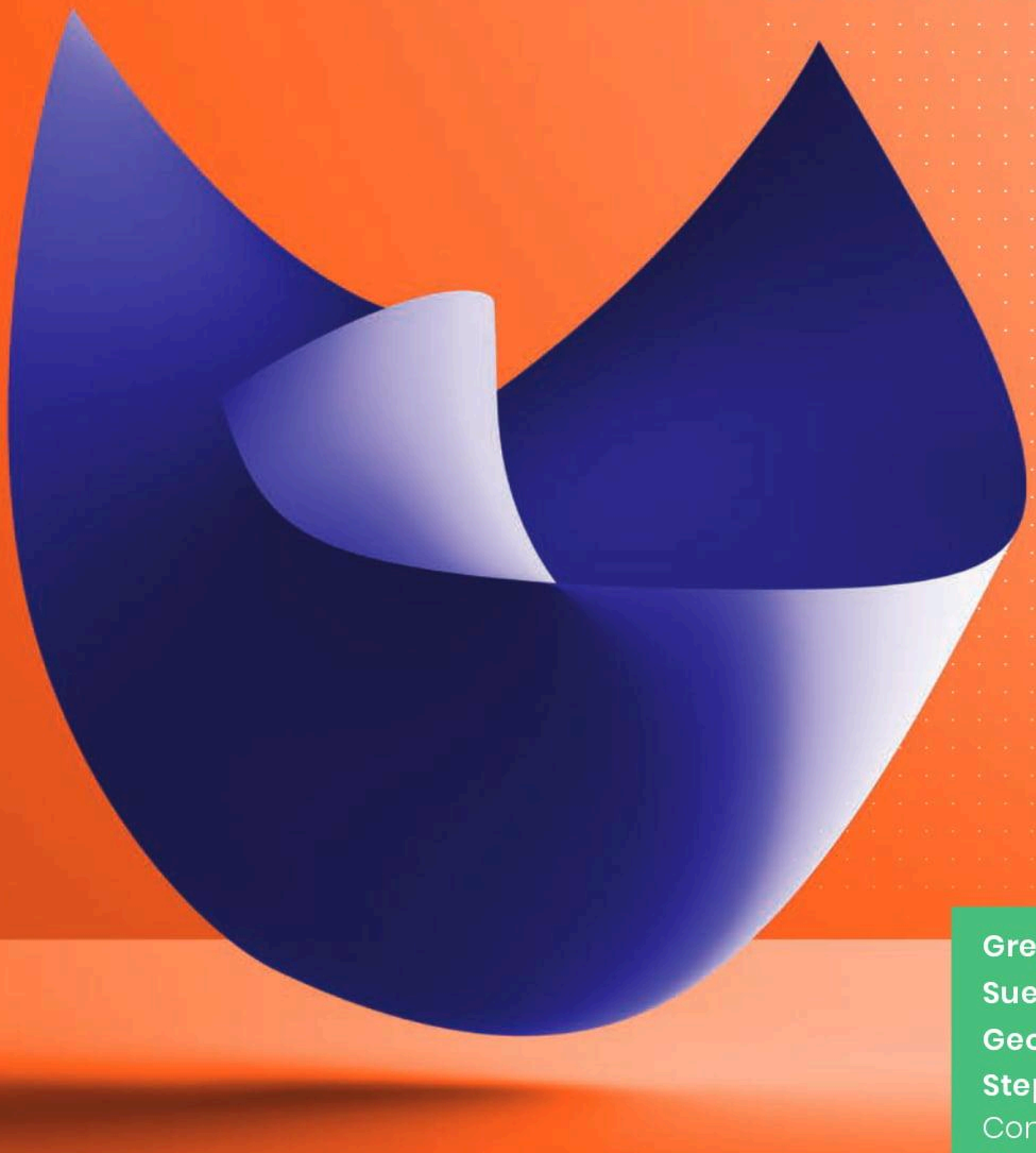
NELSON

VICmaths

VCE UNITS ③ + ④



Greg Neal
Sue Garner
George Dimitriadis
Stephen Swift
Contributing author:
Neale Woods

specialist mathematics 12

Nelson

Nelson VICmaths Specialist Mathematics 12
1st Edition
Greg Neal
Sue Garner
George Dimitriadis
Stephen Swift
ISBN 9780170448543

Publisher: Robert Yen, Dirk Strasser
Project editor: Tanya Smith
Editors: Karen Chin, Kim Wilson
Series text design: Alba Design (Rina Gargano)
Series cover design: Watershed Art & Design (Leigh Ashforth)
Series designer: Nikita Bansal
Permissions researcher: Liz McShane
Production controller: Karen Young
Typeset by: KGL

Acknowledgements
Selected VCE Examination questions and extracts from the VCE Study Designs are copyright Victorian Curriculum and Assessment Authority (VCAA), reproduced by permission. VCE® is a registered trademark of the VCAA. The VCAA does not endorse this product and makes no warranties regarding the correctness or accuracy of this study resource. To the extent permitted by law, the VCAA excludes all liability for any loss or damage suffered or incurred as a result of accessing, using or relying on the content. Current VCE Study Designs, past VCE exams and related content can be accessed directly at www.vcaa.vic.edu.au.

TI-Nspire: Images used with permission by Texas Instruments, Inc
Casio ClassPad: Images used with permission by Shriro Australia Pty. Ltd.

For product information and technology assistance,
in Australia call **1300 790 853**;
in New Zealand call **0800 449 725**

For permission to use material from this text or product, please email
aust.permissions@cengage.com

National Library of Australia Cataloguing-in-Publication Data
A catalogue record for this book is available from the National Library of Australia.

Cengage Learning Australia
Level 7, 80 Dorcas Street
South Melbourne, Victoria Australia 3205

Cengage Learning New Zealand
Unit 4B Rosedale Office Park
331 Rosedale Road, Albany, North Shore 0632, NZ

For learning solutions, visit **cengage.com.au**

Printed in China by 1010 Printing International Limited.
1 2 3 4 5 6 7 25 24 23 22 21

Contents

To the student

Nelson VICmaths is your best friend when it comes to studying Specialist Mathematics in Year 12. It has been written to help you maximise your learning and success this year. Every explanation, every exam hack and every worked example has been written with the exams in mind.

The 3 steps to mastering each topic

STEP 1
Study each Worked example

STEP 2
Complete the Matched example in the Mastery Workbook

STEP 3
Do the Mastery questions in the exercise that are linked to the Worked example

To the teacher

Now there's a better way to VCE maths mastery.

Nelson VICmaths 11–12 is a new VCE mathematics series that is backed by research into the science of learning. The design and structure of the series have been informed by teacher advice and evidence-based pedagogy, with the focus on preparing VCE students for their exams and maximising their learning achievement.

- Using **backwards learning design,** this series has been built by analysing past VCE exam questions and ensuring that all theory and examples are precisely mapped to the VCE Study Design.
- To reduce the **cognitive load** for learners, explanations are clear and concise, using the technique of **chunking** text with accompanying diagrams and infographics.
- The student book and workbook combination has been designed for **mastery** of the learning content.
- The exercise structure of **Recap, Mastery** and **Exam Practice** leads students from procedural fluency to **higher-order thinking** using the learning technique of **interleaving**.
- **Exam practice** includes exam-style questions and graded past VCE exam questions with success percentages based on **VCAA performance data**.
- The cumulative structure of Exercise **Recaps** and chapter-based **Cumulative examinations** is built on the learning and memory techniques of **spacing** and **retrieval**.

About the authors

Greg Neal has taught for over 40 years and was Head of Mathematics at Ballarat High School. He has been a VCAA examination assessor for Mathematical Methods for over 20 years and has expertise with CAS technology. Greg has written a range of Nelson resources for Further Mathematics, Mathematical Methods and Specialist Mathematics.

Sue Garner has taught senior mathematics at Ballarat Grammar School for 30 years. She has been a VCAA examination assessor for Mathematical Methods and Specialist Mathematics for many years and works for the VCAA in the role of Curriculum Manager for VCE Coursework. Sue has expertise with ClassPad technology and regularly presents workshops on CAS.

George Dimitriadis teaches at Montmorency Secondary College and has taught for 40 years. He is a VCAA Examination Development panelist and examination assessor for Specialist Mathematics. George has extensive experience in writing textbooks for Years 7 to 12 and contributing to mathematics enrichment blogs.

Stephen Swift is the lead author of *Nelson Senior Maths* (Australian Curriculum) and *Nelson QMaths* (QLD) for Mathematical Methods and Specialist Mathematics. He has taught all levels of mathematics in several states for 40 years and was Head of Mathematics at Wellington Point State High School in Brisbane.

Contributing authors

Neale Woods coordinated the *Using CAS* content. He is a CAS specialist who often presents at conferences and conducts technology workshops for teachers and students. Neale has been teaching mathematics for over 45 years, including for Virtual Schools Victoria.

Darren Smyth (maffsguru.com) created the VCE question analysis videos.

Study Design grid

Area of study	***Nelson VICmaths Specialist Mathematics 12* chapter**	
1 Discrete mathematics		
Logic and proof	**3**	Logic and proof
2 Functions, relations and graphs		
	2	Rational functions
3 Algebra, number and structure		
Complex numbers	**4**	Complex numbers
4 Calculus		
Differential calculus and integral calculus	**5**	Differentiation
	7	Integration
	8	Areas and volumes of integration
Differential equations	**9**	Differential equations
Kinematics: rectilinear motion	**10**	Kinematics
5 Space and measurement		
Vectors	**1**	Vectors
Vectors and cartesian equations	**6**	Vector equations
Vector calculus	**11**	Vector calculus
6 Data analysis, probability and statistics		
Distribution of linear combinations of random variables	**12**	Random variables and hypothesis testing
Distribution of the sample mean	**12**	Random variables and hypothesis testing
Confidence intervals for the population mean	**12**	Random variables and hypothesis testing
Hypothesis testing for a population mean with a sample drawn from a normal distribution of known variance, or for a large sample	**12**	Random variables and hypothesis testing

About this book

In each chapter

Study Design coverage and extracts are shown at the start of the chapter, along with a listing of **Nelson MindTap chapter resources**.

Study Design coverage

AREA OF STUDY 2: FUNCTIONS, RELATIONS AND GRAPHS

- rational functions and the expression of rational functions of low degree as sums of partial fractions
- graphs of rational functions of low degree, their asymptotic behaviour, and the nature and location of stationary points and points of inflection
- graphs of simple quotient functions, their asymptotic behaviour, and the nature and location of stationary points and points of inflection.

VCE Mathematics Study Design 2023–2027 p. 110, © VCAA 2022

Video playlists (9):

2.1 The conic sections
2.2 Rational functions and partial fractions
2.3 Graphing rational functions
2.4 Graphing quotient functions
2.5 Absolute value functions
2.6 Reciprocal circular functions
2.7 Trigonometric identities
2.8 Inverse circular functions
VCE question analysis Rational functions

Worksheets (7):

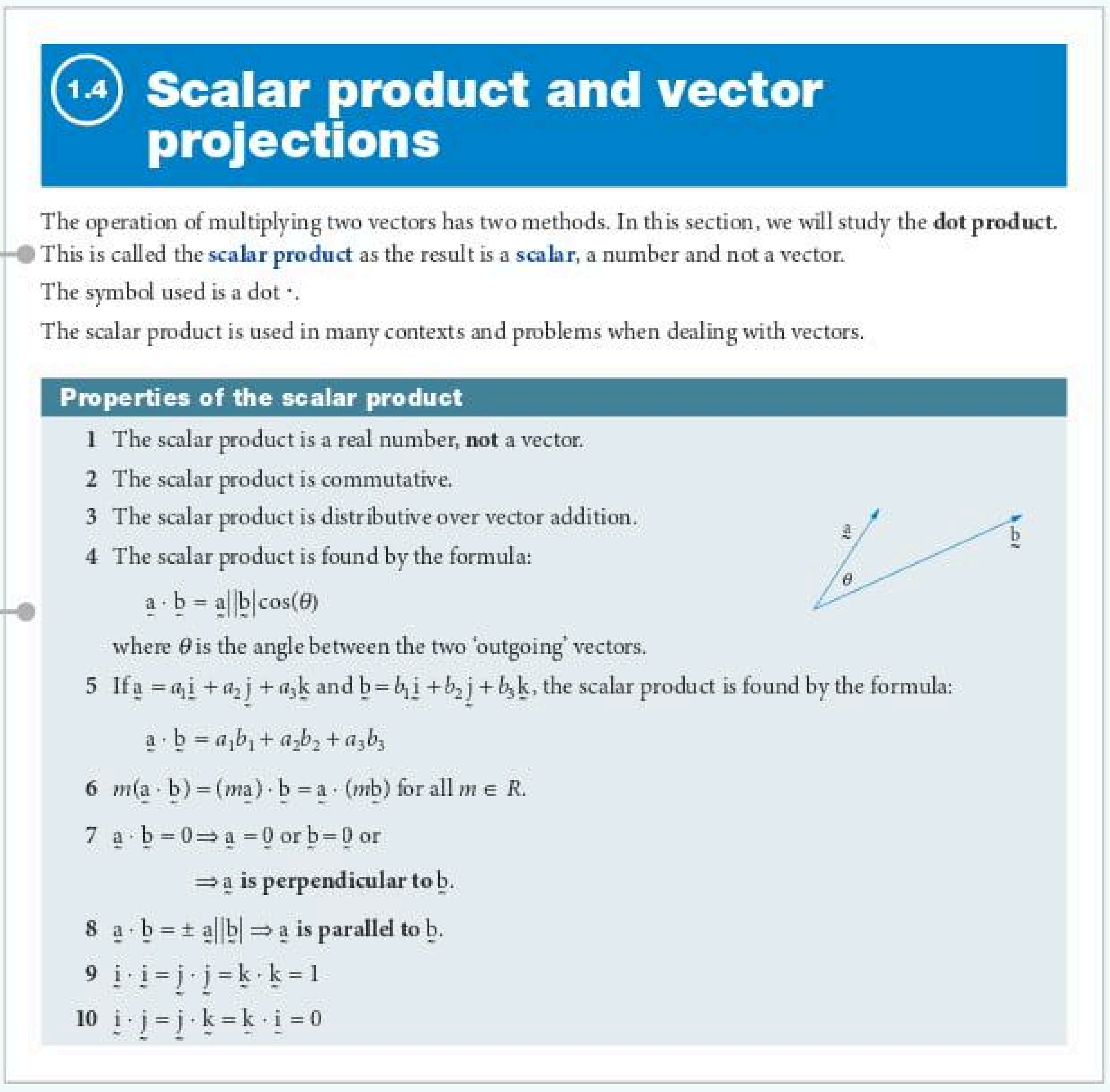
2.5 Absolute value • Absolute value functions • Absolute value inequalities • Transformations of absolute values and hyperbolas
2.7 Simplifying periodic functions • Trigonometric identities
2.8 Inverse functions

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

Important words and phrases are printed in blue and listed in the **Glossary and index** at the back of the book.

Important facts and formulas are highlighted in a shaded box.



1.4 Scalar product and vector projections

The operation of multiplying two vectors has two methods. In this section, we will study the **dot product.** This is called the **scalar product** as the result is a **scalar**, a number and not a vector.

The symbol used is a dot ·.

The scalar product is used in many contexts and problems when dealing with vectors.

Properties of the scalar product

1 The scalar product is a real number, **not** a vector.
2 The scalar product is commutative.
3 The scalar product is distributive over vector addition.
4 The scalar product is found by the formula:

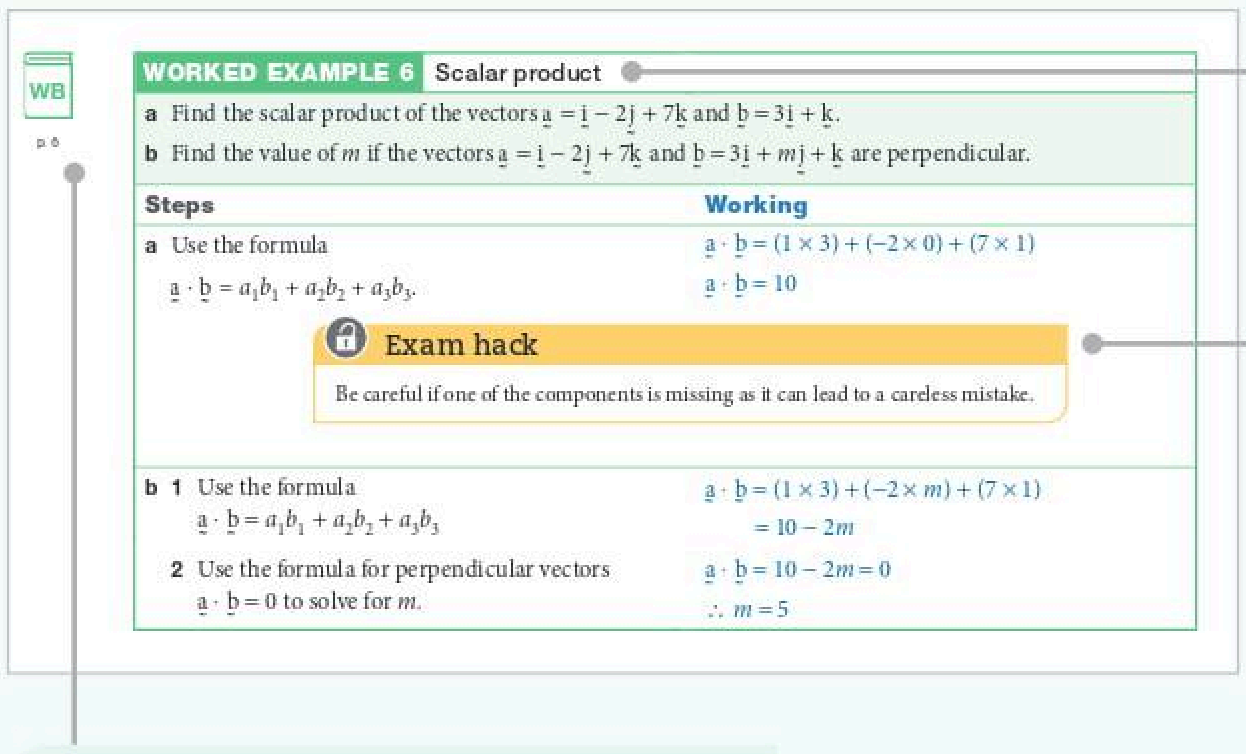
$$\underset{\sim}{a}\cdot\underset{\sim}{b} = \underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta)$$

where θ is the angle between the two 'outgoing' vectors.

5 If $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$ and $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}$, the scalar product is found by the formula:

$$\underset{\sim}{a}\cdot\underset{\sim}{b} = a_1b_1 + a_2b_2 + a_3b_3$$

6 $m(\underset{\sim}{a}\cdot\underset{\sim}{b}) = (m\underset{\sim}{a})\cdot\underset{\sim}{b} = \underset{\sim}{a}\cdot(m\underset{\sim}{b})$ for all $m \in R$.
7 $\underset{\sim}{a}\cdot\underset{\sim}{b} = 0 \Rightarrow \underset{\sim}{a} = \underset{\sim}{0}$ or $\underset{\sim}{b} = \underset{\sim}{0}$ or
$\Rightarrow \underset{\sim}{a}$ **is perpendicular to** $\underset{\sim}{b}$.
8 $\underset{\sim}{a}\cdot\underset{\sim}{b} = \pm\,\underset{\sim}{a}||\underset{\sim}{b}| \Rightarrow \underset{\sim}{a}$ **is parallel to** $\underset{\sim}{b}$.
9 $\underset{\sim}{i}\cdot\underset{\sim}{i} = \underset{\sim}{j}\cdot\underset{\sim}{j} = \underset{\sim}{k}\cdot\underset{\sim}{k} = 1$
10 $\underset{\sim}{i}\cdot\underset{\sim}{j} = \underset{\sim}{j}\cdot\underset{\sim}{k} = \underset{\sim}{k}\cdot\underset{\sim}{i} = 0$



WORKED EXAMPLE 6 Scalar product

a Find the scalar product of the vectors $\underset{\sim}{a} = \underset{\sim}{i} - 2\underset{\sim}{j} + 7\underset{\sim}{k}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + \underset{\sim}{k}$.
b Find the value of m if the vectors $\underset{\sim}{a} = \underset{\sim}{i} - 2\underset{\sim}{j} + 7\underset{\sim}{k}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + m\underset{\sim}{j} + \underset{\sim}{k}$ are perpendicular.

Steps	Working
a Use the formula $\underset{\sim}{a}\cdot\underset{\sim}{b} = a_1b_1 + a_2b_2 + a_3b_3$.	$\underset{\sim}{a}\cdot\underset{\sim}{b} = (1\times3) + (-2\times0) + (7\times1)$ $\underset{\sim}{a}\cdot\underset{\sim}{b} = 10$
b **1** Use the formula $\underset{\sim}{a}\cdot\underset{\sim}{b} = a_1b_1 + a_2b_2 + a_3b_3$	$\underset{\sim}{a}\cdot\underset{\sim}{b} = (1\times3) + (-2\times m) + (7\times1)$ $= 10 - 2m$
2 Use the formula for perpendicular vectors $\underset{\sim}{a}\cdot\underset{\sim}{b} = 0$ to solve for m.	$\underset{\sim}{a}\cdot\underset{\sim}{b} = 10 - 2m = 0$ $\therefore m = 5$

Exam hack

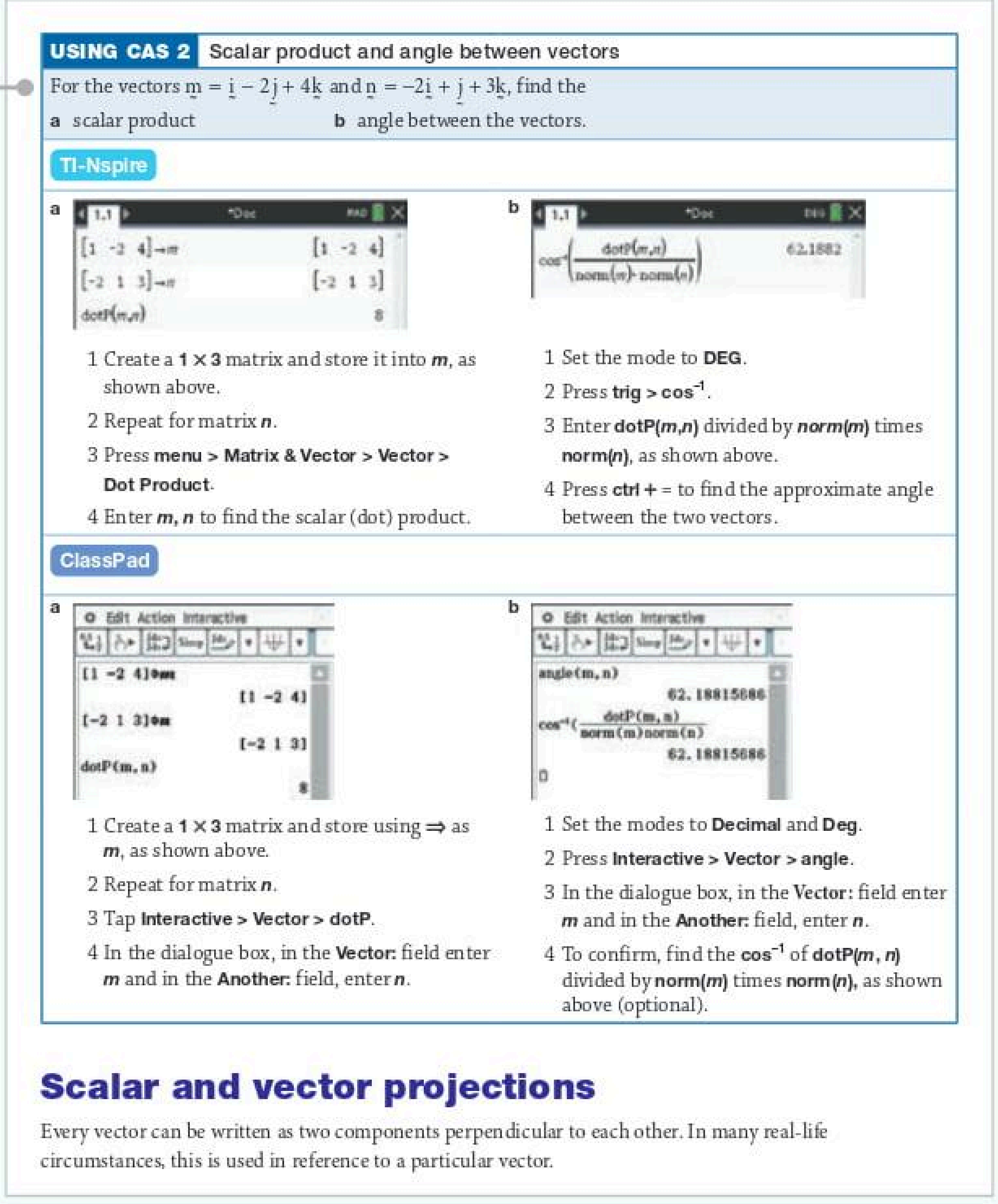
Be careful if one of the components is missing as it can lead to a careless mistake.

Worked examples are explained clearly step-by-step, with the mathematical working shown on the right-hand side.

Exam hacks highlight valuable exam hints and common student errors.

Links to scaffolded matched examples in the Mastery Workbook (WB).

Using CAS provides clear instructions for TI-Nspire and Casio ClassPad calculators.



USING CAS 2 Scalar product and angle between vectors

For the vectors $\underset{\sim}{m} = \underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}$ and $\underset{\sim}{n} = -2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$, find the

a scalar product **b** angle between the vectors.

TI-Nspire

a

```
[1 -2 4]→m        [1 -2 4]
[-2 1 3]→n        [-2 1 3]
dotP(m,n)                8
```

1 Create a **1 × 3** matrix and store it into ***m***, as shown above.
2 Repeat for matrix ***n***.
3 Press **menu > Matrix & Vector > Vector > Dot Product**.
4 Enter ***m***, ***n*** to find the scalar (dot) product.

b

$\cos^{-1}\left(\dfrac{\text{dotP}(m,n)}{\text{norm}(m)\cdot\text{norm}(n)}\right)$ 62.1882

1 Set the mode to **DEG**.
2 Press **trig > cos^{-1}**.
3 Enter **dotP(*m*,*n*)** divided by ***norm(m)*** times **norm(*n*)**, as shown above.
4 Press **ctrl +** = to find the approximate angle between the two vectors.

ClassPad

a

```
[1 -2 4]⇒m
                [1 -2 4]
[-2 1 3]⇒n
                [-2 1 3]
dotP(m,n)
                       8
```

1 Create a **1 × 3** matrix and store using ⇒ as ***m***, as shown above.
2 Repeat for matrix ***n***.
3 Tap **Interactive > Vector > dotP**.
4 In the dialogue box, in the **Vector:** field enter ***m*** and in the **Another:** field, enter ***n***.

b

```
angle(m,n)
                62.18815686
cos⁻¹(dotP(m,n)/(norm(m)norm(n)))
                62.18815686
```

1 Set the modes to **Decimal** and **Deg**.
2 Press **Interactive > Vector > angle**.
3 In the dialogue box, in the **Vector:** field enter ***m*** and in the **Another:** field, enter ***n***.
4 To confirm, find the **cos^{-1}** of **dotP(*m*, *n*)** divided by **norm(*m*)** times **norm(*n*)**, as shown above (optional).

Scalar and vector projections

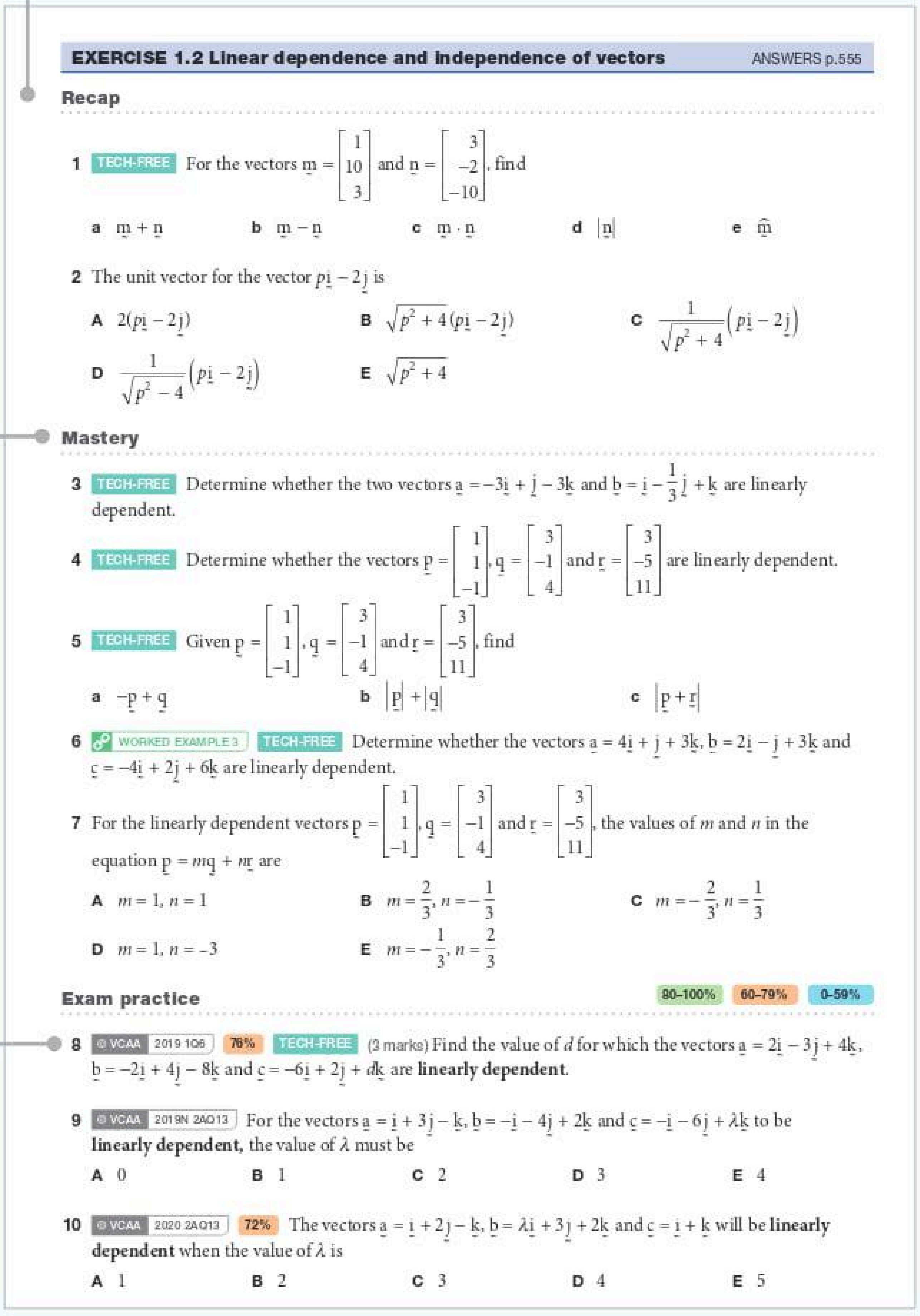
Every vector can be written as two components perpendicular to each other. In many real-life circumstances, this is used in reference to a particular vector.

Graded exercises, including **Recap, Mastery** and **Exam practice** questions, are linked to worked examples and **Using CAS**, and include past VCE exam and exam-style questions.

Recap questions revise skills from the previous exercise, **Mastery** questions provide skill practice, while **Exam practice** applies learned skills to VCE exam and exam-style problems.

© VCAA 2019 1Q6 76% TECH-FREE

Past VCE exam questions are clearly tagged and graded by colour-coded success percentages based on VCAA data, presented in the order: Exam 1, Exam 2A, Exam 2B.



EXERCISE 1.2 Linear dependence and independence of vectors ANSWERS p.555

Recap

1 TECH-FREE For the vectors $\underset{\sim}{m} = \begin{bmatrix} 1 \\ 10 \\ 3 \end{bmatrix}$ and $\underset{\sim}{n} = \begin{bmatrix} 3 \\ -2 \\ -10 \end{bmatrix}$, find

a $\underset{\sim}{m} + \underset{\sim}{n}$ **b** $\underset{\sim}{m} - \underset{\sim}{n}$ **c** $\underset{\sim}{m} \cdot \underset{\sim}{n}$ **d** $|\underset{\sim}{n}|$ **e** $\hat{\underset{\sim}{m}}$

2 The unit vector for the vector $p\underset{\sim}{i} - 2\underset{\sim}{j}$ is

A $2(p\underset{\sim}{i} - 2\underset{\sim}{j})$ **B** $\sqrt{p^2 + 4}(p\underset{\sim}{i} - 2\underset{\sim}{j})$ **C** $\dfrac{1}{\sqrt{p^2 + 4}}\left(p\underset{\sim}{i} - 2\underset{\sim}{j}\right)$

D $\dfrac{1}{\sqrt{p^2 - 4}}\left(p\underset{\sim}{i} - 2\underset{\sim}{j}\right)$ **E** $\sqrt{p^2 + 4}$

Mastery

3 TECH-FREE Determine whether the two vectors $\underset{\sim}{a} = -3\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} - \dfrac{1}{3}\underset{\sim}{j} + \underset{\sim}{k}$ are linearly dependent.

4 TECH-FREE Determine whether the vectors $\underset{\sim}{p} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\underset{\sim}{q} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ and $\underset{\sim}{r} = \begin{bmatrix} 3 \\ -5 \\ 11 \end{bmatrix}$ are linearly dependent.

5 TECH-FREE Given $\underset{\sim}{p} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\underset{\sim}{q} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ and $\underset{\sim}{r} = \begin{bmatrix} 3 \\ -5 \\ 11 \end{bmatrix}$, find

a $-\underset{\sim}{p} + \underset{\sim}{q}$ **b** $|\underset{\sim}{p}| + |\underset{\sim}{q}|$ **c** $|\underset{\sim}{p} + \underset{\sim}{r}|$

6 WORKED EXAMPLE 3 TECH-FREE Determine whether the vectors $\underset{\sim}{a} = 4\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{c} = -4\underset{\sim}{i} + 2\underset{\sim}{j} + 6\underset{\sim}{k}$ are linearly dependent.

7 For the linearly dependent vectors $\underset{\sim}{p} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\underset{\sim}{q} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ and $\underset{\sim}{r} = \begin{bmatrix} 3 \\ -5 \\ 11 \end{bmatrix}$, the values of m and n in the equation $\underset{\sim}{p} = m\underset{\sim}{q} + n\underset{\sim}{r}$ are

A $m = 1, n = 1$ **B** $m = \dfrac{2}{3}, n = -\dfrac{1}{3}$ **C** $m = -\dfrac{2}{3}, n = \dfrac{1}{3}$

D $m = 1, n = -3$ **E** $m = -\dfrac{1}{3}, n = \dfrac{2}{3}$

Exam practice 80–100% 60–79% 0–59%

8 © VCAA 2019 1Q6 76% TECH-FREE (3 marks) Find the value of d for which the vectors $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}$, $\underset{\sim}{b} = -2\underset{\sim}{i} + 4\underset{\sim}{j} - 8\underset{\sim}{k}$ and $\underset{\sim}{c} = -6\underset{\sim}{i} + 2\underset{\sim}{j} + d\underset{\sim}{k}$ are **linearly dependent**.

9 © VCAA 2019N 2AQ13 For the vectors $\underset{\sim}{a} = \underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{b} = -\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{c} = -\underset{\sim}{i} - 6\underset{\sim}{j} + \lambda\underset{\sim}{k}$ to be **linearly dependent**, the value of λ must be

A 0 **B** 1 **C** 2 **D** 3 **E** 4

10 © VCAA 2020 2AQ13 72% The vectors $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{b} = \lambda\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{c} = \underset{\sim}{i} + \underset{\sim}{k}$ will be **linearly dependent** when the value of λ is

A 1 **B** 2 **C** 3 **D** 4 **E** 5

KEY

2021	2021 exam year
<year>N	Northern hemisphere exam
<year>S	VCAA sample exam
TECH-FREE	Technology not permitted
1	Exam 1 (tech-free)
2A	Exam 2 Section A (multiple-choice)
2B	Exam 2 Section B

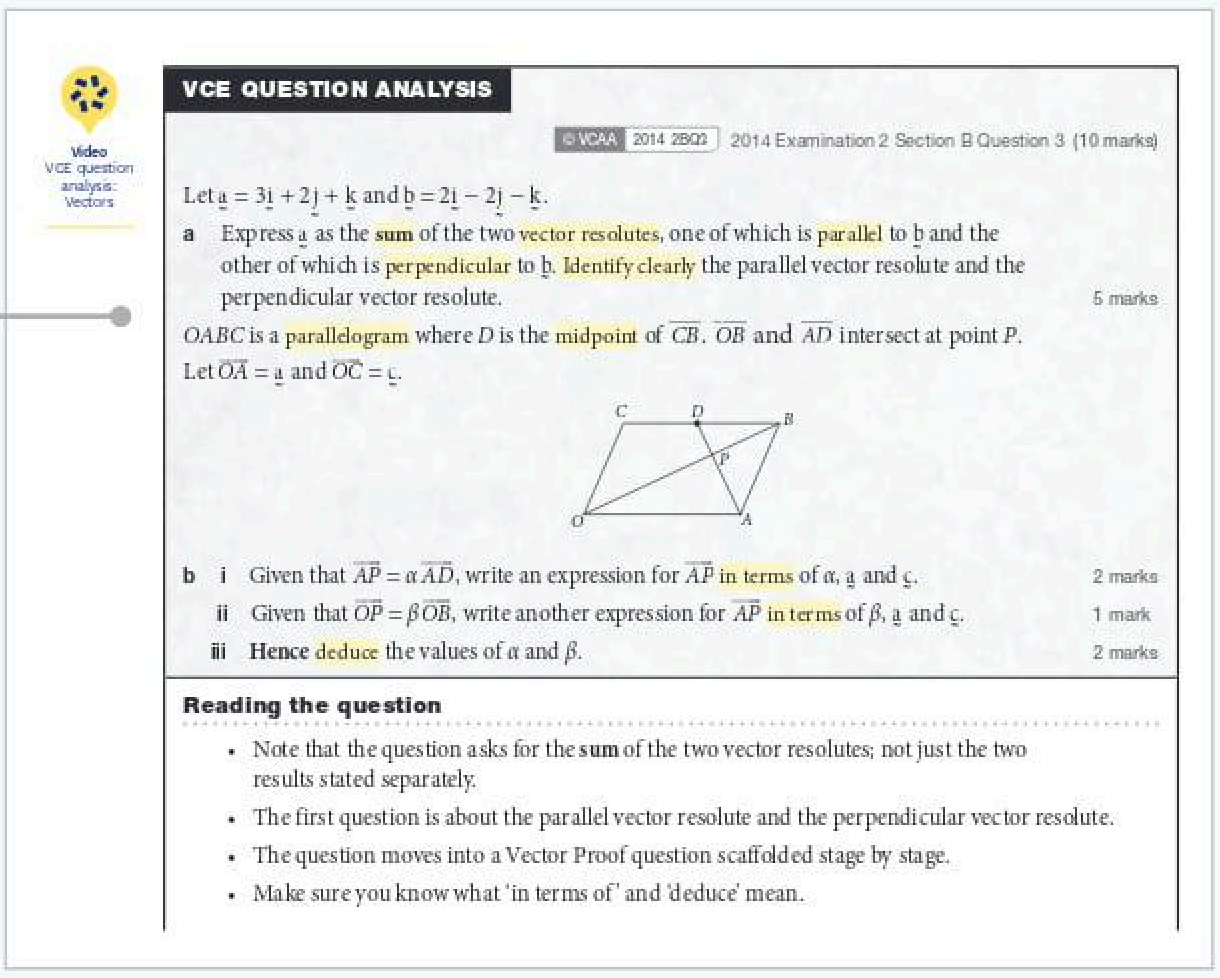
For multiple-choice questions and other 1-mark questions, the success percentage is the percentage of students who answered correctly. For questions worth more than 1 mark, the success percentage is the mean percentage result scored by students.

Success percentage range	Example
80–100% (straightforward)	83%
60–79% (standard)	62%
0–59% (complex)	45%

Note: Questions from Northern hemisphere and VCAA sample exams do not have success percentages.

At the end of each chapter

VCE question analysis leads students through a past VCE exam question that exemplifies the chapter, discussing how to approach the question, providing advice on interpreting the question, common student errors, and statistics on student performance.



Video
VCE question analysis: Vectors

VCE QUESTION ANALYSIS

©VCAA 2014 2BQ3 2014 Examination 2 Section B Question 3 (10 marks)

Let $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$.

a Express $\underset{\sim}{a}$ as the **sum** of the two vector resolutes, one of which is parallel to $\underset{\sim}{b}$ and the other of which is perpendicular to $\underset{\sim}{b}$. Identify clearly the parallel vector resolute and the perpendicular vector resolute. 5 marks

$OABC$ is a parallelogram where D is the midpoint of $\overline{CB}$. $\overline{OB}$ and $\overline{AD}$ intersect at point P.

Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.

b i Given that $\overrightarrow{AP} = \alpha\overrightarrow{AD}$, write an expression for $\overrightarrow{AP}$ in terms of α, $\underset{\sim}{a}$ and $\underset{\sim}{c}$. 2 marks

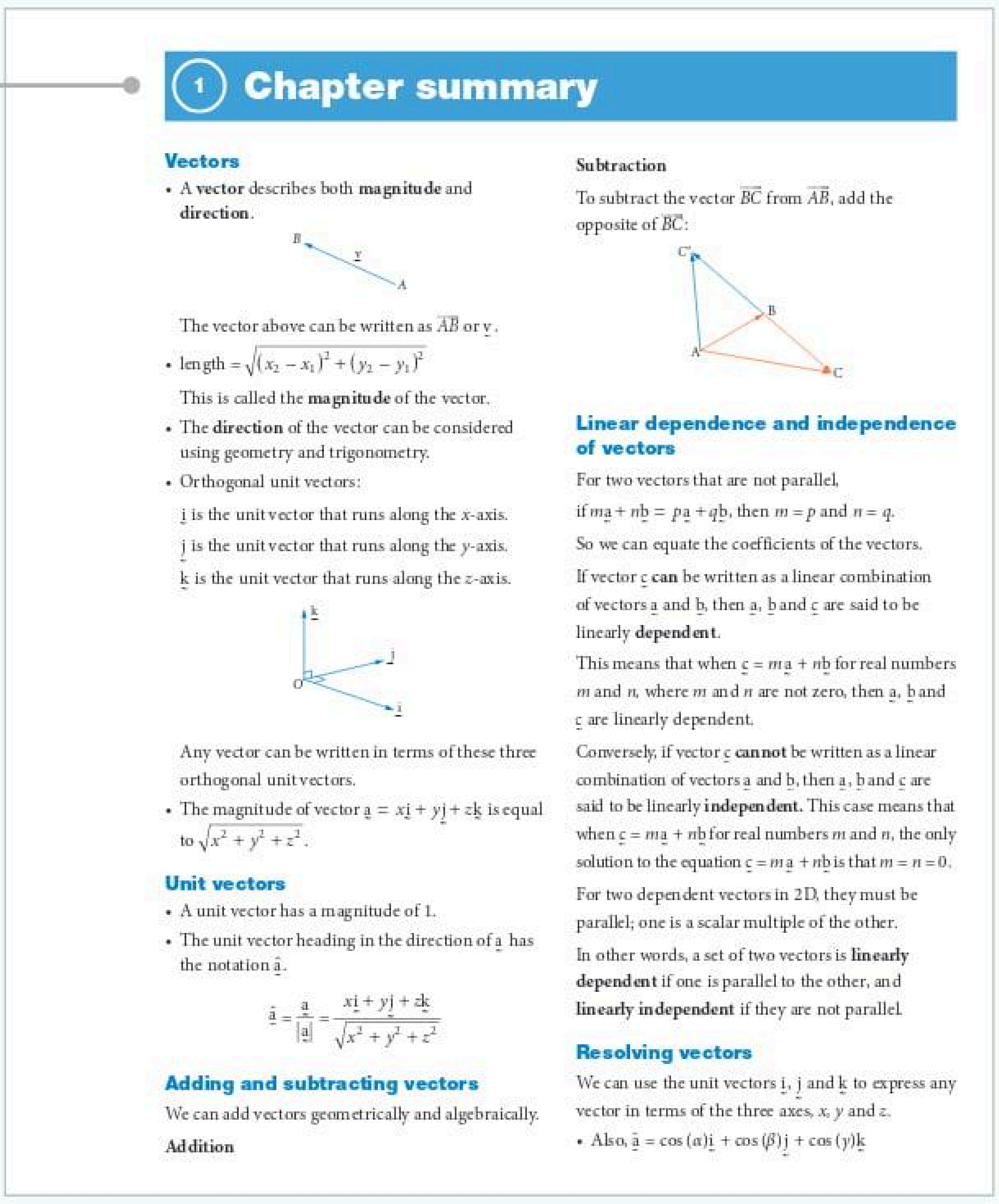
ii Given that $\overrightarrow{OP} = \beta\overrightarrow{OB}$, write another expression for $\overrightarrow{AP}$ in terms of β, $\underset{\sim}{a}$ and $\underset{\sim}{c}$. 1 mark

iii **Hence** deduce the values of α and β. 2 marks

Reading the question

- Note that the question asks for the **sum** of the two vector resolutes; not just the two results stated separately.
- The first question is about the parallel vector resolute and the perpendicular vector resolute.
- The question moves into a Vector Proof question scaffolded stage by stage.
- Make sure you know what 'in terms of' and 'deduce' mean.

Chapter summary for easy reference.



① **Chapter summary**

Vectors

- A **vector** describes both **magnitude** and **direction**.

The vector above can be written as $\overrightarrow{AB}$ or $\underset{\sim}{v}$.

- length $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

This is called the **magnitude** of the vector.

- The **direction** of the vector can be considered using geometry and trigonometry.
- Orthogonal unit vectors:

$\underset{\sim}{i}$ is the unit vector that runs along the x-axis.

$\underset{\sim}{j}$ is the unit vector that runs along the y-axis.

$\underset{\sim}{k}$ is the unit vector that runs along the z-axis.

Any vector can be written in terms of these three orthogonal unit vectors.

- The magnitude of vector $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ is equal to $\sqrt{x^2 + y^2 + z^2}$.

Unit vectors

- A unit vector has a magnitude of 1.
- The unit vector heading in the direction of $\underset{\sim}{a}$ has the notation $\hat{\underset{\sim}{a}}$.

$$\hat{\underset{\sim}{a}} = \frac{\underset{\sim}{a}}{|\underset{\sim}{a}|} = \frac{x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}}{\sqrt{x^2 + y^2 + z^2}}$$

Adding and subtracting vectors

We can add vectors geometrically and algebraically.

Addition

Subtraction

To subtract the vector $\overrightarrow{BC}$ from $\overrightarrow{AB}$, add the opposite of $\overrightarrow{BC}$:

Linear dependence and independence of vectors

For two vectors that are not parallel,

if $m\underset{\sim}{a} + n\underset{\sim}{b} = p\underset{\sim}{a} + q\underset{\sim}{b}$, then $m = p$ and $n = q$.

So we can equate the coefficients of the vectors.

If vector $\underset{\sim}{c}$ **can** be written as a linear combination of vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, then $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are said to be linearly **dependent**.

This means that when $\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$ for real numbers m and n, where m and n are not zero, then $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are linearly dependent.

Conversely, if vector $\underset{\sim}{c}$ **cannot** be written as a linear combination of vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, then $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are said to be linearly **independent.** This case means that when $\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$ for real numbers m and n, the only solution to the equation $\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$ is that $m = n = 0$.

For two dependent vectors in 2D, they must be parallel; one is a scalar multiple of the other.

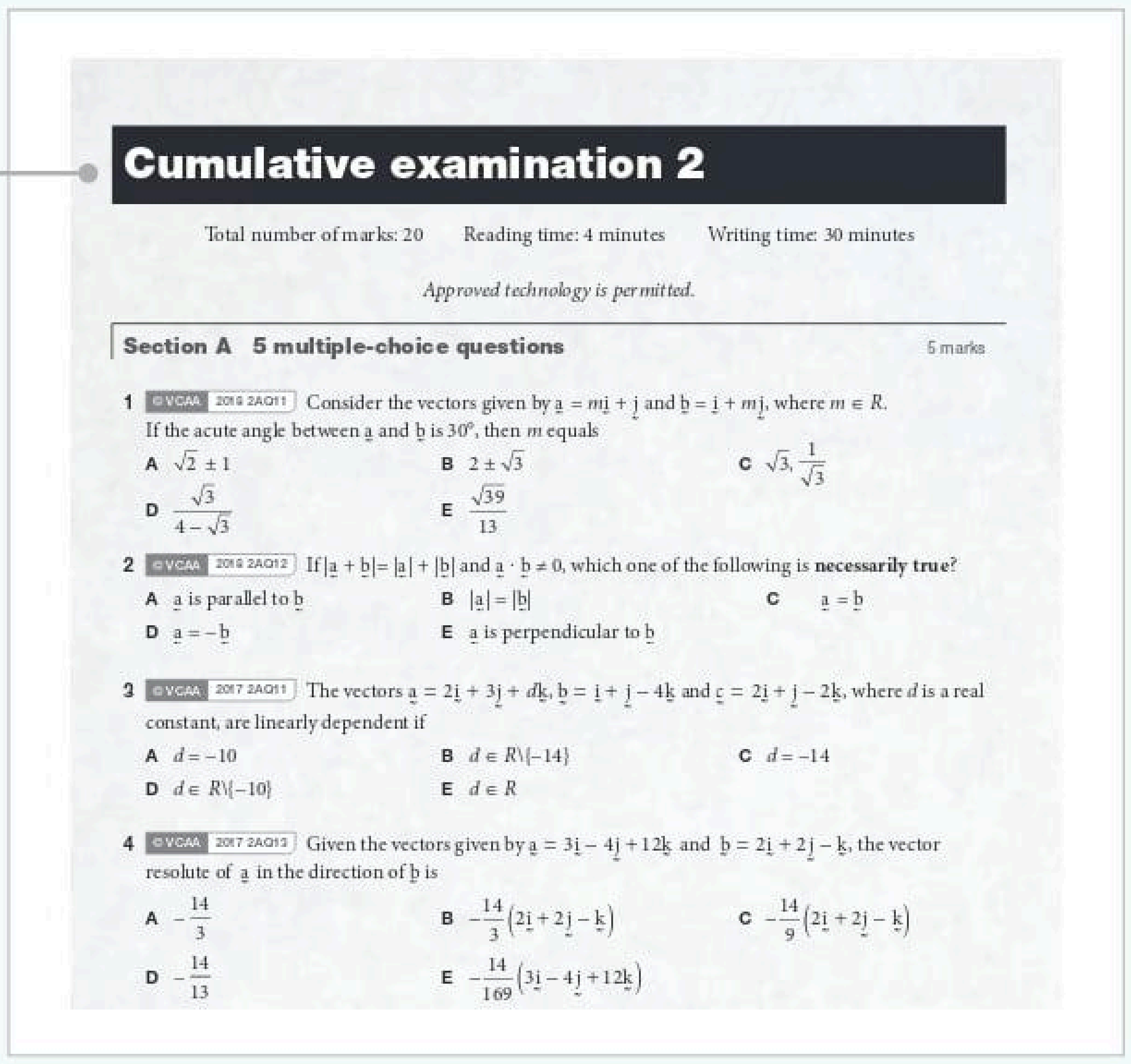
In other words, a set of two vectors is **linearly dependent** if one is parallel to the other, and **linearly independent** if they are not parallel.

Resolving vectors

We can use the unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ to express any vector in terms of the three axes, x, y and z.

- Also, $\hat{\underset{\sim}{a}} = \cos(\alpha)\underset{\sim}{i} + \cos(\beta)\underset{\sim}{j} + \cos(\gamma)\underset{\sim}{k}$

Cumulative examinations 1 and **2** are mini-exams based on the format of the VCE examinations 1 and 2, with around 50% of questions focusing on the chapter in which they appear.



Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions 5 marks

1 ©VCAA 2018 2AQ11 Consider the vectors given by $\underset{\sim}{a} = m\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{b} = \underset{\sim}{i} + m\underset{\sim}{j}$, where $m \in R$. If the acute angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is 30°, then m equals

A $\sqrt{2} \pm 1$
B $2 \pm \sqrt{3}$
C $\sqrt{3}, \frac{1}{\sqrt{3}}$
D $\frac{\sqrt{3}}{4 - \sqrt{3}}$
E $\frac{\sqrt{39}}{13}$

2 ©VCAA 2018 2AQ12 If $|\underset{\sim}{a} + \underset{\sim}{b}| = |\underset{\sim}{a}| + |\underset{\sim}{b}|$ and $\underset{\sim}{a} \cdot \underset{\sim}{b} \neq 0$, which one of the following is **necessarily true**?

A $\underset{\sim}{a}$ is parallel to $\underset{\sim}{b}$
B $|\underset{\sim}{a}| = |\underset{\sim}{b}|$
C $\underset{\sim}{a} = \underset{\sim}{b}$
D $\underset{\sim}{a} = -\underset{\sim}{b}$
E $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$

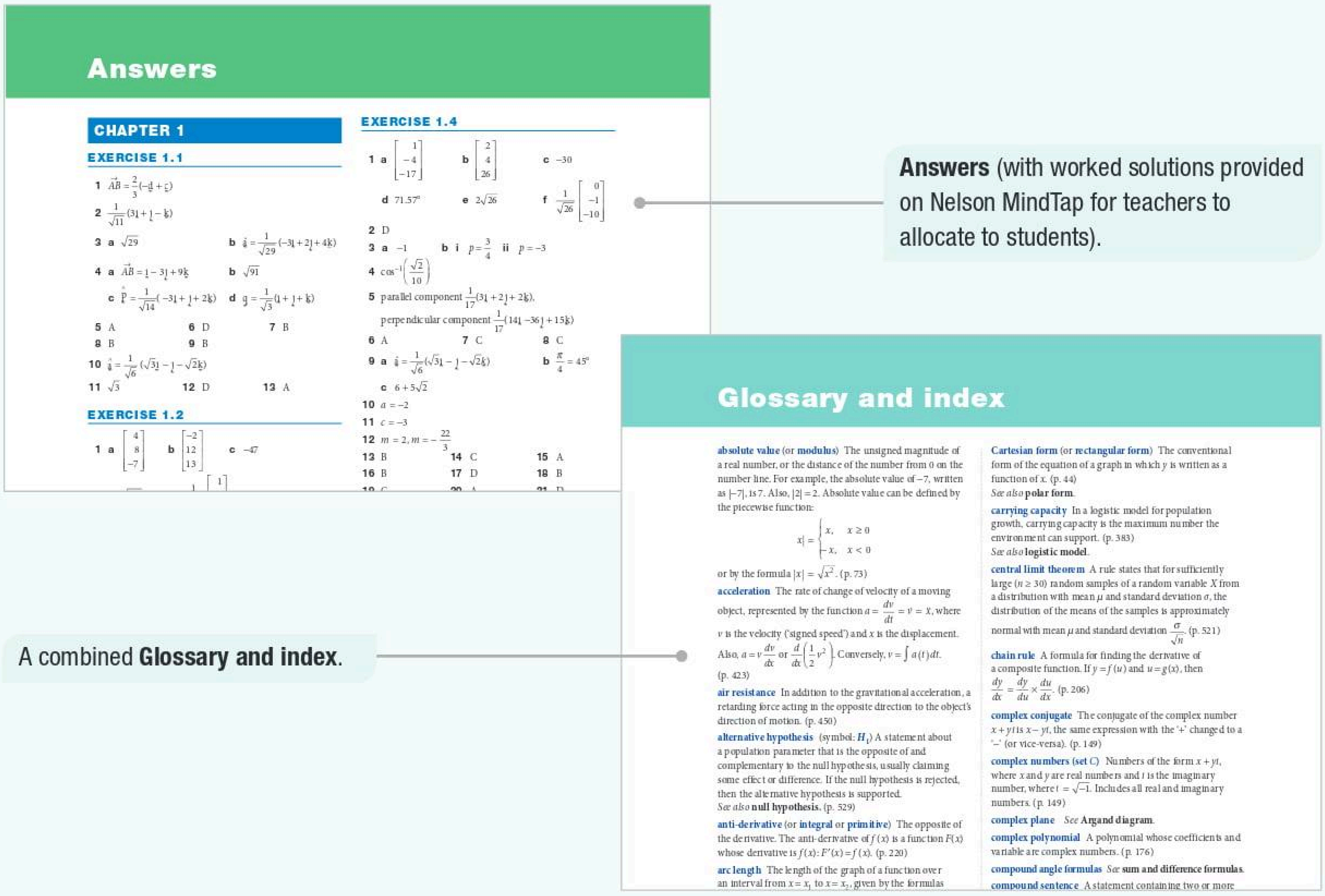
3 ©VCAA 2017 2AQ11 The vectors $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + d\underset{\sim}{k}$, $\underset{\sim}{b} = \underset{\sim}{i} + \underset{\sim}{j} - 4\underset{\sim}{k}$ and $\underset{\sim}{c} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$, where d is a real constant, are linearly dependent if

A $d = -10$
B $d \in R\backslash\{-14\}$
C $d = -14$
D $d \in R\backslash\{-10\}$
E $d \in R$

4 ©VCAA 2017 2AQ13 Given the vectors given by $\underset{\sim}{a} = 3\underset{\sim}{i} - 4\underset{\sim}{j} + 12\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, the vector resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is

A $-\frac{14}{3}$
B $-\frac{14}{3}\left(2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right)$
C $-\frac{14}{9}\left(2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right)$
D $-\frac{14}{13}$
E $-\frac{14}{169}\left(3\underset{\sim}{i} - 4\underset{\sim}{j} + 12\underset{\sim}{k}\right)$

At the end of the book



Answers

CHAPTER 1

EXERCISE 1.1

1 $\overrightarrow{AB} = \frac{2}{3}(-\underset{\sim}{a} + \underset{\sim}{c})$

2 $\frac{1}{\sqrt{11}}(3\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k})$

3 a $\sqrt{29}$ **b** $\hat{\underset{\sim}{a}} = \frac{1}{\sqrt{29}}(-3\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k})$

4 a $\overrightarrow{AB} = \underset{\sim}{i} - 3\underset{\sim}{j} + 9\underset{\sim}{k}$ **b** $\sqrt{91}$

c $\hat{\underset{\sim}{p}} = \frac{1}{\sqrt{14}}(-3\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k})$ **d** $\hat{\underset{\sim}{q}} = \frac{1}{\sqrt{3}}(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})$

5 A **6** D **7** B

8 B **9** B

10 $\hat{\underset{\sim}{a}} = \frac{1}{\sqrt{6}}(\sqrt{3}\underset{\sim}{i} - \underset{\sim}{j} - \sqrt{2}\underset{\sim}{k})$

11 $\sqrt{3}$ **12** D **13** A

EXERCISE 1.2

1 a $\begin{bmatrix} 4 \\ 8 \\ -7 \end{bmatrix}$ **b** $\begin{bmatrix} -2 \\ 12 \\ 13 \end{bmatrix}$ **c** −47

EXERCISE 1.4

1 a $\begin{bmatrix} 1 \\ -4 \\ -17 \end{bmatrix}$ **b** $\begin{bmatrix} 2 \\ 4 \\ 26 \end{bmatrix}$ **c** −30

d 71.57° **e** $2\sqrt{26}$ **f** $\frac{1}{\sqrt{26}}\begin{bmatrix} 0 \\ -1 \\ -10 \end{bmatrix}$

2 D

3 a −1 **b i** $p = \frac{3}{4}$ **ii** $p = -3$

4 $\cos^{-1}\left(\frac{\sqrt{2}}{10}\right)$

5 parallel component $\frac{1}{17}(3\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k})$,

perpendicular component $\frac{1}{17}(14\underset{\sim}{i} - 36\underset{\sim}{j} + 15\underset{\sim}{k})$

6 A **7** C **8** C

9 a $\hat{\underset{\sim}{a}} = \frac{1}{\sqrt{6}}(\sqrt{3}\underset{\sim}{i} - \underset{\sim}{j} - \sqrt{2}\underset{\sim}{k})$ **b** $\frac{\pi}{4} = 45°$

c $6 + 5\sqrt{2}$

10 $a = -2$

11 $c = -3$

12 $m = 2, m = -\frac{22}{3}$

13 B **14** C **15** A

16 B **17** D **18** B

Glossary and index

absolute value (or **modulus**) The unsigned magnitude of a real number, or the distance of the number from 0 on the number line. For example, the absolute value of −7, written as $|-7|$, is 7. Also, $|2| = 2$. Absolute value can be defined by the piecewise function:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

or by the formula $|x| = \sqrt{x^2}$. (p. 73)

acceleration The rate of change of velocity of a moving object, represented by the function $a = \frac{dv}{dt} = \dot{v} = \ddot{x}$, where v is the velocity ('signed speed') and x is the displacement. Also, $a = v\frac{dv}{dx}$ or $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$. Conversely, $v = \int a(t)\,dt$. (p. 423)

air resistance In addition to the gravitational acceleration, a retarding force acting in the opposite direction to the object's direction of motion. (p. 450)

alternative hypothesis (symbol: H_1) A statement about a population parameter that is the opposite of and complementary to the null hypothesis, usually claiming some effect or difference. If the null hypothesis is rejected, then the alternative hypothesis is supported.
See also **null hypothesis.** (p. 529)

anti-derivative (or **integral** or **primitive**) The opposite of the derivative. The anti-derivative of $f(x)$ is a function $F(x)$ whose derivative is $f(x)$: $F'(x) = f(x)$. (p. 220)

arc length The length of the graph of a function over an interval from $x = x_1$ to $x = x_2$, given by the formulas

Cartesian form (or **rectangular form**) The conventional form of the equation of a graph in which y is written as a function of x. (p. 44)
See also **polar form.**

carrying capacity In a logistic model for population growth, carrying capacity is the maximum number the environment can support. (p. 383)
See also **logistic model.**

central limit theorem A rule states that for sufficiently large ($n \geq 30$) random samples of a random variable X from a distribution with mean μ and standard deviation σ, the distribution of the means of the samples is approximately normal with mean μ and standard deviation $\frac{\sigma}{\sqrt{n}}$. (p. 521)

chain rule A formula for finding the derivative of a composite function. If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$. (p. 206)

complex conjugate The conjugate of the complex number $x + yi$ is $x - yi$, the same expression with the '+' changed to a '−' (or vice-versa). (p. 149)

complex numbers (set C) Numbers of the form $x + yi$, where x and y are real numbers and i is the imaginary number, where $i = \sqrt{-1}$. Includes all real and imaginary numbers. (p. 149)

complex plane *See* **Argand diagram.**

complex polynomial A polynomial whose coefficients and variable are complex numbers. (p. 176)

compound angle formulas *See* **sum and difference formulas.**

compound sentence A statement containing two or more

Answers (with worked solutions provided on Nelson MindTap for teachers to allocate to students).

A combined **Glossary and index**.

Nelson MindTap

Video playlists

Worksheets

Skillsheets

Nelson MindTap is an online learning space that provides students with tailored learning experiences. Access tools and content that make learning simpler yet smarter to help you achieve VCE maths mastery.

Nelson MindTap includes an eText with integrated interactives and online assessment. Margin links in the student book signpost multimedia student resources found on MindTap.

Nelson MindTap for students:

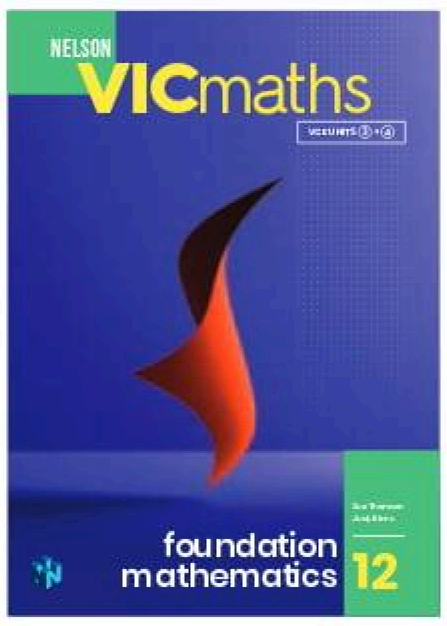
- **Watch** video tutorials featuring expert teacher advice to unpack new concepts and develop your understanding.
- **Revise** using quizzes, worksheets and skillsheets to practise your skills and build your confidence.
- **Navigate** your own path, accessing the content, analytics and support as you need it.

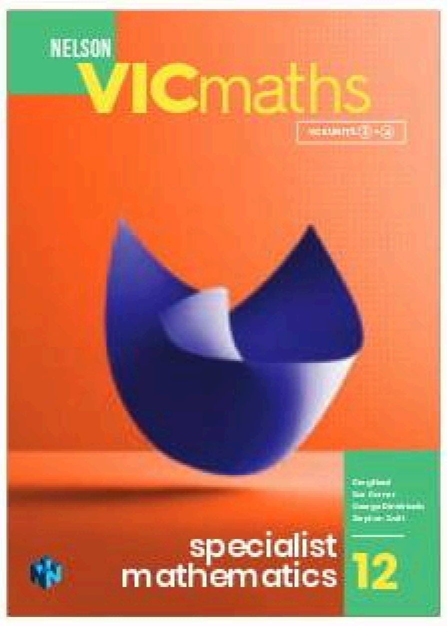
Nelson MindTap for teachers*:

- Tailor content to different learning needs – assign directly to the student, or the whole class.
- Monitor progress using assessment tools like Gradebook and Reports.
- Integrate content and assessment directly within your school's LMS for ease of access.
- Access topic tests, teaching plans and worked solutions to each exercise set.

*Complimentary access to these resources is only available to teachers who use this book as part of a class set, book hire or booklist. Contact your Cengage Education Consultant for information about access and conditions.

Nelson VICmaths 11–12 series





Companion resources

Mastery Workbook

- Step-by-step scaffolding to guide students towards mastery of the course content.
- Write-in support that encourages students to show working.
- Matched examples for *every* worked example in the student book.
- Full integration between student book and workbook.
- Answers (worked solutions for teachers on Nelson MindTap)

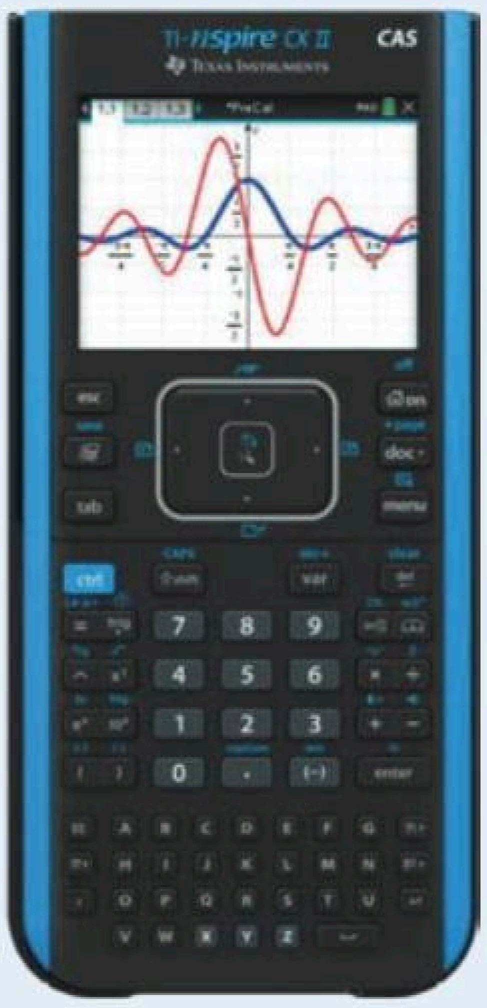
Exam*plus*

- Create and simulate exam-like conditions in minutes.
- Save time with an extensive bank of filterable and difficulty-graded questions for Year 12.
- Over 1000 past VCAA and unseen exam-style questions and solutions all in the one place.
- Extensively researched and user-tested.

TI-Nspire CAS introduction

The latest TI-Nspire model is TI-Nspire CX II CAS. When purchasing a new handheld, you also gain access to the student software. If you purchase a used handheld, then you can pay an additional fee for the student software. Alternatively, you can connect your handheld to your computer using the TI-Nspire™ CX II Connect web-based app, which enables you to perform a variety of functions such as screen captures, file transfers and operating system updates.

Note that there is also TI-Nspire non-CAS technology available. It is **vital** that you use the CAS technology.



TI-Nspire CX II CAS

Student book instructions

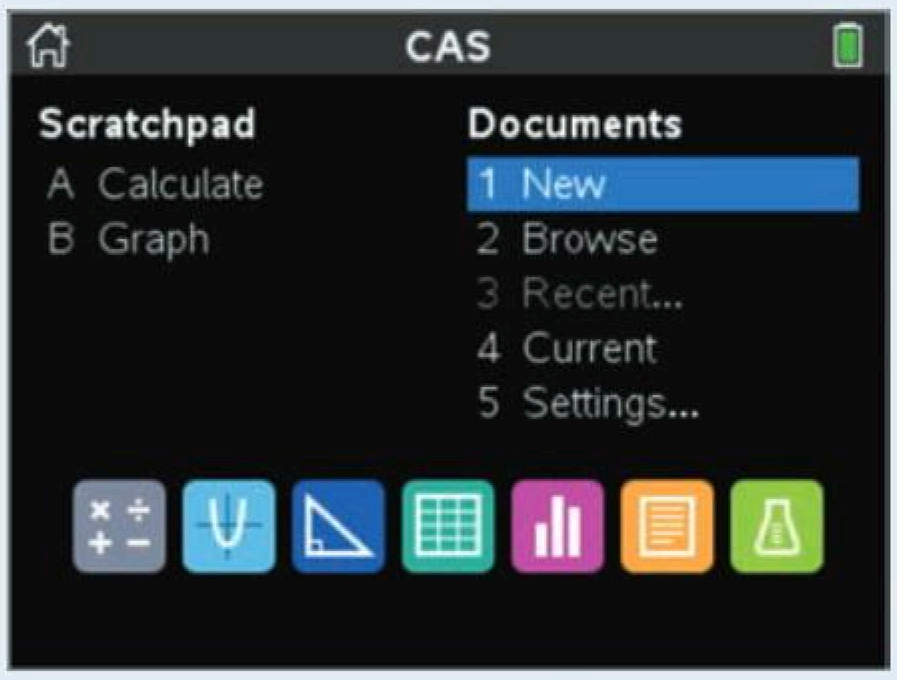
The instructions in this student book use words instead of symbols. Most keys on the keypad are clearly labelled with a word or an abbreviation. Four words that are used to represent less obvious keys are:

Word	Key
home	⌂ on
catalog	📖
template	⊞
Scratchpad	🖩

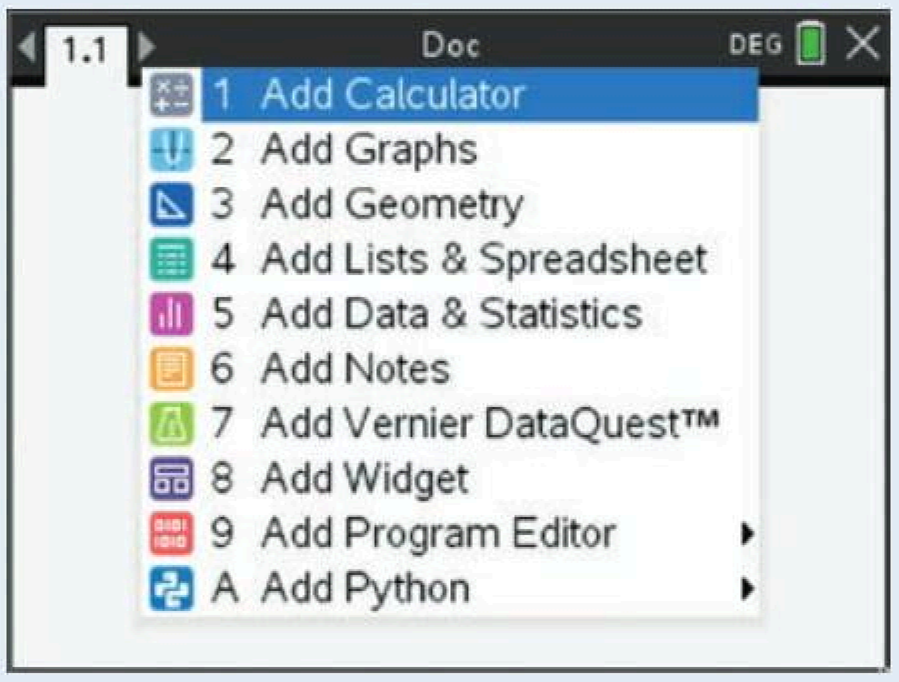
Several keys have a second function highlighted in blue above the key. For example, press **ctrl** + $\mathbf{x^2}$ to access the square root function $\sqrt{\ }$.

Applications

The applications available are outlined below.



Press **home** to view the home page. The **Scratchpad** options on the left are available for quick calculations and graphing. The **Document** options on the right are used for navigation. The seven icons on the bottom are the main applications.



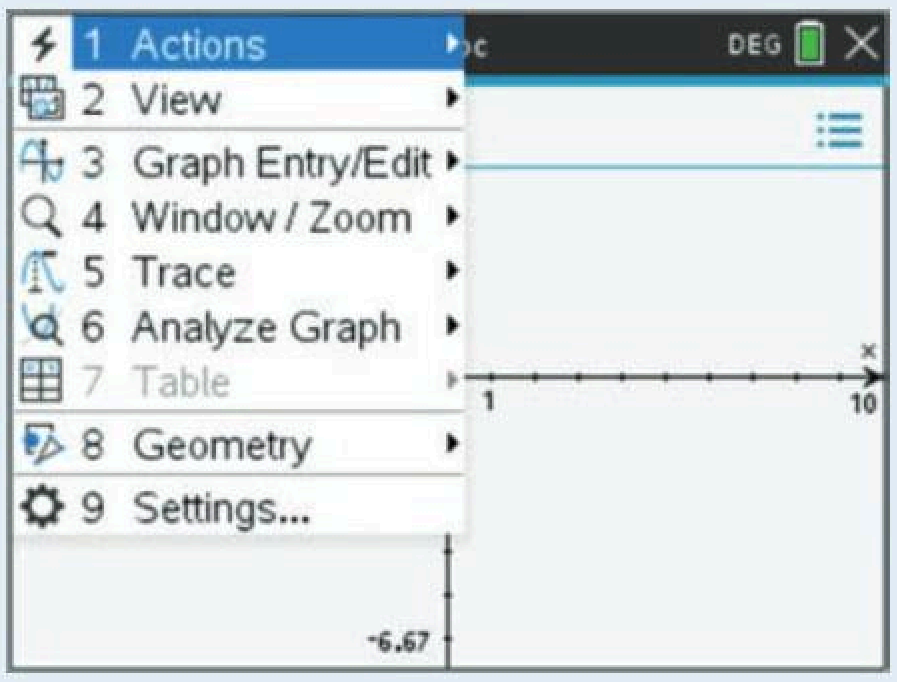
When you select **Documents > New** from the home page, a list of the seven applications plus three additional menu options will be displayed. From any application, press **ctrl + I** (for insert) to display this list and add a new page to the document.

Menus

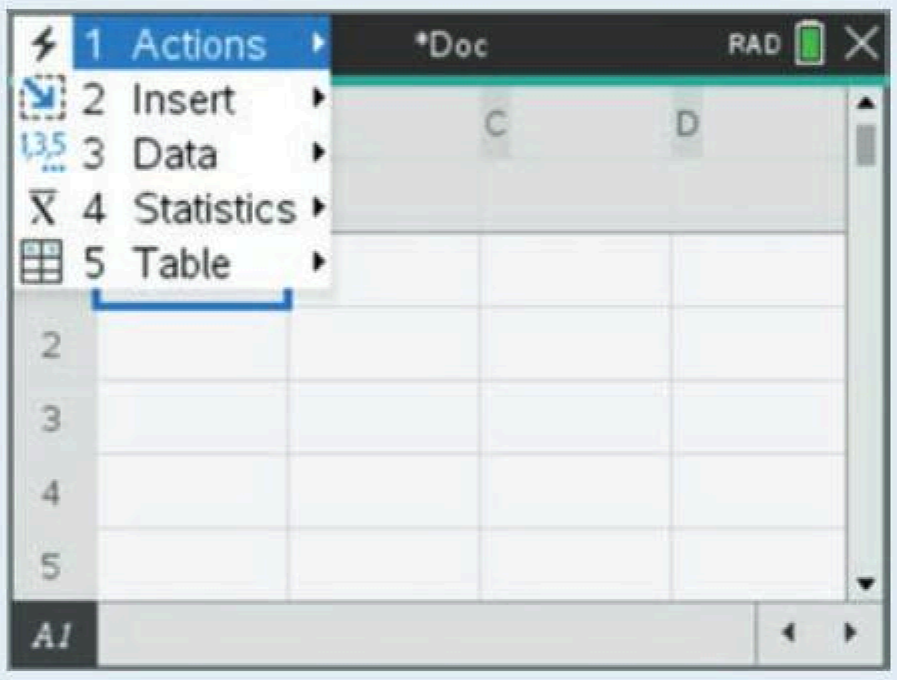
The instructions in this student book primarily use the **Calculator**, **Graphs**, **Lists & Spreadsheet** and **Data & Statistics** applications (see **Hints** on page xviii). The following figures show the initial menu options for these four applications. These menu options link to submenu options, which are not shown below. On the handheld and software, the applications are referred to as **Documents**. In the student book instructions, the applications are referred to as **pages** of the document (e.g. Add a **Graphs** page).

Calculator

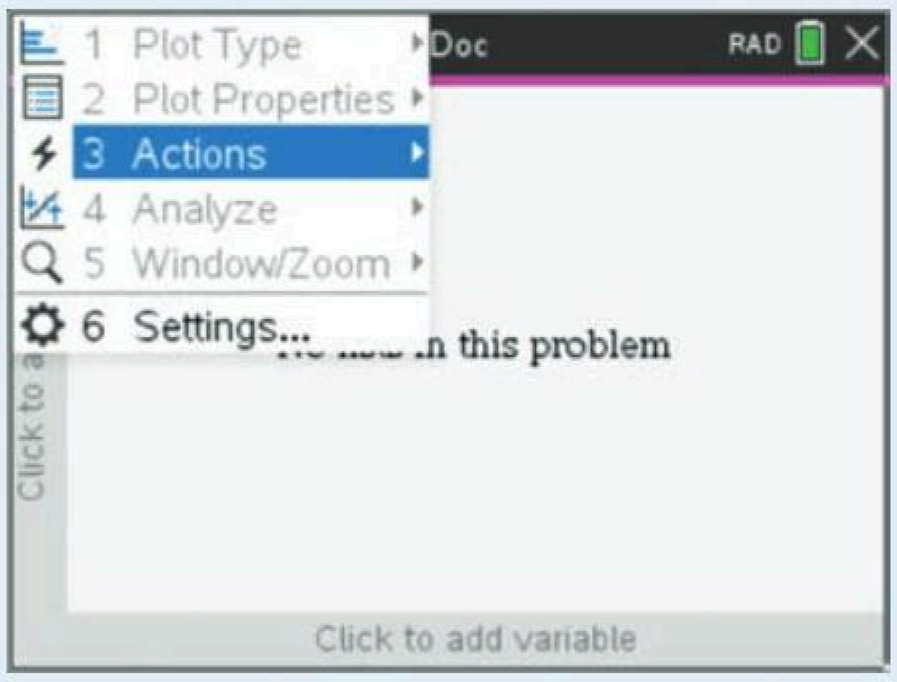
Graphs



Lists & Spreadsheet



Data & Statistics



If your document contains more than one page, move among them by clicking on the numbered tabs at the top of the screen. Alternatively, press **ctrl + left arrow** or **ctrl + right arrow**. To view all the pages of a document, press **ctrl + up arrow**.

All menu and submenu options include numbers. The student book instructions do not include numbers. For example, the instruction in a **Calculator** page for clearing all calculations is 'press **menu > Actions > Clear History**'. The shortcut is 'press **menu > 1 > 5**'. For efficiency, you are encouraged to learn the sequence of numbers for frequently used commands.

Document settings

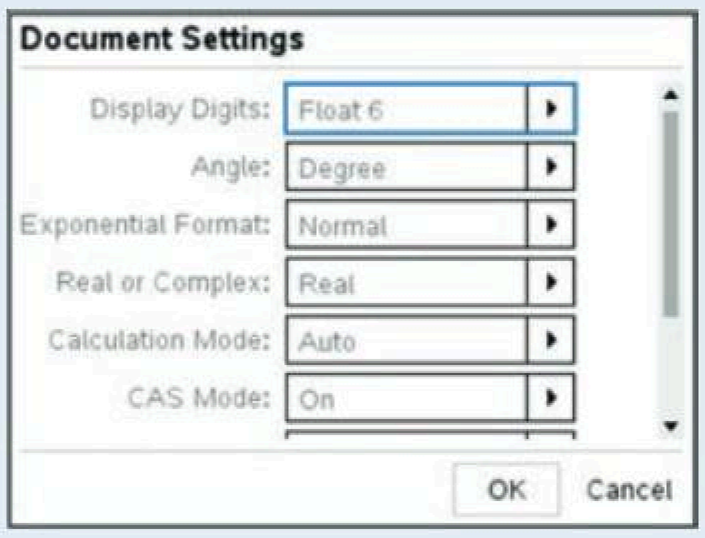
The document settings can be accessed in the following ways:

1 From the **home** page, press **Settings > Document Settings**.

2 From a document page, press the **doc** key or click on **Doc** at the top of the page, then select **Settings & Status > Document Settings**.

3 Click on the **battery icon** in the top right-hand corner of the page, then select **Document Settings**.

The document settings shown below are the primary ones you will be using.


Document Settings

Display Digits: Float 6

Angle: Degree

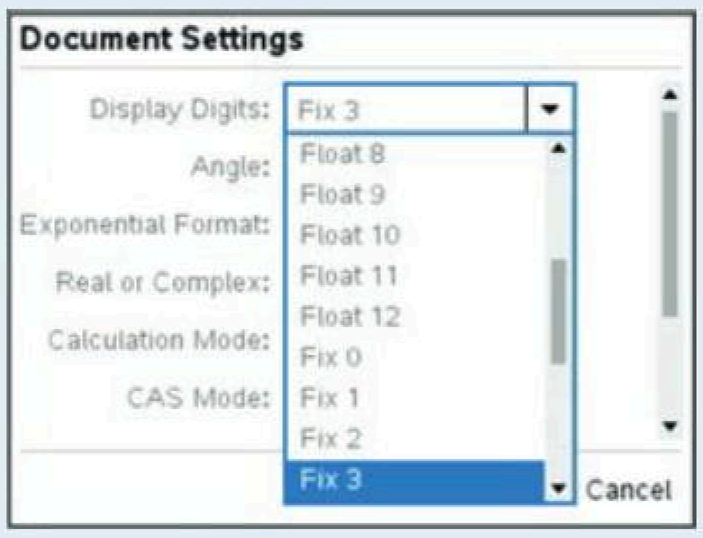
Exponential Format: Normal

Real or Complex: Real

Calculation Mode: Auto

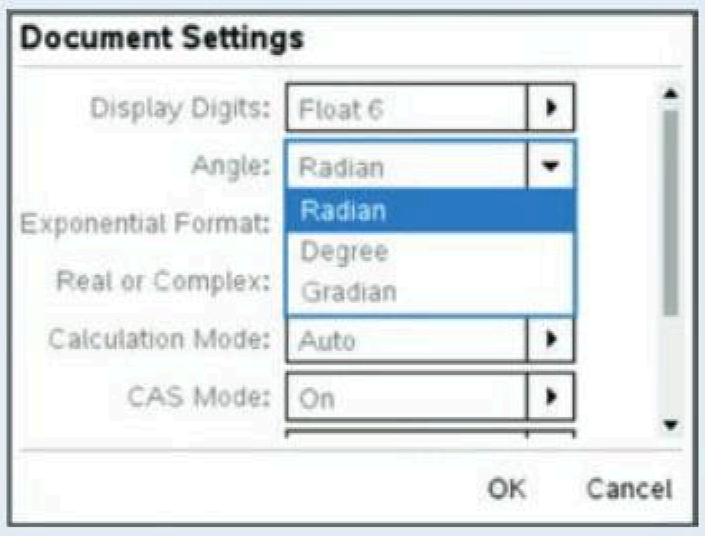
CAS Mode: On

The screen above shows the default document settings. The **Display Digits** field is set to **Float 6**, which means up to six significant figures will be displayed.


Document Settings

Display Digits: Fix 3

Click in any field to display the options. The screen above shows the **Display Digits** options. Scroll down to select a specific number of decimal places.


Document Settings

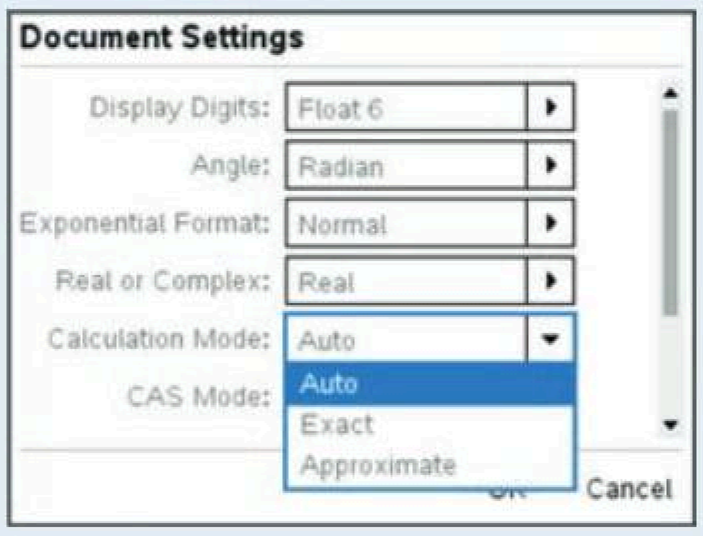
Display Digits: Float 6

Angle: Radian

Calculation Mode: Auto

CAS Mode: On

There are three **Angle** options shown above. Select either **Radian** or **Degree**. These two angle settings can be toggled at any time by clicking on **DEG** or **RAD** in the top right-hand corner of the screen.


Document Settings

Display Digits: Float 6

Angle: Radian

Exponential Format: Normal

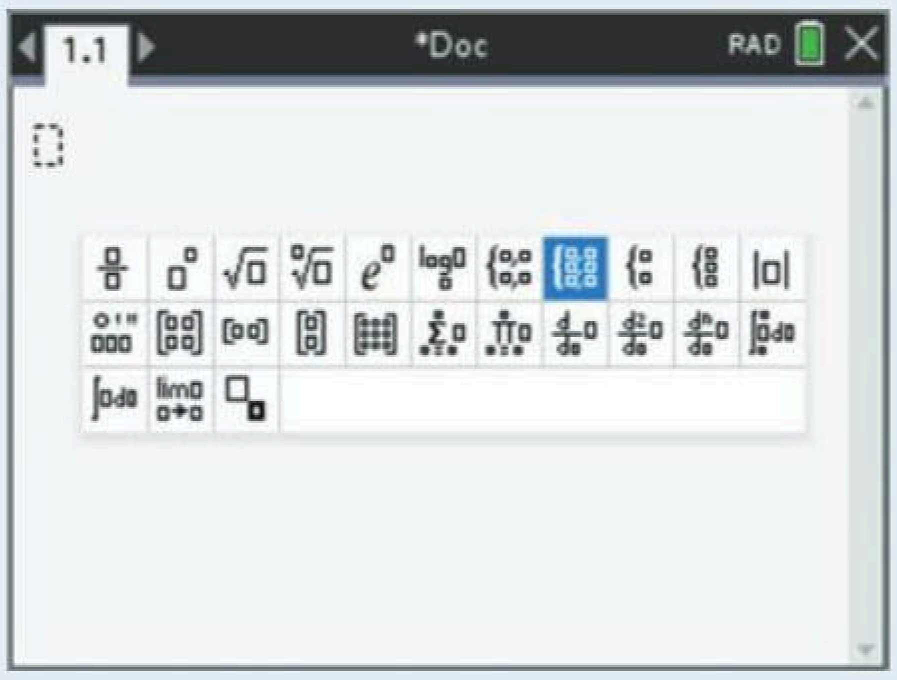
Real or Complex: Real

Calculation Mode: Auto

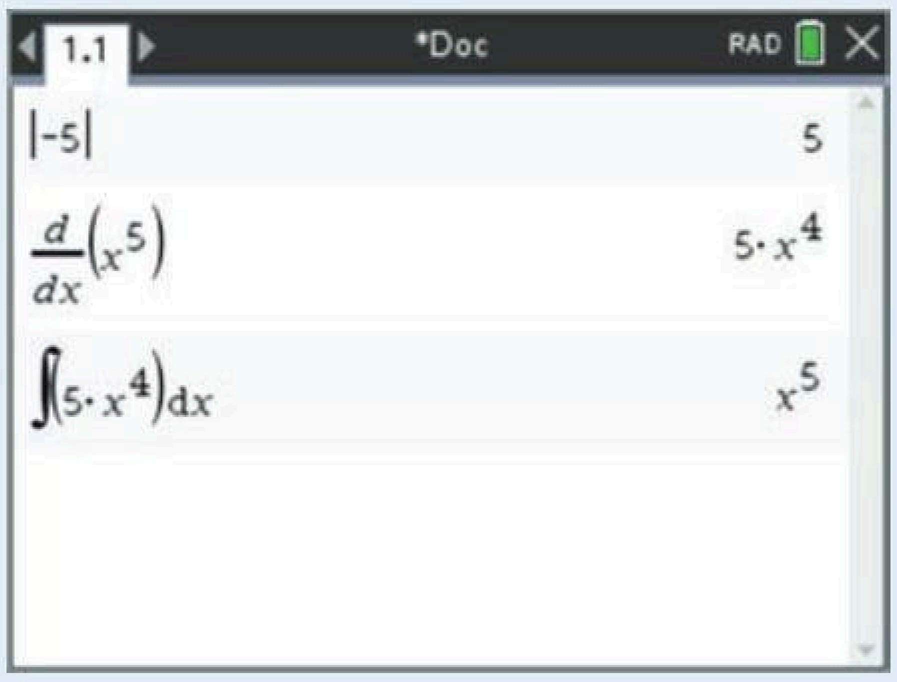
There are three **Calculation Mode** options shown above. It is recommended that you keep the default setting of **Auto**. If you require an approximate or decimal answer, press **ctrl + enter** or include a decimal point in your calculation.

Templates

The **template** key is located to the right of the **9** key.



Press **template** to view the template options. Most templates will be directly inserted but for some, you will be prompted with a dialogue box. For example, after selecting the **3 piecewise function template,** you will be prompted for the number of function pieces.



$|-5|$ 5

$\frac{d}{dx}\left(x^5\right)$ $5\cdot x^4$

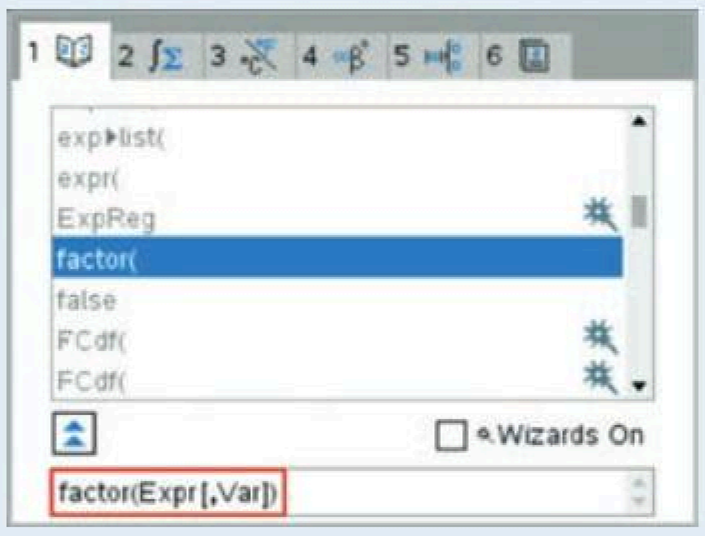
$\int\left(5\cdot x^4\right)dx$ x^5

The screen above shows three examples using templates: an **absolute value**, a **derivative** and an **integral**. Many of the templates can be accessed using keys on the keyboard, for example, fraction, square root etc.

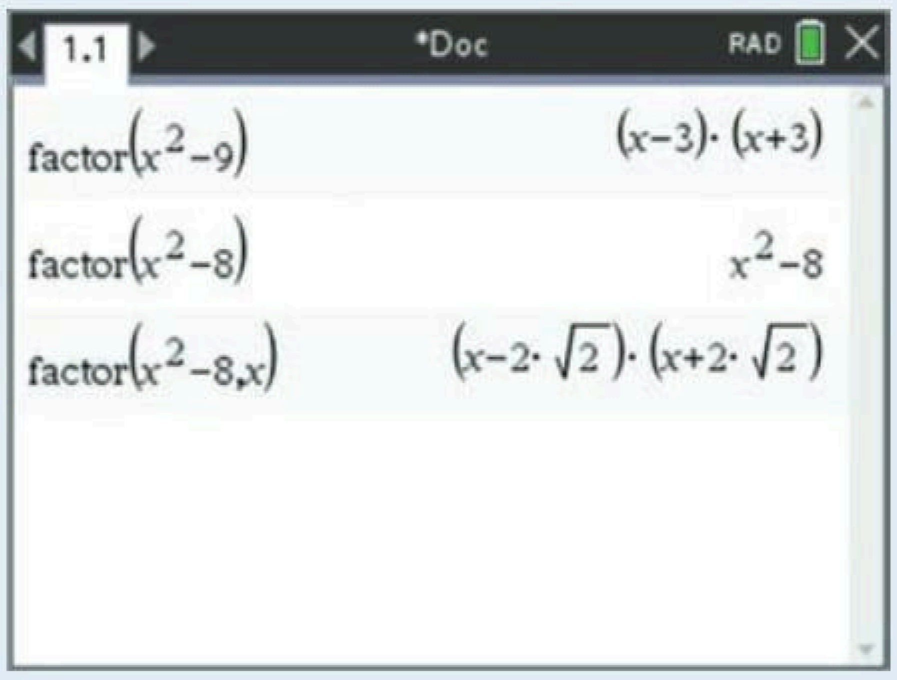
Catalog

The **catalog** key is located to the right of the **template** key. In addition to using the menus and submenus, you can access all the commands using the catalog. The advantage of using the catalog is that it shows the parameters required for each command at the bottom of the screen. Optional parameters appear in square brackets.

For example:



Press **catalog** and ensure tab **1** is selected. Press **F** to jump to the commands starting with F. Scroll down to **factor(**. The parameters for the **factor** command appear at the bottom of the screen (see **red** rectangle). **Expr** is a required parameter. The square brackets around **Var** means it is an optional parameter.



$\text{factor}(x^2-9)$ $(x-3)\cdot(x+3)$

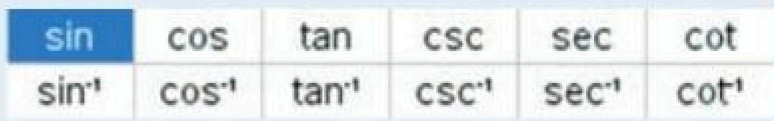
$\text{factor}(x^2-8)$ x^2-8

$\text{factor}(x^2-8,x)$ $(x-2\cdot\sqrt{2})\cdot(x+2\cdot\sqrt{2})$

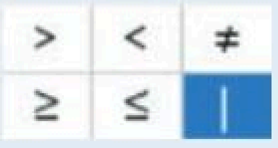
When you **factor** the expression $x^2 - 9$, the solution is $(x - 3)(x + 3)$. When you factor $x^2 - 8$, the expression remains unchanged. However, if you include the optional parameter, **x**, the expression will factorise to $\left(x-2\sqrt{2}\right)\left(x+2\sqrt{2}\right)$.

Symbols

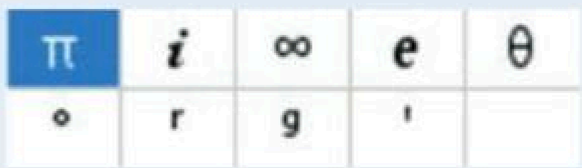
All symbols are listed at the end of the **catalog**. To access the symbol palette, press **ctrl + catalog.** Frequently used symbols are available in mini-palettes by pressing the keys shown below.



Press **trig** to access the trigonometry functions.



Press **ctrl + =** to access the inequalities and the **constraint** symbol.

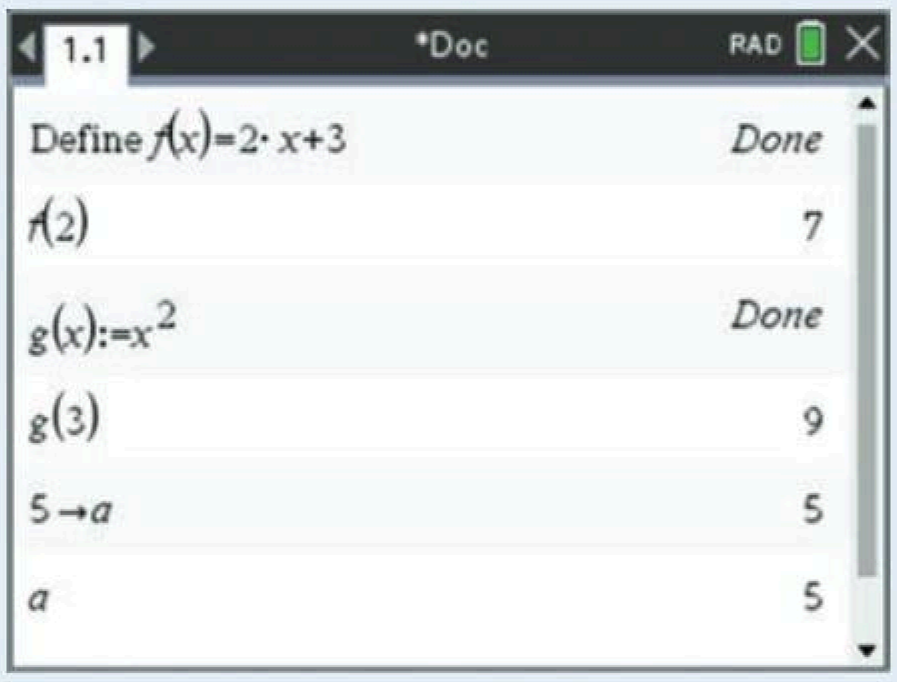


Press π to access commonly used symbols.

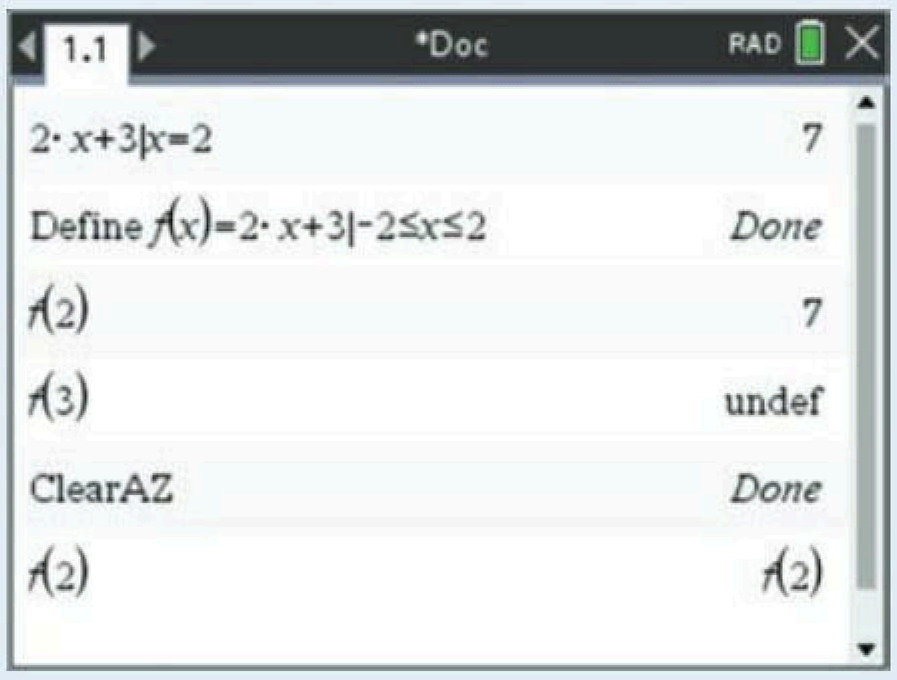
Hints

The instructions below provide a few hints to assist with using the four main applications.

Calculator



Define $f(x)=2\cdot x+3$	*Done*
$f(2)$	7
$g(x):=x^2$	*Done*
$g(3)$	9
$5\to a$	5
a	5



$2\cdot x+3\vert x=2$	7
Define $f(x)=2\cdot x+3\vert -2\leq x\leq 2$	*Done*
$f(2)$	7
$f(3)$	undef
ClearAZ	*Done*
$f(2)$	$f(2)$

There are a few options available to define or store functions and variables.

1 Press **menu > Actions > Define.**

2 Press **ctrl + template** for the **:=** symbol.

3 Press **ctrl + var** for the **store** symbol.

Press **var** to view the list of defined and stored variables.

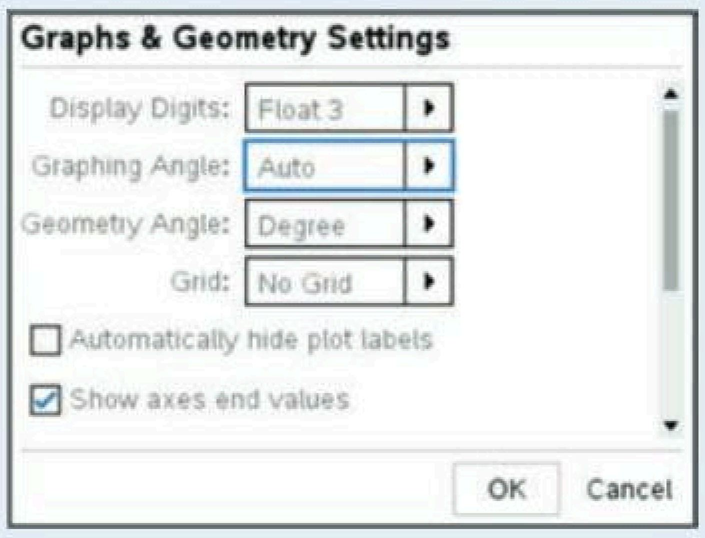
Press **ctrl + =** to access the **constraint** symbol (|).

1 Use **constraint** for substitution.

2 Use **constraint** to limit the domain of a function.

Press **menu > Actions > Clear a-z** to delete defined variables.

Graphs


Graphs & Geometry Settings

Display Digits: Float 3

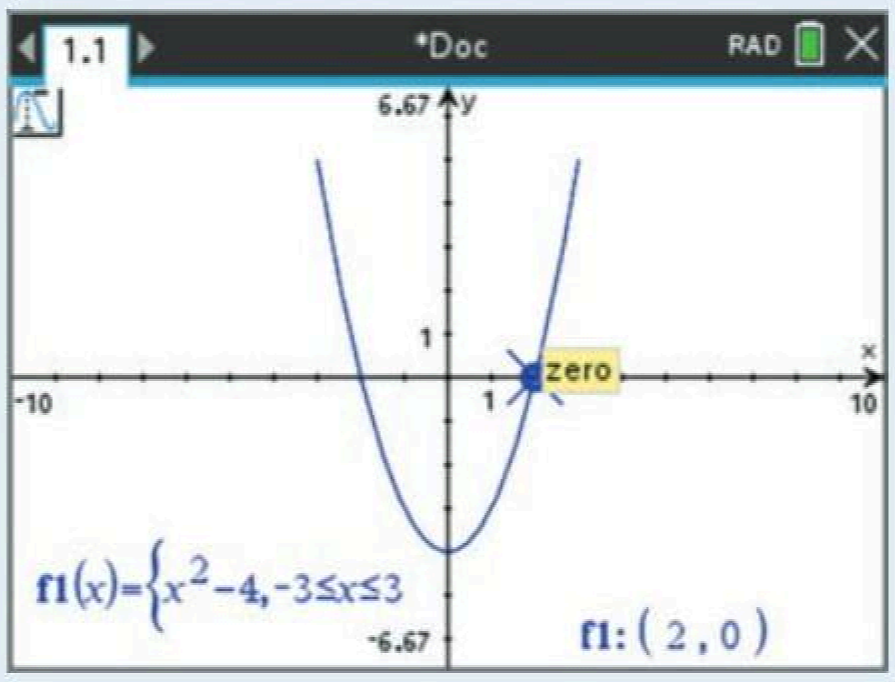
Graphing Angle: Auto

Geometry Angle: Degree

Grid: No Grid

☐ Automatically hide plot labels

☑ Show axes end values



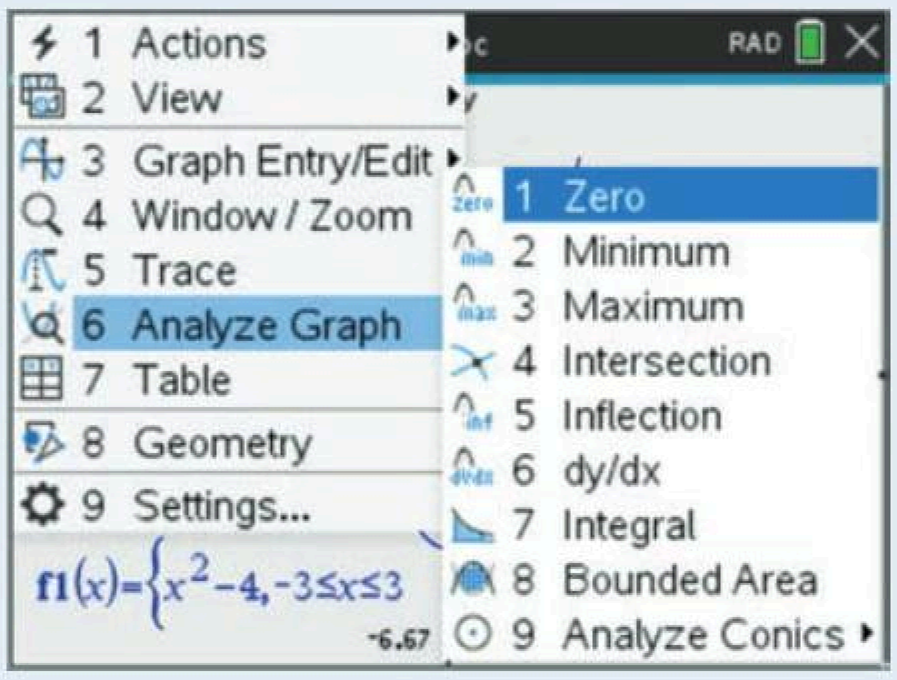
$f1(x)=\{x^2-4,-3\leq x\leq 3$

f1: (2 , 0)

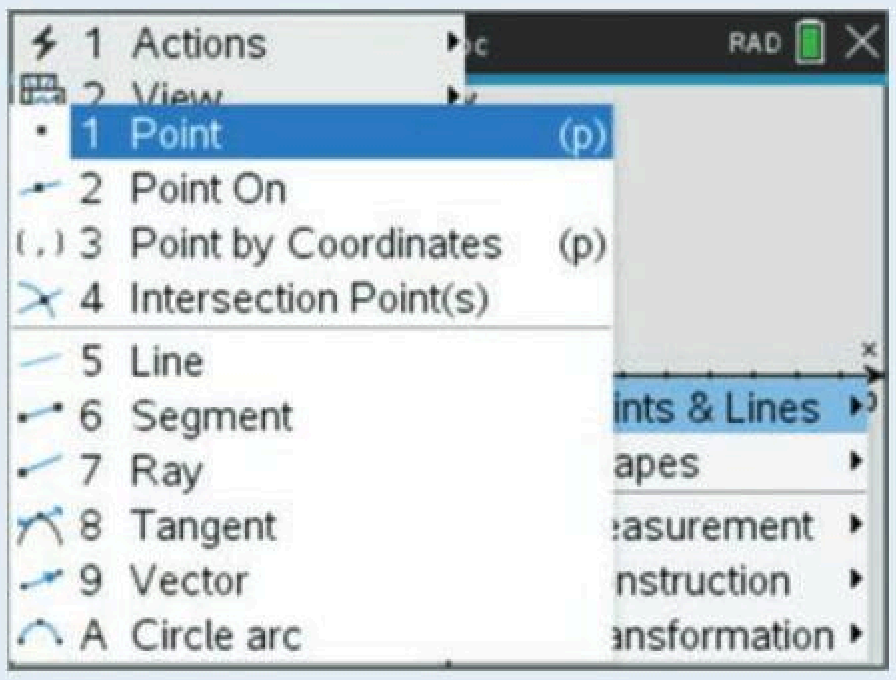
Press **menu > Settings** to view and/or change the settings for the **Graphs** and **Geometry** applications. The default setting for **Display Digits** is **Float 3** so change this if you need greater accuracy. The **Graphing Angle** is set to **Auto** but can be changed to **Degree** or **Radian**.

The **constraint** command can also be used in a **Graphs** page to specify a domain.

Press **menu > Trace > Graph Trace** to identify points on a graph.

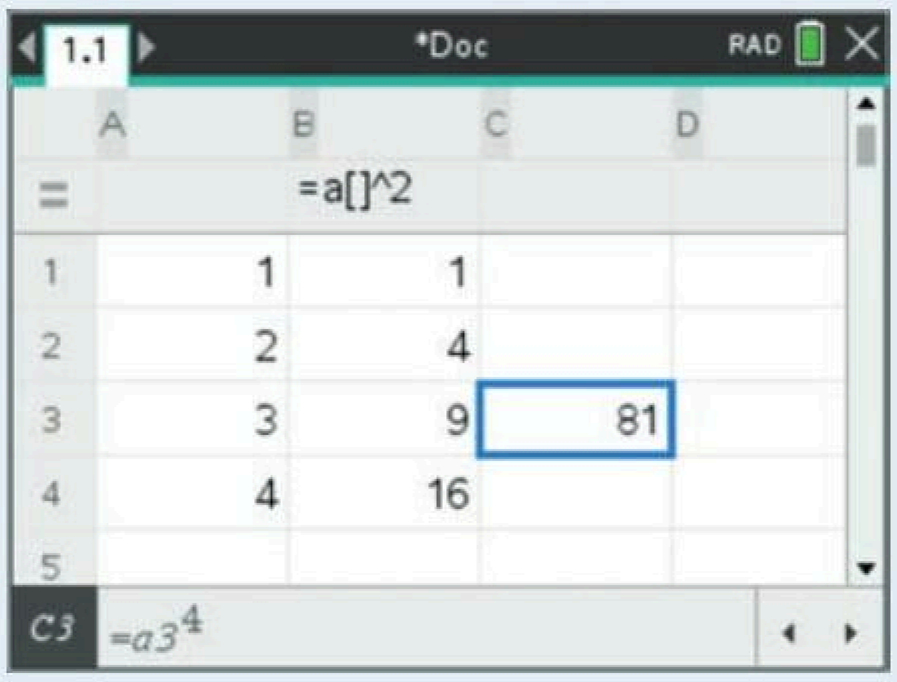


A second option for identifying key points on a graph is to press **menu > Analyze Graph**. These options prompt you for a **lower bound** and **upper bound** to locate the point.



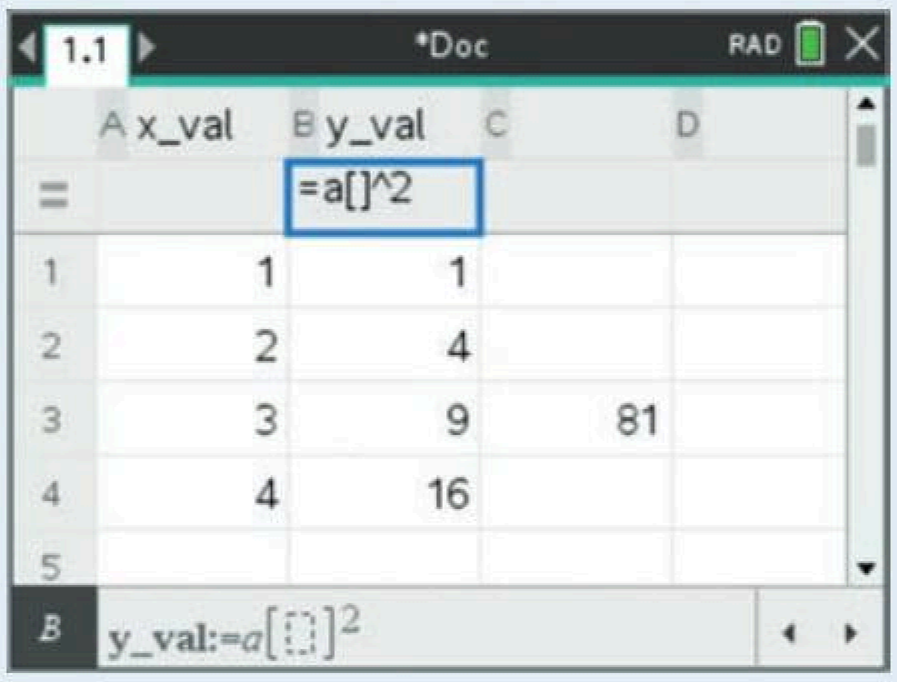
A third option is to press **menu > Geometry > Points & Lines**. Locate points of intersection or place points on the graph and move them along the graph. Click on a coordinate to manually change a value.

Lists & Spreadsheet



	A	B	C	D
=		=a[]^2		
1	1	1		
2	2	4		
3	3	9	81	
4	4	16		
5				

C3 $=a3^4$



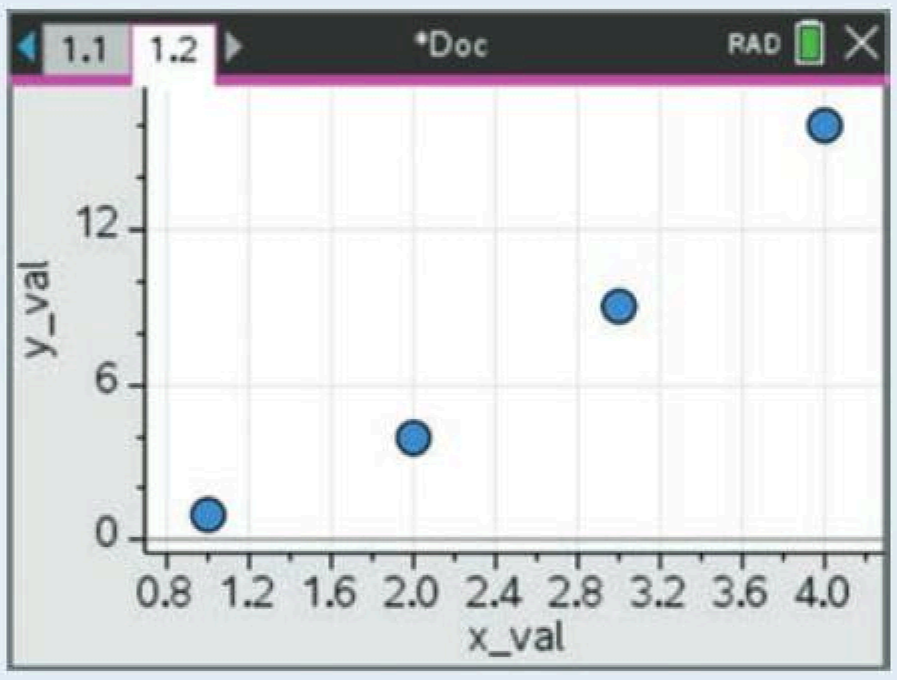
	A x_val	B y_val	C	D
=		=a[]^2		
1	1	1		
2	2	4		
3	3	9	81	
4	4	16		
5				

B y_val:=a[⬚]^2

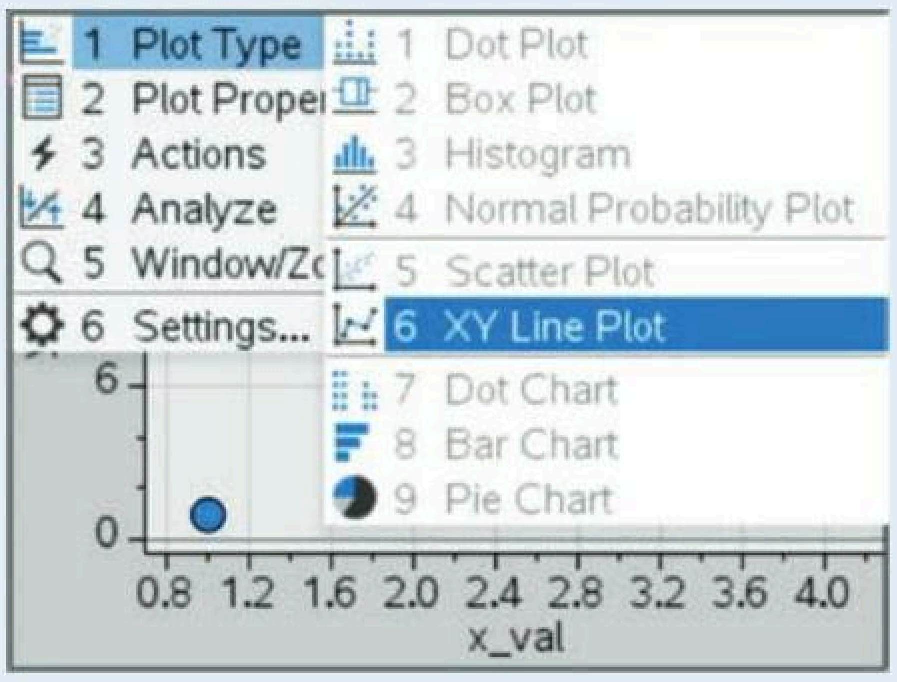
The columns in the **Lists & Spreadsheet** application can be used for lists. Above, the values in the list in column **B** are the squares of the values of the list in column **A**. It can also be used as a spreadsheet. The value in cell **C3** is the value in cell **A3** raised to the power of 4.

Calculations using lists can be completed using the column headings **A**, **B**, **C** etc. Lists need to be labelled to be used in other applications. List **A** above has been labelled **x_val** and list **B** labelled **y_val**. List names cannot have spaces so spaces are replaced by asterisks. Alternatively, press **ctrl + space bar** to insert the **underscore** character instead of a space.

Data & Statistics



The **Data & Statistics** application is reliant on lists generated in other applications. It is designed for ungrouped data. The plot above displays the **x_val** and **y_val** lists from the **Lists & Spreadsheet** application.



Press **menu > Plot Type** to view the various graphing options. Options 1 to 4 are for univariate data (one list). Options 5 and 6 are for bivariate data (two lists). Options 7 to 9 are for categorical data (one list).

Operating systems

Ensure the latest operating system is installed on your handheld and software.

Installing the latest operating system is relatively straightforward. Using the USB cable provided, connect the handheld to a computer with the student or computer link software installed. Select **Help > Check for OS Updates**. If you see a message that a new OS is available, follow the links to install it. Alternatively, go to the TI website at https://education.ti.com/ to download the latest operating system. Select **Tools > Install OS** then select the downloaded file.

To determine the version of your operating system, press **home > Settings > Status**. At the time of publication, the operating system for the CX II is version **5.4.0.259**.

Handheld Status

Batteries: 100%

Version: 5.4.0.259

Storage Capacity: 92.3 MB

Storage Available: 88.0 MB

Network: Wireless client is not attached.

Login: Not logged in.

Casio ClassPad introduction

The latest model of the Casio ClassPad is the fx-CP400. The connectivity software Screen Receiver, Share Assistant and Program Link Software can be downloaded for free. The ClassPad Manager software emulator is a separate program available at an additional cost.

Casio ClassPad

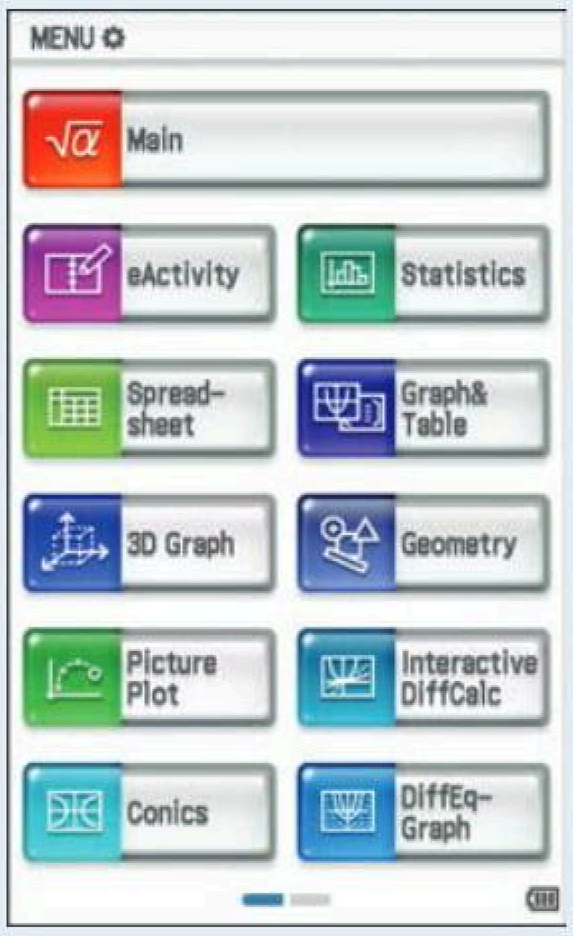
Student book instructions

The instructions in this student book use words instead of symbols. ClassPad tools are located at the top of the screen. These tools vary with each application. Initially, these instructions will show a tool enclosed in a red rectangle with the corresponding word highlighted in red. Examples are shown here.

Word	Tool
Graph	
View Window	
Table	
Table Input	

Applications

The applications available are outlined below.

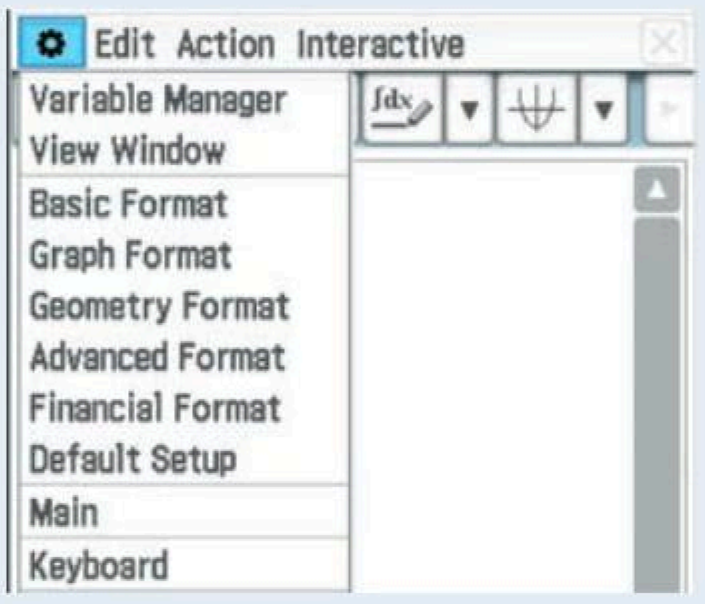


Tap **Menu** to view the applications.

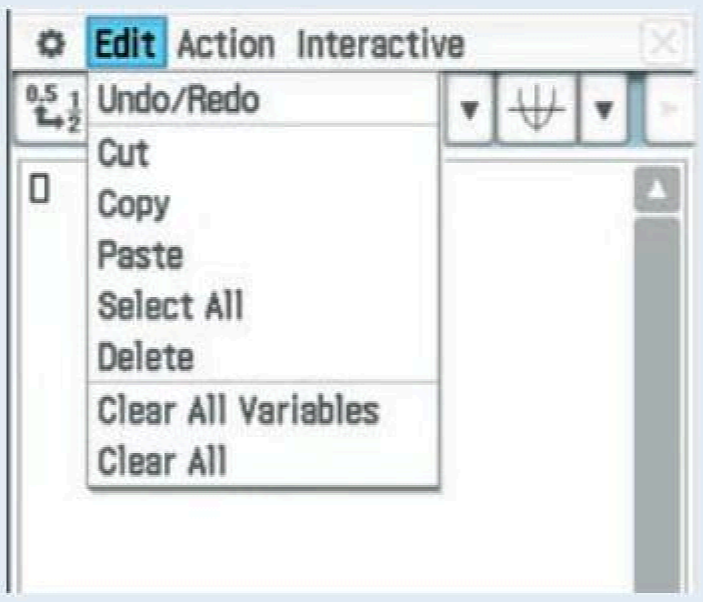
Slide the scroll bar at the bottom of the screen to access the full list.

Menus

The instructions in this student book will primarily use the **Main, Statistics, Geometry, Graph&Table** and **Conics** applications (see **Applications** on page xxiv). All these applications have the ⚙ (systems) and **Edit** menus available at the top left of the screen.

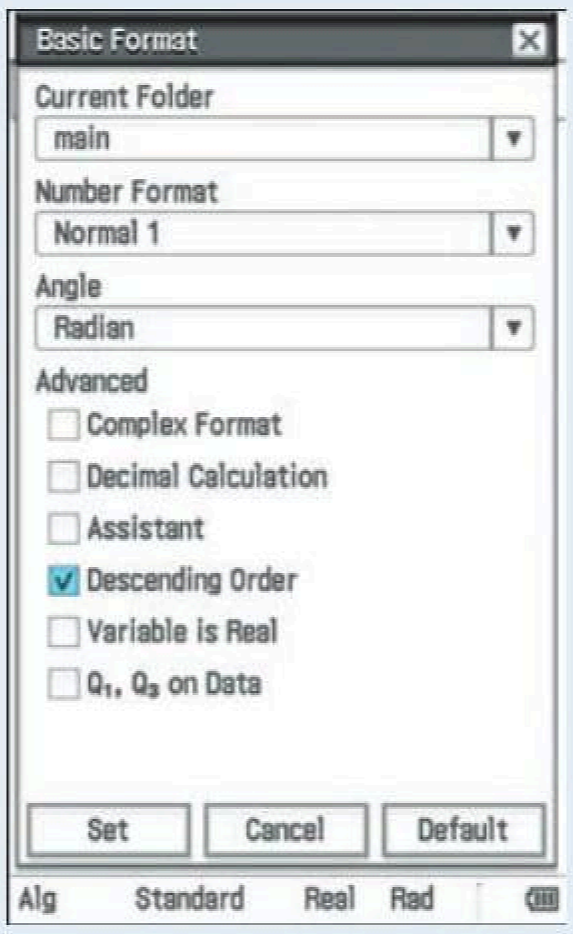


Tap ⚙ to view the system menu. The menu options allow you to manage variables and format applications.



Tap **Edit** to view the edit menu. The menu options allow you to cut, copy, paste and delete screen content and clear variables. The **Edit** menu varies with each application.

Document settings



Basic Format

Current Folder
main

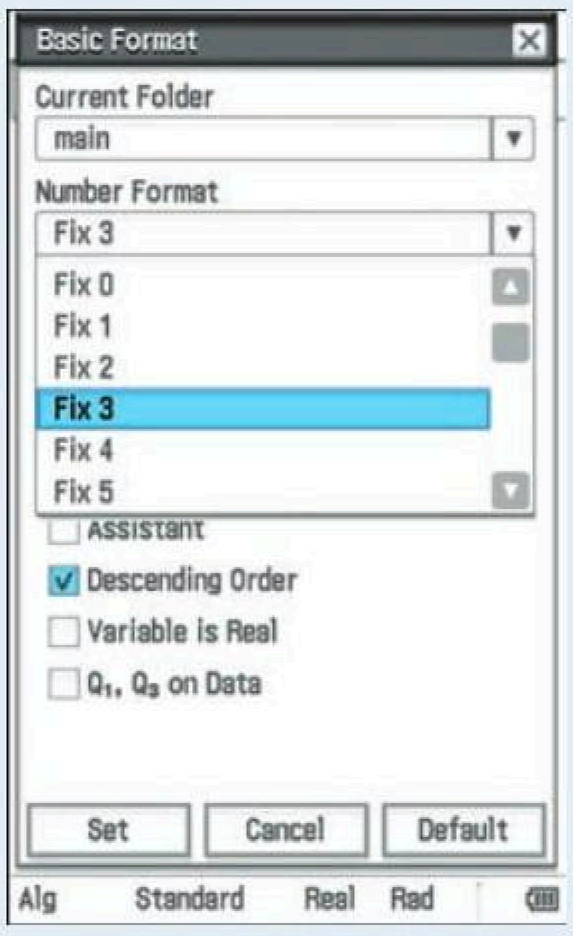
Number Format
Normal 1

Angle
Radian

Advanced
☐ Complex Format
☐ Decimal Calculation
☐ Assistant
☑ Descending Order
☐ Variable is Real
☐ Q_1, Q_3 on Data

Alg Standard Real Rad

Tap ⚙ > **Basic Format**. The screen above shows the default document settings. The **Number Format** default field is set to **Normal 1**.



Basic Format

Current Folder
main

Number Format
Fix 3

☐ Assistant
☑ Descending Order
☐ Variable is Real
☐ Q_1, Q_3 on Data

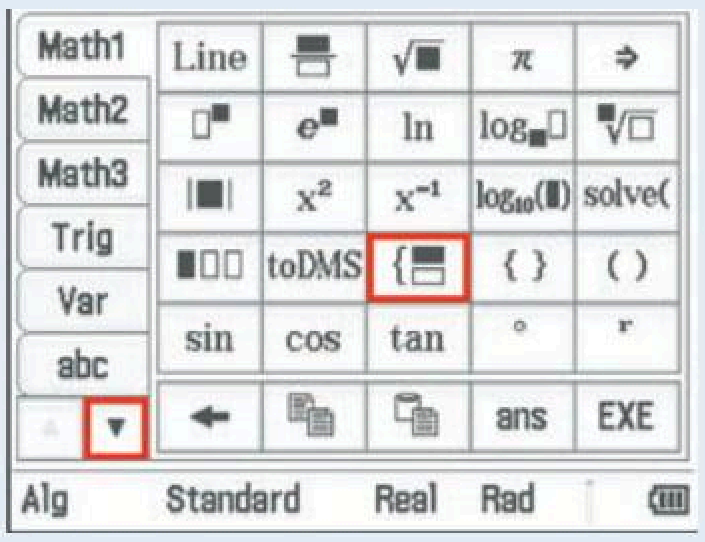
Alg Standard Real Rad

Tap the **Number Format** field to display the options. The screen above shows the **Display Digits** options. Scroll down to fix a specific number of decimal places.

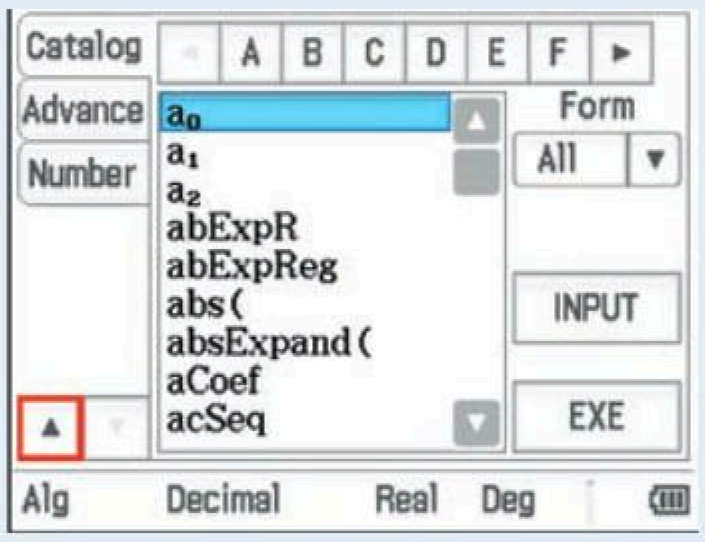
The settings **Standard/Decimal**, **Real/Cplx** and **Rad/Deg/Gra** can be toggled at the bottom of the screen. The recommended settings are **Standard**, **Real** and **Rad**.

Keyboard

There are nine soft keyboards available.

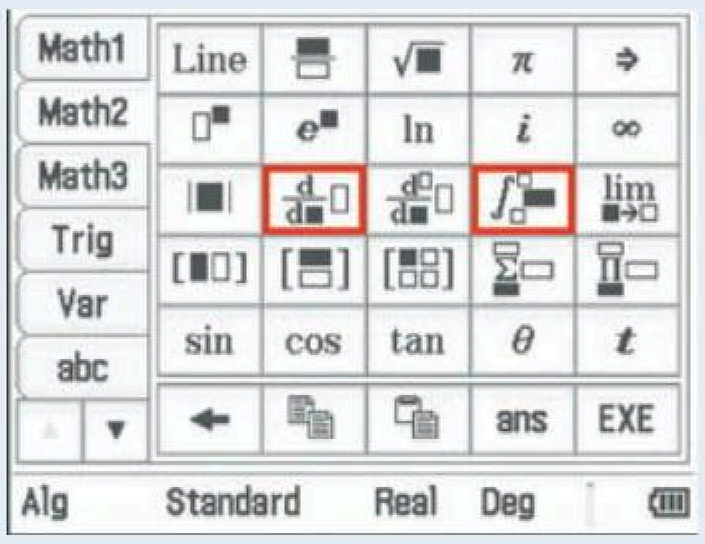


Press **Keyboard** to view the soft keyboards. The **Math1** soft keyboard is shown above with the **simultaneous equation** template highlighted in red. Tap the left tabs to access the other keyboards. Press the **down arrow** to view the second screen.

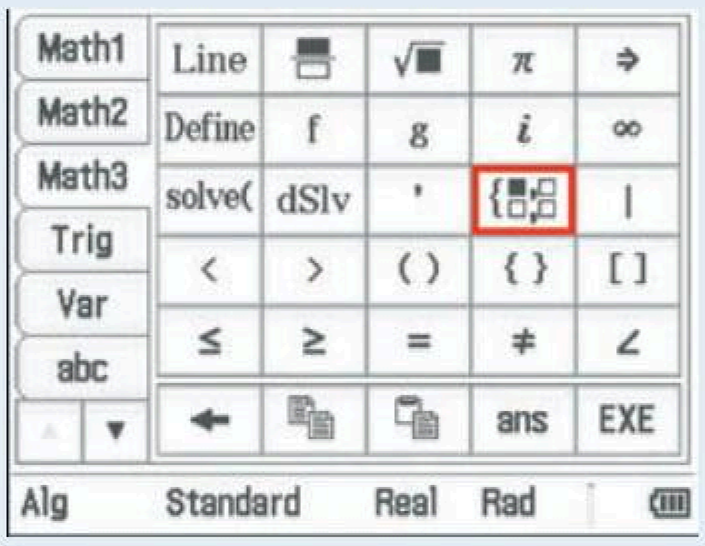


a_0
a_1
a_2
abExpR
abExpReg
abs(
absExpand(
aCoef
acSeq

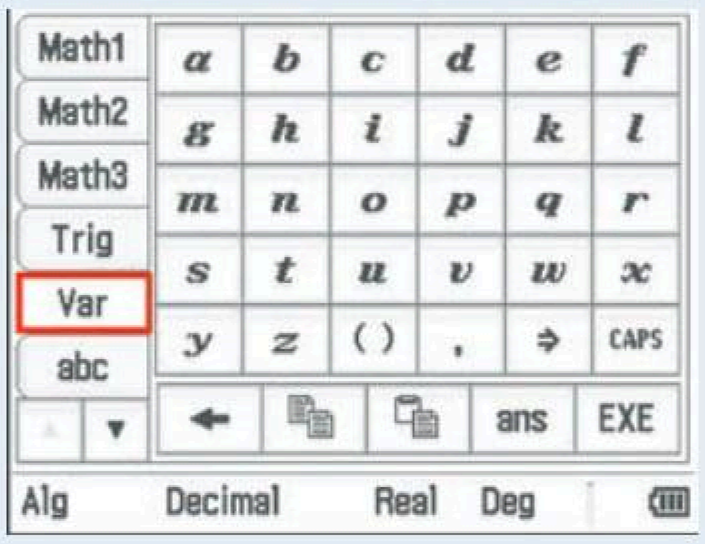
All functions can be accessed from **Catalog**. Tap on the letters at the top to jump through the list. Press the **up arrow** to return to the first screen.



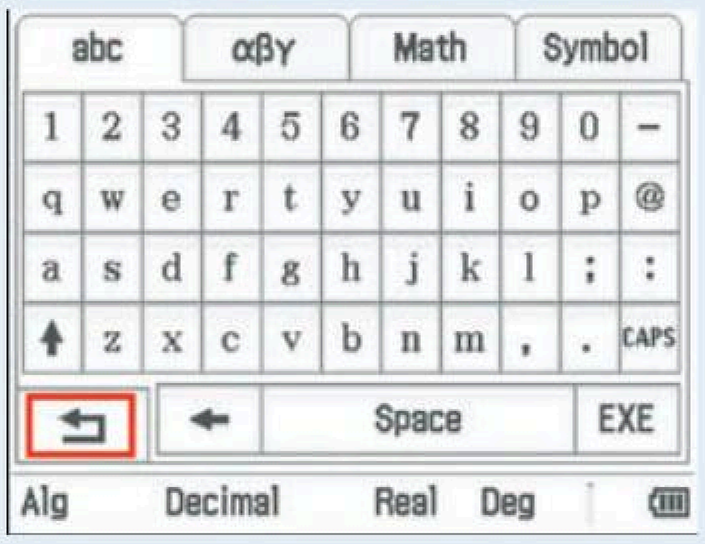
Tap **Math2** to display the soft keyboard. The **derivative** and **integral** templates are highlighted in red.



The **Math3** template is shown above with the **piecewise** template highlighted in red. Use this keyboard for inequalities.



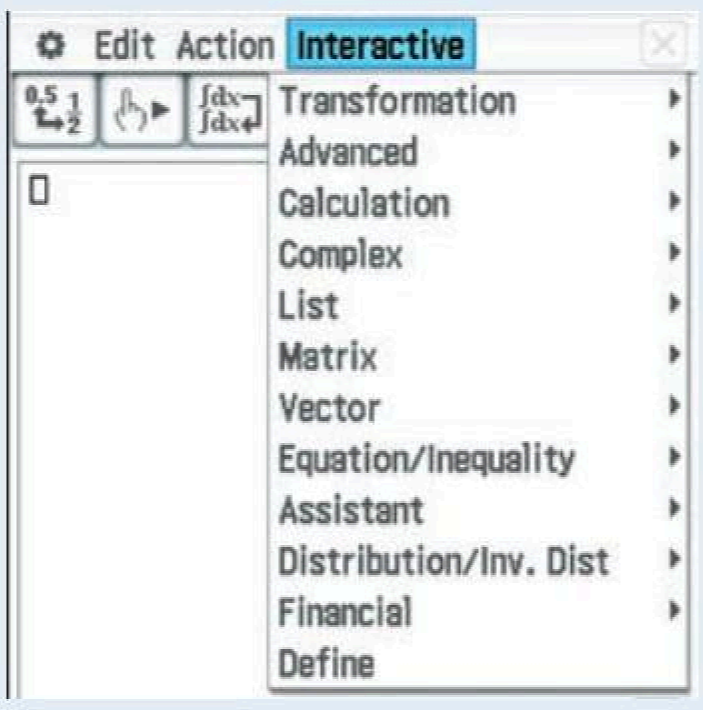
Tap **Var** to access the variables. Use variables, not letters, in your algebraic calculations.



Tap **abc** to access letters and symbols. Use letters to name functions and matrices, etc. Tap the tabs at the top of the screen to access the range of symbols. Press **back** to return to the main screen.

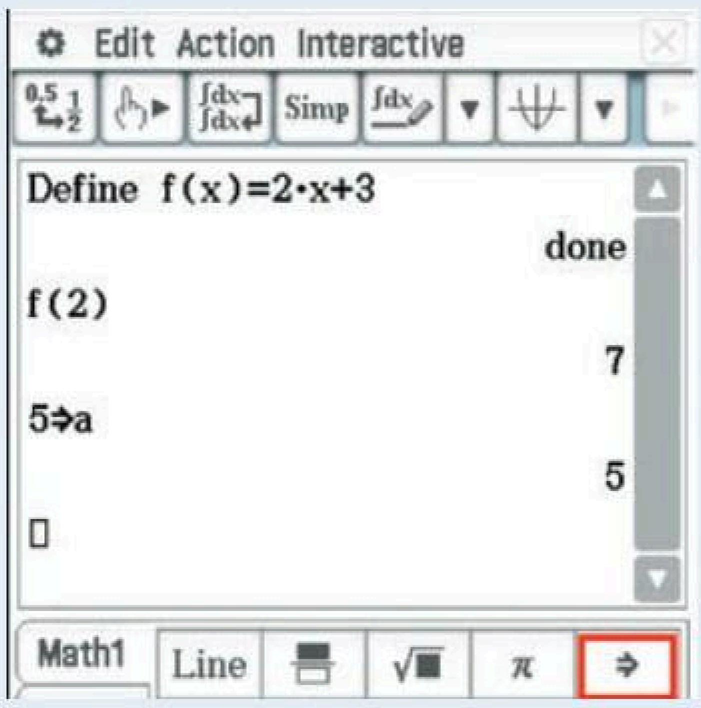
Applications

Main



In **Main**, there are two menus available to enter functions, **Active** and **Interactive**. With the **Interactive** menu, users are prompted with a dialogue box to enter the parameters. With few exceptions, the instructions in this student book are written using the **Interactive** menu.

For **derivative** and **integral** it is easier to use **Interactive > Calculation > diff or summa** (∫) rather than the icons in **Math2**.



```
Define f(x)=2·x+3
                      done
f(2)
                         7
5⇒a
                         5
```

To define a function, enter and highlight the expression then tap **Interactive > Define**. The **store** arrow is available from the **Math** and **Trig** soft keyboards. **Store** can be used to store values.

Use the **abc** soft keyboard to label values.

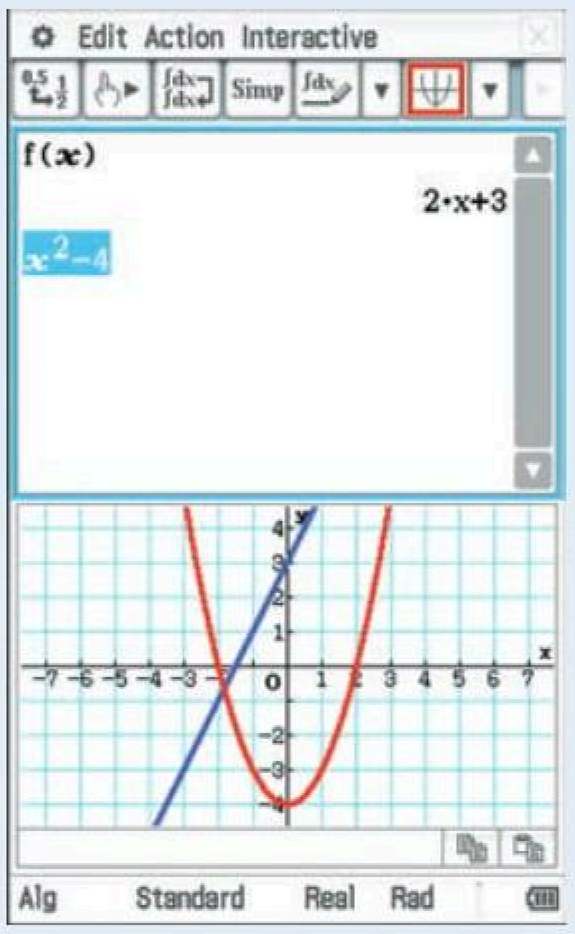
All algebraic working uses **Var** and not **abc**.

Define a function will be over-saved with a new defined function.

In contrast, stored variables need to be cleared using **Variable Manager**.

Graphing

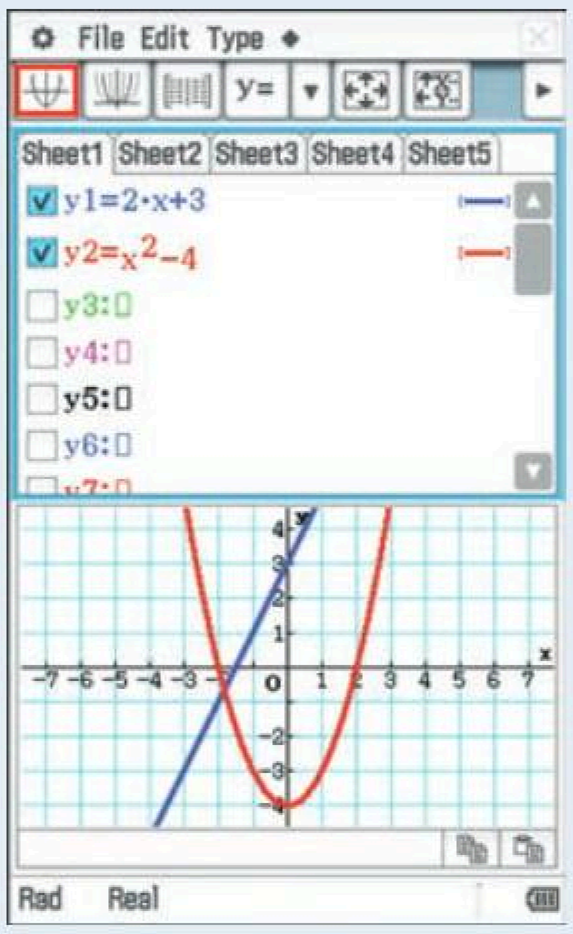
There are two main options for graphing functions and relations.



In **Main**, tap **Graph** to open the graph window. Enter and highlight the function or expression and drag it into the **Graph** window from either the right or left side.

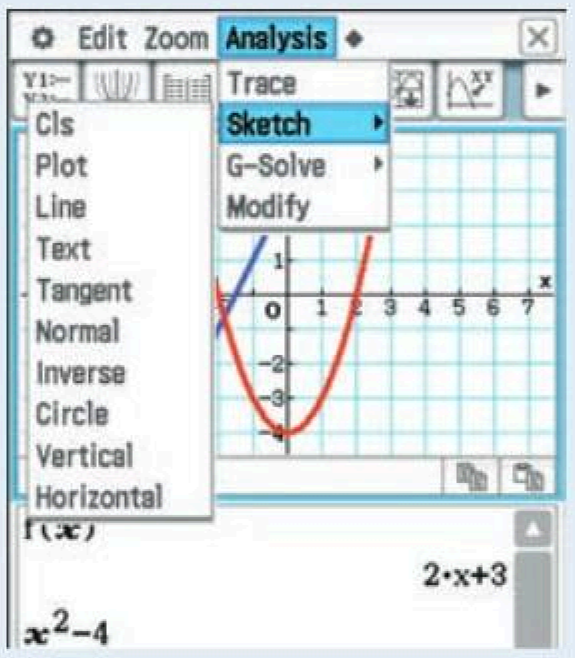
If the function has a restricted domain, it must be dragged from the left side of the screen.

The significance of **Main** is that the equations are on the top and corresponding graphs are on the bottom of the twin screens.

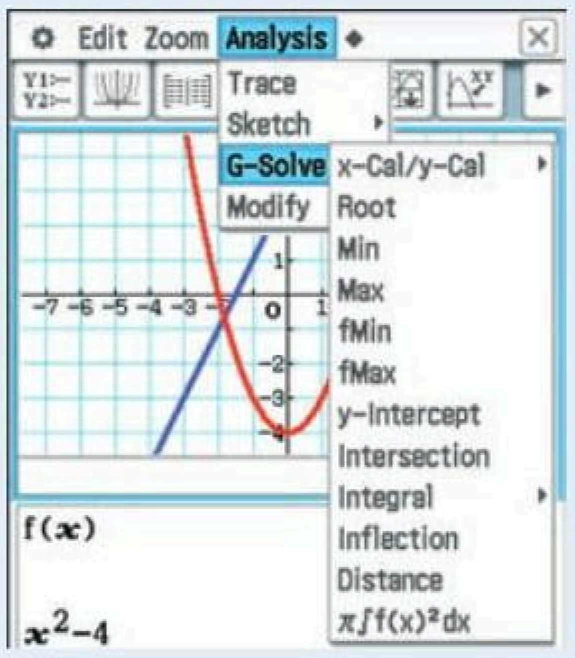


Graph&Table is used much less frequently because it requires rearrangement in terms of **y=...**

Tap **Menu > Graph&Table**. Enter the function then tap **Graph**. The instructions in this student book use the **Main** option in preference to the **Graph&Table** option.

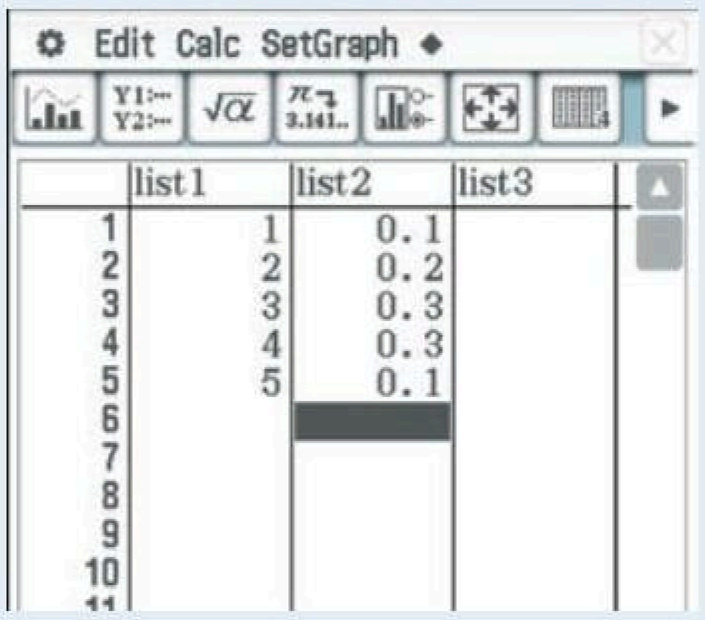


There are several menu options available to analyse graphs. Tap **Analysis > Trace** and press the **arrow** keys to move along the graph. Tap **Analysis > Sketch** for the options shown above.



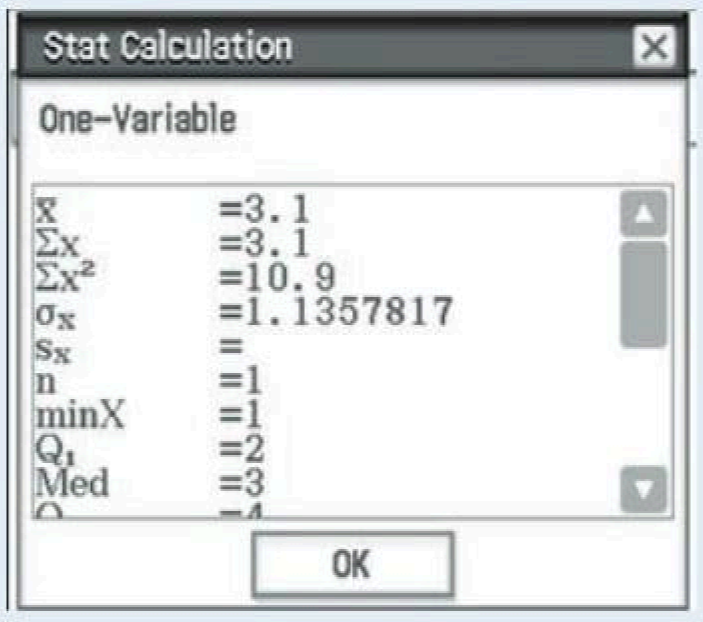
Tap **Analysis > G-Solve** to identify key features of a graph. For example, select **Root** to find the *x*-intercepts and select **Intersection** to locate points of intersection of two graphs. **G-Solve** gives decimal values. Use **Main** to calculate exact values.

Statistics



	list1	list2	list3
1	1	0.1	
2	2	0.2	
3	3	0.3	
4	4	0.3	
5	5	0.1	
6			
7			
8			
9			
10			

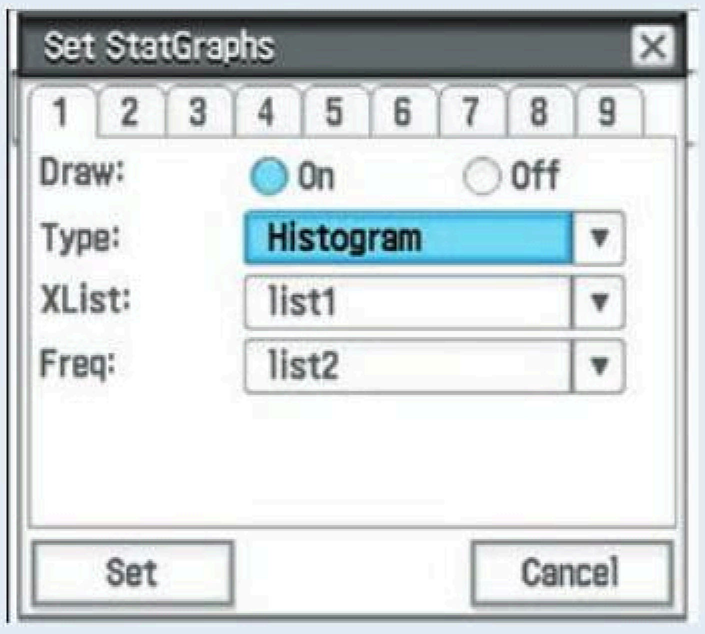
Tap **Menu > Statistics** to open the Statistics application. The default settings show **list1**, **list2**, **list3** etc. If required, tap on the list name to enter a new heading.



One-Variable

```
x̄      =3.1
Σx     =3.1
Σx²    =10.9
σx     =1.1357817
sx     =
n      =1
minX   =1
Q₁     =2
Med    =3
```

Tap **Calc** to view the Calculation menu options. Tap **One-Variable** for the dialogue box used to calculate statistical analysis of a list. The screen above displays the one variable statistics for **list1** for the frequencies in **list2**.



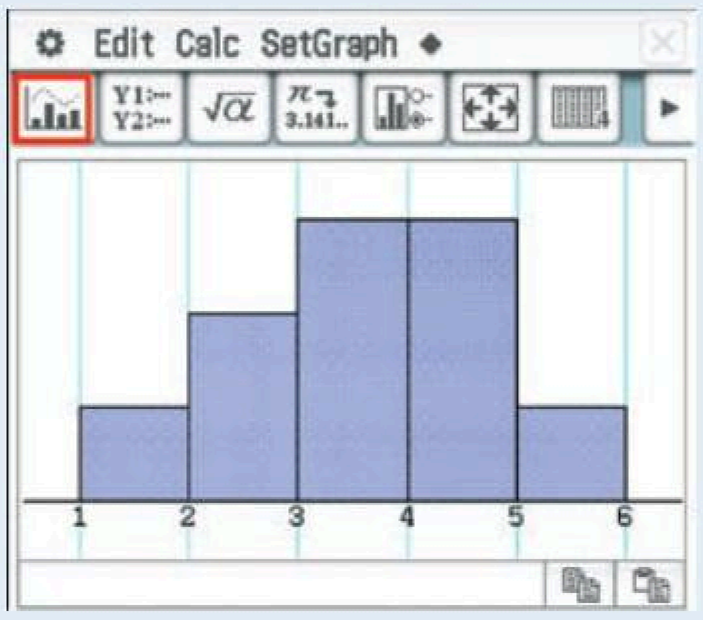
Draw: ● On ○ Off

Type: Histogram

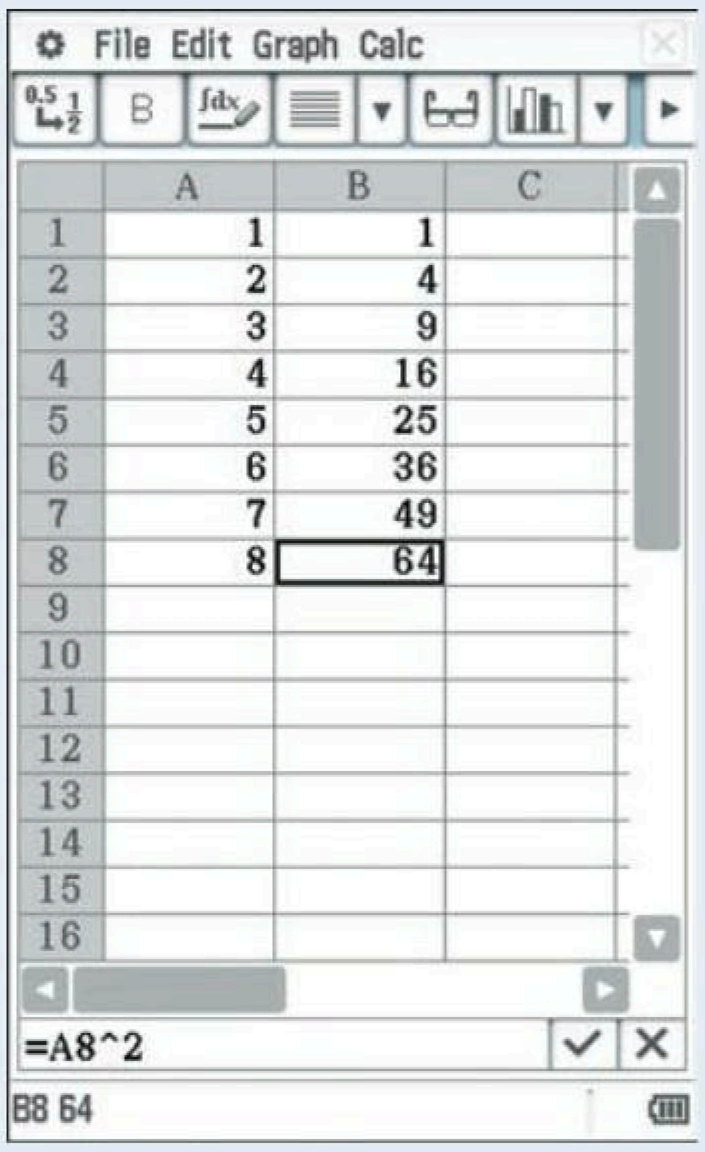
XList: list1

Freq: list2

Tap **SetGraph** to set up the statistics graphs. Tap **Type:** to view the graphing options for statistical graphs. When finished, tap **Set**.



Tap **Graph** to display the statistical graph. The graph above displays a histogram of the **list1** values for the frequencies in **list2**.

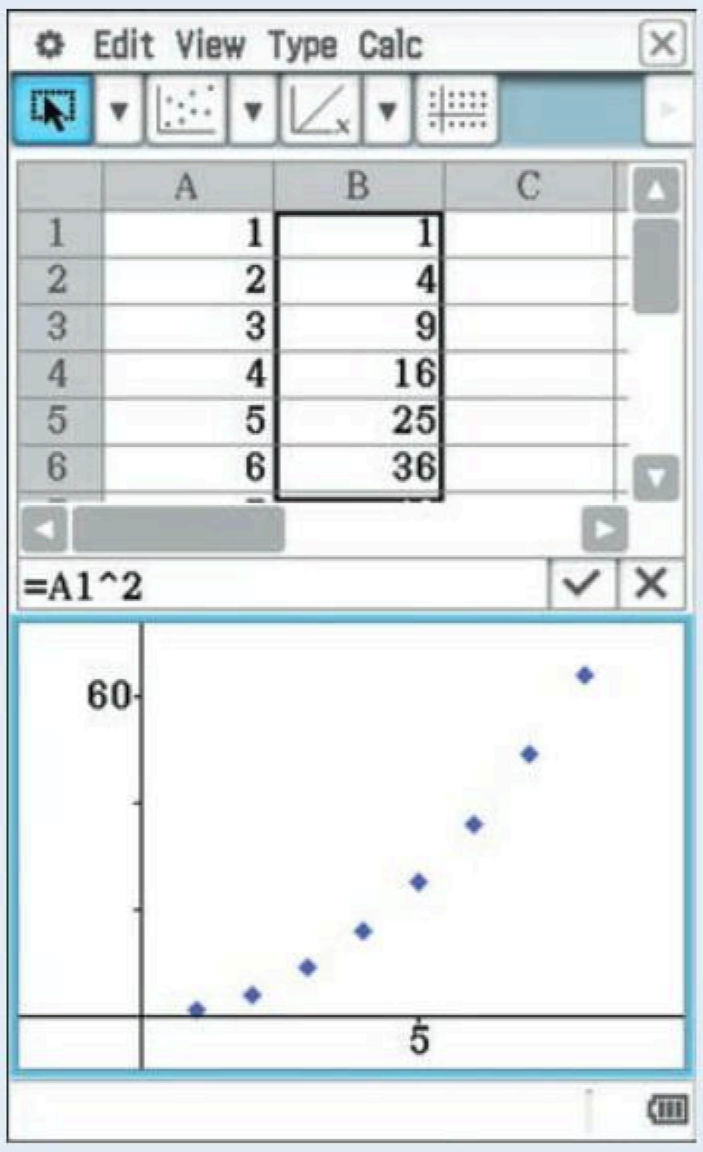


	A	B	C
1	1	1	
2	2	4	
3	3	9	
4	4	16	
5	5	25	
6	6	36	
7	7	49	
8	8	64	
9			
10			
11			
12			
13			
14			
15			
16			

=A8^2

B8 64

Tap **Menu > Spreadsheet** to open the Spreadsheet application. Values and formulas are entered using standard spreadsheet techniques. Column **A** shows the numbers 1 to 8 and Column **B** shows the square of the numbers.



	A	B	C
1	1	1	
2	2	4	
3	3	9	
4	4	16	
5	5	25	
6	6	36	

=A1^2

Highlight the required data and tap **Calc** to access the range of statistical functions.

Highlight the data and tap **Graph** to access the range of graphs available. The graph above shows a **Scatterplot** of the squared values.

Operating systems

Ensure you have the latest operating system installed on your handheld.

Tap **Settings** (located in the bottom left corner of the screen) then select **Version**. At the time of publication, the latest version is **02.01.7001**.

To download the latest operating system, go to the Australian Shriro website at www.casio.edu.shriro.com.au/classpad.php. Using the USB cable provided, connect the handheld to a computer. Start the installation program and follow the prompts. Some of these prompts will be on the computer and others will be on the handheld.

ClassPad II

Version 02.01.7001.0000

CHAPTER

1

VECTORS

Study Design coverage

AREA OF STUDY 5: SPACE AND MEASUREMENT

Vectors

- addition and subtraction of vectors and their multiplication by a scalar, position vectors
- linear dependence and independence of a set of vectors and geometric interpretation
- magnitude of a vector, unit vector, the orthogonal unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$
- resolution of a vector into rectangular components
- scalar (dot) product of two vectors, deduction of dot product for the $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ vector system and its use to find scalar resolute and vector resolute
- vector (cross) product of two vectors in three dimensions, including the determinant form
- parallel and perpendicular vectors
- vector proofs of simple geometric results, such as ‘the diagonals of a rhombus are perpendicular’, ‘the medians of a triangle are concurrent’ and ‘the angle subtended by a diameter in a circle is a right angle’.

VCE Mathematics Study Design 2023–2027 p. 112, © VCAA 2022

Video playlists (8):

1.1	Operations with vectors
1.2	Linear dependence and independence of vectors
1.3	Resolving vectors
1.4	Scalar product and vector projections
1.5	Vector product
1.6	Parallel and perpendicular vectors
1.7	Vector proofs of geometric results
VCE question analysis	Vectors

Worksheets (12):

1.1	Representing vectors • Unit vectors • Scalar multiplication
1.4	Scalar product • Component form of the dot product • Properties of the dot product • Projections
1.5	Vector product
1.6	Parallel and perpendicular vectors • Resultant and unit vectors in 3D space
1.7	Proofs using vectors • Geometry proofs using vectors

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

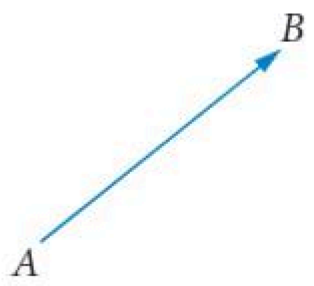
1.1 Operations with vectors

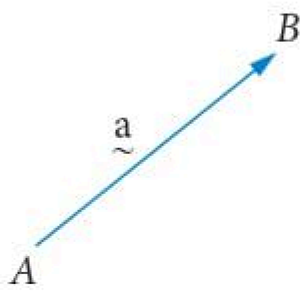
Video playlist Operations with vectors

Worksheet Representing vectors

A **vector** is defined as having a magnitude and a direction.
The vector $\overrightarrow{AB}$ is a line of magnitude $|\overrightarrow{AB}|$ and direction starting at A and finishing at B.

A vector can also be expressed with the tilde symbol ‘~’, for example, $\underset{\sim}{a}$.





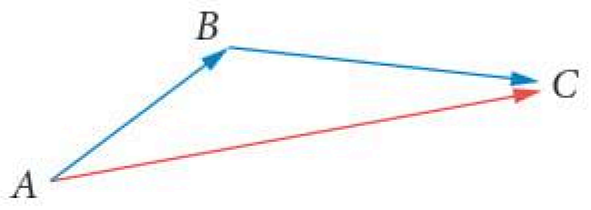
Adding and subtracting vectors

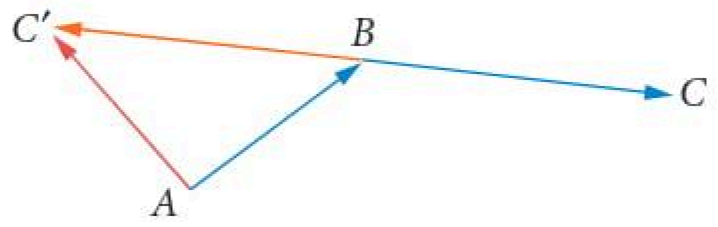
We **add** vectors by placing them ‘head to tail’.

In this diagram, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

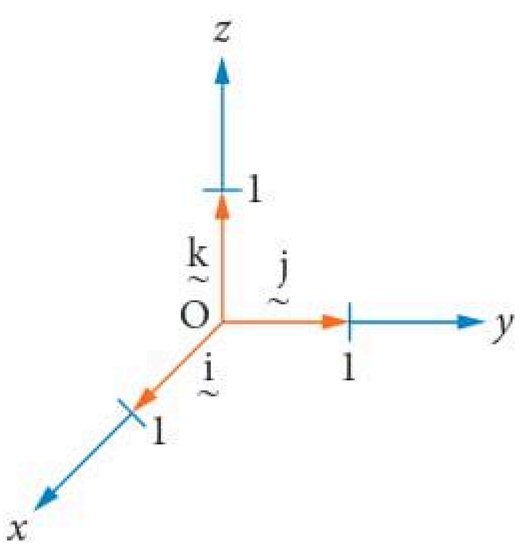
We **subtract** vectors by reversing the direction of the vector that is subtracted.

In this diagram, $\overrightarrow{AB} - \overrightarrow{BC} = \overrightarrow{AB} + \overrightarrow{BC'} = \overrightarrow{AC'}$.





Position vectors



Vectors can be written with reference to a certain point, usually the origin, O.

$\underset{\sim}{i}$ is the **unit vector** in the x direction.

$\underset{\sim}{j}$ is the unit vector in the y direction.

$\underset{\sim}{k}$ is the unit vector in the z direction.

Consider the three position vectors: $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$, $\underset{\sim}{b} = \underset{\sim}{i} - 4\underset{\sim}{j}$, $\underset{\sim}{c} = -3\underset{\sim}{i} + \underset{\sim}{j}$

Adding and subtracting the matching terms, we get

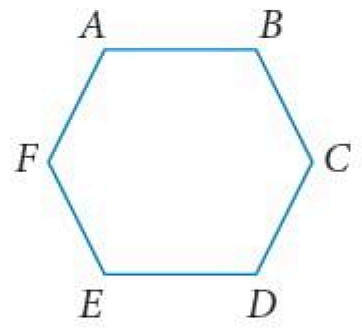
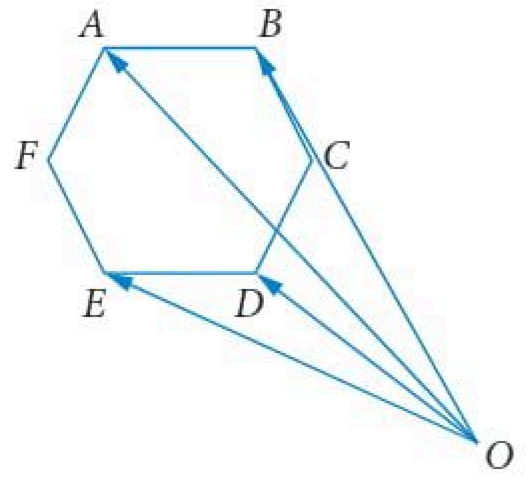
$\underset{\sim}{a} + \underset{\sim}{b} = 3\underset{\sim}{i} - \underset{\sim}{j}$

$\underset{\sim}{b} + 2\underset{\sim}{c} = -5\underset{\sim}{i} - 2\underset{\sim}{j}$

$3\underset{\sim}{a} - \underset{\sim}{b} + \underset{\sim}{c} = 2\underset{\sim}{i} + 14\underset{\sim}{j}$

> Another way of expressing the position vector $x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ is (x, y, z) or $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

WB
p. 1

WORKED EXAMPLE 1	**Position vectors**
For the given diagram of a regular hexagon, express $\overrightarrow{AB}$ in terms of $\underset{\sim}{e}$ and $\underset{\sim}{d}$, where $\overrightarrow{OA}$ is the position vector $\underset{\sim}{a}$.	
Steps	**Working**
1 Label the hexagon, adding a chosen origin, O.  	
2 Express $\overrightarrow{AB}$ in terms of $\underset{\sim}{e}$ and $\underset{\sim}{d}$.	$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$ Also $\overrightarrow{AB} = \overrightarrow{ED}$ (sides are parallel). So $\overrightarrow{AB} = \overrightarrow{ED} = \overrightarrow{EO} + \overrightarrow{OD}$ giving $\overrightarrow{AB} = -\underset{\sim}{e} + \underset{\sim}{d}$.

Unit vectors

Worksheet
Unit vectors

A unit vector has a magnitude of 1 unit.

The unit vector of $\underset{\sim}{a}$ is given by $\hat{\underset{\sim}{a}} = \dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$.

> $\hat{\underset{\sim}{a}}$ is pronounced 'a-hat'.

The magnitude of vector $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ is $|\underset{\sim}{a}| = \sqrt{x^2 + y^2 + z^2}$.

For example, for the position vector $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$, the magnitude of $\underset{\sim}{a}$ is

$$|\underset{\sim}{a}| = \sqrt{2^2 + 3^2 + (-1)^2} = \sqrt{14}$$

So the unit vector $\hat{\underset{\sim}{a}} = \dfrac{1}{\sqrt{14}}(2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k})$.

CAS can be used to find unit vectors and to perform operations with vectors.

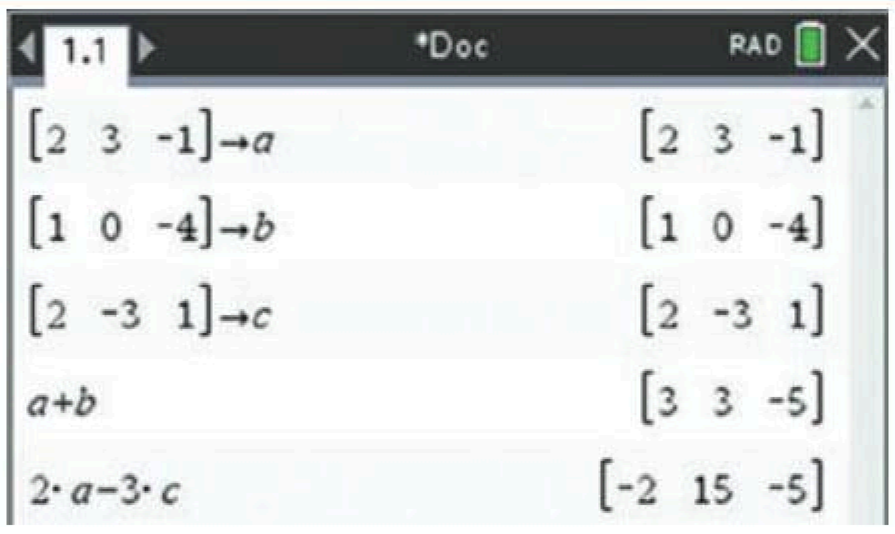
USING CAS 1 Operations with vectors

Given the three position vectors: $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{b} = \underset{\sim}{i} - 4\underset{\sim}{k}$, $\underset{\sim}{c} = 2\underset{\sim}{i} - 3\underset{\sim}{i} + \underset{\sim}{j}$, find

a $\underset{\sim}{a} + \underset{\sim}{b}$ and $2\underset{\sim}{a} - 3\underset{\sim}{c}$. **b** the magnitude $|\underset{\sim}{c}|$ and unit vector $\hat{\underset{\sim}{c}}$.

TI-Nspire

a



$[2\ 3\ -1] \to a$ $[2\ 3\ -1]$

$[1\ 0\ -4] \to b$ $[1\ 0\ -4]$

$[2\ -3\ 1] \to c$ $[2\ -3\ 1]$

$a+b$ $[3\ 3\ -5]$

$2 \cdot a - 3 \cdot c$ $[-2\ 15\ -5]$

1 Press the **template** key and select the **3×3** matrix template.

2 In the dialogue box, select **1** row and **3** columns*.

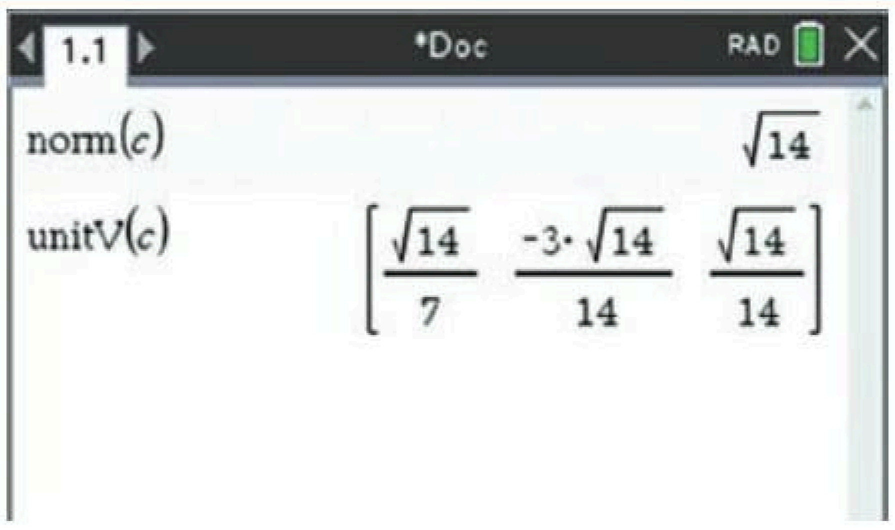
3 Enter the $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ components for $\underset{\sim}{a}$ into the template.

4 Press **ctrl + var** and store the matrix into ***a***.

5 Repeat to store the other two matrices into ***b*** and ***c***.

6 Enter the additions and multiplications as shown above.

b



norm(c) $\sqrt{14}$

unitV(c) $\left[\frac{\sqrt{14}}{7}\ \ \frac{-3\cdot\sqrt{14}}{14}\ \ \frac{\sqrt{14}}{14}\right]$

1 Press **menu > Matrix & Vector > Norms > Norm**.

2 Enter the stored matrix ***c*** to find the magnitude.

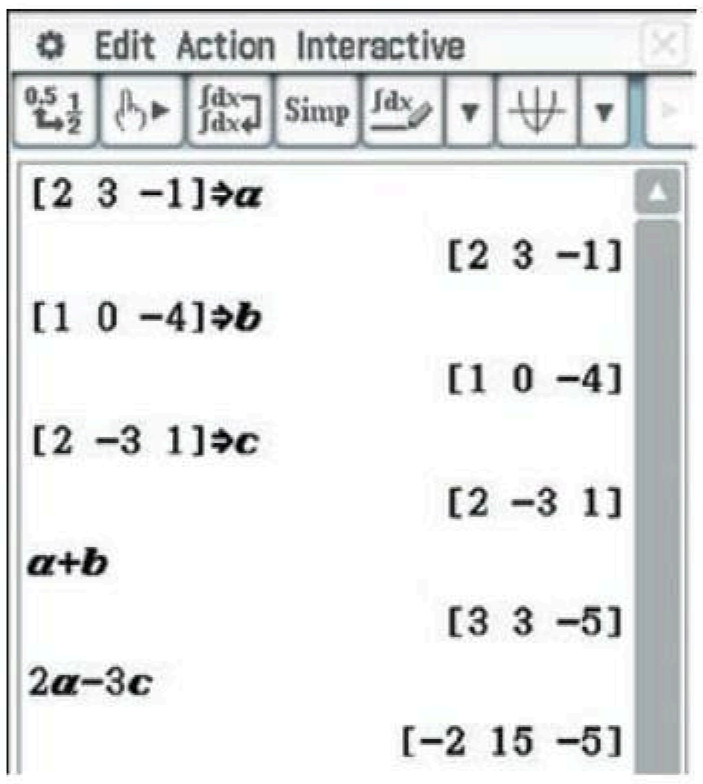
3 Press **menu > Matrix & Vector > Vector > Unit Vector**.

4 Enter the stored matrix ***c*** to find the unit vector.

* Vectors can also be stored as column matrices.

ClassPad

a



$[2\ 3\ -1] \Rightarrow a$ $[2\ 3\ -1]$

$[1\ 0\ -4] \Rightarrow b$ $[1\ 0\ -4]$

$[2\ -3\ 1] \Rightarrow c$ $[2\ -3\ 1]$

$a+b$ $[3\ 3\ -5]$

$2a-3c$ $[-2\ 15\ -5]$

1 Open the **Keyboard** and tap **Math2**.

2 Tap *twice* on the **1×2** matrix template to create a **1×3** matrix*.

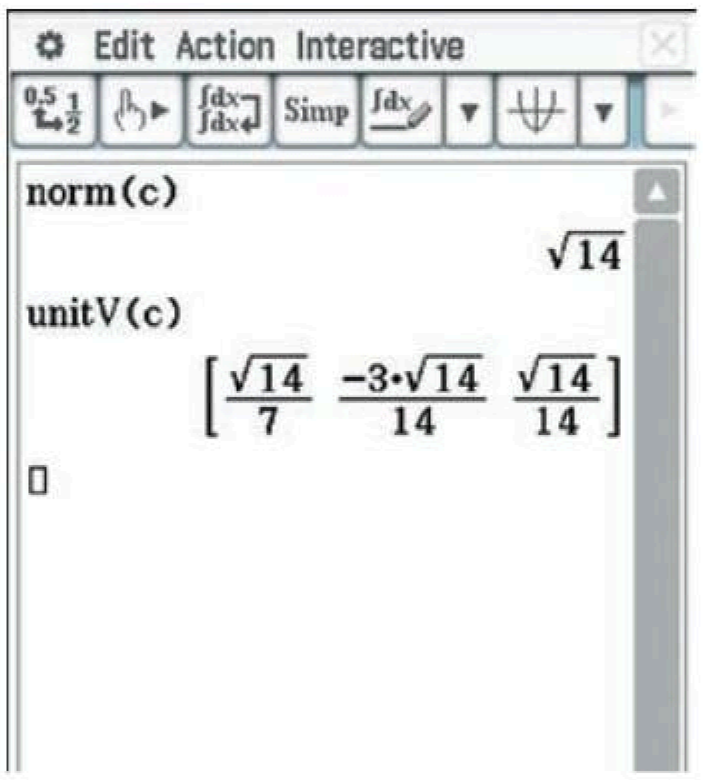
3 Enter the $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ components for $\underset{\sim}{a}$ into the template.

4 Tap ⇒ to store into matrix ***a***.

5 Repeat to store the other two matrices into ***b*** and ***c***.

6 Enter the additions and multiplications as shown above.

b



norm(c) $\sqrt{14}$

unitV(c) $\left[\frac{\sqrt{14}}{7}\ \ \frac{-3\cdot\sqrt{14}}{14}\ \ \frac{\sqrt{14}}{14}\right]$

1 Press **Interactive > Vector > norm**.

2 In the dialogue box, enter the stored matrix ***c*** to find the magnitude.

3 Press **Interactive > Vector > unitV**.

4 In the dialogue box, enter the stored matrix ***c*** to find the unit vector.

* Vectors can also be stored as column matrices.

WB

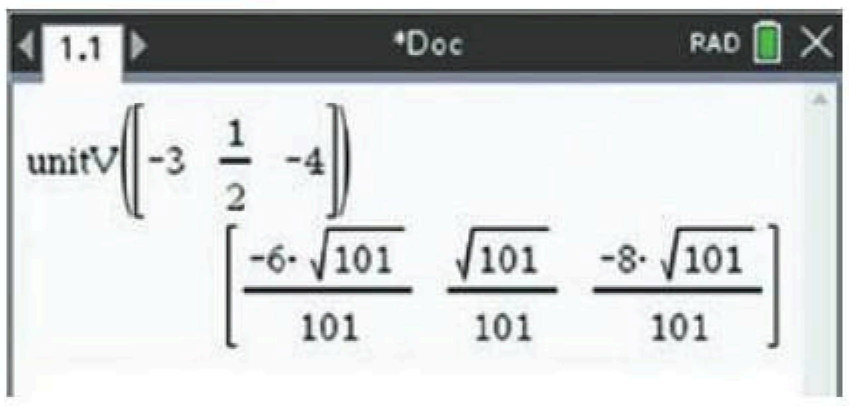
p. 2

WORKED EXAMPLE 2 Unit vectors

Find the unit vector $\hat{\underset{\sim}{b}}$, where $\underset{\sim}{b} = -3\underset{\sim}{i} + \frac{1}{2}\underset{\sim}{j} - 4\underset{\sim}{k}$.

Steps	**Working**
1 Find the magnitude of $\underset{\sim}{b}$.	$\lvert\underset{\sim}{b}\rvert = \sqrt{(-3)^2 + \left(\frac{1}{2}\right)^2 + (-4)^2} = \sqrt{\frac{101}{4}}$
2 Write down $\hat{\underset{\sim}{b}}$ using the formula $\hat{\underset{\sim}{b}} = \frac{\underset{\sim}{b}}{\lvert\underset{\sim}{b}\rvert}$.	$\hat{\underset{\sim}{b}} = \frac{1}{\sqrt{\frac{101}{4}}}\left(-3\underset{\sim}{i} + \frac{1}{2}\underset{\sim}{j} - 4\underset{\sim}{k}\right)$ $\therefore\ \hat{\underset{\sim}{b}} = \frac{2}{\sqrt{101}}\left(-3\underset{\sim}{i} + \frac{1}{2}\underset{\sim}{j} - 4\underset{\sim}{k}\right)$

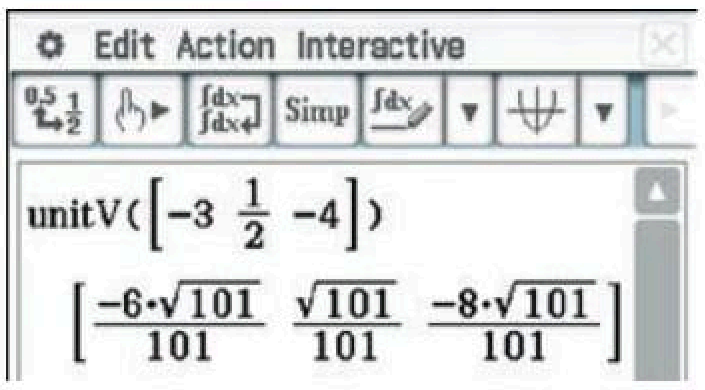
TI-Nspire



$\mathrm{unitV}\left(\begin{bmatrix}-3 & \frac{1}{2} & -4\end{bmatrix}\right)$

$\begin{bmatrix}\frac{-6\cdot\sqrt{101}}{101} & \frac{\sqrt{101}}{101} & \frac{-8\cdot\sqrt{101}}{101}\end{bmatrix}$

ClassPad



$\mathrm{unitV}\left(\begin{bmatrix}-3 & \frac{1}{2} & -4\end{bmatrix}\right)$

$\begin{bmatrix}\frac{-6\cdot\sqrt{101}}{101} & \frac{\sqrt{101}}{101} & \frac{-8\cdot\sqrt{101}}{101}\end{bmatrix}$

Worksheet
Scalar multiplication

Scalar multiplication

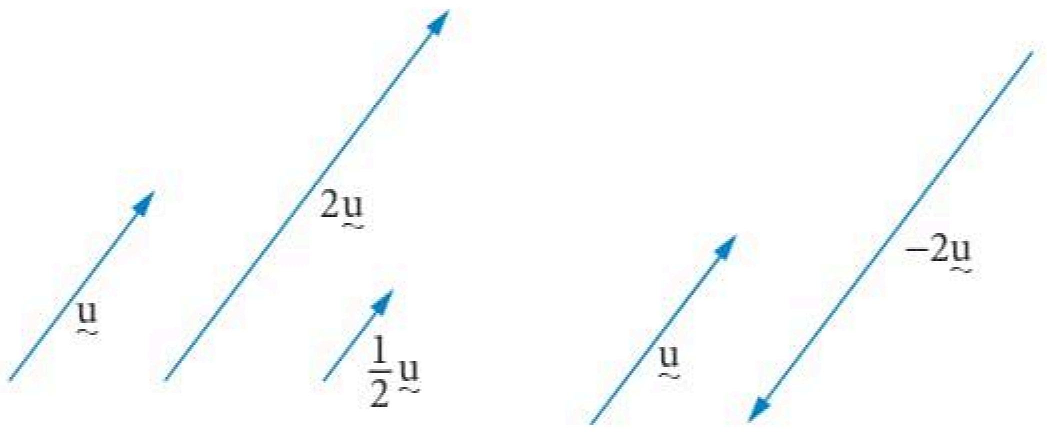
Multiplication by a real number (scalar) changes the magnitude, but not the direction, of the vector.

Consider the vector $\underset{\sim}{u} = \underset{\sim}{i} + \underset{\sim}{j}$.

$2\underset{\sim}{u}$ is twice the length of $\underset{\sim}{u}$.

$\frac{1}{2}\underset{\sim}{u}$ is half the length of $\underset{\sim}{u}$.

When $\underset{\sim}{u}$ is multiplied by -2, the vector $\underset{\sim}{u}$ has double the length, but in the opposite direction.

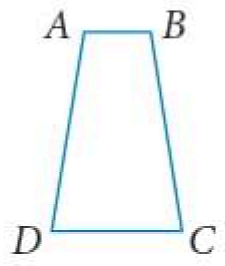


EXERCISE 1.1 Operations with vectors

ANSWERS p. 555

Mastery

1 WORKED EXAMPLE 1 TECH-FREE For the trapezium shown, if $\overrightarrow{AB} = \frac{2}{3}\overrightarrow{DC}$, express $\overrightarrow{AB}$ in terms of $\underset{\sim}{c}$ and $\underset{\sim}{d}$, where $\overrightarrow{OC} = \underset{\sim}{c}$ and $\overrightarrow{OD} = \underset{\sim}{d}$.



2 WORKED EXAMPLE 2 TECH-FREE Find the unit vector $\hat{\underset{\sim}{b}}$, where $\underset{\sim}{b} = 3\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$.

3 TECH-FREE

a Find the magnitude of vector $\underset{\sim}{a}$, where $\underset{\sim}{a} = -3\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}$.

b Hence find the unit vector $\hat{\underset{\sim}{a}}$.

4 TECH-FREE Given $\underset{\sim}{p} = -3\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{q} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$, $\overrightarrow{OA} = 4\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$ and $\overrightarrow{OB} = 5\underset{\sim}{i} - \underset{\sim}{j} + 6\underset{\sim}{k}$, find

a $\overrightarrow{AB}$ **b** $\left|\overrightarrow{AB}\right|$ **c** $\hat{\underset{\sim}{p}}$ **d** $\hat{\underset{\sim}{q}}$

5 Given $\underset{\sim}{m} = (-6, 4, -2)$, $\underset{\sim}{z} = (0, 3, -4)$, $\underset{\sim}{d} = (1, 2, 5)$ and $\underset{\sim}{e} = (3, -2, -1)$, $\underset{\sim}{m} + \underset{\sim}{e}$ equals

A $(-3, 2, -3)$ **B** $(-6, 7, -6)$ **C** $(-5, 6, -3)$ **D** $(1, 5, 1)$ **E** $(1, 2, 5)$

6 Given $\underset{\sim}{m} = (-6, 4, -2)$, $\underset{\sim}{z} = (0, 3, -4)$, $\underset{\sim}{d} = (1, 2, 5)$ and $\underset{\sim}{e} = (3, -2, -1)$, $2\underset{\sim}{z} - 3\underset{\sim}{d}$ equals

A $(-3, 2, -3)$ **B** $(-6, 7, -6)$ **C** $(-5, 6, -3)$ **D** $(-3, 0, -23)$ **E** $(-2, 0, 6)$

7 Given $\underset{\sim}{m} = (-6, 4, -2)$, $\underset{\sim}{z} = (0, 3, -4)$, $\underset{\sim}{d} = (1, 2, 5)$ and $\underset{\sim}{e} = (3, -2, -1)$, $-3\underset{\sim}{m}$ equals

A $(-18, 12, -6)$ **B** $(18, -12, 6)$ **C** $(0, 9, -12)$ **D** $(-3, 0, -23)$ **E** $(-2, 0, 6)$

8 For the vector $\underset{\sim}{m} = -3\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}$, the unit vector $\widehat{\underset{\sim}{m}}$ is

A $\frac{1}{2\sqrt{2}}\left(-3\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}\right)$ **B** $\frac{1}{\sqrt{26}}\left(-3\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}\right)$ **C** $\frac{1}{8}\left(-3\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}\right)$

D $8(-3\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k})$ **E** $26(-3\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k})$

9 Using CAS 1 Given $\underset{\sim}{m} = -3\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}$, the vector three times the magnitude of $\widehat{\underset{\sim}{m}}$ in the direction of $\underset{\sim}{m}$ equals

A $\frac{3}{2\sqrt{2}}\left(-3\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}\right)$ **B** $\frac{3}{\sqrt{26}}\left(-3\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}\right)$ **C** $\frac{3}{8}\left(-3\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}\right)$

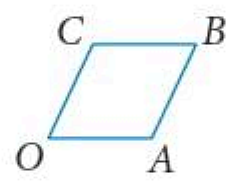
D $24(-3\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k})$ **E** $26(-3\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k})$

Exam practice

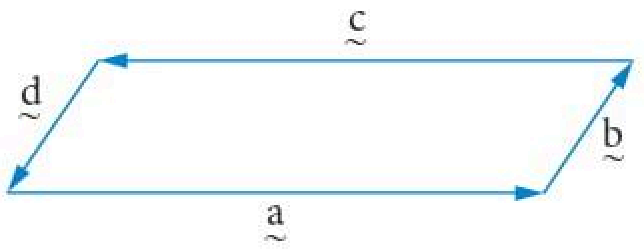
80–100% 60–79% 0–59%

10 © VCAA 2014 1Q1a 79% TECH-FREE (1 mark) Consider the vector $\underset{\sim}{a} = \sqrt{3}\underset{\sim}{i} - \underset{\sim}{j} - \sqrt{2}\underset{\sim}{k}$, where $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors in the positive directions of the *x*, *y* and *z* axes respectively. Find the unit vector in the direction of $\underset{\sim}{a}$.

11 © VCAA 2015 1Q1a 71% TECH-FREE (1 mark) Consider the rhombus *OABC* shown, where $\overrightarrow{OA} = a\underset{\sim}{i}$ and $\overrightarrow{OC} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$, and *a* is a positive real constant. Find *a*.



12 © VCAA 2006 2AQ15 74% In the parallelogram shown, $|\underset{\sim}{a}| = 2|\underset{\sim}{b}|$.



Which one of the following statements is true?

A $\underset{\sim}{a} = 2\underset{\sim}{b}$

B $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{c} + \underset{\sim}{d}$

C $\underset{\sim}{b} - \underset{\sim}{d} = 0$

D $\underset{\sim}{a} + \underset{\sim}{c} = 0$

E $\underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{c} - \underset{\sim}{d}$

13 © VCAA 2018 2AQ12 36% If $|\underset{\sim}{a} + \underset{\sim}{b}| = |\underset{\sim}{a}| + |\underset{\sim}{b}|$ and $\underset{\sim}{a} \cdot \underset{\sim}{b} \neq 0$, which one of the following is **necessarily true**?

A $\underset{\sim}{a}$ is parallel to $\underset{\sim}{b}$ **B** $|\underset{\sim}{a}| = |\underset{\sim}{b}|$

C $\underset{\sim}{a} = \underset{\sim}{b}$ **D** $\underset{\sim}{a} = -\underset{\sim}{b}$

E $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$

Video playlist
Linear dependence and independence of vectors

1.2 Linear dependence and independence of vectors

If vector $\underset{\sim}{c}$ can be written as a linear combination of vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, then $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are said to be **linearly dependent**. This means that when $\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$ for real numbers m and n, where m and n are not zero, $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are linearly dependent.

Conversely, if vector $\underset{\sim}{c}$ *cannot* be written as a linear combination of vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, then $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are said to be **linearly independent**.

A set of vectors is linearly dependent if at least one of its vectors can be expressed as a linear combination of the other vectors.

Conversely, a set of vectors is linearly independent if no vector can be expressed as a linear combination of the other vectors.

A set of vectors is linearly independent if it is not linearly dependent.

For the special case of two vectors, they are linearly dependent iff (‘if and only if’) they are parallel.

$\underset{\sim}{a}$ and $\underset{\sim}{b}$ are linearly dependent iff $\underset{\sim}{a} = m\underset{\sim}{b}$, where m is a scalar.

$\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are linearly dependent iff $\underset{\sim}{a} = m\underset{\sim}{b} + n\underset{\sim}{c}$, where m and n are scalars.

WB
p. 3

WORKED EXAMPLE 3 Linear dependence

Consider the set of four vectors $\underset{\sim}{p} = -3\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{q} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$, $\underset{\sim}{r} = 3\underset{\sim}{i} + 3\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{s} = \underset{\sim}{i} - 3\underset{\sim}{j} + 3\underset{\sim}{k}$.

a If $\underset{\sim}{q} = m\underset{\sim}{r}$, determine the value of m such that the two vectors $\underset{\sim}{q}$ and $\underset{\sim}{r}$ are linearly dependent.

b Determine whether the three vectors $\underset{\sim}{p}$, $\underset{\sim}{q}$ and $\underset{\sim}{s}$ are linearly dependent.

Steps	**Working**
a Set up the statement $\underset{\sim}{q} = m\underset{\sim}{r}$, where $\underset{\sim}{q} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\underset{\sim}{r} = \begin{bmatrix}3\\3\\3\end{bmatrix}$.	$\underset{\sim}{q} = m\underset{\sim}{r}$ $\begin{bmatrix}1\\1\\1\end{bmatrix} = m\begin{bmatrix}3\\3\\3\end{bmatrix}$ We can see that $m = \frac{1}{3}$, $\therefore \underset{\sim}{q} = \frac{1}{3}\underset{\sim}{r}$ giving linearly dependent (and parallel) vectors.
b 1 Set up the statement $\underset{\sim}{p} = m\underset{\sim}{q} + n\underset{\sim}{s}$, where $\underset{\sim}{p} = \begin{bmatrix}-3\\1\\2\end{bmatrix}$, $\underset{\sim}{q} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\underset{\sim}{s} = \begin{bmatrix}1\\-3\\3\end{bmatrix}$.	$\underset{\sim}{p} = m\underset{\sim}{q} + n\underset{\sim}{s}$ $\begin{bmatrix}-3\\1\\2\end{bmatrix} = m\begin{bmatrix}1\\1\\1\end{bmatrix} + n\begin{bmatrix}1\\-3\\3\end{bmatrix}$
2 Equate $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ terms.	$\begin{array}{ccc}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ -3 = m + n & 1 = m - 3n & 2 = m + 3n\end{array}$
3 Solve for m and n. Substitute the values found for m and n and see if they hold for the components of $\underset{\sim}{i}$.	$1 = m - 3n$ $2 = m + 3n$ $m = \frac{3}{2}, n = \frac{1}{6}$ Check by substituting into $-3 = m + n$: $-3 \neq \frac{3}{2} + \frac{1}{6}$
Solutions do **not** hold for all three equations $\therefore$ linearly independent.	There are no values of m and n for this set of equations. $\underset{\sim}{p}$, $\underset{\sim}{q}$ and $\underset{\sim}{s}$ are **not linearly dependent.**

EXERCISE 1.2 Linear dependence and independence of vectors

ANSWERS p.555

Recap

1 TECH-FREE For the vectors $\underset{\sim}{m} = \begin{bmatrix} 1 \\ 10 \\ 3 \end{bmatrix}$ and $\underset{\sim}{n} = \begin{bmatrix} 3 \\ -2 \\ -10 \end{bmatrix}$, find

a $\underset{\sim}{m} + \underset{\sim}{n}$ **b** $\underset{\sim}{m} - \underset{\sim}{n}$ **c** $\underset{\sim}{m} \cdot \underset{\sim}{n}$ **d** $|\underset{\sim}{n}|$ **e** $\hat{\underset{\sim}{m}}$

2 The unit vector for the vector $p\underset{\sim}{i} - 2\underset{\sim}{j}$ is

A $2(p\underset{\sim}{i} - 2\underset{\sim}{j})$ **B** $\sqrt{p^2 + 4}(p\underset{\sim}{i} - 2\underset{\sim}{j})$ **C** $\frac{1}{\sqrt{p^2 + 4}}(p\underset{\sim}{i} - 2\underset{\sim}{j})$

D $\frac{1}{\sqrt{p^2 - 4}}(p\underset{\sim}{i} - 2\underset{\sim}{j})$ **E** $\sqrt{p^2 + 4}$

Mastery

3 TECH-FREE Determine whether the two vectors $\underset{\sim}{a} = -3\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} - \frac{1}{3}\underset{\sim}{j} + \underset{\sim}{k}$ are linearly dependent.

4 TECH-FREE Determine whether the vectors $\underset{\sim}{p} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\underset{\sim}{q} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ and $\underset{\sim}{r} = \begin{bmatrix} 3 \\ -5 \\ 11 \end{bmatrix}$ are linearly dependent.

5 TECH-FREE Given $\underset{\sim}{p} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\underset{\sim}{q} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ and $\underset{\sim}{r} = \begin{bmatrix} 3 \\ -5 \\ 11 \end{bmatrix}$, find

a $-\underset{\sim}{p} + \underset{\sim}{q}$ **b** $|\underset{\sim}{p}| + |\underset{\sim}{q}|$ **c** $|\underset{\sim}{p} + \underset{\sim}{r}|$

6 WORKED EXAMPLE 3 TECH-FREE Determine whether the vectors $\underset{\sim}{a} = 4\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{c} = -4\underset{\sim}{i} + 2\underset{\sim}{j} + 6\underset{\sim}{k}$ are linearly dependent.

7 For the linearly dependent vectors $\underset{\sim}{p} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\underset{\sim}{q} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ and $\underset{\sim}{r} = \begin{bmatrix} 3 \\ -5 \\ 11 \end{bmatrix}$, the values of m and n in the equation $\underset{\sim}{p} = m\underset{\sim}{q} + n\underset{\sim}{r}$ are

A $m = 1, n = 1$ **B** $m = \frac{2}{3}, n = -\frac{1}{3}$ **C** $m = -\frac{2}{3}, n = \frac{1}{3}$

D $m = 1, n = -3$ **E** $m = -\frac{1}{3}, n = \frac{2}{3}$

Exam practice

80–100% 60–79% 0–59%

8 © VCAA 2019 1Q6 76% TECH-FREE (3 marks) Find the value of d for which the vectors $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}$, $\underset{\sim}{b} = -2\underset{\sim}{i} + 4\underset{\sim}{j} - 8\underset{\sim}{k}$ and $\underset{\sim}{c} = -6\underset{\sim}{i} + 2\underset{\sim}{j} + d\underset{\sim}{k}$ are **linearly dependent**.

9 © VCAA 2019N 2AQ13 For the vectors $\underset{\sim}{a} = \underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{b} = -\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{c} = -\underset{\sim}{i} - 6\underset{\sim}{j} + \lambda\underset{\sim}{k}$ to be **linearly dependent,** the value of λ must be

A 0 **B** 1 **C** 2 **D** 3 **E** 4

10 © VCAA 2020 2AQ13 72% The vectors $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{b} = \lambda\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{c} = \underset{\sim}{i} + \underset{\sim}{k}$ will be **linearly dependent** when the value of λ is

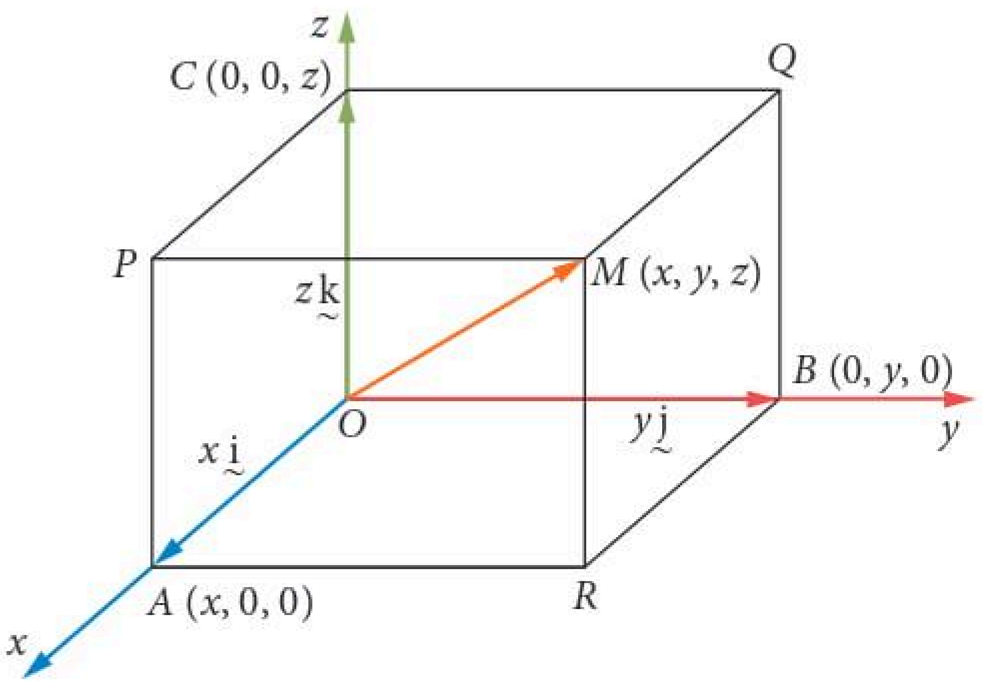
A 1 **B** 2 **C** 3 **D** 4 **E** 5

Video playlist Resolving vectors

1.3 Resolving vectors

The process of writing a vector as a linear combination of vectors in particular directions is called **resolution**.

Orthogonal components are at right angles to each other. To represent a vector in space, we resolve the vector along the three mutually perpendicular axes as shown below.



The vector $\overrightarrow{OM}$ can be resolved along the three axes as shown. Here, OM is the diagonal of the parallelepiped with edges OA, OB and OC along the three perpendicular axes.

From the above figure, we say that

$\overrightarrow{OA} = x\underset{\sim}{i}$

$\overrightarrow{OB} = y\underset{\sim}{j}$

$\overrightarrow{OC} = z\underset{\sim}{k}$

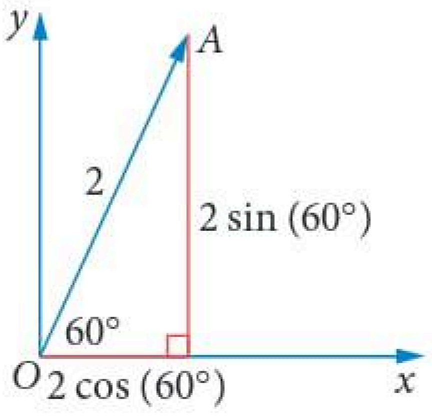
The vector $\overrightarrow{OM}$ can be represented as $\overrightarrow{OM} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$. This is known as the **component form** of a vector.

It is easy to write a vector in component form if we know the beginning and endpoints of the vector.

Consider a vector starting at (0, 0, 0) and finishing at (2, 3, 4); we write the vector as $2\underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k}$.

Consider a vector $\overrightarrow{OA}$ of length 2 units at an angle of 60° to the x-axis. We know from trigonometry that the x- and y-components of the vector depend on cosine and sine.

$$\overrightarrow{OA} = (2\cos(60°))\underset{\sim}{i} + (2\sin(60°))\underset{\sim}{j}$$
$$= \underset{\sim}{i} + \sqrt{3}\underset{\sim}{j}$$



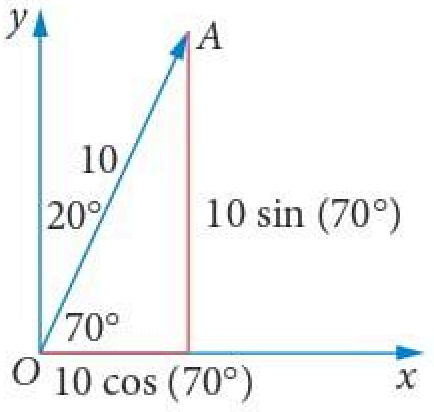
WB p. 4

WORKED EXAMPLE 4 Resolving vectors

a Write the vector $\overrightarrow{OM}$ in component form given the points $O(0, 0, 0)$ and $M(-2, 3, -5)$.

b Write the vector $\overrightarrow{AB}$ in component form given the points $A(1, 2, -5)$ and $B(-1, 3, -8)$.

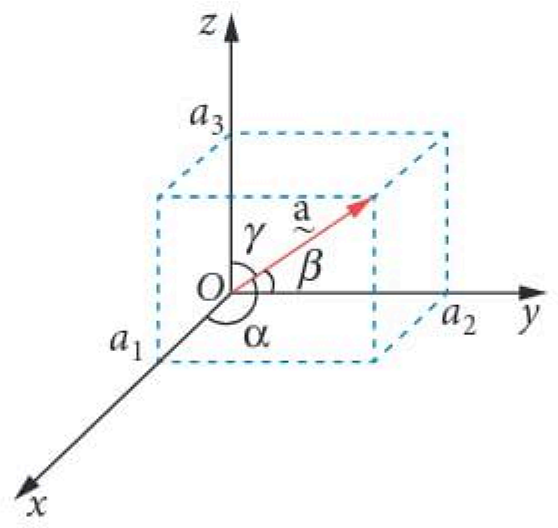
c Write a vector of magnitude 10 units at an angle of 20° to the y-axis in 2D component form.

Steps	Working
a Write the points $O(0, 0, 0)$ and $M(-2, 3, -5)$ using $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$.	$\overrightarrow{OM} = -2\underset{\sim}{i} + 3\underset{\sim}{j} - 5\underset{\sim}{k}$
b Use the distance and direction between each component of the points $A(1, 2, -5)$ and $B(-1, 3, -8)$ to write in component form.	$\overrightarrow{AB} = (-1-1)\underset{\sim}{i} + (3-2)\underset{\sim}{j} + (-8-[-5])\underset{\sim}{k}$ $= -2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$
c Draw a diagram using the information given.  	$\overrightarrow{OA} = (10\cos(70°))\underset{\sim}{i} + (10\sin(70°))\underset{\sim}{j}$

In the case where we are given the magnitude and direction of the vector in a 3D context; to express a vector in component form, the following formulas apply.

Given the vector $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$ makes angles, respectively, of α, β, γ to the positive directions of the x, y and z axes, then

$$\cos(\alpha) = \frac{a_1}{|\underset{\sim}{a}|} \qquad \cos(\beta) = \frac{a_2}{|\underset{\sim}{a}|} \qquad \cos(\gamma) = \frac{a_3}{|\underset{\sim}{a}|}$$

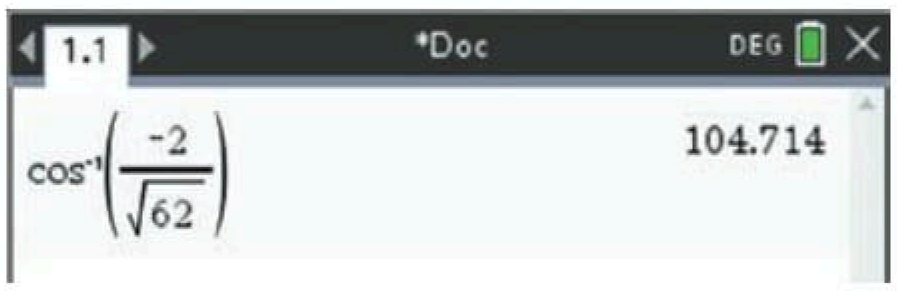


WORKED EXAMPLE 5 Angle between vectors and axes

Find the angle between the vector $\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + 7\underset{\sim}{k}$ and the y-axis. Write your answer in degrees to two decimal places.

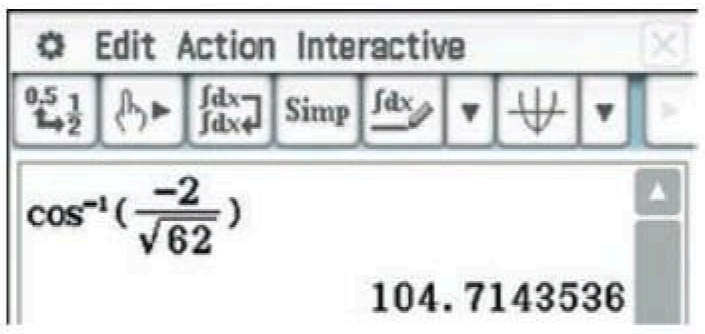
Steps	**Working**
1 Identify a_2 and $\|\underset{\sim}{a}\|$.	$a_2 = -2$ $\|\underset{\sim}{a}\| = \sqrt{3^2 + (-2)^2 + 7^2} = \sqrt{62}$
2 Use the formula $\cos(\beta) = \dfrac{a_2}{\|\underset{\sim}{a}\|}$, where β is the angle between the vector and the y-axis.	$\cos(\beta) = \dfrac{-2}{\sqrt{62}}$ $\therefore\ \beta = 104.71°$

TI-Nspire



$\cos^{-1}\left(\frac{-2}{\sqrt{62}}\right)$ 104.714

ClassPad



$\cos^{-1}\left(\frac{-2}{\sqrt{62}}\right)$

104.7143536

WB

p. 5

EXERCISE 1.3 Resolving vectors

ANSWERS p. 555

Recap

1 TECH-FREE Let $\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$.
Are these two vectors linearly dependent or linearly independent?

2 For the vectors $\underset{\sim}{p} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\underset{\sim}{q} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ and $\underset{\sim}{r} = \begin{bmatrix} 3 \\ -5 \\ 11 \end{bmatrix}$, which statement is **false**?

A The three vectors are linearly dependent.

B $\underset{\sim}{p}$ has coefficients 1, 1 and –1.

C $\hat{\underset{\sim}{q}} = \frac{1}{2\sqrt{2}}\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$

D $\hat{\underset{\sim}{p}} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

E $-2\underset{\sim}{p} + \underset{\sim}{r} = \begin{bmatrix} 1 \\ -7 \\ 14 \end{bmatrix}$

Mastery

3 TECH-FREE Let $\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{b} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$. Find

a $\underset{\sim}{a} + \underset{\sim}{b}$

b $2\underset{\sim}{a} - 3\underset{\sim}{b}$

4 WORKED EXAMPLE 4 TECH-FREE Write the vector $\overrightarrow{OM}$ in component form given the points $O\begin{bmatrix}0\\0\\0\end{bmatrix}$ and $M\begin{bmatrix}2\\3\\-1\end{bmatrix}$.

5 TECH-FREE Write the vector $\overrightarrow{AB}$ in component form given the points $A\begin{bmatrix}3\\2\\1\end{bmatrix}$ and $B\begin{bmatrix}-3\\1\\10\end{bmatrix}$.

6 TECH-FREE Write a vector of magnitude 2 units at an angle of 30° to the x-axis in 2D component form.

7 WORKED EXAMPLE 5 TECH-FREE Let $\underset{\sim}{b} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$. Find

a its magnitude.

b the angle the vector makes with the x, y and z axes, respectively. Write your answers to the nearest degree.

8 Given $\underset{\sim}{m} = (1, 4, -2)$ and $\underset{\sim}{e} = (3, 2, 1)$, $\underset{\sim}{e} - 2\underset{\sim}{m}$ equals

A $(1, -6, 5)$ **B** $(1, -6, 6)$ **C** $(7, 8, 0)$ **D** $(-5, 0, -4)$ **E** $(4, 6, -1)$

9 Given $\underset{\sim}{m} = (-6, 4, -2)$, $\underset{\sim}{d} = (1, 2, 5)$ and $\underset{\sim}{e} = (3, -2, -1)$, $\underset{\sim}{m} + \underset{\sim}{e} + \underset{\sim}{d}$ equals

A $(-5, 6, 3)$ **B** $(4, 0, 4)$ **C** $(-5, 6, -3)$ **D** $(-2, 0, 2)$ **E** $(-2, 4, 2)$

Exam practice

80–100% 60–79% 0–59%

10 © VCAA 2014 1Q1 TECH-FREE (3 marks) Consider the vector $\underset{\sim}{a} = \sqrt{3}\underset{\sim}{i} - \underset{\sim}{j} - \sqrt{2}\underset{\sim}{k}$, where $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors in the positive directions of the x, y and z axes, respectively.

a 79% Find the unit vector in the direction of $\underset{\sim}{a}$. 1 mark

b 42% Find the acute angle that $\underset{\sim}{a}$ makes with the positive direction of the x-axis. 2 marks

11 TECH-FREE (6 marks) Write each vector as a linear combination of $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$.

a $(3, -5, -3)$ **b** $A(1, 2, 4)$ to $B(-1, 2, 6)$ **c** $\begin{bmatrix}-3\\0\\2\end{bmatrix}$

1.4 Scalar product and vector projections

The operation of multiplying two vectors has two methods. In this section, we will study the **dot product.** This is called the **scalar product** as the result is a **scalar**, a number and not a vector.

The symbol used is a dot •.

The scalar product is used in many contexts and problems when dealing with vectors.

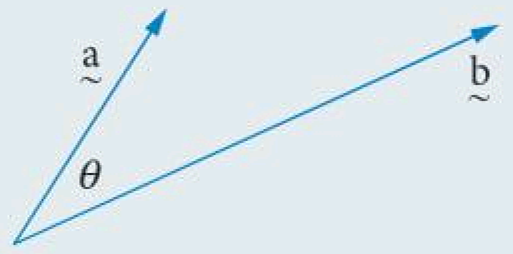
Video playlist
Scalar product and vector projections

Worksheets
Scalar product

Component form of the dot product

Properties of the dot product

Properties of the scalar product



1. The scalar product is a real number, **not** a vector.
2. The scalar product is commutative.
3. The scalar product is distributive over vector addition.
4. The scalar product is found by the formula:

   $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta)$

   where θ is the angle between the two 'outgoing' vectors.
5. If $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$ and $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}$, the scalar product is found by the formula:

   $\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1b_1 + a_2b_2 + a_3b_3$
6. $m(\underset{\sim}{a} \cdot \underset{\sim}{b}) = (m\underset{\sim}{a}) \cdot \underset{\sim}{b} = \underset{\sim}{a} \cdot (m\underset{\sim}{b})$ for all $m \in R$.
7. $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0 \Rightarrow \underset{\sim}{a} = \underset{\sim}{0}$ or $\underset{\sim}{b} = \underset{\sim}{0}$ or

   $\Rightarrow \underset{\sim}{a}$ **is perpendicular to** $\underset{\sim}{b}$.
8. $\underset{\sim}{a} \cdot \underset{\sim}{b} = \pm\ \underset{\sim}{a}||\underset{\sim}{b}| \Rightarrow \underset{\sim}{a}$ **is parallel to** $\underset{\sim}{b}$.
9. $\underset{\sim}{i} \cdot \underset{\sim}{i} = \underset{\sim}{j} \cdot \underset{\sim}{j} = \underset{\sim}{k} \cdot \underset{\sim}{k} = 1$
10. $\underset{\sim}{i} \cdot \underset{\sim}{j} = \underset{\sim}{j} \cdot \underset{\sim}{k} = \underset{\sim}{k} \cdot \underset{\sim}{i} = 0$

Useful reminders when finding the dot product

- Select the easiest component formula, $\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1b_1 + a_2b_2 + a_3b_3$, when both vectors are given.
- Use the formula $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos(\theta)$, when there is an angle given or an angle to be found.
- The rule for perpendicular vectors, $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$, is the rule most often used in a variety of contexts.
- The rule for parallel vectors, $\underset{\sim}{a} \cdot \underset{\sim}{b} = \pm|\underset{\sim}{a}||\underset{\sim}{b}|$, has two options:

  positive $\underset{\sim}{a}||\underset{\sim}{b}|$ is for when $\cos(\theta) = 1$, meaning parallel in the same direction.

  negative $|\underset{\sim}{a}||\underset{\sim}{b}|$ is for when $\cos(\theta) = -1$, meaning parallel in the opposite direction.

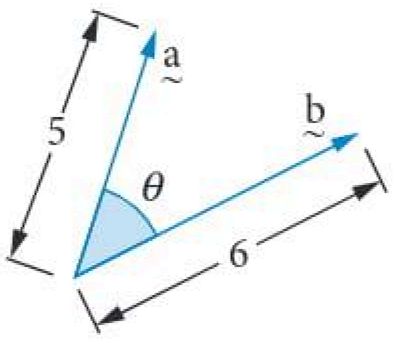
Don't forget to use this most helpful rule.

For example, to find the scalar product of vectors of magnitudes 5 and 6 at an angle of 60° to each other, there is an angle involved, so use the formula

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos(\theta)$

Substituting the values gives the scalar product

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 5 \times 6 \times \cos(60°) = 15$



WB
p. 6

WORKED EXAMPLE 6 Scalar product

a Find the scalar product of the vectors $\underset{\sim}{a} = \underset{\sim}{i} - 2\underset{\sim}{j} + 7\underset{\sim}{k}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + \underset{\sim}{k}$.

b Find the value of m if the vectors $\underset{\sim}{a} = \underset{\sim}{i} - 2\underset{\sim}{j} + 7\underset{\sim}{k}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + m\underset{\sim}{j} + \underset{\sim}{k}$ are perpendicular.

Steps	**Working**
a Use the formula $\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1b_1 + a_2b_2 + a_3b_3$.	$\underset{\sim}{a} \cdot \underset{\sim}{b} = (1 \times 3) + (-2 \times 0) + (7 \times 1)$ $\underset{\sim}{a} \cdot \underset{\sim}{b} = 10$
b 1 Use the formula $\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1b_1 + a_2b_2 + a_3b_3$	$\underset{\sim}{a} \cdot \underset{\sim}{b} = (1 \times 3) + (-2 \times m) + (7 \times 1)$ $= 10 - 2m$
2 Use the formula for perpendicular vectors $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$ to solve for m.	$\underset{\sim}{a} \cdot \underset{\sim}{b} = 10 - 2m = 0$ $\therefore m = 5$

Exam hack

Be careful if one of the components is missing as it can lead to a careless mistake.

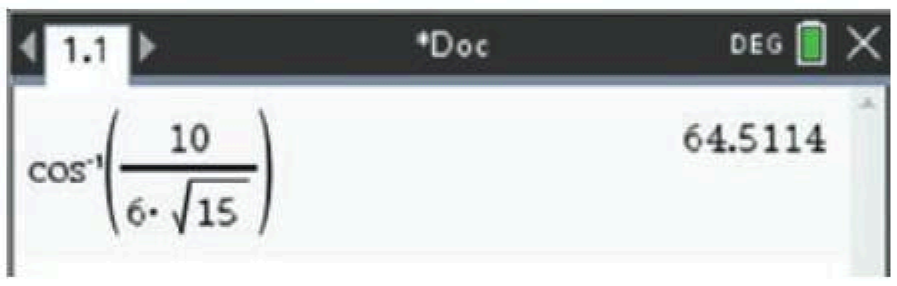
WB
p. 7

WORKED EXAMPLE 7 Angle between vectors

Find the angle between the vectors $\underset{\sim}{a} = \underset{\sim}{i} - 2\underset{\sim}{j} + 7\underset{\sim}{k}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + \underset{\sim}{k}$, expressed in degrees correct to two decimal places.

Steps	**Working**
1 Evaluate $\underset{\sim}{a} \cdot \underset{\sim}{b}$ using the formula $\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1b_1 + a_2b_2 + a_3b_3$	$\underset{\sim}{a} \cdot \underset{\sim}{b} = (1 \times 3) + (-2 \times 0) + (7 \times 1)$ $\underset{\sim}{a} \cdot \underset{\sim}{b} = 10$
2 Evaluate $\underset{\sim}{a}\|\|\underset{\sim}{b}\|$.	$\|\underset{\sim}{a}\|\|\underset{\sim}{b}\| = \sqrt{1^2 + (-2)^2 + 7^2} \times \sqrt{3^2 + 1^2}$ $\therefore \underset{\sim}{a}\|\|\underset{\sim}{b}\| = 6\sqrt{15}$
3 Use the formula $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{a}\|\|\underset{\sim}{b}\| \cos(\theta)$ to find θ.	$\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{a}\|\|\underset{\sim}{b}\| \cos(\theta)$ $\therefore 10 = 6\sqrt{15} \cos(\theta)$ $\cos(\theta) = \frac{10}{6\sqrt{15}}$ $\theta = 64.51°$

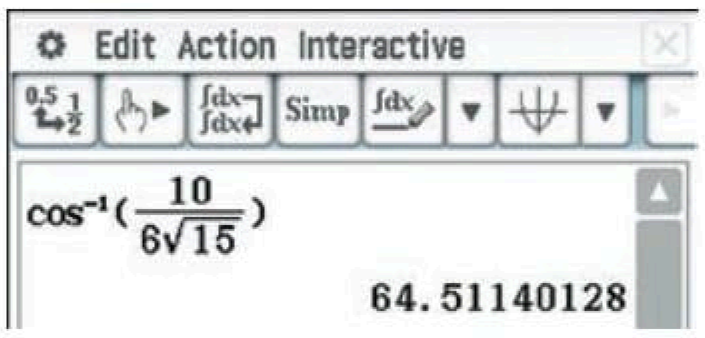
TI-Nspire



1.1 *Doc DEG

$\cos^{-1}\left(\frac{10}{6\cdot\sqrt{15}}\right)$ 64.5114

ClassPad



$\cos^{-1}\left(\frac{10}{6\sqrt{15}}\right)$

64.51140128

To find the angle between two vectors, it is useful to rearrange the formula $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos(\theta)$.

The angle between two vectors

$$\cos(\theta) = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

USING CAS 2 Scalar product and angle between vectors

For the vectors $\underset{\sim}{m} = \underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}$ and $\underset{\sim}{n} = -2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$, find the

a scalar product **b** angle between the vectors.

TI-Nspire

a

```
[1 -2 4]→m          [1 -2 4]
[-2 1 3]→n          [-2 1 3]
dotP(m,n)                  8
```

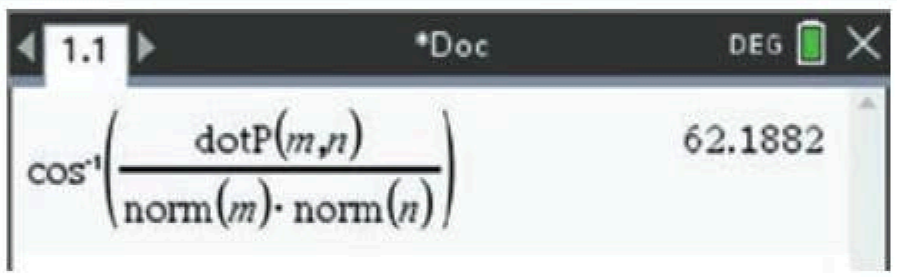
1 Create a **1 × 3** matrix and store it into ***m***, as shown above.

2 Repeat for matrix ***n***.

3 Press **menu > Matrix & Vector > Vector > Dot Product**.

4 Enter ***m***, ***n*** to find the scalar (dot) product.

b



$$\cos^{-1}\left(\frac{\text{dotP}(m,n)}{\text{norm}(m)\cdot\text{norm}(n)}\right) \quad 62.1882$$

1 Set the mode to **DEG**.

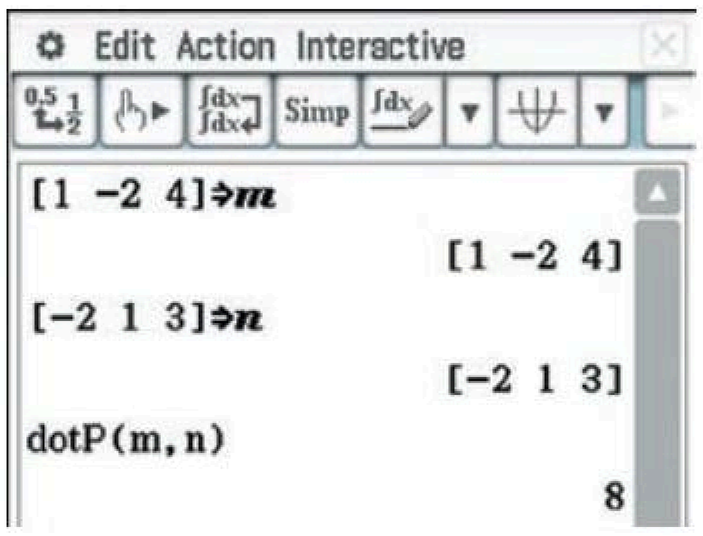
2 Press **trig > cos^{-1}**.

3 Enter **dotP(*m*,*n*)** divided by ***norm*(*m*)** times **norm(*n*)**, as shown above.

4 Press **ctrl +** = to find the approximate angle between the two vectors.

ClassPad

a



```
[1 -2 4]⇒m
                [1 -2 4]
[-2 1 3]⇒n
                [-2 1 3]
dotP(m,n)
                       8
```

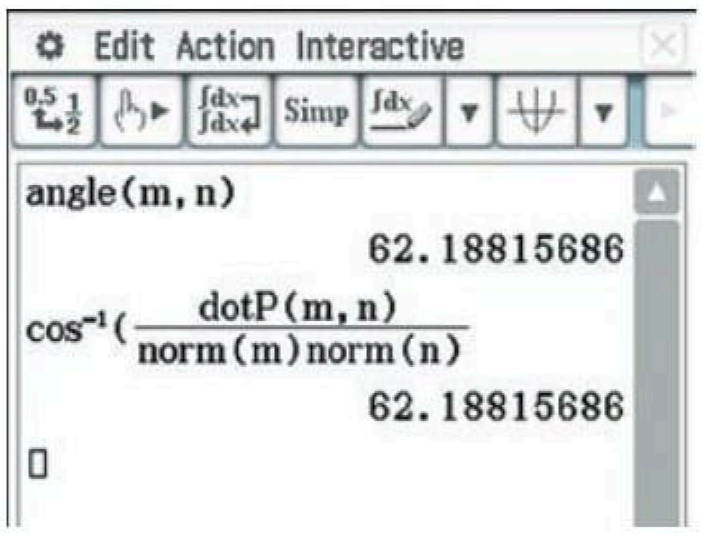
1 Create a **1 × 3** matrix and store using ⇒ as ***m***, as shown above.

2 Repeat for matrix ***n***.

3 Tap **Interactive > Vector > dotP**.

4 In the dialogue box, in the **Vector:** field enter ***m*** and in the **Another:** field, enter ***n***.

b



angle(m, n)

$$62.18815686$$

$$\cos^{-1}\left(\frac{\text{dotP}(m,n)}{\text{norm}(m)\text{norm}(n)}\right)$$

$$62.18815686$$

1 Set the modes to **Decimal** and **Deg**.

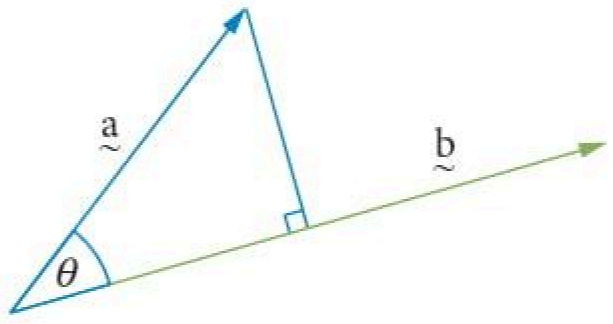
2 Press **Interactive > Vector > angle**.

3 In the dialogue box, in the **Vector:** field enter ***m*** and in the **Another:** field, enter ***n***.

4 To confirm, find the **cos^{-1}** of **dotP(*m*, *n*)** divided by **norm(*m*)** times **norm(*n*),** as shown above (optional).

Scalar and vector projections

Every vector can be written as two components perpendicular to each other. In many real-life circumstances, this is used in reference to a particular vector.



Imagine a plane flying in the air and the wind picks up, affecting the plane's trajectory. This wind can be represented as $\underset{\sim}{a}$ and the plane as $\underset{\sim}{b}$. How much of the wind affects the course of the plane, both in magnitude and direction?

$\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}$ is the **scalar projection** of the wind affecting the plane.

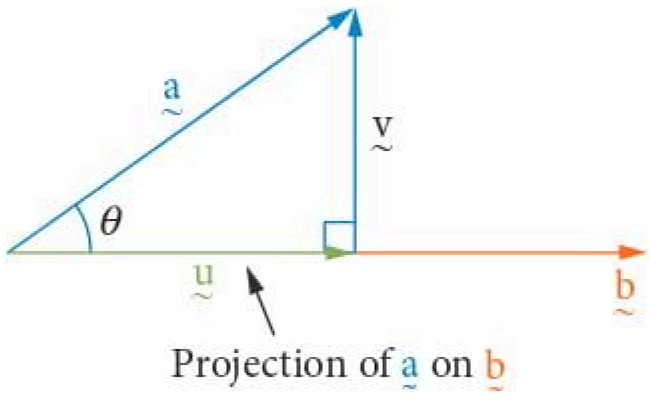
$(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$ is the **vector projection** of the wind affecting the plane.

$\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}$	Scalar projection of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ (**or** scalar projection of $\underset{\sim}{a}$ **on** $\underset{\sim}{b}$)
$(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$	Vector projection of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ (**or** vector projection of $\underset{\sim}{a}$ **on** $\underset{\sim}{b}$)
$\underset{\sim}{a} - (\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$	Vector projection of $\underset{\sim}{a}$ perpendicular to the direction of $\underset{\sim}{b}$

Projection of vectors

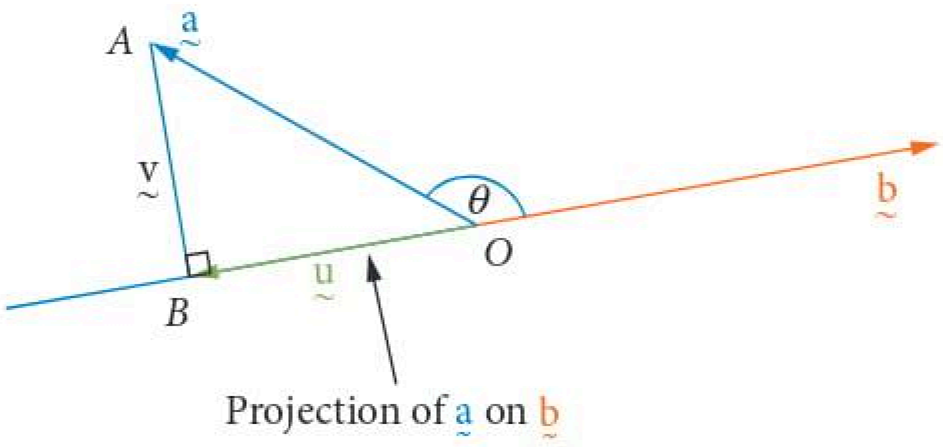
Worksheet Projections

Any vector $\underset{\sim}{a}$ can be written as the sum of two vectors, one perpendicular to a given vector $\underset{\sim}{b}$ and one parallel to $\underset{\sim}{b}$.



In the diagram, $\underset{\sim}{a} = \underset{\sim}{u} + \underset{\sim}{v}$, where $\underset{\sim}{u}$ is the vector projection (or **vector resolute**) of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ (or in the direction of $\underset{\sim}{b}$), and $\underset{\sim}{v}$ is the vector projection of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$.

The scalar projection (or **scalar resolute**) of vector $\underset{\sim}{a}$ on $\underset{\sim}{b}$ is the **magnitude** of the vector projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$. If the angle between the two vectors is obtuse, then the scalar projection is negative and the vector projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ is in the opposite direction to $\underset{\sim}{b}$, as shown by $\underset{\sim}{u}$ in the second diagram.



Scalar and vector projections

The **scalar projection** of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ is $\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}$.

The **vector projection** of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ is $\left(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$.

The **vector projection** of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$ is $\underset{\sim}{a} - \left(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$.

WB p. 8

WORKED EXAMPLE 8 Vector projections

Find the vector projections of $\underset{\sim}{a} = 3\underset{\sim}{i} + 6\underset{\sim}{j} - 6\underset{\sim}{k}$ parallel and perpendicular to $\underset{\sim}{b} = 4\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$.

Steps	Working
1 Find $\hat{\underset{\sim}{b}} = \dfrac{\underset{\sim}{b}}{\lvert\underset{\sim}{b}\rvert}$.	$\lvert\underset{\sim}{b}\rvert = \sqrt{4^2 + 2^2 + (-2)^2}$ $= 2\sqrt{6}$ $\hat{\underset{\sim}{b}} = \dfrac{1}{2\sqrt{6}}\left(4\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}\right)$ $= \dfrac{1}{\sqrt{6}}\left(2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}\right)$

2 Calculate the vector projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$.

$$\left(\underset{\sim}{a}\cdot\hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$$
$$=\left[\left(3\underset{\sim}{i}+6\underset{\sim}{j}-6\underset{\sim}{k}\right)\cdot\frac{1}{\sqrt{6}}\left(2\underset{\sim}{i}+\underset{\sim}{j}-\underset{\sim}{k}\right)\right]\frac{1}{\sqrt{6}}\left(2\underset{\sim}{i}+\underset{\sim}{j}-\underset{\sim}{k}\right)$$
$$=\frac{18}{\sqrt{6}}\,\frac{1}{\sqrt{6}}\left(2\underset{\sim}{i}+\underset{\sim}{j}-\underset{\sim}{k}\right)$$
$$=6\underset{\sim}{i}+3\underset{\sim}{j}-3\underset{\sim}{k}$$

3 Find the vector projection of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$.

$$\underset{\sim}{a}-\left(6\underset{\sim}{i}+3\underset{\sim}{j}-3\underset{\sim}{k}\right)=-3\underset{\sim}{i}+3\underset{\sim}{j}-3\underset{\sim}{k}$$

EXERCISE 1.4 Scalar product and vector projections

ANSWERS p. 555

Recap

1 TECH-FREE For the vectors $\underset{\sim}{m}=\begin{bmatrix}1\\0\\3\end{bmatrix}$ and $\underset{\sim}{n}=\begin{bmatrix}0\\-2\\-10\end{bmatrix}$, find

- **a** $\underset{\sim}{m}+2\underset{\sim}{n}$
- **b** $2\underset{\sim}{m}-2\underset{\sim}{n}$
- **c** $\underset{\sim}{m}\cdot\underset{\sim}{n}$
- **d** the angle that $\underset{\sim}{m}$ makes with the x-axis
- **e** $\left|\underset{\sim}{n}\right|$
- **f** $\hat{\underset{\sim}{n}}$

2 The value of m such that the vector $\underset{\sim}{a}=\frac{1}{m}\left(-2\underset{\sim}{i}+5\underset{\sim}{j}\right)$ is a unit vector in the direction of $\underset{\sim}{a}$ is

A $-\sqrt{29}$ **B** $\frac{1}{\sqrt{29}}$ **C** $\sqrt{7}$ **D** $\sqrt{29}$ **E** 29

Mastery

3 WORKED EXAMPLE 6 TECH-FREE

- **a** Find the scalar product of the vectors $\underset{\sim}{a}=\underset{\sim}{i}-2\underset{\sim}{j}$ and $\underset{\sim}{b}=3\underset{\sim}{i}+2\underset{\sim}{j}$.
- **b** Find the value of p if the vectors $\underset{\sim}{a}=\underset{\sim}{i}-2\underset{\sim}{j}$ and $\underset{\sim}{b}=3\underset{\sim}{i}+2p\underset{\sim}{j}$ are
  - **i** perpendicular
  - **ii** parallel.

4 WORKED EXAMPLE 7 TECH-FREE Find the angle between the vectors $\underset{\sim}{a}=\underset{\sim}{i}-2\underset{\sim}{j}$ and $\underset{\sim}{b}=3\underset{\sim}{i}+\underset{\sim}{j}$, written in the form $\cos^{-1}\left(\frac{p}{10}\right)$, where $p\in R$

5 WORKED EXAMPLE 8 TECH-FREE Find the vector projections of $\underset{\sim}{a}=\underset{\sim}{i}-2\underset{\sim}{j}+\underset{\sim}{k}$ parallel and perpendicular to $\underset{\sim}{b}=3\underset{\sim}{i}+2\underset{\sim}{j}+2\underset{\sim}{k}$.

6 The scalar product of the vectors $\underset{\sim}{a}=3\underset{\sim}{i}-2\sqrt{3}\underset{\sim}{j}$ and $\underset{\sim}{b}=\sqrt{3}\underset{\sim}{i}+2\underset{\sim}{j}$ is

A $-\sqrt{3}$ **B** -1 **C** 1 **D** $\sqrt{3}$ **E** $3\sqrt{3}$

7 The angle between the vectors $\underset{\sim}{a}=3\underset{\sim}{i}-2\sqrt{3}\underset{\sim}{j}$ and $\underset{\sim}{b}=\sqrt{3}\underset{\sim}{i}+2\underset{\sim}{j}$, correct to two decimal places is

A 75.04° **B** 77.40° **C** 81.79° **D** 98° **E** 98.21°

8 Using CAS 2 For the vectors $\underset{\sim}{m}=\underset{\sim}{i}-\underset{\sim}{j}+\underset{\sim}{k}$ and $\underset{\sim}{n}=-2\underset{\sim}{i}-\underset{\sim}{j}+3\underset{\sim}{k}$, the dot product $\underset{\sim}{m}\cdot\underset{\sim}{n}$ is

A $\sqrt{2}$ **B** $\sqrt{3}$ **C** 2 **D** $\sqrt{14}$ **E** 6

Exam practice

80–100% 60–79% 0–59%

9 © VCAA 2014 1Q1 TECH-FREE (5 marks) Consider the vector $\underset{\sim}{a} = \sqrt{3}\underset{\sim}{i} - \underset{\sim}{j} - \sqrt{2}\underset{\sim}{k}$, where $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors in the positive directions of the x, y and z axes, respectively.

a 79% Find the unit vector in the direction of $\underset{\sim}{a}$. 1 mark

b 42% Find the acute angle that $\underset{\sim}{a}$ makes with the positive direction of the x-axis. 2 marks

c 78% The vector $\underset{\sim}{b} = 2\sqrt{3}\underset{\sim}{i} + m\underset{\sim}{j} - 5\underset{\sim}{k}$. Given that $\underset{\sim}{b}$ is perpendicular to $\underset{\sim}{a}$, find the value of m. 2 marks

10 © VCAA 2017 1Q5 45% TECH-FREE (4 marks) Relative to a fixed origin, the points B, C and D are defined respectively by the position vectors $\underset{\sim}{b} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{c} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{d} = a\underset{\sim}{i} - 2\underset{\sim}{j}$, where a is a real constant. Given that the magnitude of angle BCD is $\frac{\pi}{3}$, find a.

11 © VCAA 2017N 1Q10 TECH-FREE (4 marks) Consider the vectors $\underset{\sim}{a} = -\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + c\underset{\sim}{j} + \underset{\sim}{k}$. Find the value of c, $c \in R$, if the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is $\frac{\pi}{3}$.

12 © VCAA 2018N 1Q2 TECH-FREE (3 marks) Let $\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + m\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$, where $m \in R$. Find the value(s) of m such that the magnitude of the vector resolute of $\underset{\sim}{a}$ parallel to $\underset{\sim}{b}$ is equal to $\sqrt{14}$.

13 © VCAA 2006 2AQ17 85% Let $\underset{\sim}{u} = \underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{v} = \underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$. The angle between the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is

A 0°
B 45°
C 30°
D 22.5°
E 90°

14 © VCAA 2018 2AQ11 80% Consider the vectors given by $\underset{\sim}{a} = m\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{b} = \underset{\sim}{i} + m\underset{\sim}{j}$, where $m \in R$. If the acute angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is 30°, then m equals

A $\sqrt{2} \pm 1$
B $2 \pm \sqrt{3}$
C $\sqrt{3}, \frac{1}{\sqrt{3}}$
D $\frac{\sqrt{3}}{4-\sqrt{3}}$
E $\frac{\sqrt{39}}{13}$

15 © VCAA 2014 2AQ16 77% Two vectors are given by $\underset{\sim}{a} = 4\underset{\sim}{i} + m\underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{b} = -2\underset{\sim}{i} + n\underset{\sim}{j} - \underset{\sim}{k}$, where m, $n \in R^+$. If $|\underset{\sim}{a}| = 10$ and $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$, then m and n respectively are

A $5\sqrt{3}, \frac{\sqrt{3}}{3}$
B $5\sqrt{3}, \sqrt{3}$
C $-5\sqrt{3}, \sqrt{3}$
D $\sqrt{93}, \frac{5\sqrt{93}}{93}$
E 5, 1

16 © VCAA 2014 2AQ15 69% If θ is the angle between $\underset{\sim}{a} = \sqrt{3}\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} - 4\underset{\sim}{j} + \sqrt{3}\underset{\sim}{k}$, then $\cos(2\theta)$ is

A $-\frac{4}{5}$
B $\frac{7}{25}$
C $-\frac{7}{25}$
D $\frac{14}{25}$
E $-\frac{24}{25}$

17 © VCAA 2020 2AQ16 69% Let $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - 4\underset{\sim}{j} + 4\underset{\sim}{k}$, where the acute angle between the vectors is θ. The value of $\sin(2\theta)$ is

A $\frac{1}{9}$
B $\frac{4\sqrt{5}}{9}$
C $\frac{4\sqrt{5}}{81}$
D $\frac{8\sqrt{5}}{81}$
E $\frac{2\sqrt{46}}{25}$

18 © VCAA 2019N 2AQ12 Given that θ is the acute angle between $\underset{\sim}{a} = -2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = -4\underset{\sim}{i} + 4\underset{\sim}{j} + 7\underset{\sim}{k}$, then $\sin(2\theta)$ is equal to

A $2\sqrt{2}$
B $\frac{4\sqrt{2}}{9}$
C $\frac{2\sqrt{2}}{9}$
D $\frac{2\sqrt{2}}{3}$
E $\frac{4\sqrt{2}}{3}$

19 © VCAA 2019N 2AQ11 The vector resolute of $\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$ that is **perpendicular** to $\underset{\sim}{b} = \underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$ is

A $-\frac{2}{3}\left(\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}\right)$
B $-\frac{2}{3}\left(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}\right)$
C $\frac{1}{3}\left(8\underset{\sim}{i} - \underset{\sim}{j} + 7\underset{\sim}{k}\right)$
D $\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}$
E $\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$

20 © VCAA 2019 2AQ12 62% The vector resolute of $\underset{\sim}{i}+\underset{\sim}{j}-\underset{\sim}{k}$ in the direction of $m\underset{\sim}{i}+n\underset{\sim}{j}+p\underset{\sim}{k}$ is $2\underset{\sim}{i}-3\underset{\sim}{j}+\underset{\sim}{k}$, where m, n and p are real constants.

The values of m, n and p can be found by solving the equations

A $\dfrac{m(m+n-p)}{m^2+n^2+p^2}=2,\ \dfrac{n(m+n-p)}{m^2+n^2+p^2}=-3,\ \text{and}\ \dfrac{p(m+n-p)}{m^2+n^2+p^2}=1$

B $\dfrac{m(m+n-p)}{m^2+n^2+p^2}=1,\ \dfrac{n(m+n-p)}{m^2+n^2+p^2}=1,\ \text{and}\ \dfrac{p(m+n-p)}{m^2+n^2+p^2}=-1$

C $m+n-p=6,\ m+n-p=-9$ and $m+n-p=-3$

D $m+n-p=3m,\ m+n-p=3n$ and $m+n-p=-3p$

E $m+n-p=2\sqrt{3},\ m+n-p=-3\sqrt{3}$ and $m+n-p=\sqrt{3}$

21 © VCAA 2006 2AQ16 49% A unit vector perpendicular to $5\underset{\sim}{i}+\underset{\sim}{j}-2\underset{\sim}{k}$ is

A $\dfrac{1}{4}\left(5\underset{\sim}{i}+\underset{\sim}{j}-2\underset{\sim}{k}\right)$

B $2\underset{\sim}{i}-4\underset{\sim}{j}+3\underset{\sim}{k}$

C $\dfrac{1}{29}\left(2\underset{\sim}{i}-4\underset{\sim}{j}+3\underset{\sim}{k}\right)$

D $\dfrac{1}{\sqrt{29}}\left(2\underset{\sim}{i}-4\underset{\sim}{j}+3\underset{\sim}{k}\right)$

E $\dfrac{1}{\sqrt{30}}\left(5\underset{\sim}{i}+\underset{\sim}{j}-2\underset{\sim}{k}\right)$

22 © VCAA 2006 2BQ2 (10 marks) Point A has position vector $\underset{\sim}{a}=-\underset{\sim}{i}-4\underset{\sim}{j}$, point B has position vector $\underset{\sim}{b}=2\underset{\sim}{i}-5\underset{\sim}{j}$, point C has position vector $\underset{\sim}{c}=5\underset{\sim}{i}-4\underset{\sim}{j}$, and point D has position vector $\underset{\sim}{d}=2\underset{\sim}{i}+5\underset{\sim}{j}$ relative to the origin O.

a 77% Show that $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are perpendicular. 2 marks

b 74% Use a vector method to find the cosine of $\angle ADC$, the angle between $\overrightarrow{DA}$ and $\overrightarrow{DC}$. 3 marks

c 35% Find the cosine of $\angle ABC$, and **hence** show that $\angle ADC$ and $\angle ABC$ are supplementary. 2 marks

Point P has position vector $\underset{\sim}{p}=2\underset{\sim}{i}$.

d 33% Use the cosine of $\angle APC$ and an appropriate trigonometric formula to prove that $\angle APC=2\angle ADC$. 3 marks

1.5 Vector product

Video playlist Vector product

Worksheet Vector product

The result of multiplying two vectors in 3D space is called the **vector product**.

It is also called a **cross product** because the symbol used is a cross, '×'. As its name suggests, the vector product is a vector, unlike the **scalar product** or **dot product**, which is a number (scalar).

The vector product of two vectors in three dimensions is a vector at right angles to both the original vectors. Its magnitude is the product of the magnitudes of the original vectors and the sine of the angle between their directions.

The vector product

If $\underset{\sim}{a}=a_1\underset{\sim}{i}+a_2\underset{\sim}{j}+a_3\underset{\sim}{k}$ and $\underset{\sim}{b}=b_1\underset{\sim}{i}+b_2\underset{\sim}{j}+b_3\underset{\sim}{k}$, then the vector product of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is:

$$\underset{\sim}{a}\times\underset{\sim}{b}=\left(a_2b_3-a_3b_2\right)\underset{\sim}{i}-\left(a_1b_3-a_3b_1\right)\underset{\sim}{j}+\left(a_1b_2-a_2b_1\right)\underset{\sim}{k}$$

WB
p. 9

WORKED EXAMPLE 9 Vector product

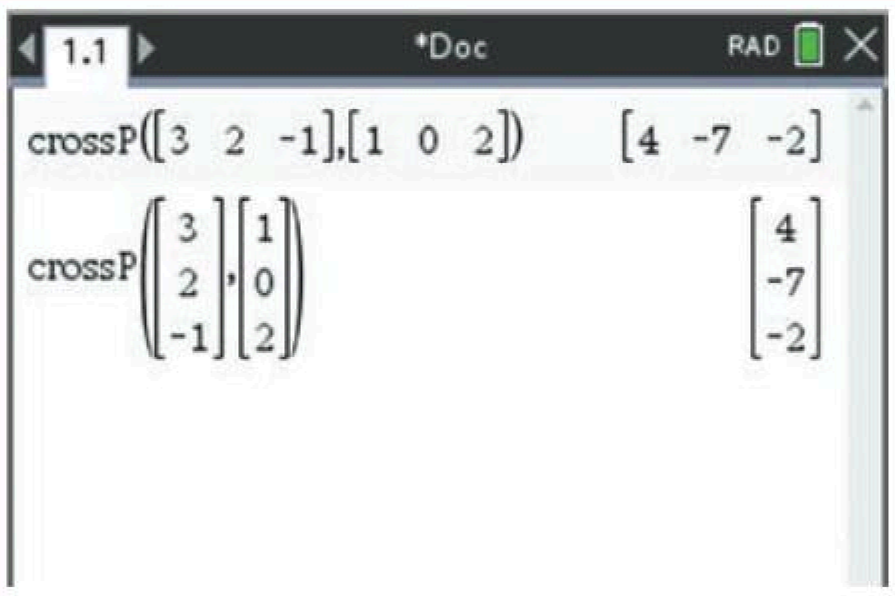
Find the vector product of the vectors $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{k}$.

Steps	**Working**
1 Identify the coefficients in both vectors.	$\begin{array}{cccc} & \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \underset{\sim}{a} & 3 & 2 & -1 \\ \underset{\sim}{b} & 1 & 0 & 2 \end{array}$
2 Apply the vector product formula.	$\underset{\sim}{a} \times \underset{\sim}{b} = (a_2b_3 - a_3b_2)\underset{\sim}{i} - (a_1b_3 - a_3b_1)\underset{\sim}{j} + (a_1b_2 - a_2b_1)\underset{\sim}{k}$ $= (2 \times 2 - (-1) \times 0)\underset{\sim}{i} - (3 \times 2 - (-1) \times 1)\underset{\sim}{j} + (3 \times 0 - 2 \times 1)\underset{\sim}{k}$
3 Simplify.	$= (4 - 0)\underset{\sim}{i} - (6 - (-1))\underset{\sim}{j} + (0 - 2)\underset{\sim}{k}$ $= 4\underset{\sim}{i} - 7\underset{\sim}{j} - 2\underset{\sim}{k}$

USING CAS 3 Vector product

Find the vector product of $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{k}$.

TI-Nspire



crossP([3 2 -1],[1 0 2]) [4 -7 -2]

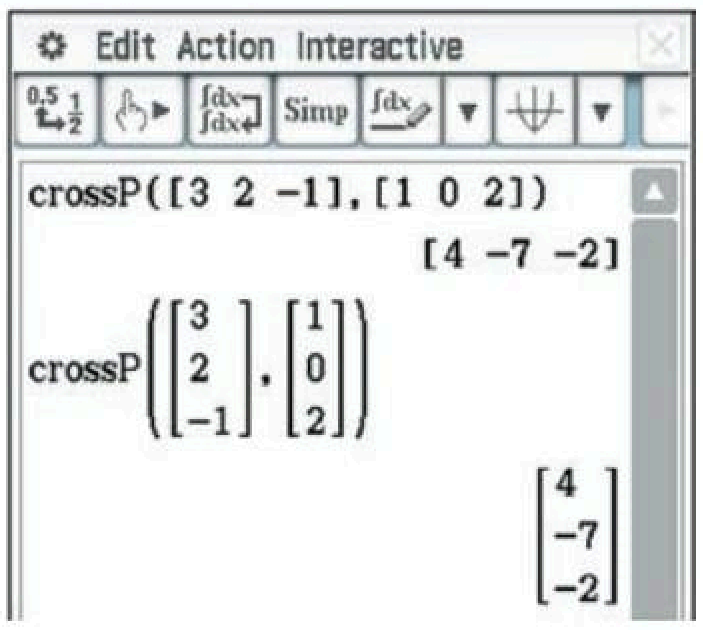
$\text{crossP}\left(\begin{bmatrix}3\\2\\-1\end{bmatrix},\begin{bmatrix}1\\0\\2\end{bmatrix}\right) \quad \begin{bmatrix}4\\-7\\-2\end{bmatrix}$

1 In a Calculator page, press **menu > Matrix & Vector > Vector > Cross Product**.

2 Press the template key then use the matrix template* to enter the two vectors, separated by a comma.

3 Press **enter**.

ClassPad



crossP([3 2 −1],[1 0 2])
[4 −7 −2]

$\text{crossP}\left(\begin{bmatrix}3\\2\\-1\end{bmatrix},\begin{bmatrix}1\\0\\2\end{bmatrix}\right)$

$\begin{bmatrix}4\\-7\\-2\end{bmatrix}$

1 Tap the **Math2** keyboard then enter and highlight the first vector using the matrix template*.

2 Tap **Interactive > Vector > crossP**.

3 In the dialogue box, copy the vector in the **Vector:** field and paste it into the **Another:** field.

4 Change the values to those for the second vector.

5 Tap **OK**.

*The vectors can be entered as either 1×3 row matrices or 3×1 column matrices.

The vector product is $4\underset{\sim}{i} - 7\underset{\sim}{j} - 2\underset{\sim}{k}$.

It helps if you notice the pattern in the order that the coefficients are used when calculating the vector product.

$$\begin{array}{cccc} & \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \underset{\sim}{a} & 3 & 2 & -1 \\ \underset{\sim}{b} & 1 & 0 & 2 \end{array}$$

For $(a_2b_3 - a_3b_2)\underset{\sim}{i}$, ignore the $\underset{\sim}{i}$ column and calculate the **determinant** of the remaining matrix:

$$\begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} = 2 \times 2 - (-1) \times 0$$

For $(a_1b_3 - a_3b_1)\underset{\sim}{j}$, ignore the $\underset{\sim}{j}$ column and calculate the determinant of the remaining matrix:

$$\begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} = 3 \times 2 - (-1) \times 1$$

For $(a_1b_2 - a_2b_1)\underset{\sim}{k}$, ignore the $\underset{\sim}{k}$ column and calculate the determinant of the remaining matrix:

$$\begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix} = 3 \times 0 - 2 \times 1$$

The determinant form of the vector product

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\underset{\sim}{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\underset{\sim}{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\underset{\sim}{k}$$

where $\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = a_2b_3 - a_3b_2$ and so on.

WORKED EXAMPLE 10 Vector product in determinant form

WB p. 10

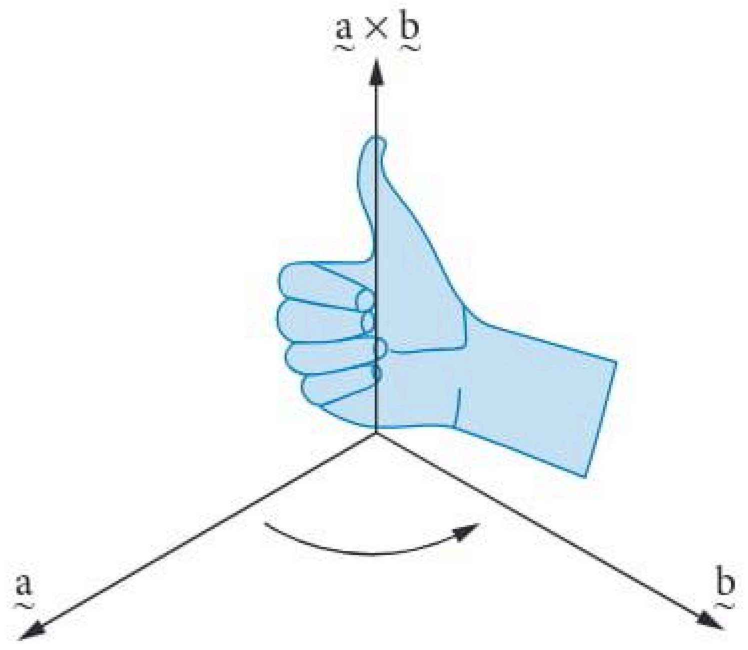
Find the vector product of $\underset{\sim}{a} = (3, -3, 1)$ and $\underset{\sim}{b} = (4, 9, 2)$.

Steps	Working
1 Identify the coefficients in both vectors.	$\begin{array}{cccc} & \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \underset{\sim}{a} & 3 & -3 & 1 \\ \underset{\sim}{b} & 4 & 9 & 2 \end{array}$
2 Apply the vector product determinant formula, using the pattern described above.	$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} -3 & 1 \\ 9 & 2 \end{vmatrix}\underset{\sim}{i} - \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix}\underset{\sim}{j} + \begin{vmatrix} 3 & -3 \\ 4 & 9 \end{vmatrix}\underset{\sim}{k}$
3 Simplify.	$= (-6 - 9)\underset{\sim}{i} - (6 - 4)\underset{\sim}{j} + (27 - (-12))\underset{\sim}{k}$ $= -15\underset{\sim}{i} - 2\underset{\sim}{j} + 39\underset{\sim}{k}$

Using determinants provides an easy way of calculating the vector product.

To view the direction of a vector product, we can use the **right-hand thumb rule.**

If the fingers of the right hand are curled in the direction of rotation from vector $\underset{\sim}{a}$ to vector $\underset{\sim}{b}$, then the thumb points in the direction of the vector product of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.



This also means that $\underset{\sim}{a} \times \underset{\sim}{b}$ and $\underset{\sim}{b} \times \underset{\sim}{a}$ are not the same and point in opposite directions.

The magnitude of the vector product is $\underset{\sim}{a} \times \underset{\sim}{b}| = |\underset{\sim}{a}||\underset{\sim}{b}|\sin(\theta)$.

Note that this is NOT the same as the scalar product $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta)$.

The vector product is used less often than the scalar product and it is only defined in 3D space.

EXERCISE 1.5 Vector product

ANSWERS p. 555

Recap

1 TECH-FREE Given $\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$, $\underset{\sim}{b} = \underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}$, $\underset{\sim}{c} = 3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{d} = 2\underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k}$, find

a the scalar projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ using the scalar product $\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}$.

b the vector projection of $\underset{\sim}{c}$ on $\underset{\sim}{d}$ using the scalar product $\underset{\sim}{c} \cdot \hat{\underset{\sim}{d}}$.

c the cosine of the angle between $\underset{\sim}{a}$ and $\underset{\sim}{c}$ using a suitable scalar product.

2 TECH-FREE Given $\underset{\sim}{a} = 2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{b} = \underset{\sim}{i} - 2\underset{\sim}{j}$, $\underset{\sim}{c} = 3\underset{\sim}{i} - \underset{\sim}{j} + n\underset{\sim}{k}$, where n is a real constant, find

a the scalar projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$.

b the vector projection of $\underset{\sim}{c}$ on $\underset{\sim}{a}$ in terms of n.

c the value of n such that vector projection of $\underset{\sim}{c}$ on $\underset{\sim}{a}$ is $2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$.

d the value of n such that $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{c}$.

Mastery

3 WORKED EXAMPLE 9 TECH-FREE Find the vector product of

a $\underset{\sim}{a} = \underset{\sim}{i} - 2\underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$

b $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k}$

4 TECH-FREE Calculate the cross product of vectors (3, 4, 7) and (4, 9, 2).

5 WORKED EXAMPLE 10 Using the determinant method, find $\underset{\sim}{a} \times \underset{\sim}{b}$ in component form if $\underset{\sim}{a} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix}$.

6 If $\underset{\sim}{w} = (1, 0, -3)$ and $\underset{\sim}{v} = (6, -3, -4)$, find

a $\underset{\sim}{v} \times \underset{\sim}{w}$

b $\underset{\sim}{w} \times \underset{\sim}{v}$

7 If $\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + 6\underset{\sim}{k}$ and $\underset{\sim}{b} = 4\underset{\sim}{i} - \underset{\sim}{j} - 6\underset{\sim}{k}$, calculate $\underset{\sim}{a} \times \underset{\sim}{b}$.

8 Calculate $\underset{\sim}{v} \times \underset{\sim}{w}$ for $\underset{\sim}{w} = (3, -1, 5)$ and $\underset{\sim}{v} = (0, 4, -2)$.

9 Calculate $\underset{\sim}{w} \times \underset{\sim}{v}$ for $\underset{\sim}{w} = (1, 6, -8)$ and $\underset{\sim}{v} = (4, -2, -1)$.

10 The vector product of the vectors $\underset{\sim}{a} = 3\underset{\sim}{i} - 2\sqrt{3}\underset{\sim}{j}$ and $\underset{\sim}{b} = \sqrt{3}\underset{\sim}{i} + 2\underset{\sim}{j}$ is

A $3\sqrt{3}$

B $2\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k}$

C $12\underset{\sim}{k}$

D $-\sqrt{3}$

E 1

Exam practice

80–100% 60–79% 0–59%

11 The vector product of the vectors $\underset{\sim}{a} = 3\underset{\sim}{i} - 2\sqrt{3}\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = \sqrt{3}\underset{\sim}{i} + 2\underset{\sim}{j}$ is

A $3\sqrt{3}$

B $-2\underset{\sim}{i} + \sqrt{3}\underset{\sim}{j} + 12\underset{\sim}{k}$

C $\sqrt{3}$

D $-\sqrt{3}$

E $2\underset{\sim}{i} - \sqrt{3}\underset{\sim}{j} - 12\underset{\sim}{k}$

12 The vector product of $\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{i} + \underset{\sim}{j}$ is

A $\underset{\sim}{k}$

B $-2\underset{\sim}{k}$

C $2\underset{\sim}{k}$

D $-2\underset{\sim}{i}$

E $2\underset{\sim}{i}$

13 If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are vectors such that $\underset{\sim}{a} \times \underset{\sim}{b} = 0$ and $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$, then

A $\underset{\sim}{a} = 0$ and $\underset{\sim}{b} = 0$

B $\underset{\sim}{a} = 0$

C at least one of $\underset{\sim}{a}$ or $\underset{\sim}{b}$ equals 0

D $\underset{\sim}{b} = 0$

E none of the above

1.6 Parallel and perpendicular vectors

Video playlist Parallel and perpendicular vectors

Worksheets Parallel and perpendicular vectors

Resultant and unit vectors in 3D space

Three important results come from the scalar product.

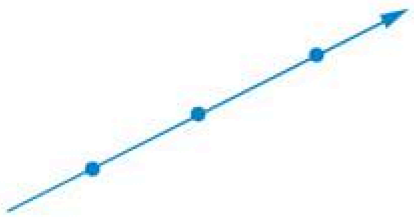
Parallel and perpendicular vectors

If $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$, then vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular.

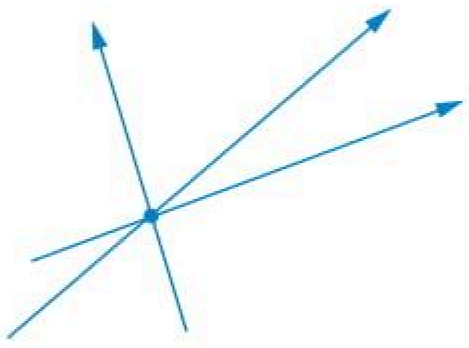
If $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|$, then vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel in the same direction.

If $\underset{\sim}{a} \cdot \underset{\sim}{b} = -|\underset{\sim}{a}||\underset{\sim}{b}|$, then vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel in the opposite direction.

If three or more points lie on the same vector, they are said to be **collinear.**



If two or more vectors go through the same point, those vectors are said to be **concurrent**.



WORKED EXAMPLE 11 Parallel vectors

WB p. 11

Consider the vectors $\underset{\sim}{a} = -3\underset{\sim}{i} - \frac{1}{2}\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$.

a Find a unit vector that is parallel, and in the opposite direction, to $\underset{\sim}{a}$.

b Find the vector that is parallel to and in the same direction as $\underset{\sim}{a}$, with the same magnitude as vector $\underset{\sim}{b}$.

Steps	Working
a 1 Find the magnitude of $\underset{\sim}{a}$.	$\lvert\underset{\sim}{a}\rvert = \sqrt{(-3)^2 + \left(-\frac{1}{2}\right)^2 + 3^2}$ $= \frac{\sqrt{73}}{2}$

2 Find the vector in the opposite direction to $\underset{\sim}{a}$.	$-\underset{\sim}{a} = 3\underset{\sim}{i} + \frac{1}{2}\underset{\sim}{j} - 3\underset{\sim}{k}$
3 We know $\lvert\underset{\sim}{a}\rvert = \frac{\sqrt{73}}{2}$.	$\frac{-a}{\lvert a\rvert} = \frac{2}{\sqrt{73}}\left(3\underset{\sim}{i} + \frac{1}{2}\underset{\sim}{j} - 3\underset{\sim}{k}\right)$
b **1** Find the magnitude of $\underset{\sim}{b}$.	$\lvert\underset{\sim}{b}\rvert = \sqrt{2^2 + 1^2 + (-1)^2}$ $= \sqrt{6}$
2 Find $\hat{\underset{\sim}{a}}$.	$\hat{\underset{\sim}{a}} = \frac{2}{\sqrt{73}}\left(-3\underset{\sim}{i} - \frac{1}{2}\underset{\sim}{j} + 3\underset{\sim}{k}\right)$
3 Multiply $\hat{\underset{\sim}{a}}$ by the magnitude of vector $\underset{\sim}{b}$.	$\sqrt{6} \times \frac{2}{\sqrt{73}}\left(-3\underset{\sim}{i} - \frac{1}{2}\underset{\sim}{j} + 3\underset{\sim}{k}\right)$ $= \frac{2\sqrt{6}}{\sqrt{73}}\left(-3\underset{\sim}{i} - \frac{1}{2}\underset{\sim}{j} + 3\underset{\sim}{k}\right)$

WB
p. 12

WORKED EXAMPLE 12 Perpendicular vectors

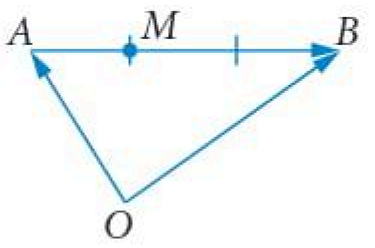
Show that $\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}$ and $\underset{\sim}{b} = -2\underset{\sim}{i} + 3\underset{\sim}{j} + 3\underset{\sim}{k}$ are perpendicular.

Steps	**Working**
1 Find the scalar product.	$\underset{\sim}{a} \cdot \underset{\sim}{b} = 3 \times (-2) + (-2) \times 3 + 4 \times 3$ $= (-6) + (-6) + 12$ $= 0$
2 State conclusion.	Since $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$, $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular.

WB
p. 13

WORKED EXAMPLE 13 Ratios of vectors

Consider the vectors $\overrightarrow{OA} = -\underset{\sim}{i} + \frac{1}{2}\underset{\sim}{j}$ and $\overrightarrow{OB} = \frac{3}{4}\underset{\sim}{i} + \underset{\sim}{j}$. Find the coordinates of M such that M is the point of trisection, closest to A, of the vector $\overrightarrow{AB}$.

Steps	**Working**
1 Find $\overrightarrow{AB}$.	$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$ $\therefore\ \overrightarrow{AB} = \underset{\sim}{i} - \frac{1}{2}\underset{\sim}{j} + \frac{3}{4}\underset{\sim}{i} + \underset{\sim}{j}$ $= \frac{7}{4}\underset{\sim}{i} + \frac{1}{2}\underset{\sim}{j}$
2 Find $\overrightarrow{AM}$.	$\overrightarrow{AM} = \frac{1}{3}\overrightarrow{AB}$ $\therefore\ \overrightarrow{AM} = \frac{1}{3}\left(\frac{7}{4}\underset{\sim}{i} + \frac{1}{2}\underset{\sim}{j}\right)$ 
3 Find $\overrightarrow{OM}$.	$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$ $\therefore\ \overrightarrow{OM} = -\underset{\sim}{i} + \frac{1}{2}\underset{\sim}{j} + \frac{7}{12}\underset{\sim}{i} + \frac{1}{6}\underset{\sim}{j}$ $= -\frac{5}{12}\underset{\sim}{i} + \frac{2}{3}\underset{\sim}{j}$
4 State the coordinates of M.	$M\left(-\frac{5}{12}, \frac{2}{3}\right)$

EXERCISE 1.6 Parallel and perpendicular vectors

ANSWERS p. 555

Recap

1 TECH-FREE Given $\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$, $\underset{\sim}{b} = \underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}$, $\underset{\sim}{c} = 3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{d} = 2\underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k}$, find

a $\underset{\sim}{a} \times \underset{\sim}{b}$
b $\underset{\sim}{c} \times \underset{\sim}{d}$

2 TECH-FREE Given $\underset{\sim}{a} = 2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{b} = \underset{\sim}{i} - 2\underset{\sim}{j}$, $\underset{\sim}{c} = 3\underset{\sim}{i} - \underset{\sim}{j} + n\underset{\sim}{k}$, where n is a real constant, find

a $\underset{\sim}{a} \times \underset{\sim}{b}$
b $\underset{\sim}{a} \times \underset{\sim}{c}$

Mastery

3 WORKED EXAMPLE 11 TECH-FREE For the vector $\underset{\sim}{a} = -3\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$, find a unit vector that is parallel, and in the opposite direction, to $\underset{\sim}{a}$.

4 TECH-FREE Consider the vectors $\underset{\sim}{a} = -3\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} - 4\underset{\sim}{k}$. Find the vector that is parallel to and in the same direction as $\underset{\sim}{a}$, with the same magnitude as vector $\underset{\sim}{b}$.

5 WORKED EXAMPLE 12 Consider the vectors $\underset{\sim}{a} = -\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$. A vector perpendicular to $\underset{\sim}{a}$, with the same magnitude as vector $\underset{\sim}{b}$ is

A $\frac{1}{\sqrt{3}}\left(-\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}\right)$
B $\sqrt{2}(2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k})$
C $\sqrt{2}(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})$
D $\frac{1}{\sqrt{3}}\left(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right)$
E $\frac{1}{\sqrt{6}}\left(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right)$

6 WORKED EXAMPLE 13 TECH-FREE Consider the vectors $\overrightarrow{OA} = -\underset{\sim}{i} + \underset{\sim}{j}$ and $\overrightarrow{OB} = \frac{3}{4}\underset{\sim}{i} + \frac{1}{4}\underset{\sim}{j}$.

a Find the vector $\overrightarrow{AB}$.

b Hence, find the coordinates of M such that M is the point of bisection of the vector $\overrightarrow{AB}$.

7 For $\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}$

a $\underset{\sim}{a} + 2\underset{\sim}{b}$ equals

A $\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}$
B $3\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$
C $3\underset{\sim}{i} - 4\underset{\sim}{j} + 5\underset{\sim}{k}$
D $4\underset{\sim}{i} - 4\underset{\sim}{j} + 5\underset{\sim}{k}$
E $5\underset{\sim}{i} - 6\underset{\sim}{j} + 9\underset{\sim}{k}$

b $\underset{\sim}{a} \cdot \underset{\sim}{b}$ equals

A 3
B 4
C 5
D 6
E 11

8 Find the scalar product of each pair of vectors.

a Vectors with magnitudes 5 and 8 at an angle of 75°

A 40
B $\sqrt{6} - \sqrt{2}$
C $10\left(\sqrt{6} - \sqrt{2}\right)$
D $\sqrt{6}$
E $\sqrt{2}\left(\frac{\sqrt{3}-1}{4}\right)$

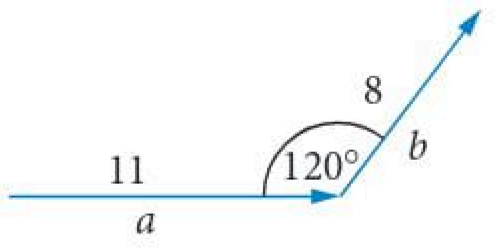
b (3, –5) and (6, 4)

A –2
B 0
C 2
D 38
E 40

c (0, –3, 7) and (3, 5, 2)

A –2
B –1
C 1
D 14
E 27

9 Given the two vectors shown on the right, $\underset{\sim}{a} \cdot \underset{\sim}{b}$ is



A -44 **B** -35.4 **C** 44

D 76.2 **E** 88

10 The dot product of $-5\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ and $3\underset{\sim}{i} - 4\underset{\sim}{j} - 7\underset{\sim}{k}$ is

A -30 **B** -16 **C** -12 **D** 0 **E** 16

Exam practice

80–100% 60–79% 0–59%

11 © VCAA 2020 1Q5 TECH-FREE (4 marks) Let $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} + m\underset{\sim}{j} - \underset{\sim}{k}$, where m is an integer. The vector resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is $-\frac{11}{18}\left(\underset{\sim}{i} + m\underset{\sim}{j} - \underset{\sim}{k}\right)$.

a 66% Find the value of m. 3 marks

b 28% Find the component of $\underset{\sim}{a}$ that is perpendicular to $\underset{\sim}{b}$. 1 mark

12 © VCAA 2019 2AQ11 66% Let point M have coordinates $(a, 1, -2)$ and let point N have coordinates $(-3, b, -1)$. If the coordinates of the midpoint of $\overrightarrow{MN}$ are $\left(-5, \frac{3}{2}, c\right)$ and a, b and c are real constants, then the values of a, b and c are respectively

A $-13, 2$ and $-\frac{1}{2}$ **B** $-2, \frac{1}{2}$ and -3 **C** $-7, -2$ and $-\frac{3}{2}$

D $-2, -\frac{1}{2}$ and -3 **E** $-7, 2$ and $-\frac{3}{2}$

13 © VCAA 2021N 2AQ13 The scalar resolute of $\underset{\sim}{a} = \underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ in the direction of $\underset{\sim}{b} = 4\underset{\sim}{i} - 4\underset{\sim}{j} + 7\underset{\sim}{k}$ is

A $-\frac{3}{2}$ **B** $\frac{1}{3}$ **C** $\frac{1}{2}$ **D** $\frac{19}{9}$ **E** $\frac{19}{6}$

14 © VCAA 2021N 2AQ13 MODIFIED The vector resolute of $\underset{\sim}{a} = \underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ in the direction of $\underset{\sim}{b} = 4\underset{\sim}{i} - 4\underset{\sim}{j} + 7\underset{\sim}{k}$ is

A $\frac{19}{81}(4\underset{\sim}{i} - 4\underset{\sim}{j} + 7\underset{\sim}{k})$ **B** $\frac{361}{81}(4\underset{\sim}{i} - 4\underset{\sim}{j} + 7\underset{\sim}{k})$ **C** $\frac{19}{81}(\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k})$

D $\frac{361}{81}(\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k})$ **E** $\frac{19\sqrt{6}}{54}(\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k})$

1.7 Vector proofs of geometric results

Video playlist
Vector proofs of geometric results

Worksheets
Proofs using vectors

Geometry proofs using vectors

Vectors are used to carry out proofs of simple geometric results. This could be as simple as using vectors to prove Pythagoras' theorem, or to show that the diagonals of a rhombus are **perpendicular**, the medians of a triangle are **concurrent**, the angle in a semicircle is a **right angle**, or that the **angle** subtended by a diameter in a circle is a **right angle**.

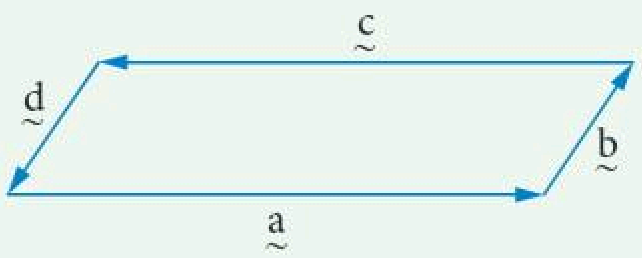
In the topic of vectors, key words recognised here are

- **perpendicular**
- **concurrent**
- **angle**
- **right angle**.

From these key words, we can see the importance of the dot product in these proofs.

WORKED EXAMPLE 14 Vector proof 1

© VCAA 2006 2AQ15 74% In the parallelogram shown on the right, $|\underset{\sim}{a}| = 2|\underset{\sim}{b}|$.



Which one of the following statements is true?

A $\underset{\sim}{a} = 2\underset{\sim}{b}$

B $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{c} + \underset{\sim}{d}$

C $\underset{\sim}{b} - \underset{\sim}{d} = 0$

D $\underset{\sim}{a} + \underset{\sim}{c} = 0$

E $\underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{c} - \underset{\sim}{d}$

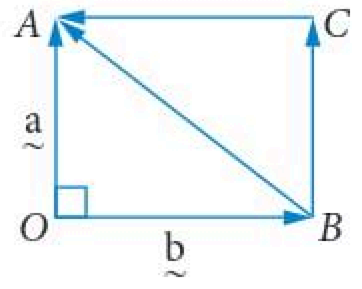
Steps	**Working**
1 Consider what is given.	$\lvert\underset{\sim}{a}\rvert = 2\lvert\underset{\sim}{b}\rvert$ Meaning also $\lvert\underset{\sim}{c}\rvert = 2\lvert\underset{\sim}{d}\rvert$.
2 Consider what we know.	$\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d} = 0$ This is none of the options.
3 Consider what is clearly wrong.	Option A: $\underset{\sim}{a} = 2\underset{\sim}{b}$ confuses vector with only its magnitude. Option B: $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{c} + \underset{\sim}{d}$ confuses with Step 2.
4 Consider vector operations.	Parallel and opposite vectors. Option D: $\underset{\sim}{a} + \underset{\sim}{c} = 0$ is correct.

WB p. 14

WORKED EXAMPLE 15 Vector proof 2

Use vectors to prove that the diagonals of a square intersect at right angles.

WB p. 15

Steps	**Working**
1 Sketch a diagram and set up the proof.	 Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$.
2 State what needs to be proved.	To prove: $\overrightarrow{BA} \cdot \overrightarrow{OC} = 0$
3 Express $\overrightarrow{BA}$ and $\overrightarrow{OC}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.	$\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA} = \underset{\sim}{a} - \underset{\sim}{b}$ $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC} = \underset{\sim}{a} + \underset{\sim}{b}$
4 Express $\overrightarrow{BA} \cdot \overrightarrow{OC}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.	$\overrightarrow{BA} \cdot \overrightarrow{OC} = (\underset{\sim}{a} - \underset{\sim}{b}) \cdot (\underset{\sim}{a} + \underset{\sim}{b})$ $= \underset{\sim}{a} \cdot \underset{\sim}{a} - \underset{\sim}{b} \cdot \underset{\sim}{b}$
5 Use $\underset{\sim}{a} \cdot \underset{\sim}{a} = \lvert\underset{\sim}{a}\rvert^2$.	$= \lvert\underset{\sim}{a}\rvert^2 - \lvert\underset{\sim}{b}\rvert^2$ $= 0$, since $OABC$ is a square and $\lvert\underset{\sim}{a}\rvert^2 = \lvert\underset{\sim}{b}\rvert^2$.
6 State the conclusion.	Hence, $\overrightarrow{BA} \cdot \overrightarrow{OC} = 0$.

Video
VCE question analysis: Vectors

VCE QUESTION ANALYSIS

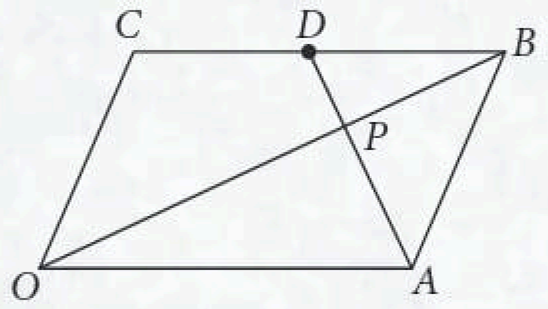
© VCAA 2014 2BQ3 2014 Examination 2 Section B Question 3 **(10 marks)**

Let $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$.

a Express $\underset{\sim}{a}$ as the **sum** of the two vector resolutes, one of which is parallel to $\underset{\sim}{b}$ and the other of which is perpendicular to $\underset{\sim}{b}$. Identify clearly the parallel vector resolute and the perpendicular vector resolute. 5 marks

$OABC$ is a parallelogram where D is the midpoint of $\overline{CB}$. $\overline{OB}$ and $\overline{AD}$ intersect at point P.

Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.



b **i** Given that $\overrightarrow{AP} = \alpha\overrightarrow{AD}$, write an expression for $\overrightarrow{AP}$ in terms of α, $\underset{\sim}{a}$ and $\underset{\sim}{c}$. 2 marks

ii Given that $\overrightarrow{OP} = \beta\overrightarrow{OB}$, write another expression for $\overrightarrow{AP}$ in terms of β, $\underset{\sim}{a}$ and $\underset{\sim}{c}$. 1 mark

iii **Hence** deduce the values of α and β. 2 marks

Reading the question

- Note that the question asks for the **sum** of the two vector resolutes; not just the two results stated separately.
- The first question is about the parallel vector resolute and the perpendicular vector resolute.
- The question moves into a Vector Proof question scaffolded stage by stage.
- Make sure you know what 'in terms of' and 'deduce' mean.

Thinking about the question

- This question has two quite distinct sections. It is unusual to have the first part of a question worth as many as 5 marks. Usually part **a** leads much more gently into the question.
- Part **a** is a skills question testing vector resolutes.
- Part **b** follows a vector proof model.
- Finishing the question with writing a vector in terms of other constants and vectors will be challenging.

Worked solution (✓ = 1 mark)

a Given $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$

$|\underset{\sim}{b}| = \sqrt{2^2 + (-2)^2 + 1^2} = 3$ so unit vector $\hat{\underset{\sim}{\mathbf{b}}} = \frac{\mathbf{1}}{\mathbf{3}}(\mathbf{2}\underset{\sim}{\mathbf{i}} - \mathbf{2}\underset{\sim}{\mathbf{j}} - \underset{\sim}{\mathbf{k}})$. ✓

Parallel component $= (\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$

$$= \frac{6-4-1}{3} \times \frac{1}{3}(2\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k})$$

$$= \frac{1}{9}(2\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k})$$

$$= \frac{\mathbf{2}}{\mathbf{9}}\underset{\sim}{\mathbf{i}} - \frac{\mathbf{2}}{\mathbf{9}}\underset{\sim}{\mathbf{j}} - \frac{\mathbf{1}}{\mathbf{9}}\underset{\sim}{\mathbf{k}} \checkmark$$

Perpendicular component $= \underset{\sim}{a} - \left(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$

$$= 3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k} - \left(\frac{2}{9}\underset{\sim}{i} - \frac{2}{9}\underset{\sim}{j} - \frac{1}{9}\underset{\sim}{k}\right) \checkmark$$

$$= 3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k} - \frac{2}{9}\underset{\sim}{i} + \frac{2}{9}\underset{\sim}{j} + \frac{1}{9}\underset{\sim}{k}$$

$$= \boldsymbol{\frac{25}{9}\underset{\sim}{i} + \frac{20}{9}\underset{\sim}{j} + \frac{10}{9}\underset{\sim}{k}} \checkmark$$

We need the sum of 2 vector resolutes

$$\boldsymbol{\underset{\sim}{a} = \frac{2}{9}\underset{\sim}{i} - \frac{2}{9}\underset{\sim}{j} - \frac{1}{9}\underset{\sim}{k} + \frac{25}{9}\underset{\sim}{i} + \frac{20}{9}\underset{\sim}{j} + \frac{10}{9}\underset{\sim}{k}} \checkmark$$

Note that the above expression simplifies to $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$ as these are the two resolutes of $\underset{\sim}{a}$.

b **i** Given $\overrightarrow{AP} = \alpha\overrightarrow{AD}$

and since $OABC$ is a parallelogram, $\overrightarrow{OA} = \overrightarrow{CB} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \overrightarrow{AB} = \underset{\sim}{c}$,

consider $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD}$.

So $\overrightarrow{AD} = \underset{\sim}{c} + \overrightarrow{BD}$

$$= \underset{\sim}{c} + \frac{1}{2}\overrightarrow{BC}$$

$$= \boldsymbol{\underset{\sim}{c} - \frac{1}{2}\underset{\sim}{a}} \quad \checkmark$$

Since $\overrightarrow{AP} = \alpha\overrightarrow{AD}$

then $\overrightarrow{AP} = \boldsymbol{\alpha\left(\underset{\sim}{c} - \frac{1}{2}\underset{\sim}{a}\right)}$ ✓

ii Given $\overrightarrow{OP} = \beta\overrightarrow{OB}$,

consider $\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB}$

$$= \underset{\sim}{c} + \underset{\sim}{a}$$

We have $\overrightarrow{OP} = \beta\overrightarrow{OB}$

$$= \beta\left(\underset{\sim}{c} + \underset{\sim}{a}\right)$$

An expression for $\overrightarrow{AP}$ is $\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP}$

$$= -\underset{\sim}{a} + \beta\left(\underset{\sim}{c} + \underset{\sim}{a}\right)$$

$$= \boldsymbol{(\beta - 1)\underset{\sim}{a} + \beta\underset{\sim}{c}} \checkmark$$

iii Equate expressions for $\overrightarrow{AP}$:

$\overrightarrow{AP} = \alpha\left(\underset{\sim}{c} - \frac{1}{2}\underset{\sim}{a}\right)$ and $\overrightarrow{AP} = (\beta - 1)\underset{\sim}{a} + \beta\underset{\sim}{c}$

So $\alpha\left(\underset{\sim}{c} - \frac{1}{2}\underset{\sim}{a}\right) = (\beta - 1)\underset{\sim}{a} + \beta\underset{\sim}{c}$ ✓

Equate $\underset{\sim}{a}$ and $\underset{\sim}{c}$ vectors:

$\underset{\sim}{a}$ gives $-\frac{1}{2}\alpha = \beta - 1$

$\underset{\sim}{c}$ gives $\alpha = \boldsymbol{\beta}$

Solving for α and β gives $\boldsymbol{\alpha = \beta = \frac{2}{3}}$. ✓

Student performance

80–100% 60–79% 0–59%

a 69% Performance was fair with this long 5-mark question which tested vector resolutes. Students often made calculation errors or did not answer the question completely, often forgetting to express $\underset{\sim}{a}$ as the sum of the two vector resolutes at the end, and losing the final mark.

b **i** 80% The first part of part **b** was done well, despite the introduction of the constant α. Some students missed that BD was in the opposite direction to $\bar{a}$.

ii 55% The success rate dramatically decreased for the rest of the question.

iii 44% Most difficult. Finding α and β by equating coefficients was challenging and required a better understanding of parts **i** and **ii**.

More exam practice is needed when dealing with parameters in vectors.

EXERCISE 1.7 Vector proofs of geometric results

ANSWERS p. 556

Recap

1 TECH-FREE Given $A(1, -5)$, $B(-4, 3)$ and $C(5, 0)$, find the magnitude of each vector.

a $\overrightarrow{AB}$ **b** $\overrightarrow{BC}$ **c** $\overrightarrow{AC}$

2 TECH-FREE Given $A(1, -5)$, $B(-4, 3)$ and $C(5, 0)$, find the coordinates of M, the point of bisection of the vector:

a $\overrightarrow{AB}$ **b** $\overrightarrow{BC}$ **c** $\overrightarrow{AC}$

Mastery

3 WORKED EXAMPLE 14 TECH-FREE The median of a triangle is a vector from a vertex to the midpoint of the opposite side. Show that the sum of the medians of a triangle equal zero.

4 TECH-FREE Show that the midpoint of the hypotenuse of a right-angled triangle is equidistant from the vertices of the triangle.

5 TECH-FREE If $ABCD$ is a parallelogram and M is the midpoint of AC, prove that M is also the midpoint of BD.

6 For $\underset{\sim}{a} = 4\underset{\sim}{i} + m\underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{b} = 8\underset{\sim}{i} - 4\underset{\sim}{j} - 6\underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{a}$

A $m = 4$ **B** $m = -4$ **C** $m = -2$ **D** $m = 2$ **E** $\underset{\sim}{a} = 2$

7 For $\underset{\sim}{a} = 4\underset{\sim}{i} + m\underset{\sim}{j} - 4\underset{\sim}{k}$ and $\underset{\sim}{b} = 8\underset{\sim}{i} - 4\underset{\sim}{j} + n\underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{a}$

A $m = 4, n = 2$ **B** $m = -4, n = -2$ **C** $m = -2, n = 8$

D $m = 2, n = -8$ **E** $m = -2, n = -8$

8 Let $\underset{\sim}{b} = -2\underset{\sim}{i} + 7\underset{\sim}{j} - \underset{\sim}{k}$. $|\underset{\sim}{b}| =$

A 2 **B** 4 **C** 54 **D** $3\sqrt{6}$ **E** $\sqrt{10}$

9 The values of p and q such that $p\begin{bmatrix}1\\2\end{bmatrix} - q\begin{bmatrix}2\\-4\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix}$ are

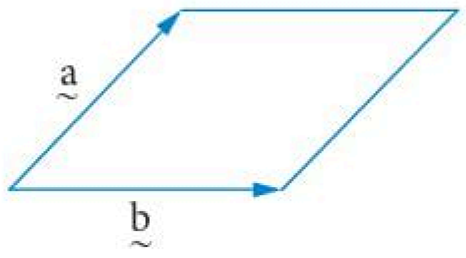
A $p = -\frac{19}{17}, q = \frac{9}{34}$ **B** $p = \frac{1}{2}, q = \frac{1}{4}$ **C** $p = \frac{1}{4}, q = \frac{1}{2}$

D $p = -\frac{1}{2}, q = -\frac{1}{4}$ **E** $p = \frac{1}{2}, q = -\frac{1}{4}$

10 The vectors $2\underset{\sim}{i}+\underset{\sim}{j}-\underset{\sim}{k}$, $\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k}$ and $3\underset{\sim}{i}-2m\underset{\sim}{j}+4\underset{\sim}{k}$, where m is a real constant, are linearly dependent when m equals

A -3 **B** -2 **C** $-\frac{5}{3}$ **D** 2 **E** 11

11 WORKED EXAMPLE 15 © VCAA 2011 2AQ10 58% The diagram on the right shows a rhombus, spanned by the two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.



It follows that

A $\underset{\sim}{a}\cdot\underset{\sim}{b}=0$

B $\underset{\sim}{a}=\underset{\sim}{b}$

C $(\underset{\sim}{a}+\underset{\sim}{b})\cdot(\underset{\sim}{a}-\underset{\sim}{b})=0$

D $|\underset{\sim}{a}+\underset{\sim}{b}|=|\underset{\sim}{a}-\underset{\sim}{b}|=0$

E $2\underset{\sim}{a}+2\underset{\sim}{b}=0$

Exam practice

80–100% 60–79% 0–59%

12 © VCAA 2013 1Q3 TECH-FREE (4 marks) The coordinates of three points are $A(-1, 2, 4)$, $B(1, 0, 5)$ and $C(3, 5, 2)$.

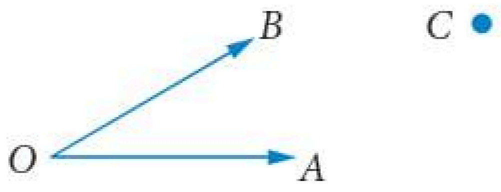
a 86% Find $\overrightarrow{AB}$. 1 mark

b 78% The points A, B and C are the vertices of a triangle. Prove that the triangle has a right angle at A. 2 marks

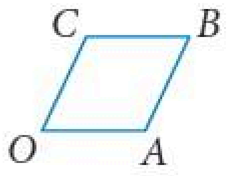
c 72% Find the length of the hypotenuse of the triangle. 1 mark

13 © VCAA 2006S 1Q2 TECH-FREE (4 marks) $OACB$ is a parallelogram with $\overrightarrow{OA}=\underset{\sim}{a}$ and $\overrightarrow{OB}=\underset{\sim}{b}$.

Given that $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{OC}$, prove that $OACB$ is a rhombus.



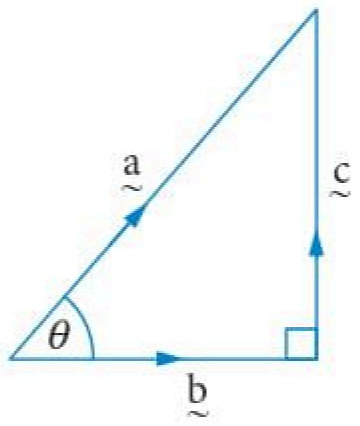
14 © VCAA 2015 1Q1 TECH-FREE (3 marks) Consider the rhombus $OABC$ shown on the right, where $\overrightarrow{OA}=a\underset{\sim}{i}$ and $\overrightarrow{OC}=\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k}$, and a is a positive real constant.



a 71% Find a. 1 mark

b 67% Show that the diagonals of the rhombus $OABC$ are perpendicular. 2 marks

15 © VCAA 2004 1IQ19 50% The right-angled triangle shown on the right has sides represented by the vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$. Which one of the following statements is **false**?



A $|\underset{\sim}{b}|^2+|\underset{\sim}{c}|^2=|\underset{\sim}{a}|^2$

B $\underset{\sim}{b}\cdot(\underset{\sim}{a}-\underset{\sim}{c})=|\underset{\sim}{b}|^2$

C $\underset{\sim}{b}\cdot(\underset{\sim}{a}-\underset{\sim}{b})=|\underset{\sim}{b}||\underset{\sim}{c}|$

D $\underset{\sim}{a}\cdot\underset{\sim}{b}=|\underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta)$

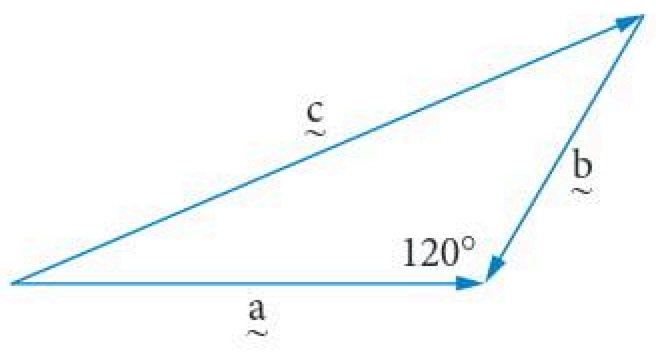
E $\underset{\sim}{a}\cdot\underset{\sim}{c}=|\underset{\sim}{a}||\underset{\sim}{c}|\sin(\theta)$

16 © VCAA 2008 2AQ17 43% P, Q and R are three collinear points with position vectors $\underset{\sim}{p}$, $\underset{\sim}{q}$ and $\underset{\sim}{r}$ respectively, where Q lies between P and R.

If $|\overrightarrow{QR}|=\frac{1}{2}|\overrightarrow{PQ}|$, then $\underset{\sim}{r}$ is equal to

A $\frac{3}{2}\underset{\sim}{q}-\frac{1}{2}\underset{\sim}{p}$ **B** $\frac{3}{2}\underset{\sim}{p}-\frac{1}{2}\underset{\sim}{q}$ **C** $\frac{3}{2}\underset{\sim}{q}-\frac{3}{2}\underset{\sim}{p}$ **D** $\frac{1}{2}\underset{\sim}{p}-\frac{3}{2}\underset{\sim}{q}$ **E** $\frac{3}{2}\underset{\sim}{p}-\frac{3}{2}\underset{\sim}{q}$

17 © VCAA 2009 2BQ17 30% Vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are shown on the right.



From the diagram, it follows that

A $|\underset{\sim}{c}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2$

B $|\underset{\sim}{c}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - |\underset{\sim}{a}||\underset{\sim}{b}|$

C $|\underset{\sim}{c}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 + |\underset{\sim}{a}\cdot\underset{\sim}{b}|$

D $|\underset{\sim}{c}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 + |\underset{\sim}{a}||\underset{\sim}{b}|$

E $|\underset{\sim}{c}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - |\underset{\sim}{a}\cdot\underset{\sim}{b}|$

18 © VCAA 2021N 2AQ11 Let $\underset{\sim}{a}$ and $\underset{\sim}{b}$ be arbitrary non-zero vectors.

Which one of the following statements is always true?

A $|\underset{\sim}{a}+\underset{\sim}{b}| \geq |\underset{\sim}{a}-\underset{\sim}{b}|$

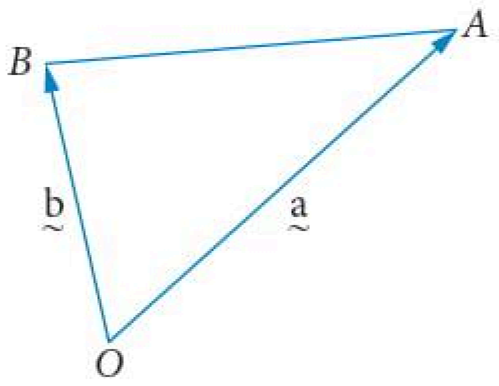
B $|\underset{\sim}{a}-\underset{\sim}{b}| \leq |\underset{\sim}{a}|+|\underset{\sim}{b}|$

C $|\underset{\sim}{a}-\underset{\sim}{b}| < |\underset{\sim}{a}|-|\underset{\sim}{b}|$

D $|\underset{\sim}{a}-\underset{\sim}{b}| < |\underset{\sim}{a}|+|\underset{\sim}{b}|$

E $|\underset{\sim}{a}-\underset{\sim}{b}| > |\underset{\sim}{a}|-|\underset{\sim}{b}|$

19 © VCAA 2010 2BQ1 (11 marks) The diagram on the right shows a triangle with vertices O, A and B. Let O be the origin, with vectors $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.



a Find the following vectors in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

- **i** 96% $\overrightarrow{MA}$, where M is the midpoint of the line segment OA. 1 mark
- **ii** 94% $\overrightarrow{BA}$. 1 mark
- **iii** 80% $\overrightarrow{AQ}$, where Q is the midpoint of the line segment AB. 1 mark

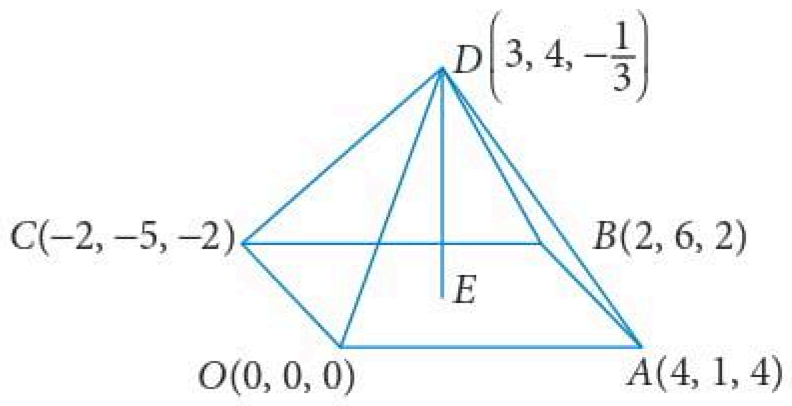
b 59% Let N be the midpoint of the line segment OB. Use a vector method to prove that the quadrilateral $MNQA$ is a parallelogram. 3 marks

Now consider the **particular** triangle OAB with $\overrightarrow{OA} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + \sqrt{3}\underset{\sim}{k}$ and $\overrightarrow{OB} = \alpha\underset{\sim}{i}$, where α, which is greater than zero, is chosen so that triangle OAB is isosceles, with $|\overrightarrow{OB}| = |\overrightarrow{OA}|$.

c 87% Show that $\alpha = 4$. 1 mark

d
- **i** 67% Find $\overrightarrow{OQ}$, where Q is the midpoint of the line segment AB. 1 mark
- **ii** 61% Use a vector method to show that $\overrightarrow{OQ}$ is perpendicular to $\overrightarrow{AB}$. 3 marks

20 © VCAA 2003 2Q3 (13 marks) $OABCD$ is a pyramid. The height of the pyramid is the length of DE, where E is the point on the base $OABC$ such that DE is perpendicular to the base.



a 48% Show that the base $OABC$ is a rhombus. 3 marks

b 65% Use a vector method to find $\angle AOC$ correct to the nearest tenth of a degree. 3 marks

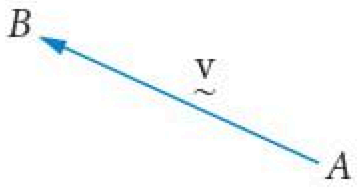
The unit vector $p\underset{\sim}{i} + q\underset{\sim}{j} + r\underset{\sim}{k}$, $p > 0$, is perpendicular to both $\overrightarrow{OA}$ and $\overrightarrow{OC}$.

c
- **i** 35% Show that $q = 0$ and find the **exact** values of p and r. 4 marks
- **ii** 3% Hence find the **exact** height of the pyramid. 3 marks

1 Chapter summary

Vectors

- A **vector** describes both **magnitude** and **direction**.



The vector above can be written as $\overrightarrow{AB}$ or $\underset{\sim}{v}$.

- length $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

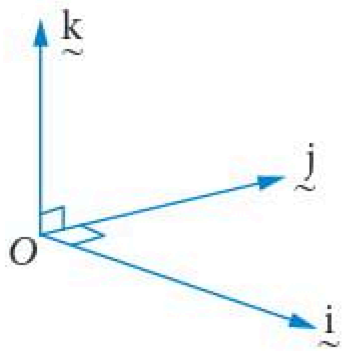
This is called the **magnitude** of the vector.

- The **direction** of the vector can be considered using geometry and trigonometry.
- Orthogonal unit vectors:

$\underset{\sim}{i}$ is the unit vector that runs along the x-axis.

$\underset{\sim}{j}$ is the unit vector that runs along the y-axis.

$\underset{\sim}{k}$ is the unit vector that runs along the z-axis.



Any vector can be written in terms of these three orthogonal unit vectors.

- The magnitude of vector $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ is equal to $\sqrt{x^2 + y^2 + z^2}$.

Unit vectors

- A unit vector has a magnitude of 1.
- The unit vector heading in the direction of $\underset{\sim}{a}$ has the notation $\hat{\underset{\sim}{a}}$.

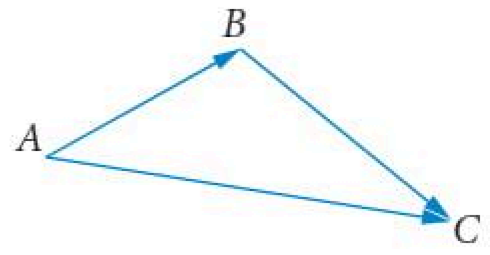
$$\hat{\underset{\sim}{a}} = \frac{\underset{\sim}{a}}{|\underset{\sim}{a}|} = \frac{x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}}{\sqrt{x^2 + y^2 + z^2}}$$

Adding and subtracting vectors

We can add vectors geometrically and algebraically.

Addition

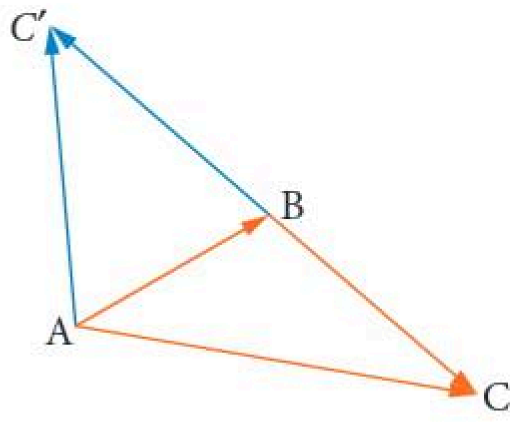
Using the triangle or parallelogram rule:



$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ (Note the common letter B.)

Subtraction

To subtract the vector $\overrightarrow{BC}$ from $\overrightarrow{AB}$, add the opposite of $\overrightarrow{BC}$:



Linear dependence and independence of vectors

For two vectors that are not parallel,

if $m\underset{\sim}{a} + n\underset{\sim}{b} = p\underset{\sim}{a} + q\underset{\sim}{b}$, then $m = p$ and $n = q$.

So we can equate the coefficients of the vectors.

If vector $\underset{\sim}{c}$ **can** be written as a linear combination of vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, then $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are said to be linearly **dependent**.

This means that when $\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$ for real numbers m and n, where m and n are not zero, then $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are linearly dependent.

Conversely, if vector $\underset{\sim}{c}$ **cannot** be written as a linear combination of vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, then $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are said to be linearly **independent.** This case means that when $\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$ for real numbers m and n, the only solution to the equation $\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$ is that $m = n = 0$.

For two dependent vectors in 2D, they must be parallel; one is a scalar multiple of the other.

In other words, a set of two vectors is **linearly dependent** if one is parallel to the other, and **linearly independent** if they are not parallel.

Resolving vectors

We can use the unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ to express any vector in terms of the three axes, x, y and z.

- Also, $\hat{\underset{\sim}{a}} = \cos(\alpha)\underset{\sim}{i} + \cos(\beta)\underset{\sim}{j} + \cos(\gamma)\underset{\sim}{k}$ where:

  α is the angle between the unit vector and the x-axis.

  β is the angle between the unit vector and the y-axis.

  γ is the angle between the unit vector and the z-axis.

This means that if $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$, then

$$\cos(\alpha) = \frac{a_1}{|\underset{\sim}{a}|}$$

$$\cos(\beta) = \frac{a_2}{|\underset{\sim}{a}|}$$

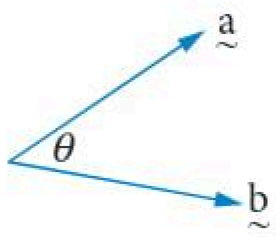
$$\cos(\gamma) = \frac{a_3}{|\underset{\sim}{a}|}$$

Scalar (dot) product

If $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$ and $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}$,

then $\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1b_1 + a_2b_2 + a_3b_{3.}$

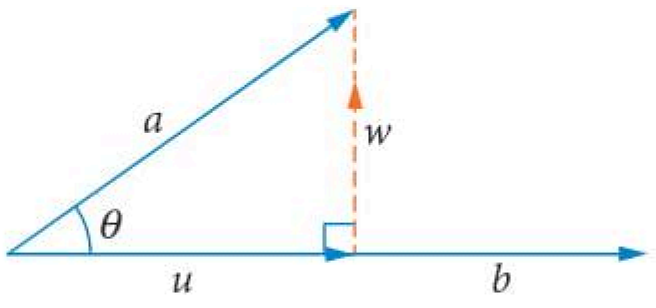
$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta)$, where θ is the angle between the outgoing vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.



- Angle between two vectors

$$\cos(\theta) = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

Scalar and vector projections



Scalar projection of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b} = \underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}$

Vector projection of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b} = \left(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$

Vector projection of $\underset{\sim}{a}$ perpendicular to the direction of $\underset{\sim}{b} = \underset{\sim}{a} - \left(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$

Parallel and perpendicular vectors

- If $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$, then vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular.
- If $\underset{\sim}{a} \cdot \underset{\sim}{b} = \pm|\underset{\sim}{a}||\underset{\sim}{b}|$, then vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel.
- Parallel in the same direction: $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|$
- Parallel in the opposite direction: $\underset{\sim}{a} \cdot \underset{\sim}{b} = -|\underset{\sim}{a}||\underset{\sim}{b}|$

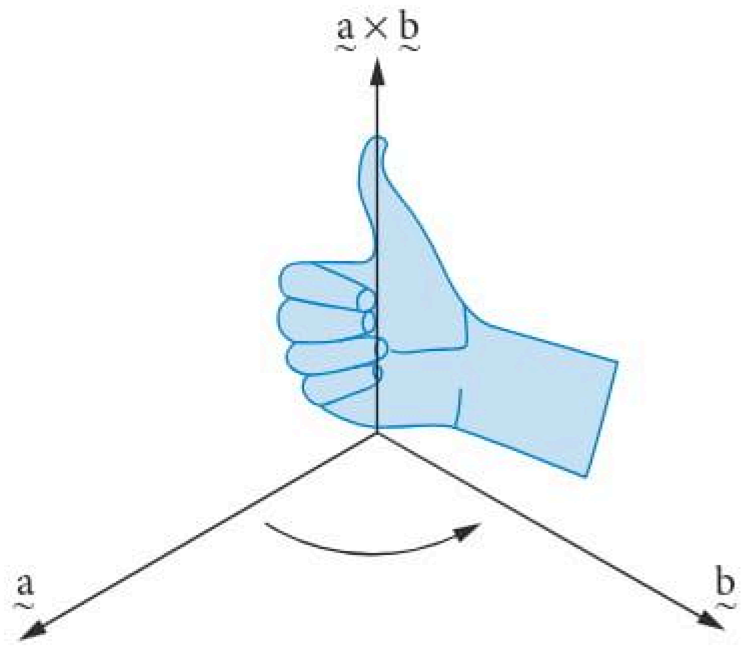
Vector product

If $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$ and $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}$, then the vector product of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is:

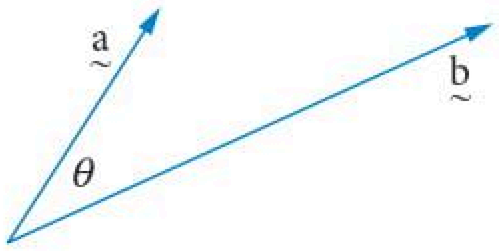
$$\underset{\sim}{a} \times \underset{\sim}{b} = (a_2b_3 - a_3b_2)\underset{\sim}{i} - (a_1b_3 - a_3b_1)\underset{\sim}{j} + (a_1b_2 - a_2b_1)\underset{\sim}{k}$$

In determinant form:

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\underset{\sim}{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\underset{\sim}{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\underset{\sim}{k}$$



- The magnitude of the vector product is $\underset{\sim}{a} \times \underset{\sim}{b}| = |\underset{\sim}{a}||\underset{\sim}{b}|\sin(\theta)$, where θ is the angle between the two 'outgoing' vectors.



Vector proofs of geometric results

- All techniques dealt with in this chapter can be included in vector proofs of simple geometric results.

Cumulative examination 1

Total number of marks: 9 Reading time: 4 minutes Writing time: 14 minutes

TECH-FREE *Technology is NOT permitted.*

1 © VCAA 2017 1Q5 (4 marks) Relative to a fixed origin, the points B, C and D are defined respectively by the position vectors $\underset{\sim}{b} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{c} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{d} = a\underset{\sim}{i} - 2\underset{\sim}{j}$, where a is a real constant.

Given that the magnitude of angle BCD is $\frac{\pi}{3}$, find a.

2 © VCAA 2019N 1Q5 (5 marks) A triangle has vertices $A(\sqrt{3} + 1, -2, 4)$, $B(1, -2, 3)$ and $C(2, -2, \sqrt{3} + 3)$.

a Find angle ABC. 3 marks

b Find the area of the triangle. 2 marks

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 © VCAA 2018 2AQ11 Consider the vectors given by $\underset{\sim}{a} = m\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{b} = \underset{\sim}{i} + m\underset{\sim}{j}$, where $m \in R$. If the acute angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is 30°, then m equals

A $\sqrt{2} \pm 1$

B $2 \pm \sqrt{3}$

C $\sqrt{3}, \dfrac{1}{\sqrt{3}}$

D $\dfrac{\sqrt{3}}{4-\sqrt{3}}$

E $\dfrac{\sqrt{39}}{13}$

2 © VCAA 2018 2AQ12 If $|\underset{\sim}{a} + \underset{\sim}{b}| = |\underset{\sim}{a}| + |\underset{\sim}{b}|$ and $\underset{\sim}{a} \cdot \underset{\sim}{b} \neq 0$, which one of the following is **necessarily true**?

A $\underset{\sim}{a}$ is parallel to $\underset{\sim}{b}$

B $|\underset{\sim}{a}| = |\underset{\sim}{b}|$

C $\underset{\sim}{a} = \underset{\sim}{b}$

D $\underset{\sim}{a} = -\underset{\sim}{b}$

E $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$

3 © VCAA 2017 2AQ11 The vectors $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + d\underset{\sim}{k}$, $\underset{\sim}{b} = \underset{\sim}{i} + \underset{\sim}{j} - 4\underset{\sim}{k}$ and $\underset{\sim}{c} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$, where d is a real constant, are linearly dependent if

A $d = -10$

B $d \in R\backslash\{-14\}$

C $d = -14$

D $d \in R\backslash\{-10\}$

E $d \in R$

4 © VCAA 2017 2AQ13 Given the vectors given by $\underset{\sim}{a} = 3\underset{\sim}{i} - 4\underset{\sim}{j} + 12\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, the vector resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is

A $-\dfrac{14}{3}$

B $-\dfrac{14}{3}\left(2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right)$

C $-\dfrac{14}{9}\left(2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right)$

D $-\dfrac{14}{13}$

E $-\dfrac{14}{169}\left(3\underset{\sim}{i} - 4\underset{\sim}{j} + 12\underset{\sim}{k}\right)$

5 © VCAA 2017 2AQ15 A body has displacement of $3\underset{\sim}{i} + \underset{\sim}{j}$ metres at a particular time. The body moves with constant velocity and two seconds later its displacement is $-\underset{\sim}{i} + 5\underset{\sim}{j}$ metres. The velocity, in m s^{-1}, of the body is

A $2\underset{\sim}{i} + 6\underset{\sim}{j}$

B $-2\underset{\sim}{i} + 2\underset{\sim}{j}$

C $-4\underset{\sim}{i} + 4\underset{\sim}{j}$

D $4\underset{\sim}{i} - 4\underset{\sim}{j}$

E $\underset{\sim}{i} + 3\underset{\sim}{j}$

Section B 2 questions

15 marks

1 © VCAA 2013 2BQ4 (12 marks) Let $\underset{\sim}{a} = -\frac{7\sqrt{3}}{3}\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} + \sqrt{3}\underset{\sim}{j} + 2\sqrt{3}\underset{\sim}{k}$.

a Find a unit vector in the direction of $\underset{\sim}{b}$. 1 mark

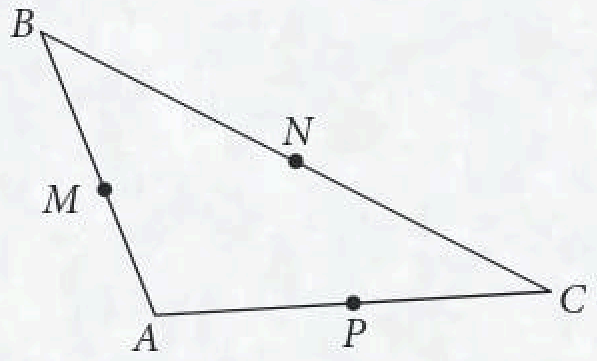
b Resolve $\underset{\sim}{a}$ into two vector components, one that is parallel to $\underset{\sim}{b}$ and one that is perpendicular to $\underset{\sim}{b}$. 3 marks

c Find the value of m such that $\underset{\sim}{c} = m\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$ makes an angle of $\frac{2\pi}{3}$ with $\underset{\sim}{b}$ and where $\underset{\sim}{c} \neq \underset{\sim}{a}$. 2 marks

d Find the angle, in degrees, that $\underset{\sim}{c}$ makes with $\underset{\sim}{a}$, correct to one decimal place. 2 marks

For the triangle ABC shown below, the midpoints of the sides are the points, M, N and P.

Let $\overrightarrow{AC} = \underset{\sim}{u}$ and $\overrightarrow{CB} = \underset{\sim}{v}$.

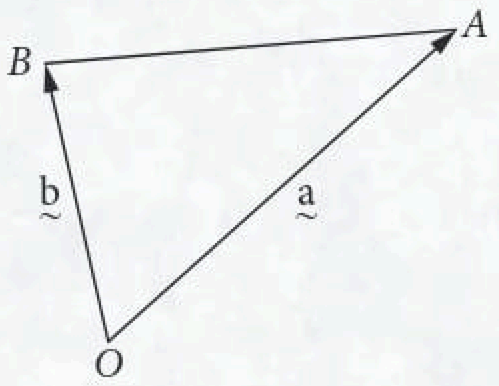


e **i** Express $\overrightarrow{AN}$ in terms of $\underset{\sim}{u}$ and $\underset{\sim}{v}$. 1 mark

ii Express $\overrightarrow{CM}$ and $\overrightarrow{BP}$ in terms of $\underset{\sim}{u}$ and $\underset{\sim}{v}$. 2 marks

iii **Hence** simplify the expression $\overrightarrow{AN} + \overrightarrow{CM} + \overrightarrow{BP}$. 1 mark

2 © VCAA 2010 2BQ1a (3 marks) The diagram below shows a triangle with vertices O, A and B. Let O be the origin, with vectors $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.



Find the following vectors in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

a $\overrightarrow{MA}$, where M is the midpoint of the line segment OA. 1 mark

b $\overrightarrow{BA}$. 1 mark

c $\overrightarrow{AQ}$, where Q is the midpoint of the line segment AB. 1 mark

CHAPTER

2 RATIONAL FUNCTIONS

*Revision of Year 11

Study Design coverage

AREA OF STUDY 2: FUNCTIONS, RELATIONS AND GRAPHS

- rational functions and the expression of rational functions of low degree as sums of partial fractions
- graphs of rational functions of low degree, their asymptotic behaviour, and the nature and location of stationary points and points of inflection
- graphs of simple quotient functions, their asymptotic behaviour, and the nature and location of stationary points and points of inflection.

VCE Mathematics Study Design 2023–2027 p. 110, © VCAA 2022

STUDY DESIGN COVERAGE

Video playlists (9):

2.1 The conic sections
2.2 Rational functions and partial fractions
2.3 Graphing rational functions
2.4 Graphing quotient functions
2.5 Absolute value functions
2.6 Reciprocal circular functions
2.7 Trigonometric identities
2.8 Inverse circular functions
VCE question analysis Rational functions

Worksheets (7):

2.5 Absolute value • Absolute value functions • Absolute value inequalities • Transformations of absolute values and hyperbolas
2.7 Simplifying periodic functions • Trigonometric identities
2.8 Inverse functions

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

2.1 The conic sections

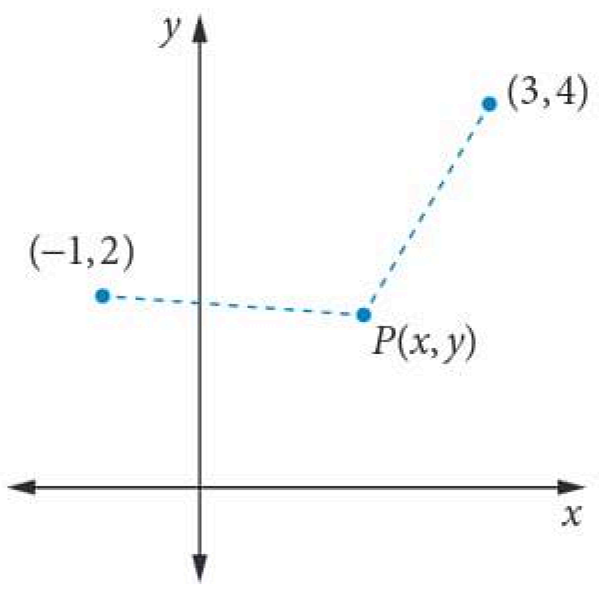
Video playlist The conic sections

We studied straight lines, parabolas, circles, ellipses and hyperbolas in Year 11. You will be familiar with the Cartesian and parametric definitions. Remember that a **locus** is the path traced out by a point under a given condition.

WB p. 16

WORKED EXAMPLE 1 Finding a locus

Find the locus of a point equidistant from the points (–1, 2) and (3, 4).

Steps	**Working**
1 Sketch the points (–1, 2) and (3, 4) and show that the point P (x, y) is equidistant from (–1, 2) and (3, 4).	
2 Equate the distance formulas.	$\sqrt{(x-(-1))^2+(y-2)^2} = \sqrt{(3-x)^2+(4-y)^2}$
3 Square both sides.	$(x+1)^2+(y-2)^2=(3-x)^2+(4-y)^2$

4 Simplify.	$x^2 + 2x + 1 + y^2 - 4y + 4 = 9 - 6x + x^2 + 16 - 8y + y^2$ $2x + 6x - 4y + 8y + 1 + 4 - 9 - 16 = 0$ $8x + 4y - 20 = 0$ $2x + y - 5 = 0$
5 State the answer.	The locus is the straight line with the equation $2x + y - 5 = 0$.

The locus is actually the perpendicular bisector of the points (–1, 2) and (3, 4).
Check that it is perpendicular to the line through the points and passes through their midpoint.

Conic section definition as a locus

A **conic section** is the shape made when a cone is intersected by a plane.

The **eccentricity** of a conic section is given by

$$e = \frac{\sin(\beta)}{\sin(\alpha)}$$

where α is the slant angle of the cone

and β is the angle between the plane and the base of the cone.

For circles: $e = 0$ ellipses: $0 < e < 1$ parabolas: $e = 1$ hyperbolas: $e > 1$.

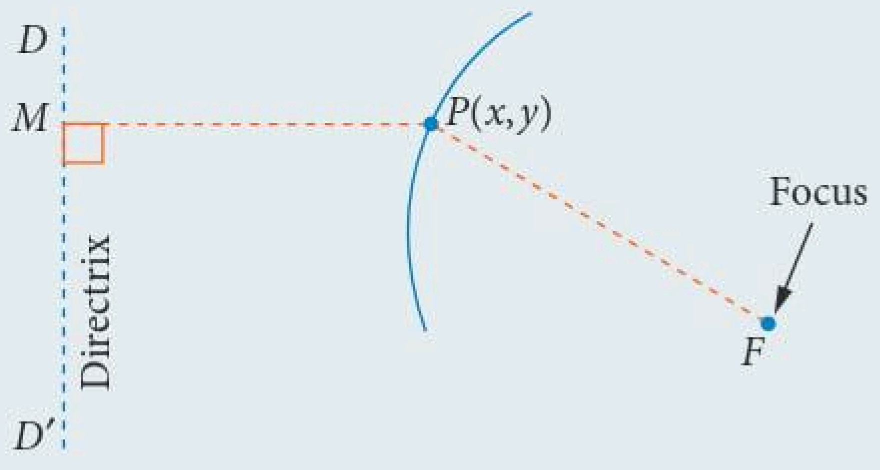
A conic section is a locus referred to a fixed point (**focus**) and line (**directrix**).

PF = distance between the points and focus.

PM = distance between the points and directrix.

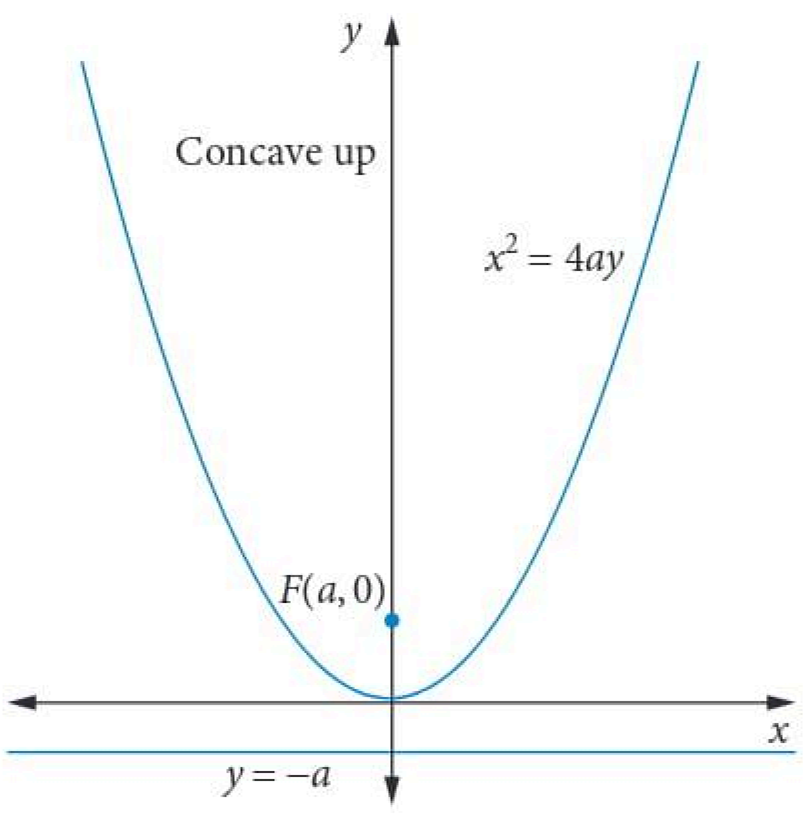
$$\frac{PF}{PM} = e$$

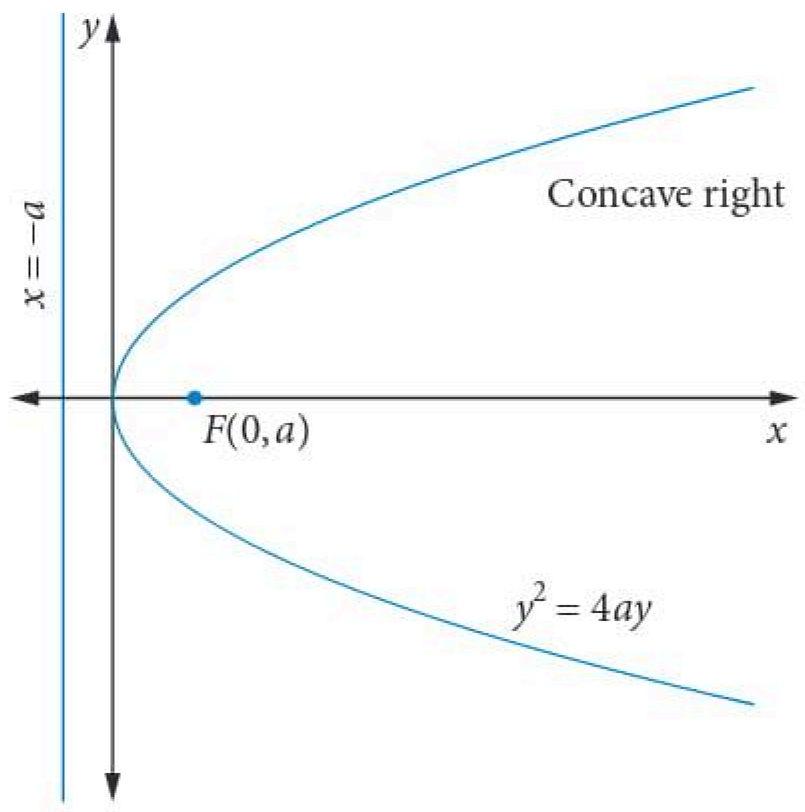
For a circle, $PM = \infty$ and $PF = r$.



We studied conic sections with directrices parallel to the axes in Year 11.

Parabolas ($e = 1$, $a > 0$)

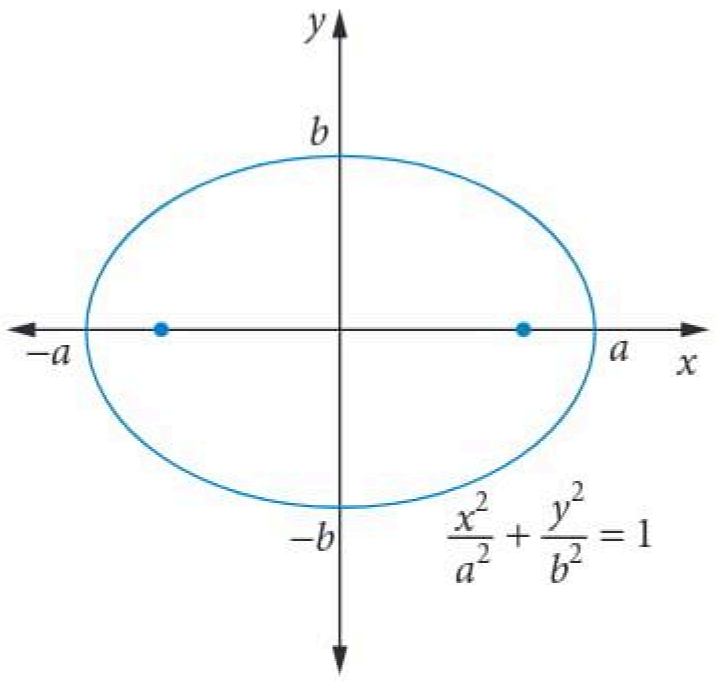


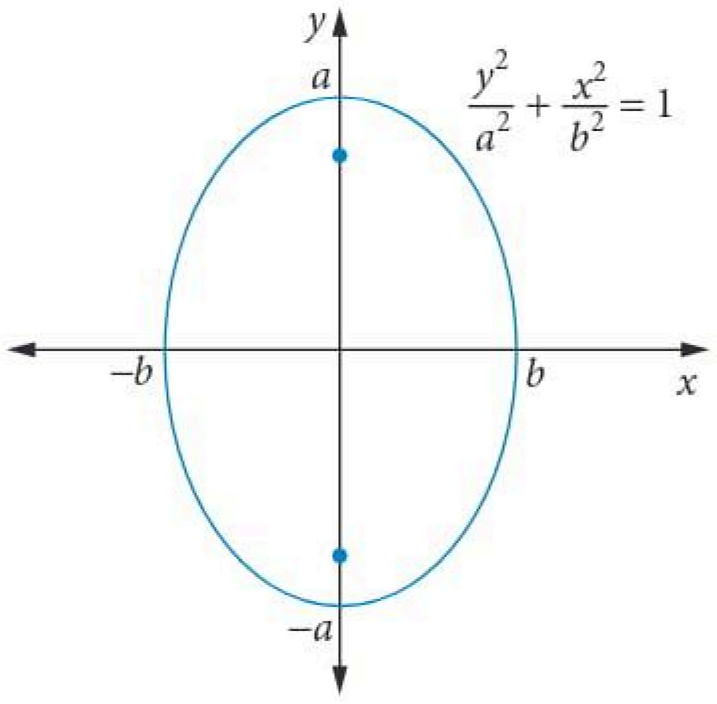


Vertex:	(0, 0)	(0, 0)
Axis:	$x = 0$	$y = 0$
Equation:	$x^2 = 4ay$	$y^2 = 4ax$

If $a < 0$, the parabola is concave down or concave left.

Ellipses ($0 < e < 1$, $a > b$)

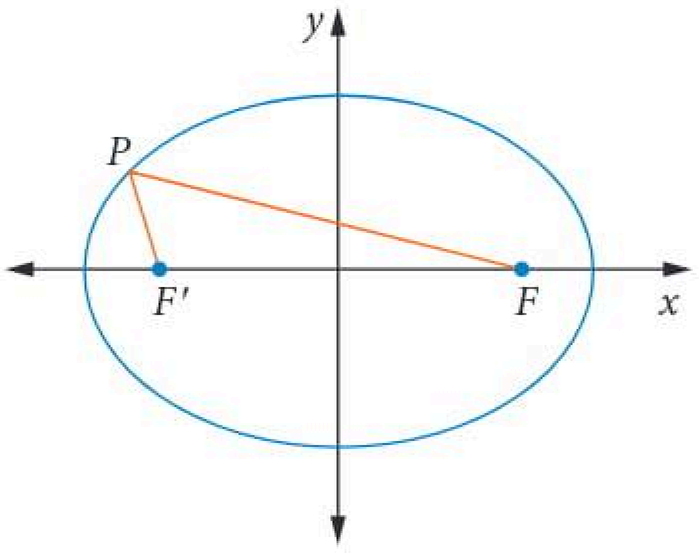




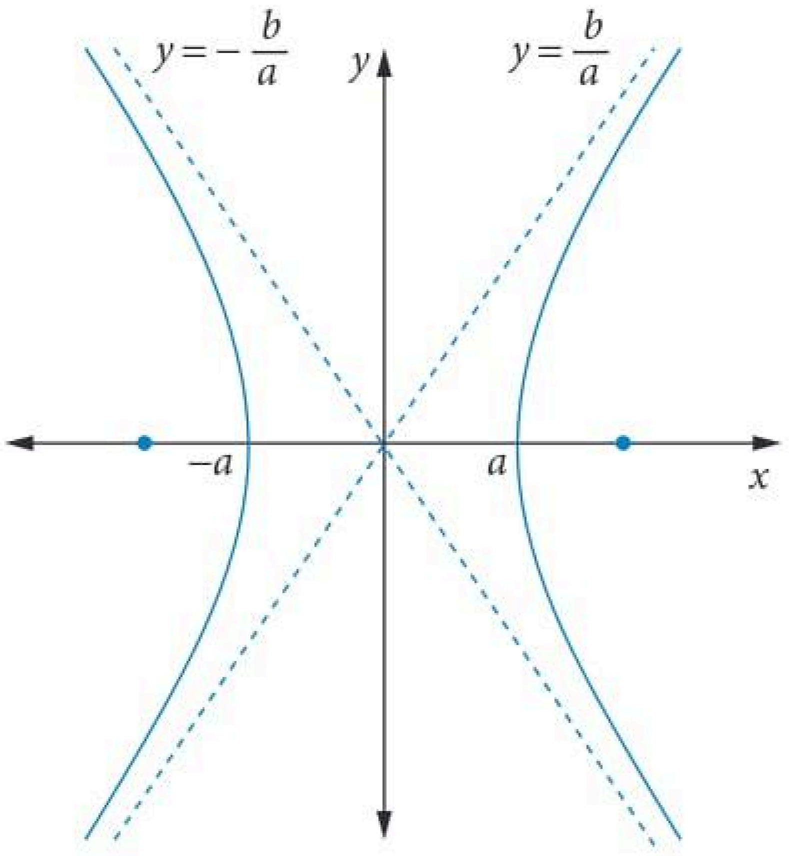
Axis lengths:	Major $2a$	Minor $2b$
Vertices:	$(\pm a, 0), (0, \pm b)$	$(0, \pm a), (\pm b, 0)$
Axis:	$x = 0$	$y = 0$
Formula:	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$

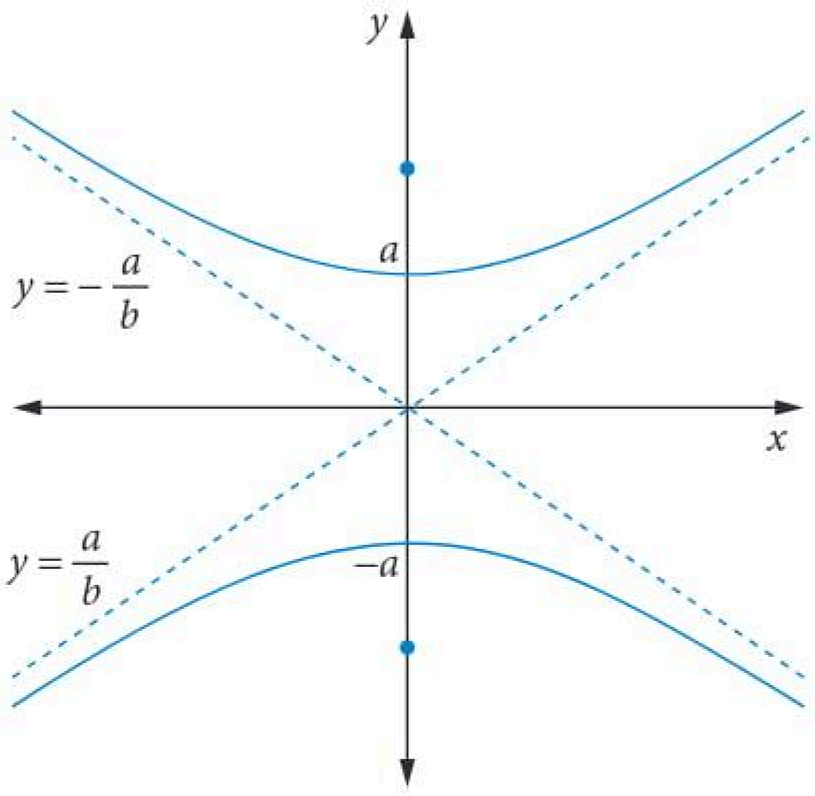
$b^2 = a^2 (1 - e^2)$, or $e^2 = 1 - \frac{b^2}{a^2}$

For any point $P (x, y)$ on an ellipse, $PF + PF' = 2a$.



