of the system of unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$.

Components of a vector.

Component form of basic operations with vectors.

If $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$, $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}$, then:

1. $a = \sqrt{a_1^2 + a_2^2 + a_3^2}$.
2. $\underset{\sim}{a} = \underset{\sim}{b} \Rightarrow a_1 = b_1,\ a_2 = b_2$ and $a_3 = b_3$.
3. $\lambda\underset{\sim}{a} = \lambda a_1\underset{\sim}{i} + \lambda a_2\underset{\sim}{j} + \lambda a_3\underset{\sim}{k}$.
4. $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel to each other: $\underset{\sim}{a} = \lambda\underset{\sim}{b}$ that is, $a_1 = \lambda b_1$, $a_2 = \lambda b_2$ and $a_3 = \lambda b_3$.
5. $\underset{\sim}{a} + \underset{\sim}{b} = (a_1 + b_1)\underset{\sim}{i} + (a_2 + b_2)\underset{\sim}{j} + (a_3 + b_3)\underset{\sim}{k}$, $\underset{\sim}{a} - \underset{\sim}{b} = (a_1 - b_1)\underset{\sim}{i} + (a_2 - b_2)\underset{\sim}{j} + (a_3 - b_3)\underset{\sim}{k}$.

   Position vector of a point $P(x, y, z)$ relative to the origin: $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$.

***Linear dependence and independence of a set of NON-ZERO vectors.**

• **Definitions:**

A set of vectors is linearly independent if there is no vector in the set that can be written as a combination of the other vectors.
A set of vectors that is not linearly independent is linearly dependent.

A set of vectors containing the zero vector is ALWAYS linearly dependent.

• Alternative definition:

A set of vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$, ... *is linearly independent if* $\gamma\underset{\sim}{a} + \alpha\underset{\sim}{b} + \beta\underset{\sim}{c} + ... \neq \underset{\sim}{0}$ *unless* $\gamma = \alpha = \beta = \cdots = 0$.

• Geometric interpretation: *A closed n-sided figure can be constructed from a set of n linearly dependent vectors.*

For example:

A triangle can be constructed from a set of three linearly dependent vectors.

A quadrilateral can be constructed from a set of four linearly dependent vectors.

***Scalar (dot) product of two vectors.**

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta)$ where θ is the angle between the vector $\underset{\sim}{a}$ and the vector $\underset{\sim}{b}$.

Note: $\underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{a}|^2$.

Algebraic properties of the scalar product.

Scalar products for the $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$ system:

If $\underset{\sim}{a} = a_1 \underset{\sim}{i} + a_2 \underset{\sim}{j} + a_3 \underset{\sim}{k}$, $\underset{\sim}{b} = b_1 \underset{\sim}{i} + b_2 \underset{\sim}{j} + b_3 \underset{\sim}{k}$, then $\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

Note: $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta) = a_1 b_1 + a_2 b_2 + a_3 b_3$.

Calculating the magnitude of the angle between two vectors.

Condition for $\underset{\sim}{a}$ and $\underset{\sim}{b}$ to be perpendicular to each other: $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$.

Vector proof of compound angle formulae in trigonometry.

***Vector and scalar resolutes.**

The *vector resolute* of vector $\underset{\sim}{a}$ in the direction of vector $\underset{\sim}{b}$ (also called the *parallel resolute* of $\underset{\sim}{a}$ along the direction of $\underset{\sim}{b}$) is given by $\underset{\sim}{u} = \left(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$.

The *scalar resolute* of vector $\underset{\sim}{a}$ in the direction parallel to vector $\underset{\sim}{b}$ is given by $u = \underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}$.

The *vector resolute* of $\underset{\sim}{a}$ in the direction *perpendicular* to $\underset{\sim}{b}$ (also called the *perpendicular resolute* of $\underset{\sim}{a}$ along the direction of $\underset{\sim}{b}$) is given by $\underset{\sim}{w} = \underset{\sim}{a} - \underset{\sim}{u}$. Note: $\underset{\sim}{a} = \underset{\sim}{u} + \underset{\sim}{w}$.

Expressing a given vector $\underset{\sim}{a}$ as a sum of two vectors, the first parallel to a stated direction and the second perpendicular to that direction.

$\underset{\sim}{a} = \text{parallel resolute along stated direction} + \text{perpendicular resolute along stated direction}$.

***Vector proofs of simple geometric theorems.**

Vector properties useful in vector proofs of simple geometric theorems.

Examples.

Step 1: *Draw a clear and well labeled diagram.*

Step 2: *Clearly state in vector form all the information (explicit and implicit) given in the problem.*

Complex numbers

*Introduction.

The imaginary number i. Notation: $i = \sqrt{-1}$. $i^2 = -1$.
A ***complex number*** is denoted by z. It has the form $z = x + iy$, where x and y are real numbers.

x is called the ***real part*** *of* z, denoted $\mathrm{Re}(z)$.

y is called the ***imaginary part*** *of* z, denoted $\mathrm{Im}(z)$.

If $\mathrm{Im}(z) = 0$, then z is real.

$z_1 = z_2$ if and only if $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

The set of complex numbers is denoted by C. *Note*: R is a subset of C.

The ***complex conjugate*** of $z = x + iy$ is denoted by $\bar{z}$. It is defined to be $\bar{z} = \mathrm{Re}(z) - i\,\mathrm{Im}(z) = x - iy$.

z is real if and only if $z = \bar{z}$. Note: $\mathrm{Re}(z) = \frac{1}{2}\left(z + \bar{z}\right)$ and $\mathrm{Im}(z) = \frac{1}{2i}\left(z - \bar{z}\right)$.

*Geometric representation of a complex number - Argand Diagrams.

A complex number $z = x + iy$ can be considered as being associated with the point $P(x, y)$ in the Cartesian plane. Such representations are called ***Argand Diagrams***.
$x + iy$ is called the ***Cartesian form*** of the complex number z.

When the Cartesian plane is used to represent a complex number, it is referred to as the ***Complex*** or ***z-Plane***. In the Complex Plane:

1. The x-axis is called the ***real axis*** and represents Re z.

2. The y-axis is called the ***imaginary axis*** and represents Im z.

The magnitude of z is called the **modulus** and is denoted by $|z|$.
It is the distance from the origin O to the point P representing z in the Complex Plane:

$|z| = |x + iy| = \sqrt{x^2 + y^2}$. $|z|^2 = z\bar{z}$.

A complex number $z = x + iy$ can also be considered as the position vector $\overrightarrow{OP}$ from the origin to the point $P(x, y)$ in the Complex Plane. (This leads to a representation called the *polar form* of a complex number).

*Basic operations with complex numbers expressed in Cartesian form.

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then:

Addition: $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = x_1 + x_2 + iy_1 + iy_2 = (x_1 + x_2) + i(y_1 + y_2)$.

Subtraction: $z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = x_1 - x_2 + iy_1 - iy_2 = (x_1 - x_2) + i(y_1 - y_2)$.

Multiplication by a real scalar: $kz_1 = k(x_1 + iy_1) = kx_1 + iky_1$.

Multiplication:

$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_1y_2 + iy_1x_2 - y_1y_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$.
Note: $z\bar{z} = (x + iy)(x - iy) = x^2 - ixy + iyx + y^2 = x^2 + y^2 = |z|^2$.

Division: $\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{z_2 \bar{z}_2} = \frac{z_1 \bar{z}_2}{|z_2|^2}$.

The 'trick' is to multiply numerator and denominator by the complex conjugate of z_2.

Note: $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$. For example, $\frac{1}{i} = -i$.

Simplifying surds: $\sqrt{ab} \neq \sqrt{a}\sqrt{b}$ if a and b are both negative.

Geometrical representation of addition, subtraction and multiplication by a scalar.

Note: $|z_1 - z_2|$ gives the distance between z_1 and z_2.
Cartesian methods for finding the square root of a complex number.

*Polar form of a complex number.

A complex number $z = x + iy$ can be defined in terms of the magnitude r and direction θ of the position

vector $\overrightarrow{OP}$ from the origin to the point $P(x, y)$.

Magnitude (modulus): Calculate from $|z|$: $r = |z|$.

Direction: Defined by the angle θ measured from the positive real axis to $\overrightarrow{OP}$.

Calculate θ by drawing an Argand Diagram and then use simple trigonometry.

$x = r\cos\theta$ and $y = r\sin\theta \Rightarrow z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = r\text{cis}(\theta)$.

Complex conjugate: If $z = r\text{cis}(\theta)$ then $\bar{z} = r\text{cis}(-\theta)$.

$\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{1}{r}\text{cis}(-\theta)$.

θ is called the ***argument*** of z and is denoted **a**rg z. *Note*: $\arg z = \theta + 2m\pi,\ \ m = 0,\ \pm 1,\ \pm 2,\ \cdots$.

Note: The argument of zero is undefined.
The single value of arg z in the interval $(-\pi, \pi]$ is called the ***Argument*** or **principle argument** of z and is denoted **A**rg(z).
When $-\pi < \theta \leq \pi$, $r\text{cis}(\theta)$ is called the ***polar form*** of the complex number z.

*Basic operations with complex numbers expressed in polar form.

If $z_1 = r_1\text{cis}(\theta_1)$ and $z_2 = r_2\text{cis}(\theta_2)$, then

Addition and Subtraction:

Convert from polar form to Cartesian form and then perform additions and subtractions.

Multiplication: $z_1 z_2 = r_1 r_2 \text{cis}(\theta_1)\text{cis}(\theta_2) = r_1 r_2 \text{cis}(\theta_1 + \theta_2)$.

Division: $\frac{z_1}{z_2} = \frac{r_1\text{cis}(\theta_1)}{r_2\text{cis}(\theta_2)} = \frac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$. *Note*: $\frac{1}{z} = \frac{1}{r}\text{cis}(-\theta)$.

Note that an integer multiple of 2π may need to be added or subtracted to convert $(\theta_1 \pm \theta_2)$ into a principle argument.
Geometric interpretation of multiplication and division of complex numbers.

*De Moivre's Theorem and its applications.

De Moivre's Theorem: $z^n = r^n \text{cis}(n\theta) = r^n[\cos(n\theta) + i\sin(n\theta)]$.

$$\frac{1}{z^n} = z^{-n} = r^{-n}\text{cis}(-n\theta) = r^{-n}[\cos(-n\theta) + i\sin(-n\theta)].$$

Calculating integer powers of a complex number and simplifying expressions written in Cartesian form:

Step 1: Express *z* in polar form.

Step 2: Apply De Moivre's Theorem.

Calculating roots of a complex number *z*:

Applications in trigonometry and proofs of trigonometric identities.

*Polynomials over *C* - two important theorems.

The Fundamental Theorem of Algebra:

Every polynomial equation $p(z) = 0$ *of degree n with complex coefficients has n complex solutions (some of which may be repeated)).*
Every polynomial $p(z)$ of *degree n can be factorised into n complex linear factors (some of which may be repeated).*

The Conjugate Root Theorem:

If a polynomial equation $p(z) = 0$ *has REAL coefficients and* $z = a + ib$ *is a solution, then* $\bar{z} = a - ib$ *is also a solution; that is, zero's (roots) of a polynomial with REAL coefficients occur in conjugate pairs.*
If a polynomial $p(z)$ *has REAL coefficients and* $(z - z_1)$ *is a linear factor, then* $(z - \overline{z_1})$ *is also a linear factor.*
Note: The conjugate root theorem IS NOT VALID if *p*(*z*) has NON-REAL coefficients.

For example, there are no conjugate pair solutions to the equation $z^3 = i \Leftrightarrow z^3 - i = 0$.

*Linear factors over *C* of polynomials up to degree three with integer coefficients.

Factorising quadratics over *C*.

Complete the square or use the quadratic formula to find the linear factors.

Special cases: $z^2 - a^2 = (z-a)(z+a)$ and $z^2 + a^2 = (z-ia)(z+ia)$.

Factorising cubics over *C*.

If a linear factor is known, long division of polynomials can be used to factorise the cubic in the form (LINEAR)(QUADRATIC). The quadratic factor can then be factorised over *C*.
Linear factors can be found using the Factor Theorem.

Special cases: $z^3 - a^3 = (z-a)(z^2 + az + a^2)$ and $z^3 + a^3 = (z+a)(z^2 - az + a^2)$.
Application of the *Conjugate Root Theorem* and the *Fundamental Theorem of Algebra*.

*Linear factors over *C* of $z^4 - a^4$ and $z^6 - a^6$ for small integer values of *a*.

$z^4 - a^4 = (z^2)^2 - (a^2)^2 = (z^2 + a^2)(z^2 - a^2) = (z-ia)(z+ia)(z-a)(z+a)$.

$z^6 - a^6 = (z^3)^2 - (a^3)^2 = (z^3 + a^3)(z^3 - a^3) = (z+a)(z-a)(z^2 - az + a^2)(z^2 + az + a^2)$.

Application of the *Conjugate Root Theorem* and the *Fundamental Theorem of Algebra*.

***Solution over *C* of polynomial equations with complex coefficients.**

Application of the *Conjugate Root Theorem* (where appropriate) and the *Fundamental Theorem of Algebra*.

Long division of polynomials (to obtain quadratic and cubic factors).

Factorisation of polynomials up to degree three with integer coefficients, $z^4 - a^4$, $z^6 - a^6$, $z^8 + 1$.

Factorisation and solutions over C of polynomial equations with complex coefficients.

REPRESENTATION OF RELATIONS IN THE COMPLEX PLANE.

***Relations in the complex plane.**

Algebraic approach: Substitute $z = x + iy$ into the given condition and re-arrange the resulting expression into a recognisable relation. This approach always gives a Cartesian equation.

Geometric approach: Interpretation of distances between points and interpretation of the polar form. This approach does not give a Cartesian equation.

Representation of Relation	**Description of Locus** (values of z that satisfy the equation) **- Geometric Approach**	**Examples**
$\operatorname{Re} z = a$, $\operatorname{Re}(z - z_1) = a$	Vertical line.	$\operatorname{Re} z = -3$
$\operatorname{Im} z = a$, $\operatorname{Im}(z - z_1) = a$	Horizontal line.	$\operatorname{Im}(z - 1 + i) = 2$
$a \operatorname{Re}(z - z_1) \pm \operatorname{Im}(z - z_2) = b$	Straight line with non-zero gradient.	$2\operatorname{Re}(z+1) - \operatorname{Im}(z - i) = 5$
$\|z - z_1\| = \|z - z_2\|$ *Note*: Coordinates of the midpoint of the line segment joining the points (x_1, y_1) and (x_2, y_2): $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$	Straight line which is the perpendicular bisector of the line segment joining the points z_1 and z_2. *Note*: $\|z - z_1\|$ and $\|z - z_2\|$ are the distances from the fixed points z_1 and z_2 to the point representing z. Therefore, the subset consists of all points z whose distance from z_1 is equal to their distance from z_2.	$\|z - 1 + i\| = \|z + 2 - 3i\|$
$\operatorname{Arg}(z) = \theta_1$ $\operatorname{Arg}(z - z_1) = \theta_1$	Ray from (but not including) the origin along the direction θ_1 (measured anti-clockwise from the positive real axis). Ray from (but not including) the point z_1 along the direction θ_1 (measured anti-clockwise from the positive real axis). Note: Terminus of ray is NOT included because $\operatorname{Arg}(0)$ is not defined.	$\operatorname{Arg}(z) = -\frac{\pi}{3}$ $\operatorname{Arg}(z+1) = \frac{3\pi}{4}$ $\operatorname{Arg}(iz) = \frac{\pi}{4}$ $\operatorname{Arg}(iz - 1) = \frac{\pi}{3}$
$\|z - z_1\| = a$, $(z - z_1)(\bar{z} - \bar{z}_1) = a^2$ $\|z - z_1\| = b\|z - z_2\|$, $b > 0$ $(b \neq 1)$	Circle with centre at z_1 and radius a.	$\|z + 1 - i\| = 2$ $(z + 1 - i)(\bar{z} + 1 + i) = 4$ $\|z - 2\| = 2\|z + 1\|$

Differential calculus

*Derivatives of $y = \tan(kx)$ and $y = \cot(kx)$.

Application of the *Quotient Rule*:

y	$\dfrac{dy}{dx}$
$\tan(kx)$	$k\sec^2(kx)$
$\cot(kx)$	$-k\mathrm{cosec}^2(kx)$
$\sec(kx)$	$k\tan(kx)\sec(kx)$
$\mathrm{cosec}(kx)$	$-k\cot(kx)\,\mathrm{cosec}(kx)$

Application of the *Quotient*, *Product* and *Chain Rules*.

*Obtaining $\dfrac{dy}{dx}$ from functions of the form $x = f(y)$.