Hyperbolas ($e > 1$)





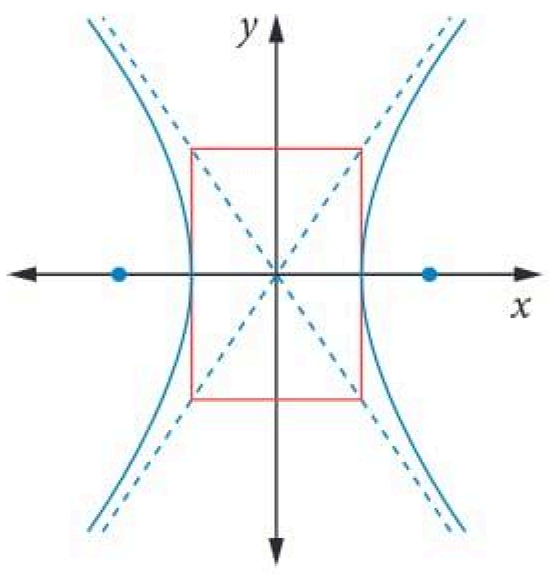
Vertices:	$(\pm a, 0)$	$(0, \pm a)$
Transverse axis:	$x = 0$	$y = 0$
Formula:	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

$b^2 = a^2 (e^2 - 1)$, or $e^2 = 1 + \frac{b^2}{a^2}$

Asymptotes:	$y = \pm\frac{b}{a}$	$y = \pm\frac{a}{b}$

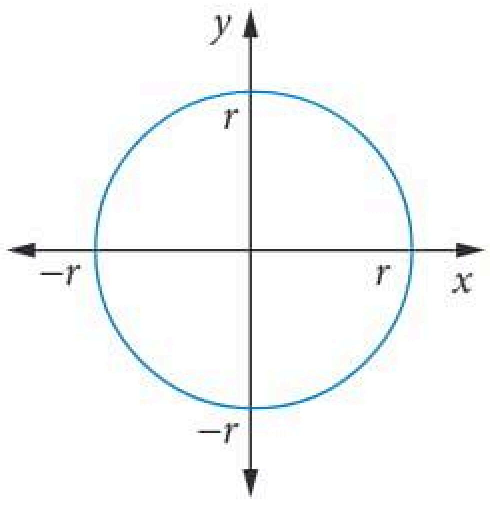
The **auxiliary rectangle**, through $(a, 0)$, $(0, b)$, $(-a, 0)$ and $(-b, 0)$ assists in sketching the asymptotes and hyperbola.

For a 'vertical' hyperbola it passes through $(0, a)$, $(b, 0)$, $(0, -a)$ and $(-b, 0)$.



Circles ($e = 0$, $a = b$)

A **circle** with radius r and centre at the origin has the equation $x^2 + y^2 = r^2$.



Centre (h, k)

The equation of a conic section with centre (h, k) has x and y replaced by $(x - h)$ and $(y - k)$ respectively in the appropriate equation.

WB

p. 17

WORKED EXAMPLE 2 **Finding an ellipse from given information**

An ellipse has its major axis parallel to the x-axis and is of length 16. A line segment at an inclination of 45° from the centre to the point $(-3, -8)$ on the ellipse has length $5\sqrt{2}$.

Find possible equations for the ellipse.

Steps	**Working**
1 Use the length of the major axis to find a.	$2a = 16$ $a = 8$
2 Write the equation of the straight line and simplify.	$y - (-8) = \tan(45°)\,(x - (-3))$ $y + 8 = 1\,(x + 3)$ $y = x - 5$
3 Substitute the centre in the equation.	$k = h - 5$
4 Write the length of the line.	$5\sqrt{2} = \sqrt{(h - (-3))^2 + (k - (-8))^2}$
5 Simplify, substitute and solve for h.	$50 = (h + 3)^2 + (h - 5 + 8)^2$ $2\,(h + 3)^2 = 50$ $h + 3 = \pm 5$ $h = 2$ or -8
6 Find the values of k.	$k = h - 5 = -3$ or -13
7 Write the possible centres.	The centre is $(2, -3)$ or $(-8, -13)$.
8 Write the general formula, substitute known values and solve for b, for both cases.	$\frac{(x - h)^2}{8^2} + \frac{(y - k)^2}{b^2} = 1$ $\frac{(-3 - 2)^2}{8^2} + \frac{(-8 + 3)^2}{b^2} = 1$ $5^2b^2 + 5^2 \times 8^2 = 8^2b^2$ $(64 - 25)b^2 = 5^2 \times 8^2$ $b = \frac{40}{\sqrt{39}}$ (negative root irrelevant)

	or $\dfrac{(-3+8)^2}{8^2}+\dfrac{(-8+13)^2}{b^2}=1$ $5^2b^2+5^2\times 8^2=8^2b^2$ $b=\dfrac{40}{\sqrt{39}}$
9 Write the possible equations.	The equation of the ellipse could be $\dfrac{(x-2)^2}{64}+\dfrac{39(y+3)^2}{1600}=1$ or $\dfrac{(x+8)^2}{64}+\dfrac{39(y+13)^2}{1600}=1$

The **parametric form** of a conic section has x and y expressed in terms of an independent variable, usually written as θ or t.

A Pythagorean identity can change circles, ellipses and hyperbolas to parametric form.

Circles and ellipses have a sum, so use $\sin^2(t)+\cos^2(t)=1$.

Hyperbolas have a difference so use $\tan^2(t)+1=\sec^2(t)$ or $1+\cot^2(t)=\operatorname{cosec}^2(t)$, written as $\sec^2(t)-\tan^2(t)=1$ or $\operatorname{cosec}^2(t)-\cot^2(t)=1$.

WB

p. 18

WORKED EXAMPLE 3 Changing from Cartesian to parametric form

Express the hyperbola $\dfrac{(y-3)^2}{81}-\dfrac{(x+1)^2}{100}=1$ in parametric form.

Steps	**Working**
1 Use $\tan^2(t)+1=\sec^2(t)$.	$\sec^2(t)-\tan^2(t)=1$
2 Satisfy the identity.	Choose $\sec^2(t)=\dfrac{(y-3)^2}{81}$, $\tan^2(t)=\dfrac{(x+1)^2}{100}$ to satisfy the identity.
3 Use the roots. Specify the values of t to complete all the points of the hyperbola.	$\sec(t)=\pm\dfrac{y-3}{9}$, $\tan(t)=\pm\dfrac{x+1}{10}$, $0\le t\le 2\pi, t\ne\dfrac{\pi}{2},\dfrac{3\pi}{2}$ ← sec and tan are undefined at $\dfrac{\pi}{2},\dfrac{3\pi}{2}$.
4 Use the positive roots.	Choose $\sec(t)=\dfrac{y-3}{9}$, $\tan(t)=\dfrac{x+1}{10}$ for simplicity.
5 Simplify to get the parametric form.	The parametric form is $\begin{cases} y=3+9\sec(t)\\ x=-1+10\tan(t)\end{cases}$

The other choices $\begin{cases} y=3-9\sec(t)\\ x=-1+10\tan(t)\end{cases}$, $\begin{cases} y=3+9\sec(t)\\ x=-1-10\tan(t)\end{cases}$ or $\begin{cases} y=3-9\sec(t)\\ x=-1-10\tan(t)\end{cases}$

give the same hyperbola but start at different points and/or traverse it in the opposite direction as t goes from 0 to 2π.

Check the form of the **parametric equations** for Worked example 3 for the identity $1+\cot^2(t)=\operatorname{cosec}^2(t)$.

The parabolas $x^2=4ay$ and $y^2=4ax$ use the substitutions $x=2at$ or $y=2at$ respectively.

WB
p. 19

WORKED EXAMPLE 4 Changing from parametric to Cartesian form

Express the parametric equations $\begin{cases} x = 4 + 3\cos(t) \\ y = -6 + 5\sin(t) \end{cases}$, $0 \le t < 2\pi$ in Cartesian form.

Steps	**Working**
1 Isolate sin (*t*) and cos (*t*).	$\cos(t) = \dfrac{x-4}{3}, \sin(t) = \dfrac{y+6}{5}$
2 Write the identity.	$\sin^2(t) + \cos^2(t) = 1$
3 Substitute the expressions.	$\left(\dfrac{y+6}{5}\right)^2 + \left(\dfrac{x-4}{3}\right)^2 = 1$
4 Simplify.	$\dfrac{(y+6)^2}{5^2} + \dfrac{(x-4)^2}{3^2} = 1$

The above example is an ellipse with centre $(4, -6)$. Its major axis is 10 units long and vertical, while the minor axis is 6 units long.

Exam hack

You must be able to change between parametric and **Cartesian forms** of conic sections.

Parametric forms of conic sections

Parametric forms of conic sections with centres at the origin, for $0 \le t < 2\pi$.

Parabolas: vertical axis $\begin{cases} x = 2at \\ y = at^2 \end{cases}$

horizontal axis $\begin{cases} y = 2at \\ x = at^2 \end{cases}$

Ellipses $(a > b)$: horizontal major axis $\begin{cases} x = a\cos(t) \\ y = b\sin(t) \end{cases}$

vertical major axis $\begin{cases} x = b\cos(t) \\ y = a\sin(t) \end{cases}$

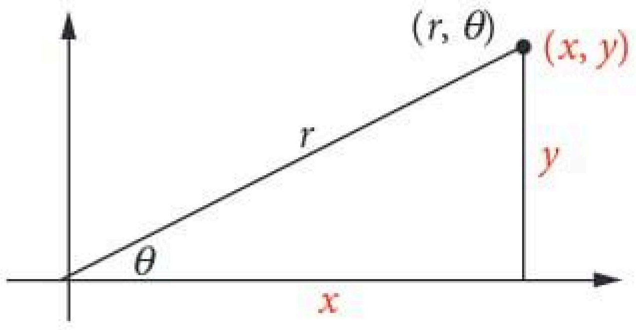
Hyperbolas: horizontal transverse axis $\begin{cases} x = a\sec(t) \\ y = b\tan(t) \end{cases}$ or $\begin{cases} x = a\,\mathrm{cosec}(t) \\ y = b\cot(t) \end{cases}$

vertical transverse axis $\begin{cases} y = a\sec(t) \\ x = b\tan(t) \end{cases}$ or $\begin{cases} y = a\,\mathrm{cosec}(t) \\ x = b\cot(t) \end{cases}$

Circles: $\begin{cases} x = r\cos(t) \\ y = r\sin(t) \end{cases}$

For the centre (h, k), replace x and y by $(x - h)$ and $(y - k)$ respectively in the equation of any of the conic sections. This adds h and k to the parametric equations for x and y, respectively.

Polar coordinates in the plane express the position of a point in terms of the distance r of a point from the origin and the anticlockwise angle θ from the x-axis.



Conversions between (x, y) and (r, θ):

$\tan(\theta) = \dfrac{y}{x}, r^2 = x^2 + y^2$

$x = r\cos(\theta), y = r\sin(\theta)$

The **polar form** of the equation of a conic section has a focus placed at the origin and the form $r=\dfrac{de}{1\pm e\cos(\theta)}$ or $r=\dfrac{de}{1\pm e\sin(\theta)}$, $0\le\theta<2\pi$ for an **ellipse**, **hyperbola** or **parabola**.

e is the eccentricity and d is the distance from the focus to the directrix. The equation with $\cos(\theta)$ gives horizontal conic sections, and $\sin(\theta)$ gives vertical ones.

For ellipses, $e<0$; for parabolas, $e=1$; for hyperbolas, $e>1$

The vertices of an ellipse are at $\theta=0, \dfrac{\pi}{2}, \pi$ and $\dfrac{3\pi}{2}$.

For a horizontal hyperbola, they are at $\theta=0$ and π; for a vertical hyperbola, they are at $\theta=\dfrac{\pi}{2}$ and $\dfrac{3\pi}{2}$.

The **polar form** of a **circle** is $r=\pm2a\sin(\theta)$ or $r=\pm2a\cos(\theta)$, where $a=$ radius. The centres are $(a,0)$, $(-a,0)$, $(0,b)$ and $(0,-a)$ for $2a\cos(\theta)$, $-2a\cos(\theta)$, $2a\sin(\theta)$ and $-2a\sin(\theta)$ respectively.

WORKED EXAMPLE 5 Changing from polar to Cartesian form

Write the equation $r=\dfrac{20}{7-3\cos(\theta)}$, $0\le\theta<2\pi$ in Cartesian form.

WB p. 20

Steps	**Working**
1 Divide by 7 to get e and determine the type of conic.	$e=\frac{3}{7}$, so it is a horizontal ellipse.
2 Find the vertices.	$r(0)=5,\ r\left(\frac{\pi}{2}\right)=\frac{20}{7},\ r(\pi)=2$ and $r\left(\frac{3\pi}{2}\right)=\frac{20}{7}$
3 Change to Cartesian coordinates.	$(5,0)\to(5,0),\ \left(\frac{20}{7},\frac{\pi}{2}\right)\to\left(0,\frac{20}{7}\right).$ $(2,\pi)\to(-2,0)$ and $\left(\frac{20}{7},\frac{3\pi}{2}\right)\to\left(0,-\frac{20}{7}\right).$
4 Find a and b.	$2a=5-(-2)=7$ $a=3.5$ $2b=\frac{20}{7}-\left(-\frac{20}{7}\right)=\frac{40}{7}$ $b=\frac{20}{7}$
5 Find the centre. We can use $b^2=a^2(1-e^2)$ to find b.	$\frac{5+(-2)}{2}=1.5,\ \frac{\frac{20}{7}+\left(-\frac{20}{7}\right)}{2}=0,$ so the centre is (1.5, 0).
6 Write and simplify the equation.	$\frac{(x-1.5)^2}{3.5^2}+\frac{(y-0)^2}{\left(\frac{20}{7}\right)^2}=1$ $\frac{4(x-1.5)^2}{49}+\frac{49y^2}{400}=1$

The horizontal form with $\cos(\theta)$ above is confirmed by the lengths of the horizontal and vertical axes, $2a=7$ and $2b=\dfrac{40}{7}\approx5.714$.

For a hyperbola, we use $b^2=a^2(e^2-1)$ to find b.

WB

p. 21

WORKED EXAMPLE 6 Changing from Cartesian to polar form

Write the equation $\frac{(y-2.5)^2}{4} - \frac{x^2}{2.25} = 1$ in polar form.

Steps	**Working**
1 State the type of conic.	It is a vertical hyperbola.
2 Use *a* and *b* to find *e*.	$e^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{2.25}{4} = \frac{6.25}{4}$ $e = 1.25$
3 Check that the centre is at a focus.	$k - ae = 2.5 - 2 \times 1.25 = 0$, so the lower focus is at the origin.
4 Find *d* and *de*.	$d = ae - \frac{a}{e} = 2.5 - 1.6 = 0.9$ $de = 0.9 \times 1.25 = 1.125$
5 Write and simplify the equation.	$r = \frac{1.125}{1 \pm 1.25\sin(\theta)} = \frac{9}{8 \pm 10\sin(\theta)}$
6 Choose the correct sign.	For +, $r\left(\frac{\pi}{2}\right) = 0.5$ and $r\left(\frac{3\pi}{2}\right) = -4.5$, which gives vertices at (0, 0.5) and (0, −4.5), which works.
7 Write the answer.	$r = \frac{9}{8 + 10\sin(\theta)}$

For $\frac{(y+2.5)^2}{4} - \frac{x^2}{2.25} = 1$, $k = -2.5$, so $k + ae = -2.5 + 2.5 = 0$, the top focus is at the origin and the polar equation would be $r = \frac{9}{8 - 10\sin(\theta)}$.

Wherever possible, use CAS to check graphs or conversions of conics.

USING CAS 1 Graphing conics

Graph each conic section.

a $\begin{cases} x = 4 + 3\cos(t) \\ y = -6 + 5\sin(t) \end{cases}, 0 \le t < 2\pi$

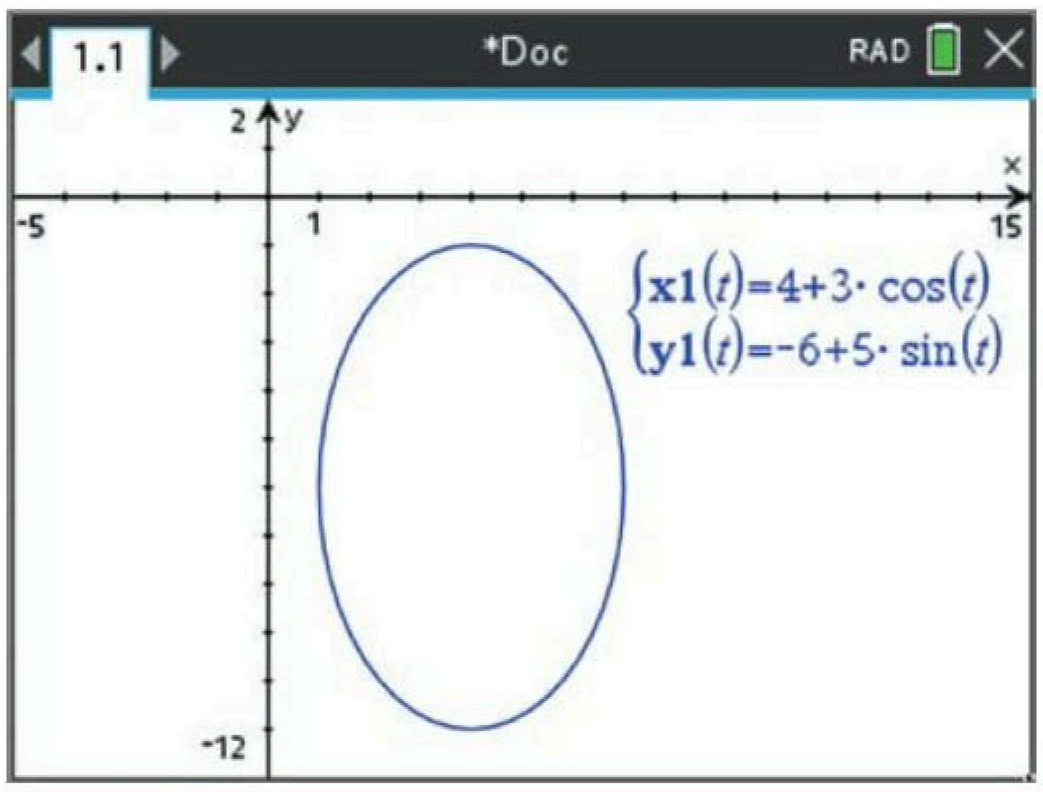
b $\frac{(y-2.5)^2}{4} - \frac{x^2}{2.25} = 1$

c $r = \frac{9}{8 + 10\sin(\theta)}$

TI-Nspire

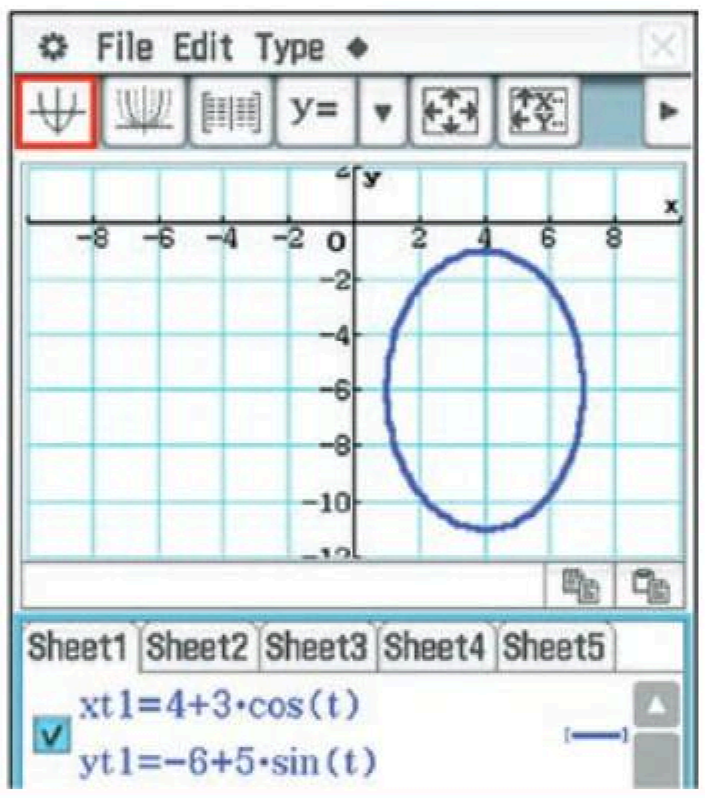
ClassPad

a



$x1(t) = 4 + 3\cdot\cos(t)$

$y1(t) = -6 + 5\cdot\sin(t)$



☑ xt1=4+3·cos(t)

yt1=−6+5·sin(t)

1 Add a **Graphs** page.

2 Press **menu > Graph Entry/Edit > Parametric**.

3 Enter the parametric equations as shown on the left.

4 Press **enter**.

5 Adjust the window settings to suit.

1 Open the **Graph&Table** application.

2 Tap **Type > ParamType**.

3 Enter the parametric equations as shown on the left.

4 Tap **Graph**.

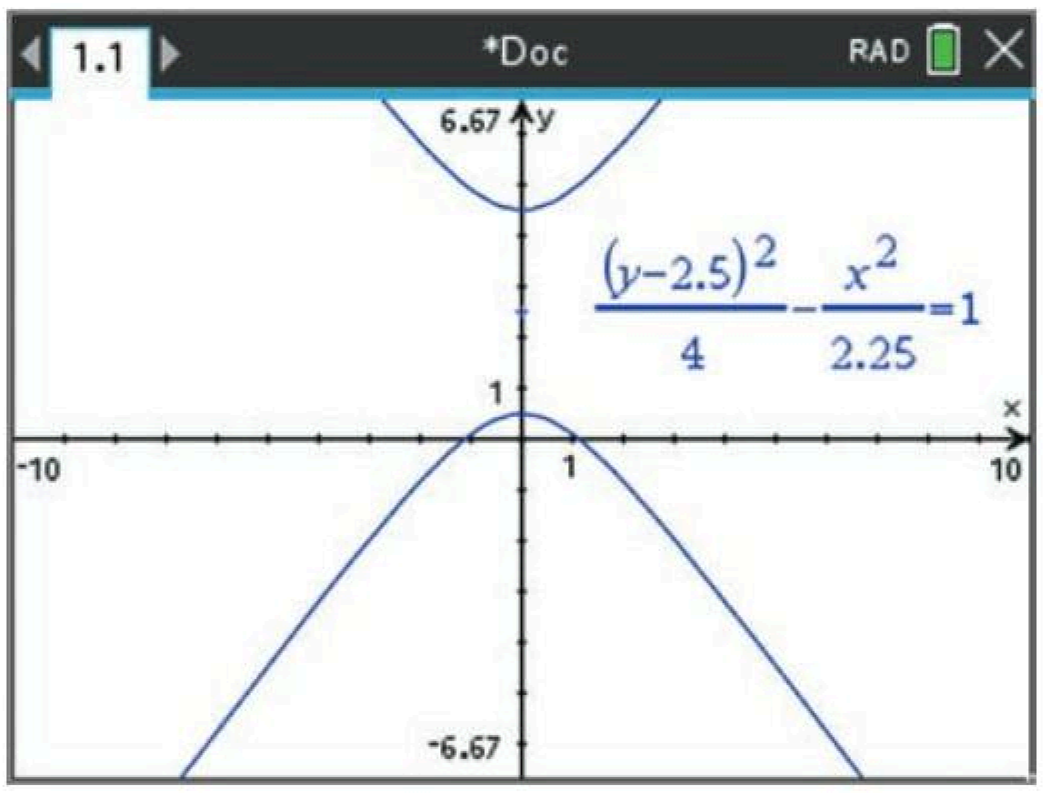
5 Adjust the window settings to suit.

Note: the windows have been swapped.

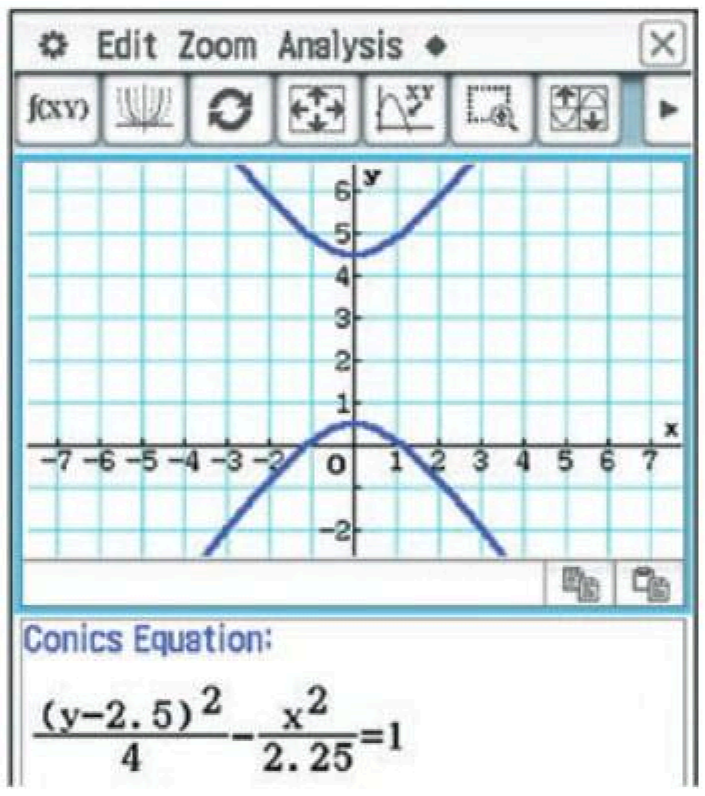
TI-Nspire

ClassPad

b



$\frac{(y-2.5)^2}{4}-\frac{x^2}{2.25}=1$



Conics Equation:

$\frac{(y-2.5)^2}{4}-\frac{x^2}{2.25}=1$

1 Delete the previous graph and reset the window settings to Standard.

2 Press **Graph Entry/Edit > Relation**.

3 Enter the relation as shown above.

4 Press **enter**.

1 Delete the previous graph.

2 Open the **Conics** application.

3 Enter the conic equation as shown above.

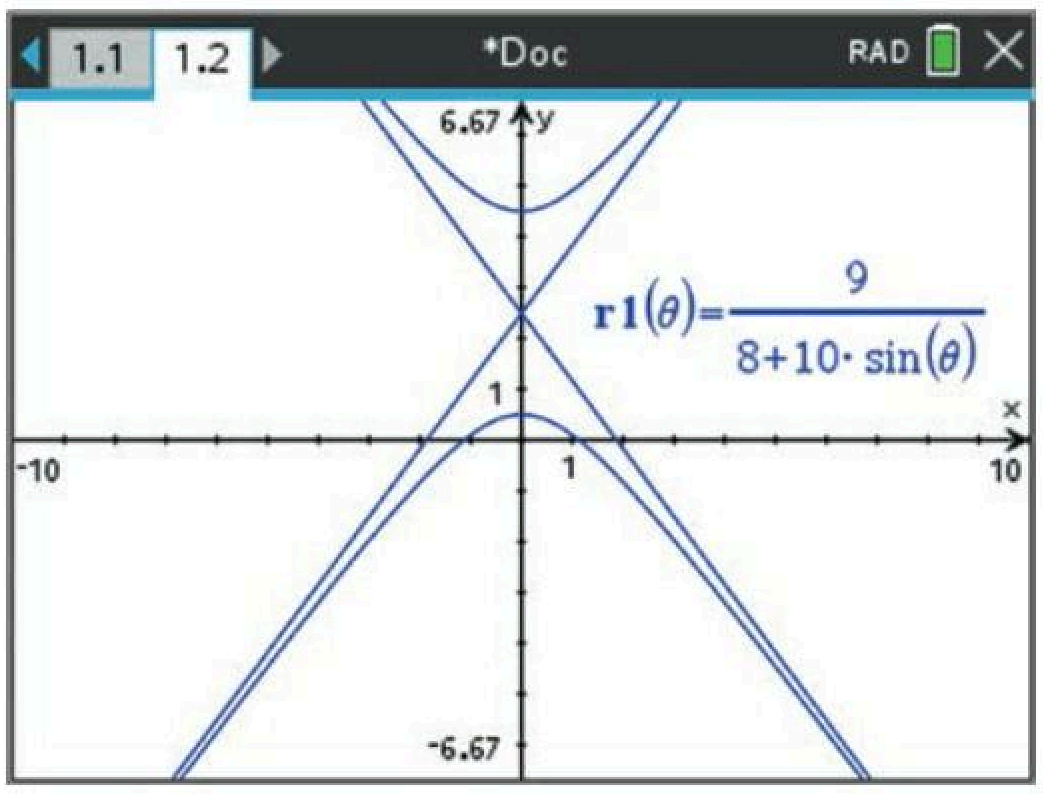
4 Tap **Graph**.

5 Adjust the window settings to suit.

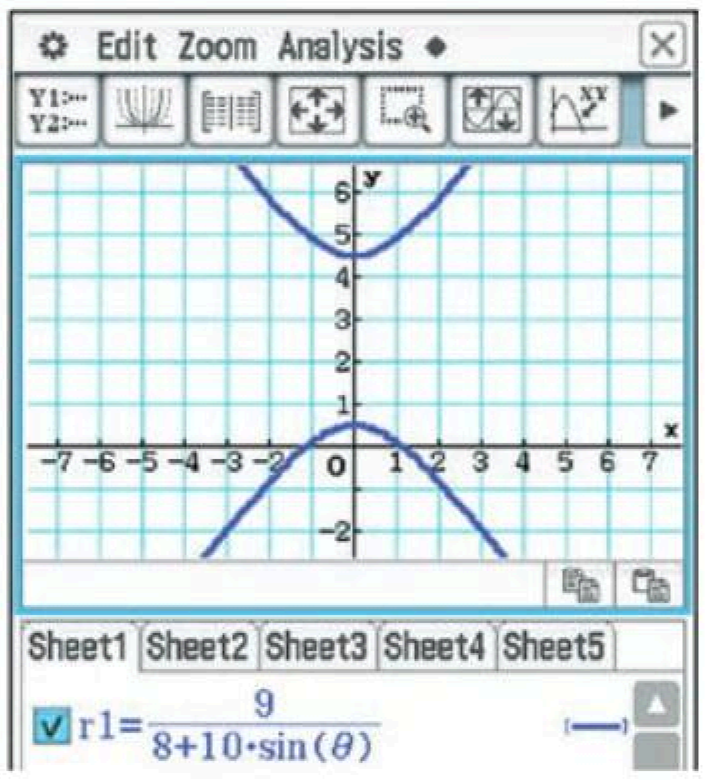
TI-Nspire

ClassPad

c



$r1(\theta)=\frac{9}{8+10\cdot\sin(\theta)}$



☑ $r1=\frac{9}{8+10\cdot\sin(\theta)}$

1 Add a new **Graphs** page.

2 Press **Graph Entry/Edit > Polar**.

3 Enter the polar equation as shown above.

4 Press **enter**.

Note: the asymptotes are included with the graph.

1 Open the **Graph&Table** application.

2 Tap **Type > r=Type**.

3 Enter the polar equation as shown above.

4 The window settings will be the same as the previous graph.

Compare the graphs of $\frac{(y-2.5)^2}{4}-\frac{x^2}{2.25}=1$ and $r=\frac{9}{8+10\sin(\theta)}$.

TI-Nspire users: switch between **1.1** to **1.2**.

ClassPad users: switch between **Conics** and **Graph&Table**, tapping **Graph** each time.

EXERCISE 2.1 The conic sections

ANSWERS p. 556

Mastery

1 WORKED EXAMPLE 1 Find the locus of each description and identify the shape.

a A point moves so that it is equidistant from the points (3, 6) and (7, –2).

b A point moves so that its distance from (0, 8) is three times its distance from (6, 4).

c A point moves so that the angle between the lines drawn from the point to (2, –5) and (6, 3) is always 90°.

2 WORKED EXAMPLE 2 A hyperbola has vertices at (2, 4) and (10, 4). One of its asymptotes has a slope of –0.75.

a Determine the Cartesian equation of the hyperbola.

b Find the number of intersections with the line $y = 0.5x + 5$.

3 An ellipse has a major axis four times as long as the minor axis. The top focus is at (2, 7) and the rightmost point is vertically above (7, –6).

a Find the Cartesian equation of the ellipse.

b Determine the number of intersections of the ellipse with the circle with equation $(x-3)^2 + (y+4)^2 = 64$.

4 WORKED EXAMPLE 3 Express each Cartesian equation in parametric form.

a $y = 12(x+5)^2 + 4$

b $(x+2)^2 + (y-6)^2 = 36$

c $\frac{(x-4)^2}{25} + \frac{(y+2)^2}{49} = 1$

d $\frac{(x+1)^2}{16} + y^2 = 1$

e $\frac{(x+5)^2}{32} - \frac{(y-4)^2}{50} = 1$

f $\frac{(y+2)^2}{400} - \frac{(x-6)^2}{256} = 1$

5 WORKED EXAMPLE 4 Express each pair of parametric equations in Cartesian form, where $0 \le t < 2\pi$ for parts **a** to **e**.

a $\begin{cases} x = 2 + 5\cos(t) \\ y = 3\sin(t) - 4 \end{cases}$

b $\begin{cases} x = 2\sec(t) - 1 \\ y = 3 + 4\tan(t) \end{cases}$

c $\begin{cases} x = 2 + 3\cot(t) \\ y = 5\operatorname{cosec}(t) \end{cases}$

d $\begin{cases} x = 6\cos(t) - 3 \\ y = 6\sin(t) + 7 \end{cases}$

e $\begin{cases} x = 3 + 4\cos(t) \\ y = 1 + 6\sin(t) \end{cases}$

f $\begin{cases} x = 3 - 0.25t^2 \\ y = 1 - 0.5t \end{cases}$

6 WORKED EXAMPLE 5 Express each Cartesian equation in polar form.

a $\frac{x^2}{36} + \frac{(y-8)^2}{100} = 1$

b $x = \frac{1}{8}y^2 - 2$

c $\frac{(y-5)^2}{16} - \frac{x^2}{9} = 1$

d $\frac{(x+3)^2}{16} + \frac{y^2}{7} = 1$

e $x^2 + (y-4)^2 = 16$

f $\frac{(x+11)^2}{25} - \frac{y^2}{96} = 1$

7 WORKED EXAMPLE 6 Express each polar equation in Cartesian form, where $0 \le \theta < 2\pi$.

a $r = \frac{13}{6 + 7\cos(\theta)}$

b $r = \frac{13}{7 + 6\sin(\theta)}$

c $r = \frac{6}{1 + \sin(\theta)}$

d $r = -10\cos(\theta)$

e $r = \frac{15}{3 + 2\sin(\theta)}$

f $r = \frac{15}{1 - 2\sin(\theta)}$

▶

Exam practice

8 © VCAA 2018 1Q9 **TECH-FREE** (3 marks) A curve is specified parametrically by

$$\underset{\sim}{r}(t) = \sec(t)\,\underset{\sim}{i} + \frac{\sqrt{2}}{2}\tan(t)\,\underset{\sim}{j},\ t \in R$$

a 80% Show that the Cartesian equation of the curve is $x^2 - 2y^2 = 1$. 2 marks

b 70% Find the x-coordinates of the points of intersection of the curve $x^2 - 2y^2 = 1$ and the line $y = x - 1$. 1 mark

9 © VCAA 2015 2AQ1 84% The ellipse $\frac{(x-2)^2}{9} + \frac{(y-3)^2}{4} = 1$ can be expressed in parametric form as

A $x = 2 + 3t$ and $y = 3 + 2\sqrt{1 + t^2}$

B $x = 2 + 3\sec(t)$ and $y = 3 + 2\tan(t)$

C $x = 2 + 9\cos(t)$ and $y = 3 + 4\sin(t)$

D $x = 3 + 2\cos(t)$ and $y = 2 + 3\sin(t)$

E $x = 2 + 3\cos(t)$ and $y = 3 + 2\sin(t)$

10 © VCAA 2016 2AQ1 61% The Cartesian equation of the relation given by $x = 3\,\text{cosec}^2(t)$ and $y = 4\cot(t) - 1$ is

A $\frac{(y+1)^2}{16} - \frac{x^2}{9} = 1$

B $(y+1)^2 = \frac{16(x+3)}{3}$

C $\frac{x^2}{9} + \frac{(y+1)^2}{16} = 1$

D $4x - 3y = 15$

E $(y+1)^2 = \frac{16(x-3)}{3}$

11 © VCAA 2016S 2AQ1 A circle with centre $(a, -2)$ and radius 5 units has equation $x^2 - 6x + y^2 + 4y = b$, where a and b are real constants. The values of a and b are respectively

A -3 and 38

B 3 and 12

C -3 and -8

D -3 and 0

E 3 and 18

12 © VCAA 2015 2AQ3 50% If both a and c are non-zero real numbers, the relation $a^2x^2 + (1 - a^2)y^2 = c^2$ **cannot** represent

A a circle.

B an ellipse.

C a hyperbola.

D a single straight line.

E a pair of straight lines.

13 © VCAA 2015 2AQ4 43% The two asymptotes of a particular hyperbola have gradients $\frac{2}{3}$ and $-\frac{2}{3}$ respectively and intersect at the point (2, 1). One branch of the hyperbola passes through the point (5, 5).

The equation of the hyperbola is

A $\frac{(x-2)^2}{4} - \frac{(y-1)^2}{9} = 1$

B $\frac{(x-2)^2}{4} - \frac{(y-1)^2}{9} = \frac{17}{36}$

C $\frac{(y-1)^2}{9} - \frac{(x-2)^2}{4} = \frac{17}{36}$

D $\frac{(y-1)^2}{4} - \frac{(x-2)^2}{9} = 3$

E $\frac{(x-2)^2}{9} - \frac{(y-1)^2}{4} = 3$

14 © VCAA 2019 2BQ1 (9 marks) A curve is defined parametrically by $x = \sec(t) + 1$, $y = \tan(t)$, where $t \in \left[0, \frac{\pi}{2}\right)$.

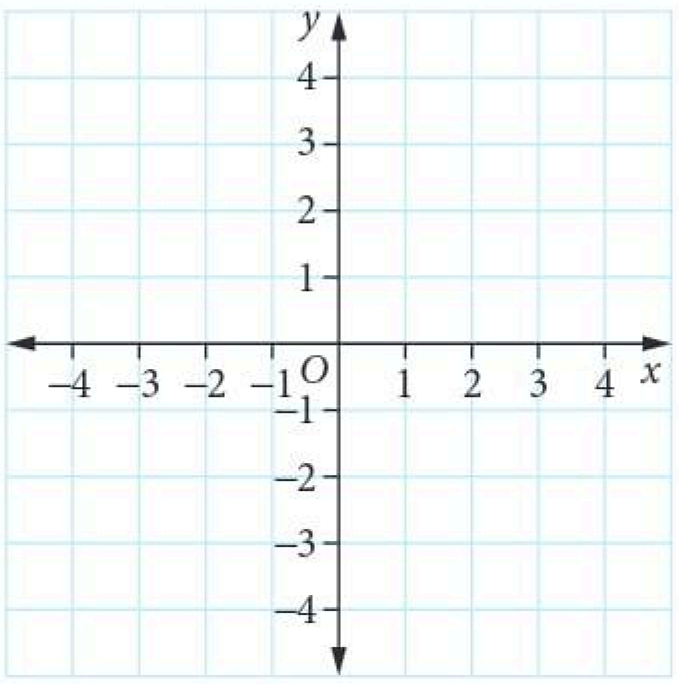
a 80% Show that the curve can be represented in Cartesian form by the rule $y = \sqrt{x^2 - 2x}$. 2 marks

b 59% State the domain and range of the relation given by $y = \sqrt{x^2 - 2x}$. 2 marks

c **i** 67% Express $\frac{dy}{dx}$ in terms of $\sin(t)$. 2 marks

ii 61% State the limiting value of $\frac{dy}{dx}$ as t approaches $\frac{\pi}{2}$. 1 mark

d 90% Copy the axes below and, in the graph, sketch the curve $y = \sqrt{x^2 - 2x}$ for $x \in [2, 4]$, labelling the endpoints with their coordinates. 2 marks



Video playlist
Rational functions and partial fractions

2.2 Rational functions and partial fractions

A **rational function** is of the form $f(x) = \frac{p(x)}{q(x)}$, where p and q are polynomials and q is not the zero polynomial. Although polynomial functions are continuous, rational functions have **discontinuities** where $q(x) = 0$. At these values, the rational function is not defined, so the function is **discontinuous**.

WB p. 22

WORKED EXAMPLE 7 Points of discontinuity

For each function, find any points of discontinuity.

a $f(x) = \dfrac{x+3}{(x+2)(x-3)}$ **b** $g(x) = \dfrac{2x^2 - 3x + 1}{x^2 - 1}$ **c** $h(x) = \dfrac{3x^2 - 5x - 2}{x^2 - 3x + 5}$

Steps	**Working**
a 1 Find when the denominator is zero.	$(x+2)(x-3) = 0$ at $x = -2$ or $x = 3$
2 State the points of discontinuity.	$f(x)$ is discontinuous at $x = -2$ and $x = 3$.
b 1 Factorise where possible.	$g(x) = \dfrac{2x^2 - 3x + 1}{x^2 - 1}$ $= \dfrac{(2x-1)(x-1)}{(x+1)(x-1)}$ ← Do not cancel $(x-1)$.
2 State the points of discontinuity.	$g(x)$ is discontinuous at $x = -1$ and $x = 1$.

c **1** Factorise where possible.	$h(x) = \dfrac{3x^2 - 5x - 2}{x^2 - 3x + 5} = \dfrac{(3x + 1)(x - 2)}{x^2 - 3x + 5}$
2 Check the discriminant of the denominator.	$\Delta = b^2 - 4ac = 9 - 20 = -11 < 0$
3 State the conclusion.	$x^2 - 3x + 5$ has no real roots, so there are no points of discontinuity.

In part **b** in Worked example 7, the common factor $(x - 1)$ cannot be cancelled if $(x - 1) = 0$ because dividing by 0 is undefined. It is better to leave it uncancelled to remind us that a discontinuity exists at $x = 1$.

Rational functions

A rational function $f(x) = \dfrac{p(x)}{q(x)}$ is classified according to the degrees of its numerator and denominator.

If **deg** $(\boldsymbol{p}) = \mathbf{0}$ (i.e. the numerator is a constant), the function is a **reciprocal function**.

If **deg** $(\boldsymbol{p}) <$ **deg** $(\boldsymbol{q})$, the function is a **simple rational function**.

If **deg** $(\boldsymbol{p}) \geq$ **deg** $(\boldsymbol{q})$, the function is a **quotient** function.

$f(x) = 0$ if and only if $p(x) = 0$ and $q(x) \neq 0$.

The roots of a rational function $f(x)$ are those of its numerator $p(x)$.
Since the numerator of a reciprocal function is a constant, it has no roots.

The vertical asymptotes of a rational function are the zeros of the denominator.

WORKED EXAMPLE 8 Roots of rational functions

WB
p. 23

Find the zeros of each function.

a $g(x) = \dfrac{2x^2 - 5x - 3}{x^2 + 5x + 4}$ **b** $h(x) = \dfrac{2x^3 + 2x^2 - 18x - 18}{3x^2 - 8x - 3}$

Steps	**Working**
a **1** Factorise where possible.	$\dfrac{2x^2 - 5x - 3}{x^2 + 5x + 4} = \dfrac{(2x + 1)(x - 3)}{(x + 1)(x + 4)}$
2 Find the roots of the numerator.	$g(x) = 0$ when $2x + 1 = 0$ or $x - 3 = 0$. $x = -\dfrac{1}{2}$ or $x = 3$
b **1** Factorise where possible.	$\dfrac{2x^3 + 2x^2 - 18x - 18}{3x^2 - 8x - 3} = \dfrac{2(x^3 + x^2 - 9x - 9)}{(3x + 1)(x - 3)}$ $= \dfrac{2[x^2(x + 1) - 9(x + 1)]}{(3x + 1)(x - 3)}$ $= \dfrac{2[(x + 1)(x^2 - 9)]}{(3x + 1)(x - 3)}$ $= \dfrac{2(x + 1)(x - 3)(x + 3)}{(3x + 1)(x - 3)}$
2 Find the roots.	$h(x) = 0$ when $x + 1 = 0$ or $x + 3 = 0$. $x = -3$ or $x = -1$

$x = 3$ is **not** a root because $\boldsymbol{f(x)}$ is undefined at $x = 3$.

If deg $(p) \geq$ deg (q), then divide the numerator by the denominator to obtain a quotient and a remainder.

If deg $(p) =$ deg (q), then the quotient is a constant, the result of dividing the leading terms.

WB
p. 24

WORKED EXAMPLE 9 Quotient and remainder where deg (p) = deg (q)

Find the quotient and remainder of $\dfrac{6x^2-2x+3}{2x^2+x-1}$.

Steps | **Working**

1 Divide the highest powers: $6x^2 \div 2x^2 = 3$.

$$\begin{array}{r} 3 \\ 2x^2+x-1\overline{\big)6x^2-2x+3} \end{array}$$

2 Multiply the quotient (3) by the divisor.

Subtract to get the remainder.

$2x^2$ doesn't go into $-5x$, so stop dividing.

$$\begin{array}{r} 3 \\ 2x^2+x-1\overline{\big)6x^2-2x+3} \\ \underline{6x^2+3x-3} \\ -5x+6 \end{array}$$

3 Write the answer.

The quotient is 3 and the remainder is $-5x+6$.

WB
p. 25

WORKED EXAMPLE 10 Quotient and remainder where deg (p) > deg (q)

Find the quotient and remainder of $\dfrac{5x^3+3x^2+2}{x+2}$.

Steps | **Working**

1 Set out as a long division, with all powers shown. Or by synthetic division:

$$\begin{array}{r|rrrr} & 5 & 3 & 0 & 2 \\ -2 & \downarrow & -10 & 14 & -28 \\ \hline & 5 & -7 & 14 & -26 \end{array}$$

$$\begin{array}{r} 5x^2-7x+14 \\ x+2\overline{\big)5x^3+3x^2+0x+2} \\ \underline{5x^3+10x^2} \\ -7x^2+0x \\ \underline{-7x^2-14x} \\ 14x+2 \\ \underline{14x+28} \\ -26 \end{array}$$

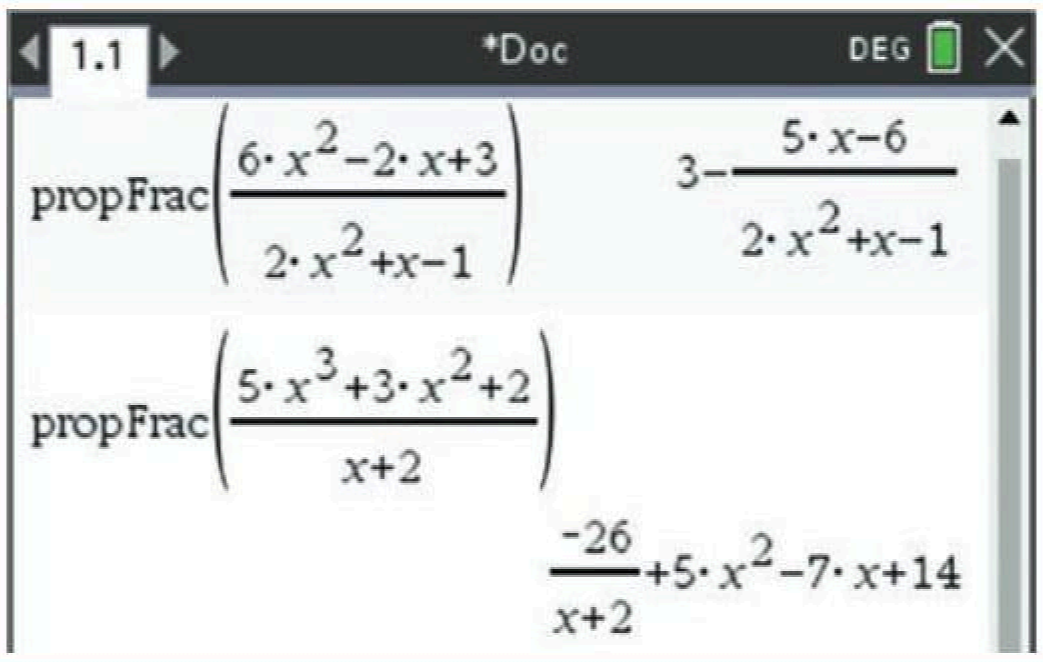
2 Write the answer.

The quotient is $5x^2-7x+14$ and the remainder is -26.

USING CAS 2 Quotients of rational functions

Verify the solutions to the 2 examples above by finding the quotients and remainders for $\dfrac{6x^2-2x+3}{2x^2+x-1}$ and $\dfrac{5x^3+3x^2+2}{x+2}$.

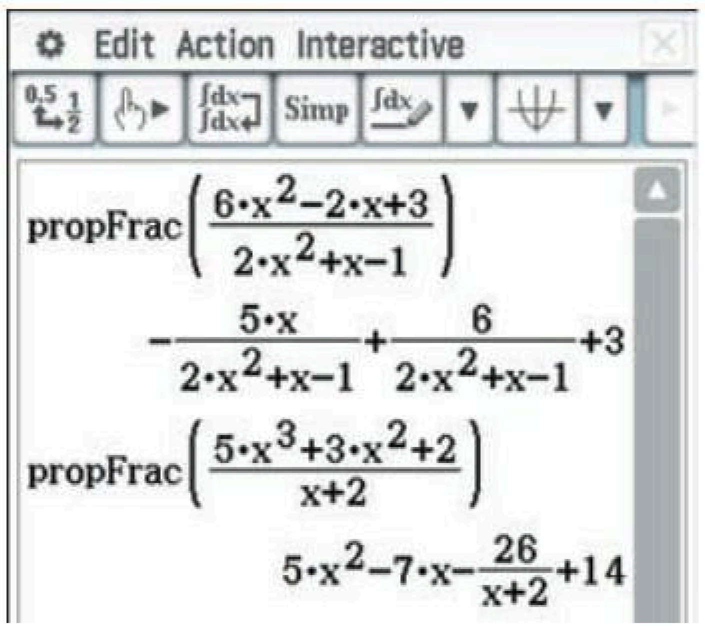
TI-Nspire



$\text{propFrac}\left(\dfrac{6\cdot x^2-2\cdot x+3}{2\cdot x^2+x-1}\right)$ $\quad 3-\dfrac{5\cdot x-6}{2\cdot x^2+x-1}$

$\text{propFrac}\left(\dfrac{5\cdot x^3+3\cdot x^2+2}{x+2}\right)$ $\quad \dfrac{-26}{x+2}+5\cdot x^2-7\cdot x+14$

ClassPad



$\text{propFrac}\left(\dfrac{6\cdot x^2-2\cdot x+3}{2\cdot x^2+x-1}\right)$

$-\dfrac{5\cdot x}{2\cdot x^2+x-1}+\dfrac{6}{2\cdot x^2+x-1}+3$

$\text{propFrac}\left(\dfrac{5\cdot x^3+3\cdot x^2+2}{x+2}\right)$

$5\cdot x^2-7\cdot x-\dfrac{26}{x+2}+14$

1 Press **menu > Number > Fraction Tools > Proper Fraction**.

2 Enter the first expression as shown.

3 Repeat for the second expression.

1 Enter and highlight the first expression.

2 Tap **Interactive > Transformation > Fraction > propFrac**.

3 Tap **OK**.

4 Repeat for the second expression.

For the first expression, notice that $2x^2 + x - 1 = (2x - 1)(x + 1)$. Separating the denominator of $\frac{-5x+6}{2x^2+x-1}$ into the smaller denominators $2x - 1$ and $x + 1$ is called expressing as **partial fractions**.
It is the reverse of changing to a common denominator. We learned how to decompose algebraic fractions into partial fractions in Year 11.

Partial fractions

- $\frac{ax+b}{(x+c)(x+d)}$ can be expressed in the form $\frac{A}{(x+c)} + \frac{B}{(x+d)}$.
- $\frac{ax+b}{(x+c)^2}$ can be expressed in the form $\frac{A}{(x+c)^2} + \frac{B}{(x+c)}$.
- $\frac{a}{(x+c)^2(x+d)}$ or $\frac{ax+b}{(x+c)^2(x+d)}$ can be expressed in the form $\frac{A}{(x+c)^2} + \frac{B}{(x+c)} + \frac{C}{(x+d)}$.
- $\frac{ax^2+bx+c}{(mx+n)(px^2+qx+r)}$, where $px^2 + qx + r$ is **irreducible** (cannot be factorised), can be expressed in the form $\frac{A}{mx+n} + \frac{Bx+C}{px^2+qx+r}$.

In some exam questions, you do not need to find the values of the denominators of partial fractions. However, you can find the values of A, B, C etc., by equating the coefficients of the denominators, substituting values of x or a combination of methods.

WORKED EXAMPLE 11 Expression as partial fractions

WB p. 26

Express each expression as partial fractions.

a $\frac{3x-1}{x^2-4x-5}$

b $\frac{10x^2-7}{(x+2)^2(x^2-2x+3)}$

Steps	**Working**
a 1 Factorise the denominator.	$\frac{3x-1}{x^2-4x-5} = \frac{3x-1}{(x-5)(x+1)}$
2 Decompose the denominator.	$= \frac{A}{x+1} + \frac{B}{x-5}$
3 Express with a common denominator.	$= \frac{A(x-5)+B(x+1)}{(x+1)(x-5)}$
4 Simplify and equate the denominators.	So $3x - 1 \equiv A(x-5) + B(x+1)$ $\equiv Ax - 5A + Bx + B$

	Method 1		Method 2	
5	Put $x = 5$.	$15 - 1 = A(0) + B(6)$	Equate coeffs.	$A + B = 3,\ -5A + B = -1$
6	Simplify.	$6B = 14$ $B = 2\frac{1}{3}$	Solve for A.	$(A + B) - (-5A + B) = 3 - (-1)$ $6A = 4$ $A = \frac{2}{3}$
7	Put $x = -1$.	$-3 - 1 = A(-6) + B(0)$	Substitute.	$\frac{2}{3} + B = 3$
8	Simplify.	$-6A = -4$ $A = \frac{2}{3}$	Solve for B.	$B = 2\frac{1}{3}$
9	Write the result.	$\frac{3x - 1}{x^2 - 4x - 5} = \frac{\frac{2}{3}}{x + 1} + \frac{2\frac{1}{3}}{x - 5}$ $= \frac{2}{3(x + 1)} + \frac{7}{3(x - 5)}$		

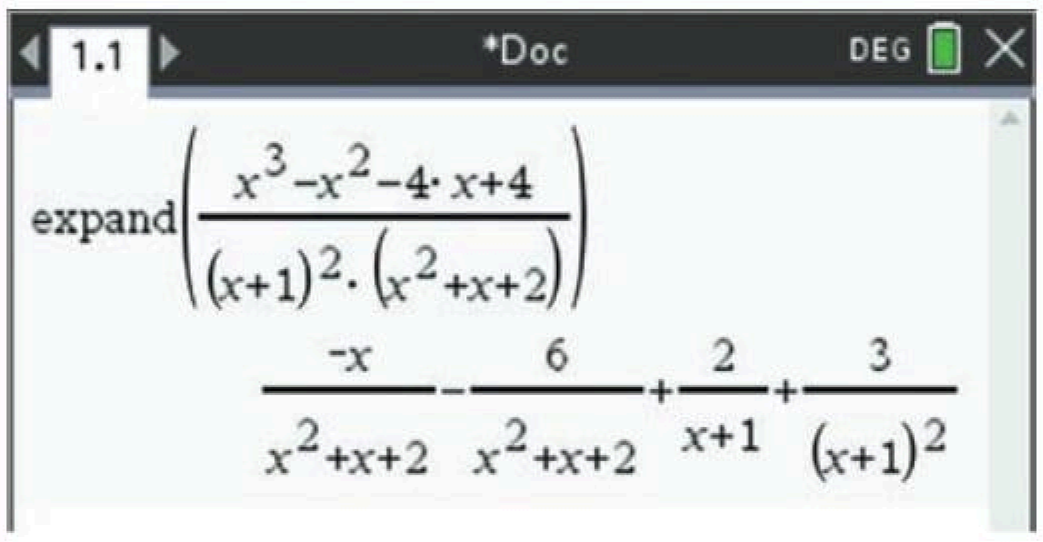
b 1	Check $x^2 - 2x + 3$.	$\Delta = (-2)^2 - 4 \times 1 \times 3 = -8 < 0$
2	Write the conclusion.	$x^2 - 2x + 3$ is an irreducible quadratic.
3	Decompose the denominator.	$\frac{10x^2 - 7}{(x + 2)^2(x^2 - 2x + 3)} = \frac{A}{(x + 2)^2} + \frac{B}{x + 2} + \frac{Cx + D}{x^2 - 2x + 3}$
4	Express with a common denominator.	$= \frac{A(x^2 - 2x + 3) + B(x + 2)(x^2 - 2x + 3) + (Cx + D)(x + 2)^2}{(x + 2)^2(x^2 - 2x + 3)}$
5	Equate numerators.	$10x^2 - 7 \equiv A(x^2 - 2x + 3) + B(x + 2)(x^2 - 2x + 3) + (Cx + D)(x + 2)^2$
6	Put $x = -2$.	$40 - 7 = A\ (4 + 4 + 3) + B \times 0 + (Cx + D) \times 0$
7	Solve for A.	$11A = 33$ $A = 3$
8	Simplify RHS denominator.	$10x^2 - 7 = 3x^2 - 6x + 9 + Bx^3 - Bx + 6B$ $+ Cx^3 + 4Cx^2 + 4Cx + Dx^2 + 4Dx + 4D$
9	Collect terms.	$7x^2 + 6x - 16$ $= x^3\ (B + C) + x^2\ (4C + D) + x(4C - B + 4D) + 6B + 4D$
10	Equate coefficients and solve.	$B + C = 0,\ 4C + D = 7,\ 4C - B + 4D = 6,\ 6B + 4D = -16$ Substitute $B = -C$: $4C + D = 7,\ 5C + 4D = 6,\ -6C + 4D = -16$ Substitute $D = 7 - 4C$: $5C + 28 - 16C = 6$ $11C = 22$, so $C = 2,\ B = -2,\ D = -1$
11	Write the result.	$\frac{10x^2 - 7}{(x + 2)^2(x^2 - 2x + 3)} = \frac{3}{(x + 2)^2} - \frac{2}{x + 2} + \frac{2x - 1}{x^2 - 2x + 3}$

If you have time in an exam, you should come back to check a partial fraction by finding the common denominator of your answer. Partial fractions can also be found using CAS.

USING CAS 3 Partial fractions

Express $\dfrac{x^3 - x^2 - 4x + 4}{(x+1)^2(x^2 + x + 2)}$ as partial fractions.

TI-Nspire



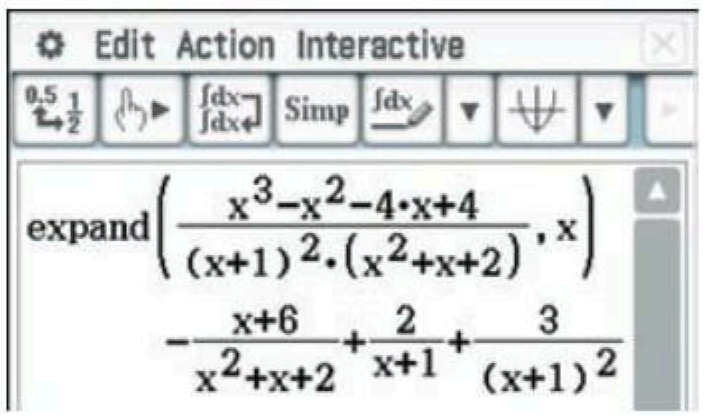
$$\text{expand}\left(\frac{x^3-x^2-4\cdot x+4}{(x+1)^2\cdot(x^2+x+2)}\right)$$

$$\frac{-x}{x^2+x+2}-\frac{6}{x^2+x+2}+\frac{2}{x+1}+\frac{3}{(x+1)^2}$$

1 Press **menu > Algebra > Expand**.

2 Enter the expression as shown above.

ClassPad



$$\text{expand}\left(\frac{x^3-x^2-4\cdot x+4}{(x+1)^2\cdot(x^2+x+2)},x\right)$$

$$-\frac{x+6}{x^2+x+2}+\frac{2}{x+1}+\frac{3}{(x+1)^2}$$

1 Enter and highlight the expression.

2 Tap **Interactive > Transformation > expand**.

3 In the dialogue box, tap **Partial Fraction**.

4 In the **Variable**: field, enter **x.**

5 Tap **OK.**

The solution expressed in partial fractions is $\dfrac{3}{(x+1)^2} + \dfrac{2}{x+1} - \dfrac{x+6}{x^2 + x + 2}$.

In some cases it is easiest to equate coefficients, and in others to substitute values. You may have a preference for one method over the other, but practise both because a combined method is often the most efficient.

EXERCISE 2.2 Rational functions and partial fractions

ANSWERS p. 557

Recap

1 The parametric form of $\dfrac{(x-3)^2}{25} + \dfrac{(y+5)^2}{49} = 4$ is

A $\begin{cases} x = 3 + 5\cos(t) \\ y = 7\sin(t) - 5 \end{cases}$

B $\begin{cases} x = 5\cos(t) - 3 \\ y = 7\sin(t) + 5 \end{cases}$

C $\begin{cases} x = 3 + 10\cos(t) \\ y = 14\sin(t) - 5 \end{cases}$

D $\begin{cases} x = 10\cos(t) - 3 \\ y = 14\sin(t) + 5 \end{cases}$

E $\begin{cases} x = 3 + 2.5\cos(t) \\ y = 3.5\sin(t) - 5 \end{cases}$

2 The Cartesian form of $r = \dfrac{12}{2 - 4\sin(\theta)}$ is

A $\dfrac{(y+4)^2}{4} - \dfrac{x^2}{12} = 1$

B $\dfrac{(x+4)^2}{4} - \dfrac{y^2}{12} = 1$

C $\dfrac{x^2}{12} + \dfrac{(y+4)^2}{4} = 1$

D $\dfrac{(x+4)^2}{4} + \dfrac{y^2}{12} = 1$

E $\dfrac{(y-4)^2}{4} - \dfrac{x^2}{12} = 1$

Mastery

3 WORKED EXAMPLE 7 Find any points of discontinuity for each function.

a $\dfrac{2x-5}{(x-1)(x+3)}$

b $\dfrac{x^3 - 3x^2 + 7}{x^2 - 5x + 6}$

c $\dfrac{x^2 - 7x + 10}{3x^2 + 11x + 6}$

d $\dfrac{2x^2 + x - 6}{x^3 + 4x^2 - 9x - 36}$

e $\dfrac{6x^3 - 6}{2x^3 - x^2 - 22x - 24}$

f $\dfrac{12x^2 + 4x - 40}{4x^3 - 32x^2 + 83x - 70}$

4 WORKED EXAMPLE 8 Find the zeros of each function.

a $\dfrac{2x-5}{(x-1)(x+3)}$

b $\dfrac{x^2-7x+10}{3x^2+11x+6}$

c $\dfrac{2x^2+x-6}{x^3+4x^2-9x-36}$

d $\dfrac{x^3-3x^2-7x+21}{x^2-5x+6}$

e $\dfrac{2x^3-5x^2-11x-4}{x^2+5x-6}$

f $\dfrac{6x^3-7x^2-7x+6}{2x^3-x^2-22x-24}$

5 WORKED EXAMPLES 9, 10 Find the quotient and remainder for each expression.

a $\dfrac{3x^2-2x+5}{x^2+x+4}$

b $\dfrac{5x-3}{2x+1}$

c $\dfrac{3x^2-x-6}{x-2}$

d $\dfrac{x^3-x^2+3x+1}{x+3}$

e $\dfrac{2x^3+3x^2-x+4}{x^2-1}$

f $\dfrac{x^3-2x^2+3x-1}{x^2+3x+2}$

6 Using CAS 2 Find the quotient and remainder for each expression.

a $\dfrac{3x^3-17x^2-2x}{3x-2}$

b $\dfrac{3x^4+14x^3-x^2+21x-2}{x+5}$

c $\dfrac{x^4+4x^3+11x^2+24x+27}{x^2+5}$

d $\dfrac{6x^5-24x^4-13x^3-8x^2+x+20}{x^2+5}$

e $\dfrac{7x^3-2x^4+3x^2+6x+13}{2x^2-x+2}$

f $\dfrac{x^5-x^4-4x^3-2x^2-2x-7}{x^3+2}$

7 WORKED EXAMPLE 11 Express each fraction as partial fractions with numerators involving *A*, *B*, *C*, etc. Do not solve for the values of the constants.

a $\dfrac{x^2+3x+4}{(x+1)(2x+1)(x+4)}$

b $\dfrac{5x-7}{6x^2+5x-6}$

c $\dfrac{3x^2-2x-3}{(2x+3)(3x^2-4x+5)}$

d $\dfrac{x^2+1}{(x-2)^2(3x-4)(4x+1)}$

e $\dfrac{3x^3+2x^2-3x-2}{(x+1)^2(2x+1)^2(3x+1)}$

f $\dfrac{2x^2+x-5}{(2x-1)^2(2x+1)(3x^2+3x+2)}$

8 Express as partial fractions.

a $\dfrac{3}{2x^2-7x+5}$

b $\dfrac{4x+5}{x^2+x-2}$

c $\dfrac{8x-11}{2x^2+5x-3}$

d $\dfrac{6x^2+15x+6}{(x-1)^2(2x+1)^2}$

e $\dfrac{2x^2+19x+29}{(x+3)^2(x-2)}$

f $\dfrac{2x+9}{(x^2+x+3)(x+2)}$

9 Using CAS 3 Express as partial fractions.

a $\dfrac{6x^2-12x+27}{(x+7)(x-2)^2}$

b $\dfrac{x^2-43x-62}{(x+2)(x+1)(x-5)}$

c $\dfrac{64-24x^2-37x}{(x+2)(x+4)(x^2-5x+7)}$

d $\dfrac{4x^2+41x-10}{(x^2-2x+8)(2x^2+3x+2)}$

e $\dfrac{9x^3-56x^2+97x-62}{(x+5)(x-3)^3}$

f $\dfrac{78x^4+367x^3+346x^2+642x+732}{(x-5)(3x+4)(5x^2-6x+4)}$

10 Express $\dfrac{x^4-3x^3+x+1}{(x+2)(x+1)}$ as a quotient and partial fractions.

Exam practice

80–100% 60–79% 0–59%

11 © VCAA 2014 1Q6a 86% TECH-FREE (1 mark)

Verify that $\frac{a}{a-4} = 1 + \frac{4}{a-4}$.

12 © VCAA 2016S 1Q10a TECH-FREE (1 mark)

Verify that $\frac{5x^3 + 12x + 4}{x^2(x^2 + 4)}$ can be written as $\frac{1}{x^2} + \frac{3}{x} + \frac{2x-1}{x^2+4}$.

13 The quotient of $2x^3 + x^2 - 3x + 2$ and $x^2 - 2$ is

A $2x + 1$ **B** $x + 4$ **C** 16 **D** $2x^2 + 5x + 7$ **E** $2x - 5$

14 The algebraic fraction $\frac{2x-4}{(x^2+2x+5)(x-2)^2}$ can be written in partial fraction form, where A, B, C, D and E are real numbers, as

A $\frac{A}{x^2+2x+5} + \frac{Bx+C}{(x-2)^2} + \frac{D}{x-2}$

B $\frac{Ax+B}{x^2+2x+5} + \frac{C}{(x-2)^2} + \frac{D}{x-2}$

C $\frac{Ax+B}{x^2+2x+5} + \frac{Cx+D}{(x-2)^2} + \frac{E}{x-2}$

D $\frac{A}{x^2+2} + \frac{B}{x+5} + \frac{C}{(x-2)^2} + \frac{D}{x-2}$

E $\frac{A}{(x^2+2x+5)^2} + \frac{B}{x^2+2x+5} + \frac{C}{(x-2)^2} + \frac{D}{x-2}$

15 © VCAA 2018 2AQ3 46% Which one of the following, where A, B, C and D are non-zero real numbers, is the partial fraction form for the expression $\frac{2x^2+3x+1}{(2x+1)^3(x^2-1)}$?

A $\frac{A}{2x+1} + \frac{B}{x-1} + \frac{C}{x+1}$

B $\frac{A}{2x+1} + \frac{B}{(2x+1)^2} + \frac{C}{(2x+1)^3} + \frac{Dx}{x^2-1}$

C $\frac{A}{2x+1} + \frac{Bx+C}{x^2-1}$

D $\frac{A}{2x+1} + \frac{B}{(2x+1)^2} + \frac{C}{x-1}$

E $\frac{A}{2x+1} + \frac{Bx+C}{(2x+1)^2} + \frac{D}{x-1}$

16 © VCAA 2020 2AQ7 26% For non-zero real constants a and b, where $b < 0$, the expression $\frac{1}{ax(x^2+b)}$ in partial fraction form with linear denominators, where A, B and C are real constants, is

A $\frac{A}{ax} + \frac{Bx+C}{x^2+b}$

B $\frac{A}{ax} + \frac{B}{x+\sqrt{b}} + \frac{C}{x-\sqrt{b}}$

C $\frac{A}{x} + \frac{B}{ax+\sqrt{|b|}} + \frac{C}{ax-\sqrt{|b|}}$

D $\frac{A}{x} + \frac{B}{x+\sqrt{|b|}} + \frac{C}{x-\sqrt{|b|}}$

E $\frac{A}{ax} + \frac{B}{\left(x+\sqrt{b}\right)^2} + \frac{C}{x+\sqrt{b}}$

17 Express $\frac{9x-20}{(x-3)^2(2x+1)}$ as partial fractions.

Video playlist Graphing rational functions

2.3 Graphing rational functions

The relationships between a function and its reciprocal (and their graphs) follow from simple considerations, such as 'as $y \to 0$, $\frac{1}{y} \to \pm\infty$' and 'as $y \to \pm\infty$, $\frac{1}{y} \to 0$'.

Reciprocal polynomial functions

For any polynomial function $p(x)$ and its reciprocal function $f(x) = \frac{1}{p(x)}$,

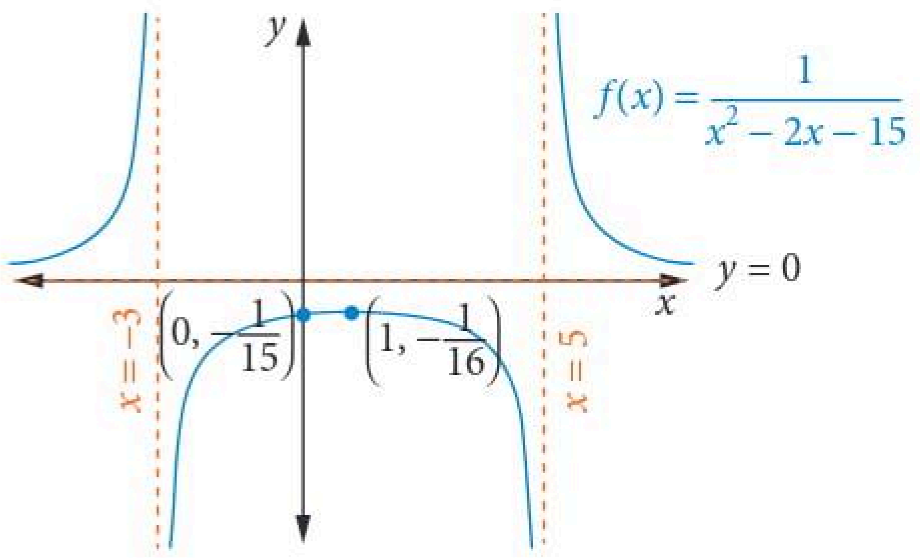
- the signs of $p(x)$ and $f(x)$ are the same
- the zeros of $p(x)$ correspond to the **vertical asymptotes** of $f(x)$
- as $x \to \pm\infty$, $p(x) \to \pm\infty$ so $f(x) \to 0$, so the x-axis ($y = 0$) is a **horizontal asymptote**
- the maxima and minima of $f(x)$ correspond to the minima and maxima of $p(x)$
- the y-intercept of $f(x)$ is the reciprocal of the y-intercept of $p(x)$
- for $p(x)$ concave up, $f(x)$ is concave down. For $p(x)$ concave down, $f(x)$ is concave up
- the domain of $f(x)$ excludes the zeros of $p(x)$.

When graphing a reciprocal function, find the vertical asymptotes by finding the zeros of the denominator. Find the y-intercept and the signs of the function between the asymptotes. Remember that reciprocal functions have a horizontal asymptote at the x-axis.

WB p. 28

WORKED EXAMPLE 12 Graphing reciprocal polynomial functions

Sketch the graph of $f(x) = \frac{1}{x^2 - 2x - 15}$.

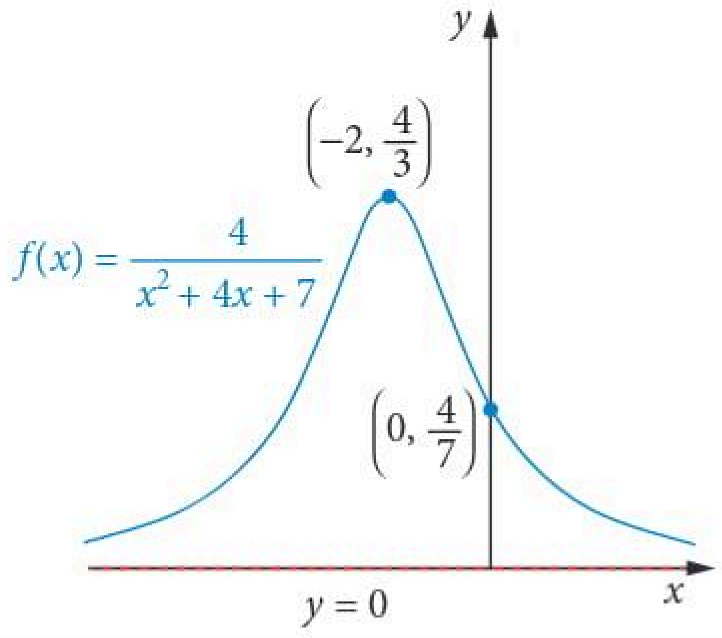
Steps	**Working**
1 Factorise $p(x) = x^2 - 2x - 15$.	$x^2 - 2x - 15 = (x - 5)(x + 3)$
2 State the important points of $p(x)$.	$p(x)$ has zeros at $x = -3$ and $x = 5$. $p(0) = -15$ Concave up as leading coefficient is positive so by quadratic symmetry, $p(x)$ has a local minimum at $(1, -16)$.
3 State the signs of $p(x)$.	For $x < -3$ or $x > 5$, $p(x) > 0$. For $-3 < x < 5$, $p(x) < 0$.
4 Use $p(x)$ to identify characteristics of $f(x)$.	$f(x)$ has vertical asymptotes at $x = -3$ and $x = 5$. $f(0) = -\frac{1}{15}$ and $f(x)$ has a local maximum at $\left(1, -\frac{1}{16}\right)$. As $x \to \pm\infty$, $p(x) \to \pm\infty$, so $f(x) \to 0$. $f(x)$ has a horizontal asymptote at $y = 0$ (the x-axis). For $x < -3$ or $x > 5$, $f(x) > 0$. For $-3 < x < 5$, $f(x) < 0$.
5 Sketch the graph, labelling all asymptotes and intercepts with their coordinates. Label the maximum.	

A reciprocal function of the form $\frac{k}{p(x)}$ is a **dilation** of the reciprocal function $\frac{1}{p(x)}$ from the x-axis by the factor k.

WORKED EXAMPLE 13 Graphing a multiple of a reciprocal polynomial function

WB p. 29

Sketch the graph of $f(x) = \frac{4}{x^2+4x+7}$.

Steps	Working
1 Consider $p(x) = x^2 + 4x + 7$.	For $p(x) = x^2 + 4x + 7$, $\Delta = -12 < 0$, so there are no real zeros, so $f(x)$ has no vertical asymptotes.
2 Complete the square for $p(x)$.	$p(x) = x^2 + 4x + 4 + 3$ $= (x+2)^2 + 3$ Therefore, $p(x)$ has a minimum at $(-2, 3)$. $f(x)$ has a maximum at $x = -2$, $f(-2) = \frac{4}{3}$.
3 State the behaviour as $x \to \pm\infty$.	As $x \to -\infty$, $p(x) \to \infty$ so $f(x) \to 0$ from above. As $x \to \infty$, $p(x) \to \infty$ so $f(x) \to 0$ from above.
4 Find the y-intercept.	$f(0) = \frac{4}{7}$
5 Sketch the graph, labelling all asymptotes and intercepts with their coordinates. Label the minimum and maximum.	

In a simple rational function, the degree of the numerator is less than the degree of the denominator. Its graph has a horizontal asymptote of $y = 0$ (the x-axis).

We can sketch a simple rational function by considering the signs of the function between the zeros and vertical asymptotes. Remember to find all intercepts and label their coordinates.

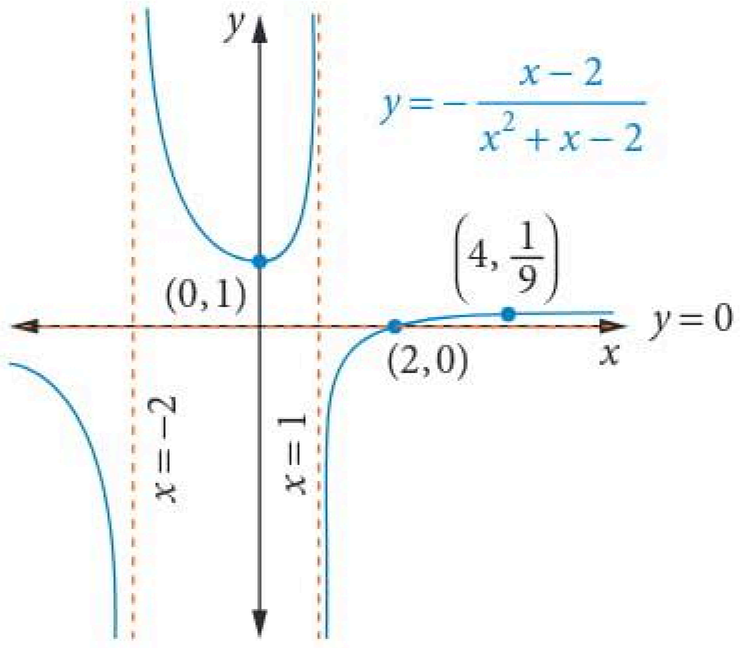
Solve $f'(x) = 0$ to check for stationary points.

WORKED EXAMPLE 14 Graphing a rational function with a linear numerator

WB p. 30

Sketch the graph of $f(x) = \frac{x-2}{x^2+x-2}$.

Steps	Working
1 Consider $p(x) = x^2 + x - 2$.	$p(x) = x^2 + x - 2 = (x-1)(x+2)$, so $f(x)$ has vertical asymptotes $x = 1$ and $x = -2$.
2 Find any zeros.	$f(2) = 0$, so there is an x-intercept at $(2, 0)$.
3 Find the y-intercept.	$f(0) = 1$, so the y-intercept is $(0, 1)$.

4 Differentiate.	$f'(x) = \dfrac{x^2 + x - 2 - 2x^2 + 3x + 2}{(x^2 + x - 2)^2}$ $= \dfrac{-x^2 + 4x}{(x-1)^2(x+2)^2}$ $= \dfrac{-x(x-4)}{(x-1)^2(x+2)^2}$
5 Find any stationary points.	$f'(x) = 0$ at $x = 0, 4$. $f'(-1) < 0, f'(0.5) > 0$, so there is a minimum at $x = 0$. $f'(3) > 0, f'(5) < 0$, so there is a maximum at $x = 4$. The stationary points are a local minimum at (0, 1) and a local maximum at $\left(4, \dfrac{1}{9}\right)$.
6 State the behaviour as $x \to \pm\infty$.	As $x \to \pm\infty$, $y \to 0$, so there is a horizontal asymptote at $y = 0$.
7 State the signs of $f(x)$ between the asymptotes and zeros.	For $x < -2, f(x) < 0$. For $-2 < x < 1, f(x) > 0$. For $1 < x < 2, f(x) < 0$. For $x > 2, f(x) > 0$.
8 Sketch the graph, labelling all asymptotes and intercepts with their coordinates. Label the maximum.	

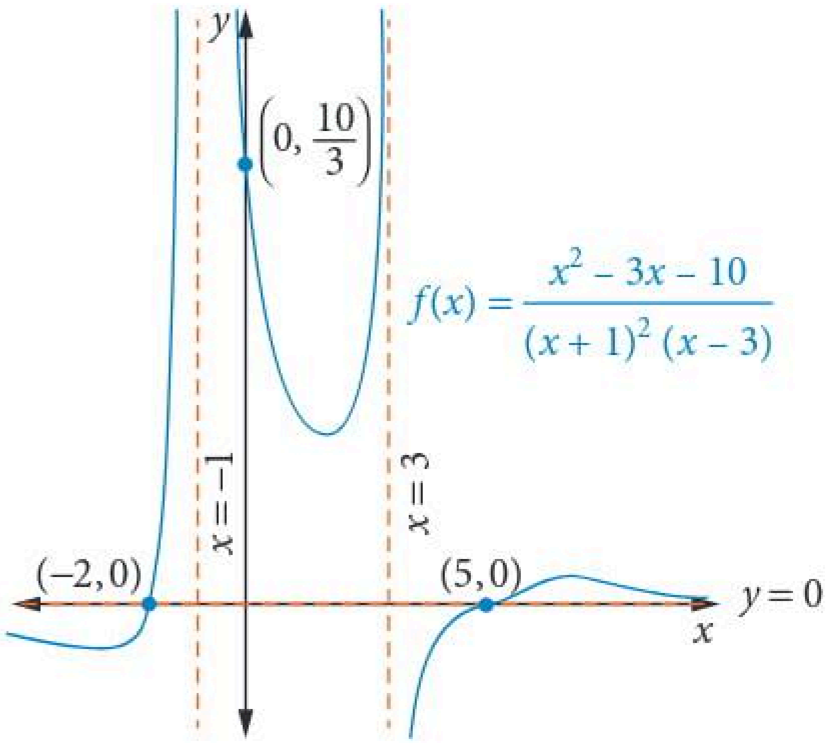
In some cases, the derivative does not help us to find the maxima and minima easily, so we just indicate where they are approximately by the shape of the curve.

WB
p. 31

WORKED EXAMPLE 15 Graphing a rational function with a quadratic numerator

Sketch the graph of $f(x) = \dfrac{x^2 - 3x - 10}{(x+1)^2(x-3)}$.

Steps	**Working**
1 Factorise.	$f(x) = \dfrac{x^2 - 3x - 10}{(x+1)^2(x-3)} = \dfrac{(x-5)(x+2)}{(x+1)^2(x-3)}$
2 State the zeros and vertical asymptotes.	There are zeros at $x = -2$ and $x = 5$ and vertical asymptotes at $x = -1$ and $x = 3$.
3 State the signs of the function.	For $x < -2, f(x) < 0$. For $-2 < x < -1, f(x) > 0$. For $-1 < x < 3, f(x) > 0$. For $3 < x < 5, f(x) < 0$. For $x > 5, f(x) > 0$.

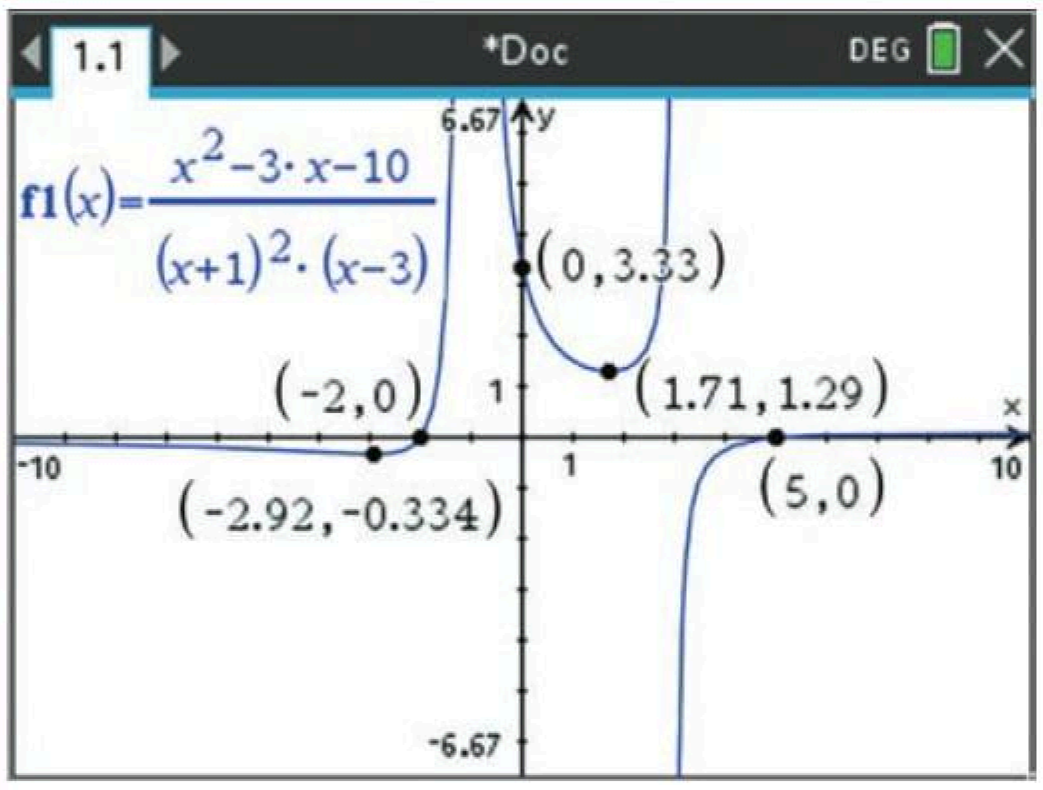
4 State the behaviour as $x \to \pm\infty$.	As $x \to -\infty, f(x) \to 0$ from below. As $x \to \infty, f(x) \to 0$ from above. $y = 0$ is a horizontal asymptote.
5 Find the y-intercept.	$f(0) = \frac{10}{3}$
6 Sketch the graph, labelling all asymptotes and intercepts with their coordinates.	

We can use CAS to graph rational functions, such as $f(x)$ from the above example.

USING CAS 4 Rational functions

Graph $f(x) = \frac{x^2 - 3x - 10}{(x+1)^2(x-3)}$.

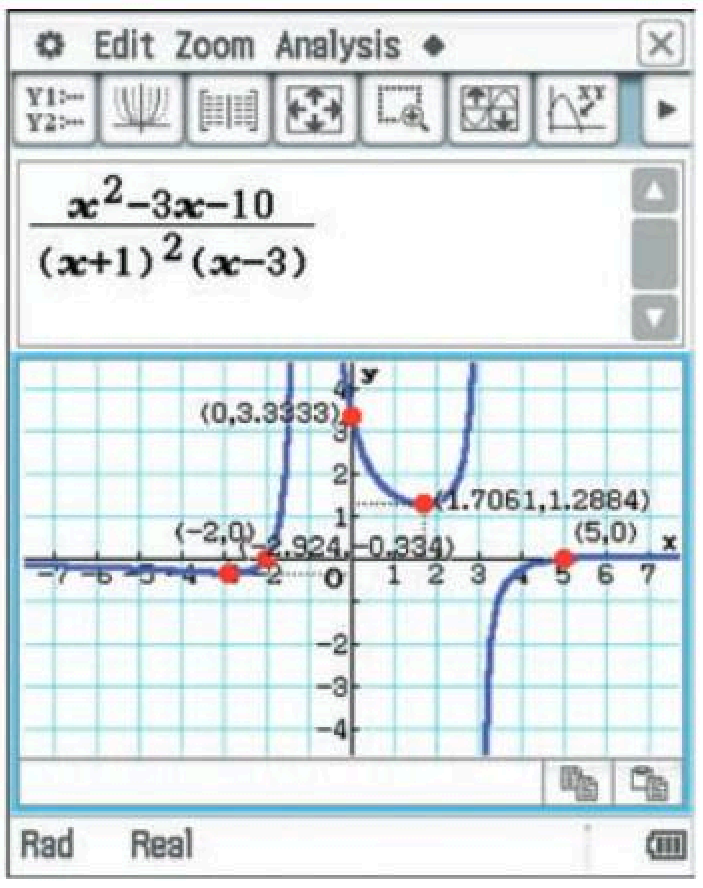
TI-Nspire



1 Add a **Graphs** page and enter the function.

2 Press **menu > Trace > Graph Trace** or press **menu > Analyse Graph** to find and label the key features of the function.

ClassPad



1 In **Main**, enter and highlight the expression.

2 Tap the **graph** icon and drag the expression down into the graph window.

3 Tap **Analysis > G-Solve** to find and label the key features of the function.

4 Press **enter** each time to label the coordinates.

EXERCISE 2.3 Graphing rational functions

ANSWERS p. 557

Recap

1 The function $f: R \to R, f(x) = \dfrac{2x^2 + 5x - 3}{(x-1)(x^2 - 2x - 8)}$ has

A zeros at $x = -\dfrac{1}{2}, 3$ and discontinuities at $x = -2, 1, 4$.

B zeros at $x = \dfrac{1}{2}, -3$ and only one discontinuity, at $x = 1$.

C zeros at $x = -\dfrac{1}{2}, 3$ and discontinuities at $x = 2, 1, -4$.

D zeros at $x = \dfrac{1}{2}, -3$ and discontinuities at $x = 1, 2\sqrt{2}$.

E zeros at $-x = -\dfrac{1}{2}, 3$ and discontinuities at $x = -2, 1, 4$.

2 As partial fractions, $\dfrac{3x-2}{(x+1)^2(x+3)}$ equals

A $\dfrac{11}{2(x+3)} - \dfrac{5}{2(x+1)}$

B $\dfrac{11}{2(x+3)} - \dfrac{5}{2(x+1)^2}$

C $\dfrac{11}{4(x+1)} - \dfrac{5}{2(x+1)^2} - \dfrac{11}{4(x+3)}$

D $\dfrac{1}{7(x+1)} - \dfrac{6}{7(x+1)^2} - \dfrac{6}{7(x+3)}$

E $\dfrac{11}{4(x+3)} + \dfrac{5}{2(x+1)^2} - \dfrac{11}{4(x+1)}$

Mastery

3 WORKED EXAMPLE 12 Sketch the graph of each function.

a $f(x) = \dfrac{1}{x-2}$

b $f(x) = \dfrac{1}{(x+3)^2}$

c $f(x) = \dfrac{1}{x^2 - 2x + 5}$

d $f(x) = \dfrac{1}{3x + 5 - 2x^2}$

e $f(x) = \dfrac{1}{(x+3)^2(x+1)}$

f $f(x) = \dfrac{1}{(x-4)(x+1)(x+3)}$

4 WORKED EXAMPLE 13 Sketch the graph of each function.

a $f(x) = \dfrac{3}{x^2 - 1}$

b $f(x) = \dfrac{6}{9 - x^2}$

c $f(x) = \dfrac{8}{x^2 - 8x + 12}$

d $f(x) = \dfrac{6}{x^2 - 6x + 10}$

e $f(x) = \dfrac{24}{(x+1)(x+4)(x-3)}$

f $f(x) = \dfrac{8}{(x-2)^2(x+2)(x+1)}$

5 WORKED EXAMPLE 14 Sketch the graph of each function, including any maxima or minima.

a $f(x) = \dfrac{2x-5}{x^2-4}$

b $f(x) = \dfrac{3x-4}{(x-2)(x-4)}$

c $f(x) = \dfrac{x-4}{(x+5)(x-3)}$

d $f(x) = \dfrac{x+5}{x^2-9}$

e $f(x) = \dfrac{x-1}{(2x-3)(x-3)}$

f $f(x) = \dfrac{x+3}{x^2-4}$

6 WORKED EXAMPLE 15 Sketch the graph of each function, showing all important features.

a $f(x) = \dfrac{x-3}{2x-x^2+8}$ **b** $f(x) = \dfrac{x+2}{x^2+2x-15}$ **c** $f(x) = \dfrac{(x+1)(x-2)}{(x-1)(x+2)(x-3)}$

d $f(x) = \dfrac{x^2-9}{(x+2)(x-4)^2}$ **e** $f(x) = \dfrac{x^2-4}{(3x-x^2+28)(x+1)}$ **f** $f(x) = \dfrac{x^2+x-20}{(x^2-9)(x-5)}$

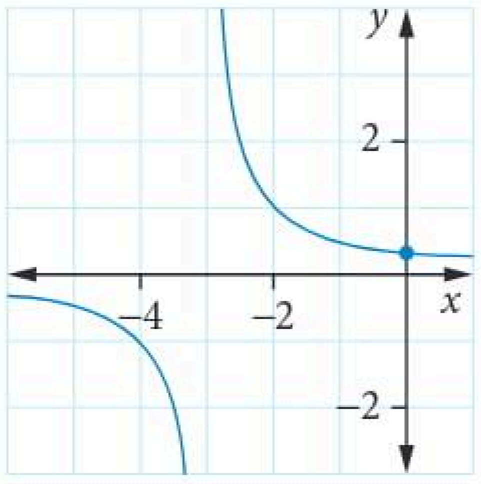
7 Using CAS 4 Sketch the graph of each function.

a $f(x) = \dfrac{1}{(x+1)(x-3)}$ **b** $f(x) = \dfrac{3x-6}{(x-4)(x+2)}$ **c** $f(x) = \dfrac{1}{x^2-6x+12}$

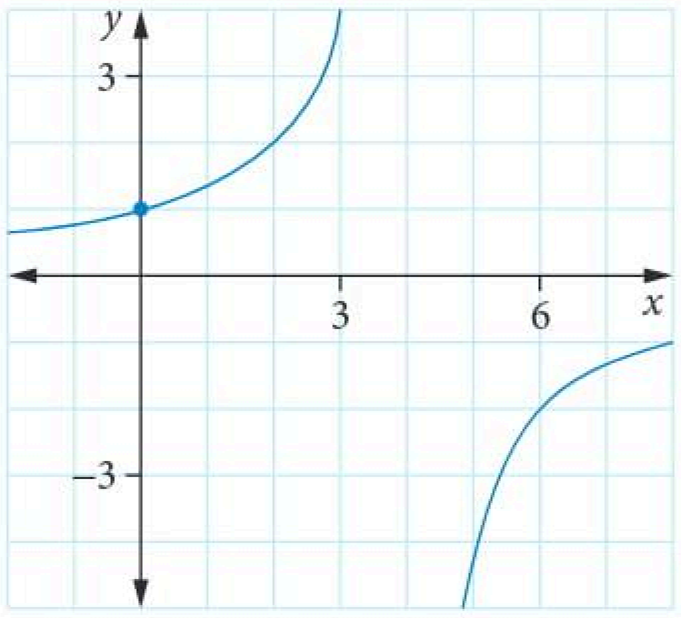
d $f(x) = \dfrac{x+5}{(x-2)^2(x+4)}$ **e** $f(x) = \dfrac{2x-3}{(2x-5)(x^2-8x+7)}$ **f** $f(x) = \dfrac{x-3}{2x^2-6x+5}$

8 State a possible rule for each function graphed.

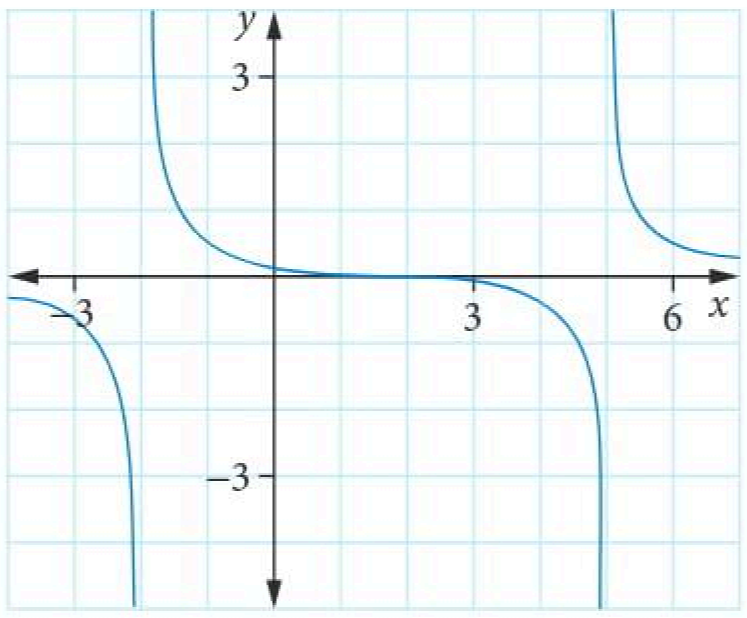
a



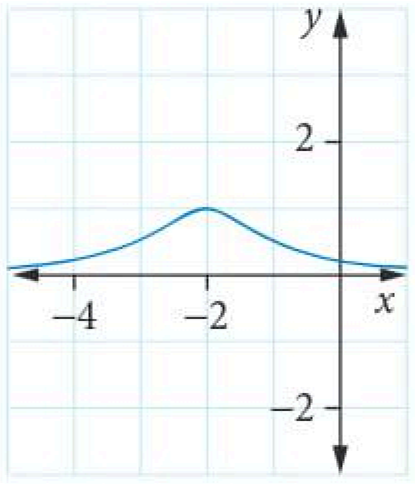
b



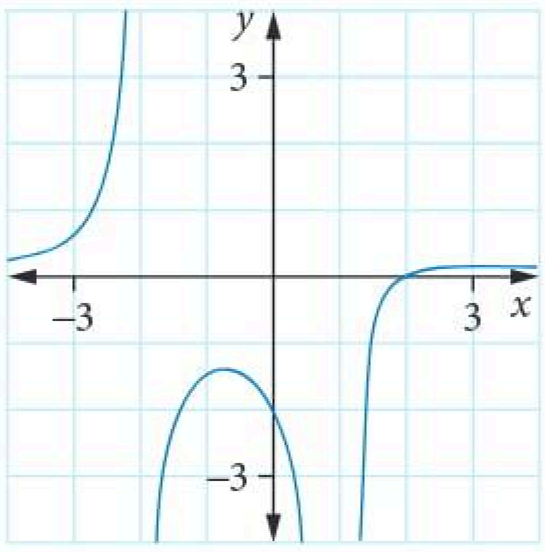
c



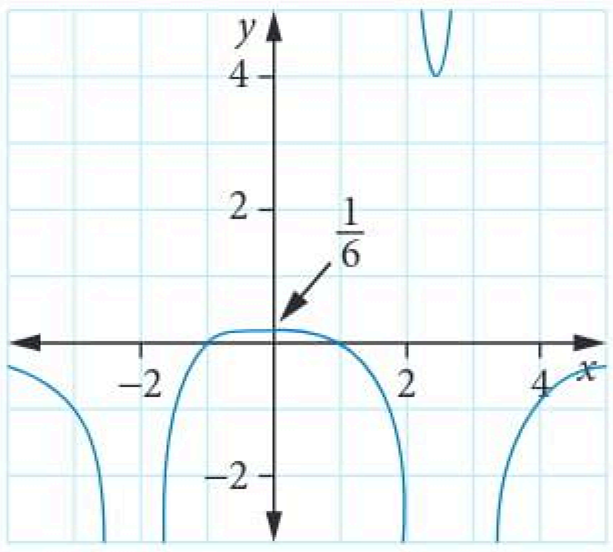
d



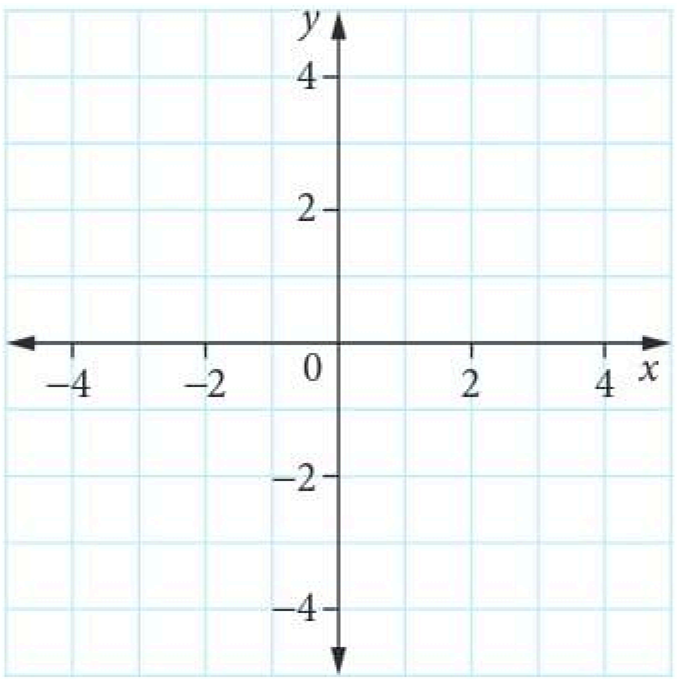
e



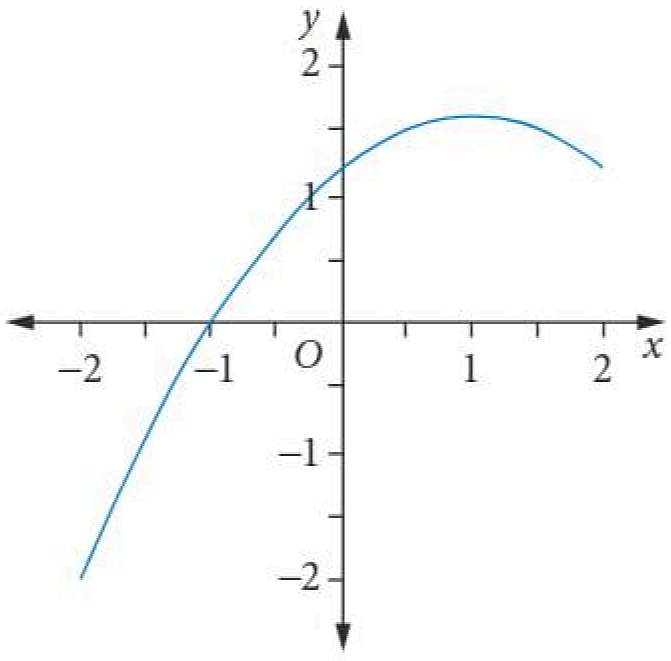
f



9 © VCAA 2018 1Q5 53% TECH-FREE (4 marks) Copy the axes below, and on them sketch the graph of $f(x) = \dfrac{x+1}{x^2-4}$, labelling any asymptotes with their equations and any intercepts with their coordinates.



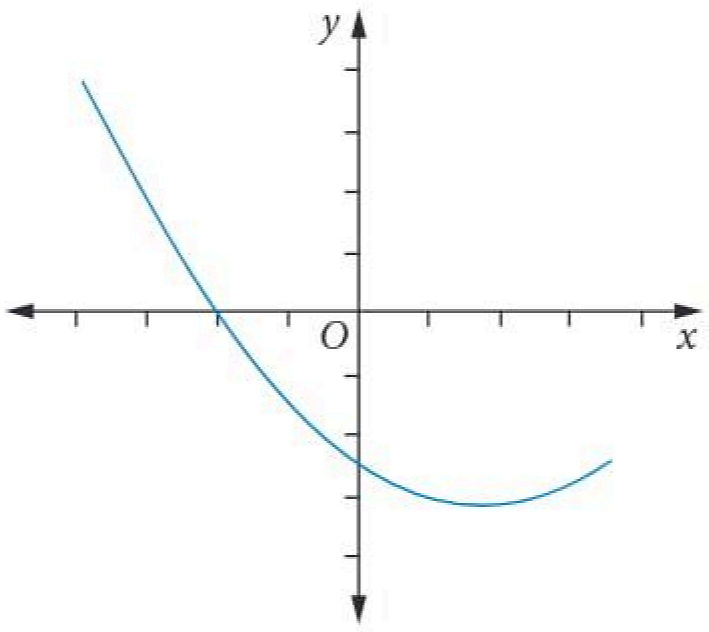
10 © VCAA 2012 2AQ3 66% The graph of $y = f(x)$ is shown below.



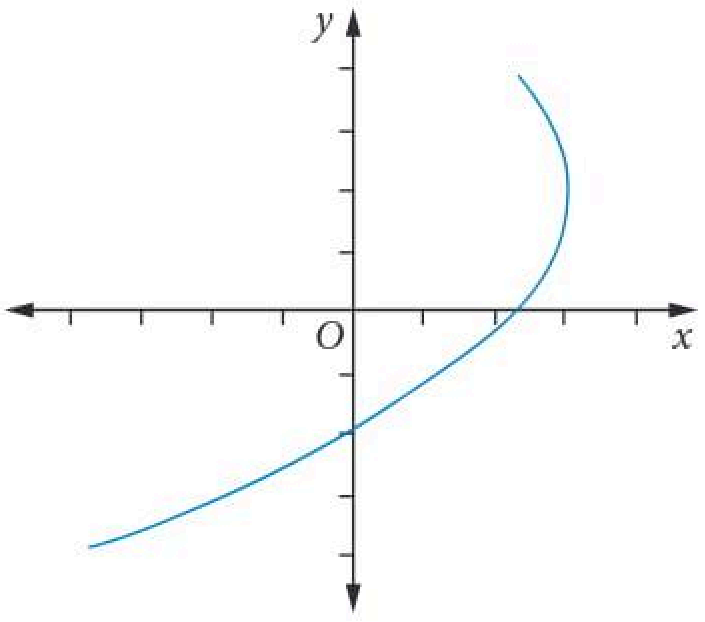
All of the following axes have the same scale as the axes in the diagram above.

The graph of $y = \dfrac{1}{f(x)}$ is best represented by

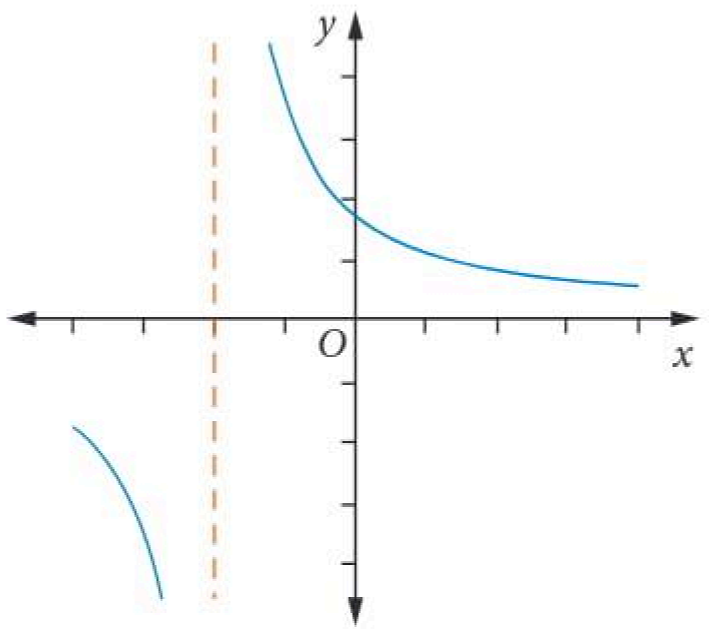
A



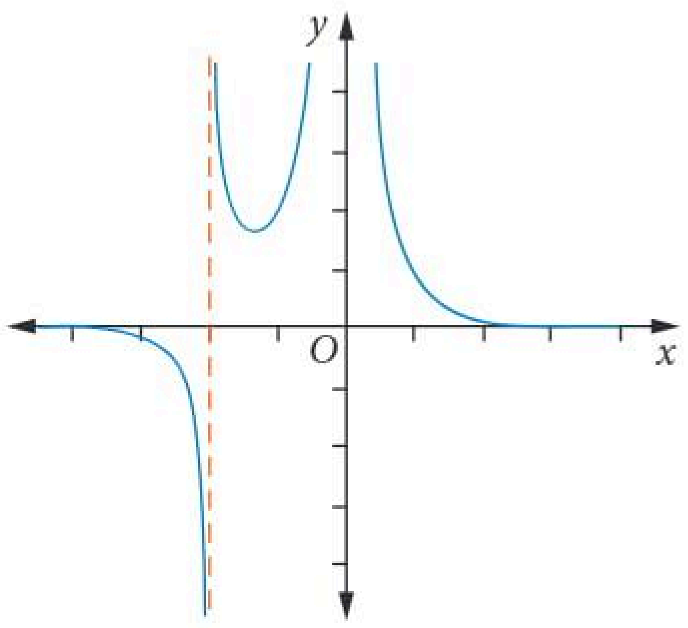
B



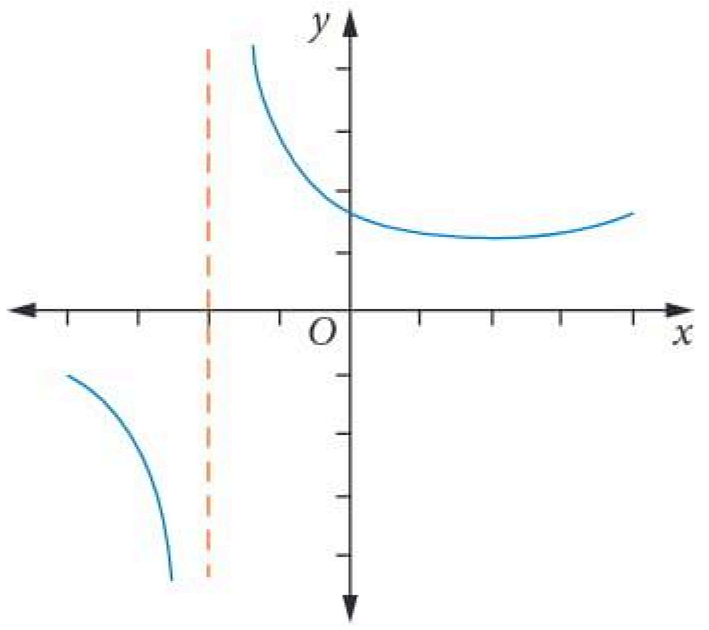
C



D



E



11 © VCAA 2013 2AQ3 47% The graph of $y = \dfrac{1}{ax^2 + bx + c}$ has asymptotes at $x = -5$, $x = 3$ and $y = 0$.

Given that the graph has one stationary point with a y-coordinate of $-\dfrac{1}{8}$, it follows that

A $a = 1, b = 2, c = -15$

B $a = \dfrac{1}{2}, b = -1, c = -\dfrac{15}{2}$

C $a = -\dfrac{1}{2}, b = -1, c = 15$

D $a = -1, b = -2, c = -15$

E $a = \dfrac{1}{2}, b = 1, c = -\dfrac{15}{2}$

2.4 Graphing quotient functions

Video playlist Graphing quotient functions

In Year 11, we learned that a quotient function with numerator and denominator of the same degree has the form

$$f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots}{b_n x^n + b_{n-1} x^{n-1} + \ldots}$$

As $x \to \pm\infty$, the x^n terms will dominate, so $f(x) \to \dfrac{a_n}{b_n} = c$, a constant.

A quotient function whose numerator and denominator have the same degree has a horizontal asymptote $y = c$, where c is the quotient of the leading coefficients.

WB
p. 32

WORKED EXAMPLE 16 Quotient function graph with a horizontal asymptote

Sketch the graph of $f(x) = \dfrac{2x^2 + 6x + 4}{x^2 - 2x + 1}$.

Steps / **Working**

1 Factorise.

$$f(x) = \frac{2(x+2)(x+1)}{(x-1)^2}$$

2 State the zeros and vertical asymptotes.

There are zeros at $x = -2$ and $x = -1$ and a vertical asymptote at $x = 1$.

3 Find the y-intercept.

$f(0) = 4$

4 Find the horizontal asymptote by dividing the leading coefficients.

$2x^2 \div x^2 = 2$, so the horizontal asymptote is $y = 2$.

5 Differentiate and simplify to find stationary points.

$$\begin{aligned} f'(x) &= \frac{(4x+6)(x-1)^2 - (2x^2+6x+4) \times 2(x-1)}{(x-1)^4} \\ &= \frac{(4x+6)(x-1) - 2(2x^2+6x+4)}{(x-1)^3} \\ &= \frac{4x^2 + 2x - 6 - 4x^2 - 12x - 8}{(x-1)^3} \\ &= \frac{-10x - 14}{(x-1)^3} \end{aligned}$$

6 Find any points where $f'(x) = 0$.

$f'(-1.4) = 0$ so there is a stationary point at $x = -1.4$.

7 Find the nature of the stationary points.

x	< -1.4	-1.4	> -1.4
$f'(x)$	–	0	+

8 State the nature of the stationary point.

There is a minimum at $\left(-1.4, -\frac{1}{12}\right)$.

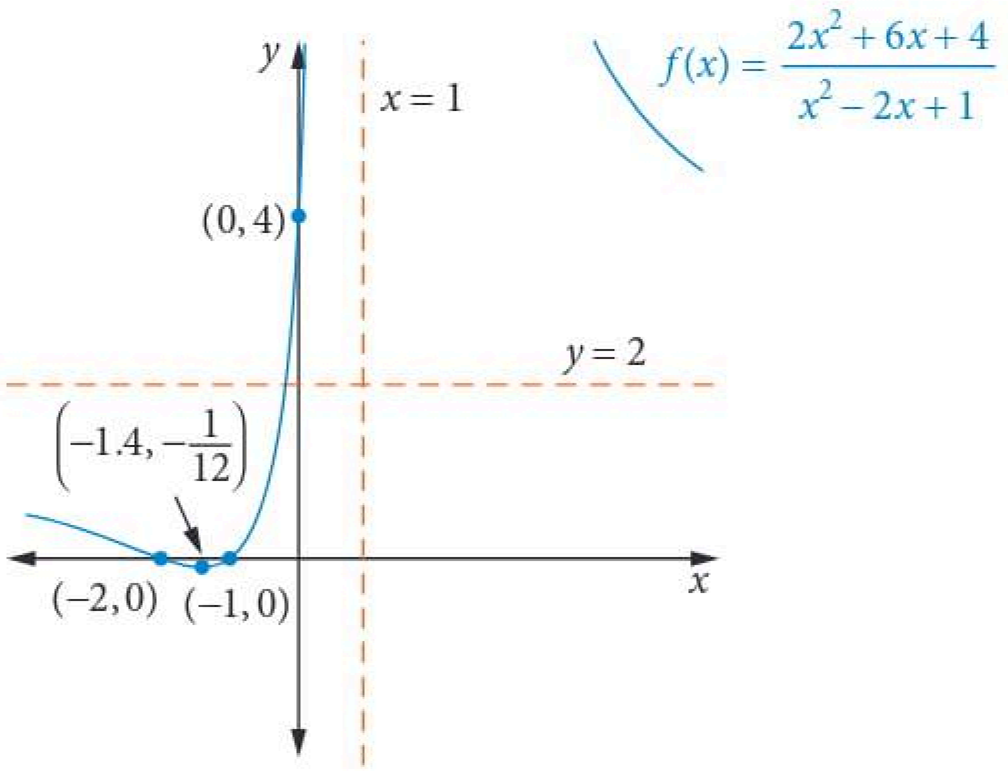
9 State the sign of $f(x)$ in each region of the graph.

For $x < -2$, $f(x)$ is positive.
For $-2 < x < -1$, $f(x)$ is negative.
For $-1 < x < 1$, $f(x)$ is positive.
For $x > 1$, $f(x)$ is positive.

10 State the behaviour as $x \to \pm\infty$ (e.g. try $f(100), f(-100)$).

As $x \to -\infty$, $f(x) < 2$, so $f(x) \to 2$ from below.
As $x \to \infty$, $f(x) > 2$, so $f(x) \to 2$ from above.

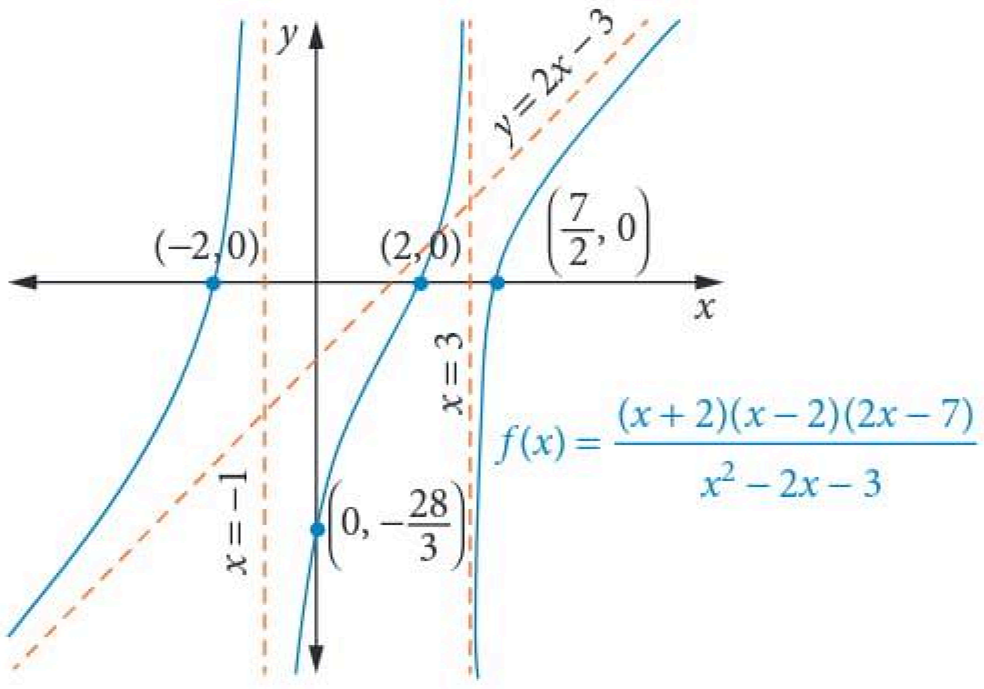
11 Sketch the graph, marking all important points.



A quotient function with deg (numerator) – deg (denominator) = 1 has an oblique (sloping) asymptote of the linear form $y = mx + c$, where $mx + c$ is the quotient of the numerator and denominator.

WORKED EXAMPLE 17 Quotient function graph with a sloping asymptote

Sketch the graph of $f(x) = \dfrac{(x+2)(x-2)(2x-7)}{x^2 - 2x - 3}$.

Steps	Working
1 Factorise.	$f(x) = \dfrac{(x+2)(x-2)(2x-7)}{(x-3)(x+1)}$
2 State the zeros and vertical asymptotes.	The zeros are at $x = -2$, $x = 2$ and $x = \dfrac{7}{2} = 3\dfrac{1}{2}$. The vertical asymptotes are $x = -1$ and $x = 3$.
3 State the signs of the function.	For $x < -2, f(x) < 0$. For $-2 < x < -1, f(x) > 0$. For $-1 < x < 2, f(x) < 0$. For $2 < x < 3, f(x) > 0$. For $3 < x < \dfrac{7}{2}, f(x) < 0$. For $x > \dfrac{7}{2}, f(x) > 0$.
4 Find the sloping asymptote.	$(x+2)(x-2)(2x-7) = 2x^3 - 7x^2 - 8x + 28$ $\dfrac{2x^3 - 7x^2 - 8x + 28}{x^2 - 2x - 3} = 2x - 3 + \dfrac{-8x + 19}{x^2 - 2x - 3}$, so there is a sloping asymptote; $y = 2x - 3$.
5 State the behaviour as $x \to \pm\infty$.	As $x \to -\infty$, $\dfrac{-8x + 19}{x^2 - 2x - 3} > 0$ so $f(x) \to y = 2x - 3$ from above. As $x \to \infty$, $\dfrac{-8x + 19}{x^2 - 2x - 3} < 0$ so $f(x) \to y = 2x - 3$ from below.
6 Find the y-intercept.	$f(0) = -\dfrac{28}{3} = -9\dfrac{1}{3}$
7 Sketch the graph.	

WB p. 33

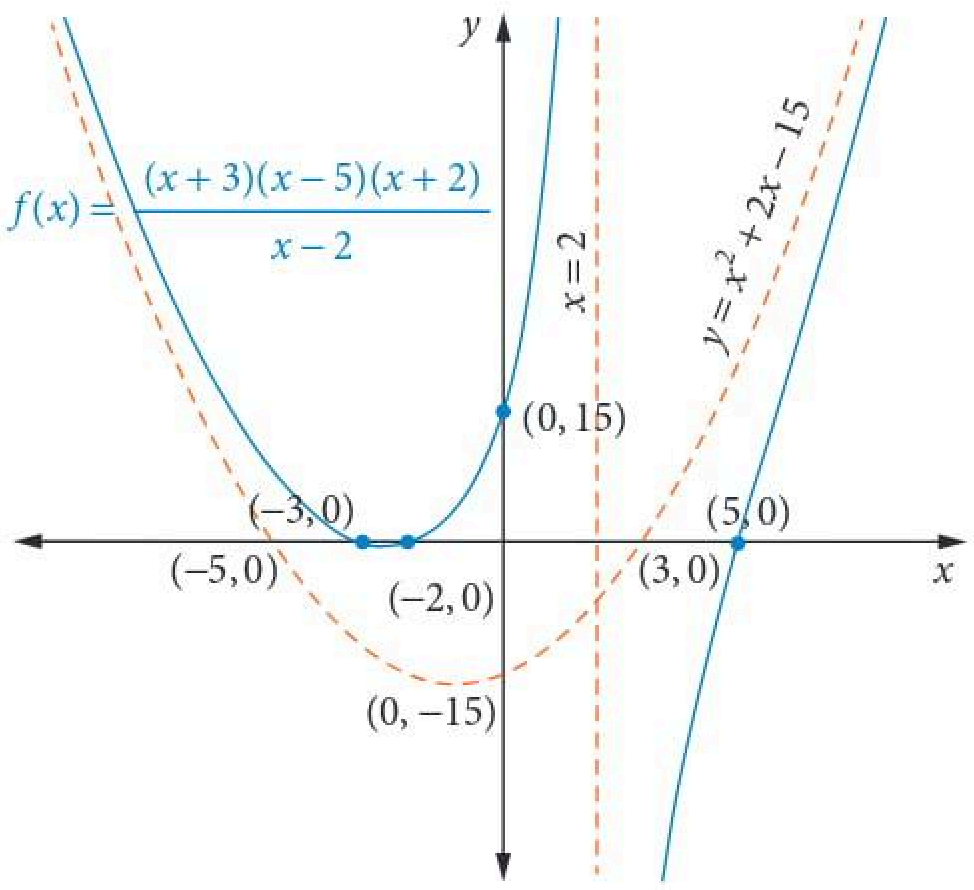
A quotient function with deg (numerator) − deg (denominator) = 2 has a curved asymptote of the quadratic form $y = ax^2 + bx + c$, where $ax^2 + bx + c$ is the quotient of the numerator and denominator.

WB

p. 34

WORKED EXAMPLE 18 Quotient function graph with a curved asymptote

Sketch the graph of $f(x) = \dfrac{(x+3)(x-5)(x+2)}{x-2}$.

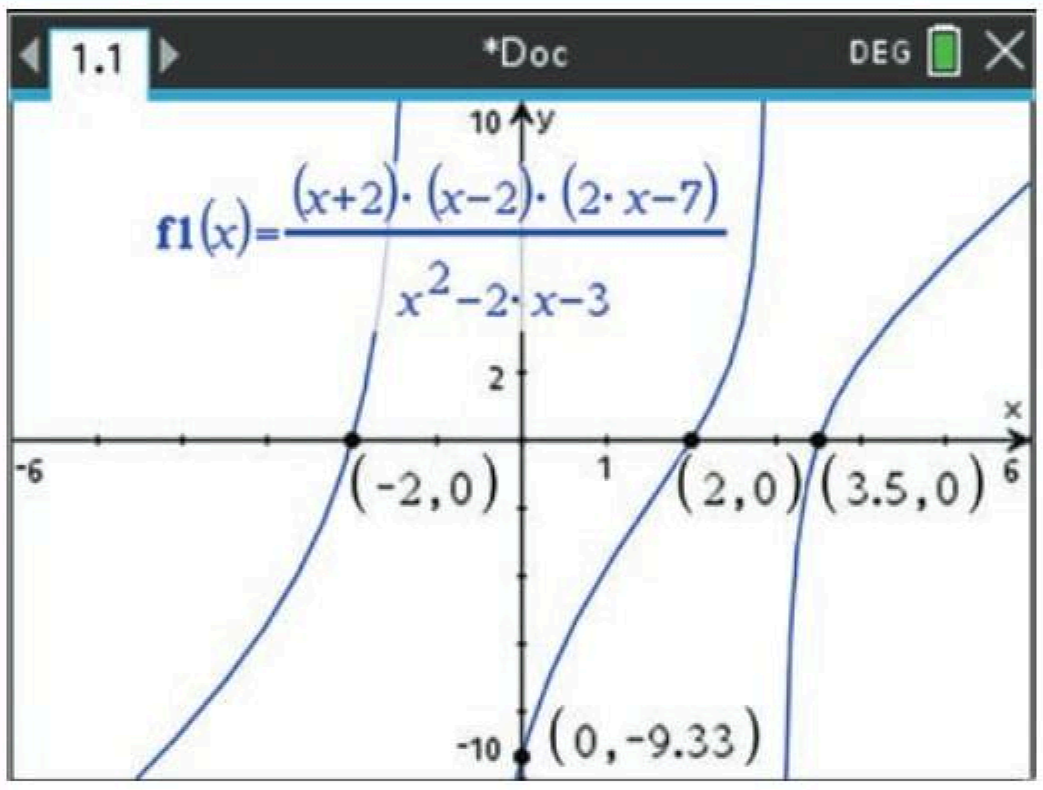
Steps	**Working**
1 State the zeros and vertical asymptotes.	The zeros are at $x = -3$, $x = 5$ and $x = -2$ and the vertical asymptote is at $x = 2$.
2 State the signs of the function.	For $x < -3, f(x) > 0$. For $-3 < x < -2, f(x) < 0$. For $-2 < x < 2, f(x) > 0$. For $2 < x < 5, f(x) < 0$. For $x > 5, f(x) > 0$.
3 Find the curved asymptote.	$(x+3)(x-5)(x+2) = x^3 - 19x - 30$ $\dfrac{x^3 - 19x - 30}{x-2} = x^2 + 2x - 15 - \dfrac{60}{x-2}$, so there is a curved asymptote; $y = x^2 + 2x - 15$.
4 State the behaviour as $x \to \pm\infty$.	As $x \to -\infty$, $-\dfrac{60}{x-2} > 0$ so $f(x) \to y = x^2 + 2x - 15$ from above. As $x \to \infty$, $-\dfrac{60}{x-2} < 0$ so $f(x) \to y = x^2 + 2x - 15$ from below.
5 Find the y-intercept.	$f(0) = 15$
6 Determine the characteristics of the curved asymptote.	$y = x^2 + 2x - 15 = (x+5)(x-3)$ has zeros at -5 and 3. The vertex is halfway between the zeros, at $x = -1$, $y = -16$. The y-intercept is -15.
7 Sketch the graph.	

We should always check for stationary points, but in more complex graphs such as in the previous two examples, they may be too difficult to find without using CAS.

USING CAS 5 Quotient functions

Graph $f(x) = \dfrac{(x+2)(x-2)(2x-7)}{x^2 - 2x - 3}$.

TI-Nspire

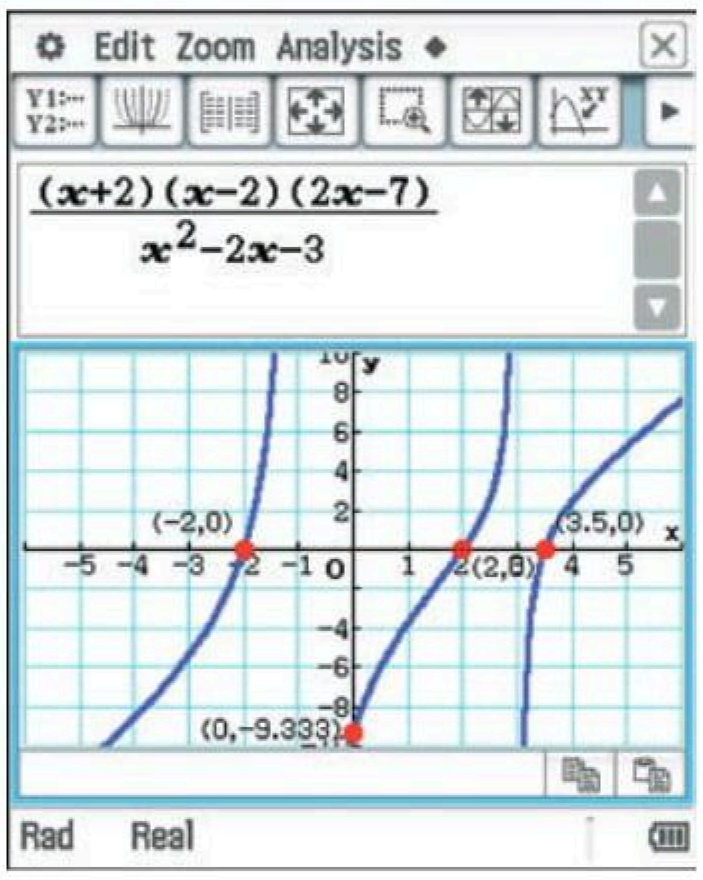


1 Add a **Graphs** page and enter the function.

2 Adjust the window settings to suit.

3 Press **menu > Trace > Graph Trace** or press **menu > Analyse Graph** to find and label the key features of the function.

ClassPad



1 In **Main**, enter and highlight the expression.

2 Tap the graph icon and drag the expression down into the graph window.

3 Adjust the window settings to suit.

4 Tap **Analysis > G-Solve** to find and label the key features of the function.

EXERCISE 2.4 Graphing quotient functions

ANSWERS p. 561

Recap

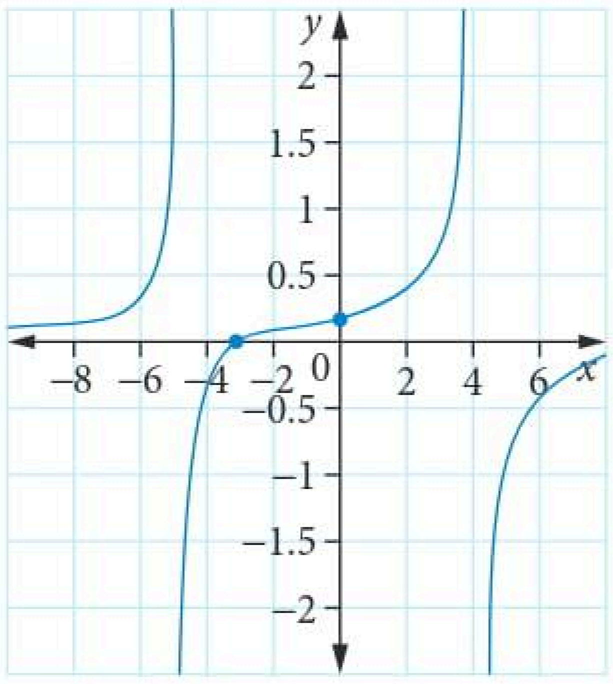
1 The asymptotes of $f(x) = \dfrac{3x+6}{(x^2-4)(x-3)}$ are

A $y = 3, x = -2, x = 2, x = 3$
B $y = 3, x = 2, x = 3$
C $y = 0, x = -2, x = 2, x = 3$
D $y = 0, x = 2, x = 3$
E $y = 0, x = -2, x = 2, x = -3$

2 The graph of a function is shown on the right.

A possible equation is:

A $f(x) = \dfrac{x-3}{(x-4)(x+5)}$
B $f(x) = \dfrac{x+3}{(x+4)(x-5)}$
C $f(x) = \dfrac{3-x}{(x+4)(x-5)}$
D $f(x) = \dfrac{x+3}{(x-4)(x+5)}$
E $f(x) = \dfrac{x-3}{(x+4)(x-5)}$



Mastery

3 WORKED EXAMPLE 16 Sketch the graph of each function, labelling all important features.

a $f(x) = \dfrac{6x-9}{2x+5}$

b $f(x) = \dfrac{4-3x}{x+1}$

c $f(x) = \dfrac{2-5x-3x^2}{(x-3)^2}$

d $f(x) = \dfrac{x^2-3x-10}{2x^2+4x-6}$

e $f(x) = \dfrac{4x^2-2x+3}{x+4-3x^2}$

f $f(x) = \dfrac{(2x-5)^2(x+5)}{(1-x)(x+2)^2}$

4 WORKED EXAMPLE 17 Sketch the graph of each function, labelling all important features.

a $f(x) = \dfrac{x^2-6x+8}{x-3}$

b $f(x) = \dfrac{2x^2-5x-3}{4-x}$

c $f(x) = \dfrac{(2x+5)(x-3)^2}{x^2+5x-6}$

d $f(x) = \dfrac{(3x^2-2x+1)(x-2)}{3+x-2x^2}$

e $f(x) = \dfrac{3(x^2-5x-6)(x+2)}{4-x^2}$

f $f(x) = \dfrac{2x^3-54}{x^2-3x-4}$

5 WORKED EXAMPLE 18 Sketch the graph of each function, labelling all important features.

a $f(x) = \dfrac{(3x+2)(x-1)(x-3)}{2-x}$

b $f(x) = \dfrac{(x+3)(x-1)^2}{x+2}$

c $f(x) = \dfrac{(3x^2-4x+2)(x-2)}{2x-5}$

d $f(x) = \dfrac{(2x^2+1)(x+2)(3-x)}{x^2+2x+1}$

e $f(x) = \dfrac{(x^2+x-6)(x^2+3)}{x^2-2x+1}$

f $f(x) = \dfrac{(x+3)(x^3-2x^2-x+2)}{x^2+6x+8}$

6 Using CAS 5 Sketch the graph of each function.

a $f(x) = \dfrac{2x^2-x}{(x+1)(x-2)}$

b $f(x) = \dfrac{4x^2-9x^3+4x-6}{(x+1)(3x^2-4x+2)}$

c $f(x) = \dfrac{4x^3-8x^2-52x+83}{(x+4)(2x-7)}$

d $f(x) = \dfrac{11x^2-x^3-12x-86}{2(x+2)(x-5)}$

e $f(x) = \dfrac{(1-x)(x^2-8)}{x+3}$

f $f(x) = \dfrac{4x^4+14x^3+15x^2+8x-9}{4x^2+6x+3}$

7 a How many zeros does $f(x) = \dfrac{2x^2-8a^2}{x+a}$ have?

b How many straight-line asymptotes does $f(x) = \dfrac{2x^2-8a^2}{x+a}$ have?

c Find the equation(s) of the straight-line asymptote.

d Sketch the graph.

8 Find any vertical, horizontal, sloping and curved asymptotes for each function.

a $f(x) = \dfrac{x+2}{x^2-2x-24}$

b $f(x) = \dfrac{3x^2+x-2}{x^2-3x+2}$

c $f(x) = \dfrac{2x^2-3x-5}{x+2}$

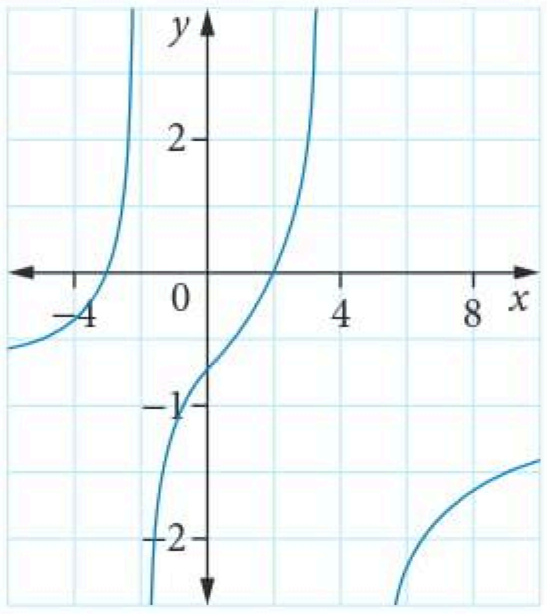
d $f(x) = \dfrac{2x^2-3x^3+x+5}{x^2-2x-8}$

e $f(x) = \dfrac{(2x-1)(x+2)(3-x)}{x+4}$

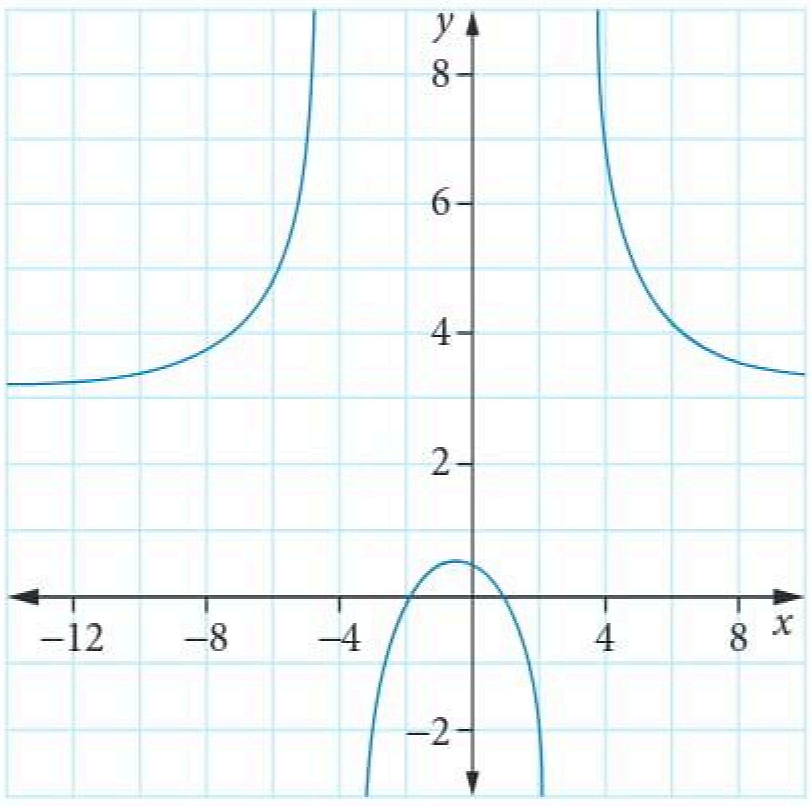
f $f(x) = \dfrac{x^4-x^3-x+1}{x^2+3x+5}$

9 State a possible function rule for each graph.

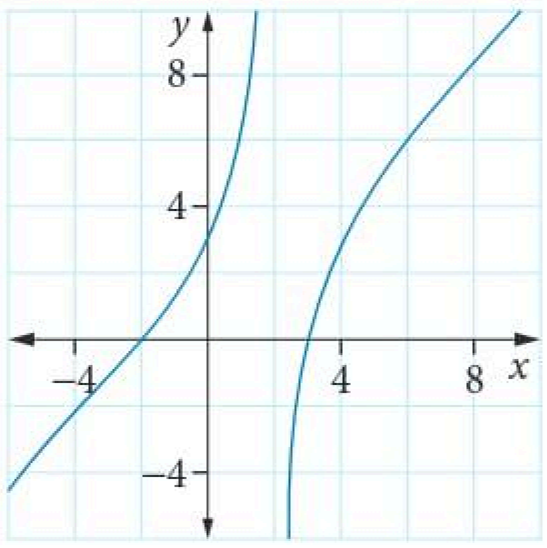
a



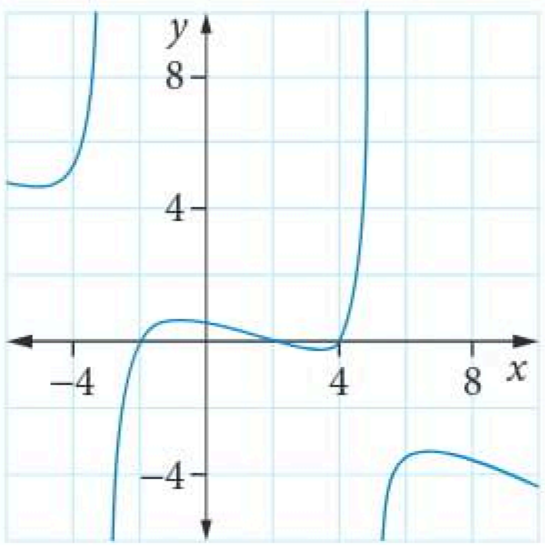
b



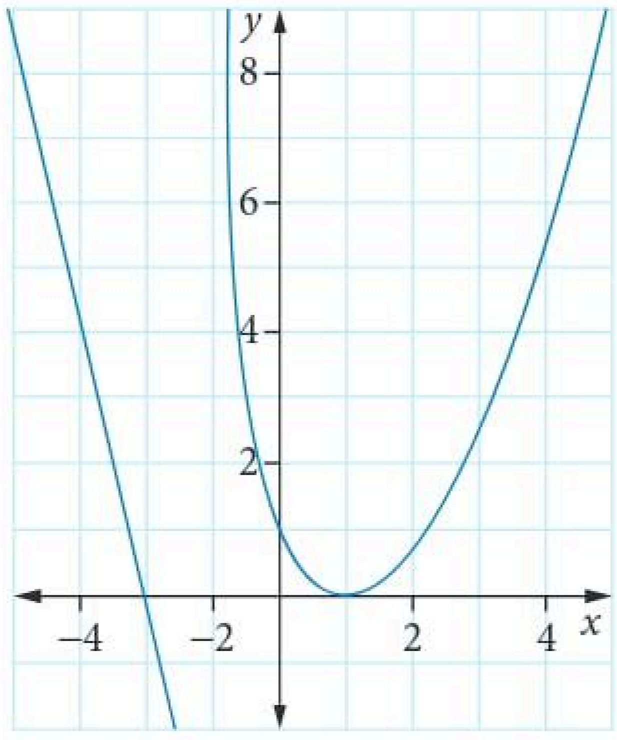
c



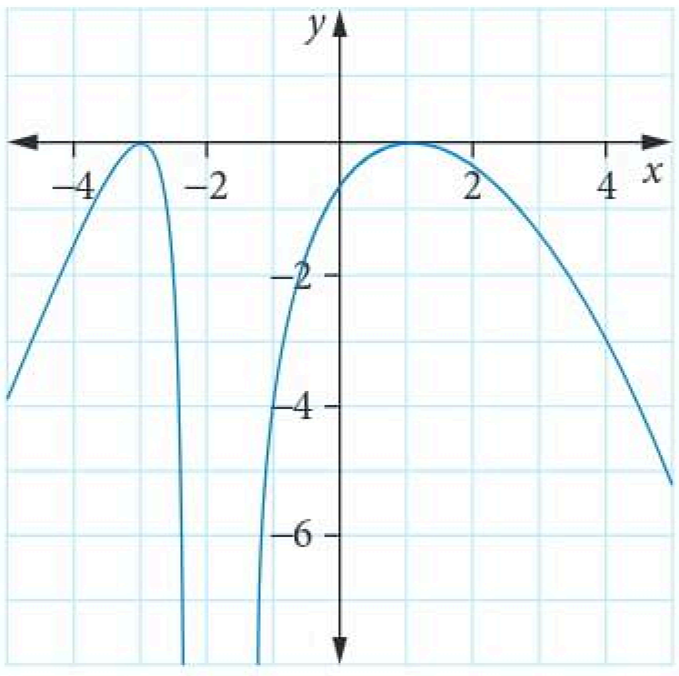
d



e



f



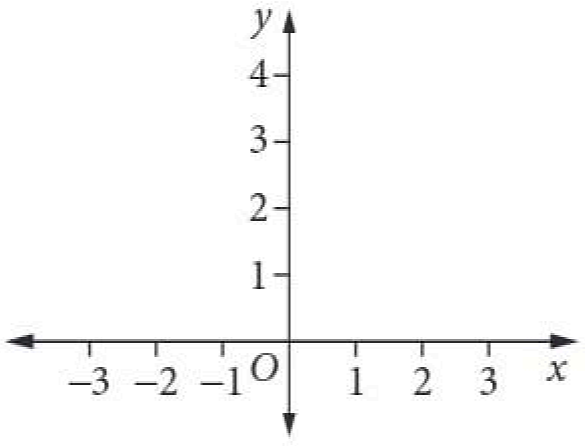
Exam practice

80–100% 60–79% 0–59%

10 © VCAA 2011 1Q3 TECH-FREE (4 marks)

a 91% Show that $f(x) = \dfrac{2x^2+3}{x^2+1}$ can be written in the form $f(x) = 2 + \dfrac{1}{x^2+1}$. 1 mark

b 63% Copy the axes below and on them sketch the graph of the relation $y = \dfrac{2x^2+3}{x^2+1}$ from -3 to 3. Label any asymptotes with their equations and label the intercepts with the axes, writing them as coordinates. 3 marks



11 © VCAA 2019 2AQ2 86% The asymptote(s) of the graph of $f(x) = \dfrac{x^2+1}{2x-8}$ has equation(s)

A $x = 4$

B $x = 4$ and $y = \dfrac{x}{2}$

C $x = 4$ and $y = \dfrac{x}{2} + 2$

D $x = 8$ and $y = \dfrac{x}{2}$

E $x = 8$ and $y = 2x + 2$

12 © VCAA 2016 2AQ3 71% The straight-line asymptote(s) of the graph of the function with rule $f(x) = \frac{x^3 - ax}{x^2}$, where a is a non-zero real constant, is given by

A $x = 0$ only.

B $x = 0$ and $y = 0$ only.

C $x = 0$ and $y = x$ only.

D $x = 0$, $x = \sqrt{a}$ and $x = -\sqrt{a}$ only.

E $x = 0$ and $y = a$ only.

13 © VCAA 2016S 2AQ3 The features of the graph of the function with rule $f(x) = \frac{x^2 - 4x + 3}{x^2 - x - 6}$ include

A asymptotes at $x = 1$ and $x = -2$.

B asymptotes at $x = 3$ and $x = -2$.

C an asymptote at $x = 1$ and a point of discontinuity at $x = 3$.

D an asymptote at $x = -2$ and a point of discontinuity at $x = 3$.

E an asymptote at $x = 3$ and a point of discontinuity at $x = -2$.

14 © VCAA 2011 2AQ1 34% The number of straight line asymptotes of the graph of $y = \frac{2x^3 + x^2 - 1}{x^2 - x - 2}$ is

A 0 **B** 1 **C** 2 **D** 3 **E** 4

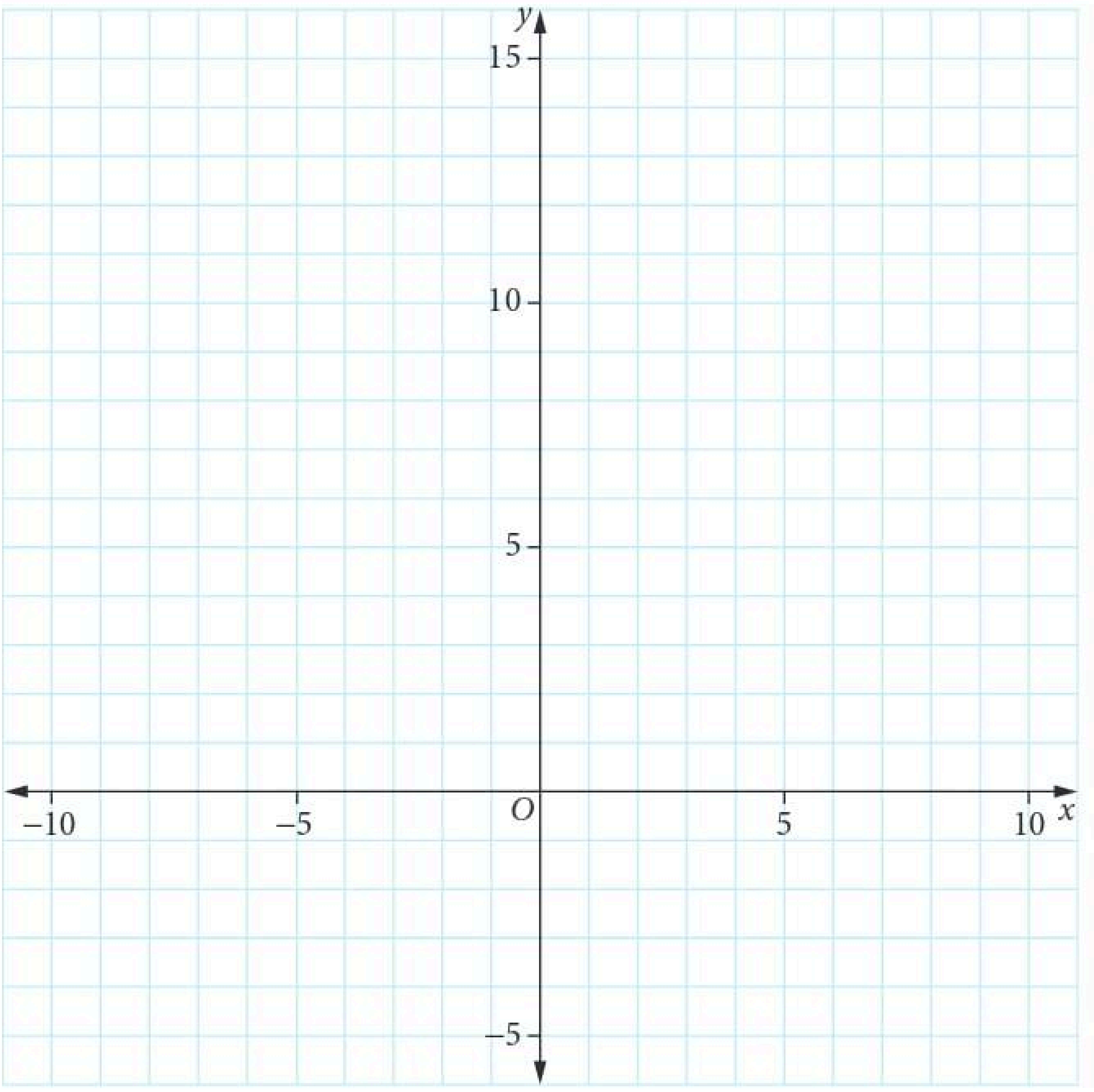
15 © VCAA 2021 2BQ1 (10 marks)

Let $f(x) = \frac{(2x - 3)(x + 5)}{(x - 1)(x + 2)}$.

a 79% Express $f(x)$ in the form $A + \frac{Bx + C}{(x - 1)(x + 2)}$, where A, B and C are real constants. 1 mark

b 82% State the equations of the asymptotes of the graph of f. 2 marks

c 48% Copy the set of axes shown below and on them sketch the graph of f. Label the asymptotes with their equations, and label the maximum turning point and the point of inflection with their coordinates, correct to two decimal places. Label the intercepts with the coordinate axes. 3 marks



d Let $g_k(x) = \dfrac{(2x-3)(x+5)}{(x-k)(x+2)}$, where k is a real constant.

i 24% For what values of k will the graph of g_k have two asymptotes? 2 marks

ii 16% Given that the graph of g_k has more than two asymptotes, for what values of k will the graph of g_k have no stationary points? 2 marks

2.5 Absolute value functions

In Year 11, we learned that $|x|$ is the **absolute value** or **modulus** of a real number x, and is the (positive) distance of x from the origin on the number line.

$$|x| = \begin{cases} x, & \text{for } x \geq 0 \\ -x, & \text{for } x < 0 \end{cases}.$$

Solve equations like $|f(x)| = k$ as 2 equations: $f(x) = k$ or $f(x) = -k$.

Solve inequalities like $|f(x)| \geq k$ as 2 inequalities: $f(x) \geq k$ **or** $f(x) \leq -k$.

Solve inequalities like $|f(x)| \leq k$ as 2 inequalities: $f(x) \leq k$ **and** $f(x) \geq -k$. This can be written as $-k \leq f(x) \leq k$.

Video playlist
Absolute value functions

Worksheets
Absolute value

Absolute value functions

Absolute value inequalities

WB
p. 35

WORKED EXAMPLE 19 Absolute value equations and inequalities

Solve

a $|2x-5| = 11$ **b** $|3x+2| \geq 8$ **c** $|x+4| < 6$

Steps	**Working**
a **1** If $|f(x)| = 11$, then $f(x) = 11$ or -11.	$2x - 5 = \pm 11$
2 Solve each equation.	$2x - 5 = 11 \Rightarrow x = 8$ $2x - 5 = -11 \Rightarrow x = -3$
3 Write the solutions.	$x = 8$ or $x = -3$
b **1** If $|f(x)| \geq 8$, then $f(x) \geq 8$ or $f(x) \leq -8$.	$3x + 2 \geq 8$ or $3x + 2 \leq -8$
2 Solve each inequality.	$3x + 2 \leq -8 \Rightarrow 3x \leq -10 \Rightarrow x \leq -\dfrac{10}{3}$ $3x + 2 \geq 8 \Rightarrow 3x \geq 6 \Rightarrow x \geq 2$
3 Write the solutions.	$x \geq 2$ or $x \leq -\dfrac{10}{3}$
c **1** If $|f(x)| < 6$, then $-6 < f(x) < 6$.	$-6 < x + 4 < 6$
2 Solve each inequality.	$-6 < x + 4 \Rightarrow -10 < x$ $x + 4 < 6 \Rightarrow x < 2$
3 Write the solution.	$x \in (-10, 2)$

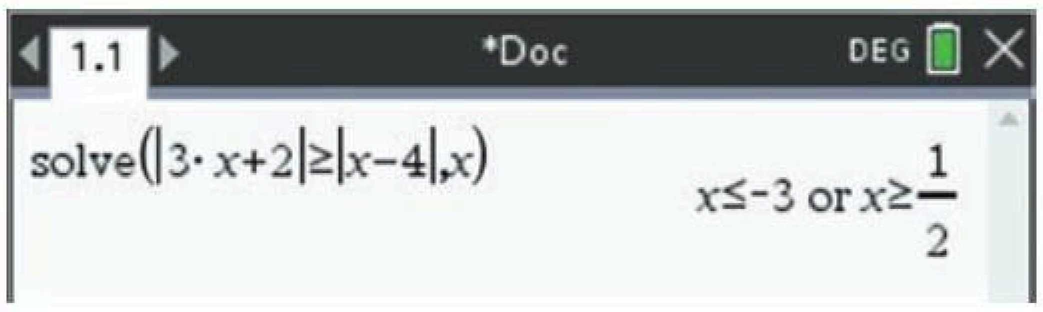
The solutions to the above inequalities can be shown in alternate forms; for example,

b $x \in \left(-\infty, -\dfrac{10}{3}\right] \cup [2, \infty)$ and **c** $\{x: -10 < x < 2\}$.

USING CAS 6 Absolute value equations and inequalities

Solve $|3x + 2| \geq |x - 4|$.

TI-Nspire



solve(|3·x+2|≥|x−4|,x) $x \leq -3$ or $x \geq \frac{1}{2}$

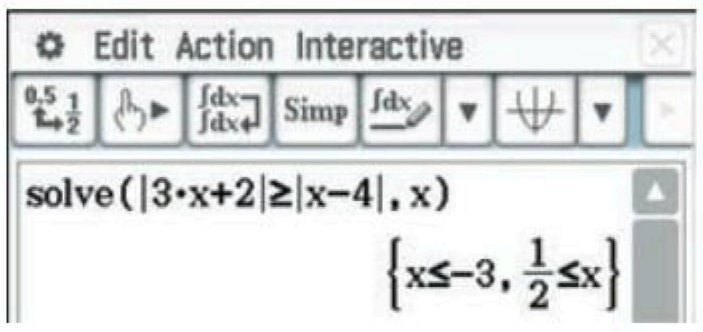
1 Press **menu > Algebra > Solve**.

2 Press the **template** key to access the **absolute value** template.

3 Press **ctrl + =** to access the ≥ sign.

4 Enter the inequality followed by, **x** as shown above.

ClassPad



solve(|3·x+2|≥|x−4|, x)

$\left\{x \leq -3, \frac{1}{2} \leq x\right\}$

1 Open the **Keyboard > Math1** or **Math2** to access the absolute value function.

2 Enter and highlight the inequality.

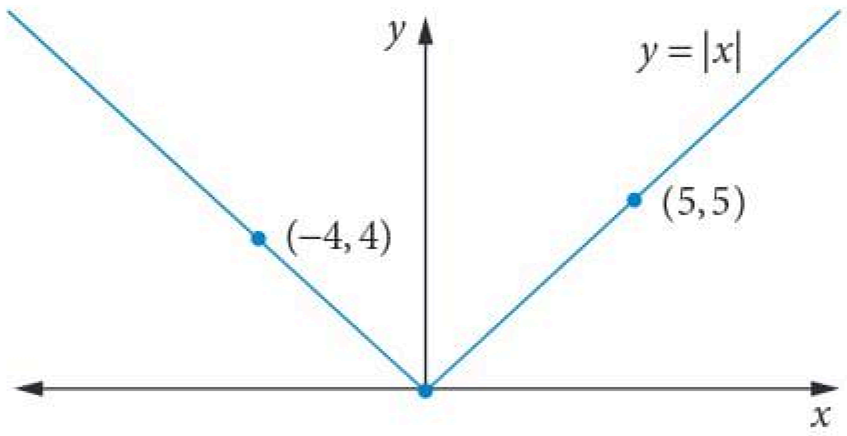
3 Tap **Math3** to access the ≥ sign.

4 Tap **Interactive > Equation/Inequality > solve > OK**.

The solution is $(-\infty, -3] \cup \left[\frac{1}{2}, \infty\right)$.

For $x \geq 0$, $|x| = x$, so the absolute value function is the same as $y = x$ for $x > 0$.

For $x < 0$, $|x| = -x$, so the graph of the absolute value function is symmetrical about the y-axis.



The graph of the absolute value function has a sharp vertex at (0, 0), called a **cusp**.

$f(x) = a|x|$ or $f(x) = |ax|$ is dilated from the x-axis by a factor of a.

For $a > 1$ or $a < -1$, it is stretched; for $-1 < a < 1$, it is shrunk.

$f(x) = |x + b|$ is translated in the x direction. For $b > 0$, it moves left; for $b < 0$, it moves right.

$f(x) = |x| + c$ is translated in the y direction. For $c > 0$, it moves up; for $c < 0$, it moves down.

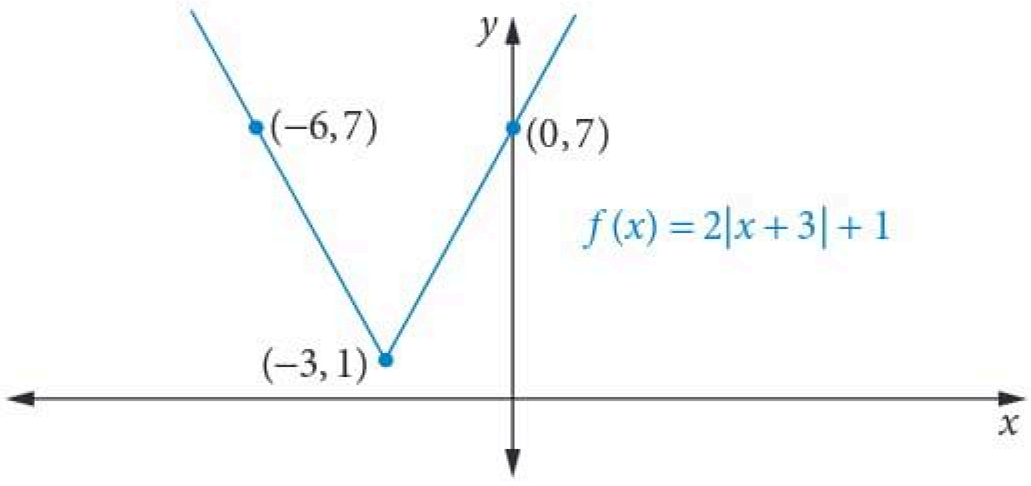
WB

p. 36

Worksheet Transformations of absolute values and hyperbolas

WORKED EXAMPLE 20 Absolute value graphs

Sketch the graph of $f(x) = 2|x + 3| + 1$.

Steps	**Working**
1 State the transformations.	The graph of $f(x) = |x|$ is translated 3 units to the left, stretched from the x-axis by the factor 2 and translated 1 unit up.
2 Sketch the graph and include a few points for reference.	

EXERCISE 2.5 Absolute value functions

ANSWERS p. 564

Recap

1 The asymptotes of $f(x) = \dfrac{39x - 2x^3 - x^2 - 43}{x^2 + 2x - 15}$ are

A $x = -5, x = 3, y = 2x + 1$

B $x = 5, x = -3, y = -1$

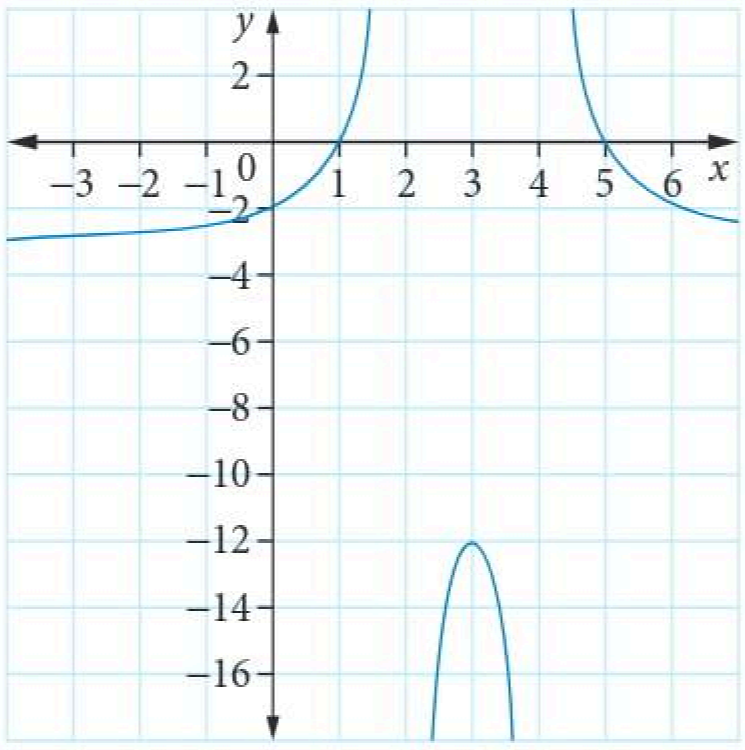
C $x = -5, x = 3, y = -2x - 1$

D $x = 5, y = -3, y = 2x - 3$

E $x = -5, x = 3, y = 3 - 2x$

2 The graph of the function $f(x)$ is shown on the right.

A possible equation for the given graph is



A $f(x) = \dfrac{3(x-1)(x-5)}{(x-2)(x-4)}$

B $f(x) = \dfrac{3(1-x)(x-5)}{(x-2)(x-4)}$

C $f(x) = \dfrac{3(x-1)(x-5)}{(x+2)(x+4)}$

D $f(x) = \dfrac{-3(x+1)(x+5)}{(x-2)(x-4)}$

E $f(x) = \dfrac{3(x+1)(x+5)}{(x-2)(x-4)}$

Mastery

3 WORKED EXAMPLE 19 Solve each expression.

a $|4x + 3| = 9$ **b** $|m - 5| = 1$ **c** $|3z + 4| = 7$

d $|3g + 2| < 8$ **e** $|2k + 6| > 14$ **f** $|2x - 1| \geq 7$

4 Using CAS 6 Solve each expression.

a $|5x - 3| < 7$ **b** $|3 - 2p| = 5 + 3p$ **c** $2|q + 4| > 3\ (q - 2)$

d $|3m - 2| \leq |5m + 7|$ **e** $2|2a + 4| = 3|7 - a|$ **f** $|x^2 - 5x + 4| > 2$

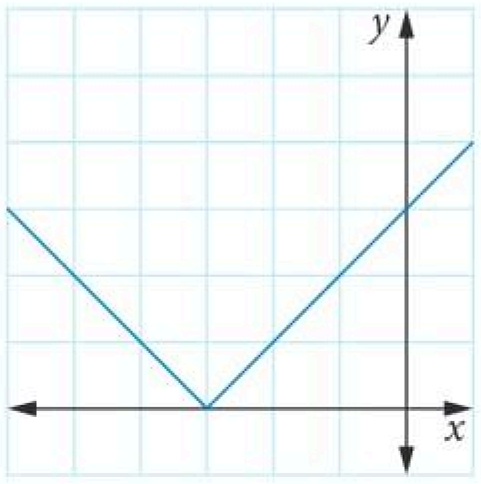
5 WORKED EXAMPLE 20 Sketch the graph of each function.

a $f(x) = |x - 3| + 1$ **b** $f(x) = |2x + 5| - 4$ **c** $f(x) = 3|4 - x| - 2$

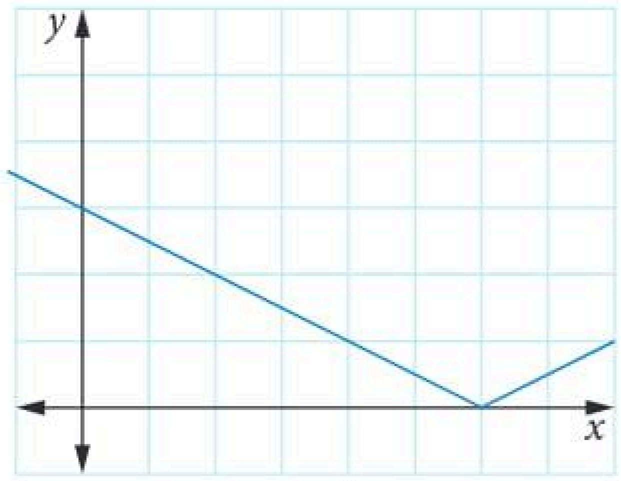
d $f(x) = 2|3x - 1| + 4$ **e** $f(x) = 6 - \dfrac{1}{2}|x + 1|$ **f** $f(x) = |x - 4| + |x + 2|$

6 State possible equations for the functions with these graphs. All are shown with unit grids.

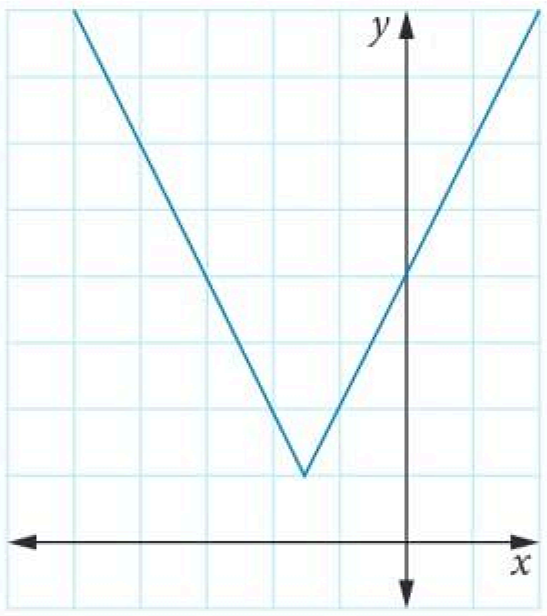
a



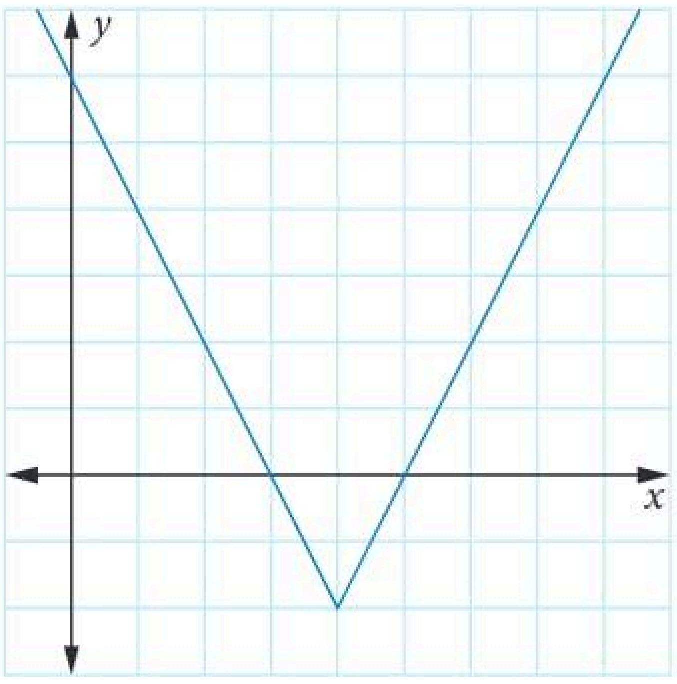
b



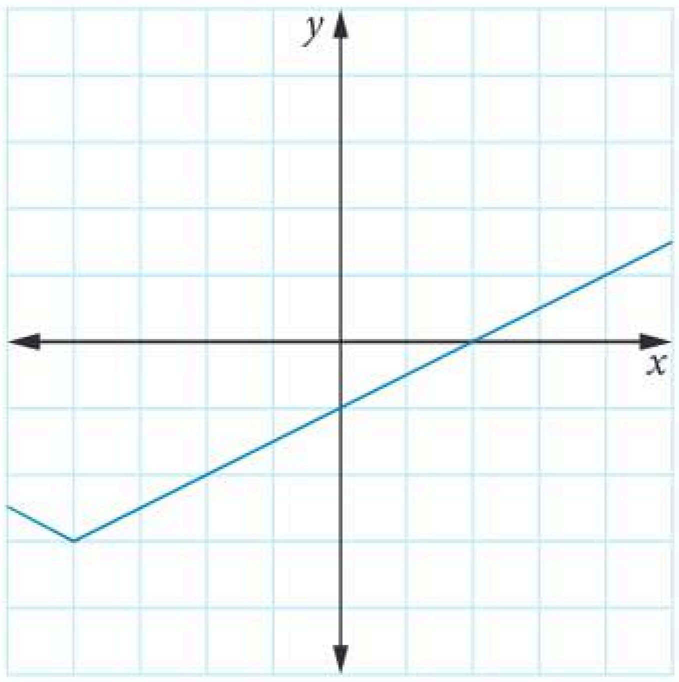
c



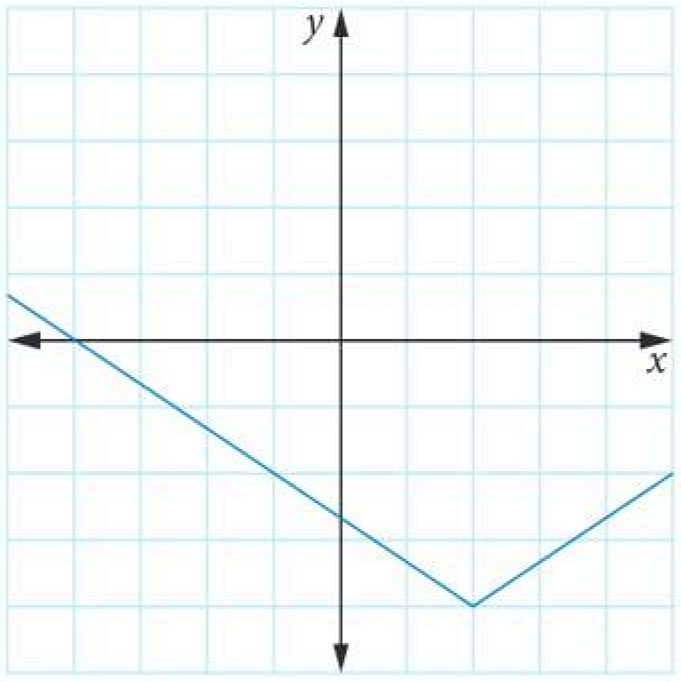
d



e



f



Exam practice

80–100% 60–79% 0–59%

7 © VCAA 2020 1Q4 46% TECH-FREE (4 marks)

Solve the inequality $3 - x > \dfrac{1}{|x\ - 4|}$ for x, expressing your answer in interval notation.

8 TECH-FREE (3 marks) The function $f(x)$ has the rule $f(x) = |2x - 3| - |3x - 1|$.

a Sketch the graph of $f(x)$ for $-5 \le x < 5$. 2 marks

b State the domain and range of f. 1 mark

9 TECH-FREE (2 marks) Find the range of $f(x) = |x - k| - |3k - x|$ in terms of k.

10 © VCAA 2021N 2AQ2 Consider the function with rule $f(x) = |x - 3| + |x + 3| - a$, where a is a real constant.

The graph of $\dfrac{1}{f(x)}$ will have three asymptotes if the set of values of a is

A $\{-3, 3\}$ **B** $\{\}$ **C** $[6, \infty)$ **D** $(-\infty, 6)$ **E** $[-3, 3]$

11 The solution(s) of $|3x - 1| \le |x + 1|$ are

A $x \le 1$ **B** $x \le 0$ or $x \ge 1$ **C** $0 \le x \le 1$ **D** $x \ge 0$ **E** $0 \le x \le \dfrac{1}{2}$

2.6 Reciprocal circular functions

Video playlist Reciprocal circular functions

Remember from Year 11 that **secant**, **cosecant** and **cotangent** are defined as $\sec(\theta) = \dfrac{1}{\cos(\theta)} = \dfrac{1}{x}$, $\operatorname{cosec}(\theta) = \dfrac{1}{\sin(\theta)} = \dfrac{1}{y}$ and $\cot(\theta) = \dfrac{1}{\tan(\theta)} = \dfrac{\cos(\theta)}{\sin(\theta)} = \dfrac{x}{y}$. We can calculate $\sec(\theta)$, $\operatorname{cosec}(\theta)$ and $\cot(\theta)$ from the corresponding values of $\cos(\theta)$, $\sin(\theta)$ and $\tan(\theta)$.

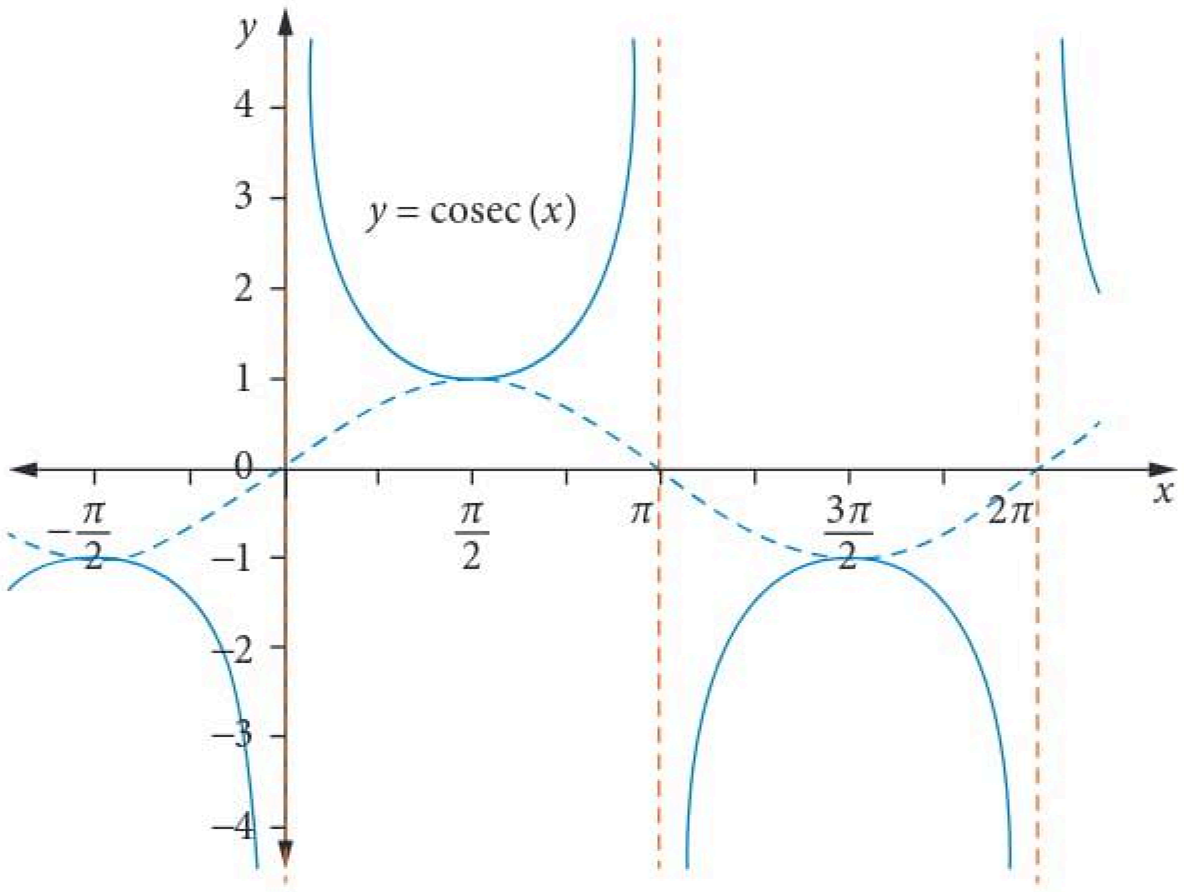
The graphs of $y = \sec(x)$, $y = \operatorname{cosec}(x)$ and $y = \cot(x)$ are related to the graphs of $y = \cos(x)$, $y = \sin(x)$ and $y = \tan(x)$ respectively in the same way as any reciprocal graphs are related to the originals.

- They have vertical asymptotes wherever the related graph has a zero.
- $y = \cot(x)$ has zeros where $y = \tan(x)$ has asymptotes.
- Maxima become minima and vice versa.
- The signs remain the same.

<table>
<tr>
<td>

$y = \operatorname{cosec}(x)$
asymptotes at $x = -\pi, 0, \pi, 2\pi, 3\pi$, etc.
$x = n\pi, n \in Z$
no zeros
domain: $R \setminus \{n\pi: n \in Z\}$
range: $(-\infty, -1] \cup [1, \infty)$
minima at $x = -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$ etc.
$\left((4n+1)\frac{\pi}{2}, 1\right), n \in Z$
maxima at $x = -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}$ etc.
$\left((4n-1)\frac{\pi}{2}, -1\right), n \in Z$

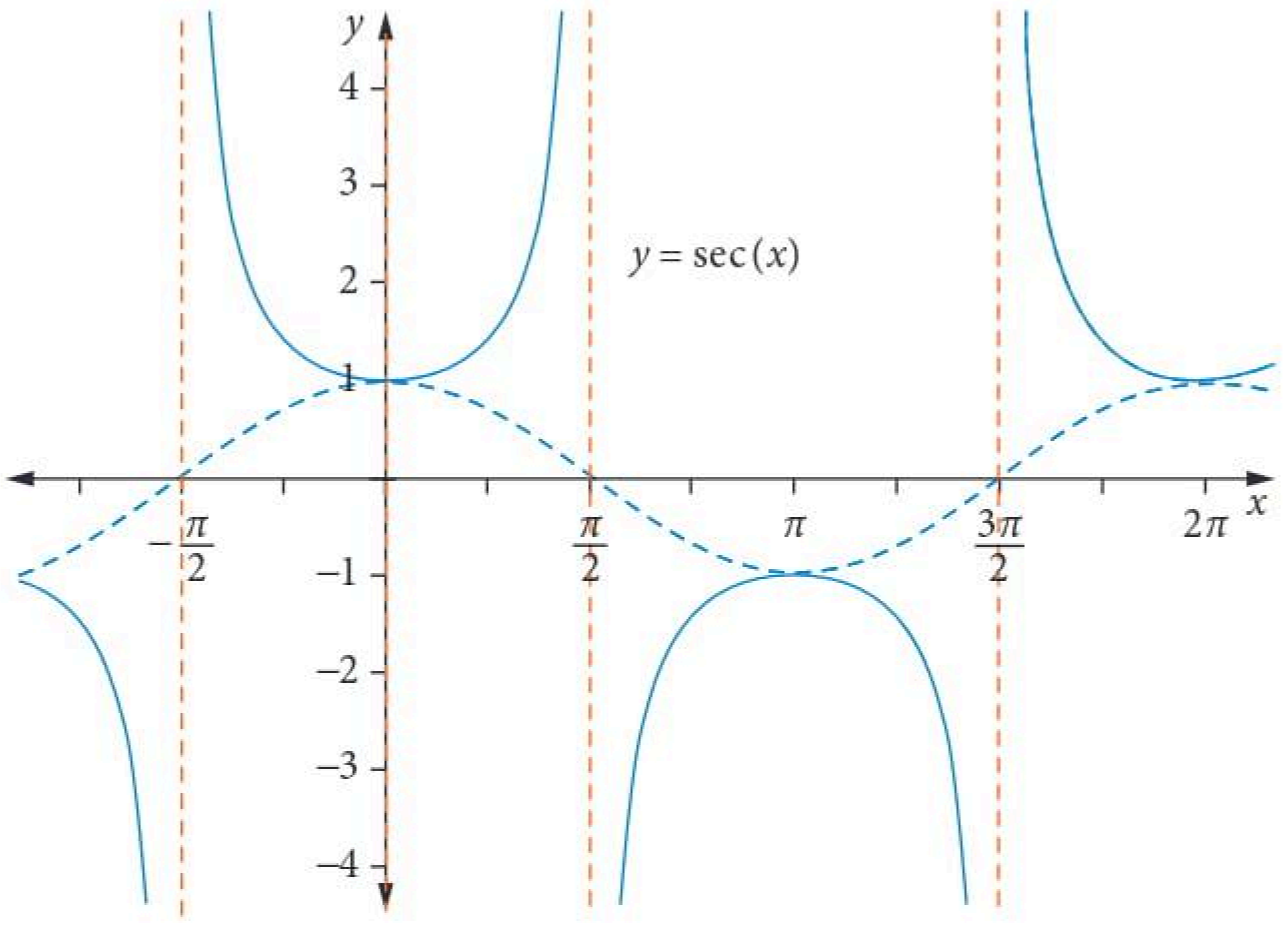
</td>
<td>



</td>
</tr>
<tr>
<td>

$y = \sec(x)$
asymptotes at $x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ etc.
$x = (2n+1)\frac{\pi}{2}, n \in Z$
no zeros
domain: $R \setminus \left\{(2n+1)\frac{\pi}{2} n \in Z\right\}$
range: $(-\infty, -1] \cup [1, \infty)$
minima at $x = -2\pi, 0, 2\pi, 4\pi$ etc.
$(2n\pi, 1), n \in Z$
maxima at $x = -\pi, \pi, 3\pi, 5\pi$ etc.
$((2n+1)\pi, -1), n \in Z$

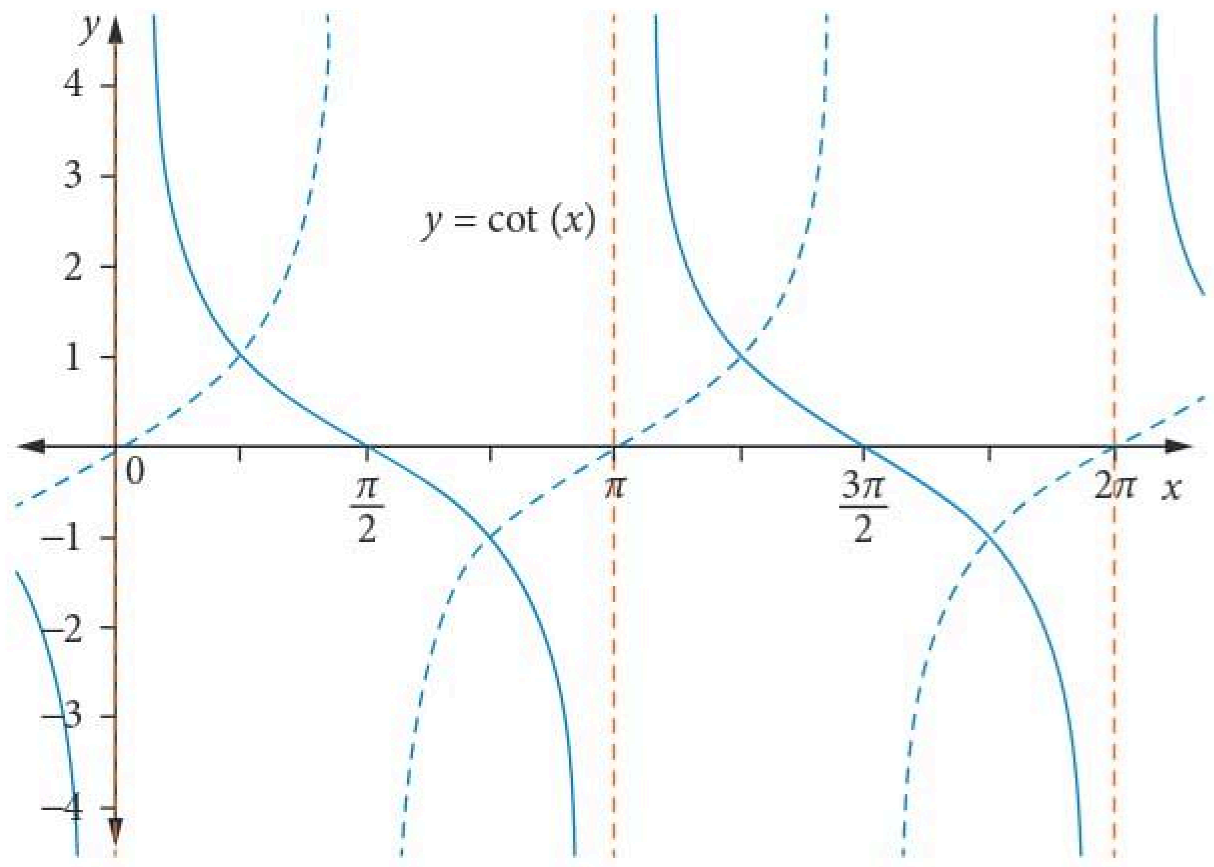
</td>
<td>



</td>
</tr>
<tr>
<td>

$y = \cot(x)$
asymptotes at $x = -\pi, 0, \pi, 2\pi, 3\pi$ etc.
$x = n\pi, n \in Z$
zeros (centres) at $x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ etc.
$x = (2n+1)\frac{\pi}{2}, n \in Z$
domain: $R\setminus\{n\pi: n \in Z\}$
range: R

</td>
<td>



</td>
</tr>
</table>

We can sketch transformations of the reciprocal circular functions by sketching the corresponding circular function first, but it is generally easier to proceed from the basic reciprocal function.

Order of transformations

Sketch the graphs of $a\,f\,[n(x+b)]+c$, for $f(x)=\sec(x)$, $f(x)=\operatorname{cosec}(x)$ and $f(x)=\cot(x)$ by transforming $f(x)$ in the following order

'HORIZONTALLY':

- Dilation from the y-axis by a factor of $\frac{1}{n}$ (period from n): shrink for $n>1$ or $n<-1$, stretch for $-1<n<1$
- **Reflection** in the y-axis if $n<0$
- **Translation** parallel to the x-axis (phase from b): left if $b>0$, right if $b<0$.

This can be written as $|n|>1$.

This can be written as $|n|<1$.

'VERTICALLY':

- Dilation from the x-axis by a factor of a: shrink for $-1<a<1$, stretch for $a>1$ or $a<-1$
- Reflection in the x-axis if $a<0$
- Translation parallel to the y-axis: up if $c>0$, down if $c<0$.

We can also sketch cot graphs by transforming the asymptotes and important points and joining them:

- Asymptotes 0, π; points $\left(\frac{\pi}{4},1\right),\left(\frac{3\pi}{4},-1\right)$.

WB p. 37

WORKED EXAMPLE 21 Graphing reciprocal circular functions

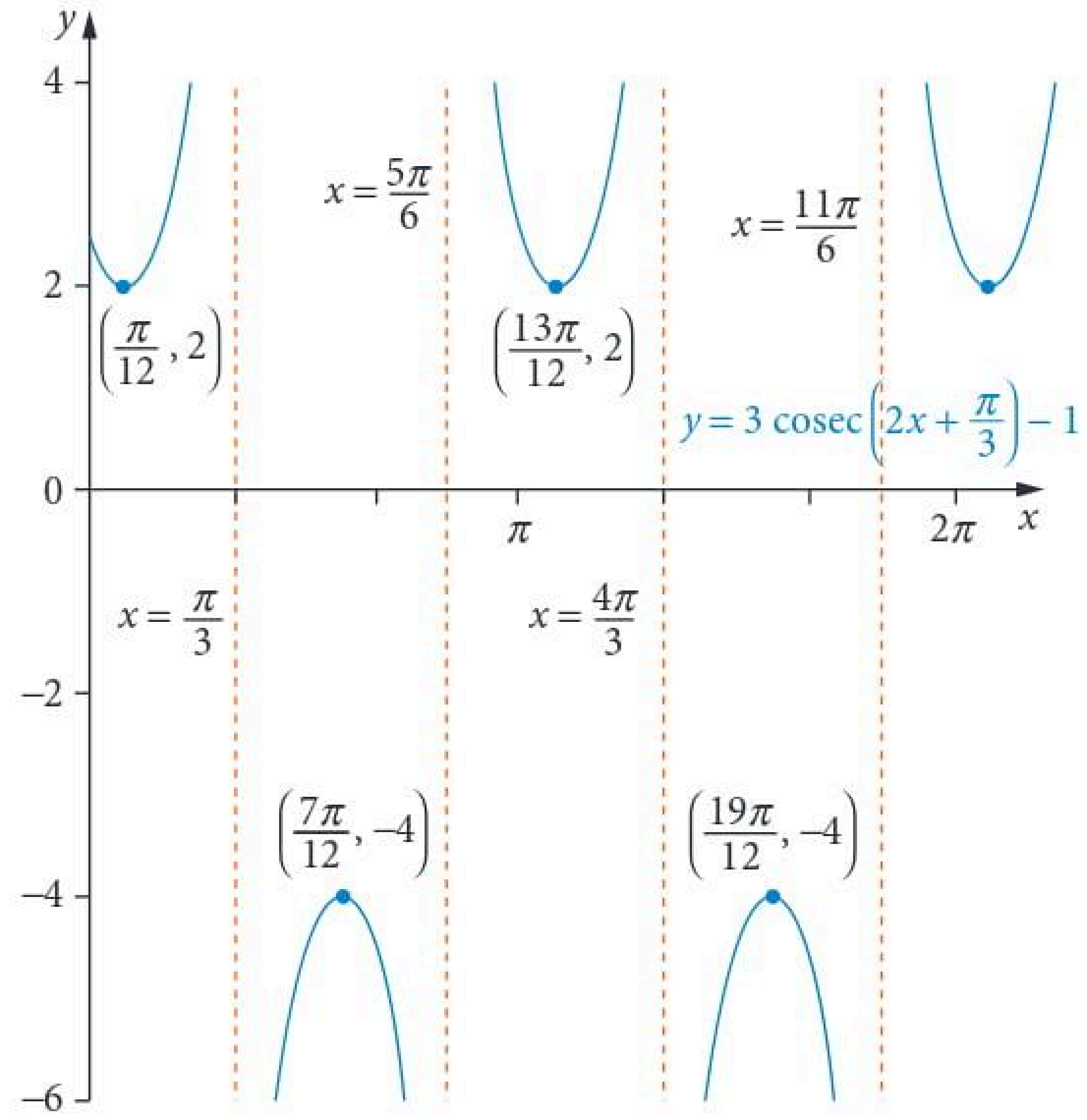
Sketch the graph of $y=3\operatorname{cosec}\left(2x+\frac{\pi}{3}\right)-1$ from $x=0$ to $x=2\pi$.

Steps	**Working**
1 Express as $a\,f\,[n(x+b)]+c$.	$y=3\operatorname{cosec}\left[2\left(x+\frac{\pi}{6}\right)\right]-1$
2 State the transformations.	Dilate the (dotted) graph of $y=\operatorname{cosec}(x)$ from the y-axis by a factor of $\frac{1}{2}$, move left by $\frac{\pi}{6}$ units. Dilate from the x-axis by a factor of 3, move down by 1 unit.
3 State the asymptotes and important points of cosec (x) (from zeros, maxima and minima of sin (x)).	cosec (x) has asymptotes at 0, π, 2π, $\pm 2n\pi$, minima at $\left(\frac{\pi}{2},1\right)\pm 2n\pi$ and maxima at $\left(\frac{3\pi}{2},-1\right)\pm 2n\pi$.
4 Transform the important points and asymptotes.	$\left(\frac{\pi}{2},1\right)\to\left(\frac{\pi}{4},1\right)\to\left(\frac{\pi}{12},1\right)\to\left(\frac{\pi}{12},3\right)\to\left(\frac{\pi}{12},2\right)$ $\left(\frac{3\pi}{2},-1\right)\to\left(\frac{3\pi}{4},-1\right)\to\left(\frac{7\pi}{12},-1\right)\to\left(\frac{7\pi}{12},-3\right)\to\left(\frac{7\pi}{12},-4\right)$ $x=0\to x=0\to x=-\frac{\pi}{6}$ $x=\pi\to x=\frac{\pi}{2}\to x=\frac{\pi}{3}$ $x=2\pi\to x=\pi\to x=\frac{5\pi}{6}$
5 State the change of period.	The period changes to π.

6 State the asymptotes and important points of the new function.

$3\,\text{cosec} = \left(2x + \frac{\pi}{3}\right) - 1$ has asymptotes at $-\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6} \pm n\pi$, minima at $\left(\frac{\pi}{12} \pm n\pi, 2\right)$ and maxima at $\left(\frac{7\pi}{12} \pm n\pi, -4\right)$.

7 Sketch the graph over the required domain, labelling important points.



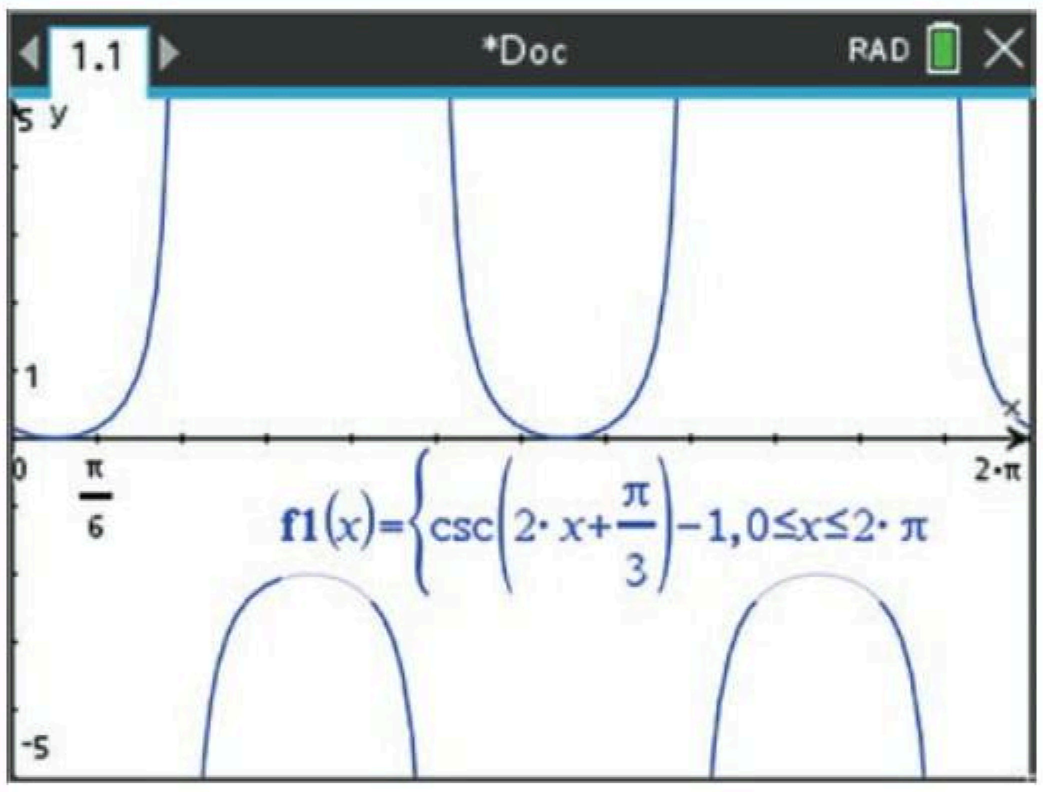
Exam hack

Remember to put *exact values* on your sketch in an examination unless advised otherwise. It is not enough to copy the graph on your calculator.

USING CAS 7 Reciprocal circular functions

Graph $y = \text{cosec}\left(2x + \frac{\pi}{3}\right) - 1$ from $x = 0$ to $x = 2\pi$.

TI-Nspire



$f1(x)=\left\{\csc\left(2\cdot x+\frac{\pi}{3}\right)-1, 0\le x\le 2\cdot\pi\right.$

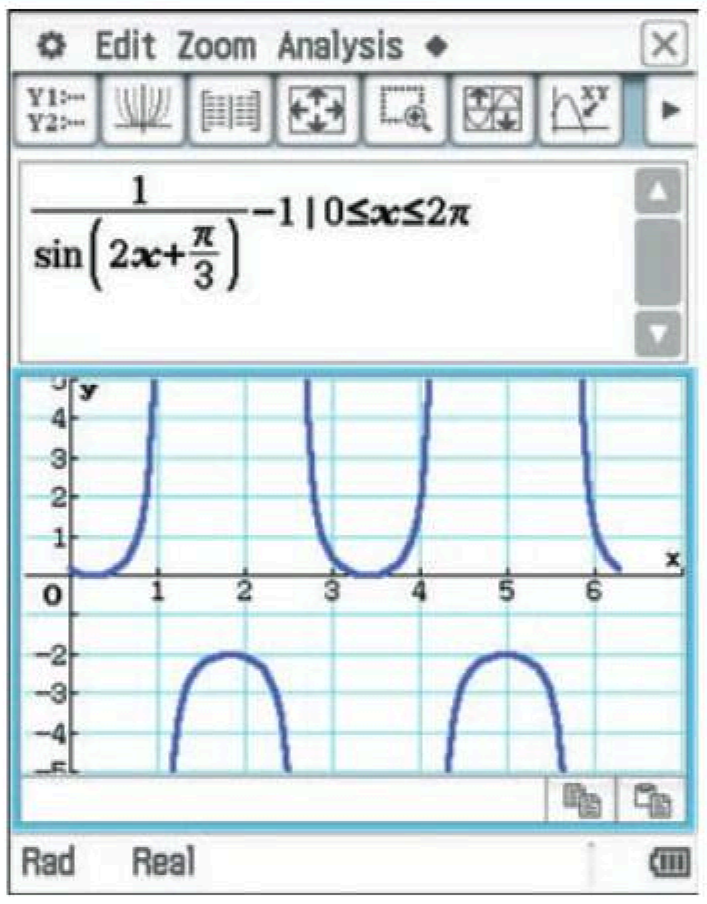
1 Enter the function as shown above.

2 Press **trig** to access the **csc** function.

3 Press **ctrl + =** to access the | and ≤ signs.

4 Adjust the window settings to suit.

ClassPad



$\frac{1}{\sin\left(2x+\frac{\pi}{3}\right)}-1\,|\,0\le x\le 2\pi$

1 Open the **Keyboard** and tap **Math3** to access the | and ≤ signs.

2 Enter, highlight and drag the expression down into the graph window for cosec using **1/sin** as shown above.

3 Adjust the window settings to suit.

WB
p. 38

WORKED EXAMPLE 22 Equations involving reciprocal circular functions

Solve $\operatorname{cosec}(2x) = 1 + 2\sin(2x)$ for all real values of x.

Steps	**Working**
1 Change cosec to sine.	$\frac{1}{\sin(2x)} = 1 + 2\sin(2x)$
2 Simplify.	$1 = \sin(2x) + 2\sin^2(2x)$
3 Write in standard quadratic form.	$2\sin^2(2x) + \sin(2x) - 1 = 0$
4 Factorise the quadratic.	$[2\sin(2x) - 1][\sin(2x) + 1] = 0$
5 Solve for $\sin(2x)$.	$\sin(2x) = \frac{1}{2}$ or $\sin(2x) = -1$
6 Solve for $2x$.	$2x = \frac{\pi}{6} + 2n\pi$ or $2x = \frac{5\pi}{6} + 2n\pi$ or $2x = \frac{3\pi}{2} + 2n\pi$ for $n \in Z$
7 Solve for x.	$x = \frac{\pi}{12} + n\pi,\ x = \frac{5\pi}{12} + n\pi$ or $x = \frac{3\pi}{4} + n\pi$ for $n \in Z$

Exam hack

If you are asked to 'show the solutions are …', they may be in a different form. In that case, you should check whether your solutions can be rewritten to be in the same form.

For the above example, you could change the solutions to $x = (12n+1)\frac{\pi}{12},\ x = (12n+5)\frac{\pi}{12}$ or $x = (4n+3)\frac{\pi}{4}$ using common denominators. You could also write them as $x = (12n-11)\frac{\pi}{12},\ x = (12n-7)\frac{\pi}{12}$ or $x = (4n-1)\frac{\pi}{4}$ by taking 2π from the solutions to $2x$.

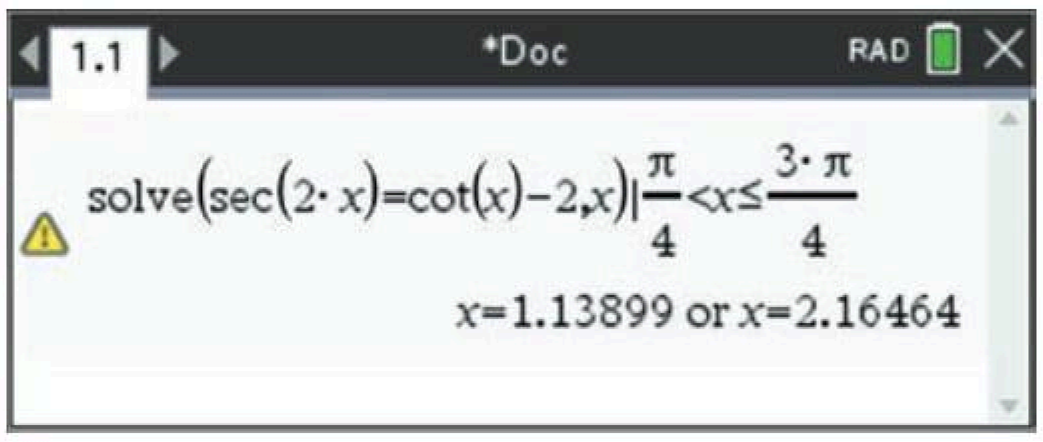
The solutions $2x = \frac{\pi}{6} + 2n\pi$ or $2x = \frac{5\pi}{6} + 2n\pi$ can be combined as $2x = n\pi + (-1)^n\frac{\pi}{6}$ or $2x = n\pi - (-1)^{n+1}\frac{\pi}{6}$, giving $x = [12n + (-1)^n]\frac{\pi}{12}$, etc.

USING CAS 8 Equations and inequalities involving reciprocal circular functions

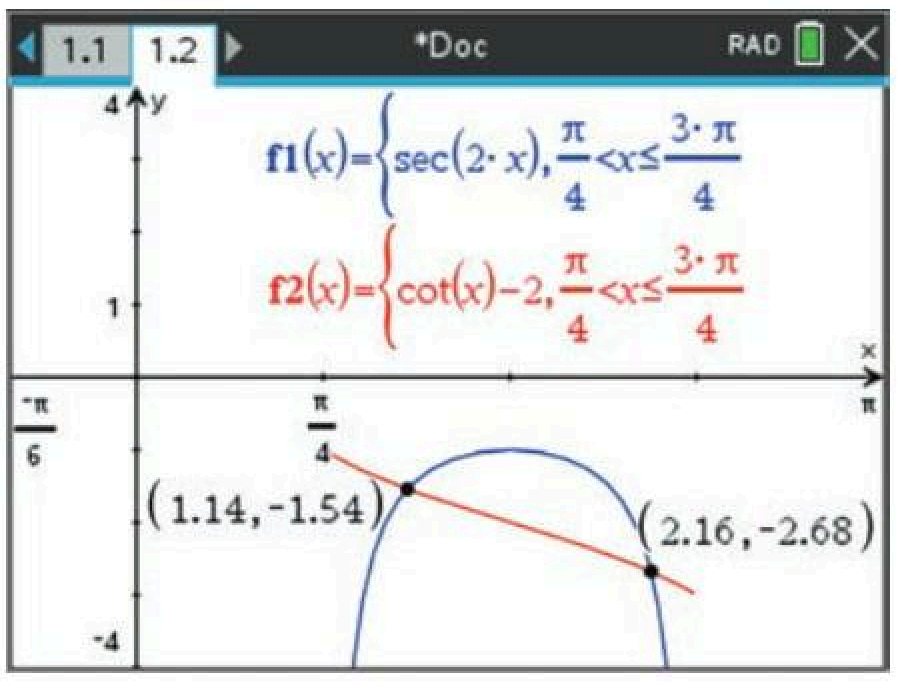
Solve $\sec(2x) \geq \cot(x) - 2$ for $x \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$, correct to three decimal places.

The *inequality* is difficult to solve using CAS, so solve the *equation* instead and consider the graphs.

TI-Nspire



```
solve(sec(2·x)=cot(x)−2,x)|π/4<x≤3·π/4
                    x=1.13899 or x=2.16464
```



$f1(x) = \{\sec(2\cdot x), \frac{\pi}{4} < x \leq \frac{3\cdot\pi}{4}$

$f2(x) = \{\cot(x) - 2, \frac{\pi}{4} < x \leq \frac{3\cdot\pi}{4}$

1 Press **menu > Algebra > Solve** and enter the equation as shown above.

2 Enter **', x'**.

3 Press **ctrl +** = to access the | and ≤ signs.

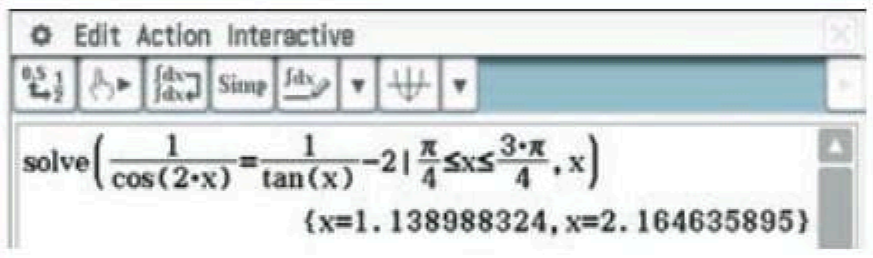
4 Add a **Graphs** page.

5 Enter the two functions over the domain $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ as shown above.

6 Adjust the window settings and axis scales to suit.

7 Press **menu > Trace** or **Analyze Graph** to determine the points of intersection.

ClassPad



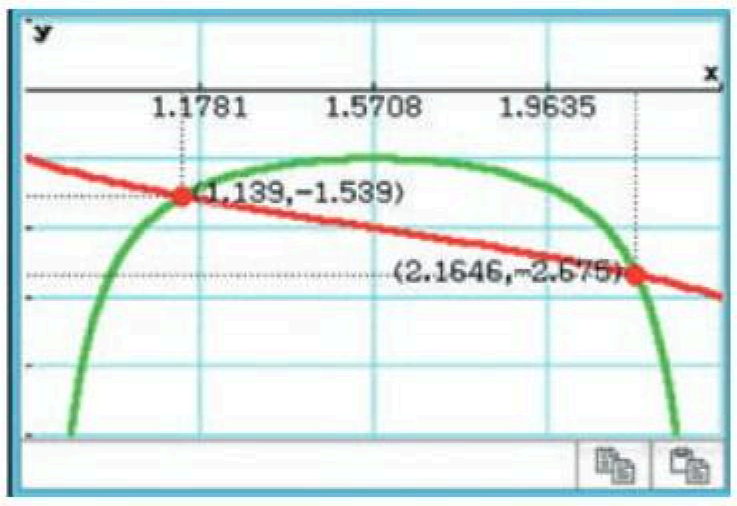
solve$\left(\frac{1}{\cos(2\cdot x)}=\frac{1}{\tan(x)}-2 \mid \frac{\pi}{4}\le x\le\frac{3\cdot\pi}{4},x\right)$

{x=1.138988324, x=2.164635895}

1 In **Main**, enter and highlight the inequality as shown above. Use **1/cos** for sec and **1/tan** for cot.

2 Open the **Keyboard** and tap **Math3** to access the | and ≤ signs.

3 Tap **Interactive > Equation/Inequality > solve > OK.**



4 In **Main**, open the graph screen and drag each side of the equation into the graph window separately over given domain.

5 Adjust the window settings to suit.

6 Tap **Analysis > G-Solve > Intersection** to find the points of intersection.

7 From the graph, decide where $\sec(2x) \ge \cot(x) - 2$.

The solution to the inequality $\sec(2x) \ge \cot(x) - 2$ is $\{x: 1.139 \le x \le 2.165\}$.

EXERCISE 2.6 Reciprocal circular functions

ANSWERS p. 565

Recap

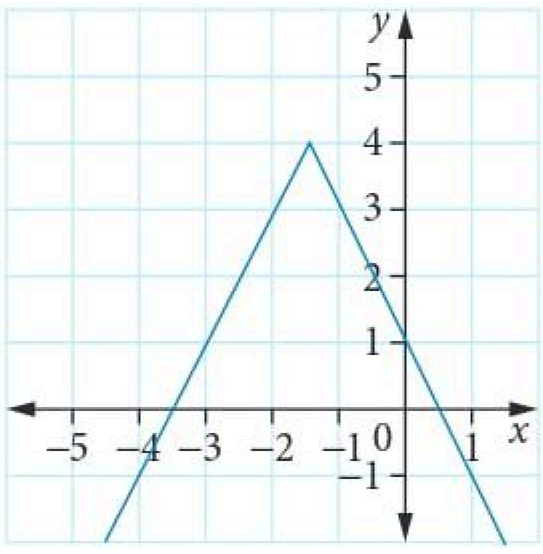
1 The solution of $|2x - 8| < 10$ is

A $x > 9$
B $x < -1$ or $x > 9$
C $-9 < x < 1$
D $-1 < x < 9$
E $x \in R$

2 The graph of the function $f(x)$ is shown.

A possible equation for the function is

A $f(x) = 4 - |2x + 3|$
B $f(x) = |2x + 3| - 4$
C $f(x) = 6 - |4x + 4|$
D $f(x) = 4 - |2x - 3|$
E $f(x) = 4 - |4x + 6|$



Mastery

3 TECH-FREE Find the exact value of

a $\cot\left(\frac{3\pi}{4}\right)$
b $\sec\left(-\frac{\pi}{3}\right)$
c $\operatorname{cosec}\left(\frac{5\pi}{6}\right)$
d $\cot\left(\frac{4\pi}{3}\right)$
e $\operatorname{cosec}\left(-\frac{3\pi}{4}\right)$
f $\sec\left(\frac{5\pi}{3}\right)$

4 Given $\operatorname{cosec}(x) = a$ and $0 \le x \le \frac{\pi}{2}$, find in terms of a

a $\sec(x)$
b $\cot(x)$
c $\sin(x) - \cos(x)$

5 If $\cot(x) = a$ and $\cos(x) = b$, find $\sec(x)$ and $\operatorname{cosec}(x)$ in terms of a and b.

6 $\frac{3\pi}{2} < x < 2\pi$ and $\sec(x) = t$. Find the other 5 trigonometric ratios in terms of t.

7 WORKED EXAMPLE 21 Graph each function for the stated domain.

a $y = 2\sec\left(\frac{x}{2} + \frac{\pi}{6}\right) + 1$ for $-2\pi \le x \le 2\pi$

b $y = 3\cot\left(\pi x - \frac{1}{2}\right) - 2$ for $0 \le x \le 3$

c $y = 2\operatorname{cosec}\left(3x - \frac{\pi}{4}\right) + 2$ for $0 \le x \le \pi$

d $y = -0.5\sec\left(2\pi\left(x - \frac{1}{4}\right)\right) + 1$ for $0 \le x \le 2$

e $y = 2\cot\left(\frac{x}{3} + \frac{\pi}{2}\right) - 3$ for $0 \le x \le 4\pi$

f $y = 4\operatorname{cosec}\left(\frac{x}{2} - \frac{\pi}{6}\right) + 2$ for $0 \le x \le 4\pi$

8 Using CAS 7 Sketch each function.

a $y = 3\cot(2x) + 1$ for $-\pi \le x \le \pi$

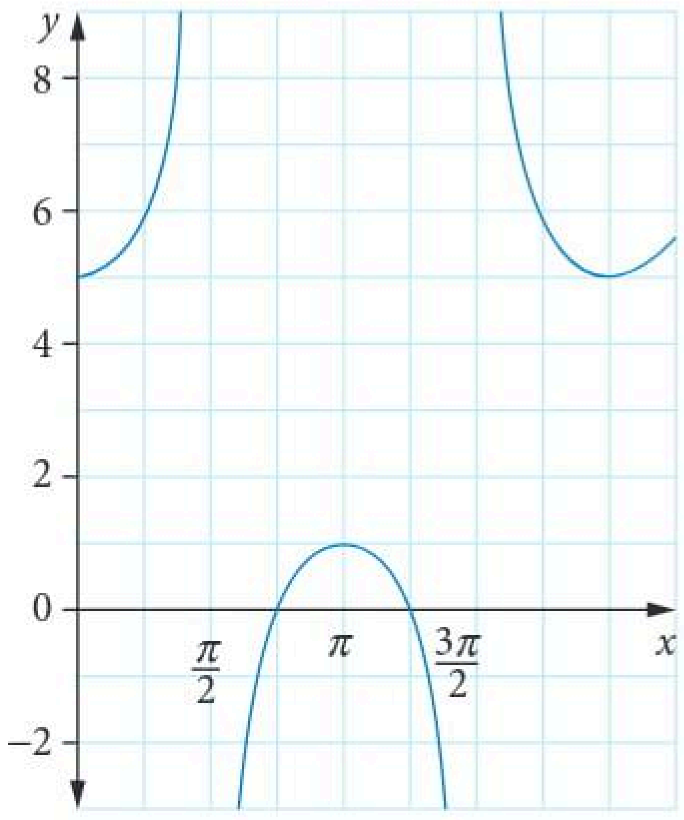
b $y = 0.8\operatorname{cosec}\left(x + \frac{\pi}{4}\right)$ for $-\pi \le x \le \pi$

c $y = 2\sec\left(\pi x - \frac{1}{2}\right) - 1$ for $0 \le x \le 2$

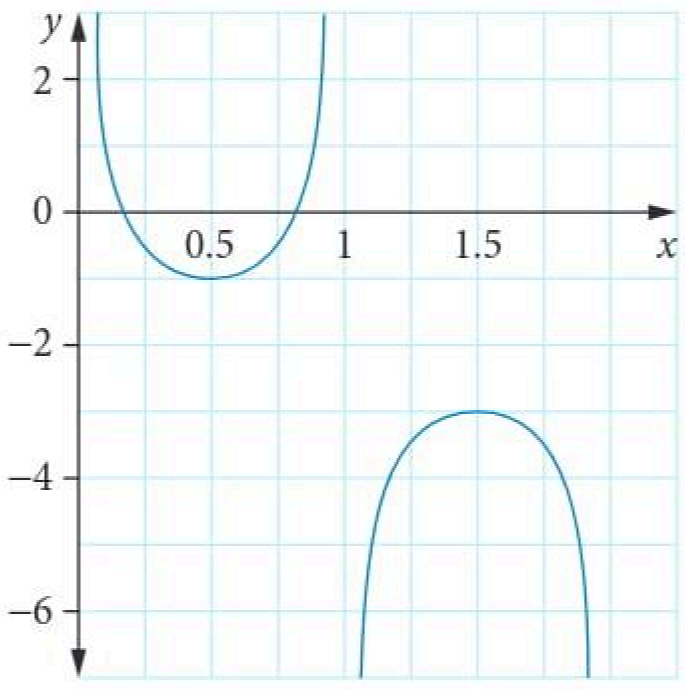
d $y = \frac{1}{2}\cot\left(\frac{x}{2} + \frac{\pi}{2}\right)$ for $0 \le x \le 4\pi$

9 Give a possible rule for each function graphed.

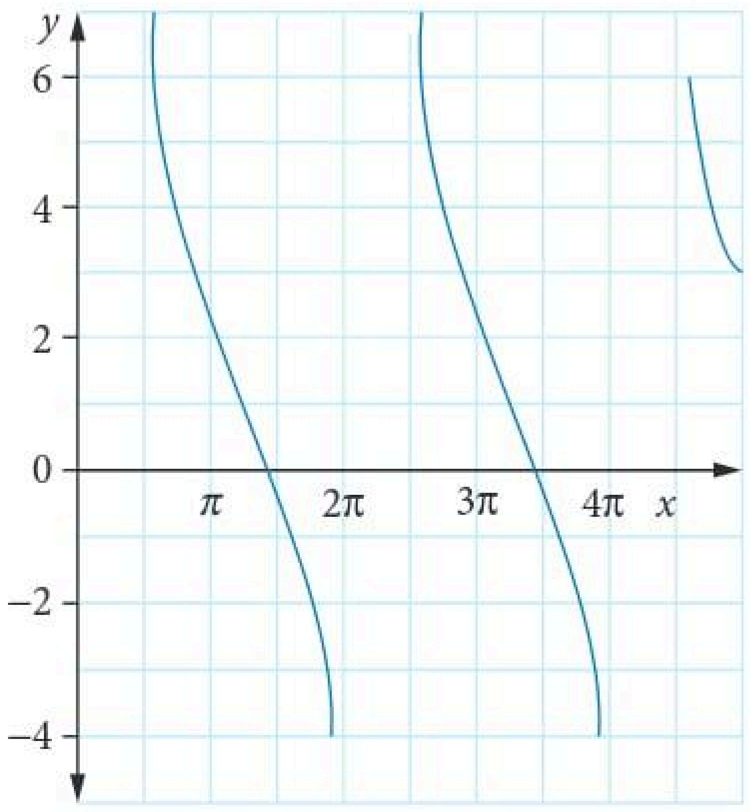
a



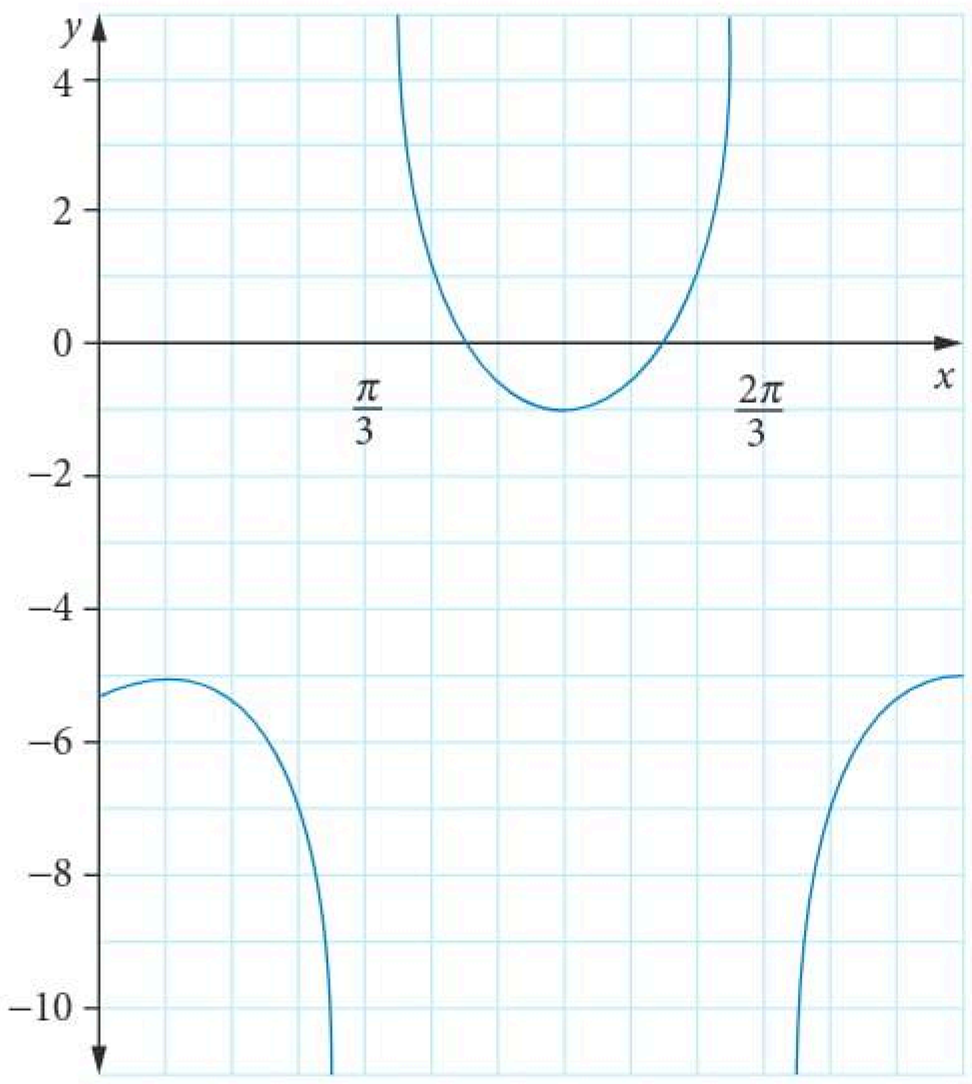
b



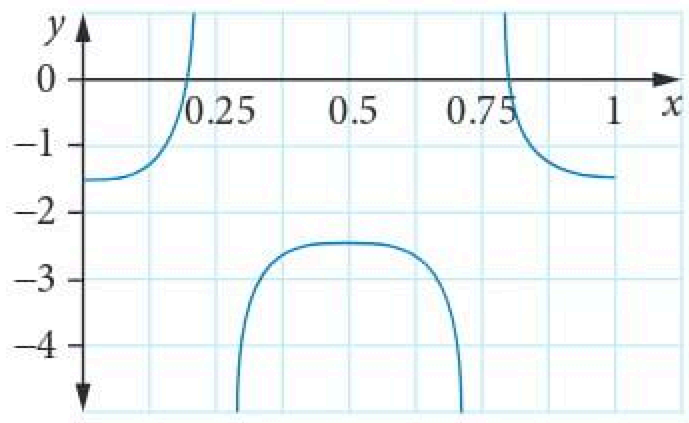
c



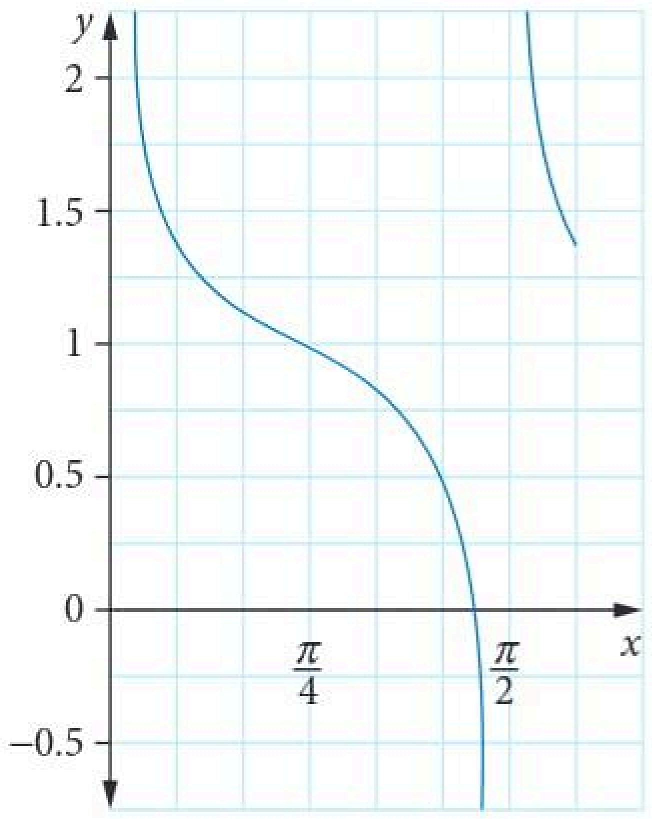
d



e



f



10 WORKED EXAMPLE 22 Solve each equation for all real values of *x*.

a $\sqrt{3}\,\text{cosec}(2x) + 2 = 0$

b $3\cot(3x) = -3\cot(3x) = -\sqrt{3}$

c $2\sec\left(\frac{x}{2} - \frac{\pi}{6}\right) + 4 = 0$

d $2 - \sec^2(\alpha) = \tan^2(\alpha) + \sec(\alpha)$

11 Solve each inequality for $0 \le x \le 2\pi$.

a $\sec(2x) > \sqrt{2}$

b $3\,\text{cosec}(x) \le -6$

c $\sqrt{3}\cot\left(2x + \frac{\pi}{4}\right) < 3$

d $\sqrt{3}(\sec(3x) - \sec^2(3x) \ge 2(\sec^2(3x) - 1)$

12 Using CAS 8 Solve each equation or inequality in the stated domain, correct to three decimal places.

a $\sqrt{3}\cot(3x) > 1$ for $0 \le x \le \frac{\pi}{2}$

b $4\sec(2x) + 6 = 0$ for $-\pi < x < \pi$

c $6\,\text{cosec}^2(x) - 24 = 0$ for $x \in R$

d $1 + \cot(2x) - 2\,\text{cosec}(2x) \ge 0$ for $0 < x < \pi$

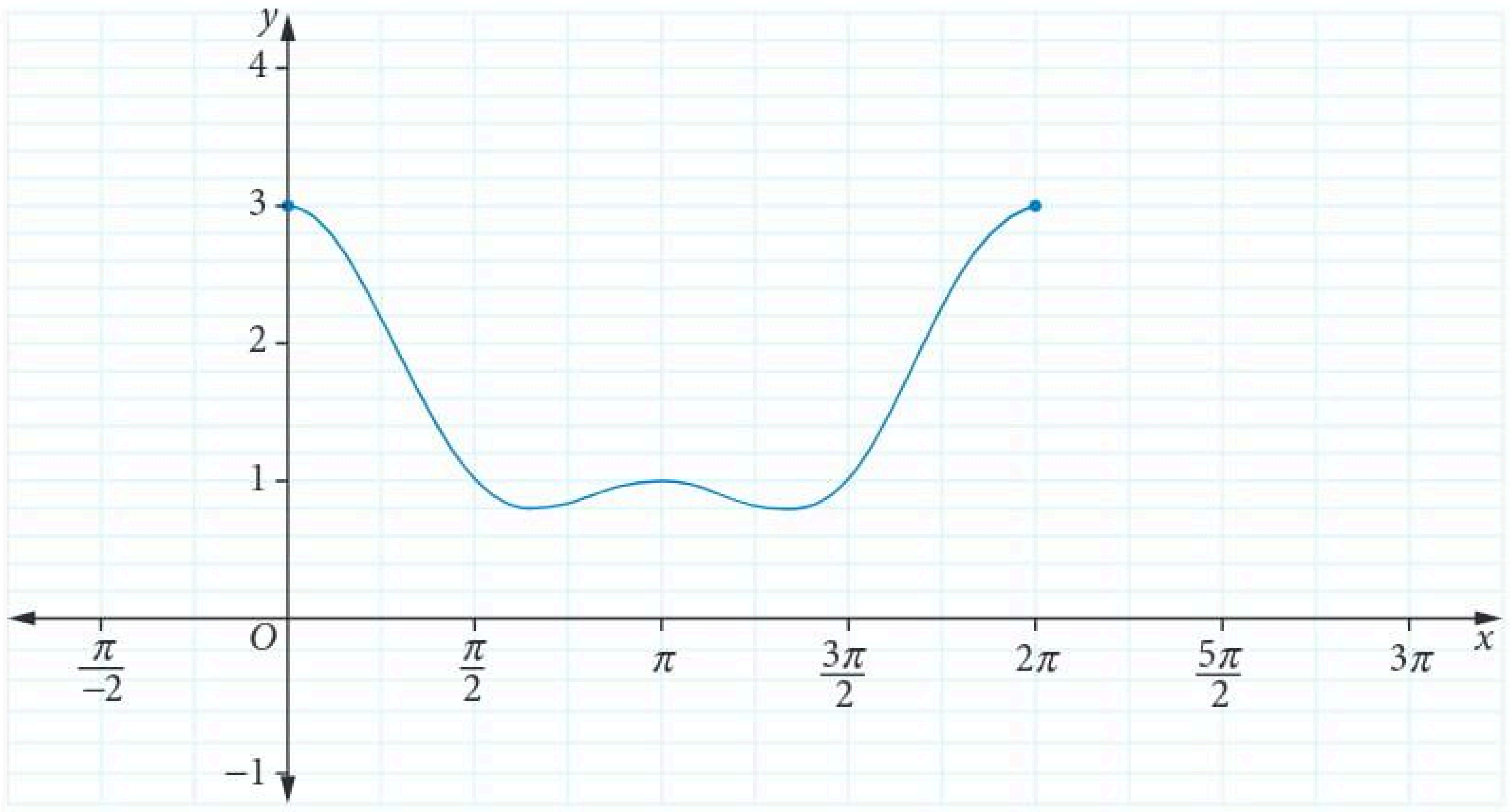
e $5\cot^2(x) + 3\sec^2(x) < 12$ for $0 \le x \le \frac{\pi}{2}$

f $2\sin(3x) + \cot(3x) \le 3$ for $0 < x < \pi$

Exam practice

80–100% 60–79% 0–59%

13 © VCAA 2019 1Q5 TECH-FREE (6 marks) The graph of $f(x) = \cos^2(x) + \cos(x) + 1$ over the domain $0 \le x \le 2\pi$ is shown below.



a **i** 86% Find $f'(x)$. 1 mark

ii 68% Hence, find the coordinates of the turning points of the graph in the interval $(0, 2\pi)$. 2 marks

b 54% Copy the above graph and on it sketch the graph of $y = \frac{1}{f(x)}$ on the set of axes above. Clearly label the turning points and endpoints of this graph with their coordinates. 3 marks

14 © VCAA 2019 2AQ3 **65%** The implied domain of the function with rule

$f(x) = 1 - \sec\left(x + \frac{\pi}{4}\right)$ is

A R

B $[0, 2]$

C $R \setminus \left\{\frac{(4n-1)\pi}{4}\right\}, n \in Z$

D $R \setminus \left\{\frac{(4n+1)\pi}{4}\right\}, n \in Z$

E $R \setminus \left\{\frac{(2n-1)\pi}{2}\right\}, n \in Z$

15 © VCAA 2010 2AQ5 **60%** For $-\frac{\pi}{2} < x < \frac{\pi}{2}$, the graphs of the two curves given by $y = 2\sec^2(x)$ and $y = 5\,|\tan(x)|$ intersect

A only at the one point (arctan (2), 10).

B only at the two points (±arctan (2), 10).

C only at the one point $\left(\arctan\left(\frac{1}{2}\right), \frac{5}{2}\right)$.

D only at the two points $\left(\pm\arctan\left(\frac{1}{2}\right), \frac{5}{2}\right)$.

E at the two points $\left(\pm\arctan\left(\frac{1}{2}\right), \frac{5}{2}\right)$, as well as at the two points (±arctan (2), 10).

16 © VCAA 2018 2AQ4 **49%** If $\cos(x) = -a$ and $\cot(x) = b$, where $a, b > 0$, then $\operatorname{cosec}(-x)$ is equal to

A $\frac{b}{a}$

B $-\frac{b}{a}$

C $-\frac{a}{b}$

D $\frac{a}{b}$

E $-ab$

17 © VCAA 2017 2AQ2 **37%** The solutions to $\cos(x) > \frac{1}{4}\operatorname{cosec}(x)$ for $x \in (0, 2\pi) \setminus \{\pi\}$ are given by

A $x \in \left(\frac{\pi}{12}, \frac{5\pi}{12}\right) \cup \left(\frac{5\pi}{12}, \frac{13\pi}{12}\right) \cup \left(\frac{17\pi}{12}, 2\pi\right)$

B $x \in \left(\frac{\pi}{12}, \frac{5\pi}{12}\right) \cup \left(\frac{13\pi}{12}, \frac{17\pi}{12}\right)$

C $x \in \left(\frac{\pi}{12}, \frac{5\pi}{12}\right) \cup \left(\frac{13\pi}{12}\right) \cup \left(\frac{13\pi}{12}, 2\pi\right)$

D $x \in \left(\frac{\pi}{12}, \frac{13\pi}{12}\right) \cup \left(\frac{17\pi}{12}, 2\pi\right)$

E $x \in \left(\frac{\pi}{12}, \frac{5\pi}{12}\right) \cup \left(\frac{13\pi}{12}\right) \cup \left(\frac{17\pi}{12}, 2\pi\right)$

2.7 Trigonometric identities

We learned the following **trigonometric identities** in Year 11.

Trigonometric identities

$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$

$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$

$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$

$\tan(x+y) = \dfrac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$

$\tan(x-y) = \dfrac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$

$\sin(2x) = 2\sin(x)\cos(x)$

$\cos(2x) = \cos^2(x) - \sin^2(x)$

$\quad$ or $2\cos^2(x) - 1$

$\quad$ or $1 - 2\sin^2(x)$

$\tan(2x) = \dfrac{2\tan(x)}{1-\tan^2(x)}$

$\sin^2(x) + \cos^2(x) = 1$

$\tan^2(x) + 1 = \sec^2(x)$

$\cot^2(x) + 1 = \text{cosec}^2(x)$

Video playlist
Trigonometric identities

Worksheets
Simplifying periodic functions

Trigonometric identities

WB
p. 39

WORKED EXAMPLE 23 Compound angles and exact values

Show that $\sin\left(\dfrac{\pi}{12}\right) = \dfrac{\sqrt{6}-\sqrt{2}}{4}$, and hence that $\text{cosec}\left(\dfrac{\pi}{12}\right) = \sqrt{6}+\sqrt{2}$.

Steps	**Working**
1 Choose an appropriate **sum and difference identity**.	$\sin\left(\dfrac{\pi}{12}\right) = \sin\left(\dfrac{\pi}{3} - \dfrac{\pi}{4}\right)$
2 Expand using $\sin(x-y)$.	$= \sin\left(\dfrac{\pi}{3}\right)\cos\left(\dfrac{\pi}{4}\right) - \cos\left(\dfrac{\pi}{3}\right)\sin\left(\dfrac{\pi}{4}\right)$
3 Simplify using exact values.	$= \dfrac{\sqrt{3}}{2} \times \dfrac{\sqrt{2}}{2} - \dfrac{1}{2} \times \dfrac{\sqrt{2}}{2}$ $= \dfrac{\sqrt{6}-\sqrt{2}}{4}$
4 Invert both sides to get $\text{cosec}\left(\dfrac{\pi}{12}\right)$.	$\text{cosec}\left(\dfrac{\pi}{12}\right) = \dfrac{4}{\sqrt{6}-\sqrt{2}}$
5 Rationalise the denominator.	$= \dfrac{4(\sqrt{6}+\sqrt{2})}{(\sqrt{6}-\sqrt{2})(\sqrt{6}+\sqrt{2})}$ $= \dfrac{4(\sqrt{6}+\sqrt{2})}{4}$
6 Simplify.	$= \sqrt{6}+\sqrt{2}$

Exam hack

'Show that' means you have to prove the result. If the required expression is of the form $\frac{\sqrt{a \pm \sqrt{b}}}{c}$, use $\cos(2A)$; but if it is of the form $\frac{\sqrt{a} \pm \sqrt{b}}{c}$ or $\frac{a \pm \sqrt{b}}{c}$, use an angle sum or difference.

We can sometimes use a double angle formula to change a trigonometric equation so that it contains only one trigonometric ratio.

WB
p. 40

WORKED EXAMPLE 24 Double angle application to trigonometric equation

Solve $2\cos(2x) = \sin(x) + 2$ for $0 \le x \le 2\pi$.

Steps	Working
1 Change to an equation with only $\sin(x)$.	$2[1 - 2\sin^2(x)] = \sin(x) + 2$
2 Expand the brackets.	$2 - 4\sin^2(x) = \sin(x) + 2$
3 Simplify.	$-4\sin^2(x) = \sin(x)$
4 Change to one side and factorise.	$\sin(x)[4\sin(x) + 1] = 0$
5 Solve for $\sin(x)$.	$\sin(x) = 0$ or $\sin(x) = -\frac{1}{4}$
6 Solve for x in the required domain.	$\sin(x) = 0 \Rightarrow x = 0, \pi, 2\pi$ $\sin(x) = -\frac{1}{4} \Rightarrow x = \pi + \sin^{-1}\left(\frac{1}{4}\right) 2\pi - \sin^{-1}\left(\frac{1}{4}\right)$ 3rd and 4th quadrants

Exam hack

Don't divide both sides by $\sin(x)$ at step 3 because if $\sin(x) = 0$ you'd be dividing by 0.

WB
p. 41

WORKED EXAMPLE 25 Identities and exact values

Given $\cos(x) = \frac{1}{4}$ and $\pi < x < 2\pi$, find the value of $\tan(x)$.

Steps	Working
1 Find $\sec(x)$.	$\sec(x) = \frac{1}{\cos(x)} = 4$
2 Now find $\tan(x)$.	$\tan^2(x) + 1 = \sec^2(x)$ $\tan^2(x) + 1 = 16$ $\tan^2(x) = 15$ $\tan(x) = \pm\sqrt{15}$
3 Find the quadrant.	Since $\cos(x) > 0$ and $\pi < x < 2\pi$, x is in the fourth quadrant.
4 Exclude the incorrect answer.	$\tan(x) < 0$, so $\tan(x) = -\sqrt{15}$

WORKED EXAMPLE 26 Identities and equations

Solve $x^3 - x[\sec^2(x) - \tan^2(x)] = 0$.

Steps	**Working**
1 Rearrange the identity $\tan^2(x) + 1 = \sec^2(x)$.	$\sec^2(x) - \tan^2(x) = 1$
2 Substitute in the equation.	$x^3 - x = 0$
3 Factorise.	$x(x^2 - 1) = 0$ $x(x - 1)(x + 1) = 0$
4 Use the null factor law.	$x = 0,\ x - 1 = 0$, or $x + 1 = 0$
5 Write the solutions.	$x = 0,\ x = 1$ or $x = -1$

We can also solve the equation $x^3 - x = 0$ by inspecting the intersections of the graphs of $y = x^3$ and $y = x$.

EXERCISE 2.7 Trigonometric identities

ANSWERS p. 567

WB p. 42

Recap

1 The graph of $f(x)$ is shown on the right.

A possible equation for the function is:

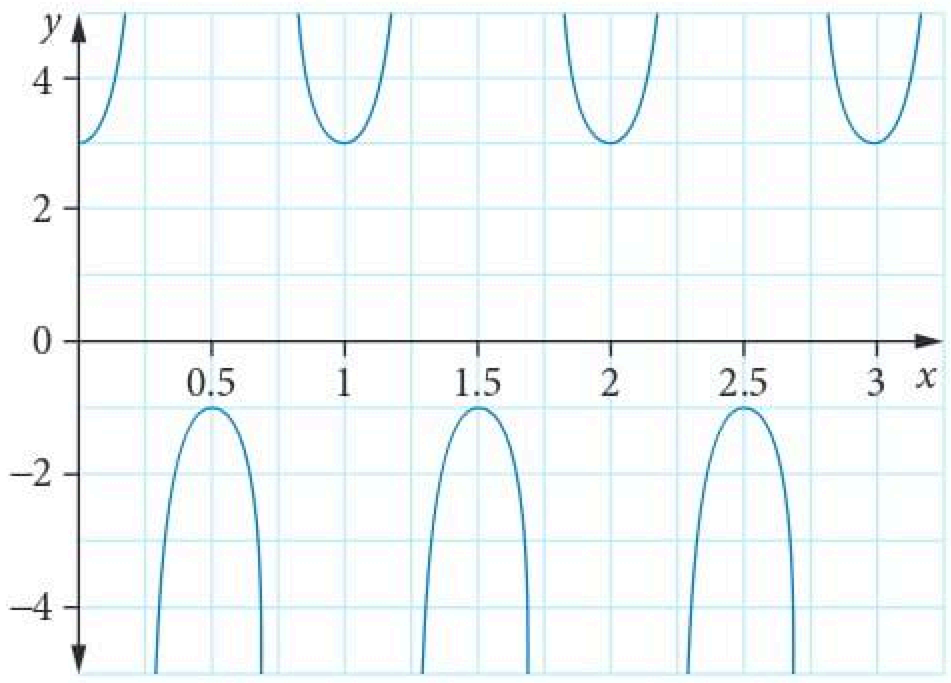
A $f(x) = 2\ \text{cosec}\,(2\pi x) + 1$

B $f(x) = 2.5\ \sec(2\pi x)$

C $f(x) = 3\ \sec(2\pi x) + 1$

D $f(x) = 2\ \sec(2\pi x) + 1$

E $f(x) = 2\ \text{cosec}\,(2\pi x) - 1$



2 The solutions of $\cos(2x) = 1 + \sin(x)$ for $0 \le x < 2\pi$ are

A $0, \dfrac{\pi}{6}$ **B** $0, \dfrac{11\pi}{6}$ **C** $\dfrac{\pi}{6}$ **D** $\dfrac{11\pi}{6}$ **E** $0, \dfrac{\pi}{6}, \dfrac{11\pi}{6}$

Mastery

3 WORKED EXAMPLE 23 Evaluate each expression in exact form.

a $\sin\left(\dfrac{7\pi}{12}\right)$ **b** $\sec\left(-\dfrac{\pi}{12}\right)$ **c** $\tan\left(\dfrac{5\pi}{12}\right)$ **d** $\cot\left(-\dfrac{3\pi}{8}\right)$

4 Show that each equation is true.

a $\text{cosec}\left(-\dfrac{\pi}{12}\right) = -\sqrt{2} - \sqrt{6}$ **b** $\cos\left(\dfrac{17\pi}{12}\right) = \dfrac{\sqrt{2} - \sqrt{6}}{4}$ **c** $\cot\left(-\dfrac{7\pi}{12}\right) = 2 - \sqrt{3}$

5 Find each exact value.

a $\text{cosec}\left(-\dfrac{\pi}{8}\right)$ **b** $\cos\left(\dfrac{\pi}{12}\right)$ **c** $\cot\left(-\dfrac{\pi}{8}\right)$

6 Show each equation is true.

a $\sin\left(\dfrac{3\pi}{8}\right) = \dfrac{\sqrt{2 + \sqrt{2}}}{2}$ **b** $\sec\left(-\dfrac{7\pi}{8}\right) = -\sqrt{4 - 2\sqrt{2}}$ **c** $\tan\left(\dfrac{5\pi}{8}\right) = -1 - \sqrt{2}$

7 WORKED EXAMPLE 24 Solve each equation for $-\pi \le x \le \pi$.

a $\sin(2x) = \cos(x)$

b $\cos(2x) = \cos(x) + 2$

▶ **8** Solve each inequality.

a $\{x : \sec^2(2x) < 2, -\pi \le x \le \pi\}$

b $\{x : 4\sin^2(3x) > 3, 0 \le x \le \pi\}$

9 Given $\sin(x+y) = a$ and $\sin(x-y) = b$, find an expression for $\sin(x)\cos(y)$.

10 Given $\cot(x) - \tan(x) = \frac{1}{2}$, find $\cot(2x)$.

11 Given $\tan(2x) = \sqrt{15}$ and $\frac{3\pi}{2} < x < 2\pi$, find the value of $\operatorname{cosec}(x)$.

12 WORKED EXAMPLE 25

a If $\operatorname{cosec}(x) = 3$, find $\cot(x)$, where x is acute.

b Given $\tan(x) = \sqrt{3+\sqrt{2}}$ and $0 < x < \pi$, find $\sec(x)$.

c Show that $\sec^2(x) + \operatorname{cosec}^2(x) = \sec^2(x)\operatorname{cosec}^2(x)$.

d Show that $\sec(2x) = \frac{1+\tan^2(x)}{1-\tan^2(x)}$.

13 WORKED EXAMPLE 26 Solve $x^4 - 4\operatorname{cosec}^2(x) = -\frac{4}{\tan^2(x)}$ for $x \in R$.

Exam practice

80–100% 60–79% 0–59%

14 © VCAA 2018 1Q7 81% TECH-FREE (3 marks) Given that $\cot(2x) + \frac{1}{2}\tan(x) = a\cot(x)$, use a suitable double angle formula to find the value of a, $a \in R$.

15 © VCAA 2016 1Q9 62% TECH-FREE (3 marks) Given that $\cos(x - y) = \frac{3}{5}$ and $\tan(x)\tan(y) = 2$, find $\cos(x + y)$.

16 The value of $\sec\left(\frac{17\pi}{12}\right)$ is

A $\frac{\sqrt{2}-\sqrt{6}}{4}$ **B** $-\sqrt{6}-\sqrt{2}$ **C** $\frac{\sqrt{6}-\sqrt{2}}{4}$ **D** $\sqrt{6}-\sqrt{2}$ **E** $\sqrt{6}+\sqrt{2}$

17 $\sin(x+y)\cos(x-y)$ is equal to

A $\sin(x)\cos(x) + \sin(y)\cos(y)$

B $\sin(x)\cos(x)[\cos^2(y) - \sin^2(y)] + \sin(y)\cos(y)[\cos^2(x) - \sin^2(x)]$

C $\sin(x)\cos(x)[\cos^2(y) - \sin^2(y)] - \sin(y)\cos(y)[\cos^2(x) - \sin^2(x)]$

D $\sin(x)\cos(x) - \sin(y)\cos(y)$

E $\sin(x)\cos(x)[\sin^2(y) - \cos^2(y)] + \sin(y)\cos(y)[\sin^2(x) - \cos^2(x)]$

18 $\cos(4\theta)$ is equal to

A $4\cos^2(\theta) - 1$

B $4\cos^4(\theta) - 1$

C $2\cos^4(\theta) - 2\cos^2(\theta) - 1$

D $4\cos^4(\theta) - 2\cos^2(\theta) + 1$

E $4\cos^4(\theta) - 2\cos^2(\theta) - 1$

19 $\cot(x) = 4$, a possible value of $\operatorname{cosec}(x)$ is

A 5 **B** $\sqrt{3}$ **C** $\sqrt{5}$ **D** $\frac{1}{\sqrt{3}}$ E $-\sqrt{3}$ ▶

▶ **20** The solutions of $\cos(2x) - \sin(x) = 0$ on the interval $[0, 2\pi)$ are

A $\frac{\pi}{6}$
B $\frac{\pi}{6}, \frac{5\pi}{6}$
C $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}$
D $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$
E $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{2}$

21 © VCAA 2012 2BQ2a 67% (2 marks) Given that $\cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{\sqrt{3}+2}}{2}$, show that $\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{2-\sqrt{3}}}{2}$.

2.8 Inverse circular functions

We learned about the **inverse circular functions** in Year 11, in which the **principal value domains** of $\sin(x)$, $\cos(x)$ and $\tan(x)$ are restricted to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, $0 \le x \le \pi$ and $-\frac{\pi}{2} < x < \frac{\pi}{2}$ respectively to make them one-to-one, so that their inverses are functions.

Video playlist Inverse circular functions

Worksheet Inverse functions

Inverse circular functions

The **inverse sine**, **inverse cosine** and **inverse tangent** functions are defined as:

- $y = \sin^{-1}(x)$ if and only if $\sin(y) = x$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.
- $y = \cos^{-1}(x)$ if and only if $\cos(y) = x$ and $0 \le y \le \pi$.
- $y = \tan^{-1}(x)$ if and only if $\tan(y) = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

$\sin^{-1}(x)$, $\cos^{-1}(x)$ and $\tan^{-1}(x)$ are also called the **arcsine** (**arcsin**), **arccosine** (**arccos**) and **arctangent** (**arctan**) functions, respectively.

The domains of $\sin^{-1}(x)$ and $\cos^{-1}(x)$ are both $[-1, 1]$.

The domain of $\tan^{-1}(x)$ is the set of real numbers, R.

For any values, $\text{arccosec}(a) = \arcsin\left(\frac{1}{a}\right)$, $\text{arcsec}(a) = \arccos\left(\frac{1}{a}\right)$ and $\text{arccot}(a) = \arctan\left(\frac{1}{a}\right)$.

WORKED EXAMPLE 27 Exact value of an inverse circular function

WB p. 43

Find the value of **a** $\cos^{-1}(-0.5)$ **b** $\text{arccot}\left(\sqrt{3}\right)$

Steps	**Working**
a 1 What value of $\cos(y) = 0.5$?	$\cos\left(\frac{\pi}{3}\right) = 0.5$
2 For what value does $\cos(y) = -0.5$ with y in the first two quadrants?	$\cos\left(\frac{2\pi}{3}\right) = -0.5$
3 Write the answer.	$\cos^{-1}(-0.5) = \frac{2\pi}{3}$
b 1 Write in terms of $\arctan(x)$.	$\text{arccot}\left(\sqrt{3}\right) = \arctan\left(\frac{1}{\sqrt{3}}\right)$
2 What angle has $\tan(x) = \frac{1}{\sqrt{3}}$?	$= \frac{\pi}{6}$

CAS shows the inverse circular functions as $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$.

USING CAS 9 Inverse circular functions

Find $\arccos\left(\frac{1}{\sqrt{2}}\right)$, $\arctan(-4)$ and $\operatorname{cosec}^{-1}(3)$.

TI-Nspire

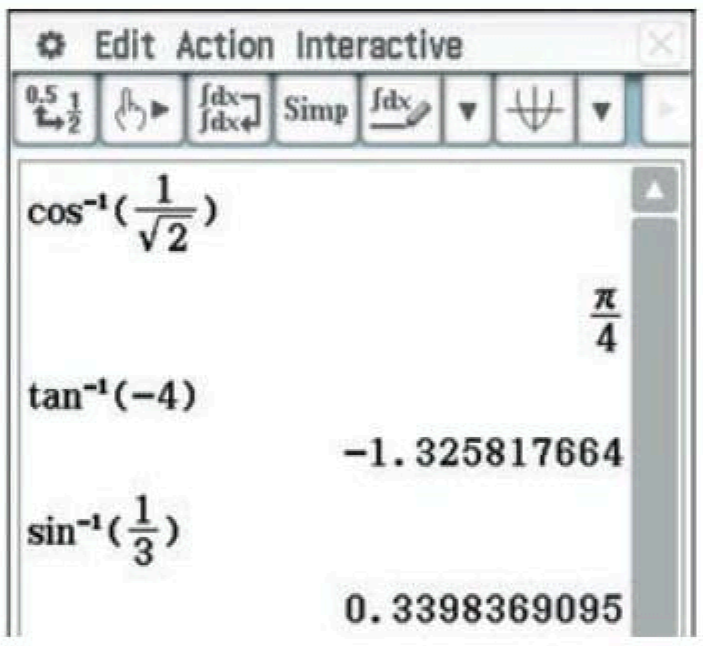
$\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ $\frac{\pi}{4}$

$\tan^{-1}(-4)$ -1.32582

$\csc^{-1}(3)$ 0.339837

Press **trig** to access the **cos⁻¹**, **tan⁻¹** and **csc⁻¹** functions.

ClassPad



$\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$\frac{\pi}{4}$

$\tan^{-1}(-4)$

-1.325817664

$\sin^{-1}\left(\frac{1}{3}\right)$

0.3398369095

1 In **Main**, open the **Keyboard > Trig** to access the **cos⁻¹**, **tan⁻¹** and **sin⁻¹** functions.

2 Enter $\sin^{-1}\left(\frac{1}{3}\right)$ for $\operatorname{cosec}^{-1}(3)$.

$\arccos\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$, $\arctan(-4) \approx -1.326$ and $\operatorname{cosec}^{-1}(3) \approx 0.3398$

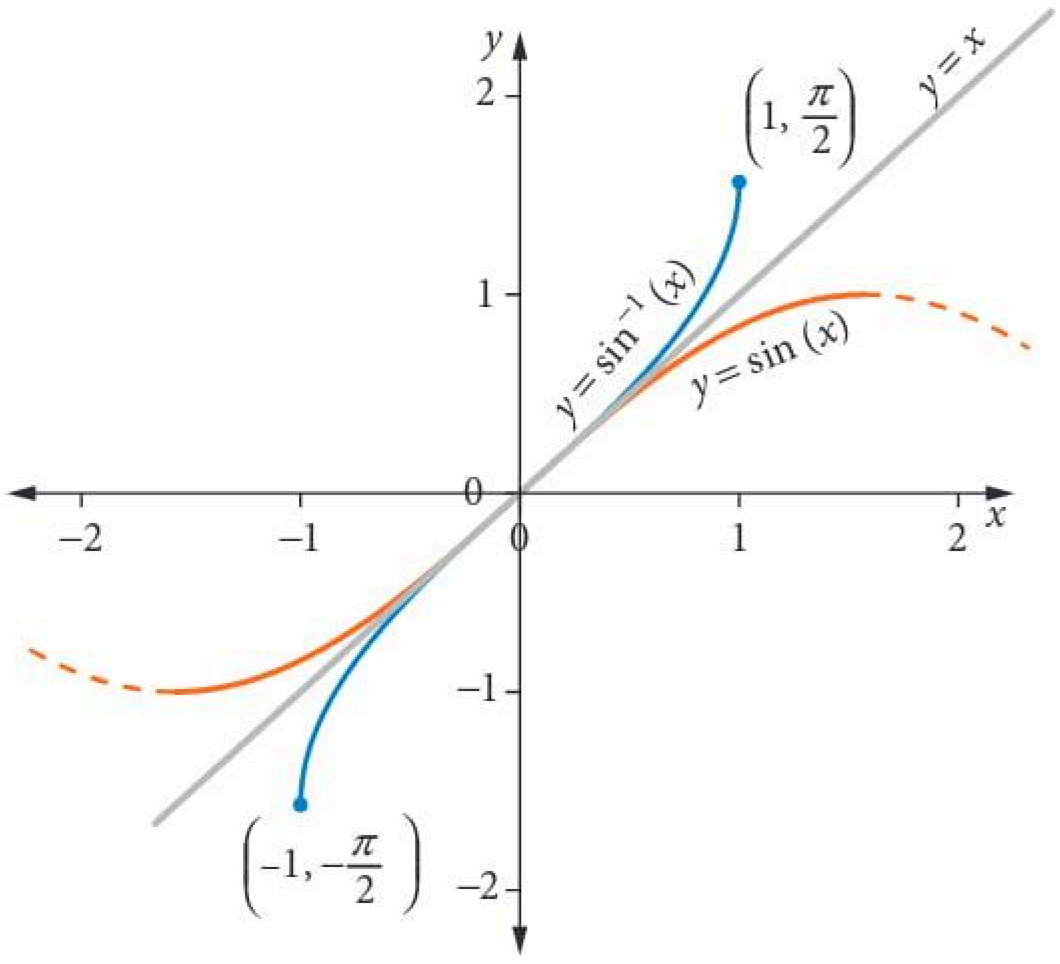
The graph of an inverse function is the reflection of the original function in the line $y = x$. This is because the values of x and y are swapped for the inverse. The matched graphs of

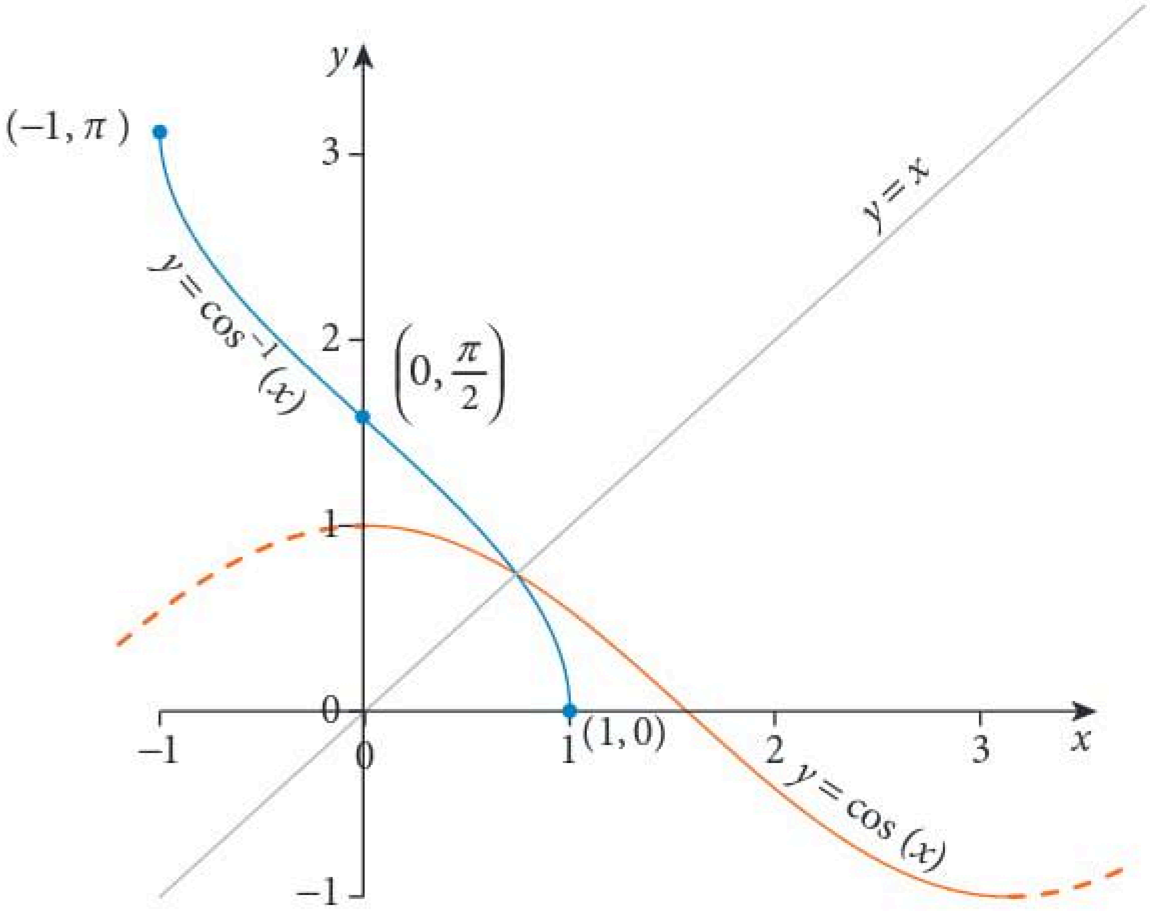
$y = \sin(x)$, $y = x$ and $y = \sin^{-1}(x)$;

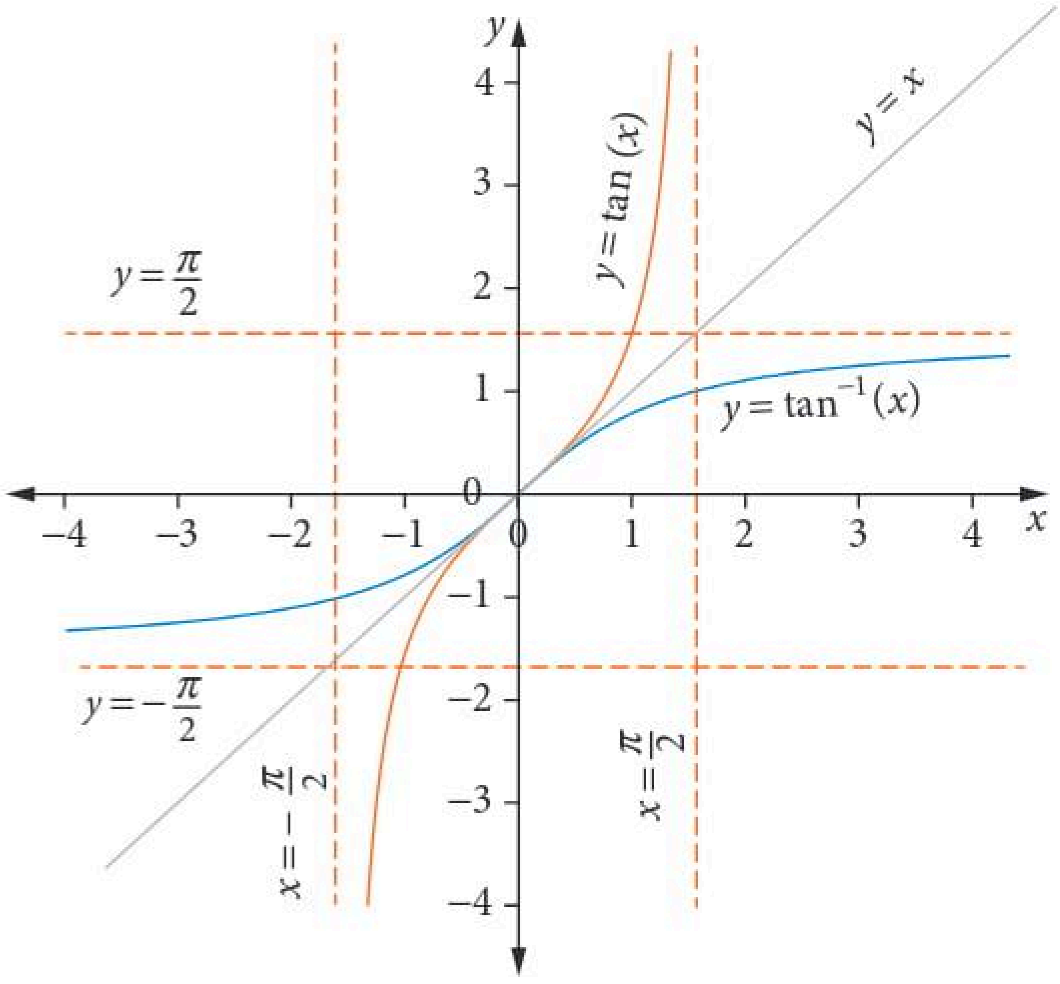
$y = \cos(x)$, $y = x$ and $y = \cos^{-1}(x)$;

$y = \tan(x)$, $y = x$ and $y = \tan^{-1}(x)$ are shown below and on the next page.

The endpoints of $\sin^{-1}(x)$ and $\cos^{-1}(x)$ are shown to indicate that these functions do not continue.







The inverse circular functions are transformed in the same manner as other functions.

Sketch the graphs of $y = a\,f\,[n\,(x + b)] + c$ for $f(x) = \sin^{-1}(x), f(x) = \cos^{-1}(x)$ and $f(x) = \tan^{-1}(x)$ by transforming $f(x)$ in the following order.

'HORIZONTALLY':

- Dilation from the y-axis by a factor of $\frac{1}{n}$ (period from n):
  shrink for $n > 1$ or $n < -1$, stretch for $-1 < n < 1$

This can be written as $|n| > 1$.

This can be written as $|n| < 1$.

- Reflection in the y-axis if $n < 0$.
- Translation parallel to the x-axis (phase from b): left if $b > 0$, right if $b < 0$.

'VERTICALLY':

- Dilation from the x-axis by a factor of a: shrink for $-1 < a < 1$, stretch for $a > 1$ or $a < -1$
- Reflection in the x-axis if $a < 0$.
- Translation parallel to the y-axis: up if $c > 0$, down if $c < 0$.

It may be easier to transform the **important points** and **asymptotes** for inverse tan graphs and join them:

asymptotes $y = \frac{\pi}{2}$, $y = -\frac{\pi}{2}$, points $\left(-1, -\frac{\pi}{4}\right)$, $(0, 0)$, $\left(1, \frac{\pi}{4}\right)$. We can also do this for inverse sin and cos.

We can find domains and ranges using simple inequalities.

WB
p. 44

WORKED EXAMPLE 28 Domain and range of inverse circular functions

What are the maximal domain and range of $y = -3\sin^{-1}\left(1 - \frac{x}{2}\right) - \pi$?

Steps	**Working**
1 Use the domain of $\sin^{-1}(A)$.	$-1 \le 1 - \frac{x}{2} \le 1$
2 Solve each side.	$-1 \le 1 - \frac{x}{2} \Rightarrow -2 \le -\frac{x}{2} \Rightarrow x \le 4$
	$1 - \frac{x}{2} \le 1 \Rightarrow -\frac{x}{2} \le 0 \Rightarrow x \ge 0$
3 Combine the solutions.	$0 \le x \le 4$
4 Use the range of $\sin^{-1}(A)$.	$-\frac{\pi}{2} \le \sin^{-1}(A) \le \frac{\pi}{2}$

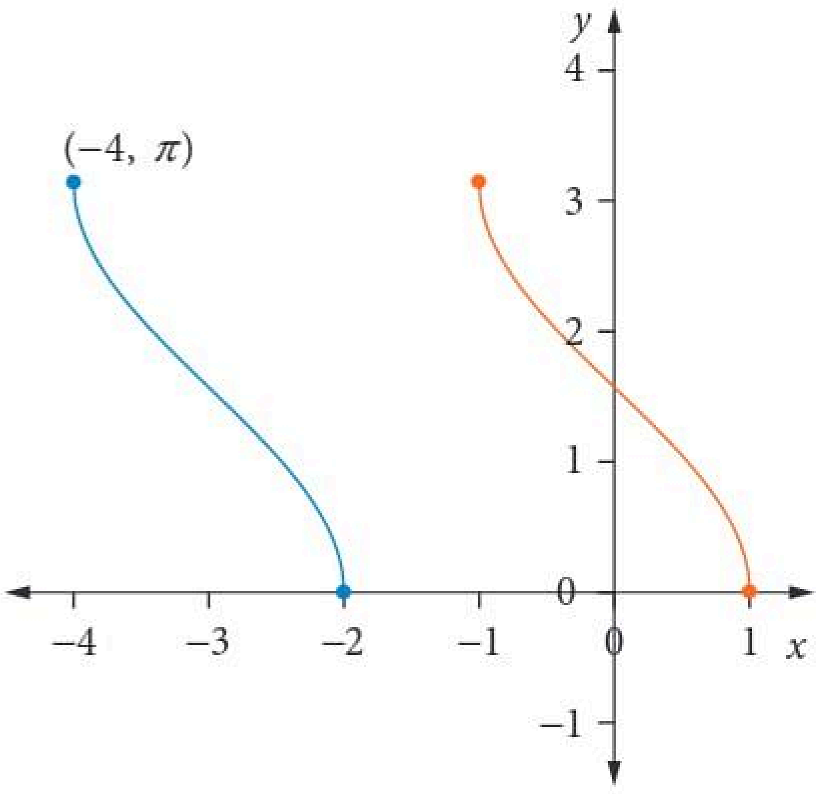
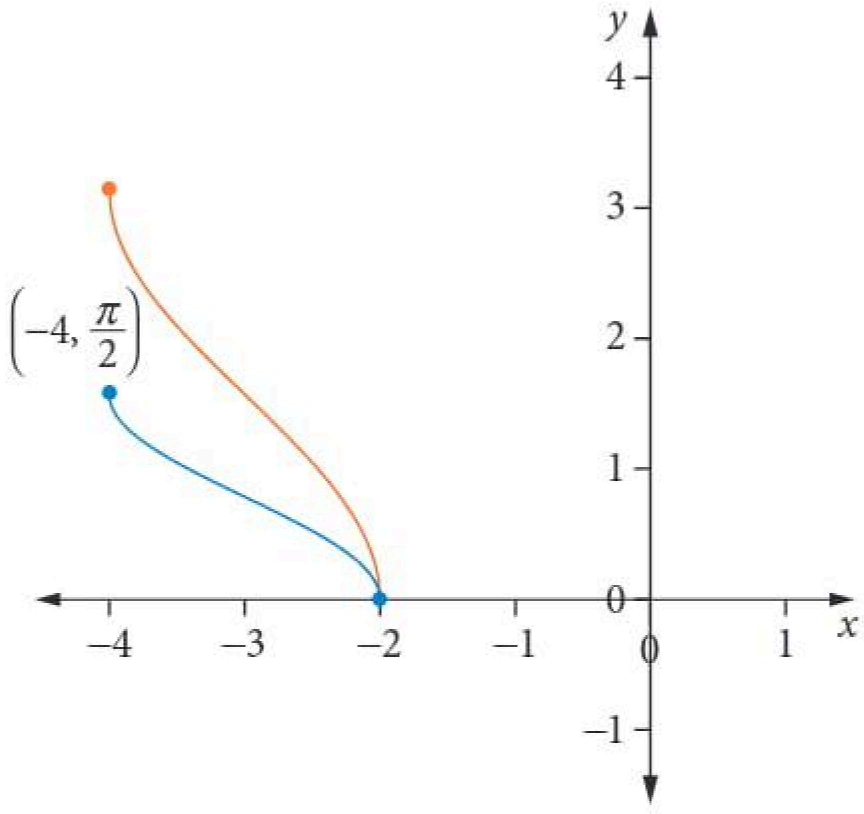
5 Multiply by −3, reversing the signs for negative multiplication.	$\frac{3\pi}{2} \ge -3\sin^{-1}(A) \ge -\frac{3\pi}{2}$
6 Subtract π.	$\frac{3\pi}{2} - \pi \ge -3\sin^{-1}(A) - \pi \ge -\frac{3\pi}{2} - \pi$ $\frac{\pi}{2} \ge -3\sin^{-1}(A) - \pi \ge -\frac{5\pi}{2}$
7 Write the answer.	The domain is $x \in [0, 4]$ and the range is $y \in \left[-\frac{5\pi}{2}, \frac{\pi}{2}\right]$.

We should include the endpoints of graphs with limited domains.

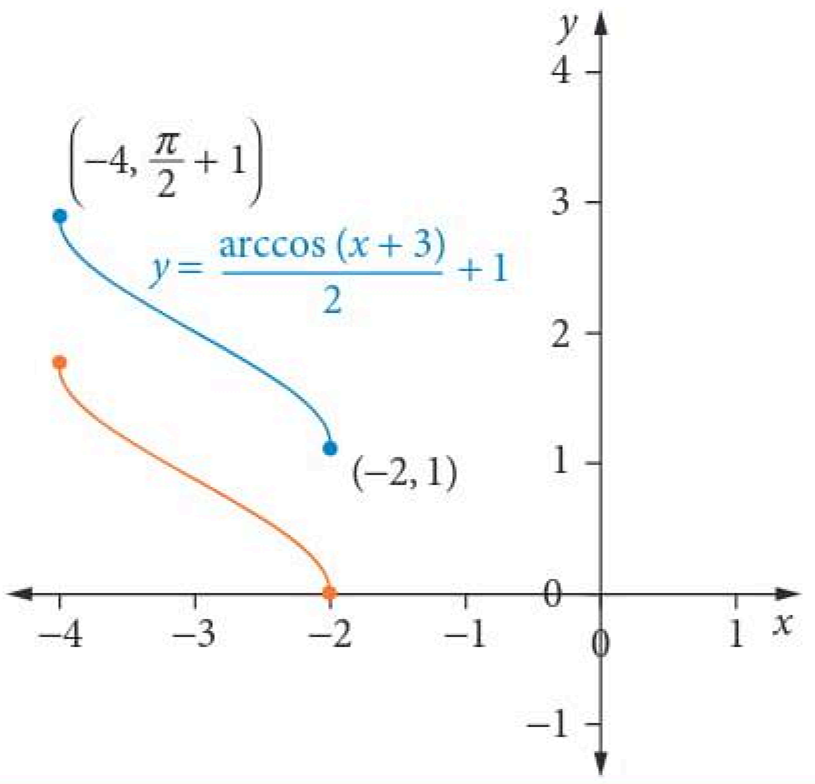
WB
p. 45

WORKED EXAMPLE 29 Transformation of inverse circular functions

Sketch the graph of $y = \frac{\arccos(x+3)}{2} + 1$.

Steps	**Working**
1 State the transformations.	Translation parallel to the x-axis: left by 3. Dilation from the x-axis by a factor of $\frac{1}{2}$. Translation parallel to the y-axis: up by 1.
2 Translate 3 left.	
3 Dilate from the x-axis by a factor of $\frac{1}{2}$.	

4 Translate 1 up.

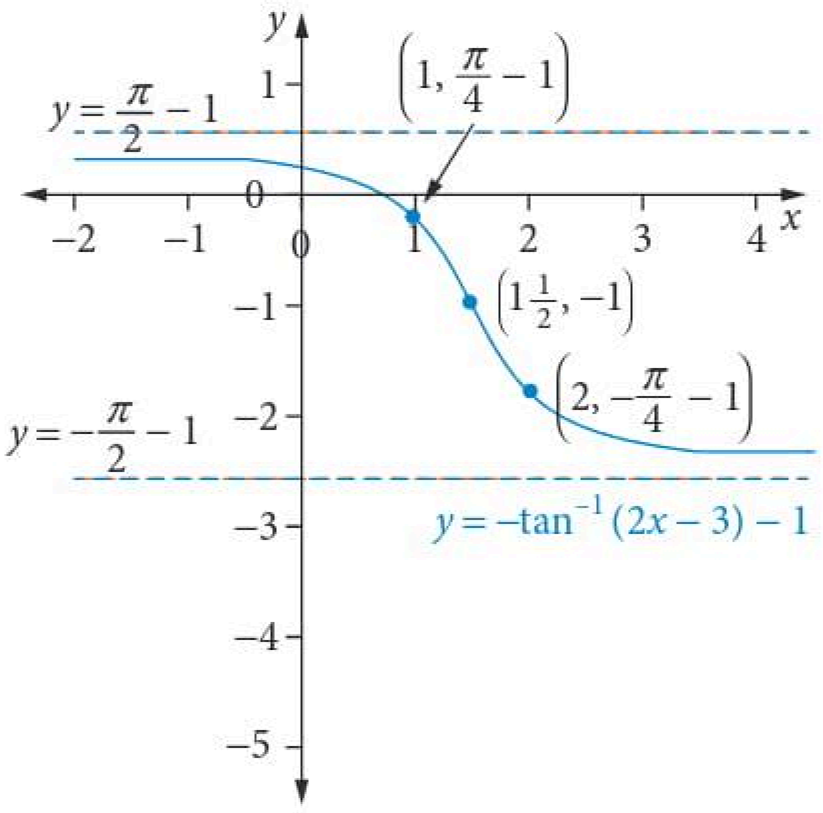


Make sure you include the equations of asymptotes where applicable.

WB
p. 46

WORKED EXAMPLE 30 Transformation of arctan function

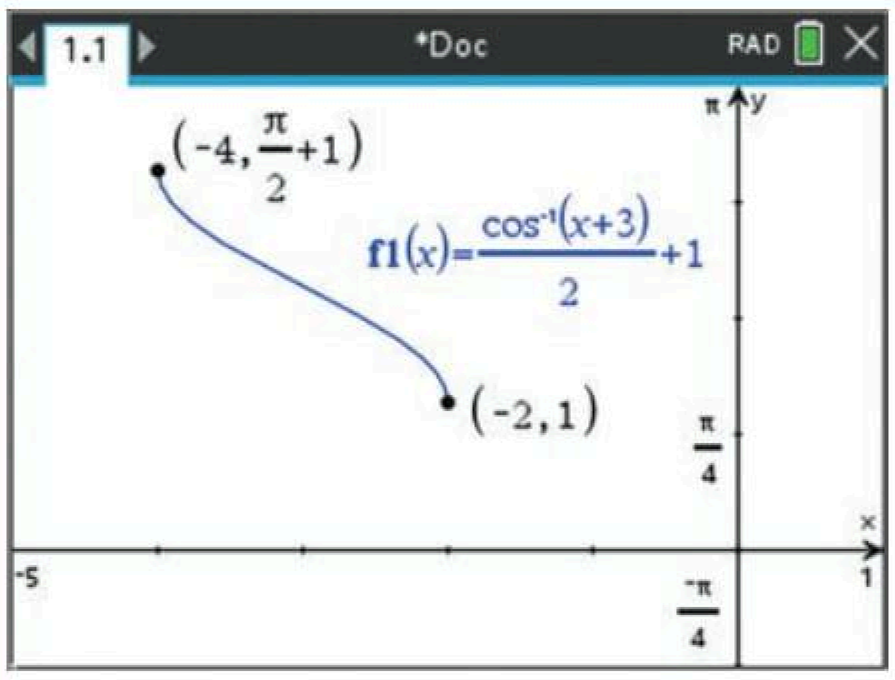
Sketch the graph of $y = -\tan^{-1}(2x - 3) - 1$.

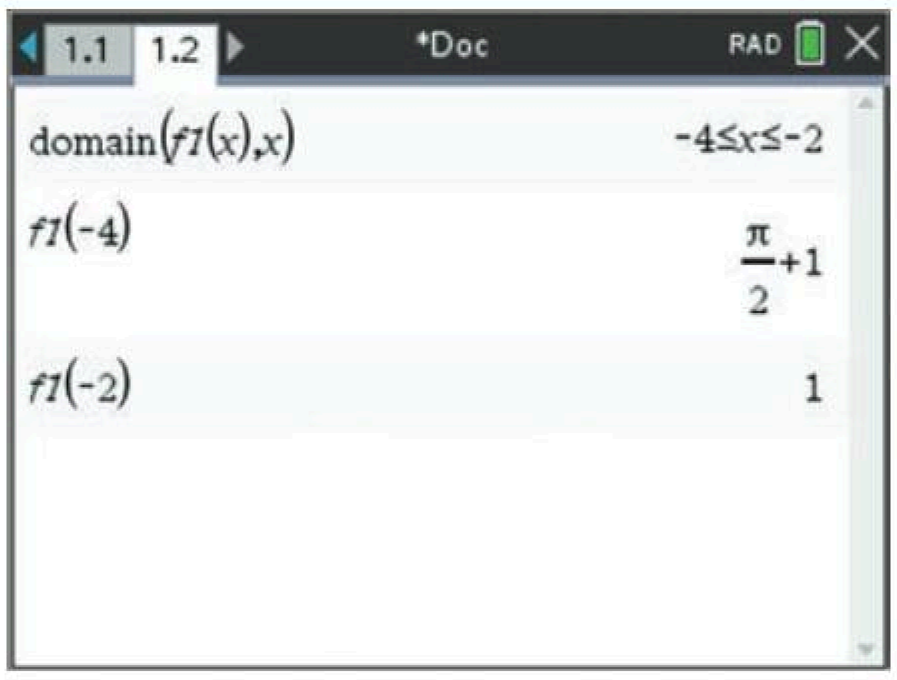
Steps	**Working**
1 Write in standard form.	$y = -\tan^{-1}\left[2\left(x - 1\frac{1}{2}\right)\right] - 1$
2 State the transformations.	Dilation from the y-axis by a factor of $\frac{1}{2}$. Translation $1\frac{1}{2}$ units right. Reflection in the x-axis. Translation 1 unit down.
3 Transform the important points and asymptotes.	$\left(-1, -\frac{\pi}{4}\right) \rightarrow \left(-\frac{1}{2}, -\frac{\pi}{4}\right) \rightarrow \left(1, -\frac{\pi}{4}\right) \rightarrow \left(1, \frac{\pi}{4}\right) \rightarrow \left(1, \frac{\pi}{4} - 1\right)$ $(0, 0) \rightarrow (0, 0) \rightarrow \left(1\frac{1}{2}, 0\right) \rightarrow \left(1\frac{1}{2}, 0\right) \rightarrow \left(1\frac{1}{2}, -1\right)$ $\left(1, -\frac{\pi}{4}\right) \rightarrow \left(\frac{1}{2}, -\frac{\pi}{4}\right) \rightarrow \left(2, -\frac{\pi}{4}\right) \rightarrow \left(2, \frac{\pi}{4}\right) \rightarrow \left(2, \frac{\pi}{4} - 1\right)$
Only the vertical changes are needed on the asymptotes.	$y = \frac{\pi}{2} \rightarrow y = -\frac{\pi}{2} \rightarrow y = -\frac{\pi}{2} - 1$ $y = -\frac{\pi}{2} \rightarrow y = \frac{\pi}{2} \rightarrow y = \frac{\pi}{2} - 1$
4 Sketch the graph.	

USING CAS 10 Graphing inverse circular functions

Graph $y = \dfrac{\cos^{-1}(x+3)}{2} + 1$, showing the domain and range.

TI-Nspire





domain$(f1(x),x)$ $\quad -4 \le x \le -2$

$f1(-4)$ $\quad \frac{\pi}{2}+1$

$f1(-2)$ $\quad 1$

1 Press **trig** to access **cos^{-1}** and enter the function as shown above.

2 Adjust the window settings to suit.

3 Press **menu > Geometry > Points & Lines > Point On**.

4 Click to place two points on the graph and alter the coordinates to find the endpoints.

5 To find the exact values of the domain and range, add a **Calculator** page.

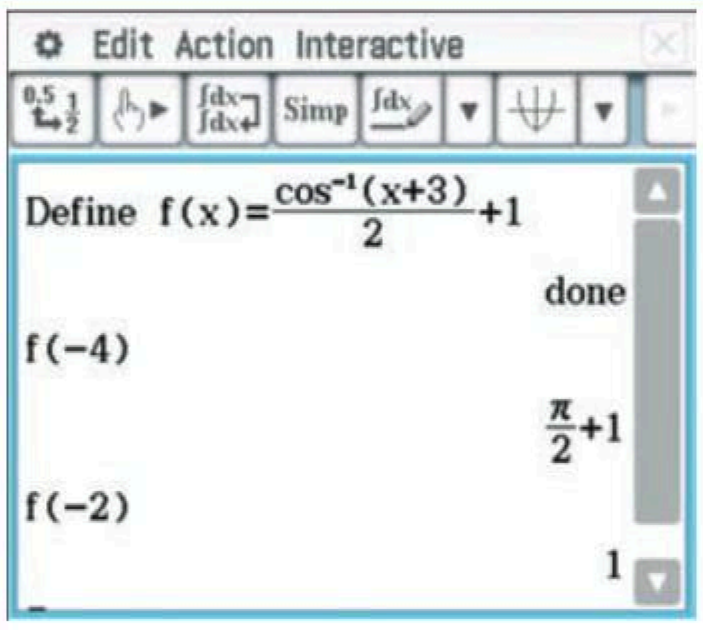
6 Press **catalog** then **D** to jump to the functions starting with d.

7 Scroll down to select **domain**.

8 Enter **f1(x), x** to determine the domain.

9 Enter **f1(–4)** and **f1(–2)** to determine the exact values of the range.

ClassPad



Define $f(x)=\frac{\cos^{-1}(x+3)}{2}+1$

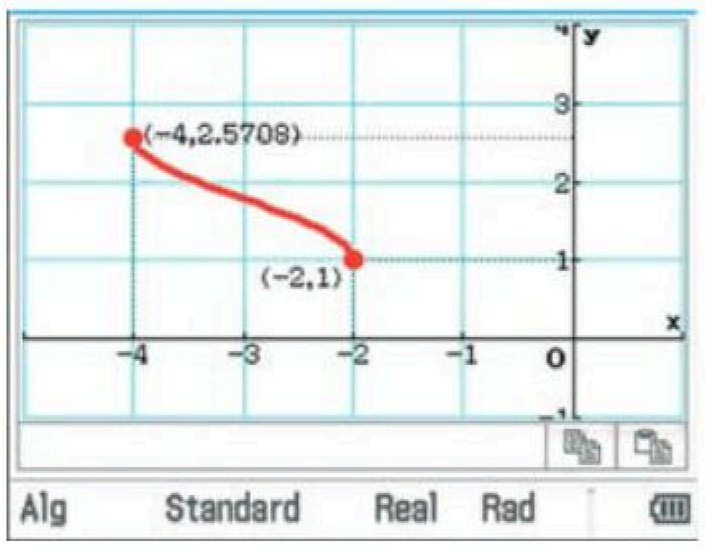
done

$f(-4)$

$\frac{\pi}{2}+1$

$f(-2)$

1



1 In **Main**, open the **Keyboard** and tap **Trig** to access **cos^{-1}**.

2 **Define** the expression, and drag into the **Graph** screen.

3 Adjust the window settings to suit.

4 Tap **Analysis > G-Solve > fMax** to determine the maximum endpoint.

5 Tap **Analysis > G-Solve > fMin** to determine the minimum endpoint.

6 To find the exact values of the range, enter in the top screen $f(-4)$ and $f(-2)$.

$f(-4) = \dfrac{\pi}{2} + 1$ and $f(-2) = 1$

Use simple inequalities to find the domain and range of a function involving the inverse circular functions with constants or other functions.

WORKED EXAMPLE 31 Implied domains and ranges of inverse circular functions

State the implied domain and range of each function.

a $f(x) = a + \arcsin^2(bx + c)$, where $a, b, c \in R$ and $b > 0$

b $y = a\sqrt{\dfrac{b}{\arctan(cx + d)}}$, where $a, b, c, d \in R$ and $b, c > 0$

Steps	**Working**
a **1** Use the domain of arcsin (x).	$-1 \le bx + c \le 1$
2 Solve for x.	$-1 - c \le bx,\ bx \le 1 - c$ $x \ge -\dfrac{1+c}{b},\ x \le \dfrac{1-c}{b}\quad (b > 0)$
3 Use the range of arcsin (x).	$-\dfrac{\pi}{2} \le \arcsin(bx + c) \le \dfrac{\pi}{2}$
4 Work towards the expression.	$0 \le (\arcsin(bx + c))^2 \le \dfrac{\pi^2}{4}$ $a \le a + (\arcsin(bx + c))^2 \le a + \dfrac{\pi^2}{4}$
5 Write the answer.	domain $= \{x \in R : -\dfrac{1+c}{b} \le x \le \dfrac{1-c}{b}$ range $= \{y \in R : a \le y \le a + \dfrac{\pi^2}{4}$
b **1** Use the domain of arctan (x).	$cx + d \in R$
2 The square root of $\dfrac{b}{\arctan(cx + d)}$ is part of the expression.	But $\dfrac{b}{\arctan(cx + d)} \ge 0$ and $b > 0$, so $\arctan(cx + d) > 0$, so $cx + d > 0$
3 Solve for x.	$cx > -d$ $x > -\dfrac{d}{c}$
4 Use the (restricted) range of arctan $(cx + d)$.	$0 < \arctan(cx + d) < \dfrac{\pi}{2}$
5 Work towards expression.	$\dfrac{1}{\arctan(cx + d)} > \dfrac{2}{\pi}$ $\dfrac{b}{\arctan(cx + d)} > \dfrac{2b}{\pi}\quad (b > 0)$ $\sqrt{\dfrac{b}{\arctan(cx + d)}} > \sqrt{\dfrac{2b}{\pi}}$
Consider $a > 0$ and $a < 0$ separately.	$a\sqrt{\dfrac{b}{\arctan(cx + d)}} > a\sqrt{\dfrac{2b}{\pi}}$ for $a > 0$ $a\sqrt{\dfrac{b}{\arctan(cx + d)}} < a\sqrt{\dfrac{2b}{\pi}}$ for $a < 0$ $y > a$ $y > a\sqrt{\dfrac{2b}{\pi}}$ for $a > 0$, $y < a\sqrt{\dfrac{2b}{\pi}}$ for $a < 0$
6 Write the answer.	Domain is $x > -\dfrac{d}{c}$, range is $\lvert y \rvert > \left\lvert a\sqrt{\dfrac{2b}{\pi}} \right\rvert$ and y is the same sign as a.

WB p. 47

Video
VCE question analysis: Rational functions

VCE QUESTION ANALYSIS

© VCAA 2017 2BQ1 2017 Examination 2 Section B Question 1 (8 marks)

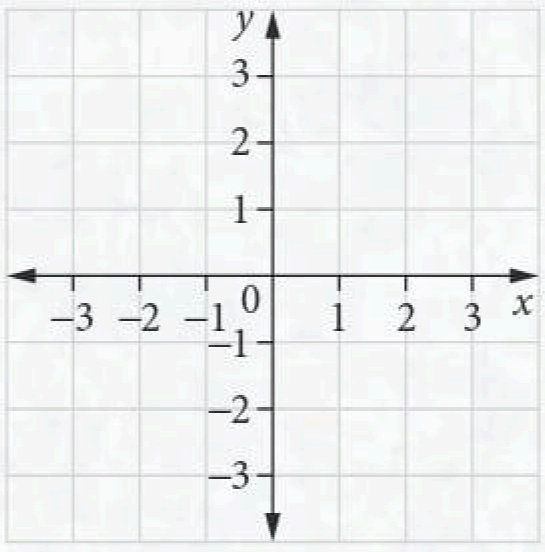
Let $f: D \to R, f(x) = \dfrac{x}{1+x^3}$, where D is the maximal domain of f.

a **i** Find the equations of any asymptotes of the graph of f. 1 mark

ii Find $f'(x)$ and state the coordinates of any stationary points of the graph of f, correct to two decimal places. 2 marks

iii Find the coordinates of any points of inflection of the graph of f, correct to two decimal places. 2 marks

b Copy the axes below, and on them sketch the graph of $f(x) = \dfrac{x}{1+x^3}$ from $x = -3$ to $x = 3$, marking all stationary points, points of inflection and intercepts with axes, labelling them with their coordinates. Show any asymptotes and label them with their equations. 3 marks



Reading the question

- The degree of the denominator is greater than the degree of the numerator.
- Need to differentiate to find stationary points.
- Use second derivative for points of inflection.

Thinking about the question

- The degrees show there is a horizontal asymptote at $y = 0$.
- The denominator will factorise, so there must be at least one vertical asymptote.
- The stationary points are at $f'(x) = 0$ and the points of inflection are at $f''(x) = 0$.
- For part **b**, use zeros, y-intercept and signs as well as the answers to part **a**.

Worked solution (✓ = 1 mark)

a **i** $f(x) = \dfrac{x}{1+x^3}$

$f(x)$ is undefined for $1 + x^3 = 0$.

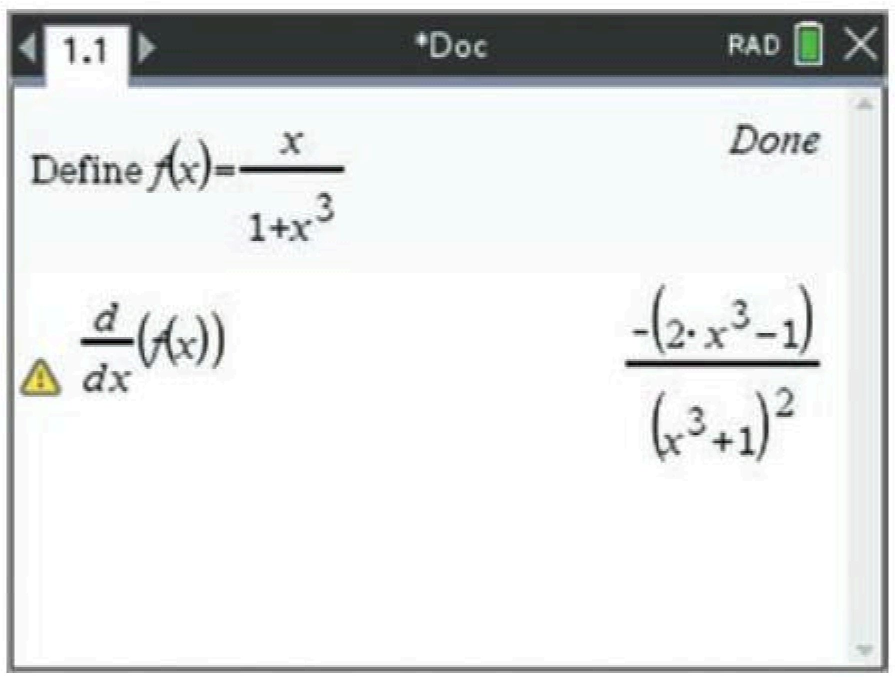
$$x^3 = -1$$

$$x = -1$$

The equations of the asymptotes are $\boldsymbol{y = 0}$ (horizontal) and $\boldsymbol{x = -1}$ (vertical). ✓

ii Use CAS to find the derivative and solve $f'(x) = 0$.

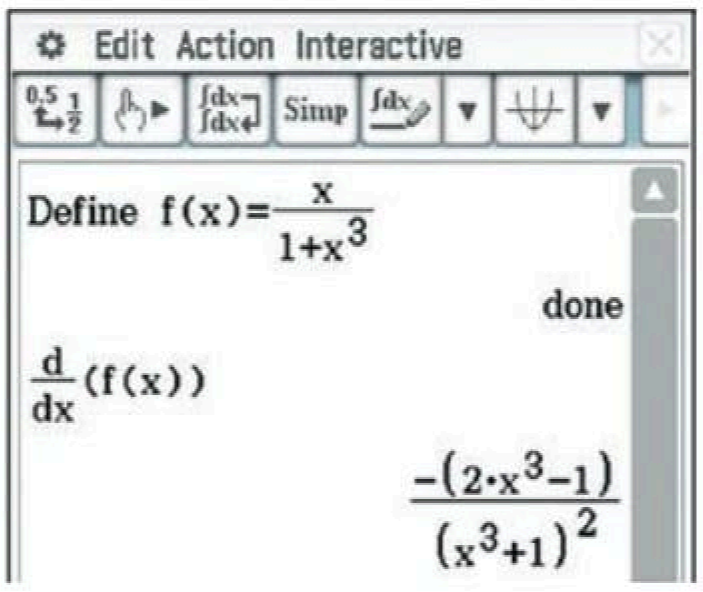
TI-Nspire



Define $f(x)=\frac{x}{1+x^3}$ Done

$\frac{d}{dx}(f(x))$ $\frac{-(2\cdot x^3-1)}{(x^3+1)^2}$

ClassPad



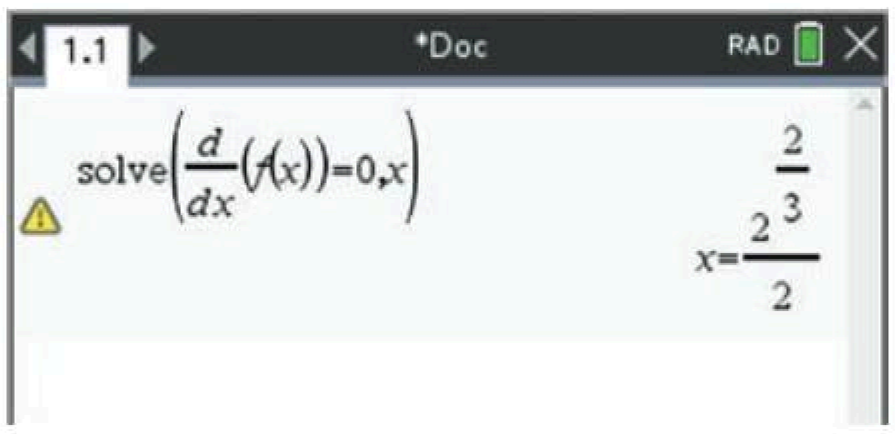
Define $f(x)=\frac{x}{1+x^3}$

done

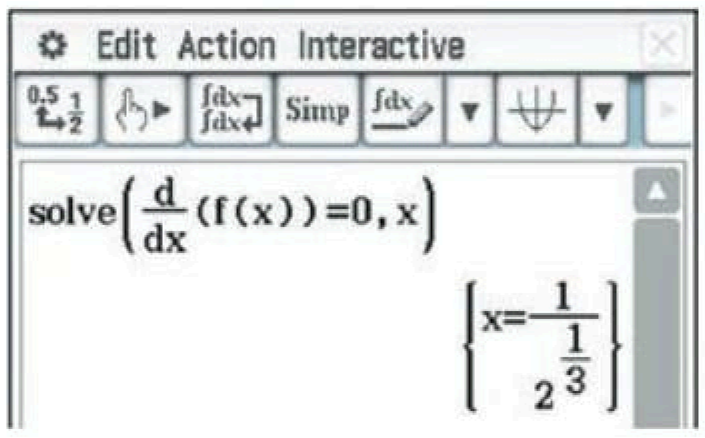
$\frac{d}{dx}(f(x))$

$\frac{-(2\cdot x^3-1)}{(x^3+1)^2}$

$$f'(x) = \frac{\mathbf{1 - 2x^3}}{\mathbf{(1 + x^3)^2}} \checkmark$$



solve$\left(\frac{d}{dx}(f(x))=0,x\right)$ $x=\frac{2^{\frac{2}{3}}}{2}$



solve$\left(\frac{d}{dx}(f(x))=0,x\right)$

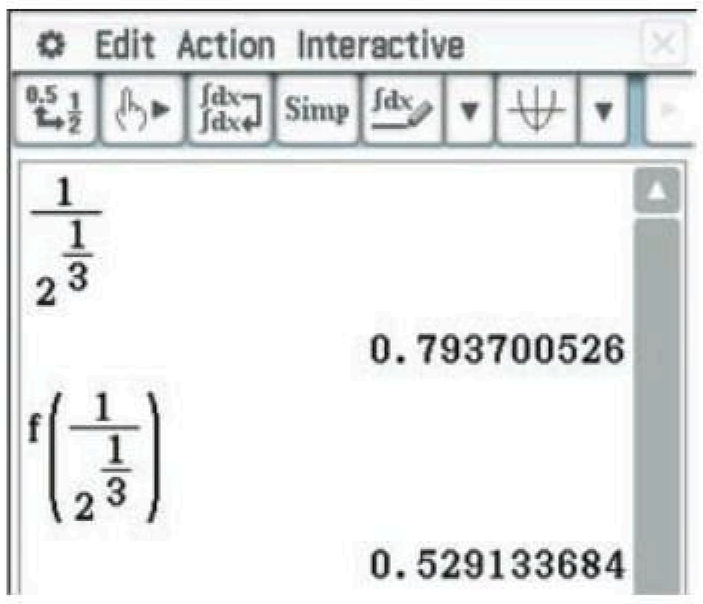
$\left\{x=\frac{1}{2^{\frac{1}{3}}}\right\}$

$f'(x) = 0$ at $x = \frac{2^{\frac{2}{3}}}{2} = \frac{1}{\sqrt[3]{2}} \approx 0.79$

Find the approximate y-coordinate using CAS.

$\frac{2^{\frac{2}{3}}}{2}$ 0.793701

$f\left(\frac{2^{\frac{2}{3}}}{2}\right)$ 0.529134



$\frac{1}{2^{\frac{1}{3}}}$

0.793700526

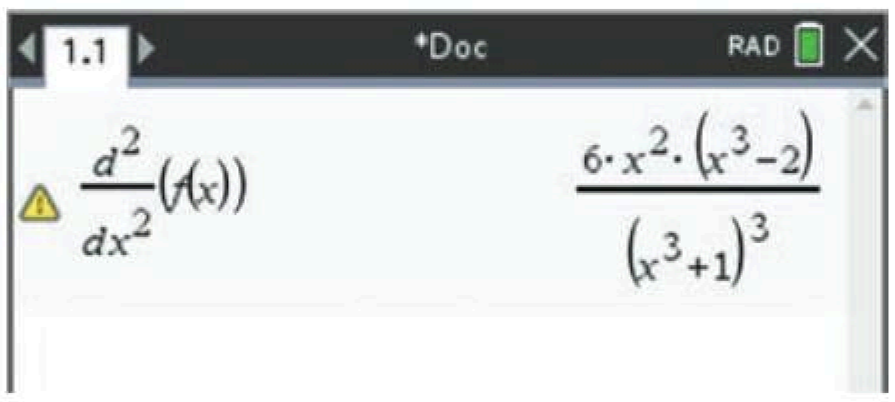
$f\left(\frac{1}{2^{\frac{1}{3}}}\right)$

0.529133684

$f(x)$ has one stationary point, at about **(0.79, 0.53)**. ✓

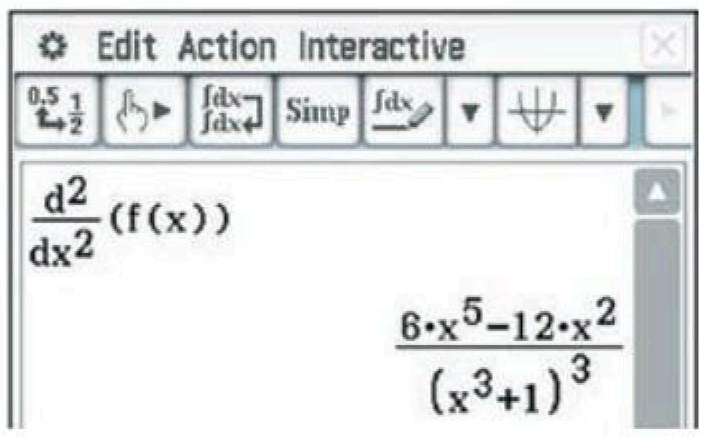
iii Use CAS to find the second derivative and solve $f''(x) = 0$.

TI-Nspire



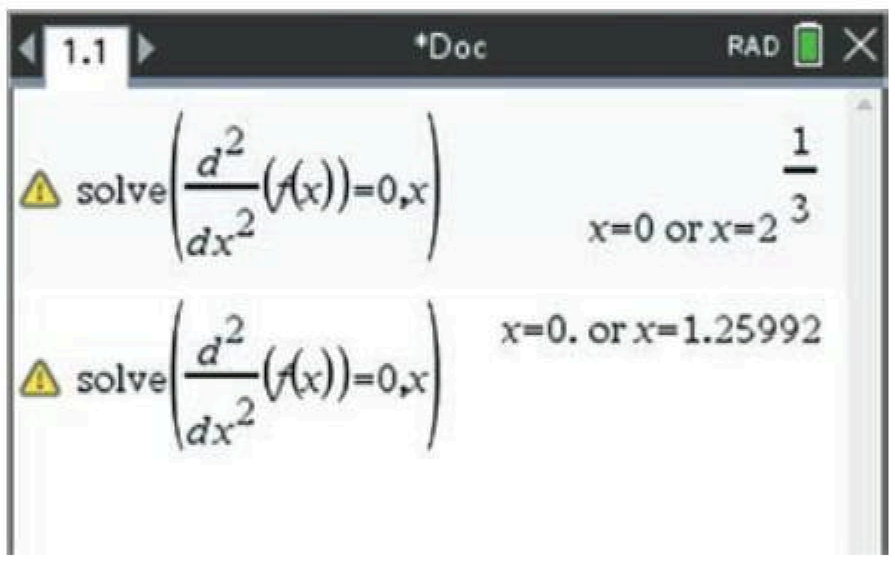
$$\frac{d^2}{dx^2}(f(x)) \quad \frac{6\cdot x^2\cdot(x^3-2)}{(x^3+1)^3}$$

ClassPad



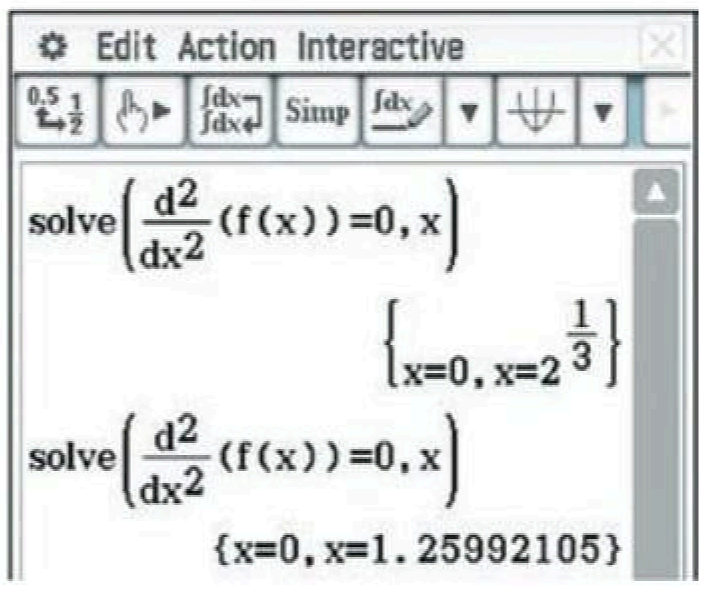
$$\frac{d2}{dx2}(f(x)) \quad \frac{6\cdot x^5-12\cdot x^2}{(x^3+1)^3}$$

$$f''(x) = \frac{6x^2(x^3-2)}{(1+x^3)^3}$$



$$\text{solve}\left(\frac{d^2}{dx^2}(f(x))=0,x\right) \quad x=0 \text{ or } x=2^{\frac{1}{3}}$$

$$\text{solve}\left(\frac{d^2}{dx^2}(f(x))=0,x\right) \quad x=0. \text{ or } x=1.25992$$



$$\text{solve}\left(\frac{d2}{dx2}(f(x))=0,x\right) \quad \left\{x=0, x=2^{\frac{1}{3}}\right\}$$

$$\text{solve}\left(\frac{d2}{dx2}(f(x))=0,x\right) \quad \{x=0, x=1.25992105\}$$

$f''(x) = 0$ at $x = 0$ and $x \approx 1.26$.

For $x < 1.26$, $f''(x) > 0$ and for $x > 1.26$, $f''(x) < 0$.

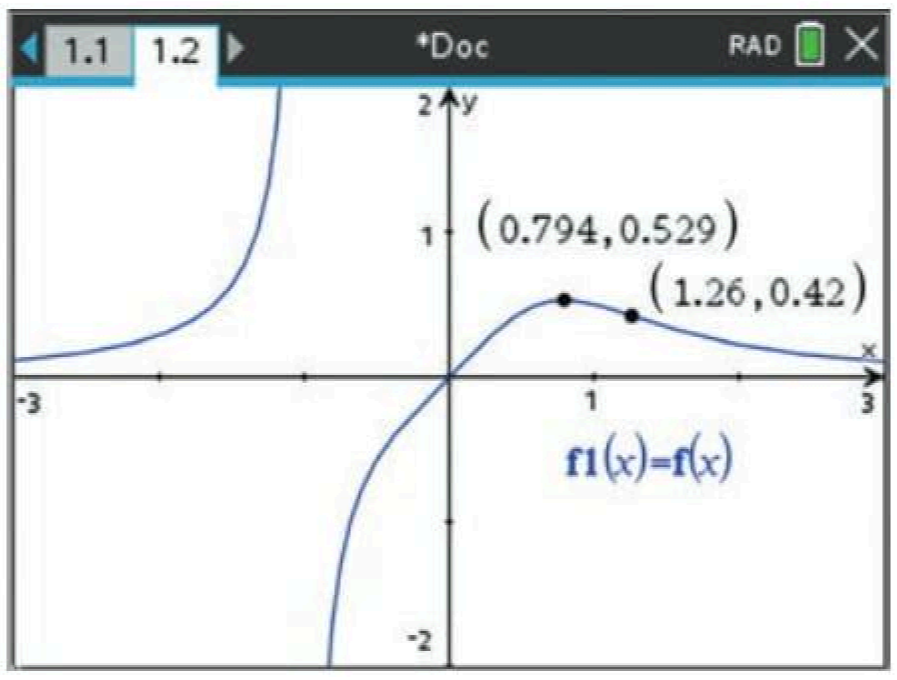
$f(1.259\,92) \approx 0.42$, $f'(1.259\,92) \approx -0.333\,333$

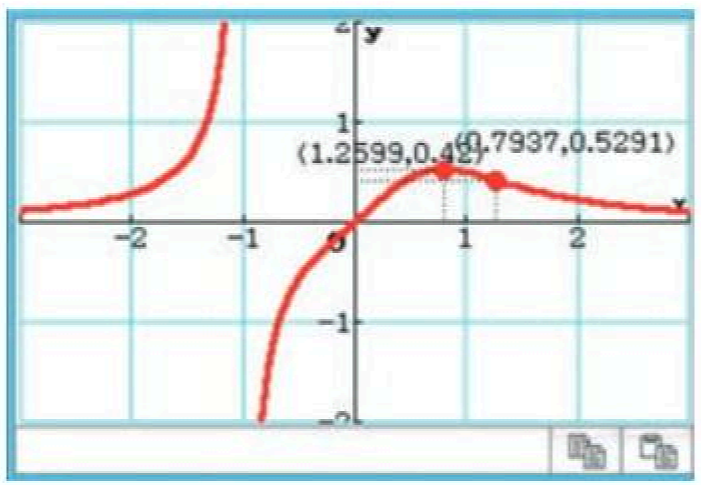
$f''(x)$ does not change sign at $x = 0$, so there isn't a point of inflection there.

There is only one point of inflection, at $\boldsymbol{x \approx 1.26}$. ✓

There is a point of inflection at about **(1.26, 0.42)**, ✓ where $f(x)$ changes from concave up to concave down on a falling curve.

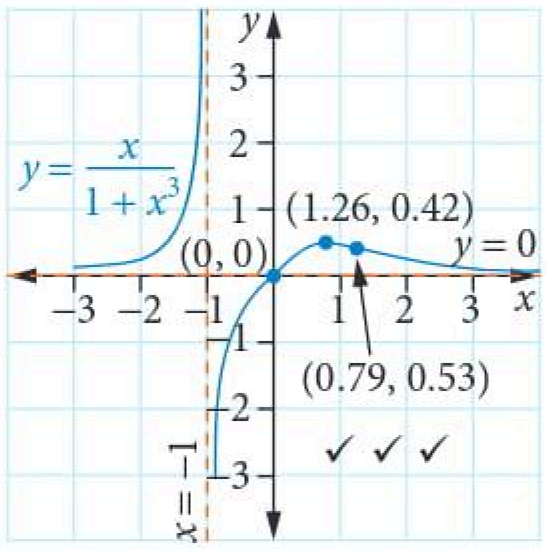
b Draw the function with CAS from $x = -3$ τo $x = 3$.





Sketch the function from $x = -3$ to $x = 3$, including the asymptotes, zeros, y-intercept, etc. ✓✓✓

Use the results of part **a**, and CAS as a guide.



Exam hack

It is not sufficient to merely copy the graph from your calculator. Make sure you include the coordinates of the relevant points and any other information.

Student performance 80–100% 60–79% 0–59%

a **i** 36% Most students found the asymptote $x = -1$ but did not identify the other asymptote $y = 0$.

ii 90% Well done.

iii 52% Some students included (0, 0) in their answer, but it is not a point of inflection as there is no change of concavity.

b 83% Generally well done but sometimes not enough points were given, or they were not marked clearly, or the graph was badly drawn.

EXERCISE 2.8 Inverse circular functions

ANSWERS p. 567

Recap

1 If $\sec(x) = 3$, then

A $\operatorname{cosec}(x) = \frac{1}{3}$ and $\tan(x) = 2\sqrt{2}$

B $\operatorname{cosec}(x) = \frac{2\sqrt{2}}{3}$ and $\tan(x) = 2\sqrt{2}$

C $\operatorname{cosec}(x) = \frac{3\sqrt{2}}{4}$ and $\tan(x) = 2\sqrt{2}$

D $\operatorname{cosec}(x) = \frac{2\sqrt{2}}{3}$ and $\tan(x) = \frac{2\sqrt{2}}{3}$

E $\operatorname{cosec}(x) = \frac{3\sqrt{2}}{4}$ and $\tan(x) = \frac{2\sqrt{2}}{3}$

2 How many solutions of $2\cos^2(x) + \tan^2(x) + \cos(x) = \sec^2(x)$ are in the domain $0 \le x \le 2\pi$?

A 2 **B** 3 **C** 4 **D** 5 **E** 6

Mastery

3 WORKED EXAMPLE 27 Evaluate each expression.

a $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ **b** $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ **c** $\arccos\left(-\frac{\sqrt{3}}{2}\right)$

d $\arcsin(1)\arcsin(-1)$ **e** $\tan^{-1}\left(-\sqrt{3}\right) + \cos^{-1}\left(\frac{1}{2}\right)$ **f** $\frac{\arccos(-1)}{\arcsin(-1)}$

4 Using CAS 9 Evaluate each expression.

a $\arctan(-1)$ **b** $\arcsin\left(-\frac{1}{2}\right)$ **c** $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

Find the values, correct to three decimal places, of

d $\arcsin(-0.75)$ **e** $\arccos(0.93)$ **f** $\tan^{-1}(5)$

5 WORKED EXAMPLE 28 Find the maximal domain and range of each function.

a $3\tan^{-1}(2x-4)+2$ **b** $-\frac{1}{3}\arcsin\left(\frac{1}{2}x+1\right)-4$ **c** $2\cos^{-1}\left(\frac{1}{3}x-2\right)+1$

d $\sin^{-1}(2x)+3$ **e** $2\arctan\left(\sqrt{3-x^2}\right)+1$ **f** $\arccos(x^2-2x+1)-2$

6 WORKED EXAMPLES 29, 30 Sketch the graph of each function, labelling it carefully.

a $y=-\frac{1}{3}\sin^{-1}(2x-\ 3)+1$ **b** $f(x)=3\tan^{-1}(2x+1)-4$

c $y=\frac{1}{\pi}\arccos\left(\frac{1}{2}x+3\right)+2$ **d** $f(x)=\frac{1}{2}\arctan\left(\frac{1}{3}x+3\right)-1$

e $y=-2\cos^{-1}(1-3x)-1$ **f** $f(x)=\frac{3}{\pi}\arcsin(1-2x)+2$

7 Using CAS 10 Sketch the graph of each function and state its domain and range.

a $f(x)=2\,\text{arcsec}\,(3x-5)$ **b** $y=0.5\arctan(2x-5)$ **c** $f(x)=3-\arcsin(x+2)$

8 WORKED EXAMPLE 31 State the implied domain and range of each function, where $a, b, c \in R$ and $b > 0$.

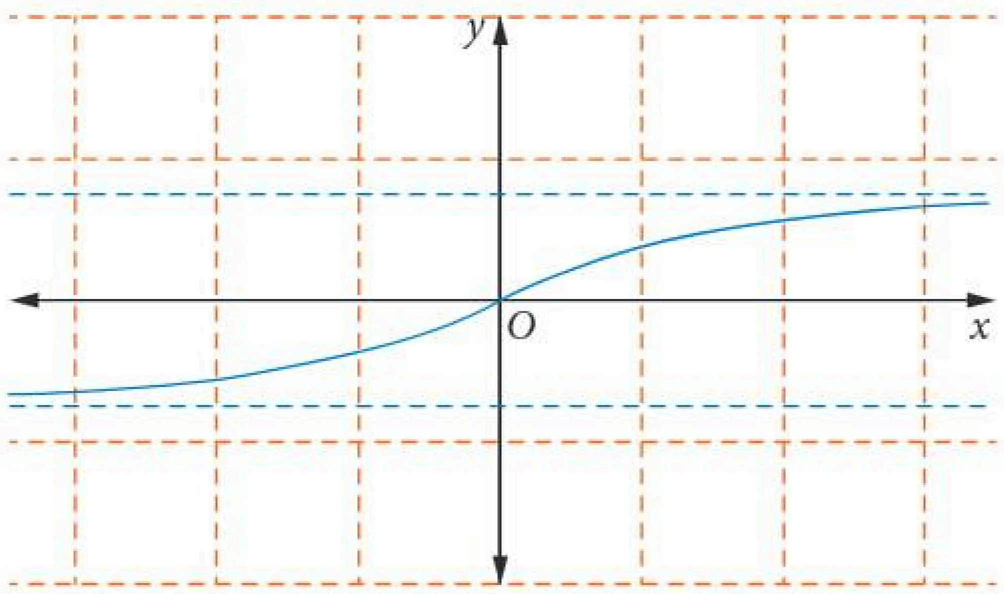
a $f(x)=a\arcsin(bx-c)$ **b** $f(x)=\sqrt{a\arccos(bx+c)}$ **c** $f(x)=\frac{a}{\arctan(c-bx)}$

d $f(x)=a\times 4c^{\arcsin(bx)}$ **e** $f(x)=x\arccos(bx-c)+a$ **f** $f(x)=\arctan^2(bx+c)-a$

Exam practice

80–100% 60–79% 0–59%

9 © VCAA 2015 1Q8bc TECH-FREE (3 marks) The graph of $f(x)=\frac{1}{2}\arctan(x)$ is shown.



a **i** 72% Write down the equations of the asymptotes. 1 mark

ii 54% Copy the graph and on it sketch the graph of f^{-1}, labelling any asymptotes with their equations. 1 mark

b 86% Find $f\left(\sqrt{3}\right)$. 1 mark

10 © VCAA 2013 1Q4 TECH-FREE (4 marks)

a 66% State the maximal domain and the range of $y=\arccos(1-2x)$. 2 marks

b 61% Sketch the graph of $y=\arccos(1-2x)$ over its maximal domain. Label the endpoints with their **coordinates**. 2 marks

11 © VCAA 2017 1Q10b 53% TECH-FREE (2 marks)

State the maximal domain and the range of $f(x)=\sqrt{\arccos\left(\frac{x}{2}\right)}$.

12 © VCAA 2012 1Q10 TECH-FREE (6 marks) Consider the functions with rules

$$f(x) = \arcsin\left(\frac{x}{2}\right) + \frac{3}{\sqrt{25x^2 - 1}} \text{ and } g(x) = \arcsin(3x) - \frac{3}{\sqrt{25x^2 - 1}}.$$

a **i** 67% Find the maximal domain of $f_1(x) = \arcsin\left(\frac{x}{2}\right)$. 1 mark

ii 33% Find the maximal domain of $f_2(x) = \frac{3}{\sqrt{25x^2 - 1}}$. 1 mark

iii 26% Find the largest set of values of $x \in R$ for which $f(x)$ is defined. 1 mark

b 20% Given that $h(x) = f(x) + g(x)$ and that $\theta = h\left(\frac{1}{4}\right)$, evaluate $\sin(\theta)$.

Give your answer in the form $\frac{a\sqrt{b}}{c}$, $a, b, c \in Z$. 3 marks

13 © VCAA 2014 1Q7a 4% TECH-FREE (1 mark) Consider $f(x) = 3x \arctan(2x)$. Write down the range of f.

14 © VCAA 2014 2AQ4 90% The domain of $\arcsin(2x - 1)$ is

A $[-1, 1]$

B $[-1, 0]$

C $[0, 1]$

D $\left[-\frac{1}{2}, \frac{1}{2}\right]$

E $\left[0, \frac{1}{2}\right]$

15 © VCAA 2013 2AQ1 88% The domain of the function with rule $f(x) = \arcsin(3x)$ is

A $[-1, 1]$

B $[-3, 3]$

C $\left[0, \frac{\pi}{3}\right]$

D $\left[-\frac{1}{3}, \frac{1}{3}\right]$

E $\left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$

16 © VCAA 2012 2AQ4 88% The domain and range of the function with rule

$f(x) = \arccos(2x - 1) + \frac{\pi}{2}$ are respectively

A $[-2, 0]$ and $[0, \pi]$

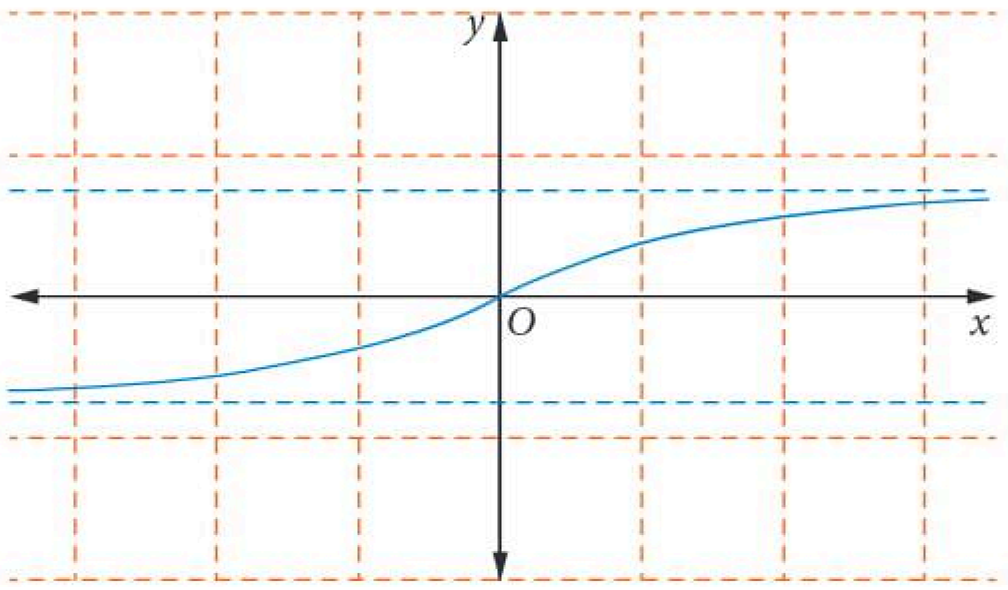
B $[-2, 0]$ and $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

C $[0, 1]$ and $[0, \pi]$

D $[0, 1]$ and $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

E $[0, \pi]$ and $[0, 1]$

17 © VCAA 2018 2AQ1 85% Part of the graph of $y = \frac{1}{2}\tan^{-1}(x)$ is shown below.



The equations of its asymptotes are

A $y = \pm\frac{1}{2}$

B $y = \pm\frac{3}{4}$

C $y = \pm 1$

D $y = \pm\frac{\pi}{2}$

E $y = \pm\frac{\pi}{4}$

18 © VCAA 2016 2AQ2 **84%** The implied domain of $y = \arccos\left(\dfrac{x-a}{b}\right)$, where $b > 0$ is

A $[-1, 1]$
B $[a - b, a + b]$
C $[a - 1, a + 1]$
D $[a, a + b\pi]$
E $[-b, b]$

19 © VCAA 2011 2AQ3 **78%** The implied domain of the function with the rule $f(x) = b + \cos^{-1}(ax)$, where $a > 0$, is

A $\left(-\dfrac{\pi}{a}, \dfrac{\pi}{a}\right)$
B $[-1, 1]$
C $\left[-\dfrac{\pi}{a}, \dfrac{\pi}{a}\right]$
D $\left(-\dfrac{1}{a}, \dfrac{1}{a}\right)$
E $\left[-\dfrac{1}{a}, \dfrac{1}{a}\right]$

20 © VCAA 2015 2AQ2 **70%** The range of the function with rule $f(x) = (2 - x)\arcsin\left(\dfrac{x}{2} - 1\right)$ is

A $[-\pi, 0]$
B $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$
C $\left[-\dfrac{(2-x)\pi}{2}, \dfrac{(2-x)\pi}{2}\right]$
D $[0, 4]$
E $[0, \pi]$

21 © VCAA 2016S 2AQ2 The maximal domain and range of the function with rule

$f(x) = 3\sin^{-1}(4x - 1) + \dfrac{\pi}{2}$ are respectively

A $[-\pi, 2\pi]$ and $\left[0, \dfrac{1}{2}\right]$
B $\left[0, \dfrac{1}{2}\right]$ and $[-\pi, 2\pi]$
C $\left[-\dfrac{3\pi}{2}, \dfrac{3\pi}{2}\right]$ and $\left[-\dfrac{1}{2}, 0\right]$
D $\left[0, \dfrac{1}{2}\right]$ and $[0, 3\pi]$
E $\left[-\dfrac{1}{2}, 0\right]$ and $[-\pi, 2\pi]$

22 © VCAA 2020 2AQ2 A function f has the rule $f(x) = \left|b\cos^{-1}(x) - a\right|$, where $a > 0$, $b > 0$ and $a < \dfrac{b\pi}{2}$.

The range of f is

A $[-a, b\pi - a]$
B $[0, b\pi - a]$
C $[a, b\pi - a]$
D $[0, b\pi + a]$
E $[a - b\pi, a]$

23 © VCAA 2018 2AQ2 **60%** Consider the function f with rule $f(x) = \dfrac{1}{\sqrt{\sin^{-1}(cx + d)}}$, where $c, d \in R$ and $c > 0$.

The domain of f is

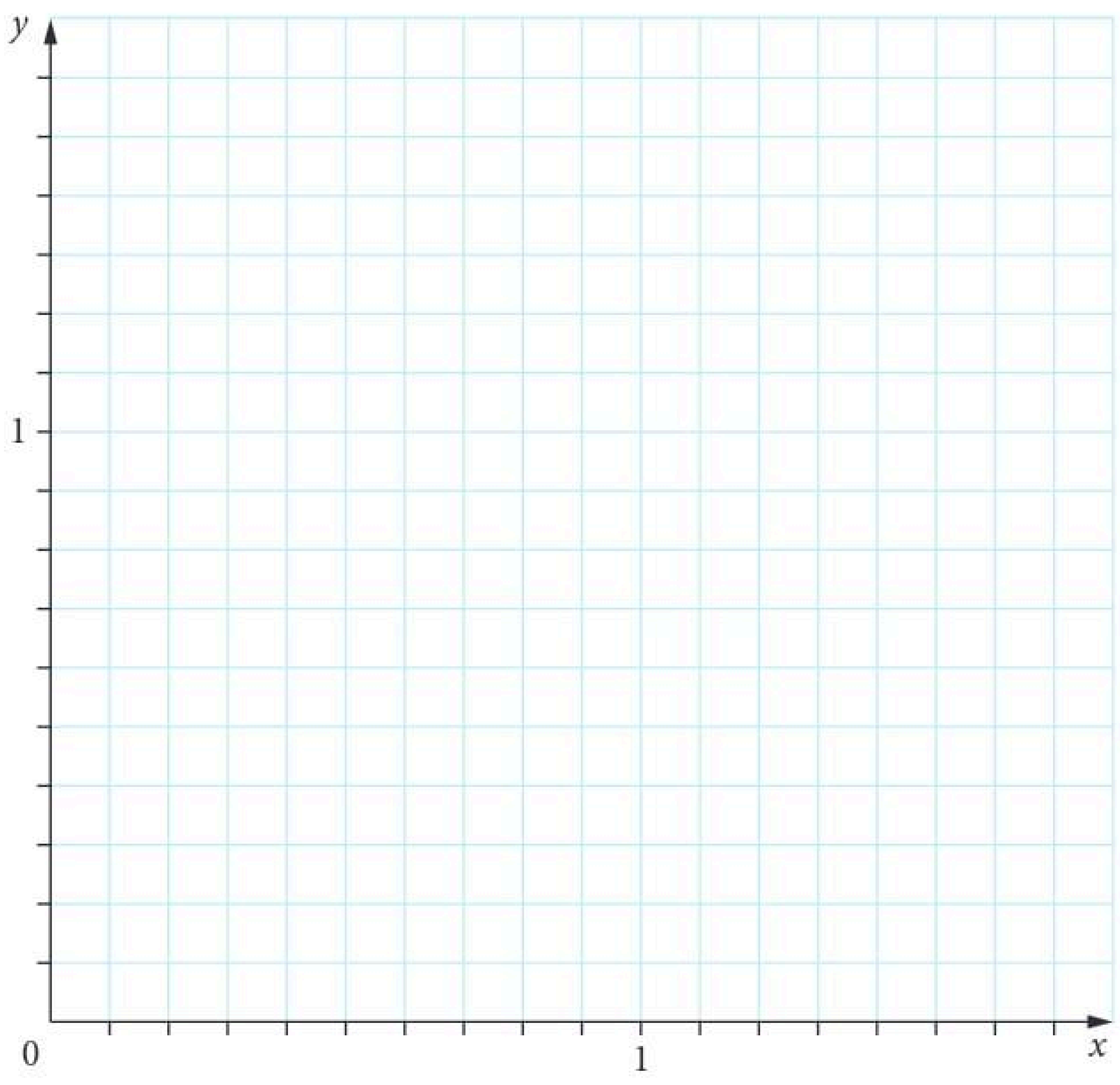
A $x > -\dfrac{d}{c}$
B $-\dfrac{d}{c} < x \le \dfrac{1-d}{c}$
C $\dfrac{-1-d}{c} \le x \le \dfrac{1-d}{c}$
D $x \in R \setminus \left\{-\dfrac{d}{c}\right\}$
E $x \in R$

24 © VCAA 2015 2BQ1bcde (7 marks) Consider $y = \sqrt{2 - \sin^2(x)}$.

a **91%** Write down the values of y where $x = 0$ and where $x = \dfrac{\pi}{2}$. 1 mark

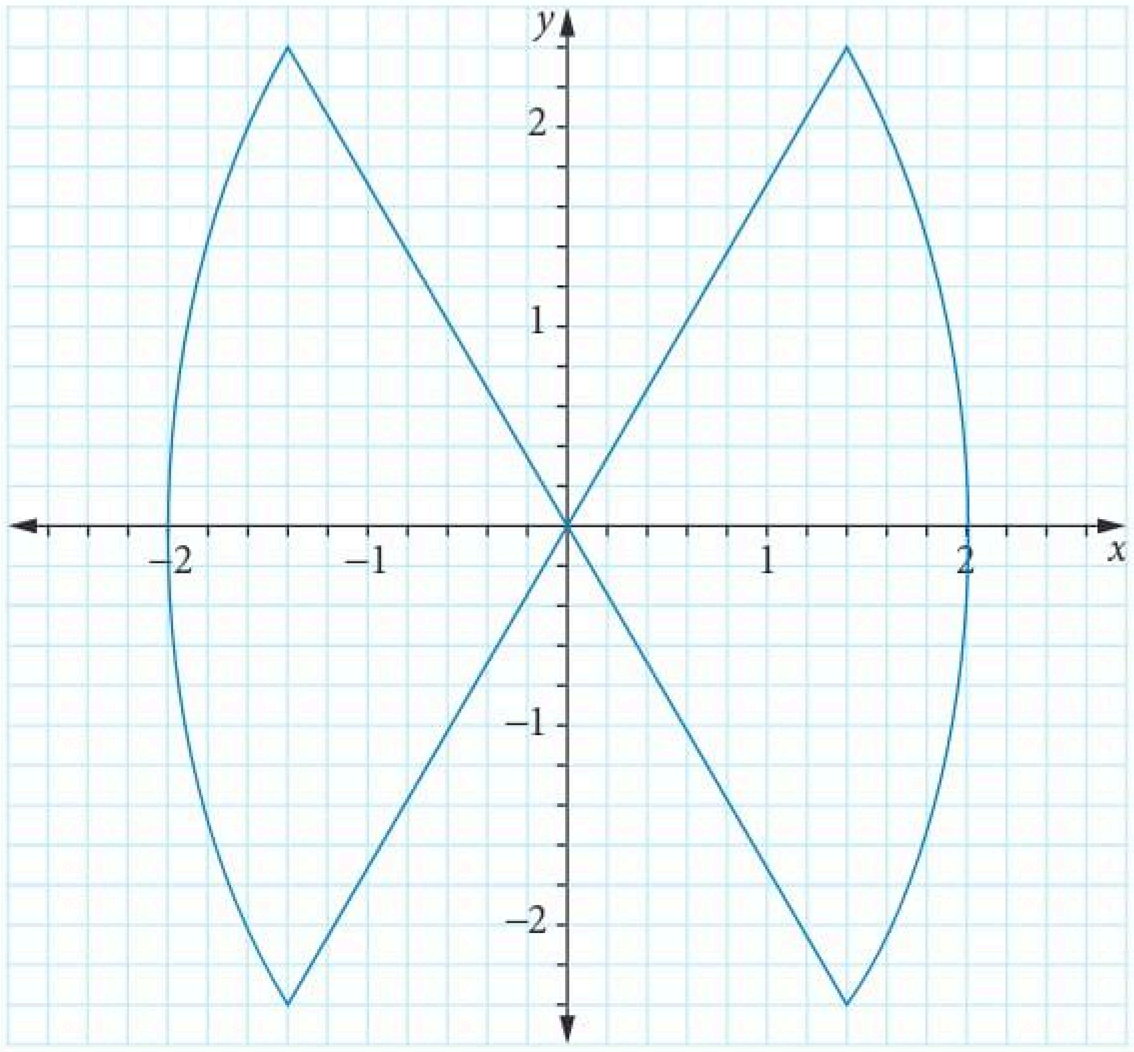
Now consider the function f with rule $f(x) = \sqrt{2 - \sin^2(x)}$ for $0 \le x \le \dfrac{\pi}{2}$.

b **79%** Find the rule for the inverse function f^{-1}, and state the domain and range of f^{-1}. 3 marks c 51% Copy the axes below, and on them sketch and label the graphs of f and f^{-1}. 2 marks



d 66% The graphs of f and f^{-1} intersect at the point $P(a, a)$. Find a, correct to three decimal places. 1 mark

25 © VCAA 2017 2BQ3 62% (2 marks) A brooch is designed using inverse circular functions to make the shape shown in the diagram below.



The edges of the brooch in the first quadrant are described by the piecewise function

$$f(x) = \begin{cases} 3\arcsin\left(\dfrac{x}{2}\right), & 0 \le x \le \sqrt{2} \\ 3\arccos\left(\dfrac{x}{2}\right), & \sqrt{2} < x \le 2 \end{cases}$$

a 74% Write down the coordinates of the corner point of the brooch in the first quadrant. 1 mark

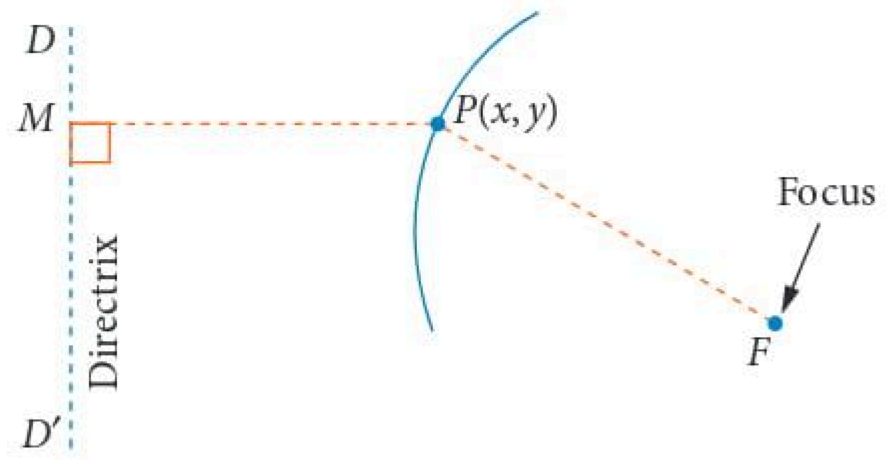
b 49% Specify the piecewise function that describes the edges in the third quadrant. 1 mark

2 Chapter summary

Conic sections

- A **conic section** is the shape made when a cone is intersected by a plane.
- The **eccentricity** of a conic section is the ratio

  $e = \dfrac{\sin(\beta)}{\sin(\alpha)}$, where α is the slant angle of the cone and β is the angle between the plane and the base of the cone.
- For circles: $e = 0$ ellipses: $0 < e < 1$
  parabolas: $e = 1$ hyperbolas: $e > 1$
- A conic section is a locus referred to a fixed point (**focus**) and line (**directrix**).

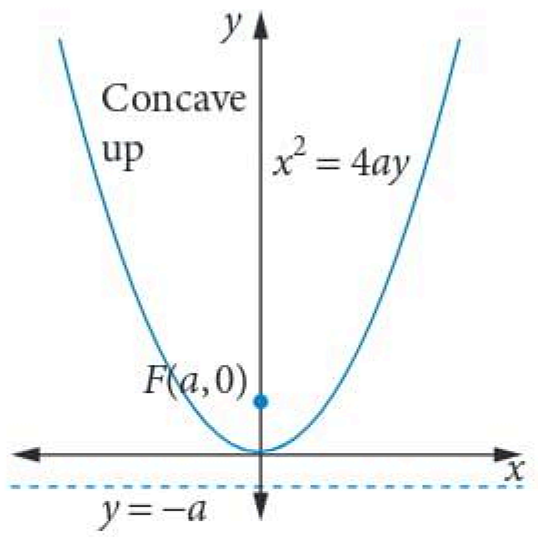


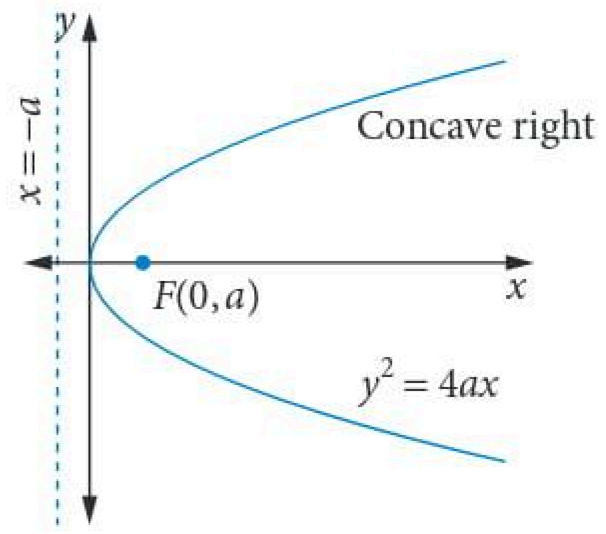
$$\frac{PF}{PM} = e$$

For a circle, $PM = \infty$ and $PF = r$.