$\dfrac{dy}{dx}$ can be obtained from functions of the form $x = f(y)$ via $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$, $\dfrac{dx}{dy} \neq 0$.

*Implicit differentiation.

When it is difficult or impossible to get y as a function of x, *implicit differentiation* can be used to find $\frac{dy}{dx}$.

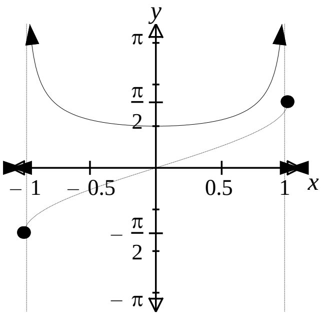
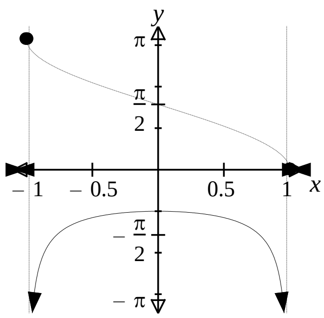
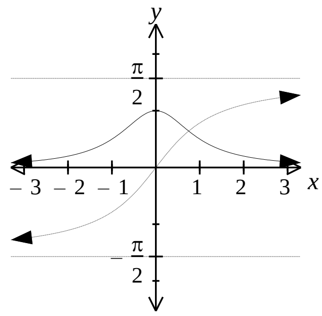
To find $\frac{dy}{dx}$:

***Step* 1:** Treat y as a (implicit) function of x and differentiate both sides of the given relationship with respect to x. The *Product*, *Quotient* and *Chain Rules* will often be necessary. They must be carefully applied.

For example, *Chain Rule*: $\dfrac{d(\text{function of } y)}{dx} = \dfrac{d(\text{function of } y)}{dy} \times \dfrac{dy}{dx}$.

***Step* 2:** Group together terms involving $\frac{dy}{dx}$ and then make $\frac{dy}{dx}$ the subject.

*Derivatives of inverse trigonometric functions.

y	$\frac{dy}{dx}$	Graph of $\frac{dy}{dx}$
$\text{Sin}^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{a^2-x^2}}$	
$\text{Cos}^{-1}\left(\frac{x}{a}\right)$	$\frac{-1}{\sqrt{a^2-x^2}}$	
$\text{Tan}^{-1}\left(\frac{x}{a}\right)$	$\frac{a}{x^2+a^2}$	

Application of the *Quotient*, *Product* and *Chain Rules*.

*Second derivatives.

The second derivative is the derivative of the derivative. Notation: $\frac{d^2y}{dx^2}$, $\frac{d}{dx}\left[\frac{dy}{dx}\right]$, $f''(x)$.

$f''(x)$ can be thought of as the gradient function of $f'(x)$. It gives the rate of change of $f'(x)$.
Concavity and its direction.
Points of inflection.

A point where a curve changes its concavity from downward to upward or vice versa is called a *point of inflection*. They are characterised by a change in sign of $f''(x)$. Such a change of sign **may** occur when

1. $f''(x)=0$.
2. $f''(x)$ fails to exist.

WARNING: $f''(x)=0$ is a necessary but NOT sufficient condition for a point of inflection to exist (*eg.* $f(x)=x^4$ has a min. turning point at x = 0). The *nature* of the solutions to $f''(x)=0$ must be tested:

1. Only solutions corresponding to TURNING POINTS of $f'(x)$ are inflection points of $f(x)$.
2. Solutions corresponding to stationary points of inflection of $f'(x)$ are NOT inflection points of $f(x)$.

Example: $f(x)=\frac{3}{20}x^5-x^4+2x^3-2$ has a point of inflection at $x=0$ but not at $x=2$.

Algebraic and CAS calculator approaches to finding points of inflection.

Finding inflection points of $f(x)$ from graphs of $f'(x)$ versus x and $f''(x)$ versus x.

The *second-derivative test* for maximum and minimum turning points of a function.

Examples:

1. Second derivative test gives a conclusion and is more efficient than the sign test: $f(x)=x^3+3x^2-x-3$, any polynomial function.

2. Second derivative test gives a conclusion but is less efficient than the sign test: $f(x)=\frac{x-1}{x^2-x+1}$.

3. Second derivative test is inconclusive and the sign test must be used: $f(x)=(x-a)^3+b$, $f(x)=(x-a)^4+b$

*Related rates (application of chain rule to rates of change).

If $\frac{dV}{dt}$ is required and a *related rate* $\frac{dr}{dt}$ is given, a ‘bridge’ must be built between the two:

$$\frac{dV}{dt}=\underbrace{\left|\,?\,\right|}_{BRIDGE}\times\frac{dr}{dt}.$$

It is clear from the chain rule that $\underbrace{\left|\,?\,\right|}_{BRIDGE}=\frac{dV}{dr}$. Therefore $\frac{dV}{dt}=\underbrace{\left|\frac{dV}{dr}\right|}_{BRIDGE}\times\frac{dr}{dt}$.

To get an expression for $\frac{dV}{dr}$, the relationship between V and r must be determined. This is done using the information (explicit and/or implicit) contained in the problem.

When attempting to set up a differential equation relating two quantities, the expression for a related rate of change will sometimes be needed but **not** explicitly given. In such cases, a related rate can usually be found by using the information given in the problem.

Integral calculus - techniques of anti-differentiation

*The anti-derivative (indefinite integral).

Definition: An anti-derivative of a function $f(x)$ is another function $F(x) + C$ whose derivative is $f(x)$:

$$\int f(x)\,dx = F(x) + C, \text{ where } \frac{dF}{dx} = f(x).$$

The anti-derivative of a function is a family of functions - each member of the family differs from the other by a constant C.

Basic Properties:

$$\left.\begin{array}{l} 1.\ \int \alpha f(x)\,dx = \alpha \int f(x)\,dx \\ 2.\ \int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx \end{array}\right\} \Rightarrow 3.\ \int \alpha f(x) + \beta g(x)\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx.$$

*Standard forms (see also the more 'standard' standard forms on the VCAA formula sheet).

$$\int x^r\,dx = \frac{1}{r+1}x^{r+1} + C \quad (r \neq -1)$$

$$\int (ax+b)^r\,dx = \frac{1}{a}\frac{1}{(r+1)}(ax+b)^{r+1} + C \quad (r \neq -1).$$

$$\int \frac{1}{x}\,dx = \log_e|x| + C$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e|ax+b| + C.$$

$$\int e^{kx}dx = \frac{1}{k}e^{kx} + C.$$

$$\int \sin(kx)\,dx = -\frac{1}{k}\cos(kx) + C$$

$$\int \cos(kx)\,dx = \frac{1}{k}\sin(kx) + C.$$

$$\int \sec^2(kx)\,dx = \frac{1}{k}\tan(kx) + C$$

$$\int \operatorname{cosec}^2(kx)\,dx = -\frac{1}{k}\cot(kx) + C.$$

*Linear substitution (change of variable) and its use to anti-differentiate expressions of the form $f(x)\,g(cx+e)$.

Particular cases include expressions of the form $f(x)(cx+e)^r$, such as $(ax+b)\sqrt{cx+e}$.

$\int (ax+b)\,g(cx+e)\,dx$ can be found by making the substitution $w = cx + e$.

<u>Note</u>: $x = \dfrac{w-e}{c}$ and $dx = \dfrac{1}{c}dw$.

<u>NB</u>: When using a substitution, you must always remember to substitute for dx as well as for $f(x)$.

***Anti-differentiation of expressions of the form $f[g(x)]\,g'(x)$ - the *Reverse Chain Rule*.**

$\int f[g(x)]\,g'(x)\,dx$ can be found by making the substitution (change of variable) $w = g(x)$:

***Step 1*:** $f[g(x)] \to f(w)$.

***Step 2*:** $\frac{dw}{dx} = g'(x) \Rightarrow dx = \frac{dw}{g'(x)}$.

***Step 3*:** Substitute from Steps 1-2: $\int f[g(x)]\,g'(x)\,dx = \int f(w)\,dw$. $f(w)$ is now a standard form.

***Step 4*:** Find $\int f(w)\,dw$. The answer will be in terms of w.

***Step 5*:** Back-substitute $w = g(x)$ to express the answer in terms of x.

In particular, $\displaystyle\int \frac{g'(x)}{g(x)}\,dx = \log_e|g(x)| + C$.

***Anti-differentiation of $\frac{1}{\sqrt{a^2 - x^2}}$ and $\frac{1}{a^2 + x^2}$, $a > 0$.**

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \text{Sin}^{-1}\left(\frac{x}{a}\right) + C, \quad \int \frac{-1}{\sqrt{a^2 - x^2}}\,dx = \text{Cos}^{-1}\left(\frac{x}{a}\right) + C, \quad \int \frac{a}{a^2 + x^2}\,dx = \text{Tan}^{-1}\left(\frac{x}{a}\right) + C.$$

Cases could involve completing the square and/or linear substitution.

Note: The aim of a substitution is to change the 'difficult' integrand into a standard form. Algebra may also be required.

Note: If more than one technique can be used to calculate an anti-derivative, answers from each technique are equivalent but will generally differ in appearance and may differ by a constant.

***Anti-differentiation of expressions of the form $\sin^m(kx)\cos^n(kx)$.**

***<u>Case 0</u>*:** $\int \sin(kx)\,dx = -\frac{1}{k}\cos(kx) + C$ and $\int \cos(kx)\,dx = \frac{1}{k}\sin(kx) + C$.

***<u>Case 1</u>*:** One of m or n is equal to 1. Use the *Reverse Chain Rule*:

$$\int \sin(kx)\cos^n(kx)\,dx = -\frac{1}{k}\frac{1}{(n+1)}\cos^{n+1}(kx) + C.$$

$$\int \sin^m(kx)\cos(kx)\,dx = \frac{1}{k}\frac{1}{(n+1)}\sin^{m+1}(kx) + C.$$

Case 2: m is odd and positive, n has any value.

Step 1: Make the replacement $\sin^m(kx) = \sin^{m-1}(kx)\sin(kx)$.

Step 2: Since m is odd, $m-1$ is even. Hence $\sin^{m-1}(kx)$ can be expressed in terms of powers of $\cos^2(kx)$ by using the trigonometric identity $\sin^2(kx) = 1-\cos^2(kx)$.

Step 3: Therefore $\int \sin^m(kx)\cos^n(kx)\,dx = \int \sin(kx)\,[\text{Sum of powers of } \cos(kx)]\,dx$.

Step 4: Use the *Reverse Chain Rule*: Substitute $w = \cos(kx)$.

Case 3: n is odd and positive, m has any value.

Step 1: Make the replacement $\cos^n(kx) = \cos^{n-1}(kx)\cos(kx)$.

Step 2: Since n is odd, $n-1$ is even. Hence $\cos^{n-1}(kx)$ can be expressed in terms of powers of $\sin^2(kx)$ by using the trigonometric identity $\cos^2(kx) = 1-\sin^2(kx)$.

Step 3: Therefore $\int \sin^m(kx)\cos^n(kx)\,dx = \int [\text{Sum of powers of } \sin(kx)]\cos(kx)\,dx$.

Step 4: Use the *Reverse Chain Rule*: Substitute $w = \sin(kx)$.

Case 4: m and n are both odd and positive.

There is a choice between ***Case 2*** and ***Case 3***. Make the choice that leads to the least amount of work.

Case 5: m and n are both even and positive.

Use the following trigonometric identities (derived from *double angle formulae*):

$\sin^2 A = \frac{1}{2}(1-\cos 2A)$ $\qquad$ $\sin A\cos A = \frac{1}{2}\sin 2A$ $\qquad$ $\sin^2 A + \cos^2 A = 1$

$\cos^2 A = \frac{1}{2}(1+\cos 2A)$

This will convert the given problem into separate ***Case 0 - Case 3*** problems.

*Anti-differentiation of rational functions.

Step 1: If the degree of the numerator is equal to or greater than the degree of the denominator, it is necessary to perform polynomial division (*eg.* polynomial long division).

Step 2: Factorise the denominator.

Note: Not necessary if the *reverse chain rule* can be used to anti-differentiate the rational function.

Possible factors: Non-repeated linear factors, repeated linear factors, irreducible quadratic factors.

Step 3: Express the rational function as a sum of partial fractions.

This reduces the rational function to a sum of standard forms. There will be partial fraction term corresponding to each type of factor in the numerator:

Factor in denominator	**Term(s) That Factor Contributes to the Partial Fraction Form**
Non-Repeated Linear Factor $(x-\alpha)$.	$\dfrac{A}{(x-\alpha)}$
n-Fold Repeated Linear Factor $(x-\alpha)^n$.	$\dfrac{A_1}{(x-\alpha)}+\dfrac{A_2}{(x-\alpha)^2}+\cdots+\dfrac{A_n}{(x-\alpha)^n}$
Non-Repeated Irreducible Quadratic (ax^2+bx+c) .	$\dfrac{Bx+C}{(ax^2+bx+c)}$

Step 4: Anti-differentiate each partial fraction term.

Special Case: Anti-differentiation of a rational function whose denominator is an irreducible quadratic.

Case 1: The numerator is a constant.

Complete the square and make a linear substitution to obtain: $(\text{Some constant})\times\left(\dfrac{a}{a^2+w^2}\right)$.

This is a standard form whose anti-derivative is: $(\text{Some constant})\times\text{Tan}^{-1}\left(\frac{w}{a}\right)$.

Case 2: The numerator is a linear function.

Re-arrange the rational function to obtain:

$$(\text{Some constant})\times\left(\frac{\text{Derivative of Irreducible Quadratic}}{\text{Irreducible Quadratic}}\right)+\frac{\text{Some constant}}{\text{Irreducible Quadratic}}.$$

The anti-derivative of the first term can be found by using the *Reverse Chain Rule*. It will involve $\log_e$.

The anti-derivative of the second term is a **Case 1** problem. It will involve Tan^{-1} .

Summary: When finding an integral that is not s standard form, either substitution or an algebraic re-arrangement or both will be required. (Alternatively, integration by recognition may be required – see below).

The relationship between the graph of a function and the graph of its anti-derivatives.

Note: There is a family of graphs - each graph differs from the other by an arbitrary vertical translation (due to the arbitrary constant of anti-differentiation).

Feature on the graphs of the anti-derivative of $f(x)$ *ie.* $F(x)$	**Feature on the graph of $f(x)$ *ie.*** $F'(x)$
Stationary point at the point $x = a$. (Use the **sign test** to determine nature).	$f(a) = 0$ ($ie. x$-interceptat $x = a$).
$F(x)$ is increasing over the interval (a,b) . $F(x)$ is decreasing over the interval (a,b) .	$f(x) > 0$ over the interval (a,b) . $f(x) < 0$ over the interval (a,b) .
Point of inflection at the point $x = a$.	Turning point at $x = a$.
Stationary point of inflection at the point $x = a$.	Turning point at $x = a$ **and** $f(a) = 0$ (*ie.* the turning point is also an x-intercept).
Either a vertical asymptote at $x = a$ or a 'vertex' at $x = a$. No conclusion can be drawn unless the limit of $F(x)$ as $x \to a$ is investigated. Examples: • The graph of $\frac{dy}{dx} = \frac{1}{x^{3/2}}$ has a vertical asymptote at $x = 0$ and the graph of $y = y(x)$ has a vertical asymptote at $x = 0$. • The graph of $\frac{dy}{dx} = \frac{\pm x}{\sqrt{1-x^2}}$ has vertical asymptote s at $x = \pm 1$ and the graph of $y = y(x)$ has a vertices at $x = \pm 1$.	Vertical asymptote at $x = a$.

Note: The particular anti-derivative function $\int_{x_0}^{x} f(x)\,dx$ can be plotted on a CAS calculator

***Anti-differentiation by recognition that** $\frac{d}{dx}[f(x)] = g(x)$ **implies that** $\int g(x)\,dx = f(x) + C$

***Definite integrals.**

Notation: $\int_a^b f(x)\,dx$. $x = a$ and $x = b$ are called the (upper and lower) integration limits or terminals.

Fundamental Theorem of Calculus : $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$, where $F(x)$ is any anti-derivative of $f(x)$.

Basic properties: 1. Change of sign when limits of integration are interchanged: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

2. The integral is zero when the upper and lower limits are equal: $\int_a^a f(x)\,dx = 0$.

3. Definite integrals can be divided into subintervals: $\int_a^b f(x)\,dx = \int_a^\beta f(x)\,dx + \int_\beta^b f(x)\,dx$.

When using a technique involving substitution, you must remember to substitute for the integral terminals as well as for dx and $f(x)$.

Integral calculus - integration and its applications

*Approximate evaluation of definite integrals using technologies such as a CAS calculator.