Cartesian equations of conic sections

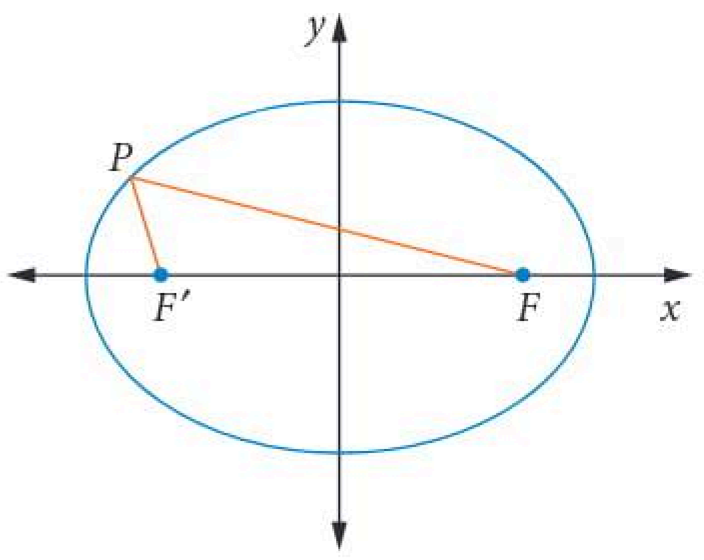
- Parabolas ($e = 1$, $a > 0$).



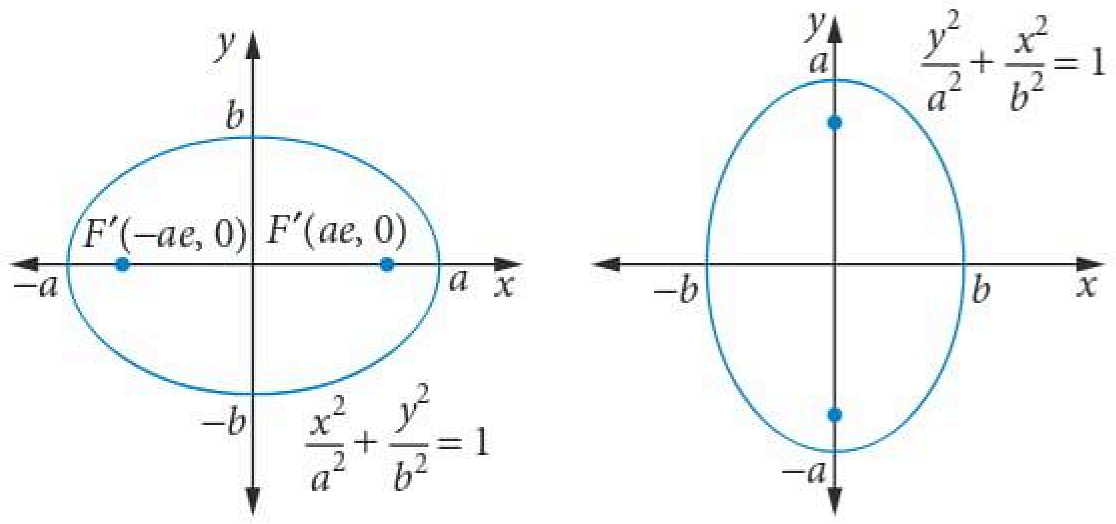


Vertex:	(0, 0)	(0, 0)
Axis:	$x = 0$	$y = 0$
Equation:	$x^2 = 4ay$	$y^2 = 4ax$

For any point $P(x, y)$ on an ellipse, $PF + PF' = 2a$



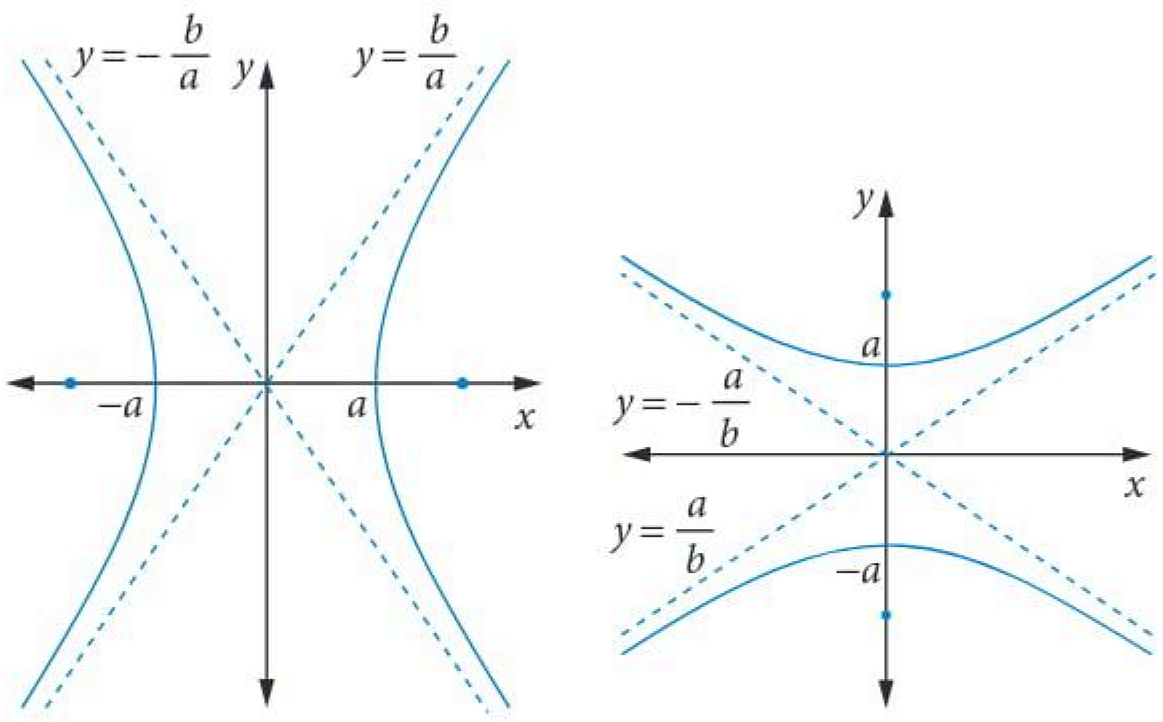
- Ellipses ($0 < e < 1$, $a > b$)



Axis lengths:	Major $2a$	Minor $2b$
Vertices:	$(\pm a, 0), (0, \pm b)$	$(0, \pm a), (\pm b, 0)$
Axis:	$x = 0$	$y = 0$
Formula:	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$

$b^2 = a^2(1 - e^2)$, or $e^2 = 1 - \dfrac{b^2}{a^2}$

- Hyperbolas ($e > 1$)

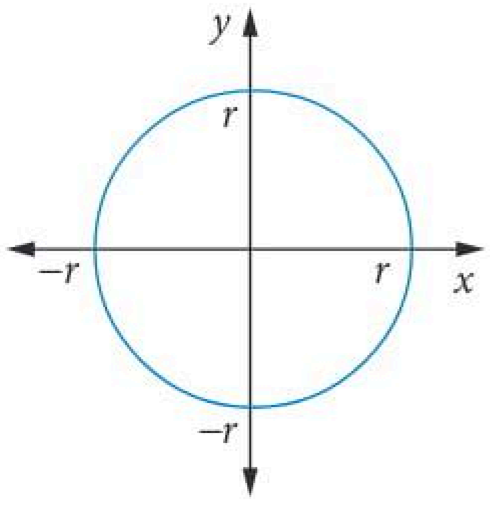


Vertices:	$(\pm a, 0)$	$(0, \pm a)$
Transverse axis:	$x = 0$	$y = 0$
Formula:	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

$b^2 = a^2(e^2 - 1)$, or $e^2 = 1 + \dfrac{b^2}{a^2}$

Asymptotes:	$y = \pm\frac{b}{a}$	$y = \pm\frac{a}{b}$

- A circle with radius r and centre at the origin has the equation $x^2 + y^2 = r^2$.



- Cartesian equations of conic sections with centre (h, k), have x and y by replaced by $(x - h)$ and $(y - k)$ respectively.

Parametric equations of conic sections

- Parametric forms of conic sections with centres at the origin, for $0 \le t < 2\pi$.
- Parabolas: vertical axis $\begin{cases} x = 2at \\ y = at^2 \end{cases}$

  horizontal axis $\begin{cases} y = 2at \\ x = at^2 \end{cases}$
- Ellipses $(a > b)$:

  horizontal major axis $\begin{cases} x = a\cos(t) \\ y = b\sin(t) \end{cases}$

  vertical major axis $\begin{cases} x = b\cos(t) \\ y = a\sin(t) \end{cases}$
- Hyperbolas:

  horizontal transverse axis

  $\begin{cases} x = a\sec(t) \\ y = b\tan(t) \end{cases}$ or $\begin{cases} x = a\operatorname{cosec}(t) \\ y = b\cot(t) \end{cases}$

  vertical transverse axis

  $\begin{cases} y = a\sec(t) \\ x = b\tan(t) \end{cases}$ or $\begin{cases} y = a\operatorname{cosec}(t) \\ x = b\cot(t) \end{cases}$
- Circles: $\begin{cases} x = r\cos(t) \\ y = r\sin(t) \end{cases}$
- For centres (h, k), add h and k to the parametric equations for x and y, respectively.

Rational functions

- A **rational function** is of the form $f(x) = \dfrac{p(x)}{q(x)}$, where p and q are polynomials and q is not the zero polynomial.
- Rational functions have **discontinuities** where the denominator $q(x) = 0$. At these values, the rational function is **not defined**.
- If **deg (numerator) = 0**, then the function is a **reciprocal function**.
- If **deg (numerator) < deg (denominator)**, then it is a **simple rational function**.
- If **deg (numerator) ≥ deg (denominator)**, then it is a **quotient function**.
- The **roots** of rational functions are those of the numerator.
- If deg (numerator) ≥ deg (denominator), then divide the numerator by the denominator to obtain a **quotient** and a **remainder**.

Partial fractions

- $\dfrac{ax + b}{(x + c)(x + d)}$ can be expressed in the form $\dfrac{A}{(x + c)} + \dfrac{B}{(x + d)}$.
- $\dfrac{ax + b}{(x + c)^2}$ can be expressed in the form $\dfrac{A}{(x + c)^2} + \dfrac{B}{(x + c)}$.
- $\dfrac{a}{(x + c)^2(x + d)}$ or $\dfrac{ax + b}{(x + c)^2(x + d)}$ can be expressed in the form $\dfrac{A}{(x + c)^2} + \dfrac{B}{(x + c)} + \dfrac{C}{(x + d)}$.
- $\dfrac{ax^2 + bx + c}{(mx + n)(px^2 + qx + r)}$, where $px^2 + qx + r$ cannot be factorised, can be expressed in the form $\dfrac{A}{mx + n} + \dfrac{Bx + C}{px^2 + qx + r}$.

Reciprocal functions

- For any polynomial function $p(x)$ and its reciprocal $f(x) = \dfrac{1}{p(x)}$,
  - the signs of $p(x)$ and $f(x)$ are the same.
  - the zeros of $p(x)$ correspond to **vertical asymptotes** of $f(x)$.
  - as $x \to \pm\infty$, $p(x) \to \pm\infty$ so $f(x) \to 0$ and the x-axis $(y = 0)$ is a **horizontal asymptote.**
  - the maxima and minima of $f(x)$ correspond to the minima and maxima of $p(x)$.
  - the y-intercept of $f(x)$ is the reciprocal of the y-intercept of $p(x)$.
  - for $p(x)$ concave up, $f(x)$ is concave down. For $p(x)$ concave down, $f(x)$ is concave up.
  - the domain of $f(x)$ excludes the zeros of $p(x)$.
- A reciprocal function of the form $\dfrac{k}{f(x)}$ is a dilation from the x-axis of the reciprocal function $\dfrac{1}{f(x)}$ by a factor of k.

Simple rational functions

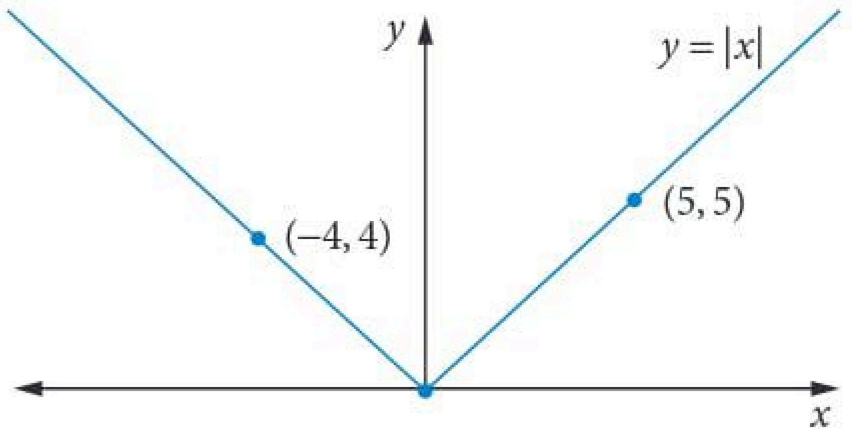
- In a simple rational function, the degree of the numerator is less than the degree of the denominator, and its graph has a horizontal asymptote of $y = 0$ (the x-axis).
- Sketch a simple rational function by considering the sign of the function between the zeros and vertical asymptotes.

Quotient functions

- A **quotient function** has the form $\frac{p(x)}{q(x)}$, where $\deg(p(x)) \geq \deg(q(x))$.
- A quotient function whose numerator and denominator are the same degree has a **horizontal asymptote** $y = c$, where c is the quotient of the leading coefficients.
- A quotient function with $\deg(\text{numerator}) - \deg(\text{denominator}) = 1$ will have an oblique asymptote of the linear form $y = mx + c$, where $mx + c$ is the quotient of the numerator and denominator.
- A quotient function with $\deg(\text{numerator}) - \deg(\text{denominator}) = 2$ will have a curved asymptote of the quadratic form $y = ax^2 + bx + c$, where $ax^2 + bx + c$ is the quotient of the numerator and denominator.

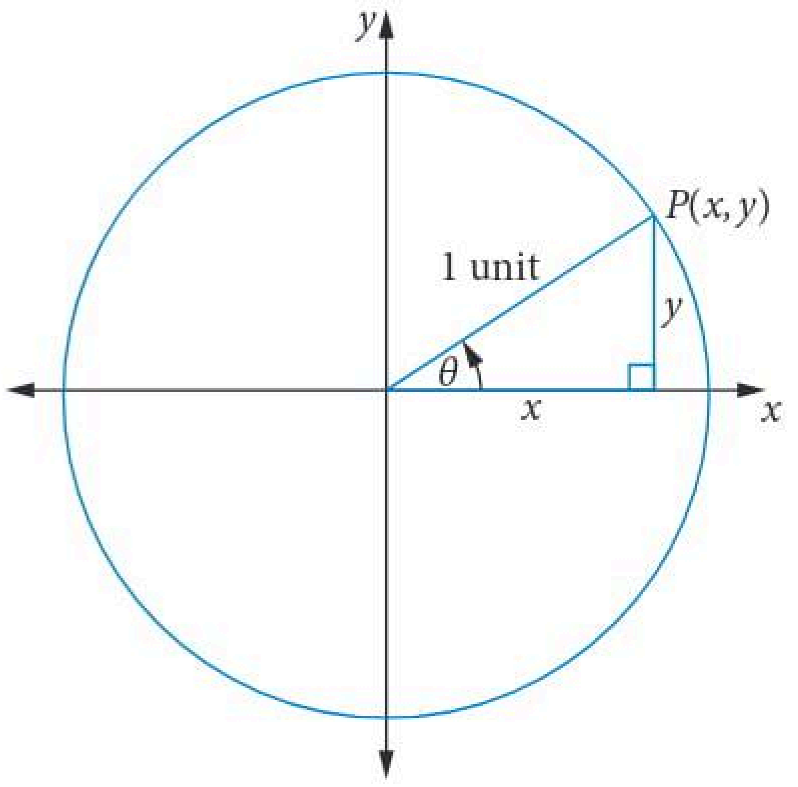
Absolute value functions

- $|x| = \begin{cases} x, & \text{for } x \geq 0 \\ -x, & \text{for } x < 0 \end{cases}$.
- $|x|$ can also be defined as $|x| = \sqrt{x^2}$.
- The graph of the absolute value function has the shape.



- $f(x) = a|x|$ or $f(x) = |ax|$ is the graph of $y = |x|$ dilated from the x-axis by factor of a.
  For $a > 1$ or $a < -1$, it is stretched; for $-1 < a < 1$, it is shrunk.
- $f(x) = |x + b|$ is translated in the x direction.
  For $b > 0$, it moves left; for $b < 0$, it moves right.
- $f(x) = |x| + c$ is translated in the y direction.
  For $c > 0$, it moves up; for $c < 0$, it moves down.

The reciprocal circular functions

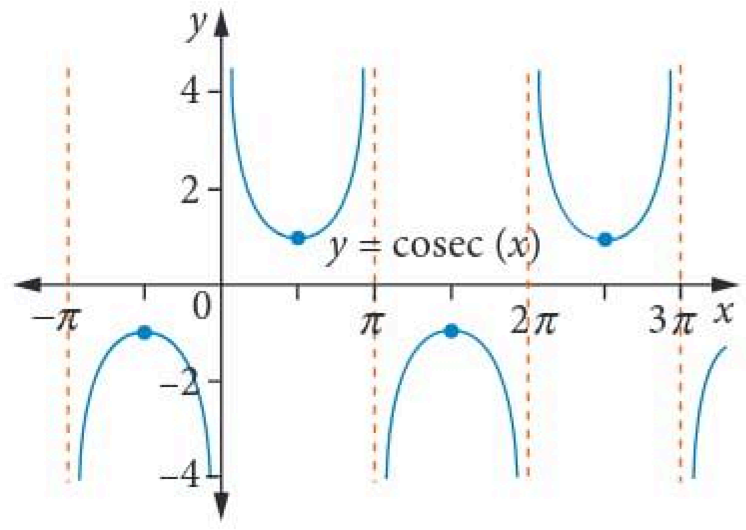


$$\operatorname{cosec}(\theta) = \frac{1}{\sin(\theta)} = \frac{1}{y}$$

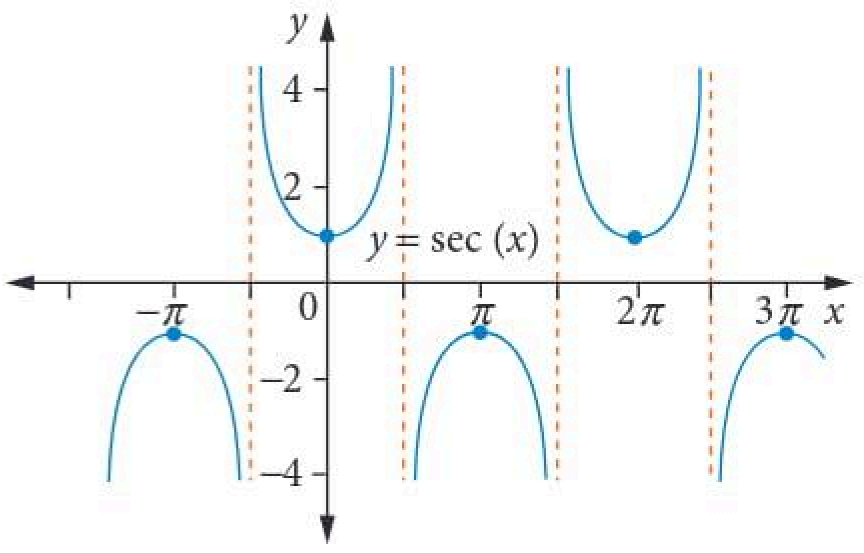
$$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{x}$$

$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)} = \frac{x}{y}$$

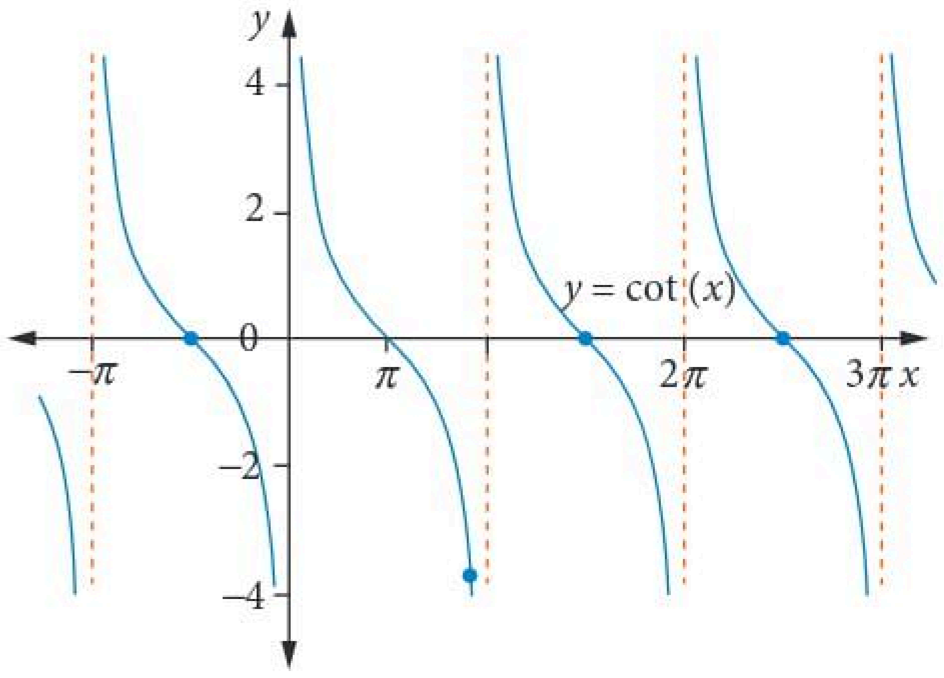
- $y = \operatorname{cosec}(x)$ has domain $R\backslash\{n\pi : n \in Z\}$, range $(-\infty, -1] \cup [1, \infty)$, vertical asymptotes $x = n\pi$ for $n \in Z$, minima at $x = (4n + 1)\frac{\pi}{2}$ for $n \in Z$ and maxima at $x = (4n - 1)\frac{\pi}{2}$ for $n \in Z$.



- $y = \sec(x)$ has domain $R \setminus \{(2n + 1)\frac{\pi}{2} : n \in Z$, range $(-\infty, -1] \cup [1, \infty)$, vertical asymptotes $x = (2n + 1)\frac{\pi}{2}$ for $n \in Z$, minima at $x = 2n\pi$ for $n \in Z$ and maxima at $x = (2n + 1)\pi$ for $n \in Z$.



- $y = \cot(x)$ has domain $R\backslash\{n\pi : n \in Z\}$, range R, vertical asymptotes $x = n\pi$ for $n \in Z$ and zeros at $x = (2n + 1)\frac{\pi}{2}$ for $n \in Z$.



- Sketch the graphs of $a\,f\,[n(x+b)]+c$, for $f(x)=\sec(x)$, $f(x)=\operatorname{cosec}(x)$ and $f(x)=\cot(x)$ by transforming $f(x)$ as follows:
  - – (horizontal) dilation from the y-axis by a factor of $\frac{1}{n}$ (period from n): shrink for $n>1$ or $n<-1$, stretch for $-1<n<1$.
  - – (horizontal) reflection in the y-axis if $n<0$
  - – (horizontal) translation parallel to the x-axis (phase from b): left if $b>0$, right if $b<0$.
  - – (vertical) dilation from the x-axis by a factor of a: shrink for $-1<a<1$, stretch for $a>1$ or $a<-1$.
  - – (vertical) reflection in the x-axis if $a<0$
  - – (vertical) translation parallel to the y-axis: up if $c>0$, down if $c<0$.

The sum and difference identities

- $\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y)$
- $\sin(x-y)=\sin(x)\cos(y)-\cos(x)\sin(y)$
- $\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y)$
- $\cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$
- $\tan(x+y)=\dfrac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$
- $\tan(x-y)=\dfrac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$

The double angle identities

- $\sin(2x)=2\sin(x)\cos(x)$
- $\cos(2x)=\cos^2(x)-\sin^2(x)$

  or $2\cos^2(x)-1$

  or $1-2\sin^2(x)$
- $\tan(2x)=\dfrac{2\tan(x)}{1-\tan^2(x)}$

The Pythagorean identities

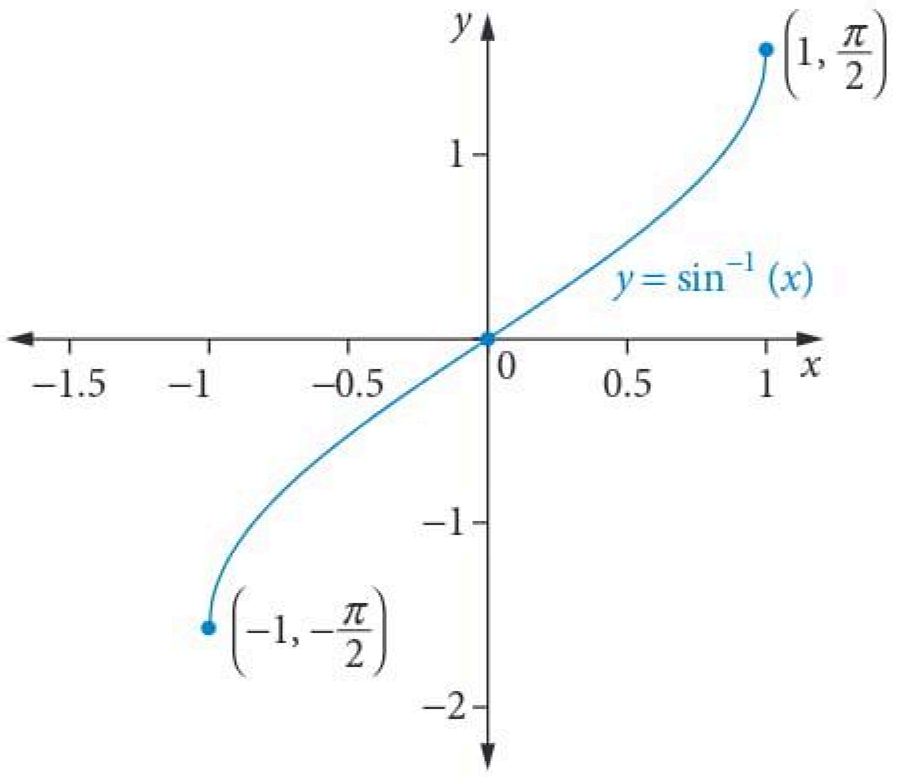
- $\sin^2(x)+\cos^2(x)=1$
- $\tan^2(x)+1=\sec^2(x)$
- $\cot^2(x)+1=\operatorname{cosec}^2(x)$

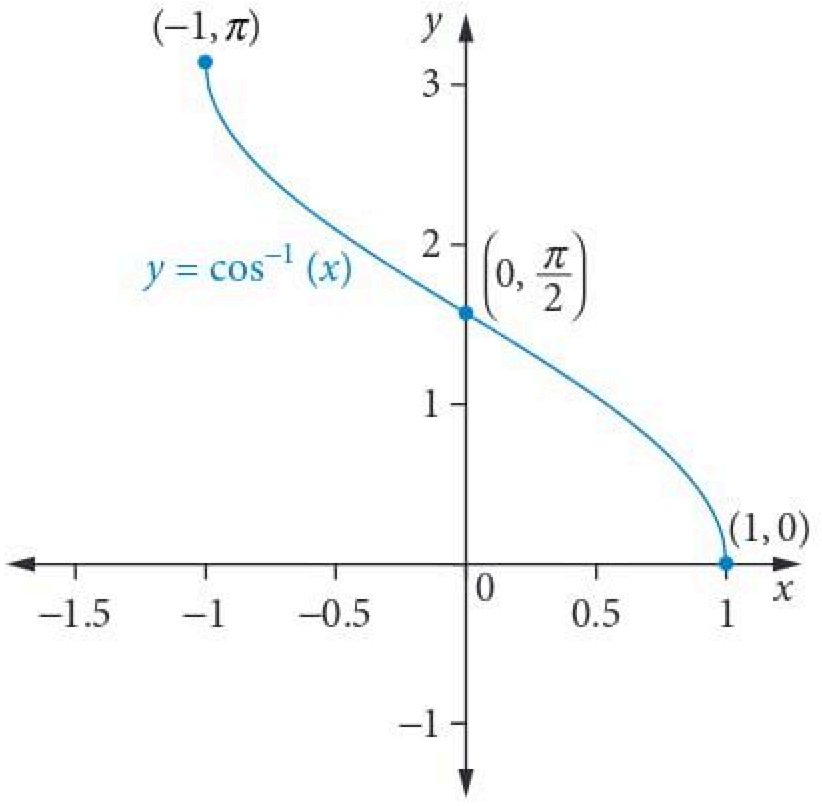
Inverse circular functions

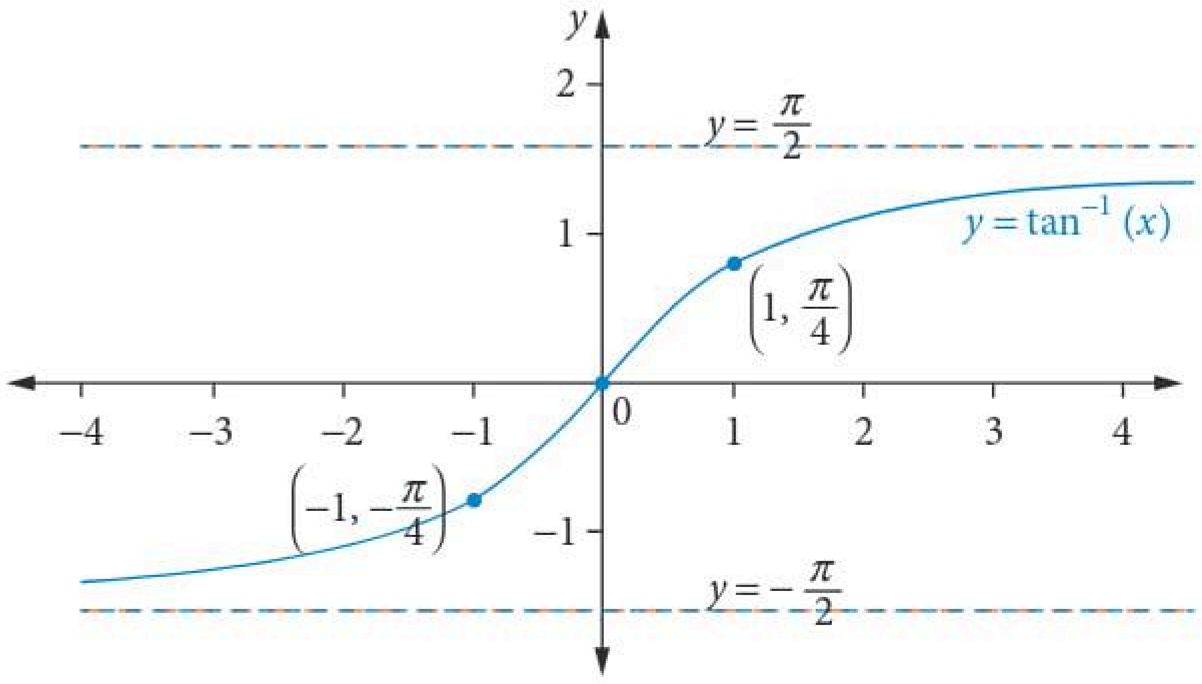
- $y=\sin^{-1}(x)$ if and only if $\sin(y)=x$ and $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$.
- $y=\cos^{-1}(x)$ if and only if $\cos(y)=x$ and $0\le y\le\pi$.
- $y=\tan^{-1}(x)$ if and only if $\tan(y)=x$ and $-\frac{\pi}{2}<y<\frac{\pi}{2}$.
- $\sin^{-1}(x)$ has domain $[-1, 1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
- $\cos^{-1}(x)$ has domain $[-1, 1]$ and range $[0, \pi]$.
- $\tan^{-1}(x)$ has domain $(-\infty, \infty)$ and range. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
- For any values, $\operatorname{arccosec}(a)=\arcsin\left(\frac{1}{a}\right)$,

  $\operatorname{arcsec}(a)=\arccos\left(\frac{1}{a}\right)$ and

  $\operatorname{arccot}(a)=\arctan\left(\frac{1}{a}\right)$.







- Sketch the graphs of $a\,f\,[n(x + b)] + c$, for $f(x) = \sin^{-1}(x), f(x) = \cos^{-1}(x)$ and $f(x) = \tan^{-1}(x)$ by transforming $f(x)$ in the following order.
- dilation from the y-axis by a factor of $\frac{1}{n}$ (period from n): shrink for $n > 1$ or $n < -1$, stretch for $-1 < n < 1$.
- reflection in the y-axis if $n < 0$
- translation parallel to the x-axis (phase from b): left if $b > 0$, right if $b < 0$.
- dilation from the x-axis by a factor of a: shrink for $-1 < a < 1$, stretch for $a > 1$ or $a < -1$.
- reflection in the x-axis if $a < 0$
- translation parallel to the y-axis: up if $c > 0$, down if $c < 0$.
- It may be easier to transform the important points and asymptotes for inverse tan graphs and join them.

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 © VCAA 2019 1Q6 (3 marks) Find the value of d for which the vectors $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}$, $\underset{\sim}{b} = -2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}$ and $\underset{\sim}{c} = -6\underset{\sim}{i} + 2\underset{\sim}{j} + d\underset{\sim}{k}$ are linearly dependent.

2 © VCAA 2013 1Q3a (1 mark) The coordinates of three points are $A\,(-1, 2, 4)$, $B\,(1, 0, 5)$ and $C\,(3, 5, 2)$. Find $\overrightarrow{AB}$.

3 © VCAA 2012 1Q2 (3 marks) Find all real solutions of the equation $2\cos(x) = \sqrt{3}\cot(x)$.

4 © VCAA 2019 1Q2 (3 marks) Find all values of x for which $x - 4| = \frac{x}{2} + 7$.

Cumulative examination 2

Total number of marks: 22 Reading time: 5 minutes Writing time: 33 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 © VCAA 2012 2AQ16 The distance between the points $P(-2, 4, 3)$ and $Q(1, -2, 1)$ is

A 7 **B** 21 **C** 31 **D** 11 **E** 49

2 © VCAA 2013 2AQ15 Let $\underset{\sim}{u} = 4\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$, $\underset{\sim}{v} = 3\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{w} = -4\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$.

Which one of the following statements is **not** true?

A $|\underset{\sim}{u}| = |\underset{\sim}{v}|$

B $|\underset{\sim}{u}| = |-\underset{\sim}{w}|$

C $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are linearly independent

D $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$

E $(\underset{\sim}{u} + \underset{\sim}{w}) \cdot \underset{\sim}{v} = 12$

3 © VCAA 2012 2AQ1 The graph with equation $y = \dfrac{1}{2x^2 - x - 6}$ has asymptotes given by

A $x = -\dfrac{3}{2}$, $x = 2$ and $y = 1$

B $x = -\dfrac{3}{2}$ and $x = 2$ only

C $x = \dfrac{3}{2}$, $x = -2$ and $y = 0$

D $x = -\dfrac{3}{2}$, $x = 2$ and $y = 0$

E $x = \dfrac{3}{2}$ and $x = -2$ only

4 © VCAA 2013 2AQ4 The graphs of $y = ax$ and $y = \arctan(bx)$ intersect exactly three times if

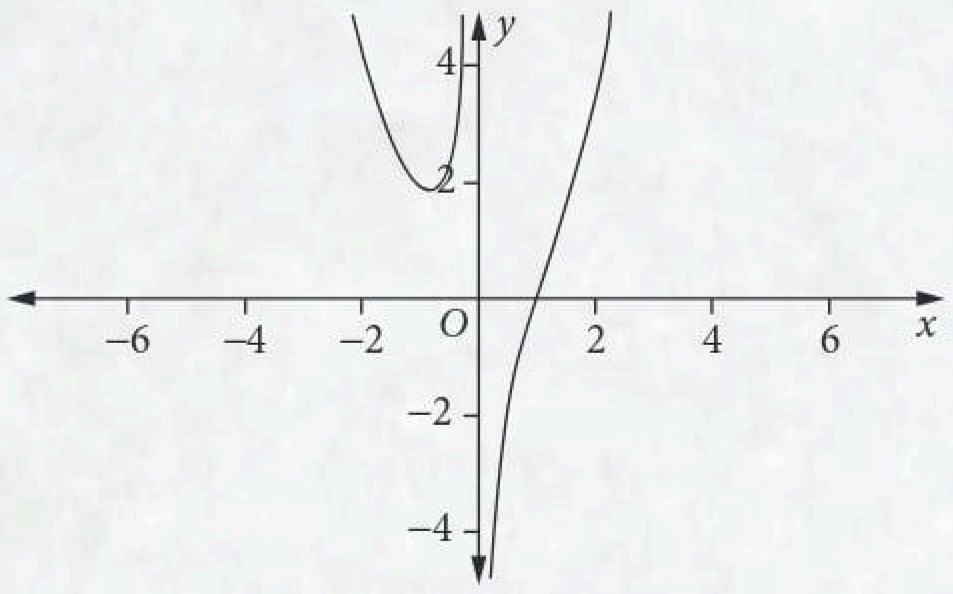
A $0 < b < a$ **B** $a < b < 0$ **C** $a = b$

D $b < a < 0$ **E** $0 < b^2 < a^2$

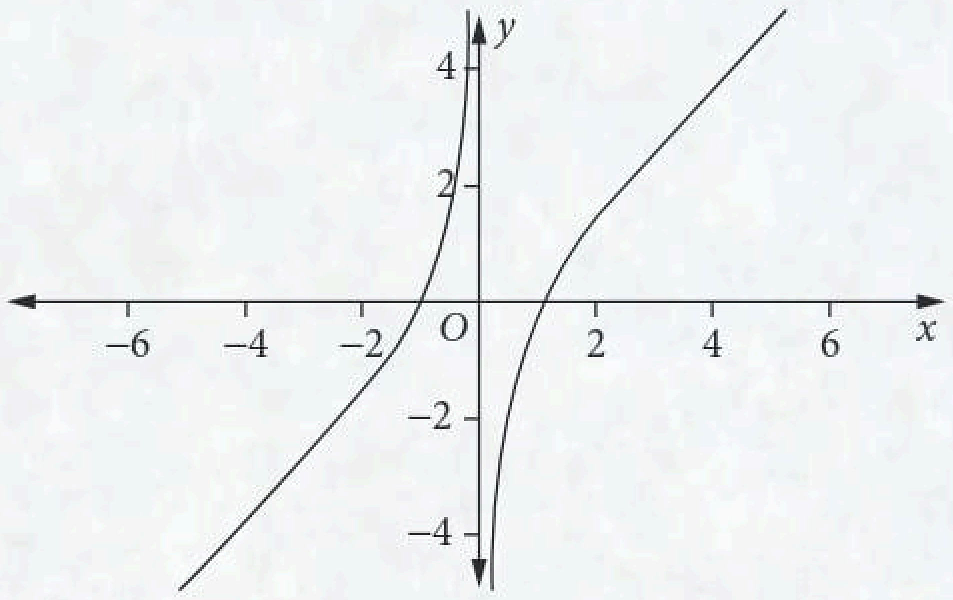
5 © VCAA 2010 2AQ3 Let $f(x) = \dfrac{x^k + a}{x}$, where k and a are real constants.

If k is an odd integer which is greater than 1 and $a < 0$, a possible graph of f could be

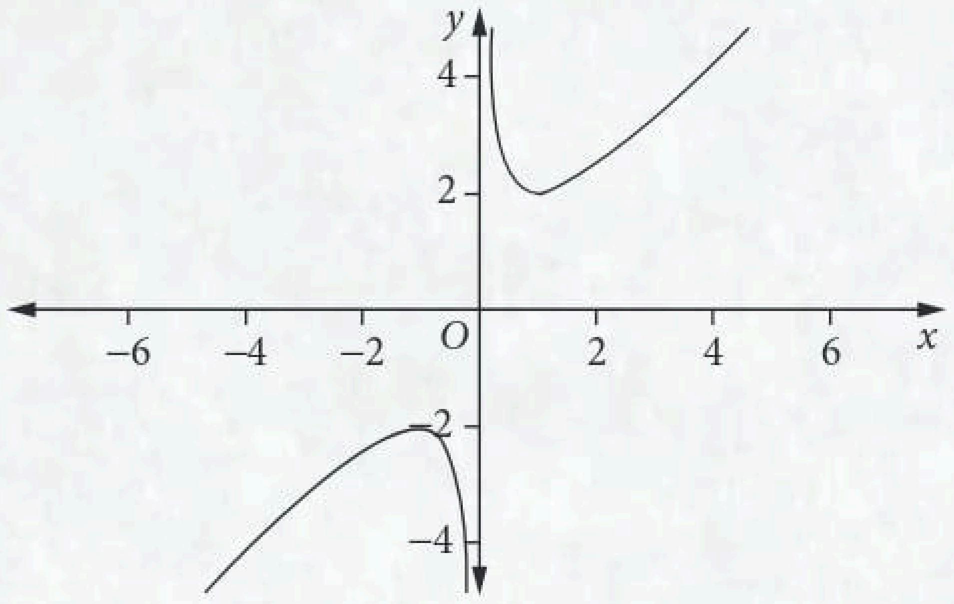
A



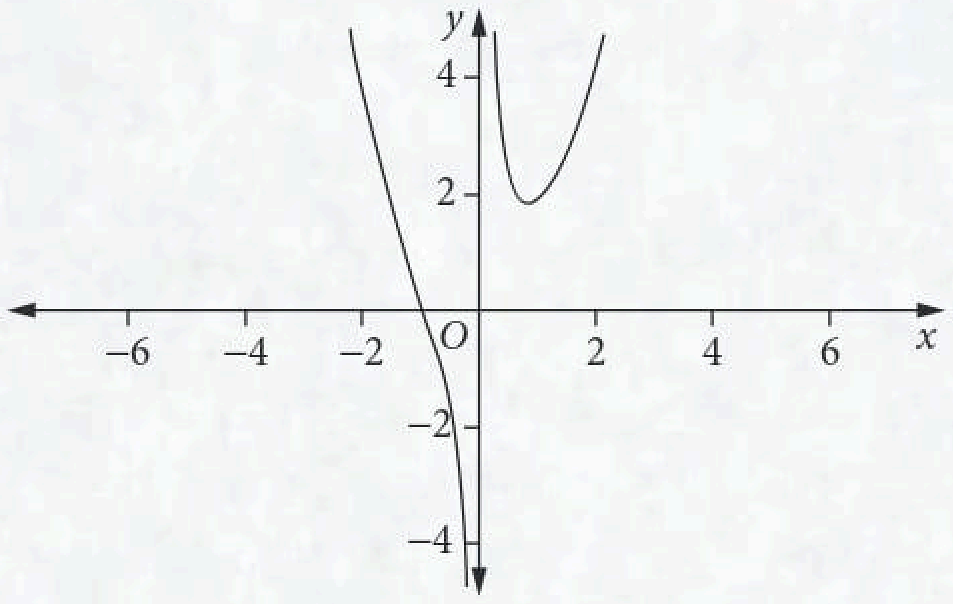
B



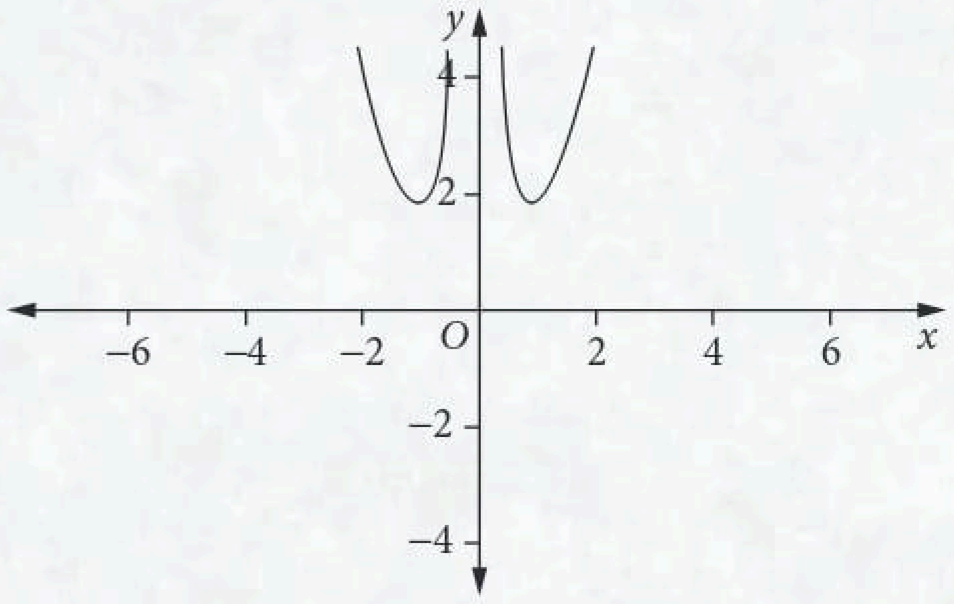
C



D



E



Section B 2 questions

17 marks

1 © VCAA 2019 2BQ4 (9 marks) The base of a pyramid is the parallelogram $ABCD$ with vertices at points $A\,(2, -1, 3)$, $B\,(4, -2, 1)$, $C\,(a, b, c)$ and $D\,(4, 3, -1)$. The apex (top) of the pyramid is located at $P\,(4, -4, 9)$.

a Find the values of a, b and c. 2 marks

b Find the cosine of the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$. 2 marks

c Find the area of the base of the pyramid. 2 marks

d Show that $6\underset{\sim}{i} + 2\underset{\sim}{j} + 5\underset{\sim}{k} + 2\underset{\sim}{j}$ is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AD}$, and hence find a unit vector that is perpendicular to the base of the pyramid. 3 marks

2 © VCAA 2014 2BQ1 (8 marks) Consider the function f with rule $f(x) = \dfrac{9}{(x+2)(x-4)}$ over its maximal domain.

a Find the coordinates of the stationary point(s). 3 marks

b State the equations of all asymptotes of the graph of f. 2 marks

c Sketch the graph of f for $x \in [-6, 6]$, showing asymptotes, the values of the coordinates of any intercepts with the axes, and the stationary point(s). 3 marks

CHAPTER

3

LOGIC AND PROOF

Note: This is a new topic introduced to the Specialist Mathematics course in 2023, so this chapter does not contain past VCE exam questions.

Study Design coverage

AREA OF STUDY 1: DISCRETE MATHEMATICS

Logic and proof

- conjecture – making a statement to be proved or disproved
- implications, equivalences and if and only if statements (necessary and sufficient conditions)
- natural deduction and proof techniques: direct proofs using a sequence of direct implications, proof by cases, proof by contradiction, and proof by contrapositive
- quantifiers ‘for all’ and ‘there exists’, examples and counter-examples
- proof by mathematical induction.

VCE Mathematics Study Design 2023–2027 p. 109, © VCAA 2022

Video playlists (6):

3.1 Conjectures
3.2 The language of proof
3.3 Direct proof
3.4 Proof by contrapositive and contradiction
3.5 Proof by mathematical induction
VCE question analysis Logic and proof

Worksheets (7):

3.1 Counterexamples
3.2 Necessary and sufficient • Converse
3.3 Quantifiers
3.4 Contrapositive
3.5 Proof by induction • Mathematical induction

Nelson MindTap

To access resources above, visit
cengage.com.au/nelsonmindtap

3.1 Conjectures

Video playlist
Conjectures

The language of logic

Logic is the study of correct reasoning using a defined set of principles and definitions and is applied in many areas including philosophy, mathematics, computer science and economics.

A study of logic requires information to be communicated clearly and accurately using a set of definitions. A **statement** (or **proposition**) is a sentence that is either true or false. Questions are not statements.

‘Water at room temperature is wet’ (true) and ‘1 + 1 = 3’ (false) are both statements, but ‘Is it hot today?’ is not a statement.

A statement that allows a conclusion (**inference**) to be drawn is a **premise**. Making an inference is drawing a conclusion based on the facts. Inferences can be made from premises, but they may not be true. For example:

‘Elephants have big ears.

My Basset Hound dog has big ears.

So my dog is an elephant.’

which is false, because the logic is not correct.

An **atomic sentence** is a single statement that cannot be broken down into simpler sentences, and a **compound sentence** contains 2 or more atomic statements.

'The sky is blue' is an atomic sentence, whereas 'Today is Monday, and tomorrow I will go to the shop' is a compound proposition because it consists of two atomic sentences, 'Today is Monday' and 'Tomorrow I will go to the shop'.

A **logical argument** consists of premises and inferences that are true and from which a general conclusion can be made.

WB

p. 48

WORKED EXAMPLE 1 Identifying logical terms

A conversation between Lee and Ami is shown below.

Lee: How are you going, Ami? 1
Ami: I'm tired. 2
Lee: That's because you work too hard. 3
Ami: To recover, I'm going to relax, but I'll be at work tomorrow. 4
Lee: Rest always helps with tiredness. 5

a Decide whether each sentence is a statement.
b Describe the statement that is not atomic.
c Describe each inference.
d Explain if there is a logical argument.

Steps	**Working**
a A statement is true or false.	1 is a question, so it is not a statement. Sentences 2 to 5 are either true or false, so they are all statements.
b Look for a sentence that can be written as two or more sentences.	Statement 4 is not atomic as it can be written as 'To recover, I'm going to relax' and 'I'll be at work tomorrow'.
c Identify a premise from which a conclusion is made.	2 is a premise. 3 is an inference from 2 because Lee concludes that Ami is tired because she works too hard. 4 is an inference from 2 because Ami concludes that relaxing for the day will solve her tiredness.
d Decide if the premises and inferences are correct and that a general conclusion is made.	Each premise and its inference are true. The general conclusion is that rest cures tiredness. Hence, the conversation is a logical argument.

Inductive and deductive reasoning

Consider a function machine that accepts a set of random numbers as input, performs hidden calculations and then outputs the answers below.

Input	Output
2.7	3
8	1
$\sqrt{11}$	5
−55	7

Based on our observation, we might conclude that the output will always be an odd number. Our reasoning is logically true, but we cannot be certain that our conclusion is correct. For instance, what would we conclude if the next output was 16?

The method of forming a generalisation based on specific examples is **inductive reasoning**.

Consider the two statements below and the conclusion drawn.

Statement 1: All circles are round. (premise)

Statement 2: This shape is round. (premise)

Conclusion: This shape is a circle. (inference)

In this example, a true conclusion is drawn from statements 1 and 2.

However, even if all premises are true, it is not guaranteed that the conclusion drawn is true.

This is demonstrated by the following example:

Statement 1: Yani is a wrestler.

Statement 2: Yani is rich.

Conclusion: All wrestlers are rich.

Any valid example can be given to disprove the conclusion. For the example above, it is sufficient to state that the wrestler, Big Ted, is actually living in poverty.

The method of forming a specific conclusion from the general case is **deductive reasoning**, which is the opposite of **inductive reasoning**.

This is illustrated by the following example:

Statement 1: All wrestlers are fit.

Statement 2: Yani is a wrestler.

Conclusion: Yani is fit.

Conclusions made from deductive reasoning are true if all premises are true.

WORKED EXAMPLE 2 Inductive and deductive reasoning

WB p. 49

Decide, with reasons, whether each conclusion is made through inductive reasoning, deductive reasoning or neither.

a Every coin I've taken out of my pocket was 10 cents. I only have coins, so all of them must be 10 cent coins.

b During summer, Karen always suffers from hay fever. It's a hot day today, so Karen has hay fever.

c When Vin does homework, he passes the subject. Vin will do Science homework, so he should pass the subject.

Steps	**Working**
a 1 Write the information consisting of statements and a conclusion.	*Statement 1*: Every coin I've taken out of my pocket was 10 cents. *Statement 2*: I only have coins. *Conclusion*: I only have 10 cent coins in my pocket.
2 Decide if the information fits the definition of inductive or deductive reasoning.	A general conclusion is made from a particular case (the observations of coins).
3 State the type of reasoning.	This is inductive reasoning.
b 1 Write the information consisting of a statement and a conclusion.	*Statement 1*: During summer, Karen always suffers from hay fever. *Statement 2*: It's a hot day today. *Conclusion*: Karen has hay fever.
2 Decide if the information fits the definition of inductive or deductive reasoning.	A specific conclusion is made, but it does not follow from the statements because it is not mentioned if it's a summer day.
3 State the type of reasoning.	This is neither inductive nor deductive reasoning.

c **1** Write the information consisting of a statement and a conclusion.	*Statement 1*: When Vin does homework, he passes the subject. *Statement 2*: Vin will do Science homework. *Conclusion*: Vin will pass Science.
2 Decide if the information fits the definition of inductive or deductive reasoning.	A specific conclusion is made from a general case.
3 State the type of reasoning.	This is deductive reasoning.

Formulating a conjecture

In 1742, the German mathematician Christian Goldbach proposed a **conjecture** that every even number greater than 2 can be expressed as the sum of 2 prime numbers. Thus, $4 = 2 + 2$, $6 = 3 + 3$, $8 = 3 + 5$ and so on. A conjecture is a theory or hypothesis, an 'educated guess' based on observation, facts and experimentation. A proof has yet to be found for Goldbach's conjecture, although this number property has been shown to be correct for integers up to 4×10^{18} using computers and other methods. Golbach's proposal is an **open conjecture**, a mathematical statement that seems to be true but a general proof has not been established. A conjecture is therefore a conclusion reached from inductive reasoning.

There have been well-known conjectures in mathematics for which proofs have been established. One is Fermat's last theorem (1637), which states that the equation $a^n + b^n = c^n$ has no integer value solutions for a, b and c for values of $n > 2$. Proofs exist that there are infinite solutions for $a + b = c$ ($n = 1$) and for $a^2 + b^2 = c^2$ ($n = 2$, Pythagoras' theorem), but it was not until 1995 that a general proof was formulated covering all values of n.

When formulating conjectures, we look for a pattern and predict a relationship that will produce more results following the same pattern, keeping in mind that our conjecture might need to change to accommodate new findings. Consider the sequence of numbers 1, 2, 3. Perhaps the sequence is arithmetic, with each term increasing by 1.

What if the sequence is 1, 2, 3, 5? Our conjecture now changes and we might consider the Fibonacci sequence, where each term is the sum of the two preceding terms. However, if the sequence is 1, 2, 3, 5, 9, the conjecture might change to stating that the nth term, $t(n)$, of the sequence is $t(n) = \frac{1}{6}n^3 - n^2 + \frac{17}{6}n - 1$, a function satisfying all 5 values.

Worksheets
Counter-examples

For a conjecture to be true, it must apply to all possible cases. However, it is sufficient to provide only one **counterexample** as proof that the conjecture is not true. For example, the conjecture that all prime numbers are odd numbers can be dismissed by providing the counterexample that the number 2 is both prime and even.

> Merely providing examples to show the conjecture works is insufficient proof that the conjecture always works.

WB
p. 50

WORKED EXAMPLE 3 Conjectures and counterexamples

For each conjecture

- **i** provide an example
- **ii** give a counterexample.

a The sum of any 2 prime numbers is an even number.

b For all integers a and b, $\sqrt{a^2 + b^2} < a + b$.

c All natural numbers are either prime or composite.

Steps	Working
a **i** State a particular case when the conjecture holds.	Let $p_1 = 5$ and $p_2 = 7$ be the two prime numbers, so $p_1 + p_2 = 12$, an even number.
ii Provide a counterexample.	Let $p_1 = 2$ and $p_2 = 3$ be the two prime numbers, so $p_1 + p_2 = 5$, an odd number.
b **i** Give an example when the conjecture holds.	Let $a = 3$ and $b = 4$. $\sqrt{a^2 + b^2} = \sqrt{9 + 16} = 5$ and $a + b = 7$ So $\sqrt{a^2 + b^2} < a + b$.
ii Give a counterexample.	Let $a = 0$ and $b = 6$. $\sqrt{a^2 + b^2} = \sqrt{0 + 36} = 6$ and $a + b = 6$ So $\sqrt{a^2 + b^2} = a + b$.
c **i** Provide an example that shows the conjecture to be true.	Let $n = 1, 2, 3, 4, 5 \ldots$ be the set of natural numbers. All values of $n > 1$ are prime or composite. $n = 7$ is prime, $n = 12$ is composite.
ii Give a counterexample.	$n = 1$ is neither prime nor composite.

WORKED EXAMPLE 4 Forming a conjecture

WB p. 51

a State a conjecture using notation that describes the sequence $\frac{9}{4}, \frac{17}{4}, \frac{25}{4} \ldots$

b Predict the next three terms of the sequence.

c Provide an informal justification of your conjecture.

Steps	Working
a **1** Look for patterns.	The dividends are odd numbers, increasing by 8. The denominators are 4. The numerators 9, 17, 25 are given by $8n + 1$, $n = 1, 2, 3$.
2 Provide a conjecture.	Hence, the nth term is $t(n) = \frac{8n + 1}{4}, n = 1, 2, 3 \ldots$
b Predict more terms.	Find the 3 terms for $n = 4, n = 5, n = 6$. The next 3 terms are $t(4) = \frac{33}{4}, t(5) = \frac{41}{4}, t(6) = \frac{49}{4}$.
c Show general relationships.	$\frac{9}{4} = 2 + \frac{1}{4}, \frac{17}{4} = 4 + \frac{1}{4}, \frac{25}{4} = 6 + \frac{1}{4} \ldots$ $\frac{8n + 1}{4} = 2n + \frac{1}{4}, n = 1, 2, 3 \ldots$ Division by 4 produces even numbers and $\frac{1}{4}$. The divisor is 4 and the fraction remainder is $\frac{1}{4}$.

EXERCISE 3.1 Conjectures

ANSWERS p. 570

Mastery

1 WORKED EXAMPLE 1 Choose which description(s) (**i** to **iv**) apply to each sentence below.

i atomic sentence **ii** compound sentence **iii** statement **iv** sentence only

a Stand up.

b The square root of 6 is 3.

c Today is hot so I'll go for a swim.

d He asked me to listen.

e Circles are round, birds sing and grass grows.

f Please sit down. What is your name?

2 WORKED EXAMPLE 2 State whether each conclusion involves inductive reasoning, deductive reasoning or neither.

a Worms always come up from the soil after it rains. It has stopped raining, so worms will appear.

b When Tim rides his bike, his dog is nearby. Next time Tim rides his bike, a dog will be nearby.

c Each of 10 times that a biscuit was taken out of a bag of Yummy biscuits, it was either square shaped or circular. Therefore, each biscuit taken out of a Yummy bag will be square or round.

3 Explain why the conclusion made for each of the following examples of inductive reasoning is false.

a All the cockatoos I have seen are white, therefore all cockatoos are white.

b All left-handed people I know use left-handed scissors, so all left-handed people use left-handed scissors.

c In every display home I've been to, I've noticed that all bedroom walls are painted white. Therefore, all display homes have bedroom walls painted white.

d Every time I go to the shops, I spend at least five minutes looking for a parking spot. This means everyone going to the shops will spend at least five minutes looking for parking.

4 Consider the following.

Statement 1: $a > b$

Statement 2: $b < c$

Conclusion: $a \geq c$

By deductive reasoning, the conclusion will be true when

A $|a - b| = |c|$ **B** $|a| > |b - c|$ **C** $|a| + |b| = |c|$

D $|a + c| < |b|$ **E** $|a - c| = 0$

5 In each of the following, explain why the conclusion reached is not valid.

a *Statement 1*: Most triangles have acute angles for all three of their interior angles.

Statement 2: This polygon has an obtuse interior angle.

Conclusion: The polygon is not a triangle.

b *Statement 1*: As the number of sides of a regular polygon approaches infinity, the polygon looks more circular.

Statement 2: This shape is a circle.

Conclusion: This shape must be a polygon.

c *Statement 1*: All polygons are two-dimensional.

Statement 2: This shape is two-dimensional.

Conclusion: This shape is a polygon.

6 WORKED EXAMPLE 3 Provide an example and a counterexample for each of the following conjectures.

- **a** All numbers of the form $6n - 1$, $n = 2, 3, 4 \ldots$ are prime numbers.
- **b** All quadrilaterals with opposite sides of equal length are rectangles.
- **c** The exterior angle of any polygon is acute.
- **d** For all numbers x such that $|x| > 0$, $x^2 > x$.
- **e** For all integers, n, if n^2 is divisible by 4, then n is also divisible by 4.

7 The expression $2a + 6 = 4a$ will be a proposition if it is written as

- **A** $2a + 6 = 4a$ when $a = 3$.
- **B** $2a + 6 = 4a$, where a is a real number.
- **C** $2a + 6 = 4a$ is an inequation when $a > 3$.
- **D** $2a + 6 = 4a$ has more than one solution.
- **E** $6 = 4a - 2a$.

8 In atomic form, 'Hey, listen. Circles are round and I am tall' is

- **A** 'Hey, listen.', 'Circles are round and I am tall.'
- **B** 'Hey, listen.', 'Circles are round.', 'I am tall.'
- **C** 'Circles are round and I am tall.'
- **D** 'Circles are round.', 'I am tall.'
- **E** 'Hey, listen. Circles are round.', 'I am tall.'

9 What inference can be drawn from 'The dog barks when the postman arrives, and the dog is now barking'?

- **A** The dog sees the postman.
- **B** The dog sees something.
- **C** The postman is approaching.
- **D** The postman has arrived.
- **E** The dog hates the postman.

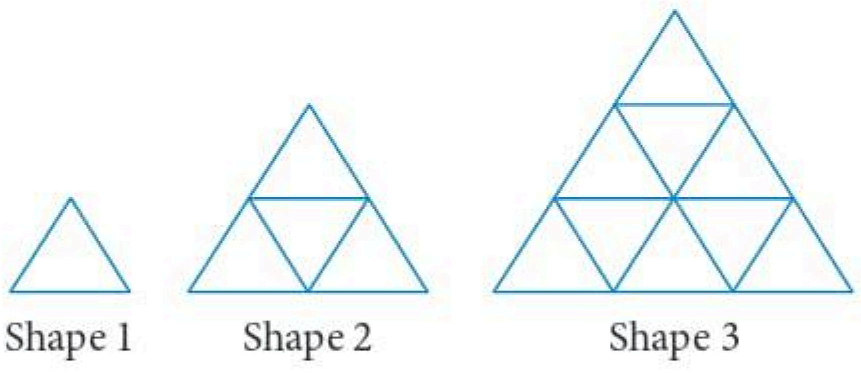
10 **a** State 2 conjectures about the sum of the infinite sequence $1 - 1 + 1 - 1 + 1 - \ldots$

- **b** Suggest a third conjecture by starting with $1 - 1 + 1 - 1 + 1 - \ldots = 1 - (1 - 1 + 1 - \ldots)$

11 WORKED EXAMPLE 4

- **a** State a conjecture using notation that describes the sequence $\frac{7}{3}, \frac{13}{3}, \frac{19}{3} \ldots$
- **b** Predict the next 3 terms of the sequence.
- **c** Provide an informal justification of your conjecture.

12 Let $f(n)$ be the total number of sides in shape n.



- **a** State the value of $f(1)$, $f(2)$ and $f(3)$.
- **b** Describe a conjecture in the form $f(n) = Af(n-1) + Bn$, (A, B constants).
- **c** Use your conjecture to predict the result for $f(7)$.

13 Which of the following is a counterexample to the conjecture, 'If p is an odd prime number, then $p + 2$ is also a prime number?'

A 3 **B** 5 **C** 7 **D** 9 **E** 11

14 Consider the conjecture, 'If n is an integer, then $n^2 + 1$ is prime'.

A value of n to use as a counterexample to the conjecture is

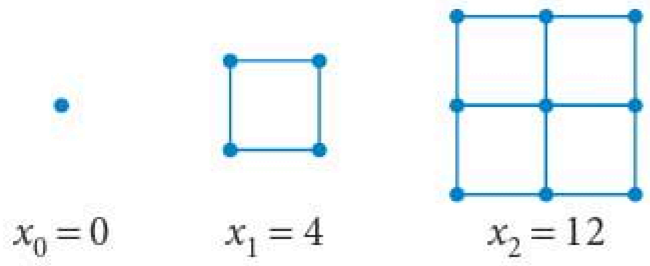
A 1 **B** 2 **C** 3 **D** 4 **E** 10

15 (3 marks) A conjecture that remained unproven for a long time is that there is only one solution to $x^a - y^b = 1$, where $x > 0$, $y > 0$, $a > 1$ and $b > 1$ are natural numbers.

a Given that $m > 0$, $n > 1$ are natural numbers, list the values of m^n up to $m = n = 4$. 2 marks

b Hence state the solution to $x^a - y^b = 1$. 1 mark

16 (6 marks)



Let x_n be the number of lines connecting adjacent vertices in a square shape with side length n.

a Find x_3, x_4 and x_5. 3 marks

b State a conjecture, x_n, in the form $x_n = Ax_{n-1} + Bn$, (A, B constants), that gives the number of lines in a shape with n sides. Use it to predict x_5. 2 marks

c Discuss how your conjecture for the explicit form changes if each shape is drawn in three dimensions. 1 mark

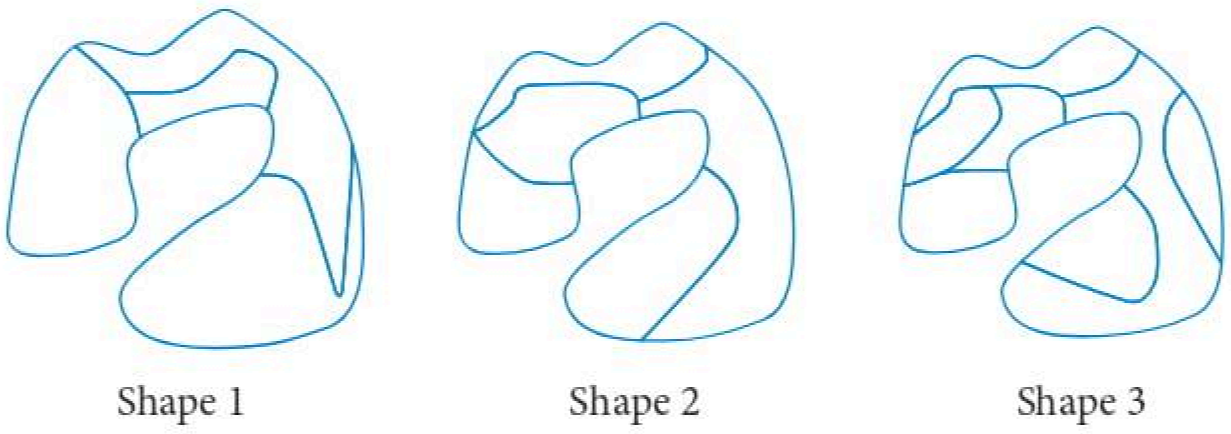
17 (4 marks) The Collatz conjecture states that for any natural number, n, successively applying the function $f(n) = \begin{cases} \dfrac{n}{2} & n \text{ even} \\ 3n + 1 & n \text{ odd} \end{cases}$ produces a certain result.

a Apply the function to the number 20. 1 mark

b Apply the function to the number 25. 1 mark

c Describe the complete conjecture. 2 marks

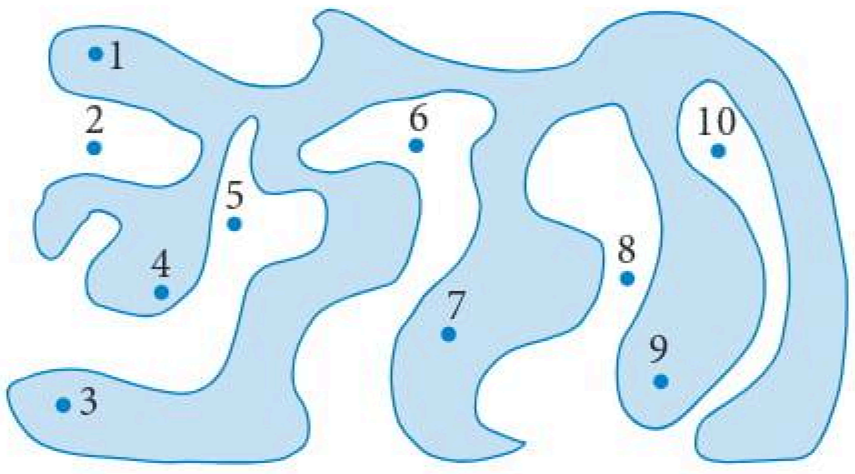
18 (4 marks) Consider the 3 shapes below.



a For each shape, determine the maximum number of colours needed to colour the regions so that no 2 adjacent regions have the same colour. 3 marks

b State a conjecture in relation to the number of colours needed. 1 mark

19 (5 marks) Look at the 10 labelled points in the shape below.



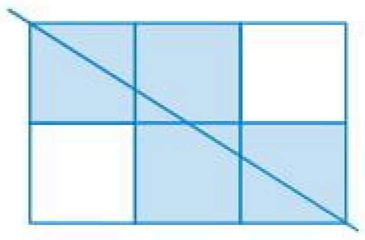
a For each point 1–10, imagine a horizontal line starting at that point and finishing anywhere outside the shape on the right.

Copy and complete the table to show the number of intersections of each horizontal line with the shape's perimeter. 3 marks

Point	Number of intersections
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

b What conjecture can be put forward involving the relationship between the number of intersections and whether the point lies inside or outside the shape? 2 marks

20 (3 marks) Below is a grid consisting of 2 rows and 3 columns, with a line drawn from one corner to another passing through 4 squares.



The table shows results for the number of squares crossed using grids of different size.

Rows	1	2	3	4	5	6	6	7	7	8	8	8	8	8	12	12
Columns	1	3	4	4	3	2	3	1	5	1	2	3	4	8	6	8
Squares crossed	1	4	6	4	7	6	6	7	11	8	8	10	8	8	12	16

a Suggest a conjecture that relates the number of squares crossed to the number of rows and columns. 2 marks

b Use your conjecture to show that a line drawn through a grid that has 187 columns and 198 rows will cross 374 squares. 1 mark

21 (4 marks) Suppose it is conjectured that for $x, y \in R$, $|x+y| \leq |x| + |y|$.

a Verify that the conjecture is true for $x = -5$ and $y = 3$. 2 marks

b Prove the conjecture for all real values, x, y. 2 marks

Video playlist
The language of proof

3.2 The language of proof

Symbols and terminology play an important role in presenting information and forming valid reasoning and providing proofs.

Connectives

Logical connectives (or **logical operators**), combine, compare, imply or negate statements. The main connectives are listed below.

Connective	Meaning	Symbol
Conjunction	AND	$\wedge$
Disjunction	OR	$\vee$
Negation	NOT	$\neg$
Equivalence	is identical to	$\equiv$
Conditional	if then	$\rightarrow$
Biconditional	if and only if	$\leftrightarrow$

WB

p. 52

WORKED EXAMPLE 5 Logical connectives

Consider the statements:

A: 'it is raining'

B: 'I stay indoors'

C: 'the sun shines'

D: 'I go for a walk'.

Translate each symbolic statement into an English sentence using correct grammar.

a $A \rightarrow B$ **b** $\neg A \rightarrow D$ **c** $\neg C \rightarrow (A \wedge B)$

d $D \rightarrow \neg A \vee C$ **e** $\neg B \leftrightarrow C$ **f** $\neg B \equiv D$

Steps	**Working**
a 1 Write the statement using the word connectives.	**If** A **then** B.
2 Write the completed sentence.	**If** it is raining, **then** I stay indoors.
b 1 Write the statement using word connectives.	**If NOT** A **then** D.
2 Write the completed sentence.	If it is not raining, then I go for a walk.
c 1 Add word connectives to the statements.	**If NOT** C **then** A **AND** B.
2 Write the completed sentence.	If the sun is not shining, then it is raining and I stay indoors.
d 1 Add word connectives.	**If** D **then NOT** A **OR** C.
2 Write the completed sentence.	If I go for a walk, then it is not raining or the sun shines.
e 1 Join statements with word connectives.	**NOT** B **if and only if** C.
2 Write the completed sentence.	I don't stay indoors if and only if the sun shines.
f 1 Join statements with word connectives.	**NOT** B **is identical to** C.
2 Write the completed sentence.	Not staying indoors is the same as going for a walk.

WORKED EXAMPLE 6 Statements using symbolic logic

Write statements with logical connectives to describe each sentence.

a A triangle has an interior angle greater than 90° if and only if it is an obtuse triangle.

b If a number is divisible by 2 and 3, then it is not divisible by 7.

Use A: divisible by 2. B: divisible by 3. C: divisible by 7.

c If today is not Tuesday or Thursday, I go to the movies but not to the shops.

Use A: today is Tuesday. B: today is not Thursday.

C: I go to the movies. D: I go to the shops.

Steps	**Working**
a **1** Write the statements to be used.	A: A triangle has an interior angle greater than 90°. B: An obtuse triangle.
2 Identify the connectives and write the statements in notation form.	Use a biconditional operator. $A \leftrightarrow B$
b **1** Decide on the connectives.	$A \wedge B$ means 'divisible by 2' and 'divisible by 3'. $\neg C$ means 'The number is not divisible by 7'.
2 Write the information in notation form.	Use a conditional operator. $A \wedge B \rightarrow \neg C$
c **1** Decide on the connectives to use.	A: Today is Tuesday. B: Today is Thursday. $\neg(A \vee B)$ means 'Today is not Tuesday or Thursday'. C is 'I go to the movies'. $\neg D$ is 'I don't go to the shops'. $C \wedge \neg D$ is 'I go to the movies and not to the shops'.
2 Write the sentence in logical form.	Combine with a conditional operator. $\neg(A \vee B) \rightarrow C \wedge \neg D$

WB

p. 53

The 'if and only if' biconditional statement

The biconditional connective $X \leftrightarrow Y$ is written as X iff Y.

It is read as 'X **if and only if** Y' and means $(X \rightarrow Y) \wedge (Y \rightarrow X)$.

'iff' is the abbreviation for 'if and only if'

$X \rightarrow Y$ is the **converse** statement to $Y \rightarrow X$ and $Y \rightarrow X$ is the converse statement to $X \rightarrow Y$.

To show $X \leftrightarrow Y$ we show both $X \rightarrow Y$ and $Y \rightarrow X$ to be true.

$X \rightarrow Y$ means 'X is a **sufficient condition** for Y'.

$Y \rightarrow X$ means 'X is a **necessary condition** for Y'.

Hence, **necessary and sufficient condition** means $X \leftrightarrow Y$.

For example, letting X: '$a + 4 = 9$' and Y: '$a = 5$', we have the following.

Read $X \rightarrow Y$ as 'if $a + 4 = 9$ then $a = 5$'.

$a + 4 = 9$ is a sufficient condition that gives $a = 5$ but it is not necessary. For instance, $3a - 7 = 8$ also gives $a = 5$.

Read $Y \rightarrow X$ as 'if $a = 5$ then $a + 4 = 9$'.

$a + 4 = 9$ is a necessary condition since $a = 5$ can lead to $a - 2 = 3$, $2a + 1 = 11$, and so on.

Thus $X \leftrightarrow Y$ can be written as '$a + 4 = 9$ iff $a = 5$'.

Worksheets
Necessary and sufficient
Converse

WB

p. 54

WORKED EXAMPLE 7 Definitions as biconditional statements

Express each definition as a biconditional statement.

a A hexagon is a 6-sided polygon.

b An acute triangle is a triangle with 3 acute angles.

Steps	Working
a **1** Find statements A and B so that $A \rightarrow B$ and $B \rightarrow A$.	A: hexagon B: 6-sided shape
2 State $A \rightarrow B$ and $B \rightarrow A$ in words.	$A \rightarrow B$ If a shape is a hexagon then it has 6 sides. $B \rightarrow A$ If a shape has 6 sides, then it is a hexagon.
3 State the biconditional statement $A \leftrightarrow B$.	A shape is a hexagon if and only if it has 6 sides.
b **1** Find statements A and B so that $A \rightarrow B$ and $B \rightarrow A$.	A: an acute triangle B: 3 acute angles
2 State $A \rightarrow B$ and $B \rightarrow A$ in words.	$A \rightarrow B$ If a triangle is an acute triangle then it has 3 acute angles. $B \rightarrow A$ If a triangle has 3 acute angles then it is an acute triangle.
3 State the biconditional statement $A \leftrightarrow B$.	A triangle is acute if and only if it has 3 acute angles.

WB

p. 55

WORKED EXAMPLE 8 'if and only if' involving divisibility

Show that for $n \in N$, n is divisible by 35 if and only if n is divisible by both 5 and 7.

Steps	Working
1 Choose statements.	X (necessary condition for Y): number divisible by 35. Y (sufficient condition for X): divisible by both 5 and 7.
2 Show $X \rightarrow Y$.	n is divisible by 35 means $n = 7 \times 5 \times m,\ m \in N$. Hence n is divisible by both 5 and 7.
3 Show $Y \rightarrow X$.	n is divisible by both 5 and 7 means n is also divisible by $5 \times 7 = 35$.

WB

p. 56

WORKED EXAMPLE 9 'if and only if' involving square numbers

Show that for $n \in N$, n^2 even $\leftrightarrow$ n is even.

Steps	Working
1 Choose statements.	X: n^2 is even. Y: n is even.
2 Show $X \rightarrow Y$.	n^2 even means $n \times n$ is even. Hence, n is even, since if n was odd, the product of 2 odd numbers would be odd.
3 Show $Y \rightarrow X$.	n is even, so $n = 2m,\ m \in N$. $n^2 = 4m^2$ so $n = 2m$. $2m$ is divisible by 2, so n is even.

Logical equivalence

Two statements are **logically equivalent** if they always produce the same truth value.

For example, statement A: 'all even numbers' and statement B: 'all multiples of 2', are equivalent, so we have $A \equiv B$.

Some common logical equivalences are the following.

Idempotent laws	$A \lor A \equiv A$	$A \land A \equiv A$
Double negation law	$\neg(\neg A) \equiv A$	
Commutative laws	$A \lor B \equiv B \lor A$	$A \land B \equiv B \land A$
Associative laws	$(A \lor B) \lor C \equiv A \lor (B \lor C)$	$(A \land B) \land C \equiv A \land (B \land C)$
Distributive laws	$A \lor (B \land C) \equiv (A \lor B) \land (A \lor C)$	$A \land (B \lor C) \equiv (A \land B) \lor (A \land C)$
De Morgan's laws	$\neg(A \lor B) \equiv \neg A \land \neg B$	$\neg(A \land B) \equiv \neg A \lor \neg B$

Proofs of these equivalences can be shown using **truth tables**, which we learned about in Year 11.

De Morgan's laws

De Morgan's laws comprise logical equivalences that illustrate the relationship between the negation, conjunction and disjunction involving two propositions.

De Morgan's laws

$\neg(A \lor B) \equiv \neg A \land \neg B$	The negation of a disjunction is the conjunction of the negations.
$\neg(A \land B) \equiv \neg A \lor \neg B$	The negation of a conjunction is the disjunction of the negations.

To demonstrate De Morgan's Laws, in Worked example **6c**, we have

A: Today is Tuesday B: Today is Thursday.

$\neg(A \lor B)$ means 'Today is not Tuesday or Thursday'.

Write $\neg A$ is 'Today is not Tuesday' and $\neg B$ is 'Today is not Thursday'.

Using $\neg(A \lor B) \equiv \neg A \land \neg B$ we can say 'Today is not Tuesday or Thursday' is equivalent to 'Today is not Tuesday and Today is not Thursday'.

A useful property that will be used later in the chapter as part of proofs is to negate a statement. Consider the negation of the statement, 'The maths problem can be solved by factorising or by using the quadratic formula'.

Let A: 'use factorising' B: 'use the quadratic formula'.

Then $\neg(A \lor B) \equiv \neg A \land \neg B$ reads, 'The problem **can't** be solved by factoring **and** it **can't** be solved using the quadratic formula'. Negation is used in De Morgan's laws.

WORKED EXAMPLE 10 Verifying De Morgan's laws

WB p. 57

a In a bag are 7 tokens numbered 1, 2, 3, 4, 5, 6, 7. Tokens 2, 4 and 5 are randomly removed and put back into the bag. Tokens 1, 2, 3 and 7 are then randomly removed.
Show that $\neg(A \lor B) \equiv \neg A \land \neg B$ is true.

b Show that $\neg(A \land B) \equiv \neg A \lor \neg B$ using the statement: 'I like swimming and I like shopping'.

c Show that $\neg(A \land B \land C) \equiv \neg A \lor \neg B \lor \neg C$.

Steps	Working
a 1 Write the statements to be used.	A: {2, 4, 5} B: {1, 2, 3, 7}
2 Find the set of numbers that satisfy $\neg(A \lor B)$.	$A \lor B$: {1, 2, 3, 4, 5, 7} Numbers in A or B. $\neg(A \lor B)$: {6}
3 Find the set of numbers that satisfy $\neg A \land \neg B$.	$\neg A$: {1, 3, 6, 7} Numbers from 1 to 7 that are not in A. $\neg B$: {4, 5, 6} Numbers from 1 to 7 that are not in B. $\neg A \land \neg B$: {6} Numbers from 1 to 7 that are not in A and not in B.

b	**1** Write the statements to use.	A: I like swimming. B: I like shopping.
	2 Express $\neg(A \wedge B)$ with words.	**Not** (I like swimming **and** I like shopping).
	3 Express $\neg(A \wedge B)$ with correct grammar.	I don't like swimming and shopping.
	4 Express $\neg A \vee \neg B$ in word form.	**not** I like swimming **or not** I like shopping.
	5 Express $\neg A \vee \neg B$ using correct grammar.	I don't like swimming or shopping.
c	**1** Write in the form $\neg(X \wedge Y)$.	$\neg(A \wedge B \wedge C) = \neg((A \wedge B) \wedge C)$
	2 Apply De Morgan's law twice.	$\neg((A \wedge B) \wedge C) = \neg(A \wedge B) \vee \neg C$ $= \neg A \vee \neg B \vee \neg C$

EXERCISE 3.2 The language of proof

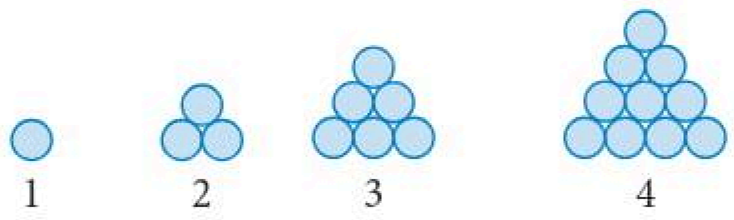
ANSWERS p. 571

Recap

1 A conjecture states that for prime numbers up to 41, the difference between any two consecutive prime numbers is an even number. A counterexample is that for two particular prime numbers in the interval [1, 41], the difference is

A 1 **B** 3 **C** 5 **D** 7 **E** 9

2 Which conjecture, $f(n)$, describes the total number of circles in shape n?



A $f(n) = f(n-1) + n, n > 1$

B $f(n+1) = f(n) + n + 1, n > 0$

C $f(n-1) = 2f(n) + n, f(1) = 1$

D $f(n) = f(n-1) + n - 1, f(1) = 1$

E $f(n+1) = (n+1)f(n) + n, f(1) = 1$

Mastery

3 WORKED EXAMPLE 5 Translate each symbolic statement into an English sentence using correct grammar.

A: winter B: it's cold C: people ski D: summer E: I swim

a $A \rightarrow (B \wedge C)$

b $(D \vee \neg B) \rightarrow E$

c $\neg(\neg C \wedge \neg B) \rightarrow \neg D \wedge \neg E$

4 WORKED EXAMPLE 6 Write statements with logical connectives to describe each sentence.

a A number is rational if and only if it can be expressed as the ratio of two integers.

b I pass tests if and only if I study hard.

c If it's raining or it's cold, I stay indoors.

5 Express the following statement in terms of statements A and B so that $A \leftrightarrow B$:

'The quadratic equation $f(x) = 0$ has two x-intercepts iff the discriminant is positive.'

6 Two sets of whole numbers, A = {numbers between 1 and 10} and B = {odd numbers from 1 to 7} are randomly selected from the first 10 natural numbers.

Use the sets to verify De Morgan's laws.

7 Use logical connectives with each set of statements to make them equivalent to statement *Z*.

- **a** *Z*: A number divisible by 6 is divisible by 2 or divisible by 3.

  A: Number divisible by 6
  B: Number divisible by 2
  C: Number divisible by 3

- **b** *Z*: A 4-sided shape that has opposite sides parallel and diagonals of different length is not a square.

  A: Opposite sides parallel
  B: A shape with 4 sides
  C: Diagonals are of different length
  D: A square

- **c** *Z*: Parallel lines do not intersect

  A: Non-intersecting lines
  B: No point of intersection
  C: Infinite points of intersection

8 Explain why the converse of $(A \wedge B) \to C$ is not always true for *A*: recurring decimal, *B*: terminating decimal and *C*: rational number.

9 State the converse and the biconditional statement for the statement, 'If points lie on the same line, then they are collinear'.

10 Which statement is the converse of the statement, 'If a quadrilateral is a square, then it is both a rectangle and a rhombus'?

- **A** If a quadrilateral is a square, then it is a rhombus or a square.
- **B** If the quadrilateral is either a rectangle or a rhombus, then it is a square.
- **C** If a quadrilateral can be both a rectangle and a rhombus, then it must be a square.
- **D** If a quadrilateral is both a rectangle and a rhombus, then it could be a square.
- **E** If a quadrilateral is both a rectangle and a rhombus, then it is a square.

11 State the converse to each statement and decide if it is always true.

- **a** If it is sunny, then it is a hot day.
- **b** When I am hungry, I go to the supermarket to buy food.
- **c** If this shape is a triangle, then it has exactly three sides.

12 WORKED EXAMPLE 7 Express each definition as a biconditional statement.

- **a** Two non-parallel lines have one point of intersection.
- **b** The square of an odd number is odd.

13 Suppose we have the conditional statement 'If the sum of two angles is 180°, then they are supplementary'.

- **a** Write the converse of the statement.
- **b** State the biconditional statement.

14 Write the converse of the statement 'A square is a figure with 4 right angles', and explain whether the biconditional statement is true.

15 WORKED EXAMPLE 8 Show that for $n \in N$, *n* is divisible by both 3 and 4 if and only if *n* is divisible 12.

16 WORKED EXAMPLE 9 Show that for $n \in N$, n^2 is odd $\leftrightarrow$ *n* is odd.

17 WORKED EXAMPLE 10 Verify one of De Morgan's laws by using the statements *A*: {whole numbers between 3 and 10} and *B*: {odd numbers from 1 to 7}, where both sets are drawn from the set of whole numbers {1, 2, 3 … 10}.

Exam practice

18 (2 marks) Consider statements P: ‘The function is differentiable’ and Q: ‘The function is continuous’.

If a function is differentiable at point c, then it is continuous at c. Explain, using P, Q and ‘necessary and sufficient’ language, whether $P \leftrightarrow Q$.

19 (2 marks) Use one of De Morgan’s laws to show that for N statements, $A_1, A_2, A_3 \dots A_N$, $\neg(A_1 \wedge A_2 \wedge \dots \wedge A_N) \equiv \neg A_1 \vee \neg A_2 \vee \dots \vee \neg A_N$.

20 (3 marks) Apply De Morgan’s laws to show each equivalence

a $\neg((A \vee \neg B) \vee C) \equiv \neg A \wedge B \wedge \neg C$ 1 mark

b $\neg(\neg(A \vee B) \wedge (\neg C \wedge \neg D)) \equiv A \vee B \vee C \vee D$ 1 mark

c $\neg(A \vee B \vee C) \equiv \neg A \wedge \neg B \vee \neg C$ 1 mark

21 (2 marks) A: ‘real numbers a, b such that $a \times b = 0$’ B: ‘$a = 0$’ C: ‘$b = 0$’

Use the statements A, B and C and logical connectives to describe the Null Factor Law:

‘If $a \times b = 0$ then either a or b is zero or both a and b are zero’.

22 (5 marks)

a By using the notation $\neg X$ for $\bar{X}$, $X \vee Y$ for $X + Y$ and $X \wedge Y$ for XY, show that

$\neg(\neg(A \wedge \neg B \vee \neg C) \vee \neg(A \wedge B \wedge C))$ can be written as $\overline{\overline{A\bar{B} + \bar{C}} + \overline{ABC}}$. 2 marks

b Given the properties $(X + Y)Z = XZ + YZ$, $XX = X$ and $X\bar{X} = 0$, show that

$\neg(\neg(A \wedge \neg B \vee \neg C) \vee \neg(A \wedge B \wedge C)) = 0$ 3 marks

Video playlist
Direct proof

3.3 Direct proof

Verifying $X \rightarrow Y$ is known as **direct proof**. It is the most common method of proof, where X (assumed to be true) is used to directly conclude that Y is true. A direct proof consists of a sequence of statements already given or which can be deduced from previous statements. The last statement is the conclusion to be proved.

WB
p. 58

WORKED EXAMPLE 11 Direct proof

a Given that even numbers are divisible by 2, prove that the sum of 2 odd numbers is even.

b Prove that if an integer n is not divisible by 3, then $n^2 - 1$ is a multiple of 3.

Steps	**Working**
a **1** State what is to be proved.	P: sum of 2 odd numbers. Q: even number. Show $P \rightarrow Q$.
2 Write the first statement using known information.	Let a and b be the 2 odd numbers. Hence for integers m, n we have $a = 2m + 1$, $b = 2n + 1$ State the sum. $a + b = 2m + 2n + 2$ $= 2(m + n + 1)$ $= 2k$, for integer k.
3 Establish the conclusion.	$2k$ is divisible by 2 (given). Hence the sum of the 2 odd numbers, $a + b$, is divisible by 2.

b **1** Write the statements to use.	A: n is not divisible by 3. B: $n^2 - 1$ is a multiple of 3.

2 Write the first statement using known information.	Not divisible by 3 means that after division, the remainder is 1 or 2.
	Hence for integer p,
	$n = 3p + 1$ or $n = 3p + 2$
	If $n = 3p + 1$, then
	$n^2 - 1 = (3p + 1)^2 - = 3(3p^2 + 2p) = 3k$, for integer k.
	If $n = 3p + 2$, then
	$n^2 - 1 = (3p + 2)^2 - 1 = 3(3p^2 + 4p + 1)$
	$= 3k$, for integer k.

3 Establish the conclusion.	Both cases give multiples of 3, so $n^2 - 1$ is a multiple of 3.

Sets of numbers

C complex numbers: of the form $a + bi$, where a, b are real numbers.

R real numbers: all integers, rational and irrational numbers.

Q rational numbers: of the form $\frac{a}{b}$, where a, b are integers but $b \neq 0$.

Z integers: … $-2, -1, 0, 1, 2, 3$ …

N natural numbers: 1, 2, 3, 4 …

Subsets are denoted with a + or –, such as R^- negative real numbers, Z^+ positive integers.

Quantifiers

Worksheet
Quantifiers

Using **quantifiers**, the statement 'You can fool some of the people all of the time' can be written as '$\exists$ people p such that $\forall$ time t, you can fool p'. Quantifiers such as $\exists$ and $\forall$ are symbols that combine mathematical statements, when the amount of a quantity relates to 'all' and 'there is'.

In direct proof $P \rightarrow Q$, statement Q is the **predicate**. It will be either true or false.

For example, for P: x is a real number Q: $x^2 = -1$, $P \rightarrow Q$ is false.

However, writing P: x is a complex number Q: $x^2 = -1$, $P \rightarrow Q$ is true, since $x = \pm i$.

We write this as '$\exists x \in C, P(x)$', where $\exists$ is the **existential quantifier** 'there exists', and the sentence reads, 'There exists an x in the set of complex numbers such that $x^2 = -1$.'

Note that '$\exists x$' doesn't necessarily mean there is *only one* value of x. It means 'there is *at least one* value'.

For example, suppose the predicate $P(x)$ is $x^2 - 3x + 2 = 0$ and the domain is all real numbers. We have '$\exists x \in R, P(x)$ with solutions $x = 1$, $x = 2$', also written as '$\exists x \in R, P(1), P(2)$', since $P(1) = 0$ and $P(2) = 0$.

Now consider values of x that satisfy $P(x)$: $x + 0 = x$. *All* real values of x are solutions, written as" $\forall x \in R, P(x)$. $\forall$ is the **universal quantifier**, meaning 'for all' or 'for every'.

In general, we write '$\forall x \in D, (P(x) \rightarrow Q(x))$', where D is the domain, but if the domain is assumed, it does not need to be included.

Quantifiers can be used in different ways. The statement '$x + y = 3, x, y \in R$' is true because for all values, x, there is a y that satisfies $x + y = 3$, and for all values, y, there is an x that satisfies $x + y = 3$. This can be expressed as, respectively, '$\forall x \exists y\ (x + y = 3)$' and '$\exists x \forall y\ (x + y = 3)$'.

Make sure that quantifiers are written in the correct order. For example, the statement 'For every natural number m there exists a natural number n such that $n = m^2$, can be written as '$\forall n \exists m\ (n = m^2)$' *but not as* '$\exists m \forall n\ (n = m^2)$'. The first statement is true but the second is false because of its claim that all natural numbers can be written as the square of a natural number.

WB
p. 59

WORKED EXAMPLE 12 Interpreting expressions containing quantifiers

Write each expression as a statement. If no domain is given, assume it is R.

a $\forall x \exists y\,(y = \sin(x))$ **b** $\forall x, y \in R, \exists z \in C\,(z = x + yi)$

Steps	**Working**
a 1 State the meaning of each quantifier and the predicate.	$\forall x$ All real numbers, x. $\exists y$ There is a y value corresponding to the given value of x. Predicate: $y = \sin(x)$
2 Form one statement.	Combine the statements to make one statement. 'For all real numbers, x, there exists a y such that y is the sine of x'.
b 1 State the meaning of each quantifier and the predicate.	$\forall x \in R$ All real numbers, x. $\forall y \in R$ All real numbers, y. $\exists z \in C$ There exists a z in the set of complex numbers. Predicate: $z = x + yi$
2 Form one statement.	Combine the statements into one statement. 'For all real numbers, x and y, there exists a z in the set of complex numbers such that $z = x + yi$'.

WB
p. 60

WORKED EXAMPLE 13 Writing statements using quantifiers

Write each expression using appropriate quantifiers. If no domain is given, it can be assumed to be R and it need not be included in the answer.

a $c = \sqrt{a^2 + b^2}$ **b** $p = mn,\ p, m, n \in N$

Steps	**Working**
a 1 Determine the domain, quantifiers and predicate.	No domain is given, so assume it is R. Every number in R qualifies to be a and b, so $\forall a, b \in R$ is needed. c depends on a and b and is real. The predicate is the formula $c = \sqrt{a^2 + b^2}$.
2 Form one statement.	Since the domain is the default domain for all variables, it can be omitted. $\forall a \forall b\ \exists c\,(c = \sqrt{a^2 + b^2})$
b 1 Determine the domain, quantifiers and predicate.	$p = mn,\ p, m, n \in N$ The domain is N and includes all values of p, m and n. All of the domain can be used as values for m and n, so we use $\forall m \in N$ and $\forall n \in N$. p is the product of m and n and depends on their values, so $\exists p$ is needed. The predicate is $p = mn$.
2 Form one statement.	$\forall m, n \in N, \exists p\,(p = mn)$

WORKED EXAMPLE 14 Decide if statements involving quantifiers are true

Decide if each statement is true or false. Assume x and y are natural numbers.

a $\forall x\,(x < 5 \rightarrow \exists y\,(x + y = 6))$ **b** $\forall x \exists y\,(y > 2 \rightarrow (y^2 = x))$

Steps	**Working**
a **1** Work from left to right and test each given condition.	$\forall x\,(x < 5)$ are the numbers, $x = 1, 2, 3, 4$. $\rightarrow \exists y\,(x + y = 6)$ means that for each x, there is a number, y, such that $x + y = 6$. This is true for all possible values of x.
2 Decide if the statement is true.	The statement is true.
b **1** Write the statement as a sentence.	For all values of x, there is a y value greater than 2 such that x is the square of the y value.
2 Decide if the statement is true.	The statement is false. A counterexample is $y = 3, x = 2, 3^2 \neq 2$.

WB p. 61

WB p. 62

WORKED EXAMPLE 15 Proof using quantifiers 1

Prove $\forall x, y \in Q((x + y) \in Q)$.

Steps	**Working**
1 Translate the statement to omit quantifiers.	For all rational values x, y, $x + y$ is rational.
2 Write as a conditional statement, $P \rightarrow Q$.	P: x and y are rational numbers. Q: $x + y$ is rational.
3 Use P to deduce Q.	Let $x = \frac{a}{b}, y = \frac{c}{d}, a, b, c, d \in Z$. $x + y = \frac{ad + bc}{bd} = \frac{m}{n}$, where $m, n \in Z$.
4 State the proof.	$x + y$ is the ratio of two integers, hence the sum of two rational numbers is rational.

Negation of a quantifier

The negation of an existential statement ('there exists') is a universal statement ('for all').

$$\neg \exists x\,(P) \equiv \forall x\,(\neg P(x))$$

Statement: Some maths students study Physics.

Negation: No maths student studies Physics.

So to prove an existential statement is false, prove that its negation is true.

The negation of a universal statement is an existential statement.

$$\neg \forall x\,(P) \equiv \exists x\,(\neg P(x))$$

Statement: All maths students study Physics.

Negation: At least one student does not study Physics.

So to prove a universal statement is false, prove that its negation is true.

Exam hack

The negation of 'all' is NOT 'none'. The negation of 'all' is 'not all', or 'some' or 'at least one'.

WB p. 63

WORKED EXAMPLE 16 Proof using quantifiers 2

Prove that each statement is false.

a There is a positive integer, n, such that $n^2 + 3n + 2$ is prime.

b Every integer is odd.

Steps	Working
a **1** Write the statement using quantifiers.	$\exists n \in N, (n^2 + 3n + 2)$ is prime.
2 Write the negation.	$\forall n \in N, (n^2 + 3n + 2)$ is not prime.
3 Show that the statement is true.	$n^2 + 3n + 2 = (n+1)(n+2)$ $n + 1 > 1$ and $n + 2 > 1$ So $(n+1)(n+2)$ is the product of two integers, each of which is greater than 1. Hence $n^2 + 3n + 2$ is not prime.
b **1** Write the statement using quantifiers.	$\forall n \in Z, \exists k \in Z, (n = 2k + 1)$
2 Write the negation.	$\exists n \in Z, \forall k \in Z, (n \neq 2k + 1)$
3 Show the statement to be true.	Provide a counterexample. $n = 4$ cannot be written as $2k + 1$.

EXERCISE 3.3 Direct proof

ANSWERS p. 571

Recap

1 $\neg(\neg A \vee (\neg B \wedge \neg C))$ is equivalent to

A $(A \wedge B) \vee C$ **B** $A \wedge \neg B \vee C$ **C** $\neg A \wedge \neg B \vee \neg C$

D $A \wedge (B \vee C)$ **E** $(\neg A \vee B) \vee C$

2 The converse of the statement 'If $k > 3$ then $k^2 > 9$' is true when

A $k < -3$ **B** $k > 3$ **C** $k < 3$ **D** $k \geq -3$ **E** $k \leq 3$

Mastery

3 Show that if n is odd, then n^4 is odd.

4 Prove that $n^3 + n$ is even for all $n \in N$.

5 Prove that if m and n are both odd integers, then $m \times n$ is also odd.

6 Use direct reasoning to show that if n is an even integer then $7n + 4$ is an even integer.

7 Prove that if $n \in N$ is even, then $2n^2 + 4n + 13$ is odd.

8 Prove $P \rightarrow Q$ given the statements P: b is divisible by a for $a, b \in N$, Q: $a \leq b$.

9 Prove $P \rightarrow Q$ for P: $3n + 2$ odd, $n \in N$, Q: n odd.

10 WORKED EXAMPLE 13 Write each sentence to include domains and quantifiers.

a The square of a positive number is positive.

b For every real number, x, there is a real number y such that $y = x + 1$.

c The area, A, of a rectangle of length x and width y is $A = xy$.

d The magnitude of a complex number is at least zero.

e The quadratic equation $ax^2 + bx + c = 0$ has complex solutions.

11 WORKED EXAMPLE 14 Decide, with reasons, if each statement is true. If it is false, provide a counterexample. In each case, let $m \in N, n \in N$.

a $\forall m \exists n (m - n = 0)$

b $\forall m \ \exists n (m - 3n = 0)$

c $\forall m (m > 6 \to \forall n (n > m \to n > 6))$

d $\forall m (m < 10 \to \forall n (n < m \to m = \sqrt{n}))$

Exam practice

12 (2 marks)

a Show that for even integers, m, n, $m - n$ is even. 1 mark

b Given the result from part **a**, prove that if $m^2 + n^2$ is even, then $m + n$ is even. 1 mark

13 (2 marks) Prove that the product of a rational number with another rational number is also a rational number.

14 (3 marks) Show that $n(n + 1)(2n + 1)$ is divisible by 6 for all $n \in N$.

15 (4 marks) Prove that if m and n are odd integers, then

a $m^2 + n^2$ is even. 2 marks

b $m^2 + n^2$ is not divisible by 4. 2 marks

16 (3 marks)

a Given a, b are even integers and c is an odd integer, verify using $a = 3$, $b = -2$ and $c = -9$ that $ab + c$ is an odd number. 1 mark

b Hence, or otherwise, deduce that for integers x, y, z, $(4xy + 2z + 1)^2$ is odd. 2 marks

17 (3 marks) Show that $\forall p \in Z \ (s = 2p + 1 \to \exists k \in Z \ (s^2 + 3s + 5 = 2k + 1))$.

18 WORKED EXAMPLE 14 (3 marks) Prove $\forall m, n \in N \ \exists k \in N \ (m^2 + n^2 + 1 = 2k)$.

19 (3 marks) Prove $\forall m \in N \ \exists a, b \in N \ (m = a^2 - b^2)$.

20 WORKED EXAMPLE 15 (4 marks) Prove each statement by first writing it as a grammatical sentence.

a $\forall n \in N \ \exists m \in N \ (n(n + 1) = 2m)$ 2 marks

b $\forall x, y \in R \ \exists z, w \in C \ (z + w \in C)$ 2 marks

21 (6 marks)

a Given that the product of 2 odd numbers is an odd number, prove that for $n \in N$, 3^n is odd. 3 marks

b Given that the product of 2 even numbers is even, explain why $(3^n + 1)^n$ is always even. 3 marks

22 (4 marks) Use the expansion $(a + b)^n = k_0 a^n b^0 + k_1 a^{n-1} b^1 + k_2 a^{n-2} b^2 + k_3 a^{n-3} b^3 + k_4 a^{n-4} b^4 + \dots + k_n a^{n-n} b^n$, ($k_1$, k_2 … k_n non-zero constants), to prove that for all odd numbers, x, x^n will be odd if $k_n = 1$.

Video playlist
Proof by contrapositive and contradiction

Worksheet
Contrapositive

3.4 Proof by contrapositive and contradiction

Proof by contrapositive

Consider these two statements:

Statement 1: 'If I don't study, I will fail the test'.

Statement 2: 'If I pass the test, then I have studied'.

The two statements are logically equivalent and each can be written as a conditional statement.

Statement 1: P: 'I don't study' Q: 'I fail the test' $P \rightarrow Q$.

Statement 2: $\neg Q$: 'I pass the test' $\neg P$: 'I study' $\neg Q \rightarrow \neg P$.

Proof by contrapositive requires taking the negation of each statement.

To prove $P \rightarrow Q$, prove $\neg Q \rightarrow \neg P$.

WB p. 64

WORKED EXAMPLE 17 Proof by contrapositive

Prove that $\forall n \in N$, if $7n + 9$ is even, then n is odd.

Steps	**Working**
1 Express the statement $P \rightarrow Q$ in the form $\neg Q \rightarrow \neg P$.	P: $7n + 9$ is even. Q: n is odd. $P \rightarrow Q$ If $7n + 9$ is even, then n is odd. $\neg Q \rightarrow \neg P$ If n is even, then $7n + 9$ is odd.
2 Prove $\neg Q \rightarrow \neg P$.	If n is even, then $7n$ is even. Hence $7n + 9$ is odd, since even + odd = odd. So $Q \rightarrow \neg P$ is true. Hence if $7n + 9$ is even, then n is odd.

WB p. 65

WORKED EXAMPLE 18 Contrapositive to prove divisibility

Prove that for every natural number, n, if n^2 is divisible by 3 then n is divisible by 3.

Steps	**Working**
1 Express the statement $P \rightarrow Q$ in the form $\neg Q \rightarrow \neg P$.	P: n^2 is divisible by 3. Q: n is divisible by 3. $P \rightarrow Q$ if n^2 is divisible by 3 then n is divisible by 3. $\neg Q \rightarrow \neg P$ if n is not divisible by 3, then n^2 is not divisible by 3.
2 Prove $\neg Q \rightarrow \neg P$.	If n is not divisible by 3, then the remainder after division is 1 or 2. Hence n is of the form $3k + 1$ or $3k + 2$ for $k \in N$. If $n = 3k + 1$, $n^2 = 3(3k^2 + 2k) + 1$, which is not divisible by 3. If $n = 3k + 2$, $n^2 = 3(3k^2 + 4k + 1) + 1$, which is not divisible by 3. Hence, $\neg Q \rightarrow \neg P$ is true, so $P \rightarrow Q$ is true.

Proof by contradiction

The method of **proof by contradiction** involves the assumption of a false conjecture that will lead to a contradiction.

The usual approach to prove $P \rightarrow Q$ is to accept P and assume $\neg Q$. The contradiction arising from $P \rightarrow \neg Q$ is proof that $P \rightarrow Q$ is true. However, give some thought to what assumptions need to be made, otherwise a contradiction may not occur.

WORKED EXAMPLE 19 Proof by contradiction involving surds

Use proof by contradiction to prove that $\sqrt{3}$ is irrational.

Steps	Working
1 Assume the statement is false and write it as an equation.	Assume that $\sqrt{3}$ is rational and can be written as $\sqrt{3} = \frac{a}{b}$, where $a, b \in Z$ have no common factors other than 1.
2 Rearrange the equation and form conclusions about the types of variables used.	$a^2 = 3b^2$ $3b^2$ is odd, so a^2 is odd. Hence a is odd. $3b^2$ is odd, so b^2 is odd. Hence b is odd.
3 Write the equation in terms of the types of variables and rearrange.	a and b are odd, so let $a = 2m + 1$, $b = 2n + 1$, $m, n \in Z$. Then $a^2 = 3b^2$ becomes $4m^2 + 4m + 1 = 12n^2 + 12n + 3$. $2(m^2 + 2m) = 2(3n^2 + 3n) + 1$.
4 State the contradiction.	$2(m^2 + 2m)$ is even and $2(3n^2 + 3n) + 1$ is odd. This is a contradiction.
5 State the conclusion.	The assumption that $\sqrt{3}$ is rational leads to a contradiction, so the original assumption that $\sqrt{3}$ is rational is false.

WB p. 66

WORKED EXAMPLE 20 Proof by contradiction

Show that for all integers a, b, c, if $a^2 + b^2 = c^2$, then either a or b is even.

Steps	Working
1 Assume the statement is false.	Assume $a^2 + b^2 = c^2$ and both a and b are odd.
2 Rewrite the expression according to the types of numbers used.	a, b are odd, so $a = 2m + 1$, $b = 2n + 1$, m, n integers $(2m + 1)^2 + (2n + 1)^2 = c^2$
3 Expand and rearrange the expression and form a conclusion.	$a^2 + b^2 = 4m^2 + 4m + 1 + 4n^2 + 4n + 1$ $= 4(m^2 + m + 1 + n^2 + n) + 2$ This is divisible by 2, so $a^2 + b^2$ is even. Hence c^2 is also even.
4 State what must be true to avoid a contradiction.	Hence $4k + 2 = c^2$. $c = 2p$, for integer p $4k + 2 = 4p^2$
5 Explain the contradiction, testing each case.	This is not possible because $4p^2$ is divisible by 4 but $4k + 2$ is not divisible by 4.
6 State the conclusion.	The assumption that both a and b are odd is wrong, so either a or b must be even.

WB p. 67

EXERCISE 3.4 Proof by contrapositive and contradiction

ANSWERS p. 572

Recap

1 If x and y are natural numbers, a counterexample to $\forall x(x > 3 \rightarrow \exists y(x + y < 10))$ is

A $x = 4, y = 5$ **B** $x = 3, y = 7$ **C** $x = 6, y = 4$

D $x = 6, y = 2$ **E** $x = 8, y = 1$

2 If $x, y, b \in R$, then the statement 'The line $y = 3x + b$ intersects the positive x-axis at $x = \frac{1}{3}(y - b)$' can be expressed as

A $\forall x\left(\exists y > b \rightarrow \left(\frac{1}{3}(y - b) > 0\right)\right)$

B $\forall x \forall y\left(\exists b \rightarrow \left(\frac{1}{3}(y - b) > 0\right)\right)$

C $\forall x \exists y \exists b\left(\frac{1}{3}(y - b) \geq 0\right)$

D $\forall y\left(\exists x > b \rightarrow \left(\frac{1}{3}(y - b) \geq 0\right)\right)$

E $\forall x \exists y\left(\exists b \rightarrow \left(\frac{1}{3}(y - b) \geq 0\right)\right)$

Mastery

3 WORKED EXAMPLE 17 Use proof by contrapositive to show that $\forall n \in N$, if $5n - 7$ is even, then n is odd.

4 The contrapositive to the statement, 'If I live in Melbourne, then I live in Australia,' is

A If I live in Melbourne, then I don't live in Australia.

B If I live in Melbourne, then I must live in Australia.

C If I live in Australia, then I live in Melbourne.

D If I don't live in Australia, then I don't live in Melbourne.

E If I don't live in Melbourne, then I don't live in Australia.

5 WORKED EXAMPLE 18 Prove by contrapositive that for every natural number, n, if n^3 is divisible by 3, then n is divisible by 3.

6 Vector $\underset{\sim}{u}$ and vector $\underset{\sim}{v}$ are parallel if $\underset{\sim}{u} = k\underset{\sim}{v}, k \in R \setminus \{0\}$.

To prove this property using proof by contradiction, what will statements P and Q be?

A P: $\underset{\sim}{u}, \underset{\sim}{v}$ parallel $\quad Q$: $\underset{\sim}{u} \neq k\underset{\sim}{v}$

B P: $\underset{\sim}{u}, \underset{\sim}{v}$ not parallel $\quad Q$: $\underset{\sim}{u} \neq k\underset{\sim}{v}$

C P: $\underset{\sim}{u}, \underset{\sim}{v}$ parallel $\quad Q$: $\underset{\sim}{u} = k\underset{\sim}{v}$

D P: $\underset{\sim}{u}, \underset{\sim}{v}$ perpendicular $\quad Q$: $\underset{\sim}{u} \neq k\underset{\sim}{v}$

E P: $\underset{\sim}{u}, \underset{\sim}{v}$ not parallel $\quad Q$: $\underset{\sim}{u} = k\underset{\sim}{v}$

7 WORKED EXAMPLE 19 Prove that $\sqrt[3]{2}$ is irrational.

8 Use proof by contrapositive to show that $\forall a, b, c \in N$, if $ab + c$ is odd, then c is odd.

9 It is required to show there do not exist integers m and n such that $15m + 25n = 1$.

a Write the statement that includes the assumption that will lead to a contradiction.

b Use your statement to complete the proof by contradiction.

10 WORKED EXAMPLE 20 Prove by contradiction that the difference of the squares of consecutive odd numbers is divisible by 4. (Assume the difference of the squares of consecutive odd numbers is not divisible by 4.)

Exam practice

11 (2 marks) Show using the contrapositive that $\forall n \in Z$, if $n^3 + 5$ is odd, then n is even.

12 (2 marks) Show using proof by contradiction that $\forall n \in N$, if $n^3 + 5$ is odd, then n is even.

13 (3 marks) Prove by contrapositive that $\forall m, n \in Z$, if 3 does not divide mn, then 3 does not divide m and 3 does not divide n.

14 (5 marks) Let n be an integer and consider the statement 'If n^2 is even, then n is even'.

a Write statements P, Q so that $P \rightarrow Q$. 2 marks

b Show the contrapositive as a statement. 2 marks

c Prove $\neg Q \rightarrow \neg P$. 1 mark

15 (3 marks) Use proof by contradiction to show that $\forall a, b \in Z$, $a^2 - 4b \neq 2$.

16 (3 marks) Apply the method of contrapositive to show that if the sum of 2 real numbers is irrational, then at least one of the 2 real numbers is irrational.

17 (3 marks) Prove using contrapositive that $\forall m, n \in Z$, mn odd $\rightarrow m$ and n odd.

18 (3 marks) Prove by contradiction that there is no smallest positive rational number.

19 (6 marks) To prove by contradiction that there is an infinite number of prime numbers, we start by assuming that there is a finite number of prime numbers, N.

Let P_N be the product of the finite number of prime numbers.

a Explain why $P_N + 1$ is a composite number. 3 marks

b By considering if there is a remainder when $P_N + 1$ is divided by a prime number, complete the proof by establishing a contradiction. 3 marks

20 (3 marks) Prove by contradiction that the sum of the squares of 2 odd numbers is even.

21 (5 marks) Consider the statement $\forall n \in N \ \exists k_1 \in N(3n + 2 = 2k_1 + 1 \rightarrow \exists k_2 \in N(n = 2k_2 + 1))$.

a Translate the statement into a grammatical sentence. 2 marks

b Prove the statement using contrapositive. 3 marks

3.5 Proof by mathematical induction

Video playlist
Proof by mathematical induction

Worksheets
Proof by induction

Mathematical induction

A widely-used method of proof in mathematics is **proof by mathematical induction**, which we studied in the Year 11 chapter, *Proof and number*. A conjecture is shown to be true for a specific value and the general case is established based on the assumption that the conjecture is true. The method is related to inductive reasoning, which we examined at the beginning of this chapter, because both methods make generalisations that are based on specific cases being true.

The method falls within the area of **discrete mathematics**, where integers comprise the number system involved.

For example, consider the statement, 'The sum of the interior angles of an n-sided polygon is $180(n - 2)$ degrees'. We know it is true for $n = 3$ (triangle), but how can it be proved that $180(n - 2)$ is true for all $n \geq 3$?

Proof by mathematical induction

Step 1 (base step): Show true for the first value of n, such as $n = 1$.

Step 2 (assumption): Assume true for $n = k$.

Step 3 (induction): Prove true for $n = k + 1$.

Step 4 (conclusion): Conclude that the statement is true for all required integers, n.

Mathematical induction works because of a 'domino effect' after we have proved that if a statement is true for $n = k$, then it is also true for the next integer, $n = k + 1$.

Because we have shown the statement to be true for $n = 1$,

then it is also true for $n = 1 + 1 = 2$,

therefore it is also true for $n = 2 + 1 = 3$,

$n = 3 + 1 = 4$, and so on for all positive integers, n.

WB

p. 68

WORKED EXAMPLE 21 Proof by induction for the sum of terms of an arithmetic sequence

Prove by induction that $4 + 9 + 14 + \ldots + (5n - 1) = \frac{n}{2}(5n + 3), n \in N$.

Steps	Working
1 Prove the base step.	Let $P(n) = \frac{n}{2}(5n + 3)$. Prove that the formula is true for $n = 1$. $4 = \frac{1}{2}(5 \times 1 + 3)$ $4 = 4$ LHS = RHS, so formula is true for $n = 1$.
2 Assume the conjecture is true for some integer $n = k$.	Assume the formula is true for $n = k$, an integer. Assume $4 + 9 + \ldots + (5k + 3) = \frac{k}{2}(5k + 3)$
3 Prove that the conjecture is true for $n = k + 1$, substituting above assumption.	Need to prove $P(k + 1) = \frac{k + 1}{2}(5(k + 1) + 3)$ $= \frac{(k + 1)(5k + 8)}{2}$ For $n = k + 1$, the sum is the kth term + the $(k + 1)$th term. The $(k + 1)$th term is $5(k + 1) - 1 = 5k + 4$. $P(k + 1) = \frac{k}{2}(5k + 3) + (5k + 4)$ $= \frac{k(5k + 3)}{2} + \frac{10k + 8}{2}$ $= \frac{5k^2 + 13k + 8}{2}$ $= \frac{(k + 1)(5k + 8)}{2}$ as required
4 State the conclusion.	The formula is true for $n = k + 1$ if it is true for $n = k$. The formula is true for $n = 1$, so by mathematical induction the formula is true for all integers $n \in N$.

WORKED EXAMPLE 22 Proof by induction of the sum of terms of triangular numbers

Prove by induction that $1 + 3 + 6 + 10 + 15 + \ldots + \frac{n}{2}(n+1) = \frac{n}{6}(n+1)(n+2)$, $n \in N$.

WB p. 69

Steps	Working
1 Prove the base step.	Let $P(n) = \frac{n}{6}(n+1)(n+2)$. Prove that the formula is true for $n = 1$. LHS = 1 RHS $= \frac{1}{6}(1+1)(1+2) = 1$ LHS = RHS, so the formula is true for $n = 1$.
2 Assume the statement is true for some integer $n = k$.	Assume the formula is true for $n = k$, an integer. $1 + 3 + \ldots + \frac{k}{2}(k+1) = \frac{k}{6}(k+1)(k+2)$
3 Prove that the conjecture is true for $n = k + 1$, substituting above assumption.	Need to prove $P(k+1) = \frac{k+1}{6}((k+1)+1)((k+1)+2)$ $= \frac{(k+1)(k+2)(k+3)}{6}$ The $(k+1)$th term is $\frac{1}{2}(k+1)(k+2)$. $P(k+1) = \frac{k}{6}(k+1)(k+2) + \frac{1}{2}(k+1)(k+2)$ $= (k+1)(k+2)\left(\frac{k}{6} + \frac{1}{2}\right)$ $= (k+1)(k+2)\left(\frac{k+3}{6}\right)$ $= \frac{(k+1)(k+2)(k+3)}{6}$ as required
4 State the conclusion.	The formula is true for $n = k + 1$ if it is true for $n = k$. The formula is true for $n = 1$, so by mathematical induction the formula is true for all $n \in N$.

WORKED EXAMPLE 23 Divisibility proof using induction

Prove by induction that for any integer $n \geq 2$, $n^3 - n$ is divisible by 3.

WB p. 70

Steps	Working
1 Prove the base step.	$P(n) = n^3 - n$ Prove true for $n = 2$: $2^3 - 2 = 6$ is divisible by 3.
2 Assume the statement is true for some integer $n = k$.	Assume the conjecture is true for $n = k$, an integer. That is, that $k^3 - k$ is divisible by 3.
3 Prove that the statement is true for $n = k + 1$.	$P(k+1) = (k+1)^3 - (k+1)$ $= k^3 + 3k^2 + 3k + 1 - k - 1$ $= k^3 + 3k^2 + 2k$ $= k(k^2 + 3k + 2)$ $= k(k+1)(k+2)$ Since $P(k+1)$ is the product of 3 consecutive integers, one of them must be a multiple of 3, so $P(k+1)$ is divisible by 3.

4 State the conclusion.	The conjecture is true for $n = k + 1$ if it is true for $n = k$, and as it is true for $n = 2$, then by mathematical induction it is true for all integers $n \geq 2$.

The method of mathematical induction is applicable to a range of mathematics topics, so formulas and properties from different areas of study may be needed.

WB

p. 71

WORKED EXAMPLE 24 Induction with trigonometry

Given $n \in N$ and $x \neq k\pi, k \in Z$, show that $\sin(x) + \sin(3x) + \sin(5x) + \ldots + \sin(2n-1)x = \dfrac{1-\cos(2nx)}{2\sin(x)}$.

Steps	**Working**
1 Prove the base step.	$P(n) = \dfrac{1-\cos(2nx)}{2\sin(x)}$ Prove true for $n = 1$. $\text{LHS} = \sin(x)$ $\text{RHS} = \dfrac{1-\cos(2x)}{2\sin(x)} = \dfrac{1-(1-2\sin^2(x))}{2\sin(x)} = \sin(x)$ LHS = RHS, so true for $n = 1$.
2 Assume the statement is true for some integer $n = k$.	Assume the conjecture is true for $n = k$, an integer. $P(k) = \dfrac{1-\cos(2kx)}{\sin(x)}$
3 Prove that the statement is true for $n = k + 1$.	Need to prove that $P(k+1) = \dfrac{1-\cos(2(k+1)x)}{2\sin(x)}$ $P(k+1) = \dfrac{1-\cos(2kx)}{2\sin(x)} + \sin[(2k+1)x]$ $= \dfrac{1-\cos(2kx) + 2\sin(x)\sin[(2k+1)x]}{2\sin(x)}$ $= \dfrac{1-\cos(2kx) - \left(\cos[(2k+2)x] - \cos(2kx)\right)}{2\sin(x)}$, using the identity $\sin A \sin B = -\dfrac{1}{2}\left(\cos(A+B) - \cos(A-B)\right)$ $= \dfrac{1-\cos[(2(k+1)x]}{2\sin(x)}$ as required
4 State the conclusion.	The formula is true for $n = k + 1$ if it is true for $n = k$, and as it is true for $n = 1$, then by mathematical induction it is true for all $n \in N$.

Video
VCE question analysis: Logic and proof

VCE QUESTION ANALYSIS

A sample question for a new topic **(11 marks)**

Consider the following conversation between Aarush and Petros. They are discussing the conjecture P: $\forall n \in N\,(f(n) \geq g(n)), f(n) = 3^n, g(n) = 2n + 1$.

Petros: 'Hi, Aarush.' 1

Aarush: 'Hi. I've been thinking a lot about the problem.' 2

Petros: 'Have you found a proof?' 3

Aarush: 'I think so.' 4

Petros: 'You always find a solution when you think a lot about the problem.' 5

a Explain why sentences 1 and 3 are not statements. 1 mark

b Explain why Statement 2 is a premise and Statement 4 is an inference. 1 mark

c Petros wants to show that the negation, $\neg P$, is always false.

  i Write $\neg P$ using quantifiers. 1 mark

  ii Petros shows that $\neg P$ is false for $n = 3$. He then claims this is a necessary and sufficient counterexample that proves statement P is true. Explain why this reasoning is incorrect. 2 marks

d Aarush thinks he has proved the conjecture by using mathematical induction. Show how this could be achieved by following the steps below.

Step 1: Verify the conjecture is true for a particular value of n.

Step 2: Make an assumption.

Step 3: Based on your assumption, conclude that the statement is always true. 3 marks

e Petros used proof by contradiction to show that for all positive integers, $n, f(n) - g(n)$ is divisible by 2. Given that 3^n is odd for all positive integers, show the proof that Petros would have provided. 3 marks

Reading the question

- Highlight key words and terms and recall their meaning.
- Appreciate the relationship between terms such as statement, premise, inference and conclusion. The connection is tested in part **b**.
- For part **d**, show each step by providing sufficient calculation and reasoning.
- The conjecture is given in quantifier notation, so ensure that the meaning of the quantifier and the domain are understood.

Thinking about the question

- Most of the questions require explanations to be given rather than calculations. This means that it is important that responses are clear and correct terminology is used.
- Since $\frac{1}{2}$ marks are not awarded, provide clear answers for all 1-mark questions. This applies particularly to part **c i** where notation is required.
- Part **c ii** requires understanding of how to express negation by changing a universal quantifier to an existential quantifier.
- In part **c ii**, reference to 'necessary and sufficient' and 'counterexample' should form part of the answer.
- In part **e**, assume the opposite of 'divisible by 2' and show that this leads to a contradiction.

Worked solution (✓ = 1 mark)

a A statement is either true or false. Sentences 1 and 3 are not true or false statements. Sentence 1 is a greeting and sentence 2 is a question. ✓

b An inference is a conclusion drawn from a premise. Statement 4 ('I think so [that I have solved the problem]') is a conclusion based on the premise Statement 2 ('I've been thinking a lot about the problem'). ✓

c **i** $\neg P$ or 'not P' is 'There exists a value of $n \in N$ such that $f(n) < g(n)$. $\exists n \in N\ (3^n < 2n + 1)$ ✓

  ii Using one value of n to show the negation is false is a necessary condition (✓) but not a sufficient one because it must be false for all values of n. (✓)

d Verify conjecture is true for $n = 1$, $3^1 \geq 2 \times 1 + 1$. ✓

Assume it is true for $n = k$, an integer, so assume $3^k \geq 2k + 1$. ✓

We need to show $3^{k+1} \geq 2(k+1) + 1$, that is, $3^{k+1} \geq 2k + 3$.

$3^{k+1} = 3(3^k) \geq 3(2k+1)$, by assumption

$3^{k+1} \geq 6k + 3 > 2k + 3$

The conjecture is true for $n = k + 1$ if it is true for $n = k$. The conjecture is true for $n = 1$, so by mathematical induction, it must be true for all values of $n \in N$. ✓

e Assume that $f(n) - g(n)$ is not divisible by 2. ✓

Then $3^n - 2n - 1 = 2k + 1$, for integer k. [$(2k+1)$ is always odd.]

$3^n - 2n - 2 = 2k$ ✓

LHS is odd (3^n is odd, $2n$ is even, 2 is even, so odd – even – even is odd).

RHS is even for all k.

This is a contradiction, so it is not true that $f(n) - g(n)$ is not divisible by 2.

Therefore $f(n) - g(n)$ is divisible by 2. ✓

EXERCISE 3.5 Proof by mathematical induction

ANSWERS p. 573

Recap

1 The contrapositive to the statement $(\neg A \vee B) \rightarrow \neg C$ is

A $\neg C \rightarrow \neg(A \wedge \neg B)$

B $\neg(A \wedge \neg B) \rightarrow \neg C$

C $\neg(A \wedge \neg B) \rightarrow C$

D $C \rightarrow (\neg A \vee \neg B)$

E $C \rightarrow (A \wedge \neg B)$

2 To prove by contradiction the statement, 'For all positive integers, n, $3^n + 1$ is not divisible by 2', the assumption to make can be

A 'For all positive integers, n, if $3^n + 1$ is even, then n is not divisible by 2.'

B 'For all positive integers, n, $3^n + 1$ is divisible by 2.'

C 'For all positive integers, n, $3^n + 1$ is not divisible by 2.'

D 'For all positive integers, n, if n is not divisible by 2, then $3^n + 1$ is odd.'

E 'For all positive integers, n, if $3^n + 1$ is odd, then it is not divisible by 2.'

Mastery

3 WORKED EXAMPLE 21 Prove by induction that $4 + 9 + 14 + \ldots + (5n - 1) = \frac{n}{2}(5n + 3), n \in N$.

4 WORKED EXAMPLE 22 Prove by induction that for all integers $n > 1$,

$1 \times 2 + 2 \times 2^2 + 3 \times 2^3 + 4 \times 2^4 + \ldots + n \times 2^n = 2\left(1 + (n-1)2^n\right)$.

5 WORKED EXAMPLE 23 Use proof by induction to show that $\forall n \in N$, $6^n + 4$ is divisible by 5.

6 Prove by induction that $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n}{6}(n + 1)(2n + 1)$.

7 Prove each of the following by induction, given $n \in N$.

a $1 + 2 + 2^2 + 2^3 + 2^4 \ldots + 2^n = 2^{n+1} - 1$

b $1 + 2 + 3 + \ldots + n = \frac{n}{2}(n + 1)$

8 Prove by mathematical induction that $\forall n \in N$, 3 divides $7^n - 4^n$.

9 Prove by induction that $\forall n \in N$, $9^n - 1$ is divisible by 8.

Exam practice

10 (3 marks) Prove by mathematical induction that for all positive whole numbers, n, $5^n + 2 \times 11^n$ is divisible by 3.

11 (3 marks) Prove by mathematical induction that for all positive integers, n, $3^{2n+1} + 2^{n+2}$ is divisible by 7.

12 (3 marks) Let $P(n) = 2^n$ and $Q(n) = n^2$, where $n \in Z^+$.
Prove by induction that $Q(n) < P(n)$ for $n > 4$.

13 (3 marks) Use induction to prove that for positive integers, n, $2^n > n$.

14 (3 marks) Use mathematical induction to prove that for all positive integers, n, $5^n + 9^n + 2$ is divisible by 4.

15 (4 marks) Use induction to prove that for positive integers $n \geq 4$, $2^n < n!$

16 WORKED EXAMPLE 24 (4 marks) Given $n \in N$ and $\sin(x) \neq 0$, use the principle of induction to prove that

$$\cos(x)\cos(2x)\cos(4x) \ldots \cos(2^{n-1}x) = \frac{\sin(2^n x)}{2^n \sin(x)}$$

Hint: Show that $P(k+1) = P(k) \times \frac{\sin(2^k x)\cos(2^k x)}{2^k \sin(x)}$ and apply $2\sin A\cos A = \sin 2A$.

17 (4 marks) A sequence is defined as $u_{n+1} = 2u_n + 1$, with $u_1 = 1$ and n is a positive integer.
Prove by induction that $u_n = 2^n - 1$.

18 (4 marks) Use the method of induction to prove $u_1 + u_2 + u_3 + \ldots + u_n = 3^n - 1$, given that the nth term is $u_n = 2 \times 3^{n-1}$.

19 (5 marks) Apply the method of induction to prove that for all natural numbers, n,

$$\frac{1}{2 \times 5} + \frac{1}{5 \times 8} + \frac{1}{8 \times 11} + \ldots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{(6n+4)}$$

20 (5 marks) The symbol $\prod$ means product, and $\prod_{i=1}^{4}(2i+1)$ means $3 \times 5 \times 7 \times 9$.

Use proof by induction to prove that $\prod_{i=2}^{n}\left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}$.

3 Chapter summary

Conjectures

- Definitions and 'rules' of logic allow for conclusions (inferences) to be drawn from premises (statements).
- A statement can be atomic or compound. An atomic statement (or sentence) consists of a single proposition, whereas a compound sentence is a combination of at least two atomic sentences.
- A logical argument comprises a set of premises that provide a logical conclusion.
- Inductive reasoning involves reaching a general conclusion based on a finite number of observations.
- Deductive reasoning draws a specific conclusion on the basis of accepting as true the more general case.
- A conjecture is a proposition that seems to be true but has not been conclusively proved, whereas a theorem is a statement that has been logically proven.
- A conjecture can be disproved solely on providing one counterexample to what is being stated to be true.

The language of proof

A set of logical connectives are used to combine statements and to show conditions that have been put in place. They include conjunction (AND), disjunction (OR) and negation (NOT).

A number of logically equivalent statements are used when proving other statements.

Of particular usefulness are De Morgan's laws, $\neg(A \vee B) \equiv \neg A \wedge \neg B$ and $\neg(A \wedge B) \equiv \neg A \vee \neg B$.

The biconditional statement A iff B (or $B \leftrightarrow A$) provides proof that $A \rightarrow B$ and $B \rightarrow A$.

Direct proof

In direct proof, true statements are used in sequence to arrive at a logical conclusion, which forms the last statement.

To indicate quantity in relation to proofs involving number systems, quantifiers are used. The existential quantifier, $\exists$ (meaning 'there exists'), is used to indicate that there is at least one value in the set of numbers that can be used to prove the statement. In the same way, the universal quantifier, $\forall$, (meaning 'for all'), is used to state that all numbers in the given set of numbers can be used as part of the proof.

Proof by contrapositive and contradiction

One method to prove that a statement is true is to use proof by contrapositive. To prove $P \rightarrow Q$ is true, we show $\neg Q \rightarrow \neg P$ is true. This means that to prove 'if P then Q', we prove 'if not Q then not P'.

When the method of proof by contradiction is used, it is assumed 'if P and $\neg Q$', which results in a contradiction in the reasoning process. This means the assumption $\neg Q$ is false, hence it must be true that $P \rightarrow \neg Q$.

Proof by mathematical induction

Step 1 (base step): Show true for the first value of n, such as $n = 1$.

Step 2 (assumption): Assume true for $n = k$.

Step 3 (induction): Prove true for $n = k + 1$.

Step 4 (conclusion): Conclude that the statement is true for all required integers, n. For example, 'The statement is true for $n = k + 1$ if it is true for $n = k$. It is true for $n = 1$, so by mathematical induction it is true for all $n \in N$.'

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 (3 marks) Consider the statement: 'If i is an imaginary number, then it is not a real number'.

a Write the converse statement. 1 mark

b Write the biconditional statement. 1 mark

c Write the contrapositive statement. 1 mark

2 (2 marks) State two conjectures about the value of the nth term of the sequence $2, 3, 5 \ldots$

3 (2 marks) For vectors $\underset{\sim}{u} = a\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{v} = 2\underset{\sim}{i} + a\underset{\sim}{j} - 2\underset{\sim}{k}$, find all non-zero values of a so that $\underset{\sim}{u}$ is perpendicular to $\underset{\sim}{u} + \underset{\sim}{v}$.

4 (3 marks)

a Write $b^2x^2 - 4b^2x + y^2 - 2y + 3b^2 + 1 = 0$ in the form $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, where h, k, a and b are constants. 2 marks

b For what values of a and b will $4x^2 - 16x + y^2 - 2y + 12 + 1 = 0$ represent a circle? 1 mark

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions 5 marks

1 Which of the following is a statement?

A Hello. Please come here.
B What is your name?
C Logic confuses me.
D $1 + 2$
E The topic of Boolean logic.

2 Which function supports the conjecture that $f(x)$ generates the terms of the sequence 5, 15, 29?

A $f(n) = 5n$
B $f(n) = 2n^2 - 3$
C $f(n) = 3n^2 + 2$
D $f(n) = n^2 - 2n + 6$
E $f(n) = n(n^2 + 3n + 1)$

3 Which statement is not equivalent to $\forall x \in R, x^2 \geq 0$?

A All real numbers have non-negative squares.
B Every real number has a square that is positive.
C The square of every real number is non-negative.
D The value x^2 is not non-negative for every real value of x.
E x^2 is not less than zero for each real number, x.

4 For what value of a is the vector $a\underset{\sim}{i} - 5\underset{\sim}{j} + \frac{2}{3}\underset{\sim}{k}$ parallel to the vector $18\underset{\sim}{i} - 60\underset{\sim}{j} + 8\underset{\sim}{k}$?

A $\frac{3}{4}$
B $\frac{1}{4}$
C $\frac{3}{2}$
D $\frac{1}{5}$
E $\frac{4}{5}$

5 The equation of an asymptote to the graph of $y = \dfrac{x^3 - 7x^2 + 10x + 1}{x^2 - 7x + 10}$ is

A $x = -5$
B $y = 2x + 1$
C $x = 1$
D $y = x$
E $x = -2$

Section B 5 questions

15 marks

1 (3 marks) Prove by induction that for all natural numbers, n, $9^n + 3$ is divisible by 4.

2 (2 marks) Prove that the greatest common divisor of a and b is a iff a divides b.

3 (3 marks) Proof by contradiction can be used to prove that the length of the hypotenuse of any right-angled triangle is less than the sum of the lengths of the other two sides.

a What statement, P, can be used that will lead to a contradiction? 1 mark

b Show that P results in a contradiction. 2 marks

4 (4 marks)

a Given $P(x) = 2x^3 + x^2 - 2x + 1$ and $Q(x) = 2x + 5$, write $\dfrac{P(x)}{Q(x)}$ in the form $R(x) + \dfrac{A}{Q(x)}$, where $R(x)$ is polynomial of degree 2 and A is a constant. 2 marks

b State the equations of the asymptotes. 2 marks

5 (3 marks) The magnitude of vector $\underset{\sim}{u} = a\underset{\sim}{i} - b\underset{\sim}{j} + \sqrt{6}\underset{\sim}{k}$ is 4.

The scalar product of $\underset{\sim}{u}$ with $\sqrt{2}\underset{\sim}{i} + \underset{\sim}{j} - \dfrac{1}{3}b\underset{\sim}{k}$ is $\sqrt{2} - \sqrt{6} - 3$.

a Find the value of a and b if they are both whole numbers. 2 marks

b Hence, find the magnitude of $\sqrt{2}\underset{\sim}{i} + \underset{\sim}{j} - \dfrac{1}{3}b\underset{\sim}{k}$. 1 mark

CHAPTER

4

COMPLEX NUMBERS

Study Design coverage

AREA OF STUDY 3: ALGEBRA, NUMBER AND STRUCTURE

Complex numbers

- De Moivre's theorem, proof for integral powers, powers and roots of complex numbers in polar form, and their geometric representation and interpretation
- the *n*th roots of unity and other complex numbers and their location in the complex plane
- factors over *C*, of polynomials; and introduction to the fundamental theorem of algebra, including its application to factorisation of polynomial functions of a single variable over *C*, for example, $z^8 + 1$, $z^2 - i$ or $z^3 - (2 - i)z^2 + z - 2 + i$
- solution over *C* of polynomial equations by completing the square, use of the quadratic factorisation and the conjugate root theorem.

VCE Mathematics Study Design 2023–2027 p. 110, © VCAA 2022

Video playlists (7):

4.1 Complex numbers
4.2 Polar form
4.3 De Moivre's theorem
4.4 Roots of unity
4.5 Factorising polynomials
4.6 Solving polynomial equations
VCE question analysis Complex numbers

Worksheets (20):

4.1 Complex numbers • Complex number operations • Complex conjugates • Complex conjugates and inverses • Addition and subtraction in the plane • The complex plane
4.2 Evaluating the modulus 1 • Evaluating the modulus 2 • Modulus and argument • Complex number conversions • Polar complex number conversions • Multiplication in the plane • Division in the plane • Complex plane graphs
4.3 Using de Moivre's theorem
4.4 Roots of complex numbers
4.5 Complex polynomials • Remainder and factor theorems
4.6 Polynomials with real coefficients • Real and imaginary factors

Nelson MindTap

To access resources above, visit
cengage.com.au/nelsonmindtap

4.1 Complex numbers

Video playlist Complex numbers

Worksheets Complex numbers
Complex number operations
Complex conjugates
Complex conjugates and inverses
Addition and subtraction in the plane
The complex plane

Remember that the **imaginary number** $i = \sqrt{-1}$, so $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$ and so on.

The **complex numbers**, ***C***, are of the form $z = a + bi$, $a, b \in R$.

Complex numbers

Complex numbers are equal if and only if the real parts are equal and the imaginary parts are equal.

For two complex numbers $z = a + bi$ and $w = c + di$, if $z = w$, then $a = c$ and $b = d$.

The **complex conjugate** of $z = a + bi$ is $\overline{z} = a - bi$.

Both $z\overline{z}$ and $z + \overline{z}$ are real, while $z - \overline{z}$ is imaginary.

The complex number $z = x + yi$ has a **real part**, **Re(*z*)** $= x$ and an **imaginary part**, **Im(*z*)** $= y$.

If $x = 0$, the number is **purely imaginary**. If $y = 0$, the number is **purely real** (or just real).

Complex numbers are shown geometrically as points or vectors in a number plane called the **Argand diagram** or **complex plane**.

The complex number $z = x + yi$ is represented as (x, y) or the position vector of the point (x, y).

The **real axis** is horizontal and is labelled $\text{Re}(z)$ or x.

The **imaginary axis** is vertical and is labelled $\text{Im}(z)$ or y.

WB

p. 73

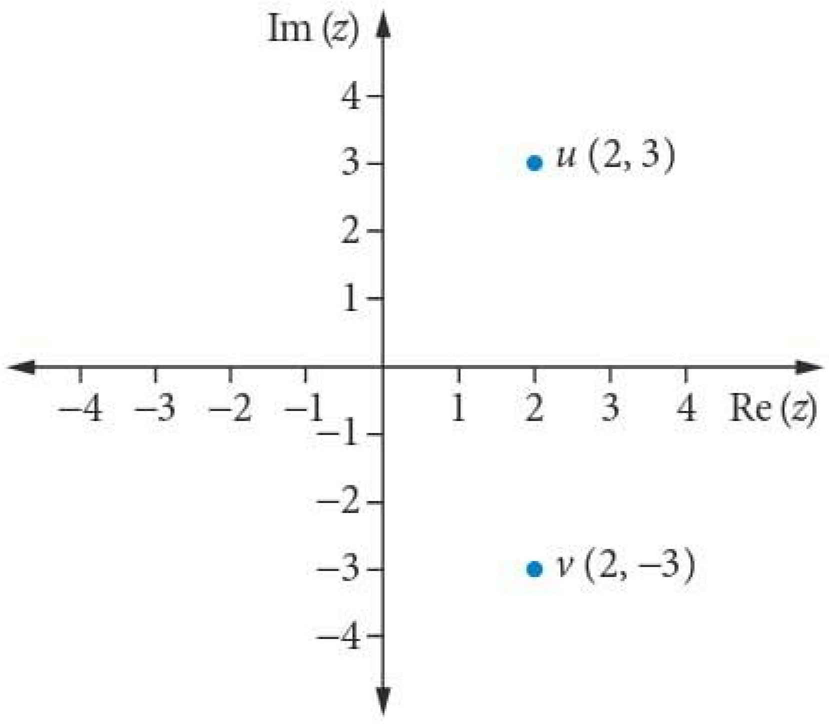
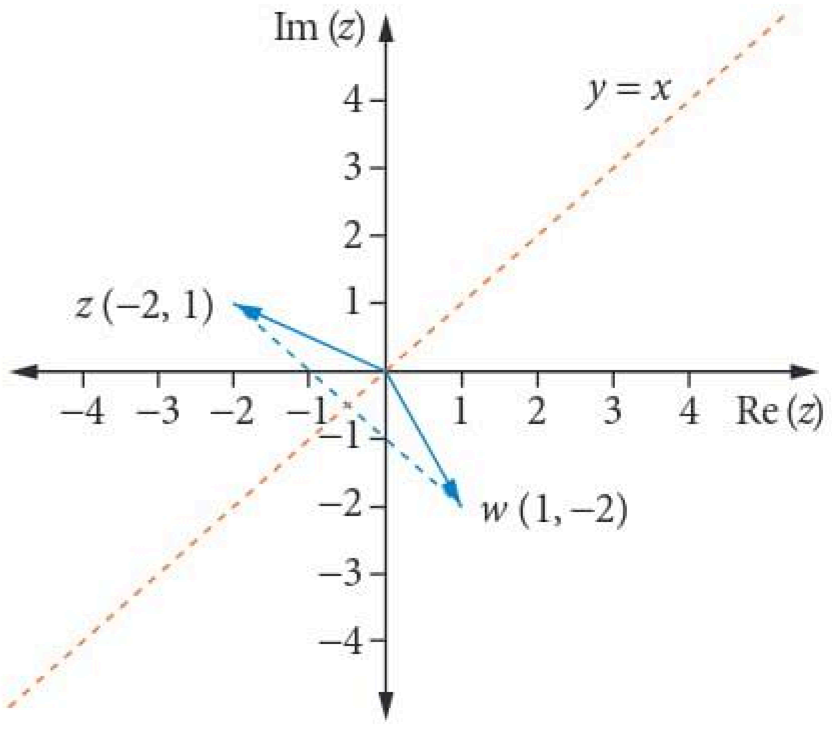
WORKED EXAMPLE 1 Complex numbers in the complex plane

a Show $u = 2 + 3i$ and $v = 2 - 3i$ as points on an Argand diagram.

b What is the relationship between u and v?

c Show $w = 1 - 2i$ and $z = -2 + i$ as vectors on an Argand diagram.

d What is the relationship between w and z?

Steps	**Working**
a **1** Write the complex numbers as points.	$u = (2, 3)$, $v = (2, -3)$
2 Plot and label the points on the plane.	
b State the relationship.	$u = \bar{v}$; they are reflections of each other in the x-axis.
c **1** Write the complex numbers as points.	$w = (1, -2)$, $z = (-2, 1)$
2 Draw as position vectors.	
d State the relationship.	They are reflections of each other in the line $y = x$.

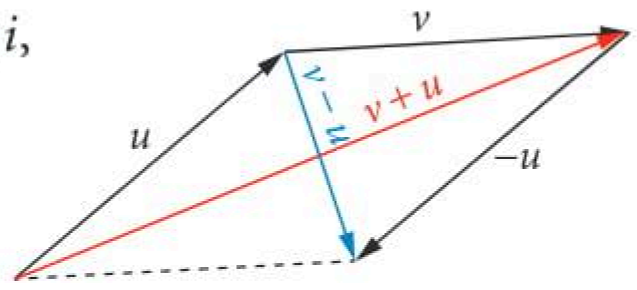
Operations

To *add* or *subtract* complex numbers, treat the real and imaginary parts separately.

To *multiply* complex numbers, expand and use $i^2 = -1$.

To *divide* complex numbers such as $\frac{z}{w}$, **realise the denominator** by multiplying by $\frac{\bar{w}}{\bar{w}}$.

Notice that $(a, b) + (c, d) = (a + c, b + d)$ and $(a + ib) + (c + id) = (a + c) + (b + d)i$, so vector addition gives the same result as complex number addition. Similarly, subtraction of vectors gives the same result as complex number subtraction. The diagram shows both.



WB

p. 75

WORKED EXAMPLE 2 Complex number operations

Simplify each expression.

a $(3 - 2i) + (-5 + 3i)$ **b** $(5 + 4i) - (2 - 3i)$ **c** $(2 - 3i) \times (-1 + 5i)$ **d** $(2 - i) \div (4 - 3i)$

Steps	**Working**
a **1** Separate the real and imaginary parts.	$(3 - 2i) + (-5 + 3i) = (3 + (-5)) + (-2 + 3)i$
2 Add the parts.	$= -2 + i$

b	**1**	Separate the real and imaginary parts.	$(5+4i)-(2-3i)=(5-2)+(4-(-3))i$
	2	Complete the subtractions.	$=3+7i$
c	**1**	Expand the brackets.	$(2-3i)\times(-1+5i)=2(-1+5i)-3i(-1+5i)$
	2	Multiply out.	$=-2+10i+3i-15i^2$
	3	Use $i^2=-1$.	$=-2+10i+3i+15$
	4	Express in standard form.	$=13+13i$
d	**1**	Write in fraction form.	$(2-i)\div(4-3i)=\frac{2-i}{4-3i}$
	2	Multiply by $\frac{\overline{w}}{\overline{w}}(=1)$ and simplify the expression.	$=\frac{2-i}{4-3i}\times\frac{4+3i}{4+3i}$ $=\frac{8+6i-4i+3}{16+9}$ $=\frac{11+2i}{25}$
	3	Express as $x+yi$.	$=\frac{11}{25}+\frac{2}{25}i$

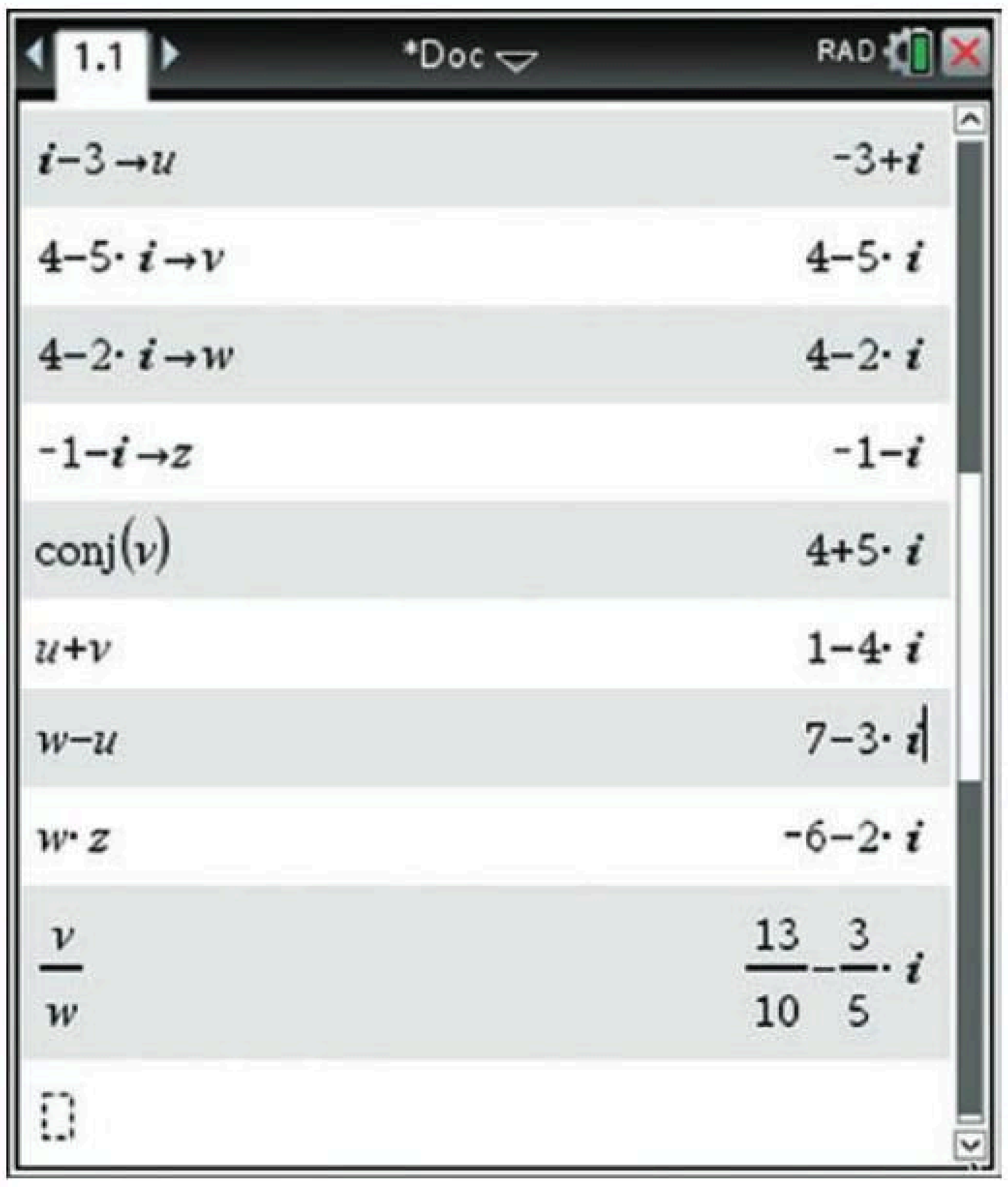
For CAS complex number calculations, set the complex number mode and use the special i provided on CAS. Do not use the letter or variable i.

USING CAS 1 Complex number operations

$u=i-3$, $v=4-5i$, $w=4-2i$ and $z=-1-i$.

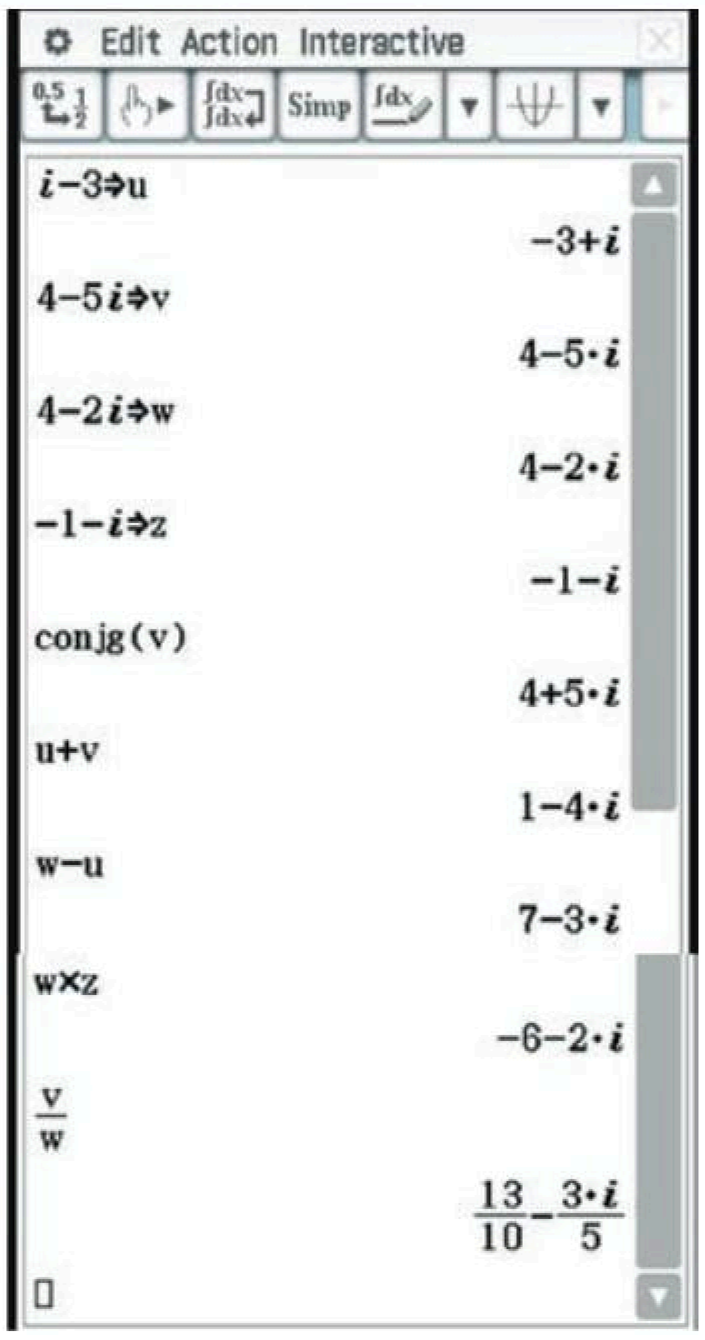
Find **a** $\overline{v}$ **b** $u+v$ **c** $w-u$ **d** wz **e** $\frac{v}{w}$

TI-Nspire



```
i-3→u          -3+i
4-5·i→v        4-5·i
4-2·i→w        4-2·i
-1-i→z         -1-i
conj(v)        4+5·i
u+v            1-4·i
w-u            7-3·i
w·z            -6-2·i
v/w            13/10-3/5·i
```

ClassPad



```
i-3⇒u
                 -3+i
4-5i⇒v
                 4-5·i
4-2i⇒w
                 4-2·i
-1-i⇒z
                 -1-i
conjg(v)
                 4+5·i
u+v
                 1-4·i
w-u
                 7-3·i
w×z
                 -6-2·i
v/w
                 13/10-3·i/5
```

1 In **Document Settings**, change the **Real or Complex:** field to **Rectangular**.

2 Press the $\boldsymbol{\pi}$ key to access $\boldsymbol{i}$ from the mini-palette to store the values of u, v, w and z.

3 Press **Menu > Complex Number Tools > Complex Conjugate** to find $\overline{v}$.

4 Complete the rest of the operations.

1 In **Main**, set the mode to **Cplx** (not Real).

2 Use the **Math2** or **Math3** keyboard to access $\boldsymbol{i}$ and enter the complex numbers.

3 Store as u, v, w and z, using the **abc** menu.

4 Highlight and tap **Interactive > Complex > conjg** to find $\overline{v}$.

5 Complete the rest of the operations.

EXERCISE 4.1 Complex numbers

ANSWERS p. 574

Mastery

1 Simplify each expression.

a i^5 **b** i^{-7} **c** i^{-1} **d** i^{14} **e** i^{-13}

2 Find the real and imaginary parts of each complex number, given $a, b, c, d, x, y \in R$.

a $z = 2i - 4$ **b** $w = \dfrac{3-7i}{5}$ **c** $z = \sqrt{3} + 4$ **d** $w = \dfrac{x-iy}{x+y}$ **e** $z = a + bi + c + di$

3 Find the complex conjugate of each number, given $a, b, c \in R$.

a $z = -5 + 6i$ **b** $w = \dfrac{5+2i}{3}$ **c** $z = i\sqrt{3} + 1$ **d** $a + ci - bi$ **e** $\dfrac{a-ib}{c+ib}$

4 WORKED EXAMPLE 1 Show each complex number as a point on an Argand diagram.

$a = 3 - 2i,\ b = -2 + 4i,\ c = 3 + 4i,\ d = -1 - 3i,\ e = 2i$

5 Show each complex number as a vector in the complex plane.

$t = -3,\ u = 4 + i,\ v = 2i - 3,\ w = -2 - i,\ z = -1 + 3i$

6 WORKED EXAMPLE 2 Simplify each expression.

a $(-6 - 2i) + (8 - 5i)$ **b** $(-4 - 4i) + (6 - 8i)$ **c** $(-1 - 7i) + (-5 + 2i)$

d $(7 + 6i) + (1 - 9i)$ **e** $(-2 + 1i) + (-2 + 1i)$

7 Simplify each expression.

a $(6 + 4i) - (-4 - 8i)$ **b** $(5 + 6i) - (1 - 5i)$ **c** $(2 + 6i) - (-7 + 1i)$

d $(-3 + 2i) - (-3 - 3i)$ **e** $(-6 - 1i) - (4 + 8i)$

8 Simplify each expression.

a $(-5 - 4i) \times (4 + 3i)$ **b** $(-6 + 4i) \times (-4 - 7i)$ **c** $(-6 - 8i) \times (-7 + 7i)$

d $(-2 + 4i) \times (-7 + 4i)$ **e** $(5 - 1i) \times (-9 - 8i)$

9 Simplify each expression.

a $(13i - 1) \div (i - 4)$ **b** $\dfrac{-2-6i}{8+6i}$ **c** $(1 - 4i) \div (8 - 6i)$

d $\dfrac{3+2i}{5i-4}$ **e** $\dfrac{7+2i}{-4-8i}$

10 Using CAS 1 Simplify each expression.

a $(8+3i)+(4i-2)$ **b** $(i-4)(5-7i)$ **c** $\left(9-\sqrt{3}i\right)\times\left(5+\sqrt{3}i\right)$

d $\dfrac{14+29i}{5i-6}$ **e** $(-9-i)\div(5-3i)$

11 **a** $w=3+i$ and $z=4-3i$. Show w, wi, z and zi as vectors on an Argand diagram.

b What is the effect of multiplying a number by i?

12 $w=2-3i$ and $z=-1+2i$. Show z, $\overline{w}$ and $\overline{w}-z$ as vectors on the complex plane.

Exam practice

80–100% 60–79% 0–59%

13 © VCAA 2016 1Q6 74% TECH-FREE (3 marks) Write $\dfrac{(1-\sqrt{3}i)^4}{1+\sqrt{3}i}$ in the form $a+bi$, where a and b are real constants.

14 © VCAA 2012 2AQ8 71% If $z=a+bi$, where both a and b are non-zero real numbers and $z\in C$, which of the following does **not** represent a real number?

A $z+\overline{z}$ **B** $|z|$ **C** $z\overline{z}$ **D** z^2-2abi **E** $(z-\overline{z})(z+\overline{z})$

15 © VCAA 2020 2AQ5 66% Given the complex number $z=a+bi$, where $a\in R\backslash\{0\}$ and $b\in R$, $\dfrac{4z\overline{z}}{(z+\overline{z})^2}$ is equivalent to

A $1+\left(\dfrac{\text{Im}(z)}{\text{Re}(z)}\right)^2$ **B** $4[\text{Re}(z)\times\text{Im}(z)]$ **C** $4([\text{Re}(z)]^2+[\text{Im}(z)]^2)$

D $4[1+(\text{Re}(z)+\text{Im}(z))^2]$ **E** $\dfrac{2\times\text{Im}(z)}{[\text{Re}(z)]^2}$

16 © VCAA 2019 2AQ4 44% The expression $i^{1!}+i^{2!}+i^{3!}+\ldots+i^{100!}$ is equal to

A 0 **B** 96 **C** $95+i$ **D** $94+2i$ **E** $98+2i$

17 © VCAA 2012 2AQ6 37% For any complex number z, the location on an Argand diagram of the complex number $u=i^3\overline{z}$ can be found by

A rotating z through $\dfrac{3\pi}{2}$ in an anticlockwise direction about the origin.

B reflecting z about the x-axis and then reflecting it about the y-axis.

C reflecting z about the y-axis and then rotating it anticlockwise through $\dfrac{\pi}{2}$ about the origin.

D reflecting z about the x-axis and then rotating it anticlockwise through $\dfrac{\pi}{2}$ about the origin.

E rotating z through $\dfrac{3\pi}{2}$ in a clockwise direction about the origin.

18 © VCAA 2020 2AQ8 34% Given that $(x+iy)^{14}=a+ib$, where $x, y, a, b\in R$, $(y-ix)^{14}$ for all values of x and y is equal to

A $-a-ib$ **B** $b-ia$ **C** $-b+ia$ **D** $-a+ib$ **E** $b+ia$

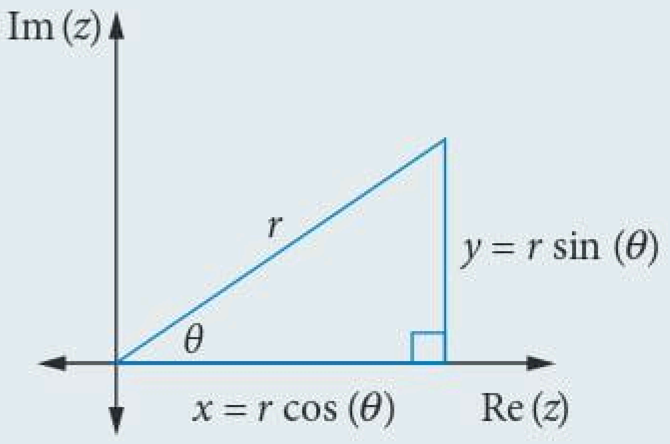
Video playlist
Polar form

Worksheets
Evaluating the modulus 1
Evaluating the modulus 2
Modulus and argument
Complex number conversions
Polar complex number conversions
Multiplication in the plane
Division in the plane

4.2 Polar form

Polar form

The **polar form** of a complex number, $z = r(\cos(\theta) + i\sin(\theta))$, is often more useful than the Cartesian (rectangular) form, $z = x + yi$. The **modulus** r and **argument** θ are given by $r = |z| = \sqrt{x^2 + y^2}$, $\tan(\theta) = \frac{y}{x}$, $x = r\cos(\theta)$ and $y = r\sin(\theta)$, as shown in the diagram.



The **principal argument** is the value of θ in the interval $(-\pi, \pi]$.

$\cos(\theta) + i\sin(\theta)$ is often abbreviated to $\mathbf{cis}\,(\boldsymbol{\theta})$.

The polar form is also called the **modulus-argument form** or **trigonometric form**.

Multiplication and division of complex numbers is simpler in polar form than in rectangular form:

$$r_1 \operatorname{cis}(\theta_1) \times r_2 \operatorname{cis}(\theta_2) = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \text{ and } \frac{r_1 \operatorname{cis}(\theta_1)}{r_2 \operatorname{cis}(\theta_2)} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2).$$

Remember: 'multiply the mods, add the args'

WB
p. 76

WORKED EXAMPLE 3 Polar form

$u = 2\operatorname{cis}\left(\frac{\pi}{6}\right)$, $v = 3 + 3\sqrt{3}i$, $w = 4\operatorname{cis}\left(-\frac{3\pi}{4}\right)$ and $z = 3i - 3$.

a Express v and z in polar form.
b Express u and w in Cartesian form.
c Find uv and wv.
d Find $\frac{v}{u}$ and $\frac{z}{u}$.

Steps			**Working**
a	**1**	Find the modulus and principal argument of v.	$\lvert v\rvert = \sqrt{9 + 9 \times 3} = 6$ $\tan(\theta) = \frac{3\sqrt{3}}{3} = \sqrt{3}$, so $\theta = \frac{\pi}{3}$ (1st quadrant)
	2	Find the modulus and principal argument of z.	$\lvert z\rvert = \sqrt{9 + 9} = 3\sqrt{2}$ $\tan(\theta) = \frac{3}{-3} = -1$, so $\theta = \frac{3\pi}{4}$ (2nd quadrant)
	3	Write the polar forms.	$v = 6\operatorname{cis}\left(\frac{\pi}{3}\right)$, $z = 3\sqrt{2}\operatorname{cis}\left(\frac{3\pi}{4}\right)$

b	**1**	Write in complete form.	$u = 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$ $w = 4\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$
	2	Substitute the values of the trig ratios and multiply out.	$u = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \sqrt{3} + i$ $w = 4\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$ $= 4\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$ $= -2\sqrt{2} - 2\sqrt{2}i$
c	**1**	Write uv in polar form, multiply the moduli and add the arguments.	$uv = 2\operatorname{cis}\left(\frac{\pi}{6}\right) \times 6\operatorname{cis}\left(\frac{\pi}{3}\right)$ $= 12\operatorname{cis}\left(\frac{\pi}{2}\right)$ $= 12i$
	2	Write wv in polar form, multiply the moduli and add the arguments.	$wv = 4\operatorname{cis}\left(-\frac{3\pi}{4}\right) \times 6\operatorname{cis}\left(\frac{\pi}{3}\right)$ $= 24\operatorname{cis}\left(-\frac{5\pi}{12}\right)$
d	**1**	Write $\frac{v}{u}$ in polar form, divide the moduli and subtract the arguments.	$\frac{v}{u} = \frac{6\operatorname{cis}\left(\frac{\pi}{3}\right)}{2\operatorname{cis}\left(\frac{\pi}{6}\right)}$ $= 3\operatorname{cis}\left(\frac{\pi}{6}\right)$
	2	Write $\frac{z}{u}$ in polar form, divide the moduli and subtract the arguments.	$\frac{z}{u} = \frac{3\sqrt{2}\operatorname{cis}\left(\frac{3\pi}{4}\right)}{2\operatorname{cis}\left(\frac{\pi}{6}\right)}$ $= \frac{3\sqrt{2}}{2}\operatorname{cis}\left(\frac{7\pi}{12}\right)$

Assuming θ is positive, multiplication of a complex number by $r\operatorname{cis}(\theta)$ rotates the number through θ (anticlockwise) and magnifies it by the factor r. Division by $r\operatorname{cis}(\theta)$ rotates it through $-\theta$ (clockwise) and multiplies it by the factor $\frac{1}{r}$.

USING CAS 2 Polar and Cartesian conversions

a Find the modulus and argument of $2 - 2i$.

b Convert $1 + \sqrt{3}i$ into polar form and $4\,\text{cis}\left(\frac{2\pi}{3}\right)$ to Cartesian form.

TI-Nspire

a

$|2-2\cdot i|$ $2\cdot\sqrt{2}$

$\text{angle}(2-2\cdot i)$ $\frac{-\pi}{4}$

1 Press **menu > Number > Complex Number Tools > Magnitude**.

2 Enter **2** – **2*i***.

3 Press **menu > Number > Complex Number Tools > Polar Angle**.

4 Enter **2 – 2*i***.

b

$(1+\sqrt{3}\cdot i)$▸Polar $e^{\frac{i\cdot\pi}{3}}\cdot 2$

$\left(\left(4\angle\frac{2\cdot\pi}{3}\right)\right)$▸Rect $-2+2\cdot\sqrt{3}\cdot i$

1 Enter **1** + **$\sqrt{3}i$**.

2 Press **menu > Number > Complex Number Tools > Convert to Polar**.

Note: the answer $e^{\frac{i\pi}{3}}\cdot 2$ equals $2\,\text{cis}\left(\frac{\pi}{3}\right)$.

3 Insert a set of brackets and enter **4**.

4 Press **ctrl** + **catalog** to access the symbol palette.

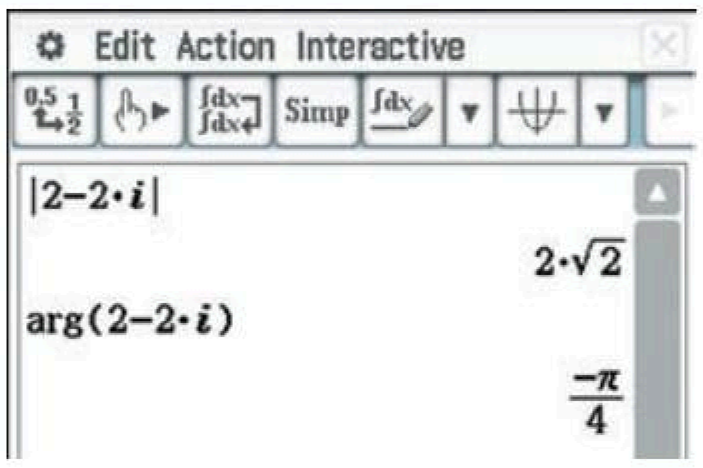
5 Select the angle symbol ∠.

6 Enter $\mathbf{\frac{2\pi}{3}}$.

7 Press **menu > Number > Complex Number Tools > Convert to Rectangular**.

ClassPad

a



$|2-2\cdot i|$

$2\cdot\sqrt{2}$

$\arg(2-2\cdot i)$

$\frac{-\pi}{4}$

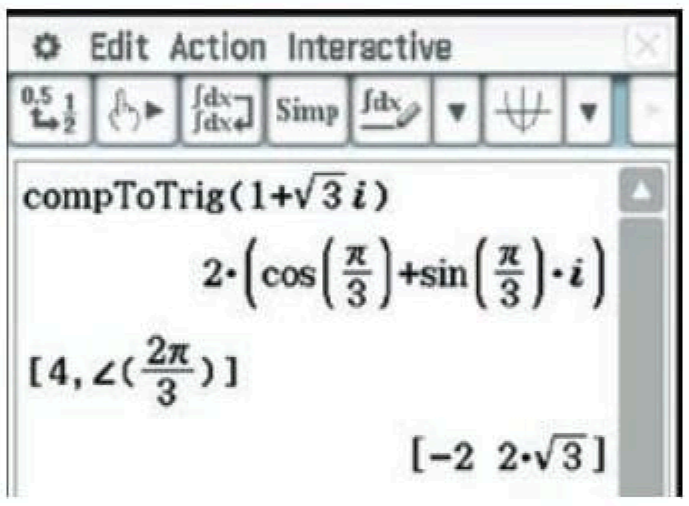
1 Open the **Keyboard** and tap **Math2** or **Math3** to access ***i***.

2 Tap on the **absolute value** template.

3 Enter and highlight $|\mathbf{2}-\mathbf{2}\boldsymbol{i}|$ and press **EXE**.

4 Enter and highlight **2** – **2*i*** and tap **Interactive > Complex > arg**.

b



compToTrig$(1+\sqrt{3}\,i)$

$2\cdot\left(\cos\left(\frac{\pi}{3}\right)+\sin\left(\frac{\pi}{3}\right)\cdot i\right)$

$[4,\angle(\frac{2\pi}{3})]$

$[-2\ \ 2\cdot\sqrt{3}]$

1 Enter and highlight **1** + **$\sqrt{3}i$**.

2 Tap **Interactive > Complex > compToTrig**.

Note: the answer $2\left(\cos\left(\frac{\pi}{3}\right)+\sin\left(\frac{\pi}{3}\right)i\right)$ equals $2\,\text{cis}\left(\frac{\pi}{3}\right)$.

3 Enter with square brackets and highlight $\left[4,\angle\left(\frac{2\pi}{3}\right)\right]$.

Open the **Keyboard > Math3** to enter ∠.

Note: the answer $\left[-2\ \ 2\cdot\sqrt{3}\right]$ equals $-2+2\sqrt{3}i$.

a The modulus equals $2\sqrt{2}$ and argument equals $-\frac{\pi}{4}$.

b $1+\sqrt{3}i = 2\,\text{cis}\left(\frac{\pi}{3}\right)$ and $4\,\text{cis}\left(\frac{2\pi}{3}\right) = -2+2\sqrt{3}i$

Straight lines

Straight lines in the complex plane are similar to those in the Cartesian plane.

The horizontal line through (a, b) parallel to the real axis has the equation $\mathrm{Im}(z) = b$ and the vertical line parallel to the imaginary axis is $\mathrm{Re}(z) = a$.

Using $z + \overline{z} = 2\,\mathrm{Re}(z)$ and $z - \overline{z} = 2i\,\mathrm{Im}(z)$ gives $z + \overline{z} = 2a$ and $z - \overline{z} = 2bi$.

So $z + \overline{z}$ is horizontal and parallel to the real axis and $z - \overline{z}$ is vertical and parallel to the imaginary axis.

The values of $z \in \{z : |z - v| = |z - u|\}$ are equidistant from u and v, so $|z - v| = |z - u|$ is the perpendicular bisector of u and v.

The vector $z - u$ for a point z on the straight line through u and v must be a multiple of $v - u$, as it is in the same direction. This means the equation of the straight line is given by $z - u = k(v - u)$, $k \in R$.

We can write this as $z = u + k(v - u)$. It has the slope $m = \dfrac{\mathrm{Im}\,(v) - \mathrm{Im}\,(u)}{\mathrm{Re}\,(v) - \mathrm{Re}\,(u)} = \dfrac{\mathrm{Im}\,(v - u)}{\mathrm{Re}\,(v - u)}$.

But $v - u = |v - u|\ \mathrm{cis}\,(v - u)$, so the equation can also be written as $z - u = k'\ \mathrm{cis}\,(v - u)$, $k' \in R$.

The new constant, $k' = k\,|v - u|$, is just a multiple of the old constant, k.

Worksheet Complex plane graphs

WB p. 77

WORKED EXAMPLE 4 Straight lines in the complex plane

a What is the equation of the line passing through $1 - 3i$ and $4 + i$?

b Express the equation in Cartesian form.

Steps			**Working**
a	**1**	Use $z - u = k(v - u)$.	$\|z - (1 - 3i)\| = k\|4 + i - (1 - 3i)\|$ for $k \in R$
	2	Simplify.	$\|z - 1 + 3i)\| = k\|3 + 4i\|$ for $k \in R$
	3	Write the answer.	The equation is $\|z - 1 + 3i\| = k\|3 + 4i\|$ for $k \in R$.
b	**1**	Write $z = x + iy$.	$\|x + iy - 1 + 3i\| = k\|3 + 4i\|$
	2	Equate real and imaginary parts.	$x - 1 = 3k$ and $y + 3 = 4k$
	3	Eliminate k.	$4x - 4 = 12k$ and $3y + 9 = 12k$ $\Rightarrow 4x - 4 = 3y + 9$ $\Rightarrow 4x - 3y - 13 = 0$
	4	Write the answer.	The Cartesian equation is $4x - 3y - 13 = 0$.

We can also find the Cartesian equation directly using $m = \dfrac{\mathrm{Im}(v) - \mathrm{Im}(u)}{\mathrm{Re}(v) - \mathrm{Re}(u)} = \dfrac{1 - (-3)}{4 - 1} = \dfrac{4}{3}$ and either $(1, -3)$ or $(4, 1)$ in the equation $y - y_1 = m(x - x_1)$.

Circles

$|z - u| = r$ is a circle with centre u and radius r, where $u \in C$ and $r \in R$. The region inside the circle is given by $|z - u| < r$ and the outside by $|z - u| > r$. To include the circumference we use $\le$ or $\ge$.

The same circle is also given by $(z - u)\left(\overline{z - u}\right) = r^2$ or $(z - u)\left(\overline{z} - \overline{u}\right) = r^2$.

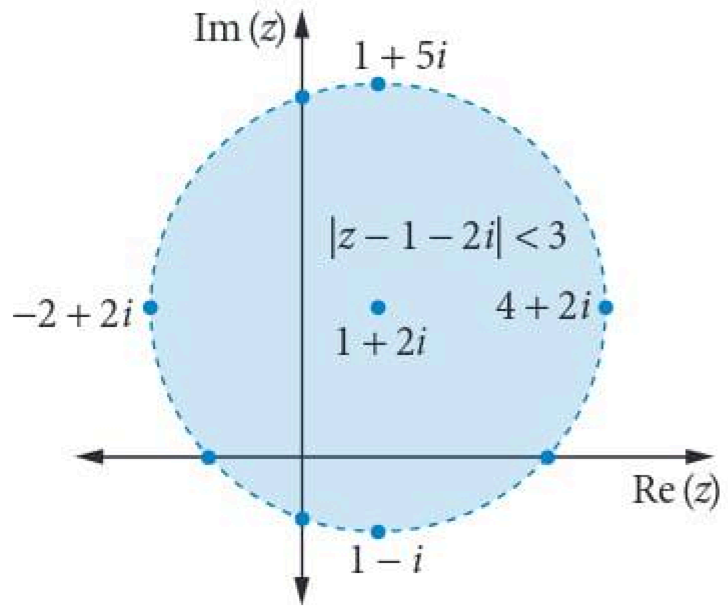
WB p. 78

WORKED EXAMPLE 5 Circles in the complex plane

a Find the equation of the circle with centre $1 + 2i$ and radius 3.

b Sketch and label the region strictly inside the circle.

Steps			**Working**
a	**1**	Use $\|z - u\| = r$.	$\|z - (1 + 2i)\| = 3$
	2	Simplify.	The equation is $\|z - 1 - 2i\| = 3$.
b	**1**	The inequality will use < for strictly inside the circle.	The label will be $\|z - 1 - 2i\| < 3$.

2 Sketch the circle with a dashed line to show the circumference is not included and shade the inside.	

Ellipses

The equation of an ellipse with foci u and v and major semi-axis length a in the complex plane is given by $|z-u|+|z-v|=2a$ (using the locus for the sum of the distances from the foci). The minor semi-axis length is given by $b^2 = a^2 - \frac{1}{4}|v-u|^2$, from point z on the minor axis.

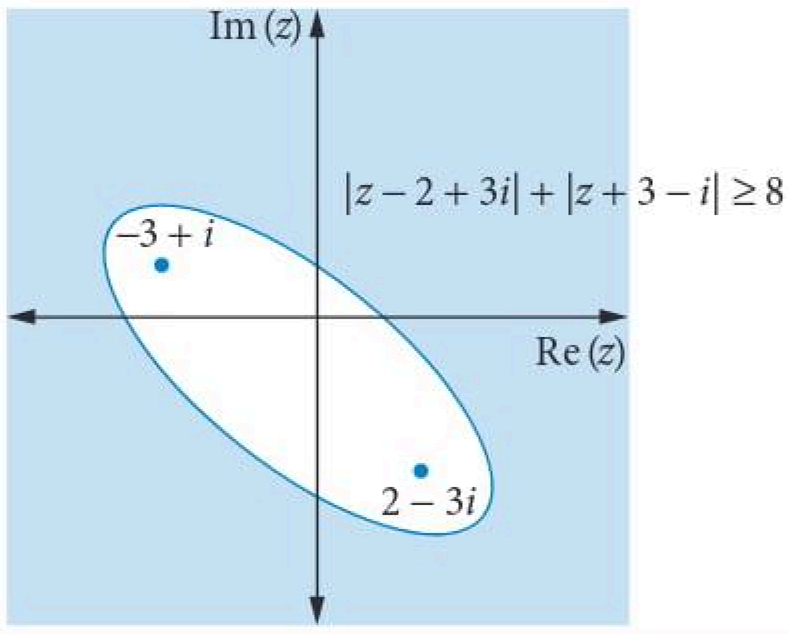
WB

p. 79

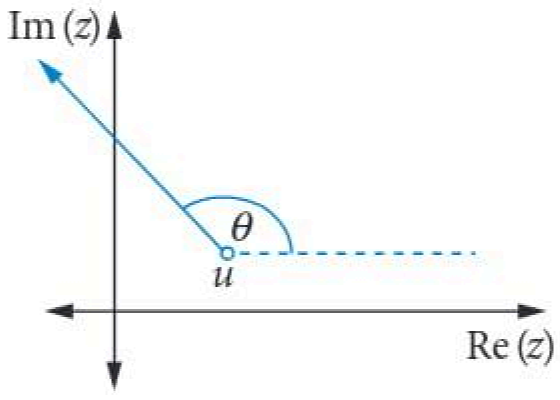
WORKED EXAMPLE 6 Ellipses in the complex plane

A region of the complex plane is given by $|z-2+3i|+|z+3-i| \geq 8$.

Describe and sketch the shape.

Steps	**Working**
1 Describe the shape.	It is an ellipse with foci $2-3i$, $-3+i$ and major semi-axis length 4. The minor semi-axis length is given by $b^2 = 16 - 0.25 \times 41 \cong 5.75$, so the minor semi-axis length is approximately 2.4.
2 Use the foci and $a = 4$. Include the circumference.	

Rays and semicircles



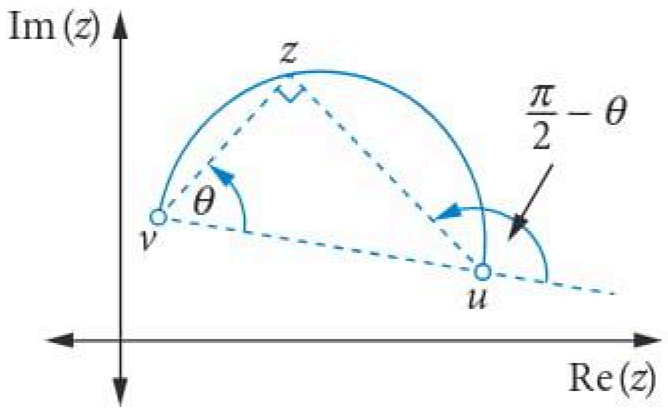
The arguments of complex numbers are also used to specify regions of the plane.

The equation $\text{Arg}(z) = \theta$ specifies a ray in the direction θ from the origin of the complex plane.

$\text{Arg}(z-u) = \theta$ starts from point u instead of the origin.

The ray is drawn with an open circle at its origin because $\text{Arg}(0)$ is undefined.

We can also specify a ray as $\{z: z-u = k \text{ cis}(\theta) \text{ for } k \in R^+\}$.



For example, $\left\{z : \text{Arg}(z-2-i) = \frac{3\pi}{4}\right\}$ and $\left\{z : z-2-i = k\left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right), k \in R^+\right\}$ both specify the ray at 135° from the point $2+i$. Using $k' = \frac{\sqrt{2}}{2}k$ simplifies the RHS to $k'(-1+i)$, so we could write it as $z = 2+i+k(i-1)$ for $k \in R^+$.

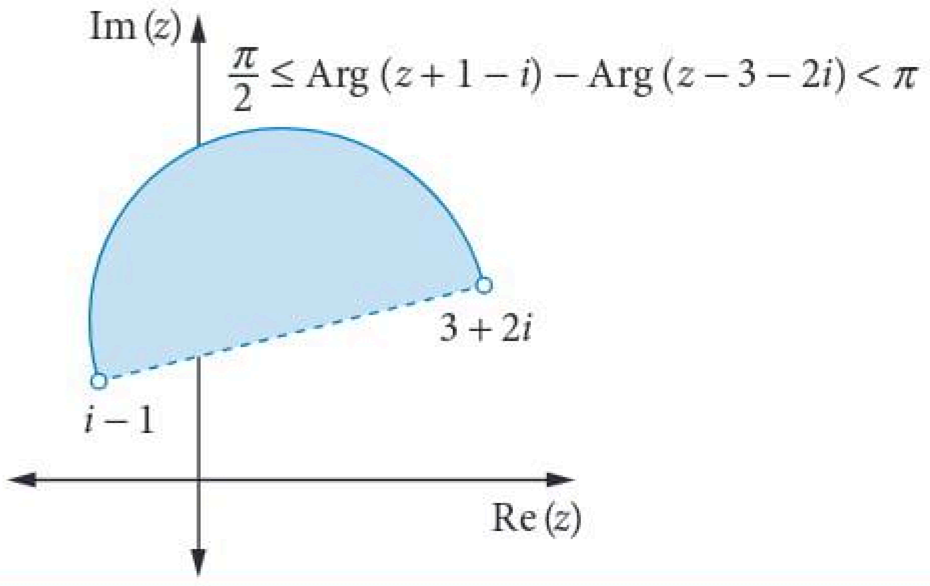
The diameter of a circle subtends an angle of $\frac{\pi}{2}$ at the circumference. This means the equation

$\text{Arg}(z-u)-\text{Arg}(z-v)=\frac{\pi}{2}$ is the semicircle on the right-hand side of the diameter from u to v.

$\text{Arg}(z-u)-\text{Arg}(z-v)=-\frac{\pi}{2}$ is the semicircle on the left-hand side of the diameter from u to v.

WORKED EXAMPLE 7 Semicircles in the complex plane

A region of the complex plane is given by $\frac{\pi}{2} \leq \text{Arg}\,(z+1-i)-\text{Arg}\,(z-3-2i)<\pi$.
Describe and sketch the shape.

Steps	**Working**
1 Describe the shape.	It is a semicircle on the diameter $i-1$ to $3+2i$, including the interior and the circumference, but not the diameter.
2 Sketch the shape.	

WB p. 80

Some problems will require us to find a combination of regions.

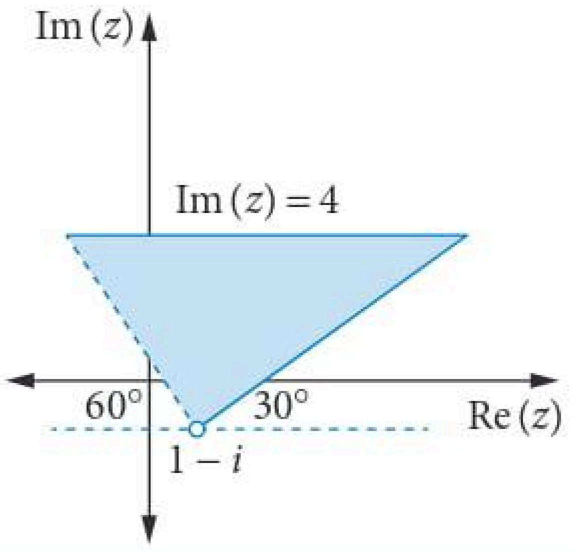
WORKED EXAMPLE 8 Combined regions

A region of the complex plane is given $\{z: z-\bar{z} \leq 8i\} \cap \left\{z: \frac{\pi}{6} \leq \text{Arg}(z-1+i) < \frac{2\pi}{3}\right\}$.

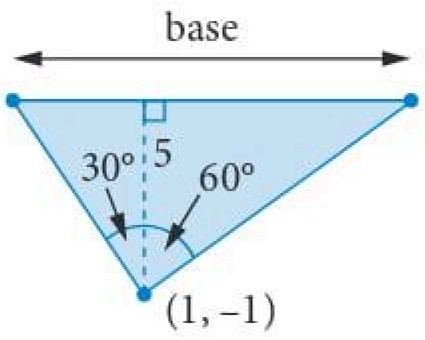
a Describe and sketch the shape.

b Find its area.

WB p. 81

Steps	**Working**
a **1** Describe the shape.	It is the intersection of two regions.
2 Consider the regions.	$z-\bar{z}=2i\,\text{Im}(z)$, so $\{z: z-\bar{z} \leq 8i\}$ is the region on or below the line $\text{Im}(z)=4$. $\frac{\pi}{6} \leq \text{Arg}(z-1+i) < \frac{2\pi}{3}$ is the triangular region between the rays from $1-i$, given by $\theta=\frac{\pi}{6}$ and $\theta=\frac{2\pi}{3}$, including the ray $\theta=\frac{\pi}{6}$.
3 Sketch the graph.	

b **1** Draw a diagram.



2 Find the height and base.

triangle height = 4 – (–1) = 5

$$\text{base} = 5\tan(30°) + 5\tan(60°)$$
$$= \frac{5\sqrt{3}}{3} + 5\sqrt{3}$$
$$= \frac{20\sqrt{3}}{3}$$

3 Find the area.

$$\text{area} = \frac{1}{2}bh$$
$$= \frac{1}{2} \times \frac{20\sqrt{3}}{3} \times 5$$
$$= \frac{50\sqrt{3}}{3} \text{ square units}$$

Exam hack

Unless otherwise stated, you must give exact answers.

EXERCISE 4.2 Polar form

ANSWERS p. 575

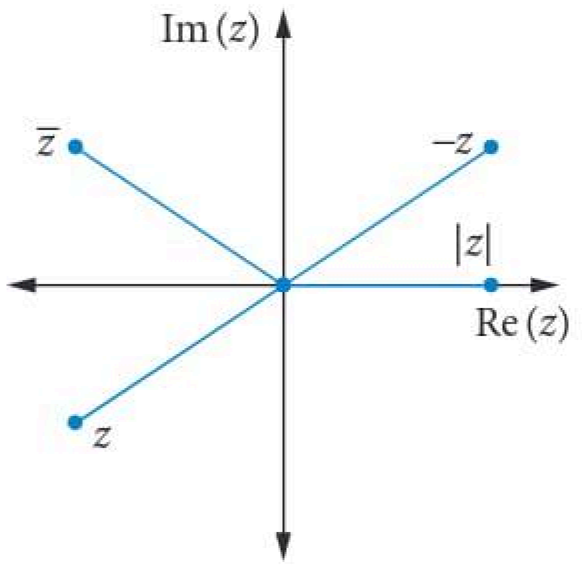
Recap

1 $w = -2 - 4i$ and $z = -1 + 4i$. The real part of $\frac{w}{z}$ is equal to:

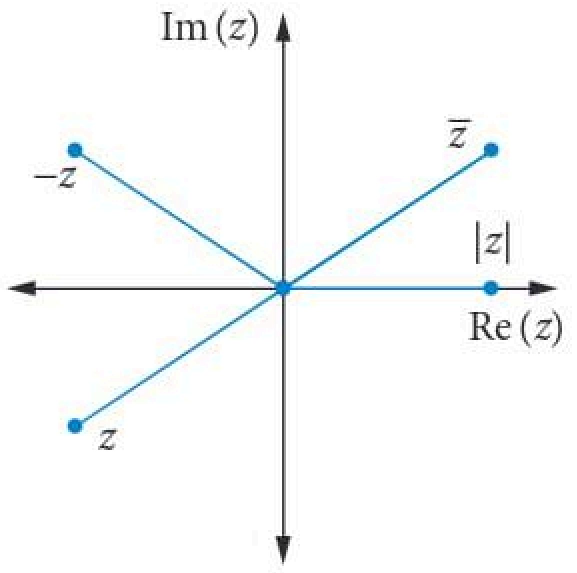
A $\frac{14}{17}$ **B** $-\frac{14}{17}$ **C** $\frac{14}{15}$ **D** $-\frac{14}{15}$ **E** $\frac{18}{17}$

2 Which diagram correctly represents the positions of z, $\bar{z}$, $|z|$ and $-z$ on an Argand diagram?

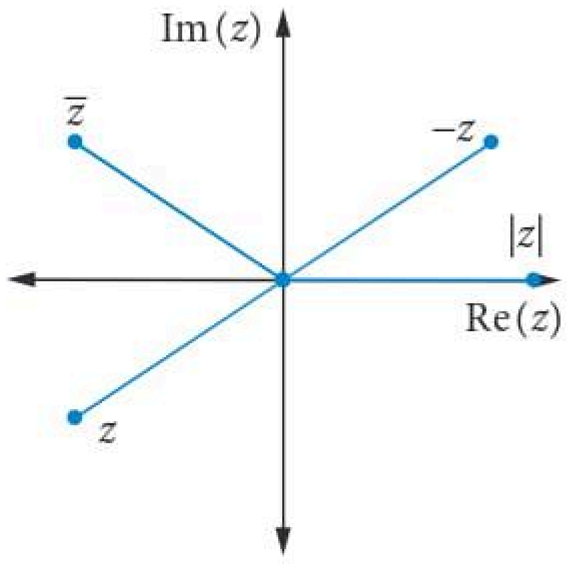
A



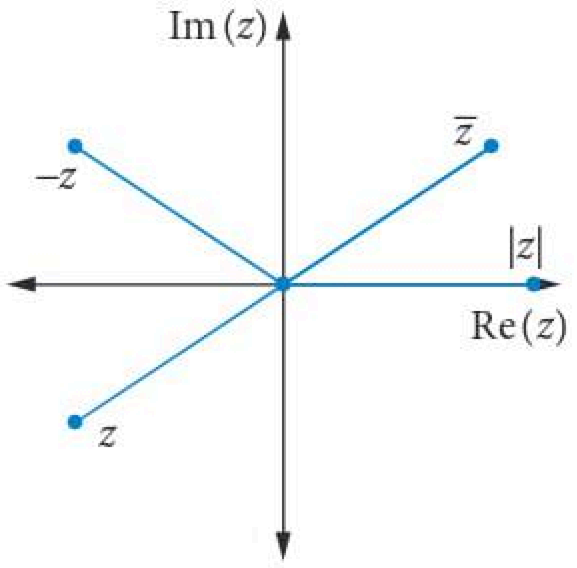
B



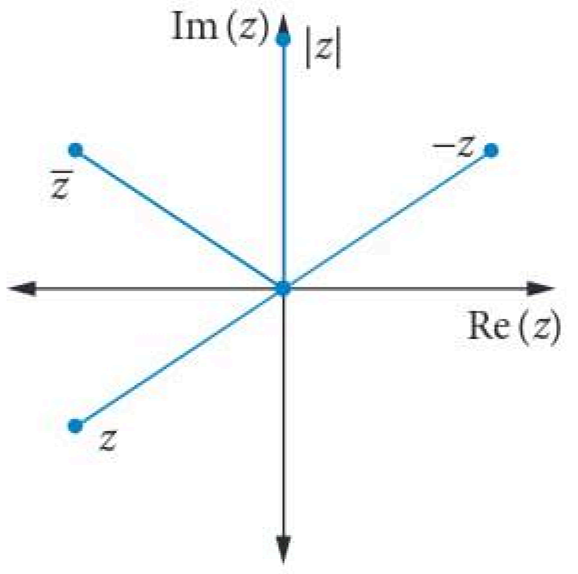
C



D



E



Mastery

3 WORKED EXAMPLE 3 Express each complex number in polar form.

a $3-3i$ **b** $\sqrt{3}+i$ **c** $2-2\sqrt{3}i$ **d** $3i$ **e** $3-4i$

4 Express each complex number in Cartesian form.

a $\left[5\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right]$ **b** $2\operatorname{cis}\left(\frac{5\pi}{4}\right)$ **c** $4\operatorname{cis}(\pi)$

d $6\operatorname{cis}\left(-\frac{\pi}{6}\right)$ **e** $\sqrt{3}\operatorname{cis}\left(\frac{2\pi}{3}\right)$

5 $t=3\operatorname{cis}\left(\frac{5\pi}{6}\right)$, $u=0.5\operatorname{cis}\left(\frac{3\pi}{4}\right)$, $v=4\operatorname{cis}\left(-\frac{\pi}{4}\right)$, $w=2\operatorname{cis}\left(\frac{\pi}{3}\right)$ and $z=5\operatorname{cis}\left(\frac{7\pi}{12}\right)$.

Find each of the following, expressing your answers with the principal argument.

a tu **b** vw **c** $\frac{z}{w}$ **d** $\frac{v}{u}$ **e** $\frac{zuv}{tw}$

6 Using CAS 2 Express each complex number in polar form.

a $4-3i$ **b** $2+i$ **c** $\sqrt{3}-i$ **d** $6+6i$ **e** $5i-12$

Express each complex number in Cartesian form.

f $4\operatorname{cis}\left(-\frac{\pi}{6}\right)$ **g** $3\operatorname{cis}\left(\frac{3\pi}{2}\right)$ **h** $5\operatorname{cis}\left(\frac{4\pi}{3}\right)$ **i** $2\operatorname{cis}\left(-\frac{3\pi}{4}\right)$ **j** $10\operatorname{cis}\left(\frac{5\pi}{12}\right)$

7 WORKED EXAMPLE 4 Find the equation of each line in polar and Cartesian form.

a The line through $2-i$ and $3+2i$.

b The perpendicular bisector of $4+2i$ and $i-3$.

c The line with slope 2 passing through $i-1$.

d The line parallel to the imaginary axis passing through $1+4i$.

e The line passing through the point $-3-2i$ in the direction $-\frac{5\pi}{6}$.

8 Describe and sketch each equation.

a $z-2-i=k(2+3i),\ k\in R$

b $z+3-2i=k(1-i),\ k\in R^{+}$

c $|z-3+i|=|z+1-2i|$

d $\operatorname{Arg}(z-1+2i)=\frac{3\pi}{4}$

e $|\ z-1\ +i\ |=k\operatorname{cis}\left(\frac{\pi}{6}\right),\ k\in R$

9 WORKED EXAMPLE 5 Write the equation of each figure in the complex plane.

a The circle with centre $3+2i$ and radius 5.

b The inside and circumference of the circle with centre $-1-i$ and radius 4.

c The outside of the circle with centre $3i-5$ and radius 2.

10 Describe and sketch each equation or inequality in the complex plane.

a $|z-4-3i|=4$ **b** $(z-3+3i)(\bar{z}-3-3i)=9$

c $|z+2-i|\geq 5$ **d** $(z+1+4i)(\bar{z}+1-4i)<25$

e $|z-2+3i|\leq 2$

11 WORKED EXAMPLE 6 Describe and sketch the region specified by

a $|z-2+i|+|z+3-i|=14$ **b** $|z-3+i|+|z+1-5i|<14$

c $|z+5+i|+|z-3-3i|\leq 24$ **d** $|z-5+3i|+|z-1-i|>18$

12 WORKED EXAMPLE 7 Describe and sketch the region specified by

a $\text{Arg}(z-4+i)-\text{Arg}(z+4+i)=\dfrac{\pi}{2}$ **b** $\text{Arg}(z-1-3i)-\text{Arg}(z+2+i)=-\dfrac{\pi}{2}$

c $0<\text{Arg}(z+3-i)-\text{Arg}(z-2+3i)\leq\dfrac{\pi}{2}$ **d** $\dfrac{\pi}{2}<\text{Arg}(z-1+i)-\text{Arg}(z-3+4i)\leq\pi$

13 Describe and sketch each equation or inequality in the complex plane.

a $|z-i+4|=3$ **b** $z-2+i=k(3+2i),\ k\in R$

c $|z-3-2i|=|z+2-3i|$ **d** $\text{Arg}(z-1-3i)=\dfrac{\pi}{4}$

e $(z-1+2i)(\bar{z}-1-2i)\leq 4$ **f** $|z+5-3i|+|z-3+3i|<20$

g $2\,\text{Re}(z)-\text{Im}(z)<1$

14 WORKED EXAMPLE 8 Describe and sketch the regions given by each inequality.

a $-2<\text{Im}(z)<3$ and $1<\text{Re}(z)<6$ **b** $\dfrac{\pi}{6}\leq\text{Arg}(z-1-i)<\dfrac{2\pi}{3}$ and $|z-1-i|<4$

c $3<|z-3+2i|\leq 5$ **d** $\text{Im}(z)<z-2i-k\left(\sqrt{3}+i\right)$

e $\text{Re}(z)\leq 1$ and $|z+2-3i|+|z-4+i|\leq 16$

Exam practice

80–100% 60–79% 0–59%

15 © VCAA 2011 1Q4 55% TECH-FREE (3 marks) Consider $z=\dfrac{1-\sqrt{3}i}{-1+i},\ z\in C$.

Find the principal argument of z in terms of $k\pi,\ k\in R$.

16 © VCAA 2015 2AQ5 81% Given $z=\dfrac{1+i\sqrt{3}}{1+i}$, the modulus and argument of the complex number z^5 are respectively

A $2\sqrt{2}$ and $\dfrac{5\pi}{6}$

B $4\sqrt{2}$ and $\dfrac{5\pi}{12}$

C $4\sqrt{2}$ and $\dfrac{7\pi}{12}$

D $2\sqrt{2}$ and $\dfrac{5\pi}{12}$

E $4\sqrt{2}$ and $-\dfrac{\pi}{12}$

17 © VCAA 2017 2AQ5 75% On an Argand diagram, a point that lies on the path defined by $|z-2+i|=|z-4|$ is

A $\left(3,-\dfrac{1}{2}\right)$

B $\left(-3,-\dfrac{1}{2}\right)$

C $\left(-3,\dfrac{3}{2}\right)$

D $\left(3,\dfrac{1}{2}\right)$

E $\left(3,-\dfrac{3}{2}\right)$

18 © VCAA 2016 2AQ5 72% If $\text{Arg}(-1+ai)=-\dfrac{2\pi}{3}$, then the real number a is

A $-\sqrt{3}$

B $-\dfrac{\sqrt{3}}{2}$

C $-\dfrac{1}{\sqrt{3}}$

D $\dfrac{1}{\sqrt{3}}$

E $\sqrt{3}$

19 © VCAA 2014 2AQ8 69% The **principal** argument of $\dfrac{-3\sqrt{2}-i\sqrt{6}}{2+2i}$ is

A $-\dfrac{13\pi}{12}$ **B** $\dfrac{7\pi}{12}$ **C** $\dfrac{11\pi}{12}$ **D** $\dfrac{13\pi}{12}$ **E** $-\dfrac{11\pi}{12}$

20 © VCAA 2018 2AQ6 58% The complex numbers z, iz and $z + iz$, where $z \in C \setminus \{0\}$, are plotted in the Argand plane, forming the vertices of a triangle.

The area of this triangle is given by

A $|z|$ **B** $|z| + |z|^2$ **C** $\dfrac{|z|^2}{2}$ **D** $|z|^2$ **E** $\dfrac{\sqrt{3}\,|z|^2}{2}$

21 © VCAA 2016S 2AQ5 On an Argand diagram, a set of points that lies on a circle of radius 2 centred at the origin is

A $\{z \in C: z\bar{z} = 2\}$ **B** $\{z \in C: z^2 = 4\}$

C $\{z \in C: \mathrm{Re}\,(z^2) + \mathrm{Im}\,(z^2) = 4\}$ **D** $\{z \in C: (z+\bar{z})^2 - (z-\bar{z})^2 = 16\}$

E $\{z \in C: (\mathrm{Re}\,(z))^2 + (\mathrm{Im}\,(z))^2 = 16\}$

22 © VCAA 2015 2AQ8 57% A relation that does **not** represent a circle in the complex plane is

A $z\bar{z} = 4$ **B** $|z + 3i| = 2\,|z - \mathrm{i}|$ **C** $|z - i| = |z + 2|$

D $|z - 1 + i| = 4$ **E** $|z| + 2\,|\bar{z}| = 4$

23 © VCAA 2016 2AQ6 57% The points corresponding to the four complex numbers given by

$z_1 = 2\,\mathrm{cis}\left(\dfrac{\pi}{3}\right)$, $z_2 = \mathrm{cis}\left(\dfrac{3\pi}{4}\right)$, $z_3 = 2\,\mathrm{cis}\left(-\dfrac{2\pi}{3}\right)$, $z_4 = \mathrm{cis}\left(-\dfrac{\pi}{4}\right)$ are the vertices of a parallelogram in the complex plane.

Which one of the following statements is **not** true?

A The acute angle between the diagonals of the parallelogram is $\dfrac{5\pi}{12}$.

B The diagonals of the parallelogram have lengths 2 and 4.

C $z_1 z_2 z_3 z_4 = 0$

D $z_1 + z_2 + z_3 + z_4 = 0$

E $1 \le |z| \le 2$ for all four of z_1, z_2, z_3, z_4

24 © VCAA 2014 2AQ9 56% The circle $|z - 3 - 2i| = 2$ is intersected exactly twice by the line given by

A $|z - i| = |z + 1|$ **B** $|z - 3 - 2i| = |z - 5|$ **C** $|z - 3 - 2i| = |z - 10i|$

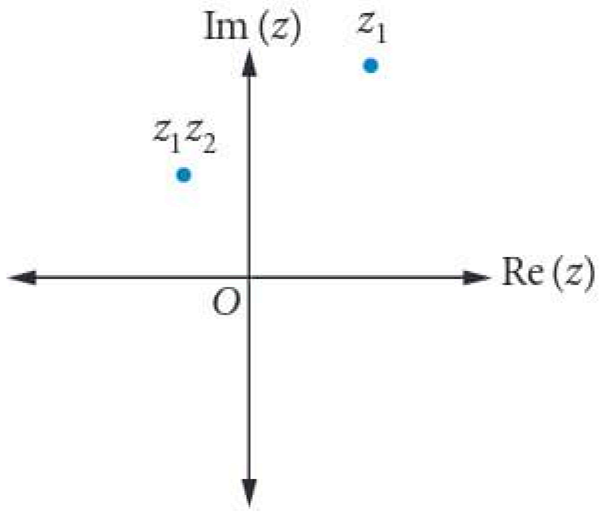
D $\mathrm{Im}\,(z) = 0$ **E** $\mathrm{Re}\,(z) = 5$

25 © VCAA 2015 2AQ9 47% Let $z_1 = r_1\,\mathrm{cis}\,(\theta_1)$ and $z_2 = r_2\,\mathrm{cis}\,(\theta_2)$, where z_1 and $z_1 z_2$ are shown in the Argand diagram given; θ_1 and θ_2 are acute angles.

A statement that is **necessarily** true is

A $r_2 > 1$ **B** $\theta_1 < \theta_2$ **C** $\left|\dfrac{z_1}{z_2}\right| > r_1$

D $\theta_1 = \theta_2$ **E** $r_1 > 1$



26 © VCAA 2015 2AQ6 43% Which one of the following relations has a graph that passes through the point $1 + 2i$ in the complex plane?

A $z\bar{z} = \sqrt{5}$ **B** $\mathrm{Arg}(z) = \dfrac{\pi}{3}$ **C** $|z - 1| = |z - 2i|$

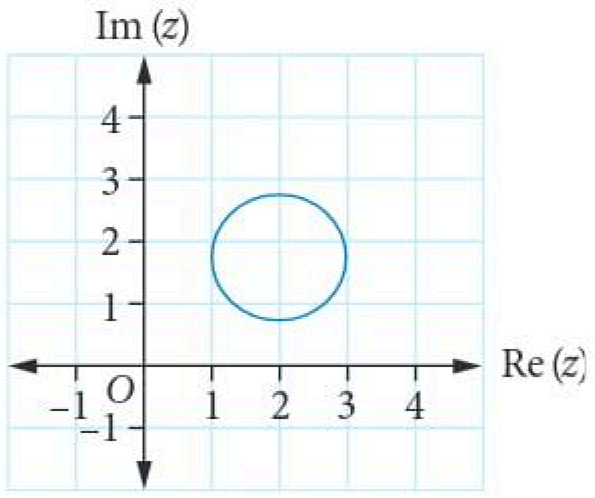
D $\mathrm{Re}\,(z) = 2\,\mathrm{Im}\,(z)$ **E** $z + \bar{z} = 2$

▶ **27** © VCAA 2018 2AQ5 **41%** Let $z = a + bi$, where $a, b \in R \setminus \{0\}$.

If $z + \frac{1}{z} \in R$, which one of the following must be **true**?

A $\text{Arg}(z) = \frac{\pi}{4}$ **B** $a = -b$ **C** $a = b$ **D** $|z| = 1$ **E** $z^2 = 1$

28 © VCAA 2021 2AQ5 **32%** The graph of the circle given by $|z - 2 - \sqrt{3}i| = 1$, where $z \in C$, is shown below.



For points on this circle, the maximum value of $|z|$ is

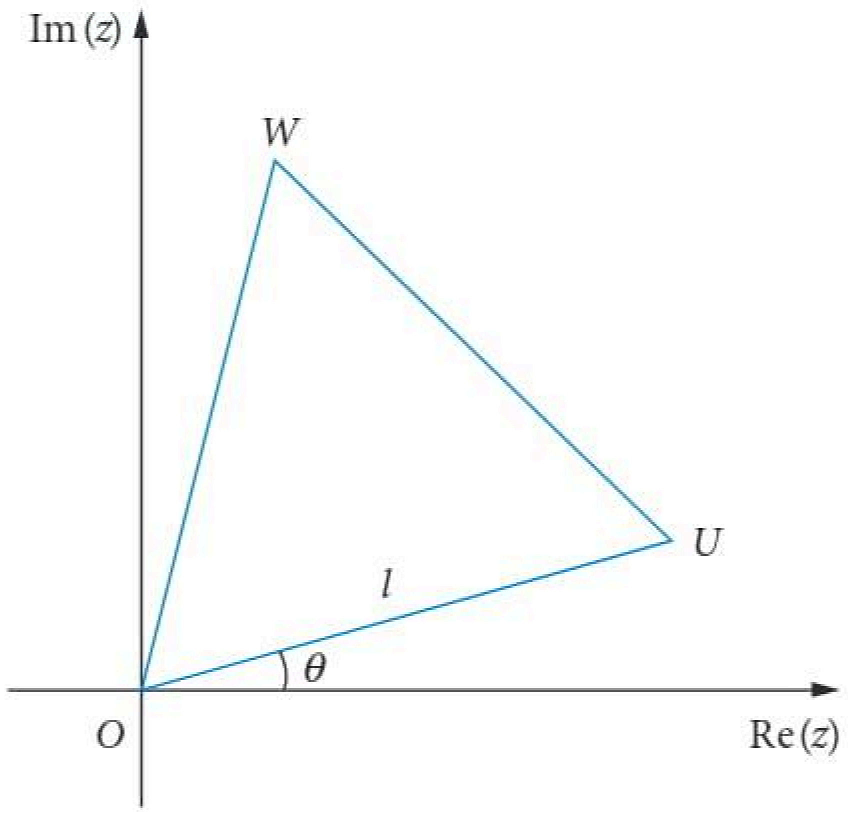
A $\sqrt{3} + 1$ **B** 3 **C** $\sqrt{13}$ **D** $\sqrt{7} + 1$ **E** 8

29 © VCAA 2021 2AQ6 **23%** If $z \in C$, $z \neq 0$ and $z^2 \in R$, then the possible values of $\arg(z)$ are

A $\frac{k\pi}{2}, k \in Z$ **B** $k\pi, k \in Z$ **C** $\frac{(2k+1)\pi}{2}, k \in Z$

D $\frac{(4k+1)\pi}{2}, k \in Z$ **E** $\frac{(4k-1)\pi}{2}, k \in Z$

30 © VCAA 2021N 2BQ2 (9 marks) Points O, U and W in the complex plane represent complex numbers 0, u and w respectively, where $\text{Arg}(u) = \theta$. When plotted on an Argand diagram, the points are the vertices of an equilateral triangle of side length l, as shown below.



a Express w in polar form in terms of l and θ. 1 mark

b Point P represents the complex number p and is the midpoint of U and W.

- **i** Express p in polar form in terms of l and θ. 2 marks
- **ii** Express p^{12} in the form $\frac{a}{b}u^n$, where a, b and n are integers. 3 marks

c Show that $\frac{w}{u} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. 1 mark

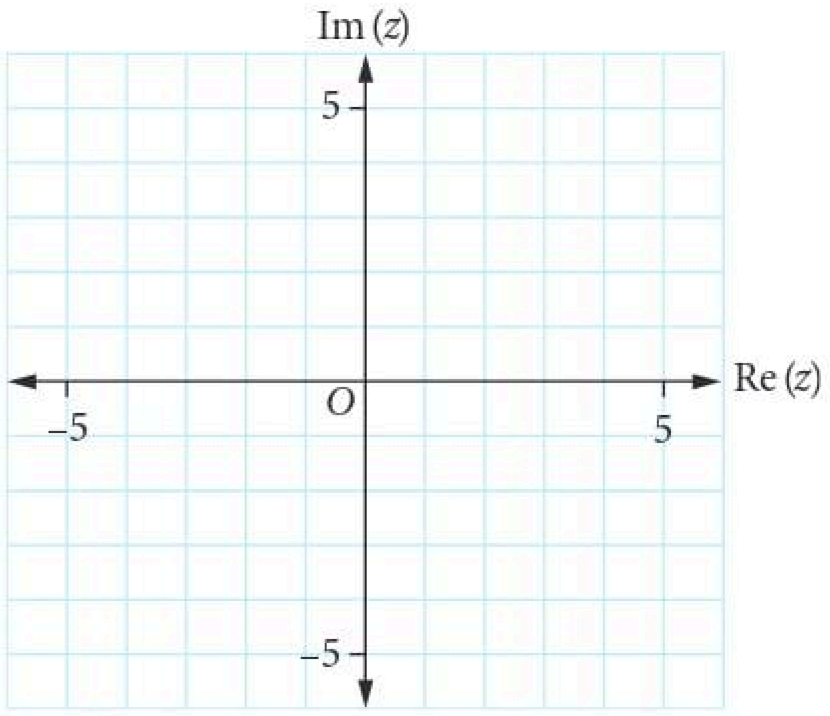
d Point U has coordinates $(a, 17)$ and point W has coordinates $(b, 41)$.

Find the values of a and b. 2 marks ▶

31 © VCAA 2020 2BQ2 (11 marks) Two complex numbers, u and v, are defined as $u = -2 - i$ and $v = -4 - 3i$.

a 84% Express the relation $|z - u| = |z - v|$ in the Cartesian form $y = mx + c$, where $m, c \in R$. 3 marks

b 83% Copy the Argand diagram and on it plot the points that represent u and v and the relation $|z - u| = |z - v|$. 2 marks



c 54% State a geometrical interpretation of the graph of $|z - u| = |z - v|$ in relation to the points that represent u and v. 2 mark

d **i** 55% Sketch the ray given by $\text{Arg}(z - u) = \frac{\pi}{4}$ on the Argand diagram in part **b**. 1 mark

ii 25% Write down the function that describes the ray $\text{Arg}(z - u) = \frac{\pi}{4}$, giving the rule in Cartesian form. 1 mark

e 40% The points representing u and v and $-5i$ lie on the circle given by $|z - z_c| = r$, where z_c is the centre of the circle and r is the radius.

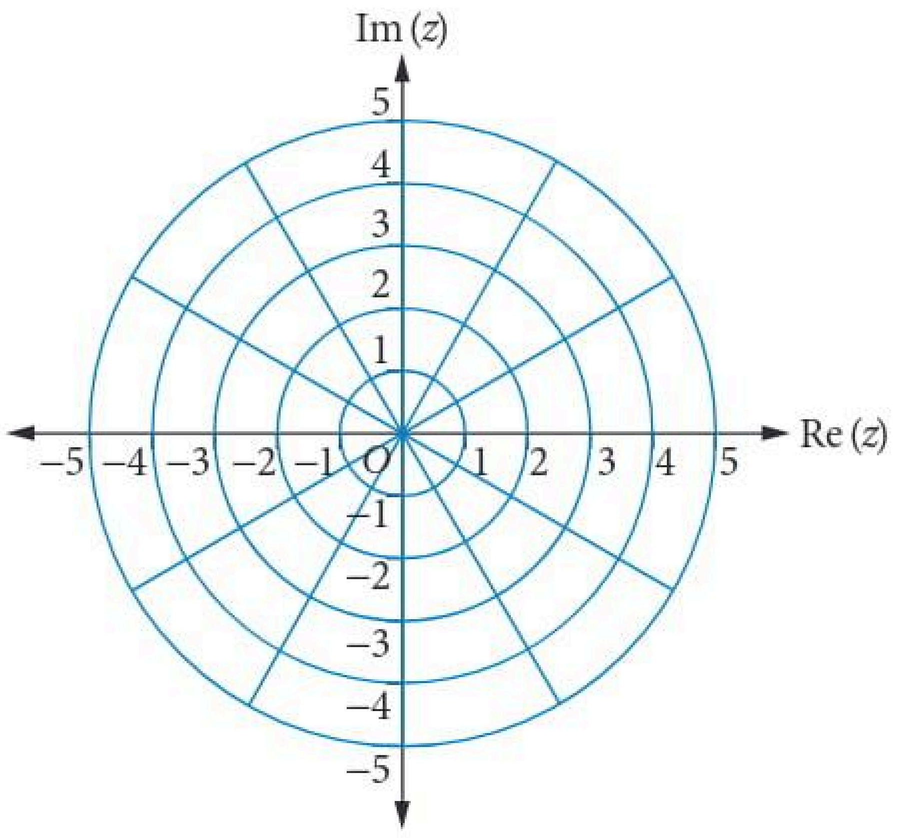
Find z_c in the form $a + ib$, where $a, b \in R$, and find the radius r. 3 marks

32 © VCAA 2018 2BQ2 (11 marks)

a 71% State the centre in the form (x, y), where $x, y \in R$, and state the radius of the circle given by $|z - (1 + 2i)| = 2$, where $z \in C$. 1 mark

b 70% By expressing the circle given by $|z + 1| = \sqrt{2}\,|z - i|$ in Cartesian form, show that this circle has the same centre and radius as the circle given by $|z - (1 + 2i)| = 2$. 2 marks

c 68% Copy the Argand diagram below, and on it graph the circle given by $|z + 1| = \sqrt{2}\,|z - i|$, labelling the intercepts with the vertical axis. 2 marks



The line given by $|z - 1| = |z - 3|$ intersects the circle given by $|z + 1| = \sqrt{2}\,|z - i|$ in two places.

d 64% Draw the line given by $|z - 1| = |z - 3|$ on the Argand diagram in part **c**. Label the points of intersection with their coordinates. 2 marks

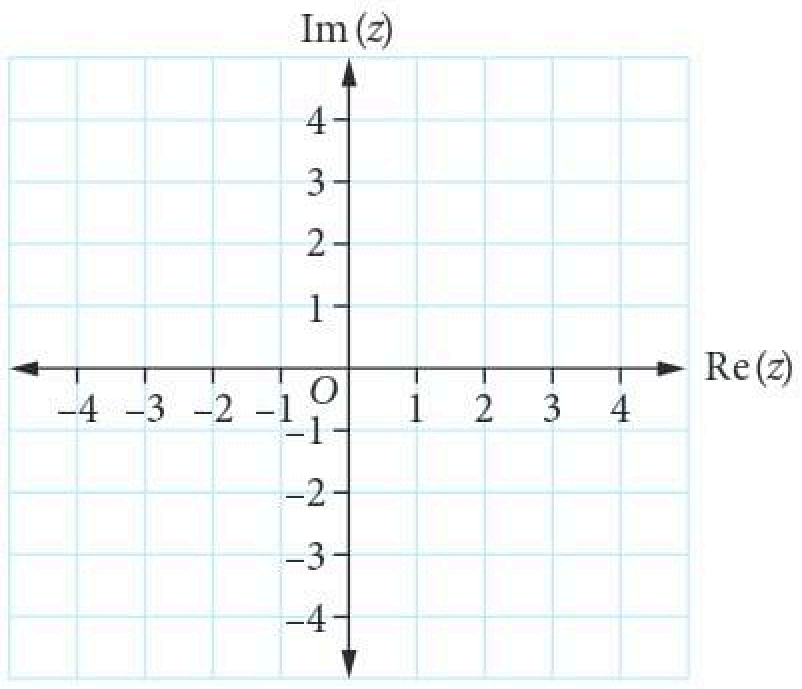
e 39% Find the area of the minor segment enclosed by an arc of the circle given by $|z + 1| = \sqrt{2}\,|z - i|$ and part of the line given by $|z - 1| = |z - 3|$. 3 marks

33 © VCAA 2016 2BQ2 (11 marks) A line in the complex plane is given by $|z-1|=|z+2-3i|$, $z \in C$.

a 79% Find the equation of this line in the form $y=mx+c$. 2 marks

b 75% Find the points of intersection of the line $|z-1|=|z+2-3i|$ with the circle $|z-1|=3$. 2 marks

c 83% Sketch both the line $|z-1|=|z+2-3i|$ and the circle $|z-1|=3$ on a copy of the Argand diagram below. 2 marks



d 46% The line $|z-1|=|z+2-3i|$ cuts the circle $|z-1|=3$ into two segments.

Find the area of the major segment. 2 marks

e 34% Sketch the ray given by $\operatorname{Arg}(z)=-\dfrac{3\pi}{4}$ on the Argand diagram in part **c**. 1 mark

f 12% Write down the range of values of α, $\alpha \in R$, for which a ray with equation $\operatorname{Arg}(z)=\alpha\pi$ intersects the line $|z-1|=|z+2-3i|$. 2 marks

34 © VCAA 2017 2BQ4 (10 marks)

a 74% Express $-2-2\sqrt{3}i$ in polar form. 1 mark

b 55% Show that the roots of $z^2+4z+16=0$ are $z=-2-2\sqrt{3}i$ and $z=-2+2\sqrt{3}i$. 1 mark

c 31% Express the roots of $z^2+4z+16=0$ in terms of $z=2-2\sqrt{3}i$. 1 mark

d 73% Show that the Cartesian form of the relation $|z|=\left|z-\left(2-2\sqrt{3}i\right)\right|$ is $x-\sqrt{3}y-4=0$. 2 marks

e 69% Copy the Argand diagram from question **32** on page 165 and on it sketch the line represented by $x-\sqrt{3}y-4=0$ and plot the roots of $z^2+4z+16=0$. 2 marks

f 1% The equation of the line passing through the two roots of $z^2+4z+16=0$ can be expressed as $|z-a|=|z-b|$, where $a, b \in C$.

Find b in terms of a. 1 mark

g 31% Find the area of the major segment bounded by the line passing through the roots of $z^2+4z+16=0$ and the major arc of the circle given by $|z|=4$. 2 marks

4.3 De Moivre's theorem

de Moivre's theorem

$$[\cos(\theta) + i\sin(\theta)]^n = \cos(n\theta) + i\sin(n\theta).$$

Video playlist De Moivre's theorem

Worksheet Using de Moivre's theorem

In this course, we only need to prove the **de Moivre's theorem** for $n \in Z$. The proof uses induction.

For $n = 1$,

LHS $= [\cos(\theta) + i\sin(\theta)]^1 = \cos(\theta) + i\sin(\theta) = \cos(1\theta) + i\sin(1\theta) =$ RHS, so it is true for $n = 1$.

Suppose it is true for $n = k$. Then,

$[\cos(\theta) + i\sin(\theta)]^k = \cos(k\theta) + i\sin(k\theta)$.

For $n = k + 1$,

$$\begin{aligned}
\text{LHS} &= [\cos(\theta) + i\sin(\theta)]^{k+1} \\
&= [\cos(\theta) + i\sin(\theta)][\cos(\theta) + i\sin(\theta)]^k \\
&= [\cos(\theta) + i\sin(\theta)][\cos(k\theta) + i\sin(k\theta)] \text{ from the assumption} \\
&= \cos(\theta)\cos(k\theta) + i\cos(\theta)\sin(k\theta) + i\sin(\theta)\cos(k\theta) - \sin(\theta)\sin(k\theta) \\
&= \cos(\theta)\cos(k\theta) - \sin(\theta)\sin(k\theta) + i[\sin(\theta)\cos(k\theta) + \cos(\theta)\sin(k\theta)] \\
&= \cos(\theta + k\theta) + i\sin(\theta + k\theta) \text{ using the expansions of } \cos(x + y) \text{ and } \sin(x + y) \\
&= \cos[(k + 1)\theta] + i\sin[(k + 1)\theta] \\
&= \text{RHS}
\end{aligned}$$

Thus, the assumption is true for $n = 1$, and if it is true for k, then it is true for $k + 1$, so by mathematical induction, it is true for $n \in N$.

For $n = 0$, LHS $= [\cos(\theta) + i\sin(\theta)]^0 = 1 = \cos(0) + i\sin(0) =$ RHS.

For $n < 0$, let $n = -m$, where $m \in N$.

Then

$$\begin{aligned}
[\cos(\theta) + i\sin(\theta)]^n &= [\cos(\theta) + i\sin(\theta)]^{-m} \\
&= \frac{1}{[\cos(\theta) + i\sin(\theta)]^m} \\
&= \frac{1}{\cos(m\theta) + i\sin(m\theta)} \text{ since it has been shown for } m \in N \\
&= \frac{1}{\cos(m\theta) + i\sin(m\theta)} \times \frac{\cos(m\theta) - i\sin(m\theta)}{\cos(m\theta) - i\sin(m\theta)} \\
&= \frac{\cos(m\theta) - i\sin(m\theta)}{\cos^2(m\theta) + \sin^2(m\theta)} \\
&= \frac{\cos(m\theta) - i\sin(m\theta)}{1} \\
&= \cos(m\theta) - i\sin(m\theta) \\
&= \cos(-m\theta) + i\sin(-m\theta) \text{ using } \cos(-x) = \cos(x) \text{ and } \sin(-x) = -\sin(x) \\
&= \cos(n\theta) + i\sin(n\theta)
\end{aligned}$$

Thus the theorem has been proven for $n \in N^+$, $n = 0$, $n \in N^-$, so it is true for $n \in Z$.

We can use de Moivre's theorem to find **powers of complex numbers**.

WB

p. 82

WORKED EXAMPLE 9 Power of a complex number

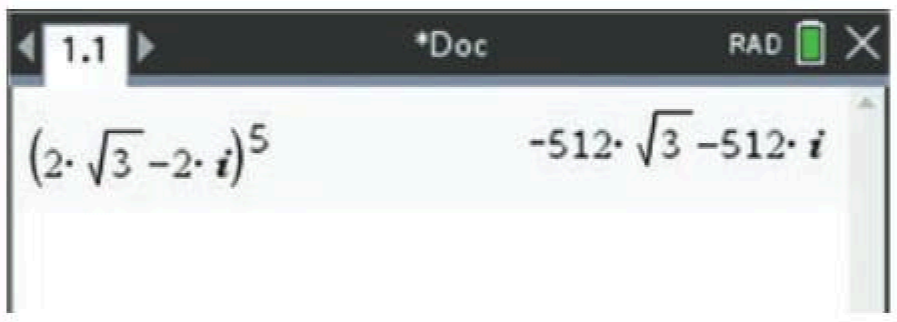
Given $z = 2\sqrt{3} - 2i$, write z in polar form and hence find z^5 and express it in Cartesian form.

Steps	Working
1 Find the modulus and argument.	$\lvert z \rvert = \sqrt{(2\sqrt{3})^2 + 2^2} = \sqrt{12+4} = 4$ $\tan(\theta) = -\frac{1}{\sqrt{3}}$ and $x > 0, y < 0$, so $\theta = -\frac{\pi}{6}$
2 Write in polar form.	$z = 4\,\text{cis}\left(-\frac{\pi}{6}\right)$
3 Find z^5 using de Moivre's theorem.	$z^5 = 4^5\left[\text{cis}\left(-\frac{\pi}{6}\right)\right]^5$ $= 4^5\,\text{cis}\left(-\frac{5\pi}{6}\right)$ $= 1024\,\text{cis}\left(-\frac{5\pi}{6}\right)$ $= 1024\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)$ $= 1024 \times \left(-\frac{\sqrt{3}}{2}\right) + i \times 1024 \times \left(-\frac{1}{2}\right)$ $= -516\sqrt{3} - 512i$

USING CAS 3 Powers of complex numbers

Simplify $\left(2\sqrt{3} - 2i\right)^5$.

TI-Nspire

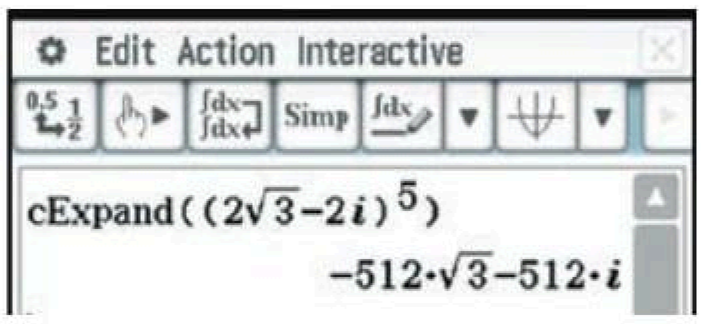


$(2\cdot\sqrt{3}-2\cdot i)^5$ $\quad -512\cdot\sqrt{3}-512\cdot i$

1 Enter the expression.

2 The expression will automatically simplify.

ClassPad



cExpand$((2\sqrt{3}-2i)^5)$

$-512\cdot\sqrt{3}-512\cdot i$

1 Enter and highlight the expression.

2 Tap **Interactive > Complex > cExpand**.

$\left(2\sqrt{3} - 2i\right)^5 = -512\sqrt{3} - 512i$

EXERCISE 4.3 De Moivre's theorem

ANSWERS p. 580

Recap

1 Which equation does NOT give a straight line or ray in the Argand plane?

A $\text{Arg}(z) = -\frac{2\pi}{3}$

B $\lvert z - 2 - 2i \rvert = \lvert z + 2 - 6i \rvert$

C $\text{Arg}(z - 3 + i) - \text{Arg}(z + 2 - i) = \frac{\pi}{2}$

D $z - 1 + 2i = k(2 - i)$

E $\text{Arg}(z - 1 - i) = \frac{\pi}{3}$

2 Equations X, Y and Z describe figures in the complex plane.

X $|z-3+2i|+|z+1+2i|=10$

Y $(z+3-2i)(\bar{z}+3+2i)=16$

Z $|z-5+3i|=|z-2+3i|$

Which of the following is correct?

A X and Y are circles and Z is a straight line.

B X and Y are ellipses and Z is a circle.

C X and Y are circles and Z is an ellipse.

D X is an ellipse, Y is a circle and Z is a straight line.

E X is a circle, Y is an ellipse and Z is a straight line.

Mastery

3 WORKED EXAMPLE 9 Convert each expression to polar form, find its modulus and argument, then write the expression in Cartesian form.

a $(1+i)^5$ **b** $\left(-\sqrt{3}-i\right)^3$ **c** $\left(2\sqrt{2}-2i\sqrt{2}\right)^5$ **d** $\left(2+2i\sqrt{3}\right)^3$ **e** $\left(-1+\sqrt{3}i\right)^4$

4 Using CAS 3 Find each expression in Cartesian form.

a $(1-2i)^4$ **b** $(2-3i)^3$ **c** $(2i-3)^5$

d $\left(0.5-0.5\sqrt{3}i\right)^7$ **e** $\left(\sqrt{6}+\sqrt{2}+\sqrt{6}i-\sqrt{2}i\right)^4$

5 Solve each equation for $k \in Z$.

a $(1-i)^k=16$ **b** $\left(\sqrt{3}-i\right)^k=-64$ **c** $\left(\sqrt{2}i-\sqrt{2}\right)^k=16\sqrt{2}-16\sqrt{2}i$

6 Convert each expression to polar form, then evaluate and express the answer in Cartesian form.

a $\left(\sqrt{3}+i\right)^5(1+i)^3$ **b** $\left(1-\sqrt{3}i\right)^3\left(\sqrt{3}-i\right)^5$ **c** $\dfrac{\left(-1-\sqrt{3}i\right)^4}{(i-1)^3}$

d $\dfrac{\left(1+\sqrt{3}i\right)^6}{(\sqrt{3}-i)^2}$ **e** $\dfrac{(2+2i)^3}{(\sqrt{3}+i)^4}$

Exam practice

80–100% 60–79% 0–59%

7 © VCAA 2010 1Q4 83% TECH-FREE (3 marks) Copy the Argand diagram from question **32** on page 165. Given that $z=1+i$, plot and label points on the diagram.

i z **ii** z^2 **iii** z^4

8 © VCAA 2018 1Q2 TECH-FREE (4 marks)

a 83% Show that $1+i=\sqrt{2}\operatorname{cis}\left(\dfrac{\pi}{4}\right)$. 1 mark

b 70% Evaluate $\dfrac{\left(\sqrt{3}-i\right)^{10}}{(1+i)^{12}}$, giving your answer in the form $a+bi$, where $a, b \in R$. 3 marks

9 © VCAA 2014 2AQ5 85% If the complex number z has modulus $2\sqrt{2}$ and argument $\dfrac{3\pi}{4}$, then z^2 is equal to

A $-8i$ **B** $4i$ **C** $-2\sqrt{2}i$ **D** $2\sqrt{2}i$ **E** $-4i$

10 © VCAA 2015 2AQ5 81% Given $z = \dfrac{1+i\sqrt{3}}{1+i}$, the modulus and argument of the complex number z^5 are respectively

A $2\sqrt{2}$ and $\dfrac{5\pi}{6}$

B $4\sqrt{2}$ and $\dfrac{5\pi}{12}$

C $4\sqrt{2}$ and $\dfrac{7\pi}{12}$

D $2\sqrt{2}$ and $\dfrac{5\pi}{12}$

E $4\sqrt{2}$ and $-\dfrac{\pi}{12}$

11 © VCAA 2013 2AQ7 80% If $z = r\,\text{cis}\,(\theta)$, then $\dfrac{z^2}{\bar{z}}$ is equivalent to

A $r^3\text{cis}\,(3\theta)$

B $r^3\text{cis}\,(-\theta)$

C $2\,\text{cis}\,(3\theta)$

D $r^3\text{cis}\,(\theta)$

E $r\,\text{cis}\,(3\theta)$

12 © VCAA 2010 2AQ9 69% Given that $z = 4\,\text{cis}\left(\dfrac{2\pi}{3}\right)$, it follows that $\text{Arg}\,(z^5)$ is

A $\dfrac{10\pi}{3}$

B $\dfrac{4\pi}{3}$

C $\dfrac{7\pi}{3}$

D $-\dfrac{\pi}{3}$

E $-\dfrac{2\pi}{3}$

13 © VCAA 2015 2AQ7 63% If $z = \sqrt{3} + 3i$, then z^{63} is

A real and negative.

B equal to a negative real multiple of i.

C real and positive.

D equal to a positive real multiple of i.

E a positive real multiple of $1 + i\sqrt{3}$.

Video playlist
Roots of unity

4.4 Roots of unity

We can use de Moivre's theorem to solve complex equations of the form $z^n = 1$, whose solutions are called the **roots of unity** (unity means '1').

In polar form, $1 = \cos(0) + i\sin(0) = \cos(\pm 2\pi) + i\sin(\pm 2\pi) = \cos(\pm 4\pi) + i\sin(\pm 4\pi) = \ldots$

We can write this as $1 = \cos(2k\pi) + i\sin(2k\pi)$ for $k \in Z$.

WB
p. 83

WORKED EXAMPLE 10 Complex roots of unity

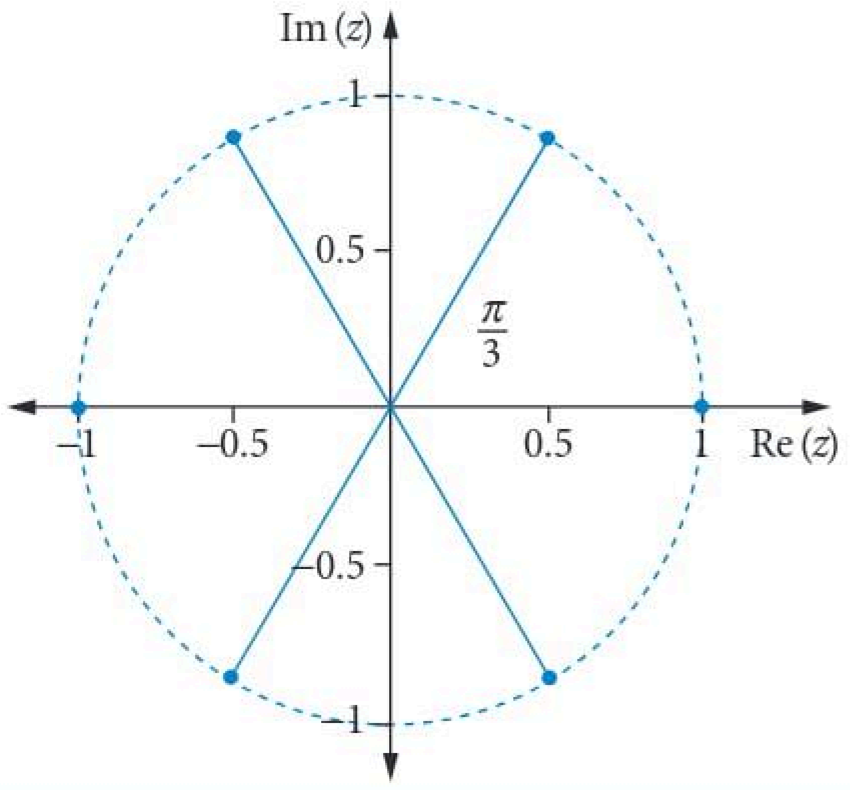
Solve $z^6 = 1$ and show the solutions on an Argand diagram.

Steps	Working
1 The modulus of z will be 1.	Let $z = \cos(\theta) + i\sin(\theta)$
2 Use de Moivre's theorem.	$z^6 = [\cos(\theta) + i\sin(\theta)]^6$ $= \cos(6\theta) + i\sin(6\theta)$
3 Write the equation with $1 = \cos(2k\pi) + i\sin(2k\pi)$.	$\cos(6\theta) + i\sin(6\theta) = \cos(2k\pi) + i\sin(2k\pi)$
4 Write the argument equation and solve for θ.	$6\theta = 2k\pi$ $\theta = \dfrac{k\pi}{3}$ for $k \in Z$
5 Write out some solutions.	$\theta = 0, \pm\dfrac{\pi}{3}, \pm\dfrac{2\pi}{3}, \pm\pi, \pm\dfrac{4\pi}{3}, \ldots$
6 Choose only the principal arguments.	$\theta = 0, \pm\dfrac{\pi}{3}, \pm\dfrac{2\pi}{3}, \pi$
7 Write the solutions.	$z = \text{cis}\,(0) = 1,\ \text{cis}\left(\pm\dfrac{\pi}{3}\right),\ \text{cis}\left(\pm\dfrac{2\pi}{3}\right), \text{cis}\,(\pi) = -1$ or $z = \text{cis}\left(\dfrac{k\pi}{3}\right)$ for $k = -2, -1, 0, 1, 2, 3$.

8 Show the roots on an Argand diagram.

Exam hack

In an exam, make sure that you choose the principal arguments to achieve full marks.



The solutions to $z^6 = 1$ are called the sixth **roots of unity**.

The same method can be used for any value of n in finding $\sqrt[n]{1}$.

Roots of unity

- The **nth roots of unity** are given by $z = \text{cis}\left(\pm\frac{2k\pi}{n}\right)$.
- For n even, $k = \left\{-\frac{n}{2} + 1 \ldots -1, 0, 1, 2 \ldots \frac{n}{2}\right\}$.
- For n odd, $k = \left\{-\frac{n-1}{2} \ldots -1, 0, 1, 2 \ldots \frac{n-1}{2}\right\}$.
- The nth roots are separated from each other by the angle $\frac{2\pi}{n}$.

We can use de Moivre's theorem for general **complex roots**.

Worksheet Roots of complex numbers

WB p. 84

WORKED EXAMPLE 11 General complex roots

Find the 5th roots of $z = -16\sqrt{2} - 16i\sqrt{2}$ and show them on an Argand diagram.

Steps	**Working**
1 Find z in polar form.	$z = 32\,\text{cis}\left(-\frac{3\pi}{4}\right)$
2 Let the roots be w and write $w^5 = z$ in general form.	$w^5 = -16\sqrt{2} - 16i\sqrt{2}$ $w^5 = 32\,\text{cis}\left(-\frac{3\pi}{4} + 2k\pi\right)$ for $k \in Z$
3 Use de Moivre's theorem.	$w = \left[32\,\text{cis}\left(-\frac{3\pi}{4} + 2k\pi\right)\right]^{\frac{1}{5}}$ $= 32^{\frac{1}{5}}\left[\text{cis}\left(-\frac{3\pi}{4} + 2k\pi\right)\right]^{\frac{1}{5}}$ $= 2\left[\text{cis}\left(-\frac{3\pi}{20} + \frac{2k\pi}{5}\right)\right]$ $= 2\,\text{cis}\left(\frac{8k\pi - 3\pi}{20}\right)$ for $k \in Z$

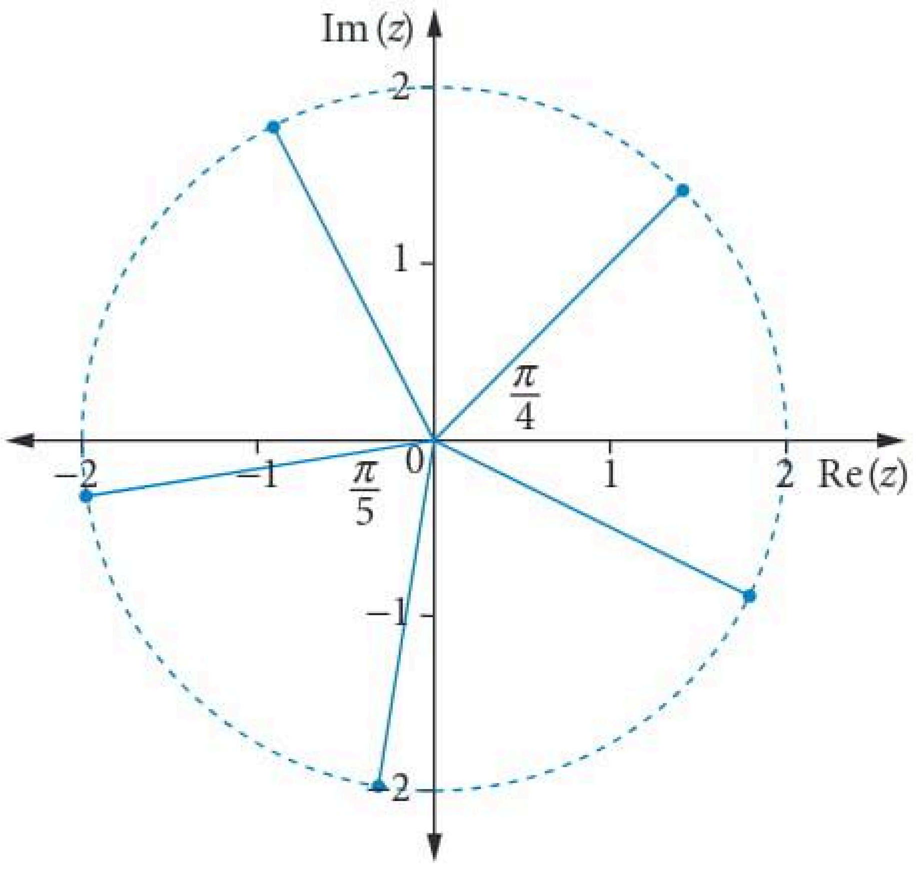
4 Choose only the principal arguments: $z = -2, -1, 0, 1, 2$ to obtain 5 equally spaced solutions, $\frac{2\pi}{5}$ apart.

$w = 2\operatorname{cis}\left(-\frac{19\pi}{20}\right)$, $w = 2\operatorname{cis}\left(-\frac{11\pi}{20}\right)$,

$w = 2\operatorname{cis}\left(-\frac{3\pi}{20}\right)$, $w = 2\operatorname{cis}\left(\frac{5\pi}{20}\right)$

$= 2\operatorname{cis}\left(\frac{\pi}{4}\right)$ or $w = 2\operatorname{cis}\left(\frac{13\pi}{20}\right)$

5 Show the roots on an Argand diagram.



Roots of a complex number

The nth roots of $r\,[\cos(\theta) + i\sin(\theta)]$ are given by

$$z = r^{\frac{1}{n}}\left[\cos\left(\frac{2k\pi + \theta}{n}\right) + i\sin\left(\frac{2k\pi + \theta}{n}\right)\right] \text{ for } k \in Z.$$

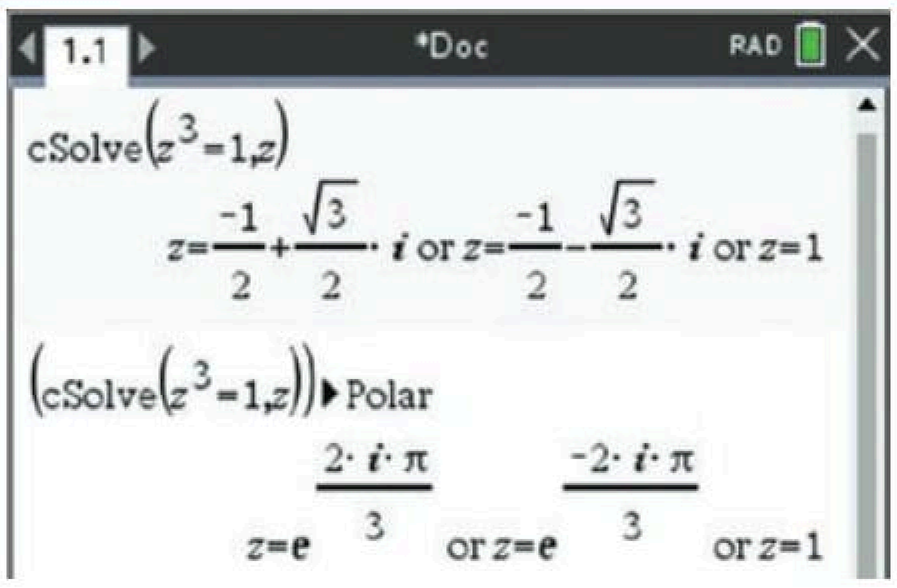
The roots have the same modulus and their arguments are separated by $\frac{2\pi}{n}$, so they are evenly spaced around the circle $|z| = r^{\frac{1}{n}}$.

The principal arguments are in the domain $(-\pi, \pi]$.

USING CAS 4 Roots of unity

Solve $z^3 = 1$ to find the roots of unity.

TI-Nspire



cSolve$(z^3=1,z)$

$$z=\frac{-1}{2}+\frac{\sqrt{3}}{2}\cdot i \text{ or } z=\frac{-1}{2}-\frac{\sqrt{3}}{2}\cdot i \text{ or } z=1$$

$(\text{cSolve}(z^3=1,z))$ ▸Polar

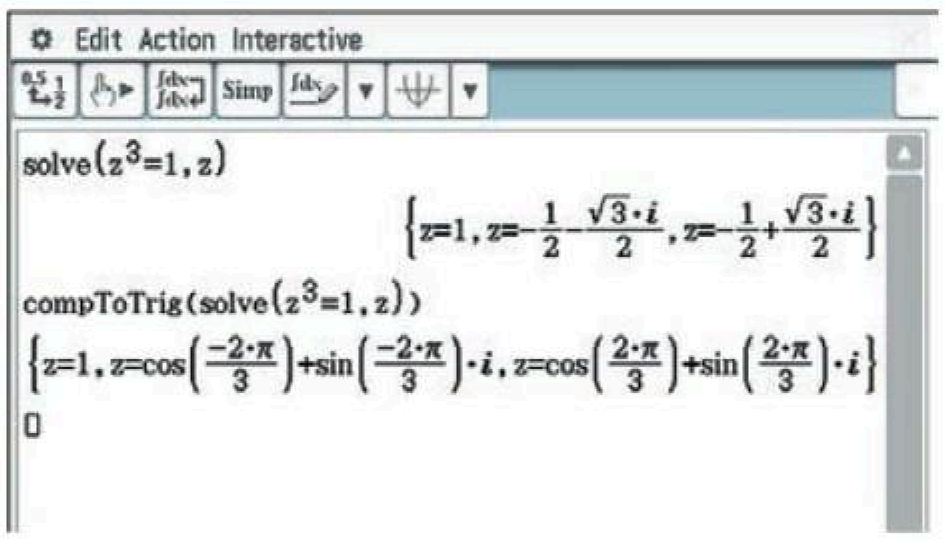
$$z=e^{\frac{2\cdot i\cdot \pi}{3}} \text{ or } z=e^{\frac{-2\cdot i\cdot \pi}{3}} \text{ or } z=1$$

1 Press **menu > Algebra > Complex > Solve**.

2 Enter z^3**=1,z**.

3 Press **menu > Number > Complex Number Tools > Convert to Polar** to express the solutions in polar form.

ClassPad



solve$(z^3=1,z)$

$$\left\{z=1, z=-\frac{1}{2}-\frac{\sqrt{3}\cdot i}{2}, z=-\frac{1}{2}+\frac{\sqrt{3}\cdot i}{2}\right\}$$

compToTrig(solve$(z^3=1,z)$)

$$\left\{z=1, z=\cos\left(\frac{-2\cdot\pi}{3}\right)+\sin\left(\frac{-2\cdot\pi}{3}\right)\cdot i, z=\cos\left(\frac{2\cdot\pi}{3}\right)+\sin\left(\frac{2\cdot\pi}{3}\right)\cdot i\right\}$$

1 Enter and highlight the equation z^3**=1**.

2 Tap **Interactive > Equation/Inequality > solve**, changing the variable to **z**.

3 Copy the equation and tap **Interactive > Complex > compToTrig** to express the solution in polar form.

The cube roots of unity are $z = 1, \operatorname{cis}\left(\frac{-2\pi}{3}\right), \operatorname{cis}\left(\frac{2\pi}{3}\right)$.

Use CAS for the roots of unity in Examination 2, but make sure you can still find powers by converting to polar form.

EXERCISE 4.4 Roots of unity

ANSWERS p. 580

Recap

1 $\left(\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)^7$ in polar form is equal to

A $\text{cis}\left(\frac{\pi}{3}\right)$ **B** $\text{cis}\left(-\frac{\pi}{3}\right)$ **C** $\text{cis}\left(-\frac{\pi}{6}\right)$ **D** $\text{cis}\left(-\frac{5\pi}{6}\right)$ **E** $\text{cis}\left(\frac{2\pi}{3}\right)$

2 $\frac{\left(\sqrt{3}-i\right)^5}{(2+2i)^3}$ in polar form is equal to

A $\sqrt{2}\,\text{cis}\left(\frac{\pi}{12}\right)$ **B** $\sqrt{2}\,\text{cis}\left(-\frac{\pi}{12}\right)$ **C** $\sqrt{2}\,\text{cis}\left(\frac{5\pi}{12}\right)$ **D** $\sqrt{2}\,\text{cis}\left(-\frac{11\pi}{12}\right)$ **E** $\sqrt{2}\,\text{cis}\left(\frac{11\pi}{12}\right)$

Mastery

3 WORKED EXAMPLE 10

a Solve $z^4 = 1$ and show the solutions on an Argand diagram.

b Find the 6th roots of unity in Cartesian form and show them on an Argand diagram.

c Solve $z^7 = 1$ and show the solutions on an Argand diagram.

4 WORKED EXAMPLE 11

a Show that $\frac{3\sqrt{2}}{2}+\frac{3\sqrt{2}}{2}i$ is a solution of $z^3 = -\frac{27\sqrt{2}}{2}+\frac{27\sqrt{2}}{2}i$ and find all the solutions in Cartesian form. Show the solutions on an Argand diagram.

b Show that $-2\sqrt{3}+2i$ is a solution of $z^4 = -128-128i\sqrt{3}$ and find the other solutions in Cartesian form. Show the solutions on an Argand diagram.

c Solve $z^3 = -8$ and show the solutions on an Argand diagram.

d Solve $z^4 = \frac{9}{2}-\frac{9\sqrt{3}}{2}i$ and show the solutions on an Argand diagram.

5 Using CAS 4 Solve equations to find the following roots.

a $\sqrt[3]{1}$ **b** $\sqrt[8]{1}$ **c** $\sqrt[4]{-4}$ **d** $\sqrt[6]{27}$ **e** $\sqrt[4]{-16}$

Exam practice

80–100% 60–79% 0–59%

6 © VCAA 2020 1Q3 54% TECH-FREE (3 marks) Find the cube roots of $\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$. Express your answers in polar form using principal values of the argument.

7 The solutions of $z^6 = 1$ are given by

A $\operatorname{cis}\left(\frac{\pi}{6}\right), \operatorname{cis}\left(\frac{\pi}{3}\right), i, \operatorname{cis}\left(\frac{2\pi}{3}\right), \operatorname{cis}\left(\frac{5\pi}{6}\right), 1$

B $\operatorname{cis}\left(\frac{\pi}{3}\right), \operatorname{cis}\left(\frac{2\pi}{3}\right), -1, \operatorname{cis}\left(\frac{4\pi}{3}\right), \operatorname{cis}\left(\frac{5\pi}{3}\right), 1$

C $i, \operatorname{cis}\left(\frac{2\pi}{3}\right), -1, \operatorname{cis}\left(\frac{4\pi}{3}\right), -i, 1$

D $\operatorname{cis}\left(\frac{2\pi}{3}\right), -i, \operatorname{cis}\left(\frac{4\pi}{3}\right), \operatorname{cis}\left(\frac{5\pi}{3}\right), i$

E $\operatorname{cis}\left(\frac{\pi}{6}\right), \operatorname{cis}\left(\frac{\pi}{3}\right), -i, \operatorname{cis}\left(\frac{2\pi}{3}\right), \operatorname{cis}\left(\frac{5\pi}{6}\right), 1$

8 © VCAA 2017 2AQ4 **53%** The solutions to $z^n = 1 + i, n \in Z^+$ are given by

A $2^{\frac{1}{2n}} \operatorname{cis}\left(\frac{\pi}{4n} + \frac{2\pi k}{n}\right), k \in R$

B $2^{\frac{1}{n}} \operatorname{cis}\left(\frac{\pi}{4n} + 2\pi k\right), k \in Z$

C $2^{\frac{1}{2n}} \operatorname{cis}\left(\frac{\pi}{4} + \frac{2\pi k}{n}\right), k \in R$

D $2^{\frac{1}{n}} \operatorname{cis}\left(\frac{\pi}{4n} + \frac{2\pi k}{n}\right), k \in Z$

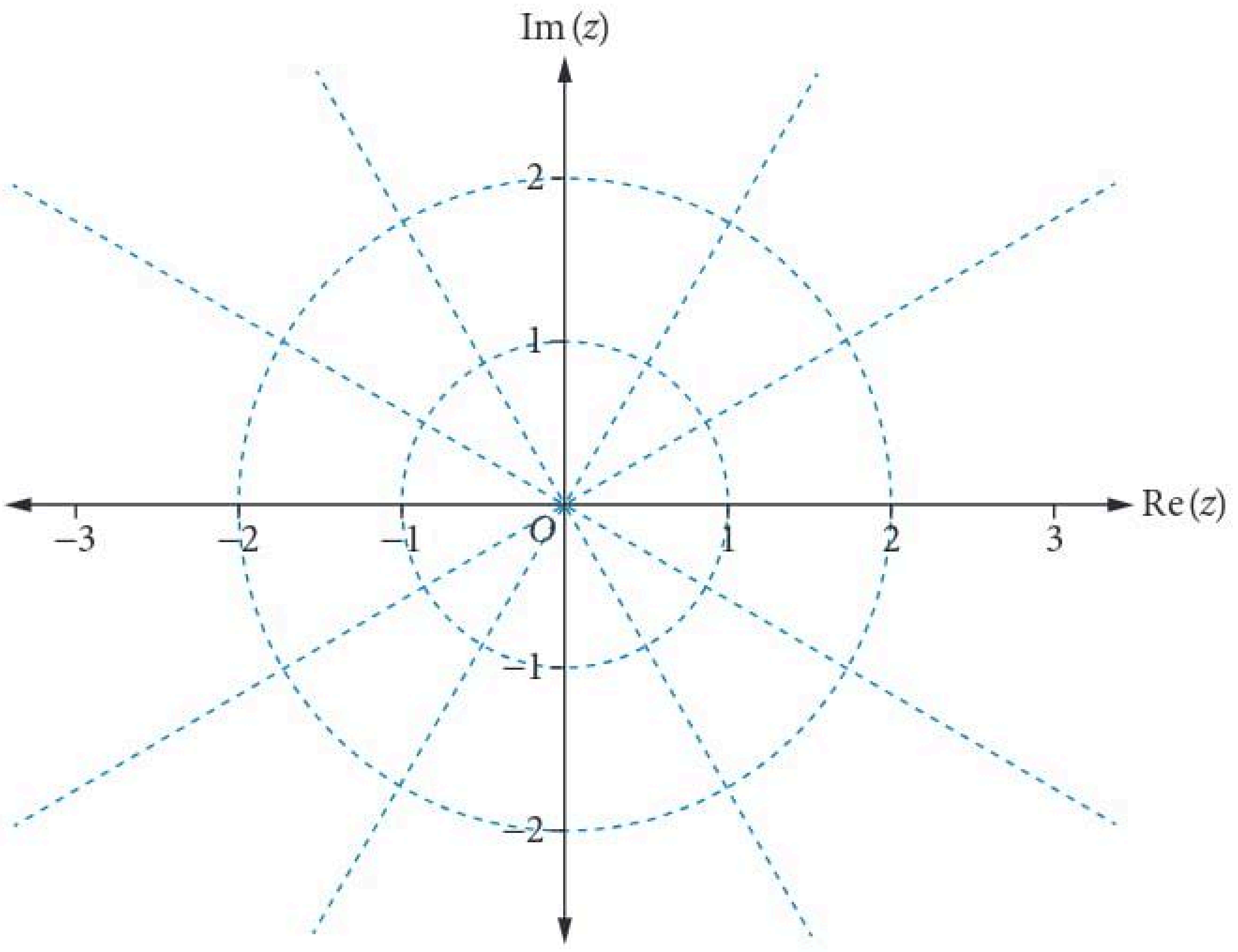
E $2^{\frac{1}{2n}} \operatorname{cis}\left(\frac{\pi}{4n} + \frac{2\pi k}{n}\right), k \in Z$

9 © VCAA 2016S 2BQ2 (11 marks) Let $u = \frac{1}{2} + \frac{\sqrt{3}}{2}i$.

a **i** Express u in polar form. 2 marks

ii Hence show that $u^6 = 1$. 1 mark

iii Copy the Argand diagram below and plot all roots of $z^6 - 1 = 0$ on it, labelling u and w, where $w = -u$. 3 marks



b **i** Copy the Argand diagram above and on it draw and label the subset of the complex plane given by $S = \{z: |z| = 1\}$. 1 mark

ii Draw and label the subset of the complex plane given by $T = \{z : |z - u| = |z + u|\}$ on your Argand diagram. 2 marks

iii Find the coordinates of the points of intersection between S and T. 2 marks

4.5 Factorising polynomials

Quadratic factorisation

We can use the **quadratic formula**, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, to factorise quadratics when the **discriminant**, $\Delta = b^2 - 4ac$, is *negative*. Since $\Delta < 0$, the roots are not real; they may be purely imaginary or complex.

For roots u and v, the quadratic equation will be of the form $k(z-u)(z-v) = 0$, where k is the coefficient of z^2. Expanding $k(z-u)(z-v)$ gives $k(z^2 - (u+v)z + uv)$. For an equation with real coefficients, k, $u+v$ and uv must all be real.

Suppose that $u = a + ib$ and $v = c + id$, where at least one root, say u, is complex.

Then $u + v = a + c + i(b + d) \in R \Rightarrow b + d = 0$, so $d = -b$.

$uv = (a + ib)(c + id) = (ac - bd) + i(bc + ad) \in R \Rightarrow bc + ad = 0$.

Substituting $d = -b$ gives $bc - ab = 0$, so $b(c - a) = 0$. For u complex, $b \neq 0$, so $c - a = 0$ and $c = a$.

This means that a real quadratic expression with real coefficients with at least one complex factor must be of the form $k(z-u)(z-v) = k(x - a - ib)(x - a + ib)$. The factors are **complex conjugates**.

Video playlist
Factorising polynomials

WB
p. 85

WORKED EXAMPLE 12 **Factorisation of quadratics with complex factors**

Factorise each quadratic expression over the complex numbers.

a $z^2 - 2z + 5$ **b** $2z^2 + 3z + 4$

Steps			**Working**
a	**1**	Complete the square.	$z^2 - 2z + 5 = z^2 - 2z + 1 - 1 + 5$ $= (z-1)^2 + 4$
	2	Use i to make a difference of squares.	$= (z-1)^2 - (-4)$ $= (z-1)^2 - (2i)^2$
	3	Complete the factorisation.	$= (z - 1 - 2i)(z - 1 + 2i)$
b	**1**	Use the quadratic formula to find the roots of $2z^2 + 3z + 4 = 0$. This can also be solved by completing the square.	For $2z^2 + 3z + 4 = 0$, $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $= \dfrac{-3 \pm \sqrt{(-3)^2 - 4 \times 2 \times 4}}{4}$ $= \dfrac{-3 \pm \sqrt{-23}}{4}$ $= \dfrac{-3 \pm \sqrt{23}i}{4}$ $= -\dfrac{3}{4} \pm \dfrac{\sqrt{23}}{4}i$
	2	Use the roots for the factorisation and simplify.	$2z^2 + 3z + 4$ $= 2\left[z - \left(-\dfrac{3}{4} + \dfrac{\sqrt{23}}{4}i\right)\right]\left[z - \left(-\dfrac{3}{4} - \dfrac{\sqrt{23}}{4}i\right)\right]$ $= 2\left(z + \dfrac{3}{4} - \dfrac{\sqrt{23}}{4}i\right)\left(z + \dfrac{3}{4} + \dfrac{\sqrt{23}}{4}i\right)$

Worksheets
Complex polynomials

Remainder and factor theorems

The factor and remainder theorems

A **complex polynomial** is similar to a real polynomial, except that the coefficients and variable are complex numbers.

A complex polynomial of degree n is of the form:

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \ldots + a_1 z^1 + a_0$$

where $a_n \neq 0$ and $z, a_n, a_{n-1}, a_{n-2} \ldots a_1, a_0 \in C$.

$a_n z^n$ is the **leading term** of the polynomial.

- If a polynomial $P(z)$ is divided by $D(z)$, then the **remainder** $R(z)$ is of degree less than $D(z)$.
- The **division identity** states that $P(z) = D(z)Q(z) + R(z)$.
- $P(z)$ is called the **dividend**, $D(z)$ the **divisor**, $Q(z)$ the **quotient** and $R(z)$ the **remainder**.
- The **remainder theorem** states that if a polynomial $P(z)$ is divided by $D(z) = z - a$ for $a \in C$, then the remainder is given by $R = P(a)$.
- The **factor theorem** states that for $a \in C$, $z - a$ is a factor of $P(z)$ if and only if $P(a) = 0$.

WB

p. 86

WORKED EXAMPLE 13 Finding a remainder

Find the remainder when $p(z) = z^4 + (2 - i)z^3 - 5z + 5 - 3i$ is divided by $z + 2 - i$.

Steps	**Working**
1 Find $p(-2 + i)$, using binomial expansions.	$p(-2+i) = (-2+i)^4 + (2-i)(-2+i)^3 - 5(-2+i) + 5 - 3i$ $= 16 - 32i + 24i^2 - 8i^3 + i^4 + (2-i)(-8 + 12i - 6i^2 + i^3)$ $+ 10 - 5i + 5 - 3i$ $= -7 - 24i + (2-i)(-2 + 11i) + 15 - 8i$ $= 8 - 32i + (-4 + 22i + 2i - 11i^2)$ $= 15 - 8i$
2 Write the answer.	The remainder is $15 - 8i$.

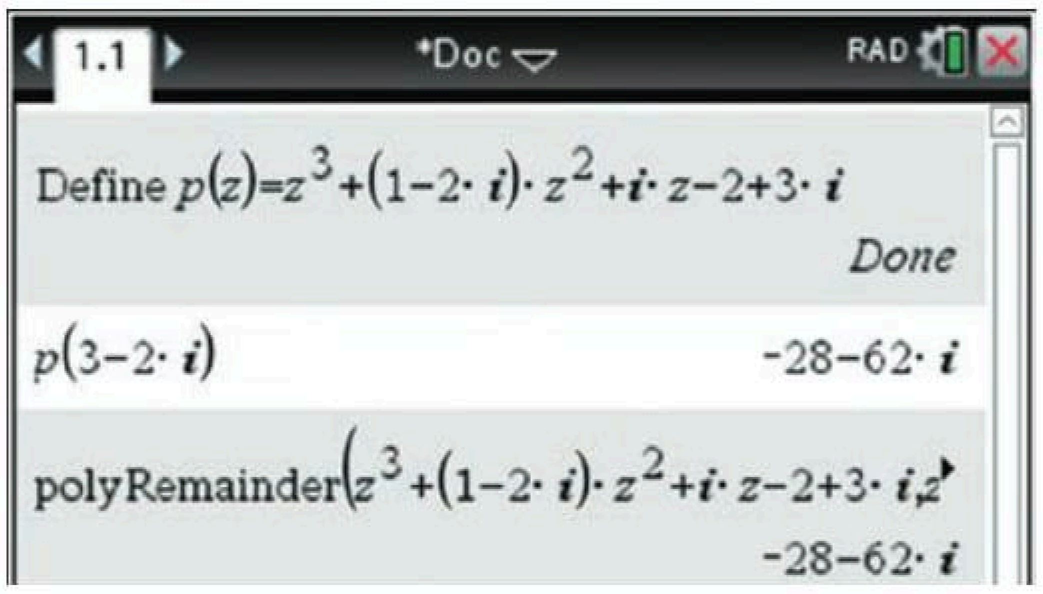
Exam hack

Use CAS to find the remainders. This is highly recommended in Examination 2, rather than finding $p(-2 + i)$ by hand.

USING CAS 5 Polynomial division remainders

Find the remainder when $z^3 + (1 - 2i)z^2 + iz - 2 + 3i$ is divided by $z - 3 + 2i$.

TI-Nspire



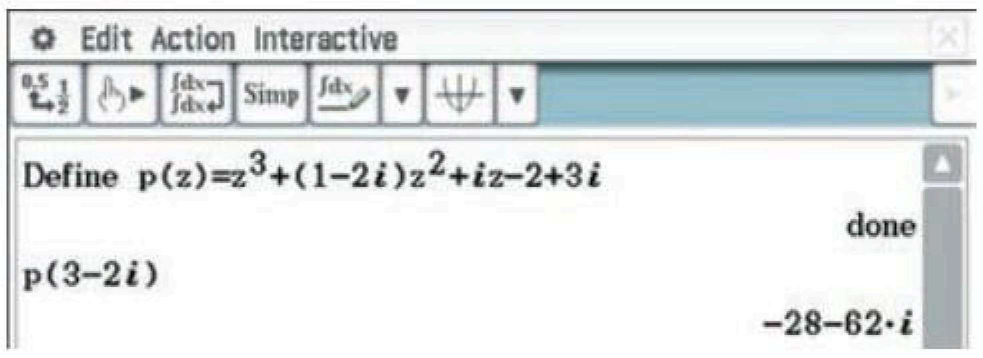
```
Define p(z)=z^3+(1-2·i)·z^2+i·z-2+3·i
                                   Done
p(3-2·i)                       -28-62·i
polyRemainder(z^3+(1-2·i)·z^2+i·z-2+3·i,z▸
                               -28-62·i
```

1 Define the polynomial, remembering to use ***i*** from the π mini-palette.

2 Enter the divisor from the factor ***p*(3 – 2*i*)**.

Alternatively, find the remainder using **Algebra > Polynomial Tools > Remainder of a Polynomial.**

ClassPad



```
Define p(z)=z^3+(1-2i)z^2+iz-2+3i
                                   done
p(3-2i)
                               -28-62·i
```

1 Define the polynomial, remembering to use ***i*** from the **Math2** or **Math3** keyboard.

2 Enter the divisor from the factor ***p*(3 – 2*i*)**.

The remainder is $-28 - 62i$.

WORKED EXAMPLE 14 **Finding factors**

Show that $z - 1 + i$ and $z - 2$ are both factors of $P(z) = z^3 - z^2 - 3(1 - i)z + 2 - 6i$ and find the remaining factor.

Steps	**Working**
1 Apply the factor theorem by showing that $P(1 - i) = 0$. **Exam hack** For 'show that' questions, you must show *detailed working*. Use the marks allocated as a guide to the amount of working or steps required.	$P(1 - i) = (1 - i)^3 - (1 - i)^2 - 3(1 - i)(1 - i) + 2 - 6i$ $= 1 - 3i + 3i^2 - i^3 - (1 - 2i + i^2) - 3(1 - 2i + i^2) + 2 - 6i$ $= 1 - 3i - 3 + i - (1 - 2i - 1) - 3(1 - 2i - 1) + 2 - 6i$ $= -2 - 2i - 1 + 2i + 1 - 3 + 6i + 3 + 2 - 6i$ $= 0$
2 Show that $P(2) = 0$ also.	$P(2) = 2^3 - 2^2 - 3(1 - i) \times 2 + 2 - 6i$ $= 8 - 4 - 6 + 6i + 2 - 6i$ $= 0$
3 State the result.	Since $P(1 - i) = 0$, $z - (1 - i)$ is a factor of $P(z)$. Since $P(2) = 0$, $z - 2$ is a factor of $P(z)$.
4 Identify the remaining factor.	Since z is a cubic where the coefficient of z^3 is 1, the last factor must be $z + a$, where $a \in C$.
5 Write $P(z)$ as a product.	$P(z) = (z - 1 + i)(z - 2)(z + a)$ $z^3 - z^2 - 3(1 - i)z + 2 - 6i = (z^2 - 3z + iz + 2 - 2i)(z + a)$
6 Find a.	Equating the constant term: $a(2 - 2i) = 2 - 6i$ $a = \dfrac{2 - 6i}{2 - 2i}$ $= \dfrac{1 - 3i}{1 - i} \times \dfrac{1 + i}{1 + i}$ $= 2 - i$
7 Write the remaining factor.	The remaining factor of $P(z)$ is $z + 2 - i$.

WB
p. 87

Use CAS to *solve* $(z - 1 + i)(z - 2)(z + a) = z^3 - z^2 - 3(1 - i)z + 2 - 6i$ to check the value of a.

The fundamental theorem of algebra

The polynomial equation $z^2 + 4 = 0$ can be written as $(z - 2i)(z + 2i) = 0$, so has 2 roots, $-2i$ and $2i$.

The polynomial equation $z^3 - z^2 - z + 1 = 0$ can be written as $(z - 1)(z - 1)(z + 1) = 0$, so it has 3 roots, 1, 1 and −1, of which 2 are equal.

The equation $z^4 - 1 = 0$ can be written as $(z - 1)(z + 1)(z - i)(z + i) = 0$, so it has 4 roots, 1, −1, i and $-i$, of which 2 are complex.

The fundamental theorem of algebra

The **fundamental theorem of algebra** states that any (non-constant) polynomial equation has at least one complex root. Therefore, any polynomial equation of degree n has n roots, some of which may be equal.

If the degree of $p(z)$ is 1, then $p(z) = mz + n$ and it has the root $-\frac{n}{m}$, so it has 1 root.

Suppose the degree of $p(z)$ is a polynomial equation $p(z) = 0$ of degree $n > 1$.

Then by the fundamental theorem it has at least one root, say $a_1 \in C$.

Then $(z - a_1)$ is a factor of $p(z)$, so

$p(z) = (z - a_1)p_1(z) = 0$, where $p_1(z)$ is a polynomial of degree $n - 1$.

If the degree of $p_1(z)$ is not zero, then it has at least one root, say $a_2 \in C$.

Then $(z - a_2)$ is a factor of $p_1(z)$, so $p(z) = (z - a_1)\ (z - a_2)p_2(z) = 0$, where $p_2(z)$ is a polynomial of degree $n - 2$.

We can continue like this until we have n roots and are left with a polynomial $p_n(z)$ of degree 0, which is a constant, say $k \in C$.

Thus $p(z) = (z - a_1)(z - a_2)(z - a_3) \ldots (z - a_n)k = 0$ and $p(z)$ has n roots.

Thus we can write any polynomial expression $p(z)$ of degree n as follows.

$p(z) = (z - a_1)(z - a_2)(z - a_3) \ldots (z - a_n)k$

We can take the factor k into one of the factors, say the first one, to get

$$p(z) = (kz - ka_1)(z - a_2)(z - a_3) \ldots (z - a_n)$$

Hence, any polynomial of degree n can be written as a product of n linear factors. Some of the factors may be complex.

Unfortunately, there are no practical methods of finding the factors for some polynomials.

Factorisation

The simplest complex polynomials are complex quadratics.

WB

p. 88

WORKED EXAMPLE 15 Factorising a complex quadratic

Factorise **a** $z^2 + i$ **b** $z^2 + (2 + 2i)z - 3 + 6i$

Steps	**Working**
a **1** Write as $z^2 = \ldots$ Write in general polar form to make it easier to find the square roots.	$z^2 = -i$ $= \text{cis}\left(\frac{3\pi}{2}\right) = \text{cis}\left(\frac{3\pi}{2} + 2k\pi\right), k \in Z$
2 Use de Moivre's theorem to find the general roots.	$z = \left(z^2\right)^{\frac{1}{2}}$ $= \left(\text{cis}\left(\frac{3\pi}{2} + 2k\pi\right)\right)^{\frac{1}{2}}, k \in Z$ $= \text{cis}\left(\frac{3\pi}{4} + k\pi\right), k \in Z$
3 Choose only the principal arguments.	$z = \text{cis}\left(-\frac{\pi}{4}\right), \text{cis}\left(\frac{3\pi}{4}\right)$ $= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$
4 Use the roots to write the answer.	$z^2 + i = \left(z - \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)\right)\left(z + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$ $= \left(z - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)\left(z + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$

b	**1**	Try the factors of $-3 + 6i$.	Let $p(z) = z^2 + (2 + 2i)z - 3 + 6i$.
			$p(3) = 3^2 + 3(2 + 2i) - 3 + 6i$
			$= 9 + 6 + 6i - 3 + 6i$
			$\neq 0$
			$p(-3) = (-3)^2 - 3(2 + 2i) - 3 + 6i$
			$= 9 - 6 - 6i - 3 + 6i$
			$= 0$
	2	Use the factor theorem.	$(z - (-3))$ is a factor of $p(z)$.
	3	Write $p(z)$ as factors.	$p(z) = (z + 3)(z + a + bi)$
	4	Expand.	$= z^2 + az + bzi + 3z + 3a + 3bi$
			$= z^2 + (a + 3 + bi)z + 3a + 3bi$
	5	Equate real and imaginary parts to solve for a and b.	$a + 3 = 2$, $b = 2$, $3a = -3$ and $3b = 6$
			$a = -1$ and $b = 2$
	6	Write the answer.	$z^2 + (2 + 2i)z - 3 + 6i = (z + 3)(z - 1 + 2i)$

Unless a polynomial has only the leading term and a constant, look for factors when factorising complex polynomials. You might be able to use grouping instead of the remainder theorem.

WORKED EXAMPLE 16 Factorising a complex cubic

WB p. 89

Factorise **a** $z^3 + 3z^2i - 2z^2 - z + 2 - 3i$ **b** $z^3 + (3 - i)z^2 + (4 - 5i)z + 2 - 4i$

Steps			**Working**
a	**1**	Group terms to get a common factor.	$z^3 + 3z^2i - 2z^2 - z + 2 - 3i$
			$= z^2(z - 2 + 3i) - 1(z - 2 + 3i)$
	2	Factorise.	$= (z^2 - 1)(z - 2 + 3i)$
			$= (z - 1)(z + 1)(z - 2 + 3i)$
b	**1**	Try some values to get $p(a) = 0$.	Let $p(z) = z^3 + (3 - i)z^2 + (4 - 5i)z + 2 - 4i$
			$p(1) = 1 + (3 - i) + (4 - 5i) + 2 - 4i = 10 - 10i$
			$p(-1) = -1 + (3 - i) - (4 - 5i) + 2 - 4i = 0$
	2	State the factor.	Since $p(-1) = 0$, $z + 1$ is a factor.
	3	Write $p(z)$ as a product of factors and expand.	$p(z) = (z + 1)(z^2 + az + b)$
			$z^3 + (3 - i)z^2 + (4 - 5i)z + 2 - 4i$
			$= z^3 + az^2 + bz + z^2 + az + b$
			$= z^3 + (a + 1)z^2 + (a + b)z + b$
	4	Use the constant term to find b.	$b = 2 - 4i$
	5	Use the z^2 term to find a.	$a + 1 = 3 - i$
			$a = 2 - i$
	6	Check using the z term.	$a + b = 2 - i + 2 - 4i = 4 - 5i$, which is correct for z.
	7	Write the partial result.	$p(z) = (z + 1)q(z)$, where $q(z) = z^2 + (2 - i)z + 2 - 4i$.
	8	Try some values to get $q(a) = 0$.	$q(-1) = 1 - (2 - i) + 2 - 4i = 1 - 3i$
			$q(i) = -1 + (2 - i)i + 2 - 4i = 2 - 2i$
			$q(2) = 4 + 2(2 - i) + 2 - 4i = 10 - 6i$
			$q(-2) = 4 - 2(2 - i) + 2 - 4i = 2 - 2i$
			$q(2i) = -4 + 2i(2 - i) + 2 - 4i = 0$

9	State the factor.	Since $q(2i) = 0$, $z - 2i$ is a factor of $q(z)$.
10	Write $q(z)$ as a product of factors and expand.	$q(z) = (z - 2i)(z + c)$ $z^2 + (2 - i)z + 2 - 4i = z^2 + (c - 2i)z - 2ic$
11	Use the constant term to find c.	$-2ic = 2 - 4i$ $c = \dfrac{2-4i}{-2i} = \dfrac{2-4i}{-2i} \times \dfrac{i}{i} = 2 + i$
12	Write the answer.	$z^3 + (3 - i)\,z^2 + (4 - 5i)z + 2 - 4i$ $= (z + 1)\,(z - 2i)(z + 2 + i)$

Exam hack

In Examination 2, you can use your calculator to find $z^2 + (2 - i)z + 2 - 4i$ by polynomial division.

EXERCISE 4.5 Factorising polynomials

ANSWERS p. 581

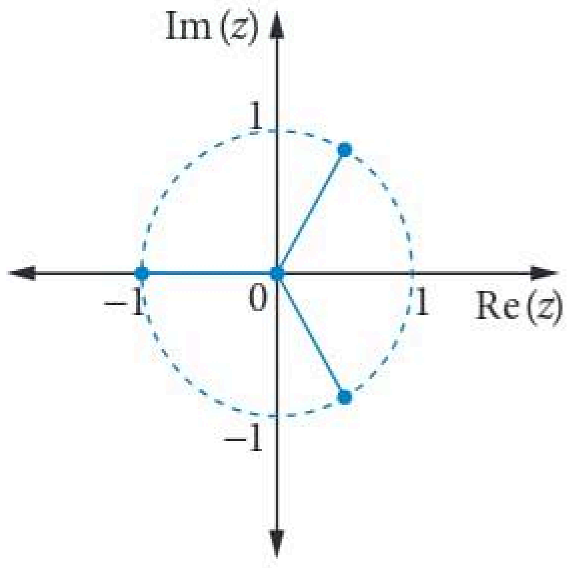
Recap

1 One of the cube roots of $-i$ is

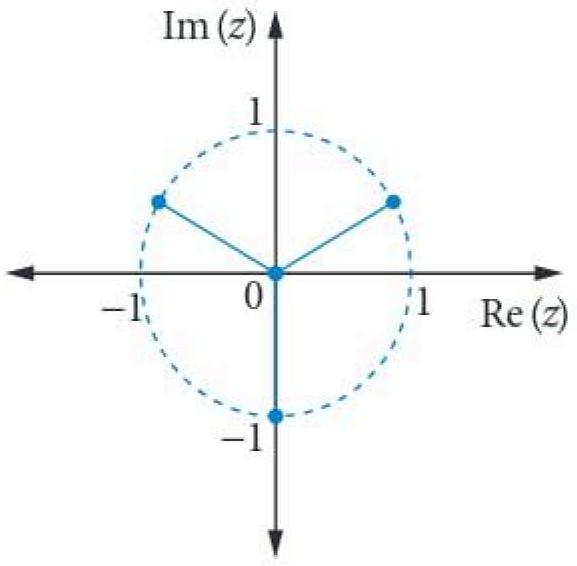
A $-i$ **B** $\frac{\sqrt{3}}{2} + \frac{1}{2}i$ **C** $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$ **D** i **E** $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$

2 Which diagram shows the solutions of $z^3 - i = 0$ on an Argand diagram?

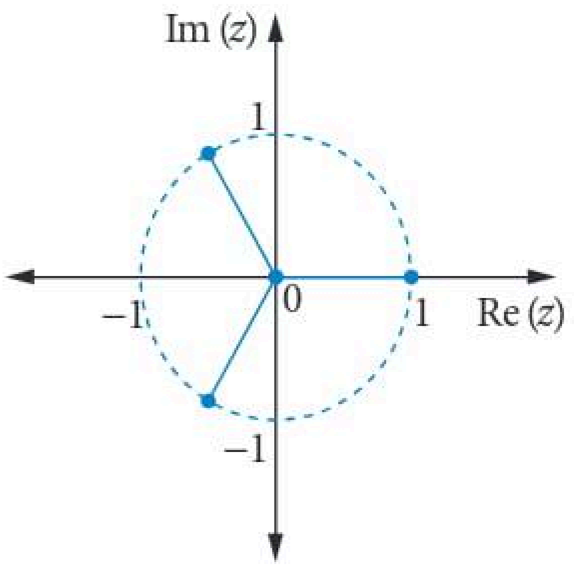
A



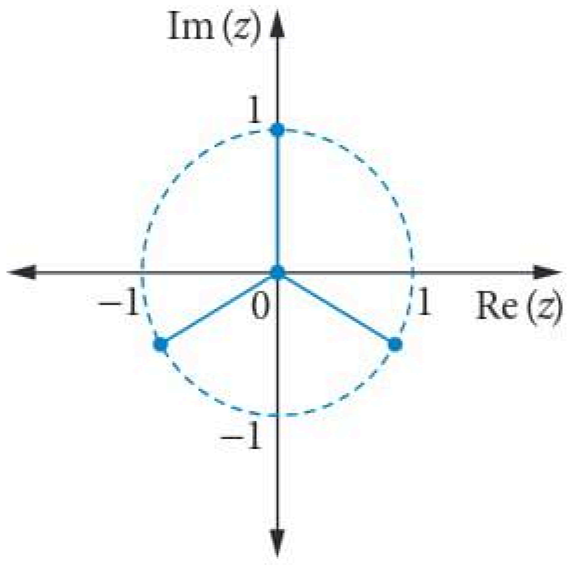
B



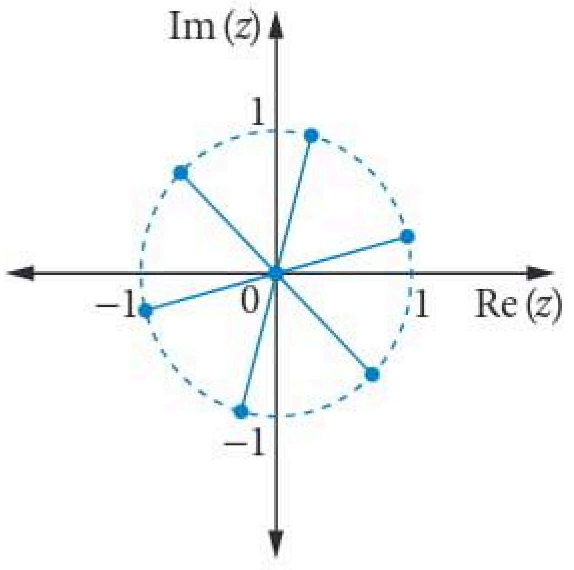
C



D



E



Mastery

3 WORKED EXAMPLE 12 Factorise each expression.

a $z^2 + 4z + 6$ **b** $z^2 - 5z + 7$ **c** $3z^2 - 2z + 1$

d $2z^2 + 3z + 5$ **e** $4z^2 - z + 1$

4 WORKED EXAMPLE 13 Find the remainder when

a $p(z) = z^2 + (2 - i)z - 3 + 2i$ is divided by $z - 2 + i$.

b $p(z) = z^3 + (1 + 3i)z^2 + (-2 + i)z + 2i - 1$ is divided by $z + 1 + 2i$.

c $p(z) = 2z^3 + (-2 - 2i)z^2 + (3 + i)z - 3i + 4$ is divided by $z - 3 - 2i$.

d $p(z) = z^4 - 3z^3 + (5 + 6i)z^2 - 3z + 4 + 6i$ is divided by $z - 2i$.

e $p(z) = z^4 + 2iz^3 - 3z^2 + (1 + i)z - 2 - 3i$ is divided by $z + 2 - 2i$.

5 Using CAS 5 Find the remainder when

a $z^3 - 3iz^2 + 2$ is divided by $z - 1$.

b $z^3 + (2 - i)z + 2 + 3i$ is divided by $z + 2i$.

c $z^3 - (3 - 2i)z^2 + 4z - 3i$ is divided by $z - 2 + i$.

d $2z^3 + 4iz^2 - 3z + i - 3$ is divided by $z + 1 + 2i$.

e $3z^3 + (2 - i)z^2 - 3z - 4 + i$ is divided by $z + 2 - i$.

6 WORKED EXAMPLE 14

a Show that $z + 1 - 2i$ is a factor of $z^2 - (1 + 2i)z - 2 + 4i$ and find the remaining factor.

b Show that $z + 2$ and $z - 3 + i$ are factors of $z^3 - iz^2 + (-5 + 5i)z - 2 + 14i$ and find the remaining factor.

c Show that $z - 1 + 2i$ and $z + 3 - i$ are factors of $z^3 + (-1 - i)z^2 - 5z + 17 - 19i$ and find the remaining factor.

d Show that $z - 2 - i$ is a factor of $3z^3 - (6 + 5i)z^2 - (1 - 4i)z - 2 - i$ and find the remaining factors.

e Show that $z + 2 - 2i$ and $z - 2 + i$ are factors of $z^4 + (2 - 2i)z^3 + (-3 + 2i)z^2 + 14zi + 12 + 4i$ and find the remaining factors.

7 WORKED EXAMPLE 15 Factorise each expression.

a $z^2 - i$ **b** $z^6 + 1$ **c** $z^4 - 1$ **d** $z^2 - \frac{1}{2} + \frac{\sqrt{3}}{2}i$ **e** $z^3 - i$

8 Factorise each expression.

a $z^2 - 2iz - 2z - 3 - 2i$ **b** $z^2 - 4z - zi + 4 + 2i$ **c** $z^2 + (3 - 2i)z + 3 + 3i$

d $z^2 + (1 + i)z + 6 - 2i$ **e** $z^2 - (5 + 2i)z + 6 + 6i$

9 WORKED EXAMPLE 16 Factorise each polynomial.

a $3z^3 - 2z^2 + iz^2 - 3z + 2 - i$

b $2z^3 + 2z^2 - 4z^2i - 2iz - 2i - 4$

c $z^3 + (1 - 3i)z^2 + (-4 - 5i)z - 4 + 2i$

d $z^3 + (3 - 2i)z^2 + (1 - 3i)z + 6 - 2i$

e $z^4 + z^3i - 4iz^2 + (-4 + 8i)z - 8 - 8i$

Exam practice

80–100% 60–79% 0–59%

10 © VCAA 2010 1Q1 75% TECH-FREE (3 marks) Consider $f(z) = z^3 + 9z^2 + 28z + 20,\ z \in C$.

Given that $f(-1) = 0$, factorise $f(z)$ over C.

11 © VCAA 2020 2AQ6 79% For the complex polynomial $P(z) = z^3 + az^2 + bz + c$ with real coefficients a, b and c, $P(-2) = 0$ and $P(3i) = 0$. The values of a, b and c are respectively

A $-2, 9, -18$ **B** $3, 4, 12$ **C** $2, 9, 18$ **D** $-3, -4, 12$ **E** $2, -9, -18$

12 What is the remainder when $z^3 - 3iz^2 + (2 - i)z - 4 + 3i$ is divided by $z - 1 + 2i$?

A $-5 + 15i$ **B** $1 + 13i$ **C** $-27 + 9i$ **D** $-1 + 11i$ **E** $-27 + 13i$

13 Which of the following are factors of $z^3 + (1 + i)z^2 + (-6 + i)z - 6i$?

I $z + i$ **II** $z + 2i$ **III** $z + 3$

A I, II and III **B** III only **C** I only **D** I and III **E** I and II

14 $p(z) = z^3 + (2i - 5)z^2 + (9 - 8i)z - 5 + 10i$. Given that $p(1 - 2i) = 0$, $p(z)$ factorises to

A $(z + 1 - 2i)(z - 2 + i)(z - 2 - i)$ **B** $(z - 1 + 2i)(z - 2 + i)(z - 2 - i)$

C $(z - 1 + 2i)(z + 2 + 3i)(z + 2 - 3i)$ **D** $(z - 1 + 2i)(z - 1)(z + 5)$

E $(z + 1 - 2i)(z - 1)(z + 5)$

Video playlist Solving polynomial equations

4.6 Solving polynomial equations

Quadratic equations

Complex roots of real quadratic equations must be complex conjugates. Quadratic equations with no z-term are solved using the roots of complex numbers. If necessary, we can always use the quadratic formula for real quadratic equations.

WB p. 90

WORKED EXAMPLE 17 Real quadratic equations with complex solutions

Solve each quadratic equation.

a $z^2 + 64 = 0$ **b** $z^2 - 2z + 3 = 0$ **c** $3z^2 + 10z + 15 = 0$

Steps			Working
a	**1**	Rearrange.	$z^2 + 64 = 0$ $z^2 = -64$ $z = \pm\sqrt{-64}$
	2	Separate −1 and simplify to get i.	$z = \pm\sqrt{64} \times \sqrt{-1}$ $= \pm 8i$
	3	Write the answer.	The solutions are $\pm 8i$.
b	**1**	Solve by completing the square. This can also be solved by using the quadratic formula.	$z^2 - 2z + 3 = 0$ $z^2 - 2z + 1 - 1 + 3 = 0$ $(z - 1)^2 + 2 = 0$ $(z - 1)^2 = -2$ $z - 1 = \pm\sqrt{-2}$
	2	Write $\sqrt{-2}$ as $\sqrt{2}i$ and solve.	$z - 1 = \pm\sqrt{2}i$ $z = 1 \pm \sqrt{2}i$
	3	Write the answer.	The solutions are $1 \pm \sqrt{2}i$.

c	**1** Solve using the quadratic formula. This can also be solved by completing the square.	$3z^2+10z+15=0$ $z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $=\frac{-10\pm\sqrt{10^2-4\times3\times15}}{2\times3}$ $=\frac{-10\pm\sqrt{-80}}{6}$
	2 Simplify the surd.	$=\frac{-10\pm4\sqrt{-5}}{6}$
	3 Write $\sqrt{-5}$ as $\sqrt{5}\,i$ and simplify.	$=\frac{-10\pm4\sqrt{5}i}{6}$ $=\frac{-5\pm2\sqrt{5}i}{3}$
	4 Write the answer.	The solutions are $z=\frac{-5\pm2\sqrt{5}i}{3}$.

If a real quadratic equation has a complex root, the other root must be its complex conjugate.

WORKED EXAMPLE 18 Finding a real quadratic equation from a complex root

WB p. 91

One of the roots of a quadratic equation with real coefficients is $z=\frac{3}{4}-2\sqrt{2}i$. Find a possible equation.

Steps	**Working**
1 Write the other root.	The other root is $\bar{z}=\frac{3}{4}+2\sqrt{2}i$.
2 Write a quadratic equation with these roots.	The equation is $(x-z)(x-\bar{z})=0$.
3 Expand and simplify.	$x^2-(z+\bar{z})x+z\bar{z}=0$
4 Find $z+\bar{z}$.	$z+\bar{z}=\frac{3}{4}-2\sqrt{2}i+\frac{3}{4}+2\sqrt{2}i$ $=\frac{6}{4}=\frac{3}{2}$
5 Find $z\bar{z}$.	$z\bar{z}=\left(\frac{3}{4}-2\sqrt{2}i\right)\left(\frac{3}{4}+2\sqrt{2}i\right)$ $=\left(\frac{3}{4}\right)^2-\left(2\sqrt{2}i\right)^2$ $=\frac{9}{16}-(-8)=\frac{137}{16}$
6 Substitute $z+\bar{z}$ and $z\bar{z}$ into the equation and simplify.	$x^2-\frac{3}{2}x+\frac{137}{16}=0$ $16x^2-24x+137=0$

Note that the equation in the answer is a possible equation. Any multiple of this equation is also a solution.

As shown in the above example, the quadratic equation $x^2-(\alpha+\beta)x+\alpha\beta=0$ has roots α and β.

We solve complex quadratic equations similarly to real quadratic equations. If the coefficient of z^2 includes i, multiply the whole equation by i before starting to solve it. If the discriminant of a quadratic equation is complex, it is easier to complete the square than to use the quadratic formula.

WB

p. 92

WORKED EXAMPLE 19 Complex quadratic equations

Solve the equations:

a $iz^2+8=0$ **b** $3iz^2+2z+i=0$ **c** $z^2+2iz-i\sqrt{3}=0$

Steps			Working
a	**1**	The coefficient of z includes i, so multiply by i.	$i^2z^2+8i=0i$ $-z^2+8i=0$
	2	Write as $z^2=\dots$	$z^2=8i$
	3	Write in general polar form.	$z^2=8\operatorname{cis}\left(\frac{\pi}{2}+2k\pi\right),\ k\in Z$
	4	Use de Moivre's theorem.	$z=\sqrt{8}\operatorname{cis}\left(\frac{\pi}{4}+k\pi\right),\ k\in Z$
	5	Use only the principal roots.	$z=2\sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4}\right),\ 2\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$
	6	Change to Cartesian form and simplify.	$=2\sqrt{2}\left(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right),\ 2\sqrt{2}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)$ $=-2-2i,\ 2+2i$
b	**1**	The coefficient of z includes i, so multiply by i.	$3i^2z^2+2iz+i^2=0$ $-3z^2+2iz-1=0$ $3z^2-2iz+1=0$
	2	Check the discriminant.	$\Delta=b^2-4ac$ $=(-2i)^2-4\times3\times1$ $=-16$
	3	The discriminant is real, so use the quadratic formula. Don't re-calculate the discriminant, just use $\sqrt{-16}=4i$.	$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $=\frac{2i\pm4i}{6}$
	4	Simplify.	$z=i,-\frac{i}{3}$
c	**1**	Check the discriminant.	$\Delta=b^2-4ac$ $=(2i)^2-4\times1\times\left(-i\sqrt{3}\right)$ $=-4+4\sqrt{3}i$
	2	The discriminant is complex, so complete the square.	$z^2+2iz-i\sqrt{3}=0$ $z^2+2iz=i\sqrt{3}$
	3	Add the square of half the z term to each side.	$z^2+2iz+i^2=i\sqrt{3}+i^2$
	4	Write the perfect square.	$(z+i)^2=-1+\sqrt{3}i$
	5	Write the RHS in general polar form.	$(z+i)^2=2\operatorname{cis}\left(\frac{2\pi}{3}+2k\pi\right),\ k\in Z$

6	Use de Moivre's theorem.	$z + i = \sqrt{2}\,\text{cis}\left(\frac{\pi}{3} + k\pi\right), k \in Z$
7	Use only the principal values.	$z + i = \sqrt{2}\,\text{cis}\left(\frac{\pi}{3}\right), \sqrt{2}\,\text{cis}\left(-\frac{2\pi}{3}\right)$
8	Simplify.	$z = \sqrt{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) - i, \sqrt{2}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) - i$ $= \frac{\sqrt{2}}{2} + \frac{\sqrt{6} - 2}{2}i, -\frac{\sqrt{2}}{2} - \frac{\sqrt{6} + 2}{2}i$

The conjugate root theorem

Many polynomial equations with real coefficients have some complex solutions.

A real cubic polynomial function has a graph that crosses the x-axis, so it must have at least one real zero. This means that a real cubic equation must have at least one real root. Similarly, other real polynomial equations with odd degree must have at least one real root.

Since the graph of a real polynomial function of even degree might not cross the x-axis, real polynomials of even degree might not have any real roots.

The **conjugate root theorem** states that if w is a complex root of a real polynomial, then its conjugate $\overline{w}$ is also a root of the polynomial.

Proof

Suppose $p(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \ldots + a_1 z^1 + a_0$, where $a_n, a_{n-1}, a_{n-2} \ldots a_1, a_0 \in R$ and that w is a complex root of the equation $p(z) = 0$.

Then $a_n w^n + a_{n-1} w^{n-1} + a_{n-2} w^{n-2} + \ldots + a_1 w^1 + a_0 = 0$,

so $\overline{a_n w^n + a_{n-1} w^{n-1} + a_{n-2} w^{n-2} + \ldots + a_1 w^1 + a_0} = \overline{0}$

$\overline{a_n w^n} + \overline{a_{n-1} w^{n-1}} + \overline{a_{n-2} w^{n-2}} + \ldots + \overline{a_1 w^1} + \overline{a_0} = 0$ as $\overline{z + w} = \overline{z} + \overline{w}$ and $\overline{0} = 0$

$\overline{a_n}\,\overline{w^n} + \overline{a_{n-1}}\,\overline{w^{n-1}} + \overline{a_{n-2}}\,\overline{w^{n-2}} + \ldots + \overline{a_1}\,\overline{w^1} + \overline{a_0} = 0$ as $\overline{zw} = \overline{z}\,\overline{w}$

$a_n \overline{w^n} + a_{n-1} \overline{w^{n-1}} + a_{n-2} \overline{w^{n-2}} + \ldots + a_1 \overline{w^1} + a_0 = 0$ as $a_n, a_{n-1}, a_{n-2} \ldots a_1, a_0 \in R$

$a_n (\overline{w})^n + a_{n-1} (\overline{w})^{n-1} + a_{n-2} (\overline{w})^{n-2} + \ldots + a_1 (\overline{w})^1 + a_0 = 0$ as $\overline{zw} = \overline{z}\,\overline{w}$.

Thus, $\overline{w}$ is also a root of the equation.

WORKED EXAMPLE 20 Finding a conjugate root

WB p. 94

One root of a polynomial equation with real coefficients is $4 - 3i$.

State another root of the polynomial.

Steps	**Working**
Give the conjugate.	By the conjugate root theorem, $4 + 3i$ is also a root.

Polynomial equations

Worksheets
Polynomials with real coefficients

Real and imaginary factors

Complex roots of cubics or higher-order polynomial equations with real coefficients occur in conjugate pairs. This means that there can only be an even number of complex roots, so polynomial equations of odd degree must have at least one real root.

WB
p. 95

WORKED EXAMPLE 21 Solving a cubic equation with real coefficients

$p(z) = 2z^3 - 11z^2 + 22z - 15$ and $p(2 - i) = 0$. Solve $p(z) = 0$.

Steps	Working
1 Use the factor theorem.	Since $p(2 - i) = 0$, one root is $2 - i$.
2 Use the conjugate root theorem.	The conjugate, $2 + i$ is a second root.
3 Write $p(z)$ as a product.	$p(z) = (z - 2 + i)(z - 2 - i)(az + b)$ $2z^3 - 11z^2 + 22z - 15 = (z^2 + 4z + 5)(az + b)$ Equating the z^3 terms: $2 = a$ Equating the constant terms: $-15 = 5b$, so $b = -3$ $p(z) = (z^2 + 4z + 5)(2z - 3)$
4 Solve the equation.	$p(z) = (z - 2 - i)(z - 2 + i)(2z - 3) = 0$ $z = 2 - i,\ z = 2 + i$ or $z = \frac{3}{2}$

Exam hack

The product of the z terms in the factors is the z term in the polynomial and the product of the constant terms is the constant in the polynomial, as shown in Step 4.

We can use the factor theorem to find some roots of a polynomial equation. Try factors of the constant term.

WB
p. 96

WORKED EXAMPLE 22 Using the factor theorem to solve a real polynomial equation

Solve $3z^4 - 3z^3 - 14z^2 = 11z + 3$.

Steps	Working
1 Write the equation as $p(z) = 0$.	Let $p(z) = 3z^4 - 3z^3 - 14z^2 - 11z - 3 = 0$.
2 Try factors of the constant -3.	$p(1) = -28,\ p(-1) = 0,\ p(3) = 0$ $z + 1$ and $z - 3$ are factors of $p(z)$.
3 Write $p(z)$ as a product.	$p(z) = (z + 1)(z - 3)(az^2 + bz + c)$ $3z^4 - 3z^3 - 14z^2 - 11z - 3 = (z^2 - 2z - 3)(az^2 + bz + c)$ Equating the z^4 terms: $3 = a$. Equating the constant terms: $-3 = -3c$, so $c = 1$ $3z^4 - 3z^3 - 14z^2 - 11z - 3 = (z^2 - 2z - 3)(3z^2 + bz + 1)$. Equating the z terms: $-11 = -2 - 3b$, so $b = 3$. Therefore, $p(z) = (z + 1)(z - 3)(3z^2 + 3z + 1)$.
4 Solve $p(z) = 0$, using the quadratic formula for the last factor.	$z = -1,\ z = 3,\ z = \frac{-3 \pm \sqrt{9 - 12}}{6} = \frac{-3 \pm \sqrt{3}i}{6}$

5	Write the solutions.	The solutions are $z = -1, 3, -\frac{1}{2} + \frac{\sqrt{3}i}{6}, -\frac{1}{2} - \frac{\sqrt{3}i}{6}$.

Exam hack

Use the z^2 and z^3 terms to check the value of b.

We can apply the same techniques to solve equations with complex coefficients, but the conjugate root theorem *does not* apply.

A polynomial with complex coefficients must have at least one complex root, but they are not necessarily all complex.

WB

p. 97

WORKED EXAMPLE 23 Solving a complex cubic equation

Solve $z^3 + (2i - 2)z^2 + (2i - 5)z + 6 - 4i = 0$.

	Steps	Working
1	Write the polynomial function.	Let $p(z) = z^3 + (2i - 2)z^2 + (2i - 5)z + 6 - 4i$.
2	Try factors of the constant $6 - 4i$.	$p(1) = 0$, $p(-2) = 0$ so $z - 1$, $z + 2$ are factors.
3	Write $p(z)$ as a product.	$p(z) = (z - 1)(z + 2)(z + a)$ $z^3 + (2i - 2)z^2 + (2i - 5)z + 6 - 4i = (z^2 + z - 2)(z + a)$ Equating the constant terms: $6 - 4i = -2a$, so $a = -3 + 2i$. Therefore, $p(z) = (z - 1)(z + 2)[z + (-3) + 2i]$.
4	Solve $p(z) = 0$.	$z = 1, z = -2$ or $z = 3 - 2i$

In this case, it is important to try all the factors of the constant $6 - 4i$, and not stop at the first one.

Otherwise, we get $p(z) = (z - 1)[z^2 + (2i - 1)z - 6 + 4i]$. If we use the quadratic formula on the second factor, we get $z = \frac{1 - 2i \pm \sqrt{21 - 20i}}{2}$, which does simplify to the correct answers, but is quite difficult.

WB

p. 98

WORKED EXAMPLE 24 Solving a complex quartic equation

Solve $6iz^4 + 4iz^2 = 7z^3 + 7z + 2i$.

	Steps	Working
1	Write the polynomial function.	Let $p(z) = 6iz^4 - 7z^3 + 4iz^2 - 7z - 2i$.
2	Try factors of the constant $2i$.	$p(1) = -14 + 8i$, $p(-1) = 14 + 8i$, $p(2) = -70 + 110i$, $p(-2) = 70 + 110i$ $p(i) = 0$, $p(-i) = 0$, so $(z + i)$ and $(z - i)$ are factors.
3	Write $p(z)$ as a product.	$p(z) = (z + i)(z - i)(az^2 + bz + c)$ $6iz^4 - 7z^3 + 4iz^2 - 7z - 2i = (z^2 + 1)(az^2 + bz + c)$ Equating the z^4 terms: $6i = a$. Equating the constant terms: $-2i = c$ $6iz^4 - 7z^3 + 4iz^2 - 7z - 2i = (z^2 + 1)(6iz^2 + bz - 2i)$. Equating the z terms: $-7 = b$ So $p(z) = (z + i)(z - i)(6iz^2 - 7z - 2i)$.

4 Solve the quadratic equation.

$$6iz^2 - 7z - 2i = 0$$

$$z = \frac{7 \pm \sqrt{49 - 4 \times 6i \times (-2i)}}{12i} = \frac{7 \pm 1}{12i} = \frac{6}{12i} \text{ or } \frac{8}{12i}$$

$$z = \frac{1}{2i} \text{ or } \frac{2}{3i}$$

$$z = \frac{i}{-2} \text{ or } \frac{2i}{-3}$$

5 Write all the solutions of $p(z) = 0$.

The solutions are $z = i$, $z = -i$, $z = -\frac{1}{2}i$, $z = -\frac{2}{3}i$.

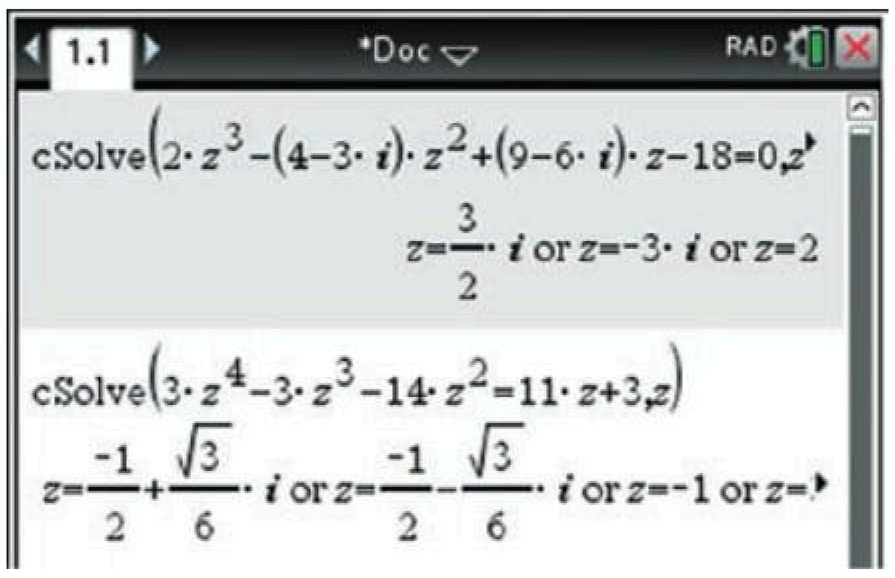
Exam hack

You might find it easier to solve $6iz^4 + 4iz^2 = 7z^3 + 7z + 2i$ if you make the leading term real and reduce the number of imaginary terms, and $i(6iz^4 + 4iz^2) = i(7z^3 + 7z + 2i)$ gives $-6z^4 - 4z^2 = 7iz^3 + 7iz - 2$, so $p(z) = 6z^4 + 7iz^3 + 4z^2 + 7iz - 2$.

USING CAS 6 Solving polynomial equations

Solve: $2z^3 - (4 - 3i)z^2 + (9 - 6i)z - 18 = 0$ and $3z^4 - 3z^3 - 14z^2 = 11z + 3$.

TI-Nspire



$\text{cSolve}\left(2\cdot z^3-(4-3\cdot i)\cdot z^2+(9-6\cdot i)\cdot z-18=0,z\right.$

$z=\frac{3}{2}\cdot i \text{ or } z=-3\cdot i \text{ or } z=2$

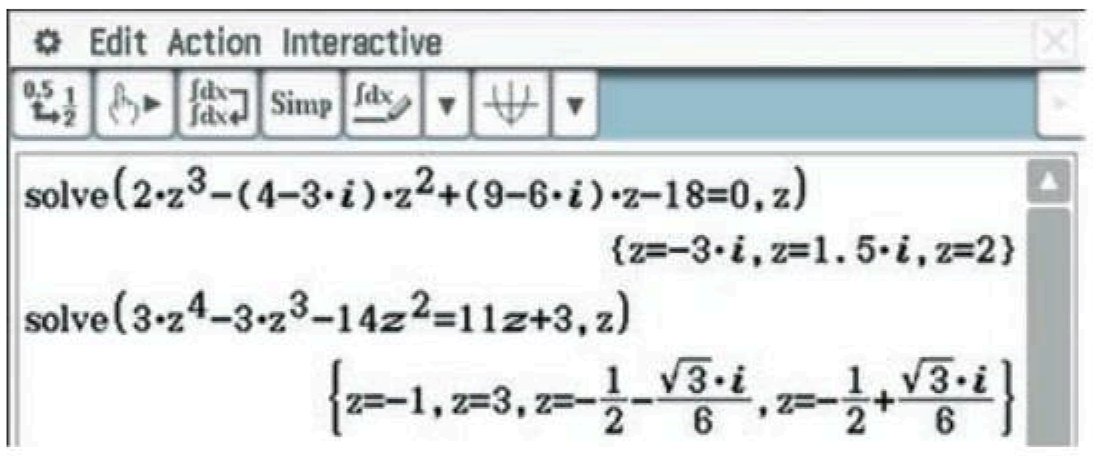
$\text{cSolve}\left(3\cdot z^4-3\cdot z^3-14\cdot z^2=11\cdot z+3,z\right)$

$z=\frac{-1}{2}+\frac{\sqrt{3}}{6}\cdot i \text{ or } z=\frac{-1}{2}-\frac{\sqrt{3}}{6}\cdot i \text{ or } z=-1 \text{ or } z=$

1 Press **menu > Algebra > Complex > Solve**.

2 Enter each equation followed by **,z**.

ClassPad



$\text{solve}(2\cdot z^3-(4-3\cdot i)\cdot z^2+(9-6\cdot i)\cdot z-18=0,z)$

$\{z=-3\cdot i, z=1.5\cdot i, z=2\}$

$\text{solve}(3\cdot z^4-3\cdot z^3-14z^2=11z+3,z)$

$\left\{z=-1, z=3, z=-\frac{1}{2}-\frac{\sqrt{3}\cdot i}{6}, z=-\frac{1}{2}+\frac{\sqrt{3}\cdot i}{6}\right\}$

1 Enter and highlight the first equation.

2 Tap **Interactive > Equation/Inequality > solve** and change the variable to ***z***.

3 Repeat for the second equation.

$$2z^3 - (4 - 3i)z^2 + (9 - 6i)z - 18 = 0 \Rightarrow z = \frac{3}{2}i, -3i, 2$$

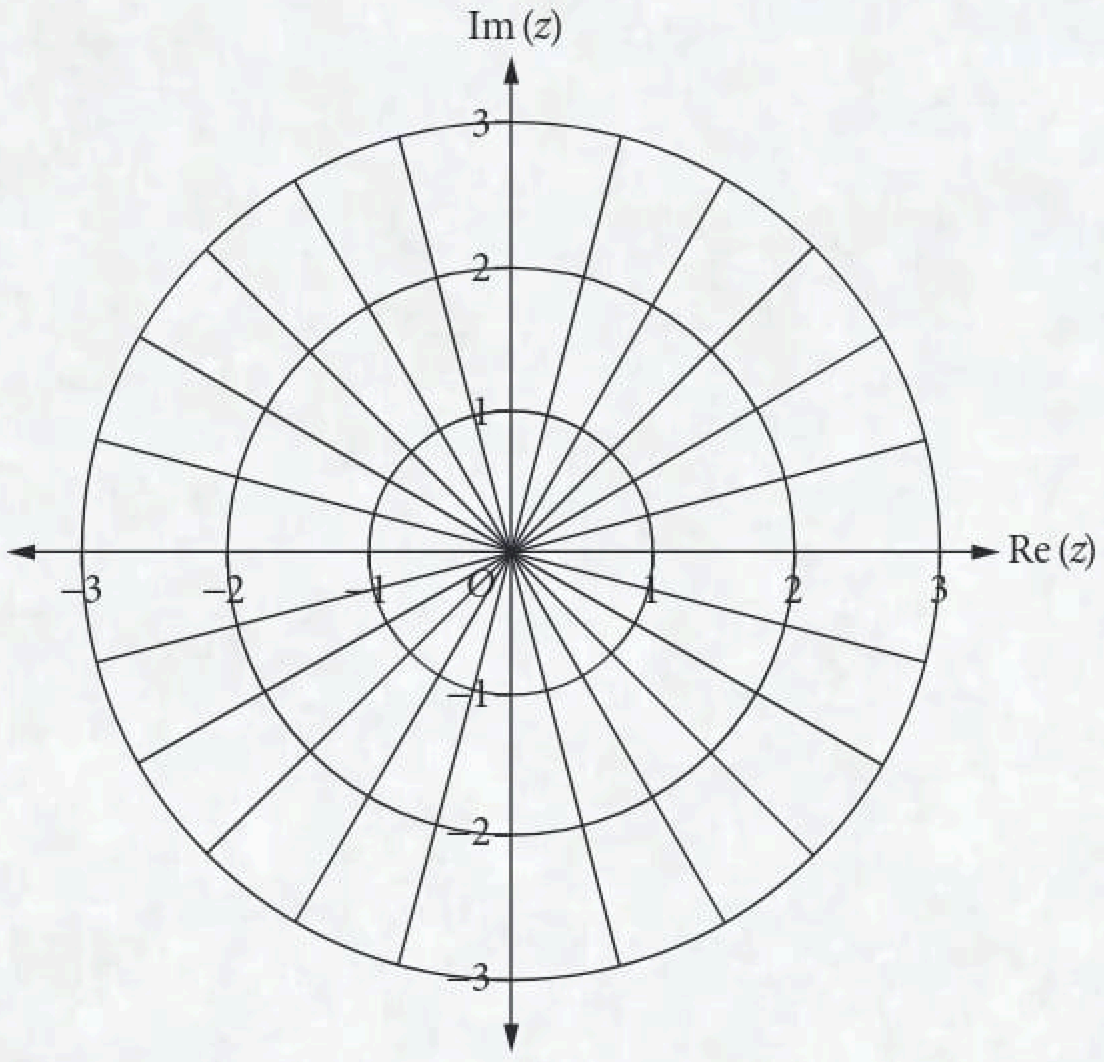
$$3z^4 - 3z^3 - 14z^2 = 11z + 3 \Rightarrow z = -1, 3, -\frac{1}{2} - \frac{\sqrt{3}}{6}i, -\frac{1}{2} + \frac{\sqrt{3}}{6}i$$

When we solve cubic equations over C with CAS, we will sometimes get answers involving sin, $\tan^{-1}$, roots of roots, etc. In this case, try switching to approximate (decimal) or exact (standard) mode to solve them. In this case, CAS is using an advanced formula beyond the scope of this course.

VCE QUESTION ANALYSIS

© VCAA 2015 2BQ2 2015 Examination 2 Section B Question 2 (12 marks)

a **i** Copy the Argand diagram below to plot and label the points $0 + 0i$ and $1 + i\sqrt{3}$. 2 marks



ii On the same Argand diagram above, sketch the line $|z-(1+i\sqrt{3})| = |z|$ and the circle $|z-2| = 1$. 2 marks

iii Use the fact that the line $|z-(1+i\sqrt{3})| = |z|$ passes through the point $z = 2$, or otherwise, to find the equation of this line in Cartesian form. 1 mark

iv Find the points of intersection of the line and the circle, expressing your answers in the form $a + ib$. 3 marks

b **i** Consider the equation $z^2 - 4\cos(\alpha)\, z + 4 = 0$, where α is a real constant and $0 < \alpha < \frac{\pi}{2}$. Find the roots z_1 and z_2 of this equation, in terms of α, expressing your answers in polar form. 3 marks

ii Find the value of α for which $\left|\operatorname{Arg}\left(\frac{z_1}{z_2}\right)\right| = \frac{5\pi}{6}$. 1 mark

Video
VCE question analysis: Complex numbers

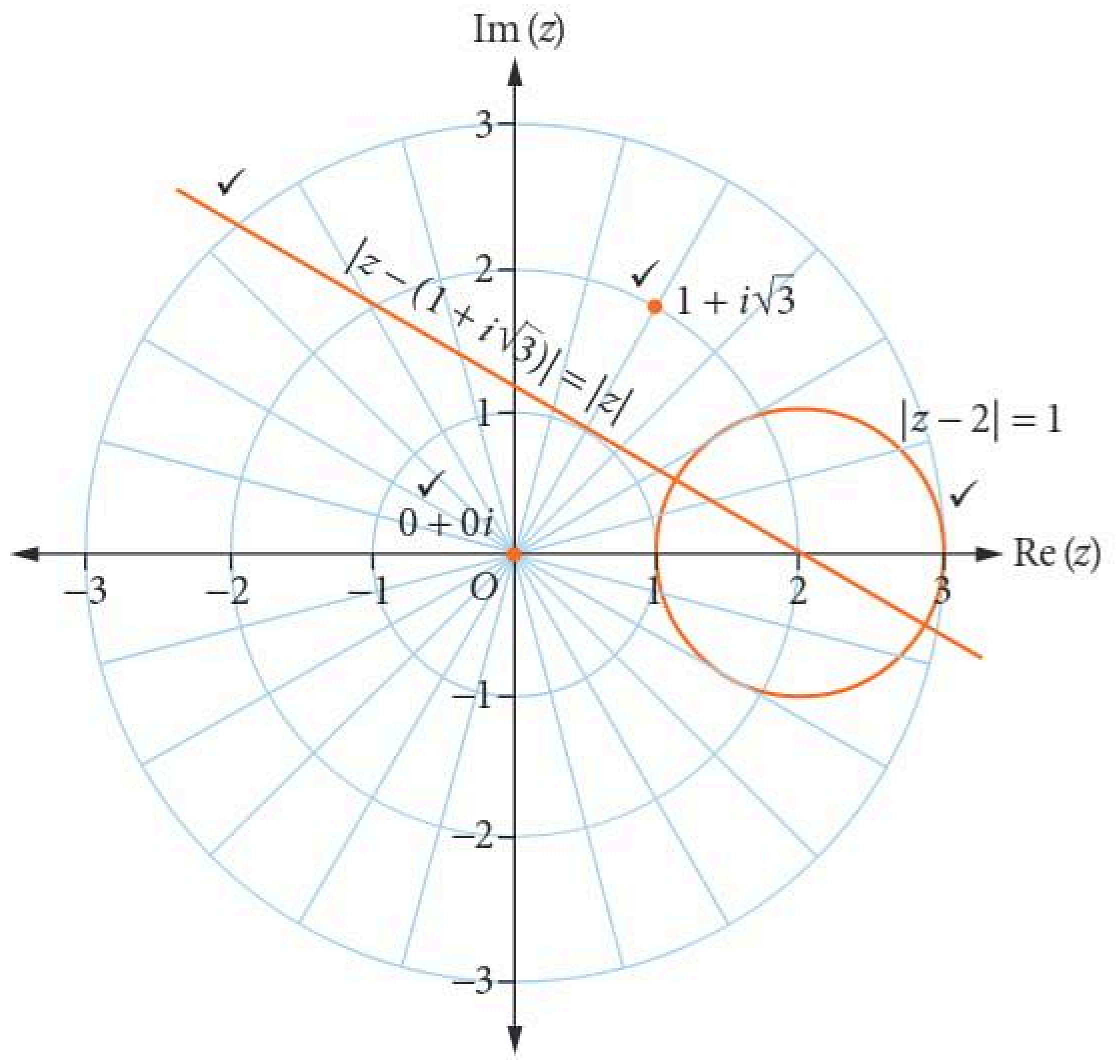
Reading the question

- Two points, a line and a circle must be shown.
- The intersections of the line and circle must be found in exact form.
- The solutions of a quadratic are required.
- The value of a is needed.

Thinking about the question

- The first point is the origin.
- $|1 + i\sqrt{3}| = 2$, so it is 2 from the origin, with real coordinate 1.
- The line $|z-(1+i\sqrt{3})| = |z|$ is the perpendicular bisector of $1 + i\sqrt{3}$ and the origin.
- The circle $|z-2| = 1$ has centre $2 + 0i$ and radius 1.
- $z^2 - 4\cos(\alpha)\, z + 4 = 0$ is solved with the quadratic formula.
- The difference between the arguments of z_1 and z_2 is $\frac{5\pi}{6}$.

Worked solution (✓ = 1 mark)



a **i** Label the origin as $0+0i$. ✓

$1+i\sqrt{3}$ has modulus 2 and argument $\tan^{-1}\left(\sqrt{3}\right)=\frac{\pi}{3}$. ✓

ii The line $z-\left(1+i\sqrt{3}\right)\Big|=|z|$ must pass through $\left(1,\frac{\pi}{3}\right)$ and is perpendicular to the radius.

Label the line. ✓

Draw the circle with radius 1 and centre 2. Label the circle. ✓

iii The line passes through (2, 0) and has inclination $\frac{\pi}{3}+\frac{\pi}{2}=\frac{5\pi}{6}$, so $m=\tan\left(\frac{5\pi}{6}\right)=-\frac{1}{\sqrt{3}}$.

The equation of the line is given by $y-y_1=m(x-x_1)$.

$$y-0=-\frac{1}{\sqrt{3}}(x-2)$$

$$\sqrt{3}y=-x+2$$

The equation in Cartesian form is $\boldsymbol{x+\sqrt{3}y-2=0}$. ✓

iv The equation of the circle is $(x-2)^2+y^2=1^2$. Substitute $y=-\frac{1}{\sqrt{3}}(x-2)$:

$(x-2)^2+\frac{1}{3}(x-2)^2=1$ ✓

$$\frac{4}{3}(x-2)^2=1$$

$$(x-2)^2=\frac{3}{4}$$

$x-2=\pm\frac{\sqrt{3}}{2}$, so $x=2+\frac{\sqrt{3}}{2}$ or $2-\frac{\sqrt{3}}{2}$ ✓

$$y=-\frac{1}{\sqrt{3}}(x-2)=-\frac{1}{\sqrt{3}}\left(2+\frac{\sqrt{3}}{2}-2\right)\text{ or }-\frac{1}{\sqrt{3}}\left(2-\frac{\sqrt{3}}{2}-2\right)$$

$$=-\frac{1}{2}\text{ or }\frac{1}{2}$$

The **points of intersection are** $\boldsymbol{2+\frac{\sqrt{3}}{2}-\frac{1}{2}i,\ 2-\frac{\sqrt{3}}{2}+\frac{1}{2}i}$. ✓

b **i** $$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4\cos(\alpha) \pm \sqrt{(4\cos(\alpha))^2 - 4 \times 1 \times 4}}{2 \times 1} \checkmark$$

$$= \frac{4\cos(\alpha) \pm \sqrt{(4\cos(\alpha))^2 - 4 \times 1 \times 4}}{2 \times 1}$$

$$= \frac{4\cos(\alpha) \pm \sqrt{16\cos^2(\alpha) - 16}}{2}$$

$$= \frac{4\cos(\alpha) \pm 4\sqrt{\cos^2(\alpha) - 1}}{2}$$

$$= 2\cos(\alpha) \pm 2\sqrt{(1 - \cos^2(\alpha)) \times (-1)} \checkmark$$

$$= 2\cos(\alpha) \pm 2\sin(\alpha)\,i$$

$$= 2(\cos(\alpha) + i\sin(\alpha)) \text{ or } 2(\cos(\alpha) - i\sin(\alpha))$$

$$= 2(\cos(\alpha) + i\sin(\alpha)) \text{ or } 2(\cos(-\alpha) + i\sin(-\alpha))$$

$$\boldsymbol{z_1 = 2\,\mathrm{cis}(\alpha),\ z_2 = 2\,\mathrm{cis}(-\alpha)} \checkmark$$

ii $\frac{z_1}{z_2} = \frac{2\,\mathrm{cis}(\alpha)}{2\,\mathrm{cis}(-\alpha)} = \mathrm{cis}(2\alpha)$, so $\left|\mathrm{Arg}\left(\frac{z_1}{z_2}\right)\right| = 2\alpha$, allowing for z_1 and z_2 to be interchanged.

$$2\alpha = \frac{5\pi}{6}$$

$$\boldsymbol{\alpha = \frac{5\pi}{12}} \checkmark$$

Student performance

80–100% 60–79% 0–59%

a **i** 81% Answered quite well, but many students did not realise $1 + i\sqrt{3}$ was on the circle of radius 2. Some students did not fully label the points.

ii 72% Few students realised the line was the perpendicular bisector of the points and some terminated the line at (2, 0) or drew it with a positive gradient. Most students graphed the circle correctly, but some were poorly drawn.

iii 62% Generally well answered. The most common error was giving the gradient as positive.

iv 57% Reasonably well answered. Common errors were answers in the incorrect form or with sign errors.

b **i** 36% Most students attempted to apply the quadratic formula or complete the square, but few found the values in polar form. The discriminant was a problem for many.

ii 23% Many students did not attempt this question. Common errors involved unsimplified expressions.

EXERCISE 4.6 Solving polynomial equations

ANSWERS p. 582

Recap

1 What is the remainder when $z^3 + (3 - i)z^2 - (2 + 3i)z + 4$ is divided by $z + 2 - i$?

A 14 **B** $4 - 30i$ **C** $10 - 10i$ **D** $16 + 6i$ **E** $18 + 12i$

2 The factors of $z^3 + (4i - 2)z^2 - (4 + 6i)z + 4$ are

A $z + 1 - i, z + 2i, z - 2i$ **B** $z - 1 + i, z + 2i$ **C** $z - 1 + i, z - 1 - i, z + 2i$

D $z + 1 - i, z - 2i$ **E** $z - 1 + i, z + 1 - i, z - 2i$

Mastery

3 WORKED EXAMPLE 17 Solve each equation.

a $z^2 - 3z + 5 = 0$ **b** $z^2 + 4z + 6 = 0$ **c** $3z^2 + 6 = 2\sqrt{2}z$

d $z^2 + \sqrt{3}z + 5 = 0$ **e** $2z^2 + 4 = 5z$ **f** $4z^2 - 6z + 4 = 0$

4 WORKED EXAMPLE 18 Find a possible real quadratic equation that has a root of

a $2 - 3i$ **b** $-1 + 2i$ **c** $2 - \sqrt{3}i$ **d** $\dfrac{1 - \sqrt{2}i}{3}$ **e** $-\dfrac{1}{2} + \dfrac{\sqrt{3}}{4}i$

5 WORKED EXAMPLE 19 Solve each equation.

a $2z^2 - 3iz + 5 = 0$ **b** $2iz^2 + 8z + 8\sqrt{3} = 0$ **c** $z^2 + 6iz = 9 + 4i$

d $z^2 + 2\sqrt{2} = 6 + \sqrt{2}zi - 4zi$ **e** $2z^2 + 3iz + 2 = 0$ **f** $3z^2 - 4iz + 4 = 0$

6 WORKED EXAMPLE 20

a $3 + 4i$ is a root of a particular polynomial equation with real coefficients. State another root of the polynomial.

b For a particular polynomial $p(z)$ with real coefficients, $p(-3 + 5i) = 0$. State two roots of the polynomial.

7 WORKED EXAMPLES 21–24 Solve each equation.

a $z^3 - 3z^2 + z - 3 = 0$ **b** $z^3 - 2z^2 + 5z = z^2 - 2z + 5$

c $3z^3 + 8z + 10 = 10z^2 + z$ **d** $z^3 + 4z^2 + 4z + 4 = z^2 + 2$

e $4z^3 + 2z = 4z^2 + 20$ **f** $3z^3 + (3 - 7i)z^2 - (4 + 7i) - 4 = 0$

g $2z^3 - 5z^2 + (1 - i)z + 8 - i = 0$ **h** $2z^3 - (7 + i)z^2 + (3 + 21i)z + 14 - 2i = 0$

i $z^3 + 3iz^2 + 12 = 3z^2 + 4z + 3iz + 18i$ **j** $3z^3 + 4zi + 4i = 2z^2 + 10z^2i + 10z$

k $z^4 - 5z^3 - 2iz^3 + 4iz^2 + 20z + 10iz - 16 - 12i$, given that one root is $3 + i$

8 Using CAS 6 Solve each equation.

a $z^3 - (5 + i)z^2 + 4z - 20 - 4i = 0$ **b** $z^2 + 3z + i = 3z^2 + iz^2 + 3$

c $z^3 + 11iz + 10 = 3z^2 + 5iz^2 + 6z$ **d** $z^4 - (5 + 3i)z^3 + (8 + 11i)z^2 - (6 + 16i)z + 10i = 0$

e $z^4 + 6i = z^3 + 6iz^3 + 5z^2 + 7z + 60iz + 108$

Exam practice

80–100% 60–79% 0–59%

9 © VCAA 2017 1Q3 69% TECH-FREE (3 marks) Let $z^3 + az^2 + 6z + a = 0$, $z \in C$, where a is a real constant. Given that $z = 1 - i$ is a solution to the equation, find all other solutions.

10 © VCAA 2016S 1Q1 TECH-FREE (3 marks)

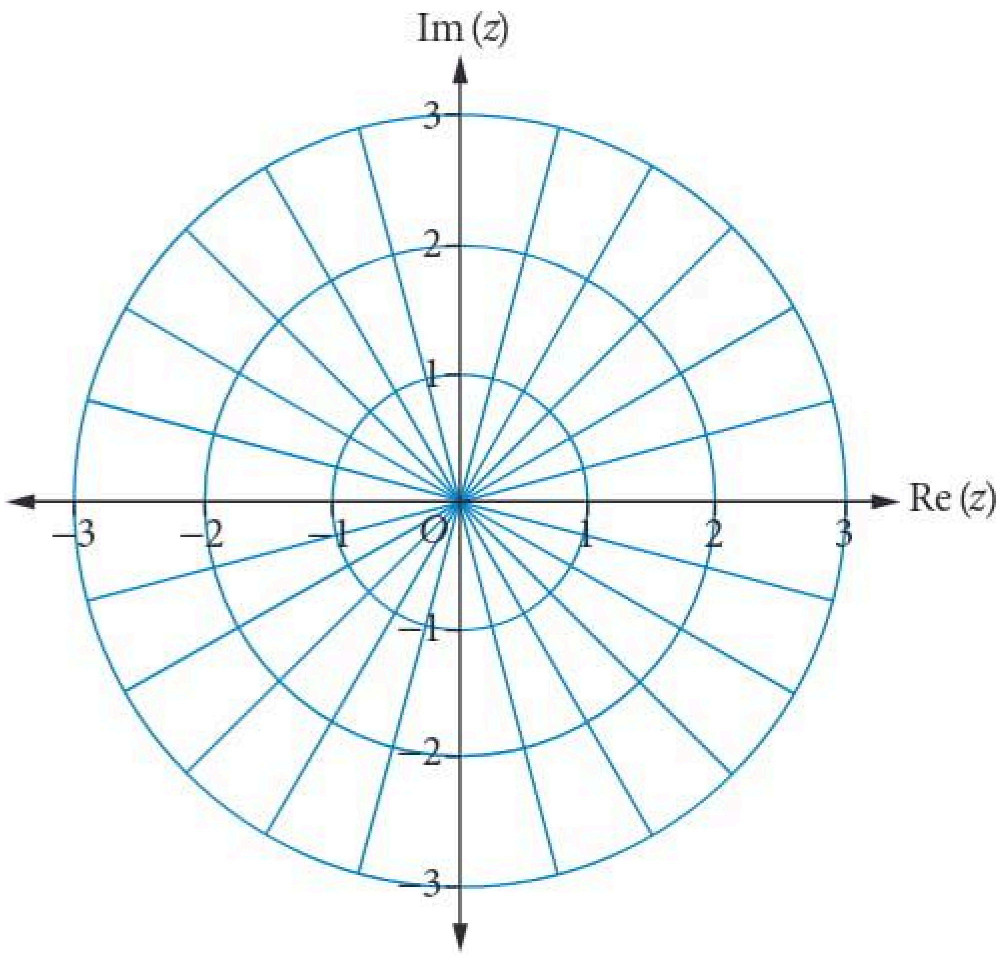
a Show that $\sqrt{5} - i$ is a solution of the equation $z^3 - \left(\sqrt{5} - i\right)z^2 + 4z - 4\sqrt{5} + 4i = 0$. 1 mark

b Find all other solutions of the equation $z^3 - \left(\sqrt{5} - i\right)z^2 + 4z - 4\sqrt{5} + 4i = 0$. 2 marks

11 © VCAA 2012 1Q3 TECH-FREE (4 marks) Consider the equation $z^3 - z^2 - 2z - 12 = 0$, $z \in C$.

a 55% Given that $z = 2\text{cis}\left(\frac{2\pi}{3}\right)$ is a root of the equation, find the other two roots in the form $a + ib$, where $a, b \in R$. 3 marks

b 33% Copy this Argand diagram and plot all of the roots clearly on it. 1 mark



12 © VCAA 2013 1Q8 50% TECH-FREE (4 marks) Find all solutions of $z^4 - 2z^2 + 4 = 0$, $z \in C$ in Cartesian form.

13 © VCAA 2016 2AQ4 68% One of the roots of $z^3 + bz^2 + cz = 0$ is $3 - 2i$, where b and c are real numbers. The values of b and c respectively are

A 6, 13 **B** 3, −2 **C** −3, 2 **D** 2, 3 **E** −6, 13

14 © VCAA 2011 2AQ6 65% The polynomial $P(z)$ has real coefficients. Four of the roots of the equation $P(z) = 0$ are $z = 0$, $z = 1 - 2i$, $z = 1 + 2i$ and $z = 3i$.

The **minimum** number of roots that the equation $P(z) = 0$ could have is

A 4 **B** 5 **C** 6 **D** 7 **E** 8

15 © VCAA 2013 2AQ8 61% The principal arguments of the solutions to the equation $z^2 = 1 + i$ are

A $\frac{\pi}{8}$ and $\frac{9\pi}{8}$ **B** $-\frac{\pi}{8}$ and $\frac{7\pi}{8}$ **C** $-\frac{7\pi}{8}$ and $\frac{\pi}{8}$ **D** $\frac{7\pi}{8}$ and $\frac{15\pi}{8}$ **E** $-\frac{3\pi}{4}$ and $\frac{\pi}{4}$

16 © VCAA 2017 2AQ3 47% The number of distinct roots of the equation $(z^4 - 1)(z^2 + 3iz - 2) = 0$, where $z \in C$, is

A 2 **B** 3 **C** 4 **D** 5 **E** 6

17 © VCAA 2014 2BQ2 (13 marks) Consider the complex number $z_1 = \sqrt{3} - 3i$.

a **i** 89% Express z_1 in polar form. 2 marks

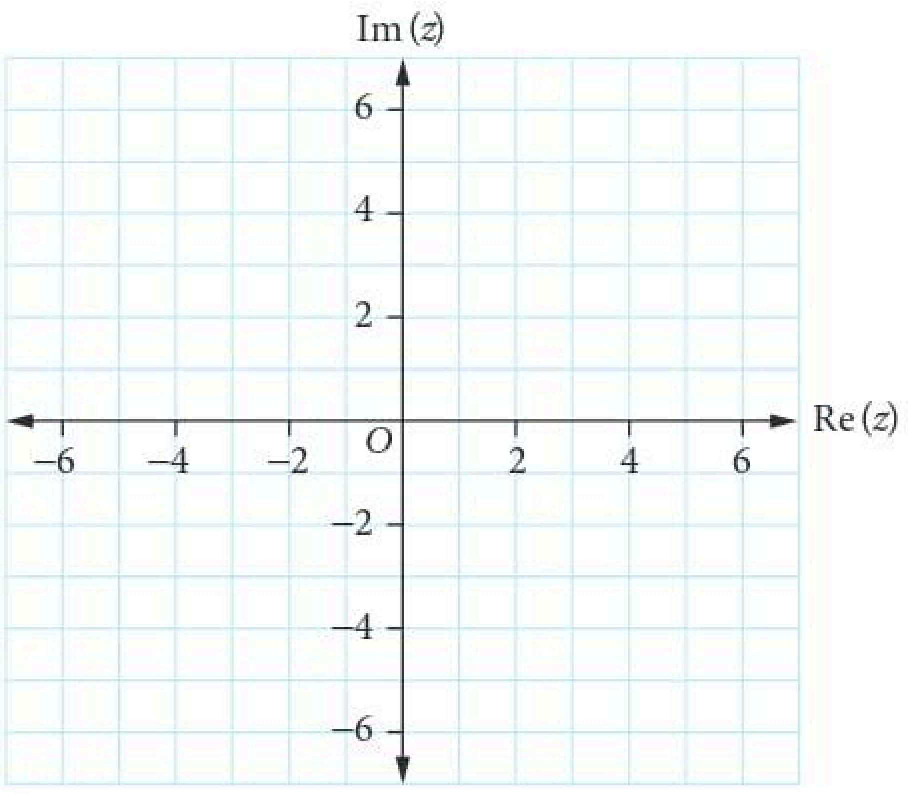
ii 67% Find $\text{Arg}\left(z_1^4\right)$. 1 mark

iii 76% Given that z_1 is one root of the equation $z^3 + 24\sqrt{3} = 0$, find the other two roots, expressing your answers in cartesian form. 2 marks

b **i** 85% Find the value of $(z_1 + 2i)(\overline{z}_1 - 2i)$, where $z_1 = \sqrt{3} - 3i$. 1 mark

ii 69% Show that the relation $(z + 2i)(\overline{z} - 2i) = 4$ can be expressed in cartesian form as $x^2 + (y + 2)^2 = 4$. 2 marks

iii 86% Copy the axes below and sketch $\{z : (z + 2i)(\overline{z} - 2i) = 4\}$ on them. 2 marks



c 16% The line joining the points corresponding to $k - 2i$ and $-(2 + k)i$, where $k < 0$ is tangent to the curve given by $\{z : (z + 2i)\}(\overline{z} - 2i) = 4\}$.