The Specialist Mathematics Examination Setting Panel has stated that Examination 2 will contain questions that can only be solved by using technology. For example, approximate evaluation of $\int_a^b f(x)\,dx$ using CAS

*Area under a curve.

$\int_a^b f(x)\,dx$ gives the *signed area* between the curve $y = f(x)$ and the x-axis, between $x = a$ and $x = b$.

Area *above* the x-axis is *positively signed*. Area *below* the x-axis is *negatively signed*.

*Calculation of areas of regions bounded by curves.

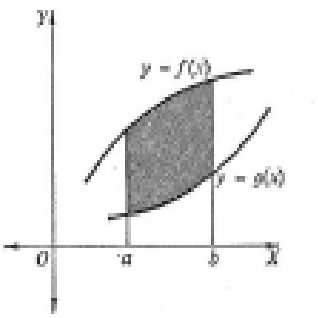
When calculating the magnitude of the area bounded by two curves $y = f(x)$ and $y = g(x)$:

Step 1**:** Find the point(s) of intersection (if any) of $y = f(x)$ and $y = g(x)$.

Step 2**:** Sketch graphs of $y = f(x)$ and $y = g(x)$. It is essential to show all points of intersection.

Step 3**:** Shade in the required area.

Step 4**:** Let f and g be continuous on $[a, b]$ such that $f(x) \geq g(x)$ over $[a, b]$.
Then the area of the region bounded by the two curves and the lines $x = a$ and $x = b$ is given by
$\int_a^b f(x) - g(x)\,dx$.



Note: the roles of upper and lower curve change when the curves intersect over the interval $a < x < b$.

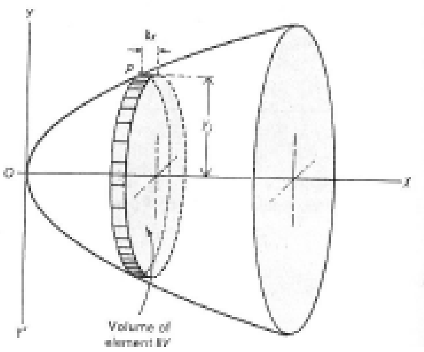
• Area of the region bounded by $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is **always** given by
$\int_a^b |f(x)|\,dx$.

* Calculation of volumes of solids of revolution formed by rotating a region about the *x*- or *y*-axis.

A curve may be rotated about either the x-axis or the y-axis to form a surface of revolution.
The volume enclosed by this surface is a solid of revolution.
Any cross-section of such a solid, taken perpendicular to the axis of rotation, will be circular.

***Rotation of the curve y = f(x) about the x-axis*:**

Vertical cross-sections of the volume are circular: $V = \int_a^b \pi y^2 dx = \int_a^b \pi [f(x)]^2 dx$.



***Rotation of the curve y = f(x) about the y-axis*:**

Horizontal cross-sections of the volume are circular: $V = \int_{y=f(a)}^{y=f(b)} \pi x^2 dy = \left| \int_{x=a}^{x=b} \pi x^2 \frac{dy}{dx} dx \right|$.

$y=f(x)$ must be used to solve for x^2 as a function of y here.

Rotation of regions not bounded by the x-axis/y-axis.

*Arc length of a curve.

The length L of the curve $y = f(x)$ from $x = a$ to $x = b$ is given by the formula $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$.

Differential equations techniques

*Verification of solutions.

To verify that a given function is a solution to a DE, you must substitute the function and all required derivatives of the function into the DE and show that equality is obtained.

General solutions contain arbitrary constants. Particular solutions do not contain arbitrary constants.

*Solution of DE's of the form $\frac{dy}{dx}=g(x)$, $\frac{d^2y}{dx^2}=g(x)$, $\frac{dy}{dx}=h(y)$ and $\frac{dy}{dx}=g(x)h(y)$.

$\frac{dy}{dx}=g(x)$	$\frac{d^2y}{dx^2}=g(x)$	$\frac{dy}{dx}=h(y)$	$\frac{dy}{dx}=g(x)h(y)$
1. Anti-differentiate to obtain the general solution: $y=\int g(x)\,dx+C=G(x)+C$. 2. Substitute given boundary conditions and solve for C. 3. Substitute value of C into general solution to obtain the particular solution. * Sometimes it is either difficult or impossible to perform the anti-differentiation in Step 1. However, if the boundary condition $y(x_0)=y_0$ is given, the solution to the DE can always be expressed as an integral: $y(x)=\int_{x_0}^{x} g(w)\,dw+y_0$.	1. Anti-differentiate: $\frac{dy}{dx}=\int g(x)\,dx+C=G(x)+C$. 2. If additional information linking $\frac{dy}{dx}$ and x is given, use it to determine the value of C. 3. Anti-differentiate again to obtain the general solution: $y=\int [G(x)+C]\,dx$ $=\int G(x)\,dx+Cx+D$. 4. Substitute given boundary conditions to obtain simultaneous equations in C and D. Solve for C and D. 5. Substitute value of C and D into general solution to obtain the particular solution.	1. Take the reciprocal of both sides: $\frac{dx}{dy}=\frac{1}{h(y)}$. 2. Anti-differentiate to obtain the general solution: $x=\int \frac{1}{h(y)}\,dy+C=H(y)+C$. 3. Re-arrange to make y the subject. 4. Substitute given boundary conditions and solve for the new constant. 5. Substitute value of new constant into general solution from Step 3 to obtain the particular solution. **Note**: If $H(y)$ involves logarithms, special care must be taken if the value of C is determined in Step 2 *before* re-arranging to make y the subject.	1. 'Separate the variables': $\frac{1}{h(y)}dy=g(x)dx$. 2. Anti-differentiate: $\int \frac{1}{h(y)}\,dy=\int g(x)\,dx$. 3. Rearrange to make y the subject (if required), solve for C if boundary conditions are given.

Families of curves - general solutions.

You are expected to be able to solve a differential equation (including differential equations with boundary conditions) using a CAS calculator.

*Applications of differential equations.

Applications include the contexts of population growth, Newton's Law of Cooling and radioactive decay. (specialised knowledge of these contexts is NOT be required).

Some problems may involve a difference of rates:

Rate of change = (rate of **increase**) – (rate of **decrease**).

One important example is the mixture of solutions in a tank when there is an inflow and an outflow.

Differential equations which cannot be solved exactly are included - in such cases a numerical solution may be required.

*Constructing differential equations with related rates (application of chain rule to rates of change).

If $\frac{dV}{dt}$ is required and a *related rate* $\frac{dr}{dt}$ is given, a link must be made between the two:

$$\frac{dV}{dt} = \underbrace{\left| \; ? \; \right|}_{Link} \times \frac{dr}{dt}.$$

It is clear from the chain rule that $\underbrace{\left| \; ? \; \right|}_{Link} = \frac{dV}{dr}$. Therefore $\frac{dV}{dt} = \underbrace{\left| \frac{dV}{dr} \right|}_{Link} \times \frac{dr}{dt}$.

To get an expression for $\frac{dV}{dr}$, a relationship between V and r must be found. This is done using the information (explicit and/or implicit) given in the problem.

When attempting to set up a differential equation relating two quantities, the expression for a related rate of change will sometimes be needed but **not** explicitly given. In such cases, a related rate can usually be found by using the information given in the problem.

Note 1: If a quantity is decreasing over time then its rate of change with respect to time is negative.

Note 2: Take care with units.

For example, if $\frac{dV}{dt}$ is given in units of l/min but V given in units of cm^3, then the conversion $1l = 1000$ cm^3 is required so that $\frac{dV}{dt}$ is in units of cm^3/min.

*Numerical solution of differential equations by Euler's method (first-order approximation).

When a differential equation cannot be solved exactly, a **numerical method** must be used to calculate an approximate solution.

Numerical methods start with an initial condition and then repeatedly perform a set sequence of steps to generate an approximate value for the solution function at a number of discrete points. Such repetition is called **iteration**.

Euler's Method is a numerical method for calculating approximate values of the solution of $\frac{dy}{dx} = f(x, y)$, given the boundary condition $y(x_0) = y_0$, at the discrete points

$$x_1 = x_0 + h,\ x_2 = x_1 + h,\ x_3 = x_2 + h,\ \ldots\ x_n = x_{n-1} + h,\ \ldots.$$

The constant amount h that the x variable increases by is called the step size or **increment**.