Find the value of k. 3 marks

18 © VCAA 2013 2BQ2 (12 marks)

a 57% Copy the Argand diagram from question **12** on the previous page and sketch $\{z : z\overline{z} = 4, z \in C\}$ **and** sketch $\{z : |z + \overline{z}| = |z - \overline{z}|, z \in C\}$ on it. 3 marks

b 40% Find all elements of $\{z : z\overline{z} = 4, z \in C\} \cap \{z : |z + \overline{z}| = |z - \overline{z}|, z \in C\}$, expressing your answer(s) in the form $a + ib$. 3 marks

c 59% One of the roots of the equation $z^4 + 16 = 0$ is $z = \sqrt{2} + i\sqrt{2}$. Write down the other roots in cartesian form. Plot and label all of these roots on the Argand diagram you drew in part **a**. 2 marks

d 60% Express $z^4 + 16$ as the product of four linear factors in terms of z. 1 mark

e 58% On the Argand diagram provided in part **a**, shade the region defined by

$$\{z : |z| \leq 2, z \in C\} \cap \{z : \text{Re}(z) \geq \sqrt{2}, z \in C\}.$$

1 mark

f 49% Find the area of the shaded region in part **e**. 2 marks

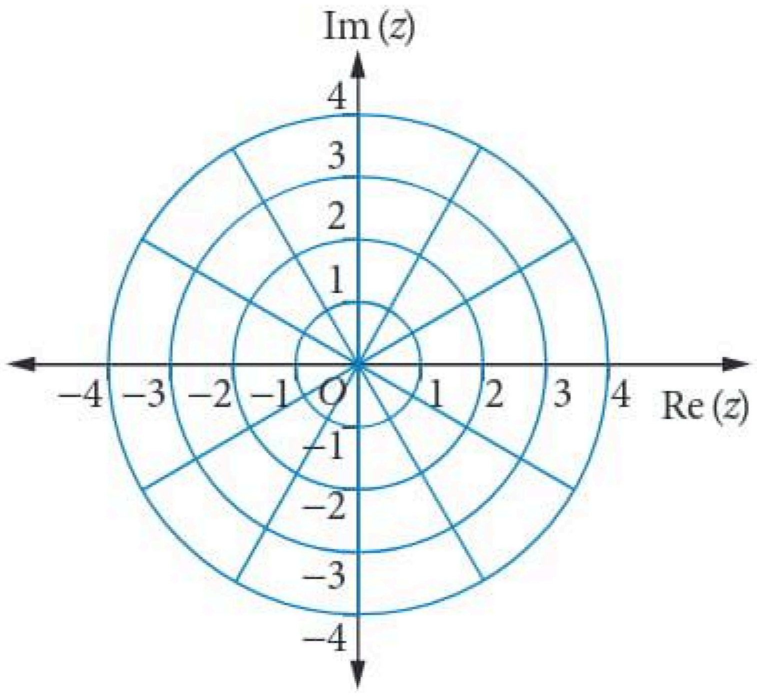
19 © VCAA 2021 2BQ2 (9 marks) The polynomial $p(z) = z^3 + \alpha z^2 + \beta z + \gamma$, where $z \in C$ and $\alpha, \beta, \gamma \in R$, can also be written as $p(z) = (z - z_1)(z - z_2)(z - z_3)$, where $z_1 \in R$ and $z_2, z_3 \in C$.

a **i** 73% State the relationship between z_2 and z_3. 1 mark

ii 43% Determine the values of α, β and γ, given that $p(2) = -13$, $|z_2 + z_3| = 0$ and $|z_2 - z_3| = 6$.

Consider the point $z_4 = \sqrt{3} + i$. 3 marks

b 45% Copy the Argand diagram and on it sketch the ray given by $\text{Arg}(z - z_4) = \dfrac{5\pi}{6}$. 2 marks



c The ray $\text{Arg}(z - z_4) = \dfrac{5\pi}{6}$ intersects the circle $|z - 3i| = 1$, dividing it into a major and a minor segment.

i 75% Sketch the circle $|z - 3i| = 1$ on the Argand diagram in part **b**. 1 mark

ii 30% Find the area of the minor segment. 2 marks

4 Chapter summary

Imaginary and complex numbers

- The **imaginary number *i*** is defined as $i = \sqrt{-1}$ and $i^2 = -1$.
- A **complex number** $z = a + ib$, where a and b are real, has a **real part** $\text{Re}(z) = x$ and an **imaginary part** $\text{Im}(z) = b$.
- Complex numbers are equal if and only if the real parts are equal and the imaginary parts are equal.

Complex conjugates

- The **complex conjugate** of $z = a + ib$, where a and b are real, is $\bar{z} = a - ib$.
- Both $z\bar{z}$ and $z + \bar{z}$ are real.
- If a quadratic equation with real coefficients has complex roots, then they are complex conjugates.

Operations with complex numbers

- To **add or subtract** complex numbers, treat the real and imaginary parts separately.
- To **multiply** complex numbers, expand and use $i^2 = -1$.
- To divide complex numbers such as $\frac{z}{w}$, **realise the denominator** by multiplying by $\frac{\bar{w}}{\bar{w}}$.

The Argand diagram

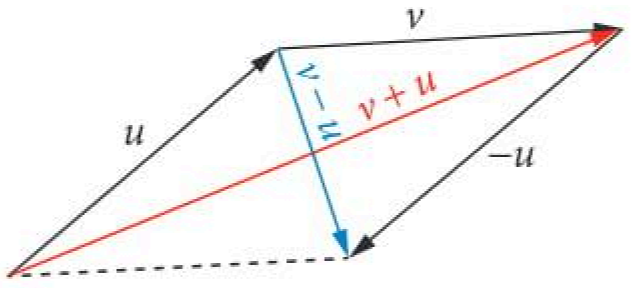
- The complex number $z = x + yi$ is represented by the point (x, y) on the Argand diagram.
- The **real axis** is horizontal and labelled $\text{Re}(z)$ or x.
- The **imaginary axis** is vertical and labelled $\text{Im}(z)$ or y.
- Complex conjugates are reflections of each other in the x-axis (real axis).
- Negatives are reflections of each other in the line $y = -x$.

Complex numbers as vectors

- The length of the vector representing $z = a + bi$ is the **modulus** of z, written $|z|$.
- $|z| = \sqrt{a^2 + b^2}$
- $z\bar{z} = |z|^2$

Operations in the plane

- We can show addition and subtraction of complex numbers using a parallelogram of vectors.



- Multiplying a complex number by i rotates it through an angle of $\frac{\pi}{2}$ (anticlockwise).
- Multiplying by $-i$ rotates the complex number through the angle $-\frac{\pi}{2}$ (clockwise).
- $z^{-1} = \frac{1}{z} = \frac{1}{|z|}\bar{z}$.

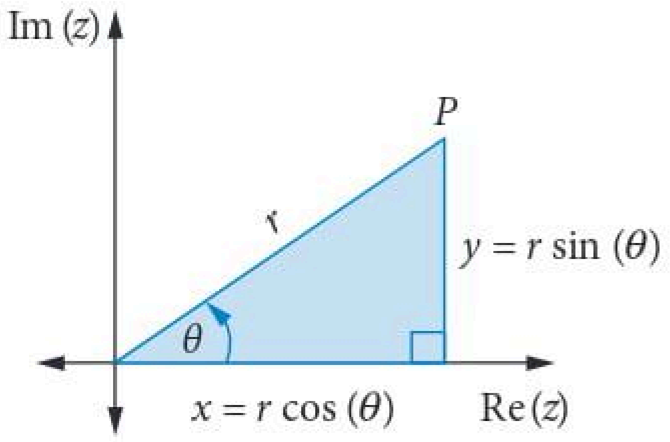
Polar form

- The complex number $x + yi$ or the vector (x, y) is in **Cartesian form** (or **rectangular form**).
- A complex number written in terms of its **modulus** $r = |z|$ and the angle θ that its vector makes with the positive direction of the x-axis is written in **polar form** (or **modulus-argument form** or **trigonometric form**), where θ is called the **principal argument,** in the interval $(-\pi, \pi]$.
- The relationship between the Cartesian and polar forms:

$x = r\cos(\theta)$ and $y = r\sin(\theta)$

$r = \sqrt{x^2 + y^2}$ and $\tan(\theta) = \frac{y}{x}$

$x + yi = r[\cos(\theta) + i\sin(\theta)] = r\,\text{cis}(\theta)$



Multiplication and division

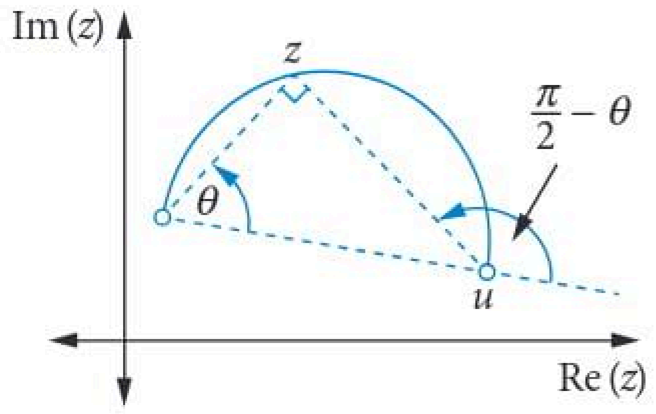
- If $z_1 = r_1 \text{ cis}(\theta_1)$ and $z_2 = r_2 \text{ cis}(\theta_2)$, then the product $z_1 z_2$ has modulus $r_1\, r_2$ and argument $(\theta_1 + \theta_2)$, that is, $r_1 \text{ cis}(\theta_1) \times r_2 \text{ cis}(\theta_2) = r_1\, r_2 \text{ cis}(\theta_1 + \theta_2)$.
- If $z_1 = r_1 \text{ cis}(\theta_1)$ and $z_2 = r_2 \text{ cis}(\theta_2)$, then the quotient $\dfrac{z_1}{z_2}$ has modulus $\dfrac{r_1}{r_2}$ and argument $(\theta_1 - \theta_2)$, that is, $\dfrac{r_1 \text{ cis}(\theta_1)}{r_2 \text{ cis}(\theta_2)} = \dfrac{r_1}{r_2} \text{cis}(\theta_1 - \theta_2)$.
- **Multiplication** of a complex number by $r \text{ cis}(\theta)$ is equivalent to (anticlockwise) **rotation** of its vector through the angle θ and **magnification** by the factor r.
- **Division** of a complex number by $r \text{ cis}(\theta)$ is equivalent to (clockwise) **rotation** of its vector through the angle $-\theta$ and **reduction** by the factor r.

Identities

- $z + \overline{z} = 2\text{Re}(z)$, $z - \overline{z} = 2\text{Im}(z)$, $\text{Re}(z \pm w) = \text{Re}(z) \pm \text{Re}(w)$, $\text{Im}(z \pm w) = \text{Im}(z) \pm \text{Im}(w)$
- $z\overline{z} = |z|^2$, $|\overline{z}| = |z|$, $|zw| = |z||w|$, $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$ and $\left|\dfrac{1}{z}\right| = \dfrac{1}{|z|}$
- $\text{Arg}(zw) = \text{Arg}(z) + \text{Arg}(w)$
- $\text{Arg}\left(\dfrac{z}{w}\right) = \text{Arg}(z) - \text{Arg}(w)$

Regions in the complex plane

- $\text{Arg}(z) = \theta$ specifies a **ray from the origin** at angle θ in the complex plane.
- $\text{Arg}[z - (x + yi)] = \theta$ specifies a **ray from $x + yi$** at angle θ in the complex plane.
- $\text{Re}(z) = a$, where $a \in R$, is a straight line parallel to the imaginary axis.
- $\text{Im}(z) = b$, where $b \in R$, is a straight line parallel to the real axis.
- $\text{Im}(z - u) = m \text{ Re}(z - u)$ or $\text{Im}(z) - \text{Im}(u) = m[\text{Re}(z) - \text{Re}(u)]$ is the straight line through u with slope m for $u \in C$ and $m \in R$. The Cartesian equation is $y - \text{Im}(u) = m[x - \text{Re}(u)]$.
- A straight line through u and v will have slope $m = \dfrac{\text{Im}(v) - \text{Im}(u)}{\text{Re}(v) - \text{Re}(u)}$ for $u, v \in C$.
- $|z - u| = |z - v|$ is the perpendicular bisector of the points u and v.
- $|z - u| = r$ is a circle with centre u and radius r, where $u \in C$ and $r \in R$.
- $(z - u)(\overline{z} - u) = r^2$ is the same circle.
- $\text{Arg}(z - u) - \text{Arg}(z - v) = \dfrac{\pi}{2}$ is the semicircle on the right-hand side of the diameter from u to v.
- $|z - u| = 2|z - v|$ for $u, v \in C$ is also a circle.
- The ellipse with foci at $u, v \in C$ and major axis of length $2a$ is given by $|z - u| + |z - v| = 2a$.
- $\text{Arg}(z - u) - \text{Arg}(z - v) = \dfrac{\pi}{2}$ is the semicircle on the right-hand side of the diameter from u to v.



De Moivre's theorem

- $[\cos(\theta) + i\sin(\theta)]^n = \cos(n\theta) + i\sin(n\theta)$.
- Use De Moivre's theorem to find **powers** of complex numbers.

Roots of unity

- Use De Moivre's theorem to solve complex equations of the form $z^n = 1$, whose solutions are called the roots of unity.
- The nth roots of unity are given by $\text{cis}\left(\pm\dfrac{2k\pi}{n}\right)$.
- For n even, $k = \left\{-\dfrac{n}{2} + 1 \ldots -1, 0, 1, 2 \ldots \dfrac{n}{2}\right\}$
- For n odd, $k = \left\{-\dfrac{n-1}{2} \ldots -1, 0, 1, 2 \ldots \dfrac{n-1}{2}\right\}$
- The nth roots are separated from each other by the angle $\dfrac{2\pi}{n}$.

General roots

- The nth roots of $r\cos(\theta) + i\sin(\theta)$ are given by $r^{\frac{1}{n}}\left[\cos\left(\dfrac{2k\pi + \theta}{n}\right) + i\sin\left(\dfrac{2k\pi + \theta}{n}\right)\right]$ for $k \in Z$.
- The roots have the same modulus and their arguments are separated by $\dfrac{2\pi}{n}$, so they are evenly spaced around the circle $|z| = r^{\frac{1}{n}}$.

Complex polynomials

- A complex polynomial of degree n is of the form $P(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \ldots + a_1 z^1 + a_0$, where $a_n \neq 0$ and $z, a_n, a_{n-1}, a_{n-2} \ldots a_1, a_0 \in C$.
- If a polynomial $P(z)$ is divided by $D(z)$, then the remainder $R(z)$ is of degree less than $D(z)$.
- The division identity states that $P(z) = D(z)Q(z) + R(z)$. $P(z)$ is called the dividend, $D(z)$ the divisor, $Q(z)$ the quotient and $R(z)$ the remainder.
- The remainder theorem states that if a polynomial $P(z)$ is divided by $D(z) = z - a$ for $a \in C$, then the remainder is given by $R = P(a)$.
- The factor theorem states that for $a \in C$, $z - a$ is a factor of $P(z)$ if and only if $P(a) = 0$.
- A polynomial equation has the form $p(z) = 0$, where $p(z)$ is a polynomial.
- The fundamental theorem of algebra states that a non-constant polynomial equation has at least one complex root.
- The polynomial equation $p(z) = 0$ of degree $n > 1$ has n roots, some of which may be equal.
- Any polynomial *can be* factorised to n linear factors.

Polynomial equations

- Quadratic equations with no z term are solved using the roots of complex numbers. If necessary, use the quadratic formula for real quadratic equations.
- If a real quadratic equation has a complex root, the other root must be the complex conjugate. The quadratic equation $x^2 - (\alpha + \beta)x + \alpha\beta = 0$ has roots α and β.
- If the coefficient of z^2 in a complex quadratic equation includes i, multiply the whole equation by i before starting to solve it. If the discriminant of a quadratic equation is complex, it is easier to complete the square than to use the quadratic formula.
- The **conjugate root theorem** states that if w is a complex root of a real polynomial, then its conjugate $\overline{w}$ is also a root of the polynomial.
- Complex roots of polynomial equations with real coefficients occur in conjugate pairs.
- Use the factor theorem to find some roots of a polynomial equation: try factors of the constant term.
- If we find enough to reduce the equation to a quadratic, we can solve the equation completely.

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 © VCAA 2021 1Q6 (4 marks) Consider the three vectors $\underset{\sim}{a} = \underset{\sim}{i} + 6\underset{\sim}{j} - 3\underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{i} - 8\underset{\sim}{j} - 5\underset{\sim}{k}$ and $\underset{\sim}{c} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + \left|1 - p^2\right|\underset{\sim}{k}$, where p is a real constant.

Find the values of p for which the three vectors are linearly independent.

2 (1 mark) What is the contrapositive of the statement 'If n^3 is divisible by 8 then n is divisible by 2'?

3 © VCAA 2019 1Q7 (5 marks)

a Show that $3 - \sqrt{3}i = 2\sqrt{3}\operatorname{cis}\left(-\frac{\pi}{6}\right)$. 1 mark

b Find $\left(3 - \sqrt{3}i\right)^3$, expressing your answer in the form $x + iy$, where $x, y \in R$. 2 marks

c Find the integer values of n for which $\left(3 - \sqrt{3}i\right)^n$ is real. 1 mark

d Find the integer values of n for which $\left(3 - \sqrt{3}i\right)^n = ai$, where a is a real number. 1 mark

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 © VCAA 2021 2AQ12 Consider the vectors $\underset{\sim}{a} = x\underset{\sim}{i} + \underset{\sim}{j}, = \underset{\sim}{b} = \underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{c} = \underset{\sim}{i} + x\underset{\sim}{j}$.

Given that θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$, and φ is the angle between $\underset{\sim}{b}$ and $\underset{\sim}{c}$, $\cos(\theta)\cos(\phi)$ is

A $\dfrac{2(1+x^2)}{1-x^2}$ **B** $\dfrac{\sqrt{2}(1-x^2)}{1+x^2}$ **C** $-\dfrac{(x+1)^2}{2(1+x^2)}$ **D** $-\dfrac{(x-1)^2}{2(1+x^2)}$ **E** $\dfrac{\sqrt{2}(1+x^2)}{1-x^2}$

2 © VCAA 2017 2AQ1 The implied domain of $f(x) = 2\cos^{-1}\left(\dfrac{1}{x}\right)$ is

A R **B** $[-1, 1]$ **C** $(-\infty, -1] \cup [1, \infty)$

D $R\backslash\{0\}$ **E** $[-1, 1]\backslash\{0\}$

3 © VCAA 2014 2AQ7 The sum of the roots of $z^3 - 5z^2 + 11z - 7 = 0$, where $z \in C$, is

A $1 + 2\sqrt{3}i$ **B** $5i$ **C** $4 - 2\sqrt{3}i$ **D** $2\sqrt{3}i$ **E** 5

4 © VCAA 2019 2AQ6 Let $z, w \in C$, where $\text{Arg}(z) = \dfrac{\pi}{2}$ and $\text{Arg}(w) = \dfrac{\pi}{4}$.

The value of $\text{Arg}\left(\dfrac{z^5}{w^4}\right)$ is

A $-\dfrac{\pi}{2}$ **B** $\dfrac{\pi}{2}$ **C** π **D** $\dfrac{5\pi}{2}$ **E** $\dfrac{7\pi}{2}$

5 © VCAA 2019 2AQ5 Let $z = x + yi$, where $x, y \in R$. The rays $\text{Arg}(z - 2) = \dfrac{\pi}{4}$ and $\text{Arg}(z - (5 + i)) = \dfrac{5\pi}{6}$, where $z \in C$, intersect on the complex plane at a point (a, b).

The value of b is

A $-\sqrt{3}$ **B** $2 - \sqrt{3}$ **C** 0 **D** $\sqrt{3}$ **E** $2 + \sqrt{3}$

Section B 2 questions

15 marks

1 © VCAA 2018 2BQ1 MODIFIED (5 marks) Consider the function $f: D \to R$, where $f(x) = 2\arcsin(x^2 - 1)$.

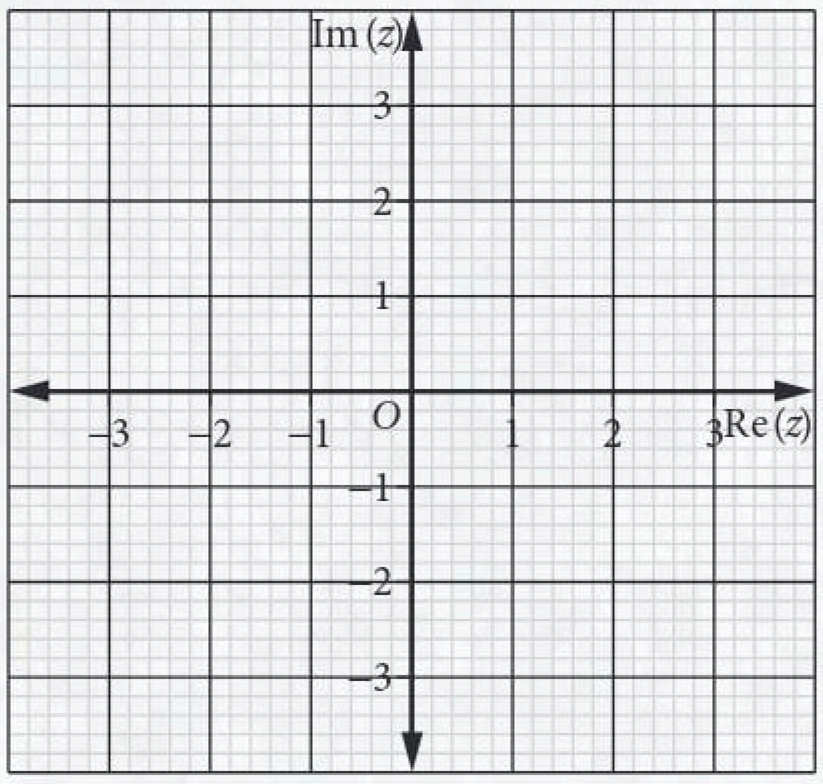
a Determine the maximal domain D and the range of f. 2 marks

b Sketch the graph of $y = f(x)$, labelling any endpoints and the y-intercept with their coordinates. 3 marks

2 © VCAA 2019 2BQ2 MODIFIED (10 marks)

a **i** Show that the solutions of $2z^2 + 4z + 5 = 0$, where $z \in C$, are $z = -1 \pm \frac{\sqrt{6}}{2}i$. 1 mark

ii Copy the Argand diagram below and on it plot the solutions of $2z^2 + 4z + 5 = 0$. 1 mark



Let $|z + m| = n$, where $m, n \in R$, represent the circle of minimum radius that passes through the solutions of $2z^2 + 4z + 5 = 0$.

b **i** Find m and n. 2 marks

ii Find the Cartesian equation of the circle $|z + m| = n$. 1 mark

iii Sketch the circle on the Argand diagram in part **a ii**. Intercepts with the coordinate axes do not need to be calculated or labelled. 1 mark

c Find all values of d, where $d \in R$, for which the solutions of $2z^2 + 4z + d = 0$ satisfy the relation $|z + m| \le n$. 2 marks

d All complex solutions of $az^2 + bz + c = 0$ have non-zero real and imaginary parts.

Let $|z + p| = q$ represent the circle of minimum radius in the complex plane that passes through these solutions, where $a, b, c, p, q \in R$.

Find p and q in terms of a, b and c. 2 marks

CHAPTER

5

DIFFERENTIATION

Study Design coverage

AREA OF STUDY 4: CALCULUS

Differential calculus and integral calculus

- derivatives of inverse circular functions
- second derivatives, use of notations $f''(x)$ and $\frac{d^2y}{dx^2}$, and their application to the analysis of graphs of functions, including points of inflection and concavity
- applications of chain rule to related rates of change and implicit differentiation; for example, implicit differentiation of the relations $x^2 + y^2 = 9$, $3xy^2 = x + y$ and $x\sin(y) + x^2\cos(y) = 1$.

VCE Mathematics Study Design 2023–2027 p. 111, © VCAA 2022

Video playlists (9):

5.1	The product, quotient and chain rules
5.2	Differentiating circular functions
5.3	Differentiating inverse circular functions
5.4	Differentiating exponential and logarithmic functions
5.5	The second derivative
5.6	Applying the second derivative
5.7	The chain rule and related rates of change
5.8	Implicit differentiation
VCE question analysis	Differentiation

Worksheets (19):

5.1	Product rule • Quotient rule • Chain rule
5.2	Derivatives of trigonometric functions • Trigonometric functions and gradient • Differentiating trigonometric functions
5.3	Derivatives of inverse trigonometric functions • Inverse trigonometric functions and gradient
5.4	Derivatives of exponential functions • Derivatives of logarithmic functions • Exponential and logarithmic functions
5.5	The second derivative • First and second derivatives
5.6	Concavity • Higher derivatives
5.7	Related rates • Related rates of change
5.8	Implicit differentiation • Curve sketching with derivatives

Nelson MindTap

To access resources above, visit
cengage.com.au/nelsonmindtap

5.1 The product, quotient and chain rules

Video playlist
The product, quotient and chain rules

Worksheets
Product rule

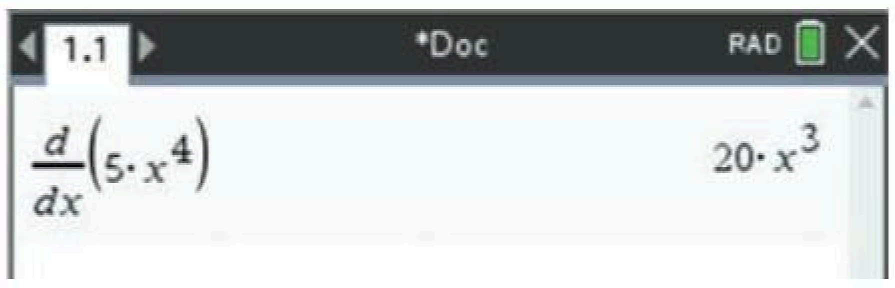
Quotient rule

Chain rule

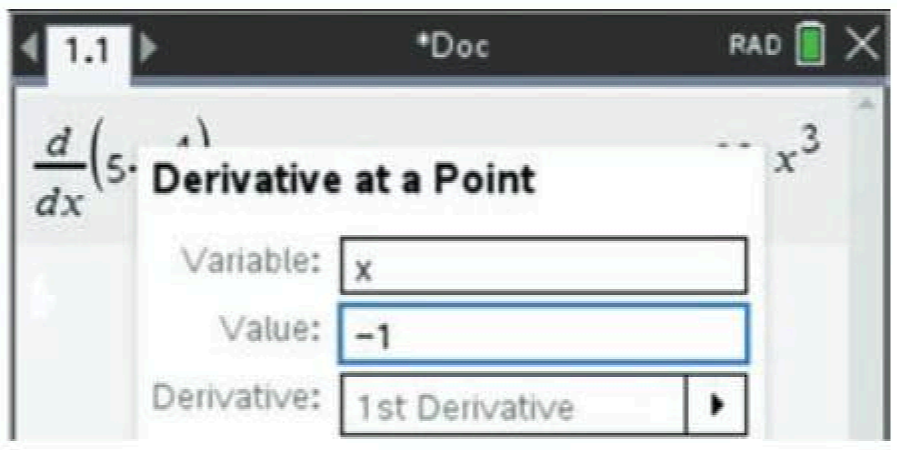
USING CAS 1 Differentiation

Find the derivative of $5x^4$ and determine the value of the derivative when $x = -1$.

TI-Nspire



$\frac{d}{dx}(5\cdot x^4)$ $20\cdot x^3$



Derivative at a Point

Variable: x

Value: −1

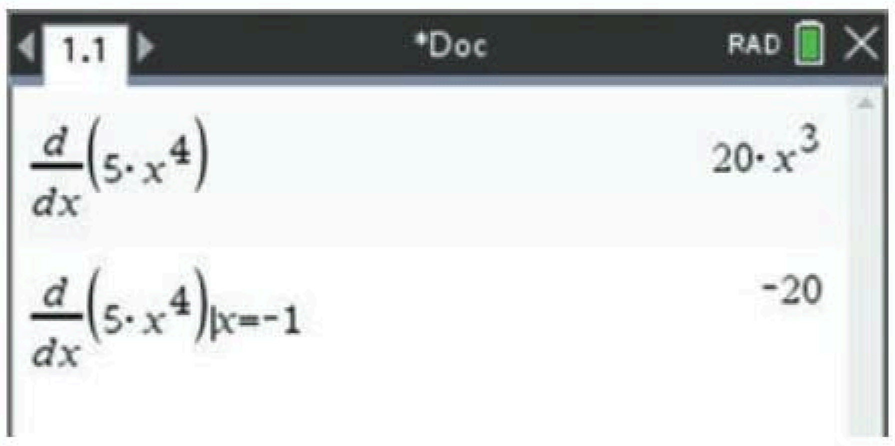
Derivative: 1st Derivative

1 Press **menu > Calculus > Derivative**.

2 In the template, enter **x** as the variable and then enter the expression.

3 Press **menu > Calculus > Derivative at a Point**.

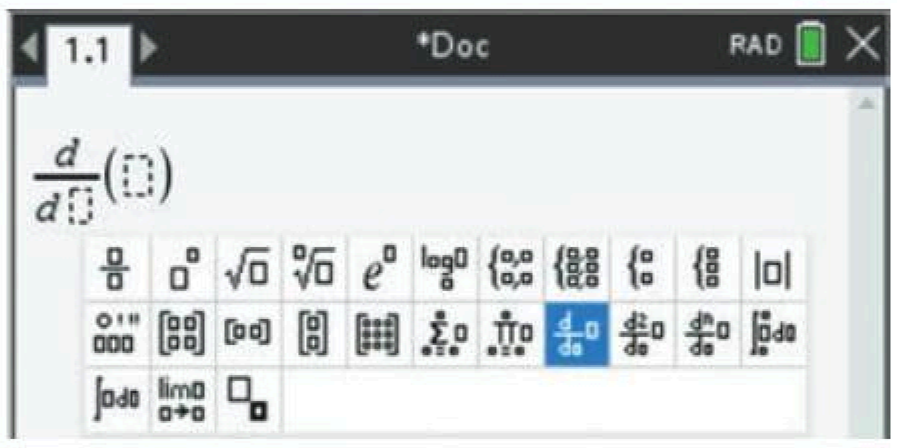
4 In the dialogue box, in the **Variable**: field enter **x** and in the **Value**: field enter **−1**.



$\frac{d}{dx}(5\cdot x^4)$ $20\cdot x^3$

$\frac{d}{dx}(5\cdot x^4)|x=-1$ -20

5 In the template, enter the expression.



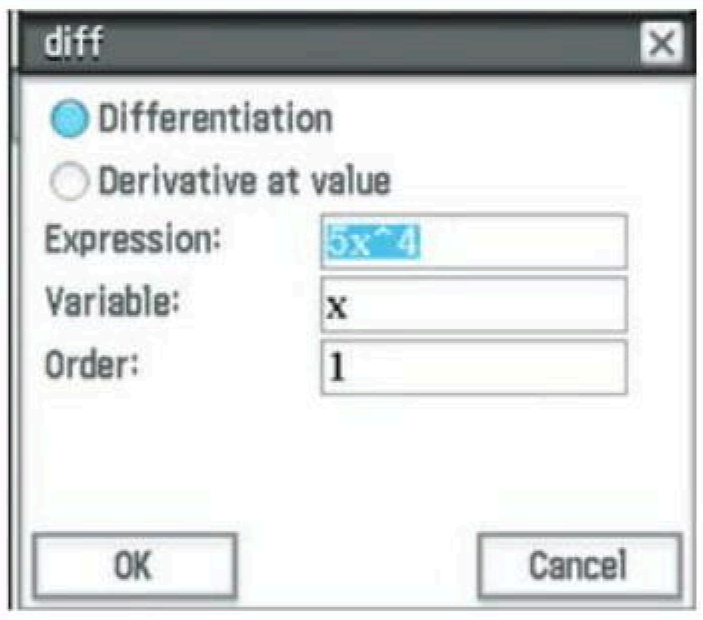
$\frac{d}{d□}(□)$

Another way of entering the derivative:

1 Press **template** and select the **derivative** template.

2 Press **Shift** + –.

ClassPad

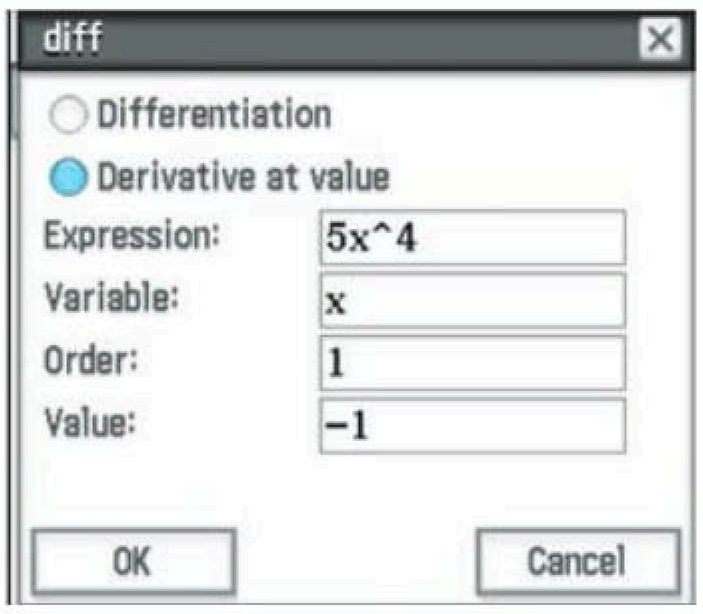


● Differentiation
○ Derivative at value
Expression: 5x^4
Variable: x
Order: 1

1 Enter and highlight the expression **5x⁴.**

2 Tap **Interactive > Calculation > diff**.

3 Tap **OK**.



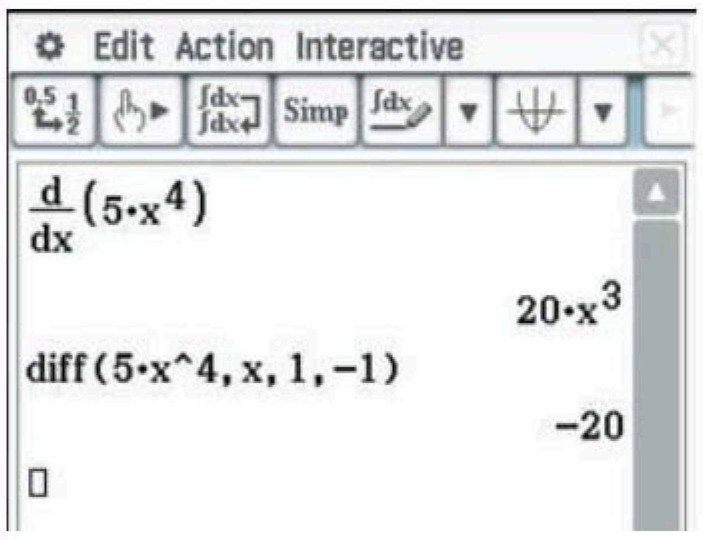
○ Differentiation
● Derivative at value
Expression: 5x^4
Variable: x
Order: 1
Value: −1

4 Highlight again, **Interactive > Calculation > diff**.

5 In the dialogue box, tap **Derivative at value**.

6 In the **Value:** field enter –**1**.

7 Tap **OK**.



$\frac{d}{dx}(5\cdot x^4)$

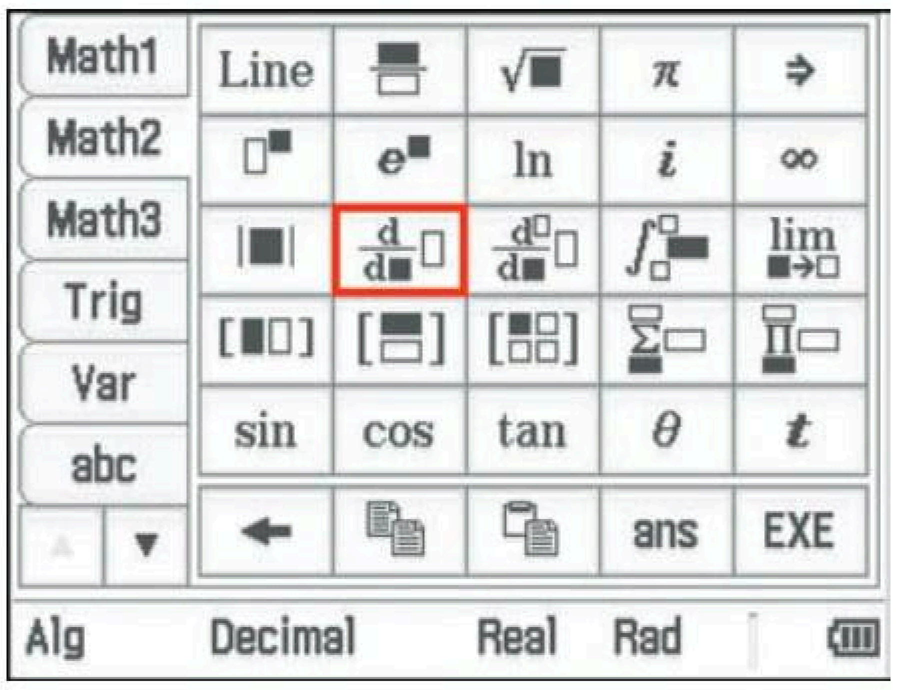
$20\cdot x^3$

diff(5·x^4, x, 1, −1)

-20

8 The first answer above is the derivative.

9 The second answer above is the derivative when $x = -1$.



Another way of entering the derivative:

1 Open the **Keyboard > Math2** and select the **derivative** template.

2 Enter the expression.

Derivative of $5x^4 = 20x^3$

The value of the derivative when $x = -1$ is -20.

The product rule

The **product rule** is used to differentiate the product of two functions $f(x) = u(x) \times v(x)$.

$f'(x) = u(x)\,v'(x) + v(x)\,u'(x)$.

The product rule

$$\frac{d}{dx}(uv) = uv' + vu'$$

$$\text{or} \quad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

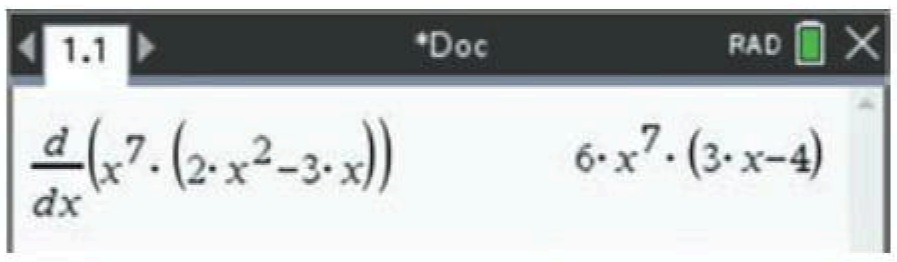
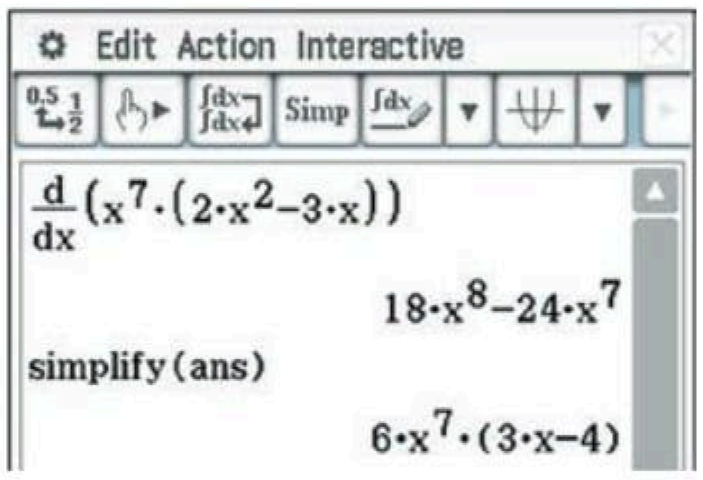
5.1

WB

p. 99

WORKED EXAMPLE 1 The product rule

Use the product rule to find $\frac{dy}{dx}$ for the function $y = x^7(2x^2 - 3x)$, then verify your answer by expanding the function first, then differentiating.

Steps	**Working**
1 Use the product rule: $\frac{d}{dx}(uv) = uv' + vu'$	Let $u = x^7$ and $v = (2x^2 - 3x)$ $u' = 7x^6$ and $v' = 4x - 3$ $\frac{dy}{dx} = x^7(4x - 3) + 7x^6(2x^2 - 3x)$
2 Simplify the answer.	$\frac{dy}{dx} = x^6(4x^2 - 3x + 14x^2 - 21x)$ $= 6x^7(3x - 4)$
TI-Nspire  $\frac{d}{dx}\left(x^7\cdot\left(2\cdot x^2-3\cdot x\right)\right)$ $6\cdot x^7\cdot(3\cdot x-4)$	**ClassPad**  $\frac{d}{dx}(x^7\cdot(2\cdot x^2-3\cdot x))$ $18\cdot x^8-24\cdot x^7$ simplify(ans) $6\cdot x^7\cdot(3\cdot x-4)$
3 Verify by expanding $y = x^7(2x^2 - 3x)$, then differentiating.	$y = x^7(2x^2 - 3x)$ $= 2x^9 - 3x^8$ $\frac{dy}{dx} = 18x^8 - 24x^7$ $= 6x^7(3x - 4)$

Exam hack

In an exam, marks are often lost if mistakes are made after an already correct answer, so it is wise to know when to stop simplifying. For the above example:

$\frac{dy}{dx} = x^6(4x^2 - 3x + 14x^2 - 21x)$

$= 6x^7(3x - 4)$ Correct: stop here.

$= 18x^8 - 4$ Incorrect ∴ you would lose the answer mark.

The quotient rule

The **quotient rule** is used to differentiate the quotient of two functions $f(x) = \frac{u(x)}{v(x)}$.

$$f'(x) = \frac{v(x)\,u'(x) - u(x)\,v'(x)}{(v(x))^2}$$

The quotient rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

or $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Exam hack

For the quotient rule, differentiate in alphabetical order, u', then v'.

WB

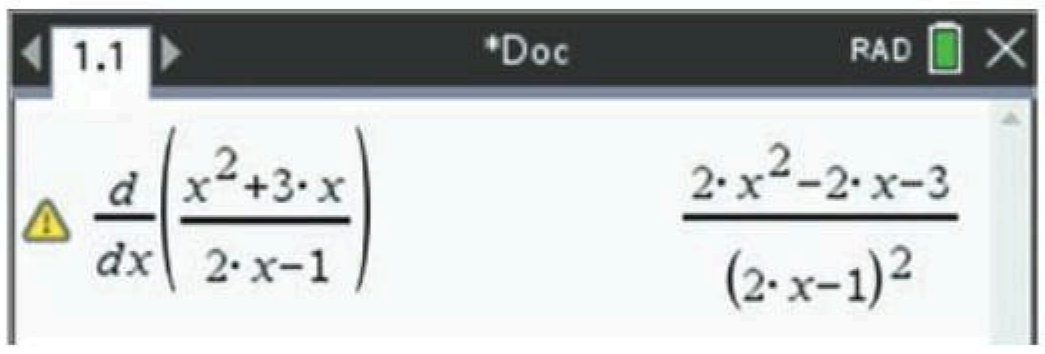
p. 100

WORKED EXAMPLE 2 The quotient rule

Find $\frac{dy}{dx}$ for the function $y = \frac{x^2+3x}{2x-1}$.

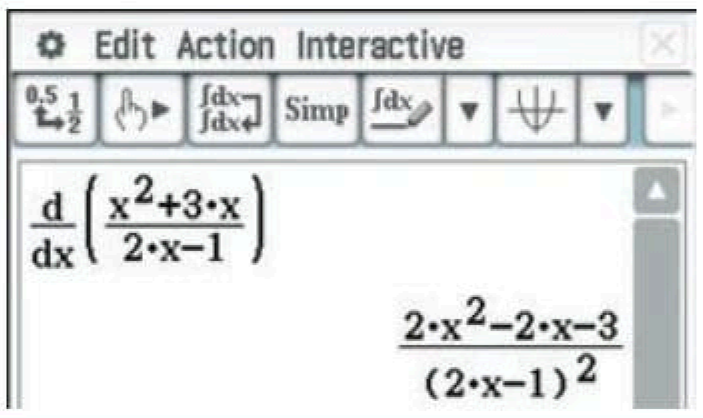
Steps	Working
1 Use the quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$	Let $u = x^2 + 3x$ and $v = 2x - 1$ $u' = 2x + 3$ and $v' = 2$ $\frac{dy}{dx} = \frac{(2x-1)(2x+3) - (x^2+3x)(2)}{(2x-1)^2}$
2 Simplify the answer.	$\frac{dy}{dx} = \frac{4x^2+4x-3-2x^2-6x}{(2x-1)^2}$ $= \frac{2x^2-2x-3}{(2x-1)^2}$

TI-Nspire



$\frac{d}{dx}\left(\frac{x^2+3\cdot x}{2\cdot x-1}\right)$ $\frac{2\cdot x^2-2\cdot x-3}{(2\cdot x-1)^2}$

ClassPad



$\frac{d}{dx}\left(\frac{x^2+3\cdot x}{2\cdot x-1}\right)$

$\frac{2\cdot x^2-2\cdot x-3}{(2\cdot x-1)^2}$

The chain rule

The **chain rule** is used to differentiate a composite function of the form $y = f(g(x))$ or $y = f \circ g(x)$.

$$\frac{dy}{dx} = f'(g(x))g'(x)$$

This rule can be seen as the derivative of the 'outer' times the derivative of the 'inner'.

The chain rule

If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

WB

p. 101

WORKED EXAMPLE 3 The chain rule

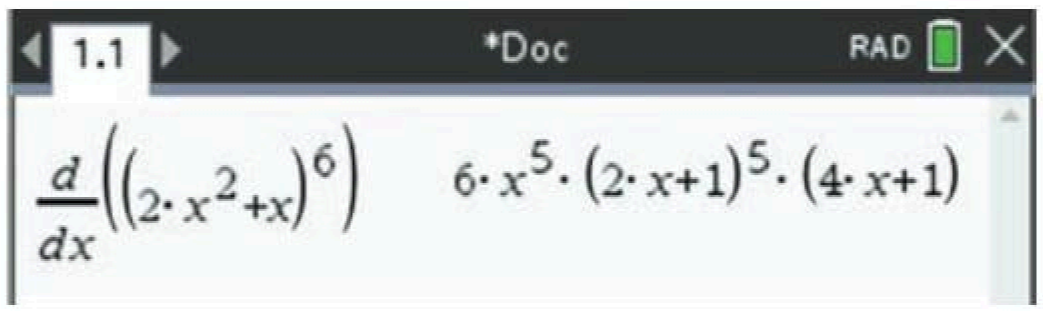
Find $\frac{dy}{dx}$ for the function $y = (2x^2 + x)^6$.

Steps	Working
1 Use the chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ The 'outer' function $y = f(u)$ is $\boldsymbol{y = u^6}$. The 'inner' function $u = g(x)$ is $\boldsymbol{u} = 2x^2 + x$.	Let $y = u^6$ and $u = 2x^2 + x$ $y' = 6u^5$ and $u' = 4x + 1$ $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 6u^5(4x+1)$

2 Replace $u = 2x^2 + x$ and simplify the answer.

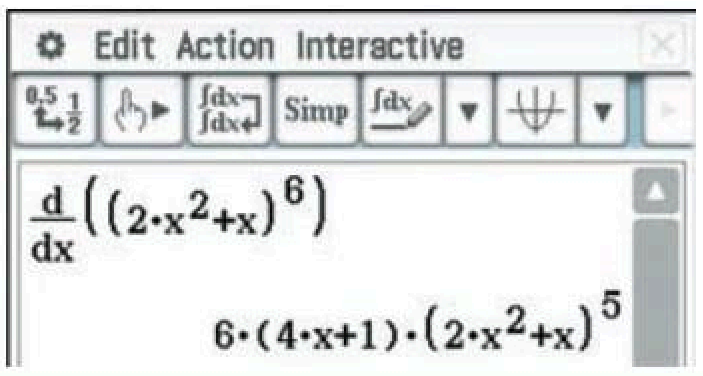
$$\frac{dy}{dx} = 6(2x^2 + x)^5 \times (4x + 1)$$
$$= 6(4x + 1)(2x^2 + x)^5$$

TI-Nspire



$\frac{d}{dx}\left(\left(2\cdot x^2+x\right)^6\right)$ $\quad 6\cdot x^5\cdot(2\cdot x+1)^5\cdot(4\cdot x+1)$

ClassPad



$\frac{d}{dx}\left(\left(2\cdot x^2+x\right)^6\right)$

$6\cdot(4\cdot x+1)\cdot(2\cdot x^2+x)^5$

Differentiation at a value

We can use the derivative to find the gradient, or instantaneous rate of change, of a curve at a particular point.

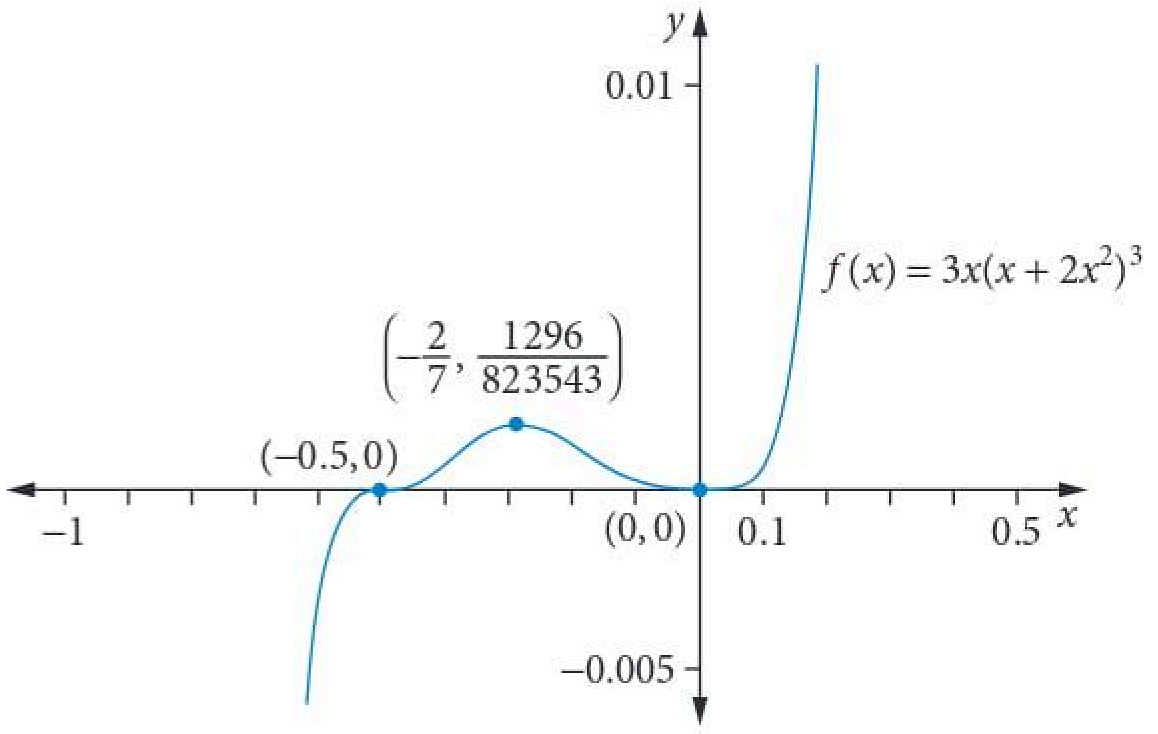
For example, we can find the gradient of the curve $f(x) = 3x(x + 2x^2)^3$ at $x = -1$.

$$f'(x) = 3x \times 3(x + 2x^2)^2 \times (1 + 4x) + (x + 2x^2)^3 \times 3.$$
$$= 3(x + 2x^2)^2[3x(1 + 4x) + (x + 2x^2)]$$
$$= 3(x + 2x^2)^2(3x + 12x^2 + x + 2x^2)$$
$$= 6x^3(1 + 2x)^2(7x + 2)$$

Exam hack

Keep the derivative in factorised form so that equating to zero for stationary points is easier than if it is in expanded form.

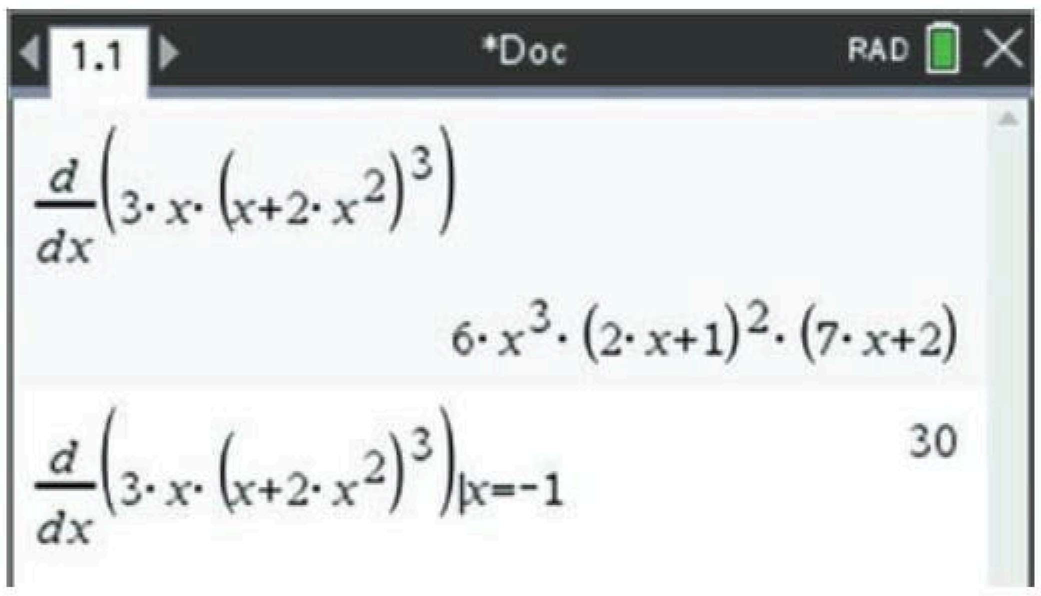
The graph of $f(x) = 3x(x + 2x^2)^3$ looks like this:



The gradient of the curve $f(x) = 3x\,(x + 2x^2)^3$ at $x = -1$ is found by substituting $x = -1$ into f'.

$\therefore\ f'(-1) = -6(-1)^2\,(-5) = 30$

TI-Nspire

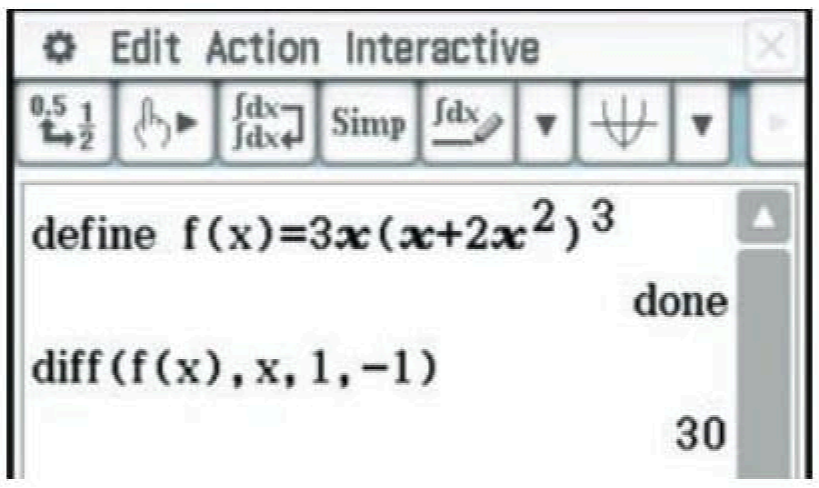


$\frac{d}{dx}\left(3\cdot x\cdot\left(x+2\cdot x^2\right)^3\right)$

$6\cdot x^3\cdot(2\cdot x+1)^2\cdot(7\cdot x+2)$

$\frac{d}{dx}\left(3\cdot x\cdot\left(x+2\cdot x^2\right)^3\right)|x=-1$ $\quad 30$

ClassPad



```
define f(x)=3x(x+2x^2)^3
                     done
diff(f(x),x,1,-1)
                       30
```

WB

p. 102

WORKED EXAMPLE 4 Gradient at a point

Find the gradient of the curve at $x = 1$ for the function $y = \dfrac{1}{x^2+1}$.

Steps	Working
1 Write y with a negative power and use the chain rule. $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$	$y = \dfrac{1}{x^2+1} = (x^2+1)^{-1}$ Let $y = u^{-1}$ and $u = x^2 + 1$ $\dfrac{dy}{du} = -u^{-2}$ and $\dfrac{du}{dx} = 2x$ $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx} = -u^{-2} \times 2x$
2 Replace $u = x^2 + 1$ and simplify the answer.	$\dfrac{dy}{dx} = -(x^2+1)^{-2} \times 2x$ $= \dfrac{-2x}{\left(x^2+1\right)^2}$
3 Substitute $x = 1$ into $\dfrac{dy}{dx}$.	$\dfrac{dy}{dx} = \dfrac{-2}{(2)^2} = -\dfrac{1}{2}$

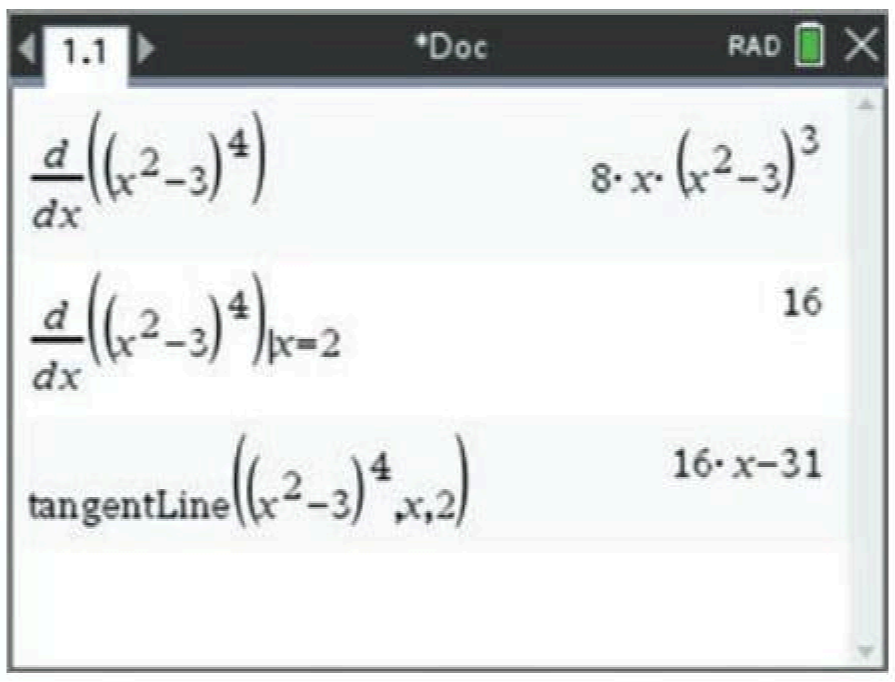
USING CAS 2 Differentiation applications

a Find $\dfrac{dy}{dx}$ for the function $y = (x^2 - 3)^4$ and the equation of the tangent at $x = 2$.

b Graph the function $y = (x^2 - 3)^4$ and the tangent to the function at $x = 2$.

TI-Nspire

a



$\frac{d}{dx}\left(\left(x^2-3\right)^4\right)$ $8\cdot x\cdot\left(x^2-3\right)^3$

$\frac{d}{dx}\left(\left(x^2-3\right)^4\right)|x=2$ 16

tangentLine$\left(\left(x^2-3\right)^4,x,2\right)$ $16\cdot x-31$

1 Press **menu > Calculus > Derivative**.

2 In the template, enter ***x*** as the variable and then enter the expression.

3 Press **menu > Calculus > Derivative at a Point**.

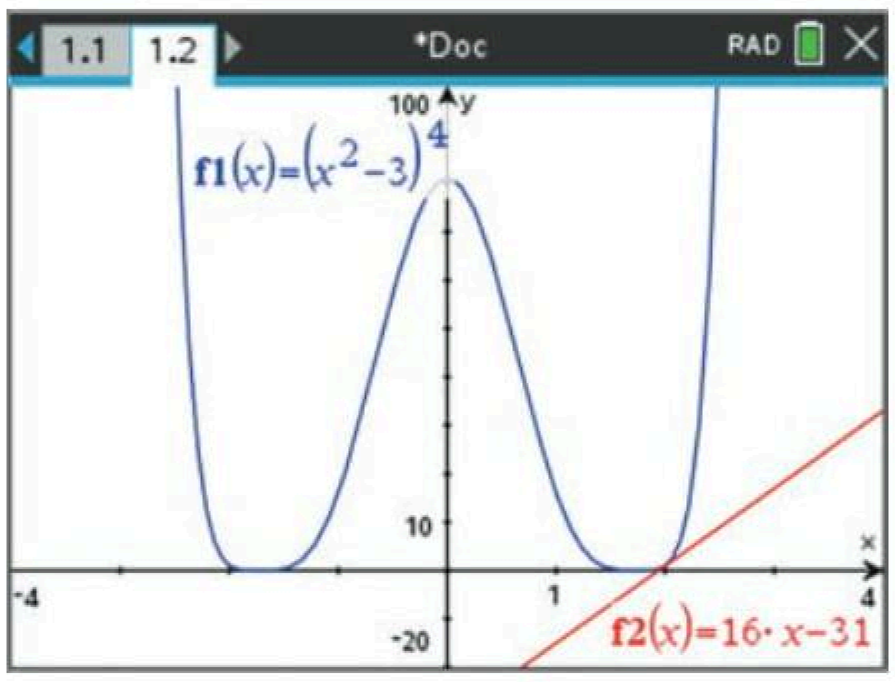
4 In the dialogue box, in the **Variable**: field enter ***x*** and in the **Value**: field enter **2**.

5 In the template, enter the expression.

6 Press **menu > Calculus > Tangent Line**.

7 Enter the expression followed by, ***x*,2**.

b



1 Add a **Graphs** page.

2 Graph the function ***f*1 (*x*) = (*x*² – 3)⁴**.

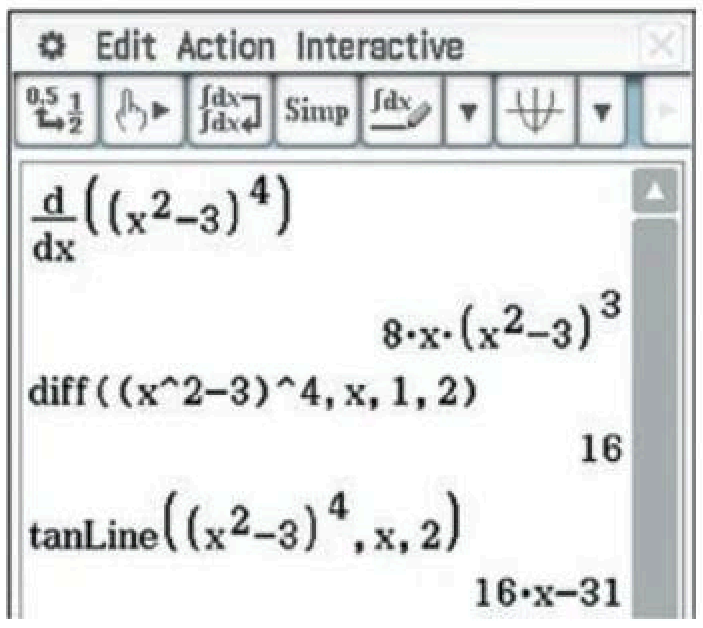
3 Adjust the window settings to suit.

4 Graph, ***f*2 (*x*) = 16*x* – 31,** which is the line tangent to the function at $x = 2$ found in part **a**.

The gradient of the tangent is $y = 16x - 31$.

ClassPad

a



```
d/dx((x^2-3)^4)
                8·x·(x^2-3)^3
diff((x^2-3)^4,x,1,2)
                           16
tanLine((x^2-3)^4,x,2)
                      16·x-31
```

1 Enter and highlight **$(x^2 - 3)^4$**.

2 Tap **Interactive > Calculation > diff.**

3 In the dialogue box, keep the default variable and tap **OK**

4 Copy and highlight **$(x^2 - 3)^4$**.

5 Tap **Interactive > Calculation > diff.**

6 In the dialogue box, tap **Derivative at value**.

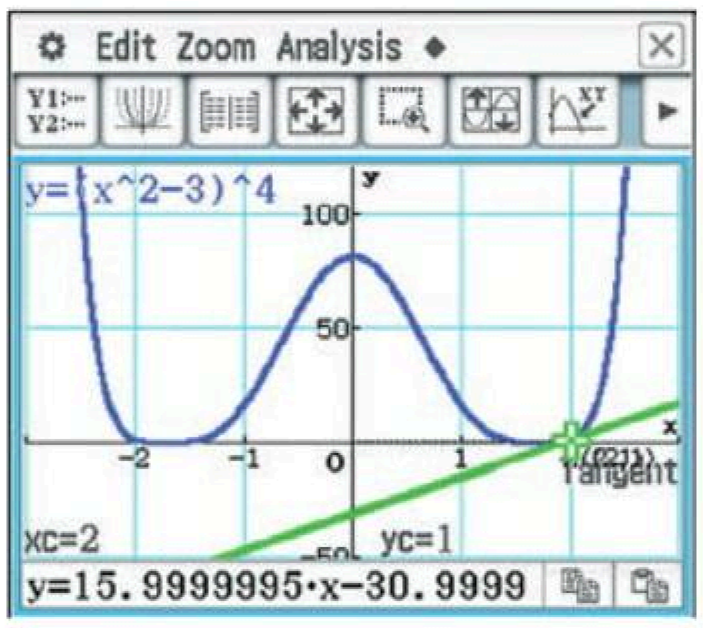
7 In the **Value:** field, enter **2** and tap **OK**.

8 Copy and highlight **$(x^2 - 3)^4$**.

9 Tap **Interactive > Calculation > line > tanLine**.

10 In the dialogue box, in the **Point:** field, enter **2** and tap **OK**.

b



```
y=15.9999995·x-30.9999
```

1 Tap the **Graph** icon to open a graphs page.

2 Drag the original equation **$(x^2 - 3)^4$** into the graph window (note the windows have been swapped).

3 In the graph window, tap **Analysis > Sketch > Tangent**.

4 Enter **2**.

5 A dialogue box will appear with a **2** in it.

6 Tap **OK** to graph the tangent line.

7 Press **EXE** and the equation of the tangent line will appear.

8 Rounding to the nearest whole number, the equation of the tangent is $y = 16x - 31$.

The gradient of the tangent is $y = 16x - 31$.

EXERCISE 5.1 The product, quotient and chain rules

ANSWERS p. 583

Mastery

1 WORKED EXAMPLE 1 TECH-FREE Find $\frac{dy}{dx}$ for the function $y = x^2(2x - 3x^4)$.

2 WORKED EXAMPLE 2 TECH-FREE Find $\frac{dy}{dx}$ for the function $y = \frac{x-3}{x+2}$.

3 WORKED EXAMPLE 3 TECH-FREE Find $f'(x)$ if $f(x) = (3 - 4x^2)^2$.

4 WORKED EXAMPLE 4 TECH-FREE Find $f'(-1)$ if $f(x) = (x + 3x^2)^{-1}$.

5 If $f(x) = \frac{2x+3}{x^2+1}$, then $f'(0)$ equals

A $-\frac{3}{2}$ **B** $-\frac{18}{25}$ **C** $\frac{3}{2}$ **D** 2 **E** 3

6 If $f(x) = 2x(x + 3x^2)$, then $f'(2)$ equals

A −64 **B** 0 **C** 56 **D** 64 **E** 80

7 Using CAS 1 The gradient of the function $f(x) = (x^2 + 3x)^{-4}$ at $x = 1$ is

A $\frac{-4(2x+3)}{(x^2+3x)^5}$ **B** $\frac{-4}{(x^2+3x)^5}$ **C** $\frac{-5}{256}$ **D** $\frac{117}{1024}$ **E** 1280

8 Differentiate each function with respect to x.

a $f(x) = 5x^7$ **b** $f(x) = \dfrac{9}{10x^{\frac{2}{3}}}$ **c** $\sqrt{3x^2+1}$

9 Find the gradient of the graph of each function at $x = 1$.

a $f(x) = 5x^7$ **b** $f(x) = \dfrac{9}{10x^{\frac{2}{3}}}$ **c** $f(x) = \sqrt{3x^2+1}$

Exam practice

10 **TECH-FREE** (3 marks) Find the derivative of the function $f: (1, \infty) \rightarrow R$, $f(x) = -\dfrac{1}{2x-2}$ at $x = 3$.

11 For the function $y = \sqrt{x}(x+3)$, $\dfrac{dy}{dx}$ equals

A $\dfrac{3}{2\sqrt{x}}(x+1)$ **B** $\dfrac{3\sqrt{x}}{2}(x+1)$ **C** $\dfrac{\sqrt{x}}{2}(3x+1)$ **D** $\dfrac{2}{\sqrt{x}}(3x-1)$ **E** $\dfrac{1}{2x^{\frac{3}{2}}}(x-3)$

12 For the function $y = \dfrac{2x+1}{x^7}$, $\dfrac{dy}{dx}$ equals

A $\dfrac{12x+7}{x}$ **B** $\dfrac{12x+7}{x^8}$ **C** $\dfrac{-12x-7}{x^8}$ **D** $\dfrac{-12x+7}{x^8}$ **E** $\dfrac{-12x-7}{x^7}$

13 For the function $y = (2x-5)^4$, $\dfrac{dy}{dx}$ equals

A $4(2x-5)^3$ **B** $2(2x-5)^4$ **C** $8(2x-5)^4$ **D** $8(2x-5)^3$ **E** $(2x-5)^3$

14 The derivative of $1 - x + \dfrac{1}{3}x^3$ is

A $1+\dfrac{1}{3}x^2$ **B** $\dfrac{1}{3}x^2$ **C** $1-x^2$ **D** $-x-x^2$ **E** x^2-1

15 Given $f(x) = x^2 - \dfrac{1}{3}x^{\frac{3}{2}}$, the value of $f'(9)$ is

A $-15\dfrac{2}{3}$ **B** $-1\dfrac{2}{3}$ **C** $12\dfrac{1}{3}$ **D** $16\dfrac{1}{2}$ **E** $19\dfrac{1}{2}$

16 The value of the gradient function of $f(x) = 2\left(\sqrt{x} - \dfrac{1}{3}\sqrt{x^3}\right)$ at $x = 1$ is

A 0 **B** $\dfrac{2}{3}$ **C** $\dfrac{7}{9}$ **D** 1 **E** $\dfrac{4}{3}$

17 The derivative of $f(x) = \left(\sqrt{3x} + \sqrt{5x^3}\right)\left(\sqrt{3x} - \sqrt{5x^3}\right)$ is

A $3x^2-5$ **B** $3-15x^2$ **C** $15+3x^2$ **D** $5-3x^2$ **E** $3x^2-15$

18 The derivative of $f(x) = \dfrac{b}{a}x^{\frac{a}{b}}$, with a and b real constants, is

A $\dfrac{1}{x^{\frac{a}{b}}}$ **B** $\dfrac{1}{x^{\frac{a}{b}}}-1$ **C** $\dfrac{1}{x^{\frac{a}{b}-1}}$ **D** $x^{\frac{a}{b}-1}$ **E** x^{ab-1}

19 The gradient function of $f(x) = 1 + x - x^2$ is

A $1-x$ **B** $2-x$ **C** $2x+1$ **D** $x-2$ **E** $1-2x$

20 If $f(x) = 2x^3 + x$, the value of $f'(-1)$ is

A -7 **B** -4 **C** 5 **D** 6 **E** 7

21 If $y = \dfrac{x^2}{x+2}$, then $\dfrac{dy}{dx}$ at $x = -1$ is equal to

A -3 **B** 0 **C** $\dfrac{1}{3}$ **D** $\dfrac{5}{9}$ **E** 2

22 If $y = \dfrac{1}{1+x}$, then $\dfrac{dy}{dx}$ is

A $-x^2$ **B** $\dfrac{x}{y}$ **C** $x + y$ **D** $\dfrac{-1}{(1+x)^2}$ **E** $\dfrac{1}{(1+x)^2}$

5.2 Differentiating circular functions

Video playlist Differentiating circular functions

Worksheets Derivatives of trigonometric functions

Trigonometric functions and gradient

Differentiating trigonometric functions

The derivatives of the circular functions

$y = \sin(x)$	$\dfrac{dy}{dx} = \cos(x)$	$y = \tan(x)$	$\dfrac{dy}{dx} = \sec^2(x)$
$y = \sin(kx)$	$\dfrac{dy}{dx} = k\cos(kx)$	$y = \tan(kx)$	$\dfrac{dy}{dx} = k\sec^2(kx)$
$y = \cos(x)$	$\dfrac{dy}{dx} = -\sin(x)$	$y = \cot(x)$	$\dfrac{dy}{dx} = -\text{cosec}^2(x)$
$y = \cos(kx)$	$\dfrac{dy}{dx} = -k\sin(kx)$	$y = \cot(kx)$	$\dfrac{dy}{dx} = -k\,\text{cosec}^2(kx)$

WORKED EXAMPLE 5 The chain rule with a circular function

WB p. 103

Find $\dfrac{dy}{dx}$ for the function $y = 3\sin^2(4x)$.

Steps	Working
1 Rewrite $y = 3\sin^2(4x)$ in chain rule form.	$y = 3\sin^2(4x) = 3[\sin(4x)]^2$
2 Use the chain rule $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ and the rule: $y = \sin(kx) \Rightarrow \dfrac{dy}{dx} = k\cos(kx)$	Let $u = \sin(4x) \Rightarrow \dfrac{du}{dx} = 4\cos(4x)$ Let $y = 3u^2 \Rightarrow \dfrac{dy}{du} = 6u$ $\therefore \dfrac{dy}{dx} = 6u \times 4\cos(4x)$
3 Substitute $u = \sin(4x)$.	$\therefore \dfrac{dy}{dx} = 6\sin(4x) \times 4\cos(4x)$
4 Simplify the answer using a double angle formula.	$\dfrac{dy}{dx} = 12\sin(8x)$

WB
p. 104

WORKED EXAMPLE 6 The product rule with a circular function

Find the gradient of the function $y = x^2 \tan(x^2 + \pi)$ at $x = 0$.

Steps	Working
1 Put $y = x^2 \tan(x^2 + \pi)$ in product rule form. Also, $y = \tan(kx) \Rightarrow \frac{dy}{dx} = k\sec^2(kx)$.	Let $u = x^2$ and $v = \tan(x^2 + \pi)$. $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = \sec^2(x^2 + \pi) \times 2x$
2 Use the product rule $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.	$\frac{dy}{dx} = x^2 \times 2x\sec^2(x^2 + \pi) + \tan(x^2 + \pi) \times 2x$ $\therefore \frac{dy}{dx} = 2x[x^2\sec^2(x^2 + \pi) + \tan(x^2 + \pi)]$
3 Substitute $x = 0$. This derivative involves the chain rule as part of the product rule.	When $x = 0$, $\frac{dy}{dx} = 0$, so the gradient of the function at $x = 0$ is 0.

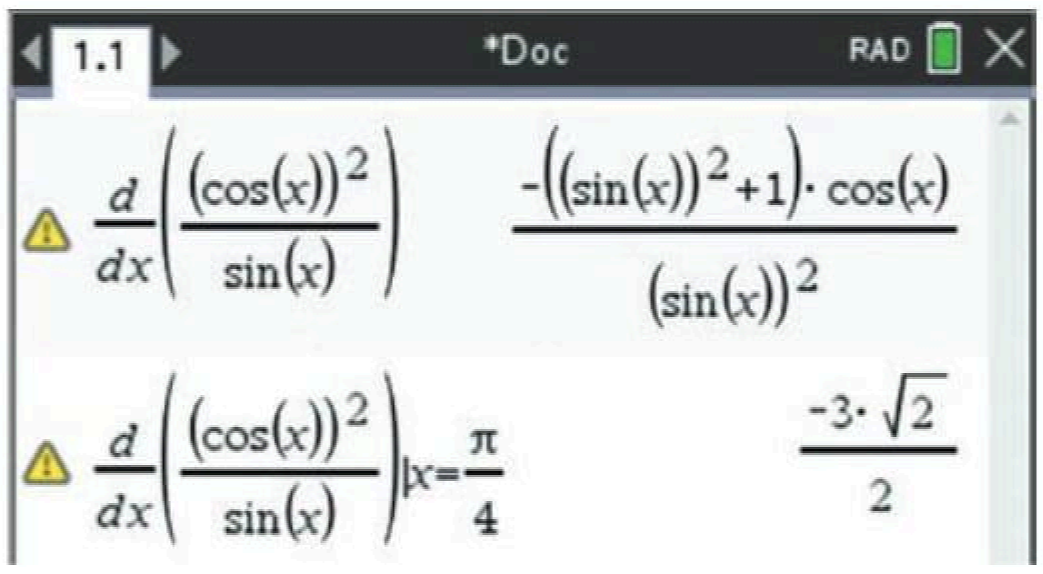
WB
p. 105

WORKED EXAMPLE 7 The quotient rule with a circular function

Find the gradient of the curve $f(x) = \frac{\cos^2(x)}{\sin(x)}$ at $x = \frac{\pi}{4}$.

Steps	Working
1 Use the quotient rule to find $f'(x)$.	$u = \cos^2(x)$ $\quad$ $v = \sin(x)$ $\frac{du}{dx} = -2\cos(x)\sin(x)$ $\quad$ $\frac{dv}{dx} = \cos(x)$ $f'(x) = \frac{[\sin(x) \times (-2\cos(x)\sin(x))] - [\cos^2(x)\cos(x)]}{\sin^2(x)}$ $= \frac{-2\sin^2(x)\cos(x) - \cos^3(x)}{\sin^2(x)}$
2 Substitute $x = \frac{\pi}{4}$.	$f'\left(\frac{\pi}{4}\right) = \frac{-2\sin^2\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) - \cos^3\left(\frac{\pi}{4}\right)}{\sin^2\left(\frac{\pi}{4}\right)}$ $= \frac{-2\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}{\left(\frac{1}{2}\right)}$ $= -\frac{3\sqrt{2}}{2}$
3 Answer the question.	The gradient of the curve $f(x) = \frac{\cos^2(x)}{\sin(x)}$ at $x = \frac{\pi}{4}$ is $-\frac{3\sqrt{2}}{2}$.

TI-Nspire

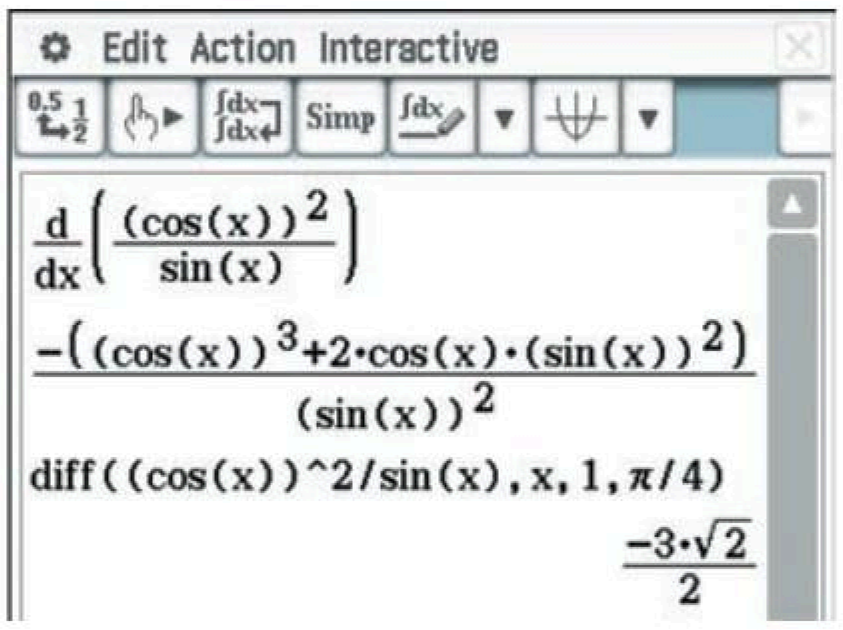


$\frac{d}{dx}\left(\frac{(\cos(x))^2}{\sin(x)}\right)$ $\frac{-((\sin(x))^2+1)\cdot\cos(x)}{(\sin(x))^2}$

$\frac{d}{dx}\left(\frac{(\cos(x))^2}{\sin(x)}\right)|x=\frac{\pi}{4}$ $\frac{-3\cdot\sqrt{2}}{2}$

Note that with CAS, $\cos^2(x)$ must be entered as $(\cos(x))^2$.

ClassPad



$\frac{d}{dx}\left(\frac{(\cos(x))^2}{\sin(x)}\right)$

$\frac{-((\cos(x))^3+2\cdot\cos(x)\cdot(\sin(x))^2)}{(\sin(x))^2}$

diff((cos(x))^2/sin(x),x,1,π/4)

$\frac{-3\cdot\sqrt{2}}{2}$

EXERCISE 5.2 Differentiating circular functions

ANSWERS p. 583

Recap

1 TECH-FREE Let $f(x) = 3(x + 1)^2$. Evaluate $f'(1)$.

2 If $f(x) = (2x + 1)(x - x^2)$, then $f'(2)$ equals

A -22 **B** -19 **C** -10 **D** -3 **E** -1

Mastery

3 WORKED EXAMPLE 5 TECH-FREE Find $\frac{dy}{dx}$ for the function $y = \sin^4(2x)$.

4 WORKED EXAMPLE 6 TECH-FREE Find the gradient of the function $y = x\cos(2x)$ at $x = \pi$.

5 For the function $y = \cos^2(2x)$, $\frac{dy}{dx}$ can be expressed as

A $2\cos^2(2x)$ **B** $\sin(4x)$ **C** $\cos(2x)\sin(2x)$

D $-\cos(2x)\sin(2x)$ **E** $-2\sin(4x)$

6 Find the gradient of the function $y = x^2\sin(2x - \pi)$ at $x = \pi$.

A $2\pi^2$ **B** -2π **C** 0 **D** $-2\pi^2$ **E** -8π

Exam practice

7 TECH-FREE (2 marks) Differentiate $y = \cos(x)\sin(x)$.

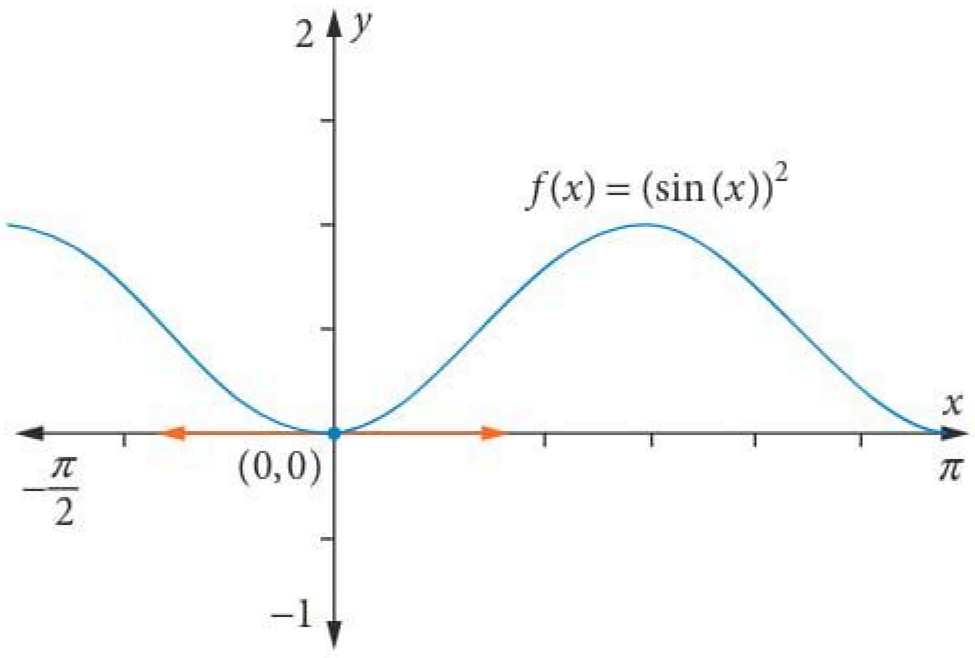
8 WORKED EXAMPLE 7 TECH-FREE (2 marks) Find $\frac{dy}{dx}$ for $y = \frac{\cos(2x)}{\sin(2x)}$.

9 TECH-FREE (2 marks) Find y' if $y = \cos(4x^3 - x)$.

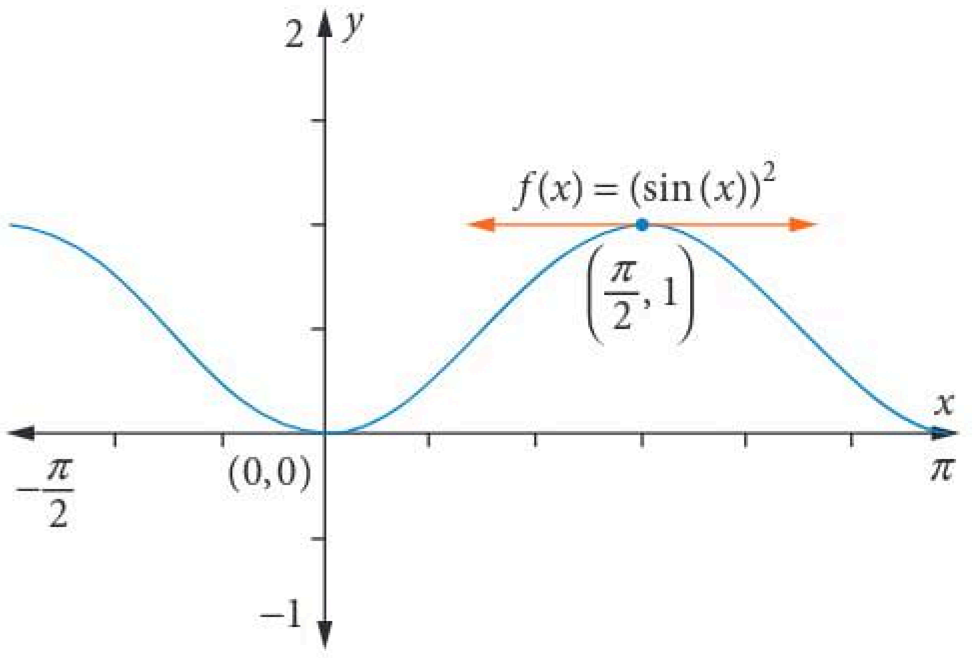
10 TECH-FREE (2 marks) Differentiate $y = 4x^2\cos(x^3)$.

11 If $f(x) = \sin^2(x)$, which one of the following graphs could represent the tangent to the graph of $y = f(x)$ at $x = \dfrac{\pi}{4}$?

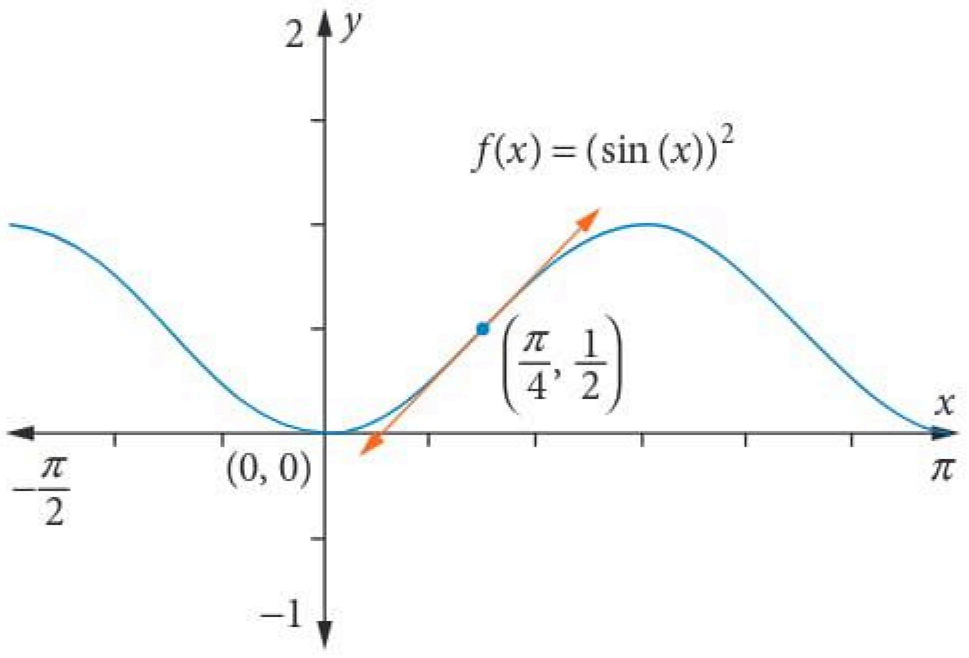
A



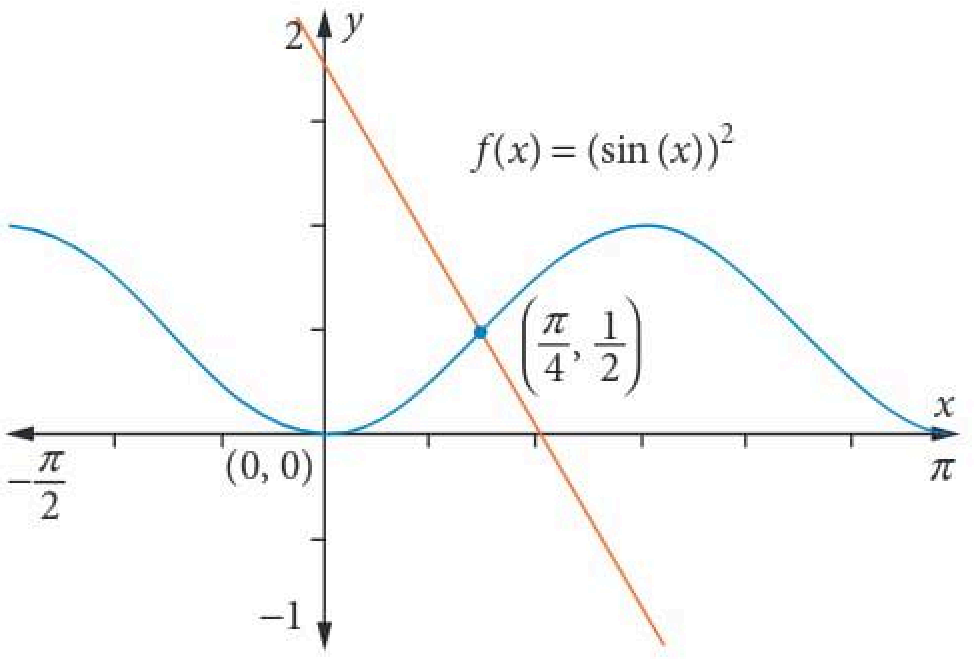
B



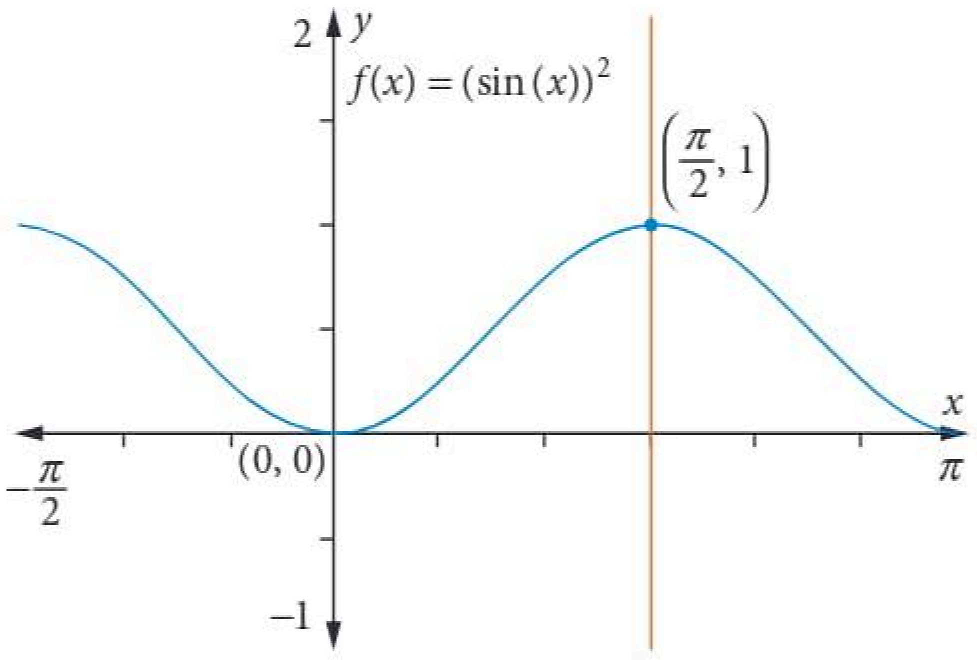
C



D



E



12 For the function $y = \tan^2(2x)$, $\dfrac{dy}{dx}$ equals

A $2\tan(2x)$
B $4\tan(2x)$
C $4\tan(2x)\sec^2(2x)$
D $4\sec^2(2x)$
E $4\tan(2x)\sec(2x)$

13 For the function $y = x\sin^2(2x)$, the gradient at $x = \dfrac{\pi}{2}$ is

A 0
B π
C $-\pi$
D undefined
E $\dfrac{\pi}{2}$

14 For the function $y = \sin^2(x)$, $\dfrac{dy}{dx}$ equals

A $\sin(2x)$
B $\cos(2x)$
C $\tan(2x)$
D $2\sin(x)$
E $2\cos(x)$

15 For the function $y = x\sin^2(x)$, the gradient at $x = \dfrac{\pi}{4}$ is

A $\dfrac{\pi}{4}$
B $\dfrac{\pi}{4} + \dfrac{1}{2}$
C $\dfrac{\pi}{4} + 1$
D 1
E 0

16 The value of $\dfrac{dy}{dx}$ at $x = 1.5$ for $y = \sqrt{1 + \cot(x)}$ is closest to

A −0.8
B −0.5
C 0.2
D 0.5
E 0.6

5.3 Differentiating inverse circular functions

The derivatives of the inverse circular functions

$y = \sin^{-1}(x)$	$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}, x \in (-1, 1)$ Note the domain $x \in (-1, 1)$, where the derivative is *not* defined at the endpoints of the graph of $y = \sin^{-1}(x)$.
$y = \cos^{-1}(x)$	$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}, x \in (-1, 1)$
$y = \tan^{-1}(x)$	$\frac{dy}{dx} = \frac{1}{1+x^2}, x \in R$
$y = \sin^{-1}\left(\frac{x}{a}\right)$	$\frac{dy}{dx} = \frac{1}{\sqrt{a^2-x^2}}, x \in (-a, a)$ Note the domain $x \in (-a, a)$, where the derivative is *not* defined at the endpoints of the graph of $y = \sin^{-1}\left(\frac{x}{a}\right)$.
$y = \cos^{-1}\left(\frac{x}{a}\right)$	$\frac{dy}{dx} = \frac{-1}{\sqrt{a^2-x^2}}, x \in (-a, a)$
$y = \tan^{-1}\left(\frac{x}{a}\right)$	$\frac{dy}{dx} = \frac{a}{a^2+x^2}, x \in R$

Video playlist Differentiating inverse circular functions

Worksheets Derivatives of inverse trigonometric functions

Inverse trigonometric functions and gradient

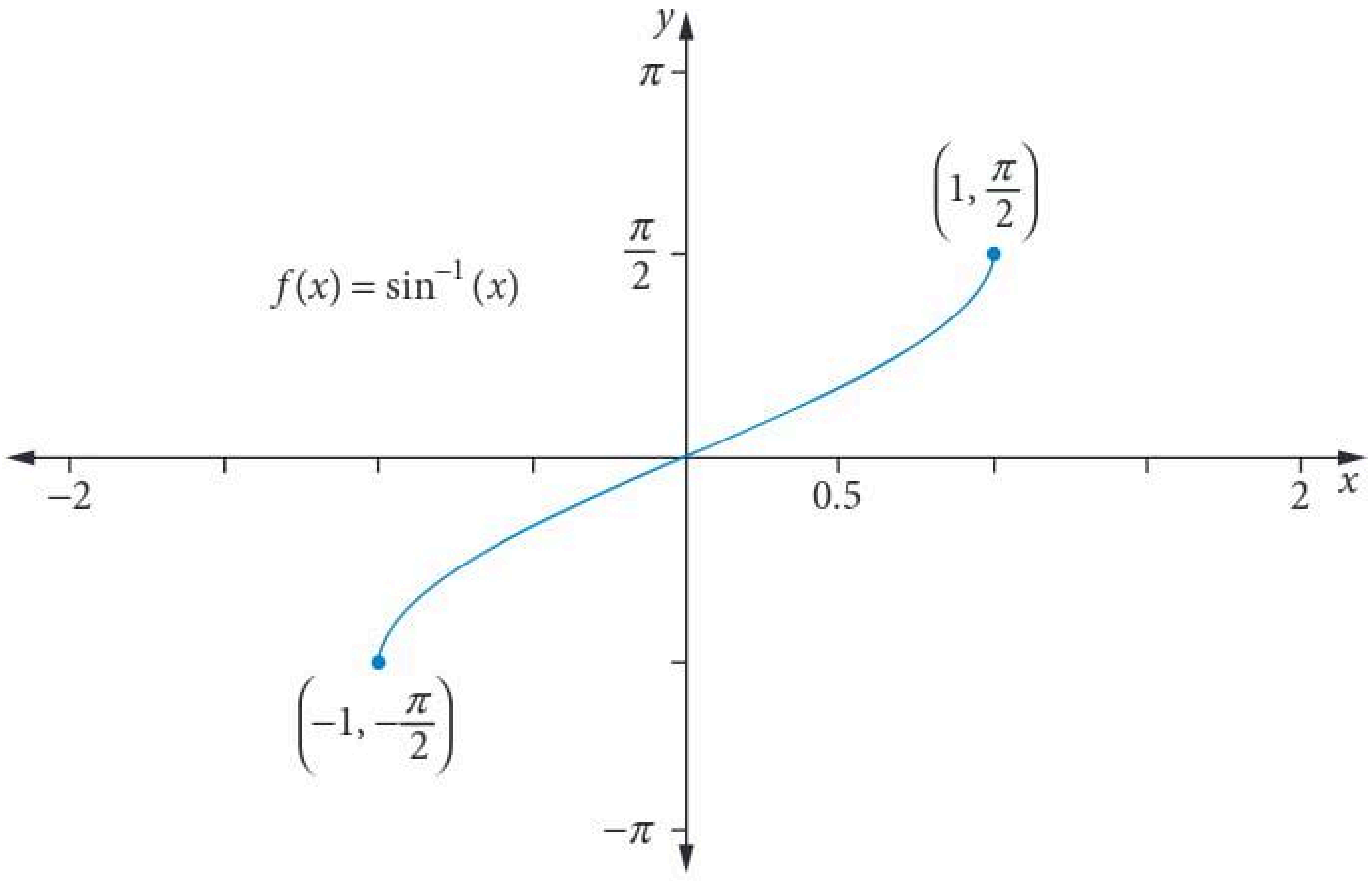
Exam hack

Note that $\sin^{-1}(x)$, $\cos^{-1}(x)$ and $\tan^{-1}(x)$ can also be written as $\arcsin(x)$, $\arccos(x)$ and $\arctan(x)$, respectively, especially in exams.

Using $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

We use this relationship to show the derivative of the inverse circular functions.

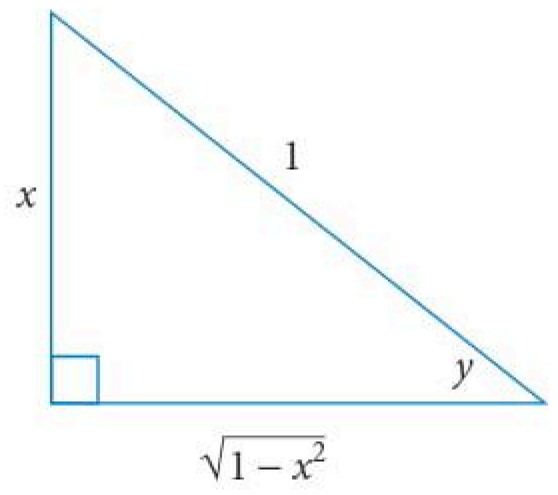
Let $y = \sin^{-1}(x)$, where $x \in [-1, 1]$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



If $y = \sin^{-1}(x)$, then $x = \sin(y)$.

Then $\dfrac{dx}{dy} = \cos(y)$.

Using $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$, we get $\dfrac{dy}{dx} = \dfrac{1}{\cos(y)}$.



Using right-angled triangle relationships and $\sin(y) = x = \dfrac{x}{1}$,

$$\cos(y) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

so $\dfrac{1}{\cos(y)} = \dfrac{1}{\sqrt{1-x^2}}$, for $y \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

Because $\dfrac{dy}{dx} = \dfrac{1}{\cos(y)}$, hence, $\dfrac{d}{dx}\left(\sin^{-1}(x)\right) = \dfrac{1}{\sqrt{1-x^2}}$ for $x \in (-1, 1)$.

We get, if $y = \sin^{-1}(x)$, then $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$, $x \in (-1, 1)$.

Generally, $\dfrac{d}{dx}\left(\sin^{-1}\left(\dfrac{x}{a}\right)\right) = \dfrac{1}{\sqrt{a^2 - x^2}}$ for $x \in (-a, a)$.

WB

p. 106

WORKED EXAMPLE 8 Differentiating inverse circular functions

Find $\dfrac{dy}{dx}$ for the function $y = \sin^{-1}\left(\dfrac{3x}{2}\right)$, giving the domain for your answer.

Steps	**Working**
1 Identify the value of a in $y = \sin^{-1}\left(\dfrac{3x}{2}\right)$.	$a = \dfrac{2}{3}$
2 Use the rule $\dfrac{d}{dx}\left(\sin^{-1}\left(\dfrac{x}{a}\right)\right) = \dfrac{1}{\sqrt{a^2 - x^2}}$.	$\dfrac{d}{dx}\left(\sin^{-1}\left(\dfrac{3x}{2}\right)\right) = \dfrac{1}{\sqrt{\left(\frac{2}{3}\right)^2 - x^2}}$
3 Simplify your answer if necessary.	$\therefore \dfrac{dy}{dx} = \dfrac{1}{\sqrt{\frac{4-9x^2}{9}}} = \dfrac{3}{\sqrt{4-9x^2}}$
4 Give the domain.	$x \in \left(-\dfrac{2}{3}, \dfrac{2}{3}\right)$

Note: The derivative can also be found using the chain rule:

$$u = 3x, \quad \frac{du}{dx} = 3$$

$$y = \sin^{-1}\left(\frac{u}{2}\right), \quad \frac{dy}{du} = \frac{1}{\sqrt{4-u^2}}$$

$$\frac{dy}{dx} = \frac{3}{\sqrt{4-(3x)^2}} = \frac{3}{\sqrt{4-9x^2}}$$

WORKED EXAMPLE 9 Differentiating inverse circular functions at a point

Find the gradient of the function $y = \arctan(2 + 3x^2)$ at $x = 2$.

Steps	Working
1 Put $y = \arctan(2+3x^2)$ in chain rule form.	Let $u = 2 + 3x^2$ and $y = \arctan(u)$.
2 Use the chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ and the rule: $y = \arctan(x) \Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$	$u = 2 + 3x^2 \Rightarrow \frac{du}{dx} = 6x$ $y = \arctan(u) \Rightarrow \frac{dy}{dx} = \frac{1}{1+u^2}$ $\therefore \frac{dy}{dx} = \frac{1}{1+u^2} \times 6x$
3 Substitute $u = 2 + 3x^2$.	$\frac{dy}{dx} = \frac{1}{1+\left(2+3x^2\right)^2} \times 6x$ $\therefore \frac{dy}{dx} = \frac{6x}{1+\left(2+3x^2\right)^2}$
4 Substitute $x = 2$.	$\therefore \frac{dy}{dx} = \frac{6(2)}{1+\left(2+3(2)^2\right)^2} = \frac{12}{197}$

WB p. 107

EXERCISE 5.3 Differentiating inverse circular functions

ANSWERS p. 584

Recap

1 TECH-FREE Find $\frac{dy}{dx}$ for $y = \frac{\sin(2x)}{x}$ at $x = \pi$.

2 The gradient of the curve with rule $y = \sin(x^2 + 1)$ at $x = 1$ is

A $2\cos(x)$ **B** $2\cos(2)$ **C** $2x\cos(x^2+1)$ **D** $2\cos(3)$ **E** $\cos(2x)$

Mastery

3 WORKED EXAMPLE 8 TECH-FREE Find $\frac{dy}{dx}$ for the function $y = \tan^{-1}\left(\frac{2x}{5}\right)$, giving the domain and range for your answer.

4 WORKED EXAMPLE 9 TECH-FREE Find the gradient of the function $y = \arcsin(2x^2 - x)$ at $x = 0$.

5 The gradient of the function $y = -\cos^{-1}(3x^2)$, including its domain is

A $\frac{6x}{\sqrt{1-9x^4}}, x \in R$ **B** $-\cos^{-1}(3x^2), x \in R$ **C** $x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$

D $\frac{6x}{\sqrt{1-9x^4}}, x \in \left[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$ **E** $\frac{6x}{\sqrt{1-9x^4}}, x \in \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

6 The gradient of the function $y = \arctan(2x^2 + 1)$ at $x = -1$ is

A -1 **B** $\tan^{-1}(3)$ **C** $\frac{2}{5}$ **D** $-\frac{2}{5}$ **E** 0

Exam practice

80–100% 60–79% 0–59%

7 © VCAA 2004 1BQ1 58% TECH-FREE (2 marks) Show that, for $0 < x < \frac{1}{2}$,

$$\frac{d}{dx}\left(\sin^{-1}\left(\sqrt{2x}\right)\right) = \frac{1}{\sqrt{2x(1-2x)}}.$$

8 © VCAA 2009 1Q10 TECH-FREE (5 marks) Let $f(x) = \frac{2}{\pi}\arcsin\left(\frac{1}{2}x+1\right) - 3$.

a 50% State the implied domain and the range of f. 2 marks

b 45% Find $f'(x)$ giving your answer in the form $f'(x) = \frac{a}{\pi\sqrt{bx(x+c)}}$, where a, b and c are integers. 3 marks

9 For the function $y = \cos^{-1}\left(\frac{x}{4}+1\right)$, $\frac{dy}{dx}$ equals

A $\frac{-1}{\sqrt{-x(x+8)}}$ **B** $\frac{-1}{\sqrt{x(x+8)}}$ **C** $\frac{-1}{\sqrt{4-x^2}}$ **D** $\frac{-1}{\sqrt{16-x^2}}$ **E** $\frac{-4}{\sqrt{1-16x^2}}$

10 For the function $y = x\cos^{-1}(2x)$, $\frac{dy}{dx}$ at $x = \frac{1}{2\sqrt{2}}$ equals

A $\frac{1}{4\sqrt{2}}$ **B** $\frac{\pi}{4}$ **C** $\frac{\pi}{4}+1$ **D** $\frac{\pi}{4}-1$ **E** $-2\sqrt{2}$

11 © VCAA 2003 1AQ4 67% If $y = \sin^{-1}\left(\frac{4}{x}\right)$, $x > 4$, then $\frac{dy}{dx}$ is equal to

A $-\frac{4}{x\sqrt{x^2-16}}$ **B** $\frac{x}{\sqrt{x^2-16}}$ **C** $-\frac{4}{x\sqrt{x^2-4}}$ **D** $-\frac{4}{\sqrt{x^2-16}}$ **E** $\frac{4}{x\sqrt{x^2-16}}$

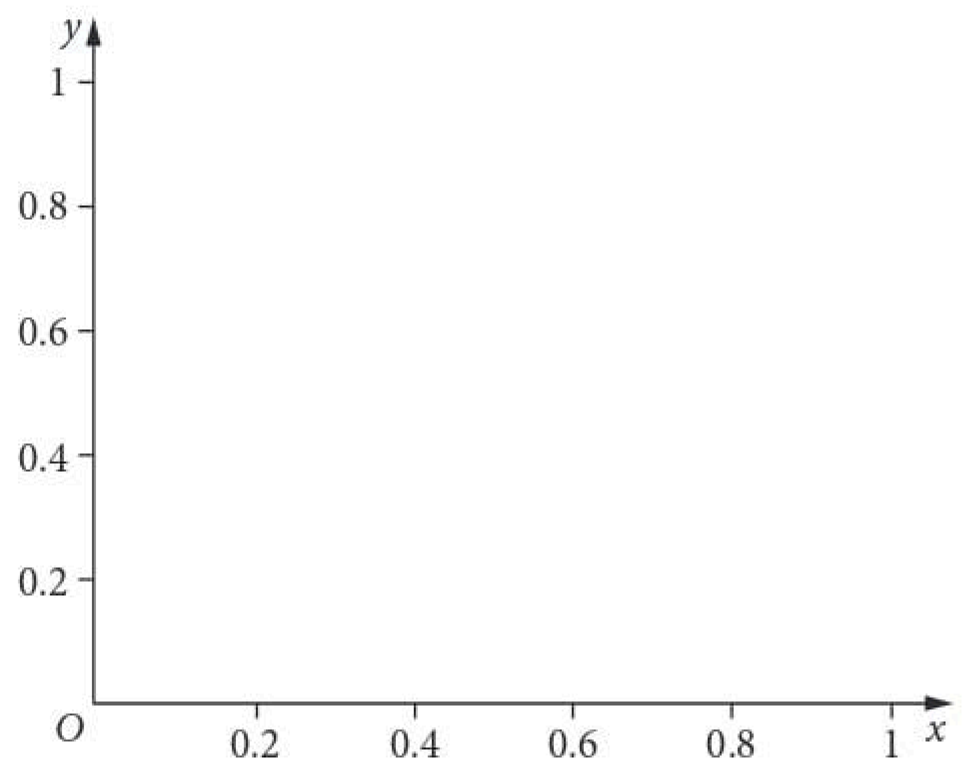
12 © VCAA 2005 1AQ5 46% If $y = \tan^{-1}(\sqrt{3x})$, then $\frac{dy}{dx}$ is equal to

A $\frac{1}{1+3x}$ **B** $\frac{\sqrt{3}}{1+3x}$ **C** $\frac{\sqrt{3}}{1+3x^2}$ **D** $\frac{1}{2\sqrt{3x}(1+3x)}$ **E** $\frac{\sqrt{3}}{2\sqrt{x}(1+3x)}$

13 © VCAA 2007 2BQ2 (4 marks) Let $f(x) = x\arctan(x)$.

a 76% Find $f'(x)$, and calculate the slope of the graph of f at $x = 0$. 2 marks

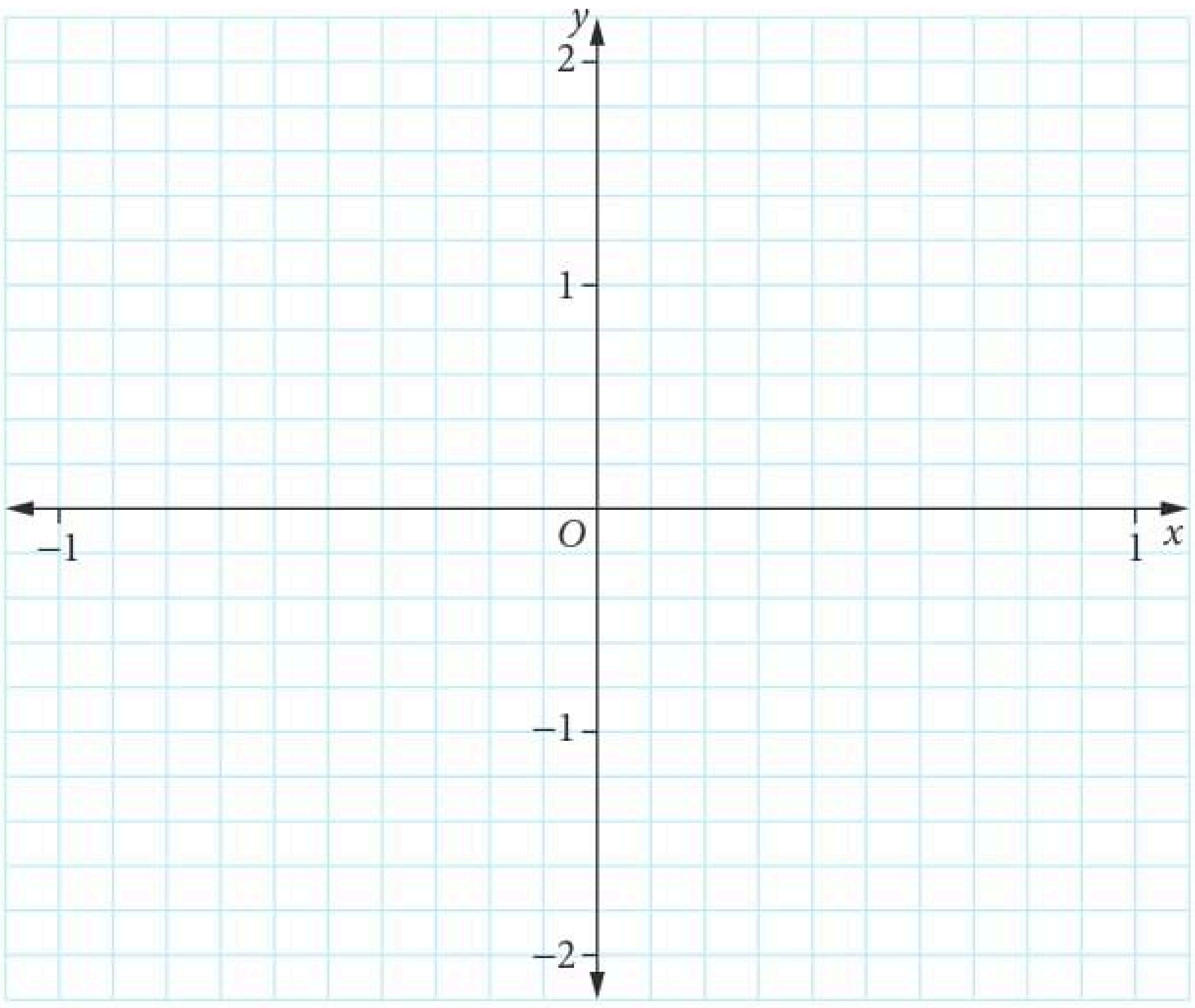
b 75% Copy the axes below, and on them sketch the curves of $y = x\arctan(x)$ and $y = \arctan(x)$ over the domain $[0, 1]$, clearly labelling each graph. 2 marks



14 © VCAA 2010 2BQ4acde (10 marks)

Consider the function f with rule $f(x) = \sin^{-1}(2x^2 - 1)$.

a 65% Copy the axes below and on them sketch the graph of the relation $y = f(x)$. Label the endpoints with their **exact coordinates** and label the x- and y-intercepts with their exact values. 3 marks

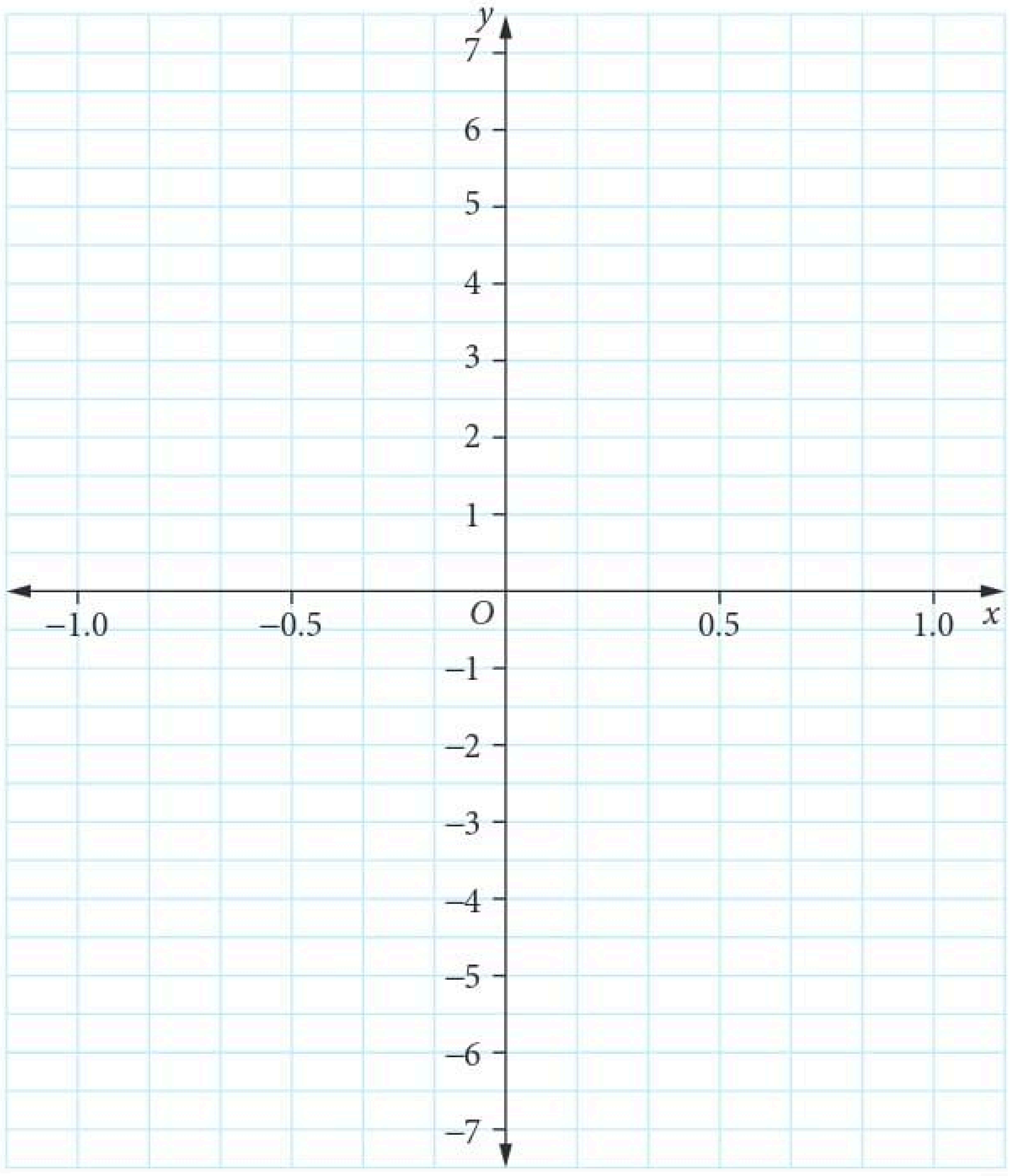


b 55% Show that $f'(x) = \dfrac{2}{\sqrt{1-x^2}}$ for $x \in (0, a)$ and find the value of a. 3 marks

c 41% Copy and complete the following to specify $f'(x)$ as a hybrid function over the maximal domain of f'. 2 marks

$$f'(x) = \begin{cases} \underline{\hspace{6em}} & \text{for } x \in \underline{\hspace{4em}} \\ \underline{\hspace{6em}} & \text{for } x \in \underline{\hspace{4em}} \end{cases}$$

d 27% Copy the axes below and on them sketch the graph of the hybrid function f', showing any asymptotes. 2 marks



Video playlist
Differentiating exponential and logarithmic functions

Worksheets
Derivatives of exponential functions

Derivatives of logarithmic functions

Exponential and logarithmic functions

5.4 Differentiating exponential and logarithmic functions

The function $f(x) = e^x$ is the only function whose derivative (and **anti-derivative**) is itself.

The derivatives of the exponential and logarithmic functions

$y = e^x$	$\frac{dy}{dx} = e^x$
$y = \log_e(x)$	$\frac{dy}{dx} = \frac{1}{x}$
$y = e^{kx}$	$\frac{dy}{dx} = ke^{kx}$
$y = \log_e(kx)$	$\frac{dy}{dx} = \frac{k}{kx} = \frac{1}{x}$

WB
p. 108

WORKED EXAMPLE 10 Differentiating exponential functions

Find $\frac{dy}{dx}$ for the function $y = x^2 e^{3x}$ at $x = 1$.

Steps	**Working**
1 Use the product rule $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.	Let $u = x^2$ and $v = e^{3x}$. $\frac{dy}{dx} = x^2 \times 3e^{3x} + e^{3x} \times 2x$
2 Simplify the answer.	$\frac{dy}{dx} = xe^{3x}(3x + 2)$
3 Substitute $x = 1$.	$\frac{dy}{dx} = e^3(3 + 2) = 5e^3$

WB
p. 109

WORKED EXAMPLE 11 Differentiating logarithmic functions

Find $\frac{dy}{dx}$ for $y = \log_e\left(3x^2 - \frac{1}{x}\right)$, where $x \neq 0$.

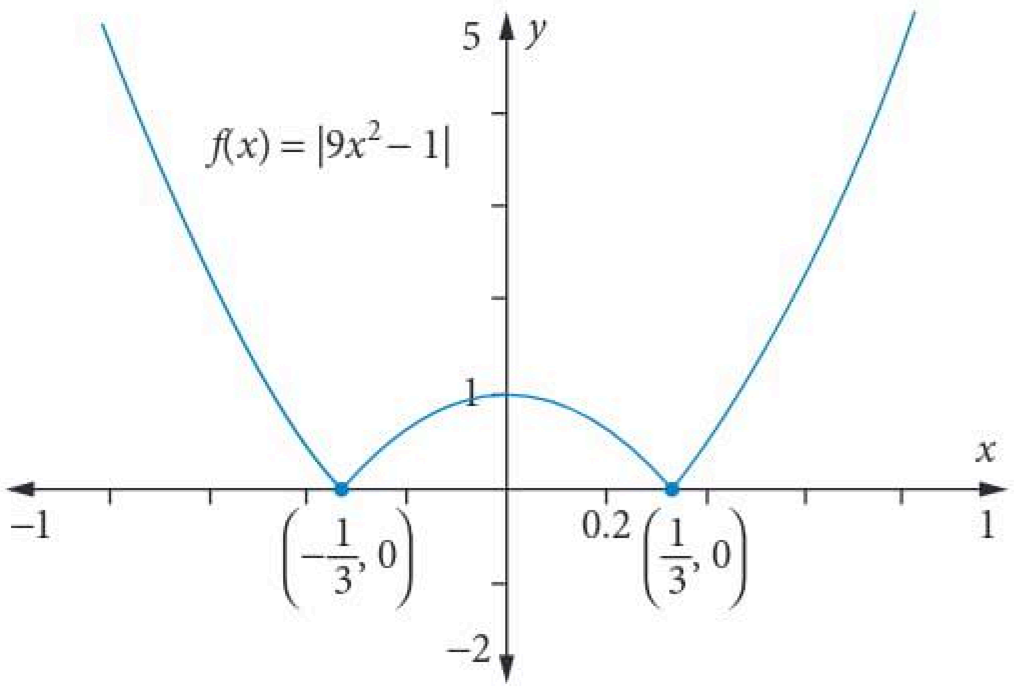
Steps	**Working**
1 Use the chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.	Let $u = 3x^2 - \frac{1}{x} \Rightarrow \frac{du}{dx} = 6x + \frac{1}{x^2}$ Let $y = \log_e(u) \Rightarrow \frac{dy}{du} = \frac{1}{u}$ $\frac{dy}{dx} = \frac{1}{u} \times \left(6x + \frac{1}{x^2}\right)$
2 Substitute $u = 3x^2 - \frac{1}{x}$.	$\therefore \frac{dy}{dx} = \frac{1}{\left(3x^2 - \frac{1}{x}\right)} \times \left(6x + \frac{1}{x^2}\right)$

3 Simplify the answer.

$$\frac{dy}{dx}=\frac{6x+\frac{1}{x^2}}{3x^2-\frac{1}{x}}\times\frac{x^2}{x^2}$$
$$=\frac{6x^3+1}{3x^4-x}$$
$$=\frac{6x^3+1}{x(3x^3-1)}$$

We can now use the product, quotient and chain rules to differentiate the full range of functions we use in Specialist Mathematics.

Consider the function $y = |9x^2 - 1|$.



This function actually equals $f(x)=\begin{cases}9x^2-1 & \text{for } x\le -\frac{1}{3}\cup x\ge\frac{1}{3}\\ -(9x^2-1) & \text{for } -\frac{1}{3}<x<\frac{1}{3}\end{cases}$

Hence the derivative of this function equals $f'(x)\begin{cases}18x & \text{for } x<-\frac{1}{3}\cup x>\frac{1}{3}\\ -18x & \text{for } -\frac{1}{3}<x<\frac{1}{3}\end{cases}$

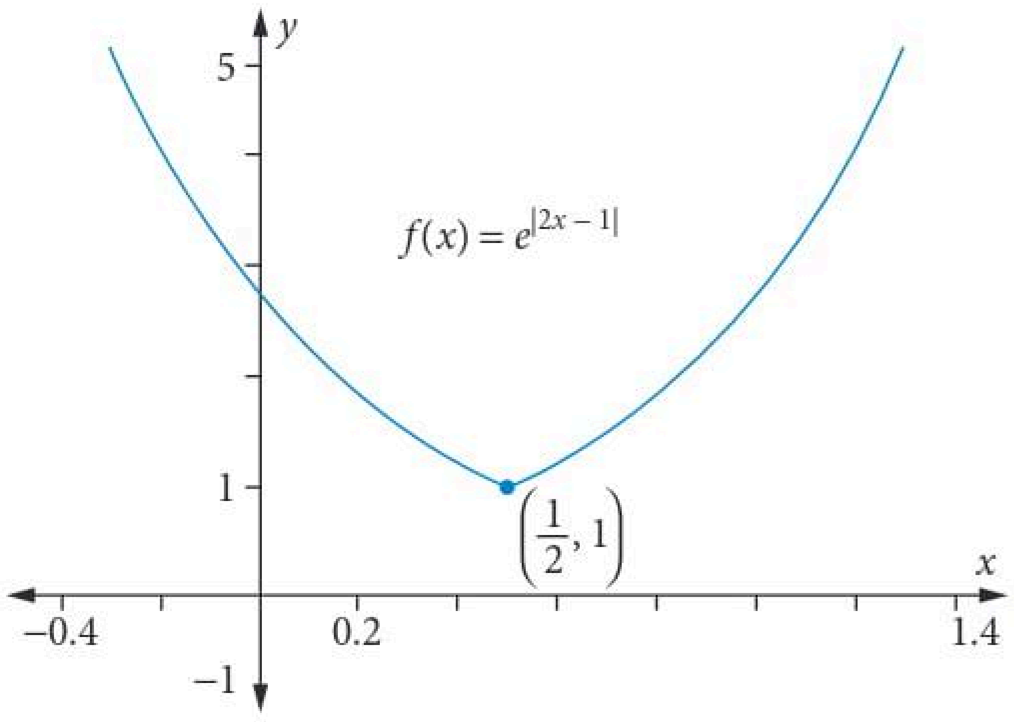
Note that the derivative does not exist at the cusps, $x=-\frac{1}{3}$ and $x=\frac{1}{3}$.

Now consider the function $y = e^{|2x-1|}$.

This function actually equals $f(x)=\begin{cases}e^{2x-1} & \text{for } x\ge\frac{1}{2}\\ e^{-(2x-1)} & \text{for } x<\frac{1}{2}\end{cases}$

Hence, the derivative of this function equals $f'(x)\begin{cases}2e^{2x-1} & \text{for } x>\frac{1}{2}\\ -2e^{-(2x-1)} & \text{for } x<\frac{1}{2}\end{cases}$

Note that the derivative does not exist at the cusp.



USING CAS 3 Differentiation: a mixture of functions

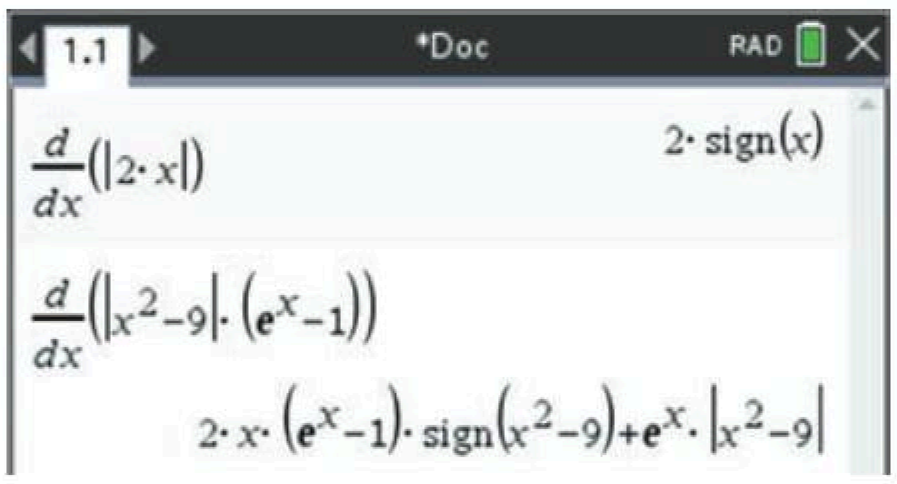
Find $\frac{dy}{dx}$ for each function.

a $y = |2x|$

b $y = |x^2 - 9|(e^x - 1)$

TI-Nspire

a, b



$\frac{d}{dx}(|2\cdot x|)$ $\quad 2\cdot \text{sign}(x)$

$\frac{d}{dx}(|x^2-9|\cdot(e^x-1))$ $\quad 2\cdot x\cdot(e^x-1)\cdot \text{sign}(x^2-9)+e^x\cdot|x^2-9|$

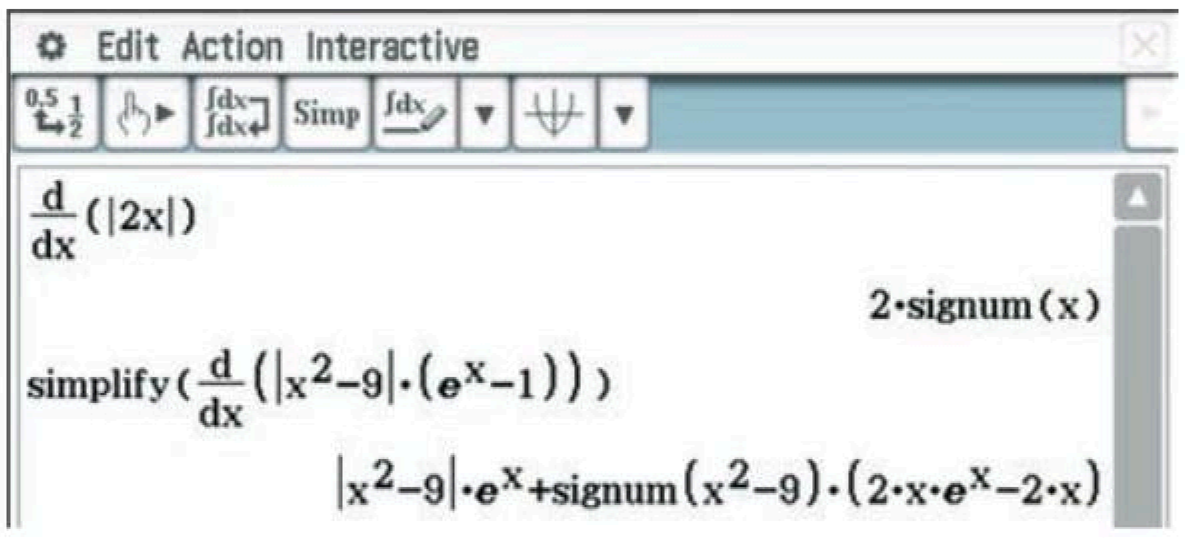
1 Press **menu > Calculus > Derivative**.

2 In the template, enter **x** as the variable and then enter the first expression.

3 Repeat for the second expression.

ClassPad

a, b



$\frac{d}{dx}(|2x|)$ $\quad 2\cdot\text{signum}(x)$

$\text{simplify}(\frac{d}{dx}(|x^2-9|\cdot(e^x-1)))$ $\quad |x^2-9|\cdot e^x+\text{signum}(x^2-9)\cdot(2\cdot x\cdot e^x-2\cdot x)$

1 Enter and highlight the first expression.

2 Tap **Interactive > Calculation > diff**.

3 Repeat for the second expression.

4 Tap **Simp** to simplify the expression.

The solutions include the 'sign' or 'signum' function. This means the solution varies depending on whether the sign of the expression inside it is positive or negative.

Exam hack

Never write the word 'sign' or 'signum' in your answer in the exam as this is CAS language, not mathematical notation.

EXERCISE 5.4 Differentiating exponential and logarithmic functions

ANSWERS p. 584

Recap

1 Find $\frac{d}{dx}\left(\arctan\left(x^2\right)\right)$.

2 © VCAA 2017 1Q6 **41%** **TECH-FREE** (3 marks) Let $f(x) = \frac{1}{\arcsin(x)}$. Find $f'(x)$ and state the largest set of values of x for which $f'(x)$ is defined.

Mastery

3 WORKED EXAMPLE 10 **TECH-FREE** Find the gradient of the each function at $x = 1$.

a $y = xe^{3x}$

b $y = x\log_e(x)$

c $y = x^2\log_e(2x)$

4 WORKED EXAMPLE 11 **TECH-FREE** Find $\frac{dy}{dx}$ for each function.

a $y = \log_e\left(\frac{1}{x} + x\right)$

b $y = \log_e(x^2 + 1)$

c $y = x^2\log_e(3x)$

5 $\frac{d}{dx}\left(e^{4x+1}\right)$ at $x = 2$ is equal to

A e^4

B e^5

C $4e^4$

D $4e^9$

E $9e^9$

6 Using CAS 3 An expression for $\frac{dy}{dx}$ in the function $y = 3\log_e(3x)(x+9)$ is

A $3\log_e(3x) + 3 + \frac{27}{x}$ **B** $3\log_e(3x) + 3x$ **C** $\frac{3\log_e(3x)}{x}$

D $3\log_e(3x)$ **E** $3\log_e\left(\frac{x}{3}\right) + \frac{27}{x}$

7 Find $\frac{dy}{dx}$ for each function.

a $y = \frac{e^x}{x^2}$ **b** $y = \frac{e^{6x}}{3x}$ **c** $y = \frac{2e^{5x}}{5x^3}$

8 Find the derivative of each function.

a $y = x^2 e^{-4x}$ **b** $y = \frac{e^{\sqrt{x}}}{x}$ **c** $y = \sin(3e^{5x})$

9 Find $f'(2)$ for each function.

a $f(x) = xe^x$ **b** $f(x) = (2x+3)e^x$ **c** $f(x) = 5x^3e^x$

10 Differentiate each function at $x = 1$.

a $y = 9e^x$ **b** $y = -e^x$ **c** $y = e^x + x^2$ **d** $y = (2e^x - 3)^6$

Exam practice

11 TECH-FREE (8 marks) Find the derivative of each function at $x = 1$.

a $y = \log_e(3x)$ **b** $y = \log_e(3x^2 + 1)$ **c** $y = \log_e(e^x)$ **d** $y = \log_e[\sin(x^2)]$

12 For $y = (2x+1)^2 e^{3x+1}$, $\frac{dy}{dx}$ equals

A $e^{3x}(6x+7)(2x+1)$ **B** $e^{3x+1}(6x+7)(2x+1)$ **C** $e^{3x+1}(6x+7)$

D $e^{3x+1}(x^2+20x+7)$ **E** $e^{3x}(12x^2+20x+7)$

Remember $\log_e e^x = x$ and $e^{\log_e(x)} = x$.

13 The gradient function for the curve $y = \log_e|1-2x|$ is

A $\frac{2}{2x-1}$ **B** $\frac{-2}{2x-1}$ **C** $-\log_e(2)$ **D** $\frac{1}{2x-1}$ **E** $\frac{1}{1-2x}$

14 The gradient of the curve $y = \log_e\left|\left(\frac{2x-1}{x}\right)\right|$ at $x = -2$ is

A $\left.\frac{1}{x(x-1)}\right|$ **B** $\frac{1}{x(2x-1)}$ **C** $\log_e(5) - \log_e(2)$

D $\frac{1}{6}$ **E** $\frac{1}{10}$

15 The gradient function of the curve $y = \log_e[\sin(x)]$ is

A $\cos(x)$ **B** $\cot(x)$ **C** $\cos(x)\sin(x)$ **D** $\tan(x)$ **E** $\log_e[\tan(x)]$

5.5 The second derivative

Video playlist
The second derivative

Worksheets
The second derivative

First and second derivatives

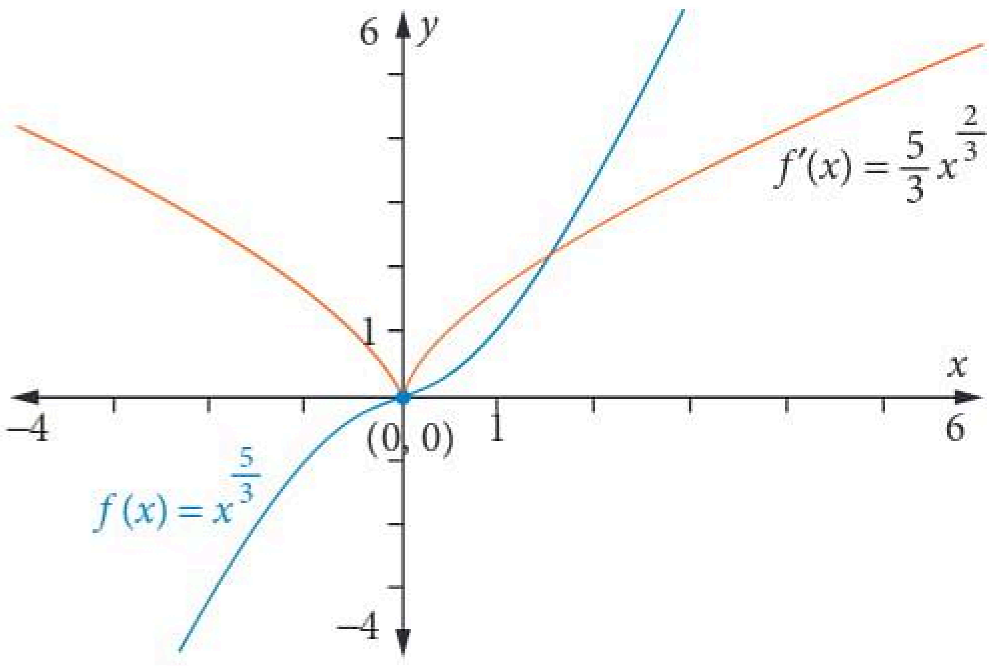
The derivative of a function is used to analyse the behaviour of that function.

The derivative of the derivative is called the **second derivative**.

For a function $y = f(x)$, the (first) derivative is written as $f'(x)$ or $\frac{dy}{dx}$ and the second derivative as $f''(x)$ or $\frac{d^2y}{dx^2}$.

If the first derivative exists at a particular point of a function, we can't assume that the second derivative also exists at that point. For example, for $f(x) = x^{\frac{5}{3}}$, $f'(x) = \frac{5}{3}x^{\frac{2}{3}}$ and $f'(0) = 0$.

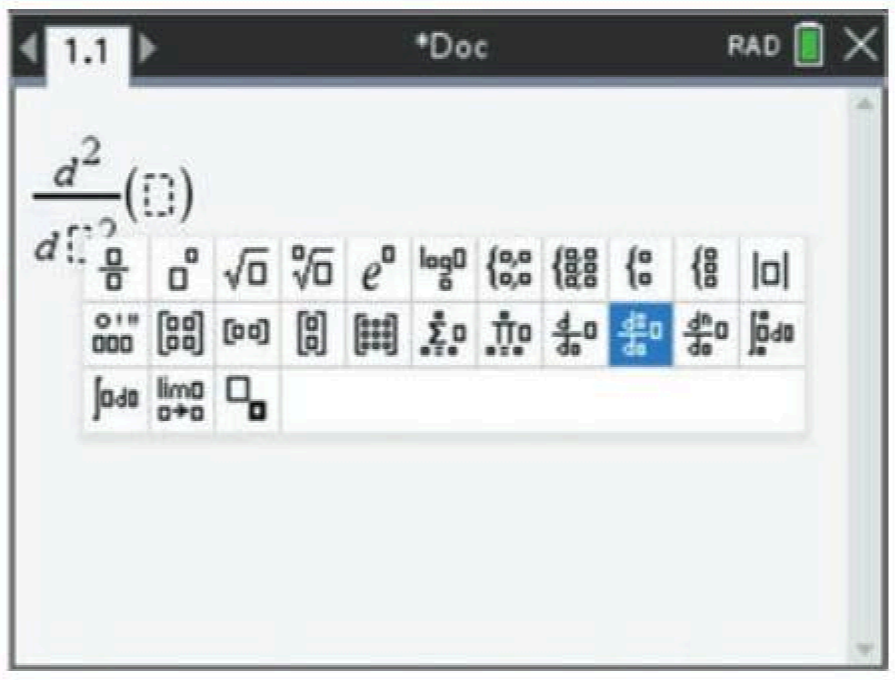
Differentiate again to get $f''(x) = \frac{10}{9}x^{-\frac{1}{3}} = \frac{10}{9x^{\frac{1}{3}}}$ where it can be clearly seen that $f''(0)$ is undefined.



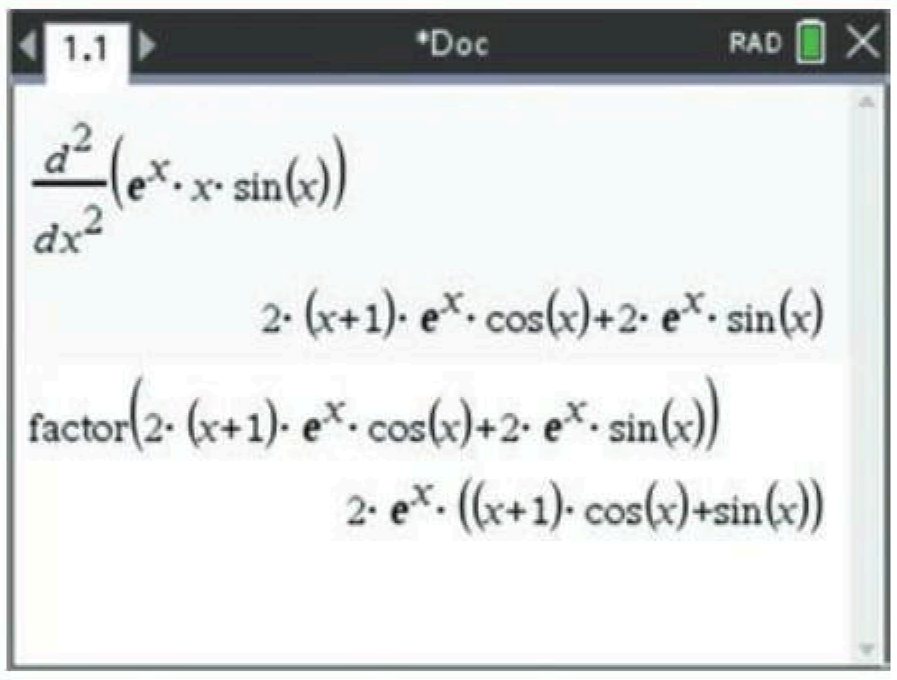
USING CAS 4 Finding the second derivative

Find the second derivative of $e^x x\sin(x)$.

TI-Nspire



$\frac{d^2}{dx^2}(\square)$



$\frac{d^2}{dx^2}(e^x \cdot x \cdot \sin(x))$

$2\cdot(x+1)\cdot e^x\cdot\cos(x)+2\cdot e^x\cdot\sin(x)$

$\text{factor}(2\cdot(x+1)\cdot e^x\cdot\cos(x)+2\cdot e^x\cdot\sin(x))$

$2\cdot e^x\cdot((x+1)\cdot\cos(x)+\sin(x))$

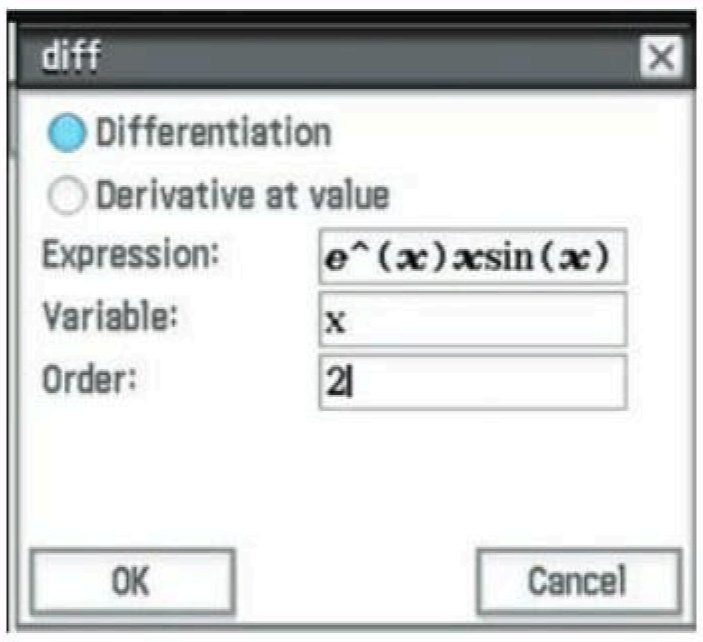
1 Press the **template** key.

2 Select the **second derivative** template.

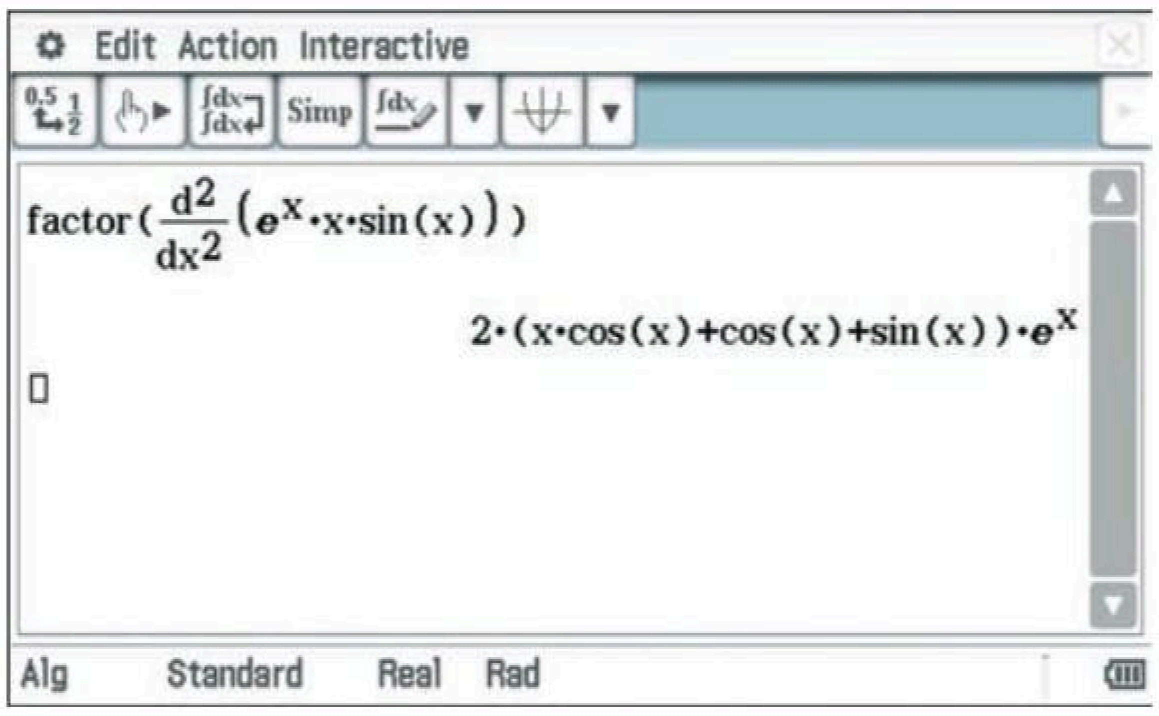
3 Enter the expression as shown above.

4 Press **menu > Algebra > factor** to simplify the expression (optional).

ClassPad



● Differentiation
○ Derivative at value
Expression: e^(x)xsin(x)
Variable: x
Order: 2



$\text{factor}(\frac{d^2}{dx^2}(e^x\cdot x\cdot\sin(x)))$

$2\cdot(x\cdot\cos(x)+\cos(x)+\sin(x))\cdot e^x$

1 Enter the expression.

2 Highlight and tap **Interactive > Calculation > diff**.

3 In the **Order:** field, change the value to **2**.

4 Tap **OK**.

5 The second derivative will be displayed.

6 Highlight the result and tap **Interactive > Transformation > factor** to factorise (optional).

7 Use **Rotate** or scroll to view the whole solution.

WORKED EXAMPLE 12 Second derivative

For $y = \log_e(3x^2)$, find

a $\dfrac{d^2y}{dx^2}$

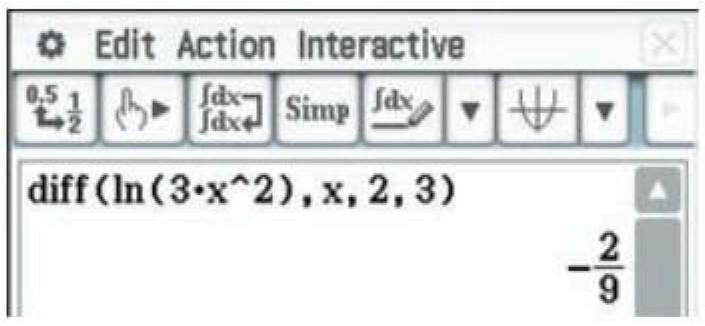
b the second derivative at $x = 3$.

Steps	**Working**
a **1** Find $\dfrac{dy}{dx}$.	$\dfrac{dy}{dx} = \dfrac{6x}{3x^2} = \dfrac{2}{x} = 2x^{-1}$
2 Find $\dfrac{d^2y}{dx^2}$.	$\dfrac{d^2y}{dx^2} = -2x^{-2} = -\dfrac{2}{x^2}$
b Substitute $x = 3$.	At $x = 3$, $\dfrac{d^2y}{dx^2} = -\dfrac{2}{9}$.

TI-Nspire

$\dfrac{d^2}{dx^2}\left(\ln\left(3\cdot x^2\right)\right)|x=3$ $\quad \dfrac{-2}{9}$

ClassPad



```
diff(ln(3·x^2),x,2,3)
```

$-\dfrac{2}{9}$

On ClassPad, the second derivative is shown by Order 2.

EXERCISE 5.5 The second derivative

ANSWERS p. 584

Recap

1 TECH-FREE Let $f(x) = \ln(x)$. The graph of f is transformed by:

- a dilation by a factor of 3 from the x-axis, followed by
- a translation of I unit horizontally to the right, followed by
- a dilation by a factor of $\dfrac{1}{2}$ from the y-axis.

Find the rule of the transformed graph.

2 © VCAA 2018N 2AQ3 The implied domain of the function with the rule $f(x) = \dfrac{3x}{\dfrac{\pi}{2} - \arccos(2 - x)}$ is

A $[1, 3]$ **B** $[-1, 1]$ **C** $[0, 1) \cup (1, 2]$ **D** $[-1, 0) \cup (0, 1]$ **E** $[1, 2) \cup (2, 3]$

Mastery

3 WORKED EXAMPLE 12 TECH-FREE Find $\dfrac{d^2v}{dt^2}$ if $v = (t + 3)(2t - 1)^2$.

WB p. 110

4 TECH-FREE For $y = \log_e(3x)$, find $\dfrac{d^2y}{dx^2}$.

5 TECH-FREE For $y = \log_e(3x^2)$, find $\dfrac{d^2y}{dx^2}$ at $x = 1$.

6 TECH-FREE For $y = \sin^{-1}(x + 1)$, find $\dfrac{d^2y}{dx^2}$ at $x = -1$.

7 Using CAS 4 Find the second derivative of $y = x^7 - 2x^5 + x^4 - x - 3$.

8 TECH-FREE Find the second derivative of each function.

a $y = x^7 - 2x^5 + 4x^4 - 7$

b $y = 5\cos(2x)$

c $y = 2x^2 - 3x + 3$

d $y = x^{-4}$

9 TECH-FREE Find $f''(x)$ for each function.

a $f(x) = \sqrt{2 - x}$

b $f(x) = \dfrac{x + 5}{3x - 1}$

10 For $f(x) = \sin(e^x)$, $f''(0)$ equals

A $\sin(1) - \cos(1)$

B $\cos(1) - \sin(1)$

C $\cos(1)$

D $e^2\cos(e^2)$

E $e^x[\cos(e^x) - \sin(e^x)]$

Exam practice

80–100% 60–79% 0–59%

11 © VCAA 2010 1Q5 51% TECH-FREE (3 marks) Given that $f(x) = \arctan(2x)$, find $f''\left(\dfrac{\pi}{2}\right)$.

12 TECH-FREE (2 marks) Find the value of x for which $\dfrac{d^2y}{dx^2} = 3$, given $y = 3x^3 - 2x^2 + 5x$.

13 TECH-FREE (2 marks) If $f(x) = x^9 - 5$, find $f''(x)$.

14 TECH-FREE (2 marks) Find $\dfrac{d^2y}{dx^2}$ if $y = 2x^5 - x^3 + 1$.

15 TECH-FREE (2 marks) $f(x) = 3x^4 - 2x^3 + 5x - 4$. Find $f'(1)$ and $f''(-2)$.

16 TECH-FREE (2 marks) $f(x) = x^4 - x^3 + 2x^2 - 5x - 1$. Find $f'(-1)$ and $f''(2)$.

17 TECH-FREE (2 marks) If $g(x) = \sqrt{x}$, find $g''(4)$.

18 TECH-FREE (2 marks) Given $h = 5t^3 - 2t^2 + t + 5$, find $\dfrac{d^2h}{dt^2}$ when $t = 1$.

19 For $y = \sin(e^x)$, $\dfrac{d^2y}{dx^2}$ equals

A $\sin(e^x)$

B $e^x\cos(e^x)$

C $x\cos(e^x)$

D $e^x[\cos(e^x) - \sin(e^x)]$

E $e^x[\cos(e^x) - e^x\sin(e^x)]$

20 For $y = \cos(e^x)$, $\dfrac{d^2y}{dx^2}$ equals

A $\sin(e^x)$

B $e^x\cos(e^x)$

C $x\cos(e^x)$

D $e^{-x}[e^x\cos(e^x) + \sin(e^x)]$

E $e^x[\cos(e^x) - e^x\sin(e^x)]$

21 For $f(x) = \cos(e^x)$, $f''(0)$ equals

A $\sin(1) - \cos(1)$

B $-\cos(1) - \sin(1)$

C $\cos(1)$

D $e^2\cos(e^2)$

E $e^x[\cos(e^x) - \sin(e^x)]$

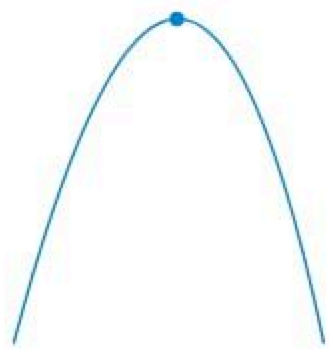
5.6 Applying the second derivative

We know that solving $f'(x) = 0$ or $\dfrac{dy}{dx} = 0$ finds stationary points on a curve.

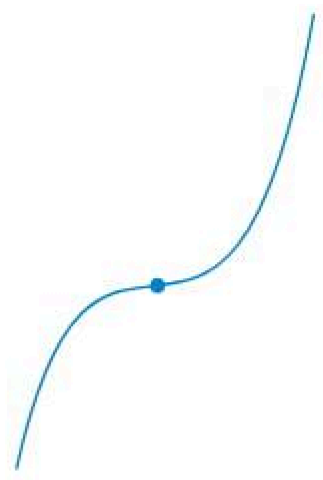
There are three types of stationary points:

Local minimum

Local maximum



Stationary point of inflection



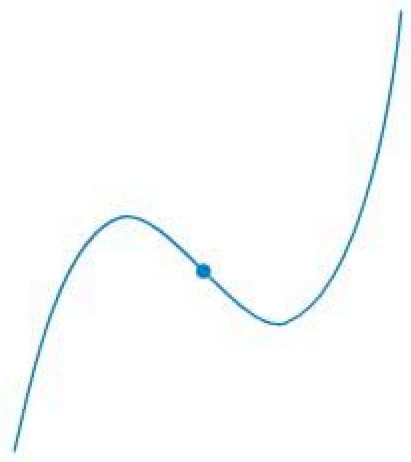
Video playlist
Applying the second derivative

Worksheet
Concavity

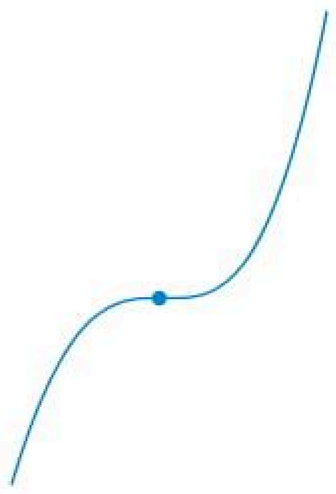
Points of inflection

A **point of inflection** on the graph of a function is where the graph 'bends' and its **concavity** changes, either from concave up to concave down, or from concave down to concave up. At a point of inflection, the curve is *neither* concave up nor down.
There are two types of points of inflection:

Non-stationary point of inflection



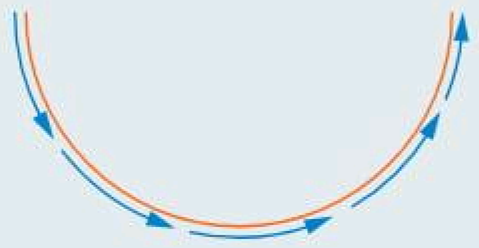
Stationary point of inflection



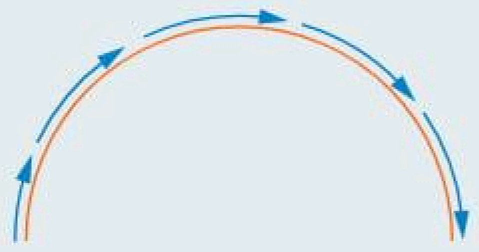
At the point of inflection, the graph is changing from concave down to concave up, as shown in the diagrams above. In the second graph, the gradient of the point of inflection is 0.

Concavity

- Where $f''(x) > 0$, the gradient of the graph of $f(x)$ is increasing, and the graph is **concave up**.



- Where $f''(x) < 0$, the gradient of the graph of $f(x)$ is decreasing, and the graph is **concave down**.



Points of inflection

- A **point of inflection** on the graph of $y = f(x)$ is a point where the concavity of the graph changes and $f''(x) = 0$.
- At this point, the curve is neither concave up or concave down.
- It is important to check that $f''(x)$ has different signs on either side of the point of inflection.

Consider the graph of the quartic function $f(x) = \frac{1}{2}(x-2)^3(x+4)$, which has points of inflection at $\left(-1, -\frac{81}{2}\right)$ and $(2, 0)$.

$f'(x) = 0$ at $x = -\frac{5}{2}, x = 2$

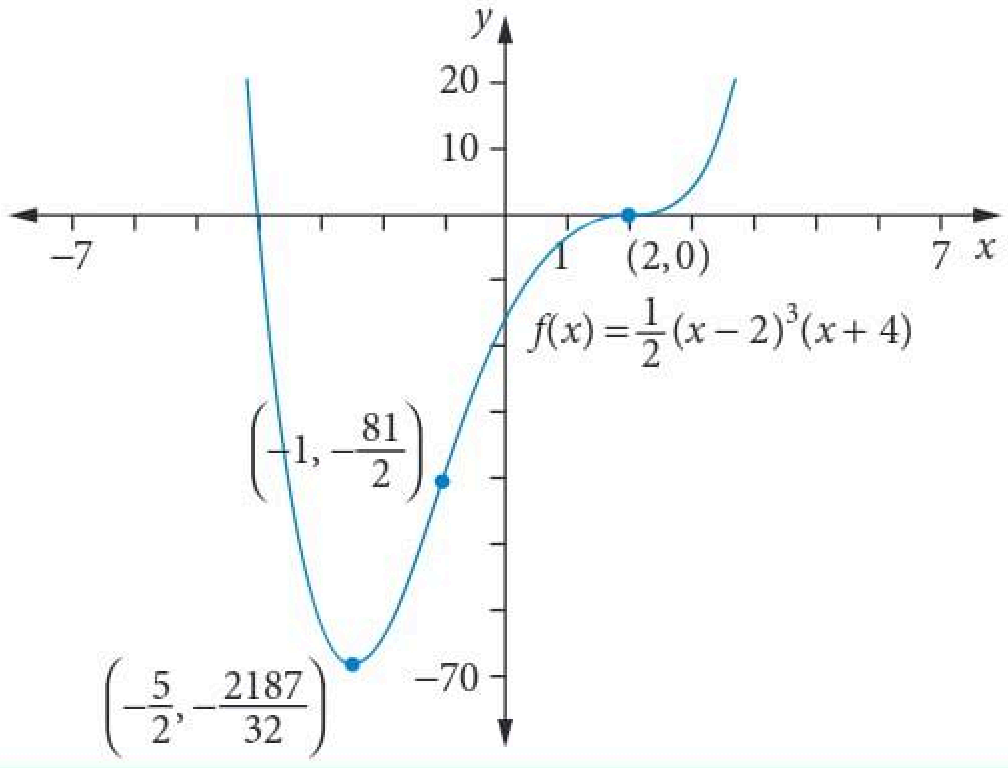
$f''(x) = 0$ at $x = -1, x = 2$

We have $f''(x) = 0$ and $f'(x) = 0$ at $x = 2$.

$\therefore$ $x = 2$ is a **stationary point of inflection**.

We also have $f''(x) = 0$ but $f'(x) \neq 0$ at $x = -1$.

$\therefore$ $x = -1$ is a **non-stationary point of inflection**.

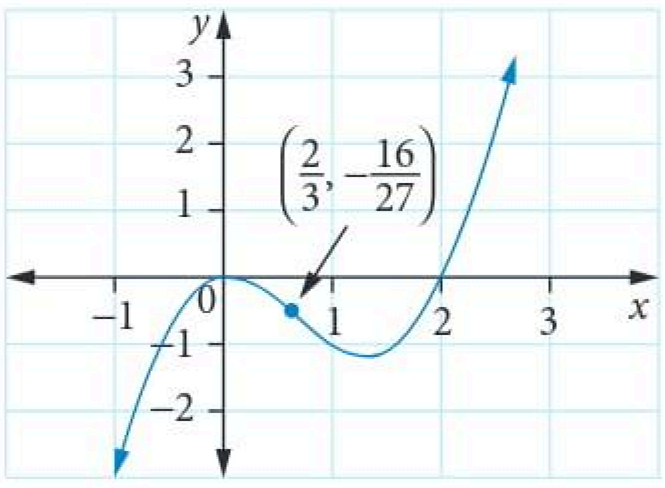


WB

p. 111

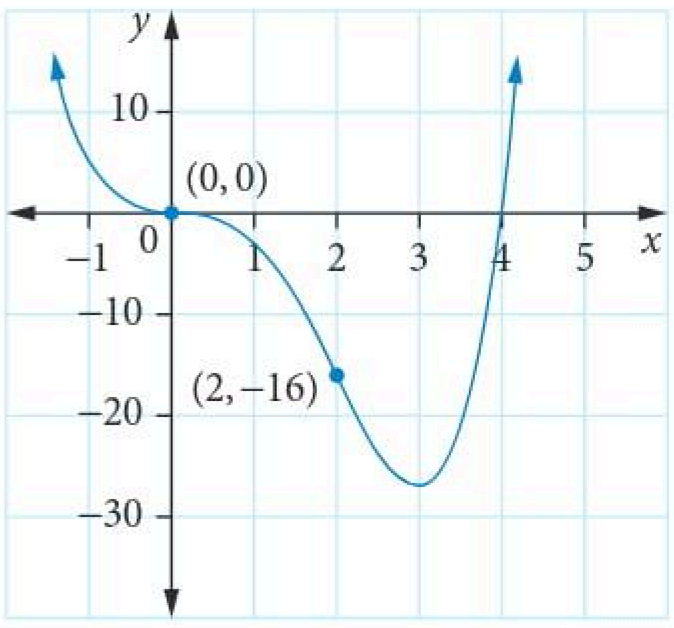
WORKED EXAMPLE 13 Points of inflection

For the graph of $f(x) = x^2(x-2)$, find any points of inflection and state whether they are stationary or non-stationary.

Steps	**Working**
1 Find $f'(x)$.	$f(x) = x^2(x-2)$ $= x^3 - 2x^2$ $\therefore f'(x) = 3x^2 - 4x$
2 Find $f''(x)$.	$f''(x) = 6x - 4$
3 Solve $f''(x) = 0$.	$f''(x) = 6x - 4 = 0$ at $x = \frac{2}{3}$
4 Check that concavity changes on either side of $x = \frac{2}{3}$, such as at $x = 0$ and $x = 1$. Use CAS to check this.	$f''(0) = -4 < 0$ (concave down) $f''(1) = 2 > 0$ (concave up) $x = \frac{2}{3}$ is a point of inflection.
5 For stationary or non-stationary, substitute $x = \frac{2}{3}$ into $f'(x)$.	$f'\left(\frac{2}{3}\right) = 3\left(\frac{2}{3}\right)^2 - 4\left(\frac{2}{3}\right)$ $= -\frac{4}{3}$ $\neq 0$ non-stationary
6 For the y value of the point, substitute $x = \frac{2}{3}$ into $f(x)$.	$f\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^3 - 2\left(\frac{2}{3}\right)^2$ $= -\frac{16}{27}$
7 Answer the question. Concave down to concave up.	$\left(\frac{2}{3}, -\frac{16}{27}\right)$ is a non-stationary point of inflection. 

WORKED EXAMPLE 14 Concavity

For the graph of $f(x) = x^3(x-4)$, find the points of inflection and comment on the change in concavity at each point.

Steps	**Working**
1 Find $f'(x)$.	$f(x) = x^3(x-4) = x^4 - 4x^3$ $f'(x) = 4x^3 - 12x^2$
2 Solve $f''(x) = 0$.	$f''(x) = 12x^2 - 24x$ $= 12x(x-2)$ $= 0$ $x = 0, x = 2$
3 Check that concavity changes on either side of $x = 0$ and $x = 2$, such as $x = -1$ and $x = 1$ for $x = 0$, and $x = 1$ and $x = 3$ for $x = 2$.	$f''(-1) = 36 > 0$ (concave up) $f''(1) = -12 < 0$ (concave down) $\therefore x = 0$ is a point of inflection. $f''(3) = 36 > 0$ (concave up) $\therefore x = 2$ is a point of inflection.
4 For stationary or non-stationary, substitute $x = 0$ and $x = 2$ into $f'(x)$.	$f'(0) = 0$ stationary $f'(2) = -16 \neq 0$ non-stationary
5 For the y values of the points, substitute $x = 0$ and $x = 2$ into $f(x)$.	$f(0) = 0$ $f(2) = -16$
6 Discuss change in concavity at each point of inflection.	 (0, 0) is a stationary point of inflection, where the graph changes from concave up to concave down. (2, –16) is a non-stationary point of inflection, where the graph changes from concave down to concave up.

WB

p. 112

The second derivative test

As the sign of the second derivative describes the concavity of the graph, we have the **second derivative test** that identifies whether a stationary point is a local minimum point or a local maximum point. This is called finding the nature of the stationary point.

We know that a stationary point occurs when $f'(x) = 0$.

Worksheet
Higher derivatives

The second derivative test for maximum and minimum points

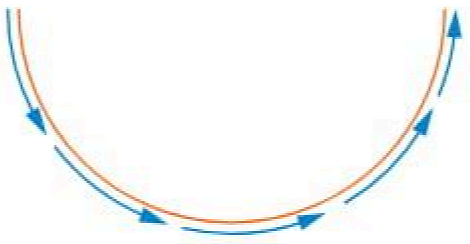
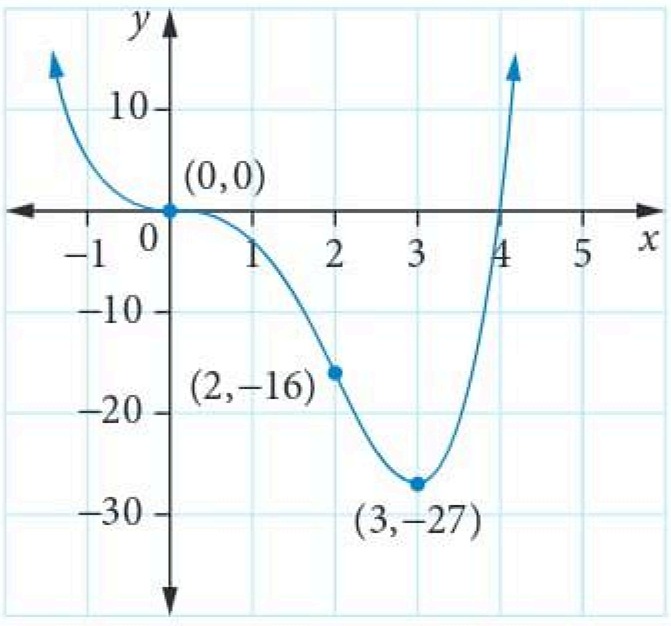
- If $f''(x) > 0$ at a stationary point, then the graph is **concave up** there so it's a local **minimum point.**
- If $f''(x) < 0$ at a stationary point, then the graph is **concave down** there so it's a local **maximum point**.

WB

p. 113

WORKED EXAMPLE 15 The second derivative test

Show that the graph of $f(x) = x^3(x - 4)$ has a stationary point at $x = 3$ and determine its nature.

Steps	**Working**
1 Find $f'(x)$.	$f(x) = x^4 - 4x^3$ $f'(x) = 4x^3 - 12x^2$
2 Solve $f'(x) = 0$.	$f'(x) = 4x^2(x - 3) = 0$ $x = 0, x = 3$ There are stationary points at $x = 0$ and $x = 3$.
3 Substitute $x = 3$ into $f(x)$.	$f(3) = -27$
4 Find $f''(x)$.	$f''(x) = 12x^2 - 24x$
5 Use the second derivative test at $x = 3$. 	$f''(3) = 36 > 0$ The graph is concave up at $x = 3$, so the stationary point is a local minimum point.
6 State the **nature** of the stationary point at $x = 3$. At (3, –27), the second derivative is positive (concave up).	 (3, –27) is a local minimum point.

EXERCISE 5.6 Applying the second derivative

ANSWERS p. 585

Recap

1 TECH-FREE Find $f'(x)$ and $f''(x)$ if $f(x) = 2x^2 - \frac{1}{4}x$.

2 The second derivative at $x = 0$ of the function with the rule $f(x) = \dfrac{3x}{\arccos(x)}$ is

A $\dfrac{24}{\pi^2}$

B $-\dfrac{24}{\pi^2}$

C 0

D $\dfrac{6}{\pi}$

E $-\dfrac{6}{\pi}$

Mastery

3 WORKED EXAMPLE 13 TECH-FREE For the graph of $f(x) = (x + 1)^2(2x - 1)$, find any points of inflection and state whether they are stationary or non-stationary.

4 WORKED EXAMPLE 14 TECH-FREE For the graph of $f(x) = (x + 1)^3(x - 4)$, find the points of inflection and comment on the change in concavity at each point.

5 WORKED EXAMPLE 15 TECH-FREE State the nature of the stationary points at $x = 1$ and $x = 4$ in the graph of $f(x) = x(4 - x)^3$.

6 In the graph of $f(x) = 3(x + 1)^2(x + 5)$, the point at $x = -\frac{7}{3}$

A changes from concave down to concave up.

B changes from concave up to concave down.

C changes from concave down to concave down.

D is a stationary point of inflection.

E is a local maximum.

7 In the graph of $f(x) = 3(x + 1)^2(x + 5)$, the point at $x = -\frac{11}{3}$

A changes from concave down to concave up.

B changes from concave up to concave down.

C changes from concave down to concave down.

D is a stationary point of inflection.

E is a local maximum.

8 In the graph of $f(x) = 3(x + 1)^2(x + 5)$, the point at $x = -1$

A changes from concave down to concave up.

B changes from concave up to concave down.

C is a local minimum.

D is a stationary point of inflection.

E is a local maximum.

9 Explain why the graph of $f(x) = x^4$ does not have a point of inflection even though $f''(x) = 0$ has a solution.

Exam practice

80–100% 60–79% 0–59%

10 TECH-FREE (3 marks) Differentiate $\frac{\cos(x)}{x}$.

11 TECH-FREE (4 marks) Find $f''(\pi)$ if $f(x) = 5x^3 \tan(3x)$.

12 © VCAA 2009 2AQ11 58% If $f'(x) > 0$ and $f''(x) < 0$ for all x over a given part of the domain of a function f, then the graph of f over this part of the domain would be a curve which

A increases, having increasing gradient with increasing x.

B decreases, having increasing gradient with increasing x.

C increases, having decreasing gradient with increasing x.

D decreases, having decreasing gradient with increasing x.

E increases, having a non-stationary point of inflection.

13 The graph of $f(x) = x^3(2 - x)$ has a non-stationary point of inflection at

A $x = 0$ **B** $x = 1$ **C** $x = \frac{2}{3}$ **D** $x = \frac{3}{2}$ **E** $x = 2$

14 © VCAA 2008 2BQ1 (7 marks) The function $f: [0, \infty) \to R$, where $f(x) = \frac{6x\sqrt{x}}{3x^2+1}$, has first and second derivatives with rules given by

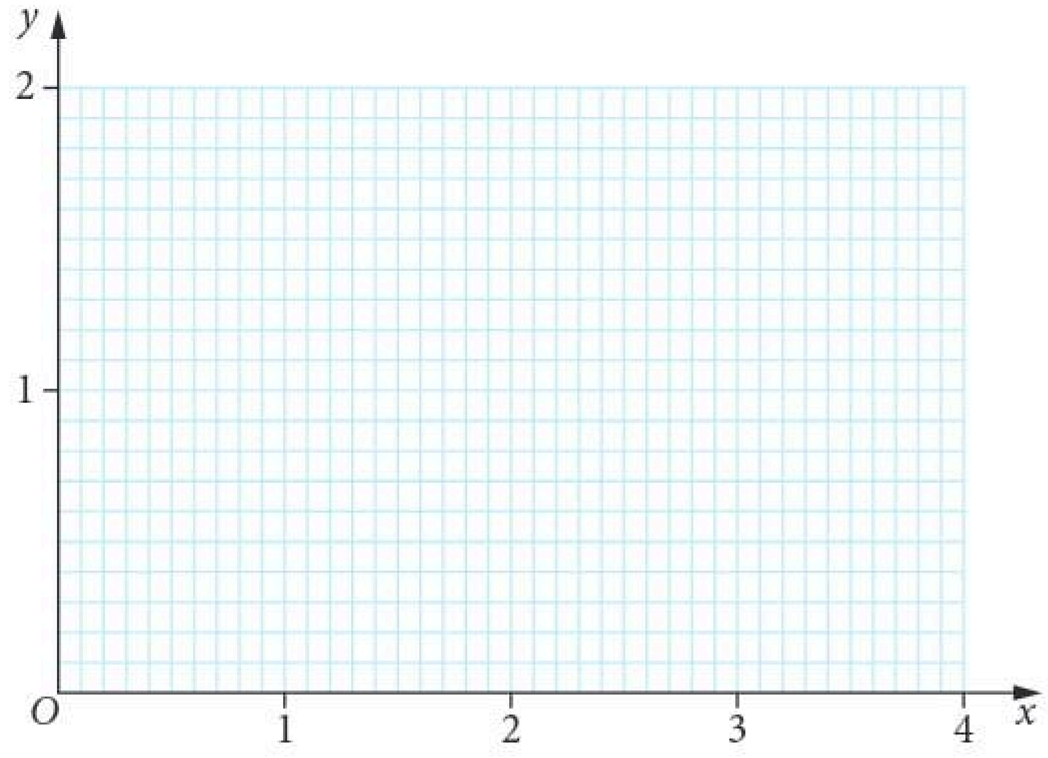
$$f'(x) = \frac{9\sqrt{x}(1-x^2)}{(3x^2+1)^2} \text{ and } f''(x) = \frac{9(9x^4-26x^2+1)}{2\sqrt{x}(3x^2+1)^3}.$$

a 75% Find the coordinates of the maximum turning point of the graph of *f* and use an appropriate test to verify its nature. 2 marks

b **i** 37% Write down a polynomial equation, which, when solved, will give the *x*-coordinates of the points of inflection of the graph of *f*. 1 mark

ii 52% Find the coordinates of the two points of inflection of the graph of *f*. Give your answers correct to one decimal place. 2 marks

c 61% Copy the axes below and on them sketch the graph of *f*, clearly indicating the location of any intercepts with the axes, the maximum turning point and the two points of inflection. 2 marks



Video playlist The chain rule and related rates of change

Worksheets Related rates

Related rates of change

5.7 The chain rule and related rates of change

We can use the chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ to solve **related rates** problems. These problems involve three or more variables.

Imagine a spherical balloon being inflated at a rate of 10 cm^3/s. Note the units suggest the variables Volume and Time. We then ask at what rate the radius is increasing instantaneously when the radius equals 2 cm. Note that the variable of Length is introduced.

This problem has three variables that are related to each other: V cm^3, t s and r cm.

Related rate problems have at least three variables to 'relate'.

Steps in solving a related rates problem

1. Write down the rate you are given: **'Given'**.
2. Write down the rate you have to find and when you have to find it: **'To find, when'**.
3. State the chain rule using three variables: **'Chain rule'**.
4. Write down a formula relating two of the variables that we then differentiate and substitute the value required: **'Differentiate'**.
5. Solve for the required rate: **'Solve'**.

To solve the balloon problem, we have:

1 **Given:**	$\frac{dV}{dt} = 10$
2 **To find, when:**	$\frac{dr}{dt} = ?$ at $r = 2$
3 **Chain rule:**	$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$
4 **Differentiate:**	Volume of sphere $V = \frac{4}{3}\pi r^3$ $\therefore \frac{dV}{dr} = 4\pi r^2$ When $r = 2$, $\frac{dV}{dr} = 4\pi(2^2)$ $= 16\pi$
5 **Solve:**	$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$ $\therefore 10 = 16\pi \times \frac{dr}{dt}$ Giving $\frac{dr}{dt} = \frac{10}{16\pi}$ $= \frac{5}{8\pi}$ cm/s

Exam hack

Students often make the mistake of placing π in the numerator rather than the denominator.

WB
p. 114

WORKED EXAMPLE 16 Related rates 1

A cube of length l metres, originally full of liquid, is leaking out of a small hole at one corner at a rate of 5 m^3/s. At what rate is the length, width and height of liquid decreasing when $l = 2$?

Steps	Working
1 Write the given information.	$\frac{dV}{dt} = 5$
2 Write what rate to find and when.	$\frac{dl}{dt} = ?$ at $l = 2$
3 State the chain rule using required variables.	$\frac{dV}{dt} = \frac{dV}{dl} \times \frac{dl}{dt}$
4 Differentiate and substitute.	volume of cube $= l^3$ $\frac{dV}{dl} = 3l^2$ At $l = 2$, $\frac{dV}{dl} = 3 \times 2^2 = 12$
5 Solve using $\frac{dV}{dt} = \frac{dV}{dl} \times \frac{dl}{dt}$.	$5 = 12 \times \frac{dl}{dt}$ $\therefore \frac{dl}{dt} = \frac{5}{12}$ m/s

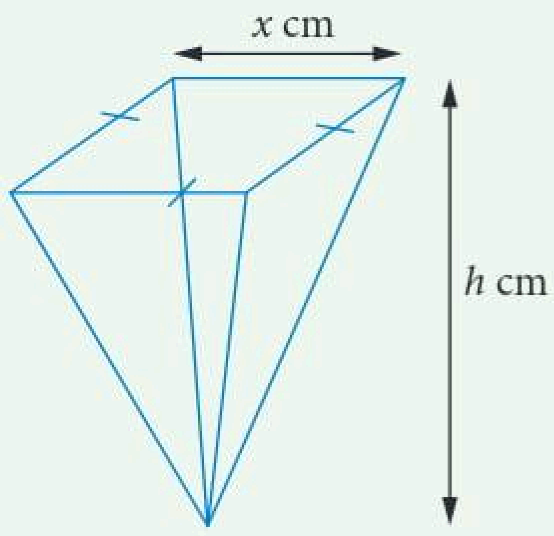
The above example has 3 variables: V, l and t.

The next example includes more than 3 variables.

WB p. 115

WORKED EXAMPLE 17 Related rates 2

A right inverted square-based pyramid of base length x cm and vertical height h cm is being filled with water at a rate of 20 cm^3/min. Given that the height of water is always 5 cm more than the base width, x, at what rate is the height of water increasing when $h = 10$ cm?



Steps	Working
1 Given information.	$\frac{dV}{dt} = 20$
2 Rate to find and when.	$\frac{dh}{dt} = ?$ at $h = 10$
3 Chain rule using required variables.	$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$
4 Differentiate, using $h = x + 5$ $\therefore x = h - 5$.	volume of pyramid $= \frac{1}{3}x^2h$ $V = \frac{1}{3}h(h-5)^2$ $\frac{dV}{dh} = \frac{1}{3}(h-5)(3h-5)$ At $h = 10$, $\frac{dV}{dh} = \frac{1}{3}(10-5)(30-5) = \frac{125}{3}$.
5 Solve, using $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$.	$20 = \frac{125}{3} \times \frac{dh}{dt}$ $\therefore \frac{dh}{dt} = \frac{12}{25}$ cm/min

EXERCISE 5.7 The chain rule and related rates of change

ANSWERS p. 585

Recap

80–100% 60–79% 0–59%

1 TECH-FREE Consider the function with the rule $f(x) = \frac{x^4 - 6x^2}{12}$.

Find the values of x at which the graph of $y = f(x)$ has a point of inflection.

2 © VCAA 2009 2AQ1 75% The graph of the function with rule $f(x) = 2x - \frac{1}{x^2}$ has

A one asymptote and a local maximum at (1, 3).

B one asymptote and a local maximum at (–1, –3).

C two asymptotes and a local maximum at (1, 3).

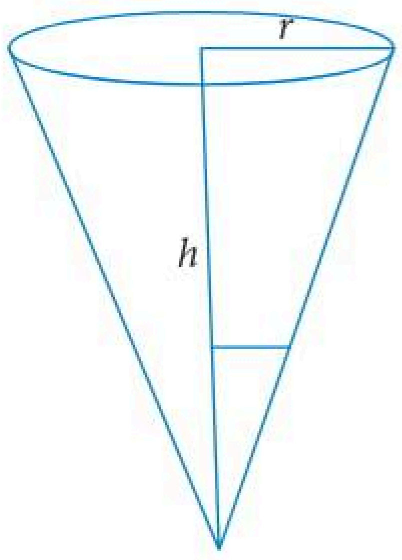
D two asymptotes and a local minimum at (–1, –3).

E two asymptotes and a local maximum at (–1, –3).

Mastery

3 WORKED EXAMPLE 16 TECH-FREE The radius of a sphere is increasing at a rate of 2 cm/min. State an expression for the rate of increase, in cm^3/min, of the volume of the sphere, in terms of the radius.

4 WORKED EXAMPLE 17 TECH-FREE A right cone, full of melted ice cream, of radius r mm and vertical height h mm, with a hole in the bottom, is draining ice cream at a rate of 10 mm^3/s. Given that the height of ice cream in the cone is always double the radius, at what rate is the radius of the ice cream decreasing when $r = 5$ mm?



Exam practice

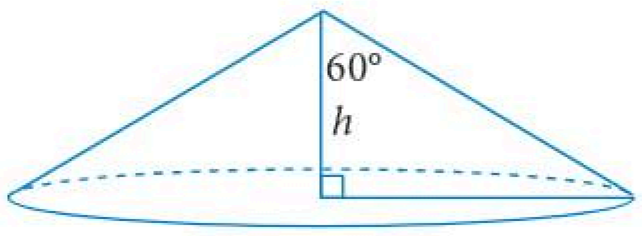
80–100% 60–79% 0–59%

5 TECH-FREE (2 marks) A right cone, whose height is always 2 cm more than the radius, is filled with water. Its height is changing at a rate of 2 cm/s. At what rate is the volume of water changing when its height is 4 cm?

6 © VCAA 2002 2AQ17 69% The radius of a sphere is increasing at a rate of 3 cm/min. When the radius is 8 cm, the rate of increase, in cm^3/min, of the volume of the sphere is

A $85\frac{1}{3}\pi$
B 256π
C $682\frac{2}{3}\pi$
D 768π
E 2048π

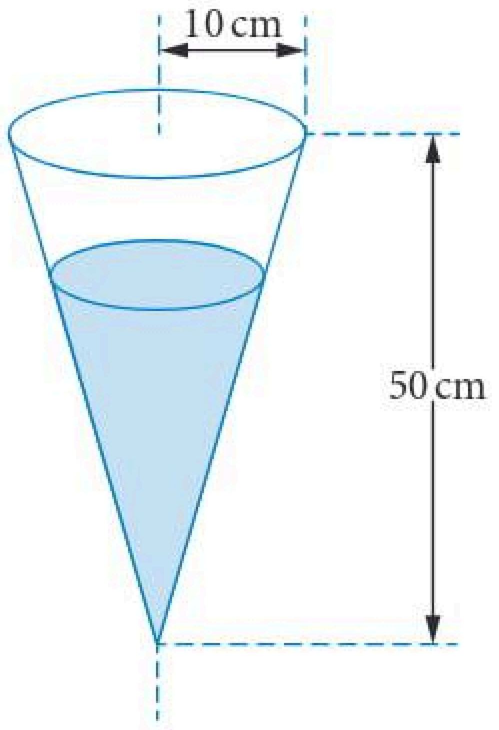
7 © VCAA 2019 2AQ10 58% Sand falls from a chute to form a pile in the shape of a right circular cone with semi-vertex angle 60°. Sand is added to the pile at a rate of 1.5 m^3 per minute.



The rate at which the height h metres of the pile is increasing, in metres per minute, when the height of the pile is 0.5 m, correct to two decimal places, is

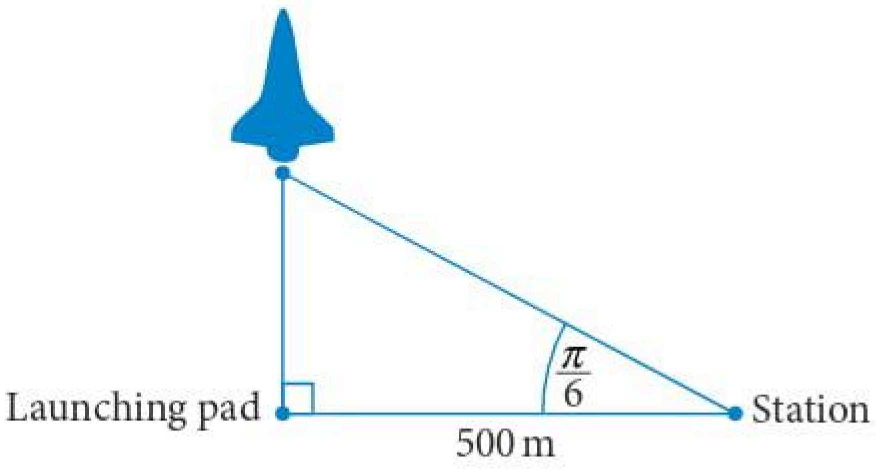
A 0.21
B 0.31
C 0.64
D 3.82
E 3.53

8 © VCAA 2003 1AQ16 35% Water is draining from a cone-shaped funnel at the constant rate of 600 cm^3/min. The cone has height 50 cm and base radius 10 cm. Let h cm be the depth of water in the funnel at time t min. The rate of **decrease** of h, in cm/min, is given by



A 12
B $\frac{100\pi}{3}$
C $\frac{15\,000}{\pi h^2}$
D $24\pi h^2$
E $\frac{18}{\pi}$

9 © VCAA 2007 2AQ12 27% An ascending space shuttle rises vertically from a launching pad. As it rises the shuttle is tracked from a station at ground level 500 metres away.



When the angle of elevation of the shuttle is $\frac{\pi}{6}$ radians from the horizontal direction, and is increasing at a rate of 0.5 radians per second, the speed of the shuttle is closest to

A 333 m s^{-1} **B** 144 m s^{-1} **C** 289 m s^{-1} **D** 577 m s^{-1} **E** 500 m s^{-1}

10 © VCAA 2004 2BQ3 (5 marks) The volume, V litres, of oil in an irregularly shaped tank, when the oil depth is h metres, is given by $V = 8000\, h \tan^{-1}(h)$.

a **i** 77% Find the exact volume of oil in the tank, in litres, when the oil depth is one metre. 1 mark

ii 48% Find the oil depth, correct to the nearest centimetre, when its volume is 10 000 litres. 1 mark

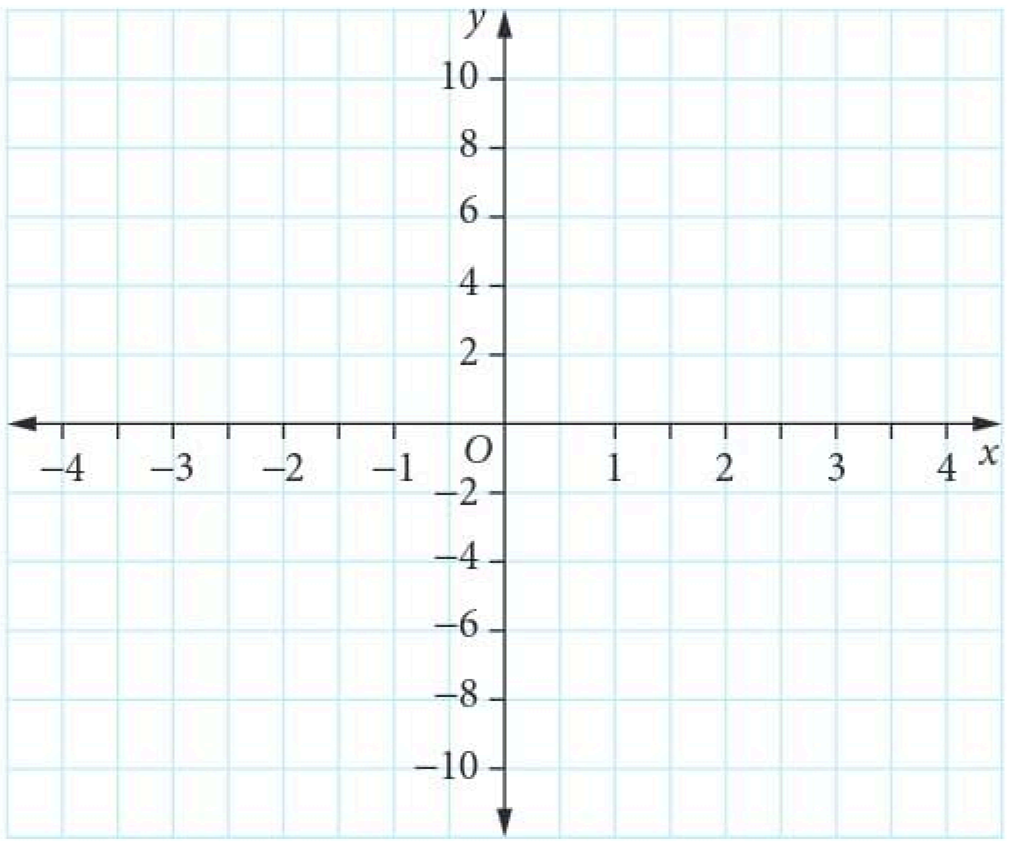
The tank is initially empty. Oil is then poured into the tank at a constant rate of 2000 litres per minute.

b 59% Find, in terms of h, an expression for the rate at which the oil depth is increasing, in metres per minute, when the depth is h metres. 3 marks

11 © VCAA 2009 2BQ4abe (7 marks) Consider the function f with rule $y = \frac{x^4 - 1}{x^2}$ over the range $-10 \le y \le 10$.

a 65% The domain of f may be expressed in the form $x \in [-a, -b] \cup [b, a]$, where $a, b > 0$. Find the values of a and b correct to one decimal place. 2 marks

b 70% Copy the set of axes below and on them sketch the graph of f for $y \in [-10, 10]$, clearly showing the location of the x-intercepts. 2 marks

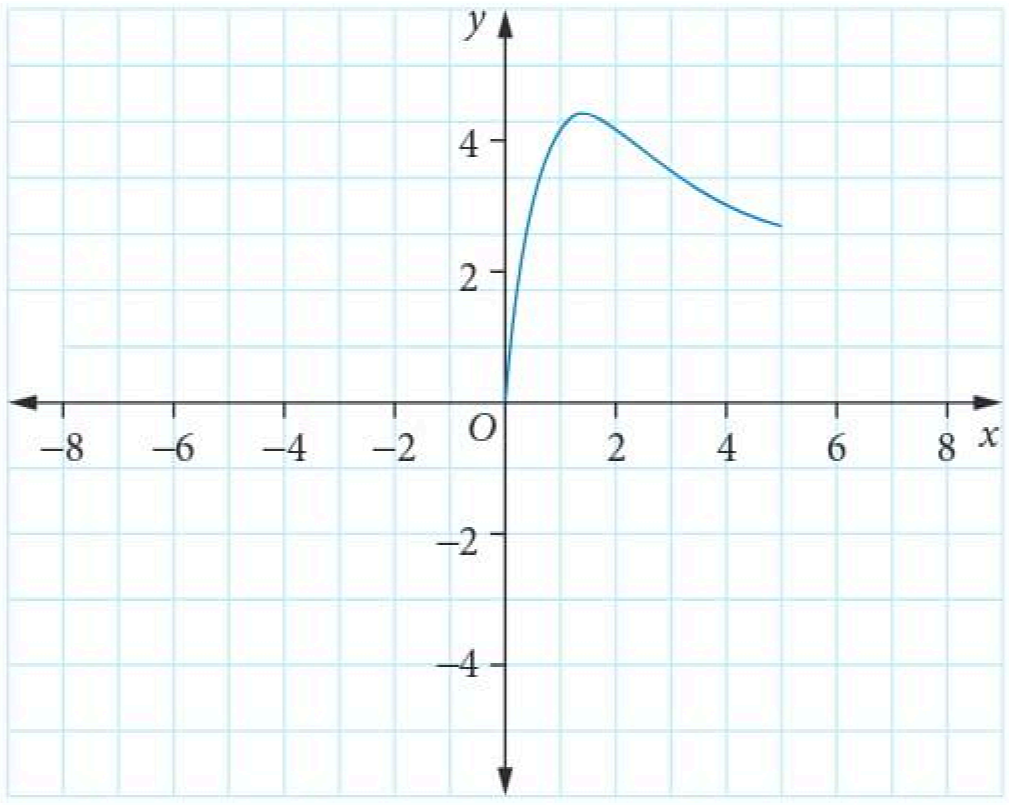


A glass with a hollow stem, and with its base at $y = -10$, is made by rotating the part of the graph of f where $x > 0$ and $y \in [-10, 10]$ about the y-axis to form a volume of revolution. Liquid is poured into the glass at a rate of 1.5 cm^3/s.

c 31% The function $y = \frac{x^4 - 1}{x^2}$ can be rearranged to give $x^2 = \frac{y + \sqrt{y^2 + 4}}{2}$.

Given that $\frac{dV}{dy} = \pi x^2$, where V cm^3 is the volume of liquid in the glass where the radius of the surface of the liquid is x cm, find the rate at which the surface is rising when it is 6 cm from the top of the glass. Give your answer in cm per second, correct to two decimal places. 3 marks

12 © VCAA 2006 2BQ1def (5 marks) The top part of a wine glass, while lying on its side, is constructed by rotating the graph of $y = \frac{6x}{\sqrt{1 + x^3}}$ from $x = 0$ to $x = 5$ about the x-axis as shown. All lengths are measured in centimetres.



At time $t = 0$ seconds, wine begins to be poured into the upright glass so that its depth (x cm in the graph) is increasing at a rate of 2 cm/s.

a 75% Given that $\frac{dy}{dx} = \frac{6 - 3x^3}{\left(1 + x^3\right)^{\frac{3}{2}}}$, find an expression for $\frac{dy}{dt}$, the rate of change of the radius of the surface of the wine with respect to time, in terms of x. 1 mark

b 28% **Hence** find an expression for the rate of change of the area, A cm^2, of the surface of the wine in the upright glass with respect to t, in terms of x. Give your answer in the form $\frac{dA}{dt} = \frac{ax^b(6 - 3x)^3}{\left(1 + x^3\right)^c}$, where a, b and c are constants. 3 marks

c 28% Find the exact value of the depth of the wine for which the area of its surface, A cm^2, is a maximum. 1 mark

13 © VCAA 2021N 2BQ1 (6 marks) A curve is defined parametrically by $x = \sec(t)$, $y = \operatorname{cosec}(t)$ where $t \in \left(0, \frac{\pi}{2}\right)$.

a Show that the curve can be represented in Cartesian form by the relation $y = \frac{x}{\sqrt{x^2 - 1}}$. 2 marks

b State the domain and range of the relation given by $y = \frac{x}{\sqrt{x^2 - 1}}$ for this curve. 2 marks

c Use the derivative of the relation to show that the gradient of the curve is negative at all points on the curve. 2 marks

Video playlist
Implicit differentiation

Worksheets
Implicit differentiation

Curve sketching with derivatives

5.8 Implicit differentiation

We have, up till now, differentiated functions that are written as $y = \ldots$ or $f(x) = \ldots$ explicitly.

Implicit differentiation is used to differentiate functions of the form where y is not the subject, and can appear more than once, such as in the equation $y^2 + 2x = xy$.

Implicit differentiation uses the chain rule to differentiate each term of the equation with respect to x.

The LHS of the above equation is $y^2 + 2x$, so we differentiate term-by-term.

For the y^2 term, using the chain rule, we differentiate with respect to y and multiply it by $\frac{dy}{dx}$.

$$\begin{aligned}\frac{d}{dx}\left(y^2\right) &= \frac{d}{dy}\left(y^2\right) \times \frac{dy}{dx}\\ &= 2y\frac{dy}{dx}\end{aligned}$$

> This process is like considering y as the 'inner' term in the chain rule, and as we don't actually know what y equals, the derivative of this 'inner' term is $\frac{dy}{dx}$.

$$\frac{d}{dx}(y^2 + 2x) = 2y\frac{dy}{dx} + 2$$

We then differentiate the RHS of the equation xy using the product rule:

$$\frac{d}{dx}(xy) = x \cdot \frac{dy}{dx} + y \cdot 1 = x\frac{dy}{dx} + y$$

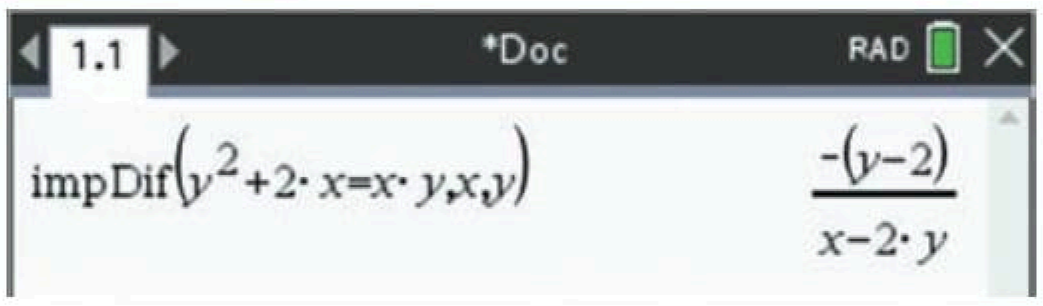
This differentiated equation now becomes

$$2y\frac{dy}{dx} + 2 = x\frac{dy}{dx} + y$$

Now make $\frac{dy}{dx}$ the subject of the equation:

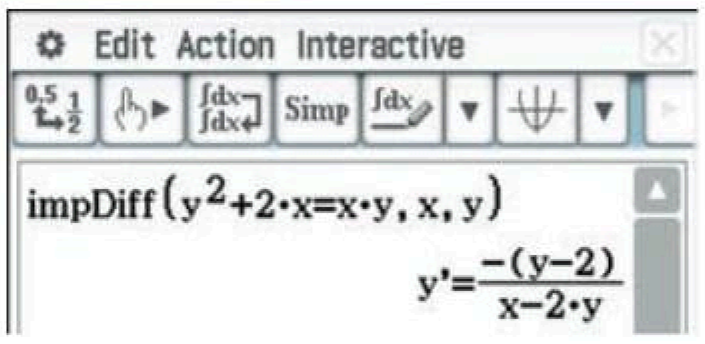
$$\begin{aligned}2y\frac{dy}{dx} - x\frac{dy}{dx} &= y - 2\\ \frac{dy}{dx}(2y - x) &= y - 2\\ \frac{dy}{dx} &= \frac{y-2}{2y-x}\end{aligned}$$

TI-Nspire



impDif$(y^2+2\cdot x=x\cdot y,x,y)$ $\quad \frac{-(y-2)}{x-2\cdot y}$

ClassPad



impDiff$(y^2+2\cdot x=x\cdot y, x, y)$

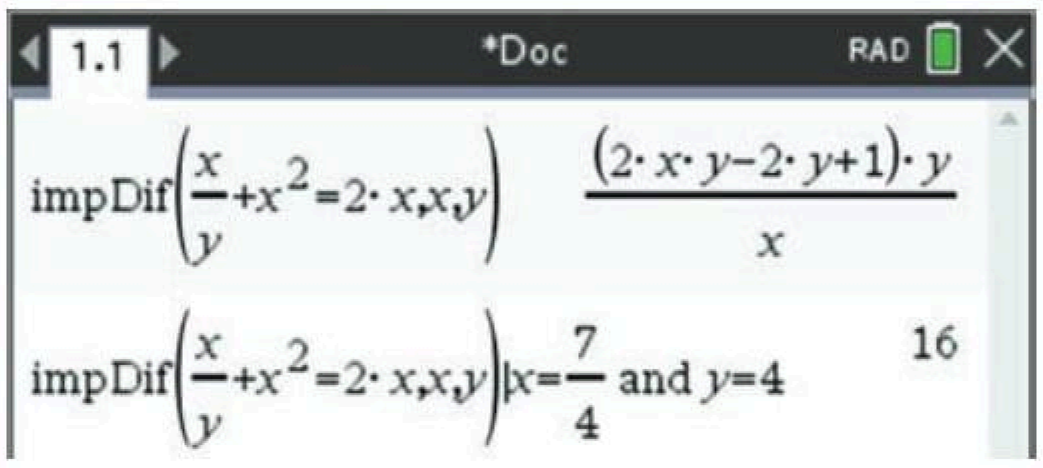
$y'=\frac{-(y-2)}{x-2\cdot y}$

See next page on how to use CAS for implicit differentiation.

USING CAS 5 Implicit differentiation

Use implicit differentiation to find the derivative of $\frac{x}{y}+x^2=2x$ and determine the gradient at the point $\left(\frac{7}{4},4\right)$.

TI-Nspire



$\text{impDif}\left(\frac{x}{y}+x^2=2\cdot x,x,y\right)$ $\quad \frac{(2\cdot x\cdot y-2\cdot y+1)\cdot y}{x}$

$\text{impDif}\left(\frac{x}{y}+x^2=2\cdot x,x,y\right)|x=\frac{7}{4} \text{ and } y=4$ $\quad 16$

1 Press **menu > Calculus > Implicit Differentiation.**

2 Enter the equation followed by **,x,y.**

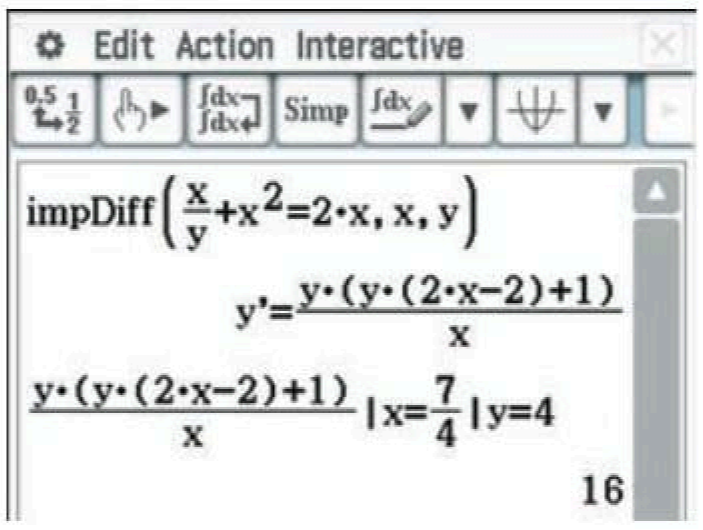
3 To find the gradient, press **=** to access the mini-palette and select **|**.

4 Enter ***x*** = $\frac{7}{4}$ **and *y* = 4.**

Note, the **and** command can be accessed from the **catalog** or manually typed in.

5 Press **enter**.

ClassPad



$\text{impDiff}\left(\frac{x}{y}+x^2=2\cdot x, x, y\right)$

$y'=\frac{y\cdot(y\cdot(2\cdot x-2)+1)}{x}$

$\frac{y\cdot(y\cdot(2\cdot x-2)+1)}{x}|x=\frac{7}{4}|y=4$

16

1 Enter the expression.

2 Highlight and tap **Interactive > Calculation > ImpDiff > OK.**

3 Highlight the right-hand side of the solution and drag to the next line.

4 Open the **Keyboard > Math3** and tap on **|**.

5 Enter ***x*** = $\frac{7}{4}$.

6 Tap on **|** again and enter ***y*** = **4**.

7 Press **EXE**.

WORKED EXAMPLE 18 Implicit differentiation

a Find $\frac{dy}{dx}$ for the equation $3xy+y^2=2$.

b Hence find $\frac{dy}{dx}$ at $y=1$.

Steps	Working
a **1** Differentiate both sides of the equation with respect to *x*, using the product rule for 3*xy*.	$\frac{d}{dx}(3xy)+\frac{d}{dx}(y^2)=\frac{d}{dx}(2)$ $\left(3x\cdot\frac{dy}{dx}+y\cdot 3\right)+2y\frac{dy}{dx}=0$
2 Collect $\frac{dy}{dx}$ terms and take out as a common factor.	$3x\frac{dy}{dx}+2y\frac{dy}{dx}=-3y$ $\frac{dy}{dx}(3x+2y)=-3y$
3 Make $\frac{dy}{dx}$ the subject.	$\frac{dy}{dx}=\frac{-3y}{3x+2y}$

Exam hack

A common mistake is to forget that the derivative of a constant on the RHS is zero.

WB

p. 116

b **1** Find x for $y = 1$.

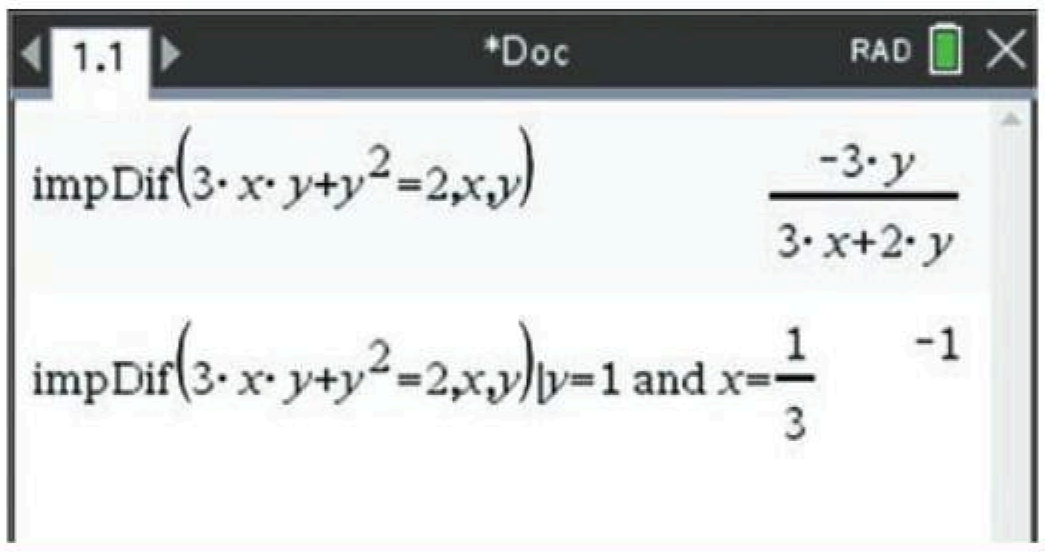
$3x \cdot 1 + 1^2 = 2$

$\therefore x = \frac{1}{3}$

2 Find $\frac{dy}{dx}$ at the point $\left(\frac{1}{3}, 1\right)$.

$$\frac{dy}{dx} = \frac{-3 \times 1}{3 \times \frac{1}{3} + 2 \times 1} = -1$$

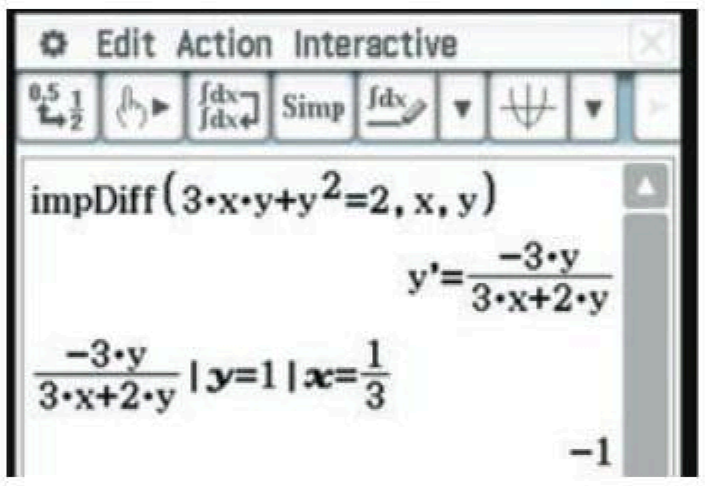
TI-Nspire



$\text{impDif}(3 \cdot x \cdot y + y^2 = 2, x, y)$ $\quad \frac{-3 \cdot y}{3 \cdot x + 2 \cdot y}$

$\text{impDif}(3 \cdot x \cdot y + y^2 = 2, x, y) | y = 1 \text{ and } x = \frac{1}{3}$ $\quad -1$

ClassPad



$\text{impDiff}(3 \cdot x \cdot y + y^2 = 2, x, y)$

$y' = \frac{-3 \cdot y}{3 \cdot x + 2 \cdot y}$

$\frac{-3 \cdot y}{3 \cdot x + 2 \cdot y} | y = 1 | x = \frac{1}{3}$

-1

Video
VCE question analysis: Differentiation

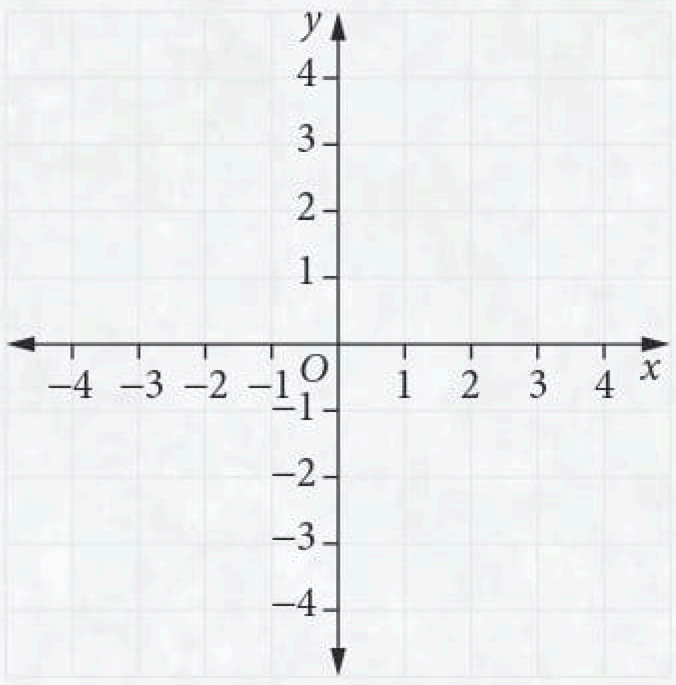
VCE QUESTION ANALYSIS

© VCAA 2018 2BQ1 2018 Examination 2 Section B Question 1 (**11 marks**)

Consider the function $f: D \to R$, where $f(x) = 2\arcsin(x^2 - 1)$.

a Determine the maximal domain D and the range of f. 2 marks

b Copy the axes below and on them sketch the graph of $y = f(x)$, labelling any endpoints and the y-intercept with their coordinates. 3 marks



c Find $f'(x)$, for $x > 0$, expressing your answer in the form

$f'(x) = \frac{A}{\sqrt{2 - x^2}}, A \in R$. 1 mark

d Write down $f'(x)$, for $x < 0$, expressing your answer in the form

$f'(x) = \frac{B}{\sqrt{2 - x^2}}, B \in R$. 1 mark

e The derivative $f'(x)$ can be expressed in the form

$f'(x) = \frac{g(x)}{\sqrt{2 - x^2}}$ **over its maximal domain.**

i Find the maximal domain of f'. 1 mark

ii Find $g(x)$, expressing your answer as a piecewise (hybrid) function. 1 mark

iii Copy the axes from part **b** again and on them sketch the graph of g. 2 marks

Reading the question

- Note that questions ask for labelling any endpoints and the y-intercept with their coordinates. Marks are often lost when students forget to write their points as coordinates.
- The questions are about endpoints, domains and range and the derivative.
- You need to understand how to write an answer in the given required form.
- You need to know what a piecewise (hybrid) function is and how to handle this type of function using both by hand and CAS techniques.

Thinking about the question

- Remember that the maximal domain is the domain that naturally occurs in the given function. Think about where the curve starts and stops.
- Don't forget the range!
- The question is quite unusual as it includes two graphs with grids provided, meaning the examiner wants precise and correct shapes, with curves going through the correct grid square.

Exam hack

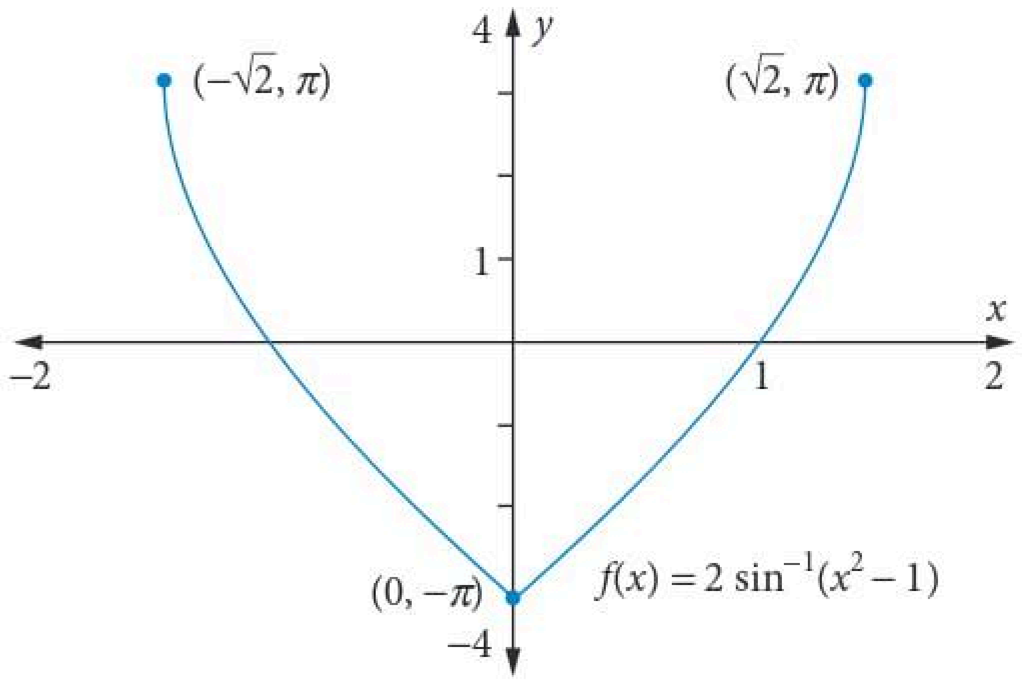
The examiners advise students to set viewing windows on CAS to a scale that closely matches the scale provided in the examination.

- Clearly $x > 0$ and $x < 0$ is important in this question.
- $f'(x) = \dfrac{g(x)}{\sqrt{2 - x^2}}$ is not automatically given by your technology, make sure that you know how to get it, either by continuing with CAS or using by-hand algebra.

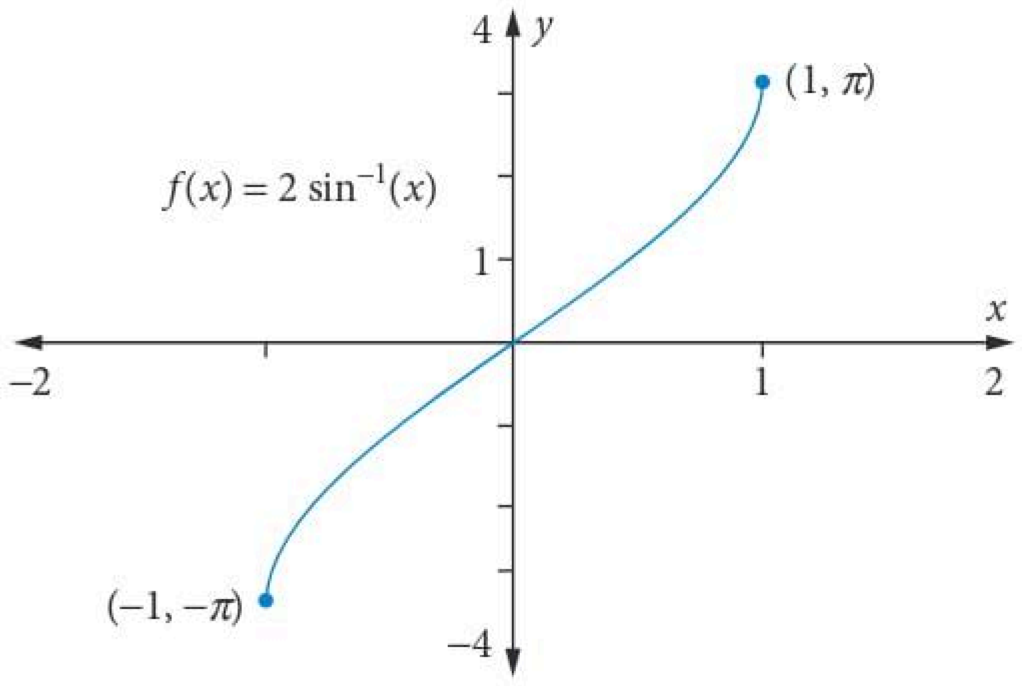
Worked solution (✓ = 1 mark)

a The inverse sine graph $f(x) = 2\arcsin(x^2 - 1)$ has a natural maximal domain and range. Start thinking what the x^2 will do with the graph.

Define the function using CAS and graph it to see what the shape looks like.

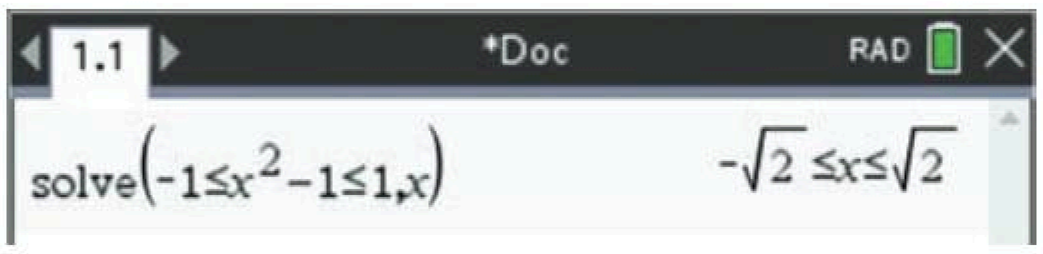


The graph of $y = 2\arcsin(x)$ has the domain $[-1, 1]$ and range $[-\pi, \pi]$.



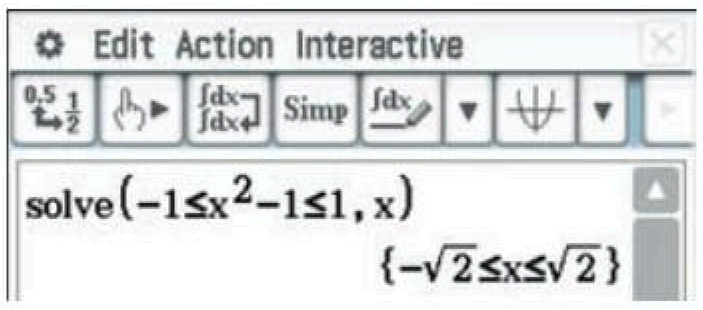
For maximal domain solve $-1 \le x^2 - 1 \le 1$, giving $\boldsymbol{-\sqrt{2} \le x \le \sqrt{2}}$. ✓

TI-Nspire



$\text{solve}(-1\le x^2-1\le 1,x)$ $\quad -\sqrt{2}\le x\le\sqrt{2}$

ClassPad

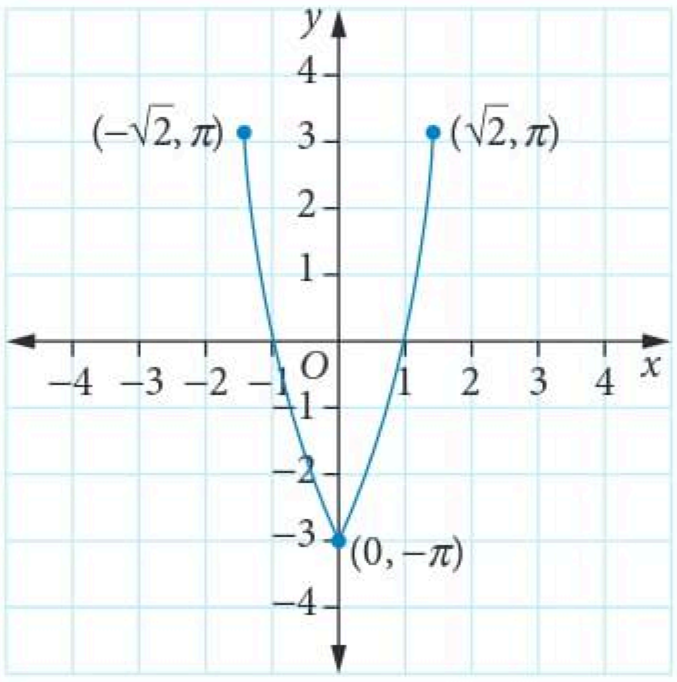


$\text{solve}(-1\le x^2-1\le 1, x)$

$\{-\sqrt{2}\le x\le\sqrt{2}\}$

Range does not change.
Maximal domain $D = \left[-\sqrt{2},\sqrt{2}\right]$ and range = $[-\pi, \pi]$. ✓

b Graph of $y = f(x)$

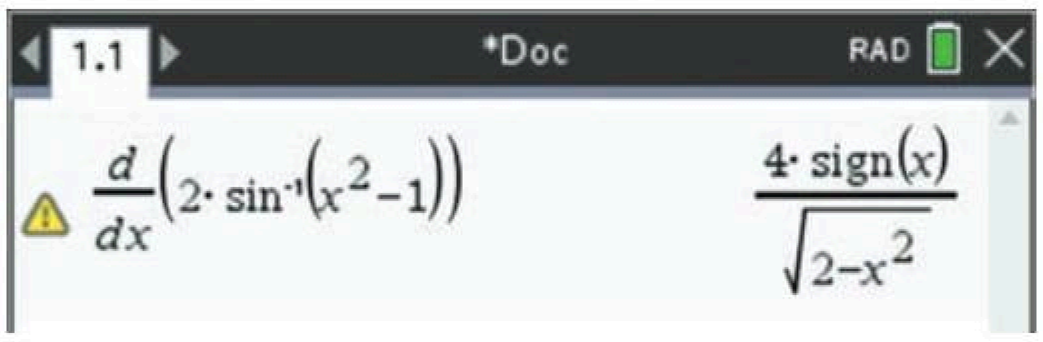


✓✓✓

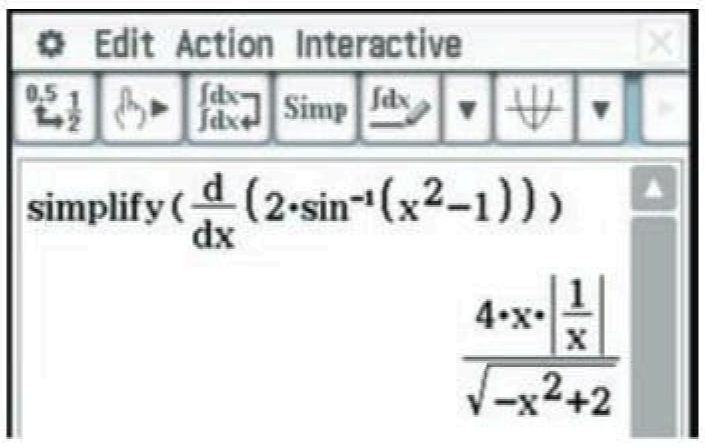
c $f'(x) = \dfrac{4x}{\sqrt{-x^2(x^2-2)}}$

Simplifying gives $f'(x) = \dfrac{4x\left|\dfrac{1}{x}\right|}{\sqrt{2-x^2}} = \dfrac{4\dfrac{x}{|x|}}{\sqrt{2-x^2}}$. For $x > 0$, cancel x and $|x|$,

expressing your answer in the required form $f'(x) = \dfrac{4}{\sqrt{2-x^2}}$. ✓



$\dfrac{d}{dx}\left(2\cdot\sin^{-1}(x^2-1)\right)$ $\quad \dfrac{4\cdot \text{sign}(x)}{\sqrt{2-x^2}}$



$\text{simplify}\left(\dfrac{d}{dx}(2\cdot\sin^{-1}(x^2-1))\right)$

$\dfrac{4\cdot x\cdot\left|\dfrac{1}{x}\right|}{\sqrt{-x^2+2}}$

The **sign(x)** means the solution is negative when $x < 0$ and positive when $x > 0$.

Therefore, $f'(x) = \dfrac{4}{\sqrt{2-x^2}}$ when $x > 0$.

For $x > 0$, $|x| = x$ so $\cancel{x}\cdot\left|\dfrac{1}{\cancel{x}}\right| = 1$.

Therefore, $f'(x) = \dfrac{4}{\sqrt{2-x^2}}$ when $x > 0$.

d From above, $f'(x) = \dfrac{4\dfrac{x}{|x|}}{\sqrt{2-x^2}}$.

For $x < 0$, $|x| = -x$ so we can cancel x and $|x|$ leaving -1, so that $f'(x) = \dfrac{-4}{\sqrt{2-x^2}}$. ✓

e **i** f' does not exist at endpoints or cusps.

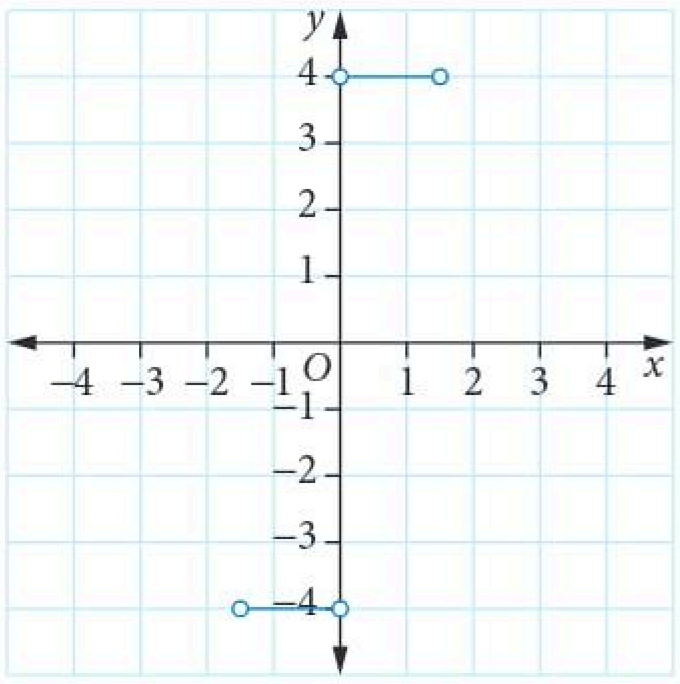
Maximal domain of f' is $\left(-\sqrt{2},\sqrt{2}\right)\setminus\{0\}$

or can be expressed as $\left(-\sqrt{2},0\right)\cup\left(0,\sqrt{2}\right)$. ✓

ii Piecewise (hybrid) function

$$g(x)=\begin{cases}4, & 0<x<\sqrt{2}\\ -4, & -\sqrt{2}<x<0\end{cases}\ ✓$$

iii The graph of $g(x)$ in the function $f'(x)=\dfrac{g(x)}{\sqrt{2-x^2}}$.



✓✓

Student performance

80–100% 60–79% 0–59%

a 76% Generally well done, with common errors including giving the domain but not the range, giving decimal approximations rather than π, and using round brackets instead of square brackets.

b 76% Students were often careless in drawing graphs, such as not labelling endpoints or the y-intercept. Exam assessors give some leeway if the curve is *slightly* outside the required grid. Make sure you practise drawing accurate graphs before your exam. The curve must be drawn in one smooth movement rather than 'feathered' in little bits.

c 80%

d 81%

e **i** 21% Students achievement reduced dramatically when a deeper analysis was required about the derivative function. Examiners identified that students did not realise that the cusp or the endpoints were not included in the domain of the derivative function. For a Specialist Maths student, this is a significant lack of knowledge.

ii 49%

iii 48% More exam practice is needed when dealing with sketching functions. A common mistake was sketching $f'(x)$ rather than $g(x)$.

EXERCISE 5.8 Implicit differentiation

ANSWERS p. 585

Recap

1 © VCAA 2018N 1Q3 TECH-FREE (3 marks) Find $\sin(t)$ given that $t=\arccos\left(\dfrac{12}{13}\right)+\arctan\left(\dfrac{3}{4}\right)$.

2 © VCAA 2016 2AQ7 37% Given that $x=\sin(t)-\cos(t)$ and $y=\dfrac{1}{2}\sin(2t)$, then $\dfrac{dy}{dx}$ in terms of t is

A $\cos(t)-\sin(t)$

B $\cos(t)+\sin(t)$

C $\sec(t)+\operatorname{cosec}(t)$

D $\sec(t)-\operatorname{cosec}(t)$

E $\dfrac{\cos(2t)}{\cos(t)-\sin(t)}$

Mastery

3 WORKED EXAMPLE 18 TECH-FREE Find $\dfrac{dy}{dx}$ for the equation $3x\sin(y)=1$.

4 TECH-FREE Find $\dfrac{dy}{dx}$ at $y=1$ for the equation $3xy^2+4x=1$.

5 Using CAS 5 Use implicit differentiation to find the derivative of $xy+x^2=2$ to determine the gradient at the point (1, 1).

▶ **6** For the equation $3y^2 + \frac{x}{y} = 1$, $\frac{dy}{dx}$ equals

A $\frac{x}{y} + 1$ **B** $\frac{y}{6y^3 - x} + 1$ **C** $\frac{y}{x - 6y^3} + 1$ **D** $\frac{y}{x - 6y^3}$ **E** $6y - \frac{x}{y} + 1$

7 For the equation $3y + \frac{x}{2y} = x$, $\frac{dy}{dx}$ equals

A $\frac{2y^2 - y}{6y^2 - x}$ **B** $\frac{2y^2 + y}{6y^2 - x}$ **C** $\frac{y}{x - 6y^2}$ **D** $\frac{y}{x - 6y^3}$ **E** $\frac{4y^2 - y}{6y^2 - x}$

Exam practice

80–100% 60–79% 0–59%

8 © VCAA 2009 1Q5 TECH-FREE (6 marks) Consider the family of curves defined by the relation $3x^3 - y^2 + kx + 5y - 2xy = 4$, where $k \in R$.

a 76% Verify that every curve in the family passes through the point (0, 4), and find the other point of intersection with the y-axis. 2 marks

b 73% Find an expression for $\frac{dy}{dx}$ in terms of x, y and k. 2 marks

c 45% **Hence** evaluate the gradient of the curve at the point (1, 1). 2 marks

9 © VCAA 2008 1Q2 68% TECH-FREE (4 marks) Given the relation $3x^2 + 2xy + y^2 = 11$, find the gradient of the **normal** to the graph of the relation at the point in the first quadrant where $x = 1$.

The word 'normal' is no longer part of the course. It means the line that is perpendicular to the tangent drawn at the same point on the graph.

10 © VCAA 2006 1Q1 TECH-FREE (4 marks) Consider the relation $2xy - 9y^2 + 9 = 0$.

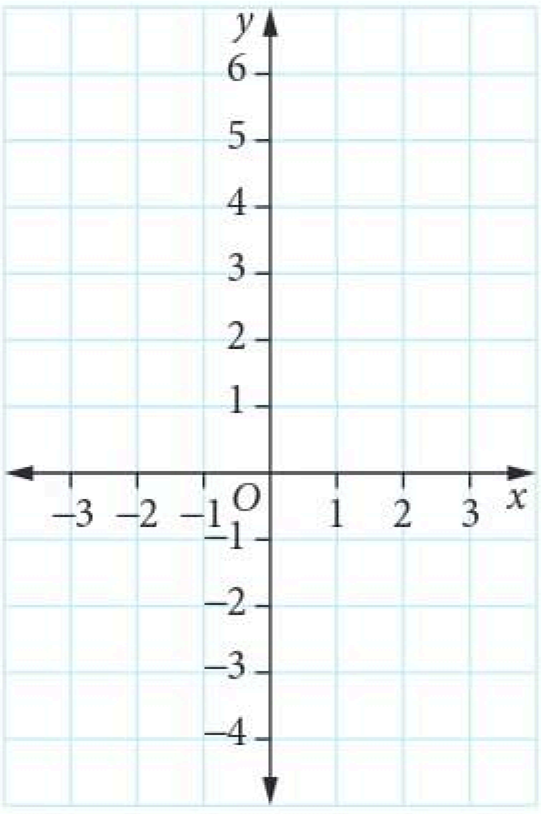
a 70% Find an expression for $\frac{dy}{dx}$ in terms of x and y. 2 marks

b 60% Hence find the exact value of $\frac{dy}{dx}$ when $y = 1$. 2 marks

11 © VCAA 2012 1Q6 62% TECH-FREE (3 marks) Find the gradient of the tangent to the curve $xy^2 + y + (\log_e(x - 2))^2 = 14$ at the point (3, 2).

12 © VCAA 2010 1Q9 TECH-FREE (6 marks)

a 63% Copy the axes below and on them sketch the graph with equation $x^2 - \frac{(y-2)^2}{4} = 1$. State all intercepts with the coordinate axes and give the equations of any asymptotes. 3 marks



b 50% Find the gradient of the curve with equation $x^2 - \frac{(y-2)^2}{4} = 1$ at the point where $x = 2$ and $y < 0$. 3 marks ▶

13 © VCAA 2018 1Q3 46% TECH-FREE (4 marks) Find the gradient of the curve with equation $2x^2 \sin(y) + xy = \frac{\pi^2}{18}$ at the point $\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$. Give your answer in the form $\frac{a}{\pi\sqrt{b} + c}$ where a, b and c are integers.

14 © VCAA 2006 2AQ8 60% The slope of the curve $2x^3 - y^2 = 7$ at the point where $y = -3$ is

A -4
B -2
C 2
D 4
E $\frac{27}{2}$

5 Chapter summary

The product rule

$$\frac{d}{dx}(uv) = uv' + vu' \quad \text{or} \quad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

The quotient rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2} \quad \text{or} \quad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

The chain rule

The chain rule is used to differentiate a composite function $y = f[g(x)]$.

If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

Derivatives of circular functions

$y = \sin(x)$	$\frac{dy}{dx} = \cos(x)$	$y = \tan(x)$	$\frac{dy}{dx} = \sec^2(x)$
$y = \sin(ax)$	$\frac{dy}{dx} = a\cos(ax)$	$y = \tan(ax)$	$\frac{dy}{dx} = a\sec^2(ax)$
$y = \cos(x)$	$\frac{dy}{dx} = -\sin(x)$	$y = \cot(x)$	$\frac{dy}{dx} = -\text{cosec}^2(x)$
$y = \cos(ax)$	$\frac{dy}{dx} = -a\sin(ax)$	$y = \cot(ax)$	$\frac{dy}{dx} = -a\,\text{cosec}^2(ax)$

$y = \sin^{-1}(x)$	$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}, x \in (-1, 1)$
$y = \cos^{-1}(x)$	$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}, x \in (-1, 1)$
$y = \tan^{-1}(x)$	$\frac{dy}{dx} = \frac{1}{1+x^2}, x \in R$
$y = \sin^{-1}\left(\frac{x}{a}\right)$	$\frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}}, x \in (-a, a)$
$y = \cos^{-1}\left(\frac{x}{a}\right)$	$\frac{dy}{dx} = \frac{-1}{\sqrt{a^2 - x^2}}, x \in (-a, a)$
$y = \tan^{-1}\left(\frac{x}{a}\right)$	$\frac{dy}{dx} = \frac{a}{a^2 + x^2}, x \in R$

Derivatives of exponential and logarithmic functions

$y = e^x$	$\frac{dy}{dx} = e^x$	$y = \log_e(x)$	$\frac{dy}{dx} = \frac{1}{x}$
$y = e^{ax}$	$\frac{dy}{dx} = ae^{ax}$	$y = \log_e(ax)$	$\frac{dy}{dx} = \frac{a}{ax} = \frac{1}{x}$

The second derivative

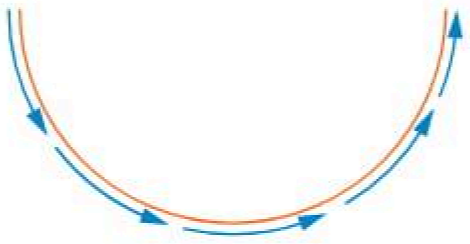
For $y = f(x)$, the first derivative is $f'(x)$ or $\frac{dy}{dx}$ and the second derivative is $f''(x)$ or $\frac{d^2y}{dx^2}$.

Points of inflection

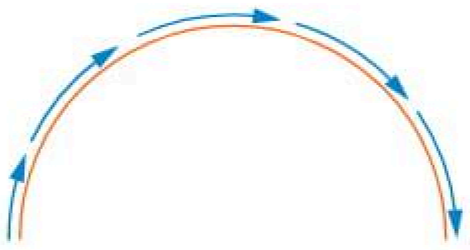
- A **point of inflection** on the graph of $y = f(x)$ is a point where the concavity of the graph changes and $f''(x) = 0$.
- $f''(x)$ has different signs on either side of the point of inflection.
- A **stationary point of inflection** is located where $f''(x) = 0$ and $f'(x) = 0$.
- A **non-stationary point of inflection** is located where $f''(x) = 0$ and $f'(x) \neq 0$.

Concavity

- Where $f''(x) > 0$, the gradient of the graph of $f(x)$ is increasing, and the graph is **concave up.**



- Where $f''(x) < 0$, the gradient of the graph of $f(x)$ is decreasing, and the graph is **concave down.**



The second derivative test

- If $f''(x) > 0$ at a stationary point, then it is a local **minimum point**.
- If $f''(x) < 0$ at a stationary point, then it is a local **maximum point**.

The chain rule and related rates of change

Related rates problems have at least three variables to 'relate'.

1 Write down the rate you are given.

2 Write down the rate you have to find.

3 Write down a formula relating two of the variables.

4 State the chain rule using three variables.

5 Solve for the required rate.

Implicit differentiation

Implicit differentiation is used to differentiate functions of the form where y is not the subject but appears more than once, such as in $y^2 + 2x = xy$.

Cumulative examination 1

Total number of marks: 11 Reading time: 5 minutes Writing time: 17 minutes

TECH-FREE *Technology is NOT permitted.*

1 © VCAA 2019N 1Q7 (5 marks) Given that $3x^2 + 2xy + y^2 = 6$, find $\frac{d^2y}{dx^2}$ at the point (1, 1).

2 © VCAA 2012 1Q10 (6 marks)

a Consider the functions with rules

$$f(x) = \left(\arcsin\left(\frac{x}{2}\right) + \frac{3}{\sqrt{25x^2 - 1}}\right) \text{ and } g(x) = \arcsin(3x) - \frac{3}{\sqrt{25x^2 - 1}}.$$

i Find the maximal domain of $f_1(x) = \arcsin\left(\frac{x}{2}\right)$. 1 mark

ii Find the maximal domain of $f_2(x) = \frac{3}{\sqrt{25x^2 - 1}}$. 1 mark

iii Find the largest set of values of $x \in R$ for which $f(x)$ is defined. 1 mark

b Given that $h(x) = f(x) + g(x)$ and that $\theta = h = \left(\frac{1}{4}\right)$, evaluate $\sin(\theta)$.

Give your answer in the form $\frac{a\sqrt{b}}{c}$, $a, b, c \in Z$. 3 marks

Cumulative examination 2

Total number of marks: 21 Reading time: 4 minutes Writing time: 32 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 © VCAA 2018 2AQ6 The complex numbers z, iz and $z + iz$, where $z \in C\backslash\{0\}$, are plotted in the Argand plane, forming the vertices of a triangle. The area of this triangle is given by

A $|z|$ **B** $|z| + |z|^2$ **C** $\frac{|z|^2}{2}$ **D** $|z|^2$ **E** $\frac{\sqrt{3}|z|^2}{2}$

2 © VCAA 2018 2AQ2 Consider the function f with the rule $f(x) = \frac{1}{\sqrt{\sin^{-1}(cx + d)}}$, where $c, d \in R$, and $c > 0$. The domain of f is

A $x > -\frac{d}{c}$ **B** $-\frac{d}{c} \leq x \leq \frac{1-d}{c}$ **C** $\frac{-1-d}{c} \leq x \leq \frac{1-d}{c}$

D $x \in R\backslash\left\{-\frac{d}{c}\right\}$ **E** $x \in R$

3 © VCAA 2017 2AQ6 Given that $\frac{dy}{dx} = e^x \arctan(y)$, the value of $\frac{d^2y}{dx^2}$ at the point $(0, 1)$ is

A $\frac{1}{2}$ **B** $\frac{3\pi}{8}$ **C** $-\frac{1}{2}$ **D** $\frac{\pi}{4}$ **E** $-\frac{\pi}{8}$

4 © VCAA 2017 2AQ10 A function f, its derivative f' and its second derivative f'' are defined for $x \in R$ with the following properties

$f(a) = 1, f(-a) = -1, \quad f(b) = -1, f(-b) = 1$

and $f''(x) = \frac{(x + a)^2(x - b)}{g(x)}$, where $g(x) < 0$.

The coordinates of any points of inflection of $|f(x)|$ are

A $(-a, 1)$ and $(b, 1)$ **B** $(b, -1)$ **C** $(-a, -1)$ and $(b, -1)$

D $(-a, 1)$ **E** $(b, 1)$

5 © VCAA 2018N 2AQ7 The gradient of the line that is **perpendicular** to the graph of the relation $3y^2 - 5xy - x^2 = 1$ at the point $(1, 2)$ is

A $-\frac{1}{12}$ **B** $\frac{12}{7}$ **C** 21 **D** $-\frac{7}{12}$ **E** $-\frac{7}{13}$

▶

Section B 2 questions

16 marks

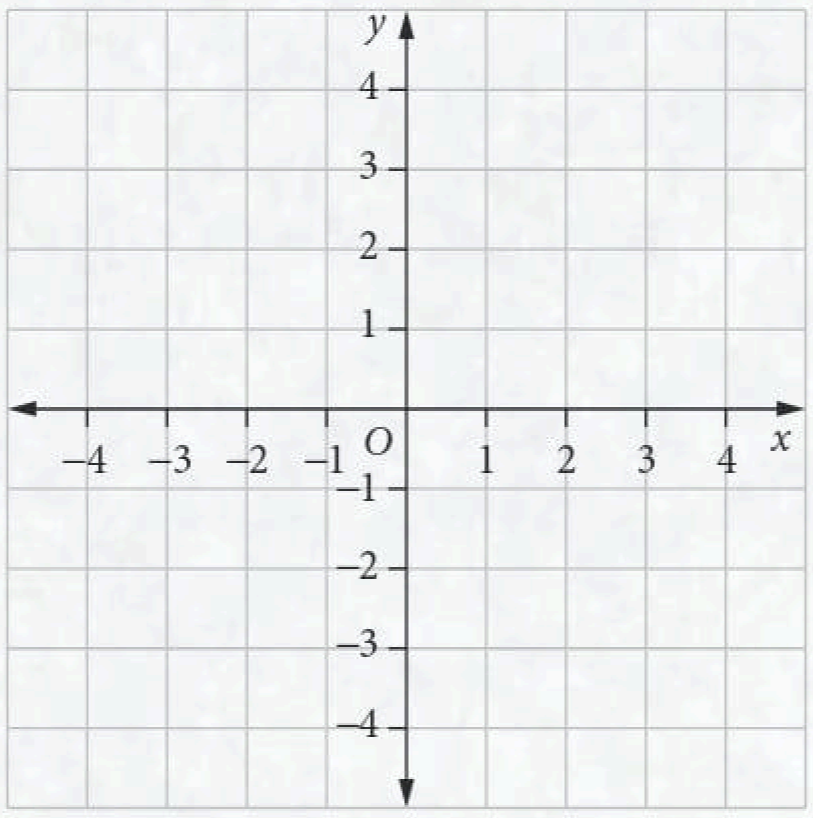
1 © VCAA 2021N 2BQ1 (8 marks) A curve is defined parametrically by $x = \sec(t)$, $y = \operatorname{cosec}(t)$ where $t \in \left(0, \frac{\pi}{2}\right)$.

a Show that the curve can be represented in Cartesian form by the relation $y = \frac{x}{\sqrt{x^2 - 1}}$. 2 marks

b State the domain and range of the relation given by $y = \frac{x}{\sqrt{x^2 - 1}}$ for this curve. 2 marks

c Use the derivative of the relation to show that the gradient of the curve is negative at all points on the curve. 2 marks

d Copy the grid below and on it sketch the graph of the relation, labelling any asymptotes with their equations. 2 marks



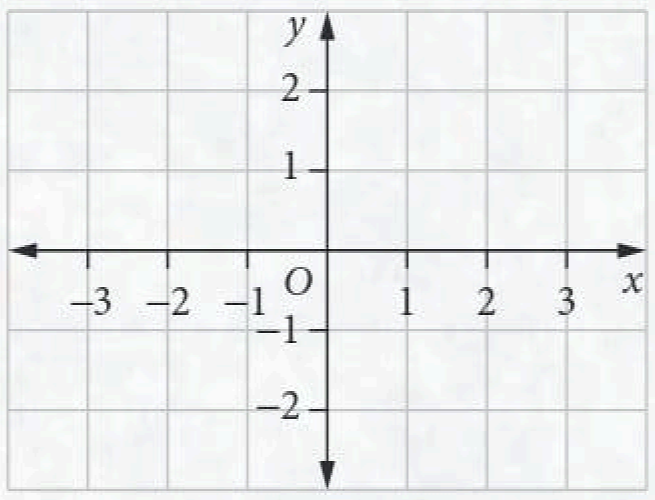
2 © VCAA 2017 2BQ1 (8 marks) Let $f: D \to R$, $f(x) = \frac{x}{1 + x^3}$, where D is the maximal domain of f.

a **i** Find the equations of any asymptotes of the graph of f. 1 mark

ii Find $f'(x)$ and state the coordinates of any stationary points of the graph of f, correct to two decimal places. 2 marks

iii Find the coordinates of any points of inflection of the graph of f, correct to two decimal places. 2 marks

b Copy the axes provided below and on them sketch the graph of $f(x) = \frac{x}{1 + x^3}$ from $x = -3$ to $x = 3$, marking all stationary points, points of inflection and intercepts with axes, labelling them with their coordinates. Show any asymptotes and label them with their equations. 3 marks



CHAPTER

6 VECTOR EQUATIONS

Note: This is a new topic introduced to the Specialist Mathematics course in 2023, so this chapter does not contain past VCE exam questions.

Study Design coverage

AREA OF STUDY 5: SPACE AND MEASUREMENT

Vector and cartesian equations

- vector equations and parametric equations of curves in two or three dimensions involving a parameter (and the corresponding cartesian equation in the two-dimensional case)
- vector equation of a straight line, given the position of two points, or equivalent information, in both two and three dimensions
- vector cross product, normal to a plane and vector, parametric and cartesian equations of a plane.

VCE Mathematics Study Design 2023–2027 p. 112, © VCAA 2022

Video playlists (5):

6.1 Curves in 2D and 3D
6.2 Vector equation of a line
6.3 Normals to a plane
6.4 Equations of planes

VCE question analysis Vector equations

Worksheets (2):

6.2 Equations of lines in space
6.4 Equations of planes

Spreadsheets (1):

6.1 3D sketcher

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

6.1 Curves in 2D and 3D

Video playlist Curves in 2D and 3D

Spreadsheet 3D sketcher

The standard parametric form of a parabola with turning point (h, k) is $x = 2at + h, y = at^2 + k$.

This is the same as the Cartesian form $4a(y - k) = (x - h)^2$.

We can write the same curve in vector form as $\underset{\sim}{r}(t) = (2at + h)\underset{\sim}{i} + (at^2 + k)\underset{\sim}{j}$, where $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are the unit vectors in the x and y directions.

In general, the **vector equation** of a curve in two dimensions will be given by $\underset{\sim}{r}(t) = f(t)\underset{\sim}{i} + g(t)\underset{\sim}{j}$, where f and g are functions of the **parameter** t. We can find the Cartesian form by eliminating t from the equation.

WORKED EXAMPLE 1 Changing from vector to Cartesian form

WB p. 117

Convert each vector equation to Cartesian form and describe the curve.

a $\underset{\sim}{r}(t) = (3t + 2)\underset{\sim}{i} + (2t - 5)\underset{\sim}{j}$

b $\underset{\sim}{r}(t) = (5\cos(t) - 1)\underset{\sim}{i} + (4\cos(t) - 3)\underset{\sim}{j}$ for $0 \le t < 2\pi$

Steps	Working
a **1** Write in parametric form.	$x = 3t + 2, y = 2t - 5$
2 Write t in terms of x.	$t = \dfrac{x-2}{3}$
3 Substitute in y and express in standard form.	$y = 2\left(\dfrac{x-2}{3}\right) - 5$ $3y = 2x - 4 - 15$ $2x - 3y - 19 = 0$
4 Describe the curve.	The curve is a straight line with gradient $\dfrac{2}{3}$ and y intercept $-6\dfrac{1}{3}$.

b **1** Write in parametric form.	$x = 5\cos(t) - 1,\ y = 4\cos(t) - 3$
2 Isolate $\cos(t)$ and $\sin(t)$.	$\cos(t) = \dfrac{x+1}{5},\ y = \dfrac{y+3}{4}$
3 Use $\cos^2(\theta) + \sin^2(\theta) = 1$.	$\cos^2(t) + \sin^2(t) = 1$
4 Substitute expressions and simplify.	$\left(\dfrac{x+1}{5}\right)^2 + \left(\dfrac{y+3}{4}\right)^2 = 1$ $\dfrac{(x+1)^2}{25} + \dfrac{(y+3)^2}{16} = 1$
5 Describe the curve.	The curve is an ellipse with centre $(-1, -3)$, horizontal axis of length 10 and vertical axis of length 8.

Exam hack

Simplify parametric equations in 2D with trigonometric functions by trying one of the trigonometric identities $\sin^2(\theta) + \cos^2(\theta) = 1$, $\tan^2(\theta) + 1 = \sec^2(\theta) = 1$ or $1 + \cot^2(\theta) = \mathrm{cosec}^2(\theta)$.

The **vector equation of a curve in three dimensions** has the same form as the vector equation of a curve in 2D, but with three functions. It is $\underset{\sim}{r}(t) = f(t)\underset{\sim}{i} + g(t)\underset{\sim}{j} + h(t)\underset{\sim}{k}$, where f, g and h are functions of the parameter t.

We may be able to describe a 3D vector curve by considering projections on the y–z, x–z and x–y planes. These will have no $\underset{\sim}{i}$, $\underset{\sim}{j}$ or $\underset{\sim}{k}$ components respectively.

WB
p. 118

WORKED EXAMPLE 2 Describing a 3D curve from its vector equation

Describe the shape of the curve given by $\underset{\sim}{r}(t) = 3\cos(t)\underset{\sim}{i} + 0.5t\underset{\sim}{j} + 3\sin(t)\underset{\sim}{k}$.

Steps	**Working**
1 Isolate the trigonometric part for the projection on the x-z plane.	$3\cos(t)\underset{\sim}{i} + 3\sin(t)\underset{\sim}{k}$
2 Identify the shape.	The projection on the x–z plane is a circle of radius 3.
3 Describe the whole vector.	As t increases from 0 to 2π, the y component increases from 0 to π, while the x and z components trace out a circle.
4 Describe the 3D shape.	The shape is a coil of radius 3 with centre along the y-axis, with a full turn every interval of π along the axis.

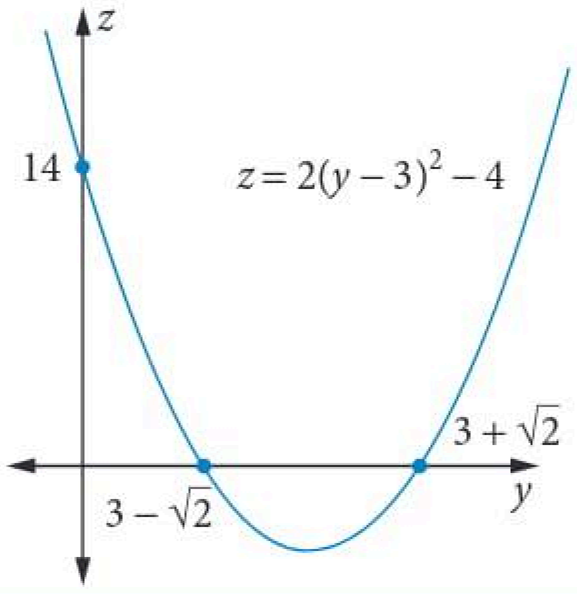
It is relatively easy to sketch the projection of a 3D curve in vector form on any of the y–z, x–z and x–y planes. We just neglect the appropriate component. Find the Cartesian form to help identify the projection. The projection on the x–y plane is perpendicular to the z-axis, and similarly for the other planes.

WB
p. 119

WORKED EXAMPLE 3 Sketching the projection of a curve on the plane of two axes

Sketch the projection of the curve $\underset{\sim}{r}(t) = (t^2 - 5t + 2)\underset{\sim}{i} + (t+3)\underset{\sim}{j} + (2t^2 - 4)\underset{\sim}{k}$ on the y–z plane.

Steps	**Working**
1 Write the projection on the y-z plane.	$(t+3)\underset{\sim}{j} + (2t^2 - 4)\underset{\sim}{k}$
2 Write in parametric form.	$y = t + 3,\ z = 2t^2 - 4$
3 Eliminate t.	$t = y - 3$ $z = 2(y-3)^2 - 4$

4 Describe the shape.	The shape is a parabola with axis parallel to the z-axis, a minimum at $(y, z) = (3, -4)$ and z-intercept 14.
5 Find the y-intercepts.	$2\,(y-3)^2 - 4 = 0$ $(y-3)^2 = 2$ $y - 3 = \pm\sqrt{2}$ $y = 3 - \sqrt{2}\,,\ 3 + \sqrt{2}$
6 Sketch the projection.	

EXERCISE 6.1 Curves in 2D and 3D

ANSWERS p. 586

1 WORKED EXAMPLE 1 Convert each vector equation to Cartesian form and describe its curve.

a $\underset{\sim}{r}(t) = (5t-3)\,\underset{\sim}{i} + (2t-4)\,\underset{\sim}{j}$

b $\underset{\sim}{r}(t) = (3\sec(t)+1)\,\underset{\sim}{i} + (2\tan(t)+2)\,\underset{\sim}{j}$ for $0 \le t < 2\pi$

c $\underset{\sim}{r}(t) = t\,\underset{\sim}{i} + (t+1)^3\,\underset{\sim}{j}$

d $\underset{\sim}{r}(t) = (5\sin(t)-3)\,\underset{\sim}{i} + (5\cos(t)+4)\,\underset{\sim}{j}$ for $0 \le t < 2\pi$

2 WORKED EXAMPLE 2 Describe the shape of the curve given by $\underset{\sim}{r}(t) = 3\cos(t)\,\underset{\sim}{i} + 0.5t\underset{\sim}{j} + t\underset{\sim}{k}$.

3 Describe the shape of the curve given by $\underset{\sim}{r}(t) = 4t\underset{\sim}{i} - t\underset{\sim}{j} + 2t\underset{\sim}{k}$.

4 WORKED EXAMPLE 3 Sketch the projection of the curve $\underset{\sim}{r}(t) = t\,\underset{\sim}{i} + (t^2-5t+4)\,\underset{\sim}{j} + (6t^2-4t+2)\,\underset{\sim}{k}$ on the x–y plane.

5 Sketch the projection of the curve $\underset{\sim}{r}(t) = (3\sin(t)+2)\,\underset{\sim}{i} + (t^2+3)\,\underset{\sim}{j} + (2\cos(t)-1)\,\underset{\sim}{k}$ perpendicular to the y-axis.

6 Express the projection of $\underset{\sim}{r}(t) = \left(2\sec(t)-1\right)\underset{\sim}{i} + \left(\sqrt{t-2}\underset{\sim}{j} + \sqrt{7-t}\right)\underset{\sim}{k}$ perpendicular to the x-axis in Cartesian form and identify the shape.

7 Describe the shape of the curve given by $\underset{\sim}{r}(t) = 3\underset{\sim}{i} + 0.5t\underset{\sim}{j} + t^2\underset{\sim}{k}$.

Exam practice

8 TECH-FREE (2 marks) Convert the Cartesian equation $y = 4\,(x-3)^2 + 2$ to 2D vector form using the parameter $t = x + 1$.

9 Which of the following points is on the curve given by $\underset{\sim}{r}(t) = (t^2-6)\,\underset{\sim}{i} + (5-t)\underset{\sim}{j} + (t^3+3)\,\underset{\sim}{k}$?

A $(2, -1, 11)$ **B** $(-5, -8, 2)$ **C** $(-2, 1, 11)$ **D** $(-2, 11, -5)$ **E** $(3, -4, -24)$

10 A curve is given by the vector equation $\underset{\sim}{r}(t) = \sqrt{t+4}\,\underset{\sim}{i} + 2t\underset{\sim}{j} + 5t^2\underset{\sim}{k}$.

Which of the following is the projection of the curve on one of the x–y, x–z or y–z planes?

A $z = 1.25y^2$ **B** $y = x^2 - 4$ **C** $z = 5\,(x+2)(x-2)$

D $z = 2.5y^2$ **E** $z = 5\,(x^2+4)^2$

Video playlist
Vector equation of a line

Worksheet
Equations of lines in space

6.2 Vector equation of a line

Remember that the **position vector** of a point A, $\overrightarrow{OA}$, is the vector from the origin O to the point A. It is usually written as the lower case vector $\underset{\sim}{a}$.

Remember that the vector from A to B, $\overrightarrow{AB}$, is given by $\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$.

A straight line through the points $A\ (3, 2, -4)$ and $B\ (-2, 3, 1)$ is in the direction of $\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a} = (-5, 1, 5)$. For any point $P\ (x, y, z)$ on the line, the vector $\overrightarrow{AP} = \underset{\sim}{p} - \underset{\sim}{a}$ will be a multiple of $\overrightarrow{AB}$.

Thus $\underset{\sim}{p} - \underset{\sim}{a} = t\overrightarrow{AB}$, where $t \in R$, so $\underset{\sim}{p} = \underset{\sim}{a} + t\overrightarrow{AB} = (3, 2, -4) + t\,(-5, 1, 5) = (3 - 5t, 2 + t, -4 + 5t),\ t \in R$.

In this case, t is a parameter for the position vector $\underset{\sim}{r}$ of a general point on the straight line, $t \in R$.

We can write this as $\underset{\sim}{r}\,(t) = (3 - 5t)\,\underset{\sim}{i} + (2 + t)\,\underset{\sim}{j} + (-4 + 5t)\,\underset{\sim}{k}$.

In parametric form, this is $x = 3 - 5t$, $y = 2 + t$, $z = -4 + 5t$.

Writing $t = \dfrac{x-3}{-5}$, $t = \dfrac{y-2}{1}$ and $t = \dfrac{z-(-4)}{5}$ gives the Cartesian form $\dfrac{x-3}{-5} = \dfrac{y-2}{1} = \dfrac{z+4}{5}$.

Equation of a line

The equation of a straight line through $A\ (a_1, a_2, a_3)$ and $B\ (b_1, b_2, b_3)$ is:

- $\underset{\sim}{r}\,(t) = \underset{\sim}{a} + t\overrightarrow{AB} = \underset{\sim}{a} + t\underset{\sim}{d}$ as a **vector equation**, where $t \in R$ and $\overrightarrow{AB} = \underset{\sim}{d} = \underset{\sim}{b} - \underset{\sim}{a}$ is the **displacement vector** in the direction of the line. Note that $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$ and similarly for $\underset{\sim}{b}$.
- $x = a_1 + td_1$, $y = a_2 + td_2$, $z = a_3 + td_3$ are the **parametric equations** of a line in 3D.
- $\dfrac{x - a_1}{b_1 - a_1} = \dfrac{y - a_2}{b_2 - a_2} = \dfrac{z - a_3}{b_3 - a_3}$ or $\dfrac{x - a_1}{d_1} = \dfrac{y - a_2}{d_2} = \dfrac{z - a_3}{d_3}$ is a Cartesian equation.
- In the vector forms, if $0 \le t \le 1$, the points are on the line segment AB,
  if $t < 0$, the points are on the line to the left of A, and
  if $t > 1$, the points are to the right of B.

WB
p. 120

WORKED EXAMPLE 4 Vector equations of straight lines

Find the vector equation and parametric form of lines through

a (3, 5) and (–2, 4) **b** (3, –4, 4) and (–2, 3, 1)

Steps	**Working**
a 1 Find the displacement vector.	$\underset{\sim}{d} = \underset{\sim}{b} - \underset{\sim}{a} = (-2, 4) - (3, 5) = (-5, -1)$
Write the vector equation.	$\underset{\sim}{r}\,(t) = \underset{\sim}{a} + t\underset{\sim}{d} = (3, 5) + t\,(-5, -1),\ t \in R$
Write the parametric form.	$\underset{\sim}{r}\,(t) = (3 - 5t)\,\underset{\sim}{i} + (5 - t)\,\underset{\sim}{j}$ or $x = 3 - 5t, y = 5 - t, t \in R$
b 1 Find the displacement vector.	$\underset{\sim}{d} = \underset{\sim}{b} - \underset{\sim}{a} = (-2, 3, 1) - (3, -4, 4) = (-5, 7, -3)$
2 Write the vector equation.	$\underset{\sim}{r}\,(\mathrm{t}) = \underset{\sim}{a} + t\underset{\sim}{d} = (3, -4, 4) + t\,(-5, 7, -3),\ t \in R$
3 Write the parametric form.	$\underset{\sim}{r}\,(t) = (3 - 5t)\,\underset{\sim}{i} + (-4 + 7t)\,\underset{\sim}{j} + (4 - 3t)\,\underset{\sim}{k}$ or $x = 3 - 5t, y = -4 - 3t, z = 4 - 3t, t \in R$

We can find the Cartesian form from the vector form or directly from information.

WORKED EXAMPLE 5 Cartesian equations of straight lines

Find the Cartesian equation of lines through

a (1, 3) and (4, 5) **b** (3, 6, 2) and (–1, –4, 0)

Steps	Working
a **1** Find the displacement vector.	$\underset{\sim}{d} = \underset{\sim}{b} - \underset{\sim}{a} = (4, 5) - (1, 3) = (3, 2)$
2 Write the Cartesian equation.	$\frac{x-1}{3} = \frac{y-3}{2}$
3 For 2D cases, it is usual to write in the standard form.	$2(x-1) = 3(y-3)$ $2x - 3y + 7 = 0$
b **1** Find the displacement vector.	$\underset{\sim}{d} = \underset{\sim}{b} - \underset{\sim}{a} = (-1, -4, 0) - (3, 6, 2) = (-4, -10, -2)$
2 Write the Cartesian equation.	$\frac{x-3}{-4} = \frac{y-6}{-10} = \frac{z-2}{-2}$
3 Simplify if possible.	$\frac{x-3}{2} = \frac{y-6}{5} = z - 2$

WB p. 121

We can use any point on the line in the Cartesian form, so $\frac{x+1}{2} = \frac{y+4}{5} = z$ (using the second point as $\underset{\sim}{a}$) is also an equation of the line in part **b** above. The equation can also be written as a *pair* of equations similar to the standard form of 2D equations. Hence, $x = 2z - 1$ and $y = 5z - 4$ also give the same straight line.

WORKED EXAMPLE 6 Vector equation of a segment

WB p. 122

a Find the vector equation of the line segment between P (1, 6, –2) and Q (4, 3, 3).

b Which of the points (2, 5, –4), $\left(3, 2, \frac{4}{3}\right)$, (5.5, 1.5, –5.5) and (2, 5, 0) are on the line segment?

Steps	Working
a **1** Find the displacement vector.	$\underset{\sim}{d} = \underset{\sim}{b} - \underset{\sim}{a} = (4, 3, 3) - (1, 6, -2) = (3, -3, 5)$
2 Write vector equation.	$\underset{\sim}{r}(t) = (1 + 3t)\underset{\sim}{i} + (6 - 3t)\underset{\sim}{j} + (-2 + 5t)\underset{\sim}{k}, t \in R, 0 \le t \le 1$
b **1** Reject the points with a value outside the range.	–4 is outside the range $-2 \le z \le 3$ and 5.5 is outside the range $1 \le x \le 4$ so (5.5, 1.5, –5.5) and (2, 5, –4) are not on the line.
2 Find the value of t for the x values of the remainder.	For $\left(3, 2, \frac{4}{3}\right)$, $1 + 3t = 3$, so $t = \frac{2}{3}$. For (2, 5, 0), $1 + 3t = 2$, so $t = \frac{1}{3}$.
3 Test the values of t.	For $t = \frac{2}{3}$, $6 - 3t = 4$ not OK and $-2 + 5t = \frac{4}{3}$ OK. For $t = \frac{1}{3}$, $6 - 3t = 5$ OK and $-2 + 5t = -\frac{1}{3}$ not OK.
4 Write the conclusion.	No points are on the line segment PQ.

Exam hack

Look for obviously wrong coordinates first in testing points.

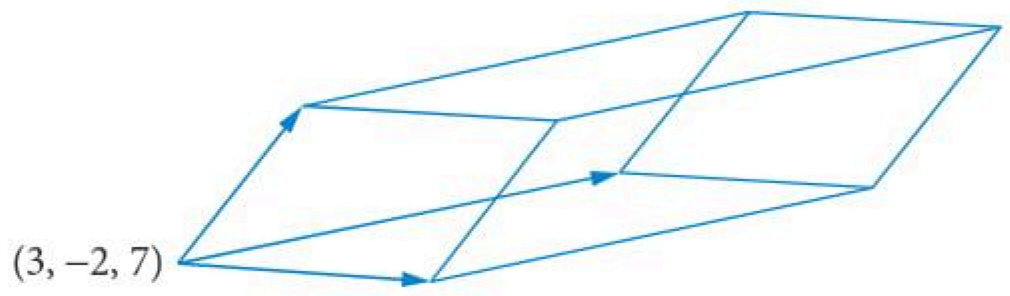
We may have to use information other than given points to find the equation of a straight line.

WB

p. 123

WORKED EXAMPLE 7 Equation of a line from given information

The vectors $3\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}$, $-5\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$ and $4\underset{\sim}{i} + 7\underset{\sim}{j} + 3\underset{\sim}{k}$ are parallel to edges of a parallelepiped. One vertex is at the point $P\,(3, -2, 7)$. Find the equation of the line through this vertex and the point Q on the parallelepiped furthest from it.

Steps	Working
1 Sketch the parallelepiped.	
2 Find the furthest point.	$\underset{\sim}{q} = (3, -2, 7) + (3, -1, 4) + (-5, 2, 1) + (4, 7, 3)$ $= (5, 6, 14)$
3 Find the displacement vector.	$\underset{\sim}{q} - \underset{\sim}{p} = (5, 6, 14) - (3, -2, 7) = (2, 8, 7)$
4 Write the equation of the line PQ.	$\underset{\sim}{r}\,(t) = (3, -2, 7) + t\,(2, 8, 7)$ or $\underset{\sim}{r}\,(t) = (3 + 2t)\,\underset{\sim}{i} + (-2 + 8t)\,\underset{\sim}{j} + (7 + 7t)\,\underset{\sim}{k},\ t \in R$

It is very easy to make arithmetic mistakes with multiple vector operations, so use CAS for examples like those above.

EXERCISE 6.2 Vector equation of a line

ANSWERS p. 586

Recap

1 The Cartesian equation of the curve given by $\underset{\sim}{r}\,(t) = (5 - 2t)\,\underset{\sim}{i} + (3 + 4t)\,\underset{\sim}{j},\ t \in R$, is

A $y = 4x + 3$ **B** $y = -2x + 9$ **C** $y = -2x - 3$ **D** $y = -2x + 5$ **E** $y = 2x + 12$

2 The projection of $\underset{\sim}{r}\,(t) = (2\sin(t) - 3)\,\underset{\sim}{i} + (5 + t)\,\underset{\sim}{j} + (7 + 2\cos(t))\,\underset{\sim}{k},\ t \in R$, onto the y-z plane is

A $(y + 3)^2 + (z - 7)^2 = 4$ **B** $z = 2\cos(y - 12)$ **C** $z = 2\cos(y - 5) - 7$

D $z = 2\cos(y + 5) + 7$ **E** $z = 2\cos(y - 5) + 7$

Mastery

3 WORKED EXAMPLE 4 Find the vector equation of the line through

a $(3, 8)$ and $(5, -2)$ **b** $(-5, 3, 8)$ and $(2, 7, 0)$ **c** $(6, -1, -4)$ and $(8, 3, 6)$

d $(5, -8, 0)$ and $(-1, -6, 1)$ **e** $(-8, 8, -8)$ and $(0, 7, -1)$ **f** $(-3, -6, -8)$ and $(5, 4, -2)$

4 Find the parametric form of the line through

a $(9, -8, 0)$ and $(1, 2, -7)$ **b** $(-4, -1, 5)$ and $(3, 8, -5)$ **c** $(-8, -4, -4)$ and $(-2, 9, 5)$

d $(2, 9, -9)$ and $(7, -6, -3)$ **e** $(-5, -7, 2)$ and $(5, 6, -8)$ **f** $(7, -1, -7)$ and $(3, -8, -9)$

5 What is the vector equation of the line through $(1, 6, -2)$ in the direction of $3\underset{\sim}{i} + 5\underset{\sim}{j} - 2\underset{\sim}{k}$?

6 WORKED EXAMPLE 5 Find the Cartesian equation of the line through

a $(6, 1)$ and $(-3, 7)$ **b** $(-2, 4)$ and $(2, 9)$ **c** $(-3, 0, -8)$ and $(2, 2, 1)$

d $(-3, -2, 8)$ and $(-4, 5, 4)$ **e** $(2, -5, -9)$ and $(7, 5, 7)$ **f** $(4, 4, 6)$ and $(-2, -2, -2)$

7 WORKED EXAMPLE 6 Find the vector equation of the line segment between

a $(5, -1, -4)$ and $(-6, 3, 8)$ **b** $(-6, 8, -2)$ and $(5, 1, -7)$ **c** $(-1, -8, 7)$ and $(2, -5, -9)$

d $(0, -1, -5)$ and $(-7, 3, 1)$ **e** $(3, 6, 3)$ and $(-1, -4, 0)$ **f** $(5, -6, 0)$ and $(9, -8, 1)$

8 State whether each point lies on the given line.

a $(13, 8, -7), -2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k} + t(-5\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k})$

b $(10, -3, -8), 6\underset{\sim}{i} - 5\underset{\sim}{j} + t(\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k})$

c $(8, 9, 13), 6\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k} + t(-\underset{\sim}{i} - 6\underset{\sim}{j} - 4\underset{\sim}{k})$

d $(3, -9, -4), -3\underset{\sim}{j} - 6\underset{\sim}{k} + t(3\underset{\sim}{i} - 6\underset{\sim}{j} + 2\underset{\sim}{k})$

e $(-13, 2, 11), -5\underset{\sim}{i} - 6\underset{\sim}{j} - 5\underset{\sim}{k} + t(-2\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k})$

f $(-6, -4, -3), 3\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k} + t(-3\underset{\sim}{i} + 5\underset{\sim}{j} + 2\underset{\sim}{k})$

9 State whether each point lies on the line segment joining $A\ (2, -4, 3)$ and $B\ (6, 4, -9)$.

a $(4, 0, 4)$ **b** $(8, 8, -16)$ **c** $(-3, -14, 17)$ **d** $(-1, -10, -7)$ **e** $(5, 2, -7)$

10 State whether each point lies on the line segment joining $A\ (-3, 4, -5)$ and $B\ (7, -11, 15)$.

a $(3, -6, 7)$ **b** $(-7, 9, -13)$ **c** $(3, -6, -7)$ **d** $(8, -14, 17)$ **e** $(1, -3, 3)$

11 WORKED EXAMPLE 7 A rectangular prism has its longest edge parallel to the y-axis and its shortest edge parallel to the z-axis. The edges are of length 20 cm, 30 cm and 50 cm. The corner nearest the origin is at (10, 30, 20). What is the vector equation of the diagonal through this corner to the furthest corner?

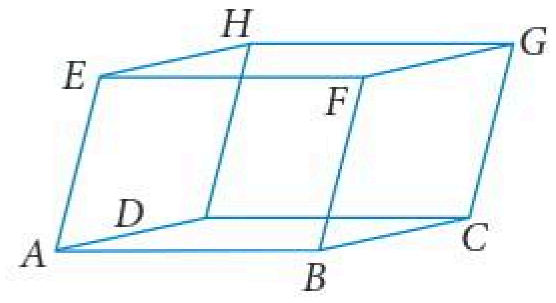
12 A line through $(-2, 4, -3)$ makes angles of 60°, –45° and 30° with the x, y and z directions. What is the vector equation of the line?

Exam practice

13 TECH-FREE (3 marks) A parallelepiped is shown on the right.

Let $\overrightarrow{AB} = \underset{\sim}{u}$, $\overrightarrow{AD} = \underset{\sim}{v}$ and $\overrightarrow{AE} = \underset{\sim}{w}$.

Express the vector equation of $\overrightarrow{BH}$ in terms of $\underset{\sim}{u}$, $\underset{\sim}{v}$, $\underset{\sim}{w}$ and $\underset{\sim}{a}$, the position vector of A.



14 TECH-FREE (3 marks) Find the Cartesian equation of the projection of $\underset{\sim}{r}(t) = (2 - 3t)\,\underset{\sim}{i} + (-3 + 2t)\,\underset{\sim}{j} + (1 + 4t)\,\underset{\sim}{k}$ perpendicular to the z-axis.

15 An equation of the line through $(2, 3, -5)$ and $(5, -3, 4)$ is

A $\underset{\sim}{r}(t) = 2\underset{\sim}{i} + 3\underset{\sim}{j} - 5\underset{\sim}{k} + t(5\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k})$

B $\underset{\sim}{r}(t) = 5\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k} + t(3\underset{\sim}{i} - 6\underset{\sim}{j} + 9\underset{\sim}{k})$

C $\dfrac{x-2}{3} = \dfrac{y-3}{6} = \dfrac{z+5}{9}$

D $\dfrac{x-5}{3} = \dfrac{y+3}{-6} = z - 4$

E $x = 5t + 2, y = -3t + 3, z = 4t - 5$

16 The equation of a line is given by $\underset{\sim}{r}(t) = 4\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k} + t(2\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k})$.

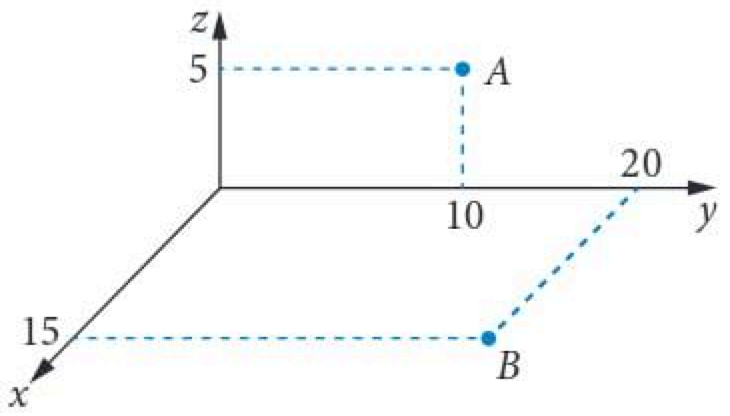
Which of the following points lies on the line?

A $(10, 11, 1)$ **B** $(0, 2, 4)$ **C** $(12, 15, -6)$ **D** $(-4, -17, 6)$ **E** $(-8, -25, 4)$

17 Which of the following points is on the line segment $\overrightarrow{AB}$?

A $(6, 18, 3)$ **B** $(12, 6, 9)$ **C** $(18, 34, -1)$

D $(-3, 6, 6)$ **E** $(9, 22, 3)$



Video playlist
Normals to a plane

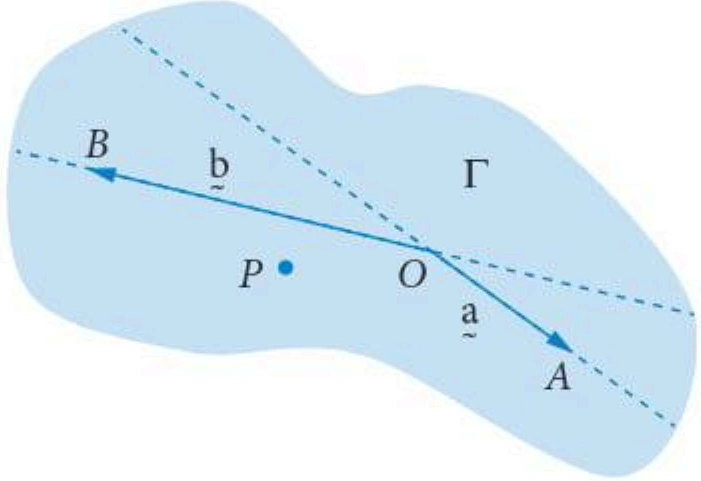
6.3 Normals to a plane

A **plane** is a flat surface extends in all directions. A **normal** is a vector perpendicular to the plane. A plane may be fixed by any three points not in a straight line on the plane. Any normals of the same plane are parallel, so they are multiples of each other. A plane has two dimensions, so the position vector of any point in the plane will be a linear combination of any two independent vectors in the plane.

Independent vectors

- No vector in a set of **independent vectors** can be written as a linear combination of the others.
- In two dimensions, the largest independent set has only two vectors. Any other vector can be written as a combination of two independent vectors. The independent vectors are in different directions.
- For independent vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ in a plane, the position vector of any point P in the plane will have a position vector given by $\underset{\sim}{p} = n\underset{\sim}{a} + m\underset{\sim}{b}$ for some $n, m \in R$.

Consider the plane Γ containing independent position vectors $\underset{\sim}{a} = 2\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} - 5\underset{\sim}{j} + 2\underset{\sim}{k}$. For $\underset{\sim}{a}$ and $\underset{\sim}{b}$ to be independent, they must be in different directions, as shown in the diagram on the right. Any point P will have a position vector given by $\underset{\sim}{p} = n\underset{\sim}{a} + m\underset{\sim}{b}$ for some $n, m \in R$.



WB
p. 124

WORKED EXAMPLE 8 Points on a plane

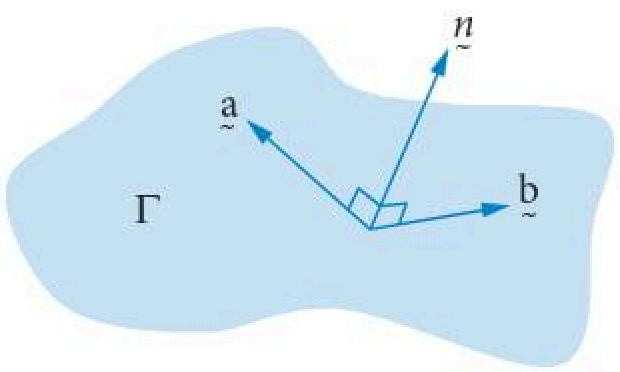
The plane Γ_1 contains position vectors $\underset{\sim}{a} = 2\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} - 5\underset{\sim}{j} + 2\underset{\sim}{k}$.
Determine whether each of the following points are on the plane.

a (13, –7, 4) **b** (14, 6, 2)

Steps	Working
a 1 Write as a combination.	$13\underset{\sim}{i} - 7\underset{\sim}{j} + 4\underset{\sim}{k} = n(2\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}) + m(3\underset{\sim}{i} - 5\underset{\sim}{j} + 2\underset{\sim}{k})$
2 Simplify.	$= 2n\underset{\sim}{i} + 4n\underset{\sim}{j} - n\underset{\sim}{k} + 3m\underset{\sim}{i} - 5m\underset{\sim}{j} + 2m\underset{\sim}{k}$ $= (2n + 3m)\underset{\sim}{i} + (4n - 5m)\underset{\sim}{j} + (2m - n)\underset{\sim}{k}$
3 Separate components and solve.	$2n + 3m = 13$ [1], $4n - 5m = -7$ [2], $2m - n = 4$ [3] $[1] + 2 \times [3] \Rightarrow 2n + 3m + 2(2m - n) = 13 + 2 \times 4$ $\Rightarrow 7m = 21$ $\Rightarrow m = 3$ Substituting in [3] gives $n = 2$.
4 Check values.	$2(2\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}) + 3(3\underset{\sim}{i} - 5\underset{\sim}{j} + 2\underset{\sim}{k})$ $= 4\underset{\sim}{i} + 8\underset{\sim}{j} - 2\underset{\sim}{k} + 9\underset{\sim}{i} - 15\underset{\sim}{j} + 6\underset{\sim}{k}$ $= 13\underset{\sim}{i} - 7\underset{\sim}{j} + 4\underset{\sim}{k}$ OK
5 Write the conclusion.	(13, –7, 4) is on Γ_1.
b 1 Write as a combination.	$14\underset{\sim}{i} + 6\underset{\sim}{j} + 2\underset{\sim}{k} = n(2\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}) + m(3\underset{\sim}{i} - 5\underset{\sim}{j} + 2\underset{\sim}{k})$
2 Simplify.	$14\underset{\sim}{i} + 6\underset{\sim}{j} + 2\underset{\sim}{k} = 2n\underset{\sim}{i} + 4n\underset{\sim}{j} - n\underset{\sim}{k} + 3m\underset{\sim}{i} - 5m\underset{\sim}{j} + 2m\underset{\sim}{k}$ $= (2n + 3m)\underset{\sim}{i} + (4n - 5m)\underset{\sim}{j} + (2m - n)\underset{\sim}{k}$

3 Separate components and solve.	$2+3m=14$ [1], $4-5m=6$ [2], $2m-n=2$ [3] $2\times[1]-[2]\Rightarrow 2(2n+3m)-(4n-5m)=2\times 14-6$ $\Rightarrow 11m=22$ $\Rightarrow m=2$ Substituting in [2] gives $n=4$.
4 Check values.	$4(2\underset{\sim}{i}+4\underset{\sim}{j}-\underset{\sim}{k})+2(3\underset{\sim}{i}-5\underset{\sim}{j}+2\underset{\sim}{k})$ $=8\underset{\sim}{i}+16\underset{\sim}{j}-4\underset{\sim}{k}+6\underset{\sim}{i}-10\underset{\sim}{j}+4\underset{\sim}{k}$ $=14\underset{\sim}{i}+6\underset{\sim}{j}$ Not OK.
5 Write the conclusion.	(14, 6, 2) is not on Γ_1.

The vector product of two vectors is itself a vector perpendicular to both the original vectors. Suppose a plane contains position vectors $3\underset{\sim}{i}-2\underset{\sim}{j}+2\underset{\sim}{k}$ and $-4\underset{\sim}{i}+5\underset{\sim}{j}-\underset{\sim}{k}$. The vector product must be perpendicular to both, so is a normal to the plane.



WORKED EXAMPLE 9 Normal to a plane

WB

p. 125

The plane Γ_1 contains position vectors $\underset{\sim}{a}=4\underset{\sim}{i}-2\underset{\sim}{j}+2\underset{\sim}{k}$ and $\underset{\sim}{b}=-4\underset{\sim}{i}+5\underset{\sim}{j}-\underset{\sim}{k}$.
Find a normal to Γ_1 and a normal unit vector.

Steps	**Working**
1 Find the vector product.	$\underset{\sim}{a}\times\underset{\sim}{b}$ $=(4\underset{\sim}{i}-2\underset{\sim}{j}+2\underset{\sim}{k})\times(-4\underset{\sim}{i}+5\underset{\sim}{j}-\underset{\sim}{k})$ $=[(-2)\times(-1)-2\times 5]\,\underset{\sim}{i}-[4\times(-1)-2\times(-4)]\,\underset{\sim}{j}+$ $[4\times 5-(-2)\times(-4)]\,\underset{\sim}{k}$ $=-8\underset{\sim}{i}-4\underset{\sim}{j}+12\underset{\sim}{k}$
2 Find the unit vector.	$\hat{\underset{\sim}{n}}=\dfrac{1}{\lvert\underset{\sim}{n}\rvert}\underset{\sim}{n}$ $=\dfrac{1}{\sqrt{(-8)^2+(-4)^2+12^2}}(-8\underset{\sim}{i}-4\underset{\sim}{j}+12\underset{\sim}{k})$ $=\dfrac{1}{\sqrt{64+16+144}}(-8\underset{\sim}{i}-4\underset{\sim}{j}+12\underset{\sim}{k})$ $\dfrac{1}{4\sqrt{14}}(-8\underset{\sim}{i}-4\underset{\sim}{j}+12\underset{\sim}{k})=\dfrac{1}{\sqrt{14}}(-2\underset{\sim}{i}-\underset{\sim}{j}+3\underset{\sim}{k})$
3 Write the answer.	$-8\underset{\sim}{i}-4\underset{\sim}{j}+12\underset{\sim}{k}$ is a normal and $\dfrac{1}{\sqrt{14}}(-2\underset{\sim}{i}-\underset{\sim}{j}+3\underset{\sim}{k})$ is a unit vector normal to the plane Γ_1.

There are only two unit normals to any plane; they are the negative of each other. In the above example, the other unit normal is $\dfrac{1}{\sqrt{14}}(2\underset{\sim}{i}+\underset{\sim}{j}-3\underset{\sim}{k})$.

Any three points A, B and C not in a straight line are sufficient to specify a plane. We can find a normal to a plane by finding the vector product of two displacement vectors, say $\overrightarrow{AB} \times \overrightarrow{AC}$.

WB

p. 126

WORKED EXAMPLE 10 Normal to a plane using three points

The points A (5, 2, –1), B (6, –6, 5) and C (1, –4, 3) are on the plane Γ_1. Find a normal to the plane.

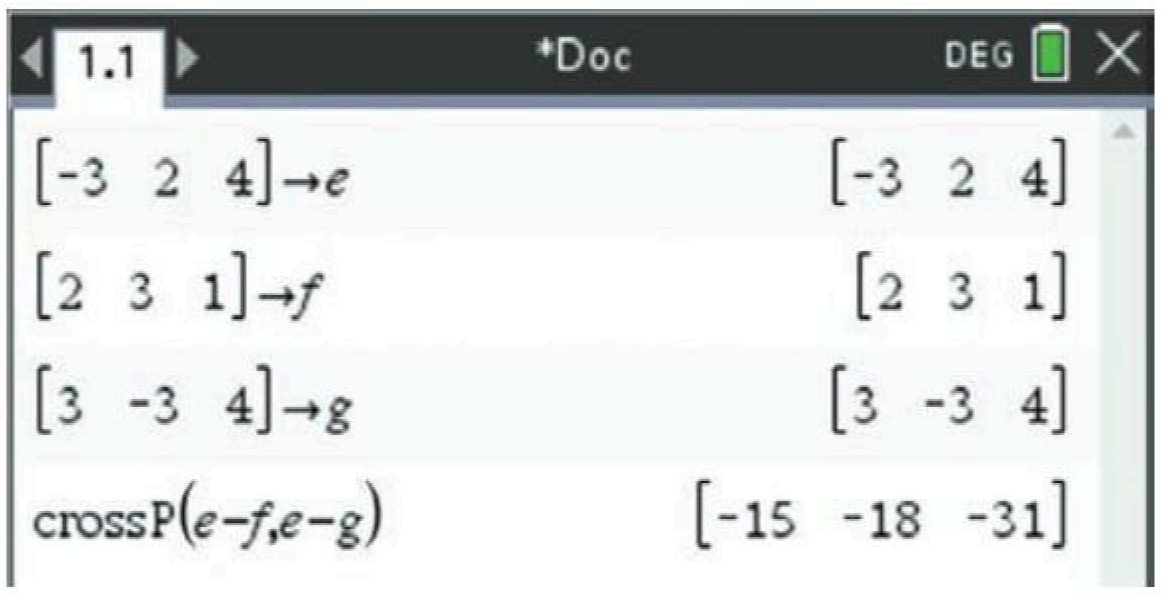
Steps	**Working**
1 Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$.	$\overrightarrow{AB} = 6\underset{\sim}{i} - 6\underset{\sim}{j} + 5\underset{\sim}{k} - (5\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}) = \underset{\sim}{i} - 8\underset{\sim}{j} + 6\underset{\sim}{k}$ $\overrightarrow{AC} = \underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k} - (5\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}) = -4\underset{\sim}{i} - 6\underset{\sim}{j} + 4\underset{\sim}{k}$
2 Find the vector product.	$\overrightarrow{AB} \times \overrightarrow{AC}$ $= (-\underset{\sim}{i} - 8\underset{\sim}{j} + 6\underset{\sim}{k}) \times (-4\underset{\sim}{i} - 6\underset{\sim}{j} + 4\underset{\sim}{k})$ $= [(-8) \times 4 - 6 \times (-6)]\underset{\sim}{i} - [1 \times 4 - 6 \times (-4)]\underset{\sim}{j} +$ $[1 \times (-6) - (-8) \times (-4)]\underset{\sim}{k}$ $= 4\underset{\sim}{i} - 28\underset{\sim}{j} - 38\underset{\sim}{k}$
3 Factorise if possible.	$= 2(2\underset{\sim}{i} - 14\underset{\sim}{j} - 19\underset{\sim}{k})$
4 Write the answer.	$\underset{\sim}{n} = 2\underset{\sim}{i} - 14\underset{\sim}{j} - 19\underset{\sim}{k}$ is a normal to the plane Γ_1.

In the above example, we can choose any two displacement vectors. We will always get a multiple of the same vector as a normal. Notice that $\underset{\sim}{n}$ is chosen to be the simplest possible. Use CAS whenever possible to calculate vector products to avoid arithmetic errors.

USING CAS 1 Normal to a plane with three points

Find a normal to the plane containing the points E (–3, 2, –4), F (2, 3, 1) and G (3, –3, 4).

TI-Nspire



```
[-3  2  4]→e            [-3  2  4]
[2  3  1]→f              [2  3  1]
[3  -3  4]→g            [3  -3  4]
crossP(e-f,e-g)     [-15  -18  -31]
```

1 Press the **template** key to create a 1 × 3 matrix.

2 Enter the given values and **store** the matrix into ***e***.

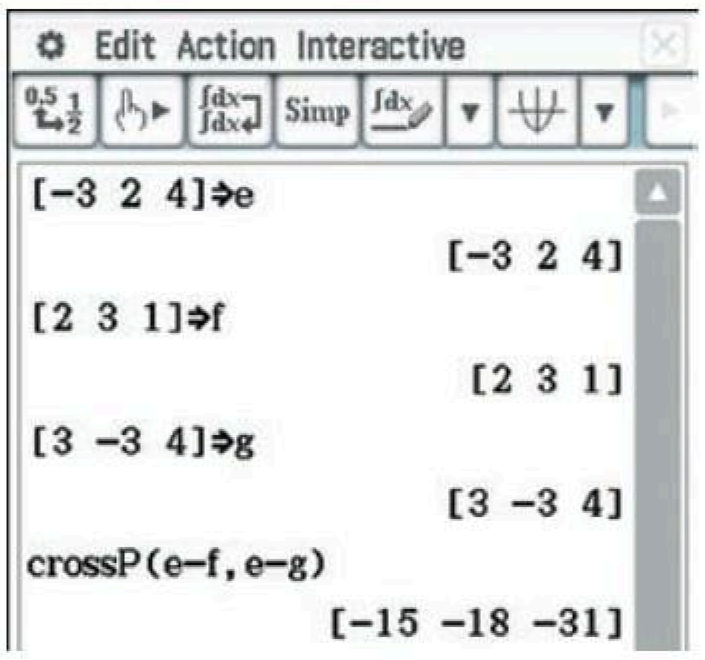
3 Repeat for matrices ***f*** and ***g***.

4 Press **menu > Matrix & Vector > Vector > Cross Product**.

5 Enter the matrices ***e–f*** and ***e–g***, separated by a comma.

6 Press **enter**.

ClassPad



```
[-3 2 4]⇒e
                [-3 2 4]
[2 3 1]⇒f
                 [2 3 1]
[3 -3 4]⇒g
                [3 -3 4]
crossP(e-f,e-g)
           [-15 -18 -31]
```

1 Open the **Keyboard > Math2** palette to create a 1 × 3 matrix, or enter using square brackets with commas between the numbers.

2 Enter the given values and **store** the matrix as ***e***.

3 Repeat for matrices ***f*** and ***g***.

4 Tap **Interactive > Vector > crossP**.

5 Enter the matrices ***e-f*** and ***e-g*** in the fields provided.

6 Tap **OK**.

$-15 - 18\underset{\sim}{j} - 31\underset{\sim}{k}$ is a normal to the plane.

Exam hack

Make sure you show the vector product you have used in your working for questions worth more than 1 mark.

EXERCISE 6.3 Normals to a plane

ANSWERS p. 587

Recap

1 The equation of the line through (2, 3, –5) and (5, –3, 4) is

A $z = 2x + y - 7$

B $\underset{\sim}{r}(t) = 2\underset{\sim}{i} - 3\underset{\sim}{j} - 5\underset{\sim}{k} + t(3\underset{\sim}{i} - 6\underset{\sim}{j} + 9\underset{\sim}{k})$

C $\underset{\sim}{r}(t) = 5\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k} + t(3\underset{\sim}{i} - 6\underset{\sim}{j} - 9\underset{\sim}{k})$

D $\underset{\sim}{r}(t) = 5\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k} + t(3\underset{\sim}{i} - 6\underset{\sim}{j} + 9\underset{\sim}{k})$

E $2x + y - 7 = 0$

2 The equation of a line is given by $\underset{\sim}{r}(t) = -2\underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k} + t(3\underset{\sim}{i} + 5\underset{\sim}{j} + 2\underset{\sim}{k})$. Which of the following points is not on the line?

A (–8, –6, –2)
B (4, 14, 4)
C (7, 19, 8)
D (1, 9, 4)
E (–11, –11, –4)

Mastery

3 WORKED EXAMPLE 8 Determine whether each point lies on the plane containing position vectors $\underset{\sim}{a} = \underset{\sim}{i} + 3\underset{\sim}{j} + 5\underset{\sim}{k}$ and $\underset{\sim}{b} = -4\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$.

a (–10, 12, 1)
b (–3, 5, 2)
c (–7, –7, 1)
d (10, 2, 16)
e (5, 13, 9)

4 WORKED EXAMPLE 9 Determine the normal to the plane containing position vectors.

a $5\underset{\sim}{i} + 4\underset{\sim}{j} - 7\underset{\sim}{k}, -7\underset{\sim}{j} - \underset{\sim}{k}$
b $4\underset{\sim}{i} + 3\underset{\sim}{j} - 3\underset{\sim}{k}, 7\underset{\sim}{i} - \underset{\sim}{j} - 3\underset{\sim}{k}$
c $2\underset{\sim}{i} - \underset{\sim}{j} - 6\underset{\sim}{k}, 5\underset{\sim}{i} + 6\underset{\sim}{j} - 5\underset{\sim}{k}$
d $6\underset{\sim}{i} + 6\underset{\sim}{j} + 5\underset{\sim}{k}, -5\underset{\sim}{i} - 2\underset{\sim}{j} - 7\underset{\sim}{k}$

5 Find a unit vector normal to the plane containing each pair of position vectors.

a $3\underset{\sim}{i} - \underset{\sim}{j}, 7\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$
b $2\underset{\sim}{i} + 6\underset{\sim}{j} + 7\underset{\sim}{k}, 4\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k}$
c $6\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}, 3\underset{\sim}{i} - \underset{\sim}{j} - 4\underset{\sim}{k}$
d $4\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}, 2\underset{\sim}{i} + 5\underset{\sim}{j} + 2\underset{\sim}{k}$

6 WORKED EXAMPLE 10 Determine the normal to the plane containing the points.

a (–4, –2, 2), (–2, –5, –3), (2, 3, 5)
b (–1, 3, –5), (2, 2, 2), (–4, –1, –5)
c (1, –2, 4), (2, 1, –3), (0, –4, 3)
d (–4, 5, –5), (0, –2, 2), (3, 2, 3)
e (–5, –3, 4), (5, 4, 2), (–1, 0, –2)

7 Determine the unit normal to the plane containing the points.

a (–3, 5, 3), (6, 3, –1), (4, –1, –5)
b (2, –2, –4), (2, –3, 3), (–4, –4, 3)
c (–4, –3, –2), (3, 4, –3), (5, 5, –4)
d (–1, 4, 2), (–5, –5, 4), (–1, 5, 1)
e (–4, –4, –4), (–6, 4, 5), (–6, 3, 3)

8 Using CAS 1 Determine the unit normal to the plane containing the points.

a (0, 3, –6), (5, 1, 4), (5, 6, 5)
b (–2, 6, –1), (5, 1, 1), (–2, 2, 5)
c (0, –6, 5), (–1, –5, –3), (6, 2, –6)
d (2, 2, –1), (–2, 3, –3), (5, –5, –1)
e (–3, –6, 3), (2, 3, 0), (–2, 6, –1)

Exam practice

9 TECH-FREE (3 marks) Find a normal to the plane containing the points (3, 0, 4), (–4, 5, 3), (–5, 6, 4).

10 A normal to the plane containing the points (−5, −2, −3), (−1, −6, 3), (6, −5, 4) is given by

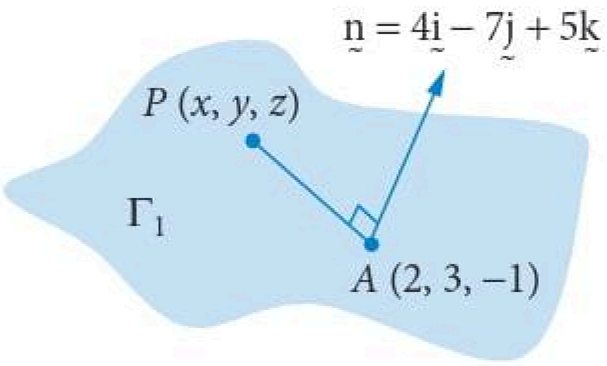
A $5\underset{\sim}{i} - 54\underset{\sim}{j} - 31\underset{\sim}{k}$
B $23\underset{\sim}{i} + 19\underset{\sim}{j} - 28\underset{\sim}{k}$
C $5\underset{\sim}{i} - 19\underset{\sim}{j} - 16\underset{\sim}{k}$
D $5\underset{\sim}{i} + 11\underset{\sim}{j} + 4\underset{\sim}{k}$
E $5\underset{\sim}{i} + 57\underset{\sim}{j} + 28\underset{\sim}{k}$

Video playlist Equations of planes

Worksheet Equations of planes

6.4 Equations of planes

Suppose the point $A\ (2, 3, -1)$ is in the plane Γ_1 and $\underset{\sim}{n} = 4\underset{\sim}{i} - 7\underset{\sim}{j} + 5\underset{\sim}{k}$ is a normal to the plane.



Then for any point $P\ (x, y, z)$ in the plane, $\overrightarrow{PA}$ is perpendicular to $\underset{\sim}{n}$.
The scalar product of perpendicular vectors is zero, so $\overrightarrow{PA} \cdot \underset{\sim}{n} = 0$.

Hence $[(x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}) - (2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k})] \cdot (4\underset{\sim}{i} - 7\underset{\sim}{j} + 5\underset{\sim}{k}) = 0.$

$$\begin{aligned}(x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}) \cdot (4\underset{\sim}{i} - 7\underset{\sim}{j} + 5\underset{\sim}{k}) - (2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}) \cdot (4\underset{\sim}{i} - 7\underset{\sim}{j} + 5\underset{\sim}{k}) &= 0\\ 4x - 7y + 5z - (2 \times 4 + 3 \times (-7) + (-1) \times 5) &= 0\\ 4x - 7y + 5z + 18 &= 0\\ 4x - 7y + 5z &= -18\end{aligned}$$

Since $P\ (x, y, z)$ can be any point in the plane, the equation $4x - 7y + 5z = -18$ must be satisfied by every point in the plane. We say that $4x - 7y + 5z = -18$ is the equation of Γ_1.

Equation of a plane

- The **equation of a plane** Γ through $A(x_0, y_0, z_0)$ with normal $\underset{\sim}{n} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$ is given by $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$, where $P\ (x, y, z)$ is any point in the plane.
- The equation can also be written as $ax + by + cz = d$, where $d = ax_0 + by_0 + cz_0$.
- The **vector equation of a plane** Γ is given by $\underset{\sim}{n} \cdot (\underset{\sim}{p} - \underset{\sim}{a}) = 0$ or $\underset{\sim}{n} \cdot \underset{\sim}{p} = k$, where $\underset{\sim}{n}$ is a normal, $\underset{\sim}{a}$ is the position vector of a particular point in the plane, $\underset{\sim}{P}$ is the position vector of a general point in the plane and $k = \underset{\sim}{n} \cdot \underset{\sim}{a}$ is a real number.

WB

p. 127

WORKED EXAMPLE 11 Equation of a plane from normal and point

a Find the equation of a plane with normal $2\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k}$ containing the point (3, −2, 7).

b Write the vector equation of a plane with normal $-3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ containing the point (2, 1, 5).

Steps	**Working**
a **1** Write the general formula.	$ax + by + cz = ax_0 + by_0 + cz_0$
2 Substitute values and simplify.	$2x + 4y + 3k = 2 \times 3 + 4 \times (-2) + 3 \times 7 = 19$
3 Write the answer.	The equation of the plane is $2x + 4y + 3k = 19$.
b **1** Write the general formula.	$\underset{\sim}{n} \cdot \underset{\sim}{p} = \underset{\sim}{n} \cdot \underset{\sim}{a}$
2 Substitute values and simplify.	$(-3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}) \cdot \underset{\sim}{p} = (-3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}) \cdot (2\underset{\sim}{i} + \underset{\sim}{j} + 5\underset{\sim}{k})$ $= -6 + 2 - 5$ $= -9$
3 Write the answer.	The vector equation is $\underset{\sim}{p} \cdot (-3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}) = -9$.

Since scalar multiplication is commutative, it doesn't matter in which order we write it.

A plane may be fixed by a point and a normal to the plane; any 3 points not in a straight line; 2 intersecting lines; or 2 parallel lines. In each case, find a normal and a point in the plane and then proceed as in the above example.

WORKED EXAMPLE 12 Equation of a plane from three points

Find the equation of a plane containing the points (1, 2, –7), (3, –5, 6) and (–4, 2, 8).

Steps	**Working**
1 Find two displacement vectors in the plane.	$\underset{\sim}{v}_1 = (3\underset{\sim}{i} - 5\underset{\sim}{j} + 6\underset{\sim}{k}) - (\underset{\sim}{i} + 2\underset{\sim}{j} - 7\underset{\sim}{k}) = 2\underset{\sim}{i} - 7\underset{\sim}{j} + 13\underset{\sim}{k}$ $\underset{\sim}{v}_2 = (-4\underset{\sim}{i} + 2\underset{\sim}{j} + 8\underset{\sim}{k}) - (\underset{\sim}{i} + 2\underset{\sim}{j} - 7\underset{\sim}{k}) = -5\underset{\sim}{i} + 15\underset{\sim}{k}$
2 Find the vector product to obtain a normal.	$(2\underset{\sim}{i} - 7\underset{\sim}{j} + 13\underset{\sim}{k}) \times (-5\underset{\sim}{i} + 15\underset{\sim}{k})$ $= (-7 \times 15 - 13 \times 0)\,\underset{\sim}{i} - (2 \times 15 - 13 \times (-5))\,\underset{\sim}{j} + (2 \times 0 - (-7) \times (-5))\,\underset{\sim}{k}$ $= -105\underset{\sim}{i} - 95\underset{\sim}{j} - 35\underset{\sim}{k}$
3 Simplify it.	$\underset{\sim}{n} = (21, 19, 7)$ is a normal to the plane.
4 Write the formula.	$ax + by + cz = ax_0 + by_0 + cz_0$
5 Substitute and simplify.	$21x + 19y + 7z = 21 \times 1 + 19 \times 2 + 7 \times (-7)$ $= 21 + 38 - 49$ $= 10$
6 Write the answer.	The equation of the plane containing the points (1, 2, –7), (3, –5, 6) and (–4, 2, 8) is $21x + 19y + 7z = 10$.

WB p. 128

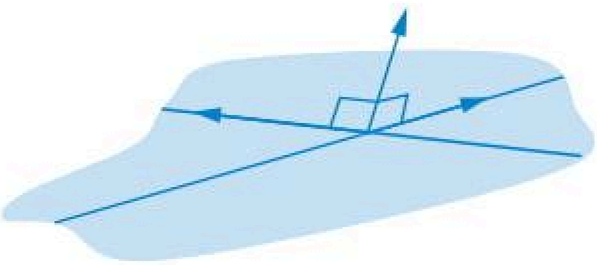
Exam hack

Verify that the other 2 points satisfy the equation to check your answer:

$21 \times 3 + 19 \times (-5) + 7 \times 6 = 63 - 95 + 42 = 10$ OK

and $21 \times (-4) + 19 \times 2 + 7 \times 8 = -84 + 38 + 56 = 10$ OK

When given 2 intersecting lines, find the intersection to get a point and the vector product of the directions to get a normal.



WORKED EXAMPLE 13 Equation of a plane from intersecting lines

WB p. 129

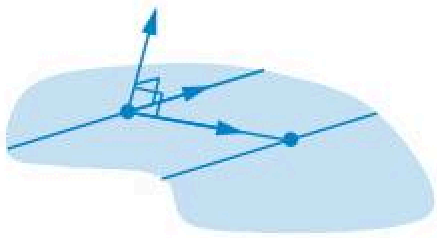
Find the equation of the plane formed by the lines $\underset{\sim}{r}_1(t) = 5\underset{\sim}{i} + 19\underset{\sim}{j} + 4\underset{\sim}{k} + t(4\underset{\sim}{i} + 7\underset{\sim}{j} + \underset{\sim}{k})$ and $\underset{\sim}{r}_2(t) = 3\underset{\sim}{i} - 23\underset{\sim}{j} + 8\underset{\sim}{k} + t(2\underset{\sim}{i} + 6\underset{\sim}{j} + 2\underset{\sim}{k})$.

Steps	**Working**
1 State the nature of the lines.	The lines have different directions, so to form a plane they must be intersecting.
2 Write in condensed form, with different parameter symbols.	$\underset{\sim}{r}_1(t) = (5 + 4t)\,\underset{\sim}{i} + (19 + 7t)\,\underset{\sim}{j} + (4 + t)\,\underset{\sim}{k}$ $\underset{\sim}{r}_2(s) = (3 + 2s)\,\underset{\sim}{i} + (-23 + 6s)\,\underset{\sim}{j} + (8 + 2s)\,\underset{\sim}{k}$
3 Choose the simplest pairs of expressions and solve.	$4 + t = 8 + 2s \Rightarrow t = 4 + 2s \quad \ldots[1]$ $5 + 4t = 3 + 2s \Rightarrow 4t = 2s - 2 \ldots[2]$ $4(4 + 2s) = 2s - 2$ $16 + 8s = 2s - 2$ $6s = -18$ $s = -3$ $t = 4 + 2s = -2$

4 Write the vectors.	$\underset{\sim}{r}_1(-2) = 5\underset{\sim}{i} + 19\underset{\sim}{j} + 19\underset{\sim}{k} + (-2)(4\underset{\sim}{i} + 7\underset{\sim}{j} + \underset{\sim}{k})$ $= -3\underset{\sim}{i} + 5\underset{\sim}{j} + 2\underset{\sim}{k}$ $\underset{\sim}{r}_2(-3) = 3\underset{\sim}{i} - 23\underset{\sim}{j} + 8\underset{\sim}{k} + (-3)(2\underset{\sim}{i} + 6\underset{\sim}{j} + 2\underset{\sim}{k})$ $= -3\underset{\sim}{i} - 41\underset{\sim}{j} + 2\underset{\sim}{k}$
5 State the intersection.	The lines intersect at $(-3, -41, 2)$.
6 Find the vector product of the directions of the lines.	$(4\underset{\sim}{i} + 7\underset{\sim}{j} + \underset{\sim}{k}) \times (2\underset{\sim}{i} + 6\underset{\sim}{j} + 2\underset{\sim}{k})$ $= (7 \times 2 - 1 \times 6)\underset{\sim}{i} - (4 \times 2 - 1 \times 2)\underset{\sim}{j} + (4 \times 6 - 7 \times 2)\underset{\sim}{k}$ $= 8\underset{\sim}{i} - 6\underset{\sim}{j} + 10\underset{\sim}{k}$
7 Write the simplest normal.	$4\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}$ is a normal to the plane.
8 Write the formula.	$ax + by + cz = ax_0 + by_0 + cz_0$
9 Substitute and simplify.	$4x - 3y + 5z = 4 \times (-3) - 3 \times (-41) + 5 \times 2 = 121$
10 Write the answer.	The equation of the plane is $4x - 3y + 5z = 121$.

Skew lines do not intersect and do not form a plane.

When given two parallel lines, choose an easy point on each line, find the equation of the line between them and find a normal using the vector product of one of the lines and the new line.



WB

p. 130

WORKED EXAMPLE 14 Equation of a plane from parallel lines

Find the equation of a plane formed by the lines $\underset{\sim}{r}_1(t) = (5 + 3t)\underset{\sim}{i} + (3 - 4t)\underset{\sim}{j} + (7 + 2t)\underset{\sim}{k}$ and $\underset{\sim}{r}_2(t) = (3 + 3t)\underset{\sim}{i} - (2 + 4t)\underset{\sim}{j} + (4 + 2t)\underset{\sim}{k}$.

Steps	Working
1 State the relationship between the lines.	The lines have the same normal vector, $3\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}$, so they are parallel.
2 Choose an easy point on each line.	$\underset{\sim}{r}_1(0) = (5 + 0)\underset{\sim}{i} + (3 - 0)\underset{\sim}{j} + (7 + 0)\underset{\sim}{k} = 5\underset{\sim}{i} + 3\underset{\sim}{j} + 7\underset{\sim}{k}$, so $(5, 3, 7)$ is on $\underset{\sim}{r}_1$. $\underset{\sim}{r}_2(-1) = (3 - 3)\underset{\sim}{i} - (2 - 4)\underset{\sim}{j} + (4 - 2)\underset{\sim}{k} = 2\underset{\sim}{j} + 2\underset{\sim}{k}$, so $(0, 2, 2)$ is on $\underset{\sim}{r}_2$.
3 Find the direction of the line.	$(5\underset{\sim}{i} + 3\underset{\sim}{j} + 7\underset{\sim}{k}) - (2\underset{\sim}{j} + 2\underset{\sim}{k})$ $= 5\underset{\sim}{i} + \underset{\sim}{j} + 5\underset{\sim}{k}$
4 Find the vector product with the direction vector of the parallel lines.	$(5\underset{\sim}{i} + \underset{\sim}{j} + 5\underset{\sim}{k}) \times (3\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k})$ $= (1 \times 2 - 5 \times (-4))\underset{\sim}{i} - (5 \times 2 - 5 \times 3)\underset{\sim}{j} + (5 \times (-4) - 1 \times 3)\underset{\sim}{k}$ $= 22\underset{\sim}{i} + 5\underset{\sim}{j} - 23\underset{\sim}{k}$
5 Write the formula.	$ax + by + cz = ax_0 + by_0 + cz_0$
6 Substitute one of the points and simplify. It doesn't matter which point you substitute, so choose the easier one of (5, 3, 7) and (0, 2, 2).	$22x + 5y - 23z = 22 \times 0 + 5 \times 2 - 23 \times 2 = -36$
7 Write the answer.	The plane is $22x + 5y - 23z = -36$.

To see whether a point is in a plane, just substitute into the equation.

WORKED EXAMPLE 15 Check if a point is on a plane

Determine whether each of the following points is on the plane $2x-3y+5z=7$.

a $(5,-4,-3)$ **b** $(-8,-1,5)$ **c** $(-6,7,8)$

Steps	Working
a **1** Substitute point in the equation and calculate the answer.	$2\times5-3\times(-4)+5\times(-3)$ $=10+12-15=7$ OK
2 Write the result.	$(5,-4,-3)$ is on the plane.
b **1** Substitute point in the equation and calculate the answer.	$2\times(-8)-3\times(-1)+5\times5$ $=-16+3+25$ $=12$ Not OK
2 Write the result.	$(-8,-1,5)$ is not on the plane.
c **1** Substitute point in the equation and calculate the answer.	$2\times(-6)-3\times7+5\times8$ $=-12-21+40$ $=7$ OK
2 Write the result.	$(-6,7,8)$ is on the plane.

WB p. 131

The angle between intersecting planes, called the **dihedral angle**, is the same as the angle between the normals. Planes must either be parallel or intersect in a straight line.

WORKED EXAMPLE 16 Intersection of planes

Find the line of intersection of the planes $x-5y+2z=4$ and $2x+3y-3z=3$.

Steps	Working
1 Choose the simplest variable to eliminate.	It is easiest to eliminate x.
2 Eliminate the variable.	$\begin{array}{rr}2\times[1] & 2x-10y+4z=8\\ -1\times[2] & -2x-3y+3z=-3\\ \hline & -13y+7z=5\end{array}$
3 Express one variable in terms of the other. Isolate the variable with the smaller coefficient.	$7z=13y+5$ $z=\frac{13}{7}y+\frac{5}{7}$
4 Express the other variable in terms of the same variable.	$x-5y+2z=4$ $x=5y-2z+4$ $=5y-2\left(\frac{13}{7}y+\frac{5}{7}\right)+4$ $=\frac{9}{7}y-\frac{18}{7}$
5 Choose the expression for t in terms of the same variable. Choose the multiple of the independent variable that eliminates fractions in the coefficients of t.	Choose $y=7t$.
6 Write the other variables in terms of t.	$x=\frac{9}{7}\times7t-\frac{18}{7}=9t-\frac{18}{7},\ z=\frac{13}{7}\times7t+\frac{5}{7}=13t+\frac{5}{7}$

WB p. 132

7 Write in vector form.	$\underset{\sim}{r}(t) = \left(-\frac{18}{7}+9t\right)\underset{\sim}{i} + 7t\underset{\sim}{j} + \left(\frac{5}{7}+13t\right)\underset{\sim}{k}$
8 Write the answer.	The planes $x-5y+2z=4$ and $2x+3y-3z=3$ intersect in the line $\underset{\sim}{r}(t) = \left(-\frac{18}{7}+9t\right)\underset{\sim}{i} + 7t\underset{\sim}{j} + \left(\frac{5}{7}+13t\right)\underset{\sim}{k}$.

In the above example, the direction vector of the line is $9\underset{\sim}{i}+7\underset{\sim}{j}+13\underset{\sim}{k}$. We could obtain a simpler value of $\underset{\sim}{a}$ by substituting a suitable value of t to get another point on the line. For example, $\underset{\sim}{r}\left(\frac{2}{7}\right) = 0\underset{\sim}{i}+2\underset{\sim}{j}+\frac{31}{7}\underset{\sim}{k}$, so $\left(0, 2, \frac{31}{7}\right)$ is on the line and we can write it as $\underset{\sim}{r}(t) = 9t\underset{\sim}{i} + (2+7t)\underset{\sim}{j} + \left(\frac{31}{7}+13t\right)\underset{\sim}{k}$, which only has one fraction.

Video
VCE question analysis: Vector equations

VCE QUESTION ANALYSIS

A sample question for a new topic (12 marks)

a Find the intersection of Γ_1: $4x-2y-5z=-3$ and Γ_2: $2x-y+3z=-7$. 4 marks

b Another line L is drawn in Γ_1 parallel to the intersection of the planes.
What is its direction vector? 1 mark

c Another line, $\overrightarrow{AB}$, is drawn directly between L and the intersection.
What is the relationship between $\overrightarrow{AB}$ and the normal to Γ_1? 1 mark

d What is the relationship between $\overrightarrow{AB}$ and L? 1 mark

e Find the equation of a line in Γ_1 parallel to the intersection and 9 units away from it. 5 marks

Reading the question

- Part **a** asks for the intersection of two planes, which will be a line.
- Part **b** asks for a second line parallel to the first.
- The second line must be in the first plane and a fixed distance from the first line.

Thinking about the question

- The planes have different normals so they are not parallel.
- One variable has to be eliminated so that each variable can be expressed in terms of a single variable. This gives the equation of the intersecting line.
- The new line L will have the same direction vector as the intersection line.
- L will be perpendicular to the normal of the plane.
- $\overrightarrow{AB}$ will be perpendicular to L and the intersection line.

Worked solution (✓ = 1 mark)

a
$$\begin{aligned} \Gamma_1: \quad & 4x-2y-5z=-3 \\ -2\times\Gamma_2: \quad & -4x+2y-6z=14\ ✓ \\ & -11z=11 \\ & \mathbf{z=-1}\ ✓ \end{aligned}$$

Substitute in Γ_2: $2x - y - 3 = -7$

$$y = 2x + 4$$

Let $\boldsymbol{x = t}$ ✓

The intersection is $\boldsymbol{t\underset{\sim}{i} + (4 + 2t)\underset{\sim}{j} - \underset{\sim}{k}}$. ✓

b The intersection of the planes is the line $t\underset{\sim}{i} + (4 + 2t)\underset{\sim}{j} - \underset{\sim}{k}$, when written as $4\underset{\sim}{j} - \underset{\sim}{k} + t(\underset{\sim}{i} + 2\underset{\sim}{j})$ has direction vector $\underset{\sim}{i} + 2\underset{\sim}{j}$, so the direction vector of the line is $\boldsymbol{\underset{\sim}{n}_L = \underset{\sim}{i} + 2\underset{\sim}{j}}$. ✓

c $\overrightarrow{AB}$ is drawn between two parallel lines in the plane, so it is also in the plane, so is perpendicular to the normal of the plane $\underset{\sim}{n}_{\Gamma_1}$. ✓

d $\overrightarrow{AB}$ is drawn directly between the line of intersection and a parallel line, so it is perpendicular to both lines, and therefore L. ✓

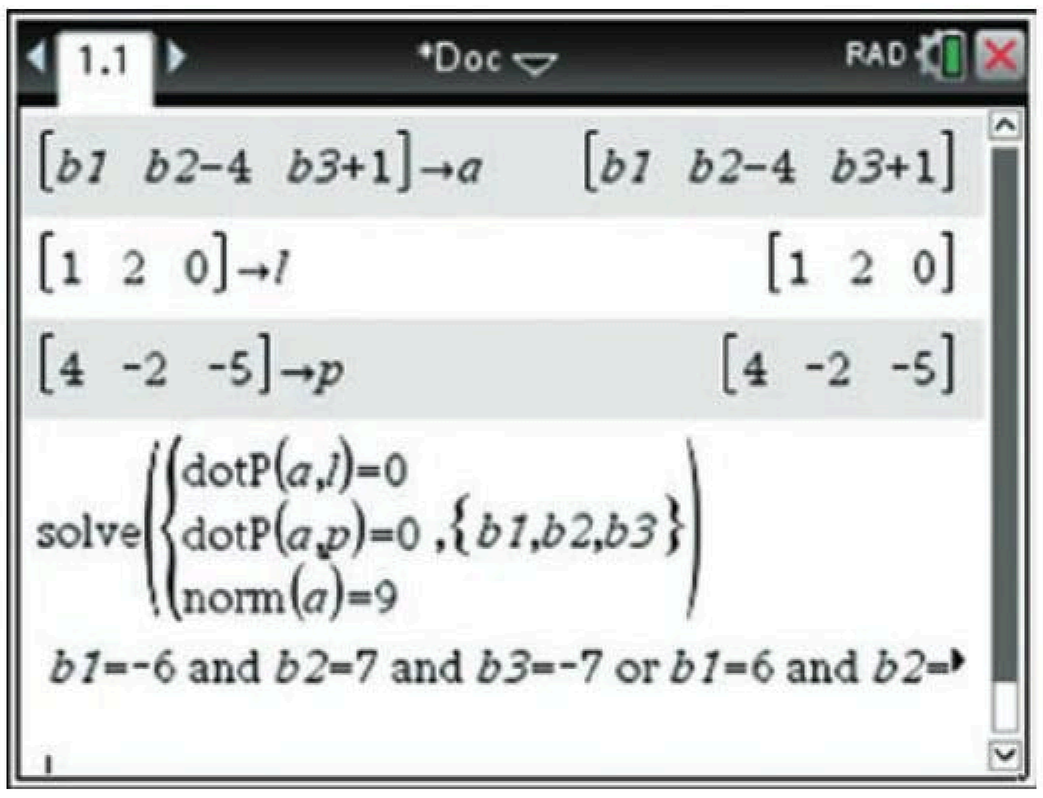
e Let A $(0, 4, -1)$ be a point on the intersection line (where $x = 0$, $y = 2x + 4$, $z = -1$). ✓

Let $B = (b_1, b_2, b_3)$

So $\overrightarrow{AB} = b_1\underset{\sim}{i} + (b_2 - 4)\underset{\sim}{j} + (b_3 + 1)\underset{\sim}{k}$, $\overrightarrow{AB} \cdot \underset{\sim}{n}_L = 0$, $\overrightarrow{AB} \cdot \underset{\sim}{n}_{\Gamma_1} = 0$ ✓ and $|AB| = 9$ ✓

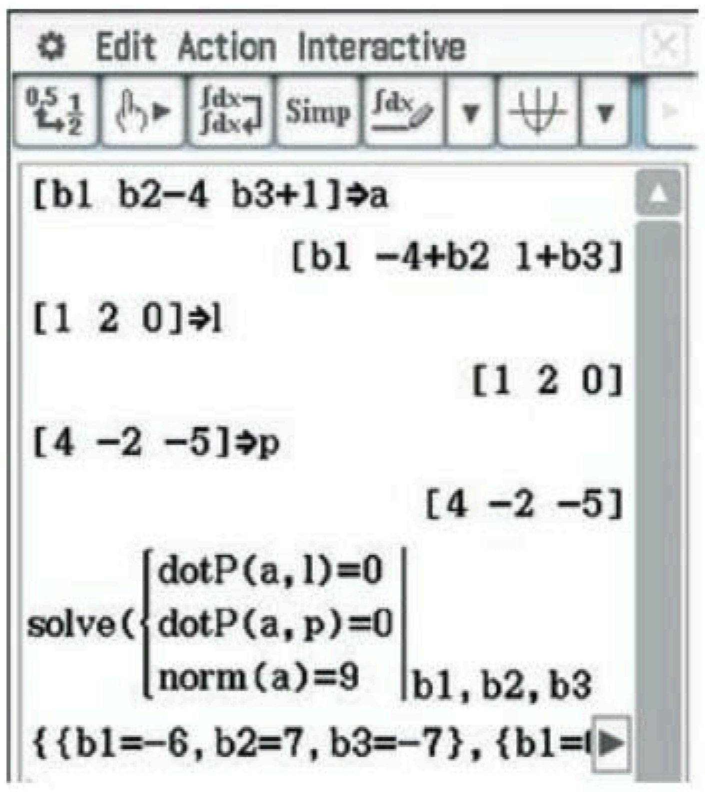
Use CAS to solve the equations.

TI-Nspire



```
[b1  b2-4  b3+1]→a        [b1  b2-4  b3+1]
[1  2  0]→l                      [1  2  0]
[4  -2  -5]→p                  [4  -2  -5]
      ⎧dotP(a,l)=0
solve ⎨dotP(a,p)=0 ,{b1,b2,b3}
      ⎩norm(a)=9
b1=-6 and b2=7 and b3=-7 or b1=6 and b2=▸
```

ClassPad



```
[b1 b2-4 b3+1]⇒a
              [b1 -4+b2 1+b3]
[1 2 0]⇒l
                      [1 2 0]
[4 -2 -5]⇒p
                    [4 -2 -5]
        ⎧dotP(a,l)=0
solve ( ⎨dotP(a,p)=0
        ⎩norm(a)=9    |b1, b2, b3
{{b1=-6, b2=7, b3=-7}, {b1=▸
```

$\boldsymbol{b_1 = -6, b_2 = 7, b_3 = -7}$ ✓

There is a second solution (6, 1, 5) on the other side of the intersection line.

The equation of a line parallel to the intersection in the plane Γ_1 and 9 units from it is

$\boldsymbol{\underset{\sim}{r}(t) = (-6 + t)\underset{\sim}{i} + (7 + 2t)\underset{\sim}{j} - 7\underset{\sim}{k}}$. ✓

EXERCISE 6.4 Equations of planes

ANSWERS p. 587

Recap

1 Which vector is a normal to the plane containing position vectors $\underset{\sim}{j} - 4k$ and $6\underset{\sim}{i} - 5\underset{\sim}{j} + 2\underset{\sim}{k}$?

A $-18\underset{\sim}{i} + 24\underset{\sim}{j} - 6\underset{\sim}{k}$

B $18\underset{\sim}{i} - 24\underset{\sim}{j} - 6\underset{\sim}{k}$

C $11\underset{\sim}{i} - 12\underset{\sim}{j} + 3\underset{\sim}{k}$

D $11\underset{\sim}{i} - 12\underset{\sim}{j} - 3\underset{\sim}{k}$

E $3\underset{\sim}{i} + 4\underset{\sim}{j} + \underset{\sim}{k}$

2 Which vector is a normal to plane that contains the points (−2, 1, −1), (1, 4, 3) and (4, 1, 4)?

A $20\underset{\sim}{i} + 9\underset{\sim}{j} - 24\underset{\sim}{k}$

B $5\underset{\sim}{i} + 3\underset{\sim}{j} - 6\underset{\sim}{k}$

C $15\underset{\sim}{i} + 29\underset{\sim}{j} - 18\underset{\sim}{k}$

D $7\underset{\sim}{i} + 9\underset{\sim}{j} - 12\underset{\sim}{k}$

E $23\underset{\sim}{i} + 9\underset{\sim}{j} - 24\underset{\sim}{k}$

Mastery

3 WORKED EXAMPLE 11 Find the equations of a plane with normal

a $5\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$ and containing the point $(-8, -6, 7)$

b $9\underset{\sim}{i} + \underset{\sim}{j} - 4\underset{\sim}{k}$ and containing the point $(1, -1, -1)$

c $4\underset{\sim}{i} + 4\underset{\sim}{j} - 7\underset{\sim}{k}$ and containing the point $(7, 4, 0)$

d $7\underset{\sim}{i} + 4\underset{\sim}{j} + 8\underset{\sim}{k}$ and containing the point $(-4, 3, -6)$

e $3\underset{\sim}{j} - 7\underset{\sim}{i}$ and containing the point $(8, -8, 0)$

4 Find the vector equations of a plane containing

a $(3, -7, 2)$ with normal $-3\underset{\sim}{i} + 3\underset{\sim}{j} - 5\underset{\sim}{k}$

b $(-8, -8, 4)$ with normal $-4\underset{\sim}{i} - 2\underset{\sim}{j} + 9\underset{\sim}{k}$

c $(1, 1, -8)$ with normal $-8\underset{\sim}{i} + 9\underset{\sim}{j} + 2\underset{\sim}{k}$

d $(3, 3, 1)$ with normal $-9\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$

e $(-1, 6, 8)$ with normal $-9\underset{\sim}{i} + 9\underset{\sim}{j} - \underset{\sim}{k}$

5 WORKED EXAMPLE 12 Find the equation of a plane containing the points

a $(-5, -6, -4), (-5, 0, -6), (4, -6, -1)$

b $(5, 2, 1), (3, -6, 0), (3, 5, 1)$

c $(-3, 0, -1), (6, 2, 3), (-5, 1, 6)$

d $(6, 4, 5), (-1, 3, -5), (0, 3, -4)$

e $(-5, 0, -1), (4, -6, 1), (5, 1, 1)$

6 WORKED EXAMPLES 13, 14 Find the equation of the plane formed by each pair of lines.

a $-2t\underset{\sim}{i} + (7-4t)\underset{\sim}{j} + (-5+5t)\underset{\sim}{k}, (-3-2t)\underset{\sim}{i} + (4-4t)\underset{\sim}{j} + (-6+5t)\underset{\sim}{k}$

b $-2\underset{\sim}{i} + (4-6t)\underset{\sim}{j} + (-11+5t)\underset{\sim}{k}, -2t\underset{\sim}{i} + (-2-6t)\underset{\sim}{j} + (-4+3t)\underset{\sim}{k}$

c $(-18-4t)\underset{\sim}{i} + (-24-7t)\underset{\sim}{j} + (-14-2t)\underset{\sim}{k}, (4-3t)\underset{\sim}{i} + (-2+3t)\underset{\sim}{j} + (8-7t)\underset{\sim}{k}$

d $(-6+7t)\underset{\sim}{i} + 2\underset{\sim}{j} + (2-2t)\underset{\sim}{k}, (4+7t)\underset{\sim}{i} + (7-2t)\underset{\sim}{k}$

e $(1-3t)\underset{\sim}{i} + (-6+8t)\underset{\sim}{j} + (-4+4t)\underset{\sim}{k}, (-8-3t)\underset{\sim}{i} + (3+8t)\underset{\sim}{j} + (-3+4t)\underset{\sim}{k}$

f $(5+t)\underset{\sim}{i} - 7t\underset{\sim}{j} + (-9-2t)\underset{\sim}{k}, (20+4t)\underset{\sim}{i} + (23+4t)\underset{\sim}{j} + (-31-6t)\underset{\sim}{k}$

g $(27-5t)\underset{\sim}{i} + (-3+t)\underset{\sim}{j} + (12-2t)\underset{\sim}{k}, (25-6t)\underset{\sim}{i} + (-2+t)\underset{\sim}{j} + (13-3t)\underset{\sim}{k}$

h $(-7+8t)\underset{\sim}{i} + 5\underset{\sim}{j} + (-5+t)\underset{\sim}{k}, (-2+8t)\underset{\sim}{i} + \underset{\sim}{j} + (2+t)\underset{\sim}{k}$

7 WORKED EXAMPLE 15 For each plane, determine if each point lies on the plane.

a $-4x - 2y + 3z = 6$: $(-1, 5, 4), (2, 3, -6), (2, 4, 2)$

b $3x - 3y + z = 10$: $(-3, -3, 10), (4, -6, 6), (1, 3, 0)$

c $3x + 6y - z = -6$: $(2, -2, 0), (6, -5, 2), (-1, -6, 39)$

d $6x - 4y + z = 3$: $(6, -4, -49), (5, -5, -7), (3, 0, -5)$

e $x - 2y + z = 11$: $(-1, -1, 8), (-3, 4, 0), (-1, -4, 4)$

f $x + 2y + 5z = 9$: $(1, -1, 2), (5, 2, 6), (3, -6, -1.2)$

8 WORKED EXAMPLE 16 Find the line of intersection of each pair of planes.

a $8y - 5x - 6z = 39, 4x + 4y - 3z = 136$

b $13x - 25y + 60z = -74, 17x - 15y - 10z = 89$

c $3x + 9y + z = -24, 15x + 13y + 5z = 8$

d $20x + 22y + 7z = -62, 4x - 7y - 10z = -58$

e $25x + 10y - 19z = -125, 19z - 20x - 8y = 119$

9 Is the line $\underset{\sim}{r}\,(\mathrm{t}) = (2+t)\,\underset{\sim}{i} + (1-t)\,\underset{\sim}{j} + (1+2t)\,\underset{\sim}{k}$ in the plane given by $x - 3y + 2z = 7$?

10 Find the equation of a plane parallel to $4x - y + 5z = 6$ and containing the point (3, 1, –3).

Exam practice

11 TECH-FREE (2 marks) A plane perpendicular to the line $(3+t)\,\underset{\sim}{i} + (1-t)\,\underset{\sim}{j} + (2+3t)\,\underset{\sim}{k}$ contains the point (3, –2, 5). Find the equation of the plane.

12 TECH-FREE (3 marks) Find the dihedral angle between the planes $3x + 4z = -2$ and $4x - 8y - z = 8$.

13 The plane through (–4, 5, –1) with normal $4\underset{\sim}{i} + 6\underset{\sim}{j} + 4\underset{\sim}{k}$ is given by

A $2x + 3y + 2z = 5$
B $13x + 6y - 22z = 0$
C $4x - 5y + z = 0$
D $2x + 3y + 2z = 0$
E $4x - 5y + z = -10$

14 The plane through the points A (–5, 3, –4), B (–6, 4, –5) and C (–4, –2, –3) is given by

A $2x + y + 3z = 5$
B $z - x = 1$
C $2x + y + 3z = -19$
D $2x - y + 3z = -15$
E $z - x = 4$

15 (10 marks)

a Show that the point A (3, –5, 7) is not on the line

$$\underset{\sim}{r}\,(\mathrm{t}) = (5-t)\,\underset{\sim}{i} + (1-3t)\,\underset{\sim}{j} + (3+4t)\,\underset{\sim}{k}$$ 1 mark

b Find the equation of the plane through A perpendicular to the given line. 2 marks

c Find the intersection B of the plane and the given line. 2 marks

d For any other point C on the given line, show that $\angle ABC = \frac{\pi}{2}$. 2 marks

e Find the general equation of a plane through a point B (b_1, b_2, b_3) and perpendicular to a given line, where B is not on the line. 3 marks

6 Chapter summary

Curves in 2D and 3D

- The **vector equation** of a curve in two dimensions is given by $\underset{\sim}{r}(t) = f(t)\underset{\sim}{i} + g(t)\underset{\sim}{j}$, where f and g are functions of the **parameter** t. The Cartesian form is found by eliminating t.
- The **vector equation of a curve in three dimensions** has the form $\underset{\sim}{r}(t) = f(t)\underset{\sim}{i} + g(t)\underset{\sim}{j} + h(t)\underset{\sim}{k}$, where f, g and h are functions of the parameter t.
- The projections on the x–y, x–z or y–z planes are obtained by omitting the other dimension.

Straight lines in 2D and 3D

- The **position vector** of a point A, $\overrightarrow{OP}$, is the vector from the origin O to the point A. It is usually written as the corresponding lower case vector $\underset{\sim}{a}$.
- The vector from A to B, $\overrightarrow{AB}$, is given by $\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$.
- The **vector equation** of a straight line through $A\ (a_1, a_2, a_3)$ and $B\ (b_1, b_2, b_3)$ is $\underset{\sim}{r}(t) = \underset{\sim}{a} + t\overrightarrow{AB} = \underset{\sim}{a} + t\underset{\sim}{d}$, where $t \in R$ and $\overrightarrow{AB} = \underset{\sim}{d} = \underset{\sim}{b} - \underset{\sim}{a}$ is the **displacement vector** in the direction of the line. $t \in R$ is the parameter for different points on the line; it is not time.
- $x = a_1 + td_1$, $y = a_2 + td_2$ and $z = a_3 + td_3$ are the **parametric equations** of a line in 3D.
- The Cartesian equation is

$$\frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3}$$

or $\dfrac{x - a_1}{d_1} = \dfrac{y - a_2}{d_2} = \dfrac{z - a_3}{d_3}$.

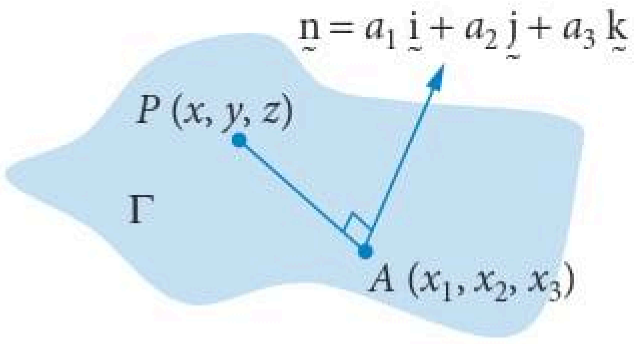
- In the vector forms, if $0 \le t \le 1$, the points are on the line segment AB, if $t < 0$, the points are on the line to the left of A, and if $t > 0$, the points are to the right of B.
- The equation can also be written as a *pair* of equations with two of the variables expressed in terms of the third, such as $x = a_1z + c_1$ and $y = a_2z + c_2$.

Normal to a plane

- A **plane** is a flat surface in 3D that extends in all directions.
- A **normal** to the plane is a vector perpendicular to the plane. Any normals of the same plane are parallel, so they are multiples of each other.
- No vector in a set of **independent vectors** can be written as a linear combination of the others.
- In two dimensions, the largest independent set has only two vectors. Any other vector can be written as a combination of two independent vectors. The independent vectors are in different directions.
- For independent vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ in a plane, the position vector of any point P in the plane will have a position vector given by $\underset{\sim}{p} = n\underset{\sim}{a} + m\underset{\sim}{b}$ for some $n, m \in R$.
- The vector product of any two independent vectors in a plane is a normal to the plane.
- A plane may be fixed by a point and a normal to the plane; any three points not in a straight line; two intersecting lines; or two parallel lines.
- There are only two unit normals to any plane; they are the negative of each other.
- Skew lines do not intersect and do not form a plane.

Equations of a plane

- The **equation of a plane** Γ through $A\ (x_0, y_0, z_0)$ with normal $\underset{\sim}{n} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$ is given by $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$, where $P\ (x, y, z)$ is any point in the plane.



- The equation of a plane can be written as $ax + by + cz = d$, where $d = ax_0 + by_0 + cz_0$.
- The **vector equation of a plane** Γ is given by $\underset{\sim}{n} \cdot (\underset{\sim}{p} - \underset{\sim}{a}) = 0$ or $\underset{\sim}{n} \cdot \underset{\sim}{p} = k$ where $\underset{\sim}{n}$ is a normal, $\underset{\sim}{a}$ is the position vector of a particular point in the plane, $\underset{\sim}{p}$ is the position vector of general points in the plane and $k = \underset{\sim}{n} \cdot \underset{\sim}{a}$ is a real number.
- The angle between intersecting planes is called the **dihedral angle**. It is the same as the angle between the normal.
- Planes must either be parallel or intersect in a straight line.

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 © VCAA 2021N 1Q1 (4 marks) Consider the function f with rule $f(x) = \frac{x^2}{2} + \sin(x)$.

a Find all values of x for which the second derivative is equal to zero. 2 marks

b Explain whether the graph of f has any points of inflection. 2 marks

2 © VCAA 2020 2BQ3d (1 mark) Let $g(x) = x^n e^{-x}$, where $n \in Z$.

Write down an expression for $g''(x)$.

3 (1 mark) A straight line passes through the points $A(-2, 3, -1)$ and $B(3, 5, -5)$.

Determine the vector form of the equation of the line.

4 (4 marks) A plane passes through the points $A(2, 0, 1)$, $B(-1, 2, 3)$ and $C(3, 1, 0)$.

a Determine a vector normal to the plane. 3 marks

b Use the result to determine the Cartesian equation of the plane. 1 mark

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 The function f is defined by the rule $f(x) = \dfrac{1}{\arccos(ax - b)}$, where $a, b \in R$ and $c > 0$.
The domain of f is

A $\left(\frac{b-1}{a}, \frac{b+1}{a}\right)$ **B** $\left(\frac{1-b}{a}, \frac{1+b}{a}\right)$ **C** $\left[-\frac{b}{a}, \frac{b}{a}\right]$ **D** $\left[\frac{b-1}{a}, \frac{b+1}{a}\right]$ **E** $\left(-\frac{b}{a}, \frac{b}{a}\right)$

2 $f(x) = x^2 e^{\arcsin(x)}$. $f'(x) =$

A $\dfrac{x^2 e^{\arcsin(x)}}{\sqrt{1-x^2}}$

B $2x\, e^{\arcsin(x)} + \dfrac{x^2 e^{\arcsin(x)}}{\sqrt{1-x^2}}$

C $\dfrac{2xe^{\arcsin(x)}}{\sqrt{1-x^2}}$

D $2x\, e^{\arcsin(x)} + \dfrac{e^{\arcsin(x)}}{\sqrt{1-x^2}}$

E $2x\, e^{\arcsin(x)} + \dfrac{x^2}{\sqrt{1-x^2}}$

3 Which equation describes the line passing through A $(-2, 5, 4)$ and B $(2, -2, 1)$?

A $(4 - 2t)\underset{\sim}{i} + (-7 + 5t)\underset{\sim}{j} + (-3 + 4t)\underset{\sim}{k}, t \in R$

B $(2 - 2t)\underset{\sim}{i} + (-2 + 5t)\underset{\sim}{j} + (1 + 4t)\underset{\sim}{k}, t \in R$

C $x = 2t + 4, y = -2t - 7, z = -3t + 1, t \in R$

D $x = -4t - 2, y = 7t + 5, z = 3t + 4, t \in R$

E $\dfrac{x+2}{2} = -\dfrac{y-5}{2} = z - 4, t \in R$

4 A line is described by the equation $\underset{\sim}{r}(t) = (4 + 3t)\underset{\sim}{i} + (2 - 3t)\underset{\sim}{j} + (-1 + 2t)\underset{\sim}{k}$
Which of the following points lie on this line?

A $(-5, 11, -5)$ **B** $(13, 11, -5)$ **C** $(10, -1, 3)$ **D** $(16, -7, 7)$ **E** $(-2, 8, -6)$

5 The vector $27i + 9j + 9k$ is normal to the plane containing the points $(3, 2, -2)$, $(1, 3, 3)$ and $(2, m, 5)$.
The value of m is

A -14 **B** -4 **C** -2 **D** 2 **E** 4

Section B 2 questions

15 marks

1 (8 marks) A function is given by the rule $y = f(x)$, where $\frac{x^2}{64} + \frac{y^2}{36} = 1; x, y \in R$ and $x, y \geq 0$.

a Find an expression for $\frac{dy}{dx}$. 1 mark

b Find an expression for the value of z, where $(z, 0)$ is the x-intercept of the tangent to $f(x)$ at the point $P(a, b)$. 2 marks

c Find an expression for $\frac{dz}{dt}$ in terms of x and $\frac{dx}{dt}$. 3 marks

d Given that $\frac{dx}{dt} = 1.5$ at $x = 2$, $\frac{dz}{dt}$ at $x = 2$. 1 mark

e What does the negative value of $\frac{dz}{dt}$ signify? 1 mark

2 (7 marks) A triangle has vertices $A(-3, -1, 2)$, $B(5, -1, -2)$ and $C(7, -5, -1)$.

a Find the equation of the plane of the triangle. 3 marks

b Find the equation of the parallel plane through $D(5, 5, 5)$. 1 mark

c Find the shortest distance from the plane of the triangle to the point D. 3 marks

CHAPTER

7

INTEGRATION

Study Design coverage

AREA OF STUDY 4: CALCULUS

Differential calculus and integral calculus

- techniques of anti-differentiation and for the evaluation of definite integrals:
  - anti-differentiation of $\frac{1}{x}$ to obtain $\log_e|x|$
  - anti-differentiation of $\frac{1}{\sqrt{a^2 - x^2}}$ and $\frac{a}{a^2 + x^2}$ for $a > 0$ by recognition that they are derivatives of corresponding inverse circular functions
  - use of the substitution $u = g(x)$ to anti-differentiate expressions
  - use of the trigonometric identities $\sin^2(ax) = \frac{1}{2}(1 - \cos(2ax))$ and $\cos^2(ax) = \frac{1}{2}(1 + \cos(2ax))$ in anti-differentiation techniques
  - anti-differentiation using partial fractions of rational functions
  - integration by parts
- numerical and symbolic integration using technology.

VCE Mathematics Study Design 2023–2027 p. 111, © VCAA 2022

Video playlists (8):

7.1	Anti-differentiation
7.2	Integrals producing inverse circular functions
7.3	Integration by substitution
7.4	Integrating circular functions
7.5	Integration by partial fractions
7.6	Further integration techniques
7.7	Integration by parts
VCE question analysis	Integration

Worksheets (12):

7.1	Indefinite integrals • Integrating functions
7.2	Inverse trigonometric functions • Integrals and inverse trigonometric functions
7.3	Integration by substitution 1 • Integration by substitution 2 • Definite integrals by substitution
7.4	Trigonometric identities • Integrals of $\sin^2 x$ and $\cos^2 x$ • Integration of $\cos^2 x$ and $\sin^2 x$
7.5	Partial fractions
7.6	Integral calculus

Nelson MindTap

To access resources above, visit
cengage.com.au/nelsonmindtap

7.1 Anti-differentiation

Video playlist Anti-differentiation

Worksheets Indefinite integrals

Integrating functions

Anti-differentiation or **integration** is the opposite process to differentiation.

The derivatives of x^2, $x^2 + 3$, $x^2 + 12$ and $x^2 - 7$ are all $2x$.

It follows that the **anti-derivative** of $2x$ is $x^2 + c$, where c represents an arbitrary constant.

In mathematical notation, this anti-derivative is written as $\int 2x\,dx = x^2 + c$.

The symbol $\int$ is an extended 'S' for summation. The function being integrated is called the **integrand** and the 'dx' indicates that the integral is 'with respect to $\boldsymbol{x}$', which is the variable in the expression.

The integral of $\frac{1}{x}$

In Mathematical Methods, Chapter 9, *Applying the exponential and logarithmic functions*, we learn that $\int \frac{1}{x}dx = \log_e(x) + c$, but for $x > 0$ because this is the domain for $\log_e(x)$.

Now we can find $\int \frac{1}{x}dx$ for $x < 0$.

If $x < 0$, then $\log_e(-x)$ is defined because $-x$ is positive.

Then $\frac{d}{dx}(\log_e(-x)) = \frac{1}{-x} \times (-1) = \frac{1}{x}$ by the chain rule.

So $\int \frac{1}{x}dx = \log_e(-x) + c$ for $x < 0$.

$$\text{So } \int \frac{1}{x}dx = \begin{cases} \log_e(x) + c & \text{for } x > 0 \\ \log_e(-x) + c & \text{for } x < 0 \end{cases}$$

But this is the same as writing $\int \frac{1}{x}dx = \log_e|x| + c, \quad x \neq 0$.

Similarly, $\int \frac{1}{ax+b}dx = \frac{1}{a}\log_e|ax+b| + c, \quad x \neq -\frac{b}{a}$.

Basic integrals

$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} + c$
$(ax+b)^n$	$\frac{(ax+b)^{n+1}}{a(n+1)} + c$
e^{ax}	$\frac{1}{a}e^{ax} + c$
$\frac{1}{x}$	$\log_e\|x\| + c$
$\frac{1}{ax+b}$	$\frac{1}{a}\log_e\|ax+b\| + c$
$\sin(ax)$	$-\frac{1}{a}\cos(ax) + c$
$\cos(ax)$	$\frac{1}{a}\sin(ax) + c$
$\sec^2(ax)$	$\frac{1}{a}\tan(ax) + c$

This means

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c.$$

WB

p. 133

WORKED EXAMPLE 1 Integration of polynomial functions

Find the anti-derivative of $5x^3 - \frac{2}{x^3} + 6$.

Steps	**Working**
1 Express all the parts of the polynomial in the form x^n.	$5x^3 - \frac{2}{x^3} + 6 = 5x^3 - 2x^{-3} + 6$

2 Anti-differentiate each term.	$\int 5x^3 - 2x^{-3} + 6\,dx = \frac{5x^4}{4} - \frac{2x^{-2}}{-2} + 6x + c$
3 Simplify, expressing the answer with positive indices.	$= \frac{5x^4}{4} + \frac{1}{x^2} + 6x + c$

WORKED EXAMPLE 2 Integrating exponential functions

WB p. 134

Find an integral of $5e^{3x} + \frac{4}{x}$.

Steps	Working
Use the formulas: $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$ and $\int \frac{1}{x}\,dx = \log_e \lvert x \rvert + c$	$\int 5e^{3x} + \frac{4}{x}\,dx = \frac{5e^{3x}}{3} + 4\log_e \lvert x \rvert$

The '+ c' is not required if the question asks for 'an integral' rather than 'the integral'.

WORKED EXAMPLE 3 Integrations that produce logarithmic functions

WB p. 134

Find the anti-derivative of $\frac{3}{2x-7}$.

Steps	Working
Use the formula: $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e \lvert ax+b \rvert + c$	$\int \frac{3}{2x-7}\,dx = \frac{3}{2}\log_e \lvert 2x-7 \rvert + c$

WORKED EXAMPLE 4 Integrating $(ax + b)^n$

WB p. 135

Find $\int 6(3x+2)^4\,dx$.

Steps	Working
Use the formula: $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$	$\int 6(3x+2)^4\,dx = \frac{6(3x+2)^5}{3 \times 5} + c$ $= \frac{2(3x+2)^5}{5} + c$

WORKED EXAMPLE 5 Integrating trigonometric functions

WB p. 135

Find $\int \cos(2x) - \sin(4x)\,dx$.

Steps	Working
Use the formulas: $\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$ and $\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$	$\int \cos(2x) - \sin(4x)\,dx$ $= \frac{1}{2}\sin(2x) + \frac{1}{4}\cos(4x) + c$

The definite integral

The quantity $\int_a^b f(x)\,dx$ is called the **definite integral** of $f(x)$ with respect to x between the values a and b. The values of a and b are the limits of integration, where a is the lower limit and b is the upper limit.

In Chapter 8, *Areas and volumes of integration*, we will examine the use of definite integrals to find the area between a function $f(x)$ and the x-axis in more detail.

Evaluating a definite integral

- Find an anti-derivative $F(x)$ of $f(x)$.
- Evaluate the integral by substituting the upper limit for x and then the lower limit for x.
- The definite integral is the difference between these two results $\int_a^b f(x)\,dx = F(b) - F(a)$.

WB

p. 136

WORKED EXAMPLE 6 Evaluating a definite integral

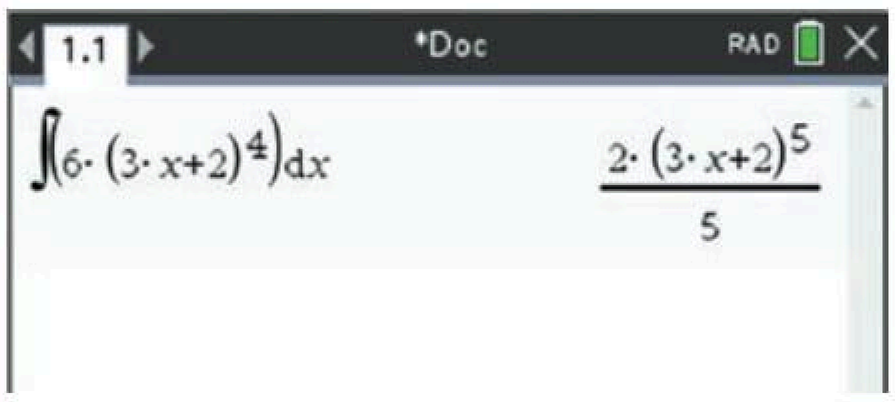
Evaluate $\int_0^4 \frac{1}{x+1}\,dx$.

Steps	Working
1 Write the anti-derivative in square brackets without the constant '*c*'. Write the limits on the right-hand bracket.	$\int_0^4 \frac{1}{x+1}\,dx = \left[\log_e\lvert x+1\rvert\right]_0^4$
2 Find $F(4) - F(0)$.	$= \log_e(4+1) - \log_e(0+1)$ $= \log_e(5) - \log_e(1) = 0$

USING CAS 1 Finding indefinite integrals

Find the **indefinite integral** $\int 6(3x+2)^4\,dx$.

TI-Nspire



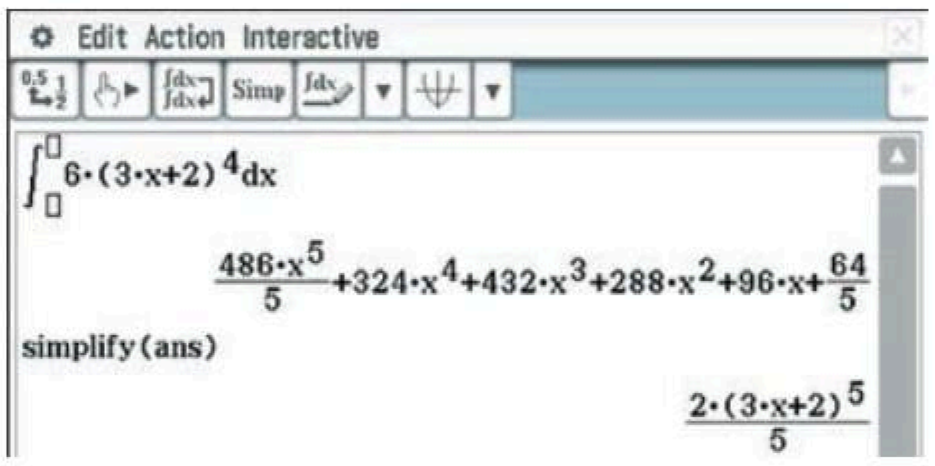
$\int (6\cdot(3\cdot x+2)^4)dx \qquad \frac{2\cdot(3\cdot x+2)^5}{5}$

1 Press **menu > Calculus > Integral**.

2 In the template, enter the expression and the variable **x**.

Note that the **Integral** template can also be accessed by pressing the template key. The shortcut is **Shift + +**.

ClassPad



$\int 6\cdot(3\cdot x+2)^4 dx$

$\frac{486\cdot x^5}{5}+324\cdot x^4+432\cdot x^3+288\cdot x^2+96\cdot x+\frac{64}{5}$

simplify(ans)

$\frac{2\cdot(3\cdot x+2)^5}{5}$

1 In **Main**, enter and highlight the expression.

2 Tap **Interactive > Calculation >** $\int$.

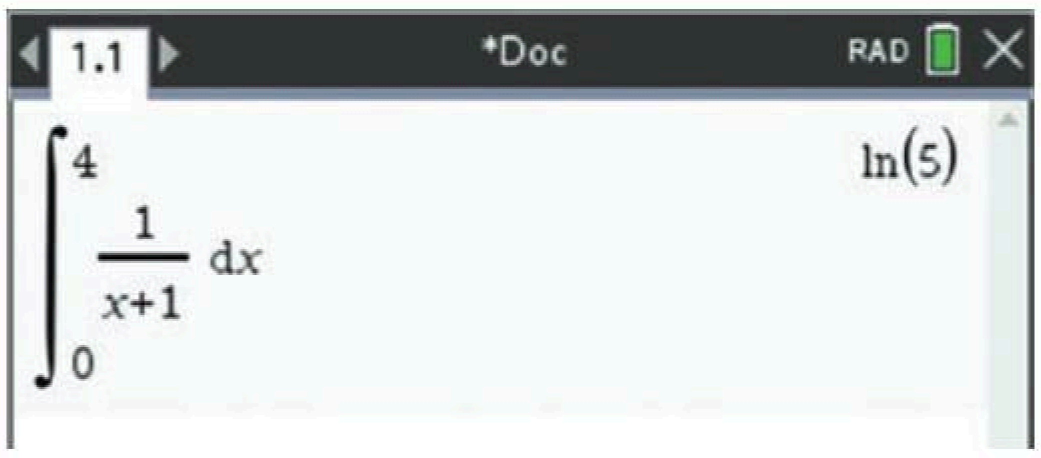
3 Tap **Simp** to factorise the solution.

Note that the **Integral** template can also be accessed by tapping **Keyboard > Math2**.

USING CAS 2 Finding definite integrals

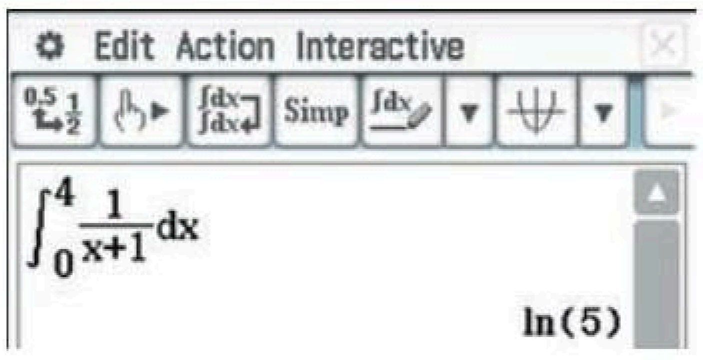
Find the definite integral $\int_0^4 \frac{1}{x+1}\,dx$.

TI-Nspire



$\int_0^4 \frac{1}{x+1}dx \qquad \ln(5)$

ClassPad



$\int_0^4 \frac{1}{x+1}dx$

$\ln(5)$

1 Press **menu > Calculus > Integral**.

2 In the template, enter the lower and upper limits, the expression and the variable **x**.

3 Press **enter**.

1 In **Main**, enter and highlight the expression.

2 Tap **Interactive > Calculation >** $\int$.

3 Select **Definite** and enter the Lower and Upper limits.

4 Tap **OK.**

EXERCISE 7.1 Anti-differentiation

ANSWERS p. 588

Mastery

1 WORKED EXAMPLE 1 TECH-FREE Find the anti-derivative of

a $10x^4 + 6x^2 - 8x + 5$ **b** $\frac{2}{x^3} - \frac{3}{x^4}$ **c** $x^{\frac{1}{2}} + x^{-\frac{2}{3}} - x^{\frac{5}{4}}$

2 WORKED EXAMPLE 2 Find an integral of

a $8e^{2x} + 4e^{-2x}$ **b** $\frac{2}{x}$ **c** $\frac{1}{3x}$

3 WORKED EXAMPLE 3 Find each integral.

a $\int \frac{1}{5x-2}\,dx$ **b** $\int \frac{6}{4x+3}\,dx$ **c** $\int \frac{8}{1-4x}\,dx$

4 WORKED EXAMPLE 4 Find each integral.

a $\int (2x+9)^3\,dx$ **b** $\int 2(4x-1)^2\,dx$ **c** $\int \frac{1}{(5x+6)^3}\,dx$

5 WORKED EXAMPLE 5 Find each integral.

a $\int \cos(10x)\,dx$ **b** $\int 3\sin(6x)\,dx$ **c** $\int 2\cos(3x+4)\,dx$

6 WORKED EXAMPLE 6 Evaluate each definite integral.

a $\int_0^2 \frac{6}{3x+2}\,dx$ **b** $\int_0^{\frac{\pi}{6}} \cos(2x)\,dx$ **c** $\int_1^2 (1-5x)^2\,dx$

7 Using CAS 1 Find the anti-derivative of each function.

a $\log_e(3x)$ **b** $x\cos(2x)$

8 Using CAS 2 Find the value of each integral correct to **three** decimal places.

a $\int_0^2 xe^x\,dx$ **b** $\int_0^{\frac{\pi}{6}} \sin\left(x^2\right)\,dx$

Exam practice

80–100% 60–79% 0–59%

9 TECH-FREE (2 marks) Find the value of a, given $\int_{-4}^{a} \frac{1}{1-2x}\,dx = \log_e\left(\sqrt{3}\right)$, $a < \frac{1}{2}$.

10 TECH-FREE (2 marks) Find $\int_3^8 \frac{1}{\sqrt{x+1}}\,dx$.

11 © VCAA 2002 1IQ14 70% The value of $\int_0^1 \frac{x^2 - 2x}{2\cos(x)}\,dx$, correct to four decimal places is

A -0.4369 **B** -0.3334 **C** -0.2622 **D** -0.0484 **E** 0.4369

12 The anti-derivative of $\sqrt[4]{x^3}$ is

A $\frac{4}{7}x^{\frac{7}{4}} + c$ **B** $\frac{3}{7}x^{\frac{7}{3}} + c$ **C** $\frac{2}{5}x^{\frac{5}{2}} + c$

D $\frac{4}{5}x^{\frac{5}{4}} + c$ **E** $\frac{7}{4}x^{\frac{7}{4}} + c$

13 The integral of $12x^3 + \frac{1}{x^3} + \sqrt{x}$ is

A $36x^2 - \frac{2}{x^4} + \frac{1}{2\sqrt{x}}$ **B** $3x^4 - \frac{1}{4x^4} + \frac{2\sqrt{x^3}}{3} + c$ **C** $3x^4 - \frac{1}{2x^2} + \frac{1}{2\sqrt{x}} + c$

D $3x^4 - \frac{1}{2x^2} + \frac{2\sqrt{x^3}}{3} + c$ **E** $3x^4 - \frac{1}{2x^2} + \frac{2\sqrt{x}}{3} + c$

14 The indefinite integral of $\frac{1}{(6x+2)^3}$ is

A $\frac{1}{6}\log_e(6x+2) + c$ **B** $-\frac{1}{2(6x+2)^2} + c$ **C** $-\frac{1}{24(6x+2)^4} + c$

D $\frac{1}{24}(6x+2)^4 + c$ **E** $-\frac{1}{12(6x+2)^2} + c$

15 The definite integral $\int_a^b \frac{10}{x}\,dx$ is equal to

A $10\log_e(b-a)$ **B** $10\log_e(b) - \log_e(a)$ **C** $\frac{10}{b^2} - \frac{10}{a^2}$

D $10\log_e\left(\frac{b}{a}\right)$ **E** $\frac{10}{b} - \frac{10}{a}$

16 $\int_n^m e^{ax}\,dx$ is equal to

A $\frac{e^{am-an}}{a}$ **B** $e^m - e^n$ **C** $\frac{e^{am} - e^{an}}{a}$ **D** $a(e^{am} - e^{an})$ **E** $\frac{e^{am}}{ae^{an}}$

17 © VCAA 2018N 2AQ9 $\int(1 - \cos(10x))\,dx$ is equivalent to

A $\int \sin^2(5x)\,dx$ **B** $\frac{1}{2}\int \sin^2(20x)\,dx$ **C** $\int \cos^2(5x)\,dx$

D $2\int \cos^2(10x)\,dx$ **E** $2\int \sin^2(5x)\,dx$

Video playlist Integrals producing inverse circular functions

Worksheets Inverse trigonometric functions

Integrals and inverse trigonometric functions

7.2 Integrals producing inverse circular functions

The integral of $\frac{1}{\sqrt{a^2 - x^2}}$

The inverse sine and cosine functions have the derivatives:

$$\frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{a^2 - x^2}}, \quad x \in (-a, a)$$

$$\frac{d}{dx}\left(\cos^{-1}\left(\frac{x}{a}\right)\right) = \frac{-1}{\sqrt{a^2 - x^2}}, \quad x \in (-a, a)$$

Integrals producing inverse sine and cosine functions

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + c, a > 0,\ x \in (-a,\ a)$$

$$\int \frac{-1}{\sqrt{a^2 - x^2}}\, dx = \cos^{-1}\left(\frac{x}{a}\right) + c, a > 0, x \in (-a,\ a)$$

Some algebraic manipulation is often required so that the integrand matches one of the formulas above.

WORKED EXAMPLE 7 Integrating $\frac{1}{\sqrt{a^2 - x^2}}$

WB
p. 137

Find $\int \frac{1}{\sqrt{25 - x^2}}\, dx$.

Steps	**Working**
1 Find the value of *a*.	$a^2 = 25$ $a = 5 \quad (a > 0)$
2 Use the formula $\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + c$ to find the integral.	$\int \frac{1}{\sqrt{25 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{5}\right) + c$

WORKED EXAMPLE 8 Integrating $\frac{-1}{\sqrt{a^2 - x^2}}$

WB
p. 138

Find $\int \frac{-3}{\sqrt{1 - 9x^2}}\, dx$.

Steps	**Working**
1 Change the integrand into the form $\frac{-1}{\sqrt{a^2 - x^2}}$.	$\int \frac{-3}{\sqrt{1 - 9x^2}}\, dx = 3\int \frac{-1}{\sqrt{1 - 9x^2}}\, dx$ $= 3\int \frac{-1}{\sqrt{9\left(\frac{1}{9} - x^2\right)}}\, dx$ $= 3\int \frac{-1}{3\sqrt{\frac{1}{9} - x^2}}\, dx$ $= \int \frac{-1}{\sqrt{\frac{1}{9} - x^2}}\, dx$
2 Find the value of *a*.	$a^2 = \frac{1}{9}$ $a = \frac{1}{3} \quad (a > 0)$
3 Use the formula $\int \frac{-1}{\sqrt{a^2 - x^2}}\, dx = \cos^{-1}\left(\frac{x}{a}\right) + c$ to find the anti-derivative.	$\int \frac{-3}{\sqrt{1 - 9x^2}}\, dx = \cos^{-1}\left(\frac{x}{\frac{1}{3}}\right) + c$ $= \cos^{-1}(3x) + c$

The integral of $\dfrac{a}{a^2+x^2}$

The inverse tangent function has the derivative $\dfrac{d}{dx}\left(\tan^{-1}\left(\dfrac{x}{a}\right)\right) = \dfrac{a}{a^2+x^2}$, $x \in R$.

Integrals producing inverse tangent functions

$$\int \frac{a}{a^2+x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right) + c, x \in R$$

WB

p. 139

WORKED EXAMPLE 9 Integrating $\dfrac{a}{a^2+x^2}$

Find $\displaystyle\int \frac{1}{64+x^2}\,dx$.

Steps	Working
1 Find the value of a.	$a^2 = 64$ $a = 8$
2 Transform the integrand into the form $\dfrac{a}{a^2+x^2}$. Multiply the integral by $\dfrac{1}{8} \times 8$.	$\displaystyle\int \frac{1}{64+x^2}\,dx = \frac{1}{8}\int \frac{8}{64+x^2}\,dx$
3 Use the formula $\displaystyle\int \frac{a}{a^2+x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right) + c$ to find the anti-derivative.	$= \dfrac{1}{8}\tan^{-1}\left(\dfrac{x}{8}\right) + c$

WB

p. 140

WORKED EXAMPLE 10 Integrating $\dfrac{1}{a^2+b^2x^2}$

Find $\displaystyle\int \frac{1}{9+4x^2}\,dx$.

Steps	Working
1 Take out a factor of 4 in the denominator.	$\displaystyle\int \frac{1}{9+4x^2}\,dx = \int \frac{1}{4\left(\frac{9}{4}+x^2\right)}\,dx$ $\displaystyle= \frac{1}{4}\int \frac{1}{\frac{9}{4}+x^2}\,dx$
2 Find the value of a.	$a^2 = \dfrac{9}{4}$ $a = \dfrac{3}{2}$
3 Transform the integrand into the form $\dfrac{a}{a^2+x^2}$. Multiply the integral by $\dfrac{1}{\frac{3}{2}} \times \dfrac{3}{2}$.	$\displaystyle\int \frac{1}{9+4x^2}\,dx = \frac{1}{4\times\frac{3}{2}}\int \frac{\frac{3}{2}}{\frac{9}{4}+x^2}\,dx$ $\displaystyle= \frac{1}{6}\int \frac{\frac{3}{2}}{\frac{9}{4}+x^2}\,dx$
4 Use the formula $\displaystyle\int \frac{a}{a^2+x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right) + c$ to find the anti-derivative.	$= \dfrac{1}{6}\tan^{-1}\left(\dfrac{x}{\frac{3}{2}}\right) + c$ $= \dfrac{1}{6}\tan^{-1}\left(\dfrac{2x}{3}\right) + c$

WORKED EXAMPLE 11 Integration involving completing the square

Find $\int \frac{1}{x^2+4x+13}\,dx$.

WB p. 141

Steps	**Working**
1 Complete the square in the denominator.	$\int \frac{1}{x^2+4x+13}\,dx = \int \frac{1}{x^2+4x+4+13-4}\,dx$ $= \int \frac{1}{(x+2)^2+9}\,dx$
2 Find the value of a.	$a^2 = 9$ $a = 3$
3 Transform the integrand into the form $\frac{a}{a^2+x^2}$. Multiply the integral by $\frac{1}{3}\times 3$.	$\int \frac{1}{(x+2)^2+9}\,dx = \frac{1}{3}\int \frac{3}{(x+2)^2+9}\,dx$
4 Use the formula: $\int \frac{a}{a^2+x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right)+c$	$= \frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right)+c$

Definite integrals producing inverse circular functions

We need to take care of the domain and range when evaluating an inverse circular function.

Domain and range of inverse circular functions

Function	Domain	Range
$y = \sin^{-1}\left(\frac{x}{a}\right)$	$[-a, a]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1}\left(\frac{x}{a}\right)$	$[-a, a]$	$[0, \pi]$
$y = \tan^{-1}\left(\frac{x}{a}\right)$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

WORKED EXAMPLE 12 Definite integrals that involve inverse trigonometric functions 1

Find $\int_0^1 \frac{3}{\sqrt{4-x^2}}\,dx$.

WB p. 142

Steps	**Working**
1 Find the value of a.	$a^2 = 4$ $a = 2$
2 Use the formula: $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)$	$\int_0^1 \frac{3}{\sqrt{4-x^2}}\,dx = 3\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1$
3 Evaluate the definite integral.	$= 3\left(\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)\right)$ $= 3\times\frac{\pi}{6} = \frac{\pi}{2}$

WB

p. 143

WORKED EXAMPLE 13 Definite integrals that involve inverse trigonometric functions 2

Find $\int_{-5}^{5} \frac{1}{25+x^2}\,dx$.

Steps	**Working**
1 Find the value of a.	$a^2 = 25$ $a = 5$
2 Transform the integrand into the form $\frac{a}{a^2+x^2}$. Multiply the integral by $\frac{1}{5} \times 5$.	$\frac{1}{5}\int_{-5}^{5} \frac{5}{25+x^2}\,dx$
3 Use the formula: $\int \frac{a}{a^2+x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right)$	$= \frac{1}{5}\left[\tan^{-1}\left(\frac{x}{5}\right)\right]_{-5}^{5}$
4 Evaluate the definite integral. The range of arctan (x) is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, therefore $\tan^{-1}(-1) = -\frac{\pi}{4}$.	$= \frac{1}{5}\left(\tan^{-1}(1) - \tan^{-1}(-1)\right)$ $= \frac{1}{5}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right)$ $= \frac{1}{5} \times \frac{\pi}{2}$ $= \frac{\pi}{10}$

EXERCISE 7.2 Integrals producing inverse circular functions

ANSWERS p. 588

Recap

1 The anti-derivative of $\frac{1}{(ax+b)^3}$, with respect to x, is

A $\frac{1}{a}\log_e(ax+b)+c$
B $3a(ax+b)^4+c$
C $\frac{-2a}{(ax+b)^2}+c$
D $\frac{-1}{2a(ax+b)^2}+c$
E $\frac{-1}{3a(ax+b)^4}+c$

2 The value of $\int_0^{\frac{\pi}{12}} \cos(2x) - \sin(4x)\,dx$ is

A $\frac{1}{8}$
B 8
C $\frac{1}{4}$
D $\frac{\pi}{6}$
E $\frac{3}{8}$

Mastery

3 WORKED EXAMPLE 7 Find each integral.

a $\int \frac{1}{\sqrt{9-x^2}}\,dx$
b $\int \frac{-4}{\sqrt{81-x^2}}\,dx$

4 WORKED EXAMPLE 8 Find each integral.

a $\int \frac{1}{\sqrt{1-4x^2}}\,dx$
b $\int \frac{-1}{\sqrt{4-9x^2}}\,dx$

5 WORKED EXAMPLE 9 Find each integral.

a $\int \frac{1}{25+x^2}\,dx$
b $\int \frac{2}{9+x^2}\,dx$

6 WORKED EXAMPLE 10 Find each integral.

a $\int \frac{1}{4+25x^2}\,dx$
b $\int \frac{4}{16+81x^2}\,dx$

7 WORKED EXAMPLE 11 Find each integral.

a $\int \frac{1}{x^2-2x+17}\,dx$
b $\int \frac{1}{x^2+6x+45}\,dx$

8 WORKED EXAMPLE 12 Evaluate each integral.

a $\int_0^{\frac{\sqrt{3}}{2}} \frac{5}{\sqrt{1-x^2}}\,dx$

b $\int_1^{\sqrt{3}} \frac{1}{1+x^2}\,dx$

9 WORKED EXAMPLE 13 Evaluate each integral.

a $\int_{-2}^{2} \frac{1}{\sqrt{16-x^2}}\,dx$

b $\int_{-\sqrt{3}}^{0} \frac{2}{9+x^2}\,dx$

Exam practice

80–100% 60–79% 0–59%

10 TECH-FREE (2 marks)

a Express $x^2 + 10x + 34$ in the form $(x+b)^2 + c$. 1 mark

b Hence find $\int \frac{1}{x^2+10x+34}\,dx$. 1 mark

11 TECH-FREE (2 marks) Find the value of $\int_{-a}^{0} \frac{1}{a^2+x^2}\,dx$.

12 © VCAA 2005 1AQ11 66% Which one of the following is an anti-derivative of $\frac{6}{\sqrt{1-4x^2}}$ for $-\frac{1}{2} \le x \le \frac{1}{2}$?

A $3\sin^{-1}(2x)$

B $6\sin^{-1}(2x)$

C $3\sin^{-1}\left(\frac{x}{2}\right)$

D $6\sin^{-1}\left(\frac{x}{2}\right)$

E $12\sin^{-1}\left(\frac{x}{2}\right)$

13 An anti-derivative of $\frac{21}{\sqrt{1-49x^2}}$ is

A $21\sin^{-1}(7x)$

B $\frac{1}{7}\sin^{-1}(7x)$

C $3\sin^{-1}\left(\frac{x}{7}\right)$

D $3\sin^{-1}(7x)$

E $63\sin^{-1}\left(\frac{x}{7}\right)$

14 The indefinite integral $\int \frac{1}{\sqrt{1-64x^2}}\,dx$ is equal to

A $\frac{1}{8}\sin^{-1}(8x)+c$

B $\sin^{-1}(8x)+c$

C $\frac{1}{8}\sin^{-1}(64x)+c$

D $\frac{1}{8}\sin^{-1}\left(\frac{x}{8}\right)+c$

E $8\sin^{-1}\left(\frac{x}{8}\right)+c$

15 The indefinite integral $\int \frac{1}{100+x^2}\,dx$ is equal to

A $\frac{1}{10}\tan^{-1}(10x)+c$

B $\frac{1}{10}\tan^{-1}\left(\frac{x}{10}\right)+c$

C $\frac{1}{10}\tan^{-1}\left(\frac{x}{100}\right)+c$

D $10\tan^{-1}\left(\frac{x}{10}\right)+c$

E $10\tan^{-1}(10x)+c$

16 The value of $\int_0^{\frac{5}{2}} \frac{1}{\sqrt{25-x^2}}\,dx$ is

A $\frac{\pi}{30}$

B $\frac{\pi}{3}$

C $\sin^{-1}\left(\frac{5}{2}\right)$

D $\sin^{-1}\left(\frac{x}{5}\right)$

E $\frac{\pi}{6}$

17 The value of $\int_{-2}^{0} \frac{1}{64+16x^2}\,dx$ is

A $\frac{\pi}{4}$

B $-\frac{\pi}{64}$

C $\frac{\pi}{64}$

D $-\frac{\pi}{128}$

E $\frac{\pi}{128}$

▶ **18** The value of $\int_{-\frac{1}{6}}^{\frac{1}{6}} \frac{1}{\sqrt{1-9x^2}}\,dx$ is

A $\frac{1}{3}\sin^{-1}\left(\frac{1}{6}\right)$ **B** $\frac{1}{3}\sin^{-1}\left(\frac{1}{2}\right)$ **C** 0 **D** $\frac{\pi}{9}$ **E** $\frac{\pi}{6}$

Video playlist
Integration by substitution

Worksheets
Integration by substitution 1

Integration by substitution 2

Definite integrals by substitution

7.3 Integration by substitution

Integration by substitution is used to find the integral of the product of two functions where one part of the integrand is the derivative of the other part of the integrand.

For example, the integral $\int (2x+1)\sqrt{x^2+x}\,dx$ can be found by substitution because if we let $u = x^2 + x$, then the derivative $\frac{du}{dx} = 2x+1$ appears in the other part of the integrand.

Note that the integral is in the form $\int g(u)\frac{du}{dx}\,dx$, where $u = x^2 + x$, $g(u) = \sqrt{x^2+x}$ and $\frac{du}{dx} = 2x+1$.

In general, let $y = \int f(x)\,dx = \int g(u)\,du$.

So $\frac{dy}{dx} = f(x)$ and $\frac{dy}{du} = g(u)$.

Using the chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\times\frac{du}{dx}\\ f(x) &= g(u)\times\frac{du}{dx}\\ \int f(x)\,dx &= \int g(u)\times\frac{du}{dx}\,dx\\ \int g(u)\,du &= \int g(u)\times\frac{du}{dx}\,dx\end{aligned}$$

Exam hack

Integration by substitution is the most common integration method used in Specialist Mathematics and is frequently examined in both Examinations 1 and 2.

Change of variable rule for integration

$$\int g(u)\times\frac{du}{dx}\,dx = \int g(u)\,du$$

WB
p. 144

WORKED EXAMPLE 14 Integration by substitution involving polynomial functions 1

Find $\int (2x+1)\sqrt{x^2+x}\,dx$.

Steps	**Working**
1 Choose a substitution for u in terms of x, which is inside the more complex function.	Let $u = x^2 + x$.
2 Find $\frac{du}{dx}$.	$\frac{du}{dx} = 2x+1$
3 Substitute u and $\frac{du}{dx}$ into the integral.	$\int (2x+1)\sqrt{x^2+x}\,dx = \int \sqrt{u}\,\frac{du}{dx}\,dx$ $= \int \sqrt{u}\,du$
4 Integrate with respect to u and then substitute $u = x^2 + x$ to express the answer in terms of x.	$\int u^{\frac{1}{2}}\,du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c$ $= \frac{2u^{\frac{3}{2}}}{3} + c$ $= \frac{2(x^2+x)^{\frac{3}{2}}}{3} + c$

Check your answer to the above example by differentiating it, using the chain rule.

WB p. 145

WORKED EXAMPLE 15 Integration by substitution involving polynomial functions 2

Find $\int 6x(x^2+2)^3\,dx$.

Steps	**Working**
1 Choose a substitution for u in terms of x, which is inside the more complex function.	Let $u = x^2 + 2$
2 Find $\frac{du}{dx}$ and write $6x$ in terms of $\frac{du}{dx}$.	$\frac{du}{dx} = 2x$ $3\frac{du}{dx} = 6x$
3 Substitute u and $3\frac{du}{dx}$ into the integral.	$\int 6x(x^2+2)^3\,dx = \int 3\frac{du}{dx} \times u^3\,dx$ $= 3\int u^3\,du$
4 Integrate with respect to u and then substitute $u = x^2 + 2$ to express the answer in terms of x.	$= \frac{3u^4}{4} + c$ $= \frac{3(x^2+2)^4}{4} + c$

WB p. 146

WORKED EXAMPLE 16 Integration by substitution involving sine and cosine functions

Find $\int \cos(x)\sin^3(x)\,dx$.

Steps	**Working**
1 Choose a substitution for u.	Let $u = \sin(x)$.
2 Find $\frac{du}{dx}$.	$\frac{du}{dx} = \cos(x)$
3 Substitute u and $\frac{du}{dx}$ into the integral.	$\int \cos(x)\sin^3(x)\,dx = \int u^3 \frac{du}{dx}\,dx$ $= \int u^3\,du$
4 Integrate with respect to u and then substitute $u = \sin(x)$ to express the answer in terms of x.	$= \frac{u^4}{4} + c$ $= \frac{\sin^4(x)}{4} + c$

Let u be the more complicated circular function.

Linear substitution

Linear substitution occurs when u is a linear function.

Linear substitution method

- Substitute the linear function, u, for the inside of the more complex part of the integrand.
- Transpose to express x in terms of u and substitute for x in the other part of the integrand.
- Find $\frac{du}{dx}$ and write an equation where $k\frac{du}{dx} = 1$.
- Substitute $k\frac{du}{dx}$ for the factor of 1.

WB

p. 147

WORKED EXAMPLE 17 Integration by linear substitution

Find $\int x\sqrt{2x-4}\,dx$.

Steps	Working
1 Choose a substitution for u in terms of x, which is inside the more complex function.	Let $u = 2x - 4$.
2 Transpose to express x in terms of u.	$x = \frac{u+4}{2}$
3 Find $\frac{du}{dx}$ and write an equation in the form $k\frac{du}{dx} = 1$.	$\frac{du}{dx} = 2$ $\frac{1}{2}\frac{du}{dx} = 1$
4 Substitute for u, x and 1 in the integral.	$\int x\sqrt{2x-4}\,dx = \int \frac{(u+4)}{2}\sqrt{u} \times \frac{1}{2}\frac{du}{dx}\,dx$ $= \frac{1}{4}\int u^{\frac{1}{2}}(u+4)\,du$ $= \frac{1}{4}\int u^{\frac{3}{2}} + 4u^{\frac{1}{2}}\,du$
5 Integrate with respect to u and then substitute $u = 2x - 4$ to express the answer in terms of x.	$= \frac{1}{4}\left(\frac{2u^{\frac{5}{2}}}{5} + \frac{8u^{\frac{3}{2}}}{3}\right) + c$ $= \frac{u^{\frac{5}{2}}}{10} + \frac{2u^{\frac{3}{2}}}{3} + c$ $= \frac{(2x-4)^{\frac{5}{2}}}{10} + \frac{2(2x-4)^{\frac{3}{2}}}{3} + c$

Definite integrals by substitution

In a definite integral that is found by substitution, it is essential to convert the lower and upper limits from x values to u values. Then substitute the limits of integration for u into the anti-derivative.

WB

p. 148

WORKED EXAMPLE 18 Substitution with definite integrals

Find the value of $\int_0^{\frac{\pi}{3}} \cos(x)\sin^3(x)\,dx$.

Steps	Working	
1 Choose a substitution for u.	Let $u = \sin(x)$.	
2 Find $\frac{du}{dx}$.	$\frac{du}{dx} = \cos(x)$	
3 Convert the limits of integration from x to u by substituting into $u = \sin(x)$.	lower limit: $x = 0$ $u = \sin(0) = 0$	upper limit: $x = \frac{\pi}{3}$ $u = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

4 Substitute u, $\frac{du}{dx}$ and the limits into the integral. Every part of the integral MUST be in terms of the substituted variable u, including limits.	$\int_0^{\frac{\pi}{3}} \cos(x)\sin^3(x)\,dx = \int_0^{\frac{\sqrt{3}}{2}} u^3 \frac{du}{dx}\,dx$ $= \int_0^{\frac{\sqrt{3}}{2}} u^3\,du$
5 Integrate with respect to u and then substitute the limits of integration into the anti-derivative.	$= \left[\frac{u^4}{4}\right]_0^{\frac{\sqrt{3}}{2}}$ $= \frac{\left(\frac{\sqrt{3}}{2}\right)^4}{4} - 0$ $= \frac{9}{64}$

WB p. 149

WORKED EXAMPLE 19 Linear substitution with definite integrals

Find the value of $\int_5^{10} \frac{x+2}{\sqrt{x-1}}\,dx$.

Steps	**Working**
1 Choose a substitution for u.	Let $u = x - 1$.
2 Transpose to express x in terms of u.	$x = u + 1$
3 Find $\frac{du}{dx}$.	$\frac{du}{dx} = 1$
4 Convert the limits from x values to u values.	lower limit: $x = 5$ $u = 5 - 1 = 4$ upper limit: $x = 10$ $u = 10 - 1 = 9$
5 Substitute u, $\frac{du}{dx}$ and the limits into the integral.	$\int_5^{10} \frac{x+2}{\sqrt{x-1}}\,dx = \int_4^9 u^{-\frac{1}{2}}(u+1+2)\frac{du}{dx}\,dx$ $= \int_4^9 u^{-\frac{1}{2}}(u+3)\,du$ $= \int_4^9 u^{\frac{1}{2}} + 3u^{-\frac{1}{2}}\,du$
6 Integrate and evaluate.	$\left[\frac{2u^{\frac{3}{2}}}{3} + 6u^{\frac{1}{2}}\right]_4^9 = \frac{2\times 9^{\frac{3}{2}}}{3} + 6\times 9^{\frac{1}{2}} - \left(\frac{2\times 4^{\frac{3}{2}}}{3} + 6\times 4^{\frac{1}{2}}\right)$ $= \frac{56}{3}$

EXERCISE 7.3 Integration by substitution

ANSWERS p. 588

Recap

1 The indefinite integral $\int \frac{1}{\sqrt{1-81x^2}}\,dx$ is equal to

A $9\sin^{-1}(x) + c$

B $\sin^{-1}(9x) + c$

C $\frac{1}{9}\sin^{-1}\left(\frac{x}{9}\right) + c$

D $\frac{1}{9}\sin^{-1}(9x) + c$

E $9\sin^{-1}\left(\frac{x}{9}\right) + c$

2 The value of $\int_0^6 \frac{1}{\sqrt{36-x^2}}\,dx$ is

A $\frac{\pi}{6}$ **B** $\frac{\pi}{3}$ **C** $\sin^{-1}\left(\frac{1}{6}\right)$ **D** π **E** $\frac{\pi}{2}$

Mastery

3 WORKED EXAMPLE 14 Find each integral.

a $\int (6x-2)\sqrt{3x^2-2x}\,dx$ **b** $\int e^x\cos\left(e^x\right)dx$

4 WORKED EXAMPLE 15 Find each integral.

a $\int 12x^2(x^3+2)^3\,dx$ **b** $\int \frac{8-20x}{\left(4x-5x^2\right)^2}\,dx$

5 WORKED EXAMPLE 16 Find each integral.

a $\int \sin(x)\cos^4(x)\,dx$ **b** $\int \frac{\log_e(x)}{3x}\,dx$

6 WORKED EXAMPLE 17 Find each integral.

a $\int x\sqrt{x+5}\,dx$ **b** $\int \frac{x}{(x-1)^3}\,dx$

7 WORKED EXAMPLE 18 Evaluate each integral.

a $\int_0^{\frac{\pi}{3}} \cos(x)\sin^2(x)\,dx$ **b** $\int_0^1 9x^2(x^3+2)^3\,dx$

8 WORKED EXAMPLE 19 Find the value of each definite integral.

a $\int_2^{14} \frac{x-1}{\sqrt{x+2}}\,dx$ **b** $\int_4^5 x(x-3)^3\,dx$

Exam practice

80–100% 60–79% 0–59%

9 © VCAA 2011 1Q6 62% TECH-FREE (2 marks) Evaluate $\int_0^1 e^x\cos(e^x)\,dx$.

10 © VCAA 2015 1Q8a 52% TECH-FREE (2 marks) Show that $\int \tan(2x)\,dx = \frac{1}{2}\log_e\left|\sec(2x)\right| + c$.

11 © VCAA 2010 1Q6 51% TECH-FREE (3 marks) Evaluate $\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \cos^2(2x)\sin(2x)\,dx$.

12 © VCAA 2017 1Q2 48% TECH-FREE (4 marks) Find $\int_1^{\sqrt{3}} \frac{1}{x(1+x^2)}\,dx$, expressing your answer in the form $\log_e\left(\sqrt{\frac{a}{b}}\right)$, where a and b are positive integers.

13 © VCAA 2013 2AQ9 85% The definite integral $\int_{e^3}^{e^4} \frac{1}{x\log_e(x)}\,dx$ can be written in the form $\int_a^b \frac{1}{u}\,du$, where

A $u=\log_e(x),\ a=\log_e(3),\ b=\log_e(4)$ **B** $u=\log_e(x),\ a=3,\ b=4$

C $u=\log_e(x),\ a=e^3,\ b=e^4$ **D** $u=\frac{1}{x},\ a=e^{-3},\ b=e^{-4}$

E $u=\frac{1}{x},\ a=e^3,\ b=e^4$

14 © VCAA 2010 2AQ14 80% Use a suitable substitution to show that the definite integral $\int_0^2 \frac{x}{\sqrt{x^2-1}}\,dx$ can be simplified to

A $\frac{1}{2}\int_{-1}^{3} u^{-\frac{1}{2}}\,du$ **B** $2\int_{-1}^{3} u^{-\frac{1}{2}}\,du$ **C** $\frac{1}{2}\int_0^2 u^{-\frac{1}{2}}\,du$ **D** $2\int_0^2 u^{-\frac{1}{2}}\,du$ **E** $\int_0^2 u^{-\frac{1}{2}}\,du$

15 © VCAA 2006 2AQ9 80% Using a suitable substitution, $\int_a^b x\left(x^2+1\right)^5 dx$ is equal to

A $\frac{1}{2}\int_a^b u^5\, du$ **B** $2\int_a^b u^5\, du$ **C** $\frac{1}{2}\int_{a^2+1}^{b^2+1} u^5\, du$ **D** $2\int_{a^2+1}^{b^2+1} u^5\, du$ **E** $\frac{1}{12}b^6 - \frac{1}{12}a^6$

16 © VCAA 2011 2AQ15 73% Using a suitable substitution, the definite integral $\int_0^{\frac{\pi}{24}} \tan(2x)\sec^2(2x)\,dx$ is equivalent to

A $\frac{1}{2}\int_0^{\frac{\pi}{24}}(u)\,du$ **B** $2\int_0^{\frac{\pi}{24}}(u)\,du$ **C** $\int_0^{2-\sqrt{3}}(u)\,du$ **D** $\frac{1}{2}\int_0^{2-\sqrt{3}}(u)\,du$ **E** $2\int_0^{2-\sqrt{3}}(u)\,du$

17 © VCAA 2016 2AQ8 73% Using a suitable substitution $\int_a^b x^3 e^{2x^4} dx$, where a and b are real constants, can be written as

A $\int_a^b \left(e^{2u}\right) du$ **B** $\int_{a^4}^{b^4} \left(e^{2u}\right) du$ **C** $\frac{1}{8}\int_a^b \left(e^{u}\right) du$

D $\frac{1}{4}\int_{a^4}^{b^4} \left(e^{2u}\right) du$ **E** $\frac{1}{8}\int_{8a^3}^{8b^3} \left(e^{u}\right) du$

18 © VCAA 2007 2AQ13 70% Using a suitable substitution, $\int_1^{e^3} \frac{[\log_e(x)]^3}{x} dx$ may be expressed completely in terms of u as

A $\int_0^3 \left(\frac{u^3}{e^u}\right) du$ **B** $\int_0^{e^3} u^3\, du$ **C** $\int_0^{\log_e(3)} u^3\, du$ **D** $\int_1^{\log_e(3)} u^3\, du$ **E** $\int_0^3 u^3\, du$

19 © VCAA 2008 2AQ16 69% Using a suitable substitution, $\int_0^{\sqrt{3}} \frac{\log_e[\arctan(x)]}{1+x^2} dx$ can be expressed completely in terms of u as

A $\int_0^{\sqrt{3}} \log_e(u)\, du$ **B** $\int_0^{\frac{\pi}{6}} \frac{\log_e(u)}{1+\tan^2(u)} du$ **C** $\int_0^{\frac{\pi}{3}} \log_e(u)\, du$

D $\int_0^{\frac{\pi}{6}} \log_e(u)\, du$ **E** $\int_0^{\frac{\pi}{3}} \frac{\log_e(u)}{1+\tan^2(u)} du$

20 © VCAA 2014 2AQ13 65% Using the substitution $u = \sqrt{x+1}$ then $\int_0^2 \frac{dx}{(x+2)\sqrt{x+1}}$ can be expressed as

A $\int_1^{\sqrt{3}} \frac{1}{\sqrt{u}\left(u^2+1\right)} du$ **B** $\int_0^2 \frac{2}{u^2+1} du$ **C** $\int_1^3 \frac{1}{\sqrt{u}\,(u+1)} du$

D $\frac{1}{4}\int_0^2 \frac{1}{u^2\left(u^2+1\right)} du$ **E** $2\int_1^{\sqrt{3}} \frac{1}{u^2+1} du$

21 © VCAA 2017 2AQ7 60% With a suitable substitution $\int_1^2 x^2\sqrt{2-x}\, dx$ can be expressed as

A $-\int_1^2 \left(4u^{\frac{1}{2}} - 4u^{\frac{3}{2}} + u^{\frac{5}{2}}\right) du$ **B** $\int_1^2 \left(4u^{\frac{1}{2}} - 4u^{\frac{3}{2}} + u^{\frac{5}{2}}\right) du$ **C** $\int_0^1 \left(-4u^{\frac{1}{2}} + 4u^{\frac{3}{2}} - u^{\frac{5}{2}}\right) du$

D $-\int_1^0 \left(4u^{\frac{1}{2}} - 4u^{\frac{3}{2}} + u^{\frac{5}{2}}\right) du$ **E** $\int_1^0 \left(4u^{\frac{1}{2}} - 4u^{\frac{3}{2}} - u^{\frac{5}{2}}\right) du$

22 © VCAA 2015 2AQ10 56% Using a suitable substitution, the definite integral $\int_0^1 x^2\sqrt{3x+1}\, dx$ is equivalent to

A $\frac{1}{9}\int_0^1 \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$ **B** $\frac{1}{27}\int_1^4 \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$ **C** $\frac{1}{9}\int_1^4 \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$

D $\frac{1}{27}\int_0^1 \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$ **E** $\frac{1}{3}\int_1^4 \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$

23 © VCAA 2012 2AQ13 **53%** Using a suitable substitution, $\int_0^{\frac{\pi}{3}} \sin^3(x)\cos^4(x)\,dx$ can be expressed in terms of u as

A $\int_0^{\frac{\pi}{3}} \left(u^6 - u^4\right) du$ **B** $\int_1^{\frac{1}{2}} \left(u^6 - u^4\right) du$ **C** $\int_{\frac{1}{2}}^{1} \left(u^6 - u^4\right) du$

D $\int_0^{\frac{\sqrt{3}}{2}} \left(u^6 - u^4\right) du$ **E** $\int_0^{\frac{\sqrt{3}}{2}} \left(u^4 - u^6\right) du$

24 © VCAA 2009 2AQ10 **23%** Let $f: [-\pi, 2\pi] \to R$, where $f(x) = \sin^3(x)$.

Using the substitution $u = \cos(x)$, the area bounded by the graph of f and the x-axis could be found by evaluating

A $-\int_{-\pi}^{2\pi} \left(1 - u^2\right) du$ **B** $3\int_{-1}^{1} \left(1 - u^2\right) du$ **C** $-3\int_0^{\pi} \left(1 - u^2\right) du$

D $3\int_1^{-1} \left(1 - u^2\right) du$ **E** $-\int_{-1}^{1} \left(1 - u^2\right) du$

25 © VCAA 2018N 2AQ8 Using a suitable substitution $\int_1^2 \left(\frac{3}{2 + (4x+1)^2}\right) dx$ can be expressed as

A $\frac{3}{4}\int_1^2 \left(\frac{1}{2+u^2}\right) du$ **B** $\frac{3}{4}\int_5^9 \left(\frac{1}{2+u^2}\right) du$ **C** $3\int_5^9 \left(\frac{1}{2+u^2}\right) du$

D $3\int_1^2 \left(\frac{1}{2+u^2}\right) du$ **E** $-12\int_9^5 \left(\frac{1}{2+u^2}\right) du$

Video playlist
Integrating circular functions

7.4 Integrating circular functions

Integration of odd powers of sin (*ax*) and cos (*ax*)

The integrals of odd powers of sin (x) and cos (x) are found by substitution.

Odd powers of sine or cosine

To find the integrals of odd powers of sine and cosine functions:

- write the integrand as
  $\sin(x) \times$ an even power of $\sin(x)$ or
  $\cos(x) \times$ an even power of $\cos(x)$
- use the Pythagorean identity $\sin^2(x) + \cos^2(x) = 1$ and substitute
  $\sin^2(x) = 1 - \cos^2(x)$ or $\cos^2(x) = 1 - \sin^2(x)$
- integrate by substitution.

WB
p. 150

WORKED EXAMPLE 20 Integrating an odd power of a sine function

Find $\int \sin^3(x)\,dx$.

Steps	**Working**
1 Write the integrand as $\sin(x) \times$ an even power of $\sin(x)$.	$\int \sin^3(x)\,dx = \int \sin(x)\sin^2(x)\,dx$
2 Substitute $\sin^2(x) = 1 - \cos^2(x)$.	$= \int \sin(x)\left[1 - \cos^2(x)\right] dx$
3 Choose a substitution for u.	Let $u = \cos(x)$.

4 Find $\frac{du}{dx}$.	$\frac{du}{dx} = -\sin(x)$ $\quad \left(-\frac{du}{dx} = \sin(x)\right)$
5 Substitute for u and $-\frac{du}{dx}$ in the integral.	$\int \sin(x)\left[1-\cos^2(x)\right]dx = \int\left(1-u^2\right)\times\left(-\frac{du}{dx}\right)dx$ $= -\int\left(1-u^2\right)du$
6 Integrate with respect to u and then substitute $u = \cos(x)$ to express the answer in terms of x.	$= -u + \frac{u^3}{3} + c$ $= -\cos(x) + \frac{\cos^3(x)}{3} + c$

When using CAS to integrate a circular function, sometimes it will give you a different answer that is equivalent to the one you found by hand.

Even powers of sin (*ax*) and cos (*ax*)

The integrals of even powers of sin (x) and cos (x) can be found using the double angle identities for cos ($2x$) to make $\cos^2(x)$ or $\sin^2(x)$ the subject.

Double angle identities

$$\cos(2x) = 2\cos^2(x) - 1 \text{ gives } \cos^2(x) = \frac{1}{2}[1+\cos(2x)]$$

$$\cos(2x) = 1 - 2\sin^2(x) \text{ gives } \sin^2(x) = \frac{1}{2}[1-\cos(2x)]$$

WORKED EXAMPLE 21 Integrating the square of a sine function

WB p. 151

Find $\int \sin^2(3x)\,dx$.

Steps	**Working**
1 Substitute $\sin^2(3x) = \frac{1}{2}[1-\cos(6x)]$.	$\int \sin^2(3x)\,dx = \frac{1}{2}\int 1-\cos(6x)\,dx$
2 Integrate.	$= \frac{1}{2}\left[x - \frac{1}{6}\sin(6x)\right] + c$ $= \frac{1}{2}x - \frac{1}{12}\sin(6x) + c$

WORKED EXAMPLE 22 Integrating the square of a cosine function

WB p. 152

Find $\int \cos^2(x)\,dx$.

Steps	**Working**
1 Substitute $\cos^2(x) = \frac{1}{2}[1+\cos(2x)]$.	$\int \cos^2(x)\,dx = \frac{1}{2}\int 1+\cos(2x)\,dx$
2 Integrate.	$= \frac{1}{2}\left[x + \frac{1}{2}\sin(2x)\right] + c$ $= \frac{1}{2}x + \frac{1}{4}\sin(2x) + c$

Worksheets
Trigonometric identities

Integrals of $\sin^2 x$ and $\cos^2 x$

Integration of $\cos^2 x$ and $\sin^2 x$

Trigonometric identities and integrals

$$\sec^2(x) = 1 + \tan^2(x)$$

$$\operatorname{cosec}^2(x) = 1 + \cot^2(x)$$

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\int \sec^2(ax)\,dx = \frac{1}{a}\tan(ax) + c$$

$$\int \operatorname{cosec}^2(ax)\,dx = -\frac{1}{a}\cot(ax) + c$$

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = -\log_e|\cos(x)| + c$$

WB p. 152

WORKED EXAMPLE 23 Integration using trigonometric identities 1

Find $\int \tan^2(2x) + 1\,dx$.

Steps	Working
Substitute $\sec^2(x) = 1 + \tan^2(x)$.	$\int \tan^2(2x) + 1\,dx = \int \sec^2(2x)\,dx$ $= \frac{1}{2}\tan(2x) + c$

WB p. 153

WORKED EXAMPLE 24 Integration using trigonometric identities 2

Find $\int \sin^2(x)\cos^2(x)\,dx$.

Steps	Working
1 Express the integrand as $[\sin(x)\cos(x)]^2$ and use the double angle rule for $\sin(2x)$. $\sin(2x) = 2\sin(x)\cos(x)$ $\frac{1}{2}\sin(2x) = \sin(x)\cos(x)$	$\int \sin^2(x)\cos^2(x)\,dx = \int [\sin(x)\cos(x)]^2\,dx$ $= \int \left[\frac{1}{2}\sin(2x)\right]^2 dx$ $= \frac{1}{4}\int \sin^2(2x)\,dx$
2 This is an even power of sine so use the formula: $\sin^2(2x) = \frac{1}{2}[1 - \cos(4x)]$	$= \frac{1}{4}\int \frac{1}{2}[1 - \cos(4x)]\,dx$ $= \frac{1}{8}\left[x - \frac{1}{4}\sin(4x)\right] + c$ $= \frac{1}{8}x - \frac{1}{32}\sin(4x) + c$

WB p. 154

WORKED EXAMPLE 25 Integration by substitution

Find $\int_0^{\frac{\pi}{3}} \sin^3(x)\cos^2(x)\,dx$.

Steps	Working
1 Write $\sin^3(x)$ as $\sin(x) \times$ an even power of $\sin(x)$.	$\int_0^{\frac{\pi}{3}} \sin^3(x)\cos^2(x)\,dx = \int_0^{\frac{\pi}{3}} \sin(x)\sin^2(x)\cos^2(x)\,dx$
2 Substitute $\sin^2(x) = 1 - \cos^2(x)$.	$= \int_0^{\frac{\pi}{3}} \sin(x)\left[1 - \cos^2(x)\right]\cos^2(x)\,dx$
3 Choose a substitution for u.	Let $u = \cos(x)$.
4 Find $\frac{du}{dx}$.	$\frac{du}{dx} = -\sin(x) \quad \left(-\frac{du}{dx} = \sin(x)\right)$

5 Convert the limits of integration from x values to u values.	lower limit: $x = 0$ $u = \cos(0) = 1$ upper limit: $x = \frac{\pi}{3}$ $u = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$
6 Substitute u, $-\frac{du}{dx}$ and the limits into the integral. Note that the new limits are not in ascending order as the original limits are.	$\int_0^{\frac{\pi}{3}} \sin^3(x)\cos^2(x)\,dx = -\int_1^{\frac{1}{2}}\left(1-u^2\right)u^2\,\frac{du}{dx}\,dx$ $= -\int_1^{\frac{1}{2}}\left(u^2-u^4\right)du$ $= \int_1^{\frac{1}{2}}\left(u^4-u^2\right)du$
7 Integrate and evaluate.	$= \left[\frac{u^5}{5}-\frac{u^3}{3}\right]_1^{\frac{1}{2}}$ $= \frac{1}{160}-\frac{1}{24}-\left(\frac{1}{5}-\frac{1}{3}\right)$ $= \frac{47}{480}$

EXERCISE 7.4 Integrating circular functions

ANSWERS p. 589

Recap

80–100% 60–79% 0–59%

1 The integral $\int \frac{x}{\sqrt{49-9x^2}}\,dx$ is equal to

A $\arcsin\left(\frac{x}{7}\right)+c$

B $\frac{1}{3}\arcsin\left(\frac{3x}{7}\right)+c$

C $-\frac{1}{9}\sqrt{49-9x^2}+c$

D $-\frac{1}{18}\sqrt{49-9x^2}+c$

E $\frac{-1}{9\sqrt{49-9x^2}}+c$

2 © VCAA 2005 1IQ15 60% An anti-derivative of $x\sqrt{3-x}$, for $x < 3$, is

A $-\frac{2x}{3}(3-x)^{\frac{3}{2}}$

B $-\frac{x^2(3-x)^{\frac{3}{2}}}{3}$

C $2(3-x)^{\frac{3}{2}}-\frac{2}{5}(3-x)^{\frac{5}{2}}$

D $-2(3-x)^{\frac{3}{2}}+\frac{2}{5}(3-x)^{\frac{5}{2}}$

E $-2(3-x)^{\frac{3}{2}}-\frac{2}{5}(3-x)^{\frac{5}{2}}$

Mastery

3 WORKED EXAMPLE 20 Find each integral.

a $\int \cos^3(x)\,dx$ **b** $\int \sin^3(2x)\,dx$ **c** $\int 4\cos^3(4x)\,dx$ **d** $\int \cos^5(x)\,dx$

4 WORKED EXAMPLE 21 Find $\int \sin^2(x)\,dx$.

5 WORKED EXAMPLE 22 Find each integral.

a $\int \sin^2(4x)\,dx$ **b** $\int \cos^2(2x)\,dx$ **c** $\int 3\cos^2(5x)\,dx$

6 WORKED EXAMPLE 23 Find $\int \tan^2(x)+1\,dx$.

7 WORKED EXAMPLE 24 Find $\int 4\sin^2(2x)\cos^2(2x)\,dx$.

8 Find $\int \sin^2(x)\cos^3(x)\,dx$ and hence evaluate $\int_0^{\frac{\pi}{6}} \sin^2(x)\cos^3(x)\,dx$.

Exam practice

80–100% 60–79% 0–59%

9 © VCAA 2014 1Q5 TECH-FREE (5 marks)

a 81% For the function with rule $f(x) = 96\cos(3x)\sin(3x)$, find the value of a such that $f(x) = a\sin(6x)$. 1 mark

b 66% Use an appropriate substitution in the form $u = g(x)$ to find an equivalent definite integral for $\int_{\frac{\pi}{36}}^{\frac{\pi}{12}} 96\cos(3x)\sin(3x)\cos^2(6x)\,dx$ in terms of u only. 3 marks

c 45% **Hence** evaluate $\int_{\frac{\pi}{36}}^{\frac{\pi}{12}} 96\cos(3x)\sin(3x)\cos^2(6x)\,dx$, giving your answer in the form $\sqrt{k},\ k \in Z$. 1 mark

10 TECH-FREE (4 marks) With a suitable substitution, $\int_0^{\frac{\pi}{6}} \sin^2(2x)\sin(2x)\,dx$ can be expressed in the form $\frac{1}{2}\int_a^b \left(1-u^2\right)du$, where a and b are positive real constants.

a Find the values of a and b. 2 marks

b Find the value of $\int_0^{\frac{\pi}{6}} \sin^2(2x)\sin(2x)\,dx$. 2 marks

11 © VCAA 2003 1IQ10 66% With a suitable substitution, $\int_0^{\frac{\pi}{6}} \cos^3(2x)\,dx$ can be expressed as

A $\frac{1}{2}\int_0^{\frac{1}{2}} \left(1-u^2\right)du$

B $\frac{1}{2}\int_0^{\frac{1}{2}} \left(u^2-1\right)du$

C $\frac{1}{2}\int_0^{\frac{\sqrt{3}}{2}} \left(1-u^2\right)du$

D $2\int_0^{\frac{1}{2}} \left(u^2-1\right)du$

E $2\int_0^{\frac{\sqrt{3}}{2}} \left(1-u^2\right)du$

12 © VCAA 2004 1IQ11 64% $\int_0^a \left[\sin^2\left(\frac{3x}{2}\right) - \cos^2\left(\frac{3x}{2}\right)\right]dx$ is equal to

A $-\frac{4}{3}\sin\left(\frac{3a}{4}\right)$

B $-\frac{1}{3}\sin(3a)$

C $\frac{1}{3}\sin(3a)$

D $\frac{1}{3}[1-\sin(3a)]$

E $-\frac{1}{3}[\cos(3a)-1]$

13 © VCAA 2017N 2AQ7 $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\sin^3(x)\cos^2(x)\right)dx$ is equivalent to

A $\int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \left(u^4-u^2\right)du$ where $u = \cos(x)$

B $-\int_{\frac{1}{\sqrt{2}}}^{\frac{1}{2}} \left(u^2-u^4\right)du$ where $u = \cos(x)$

C $-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(u^2-u^4\right)du$ where $u = \sin(x)$

D $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(u^2-u^4\right)du$ where $u = \sin(x)$

E $-\int_{\frac{1}{\sqrt{2}}}^{\frac{1}{2}} \left(u^2-u^4\right)du$ where $u = \sin(x)$

14 An anti-derivative of $\sin^3(6x)$ is

A $\frac{\sin^4(6x)}{4}$

B $\frac{1}{3}\cos^3(6x) - \cos(6x)$

C $\frac{1}{18}\cos^3(6x) - \frac{1}{6}\cos(6x)$

D $\frac{\sin^4(6x)}{24}$

E $-\frac{1}{18}\cos^3(6x) + \frac{1}{6}\cos(6x)$

15 An anti-derivative of $\sin^2\left(\frac{x}{2}\right)$ is

A $\frac{\sin^3\left(\frac{x}{2}\right)}{3}$ **B** $\frac{x}{2} - \sin(x)$ **C** $x - \sin(x)$ **D** $\frac{2\sin^3\left(\frac{x}{2}\right)}{3}$ **E** $\frac{x}{2} - \frac{1}{2}\sin(x)$

16 $\int \sin^2(2x)\cos^3(2x)\,dx$ is equal to

A $\frac{\sin^3(2x)}{3} - \frac{\sin^5(2x)}{5} + c$ **B** $\frac{2\sin^3(2x)}{3} - \frac{2\sin^5(2x)}{5} + c$ **C** $\frac{\sin^3(4x)}{12} + c$

D $\frac{\sin^3(2x)\cos^4(2x)}{6} + c$ **E** $\frac{\sin^3(2x)}{6} - \frac{\sin^5(2x)}{10} + c$

17 The value of $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan^2(x) + 1\,dx$ is

A $\frac{2\sqrt{3}}{3}$ **B** $\frac{8}{3} - \frac{\pi}{6}$ **C** $\frac{27\sqrt{3}-3}{9} + \frac{\pi}{6}$ **D** $\frac{8}{3}$ **E** $-\frac{2\sqrt{3}}{3}$

7.5 Integration by partial fractions

Video playlist Integration by partial fractions

Worksheet Partial fractions

We used partial fractions in Chapter 2 *Rational functions* to graph rational fractions. We can also use partial fractions to integrate rational fractions.

Partial fractions with two linear factors

The partial fractions for fractions with two linear factors in the denominator take the form

$$\frac{f(x)}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-b)}$$

WB p. 155

WORKED EXAMPLE 26 Integration by partial fractions with two linear factors

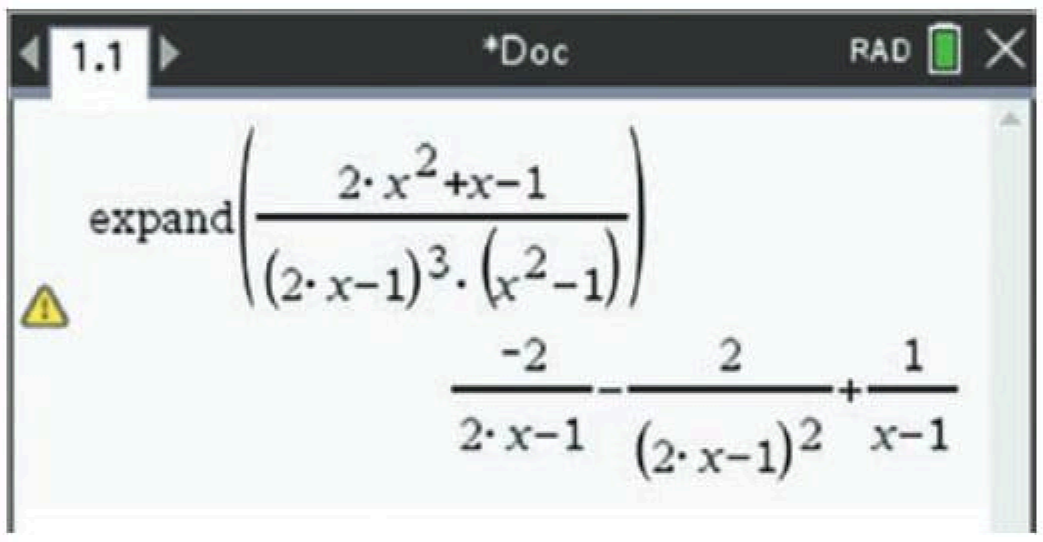
Find $\int \frac{x-5}{(x+1)(x-1)}\,dx$.

Steps	**Working**
1 Write $\frac{x-5}{(x+1)(x-1)}$ as the sum of two partial fractions.	$\frac{x-5}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$
2 Write with a single denominator and equate the numerators.	$\frac{x-5}{(x+1)(x-1)} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)}$ $x - 5 = A(x-1) + B(x+1)$
3 Solve for A and B by equating coefficients.	$A + B = 1$ ...[1] $-A + B = -5$ ...[2] [1] + [2]: $2B = -4$ $B = -2$ Sub in [1]: $A - 2 = 1$ $A = 3$
4 Substitute the values of A and B into the partial fractions.	$\frac{x-5}{(x+1)(x-1)} = \frac{3}{x+1} - \frac{2}{x-1}$
5 Use the partial fractions to find the integral.	$\int \frac{3}{(x+1)} - \frac{2}{x-1}\,dx$ $= 3\log_e\lvert x+1\rvert - 2\log_e\lvert x-1\rvert + c$

USING CAS 3 Partial fractions

Express $\dfrac{2x^2+x-1}{(2x-1)^3\left(x^2-1\right)}$ as partial fractions.

TI-Nspire



$$\text{expand}\left(\frac{2\cdot x^2+x-1}{(2\cdot x-1)^3\cdot\left(x^2-1\right)}\right)$$

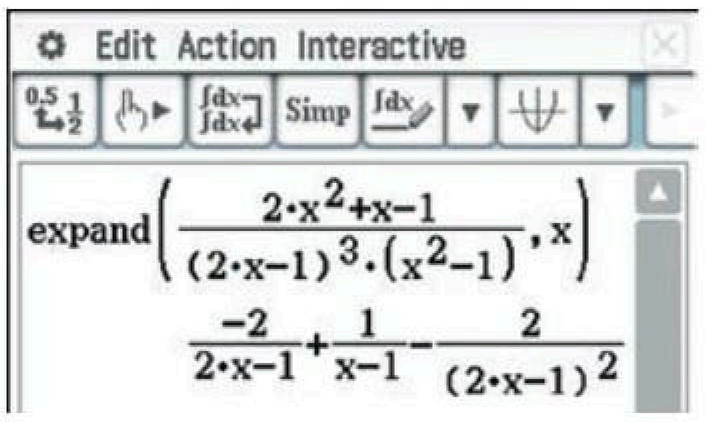
$$\frac{-2}{2\cdot x-1}-\frac{2}{(2\cdot x-1)^2}+\frac{1}{x-1}$$

1 Press **menu > Algebra > Expand**.

2 Enter the expression.

3 Press **enter**.

ClassPad



$$\text{expand}\left(\frac{2\cdot x^2+x-1}{(2\cdot x-1)^3\cdot\left(x^2-1\right)},x\right)$$

$$\frac{-2}{2\cdot x-1}+\frac{1}{x-1}-\frac{2}{(2\cdot x-1)^2}$$

1 In **Main**, enter and highlight the expression.

2 Tap **Interactive > Transformation > expand**.

3 Tap **Partial Fraction**.

4 In the **Variable**: field, enter **x**.

5 Tap **OK**.

WB

p. 156

WORKED EXAMPLE 27 Definite integration by partial fractions

Find the exact value of $\displaystyle\int_6^7 \frac{6x-18}{x^2-7x+10}\,dx$.

Steps	**Working**
1 Factorise the denominator.	$\dfrac{6x-18}{x^2-7x+10}=\dfrac{6x-18}{(x-5)(x-2)}$
2 Write as partial fractions.	$\dfrac{6x-18}{(x-5)(x-2)}=\dfrac{A}{x-5}+\dfrac{B}{x-2}$
3 Write with a single denominator and equate the numerators.	$\dfrac{6x-18}{(x-5)(x-2)}=\dfrac{A(x-2)+B(x-5)}{(x-5)(x-2)}$ $6x-18=A(x-2)+B(x-5)$
4 An alternative to equating coefficients is to substitute values of x that eliminate A or B.	Let $x=2$. Let $x=5$. $-3B=-6$ $3A=12$ $B=2$ $A=4$
5 Rewrite the integral into partial fractions.	$\displaystyle\int_6^7 \frac{6x-18}{(x-5)(x-2)}\,dx=\int_6^7 \frac{4}{x-5}+\frac{2}{x-2}\,dx$
6 Evaluate the definite integral and simplify using logarithm laws.	$=\left[4\log_e\lvert x-5\rvert+2\log_e\lvert x-2\rvert\right]_6^7$ $=4\log_e(2)+2\log_e(5)-[4\log_e(1)+2\log_e(4)]$ $=\log_e(2^4)+\log_e(5^2)-0-\log_e(4^2)$ $=\log_e\left(\dfrac{16\times 25}{16}\right)$ $=\log_e(25)$

Partial fractions with repeated linear factors

The partial fractions for fractions with a repeated linear factor in the denominator take the form

$$\frac{f(x)}{(x-a)^2} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2}$$

WORKED EXAMPLE 28 Integrating partial fractions with repeated linear factors

Find $\int \frac{x-3}{(x+2)^2}\, dx$.

Steps	**Working**
1 Write $\frac{x-3}{(x+2)^2}$ as the sum of two partial fractions.	$\frac{x-3}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$
2 Write with a single denominator and equate the numerators.	$\frac{x-3}{(x+2)^2} = \frac{A(x+2)+B}{(x+2)^2}$ $x-3 = A(x+2)+B$
3 Solve for A and B by equating coefficients.	$A = 1$ $2A + B = -3$ $2 + B = -3$ $B = -5$
4 Integrate the partial fractions.	$\int \frac{x-3}{(x+2)^2}\, dx = \int \frac{1}{x+2} - \frac{5}{(x+2)^2}\, dx$ $= \int \frac{1}{x+2} - 5(x+2)^{-2}\, dx$ $= \log_e\lvert x+2\rvert + 5(x+2)^{-1} + c$ $= \log_e\lvert x+2\rvert + \frac{5}{(x+2)} + c$

WB

p. 157

EXERCISE 7.5 Integration by partial fractions

ANSWERS p. 589

Recap

80–100% 60–79% 0–59%

1 With a suitable substitution, $\int_0^{\frac{\pi}{12}} \cos^3(4x)\, dx$ can be expressed as

A $\frac{1}{4}\int_0^{\frac{\pi}{12}} (1-u^2)\, du$

B $\frac{1}{4}\int_0^{\frac{\sqrt{3}}{2}} (1-u^2)\, du$

C $\int_0^{\frac{\sqrt{3}}{2}} (1-u^2)\, du$

D $-\frac{1}{4}\int_1^{\frac{1}{2}} (1-u^2)\, du$

E $\int_0^{\frac{\pi}{12}} (1-u^2)\, du$

2 © VCAA 2004 1IQ12 59% With a suitable substitution, $\int_0^{\frac{\pi}{3}} \cos^2(x)\sin^3(x)\, dx$ can be expressed as

A $\int_{\frac{1}{2}}^{1} u^2(1-u^2)\, du$

B $\int_1^{\frac{1}{2}} u^2(1-u^2)\, du$

C $\int_0^{\frac{\pi}{3}} u^2(1-u^2)\, du$

D $-\int_0^{\frac{\pi}{3}} u^2(1-u^2)\, du$

E $-\int_0^{\frac{\sqrt{3}}{2}} u^2(1-u^2)\, du$

Mastery

3 WORKED EXAMPLE 26 Find

a $\displaystyle\int \frac{8x-18}{(x+3)(x-4)}\,dx$ **b** $\displaystyle\int \frac{2x+8}{(x+2)(x-2)}\,dx$ **c** $\displaystyle\int \frac{8x+5}{(2x+1)(x+1)}\,dx$

4 Using CAS 3 Express $\dfrac{3x^2+10x+3}{(3x+1)^2(x^2-9)}$ in partial fraction form.

5 WORKED EXAMPLE 27 Find the exact value of $\displaystyle\int_4^5 \frac{13-x}{x^2-x-6}\,dx$.

6 WORKED EXAMPLE 28 Find

a $\displaystyle\int \frac{3x-2}{(x-1)^2}\,dx$ **b** $\displaystyle\int \frac{1-2x}{(x+4)^2}\,dx$

Exam practice

80–100% 60–79% 0–59%

7 © VCAA 2013 1Q2 71% TECH-FREE (4 marks) Evaluate $\displaystyle\int_0^1 \frac{x-5}{x^2-5x+6}\,dx$.

8 © VCAA 2011 1Q1 63% TECH-FREE (3 marks) Find an anti-derivative of $\dfrac{1+x}{9-x^2}$, $x \in R \setminus \{-3, 3\}$.

9 © VCAA 2002 1IQ10 83% $\dfrac{3x+4}{(x-4)^2}$ expressed in partial fractions has the form

A $\dfrac{A}{(x+4)}+\dfrac{B}{(x-4)}$

B $\dfrac{A}{(x-4)}+\dfrac{B}{(x-4)}$

C $\dfrac{A}{(x-4)}+\dfrac{B}{(x-4)^2}$

D $\dfrac{A}{(x-4)}+\dfrac{Bx+C}{(x-4)^2}$, $B \neq 0$

E $\dfrac{A}{(x-4)^2}+\dfrac{B}{(x-4)^2}$

10 © VCAA 2003 1IQ13 57% Which one of the following is an anti-derivative of $\dfrac{3}{x(3-x)}$ for $0 < x < 3$?

A $3[\log_e(x) - \log_e(3-x)]$

B $\log_e(x) - \log_e(3-x)$

C $\log_e(x-3) - \log_e(x)$

D $3[\log_e(x) + \log_e(3-x)]$

E $\log_e(x) + \log_e(3-x)$

11 © VCAA 2004 1IQ13 53% An anti-derivative of $\dfrac{2}{(3-x)^2} - \dfrac{1}{3-x}$, for $x < 3$, is

A $\log_e(x-3) - \dfrac{2}{x-3}$

B $\log_e(x-3) + \dfrac{2}{x-3}$

C $\log_e(3-x) - \dfrac{2}{3-x}$

D $\log_e(3-x) + \dfrac{2}{3-x}$

E $-\log_e(3-x) + \dfrac{2}{3-x}$

12 © VCAA 2007 2AQ9 38% The algebraic fraction $\dfrac{x}{3(x+c)^2}$, where c is a non-zero number, can be written in partial fraction form, where A and B are real numbers, as

A $\dfrac{A}{x+c}+\dfrac{B}{x+c}$

B $\dfrac{A}{3x+c}+\dfrac{B}{(x+c)^2}$

C $\dfrac{A}{3x+c}+\dfrac{B}{x+c}$

D $\dfrac{A}{x+c}+\dfrac{B}{(x+c)^2}$

E $\dfrac{A}{3(x+c)}+\dfrac{B}{x+c}$

13 © VCAA 2017N 2AQ5 Given that A, B, C and D are non-zero rational numbers, the expression $\dfrac{3x+1}{x(x-2)^2}$ can be expressed in partial fraction form as

A $\dfrac{A}{x}+\dfrac{B}{(x-2)}$

B $\dfrac{A}{x}+\dfrac{B}{(x-2)^2}$

C $\dfrac{A}{x}+\dfrac{B}{(x-2)}+\dfrac{C}{(x-2)^2}$

D $\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{(x-2)}$

E $\dfrac{A}{x}+\dfrac{Bx}{(x-2)}+\dfrac{Cx+D}{(x-2)^2}$

14 The value of $\displaystyle\int_{-1}^{2}\frac{3x+7}{(x+2)^2}\,dx$ is

A $3\log_e(2)+\dfrac{3}{4}$

B $3\log_e(4)+\dfrac{189}{64}$

C $\log_e(64)+\dfrac{3}{4}$

D $\log_e(4)+\dfrac{3}{4}$

E $-\dfrac{51}{16}$

15 An anti-derivative of $\dfrac{x-11}{x^2+3x-4}$ is

A $\log_e|(x+4)^3(x-1)^2|$

B $\log_e|(x+4)^3|-\log_e|(x-1)^2|$

C $3\log_e|x+4|+2\log_e|x+1|$

D $3\log_e|x-4|-2\log_e|x+1|$

E $\log_e\left|\dfrac{(x-1)^2}{(x+4)^3}\right|$

16 The value of $\displaystyle\int_{1}^{2}\frac{1}{x^2-4x-5}\,dx$ is

A $\dfrac{1}{6}\log_e(2)$

B $-\dfrac{1}{6}\log_e(2)$

C $\dfrac{1}{6}\log_e\left(\dfrac{3}{4}\right)$

D $6\log_e\left(\dfrac{1}{2}\right)$

E $-6\log_e\left(\dfrac{1}{2}\right)$

7.6 Further integration techniques

Video playlist Further integration techniques

Worksheet Integral calculus

WB p. 158

Integration by recognition

Integration by recognition involves differentiating a function and then finding the anti-derivative of another function by recognising the derivative.

WORKED EXAMPLE 29 Integration by recognition

Find the derivative of $e^{\sqrt{x}}$ and hence find $\displaystyle\int\frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$.

Steps	**Working**
1 Find the derivative.	$y=e^{\sqrt{x}}$ $\dfrac{dy}{dx}=e^{\sqrt{x}}\times\dfrac{1}{2}x^{-\frac{1}{2}}=\dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$
2 Write the corresponding integral.	$\displaystyle\int\frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx=e^{\sqrt{x}}$

3 Find the given integral.	$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx$ $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2e^{\sqrt{x}} + c$

WB

p. 159

WORKED EXAMPLE 30 Integration by splitting the numerator

Find $\int \frac{2+x}{1+x^2}\,dx$.

Steps	**Working**
1 Split the numerator to create two integrals.	$\int \frac{2+x}{1+x^2}\,dx = \int \frac{2}{1+x^2}\,dx + \int \frac{x}{1+x^2}\,dx$
2 The first integral produces $\tan^{-1}(x)$. Choose a suitable substitution for u in the second integral.	Let $u = 1 + x^2$. $\frac{du}{dx} = 2x \Rightarrow \frac{1}{2}\frac{du}{dx} = x$
3 Find the integrals.	$\int \frac{2+x}{1+x^2}\,dx = 2\int \frac{1}{1+x^2}\,dx + \frac{1}{2}\int \frac{1}{u}\frac{du}{dx}\,dx$ $= 2\int \frac{1}{1+x^2}\,dx + \frac{1}{2}\int \frac{1}{u}\,du$ $= 2\tan^{-1}(x) + \frac{1}{2}\log_e\|u\| + c$ $= 2\tan^{-1}(x) + \frac{1}{2}\log_e\left\|1+x^2\right\| + c$

EXERCISE 7.6 Further integration techniques

ANSWERS p. 589

Recap

1 Given that A, B, C and D are non-zero rational numbers, the expression $\frac{5x+1}{x(2x+1)^2}$ can be expressed in partial fraction form as

A $\frac{A}{x} + \frac{B}{(2x+1)}$

B $\frac{A}{x} + \frac{B}{(2x+1)^2}$

C $\frac{A}{x} + \frac{Bx+C}{(2x+1)^2}$

D $\frac{A}{x} + \frac{B}{(2x+1)} + \frac{Cx+D}{(2x+1)^2}$

E $\frac{A}{x} + \frac{B}{(2x+1)} + \frac{C}{(2x+1)^2}$

2 If $\int_0^2 \frac{x-3}{x^2+4x+3}\,dx = a\log_e(b) - b\log_e(a)$, the values of a and b are

A $a = 125, b = 243$

B $a = 3, b = 5$

C $a = 243, b = 125$

D $a = -3, b = 3$

E $a = -1, b = 15$

Mastery

3 WORKED EXAMPLE 29

a Find the derivative of $\sin^{-1}\left(\sqrt{x}\right)$.

b Hence find $\int \frac{1}{\sqrt{x-x^2}}\,dx$.

4 WORKED EXAMPLE 30 Find

a $\int \frac{3+2x}{9+x^2}\,dx$

b $\int \frac{2+x}{\sqrt{9-x^2}}\,dx$

Exam practice

80–100% 60–79% 0–59%

5 © VCAA 2004 II1Q1 TECH-FREE (4 marks)

a 58% Show that, for $0 < x < \frac{1}{2}$, $\frac{d}{dx}\left(\sin^{-1}\left(\sqrt{2x}\right)\right) = \frac{1}{\sqrt{2x(1-2x)}}$. 2 marks

b 69% **Hence** find the exact value of $\int_{\frac{1}{8}}^{\frac{1}{4}} \frac{1}{\sqrt{2x(1-2x)}}\,dx$. 2 marks

6 © VCAA 2012 1Q1 54% TECH-FREE (2 marks) Find the anti-derivative of $\frac{6+x}{x^2+4}$.

7 © VCAA 2006 1Q8 42% TECH-FREE (4 marks) Find the anti-derivative of $\frac{2+6x}{\sqrt{4-x^2}}$.

8 TECH-FREE (4 marks)

a Show $\frac{d\left(\tan^{-1}\left(\sqrt{x}\right)\right)}{dx} = \frac{1}{2\sqrt{x}\,(1+x)}$. 2 marks

b Hence find $\int \frac{1}{\left(x^{\frac{1}{2}} + x^{\frac{3}{2}}\right)}\,dx$. 2 marks

9 TECH-FREE (3 marks) Find $\int \frac{1+5x}{4+25x^2}\,dx$.

10 © VCAA 2021 1Q2 70% TECH-FREE (3 marks)

Evaluate $\int_0^1 \frac{2x+1}{x^2+1}\,dx$.

7.7 Integration by parts

Video playlist Integration by parts

Integration by parts is a special method used to integrate the product of two functions.

The formula for integration by parts is an integration version of the product rule for differentiation.

$$\frac{d(u \cdot v)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$u \cdot v = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

$$\int u\frac{dv}{dx}\,dx = u \cdot v - \int v\frac{du}{dx}\,dx$$

The function u is chosen so that $\frac{du}{dx}$ is simpler than u.

Integration by parts

$$\int u\frac{dv}{dx}\,dx = u \cdot v - \int v\frac{du}{dx}\,dx$$

This formula can also be written as

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

WB

p. 160

WORKED EXAMPLE 31 Integration by parts

Find $\int x \sin(2x)\,dx$.

Steps	**Working**
1 Identify $f(x)$ and $g'(x)$, where $f'(x)$ is simpler than $f(x)$.	$f(x) = x \qquad g(x) =$ $f'(x) = \qquad g'(x) = \sin(2x)$
2 Find $f'(x)$ and $g(x)$.	$f(x) = x \qquad g(x) = -\frac{1}{2}\cos(2x)$ $f'(x) = 1 \qquad g'(x) = \sin(2x)$
3 Substitute in the formula: $\int f(x)g'(x)\,dx$ $= f(x)g(x) - \int f'(x)g(x)\,dx$	$\int x\sin(2x)\,dx$ $= x \times \left(-\frac{1}{2}\cos(2x)\right) - \int 1 \times \left(-\frac{1}{2}\cos(2x)\right)dx$ $= \frac{1}{4}\sin(2x) - \frac{1}{2}x\cos(2x) + c$

WB

p. 161

WORKED EXAMPLE 32 Two-step integration by parts

Find $\int x^2 e^x\,dx$.

Steps	**Working**
1 Identify $f(x)$ and $g'(x)$, where $f(x)$ is the function with the simpler derivative.	$f(x) = x^2 \qquad g(x) =$ $f'(x) = \qquad g'(x) = e^x$
2 Find $f'(x)$ and $g(x)$.	$f(x) = x^2 \qquad g(x) = e^x$ $f'(x) = 2x \qquad g'(x) = e^x$
3 Substitute in the formula: $\int f(x)g'(x)\,dx$ $= f(x)g(x) - \int f'(x)g(x)\,dx$	$\int x^2 e^x\,dx = x^2 \times e^x - \int 2x \times e^x\,dx$ $= x^2 e^x - 2\int xe^x\,dx$
4 We need to integrate by parts again to find $\int xe^x\,dx$. Repeat the process.	$f(x) = x \qquad g(x) = e^x$ $f'(x) = 1 \qquad g'(x) = e^x$ $\int xe^x\,dx = x \times e^x - \int 1 \times e^x\,dx$ $= xe^x - e^x$
5 Substitute this result for the integral in Step 3.	$\int x^2 e^x\,dx = x^2 e^x - 2\left(xe^x - e^x\right) + c$ $= x^2 e^x - 2xe^x + 2e^x + c$

WORKED EXAMPLE 33 Integration by parts involving inverse trigonometric functions

Find $\int \arctan(x)\,dx$.

Steps	Working
1 Let $g'(x) = 1$.	$f(x) = \arctan(x) \qquad g(x) =$ $f'(x) = \qquad g'(x) = 1$
2 Find $f'(x)$ and $g(x)$.	$f(x) = \arctan(x) \qquad g(x) = x$ $f'(x) = \frac{1}{1+x^2} \qquad g'(x) = 1$
3 Substitute into the integration by parts formula.	$\int \arctan(x)\,dx = x\arctan(x) - \int \frac{x}{1+x^2}\,dx$
4 Find $\int \frac{x}{1+x^2}\,dx$ by substitution.	Let $u = 1 + x^2$ $\frac{du}{dx} = 2x$ $x = \frac{1}{2}\frac{du}{dx}$ $\int \frac{x}{1+x^2}\,dx =$ $\frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\log_e\left(1+x^2\right)$
5 Substitute the integral found into the integral equation in Step 3.	$\int \arctan(x)\,dx = x\arctan(x) - \frac{1}{2}\log_e(1+x^2) + c$

WB p. 162

WORKED EXAMPLE 34 Integration by parts involving exponential functions

WB p. 163

Find $\int e^x \cos(x)\,dx$

Steps	Working
1 Let $f(x) = e^x$ and $g'(x) = \cos(x)$.	$f(x) = e^x \qquad g(x) =$ $f'(x) = \qquad g'(x) = \cos(x)$
2 Find $f'(x)$ and $g(x)$.	$f(x) = e^x \qquad g(x) = \sin(x)$ $f'(x) = e^x \qquad g'(x) = \cos(x)$
3 Substitute into the integration by parts formula.	$\int e^x \cos(x)\,dx = e^x \sin(x) - \int e^x \sin(x)\,dx$
4 We need to integrate by parts again to find $\int e^x \sin(x)\,dx$. Repeat the process.	$f(x) = e^x \qquad g(x) = -\cos(x)$ $f'(x) = e^x \qquad g'(x) = \sin(x)$ $\int e^x \sin(x)\,dx = -e^x\cos(x) + \int e^x \cos(x)\,d$
5 Substitute into the integral equation in Step 3 and solve.	$\int e^x \cos(x)\,dx$ $= e^x\sin(x) - \left(-e^x\cos(x) + \int e^x\cos(x)\,dx\right)$ $\int e^x \cos(x)\,dx$ $= e^x\sin(x) + e^x\cos(x) - \int e^x\cos(x)\,dx$ $2\int e^x\cos(x)\,dx = e^x\sin(x) + e^x\cos(x)$ $\int e^x\cos(x)\,dx = \frac{1}{2}e^x\sin(x) + \frac{1}{2}e^x\cos(x) + c$

Video
VCE question analysis: Integration

VCE QUESTION ANALYSIS

© VCAA 2007 2BQ2 2007 Examination 2 Section B Question 2 (12 marks)

Let $f(x) = x \arctan(x)$.

a Find $f'(x)$, and calculate the slope of the graph of f at $x = 0$. 2 marks

b Sketch the curves $y = x \arctan(x)$ and $y = \arctan(x)$ over the domain [0, 1], clearly labelling each graph. 2 marks

c **i** Write down a definite integral which gives the area enclosed by the graphs of $y = x \arctan(x)$, $y = 0$ and $x = 1$. 1 mark

ii Find the area defined in part **i** correct to three decimal places. 1 mark

d Use the result for $f'(x)$ in part **a** to show that an anti-derivative of $\arctan(x)$ is $x \arctan(x) - \frac{1}{2}\log_e(1 + x^2)$. 2 marks

e Use the anti-derivative given in part **d** to find the exact area enclosed by the graphs of $y = \arctan(x)$, $y = 0$ and $x = 1$. 2 marks

f Find the area enclosed by the curves $y = x \arctan(x)$ and $y = \arctan(x)$. Give the answer correct to two decimal places. 2 marks

Reading the question

- Highlight the domain of the functions in the sketch graph.
- Highlight the required answer. The answers to parts of this question require an integral equation, a graph, a value in exact form or a decimal approximation to a given number of decimal places.
- Highlight the instructions for the question. 'Use a previous answer to show or find' indicates the questions must be answered using this result as the starting point.

Thinking about the question

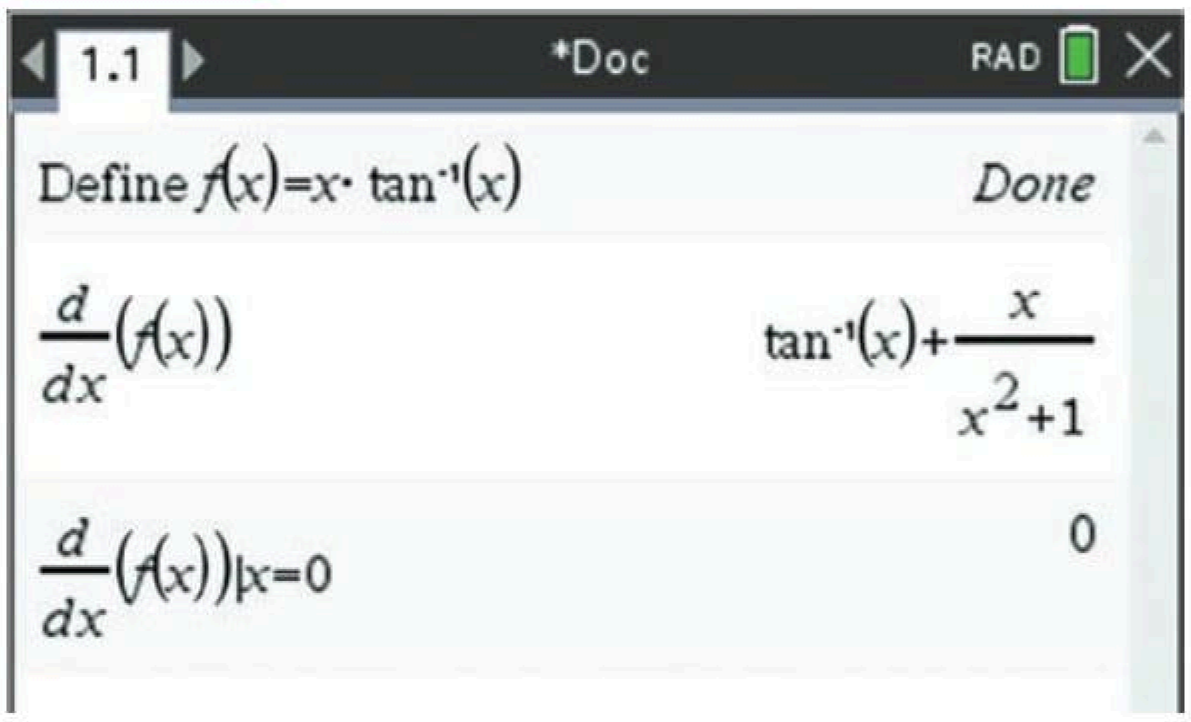
- The question requires the use of differentiation and anti-differentiation.
- You will need to understand how the derivative of a function relates to the original function.
- You will need to be able to use the anti-derivative to calculate areas. These areas may be between two curves or between a curve and the x-axis.
- You will also need to find definite and indefinite integrals using 'integration by recognition'.

Worked solution (✓ = 1 mark)

a $f'(x) = \tan^{-1}(x) + \frac{x}{x^2+1}$ ✓

Slope $= f'(0) = 0$ ✓

TI-Nspire

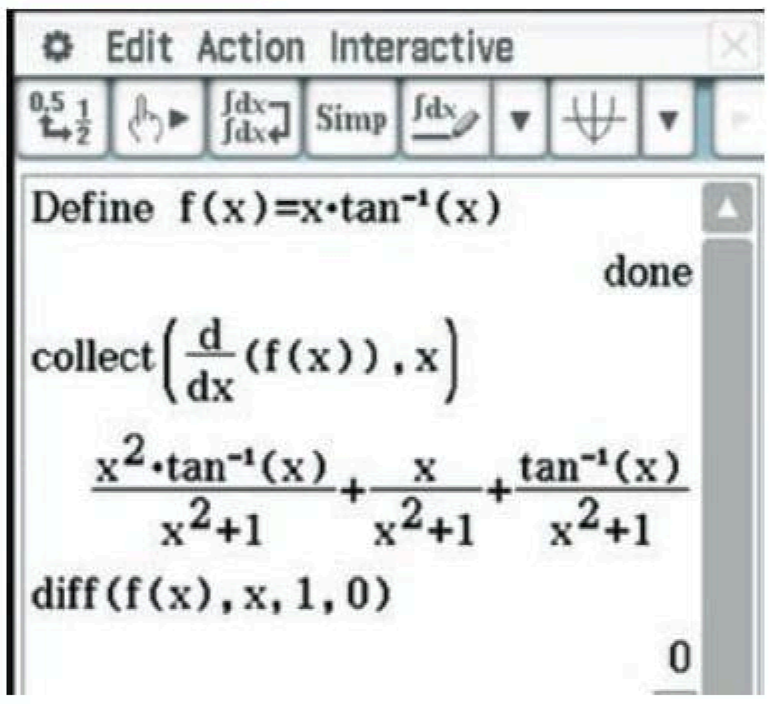


Define $f(x)=x\cdot\tan^{-1}(x)$ *Done*

$\frac{d}{dx}(f(x))$ $\tan^{-1}(x)+\frac{x}{x^2+1}$

$\frac{d}{dx}(f(x))|x=0$ 0

ClassPad



Define f(x)=x·tan⁻¹(x)

done

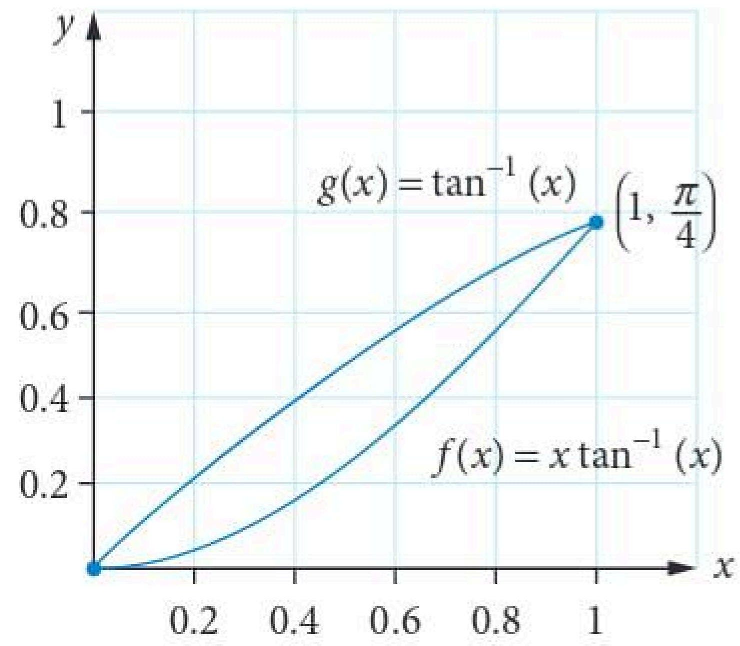
collect$\left(\frac{d}{dx}(f(x)),x\right)$

$\frac{x^2\cdot\tan^{-1}(x)}{x^2+1}+\frac{x}{x^2+1}+\frac{\tan^{-1}(x)}{x^2+1}$

diff(f(x),x,1,0)

0

b

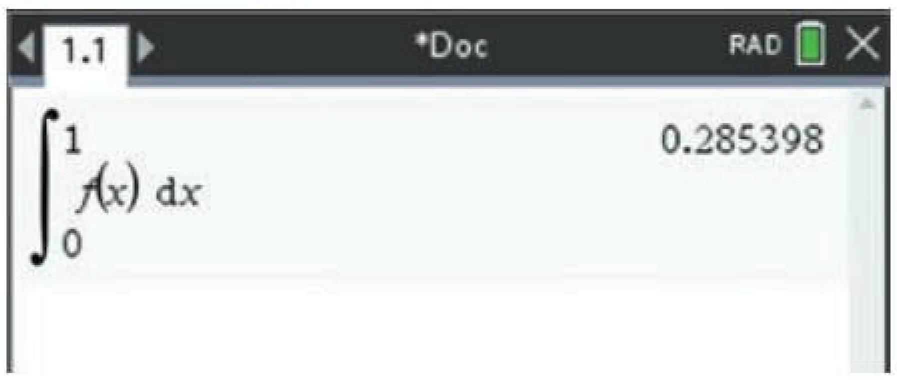


Graph of $f(x)$ ✓ Graph of $g(x)$ ✓

c **i** Area $= \int_0^1 x\tan^{-1}(x)\,dx$ ✓

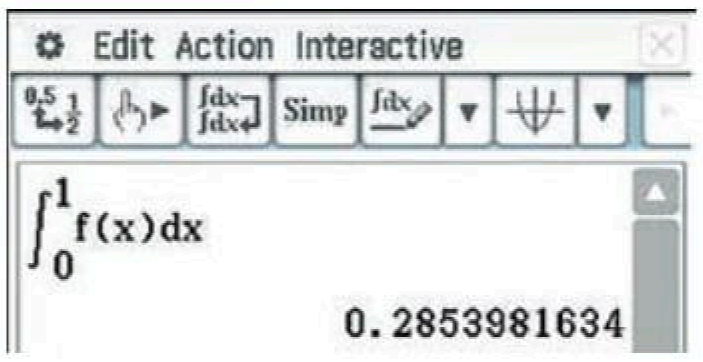
ii Area = **0.285 units^2** ✓

TI-Nspire



$\int_0^1 f(x)\,dx$ 0.285398

ClassPad



$\int_0^1 f(x)dx$

0.2853981634

d

$$\frac{d(x\tan^{-1}(x))}{dx} = \tan^{-1}(x) + \frac{x}{x^2+1} \text{ from part } \mathbf{a}$$

$$x\tan^{-1}(x) = \int\left(\tan^{-1}(x) + \frac{x}{x^2+1}\right)dx \checkmark$$

$$x\tan^{-1}(x) = \int \tan^{-1}(x)\,dx + \int \frac{x}{x^2+1}\,dx$$

$$\int \tan^{-1}(x)\,dx = x\tan^{-1}(x) - \int \frac{x}{x^2+1}\,dx$$

Let $u = x^2 + 1$ $\quad \frac{du}{dx} = 2x, \quad \frac{1}{2}\frac{du}{dx} = x$

$$\int \tan^{-1}(x)\,dx = x\tan^{-1}(x) - \frac{1}{2}\int \frac{1}{u}\,du$$

$$\int \tan^{-1}(x)\,dx = x\tan^{-1}(x) - \frac{1}{2}\log_e|u|, \quad u = x^2 + 1 > 0 \text{ so } |u| = u$$

$$\boldsymbol{\int \tan^{-1}(x)\,dx = x\tan^{-1}(x) - \frac{1}{2}\log_e(x^2+1)} \checkmark$$

e Area $= \int_0^1 \tan^{-1}(x)\,dx$

$$= \left[x\tan^{-1}(x) - \frac{1}{2}\log_e(x^2+1)\right]_0^1 \checkmark$$

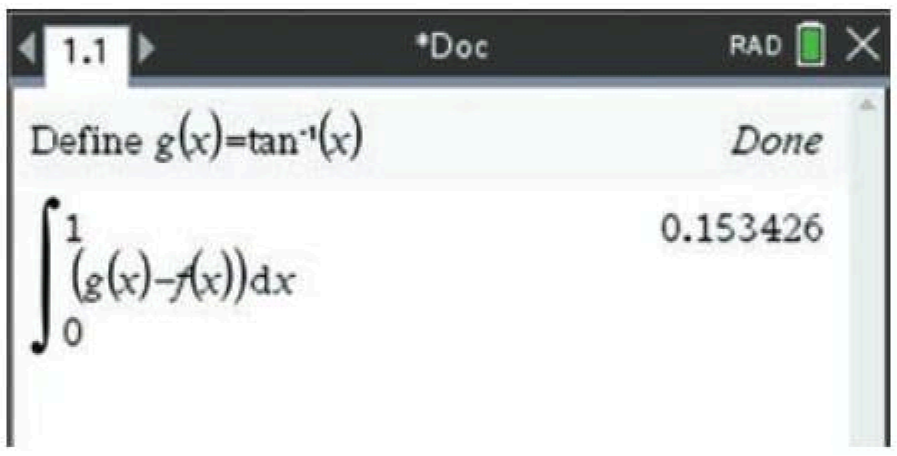
$$= 1\tan^{-1}(1) - \frac{1}{2}\log_e(1^2+1) - \left[0 - \frac{1}{2}\log_e(1)\right]$$

$$\boldsymbol{= \frac{\pi}{4} - \frac{1}{2}\log_e(2) \text{ units}^2} \checkmark$$

f Area $= \int_0^1 \left(\tan^{-1}(x) - x\tan^{-1}(x)\right) dx$ ✓

Area = 0.15 unit^2 ✓

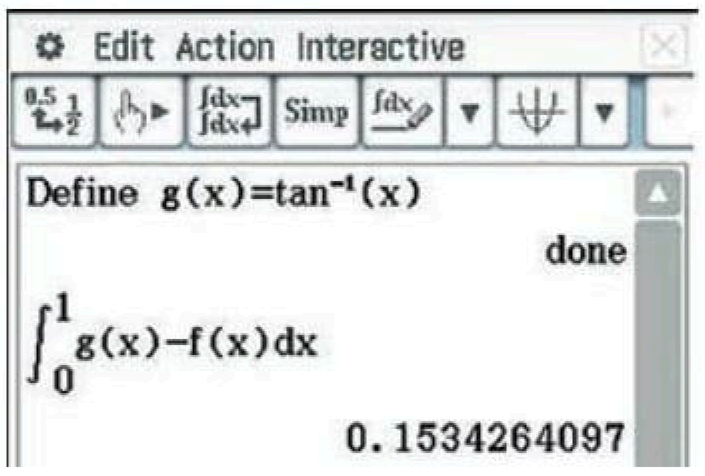
TI-Nspire



Define $g(x)=\tan^{-1}(x)$ Done

$\int_0^1 (g(x)-f(x))dx$ 0.153426

ClassPad



Define $g(x)=\tan^{-1}(x)$

done

$\int_0^1 g(x)-f(x)dx$

0.1534264097

Student performance

80–100% 60–79% 0–59%

a 76% Students answered this question well.

b 75% Some students lost marks for drawing beyond the domain, incorrect concavity, wrong intersection points and not labelling the graphs.

c **i** 74% Some students misread the question and gave an integral for the area between the two curves.

ii 67% Students who answered part **i** correctly were able to follow through with the calculation of the definite integral.

d 65% Most students correctly separated the integral into two parts. Some errors were made with the integration by substitution.

e 72% Some students neglected to evaluate $\tan^{-1}(1)$ as $\frac{\pi}{4}$.

f 71% Some students reproduced the answer to part **c** and did not calculate the area between the two curves.

EXERCISE 7.7 Integration by parts

ANSWERS p. 589

Recap

1 If $\frac{d\left(\tan^{-1}\left(\sqrt{x}\right)\right)}{dx} = \frac{1}{2\sqrt{x}\,(x+1)}$, then $\int \frac{1}{x^{\frac{3}{2}} + x^{\frac{1}{2}}}\,dx$ is equal to

A $\tan^{-1}(\sqrt{x}) + c$

B $2\tan^{-1}(\sqrt{x}) + c$

C $\frac{1}{2}\tan^{-1}(\sqrt{x}) + c$

D $2\tan^{-1}(x^{\frac{3}{2}}) + c$

E $\frac{1}{5x^{\frac{5}{2}} + 3x^{\frac{3}{2}}} + c$

2 $\int \frac{a+bx}{4+x^2}\,dx$ is equal to

A $a\tan^{-1}\left(\frac{x}{2}\right) + \int \frac{bx}{x^2+4}\,dx + c$

B $\int \frac{a}{4+x^2}\,dx + b\log_e\left(x^2+4\right) + c$

C $\frac{a}{2}\tan^{-1}\left(\frac{x}{2}\right) + \int \frac{b}{u}\,du + c$

D $\frac{a}{2}\tan^{-1}\left(\frac{x}{2}\right) + \int \frac{b}{2u}\,du + c$

E $a\tan^{-1}\left(\frac{x}{2}\right) + \int \frac{b}{x^2+4}\,dx + c$

Mastery

3 WORKED EXAMPLE 31 Find each anti-derivative.

a $\int x\sin(x)\,dx$ **b** $\int x\cos(3x)\,dx$ **c** $\int xe^x\,dx$ **d** $\int x\sec^2(x)\,dx$

4 WORKED EXAMPLE 32 Find each anti-derivative.

a $\int x^2e^{2x}\,dx$ b $\int x^2\sin(x)\,dx$ c $\int x^2\cos(x)\,dx$

5 WORKED EXAMPLE 33 Find each anti-derivative.

a $\int \arcsin(x)\,dx$ b $\int \arccos(x)\,dx$ c $\int \log_e(x)\,dx$

6 WORKED EXAMPLE 34 Find each anti-derivative.

a $\int e^x\cos(2x)\,dx$ b $\int e^x\sin(x)\,dx$ c $\int e^{2x}\sin(x)\,dx$

Exam practice

7 TECH-FREE (3 marks) Find the value of $\int_0^{\frac{\pi}{8}} x\sin(4x)\,dx$.

8 TECH-FREE (3 marks) Find the value of $\int_0^2 x^2e^{4x}\,dx$.

9 TECH-FREE (3 marks) Find the value of $\int_0^{\frac{1}{4}} \arcsin(2x)\,dx$.

10 TECH-FREE (3 marks) Find the value of $\int_0^{\frac{\pi}{8}} e^x\sin(2x)\,dx$.

11 $\int 2x^2\cos\left(\frac{x}{2}\right)dx$ is equal to

A $4x^2\sin\left(\frac{x}{2}\right)-\frac{4}{3}\int x^3\sin\left(\frac{x}{2}\right)dx$

B $x^2\sin\left(\frac{x}{2}\right)-2\int x\sin\left(\frac{x}{2}\right)dx$

C $4x^2\sin\left(\frac{x}{2}\right)-8\int x\sin\left(\frac{x}{2}\right)dx$

D $4x^2\sin\left(\frac{x}{2}\right)+8\int x\cos\left(\frac{x}{2}\right)dx$

E $4x^2\sin\left(\frac{x}{2}\right)+8\int x\sin\left(\frac{x}{2}\right)dx$

12 $\int 8x^3e^{4x}\,dx$ is equal to

A $24x^2e^{4x}-24\int x^2e^{4x}\,dx$

B $2x^3e^{4x}-6\int x^2e^{4x}\,dx$

C $32x^3e^{4x}-24\int x^2e^{4x}\,dx$

D $8x^3e^{4x}-24\int x^2e^{4x}\,dx$

E $8x^3e^{4x}-2\int x^2e^{4x}\,dx$

7 Chapter summary

Basic anti-derivatives

$f(x)$	$\int f(x)\,dx$
x^n	$\frac{x^{n+1}}{n+1}+c$
$(ax+b)^n$	$\frac{(ax+b)^{n+1}}{a(n+1)}+c$
e^{ax}	$\frac{1}{a}e^{ax}+c$
$\frac{1}{x}$	$\log_e\lvert x\rvert+c$
$\frac{1}{ax+b}$	$\frac{1}{a}\log_e\lvert ax+b\rvert+c$
$\sin(ax)$	$-\frac{1}{a}\cos(ax)+c$
$\cos(ax)$	$\frac{1}{a}\sin(ax)+c$
$\sec^2(ax)$	$\frac{1}{a}\tan(ax)+c$

The indefinite integral

For a function $F(x)$,

$$\int f(x)\,dx = F(x)+c, \text{ where } F'(x)=f(x).$$

The definite integral

- Find an anti-derivative $F(x)$ of $f(x)$.
- $\int_a^b f(x)\,dx = F(b)-F(a)$

The integral of $\frac{1}{\sqrt{a^2-x^2}}$

- $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+c,$
  $a>0, x\in(-a,a)$
- $\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right)+c,$
  $a>0, x\in(-a,a)$

The integral of $\frac{a}{a^2+x^2}$

- $\int \frac{a}{a^2+x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right)+c,$
  $x\in R$

Integration by substitution

$$\int g(u)\frac{du}{dx}\,dx = \int g(u)\,du$$

One part of the integrand is the derivative of the other part of the integrand u.

Linear substitution method

- Substitute u for the inside of the more complex part of the integrand.
- Transpose to express x in terms of u and substitute for x in the other part of the integrand.
- Find $\frac{du}{dx}$ and write an equation where $k\frac{du}{dx}=1$.
- Substitute $k\frac{du}{dx}$ for the factor of 1.

Integrals of odd powers of sine and cosine functions

- Write the integrand as
  $\sin(x)\times$ an even power of $\sin(x)$ or
  $\cos(x)\times$ an even power of $\cos(x)$.
- Substitute: $\sin^2(x)=1-\cos^2(x)$ or $\cos^2(x)=1-\sin^2(x)$.
- Integrate by substitution.

Integrals of even powers of sine and cosine functions

Use the formulas:

$$\cos^2(x)=\frac{1}{2}[1+\cos(2x)]$$

$$\sin^2(x)=\frac{1}{2}[1-\cos(2x)]$$

Partial fractions

$$\frac{f(x)}{(x-a)(x-b)}=\frac{A}{(x-a)}+\frac{B}{(x-b)}$$

$$\frac{f(x)}{(x-a)^2}=\frac{A}{(x-a)}+\frac{B}{(x-a)^2}$$

Integration by parts

$$\int u\frac{dv}{dx}\,dx = u\cdot v-\int v\frac{du}{dx}\,dx$$

This formula can also be written as

$$\int f(x)g'(x)\,dx = f(x)g(x)-\int f'(x)g(x)\,dx.$$

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 © VCAA 2019N 1Q4 (3 marks) Evaluate $\int_{e^3}^{e^4} \frac{1}{x\log_e(x)}\,dx$.

2 (3 marks) Find $\int \frac{1+8x}{1+16x^2}\,dx$.

3 (4 marks) Express $\frac{1+\sqrt{3}i}{-2\sqrt{3}+2i}$ in polar form.

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 © VCAA 2018 2AQ3 Given that A, B, C and D are non-zero rational numbers, the expression

$\dfrac{2x^2+3x+1}{(2x+1)^3\left(x^2-1\right)}$ can be expressed in partial fraction form as

A $\dfrac{A}{2x+1}+\dfrac{B}{x-1}+\dfrac{C}{x+1}$

B $\dfrac{A}{2x+1}+\dfrac{B}{(2x+1)^2}+\dfrac{C}{(2x+1)^3}+\dfrac{Dx}{x^2-1}$

C $\dfrac{A}{2x+1}+\dfrac{Bx+C}{x^2-1}$

D $\dfrac{A}{2x+1}+\dfrac{B}{(2x+1)^2}+\dfrac{C}{x-1}$

E $\dfrac{A}{2x+1}+\dfrac{Bx+C}{(2x+1)^2}+\dfrac{D}{x-1}$

2 © VCAA 2018 2AQ8 Using a suitable substitution, $\int_0^{\frac{\pi}{6}}\tan^2(x)\sec^2(x)\,dx$ can be expressed as

A $\int_0^{\frac{1}{\sqrt{3}}}\left(u^4+u^2\right)du$

B $\int_1^{\frac{2}{\sqrt{3}}}\left(u^4+u^2\right)du$

C $\int_0^{\frac{1}{\sqrt{3}}}u\,du$

D $\int_0^{\frac{\pi}{6}}u^2\,du$

E $\int_0^{\frac{1}{\sqrt{3}}}u^2\,du$

3 © VCAA 2002 1IQ12 An anti-derivative of $\dfrac{e^{2x}}{2e^{2x}-1}$ $\left(\text{for } e^{2x}>\dfrac{1}{2}\right)$ is

A $\dfrac{1}{2}\left(x-e^{2x}\right)$

B $4\log_e(2e^{2x}-1)$

C $2\log_e(2e^{2x}-1)$

D $\dfrac{1}{2}\log_e\left(2e^{2x}-1\right)$

E $\dfrac{1}{4}\log_e\left(2e^{2x}-1\right)$

4 If $z^2=16\left[\cos\left(\dfrac{2\pi}{3}\right)+i\sin\left(\dfrac{2\pi}{3}\right)\right]$ then z is equal to

A $-4+4\sqrt{3}i$ or $4-4\sqrt{3}i$

B $-2-2\sqrt{3}i$ or $2+2\sqrt{3}i$

C $-2\sqrt{3}+2i$ or $2\sqrt{3}-2i$

D $2\sqrt{3}+2i$ or $2\sqrt{3}-2i$

E $-1+\sqrt{3}i$ or $1-1\sqrt{3}i$

5 The distance between the points (3, 4, −2) and (1, 2, 1) is

A $\sqrt{17}$ **B** 3 **C** $\sqrt{21}$ **D** $\sqrt{53}$ **E** 7

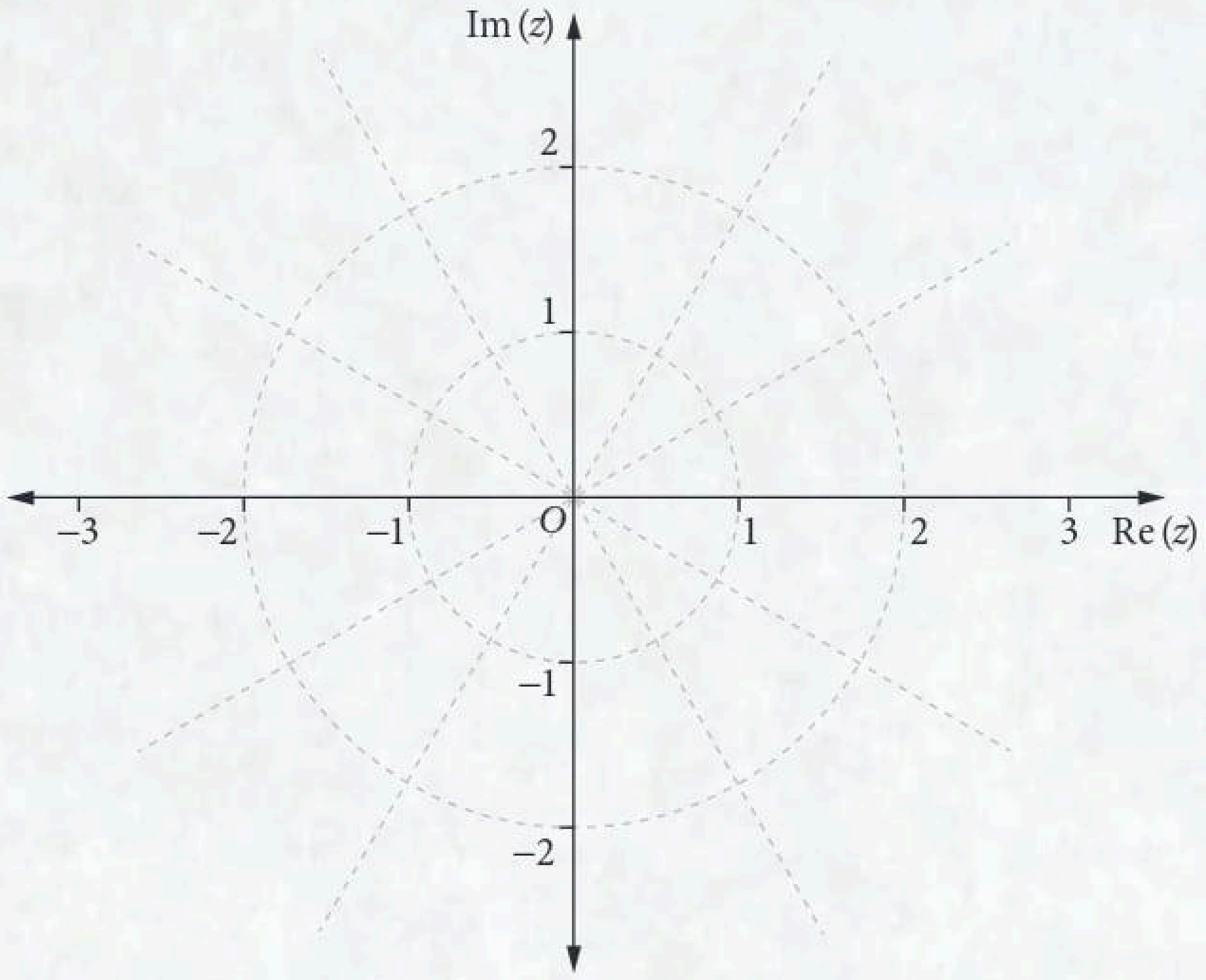
Section B 3 questions

15 marks

1 © VCAA 2007 2BQ1 (5 marks)

a Express $z_1=-\sqrt{3}+i$ in polar form. 2 marks

b Copy the Argand diagram and on it plot and label z_1. 1 mark



c By solving $z^2 - 2\sqrt{3}z + 4 = 0$ algebraically, show that the roots of this equation are $z = \sqrt{3} + i$ and $z = \sqrt{3} - i$. 2 marks

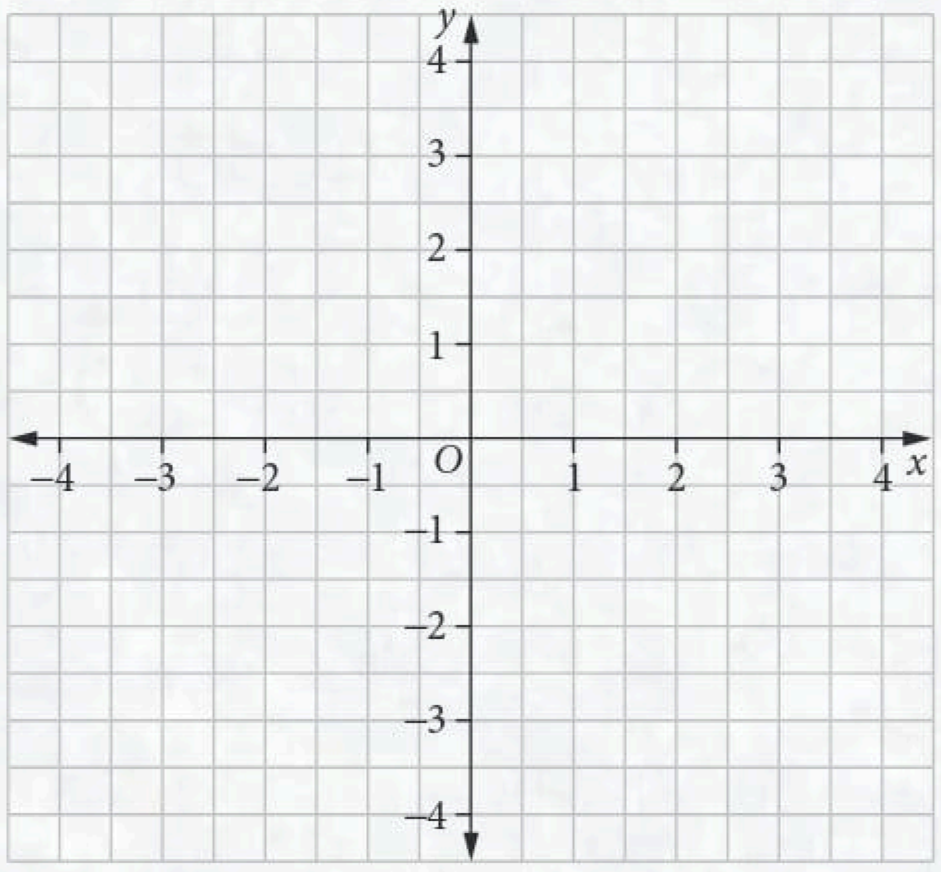
2 © VCAA 2021N 2BQ1 (8 marks) A curve is defined parametrically by $x = \sec(t)$, $y = \operatorname{cosec}(t)$, where $t \in \left[0, \frac{\pi}{2}\right]$.

a Show that the curve can be represented in cartesian form by the relation $y = \frac{x}{\sqrt{x^2 - 1}}$. 2 marks

b State the domain and range of the relation given by $y = \frac{x}{\sqrt{x^2 - 1}}$ for this curve. 2 marks

c Use the derivative of the relation to show that the gradient of the curve is negative at all points on the curve. 2 marks

d Copy the axes below and on them sketch the graph of the relation, labelling any asymptotes with their equations. 2 marks



3 (2 marks)

a Using a suitable substitution, write in terms of the variable u, an equivalent integral for $\int \frac{x}{\sqrt{x^2 - 1}}\, dx$. 1 mark

b Find an anti-derivative of $\frac{x}{\sqrt{x^2 - 1}}$ using the substitution in part **a**. 1 mark

CHAPTER

8

AREAS AND VOLUMES OF INTEGRATION

Study Design coverage

AREA OF STUDY 4: CALCULUS

Differential calculus and integral calculus

- the relationship between the graph of a function and the graphs of its anti-derivative functions
- application of integration, areas of regions bounded by curves, arc lengths for parametrically determined curves, surface area of solids of revolution, volumes of solids of revolution of a region about either coordinate axis.

VCE Mathematics Study Design 2023–2027 p. 111, © VCAA 2022

Video playlists (6):

- **8.1** Areas under curves
- **8.2** The graph of the anti-derivative
- **8.3** Volumes of solids of revolution
- **8.4** Arc lengths of curves
- **8.5** Surface area of solids of revolution
- **VCE question analysis** Areas and volumes of integration

Worksheets (9):

- **8.1** Calculating physical areas • Areas of integration • Areas between curves 1 • Areas between curves 2 • Calculating areas between curves • Sums and differences of areas
- **8.3** Volumes of integration • Volumes • Solids of revolution

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

8.1 Areas under curves

Video playlist Areas under curves

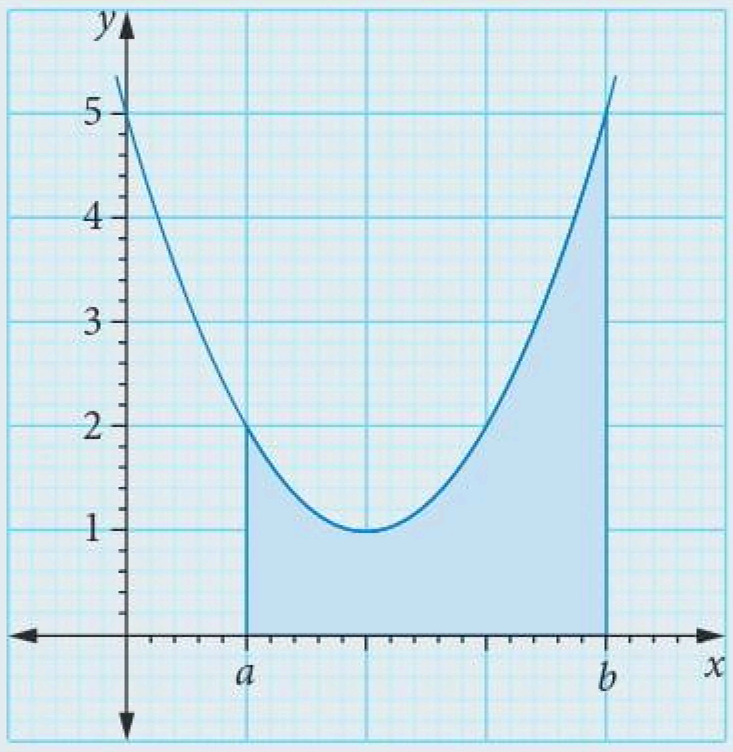
Worksheets Calculating physical areas

Areas of integration

Definite integrals

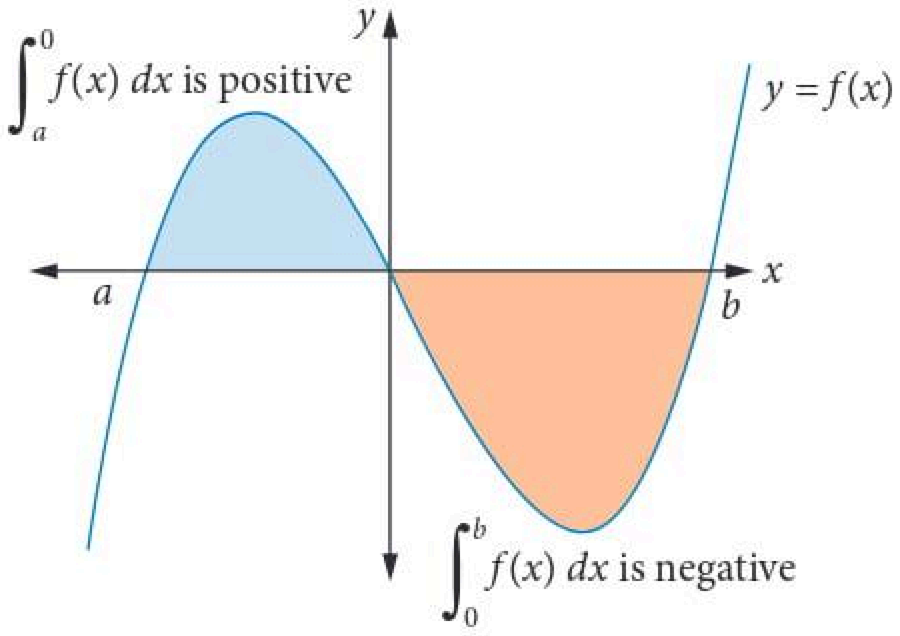
The definite integral $\int_a^b f(x)\,dx = F(a) - F(b)$, where $F(x)$ is an anti-derivative of $y = f(x)$, is the area between the curve $f(x)$, the x-axis and the lines $x = a$ and $x = b$, where $f(x) \geq 0$ for all $x \in [a,b]$.

This relationship between the anti-derivative and the area under a curve is called the **fundamental theorem of calculus**.



The area between a curve and the *x*-axis

When calculating areas, the integral representing the area below the x-axis produces a negative answer and the integral representing the area above the x-axis produces a positive answer.



The area shown can be written as

$$A = \int_a^0 f(x)\,dx - \int_0^b f(x)\,dx \quad \text{or}$$

$$A = \int_a^0 f(x)\,dx + \int_b^0 f(x)\,dx \quad \text{or}$$

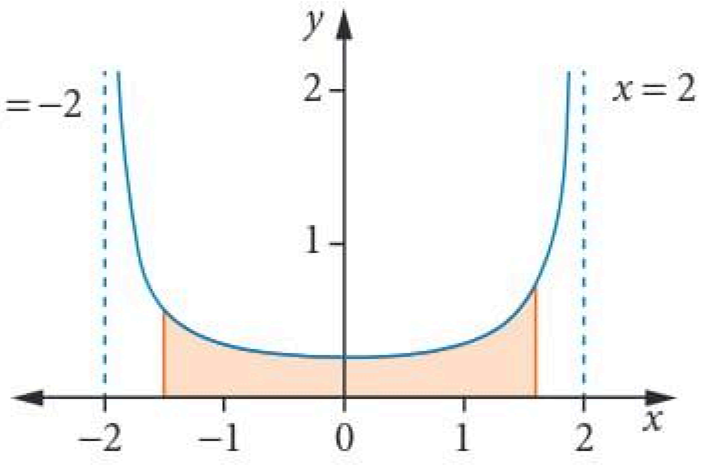
$$A = \int_a^0 f(x)\,dx + \left|\int_0^b f(x)\,dx\right|.$$

WB

p. 164

WORKED EXAMPLE 1 Finding the area between a curve and the *x*-axis

Find the area between the curve $f(x) = \dfrac{1}{\sqrt{4-x^2}}$ and the *x*-axis from $x = -\sqrt{3}$ to $x = \sqrt{3}$.

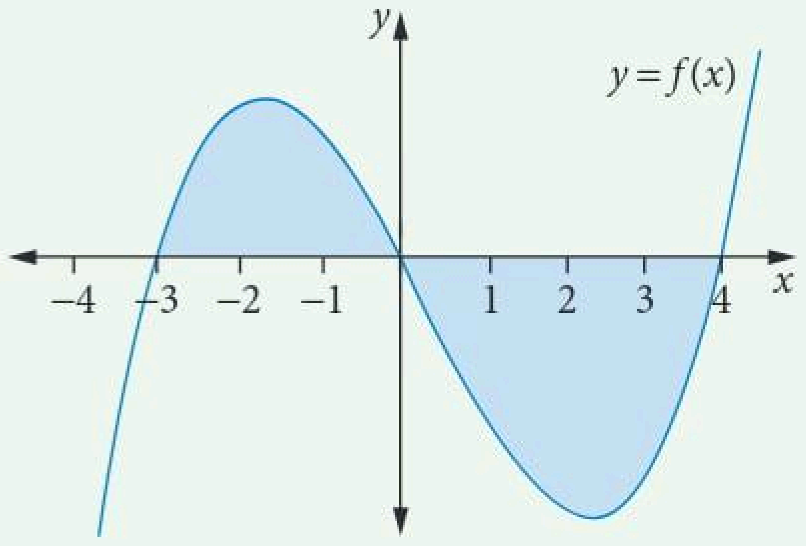
Steps	**Working**
1 Graph the function and identify the area required.	
2 Write a definite integral that represents the area and simplify the integral by symmetry.	$\text{Area} = \int_{-\sqrt{3}}^{\sqrt{3}} \dfrac{1}{\sqrt{4-x^2}}\,dx$ $= 2\int_{0}^{\sqrt{3}} \dfrac{1}{\sqrt{4-x^2}}\,dx$
3 Find the anti-derivative and evaluate the definite integral.	$= 2\left[\sin^{-1}\left(\dfrac{x}{2}\right)\right]_0^{\sqrt{3}}$ $= 2\left(\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right) - \sin^{-1}(0)\right)$ $= 2 \times \dfrac{\pi}{3}$ $\text{Area} = \dfrac{2\pi}{3}$ square units

WB

p. 165

WORKED EXAMPLE 2 Finding the area bounded by a curve and the *x*-axis

Find the area bounded by the graph of $f(x) = x(x-4)(x+3)$ and the *x*-axis.

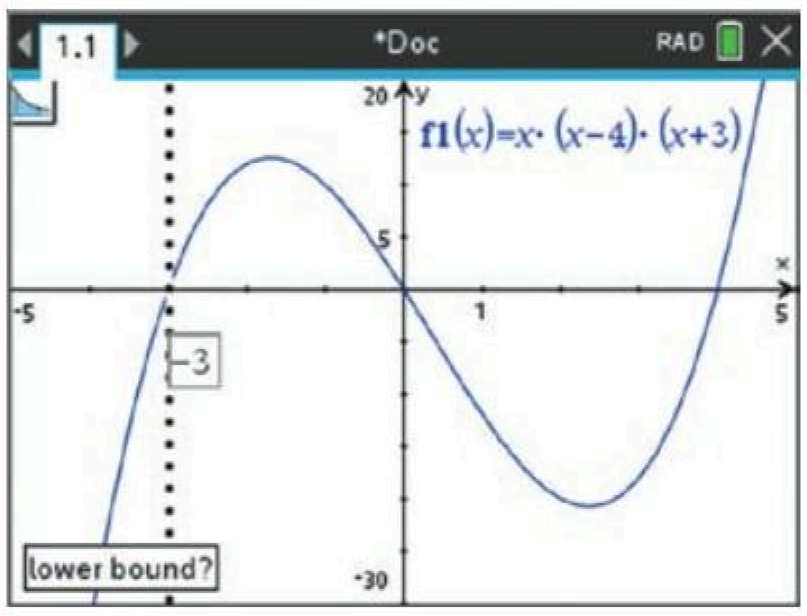


Steps	**Working**
1 Write a definite integral that represents the area.	$\text{Area} = \int_{-3}^{0} x(x-4)(x+3)\,dx - \int_{0}^{4} x(x-4)(x+3)\,dx$
2 Find the anti-derivative and evaluate the definite integral.	$= \int_{-3}^{0} x^3 - x^2 - 12x\,dx - \int_{0}^{4} x^3 - x^2 - 12x\,dx$ $= \left[\dfrac{x^4}{4} - \dfrac{x^3}{3} - 6x^2\right]_{-3}^{0} - \left[\dfrac{x^4}{4} - \dfrac{x^3}{3} - 6x^2\right]_{0}^{4}$ $= -\dfrac{81}{4} - 9 + 54 - 64 + \dfrac{64}{3} + 96$ $= 78\dfrac{1}{12}$ units^2

USING CAS 1 Finding the area bounded by a curve and the *x*-axis

Find the area between $f(x) = x(x - 4)(x + 3)$ and the *x*-axis.

TI-Nspire

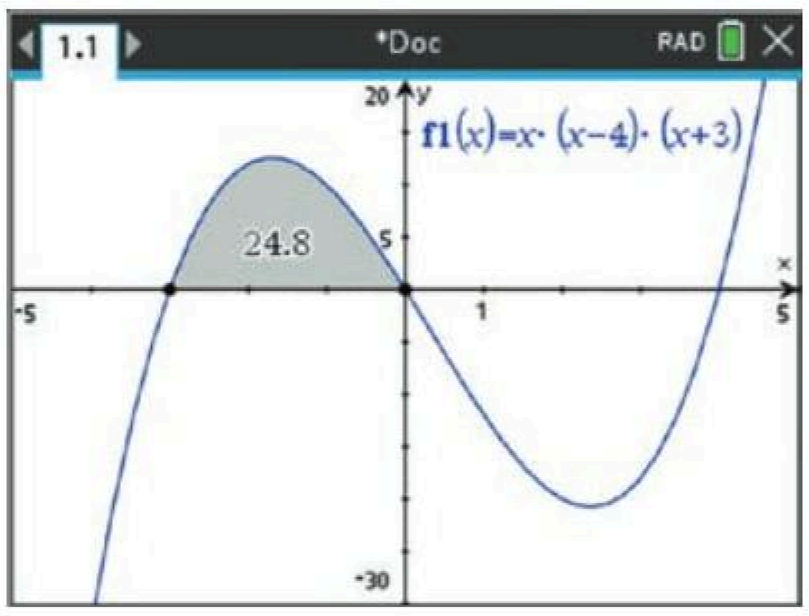


1 Add a **Graphs** page and graph the function.

2 Adjust the window settings to suit.

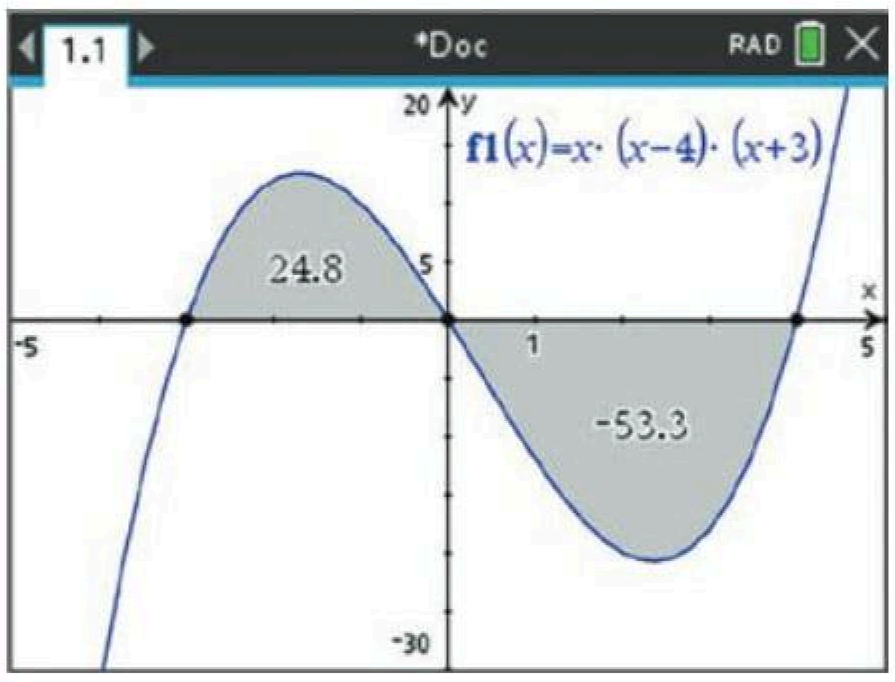
3 Press **menu > Analyze Graph > Integral**.

4 When prompted for the **lower bound?**, enter **–3**.



5 When prompted for the **upper bound?**, enter **0**.

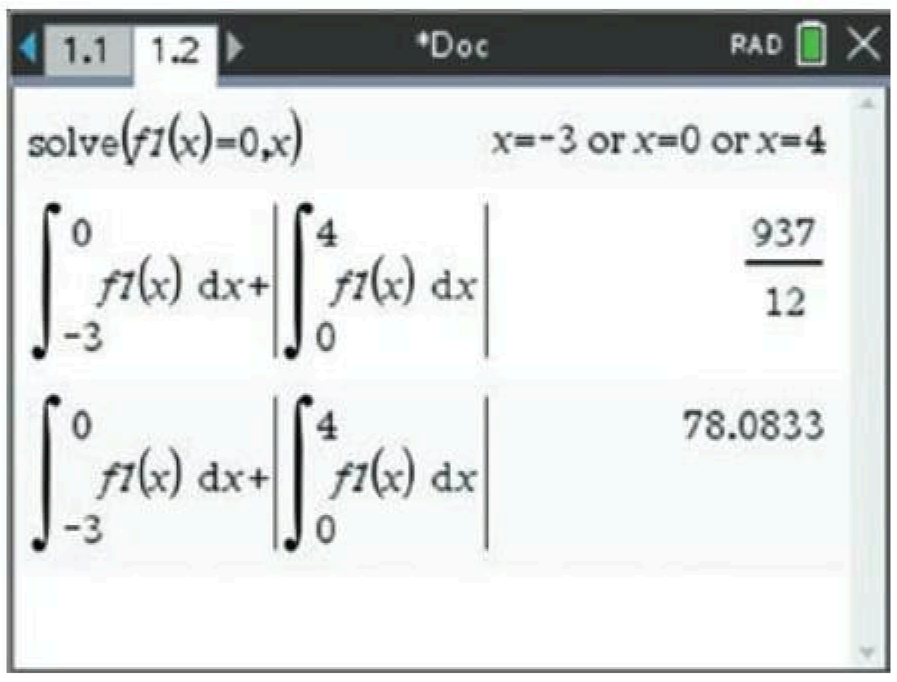
6 The area between the curve and the *x*-axis from –3 and 0 will be displayed.



7 Repeat to find the area between the curve and the *x*-axis from 0 to 4. This value is negative as it is beneath the *x*-axis.

8 The total area is 24.8 + 53.3 = 78.1.

9 To find the exact area, add a **Calculator** page.



solve$(f1(x)=0,x)$ $x=-3$ or $x=0$ or $x=4$

$$\int_{-3}^{0} f1(x)\,dx + \left|\int_{0}^{4} f1(x)\,dx\right| \qquad \frac{937}{12}$$

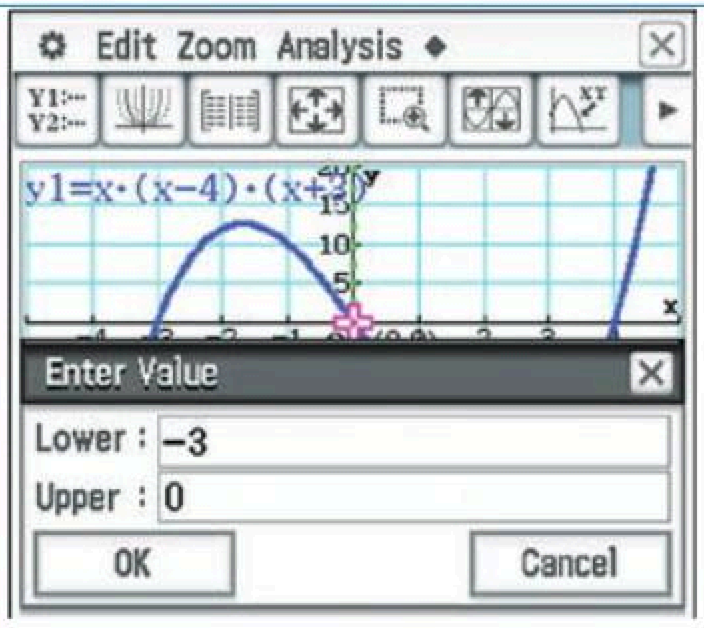
$$\int_{-3}^{0} f1(x)\,dx + \left|\int_{0}^{4} f1(x)\,dx\right| \qquad 78.0833$$

10 Solve ***f*1(*x*) = 0** to determine the *x*-intercepts.

11 Add the integral from –3 to 0 to the absolute value of the integral from 0 to 4, as shown above.

12 Press **ctrl + =** to confirm the approximate answer from the **Graphs** page.

ClassPad



Lower : −3

Upper : 0

1 Open the **Graph&Table** application and graph the function.

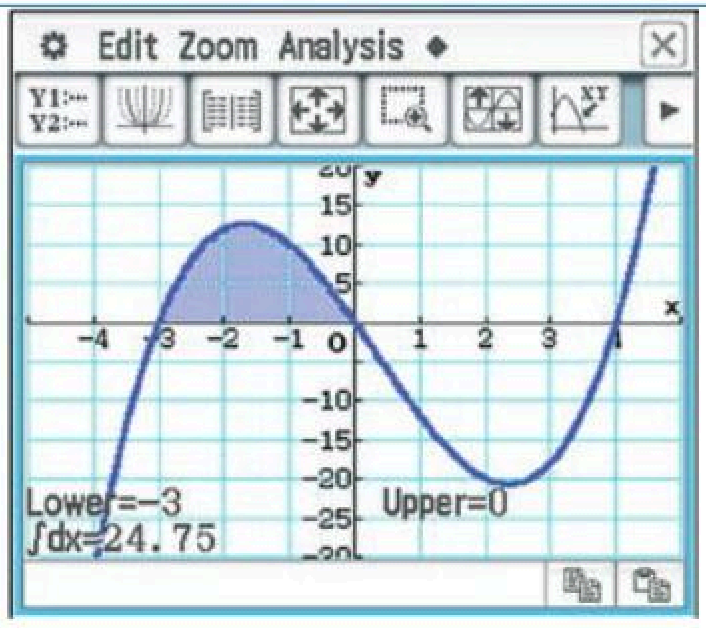
2 Adjust the window settings to suit.

3 Tap **Analysis > G-Solve > Integral > ∫dx**.

4 Enter **– 3**.

5 The **Enter Value** dialogue box will be displayed with **– 3** in the **Lower:** field.

6 Enter **0** in the **Upper:** field and tap **OK**.

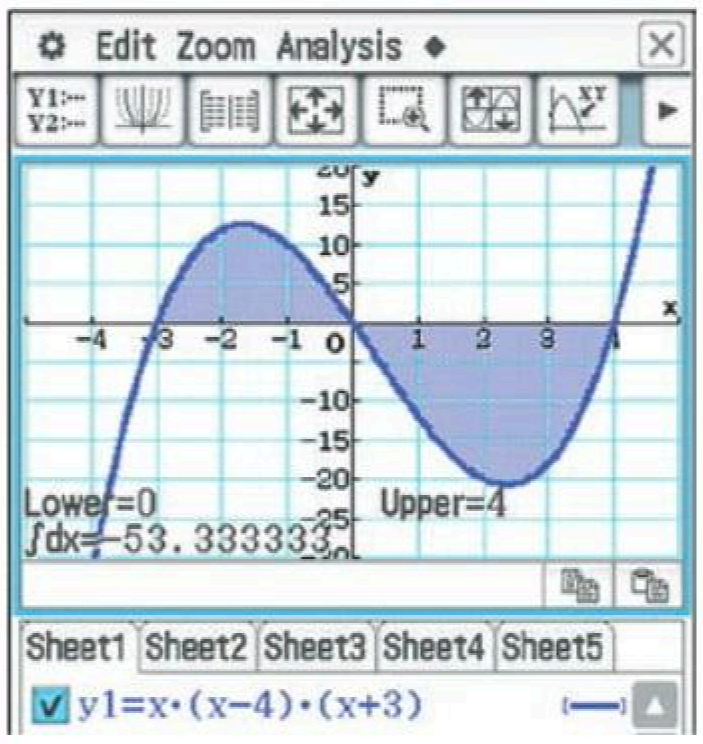


Lower=−3

∫dx=24.75

Upper=0

7 The area between the curve and the *x*-axis from –3 and 0 will be displayed.



Lower=0

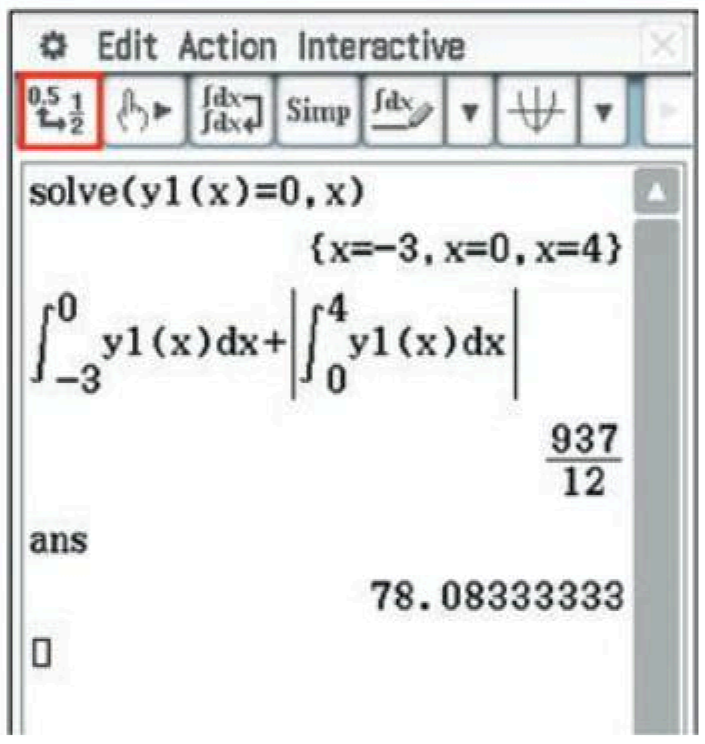
∫dx=−53.333333

Upper=4

☑ y1=x·(x−4)·(x+3)

8 Repeat to find the area between the curve and the x-axis from 0 to 4. This value is negative as it is beneath the x-axis.

9 The total area is $24.75 + 53.33 = 78.08$.



```
solve(y1(x)=0,x)
                {x=-3,x=0,x=4}
```

$$\int_{-3}^{0} y1(x)dx + \left|\int_{0}^{4} y1(x)dx\right|$$

$$\frac{937}{12}$$

```
ans
                   78.08333333
```

10 To find the exact area, tap **Main**.

11 Solve **y1(x) = 0** to determine the x-intercepts (use the letter y, not the variable $\boldsymbol{y}$).

12 Add the integral from −3 to 0 to the absolute value of the integral from 0 to 4, as shown above.

13 Tap **Convert** to confirm the approximate answer from the **Graph&Table** application.

The area is $\frac{937}{12} \approx 78.1$ square units.

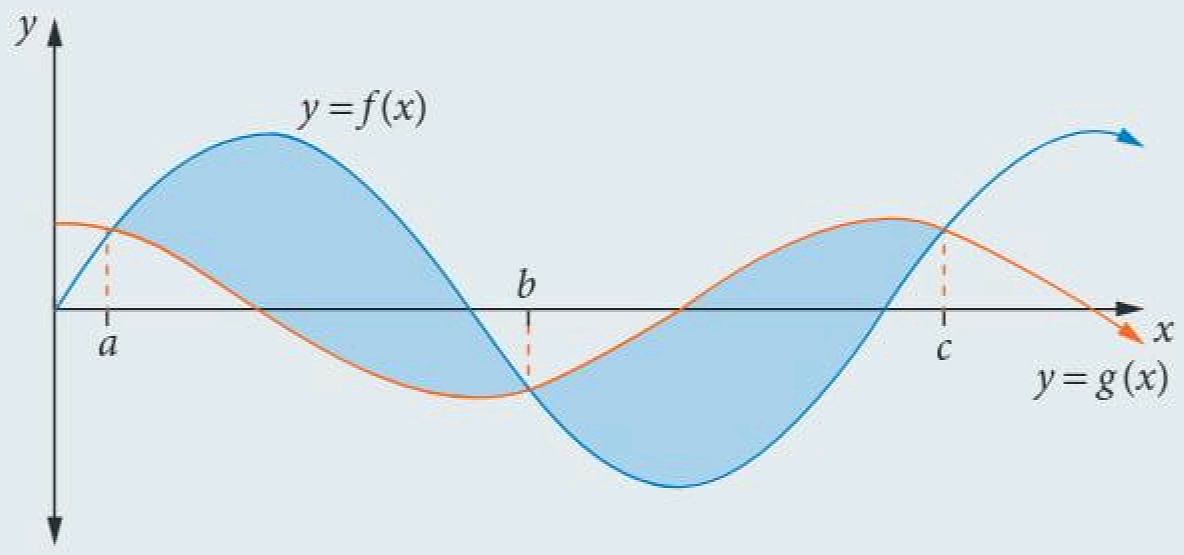
The area between two curves

Worksheets
Areas between curves 1
Areas between curves 2
Calculating areas between curves
Sums and differences of areas

Consider two functions $y = f(x)$ and $y = g(x)$ that intersect at the points where $x = a$, $x = b$ and $x = c$, as shown. The area bounded by the two curves can be found by calculating the difference of the definite integral of the upper function and the lower function using the intersection points as the limits of integration.

Area between curves

Area between two curves $= \int (\text{upper curve} - \text{lower curve})\, dx$



$$\text{Area} = \int_a^b f(x) - g(x)\,dx + \int_b^c g(x) - f(x)\,dx$$

WORKED EXAMPLE 3 Finding the area between two curves

Find the area bounded by the functions $f(x) = 8 - x^2$ and $g(x) = 2x$.

Steps | **Working**

1 Find the x-coordinates of the intersection points between the two functions.

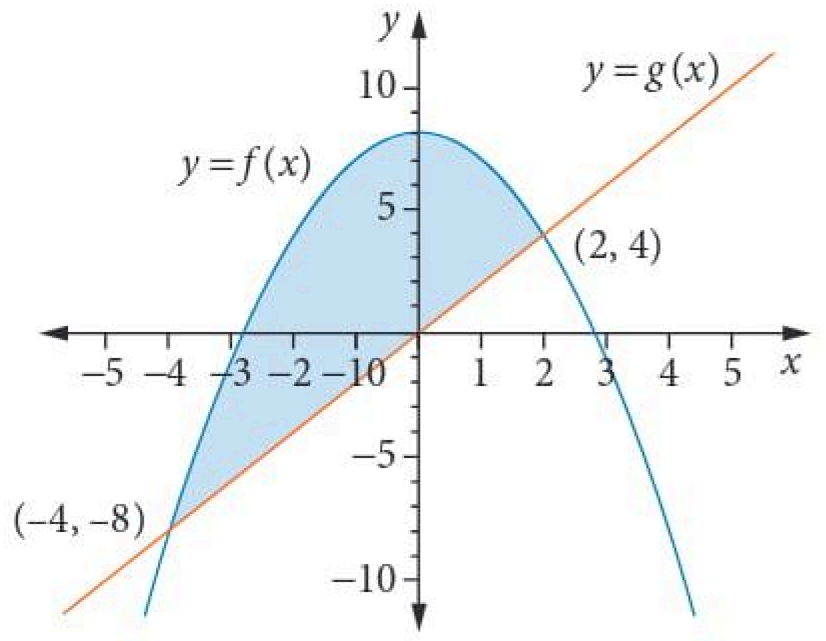
$2x = 8 - x^2$

$x^2 + 2x - 8 = 0$

$(x + 4)(x - 2) = 0$

$x = -4, x = 2$

2 Sketch the graphs of f and g on the same axes.



3 Area between two curves

$= \int (\text{top curve} - \text{bottom curve})\, dx$

$$\text{Area} = \int_{-4}^{2} f(x) - g(x)\, dx$$

$$= \int_{-4}^{2} \left(8 - x^2 - 2x\right) dx$$

$$= \left[8x - \frac{x^3}{3} - x^2\right]_{-4}^{2}$$

$$= 16 - \frac{8}{3} - 4 + 32 - \frac{64}{3} + 16$$

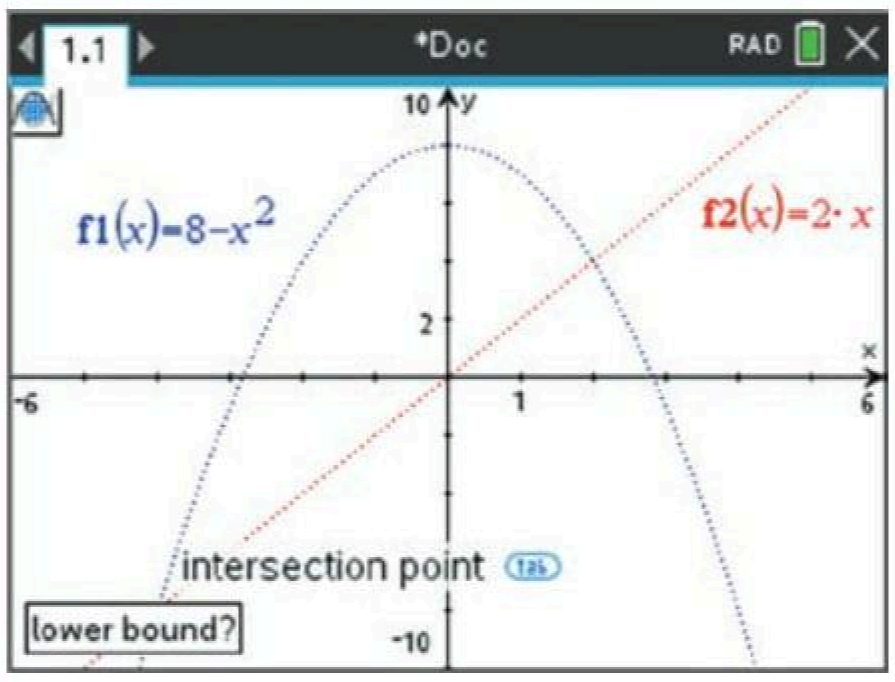
$$= 36 \text{ units}^2$$

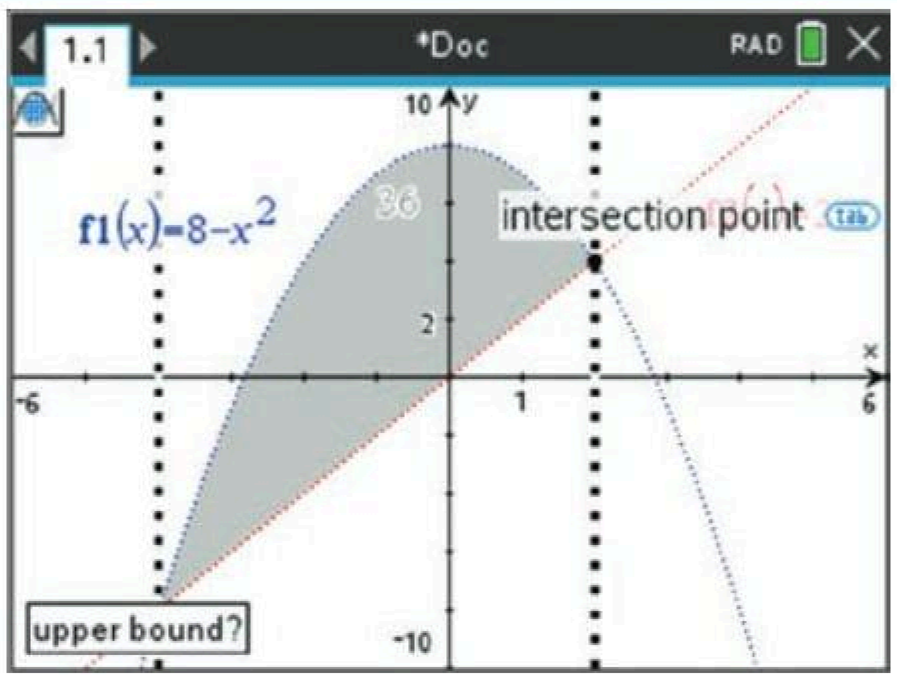
WB
p. 166

USING CAS 2 Finding the area between two curves

Find the area between the two functions $f(x) = 8 - x^2$ and $g(x) = 2x$.

TI-Nspire





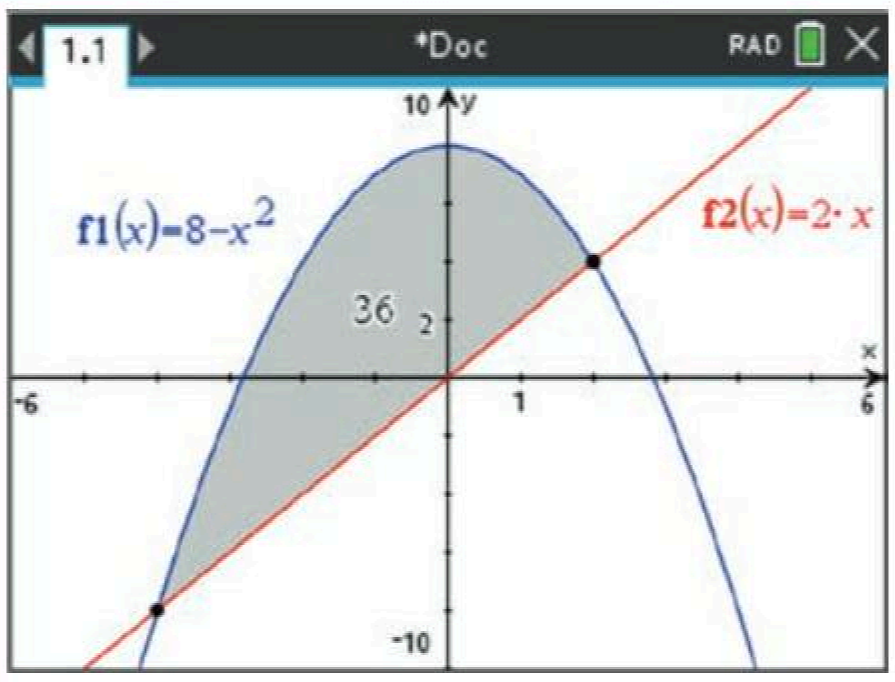
1 Add a **Graphs** page and graph the functions.

2 Adjust the window settings to suit.

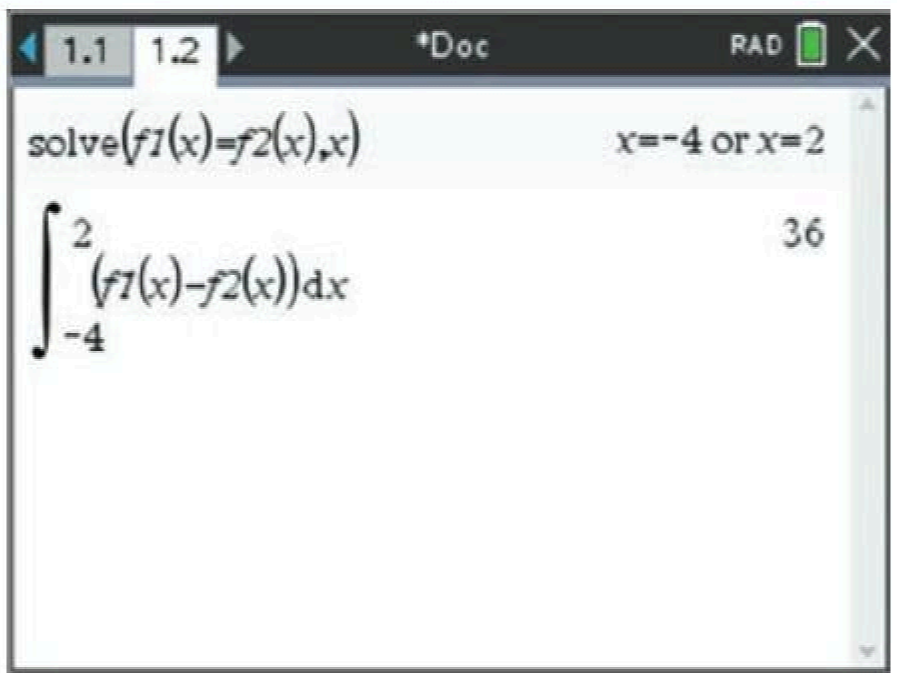
3 Press **menu > Analyze Graph > Bounded Area**.

4 When prompted for the **lower bound?**, move the cursor to select the first intersection point.

5 When prompted for the **upper bound?**, move the cursor and select the second intersection point.



6 The area between the two curves will be displayed.



```
solve(f1(x)=f2(x),x)          x=-4 or x=2
```

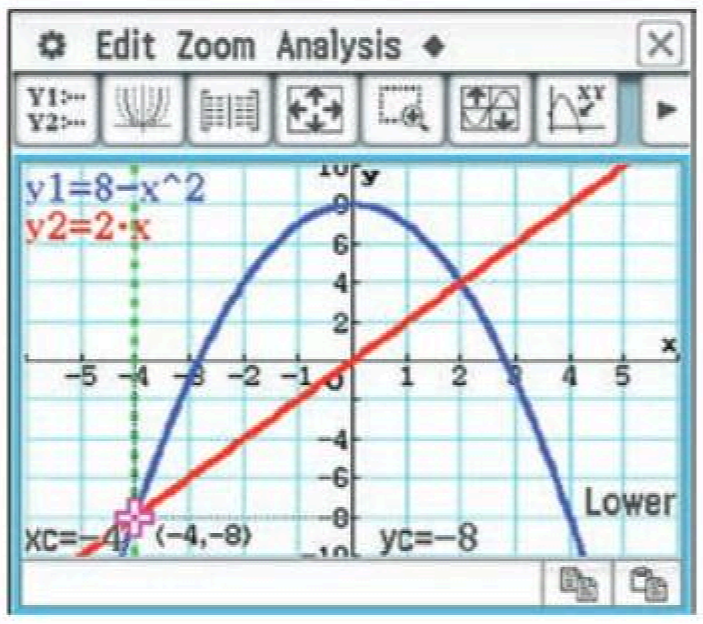
$$\int_{-4}^{2} (f1(x)-f2(x))\,dx \qquad 36$$

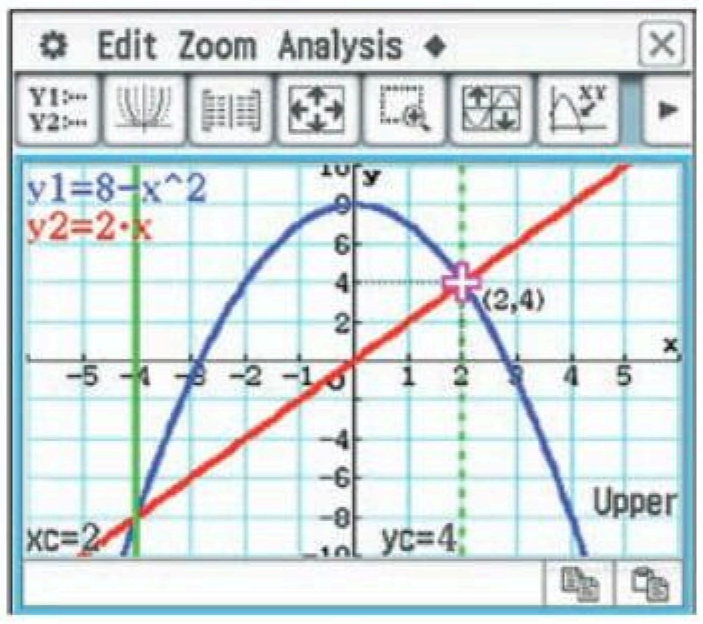
7 To find the area, add a **Calculator** page.

8 Solve ***f*1(*x*)=*f*2(*x*)** to determine the *x*-coordinates of the points of intersection.

9 Find the integral of the difference between the two functions from –4 to 2 as shown above.

ClassPad





1 Open the **Graph&Table** application and graph the functions.

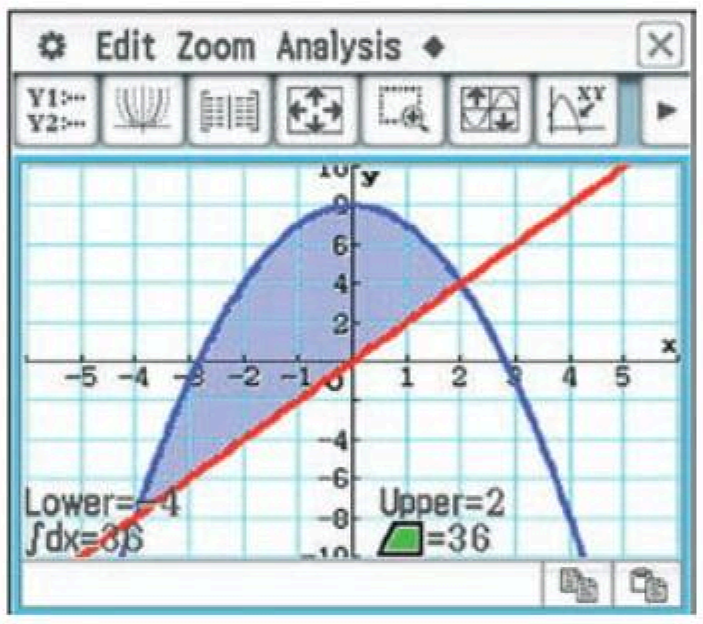
2 Adjust the window settings to suit.

3 Tap **Analysis > G-Solve > Integral > ∫dx Intersection**.

4 Press **ENTER** to label the first point of intersection.

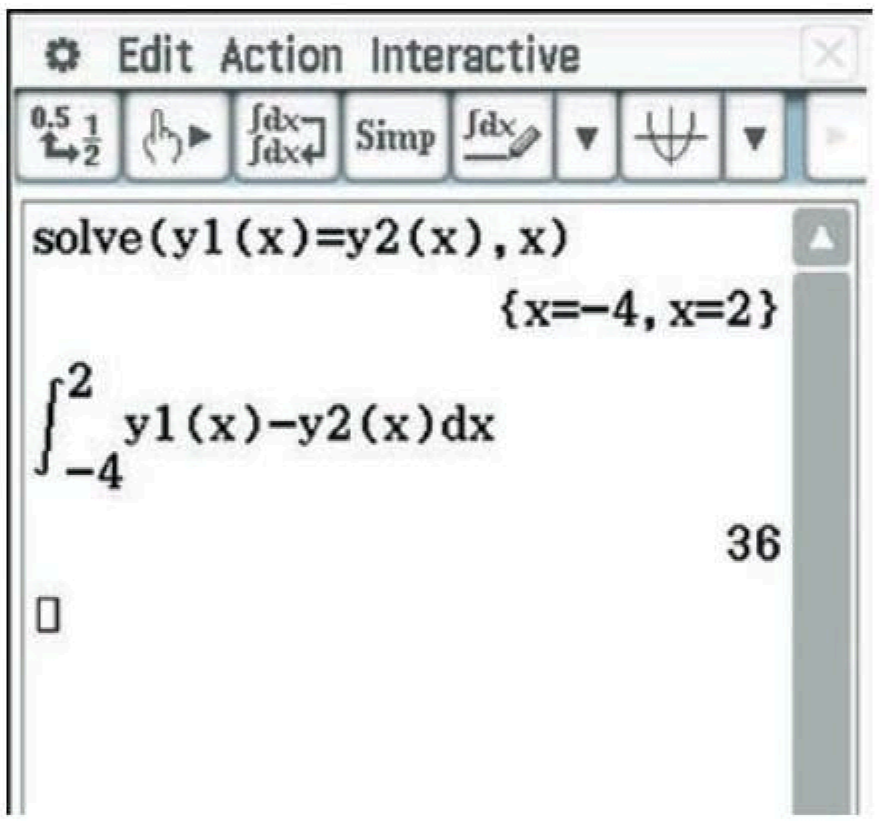
5 The cursor will jump to the second point of intersection.

6 Press **ENTER** to label this point.



7 The area between the two curves will be displayed.

8 To confirm the area, tap **Main**.



```
solve(y1(x)=y2(x),x)
                    {x=-4, x=2}
```

$$\int_{-4}^{2} y1(x)-y2(x)\,dx$$

36

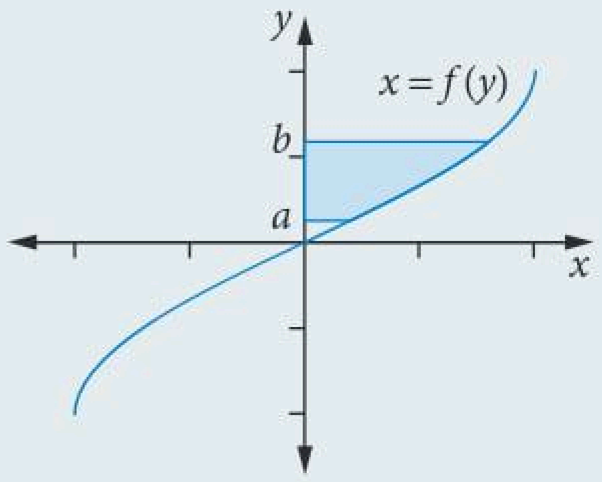
9 Solve ***y*1(*x*)=*y*2(*x*)** to determine the *x*-coordinates of the points of intersection (use the letter *y*, not the variable ***y***).

10 Find the integral of the difference between the two functions from –4 to 2 as shown above.

The area is 36 square units.

The area between a curve and the *y*-axis

Area between a curve and the *y*-axis



For the function $x = f(y)$, the shaded area between the function and the *y*-axis from $y = a$ to $y = b$ is given by the definite integral $\int_a^b f(y)\,dy$.

- Transpose the function to express *x* in terms of *y*.
- Find the definite integral of this function with respect to *y* using *y* values as the limits of integration.

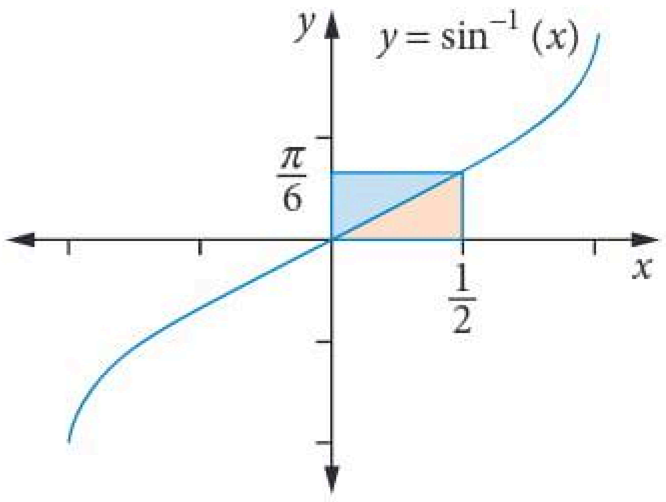
WORKED EXAMPLE 4 Applying the area between a curve and the *y*-axis

WB p. 167

Find the area bounded by the function $f(x) = \sin^{-1}(x)$ and the lines $y = 0$ and $x = \frac{1}{2}$.

Steps | **Working**

1 Graph the function and identify the area required.



The required area is shaded pink.

2 Write a definite integral for the required area.

$$\text{Area} = \int_0^{\frac{1}{2}} \sin^{-1}(x)\,dx$$

3 This area can be calculated by finding the (blue) area between $f(x)$ and the *y*-axis and subtracting it from the area of a rectangle.

Transpose to make *x* the subject and convert the upper and lower limits from *x* values to *y* values.

$y = \sin^{-1}(x) \Rightarrow x = \sin(y)$

lower limit:	upper limit:
$x = 0$	$x = \frac{1}{2}$
$y = \sin^{-1}(0) = 0$	$y = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

4 Find the area between $f(x)$ and the *y*-axis.

Area between $f(x)$ and the *y*-axis

Every part of the definite integral must be converted to *y* values.

$$= \int_0^{\frac{\pi}{6}} \sin(y)\,dy$$

$$= \left[-\cos(y)\right]_0^{\frac{\pi}{6}}$$

$$= -\cos\left(\frac{\pi}{6}\right) + \cos(0)$$

$$= 1 - \frac{\sqrt{3}}{2}$$

5 Subtract this area from the area of the rectangle.

Area between $f(x)$ and the x-axis (orange)

$$= \frac{\pi}{6} \times \frac{1}{2} - \left(1 - \frac{\sqrt{3}}{2}\right)$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 \text{ units}^2$$

Exam hack

When finding the area, always sketch the graph of the function or functions and shade the required area. The mathematical equation for 'Area =' must be a definite integral that produces the necessary area.

EXERCISE 8.1 Areas under curves

ANSWERS p. 590

Mastery

1 WORKED EXAMPLE 1

a Find the area between the curve $f(x) = \frac{1}{\sqrt{9 - x^2}}$, the x-axis, the y-axis and the line $x = \frac{3\sqrt{3}}{2}$.

b Find the area between the curve $f(x) = \frac{x}{\sqrt{1 + x^2}}$, the x-axis and the line $x = \sqrt{3}$.

2 WORKED EXAMPLE 2

a Find the area bounded by the function $f: [0, \pi] \to R, f(x) = \sin^3(3x)$.

b Find the area between the curve $f(x) = \frac{x}{\sqrt{1 + x^2}}$, the x-axis and the lines $x = -\sqrt{3}$ and $x = \sqrt{3}$.

3 Using CAS 1

a Find the area between the curve $f(x) = x^2\sqrt{9 + x^2}$, the x-axis and the lines $x = -1.5$ and $x = 2.5$, correct to two decimal places.

b For the function $f(x) = \frac{x^3 - 1}{\sqrt{4 + x}}$, where $x > -4$, find the area between the curve, the x-axis and the lines $x = -2$ and $x = 1$, correct to two decimal places.

4 WORKED EXAMPLE 3

a Find the area bounded by the functions $f(x) = \sin(x)$ and $g(x) = -\cos(x)$ between $x = 0$ and $x = 2\pi$.

b Find the area bounded by the functions $f(x) = -x$ and $g(x) = \sqrt{2 - x}$.

5 Using CAS 2

a Find the area bounded by the functions $f(x) = \sin(2x)$ and $g(x) = -\cos(x)$ over the domain $x \in [-2, 1]$, correct to two decimal places.

b Find the area bounded by the functions $f(x) = \frac{4}{x^2 + 3}$ and $g(x) = \frac{1}{x}$, correct to three decimal places.

6 WORKED EXAMPLE 4

a Find the area bounded by the function $f(x) = \sin^{-1}(2x)$ and the lines $x = 0$, $y = 0$ and $x = \frac{1}{2}$.

b Find the area bounded by the function $f(x) = \log_e(x)$, the x-axis and the line $x = e$.

7 © VCAA 2011 1Q3c 73% TECH-FREE (3 marks) Find the area enclosed by the relation $y = \frac{2x^2+3}{x^2+1}$, the x-axis and the lines $x = -1$ and $x = 1$.

8 © VCAA 2009 1Q8 TECH-FREE (4 marks)

a 77% Show that $f(x) = \frac{2+x^2}{4-x^2}$ can be written in the form $f(x) = -1 + \frac{6}{4-x^2}$. 1 mark

b 53% Find the exact area enclosed by the graph of $f(x) = \frac{2+x^2}{4-x^2}$, the x-axis, and the lines $x = -1$ and $x = 1$. 3 marks

9 © VCAA 2012 1Q7 54% TECH-FREE (3 marks) Consider the curve with equation $y = (x-1)\sqrt{2-x}$, $1 \le x \le 2$. Calculate the area of the region enclosed by the curve and the x-axis.

10 © VCAA 2006 1Q3b 49% TECH-FREE (4 marks) Find the exact area bounded by $y = \frac{36}{2x^2-18}$, the x-axis and the lines $x = -2$ and $x = 2$.

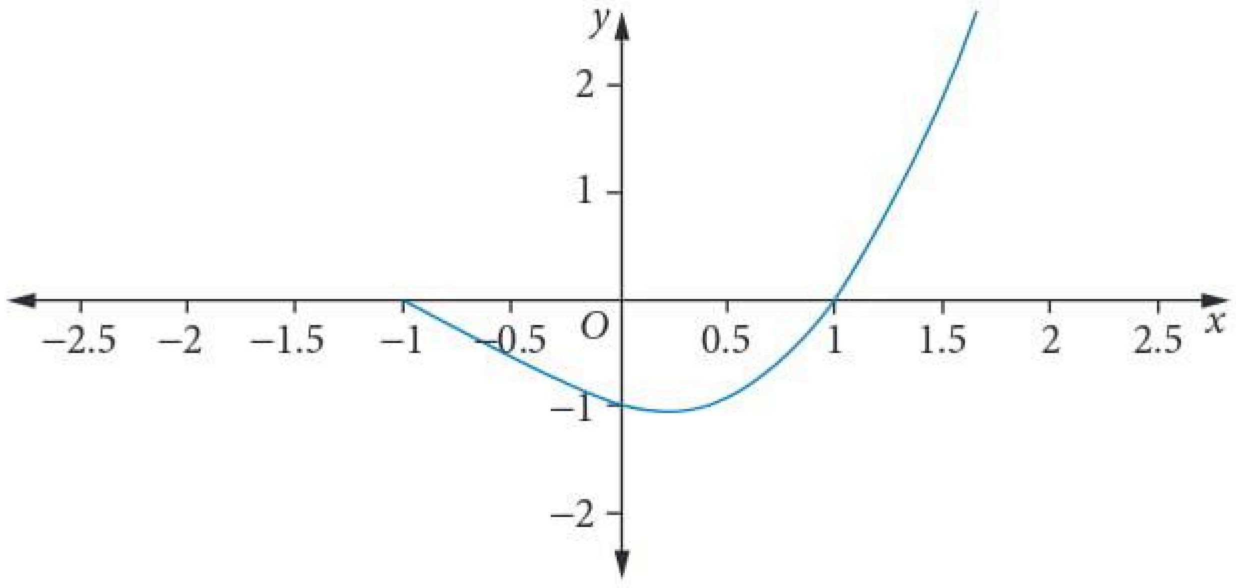
11 © VCAA 2014 1Q7 TECH-FREE (5 marks) Consider $f(x) = 3x \arctan(2x)$.

a 4% Write down the range of f. 1 mark

b 91% Show that $f'(x) = 3\arctan(2x) + \frac{6x}{1+4x^2}$. 1 mark

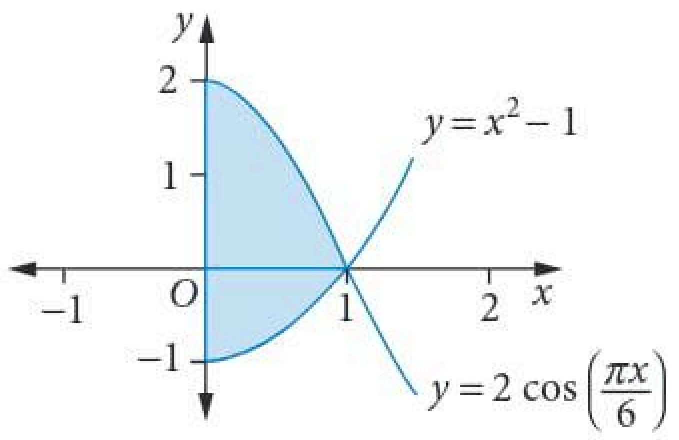
c 51% **Hence** evaluate the area enclosed by the graph of $g(x) = \arctan(2x)$, the x-axis and the lines $x = \frac{1}{2}$ and $x = \frac{\sqrt{3}}{2}$. 3 marks

12 © VCAA 2010 1Q10 47% TECH-FREE (4 marks) Part of the graph with equation $y = \left(x^2-1\right)\sqrt{x+1}$ is shown below.



Find the area that is bounded by the curve and the x-axis. Give your answer in the form $\frac{a\sqrt{b}}{c}$, where a, b and c are integers.

13 © VCAA 2003 1IQ12 77% The shaded region in the diagram on the right is bounded by the y-axis, and the curves with equations $y = x^2 - 1$ and $y = 2\cos\left(\frac{\pi x}{2}\right)$. The **exact** value of the area of the shaded region is



A $\frac{10}{3}$

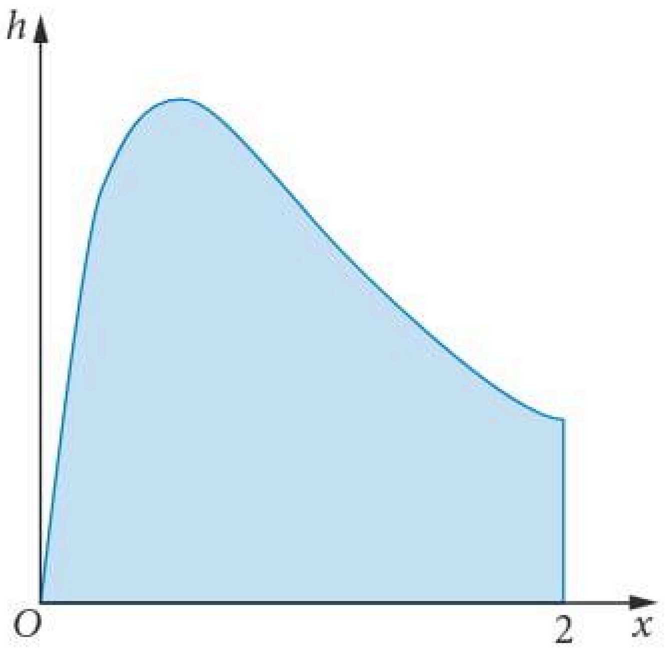
B $\frac{14}{3}$

C $\frac{4}{\pi} - \frac{4}{3}$

D $\frac{4}{\pi} - \frac{2}{3}$

E $\frac{4}{\pi} + \frac{2}{3}$

14 © VCAA 2005 2Q3 (13 marks) A council is planning to construct a fence at a park from pre-made panels. One side of each fence panel needs to be painted. To determine the amount of paint needed, the area of one side of each panel needs to be calculated. Each panel is 2 metres wide. An artist's sketch of a panel is given below. Estimates of the height of one panel are made from the artist's sketch. Taking x metres as the distance from the left-hand end of a panel and h metres as an estimate of the height, these estimates are summarised in the table.



x	h
0.0	0.00
0.5	1.60
1.0	1.25
1.5	0.85
2.0	0.55

a 69% Estimate the area of one side of a panel by approximating the shape of a panel using four trapeziums of equal width. 2 marks

A mathematician examines the artist's sketch and decides that the height of each panel can be modelled by the function

$$h(x) = \frac{10x}{\left(x^2+1\right)(3x+1)},\ x \in [0,2]$$

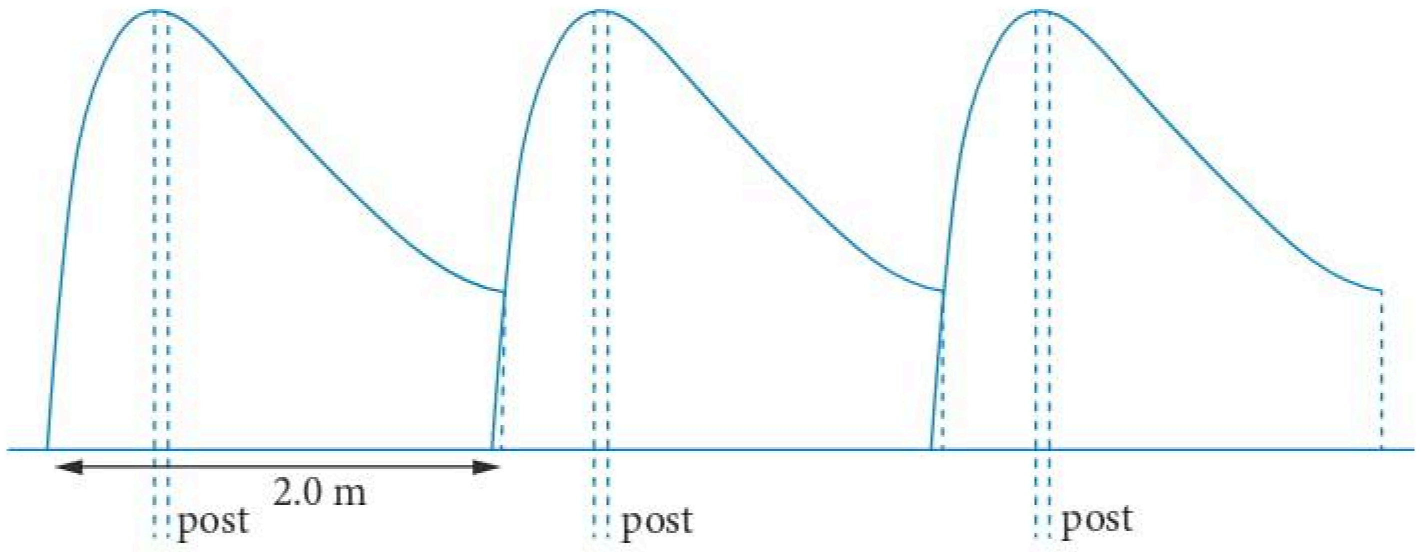
b 71% Given that $\dfrac{10x}{\left(x^2+1\right)(3x+1)}$ can be written in the form $\dfrac{x+A}{x^2+1}+\dfrac{B}{3x+1}$,

find the values of A and B. 3 marks

c 38% Write down a definite integral which represents the area of one side of a pre-made panel according to the mathematician's model. **Hence** use calculus to find this area correct to two decimal places. 5 marks

The fence is to be constructed by overlapping the panels. The fence will be 100 metres long.

d 25% To build the fence, the pre-made panels are overlapped and secured to upright posts. An artist's sketch of three panels and posts of the fence is given below.



According to the mathematician's model, what is the minimum number of panels required? 3 marks

15 © VCAA 2003 2Q4 (14 marks) Consider the function $f\colon [0, 3) \to R$, where

$$f(x) = -2 + 2\sec\left(\frac{\pi x}{6}\right)$$

a 85% Evaluate $f(2)$. 1 mark

Let f^{-1} be the inverse function of f.

b 74% Sketch the graphs of f and f^{-1}, showing their points of intersection. 2 marks

c 70% The rule for f^{-1} can be written as $f^{-1}(x) = a\cos^{-1}\left(\frac{2}{x+2}\right)$. Find the **exact** value of a. 2 marks

Let A be the magnitude of the area enclosed by the graphs of f and f^{-1}.

d 53% Write a definite integral expression for A and evaluate it correct to three decimal places. 2 marks

e **i** 48% Given that $0 < kx < \frac{\pi}{2}$, show that the derivative of $\log_e\left[\frac{1+\sin(kx)}{\cos(kx)}\right]$ is $k\sec(kx)$. 3 marks

ii 18% Hence find the **exact** value of A, the magnitude of the area enclosed by the graphs of f and f^{-1}. 4 marks

16 © VCAA 2002 2Q4a-d (13 marks) Consider the function $f: D \rightarrow R$, where $f(x) = \log_e(4 - x^2)$ and D is the largest possible domain for which f is defined.

a 69% Find D. 1 mark

b 75% Sketch the graph of f, labelling all the key features. 3 marks

Let A be the magnitude of the area enclosed by the graph of f, the coordinate axes and the line $x = 1$.

c 37% Without evaluating A, use the graph of f to show that $\log_e(3) < A < \log_e(4)$. 2 marks

d **i** 75% Differentiate $x\log_e(4 - x^2)$. 2 marks

ii 22% Find an anti-derivative of $\frac{x^2}{4 - x^2}$. 3 marks

iii 18% Hence find the exact value of A in the form $a + b\log_e(c)$, where a, b and c are integers. 2 marks

8.2 The graph of the anti-derivative

Video playlist The graph of the anti-derivative

Suppose $F(x) + c$ is the anti-derivative of $f(x)$.

$$\int f(x)\,dx = F(x) + c \qquad \text{or} \qquad f(x) = F'(x)$$

The graph of the anti-derivative $y = F(x)$ can be found using the fact that the function values for $f(x)$ represent the gradient of $F(x)$.

Properties of a function *f*(*x*) and its anti-derivative *F*(*x*)

The gradient function $f(x)$	The anti-derivative $F(x)$	Shape of $F(x)$
$f(x) = 0$: the graph intersects the x-axis	$F(x)$ has zero gradient: the graph has a stationary point	—
$f(x) > 0$: the graph is above the x-axis	$F(x)$ has a positive gradient: the graph is increasing	╱
$f(x) < 0$: the graph is below the x-axis	$F(x)$ has a negative gradient: the graph is decreasing	╲

Draw a slope graph below the graph of the function $f(x)$ using straight lines to represent the slope of the curve in each section. This slope can then be transformed into a smooth curve and moved into position on the grid provided.

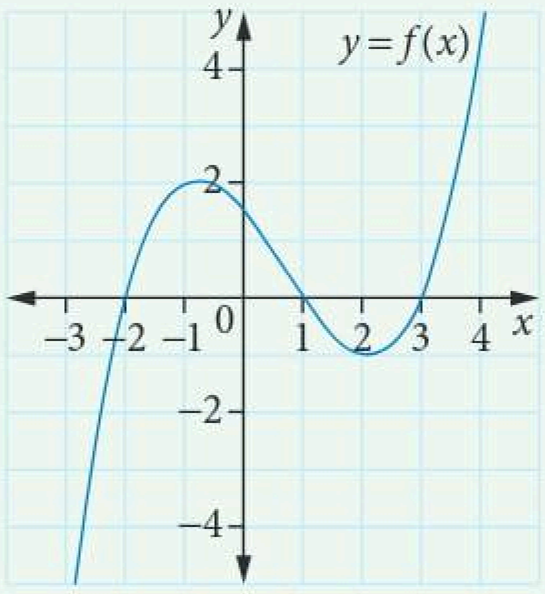
This method will determine the shape of the anti-derivative $F(x) + c$. However, additional information will need to be provided to determine the value of the arbitrary constant c.

WB

p. 168

WORKED EXAMPLE 5 Sketching a possible anti-derivative function

Sketch the graph of a possible anti-derivative for $y = f(x)$.

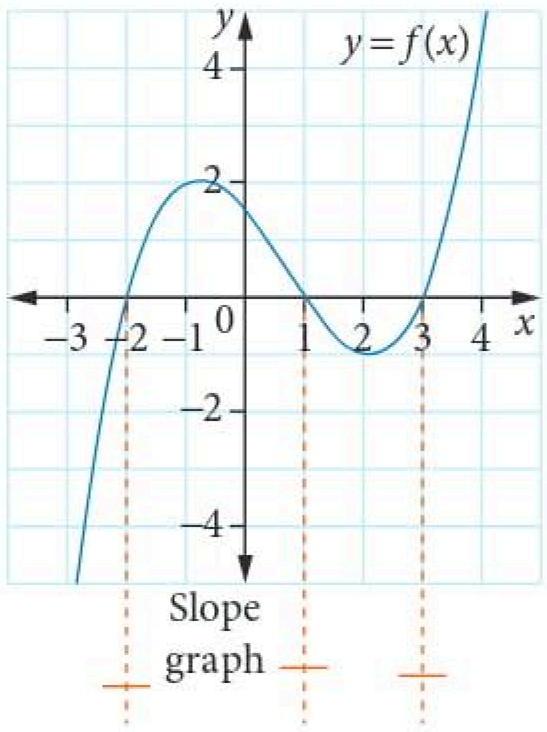


Steps **Working**

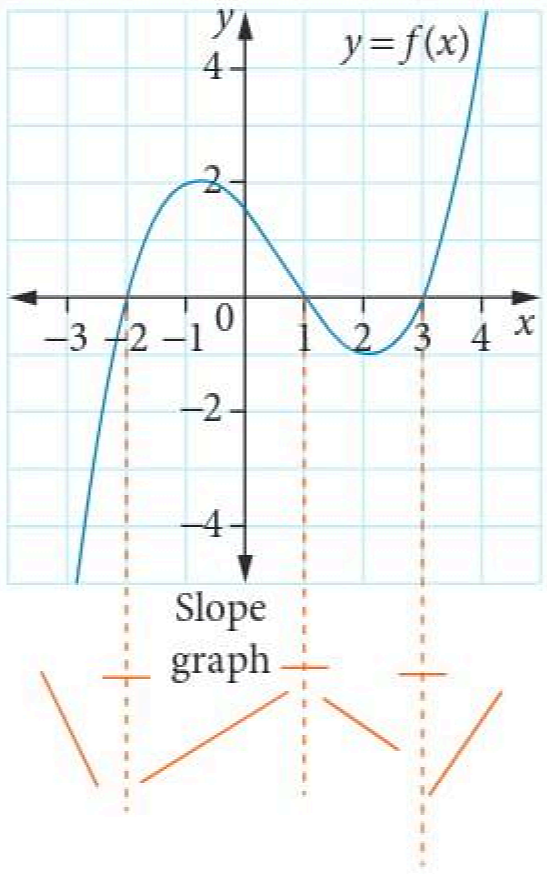
1 $y = f(x)$ has three x-intercepts at $x = -2$, $x = 1$ and $x = 3$.

These represent the three stationary points with a gradient of zero.

Draw horizontal lines below each x-intercept, for the slope graph.

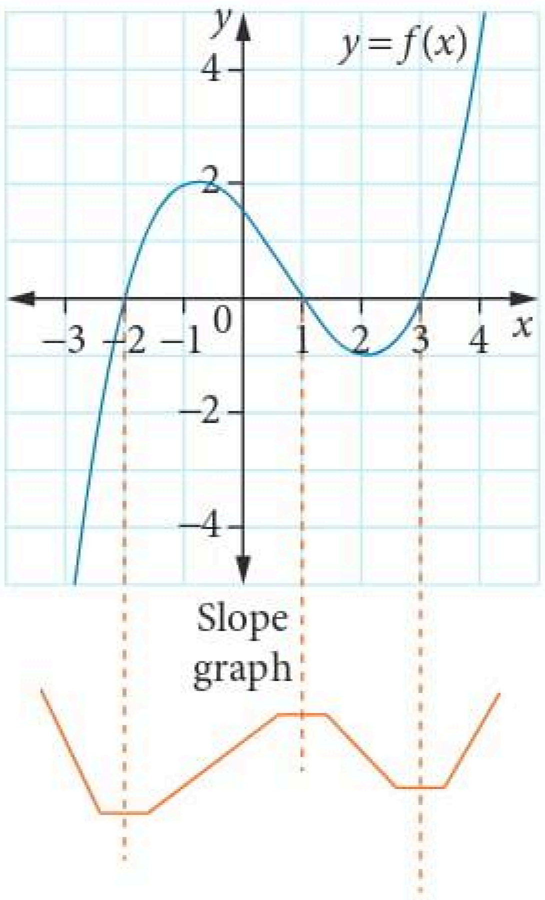


2 Draw an increasing line in the section of the slope graph, where $f(x)$ is above the x-axis and a decreasing line in the section where $f(x)$ is below the x-axis.



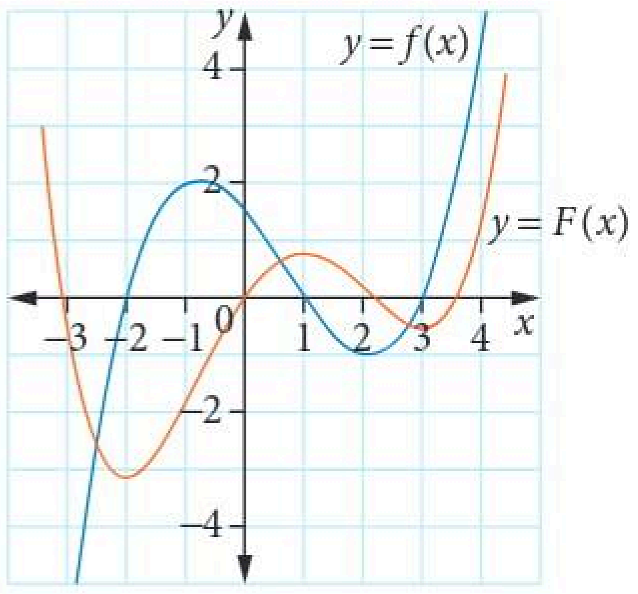
3 Move the horizontal lines on the slope graph to make the slope graph continuous.

The given graph looks like a positive cubic function (of degree 3) so the graph of the anti-derivative should look like a positive quartic function (of degree 4).



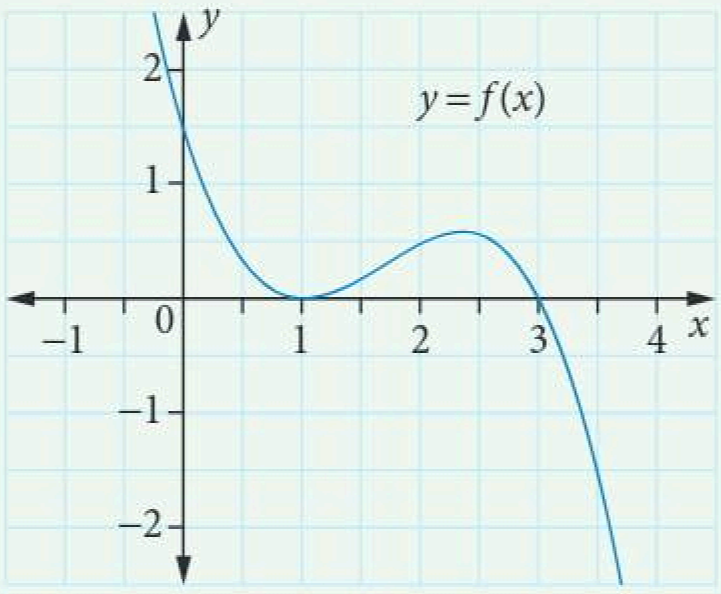
4 Draw a smooth curve from the slope graph and translate the graph upwards to any position on the axes above.

The value of the constant is not known, so any vertical translation can be used.

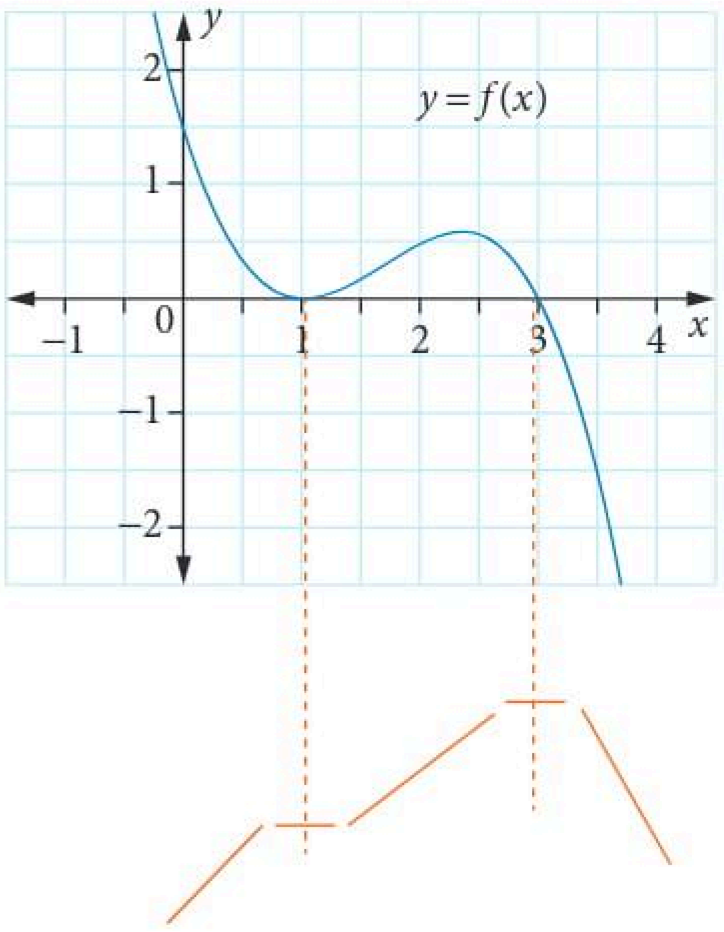
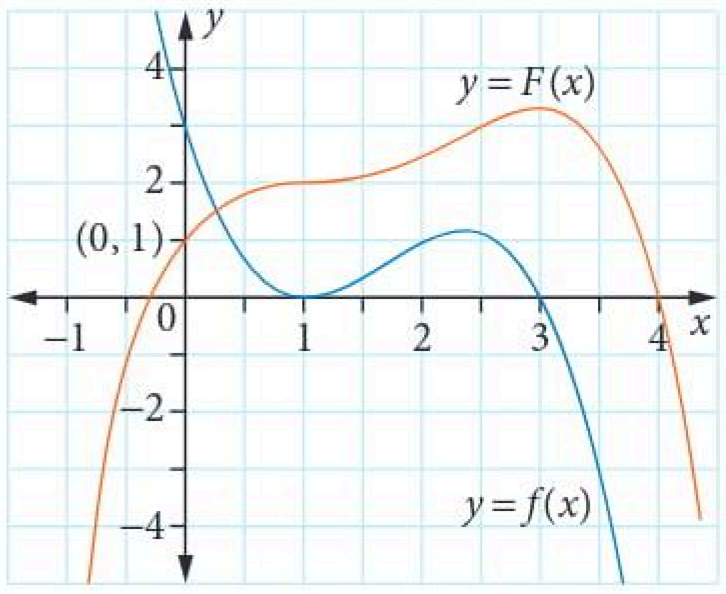


WORKED EXAMPLE 6 Sketching the anti-derivative function

Sketch the graph of the anti-derivative for $y = f(x)$, where $\int f(x)\,dx = F(x)$, given $F(0) = 1$.



WB
p. 170

Steps	**Working**
1 $y = f(x)$ has two x-intercepts at 1 and 3. These represent the two stationary points with a gradient of zero. Draw a horizontal line below each x-intercept. Add the slope lines below the graph obtained from the gradient function $f(x)$.	
2 As $F(0) = 1$, the graph of the anti-derivative $F(x)$ passes through the point (0, 1). Draw a smooth curve from the slope graph and translate the graph upwards so that it passes through the point (0, 1).	

EXERCISE 8.2 The graph of the anti-derivative

ANSWERS p. 591

Recap

1 The region bounded by the y-axis, and the curves with equations $y = x - 4$ and $y = 2\sin\left(\frac{\pi x}{4}\right)$ is

A $16\pi + 8$ **B** $\frac{16}{\pi} - 8$ **C** $\frac{\pi}{2} - 8$ **D** $\frac{16}{\pi} + 8$ **E** $\frac{\pi}{2} + 8$

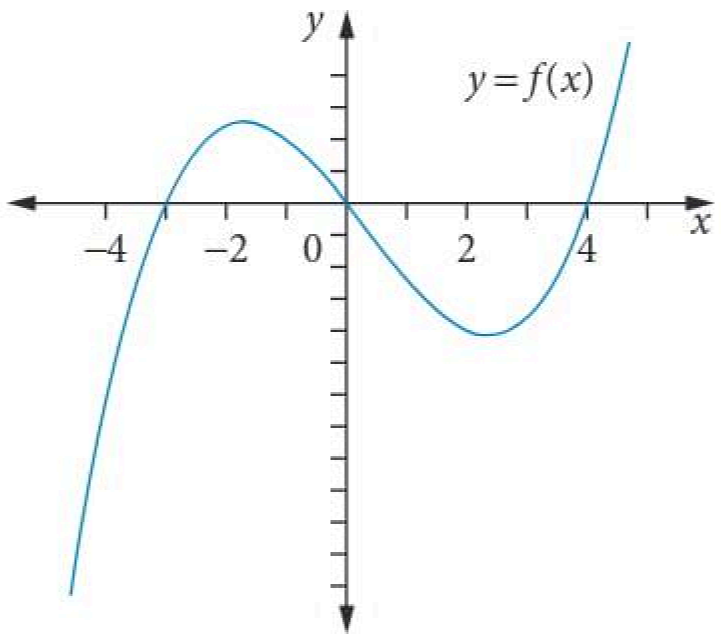
2 The area of the region enclosed by the curve $f(x) = (x-1)\sqrt{(5-x)}$ and the x-axis is

A $\frac{128}{15}$ **B** $\frac{4}{15}$ **C** $\frac{8\sqrt{2}}{15}$ **D** $\frac{4\sqrt{15}}{15}$ **E** $\frac{4\sqrt{2}}{15} + \frac{4}{15}$

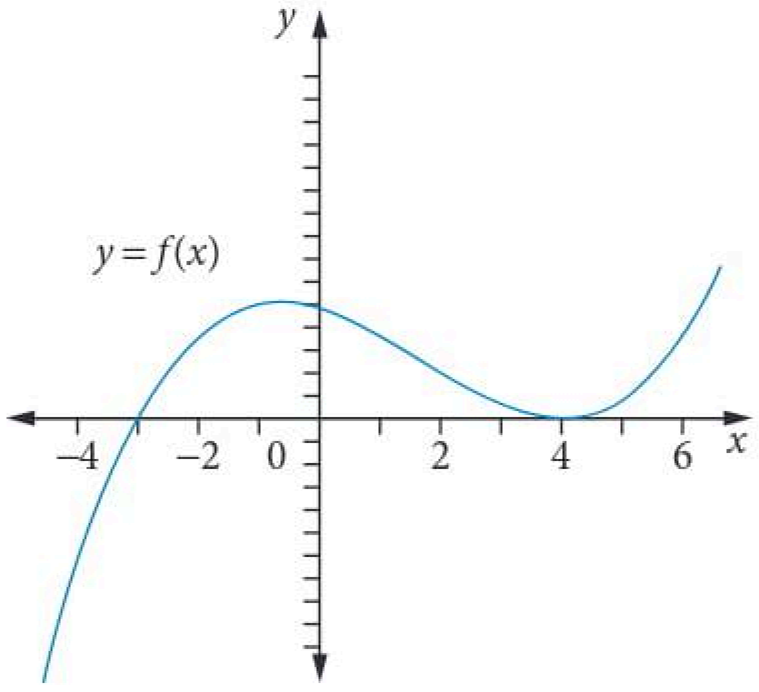
Mastery

3 WORKED EXAMPLE 5 Sketch a possible anti-derivative for $y = f(x)$.

a

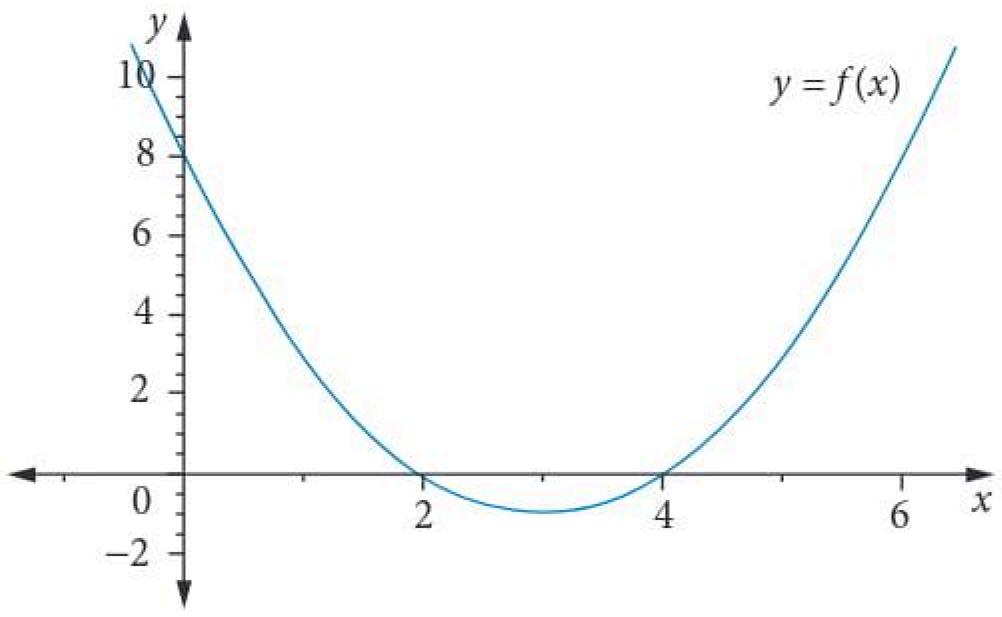


b

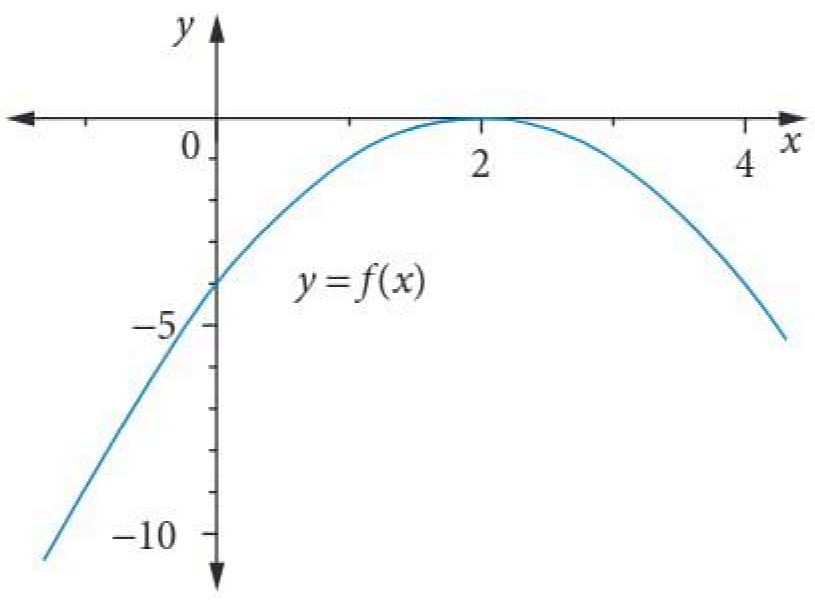


4 WORKED EXAMPLE 6

a Sketch the anti-derivative for $y = f(x)$, where $\int f(x)\,dx = F(x)$, given $F(0) = 2$.



b Sketch the anti-derivative for $y = f(x)$, where $\int f(x)\,dx = F(x)$, given $F(0) = 5$.

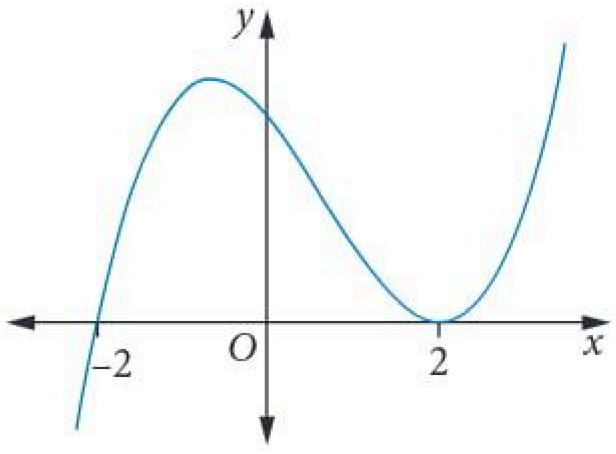


Exam practice

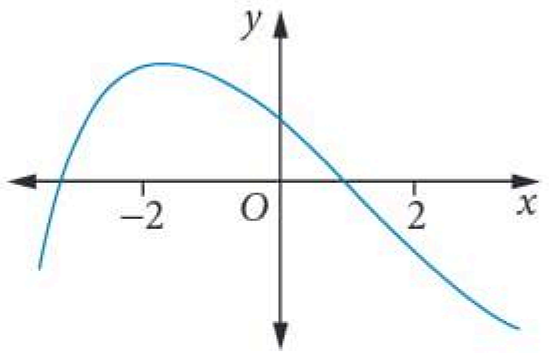
80–100% 60–79% 0–59%

5 © VCAA 2003 2AQ14 85% Part of the graph of the derivative of a function *f* is shown on the right.

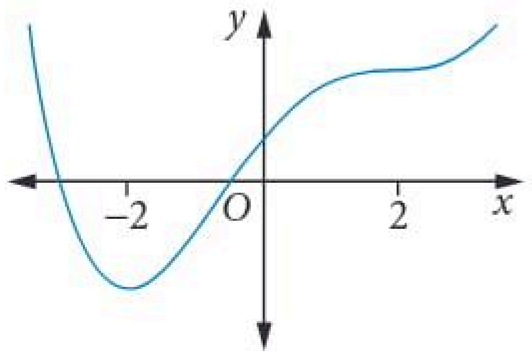
Which one of the following could be the graph of the function *f*?



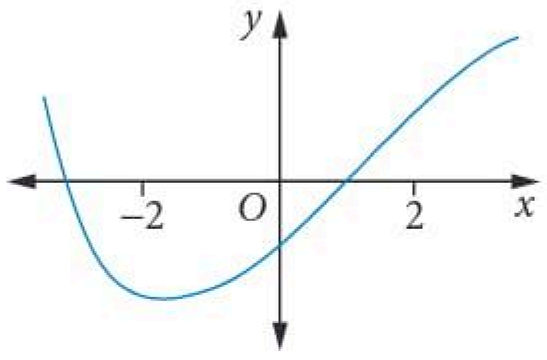
A



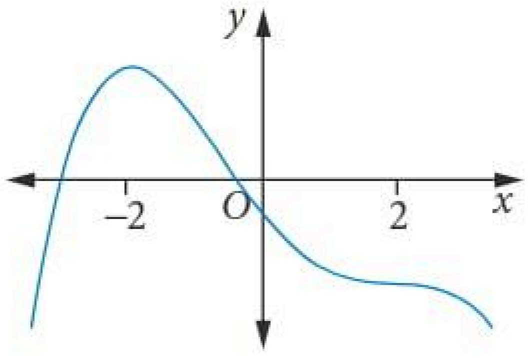
B



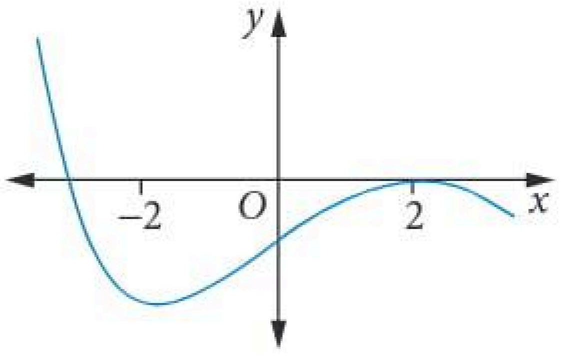
C



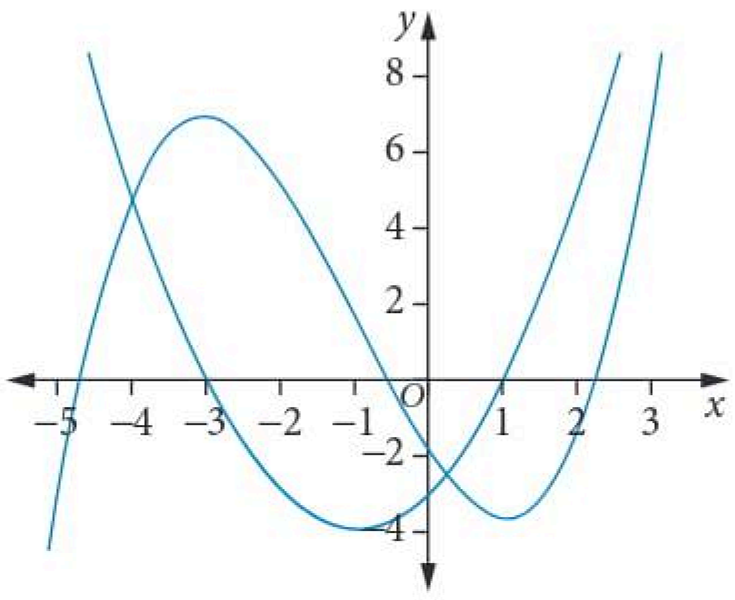
D



E



6 © VCAA 2008 2AQ12 45% The graph of a function *f*, together with one of its anti-derivative functions is shown below.



The value of $\int_{-3}^{0} f(x)\,dx$ is closest to

A −9 **B** −7 **C** −5 **D** 5 **E** 9

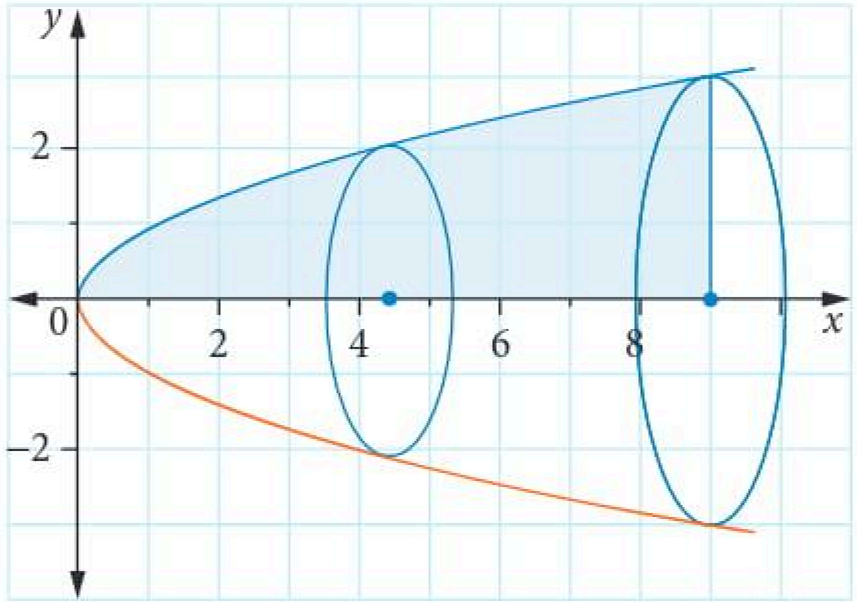
8.3 Volumes of solids of revolution

Video playlist Volumes of solids of revolution

Worksheets Volumes of integration

Volumes

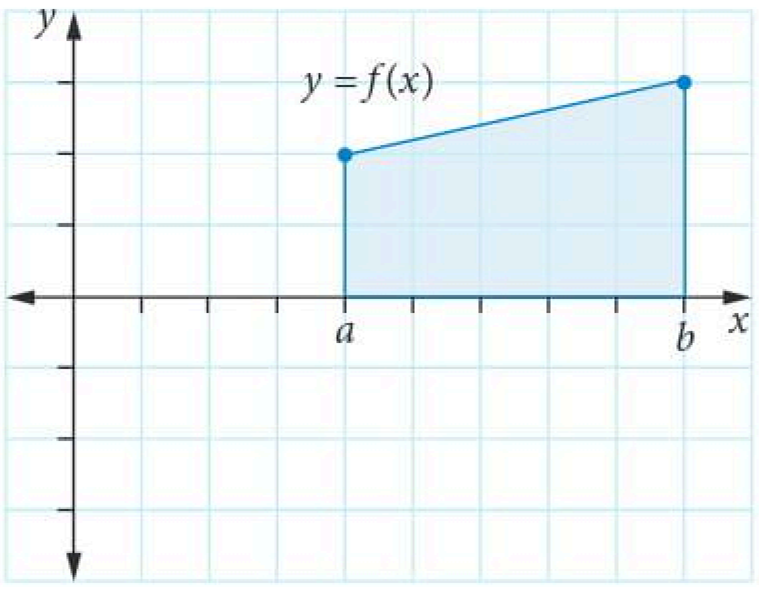
Solids of revolution



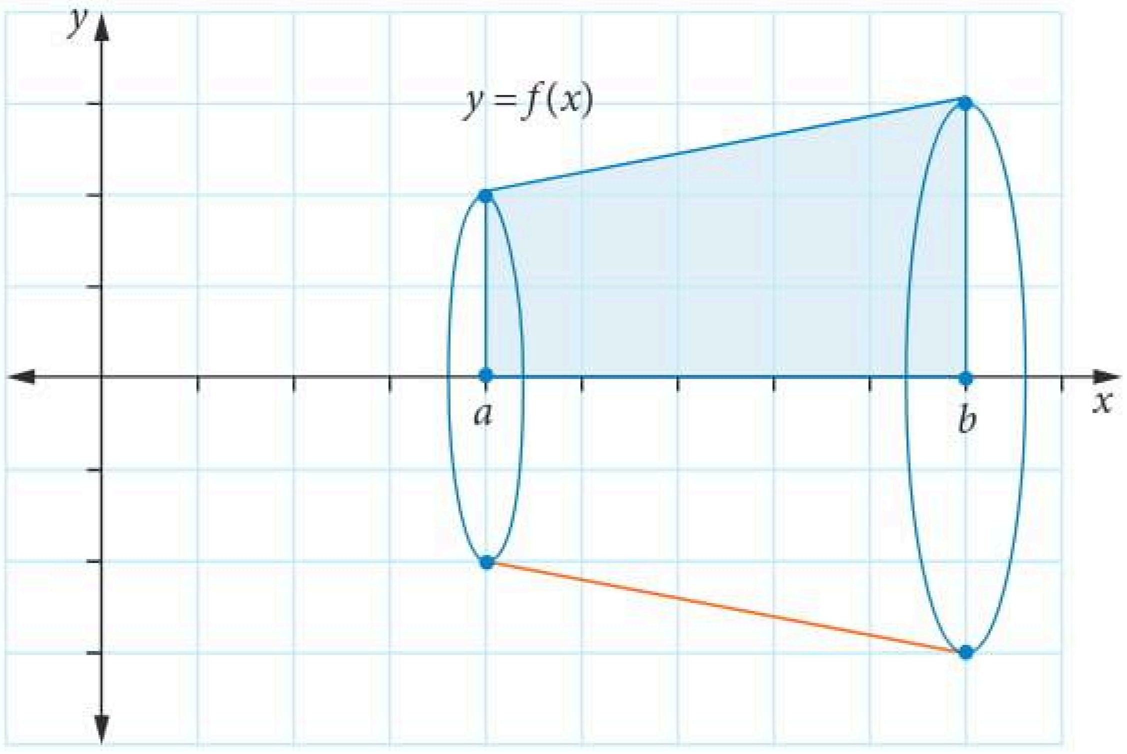
When a curve is rotated about an axis, the 3D shape formed is called a **solid of revolution**. In the diagram, the blue region has been rotated about the *x*-axis to form a cone-like solid. Integration can be used to find the formulas for the volumes of many common objects like cylinders, spheres and cones.

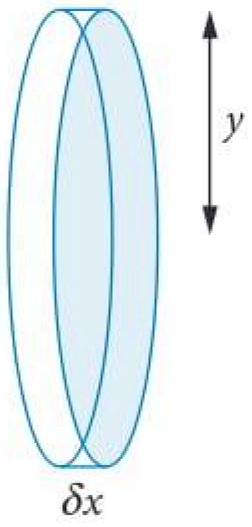
Solids of revolution about the x-axis

Consider the curve $y = f(x)$. The area between the curve and the x-axis from $x = a$ to $x = b$ is shown.



If this area is rotated about the x-axis, it produces the volume of a solid of revolution.





If we consider a small section of the volume with a thickness of δx, the shape will be approximately cylindrical with a radius of y.

The volume of the cylinder given by the formula $V = \pi r^2 h$ will be

$$\delta V = \pi y^2 \delta x$$

The sum of all the cylinders from $x = a$ to $x = b$ can be found from the definite integral

$$V = \int_a^b \pi y^2 dx$$

As π is a constant, we can place it outside the integral.

Volumes of solids of revolution

The volume of a solid of revolution about the x-axis from $x = a$ to $x = b$ is given by

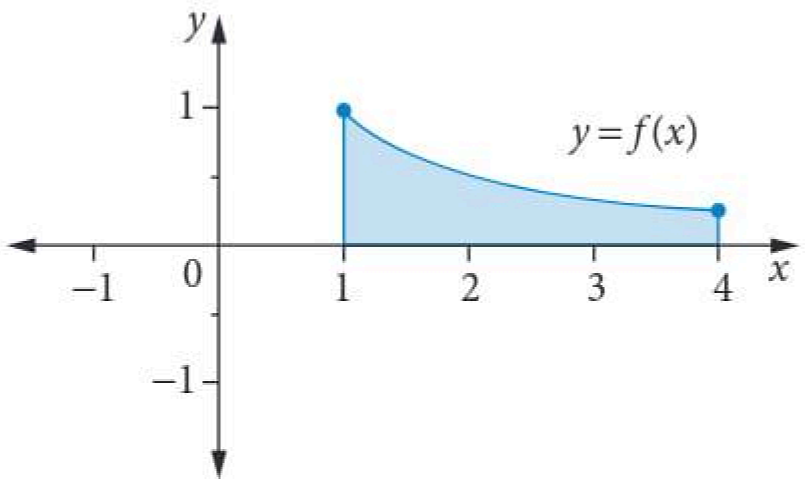
$$V = \pi \int_a^b y^2 dx$$

WB

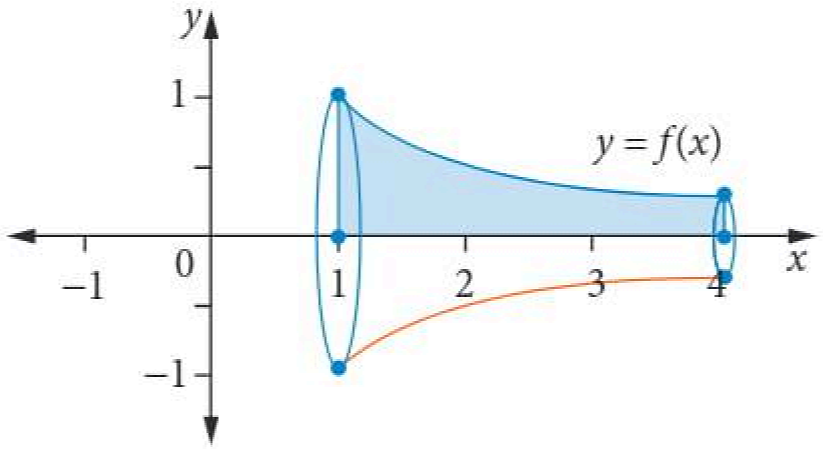
p. 171

WORKED EXAMPLE 7 Volumes of solids of revolution

Find the volume generated when the area between the function $y = \dfrac{1}{x}$ and the x-axis from $x = 1$ to $x = 4$ is rotated about the x-axis.

Steps	**Working**
1 Graph the function and identify the area required.	

2 Rotate the curve in a circular motion about the x-axis to create a solid of revolution.



3 Write the formula for the volume of a solid of revolution about the x-axis.
Substitute $y = \frac{1}{x}$ using $x = 1$ and $x = 4$ as the limits of integration.

$$V = \int_a^b \pi y^2 dx$$

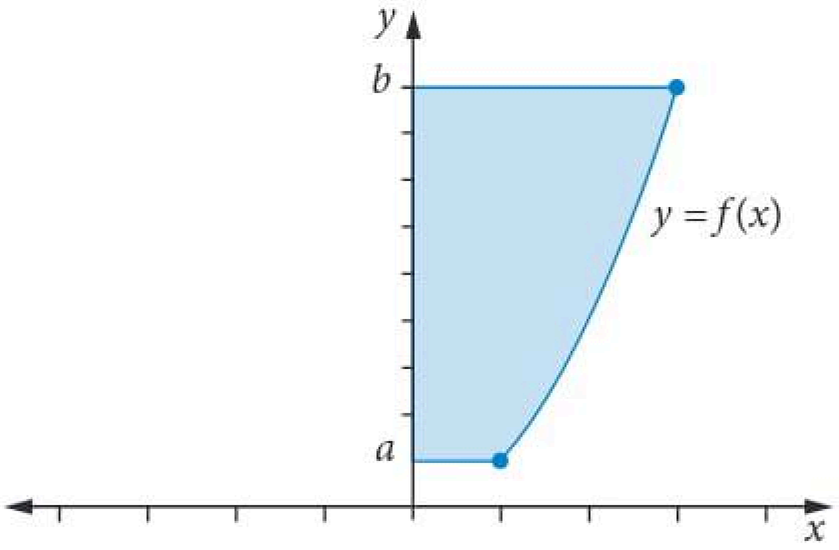
$$V = \int_1^4 \pi \left(\frac{1}{x}\right)^2 dx$$

4 Evaluate the definite integral.

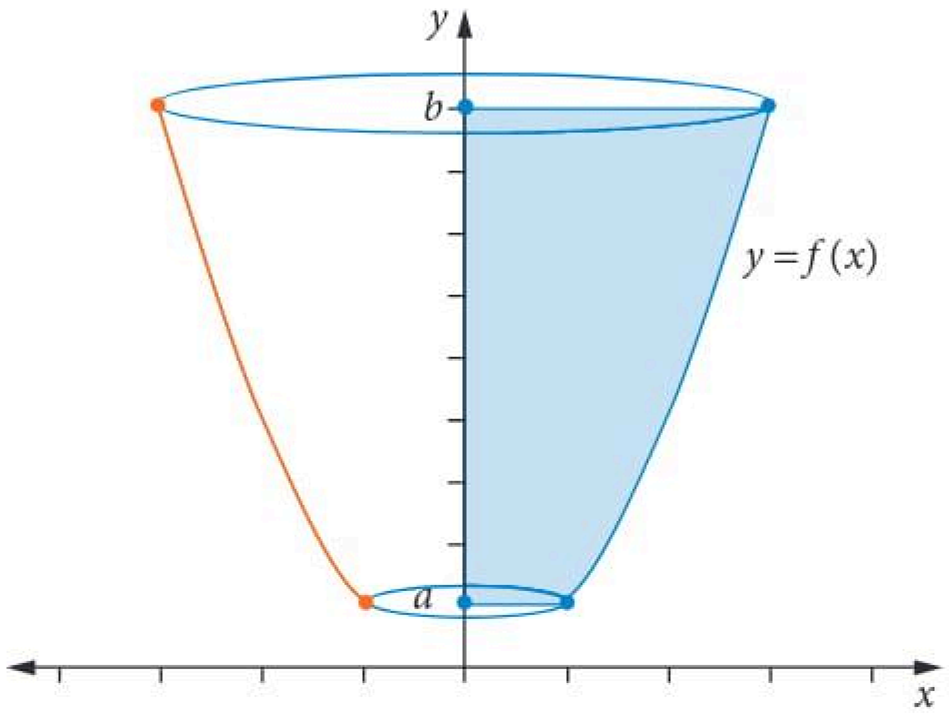
$$\begin{aligned} V &= \pi \int_1^4 \frac{1}{x^2} dx \\ &= \pi \left[\frac{x^{-1}}{-1}\right]_1^4 \\ &= \pi \left[\frac{-1}{x}\right]_1^4 \\ &= \pi \left(-\frac{1}{4} + 1\right) \\ &= \frac{3\pi}{4} \text{ units}^3 \end{aligned}$$

Solids of revolution about the *y*-axis

Consider the curve $y = f(x)$. The area between the curve and the y-axis from $y = a$ to $y = b$ is shown.



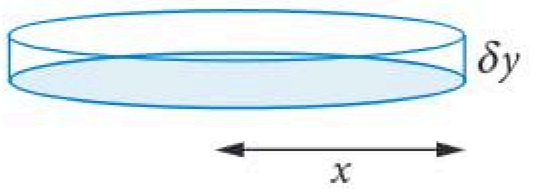
If this area is rotated about the y-axis, it produces a solid of revolution.



If we consider a small section of the volume with a thickness of δy, the shape will be approximately cylindrical with a radius of x.

The volume of the cylinder will be $\delta V = \pi x^2 \delta y$.

The sum of all the cylinders from $y = a$ to $y = b$ can be found from the definite integral $V = \pi\int_a^b x^2 dy$.



Volumes of solids of revolution about the *y*-axis

The volume of a solid of revolution about the y-axis from $y = a$ to $y = b$ is given by

$$V = \pi\int_a^b x^2 dy$$

WB

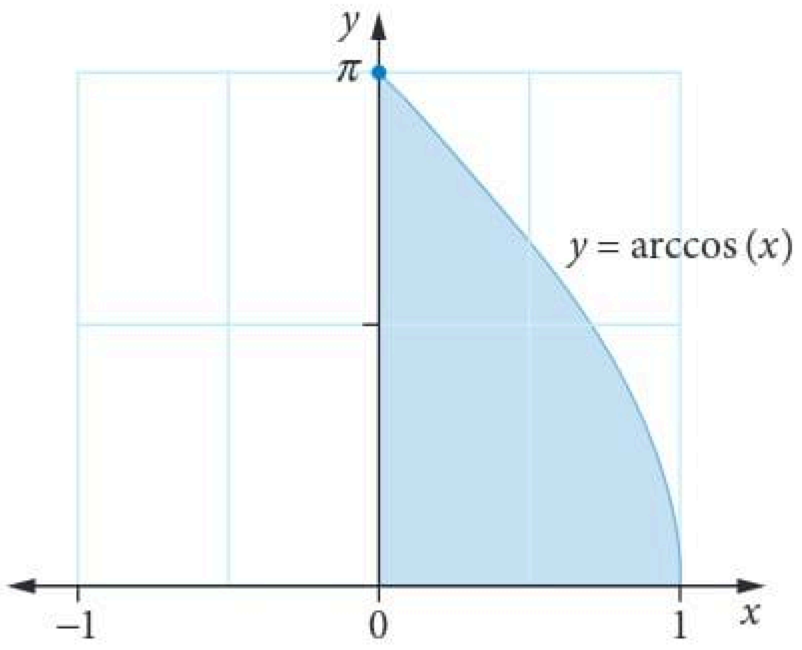
p. 172

WORKED EXAMPLE 8 Finding the volume of a solid of revolution about the *y*-axis

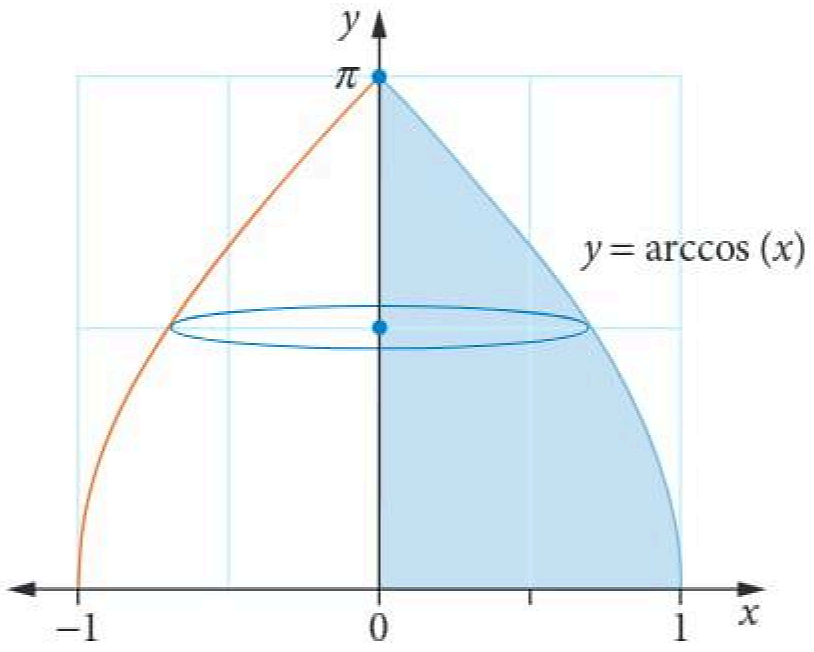
Find the volume generated when the area between the function $y = 2 \arccos(x)$, $x \in [0, 1]$ and the y-axis is rotated about the y-axis.

Steps **Working**

1 Graph the function and identify the area required.



2 Rotate the curve in a circular motion about the y-axis to create a solid of revolution.



3 Write the formula for the volume of a solid of revolution about the y-axis.

$$V = \pi\int_a^b x^2 dy$$

4 Transpose the equation to make x the subject.

$$y = 2\cos^{-1}(x)$$

$$\frac{y}{2} = \cos^{-1}(x)$$

$$x = \cos\left(\frac{y}{2}\right)$$

5 Substitute into the volume formula using $y = 0$ and $y = \pi$ as the lower and upper limits of integration.

$$V = \int_0^\pi \pi\cos^2\left(\frac{y}{2}\right) dy$$

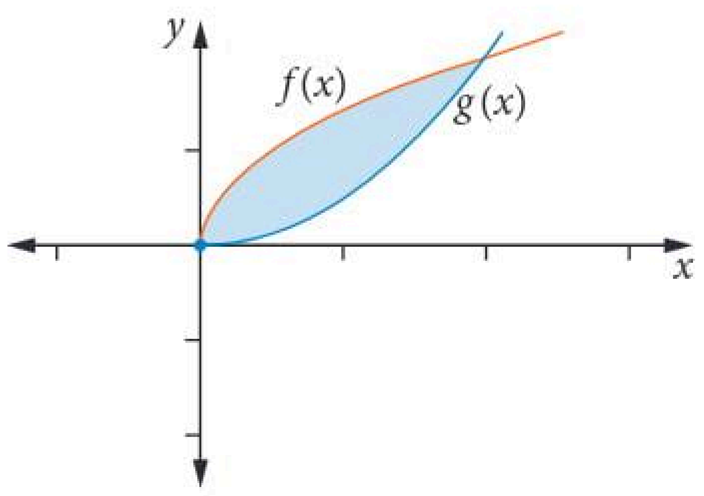
6 Use the double angle rule $\cos^2(x) = \frac{1}{2}[\cos(2x)+1]$ to find the anti-derivative.	$= \pi \int_0^{\pi} \frac{1}{2}[\cos(y)+1]\,dy$ $= \frac{\pi}{2}[\sin(y)+y]_0^{\pi}$ $= \frac{\pi}{2}(\sin(\pi)+\pi-\sin(0)-0)$ $= \frac{\pi^2}{2}$ units^3

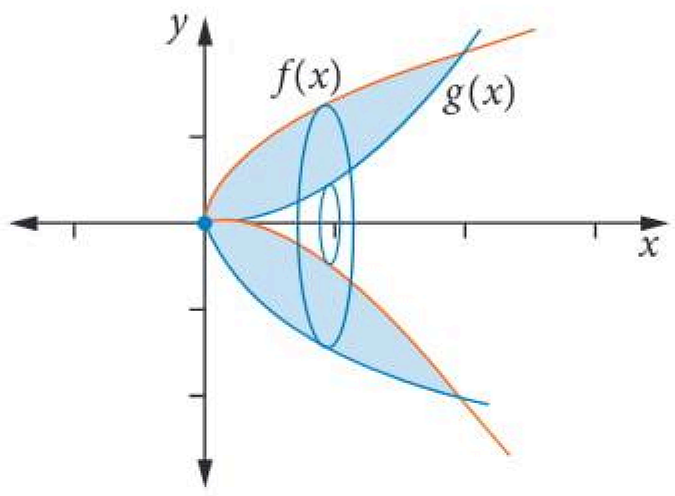
Exam hack

When an area between a curve and the y-axis is rotated about the y-axis to form a solid of revolution, the lower and upper limits of integration are the lowest and highest y values of the solid. A sketch of the volume formed will help identify the limits.

Revolving regions bounded by two curves

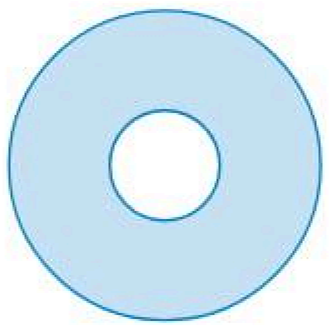
Consider the area bounded by two curves $f(x)$ and $g(x)$, which is rotated about the x-axis.





In the diagram above, the rotated function $f(x)$ will form the outer skin of the solid and the function $g(x)$ will form the inner skin.

The cross-section of this volume will be hollow and ring shaped as shown here.



Revolving regions bounded by two curves

When the area bounded between $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ is rotated about the x-axis, the volume of the solid of revolution is given by

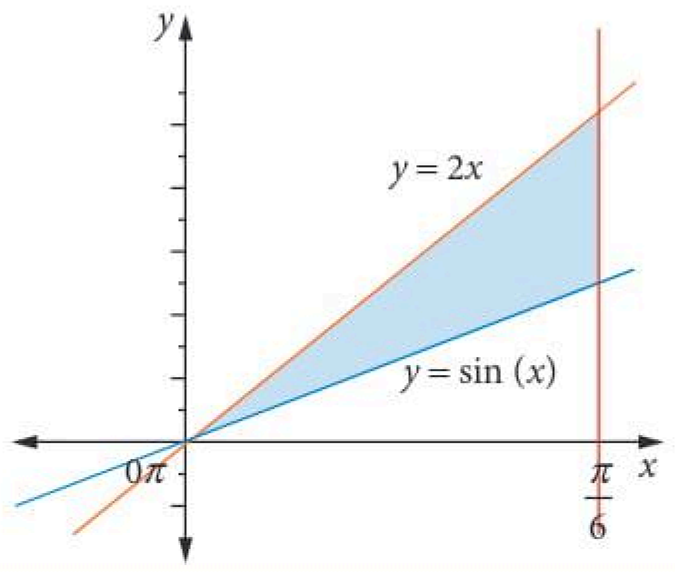
$$V = \pi \int_a^b \left\{[f(x)]^2 - [g(x)]^2\right\} dx$$

where $f(x) > g(x)$ for $x \in [a, b]$.

WORKED EXAMPLE 9 Revolving regions bounded by two curves

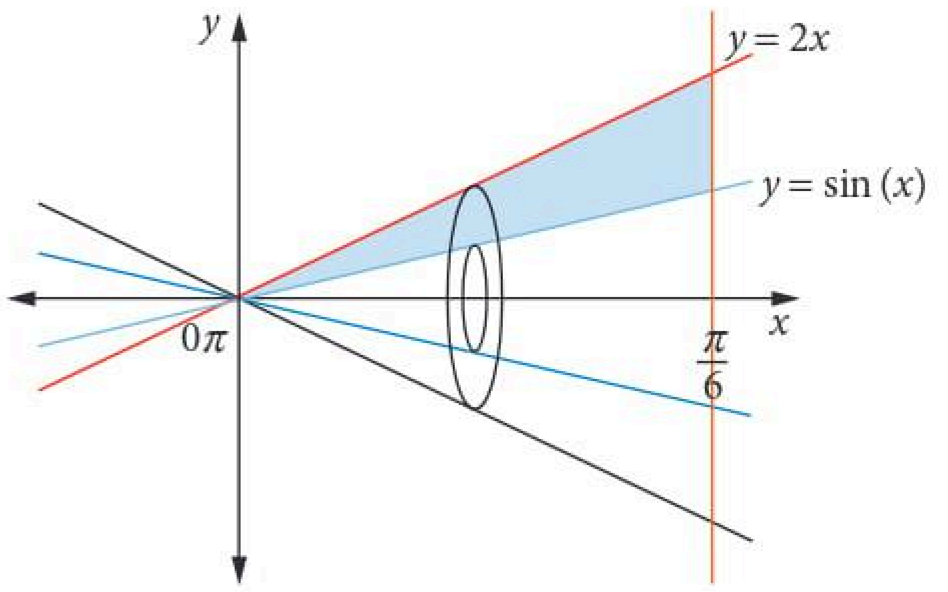
WB p. 173

The area enclosed by $y = \sin(x)$, $y = 2x$ and the line $x = \frac{\pi}{6}$ is rotated about the x-axis to form a solid of revolution. Find the volume of the solid.

Steps	**Working**
1 Graph the function and identify the area required.	

2 The function, when rotated about the x-axis, will form a hollow solid.

Write a definite integral to represent this volume.



$$V = \pi\int_0^{\frac{\pi}{6}} (2x)^2 - \sin^2(x)\,dx$$

3 Integrate and evaluate.

$$= \pi\int_0^{\frac{\pi}{6}} (2x)^2 - \frac{1}{2}\left[1 - \cos(2x)\right] dx$$

$$= \pi\int_0^{\frac{\pi}{6}} 4x^2 - \frac{1}{2} + \frac{1}{2}\cos(2x)\,dx$$

$$= \pi\left[\frac{4x^3}{3} - \frac{1}{2}x + \frac{1}{4}\sin(2x)\right]_0^{\frac{\pi}{6}}$$

$$= \pi\left(\frac{\pi^3}{162} - \frac{\pi}{12} + \frac{1}{4}\sin\left(\frac{\pi}{3}\right)\right)$$

$$= \frac{\pi^4}{162} - \frac{\pi^2}{12} + \frac{\pi\sqrt{3}}{8}\text{ units}^3$$

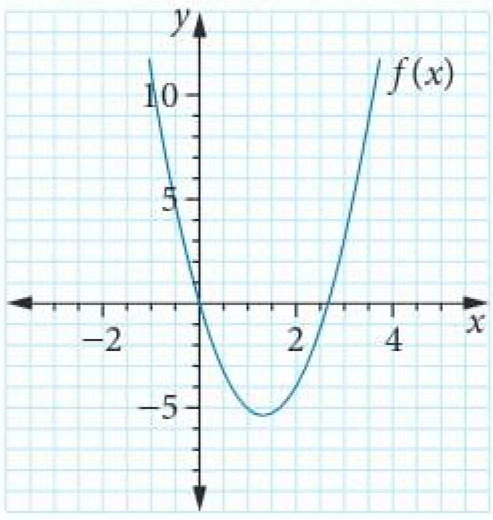
EXERCISE 8.3 Volumes of solids of revolution

ANSWERS p. 591

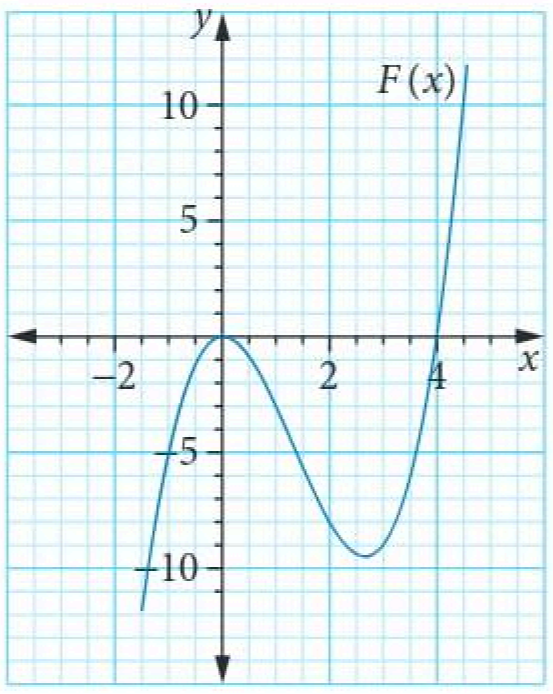
Recap

1 The graph of $f(x)$ is shown on the right, where $f(x) = F'(x)$.

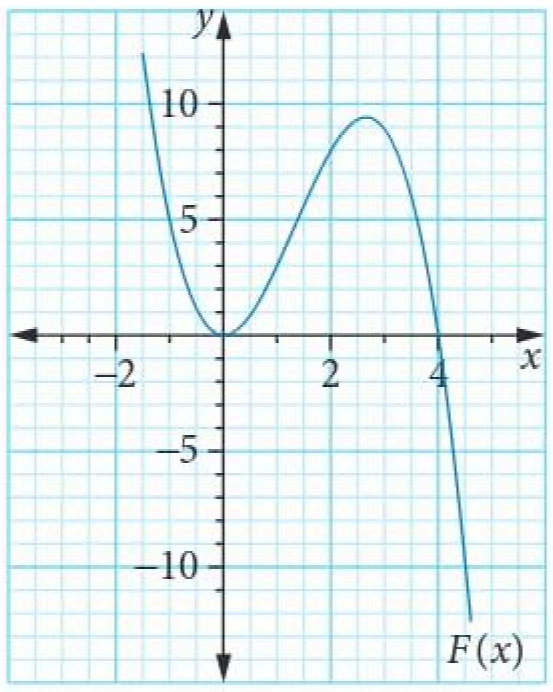
A possible graph of the function F is



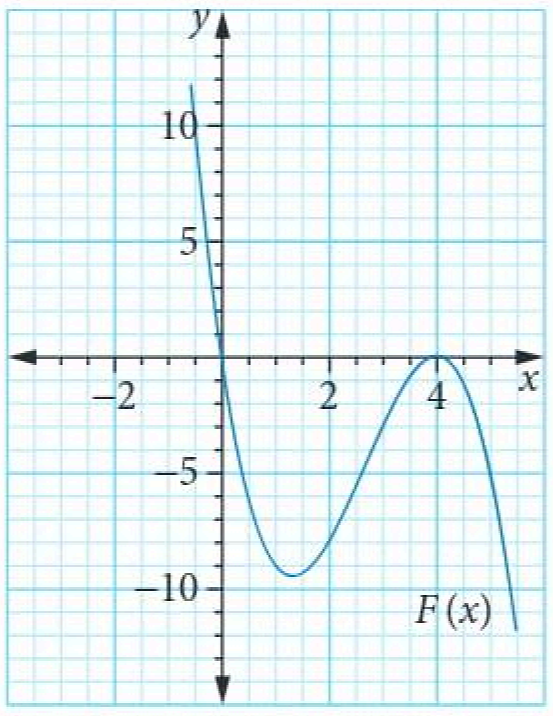
A



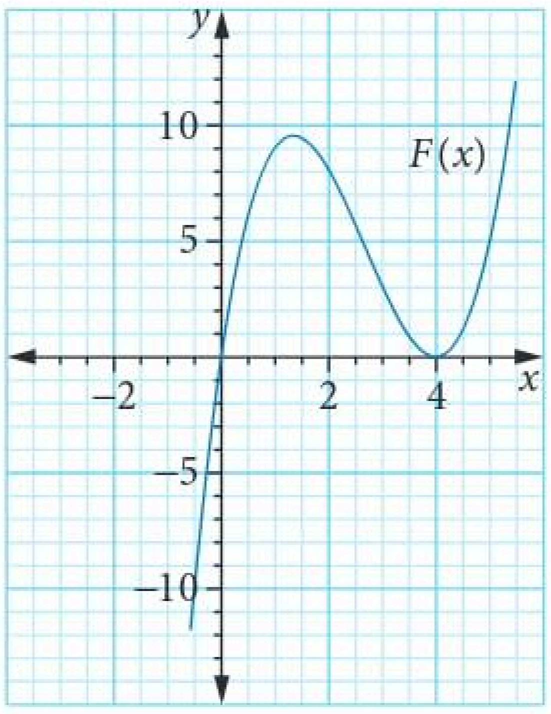
B



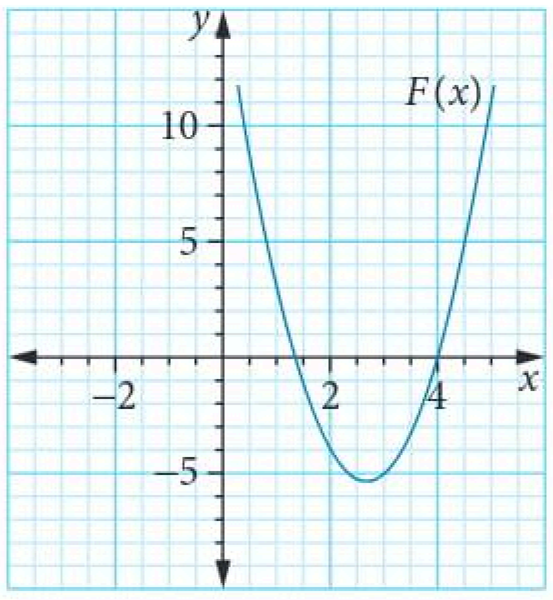
C



D



E

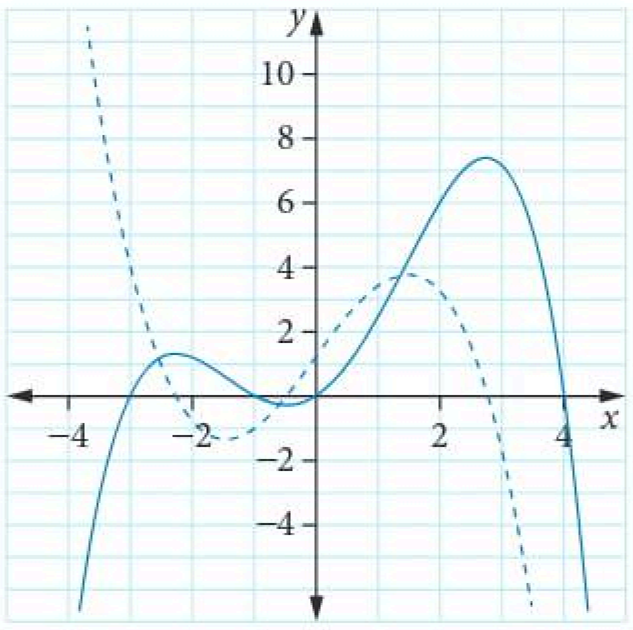


2 The graph of a function *f*, together with one of its anti-derivative functions is shown on the right.

The value of $\int_{-3}^{2} f(x)\,dx$ is closest to

A -6 **B** 5 **C** 6

D 8 **E** 10



Mastery

3 WORKED EXAMPLE 7

a The area bounded by the coordinate axes, the function $f(x) = \dfrac{1}{\sqrt{4 - x^2}}$ and the line $x = 1$ is rotated about the *x*-axis to form a solid of revolution. Find the volume of the solid.

b The area bounded by the *x*-axis and the function $f(x) = \sqrt{9 - x^2}$ is rotated about the *x*-axis to form a solid of revolution. Find the volume of the solid.

4 WORKED EXAMPLE 8

a The area bounded by the curve $y = \sin^{-1}(x)$, the *y*-axis and the line $y = \dfrac{\pi}{3}$ is rotated about the *y*-axis to form a solid of revolution. Find the exact volume of the solid.

b The area bounded by the curve $f: [-2, 2] \to R, f(x) = x^3$ and the *y*-axis is rotated about the *y*-axis to form a solid of revolution. Find the exact volume of the solid.

5 WORKED EXAMPLE 9 The area bounded by the functions $f(x) = x^2$ and $g(x) = 2x$ is rotated about the *x*-axis to form a volume of revolution. Find the volume of the solid.

Exam practice

80–100% 60–79% 0–59%

6 © VCAA 2006 1Q6 TECH-FREE (4 marks) The region in the first quadrant enclosed by the coordinate axes, the graph with equation $y = e^{-x}$ and the straight line $x = a$ where $a > 0$, is rotated about the *x*-axis to form a solid of revolution.

a 70% Express the volume of the solid of revolution as a definite integral. 1 mark

b 60% Calculate the volume of the solid of revolution, in terms of *a*. 1 mark

c 64% Find the exact value of *a* if the volume is $\dfrac{5\pi}{18}$ cubic units. 2 marks

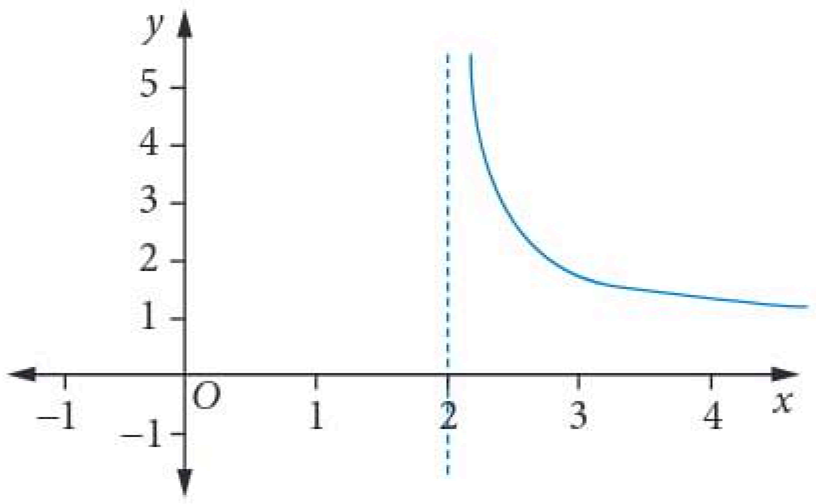
7 © VCAA 2019 1Q8 68% TECH-FREE (4 marks) Find the volume of the solid of revolution formed when the graph of $y = \sqrt{\dfrac{1 + 2x}{1 + x^2}}$ is rotated about the *x*-axis over the interval $[0, 1]$.

8 © VCAA 2011 1Q11 62% TECH-FREE (3 marks) The region in the first quadrant enclosed by the curve $y = \sin(x)$, the line $y = 0$ and the line $x = \dfrac{\pi}{6}$ is rotated about the *x*-axis. Find the volume of the resulting solid of revolution.

9 © VCAA 2014 1Q6 TECH-FREE (5 marks)

a 86% Verify that $\frac{a}{a-4} = 1 + \frac{4}{a-4}$. 1 mark

Part of the graph of $y = \frac{x}{\sqrt{(x^2 - 4)}}$ is shown on the right.

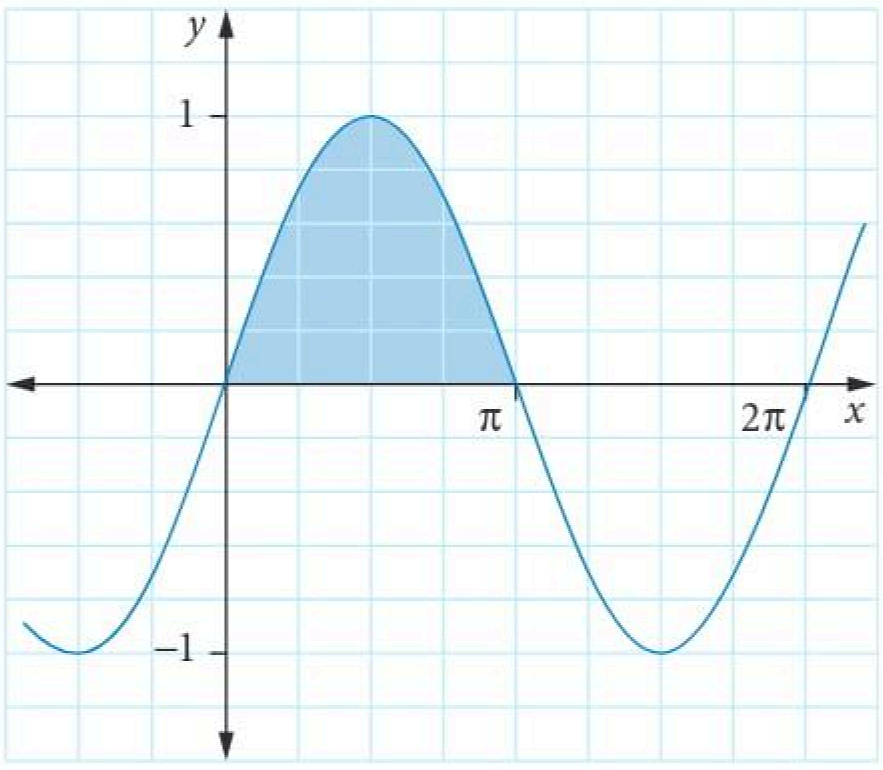


b 55% The region enclosed by the graph of $y = \frac{x}{\sqrt{(x^2 - 4)}}$ and the lines $y = 0$, $x = 3$ and $x = 4$ is rotated about the x-axis.

Find the volume of the resulting solid of revolution. 4 marks

10 © VCAA 2021 1Q4 TECH-FREE (4 marks)

a 69% The shaded region in the diagram below is bounded by the graph of $y = \sin(x)$ and the x-axis between the first two non-negative x-intercepts of the curve, that is, the interval $[0, \pi]$. The shaded region is rotated about the x-axis to form a solid of revolution.

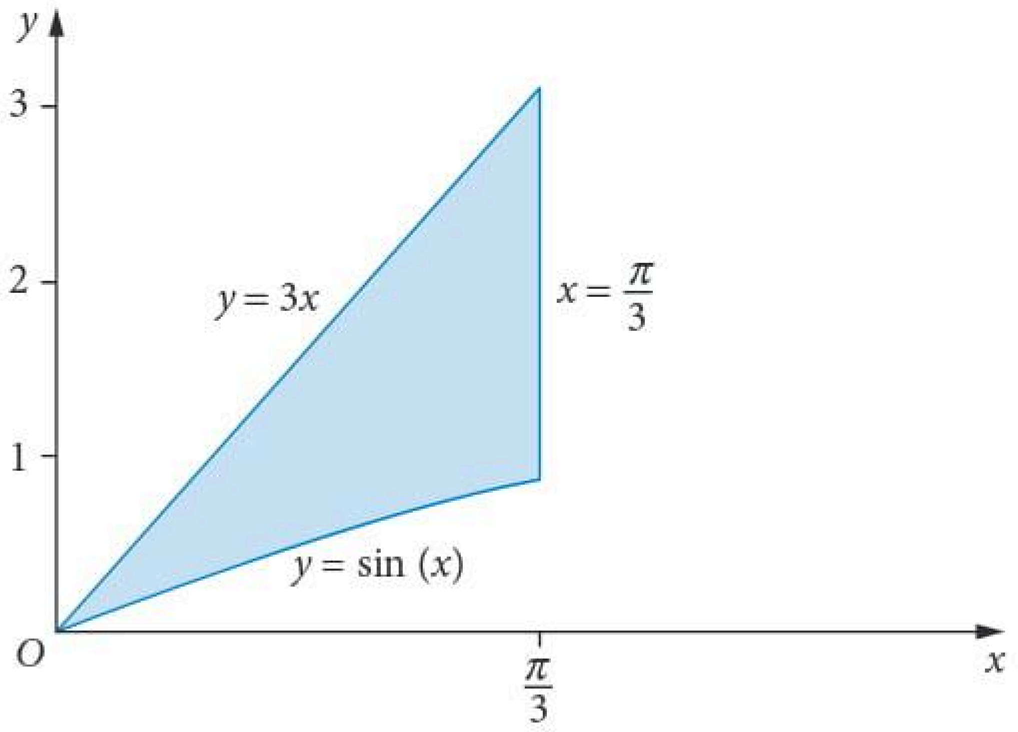


Find the volume, V_s, of the solid formed. 3 marks

b 30% Now consider the function $y = \sin(kx)$, where k is a positive real constant. The region bounded by the graph of the function and the x-axis between the first two non-negative x-intercepts of the graph is rotated about the x-axis to form a solid of revolution.

Find the volume of this solid in terms of V_s. 1 mark

11 © VCAA 2013 1Q9 53% TECH-FREE (4 marks) The shaded region below is enclosed by the graph of $y = \sin(x)$ and the lines $y = 3x$ and $x = \frac{\pi}{3}$. This region is rotated about the x-axis.



Find the volume of the resulting solid of revolution.

12 © VCAA 2015 1Q5 53% TECH-FREE (3 marks) Find the volume generated when the region bounded by the graph of $y = 2x^2 - 3$, the line $y = 5$ and the y-axis is rotated about the y-axis.

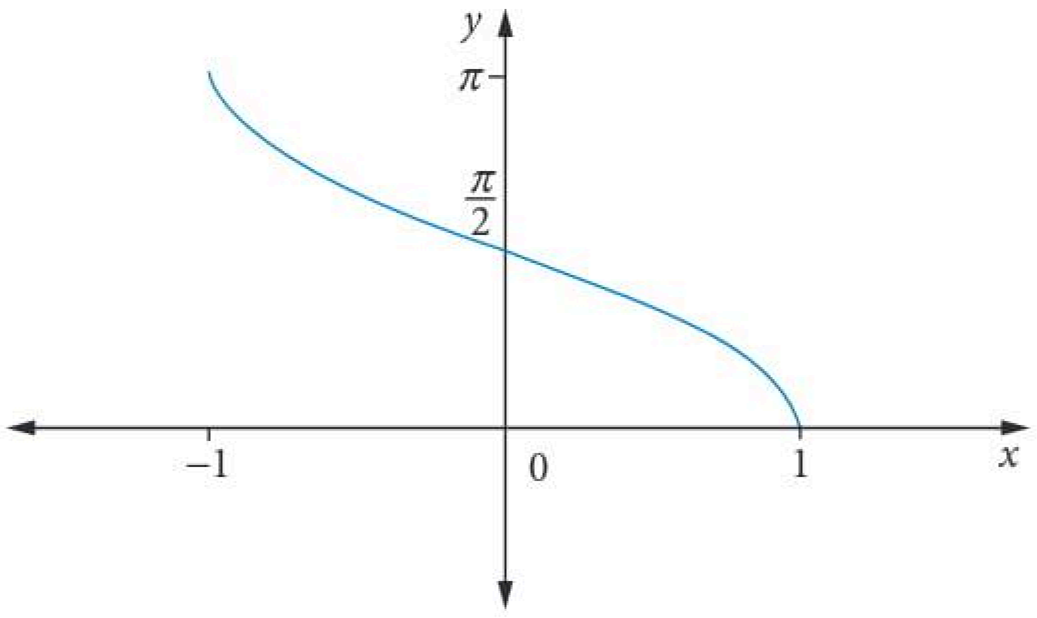
13 © VCAA 2017 1Q10 TECH-FREE (7 marks)

a 85% Show $\frac{d}{dx}\left(x\arccos\left(\frac{x}{a}\right)\right) = \arccos\left(\frac{x}{a}\right) - \frac{x}{\sqrt{a^2 - x^2}}$, where $a > 0$. 1 mark

b 53% State the maximal domain and the range of $f(x) = \sqrt{\arccos\left(\frac{x}{2}\right)}$. 2 marks

c 35% Find the volume of the solid of revolution generated when the region bounded by the graph of $y = f(x)$, and the lines $x = -2$ and $y = 0$, is rotated about the x-axis. 4 marks

14 © VCAA 2008 1Q9 TECH-FREE (4 marks) The graph of $y = \cos^{-1}(x)$, $x \in [-1, 1]$, is shown below.



a 28% Find the area bounded by the graph shown above, the x-axis and the line with equation $x = -1$. 1 mark

b 51% Find the exact volume of the solid of revolution formed if the graph shown above is rotated about the y-axis. 3 marks

15 © VCAA 2013 2AQ10 72% The region bounded by the lines $x = 0$, $y = 3$ and the graph of $y = x^{\frac{4}{3}}$, where $x \geq 0$, is rotated about the **y-axis** to form a solid of revolution.

The volume of this solid is

A $\frac{81\pi 3^{\frac{2}{3}}}{11}$ **B** $\frac{12\pi 3^{\frac{3}{4}}}{7}$ **C** $\frac{27\pi 3^{\frac{1}{3}}}{7}$ **D** $\frac{18\pi 3^{\frac{1}{2}}}{5}$ **E** $\frac{6\pi 3^{\frac{1}{2}}}{5}$

16 © VCAA 2007 2AQ10 62% The curve given by $y = \sin^{-1}(2x)$, where $0 \leq x \leq \frac{\pi}{2}$, is rotated about the y-axis to form a solid of revolution.

The volume of the solid may be found by evaluating

A $\frac{\pi}{4}\int_0^{\frac{\pi}{2}} 1 - \cos(2y)\,dy$ **B** $\frac{\pi}{8}\int_0^{\frac{1}{2}} 1 - \cos(2y)\,dy$ **C** $\frac{\pi}{8}\int_0^{\frac{\pi}{2}} 1 - \cos(2y)\,dy$

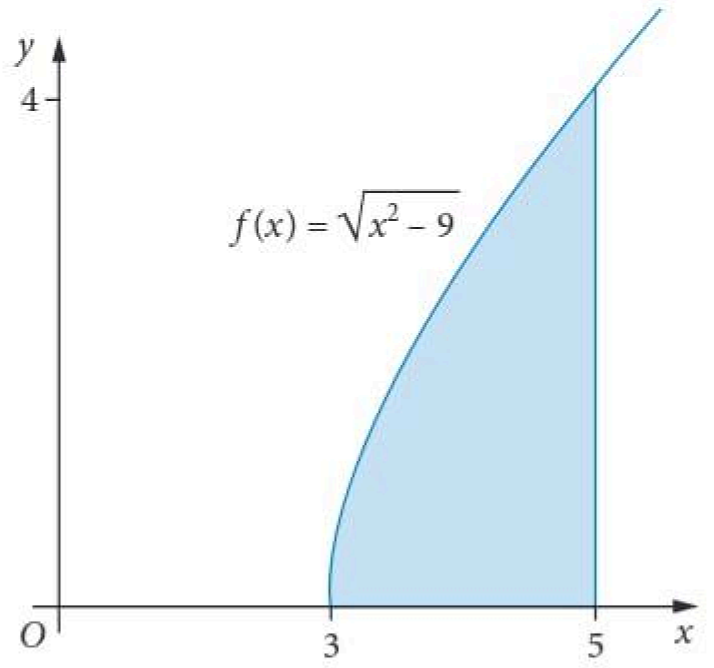
D $\frac{1}{8}\int_0^{\frac{\pi}{2}} 1 - \cos(2y)\,dy$ **E** $\frac{\pi}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 - \cos(2y)\,dy$

17 © VCAA 2012 2AQ12 48% The volume of the solid of revolution formed by rotating the graph of $y = \sqrt{9 - (x-1)^2}$ about the x-axis is given by

A $4\pi(3)^2$ **B** $\pi\int_{-3}^{3}\left(9 - (x-1)^2\right)\,dx$ **C** $\pi\int_{-2}^{4}\left(\sqrt{9 - (x-1)^2}\right)\,dx$

D $\pi\int_{-2}^{4}\left(9 - (x-1)^2\right)^2\,dx$ **E** $\pi\int_{-4}^{2}\left(9 - (x-1)^2\right)\,dx$

18 © VCAA 2001 2AQ16 16% The graph of $f: [3, \infty) \to R$, where $f(x) = \sqrt{x^2 - 9}$, is shown below. The shaded region is bounded by this graph, the x-axis, and the line with equation $x = 5$.



The shaded region is rotated about the y-axis to form a solid of revolution. The volume of this solid, in cubic units, is given by

A $\pi \int_0^4 (y^2 - 9)\, dy$

B $\pi \int_0^4 (34 - y^2)\, dy$

C $\pi \int_0^4 (y^2 + 9)\, dy$

D $\pi \int_0^4 (16 - y^2)\, dy$

E $\pi \int_0^4 \left(5 - \sqrt{y^2 + 9}\right) dy$

19 © VCAA 2014 2BQ1d (3 marks) Consider the function f with rule $f(x) = \dfrac{9}{(x+2)(x-4)}$ over its maximal domain.

The region bounded by the coordinate axes, the graph of f and the line $x = 3$, is rotated about the x-axis to form a solid of revolution.

a 90% Write down a definite integral in terms of x that gives the volume of this solid of revolution. 2 marks

b 77% Find the volume of this solid, correct to two decimal places. 1 mark

20 © VCAA 2013 2BQ1d (3 marks) The region in the first quadrant enclosed by the graph of $\dfrac{(x-1)^2}{9} + \dfrac{y^2}{4} = 1$, the x-axis, and the lines $x = 1$ and $x = 3$ is rotated about the x-axis to form a solid of revolution.

a 86% Write down a definite integral, in terms of x, that gives the volume of this solid of revolution. 2 marks

b 71% Find the volume of this solid of revolution. 1 mark

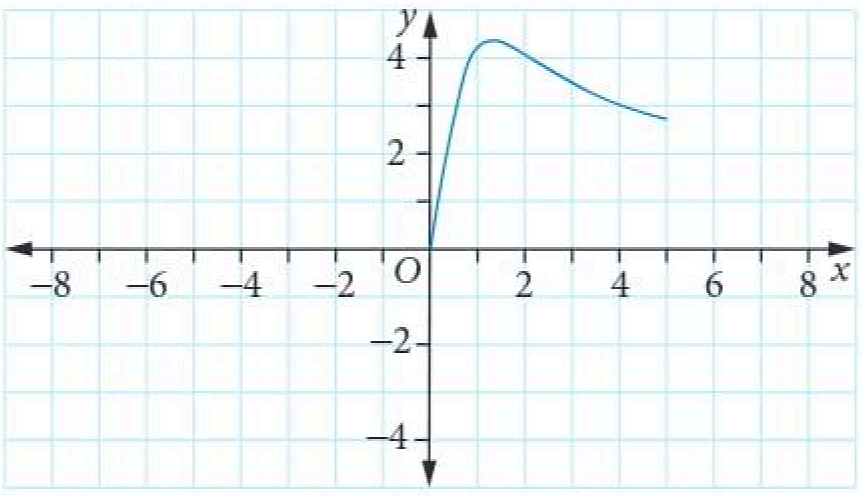
21 © VCAA 2015 2BQ1f (3 marks) Consider the function f with rule $f(x) = \sqrt{2 - \sin^2(x)}$ for $0 \le x \le \dfrac{\pi}{2}$.

The region bounded by the graph of f, the coordinate axes and the line $x = 1$ is rotated about the x-axis to form a solid of revolution.

a 84% Write down a definite integral in terms of x that gives the volume of this solid of revolution. 2 marks

b 71% Find the volume of this solid, correct to one decimal place. 1 mark

22 © VCAA 2006 2BQ1a-c (5 marks) The top part of a wine glass, while lying on its side, is constructed by rotating the graph of $y = \dfrac{6x}{\sqrt{1+x^3}}$ from $x = 0$ to $x = 5$ about the x-axis as shown below.



All lengths are measured in centimetres.

a 84% Write down a definite integral which represents the volume, V cm^3, of the glass. 2 marks

b 64% Use the substitution $u = 1 + x^3$ to write down a definite integral which represents the volume of the glass in terms of u. 2 marks

c 52% Find the value of V correct to the nearest cm^3. 1 mark

23 © VCAA 2012 2BQ1c 67% (2 marks) The region bounded by the curve $(x-1)^2 - \dfrac{y^2}{4} = 1$ and the line $x = 3$ is rotated about the x-axis. Find the volume of the solid formed.

24 © VCAA 2010 2BQ4a-b (6 marks) Consider the function f with rule $f(x) = \sin^{-1}(2x^2 - 1)$.

a 65% Sketch the graph of the relation $y = f(x)$. Label the endpoints with their **exact coordinates** and label the x- and y-intercepts with their exact values. 3 marks

b i 60% Write down a **definite integral** in terms of y, which when evaluated will give the volume of the solid of revolution formed by rotating the graph about the y-axis. 2 marks

ii 43% Find the exact value of the definite integral in part **i**. 1 mark

25 © VCAA 2009 2BQ4a-d (8 marks) Consider the function f with rule $y = \dfrac{x^4 - 1}{x^2}$ over the range $-10 \le y \le 10$.

a 65% The domain of f may be expressed in the form $x \in [-a, -b] \cup [b, a]$, where $a, b > 0$. Find the values of a and b correct to one decimal place. 2 marks

b 70% Sketch the graph of f for $y \in [-10, 10]$, clearly showing the location of the x-intercepts. 2 marks

The rule relating x to y may be rearranged to give $x^4 - yx^2 - 1 = 0$.

c 46% Show that $x^2 = \dfrac{y + \sqrt{y^2 + 4}}{2}$, giving reasons for rejecting any solutions. 2 marks

A glass with a hollow stem, and with its base at $y = -10$, is made by rotating the part of the graph of f where $x > 0$ and $y \in [-10, 10]$ about the y-axis to form a solid of revolution.

d i 57% Write down a definite integral which, when evaluated, would give the volume of the glass. 1 mark

ii 45% If the x- and y-coordinates measure lengths in centimetres, find the volume of the glass in cm^3, correct to one decimal place. 1 mark

26 © VCAA 2008 2BQ1d (5 marks) The graph of $f(x) = \dfrac{6x\sqrt{x}}{(3x^2+1)}$ is rotated about the x-axis between $x = 0$ and $x = \dfrac{1}{\sqrt{3}}$ to form a solid of revolution with volume V.

a 74% Show that $V = 2\pi\displaystyle\int_0^{\frac{1}{\sqrt{3}}} \frac{18x^3}{(3x^2+1)^2}\,dx$. 1 mark

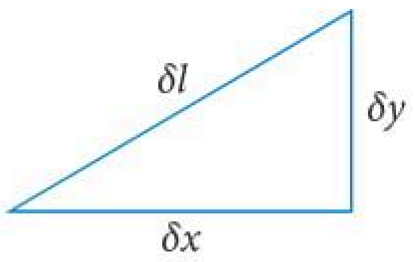
b 42% Use the substitution $u = 3x^2 + 1$ to express V in the form $2\pi\displaystyle\int_a^b \left(\frac{c}{u} + \frac{d}{u^2}\right) du$. 2 marks

c 35% **Hence**, by using an appropriate anti-derivative, find V in exact form. 2 marks

Video playlist
Arc lengths of curves

8.4 Arc lengths of curves

Integration can be used to find the **arc length** of a curve.



Consider the curve of a function $y = f(x)$ that is continuous over the interval $[x_1, x_2]$. The curve can be considered to be made up of lots of very small straight lines.

A very small section of this curve is taken as shown in the diagram.

For this small section, the length of the curve is δl and the increase in x and y are δx and δy.

Using Pythagoras' theorem:
$$(\delta l)^2 = (\delta x)^2 + (\delta y)^2$$
$$\delta l = \sqrt{(\delta x)^2 + (\delta y)^2}$$

This can be simplified further to
$$\delta l = \sqrt{(\delta x)^2\left(1 + \frac{(\delta y)^2}{(\delta x)^2}\right)}$$
$$= \sqrt{\left(1 + \frac{(\delta y)^2}{(\delta x)^2}\right)} \times \delta x$$

The sum of all these individual arc lengths can be found by integration.

Arc length

The arc length, l, of the graph of the function $y = f(x)$ over the interval $x = x_1$ to $x = x_2$ is given by the formulas

$$l = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{or} \quad l = \int_{x_1}^{x_2} \sqrt{1 + [f'(x)]^2}\, dx$$

WB
p. 174

WORKED EXAMPLE 10 The arc length of a curve

Find the arc length of the curve $y = (x-1)^{\frac{3}{2}}$ between $x = 1$ and $x = 5$.

Steps	Working
1 Find $\frac{dy}{dx}$ and $\left(\frac{dy}{dx}\right)^2$.	$\frac{dy}{dx} = \frac{3}{2}(x-1)^{\frac{1}{2}}$ $\left(\frac{dy}{dx}\right)^2 = \frac{9}{4}(x-1)$
2 Use the arc length formula.	$l = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ $= \int_1^5 \sqrt{\left(1 + \frac{9}{4}(x-1)\right)}\, dx$ $= \int_1^5 \sqrt{\left(\frac{9x-5}{4}\right)}\, dx$ $= \int_1^5 \frac{1}{2}(9x-5)^{\frac{1}{2}}\, dx$
3 Find the anti-derivative using the formula: $\int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$	$= \frac{1}{2}\left[\frac{(9x-5)^{\frac{3}{2}}}{9 \times \frac{3}{2}}\right]_1^5$ $= \frac{1}{27}\left[(9x-5)^{\frac{3}{2}}\right]_1^5$

4 Evaluate the definite integral.

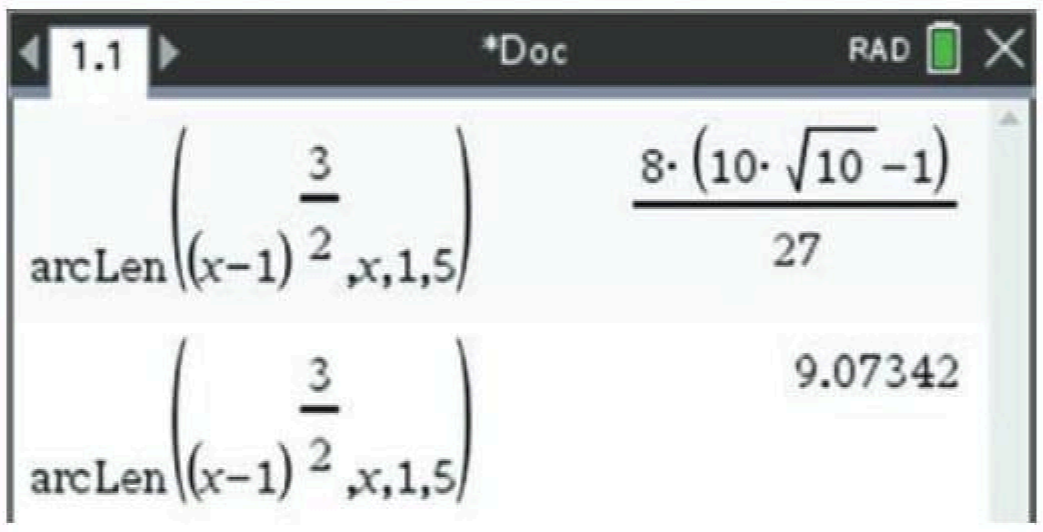
$$= \frac{1}{27}\left(40^{\frac{3}{2}} - 4^{\frac{3}{2}}\right)$$

$$= \frac{80}{27}\sqrt{10} - \frac{8}{27} \text{ units}$$

USING CAS 3 Calculating the length of a curve

Calculate the arc length of the curve $y = (x - 1)^{\frac{3}{2}}$ between $x = 1$ and $x = 5$.

TI-Nspire



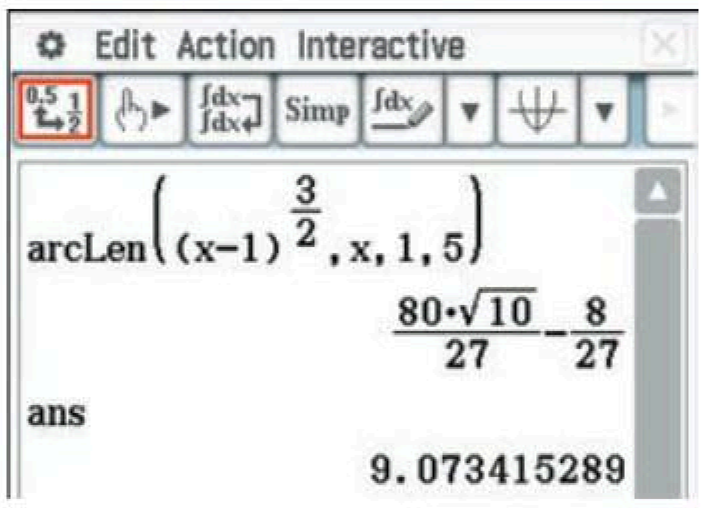
$\text{arcLen}\left((x-1)^{\frac{3}{2}},x,1,5\right)$ $\quad \frac{8\cdot\left(10\cdot\sqrt{10}-1\right)}{27}$

$\text{arcLen}\left((x-1)^{\frac{3}{2}},x,1,5\right)$ $\quad 9.07342$

1 Press **menu > Calculus > Arc Length**.

2 Enter the expression followed by **x,1,5** as shown above.

ClassPad



$\text{arcLen}\left((x-1)^{\frac{3}{2}},x,1,5\right)$

$\frac{80\cdot\sqrt{10}}{27}-\frac{8}{27}$

ans

9.073415289

1 In **Main**, enter and highlight the expression.

2 Tap **Interactive > Calculation > line > arcLen**.

3 In the dialogue box, enter the **Start** and **End** values.

4 Tap **OK**.

5 Tap **Convert** to change the answer to decimal.

The arc length is $\frac{80}{27}\sqrt{10} - \frac{8}{27}$ units.

The length of a curve written in parametric form

Consider the parametric equations for a curve $x = f(t)$ and $y = g(t)$. The length of the curve for parametric functions is found below using Pythagoras' rule.

$$(\delta l)^2 \approx (\delta x)^2 + (\delta y)^2$$

$$\frac{(\delta l)^2}{(\delta t)^2} \approx \frac{(\delta x)^2}{(\delta t)^2} + \frac{(\delta y)^2}{(\delta t)^2}$$

$$\left(\frac{\delta l}{\delta t}\right)^2 \approx \left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2$$

$$\frac{\delta l}{\delta t} \approx \sqrt{\left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2}$$

$$\delta l \approx \sqrt{\left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2}\,\delta t$$

The sum of all these individual arc lengths can be found by integration.

Arc length in parametric form

The arc length, l, of the graph of the parametric functions $x = f(t)$ and $y = g(t)$ over the interval $t = t_1$ to $t = t_2$ is given by the formulas

$$l = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \quad \text{or} \quad l = \int_{t_1}^{t_2} \sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2}\, dt$$

WB p. 175

WORKED EXAMPLE 11 Arc length in parametric form

Find the arc length for the curve defined by the parametric function

$$x = 2\cos(2t),\ y = 2\sin(2t),\ t \in \left[0, \frac{\pi}{6}\right]$$

Steps	Working
1 Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$.	$\frac{dx}{dt} = -4\sin(2t)$ $\frac{dy}{dt} = 4\cos(2t)$
2 Substitute into the formula $l = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$	$l = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ $= \int_0^{\frac{\pi}{6}} [-4\sin(2t)]^2 + [4\cos(2t)]^2\, dt$ $= \int_0^{\frac{\pi}{6}} 16\sin^2(2t) + 16\cos^2(2t)\, dt$
3 Simplify using the identity $\sin^2(x) + \cos^2(x) = 1$	$= \int_0^{\frac{\pi}{6}} \sqrt{16\left(\sin^2(2t) + \cos^2(2t)\right)}\, dt$
4 Evaluate the definite integral.	$= \int_0^{\frac{\pi}{6}} 4\, dt$ $= [4t]_0^{\frac{\pi}{6}}$ $= 4 \times \frac{\pi}{6} - 0$ $= \frac{2\pi}{3}$ units

EXERCISE 8.4 Arc lengths of curves

ANSWERS p. 592

Recap

80–100% 60–79% 0–59%

1 The area bounded by the coordinate axes, the function $f(x) = \frac{1}{\sqrt{25 - x^2}}$ and the line $x = 4$ is rotated about the x-axis to form a solid of revolution. The volume of the solid formed is

A $\pi \sin^{-1}\left(\frac{4}{5}\right)$

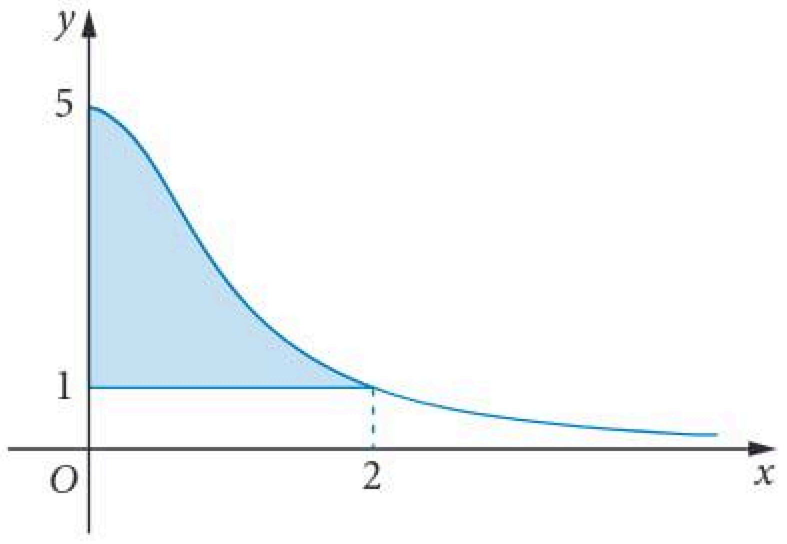
B $\sin^{-1}\left(\frac{4\pi}{5}\right)$

C $\frac{\log_e(3\pi)}{5}$

D $\frac{\pi}{20}$

E $\frac{\pi \log_e(3)}{5}$

2 © VCAA 2005 1IQ13 71% The graph of $f: [0, \infty) \rightarrow R$, where $f(x) = \frac{5}{x^2 + 1}$, is shown. The shaded region is bounded by the graph of f, the y-axis and the line with equation $y = 1$.



The shaded region is rotated about the x-axis to form a solid of revolution. The volume of the solid, in cubic units, is given by

A $\pi\int_0^2\left(\frac{5}{x^2+1}\right)^2-1\,dx$

B $\pi\int_0^2\left(\frac{5}{x^2+1}-1\right)^2dx$

C $\pi\int_0^2\frac{5}{x^2+1}-1\,dx$

D $\pi\int_0^2\left(\frac{5}{x^2+1}\right)^2dx$

E $\pi\int_0^2\left(\frac{4}{x^2+1}\right)^2dx$

Mastery

3 WORKED EXAMPLE 10

a Find the length of the curve $y = 4x^{\frac{3}{2}}$ between $x = 1$ and $x = 9$.

b Find the length of the curve $y^2 = x^3$ between $x = 0$ and $x = 1$.

4 Using CAS 3

a Find the arc length, correct to two decimal places, along the curve $f(x) = xe^x$ between $x = 1$ and $x = 4$.

b Find the arc length, correct to two decimal places, along the curve $f(x) = e^x\sin(x)$ between $x = 0$ and $x = 2$.

5 WORKED EXAMPLE 11

a Find the arc length for the curve defined by the parametric function

$x = 3\cos(t), y = 3\sin(t), t \in \left[0, \frac{\pi}{3}\right]$.

b Find the arc length for the curve defined by the parametric function

$x = 3t + 1, y = 2t^{\frac{3}{2}}, t \in [0,1]$.

Exam practice

6 © VCAA 2016 1Q7 63% TECH-FREE (4 marks) Find the arc length of the curve $y = \frac{1}{3}\left(x^2+2\right)^{\frac{3}{2}}$ from $x = 0$ to $x = 2$.

7 © VCAA 2019N 1Q8 TECH-FREE (4 marks) Find the length of the arc of the curve $y = \frac{x^4}{4} + \frac{1}{8x^2} + 3$ from $x = 1$ to $x = 2$. Give your answer in the form $\frac{a}{b}$, where a and b are positive integers.

8 © VCAA 2018N 1Q7 TECH-FREE (4 marks)

a Find $\frac{d}{dx}\left(\left(1-x^2\right)^{\frac{1}{2}}\right)$. 2 marks

b Hence find the length of the curve specified by $y = \sqrt{1-x^2}$ from $x = \frac{1}{2}$ to $x = \frac{\sqrt{3}}{2}$. Give your answer in the form $k\pi$, $k \in R$. 2 marks

9 © VCAA 2017N 1Q11 TECH-FREE (4 marks) Find the length of the curve specified parametrically by $x = a\theta - a\sin(\theta)$, $y = a - a\cos(\theta)$ from $\theta = \frac{2\pi}{3}$ to $\theta = 2\pi$, where $a \in R^+$. Give your answer in terms of a.

10 © VCAA 2019 2AQ7 70% The length of the curve defined by the parametric equations $x = 3\sin(t)$ and $y = 4\cos(t)$ for $0 \le t \le \pi$ is given by

A $\int_0^{\pi} \sqrt{9\cos^2(t) - 16\sin^2(t)}\, dt$

B $\int_0^{\pi} \sqrt{9 + 7\sin^2(t)}\, dt$

C $\int_0^{\pi} \sqrt{1 + 16\sin^2(t)}\, dt$

D $\int_0^{\pi} \left(3\cos(t) - 4\sin(t)\right) dt$

E $\int_0^{\pi} \sqrt{3\cos^2(t) + 4\sin^2(t)}\, dt$

11 The arc length of the curve $y = x^3$ between $x = 1$ and $x = 3$ is given by

A $\int_1^3 \sqrt{1 + 3x^4}\, dx$

B $\int_1^3 \sqrt{1 + x^6}\, dx$

C $\int_1^3 1 + 9x^4\, dx$

D $3^3 - 1^3$

E $\int_1^3 \sqrt{1 + 9x^4}\, dx$

12 The arc length of the curve $y = \sqrt{x^3}$ between $x = 0$ and $x = 8$ is given by

A $\int_0^8 \frac{\sqrt{1 + 9x}}{4}\, dx$

B $\int_0^8 \frac{\sqrt{4 + 9x}}{4}\, dx$

C $\frac{1}{2}\int_0^8 \sqrt{4 + 9x}\, dx$

D $\int_0^8 \sqrt{1 + \frac{4}{9x^{\frac{2}{3}}}}\, dx$

E $\int_0^8 \sqrt{1 + x^3}\, dx$

13 The arc length of the curve $y = \sqrt{x}$ between $x = 2$ and $x = 4$ is given by

A $\frac{1}{2}\int_2^4 \sqrt{\frac{4x + 1}{x}}\, dx$

B $\frac{1}{2}\int_2^4 \sqrt{4 + x}\, dx$

C $\int_2^4 \sqrt{1 + x}\, dx$

D $\int_2^4 \sqrt{1 + \frac{x}{4}}\, dx$

E $\int_2^4 \sqrt{\frac{x + 4}{4x}}\, dx$

14 The approximate arc length of the curve $y = e^x$ between $x = 1$ and $x = e$, correct to two decimal places, is equal to

A 6.85 **B** 11.45 **C** 12.85 **D** 13.26 **E** 15.85

15 The length of the curve $y = 2x^{\frac{3}{2}}$ between $x = 0$ and $x = 1$ is equal to

A $\frac{2\sqrt[3]{10}}{27} - \frac{2}{27}$

B $\frac{2}{27}\left(\sqrt{10^3} - 1\right)$

C $\frac{2\sqrt{10}}{27} - 1$

D $\frac{2\sqrt{10}}{27} - \frac{2}{27}$

E $\frac{2}{9}$

16 The arc length for the curve defined by the parametric function

$x = 4\cos(2t)$, $y = 4\sin(2t)$, $t \in \left[0, \frac{\pi}{3}\right]$ is given by the definite integral

A $\int_0^{\frac{\pi}{3}} \sqrt{16\cos^2(2t) + 16\sin^2(2t)}\, dt$

B $\int_0^{\frac{\pi}{3}} \sqrt{64\cos^2(2t) + 64\sin^2(2t)}\, dt$

C $\int_0^{\frac{\pi}{3}} \sqrt{8\cos(2t) - 8\sin(2t)}\, dt$

D $\int_0^{\frac{\pi}{3}} 16\cos^2(2t) + 16\sin^2(2t)\, dt$

E $\int_0^{\frac{\pi}{3}} 4\cos^2(2t) + 4\sin^2(2t)\, dt$

17 (3 marks) Determine the length of the curve $x = \frac{2}{3}(y - 1)^{\frac{3}{2}}$ between $y = 1$ and $y = 4$.

18 © VCAA 2016 2BQ1de (3 marks) A glass is to be modelled by rotating the curve that is the part of the graph of $f(x)=\dfrac{4+x^2+x^3}{x}$ where $x \in [-3, -0.5]$, about the y-axis, to form a solid of revolution.

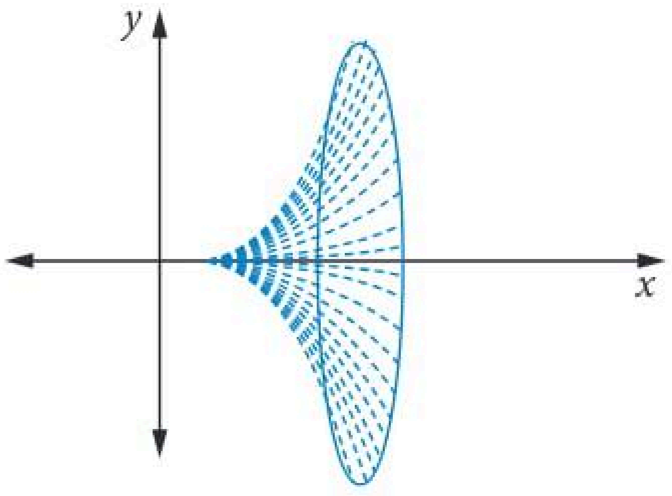
a **i** 73% Write down a definite integral, in terms of x, which gives the length of the curve to be rotated. 1 mark

ii 74% Find the length of this curve, correct to two decimal places. 1 mark

b 34% The volume of the solid formed is given by $V = a\int_c^b x^2\,dy$. Find the values of a, b and c. Do **not** attempt to evaluate this integral. 1 mark

8.5 Surface areas of solids of revolution

Video playlist
Surface areas of solids of revolution



How do we find the surface area of a curve that is rotated about an axis?

Each small section of curve that is rotated about the x- or y-axis forms a cylinder.

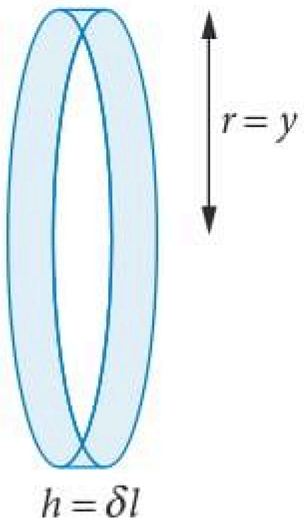
The surface area S of the cylinder can be found using the formula $S = 2\pi rh$, where r is the radius of revolution and h is the length of the curve being rotated.

Solids of revolution about the *x*-axis

For a solid of revolution rotated about the x-axis, the radius of the cylinder is $r = y$ or $f(x)$ and the height of the cylinder is $h = \delta l$, the arc length of the small section of the curve.

But $\delta l = \sqrt{1+\left(\dfrac{\delta y}{\delta x}\right)^2}\,\delta x$.

$$S = 2\pi rh$$



Let δS be the area of the small section of the surface of revolution.

$$\begin{aligned}\text{Then }\ \delta S &= 2\pi rh\\ &= 2\pi y\,\delta l\\ &= 2\pi y\sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2}\,\delta x\end{aligned}$$

The sum of all these individual surface areas δS can be found by integration.

$$\begin{aligned}S &= \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx\\ &= 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx\end{aligned}$$

Surface area of a solid of revolution

If $f(x)$ is a smooth non-negative function in the interval $[a, b]$, the surface area generated by rotating the curve $y = f(x)$ about the x-axis is given by

$$S = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\text{or}\quad S = \int_a^b 2\pi f(x)\sqrt{1+\left(f'(x)\right)^2}\,dx.$$

WB
p. 176

WORKED EXAMPLE 12 Surface area of a solid of revolution

Find the surface area generated by rotating the function $f:[0,1]\to R,\ f(x)=2x^3$ about the x-axis.

Steps	Working
1 Find $f'(x)$.	$f'(x)=6x^2$
2 Substitute into the formula: $S=\int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$	$S=\int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$ $=\int_0^1 2\pi\times 2x^3\sqrt{1+\left(6x^2\right)^2}\,dx$ $=\pi\int_0^1 4x^3\sqrt{1+36x^4}\,dx$
3 Integrate by substitution.	Let $u=1+36x^4$ $\frac{du}{dx}=144x^3,\quad \frac{1}{36}\frac{du}{dx}=4x^3$ When $x=0,\ u=1$ When $x=1,\ u=37$ $S=\frac{\pi}{36}\int_1^{37}\sqrt{u}\,\frac{du}{dx}\,dx$ $=\frac{\pi}{36}\int_1^{37}u^{\frac{1}{2}}\,du$ $=\frac{\pi}{36}\left[\frac{2u^{\frac{3}{2}}}{3}\right]_1^{37}$ $=\frac{\pi}{54}\left(37^{\frac{3}{2}}-1^{\frac{3}{2}}\right)$ $=\frac{\pi}{54}\left(37\sqrt{37}-1\right)$ sq. units

Solids of revolution about the *y*-axis

Surface area of revolution about the *x*-axis

If $g(y)$ is a smooth curve in the interval $[c,d]$, the surface area generated by rotating the curve $x=g(y)$ about the y-axis is given by

$$S=\int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

or $$S=\int_c^d 2\pi g(y)\sqrt{1+\left(g'(y)\right)^2}\,dy$$

WORKED EXAMPLE 13 Surface area of a solid of revolution about the y-axis

Find the surface area generated when the function $y = x^2$ on the interval $\left[0, \sqrt{2}\right]$ is rotated about the y-axis.

Steps	**Working**
1 Find x as a function of y, $\frac{dx}{dy}$ and the new limits of integration.	$y = x^2$ $x = y^{\frac{1}{2}}$ $\frac{dx}{dy} = \frac{1}{2}y^{-\frac{1}{2}} = \frac{1}{2\sqrt{y}}$ When $x = 0$, $y = 0^2 = 0$ When $x = \sqrt{2}$, $y = \left(\sqrt{2}\right)^2 = 2$
2 Substitute into the formula: $S = \int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$	$S = \int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$ $= \int_0^2 2\pi \times \sqrt{y}\sqrt{1+\left(\frac{1}{2\sqrt{y}}\right)^2}\,dy$ $= 2\pi\int_0^2 \sqrt{y}\sqrt{1+\frac{1}{4y}}\,dy$ $= 2\pi\int_0^2 \sqrt{y}\,\frac{\sqrt{4y+1}}{2\sqrt{y}}\,dy$ $= \pi\int_0^2 \sqrt{4y+1}\,dy$ $= \pi\int_0^2 (4y+1)^{\frac{1}{2}}\,dy$
3 Evaluate the integral.	$= \pi\left[\frac{(4y+1)^{\frac{3}{2}}}{4\times\frac{3}{2}}\right]_0^2$ $= \frac{\pi}{6}\left[(4y+1)^{\frac{3}{2}}\right]_0^2$ $= \frac{\pi}{6}(27-1)$ $= \frac{13\pi}{3}$ unit^2

WB
p. 177

Surface area of a solid of revolution in parametric form

Surface area of a solid of revolution in parametric form

The formulas for the surface area of a parametric equation $f\left(x(t), y(t)\right), t \in [t_1, t_2]$ are:

rotation about the x-axis

$$S = \int_{t_1}^{t_2} 2\pi y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\text{or } S = \int_{t_1}^{t_2} 2\pi y(t)\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt$$

rotation about the y-axis

$$S = \int_{t_1}^{t_2} 2\pi x(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\text{or } S = \int_{t_1}^{t_2} 2\pi x(t)\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt$$

WB

p. 178

WORKED EXAMPLE 14 Surface area of a solid of revolution in parametric form

Find the surface area generated by rotating the function with parametric equations $x(t) = 3\cos(t)$ and $y(t) = 3\sin(t)$, $t \in \left[0, \frac{\pi}{2}\right]$ about the x-axis.

Steps	**Working**
1 Find $x'(t)$ and $y'(t)$.	$x'(t) = -3\sin(t)$ $y'(t) = 3\cos(t)$
2 Substitute into the formula: $S = \int_{t_1}^{t_2} 2\pi y(t)\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt$	$S = \int_{t_1}^{t_2} 2\pi y(t)\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt$ $= \int_0^{\frac{\pi}{2}} 2\pi \times 3\sin(t)\sqrt{\left(-3\sin(t)\right)^2 + \left(3\cos(t)\right)^2}\, dt$ $= 6\pi \int_0^{\frac{\pi}{2}} \sin(t)\sqrt{9\sin^2(t) + 9\cos^2(t)}\, dt$ $= 18\pi \int_0^{\frac{\pi}{2}} \sin(t)\, dt$
3 Evaluate the integral.	$= 18\pi\left[-\cos(t)\right]_0^{\frac{\pi}{2}}$ $= -18\pi\left(\cos\left(-\frac{\pi}{2}\right) - \cos(0)\right)$ $= 18\pi \text{ units}^2$

VCE QUESTION ANALYSIS

© VCAA 2017 2BQ1 2017 Examination 2 Section B Question 1 (11 marks)

Let $f: D \to R,\ f(x) = \dfrac{x}{1+x^3}$, where D is the maximal domain of f.

a **i** Find the equations of any asymptotes of the graph of f. 1 mark

ii Find $f'(x)$ and state the coordinates of any stationary points of the graph of f, correct to two decimal places. 2 marks

iii Find the coordinates of any points of inflection of the graph of f, correct to two decimal places. 2 marks

b Sketch the graph of $f(x) = \dfrac{x}{1+x^3}$ from $x = -3$ to $x = 3$ marking all stationary points, points of inflection and intercepts with axes, labelling them with their coordinates. Show any asymptotes and label them with their equations. 3 marks

c The region S, bounded by the graph of $f(x) = \dfrac{x}{1+x^3}$, the x-axis and the line $x = 3$, is rotated about the x-axis to form a solid of revolution. The line $x = a$, where $0 < a < 3$, divides the region S into two regions such that, when the two regions are rotated about the x-axis, they generate solids of equal volume.

i Write down an equation involving definite integrals that can be used to determine a. 2 marks

ii Hence, find the value of a, correct to two decimal places. 1 mark

Video
VCE question analysis: Areas and volumes of integration

Reading the question

- Highlight the required answer in each part. This may be an equation, a graph or a value.
- Highlight the number of decimal places required where the answer needs to be approximated.
- Highlight the labelling required on the graph.

Thinking about the question

- The question requires the use of differentiation and anti-differentiation.
- You will need to understand how the derivative of a function is used to find stationary points and points of inflection.
- You will need to know the concavity conditions required for a point of inflection.
- You will need to understand how to calculate the volume of a solid of revolution.

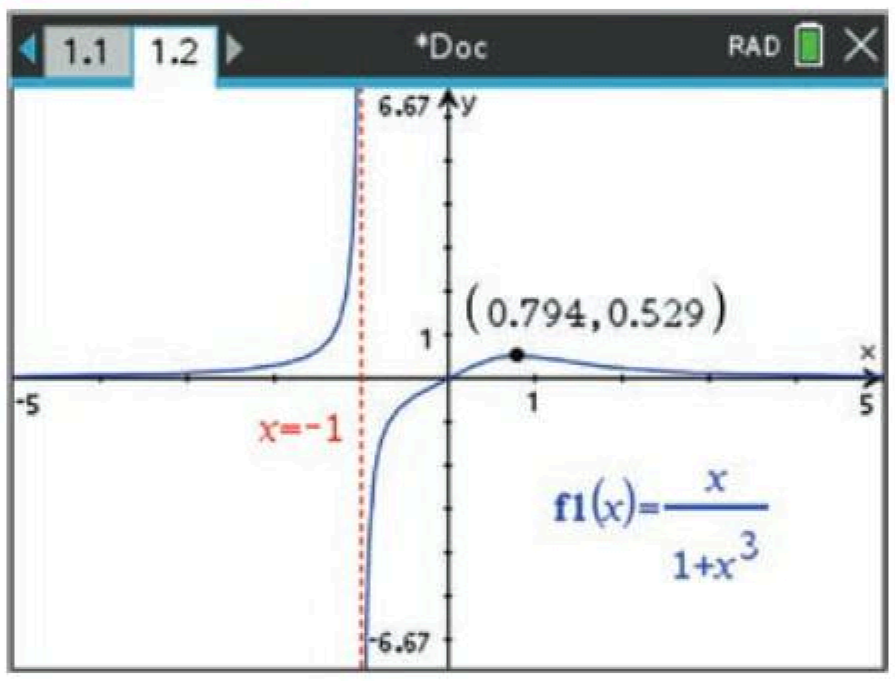
Worked solution (✓ = 1 mark)

a **i** Horizontal asymptote $y = 0$

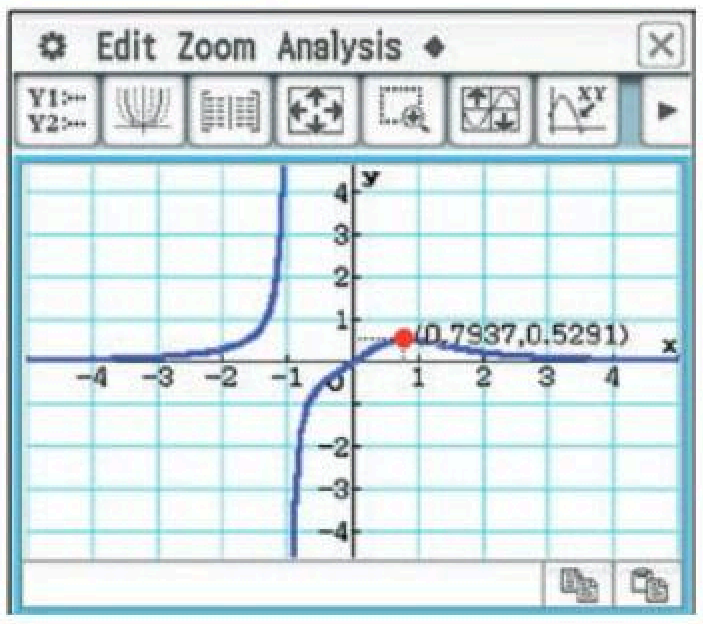
$$\begin{aligned} 1 + x^3 &\neq 0 \\ x^3 &\neq -1 \\ x &\neq -1 \end{aligned}$$

Vertical asymptote $\boldsymbol{x = -1}$ ✓

TI-Nspire



ClassPad



ii $f'(x) = \dfrac{1-2x^3}{\left(x^3+1\right)^2}$

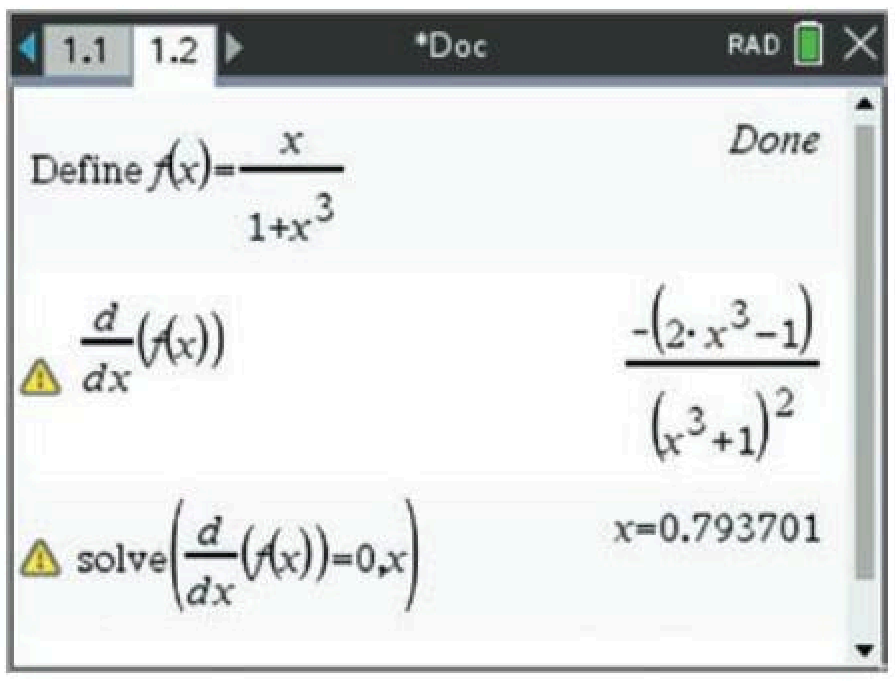
$f'(x) = 0$

$\boldsymbol{x \approx 0.79}$ ✓

$f(0.79) \approx 0.53$

Stationary point is **(0.79, 0.53)**. ✓

TI-Nspire

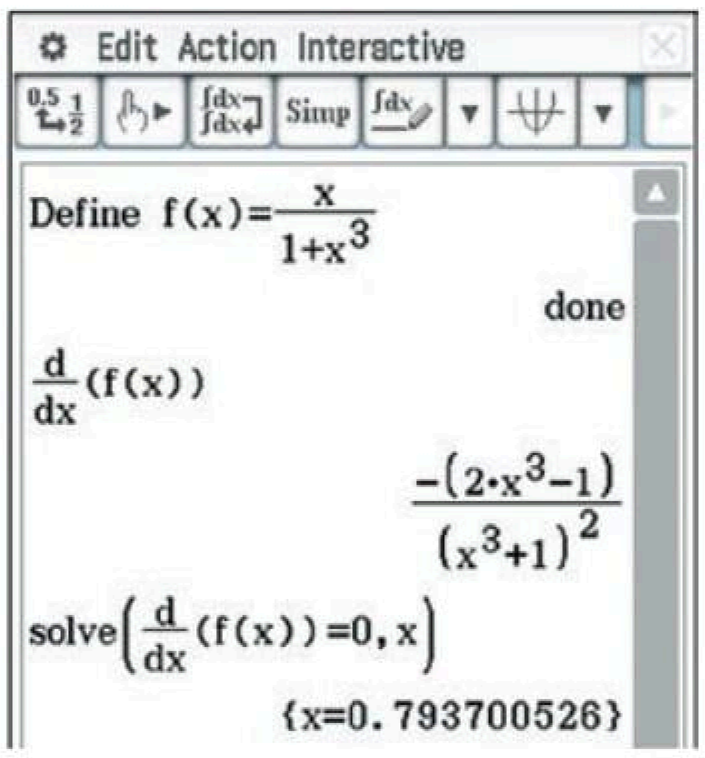


Define $f(x)=\dfrac{x}{1+x^3}$ Done

$\dfrac{d}{dx}(f(x))$ $\dfrac{-(2\cdot x^3-1)}{(x^3+1)^2}$

solve$\left(\dfrac{d}{dx}(f(x))=0,x\right)$ $x=0.793701$

ClassPad



Define $f(x)=\dfrac{x}{1+x^3}$

done

$\dfrac{d}{dx}(f(x))$

$\dfrac{-(2\cdot x^3-1)}{(x^3+1)^2}$

solve$\left(\dfrac{d}{dx}(f(x))=0,x\right)$

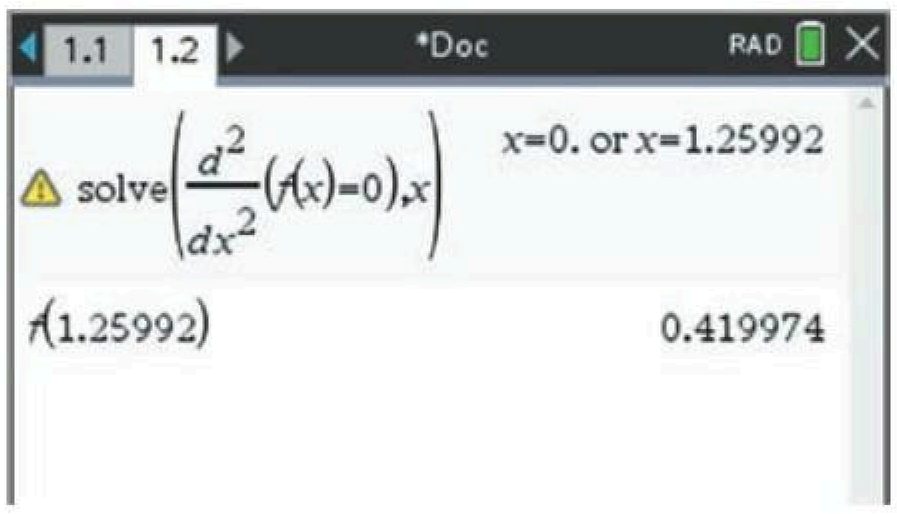
$\{x=0.793700526\}$

iii $f''(x) = 0$

$\boldsymbol{x = 0, 1.26.}$ ✓

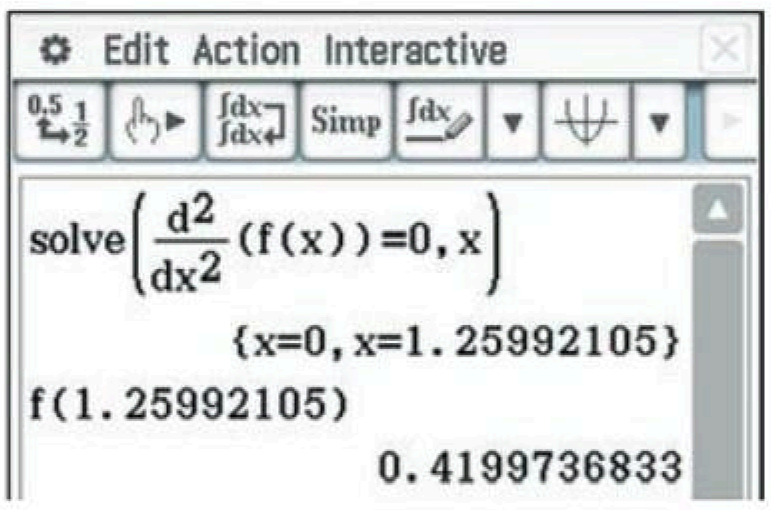
The point (0, 0) is not a point of inflection as $f''(x) < 0$ where $x < 0$ and $x > 0$ (no change in concavity).

Point of inflection is **(1.26, 0.42).** ✓



solve$\left(\dfrac{d^2}{dx^2}(f(x))=0,x\right)$ $x=0.$ or $x=1.25992$

$f(1.25992)$ 0.419974



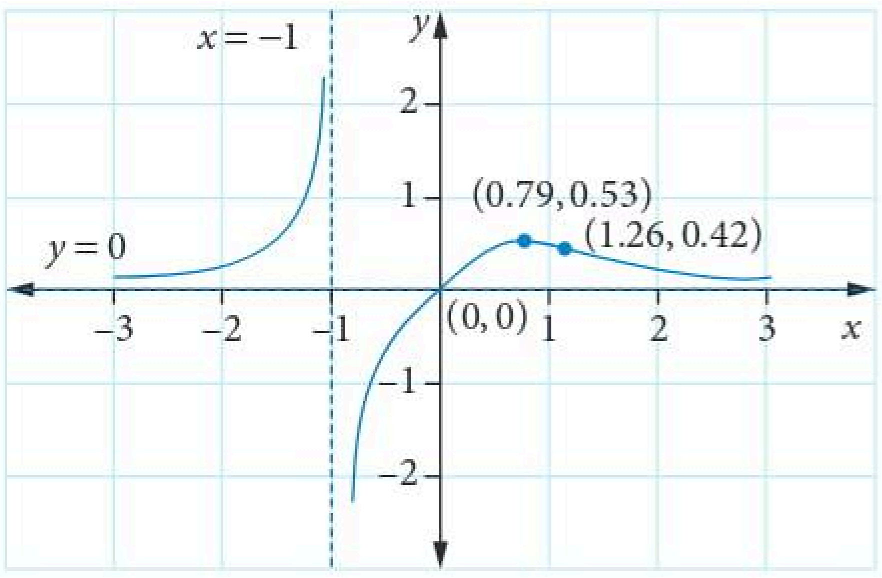
solve$\left(\dfrac{d^2}{dx^2}(f(x))=0,x\right)$

$\{x=0, x=1.25992105\}$

$f(1.25992105)$

0.4199736833

b



Correct shape with asymptotes ✓

Correct turning point coordinate ✓

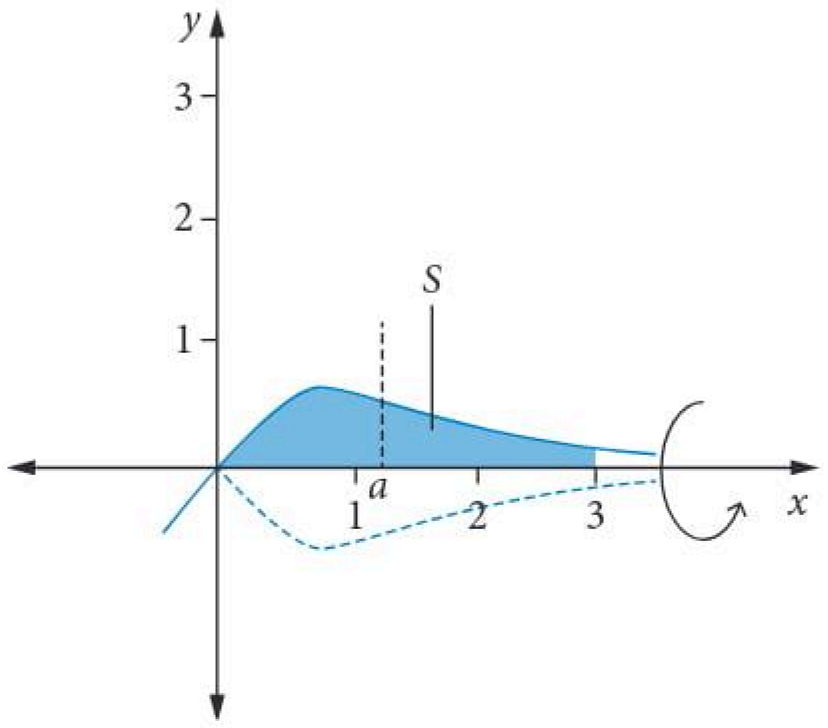
Correct point of inflection coordinates ✓

c **i** $\pi\int_0^a (f(x))^2 dx = \pi\int_a^3 (f(x))^2 dx$

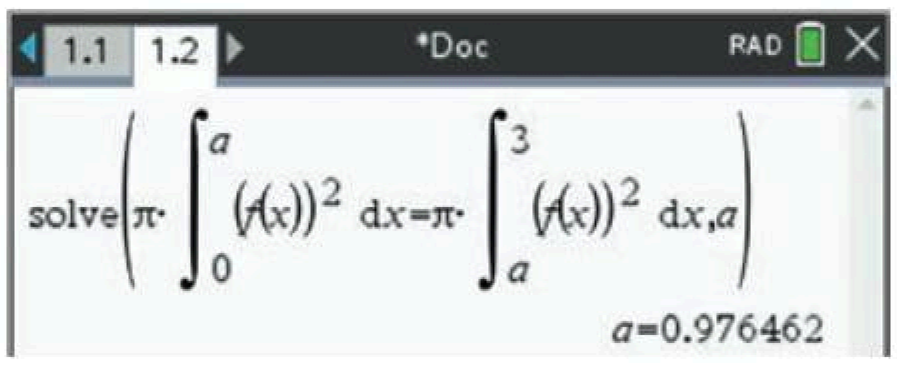
Correct left-hand side equation ✓

Correct right-hand side equation ✓

ii $a = 0.98$ ✓



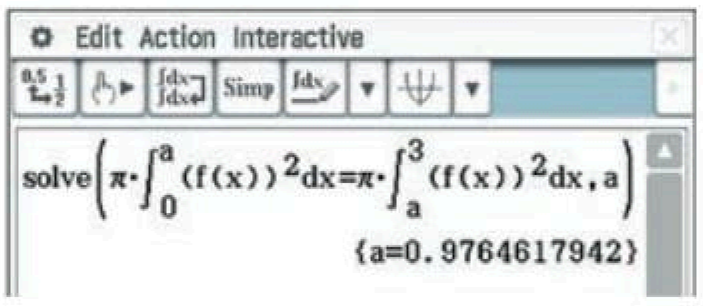
TI-Nspire



$\text{solve}\left(\pi\cdot\int_0^a (f(x))^2\ dx=\pi\cdot\int_a^3 (f(x))^2\ dx,a\right)$

$a=0.976462$

ClassPad



$\text{solve}\left(\pi\cdot\int_0^a (f(x))^2 dx=\pi\cdot\int_a^3 (f(x))^2 dx,a\right)$

$\{a=0.9764617942\}$

Student performance

80–100% 60–79% 0–59%

a **i** 36% Most students were able to write $x = -1$ but omitted $y = 0$.

ii 90% Well done.

iii 52% Also well done. A common error was to include the point (0, 0), which is another point where $f''(x) = 0$, but it is not a point of inflection.

b 83% Generally well done. In some cases, the shape of the graph was unclear and points were not labelled or positioned precisely.

c **i** 78% A common mistake was forgetting to square $f(x)$ or include π.

ii 65% Most students who obtained the correct answer in part **i** were able to achieve the correct answer in part **ii**.

EXERCISE 8.5 Surface areas of solids of revolution

ANSWERS p. 592

Recap

1 The arc length of the curve $y = \sqrt{x^5}$ between $x = 1$ and $x = 2$ is given by

A $\int_1^2 \frac{\sqrt{1+25x^3}}{4}\,dx$

B $\int_1^2 \frac{\sqrt{1+25x}}{4}\,dx$

C $\frac{1}{2}\int_1^2 \sqrt{4+25x^3}\,dx$

D $\int_1^2 \sqrt{1+\frac{5x^{\frac{3}{2}}}{2}}\,dx$

E $\int_1^2 \sqrt{1+x^3}\,dx$

2 The arc length for the curve defined by the parametric function

$x = 3\cos(3t)$, $y = 3\sin(3t)$, $t \in \left[0, \frac{\pi}{3}\right]$ is given by the definite integral

A $\int_0^{\frac{\pi}{3}} \sqrt{9\cos^2(3t)+9\sin^2(3t)}\ dt$

B $9\int_0^{\frac{\pi}{3}} \sqrt{\cos^2(3t)+\sin^2(3t)}\ dt$

C $\int_0^{\frac{\pi}{3}} \sqrt{81\cos(3t)-81\sin(3t)}\ dt$

D $\int_0^{\frac{\pi}{3}} 27\cos^2(3t)+27\sin^2(3t)\,dt$

E $3\int_0^{\frac{\pi}{3}} \sqrt{\cos^2(3t)+\sin^2(3t)}\ dt$

Mastery

3 WORKED EXAMPLE 12 Find the surface area generated by revolving the curve of each function about the x-axis.

a $y = 3x, 0 \leq x \leq 1$ **b** $y = \sqrt{9 - x^2}, -2 \leq x \leq 2$ **c** $y = \sqrt{x}, 0 \leq x \leq 6$

4 WORKED EXAMPLE 13 Find the surface area generated by revolving the curve of each function about the y-axis.

a $y = 5x - 2, y \in [0, 3]$ **b** $x = y^3, y \in [0, 1]$ **c** $y = x^2, x \in [1, 3]$

5 WORKED EXAMPLE 14 Determine the surface area generated by revolving each pair of parametric functions about an axis.

a $x(t) = 3t - 4, y(t) = t, t \in [0, 2]$ rotated about the x-axis.

b $x(t) = 5t + 1, y(t) = 2t - 1, t \in [1, 3]$ rotated about the y-axis.

c $x(t) = 4\cos(2t), \ y(t) = 4\sin(2t), t \in \left[0, \frac{\pi}{4}\right]$ rotated about the x-axis.