• Solution of $\frac{dy}{dx} = f(x, y)$ given $y(x_0) = y_0$:

$\frac{dy}{dx}$ gives the slope of the solution function $y = y(x)$ at any point, $y(x_0) = y_0$ gives a point on $y = y(x)$. Using this information a tangent to $y = y(x)$ at the point (x_0, y_0) can be constructed and $y = y(x)$ can be approximated by this line. At $x = x_1 = x_0 + h$ the value $y = y(x_1) = y_1$ can be approximated using the value found from the tangent line. A new tangent line can then be constructed using $\frac{dy}{dx}$ and the point (x_1, y_1) (the point found from the previous tangent line approximation). Continuing in this way, a piece-wise linear approximation to the solution curve $y = y(x)$ can be constructed.

At the point where $x_{n+1} = x_n + h$: $y(x_{n+1}) \approx y(x_n) + hf(y_n, x_n)$.

• Accuracy of the approximation:

* When the solution curve is **concave down** over the interval $[x_0,\ x_{n+1}]$ of interest then Euler's method **over**estimates the exact value of y.
* When the solution curve is **concave up** over the interval $[x_0,\ x_{n+1}]$ of interest then Euler's method **under**estimates the exact value of y.

• Running Euler's Method on the CAS calculator when $\frac{dy}{dx} = f(x, y)$ where $y(x_0) = y_0$:

On the TI-Nspire the command: euler$\left(f(x,y), x, y, \{x_0, x_n\}, y_0, h\right)$ gives the approximate value of y when $x = x_n$ for a step-size of h.

On the Classpad the command entered in the 'Sequence app is: $\begin{array}{l} a_{n+1} = a_n + h \\ b_{n+1} = b_n + h \times f'(a_n, b_n) \end{array}$

***Direction (slope) field for a differential equation.**

• Consider the first order DE $\frac{dy}{dx} = f(x, y)$ with solution $y = y(x)$. We can gain insight into the solution by interpreting the value of $f(x, y)$ at each point (x, y) as giving the GRADIENT (slope) of the curve $y = y(x)$ at each point (x, y).

Given a first order DE $\frac{dy}{dx} = f(x, y)$ we can draw a short line segment starting at a point (x, y) whose slope is $f(x, y)$. We can do this for a number of representative points over a lattice in a chosen region of the xy-plane.
Such a sketch is called the **direction (slope) field** of the DE.

Differential equations kinematics (straight line motion)

*Revision of Straight Line Motion in One Dimension.

In straight line motion problems, an object is restricted to movement in one dimension only. Examples of such movement are left/right, and up/down.
In straight line motion, the direction of a vector quantity is specified by its sign (positive or negative).

- Position, Displacement and Distance.

Displacement = change in position = $x_{final} - x_{initial} = \Delta x$.

In straight line motion motion, $\begin{smallmatrix} +\text{ve} \\ -\text{ve} \end{smallmatrix}$ displacement means that final position is to the $\begin{smallmatrix} \text{right} \\ \text{left} \end{smallmatrix}$ of initial position.

Distance *is a measure of the length of the path taken during the change in position of an object. It is a scalar.*
When finding distance travelled a sketch graph will usually be needed if there are direction changes. The magnitude of displacement is NOT distance.

- Velocity and Speed.

Velocity *is a measure of the rate of change of position (displacement) with respect to time. It is a vector.*

Average velocity: $v_{average} = \dfrac{\text{Change in position}}{\text{Change in time}} = \dfrac{\text{Displacement}}{\text{Change in time}} = \dfrac{x_{final} - x_{initial}}{t_{final} - t_{initial}} = \dfrac{\Delta x}{\Delta t}$.

Instantaneous velocity: $v = \dfrac{dx}{dt}$.

Speed *is a measure of the rate of change of distance with respect to time. It is a scalar.*

Average speed $= \dfrac{\text{Distance travelled}}{\text{Change in time}}$.

Instantaneous speed = magnitude of instantaneous velocity.

Acceleration *is the rate of change of velocity with respect to time. It is a vector.*

Average acceleration: $a_{average} = \dfrac{\text{Change in velocity}}{\text{Change in time}} = \dfrac{v_{final} - v_{initial}}{t_{final} - t_{initial}} = \dfrac{\Delta v}{\Delta t}$.

Instantaneous acceleration: $a = \dfrac{dv}{dt}$.

In straight line motion problems you should always:

1. Specify the positive and negative direction.
2. Specify an origin from which position is measured.

It is often helpful to draw an appropriate diagram or graph that describes the motion of the object. This lets the situation be visualised and helps organise the given information.

• Position, displacement and distance (when finding distance travelled a sketch graph will usually be needed if there are direction changes. The magnitude of displacement is NOT distance).

• $v = \frac{dx}{dt}$, $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$, Speed $= \lvert v \rvert$.
• Change in position: $\Delta x = x(t_2) - x(t_1) = \int_{t_1}^{t_2} v(t)\, dt$. • Position: $x = \int_{t_1}^{t} v(t)\, dt + x(t_1)$ (the integral solution).
• Change in velocity: $\Delta v = v(t_2) - v(t_1) = \int_{t_1}^{t_2} a(t)\, dt$. • Velocity: $v = \int_{t_1}^{t} a(t)\, dt + v(t_1)$ (the integral solution).
• Distance travelled from $t = t_1$ to $t = t_2$: $\int_{t_1}^{t_2} \lvert v(t) \rvert\, dt =$ magnitude of area between the graph of $v = v(t)$ and the t-axis from $t = t_1$ to $t = t_2$.
• Average velocity: $\bar{v} = \frac{\Delta x}{\Delta t}$. • Average speed: $\frac{\text{Distance}}{\Delta t}$. • Average acceleration: $\bar{a} = \frac{\Delta v}{\Delta t}$.

• You must always:

1. Specify the positive and negative direction.
2. Specify an origin from which position is measured.

***Analysis of straight line motion under <u>constant</u> acceleration using formulae (these formulae are NOT specifically in the VCE Study Design and are NOT given on the VCAA formula sheet).**

- When the acceleration of an object is **constant** (that is, acceleration does NOT change over time):

$$v = u + at, \qquad s = \frac{1}{2}(u+v)t, \qquad s = ut + \frac{1}{2}at^2, \qquad s = vt - \frac{1}{2}at^2, \qquad 2as = v^2 - u^2$$

a is the **constant** acceleration of the object.

v is the **final** velocity of the object. u is the **initial** velocity of the object.

s is the displacement of the object during the part of its motion under consideration.

For a given problem, defining a convenient origin relative to which position is measured often enables the displacement of the object to be more easily seen.

t is the time taken by the object to complete the part of its motion under consideration.

<u>NOTE</u>: The terms 'deceleration' and 'retardation' mean **NEGATIVE** acceleration, that is, acceleration that is in a direction opposite to the motion.

- Acceleration due to gravity.

A common situation is an object released or projected vertically into the air with an initial velocity. When an object is projected vertically in the air its speed decreases as it rises. When an object is let to fall vertically in the air its speed increases as it approaches the Earth. Both the rise and fall of objects is governed by the gravitational attraction of the Earth.

An object released or projected vertically into the air with an initial velocity is thereafter subject to constant acceleration due only to gravity UNDER THE ASSUMPTION THAT RETARDATION DUE TO AIR RESISTANCE IS NEGLIGIBLE. The five straight line motion formulae can therefore be applied.

The direction of the acceleration due to gravity is always **towards the Earth to the ground.**

***Velocity-Time Graphs and Their Use.**

Information obtainable from a *velocity - time* graph:

Velocity: read directly from graph.

Acceleration at time $t = t_1$: gradient of curve at $t = t_1$.

Change in position (displacement) between times $t = t_1$ and $t = t_2$: **signed area** between the curve and the t-axis, between $t = t_1$ and $t = t_2$.

Distance travelled between times $t = t_1$ and $t = t_2$: **magnitude** of the **area** between the curve and the t-axis, between $t = t_1$ and $t = t_2$.

***Constructing and Solving Differential Equations in Kinematics.**

- Notation: $x(t)$ means position as a function of time.

$x'(t)$ or $\dot{x}$ means $\dfrac{dx}{dt}$. This is velocity as a function of time, $v(t)$.

$x''(t)$ or $\ddot{x}$ means $\dfrac{d}{dt}\left[\dfrac{dx}{dt}\right] = \dfrac{d^2x}{dt^2}$. This is acceleration as a function of time, $a(t)$.

- Solving DE's for position, velocity, acceleration and time, subject to given boundary conditions.

The different derivative forms for acceleration: $a = \dfrac{d^2x}{dt^2} = \dfrac{dv}{dt} = v\dfrac{dv}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$.