Exam practice

6 TECH-FREE (3 marks) Find the surface area formed when $f(x) = 2\sqrt{2 - x}$, between -2 and 0 is rotated about the x-axis.

7 TECH-FREE (3 marks) Find the surface area formed when $f(x) = \sqrt{x - 4}$, between 4 and 5 is rotated about the x-axis.

8 TECH-FREE (3 marks) Find the surface area formed by rotating the curve $f(x) = 4x^2, x \in [0, 1]$ about the y-axis.

9 TECH-FREE (3 marks) Determine the surface area generated by revolving the parametric function, $x(t) = 4t - 3, y(t) = 3t + 1, t \in [0, 3]$, about the x-axis.

10 The function $h: [1, e^3] \to R, h(x) = \log_e(x)$ is rotated about the y-axis to form a solid of revolution. The surface area of the solid is given by the integral equation

A $\int_0^3 2\pi e^y \left(1 + e^{2y}\right) dy$

B $\int_1^{e^3} 2\pi \log_e(x)\left(1 + \frac{1}{x^2}\right)^{\frac{1}{2}} dx$

C $\int_0^3 2\pi \log_e(y)\sqrt{1 + \frac{1}{x^2}}\, dy$

D $\int_1^{e^3} 2\pi e^y \left(1 + \frac{1}{y^2}\right)^{\frac{1}{2}} dy$

E $\int_0^3 2\pi e^y \left(1 + e^{2y}\right)^{\frac{1}{2}} dy$

11 The function $g: \left[\frac{\pi}{6}, \frac{\pi}{3}\right] \to R, g(x) = \cos(2x)$ is rotated about the x-axis to form a solid of revolution.

The surface area of the solid is given by the integral

A $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\pi \cos(2x)\sqrt{1 + 4\sin^2(2x)}\, dx$

B $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\pi \sin(2x)\left(1 + \cos^2(2x)\right) dx$

C $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\pi \cos(x)\sqrt{1 + \sin^2(x)}\, dx$

D $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\pi \cos(2x)\sqrt{1 - 4\sin^2(2x)}\, dx$

E $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\pi \cos(2x)\sqrt{\left(1 - \sin^2(2x)\right)}\, dx$

12 (3 marks) A group of engineering students is designing a parabolic satellite dish whose shape will be formed by rotating the curve $y = ax^2$ about the y-axis. The dish is to have a diameter of 6 metres and a maximum depth of 2 metres.

a Find the value of a. 1 mark

b Express the surface area as a definite integral. 1 mark

c Find the surface area of the dish. 1 mark

13 (3 marks) The surface of a football is obtained by rotating half an ellipse with the parametric equations $x = \sqrt{2}\cos(t),\ y = \sin(t), t \in [0, \pi]$ about the x-axis.

a Find the cartesian equation of the ellipse. 1 mark

b Express the surface area as a definite integral. 1 mark

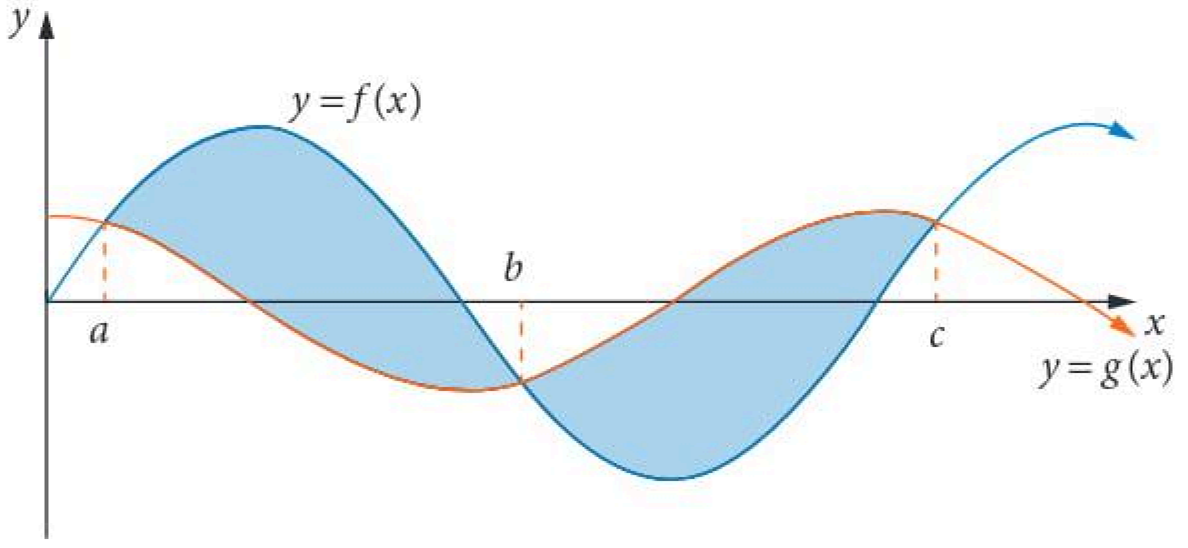
c Find the surface area of the dish. 1 mark

8 Chapter summary

Area between a curve and the *x*-axis

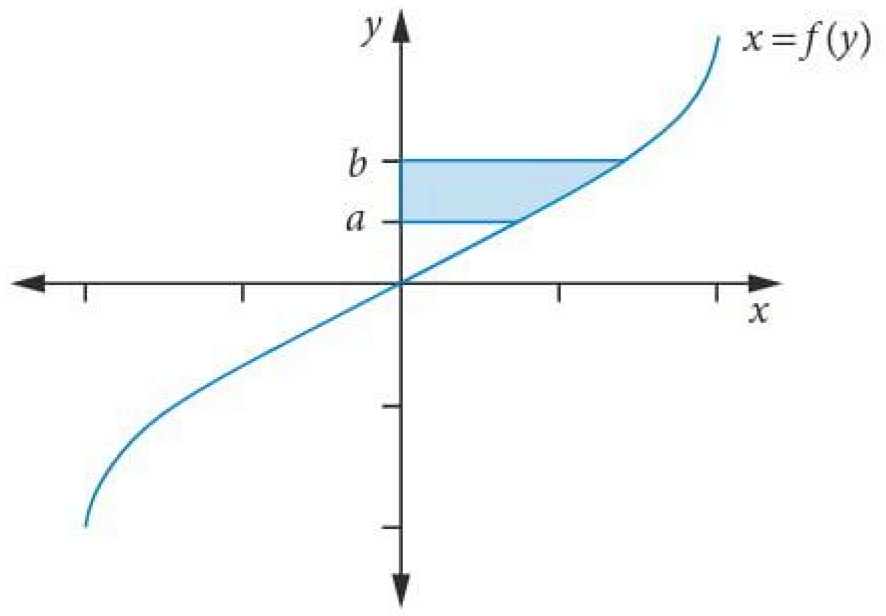
- The definite integral $\int_a^b f(x)\,dx = F(a) - F(b)$ is the area between the curve $f(x)$ and the x-axis from $x = a$ to $x = b$, where $f(x) \geq 0$ for all $x \in [a,b]$.

The area between two curves



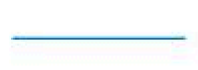
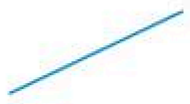
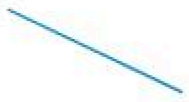
- The area bounded by the two curves can be found by calculating the definite integral of the top function – the bottom function using the intersection points as the limits of integration.
- Area between two curves = $\int (\text{upper curve} - \text{lower curve})\,dx$
- Area = $\int_a^b f(x) - g(x)\,dx + \int_b^c g(x) - f(x)\,dx$

The area between a curve and the *y*-axis



- For the function $x = f(y)$, the shaded area between the function and the y-axis is given by the definite integral $\int_0^b f(y)\,dy$.
- To find the area between the function and the y-axis by integration:
  - transpose the function to express x in terms of y
  - find the definite integral of this function with respect to y using y values as the limits of integration.

Properties of a function *f*(*x*) and its anti-derivative *F*(*x*)

The gradient function $f(x)$	The anti-derivative $F(x)$	Shape of $F(x)$
$f(x) = 0$: the graph intersects the x-axis	$F(x)$ has zero gradient: the graph has a stationary point	
$f(x) > 0$: the graph is above the x-axis	$F(x)$ has a positive gradient: the graph is increasing	
$f(x) < 0$: the graph is below the x-axis	$F(x)$ has a negative gradient: the graph is decreasing	

The volume of a solid of revolution about the *x*-axis

$$V = \pi\int_a^b y^2 dx$$

The volume of a solid of revolution about the *y*-axis

$$V = \pi\int_a^b x^2 dy$$

The length of a curve

The **arc length**, l, of the graph of the function $y = f(x)$ over the interval $x = x_1$ to $x = x_2$ is

$$l = \int_{x1}^{x2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

or

$$l = \int_{x1}^{x2} \sqrt{1+[f'(x)]^2}dx$$

The arc length, l, of the graph of the parametric functions $x = f(t)$ and $y = g(t)$ over the interval $t = t_1$ to $t = t_2$ is

$$l = \int_{t_1}^{t2} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

or

$$l = \int_{t_1}^{t2} \sqrt{[x'(t)]^2+[y'(t)]^2}dt$$

The surface area of a solid of revolution about the *x*-axis

$$S = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

or

$$S = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\ dx$$

The surface area of a solid of revolution about the *y*-axis

$$S = \int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\ dy \text{ or}$$

or

$$S = \int_c^d 2\pi g(y)\sqrt{1+(g'(y))^2}\ dy$$

The surface area of a solid of revolution in parametric form

For $f(x(t), y(t))$, $t \in [t_1, t_2]$:

rotation about the *x*-axis

$$S = \int_{t_1}^{t2} 2\pi y(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

or

$$S = \int_{t_1}^{t2} 2\pi y(t)\sqrt{(x'(t))^2+(y'(t))^2}\ dt$$

rotation about the *y*-axis

$$S = \int_{t_1}^{t2} 2\pi x(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

or

$$S = \int_{t_1}^{t2} 2\pi x(t)\sqrt{(x'(t))^2+(y'(t))^2}\ dt$$

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 30 minutes

TECH-FREE *Technology is NOT permitted.*

1 © VCAA 2018 1Q9 (5 marks)

A curve is specified parametrically by $\underset{\sim}{r}(t) = \sec(t)\underset{\sim}{i} + \frac{\sqrt{2}}{2}\tan(t)\underset{\sim}{j}, t \in R$.

a Show that the cartesian equation of the curve is $x^2 - 2y^2 = 1$. 2 marks

b Find the x-coordinates of the points of intersection of the curve $x^2 - 2y^2 = 1$ and the line $y = x - 1$. 1 mark

c Find the volume of the solid of revolution formed when the region bounded by the curve and the line is rotated about the x-axis. 2 marks

2 (3 marks) Evaluate $\int_e^{e^4} \frac{1}{2x\log_e(x)}\,dx$.

3 (2 marks) A triangle has vertices $A\left(\sqrt{3}+1, 4\right)$, $B\,(1, 3)$ and $C\,(2, \sqrt{3}+3)$. Find angle ABC.

Cumulative examination 2

Total number of marks: 21 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 © VCAA 2018 2AQ7 A curve is described parametrically by $x = \sin(2t)$, $y = 2\cos(t)$, for $0 \le t \le 2\pi$.

The length of the curve is closest to

A 9.2 **B** 9.5 **C** 12.2 **D** 12.5 **E** 38.3

2 © VCAA 2019N 2AQ6 $P(z)$ is a polynomial of degree n with real coefficients where $z \in C$. Three of the roots of the equation $P(z) = 0$ are $z = 3 - 2i$, $z = 4$ and $z = -5i$. The smallest possible value of n is

A 3 **B** 4 **C** 5 **D** 6 **E** 7

3 © VCAA 2019N 2AQ9 With a suitable substitution $\int_1^2 \sqrt{5x-1}\,dx$ can be expressed as

A $5\int_1^2 \sqrt{u}\,du$

B $\frac{1}{5}\int_1^2 \sqrt{u}\,du$

C $5\int_4^9 \sqrt{u}\,du$

D $\frac{1}{5}\int_4^9 \sqrt{u}\,du$

E $5\int_4^9 \sqrt{5u-1}\,du$

4 © VCAA 2012 2AQ12 MODIFIED The volume of the solid of revolution formed by rotating the graph of $y = \sqrt{25-(x-1)^2}$ about the x-axis is given by

A $4\pi(5)^2$

B $\pi\int_{-5}^{5} 25-(x-1)^2\,dx$

C $\pi\int_{-4}^{6} \sqrt{25-(x-1)^2}\,dx$

D $\pi\int_{-4}^{6} \left(25-(x-1)^2\right)^2 dx$

E $\pi\int_{-6}^{4} 25-(x-1)^2\,dx$

5 © VCAA 2016 2AQ2 The implied domain of $y = \arccos\left(\frac{x-a}{b}\right)$, where $b > 0$ is

A $[-1, 1]$

B $[a-b, a+b]$

C $[a-1, a+1]$

D $[a, a+b\pi]$

E $[-b, b]$

Section B 2 questions

16 marks

1 © VCAA 2018N 2BQ1 (10 marks) Consider the function f with rule $f(x) = 10 \arccos(2 - 2x)$.

a Sketch the graph of f over its maximal domain. Label the endpoints with their coordinates. 3 marks

A vase is to be modelled by rotating the graph of f about the y-axis to form a solid of revolution, where units of measurement are in centimetres.

b **i** Write down a definite integral in terms of y that gives the volume of the vase. 2 marks

ii Find the volume of the vase in cubic centimetres. 1 mark

c Water is poured into the vase at a rate of 20 cm^3s^{-1}.

Find the rate, in centimetres per second, at which the depth of the water is changing when the depth is 5π cm. 3 marks

d The vase is placed on a table. A bee climbs from the bottom of the outside of the vase to the top of the vase.

What is the minimum distance the bee will need to travel? Give your answer in centimetres, correct to one decimal place. 1 mark

2 © VCAA 2015N 2BQ1 (6 marks) Consider $y = \sqrt{2 - \sin^2(x)}$.

a Use the relation $y^2 = 2 - \sin^2(x)$ to find $\frac{dy}{dx}$, in terms of x and y. 1 mark

b **i** Write down the values of y where $x = 0$ and where $x = \frac{\pi}{2}$. 1 mark

ii Write down the values of $\frac{dy}{dx}$ where $x = 0$ and where $x = \frac{\pi}{2}$. 1 mark

Now consider the function f with rule $f(x) = \sqrt{2 - \sin^2(x)}$ for $0 \le x \le 2\pi$.

c Find the rule for the inverse function f^{-1}, and state the domain and range of f^{-1}. 3 marks

CHAPTER

9

DIFFERENTIAL EQUATIONS

Study Design coverage

AREA OF STUDY 4: CALCULUS

Differential equations

- formulation of differential equations from contexts in, for example, chemistry, biology and economics, in situations where rates are involved (including some differential equations whose analytic solutions are not required, but can be solved numerically using technology)
- the logistic differential equation
- verification of solutions of differential equations and their representation using direction (slope) fields
- solution of simple differential equations of the form $\frac{dy}{dx} = f(x)$, $\frac{dy}{dx} = g(y)$ and in general differential equations of the form $\frac{dy}{dx} = f(x)\,g(y)$ using separation of variables and differential equations of the form $\frac{d^2y}{dx^2} = f(x)$
- numerical solution by Euler's method (first order approximation).

VCE Mathematics Study Design 2023–2027 p. 111, © VCAA 2022

Video playlists (8):

9.1 Differential equations

9.2 Solving $\frac{dy}{dx} = f(x)$

9.3 Solving $\frac{d^2y}{dx^2} = f(x)$

9.4 Solving $\frac{dy}{dx} = f(y)$

9.5 Separation of variables $\frac{dy}{dx} = f(x)g(y)$

9.6 Slope fields

9.7 Euler's method

VCE question analysis Differential equations

Worksheets (4):

9.2 Simple differential equations

9.5 Solving $\frac{dy}{dx} = f(x)g(y)$ • Differential equations and exponentials • Differential equations

Nelson MindTap

To access resources above, visit
cengage.com.au/nelsonmindtap

9.1 Differential equations

Video playlist
Differential equations

A **differential equation** is an equation that involves a function and its derivative(s). The solution to the equation is the function.

The **order** of a differential equation is its highest derivative and the **degree** is the highest power of the highest derivative.

$\frac{dy}{dx}$ is order 1, $\frac{d^2y}{dx^2}$ is order 2, $\frac{d^ny}{dx^n}$ is order n.

$\frac{dy}{dx} = 5xy$ is of order 1, degree 1, and y is the function.

$\frac{d^2x}{dt^2} - \frac{dx}{dt} + 4 = 0$ is of order 2, degree 1, and x is the function.

$\left(\frac{dP}{ds}\right)^4 - \left(\frac{d^2P}{ds^2}\right)^3 + P = 0$ is of order 2, degree 3, and P is the function.

Differential equations are used to model many real-life situations:

- $v = \dfrac{dx}{dt}$ (velocity v is rate of change of displacement with respect to time) [order 1, degree 1]
- $F = m\dfrac{d^2x}{dt^2}$ (applied force is mass × acceleration) [order 2, degree 1]
- $\dfrac{dT}{dt} = kT_s$ (rate of change of temperature, T, is **proportional to** its surrounding temperature, T_s) [order 1, degree 1]
- $\dfrac{dP}{dt} = kP$ (rate of change of population growth, P, is proportional to the current population) [order 1, degree 1].

The differential equation $\dfrac{d^2x}{dt^2} = 2$ can be solved by anti-differentiation to obtain $x = t^2 + c_1t + c_2$, where c_1 and c_2 are constants. However, there are differential equations such as $\dfrac{dy}{dx} = x + y$, where a numerical approach must be used for an approximation to the solution.

WB
p. 179

WORKED EXAMPLE 1 Finding the differential equation

Express the information as a differential equation using the variables given.

a The rate of change of volume, V cm^3, of a sphere with respect to its radius, r cm, is proportional to the square of its radius.

b The rate of change of the period, P seconds, of a pendulum with respect to its length, L cm, is **inversely proportional to** the square root of its length.

c The rate of decrease in the mass M kg with respect to time, t minutes, of a chemical of initial mass 2 kg undergoing a chemical reaction is proportional to the product of its current mass and the decrease in mass.

Steps	**Working**
a 1 Express the rate of change in terms of the given variables.	Volume is changing with respect to the radius. This is $\dfrac{dV}{dr}$.
2 'Proportional' means the rate of change is the product of a constant with some function of the **independent variable**.	Let the constant of proportionality be k. Then $\dfrac{dV}{dr} = kr^2$.
b 1 Express the rate of change in terms of the given variables.	The period is changing with respect to its length. This is $\dfrac{dP}{dL}$.
2 'Inversely proportional' means the rate of change is the ratio of a constant and some function of the independent variable.	Let the constant of proportionality be k. Then $\dfrac{dP}{dL} = \dfrac{k}{\sqrt{L}}$.
c 1 Express the rate of change in terms of the given variables.	The mass is changing with respect to time. This is $\dfrac{dM}{dt}$.
2 Express the rate of change in terms of the given variables.	For the mass M at time t, the decrease in mass is $2 - M$. The rate of change equation is $\dfrac{dM}{dt} = kM(2 - M)$, where k is the constant of proportionality.

Verifying a solution to a differential equation

A solution can be shown to satisfy a differential equation by substituting the solution into the equation. This will involve differentiating the solution before substitution.

Exam hack

When **verifying a solution by substitution**, show enough working until it is clear that the left-hand side of the equation is the same as the right-hand side.

WORKED EXAMPLE 2 Verifying a solution to a differential equation

WB p. 180

Verify that $y = \sin(x)$ is a solution to the second-order differential equation $\frac{d^2y}{dx^2} + y = 0$.

Steps	**Working**
1 Find an expression for $\frac{d^2y}{dx^2}$ in terms of y.	Differentiate $y = \sin(x)$ twice. $\frac{dy}{dx} = \cos(x)$ $\frac{d^2y}{dx^2} = -\sin(x)$ $\frac{d^2y}{dx^2} = -y$, since $y = \sin(x) \Rightarrow -y = -\sin(x)$.
2 Substitute for $\frac{d^2y}{dx^2}$ in the left-hand side and simplify.	Left-hand side is $-y + y = 0$.
3 State the conclusion.	LHS = RHS, so $y = \sin(x)$ is a solution to $\frac{d^2y}{dx^2} + y = 0$.

Note that we could also write the left-hand side in terms of x to obtain $\frac{d^2y}{dx^2} + y = -\sin(x) + \sin(x) = 0$.

WORKED EXAMPLE 3 Verifying a solution to find the constant

WB p. 181

Determine the values of the constant m in $y = e^{mx}$ that satisfy $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = 0$.

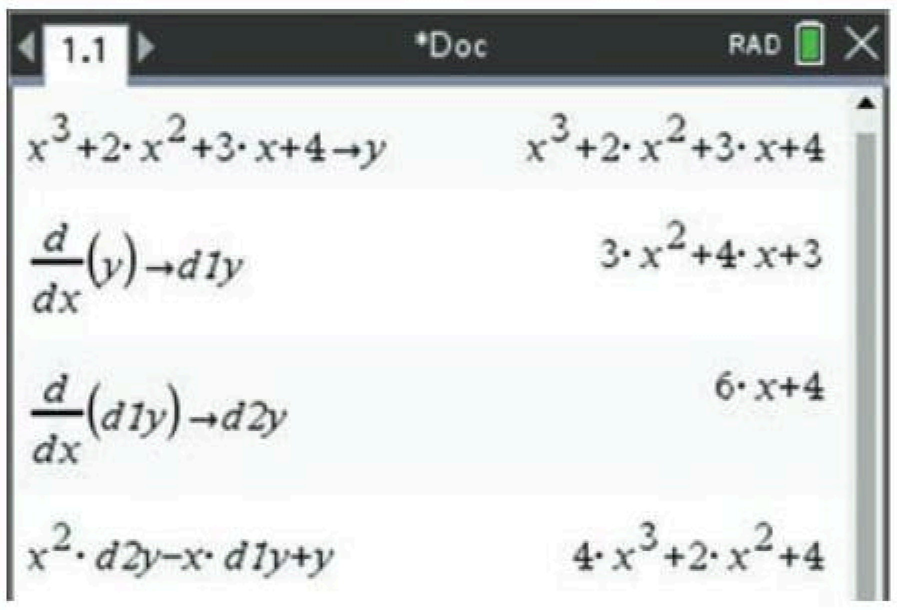
Steps	**Working**
1 Use $y = e^{mx}$ to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.	$y = e^{mx}$ $\frac{dy}{dx} = me^{mx}$ $\frac{d^2y}{dx^2} = m^2e^{mx}$
2 Substitute the information found in the differential equation.	$m^2e^{mx} + me^{mx} - 12e^{mx} = 0$
3 Take out a common factor and simplify.	$e^{mx}(m^2 + m - 12) = 0$ $m^2 + m - 12 = 0$, since $e^{mx} \neq 0$ for all values of x.
4 Solve the quadratic equation by factorising.	$(m + 4)(m - 3) = 0$ $m = -4$, $m = 3$

USING CAS 1 Verifying solutions to differential equations

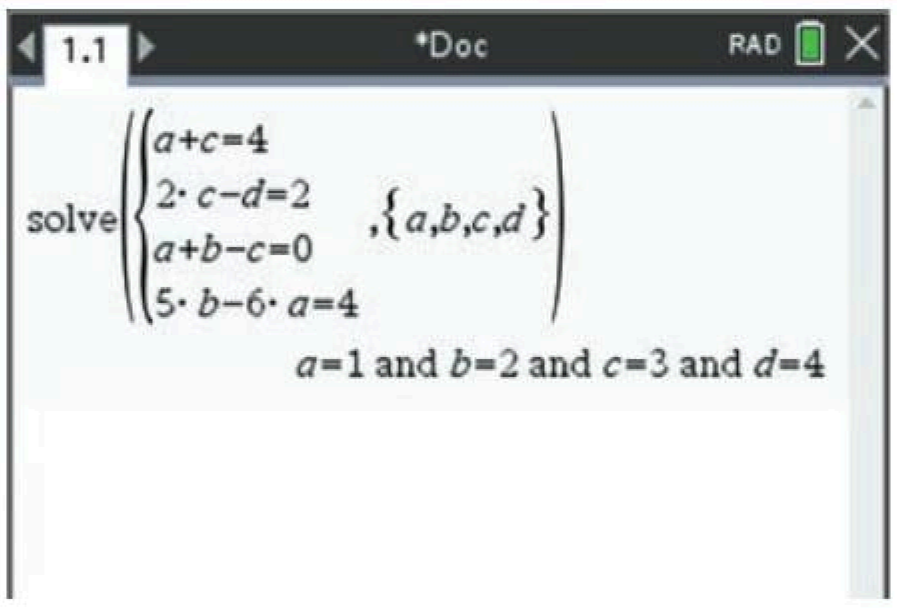
For $y = x^3 + 2x^2 + 3x + 4$, determine the values for a, b, c and d so that

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = (a+c)x^3 + (2c-d)x^2 + (a+b-c)x + (5b-6a)$$

TI-Nspire



```
x^3+2·x^2+3·x+4→y            x^3+2·x^2+3·x+4
d/dx(y)→d1y                  3·x^2+4·x+3
d/dx(d1y)→d2y                6·x+4
x^2·d2y−x·d1y+y              4·x^3+2·x^2+4
```



```
solve({a+c=4, 2·c−d=2, a+b−c=0, 5·b−6·a=4}, {a,b,c,d})
        a=1 and b=2 and c=3 and d=4
```

1 Enter the expression and store it in ***y***.

2 Find the derivative ***y*** and store it in ***d1y***.

3 Find the derivative of ***d1y*** and store it in ***d2y***.

4 Enter the differential equation using ***x***, ***y***, ***d1y*** and ***d2y***.

5 The simplified polynomial in terms of x will be displayed.

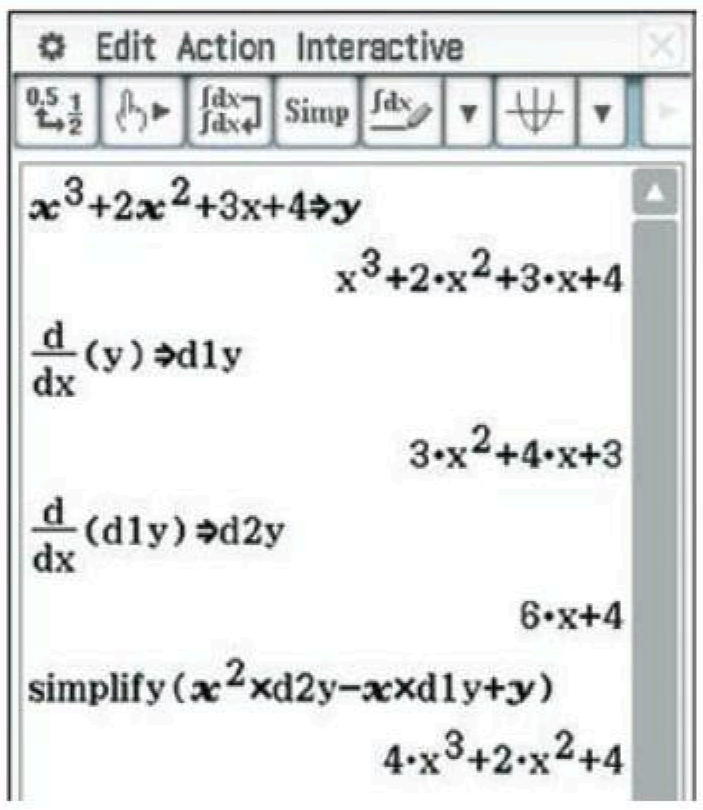
6 Press **menu** > **Algebra** > **Solve System of Equations** > **Solve System of Equations**.

7 In dialogue box, set the **Number of equations:** field to **4**.

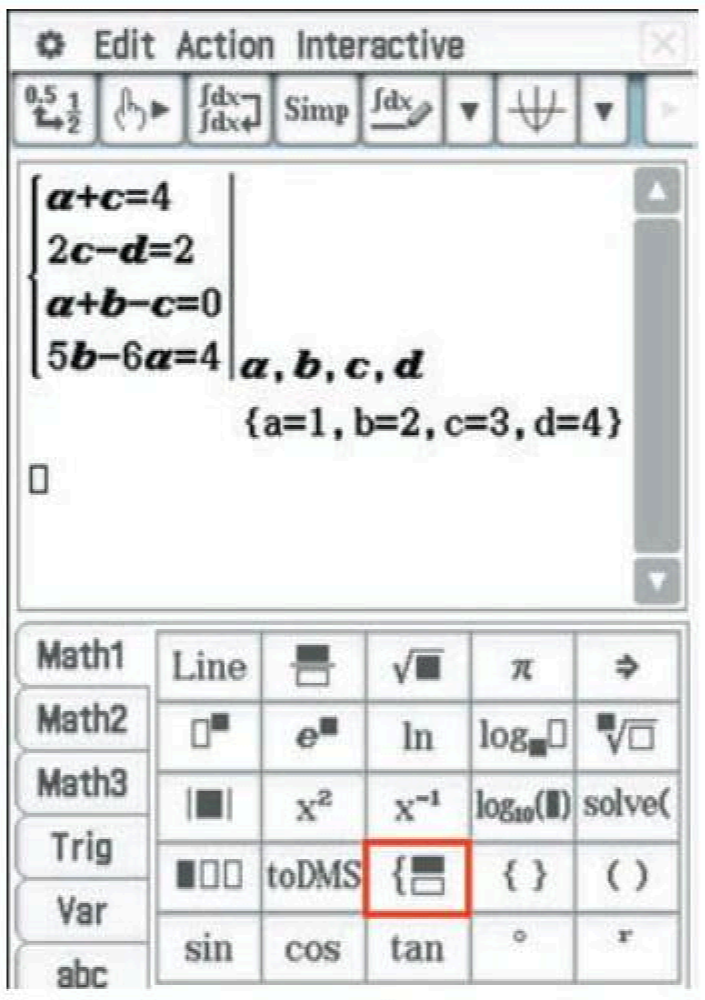
8 Change the **Variables:** field to ***a***,***b***,***c***,***d***.

9 Equate the coefficients of the original equation to set up and solve 4 simultaneous equations as shown above (*see note on following page).

ClassPad



```
x^3+2x^2+3x+4⇒y
                 x^3+2·x^2+3·x+4
d/dx(y)⇒d1y
                 3·x^2+4·x+3
d/dx(d1y)⇒d2y
                 6·x+4
simplify(x^2×d2y−x×d1y+y)
                 4·x^3+2·x^2+4
```



```
{a+c=4
 2c−d=2
 a+b−c=0
 5b−6a=4 |a, b, c, d
        {a=1, b=2, c=3, d=4}
```

1 Enter the expression and store it in **y**.

2 Find the derivative y and store it in **d1y**.

3 Find the derivative of **d1y** and store it in **d2y**.

4 Enter and highlight the differential equation using **x, y, d1y** and **d2y**.

5 Tap **Simp** and the polynomial in terms of x will be displayed.

6 Open the **Keyboard** > **Math1** and tap to insert the **simultaneous equations** template.

7 Tap the template *twice* more to create four lines for the equations.

8 Equate the coefficients of the original equation to set up 4 simultaneous equations as shown above using **Var** for a, b, c, d (*see note on following page).

9 In the lower right-hand corner of the template, enter the variables a, b, c, d.

* To determine the set of 4 simultaneous equations in terms of a, b, c and d, equate the coefficients as shown below:

$$4x^2 + 2x^2 + 4 = (a + c)x^3 + (2c - d)x^2 + (a + b - c)x + (5b - 6a)$$

This gives:

$a + c = 4$

$2c - d = 2$

$a + b - c = 0$

$5b - 6a = 4$

EXERCISE 9.1 Differential equations

ANSWERS p. 593

Mastery

1 State the order and the degree of each differential equation.

a $3\dfrac{d^2y}{dx^2} - 2\left(\dfrac{dy}{dx}\right)^2 + y = 0$

b $\left(\dfrac{dx}{dt}\right)\left(\dfrac{d^2t}{dx^2}\right)^2 = 1$

c $\left(\dfrac{dA}{dp} + A\right)\left(\dfrac{dA}{dp} - A\right) = 0$

Exam hack

The order and degree of a differential equation are whole numbers, so some arrangement of the equation may be necessary to make things clearer. Write

$\dfrac{dy}{dx} = \left(\dfrac{d^2y}{dx^2}\right)^{\frac{3}{2}}$ as $\left(\dfrac{dy}{dx}\right)^2 = \left(\dfrac{d^2y}{dx^2}\right)^3$,

not as $\dfrac{dy}{dx} = \dfrac{d^3y}{dx^3}$.

2 WORKED EXAMPLE 1 Using the variables given, write each statement as a differential equation.

a The rate of change of kinetic energy, K joules, of a moving object with respect to its speed, S m/s, is proportional to its speed.

b The rate of change of the brightness, B watts, of a candle with respect to its distance, r, in centimetres from a point is inversely proportional to the cube of the distance from the source.

c The rate of change of heat conduction, H°C, of an object with respect to time, t s, is proportional to its cross-sectional area A and inversely proportional to its thickness, T cm.

d The rate of change in the rate of change of distance, x, with respect to time, t, of a falling object is constant.

Exam hack

Taking the proportionality constant to be k, 'proportional to' means '$= k \times$ expression' and 'inversely proportional to' means '$= k \times \dfrac{1}{\text{expression}}$'.

3 WORKED EXAMPLE 2 TECH-FREE Verify that $y = \cos(2x)$ is a solution to the second-order differential equation $\dfrac{d^2y}{dx^2} + 4y = 0$.

4 TECH-FREE Verify that $y = x^k$ (k is a constant), is a solution to the differential equation $x\dfrac{d^2y}{dx^2} - (k-1)\dfrac{dy}{dx} = 0$.

5 TECH-FREE G is the gravitational force between two objects in space and R is the distance between them. The rate of change of G with respect to R is inversely proportional to the cube of R.

a Write this information as a differential equation.

b Verify that $R\dfrac{d^2G}{dR^2} + 3\dfrac{dG}{dR} = 0.$

6 WORKED EXAMPLE 3 Determine the values of the constant m in $y = e^{mx}$ that satisfy $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} - 12y = 0.$

7 Using CAS 1 If $y = x^3 - x^2 - 2x + 1$, determine the values for a, b, c and d when

$$x\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = (b+c)x^3 + (2a-b)x^2 + (7b+d)x + (a+b+c-d).$$

Exam hack

When substituting $y = e^{mx}$ in Question **6**, a fast way is to associate a variable, say, p, with the order of each term.

Thus $\dfrac{d^2x}{dt^2} - \dfrac{dx}{dt} + x = 0$ becomes

$p^2 - p + 1 = 0.$

Exam practice

80–100% 60–79% 0–59%

8 TECH-FREE (2 marks) Determine the value of the constant k if $y = x\sin(x)$ satisfies the differential equation $\dfrac{d^2y}{dx^2} + y = k\cos(x).$

9 TECH-FREE (2 marks) Find all values of the constant c if $y = c\log_e(x)$ satisfies the differential equation $\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = 0.$

10 © VCAA 2009 1Q6 81% TECH-FREE (2 marks) Find all real values of m such that $y = e^{mx}$ is a solution of $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} - 10y = 0.$

11 © VCAA 2005 1IIQ2 67% TECH-FREE (4 marks) $y = e^{2x}\cos(x)$ is a solution of the differential equation $\dfrac{d^2y}{dx^2} + k\dfrac{dy}{dx} + y = -2e^{2x}\sin(x)$, where $k \in R$.

Find the value of k.

12 © VCAA 2011 1Q2 61% TECH-FREE (3 marks) Find the value of the real constant k given that kxe^{2x} is a solution of the differential equation $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 5y = e^{2x}(15x+6).$

13 © VCAA 2003 1IIQ3 49% TECH-FREE (3 marks) $y = xe^{3x}$ is a solution of the differential equation $\dfrac{d^2y}{dx^2} + m\dfrac{dy}{dx} + ny = 0$, where $m, n \in R$.

Find the values of m and n.

14 © VCAA 2004 1IQ28 76% Which one of the following differential equations is satisfied by $y = \sin(2x)$?

A $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + y = 4\cos(2x)$

B $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 4y = 4\cos(2x)$

C $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 4y = 4\cos(2x)$

D $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 4y = 4\cos(2x)$

E $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 4y = 4\cos(2x)$

▶ **15** © VCAA 2015 2AQ14 76% A differential equation that has $y = x\sin(x)$ as a solution is

A $\dfrac{d^2y}{dx^2} + y = 0$

B $x\dfrac{d^2y}{dx^2} + y = 0$

C $\dfrac{d^2y}{dx^2} + y = -\sin(x)$

D $\dfrac{d^2y}{dx^2} + y = -2\cos(x)$

E $\dfrac{d^2y}{dx^2} + y = 2\cos(x)$

16 The rate of change of a function $y(x)$ with respect to x is proportional to the product of the dependent variable and inversely proportional to the natural log of the independent variable. This can be written as

A $\dfrac{dy}{dx} = \dfrac{k}{y\log_e(x)}$

B $\dfrac{dy}{dx} = \dfrac{1}{ky\log_e(x)}$

C $\dfrac{dy}{dx} = k\dfrac{x}{\log_e(y)}$

D $\dfrac{dy}{dx} = k\dfrac{y}{\log_e(x)}$

E $\dfrac{dy}{dx} = k\dfrac{1}{x\log_e(y)}$

Exam hack

Use only one proportionality constant for expressions involving both 'proportional to' and 'inversely proportional to'.

17 © VCAA 2004 1IQ29 69% A jug of water at a temperature of 20°C is placed in a refrigerator. The temperature inside the refrigerator is maintained at 4°C.

When the jug has been in the refrigerator for t minutes, the temperature of the water in the jug is y°C. The rate at which the water's temperature decreases is proportional to the difference between it and the temperature inside the refrigerator.

If k is a positive constant, a differential equation involving y and t is

A $\dfrac{dy}{dt} = -k(y-20);\ t=0,\ y=4$

B $\dfrac{dy}{dt} = -k(y+4);\ t=0,\ y=20$

C $\dfrac{dy}{dt} = -k(y-4);\ t=0,\ y=16$

D $\dfrac{dy}{dt} = -k(y+4);\ t=0,\ y=24$

E $\dfrac{dy}{dt} = -k(y-4);\ t=0,\ y=20$

18 © VCAA 2007 2AQ14 37% The rate at which a type of bird flu spreads throughout a population of 1000 birds in a certain area is proportional to the product of the number N of infected birds and the number of birds still **not** infected after t days. Initially, two birds in the population are found to be infected.

A differential equation, the solution of which models the number of infected birds after t days, is

A $\dfrac{dN}{dt} = k\dfrac{(1000-N)}{1000}$

B $\dfrac{dN}{dt} = k(N-2)(1000-N)$

C $\dfrac{dN}{dt} = kN(1000-N)$

D $\dfrac{dN}{dt} = kN[1000-(N+2)]$

E $\dfrac{dN}{dt} = k(N+2)(1000-N)$

▶

19 © VCAA 2017N 2Q9 The gradient of the tangent to a curve at any point $P(x, y)$ is half the gradient of the line segment joining P and the point $Q(-1, 1)$.

The coordinates of points on the curve satisfy the differential equation

A $\frac{dy}{dx} = \frac{y+1}{2(x-1)}$ **B** $\frac{dy}{dx} = \frac{2(y-1)}{x+1}$ **C** $\frac{dy}{dx} = \frac{x-1}{2(y+1)}$

D $\frac{dy}{dx} = \frac{2(x-1)}{y+1}$ **E** $\frac{dy}{dx} = \frac{y-1}{2(x+1)}$

20 © VCAA 2011 2AQ16 25% The gradient of the normal to a curve at any point $P(x, y)$ is twice the gradient of the line joining P and the point $Q(1, 1)$.

The coordinates of points on the curve satisfy the differential equation

A $\frac{dy}{dx} + \frac{x-1}{2(y-1)} = 0$ **B** $\frac{dy}{dx} - \frac{x-1}{2(y-1)} = 0$

C $\frac{dy}{dx} + \frac{2(y-1)}{x-1} = 0$ **D** $\frac{dy}{dx} + \frac{2(x-1)}{y-1} = 0$ **E** $\frac{dy}{dx} - \frac{2(y-1)}{x-1} = 0$

The word 'normal' is no longer part of the course. It means the line that is perpendicular to the tangent drawn at the same point on the graph.

9.2 Solving $\frac{dy}{dx} = f(x)$

Video playlist Solving $\frac{dy}{dx} = f(x)$

Worksheet Simple differential equations

One type of first-order, first-degree differential equation is $\frac{dy}{dx} = f(x)$.
The solution is found by anti-differentiating $f(x)$.

Solution to $\frac{dy}{dx} = f(x)$

$\frac{dy}{dx} = f(x)$ has the solution $y = \int f(x)\,dx + c$,
where c is a constant of integration.

WB p. 182

WORKED EXAMPLE 4 General solution to a first-order, first-degree differential equation

Find the general solution to each differential equation.

a $\frac{dy}{dx} = (1-2x)^3$ **b** $\frac{dy}{dt} = 6e^{-3t} + 4e^{2t}$ **c** $\frac{dp}{dy} = \frac{1}{\sqrt{9-y^2}}$

Steps	**Working**
a Find the anti-derivative and simplify.	$\int (1-2x)^3 dx = \frac{(1-2x)^4}{4 \times (-2)} + c$ $y = -\frac{1}{8}(1-2x)^4 + c$
b Find the anti-derivative and simplify.	$\int 6e^{-3t} + 4e^{2t} dt = -2e^{-3t} + 2e^{2t} + c$ $y = 2e^{2t} - 2e^{-3t} + c$
c Use $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$.	$\int \frac{1}{\sqrt{3^2 - y^2}} dy = \sin^{-1}\left(\frac{y}{3}\right) + c$ $p = \sin^{-1}\left(\frac{y}{3}\right) + c$

WORKED EXAMPLE 5 Particular solution to a first-order, first-degree differential equation

Find the solution to each differential equation.

a $\frac{dy}{dx} = \frac{x-2}{x^2-4x}$ if $y(2) = 1$

b $\frac{dy}{dx} = \frac{1}{x^2+5x+6}$ if $y(1) = 0$

c $\frac{dy}{dx} = \cos^2(x)$ if $y\left(\frac{\pi}{6}\right) = 0$

Steps	**Working**
a 1 Use $\int \frac{f'(x)}{f(x)}dx = \log_e\lvert f(x)\rvert + c$.	$y = \int\left(\frac{x-2}{x^2-4x}\right)dx$ $= \frac{1}{2}\int\left(\frac{2x-4}{x^2-4x}\right)dx$ $= \frac{1}{2}\log_e\lvert x^2-4x\rvert + c$
2 Use $y(2) = 1$ to find c.	$1 = \frac{1}{2}\log_e\lvert 2^2 - 4\times 2\rvert + c$ $c = 1 - \log_e(2)$
3 Write the answer.	$y = \frac{1}{2}\log_e\lvert x^2-4x\rvert + 1 - \log_e(2)$
b 1 Factorise x^2+5x+6 to change $\frac{1}{x^2+5x+6}$ to partial fractions.	$x^2+5x+6 = (x+3)(x+2)$ $\frac{1}{x^2+5x+6} = \frac{A}{x+3} + \frac{B}{x+2}$ $= \frac{A(x+2)+B(x+3)}{(x+2)(x+3)}$
2 Equate the numerators.	$A(x+2)+B(x+3) = 1$
3 Substitute $x = -3$ and $x = -2$ to find A and B.	$A = -1$ and $B = 1$
4 Rewrite the integral.	$y = \int\frac{1}{x^2+5x+6}dx$ $= \int\frac{1}{x+2} - \frac{1}{x+3}dx$
5 Use $\int\frac{1}{x}dx = \log_e\lvert x\rvert + c$.	$= \log_e\lvert x+2\rvert - \log_e\lvert x+3\rvert + c$
6 Use $y(1) = 0$ to find c.	$0 = \log_e(3) - \log_e(4) + c$ $c = \log_e(4) - \log_e(3) = \log_e\left(\frac{4}{3}\right)$
7 Write the anti-derivative.	$y = \log_e\lvert x+2\rvert - \log_e\lvert x+3\rvert + \log_e\left(\frac{4}{3}\right)$
8 Simplify using logarithmic laws.	$y = \log_e\left\lvert\frac{4(x+2)}{3(x+3)}\right\rvert$

WB

p. 183

c **1** Use the formula $\cos(2x) = 2\cos^2(x) - 1$ to rewrite the integral and then find it.

$$y = \int \cos^2(x)\,dx$$
$$= \frac{1}{2}\int [\cos(2x) + 1]\,dx$$
$$= \frac{1}{4}\sin(2x) + \frac{1}{2}x + c$$

2 Substitute $y\left(\frac{\pi}{6}\right) = 0$ to find c.

$$0 = \frac{1}{4}\sin\left(2 \times \frac{\pi}{6}\right) + \frac{1}{2} \times \frac{\pi}{6} + c$$
$$c = -\frac{\sqrt{3}}{8} - \frac{\pi}{12}$$

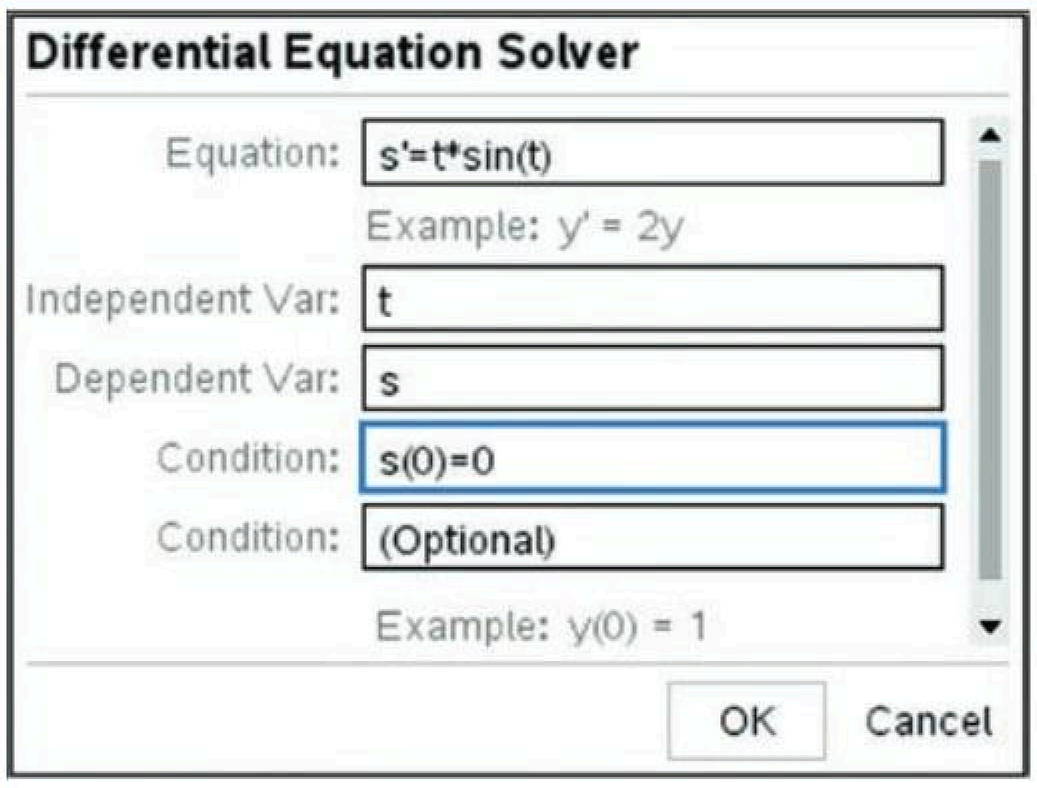
3 Write the answer.

$$y = \frac{1}{4}\sin(2x) + \frac{1}{2}x - \frac{\sqrt{3}}{8} - \frac{\pi}{12}$$

USING CAS 2 Solving $\frac{dy}{dx} = f(x)$

The speed v m/s of a particle at time t seconds is given by $v = t\sin(t)$. If the particle started from rest, how far has it travelled after 3 seconds, correct to two decimal places?

TI-Nspire


Differential Equation Solver

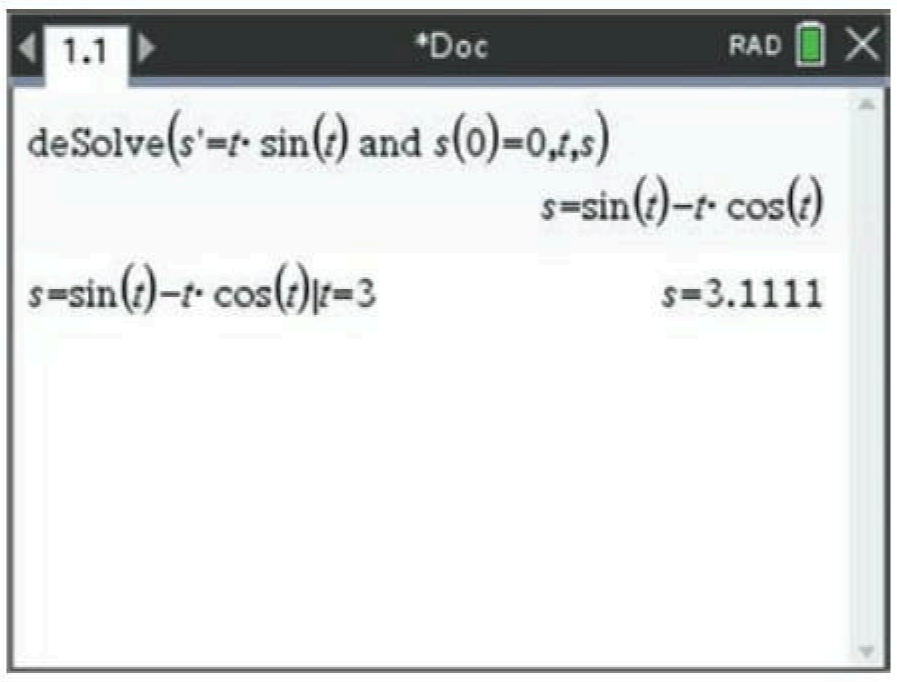
Equation: s'=t*sin(t)

Independent Var: t

Dependent Var: s

Condition: s(0)=0

Condition: (Optional)


```
deSolve(s'=t· sin(t) and s(0)=0,t,s)
                              s=sin(t)−t· cos(t)
s=sin(t)−t· cos(t)|t=3                  s=3.1111
```

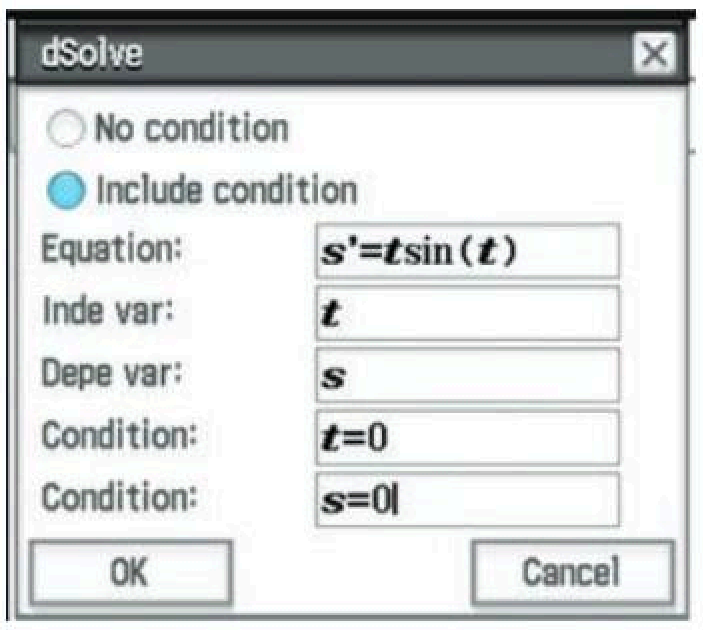
1 Press **menu** > **Calculus** > **Differential Equation Solver**.

2 In the dialogue box, enter the equations and variables as shown above. Press **π** to open the mini-palette to access the ′ symbol for **s**′.

3 The general solution will be displayed.

4 Substitute ***t*** = **3** into the general solution to find the distance travelled after 3 seconds.

ClassPad


dSolve

○ No condition

● Include condition

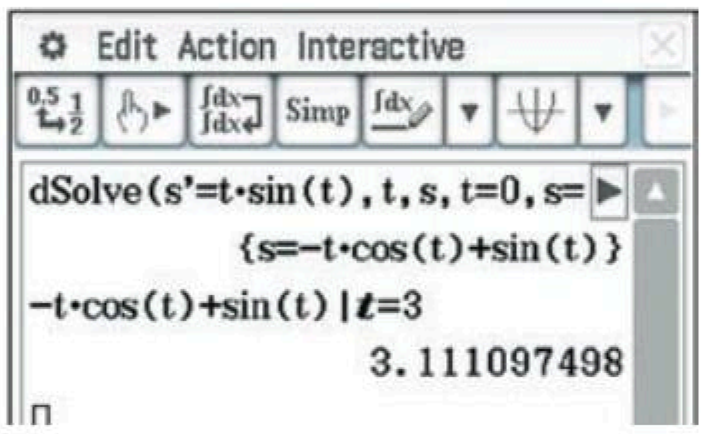
Equation: s'=tsin(t)

Inde var: t

Depe var: s

Condition: t=0

Condition: s=0


```
dSolve(s'=t·sin(t),t,s,t=0,s=
                  {s=−t·cos(t)+sin(t)}
−t·cos(t)+sin(t)|t=3
                           3.111097498
```

1 In **Main**, enter and highlight the equation. Open **Keyboard** > **Math3** to access the ′ symbol for **s′**.

2 Tap **Interactive** > **Advanced** > **dSolve**.

3 In the dialogue box, enter the variables as shown on the previous page.

4 The general solution will be displayed.

5 Substitute $\boldsymbol{t = 3}$ into the general solution to find the distance travelled after 3 seconds.

After 3 seconds, the particle will travel approximately 3.11 metres.

EXERCISE 9.2 Solving $\frac{dy}{dx} = f(x)$

ANSWERS p. 593

Recap

80–100% 60–79% 0–59%

1 The order and degree of the differential equation $\left(\frac{dy}{dx}\right)^4 - \left(\frac{d^2y}{dx^2}\right)^3 + y = 0$ are, respectively

A 1, 4 **B** 2, 1 **C** 2, 3 **D** 3, 4 **E** 4, 2

2 © VCAA 2002 1IQ16 66% TECH-FREE If $y = e^{kx}$ satisfies the differential equation $\frac{d^2y}{dx^2} = 5\frac{dy}{dx} - 6y$, the possible values for k are

A −6 and 1

B −1 and 6

C −5 and 6

D −3 and −2

E 2 and 3

Exam hack

Solving a second-order differential equation of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ when given $y = e^{kx}$ will produce a quadratic equation in k, which can be solved for k.

Mastery

3 WORKED EXAMPLE 4 State the general solution to each differential equation.

a $\frac{dy}{dx} = 2\cos(2x) - \sin(x)$

b $\frac{dx}{dt} = \frac{1}{9+t^2}$

c $\frac{dP}{df} = 2e^{2f} - 1$

4 WORKED EXAMPLE 5 Solve each differential equation.

a $\frac{dy}{dx} = \frac{3x^2+1}{x^3+x}, y(1) = 0$

b $\frac{dx}{ds} = \frac{3}{3s+2} + \frac{4}{4s+1}, x(0) = \log_e(4)$

c $\frac{df}{dx} = 2\sin^2(2x) - 1,\ f\left(\frac{\pi}{8}\right) = \frac{1}{4}$

Exam hack

Generally, powers of trigonometric functions should first be expressed in terms of trigonometric functions with power 1 using $\sin^2(x) = \frac{1}{2}[1 - \cos(2x)]$ and $\cos^2(x) = \frac{1}{2}[1 + \cos(2x)]$.

5 Using CAS 2 The speed v cm/s of an object at time t seconds is given by the differential equation $v = \sin(t)\cos(3t)$, $x(0) = 0$. Correct to two decimal places, how far has it moved after 0.4 seconds?

6 © VCAA 2010 1Q7 53% TECH-FREE (3 marks) Consider the differential equation

$\frac{d^2y}{dx^2} = \frac{4x}{(1-x^2)^2}$, $-1 < x < 1$, for which $\frac{dy}{dx} = 3$ when $x = 0$ and $y = 4$ when $x = 0$.

Given that $\frac{d}{dx}\left(\frac{2}{1-x^2}\right) = \frac{4x}{(1-x^2)^2}$, find the solution of this differential equation.

7 © VCAA 2016S 1Q7 TECH-FREE (4 marks) Solve the differential equation $\frac{dy}{dx} = \frac{y}{x^2}$ for y, given that when $x = 1$, $y = -1$.

8 TECH-FREE (2 marks) Determine the solution to $\frac{dx}{dy} = \frac{1}{\sqrt{4-9y^2}}$ if $y = 0$ when $x = 1$.

9 TECH-FREE (2 marks) Determine the solution to $\frac{dy}{dx} = \frac{3}{x^2+4x+6}$ if $y(0) = 2$.

10 TECH-FREE (2 marks) Show that the solution to $y' = \frac{e^x}{1+e^x}$, $y(0) = \log_e(2e)$, is $y = \log_e(1+e^x)+1$.

11 © VCAA 2006 1Q2 68% TECH-FREE (4 marks) Solve the differential equation $\frac{dy}{dx} = x\sqrt{x^2-16}$, $x \geq 4$, given that $y = 13$ when $x = 5$.

12 TECH-FREE (2 marks) The gradient of a point on a curve is given by $\frac{dy}{dx} = 2x\sqrt{9+x^2}$ and it passes through the point (0, 20). Find the equation of the curve.

13 TECH-FREE (5 marks) The acceleration due to gravity on the surface of the moon is such that the speed of some green cheese thrown upward by an astronaut with a strong arm is given by $v = 25 - 1.6t$, where v is in m s^{-1} and t is in seconds. Assume that the initial height is 0 m.

a Use the derivative to find the height after t seconds. 2 marks

b Find the height after 4 seconds. 1 mark

c Use a quadratic equation to show that the cheese never reaches a height of 200 metres. 2 marks

14 TECH-FREE (3 marks) The rate of change of distance, x cm, of an object with respect to time, t seconds, is proportional to the time taken to travel that distance.

a State this information as a differential equation. 1 mark

b Calculate the distance travelled by the particle in 10 s, given that $x(2) = 10$ and $x(4) = 28$. 2 marks

15 Which of the following is of the form $\frac{dy}{dx} = f(x)$?

A $\sqrt{\frac{d^2x}{dt^2}} = x^2$

B $\sin(x)\frac{dx}{dt} = 1+t$

C $\frac{1}{\frac{dt}{dx}} = x$

D $\sqrt{\log_e(t) - \frac{dx}{dt}} = 0$

E $\left(\frac{dx}{dt}\right)^2 = \frac{x}{t}$

16 The general solution to $\frac{dy}{dp} = -e^{-\frac{2}{3}p}$ is

A $y = \frac{2}{3}e^{\frac{2}{3}p} + c$

B $y = e^{-\frac{2}{3}p} + c$

C $y = \frac{3}{2}e^{-\frac{2}{3}p} + c$

D $y = -\frac{3}{2}e^{-\frac{2}{3}p} + c$

E $y = \frac{2}{3}e^{-\frac{2}{3}p} + c$

17 The solution to $\frac{dx}{dt} - \frac{1}{2}\sin\left(\frac{1}{4}t\right) = 0,\ x(2\pi) = 0$, is

A $x = \frac{1}{2}\cos\left(\frac{1}{4}t\right)$

B $x = \pi - 2\cos\left(\frac{1}{4}t\right)$

C $x = \frac{1}{4}\cos\left(\frac{1}{4}t\right)$

D $x = 2\pi + \cos\left(\frac{1}{4}t\right)$

E $x = -2\cos\left(\frac{1}{4}t\right)$

18 $\frac{dP}{dt}$ is proportional to $\sqrt{t}$, $P(0) = 1$ and $P(1) = 3$. The constant of proportionality and the constant of integration are, respectively,

A 1, 2 **B** 1, 4 **C** 3, 1 **D** 3, 5 **E** 4, 0

19 © VCAA 2003 1IQ15 59% If $f'(x) = 2\sin^2\left(\frac{x}{2}\right) - 1$ and $f\left(\frac{\pi}{2}\right) = 0$, then $f(x)$ is equal to

A $-\sin(x)$

B $1 - \sin(x)$

C $\sin(x) + 1$

D $\sin(x) - 1$

E $-1 - \sin(x)$

20 (3 marks) Consider the function $y = f(x)$, $\frac{dy}{dx} = \frac{c_1 x}{\sqrt{c_2 - x^2}}$, where c_1 and c_2 are positive constants.

Given $y(0) = 4$ and $y(2) = 2\sqrt{3}$ and the constant of integration is zero, express the solution to the differential equation in the form $y = f(x)$.

21 (3 marks) The population, P, of a small town is increasing at a rate given by the differential equation $\frac{dP}{dt} = kAe^{kt}$, where A and k are constants.

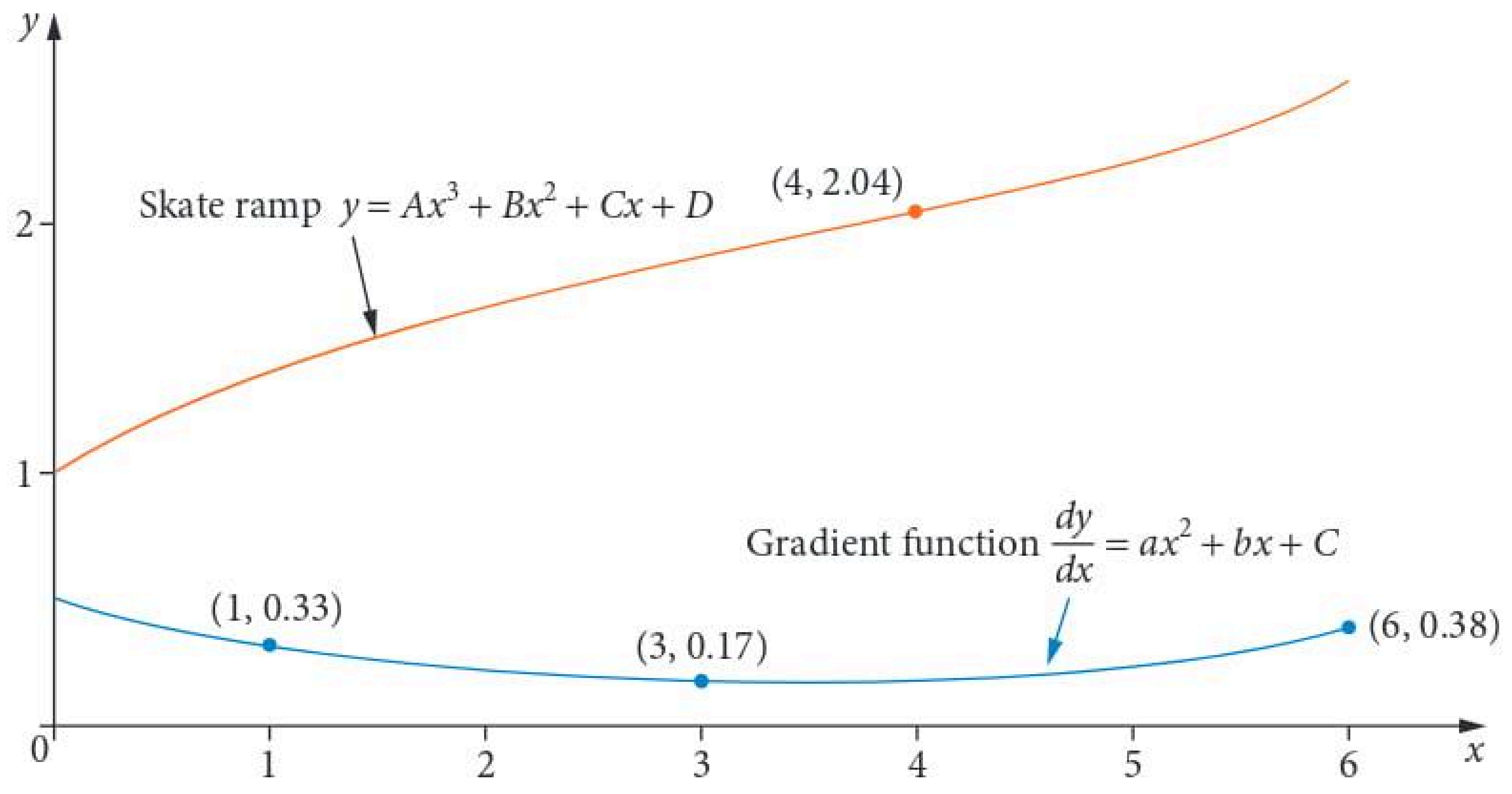
The present population is 4000 and it is predicted that the population will be 21 408 in 10 years and 99 428 in 20 years. What is the predicted population in 50 years?

Exam hack

When there are 3 constants to be found and only 2 equations, dividing one equation by the other equation may eliminate one of the constants.

22 (8 marks) The shape of a skate ramp is shown below and can be described by the function $y = Ax^3 + Bx^2 + Cx + D$, where A, B, C and D are constants.

The graph of its gradient function $\frac{dy}{dx} = ax^2 + bx + c$, where a, b and c are constants, is also shown.



a Determine the equation of the gradient function. 2 marks

b Using the equation from part **a**, at which point on the ramp is the point of inflection? Give the exact values. 2 marks

c Find the equation that describes the shape of the ramp. 2 marks

d State the relationship that must exist between A, B, C and D if there is to be a stationary point on the graph of y. 2 marks

Video playlist Solving $\frac{d^2y}{dx^2} = f(x)$

9.3 Solving $\frac{d^2y}{dx^2} = f(x)$

The general solution to the second-order differential equation $\frac{d^2y}{dx^2} = f(x)$ is obtained by anti-differentiating $f(x)$ twice to obtain $y = g(x) + c_1x + c_2$, where $g''(x) = f(x)$ and c_1 and c_2 are **constants of integration**.

WB p. 184

WORKED EXAMPLE 6 Anti-differentiating twice

Determine the general solution to y for $\frac{d^2y}{dx^2} = 12x + \cos(x)$.

Steps	Working
1 Anti-differentiate to obtain $\frac{dy}{dx}$, and include the constant of integration.	$\frac{d^2y}{dx^2} = 12x + \cos(x)$ $\frac{dy}{dx} = 6x^2 + \sin(x) + c_1$
2 Anti-differentiate $\frac{dy}{dx}$ to obtain the equation for y. Include the second constant of integration.	$y = 2x^3 - \cos(x) + c_1x + c_2$

The particular solution

To find the **particular solution**, information must be provided to determine the values of all constants of integration.

Exam hack

If you have the required information, work out the first constant of integration before anti-differentiating and adding the second constant. This will avoid ending up with one equation and two unknowns (constants of integration).

WB
p. 185

WORKED EXAMPLE 7 Finding the velocity from acceleration

A rocket is constructed so that its acceleration for a short time after take-off is given by $a = 10t + 2\ \mathrm{m\,s^{-2}}$, where t is in seconds. It takes off vertically from a height of 100 metres above sea level. Find its velocity and height at 4 seconds.

Steps	**Working**
1 The anti-derivative of velocity gives displacement. Write the differential equation and integrate it. Simplify.	$\frac{dv}{dt} = 10t + 2$ $v = \int (10t + 2)\,dt$ $= 5t^2 + 2t + c$
2 Use $v(0) = 0$ to find c.	$c = 0$
3 Write the equation and substitute $t = 4$ to find the velocity at 4 s.	$v = 5t^2 + 2t$ $v(4) = 88$ m/s
4 Find the height function by integrating the velocity.	$\frac{dh}{dt} = 5t^2 + 2t$ $h = \frac{5}{3}t^3 + t^2 + k$
5 Use $h(0) = 100$ to find k.	$k = 100$
6 Write the equation.	$h = \frac{5}{3}t^3 + t^2 + 100$
7 Substitute $t = 4$ to find the height at 4 s.	$h(4) = \frac{5}{3} \times 64 + 16 + 100$ $= 222\frac{2}{3}$
8 Write the answer.	The velocity and height after 4 s are 88 m/s and $222\frac{2}{3}$ m above sea level, respectively.

USING CAS 3 Solving $\frac{d^2y}{dx^2} = f(x)$

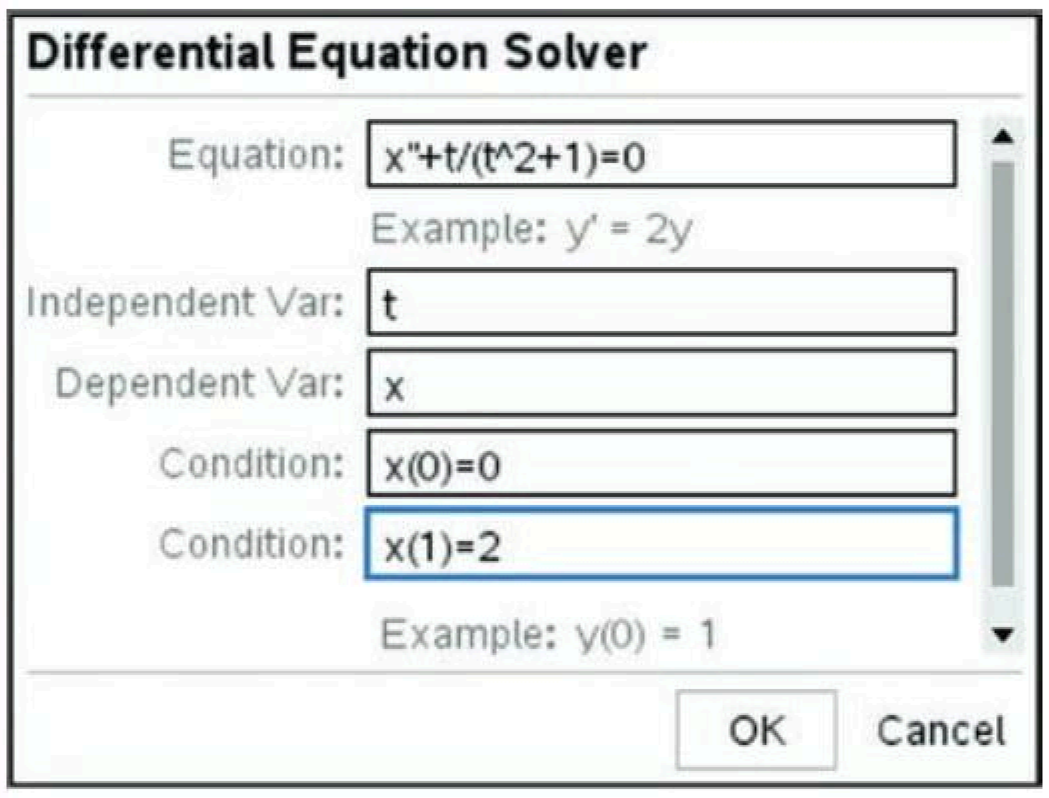
A particle starts from rest and its displacement, x cm, from O, at time t s is given by the differential equation $\frac{d^2x}{dt^2} + \frac{t}{t^2+1} = 0$, with $x(0) = 0$ and $x(1) = 2$.

a State the solution to the differential equation in the form $x = At + B\tan^{-1}(t) + Ct\ln(t^2+1)$, where A, B and C are constants.

b Find the next time the particle is at the origin, correct to two decimal places.

TI-Nspire

a, b



Differential Equation Solver

Equation: x"+t/(t^2+1)=0

Example: y' = 2y

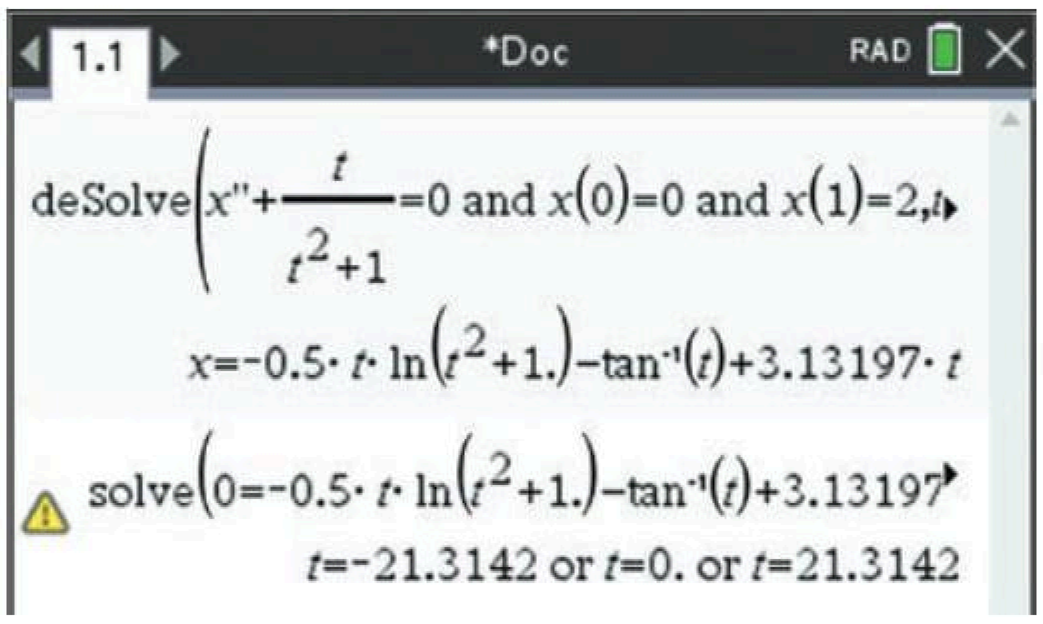
Independent Var: t

Dependent Var: x

Condition: x(0)=0

Condition: x(1)=2

Example: y(0) = 1



$\text{deSolve}\left(x''+\frac{t}{t^2+1}=0 \text{ and } x(0)=0 \text{ and } x(1)=2,t\right.$

$x=-0.5\cdot t\cdot \ln(t^2+1.)-\tan^{-1}(t)+3.13197\cdot t$

$\text{solve}\left(0=-0.5\cdot t\cdot \ln(t^2+1.)-\tan^{-1}(t)+3.13197\right.$

$t=-21.3142 \text{ or } t=0. \text{ or } t=21.3142$

1 Press **menu** > **Calculus** > **Differential Equation Solver**.

2 In the dialogue box, enter the equations and variables as shown above. Enter the ‘symbol twice for ***x***′.

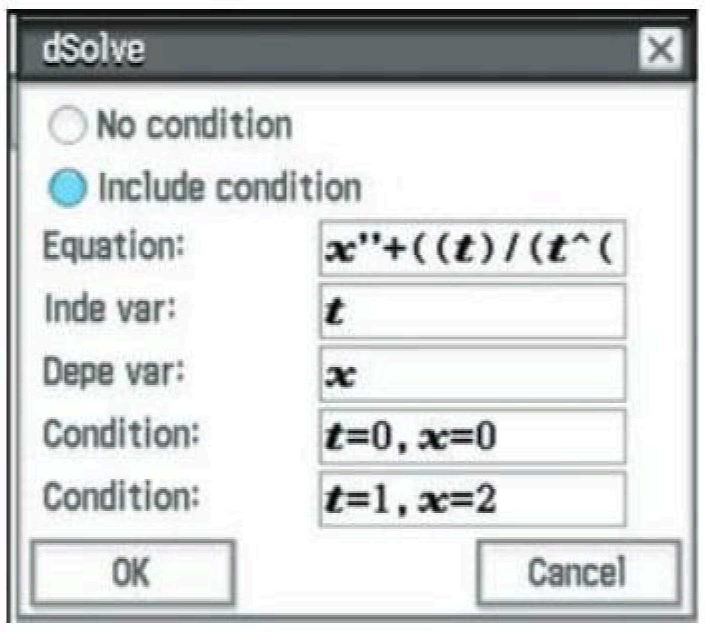
3 The general solution will be displayed.

4 Substitute ***x*** = **0** into the general solution and solve the equation for ***t***.

5 Select the positive solution.

ClassPad

a, b



○ No condition

● Include condition

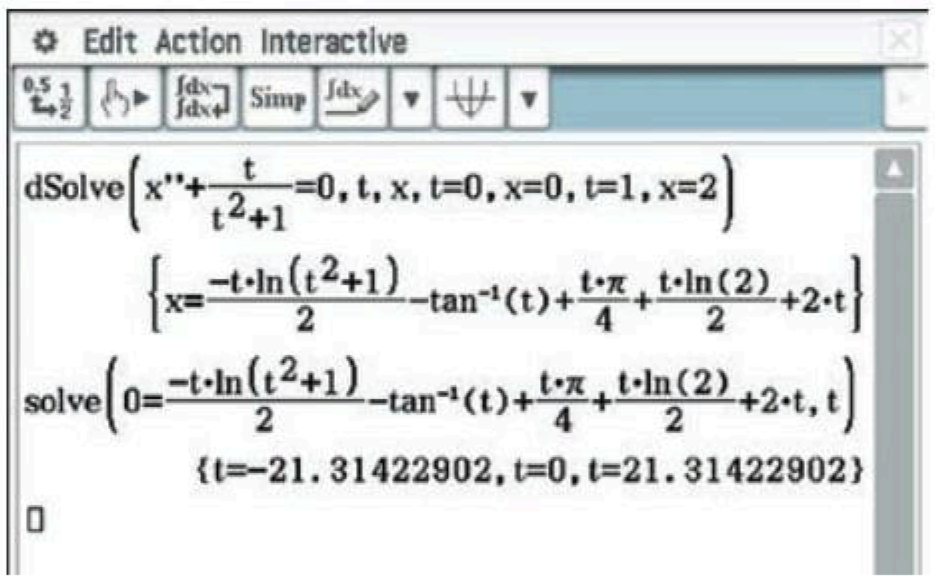
Equation: x''+((t)/(t^(

Inde var: t

Depe var: x

Condition: t=0, x=0

Condition: t=1, x=2



$\text{dSolve}\left(x''+\frac{t}{t^2+1}=0, t, x, t=0, x=0, t=1, x=2\right)$

$\left\{x=\frac{-t\cdot\ln(t^2+1)}{2}-\tan^{-1}(t)+\frac{t\cdot\pi}{4}+\frac{t\cdot\ln(2)}{2}+2\cdot t\right\}$

$\text{solve}\left(0=\frac{-t\cdot\ln(t^2+1)}{2}-\tan^{-1}(t)+\frac{t\cdot\pi}{4}+\frac{t\cdot\ln(2)}{2}+2\cdot t, t\right)$

$\{t=-21.31422902, t=0, t=21.31422902\}$

1 In **Main**, enter and highlight the equation. Open the **Keyboard** > **Math3** to tap the ′symbol twice for ***x***′.

2 Tap **Interactive** > **Equation/Inequality** > **dSolve**.

3 Tap **Include condition**, enter ***t*** = **0**, ***x*** = **0** and ***t*** = **1**, ***x*** = **2**.

4 Tap **OK**.

5 The general solution will be displayed.

6 Substitute ***x*** = **0** into the general solution and solve the equation for ***t***.

7 Select the positive solution.

a The solution to the differential equation is $x = 3.132t - \tan^{-1}(t) - 0.5t\ln(t^2+1)$.

b The next time the particle is at the origin is when $t = 21.31$ seconds.

Solution to $\frac{d^2y}{dx^2} = f(x)$

To solve $\frac{d^2y}{dx^2} = f(x)$, first anti-differentiate to obtain $\frac{dy}{dx} = \int f(x)\,dx = h(x) + c_1$, where $h'(x) = f(x)$.

Anti-differentiate a second time to get $y = \int h(x)\,dx + \int c_1 dx = g(x) + c_1 x + c_2$, where $g'(x) = h(x)$

and c_1 and c_2 are constants of integration.

EXERCISE 9.3 Solving $\frac{d^2y}{dx^2} = f(x)$

ANSWERS p. 593

Recap

1 The solution to $\frac{dy}{dx} = \frac{2x^3 - 1}{x^2 + x + 1}$, $y(1) = 0$ is

A $y = x^2 - 2x - 1 + \log_e\left(2x^3 - 1\right)$

B $y = (x-1)^2 + \log_e\left(x^2 + x + 1\right)$

C $y = 2x - 1 + \log_e\left(x^2 + x + 1\right)$

D $y = x^2 - 2x - 1 + \log_e\left(x^2 + x + 1\right)$

E $y = 2x + 1 + \log_e\left(x^2 + x + 1\right)$

2 © VCAA 2010 2AQ13 65% The amount of a drug, x mg, remaining in a patient's bloodstream t hours after taking the drug is given by the differential equation $\frac{dx}{dt} = -0.15x$.

The number of hours needed for the amount x to halve is

A $2\log_e\left(\frac{20}{3}\right)$

B $\frac{20}{3}\log_e(2)$

C $2\log_e(15)$

D $15\log_e\left(\frac{3}{2}\right)$

E $\frac{3}{2}\log_e(200)$

Exam hack

Information is not provided in this question to evaluate the constant of integration (the initial amount of the drug), so write it as a coefficient of the exponential so that it will cancel when known information is substituted.

Mastery

3 WORKED EXAMPLE 6 Determine the general solution to y for $\frac{d^2y}{dx^2} = e^x - \sin(x)$.

4 WORKED EXAMPLE 7 A balloon is released at ground level and for the first 20 seconds of its motion rises vertically with an acceleration given by $a = e^{0.1t}$. Calculate its height after 10 seconds, to the nearest metre.

5 TECH-FREE Solve $\frac{d^2y}{dx^2} = -\frac{1}{x^2}$, given that $\frac{dy}{dx} = 2.5$ when $x = 2$, and $y = 1$ when $x = 1$.

6 Using CAS 3 A skydiver's motion as she falls with an open parachute satisfies the equation $m\frac{d^2x}{dt^2} + k\frac{dx}{dt} = 0$, where m kg is her mass, x m is the distance fallen, t is the time in seconds and k is a constant relating to air resistance.

a Using $m = 65$ kg and $k = 2$, write the solution to the differential equation in the form $x = A(1 - e^{Bt})$

b How long, to the nearest second, will it take the skydiver to fall a distance of 500 m?

c What distance will the skydiver cover if she can fall indefinitely?

7 **TECH-FREE** (4 marks) The population, P, of a country town is declining according to the model $\frac{d^2P}{dt^2} - 50e^{-0.1t} = 0$, where using $t = 0$ years gives the current population.

a Solve the differential equation given that $P(3) = 3707$ and $P(10) = 1849$.

Give the answer in the form $P = Ae^{-kt} + Bt$, where A, k and B are constants, and A and B are rounded to the nearest integer. 2 marks

b What is the current population of the town? 1 mark

c When is the rate of change in the decrease in population 499 people/year? 1 mark

8 **TECH-FREE** (2 marks) Solve $\frac{d^2y}{dx^2} = 6ax + 2b$ (with a, b constants) given that the solution contains the coordinates $(0, -1)$, $(1, 1)$, $(-1, -5)$ and $(2, 7)$.

9 **TECH-FREE** (2 marks) If $\frac{d^2y}{dx^2} = e^x$, show that $\frac{d^2y}{dx^2} = y - c_1x + c_2$, where c_1 and c_2 are constants of integration.

10 **TECH-FREE** (2 marks) The points $(0, \log_e(2))$ and $(1, \log_e(6))$ satisfy the solution to

$$\frac{d^2y}{dx^2} = -\frac{1}{(x+1)^2} - \frac{1}{(x+2)^2}$$

Express y as a function of x.

11 **TECH-FREE** (3 marks) The acceleration of an object is proportional to the square of the time, t seconds, it takes to move x cm. The object's initial speed is 3 cm/s and its initial position is 10 cm from a fixed point. After 1 second, the object's position is 15 cm from the fixed point. Calculate the value of the constant of proportionality.

12 The intensity, I, of light at a distance R from the light source satisfies $\frac{d^2I}{dR^2} + \frac{1}{R^4} = 0$.

Given that $I(1) = 1$ and $I(2) = 1\frac{17}{24}$, the intensity when $R = 4$ is

A $1\frac{5}{16}$

B $2\frac{5}{32}$

C $2\frac{11}{24}$

D $3\frac{11}{32}$

E $3\frac{11}{16}$

13 If $\frac{d^2x}{dt^2} = \sqrt{t}$, the equation for x can be written as

A $x = \frac{2}{15}\sqrt{t^3} + c_1t + c_2$

B $x = \frac{4}{15}\sqrt{t^3} + c_1t^2 + c_2$

C $x = \frac{5}{15}\sqrt{t^5} + c_1t^2 + c_2t$

D $x = \frac{4}{15}\sqrt{t} + c_1t^2 + c_2$

E $x = \frac{4}{15}\sqrt{t^5} + c_1t + c_2$

14 If c_1 and c_2 are, respectively, the first and second constants of integration in the solution to $\frac{d^2y}{dx^2} = \cos(x)$, where $y'(0) = 1$ and $y(0) = -2$, which statement must be true?

A $c_1 c_2 = 1$

B $c_1 + c_2 = 0$

C $2c_1 - c_2 = 2$

D $c_1 = c_2$

E $c_1 - c_2 = 1$

15 The general solution to $e^t \dfrac{d^2x}{dt^2} - e^{-t} = 0$ is

A $x = \frac{1}{2}e^t + c_1t + c_2$

B $x = -\frac{1}{4}e^{2t} - c_1t + c_2$

C $x = \frac{1}{4}e^{-2t} + c_1t + c_2$

D $x = \frac{1}{4}e^{-t} + c_1t - c_2$

E $x = \frac{1}{2}e^{-2t} + c_1t^2 + c_2$

16 If the general solution to $\dfrac{d^2y}{dx^2} = ax + b$ is $y = Ax^3 + Bx^2 + Cx + D$ (a, b, A, B, C and D are constants), the relationship between a, b, A and B is

A $a + b = 2A + B$

B $a - b = A + B$

C $\frac{a}{b} = \frac{3A}{B}$

D $\frac{a}{b} = \frac{2A}{3B}$

E $\frac{a}{2b} = \frac{A}{B}$

17 (6 marks) Tom throws a pebble with initial speed 2 m/s down a vertical mine shaft of depth 20 metres. The pebble accelerates with a constant acceleration of 10 m/s^2.

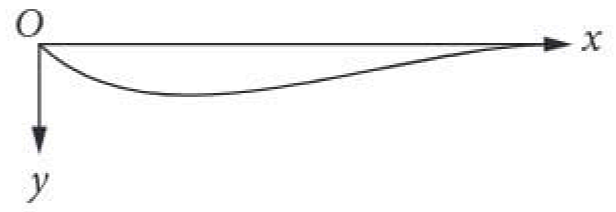
a State the differential equation for the acceleration using v for speed and t for time. 1 mark

b Obtain the equation for the speed of the pebble. 2 marks

c Obtain the equation for the distance x m the object falls after t seconds. 2 marks

d Find how long the pebble takes to hit the bottom of the mine shaft. 1 mark

18 © VCAA 2018N 2BQ5 (9 marks) A horizontal beam is supported at its endpoints, which are 2 m apart. The deflection y m of the beam measured downwards at a distance x m from the support at the origin O is given by the differential equation $80\dfrac{d^2y}{dx^2} = 3x - 4$.



a Given that both the inclination, $\dfrac{dy}{dx}$, and the deflection, y, of both the beam from the horizontal at $x = 2$ are zero, use the differential equation above to show that $80y = \frac{1}{2}x^3 - 2x^2 + 2x$. 2 marks

b Find the angle of inclination of the beam to the horizontal at the origin O. Give your answer as a positive acute angle in degrees, correct to one decimal place. 2 marks

c Find the value of x, in metres, where the maximum deflection occurs, and find the maximum deflection, in metres. 3 marks

d Find the maximum angle of inclination of the beam to the horizontal in the part of the beam where $x \geq 1$. Give your answer as a positive acute angle in degrees, correct to one decimal place. 2 marks

19 (5 marks) A three-dimensional object depicting an ice cream consists of a hemisphere of radius R resting on an inverted cone of base radius R.

The relationship between the volume, V, of the object and its radius, R, is described by $\dfrac{d^2V}{dR^2} = kR$, where k is a constant.

a Given $V'(2) = 20\pi$, and $V(3) = 45\pi$, express V as a function of R in the form $V = CR^3$, where C is a constant. 2 marks

b Use the answer to part **a** to deduce that the height, H, of the cone is $3R$. 1 mark

c Determine the values of R for which $V''(R) - V'(R) = 0$. 2 marks

Video playlist
Solving
$\frac{dy}{dx} = f(y)$

9.4 Solving $\frac{dy}{dx} = f(y)$

$\frac{dy}{dx} = f(y)$ is a first-order differential equation with degree 1. It gives the rate of change as a function of the **dependent variable**, y.

Writing $\frac{dy}{dx} = f(y)$ as $\frac{1}{f(y)}\frac{dy}{dx} = 1$ allows the integral of both sides of the equation to be taken.

$$\int \frac{1}{f(y)}\frac{dy}{dx}\,dx = \int 1\,dx$$

$$x = \int \frac{1}{f(y)}\,dy$$

WB
p. 186

WORKED EXAMPLE 8 Solving $\frac{dy}{dx} = f(y)$

Solve each differential equation.

a $\frac{dy}{dx} = \frac{1}{2y+1}$

b $2\frac{dx}{dt} = 4 + x^2$

c $\frac{dQ}{dt} = Q,\ Q(0) = 5$

Steps	**Working**
a **1** Transpose so that the same variables are together and show as an integral.	$\frac{dy}{dx} = \frac{1}{2y+1}$ $(2y+1)\frac{dy}{dx} = 1$ $\int (2y+1)\frac{dy}{dx}\,dx = \int 1\,dx$ $\int (2y+1)\,dy = \int 1\,dx$
2 Anti-differentiate both sides.	$y^2 + y = x$
3 Solve for y and add a constant of integration.	Complete the square. $y^2 + y = x$ $\left(y + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = x$ $\left(y + \frac{1}{2}\right)^2 = \frac{1}{4} + x$ $y = \frac{-1 \pm \sqrt{4x+1}}{2} + c$
b **1** Transpose so that the same variables are together and show as an integral.	$2\frac{dx}{dt} = 4 + x^2$ $\int \frac{2}{4+x^2}\,dx = \int 1\,dt$
2 Anti-differentiate both sides.	$\tan^{-1}\left(\frac{x}{2}\right) = t$

3 Solve for x and add a constant of integration.	$\frac{x}{2} = \tan(t)$ $x = 2\tan(t) + c$
c **1** Transpose so that the same variables are together and show as an integral.	$\frac{dQ}{dt} = Q$ $\int \frac{1}{Q}\,dQ = \int 1\,dt$
2 Anti-differentiate both sides.	$\log_e \lvert Q \rvert = t + c$
3 Calculate the value of the constant.	$Q(0) = 5$, so $\log_e(5) = 0 + c$ $c = \log_e(5)$
4 Substitute the constant into the general solution and solve for the dependent variable.	$\log_e \lvert Q \rvert = t + \log_e(5)$ $\log_e \lvert Q \rvert - \log_e(5) = t$ $\log_e \left\lvert \frac{Q}{5} \right\rvert = t$ $\frac{Q}{5} = e^t$ $Q = 5e^t$

Newton's law of cooling

A well-known first-order differential equation is **Newton's law of cooling**. It states that the rate at which an object loses heat is proportional to the temperature difference between the object and its surroundings.

It can be expressed as $\frac{dT}{dt} = -k(T - T_s)$, where T is the temperature of the object at time t, T_s is the temperature of the object's surroundings, and k is a proportionality constant.

Exam hack

Knowing that the general solution to Newton's law of cooling is $T = T_s + T_0 e^{-kt}$ (where T_0 is the initial temperature difference between the object and its surrounding temperature), can speed up computations.

WORKED EXAMPLE 9 Newton's law of cooling

WB
p. 187

A steel rod is heated to 100°C and then begins to cool according to Newton's law of cooling. The surrounding temperature is 25°C and the bar takes 10 min to drop to 55°C.

Find, to the nearest minute, how long it will take for the temperature of the rod to drop to 40°C.

Steps	Working
1 Write the required differential equation, including known information.	$\frac{dT}{dt} = -k(T - T_s)$, with $T_s = 25$, $\frac{dT}{dt} = -k(T - 25)$.
2 Write the integral and anti-differentiate.	$\int \frac{dT}{T - 25} = -\int k\,dt$ $\log_e \lvert T - 25 \rvert = -kt + c$ $T - 25 = e^{-kt + c}$ $T = 25 + T_0 e^{-kt}$, where $T_0 = e^c$.

3 Calculate the value of the constant using the initial temperature.	When $t = 0$, $T = 100$. $100 = 25 + T_0 \Rightarrow T = 75$ $T = 25 + 75e^{-kt}$
4 Calculate the value of the remaining constant using the temperature at 10 min. Do not round your value of k so that you can reuse it without losing accuracy. Store it on your calculator or write it down with many decimal places.	When $t = 10$, $T = 55$. $55 = 25 + 75e^{-10k}$ $k = -\frac{1}{10}\log_e\left(\frac{55-25}{75}\right) \approx 0.0916...$ $T = 25 + 75e^{-0.0916t}$
5 Find the time required for the rod's temperature to drop to the given value.	For $T = 40$, $40 = 25 + 75e^{-0.0916t}$ $t = -\frac{1}{0.0916}\log_e\left(\frac{40-25}{75}\right) \approx 18$
6 State the answer.	It will take 18 minutes for the rod's temperature to drop to 40°C.

Growth and decay

Natural elements such as uranium and carbon gradually transform into different substances. The time they take to do this can be modelled mathematically. In a similar way, human and animal population growth can be described by $\frac{dN}{dt} = f(N)$, where the rate of change of the population is a function of the current population.

Exam hack

In application questions, if the rate of change of the function represents a decrease, remember to include a negative by writing $\frac{dy}{dx} = -f(y)$.

WB
p. 188

WORKED EXAMPLE 10 Find the age of a substance using the rate of decay differential equation

The rate of decay of radioactive material depends on the amount of material present at the time. The half-life is the time taken for half of a given amount of material to decay. The half-life of carbon-14 is 5730 years. A charcoal sample contains 0.05 g of carbon-14 and it is estimated that 0.27 g of carbon-14 was present at the time the charcoal was formed. Use a differential equation to find the age when the charcoal was formed. Give your answer to the nearest year.

Steps	**Working**
1 Choose the variables and write the equation.	Let M = mass of material in grams and t = time in years.
Decreasing gives a negative rate of change.	$\frac{dM}{dt} = -kM$
Put M on the left.	$\frac{1}{M}\frac{dM}{dt} = -k$
2 Integrate both sides with respect to time and make M the subject.	$\int \frac{1}{M}\frac{dM}{dt}\,dt = \int -k\,dt$ $\ln\lvert M\rvert = -kt + c$ $M = e^{-kt+c} = M_0e^{-kt}$
M_0 is the initial value of M.	$M_0 = 0.27$

To find k, use the half-life.	At $t = 5730$, $M = \frac{1}{2}M_0$ $\frac{1}{2}M_0 = M_0e^{-5730k}$ $e^{-5730k} = \frac{1}{2}$ $-5730k = \ln\left(\frac{1}{2}\right) = -\ln(2)$ $k = \frac{\ln(2)}{5730}$
3 Write the equation for M.	$M = 0.27e^{-kt}$, where $k = \frac{\ln(2)}{5730}$
4 Substitute $M = 0.05$ to find t, the time when the charcoal was formed.	$0.05 = 0.27e^{-kt}$ $e^{-kt} = \frac{0.05}{0.27}$ $-kt = \ln\left(\frac{0.05}{0.27}\right)$ $t = -\ln\left(\frac{0.05}{0.27}\right) \div \frac{\ln(2)}{5730}$ $= 13\,940.857\ldots$
5 Write the answer.	The age to the nearest year is 13 941 years.

WORKED EXAMPLE 11 Apply the growth model to predict the population

WB p. 189

The population of a small town is increasing at a rate proportional to the number of people living there at that time. In 2010, there were 500 residents, and in 2015 there were 642 people in total.

a Write the differential equation using P for population, t for time and k as the constant of proportionality.

b Solve the differential equation.

c Calculate the predicted population for 2025.

Steps	Working
a Use the general form $\frac{dy}{dx} \propto f(y)$ with the variables given.	$\frac{dP}{dt} = kP$
b **1** Anti-differentiate.	Transpose and integrate. $\int \frac{dP}{P} = \int k\,dt$ $\log_e\lvert P\rvert = kt + c$
2 Find the value of the constant of integration. Take 2010 as the start ($t = 0$).	When $t = 0$, $P = 500$. $\log_e(500) = c$
3 Determine the value of the constant of proportionality.	Year 2015 is $t = 5$ and $P = 642$. $\log_e\lvert P\rvert = kt + \log_e(500)$ $\log_e(642) = 5k + \log_e(500)$ $k = \frac{1}{5}\log_e\left(\frac{642}{500}\right) \approx 0.04999\ldots$

4 Transpose to make P the subject.	$\log_e \lvert P\rvert = 0.049\,99t + \log_e(500)$ $\log_e \left\lvert \frac{P}{500} \right\rvert = 0.049\,99t$ $P = 500e^{0.049\,99t}$
c 1 Substitute the value for t in the equation.	Year 2025 is $t = 15$. $P = 500e^{0.049\,99 \times 15} \approx 1058$
2 State the answer.	There will be 1058 people in 2025.

Inflow/outflow

The rate of change of the amount of a dissolved substance, M, in a tank as liquid enters and exits the tank at given rates can be modelled by $\frac{dM}{dt} = f(M) =$ **inflow rate** – **outflow rate**.

WB

p. 190

WORKED EXAMPLE 12 Solving the inflow/outflow differential equation

A tank initially contains 20 litres of water in which 8 grams of chocolate flavouring is dissolved. Chocolate flavouring containing 1 g/L is then added at 4 L/min, and liquid is removed from the tank at 4 L/min.

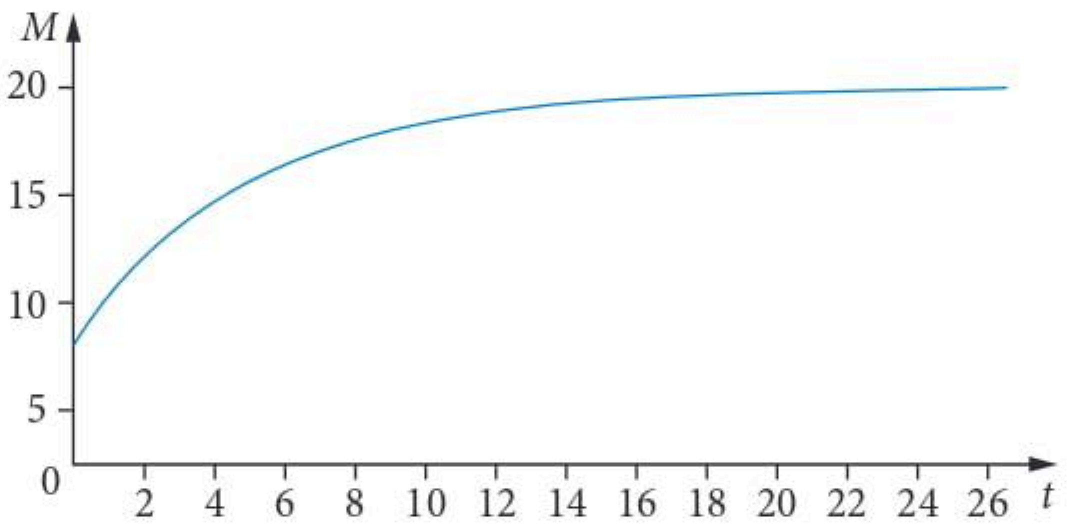
a Form the differential equation that shows the rate of change of chocolate flavouring in the tank.

b Solve the differential equation.

c Correct to one decimal place, find the amount of chocolate flavouring in the tank after 10 minutes.

d Sketch the graph of M against t and describe what happens to the amount of chocolate flavouring in the tank as time increases.

Steps	**Working**
a 1 Determine inflow rate and outflow rate.	Chocolate concentration is 1 g/L and liquid enters at 4 L/min. Inflow rate is $1 \times 4 = 4$ g/min. Let M be the amount of flavouring in the tank at time t. Each litre in the tank has $\frac{M}{20}$ g of flavouring. For 4 L/min being removed, outflow is $4 \times \frac{M}{20} = \frac{M}{5}$ g/L.
2 Use rate = inflow rate – outflow rate.	$\frac{dM}{dt} = 4 - \frac{M}{5} = \frac{20 - M}{5}$
b 1 Transpose and integrate.	$\int \frac{5}{20 - M}\, dM = \int 1\, dt$ $-5 \log_e \lvert 20 - M\rvert = t + c$
2 Evaluate the constant.	Use $t = 0$, $M = 8$ to find c. $-5 \log_e(20 - 8) = 0 + c$ $c = -5 \log_e(12)$
3 Write the equation with M as the subject.	$-5 \log_e \lvert 20 - M\rvert = t - 5 \log_e(12)$ $-5\log_e \left\lvert \frac{20 - M}{12} \right\rvert = t$ $\frac{20 - M}{12} = e^{-\frac{1}{5}t}$ $M = 20 - 12e^{-\frac{1}{5}t}$
c Use the equation to find the amount of dye for the given time.	When $t = 10$, $M = 20 - 12e^{-\frac{1}{5} \times 10} \approx 18.4$ grams.

d 1 Use CAS to draw the graph for positive values of t.	
2 Describe the behaviour of the graph for large values of t.	As t increases, the value of M approaches 20 g, which is a horizontal asymptote. The limiting case can be written as $\lim\limits_{t\to\infty}\left(20-12e^{-\frac{1}{5}t}\right)=20$.

The logistic model

A better population model than $\dfrac{dP}{dt}=kP$ is the **logistic model**, $\dfrac{dP}{dt}=kP\left(1-\dfrac{P}{K}\right)$. It takes into account the maximum possible population (the **carrying capacity**, K) that can be sustained. As before, P is the population at time t and k is a constant of proportionality. The solution is $P=\dfrac{K}{1+\left(\frac{K-P_0}{P_0}\right)e^{-kt}}$, where P_0 is the initial population and k can be evaluated from given conditions.

WORKED EXAMPLE 13 **Apply the logistic model to a population of fish**

WB p. 191

A fish farm is stocked with 100 trout and it is estimated that it can sustain a population of 1000. After 2 years, it is estimated that the number of trout has doubled.

a State the logistic equation for the number of trout, P, at time t years.

b Solve the equation for P.

c Use the logistic equation to find, correct to two decimal places, how long it will take for the trout population to reach 600.

d Sketch the graph of P against t and explain the significance of the value of the carrying capacity.

Steps	Working
a Substitute known information into $\dfrac{dP}{dt}=kP\left(1-\dfrac{P}{K}\right)$.	The carrying capacity is $K=1000$. $\dfrac{dP}{dt}=kP\left(1-\dfrac{P}{K}\right)$ $=kP\left(1-\dfrac{P}{1000}\right)$ $=\dfrac{kP(1000-P)}{1000}$
b 1 Group the variables P and t, use partial fractions and write in integral form.	$\displaystyle\int\frac{1000}{P(1000-P)}\,dP=\int k\,dt$ $\displaystyle\int\left(\frac{1}{P}+\frac{1}{1000-P}\right)dP=\int k\,dt$
2 Anti-differentiate.	$\log_e\lvert P\rvert-\log_e\lvert 1000-P\rvert=kt+c$ $\log_e\left\lvert\dfrac{P}{1000-P}\right\rvert=kt+c$

3 Evaluate the constant of integration.

When $t = 0$, $P = 100$.

$$c = \log_e\left|\frac{100}{1000 - 100}\right|$$
$$= \log_e\left(\frac{1}{9}\right)$$
$$= -\log_e(9)$$

$$\log_e\left|\frac{P}{1000 - P}\right| = kt - \log_e(9)$$

$$\log_e\left|\frac{9P}{1000 - P}\right| = kt$$

4 Evaluate the constant of proportionality.

When $t = 2$, $P = 200$.

$$\log_e\left(\frac{1800}{800}\right) = 2k$$
$$k \approx 0.405$$

5 Write the equation with P as the subject.

$$\frac{9P}{1000 - P} = e^{0.405t}$$

$$P(9 + e^{0.405t}) = 1000e^{0.405t}$$

$$P = \frac{1000e^{0.405t}}{9 + e^{0.405t}} = \frac{1000}{1 + 9e^{-0.405t}}$$

c Substitute the given value of P and solve for t algebraically or use SOLVE in CAS.

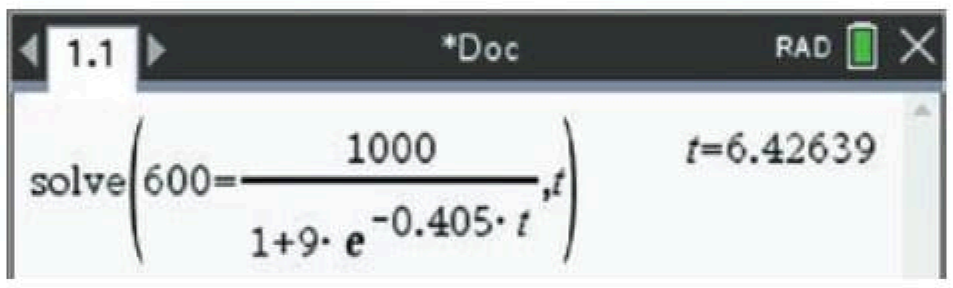
$$600 = \frac{1000}{1 + 9e^{-0.405t}}$$

$$e^{-0.405t} = \frac{1}{9}\left(\frac{1000}{600} - 1\right) = \frac{2}{27}$$

$$-0.405t = \log_e\left(\frac{2}{27}\right)$$

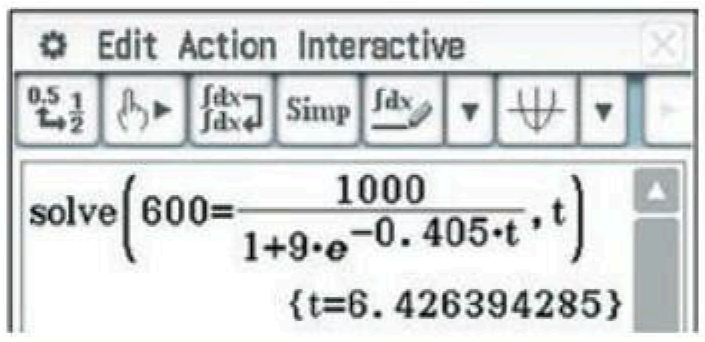
$$t \approx 6.43 \text{ years}$$

TI-Nspire



$\text{solve}\left(600=\frac{1000}{1+9\cdot e^{-0.405\cdot t}},t\right)$ $t=6.42639$

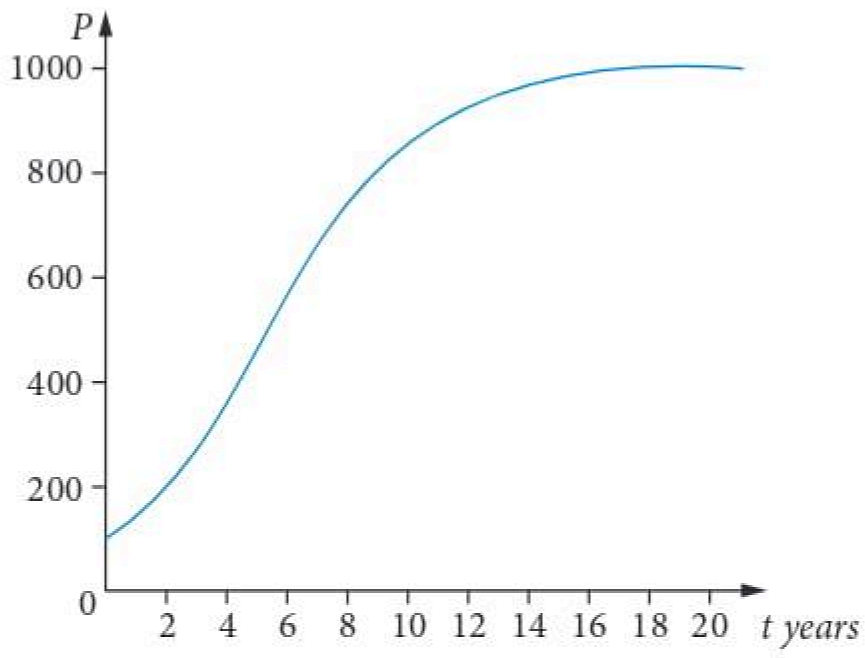
ClassPad



$\text{solve}\left(600=\frac{1000}{1+9\cdot e^{-0.405\cdot t}},t\right)$

$\{t=6.426394285\}$

d 1 Use CAS to sketch the graph.



2 Describe the graph in terms of intercepts and asymptotes.

The graph starts at 100 (the initial number of trout). This increases exponentially to approach a limiting value (asymptote) of 1000, which is the carrying capacity.

Application models of $\frac{dy}{dx} = f(y)$

Growth and decay

$\frac{dN}{dt} = f(N)$ for growth, $\frac{dN}{dt} = -f(N)$ for decline

Newton's law of cooling

$\frac{dT}{dt} = -k(T - T_s)$, where T is the temperature at time t, T_s is the temperature of surroundings, k is a constant

Inflow/outflow

$\frac{dM}{dt}$ = inflow rate − outflow rate, where M is the amount of a dissolved substance

Logistic model

$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$, K is the **carrying capacity**, where P is the population at time t and k is a constant

The solution is $P = \dfrac{K}{1 + \left(\frac{K - P_0}{P_0}\right)e^{-kt}}$, where P_0 is the initial population.

EXERCISE 9.4 Solving $\frac{dy}{dx} = f(y)$

ANSWERS p. 594

Recap

80–100% 60–79% 0–59%

1 The function $f(x)$ defined as $f''(x) = 6x - 2$ and with $f'(1) = 6$ and $f(-1) = -8$ is

A $f'(x) = x^3 + x^2 - x + 1$

B $f'(x) = 3x^3 - x^2 - 5x - 1$

C $f'(x) = x^3 + 2x^2 - 5x - 1$

D $f'(x) = x^3 - x^2 + 5x - 1$

E $f'(x) = 3x^3 - 2x^2 + x - 1$

2 © VCAA 2011 2AQ14 69% If $f''(x) = 2e^x \sin(x)$, $f'(0) = 0$ and $f(0) = 0$, then $f(x)$ equals

A $-e^x(\cos(x) + \sin(x))$

B $-e^x(\cos(x) - \sin(x)) + 1$

C $-e^x\cos(x)$

D $x - e^x\cos(x) + 1$

E $x - e^x\cos(x)$

Mastery

3 WORKED EXAMPLE 8 Solve each differential equation.

a $\frac{dy}{dx} = \frac{1}{3y}$

b $\frac{dx}{dt} = \sqrt{9 - x^2}$

c $\frac{dN}{dt} = -2N, N(0) = 1$

4 WORKED EXAMPLE 9 The rate at which a hot object cools is approximately proportional to the difference between its temperature and the temperature of the surrounding. A cup of black coffee is made with boiling water. After 2 minutes, its temperature has dropped to 80°C. The room is at 25°C. Use a differential equation to find how much longer (to the nearest minute) it will take to drop to 50°C.

5 WORKED EXAMPLE 10 The amount of carbon-14 remaining in a sample of bone is 20% of the original amount assumed to be present. Carbon-14 has a half-life of 5730 years and decays at a rate proportional to the quantity present. Use a differential equation to find an equation for the percentage quantity present after time t and hence find the age of the bone.

6 WORKED EXAMPLE 11 The population of rabbits in a habitat is increasing at a rate proportional to the number of rabbits present. In 2020 there were 100 rabbits, and in 2022 there were 150 rabbits. Assuming that this rate continues, how many rabbits will be in the habitat in 2028?

7 WORKED EXAMPLE 12 A vat has 100 litres of water in which 10 grams of salt is dissolved. Salty water is poured into the vat at 5 L/min, with each litre containing 3 grams of salt. Salt water is released from the vat at 5 L/min.

a State the differential equation and find, to the nearest gram, the amount of salt in the tank after 4 minutes.

b What is the limiting case for the amount of salt in the vat?

8 WORKED EXAMPLE 13 A habitat is originally home to 500 kangaroos. Three years later, their number increased to 800, and research has established that up to 2000 kangaroos can survive in the area.

a Use the logistic equation to find, correct to two decimal places, how long it will take for the kangaroo population to reach 1500.

b Sketch the graph of the kangaroo population from 0 to 28 years.

Exam practice

80–100% 60–79% 0–59%

9 © VCAA 2017N 1Q7a TECH-FREE (3 marks) Let $\frac{dy}{dx} = (4 - y)^2$.

Express y in terms of x, where $y(0) = 3$.

10 © VCAA 2013 1Q5 TECH-FREE (5 marks) A container of water is heated to boiling point (100°C) and then placed in a room that has a constant temperature of 20°C. After five minutes the temperature of the water is 80°C.

a 73% Use Newton's law of cooling $\frac{dT}{dt} = -k(T - 20)$, where T°C is the temperature of the water at time t minutes after the water is placed in the room, to show that $e^{-5k} = \frac{3}{4}$. 2 marks

b 60% Find the temperature of the water 10 minutes after it is placed in the room. 3 marks

11 The general solution to $3\frac{dy}{dx} = \frac{1}{y^2}$ is

A $y^2 = x^{\frac{1}{3}} + c$

B $y^2 = x^3 + c$

C $y^3 = x + c$

D $y = x^{\frac{1}{3}} + c$

E $y = x^{\frac{3}{2}} + c$

12 A particular solution to $\frac{d\theta}{dx} = \frac{1}{\cos^2(\theta)}$ is $A\theta + \sin(2\theta) = Bx$, where A and B are constants. The values of A and B are, respectively

A 1, 3

B 2, 3

C 2, 4

D 3, 1

E 3, 3

13 The rate of change in the diameter, D cm, of tree trunks with respect to time, t years, is proportional to the square root of their diameter. The general solution is

A $t = \sqrt{D} + c$

B $t = \frac{1}{2}\sqrt{D} + c$

C $t = \sqrt{2D} + c$

D $t = 2\sqrt{D} + c$

E $t = \sqrt{\frac{1}{2}D + c}$

14 A tank has 1000 L of water mixed with 5 kg of paint colour. Water pours in at 50 L/min and mixture is removed at the same rate. If M is the amount of paint colour in the tank at time t, the differential equation is

A $\frac{dM}{dt} = \frac{M}{40}$ **B** $\frac{dM}{dt} = -\frac{M}{20}$ **C** $\frac{dM}{dt} = -\frac{M}{1000}$

D $\frac{dM}{dt} = 5 - \frac{M}{1000}$ **E** $\frac{dM}{dt} = 200 - \frac{M}{40}$

15 © VCAA 2006 2AQ10 47% A chemical dissolves in a pool at a rate equal to 5% of the amount of **undissolved** chemical. Initially, the amount of undissolved chemical is 8 kg and after t hours, x kilograms has dissolved.

The differential equation that models this process is

A $\frac{dx}{dt} = \frac{x}{20}$ **B** $\frac{dx}{dt} = \frac{8-x}{20}$ **C** $\frac{dx}{dt} = \frac{x-8}{20}$ **D** $\frac{dx}{dt} = -\frac{x}{20}$ **E** $\frac{dx}{dt} = 8 - \frac{x}{20}$

16 © VCAA 2008 2AQ10 65% The volume of water, V m^3, in a cylindrical tank when it is filled to a depth of h metres is given by $V = 4h$. Water flows into the tank at a rate of 0.2 m^3/min and leaks out at a rate of $0.01\sqrt{h}$ m^3/min.

The differential equation, which when solved would enable h to be expressed in terms of t, is

A $\frac{dh}{dt} = 0.2 - 0.02\sqrt{h}$ **B** $\frac{dh}{dt} = 4\left(0.2 - 0.01\sqrt{h}\right)$ **C** $\frac{dh}{dt} = \frac{20-\sqrt{h}}{400}$

D $\frac{dh}{dt} = \frac{400}{20-\sqrt{h}}$ **E** $\frac{dh}{dt} = 20 - \frac{400}{\sqrt{h}}$

17 © VCAA 2014 2AQ10 64% A large tank initially holds 1500 L of water in which 100 kg of salt is dissolved. A solution containing 2 kg of salt per litre flows into the tank at a rate of 8 L per minute. The mixture is stirred continuously and flows out of the tank through a hole at a rate of 10 L per minute.

The differential equation for Q, the number of kilograms of salt in the tank after t min, is given by

A $\frac{dQ}{dt} = 16 - \frac{5Q}{750-t}$ **B** $\frac{dQ}{dt} = 16 - \frac{5Q}{750+t}$ **C** $\frac{dQ}{dt} = 16 + \frac{5Q}{750-t}$

D $\frac{dQ}{dt} = \frac{100Q}{750-t}$ **E** $\frac{dQ}{dt} = 8 - \frac{Q}{1500-2t}$

18 © VCAA 2013 2AQ13 57% Water containing 2 grams of salt per litre flows at the rate of 10 litres per minute into a tank that initially contained 50 litres of pure water. The concentration of salt in the tank is kept uniform by stirring and the mixture flows out of the tank at the rate of 6 litres per minute.

If Q grams is the amount of salt in the tank t minutes after the water begins to flow, the differential equation relating Q to t is

A $\frac{dQ}{dt} = 20 - \frac{3Q}{25+2t}$ **B** $\frac{dQ}{dt} = 10 - \frac{3Q}{25+2t}$ **C** $\frac{dQ}{dt} = 20 - \frac{3Q}{25-2t}$

D $\frac{dQ}{dt} = 10 - \frac{3Q}{25-2t}$ **E** $\frac{dQ}{dt} = 20 - \frac{3Q}{25}$

19 © VCAA 2009 2AQ13 38% A fish tank initially has 4 kg of salt dissolved in 100 litres of water. It is decided that this concentration is too high for saltwater fish to be kept, and so fresh water is mixed in at 10 litres per minute, while 10 litres of the mixture is removed per minute.

If x kg per litre is the **concentration** of the saltwater solution in the tank t seconds after fresh water is first added, the differential equation for x would be

A $10\frac{dx}{dt}+x=0$ **B** $\frac{dx}{dt}-10x=0$ **C** $100\frac{dx}{dt}+x=0$

D $\frac{dx}{dt}-100x=0$ **E** $100\frac{dx}{dt}-x=0$

20 A particular logistic model is $\frac{dP}{dt}=0.005(0.05P-P^2)$. The carrying capacity is

A 4 **B** 5 **C** 16 **D** 20 **E** 100

21 (3 marks) A furnace is switched on at 9 am. Heat is supplied at a constant rate; but as the furnace temperature increases, heat is lost at a rate determined by the difference between its temperature and the surrounding temperature. The temperature of both was 20°C at 9 am. At 9.30 am the furnace was at 200°C and by 10 am it was at 350°C. The minimum operating temperature is 800°C. Use a differential equation to find the time when the furnace was ready for use and the highest temperature it could be expected to reach.

Exam hack

Write the differential equation in the form that describes the required model.

22 (3 marks) A piece of very hot iron was dropped into some cold water at 15°C. After 10 minutes, the iron had cooled to 45°C, and after a further 3 minutes it was at 30°C. The rate of cooling is proportional to the temperature difference between the iron and the water. Find the initial temperature of the iron, correct to the nearest degree.

23 (4 marks) A remote island is used to set up a breeding population of bilbies. Initially, 10 bilbies are released on the island, which has an ideal environment and is thought to be able to sustain a population of 600 bilbies. After 2 years, the population has grown to 30 bilbies. Use the logistic model to find

a when the population will reach 200 2 marks

b the number of bilbies that could be taken back to the mainland in each following year while maintaining the population at 200. 2 marks

24 (3 marks) The completion of a particular chemical reaction involving a catalyst follows the logistic model. The reaction between molecules of the reactants occurs when they are adsorbed onto the catalyst sufficiently close to one another to react. As the reaction proceeds, this is less and less likely to occur. When first measured, 5% of the reaction has taken place; and after 5 minutes, another 5% has taken place. How long is it until the reaction proceeds to completion (99% reacted)?

25 © VCAA 2005 2Q1 (14 marks) A large tank initially contains 10 litres of contaminated water. Clean water that contains a **variable concentration** of a purifying chemical is pumped into the tank at a rate of 20 litres per minute.

After t min, the concentration of the chemical in the water being pumped into the tank is $\frac{2}{1+t^2}$ grams per litre.

The solution in the tank is mixed and pumped out of the tank at a rate of 10 litres per minute.

a 20% If x grams is the amount of purifying chemical in the solution at time t min, write an expression for the concentration of the chemical in the tank, in grams per litre, at time t. 1 mark

b 44% Show that the rate of increase of chemical in the tank satisfies the differential equation

$$\frac{dx}{dt}+\frac{x}{1+t}=\frac{40}{1+t^2}.$$

3 marks

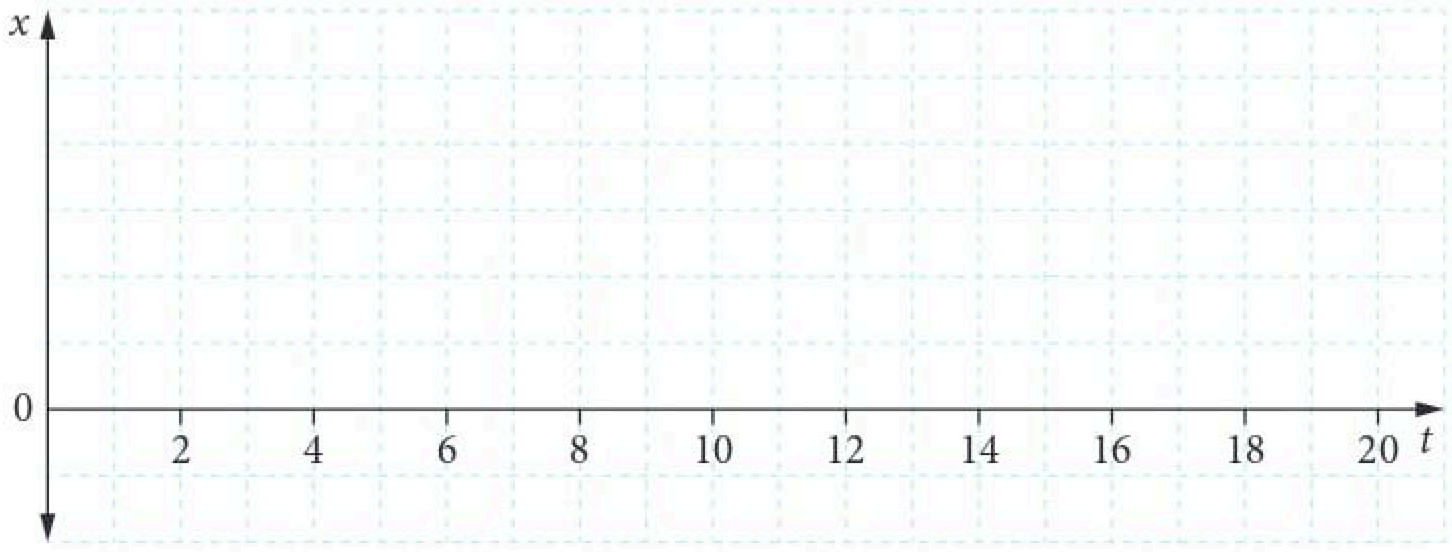
c **i** 54% If $x=\frac{40}{1+t}\tan^{-1}(t)+\frac{20}{1+t}\log_e\left(1+t^2\right)$, show that

$$\frac{dx}{dt}=\frac{40}{1+t^2}-\frac{40\tan^{-1}(t)}{(1+t)^2}-\frac{20\log_e\left(1+t^2\right)}{(1+t)^2}.$$

4 marks

ii 28% **Hence** verify that x satisfies the differential equation given in part **b**. 1 mark

d 65% Copy the axes below and sketch the graph of $x=\frac{40}{1+t}\tan^{-1}(t)+\frac{20}{1+t}\log_e\left(1+t^2\right)$ for $0\le t\le 20$. Label any stationary points with their coordinates correct to two decimal places.



3 marks

e **i** 54% The purifying chemical is only effective if there are at least 15 grams of it present in the tank. Find, correct to three decimal places, the value of t when the purifying chemical first becomes effective. 1 mark

ii 51% **Hence** find for how long the purifying chemical is effective correct to two decimal places. 1 mark

26 © VCAA 2010 2BQ3 (12 marks) The population of a town is initially 20 000 people. This population would increase at a rate of one per cent per year, except that there is a steady flow of people arriving at and leaving from the town. The population P after t years may be modelled by the differential equation $\frac{dP}{dt}=\frac{P}{100}-k$ with the initial condition $P=20\,000$ when $t=0$, where k is the number of people leaving per year minus the number of people arriving per year.

a 9% Verify by substitution that for $k=800$, $P=20\,000(4-3e^{0.01t})$ satisfies both the differential equation and the initial condition. 3 marks

b 82% For $k=800$, find the time taken for the population to decrease to zero. Give your answer correct to the nearest whole year. 2 marks

The differential equation which models the population growth can be expressed as $\frac{dt}{dP}=\frac{100}{P-100k}$ with $P=20\,000$ when $t=0$.

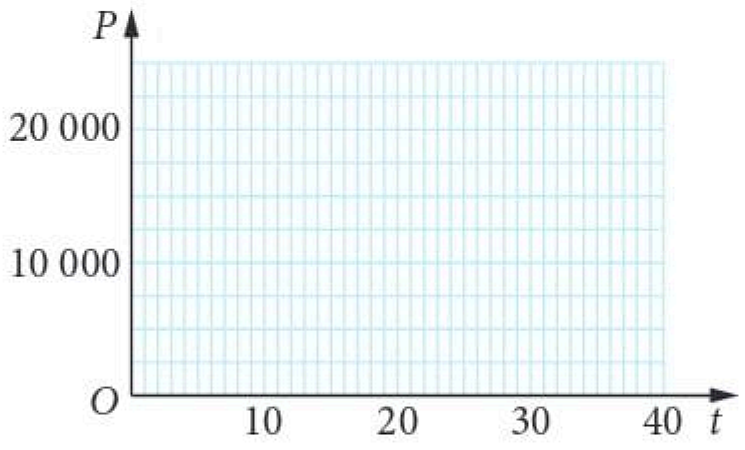
c 73% Show by integration that for $k<0$, the solution of this differential equation is $P=(20\,000-100k)e^{0.01t}+100k$. 2 marks

It can be shown that the solution given in part **c** is valid for all real values of k.

d 51% Copy the set of axes and for each of the values

i $k = 800$ **ii** $k = 200$ **iii** $k = 100$

sketch the graph of P versus t on it while the population exists, for $0 \le t \le 40$.



3 marks

e **i** 57% Find the value of k if the population has increased to 22 550 after twelve years. 1 mark

ii 17% Use the definition of k to interpret your answer to part **i** in the context of the population model. 1 mark

27 © VCAA 2011 2BQ5 (12 marks) A tank initially contains 10 litres of pure water with no chemical present. Water containing a variable concentration of chemical, $e^{-0.2t}$ grams per litre where $t \ge 0$, flows in at the rate of 20 litres per minute. The solution of water and chemical, which is kept uniform by stirring, flows out at the rate of 10 litres per minute.

a 29% If x grams is the amount of chemical in the tank at time t minutes, write down, in terms of x and t, an expression for the **concentration** of chemical in the tank in grams per litre. 1 mark

b 50% Show that the differential equation governing the rate of increase of x with respect to t is

$$\frac{dx}{dt} + \frac{x}{1+t} = 20e^{-0.2t}$$

2 marks

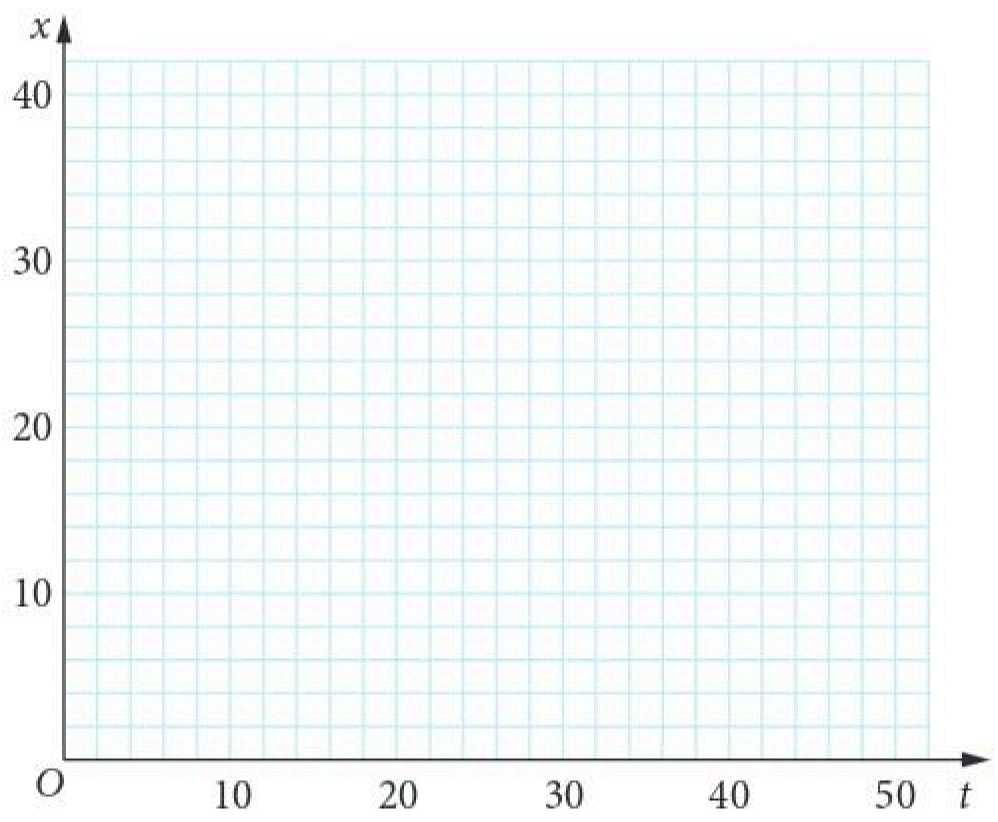
It can be shown that $x(t) = \frac{600}{t+1} - \frac{100e^{-0.2t}(t+6)}{t+1}$.

c 40%

i Find $\frac{dx}{dt}$. 1 mark

ii Verify by substitution that $x(t) = \frac{600}{t+1} - \frac{100e^{-0.2t}(t+6)}{t+1}$ satisfies **both** the **differential equation** and the **initial condition**. 3 marks

d 65% Copy the axes below, and sketch the graph of $x(t)$ for $0 \le t \le 50$, stating the **coordinates** of the turning point, correct to one decimal place.



3 marks

e 12% Find the amount of chemical which has **flowed out of the tank** over the first ten minutes. Give your answer in grams, correct to one decimal place. 2 marks

28 © VCAA 2003 2Q5 (14 marks) Nick takes a bottle of milk from the refrigerator for baby Alex. To heat the bottle, Nick puts it in a saucepan of continuously boiling water. Let y°C be the temperature of the milk at time t minutes after the baby's bottle is placed in the boiling water.

A differential equation that models the increase in temperature of the milk while the bottle is in the boiling water is $\frac{dy}{dt} = a(100 - y)$, where $a > 0$.

a 66% The milk's temperature when the bottle is put into the boiling water is 5°C.

Solve the differential equation to show that $y = 100 - 95e^{-at}$ for $0 \le t \le T$, where T is the time when Nick takes the bottle out of the boiling water. 4 marks

When Nick takes the bottle out of the boiling water at time T, the temperature of the milk is 48°C. He realises that this is too hot to give to baby Alex and so he puts the bottle into cold water. The temperature of the cold water is 10°C and the milk cools according to Newton's law of cooling:

$$\frac{dy}{dt} = -b(y - 10), \text{ where } b > 0$$

b 19% Verify, by differentiation, that for $t \ge T$, $y = 10 + Ae^{-b(t - T)}$, and evaluate A. 3 marks

Nick lets the milk cool to a temperature of 36°C to give to baby Alex. It takes three times as long for the milk to cool to this temperature from 48°C as it previously took to heat up from 5°C to 48°C.

c 34% Sketch a graph of y in terms of t from when the baby's bottle is put into the boiling water to when the milk is ready to give to baby Alex. 3 marks

d 15% Find the ratio $\frac{a}{b}$ correct to three significant figures. 4 marks

29 © VCAA 2017N 2BQ3 (9 marks) Bacteria are spreading over a Petri dish at a rate modelled by the differential equation

$$\frac{dP}{dt} = \frac{P}{2}(1 - P), 0 < P < 1$$

where P is the **proportion** of the dish covered after t hours.

a i Express $\frac{2}{P(1 - P)}$ in partial fraction form. 1 mark

ii Hence show by integration that $\frac{t - c}{2} = \log_e\left(\frac{P}{1 - P}\right)$, where c is a constant of integration. 2 marks

iii If half of the Petri dish is covered by the bacteria at $t = 0$, express P in terms of t. 2 marks

After one hour, a toxin is added to the Petri dish, which harms the bacteria and reduces their rate of growth. The differential equation that models the rate of growth is now

$$\frac{dP}{dt} = \frac{P}{2}(1 - P) - \frac{\sqrt{P}}{20} \text{ for } t \ge 1.$$

b Find the limiting value of P, which is the maximum possible proportion of the Petri dish that can now be covered by the bacteria. Give your answer correct to three decimal places. 2 marks

c The total time, T hours, measured from time $t = 0$, needed for the bacteria to cover 80% of the Petri dish is given by

$$T = \int_q^r \left(\frac{1}{\frac{P}{2}(1 - P) - \frac{\sqrt{P}}{20}} \right) dP + s$$

where q, r and $s \in R$.

Find the values of q, r and s, giving the value of q correct to two decimal places. 2 marks

Video playlist
Separation of variables
$\frac{dy}{dx} = f(x)\, g(y)$

Worksheets
Solving
$\frac{dy}{dx} = f(x)\, g(y)$

Differential equations and exponentials

Differential equations

9.5 Separation of variables $\frac{dy}{dx} = f(x)g(y)$

A first-order, degree 1, differential equation of the form $\frac{dy}{dx} = f(x)g(y)$ allows the variables to be separated and anti-differentiation to be performed.

$$\begin{aligned} \frac{dy}{dx} &= f(x)g(y) \\ \frac{1}{g(y)}\frac{dy}{dx} &= f(x) \end{aligned}$$

This is called **separation of variables** because we have the y-terms on the left and the x-terms on the right.

We also need to be careful that $g(y) = 0$ is not a solution of the differential equation.

Integrate both sides with respect to x:

$$\begin{aligned} \int \frac{1}{g(y)}\frac{dy}{dx}\,dx &= \int f(x)\,dx \\ \int \frac{1}{g(y)}\,dy &= \int f(x)\,dx \end{aligned}$$

WB
p. 193

WORKED EXAMPLE 14 Separation of variables

Solve $\frac{dy}{dx} = (2x - 3)(3y + 4)$.

Steps	**Working**
1 Separate the variables.	$\frac{1}{3y+4}\frac{dy}{dx} = 2x - 3$
2 Integrate both sides.	$\int \frac{1}{3y+4}\frac{dy}{dx}\,dx = \int (2x-3)\,dx$ $\int \frac{1}{3y+4}\,dy = \int (2x-3)\,dx$ $\frac{1}{3}\log_e \lvert 3y+4 \rvert = x^2 - 3x + c$
3 Make y the subject.	$\log_e \lvert 3y+4 \rvert = 3x^2 - 9x + 3c$ $3y + 4 = e^{3x^2-9x+3c}$ $3y = e^{3x^2-9x}e^{3c} - 4$ $= Ae^{3x^2-9x} - 4$ where $A = e^{3c}$ $y = ke^{3x^2-9x} - \frac{4}{3}$ where $k = \frac{A}{3}$ is another constant.

WORKED EXAMPLE 15 Separation of variables problem

The rate at which a new social network expands is proportional to both the size of the network and the time since its establishment. Two weeks after the establishment of a new network, there are 30 members and after another week there are 120. How long will it take to pass the 10 000-member milestone?

Steps	Working
1 Choose variables and write the rate equation for m.	Let m be the number of members and t = time in weeks. $\frac{dm}{dt} = kmt$
2 Separate variables and integrate both sides.	$\frac{1}{m}\frac{dm}{dt} = kt$ $\int \frac{1}{m}\frac{dm}{dt}\,dt = \int kt\,dt$
3 Make m the subject.	$\ln(m) = \frac{1}{2}kt^2 + c$ $m = e^{\frac{1}{2}kt^2+c}$ $= e^{\frac{1}{2}kt^2}\,e^c$
4 Rename constants to simplify.	$m = Ae^{\frac{1}{2}kt^2}$, where $A = e^c$
5 Use $m(2) = 30$ and $m(3) = 120$ to find k.	$30 = Ae^{\frac{1}{2}k(2^2)}$ and $120 = A\,e^{\frac{1}{2}k(3^2)}$ $30 = Ae^{2k}$ and $120 = Ae^{\frac{9k}{2}}$
6 Divide the second equation by the first.	$4 = e^{\frac{5}{2}k}$ $\ln(4) = \frac{5k}{2}$ $k = \frac{2\ln(4)}{5} = 0.4\ln(4)$
7 Write the equation for m.	$m = Ae^{\frac{1}{2}[0.4\ln(4)]t^2} = Ae^{[0.2\ln(4)]t^2}$
8 Use $m(2) = 30$ to find A.	$30 = Ae^{[0.2\ln(4)]2^2}$ $= Ae^{0.8\ln(4)}$ $= Ae^{\ln\left(4^{0.8}\right)}$ $= A(4^{0.8})$ $A = 30 \div 4^{0.8} = 9.896\ldots$ $m = 9.896\ldots e^{[0.2\ln(4)]t^2}$

WB p. 194

9 Substitute $m = 10\,000$ to find t.

$$\begin{aligned} 10\,000 &= 9.896\ldots e^{[0.2\ln(4)]t^2} \\ \frac{10\,000}{9.896\ldots} &= e^{[0.2\ln(4)]t^2} \\ \ln(1010.477\ldots) &= [0.2\ln(4)]t^2 \\ t^2 &= \frac{\ln(1010.477\ldots)}{0.2\ln(4)} = 24.9520\ldots \\ t &= 4.995\ldots \end{aligned}$$

10 Write the answer.

It will take about 5 weeks to get to 10 000 members.

Separation of variables

$\dfrac{dy}{dx} = f(x)g(y)$ is solved by finding

$\displaystyle\int \frac{1}{g(y)}\,dy = \int f(x)\,dx + c$, where c is a constant.

EXERCISE 9.5 Separation of variables $\frac{dy}{dx} = f(x)g(y)$

ANSWERS p. 595

Recap

80–100% 60–79% 0–59%

1 © VCAA 2015 2AQ12 76% Given that $\dfrac{dy}{dx} = 1 - \dfrac{y}{3}$ and $y = 4$ when $x = 2$, then

A $y = e^{\frac{-(x-2)}{3}} - 3$

B $y = e^{\frac{-(x-2)}{3}} + 3$

C $y = 4e^{\frac{-(x-2)}{3}}$

D $y = e^{\frac{4(y-x-2)}{3}}$

E $y = e^{\frac{(x-2)}{3}} + 3$

2 © VCAA 2005 1IQ19 69% Given that $\dfrac{dy}{dx} = y^2 + 1$ and that $y = 1$ at $x = 0$, then

A $y = \tan\left(x - \dfrac{\pi}{4}\right)$

B $y = \tan\left(x + \dfrac{\pi}{4}\right)$

C $y = \log_e\left(\dfrac{y^2+1}{2}\right)$

D $y = \dfrac{1}{3}y^3 + y - \dfrac{1}{3}$

E $y = y^2x + x + 1$

Mastery

3 WORKED EXAMPLE 14 Solve the differential equation $\dfrac{dy}{dx} = 5x(3y + 8)$.

4 WORKED EXAMPLE 15 The speed $\dfrac{dx}{dt}$ m/s of an object x m from a fixed point is given by $\dfrac{dx}{dt} = kxt^2$, where k is a constant.

a Find x as a function of t.

b Find the values of k and the constant of integration, c, given that $x(0) = 2$ and $x(3) = 2e$.

c How long (to the nearest second) will it take the object to be 100 metres from the fixed point?

5 TECH-FREE Solve $\frac{dy}{dt} = y(3y+2)t$ if the initial value of y is 6.

6 TECH-FREE Solve $e^{2x+y} - 2e^{2x-y}\frac{dy}{dx} = 0$ if the initial value of y is 6.

7 TECH-FREE

a Solve $\frac{d\theta}{dt} = \frac{1}{(1+t)\sin(\theta)}$ given that $\theta(0) = 0$.

b Show that $0 \le t \le e^2 - 1$.

8 TECH-FREE Solve the differential equation $\frac{dy}{dx} = \frac{ky+1}{kx+1}$ (k is a constant), and show that $y = A(1+kx) - B$, where A and B are constants.

9 TECH-FREE If the curve of the solution to $\sqrt{x}\frac{dy}{dx} = \frac{1}{\sqrt{y}}$ passes through the origin, state the value of the constant A in $y = \sqrt[3]{Ax}$.

Exam practice

80–100% 60–79% 0–59%

10 © VCAA 2018 1Q8 TECH-FREE (4 marks) A tank initially holds 16 L of water in which 0.5 kg of salt has been dissolved. Pure water then flows into the tank at a rate of 5 L per minute. The mixture is stirred continuously and flows out of the tank at a rate of 3 L per minute.

a 44% Show that the differential equation for Q, the number of kilograms of salt in the tank after t minutes, is given by 1 mark

$$\frac{dQ}{dt} = -\frac{3Q}{16+2t}$$

b 52% Solve the differential equation given in part **a**. to find Q as a function of t.

Express your answer in the form $Q = -\frac{a}{(16+2t)^{\frac{b}{c}}}$, where a, b and c are positive integers. 3 marks

11 TECH-FREE (3 marks) Solve the differential equation $\frac{dy}{dx} = \frac{y}{x^2}$ for y, given that when $x = 1$, $y = -1$.

12 © VCAA 2016 1Q10 46% TECH-FREE (5 marks) Solve the differential equation $\sqrt{2-x^2}\frac{dy}{dx} = \frac{1}{2-y}$, given that $y(1) = 0$. Express y as a function of x.

13 If the constant of integration in the solution to $\frac{dx}{dt} = \frac{e^t}{2x}$ is zero, the relationship between x and t is

A $x^2 + 2e^t = 0$ **B** $2x^2 - e^{-t} = 0$ **C** $x^2 - e^t = 0$ **D** $x^2 + e^{-t} = 0$ **E** $x^2 - e^{2t} = 0$

14 A solution to $\frac{dy}{dx} = \frac{x}{y}$ is

A $x^2 - y^2 = c$ **B** $x + y^2 = c$ **C** $x^2 + y^2 = c$ **D** $x + y = c$ **E** $x^2 - y = c$

▶ **15** The rate of change of distance of a particle with respect to time, t, is four times the square of the distance x and the cube of the time taken. If $x(1) = 1$, the equation for distance is

A $x = \dfrac{1}{3-2t^4}$ **B** $x = \dfrac{2}{1+t^4}$ **C** $x = \dfrac{1}{1-t^4}$

D $x = -\dfrac{1}{1-2t^4}$ **E** $x = \dfrac{1}{2-t^4}$

16 The differential equation $\dfrac{dy}{d\theta} = 2e^y \cos^2(\theta)$ can be solved using

A $\int e^y\, dy = \int 1 + \cos(\theta)\, d\theta$ **B** $\int e^{-y}\, dy = \int 1 + 2\cos(\theta)\, d\theta$

C $\int e^{-y}\, dy = \int 1 - \cos(2\theta)\, d\theta$ **D** $\int e^{-y}\, dy = \int 1 + \cos(2\theta)\, d\theta$

E $\int e^y\, dy = \int 2\left[1 + \cos(2\theta)\right] d\theta$

17 The point $\left(\dfrac{\pi}{6}, 3\right)$ satisfies the differential equation $\dfrac{dy}{dx} = \dfrac{1}{y\sqrt{1-x^2}}$. The constant of integration is

A 3.5 **B** 4 **C** 5 **D** 5.5 **E** 6

18 © VCAA 2018 2AQ9 69% A solution to the differential equation $\dfrac{dy}{dx} = \dfrac{2}{\sin(x+y) - \sin(x-y)}$ can be obtained from

A $\int 1\, dx = \int 2\sin(y)\, dy$ **B** $\int \cos(y)\, dx = \int \operatorname{cosec}(x)\, dx$

C $\int \cos(x)\, dx = \int \operatorname{cosec}(y)\, dy$ **D** $\int \sec(x)\, dx = \int \sin(y)\, dy$

E $\int \sec(x)\, dx = \int \operatorname{cosec}(y)\, dy$

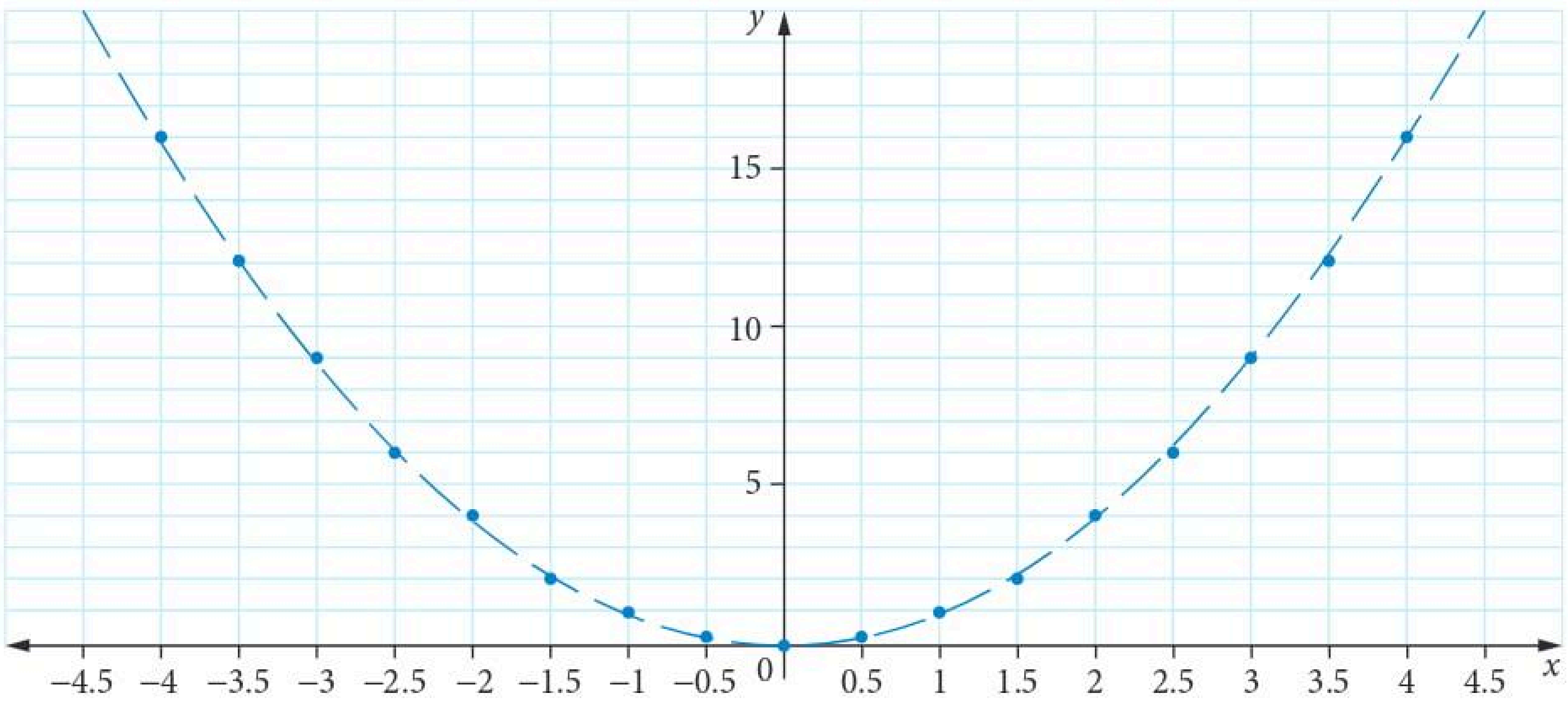
Exam hack

The sum or difference of trigonometric functions should be expressed as products so that separation of variables is made possible.

9.6 Slope fields

The graph shows tangent lines for the gradient function $\frac{dy}{dx} = 2x$.

x	Gradient
−4	−8
−3	−6
−2	−4
−1	−2
0	0
1	2
2	4
3	6
4	8



Video playlist
Slope fields

The shape is a parabola with equation $y = x^2$, which is one solution to $\frac{dy}{dx} = 2x$.

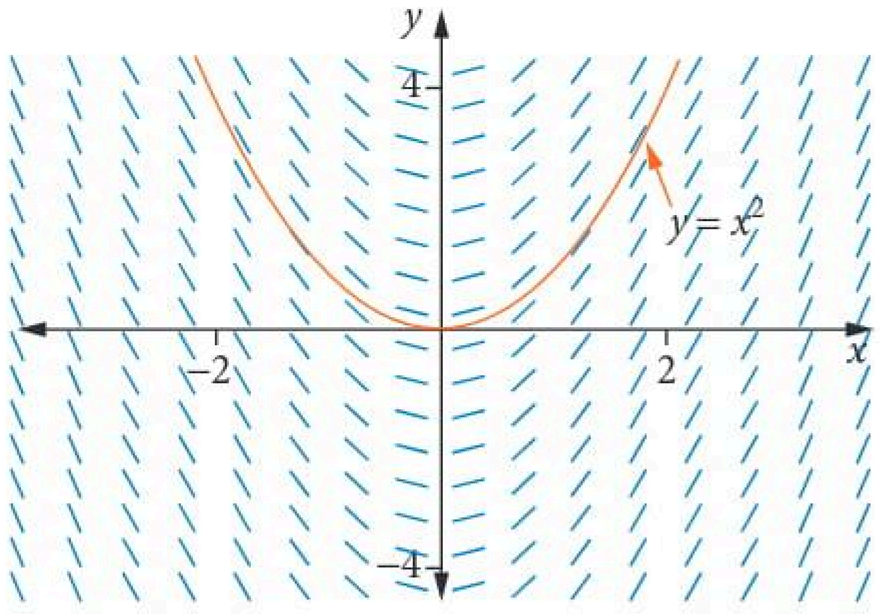
However, it is only one out of a family of curves because $\frac{dy}{dx} = 2x$ satisfies $y = x^2 + c$, where c is a constant.

A **slope field** (or **gradient field** or **direction field**) displays a family of solutions for various values of c.

Below is the slope field for $\frac{dy}{dx} = 2x$ and the solution function, $y = x^2$, for $c = 0$.

Each gradient line in a column has the same slope because in the derivative function, $\frac{dy}{dx}$ is a function of x.

If $\frac{dy}{dx}$ is a function of y, each gradient line in a row will have the same slope.



Note that the tangent lines can also be shown as arrowed lines.

The pattern of the derivatives indicates the shape of the solution function, in this case, the family of parabolae $y = x^2 + c$.

WB

p. 195

WORKED EXAMPLE 16 Sketching a slope field

a Sketch the slope field for $\frac{dy}{dx} = 0.5y$ for $-3 \le y \le 3$.

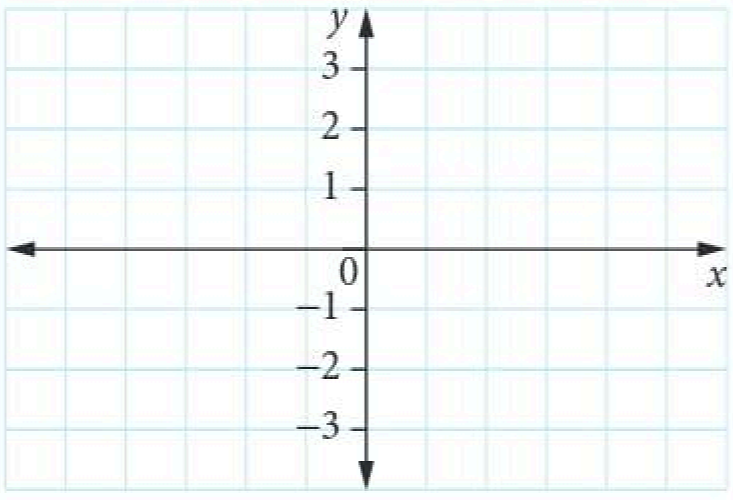
b Describe what type of function y must be.

Steps	**Working**
a 1 Complete a table of $\frac{dy}{dx}$ values.	Evaluate $\frac{dy}{dx}$ using $\frac{dy}{dx} = 0.5y$.

y	-3	-2	-1	0	1	2	3
$\frac{dy}{dx} = 0.5y$	-1.5	-1	-0.5	0	0.5	1	1.5

2 Draw the grid and decide if the slopes will be the same vertically or horizontally.

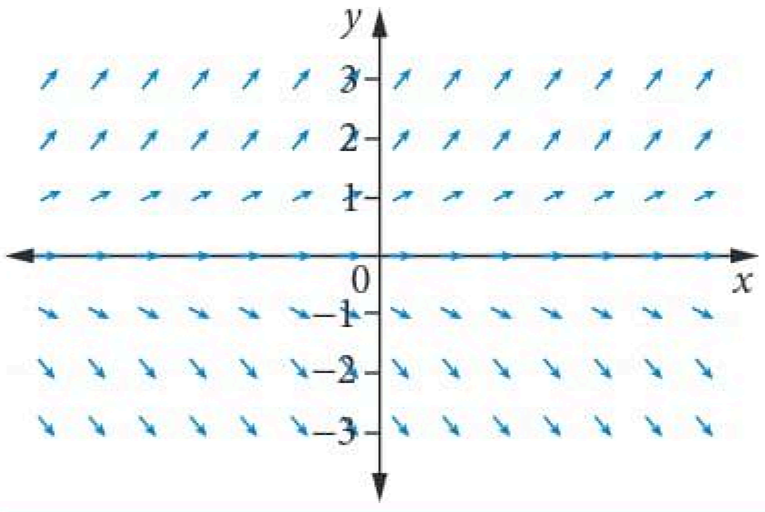
$\frac{dy}{dx}$ is a function of y only, so each gradient line will have the same slope horizontally.



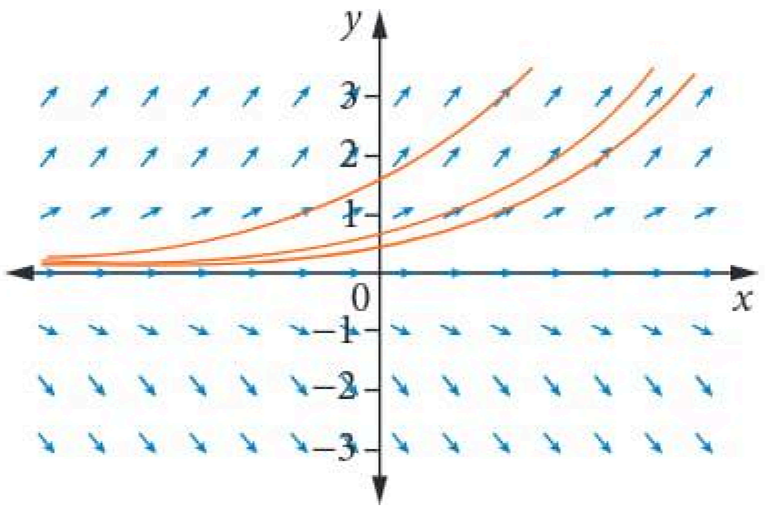
3 Sketch the slope lines.

The first gradient is -1.5 at $y = -3$ in the third row below the x-axis. Lines of gradient -1.5 are drawn for all x values in this row.

Repeat this for each of the other y values and their gradients.



b 1 Connect the pattern of arrows to show the general shape of the anti-derivative function.



2 Describe the type of curve.

The curves appear to be exponential, since the gradient rises sharply as x increases.

The use of CAS is a quick and efficient way of sketching slope fields.

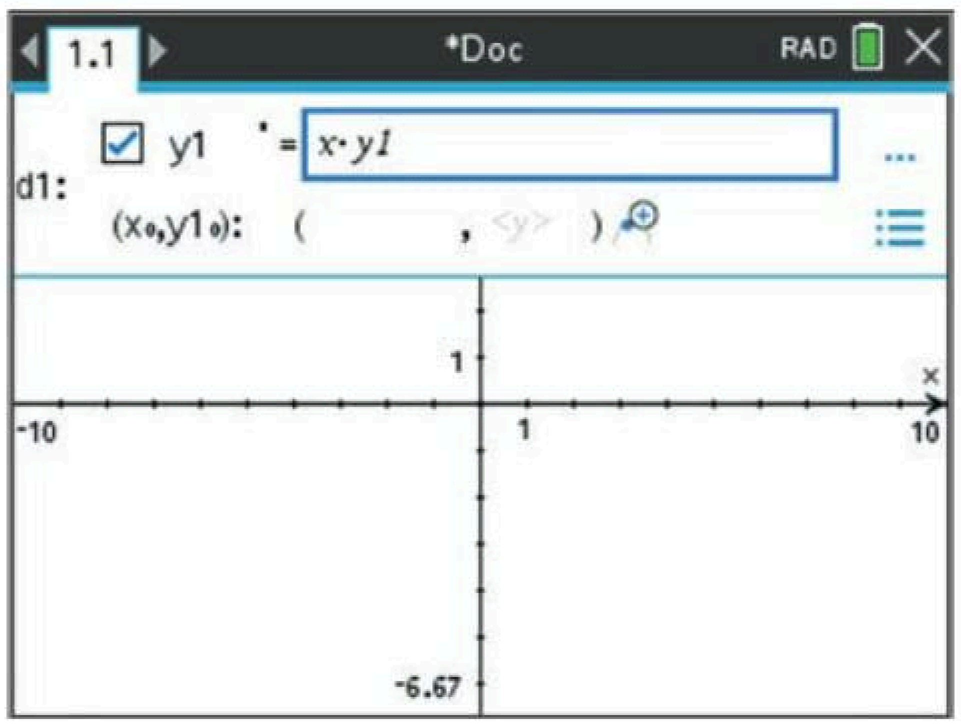
USING CAS 4 Slope fields

a Sketch the slope fields for $\frac{dy}{dx} = xy$.

b Show the solution for $\frac{dy}{dx} = xy$ if $y(2) = 5$.

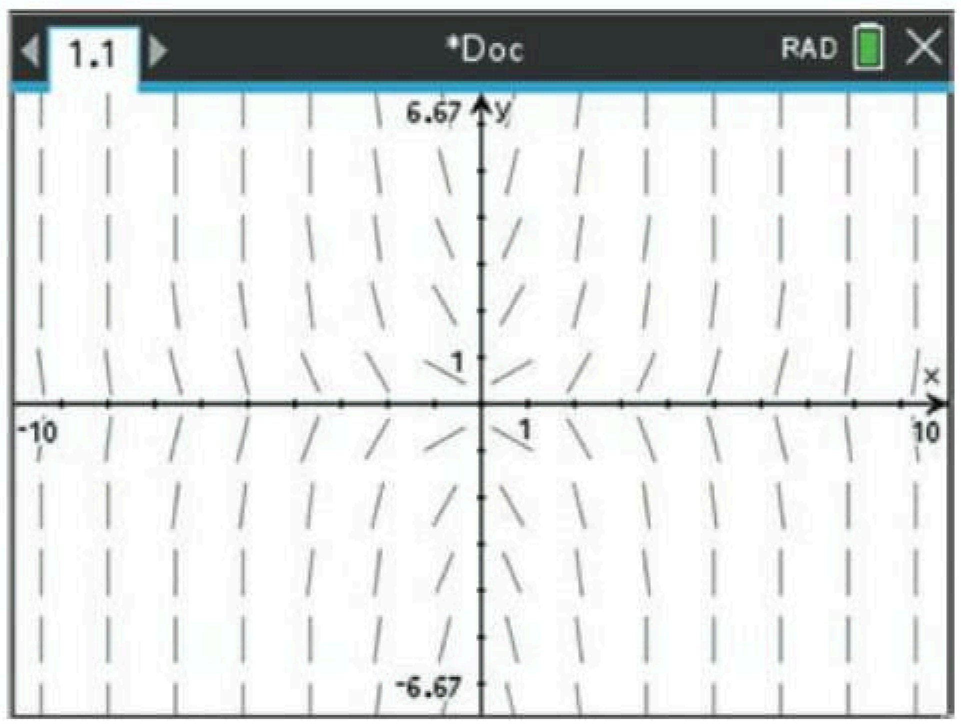
TI-Nspire

a



☑ y1' = x·y1

(x0,y10): (,)



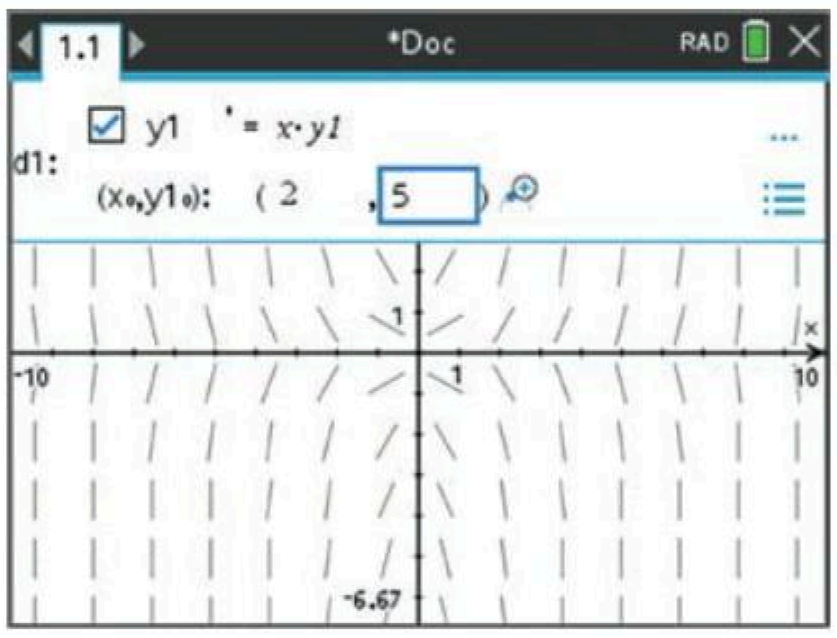
1 Add a **Graphs** page.

2 Press **menu** > **Graph Entry/Edit** > **Diff Eq**.

3 In the **y1** field, enter the expression **x** × **y1**.

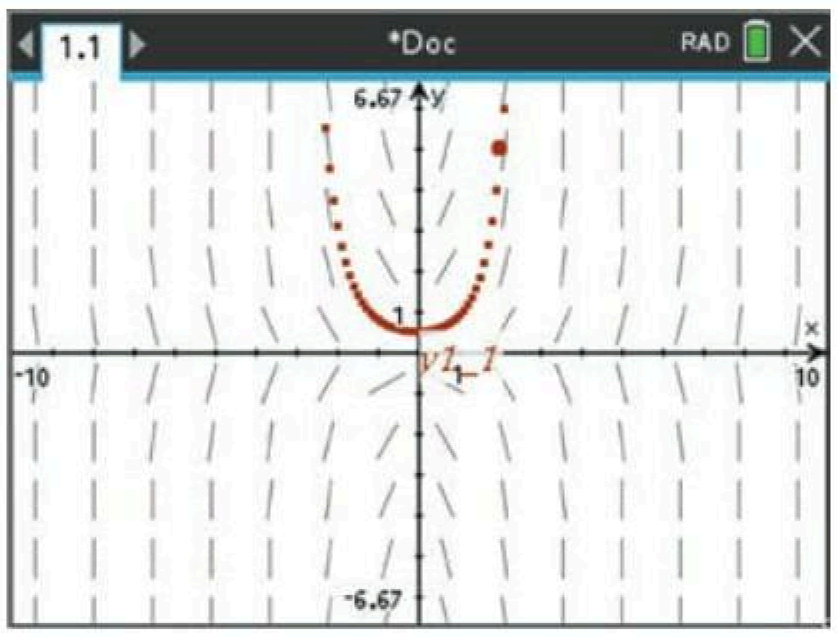
4 The slope field will be displayed.

b



☑ y1' = x·y1

(x0,y10): (2 , 5)



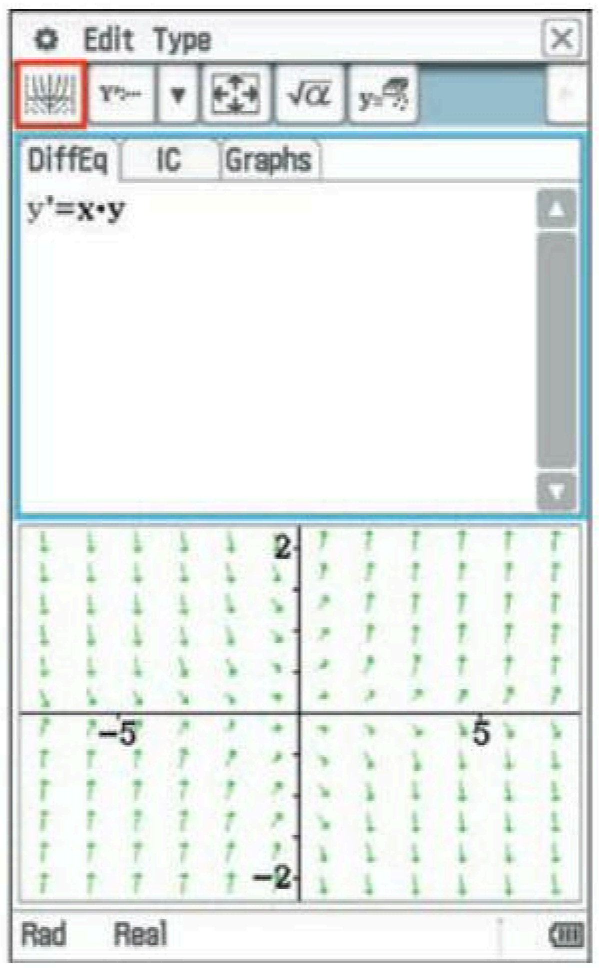
1 Press **tab** then the **up arrow** to return to the **y1** field.

2 In the **(x_0,y1$_0$)** field, enter the initial conditions **2** and **5**.

3 The slope field with a plot through the point (2, 5) will be displayed.

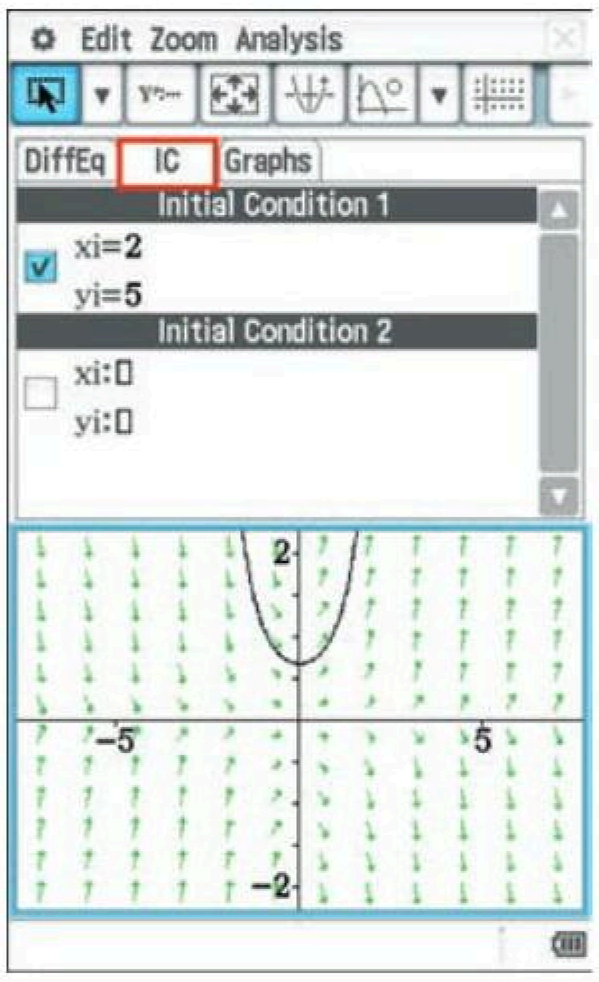
ClassPad

a



y'=x•y

b



Initial Condition 1

☑ xi=2

yi=5

Initial Condition 2

☐ xi:□

yi:□

1 Tap **Menu** and select the **DiffEq-Graph** application.

2 Tap **Math3** to find **y′** =, enter ***xy***. Using **Var** there is no need for the × (multiply) sign.

3 Tap **Graph** to display the slope field.

1 Tap the **IC** tab.

2 In the **Initial Condition 1** section, enter **xi=2** and **yi=5**.

3 Graph to display the slope field with a plot through the point (2, 5).

Exam hack

For differential equations of the form $\frac{dy}{dx} = f(x, y)$, estimate the gradient by keeping one variable constant.

First think of $\frac{dy}{dx} = xy^2$ as $\frac{dy}{dx} = kx$ and then as $\frac{dy}{dx} = ky^2$.

EXERCISE 9.6 Slope fields

ANSWERS p. 595

Recap

1 Given $\frac{dy}{dx} = e^x\sqrt{y}$ and $y(0) = 9$, then

A $y = \frac{1}{4}\left(e^x + 5\right)^2$

B $y = \frac{1}{2}\left(e^x - 5\right)^2$

C $y = \frac{1}{3}\left(e^x + 1\right)^2$

D $y = \frac{1}{4}\left(e^x - 5\right)$

E $y = -\frac{1}{2}\left(e^x + 5\right)^2$

2 For $\frac{dx}{dt} = x\cos(t)$ with $x\left(\frac{\pi}{2}\right) = e^2$, the value of the constant of integration in the solution is

A -1

B $\frac{1}{4}$

C $\frac{1}{2}$

D 1

E 2

Mastery

3 WORKED EXAMPLE 16

a Sketch the slope field for $\frac{dy}{dx} = 3x^2$ for $-5 \le x \le 5$.

b Describe what type of function y must be.

4 Using CAS 5 Sketch the slope fields for $\frac{dy}{dx} = x + y$ and show the solution when $y(2) = 1$.

Exam practice

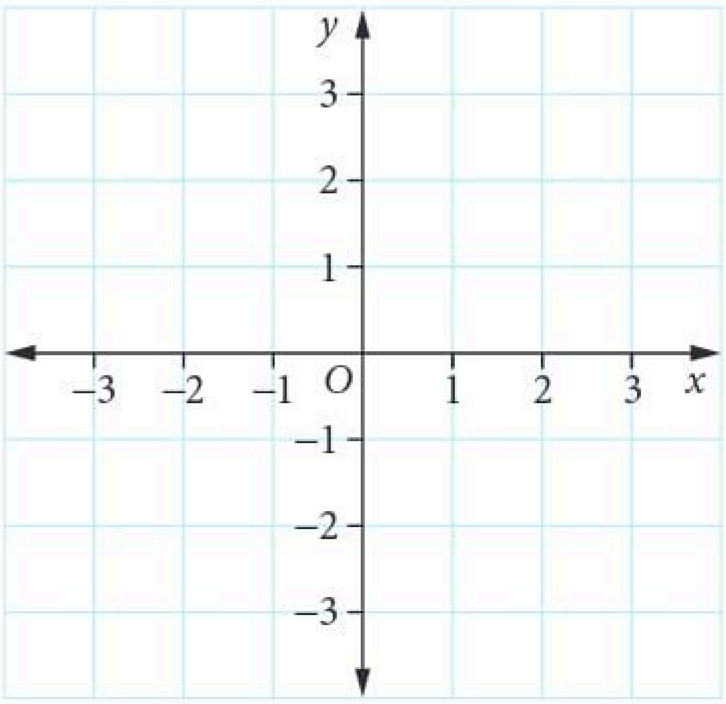
80–100% 60–79% 0–59%

5 TECH-FREE (2 marks) Draw the slope field for the function $\frac{dy}{dx} = 3\cos(x^2)$ and show the particular solution for $y(0) = 3$.

6 © VCAA 2007 1Q8 TECH-FREE (6 marks)

a 38% Copy the axes below and on them sketch the slope field of the differential equation

$\frac{dy}{dx} = \frac{1+y^2}{2}$ for $y = -2, -1, 0, 1, 2$ at each of the values $x = -2, -1, 0, 1, 2$.



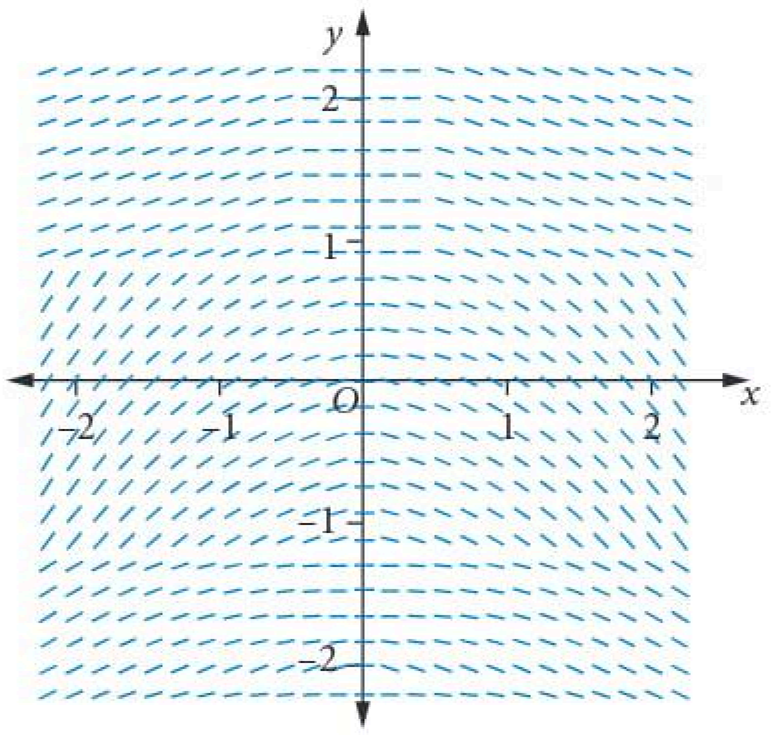
2 marks

b 59% If $y = -1$ when $x = 0$, solve the differential equation given in part **a** to find y in terms of x. 3 marks

c 13% Sketch the graph of the solution curve found in part **b** on the slope field in part **a**. 1 mark

7 © VCAA 2017 1Q8 TECH-FREE (4 marks) A slope field representing the differential equation

$\frac{dy}{dx} = \frac{-x}{1+y^2}$ is shown below.



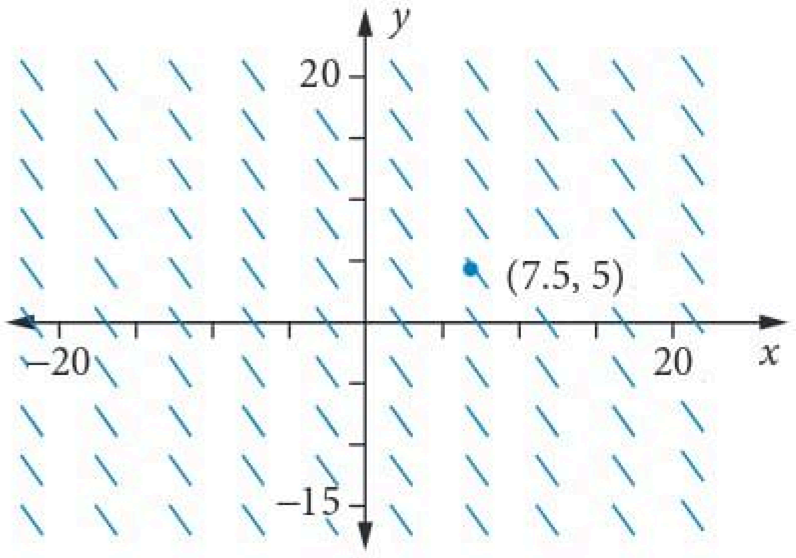
a 32% Sketch the solution curve of the differential equation corresponding to the condition $y(-1) = 1$ on the slope field above and, hence, estimate the positive value of x when $y = 0$. Give your answer correct to one decimal place.

2 marks

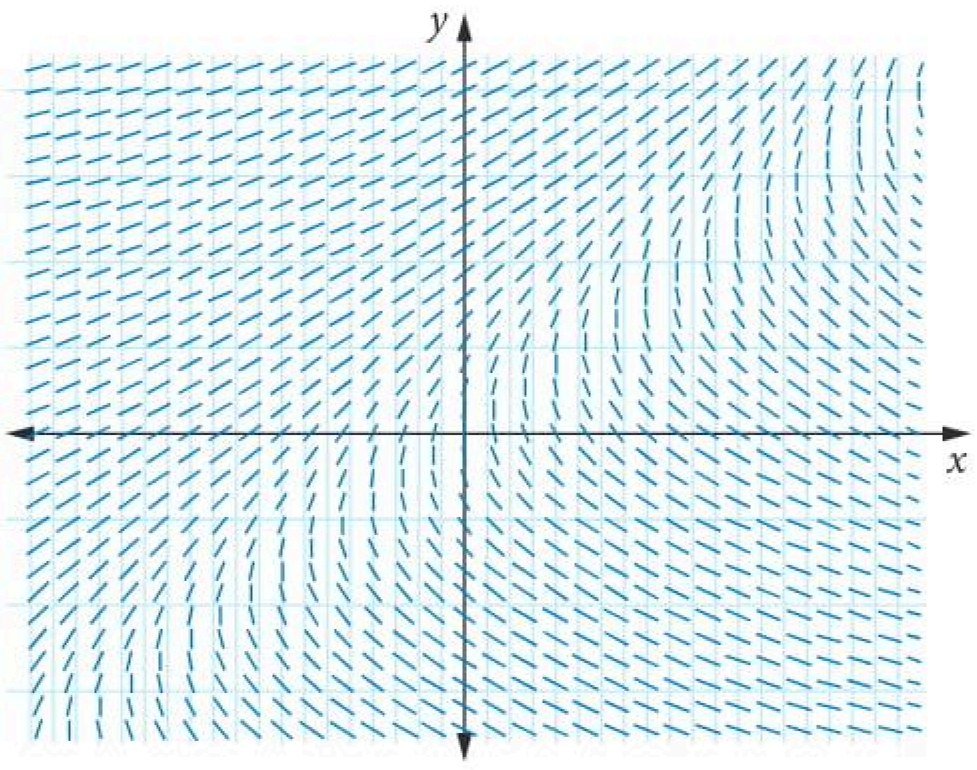
b 61% Solve the differential equation $\frac{dy}{dx} = \frac{-x}{1+y^2}$ with the condition $y(-1) = 1$.

Express your answer in the form $ay^3 + by + cx^2 + d = 0$, where a, b, c and d are integers. 2 marks

8 TECH-FREE (2 marks) Find the particular solution to the slope field shown below.



9 © VCAA 2014 2AQ14 72% The differential equation that is best represented by the direction field on the right is



A $\frac{dy}{dx} = \frac{1}{x-y}$

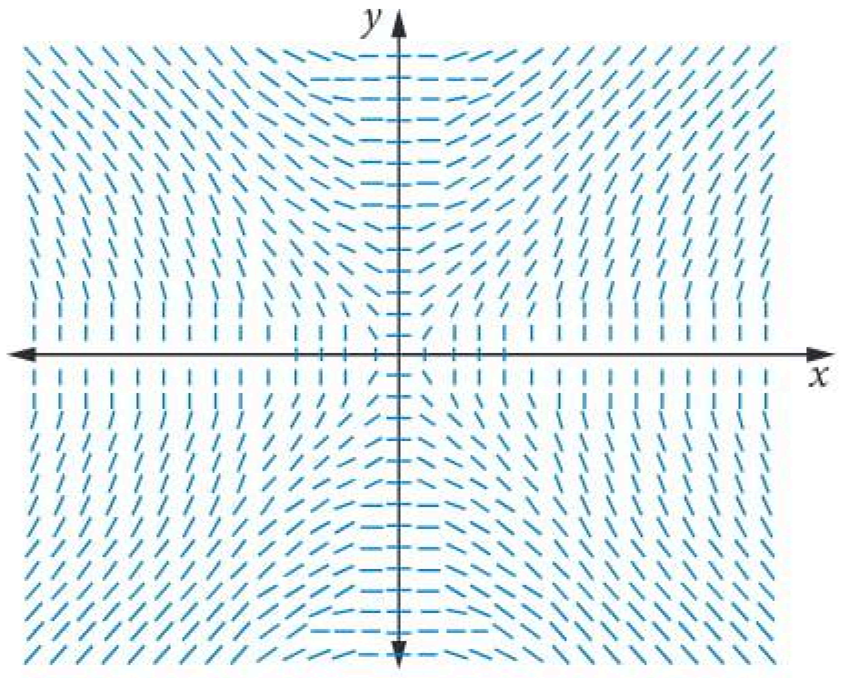
B $\frac{dy}{dx} = y - x$

C $\frac{dy}{dx} = \frac{1}{y-x}$

D $\frac{dy}{dx} = x - y$

E $\frac{dy}{dx} = \frac{1}{y+x}$

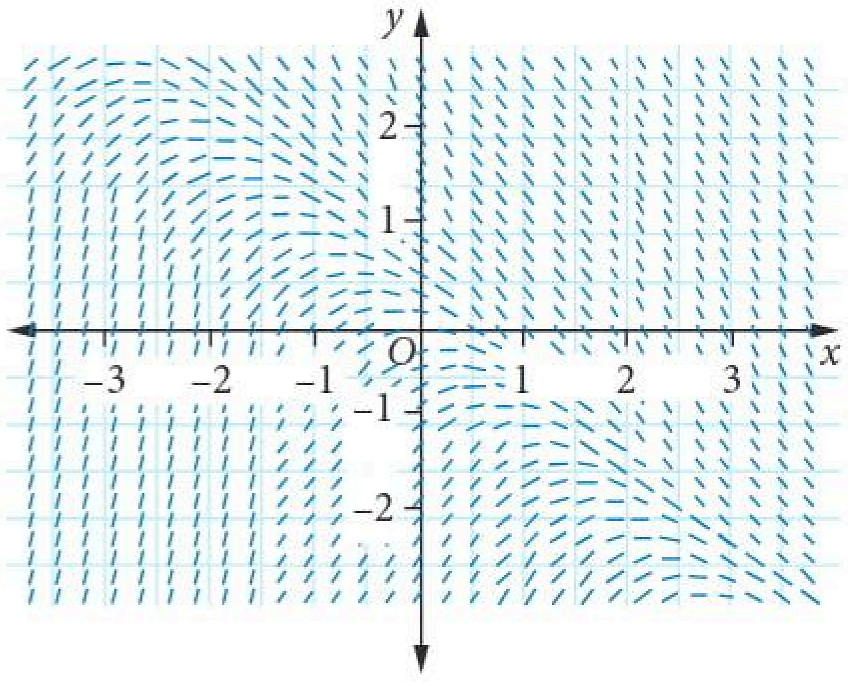
10 © VCAA 2013 2AQ12 **67%**



The differential equation that best represents the above direction field is

A $\frac{dy}{dx} = x^2 - y^2$ **B** $\frac{dy}{dx} = y^2 - x^2$ **C** $\frac{dy}{dx} = \frac{y}{x}$ **D** $\frac{dy}{dx} = -\frac{x}{y}$ **E** $\frac{dy}{dx} = \frac{x}{y}$

11 © VCAA 2016 2AQ10 **65%**



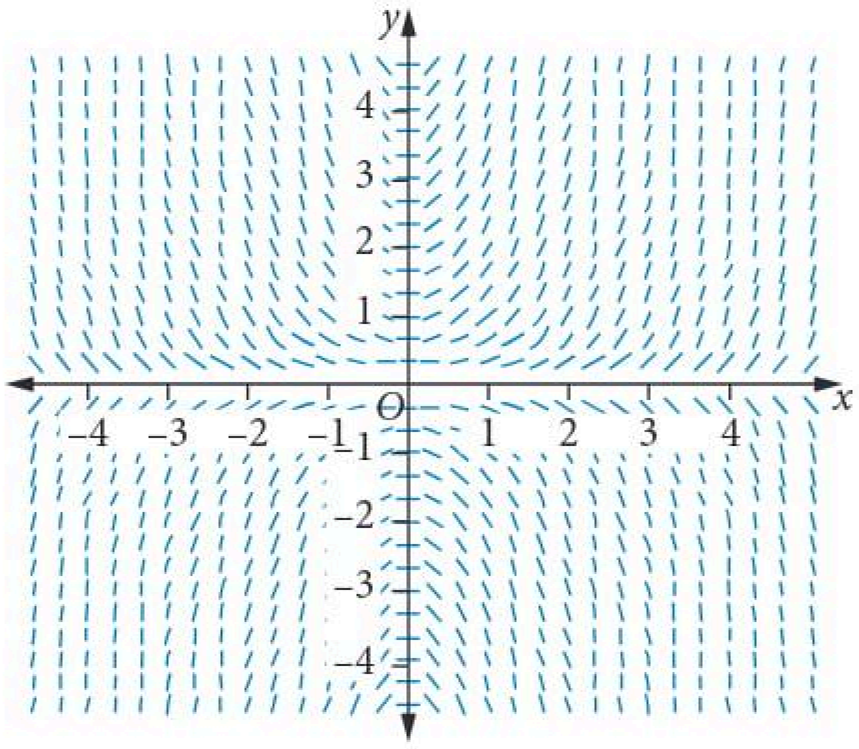
The direction field for the differential equation $\frac{dy}{dx} + x + y = 0$ is shown above.

A solution to this differential equation that includes (0, −1) could also include

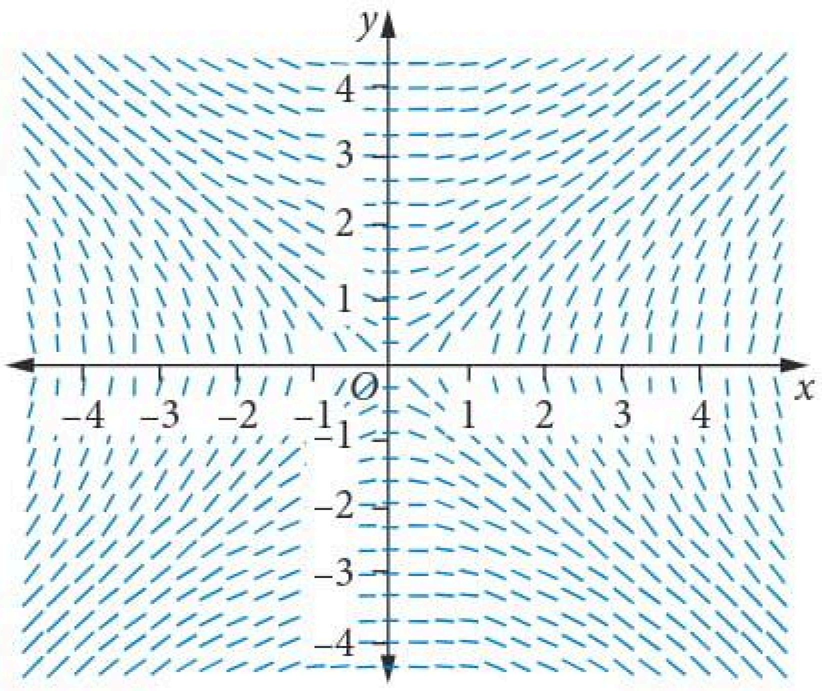
A (3, −1)
B (3.5, −2.5)
C (−1.5, −2)
D (2.5, −1)
E (2.5, 1)

12 © VCAA 2012 2AQ10 **63%** The diagram that best represents the direction field of the differential equation $\frac{dy}{dx} = xy$ is

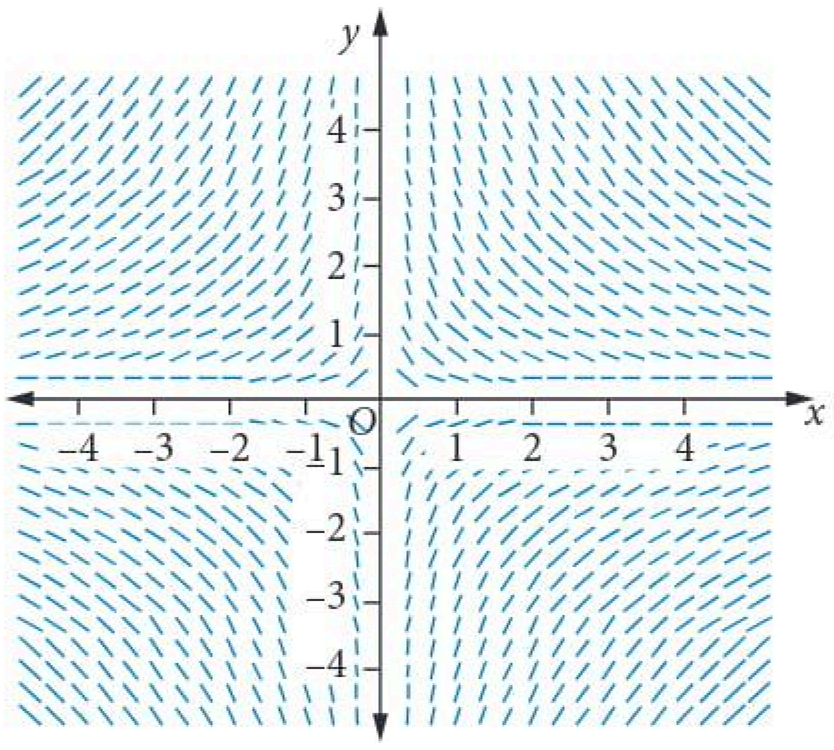
A



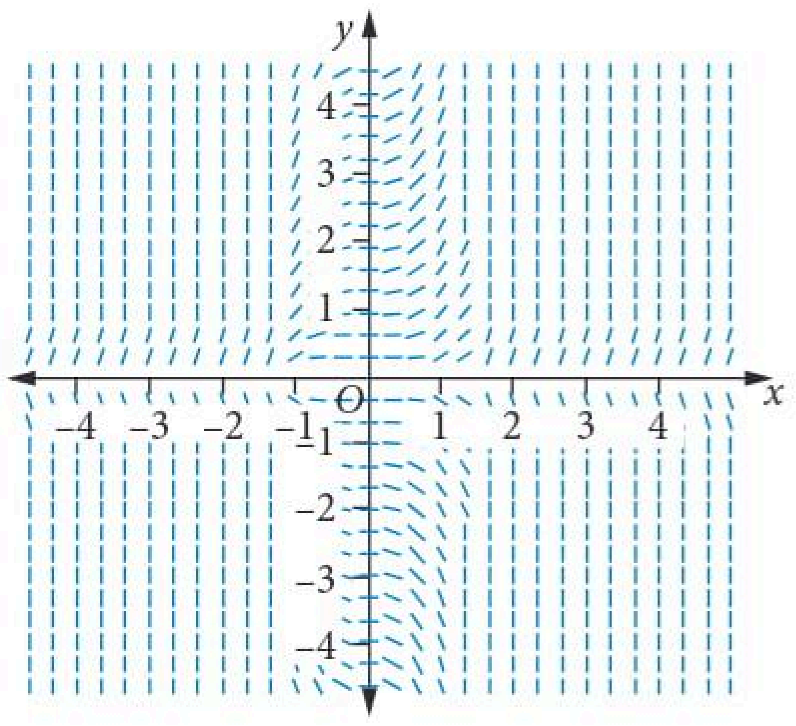
B



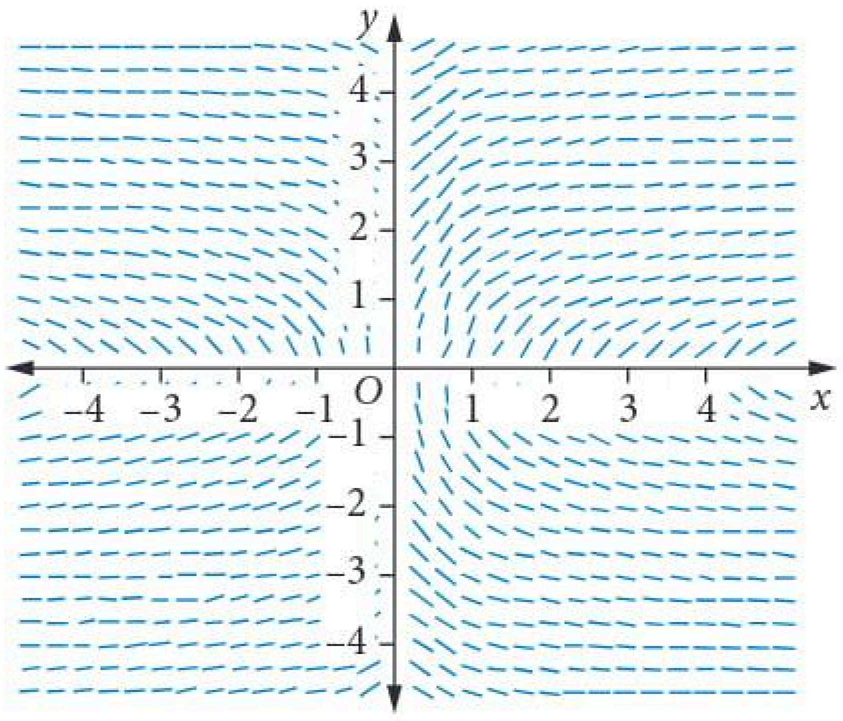
C



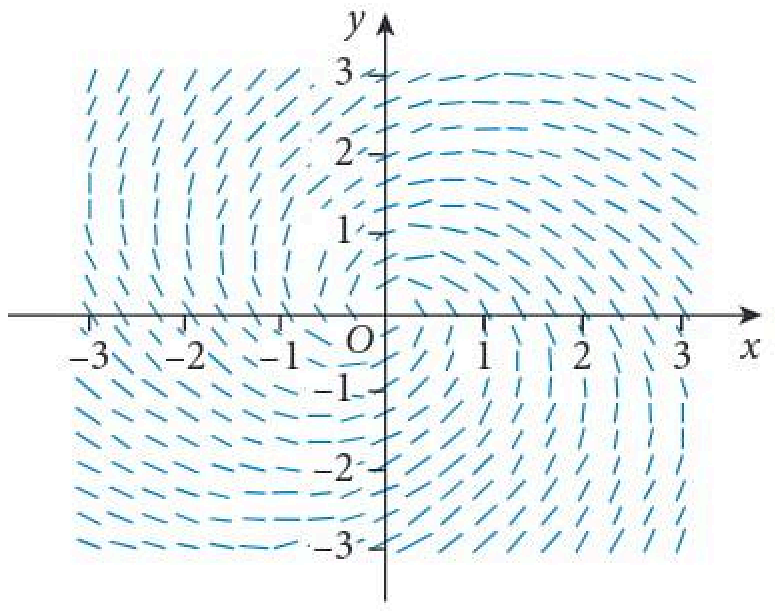
D



E



13 © VCAA 2011 2AQ17 52%



The differential equation which best represents the above direction field is

A $\frac{dy}{dx} = \frac{y - 2x}{2y + x}$

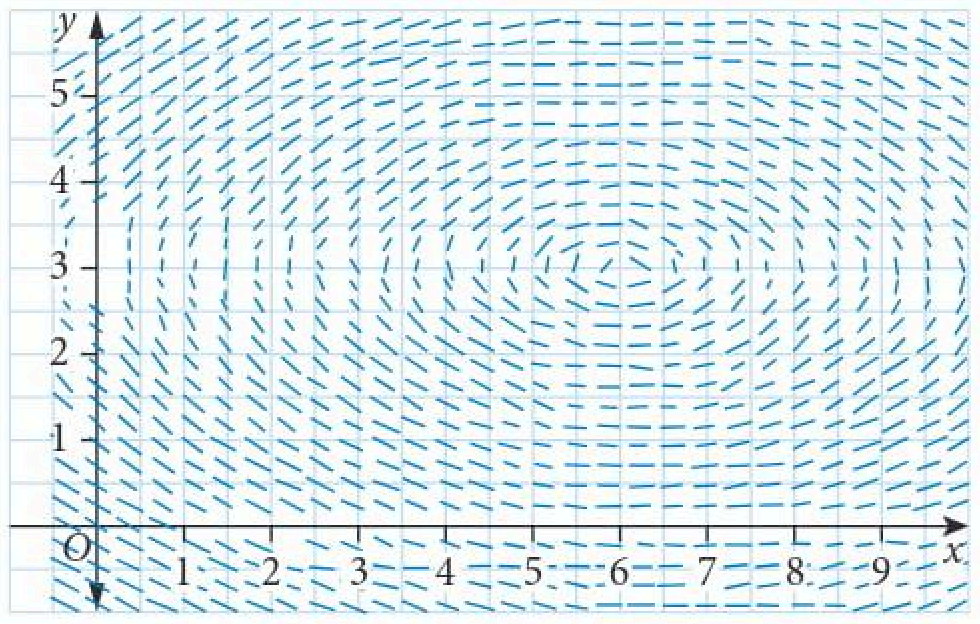
B $\frac{dy}{dx} = \frac{2x - y}{y - 2x}$

C $\frac{dy}{dx} = \frac{2y - x}{y + 2x}$

D $\frac{dy}{dx} = \frac{y - 2x}{2y - x}$

E $\frac{dy}{dx} = \frac{x - 2y}{2y + x}$

14 © VCAA 2009 2AQ9 51%



The direction field above could be that of the differential equation

A $\frac{dy}{dx} = \frac{(x-6)^2}{36} + \frac{(y-3)^2}{9}$

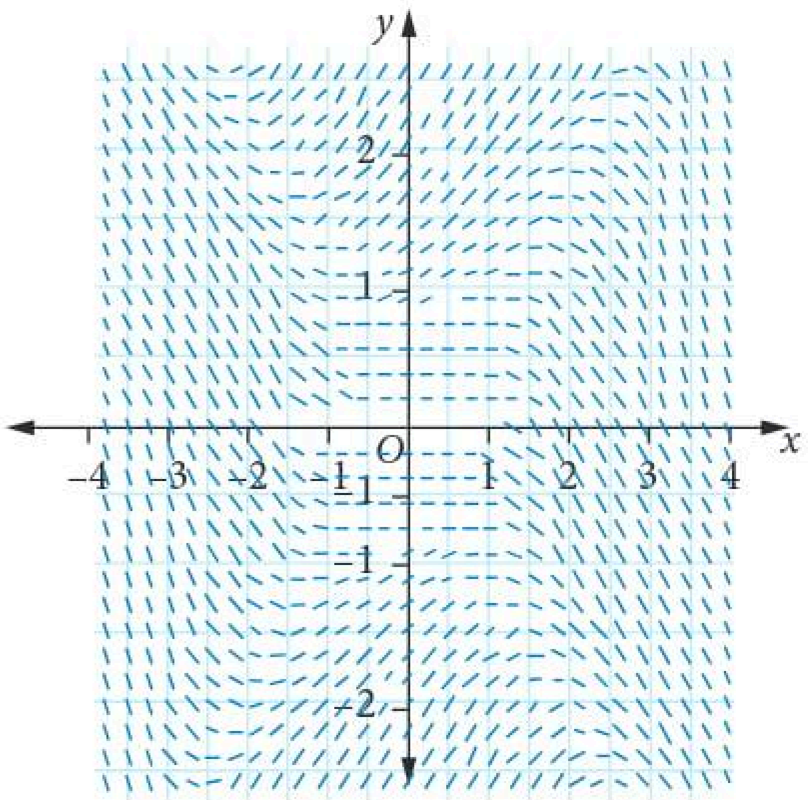
B $\frac{dy}{dx} = \frac{6 - x}{4(y - 3)}$

C $\frac{dy}{dx} = \frac{6 + x}{4(y + 3)}$

D $\frac{dy}{dx} = \frac{6 - x}{4(y + 3)}$

E $\frac{dy}{dx} = \frac{(x+6)^2}{36} + \frac{(y+3)^2}{9}$

15 © VCAA 2015 2AQ13 47%

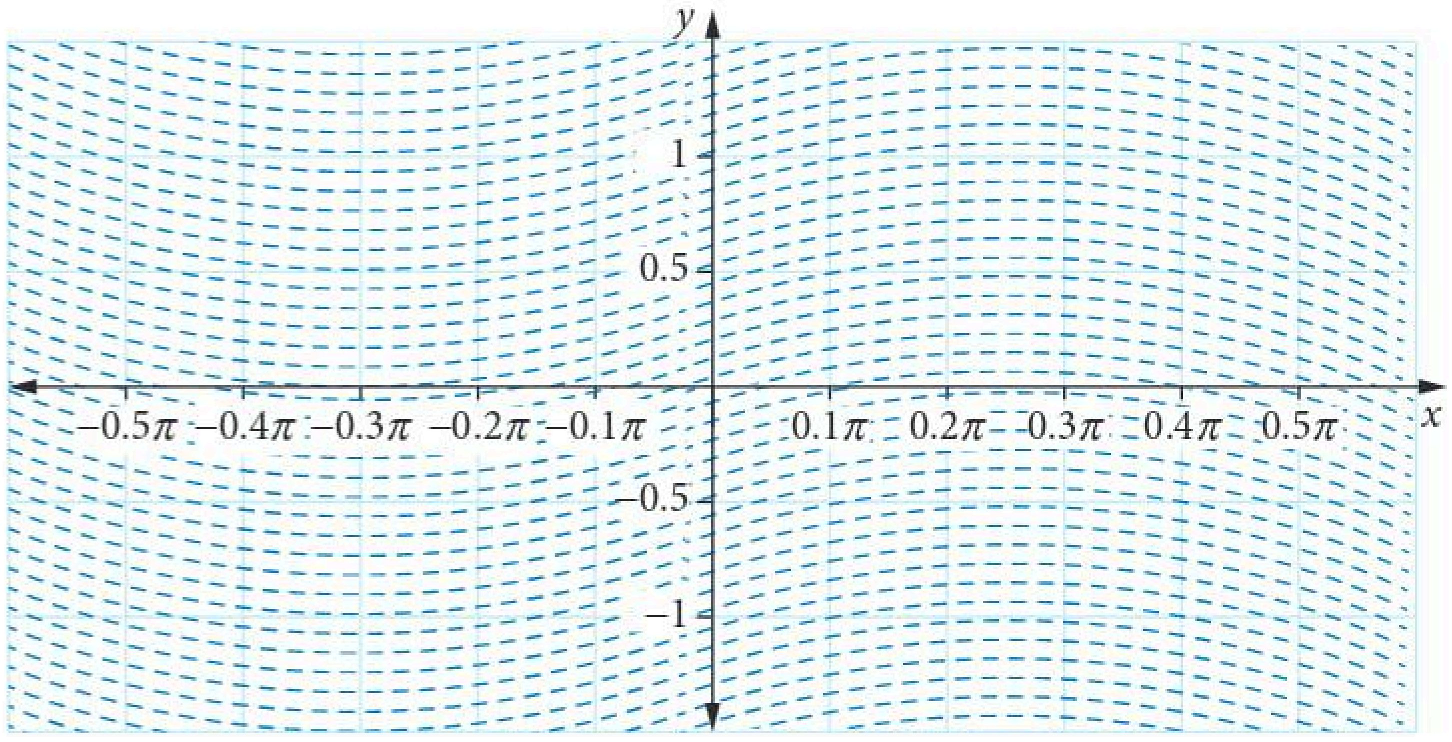


The direction field for a certain differential equation is shown above.

The solution curve to the differential equation that passes through the point (–2.5, 1.5) could also pass through

A (0, 2)
B (1, 2)
C (3, 1)
D (3, –0.5)
E (–0.5, 2)

16 © VCAA 2008 2AQ9 40%

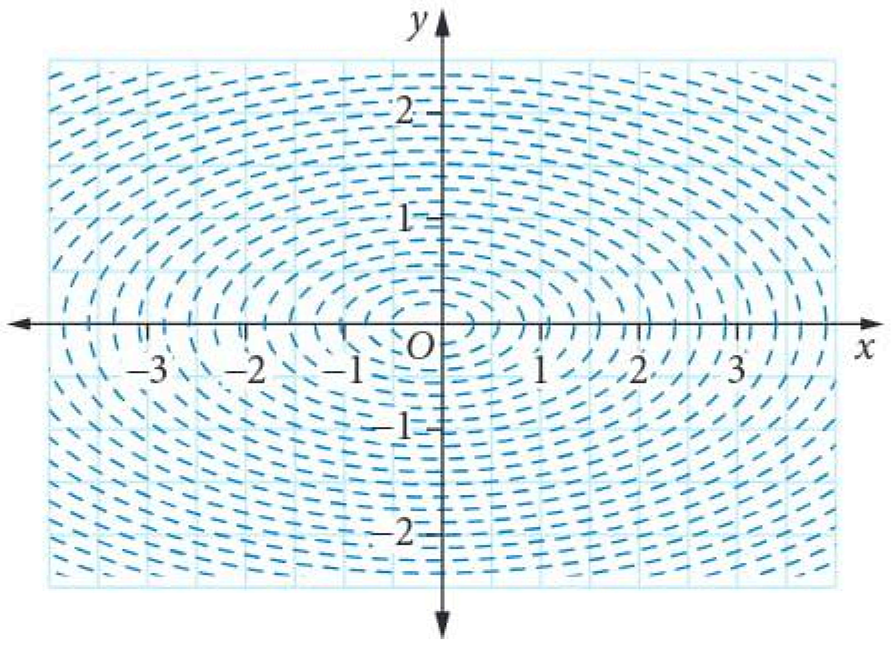


The direction (slope) field for a certain first-order differential equation is shown above.

The differential equation could be

A $\frac{dy}{dx} = \sin(2x)$
B $\frac{dy}{dx} = \cos(2x)$
C $\frac{dy}{dx} = \cos\left(\frac{1}{2}y\right)$
D $\frac{dy}{dx} = \sin\left(\frac{1}{2}y\right)$
E $\frac{dy}{dx} = \cos\left(\frac{1}{2}x\right)$

17 © VCAA 2006 2AQ11 40%

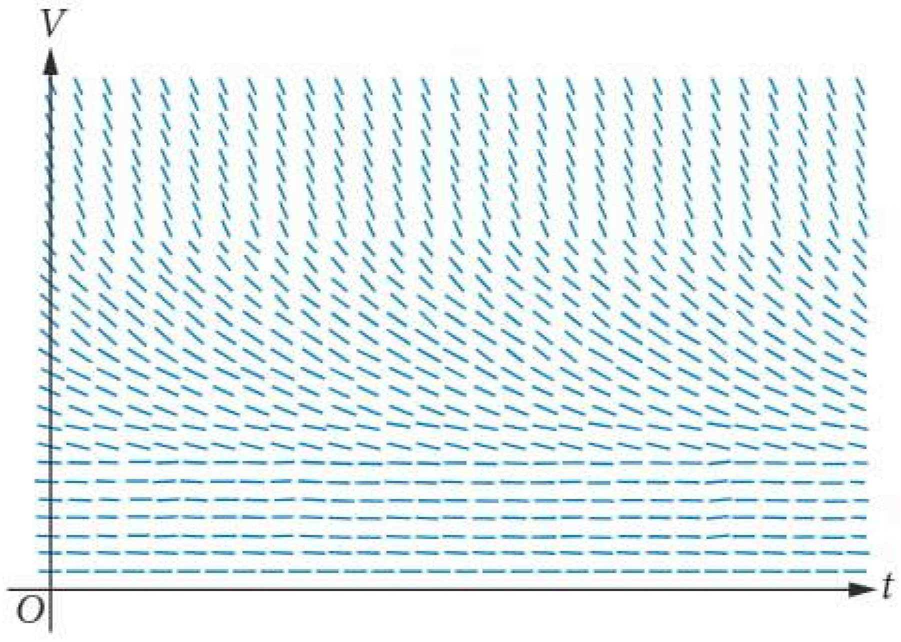


The direction (slope) field for a certain first-order differential equation is shown above.

The differential equation could be

A $\frac{dy}{dx} = \frac{x^2}{2} + y^2$
B $\frac{dy}{dx} = x^2 + \frac{y^2}{2}$
C $\frac{dy}{dx} = -\frac{x}{2y}$
D $\frac{dy}{dx} = -\frac{y}{2x}$
E $\frac{dy}{dx} = \frac{x}{2y}$

18 © VCAA 2010 2AQ11 40% A direction field for the volume of water, V megalitres, in a reservoir t years after 2010 is shown below.



According to this model, for $k > 0$, $\frac{dV}{dt}$ is equal to

A $-kt^2$

B $\frac{k}{V}$

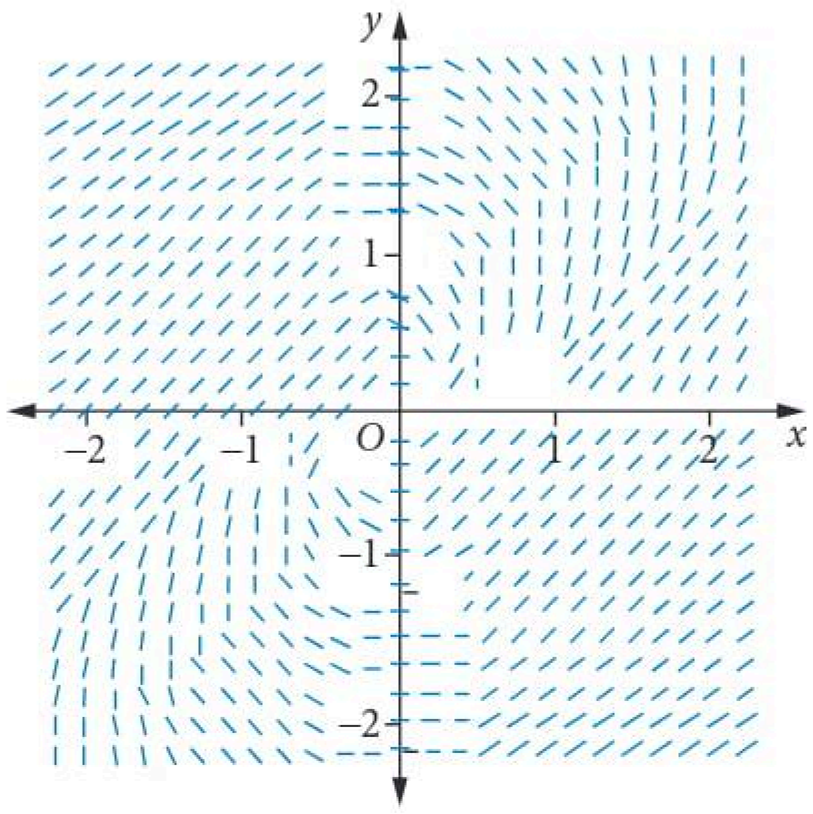
C $-kV^2$

D kV^2

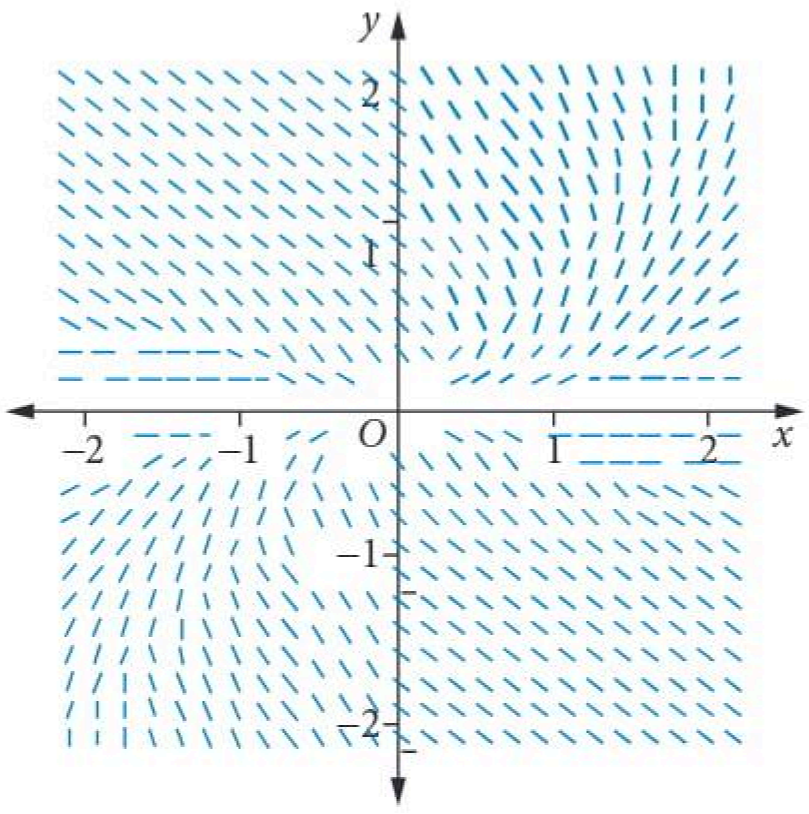
E $-\frac{k}{V}$

19 © VCAA 2020 2AQ9 35% $P(x, y)$ is a point on a curve. The x-intercept of a tangent to point $P(x, y)$ is equal to the y value at P. Which one of the following slope fields best represents this curve?

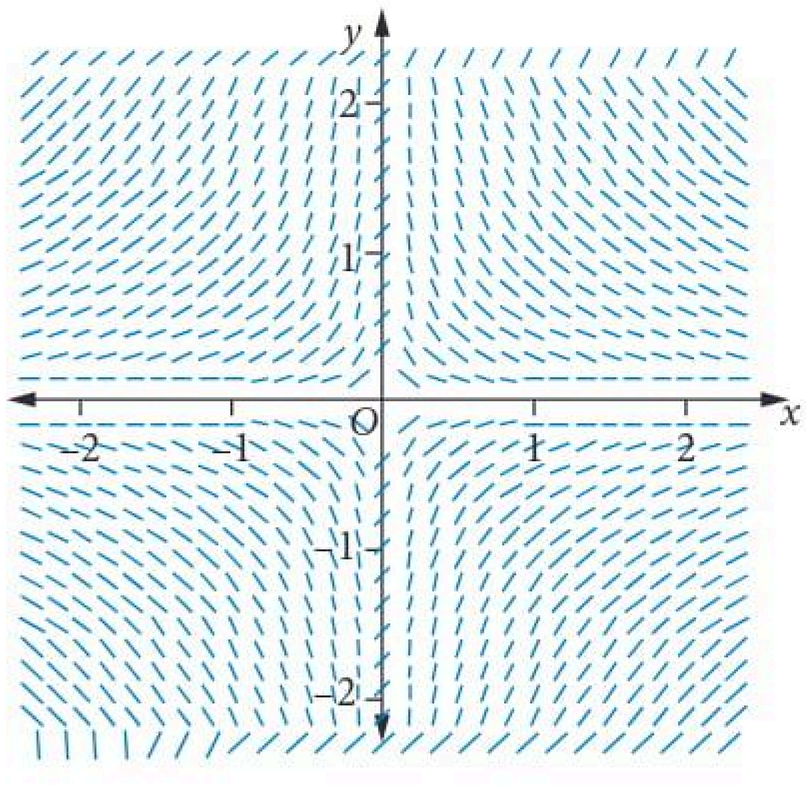
A



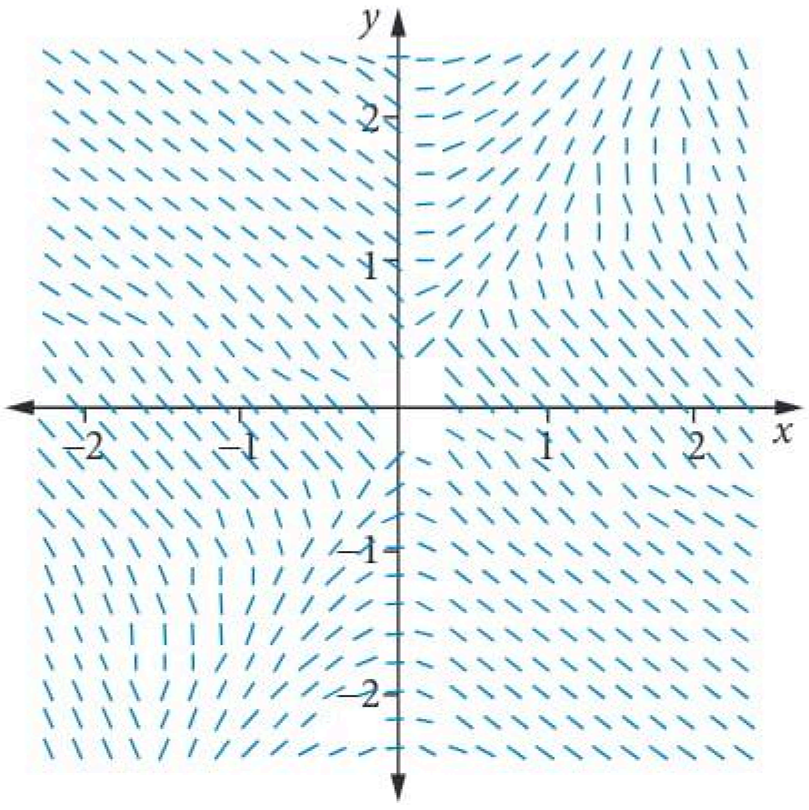
B



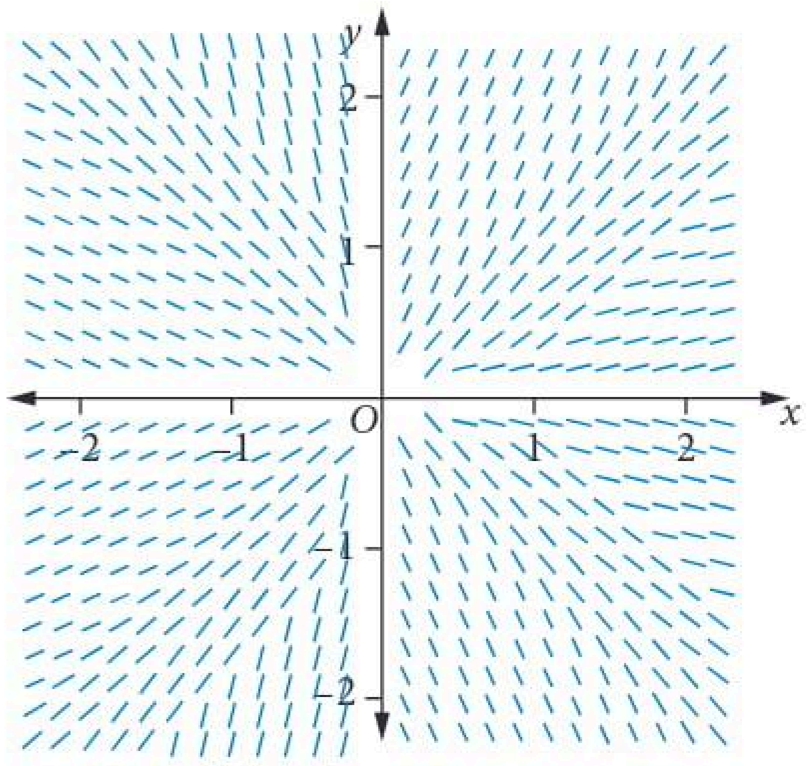
C



D



E



20 © VCAA 2017N 2AQ8 The differential equation that best represents the direction field is

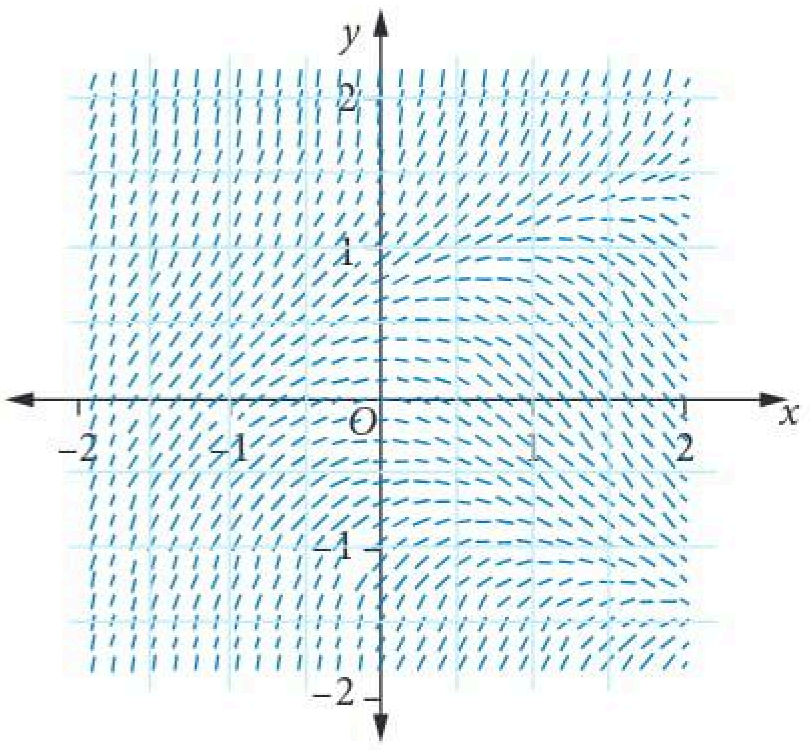
A $\frac{dy}{dx} = x - y^2$

B $\frac{dy}{dx} = y - x$

C $\frac{dy}{dx} = y^2 - x^2$

D $\frac{dy}{dx} = y^2 - x$

E $\frac{dy}{dx} = y + x$



21 © VCAA 2013S 2AQ20 The direction (slope) field for a certain first-order differential equation is shown. A solution to this differential equation could be

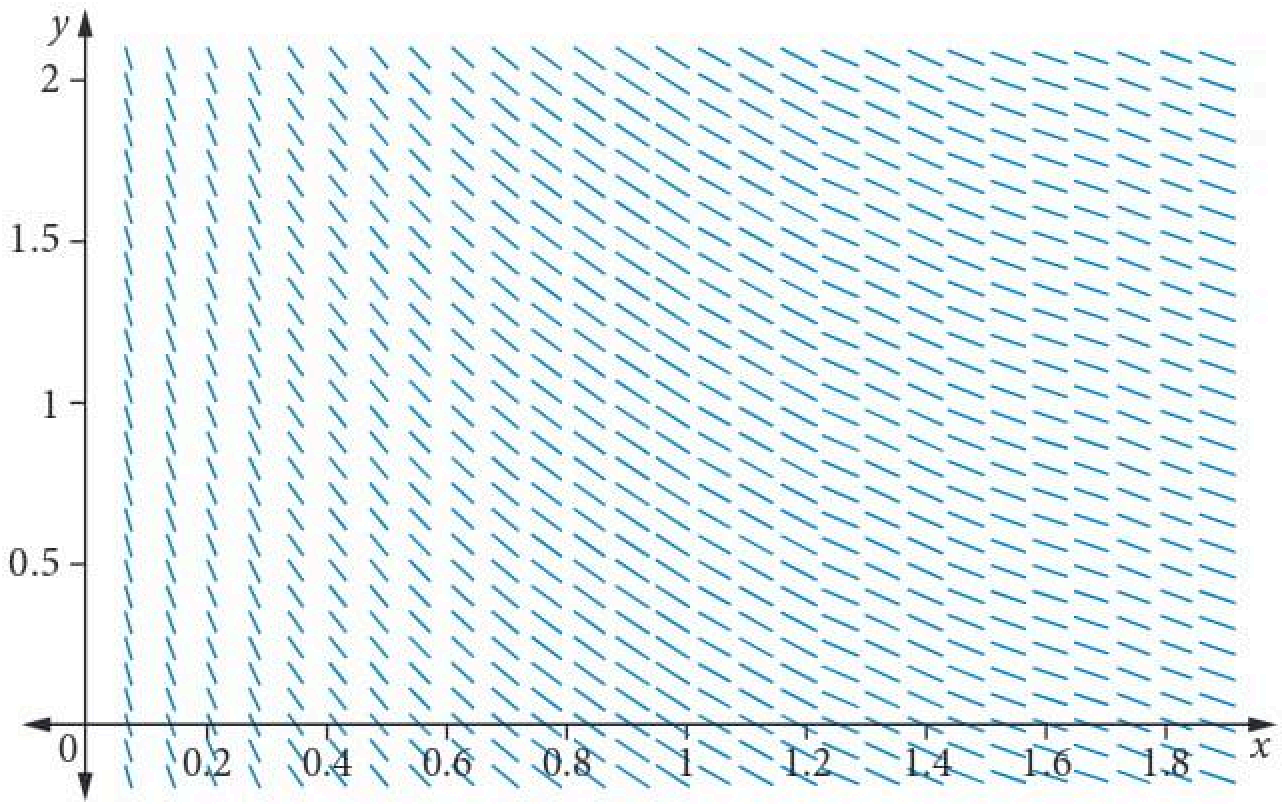
A $y = \frac{1}{x}$

B $y = -\log_e(x)$

C $y = e^{-x}$

D $y = \frac{1}{x^2}$

E $y = \log_e(-x)$



Video playlist
Euler's method

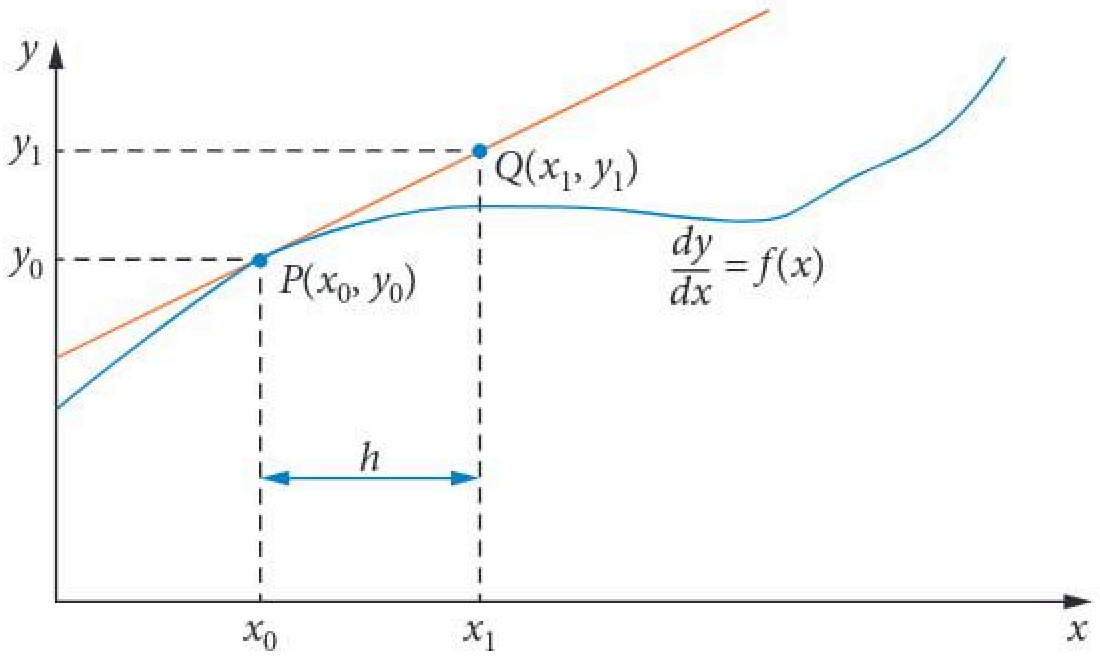
9.7 Euler's method

Euler's method provides a sequence of numerical approximations to the solution of a first-order differential equation when standard methods of anti-differentiation do not work.

In the graph below, P is a point on the curve $\frac{dy}{dx} = f(x)$ and its tangent line passes through a close point, Q.

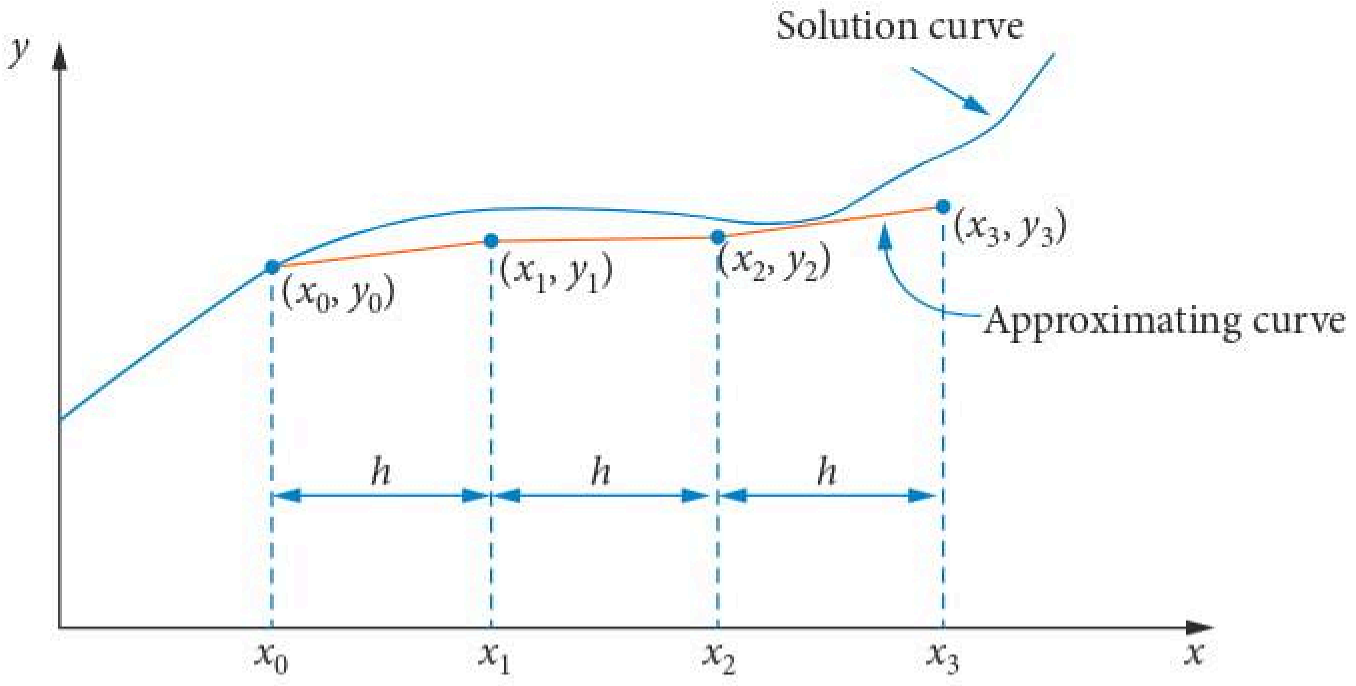
The gradient of the tangent line is $\frac{y_1 - y_0}{x_1 - x_0}$. Taking this as an approximation for $\frac{dy}{dx}$, we have $\frac{y_1 - y_0}{x_1 - x_0} = f(x_0)$.

This gives $y_1 = y_0 + (x_1 - x_0)\,f(x_0)$ or $y_1 = y_0 + h\,f(x_0)$, where h is the **step size**.



The x values are x_0, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$ … and the sequence of approximations for y are $y_1 = y_0 + h\,f(x_0)$, $y_2 = y_1 + h\,f(x_1)$, $y_3 = y_2 + h\,f(x_2)$ …

The set of coordinates (x_0, y_0), (x_1, y_1), (x_2, y_2) … lie on the approximating curve to the solution of $\frac{dy}{dx} = f(x)$. The start coordinates (x_0, y_0) is the **initial condition**.



Euler's formula can be expressed in **iterative form**.

Euler's method

$x_{n+1} = x_n + h$

$y_{n+1} = y_n + hf(x_n)$ where $f(x_n) = \frac{dy}{dx}$ at x_n

$n = 0, 1, 2, 3$ … with (x_0, y_0) given.

Accuracy in the approximations will increase by using a smaller step size or by increasing the number of iterations.

WB
p. 196

WORKED EXAMPLE 17 Euler's method

Use a step size of $h = 0.1$ with initial condition $(0, 1)$ to find the first three iterations for the solution to $\frac{dy}{dx} = 2x$.

Steps	**Working**
1 Use the initial condition to find the coordinates for the first iteration by substituting into Euler's formula.	$h = 0.1, x_0 = 0, y_0 = 1, \frac{dy}{dx} = 2x$ The first iteration uses $n = 0$. $y_1 = y_0 + h\frac{dy}{dx}$ $= y_0 + h(2x_0)$ $= 1 + (0.1 \times 0) = 1$ $x_1 = x_0 + h = 0 + 0.1 = 0.1$ So $(x_1, y_1) = (0.1, 1)$.

2 Repeat the process to work out the remaining iterations.	For the second iteration, $n = 1$. $y_2 = y_1 + h\dfrac{dy}{dx}$ $= y_1 + h(2x_1)$ $= 1 + 0.1 \times (2 \times 0.1) = 1.02$ $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$ So $(x_2, y_2) = (0.2, 1.02)$. For the third iteration, $n = 2$. $y_3 = y_2 + h(2x_2)$ $= 1.02 + 0.1(2 \times 0.2)$ $= 1.06$ $x_3 = x_2 + h = 0.2 + 0.1 = 0.3$ So $(x_3, y_3) = (0.3, 1.06)$.

WB
p. 197

WORKED EXAMPLE 18 Comparing Euler's method to the exact answer

a Use calculus to solve $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$ given that when $x = 0$, $y = 2$.

b Use a step size of $h = 0.2$ with initial condition $(0, 2)$ to obtain a series of approximations in the interval $0 \le x \le 1$ to solve $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$. State answers correct to three decimal places.

c Find the percentage error, to the nearest whole number, between the last approximation and the exact value.

Steps	**Working**
a Anti-differentiate the function and find the constant of integration using the initial condition.	$\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$ $y = \tan^{-1}(x) + c$ $(0, 2)$ means $2 = \tan^{-1}(0) + c \Rightarrow c = 2$ $y = \tan^{-1}(x) + 2$
b **1** Write down the known information and determine the number of iterations.	$h = 0.2, x_0 = 0, y_0 = 2, \dfrac{dy}{dx} = \dfrac{1}{1+x^2}$ A step size of 0.2 starting from 0 to 1 means $\dfrac{1}{0.2} = 5$ iterations.
2 Write the general form for the solution.	$x_{n+1} = x_n + 0.2$ $y_{n+1} = y_n + 0.2\left(\dfrac{1}{1+x^2}\right), n = 0, 1, 2, 3, \ldots$
3 Calculate each iteration up to the required number.	First iteration, $n = 0$. $x_0 = 0, y_0 = 2, h = 0.2$ $y_1 = y_0 + h\left(\dfrac{1}{1+x_0^2}\right) = 2 + 0.2\left(\dfrac{1}{1+0^2}\right) = 2.2$ $(x_1, y_1) = (0.2, 2.2)$

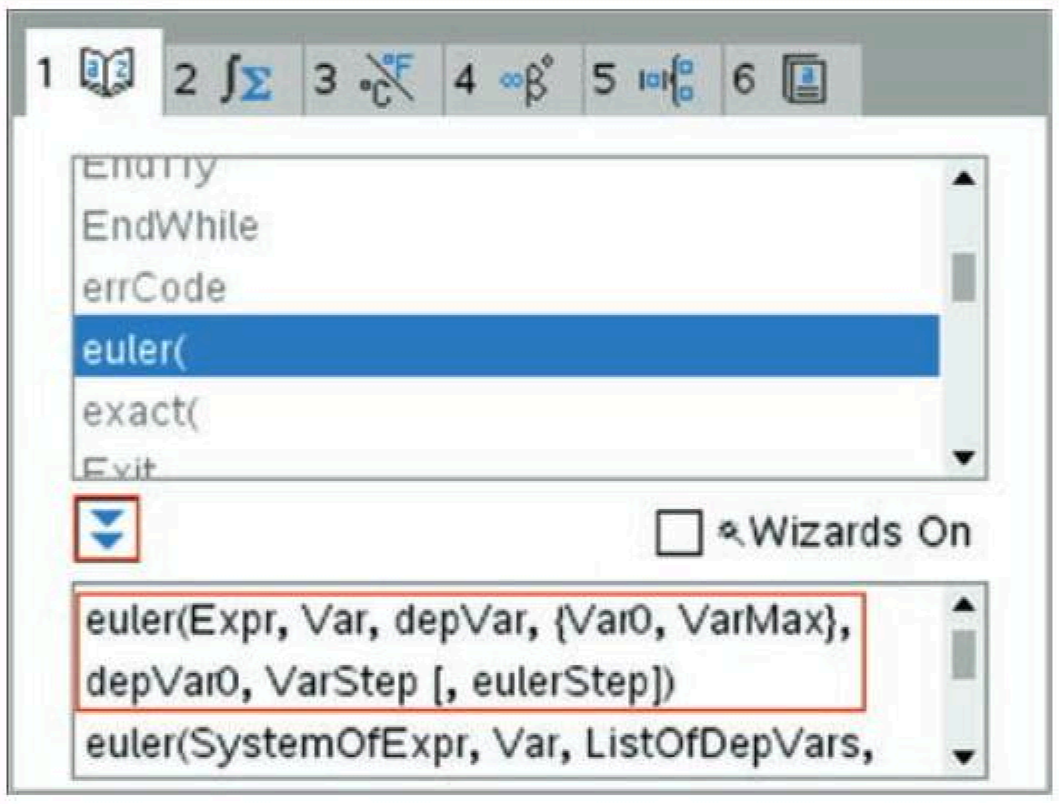
	Second iteration, $n = 1$. $x_1 = 0.2, y_1 = 2.2, h = 0.2$ $y_2 = y_1 + 0.2\left(\dfrac{1}{1+x_1^2}\right) = 2.2 + 0.2\left(\dfrac{1}{1+0.2^2}\right)$ $= 2.392$ $(x_2, y_2) = (0.4, 2.392)$
	Third iteration, $n = 2$. $x_2 = 0.4, y_2 = 2.392, h = 0.2$ $y_3 = 2.392 + 0.2\left(\dfrac{1}{1+0.4^2}\right) = 2.565$ $(x_3, y_3) = (0.6, 2.565)$
	Fourth iteration, $n = 3$. $x_3 = 0.6, y_3 = 2.565, h = 0.2$ $y_4 = 2.565 + 0.2\left(\dfrac{1}{1+0.6^2}\right) = 2.712$ $(x_4, y_4) = (0.8, 2.711)$
	Fifth iteration, $n = 4$. $x_4 = 0.8, y_4 = 2.712, h = 0.2$ $y_5 = 2.712 + 0.2\left(\dfrac{1}{1+0.8^2}\right) = 2.834$ $(x_5, y_5) = (1.0, 2.834)$
c 1 Find the exact value.	$y(1.0) = \tan^{-1}(1.0) + 2$ ≈ 2.786
2 Calculate the percentage error.	$\text{percentage error} = \dfrac{2.834 - 2.786}{2.786} \times 100\%$ $= 1.7229\ldots\%$ $\approx 2\%$

The use of CAS can significantly speed up the repetitive process of iteration, especially when many iterations are involved.

USING CAS 5 Euler's method

Use $h = 0.5$ with Euler's formula to determine the first 5 iterations for $\dfrac{dy}{dx} = \log_e(x), y(1) = 2$.

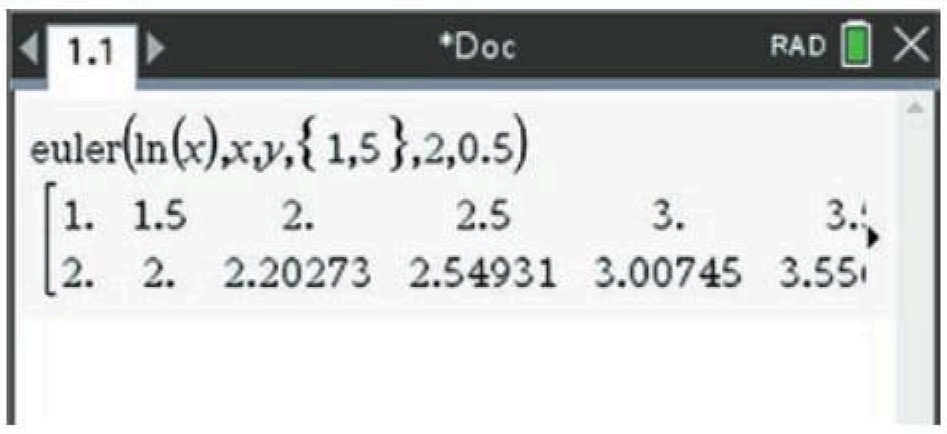
TI-Nspire



☐ Wizards On

euler(Expr, Var, depVar, {Var0, VarMax}, depVar0, VarStep [, eulerStep])

euler(SystemOfExpr, Var, ListOfDepVars,



```
euler(ln(x),x,y,{1,5},2,0.5)
[1.  1.5   2.       2.5      3.       3.
 2.  2.    2.20273  2.54931  3.00745  3.55
```

1 Press **catalog** > **E** to jump to the functions starting with **e**.

2 Scroll down and select **euler**.

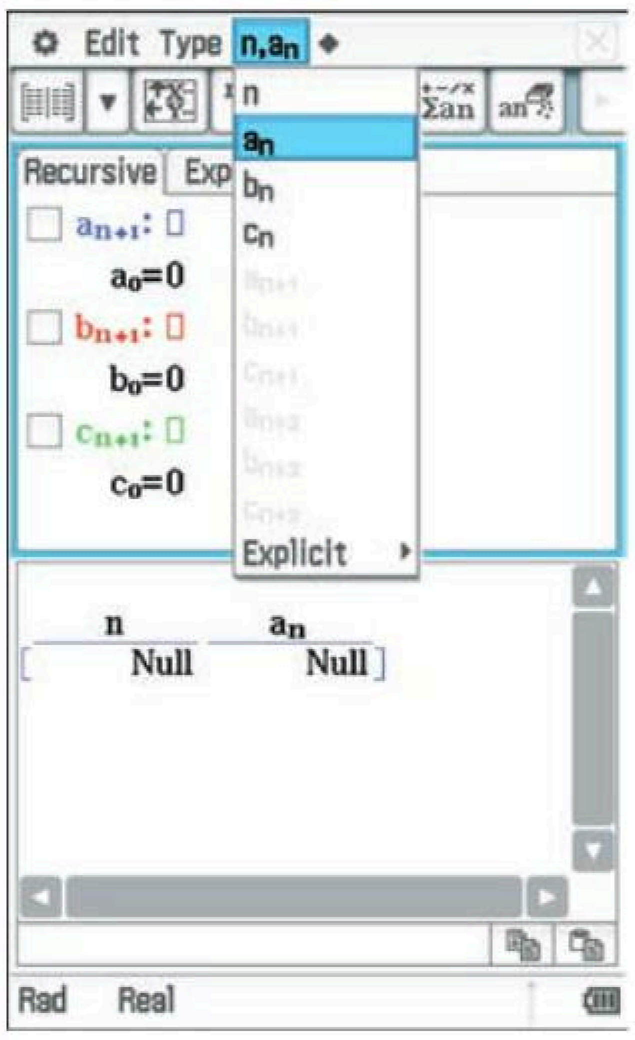
3 Click on the **arrow** to expand the display at the bottom of the screen to see the order in which to enter the parameters.

4 Enter the parameters as shown above, which are: Expr:**ln(x),** Var:**x,** depVar:**y,** {Var0:**1,** VarMax:**5**}, depVar0:**2**, VarStep:**0.5**.

5 The first 5 iterations will be displayed.

The first 5 iterations are 2, 2.203, 2.549, 3.008, 3.557.

ClassPad



a_{n+1}: □ $a_0=0$

b_{n+1}: □ $b_0=0$

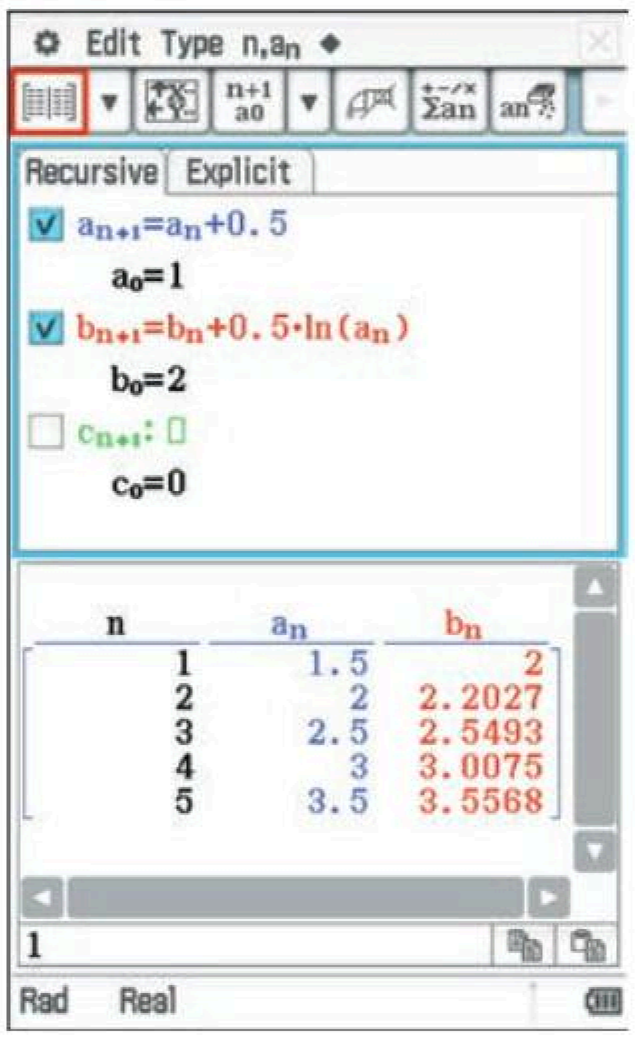
c_{n+1}: □ $c_0=0$

n	a_n
Null	Null

1 Tap **Menu** and select the **Sequence** application.

2 Tap **n,a$_n$** to display the drop down menu.

3 Select **a$_n$**.



☑ $a_{n+1}=a_n+0.5$

$a_0=1$

☑ $b_{n+1}=b_n+0.5\cdot\ln(a_n)$

$b_0=2$

☐ c_{n+1}: □

$c_0=0$

n	a_n	b_n
1	1.5	2
2	2	2.2027
3	2.5	2.5493
4	3	3.0075
5	3.5	3.5568

4 Complete the **a$_{n+1}$** formula as shown.

5 Set **a$_0$** = **1**.

6 Tap **n,a$_n$** and select **b$_n$**.

7 Complete the **b$_{n+1}$** formula as shown.

8 Set **b$_0$** = **2**.

9 Tap **Table** to display the table of iterations in the lower window.

The first 5 iterations are 2, 2.203, 2.549, 3.008, 3.557.

Euler's formula

Euler's formula in **iterative form** is

$$x_{n+1}=x_n+h,\ y_{n+1}=y_n+h\frac{dy}{dx},\ n=0,1,2,3\ \ldots,\text{ where } h \text{ is the step size.}$$

Video
VCE question analysis: Differential equations

VCE QUESTION ANALYSIS

© VCAA 2016 2BQ3 2016 Examination 2 Section B Question 3 (11 marks)

A tank initially has 20 kg of salt dissolved in 100 L of water. Pure water flows into the tank at a rate of 10 L/min. The solution of salt and water, which is kept uniform by stirring, flows out of the tank at a rate of 5 L/min.

If x kilograms is the amount of salt in the tank after t minutes, it can be shown that the differential equation relating x and t is $\frac{dx}{dt}+\frac{x}{20+t}=0$.

a Solve this differential equation to find x in terms of t. 3 marks

A second tank initially has 15 kg of salt dissolved in 100 L of water. A solution of $\frac{1}{60}$ kg of salt per litre flows into the tank at a rate of 20 L/min. The solution of salt and water, which is kept uniform by stirring, flows out of the tank at a rate of 10 L/min.

b If y kilograms is the amount of salt in the tank after t minutes, write down an expression for the **concentration**, in kg/L, of salt in the second tank at time t. 1 mark

c Show that the differential equation relating y and t is $\frac{dy}{dt}+\frac{y}{10+t}=\frac{1}{3}$. 2 marks

d Verify by differentiation and substitution into the left side that $y = \dfrac{t^2 + 20t + 900}{6(10 + t)}$ satisfies the **differential equation in part c**. Verify that the given solution for y also satisfies the **initial condition**. 3 marks

e Find when the **concentration** of salt in the second tank reaches 0.095 kg/L. Give your answer in minutes, correct to two decimal places. 2 marks

Reading the question

- The question is generally typical of the standard approach needed to set up and solve an inflow–outflow problem. Keep in mind how to calculate the inflow rate and outflow rate for the two tanks from the information provided.
- Read the question carefully, particularly parts **b** and **e** which have **concentration** highlighted to reinforce that the rate of change of concentration is not what is being asked.
- Part **d** will require differentiation using the quotient rule, but CAS can be used to speed up and check calculations.
- When verifying the solution given in part **d**, show clearly that, after substitution and simplifying, the left-hand side is the same as the right-hand side.

Thinking about the question

- Solving part **a** will require separation of variables and a constant of integration that will have to be found using the initial conditions.
- With the exception of part **b**, the questions are more than 1 mark, so working needs to be shown for each.
- Decide for which questions a calculator is required. It is necessary for part **e** and optional for the remaining questions.
- Understand the difference between finding expressions for the concentration and finding expressions for the rate.

Worked solution (✓ = 1 mark)

a $\int \frac{1}{x}\,dx = -\int \frac{1}{20+t}\,dt$

Anti-differentiate both sides.

$\mathbf{\log_e|x| = -\log_e|20+t| + c}$ ✓

As x and t are always positive, we don't need the || symbols.

$t = 0,\ x = 20$

$\log_e(20) = -\log_e(20) + c$

$c = 2\log_e(20) = \mathbf{\log_e(400)}$ ✓

Write the solution with x in terms of t.

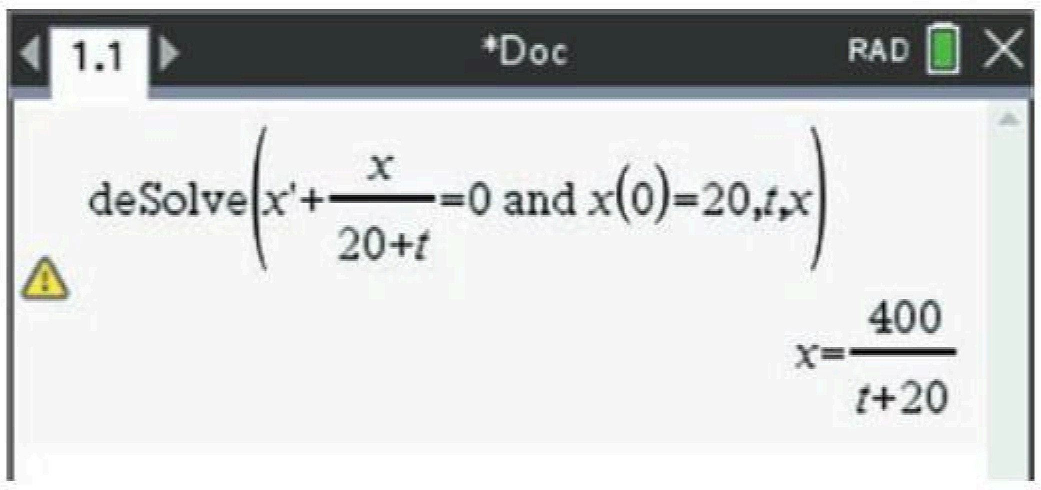
$\log_e(x) = -\log_e|20+t| + \log_e(400)$

$\log_e(x(20+t)) = \log_e(400)$

$x(20+t) = 400$

$$\mathbf{x = \frac{400}{20+t}} \checkmark$$

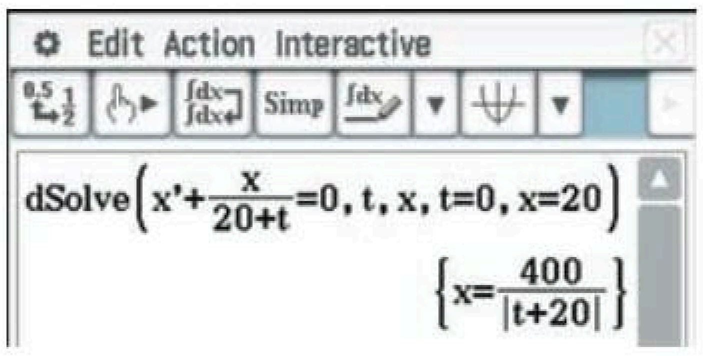
TI-Nspire



$\text{deSolve}\left(x'+\frac{x}{20+t}=0 \text{ and } x(0)=20,t,x\right)$

$x=\frac{400}{t+20}$

ClassPad



$\text{dSolve}\left(x'+\frac{x}{20+t}=0, t, x, t=0, x=20\right)$

$\left\{x=\frac{400}{|t+20|}\right\}$

✓ for correct CAS function

✓ for correct conditions

✓ for correct answer

b Change in volume = inflow – outflow

$$= 20 - 10 = 10 \text{ L}$$

The initial volume is 100 L.

The volume at time t is $100 + 10t$.

Concentration is mass per unit volume.

The mass of salt in the tank is y kg.

The concentration of salt at time t is $\boldsymbol{\dfrac{y}{100+10t}}$ **kg/L.** ✓

c Find inflow rate – outflow rate.

inflow rate: $20 \times \dfrac{1}{60} = \dfrac{1}{3}$ kg/min

outflow rate: $\mathbf{10} \times \boldsymbol{\dfrac{y}{100+10t} = \dfrac{y}{10+t}}$ **kg/min** ✓

Find the rate of change.

$$\boldsymbol{\frac{dy}{dt} = \frac{1}{3} - \frac{y}{10+t}} \textbf{ kg/min} \text{ or } \boldsymbol{\frac{dy}{dt} + \frac{y}{10+t} = \frac{1}{3}} \checkmark$$

d Obtain an expression for $\dfrac{dy}{dt}$ in terms of t only.

$$y = \frac{t^2 + 20t + 900}{6(10+t)}$$

Use the quotient rule or CAS to differentiate.

$$\frac{dy}{dt} = \boldsymbol{\frac{t^2 + 20t - 700}{6(10+t)^2}} \checkmark$$

Substitute into the LHS of the equation and show that it is the same as the RHS.

$$\text{LHS} = \frac{dy}{dt} + \frac{y}{10+t}$$

$$= \frac{t^2 + 20t - 700}{6(10+t)^2} + \frac{1}{10+t}\left(\frac{t^2 + 20t + 900}{6(10+t)}\right)$$

$$= \frac{2(t+10)^2}{6(10+t)^2}$$

$$= \boldsymbol{\frac{1}{3}}$$

$$= \text{RHS} \checkmark$$

Verify the initial condition.

Initially, $y = 15$ from part **a**.

$$y = \frac{t^2 + 20t + 900}{6(10+t)}$$

$$t = 0,\ y = \frac{900}{60} = \mathbf{15} \checkmark$$

e Concentration $= \dfrac{\text{mass}}{\text{volume}}$

For the second tank, the concentration is

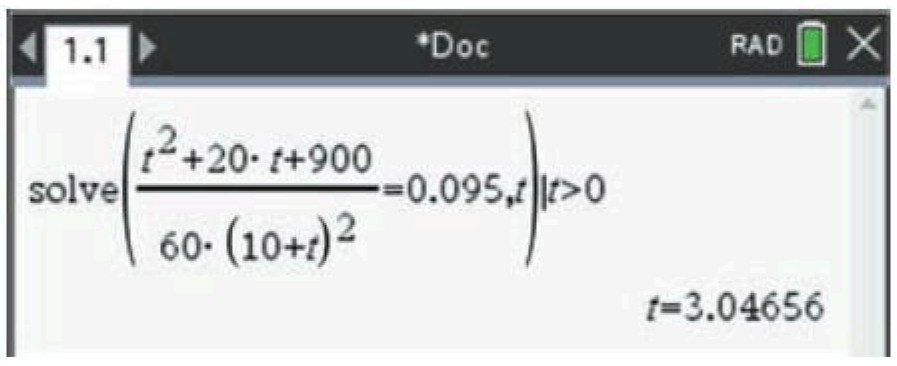
$$\frac{t^2 + 20t + 900}{6(10+t)} \times \frac{1}{100+10t}$$

$$= \boldsymbol{\frac{t^2 + 20t + 900}{60(10+t)^2}} \checkmark$$

Solve the equation

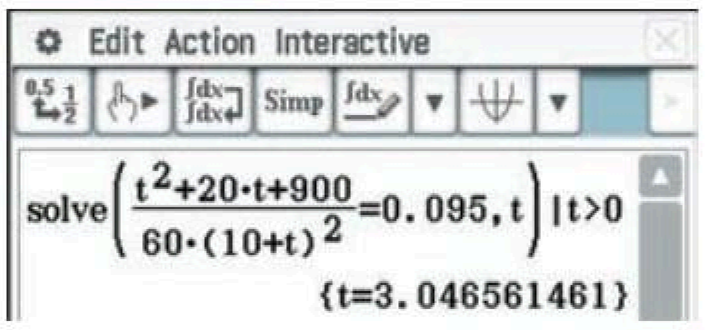
$$\frac{t^2 + 20t + 900}{60(10+t)^2} = 0.095$$

TI-Nspire



```
solve( (t^2+20·t+900)/(60·(10+t)^2) =0.095,t)|t>0
                                        t=3.04656
```

ClassPad



```
solve( (t^2+20·t+900)/(60·(10+t)^2) =0.095,t) |t>0
                                {t=3.046561461}
```

It will take **3.05 min** for the **concentration** to be 0.095 kg/L. ✓

Student performance

80–100% 60–79% 0–59%

The overall percentage average for this challenging question is 43%, with only part **a** having a score above 50%.

a 56% Common errors were not separating the variables correctly, forgetting the constant of integration or adding the constant to both sides of the equation.

b 34% Many students did not understand the question, and often deduced $\frac{dy}{dt}$ instead.

c 45% Students struggled with this 'show that' question, with some stating with the result rather than trying to prove it. Many did not realise that the answer to part **b** was related to this question.

d 38% Most students could find the derivative, but not verify the solution satisfied the initial conditions.

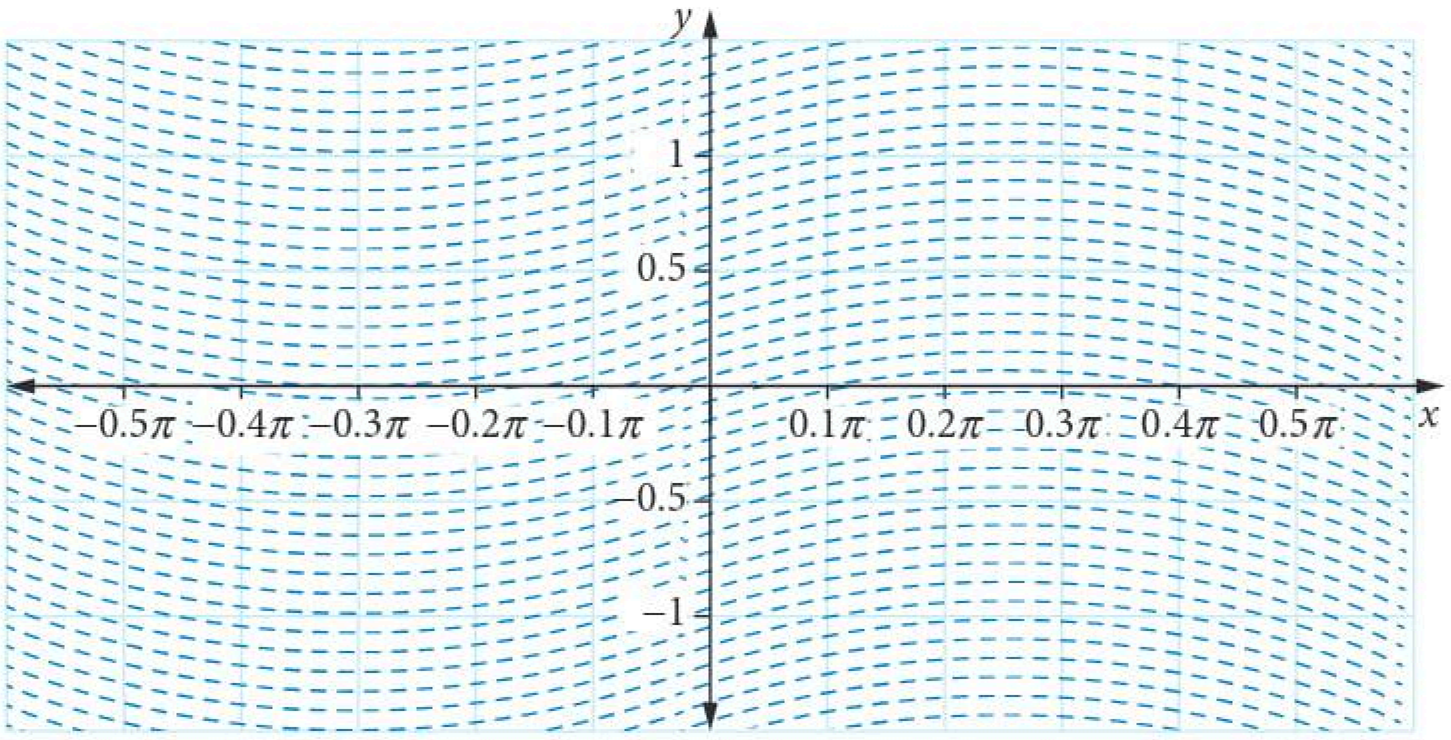
e 27% Quite challenging for students and there were many non-attempts.

EXERCISE 9.7 Euler's method

ANSWERS p. 596

Recap

1 © VCAA 2016S 2AQ7



The direction (slope field) for a certain first-order differential equation is shown above.

The differential equation could be

A $\dfrac{dy}{dx} = \sin(2x)$

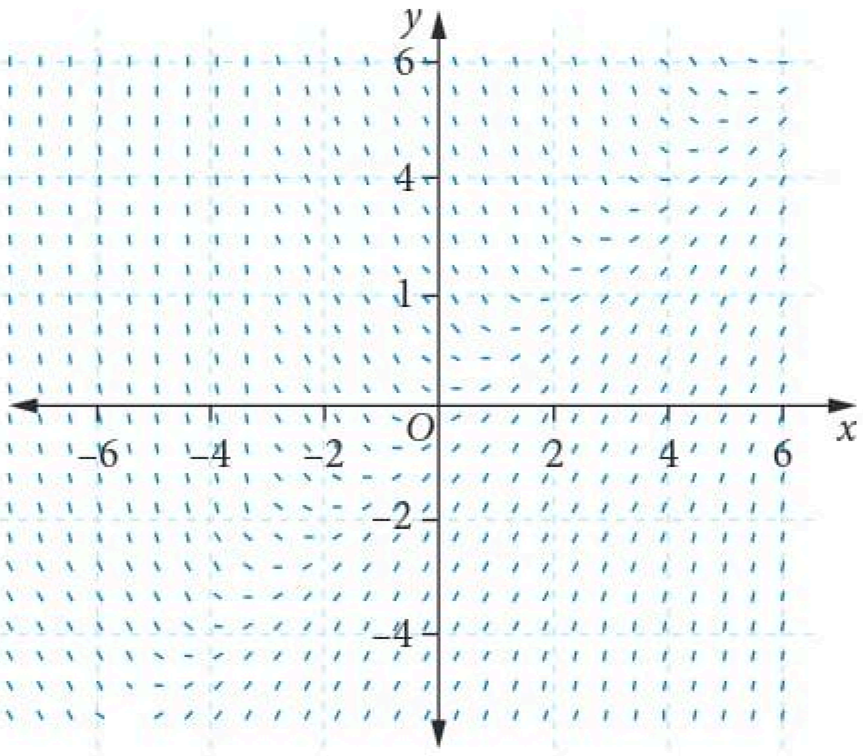
B $\dfrac{dy}{dx} = \cos(2x)$

C $\dfrac{dy}{dx} = \cos\left(\dfrac{1}{2}y\right)$

D $\dfrac{dy}{dx} = \sin\left(\dfrac{1}{2}y\right)$

E $\dfrac{dy}{dx} = \cos\left(\dfrac{1}{2}x\right)$

▶ **2**



The direction field shown above is for the differential equation $\frac{dy}{dx} = x + y$.

A solution to this differential equation that includes (0, −1) could also include

A (−2, 0) **B** (2, 0) **C** (−1, 0) **D** (−2, −1) **E** (1, 2)

Mastery

3 WORKED EXAMPLE 17 Use a step size of $h = 0.4$ and the initial condition (0, 5) to find the first 5 iterations for the solution to $\frac{dy}{dx} = 3x^2$.

4 WORKED EXAMPLE 18

a Use calculus to solve $\frac{dy}{dx} = 2\cos(2x)$, given that when $x = 0$, $y = 4$.

b Use $h = \frac{\pi}{4}$ with initial condition $(0, 4)$ to obtain a series of approximations for the solution to $\frac{dy}{dx} = 2\cos(2x)$ in the interval $0 \le x \le \pi$. State your answers correct to three decimal places.

c Find, to the nearest whole number, the percentage error between the first approximation and the exact value.

5 Using CAS 5 Use $h = 0.5$ with Euler's formula to determine the y value of the 10th approximation to the solution for $\frac{dy}{dx} = x\log_e(x)$, $y(1) = 0$. Give the answer correct to one decimal place.

Exam practice

80–100% 60–79% 0–59%

6 © VCAA 2007 1Q7 TECH-FREE (4 marks)

a 50% Use Euler's method to find y_2 if $\frac{dy}{dx} = \frac{1}{x}$, given that $y_0 = y(1) = 1$ and $h = 0.1$.

Express your answer as a fraction. 2 marks

b 47% Solve the differential equation given in part **a** to find the value of y which is estimated by y_2.

Express your answer in the form $\log_e(a) + b$, where a and b are positive real constants. 2 marks

7 © VCAA 2009 1Q9 TECH-FREE (5 marks) Let $\frac{dy}{dx} = (y + 2)^2 + 4$ and $y_0 = y(0) = 0$.

a 50% Solve the differential equation above, giving y as a function of x. 3 marks

b 47% Apply Euler's method to find y_1, using a step size of 0.1. 2 marks ▶

8 © VCAA 2019N 2BQ10 Euler's method, with a step size of 0.1, is used to approximate the solution of the differential equation $\frac{1}{y}\frac{dy}{dx} = \cos(x)$, with $y = 2$ when $x = 0$.

When $x = 0.2$, the value obtained for y, correct to four decimal places, is

A 2.2000 **B** 2.3089 **C** 2.3098 **D** 2.4189 **E** 2.4199

9 © VCAA 2005 1IQ17 80% Euler's method, with a step size of 0.1, is used to solve the differential equation $\frac{dy}{dx} = e^{-x}$, with initial condition $y = 1$ at $x = 2$.

When $x = 2.2$, the value obtained for y, correct to four decimal places, is

A 1.0122 **B** 1.0222 **C** 1.0233 **D** 1.0258 **E** 1.0271

10 © VCAA 2010 2AQ12 77% Let $\frac{dy}{dx} = \frac{x+2}{x^2+2x+1}$ and $(x_0, y_0) = (0, 2)$.

Using Euler's method, with a step size of 0.1, the value of y_1 correct to two decimal places is

A 0.17 **B** 0.20 **C** 1.70 **D** 2.17 **E** 2.20

11 © VCAA 2014 2AQ11 77% Let $\frac{dy}{dx} = x^3 - xy$ and $y = 2$ when $x = 1$.

Using Euler's method with a step size of 0.1, the approximation to y when $x = 1.1$ is

A 0.9 **B** 1.0 **C** 1.1 **D** 1.9 **E** 2.1

12 © VCAA 2013 2AQ11 70% Consider the differential equation $\frac{dy}{dx} = \frac{1}{3+3x+x^2}$, with $y_0 = 1$ when $x_0 = 0$.

Using Euler's method with a step size of 0.1, the value of y_2, correct to three decimal places, is

A 1.033 **B** 1.063 **C** 1.064 **D** 1.065 **E** 1.066

13 © VCAA 2008 2AQ11 63% When Euler's method, with a step size of 0.2, is used to solve the differential equation $\frac{dy}{dx} = 2\tan^{-1}(x+1)$ with $x_0 = 0$ and $y_0 = 1$, the value of y_2 would be given by

A $1 + 0.1\pi$ **B** $1 + 0.4\tan^{-1}(1.2)$

C $0.1\pi + 0.4\tan^{-1}(1.2)$ **D** $1 + 0.1\pi + 0.4\tan^{-1}(1.2)$

E $1 + 0.4\tan^{-1}(1.2) + 0.4\tan^{-1}(1.4)$

14 © VCAA 2003 1IQ17 60% Euler's method, with a step size of 0.2, is used to solve the differential equation $\frac{dy}{dx} = \cos\left(\frac{x}{2}\right)$, with initial condition $y = 2$ when $x = 0$.

The approximation obtained for y when $x = 0.4$ is given by

A $2 + 0.4\cos(0.1)$ **B** $2 + 0.4\cos(0.2)$

C $2.2 + 0.2\cos(0.1)$ **D** $2.2 + 0.2\cos(0.2)$

E $2 + 0.2\cos(0.1) + 0.2\cos(0.2)$

15 © VCAA 2020 2AQ12 59% If $\frac{dy}{dx} = e^{\cos(x)}$ and $y_0 = e$ when $x_0 = 0$, then using Euler's formula with step size 0.1, y_3 is equal to

A $e + 0.1\left(1 + e^{\cos(0.1)}\right)$ **B** $e + 0.1\left(1 + e^{\cos(0.1)} + e^{\cos(0.2)}\right)$

C $e + 0.1\left(e + e^{\cos(0.1)} + e^{\cos(0.2)}\right)$ **D** $e + 0.1\left(e^{\cos(0.1)} + e^{\cos(0.2)} + e^{\cos(0.3)}\right)$

E $e + 0.1\left(e + e^{\cos(0.1)} + e^{\cos(0.2)} + e^{\cos(0.3)}\right)$

▶ **16** © VCAA 2012 2AQ9 59% Euler's formula is used to find y_2, where $\frac{dy}{dx} = \cos(x), x_0 = 0, y_0 = 1$ and $h = 0.1$.

The value of y_2 correct to four decimal places is

A 1.1000 and this is an underestimate of $y(0.2)$.

B 1.1995 and this is an overestimate of $y(0.2)$.

C 1.1995 and this is an underestimate of $y(0.2)$.

D 1.2975 and this is an underestimate of $y(0.2)$.

E 1.2975 and this is an overestimate of $y(0.2)$.

17 © VCAA 2011 2AQ18 51% The amount of chemical x in a tank at time t is given by the differential equation $\frac{dx}{dt} = -\frac{10}{10-t}$ and when $t = 0$, $x_0 = 5$. Euler's method is used with a step size of 0.5 in the values of t.

The value of x correct to two decimal places when $t = 1$ is found to be

A 3.95 **B** 3.97 **C** 4.50 **D** 5.50 **E** 6.03

18 © VCAA 2017 2AQ9 45% Consider $\frac{dy}{dx} = 2x^2 + x + 1$, where $y(1) = y_0 = 2$.

Using Euler's method with a step size of 0.1, an approximation to $y(0.8) = y_2$ is given by

A 0.94 **B** 1.248 **C** 1.6 **D** 2.4 **E** 2.852

19 © VCAA 2002 2Q4e (4 marks) Suppose Euler's method is used to solve the differential equation $\frac{dy}{dx} = \log_e(4 - x^2)$, with a step size of 0.05 and initial condition $y = 0$ when $x = 0$.

a 35% Use Euler's method to express y_{20} in terms of y_{19}. 2 marks

b 28% Given that $y_{19} = 1.2464$, find y_{20}, giving your answer to four decimal places. 1 mark

c 2% Why is y_{20} an estimate of A? 1 mark

9 Chapter summary

Types of differential equations

A **differential equation** has at least one derivative, such as $\frac{dx}{dt} = 2t + 1$. In this case, t is the **independent variable** and x is the **dependent variable**. The **order** is the highest derivative and the **degree** is the highest power of the highest derivative.

$\left(\frac{dy}{dx}\right)^2 - \frac{d^2y}{dx^2} + y = 0$ is of order 2, degree 1.

The usual method for solving a differential equation is by anti-differentiation, but a numerical approach can be used to obtain an approximation to the solution when standard algebraic methods cannot be used.

Verifying a solution to a differential equation

To show that a given solution satisfies the differential equation, substitute it into the equation until it is obvious that the left-hand side and right-hand side of the differential equation have the same value.

For example, $y = e^{2x}$ is a solution to $\frac{dy}{dx} - 2y = 0$ because differentiating $y = e^{2x}$ gives $\frac{dy}{dx} = 2e^{2x}$ or $\frac{dy}{dx} = 2y$, since $y = e^{2x}$. Hence, $\frac{dy}{dx} - 2y = 0$ as required.

First-order differential equations

1 $\frac{dy}{dx} = f(x)$ has the general solution $y = \int f(x) + c$, where c is a constant of integration.

$\frac{dy}{dx} = e^x, y(0) = 2$ has solution $y = e^x + 1$.

2 $\frac{dy}{dx} = f(y)$ has the rate of change as a function of the dependent variable. The general solution is $x = g(y) + c$, where $g'(y) = f(y)$ and c is a constant of integration.

Examples:

○ **Newton's law of cooling,** $\frac{dT}{dt} = -k(T - T_s)$, which states that the rate of heat loss is proportional to the temperature difference between an object and its surroundings.

○ **Growth and decay** involving radioactive elements, bacteria and populations.

○ **Inflow/outflow**, the rate of change of the mass of a substance in a solution is modelled by $\frac{dM}{dt} = f(M) = \text{inflow rate} - \text{outflow rate}$.

○ **The logistic model** $\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$ has solution $P = \frac{K}{1 + \left(\frac{K - P_0}{P_0}\right)e^{-kt}}$, where P is the population at time t, P_0 is the initial population, K is the carrying capacity and k is a constant of proportionality.

3 $\frac{dy}{dx} = f(x)g(y)$ is solved by separation of variables that involves rewriting the differential equation as:

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

We need to be careful that $g(y) = 0$ is not a solution of the differential equation.

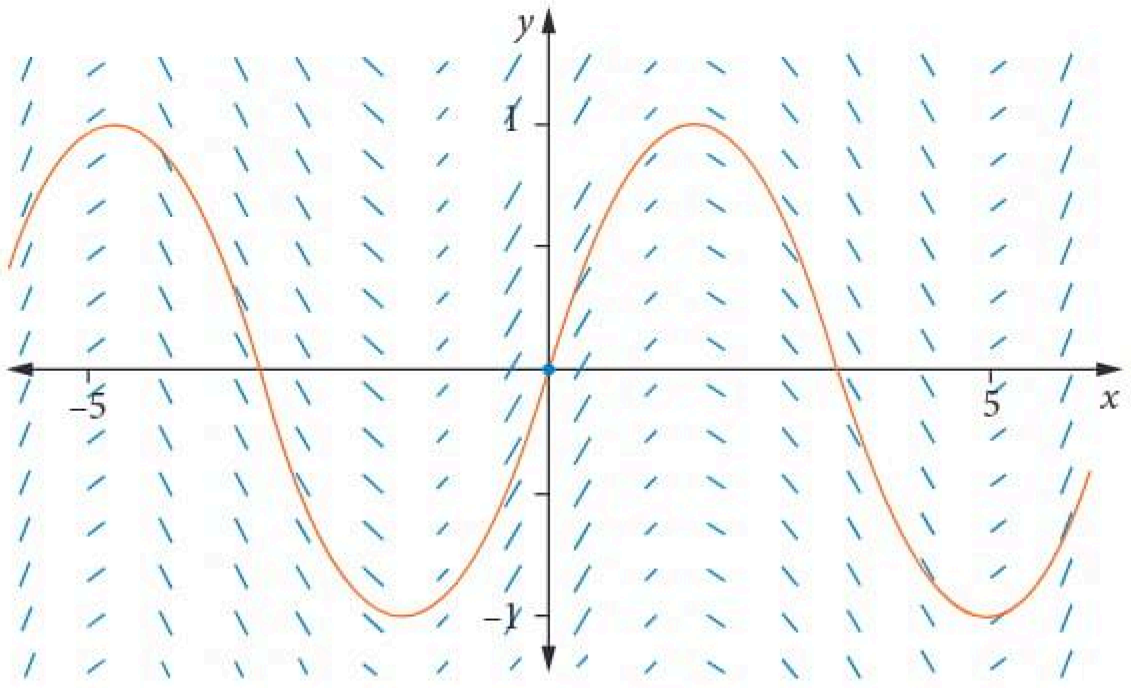
$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

Second-order differential equations

The general solution to the second-order differential equation $\frac{d^2y}{dx^2} = f(x)$ is $y = g(x) + c_1x + c_2$, where $g''(x) = f(x)$ and c_1 and c_2 are constants of integration.

Slope fields

A **slope field** (or **gradient field** or **direction field**) displays a family of solutions to a first-order differential equation by showing segments of its gradient whose overall pattern represents the solution curve.



The slope field for $\frac{dy}{dx} = \cos(x), y(0) = 0$.

Euler's method

When algebraic methods of anti-differentiation cannot be applied to solve a first-order differential equation, Euler's method is used to generate a sequence of approximations to the solution of $\frac{dy}{dx} = f(x)$ that has an **initial condition** (x_0, y_0).

Euler's formula in **iterative form** is

$$x_{n+1} = x_n + h, y_{n+1} = y_n + hf(x),$$

where $f(x) = \frac{dy}{dx}$, $n = 0, 1, 2, 3 \ldots$

Accuracy in the approximations will increase by using a smaller step size, h, or by increasing the number of iterations.

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 © VCAA 2012 1Q5 (3 marks) Let $y = \arctan(2x)$.

Find the value of a given that $\frac{d^2y}{dx^2} = ax\left(\frac{dy}{dx}\right)^2$, where a is a real constant.

2 (2 marks) Solve the differential equation $\frac{dQ}{dt} = \frac{2t^3}{Q}$, given that $t(0) = 1$.

State the answer in the form $aQ^2 + bt^4 + c = 0$, where a, b and c are constants.

3 (2 marks) Provide a counterexample to each statement.

a $\forall x, y \in R(x^2 = y^2) \rightarrow x = y$. 1 mark

b If $n^2 - 1$ is divisible by 5, then $n - 1$ is divisible by 5. 1 mark

4 (3 marks) The graph of the quadratic function $f(x) = (x - 1)^2 + 5$ is reflected in the line $y = 6$ to obtain the function $g(x)$.

a State the equation of $g(x)$. 1 mark

b Calculate the area enclosed by the graph of $f(x)$ and the graph of $g(x)$ in $[a, b]$, where a and b are the x-coordinates of the points of intersection of the two functions. 2 marks

Cumulative examination 2

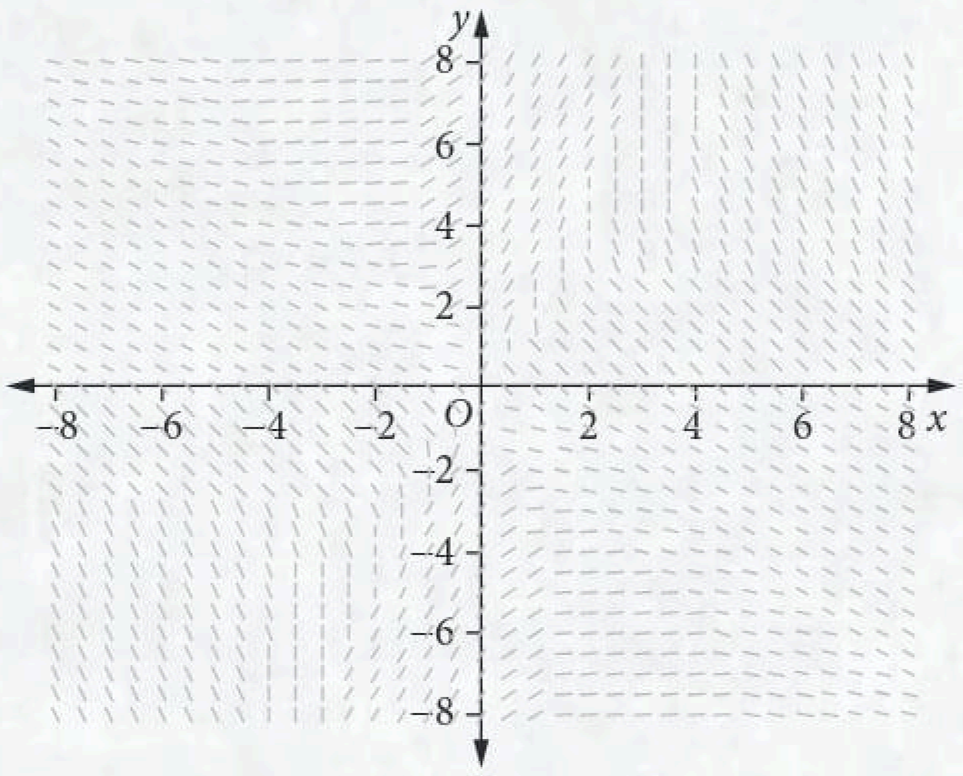
Total number of marks: 17 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 © VCAA 2018 2AQ10



The differential equation that best represents the direction field above is

A $\dfrac{dy}{dx} = \dfrac{2x + y}{y - 2x}$

B $\dfrac{dy}{dx} = \dfrac{x + 2y}{2x - y}$

C $\dfrac{dy}{dx} = \dfrac{2x - y}{x + 2y}$

D $\dfrac{dy}{dx} = \dfrac{x - 2y}{y - 2x}$

E $\dfrac{dy}{dx} = \dfrac{2x + y}{2y - x}$

2 © VCAA 2011 2AQ18 The amount of chemical x in a tank at time t is given by the differential equation $\dfrac{dx}{dt} = -\dfrac{10}{10 - t}$ and when $t = 0$, $x_0 = 5$. Euler's method is used with a step size of 0.5 in the values of t.

The value of x correct to two decimal places when $t = 1$ is found to be

A 3.95 **B** 3.97 **C** 4.50 **D** 5.50 **E** 6.03

3 © VCAA 2002 1IQ30 A 150 litre cylinder of air contains 20% oxygen. The amount of oxygen in the cylinder is to be increased by pumping in pure oxygen at a constant rate of 10 litres/minute, while removing the uniformly mixed air at the same rate.

If P litres is the volume of oxygen in the cylinder at time t minutes after the pumping begins, a differential equation for P in terms of t is

A $\dfrac{dP}{dt} = 8P;\ t = 0,\ P = 30$

B $\dfrac{dP}{dt} = 8P;\ t = 0,\ P = 150$

C $\dfrac{dP}{dt} = 30 + 10t;\ t = 0,\ P = 30$

D $\dfrac{dP}{dt} = 10 - \dfrac{P}{15};\ t = 0,\ P = 30$

E $\dfrac{dP}{dt} = 10 - \dfrac{P}{15};\ t = 0,\ P = 150$

▶

4 Let P: even numbers are divisible by 2

Q: 6 is an even number

The statement $\neg(P \wedge \neg Q)$ is equivalent to

A Even numbers are not even and 6 is an odd number.

B Odd numbers are not divisible by 2 or 6 is an even number.

C Even numbers are not divisible by 2 or 6 is an even number.

D If even numbers are divisible by 2, then 6 is an even number.

E If 6 is not even, then even numbers are not divisible by 2.

5 With the substitution $u = 2x - 4$, $\int_0^2 e^{2x-4}\,dx$ becomes

A $2\int_0^2 e^u\,du$ **B** $\frac{1}{2}\int_{-4}^0 e^u\,du$ **C** $2\int_{-4}^2 e^u\,du$

D $\frac{1}{2}\int_0^2 e^u\,du$ **E** $2\int_{-4}^2 e^u\,du$

Section B 2 questions

12 marks

1 © VCAA 2013 2BQ3a-di (7 marks) The number of mobile phones, N, owned in a certain community after t years, may be modelled by $\log_e(N) = 6 - 3e^{-0.4t}$, $t \geq 0$.

a Verify by substitution that $\log_e(N) = 6 - 3e^{-0.4t}$ satisfies the differential equation

$$\frac{1}{N}\frac{dN}{dt} + 0.4\log_e(N) - 2.4 = 0.$$

2 marks

b Find the initial number of phones, N, owned in a certain community.

Give your answer correct to the nearest integer. 1 mark

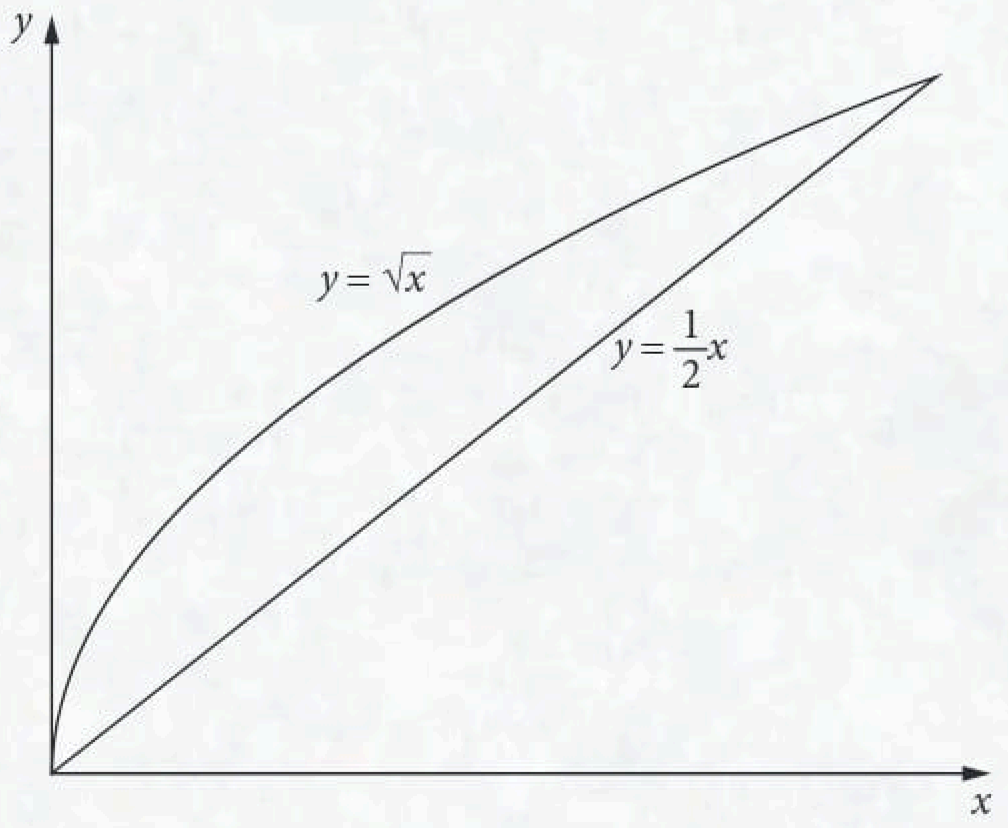
c Using this mathematical model, find the limiting number of mobile phones that would eventually be owned in the community.

Give your answer correct to the nearest integer. 2 marks

The differential equation in part **a** can also be written in the form $\frac{dN}{dt} = 0.4N(6 - \log_e(N))$.

d Find $\frac{d^2N}{dt^2}$ in terms of N and $\log_e(N)$. 2 marks

2 (5 marks) A region is defined by the area bounded by the curves with equations $y = \sqrt{x}$ and $y = \frac{1}{2}x$, as shown below.



a State the coordinates of the points of intersection of the two curves. 1 mark

b Determine the area of the region bounded by the two curves. 2 marks

c The region is rotated about the x-axis. What is the volume of the solid produced? 2 marks

CHAPTER

10

KINEMATICS

Study Design coverage

AREA OF STUDY 4: CALCULUS

Kinematics: rectilinear motion

- use of velocity–time graphs to describe and analyse rectilinear motion
- application of differentiation, anti-differentiation and solution of differential equations to rectilinear motion of a single particle, including the different derivative forms for acceleration $a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

VCE Mathematics Study Design 2023–2027 p. 111, © VCAA 2022

Video playlists (5):

10.1 Straight-line motion
10.2 Velocity–time graphs
10.3 The equations of kinematics
10.4 Variable acceleration
VCE question analysis Kinematics

Nelson MindTap

To access resources above, visit
cengage.com.au/nelsonmindtap

10.1 Straight-line motion

Video playlist
Straight-line motion

Kinematics examines the motion of an object in a straight line (**rectilinear motion**), where the object is considered to be a point without mass, shape or size.

Displacement, velocity and acceleration

Displacement is the change of position of an object in relation to a fixed reference point. We can think of it as the direct distance from where the object starts to where it stops.

The **velocity** of an object is the rate of change of its displacement with respect to time.

Acceleration is the rate of change of the velocity with respect to time.

Displacement, velocity and acceleration are **vectors** because they are defined by requiring **magnitude** and **direction.** For example, a displacement of 3 km *north,* a velocity of 5 metres per second *south* and an acceleration of −3 cm per second per second *west.*

The units for velocity are cm/s (cm s^{-1}), m/s (m s^{-1}) and km/h (km h^{-1}), and the units for acceleration are m/s^2 (m s^{-2}) and km/h^2 (km h^{-2}).

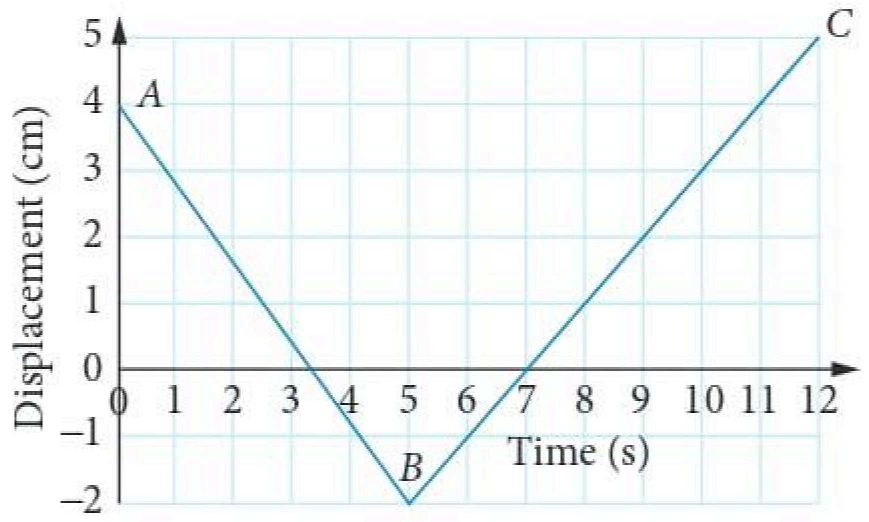
The **average velocity** of an object is its displacement divided by the total time taken during its motion.

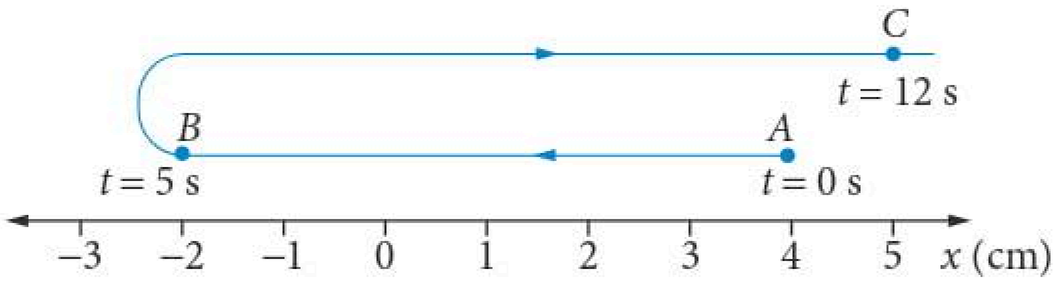
Distance and speed

Scalar quantities do not have a direction. The associated scalar quantities for displacement and velocity are **distance** and **speed** respectively, where distance is the magnitude of the displacement and speed is the magnitude of the velocity.

The **average speed** of an object is the total distance travelled divided by the total time taken.

The **displacement–time graph** and the **position–time line** shown below are ways of illustrating the straight line motion of an object.





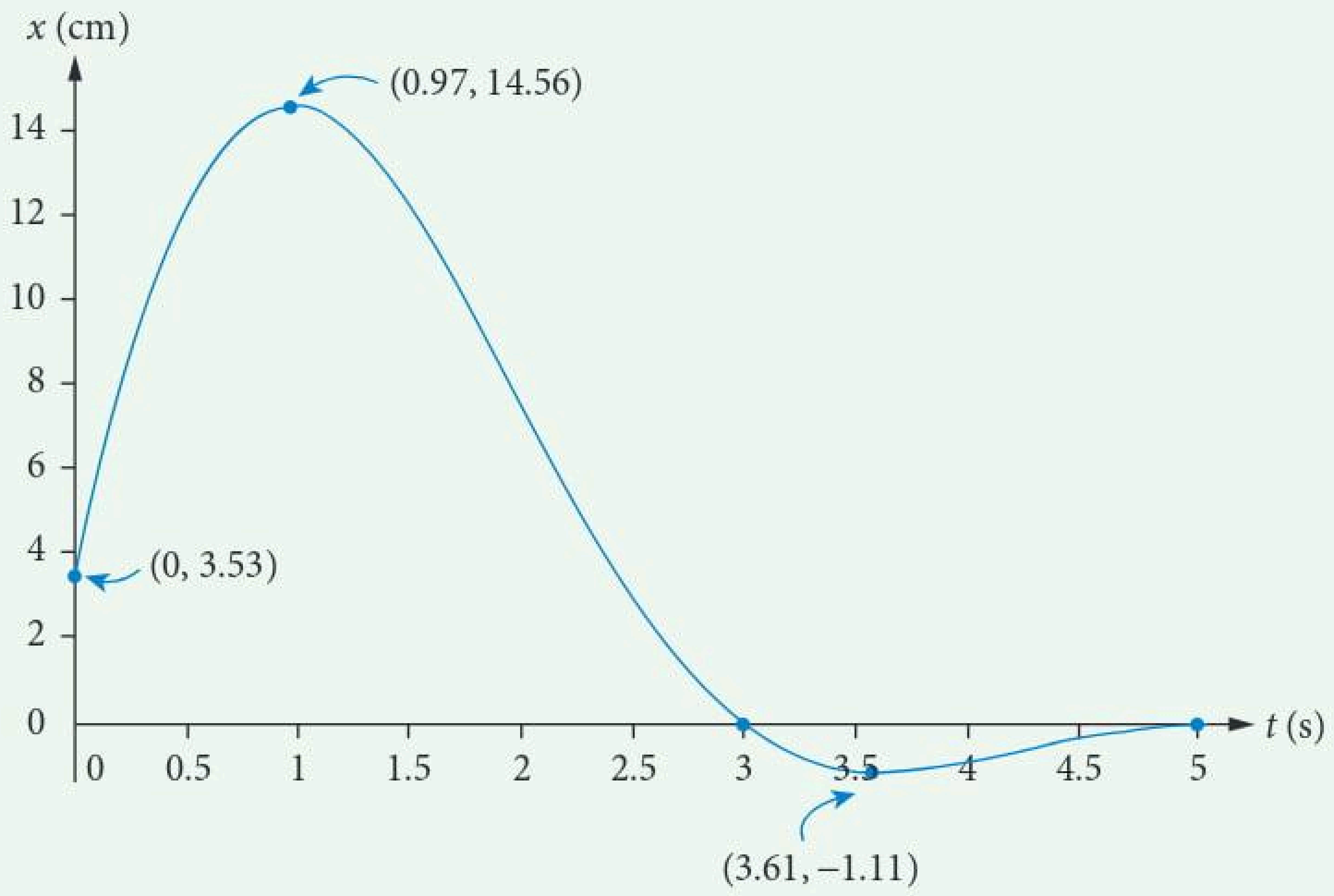
- The object takes 5 seconds to move from position *A* to *B*.
- It takes another 7 seconds to go from *B* to *C*.
- In relation to the start position (4 cm from the origin), the total distance travelled is $6 + 7 = 13$ cm and the displacement is $5 - 4 = 1$ cm (to the right of the start position).
- The average speed of the object in going from *A* to *B* is $\frac{6}{5} = 1.2$ cm/s, and the average speed in going from *B* to *C* is $\frac{7}{7} = 1$ cm/s.
- The average velocity in going from *A* to *B* to *C* is $\frac{5-4}{12} \approx 0.08$ cm/s.

WB

p. 198

WORKED EXAMPLE 1 Average speed and velocity

A particle moves in a straight line such that its position *x* cm from a reference point *O* after *t* seconds is shown in this displacement–time graph.



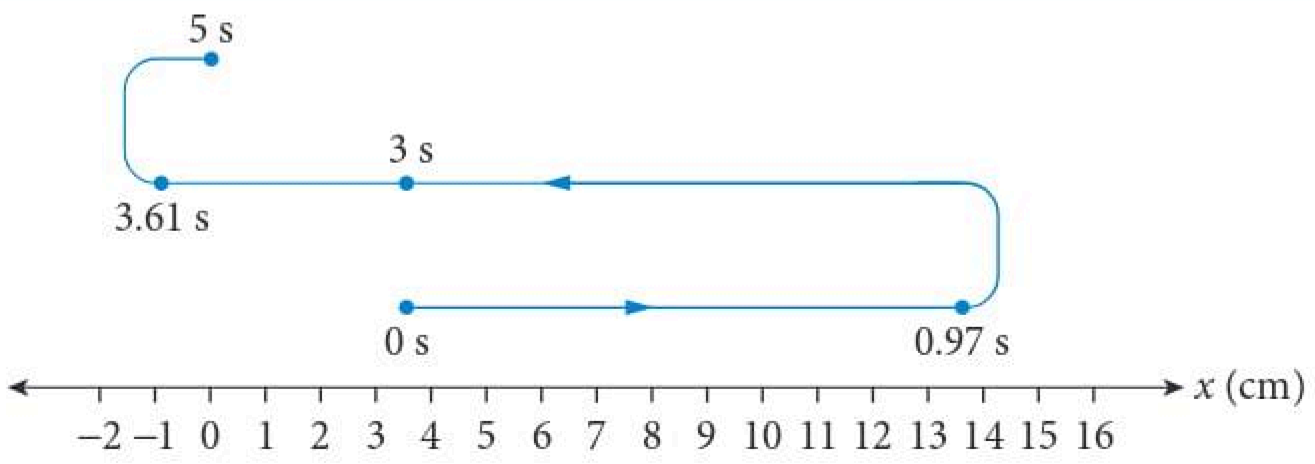
a Illustrate the motion of the particle as a position–time line in the interval [0, 5].

b Calculate the average speed in the interval [0, 5].

c Determine the average velocity in the interval [0, 5].

Give answers to two decimal places.

Steps	**Working**
a **1** Determine the initial displacement.	At $t = 0$, the particle is 3.53 cm from the origin.
2 Describe the motion for each part of the graph.	The particle moves away from the origin on the positive side for 0.97 s until it is at 14.56 cm. It reverses direction and reaches the origin after another 2.03 seconds ($t = 3$). The particle continues to move in the negative direction away from the origin for another 0.61 seconds, reverses direction when it is at 1.11 cm from the origin. It then returns to the origin after 1.39 more seconds ($t = 5$).

3 Draw the position–time line.	
b average speed $= \frac{\text{total distance}}{\text{total time}}$	Total distance is 14.56 – 3.53 + 14.56 + 1.11 + 1.11 = 27.81 cm. Total time is 5 seconds. $\frac{\text{total distance}}{\text{total time}} = \frac{27.81}{5} = 5.56 \text{ cm/s}$
c average velocity $= \frac{\text{displacement}}{\text{time taken}}$	In [0, 5], the particle starts at 3.53 cm and ends at 0 cm. Displacement is 0 – 3.53 = –3.53 cm. Total time is 5 seconds. average velocity $= \frac{-3.53}{5} = -0.71 \text{ cm/s}$

WORKED EXAMPLE 2 Average speed, velocity and direction

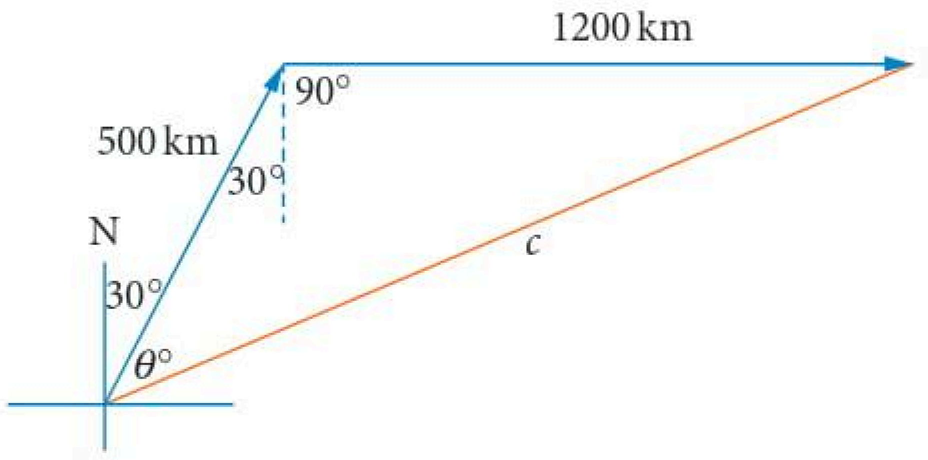
An aeroplane flies in the direction 030°T for 1 h 40 min at a constant speed of 300 km/h and then flies due east for a further 3 hours at a constant speed of 400 km/h.

a Find the average speed of the plane.

b Calculate the magnitude of the average velocity.

c In which direction is the average velocity vector?

Give answers to two decimal places.

Steps	**Working**
a **1** Show the information as a vector diagram.	300 km/h for 1 h 40 min $\left(1\frac{2}{3}\text{ h}\right)$ is a distance of $300 \times 1\frac{2}{3} = 500$ km. 400 km/h for 3 h is a distance of 1200 km. 
2 Calculate the average speed.	Total distance is 500 + 1200 = 1700 km. Total time is 4 h 40 min $\left(4\frac{2}{3}\text{ h}\right)$. average speed $= \frac{\text{distance}}{\text{time}}$ $= \frac{1700}{4\frac{2}{3}}$ = 364.29 km/h

10.1

WB p. 200

b	**1**	Find the displacement using the cosine rule.	$c^2 = a^2 + b^2 - 2ab\cos(\theta°)$. $a = 500,\ b = 1200,\ \theta° = 120°$ $c^2 = 500^2 + 1200^2 - 2 \times 500 \times 1200 \times \cos(120°)$ $= 2\,290\,000$ $c = 1513.27\,\text{km}$
	2	Calculate the magnitude of the average velocity.	The plane has flown 1513.27 km in $4\frac{2}{3}$ h. $\text{average velocity} = \frac{\text{displacement}}{\text{time}}$ $= \frac{1513.27}{4\frac{2}{3}}$ $= 324.27$ km/h
c	**1**	Use the sine rule to calculate angle $\theta°$ and use it to find the direction of the velocity vector.	$\frac{\sin(120°)}{c} = \frac{\sin(\theta°)}{1200}$ $\theta° = \sin^{-1}\left[\frac{1200\sin(120°)}{1513.27}\right] = 43.37°$
	2	State the direction.	The velocity vector has direction 30° + 43.37° = 073.37°T.

Note that in the above worked example, the average speed of the aeroplane, 364.29 km/h, is not the same as the magnitude of the average velocity, 324.27 km/h.

Exam hack

Speed is the magnitude of the velocity, but average speed is not the same as the magnitude of the average velocity.

Differentiation and anti-differentiation

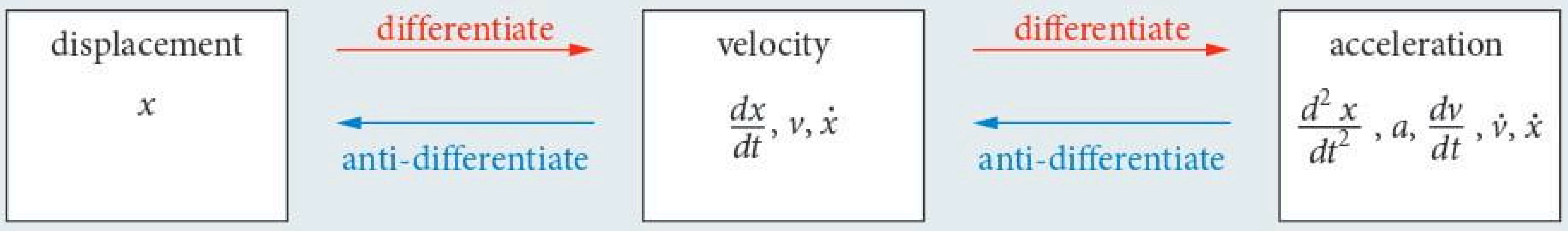
Differentiation provides the **rate of change** of a **dependent variable** with respect to the **independent variable**. For example, for $x = f(t)$, x is the displacement and the dependent variable, t is the time and the independent variable and $\frac{dx}{dt}$ gives the instantaneous velocity. The symbol for velocity is v or $\dot{x}$.

Acceleration is the rate of change of velocity. It is the result of differentiating speed, so $\frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$.

Acceleration may also be written as a, $\frac{dv}{dt}$, $\dot{v}$ or $\ddot{x}$.

Anti-differentiation of acceleration gives the velocity, and anti-differentiation of velocity gives displacement.

Displacement, velocity and acceleration



$$v = \int a\,dt,\ x = \int v\,dt$$

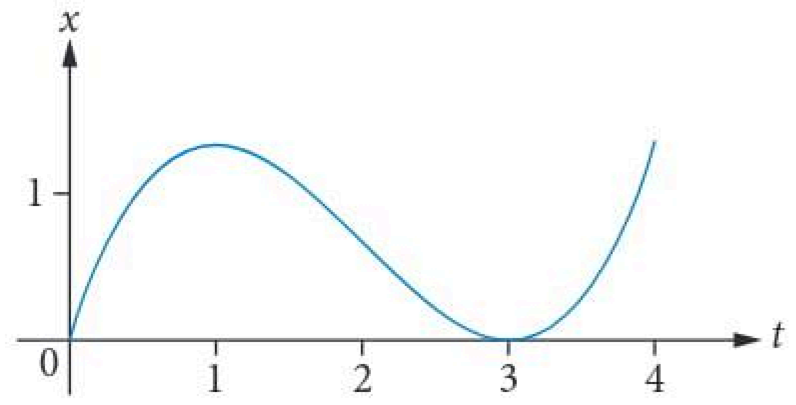
WORKED EXAMPLE 3 Finding velocity and acceleration from displacement

In relation to a fixed point, the position x cm of a particle moving horizontally at time t seconds is described by the function $x = \frac{1}{3}t^3 - 2t^2 + 3t$.

a Determine when the velocity of the particle is zero and its position at that time.

b Find at what time the acceleration is zero and the particle's velocity at this time.

c Calculate the distance that the particle will travel during the first 4 seconds.

Steps	**Working**
a 1 Differentiate the equation to obtain the velocity function.	$x = \frac{1}{3}t^3 - 2t^2 + 3t$ $\frac{dx}{dt} = t^2 - 4t + 3$
2 Let the equation for velocity be zero and solve for t.	$t^2 - 4t + 3 = 0$ $(t - 1)(t - 3) = 0$ $t = 1, t = 3$ The velocity of the particle will be zero at $t = 1$ s and $t = 3$ s.
3 Determine the position using these t values.	Use $x = \frac{1}{3}t^3 - 2t^2 + 3t$. When $t = 1$, $x = \frac{1}{3} - 2 + 3 = 1\frac{1}{3}$ cm. When $t = 3$, $x = 9 - 18 + 9 = 0$ cm.
b 1 Differentiate the velocity equation to obtain the acceleration function.	$\frac{dx}{dt} = t^2 - 4t + 3$ $\frac{d^2x}{dt^2} = 2t - 4$
2 Use the equation and the given value for acceleration to find the value for t.	$\frac{d^2x}{dt^2} = 0 \Rightarrow 2t - 4 = 0, t = 2$
3 Find the particle's velocity by substituting the value for t.	$t = 2, \frac{dx}{dt} = t^2 - 4t + 3 = -1$ cm/s
c Use the graph of the position function to find the distance in each section. Turning points occur when the velocity is zero.	 At $t = 1$ and $t = 4$, $x = 1\frac{1}{3}$ cm. Distance travelled in [0, 3] is $1\frac{1}{3} + 1\frac{1}{3} = 2\frac{2}{3}$ cm. Distance travelled in [3, 4] is $1\frac{1}{3}$ cm. Total distance is $2\frac{2}{3} + 1\frac{1}{3} = 4$ cm.

WB
p. 201

WB

p. 202

WORKED EXAMPLE 4 Velocity and position from acceleration

The acceleration a m/s^2 of a particle at time t s is given by the function $a = 6t + 2$.

a Obtain the equation for the velocity v m/s of the particle given that $v = 13$ when $t = 2$.

b Find the equation for the position x m of the particle given that $x = 4$ when $t = 1$.

c Find the exact value of the acceleration when the particle's velocity is 0.

Steps	**Working**
a Anti-differentiate the equation for acceleration with respect to t to obtain the velocity function. Then find the value of the constant of integration.	$a = 6t + 2$ $v = 3t^2 + 2t + c$ $v = 13$ when $t = 2$. $13 = 16 + c$, so $c = -3$. $\therefore v = 3t^2 + 2t - 3$
b Anti-differentiate the velocity equation with respect to t to obtain the position function and find the value of the constant of integration.	$v = 3t^2 + 2t - 3$ $x = t^3 + t^2 - 3t + c$ $x = 4$ when $t = 1$. $4 = -1 + c$, so $c = 5$. $\therefore x = t^3 + t^2 - 3t + 5$
c **1** Find the value of t when the velocity is zero.	Let $v = 3t^2 + 2t - 3 = 0$ and solve for t. $3t^2 + 2t - 3 = 0 \Rightarrow t = \frac{-1+\sqrt{10}}{3}$
2 Substitute into the equation for acceleration.	$a = 6t + 2$ $= 6 \times \left(\frac{-1+\sqrt{10}}{3}\right) + 2$ $= 2\sqrt{10}$ m/s^2

Other forms of acceleration

Acceleration can be represented in different derivative forms, depending upon the problem to be solved. From the chain rule for differentiation, $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt}$ and so $\frac{dv}{dt} = v\frac{dv}{dx}$, where v is velocity and x is displacement. Also, $v\frac{dv}{dx} = \frac{d}{dv}\left(\frac{1}{2}v^2\right)\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

These last two forms can be used when acceleration is not required as a function of time.

Different forms of acceleration

$$a = \frac{d^2x}{dt^2}, \quad a = \frac{dv}{dt}, \quad a = v\frac{dv}{dx}, \quad a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

Exam hack

Decide from the question given what form, or how, the acceleration is to be used.

For example, if $a = t^2$, where t is time, use $\frac{d^2x}{dt^2} = t^2$ or $\frac{dv}{dt} = t^2$.

WORKED EXAMPLE 5 Position from acceleration

The acceleration a cm s^{-2} of an object moving in a straight line with positive velocity v cm/s when x cm from the origin is given by $a = 4(x+1)^3$. Find the position, x cm, of the object at time t seconds given that $v = \sqrt{2}$ when $x = 0$, and when $t = 0$, $x = 0$.

Steps	Working
1 Decide which form of acceleration can be used.	Acceleration is given as a function of x, so use $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 4(x+1)^3$.
2 Anti-differentiate with respect to x and find the value of the constant of integration.	$\frac{1}{2}v^2 = (x+1)^4 + c$ When $x = 0, v = \sqrt{2}$. $1 = 1 + c \Rightarrow c = 0$ So $\frac{1}{2}v^2 = (x+1)^4$.
3 Write v as $\frac{dx}{dt}$.	$\frac{dx}{dt} = \sqrt{2}(x+1)^2$ (positive square root since v is positive)
4 Anti-differentiate and find the value of the constant of integration.	$\int \frac{dx}{(x+1)^2} = \int \sqrt{2}dt + c$ $-\frac{1}{x+1} = \sqrt{2}t + c$ When $t = 0, x = 0$. $-1 = 0 + c$ $c = -1$ So $-\frac{1}{x+1} = \sqrt{2}t - 1$.
5 Solve for x.	$-(x+1) = \frac{1}{\sqrt{2}t - 1}$ $x = \frac{\sqrt{2}t}{1 - \sqrt{2}t}$

WB p. 203

WORKED EXAMPLE 6 Velocity as a function of position

The velocity, v km/h, of a particle x km from a fixed point is $v = e^{kx}$, where k is a constant.

a Determine the value of k if the acceleration is 36 km h^{-2} when the velocity is 2 km/h.

b Obtain an expression for the position of the particle at any time given that its initial position is zero.

Steps	Working
a **1** Use the equation for velocity to find an expression for acceleration.	Use $v = e^{kx}$ to write acceleration a as $v\frac{dv}{dx}$. $\frac{dv}{dx} = ke^{kx}$ $= kv$ $v\frac{dv}{dx} = v \times kv = kv^2$ $a = kv^2$

WB p. 204

2 Evaluate k using the given information.	Use $a = 36$ when $v = 2$. $36 = 4k$ $k = 9$
b **1** Anti-differentiate velocity to obtain displacement.	$\frac{dx}{dt} = e^{9x}$ $\int e^{-9x}\,dx = \int 1\,dt$ $-\frac{1}{9}e^{-9x} = t + c$
2 Find the constant.	When $t = 0$, $x = 0$. $-\frac{1}{9} \times 1 = 0 + c \Rightarrow c = -\frac{1}{9}$
3 Write displacement as a function of time.	$-\frac{1}{9}e^{-9x} = t - \frac{1}{9}$ $e^{-9x} = -9t + 1$ $x = -\frac{1}{9}\log_e(1 - 9t),\ 0 \le t < \frac{1}{9}$

USING CAS 1 Displacement, velocity and acceleration

The displacement, x m, at time t s of a particle from O is given by the equation $x = 1.5 - t^{\sin(t)}$, $0 \le t \le 5$.

a State the maximum distance of the particle from O.

b At what time(s) is the particle at O?

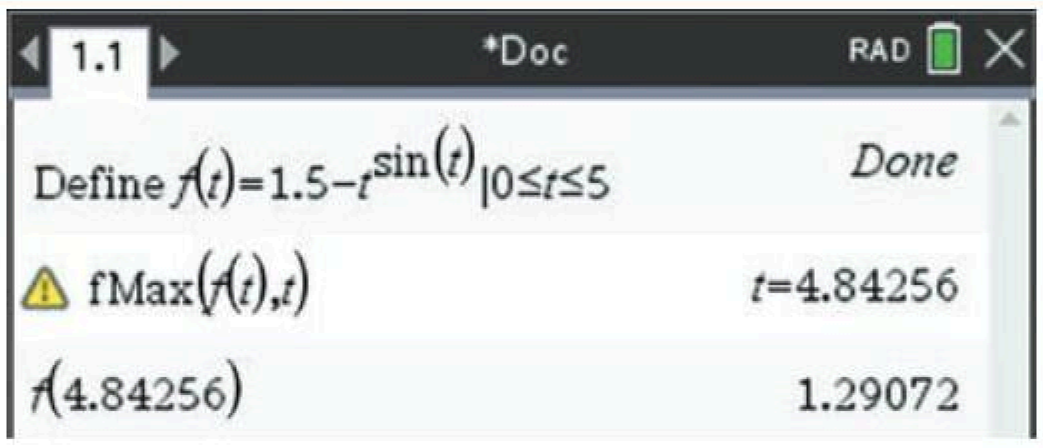
c When does the particle's velocity become 1 m s^{-1} for the second time?

d Calculate the particle's acceleration after 4 seconds.

Give all answers correct to two decimal places.

TI-Nspire

a



Define $f(t)=1.5-t^{\sin(t)}|0\le t\le 5$ Done

fMax$(f(t),t)$ $t=4.84256$

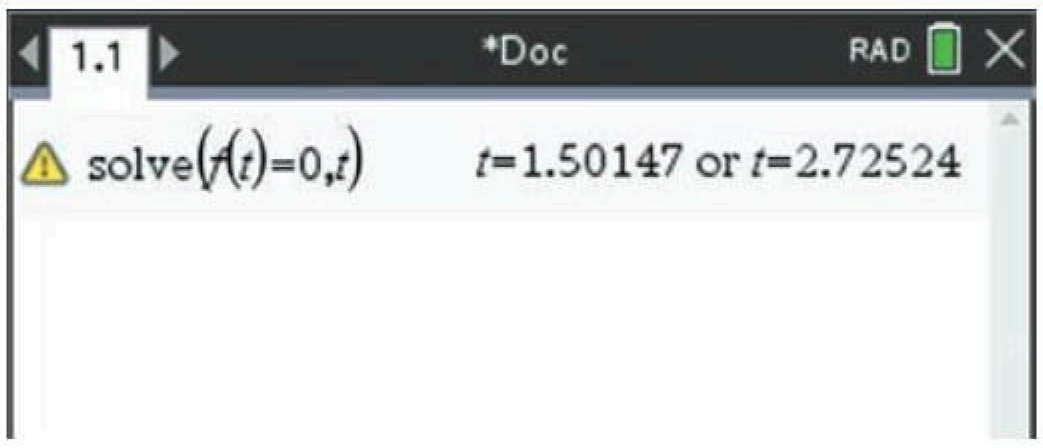
$f(4.84256)$ 1.29072

1 Define the function over the domain [0, 5].

2 Press **menu > Calculus > Function Maximum** and enter **f(t),t**.

3 Substitute the answer into the function to find the value of the function at this point.

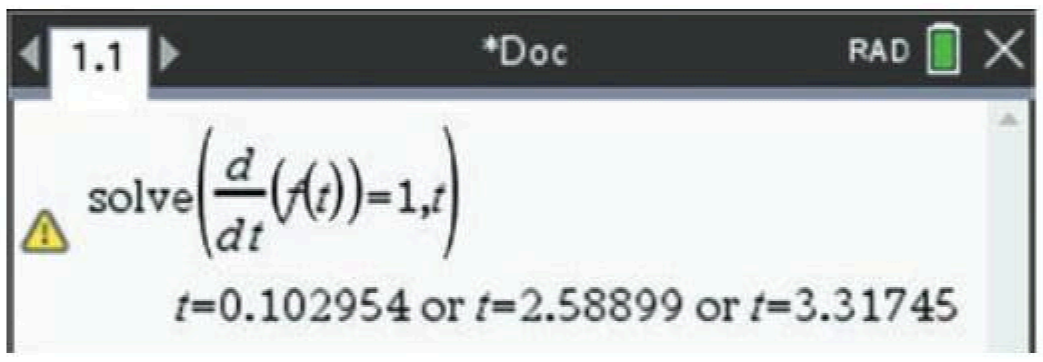
b



solve$(f(t)=0,t)$ $t=1.50147$ or $t=2.72524$

Set the function equal to **0** and **solve**.

c

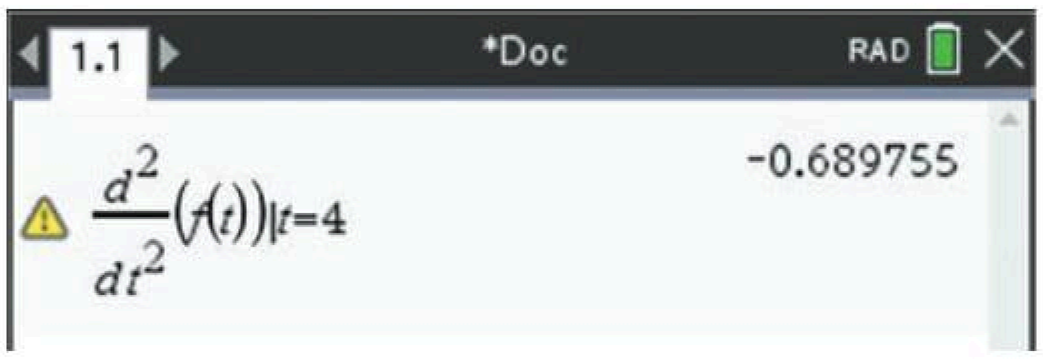


solve$\left(\frac{d}{dt}(f(t))=1,t\right)$

$t=0.102954$ or $t=2.58899$ or $t=3.31745$

Set the **derivative** of the function equal to **1** and **solve**.

d

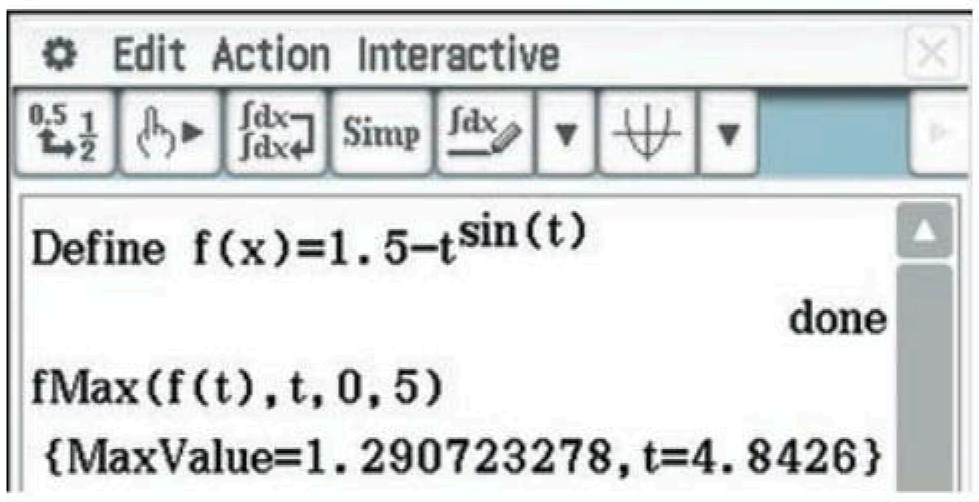


$\frac{d^2}{dt^2}(f(t))|t=4$ -0.689755

Find the **second derivative** of the function when $t = 4$.

ClassPad

a



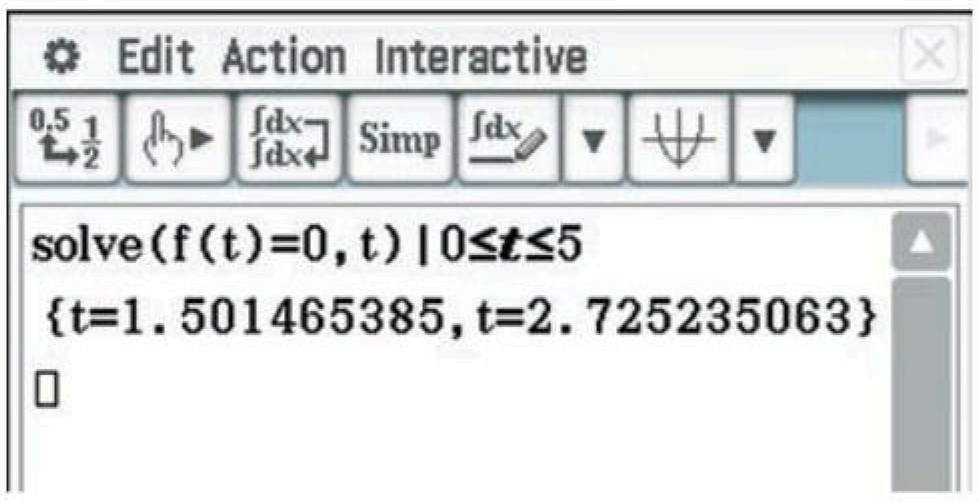
```
Define f(x)=1.5−t^sin(t)
                              done
fMax(f(t),t,0,5)
 {MaxValue=1.290723278,t=4.8426}
```

1 In **Main**, define and highlight the function.

2 Tap **Interactive > Calculation > fMin/fMax > fMax**.

3 In the dialogue box, enter ***t*** for the variable, **0** for the start and **5** for the end.

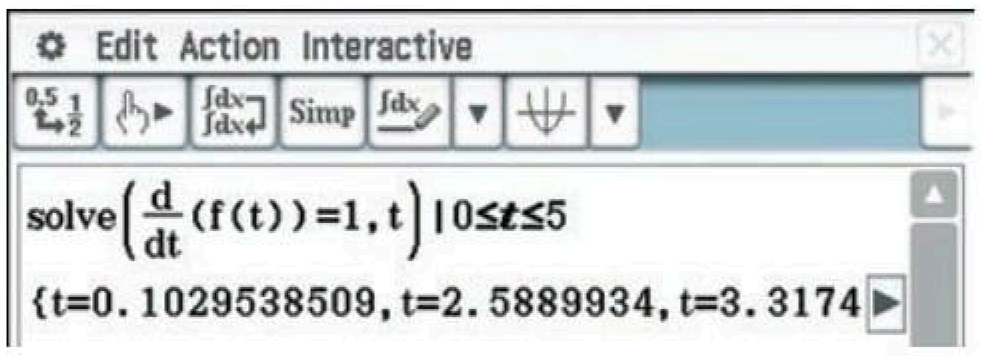
b



```
solve(f(t)=0,t)|0≤t≤5
 {t=1.501465385,t=2.725235063}
```

Set the function equal to **0** and tap **Transformation > Equation/Inequality > solve**. Include the domain [0, 5].

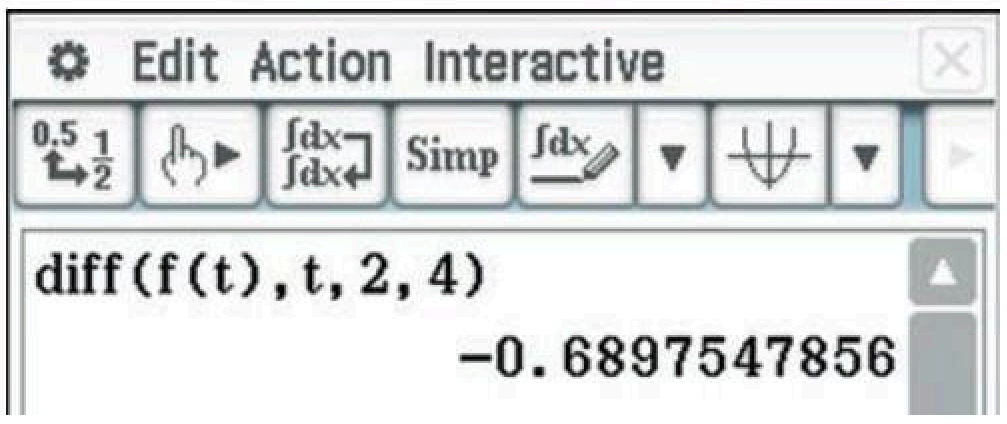
c



```
solve(d/dt(f(t))=1,t)|0≤t≤5
{t=0.1029538509,t=2.5889934,t=3.3174▶
```

Set the **derivative** of the function equal to **1** and tap **Transformation > Equation/Inequality > solve**. Include the domain [0, 5].

d



```
diff(f(t),t,2,4)
                  −0.6897547856
```

Find the **second derivative** of the function when ***t* = 4**.

a Over the domain [0, 5], the maximum value of the function is 1.29 m.

b The particle is at O when $t = 1.50$ s and $t = 2.73$ s.

c The second time the velocity of the particle is equal to 1 is when $t = 2.59$ s.

d After 4 s, the acceleration of the particle is -0.69 m s^{-2}.

EXERCISE 10.1 Straight-line motion

ANSWERS p. 596

Mastery

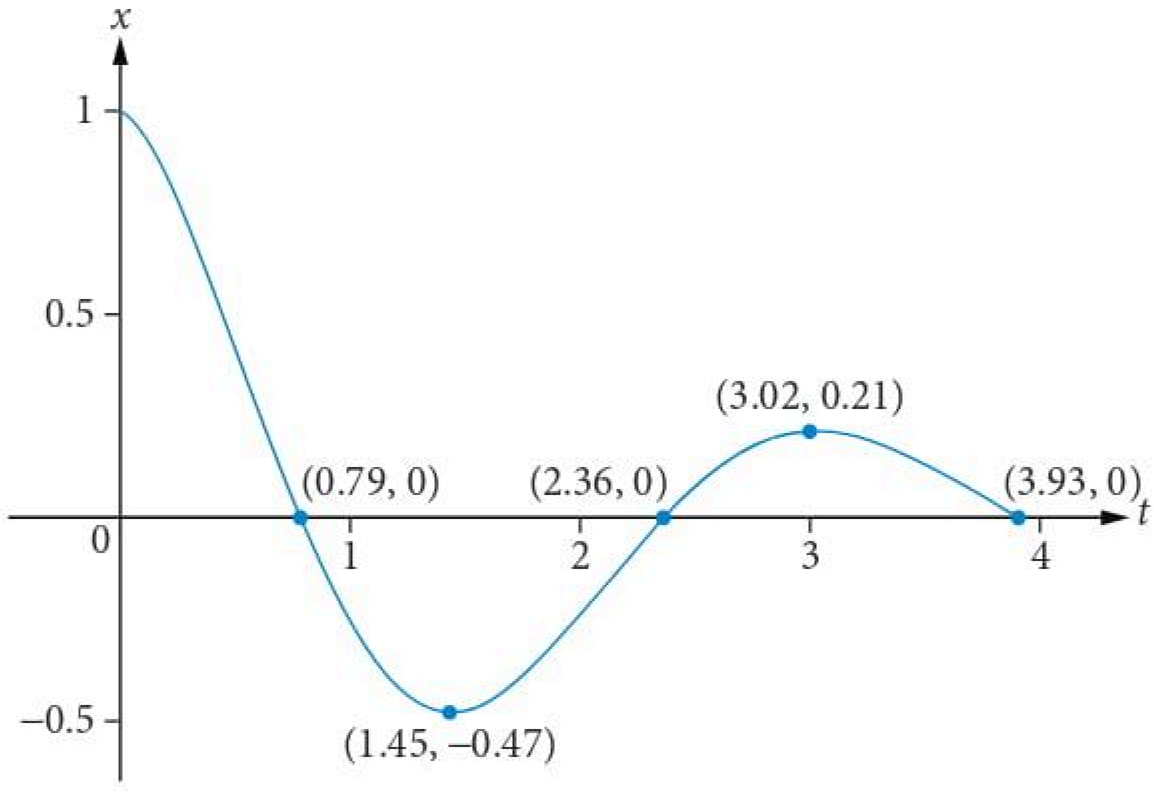
1 WORKED EXAMPLE 1 Below is the graph of the position x m of a particle at time t minutes in the interval [0, 3.93].

a For the given interval, show the motion of the particle as a position–time line.

b Calculate the average speed in the given interval.

c Determine the average velocity in the given interval.

Round answers to two decimal places.



2 WORKED EXAMPLE 2 A ship leaves port and sails due north for 5 hours at a steady speed of 12 km/h. It then immediately changes direction and sails in the direction 30°T for a further 4 h 20 min at 9 km/h.

a State the ship's average speed.

b Calculate the magnitude of the average velocity.

c In which direction, correct to two decimal places, is the average velocity vector?

Give answers to two decimal places.

3 WORKED EXAMPLE 3 The position, x cm, of a particle at time t s is described by the function $x = \frac{10}{3}t^3 + \frac{9}{2}t^2 - 9t$.

a Find when the particle's velocity is zero and its position at that time.

b Find when the particle's acceleration will be 10 cm/s^2 and its velocity at that time.

c Calculate the distance that the particle will travel during the first 2 seconds.

4 WORKED EXAMPLE 4 The acceleration, a m/s^2, of a particle at time t seconds is given by the function $a = 2t - 8$.

a Obtain the equation for the velocity, v m/s, of the particle given that $v = 9$ when $t = 1$.

b Find the position of the particle when its velocity is zero given that when $t = 0$, $x = 1$.

5 WORKED EXAMPLE 5 An object moving in the positive direction also has positive velocity. If its acceleration is $a = 3x$ when it is x cm from the origin, obtain an expression for the position at time t s, given that $v = 0$ when $x = 0$, and $x = 1$ when $t = 0$.

6 WORKED EXAMPLE 6 The velocity, v m/s, of a particle x m from a fixed point is $v = \sqrt{1 - kx^2}$, where k is a constant.

a Find the value of k if the acceleration is -8 m/s^2 when it is 1 m from a fixed point.

b Obtain an expression for the position of the particle at any time given that its initial position is zero.

7 Using CAS 1 The position, x m, at time t s of an object from O is found using the equation $x = t\cos(2t)$, $0 \le t \le 5$.

a Calculate the difference between the object's maximum and minimum distance from O. Give the answer to two decimal places.

b State the initial velocity.

c How long after the start does the object take to return to O for the first time? Give the answer correct to one decimal place.

d Find the first time that the magnitude of the object's velocity is the same as the magnitude of its position. Give your answer correct to one decimal place.

e Determine when the object's acceleration is 15 m s^{-2}. Give the answer to one decimal place.

Exam practice

80–100% 60–79% 0–59%

8 TECH-FREE (2 marks) The position, x mm, of a particle at time t seconds is given by $x = A\cos(kt) + B\sin(kt)$, where A, B and k are constants. Show that the particle's acceleration is proportional to its position.

9 TECH-FREE (3 marks) The acceleration, $\frac{d^2x}{dt^2}$, of an object at time t is given by $\frac{d^2x}{dt^2} = \sin(kt)$, where k is a positive real number. Calculate the value of the constants a, b and c if the position of the object is of the form $x = a\sin(kt) + bt + c$, $\frac{dx}{dt} = \frac{1}{k}$ when $t = 0$ and $x = -\frac{1}{k^2}$ when $t = \frac{\pi}{2k}$.

10 **TECH-FREE** (2 marks) The position, x metres, at time t seconds of a moving object is described by $x = a\cos(t) + b\sin(t)$, where a and b are constants. Determine the values of a and b if its speed is $\frac{3-\sqrt{3}}{2}$ m/s when $t = \frac{\pi}{3}$ s and its acceleration is $\sqrt{2}$ m/s^2 when $t = \frac{\pi}{4}$ seconds.

11 **TECH-FREE** (2 marks) The velocity, v cm/h, at time t hours of a particle starting from rest at the origin is $v = te^t$.

a Show that its acceleration at t hours is $(1+t)e^t$ cm/h^2. 1 mark

b Use the result from part **a** to show that the position, x cm, of the particle is $x = v - e^t + 1$. 1 mark

12 © VCAA 2012 1Q8 **57%** **TECH-FREE** (3 marks)

The velocity, v m/s, of a body when it is x metres from a fixed point O is given by $v = \frac{2x}{\sqrt{1+x^2}}$.

Find an expression for the acceleration of the body in terms of x in simplest form.

13 © VCAA 2009 1Q7 **50%** **TECH-FREE** (4 marks)

A mass has acceleration a m s^{-2} given by $a = v^2 - 3$, where v m s^{-1} is the velocity of the mass when it has a displacement of x metres from the origin.

Find v in terms of x, given that $v = -2$ where $x = 1$.

14 A cyclist rides due north for 2 h at 6 km/h and then due east for 2 h at 8 km/h. The cyclist's average velocity was

A $\sqrt{40}$ km/h, 069°T

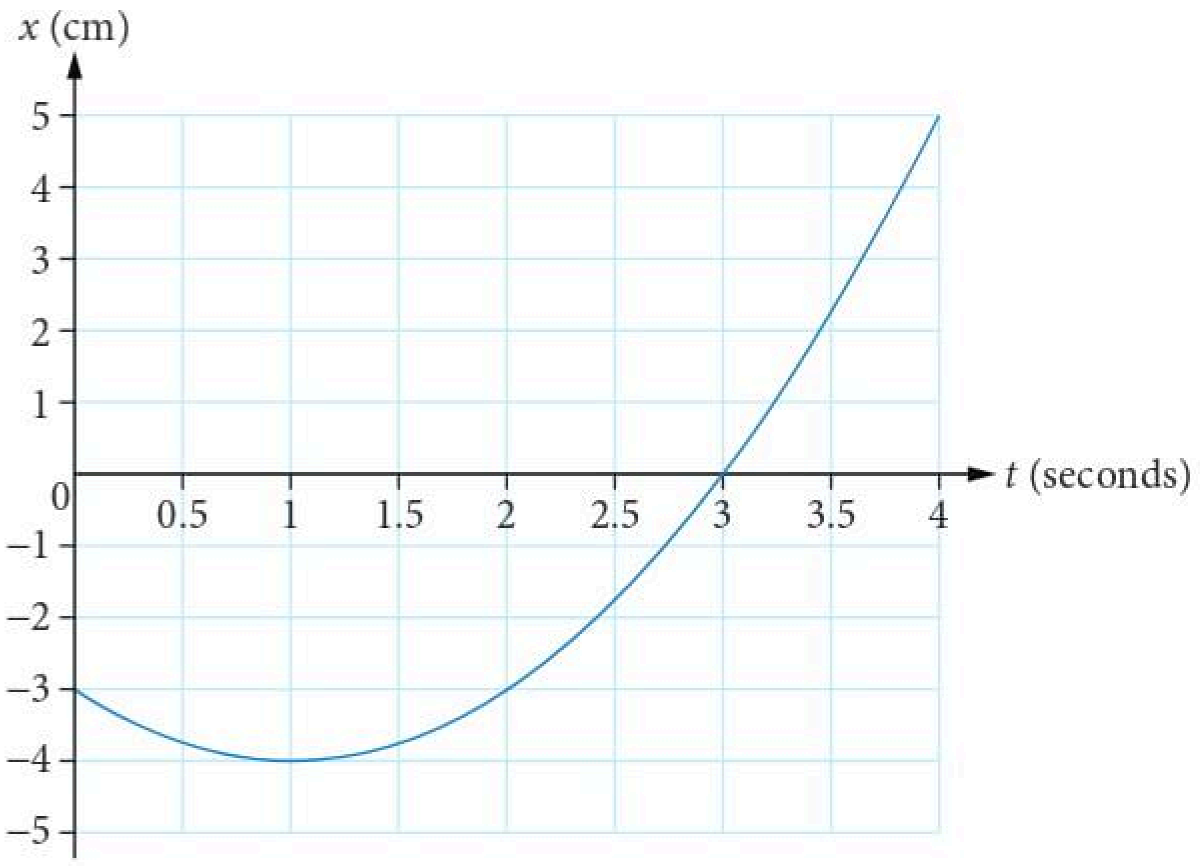
B 20 km/h, 069°T

C –5 km/h, 053°T

D 5 km/h, 037°T

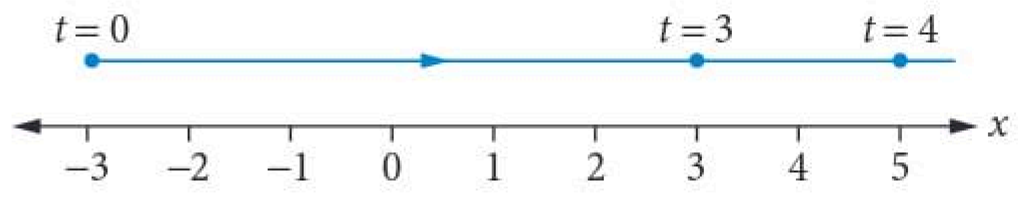
E 5 km/h, 053°T

15 The position of an object in the interval [0, 4] is shown.

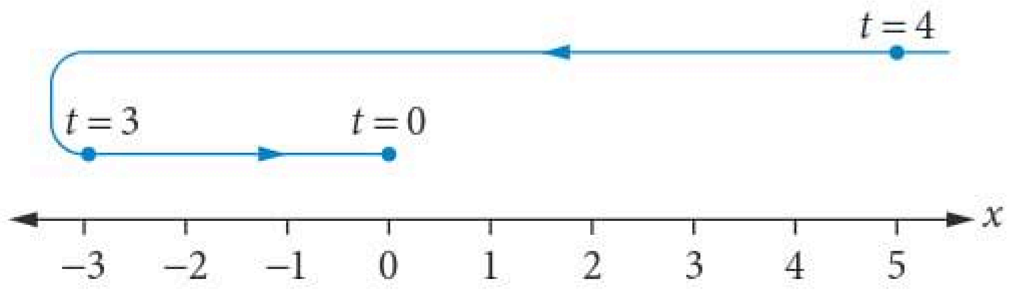


The position–time line for the interval [0, 4] is

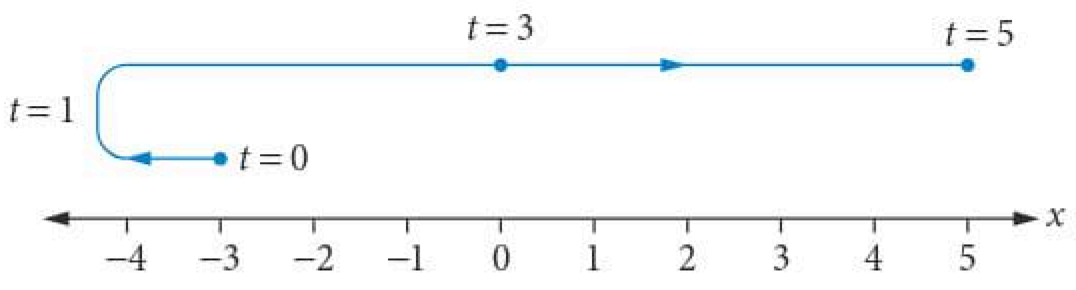
A



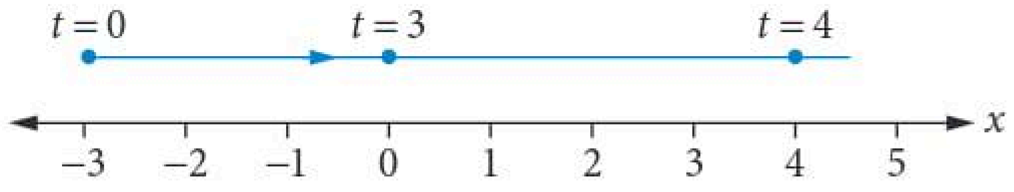
B



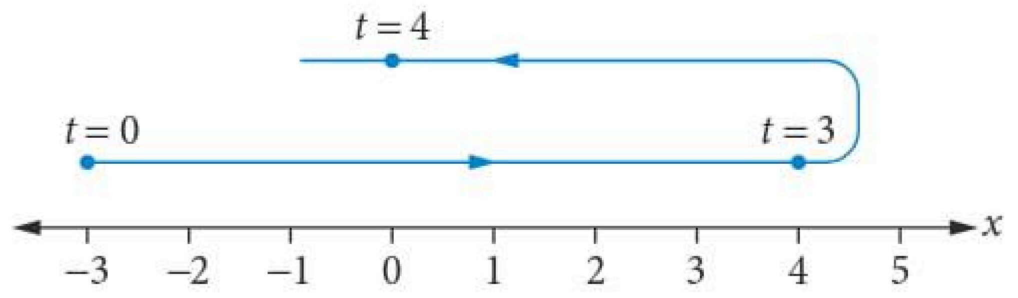
C



D



E



▶ **16** A moving object's motion is described by $x = \log_e(t^2) - t^2$, where x mm is its position in relation to a fixed point and t seconds is time. The object's speed after 0.5 seconds is

A −3 mm/s **B** −2 mm/s **C** 1 mm/s **D** 2 mm/s **E** 3 mm/s

17 © VCAA 2008 2AQ20 70% The velocity, v m s^{-1}, of a body which is moving in a straight line, when it is x m from the origin, is given by $v = \sin^{-1}(x)$.

The acceleration of the body in m s^{-2} is given by

A $-\cos^{-1}(x)$

B $-\dfrac{\cos^{-1}(x)}{\sin^2(x)}$

C $-\cot(x)\operatorname{cosec}^2(x)$

D $\dfrac{1}{\sqrt{1-x^2}\sin(x)}$

E $\dfrac{\sin^{-1}(x)}{\sqrt{1-x^2}}$

18 © VCAA 2010 2AQ20 67% The acceleration, a m s^{-2}, of a particle moving in a straight line is given by $a = \dfrac{v}{\log_e(v)}$, where v is the velocity of the particle in m s^{-1} at time t seconds. The initial velocity of the particle is 5 m s^{-1}.

The velocity of the particle, in terms of t, is given by

A $v = e^{2t}$

B $v = e^{2t} + 4$

C $v = e^{\sqrt{2t}+\log_e(5)}$

D $v = e^{\sqrt{2t+(\log_e 5)^2}}$

E $v = e^{-\sqrt{2t+(\log_e 5)^2}}$

19 © VCAA 2013 2AQ18 65% A particle moves in a straight line such that its acceleration is given by $a = v^2 - 1$, where v is its velocity and x is its displacement from a fixed point.

Given that $v = \sqrt{2}$ when $x = 0$, the velocity v in terms of x is

A $v = \sqrt{2+x}$

B $v = 1 + |x+1|$

C $v = \sqrt{2+x^2}$

D $v = \sqrt{1+(1+x)^2}$

E $v = \sqrt{1+(x-1)^2}$

20 © VCAA 2011 2AQ22 61% A particle moves in a straight line. At time t seconds, where $t \geq 0$, its displacement, x metres from the origin, and its velocity, v metres per second, are such that $v = 25 + x^2$.

If $x = 5$ initially, then t is equal to

A $25x + \dfrac{x^3}{3}$

B $25x + \dfrac{x^3}{3} - \dfrac{500}{3}$

C $\dfrac{1}{5}\tan^{-1}\left(\dfrac{x}{5}\right) + 5$

D $\tan^{-1}\left(\dfrac{x}{5}\right) - \dfrac{\pi}{4}$

E $\dfrac{1}{5}\tan^{-1}\left(\dfrac{x}{5}\right) - \dfrac{\pi}{20}$

21 © VCAA 2004 1IQ27 59% A particle is moving in a straight line in such a way that its displacement, x metres, from a fixed origin at time t seconds is given by $x = 2.5t + 9\cos\left(\dfrac{t}{2}\right)$, $t \geq 0$.

If the velocity of the particle at time t seconds is v metres per second, then the minimum value of v is

A −6.5 **B** −2 **C** 0 **D** 2.5 **E** 7

22 © VCAA 2011 2AQ20 59% A body moves in a straight line such that its velocity, v m s^{-1}, is given by $v = 2\sqrt{1-x^2}$, where x metres is its displacement from the origin.

The acceleration of the body in m s^{-2} is given by

A $\dfrac{-2x}{\sqrt{1-x^2}}$ **B** $-2x$ **C** $\dfrac{2x}{\sqrt{1-x^2}}$ **D** $2(1-2x)$ **E** $-4x$ ▶

23 © VCAA 2014 2AQ21 52% The acceleration, in m s^{-2}, of a particle moving in a straight line is given by $-4x$, where x metres is its displacement from a fixed origin O.

If the particle is at rest where $x = 5$, the speed of the particle, in m s^{-1}, where $x = 3$ is

A 8 **B** $8\sqrt{2}$ **C** 12 **D** $4\sqrt{2}$ **E** $2\sqrt{34}$

24 © VCAA 2008 2AQ22 45% A body moves in a straight line so that at time t, its velocity is v and its acceleration is a, where $a = f(v)$. Given that $v = v_0$ when $t = t_0$, and $v = v_1$ when $t = t_1$, it follows that

A $v_1 = \int_{t_0}^{t_1} f(v)dv + v_0$ **B** $t_1 = \int_{v_0}^{v_1} \frac{1}{f(v)} dv + t_0$ **C** $t_1 = \int_{v_0}^{v_1} \frac{1}{f(v)} + t_0\, dv$

D $t_1 = \int_{v_0}^{v_1} f(v)dv + t_0$ **E** $v_1 = \int_{t_0}^{t_1} \frac{1}{f(v)} dv + v_0$

25 © VCAA 2012 2AQ19 44% A body is moving in a straight line and, after t seconds, it is x m from the origin and travelling at v m s^{-1}.

Given that $v = x$, and that $t = 3$ where $x = -1$, the equation for x in terms of t is

A $x = e^{t-3}$ **B** $x = -e^{3-t}$ **C** $x = \sqrt{2t-5}$ **D** $x = -\sqrt{2t-5}$ **E** $x = -e^{t-3}$

26 © VCAA 2004 1IQ30 42% A particle moves in a straight line. When its displacement from a fixed origin is x m, its velocity is v m/s and its acceleration is a m/s^2.

Given that $a = 16x$, and that $v = -5$ when $x = 0$, the relation between v and x is

A $v = -4x - 5$ **B** $v = 8x^2 - 5$ **C** $v = -\sqrt{25 + 16x^2}$

D $v = \sqrt{25 + 16x^2}$ **E** $v = -\sqrt{25 + 32x^2}$

27 © VCAA 2003 1IQ28 37% A particle moves with constant acceleration in a straight line so that at time t, $t \geq 0$, its velocity is v and its displacement from a fixed point on the line is x.

Which one of the following equations could **not** be true?

A $t = v - 1$ **B** $t = x^2 - 1$ **C** $x = t^2 - 1$ **D** $x = v^2 - 1$ **E** $v = t - 1$

28 © VCAA 2006 2AQ12 35% A particle moves in a straight line such that its velocity, v, is given by $v = \sin(2x)$, when at a displacement of x from the origin O.

The acceleration of the particle is given by

A $2\cos(2x)$ **B** $\sin(2x)\cos(2x)$ **C** $-\frac{1}{2}\cos(2x)$ **D** $\sin(4x)$ **E** $2\cos(x)$

29 © VCAA 2003 1IQ29 30% At time t s, $t \geq 0$, the velocity, v m/s, of a particle moving in a straight line is given by $v = \cos(t) + \sqrt{3}\sin(t) - 1$. For what value of t does the particle first attain its maximum **speed** of 3 m/s?

A $\frac{\pi}{6}$ **B** $\frac{\pi}{3}$ **C** $\frac{7\pi}{6}$ **D** $\frac{4\pi}{3}$

E The particle never attains a speed of 3 m/s.

30 © VCAA 2005 1MQ20 25% A particle travels in a straight line with velocity v at time t.

If the velocity of the particle is given by $v = \frac{2}{\sqrt{1-x^2}}$, for $0 < x < 1$, then the acceleration is given by

A $2\sin^{-1}(x)$ **B** $\frac{4\sin^{-1}(x)}{\sqrt{1-x^2}}$ **C** $\frac{4x}{\left(1-x^2\right)^2}$ **D** $\frac{2x}{\left(1-x^2\right)^2}$ **E** $\frac{2x}{\left(1-x^2\right)^{\frac{3}{2}}}$

31 © VCAA 2017N 2AQ17 The acceleration, a m s^{-2}, of a particle moving in a straight line is given by $a = v^2 + 1$, where v is the velocity of the particle at any time t. The initial velocity of the particle when at origin O is 2 m s^{-1}.

The displacement of the particle from O when its velocity is 3 m s^{-1} is

A $\log_e(2)$ **B** $\frac{1}{2}\log_e\left(\frac{10}{3}\right)$ **C** $\frac{1}{2}\log_e(2)$ **D** $\frac{1}{2}\log_e\left(\frac{5}{2}\right)$ **E** $\log_e\left(\frac{4}{5}\right)$

32 © VCAA 2012 2BQ3 (11 marks) A car accelerates from rest. Its speed after T seconds is V m s^{-1}, where

$$V = 17\tan^{-1}\left(\frac{\pi T}{6}\right),\ T \geq 0$$

a 58% Write down the limiting speed of the car as $T \to \infty$. 1 mark

b 82% Calculate, correct to the nearest 0.1 m s^{-2}, the acceleration of the car when $T = 10$. 1 mark

c 75% Calculate, correct to the nearest second, the time it takes for the car to accelerate from rest to 25 m s^{-1}. 2 marks

After accelerating to 25 m s^{-1}, the car stays at this speed for 120 seconds and then begins to decelerate while braking. The speed of the car t seconds after the brakes are first applied is v m s^{-1}, where $\frac{dv}{dt} = \frac{1}{100}(145 - 2t)$, until the car comes to rest.

d **i** 69% Find v in terms of t. 2 marks

ii 68% Find the time, in seconds, taken for the car to come to rest while braking. 2 marks

e **i** 44% Write down the expressions for the distance travelled by the car during each of the three stages of its motion. 2 marks

ii 30% Find the total distance travelled; from when the car starts to accelerate to when it comes to rest.

Give your answer in metres, correct to the nearest metre. 1 mark

Video playlist
Velocity-time graphs

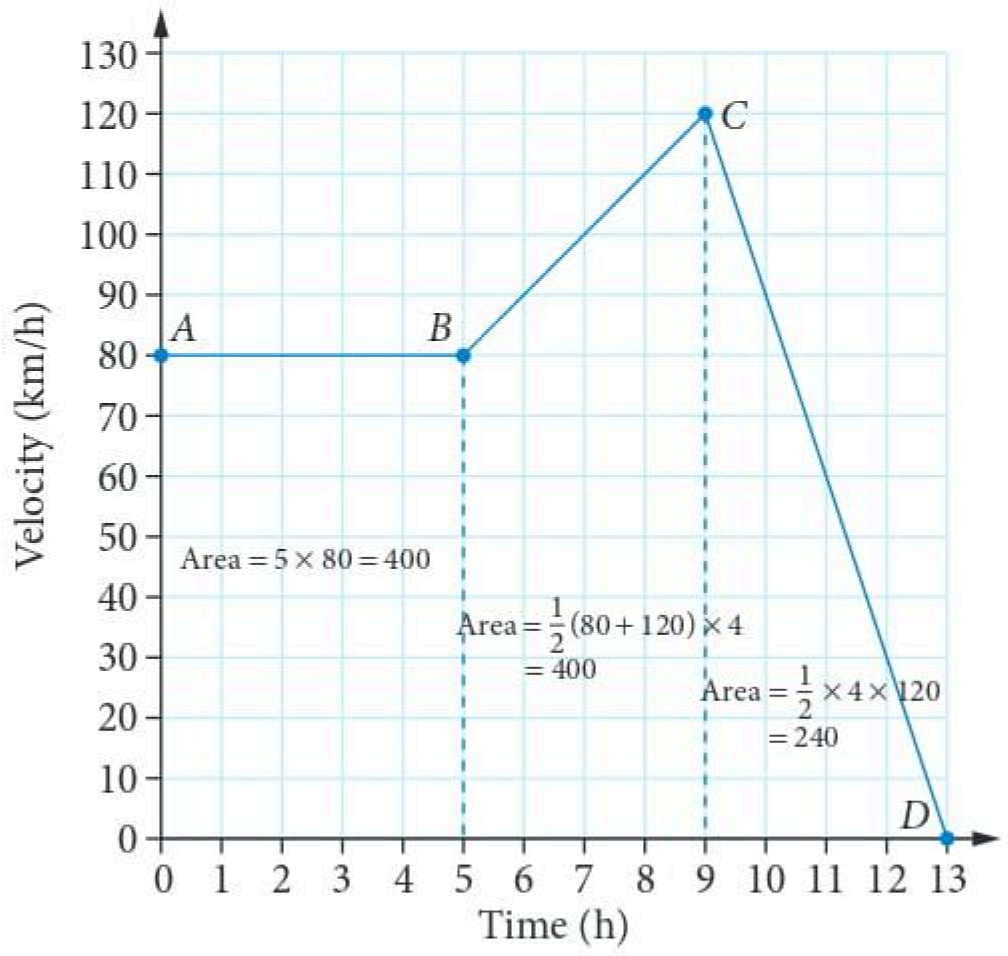
10.2 Velocity–time graphs

A **velocity–time graph** provides a visual representation of straight-line motion by showing the velocity of an object at different times.

Constant acceleration

If there is a constant acceleration in given time intervals, then acceleration is $\frac{\text{change in velocity}}{\text{change in time}}$. We write $a = \frac{v_2 - v_1}{t_2 - t_1}$, where a is acceleration, $v_2 - v_1$ is the change in velocity and $t_2 - t_1$ is the change in time.

In the velocity–time graph shown, the horizontal line from A to B has zero gradient and represents a constant velocity of 80 km/h. Since there is no change in the velocity as the object moves from A to B, its acceleration in this section is zero. The line from B to C has gradient 10, so there is a constant increase in velocity of 10 km/h each hour. The acceleration is 10 km/h per hour, or 10 km h^{-2}. Line CD has gradient –30 and indicates that the velocity is uniformly decreasing 30 km/h each hour. The acceleration is –30 km/h per hour, or –30 km h^{-2}.



Area under a velocity–time graph

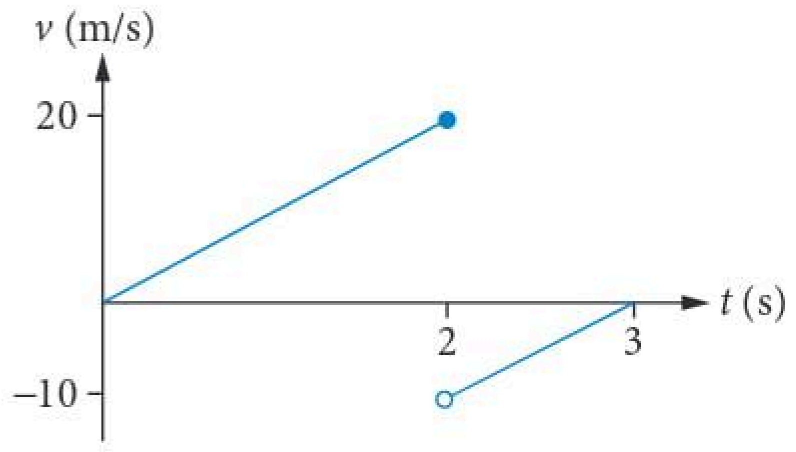
The total area under the velocity–time graph represents the distance travelled, and the signed area represents displacement.

The area under the graph on the previous page is 1040 square units, which represents a distance of 1040 km.

WB
p. 205

WORKED EXAMPLE 7 Velocity–time graph 1

A ball 20 metres above the ground begins to fall. It hits the ground after 2 seconds and rebounds with half of its impact speed. Taking the gravitational acceleration as 10 m/s^2, show the information as a velocity–time graph for $0 \le t \le 3$.

Steps	**Working**
1 Take 'down' to be positive velocity. Determine the velocity on impact.	The acceleration is constant and is the change in velocity, v, with respect to time. The initial velocity is zero. Substitute into $a = \frac{v_2 - v_1}{t_2 - t_1}$. The impact velocity is $10 = \frac{v}{2}$, so $v = 20$ m/s. The first section of the graph is a straight line starting at the origin that has gradient 10 (the acceleration).
2 Work out the time taken to reach maximum height after the rebound. Note that the velocity is now negative.	The rebound velocity is $-\frac{20}{2} = -10$ m/s. The acceleration is 10 m/s^2, so the time taken is $t = \frac{v}{a}$ $= \frac{10}{10} = 1$ s At maximum height, the velocity is zero, so the second section of the graph is a straight line starting at (2, −10) and finishing at (3, 0).
3 Show this information on the graph.	

WB

p. 206

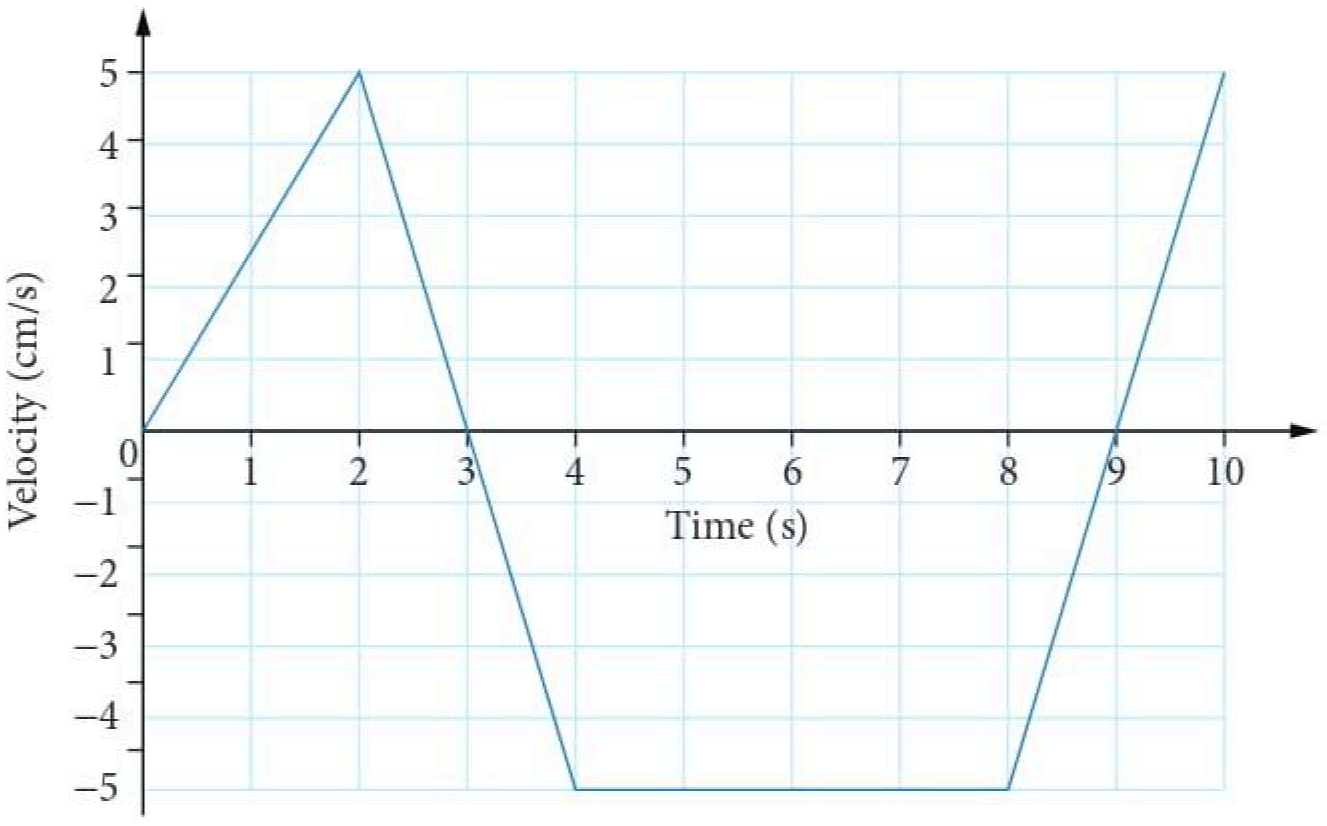
WORKED EXAMPLE 8 Velocity–time graph 2

A ball starting from rest uniformly accelerates and takes 2 seconds to reach a velocity of 5 cm/s. It takes another 2 seconds to change its velocity from 5 cm/s to –5 cm/s, which it maintains for a further 4 seconds. The ball takes another 2 seconds to uniformly change its velocity from –5 cm/s to 5 cm/s.

a Show the velocity–time graph that describes the motion of the ball.

b Find the total distance travelled.

c Determine the ball's displacement.

Steps	**Working**
a **1** Determine the shape of the velocity–time graph for each section.	A uniform increase is shown as a straight line with positive gradient. The ball starts at (0, 0) and finishes at (2, 5). A uniform decrease over 2 seconds is a line with negative gradient starting at (2, 5) and finishing at (4, –5). A constant speed of –5 cm/s for 4 seconds is represented by a horizontal line from 4 to 8 seconds starting at –5 on the vertical axis. For a uniform increase during the last 2 seconds, use a straight line with positive gradient starting at (8, –5) and finishing at (10, 5).
2 Sketch the velocity–time graph.	
b **1** Calculate the area between the graph and the horizontal axis.	Area from $t = 0$ to $t = 3$ is $\frac{1}{2} \times 3 \times 5 = 7.5$. Area from $t = 3$ to $t = 9$ is $\frac{1}{2}(4 + 6) \times 5 = 25$. Area from $t = 9$ to $t = 10$ is $\frac{1}{2} \times 1 \times 5 = 2.5$. Total area is 35 square units.
2 State the distance travelled by the ball.	The ball's total distance was 35 cm.
c Calculate the signed area between the graph and the horizontal axis.	Signed area above is $7.5 + 2.5 = 10$. Signed area below is -25. Displacement is $10 + (-25) = -15$ cm.

WB
p. 207

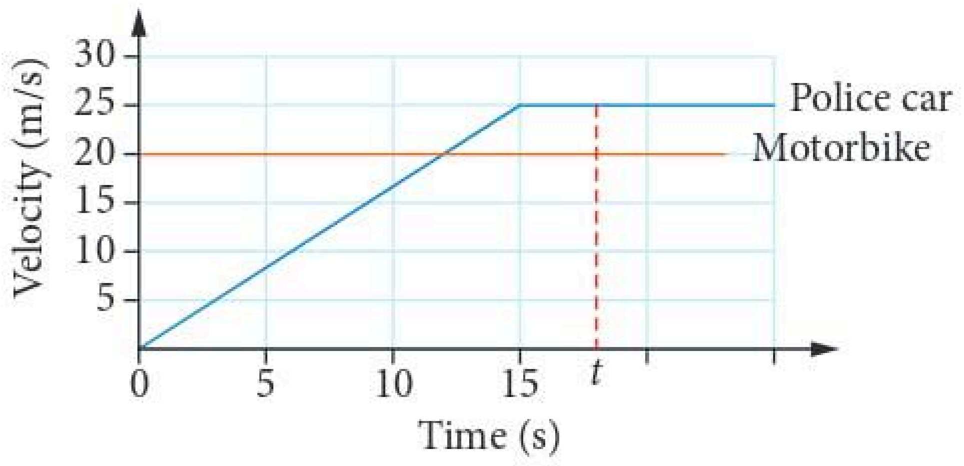
WORKED EXAMPLE 9 Velocity–time graph of two moving objects

A motorbike passes a police station at a speed of 72 km/h, which is maintained. A police car immediately leaves the station in pursuit and uniformly accelerates to reach a speed of 90 km/h in 15 seconds, which it maintains until it catches up to the bike adjacent to a park.

a Show the information as a velocity–time graph.

b Determine how long the motorbike and police car take to reach the park.

c How far is the park from the police station?

Steps	**Working**
a **1** Convert units from km/h to m/s. **Exam hack** To convert m/s to km/h, multiply by 3.6 and to convert km/h to m/s, divide by 3.6.	72 km/h is $\frac{72}{3.6} = 20$ m/s. 90 km/h is $\frac{90}{3.6} = 25$ m/s.
2 Determine the shape of the graph for the bike.	A constant speed of 20 m/s means a horizontal line starting at 20 on the vertical axis.
3 Determine the shape of the graph for the police car.	A uniform increase in 15 seconds is a straight line starting at the origin that has positive gradient of $\frac{25}{15} = \frac{5}{3}$. It finishes at (15, 25), at which point a horizontal line represents a constant speed.
4 Sketch the velocity–time graph.	Let t be the time at which the police car catches up to the bike. 
b **1** Calculate the area under the graph for the bike and police car.	Find the area under each graph from 0 s to t s. bike: $20t$ (rectangle) car: $\frac{1}{2}(2t - 15) \times 25$ (trapezium)
2 Find the value of t by equating and solving the two expressions for area.	The police car catches up to the bike when the two areas are the same. $20t = \frac{1}{2}(2t - 15) \times 25$ $t = 37.5$ s The police car and the motorbike reach the park after 37.5 s.
c Use the value of t found to calculate the distance required.	$20t$ and $\frac{1}{2}(2t - 15) \times 25$ represent the distance travelled by the bike and police, respectively. Substitute $t = 37.5$ into either expression to get 750 m.

Variable acceleration

For non-linear curves describing velocity as a function of time, the area under the graph can be found by integration using calculus by hand or by technology.

Displacement as an integral

For the velocity function $v(t)$ in the interval $[a, b]$, displacement is found using $\int_a^b v(t)\,dt$.

WB

p. 208

WORKED EXAMPLE 10 Terminal velocity of a falling object

A body initially at rest begins to fall vertically with acceleration $a = g - kv$, where v m s^{-1} is its velocity, $g = 9.8$ is a constant and k is another constant.

a Express the velocity, v, as a function of time, t.

b Find the value of the constant, k, correct to two decimal places given that $v(10) = 62$.

c State the **terminal velocity**, giving your answer correct to the nearest integer.

d How far (correct to one decimal place) has the object fallen in the time interval [2, 5]?

Steps	Working
a **1** Write the acceleration in the form $\frac{dv}{dt}$.	$\frac{dv}{dt} = g - kv$
2 Use separation of variables and integrate, including the constant of integration.	$\int \frac{1}{g - kv}\frac{dv}{dt}\,dt = \int 1\,dt$ $-\frac{1}{k}\int \frac{-k}{g - kv}\,dv = \int 1\,dt$ $-\frac{1}{k}\log_e\lvert g - kv\rvert = t + c$
3 Determine the value of the constant of integration.	$t = 0, v = 0$ $c = -\frac{1}{k}\log_e g$ $-\frac{1}{k}\log_e\lvert g - kv\rvert = t - \frac{1}{k}\log_e g$
4 Write the velocity in terms of time.	$\frac{1}{k}\log_e\left\lvert\frac{g}{g - kv}\right\rvert = t$ $\frac{g - kv}{g} = e^{-kt}$ $v = \frac{g}{k}(1 - e^{-kt})$
b Find the other constant from the information given.	$t = 10, v = 62$ $62 = \frac{g}{k}(1 - e^{-10k})$ Solve for k. $k = 0.10$, to two decimal places.
c Terminal velocity is the value that v approaches as t approaches infinity.	$v = \frac{g}{k}(1 - e^{-kt})$ As $t \to \infty$, $v \to \frac{g}{k}(1 - 0) = \frac{g}{k}$ The terminal velocity is $\frac{g}{k} = \frac{9.8}{0.10} = 98$ m s^{-1}.
d **1** Distance fallen is the integral of the velocity function in the given interval.	$\int_2^5 \frac{g}{k}(1 - e^{-kt})\,dt = \int_2^5 98(1 - e^{-0.1t})\,dt$
2 State the answer to the required accuracy.	The body has fallen 280.7 metres.

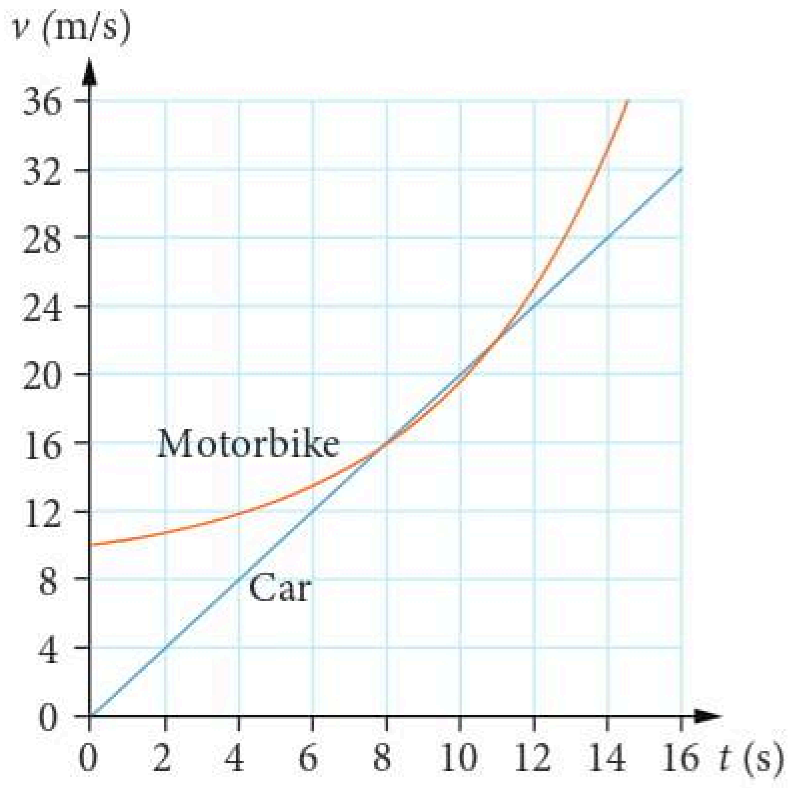
WORKED EXAMPLE 11 Velocity–time graph of car and motorbike

A car is parked 50 metres from a blind corner at the start of a long straight road. The car starts moving away from it just as a motorbike enters the straight at the corner. The speed of the motorbike is given by $v = 8.5 + 1.5e^{0.2t}$ m/s and the speed of the car is $v = 2t$ m/s.

a Sketch a velocity–time graph for $t = 0$ to 15 seconds.

b Use CAS to find when and where the motorbike passes the car, correct to the nearest metre.

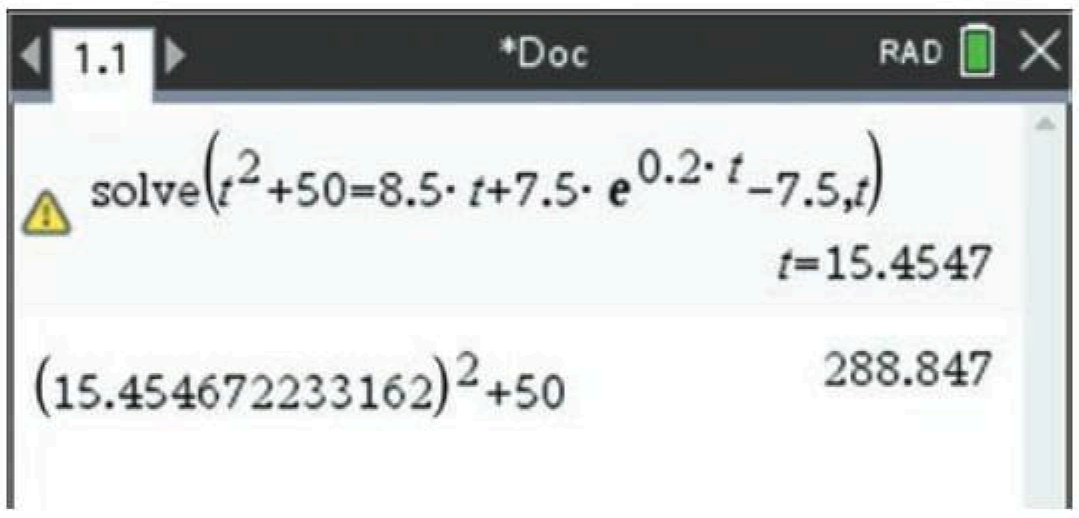
WB p. 210

Steps	**Working**
a 1 State nature of car v–t graph.	Car graph is a straight line through (0, 0), (10, 20).
2 State nature of motorbike graph.	Bike graph is an exponential curve through (0, 10), (5, 12.577…), (10, 19.583…)
3 Sketch the graphs.	
b 1 Find the distance of the car from the corner.	$x_{\text{car}} = \int 2t\ dt = t^2 + c$ When $t = 0$, $x = 50$, so $c = 50$ $x_{\text{car}} = t^2 + 50$
2 Find the distance of the motor bike from the corner.	$x = \int 8.5 + 1.5e^{0.2t}\ dt = 8.5t + 7.5e^{0.2t} + c$ When $t = 0$, $x = 0$, so $c = -7.5$ $x_{\text{bike}} = 8.5t + 7.5e^{0.2t} - 7.5$

3 Use CAS to solve the two equations simultaneously:

car distance = motor bike distance

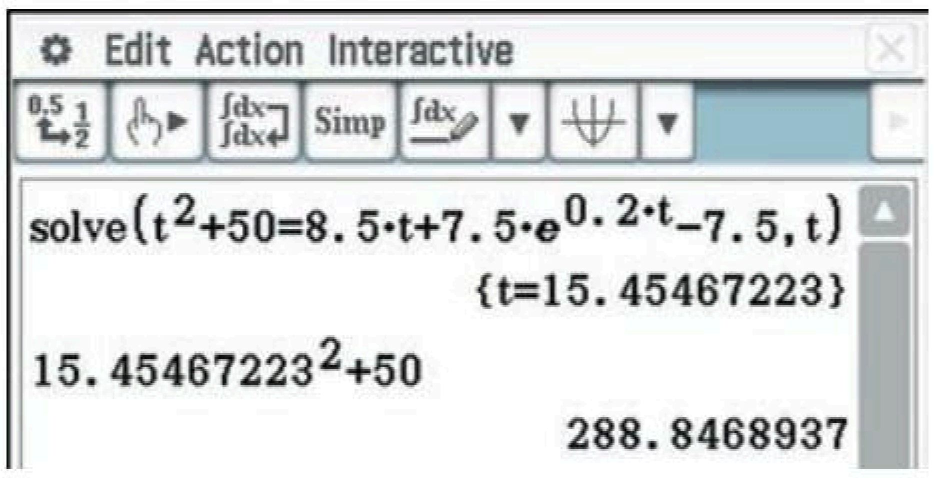
TI-Nspire



$\text{solve}(t^2+50=8.5\cdot t+7.5\cdot e^{0.2\cdot t}-7.5,t)$ $t=15.4547$

$(15.454672233162)^2+50$ 288.847

ClassPad



$\text{solve}(t^2+50=8.5\cdot t+7.5\cdot e^{0.2\cdot t}-7.5,t)$ $\{t=15.45467223\}$

15.45467223^2+50 288.8468937

4 Write the time.	$t \approx 15.5$ s
5 Find the distance.	$x = 15.5^2 + 50 \approx 290$ m
6 Write the answer.	The motorbike passes the car about 15.5 seconds after entering the straight road, about 290 metres down the road.

Velocity, acceleration and the area under the graph

- Acceleration a is the rate of change of velocity with common units m/s^2, cm/s^2 and km/h^2.
- The gradient at a given point on a velocity–time graph is the acceleration.
- The total area under the velocity–time graph is the distance.
- The total signed area under the velocity–time graph is the displacement.
- Constant velocity means zero acceleration.

EXERCISE 10.2 Velocity–time graphs

ANSWERS p. 597

Recap

80–100% 60–79% 0–59%

1 © VCAA 2009 2AQ20 62% A body is moving in a straight line. Its velocity v m s^{-1} is given by $v = x$ when it is x metres from the origin at time t seconds.

Given $x = 1$ when $t = 3$, the rule relating x to t is given by

A $x = \frac{1}{3}t$ **B** $x = \log_e(t - 3)$ **C** $x = \log_e(t) - 3$

D $x = e^{t-3} + 1$ **E** $x = e^{t-3}$

2 The velocity, v m/s at time t seconds, of a particle starting from the origin is $v(t) = 6t^2 + 1$.

What is the position of the particle when its acceleration is 6 m/s^2?

A –0.25 m **B** 0.5 m **C** 0.75 m

D 1.25 m **E** 1.5 m

Mastery

3 WORKED EXAMPLES 7, 8 TECH-FREE A toy car's initial velocity is 6 cm/s. It maintains this for 2 seconds and then uniformly reduces its velocity to –6 cm/s in 2 seconds. The car then takes 1 second to uniformly increase its velocity to zero and takes 2 more seconds to uniformly reach a velocity of 4 cm/s before taking 1 second to reduce its velocity to zero at a constant rate.

a Show the velocity–time graph that describes the motion of the toy car.

b Find the total distance travelled.

c Determine the toy car's displacement.

4 WORKED EXAMPLE 9 TECH-FREE Two joggers begin a friendly race. The first jogger uniformly increases his speed to 7 m/s in 50 seconds and maintains this speed for the rest of the race. The second jogger takes 20 seconds to uniformly increase her speed to 6 m/s and then maintains this speed for the rest of the race, which ends as a dead heat.

a Show the information as a velocity–time graph.

b Determine how long it took to finish the race.

c Find the length of the race.

5 WORKED EXAMPLE 10 A body falls vertically from rest with acceleration $a = g - v^2$, where $g = 9.8$.

- **a** Express the velocity, v, as a function of time, t in the form $v = A\left(\dfrac{e^{2At} - B}{e^{2At} + B}\right)$, where A and B are constants.
- **b** Find the terminal velocity, correct to two decimal places.
- **c** Calculate, correct to two decimal places, the distance the object has fallen in the first 20 seconds.

6 WORKED EXAMPLE 11 Two mice leave their mouse hole at the same time. The velocity of the first mouse is given by $v = 1 - e^{-t}$ m/s and the velocity of the second mouse is $v = \sin(t)$ m/s.

- **a** Show the motion of the two mice as a velocity–time graph in the interval [0, 3].
- **b** When the two mice have the same velocity, which one has travelled further, and by how much? Give your answer correct to the nearest centimetre.

Exam practice

80–100% 60–79% 0–59%

7 © VCAA 2020 2AQ3 68% A train is travelling from Station A to Station B. The train starts from rest at Station A and travels with constant acceleration for 30 seconds until it reaches a velocity of 10 m s^{-1}. It then travels at this velocity for 200 seconds. The train then slows down, with constant acceleration, and stops at Station B having travelled for 260 seconds in total. Let v m s^{-1} be the velocity of the train at time t seconds. The velocity v as a function of t is given by

A $v(t) = \begin{cases} \frac{1}{3}t & 0 \le t \le 30 \\ 10 & 30 < t \le 230 \\ \frac{1}{3}(260 - t) & 230 < t \le 260 \end{cases}$

B $v(t) = \begin{cases} \frac{1}{3}t & 0 \le t \le 30 \\ 10 & 30 < t \le 230 \\ \frac{1}{3}(230 - t) & 230 < t \le 260 \end{cases}$

C $v(t) = \begin{cases} 3t & 0 \le t \le 30 \\ 10 & 30 < t \le 230 \\ 3(230 - t) & 230 < t \le 260 \end{cases}$

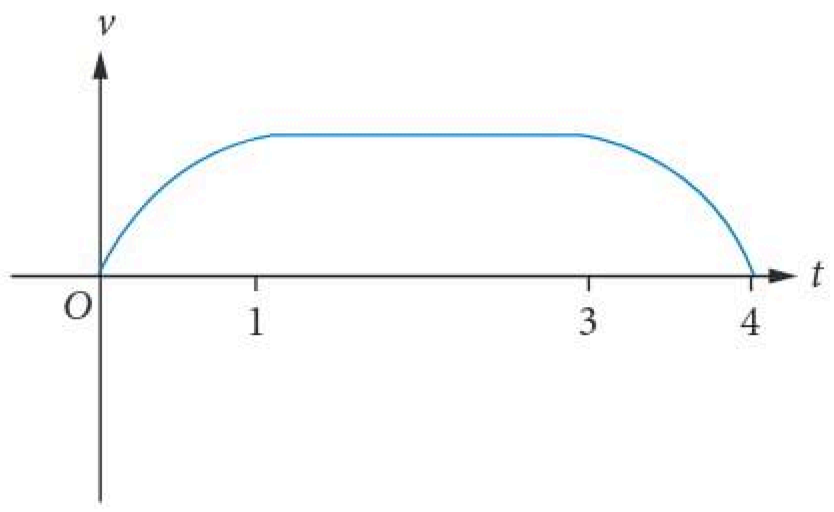
D $v(t) = \begin{cases} 3t & 0 \le t \le 30 \\ 10 & 30 < t \le 230 \\ 3(260 - t) & 230 < t \le 260 \end{cases}$

E $v(t) = \begin{cases} \frac{1}{3}t & 0 \le t \le 30 \\ 10 & 30 < t \le 200 \\ \frac{1}{3}(230 - t) & 200 < t \le 230 \end{cases}$

8 © VCAA 2020 2AQ17 58% The velocity, v m s^{-1}, of a particle at time $t \ge 0$ seconds and at position $x \ge 1$ metres from the origin is $v = \frac{1}{x}$. The acceleration of the particle, in m s^{-2}, when $x = 2$ is

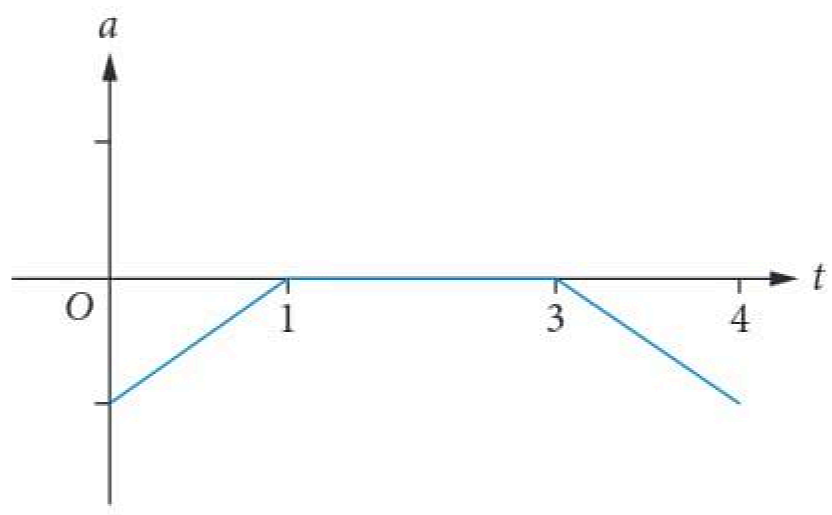
A $-\frac{1}{4}$ **B** $-\frac{1}{8}$ **C** $\frac{1}{8}$ **D** $\frac{1}{2}$ **E** $\frac{1}{4}$

9 © VCAA 2003 1IQ30 58% TECH-FREE The following is the velocity–time graph of a racing car over a short course.

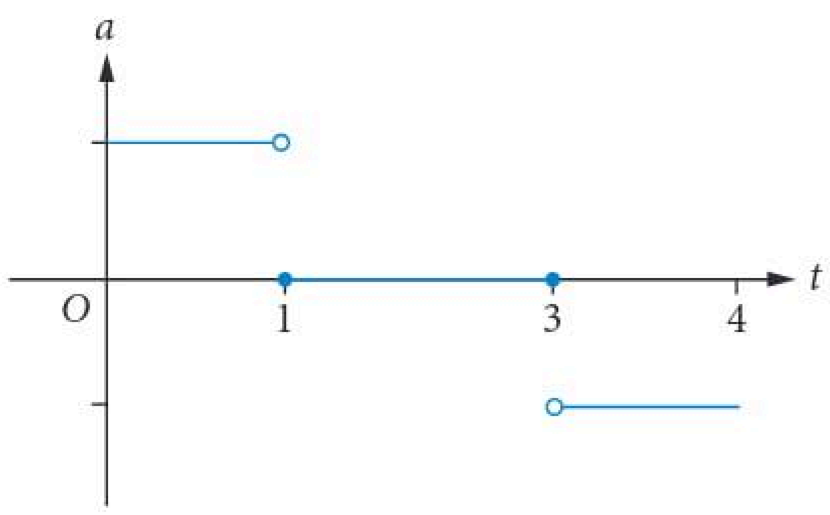


Which one of the following could be the acceleration–time graph of the car's motion?

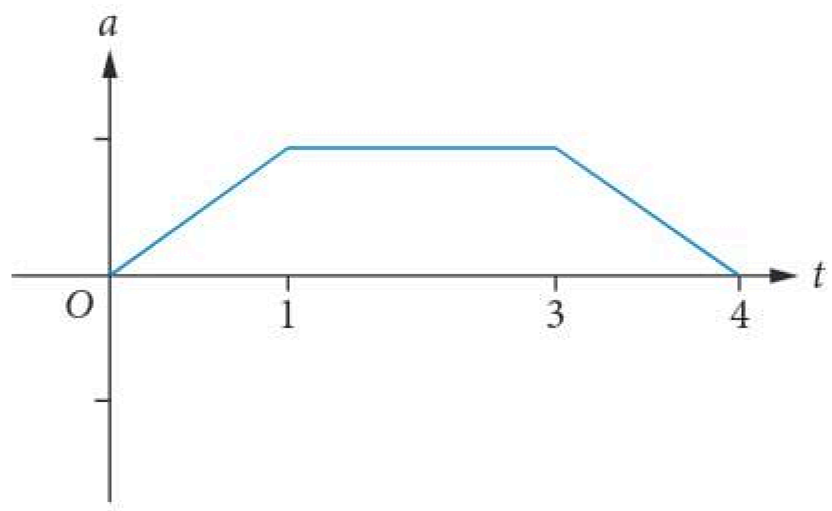
A



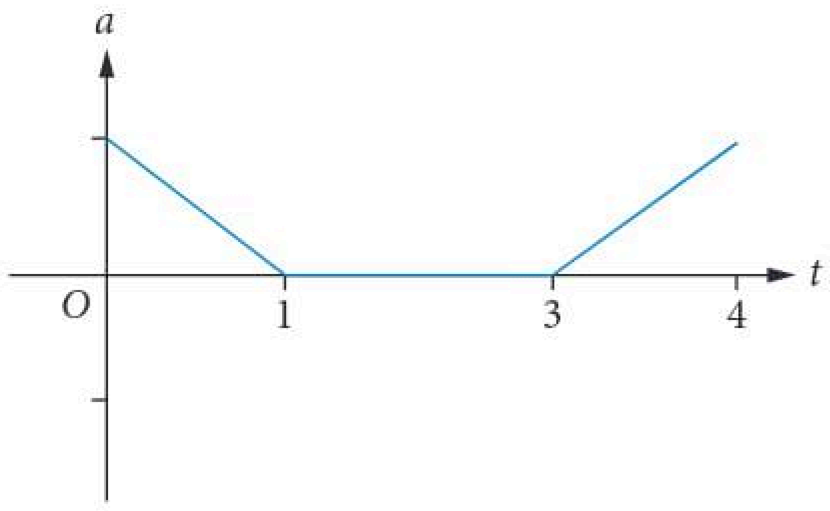
B



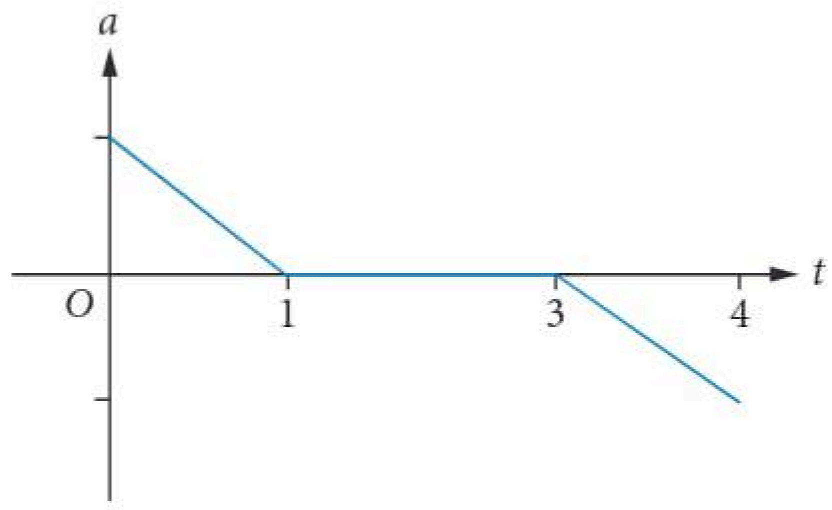
C



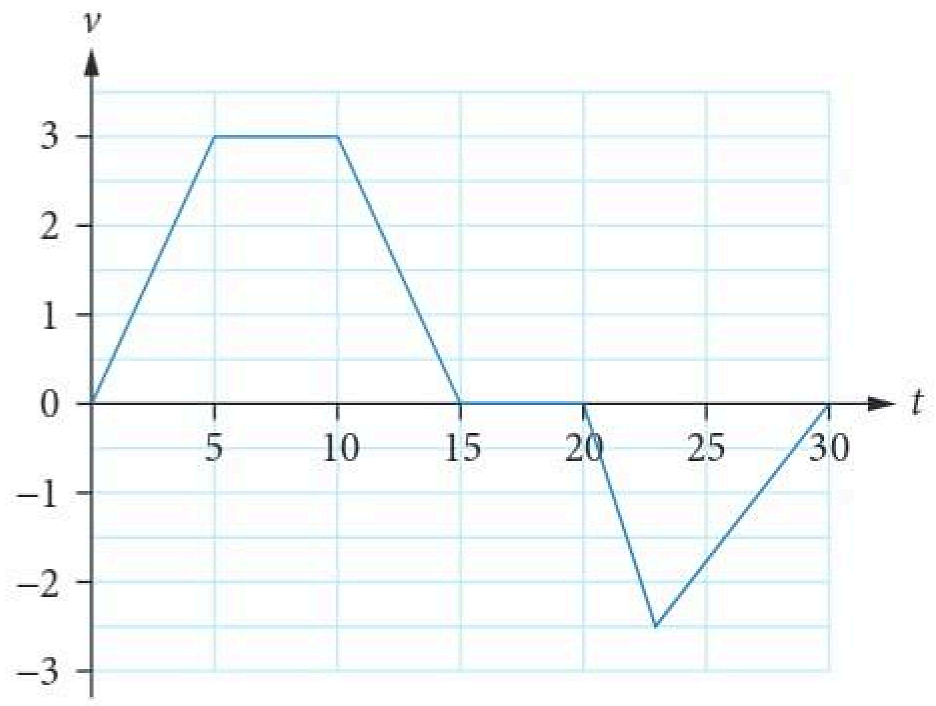
D



E



10 © VCAA 2011 2AQ19 71% The motion of a lift (elevator) in a shopping centre is given by the velocity–time graph below, where time t is in seconds, and the velocity of the lift is v metres per second. For $v > 0$, the lift is moving upwards.



The graph shows that at the end of 30 seconds, the position of the lift is

A 17.5 metres above its starting level.

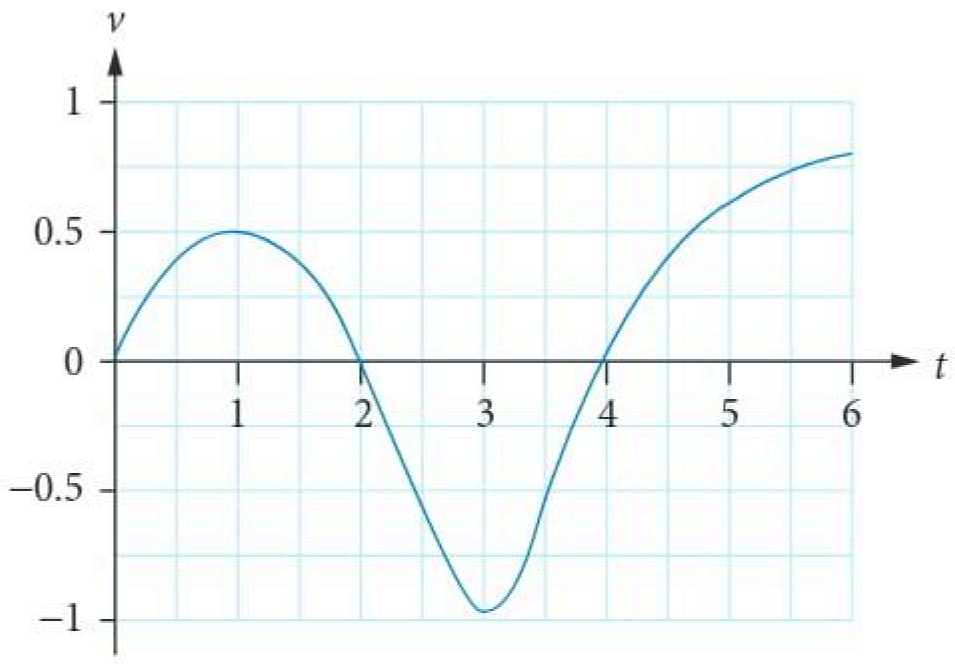
B 5 metres above its starting level.

C at the same position as its starting level.

D 5 metres below its starting level.

E 17.5 metres below its starting level.

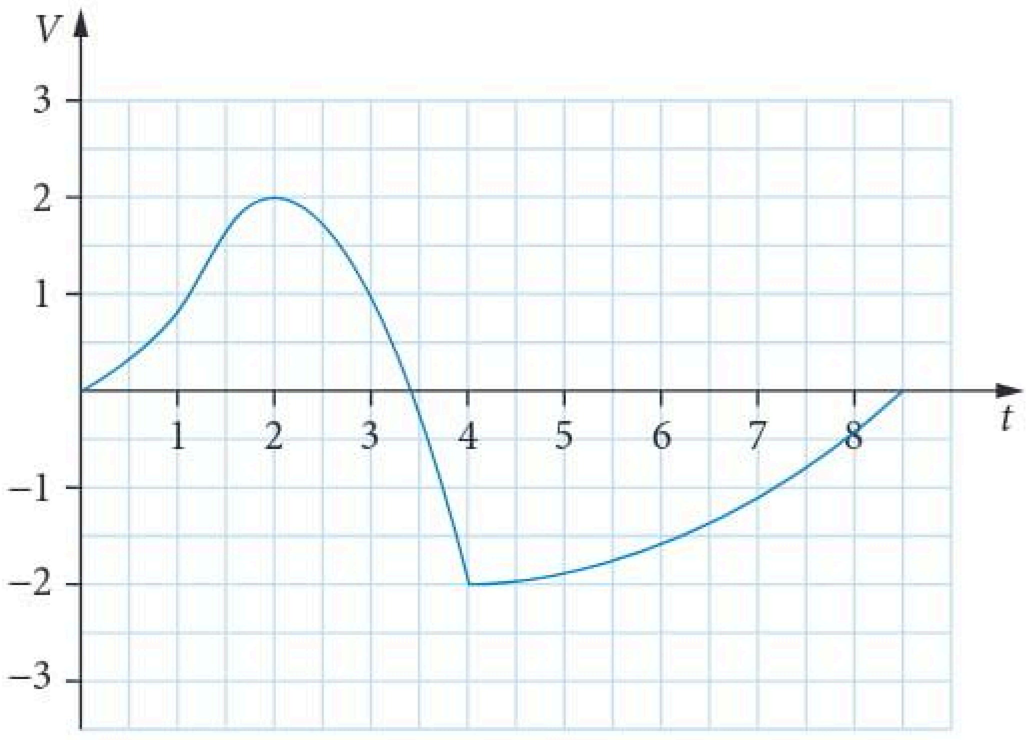
11 © VCAA 2015 2AQ11 66% The velocity–time graph for a body moving along a straight line is shown below.



The body first returns to its initial position within the time interval

A (0, 0.5) **B** (0.5, 1.5) **C** (1.5, 2.5) **D** (2.5, 3.5) **E** (3.5, 5)

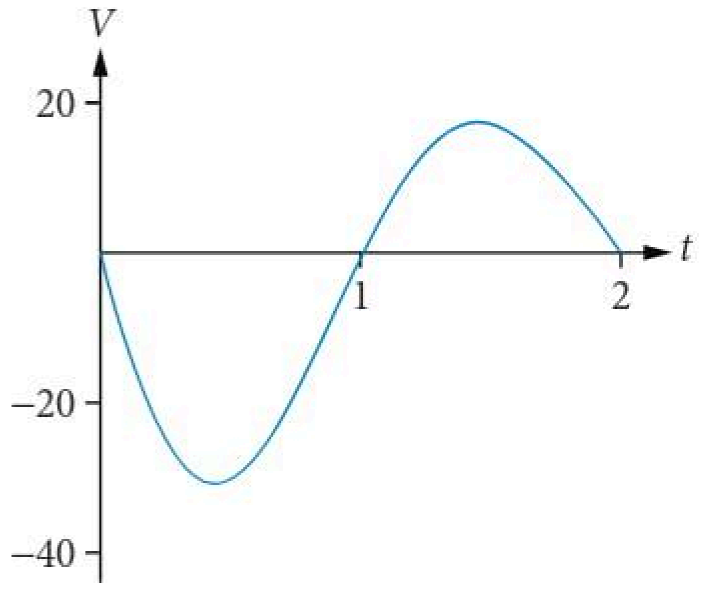
12 The velocity–time graph of a particle moving along a straight line is shown below.



In which interval does the particle return to its initial position?

A (4, 5) **B** (6, 7) **C** (5.5, 6) **D** (6.5, 7.5) **E** (7, 8)

13 © VCAA 2012 2AQ18 60% The velocity–time graph for the first 2 seconds of the motion of a particle that is moving in a straight line with respect to a fixed point is shown below.

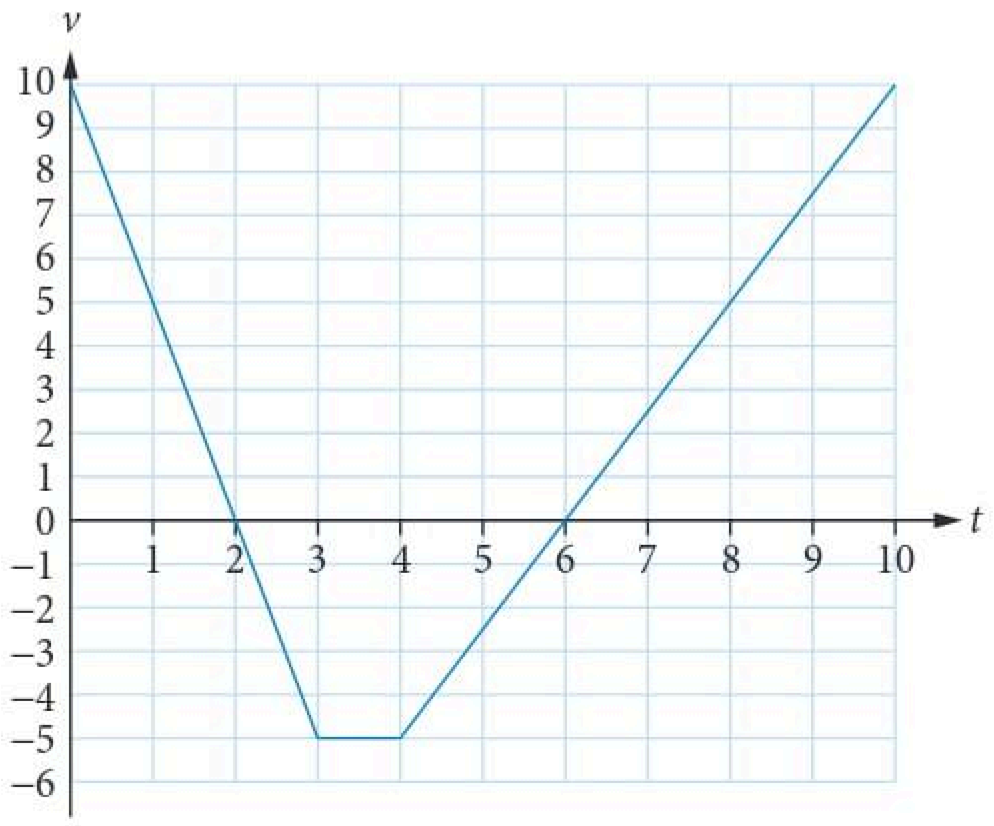


The particle's velocity v is measured in cm/s. Initially, the particle is x_0 cm from the fixed point. The distance travelled by the particle in the first 2 seconds of its motion is given by

A $\int_0^2 v\,dt$ **B** $\int_0^2 v\,dt + x_0$ **C** $\int_1^2 v\,dt - \int_0^1 v\,dt$

D $\left|\int_0^2 v\,dt\right|$ **E** $\int_1^2 v\,dt - \int_0^1 v\,dt + x_0$

14 © VCAA 2009 2AQ22 53% The velocity–time graph below shows the motion of a body travelling in a straight line, where v m s^{-1} is its velocity after t seconds.



After 10 seconds, the distance of the body from its starting point is

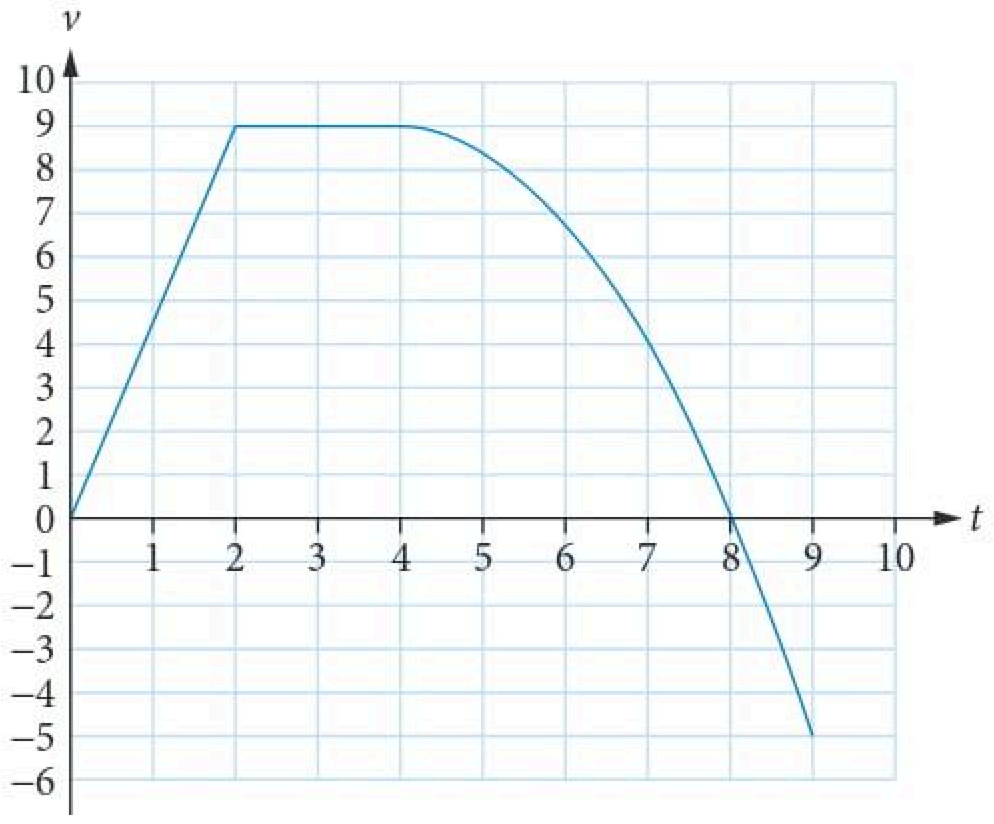
A 10 m **B** 17.5 m **C** 20 m **D** 42.5 m **E** 47.5 m

15 © VCAA 2005 1IQ21 51% A particle travelling in a straight line has velocity v m/s at time t s. Its acceleration is given by $\frac{dv}{dt} = \frac{3}{v^2 - 9}$. The time taken, in seconds, for the velocity to decrease from 2 m/s to 1 m/s is given by

A $\int_1^2 \frac{v^2 - 9}{3} dv$ **B** $\int_2^1 \frac{v^2 - 9}{3} dv$ **C** $\int_2^1 \frac{3}{v^2 - 9} dt$

D $\int_2^1 \frac{3}{v^2 - 9} dv$ **E** $\int_1^2 \frac{3}{v^2 - 9} dv$

16 © VCAA 2014 2AQ22 49% The velocity–time graph below shows the motion of a body travelling in a straight line, where v m s^{-1} is its velocity after t seconds.



The velocity of the body over the time interval $t \in [4,9]$ is given by $v(t) = -\frac{9}{16}(t - 4)^2 + 9$.

The distance, in metres, travelled by the body over nine seconds is closest to

A 45.6 **B** 47.5 **C** 48.6 **D** 51.0 **E** 53.4

17 (6 marks) A body falls vertically from rest with acceleration $a = g - kv^2$, where $g = 9.8$ and k is a constant.

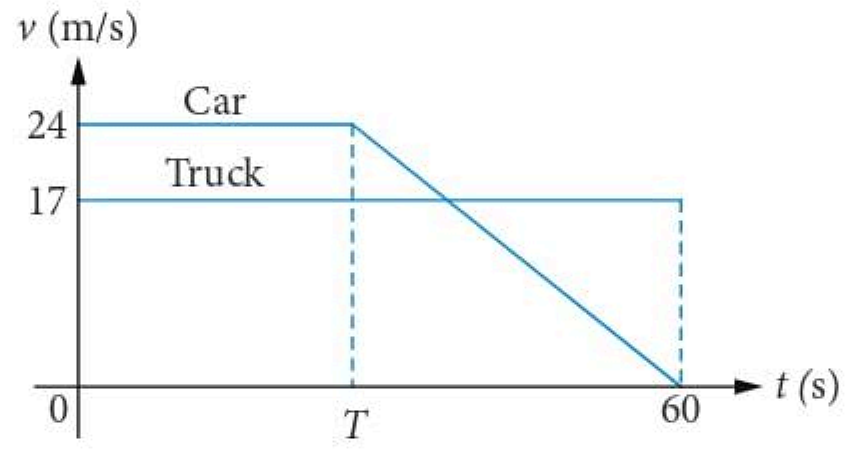
a Express the velocity, v, as a function of distance fallen, x. 2 marks

b Find the value of the constant, k, to one decimal place, given that when the object has fallen 5 m its velocity is 6.5 m s^{-1}. 2 marks

c Determine the velocity as the distance fallen by the object becomes very large. 2 marks

18 © VCAA 2002 1IIQ2 61% (3 marks) A car travelling at 24 m/s overtakes a truck travelling at a constant speed of 17 m/s along a straight road. T seconds later, the car decelerates uniformly to rest. The truck continues at constant speed and it passes the car at the instant that the car comes to a stop. This is exactly 60 seconds after the car had passed the truck. The velocity–time graph representing this situation is shown.

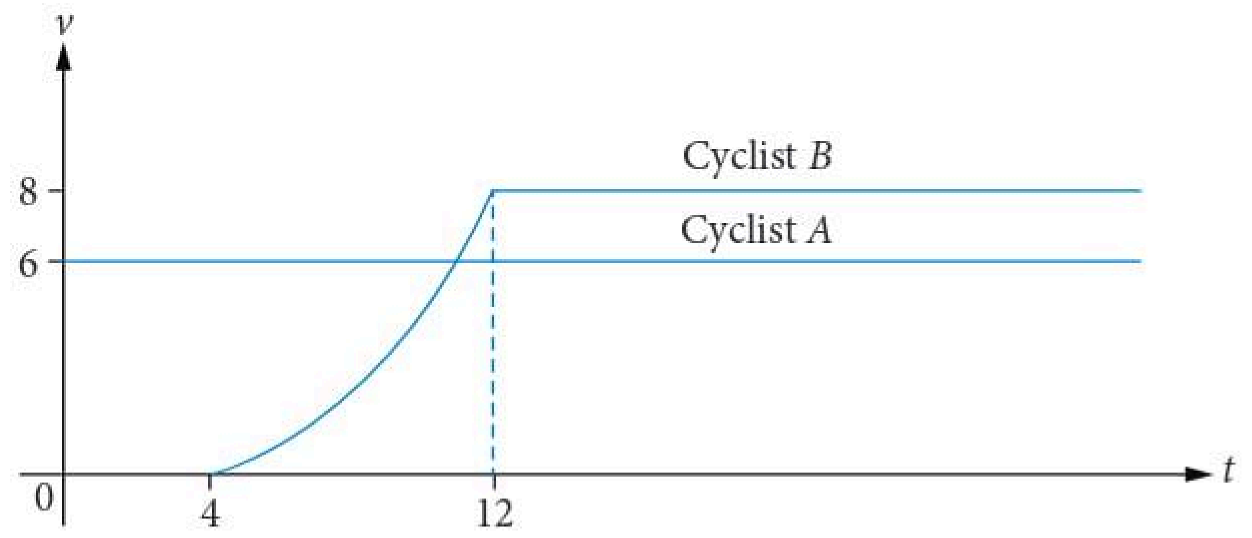
Find T.



19 © VCAA 2005 1IIQ4 (4 marks) At time $t = 0$, cyclist A, travelling at a speed of 6 m/s along a straight bicycle path, passes cyclist B, who is stationary. Four seconds later, at $t = 4$, cyclist B accelerates in the direction of cyclist A for 8 seconds in such a way that her speed, v m/s, is given by $v = (t - 4)\tan\left(\frac{\pi}{48}t\right)$.

a 56% Show that cyclist B accelerates to a speed of 8 m/s. 1 mark

Cyclist B then maintains her speed of 8 m/s. The velocity–time graph that represents this situation is shown.



b 37% Find the time at which cyclist B passes cyclist A, correct to the nearest tenth of a second. 3 marks

20 (2 marks) The velocity, v, of two objects at time t are $v = t\sin(t)$ and $v = \log_e(1 + t)$. Find the value of t when they will have first travel the same distance. Give your answer correct to one decimal place.

21 © VCAA 2007 2BQ5 (10 marks) A car travelling at 20 m s^{-1} passes a stationary police car, and then decelerates so that its velocity, v m s^{-1}, at time t seconds after passing the police car, is given by $v = 20 - 2\tan^{-1}(t)$.

a 70% After how many seconds will the car's speed be 17 m s^{-1}? Give your answer correct to one decimal place. 1 mark

b 31% Explain why v will never equal 16. 1 mark

c 57% Write down a definite integral that gives the distance, x metres, travelled by the car after T seconds. 1 mark

Three seconds later, the police car starts to chase the passing car, which has a polluting exhaust pipe. The police car accelerates so that its velocity, v m s^{-1}, at time t seconds after the polluting car passed it, is given by $v = 13\cos^{-1}\left(\frac{13 - 2t}{7}\right)$ for $t \in [3, 8]$.

d 49% Write down an expression which gives how far the polluting car is ahead of the police car when $t = 8$ seconds.

Find this distance in metres, correct to one decimal place. 3 marks

After accelerating for five seconds the police car continues at a constant velocity.

e 22% At time $t = T_c$ the police car catches the polluting car. Write an equation which, when solved, gives the value of T_c. 3 marks

f 8% Find T_c correct to the nearest second. 1 mark

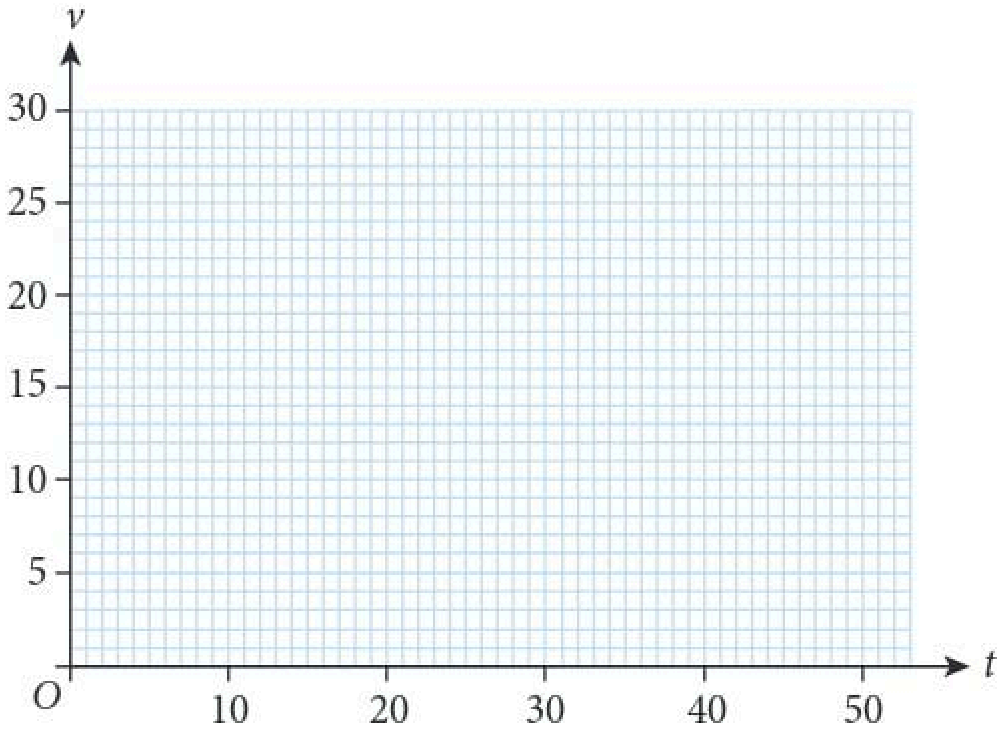
22 © VCAA 2009 2BQ1 (12 marks) A car accelerates from rest at traffic light A to a velocity of 27 m s^{-1} in nine seconds. During this period of acceleration its velocity v m s^{-1} after t seconds, is given by

$$v = t^{\frac{1}{2}} \text{ for } 0 \le t \le 9$$

The car then travels at a constant velocity of 27 m s^{-1} for another thirty seconds, and finally decelerates until it comes to rest at traffic light B. During **deceleration** [negative acceleration], its velocity v m s^{-1} is given by

$$v = 27\cos\left(\frac{\pi}{24}(t-39)\right) \text{ for } 39 \le t \le 51$$

a 77% Copy the axes below, and on them draw a velocity–time graph which shows the motion of the car as it travels from traffic light A to traffic light B. 2 marks



b 78% Calculate the distance travelled by the car during the first nine seconds of its motion. 2 marks

c 75% Calculate, correct to the nearest 0.1 m, the distance travelled by the car while it is decelerating. 2 marks

d 49% Calculate, correct to the nearest 0.1 m s^{-1}, the average speed of the car as it travels from traffic light A to traffic light B. 1 mark

The speed limit on this road is $\frac{200}{9}$ m s^{-1} (80 kilometres per hour).

e 70% Find the time interval $t_1 < t < t_2$ for which the car exceeds the speed limit.

Give your answers for t_1 and t_2 correct to the nearest 0.1 seconds. 2 marks

f 31% Just as the car begins to accelerate away from traffic light A, a motorcycle travelling at a constant 20 ms^{-1} passes the car. Find the time, correct to the nearest 0.1 seconds, and the distance, correct to the nearest metre, for the car to overtake the motorcycle. 3 marks

10.3 The equations of kinematics

Video playlist The equations of kinematics

The **equations of kinematics** are useful relationships between time, distance, velocity and acceleration when the acceleration is constant.

1 $v = u + at$

2 $s = ut + \frac{1}{2}at^2$

3 $v^2 = u^2 + 2as$

4 $s = \frac{1}{2}(u + v)t$

Exam hack

Negative acceleration is also referred to as deceleration or **retardation**. For $a = -2$ m s^{-2}, we say acceleration is -2 m s^{-2}, deceleration is 2 m s^{-2} and retardation is 2 m s^{-2}.

Here, a is acceleration, t is time, s is displacement, u is initial velocity and v is final velocity.

WORKED EXAMPLE 12 Constant retardation

A car reduces its speed from 25 m/s to 20 m/s over a distance of 25 metres. Assuming constant retardation, how long will it take the car to come to a complete stop?

Steps	**Working**
1 List the known values.	$u = 25,\ v = 20,\ s = 25$
2 Decide the formula to use first.	Use $v^2 = u^2 + 2as$ to find a. $20^2 = 25^2 + 2a \times 25 \Rightarrow a = -4.5\ \text{m/s}^2$
3 Find the time for the car's speed to go from 25 m/s to 20 m/s.	$v = u + at$ $20 = 25 - 4.5t \Rightarrow t = 1\frac{1}{9}$ s
4 Find the time for the car's speed to go from 20 m/s to 0 m/s.	$v = u + at$, with $v = 0$ because the car's final speed is 0. $0 = 25 + (-4.5)t \Rightarrow t = 5\frac{5}{9}$ s
5 State the total time taken.	The car takes $1\frac{1}{9} + 5\frac{5}{9} = 6\frac{2}{3}$ s to come to rest.

WB p. 212

WORKED EXAMPLE 13 Motion of two objects moving simultaneously

In a car rally, a Ferrari passes the starting point with speed 216 km/h, which it maintains for the remainder of the rally. Ten seconds later, a Lamborghini starts from rest and uniformly accelerates over a distance of 400 m to reach a speed of 234 km/h, which it then maintains for the remainder of the race.

a Determine when the Lamborghini catches up with the Ferrari.

b The Lamborghini finishes the rally after 85 seconds and then uniformly decelerates and stops in 25 seconds. To the nearest metre, what is its total distance from start to when it comes to a stop?

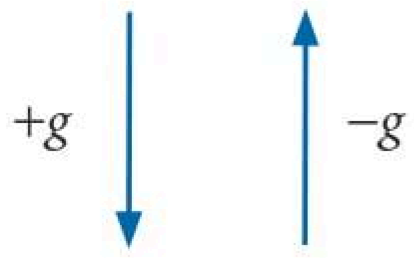
WB p. 213

Steps	**Working**
a 1 Convert km/h to m/s.	$216\ \text{km/h} = \frac{216}{3.6} = 60\ \text{m/s}$ $234\ \text{km/h} = \frac{234}{3.6} = 65\ \text{m/s}$
2 Work out the time taken for the Lamborghini to reach its maximum speed.	Use $s = \frac{1}{2}(u + v)t$ with $s = 400,\ u = 0,\ v = 65$. $400 = \frac{1}{2}(0 + 65)t \Rightarrow t = 12\frac{4}{13}$ s
3 Set up the equation for the distance travelled by each car.	Let T be the time when the two cars meet. Distance travelled by the Ferrari is $60T$. Distance travelled by the Lamborghini is $400 + \left(T - 12\frac{4}{13}\right) \times 65.$
4 The distance is the same when both cars meet.	Solve for T. $60T = 400 + \left(T - 12\frac{4}{13}\right) \times 65$ $T = 80$ The Lamborghini caught up with the Ferrari after 80 seconds.

b **1** Find the length of the rally.	Use $400 + \left(T - 12\frac{4}{13}\right) \times 65$ with $T = 85$ to get 5125 m.
2 Calculate the distance that the Lamborghini travels from its maximum speed to when it stops.	Use $s = \frac{1}{2}(u + v)t$ with $u = 65$, $v = 0$, $t = 25$ to find s. $s = \frac{1}{2}(65 + 0) \times 25 = 812.5\,\text{m}$
3 State the total distance that the Lamborghini travels.	Total distance is the distance of the race plus the distance taken to stop. This is $5125 + 812.5 = 5937.5$ metres. The distance from start to finish is 5938 metres.

Motion involving gravitational acceleration

The Earth's **gravitational acceleration**, g, is a constant approximated to 9.8 m s^{-2}, 9.81 m s^{-2} or 10 m s^{-2}. However, this acceleration may be affected by forces such as **air resistance**. By convention, in the equation of motion we normally use $+g$ for a falling object and $-g$ for upward motion, but we can also define 'up' to be negative motion and 'down' as positive as long as we are consistent within the problem. This aligns with the observation that a falling object increases in speed (positive), whilst an object moving upward slows over time (negative).



Gravitational acceleration is constant, so in the absence of air resistance the equations of kinematics can be used.

WB
p. 214

WORKED EXAMPLE 14 Upward motion under gravity

A cricket ball rises vertically with an initial velocity of 15 m/s. Show that by taking $g = 10$ m/s^2, the time it takes for the ball to reach half its maximum height is $\frac{3}{4}(2 - \sqrt{2})$ seconds.

Steps	**Working**
1 Find the time taken to reach the maximum height. Take upward motion to be positive.	$a = -10$, $u = 15$, $v = 0$ $v = u + at$ $0 = 15 - 10t$ $t = 1.5$ s
2 Calculate the maximum height reached by the ball.	$s = \frac{1}{2}(u + v)t$ $= \frac{1}{2}(15 + 0) \times 1.5$ $= 11.25$ m

3 Determine the time taken to reach half the maximum height.	$a = -10,\ u = 15,\ s = \frac{11.25}{2} = 5.625$ Use $s = ut + \frac{1}{2}at^2$ and solve for t. $5.625 = 15t + \frac{1}{2}(-10)t^2$ $t^2 - 3t + 1.125 = 0$ Solving this quadratic equation gives $\frac{3}{4}(2 - \sqrt{2})$ as the smaller value of the two solutions for t. This represents the time taken on the upward journey. (The second solution is the time taken for the ball to come halfway through its downward journey.)

WORKED EXAMPLE 15 Two bodies moving in opposite direction

WB
p. 215

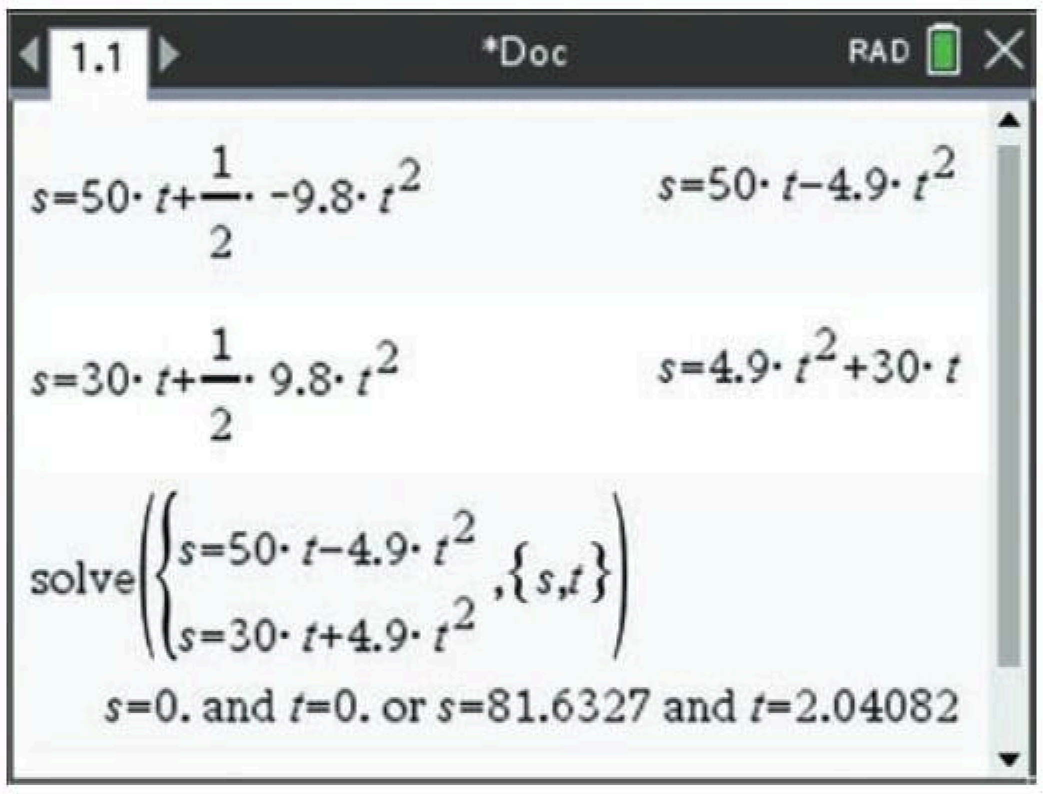
Tara is in a hot air balloon that is rising at 3 m/s. When the balloon is 200 metres above ground, she accidentally drops her bracelet over the side. Neglecting air resistance and using $g = 9.8\ \text{m/s}^2$, find how long it will take for the bracelet to hit the ground. Give your answer correct to two decimal places.

Steps	**Working**
Take upward motion to be positive. Then the acceleration and displacement will be negative. (Alternatively, take downward motion to be positive, with all variables changing sign).	$a = -9.8,\ u = 3,\ s = -200$ $s = ut + \frac{1}{2}at^2$ $-200 = 3t - 4.9t^2$ $4.9t^2 - 3t - 200 = 0$ Solve for t. Take the positive value for t. $t = 6.70221\ ...$ Time taken = 6.70 seconds.

USING CAS 2 Displacement, velocity and gravitational acceleration

Helga releases a toy rocket from ground level that rises vertically with an initial velocity of 50 m/s. At the same time, Quentin, from 100 m above, throws a ball vertically downward with an initial velocity of 30 m/s. The rocket and ball collide. Calculate, using $g = 9.8$, when and where the two objects collide. State your answers correct to two decimal places.

TI-Nspire



1.1 *Doc RAD

$s=50\cdot t+\frac{1}{2}\cdot -9.8\cdot t^2$ $\quad s=50\cdot t-4.9\cdot t^2$

$s=30\cdot t+\frac{1}{2}\cdot 9.8\cdot t^2$ $\quad s=4.9\cdot t^2+30\cdot t$

$\text{solve}\left(\begin{cases} s=50\cdot t-4.9\cdot t^2 \\ s=30\cdot t+4.9\cdot t^2 \end{cases},\{s,t\}\right)$

$s=0.$ and $t=0.$ or $s=81.6327$ and $t=2.04082$

1 Use $s = ut + \frac{1}{2}at^2$ to set up an equation for the rocket when $u = 50$ and $a = -9.8$.

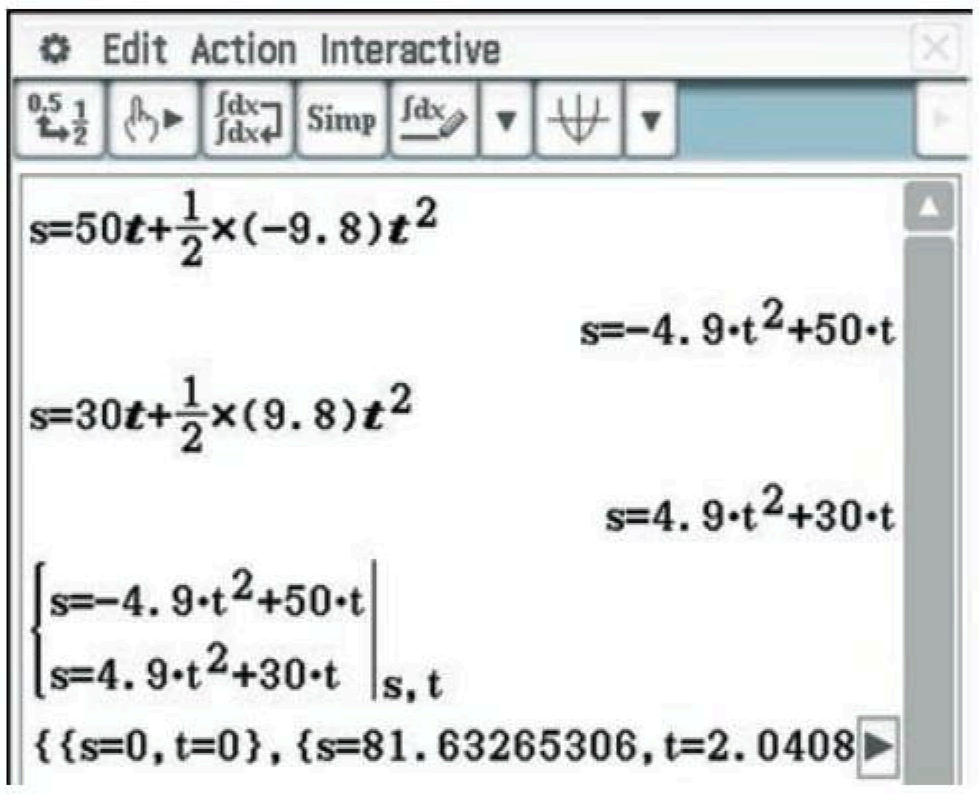
2 Set up a second equation for the ball when $u = 30$ and $a = 9.8$.

3 Press **menu > Algebra > Solve System of Equations > Solve System of Equations**.

4 Select two equations with variables **s** and ***t***.

5 Enter the equations into the template as shown above.

ClassPad



$s=50t+\frac{1}{2}\times(-9.8)t^2$

$s=-4.9\cdot t^2+50\cdot t$

$s=30t+\frac{1}{2}\times(9.8)t^2$

$s=4.9\cdot t^2+30\cdot t$

$\begin{cases} s=-4.9\cdot t^2+50\cdot t \\ s=4.9\cdot t^2+30\cdot t \end{cases}\Big|_{s,t}$

$\{\{s=0,t=0\},\{s=81.63265306,t=2.0408$

1 Use $s = ut + \frac{1}{2}at^2$ to set up an equation for the rocket when $u = 50$ and $a = -9.8$.

2 Set up a second equation for the ball when $u = 30$ and $a = 9.8$.

3 Open the **Keyboard > Math1** and select the **simultaneous equations** template.

4 Enter the two equations and the variables **s** and ***t*** as shown above.

The objects collide after 2.04 seconds at a height of 81.63 metres above ground.

WB
p. 216

WORKED EXAMPLE 16 Upward motion and differential equations

At ground level, an object is propelled vertically upward with speed 80 m/s. As the object ascends, it experiences retardation proportional to its speed, v m/s.

Take the gravitational acceleration as $g = 9.8$ m/s^2.

a State the differential equation involving v and time t that describes the situation.

b Solve the equation in part **a** to express v in terms of t, given that the velocity of the object after 2 s in 54 m/s.

c What is the maximum height reached, and when does this occur? Give the answers correct to two decimal places.

Steps	**Working**
a Choose the form of acceleration that has the required variables. The resultant acceleration consists of the gravitational acceleration and the retardation. Both act in the negative direction.	The gravitational acceleration is $-g$ and the retardation is $-kv$, where k is a positive constant. $\frac{dv}{dt} = -g - kv$

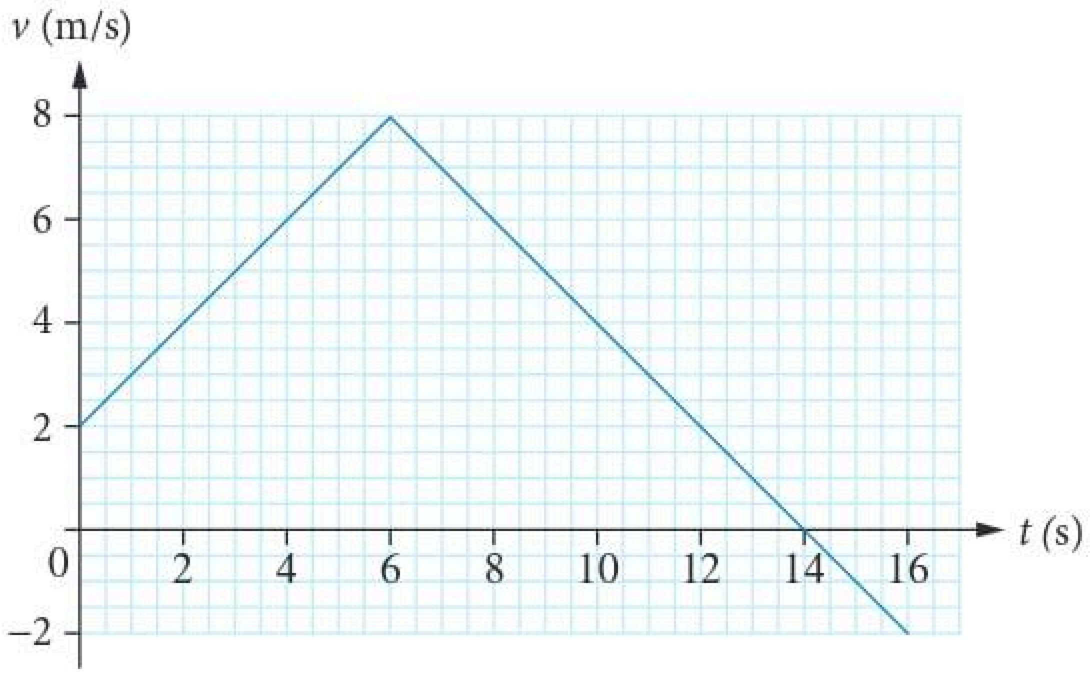
b **1** Write the integrands in the right form and anti-differentiate.	$-\frac{1}{k}\int \frac{k}{g+kv}\,dv = \int 1\,dt$ $-\frac{1}{k}\log_e\lvert g+kv\rvert = t+c$
2 Find the constant of integration, substitute into the equation and make v the subject of the equation.	$t = 0$ s, $v = 80$ m/s $c = -\frac{1}{k}\log_e\lvert g+80k\rvert$ $-\frac{1}{k}\log_e\lvert g+kv\rvert = t-\frac{1}{k}\log_e\lvert g+80k\rvert$ $t = \frac{1}{k}\log_e\lvert g+80k\rvert-\frac{1}{k}\log_e\lvert g+kv\rvert$ $= \frac{1}{k}\log_e\left\lvert\frac{g+80k}{g+kv}\right\rvert$ $kt = \log_e\left\lvert\frac{g+80k}{g+kv}\right\rvert$ $e^{kt} = \frac{g+80k}{g+kv}$ $\frac{g+kv}{g+80k} = e^{-kt}$ $g+kv = e^{-kt}\left(g+80k\right)$ $kv = e^{-kt}\left(g+80k\right)-g$ $v = \frac{1}{k}\left((g+80k)e^{-kt}-g\right)$ $v = \frac{1}{k}\left((9.8+80k)e^{-kt}-9.8\right)$
3 Determine the value of k using CAS.	$t = 2$ s, $v = 54$ m/s $54 = \frac{1}{k}\left((9.8+80k)e^{-2k}-9.8\right)$ $k = 0.0479\ldots$ $v = \frac{1}{k}\left((9.8+80k)e^{-kt}-9.8\right)$ where $k = 0.0479\ldots$
c **1** The maximum height occurs when the velocity is zero. Solve the equation to find t.	$0 = \frac{1}{k}\left((9.8+80k)e^{-kt}-9.8\right)$ where $k = 0.0479\ldots$ $t = 6.8899\ldots$ The object reaches maximum height after 6.89 s.
2 Distance travelled is the integral of the velocity equation in the given time interval.	$\int_0^{6.89}\frac{1}{k}\left((9.8+80k)e^{-kt}-9.8\right)dt = 260.4676\ldots$ The object reaches a maximum height of 260.47 m.

EXERCISE 10.3 The equations of kinematics

ANSWERS p. 597

Recap

1 The velocity–time graph of a particle's motion during the first 16 seconds is shown below.



The total distance the particle travelled at the end of 16 seconds is

A 44 m **B** 48 m **C** 56 m **D** 60 m **E** 64 m

2 © VCAA 2015 2AQ22 42% A ball is thrown vertically up with an initial velocity of $7\sqrt{6}$ m s^{-1}, and is subject to gravity and air resistance. The acceleration of the ball is given by $\ddot{x} = -(9.8 + 0.1v^2)$, where x metres is its vertical displacement, and v m s^{-1} is its velocity at time t seconds.

The time taken for the ball to reach its maximum height is

A $\frac{\pi}{3}$ **B** $\frac{5\pi}{21\sqrt{2}}$ **C** $\log_e(4)$

D $\frac{10\pi}{21\sqrt{2}}$ **E** $10\log_e(4)$

Mastery

3 WORKED EXAMPLE 12 During a given time interval, a truck starts from rest and with a constant acceleration of 5 m/s^2 reaches a speed of 50 m/s. A van starts from rest at the same time and with a constant acceleration of 8 m/s^2 reaches a speed of 40 m/s. Which vehicle travels further in 10 seconds, and by how much?

4 WORKED EXAMPLE 13 A train leaves the station and uniformly accelerates for 15 seconds until it reaches its maximum speed of 126 km/h, which it maintains. At 15 seconds, a car passes the station at 90 km/h for 5 seconds, moving parallel to the direction of the train. The car then uniformly accelerates until it reaches a speed of 144 km/h, which it maintains until it catches up to the train.

Calculate when the car catches up to the train, and its distance from the station at this time.

5 WORKED EXAMPLE 14 A ball 105 m above ground is projected vertically downward with initial velocity 10 m/s.

Taking the gravitational acceleration to be 10 m/s^2, show that the time, t_1, taken for the ball to fall one third of its initial height satisfies $t_2 - t_1 = \sqrt{2}(\sqrt{11} - 2)$, where t_2 is the time taken for the ball to hit the ground.

6 An object is thrown vertically downward with initial speed V from a height H and hits the ground with speed V_1. Show that the speed, V_2, of the object at a height $\frac{1}{2}H$ above the ground satisfies the equation $V_2 = \sqrt{V_1^2 - gH}$, where g is the gravitational acceleration.

7 WORKED EXAMPLE 15 A tourist is in a hot air balloon that is rising at a constant speed. When the balloon is 300 m above the ground, the tourist drops a marble over the side of the balloon. Neglecting air resistance and using $g = 9.8$ m/s^2, find the speed of the balloon if the marble hits the ground in 8 seconds. Give your answer correct to two decimal places.

8 Using CAS 2 Two toy rockets are released 2 s apart and move vertically upward. The first rocket has initial velocity 20 m/s and the second has initial velocity 40 m/s. Using $g = 9.81$ m/s^2, find when and where the rockets meet. State your answers correct to two decimal places.

9 WORKED EXAMPLE 16 From a hovering helicopter, a ball is released to fall vertically down. The air resistance on the ball is proportional to the square of its velocity.

Take the gravitational acceleration as $g = 10$ m/s^2 and use the constant of proportionality accurate to three decimal places.

a Give the differential equation involving the velocity, v m/s, of the descending ball when it has fallen x m.

b Express v as a function of x, given $v\,(10) = 14.21$.

c Find how far, to the nearest metre, has the ball fallen when its velocity is 30 m s^{-1}.

Exam practice

80–100% 60–79% 0–59%

10 At the end of 5 seconds, an object's change in velocity was 10 cm/s. If the object started from rest, its distance during this time was

A 15 cm
B 20 cm
C 25 cm
D 30 cm
E 35 cm

11 A ball is projected vertically upward with initial velocity $2g$ m/s, where g is the gravitational acceleration. The total time taken for the ball to return to its point of projection is

A 2 s
B 4 s
C 5 s
D 6 s
E 8 s

12 A particle starting from rest moves with constant acceleration a m/s^2 and after time t seconds travels a distance of s_1 metres to reach a final velocity of v m/s. A second particle, also starting from rest, moves with constant acceleration $3a$ m/s^2 and after time t seconds has a final velocity of $2v$ m/s and travels s_2 metres. The relationship between s_1 and s_2 is

A $3s_1 = 4s_2$
B $4s_1 = 3s_2$
C $3s_1 = 2s_2$
D $2s_1 = 3s_2$
E $2s_1 = s_2$

13 Two trains head towards each other after leaving stations A and B, which are s km apart. The acceleration of the first train is a m/s^2. If the trains meet S km from station A, the acceleration of the second train will be

A $\dfrac{a(S+s)}{S}$
B $\dfrac{a(S+s)}{S-s}$
C $\dfrac{a(S+s)}{s}$
D $\dfrac{a(S-s)}{S+s}$
E $\dfrac{a(S-s)}{S}$

14 © VCAA 2010 2AQ19 81% An object is moving in a northerly direction with a constant acceleration of 2 m s^{-2}. When the object is 100 m due north of its starting point, its velocity is 30 m s^{-1} in the northerly direction. The exact initial velocity of the object could have been

A $10\sqrt{5}$ m s^{-1}
B $5\sqrt{10}$ m s^{-1}
C $10\sqrt{7}$ m s^{-1}
D $-10\sqrt{7}$ m s^{-1}
E $7\sqrt{10}$ m s^{-1}

15 © VCAA 2004 1IQ25 A balloon is rising vertically at a constant speed of 21 metres per second. A stone is dropped from the balloon when it is h metres above the ground. The stone strikes the ground 10 seconds later.

Assuming that air resistance is negligible, the value of h is

A 210
B 280
C 490
D 700
E 770

16 © VCAA 2013 2AQ19 57% A tourist in a hot air balloon, which is rising at 2 m/s, accidentally drops a camera over the side and it falls 100 metres to the ground.

Neglecting the effect of air resistance on the camera, the time taken for the camera to hit the ground, correct to the nearest tenth of a second, is

A 4.3 s
B 4.5 s
C 4.7 s
D 4.9 s
E 5.0 s

17 (2 marks) A ball is projected vertically with an initial speed of u m/s. Show that the ball will reach a height of at least D metres if $u \geq 2\sqrt{5D}$ m/s.

18 (2 marks) Show that the time taken, t_1, for an object starting from rest to vertically fall a given distance is $t_2\sqrt{2}$, where t_2 is the time taken for the object to fall half of the distance.

19 (2 marks) The base of a hot air balloon 400 metres in the air is rising with speed 18 km/h when chewing gum stuck to the base falls off. To one decimal place, find the time taken for the gum to reach the ground.

20 © VCAA 2006S 2BQ2a (2 marks) A dragster racing car accelerates uniformly over a straight line course and completes a 'standing' (that is, starting from rest) 400 metres in eight seconds.

a Find the acceleration (in m/s^2) of the dragster over the 400 metres. 1 mark

b Show that the dragster reaches a speed of 100 m/s at the end of the 400 metre course. 1 mark

21 (6 marks) For this question, we will derive the four equations of kinematics. Let a be the acceleration, t the change in time, s the displacement, u the initial velocity and v the final velocity.

a Use acceleration $= \dfrac{\text{change in velocity}}{\text{change in time}}$ to obtain $v = u + at$. 1 mark

b In terms of u, v and t, write an expression for average velocity and use it together with average velocity $= \dfrac{\text{displacement}}{\text{change in time}}$ to obtain $s = \dfrac{1}{2}(u+v)t$. 2 marks

c **i** Make t the subject in $v = u + at$. 1 mark

ii Substitute $v = u + at$ into $s = \dfrac{1}{2}(u+v)t$. 1 mark

iii Substitute your answer to part **i** into $s = \dfrac{1}{2}(u+v)t$, then simplify the expression to obtain $v^2 = u^2 + 2as$. 1 mark

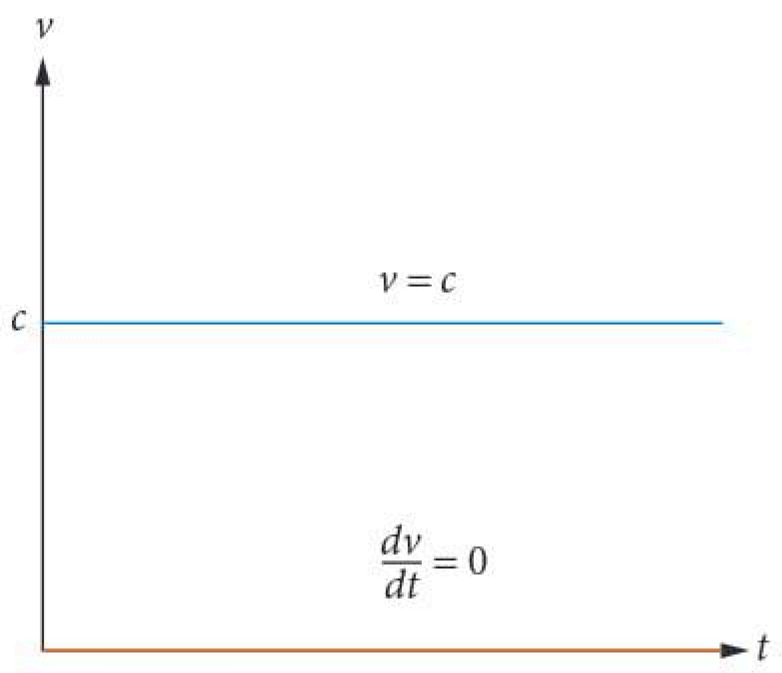
10.4 Variable acceleration

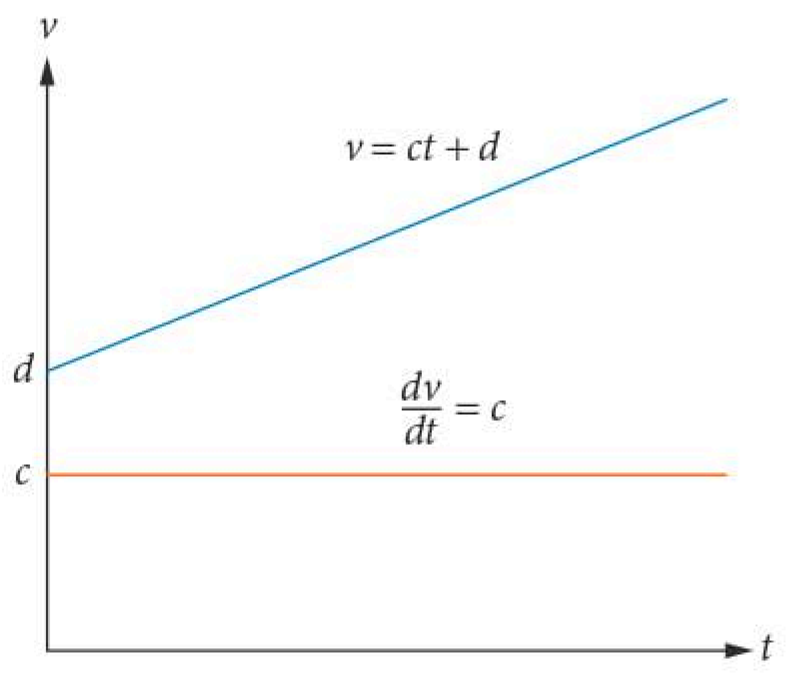
Video playlist
Variable acceleration

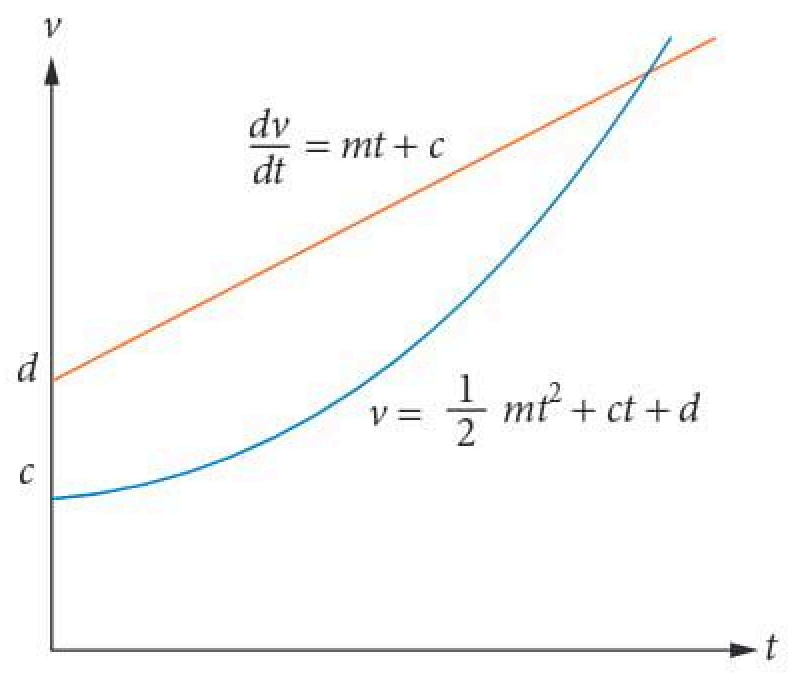
When we have acceleration as a function of time, $\dfrac{dv}{dt} = f(t)$, anti-differentiation gives the velocity, v.

Zero acceleration means $f(t) = 0$, hence v will be constant. For constant (non-zero) acceleration, $f(t) = c$ and v is a linear function representing uniform change in velocity.

There is **variable acceleration** when $f(t)$ is not a constant (including zero), such as $f(t) = mt + c$ that has the velocity function $v(t) = \dfrac{1}{2}mt^2 + ct + d$. The graphs below illustrate these three possibilities.







The area under a velocity–time graph gives distance and the signed area represents displacement. Calculus or CAS can be used to determine velocity by anti-differentiating the acceleration function, and distance or displacement is found by anti-differentiating the velocity function.

WORKED EXAMPLE 17 Uniform increase in acceleration

A car initially travelling at 10 m/s uniformly increases its acceleration from 4 m s^{-2} to 10 m s^{-2} in 30 seconds. Find the car's velocity and displacement at 30 seconds.

WB p. 218

Steps	**Working**
1 Uniform increase in acceleration is a linear function of the form $\frac{dv}{dt} = mt + c$. Find the gradient m and intercept c.	$m = \frac{10-4}{30} = 0.2$ $\frac{dv}{dt} = 0.2t + c$ When $t = 0$, $\frac{dv}{dt} = 4$, so $c = 4$. $\frac{dv}{dt} = 0.2t + 4$
2 Anti-differentiate to find the velocity function.	$v = 0.1t^2 + 4t + c$ When $t = 0$, $v = 10$, so $c = 10$. $v = 0.1t^2 + 4t + 10$
3 Find the velocity at 30 seconds.	When $t = 30$, $v = 220$ m/s.
4 Anti-differentiate to find the displacement function.	$x = \frac{1}{30}t^3 + 2t^2 + 10t + c$ When $t = 0$, $x = 0$, so $c = 0$. $x = \frac{1}{30}t^3 + 2t^2 + 10t$
5 Find the displacement by finding the area under the velocity curve.	$\left[\frac{1}{30}t^3 + 2t^2 + 10t\right]_0^{30} = 3000$ The car has travelled 3000 m.

WORKED EXAMPLE 18 Constant and uniform acceleration increase

A particle with initial speed 1 cm/s is moving with acceleration 3 cm s^{-2}, which it maintains for 10 seconds. It then takes 5 seconds to uniformly increase its acceleration to 13 cm s^{-2}. The particle's acceleration is then uniformly reduced to zero during a further 10 seconds.

a Sketch the acceleration–time graph.

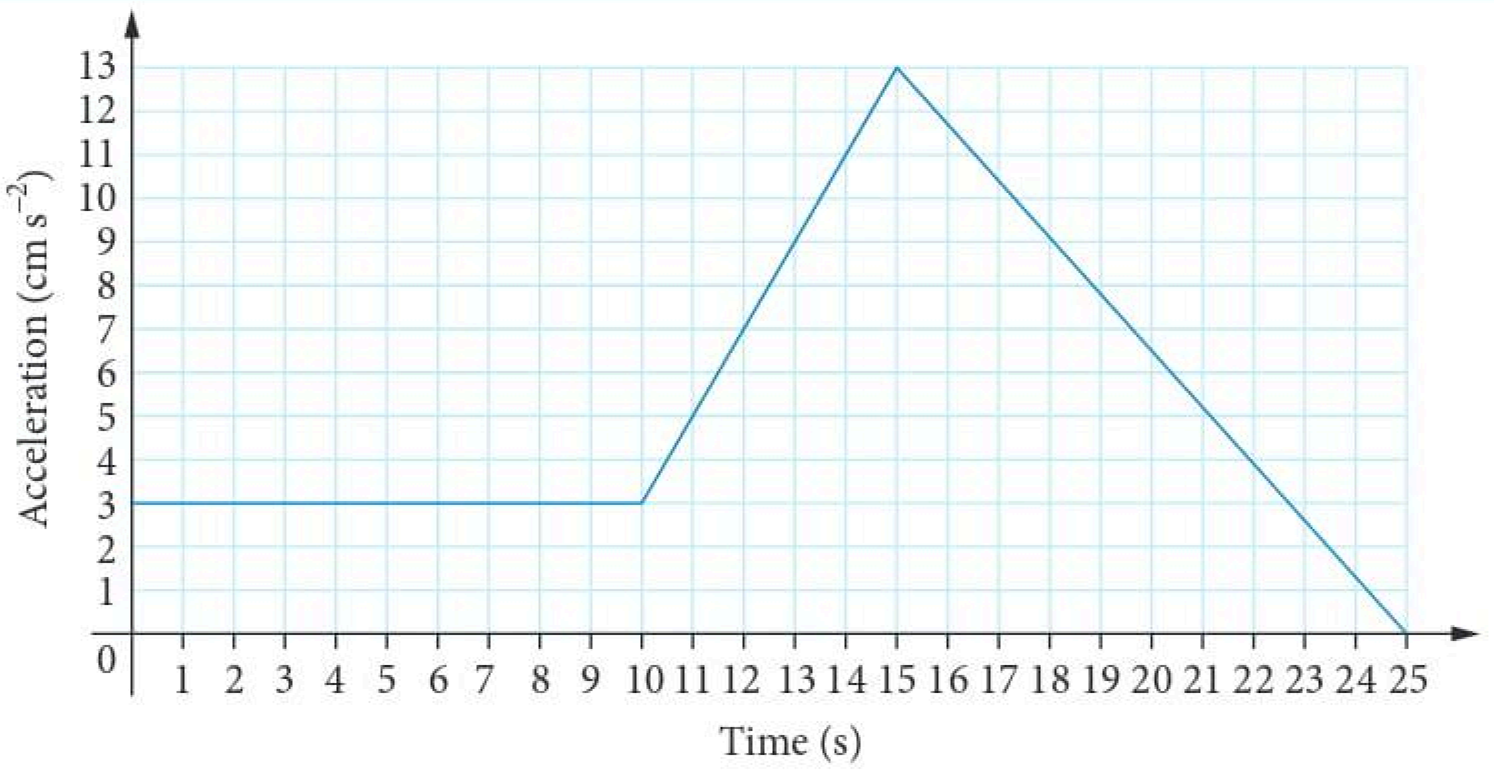
b Sketch the velocity–time graph.

c Calculate the total distance the particle has travelled.

WB p. 219

Steps	**Working**
a **1** Describe the acceleration in each required section.	Initially, the acceleration is 3 cm s^{-2} for 10 s. This is shown as a horizontal line passing through 3 on the vertical axis and starting at $t = 0$ and finishing at $t = 10$. In the next section, it takes 5 s for the acceleration to increase from 3 cm s^{-2} to 13 cm s^{-2}. This is a straight line of gradient $\frac{13-3}{15-10} = 2$ from (10, 3) to (15, 13), with equation $\frac{dv}{dt} = 2t - 17$. In the last section, the acceleration reduces from 13 cm s^{-2} to 0 cm s^{-2} in 10 s. This is a straight line of gradient $\frac{0-13}{25-15} = -1.3$ from (15, 13) to (25, 0), with equation $\frac{dv}{dt} = -1.3t + 32.5$.

2 Show the information as an acceleration–time graph.



b 1 Obtain expressions by anti-differentiation that describe the velocity in each required section.

Initially, in [0, 10], $\frac{dv}{dt} = 3$, so $v = 3t + c$.

$t = 0, v = 1 \Rightarrow c = 1$

So $v = 3t + 1$ in [0, 10].

In [10, 15], $\frac{dv}{dt} = 2t - 17$, so $v = t^2 - 17t + c$.

From the previous interval [0, 10], for $t = 10$,

$v = 3t + 1 \Rightarrow v = 31$.

So $t = 10$, $v = 31$ with $v = t^2 - 17t + c$ gives

$31 = 100 - 170 + c \Rightarrow c = 101$.

$v = t^2 - 17t + 101$

In [15, 25], $\frac{dv}{dt} = -1.3t + 32.5$, so $v = -0.65t^2 + 32.5t + c$.

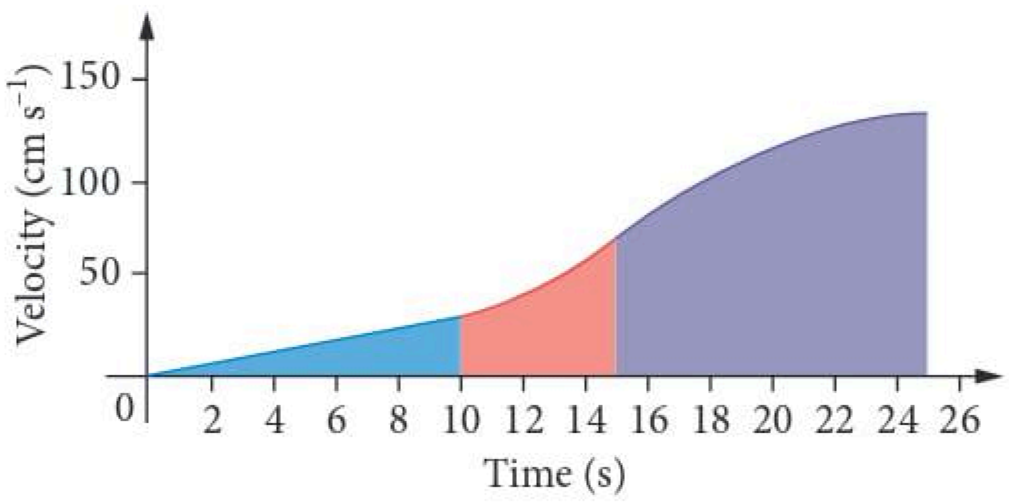
To find c, use $t = 15$ and $v = t^2 - 17t + 101 = 71$ from the previous section.

$t = 15, v = 71$

Then for $v = -0.65t^2 + 32.5t + c$,

$$71 = -0.65(15)^2 + 32.5(15) + c$$
$$c = -270.25$$
$$v = -0.65t^2 + 32.5t - 270.25$$

2 Show the information as a velocity–time graph.



c To find the total distance, use calculus by hand or CAS to find the area under each section of the velocity–time graph.

In [0, 10], the area is $\int_0^{10} (3t + 1)\, dt = 160$.

In [10, 15], the area is $\int_{10}^{15} (t^2 - 17t + 101)\, dt = 234\frac{1}{6}$.

In [15, 25], the area is $\int_{15}^{25} (-0.65t^2 + 32.5t - 270.25)\, dt = 1143\frac{1}{3}$.

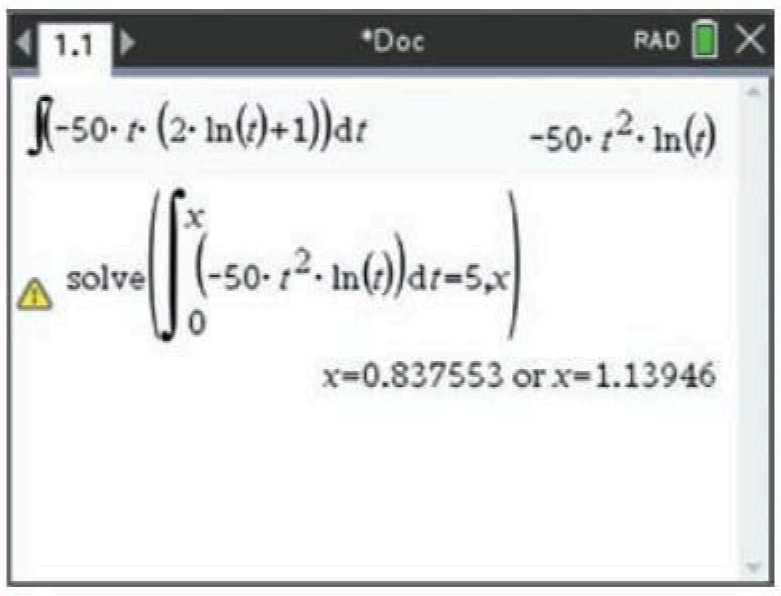
Total area is 1537.5.

The distance travelled by the particle is 1537.5 cm.

USING CAS 3 Find the time taken for a given distance

The acceleration a cm/s^2 of a particle starting from rest in the interval $0 \le t \le 1$ is given by $a = -50t[2\log_e(t) + 1]$. Calculate, to two decimal places, how long it will take for the particle to move 5 cm from its start position.

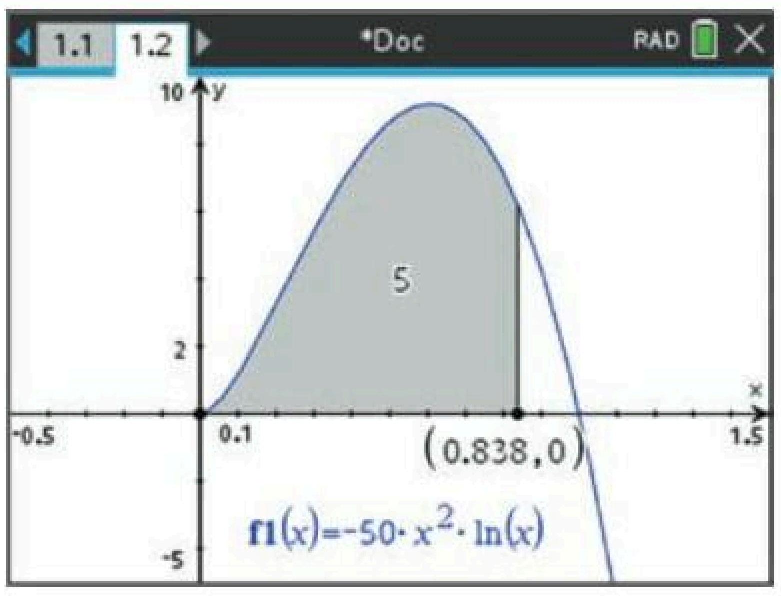
TI-Nspire



$\int(-50 \cdot t \cdot (2 \cdot \ln(t)+1))dt$ $\quad -50 \cdot t^2 \cdot \ln(t)$

$\text{solve}\left(\int_0^x (-50 \cdot t^2 \cdot \ln(t))dt=5,x\right)$

$x=0.837553$ or $x=1.13946$



1 Press **menu > Calculus > Integral**.

2 Find the indefinite integral of the acceleration to determine an expression for the velocity. The object is starting from rest, so $c = 0$.

3 Set the definite integral of the velocity from **0** to ***x*** equal to **5** and solve for ***x***.

4 As the domain is $0 \le t \le 1$, select the first solution of $x = 0.837553$.

5 Add a **Graphs** page and graph the velocity function.

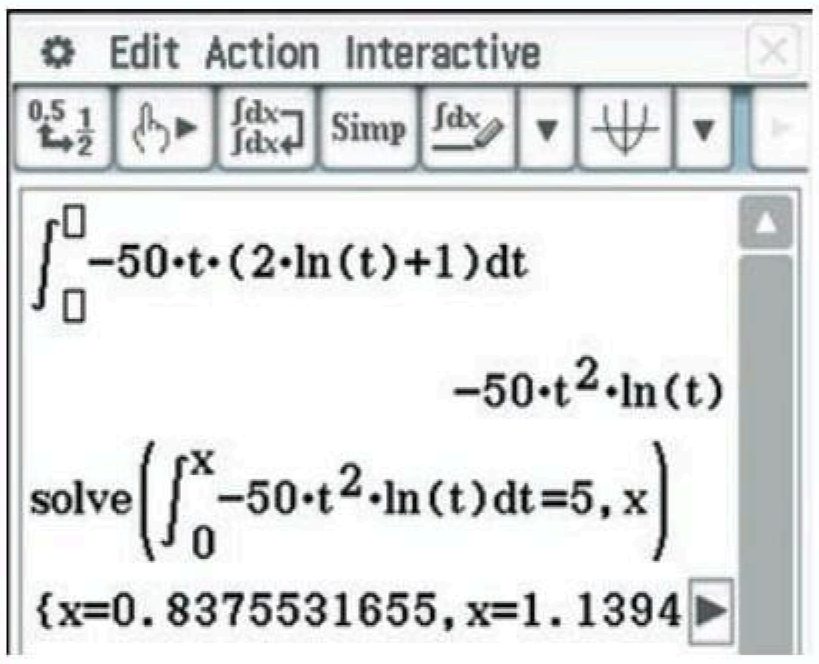
6 Adjust the window settings to suit.

7 Press **menu > Analyze Graph > Integral**.

8 Find the integral from **0** to **0.837553**.

9 The area under the velocity curve is equal to **5**, which verifies the solution from the **Calculator** page.

ClassPad

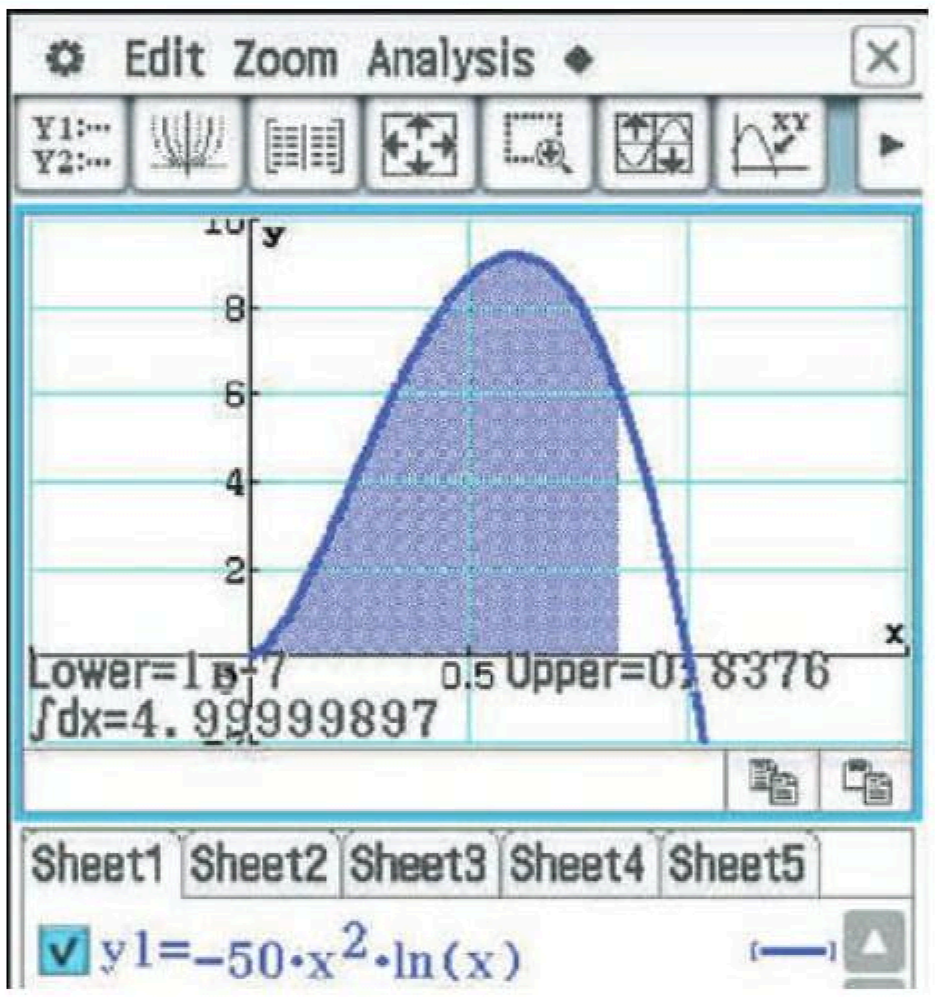


$\int_{\square}^{\square} -50 \cdot t \cdot (2 \cdot \ln(t)+1)dt$

$-50 \cdot t^2 \cdot \ln(t)$

$\text{solve}\left(\int_0^x -50 \cdot t^2 \cdot \ln(t)dt=5, x\right)$

$\{x=0.8375531655, x=1.1394$



Lower=1E-7 Upper=0.8376

∫dx=4.99999897

1 In **Main**, enter and highlight the expression.

2 Tap **Interactive > Calculation > ∫**.

3 Find the indefinite integral of the acceleration to determine an expression for the velocity. The object is starting from rest, so $c = 0$.

4 Set the definite integral of the velocity from **0** to ***x*** equal to **5** and solve for ***x***.

5 As the domain is $0 \le t \le 1$, select the first solution of $x = 0.837553$.

6 Graph the velocity function.

7 Adjust the window settings to suit.

8 Press **Analysis > G-Solve > Integral > ∫dx**.

9 Find the integral from **0.0000001** to **0.837553** (the function is not defined at $x = 0$ so enter a positive value close to 0).

10 The area under the velocity curve is equal to **5**, which verifies the solution from the screen in **Main**.

It will take the particle 0.84 s to move 5 metres.

Video
VCE question analysis: Kinematics

VCE QUESTION ANALYSIS

© VCAA | 2017 2BQ2 | 2017 Examination 2 Section B Question 2 **(10 marks)**

A helicopter is hovering at a constant height above a fixed location. A skydiver falls from rest for two seconds from the helicopter. The skydiver is subject only to gravitational acceleration and air resistance is negligible for the first two seconds. Let downward displacement be positive.

a Find the distance, in metres, fallen in the first two seconds. 2 marks

b Show that the speed of the skydiver after two seconds is 19.6 m s^{-1}. 1 mark

After two seconds, air resistance is significant and the acceleration of the skydiver is given by $a = g - 0.01v^2$.

c Find the limiting (terminal) velocity, in m s^{-1}, that the skydiver would reach. 1 mark

d **i** Write down an expression involving a definite integral that gives the time taken for the skydiver to reach a speed of 30 m s^{-1}. 2 marks

ii Hence, find the time, in seconds, taken to reach a speed of 30 m s^{-1}, correct to the nearest tenth of a second. 1 mark

e Write down an expression involving a definite integral that gives the distance through which the skydiver falls to reach a speed of 30 m s^{-1}. Find this distance, giving your answer in metres, correct to the nearest metre. 3 marks

Reading the question

- Parts **a** and **b** do not involve air resistance, so there is constant acceleration and the constant acceleration equations of kinematics can be applied.
- Parts **c** to **e** involve air resistance, so there is variable acceleration.
- The definite integrals must include limits of integration.
- Keep in mind the accuracy expected in the answers in terms of decimal places.

Thinking about the question

- Part **a** can be answered using the appropriate kinematics equation and the answer can be used to find the required speed in part **b**.
- To find limiting velocity, the equation given in part **b** must first be solved to express v as a function of t.
- A calculator is needed to find the definite integrals in parts **d** and **e**.
- The total time includes the 2 seconds of falling without air resistance.

Worked solution (✓ = 1 mark)

a Use $s = ut + \frac{1}{2}at^2$ ✓

$a = g \text{ m s}^{-2}, u = 0 \text{ m s}^{-1}, t = 2 \text{ s}$

$s = \frac{1}{2} \times 9.8 \times 2^2 = 19.6$

After 2 s, the distance fallen is **19.6 m.** ✓

b $v = u + at$

$a = g \text{ m s}^{-2}, u = 0 \text{ m s}^{-1}, t = 2 \text{ s}$

$v = 0 + 9.8 \times 2$

$= \mathbf{19.6\ m\ s^{-1}}$ ✓

c Terminal velocity occurs when there is zero acceleration.

$a = g - 0.01v^2$

$a = 0 \text{ m s}^{-2}$

$g - 0.01v^2 = 0$

$v = \sqrt{100g} = 10\sqrt{g}$

The terminal velocity is $\mathbf{10\sqrt{g}}$ **m s**$^{-1}$ or $\mathbf{10\sqrt{9.8} = 14\sqrt{5}}$ **m s**$^{-1}$. ✓

OR Terminal velocity can be found by letting t approach infinity.

This requires v to be expressed as a function of t.

$$\frac{dv}{dt} = g - 0.01v^2$$

$$\int \frac{1}{g - 0.01v^2}\,dv = \int 1\,dt$$

$$\frac{1}{2\sqrt{g}}\int\left(\frac{1}{\sqrt{g} - 0.1v} + \frac{1}{\sqrt{g} + 0.1v}\right)dv = \int 1\,dt \qquad \text{using partial fractions}$$

$$\frac{5}{\sqrt{g}}\log_e\left|\frac{\sqrt{g} + 0.1v}{\sqrt{g} - 0.1v}\right| = t + c$$

$t = 0,\ v = 0 \Rightarrow c = 0$

$$\frac{\sqrt{g} + 0.1v}{\sqrt{g} - 0.1v} = e^{0.2\sqrt{g}t}$$

$$v = 10\sqrt{g}\left(\frac{e^{0.2\sqrt{g}t} - 1}{e^{0.2\sqrt{g}t} + 1}\right)$$

As $t \to \infty$, $v \to 10\sqrt{g}\left(\frac{e^{0.2\sqrt{g}t}}{e^{0.2\sqrt{g}t}}\right) = 10\sqrt{g}$

The terminal velocity is $\mathbf{10\sqrt{g}}$ **m s**$^{-1}$**.** ✓

d **i** When $v = 19.6$, $t = 2$.

For v in $[19.6, 30]$, $\frac{dv}{dt} = g - 0.01v^2$ ✓

Anti-differentiate:

$$t = \int_{19.6}^{30} \frac{1}{g - 0.01v^2}\,dv$$

Total time is $\boldsymbol{\int_{19.6}^{30} \frac{1}{g - 0.01v^2}\,dv + 2}$. ✓

ii Use technology.

$$\int_{19.6}^{30} \frac{1}{g - 0.01v^2}\,dv + 2 = 3.80 + 2 = 5.80$$

The total time is **5.8 s.** ✓

e When $t = 2$ s, $v = 19.6$ m s^{-1}.

The limits of integration are [19.6, 30].

Express v as a function of x.

Use $a = v\frac{dv}{dx}$ ✓

$$v\frac{dv}{dx} = g - 0.01v^2$$

$$\frac{dv}{dx} = \frac{g - 0.01v^2}{v}$$

$$x = \int_{19.6}^{30} \frac{v}{g - 0.01v^2}\,dv \approx 100.397$$

The distance fallen in the first 2 seconds is 19.6 m (from part **a**).

Total distance is $\int_{19.6}^{30} \frac{v}{g - 0.01v^2}\,dv + 19.6$ ✓

The total distance is 100.4 + 19.6 = **120 m.** ✓

Student performance 80–100% 60–79% 0–59%

a 91% Most students correctly used an appropriate constant acceleration formula.

b 93% Very well done. Most students were able to find the correct value of v.

c 48% Some students did not write their answers in exact form.

d **i** 39% A very challenging question with many non-attempts. Many students made incorrect assumptions such as integrating from the start of the motion ($t = 0$) or writing an answer that did not take into account the first 2 seconds.

ii 25% Poorly done.

e 29% The errors that occurred in part **d** also occurred here.

EXERCISE 10.4 Variable acceleration

ANSWERS p. 598

Recap

1 Particle A starts from rest and travels for 10 seconds with a constant acceleration of 3 m s^{-2}. At the same time, particle B has initial velocity 12 m s^{-1} and is moving with twice the acceleration as particle A.

The time needed for particle B to reach the velocity that particle A reached after 10 s is

A 7 s **B** 3 s **C** 10 s **D** 12 s **E** 15 s

2 An object 100 metres above ground level is propelled vertically with speed 10 m/s. Taking $g = 10$ m/s^2, the time taken in seconds for the object to hit the ground is

A 2 **B** $\sqrt{21}$ **C** $-1 + \sqrt{21}$

D $1 + \sqrt{21}$ **E** $2 + \sqrt{21}$

Mastery

3 WORKED EXAMPLE 17 An out-of-control toy rocket initially travelling at 6 m/s uniformly decreases its acceleration from 5 m s^{-2} to −4 m s^{-2} in 12 seconds. Find the rocket's velocity and displacement after 40 seconds.

4 WORKED EXAMPLE 18 A high-pressure water pump is releasing water from a dam. Initially, the speed of water passing through the pipes is 20 cm s^{-1}. During the first 2 seconds, the acceleration of the water is uniformly reduced from 16 cm s^{-2} to 6 cm s^{-2}. This acceleration remains constant for the next 3 seconds and then three more seconds pass for the acceleration to uniformly reduce to 0 cm s^{-2}.

a Sketch the acceleration–time graph.

b Sketch the velocity–time graph.

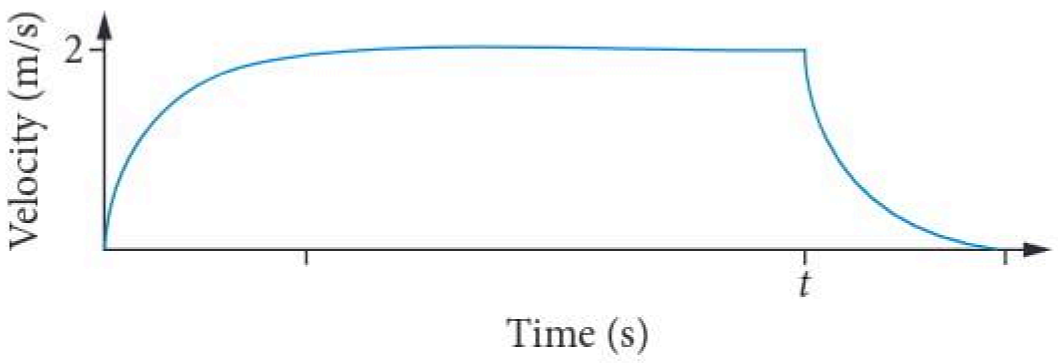
c A tracker ball is placed in the pipes at the start to monitor the movement of water. Calculate the total distance that the tracker ball has travelled at the end of 8 seconds.

5 Using CAS 3 The acceleration, a m/s^2, of a particle starting from rest is

$$a = \frac{\sin(t) - \cos(t)}{e^t}$$

Calculate, to two decimal places, how long it will take for the particle to move 0.5 metres from its start position and its velocity at that time.

6 The area under the curve in the velocity–time graph below is 40 square units and each of the two curves describes a quarter circle. Determine the value of t.

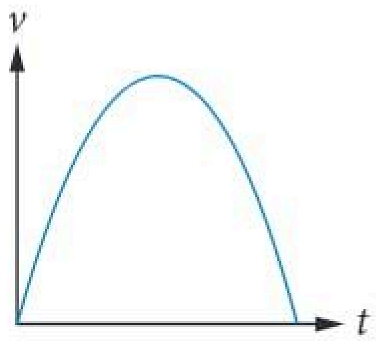


7 The acceleration, $\frac{dv}{dt}$, of an object can be described by a linear equation. Find the equation $a(t)$ if $a(6) = -3.5$ and $a(-3) = 2.5$.

Exam practice

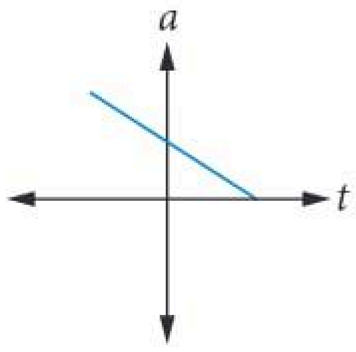
80–100% 60–79% 0–59%

8 The velocity–time graph of a particle is shown.

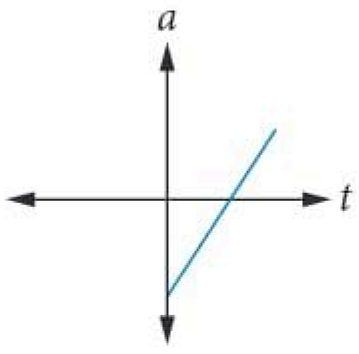


Which graph is closest to representing the acceleration–time graph?

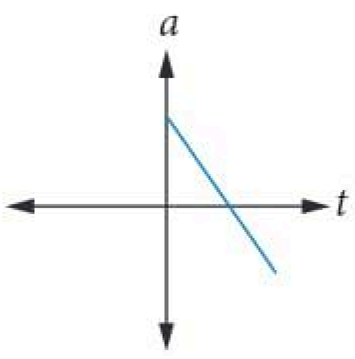
A



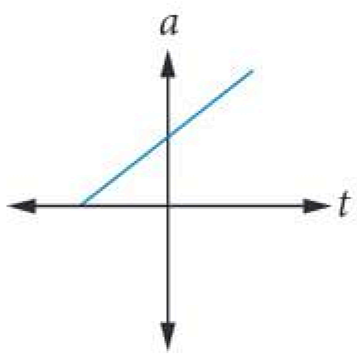
B



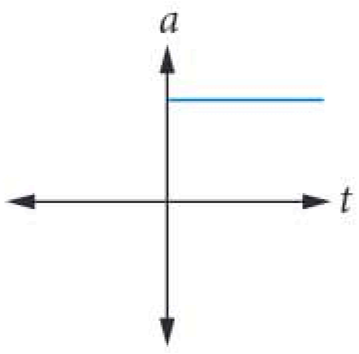
C



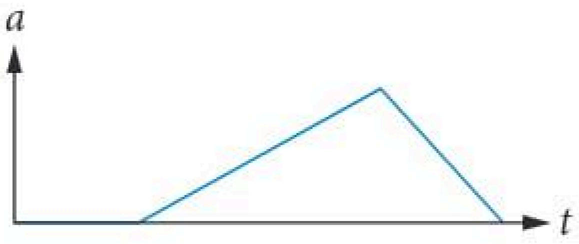
D



E

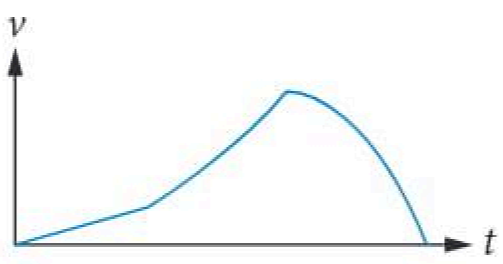


9 The acceleration–time graph of a moving particle is shown below.

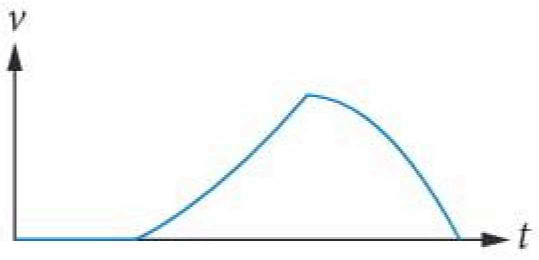


Which graph is closest to representing the velocity–time graph?

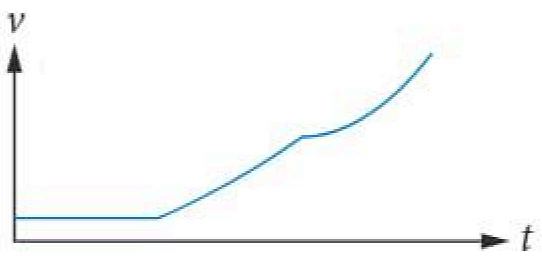
A



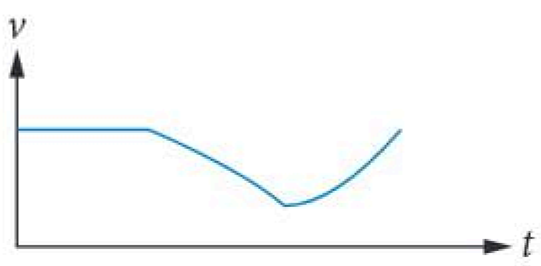
B



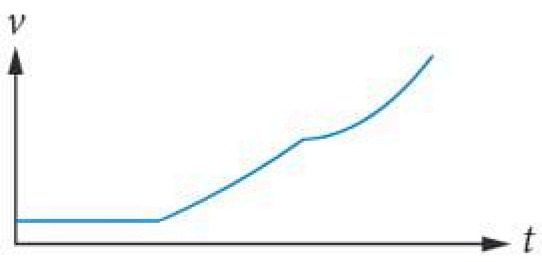
C



D



E



10 The velocity of an object is given by $v = 5t^2 + t$. The change in its acceleration in the time interval [1, 2] is

A 10 **B** 11 **C** 16 **D** 20 **E** 21

11 Object 1 moves with variable acceleration a_1 given by $a_1 = \dfrac{1}{t+1}$. The variable acceleration, a_2, of object 2 is defined by $a_2 = \dfrac{1}{t^2+1}$. If both objects start from rest at the origin, the relationship between the velocity, v_1, of object 1 and the velocity, v_2, of object 2 is

A $e^{v_2} = \tan(v_1)$ **B** $\log(v_1) = \tan(v_2)$ **C** $\log(v_2) = \tan(v_1)$

D $e^{v_1} = \tan(v_2) + 1$ **E** $e^{v_1}\tan(v_2) = 1$

12 Which of the following functions for velocity as a function of time, $v(t)$, does not indicate constant acceleration?

A $v = \dfrac{1-t^2}{1+t}$ **B** $v = \dfrac{1}{t}\displaystyle\int_0^t 2x^2\,dx$ **C** $v = \sqrt{1-t^2}\,\dfrac{d}{dt}\left(\sin^{-1}(t)\right)$

D $v = (1-\sqrt{t})(1+\sqrt{t})$ **E** $v = \cos(2t) - \dfrac{\cos^2(t)-\sin^2(t)}{\cos^2(t)+\sin^2(t)}$

13 © VCAA 2015 2AQ20 60% An object is moving in a straight line, initially at 5 m s^{-1}. Sixteen seconds later, it is moving at 11 m s^{-1} in the **opposite** direction to its initial velocity.

Assuming that the acceleration of the object is constant, after 16 seconds the distance, in metres, of the object from its starting point is

A 24 **B** 48 **C** 73 **D** 96 **E** 128

14 © VCAA 2019N 2AQ15 A lift accelerates from rest at a constant rate until it reaches a speed of 3 m s^{-1}. It continues at this speed for 10 seconds and then decelerates at a constant rate before coming to rest. The total travel time for the lift is 30 seconds.

The total distance, in metres, travelled by the lift is

A 30 **B** 45 **C** 60 **D** 75 **E** 90

15 © VCAA 2019N 2AQ17 A ball is thrown vertically upwards with an initial velocity of $7\sqrt{6}$ m s^{-1}, and is subject to gravity and air resistance. The acceleration of the ball is given by $\ddot{x} = -(9.8 + 0.1v^2)$, where v m s^{-1} is its velocity when it is at a height of x m above ground level.

The maximum height, in metres, reached by the ball is

A $5\log_e(4)$ **B** $\log_e(\sqrt{31})$ **C** $\dfrac{5\pi\sqrt{2}}{21}$

D $5\log_e(2)$ **E** $\dfrac{7\pi\sqrt{2}}{3}$

16 (4 marks) A projectile is fired vertically upwards with initial velocity 20 m/s. In addition to gravitational acceleration, the projectile experiences an additional retardation proportional to time.

a Using $g = 10$ m/s^2, state the equation for acceleration. 1 mark

b Give the equation for the velocity. 1 mark

c Find, correct to one decimal place, the proportionality constant if the maximum height of the projectile above its starting point is 19.87 metres. 2 marks

17 (2 marks) The acceleration, a m/s^2, of an object as a function of its distance x m from the origin is $a = e^{2x}$. If the object starts at the origin with initial velocity 1 m/s and its velocity is always positive, find the equation for the acceleration as a function of time.

18 (3 marks) The acceleration of two particles starting from a fixed reference point is given by $a_1 = e^t$ and $a_2 = -e^{t-2}$, with respective velocity functions $v_1(t)$ and $v_2(t)$ satisfying $v_1(0) = 1$ and $v_2(2) = 4$. Show that when the two particles have the same velocity, $d_1 + e^2 d_2 = 5e^2 \log_e\left(\dfrac{5e^2}{1+e^2}\right)$, where d_1 and d_2 are the respective distances travelled by the two particles.

19 © VCAA 2004 2Q1a,b (6 marks) The velocity–time graph below shows the velocity of a lift as it travels from the first floor to the twelfth floor of a tall building during T s of its motion.

The velocity v m/s at time t s for $0 \le t \le 2$ is given by $v = t^2(3 - t)$. After the first two seconds, the lift moves with a constant velocity of 4 m/s for a time, and then decelerates to rest in the final two seconds.

The acceleration of the lift is a m/s^2 at time t s, and the velocity–time graph is symmetrical about $t = \frac{1}{2}T$.

a **i** 84% Express a in terms of t for the first two seconds of the motion of the lift. 1 mark

ii 74% Hence find the maximum acceleration of the lift during the first two seconds of its motion. 2 marks

b 59% Given that the total distance travelled by the lift during its journey is 41 metres, use calculus to find the exact value of T. 3 marks

20 © VCAA 2012 2BQ5 (13 marks) At her favourite fun park, Katherine's first activity is to slide down a 10 m long straight slide. She starts from rest at the top and accelerates at a constant rate, until she reaches the end of the slide with a velocity of 6 m s^{-1}.

a 73% How long, in seconds, does it take Katherine to travel down the slide? 1 mark

When at the top of the slide, which is 6 metres above the ground, Katherine throws a chocolate vertically upwards. The chocolate travels up and then descends past the top of the slide to land on the ground below. Assume that the chocolate is subject only to gravitational acceleration and that air resistance is negligible.

b 53% If the initial speed of the chocolate is 10 m/s, how long, correct to the nearest tenth of a second, does it take the chocolate to reach the ground? 2 marks

c 27% Assume that it takes Katherine four seconds to run from the end of the slide to where the chocolate lands.

At what velocity would the chocolate need to be propelled upwards, if Katherine were to immediately slide down the slide and run to reach the chocolate just as it hits the ground? Give your answer in m s^{-1}, correct to one decimal place. 2 marks

Katherine's next activity is to ride a mini speedboat. To stop at the correct boat dock, she needs to stop the engine and allow the boat to be slowed by air and water resistance.

At time t seconds after the engine has been stopped, the acceleration of the boat, a m s^{-2}, is related to its velocity, v m s^{-1}, by

$$a = -\frac{1}{10}\sqrt{196 - v^2}$$

Katherine stops the engine when the speedboat is travelling at 7 m/s.

d **i** 68% Find an expression for v in terms of t. 3 marks

ii 61% Find the time it takes the speedboat to come to rest.

Give your answer in seconds in terms of π. 2 marks

iii 51% Find the distance it takes the speedboat to come to rest, from when the engine is stopped.

Give your answer in metres, correct to one decimal place. 3 marks

10 Chapter summary

Kinematics deals with **rectilinear motion** involving motion in a straight line.

Distance is the length of the path taken by an object.

Displacement is the direct distance from the start of the path to the final position. It is the change of position.

The distance travelled by an object may not be the same as its displacement.

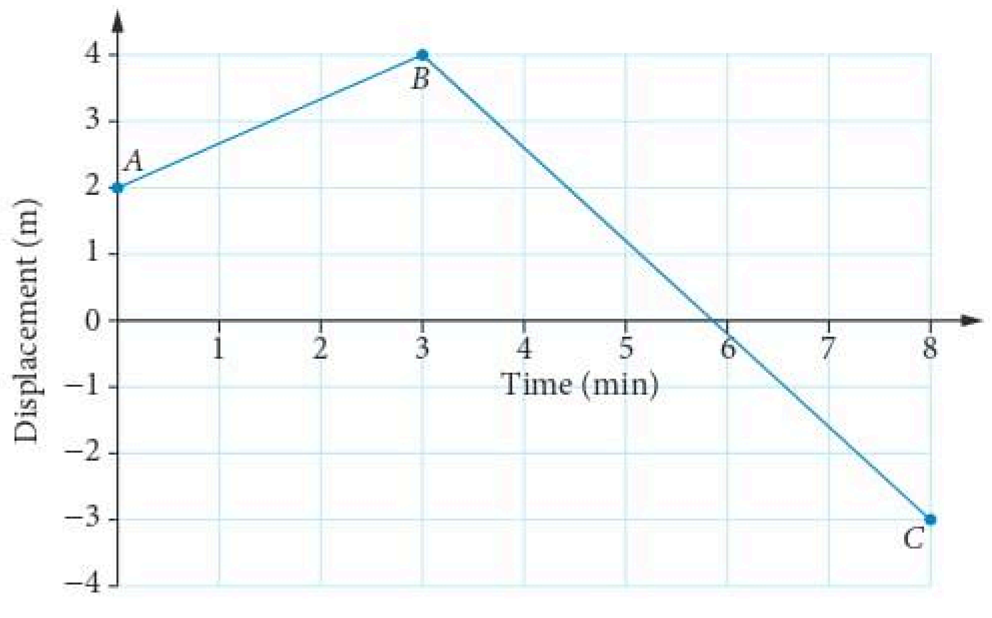
Speed is the rate of change of distance, $(x_2 - x_1)$, with respect to the time taken, $(t_2 - t_1)$.

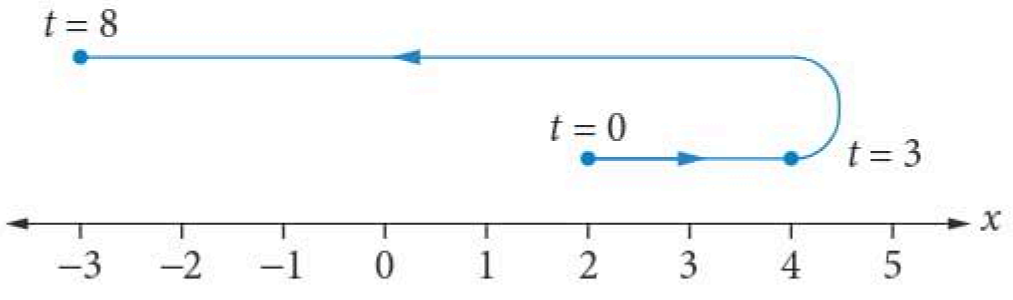
Velocity is the rate of change of displacement with respect to time.

$$\text{average speed} = \frac{\text{distance}}{\text{time}}$$

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}}$$

The **displacement–time graph** and the **position–time line** show straight line motion by indicating path length, direction and the time taken.





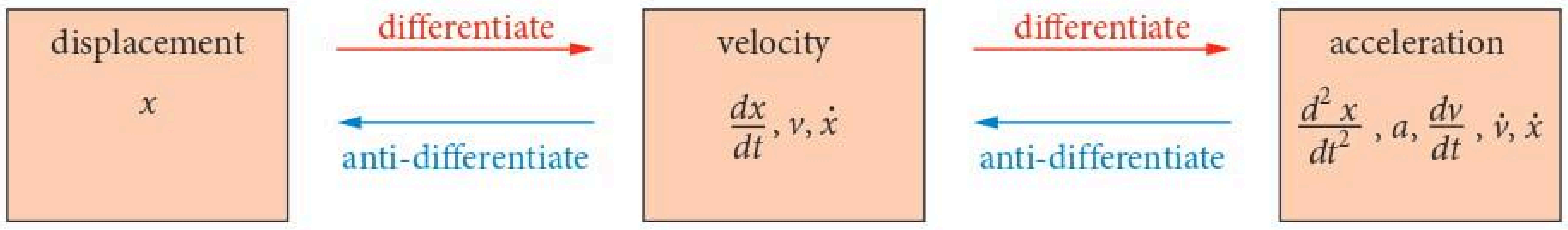
Acceleration is $\frac{\text{change in velocity}}{\text{change in time}}$, or the rate of change of speed (or velocity) with respect to time.

Negative acceleration is also referred to as **deceleration** and **retardation**.

Differentiation and anti-differentiation

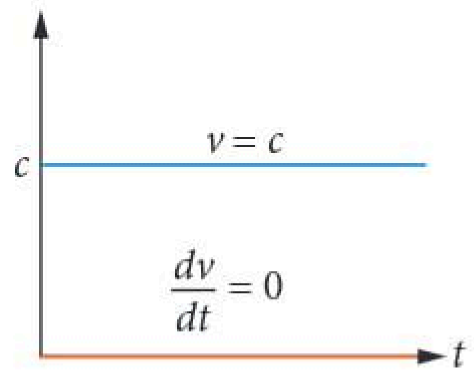
Differentiating velocity, $\frac{d}{dt}\left(\frac{dx}{dt}\right)$ gives acceleration, $\frac{d^2x}{dt^2}$. Acceleration may also be written as a, $\frac{dv}{dt}$, $\dot{v}$, $\ddot{x}$, $v\frac{dv}{dx}$ or $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

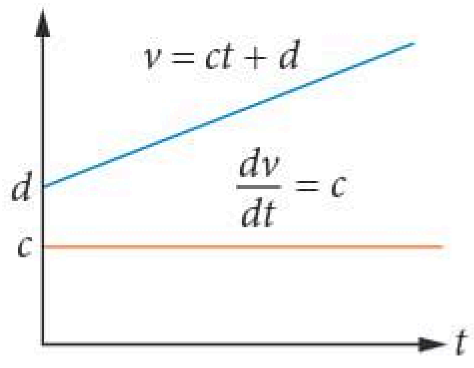
Anti-differentiation of acceleration produces speed (or velocity) and anti-differentiation of speed (or velocity) gives distance (or displacement).

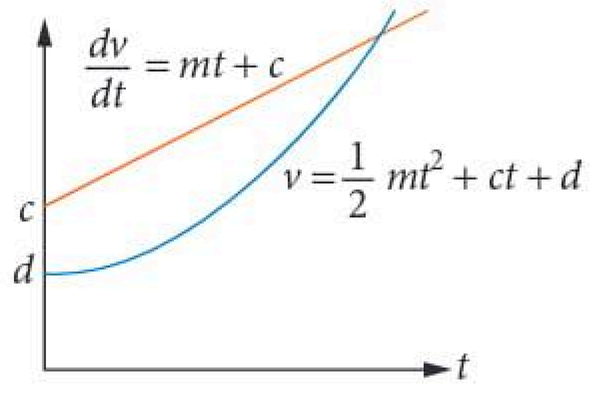


Acceleration–time graphs and velocity–time graphs

Constant velocity implies zero acceleration, and zero acceleration implies constant velocity.







The total area under a velocity–time graph gives the distance travelled and the signed area represents displacement.

For **constant acceleration**, $a = \frac{v_2 - v_1}{t_2 - t_1}$, where $v_2 - v_1$ is the change in velocity and $t_2 - t_1$ is the change in time.

For the velocity function $v(t)$ in the interval $[a, b]$, distance is found using $\int_a^b v(t)\,dt$.

Constant acceleration

For constant, (uniform) acceleration, there are four **equations of kinematics**.

1 $v = u + at$

2 $s = ut + \frac{1}{2}at^2$

3 $v^2 = u^2 + 2as$

4 $s = \frac{1}{2}(u + v)t$

where a is acceleration, t is time, s is displacement, u is initial velocity, v is final velocity.

The Earth's **gravitational acceleration**, g, is approximately 9.81 m/s^2. It appears in problems involving vertical motion, where 'vertical down' usually represents positive gravitational acceleration. Since g is a constant, the equations of kinematics can be applied to motion involving gravity if no external forces such as air resistance exist to affect the acceleration.

Variable acceleration

When acceleration is not constant, calculus can be used to provide relationships between the displacement, velocity and acceleration functions and to obtain an estimate of the solution.

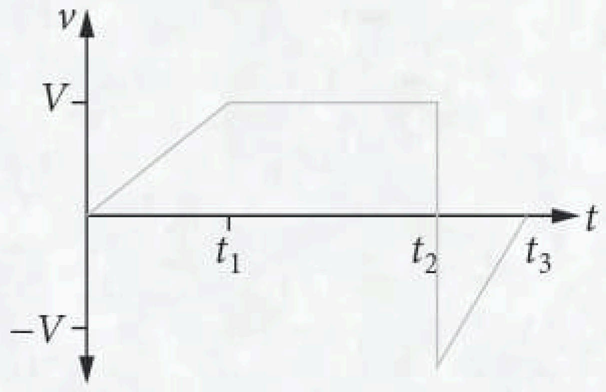
Cumulative examination 1

Total number of marks: 10   Reading time: 4 minutes   Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 (1 mark) The graph below describes the velocity, v, of a particle at time, t.

Write, in terms of t_1 and t_2, the total distance travelled by the particle after t_3 seconds.



2 © VCAA 2015 1Q6 (4 marks) The acceleration a m s^{-2} of a body moving in a straight line in terms of the velocity v m s^{-1} is given by $a = 4v^2$.

Given that $v = e$ when $x = 1$, where x is the displacement of the body in metres, find the velocity of the body when $x = 2$.

3 © VCAA 2005 1IIQ2 (4 marks) $y = e^{2x}\cos(x)$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} + k\frac{dy}{dx} + y = -2e^{2x}\sin(x), \quad \text{where } k \in R.$$

Find the value of k.

4 (1 mark) If $\underset{\sim}{a} = 3\underset{\sim}{i} + 5\underset{\sim}{j}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - 4\underset{\sim}{j}$, find the value of the constant k so that $k(6\underset{\sim}{i} - \underset{\sim}{j})$ is parallel to $2\underset{\sim}{a} + 3\underset{\sim}{b}$.

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 © VCAA 2009 2AQ12 The velocity v m s^{-1} of a body which is moving in a straight line, when it is x m from the origin, is a function of x such that $v = f(x)$.

The acceleration of the body in m s^{-2} is given by

A $f'(x)$ **B** $f'(v)$ **C** $xf'(x)$ **D** $f'\left(\frac{1}{2}x^2\right)$ **E** $f(x)f'(x)$

2 © VCAA 2018 2AQ17 A tourist standing in the basket of a hot air balloon is ascending at 2 m s^{-1}. The tourist drops a camera over the side when the balloon is 50 metres above the ground.

Neglecting air resistance, the time in seconds, correct to the nearest tenth of a second, taken for the camera to hit the ground is

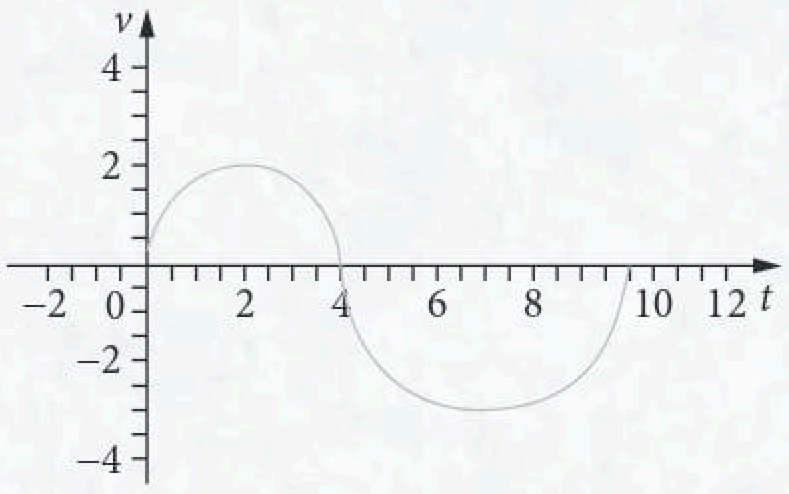
A 2.3 **B** 2.4 **C** 3.0 **D** 3.2 **E** 3.4

3 © VCAA 2018N 2AQ17 An object travels in a straight line relative to an origin O.

At time t seconds its velocity, v m/s, is given by

$$v(t) = \begin{cases} \sqrt{4-(t-2)^2}, & 0 \le t \le 4 \\ -\sqrt{9-(t-7)^2}, & 4 < t \le 10 \end{cases}$$

The graph of $v(t)$ is shown below.



The object will be back at its initial position when t is closest to

A 4.0 **B** 6.5 **C** 6.7 **D** 6.9 **E** 7.0

4 If $f'(x) = x\cos(kx^2)$ for the function $f(x) = \sin(kx^2)$, then the value of k is

A 1 **B** 2 **C** 3 **D** 4 **E** 5

5 The sum of the solutions to $3\sin(x) - \sqrt{3}\cos(x) = 0$ in $[-2\pi, 2\pi]$ is

A $-\frac{4\pi}{3}$ **B** 8π **C** $\frac{5\pi}{6}$ **D** $-\frac{2\pi}{3}$ **E** $\frac{11\pi}{6}$

Section B 4 questions

15 marks

1 (4 marks) A particle's acceleration, a m s^{-2}, is described by the equation $a = -\frac{1}{x^2}$, where x metres is the displacement of the particle from a fixed point, O.

a Express the velocity of the particle in terms of displacement in the form $v = \sqrt{A + \frac{B}{x}}$, where A and B are constants. 2 marks

b Find, correct to two decimal places, the displacement of the particle from O when the velocity is in the opposite direction to the acceleration. 2 marks

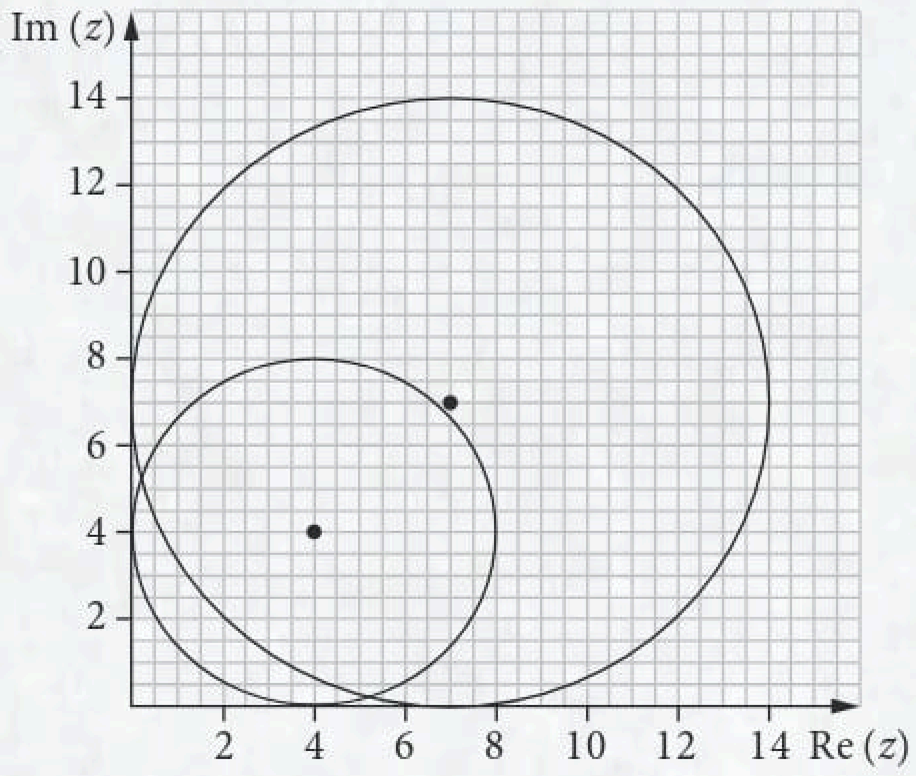
2 (4 marks) From a set of traffic lights on a motorway, car A accelerates uniformly from rest and after 30 seconds reaches a speed of 90 km/h, which it maintains. At the same time as car A begins to move, car B passes it with a uniform speed of 72 km/h, which it maintains.

a How many seconds does car A take to catch up to car B, and at what distance from the traffic lights does this occur? 2 marks

b When the two cars meet, car A continues at 90 km/h for a certain time before decelerating uniformly to a stop at a second set of traffic lights that are 5.5 km from the first set. Meanwhile, car B maintains a constant speed of 72 km/h until it passes the second set of lights. How much sooner than car B did car A arrive at the second set of lights, and how far behind was car B at this time? 2 marks

▶

▶ **3** (3 marks) Two circles are shown in the Argand diagram below.



a State in complex number form the equation of each circle. 1 mark

b Find, correct to three decimal places, the two points of intersection, *A* and *B*, of the two circles. 1 mark

c Write the equation, in complex number form, of the straight line connecting *A* to *B*. 1 mark

4 (4 marks)

a In the equation $y = x + a + \frac{1}{x^2 + ax + b}$, what must be the relationship between *a* and *b* for the graph of $y = x + a + \frac{1}{x^2 + ax + b}$ to have vertical asymptotes? 1 mark

b Express $\frac{1}{x^2 + ax + b}$ in partial fraction form using the values $a = 5$, $b = 6$. 1 mark

c State the equation of each asymptote and indicate its type. 2 marks

CHAPTER

11

VECTOR CALCULUS

Study Design coverage

AREA OF STUDY 5: SPACE AND MEASUREMENT

Vector calculus

- position vector as a function of time and sketching the corresponding path given the function, including circles, ellipses and hyperbolas in cartesian or parametric forms
- the positions of two particles each described as a vector function of time, and whether their paths cross or if the particles meet
- differentiation and anti-differentiation of a vector function with respect to time and applying vector calculus to motion in a plane and in three dimensions.

VCE Mathematics Study Design 2023–2027 p.112, © VCAA 2022

Video playlists (4):

11.1 Position vectors
11.2 Differentiating and integrating vectors
11.3 Applying vectors to motion
VCE question analysis Vector calculus

Worksheets (5):

11.1 Parametric and Cartesian equations
11.2 Derivatives of vectors • Integrals of vector functions • Position, velocity and acceleration
11.3 Projectile motion

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

11.1 Position vectors

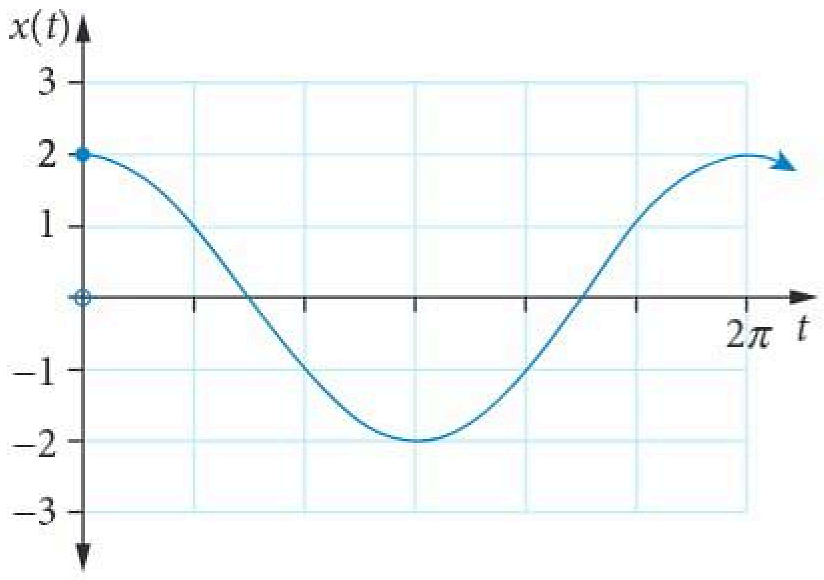
Video playlist Position vectors

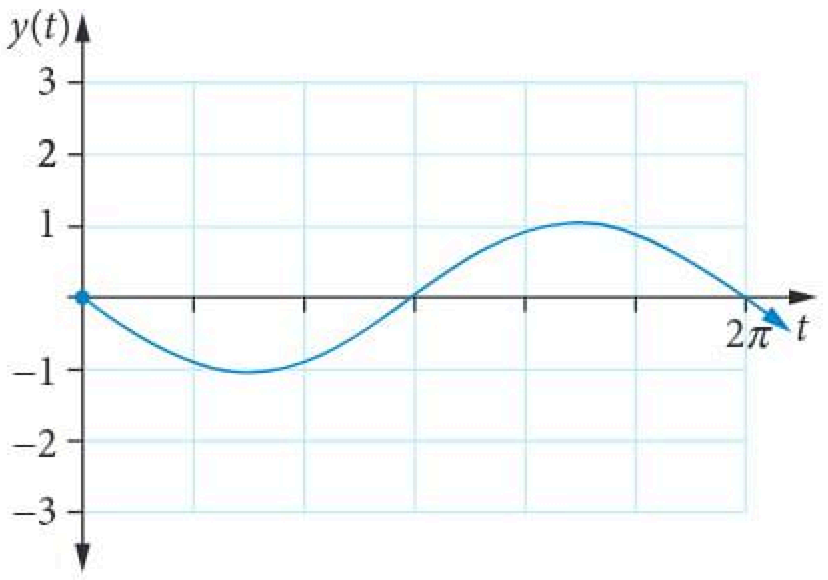
The motion of a particle in space can be measured using vectors. We use a position vector with respect to the origin in terms of the time, t. The variable t in this situation is called a **parameter**.

We describe the position vector in two dimensions as $\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j}$ and in three dimensions as $\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j} + z(t)\underset{\sim}{k}$.

For example, the path of a particle could be given by $\underset{\sim}{r}(t) = 2\cos(t)\underset{\sim}{i} - \sin(t)\underset{\sim}{j}$ for $t \geq 0$.

In this case, $x(t) = 2\cos(t)$ and $y(t) = -\sin(t)$, both for $t \geq 0$.





For $t \geq 0$: $x \in [-2, 2]$ and $y \in [-1, 1]$

Another way to visualise what is happening is setting up a table of values, again with t as the parameter.

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$x(t)$	2	0	–2	0	2
$y(t)$	0	–1	0	1	0

Students should get used to sketching x–t and y–t graphs to find the domain and range of the subsequent Cartesian form.

$\underset{\sim}{r}(t)$ is called the **vector function,** which describes the displacement of a particle from a chosen origin, in terms of time, t.

Exam hack

Always consider the correct domain and range, as this is an area where students frequently lose marks.

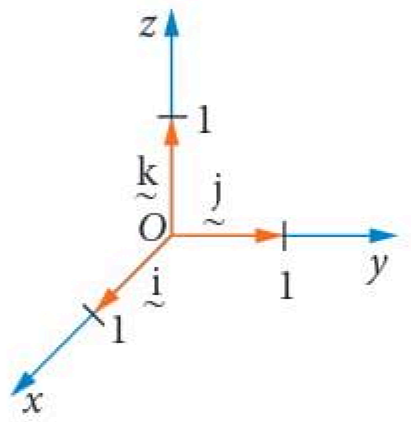
To find the **distance** of a particle from the origin, find the **magnitude** of the displacement, written as $|\underset{\sim}{r}(t)|$.

In two dimensions: $|\underset{\sim}{r}(t)| = \sqrt{(x(t))^2 + (y(t))^2}$.

In three dimensions: $|\underset{\sim}{r}(t)| = \sqrt{(x(t))^2 + (y(t))^2 + (z(t))^2}$.

Remember that for position vectors

- $\underset{\sim}{i}$ is the unit vector in the x direction.
- $\underset{\sim}{j}$ is the unit vector in the y direction.
- $\underset{\sim}{k}$ is the unit vector in the z direction.



Another way of expressing the position vector $x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ is (x, y, z) or $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

WB
p. 221

WORKED EXAMPLE 1 Position vectors

a Find the position of a particle with position vector function $\underset{\sim}{r}(t) = (3t)\underset{\sim}{i} + (2t^2)\underset{\sim}{j}$ at $t = 0$, $t = 1$ and $t = -2$.

b Find the distance of the particle from the origin at $t = 4$.

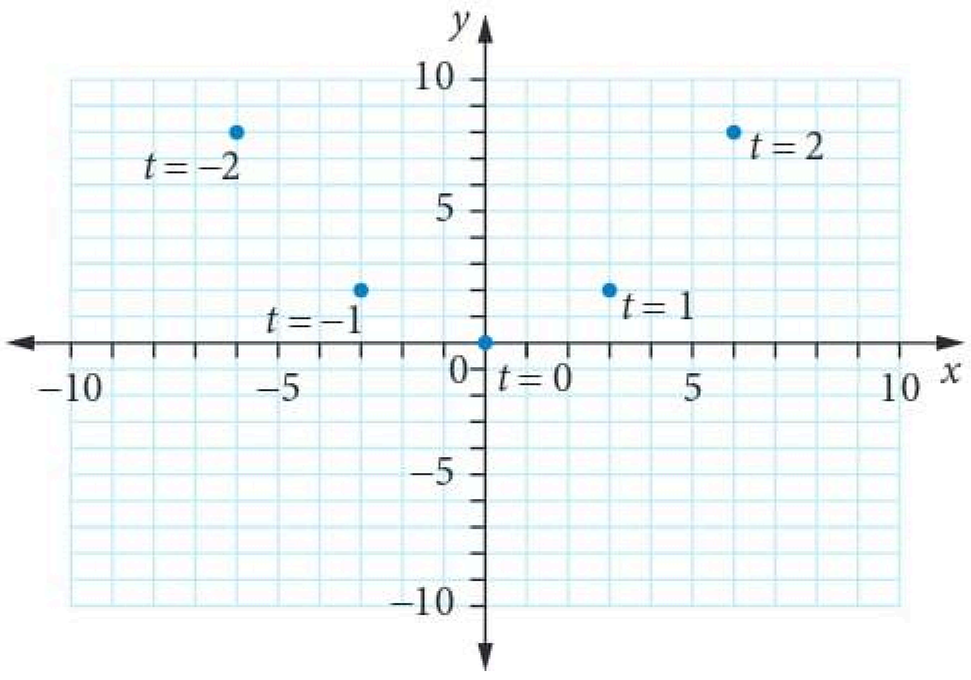
Steps	**Working**
a Substitute $t = 0$, $t = 1$ and $t = -2$.	$\underset{\sim}{r}(0) = 0\underset{\sim}{i} + 0\underset{\sim}{j}$ $\underset{\sim}{r}(1) = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ $\underset{\sim}{r}(-2) = -6\underset{\sim}{i} + 8\underset{\sim}{j}$
b Use distance $= |\underset{\sim}{r}(t)| = \sqrt{(x(t))^2 + (y(t))^2}$.	$|\underset{\sim}{r}(t)| = \sqrt{9t^2 + 4t^4}$ $\therefore\ |\underset{\sim}{r}(4)| = \sqrt{12^2 + 32^2}$ The distance from origin is $4\sqrt{73}$.

Finding Cartesian equations

Complete a table of values for the vector function $\underset{\sim}{r}(t) = (3t)\underset{\sim}{i} + (2t^2)\underset{\sim}{j}$.

t	-2	-1	0	1	2
$\underset{\sim}{r}(t)$	$-6\underset{\sim}{i} + 8\underset{\sim}{j}$	$-3\underset{\sim}{i} + 2\underset{\sim}{j}$	$0\underset{\sim}{i} + 0\underset{\sim}{j}$	$3\underset{\sim}{i} + 2\underset{\sim}{j}$	$6\underset{\sim}{i} + 8\underset{\sim}{j}$

On a Cartesian (x–y) plane, we can plot points where the particle is at particular times and see an outline of the shape of the particle's path.

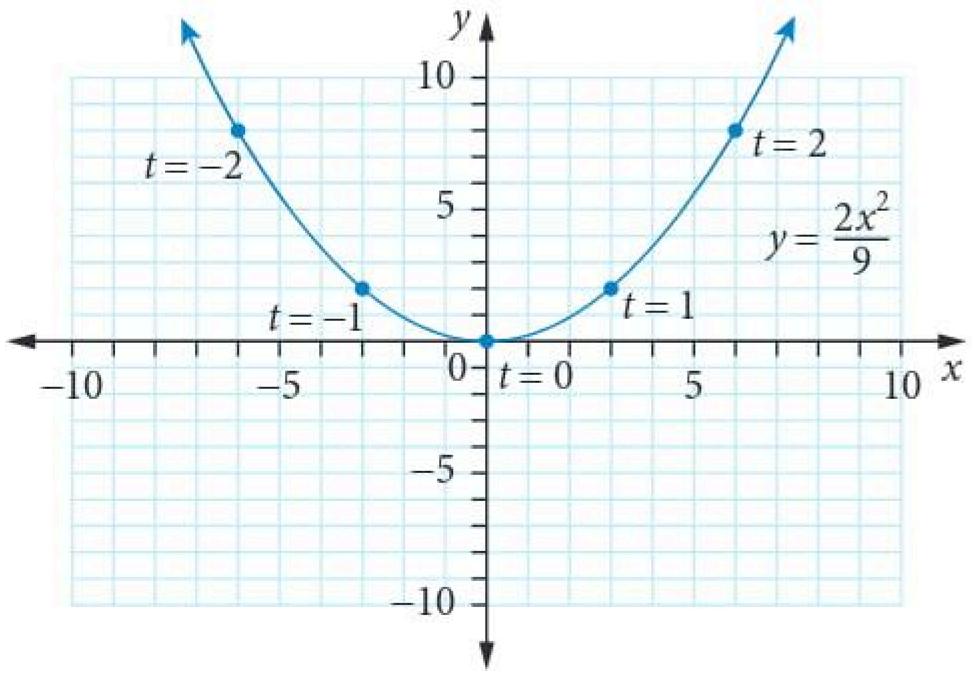


The general shape of these points may form a parabola.

We can prove this by setting up equations for x and y in terms of the parameter t, then eliminate t.

From the vector function, let $x = 3t$ and $y = 2t^2$.

From the first equation, $t = \frac{x}{3}$, so $y = 2\left(\frac{x}{3}\right)^2$, giving the parabola $y = \frac{2x^2}{9}$.

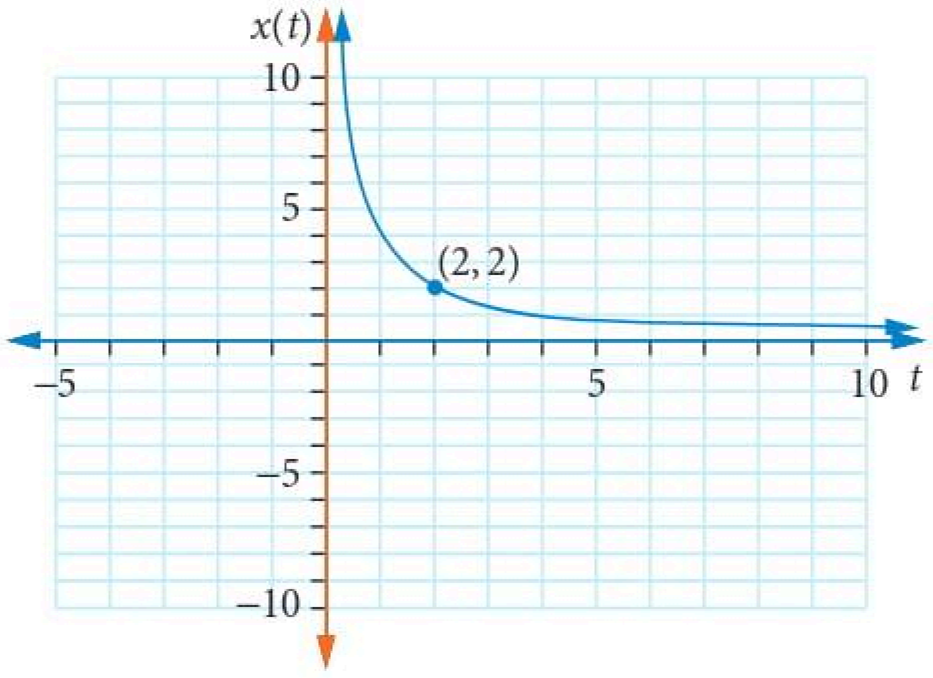
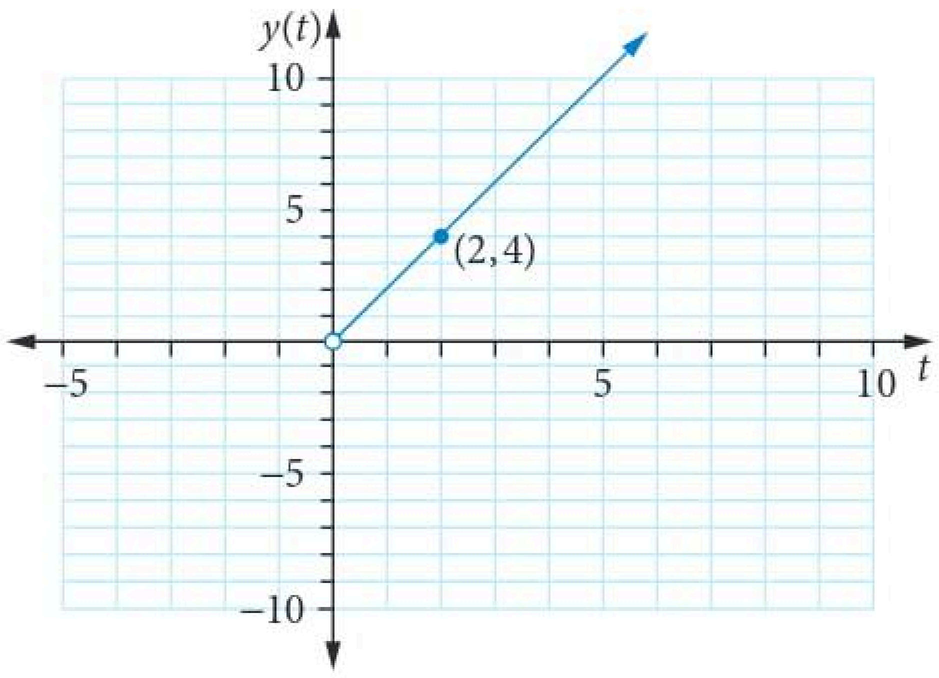


WB
p. 222

WORKED EXAMPLE 2 Cartesian equations

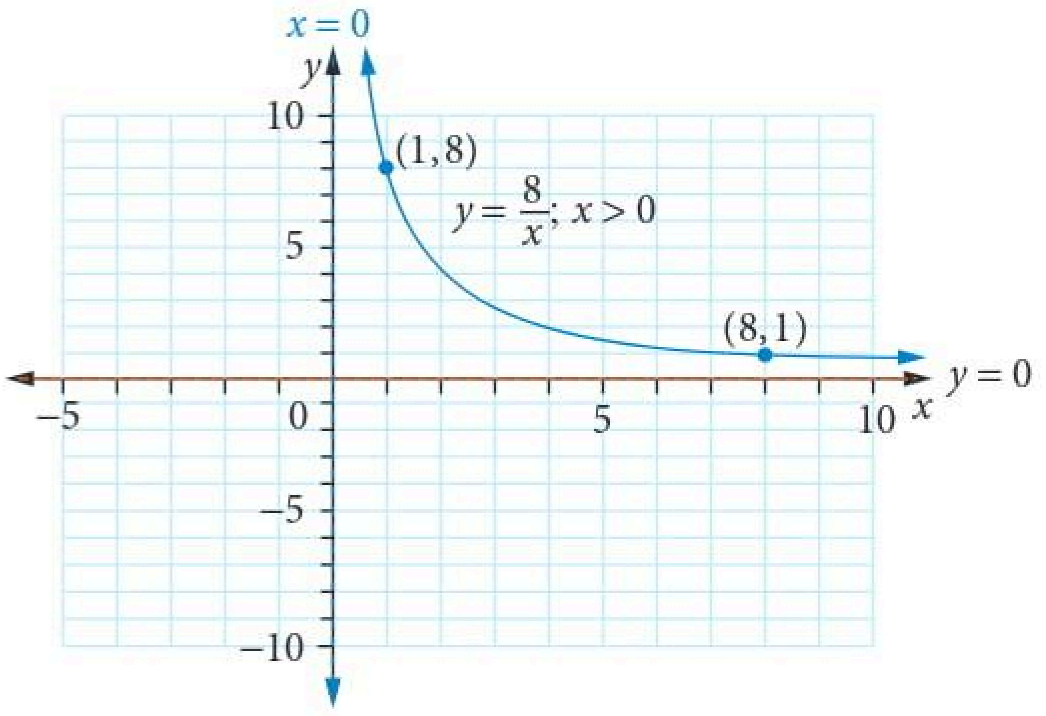
a Find the Cartesian equation of the vector function $\underset{\sim}{r}(t) = \left(\frac{4}{t}\right)\underset{\sim}{i} + (2t)\underset{\sim}{j}$ for $t > 0$.

b Give the domain and range of the Cartesian equation and sketch its graph.

Steps	**Working**
a 1 Use the vector function to write x and y as functions of t.	$x = \frac{4}{t}, y = 2t$
2 Eliminate t.	From x: $t = \frac{4}{x}$ so $y = 2\left(\frac{4}{x}\right) = \frac{8}{x}$. Cartesian equation: $y = \frac{8}{x}$
b 1 Consider $x = \frac{4}{t}, y = 2t$ for $t > 0$ to find the domain and range of $y = \frac{8}{x}$. Sketch graphs of x–t and y–t.	 giving $x > 0$.  giving $y > 0$.

2 State the domain and range of $y = \frac{8}{x}$ and sketch its graph.

For $y = \frac{8}{x}$: domain = $x > 0$, range = $y > 0$.



Worksheet Parametric and Cartesian equations

Sketching paths in Cartesian and parametric forms

Consider the position vector $\underset{\sim}{r}(t) = 2\cos(t)\underset{\sim}{i} - 4\sin(t)\underset{\sim}{j}, t \geq 0$.

To find its Cartesian equation, let $x = 2\cos(t)$ and $y = -4\sin(t)$.

Make $\cos(t)$ and $\sin(t)$ the subject of the above equations and use the trigonometric identity $\cos^2(t) + \sin^2(t) = 1$ in order to eliminate t.

Exam hack

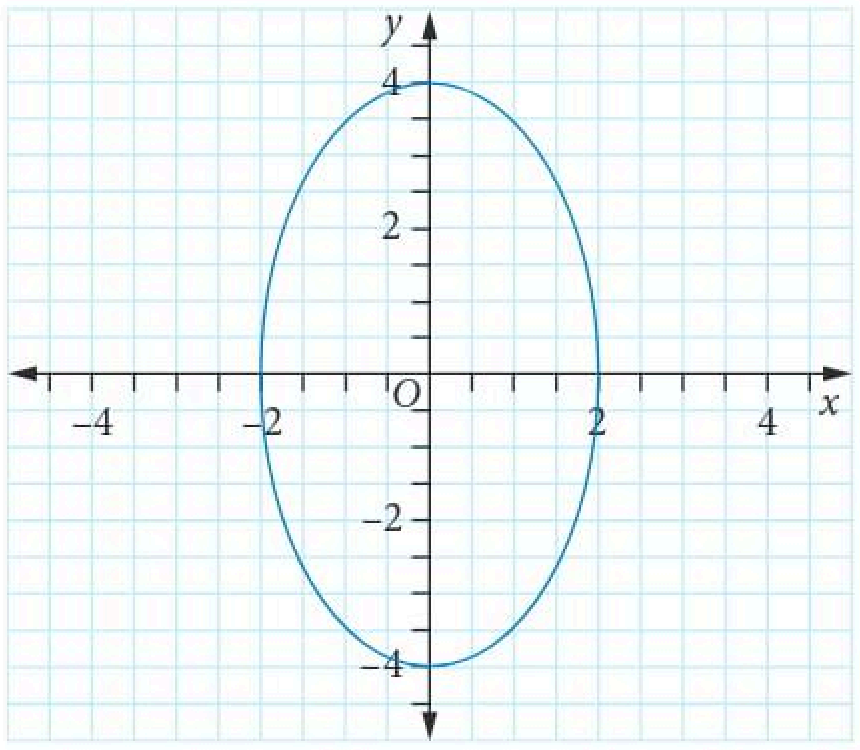
In an exam, you must write down the particular trigonometric identity used to eliminate t.

Rearrange to get $\cos(t) = \frac{x}{2}$ and $\sin(t) = -\frac{y}{4}$.

$\cos^2(t) + \sin^2(t) = 1$ gives $\left(\frac{x}{2}\right)^2 + \left(-\frac{y}{4}\right)^2 = 1$.

This is an ellipse with equation $\frac{x^2}{4} + \frac{y^2}{16} = 1$.

For $t \geq 0$, $x \in [-2, 2]$ and $y \in [-4, 4]$.



WB p. 223

WORKED EXAMPLE 3 Cartesian graphs

Sketch the Cartesian graph of the vector function $\underset{\sim}{r}(t) = (\sec(t))\underset{\sim}{i} + (2\tan(t))\underset{\sim}{j}, t \geq 0$.

Steps	Working
1 Write two equations for x and y and rearrange them into a possible trigonometric identity.	$x = \sec(t)$ $y = 2\tan(t)$, giving $\tan(t) = \frac{y}{2}$.
2 Use the trigonometric identity $1 + \tan^2(t) = \sec^2(t)$.	$1 + \tan^2(t) = \sec^2(t)$ gives $1 + \left(\frac{y}{2}\right)^2 = x^2$. Rearrange to get $x^2 - \frac{y^2}{4} = 1$.

3 Check the domain and range for $t \geq 0$.

Sketch the graph of x–t and y–t.

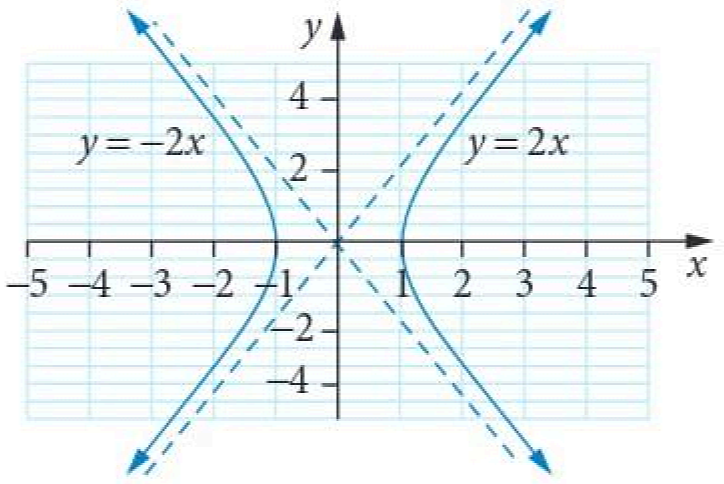
$x = \sec(t)$ for $t \geq 0$

$y = 2\tan(t)$ for $t \geq 0$

$\therefore x \geq 1, x \leq -1$

giving $y \in R$.

4 Sketch the graph of $x^2 - \frac{y^2}{4} = 1$, a non-rectangular hyperbola with asymptotes at $y = \pm 2x$.



Note how the graph fits the domain and range found of $x \geq 1, x \leq -1$ and $y \in R$.

EXERCISE 11.1 Position vectors

ANSWERS p. 599

Mastery

1 WORKED EXAMPLE 1 **TECH-FREE** Find the position of a particle with position vector function $\underset{\sim}{r}(t) = (3t+1)\underset{\sim}{i} + (2t)\,\underset{\sim}{j}$ at $t = 0$ and $t = 1$.

2 WORKED EXAMPLE 2 **TECH-FREE** Find the Cartesian equation of the vector function $\underset{\sim}{r}(t) = \left(3t^2\right)\underset{\sim}{i} + (t)\,\underset{\sim}{j}$ for $t \geq 0$.

3 WORKED EXAMPLE 3 **TECH-FREE**

- **a** Find the Cartesian equation, and the domain, of the vector function $\underset{\sim}{r}(t) = (\sin(t))\,\underset{\sim}{i} + (\cos(2t))\,\underset{\sim}{j}$ for $t \geq 0$.
- **b** Sketch the Cartesian graph of the vector function $\underset{\sim}{r}(t) = (\sin(t))\,\underset{\sim}{i} + (\cos(2t))\,\underset{\sim}{j}$ for $t \geq 0$.

4 **TECH-FREE**

- **a** Find the Cartesian equation, and the domain, of the vector function $\underset{\sim}{r}(t) = (\sin(t))\,\underset{\sim}{i} + (-\cos(t))\,\underset{\sim}{j},\ t \geq 0$.
- **b** Sketch the graph of the vector function $\underset{\sim}{r}(t) = (\sin(t))\,\underset{\sim}{i} + (-\cos(t))\,\underset{\sim}{j},\ t \geq 0$.

5 The position of a particle described by the vector function $\underset{\sim}{r}(t) = (-t)\underset{\sim}{i} + (t^2)\,\underset{\sim}{j}$ at $t = 1$ is

A $0\underset{\sim}{i} + 0\underset{\sim}{j}$ **B** $\underset{\sim}{i} + \underset{\sim}{j}$ **C** $-\underset{\sim}{i} - \underset{\sim}{j}$ **D** $-\underset{\sim}{i} + \underset{\sim}{j}$ **E** $\underset{\sim}{i} - \underset{\sim}{j}$

6 The position of a particle described by the vector function $\underset{\sim}{r}(t) = (-t)\underset{\sim}{i} + \left(t^2\right)\underset{\sim}{j}$, at $t = -1$ is

A $0\underset{\sim}{i} + 0\underset{\sim}{j}$ **B** $\underset{\sim}{i} + \underset{\sim}{j}$ **C** $-\underset{\sim}{i} - \underset{\sim}{j}$ **D** $-\underset{\sim}{i} + \underset{\sim}{j}$ **E** $\underset{\sim}{i} - \underset{\sim}{j}$

7 The Cartesian equation of the vector function $\underset{\sim}{r}(t) = \left(\frac{t}{5}\right)\underset{\sim}{i} + \left(2t^2\right)\underset{\sim}{j}$ is

A $t = \frac{x}{5}$ **B** $y = 2t^2$ **C** $y = \frac{2x^2}{5}$ **D** $y = 50x^2$ **E** $y = 2x^2$

8 This Cartesian graph could have the vector equation

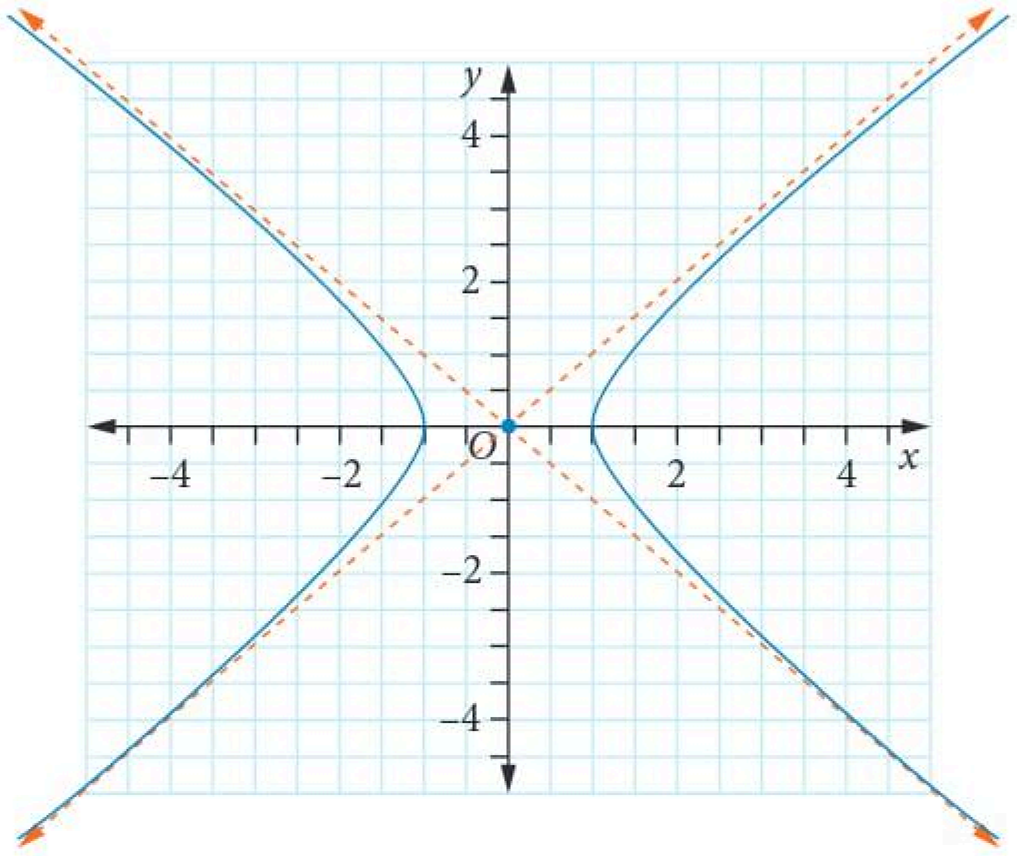
A $\underset{\sim}{r}(t) = (\sec(t))\underset{\sim}{i} + (\tan(t))\underset{\sim}{j}, t \in R$

B $\underset{\sim}{r}(t) = (\sec(2t))\underset{\sim}{i} + (\tan(t))\underset{\sim}{j}, t \in R$

C $\underset{\sim}{r}(t) = (\sec(t))\underset{\sim}{i} + (\tan(2t))\underset{\sim}{j}, t \in R$

D $\underset{\sim}{r}(t) = (\sec(t))\underset{\sim}{i} + (\cos(t))\underset{\sim}{j}, t \in R$

E $\underset{\sim}{r}(t) = (\sin(2t))\underset{\sim}{i} + (\cos(2t))\underset{\sim}{j}, t \in R$



Exam practice

80–100% 60–79% 0–59%

9 © VCAA 2018 1Q9 TECH-FREE (5 marks) A curve is specified parametrically

$\underset{\sim}{r}(t) = \sec(t)\underset{\sim}{i} + \frac{\sqrt{2}}{2}\tan(t)\underset{\sim}{j}, t \in R.$

a 80% Show that the cartesian equation of the curve is $x^2 - 2y^2 = 1$. 2 marks

b 70% Find the x-coordinates of the points of intersection of the curve $x^2 - 2y^2 = 1$ and the line $y = x - 1$. 1 mark

c 30% Find the volume of the solid of revolution formed when the region bounded by the curve and the line is rotated about the x-axis. 2 marks

10 © VCAA 2006 1Q7 TECH-FREE (4 marks) The position vector of a moving particle is given by $\underset{\sim}{r}(t) = \sqrt{t-2}\underset{\sim}{i} + 2t\underset{\sim}{j}$ for $2 \le t \le 6$.

a 77% Find the cartesian equation of the path followed by the particle. 2 marks

b 42% Sketch the path of the particle. 2 marks

11 © VCAA 2006 2AQ3 88% The position vector of a particle at time $t \ge 0$ is given by $\underset{\sim}{r}(t) = (1+t)\underset{\sim}{i} + (1-t)\underset{\sim}{j}$. The path of the particle has equation

A $y = x - 2$ **B** $y = x + 2$ **C** $y = -x - 2$ **D** $y = -x + 2$ **E** $y = x - 1$

12 © VCAA 2003 1IQ22 39% Let $\underset{\sim}{i}$ and $\underset{\sim}{j}$ be unit vectors in the east and north directions respectively. At time t, $t \ge 0$, the position vector of particle L is given by $\underset{\sim}{r} = (5t - 8)\underset{\sim}{i} + \left(t^2 - 5t + 6\right)\underset{\sim}{j}$, and the position vector of particle M is given by $\underset{\sim}{s} = (t^2 - t)\underset{\sim}{i} + (3 - t)\underset{\sim}{j}$.
Particle L is directly north of particle M at time

A 0 **B** 1 **C** 2 **D** 3 **E** 4

11.2 Differentiating and integrating vectors

Differentiating vector functions

From Chapter 10 *Kinematics*, we know that:

- displacement $= x$
- velocity $= v = \dfrac{dx}{dt}$
- acceleration $= a = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$

Displacement, velocity and acceleration vectors

Given the position vector $\underset{\sim}{\mathrm{r}}(t) = x\underset{\sim}{\mathrm{i}} + y\underset{\sim}{\mathrm{j}} + x\underset{\sim}{\mathrm{k}}$,

$$\text{velocity} = \underset{\sim}{\dot{\mathrm{r}}}(t) = \frac{dx}{dt}\underset{\sim}{\mathrm{i}} + \frac{dy}{dt}\underset{\sim}{\mathrm{j}} + \frac{dz}{dt}\underset{\sim}{\mathrm{k}}$$

$$\text{acceleration} = \underset{\sim}{\ddot{\mathrm{r}}}(t) = \frac{d^2x}{dt^2}\underset{\sim}{\mathrm{i}} + \frac{d^2y}{dt^2}\underset{\sim}{\mathrm{j}} + \frac{d^2z}{dt^2}\underset{\sim}{\mathrm{k}}$$

Video playlist Differentiating and integrating vectors

Worksheet Derivatives of vectors

The tangent to the graph of a displacement function will give the **direction of motion** of the particle at that instant. But the instantaneous rate of change of displacement is velocity, so we need to consider the **velocity vector** to find that direction.

Exam hack

When an exam question asks for the angle of direction of a curve, a common mistake is to use the displacement vector, but when we are dealing with a curved direction we need to consider the **tangent** to the curve, which is the velocity vector.

To find the direction of motion, differentiate the displacement and use the velocity vector to find the angle required.

WORKED EXAMPLE 4 Velocity and acceleration

WB p. 224

a Find the velocity and acceleration of an object with position vector $\underset{\sim}{\mathrm{r}}(t) = 2e^{2t}\underset{\sim}{\mathrm{i}} + \sin(3t)\underset{\sim}{\mathrm{j}} + 3t\underset{\sim}{\mathrm{k}}$.

b Hence find the velocity and acceleration at $t = \pi$.

Steps	**Working**
a **1** Find $\underset{\sim}{\dot{\mathrm{r}}}(t)$.	$\text{velocity} = \underset{\sim}{\dot{\mathrm{r}}}(t) = 4e^{2t}\underset{\sim}{\mathrm{i}} + 3\cos(3t)\underset{\sim}{\mathrm{j}} + 3\underset{\sim}{\mathrm{k}}$
2 Find $\underset{\sim}{\ddot{\mathrm{r}}}(t)$.	$\text{acceleration} = \underset{\sim}{\ddot{\mathrm{r}}}(t) = 8e^{2t}\underset{\sim}{\mathrm{i}} - 9\sin(3t)\underset{\sim}{\mathrm{j}}$
b Substitute $t = \pi$ into $\underset{\sim}{\dot{\mathrm{r}}}(t)$ and $\underset{\sim}{\ddot{\mathrm{r}}}(t)$.	$\text{velocity} = \underset{\sim}{\dot{\mathrm{r}}}(\pi) = 4e^{2\pi}\underset{\sim}{\mathrm{i}} + 3\cos(3\pi)\underset{\sim}{\mathrm{j}} + 3\underset{\sim}{\mathrm{k}}$ $\therefore \underset{\sim}{\dot{\mathrm{r}}}(\pi) = 4e^{2\pi}\underset{\sim}{\mathrm{i}} - 3\underset{\sim}{\mathrm{j}} + 3\underset{\sim}{\mathrm{k}}$ $\text{acceleration} = \underset{\sim}{\ddot{\mathrm{r}}}(\pi) = 8e^{2\pi}\underset{\sim}{\mathrm{i}} - 9\sin(3\pi)\underset{\sim}{\mathrm{j}}$ $\therefore \underset{\sim}{\ddot{\mathrm{r}}}(\pi) = 8e^{2\pi}\underset{\sim}{\mathrm{i}}$

Speed is the magnitude of the velocity, so to find speed we find the velocity first.

WB

p. 225

WORKED EXAMPLE 5 Speed

a Find the speed function from the velocity vector $\dot{\underset{\sim}{r}}(t) = 4e^{2t}\underset{\sim}{i} + 3\cos(3t)\underset{\sim}{j} + 3\underset{\sim}{k}$.

b Hence find the speed at $t = \pi$.

Steps	**Working**
a Find the magnitude of $\dot{\underset{\sim}{r}}(t)$.	$\lvert\dot{\underset{\sim}{r}}(t)\rvert = \sqrt{\left(4e^{2t}\right)^2 + \left(3\cos(3t)\right)^2 + 3^2}$ $\therefore \lvert\dot{\underset{\sim}{r}}(t)\rvert = \sqrt{16e^{4t} + 9\cos^2(3t) + 9}$
b Substitute $t = \pi$ into $\lvert\dot{\underset{\sim}{r}}(t)\rvert$.	$\lvert\dot{\underset{\sim}{r}}(\pi)\rvert = \sqrt{16e^{4\pi} + 9\cos^2(3\pi) + 9}$ $= \sqrt{16e^{4\pi} + 9 + 9}$ $\therefore \lvert\dot{\underset{\sim}{r}}(\pi)\rvert = \sqrt{16e^{4\pi} + 18}$

Worksheets
Integrals of vector functions

Position, velocity and acceleration

Integrating vector functions

We know that:

- velocity is the anti-derivative of acceleration.
- displacement is the anti-derivative of velocity.

When we integrate with vectors, the constant of integration, ‘+ c’ becomes ‘+ $\underset{\sim}{c}$’, a constant in vector form.

For example, if we are given the velocity vector of an object, $\dot{\underset{\sim}{r}}(t) = 2\underset{\sim}{i} + \sin(3t)\underset{\sim}{j} - \underset{\sim}{k}$ and we know that at $t = 0$, the displacement of the object is $-2\underset{\sim}{i} + \underset{\sim}{k}$, we can find the displacement vector of the object.

Given $\dot{\underset{\sim}{r}}(t) = 2\underset{\sim}{i} + \sin(3t)\,j - \underset{\sim}{k}$,

$$\underset{\sim}{r}(t) = \int\left(2\underset{\sim}{i} + \sin(3t)\underset{\sim}{j} - \underset{\sim}{k}\right)dt$$

$$= 2t\underset{\sim}{i} - \frac{1}{3}\cos(3t)\underset{\sim}{j} - t\underset{\sim}{k} + \underset{\sim}{c}$$

Given $\underset{\sim}{r}(0) = -2\underset{\sim}{i} + \underset{\sim}{k}$,

$$-2\underset{\sim}{i} + \underset{\sim}{k} = 0\underset{\sim}{i} - \frac{1}{3}\cos(0)\underset{\sim}{j} - 0\underset{\sim}{k} + \underset{\sim}{c}$$

$$-2\underset{\sim}{i} + \underset{\sim}{k} = -\frac{1}{3}\underset{\sim}{j} + \underset{\sim}{c}$$

$$\underset{\sim}{c} = -2\underset{\sim}{i} + \frac{1}{3}\underset{\sim}{j} + \underset{\sim}{k}$$

$$\therefore \underset{\sim}{r}(t) = 2t\underset{\sim}{i} - \frac{1}{3}\cos(3t)\underset{\sim}{j} - t\underset{\sim}{k} - 2\underset{\sim}{i} + \frac{1}{3}\underset{\sim}{j} + \underset{\sim}{k}$$

Collecting $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$ terms: $\underset{\sim}{r}(t) = (2t - 2)\underset{\sim}{i} + \left(-\frac{1}{3}\cos(3t) + \frac{1}{3}\right)\underset{\sim}{j} + (-t + 1)\underset{\sim}{k}$.

We can then find displacement at any time.

For example, at $t = 2\pi$, $\underset{\sim}{r}(2\pi) = (4\pi - 2)\underset{\sim}{i} + (1 - 2\pi)\underset{\sim}{k}$.

WB

p. 226

WORKED EXAMPLE 6 Velocity, acceleration, displacement

The acceleration in m/s^2 of a particle is $\ddot{\underset{\sim}{r}}(t) = 6t\underset{\sim}{i} + \sin(t)\underset{\sim}{j} - 2\underset{\sim}{k}$.

Find the velocity and displacement of the particle, given that initially the particle has a velocity of $3\underset{\sim}{i}$ m/s and a position of $2\underset{\sim}{j}$ m.

Steps	**Working**
1 Integrate to find $\dot{\underset{\sim}{r}}(t)$.	$\ddot{\underset{\sim}{r}}(t) = 6t\underset{\sim}{i} + \sin(t)\underset{\sim}{j} - 2\underset{\sim}{k}$ $\dot{\underset{\sim}{r}}(t) = \int\left(6t\underset{\sim}{i} + \sin(t)\underset{\sim}{j} - 2\underset{\sim}{k}\right)dt$ $= 3t^2\underset{\sim}{i} - \cos(t)\underset{\sim}{j} - 2t\underset{\sim}{k} + \underset{\sim}{c}$

2 Substitute the initial velocity to find $\underset{\sim}{c}$. At $t = 0$, $\dot{\underset{\sim}{r}}(0) = 3\underset{\sim}{i}$.	$\dot{\underset{\sim}{r}}(0) = 0\underset{\sim}{i} - \underset{\sim}{j} - 0\underset{\sim}{k} + \underset{\sim}{c} = 3\underset{\sim}{i}$. $\therefore \underset{\sim}{c} = 3\underset{\sim}{i} + \underset{\sim}{j}$
3 Find the expression for $\dot{\underset{\sim}{r}}(t)$.	$\dot{\underset{\sim}{r}}(t) = \left(3t^2 + 3\right)\underset{\sim}{i} + \left(1 - \cos(t)\right)\underset{\sim}{j} - 2t$
4 Integrate to find $\underset{\sim}{r}(t)$. We use a different vector constant $+\underset{\sim}{d}$ as we have already used $+\underset{\sim}{c}$.	$\underset{\sim}{r}(t) = \int\left(\left(3t^2 + 3\right)\underset{\sim}{i} + \left(1 - \cos(t)\right)\underset{\sim}{j} - 2t\underset{\sim}{k}\right)dt$ $= \left(t^3 + 3t\right)\underset{\sim}{i} + \left(t - \sin(t)\right)\underset{\sim}{j} - t^2\underset{\sim}{k} + \underset{\sim}{d}$
5 Substitute the initial displacement to find $\underset{\sim}{d}$. At $t = 0$, $\underset{\sim}{r}(0) = 2\underset{\sim}{j}$.	$\underset{\sim}{r}(0) = 0\underset{\sim}{i} + 0\underset{\sim}{j} - 0\underset{\sim}{k} + \underset{\sim}{d} = 2\underset{\sim}{j}$ $\therefore \underset{\sim}{d} = 2\underset{\sim}{j}$
6 Find the expression for $\underset{\sim}{r}(t)$.	$\underset{\sim}{r}(t) = \left(t^3 + 3t\right)\underset{\sim}{i} + \left(t - \sin(t) + 2\right)\underset{\sim}{j} - t^2\underset{\sim}{k}$

EXERCISE 11.2 Differentiating and integrating vectors

ANSWERS p. 599

Recap

1 a TECH-FREE Find the Cartesian equation of the vector function $\underset{\sim}{r}(t) = 3\sin(2t)\underset{\sim}{i} + 3\cos(2t)\underset{\sim}{j}$ for $t > 0$.

b State the domain and range of the Cartesian equation.

2 The position of a particle described by the vector function $\underset{\sim}{r}(t) = (2t)\underset{\sim}{i} + \left(-t^2\right)\underset{\sim}{j}$ at $t = 1$ is

A $0\underset{\sim}{i} + 0\underset{\sim}{j}$ **B** $2\underset{\sim}{i} + 2\underset{\sim}{j}$ **C** $2\underset{\sim}{i} - \underset{\sim}{j}$ **D** $2\underset{\sim}{i} + \underset{\sim}{j}$ **E** $2\underset{\sim}{i} - 4\underset{\sim}{j}$

Mastery

3 WORKED EXAMPLE 4 TECH-FREE Find the velocity and acceleration of an object with position vector $\underset{\sim}{r}(t) = 3t^2\underset{\sim}{i} - t\underset{\sim}{j} + (t+1)^2\underset{\sim}{k}$ at $t = 3$.

4 WORKED EXAMPLE 5 TECH-FREE Find the speed at $t = 1$ given the velocity vector $\dot{\underset{\sim}{r}}(t) = 3t\underset{\sim}{i} - t\underset{\sim}{j} + (t+1)\underset{\sim}{k}$.

5 WORKED EXAMPLE 6 TECH-FREE The acceleration in m/s^2 of a particle is $\ddot{\underset{\sim}{r}}(t) = -2t\underset{\sim}{i} + \cos(t)\underset{\sim}{j} + 3t^2\underset{\sim}{k}$.

Find the velocity of the particle, given that initially the particle has a velocity of $3\underset{\sim}{i} + 2\underset{\sim}{j}$ m/s.

6 The velocity in m/s of a particle is $\dot{\underset{\sim}{r}}(t) = \sin(2t)\underset{\sim}{i} + \cos(t)\underset{\sim}{j}$.

The displacement of the particle, given that initially the particle starts from the origin, is

A $\underset{\sim}{r}(t) = \left(\frac{1}{2} - \frac{1}{2}\cos(2t)\right)\underset{\sim}{i} + \sin(t)\underset{\sim}{j}$

B $\underset{\sim}{r}(t) = \left(-\frac{1}{2}\cos(2t)\right)\underset{\sim}{i} + \sin(t)\underset{\sim}{j} + \frac{1}{2}$

C $\underset{\sim}{r}(t) = \left(\frac{1}{2} + \cos(2t)\right)\underset{\sim}{i} + \sin(t)\underset{\sim}{j}$

D $\underset{\sim}{r}(t) = 2\cos(2t)\underset{\sim}{i} - \sin(t)\underset{\sim}{j}$

E $\underset{\sim}{r}(t) = \frac{1}{2}\cos(2t)\underset{\sim}{i} - \sin(t)\underset{\sim}{j}$

7 A particle that moves with the position vector $\underset{\sim}{r}(t) = 2t\,\underset{\sim}{i} + \cos(t)\underset{\sim}{j}$. Its velocity vector at $t = \frac{\pi}{2}$ is

A $2\underset{\sim}{i} - \underset{\sim}{j}$ **B** $2\underset{\sim}{i} + \underset{\sim}{j}$ **C** $-2\underset{\sim}{i} + \underset{\sim}{j}$ **D** $\pi\underset{\sim}{i} - \underset{\sim}{j}$ **E** $\left(\frac{\pi}{2}\right)\underset{\sim}{i} - \underset{\sim}{j}$

▶ **8** The speed of a particle with the velocity vector $\dot{\underset{\sim}{r}}(t) = 2\underset{\sim}{i} - \left(t^2 - 1\right)\underset{\sim}{j} + t\underset{\sim}{k}$ at $t = 2$ is

A $\sqrt{4 + \left(t^2 - 1\right)^2 + (t)^2}$ **B** $2\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}$ **C** $\sqrt{15}$

D $\sqrt{13}$ **E** $\sqrt{17}$

Exam practice

80–100% 60–79% 0–59%

9 © VCAA 2007 1Q6 TECH-FREE (6 marks) A particle moves so that its velocity at time t is given by $\underset{\sim}{v}(t) = -4\sin(2t)\underset{\sim}{i} + 6\cos(2t)\underset{\sim}{j}$ for $0 \le t \le \frac{\pi}{2}$.

a 81% Given that $\underset{\sim}{r}(0) = 2\underset{\sim}{i}$, find the position vector $\underset{\sim}{r}(t)$ of the particle at any time t. 2 marks

b 71% Find the cartesian equation of the path followed by the particle. 2 marks

c 44% Sketch the path followed by the particle. 2 marks

10 © VCAA 2013 1Q7 TECH-FREE (6 marks) The position vector $\underset{\sim}{r}(t)$ of a particle moving relative to an origin O at time t seconds is given by $\underset{\sim}{r}(t) = 4\sec(t)\underset{\sim}{i} + 2\tan(t)\underset{\sim}{j},\ t \in \left[0, \frac{\pi}{2}\right)$ where the components are measured in metres.

a 82% Show that the cartesian equation of the path of the particle is $\frac{x^2}{16} - \frac{y^2}{4} = 1$. 2 marks

b 44% Sketch the path of the particle, labelling any asymptotes with their equations. 2 marks

c 39% Find the speed of the particle, in m s^{-1}, when $t = \frac{\pi}{4}$. 2 marks

11 © VCAA 2012 1Q9 TECH-FREE (7 marks) The position of a particle at time t is given by $\underset{\sim}{r}(t) = \left(2\sqrt{t^2 + 2} - t^2\right)\underset{\sim}{i} + \left(2\sqrt{t^2 + 2} + 2t\right)\underset{\sim}{j},\ t \ge 0.$

a 73% Find the velocity of the particle at time t. 1 mark

b 58% Find the speed of the particle at time $t = 1$ in the form $\frac{a\sqrt{b}}{c}$, where a, b and c are positive integers. 2 marks

c 41% Show that at time $t = 1$, $\frac{dy}{dx} = \frac{1 + \sqrt{3}}{1 - \sqrt{3}}$. 2 marks

d 32% Find the angle, in terms of π, between the vector $-\sqrt{3}\underset{\sim}{i} + \underset{\sim}{j}$ and the vector $\underset{\sim}{r}(t)$ at time $t = 0$. 2 marks

12 © VCAA 2010 1Q8 TECH-FREE (5 marks) The path of a particle is given by $\underset{\sim}{r}(t) = t\sin(t)\underset{\sim}{i} - t\cos(t)\underset{\sim}{j},\ t \ge 0.$

The particle leaves the origin at $t = 0$ and then spirals outwards.

a 55% Show that the **second time** the particle crosses the x-axis after leaving the origin occurs when $t = \frac{3\pi}{2}$. 1 mark

b 39% Find the speed of the particle when $t = \frac{3\pi}{2}$. 3 marks

Let θ be the acute angle at which the path of the particle crosses the x-axis.

c 20% Find tan (θ) when $t = \frac{3\pi}{2}$. 1 mark

13 © VCAA 2007 1Q9 28% TECH-FREE (3 marks) A particle moves in the cartesian plane with position vector $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$, where x and y are functions of t. If its velocity vector is $\underset{\sim}{v} = -y\underset{\sim}{i} + x\underset{\sim}{j}$, find the acceleration vector of the particle in terms of the position vector $\underset{\sim}{r}$. ▶

14 © VCAA | 2005 1IQ25 | 80% The position vector of a particle at time t is given by $\underset{\sim}{r}(t) = (3t^2 - 2)\underset{\sim}{i} - (7 - 5t)\underset{\sim}{j} - 4\underset{\sim}{k}$. The velocity vector of the particle at time t is given by

A $6t\underset{\sim}{i} + 5\underset{\sim}{j}$

B $6t\underset{\sim}{i} - 5\underset{\sim}{j}$

C $6t\underset{\sim}{i} - 5\underset{\sim}{j} - 4\underset{\sim}{k}$

D $(6t - 2)\underset{\sim}{i} - 2\underset{\sim}{j} - 4\underset{\sim}{k}$

E $(t^3 - 2t)\underset{\sim}{i} - \left(7t - \frac{5}{2}t^2\right)\underset{\sim}{j} - 4t\underset{\sim}{k}$

15 © VCAA | 2010 2AQ4 | 75% The position vector of a particle at time $t \geq 0$ is given by $\underset{\sim}{r}(t) = \sin(t)\underset{\sim}{i} + \cos(2t)\underset{\sim}{j}$. The path of the particle has the cartesian equation

A $y = 2x^2 - 1$

B $y = 1 - 2x^2$

C $y = \sqrt{1 - x^2}$

D $y = \sqrt{x^2 - 1}$

E $y = 2x\sqrt{1 - x^2}$

16 © VCAA | 2003 1AQ24 | 72% The acceleration, in m/s^2, of a particle at time t s, $t \geq 0$, is given by $\ddot{\underset{\sim}{r}} = \cos(t)\underset{\sim}{i} - \sin(t)\underset{\sim}{j}$. The initial velocity of the particle was $\left(\underset{\sim}{i} + \underset{\sim}{j}\right)$ m/s.

The velocity of the particle, in m/s, at time t s is given by

A $-\sin(t)\underset{\sim}{i} - \cos(t)\underset{\sim}{j}$

B $\sin(t)\underset{\sim}{i} + \cos(t)\underset{\sim}{j}$

C $[\sin(t) + 1]\underset{\sim}{i} + \cos(t)\underset{\sim}{j}$

D $\sin(t)\underset{\sim}{i} + [\cos(t) + 1]\underset{\sim}{j}$

E $[1 - \sin(t)]\underset{\sim}{i} + [2 - \cos(t)]\underset{\sim}{j}$

17 © VCAA | 2018N 2AQ13 | Let $\underset{\sim}{i}$ and $\underset{\sim}{j}$ be unit vectors in the east and north directions respectively. At time t, $t \geq 0$, the position of particle A is given by $\underset{\sim}{r}_A = \left(t^2 - 5t + 6\right)\underset{\sim}{i} + (5t - 8)\underset{\sim}{j}$ and the position of particle B is given by $\underset{\sim}{r}_B = (3 - t)\underset{\sim}{i} + \left(t^2 - t\right)\underset{\sim}{j}$. Particle A will be directly east of particle B when t equals

A 1

B 2

C 1 and 2

D 2 and 4

E 4

18 © VCAA | 2011 2AQ13 | 71% The position of a particle at time t is given by $\underset{\sim}{r}(t) = \left(\sqrt{t - 2}\right)\underset{\sim}{i} + (2t)\underset{\sim}{j}$ for $t \geq 2$. The cartesian equation of the path of the particle is

A $y = 2x^2 + 4, x \geq 2$

B $y = 2x^2 + 2, x \geq 2$

C $y = 2x^2 + 4, x \geq 0$

D $y = \sqrt{\dfrac{x - 4}{2}}, x \geq 4$

E $y = 2x^2 + 2, x \geq 0$

19 © VCAA | 2018 2AQ13 | 65% The position vector of a particle that is moving along a curve at time t is given by $\underset{\sim}{r}(t) = 3\cos(t)\underset{\sim}{i} + 4\sin(t)\underset{\sim}{j}, t \geq 0$.

The **first** time when the speed of the particle is a minimum is

A 3

B $\dfrac{\pi}{2}$

C $\tan^{-1}\left(\dfrac{4}{3}\right)$

D $\dfrac{3\pi}{2}$

E 9

20 © VCAA | 2006 2AQ14 | 50% The position vector of a particle at time t seconds, $t \geq 0$, is given by $\underset{\sim}{r} = (3 - t)\underset{\sim}{i} - 6\sqrt{t}\underset{\sim}{j} + 5\underset{\sim}{k}$. The direction of motion of the particle when $t = 9$ is

A $-6\underset{\sim}{i} - 18\underset{\sim}{j} + 5\underset{\sim}{k}$

B $-\underset{\sim}{i} - \underset{\sim}{j}$

C $-6\underset{\sim}{i} - \underset{\sim}{j}$

D $-\underset{\sim}{i} - \underset{\sim}{j} + 5\underset{\sim}{k}$

E $-13.5\underset{\sim}{i} - 108\underset{\sim}{j} + 45\underset{\sim}{k}$

21 © VCAA | 2019 2AQ15 | 35% A particle is moving along the x-axis with velocity $\underset{\sim}{v} = u\underset{\sim}{i}$, where u is a real constant. At time $t = 0$, a force acts on the particle, causing it to accelerate with acceleration $\underset{\sim}{a} = \alpha\underset{\sim}{j}$, where α is a negative real constant.

Which one of the following statements correctly describes the motion of the particle for $t > 0$?

A The particle slows down, stops momentarily and then begins to move in the opposite direction to its original motion.

B The particle continues to travel along the x-axis with decreasing speed.

C The particle travels parallel to the y-axis.

D The particle moves along a circular arc.

E The particle moves along a parabola.

Video playlist
Applying vectors to motion

11.3 Applying vectors to motion

We can apply vector functions to motion in a plane. Imagine two aeroplanes in the sky where their paths may cross. What is of vital interest is if the paths cross at different times or at the same time, as crossing at the same time would indicate that they crash/collide.

Two particles **collide** when they are at the **same position** at the **same time**.

Two particles **cross** when they share **common points** at different times, in other words their paths cross at different times.

When we solve *x* and *y* components of a position vector, it is possible to get *t* values that are the same or different. When they are the same, this is where the particles collide; when they are different, this is where the paths cross at different times.

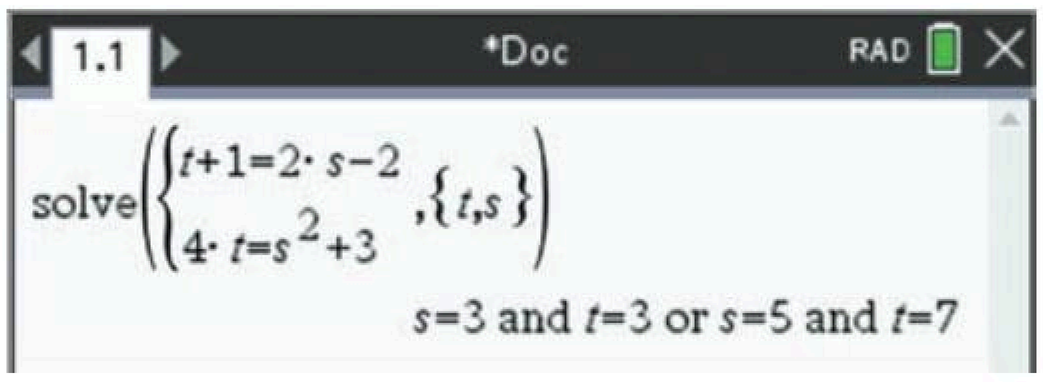
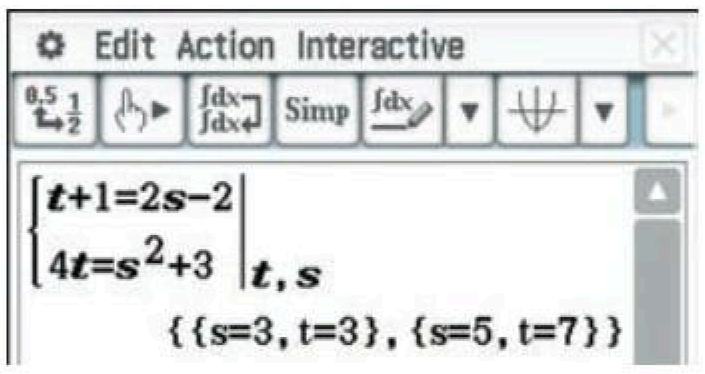
WB
p. 227

WORKED EXAMPLE 7 Vectors in motion 1

The motion of two particles is given by the vector functions $\underset{\sim}{r}_1(t) = (t+1)\underset{\sim}{i} + (4t)\underset{\sim}{j}$ and $\underset{\sim}{r}_2(t) = (2t-2)\underset{\sim}{i} + \left(t^2+3\right)\underset{\sim}{j}$, where $t \geq 0$.

a Find the point at which the particles collide.

b Find the point(s) at which their paths cross.

Steps	**Working**
a 1 Equate $\underset{\sim}{i}$ and $\underset{\sim}{j}$ coefficients and solve for *t*.	Solving $t+1=2t-2$ gives $t=3$. Solving $4t=t^2+3$ gives $t=1$ and $t=3$.
2 Substitute the same value of *t* into the vector functions.	The particles are at the same point when $t=3$, so $\underset{\sim}{r}_1(3) = 4\underset{\sim}{i} + 12\underset{\sim}{j}$.
3 Conclude whether the particles collide or cross.	The particles collide at the point (4, 12).
b 1 Write the two vectors using a different variable, *s*, for time in $\underset{\sim}{r}_2$.	$\underset{\sim}{r}_1(t) = (t+1)\underset{\sim}{i} + (4t)\underset{\sim}{j}$ and $\underset{\sim}{r}_2(s) = (2s-2)\underset{\sim}{i} + \left(s^2+3\right)\underset{\sim}{j}$
2 Equate $\underset{\sim}{i}$ and $\underset{\sim}{j}$ coefficients and solve simultaneously for *t* and *s*.	$t+1=2s-2$ and $4t=s^2+3$ Solving simultaneously gives two solutions $s=3$, $t=3$ and $s=5$, $t=7$.
TI-Nspire  $\text{solve}\left(\begin{cases} t+1=2\cdot s-2 \\ 4\cdot t=s^2+3 \end{cases},\{t,s\}\right)$ s=3 and t=3 or s=5 and t=7	**ClassPad**  $\begin{cases} t+1=2s-2 \\ 4t=s^2+3 \end{cases}\Big\vert_{t,s}$ {{s=3, t=3}, {s=5, t=7}}
3 Substitute the values into the vector functions and write the answer.	$s=3$, $t=3$ gives the point (4, 12) as before. $s=5$, $t=7$ gives the point (8, 28).
4 Conclude whether the particles collide or cross.	The particles' paths cross at the points (4, 12) and (8, 28). Cross at (8, 28). Cross and collide at (4, 12).

Circular motion

In all motion problems involving vector functions, it is important to identify where the path of the particle starts and in what direction it is moving.

Sometimes the path starts at a certain point and moves in one direction, then stopping or continuing theoretically to infinity.

Sometimes the path starts at a certain point, moves in one direction, then returns back on the same path, moving in a rhythmic fashion. An example of this is the movement in a pendulum clock.

When the path of a moving object is a circle, it is important to identify where the object starts, when $t = 0$, and the direction in which it is moving.

Remember that when finding the Cartesian equation of **parametric equations** we need to consider the x–t graph and the y–t graph to identify where the Cartesian graph exists.

WB
p. 228

WORKED EXAMPLE 8 Vectors in motion 2

The position vector of a particle at time t is given by $\underset{\sim}{r}(t) = (3\sin(2t))\underset{\sim}{i} + (3\cos(2t))\underset{\sim}{j}$, where $t \geq 0$.

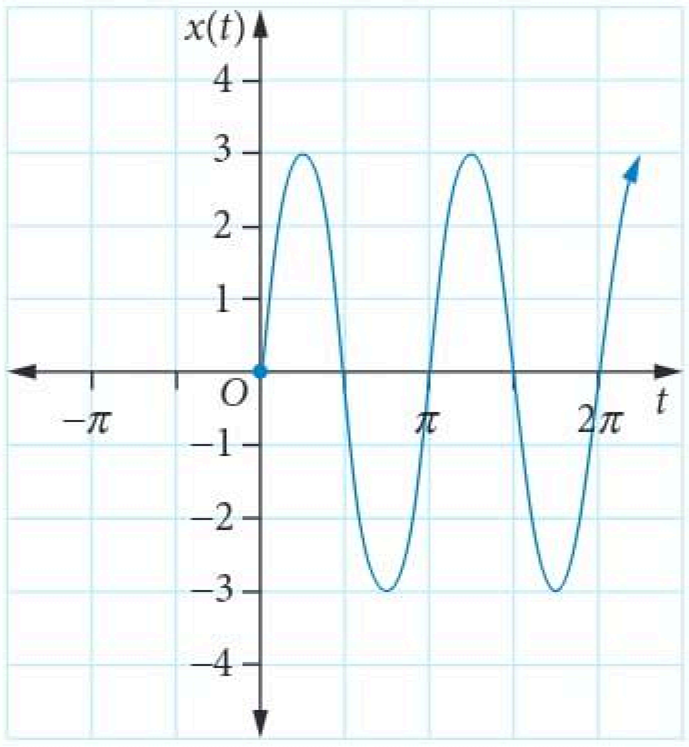
a Find the Cartesian equation of the path of the particle.

b Sketch the graph of the path.

Steps	**Working**
a 1 Write parametric equations for x and y.	$x = 3\sin(2t)$, giving $\sin(2t) = \frac{x}{3}$ $y = 3\cos(2t)$, giving $\cos(2t) = \frac{y}{3}$
2 Identify the trigonometric identity to be used.	$\sin^2(2t) + \cos^2(2t) = 1$
3 Find the Cartesian equation.	$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$ $\therefore x^2 + y^2 = 9$, a circle with centre $(0, 0)$ and radius $= 3$.
4 Identify where $t = 0$ and the direction of the path.	$\underset{\sim}{r}(0) = (3\sin(0))\underset{\sim}{i} + (3\cos(0))\underset{\sim}{j}$ $\therefore \underset{\sim}{r}(0) = 3\underset{\sim}{j}$

b 1 Identify where the Cartesian graph exists.

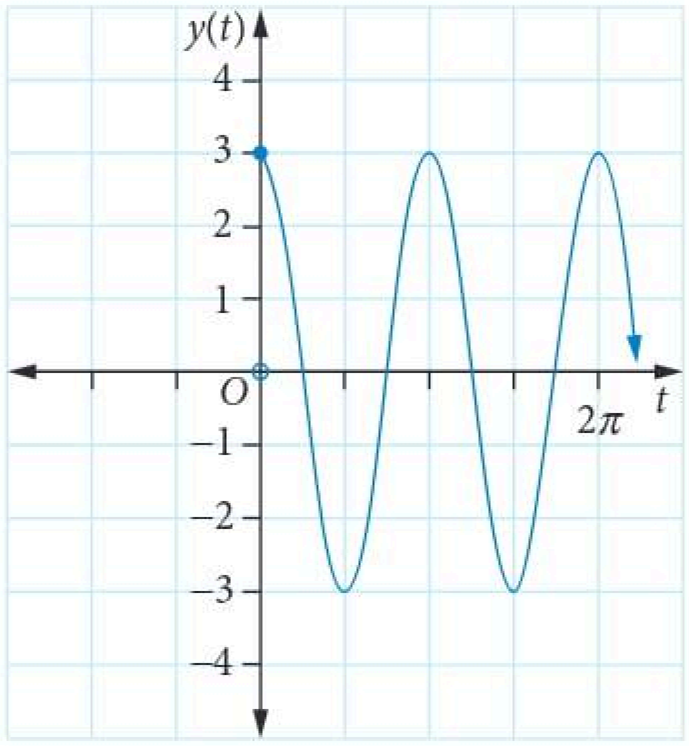
Sketch the graph of $x = 3\sin(2t)$ for $t \geq 0$.



The direction of travel is from the starting point where $x = 0$, with x values increasing.

domain: $x \in [-3, 3]$

Sketch the graph of $y = 3\cos(2t)$ for $t \geq 0$.

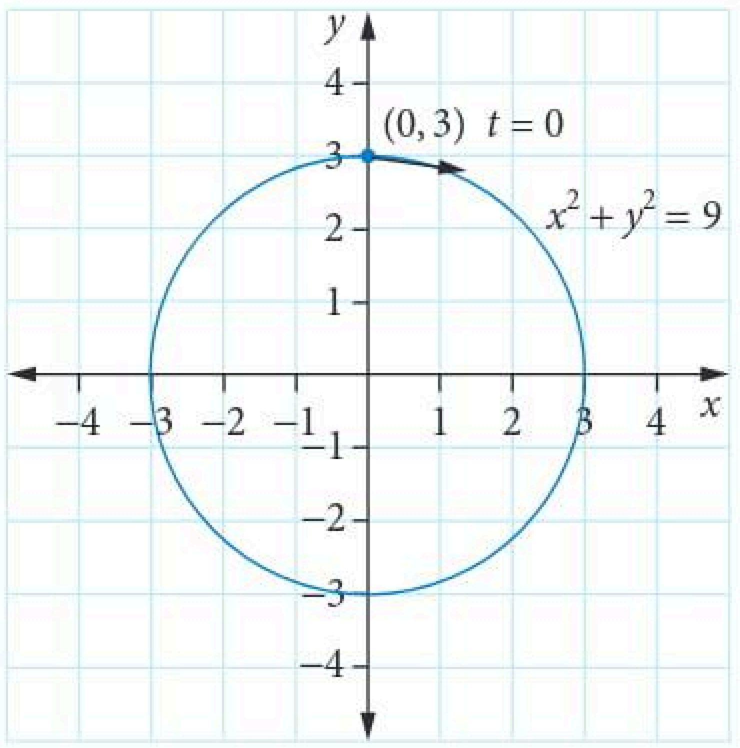


The direction of travel is from the starting point where $y = 3$, with y values decreasing.

range: $y \in [-3, 3]$

2 Sketch the Cartesian graph $x^2 + y^2 = 9$.

The particle starts at (0, 3) and travels around the circle in a clockwise direction.



Worksheet
Projectile motion

Projectile motion

A projectile is a body moving under the influence of gravity, with acceleration $\underset{\sim}{a} = -g\underset{\sim}{j}$, where g is the acceleration due to gravity and $\underset{\sim}{j}$ is the unit vector heading upwards.

For a projectile moving in two dimensions, with initial velocity V m/s at an angle of θ to the positive direction of the x-axis, the velocity can be found by anti-differentiating the acceleration.

$$\underset{\sim}{a} = -g\underset{\sim}{j}$$

$$\underset{\sim}{v} = \int \left(-g\underset{\sim}{j}\right) dt$$

$$\therefore \underset{\sim}{v} = -gt\underset{\sim}{j} + \underset{\sim}{c}$$

Exam hack

Remember that the $+ c$ needs to be with a tilde $+ \underset{\sim}{c}$.

To find $+\underset{\sim}{c}$, we know that the initial velocity must be $\underset{\sim}{v}(0) = V\cos(\theta)\underset{\sim}{i} + V\sin(\theta)\underset{\sim}{j}$, where V is a constant.

This gives $\underset{\sim}{c} = V\cos(\theta)\underset{\sim}{i} + V\sin(\theta)\underset{\sim}{j}$

$$\underset{\sim}{v} = -gt\underset{\sim}{j} + V\cos(\theta)\underset{\sim}{i} + V\sin(\theta)\underset{\sim}{j}$$

Collect for $\underset{\sim}{i}, \underset{\sim}{j}$ terms:

$$\therefore \underset{\sim}{v} = V\cos(\theta)\underset{\sim}{i} + \left(V\sin(\theta) - gt\right)\underset{\sim}{j}$$

Exam hack

Note that $V\sin(\theta) - gt$ is a special case of the constant acceleration formula $v = u + at$.

The displacement is found in a similar way by anti-differentiating the velocity. Once the displacement is known, the Cartesian equation of the pathway can be found.

WB

p. 230

WORKED EXAMPLE 9 Vectors in motion 3

The position, $\underset{\sim}{r}(t)$, of a projectile at time t seconds is given by $\underset{\sim}{r}(t) = (100t)\underset{\sim}{i} + \left(200t - 4.9t^2\right)\underset{\sim}{j}$, where $t \geq 0$. The object is initially on the ground, and the distance is in metres.

a Find the time taken to return to the ground.

b Find the maximum height reached.

c Find the initial speed of the object.

Give all answers correct to two decimal places.

Exam hack

Note that $200t - 4.9t^2$ is a special case of the constant acceleration formula $s = ut + \frac{1}{2}at^2$.

Steps	**Working**
a The projectile is at ground level when the vertical component of its position equals zero.	Solve $200t - 4.9t^2 = 0$ $\therefore t = 0,\ t = \frac{200}{4.9}$ $t = \frac{200}{4.9} \approx 40.816$ The time taken to return to the ground is 40.82 seconds.

b The maximum height reached is when the vertical component of velocity equals zero.

Differentiate the vertical component of displacement and solve for zero.

$y = 200t - 4.9t^2$

$\frac{dy}{dt} = 200 - 9.8t = 0$

$t = \frac{200}{9.8} \approx 20.408\ldots$

At $t = 20.408\ldots$, $y = 2040.8163\ldots$

The maximum height reached is 2040.82 m.

c **1** Find the velocity vector of the particle at $t = 0$.

$\underset{\sim}{r}(t) = (100t)\underset{\sim}{i} + \left(200t - 4.9t^2\right)\underset{\sim}{j}$

$\therefore \dot{\underset{\sim}{r}}(t) = 100\underset{\sim}{i} + (200 - 9.8t)\underset{\sim}{j},$

giving $\dot{\underset{\sim}{r}}(0) = 100\underset{\sim}{i} + 200\underset{\sim}{j}$.

2 Find the speed at $t = 0$.

$\dot{\underset{\sim}{r}}(0)| = \sqrt{(100)^2 + (200)^2} \approx 223.6067$

The initial speed of the object is 223.61 m/s.

VCE QUESTION ANALYSIS

© VCAA 2018 2BQ4 2018 Examination 2 Section B Question 4 (10 marks)

Video
VCE question analysis: Vector calculus

Two yachts, A and B, are competing in a race and their position vectors on a certain section of the race after time t hours are given by

$$\underset{\sim}{r}_A(t) = (t+1)\underset{\sim}{i} + (t^2 + 2t)\underset{\sim}{j} \text{ and } \underset{\sim}{r}_B(t) = t^2\underset{\sim}{i} + (t^2 + 3)\underset{\sim}{j},\ t \geq 0$$

where displacement components are measured in kilometres from a given reference buoy at origin O.

a Find the cartesian equation of the path for each yacht. 2 marks

b Show that the two yachts will not collide if they follow these paths. 2 marks

c Find the coordinates of the point where the paths of the two yachts cross.
Give your coordinates correct to three decimal places. 2 marks

One of the rules for the race is that the yachts are not allowed to be within 0.2 km of each other. If this occurs there is a time penalty for the yacht that is travelling faster.

d For what values of t is yacht A travelling faster than yacht B? 2 marks

e If yacht A does not alter its course, for what period of time will yacht A be within 0.2 km of yacht B? Give your answer in **minutes**, correct to one decimal place. 2 marks

Reading the question

- The question is centred around the concept of objects colliding or crossing paths.
- The first question asks for the Cartesian equations given the parametric equations.
- The question moves into a distance and velocity question using the parametric forms of the yachts.
- Make sure you consider the domain, $t \geq 0$, of the parameter t for a substantial portion of the question, particularly in parts **c** and **d**.

Thinking about the question

- The question is in two quite distinct sections. The first is the usual consideration of a parameter, *t*, converted into its relevant Cartesian equation. Remember that for finding the Cartesian equation, you can use trigonometric identities or methods of substituting *t* given polynomials in *t*. This question uses the latter.
- The Cartesian equation is required for each yacht.
- The second section involves considering the movement of the yachts.
- Finishing the question with a more difficult analysis of kinematics, considering speed rather than velocity and distance, using time, *t*, again as the parameter.

Worked solution (✓ = 1 mark)

a Given $\underset{\sim}{r}_A(t) = (t+1)\underset{\sim}{i} + (t^2+2t)\underset{\sim}{j}$ and $\underset{\sim}{r}_B(t) = t^2\underset{\sim}{i} + (t^2+3)\underset{\sim}{j}$, $t \geq 0$.

Yacht A: Let $x = t + 1$ and $y = t^2 + 2t$.

Eliminate t by letting $t = x - 1$ and substituting into $y = t^2 + 2t$,

giving $y = (x-1)^2 + 2(x-1)$.

Simplify to get $\boldsymbol{y = x^2 - 1}$ **for yacht A.** ✓

Yacht B: Let $x = t^2$ and $y = t^2 + 3$.

Eliminate t by equating t^2 terms,

giving $\boldsymbol{y = x + 3}$ **for yacht B.** ✓

b Equate x and y terms for

x terms: $t + 1 = t^2$

y terms: $t^2 + 2t = t^2 + 3$

Solve x terms for t getting $t^2 - t - 1 = 0$.

$\therefore\ t = \dfrac{1 \pm \sqrt{5}}{2}$, so for $t \geq 0$, $t = \dfrac{1+\sqrt{5}}{2}$

Solve y terms for t, getting $2t = 3$.

$\therefore\ t = \dfrac{3}{2}$ ✓

$\boldsymbol{\dfrac{1+\sqrt{5}}{2} \neq \dfrac{3}{2}}$ **so the two yachts will not collide.** ✓

c Solve the Cartesian equations $y = x^2 - 1$ for yacht A and $y = x + 3$ for yacht B.

$$x^2 - 1 = x + 3$$

$$\therefore\ x^2 - x - 4 = 0$$

$$\therefore\ x = \frac{1 \pm \sqrt{17}}{2}$$ ✓

For $t \geq 0$ in the equation $x = t + 1$, the only possible solution is $x = \dfrac{1+\sqrt{17}}{2}$.

To three decimal places and substituting for y, the answer is **(2.562, 5.562)**. ✓

d $\underset{\sim}{r}_A(t) = (t+1)\underset{\sim}{i} + (t^2+2t)\underset{\sim}{j}$ and $\underset{\sim}{r}(t) = t^2\underset{\sim}{i} + (t^2+3)\underset{\sim}{j}$.

$\therefore\ \underset{\sim}{r}'_A(t) = \underset{\sim}{i} + (2t+2)\underset{\sim}{j}$ and $\underset{\sim}{r}'_B(t) = 2t\underset{\sim}{i} + 2t\underset{\sim}{j}$.

We want yacht A travelling faster than yacht B.

Considering speed, we get $\left|\underset{\sim}{r}'_A(t)\right| = \sqrt{1 + (2t+2)^2}$. ✓

and $\left|\underset{\sim}{\mathrm{r}}'_B(t)\right| = \sqrt{4t^2+4t^2} = \sqrt{8t^2}$.

Solving $\sqrt{1+(2t+2)^2} > \sqrt{8t^2}$

gives $-\frac{1}{2} < t < \frac{5}{2}$.

Considering the domain $t \geq 0$, $\mathbf{0 \leq t < \frac{5}{2}}$ ✓

e We need yacht A to be within 0.2 km of yacht B.

We need distance between vectors, distance = $\left|\underset{\sim}{\mathrm{r}}_B(t) - \underset{\sim}{\mathrm{r}}_A(t)\right| < 0.2$.

$$\begin{aligned}|t^2\underset{\sim}{\mathrm{i}} + (t^2+3)\underset{\sim}{\mathrm{j}} - ((t+1)\underset{\sim}{\mathrm{i}} + (t^2+2t)\underset{\sim}{\mathrm{j}})| &= |(t^2-(t+1))\underset{\sim}{\mathrm{i}} + ((t^2+3)-(t^2+2t))\underset{\sim}{\mathrm{j}}| \\ &= |(t^2-t-1))\underset{\sim}{\mathrm{i}} + (3-2t))\underset{\sim}{\mathrm{j}}| \\ &= \sqrt{\left(t^2-t-1\right)^2+(3-2t)^2} \quad ✓\end{aligned}$$

Solve $\sqrt{\left(t^2-t-1\right)^2+(3-2t)^2} < 0.2$

gives $1.5288\ldots < t < 1.5973\ldots$

Period of time required, so

$1.5973 - 1.5288 = 0.0685\ldots$ hours

$= \mathbf{4.1}$ **minutes** ✓

Student performance

80–100% 60–79% 0–59%

a 85% A skills question that students will have practised many times.

b 76% Parts **b** and **c** test student understanding of the difference between when two objects are in the same position at the same time, and in the same position at a different time.

c 55% Part **c** required a 'show that' question where the paths of the two yachts cross. The difference between colliding at the same time and crossing paths at a different time can be confusing for students. Some students gave two answers because they missed the fact that $t \geq 0$ or that coordinates had to be written correct to three decimal places.

d 41% This was a challenging question that confused many students. One careless error was to give negative values for time.

e 37% The most difficult question required a knowledge of the formula of the distance between two vectors: $\left|\underset{\sim}{r}_B(t) - \underset{\sim}{r}_A(t)\right|$ and then to solve that for < 0.2. There were many non-attempts, or answers in terms of hours rather than minutes.

More exam practice is needed when dealing with distance between vectors, and understanding the difference between velocity and speed. Remember to work with decimals with more places than what is required in the final answer, to avoid rounding error.

EXERCISE 11.3 Applying vectors to motion

ANSWERS p. 600

Recap

1 TECH-FREE The position vector of a particle at time t is given by $\underset{\sim}{\mathrm{r}}(t) = (-\cos(3t))\underset{\sim}{\mathrm{i}} + (-\sin(3t))\underset{\sim}{\mathrm{j}}$, where $t \geq 0$. Find the Cartesian equation of the path and the velocity vector of the particle.

2 TECH-FREE If the acceleration of a particle moving in two dimensions is $\ddot{\underset{\sim}{\mathrm{r}}}(t) = 6t\underset{\sim}{\mathrm{i}} + \cos(t)\mathrm{j}$ m/s^2, find the velocity of the particle given that initially the particle has a velocity of $\underset{\sim}{\mathrm{i}} + \underset{\sim}{\mathrm{j}}$ m/s.

Mastery

3 WORKED EXAMPLE 7 **TECH-FREE** The motion of two particles is given by the vector functions $\underset{\sim}{r}_1(t) = (-t)\underset{\sim}{i} + (4t)\underset{\sim}{j}$ and $\underset{\sim}{r}_2(t) = (2t-3)\underset{\sim}{i} + \left(t^2+3\right)\underset{\sim}{j}$, where $t \in R$. Find the point at which the particles collide.

4 WORKED EXAMPLE 8 **TECH-FREE** The position vector of a particle at time t is given by $\underset{\sim}{r}(t) = 3t\underset{\sim}{i} + (t-1)\underset{\sim}{j}$, where $t \geq 0$. Find the Cartesian equation of the path of the particle.

5 WORKED EXAMPLE 9 **TECH-FREE** The position, $\underset{\sim}{r}(t)$, of a projectile at time t seconds is given by $\underset{\sim}{r}(t) = 10t\,\underset{\sim}{i} + \left(20t - 4.9t^2\right)\underset{\sim}{j}$, where $t \geq 0$. The object is initially on the ground, and distance is in metres. Find the initial speed of the object.

6 The motion of two particles is given by the vector functions $\underset{\sim}{r}_1(t) = 3t\,\underset{\sim}{i} + (t+1)\underset{\sim}{j}$ and $\underset{\sim}{r}_2(t) = \left(t^2-18\right)\underset{\sim}{i} + (2t-5)\underset{\sim}{j}$, where $t \geq 0$. The point at which the particles collide is

A $(6, -3)$
B $(6, 0)$
C $(-9, -2)$
D $(18, 7)$
E $(7, 18)$

7 The motion of two particles is given by the vector functions $\underset{\sim}{r}_1(t) = t\,\underset{\sim}{i} + (t+1)\underset{\sim}{j}$ and $\underset{\sim}{r}_2(t) = \left(t^2-2\right)\underset{\sim}{i} - (t-5)\underset{\sim}{j}$, where $t \geq 0$.

The point at which the particles collide is

A $(-1, 0)$
B $(2, 0)$
C $(2, 3)$
D $(-1, 0)$
E $(3, 2)$

8 WORKED EXAMPLE 9 The position, $\underset{\sim}{r}(t)$, of a projectile at time t seconds is given by $\underset{\sim}{r}(t) = 10t\underset{\sim}{i} + \left(30t - 4.9t^2\right)\underset{\sim}{j}$, where $t \geq 0$. The particle is initially on the ground, and distance is in metres. The time taken to return to the ground, in seconds, is closest to

A 0
B 6
C 7
D 10
E 30

Exam practice

80–100% 60–79% 0–59%

9 © VCAA 2019 1Q4 74% **TECH-FREE** (3 marks) The position vectors of two particles A and B at time t seconds after they have started moving are given by

$$\underset{\sim}{r}_A(t) = \left(t^2-1\right)\underset{\sim}{i} + \left(a + \frac{t}{3}\right)\underset{\sim}{j} \text{ and } \underset{\sim}{r}_B(t) = \left(t^3 - t\right)\underset{\sim}{i} + \left(\arccos\left(\frac{t}{2}\right)\right)\underset{\sim}{j}$$

respectively, where a is a real constant and $0 \leq t \leq 2$.

Find the value of a if the particles collide after they have started moving.

10 © VCAA 2018 1Q10 26% **TECH-FREE** (5 marks) The position vector of a particle moving along a curve at time t seconds is given by $\underset{\sim}{r}(t) = \frac{t^3}{3}\underset{\sim}{i} + \left(\arcsin(t) + t\sqrt{1-t^2}\right)\underset{\sim}{j}$, $0 \leq t \leq 1$, where distances are measured in metres.

The distance d metres that the particle travels along the curve in three-quarters of a second is given by

$$d = \int_0^{\frac{3}{4}} \left(at^2 + bt + c\right) dt$$

Find a, b and c, where $a, b, c \in Z$.

11 © VCAA 2008 2AQ13 72% A cricket ball is hit from an origin at ground level so that its position vector at time t is given by $\underset{\sim}{r}(t) = 15t\underset{\sim}{i} + \left(20t - 5t^2\right)\underset{\sim}{j}$ for $t \geq 0$, where $\underset{\sim}{i}$ is a unit vector in the forward direction and $\underset{\sim}{j}$ is a unit vector vertically up. When the cricket ball reaches its maximum height, its position vector is

A $\underset{\sim}{r} = 20\underset{\sim}{i} + 30\underset{\sim}{j}$
B $\underset{\sim}{r} = 15\underset{\sim}{i} + 20\underset{\sim}{j}$
C $\underset{\sim}{r} = 60\underset{\sim}{i}$
D $\underset{\sim}{r} = 30\underset{\sim}{i} + 10\underset{\sim}{j}$
E $\underset{\sim}{r} = 30\underset{\sim}{i} + 20\underset{\sim}{j}$

12 © VCAA 2017N 2AQ17 The acceleration, a m s^{-2}, of a particle moving in a straight line is given by $a = v^2 + 1$, where v is the velocity of the particle at any time t. The initial velocity of the particle when at origin O is 2 m s^{-1}.

The displacement of the particle from O when its velocity is 3 m s^{-1} is

A $\log_e(2)$
B $\frac{1}{2}\log_e\left(\frac{10}{3}\right)$
C $\frac{1}{2}\log_e(2)$
D $\frac{1}{2}\log_e\left(\frac{5}{2}\right)$
E $\log_e\left(\frac{4}{5}\right)$

13 © VCAA 2021N 2AQ12 A particle has position vector $\underset{\sim}{r}(t) = t^3\underset{\sim}{i} + 3t^2\underset{\sim}{j}$. The total distance travelled by the particle from $t = 1$ to $t = 4$, correct to two decimal places, is

A 17.37
B 36.05
C 65.89
D 77.42
E 78.26

14 © VCAA 2011 2BQ2 (11 marks) A golfer hits a ball at time $t = 0$ seconds from an origin O, aiming at a hole which is 200 metres away at the end of a horizontal fairway. The initial velocity of the ball is given by $\underset{\sim}{v}_0 = 35\underset{\sim}{i} + 5\underset{\sim}{j} + 24.5\underset{\sim}{k}$, where $\underset{\sim}{i}$ is a unit vector in the direction of the hole, $\underset{\sim}{j}$ is a horizontal unit vector to the left perpendicular to $\underset{\sim}{i}$, and $\underset{\sim}{k}$ is a unit vector vertically up. Velocity components are measured in metres per second. The ball, once in the air, is subject only to gravitational acceleration.

a 51% Given that the acceleration of the ball is $a(t) = -9.8\underset{\sim}{k}$, **show by integration** that the position vector of the ball t seconds after the golfer hits it is $\underset{\sim}{r}(t) = 35t\underset{\sim}{i} + 5t\underset{\sim}{j} + \left(24.5t - 49t^2\right)\underset{\sim}{k}$. 2 marks

b 80% Show that the ball is in the air for 5 seconds. 1 mark

c 72% Find the maximum height, in metres, reached by the ball. 2 marks

d 71% Find the **speed** of the ball when it hits the ground. Give your answer in metres per second, correct to the nearest integer. 3 marks

e 44% Find the distance **from the hole** to where the ball hits the ground. Give your answer correct to the nearest metre. 3 marks

15 © VCAA 2009 2BQ3 (8 marks) A child seated on a mat slides down a spiral-shaped water slide. At time t seconds after starting to slide, the position vector of the centre of the mat relative to an origin O at ground level is given by $\underset{\sim}{r}(t) = 5\sin\left(\frac{\pi}{6}t\right)\underset{\sim}{i} + 5\cos\left(\frac{\pi}{6}t\right)\underset{\sim}{j} + \left(24.5 - \frac{t^2}{8}\right)\underset{\sim}{k}$

where $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are perpendicular horizontal unit vectors and $\underset{\sim}{k}$ is a unit vector in the vertical direction. Displacement components are measured in metres.

a 80% Find the height, in metres, of the start of the water slide above the ground. 1 mark

b 80% Show that the time taken to slide down from the top to the bottom of the slide, which is at ground level, is 14 seconds. 1 mark

c 40% Find the time, in seconds, taken for the child to complete the first loop of the spiral that is vertically below the starting point. 1 mark

d 79% Find $\dot{\underset{\sim}{r}}(t)$, the velocity of the child at time t. 1 mark

e 60% Find the speed of the child when ground level is reached. Give your answer correct to the nearest 0.1 metres per second. 2 marks

f 45% Show that the magnitude of the child's acceleration is constant. 2 marks

16 © VCAA 2007 2BQ4 (12 marks) An aircraft approaching an airport with velocity $\underset{\sim}{v} = 30\underset{\sim}{i} - 40\underset{\sim}{j} - 4\underset{\sim}{k}$ is observed on the control tower radar screen at time $t = 0$ seconds. Ten seconds later it passes over a navigation beacon with position vector $-500\underset{\sim}{i} + 2500\underset{\sim}{j}$ relative to the base of the control tower, at an altitude of 200 metres. Let $\underset{\sim}{i}$ and $\underset{\sim}{j}$ be horizontal orthogonal unit vectors and let $\underset{\sim}{k}$ be a unit vector in the vertical direction. Displacement components are measured in metres.

a 67% Show that the position vector of the aircraft relative to the base of the control tower at time t is given by $\underset{\sim}{r}(t) = (30t - 800)\underset{\sim}{i} + (2900 - 40t)\underset{\sim}{j} + (240 - 4t)\underset{\sim}{k}$. 3 marks

b 60% When does the aircraft land and how far (correct to the nearest metre) from the base of the control tower is the point of landing? 3 marks

c 13% At what angle from the runway, correct to the nearest tenth of a degree, does the aircraft land? 2 marks

d 24% At what time, correct to the nearest second, is the aircraft closest to the base of the control tower? 2 marks

e 22% What distance does the aircraft travel from the time it is observed on the radar screen to the time it lands? Give your answer correct to the nearest metre. 2 marks

17 © VCAA 2021N 2BQ4 (12 marks) The position vector of a golf ball t seconds after it is hit is modelled by $\underset{\sim}{r}(t) = at\underset{\sim}{i} + \left(bt - 5t^2\right)\underset{\sim}{j}$, where $a, b \in Z$, $t \geq 0$ and where $\underset{\sim}{i}$ is the unit vector in the positive horizontal direction and $\underset{\sim}{j}$ is the unit vector in the positive vertical direction.

Displacement components are measured in metres.

The golf ball is hit from ground level and, after 4 seconds, it is 4 metres above ground level and descending as it passes over the top of a tree. The tree is at a horizontal distance of 160 metres from where the ball was hit.

a Show that $a = 40$ and $b = 21$. 2 marks

b What acute angle to the horizontal direction does the tangent to the path of the golf ball make as the golf ball passes over the top of the tree? Give your answer in degrees, correct to one decimal place. 2 marks

c What is the maximum height in metres, reached by the golf ball? 2 marks

d Find the cartesian equation of the path of the golf ball, giving your answer in the form $cy = dx - x^2$, where c and d are integers. 2 marks

A person is standing at a horizontal distance of 163 m from where the golf ball was hit. The golf ball passes directly over the head of this person.

e If the person is 1.8 metres tall, how far directly above the person's head does the ball pass? Give your answer in metres, correct to two decimal places. 2 marks

f How far, in metres, is the golf ball from this person's feet when it hits the ground? 2 marks

11 Chapter summary

Position vector

- We describe the position vector in two dimensions as $\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j}$
- In three dimensions: $\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j} + z(t)\underset{\sim}{k}$

Differentiating and integrating vector

- Given the position vector $\underset{\sim}{r}(t) = x\underset{\sim}{i} + y\underset{\sim}{j} + x\underset{\sim}{k}$

  $\text{velocity} = \dot{\underset{\sim}{r}}(t) = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$

  $\text{acceleration} = \ddot{\underset{\sim}{r}}(t) = \frac{d^2x}{dt^2}\underset{\sim}{i} + \frac{d^2y}{dt^2}\underset{\sim}{j} + \frac{d^2z}{dt^2}\underset{\sim}{k}$

- Velocity is the anti-derivative of acceleration.
- Displacement is the anti-derivative of velocity.
- When we integrate with vectors, the constant of integration, '+ c' becomes '+ $\underset{\sim}{c}$', a constant in vector form.

Applying vectors to motion

- Two particles **collide** when they are at the **same position** at the **same time**.
- Two particles **cross** when they share **common points** at different times, in other words, their paths cross at different times.

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 © VCAA 2018 1Q2 (4 marks)

a Show that $1 + i = \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$. 1 mark

b Evaluate $\frac{\left(\sqrt{3} - i\right)^{10}}{(1 + i)^{12}}$, giving your answer in the form $a + bi$, where $a, b \in R$. 3 marks

2 (3 marks) A particle with a position vector $\underset{\sim}{r}$ at any time t is given by

$$\underset{\sim}{r} = \sin(t)\underset{\sim}{i} + \cos(t)\underset{\sim}{j} + t^2\underset{\sim}{k}$$

Distances are measured in metres and time is measured in seconds.

Find the speed of the particle at $t = \frac{\pi}{2}$.

3 © VCAA 2019N 1Q4 (3 marks) Evaluate $\int_{e^3}^{e^4} \frac{1}{x\log_e(x)}\,dx$.

Cumulative examination 2

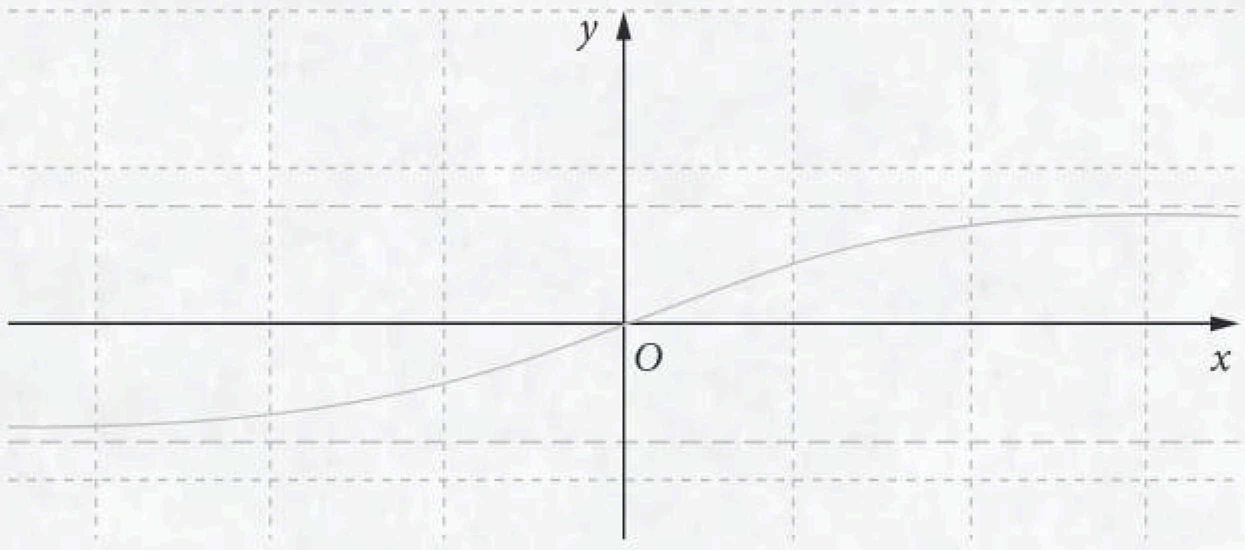
Total number of marks: 24 Reading time: 5 minutes Writing time: 36 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 © VCAA 2018 2AQ1 Part of the graph of $y=\frac{1}{2}\tan^{-1}(x)$ is shown below.



The equations of its asymptotes are

A $y=\pm\frac{1}{2}$ **B** $y=\pm\frac{3}{4}$ **C** $y=\pm1$ **D** $y=\pm\frac{\pi}{2}$ **E** $y=\pm\frac{\pi}{4}$

2 © VCAA 2018 2AQ5 Let $z=a+bi$, where $a, b\in R\backslash\{0\}$. If $z+\frac{1}{z}\in R$, which one of the following must be **true**?

A $\text{Arg}(z)=\frac{\pi}{4}$ **B** $a=-b$ **C** $a=b$ **D** $|z|=1$ **E** $z^2=1$

3 © VCAA 2018 2AQ8 Using a suitable substitution, $\int_0^{\frac{\pi}{6}}\tan^2(x)\sec^2(x)\,dx$ can be expressed as

A $\int_0^{\frac{1}{\sqrt{3}}}(u^4+u^2)\,du$

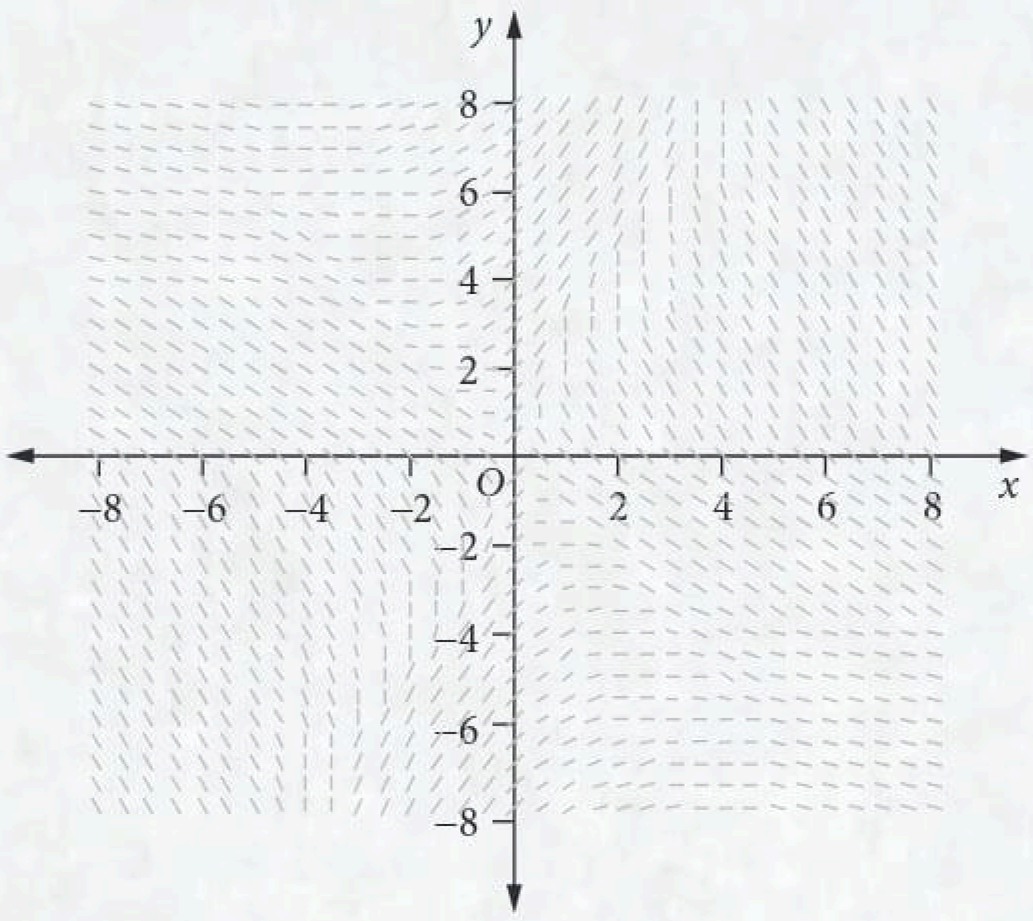
B $\int_1^{\frac{2}{\sqrt{3}}}(u^4+u^2)\,du$

C $\int_0^{\frac{1}{\sqrt{3}}}u\,du$

D $\int_0^{\frac{\pi}{6}}u^2\,du$

E $\int_0^{\frac{1}{\sqrt{3}}}u^2\,du$

4 © VCAA 2018 2AQ10



The differential equation that best represents the direction field above is

A $\frac{dy}{dx} = \frac{2x + y}{y - 2x}$

B $\frac{dy}{dx} = \frac{x + 2y}{2x - y}$

C $\frac{dy}{dx} = \frac{2x - y}{x + 2y}$

D $\frac{dy}{dx} = \frac{x - 2y}{y - 2x}$

E $\frac{dy}{dx} = \frac{2x + y}{2y - x}$

5 © VCAA 2017 2AQ12 Let $\underset{\sim}{r}(t) = (1 - \sqrt{a}\sin(t))\underset{\sim}{i} + \left(1 - \frac{1}{b}\cos(t)\right)\underset{\sim}{j}$ for $t \geq 0$ and $a, b \in R^+$ be the path of a particle moving in the cartesian plane. The path of the particle will always be a circle if

A $ab^2 = 1$

B $a^2b = 1$

C $ab^2 \neq 1$

D $ab = 1$

E $a^2b \neq 1$

Section B 2 questions

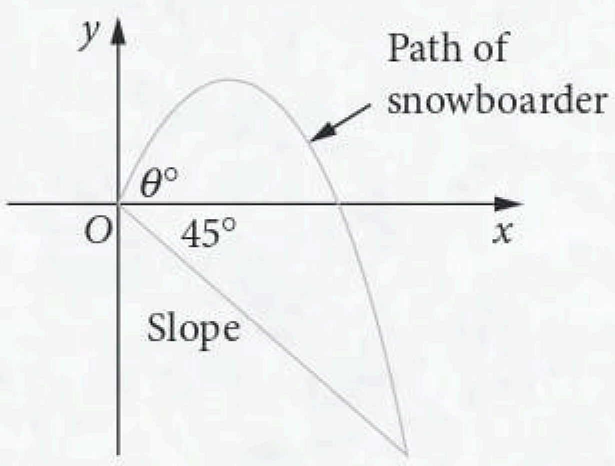
19 marks

1 © VCAA 2019N 2BQ4 (10 marks) A snowboarder at the Winter Olympics leaves a ski jump at an angle of θ degrees to the horizontal, rises up in the air, performs various tricks and then lands at a distance down a straight slope that makes an angle of 45° to the horizontal, as shown below.

Let the origin O of a cartesian coordinate system be at the point where the snowboarder leaves the jump, with a unit vector in the positive x direction being represented by $\underset{\sim}{i}$ and a unit vector in the positive y direction being represented by $\underset{\sim}{j}$. Distances are measured in metres and time is measured in seconds.

The position vector of the snowboarder t seconds after leaving the jump is given by

$$\underset{\sim}{r}(t) = (6t - 0.01t^3)\underset{\sim}{i} + (6\sqrt{3}t - 4.9t^2 + 0.01t^3)\underset{\sim}{j},\ t \geq 0$$



a Find the angle θ. 2 marks

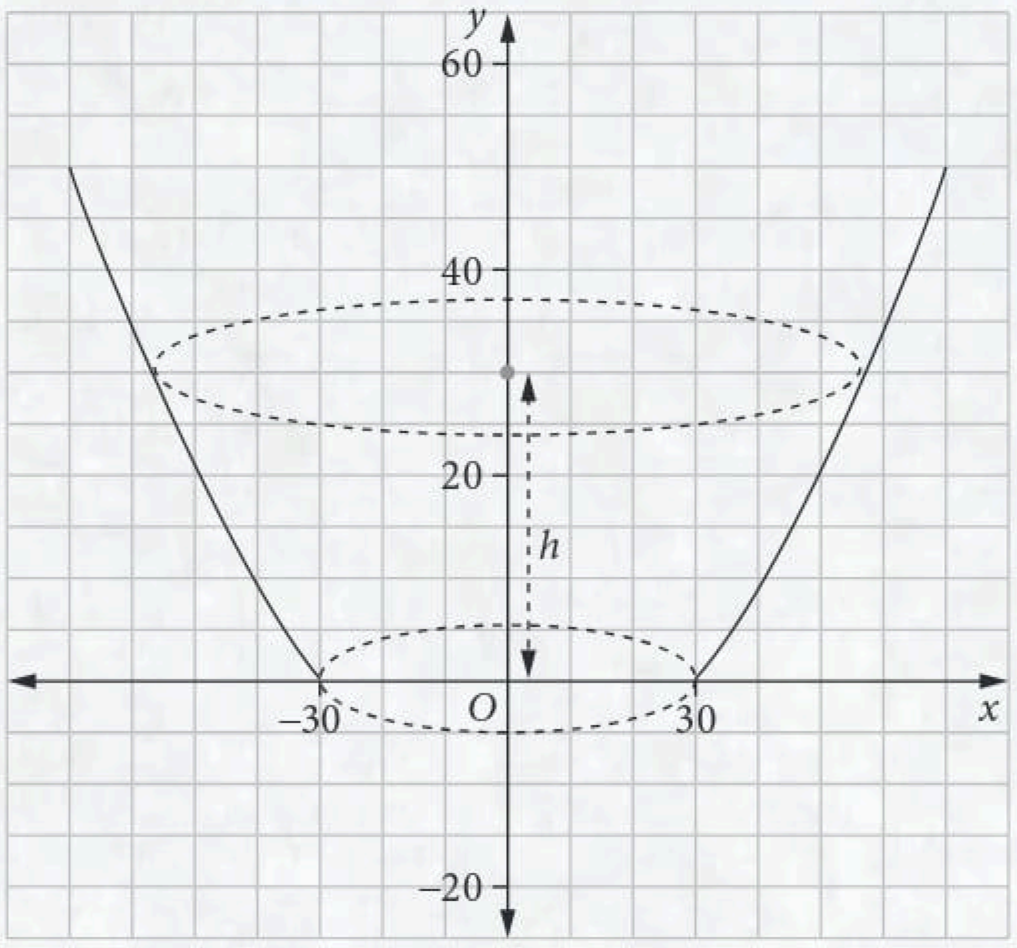
b Find the speed, in m/s, of the snowboarder when she leaves the jump at O. 1 mark

c Find the maximum height above O reached by the snowboarder. Give your answer in metres, correct to one decimal place. 2 marks

d Show that the time spent in the air by the snowboarder is $\frac{60\left(\sqrt{3} + 1\right)}{49}$ seconds. 3 marks

e Find the total distance the snowboarder travels while airborne. Give your answer in metres, correct to two decimal places. 2 marks

2 © VCAA 2019N 2BQ3 (9 marks)



The vertical cross-section of a barrel is shown above. The radius of the circular base (along the x-axis) is 30 cm and the radius of the circular top is 70 cm. The curved sides of the cross-section shown are parts of the parabola with the rule $y = \frac{x^2}{80} - \frac{45}{4}$. The height of the barrel is 50 cm.

a **i** Show that the volume of the barrel is given by $\pi \int_0^{50} (900 + 80y)\, dy$. 1 mark

ii Find the volume of the barrel in cubic centimetres. 1 mark

The barrel is initially full of water. Water begins to leak from the bottom of the barrel such that

$\frac{dV}{dt} = \frac{-8000\pi\sqrt{h}}{A}$ cm^3/s, where after t seconds the depth of the water is h cm, the volume of water remaining in the barrel is V cm^3 and the uppermost surface area of the water is A cm^2.

b Show that $\frac{dV}{dt} = \frac{-400\sqrt{h}}{4h + 45}$. 2 marks

c Find $\frac{dh}{dt}$ in terms of h, Express your answer in the form $\frac{-a\sqrt{h}}{\pi(b + ch)^2}$, where a, b and c are positive integers. 3 marks

d Using a definite integral in terms of h, find the **time**, **in hours**, correct to one decimal place, taken for the barrel to empty. 2 marks

CHAPTER

12 RANDOM VARIABLES AND HYPOTHESIS TESTING

Study Design coverage

AREA OF STUDY 6: DATA ANALYSIS, PROBABILITY AND STATISTICS

Distribution of linear combinations of random variables

- for n independent identically distributed random variables $X_1, X_2 \ldots X_n$ each with mean μ and variance σ^2:
  - $E(X_1 + X_2 + \ldots + X_n) = n\mu$
  - $Var(X_1 + X_2 + \ldots + X_n) = n\sigma^2$
- for n independent random variables $X_1, X_2 \ldots X_n$ and real numbers $a_1, a_2 \ldots a_n$:
  - $E(a_1X_1 + a_2 X_2 + \ldots + a_n X_n) = a_1 E(X_1) + a_2 E(X_2) + \ldots + a_n E(X_n)$
  - $Var(a_1 X_1 + a_2 X_2 + \ldots + a_n X_n) = a_1^2 \, Var(X_1) + a_2^2 \, Var(X_2) + \ldots + a_n^2 \, Var(X_n)$
- for n normally distributed independent random variables $X_1, X_2 \ldots X_n$ and real numbers $a_1, a_2 \ldots a_n$ the random variable $a_1X_1 + a_2 X_2 + \ldots + a_n X_n$ is also normally distributed.

Distribution of the sample mean

- the concept of the sample mean $\bar{X}$ as a random variable whose value varies between samples where X is a random variable with mean μ and the standard deviation σ
- simulation of repeated random sampling, from a variety of distributions and a range of sample sizes, to illustrate properties of the distribution of $\bar{X}$ across samples of a fixed size n including its mean μ, its standard deviation $\frac{\sigma}{\sqrt{n}}$ (where μ and σ are the mean and standard deviation of X respectively) and its approximate normality if n is large.

Confidence intervals for the population mean

- determination of confidence intervals for means and the use of simulation to illustrate variations in confidence intervals between samples and to show that the likelihood of a confidence interval containing μ depends on the level of confidence chosen in the determination of the interval
- construction of an approximate confidence interval, $\left(\bar{x} - z\frac{\sigma}{\sqrt{n}}, \ \bar{x} + z\frac{\sigma}{\sqrt{n}}\right)$ where σ is the population standard deviation and z is the appropriate quantile for the standard normal distribution or construction of an approximate confidence interval $\left(\bar{x} - z\frac{s}{\sqrt{n}}, \ \bar{x} + z\frac{s}{\sqrt{n}}\right)$ where s is the sample standard deviation and z is the appropriate quantile for the standard normal distribution, and n is large ($n \geq 30$ in many practical contexts).

Hypothesis testing for a population mean with a sample drawn from a normal distribution of known variance, or for a large sample

- concepts of null hypothesis, H_0, and alternative hypotheses, H_1, test statistic
- level of significance and p-value
- formulation of hypotheses and making a decision concerning a population mean based on:
  - a random sample from a normal population of known variance
  - a large random sample from any population
- 1-tail and 2-tail tests
- interpretation of the results of a hypothesis test in the context of the problem
- hypothesis test, relating the formulation, conduct, errors and results in terms of conditional probability.

VCE Mathematics Study Design 2023–2027 pp. 113–114, © VCAA 2022

Video playlists (7):

12.1	Linear combinations of random variables
12.2	Independent normal random variables
12.3	Sample means and simulations
12.4	Confidence intervals for the population mean
12.5	Hypothesis testing related to the mean
12.6	Errors in hypothesis testing
VCE question analysis	Random variables and hypothesis testing

Worksheets (4):

12.3	Mean, variance and standard deviation
12.4	Normal distributions • Confidence intervals • Polls and levels of confidence

Nelson MindTap

To access resources above, visit
cengage.com.au/nelsonmindtap

Video playlist Linear combinations of random variables

12.1 Linear combinations of random variables

Random variables have a probability distribution that can be used to predict the likelihood of a particular outcome occurring. The total value of homes that a real estate agent sells in a month, for example, will have a probability distribution which can be used to predict the probability that an agent will sell more than a particular value in a given month. If the total monthly commission the agent earns is a linear function of the value of the homes sold, then properties of the distribution like the mean and variance of the commission can also be determined. The monthly commission is a linear combination of the random variable, the value of homes sold.

The mean of $aX + b$

The mean or expected value of a random variable X is the long-term average value of X. If the random variable X is the height of preschool-aged girls in Victoria, then $\mathrm{E}(X)$ would be the average height of all preschool-aged girls in Victoria.

There are two formulas for the expected value of the random variable X, depending on whether the random variable is discrete or continuous. (This is covered in detail in Mathematical Methods Unit 4).

Discrete	Continuous
$\mathrm{E}(X) = \sum x \cdot p(x)$	$\mathrm{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$

The mean of $aX + b$

Consider a random variable X with an expected value $\mathrm{E}(X)$.

A random variable Y is a linear transformation of X, where $Y = aX + b$.

$$\mathrm{E}(Y) = \mathrm{E}(aX + b) = a\,\mathrm{E}(X) + b$$

WB p. 231

WORKED EXAMPLE 1 **Finding the mean of $aX + b$ for a discrete probability distribution**

The probability distribution of a discrete random variable X is shown.

x	0	1	2	3
$Pr(X = x)$	0.4	0.3	0.2	0.1

a Find the mean of X.

b Find $\mathrm{E}(3X + 2)$.

Steps	**Working**
a Use the formula $\mathrm{E}(X) = \sum x \cdot p(x)$. Rewrite the table above using columns. Add a column with the heading $x \times p(x)$ and calculate the product of the x values and their probabilities. The total of the $x \times p(x)$ column is the expected value of X or $\mathrm{E}(X)$.	see table below $\mathrm{E}(X) = \sum x \cdot p(x)$ The mean or $\mathrm{E}(X) = 1$.

x	$p(x)$	$x \times p(x)$
0	0.4	0
1	0.3	0.3
2	0.2	0.4
3	0.1	0.3
Total		1.0

b Find E(3X + 2) by substituting into the formula E(aX + b) = a E(X) + b with a = 3 and b = 2.	$E(3x + 2) = 3 \times 1 + 2$ $= 5$

WORKED EXAMPLE 2 **Finding the mean of *aX* + *b* for a continuous probability distribution**

The probability density function for a continuous random variable X is given by

$$f(x) = \begin{cases} 4x^3 & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

a Find E(X).

b Find E(10X – 4).

Steps	**Working**
a **1** Use the formula $\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)dx$ and simplify.	$E(X) = \int_0^1 x \times 4x^3\ dx$ $= \int_0^1 4x^4\ dx$
2 Evaluate the integral.	$E(X) = \left[\dfrac{4x^5}{5}\right]_0^1$ $= \dfrac{4(1)^5}{5} - 0$ $= \dfrac{4}{5}$
b Substitute a = 10 and b = –4 into the formula E(aX + b) = a E(X) + b.	$E(10X - 4) = 10 \times E(X) - 4$ $= 10 \times \dfrac{4}{5} - 4$ $= 4$

WB
p. 232

The variance of *aX* + *b*

The variance and standard deviation of a random variable X measure the spread of the variable about the mean.

The variance of a probability density function

The formulas for the variance and standard deviation are:

$$\text{Var}(X) = E(X^2) - \mu^2 \qquad \text{SD}(X) = \sqrt{\text{Var(X)}}$$

where $\mu = E(X)$

There are different formulas for E(X^2), depending on whether the distribution of the random variable is discrete or continuous.

Discrete	Continuous
$E(X^2) = \sum x^2 \cdot p(x)$	$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx$

The variance of *aX* + *b*

A random variable Y is a linear transformation of X, where $Y = aX + b$.

The variance of Y is

$$\text{Var}(aX + b) = a^2\ \text{Var}(X)$$

In the formula $Y = aX + b$, the addition of b increases all the data values by b and does not influence the spread.

WB
p. 233

WORKED EXAMPLE 3 **Finding the variance of $aX + b$ for a continuous probability density function**

The probability density function for a continuous random variable X is given by

$$f(x) = \begin{cases} 4x^3 & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the variance and hence find $\text{Var}(5X + 10)$.

Steps	**Working**
1 Write the formula for the variance and calculate $E(X)$ and $E(X^2)$. The mean μ or $E(X)$ was calculated in Worked example 2.	$\text{Var}(X) = E(X^2) - \mu^2$ $E(X) = \int_0^1 x \times 4x^3\, dx = \frac{4}{5}$
2 $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$	$E(X^2) = \int_0^1 x^2 \times 4x^3\, dx$ $= \int_0^1 4x^5\, dx$ $= \left[\frac{4x^6}{6}\right]_0^1$ $= \frac{2(1)^6}{3} - 0$ $= \frac{2}{3}$
3 Substitute in the variance formula from Step 1.	$\text{Var}(X) = \frac{2}{3} - \left(\frac{4}{5}\right)^2$ $= \frac{2}{3} - \frac{16}{25} = \frac{2}{75}$
4 Substitute into the formula $\text{Var}(aX + b) = a^2\,\text{Var}(X)$.	$\text{Var}(5X + 10) = 5^2\,\text{Var}(X)$ $= 25 \times \frac{2}{75}$ $= \frac{2}{3}$

The mean and variance of a linear combination of two independent random variables

Sums vs multiples

Consider the weight of 3 boxes, $(x_1 + x_2 + x_3)$, compared with the weight of a box that is 3 times bigger $(3X)$, where the random variable X represents the weight of a box.

Mean:	weight of 3 boxes:	$E(x_1 + x_2 + x_3) = 3E(X)$
	weight of a box 3 times as big:	$E(3X) = 3E(X)$
Variance:	weight of 3 boxes:	$\text{Var}(x_1 + x_2 + x_3) = 3\text{Var}(X)$
	weight of a box 3 times as big:	$\text{Var}(3X) = 3^2 \times \text{Var}(X)$

E($aX + bY$) and Var($aX + bY$) for sums and multiples

For two independent random variables X and Y,

$$\mathrm{E}(aX + bY) = a\,\mathrm{E}(X) + b\,\mathrm{E}(Y)$$

$$\mathrm{Var}(aX + bY) = a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y)$$

For the sum of n variables of type X, $\left(\sum_{i=1}^{n} x_i\right)$ and m variables of type Y, $\left(\sum_{j=1}^{m} y_j\right)$,

$$\mathrm{E}(x_1 + x_2 + \ldots + x_n + y_1 + y_2 + y_3 + \ldots + y_m) = n\,\mathrm{E}(X) + m\mathrm{E}(Y) \text{ and}$$

$$\mathrm{Var}(x_1 + x_2 + x_3 + \ldots + x_n + y_1 + y_2 + y_3 + \ldots + y_m) = n\,\mathrm{Var}(X) + m\mathrm{Var}(Y)$$

WORKED EXAMPLE 4 Finding the mean and variance of the sum of two variables

WB
p. 234

At Franken's furniture, the time taken to build a dining table, X hours, is a continuous random variable with a mean time of 15 hours and a variance of 9 hours, and the time taken to stain and polish the table, Y hours, is a continuous random variable with a mean time of 6 hours and a variance of 4 hours. Find the mean and variance of the time taken to finish a dining table.

Steps	**Working**
1 The total time taken to finish the table is the sum of the time to build the table and the time to stain and polish the table.	total time $= X + Y$
2 Substitute $a = 1$ and $b = 1$ into the formula $\mathrm{E}(aX + bY) = a\,\mathrm{E}(X) + b\mathrm{E}(Y)$ to find the mean time.	$\mathrm{E}(X + Y) = \mathrm{E}(X) + \mathrm{E}(Y)$ $= 15 + 6$ $= 21$ h
3 Substitute $a = 1$ and $b = 1$ into the formula $\mathrm{Var}(aX + bY) = a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y)$ to find the variance.	$\mathrm{Var}(1X + 1Y) = 1^2\mathrm{Var}(X) + 1^2\mathrm{Var}(Y)$ $= 9 + 4$ $= 13$ h

WORKED EXAMPLE 5 Finding E($aX + bY$) and Var($aX + bY$)

WB
p. 235

Two independent random variables, X and Y, have means of 30 and 20 and variances of 5 and 8 respectively. If $Z = 4X + 2Y$, find the mean and variance of Z.

Steps	**Working**
1 To find the mean of Z, use the formula $\mathrm{E}(aX + bY) = a\,\mathrm{E}(X) + b\,\mathrm{E}(Y)$.	For $Z = 4X + 2Y$, $\mathrm{E}(Z) = \mathrm{E}(4X + 2Y)$ $= 4\,\mathrm{E}(X) + 2\,\mathrm{E}(Y)$ $= 4 \times 30 + 2 \times 20$ $= 160$
2 To find the variance of Z, use the formula $\mathrm{Var}(aX + bY) = a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y)$.	For $Z = 4X + 2Y$, $\mathrm{Var}(Z) = \mathrm{Var}(4X + 2Y)$ $= 4^2\,\mathrm{Var}(X) + 2^2\,\mathrm{Var}(Y)$ $= 4^2 \times 5 + 2^2 \times 8$ $= 112$

WB

p. 236

WORKED EXAMPLE 6 Finding the mean and variance of the sum of *n* variables of type *X* and *m* variables of type *Y*

Strawberries grown on a berry farm have a mean mass of 15 g with a standard deviation of 3 g. Raspberries grown on the same farm have a mean mass of 5 g with a standard deviation of 1 g.

The masses of the strawberries are independent of the masses of the raspberries.

Find the mean mass and standard deviation, in grams, of a set of four of these strawberries and two of these raspberries.

Steps	**Working**
a **1** Find the mean using the formula $\mathrm{E}(x_1 + x_2 + \ldots + x_n + y_1 + y_2 + \ldots + y_m)$ $= n\,\mathrm{E}(X) + m\mathrm{E}(Y)$, where $n = 4$ and $m = 2$.	Let $X =$ the mass (g) of strawberries $Y =$ the mass (g) of raspberries $\mathrm{E}(x_1 + x_2 + x_3 + x_4 + y_1 + y_2)$ $= 4\ \mathrm{E}(X) + 2\mathrm{E}(Y)$ $= 4 \times 15 + 2 \times 5$ $= 70\,\mathrm{g}$
2 Find the variance using the formula $\mathrm{Var}(x_1 + x_2 + x_3 + \ldots + x_n + y_1 + y_2 + y_3 + \ldots + y_m)$ $= n\,\mathrm{Var}(X) + m\mathrm{Var}(Y)$, where $n = 4$ and $m = 2$.	$\mathrm{Var}(x_1 + x_2 + x_3 + x_4 + y_1 + y_2)$ $= 4\ \mathrm{Var}(X) + 2\mathrm{Var}(Y)$ $= 4(\sigma_X)^2 + 2(\sigma_Y)^2$ $= 4 \times 3^2$ $= 2 \times 1^2$ $= 38$
3 Find the standard deviation.	standard deviation $= \sqrt{38}$

EXERCISE 12.1 Linear combinations of random variables

ANSWERS p. 600

Mastery

1 WORKED EXAMPLE 1 The discrete random variable X has the probability distribution shown below in the table.

x	0	1	2	3	4
$Pr(X = x)$	0.4	0.1	0.1	0.2	0.2

a Find $\mathrm{E}(X)$.

b Find $\mathrm{E}(10X + 3)$.

2 WORKED EXAMPLE 2 A continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} \dfrac{1}{288}(12x - x^2) & 0 \le x \le 12 \\ 0 & \text{otherwise} \end{cases}$$

a Find the expected value of X.

b Find the expected value of $\dfrac{1}{3}X + 2$.

3 WORKED EXAMPLE 3 A continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} \dfrac{x}{8} & 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

a Find $\mathrm{Var}(X)$.

b Find $\mathrm{Var}(18X + 14)$.

4 WORKED EXAMPLE 4 When Matt cleans his house, he dusts and vacuums. The time that he takes to dust the house is a continuous random variable with a mean of 30 minutes and a variance of 25 minutes. The time that Matt takes to vacuum the house is another continuous random variable with a mean of 40 minutes and a variance of 64 minutes. Find the mean and the variance of the time that Matt takes to clean his house.

5 WORKED EXAMPLE 5 Two independent random variables X and Y have means of 50 and 15 and variances of 10 and 20. If $Z = 3X + 4Y$, find the mean and variance of Z.

6 WORKED EXAMPLE 6 Philippa prepares her vegetable garden for planting by first digging the soil and then adding organic compost. The time taken to dig each garden bed is a random variable with a mean of 20 minutes and a standard deviation of 4 minutes. The time taken to add compost to each garden bed is also a random variable with a mean of 8 minutes and a standard deviation of 2 minutes. The time taken to dig the garden bed is independent of the time taken to add organic compost to the garden bed. If the random variable Z represents the time Philippa takes to prepare three garden beds for planting, find

a $\text{E}(Z)$ **b** $\text{Var}(Z)$ **c** $\text{SD}(Z)$

Exam practice

80–100% 60–79% 0–59%

7 © VCAA 2017N 1Q9a TECH-FREE (2 marks) The random variables X and Y are independent with $\mu_X = 4$, $\text{Var}(X) = 36$ and $\mu_Y = 3$, $\text{Var}(Y) = 25$.

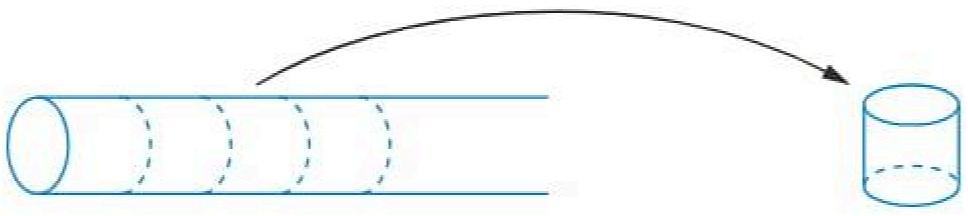
The random variable Z is such that $Z = 2X + 3Y$.

a Find $\text{E}(Z)$. 1 mark

b Find the standard deviation of Z. 1 mark

8 TECH-FREE (2 marks) Two independent random variables X and Y have means of 11 and 7 and variances of 6 and 4 respectively. Find the mean and variance of $9X - 3Y$.

9 © VCAA 2019 1Q3a TECH-FREE (3 marks) A machine produces chocolate in the form of a continuous cylinder of radius 0.5 cm. Smaller cylindrical pieces are cut parallel to its end, as shown in the diagram below. The lengths of the pieces vary with a mean of 3 cm and a standard deviation of 0.1 cm.



a 89% Find the expected volume of a piece of chocolate in cm^3. 1 mark

b 30% Find the variance of the volume of a piece of chocolate in cm^6. 1 mark

c 50% Find the expected surface area of a piece of chocolate in cm^2. 1 mark

10 © VCAA 2016 2AQ18 61% Oranges grown on a citrus farm have a mean mass of 204 grams with a standard deviation of 9 grams. Lemons grown on the same farm have a mean mass of 76 grams with a standard deviation of 3 grams.

The masses of the lemons are independent of the masses of the oranges.

The mean mass and standard deviation, in grams, respectively of a set of three of these oranges and two of these lemons are

A $764,\ 3\sqrt{29}$ **B** $636,\ 12$ **C** $764,\ \sqrt{33}$

D $636,\ 3\sqrt{10}$ **E** $636,\ 33$

11 © VCAA 2017N 2AQ18 X is a random variable with a mean of 5 and a standard deviation of 4, and Y is a random variable with a mean of 3 and a standard deviation of 2.

If X and Y are independent random variables and $Z = X - 2Y$, then Z will have mean μ and standard deviation σ given by

A $\mu = -1, \sigma = 0$
B $\mu = -1, \sigma = 4\sqrt{2}$
C $\mu = 2, \sigma = 8$
D $\mu = 2, \sigma = 4\sqrt{2}$
E $\mu = -1, \sigma = 2\sqrt{6}$

12 © VCAA 2021 2AQ19 51% The mean unscaled score for a certain assessment task is 25 and the variance is 36. The scores are scaled so that the mean score is 30 and the variance is 49. Let S be the scaled scores, to the nearest integer, and let X be the unscaled scores.

If the scaling function takes the form $S = mX + n$, where $m \in R^+$ and $n \in R$, then a score of 32 would be scaled to

A 22
B 34
C 36
D 38
E 40

13 X is a random variable with a mean of 12 and a variance of 9. The variance of the variable $Y = 4X + 12$ is

A 36
B 144
C 156
D 192
E 204

14 X is a random variable with a mean of 36 and a standard deviation of 4. The variance of the variable $Y = 2X - 5$ is

A 3
B 8
C 16
D 64
E 67

15 Two independent random variables X and Y have means of 15 and 10 and variances of 4 and 2. The mean and variance of $5X + 2Y$ is

A 95 and 24
B 415 and 108
C 25 and 6
D 95 and 6
E 95 and 108

Use the information below for Questions 16 and 17.

The mean price per punnet of strawberries and blueberries is \$1.50 and \$2.50 and the variance is \$0.60 and \$0.80, respectively.

16 The mean price of four punnets of strawberries and five punnets of blueberries is

A \$1.40
B \$4.00
C \$17.50
D \$18.50
E \$86.50

17 The variance of the price of four punnets of strawberries and five punnets of blueberries is

A \$1.40
B \$5.40
C \$6.40
D \$18.50
E \$29.60

18 (5 marks) A random variable X has the probability density function

$$f(x) = \begin{cases} \dfrac{x}{2} & \text{if } x \in [0, 2] \\ 0 & \text{otherwise} \end{cases}$$

Find

a $\mathrm{E}(X)$ 1 mark
b $\mathrm{E}(3X + 6)$ 1 mark
c $\mathrm{Var}(X)$ 1 mark
d $\mathrm{Var}(3X + 6)$ 2 marks

19 (3 marks) A random variable Y has probability density function

$$f(y) = \begin{cases} \frac{3}{32}(4y - y^2) & 0 \leq y \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

Find the variance of $Z = 3Y - 11$.

20 (2 marks) Each school day, Sam walks from his house to the bus stop and then catches the bus to school.

The time taken to walk to the bus stop is a continuous random variable X_1 with a mean of 10 minutes and a variance of 1 minutes. The time taken by the bus is a continuous random variable X_2 with a mean of 20 minutes and a variance of 5 minutes. If X_1 and X_2 are independent, find the mean and variance of

a the total time to get to school. 1 mark

b the total travel time for the school week (assume that the times for the return trip from school to home have the same mean and variance). 1 mark

12.2 Independent normal random variables

Video playlist Independent normal random variables

A linear combination of two independent normal random variables will also have a normal distribution.

The mean and variance of *aX* + *b* for normally distributed variables

If X is normally distributed with a mean of of μ_X and a variance of $(\sigma_X)^2$ and Y is an independent normally distributed random variable with a mean of μ_Y and a variance of $(\sigma_Y)^2$, then $aX + bY$ is also normally distributed with

Mean	$\text{E}(aX + bY) = a\mu_X + b\mu_Y$
Variance	$\text{Var}(aX + bY) = a^2(\sigma_X)^2 + b^2(\sigma_Y)^2$

WORKED EXAMPLE 7 The sum of two normally distributed random variables

WB p. 237

A Year 10 class sits a Maths test and an English test. The scores on the Maths test are normally distributed with a mean of 65 and a standard deviation of 5, and the scores on the English test are normally distributed with a mean of 60 and a standard deviation of 10.

If the test scores are independent, find

a the mean and variance of the total score of the two tests.

b the probability, correct to three decimal places, that a student will score better on the Maths test than on the English test.

Steps	**Working**
a 1 Let X = the Maths test score and Y = the English test score.	$\mu_X = 65, \sigma_X = 5$ $\mu_Y = 60, \sigma_Y = 10$
2 Use the formula $\text{E}(aX + bY) = a\mu_X + b\mu_Y$ to find the mean of the total score.	$\text{E}(X + Y) = \mu_X + \mu_Y$ $= 65 + 60$ $= 125$
3 Use the formula $\text{Var}(aX + bY) = a^2(\sigma_X)^2 + b^2(\sigma_Y)^2$ to find the variance of the total score.	$\text{Var}(X + Y) = (\sigma_X)^2 + (\sigma_Y)^2$ $= 5^2 + 10^2$ $= 125$

b **1** Write a probability equation.

$\Pr(X > Y) = \Pr(X - Y > 0)$

2 Find the mean and variance of $X - Y$ by using the formulas for mean and variance mentioned in part **a**.

$$\begin{aligned}E(X - Y) &= \mu_X + (-1)\mu_Y\\ &= 65 - 60\\ &= 5\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(X - Y) &= (\sigma_X)^2 + (-1)^2(\sigma_Y)^2\\ &= 5^2 + 10^2\\ &= 125\end{aligned}$$

3 Find the standard deviation.

standard deviation $= \sqrt{125} = 5\sqrt{5}$

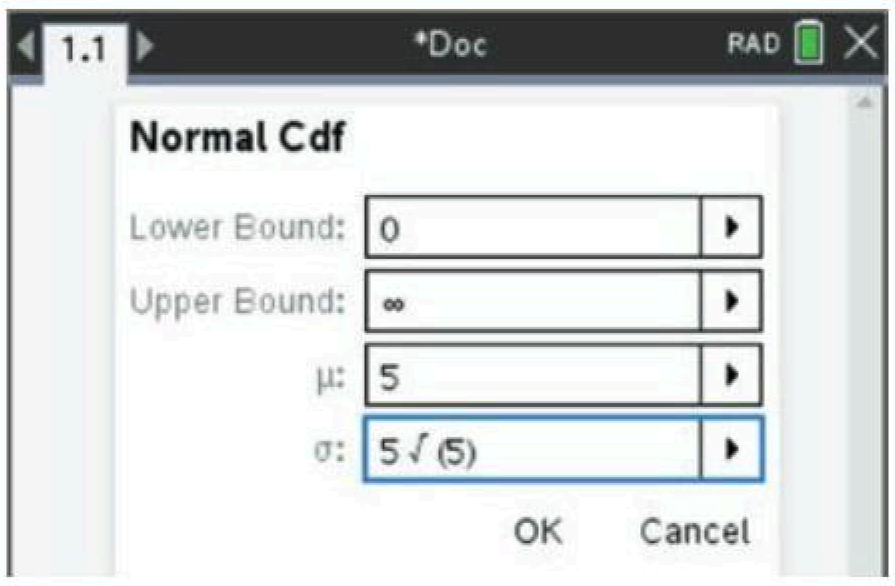
4 Calculate the normal probability using CAS (see below).

$\Pr(X - Y > 0) \approx 0.673$

USING CAS 1 Finding normal probabilities

A normally distributed random variable X, has a mean of 5 and a standard deviation of $5\sqrt{5}$. Calculate $\Pr(X > 0)$.

TI-Nspire



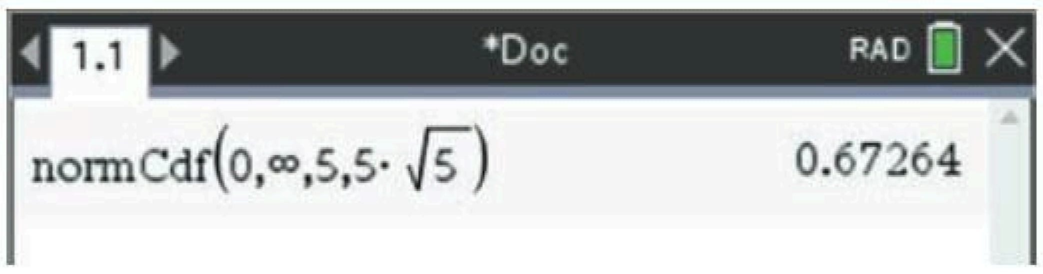
Normal Cdf

Lower Bound: 0

Upper Bound: ∞

μ: 5

σ: 5√(5)



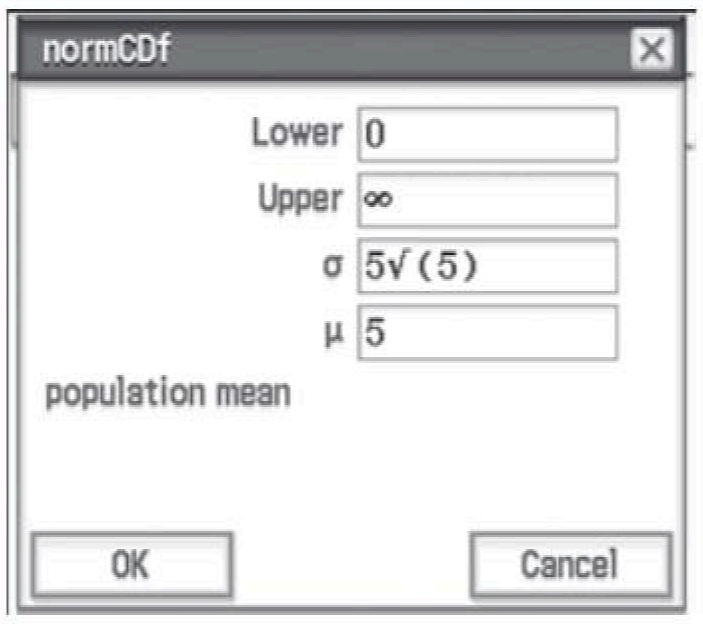
normCdf$(0,\infty,5,5\cdot\sqrt{5})$ 0.67264

1 Press **menu > Probability > Distributions > Normal Cdf**.

2 In the dialogue box, enter **0, ∞. 5, 5**$\sqrt{5}$ as shown above. Press π to access the mini-palette for the ∞ symbol.

3 The normal probability value will be displayed.

ClassPad



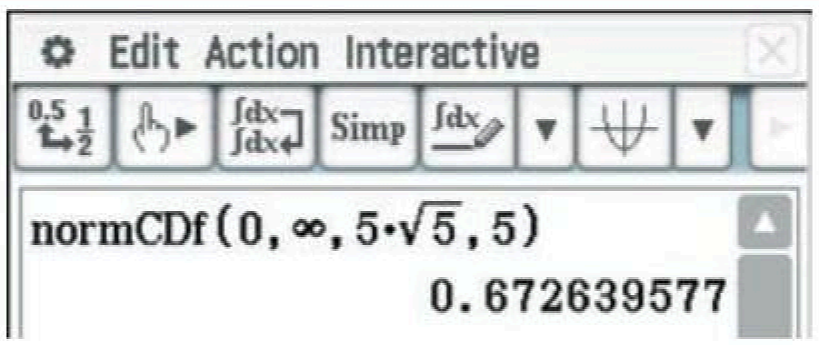
Lower 0

Upper ∞

σ 5√(5)

μ 5

population mean



normCDf$(0, \infty, 5\cdot\sqrt{5}, 5)$

0.672639577

1 In **Main,** tap **Interactive > Distribution/Inv. Dist > Continuous > normCDf**.

2 In the dialogue box, enter **0**, ∞, $5\sqrt{5}$, **5** as shown above. Open Keyboard and tap **Math2** or **Math3** for the ∞ symbol.

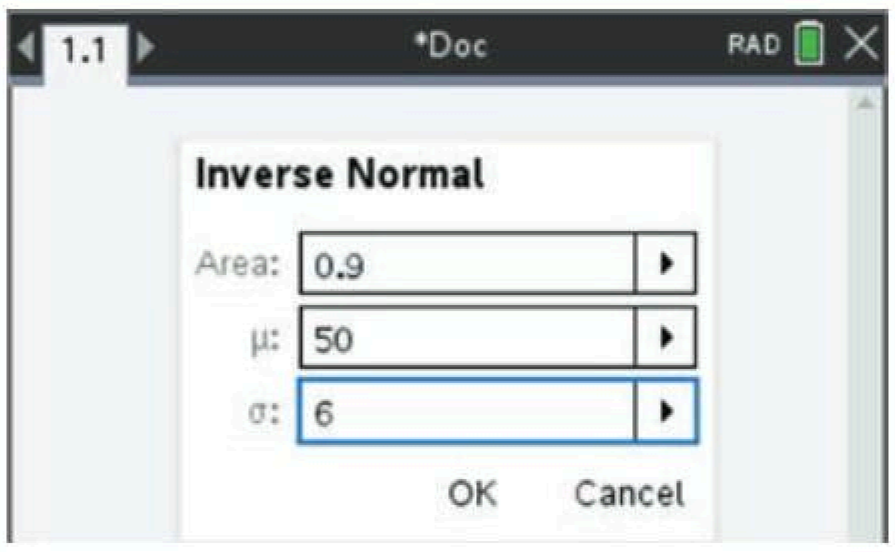
3 The normal probability value will be displayed.

$\Pr(X - Y > 0) \approx 0.673$

USING CAS 2 Finding the inverse probability for a normal distribution

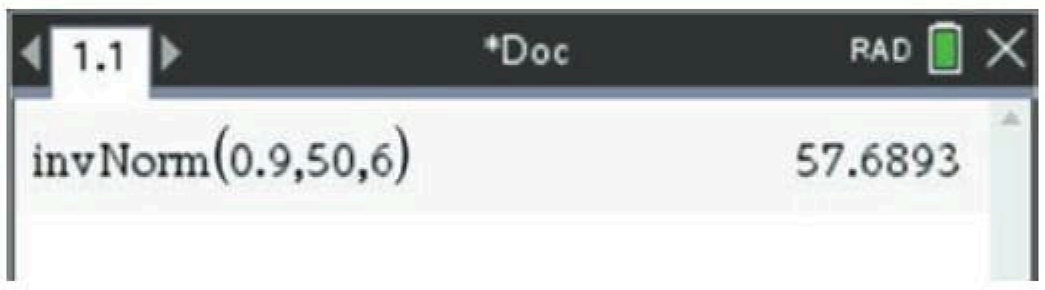
For a normal probability distribution with a mean of 50 and a standard deviation of 6, calculate the value of x such that $\Pr(X < x) = 0.9$, correct to three decimal places.

TI-Nspire



Inverse Normal

Area: 0.9
μ: 50
σ: 6



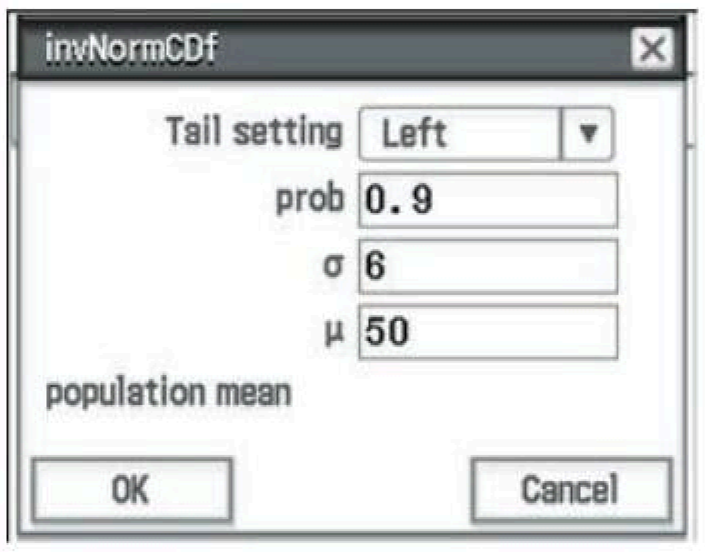
```
invNorm(0.9,50,6)                 57.6893
```

1 Press **menu > Probability > Distributions > Inverse Normal**.

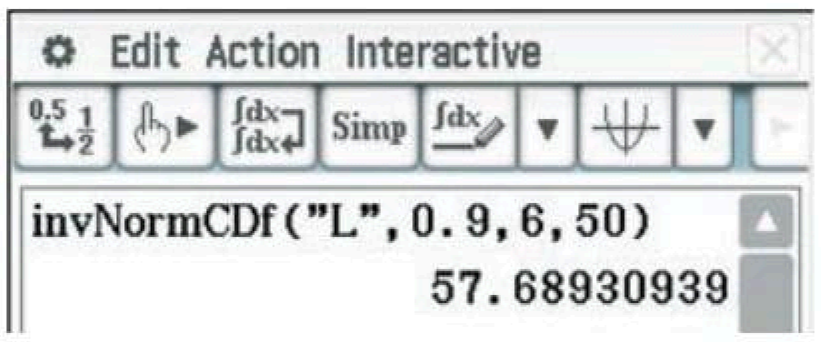
2 In the dialogue box, enter **0.9**, **50**, **6** as shown above.

3 The value of x will be displayed.

ClassPad



Tail setting Left
prob 0.9
σ 6
μ 50
population mean



```
invNormCDf("L",0.9,6,50)
                 57.68930939
```

1 Tap **Interactive > Distribution/Inv. Dist > Inverse > InvNormCDf**.

2 In the dialogue box, keep the **Tail setting** on **Left** and enter **0.9**, **6**, **50** as shown above.

3 The value of x will be displayed.

The value of x is 57.69.

WORKED EXAMPLE 8 The sum of n items selected from a normal distribution

WB
p. 238

The volume of spring water dispensed by a machine into bottles varies normally with a mean of 240 mL and a standard deviation of 5 mL. The spring water is sold in packs of six bottles. Let the normal random variable X represent the total volume of water (mL) in a pack of six bottles.

a Find the mean and standard deviation of X.

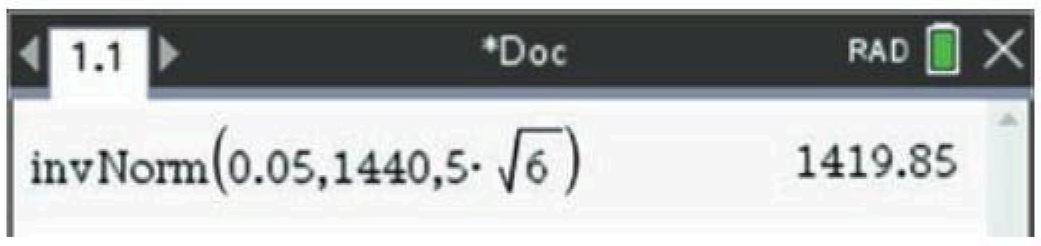
b Find $\Pr(X < 1450)$, correct to three decimal places.

c Six packs of spring water are rejected if their total volume is less than x mL. Find the value of x correct to the nearest mL if 5% of packs are rejected.

Steps	**Working**
a 1 Find the mean using the formula $E(x_1 + x_2 + x_3 + \ldots + x_n)$ $= nE(X)$ where $n = 6$.	$E(x_1 + x_2 + x_3 + \ldots + x_6)$ $= 6\,E(X)$ $= 6 \times 240$ $= 1440$

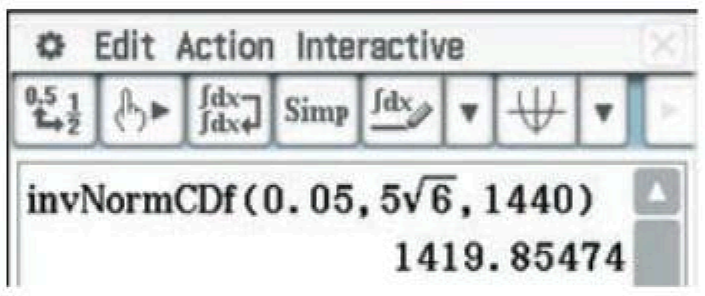
2 Find the standard deviation using the formulas $\text{Var}(x_1 + x_2 + x_3 + \ldots + x_n) = n\text{Var}(X)$ where $n = 6$ and $\text{SD}(X) = \sqrt{\text{Var}(X)}$.	$\text{Var}(x_1 + x_2 + x_3 + \cdots + x_6)$ $= 6\,\text{Var}(X)$ $= 6 \times 5^2$ $= 150$ $\text{SD(X)} = \sqrt{150} = 5\sqrt{6}$
b Calculate the normal probability using CAS.	$\Pr(X < 1450) = 0.793$
c Write the probability equation and use the inverse cumulative normal on CAS to solve.	$\Pr(X < x) = 0.05$

TI-Nspire



```
invNorm(0.05,1440,5·√6)        1419.85
```

ClassPad



```
invNormCDf(0.05,5√6,1440)
                 1419.85474
```

$x = 1420$ mL

Six packs of spring water are rejected if their total volume is less than 1420 mL.

EXERCISE 12.2 Independent normal random variables

ANSWERS p. 600

Recap

1 The random variables X and Y are independent with $\mu_X = 16$, $\text{Var}(X) = 16$ and $\mu_Y = 4$, $\text{Var}(Y) = 9$.

The random variable Z is such that $Z = 2X + 3Y$. The variance of Z is

A 20 **B** 25 **C** 59 **D** 68 **E** 145

2 Avocados sold at the market have a mean mass of 80 grams and a standard deviation of 4 grams. The avocados are sold in boxes of four.

The mean mass and standard deviation, in grams, of a box of avocados are

A 20, 1 **B** 320, 16 **C** 320, 8 **D** 320, 64 **E** 80, 2

Mastery

3 WORKED EXAMPLE 7 A factory produces nuts and bolts. The weight of the bolts is normally distributed with a mean of 15 g and a standard deviation of 0.4 g. The weight of the nuts is normally distributed with a mean of 5 g and a standard deviation of 0.2 g.

a Find the mean and the standard deviation of the combined mass of a nut and a bolt.

b Find the probability, correct to three decimal places, that the combined mass is less than 20.5 g.

4 Using CAS 1 Two groups of Year 7 students run a lap of the school oval. The recorded times for a lap are normally distributed with a mean and standard deviation of $\mu_A = 650$ seconds and $\sigma_A = 25$ s for group A and $\mu_B = 660$ s and $\sigma_B = 30$ s for group B.

If the times are independent, find

a the mean and variance of the difference in times for groups A and B.

b the probability, correct to three decimal places, that a student in group B will record a better lap time than a student in group A.

5 Using CAS 2 For a normal probability distribution with a mean of 440 and a standard deviation of 35, calculate the value of x such that $\Pr(X < x) = 0.95$, correct to three decimal places.

6 WORKED EXAMPLE 8 The volume of a double espresso dispensed at a cafe varies normally with a mean of 75 mL and a standard deviation of 5 mL. Maryam purchases four double espressos each day. Let the normal random variable X represent the total volume of coffee (mL) Maryam consumes each day.

a Find the mean and standard deviation of X.

b Find $\Pr(X > 290)$, correct to three decimal places.

c Maryam has trouble sleeping if she consumes more than x mL each day. Find the value of x, correct to the nearest mL, if Maryam has trouble sleeping on 15% of her evenings.

Exam practice

80–100% 60–79% 0–59%

7 © VCAA 2017 1Q4 48% TECH-FREE (3 marks) The volume of soft drink dispensed by a machine into bottles varies normally with a mean of 298 mL and a standard deviation of 3 mL. The soft drink is sold in packs of four bottles. Find the approximate probability that the mean volume of soft drink per bottle in a randomly selected four-bottle pack is less than 295 mL. Give your answer correct to three decimal places.

8 © VCAA 2018 2AQ20 56% The scores on the Mathematics and Statistics tests, expressed as percentages, in a particular year were both normally distributed. The mean and the standard deviation of the Mathematics test scores were 71 and 10, respectively, while the mean and the standard deviation of the Statistics test scores were 75 and 7, respectively.

Assuming the sets of test scores were independent of each other, the probability, correct to four decimal places, that a randomly chosen Mathematics score is higher than a randomly chosen Statistics score is

A 0.2877 **B** 0.3716 **C** 0.4070 **D** 0.7123 **E** 0.9088

9 © VCAA 2017 2AQ18 42% U and V are independent normally distributed random variables, where U has a mean of 5 and a variance of 1, and V has a mean of 8 and a variance of 1. The random variable W is defined by $W = 4U - 3V$.

In terms of the standard normal variable Z, $\Pr(W > 5)$ is equivalent to

A $\Pr\left(Z > \frac{9\sqrt{7}}{7}\right)$ **B** $\Pr(Z < 1.8)$ **C** $\Pr\left(Z < \frac{9\sqrt{7}}{7}\right)$

D $\Pr(Z > 0.2)$ **E** $\Pr(Z > 1.8)$

10 © VCAA 2018N 2AQ20 A farm grows oranges and lemons. The oranges have a mean mass of 200 grams with a standard deviation of 5 grams and the lemons have a mean mass of 70 grams with a standard deviation of 3 grams.

Assuming masses for each type of fruit are normally distributed, what is the probability, correct to four decimal places, that a randomly selected orange will have at least three times the mass of a randomly selected lemon?

A 0.0062 **B** 0.0828 **C** 0.1657 **D** 0.8343 **E** 0.9172

11 © VCAA 2019N 2AQ20 Nitrogen oxide emissions for a certain type of car are known to be normally distributed with a mean of 0.875 g/km and a standard deviation of 0.188 g/km.

For two randomly selected cars, the probability that their nitrogen oxide emissions differ by more than 0.5 g/km is closest to

A 0.030 **B** 0.060 **C** 0.960 **D** 0.970 **E** 0.977

12 © VCAA 2018N 2AQ19 A local supermarket sells apples in bags that have **negligible mass**. The stated mass of a bag of apples is 1 kg.

The mass of this particular type of apple is known to be normally distributed with a mean of 115 g and a standard deviation of 7 g. A particular bag contains nine randomly selected apples.

The probability that the nine apples in this bag have a total mass of less than 1 kg is

A 0.0478 **B** 0.1132 **C** 0.4265 **D** 0.5373 **E** 0.9522

13 © VCAA 2021 2AQ20 43% An office has two coffee machines that operate independently of each other. The time taken for each machine to produce a cup of coffee is normally distributed with a mean of 30 seconds and a standard deviation of 5 seconds. On a particular morning, a cup of coffee is produced from each machine.

The probability that the time taken by each coffee machine to produce one cup of coffee will differ by less than 3 seconds is closest to

A 0.164 **B** 0.236 **C** 0.329 **D** 0.451 **E** 0.671

Use the following information to answer Questions 14 and 15.

Two independent normal random variables X and Y have means of 14 and 24 and standard deviations of 3 and 6, respectively.

14 The standard deviation of $X + Y$ is

A 9 **B** $3\sqrt{5}$ **C** 45 **D** 38 **E** 3

15 The value of $\Pr(Y > 45 - X)$, correct to three decimal places, is

A 0.852 **B** 0.218 **C** 0.782 **D** 0.314 **E** 0.148

16 Two independent normal random variables X and Y have standard deviations of 10 and 4 respectively. The standard deviation of $X - Y$ is

A $\sqrt{6}$ **B** $2\sqrt{21}$ **C** 6 **D** 84 **E** $2\sqrt{29}$

17 X is a normal random variable with a mean of 125 and a standard deviation of 10 and Y is a normal random variable with a mean of 25 and a standard deviation of 5. If X and Y are independent, the mean and standard deviation of $2X - 4Y$ are

A 350 and $20\sqrt{2}$ **B** 150 and $5\sqrt{5}$ **C** 150 and 0

D 150 and $20\sqrt{2}$ **E** 135 and 30

18 Two independent normal random variables X and Y have means of 20 and 40 and standard deviations of 4 and 12, respectively. Correct to three decimal places, $\Pr(X + Y < 55)$ is equal to

A 0.346 **B** 0.415 **C** 0.384 **D** 0.106 **E** 0.654

19 (4 marks) Each weekday morning, Frankie drives to the station and then catches the train to work. The time taken for the car trip has a mean of 20 minutes and a standard deviation of 4 minutes. The time taken for the train trip has a mean of 15 minutes and a standard deviation of 2 minutes. The time to drive to work and the time to travel by train are independent variables.

a Find the mean and standard deviation of the total travel time to work. 2 marks

b Find the probability, correct to three decimal places, that Frankie takes less than 37 minutes to get to work. 2 marks

20 (3 marks) A deluxe car wash comprises of a wash and vacuum. The times for the wash and vacuum are independent normally distributed variables. If the time for the car wash has a mean of 18 minutes and a standard deviation of 3 minutes, and the time for the vacuum has a mean of 12 minutes and a standard deviation of 2 minutes, find the probability, correct to three decimal places, that 3 cars can be washed and vacuumed in less than 100 minutes.

12.3 Sample means and simulations

Now we will examine the properties of the sampling distribution of the sample means. We can use CAS to generate random samples from a normally distributed population. This will ensure that the population mean is known and we can investigate properties of the sample mean.

The sampling distribution of the sample means

Video playlist Sample means and simulations

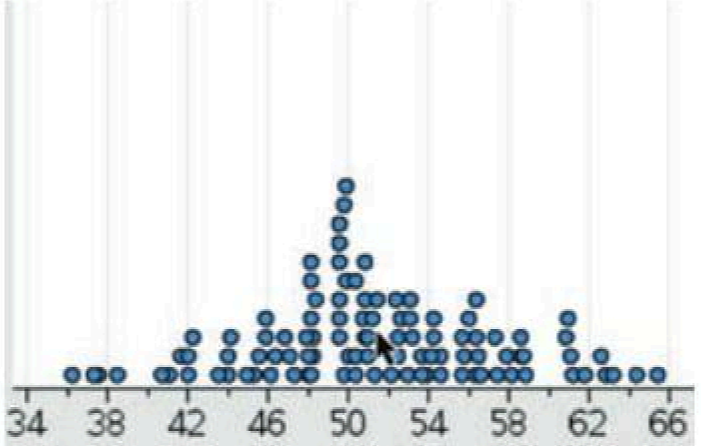
Worksheet Mean, variance and standard deviation

One hundred random samples of 20 integers are taken from the set {1, 2, 3 … 100} and the mean of each sample is calculated. The mean of the population is 50.5.

Each of the 100 samples has a different mean and these means are shown in the dotplot.

The means of the samples are symmetrically spread about the centre, which is the mean of all the sample means.

In this simulation, the mean of the sample means is 51.2, which is approximately equal to the mean of the population (50.5).

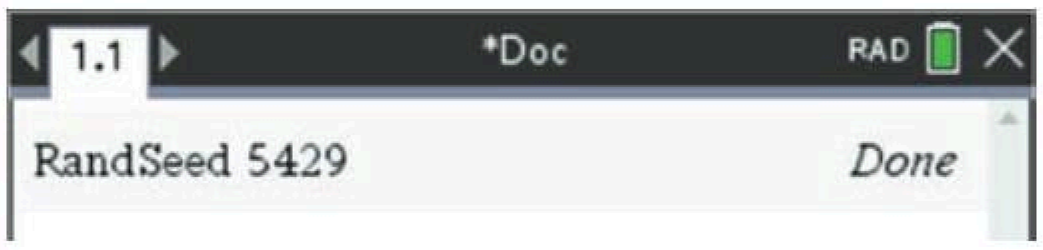


USING CAS 3 The distribution of the sample means by simulation

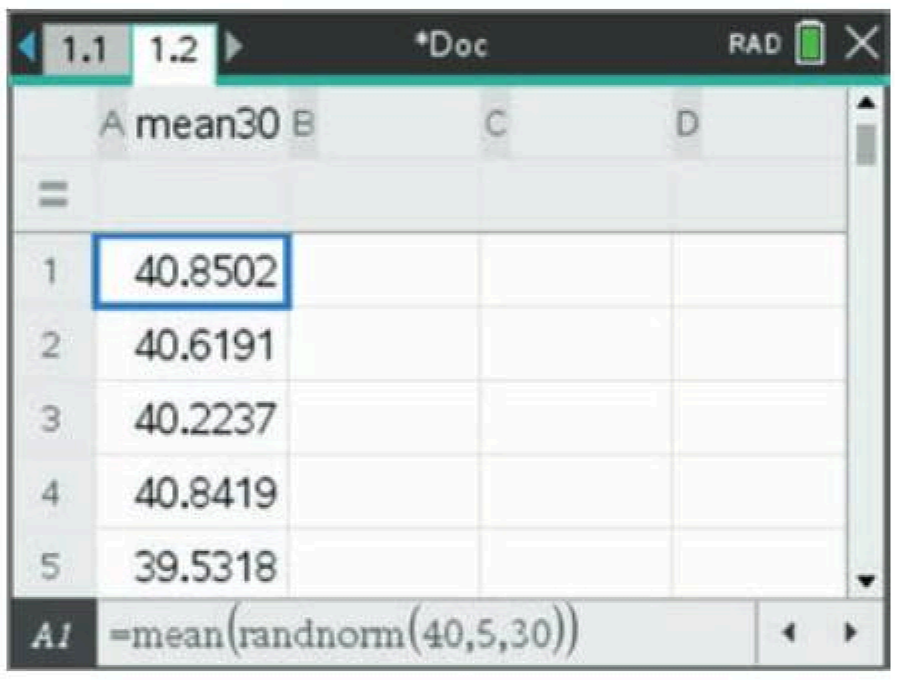
Generate 100 samples of size 30 from a population that is normally distributed with a mean of 40 and a standard deviation of 5.