Vector functions

*Position vector as a function of time.

The position vector of a point $P(x, y)$ is given by the vector from the origin O to P: $\overrightarrow{OP} = \underset{\sim}{r} = x\,\underset{\sim}{i} + y\,\underset{\sim}{j}$.

If the position of P depends on time, then x and y will be functions of time: $x = x(t)$ and $y = y(t)$.

Hence the position vector of the point P will also be a function of time: $\underset{\sim}{r}(t) = x(t)\,\underset{\sim}{i} + y(t)\,\underset{\sim}{j}$.

*Deriving the cartesian equation of the path specified by the position vector $\underset{\sim}{r}(t) = x(t)\,\underset{\sim}{i} + y(t)\,\underset{\sim}{j}$.

The 'head' of $\underset{\sim}{\boldsymbol{r}}(t)$ traces out the path followed by P. This path is therefore specified by the set of equations

$$x = x(t) \quad \ldots.\ (1)$$

$$y = y(t) \quad \ldots.\ (2)$$

Equations (1) and (2) give the coordinates of P as a function of time.

The variable t is known as the *parameter*. Equations (1) and (2) are therefore called *parametric equations*.

It is possible to derive the cartesian equation of the path specified by a set of parametric equations. To do this, the parametric equations must be combined in such a way that the parameter t gets eliminated.

The domain and range of the cartesian equation are restricted by the set of values taken by the parameter t:

Domain - determined by the range of the function $x = x(t)$.

Range - determined by the range of the function $y = y(t)$.

<u>Note</u>: $\underset{\sim}{\boldsymbol{r}}(t) = x(t)\,\underset{\sim}{\boldsymbol{i}} + y(t)\,\underset{\sim}{\boldsymbol{j}}$ gives

- A description of the path followed by P.
- The location of P on this path at any time t.

- Parametric form of arc length formula: $L = \displaystyle\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

*Sketching the path specified by $\underset{\sim}{r}(t)$.

Paths will include lines, parabolas (including the 'sideways parabola'), circles and ellipses.

*Conditions for two particles to meet.

Two particles with position vectors $\underset{\sim}{r}(t) = x_r(t)\,\underset{\sim}{i} + y_r(t)\,\underset{\sim}{j}$ and $\underset{\sim}{s}(t) = x_s(t)\,\underset{\sim}{i} + y_s(t)\,\underset{\sim}{j}$ respectively will meet (collide) only if $\underset{\sim}{r}(t) = \underset{\sim}{s}(t)$ for some value of *t*. This will occur only if the two equations

$$x_r(t) = x_s(t)\,, \qquad y_r(t) = y_s(t)$$

(obtained by equating $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components) have a COMMON solution or solutions for *t*.

<u>Note</u>: 1. If two particles meet, their paths ALWAYS intersect.

2. If the paths of two particles intersect, the particles themselves DO NOT ALWAYS meet.

*Differentiation and anti-differentiation of a vector with respect to time.

Notation: $\dot{\underset{\sim}{r}}(t) = \dfrac{d\,\underset{\sim}{r}(t)}{dt}$, $\ddot{\underset{\sim}{r}}(t) = \dfrac{d^2\,\underset{\sim}{r}(t)}{dt^2}$.

If $\underset{\sim}{r}(t) = x(t)\,\underset{\sim}{i} + y(t)\,\underset{\sim}{j}$, then:

1. $\dot{\underset{\sim}{r}}(t) = \dot{x}(t)\,\underset{\sim}{i} + \dot{y}(t)\,\underset{\sim}{j} = \dfrac{dx}{dt}\underset{\sim}{i} + \dfrac{dy}{dt}\underset{\sim}{j}$.

2. $\ddot{\underset{\sim}{r}}(t) = \ddot{x}(t)\,\underset{\sim}{i} + \ddot{y}(t)\,\underset{\sim}{j} = \dfrac{d^2x}{dt^2}\underset{\sim}{i} + \dfrac{d^2y}{dt^2}\underset{\sim}{j}$.

3. $\underset{\sim}{r}(t) = \int \dot{\underset{\sim}{r}}(t)\,dt = \int \dot{x}(t)\,dt\,\underset{\sim}{i} + \int \dot{y}(t)\,dt\,\underset{\sim}{j} + \underset{\sim}{C}$, where $\underset{\sim}{C}$ is an arbitrary constant vector:. $\underset{\sim}{C} = C_1\,\underset{\sim}{i} + C_2\,\underset{\sim}{j}$

4. $\dot{\underset{\sim}{r}}(t) = \int \ddot{\underset{\sim}{r}}(t)\,dt = \int \ddot{x}(t)\,dt\,\underset{\sim}{i} + \int \ddot{y}(t)\,dt\,\underset{\sim}{j} + \underset{\sim}{C}$, where $\underset{\sim}{C}$ is an arbitrary constant vector.

Along the curve defined by $\underset{\sim}{r}(t) = x(t)\,\underset{\sim}{i} + y(t)\,\underset{\sim}{j}$: $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx}$.

Distance travelled along a curve: $L = \displaystyle\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$.

*Kinematics in two and three dimensions.

Velocity and acceleration for motion along a curve specified by the position vector $\underset{\sim}{r}(t)$.

$\dot{\underset{\sim}{r}}(t)$ gives velocity as a function of time.

Note: $\dot{\underset{\sim}{r}}(t)$ is always tangent to the curve (NOT $\underset{\sim}{r}(t)$) and is always in the direction of motion.

$|\dot{\underset{\sim}{r}}(t)|$ gives speed as a function of time.

$\ddot{\underset{\sim}{r}}(t)$ gives acceleration as a function of time.

Finding the position, velocity and acceleration of an object moving along a curve.

Finding the Cartesian equation of the path followed by an object.

Eg. Projectile motion.. Angle at impact of a projectile. Shortest distance of an object from a point.

Dynamics

*Dynamics - objects are **moving** under the action of forces.*

*Forces in two dimensions.

A force is something that pushes or pulls. It is a vector. Forces result from the *interaction between objects*.

Units:

SI unit: Newton (N). A net force of 1 N causes a mass of 1 kg to accelerate at 1 ms^{-2}.

Alternative unit: Kilogram weight (kg wt). A force of 1 kg wt is the force exerted by gravity on a mass of 1kg.

Conversion of units: $1\text{ kg wt} = g\text{ N},\quad 1\text{ N} = \frac{1}{g}\text{ kg wt}$.

*Types of forces.

- Force due to gravity: weight $W = mg$.

  The direction of W is down.

- Force due to contact with a surface: Normal Reaction Force R .

  The direction of R is perpendicular to the surface.

- Force due to sliding friction.

  The direction of sliding friction is always opposite to the motion of the object at the point(s) of contact.

- Forces exerted by ropes, strings, chains *etc*.: tension force T .
- Forces due to air resistance, water resistance *etc*.: drag force.

  The direction of the drag force on an object is always opposite to the direction of the velocity of that object.

  The magnitude of the drag force on an object depends on the speed of that object.

- Driving forces due to engines *etc*.
- Resolving a force in a given direction.
- Net force acting on an object: $\boldsymbol{F}_{net} = \text{vector sum of all individual forces acting on the object}$.

*Newton's Laws of Motion.

***1st Law*:** If there is no net force acting on an object, then the velocity of that object does not change.

***2nd Law*:** If there is a net force acting on an object, then $\boldsymbol{F}_{net} = m\boldsymbol{a}$ (assuming that mass is constant).

Note: $\boldsymbol{F}_{net} = m\boldsymbol{a} \Rightarrow \boldsymbol{a} = \dfrac{\boldsymbol{F}_{net}}{m}$.

***3rd Law*:** The *force exerted by object A upon object B* is equal in magnitude and opposite in direction to the *force exerted by object B upon object A.*

*Applying Newton's Laws of Motion to a system.

***Step 1*:** Draw a clear diagram showing all objects comprising the system.

***Step 2*:** Identify all the forces acting on each object comprising the system.
Use clearly labeled arrows to represent these forces.

***Step 3*:** Where necessary, resolve each force into two perpendicular components.

Direction of components:

For objects in contact with a surface, choose the two components to be in directions parallel to the surface and perpendicular to the surface.
Otherwise, choose the two components to be in the vertical and horizontal direction.

***Step 4*:** For each direction, write down an expression for the net force acting on each object in terms of the individual forces acting on that object.

***Step 5*:** Apply Newton's *2nd Law* to get the equations of motion for each object.

All examples are restricted to rectilinear (straight line) motion and include motion on an inclined plane and motion of connected objects.

Note: $\boldsymbol{F}_{net} = m\boldsymbol{a} \Rightarrow \boldsymbol{a} = \dfrac{\boldsymbol{F}_{net}}{m}$.

• When $\boldsymbol{F}_{net}$ is **constant**, acceleration is constant - straight line motion formulae or a differential equation can be used.

• When $\boldsymbol{F}_{net}$ is **variable**, acceleration is NOT constant - a differential equation involving acceleration must be set up and solved.

• Interpreting the reading of a spring balance or a set of bathroom scales:

The reading of a spring balance gives the tension force in units of kg wt.

The reading of a set of bathroom scales gives the normal reaction force in units of kg wt.

*Equilibrium.

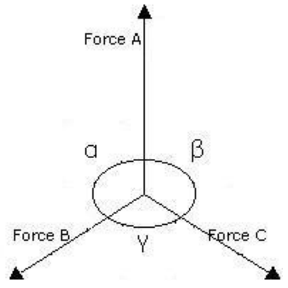
- Definition: $\boldsymbol{F}_{net} = 0 \Rightarrow \boldsymbol{a} = 0 \Rightarrow$ velocity is constant (does not change).

- Geometric meaning of $\boldsymbol{F}_{net} = 0$:

  The vector sum of all the individual forces acting on the object forms a closed polygon.

- Equilibrium situations involving **three coplanar forces -** *Lami's Theorem*:

Let force A, force B and force C be three coplanar forces with magnitudes A, B and C respectively acting on an object. Let the angle between these forces be α, γ and β (so that γ, β and α are the angles directly opposite force A, force B and force C respectively):



If the object is in equilibrium then $\dfrac{A}{\sin\gamma} = \dfrac{B}{\sin\beta} = \dfrac{C}{\sin\alpha}$.

Note: When using Lami's Theorem to find the angle between two forces, the *ambiguous case* may arise.

*Momentum.

Definition: $p = mv = m\dfrac{dr}{dt}$. Momentum is a vector. SI unit: kg ms^{-1}.

Probability and Statistics.

*Linear combinations of random variables.

- For random variables X and Y:

$E(aX+b)=aE(X)+b$;

$E(aX+bY)=aE(X)+bE(Y)$;

$\text{Var}(aX+b)=a^2\text{Var}(X)$.

- For <u>independent</u> random variables X and Y:

$\text{Var}(aX+bY)=a^2\text{Var}(X)+b^2\text{Var}(Y)$.

- For <u>independent</u> random variables X and Y with <u>normal distributions</u>:

$aX+bY$ has a normal distribution.

Note: There is a difference between the random variable mX and the random variable $X_1+X_2+\ldots+X_m$ (that is, $mX \neq X_1+X_2+\ldots+X_m$) where X_1, X_2, $\ldots X_m$ are independent copies of X (a context for seeing this is given in Non-Routine Question 1):

• They have different variances. For example, $\text{Var}(2X)=4\text{Var}(X)$ and $\text{Var}(X_1+X_2)=2\text{Var}(X)$ (using the above formulae).

• In general their probability density functions belong to different 'families' (one of the few exceptions is if X has a normal distribution - then they both have a normal distribution *but with different variances*).
For example, let X be a random variable with pdf given by

*Sample means.

Let X be a random variable defined on a population with mean μ and sd σ. Let a random sample be chosen from the population.

- The sample mean $\bar{X}$ is a random variable whose value varies between samples.
- Simulation of repeated random sampling from a variety of distributions and a range of sample sizes.
- Properties of the distribution of $\bar{X}$ across samples of a fixed size *n*.
- When the sample (large or small) is drawn from a <u>normal distribution</u> of mean μ and sd σ then $\bar{X}$ is normal:

$\bar{X} \sim \text{Norm}\left(\mu, \dfrac{\sigma}{\sqrt{n}}\right)$.

- When the sample is <u>large</u> ($n \geq 50$) and drawn from any distribution of mean μ and sd σ then $\bar{X}$ is approximately normal:

$\bar{X} \sim \text{Norm}\left(\mu, \dfrac{\sigma}{\sqrt{n}}\right)$.

- When the sample is <u>large</u> ($n \geq 50$) and drawn from a population with any distribution with <u>unknown</u> sd, *s* is a sufficiently accurate estimate of the population standard deviation σ (that is, $\sigma \approx s$ for large samples):

$\overline{X} \sim \text{Norm}\left(\mu_{\overline{X}} = \mu,\ \sigma_{\overline{X}} = \dfrac{s}{\sqrt{n}}\right)$ where s is the sample sd.

Calculation of s (using either formula or calculator) is **NOT** required.

It is essential that you understand the conceptual difference between the standard deviations σ ,

$\sigma_{\overline{X}} = \dfrac{\sigma}{\sqrt{n}}$ **and s:•** σ (sd of the random variable X defined on a population).

• $\sigma_{\overline{X}} = \dfrac{\sigma}{\sqrt{n}}$ (sd deviation of the random variable $\overline{X}$ (the mean of a random sample drawn from the population)). • s (sd of a given sample).

• z-values for $\overline{X}$: $Z = \dfrac{\overline{X} - \mu_{\overline{X}}}{\sigma_{\overline{X}}}$.

*Confidence Intervals for the Mean of a Population.

- The value of the sample mean $\bar{x}$ can be used to estimate the population mean μ .
- An interval that we are reasonably sure contains the population mean μ is called a **confidence interval**.
- Variations in confidence intervals between samples.
- μ Construction of an approximate $100(1-\alpha)$ % confidence interval for the unknown mean μ of a population:
- Approximate 95% confidence intervals :

***Sample** (large or small) drawn from a normal distribution of known sd σ : $\left(\bar{x} - 1.96\dfrac{\sigma}{\sqrt{n}},\ \bar{x} + 1.96\dfrac{\sigma}{\sqrt{n}}\right)$.

Large sample drawn from any distribution of unknown variance: $\left(\bar{x} - 1.96\dfrac{s}{\sqrt{n}},\ \bar{x} + 1.96\dfrac{s}{\sqrt{n}}\right)$.

- Using the CAS calculator to directly get a confidence interval.

*Hypothesis testing for a population mean with a sample drawn from a normal distribution of known variance or for a large sample drawn from a distribution of unknown variance.

A statistical hypothesis is a statement about the probability distribution associated with the population or model. In Specialist Mathematics this will reduce to a statement about the population mean μ .

- Formulation of a null hypothesis and an alternative hypothesis:

  * The *null hypothesis* H_0 is the hypothesis under test and is usually a ‘no effect’ statement: $\mu = \mu_o$.

  * The *alternative hypothesis* H_1 is the ‘effect’ hypothesis:

  $\mu \neq \mu_o$ (two-sided alternative).

  $\mu > \mu_o$ (one-sided alternative).

  $\mu < \mu_o$ (one-sided alternative).

• p-values for hypothesis testing related to the mean:

p-value $=$ Pr (value of $\bar{x}$ as favourable to H_1 as the one observed | H_0 true).

H_0	H_1	Favourable to H_1
$\mu = \mu_o$	$\mu \neq \mu_o$	$\lvert \bar{X} - \mu_o \rvert$ large
$\mu = \mu_o$	$\mu > \mu_o$	$\bar{X}$ large
$\mu = \mu_o$	$\mu < \mu_o$	$\bar{X}$ small

The smaller the value of p, that is, the less likely it is to obtain the observed value of $\bar{x}$ under H_0, the greater the evidence against H_0.

• Critical values for the standard normal distribution (these values are not given on the VCAA formula sheet).

• Confidence intervals for hypothesis testing related to the mean.

• Errors in hypothesis testing.

*** Type 1 error** Reject H_0 when H_0 is true.

Pr (Type 1 error) $=$ Pr (reject $H_0 \mid H_0$ is true) $= \alpha$.

α is the significance level of the test.

*** Type 2 error** Accept H_0 when H_1 is true.

Pr (Type 2 error) $=$ Pr (accept $H_0 \mid H_1$ is true).

A type I error (false-positive) occurs if an investigator rejects a null hypothesis that is actually true in the population; a type II error (false-negative) occurs if the investigator fails to reject a null hypothesis that is actually false in the population. Although type I and type II errors can never be avoided entirely, the investigator can reduce their likelihood by increasing the sample size (the larger the sample, the lesser is the likelihood that it will differ substantially from the population).

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