Calculate the mean of each sample and plot the distribution of the sample means. Answers may vary.

TI-Nspire



RandSeed 5429 Done



	A mean30	B	C	D
=				
1	40.8502			
2	40.6191			
3	40.2237			
4	40.8419			
5	39.5318			

A1 =mean(randnorm(40,5,30))

1 Press **menu > Probability > Random > Seed**.

2 Enter any number of your own choice to set a starting point for random number calculations.

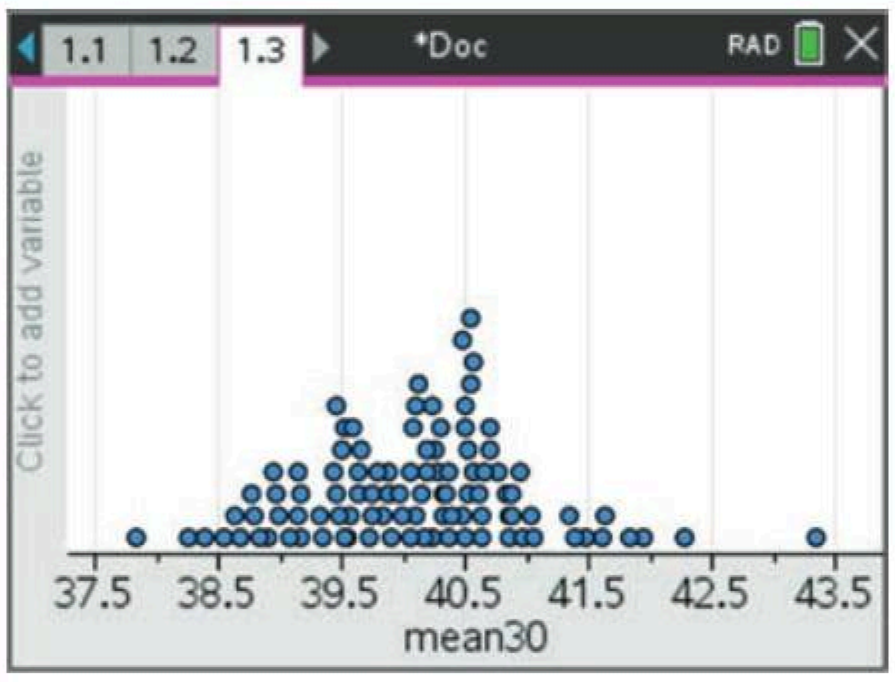
3 Add a **Lists & Spreadsheet** page.

4 Enter a suitable heading for column **A** such as **mean30**.

5 In the cell **A1**, enter the formula **=mean(randNorm(40,5,30))**. Press **catalog** to access the functions.

6 Move the cursor to cell **A1** and press **menu > Data > Fill**.

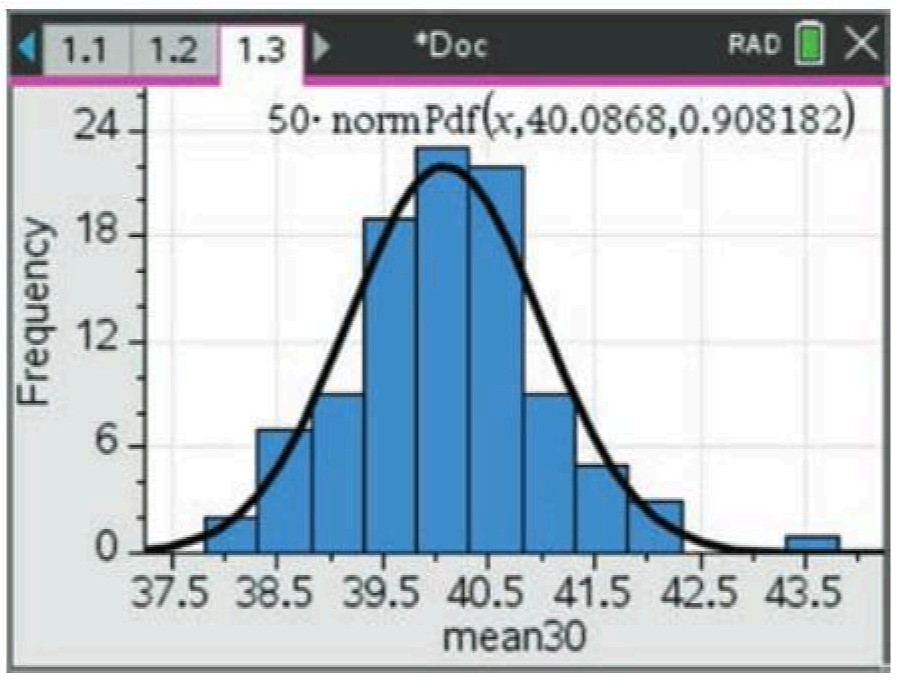
7 Press the **down arrow** to fill 100 cells of the column.



8 Add a **Data & Statistics** page.

9 On the horizontal access, select the variable **mean30**.

10 A scatterplot of the sample means will be displayed.

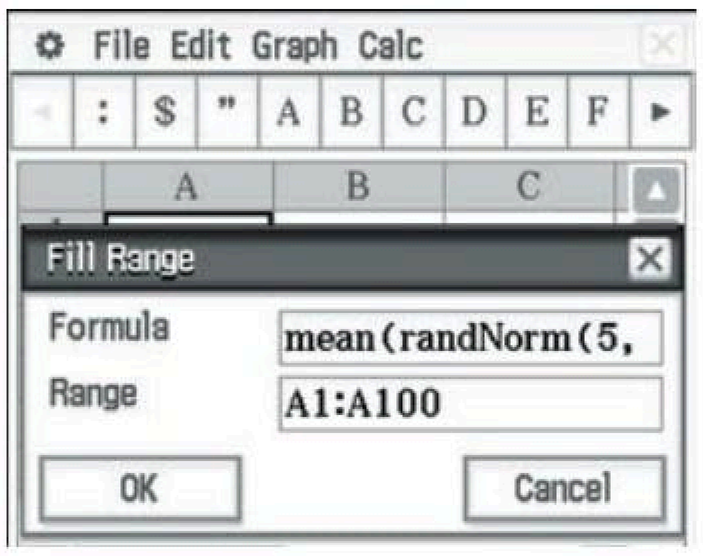


11 Press **menu > Plot Type > Histogram**.

12 Press **menu > Analyze > Show Normal PDF**.

13 The normal curve for the average mean and standard deviation will be displayed with the histogram.

ClassPad



Formula mean(randNorm(5,

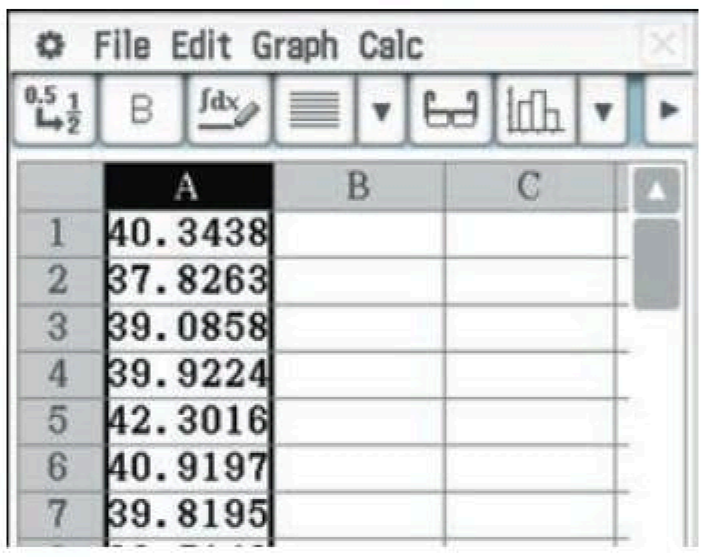
Range A1:A100

1 Open the **Spreadsheet** application.

2 Tap **Edit > Fill > Fill Range**.

3 In the dialogue box **Formula** field, enter **mean(randNorm(5,40,30))**. Open the **Keyboard > Catalog** to access the **randNorm** function.

4 For the **Range** field, enter **A1:A100**.

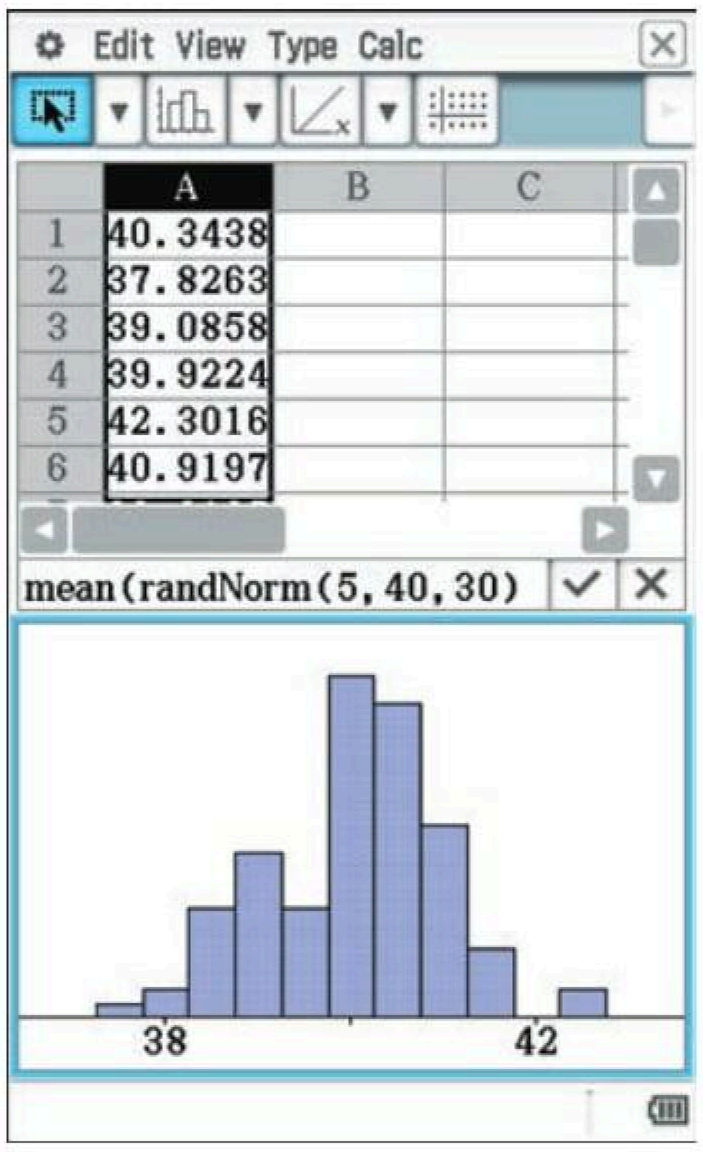


	A	B	C
1	40.3438		
2	37.8263		
3	39.0858		
4	39.9224		
5	42.3016		
6	40.9197		
7	39.8195		

5 100 sample means will be displayed in column **A**.

6 Tap **A** to highlight the whole column.

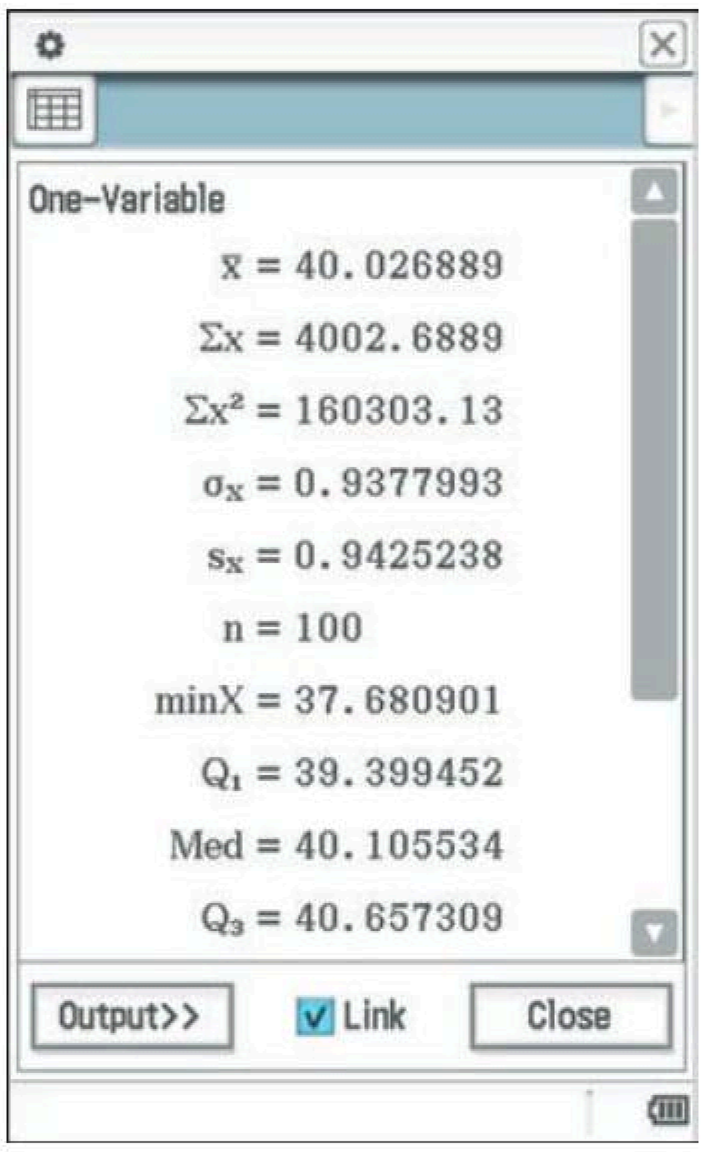
7 Tap **Graph > Histogram**.



	A	B	C
1	40.3438		
2	37.8263		
3	39.0858		
4	39.9224		
5	42.3016		
6	40.9197		

```
mean(randNorm(5, 40, 30)
```

8 A histogram of the sample means will be displayed in the lower window.



```
One-Variable
          x̄ = 40.026889
         Σx = 4002.6889
        Σx² = 160303.13
         σx = 0.9377993
         sx = 0.9425238
          n = 100
       minX = 37.680901
         Q₁ = 39.399452
        Med = 40.105534
         Q₃ = 40.657309
```

9 Tap to highlight the upper window.

10 Tap **Calc > One-Variable**.

11 The one variable statistics for the sample means will be displayed in the lower window.

12 Tap **Resize** to display them in a full screen.

The mean of the sampling distribution of sample means

The sampling distribution of the sample means contains the means ($\overline{x}$) of all possible samples of size n from a population.

The mean of the sample means ($\mu_{\overline{x}}$) is equal to the mean of the population (μ).

$$\mu_{\overline{x}} = \mu$$

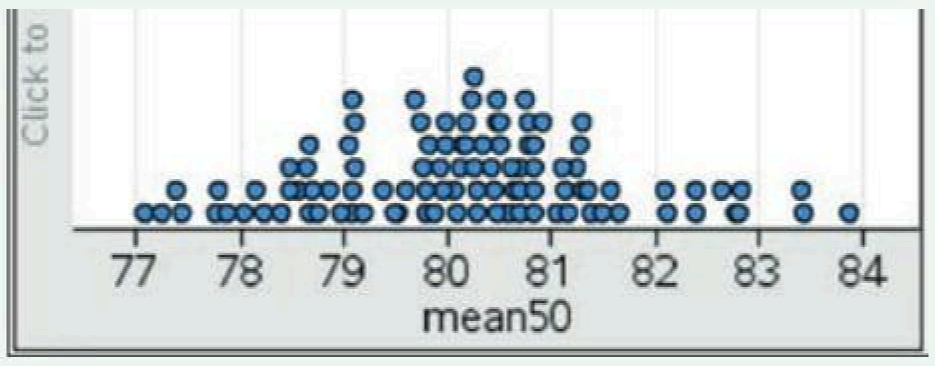
There is some variability in the sample means and the standard deviation is a measure of the variability.

The standard deviation of the sample means is the **standard error** of the sample mean.

WORKED EXAMPLE 9 Finding probabilities from a simulated sample

WB p. 239

A random sample is taken from a population that is normally distributed with a mean of 80 and a standard deviation of 10. Using simulation, 100 of these random samples is generated and the results are shown in the dotplot.
Find the probability that a sample contains a mean greater than 82.



Steps	**Working**
Count the number of sample means greater than 82. The probability is a fraction of the total number of samples.	10 of the samples contain sample means greater than 82. Probability of a mean greater than 82 $= \frac{10}{100} = \frac{1}{10}$

WORKED EXAMPLE 10 Finding the mean of the sample means using simulation

WB p. 240

The amount of time spent in the supermarket shopping for groceries is normally distributed with a mean of 30 minutes and a standard deviation of 6 minutes.

a Simulate 50 samples of size 40 and calculate the sample means for each sample. Display the results as a dotplot or histogram.

b Find the average of the sample means for the samples of size 40.

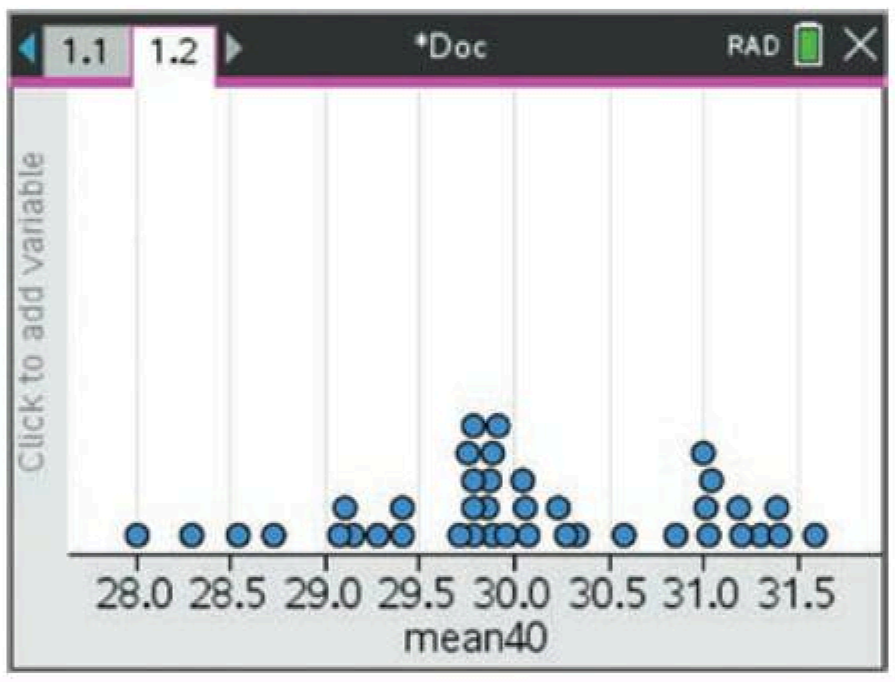
Steps **Working**

a Generate the means of the samples with CAS using the randNorm command.

TI-Nspire

Cell formula = mean(randnorm(30,6,40)).

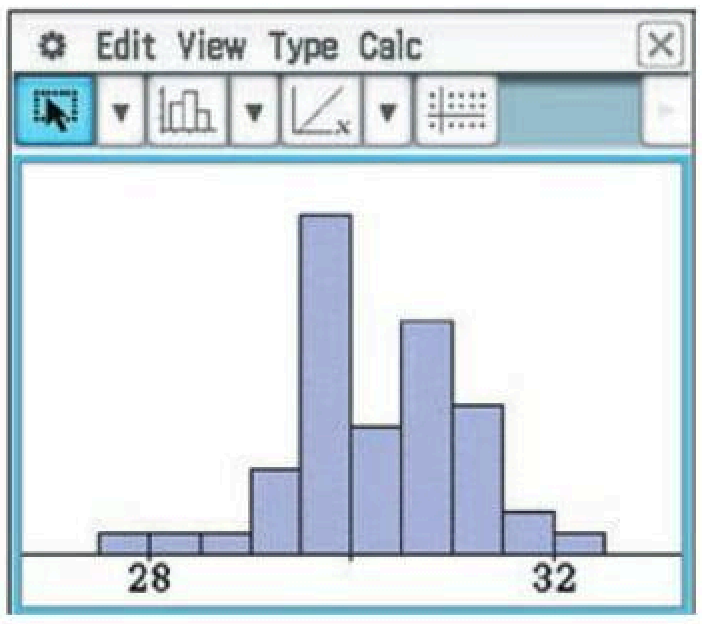
Copy for 50 cells.



ClassPad

Cell formula = mean(randNorm(6, 30,40)).

Copy for 50 cells.



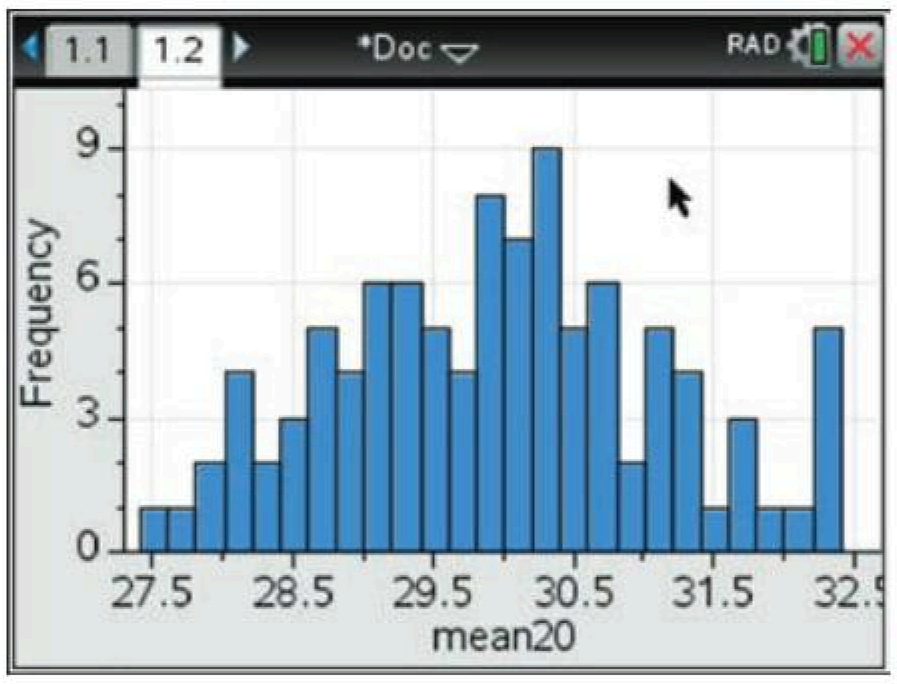
b Calculate one-variable statistics for the data in column A. Answers may vary slightly.	mean of $\overline{x}$ = 29.95 mean of $\overline{x}$ = 29.85

Larger samples and the sampling distribution of the sample means

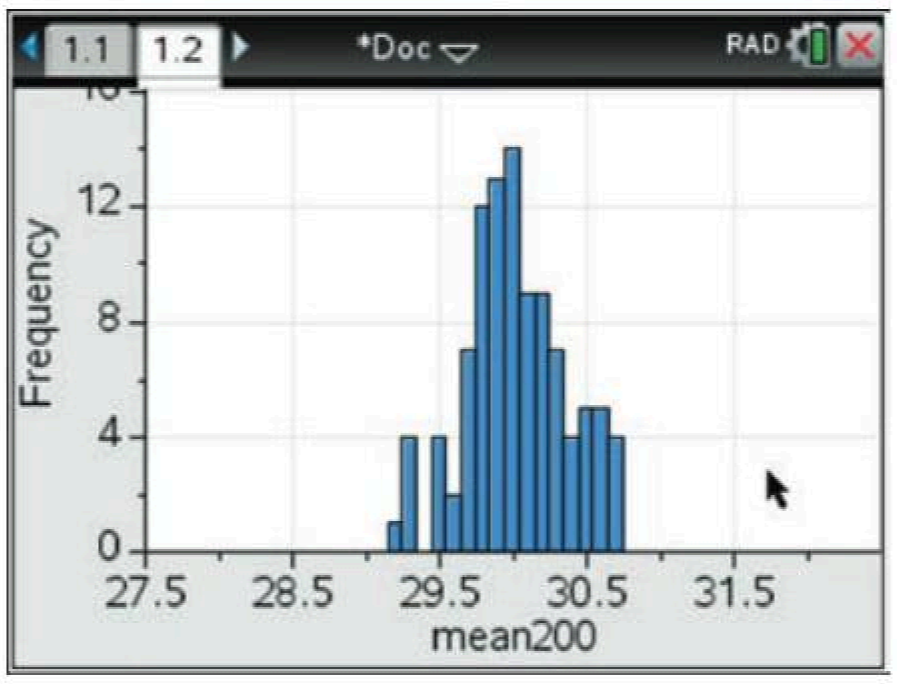
When a random sample is taken from a population, the reliability of the sample **statistics** is influenced by the sample size. This effect can be seen when the mean is calculated with samples of different sizes.

For example, in the situation below, samples are taken from a normally distributed population with a mean of 30 and a standard deviation of 5.

Sample means with a sample size of 20



Sample means with a sample size of 200



Sample size	Mean of $\bar{x}$	Standard deviation of $\bar{x}$
20	29.96	1.174
200	30.01	0.346

Large sample sizes

The effect of a larger sample size is

- the sample mean is closer to the population mean
- there is less variability in the means.

A larger sample size produces less error and a more reliable estimate of the population mean.

WB
p. 241

WORKED EXAMPLE 11 **The mean and standard deviation of sample means for different sample sizes**

Random samples of size n are taken from a population that is normally distributed with a mean of 160 and a standard deviation of 20. Complete 100 simulations for sample sizes of 30 and 150 and determine the mean and standard deviation of the sample means.

Steps **Working**

1 Generate the means of the samples by CAS using the randNorm command.

Use $n = 30$ and $n = 150$ in the formulas.

TI-Nspire

Cell formula = mean(randnorm(160,20,n)).

Copy for 100 cells.

ClassPad

Cell formula = mean(randNorm(20,160,n)).

Copy for 100 cells.

2 Answers may vary slightly.

Sample size	Mean of $\bar{x}$	Standard deviation of $\bar{x}$
30	159.50	3.385
150	159.77	1.621

The standard error for the sample means

The standard deviation of the sample means is also called the **standard error** of a sample mean. This standard error of $\bar{x}$ is written as $\mathrm{SE}(\bar{x})$ or $\sigma_{\bar{x}}$. The standard error indicates how much the sample mean $\bar{x}$, for a particular sample size, deviates from the population mean μ.

The standard error formula

The standard error for a sample mean is given by

$$\mathrm{SE}(\bar{x}) = \frac{\sigma}{\sqrt{n}}$$

where σ is the population standard deviation and n is the sample size.

We are unlikely to know the population standard deviation, so the sample standard deviation is used as an approximation of the population standard deviation.

WB p. 242

WORKED EXAMPLE 12 The standard error of the sample mean

A random sample of 200 pears is taken at a fruit cannery. The weight of the pears is normally distributed with a mean weight of 70 g and a standard deviation of 8 g. Find the standard error of the sample mean.

Steps	Working
Use the formula $\mathrm{SE}(\bar{x}) = \frac{\sigma}{\sqrt{n}}$	$\mathrm{SE}(\bar{x}) = \frac{8}{\sqrt{200}} \approx 0.566$

EXERCISE 12.3 Sample means and simulations

ANSWERS p. 600

Recap

Use the following information to answer Questions 1 and 2.

X and Y are independent normally distributed random variables where X has mean 75 and standard deviation 10 and Y has mean 50 and standard deviation 5.

1 The mean and standard deviation of $X + 4Y$ is

A 275 and $20\sqrt{5}$
B 275 and $10\sqrt{5}$
C 125 and 15
D 275 and 30
E 275 and $\sqrt{30}$

2 The value of $\Pr(X + 4Y > 290)$, correct to three decimal places, is

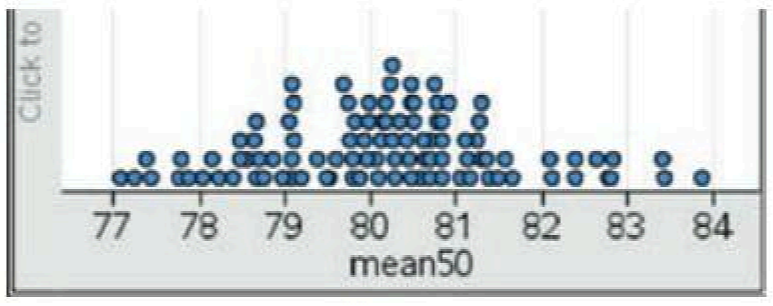
A 0.003
B 0.012
C 0.045
D 0.089
E 0.251

Mastery

3 Using CAS 3 Generate 50 samples of size 25 from a population that is normally distributed with a mean of 145 and a standard deviation of 12.

Calculate the mean of each sample and plot the distribution of the sample means.

4 WORKED EXAMPLE 9 A random sample is taken from a population that is normally distributed with a mean of 80 and a standard deviation of 10. Using simulation, 100 of these random samples is generated and the results are shown in the dotplot.



Find the probability that a sample contains a mean less than 78.

5 WORKED EXAMPLE 10 The amount of time a person spends standing, in a normal working day, is normally distributed with a mean of 400 minutes and a standard deviation of 30 minutes.

a Simulate 50 samples of size 80 and calculate the sample means for each sample. Display the results as a dotplot or histogram.

b Find the average of the sample means for samples of size 80.

6 WORKED EXAMPLE 11 Random samples of size n are taken from a population that is normally distributed with a mean of 250 and a standard deviation of 50. Complete 100 simulations for sample sizes of 20 and 200 and determine the mean and standard deviation of the sample means.

Copy and complete the table below.

Sample size	Mean of $\bar{x}$	Standard deviation of $\bar{x}$
20		
200		

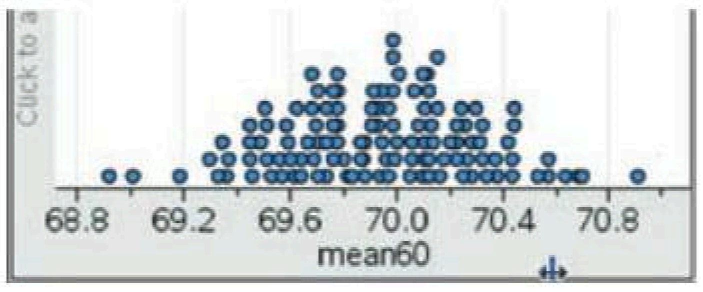
7 WORKED EXAMPLE 12 A random sample of 100 eggs is taken at a free-range farm. The weight of the eggs is normally distributed with a mean weight of 12 g and a standard deviation of 1.5 g. Find the standard error in the sample mean.

Exam practice

80–100% 60–79% 0–59%

Use the following information to answer Questions 8 and 9.

The weight of hatched chicks is normally distributed with a mean weight of 70 g and a standard deviation of 3 g. A simulation of 120 samples of size 60 is completed and the sample means are shown.



8 The probability of a sample having a sample mean less than 69.2 is

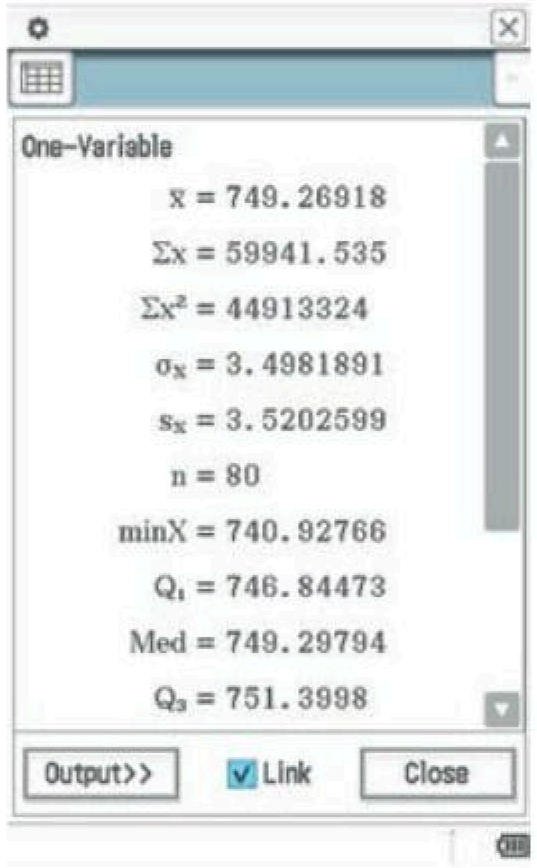
A $\frac{1}{40}$ **B** $\frac{3}{100}$ **C** $\frac{3}{70}$ **D** $\frac{1}{20}$ **E** $\frac{1}{3}$

9 The probability of a sample having a sample mean greater than x is $\frac{1}{10}$. The value of x is

A 70 **B** 70.2 **C** 70.4 **D** 70.6 **E** 70.8

Use the following information to answer Questions 10, 11 and 12.

Random samples of size 50 are taken from a population that is normally distributed with a mean of a and a standard deviation of b, and the mean of the samples is calculated. This process is repeated by simulation x times and the one-variable statistics for the sample means is shown.



```
One-Variable
    x̄ = 749.26918
   Σx = 59941.535
  Σx² = 44913324
   σx = 3.4981891
   sx = 3.5202599
    n = 80
 minX = 740.92766
   Q₁ = 746.84473
  Med = 749.29794
   Q₃ = 751.3998
```

10 The number of simulations is

A 50 **B** 80 **C** 100

D 150 **E** 749

11 The value of the population mean, a, is

A 3.5 **B** 50 **C** 80

D 741 **E** 750

12 The standard deviation of the population is

A 3.5 **B** 3.52 **C** 20

D 25 **E** 31.5

13 A random sample of size 64 is taken from a normally distributed population with a mean of 40 and a standard deviation of 8. The standard error for the sample means is

A 1 **B** 2 **C** 4 **D** 5 **E** 8

14 (2 marks) A random sample of 50 items is taken from a normally distributed population.

The sample mean is 29.95 and the standard deviation of the sample means of size 50 is 0.566.

Find, to the nearest integer,

a the population mean. 1 mark

b the population standard deviation. 1 mark

15 (2 marks) The birth weight of babies is normally distributed with a mean of μ and a standard deviation of σ. A random sample of 100 babies is taken and the birth weights are recorded. If the sample mean is 3.49 kg and the standard deviation of the sample means of size 100 is 0.05 kg, find, correct to one decimal place,

a the population mean. 1 mark

b the population standard deviation. 1 mark

16 (3 marks) The volume of soft drink in a can is normally distributed with a mean of 345 ml and a standard deviation of 2 ml.

a Simulate 50 sample means by taking random samples of size 60. Present the summary as a dotplot or histogram. 1 mark

b Find the mean of the sample means and the standard deviation of the sample means. 2 marks

12.4 Confidence intervals for the population mean

Video playlist Confidence intervals for the population mean

A **parameter** is a characteristic value of a particular population, such as a mean. A statistic is an estimate of a population parameter found from a sample. In most cases, we cannot be absolutely certain of the value of a parameter because we cannot obtain values from a whole population, so we must use a statistic instead.

Point estimates and interval estimates

A sample statistic is used to estimate a population parameter.

A **point estimate of a parameter** is a single value obtained from a sample. The statistic used is called an **estimator** of the parameter.

An **interval estimate of a parameter** is an interval that is likely to include the value of the parameter.

Suppose we need to find the amount of time Year 12 students spend on social media per day and we find the mean daily social media use from a sample of 50 Year 12 students. This sample mean is a point estimate of the population mean. An interval estimate for the daily social media use of Year 12 students is an interval that is likely to contain the population mean.

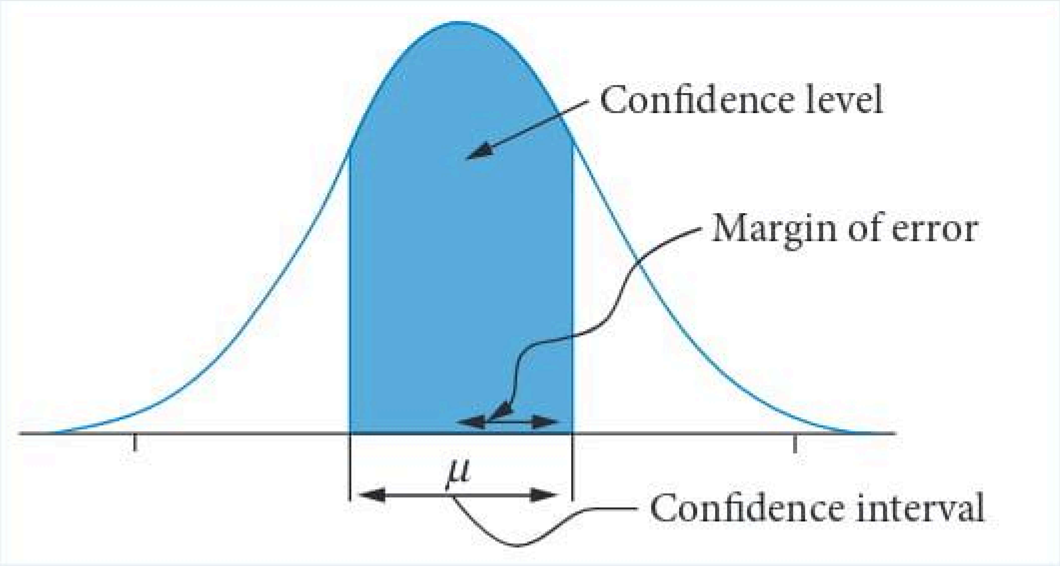
The observed interval will vary from sample to sample. The frequency with which the interval contains the population parameter is given by the **confidence level**.

For example, if we take 100 samples of size n from a population to estimate the population mean μ and calculate 95% **confidence intervals** for the sample mean for each sample, then about 95 of these intervals will contain the population mean.

Confidence intervals and the margin of error

For a confidence interval symmetric about the mean in a distribution:

- the confidence level is equivalent to the probability that the population parameter being estimated lies within the confidence interval.
- the **margin of error** is the distance of the ends of the interval from the mean. It is half of the confidence interval.



Confidence intervals from a simulation

In the confidence interval (a, b), a is the lower quantile and b is the upper quantile.

The confidence level of a confidence interval is equal to the proportion of values in the distribution that lie between the upper and lower quantiles.

If a confidence interval of (a, b) has a confidence level of 95%, then 95% of the values in the distribution will be included in this interval.

Confidence intervals and confidence levels

For a confidence interval (a, b) with a confidence level of c:

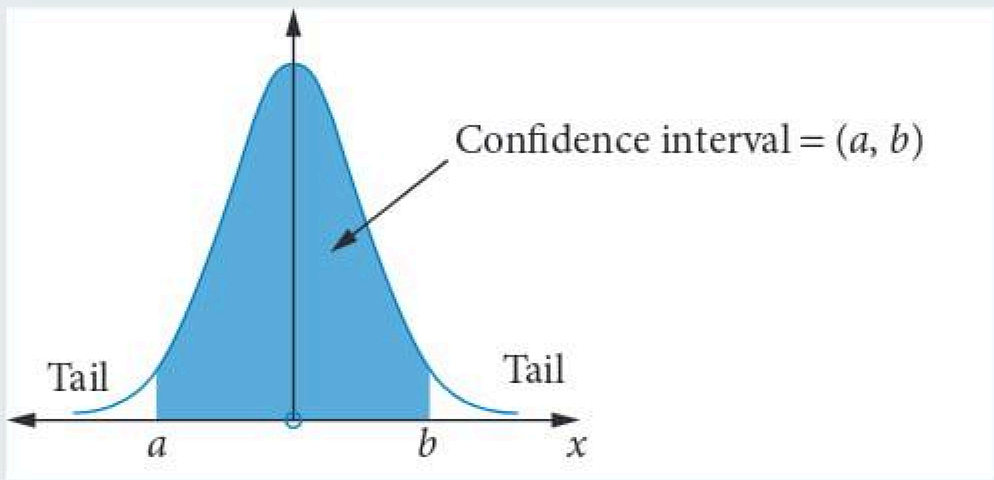
Confidence interval:

$$\Pr(a < X < b) = c$$

Tails:

The tails represent the values in the distribution that are less than a or greater than b.

$$\Pr(X < a) = \Pr(X > b) = \frac{1-c}{2}$$



Confidence intervals of a normally distributed variable

Worksheet Normal distributions

The inverse cumulative normal distribution can be used to determine the confidence interval at any confidence level for a normally distributed variable.

WB p. 243

WORKED EXAMPLE 13 Finding a confidence interval for a normally distributed variable

The time, in minutes, that Year 12 students spend on social media is a normally distributed random variable with a mean of 80 min and a standard deviation of 9 min.

Find a 95% confidence interval, correct to one decimal place, for this variable.

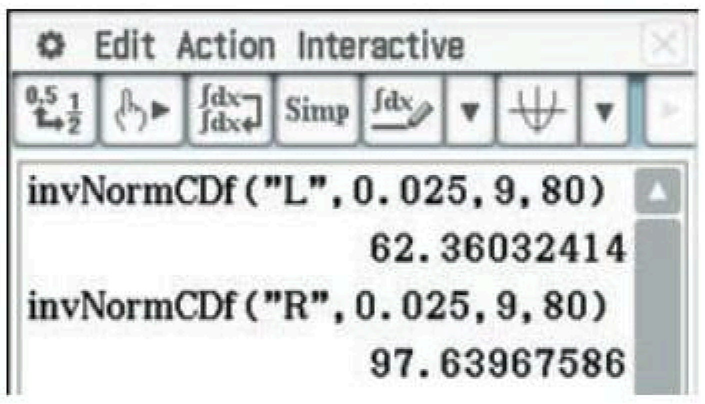
Steps	Working
1 Determine the percentage of values in each tail.	The confidence interval contains 95% of the sample. The proportion in each tail $= \frac{1-0.95}{2} = 0.025$
2 Use the inverse cumulative normal distribution to determine the quantiles of the confidence interval.	$\Pr(X < \text{lower quantile}) = 0.025$ $\Pr(X < \text{upper quantile}) = 1 - 0.025$ $= 0.975$

3 Use CAS to determine the quantiles of the confidence interval.

TI-Nspire

```
invNorm(0.025,80,9)        62.3603
invNorm(0.975,80,9)        97.6397
```

ClassPad



```
invNormCDf("L",0.025,9,80)
                 62.36032414
invNormCDf("R",0.025,9,80)
                 97.63967586
```

95% confidence interval = (62.4, 97.6)

Confidence intervals for the population mean

The Tasmanian farmers fruit growers' association needs to determine the mean weight of golden delicious apples grown in their state in this year's harvest. They take random samples of 100 apples from randomly selected orchards and use this sample to estimate the mean weight of every golden delicious apple in this year's Tasmanian harvest.

This one sample will have a mean of $\overline{x}$ and a standard deviation of s.

The sampling distribution of the sample means of size 100 is the set of every possible sample mean and is represented by $\overline{X} = \{\overline{x}_1, \overline{x}_2, \overline{x}_3 \ldots \overline{x}_k\}$. In this set, $\overline{x}_1$ is the mean of sample 1, $\overline{x}_2$ is the mean of sample 2 and so on. The standard deviation of the set of sample means of size 100, written as $\sigma_{\overline{X}}$, is a measure of the spread of the sample means about the mean. The properties of this sampling distribution enable predictions to be made about the population mean μ.

The **central limit theorem** states that the distribution of the sample means taken from any kind of distribution is approximately normally distributed. The larger the sample size, n, the closer the approximation to the normal distribution. $n = 30$ is generally considered large enough to assume the sampling distribution is normal.

The mean and standard error of the sample means

If X is any random variable with mean μ and standard deviation σ, then the sampling distribution of sample means $\overline{X}$, of size n, will be approximately normal, provided n is sufficiently large.

The mean of the sample means $E(\overline{X}) = \mu$.

The standard deviation or standard error is $SE\left(\overline{X}\right) = \sigma_{\overline{X}} = \dfrac{\sigma}{\sqrt{n}}$.

Exam hack

If the population X is normally distributed, then the sampling distribution of the sample means, *of any size* n, is also normal.

WB

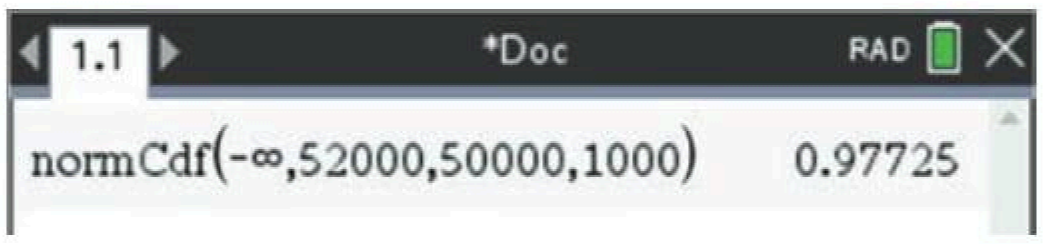
p. 244

WORKED EXAMPLE 14 Finding the probability for the sampling distribution of the sample means

The life of LED light globes is normally distributed with a mean of 50 000 hours and a standard deviation of 5000 hours. What is the probability that the mean life of a sample of 25 LED light globes will be less than 52 000 hours (correct to three decimal places)?

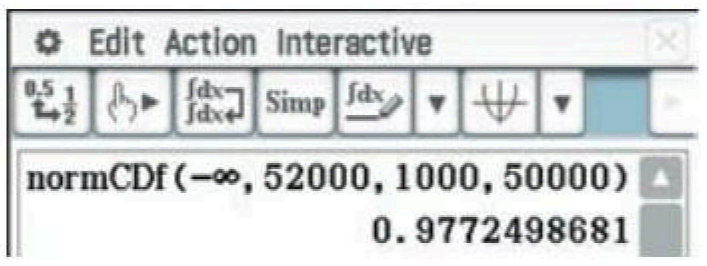
Steps	**Working**
1 Estimate standard deviation of the sampling distribution.	$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{5000}{\sqrt{25}} = \frac{5000}{5} = 1000$
2 Since the population is normal, the sampling distribution of the sample means is also normal.	The sampling distribution is approximately normal.
3 Find the probability using CAS.	

TI-Nspire



normCdf(-∞,52000,50000,1000) 0.97725

ClassPad



normCDf(−∞, 52000, 1000, 50000)
0.9772498681

$\Pr\left(\bar{X} < 52000\right) = 0.977$

WB

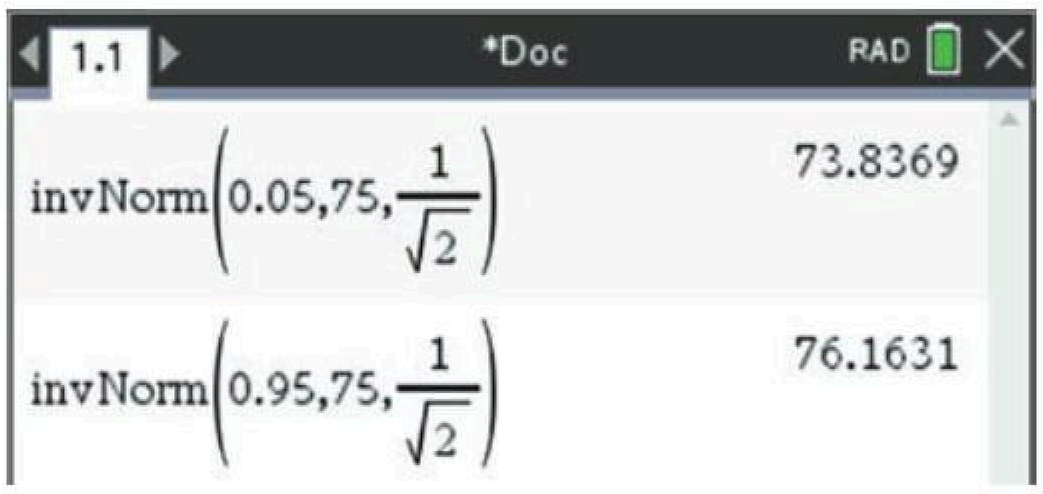
p. 245

WORKED EXAMPLE 15 Finding a confidence interval for the population mean

A random sample of size 50 is taken from a population that is normally distributed with a standard deviation of 5. If the mean of the sample is 75, find a 90% confidence interval for the population mean (answer correct to 2 decimal places).

Steps	**Working**
1 Estimate the standard deviation of the sampling distribution.	$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{5}{\sqrt{50}} = \frac{1}{\sqrt{2}}$
2 Since the population is normal, the sampling distribution of the sample means is also normal.	The sampling distribution is approximately normal.
3 Determine the percentage of values in each tail.	The confidence interval contains 90% of the sample. The proportion in each tail $= \frac{1 - 0.90}{2} = 0.05$.
4 Use the inverse cumulative normal distribution to determine the quantiles of the confidence interval.	$\Pr(X < \text{lower quantile}) = 0.05$ $\Pr(X < \text{upper quantile}) = 1 - 0.05$ $= 0.95$
5 Use CAS to determine the quantiles of the confidence interval. Enter the sample mean as the mean of the sampling distribution.	

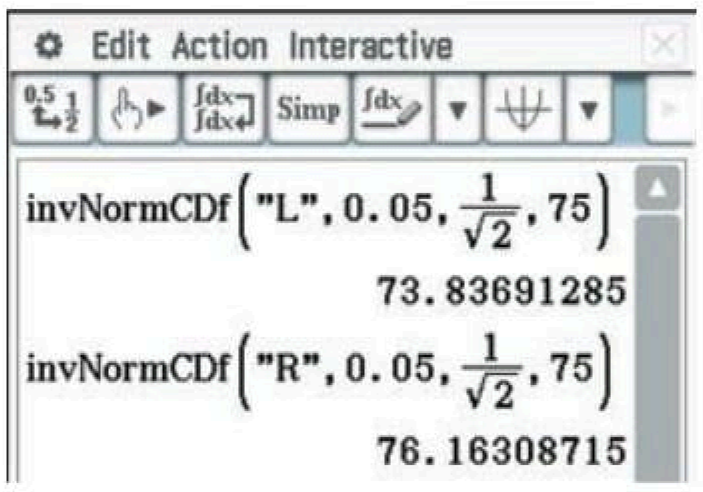
TI-Nspire



$\text{invNorm}\left(0.05,75,\frac{1}{\sqrt{2}}\right)$ 73.8369

$\text{invNorm}\left(0.95,75,\frac{1}{\sqrt{2}}\right)$ 76.1631

ClassPad



$\text{invNormCDf}\left(\text{"L"}, 0.05, \frac{1}{\sqrt{2}}, 75\right)$
73.83691285

$\text{invNormCDf}\left(\text{"R"}, 0.05, \frac{1}{\sqrt{2}}, 75\right)$
76.16308715

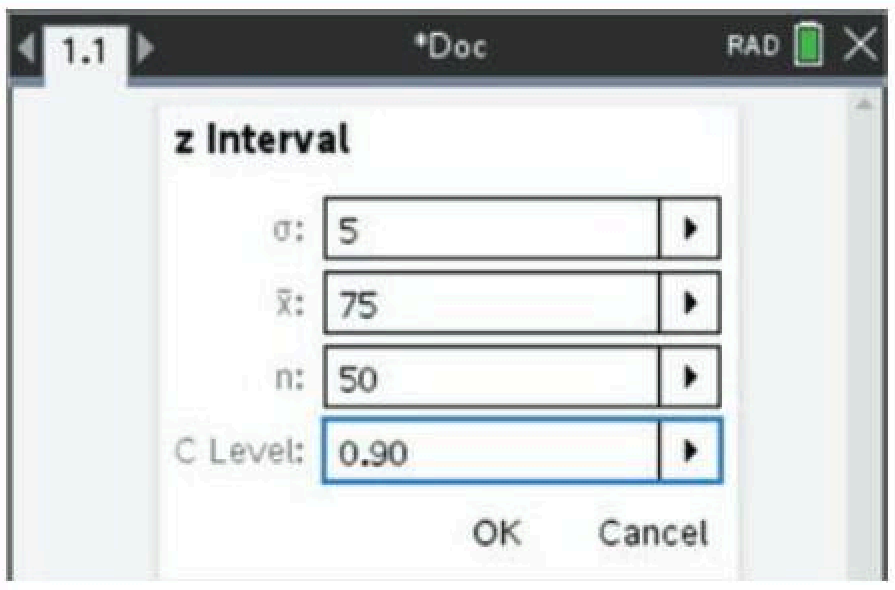
90% confidence interval = (73.84, 76.16)

CAS has a built-in feature that can be used to find the confidence interval for the population mean from a sample of a given size such as the previous example.

USING CAS 4 The confidence interval for the population mean

From Worked example **15**, a random sample of size 50 is taken from a population that is normally distributed with a mean of 75 and a standard deviation of 5. Confirm the 90% confidence interval for the population mean.

TI-Nspire



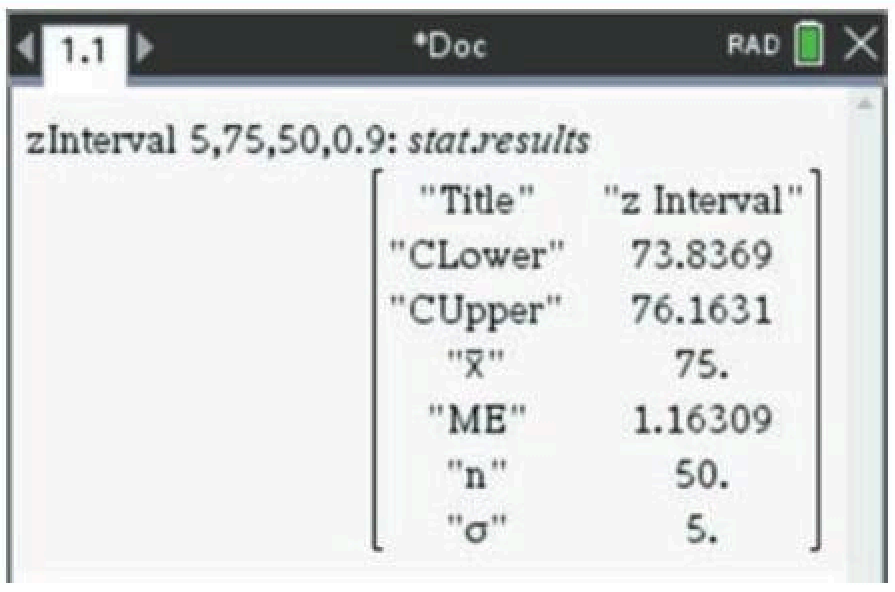
z Interval

σ: 5

x̄: 75

n: 50

C Level: 0.90



```
zInterval 5,75,50,0.9: stat.results
   "Title"    "z Interval"
   "CLower"    73.8369
   "CUpper"    76.1631
     "x̄"        75.
    "ME"       1.16309
     "n"        50.
     "σ"        5.
```

1 Press **menu > Statistics > Confidence Intervals > z Interval**.

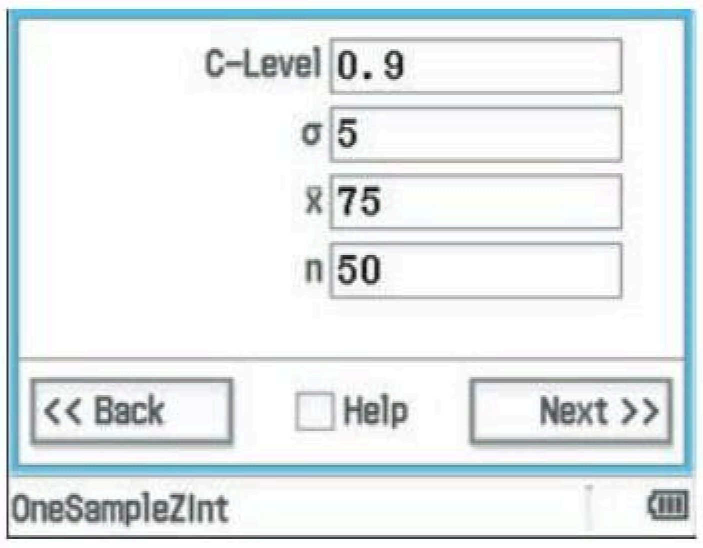
2 On the next screen, change the **Data Input Method:** setting from **Data** to **Stats**.

3 In the dialogue box, enter the values as shown above.

4 The **z Interval** statistics will be displayed.

5 The **CLower** and **CUpper** values specify the confidence interval, which is (73.84, 76.16).

ClassPad

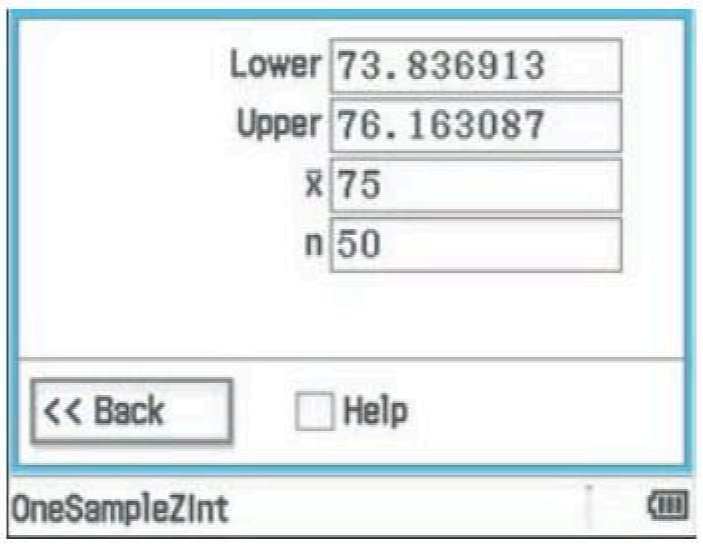


C-Level 0.9

σ 5

x̄ 75

n 50



Lower 73.836913

Upper 76.163087

x̄ 75

n 50

1 Tap to open the **Statistics** application.

2 Tap **Calc > Interval**.

3 On the next screen, tap **Variable** then tap **Next**.

4 In the dialogue box displayed in the lower screen, enter the values as shown above, then tap **Next**.

5 The **Z Interval** statistics will be displayed.

6 The **Lower** and **Upper** values designate the confidence interval, which is (73.84, 76.16).

The confidence interval is (73.84, 76.16).

Confidence intervals for the population mean and the margin of error

The confidence interval for the population mean is $\left(\bar{x} - z\frac{\sigma}{\sqrt{n}}, \bar{x} + z\frac{\sigma}{\sqrt{n}}\right)$ when the population standard deviation σ is known.

When σ is not known, the confidence interval is $\left(\bar{x} - z\frac{s}{\sqrt{n}}, \bar{x} + z\frac{s}{\sqrt{n}}\right)$,

where s is the sample standard deviation, $\bar{x}$ is the sample mean and z is the appropriate quantile for the standard normal distribution.

The margin of error $E = z\frac{s}{\sqrt{n}}$.

Worksheets
Confidence intervals

Polls and levels of confidence

z-values for common confidence levels

The z-values vary for different confidence levels and can be calculated from an inverse cumulative standard normal distribution $Z \sim N(0, 1)$.

The most common confidence levels are 90%, 95% and 99%.

Confidence level	Proportion in each tail	z
99%	0.005	2.58
95%	0.025	1.96
90%	0.05	1.64

95% confidence intervals

The 95% confidence interval for the population mean is

$$\left(\bar{x} - 1.96\frac{s}{\sqrt{n}}, \bar{x} + 1.96\frac{s}{\sqrt{n}}\right)$$

WB p. 246

WORKED EXAMPLE 16 Confidence interval formula

At a strawberry farm, the mass of a random sample of 100 strawberries is recorded. The sample has a mean of 5.2 grams and a standard deviation of 1 grams. Find a 95% confidence interval for the mean mass of strawberries at the farm (correct to three decimal places).

Steps	**Working**
1 Estimate the standard error for the sampling distribution using the sample standard deviation, s, as an estimate of the population standard deviation σ.	Use $s = 1$ to find the standard error of the sample means. $s_{\bar{X}} = \frac{s}{\sqrt{n}} = \frac{1}{\sqrt{100}} = \frac{1}{10}$
2 Since $n \geq 30$, we can use the normal distribution.	The sampling distribution is approximately normal as $n = 100 \geq 30$.
3 Determine the z-value used for a 95% confidence interval.	Use $z = 1.96$ for a 95% confidence interval.
4 Substitute into the formula for the confidence interval.	$\left(\bar{x} - 1.96\frac{s}{\sqrt{n}}, \bar{x} + 1.96\frac{s}{\sqrt{n}}\right)$ $= \left(5.2 - 1.96 \times \frac{1}{10}, 5.2 + 1.96 \times \frac{1}{10}\right)$ $= (5.004, 5.396)$ The 95% confidence interval for the mean of the mass of strawberries at the farm is (5.004, 5.396).

WB p. 247

WORKED EXAMPLE 17 Finding the sample size *n*, for a given margin of error

A population has a standard deviation of 20. Determine the smallest sample size, n, needed so that the sample mean is within 5 of the population mean at a 95% confidence level.

Steps	**Working**
Use the margin of error formula to calculate n.	margin of error = 5 Use the population standard deviation 20 for s and $z = 1.96$ for a 95% confidence level. $M = z\frac{s}{\sqrt{n}} \leq 5$ $1.96 \times \frac{20}{\sqrt{n}} \leq 5$ $\sqrt{n} \geq \frac{1.96 \times 20}{5} = 7.84$ $n \geq 7.84^2 = 61.4656$ $n = 62$

WORKED EXAMPLE 18 Finding the sample mean and size, for a given confidence interval

The standard deviation of the weights of apples picked on an orchard is 6.2 g. From the results of a sample of *n* apples, a 95% confidence interval for the mean weight of apples was calculated to be (84.48, 87.52). Calculate the mean score and the size of this random sample.

Steps	**Working**
1 Calculate the sample mean by finding the median of the confidence interval.	$\bar{x} = \dfrac{84.48 + 87.52}{2} = 86$
2 Calculate the margin of error and substitute into the formula $M = z\dfrac{\sigma}{\sqrt{n}}$ using $z = 1.96$ and $\sigma = 6.2$.	$M = 87.52 - 86 = 1.52$ $1.52 = 1.96 \times \dfrac{6.2}{\sqrt{n}}$
3 Solve the equation for *n* using CAS.	$n = 64$ The sample mean of the sample of 64 apples is 86 g.

WB
p. 248

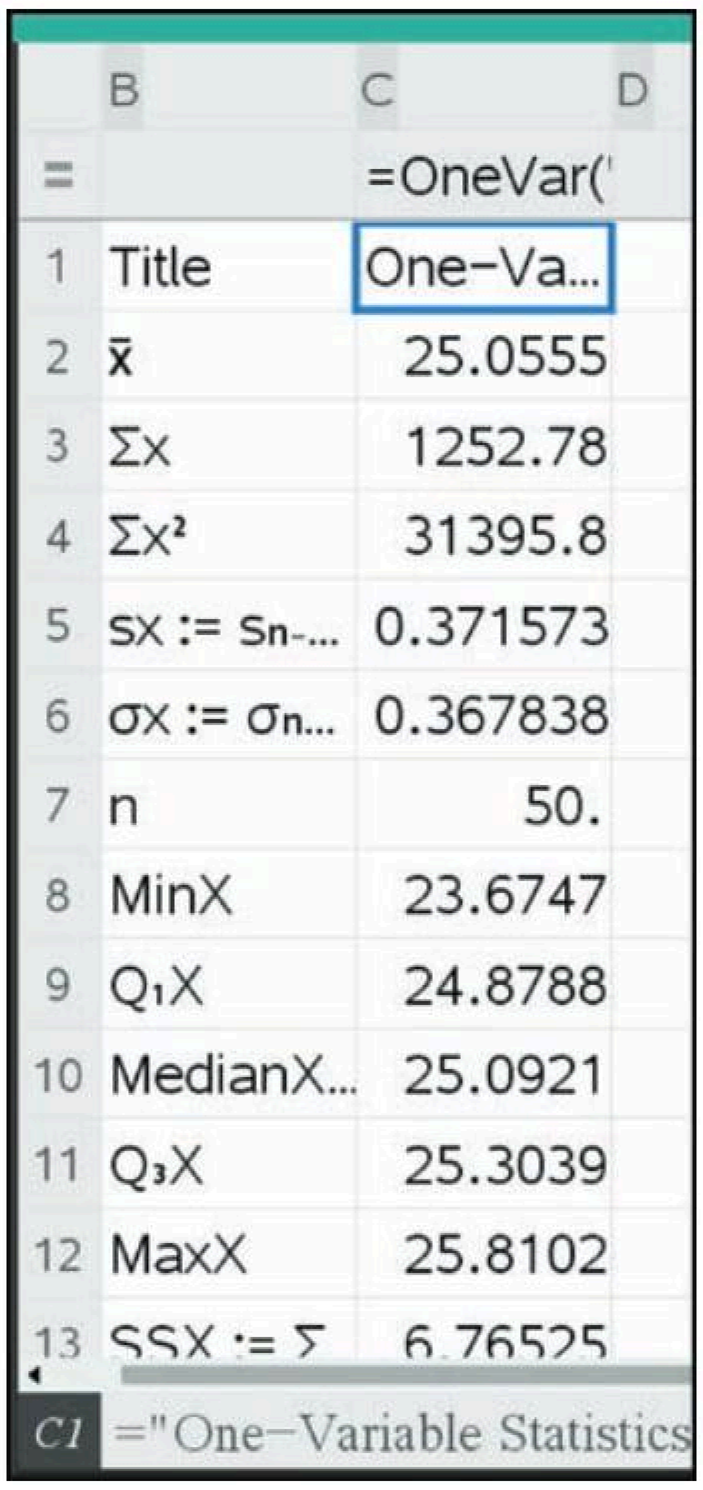
EXERCISE 12.4 Confidence intervals for the population mean

ANSWERS p. 601

Recap

The following information refers to Questions 1 and 2.

Random samples of size 100 are taken from a population that is normally distributed with a mean of *a* and a standard deviation of *b*, and the mean of the samples is calculated. This process is repeated by simulation *x* times and the one-variable statistics for the sample means is shown.



	B	C	D
=		=OneVar('	
1	Title	One-Va...	
2	$\bar{x}$	25.0555	
3	Σx	1252.78	
4	Σx²	31395.8	
5	sx := sn-...	0.371573	
6	σx := σn...	0.367838	
7	n	50.	
8	MinX	23.6747	
9	Q₁X	24.8788	
10	MedianX...	25.0921	
11	Q₃X	25.3039	
12	MaxX	25.8102	
13	SSX := Σ	6.76525	

C1 ="One−Variable Statistics

1 The value of the population mean, *a*, is closest to

A 25
B 25.1
C 50
D 100
E 1253

2 The standard deviation of the population is closest to

A 0.37
B 0.38
C 3.7
D 10
E 25

Mastery

3 WORKED EXAMPLE 13 The time, in minutes, that customers spend at the art gallery is found to be a normally distributed random variable with a mean of 60 minutes and a standard deviation of 15 minutes.

a Find a 95% confidence interval, correct to one decimal place, for this variable.

b Find the 99% confidence interval for the normal variable *X* with mean 55 and standard deviation 18.

4 WORKED EXAMPLE 14

a The resting pulse rates for adults is normally distributed with mean 70 bpm and standard deviation 10 bpm. Find the probability, correct to four decimal places, that a sample of 36 adults will have a resting pulse rate greater than 68 bpm.

b The number of hours of battery life for a particular type of laptop is found to be normally distributed with mean 650 minutes and standard deviation 45 minutes. Find the probability, correct to four decimal places, that a sample of 81 laptops will have a battery life less than 660 min.

5 WORKED EXAMPLE 15

a A random sample of size 100 is taken from a population that is normally distributed with a standard deviation of 16. If the mean of the sample is 45, find a 90% confidence interval for the population mean (answer correct to 2 decimal places).

b The mean of a sample of size 50 is 28. Determine a 95% confidence interval for the population mean if the standard deviation of the population is 9.

6 Using CAS 4 A random sample of size 1200 is taken from a population that is normally distributed with a standard deviation of 8.2. If the mean of the sample is 33.4, find a 90% confidence interval for the population mean (answer correct to three decimal places).

7 WORKED EXAMPLE 16 At an apple orchard, the weight of a random sample of 50 apples is recorded. The sample has a mean of 85 g and a standard deviation of 6 g. Find a 95% confidence interval for the mean weight of apples at the farm (answer correct to one decimal place).

8 WORKED EXAMPLE 17

a A population has a standard deviation of 30. Determine the smallest sample size, n, needed so that the sample mean is within 7 of the population mean at a 95% confidence level.

b A population has a standard deviation of 20. Determine the smallest sample size, n, needed so that the sample mean is within 4 of the population mean at a 90% confidence level.

9 WORKED EXAMPLE 18 A machine fills cups with soft drink. The machine cannot fill each cup with exactly the same amount and the standard deviation of the amount of liquid dispensed each time is 2.5 mL. From the results of a sample of n cups of soft drink, a 95% confidence interval for the mean volume of soft drink dispensed was calculated to be (249.3, 250.7). Calculate the mean volume and the size of this random sample.

Exam practice

80–100% 60–79% 0–59%

10 © VCAA 2016 1Q2 44% TECH-FREE (3 marks) A farmer grows peaches, which are sold at a local market. The mass, in grams, of peaches produced on this farm is known to be normally distributed with a variance of 16. A bag of 25 peaches is found to have a total mass of 2625 g.

Based on this sample of 25 peaches, calculate an approximate 95% confidence interval for the mean mass of all peaches produced on this farm. Use an integer multiple of the standard deviation in your calculations.

11 © VCAA 2018N 1Q4 TECH-FREE (4 marks) Throughout this question, use an integer multiple of standard deviations in calculations. The standard deviation of all scores on a particular test is 21.0.

a From the results of a random sample of n students, a 95% confidence interval for the mean score for all students was calculated to be (44.7, 51.7). Calculate the mean score and the size of this random sample. 2 marks

b Determine the size of another random sample for which the endpoints of the 95% confidence interval for the population mean of the particular test would be 1.0 either side of the sample mean. 2 marks

12 © VCAA 2019N 1Q3 **TECH-FREE** (4 marks) The number of cars per day making a U-turn at a particular location is known to be normally distributed with a standard deviation of 17.5. In a sample of 25 randomly selected days, a total of 1450 cars were observed making the U-turn.

a Based on this sample, calculate an approximate 95% confidence interval for the number of cars making the U-turn each day. Use an integer multiple of the standard deviation in your calculations. 3 marks

b The average number of U-turns made at the location is actually 60 per day. Find an approximation, correct to two decimal places, for the probability that on 25 randomly selected days the average number of U-turns is less than 53. 1 mark

13 **TECH-FREE** (3 marks) A random sample of 100 solar lights produced by a company were tested to determine the number of hours of illumination after a full day of charging. The mean number of hours was 5.5 with a standard deviation of 0.5. Find a 95% confidence interval for the average number of hours of illumination for all the lights produced by the company. Use an integer multiple of the standard deviation in your calculations.

14 © VCAA 2016 2AQ19 78% A random sample of 100 bananas from a given area has a mean mass of 210 g and a standard deviation of 16 g.

Assuming the standard deviation obtained from the sample is a sufficiently accurate estimate of the population standard deviation, an approximate 95% confidence interval for the mean mass of bananas produced in this locality is given by

A (178.7, 241.3) **B** (206.9, 213.1) **C** (209.2, 210.8)
D (205.2, 214.8) **E** (194, 226)

15 © VCAA 2016 2AQ20 68% The lifetime of a certain brand of batteries is normally distributed with a mean lifetime of 20 hours and a standard deviation of 2 hours. A random sample of 25 batteries is selected.

The probability that the mean lifetime of this sample of 25 batteries exceeds 19.3 hours is

A 0.0401 **B** 0.1368 **C** 0.6103 **D** 0.8632 **E** 0.9599

16 © VCAA 2018 2AQ19 57% The gestation period of cats is normally distributed with mean $\mu = 66$ days and variance $\sigma^2 = \frac{16}{9}$. The probability that a sample of 5 cats chosen at random has an average gestation period greater than 65 days is closest to

A 0.5000 **B** 0.7131 **C** 0.7734 **D** 0.8958 **E** 0.9532

17 © VCAA 2017 2AQ19 44% A confidence interval is to be used to estimate the population mean μ based on a sample mean $\bar{x}$. To decrease the width of a confidence interval by 75%, the sample size must be multiplied by a factor of

A 2 **B** 4 **C** 9 **D** 16 **E** 25

18 © VCAA 2017N 2AQ20 The mass of suspended matter in the air in a particular locality is normally distributed with a mean of μ micrograms per cubic metre and a standard deviation of $\sigma = 8$ micrograms per cubic metre. The mean of 100 randomly selected air samples was found to be 40 micrograms per cubic metre.

Based on this, a 90% confidence interval for μ, correct to two decimal places, is

A (38.68, 41.32) **B** (26.84, 53.16) **C** (38.43, 41.57)
D (24.32, 55.68) **E** (37.93, 42.06)

19 © VCAA 2018N 2AQ18 The heights of all six-year-old children in a given population are normally distributed. The mean height of a random sample of 144 six-year-old children from this population is found to be 115 cm. If a 95% confidence interval for the mean height of all six-year-old children is calculated to be (113.8, 116.2) cm, the standard deviation used in this calculation is closest to

A 1.20 **B** 7.35 **C** 15.09 **D** 54.02 **E** 88.13

20 © VCAA 2019N 2AQ18 Consider a random variable X with probability density function

$$f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

If a large number of samples, each of size 100, is taken from this distribution, then the distribution of the sample means, $\overline{X}$, will be approximately normal with mean $\mathrm{E}\left(\overline{X}\right) = \frac{2}{3}$ and standard deviation $\mathrm{SD}\left(\overline{X}\right)$ equal to

A $\frac{\sqrt{2}}{60}$ **B** $\frac{\sqrt{2}}{6}$ **C** $\frac{1}{180}$ **D** $\frac{1}{18}$ **E** $\frac{\sqrt{2}}{30}$

21 © VCAA 2017N 2AQ19 The petrol consumption of a particular model of car is normally distributed with a mean of 12 L/100 km and a standard deviation of 2 L/100 km.

The probability that the average petrol consumption of 16 such cars exceeds 13 L/100 km is closest to

A 0.0104 **B** 0.0193 **C** 0.0228 **D** 0.3085 **E** 0.3648

22 © VCAA 2021 2AQ18 65% A scientist investigates the distribution of the masses of fish in a particular river. A 95% confidence interval for the mean mass of a fish, in grams, calculated from a random sample of 100 fish is (70.2, 75.8).

The sample mean divided by the population standard deviation is closest to

A 1.3 **B** 2.6 **C** 5.1 **D** 10.2 **E** 13.0

23 Samples of size 50 were taken from a normally distributed population with a mean of 55 and a standard deviation of 10. The standard deviation of the sampling distribution is

A $\frac{\sqrt{2}}{2}$ **B** 5 **C** $\sqrt{2}$ **D** $\sqrt{10}$ **E** $\frac{11\sqrt{2}}{2}$

24 In a fishing competition at a large lake, 60 trout are caught and weighed. The trout have a mean weight of 1.1 kg and a standard deviation of 0.2 kg. The 95% confidence interval for the mean weight of trout in the lake is

A (1.05, 1.15) **B** (0.9, 1.3) **C** (0.7, 1.5) **D** (1.06, 1.14) **E** (0.95, 1.25)

25 A population has a standard deviation of 32. Determine the sample size needed such that the 95% confidence level of the population mean is within 4 of the sample mean.

A 50 **B** 61 **C** 125 **D** 246 **E** 342

26 A random sample of 100 is taken from a population. The sample is normally distributed with a mean of 18 and a standard deviation of 3. The margin of error in a 95% confidence interval is

A 0.059 **B** 0.424 **C** 0.54 **D** 0.588 **E** 3.528

27 © VCAA 2018N 2BQ7b (1 mark) According to medical records, the blood pressure of the general population of males aged 35 to 45 years is normally distributed with a mean of 128 and a standard deviation of 14. Researchers suggested that male teachers had higher blood pressures than the general population of males.

▶ To investigate this, a random sample of 49 male teachers from this age group was obtained and found to have a mean blood pressure of 133.

Find a 90% confidence interval for the mean blood pressure of all male teachers aged 35 to 45 years using a standard deviation of 14. Give your answers correct to the nearest integer.

28 (2 marks) At a large hardware chain, the amount of money spent by customers on the weekend is a random variable with a standard deviation of \$45. The mean of a random selection of 50 customer receipts is \$125. Find the 90% confidence interval for the average amount of money spent by all the customers on the weekend. Express the quantiles to the nearest dollar.

29 (2 marks) A random sample of 49 Victorian Year 11 students were surveyed to find the number of hours of part-time work they completed each week during a school term. If the sample results are normally distributed with a mean of 5.5 hours and a standard deviation of 1.5 hours, determine the margin of error in a 90% confidence interval, correct to two decimal places.

30 (2 marks) The average growth of children between the ages of 2 years and 3 years is thought to be 9 cm with a standard deviation of 1 cm. How many results would need to be obtained to establish this with a margin of error of 2 cm at a 95% confidence level?

12.5 Hypothesis testing related to the mean

Video playlist Hypothesis testing related to the mean

Statistical inference is the process of drawing conclusions about a population based on the properties of a sample.

A test given to all Year 7 students in Victoria has a mean of 30 and a standard deviation of 6. Researchers claim that students who watch video tutorials achieve better than average scores. A random sample of 50 Year 7 students, who watched video tutorials, was tested and the mean score on the test is found to be 33.

In this section we will examine how we can test the researchers' claim.

The null and alternative hypothesis

Hypothesis testing is a systematic way to test a claim about a parameter in the population using data measured in a sample. There are two hypotheses.

Hypothesis testing

H_0: The null hypothesis

This is a statement about a population parameter (the population mean) that is assumed to be true.

The null hypothesis is the starting point. We test to see if the parameter stated in the null hypothesis is likely to be true.

H_1: The alternative hypothesis

This statement reflects the claim being made and is the opposite of the null hypothesis. The value of the population parameter is either less than, greater than or not equal to the stated value in the null hypothesis.

The null and alternative hypotheses must be mutually exclusive and exhaustive.

For the example of the Victorian Year 7 students who watch video tutorials:

H_0: $\mu = 30$

The null hypothesis states that the mean test result for Victorian Year 7 students who watch video tutorials will be 30 (that is 'no effect').

H_1: $\mu > 30$

The alternative hypothesis states the mean test result for Victorian Year 7 students who watch video tutorials will be greater than 30 (that is 'positive effect'). This hypothesis always relates to the claim being made.

WB

p. 249

WORKED EXAMPLE 19 The null and alternative hypothesis 1

The monthly weight gain of turkeys has a mean of 1.5 kg and a standard deviation of 0.2 kg. A poultry farm claims their turkeys gain more weight on their new high protein feed. Write the null and alternative hypotheses.

Steps	**Working**
1 The null hypothesis is a statement about the population that is assumed to be true.	null hypothesis H_0: $\mu = 1.5$
2 The alternative hypothesis is a statement about the population mean that contradicts the null hypothesis. The claim that the turkeys gain more weight indicates that the direction of the alternative hypothesis must be $\mu >$ the population mean.	alternative hypothesis H_1: $\mu > 1.5$

WB

p. 250

WORKED EXAMPLE 20 The null and alternative hypothesis 2

Excite bubble gum claims that their chewing gum holds its flavour for an average of 30 min. A rival bubble gum company disputes this claim. Write the null and alternative hypotheses.

Steps	**Working**
1 The null hypothesis is a statement about the population mean that is assumed to be true.	null hypothesis H_0: $\mu = 30$
2 The alternative hypothesis is a statement about the population mean that contradicts the null hypothesis. As the rival company disputes the claim, the alternative hypothesis is $\mu \neq$ the population mean.	alternative hypothesis H_1: $\mu \neq 30$

The procedure for a hypothesis test

- Identify the population of interest and the parameters: population mean (μ) and population standard deviation (σ).
- Identify the sample and the statistics: sample size (n), sample mean ($\bar{x}$) and sample standard deviation (s).
- Write the null and alternative hypothesis.
- Find the z- and ***p*-values** for the sample mean.
- Make a decision to either reject or fail to reject the null hypothesis.
- Write the conclusion.

p-values for hypothesis testing related to the mean

The null hypothesis is the statement being tested. Usually, the null hypothesis is a statement of 'no effect' or 'no difference'.

If the null hypothesis is disproved (rejected), then the alternative hypothesis 'is supported' and can be used to describe the population parameter.

Based on the sample data, the testing method will help determine whether to reject the null hypothesis or fail to reject the null hypothesis.

The p-value is used to make that determination.

The *p*-value

The p-value is the probability of obtaining the value in the sample or a more extreme value, assuming the null hypothesis is true.

We use a one-sample z test to compare the mean in one sample to the expected norm, which in our case is the population mean. The z test can only be used when the population is normally distributed or the size of the population is large and the standard deviation is known.

The mean and standard deviation formulas for calculating the *p*-value

When calculating the p-value, with a one-sample z test, a cumulative normal distribution is used.

mean = population mean μ, or $\mu_{\overline{X}} = \mu$

standard deviation (standard error) $\sigma_{\overline{X}} = \dfrac{\sigma}{\sqrt{n}}$

If σ is not known, then use $s_{\overline{X}} = \dfrac{s}{\sqrt{n}}$ in place of $\sigma_{\overline{X}}$.

Significance levels for making decisions

The decision to reject the null hypothesis or fail to reject the null hypothesis is made using the p-value.

The value at which the null hypothesis is rejected is called the **significance level** and this is represented by the Greek letter α.

Common significance levels (α)

If the p-value is less than α, we reject the null hypothesis and support the alternative hypothesis. The low p-value indicates that the observed data are extremely unlikely if the null hypothesis is true.

If the p-value is greater than α, we fail to reject the null hypothesis.

The 'significant' here means the observed difference is probably not due to chance.

Common levels used for α are 0.05 and 0.01.

- p-value > α: the observed difference is 'not significant'
- p-value < α: the observed difference is 'significant'

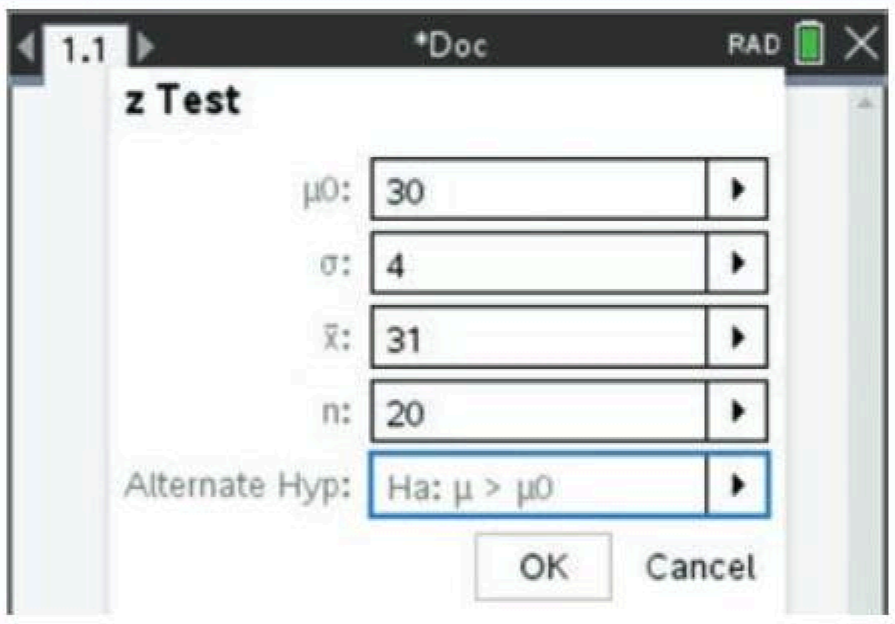
(to support the null hypothesis)

If the significance level α is not given, use 0.05.

USING CAS 5 ***z* tests**

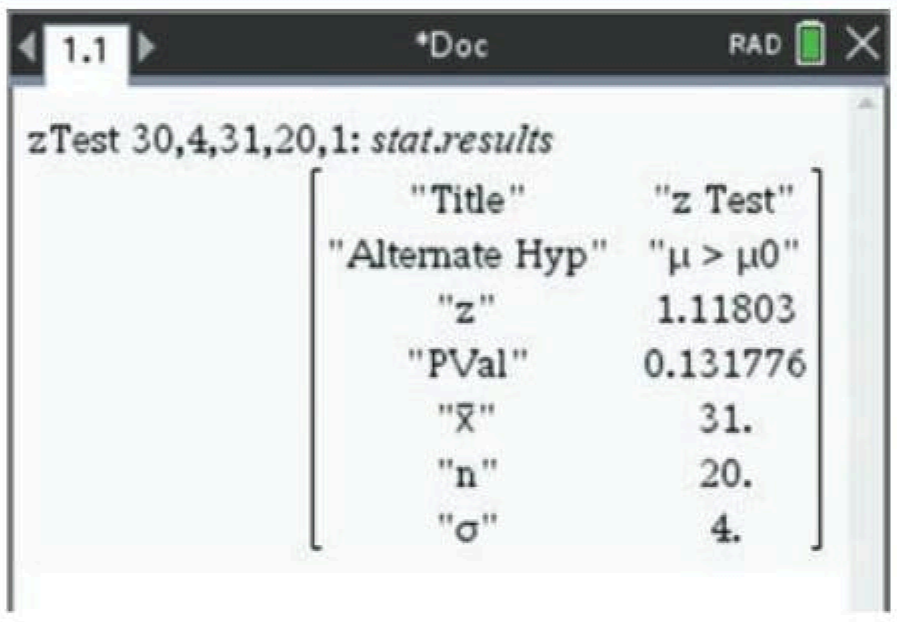
A set of data has a mean of 30 and a standard deviation of 4. If a random sample of size 20 is made, determine the probability that the mean of this sample is greater than 31.

TI-Nspire



z Test

μ0: 30
σ: 4
x̄: 31
n: 20
Alternate Hyp: Ha: μ > μ0



```
zTest 30,4,31,20,1: stat.results
        "Title"            "z Test"
        "Alternate Hyp"    "μ > μ0"
        "z"                1.11803
        "PVal"             0.131776
        "x̄"                31.
        "n"                20.
        "σ"                4.
```

1 Press **menu > Statistics > Stat Tests > *z* Test**.

2 On the next screen, change the **Data Input Method:** setting from **Data** to **Stats**.

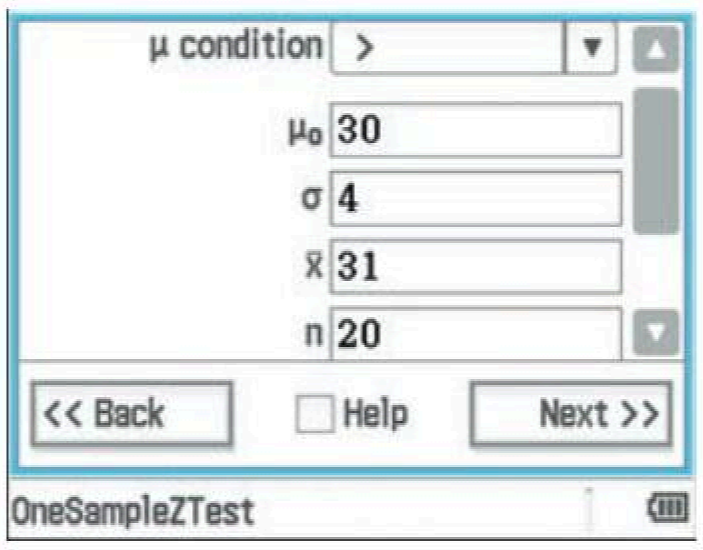
3 In the dialogue box, enter the values as shown.

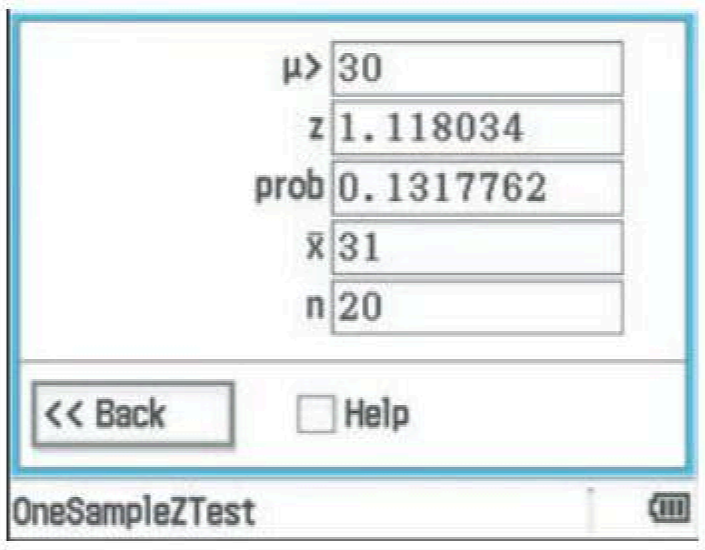
4 Change the **Alternate Hyp:** setting to Ha: $\mu > \mu$**0**.

5 The **zTest** data will be displayed.

6 The **PVal** specifies the probability, which is **0.13**.

ClassPad


μ condition >
μ_0 30
σ 4
$\bar{x}$ 31
n 20


μ> 30
z 1.118034
prob 0.1317762
$\bar{x}$ 31
n 20

1. Tap to open the **Statistics** application.
2. Tap **Calc > Test**.
3. On the next screen, tap **Variable,** then tap **Next**.
4. In the dialogue box, change the μ **condition** to **>**.
5. Enter the values as shown above, then tap **Next**.
6. The **z Test** statistics will be displayed.
7. The **prob** value designates the probability, which is 0.13.

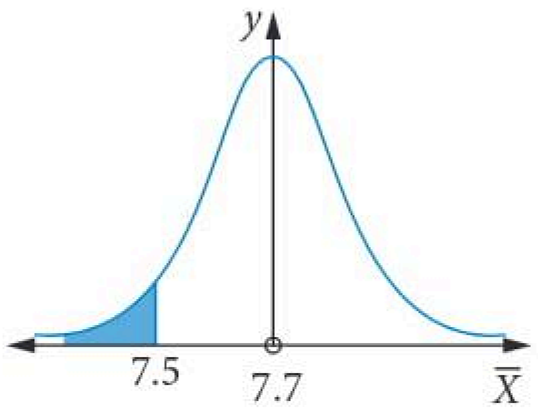
The probability that the mean of the randomly selected sample is greater than 31 is 0.13.

WB

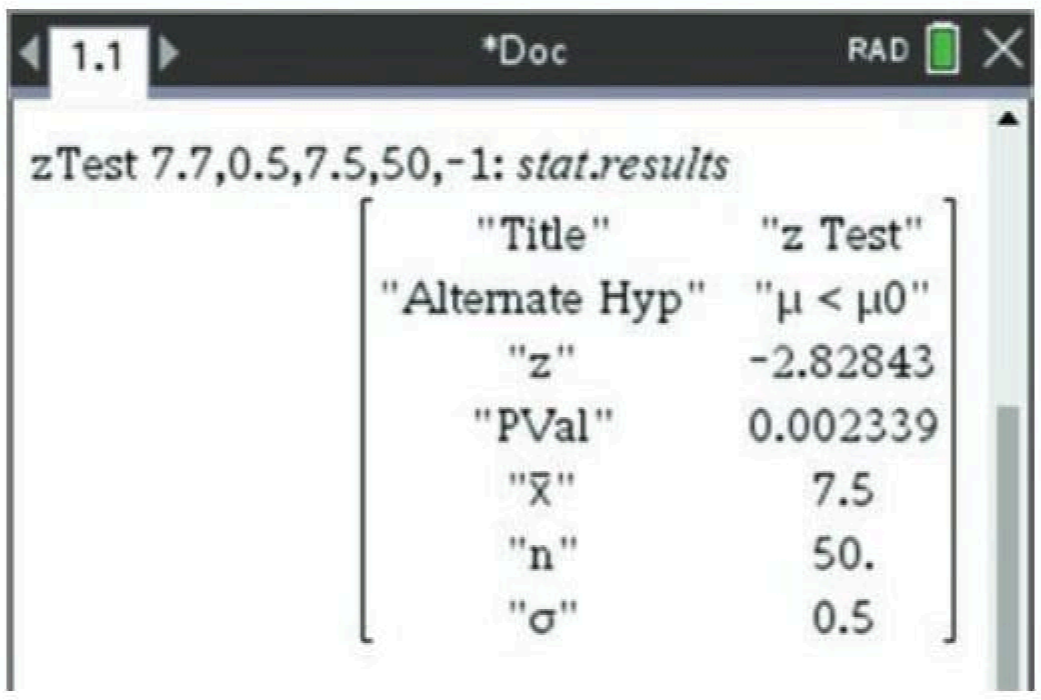
p. 251

WORKED EXAMPLE 21 One-tailed test

The number of hours of sleep an adult has each night is normally distributed with a mean of 7.7 hours and a standard deviation of 0.5 hours. The number of hours of sleep for 50 randomly selected adults who are snorers produced a mean of 7.5 hours. Test the hypothesis that snorers sleep less than the general population. Test at the 5% significance level.

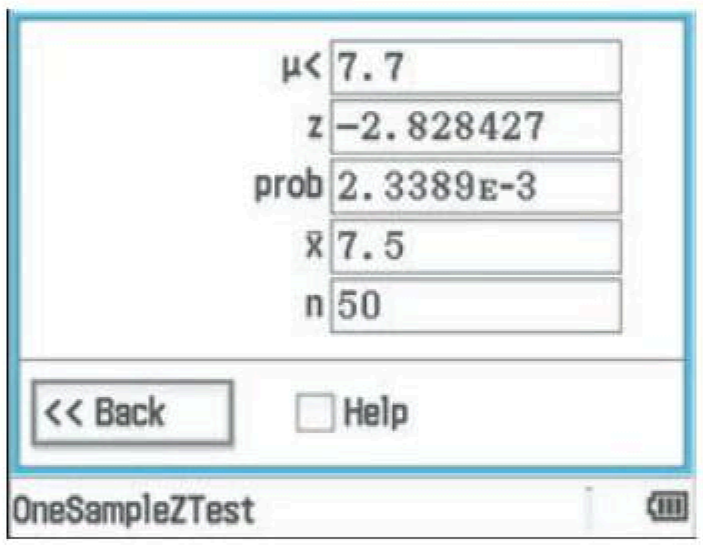
Steps	**Working**
1 Write the null hypothesis.	Null hypothesis H_0: $\mu = 7.7$. The mean number of hours of sleep that adult snorers have per night is equal to 7.7 h.
2 Write the alternative hypothesis.	Alternative hypothesis H_1: $\mu < 7.7$. The mean number of hours of sleep adult snorers have per night is less than 7.7 h.
3 Find the p-value. Assume that H_0 is true. Find the probability of obtaining the sample mean or a value more extreme in the direction of the alternative hypothesis. As H_1 is $\mu < 7.7$ and the observed mean in the sample is 7.5, the observed value or a more extreme value is $\bar{x} < 7.5$.	$p = \Pr\left(\bar{x} < 7.5 \mid \mu = 7.7\right)$ 
4 Complete a one-sample z test. The mean used is the population mean. The z test can be completed using CAS.	$\mu = 7.7$

TI-Nspire



```
zTest 7.7,0.5,7.5,50,-1: stat.results
        "Title"          "z Test"
        "Alternate Hyp"  "μ < μ0"
        "z"              -2.82843
        "PVal"           0.002339
        "x̄"              7.5
        "n"              50.
        "σ"              0.5
```

ClassPad



```
  μ< 7.7
   z -2.828427
prob 2.3389E-3
   x̄ 7.5
   n 50
```

Write the p-value.

$\Pr\left(\bar{x} < 7.5 \,|\, \mu = 7.7\right) = 0.0023$

$p = 0.0023$

5 Make a decision about the null hypothesis for the 5% level of significance.

Reject if $p < 0.05$.

Fail to reject if $p > 0.05$.

The chance of getting a mean less than 7.5 is 0.0023 if H_0 is true. Since $p < 0.05$, we reject the null hypothesis and therefore the result is significant.

6 Make a conclusion about the claim.

The mean number of hours of sleep that adult snorers have per night is significantly less than 7.7 hours ($z = -2.828$, $p = 0.0023$, $\bar{x} = 7.5$, $\sigma = 0.5$, $n = 50$).

There is enough evidence to support the claim that snorers get less sleep than the general public.

One-tailed and two-tailed tests

In the previous example, a **one-tailed test** was performed to test a 'less than' claim, but a **two-tailed test** is required for a 'not equal to' claim.

p-values for one- and two-tailed tests

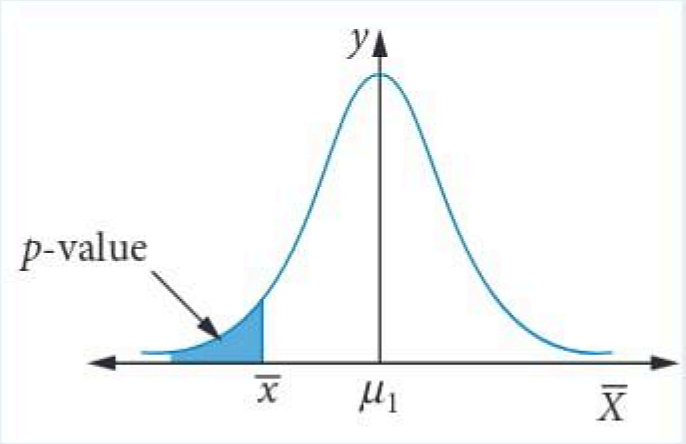
One-tailed test

A 'less than' (<) or 'greater than' (>) sign in the alternative hypothesis indicates that a one-tailed test is required.

In this one-tailed test,

- the null hypothesis H_0: $\mu = \mu_1$
- the alternative hypothesis H_1: $\mu < \mu_1$
- p-value $= \Pr\left(\bar{X} < \bar{x} \,\middle|\, \mu = \mu_1\right)$

where $\bar{x}$ is the sample mean and μ_1 is the mean of the population.



Two-tailed test

A two-tailed test looks for any change in a parameter. The presence of a not equal to sign (≠) in the alternative hypothesis indicates that a two-tailed test is required.

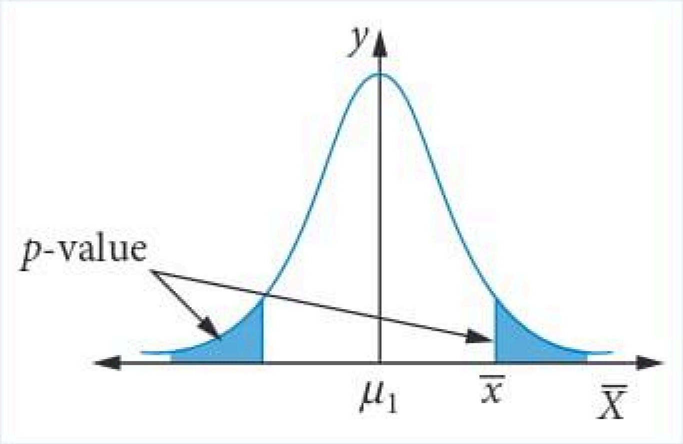
In this two-tailed test,

- the null hypothesis H_0: $\mu = \mu_1$
- the alternative hypothesis H_1: $\mu \neq \mu_1$
- p-value $= 2 \times \Pr\left(\bar{X} > \bar{x} \,\middle|\, \mu = \mu_1\right)$, for $\bar{x} > \mu_1$

or

- p-value $= 2 \times \Pr\left(\bar{X} < \bar{x} \,\middle|\, \mu = \mu_1\right)$ for $\bar{x} < \mu_1$

where $\bar{x}$ is the sample mean and μ_1 is the mean of the population.

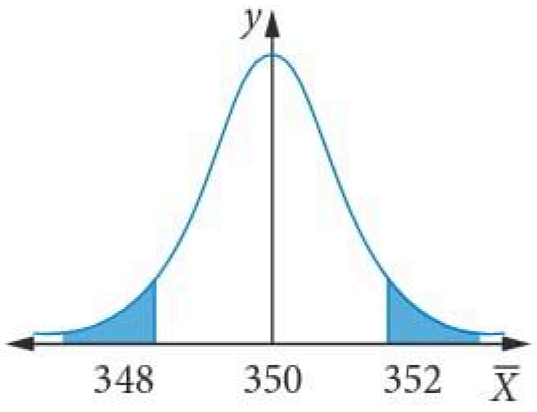


WB

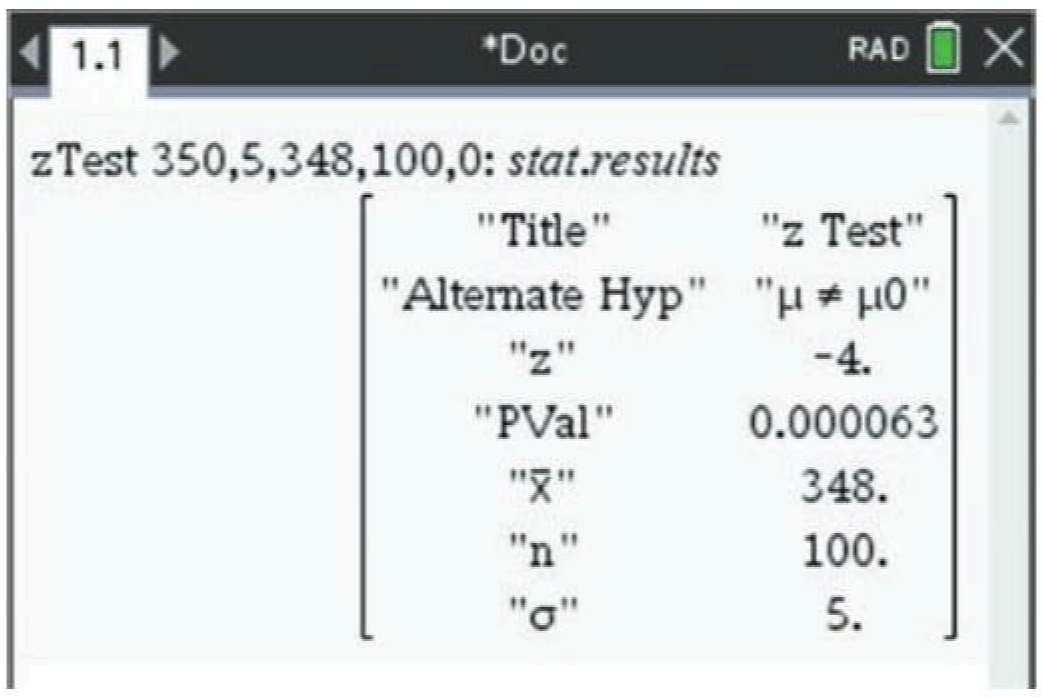
p. 253

WORKED EXAMPLE 22 Two-tailed test

A company sells fruit juice and sets their machines up to dispense the same amount of juice in each bottle. The mean and standard deviation of the volume of juice in each bottle are 350 ml and 5 ml, respectively. A consumer group claims there is not 350 ml of juice in each bottle. They then test a sample of 100 bottles and find the average volume to be 348 ml. Test the consumer groups claim at the 1% significance level.

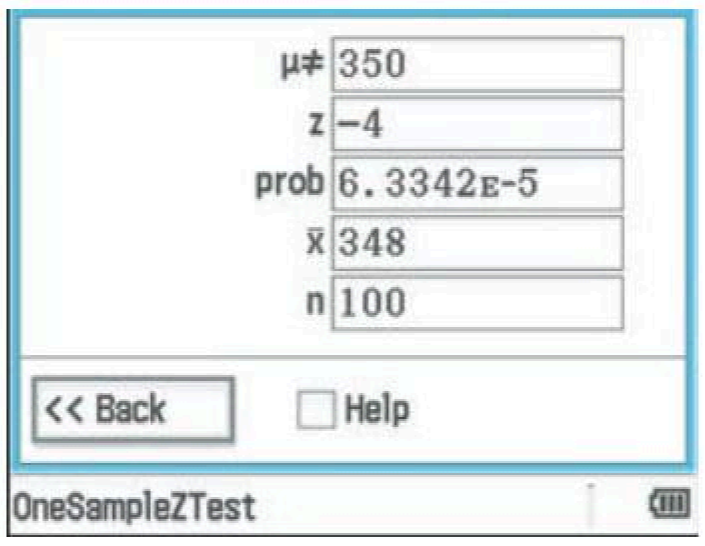
Steps	**Working**
1 Write the null hypothesis.	null hypothesis H_0: $\mu = 350$
2 Write the alternative hypothesis. The alternative hypothesis is that there is not 350 ml in a bottle.	alternative hypothesis H_1: $\mu \neq 350$
3 Assume that H_0 is true. Find the probability of obtaining the sample mean or a value more extreme in the direction of the alternative hypothesis. As H_1 is $\mu \neq 350$, a two-tailed test is required. We calculate the probability of the observed value $\bar{x} = 348$, or a more extreme value is $(\bar{x} < 348)$.	$p = 2 \times \Pr\left(\bar{X} < 348 \mid \mu = 350\right)$ 
4 Complete a one-sample z test. The mean used is the population mean. The p-value is found using CAS. Input the data using CAS.	$\mu = 350$ $p = 0.000\,063$

TI-Nspire



```
1.1                *Doc                RAD
zTest 350,5,348,100,0: stat.results
        "Title"            "z Test"
        "Alternate Hyp"    "μ ≠ μ0"
        "z"                -4.
        "PVal"             0.000063
        "x̄"                348.
        "n"                100.
        "σ"                5.
```

ClassPad



```
   μ≠ 350
    z −4
 prob 6.3342E-5
    x̄ 348
    n 100
<< Back    ☐ Help
OneSampleZTest
```

Steps	**Working**
5 Make a decision about the null hypothesis. When p-value < 0.01, the observed difference is significant.	The chance of getting a mean less than 348 or greater than 352 is 0.000 063 if H_0 is true. We reject the null hypothesis $\mu = 350$ because p is less than 0.01 and therefore the result is significant.
6 Make a conclusion about the claim. Since the null hypothesis H_0 is rejected and the alternative hypothesis is H_1: $\mu \neq 350$, we need to decide if $\mu > 350$ or $\mu < 350$. The sample mean $\bar{x} = 348$ is less than 350, therefore we choose $\mu < 350$.	The mean volume of juice in the bottle is significantly less than 350 ml ($z = -4$, $p = 0.000\,063$, $\bar{x} = 348$, $\sigma = 5$, $n = 100$). There is enough evidence to support the claim that the average volume of juice in the bottle is less than 350 ml.

Critical values for a given significance level

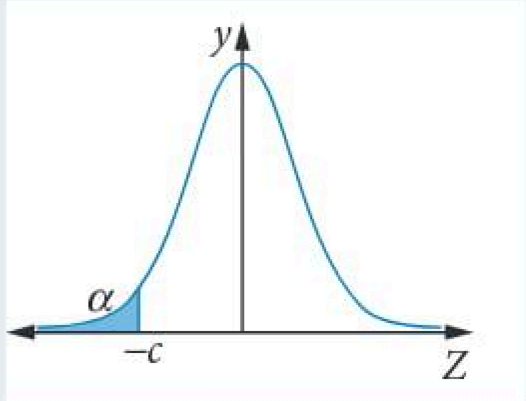
In hypothesis testing, there are two ways to determine if there is enough evidence to reject or fail to reject the null hypothesis.

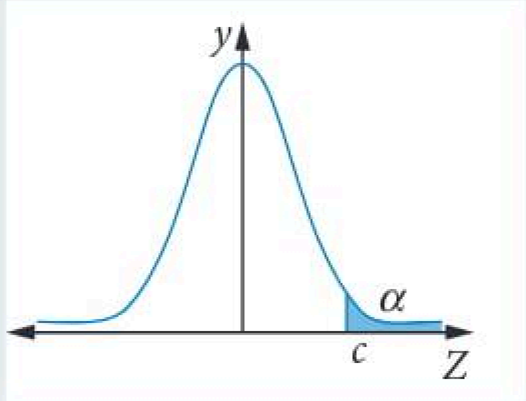
The first way is to calculate the p-value and compare it with the level of significance α, which we have covered in detail in the previous section.

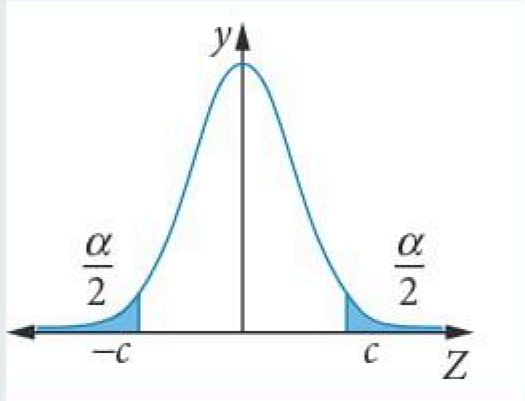
The second way is to compare the test statistic (z) with a critical value determined from the level of significance α.

If the test statistic is more extreme than the critical value, the null hypothesis is rejected. The values where it would be rejected is called the rejection region.

Critical z values for common significance levels







Left tail	
α	$z = -c$
0.01	−2.326
0.05	−1.645
0.1	−1.282

Right tail	
α	$z = c$
0.01	2.326
0.05	1.645
0.1	1.282

Two tail	
α	$z = \pm c$
0.01	± 2.576
0.05	± 1.960
0.1	± 1.645

When σ is known, the z-value for the sample statistic is given by

$$z = \frac{\bar{x} - \mu}{\sigma_{\bar{X}}} \text{ where } \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$$

When σ is not known, use $s_{\bar{X}} = \frac{s}{\sqrt{n}}$ in place of $\sigma_{\bar{X}}$.

Exam hack

Memorise the critical z-values for one- and two-tailed tests at the 0.01, 0.05 and 0.1 levels of significance. They may be required in Exam 1 in questions on hypothesis testing.

The inverse cumulative normal distribution can be used to find the critical z-value.

Let the critical z-value $= c$.

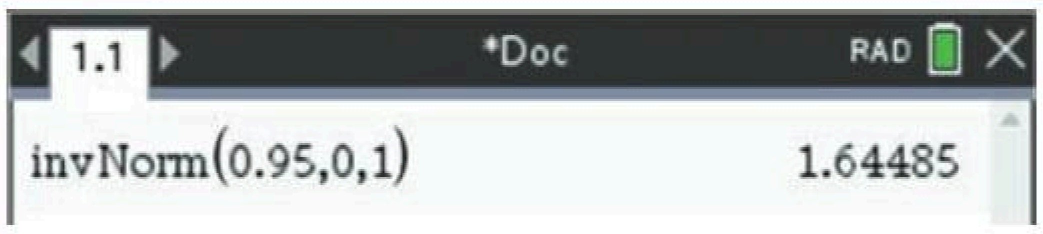
For a significance level $\alpha = 0.05$ with a right one-tailed test

$\Pr(z > c) = 0.05$

USING CAS 6 Finding the critical z-value

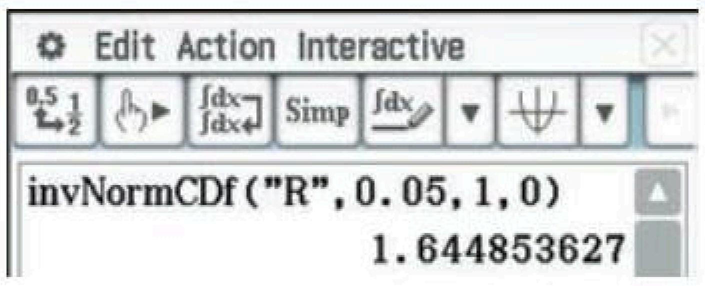
Find the critical z-value for a one-tailed test with a significance level of 0.05.

TI-Nspire



```
invNorm(0.95,0,1)                1.64485
```

ClassPad



```
invNormCDf("R",0.05,1,0)
                 1.644853627
```

For $\Pr(z > c) = 0.05$, $c = 1.64485$.

WB

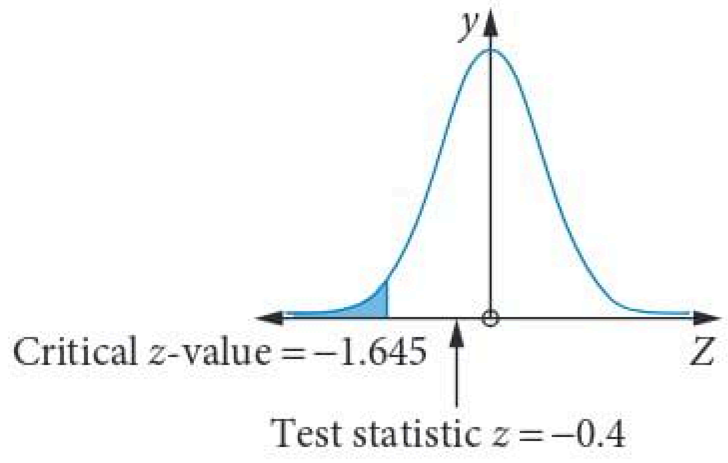
p. 254

WORKED EXAMPLE 23 Testing a null hypothesis using a critical *z*-value

YoGood sells yoghurt in containers with an advertised mean volume of 250 mL. Their competitors claim that on average the containers contain less yoghurt than advertised. They take a random sample of 16 containers and find the mean volume is 248 mL. The volume of yoghurt in all containers is normally distributed with a standard deviation of 20 mL.

a State appropriate null and alternative hypotheses for the volume.

b The p-value for this test is given by the expression $\Pr(Z < z)$, where Z has the standard normal distribution. Find the value of z and hence determine whether the null hypothesis should be rejected at the 0.05 level of significance.

Steps	**Working**
a **1** Write the null hypothesis.	$H_0: \mu = 250$
2 Write the alternative hypothesis.	$H_1: \mu < 250$
b **1** Find the value of the test statistic z. Use $z = \frac{\bar{x}-\mu}{\sigma_{\bar{X}}}$ where $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$	$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{20}{\sqrt{16}} = \frac{20}{4} = 5$ Substitute $\mu = 250$, $\sigma_{\bar{X}} = 5$ $z = \frac{\bar{x}-\mu}{\sigma_{\bar{X}}} = \frac{248-250}{5} = -0.4$
2 Compare the test statistic with the critical z-value for the significance level of 0.05.	For $\alpha = 0.05$, the critical z value $= -1.645$ and the test statistic z is $-$ 0.4. 
3 Make a decision about the null hypothesis.	The test statistic $z = -0.4$ is not less than the critical z-value $= -1.645$ and lies outside the rejection region. We fail to reject the null hypothesis $\mu = 250$ mL at the 0.05 significance level. $(z = -0.4, \ \bar{x} = 248, \sigma = 20, n = 16)$

WB

p. 255

WORKED EXAMPLE 24 Finding critical values of the sample mean

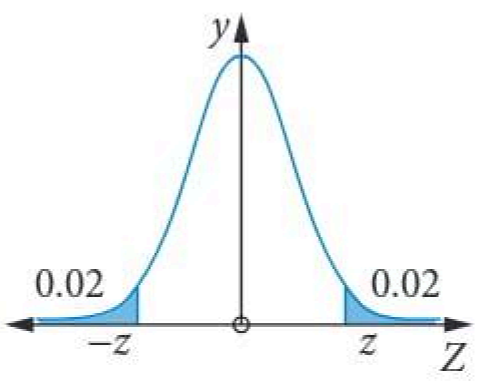
A normally distributed population has a mean of 45 and a standard deviation of 5.

Consider the hypotheses $H_0: \mu = 45$, $H_1: \mu \neq 45$

a Find the critical z-values at the 0.04 significance level, correct to three decimal places.

b A random sample of size 25 is taken from this population. The random variable $\bar{X}$ represents the sampling distribution of the sample means.

Find the critical values of $\bar{X}$ at which the null hypothesis would be rejected.

Steps	**Working**
a **1** Draw the standard normal distribution, label the critical z-values and the associated probabilities. A two-tailed test is required for $\mu \neq 45$.	

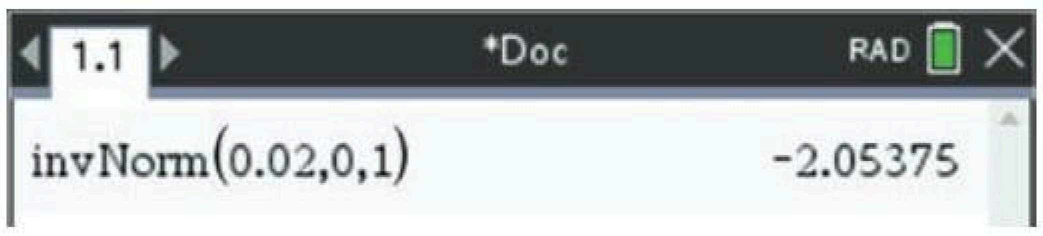
2 Use an inverse cumulative normal distribution on CAS to find z.

In the two-tailed test, we reject the null hypothesis $\mu = 45$ if the test statistic is in the rejection region shaded in the standard normal distribution.

$\Pr(Z < -2.054) = 0.02$ and $\Pr(Z > 2.054) = 0.02$

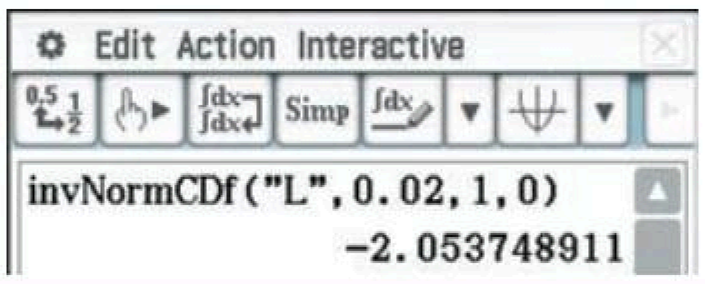
Critical z-values are -2.054 and 2.054.

TI-Nspire



```
invNorm(0.02,0,1)          -2.05375
```

ClassPad



```
invNormCDf("L",0.02,1,0)
                -2.053748911
```

b **1** Find the value of $\sigma_{\overline{X}}$.

$$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} = \frac{5}{\sqrt{25}} = 1$$

2 $\overline{X}$ is the sampling distribution of the sample means of size 25 with a mean of 45 and a standard error of 1.

The value of c corresponds to the critical z-value of 2.054.

Substitute into $z = \dfrac{\bar{x} - \mu}{\sigma_{\overline{X}}}$.

Right tail

Let the critical value = c.

$\Pr(Z > 2.054) = \Pr(\overline{X} > c) = 0.02$

$$2.054 = \frac{c - 45}{1}$$

$$c = 47.054$$

The value of d corresponds to the critical z-value of -2.054.

Substitute into $z = \dfrac{\bar{x} - \mu}{\sigma_{\overline{X}}}$.

Left tail

Let the critical value = d.

$\Pr(Z < -2.054) = \Pr(\overline{X} < d) = 0.02$

$$-2.054 = \frac{d - 45}{1}$$

$$d = 42.946$$

Reject the null hypothesis if the sample mean is less than 42.946 or greater than 47.054.

> The critical values of the sample mean can also be found using CAS by finding the 96% confidence interval with mean = 45, standard deviation = 5, sample size = 25 and confidence level = 0.96.

EXERCISE 12.5 Hypothesis testing related to the mean

ANSWERS p. 601

Recap

1 A random sample of 150 mangoes from a given area has a mean mass of 220 g and a standard deviation of 15 g.

Assuming the standard deviation obtained from the sample is a sufficiently accurate estimate of the population standard deviation, an approximate 90% confidence interval for the mean mass of mangoes produced in this locality is given by

A (205, 235)
B (217.9, 222.1)
C (218, 222)
D (218.8, 231.2)
E (216, 224)

2 The heights of all four-year-old children in a given population are normally distributed. The mean height of a random sample of 64 four-year-old children from this population is found to be 102 cm. If a 95% confidence interval for the mean height of all six-year-old children is calculated to be (101.4, 102.6) cm, the standard deviation used in this calculation is closest to

A 0.3
B 0.6
C 1.2
D 2.0
E 2.4

Mastery

3 WORKED EXAMPLE 19

a The average time spent waiting at a doctor's surgery is 40 minutes. The practice introduces a new system of warning patients by SMS in advance if the doctors are running late and they claim that this has reduced the average waiting time. Write the null and alternative hypotheses.

b The average mark for a secondary school entrance exam is 75. A private company runs entrance exam preparation courses and they claim that students who do their courses perform on average better on the entrance exam. Write the null and alternative hypotheses.

4 WORKED EXAMPLE 20

a The average weight of a standard chocolate bar produced by Wonka's Chocolates is 250 g. The company purchased a new machine and the supervisor claims the mean weight of the standard chocolate bars has changed. Write the null and alternative hypotheses.

b The residents of a small town want to evaluate changes in the pH of the water in a river that runs behind their homes. For the last 35 years, the average pH has been 5.5, however over the last few years, a lot of construction has been occurring in the area and the residents believe the mean pH of the river has changed. Write the null and alternative hypotheses.

5 Using CAS 5

a A sample of size 80 is taken from a population with mean 120 and standard deviation 25. Find the probability, correct to 2 decimal places, that the sample mean is less than 118.

b A sample of size 40 is taken from a population with mean 1000 and standard deviation 200. Find the probability, correct to 2 decimal places, that the sample mean is within 50 of the population mean.

6 WORKED EXAMPLE 21 A manufacturer produces light bulbs that, when tested, have a life with a mean of 3000 hours and a standard deviation of 500 hours. A large retail store that uses the bulbs believe they burn on average for less than 3000 hours. The store tests a random sample of 100 bulbs and finds the mean of the sample is 2800 hours.

a Write the null and alternative hypotheses.

b Calculate the z- and p-values.

c What is your conclusion about the retailer's claim?

7 WORKED EXAMPLE 22 A manufacturer produces drill bits with a mean life of 580 hours and a standard deviation of 30 hours. A construction company that uses the drill bits disputes the manufacturer's claim that the bits will last an average of 580 hours. The company draws a sample of 100 bits and finds the mean is 577 hours.

a Write the null and alternative hypotheses.

b Calculate the z- and p-values.

c What is your conclusion about the construction company's claim?

8 Using CAS 6

a Consider the hypotheses, H_0: $\mu = 28$, H_1: $\mu < 28$. Find the critical z-value, correct to three decimal places, at which the null hypothesis is rejected at the 0.02 significance level.

b Consider the hypotheses, H_0: $\mu = 240$, H_1: $\mu \neq 240$. Find the critical z-values, correct to three decimal places, at which the null hypothesis is rejected at the 0.01 significance level.

▶ **9** WORKED EXAMPLE 23 A company makes meat pies with a mean weight of 175 grams. Their competitors claim that on average their pies are smaller than their advertised weight. They take a random sample of 25 pies and find the mean weight is 170 grams. The weight of all meat pies produced by the company is normally distributed with a standard deviation of 10 grams.

a State appropriate null and alternative hypotheses for the weight.

b The p-value for this test is given by the expression $\Pr(Z < z)$, where Z has the standard normal distribution. Find the value of z and hence determine whether the null hypothesis should be rejected at the 0.05 level of significance.

10 WORKED EXAMPLE 24 A normally distributed population has a mean of 80 and a standard deviation of 10.

For the hypotheses

$H_0: \mu = 80$

$H_1: \mu \neq 80$

a find the critical z-values at the 0.02 significance level correct to three decimal places.

A random sample of size 25 is taken from this population. The random variable $\bar{X}$ represents the sampling distribution of the sample means.

b Find the critical values of $\bar{X}$ at which the null hypothesis would be rejected.

Exam practice

80–100% 60–79% 0–59%

11 © VCAA 2021 1Q3 (5 marks) A company produces a particular type of light globe called Shiny. The company claims that the lifetime of these globes is normally distributed with a mean of 200 weeks and it is known that the standard deviation of the lifetime of Shiny globes is 10 weeks. Customers have complained, saying Shiny globes were lasting less than the claimed 200 weeks. It was decided to investigate the complaints. A random sample of 36 Shiny globes was tested and it was found that the mean lifetime of the sample was 195 weeks.

Use $\Pr(-1.96 < Z < 1.96) = 0.95$ and $\Pr(-3 < Z < 3) = 0.9973$ to answer the following questions.

a 72% Write down the null and alternative hypotheses for the one-tailed test that was conducted to investigate the complaints. 1 mark

b **i** 48% Determine the p-value, correct to three decimal places, for the test. 2 marks

ii 49% What should the company be told if the test was carried out at the 1% level of significance? 1 mark

c 56% The company decided to produce a new type of light globe called Globeplus. Find an approximate 95% confidence interval for the mean lifetime of the new globes if a random sample of 25 Globeplus globes is tested and the sample mean is found to be 250 weeks. Assume that the standard deviation of the population is 10 weeks. Give your answer correct to two decimal places. 1 mark

12 © VCAA 2017N 1Q9b TECH-FREE (2 marks) The random variable X has a mean and variance given by $\mu_X = 4$ and $\text{Var}(X) = 36$.

Researchers have reason to believe that the mean of X has decreased. They collect a random sample of 64 observations of X and find that the sample mean is $\bar{X} = 3.8$.

a State the null hypothesis and the alternative hypothesis that should be used to test that the mean has decreased. 1 mark

b Calculate the mean and standard deviation for a distribution of sample means, $\bar{X}$, for samples of 64 observations. 1 mark ▶

13 © VCAA 2017 2AQ20 77% In a one-sided statistical test at the 5% level of significance, it would be concluded that

A H_0 should not be rejected if $p = 0.04$
B H_0 should be rejected if $p = 0.06$
C H_0 should be rejected if $p = 0.03$
D H_0 should not be rejected if $p \neq 0.05$
E H_0 should not be rejected if $p = 0.01$

Use the following information to answer Questions 14 and 15.

The weight of redfin in a large lake is normally distributed with a mean of 1.5 kg and a standard deviation of 0.4 kg. An angling club believes that pollution in the lake has reduced the size of the fish. They take a random sample of 80 fish and find the average weight of the fish is 1.4 kg.

14 The null hypothesis is

A H_0: $\mu = 1.4$
B H_0: $\mu \neq 1.5$
C H_0: $\mu < 1.4$
D H_0: $\mu = 1.5$
E H_0: $\mu > 1.5$

15 The alternative hypothesis is

A H_1: $\mu = 1.5$
B H_1: $\mu < 1.5$
C H_1: $\mu = 1.4$
D H_1: $\mu < 1.4$
E H_1: $\mu \neq 1.5$

Use the following information to answer Questions 16 and 17.

The drying time of a certain brand of paint has a mean of 90 minutes and a standard deviation of 10 minutes. A painting contractor claims the addition of an agent will reduce the average drying time. The drying time of 60 modified paint tins was found to have a mean of 87 minutes.

16 The null and alternative hypotheses are

A H_0: $\mu < 90$, H_1: $\mu = 90$
B H_0: $\mu = 90$, H_1: $\mu = 87$
C H_0: $\mu = 90$, H_1: $\mu < 87$
D H_0: $\mu = 87$, H_1: $\mu < 87$
E H_0: $\mu = 90$, H_1: $\mu < 90$

17 The *p*-value for the hypothesis test is

A 0.0101
B 0.0011
C 0.0201
D 0.05
E 0.00023

Use the following information to answer Questions 18 and 19.

The time taken for the express train to travel between two major centres is a normally distributed continuous random variable with a mean of 95 minutes and a standard deviation of 4 minutes. The transport authority makes some modifications to the crossings on the route and would like to know if the travel time has changed. They take a random sample of 50 trips and find the mean travel time is 93 minutes.

18 The null and alternative hypotheses are

A H_0: $\mu < 95$, H_1: $\mu = 93$
B H_0: $\mu = 95$, H_1: $\mu = 93$
C H_0: $\mu = 95$, H_1: $\mu < 93$
D H_0: $\mu = 95$, H_1: $\mu \neq 95$
E H_0: $\mu = 95$, H_1: $\mu < 95$

19 The *p*-value for the hypothesis is

A 0.0015
B 0.0092
C 0.0184
D 0.0001
E 0.0231

▶ **20** (6 marks) A health centre finds that the weight loss for people that complete their ten-week program is normally distributed with a mean of 0.65 kg per week and a standard deviation of 0.05 kg per week. A group of 45 randomly selected people who complete the ten-week program also use a diet app and they find their average weight loss is 0.67 kg per week. They claim the use of the app improves their weight loss.

a Write the null and alternative hypotheses. 2 marks

b Calculate the z- and p-values. 2 marks

c State the decision. 1 mark

d What is your conclusion about the claim made about the effectiveness of the app? 1 mark

21 (6 marks) The resting pulse rate of primary school-age children taken from data over a 30-year period has a mean of 65 beats per minute and a standard deviation of 10 beats per minute.

The medical association believes that changes in lifestyle have also affected the rest pulse of children. They take a random sample of 100 primary school-age children and find their average rest pulse is 68 beats per minute.

a Write the null and alternative hypotheses. 2 marks

b Calculate the z- and p-values. 2 marks

c State the decision. 1 mark

d What is your conclusion about the claim made about the changes in lifestyle? 1 mark

22 (4 marks) The scores on an IQ test are normally distributed with a mean of 100 and a standard deviation of 15. An organisation claims that their courses will on average improve candidate's scores on an IQ test. They take a random sample of 40 people who have completed the course and find their average IQ is 101. Test the claim made by the organisation.

23 © VCAA 2017 2BQ6 (9 marks) A dairy factory produces milk in bottles with a nominal volume of 2 L per bottle. To ensure most bottles contain at least the nominal volume, the machine that fills the bottles dispenses volumes that are normally distributed with a mean of 2005 mL and a standard deviation of 6 mL.

a 57% Find the percentage of bottles that contain at least the nominal volume of milk, correct to one decimal place. 1 mark

Bottles of milk are packed in crates of 10 bottles, where the nominal total volume per crate is 20 L.

b 68% Show that the total volume of milk contained in each crate varies with a mean of 20 050 mL and a standard deviation of $6\sqrt{10}$ mL. 2 marks

c 61% Find the percentage, correct to one decimal place, of crates that contain at least the nominal volume of 20 L. 1 mark

d 37% Regulations require at least 99.9% of crates to contain at least the nominal volume of 20 L.

Assuming the mean volume dispensed by the machine remains 2005 mL, find the maximum allowable standard deviation of the bottle-filling machine needed to achieve this outcome. Give your answer in millilitres, correct to one decimal place. 3 marks

e 55% A nearby dairy factory claims the milk dispensed into its 2 L bottles varies normally with a mean of 2005 mL and a standard deviation of 2 mL.

When authorities visit the nearby dairy factory and check a random sample of 10 bottles of milk, they find the mean volume to be 2004 mL.

Assuming that the standard deviation of 2 mL is correct, carry out a one-sided statistical test and determine, **stating a reason**, whether the nearby dairy's claim should be accepted at the 5% level of significance. 2 marks ▶

24 © VCAA 2016 2BQ6 (10 marks) The mean level of pollutant in a river is known to be 1.1 mg/L with a standard deviation of 0.16 mg/L.

a 86% Let the random variable $\overline{X}$ represent the mean level of pollutant in the measurements from a random sample of 25 sites along the river.

Write down the mean and standard deviation of $\overline{X}$. 2 marks

After a chemical spill, the mean level of pollutant from a random sample of 25 sites is found to be 1.2 mg/L.

To determine whether this sample provides evidence that the mean level of pollutant has increased, a statistical test is carried out.

b 78% Write down suitable hypotheses H_0 and H_1 to test whether the mean level of pollutant has increased. 2 marks

c **i** 70% Find the p-value for this test, correct to four decimal places. 2 marks

ii 65% State with a reason whether the sample supports the contention that there has been an increase in the mean level of pollutant after the spill. Test at the 5% level of significance. 1 mark

d 43% For this test, what is the smallest value of the sample mean that would provide evidence that the mean level of pollutant has increased? That is, find $\overline{x}_c$ such that $\Pr\left(\overline{X} > \overline{x}_c | \mu = 1.1\right) = 0.05$. Give your answer correct to three decimal places. 1 mark

e 46% Suppose that for a level of significance of 2.5%, we find that $\overline{x}_c = 1.163$. That is,

$\Pr\left(\overline{X} > 1.163 | \mu = 1.1\right) = 0.025$.

If the mean level of pollutant in the river, μ, is in fact 1.2 mg/L after the spill, find $\Pr\left(\overline{X} < 1.163 | \mu = 1.2\right)$. Give your answer correct to three decimal places. 1 mark

25 © VCAA 2019 2BQ6 (9 marks) A company produces packets of noodles. It is known from past experience that the mass of a packet of noodles produced by one of the company's machines is normally distributed with a mean of 375 grams and a standard deviation of 15 grams. To check the operation of the machine after some repairs, the company's quality control employees select two independent random samples of 50 packets and calculate the mean mass of the 50 packets for each random sample.

a 45% Assume that the machine is working properly. Find the probability that at least one random sample will have a mean mass between 370 g and 375 g. Give your answer correct to three decimal places. 2 marks

b 21% Assume that the machine is working properly. Find the probability that the means of the two random samples differ by less than 2 g. Give your answer correct to three decimal places. 3 marks

To test whether the machine is working properly after the repairs and is still producing packets with a mean mass of 375 g, the two random samples are combined and the mean mass of the 100 packets is found to be 372 g. Assume that the standard deviation of the mass of the packets produced is still 15 g. A two-tailed test at the 5% level of significance is to be carried out.

c 66% Write down suitable hypotheses H_0 and H_1 for this test. 1 mark

d 59% Find the p-value for the test, correct to three decimal places. 1 mark

e 59% Does the mean mass of the sample of 100 packets suggest that the machine is working properly at the 5% level of significance for a two-tailed test? Justify your answer. 1 mark

f 36% What is the smallest value of the mean mass of the sample of 100 packets for H_0 to be **not** rejected? Give your answer correct to one decimal place. 1 mark

26 © VCAA 2018N 2BQ7 (4 marks) According to medical records, the blood pressure of the general population of males aged 35 to 45 years is normally distributed with a mean of 128 and a standard deviation of 14. Researchers suggested that male teachers had higher blood pressures than the general population of males.

To investigate this, a random sample of 49 male teachers from this age group was obtained and found to have a mean blood pressure of 133.

a State **two** hypotheses and perform a statistical test at the 5% level to determine if male teachers belonging to the 35 to 45 years age group have higher blood pressures than the general population of males. Clearly state your conclusion with a reason. 3 marks

b Find a 90% confidence interval for the mean blood pressure of all male teachers aged 35 to 45 years using a standard deviation of 14. Give your answers correct to the nearest integer. 1 mark

27 © VCAA 2019N 2BQ6 (9 marks) A paint company claims that the mean time taken for its paint to dry when motor vehicles are repaired is 3.55 hours, with a standard deviation of 0.66 hours.

Assume that the drying time for the paint follows a normal distribution and that the claimed standard deviation value is accurate.

a Let the random variable $\overline{X}$ represent the mean time taken for the paint to dry for a random sample of 36 motor vehicles. Write down the mean and standard deviation of $\overline{X}$. 2 marks

At a crash repair centre, it was found that the mean time taken for the paint company's paint to dry on 36 randomly selected vehicles was 3.85 hours. The management of this crash repair centre was not happy and believed that the claim regarding the mean time taken for the paint to dry was too low. To test the paint company's claim, a statistical test was carried out.

b Write down suitable null and alternative hypotheses H_0 and H_1 respectively to test whether the mean time taken for the paint to dry is longer than claimed. 1 mark

c Write down an expression for the p-value of the statistical test and evaluate it correct to three decimal places. 2 marks

d Using a 1% level of significance, state with a reason whether the crash repair centre is justified in believing that the paint company's claim of a mean time taken for its paint to dry of 3.55 hours is too low. 1 mark

e At the 1% level of significance, find the set of sample mean values that would support the conclusion that the mean time taken for the paint to dry exceeded 3.55 hours. Give your answer in hours, correct to three decimal places. 2 marks

f If the true mean time taken for the paint to dry is 3.83 hours, find the probability that the paint company's claim is not rejected at the 1% level of significance, assuming the standard deviation for the paint to dry is still 0.66 hours. Give your answer correct to two decimal places. 1 mark

Video playlist
Errors in hypothesis testing

12.6 Errors in hypothesis testing

No hypothesis test is 100% certain. As the test is based on probabilities, there is always a chance of making an error. There are two types of errors.

Type I and II errors

A **type I error** occurs when we reject the null hypothesis when it is true.

The probability of making a Type I error is the α value as this is the (significance) level at which the null hypothesis is rejected.

A **type II error** occurs when we fail to reject the null hypothesis when it is false.

	H_0 True	H_0 False
Reject H_0	Type I error	Correct rejection
Fail to reject H_0	Correct decision	Type II error

A type I error is also called a **false positive** and a type II error a **false negative**.

Type 1 errors are also called **alpha errors** as the probability of a type 1 error occurring is equal to the significance level, α.

WB
p. 256

WORKED EXAMPLE 25 Type I and II errors

A car that takes unleaded fuel has a fuel consumption rate of 10 L/100 km.

A company claims that their additive will improve the fuel consumption.

a Write the null and alternative hypotheses.

b Describe a type I error and the impact in this situation.

c Describe a type II error and the impact in this situation.

Steps	Working
1 Write the null and alternative hypotheses.	H_0: $\mu = 10$ L/100 km The additive will not improve the fuel consumption. H_1: $\mu < 10$L/100 km The additive will improve the fuel consumption.
2 A type I error occurs when we reject the null hypothesis when it is true.	The null hypothesis H_0: $\mu = 10$ L/100 km is rejected when it is true. This would mean that you conclude the additive would improve the fuel consumption when it really does not. Consumers will therefore be wasting their money on the product.
3 A type II error occurs when we fail to reject the null hypothesis when it is false.	A type II error occurs if the null hypothesis is not rejected when it is false. This would mean that you conclude the additive would have not improved the fuel consumption in the car when in fact it would. Therefore, consumers are missing out on an effective product.

VCE QUESTION ANALYSIS

© VCAA 2018 2BQ6 2018 Examination 2 Section B Question 6 (8 marks)

The heights of mature water buffaloes in northern Australia are known to be normally distributed with a standard deviation of 15 cm. It is claimed that the mean height of the water buffaloes is 150 cm.

To decide whether the claim about the mean height is true, rangers selected a random sample of 50 mature water buffaloes. The mean height of this sample was found to be 145 cm.

A one-tailed statistical test is to be carried out to see if the sample mean height of 145 cm differs significantly from the claimed population mean of 150 cm.

Let $\overline{X}$ denote the mean height of a random sample of 50 mature water buffaloes.

a State suitable hypotheses H_0 and H_1 for the statistical test. 1 mark

b Find the standard deviation of $\overline{X}$. 1 mark

c Write down an expression for the p-value of the statistical test and evaluate your answer correct to four decimal places. 2 marks

d State with a reason whether H_0 should be rejected at the 5% level of significance. 1 mark

e What is the smallest value of the sample mean height that could be observed for H_0 to be not rejected? Give your answer in centimetres, correct to two decimal places. 1 mark

f If the true mean height of all mature water buffaloes in northern Australia is in fact 145 cm, what is the probability that H_0 will be accepted at the 5% level of significance? Give your answer correct to two decimal places. 1 mark

g Using the observed sample mean of 145 cm, find a **99% confidence interval** for the mean height of all mature water buffaloes in northern Australia. Express the values in your confidence interval in centimetres, correct to one decimal place. 1 mark

Video
VCE question analysis: Random variables and hypothesis testing

Reading the question

- Highlight the population parameters: mean and standard deviation.
- Highlight the sample statistics: mean and sample size.
- Identify the claim for the hypothesis.
- The level of significance should also be highlighted as this will determine the stage at which you reject or fail to reject the null hypothesis.

Thinking about the question

- The question requires an understanding of the sampling distribution of the sample means.
- You will need to understand how to identify the parameters and statistics in the question and how these are used to complete a hypothesis test.
- You will need to know how a confidence interval is calculated and understand how to interpret the confidence interval.
- You will also need to know how to calculate a p-value and a critical z-value and understand how they are used to either reject or fail to reject the null hypothesis.

Worked solution (✓ = 1 mark)

a $H_0: \mu = 150$

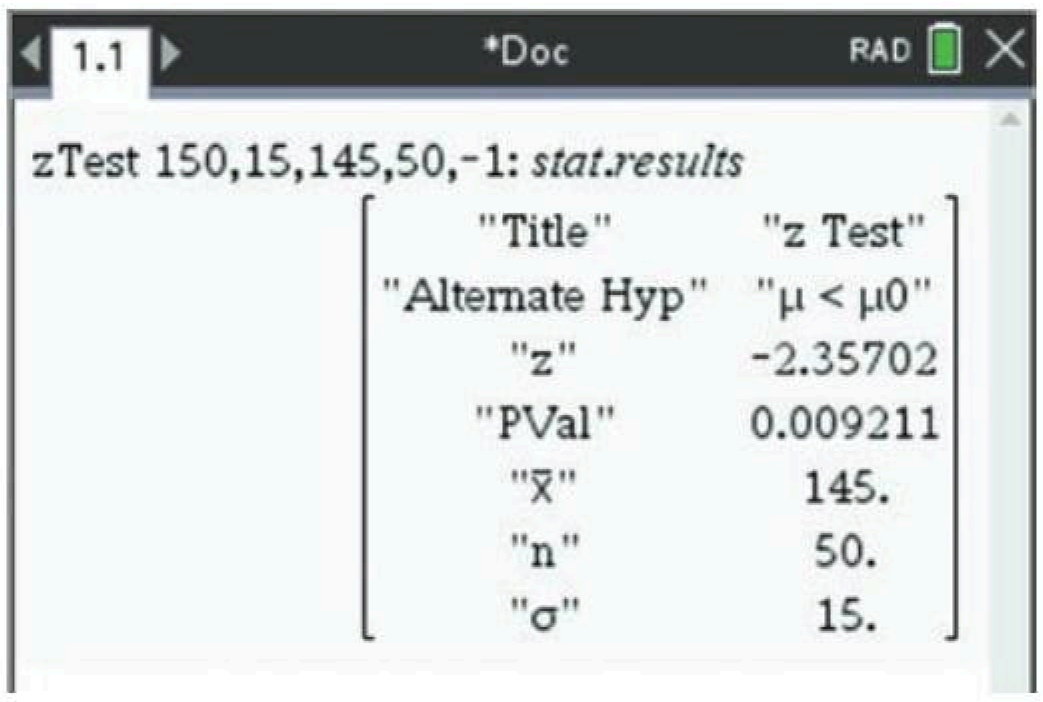
$H_1: \mu < 150$ ✓

b $\sigma_{\overline{X}} = \dfrac{\sigma}{\sqrt{n}} = \dfrac{15}{\sqrt{50}} = \dfrac{3}{\sqrt{2}} = \dfrac{3\sqrt{2}}{2}$ ✓

c p-value $= \Pr\left(\overline{X} < 145 | \mu = 150\right)$ ✓

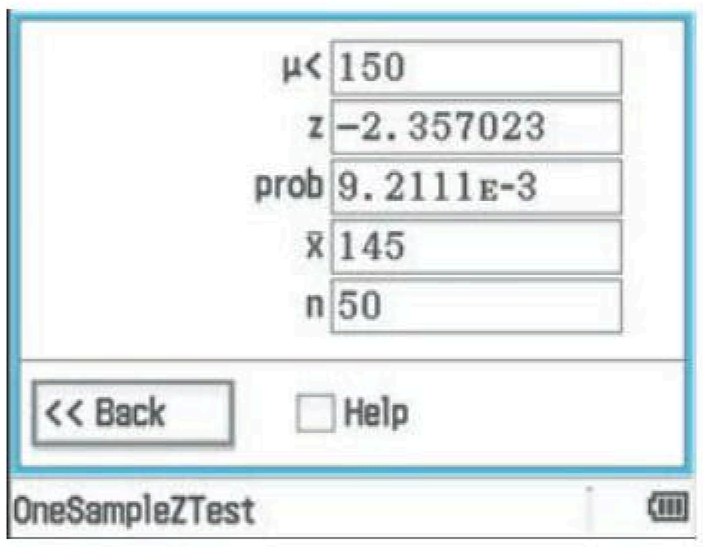
$p = 0.0092$ ✓

TI-Nspire



```
zTest 150,15,145,50,-1: stat.results
          "Title"          "z Test"
          "Alternate Hyp"  "μ < μ0"
          "z"              -2.35702
          "PVal"           0.009211
          "x̄"              145.
          "n"              50.
          "σ"              15.
```

ClassPad



```
  μ<  150
   z  -2.357023
prob  9.2111E-3
   x̄  145
   n  50
```

d The chance of a sample mean less than 145 cm if H_0 is true is 0.0092.

Reject H_0 at the 5% significance level as $p = 0.0092$, which is less than 0.05. ✓

e H_0 will not be rejected if $p > 0.05$.

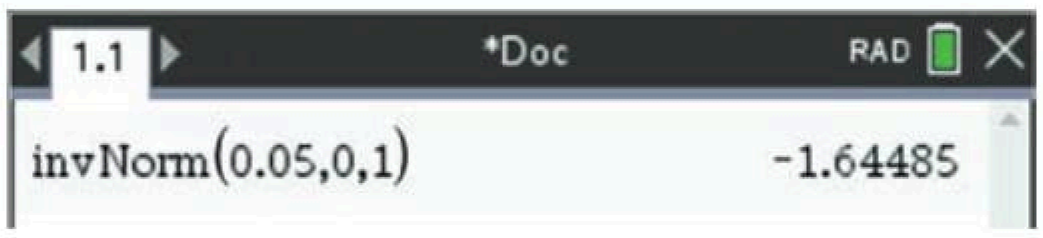
If $p = 0.05$, $z = -1.64485$

$$\frac{\overline{x} - \mu}{\sigma_{\overline{X}}} > -1.64485...$$

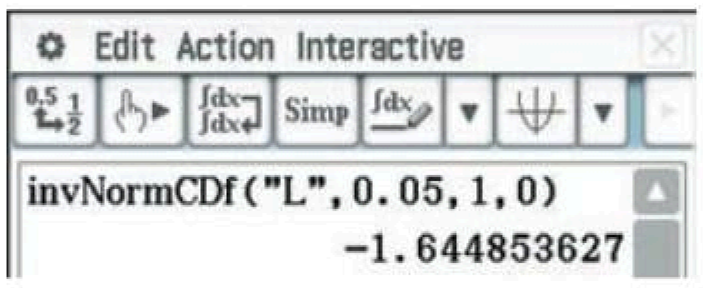
$$\frac{\overline{x} - 150}{\frac{3\sqrt{2}}{2}} > -1.64485$$

$$\overline{x} > 146.51$$

Smallest sample mean to not be rejected is **146.51 cm.** ✓



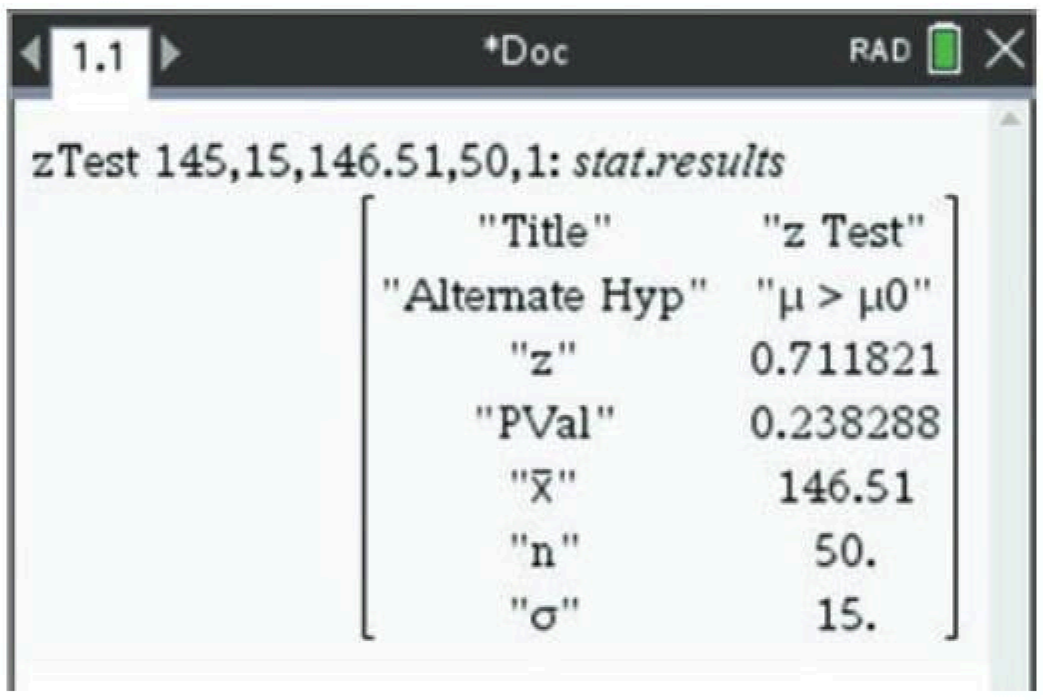
```
invNorm(0.05,0,1)                    -1.64485
```



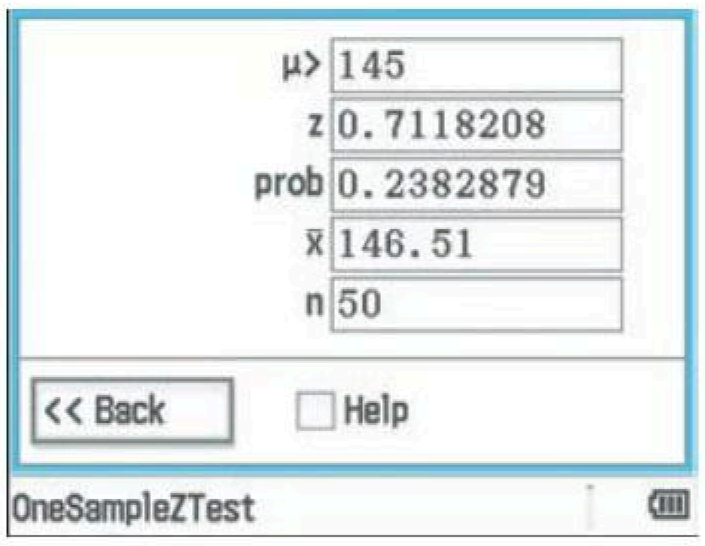
```
invNormCDF("L",0.05,1,0)
                    -1.644853627
```

f H_0 will not be rejected at the 5% significance level if $\overline{x} > 146.51$.

$\Pr\left(\overline{X} > 146.51 | \mu = 145\right) = \mathbf{0.24}$ ✓

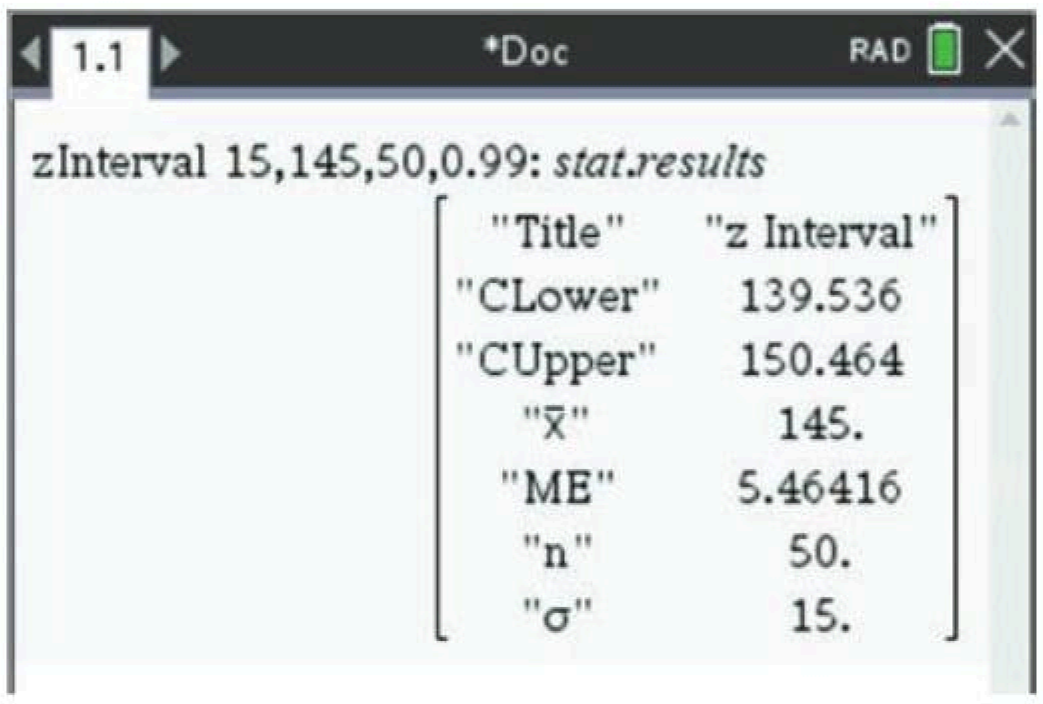


```
zTest 145,15,146.51,50,1: stat.results
          "Title"          "z Test"
          "Alternate Hyp"  "μ > μ0"
          "z"              0.711821
          "PVal"           0.238288
          "x̄"              146.51
          "n"              50.
          "σ"              15.
```

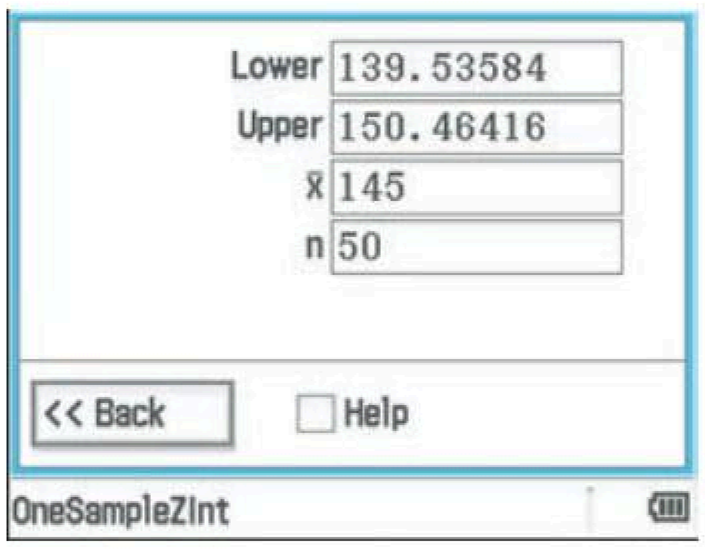


```
  μ>  145
   z  0.7118208
prob  0.2382879
   x̄  146.51
   n  50
```

g 99% CI = **(139.5, 150.5)** ✓



```
zInterval 15,145,50,0.99: stat.results
  "Title"    "z Interval"
  "CLower"    139.536
  "CUpper"    150.464
    "x̄"       145.
   "ME"       5.46416
    "n"       50.
    "σ"       15.
```



Lower 139.53584
Upper 150.46416
x̄ 145
n 50

Student performance

80–100% 60–79% 0–59%

a 70% Well done, but there was careless notation such as $H_0 = 150$. Some students did not understand deeply the nature of a one-tailed test, with answers such as H_1: $\mu \neq 150$.

b 84%

c 69% Some students did not write p to the four decimal places. Some students were careless with notation, writing calculator output rather than mathematical notation, including the incorrect $p = \Pr\left(\overline{X} < 145 \middle| \mu = 150\right)$.

d 76% Well done, but some students did not give a reason for their conclusion as required, or believed that $0.0092 > 0.05$.

e 48%

f 11% Only a few students answered this part.

g 52% This 1-mark question was well done, but calculation errors caused some students to miss out on the mark. Some students used a 95% confidence interval rather than 99%. Students are always advised to read the question carefully.

EXERCISE 12.6 Errors in hypothesis testing

ANSWERS p. 602

Recap

1 In a two-sided statistical test at the 5% level of significance, it would be concluded that

A H_0 should not be rejected if $p = 0.01$

B H_0 should be rejected if $p = 0.05$

C H_0 should be rejected if $p = 0.95$

D H_0 should not be rejected if $p \neq 0.05$

E H_0 should be rejected if $p = 0.01$

2 The curing time for a concrete slab is normally distributed with a mean of 160 hours and a standard deviation of 40 hours. A contractor claims the addition of an agent will reduce the average curing time. The mean curing time of 60 slabs that have the extra agent in the mix concrete are found to be 150 hours. A statistical test is carried out at the 5% significance level. The p value for this test is

A 0.0135 **B** 0.025 **C** 0.0264 **D** 0.0426 **E** 0.05

Mastery

3 WORKED EXAMPLE 25

a A potato farmer supplies potatoes for making chips that have a mean length of 12 cm with a standard deviation of 1 cm. An inspector takes a random sample of potatoes from the truck and measures the length of the potatoes. The chip company will reject the truck load of potatoes if the average length is less than 11.5 cm.

- **i** Write the null and alternative hypotheses.
- **ii** Describe a type I error and the impact in this situation.
- **iii** Describe a type II error and the impact in this situation.

b In the metropolitan area, the response time for an ambulance has a mean of 13.5 min and a standard deviation of 3.2 min. The director of emergency response initiates some changes to the call centre to improve the ambulance response.

- **i** Write the null and alternative hypotheses.
- **ii** Describe a type I error and the impact in this situation.
- **iii** Describe a type II error and the impact in this situation.

Exam practice

4 A type I error can occur

A when the null hypothesis is true but is not rejected.
B when the alternative hypothesis is false but is not rejected.
C when the null hypothesis is true and the alternative hypothesis is false.
D when the null hypothesis is true but is rejected.
E when null hypothesis is false but is rejected.

5 A type II error can occur

A when the null hypothesis is false but is not rejected.
B when the alternative hypothesis is false but is not rejected.
C when the null hypothesis is false and the alternative hypothesis is true.
D when the null hypothesis is true but is rejected.
E when the null hypothesis is false but is rejected.

6 (11 marks) A tyre company produces a steel belted radial tyre designed for normal road use. The life of the tyre in kilometres is a normally distributed variable with a mean of 40 000 and a standard deviation of 5000. The RACV claim that the mean life of the tyres will be improved if the maximum speed is reduced from 110 km/h to 90 km/h. They select a random sample of 100 drivers who reduce their maximum speed to 90 km/h and find the mean life of their tyres is 41 000 km.

a State the null and alternative hypotheses. 2 marks
b Find the z and p values. 2 marks
c Describe a type I error and the impact of the error in this situation. 2 marks
d Describe a type II error and the impact of the error in this situation. 2 marks
e What is the decision? 1 mark
f What is the conclusion about the claim made by the RACV? 1 mark
g What error do you run the risk of making? 1 mark

7 (9 marks) The battery life of a brand of smartphone is normally distributed with a mean of 7 h and a standard deviation of 1 h. The phone's software receives an update that claims to improve the battery life of the phone. A random sample of 50 people record the battery life of the phone and the average is found to be 7.5 h.

a State the null and alternative hypotheses. 2 marks
b Find the z and p values. 2 marks
c Describe a type I error and the impact of the error in this situation. 2 marks
d Describe a type II error and the impact of the error in this situation. 2 marks
e What is the conclusion about the claim made about the phone's software update? 1 mark

12 Chapter summary

Linear combinations of random variables

Discrete	Continuous
$\mathrm{E}(X) = \sum x \cdot p(x)$	$\mathrm{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$

A random variable Y is a linear transformation of X, where $Y = aX + b$.

$\mathrm{E}(aX + b) = a\,\mathrm{E}(X) + b$.

- The formulas for the variance and standard deviation are:

  $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2 \qquad \mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)}$

  where $\mu = \mathrm{E}(X)$.
- For two independent random variables X and Y,

  $\mathrm{E}(aX + bY) = a\mathrm{E}(X) + b\,\mathrm{E}(Y)$

  $\mathrm{Var}(aX + bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$
- For the sum of n variables of type X, $\left(\sum_{i=1}^{n} x_i\right)$ and m variables of type Y, $\left(\sum_{j=1}^{m} y_j\right)$,

  $\mathrm{E}(x_1 + x_2 + \ldots + x_n + y_1 + y_2 + y_3 + \ldots + y_m)$

  $= n\,\mathrm{E}(X) + m\mathrm{E}(Y)$

  $\mathrm{Var}(x_1 + x_2 + x_3 + \ldots + x_n + y_1 + y_2 + y_3 + \ldots + y_m)$

  $= n\,\mathrm{Var}(X) + m\mathrm{Var}(Y)$

Independent normal random variables

- If X is normally distributed with a mean of μ_X and a variance of $(\sigma_X)^2$ and Y is an independent normally distributed random variable with a mean of μ_Y and a variance of $(\sigma_Y)^2$, then $aX + bY$ is also normally distributed with

  mean $\quad \mathrm{E}(aX + b\mathrm{Y}) = a\mu_X + b\mu_Y$

  variance $\quad \mathrm{Var}(aX + b\mathrm{Y}) = a^2(\sigma_X)^2 + b^2(\sigma_Y)^2$

The sampling distribution of the sample means

- The sampling distribution of the sample means contains the means $(\bar{x})$ of all possible samples of size n from a population.
- The mean of the sample means $(\mu_{\bar{x}})$ is equal to the mean of the population (μ).

  $$\mu_{\bar{x}} = \mu$$
- The standard deviation of the sample means is the standard error of the sample mean.

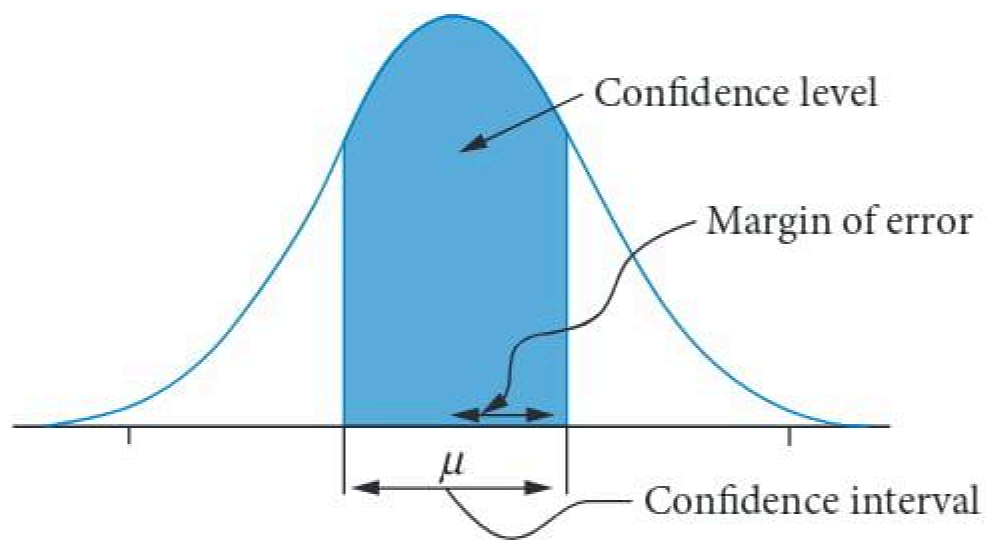
  A larger sample size produces less error and a more reliable estimate of the population mean: the sample mean is closer to the population mean and there is less variability in the means.
- The standard error for a sample mean $\mathrm{SE}(\bar{x}) = \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$, where σ is the population standard deviation and n is the sample size.

Confidence intervals for the population mean

- A **point estimate** of a parameter is a single value obtained from a sample. The statistic used is called an **estimator** of the parameter.
- An **interval estimate** of a parameter is an interval that is likely to include the value of the parameter. An interval estimate for a population parameter is called a confidence interval.
- For a confidence interval symmetric about the mean in a distribution:

  the confidence level is equivalent to the probability that the population parameter being estimated lies within the confidence interval.

  the margin of error is the distance of the ends of the interval from the mean. It is half of the confidence interval.



The central limit theorem

- If X is any random variable with mean μ and standard deviation σ, then the sampling distribution of sample means $\bar{X}$, of size n, will be approximately normal with mean $\mathrm{E}(\bar{X})$ and standard deviation $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$.
- This is the case provided n is not small. $(n \geq 30)$
- For a sample with a mean $\bar{x}$ and a standard deviation s, $\bar{x}$ is an estimator of $\mathrm{E}(\bar{X})$.
- When σ is not known, s is used as an estimator of σ.

- The standard deviation of the sampling distribution is now written as $s_{\overline{X}} = \frac{s}{\sqrt{n}}$.

Confidence intervals for the population mean and the margin of error

- The confidence interval for the population mean is $\left(\overline{x} - z\frac{\sigma}{\sqrt{n}}, \overline{x} + z\frac{\sigma}{\sqrt{n}}\right)$ when the population standard deviation σ is known.
- When σ is not known, the confidence interval is $\left(\overline{x} - z\frac{s}{\sqrt{n}}, \overline{x} + z\frac{s}{\sqrt{n}}\right)$, where s is the sample standard deviation, $\overline{x}$ is the sample mean and z is the appropriate quantile for the standard normal distribution.
- The margin of error $M = z\frac{s}{\sqrt{n}}$.

Hypothesis testing: the null and alternative hypotheses

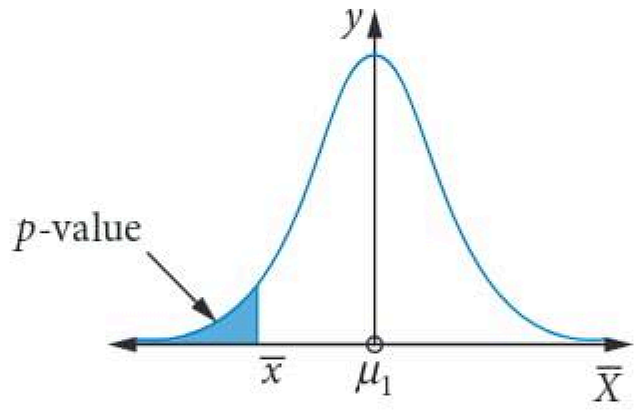
- Hypothesis testing is a systematic way to test a claim about a parameter in the population using data measured in a sample.
- H_0: The null hypothesis is a statement about a population parameter (the population mean) that is assumed to be true. We test to see if the parameter stated in the null hypothesis is likely to be true.
- H_1: The alternative hypothesis is a statement that reflects the claim being made and is the opposite of the null hypothesis. It states that the value of the population parameter is less than, greater than or not equal to the stated value in the null hypothesis.
- The p value is the probability of obtaining the value in the sample or a more extreme value, assuming that the null hypothesis is true.
- When calculating the p-value, with a one-sample z test, a cumulative normal distribution is used.

  mean = population mean μ

  standard deviation $\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}}$
- α is the significance level, the value (usually 0.05) at which the null hypothesis is rejected.
- If the p-value is less than α, the null hypothesis is rejected.
- If the p-value is greater than α, we fail to reject the null hypothesis.

One-tailed test

A < or > in the alternative hypothesis indicates that a one-tailed test is required.



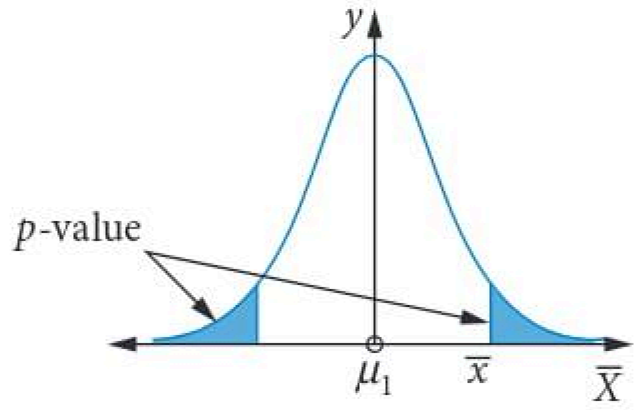
- The null hypothesis H_0: $\mu = \mu_1$
- The alternative hypothesis H_1: $\mu < \mu_1$
- p-value $= \Pr\left(\overline{X} < \overline{x} \mid \mu = \mu_1\right)$

where $\overline{x}$ is the sample mean and μ_1 is the mean of the population.

Two-tailed test

A ≠ sign in the alternative hypothesis indicates a two-tailed test is required.

- The null hypothesis H_0: $\mu = \mu_1$



- The alternative hypothesis H_1: $\mu \neq \mu_1$
- p-value $= 2 \times \Pr\left(\overline{X} > \overline{x} \mid \mu = \mu_1\right)$, for $\overline{x} > \mu_1$

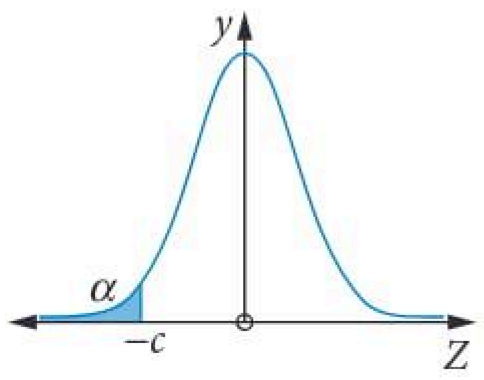
  or
- p-value $= 2 \times \Pr\left(\overline{X} < \overline{x} \mid \mu = \mu_1\right)$ for $\overline{x} < \mu_1$

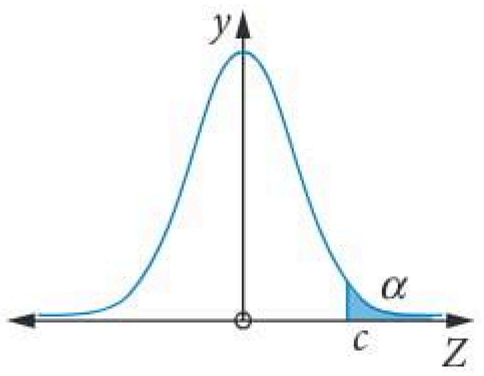
where $\overline{x}$ is the sample mean and μ_1 is the mean of the population.

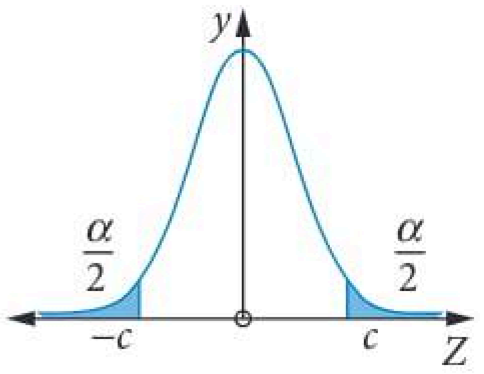
Critical values for a given significance level

A null hypothesis can also be tested by comparing a test statistic (z) with a critical value determined from the significance level α.

If the test statistic is more extreme than the critical value, the null hypothesis is rejected.







Left tail	
α	$z = -c$
0.01	−2.326
0.05	−1.645
0.1	−1.282

Right tail	
α	$z = c$
0.01	2.326
0.05	1.645
0.1	1.282

Two tail	
α	$z = \pm c$
0.01	± 2.576
0.05	± 1.960
0.1	± 1.645

When σ is known, the z value for the sample statistic is given by

$$z = \frac{\bar{x} - \mu}{\sigma_{\bar{X}}} \text{ where } \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$$

When σ is not known, use $s_{\bar{X}} = \frac{s}{\sqrt{n}}$ in place of $\sigma_{\bar{X}}$.

Errors in hypothesis testing

- A type I error occurs when we reject the null hypothesis and it is true.
- The probability of making this type of error is the α value as this is the level at which the null hypothesis is rejected.
- A type II error occurs when we fail to reject the null hypothesis and it is false.

	H_0 True	H_0 False
Reject H_0	Type I error	Correct rejection
Fail to reject H_0	Correct decision	Type II error

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 © VCAA 2018 1Q4 (4 marks) X and Y are independent random variables. The mean and the variance of X are both 2, while the mean and the variance of Y are 2 and 4, respectively.

Given that a and b are integers, find the values of a and b if the mean and the variance of $aX + bY$ are 10 and 44, respectively.

2 (4 marks)

a Show that $1 - i = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$. 1 mark

b Evaluate $\frac{\left(\sqrt{3}+i\right)^{10}}{(1-i)^{12}}$, giving your answer in the form $a + bi$, where $a, b \in R$. 3 marks

3 (2 marks) Find the volume of the solid of revolution formed when the region bounded by the curve $\frac{x^2}{4} + y^2 = 1$ and the x-axis is rotated about the x-axis.

Cumulative examination 2

Total number of marks: 20   Reading time: 4 minutes   Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 © VCAA 2018 2AQ19 The gestation period of cats is normally distributed with mean $\mu = 66$ days and variance $\sigma^2 = \frac{16}{9}$. The probability that a sample of five cats chosen at random has an average gestation period greater than 65 days is closest to

A 0.5000 **B** 0.7131 **C** 0.7734 **D** 0.8958 **E** 0.9532

2 © VCAA 2018 2AQ20 The scores on the Mathematics and Statistics tests, expressed as percentages, in a particular year were both normally distributed. The mean and the standard deviation of the Mathematics test scores were 71 and 10, respectively, while the mean and the standard deviation of the Statistics test scores were 75 and 7, respectively.

Assuming the sets of test scores were independent of each other, the probability, correct to four decimal places, that a randomly chosen Mathematics score is higher than a randomly chosen Statistics score is

A 0.2877 **B** 0.3716 **C** 0.4070 **D** 0.7123 **E** 0.9088

3 The curve given by $x = 2\cos(t) - 1$ and $y = 3\sin(t) + 1$ can be expressed in cartesian form as

A $\frac{(x+1)^2}{4} + \frac{(y-1)^2}{9} = 1$

B $\frac{(x-1)^2}{4} + \frac{(y+1)^2}{9} = 1$

C $\frac{(x+1)^2}{4} - \frac{(y-1)^2}{9} = 1$

D $\frac{(x+1)^2}{2} + \frac{(y-1)^2}{3} = 1$

E $\frac{(y+1)^2}{4} - \frac{(x-1)^2}{9} = 1$

4 $P(z)$ is a polynomial of degree n with real coefficients, where $z \in C$. Three of the roots of the equation $P(z) = 0$ are $z = 3 - 2i$, $z = 4 + 3i$ and $z = -5i$. The smallest possible value of n is

A 3 **B** 4 **C** 5 **D** 6 **E** 7

5 Given that θ is the acute angle between the vectors $\underset{\sim}{a} = 2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = 4\underset{\sim}{i} + 4\underset{\sim}{j} - 7\underset{\sim}{k}$, then sin (θ) is equal to

A $\frac{1}{3}$ **B** $\frac{2\sqrt{2}}{9}$ **C** $\frac{2\sqrt{2}}{3}$ **D** $\frac{\sqrt{2}}{3}$ **E** $\frac{2}{3}$

Section B 2 questions

15 marks

1 © VCAA 2018N 2BQ6 (6 marks) A coffee machine dispenses coffee concentrate and hot water into a 200 mL cup to produce a long black coffee. The volume of coffee concentrate dispensed varies normally with a mean of 40 mL and a standard deviation of 1.6 mL.

Independent of the volume of coffee concentrate, the volume of water dispensed varies normally with a mean of 150 mL and a standard deviation of 6.3 mL.

a State the mean and the standard deviation, in millilitres, of the total volume of liquid dispensed to make a long black coffee. 2 marks

b Find the probability that a long black coffee dispensed by the machine overflows a 200-mL cup. Give your answer correct to three decimal places. 2 marks

c Suppose that the standard deviation of the volume of water dispensed by the machine can be adjusted, but that the mean volume of water dispensed and the standard deviation of the volume of coffee concentrate dispensed cannot be adjusted.

Find the standard deviation of the volume of water dispensed that is needed for there to be only a 1% chance of a long black coffee overflowing a 200-mL cup. Give your answer in millilitres, correct to two decimal places. 2 marks

2 © VCAA 2016 2BQ1 (9 marks)

a Find the stationary point of the graph of $f(x) = \dfrac{4 + x + x^3}{x}$, $x \in R \setminus \{0\}$. Express your answer in coordinate form, giving values correct to two decimal places. 1 mark

b Find the point of inflection of the graph given in part **a**. Express your answer in coordinate form, giving values correct to two decimal places. 2 marks

c Sketch the graph of $f(x) = \dfrac{4 + x + x^3}{x}$ for $x \in [-3, 3]$, labelling the turning point and the point of inflection with their coordinates, correct to two decimal places. 3 marks

A glass is to be modelled by rotating the curve that is the part of the graph where $x \in [-3, -0.5]$, about the y-axis, to form a solid of revolution.

d **i** Write down a definite integral, in terms of x, which gives the length of the curve to be rotated. 1 mark

ii Find the length of this curve, correct to two decimal places. 1 mark

e The volume of the solid formed is given by $V = a\int_c^b x^2\, dy$. Find the values of a, b and c. Do **not** attempt to evaluate this integral. 1 mark

Answers

CHAPTER 1

EXERCISE 1.1

1 $\overrightarrow{AB} = \frac{2}{3}(-\underset{\sim}{d} + \underset{\sim}{c})$

2 $\frac{1}{\sqrt{11}}(3\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k})$

3 **a** $\sqrt{29}$ **b** $\hat{\underset{\sim}{a}} = \frac{1}{\sqrt{29}}(-3\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k})$

4 **a** $\overrightarrow{AB} = \underset{\sim}{i} - 3\underset{\sim}{j} + 9\underset{\sim}{k}$ **b** $\sqrt{91}$

c $\hat{\underset{\sim}{P}} = \frac{1}{\sqrt{14}}(-3\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k})$ **d** $\underset{\sim}{q} = \frac{1}{\sqrt{3}}(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})$

5 A **6** D **7** B

8 B **9** B

10 $\hat{\underset{\sim}{a}} = \frac{1}{\sqrt{6}}(\sqrt{3}\underset{\sim}{i} - \underset{\sim}{j} - \sqrt{2}\underset{\sim}{k})$

11 $\sqrt{3}$ **12** D **13** A

EXERCISE 1.2

1 **a** $\begin{bmatrix} 4 \\ 8 \\ -7 \end{bmatrix}$ **b** $\begin{bmatrix} -2 \\ 12 \\ 13 \end{bmatrix}$ **c** -47

d $\sqrt{113}$ **e** $\frac{1}{\sqrt{110}}\begin{bmatrix} 1 \\ 10 \\ 3 \end{bmatrix}$

2 C

3 linearly dependent

4 linearly dependent

5 **a** $\begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix}$ **b** $\sqrt{3} + \sqrt{26}$ **c** $2\sqrt{33}$

6 not linearly dependent

7 B **8** 16 **9** E **10** E

EXERCISE 1.3

1 linearly dependent

2 C

3 **a** $\underset{\sim}{a} + \underset{\sim}{b} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}$

b $2\underset{\sim}{a} - 3\underset{\sim}{b} = \underset{\sim}{i} + \underset{\sim}{j} - 6\underset{\sim}{k}$

4 $\overrightarrow{OM} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$

5 $\overrightarrow{AB} = -\ 6\underset{\sim}{i} - \underset{\sim}{j} + 9\underset{\sim}{k}$

6 $\sqrt{3}\underset{\sim}{i} + \underset{\sim}{j}$

7 **a** $\underset{\sim}{b}| = \sqrt{6}$ **b** 66°, 114°, 35°

8 A

9 E

10 **a** $\hat{\underset{\sim}{a}} = \frac{1}{\sqrt{6}}(\sqrt{3}\underset{\sim}{i} - \underset{\sim}{j} - \sqrt{2}\underset{\sim}{k})$ **b** $\frac{\pi}{4} = 45°$

11 **a** $3\underset{\sim}{i} - 5\underset{\sim}{j} - 3\underset{\sim}{k}$ **b** $-2\underset{\sim}{i} + 2\underset{\sim}{k}$

c $-3\underset{\sim}{i} + 2\underset{\sim}{k}$

EXERCISE 1.4

1 **a** $\begin{bmatrix} 1 \\ -4 \\ -17 \end{bmatrix}$ **b** $\begin{bmatrix} 2 \\ 4 \\ 26 \end{bmatrix}$ **c** -30

d 71.57° **e** $2\sqrt{26}$ **f** $\frac{1}{\sqrt{26}}\begin{bmatrix} 0 \\ -1 \\ -10 \end{bmatrix}$

2 D

3 **a** -1 **b** **i** $p = \frac{3}{4}$ **ii** $p = -3$

4 $\cos^{-1}\left(\frac{\sqrt{2}}{10}\right)$

5 parallel component $\frac{1}{17}(3\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k})$,

perpendicular component $\frac{1}{17}(14\underset{\sim}{i} - 36\underset{\sim}{j} + 15\underset{\sim}{k})$

6 A **7** C **8** C

9 **a** $\hat{\underset{\sim}{a}} = \frac{1}{\sqrt{6}}(\sqrt{3}\underset{\sim}{i} - \underset{\sim}{j} - \sqrt{2}\underset{\sim}{k})$ **b** $\frac{\pi}{4} = 45°$

c $6 + 5\sqrt{2}$

10 $a = -2$

11 $c = -3$

12 $m = 2, m = -\frac{22}{3}$

13 B **14** C **15** A

16 B **17** D **18** B

19 C **20** A **21** D

22 **a** Proof: see worked solutions **b** $\cos\angle ADC = \frac{4}{5}$

c Proof: see worked solutions

d Proof: see worked solutions

$\cos(\angle APC) = \cos(2\angle ADC) = \frac{7}{25}$

EXERCISE 1.5

1 **a** $-\frac{12}{\sqrt{21}}$ **b** $\frac{2}{17}(2\underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k})$ **c** $-\frac{\sqrt{6}}{42}$

2 **a** $-\frac{2}{\sqrt{5}}$ **b** $\frac{4-n}{9}(2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k})$

c $n = -5$ **d** $n = 4$

3 **a** $\underset{\sim}{a} \times \underset{\sim}{b} = 8\underset{\sim}{k}$ **b** $\underset{\sim}{a} \times \underset{\sim}{b} = -10\underset{\sim}{i} + 14\underset{\sim}{j} + 2\underset{\sim}{k}$

4 $(-55, 22, 11)$

5 $7\underset{\sim}{i} - 19\underset{\sim}{j} - 11\underset{\sim}{k}$

6 **a** $(9, 14, 3)$ **b** $(-9, -14, -3)$

7 $18\underset{\sim}{i} + 42\underset{\sim}{j} + 5\underset{\sim}{k}$

8 $(18, -6, -12)$

9 $(-22, -31, -26)$

10 C **11** B **12** C **13** C

EXERCISE 1.6

1 **a** $-2\underset{\sim}{i} - 11\underset{\sim}{j} - 5\underset{\sim}{k}$ **b** $7\underset{\sim}{i} + 13\underset{\sim}{j} - 4\underset{\sim}{k}$

2 **a** $-2\underset{\sim}{i} - \underset{\sim}{j} - 6\underset{\sim}{k}$ **b** $(2n - 1)\underset{\sim}{i} - (2n + 3)\underset{\sim}{j} - 8\underset{\sim}{k}$

3 $\frac{1}{\sqrt{19}}(3\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k})$

4 $\sqrt{\frac{21}{19}}\left(-3\underset{\sim}{i}-\underset{\sim}{j}+3\underset{\sim}{k}\right)$

5 C

6 **a** $\overrightarrow{AB}=\frac{7}{4}\underset{\sim}{i}-\frac{3}{4}\underset{\sim}{j}$ **b** $M\left(\frac{7}{8},-\frac{3}{8}\right)$

7 **a** E **b** E

8 **a** C **b** A **c** B

9 A **10** B

11 **a** $m=4$ **b** $\frac{47}{18}\underset{\sim}{i}-\frac{5}{9}\underset{\sim}{j}+\frac{7}{18}\underset{\sim}{k}$

12 E **13** D **14** A

EXERCISE 1.7

1 **a** $\sqrt{89}$ **b** $3\sqrt{10}$ **c** $\sqrt{41}$

2 **a** $M\left(-\frac{3}{2},-1\right)$ **b** $M\left(\frac{1}{2},\frac{3}{2}\right)$ **c** $M\left(3,-\frac{5}{2}\right)$

3, 4, 5 Proof: see worked solutions

6 C **7** E **8** D **9** B

10 C **11** C

12 **a** $\overrightarrow{AB}=2\underset{\sim}{i}-2\underset{\sim}{j}+\underset{\sim}{k}$

b Proof: see worked solutions

c $\sqrt{38}$

13 Proof: see worked solutions

14 **a** $a=\sqrt{3}$ **b** Proof: see worked solutions

15 C **16** A **17** D **18** B

19 **a** **i** $\frac{1}{2}a$ **ii** $\underset{\sim}{a}-\underset{\sim}{b}$ **iii** $\frac{1}{2}\underset{\sim}{b}-\frac{1}{2}\underset{\sim}{a}$

b $\overline{MN}=\frac{1}{2}\underset{\sim}{b}-\frac{1}{2}\underset{\sim}{a}=\overline{AQ}, \overline{NQ}=\frac{1}{2}\underset{\sim}{b}+\frac{1}{2}(\underset{\sim}{a}-\underset{\sim}{b})=\frac{1}{2}\underset{\sim}{a}=\overline{MA}$

c $\left|\overline{OA}\right|=\sqrt{9+4+3}=4$

d **i** $\overline{OQ}=\frac{7}{2}\underset{\sim}{i}+\underset{\sim}{j}+\frac{\sqrt{3}}{2}\underset{\sim}{k}$ or $\frac{1}{2}(\underset{\sim}{a}+\underset{\sim}{b})$

ii $\left|\overline{OA}\right|=\left|\overline{OB}\right|, \overline{OQ}\cdot\overline{AB}=\frac{7}{2}-2-\frac{3}{2}=0$

or $\overline{AB}\cdot\overline{AB}=\frac{1}{2}(\underset{\sim}{b}\cdot\underset{\sim}{b}-\underset{\sim}{a}\cdot\underset{\sim}{a})=\frac{1}{2}\left(|\underset{\sim}{b}|^2-|\underset{\sim}{a}|^2\right)=0$

20 **a** Proof: see worked solutions

b 109.5°

c **i** $p=\frac{1}{\sqrt{2}}, r=-\frac{1}{\sqrt{2}}$ **ii** $\frac{10}{3\sqrt{2}}$ or $\frac{5\sqrt{2}}{3}$

CUMULATIVE EXAMINATION 1

1 $a=-2$

2 **a** $\frac{\pi}{6}$ **b** 1

CUMULATIVE EXAMINATION 2

Section A

1 C 80% **2** A 36% **3** C 76%

4 C 78% **5** B 71%

Section B

1 **a** $\hat{\underset{\sim}{b}}=\frac{1}{4}\left(\underset{\sim}{i}+\sqrt{3}\underset{\sim}{j}+2\sqrt{3}\underset{\sim}{k}\right)$ 80%

b parallel component $=-\frac{\sqrt{3}}{3}\underset{\sim}{i}-\underset{\sim}{j}-2\underset{\sim}{k}$ 56%

perpendicular component $=-2\sqrt{3}\underset{\sim}{i}+2\underset{\sim}{j}$ 56%

c $m=\frac{\sqrt{3}}{3}$ 74%

d $\theta=75.5°$ 58%

e **i** $\overline{AN}=\underset{\sim}{u}+\frac{1}{2}\underset{\sim}{v}$ 92%

ii $\overrightarrow{BP}=-\frac{1}{2}\underset{\sim}{u}-\underset{\sim}{v}$ 75%

iii $\underset{\sim}{0}$ 10%

2 **a** $\overrightarrow{MA}=\frac{1}{2}\underset{\sim}{a}$ 96%

b $\overrightarrow{BA}=\underset{\sim}{a}-\underset{\sim}{b}$ 94%

c $\overrightarrow{AQ}=\frac{1}{2}\underset{\sim}{b}-\frac{1}{2}\underset{\sim}{a}$ 80%

CHAPTER 2

EXERCISE 2.1

1 **a** $y=\frac{1}{2}x-\frac{1}{2}$; a straight line

b $(x-6.75)^2+(y-3.5)^2=7.3125$; a circle

c $\frac{(y+1)^2}{45}-\frac{(x-4)^2}{45}=1$; a vertical hyperbola

2 **a** $\frac{(x-6)^2}{16}-\frac{(y-4)^2}{9}=1$ **b** 2

3 **a** $\frac{(y+5\sqrt{15}-7)^2}{400}+\frac{(x-2)^2}{25}=1$ **b** 4 (by graphs)

4 **a** $\begin{cases}x=6t-5\\y=3t^2+4\end{cases}$ **b** $\begin{cases}x=6\cos(t)-2\\y=6\sin(t)+6\end{cases}$

c $\begin{cases}x=5\cos(t)+4\\y=7\sin(t)-2\end{cases}$

d $\begin{cases}x=4\sqrt{2}\sec(t)-5\\y=4\sqrt{2}\tan(t)+4\end{cases}$ or $\begin{cases}x=4\sqrt{2}\operatorname{cosec}(t)-5\\y=5\sqrt{2}\cot(t)+4\end{cases}$

e $\begin{cases}x=4\cos(t)-1\\y=\sin(t)\end{cases}$

f $\begin{cases}y=20\sec(t)-2\\x=16\tan(t)+6\end{cases}$ or $\begin{cases}y=20\operatorname{cosec}(t)-2\\x=16\cot(t)+6\end{cases}$

5 **a** $\frac{(x-2)^2}{25}+\frac{(y+4)^2}{9}=1$

b $\frac{(x+1)^2}{4}-\frac{(y-3)^2}{16}=1$

c $\frac{y^2}{25}-\frac{(x-2)^2}{9}=1$

d $(x+3)^2+(y-7)^2=36$

e $\frac{(x-3)^2}{16}+\frac{(y-1)^2}{36}=1$

f $(y-1)^2=-(x-3)$

6 **a** $r = \dfrac{18}{5 - 4\sin(\theta)}$

b $r = \dfrac{4}{1 - \cos(\theta)}$

c $r = \dfrac{9}{4 + 5\sin(\theta)}$

d $r = \dfrac{7}{4 + 3\cos(\theta)}$

e $r = 8\sin(\theta)$

f $r = \dfrac{96}{5 - 11\cos(\theta)}$

7 **a** $= \dfrac{(x-7)^2}{36} - \dfrac{y^2}{13} = 1$

b $\dfrac{x^2}{13} + \dfrac{(y+6)^2}{49} = 1$

c $12y = 36 - x^2$

d $(x+5)^2 + y^2 = 25$

e $\dfrac{x^2}{45} + \dfrac{(y+6)^2}{81} = 1$

f $\dfrac{(y+5)^2}{25} - \dfrac{x^2}{75} = 1$

8 **a** Substitute into the identity $\tan^2(t) + 1 = \sec^2(t)$.

b $(1, 0), (3, 2)$

9 E **10** E **11** B **12** D **13** D

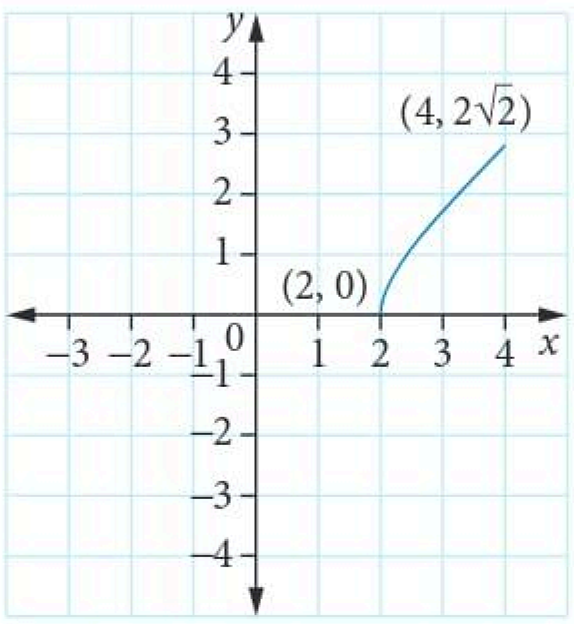
14 **a** Square $(x - 1)$ and y and use $\tan^2(t) + 1 = \sec^2(t)$ to get $y^2 + 1 = (x - 1)^2$.

Rearrange to get $y = \sqrt{x^2 - 2x}$, $(y \geq 0)$.

b $x \in [2, \infty), y \in [0, \infty)$

c **i** $\dfrac{1}{\sin(t)}$ **ii** 1

d



EXERCISE 2.2

1 C **2** A

3 **a** −3, 1 **b** 2, 3 **c** $-\dfrac{2}{3}, -3$

d −4, −3, 3 **e** $-2, -\dfrac{3}{2}, 4$

f $2, \dfrac{5}{2}, \dfrac{7}{2}$

4 **a** $\dfrac{5}{2}$ **b** 2, 5 **c** $-2, \dfrac{3}{2}$

d $-\sqrt{7}, \sqrt{7}, 3$ **e** −1, −0.5, 4

f $-1, \dfrac{2}{3}, \dfrac{3}{2}$

5 **a** $3, -5x - 7$ **b** $\dfrac{5}{2}, -\dfrac{11}{2}$

c $3x + 5, 4$ **d** $x^2 - 4x + 15, -44$

e $2x + 3, x + 7$ **f** $x - 5, 16x + 9$

6 **a** $x^2 - 5x - 4, -8$

b $3x^3 - x^2 + 4x + 1, -7$

c $x^2 + 4x + 6, 4x - 3$

d $6x^3 - 24x^2 - 43x + 112, 216x - 450$

e $3x - x^2 + 4, 4x + 5$ **f** $x^2 - x - 4, 1 - 4x^2$

7 **a** $\dfrac{A}{x+1} + \dfrac{B}{2x+1} + \dfrac{C}{x+4}$ **b** $\dfrac{A}{3x-2} + \dfrac{B}{2x+3}$

c $\dfrac{A}{2x+3} + \dfrac{Bx+C}{3x^2-4x+5}$

d $\dfrac{A}{(x-2)^2} + \dfrac{B}{x-2} + \dfrac{C}{3x-4} + \dfrac{D}{4x+1}$

e $\dfrac{A}{x+1} + \dfrac{B}{(2x+1)^2} + \dfrac{C}{2x+1} + \dfrac{D}{3x+1}$

f $\dfrac{A}{(2x-1)^2} + \dfrac{B}{2x-1} + \dfrac{C}{2x+1} + \dfrac{Dx+E}{3x^2+3x+2}$

8 **a** $\dfrac{2}{2x-5} - \dfrac{1}{x-1}$ **b** $\dfrac{1}{x+2} + \dfrac{3}{x-1}$

c $\dfrac{5}{x+3} - \dfrac{2}{2x-1}$

d $\dfrac{3}{(x-1)^2} - \dfrac{1}{x-1} + \dfrac{2}{2x+1}$

e $\dfrac{2}{(x+3)^2} - \dfrac{1}{x+3} + \dfrac{3}{x-2}$

f $\dfrac{1}{x+2} + \dfrac{-x+3}{x^2+x+3}$

9 **a** $\dfrac{3}{(x-2)^2} + \dfrac{1}{x-2} + \dfrac{5}{x+7}$

b $\dfrac{4}{x+2} - \dfrac{6}{x-5} + \dfrac{3}{x+1}$

c $\dfrac{1}{x+2} + \dfrac{2}{x+4} - \dfrac{3x-1}{x^2-5x+7}$

d $\dfrac{5-2x}{2x^2+3x+2} + \dfrac{x-7}{x^2-2x+8}$

e $\dfrac{3}{x-3} + \dfrac{1}{(x-3)^2} - \dfrac{4}{(x-3)^3} + \dfrac{6}{x+5}$

f $\dfrac{1}{(3x+4)^2} - \dfrac{2}{3x+4} + \dfrac{3}{x-5} - \dfrac{3x+5}{5x^2-6x+4}$

10 $x^2 - 6x + 16 + \dfrac{2}{x+1} - \dfrac{37}{x+2}$

11 Change the RHS to a common denominator.

12 Change the fractions to a common denominator.

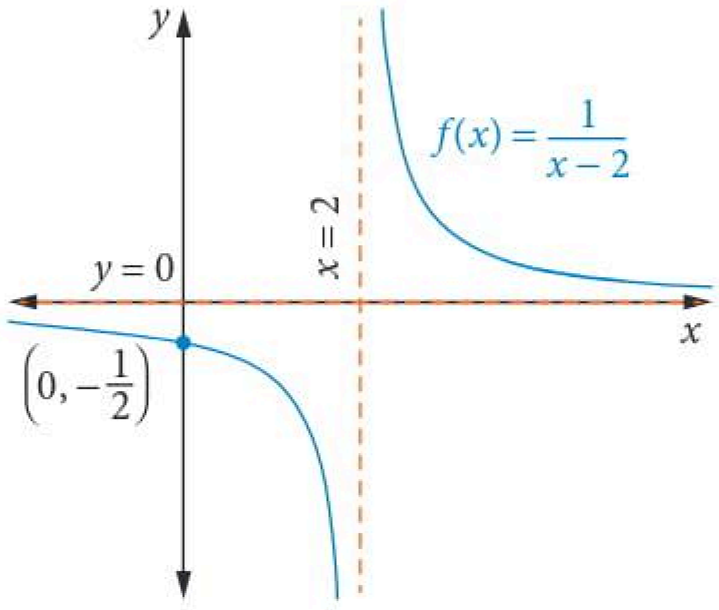
13 A **14** B **15** D **16** D

17 $\dfrac{1}{(x-3)^2} + \dfrac{1}{x-3} - \dfrac{2}{2x+1}$

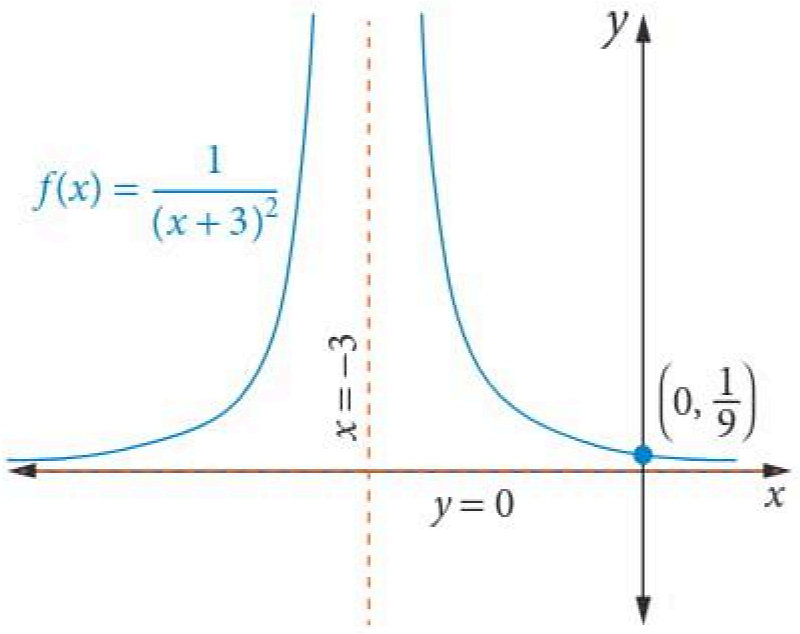
EXERCISE 2.3

1 E **2** C

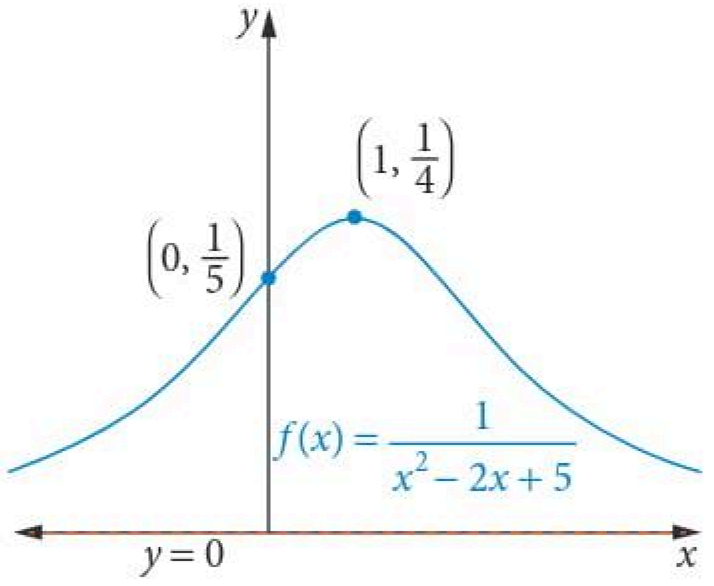
3 **a**



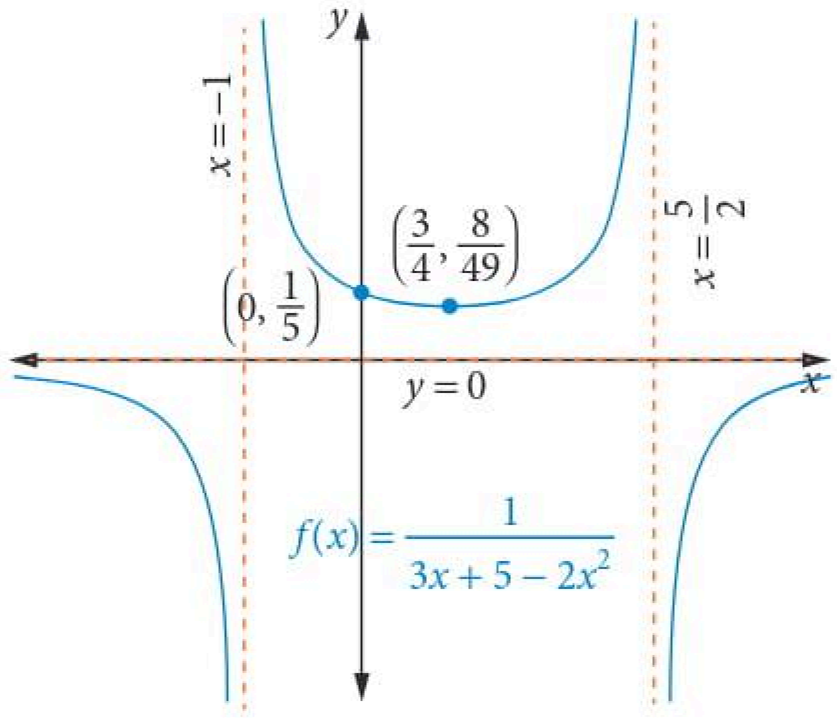
b



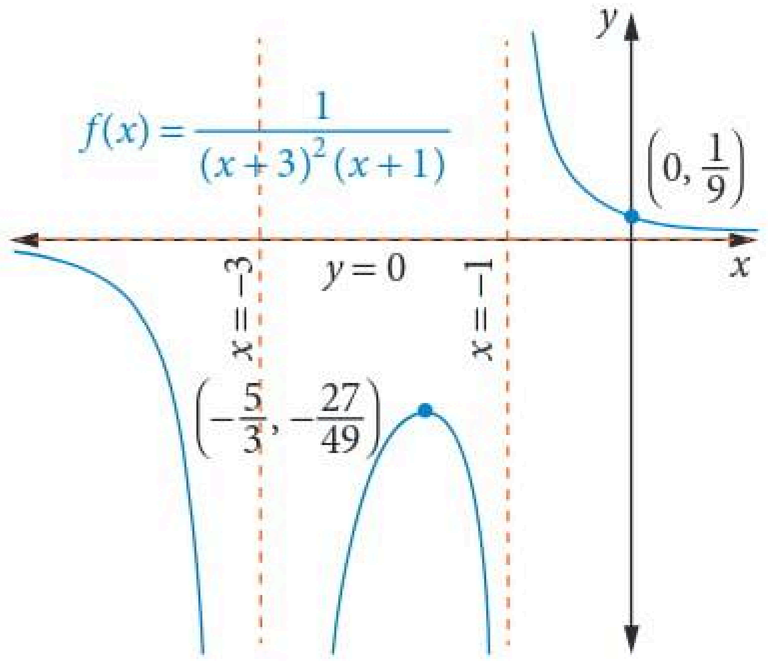
c



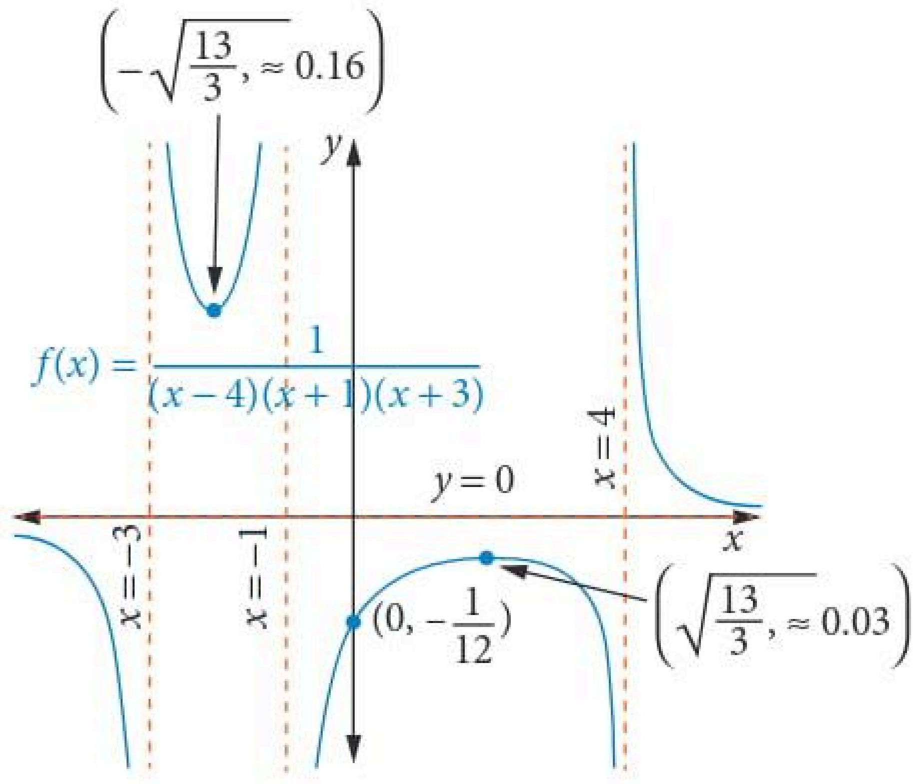
d



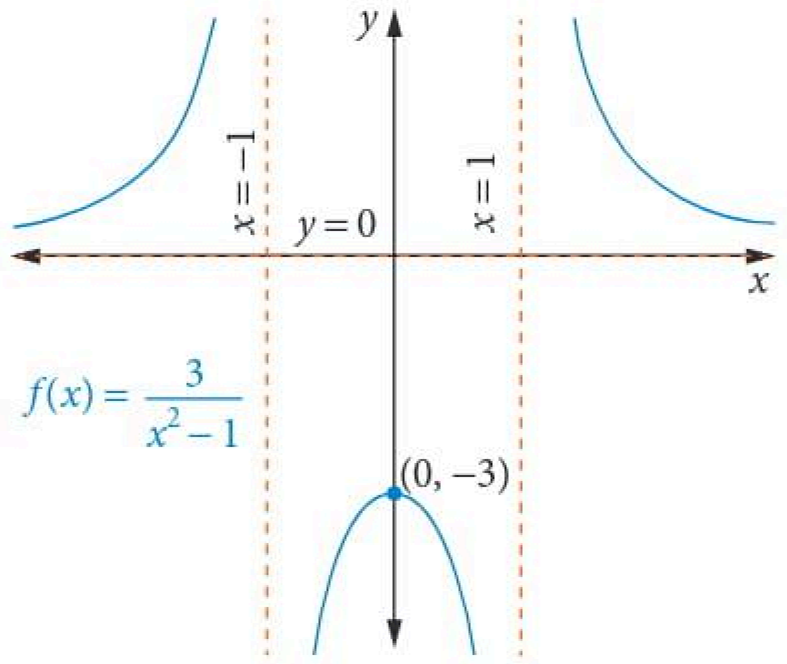
e



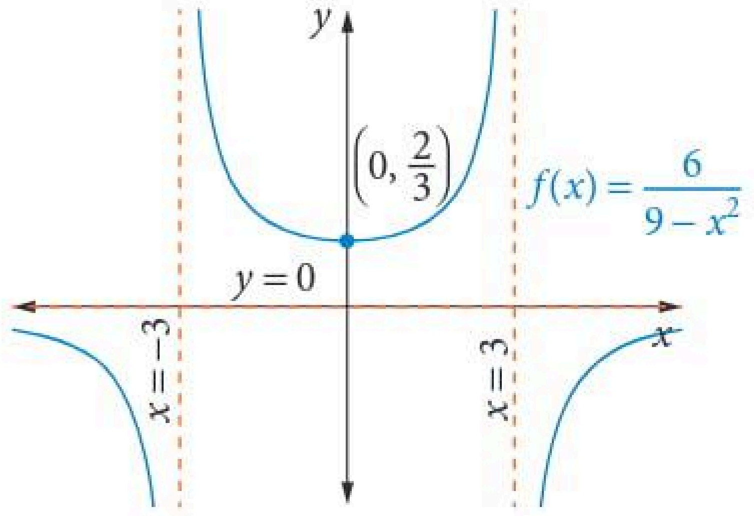
f



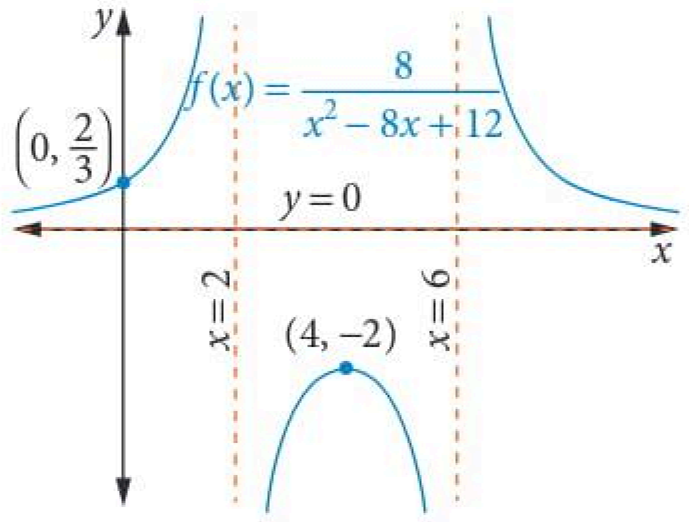
4 a



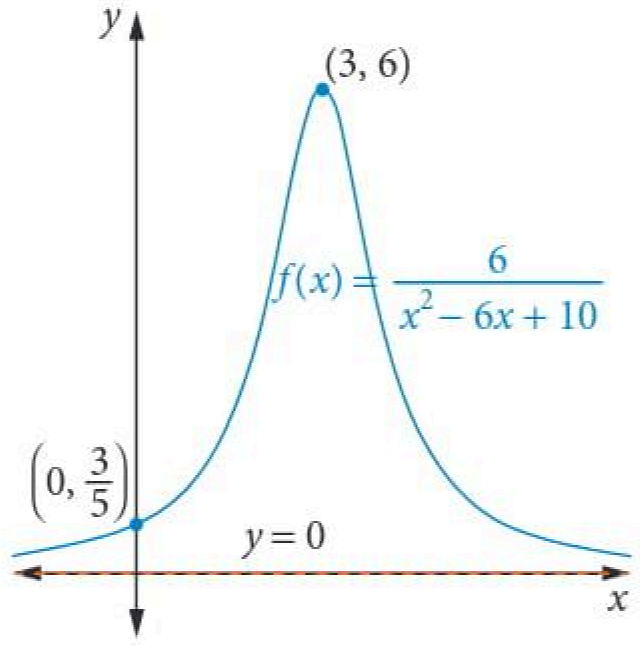
b



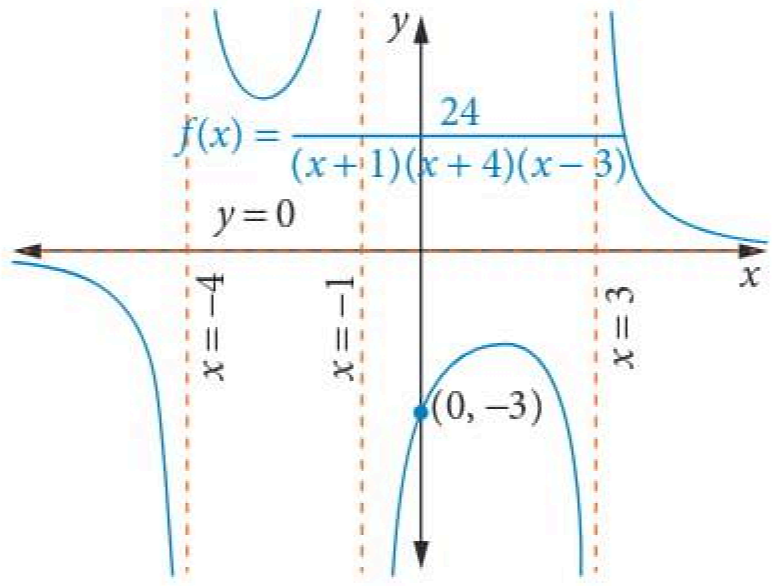
c



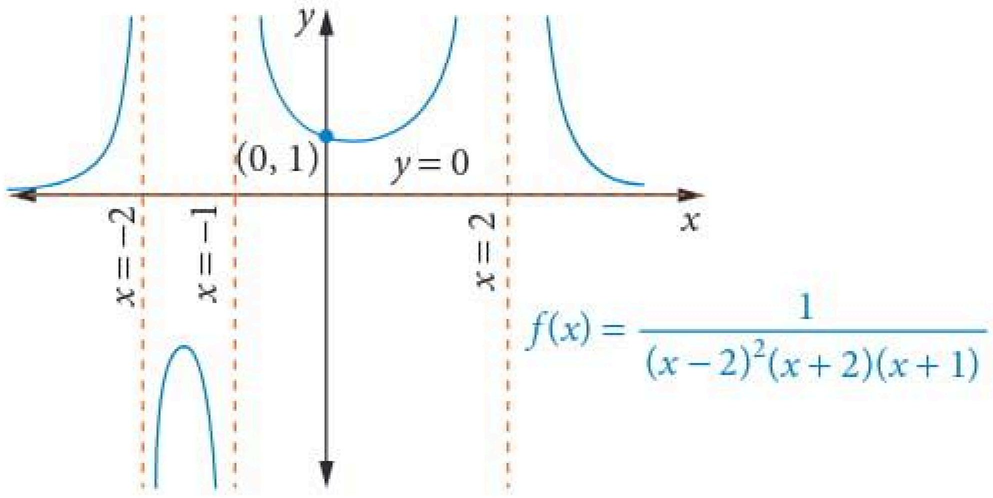
d



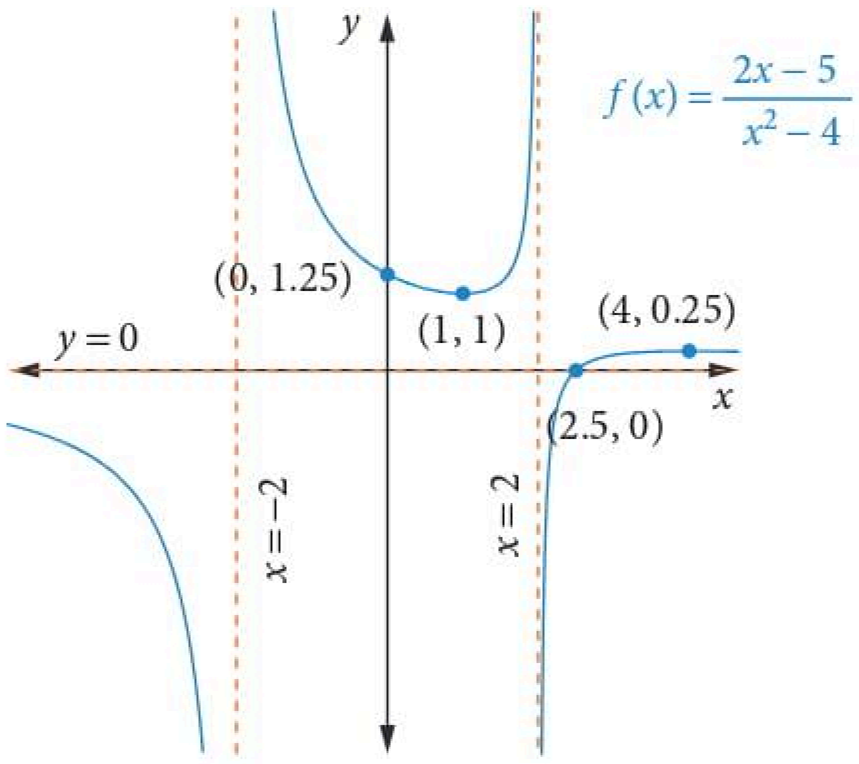
e Minimum and maximum at $x = \frac{-(\sqrt{37}+2)}{3}$ and $x = \frac{\sqrt{37}-2}{3}$



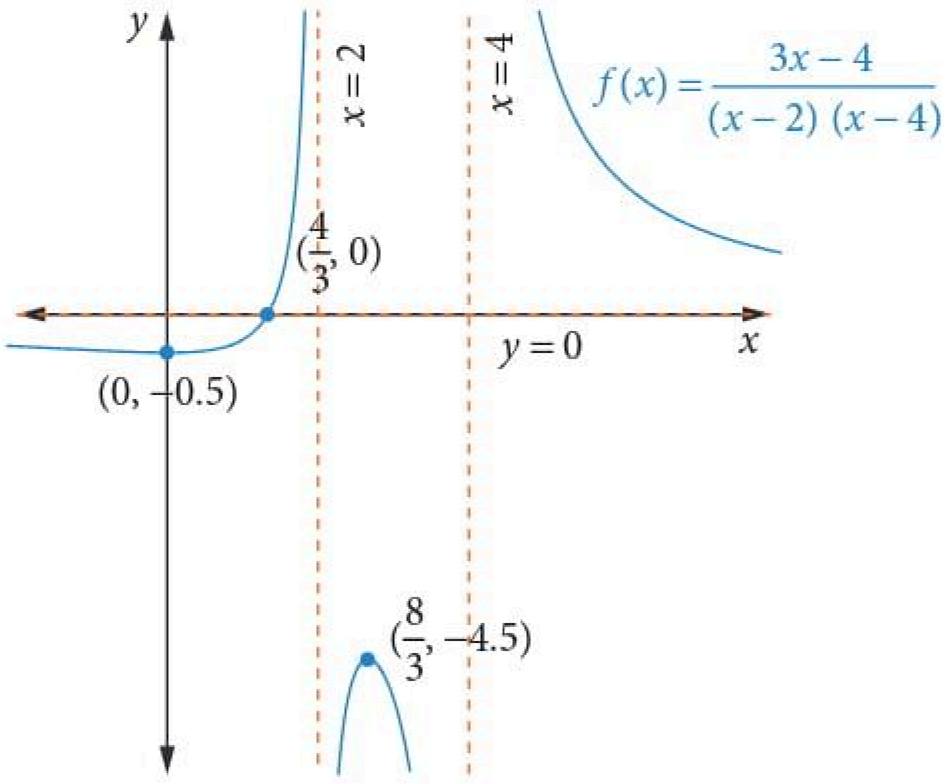
f Minimum and maximum at $x = \frac{-(\sqrt{57}+5)}{8}$ and $x = \frac{\sqrt{57}-5}{8}$



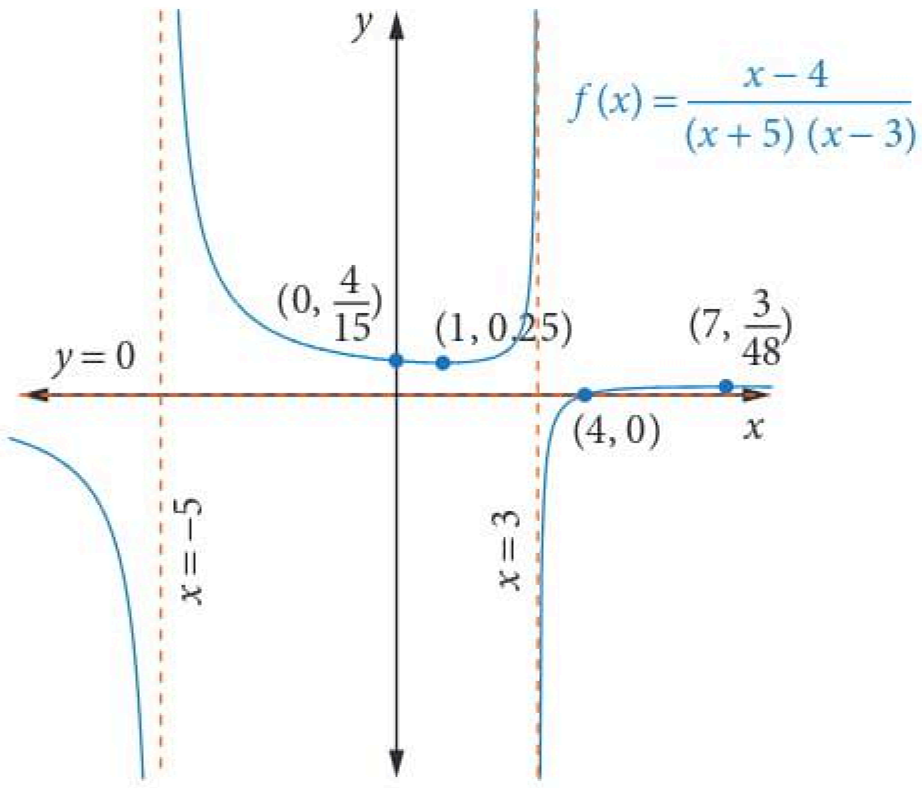
5 a



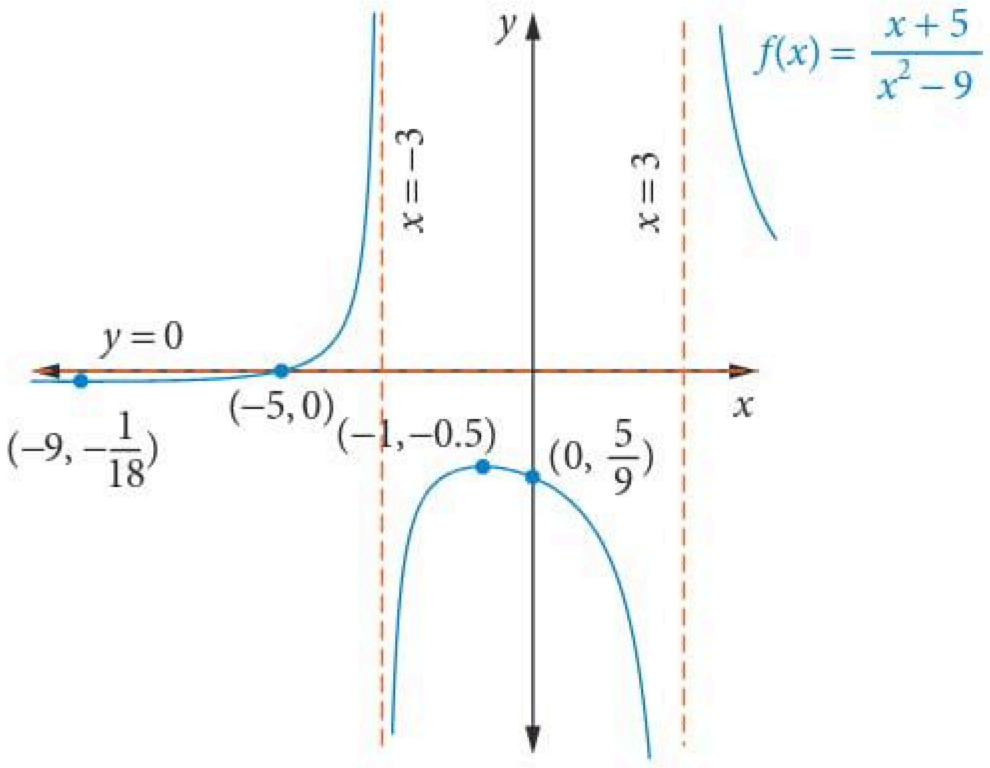
b



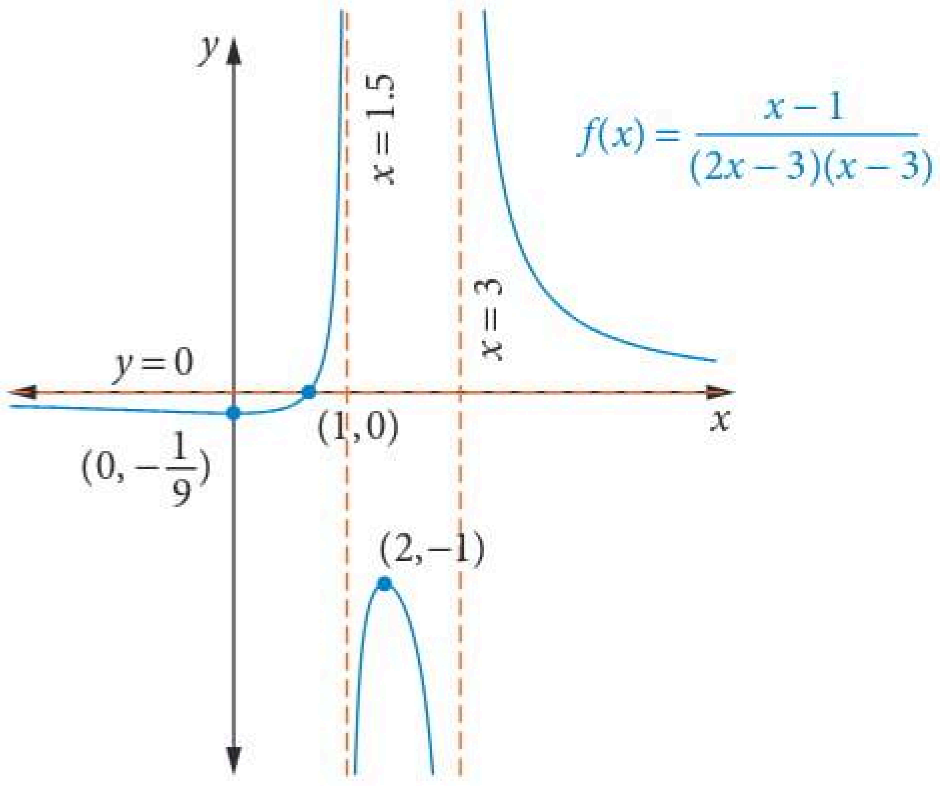
c



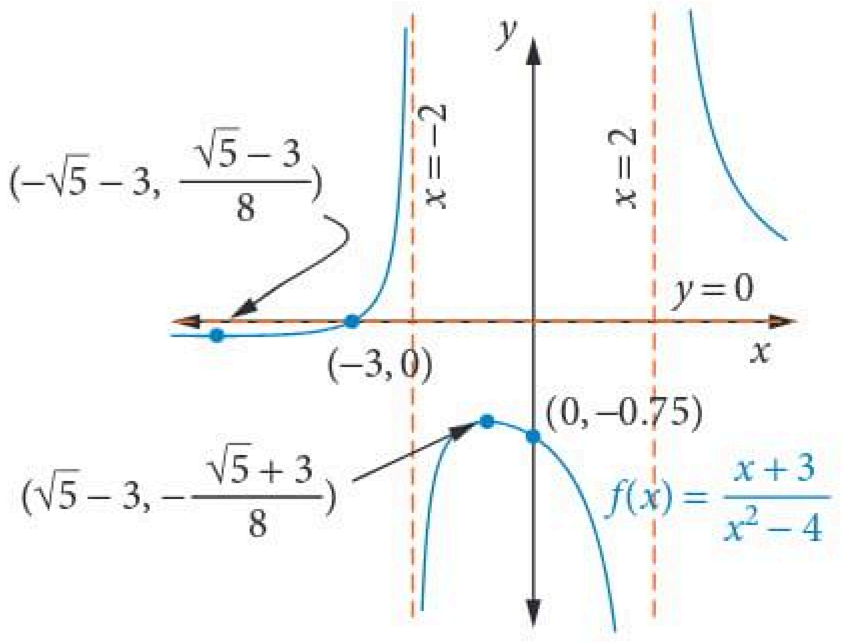
d



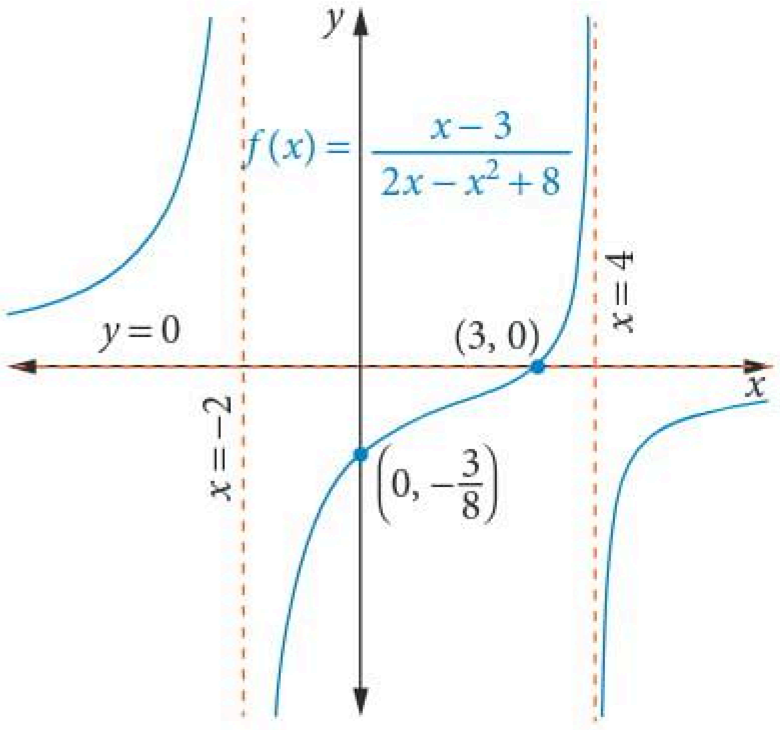
e



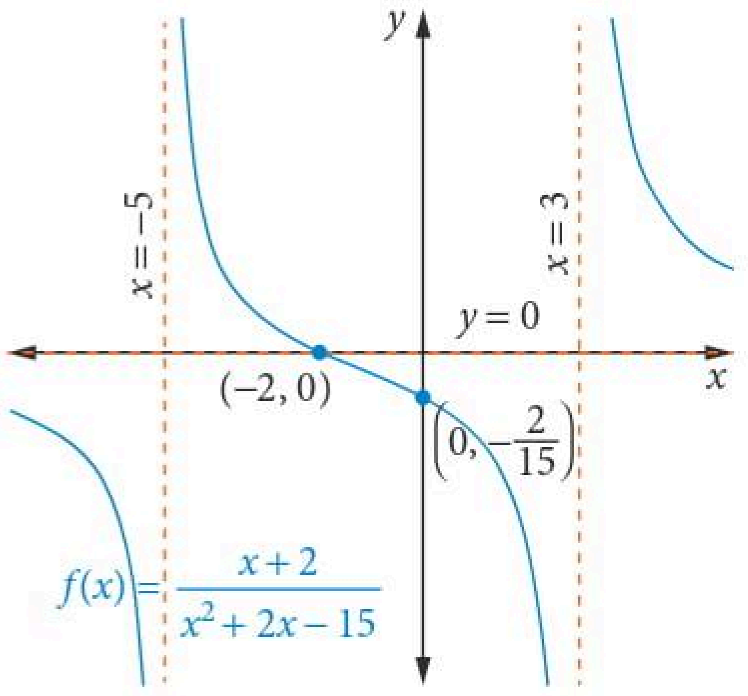
f



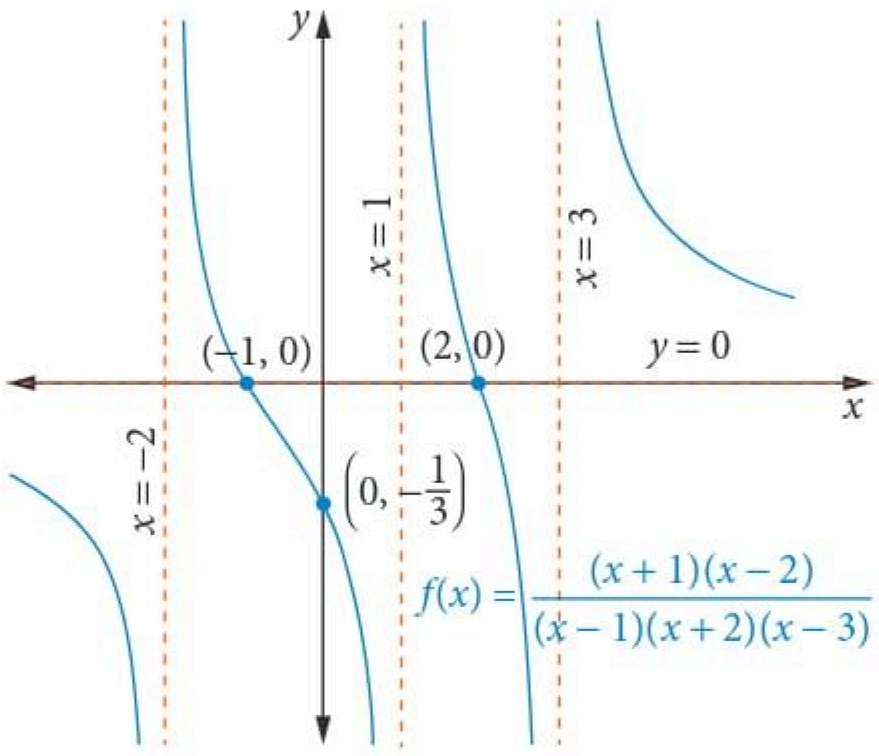
6 a



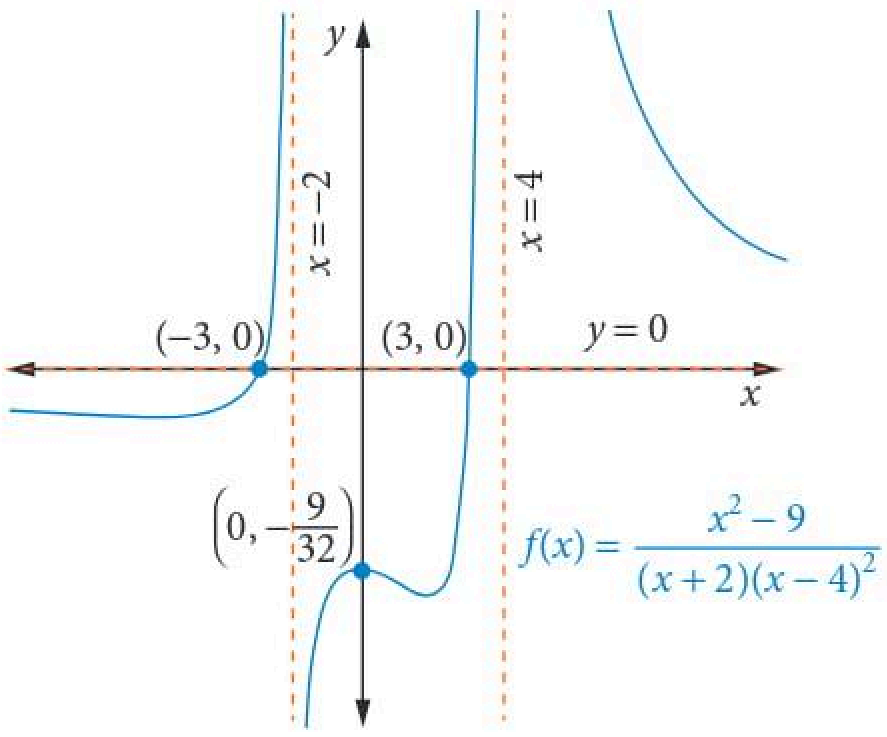
b



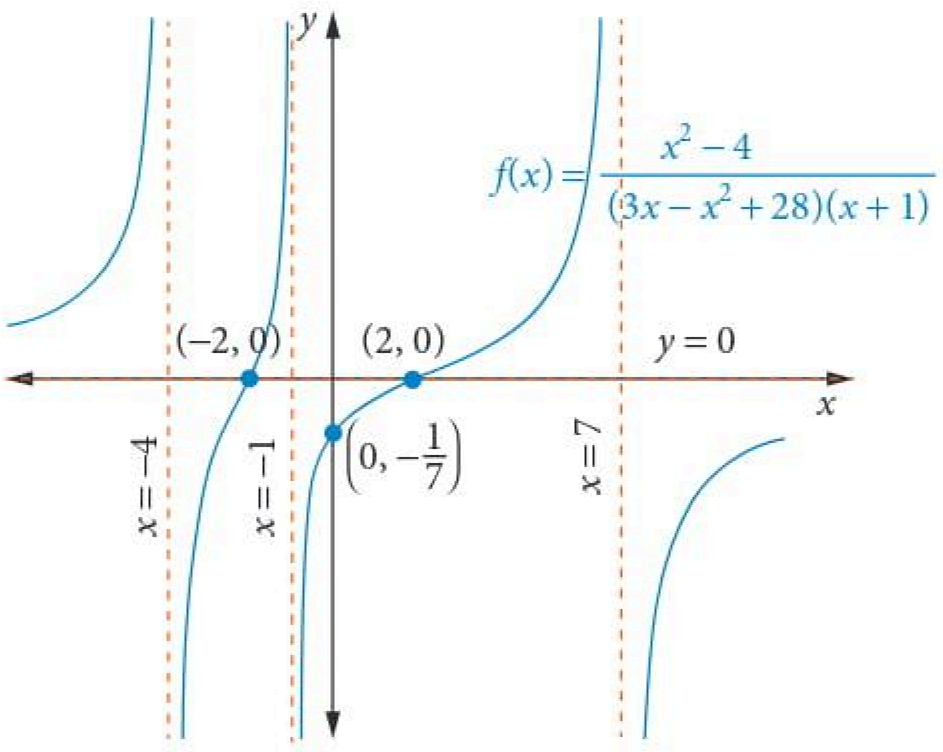
c



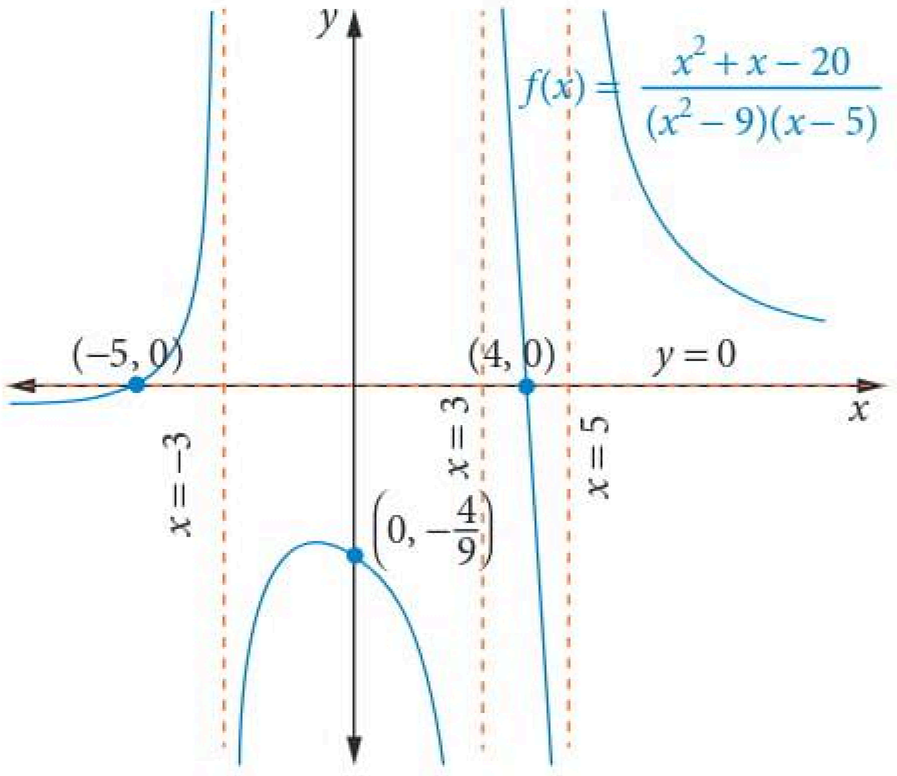
d Unmarked minimum and maximum at $-\sqrt{15} - 2$ and $x = \sqrt{15} - 2$.



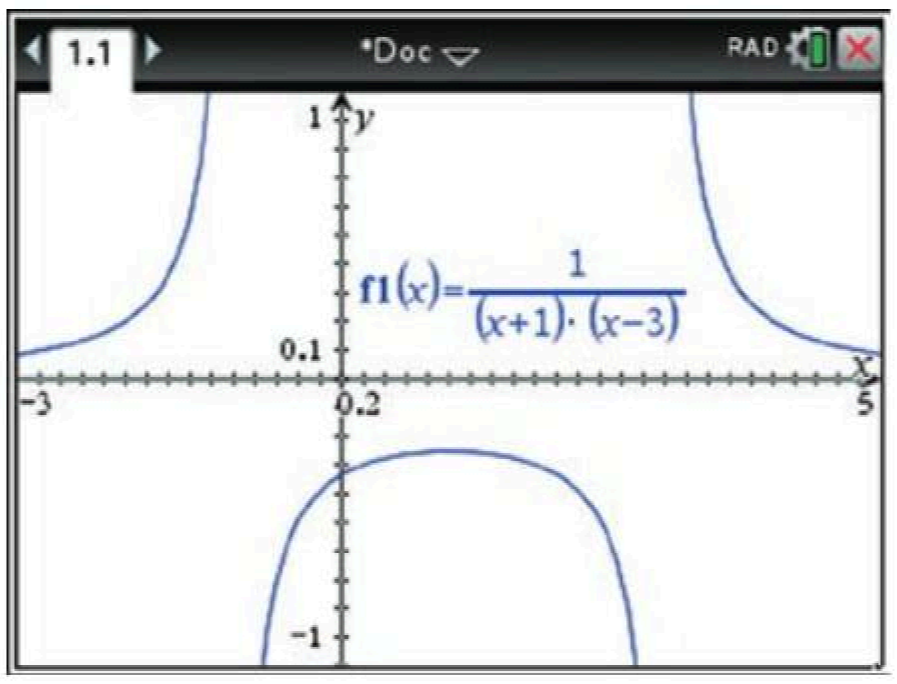
e

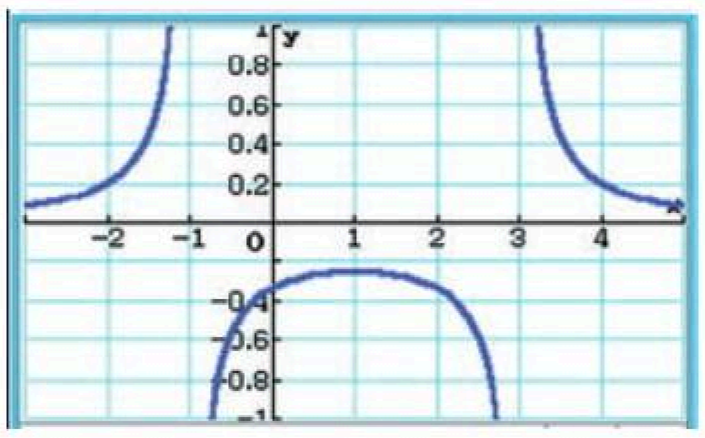


f

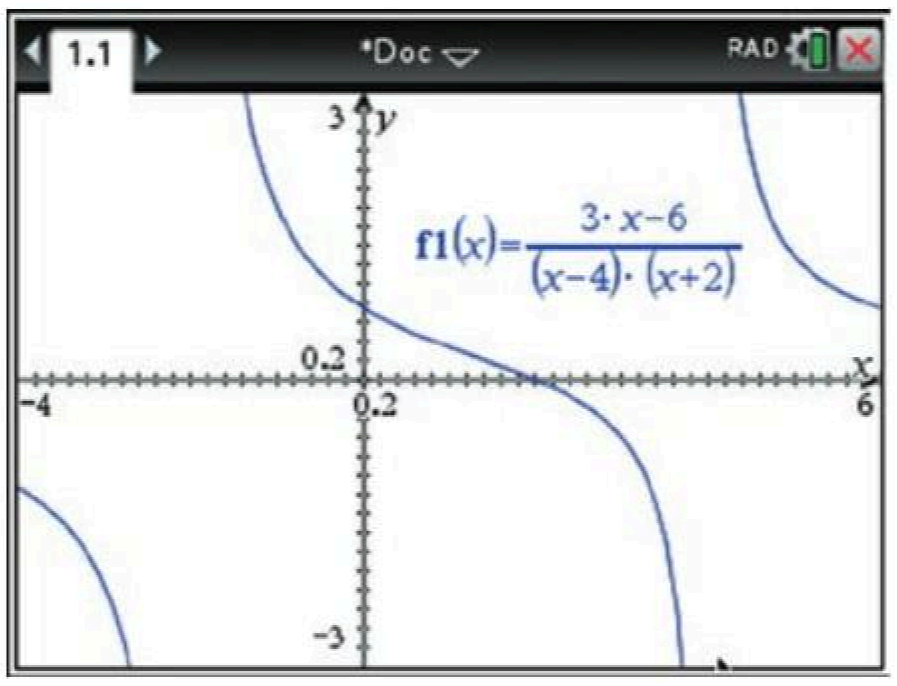


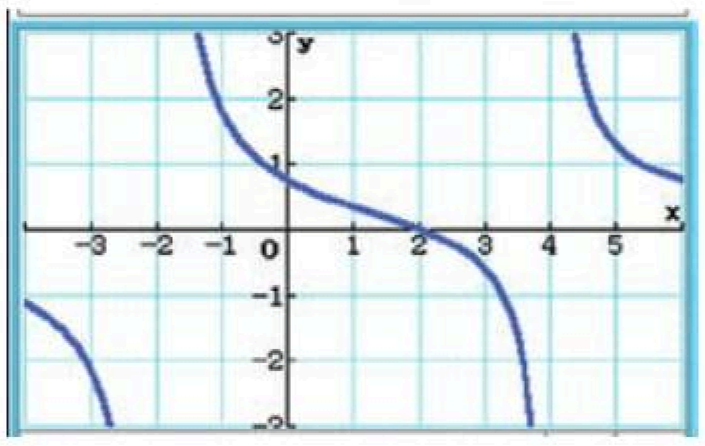
7 a



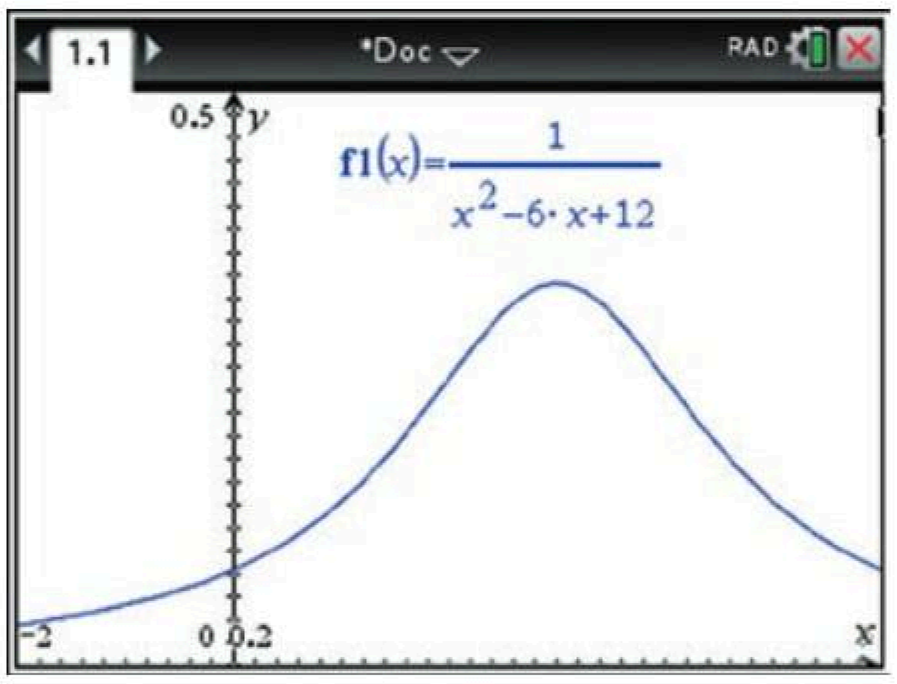


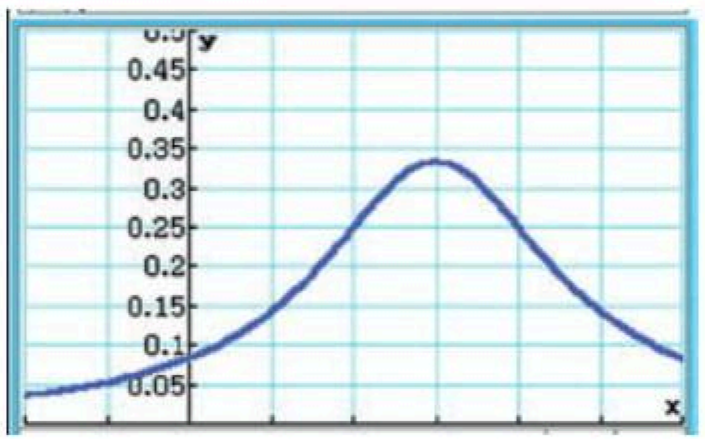
b



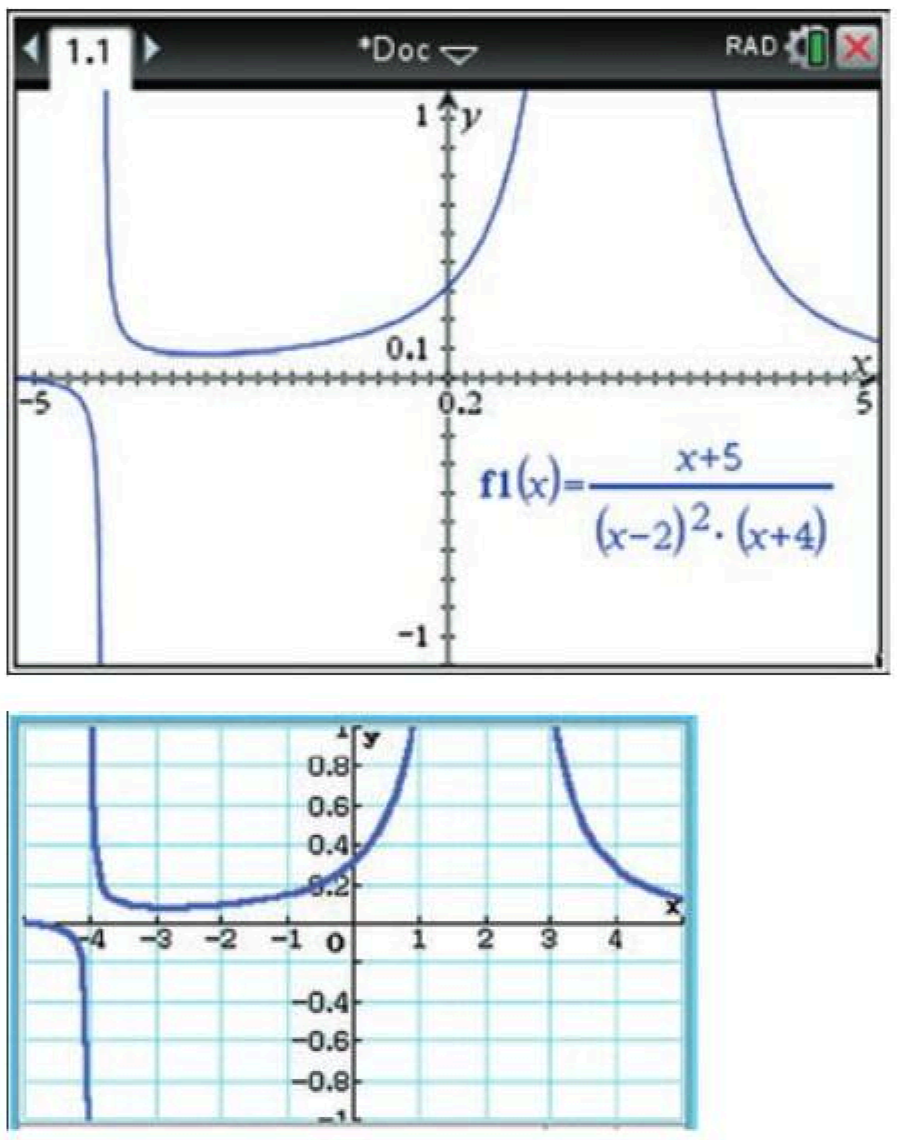


c



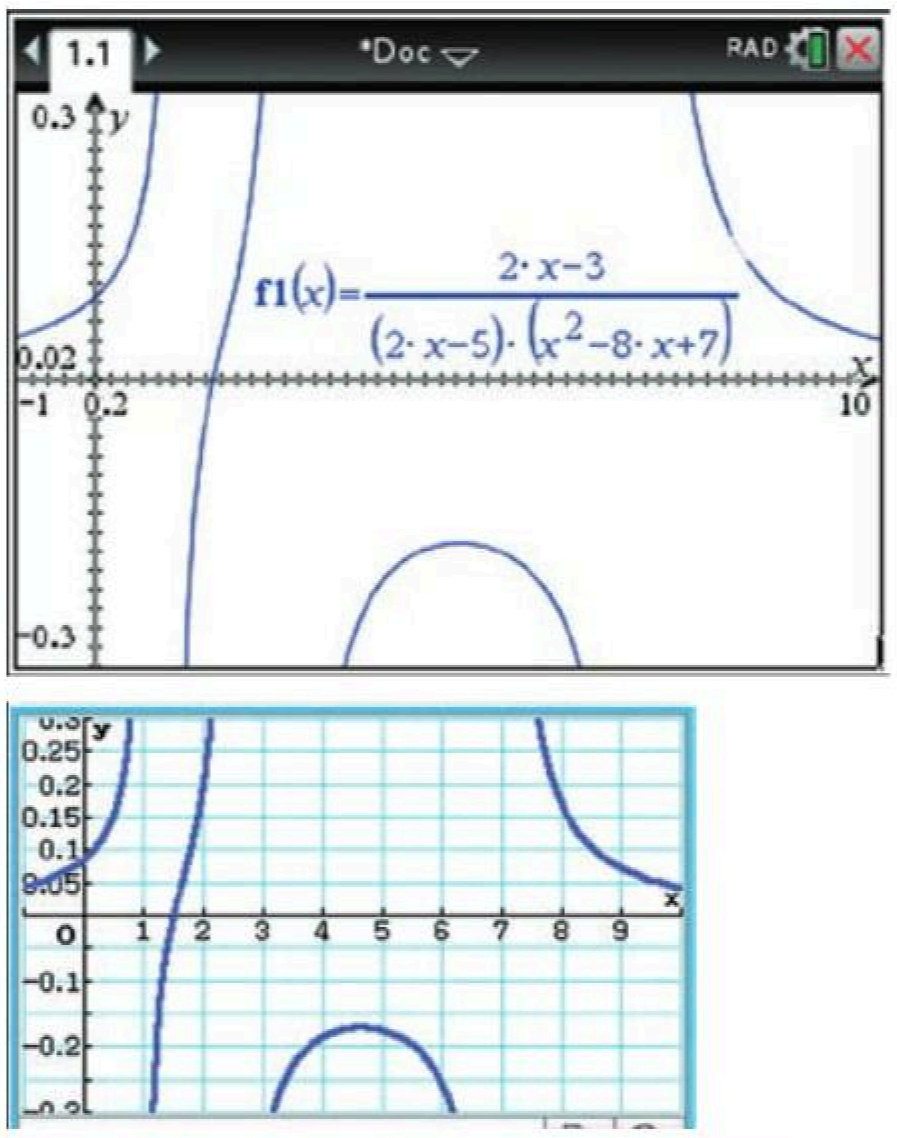


d



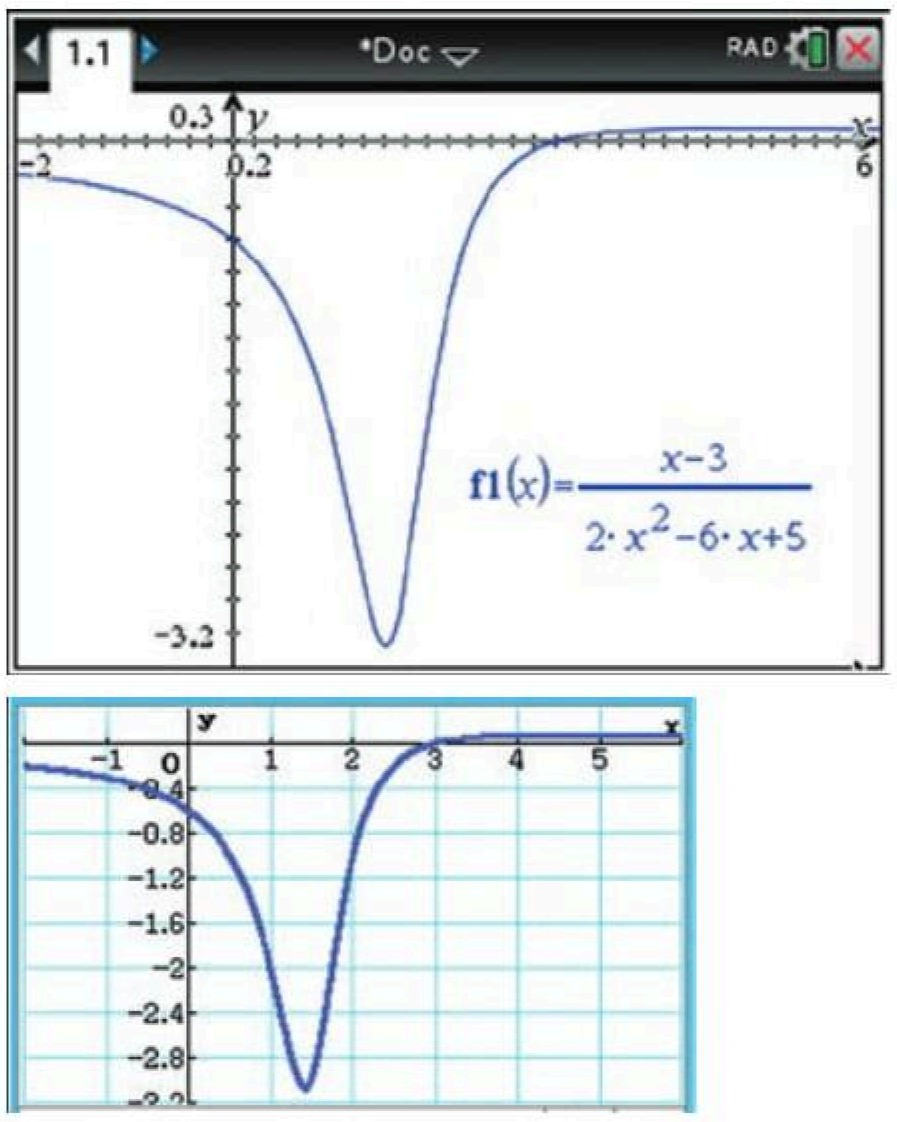
$f1(x)=\frac{x+5}{(x-2)^2\cdot(x+4)}$

e



$f1(x)=\frac{2\cdot x-3}{(2\cdot x-5)\cdot(x^2-8\cdot x+7)}$

f



$f1(x)=\frac{x-3}{2\cdot x^2-6\cdot x+5}$

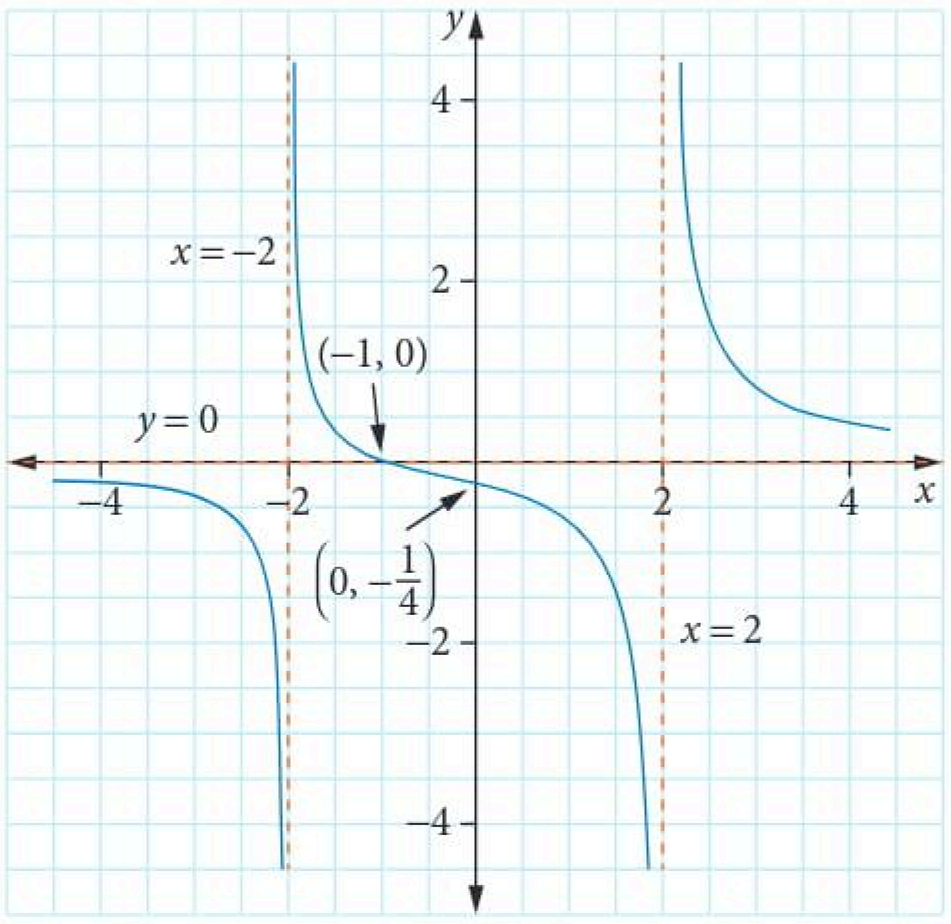
8 **a** $f(x)=\frac{1}{x+3}$ **b** $f(x)=-\frac{4}{x-4}$

c $f(x)=\frac{x-2}{(x+2)(x-5)}$ **d** $f(x)=\frac{1}{x^2+4x+5}$

e $f(x)=\frac{2(x-2)}{(x-1)^2(x+2)}$

f $f(x)=-\frac{4(x+1)(x-1)}{(x+2)^2(x-2)(x-3)}$

9

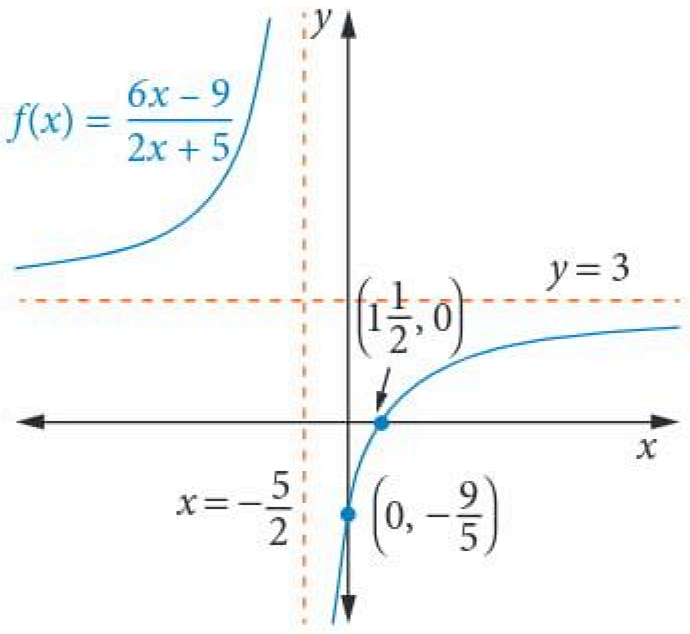


10 E **11** E

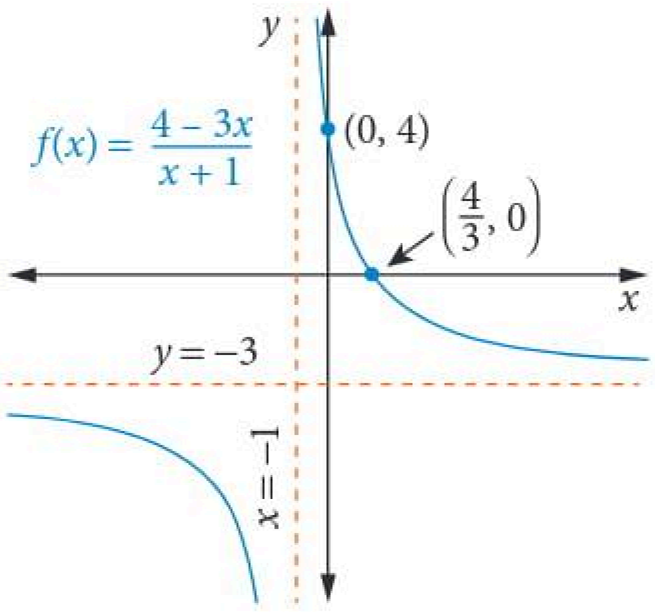
EXERCISE 2.4

1 C **2** D

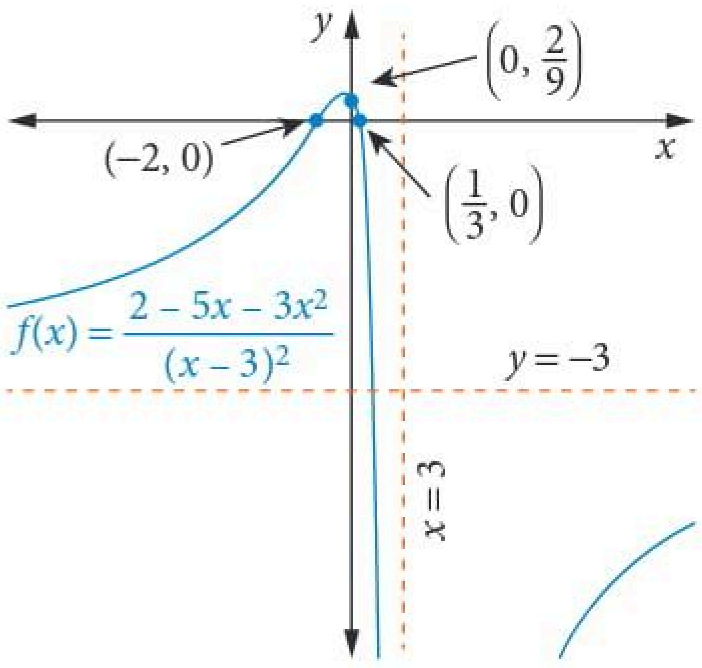
3 **a**



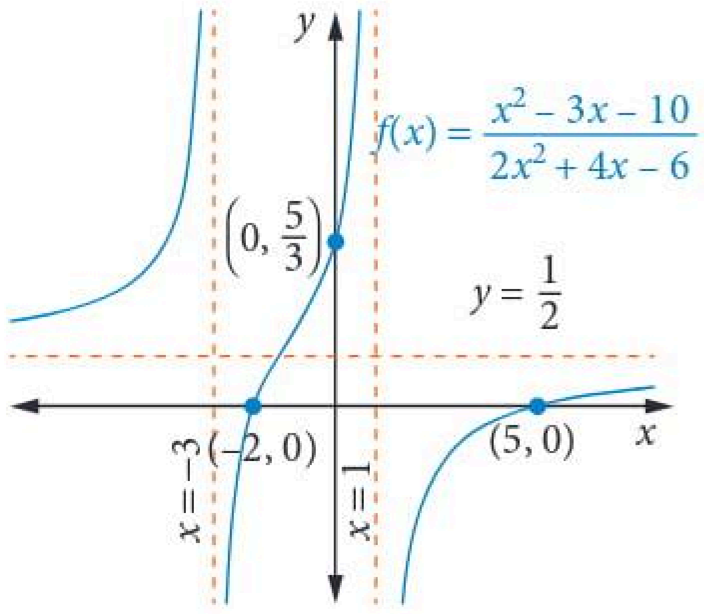
b



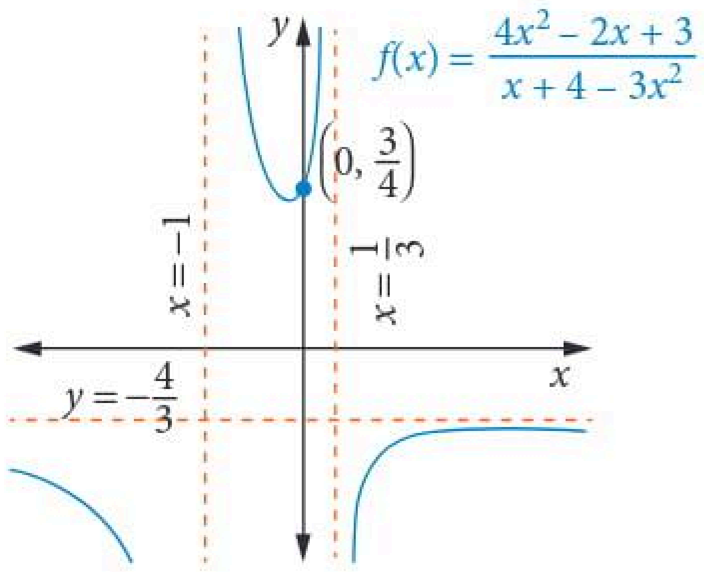
c



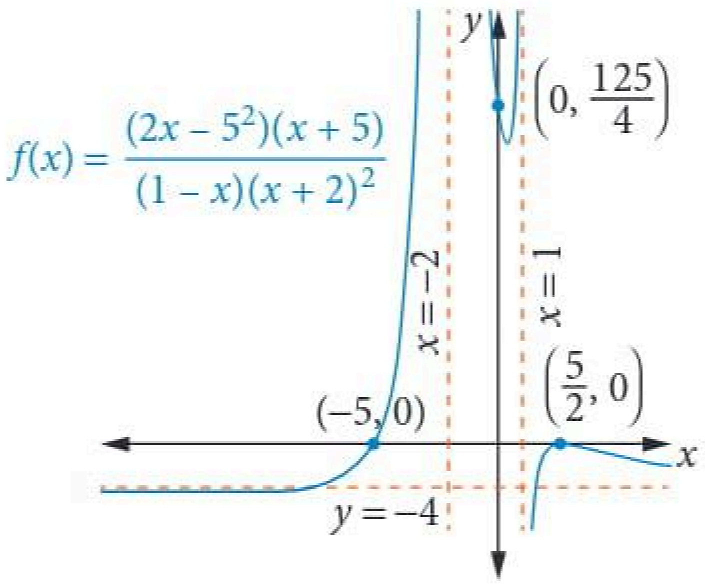
d



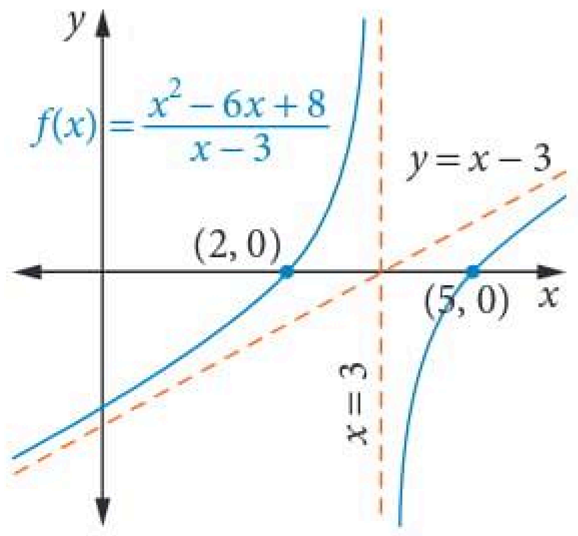
e



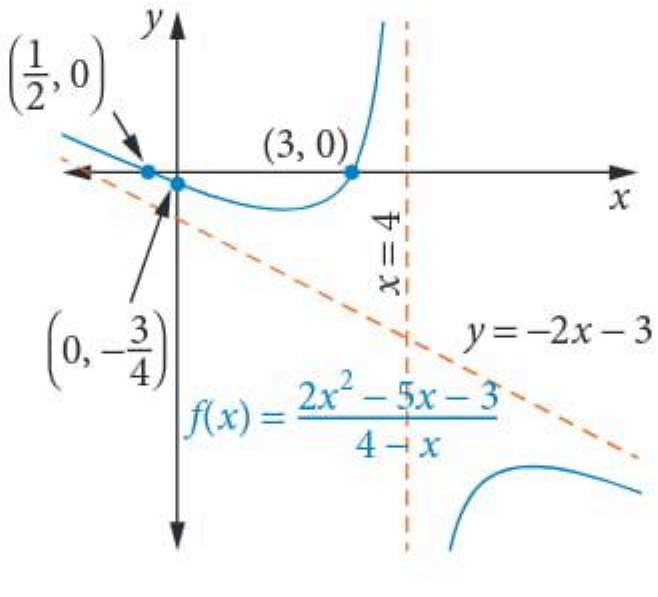
f



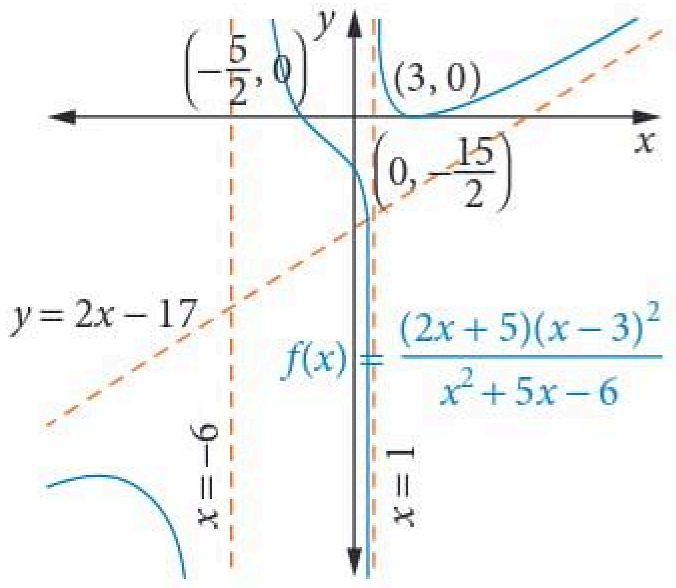
4 a



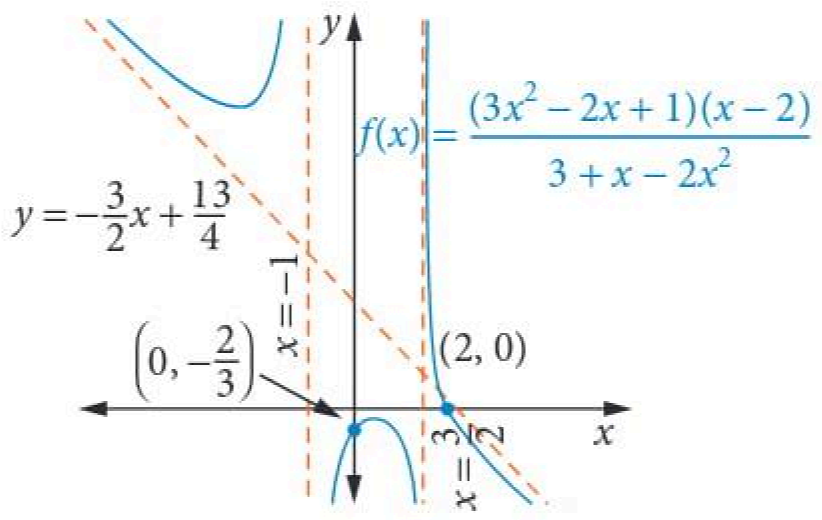
b



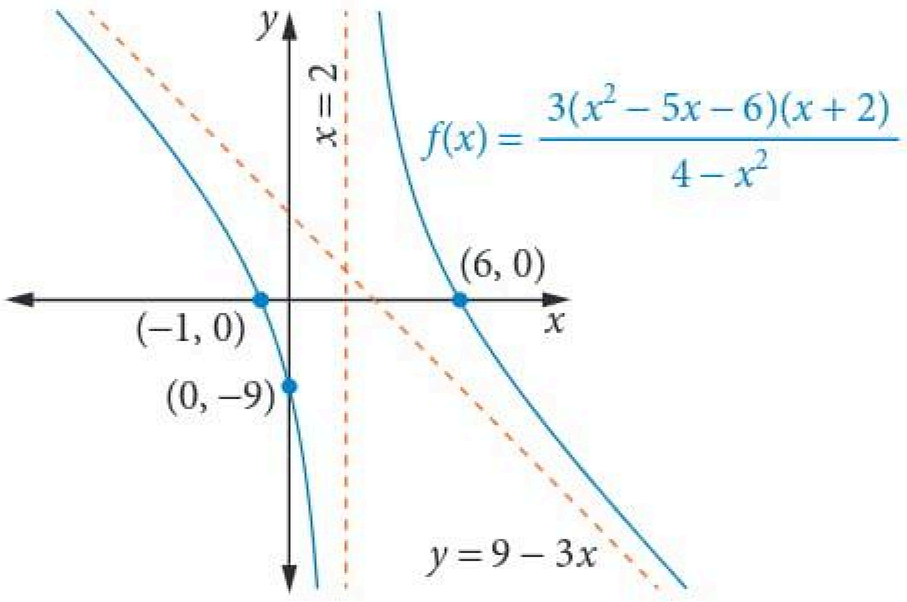
c



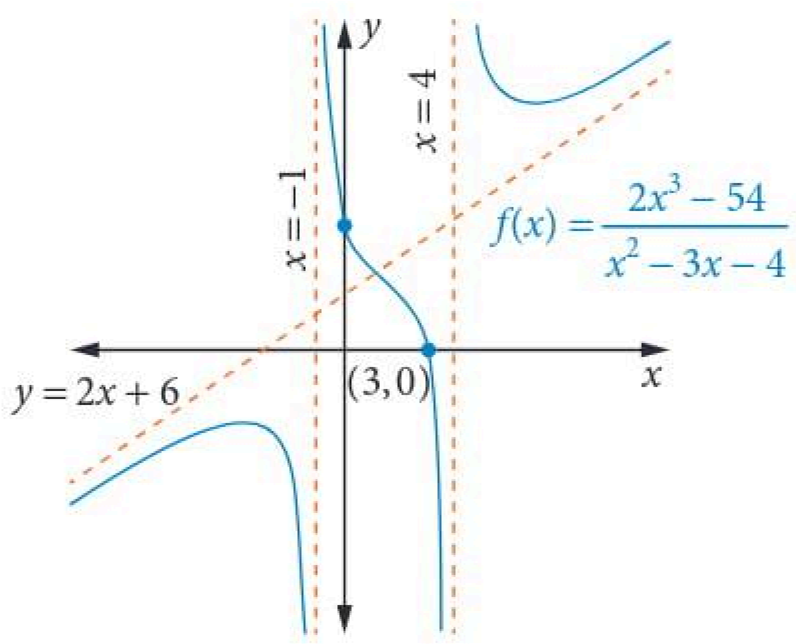
d



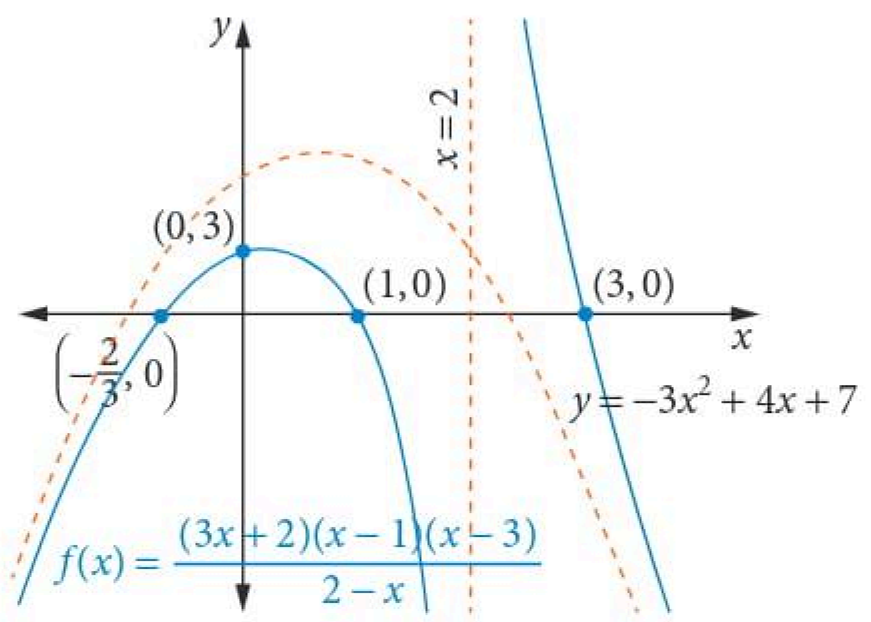
e



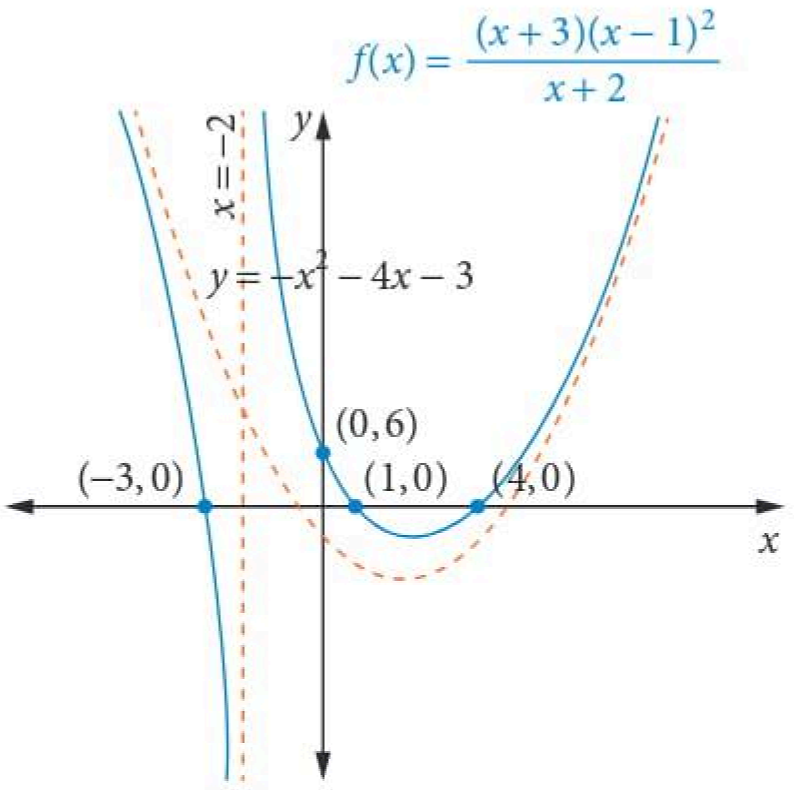
f



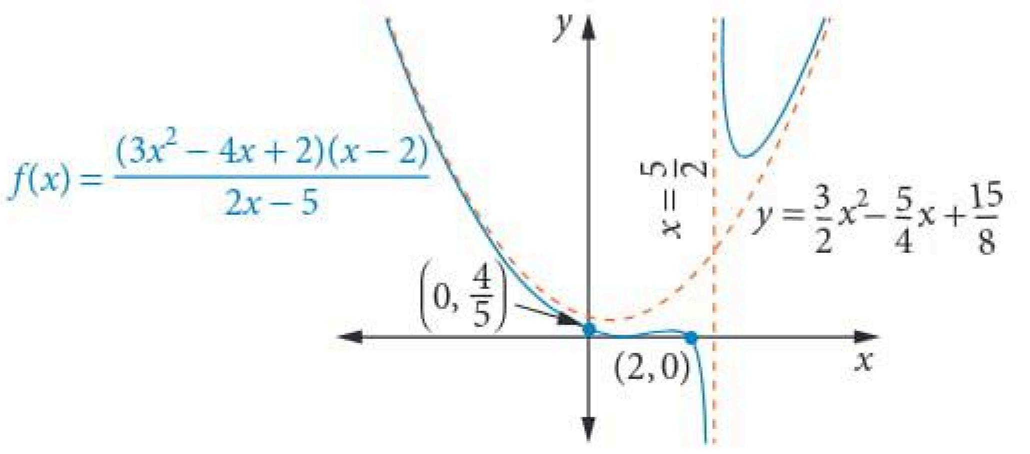
5 a



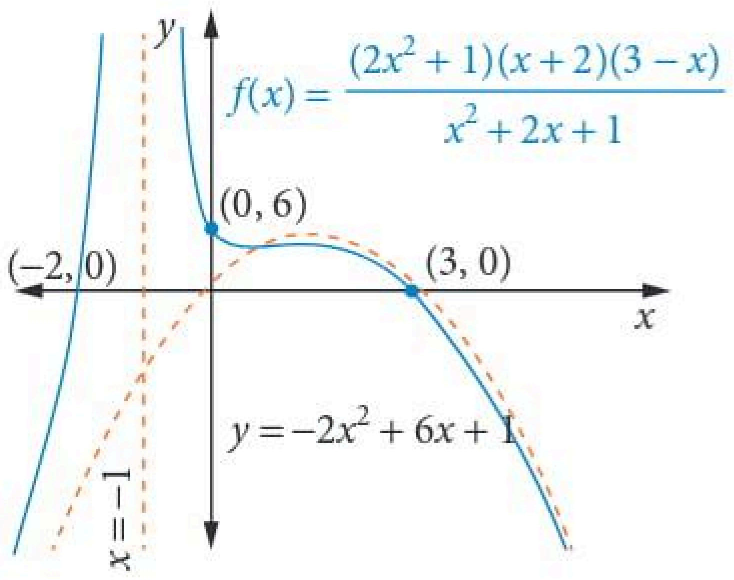
b



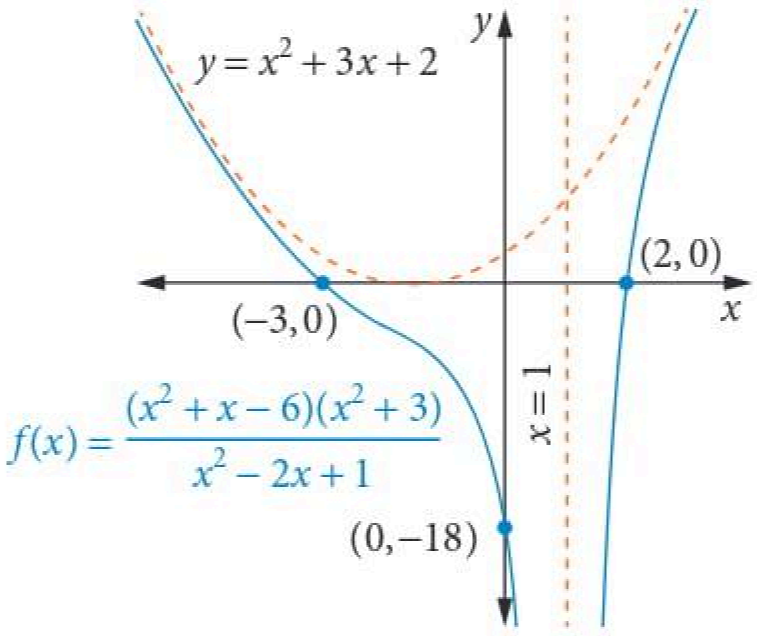
c


$f(x)=\dfrac{(3x^2-4x+2)(x-2)}{2x-5}$
$x=\frac{5}{2}$
$y=\frac{3}{2}x^2-\frac{5}{4}x+\frac{15}{8}$
$\left(0,\frac{4}{5}\right)$
$(2,0)$
x
y

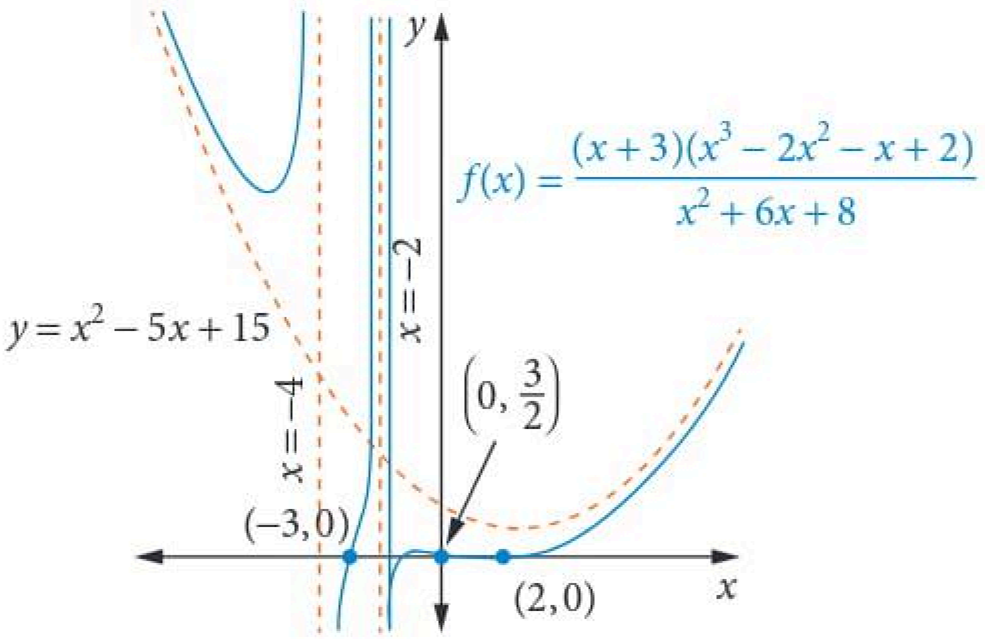
d


$f(x)=\dfrac{(2x^2+1)(x+2)(3-x)}{x^2+2x+1}$
$(0,6)$
$(-2,0)$
$(3,0)$
$y=-2x^2+6x+1$
$x=-1$
x
y

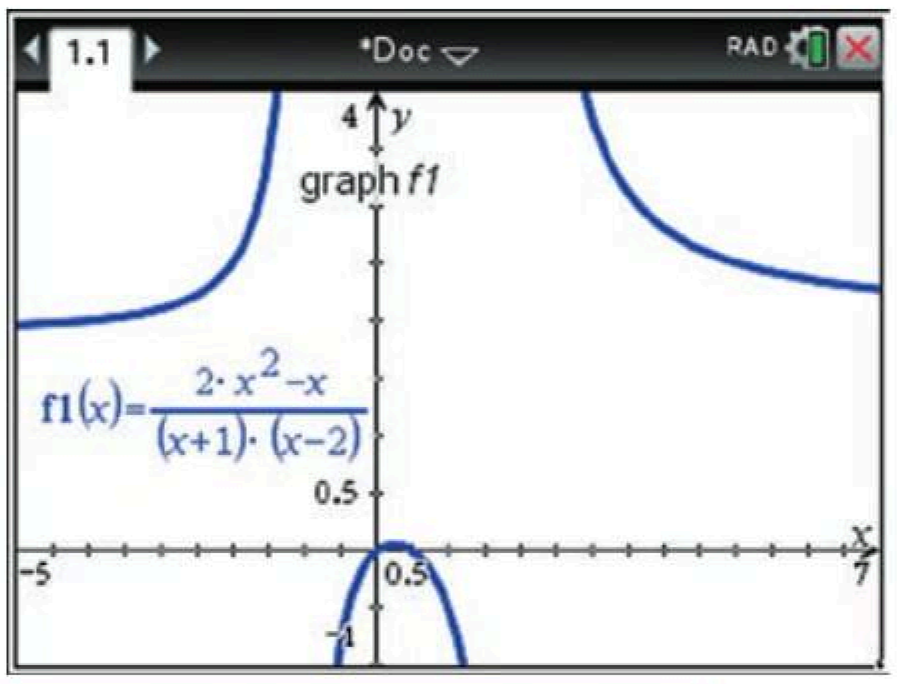
e

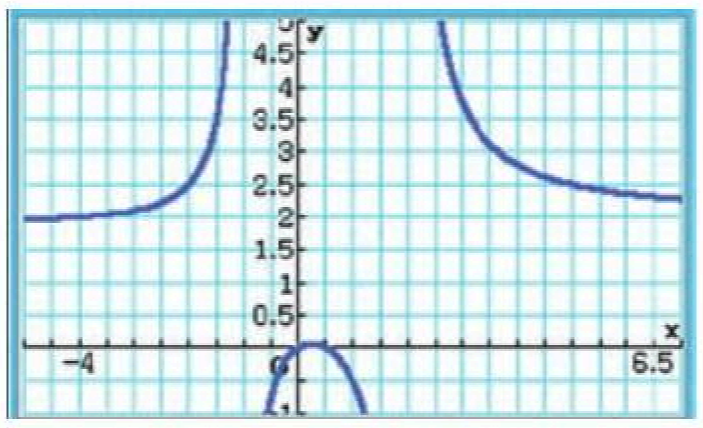

$y=x^2+3x+2$
$(2,0)$
$(-3,0)$
$f(x)=\dfrac{(x^2+x-6)(x^2+3)}{x^2-2x+1}$
$x=1$
$(0,-18)$
x
y

f

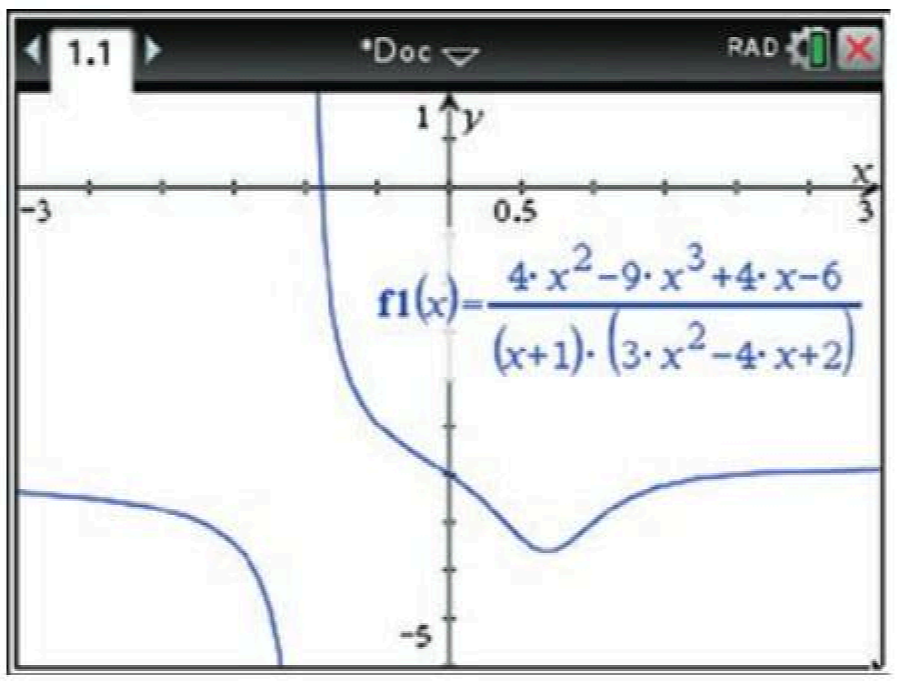

$f(x)=\dfrac{(x+3)(x^3-2x^2-x+2)}{x^2+6x+8}$
$y=x^2-5x+15$
$x=-4$
$x=-2$
$\left(0,\frac{3}{2}\right)$
$(-3,0)$
$(2,0)$
x
y

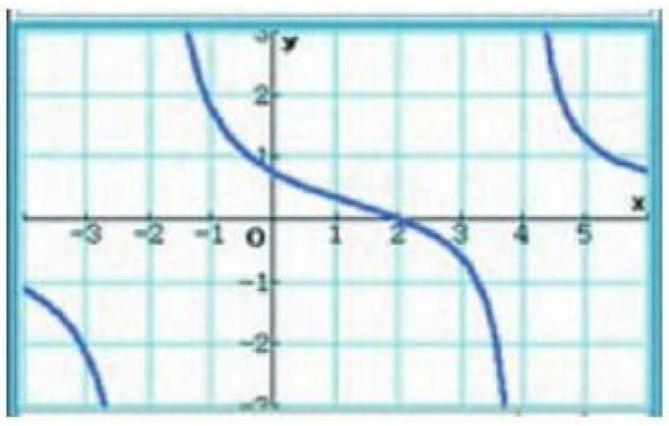
6 a


1.1
*Doc
RAD
4
graph f1
$f1(x)=\dfrac{2\cdot x^2-x}{(x+1)\cdot(x-2)}$
0.5
-5
0.5
7
-1
x
y

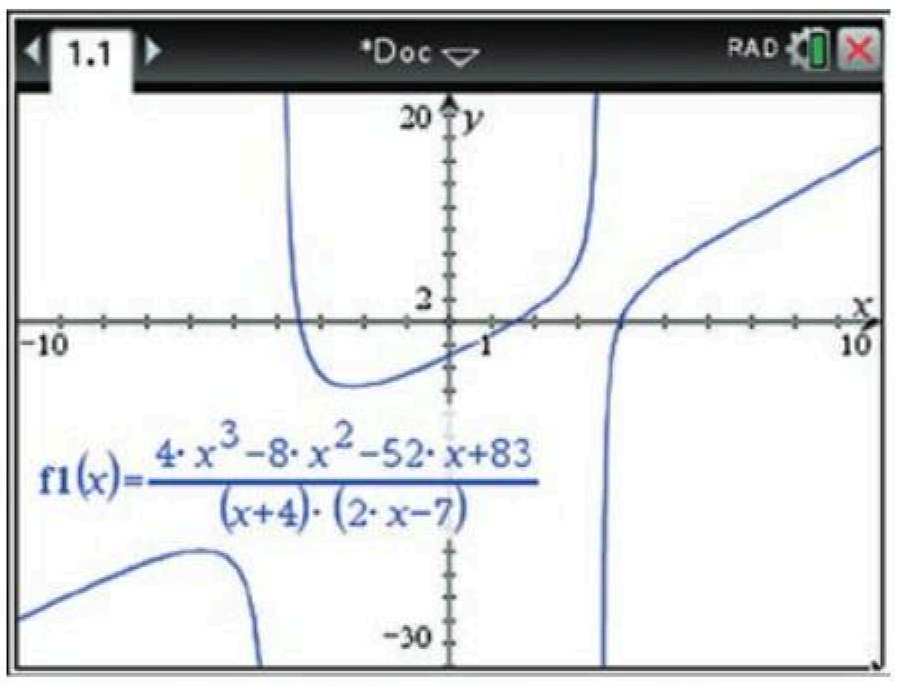

4.5
4
3.5
3
2.5
2
1.5
1
0.5
-4
0
6.5
x
y

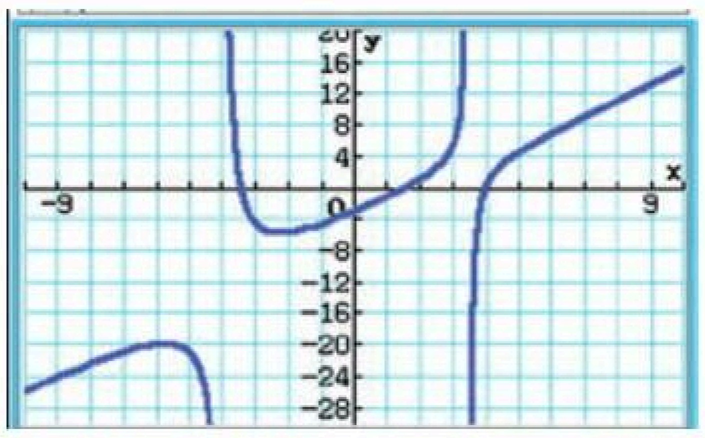
b


1.1
*Doc
RAD
1
-3
0.5
3
-5
$f1(x)=\dfrac{4\cdot x^2-9\cdot x^3+4\cdot x-6}{(x+1)\cdot(3\cdot x^2-4\cdot x+2)}$
x
y

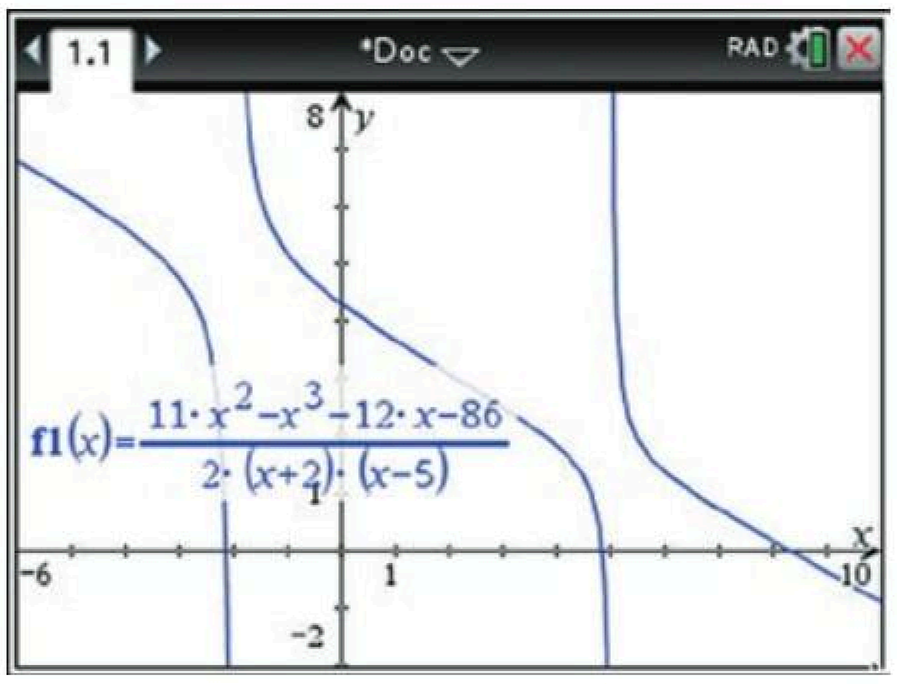

2
1
-1
-2
-3
-2
-1
0
1
2
3
4
5
x
y

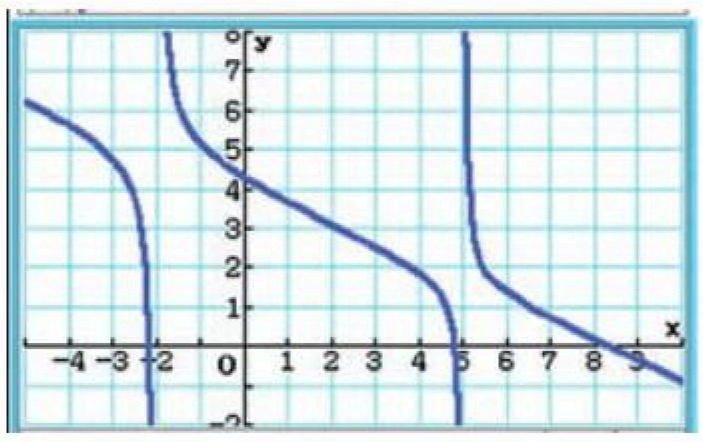
c


1.1
*Doc
RAD
20
2
-10
1
10
-30
$f1(x)=\dfrac{4\cdot x^3-8\cdot x^2-52\cdot x+83}{(x+4)\cdot(2\cdot x-7)}$
x
y

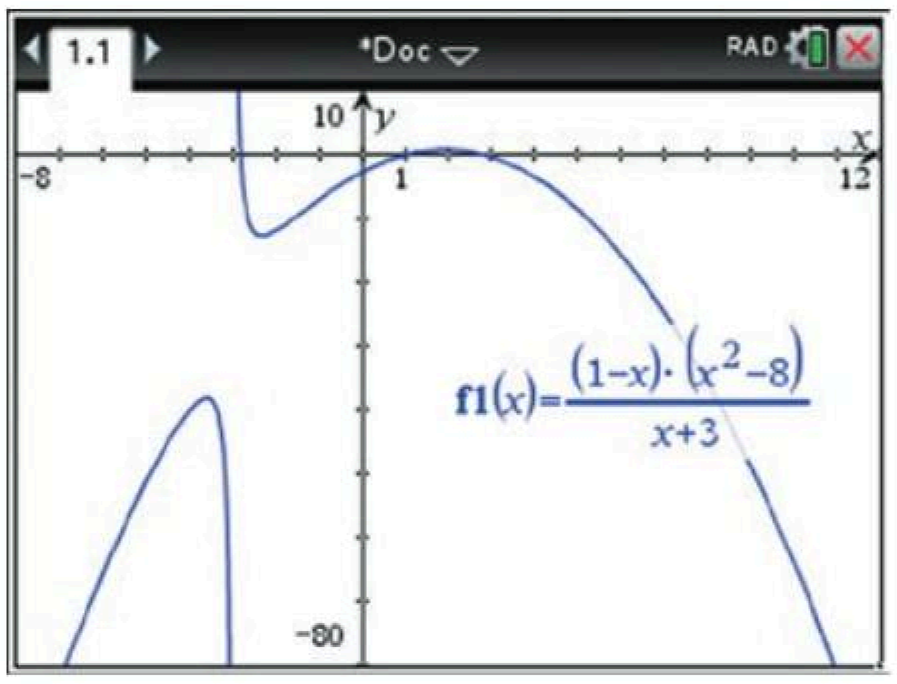

16
12
8
4
0
-8
-12
-16
-20
-24
-28
-9
9
x
y

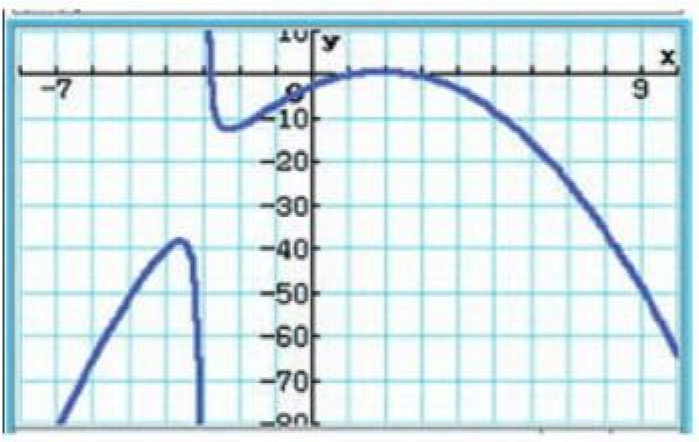
d


1.1
*Doc
RAD
8
1
-6
1
10
-2
$f1(x)=\dfrac{11\cdot x^2-x^3-12\cdot x-86}{2\cdot(x+2)\cdot(x-5)}$
x
y

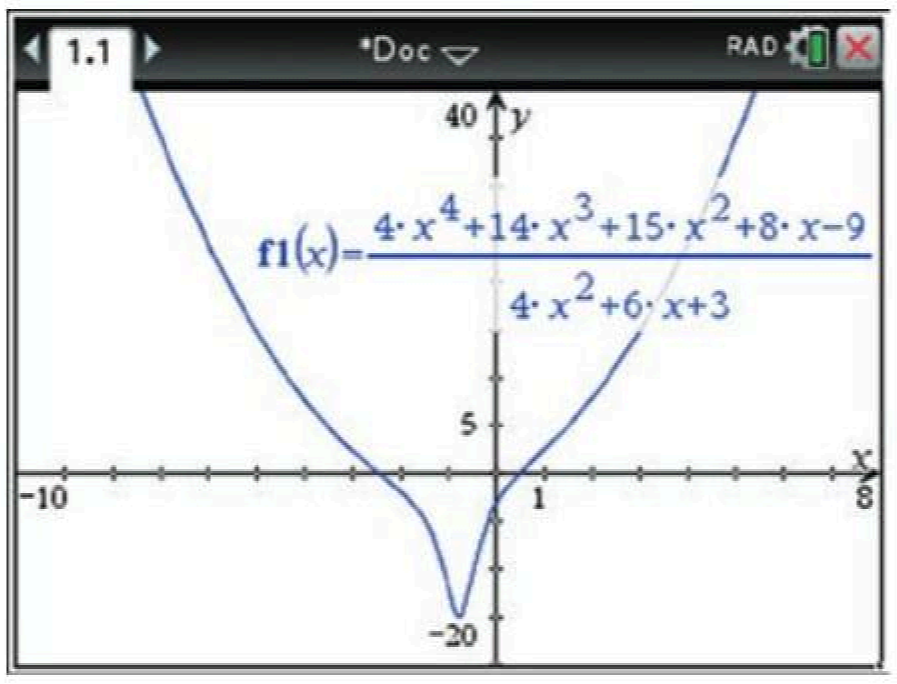

7
6
5
4
3
2
1
-4
-3
-2
0
1
2
3
4
5
6
7
8
9
x
y

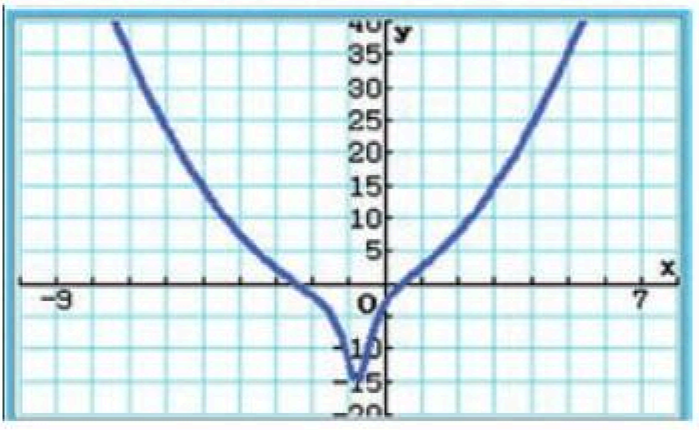
e





f



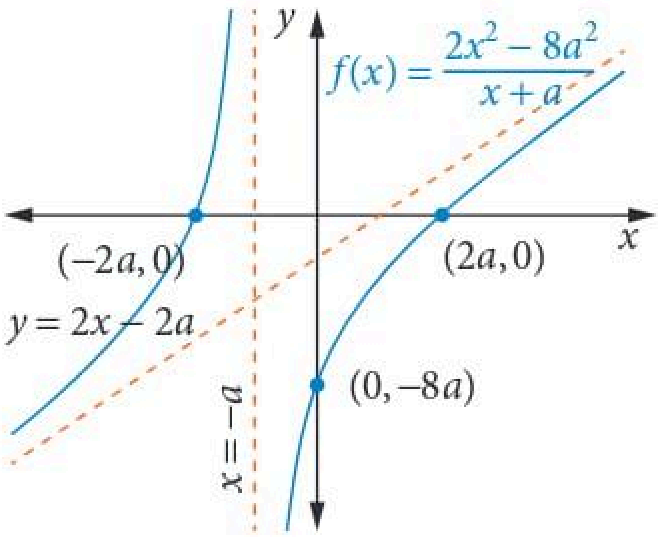


7 **a** 2 (at $-2a$ and $2a$)

b 2 (1 vertical and 1 sloping)

c $x = -a$ and $y = 2x - 2a$

d



8 **a** $y = 0, x = -4, x = 6$

b $y = 3, x = 1, x = 2$

c $x = -2, y = 2x - 7$

d $x = -2, x = 4, y = -3x - 4$

e $x = -4, y = -2x^2 + 11x - 33$

f $y = x^2 - 4x + 7$

9 **a** $f(x) = -\dfrac{(x-2)(x+3)}{(x+2)(x-4)}$

b $f(x) = \dfrac{3(x-1)(x+2)}{(x+4)(x-3)}$

c $f(x) = \dfrac{(x-3)(x+2)}{x-2}$

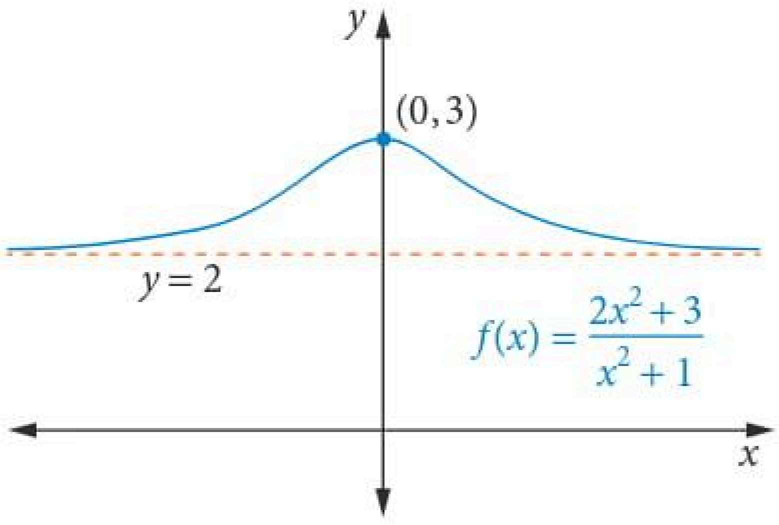
d $f(x) = -\dfrac{(x-2)(x-4)(x+2)}{2(x+3)(x-5)}$

e $f(x) = \dfrac{(x-1)^2(x+2)}{2(x+1)}$

f $f(x) = -\dfrac{(x-1)^2(x+3)^2}{4(x+2)^2}$

10 **a** Change the second expression to a common factor.

b

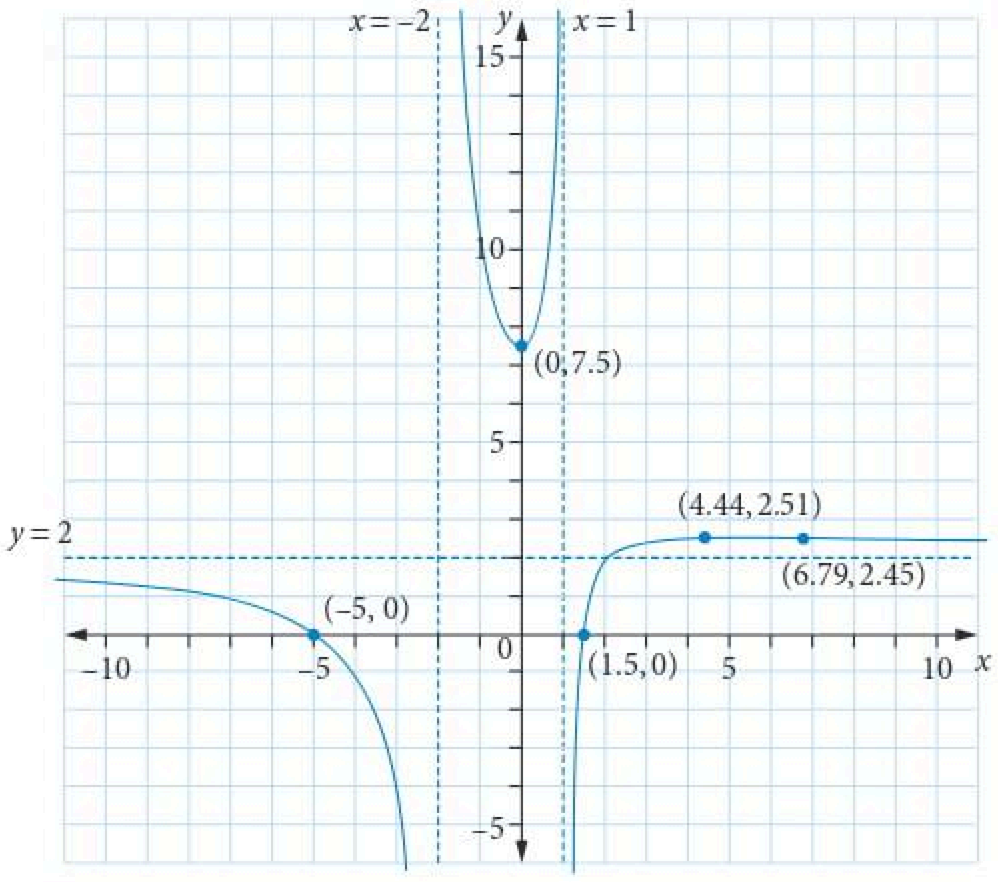


11 C **12** C **13** D **14** D

15 **a** $f(x) = 2 + \dfrac{5x - 11}{(x+1)(x+2)}$

b $x = 1, x = -2, y = 2$

c



d **i** $k = -2, k = -5, k = \dfrac{3}{2}$

ii $k < -5$ or $k > \dfrac{3}{2}$

EXERCISE 2.5

1 E

2 B

3 **a** $x = -3, x = \dfrac{3}{2}$

b $m = 4, m = 6$

c $z = -\dfrac{11}{3}, z = 1$

d $-\dfrac{10}{3} < g < 2.$

e $k < -10$ or $k > 4$

f $x \le -3$ or $x \ge 4$

4 **a** $-\dfrac{4}{5} < x < 2$

b $p = -\dfrac{2}{5}$

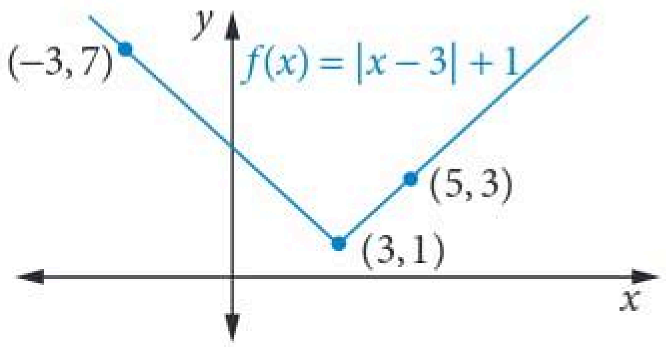
c $q < 14$

d $m \le -\dfrac{9}{2}$ or $m \ge -\dfrac{5}{8}$, so $m \in R$.

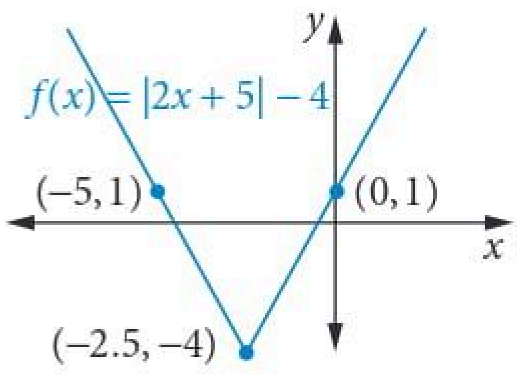
e $a = -29, a = \dfrac{13}{7}$

f $2 < x < 3$ or $x < \dfrac{5 - \sqrt{17}}{2}$ or $x > \dfrac{5 + \sqrt{17}}{2}$

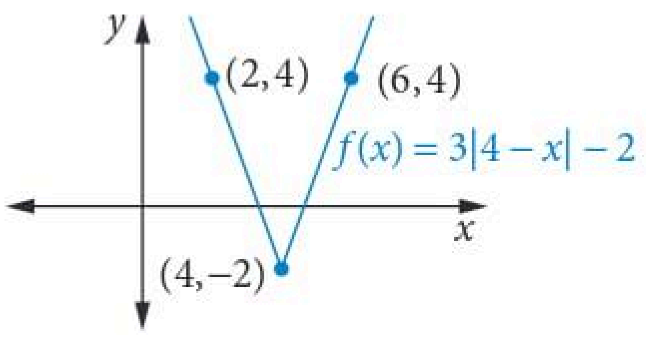
5 a



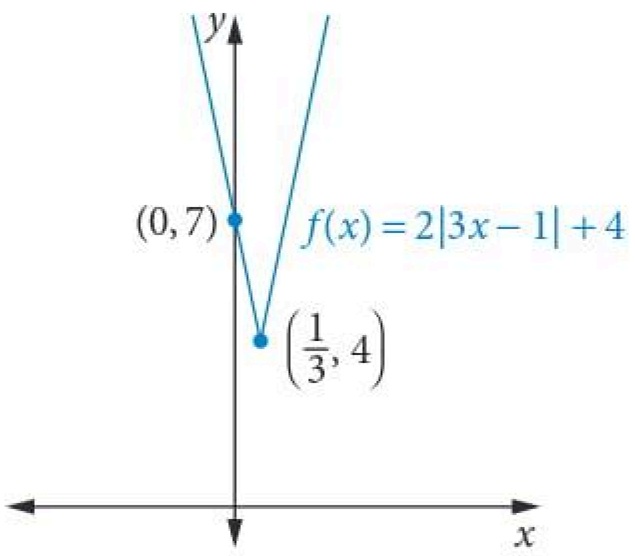
b



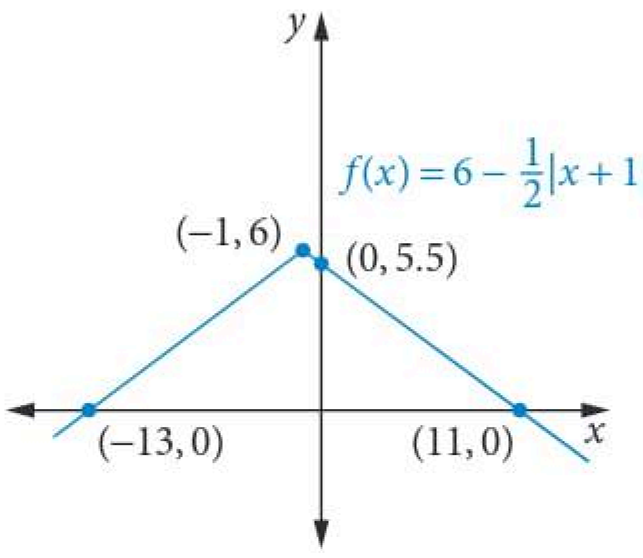
c



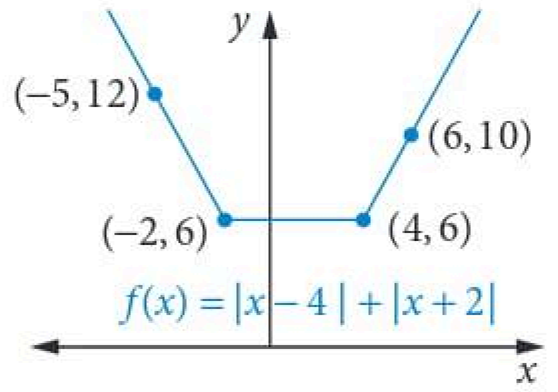
d



e



f



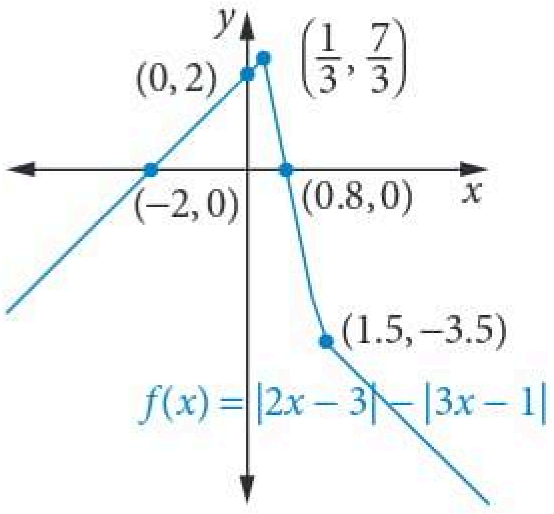
6 a $f(x) = |x + 3|$ **b** $f(x) = \frac{1}{2}|x - 6|$

c $f(x) = 2\left|x + \frac{3}{2}\right| + 1$ **d** $f(x) = 2\,|\,x - 4\,| - 2$

e $f(x) = \frac{1}{2}\,|\,x + 4\,| - 3$ **f** $f(x) = \frac{2}{3}\,|\,x - 2\,| - 4$

7 $x \in \left(-\infty, \frac{7 - \sqrt{5}}{2}\right)$

8 a



b domain: R, range: $\left\{x \in R : x \leq \frac{7}{3}\right\}$

9 $-2k \leq x \leq 2k$

10 C **11** C

EXERCISE 2.6

1 D **2** A

3 a −1 **b** 2 **c** 2

d $\frac{1}{\sqrt{3}}$ **e** $-\sqrt{2}$ **f** 2

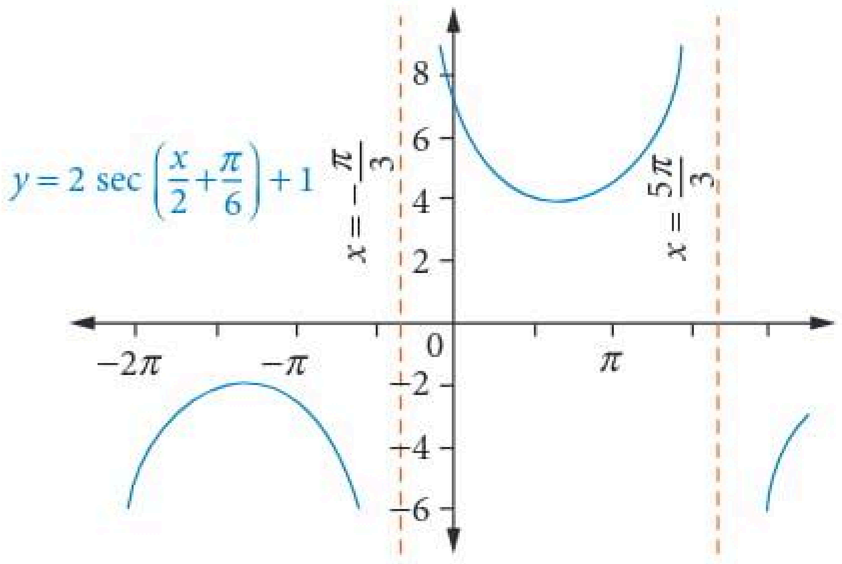
4 a $\pm\frac{a}{\sqrt{a^2 - 1}}$ **b** $\pm\sqrt{a^2 - 1}$ **c** $\frac{1 \pm \sqrt{a^2 - 1}}{a}$

5 $\sec(x) = \frac{1}{b}$, $\operatorname{cosec}(x) = \frac{a}{b}$

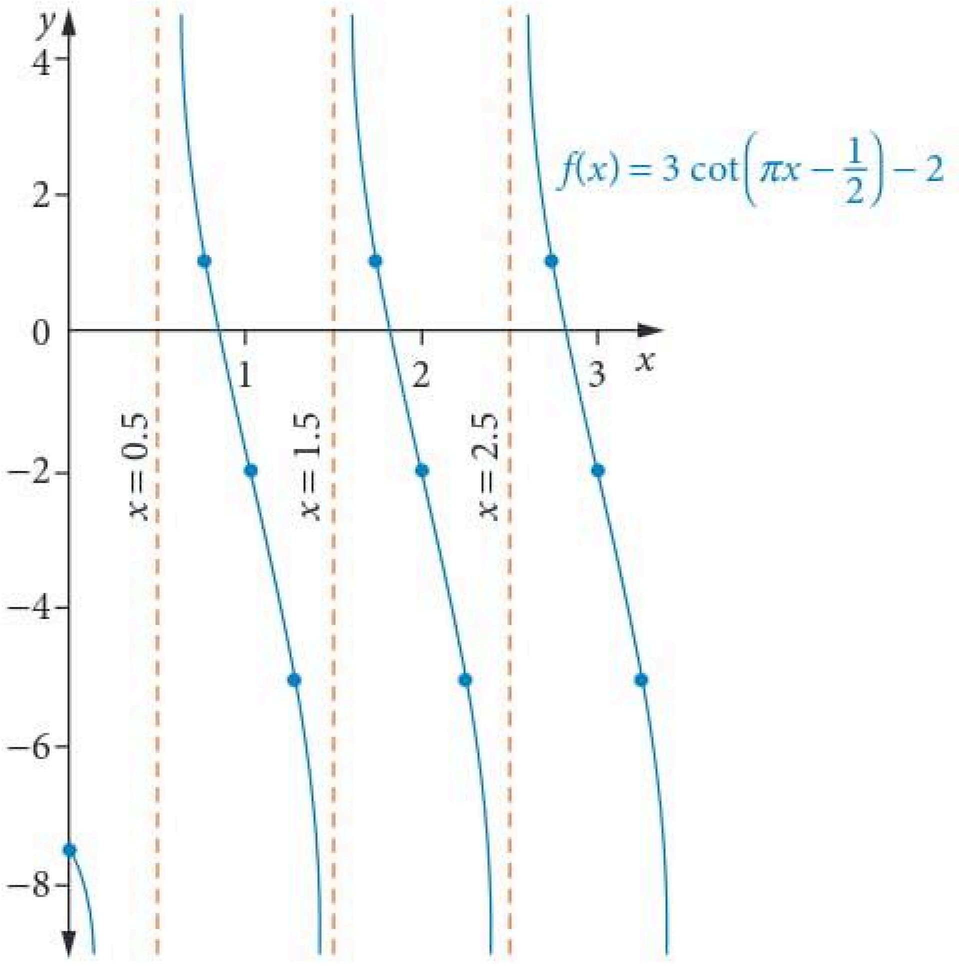
6 $\sin(x) = -\frac{\sqrt{t^2 - 1}}{t}$, $\cos(x) = \frac{1}{t}$, $\tan(x) = -\sqrt{t^2 - 1}$,

$\cot(x) = -\frac{1}{\sqrt{t^2 - 1}}$, $\operatorname{cosec}(x) = -\frac{t}{\sqrt{t^2 - 1}}$

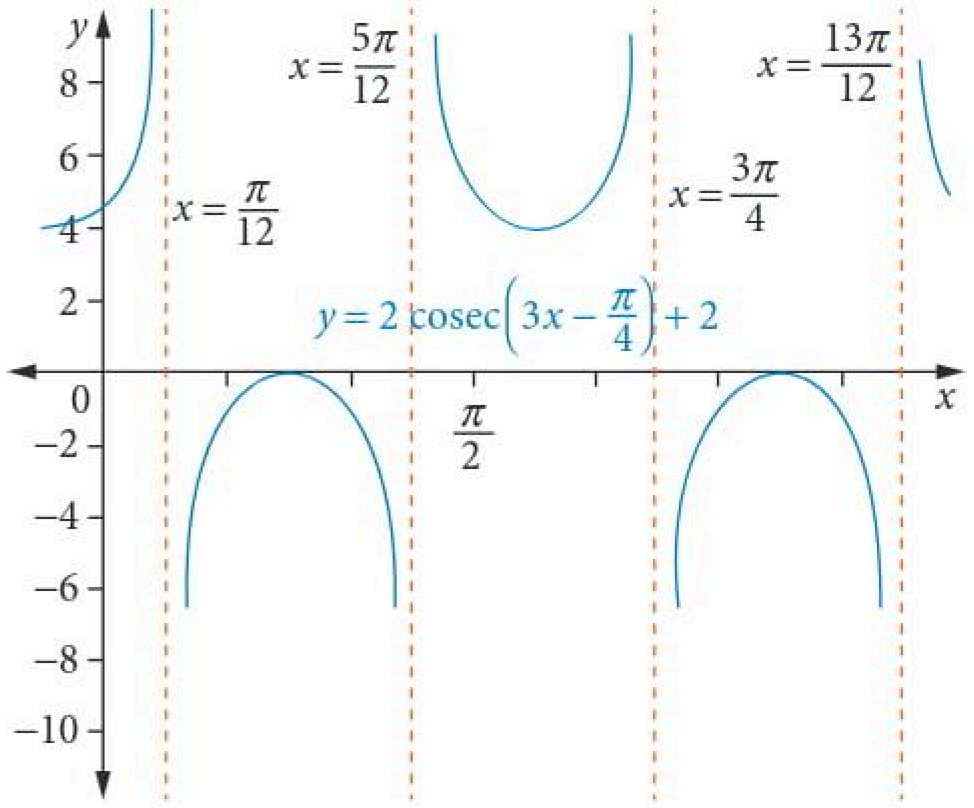
7 a



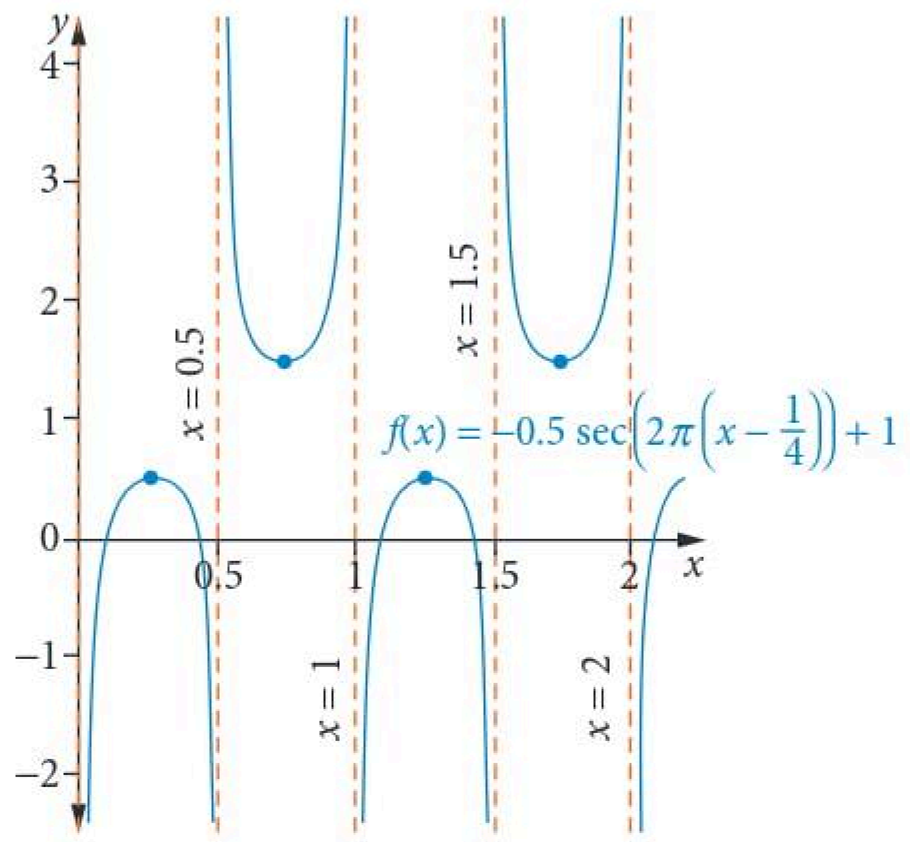
b



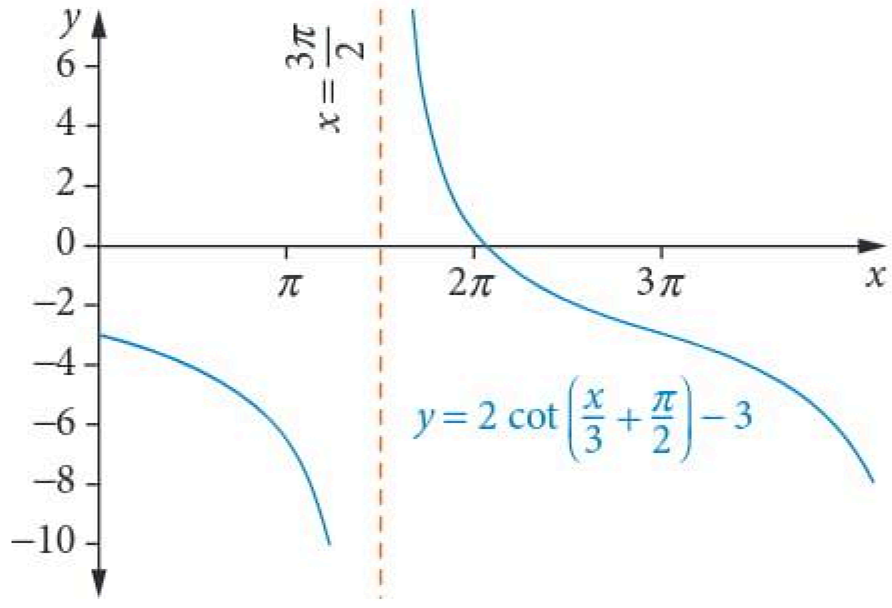
c



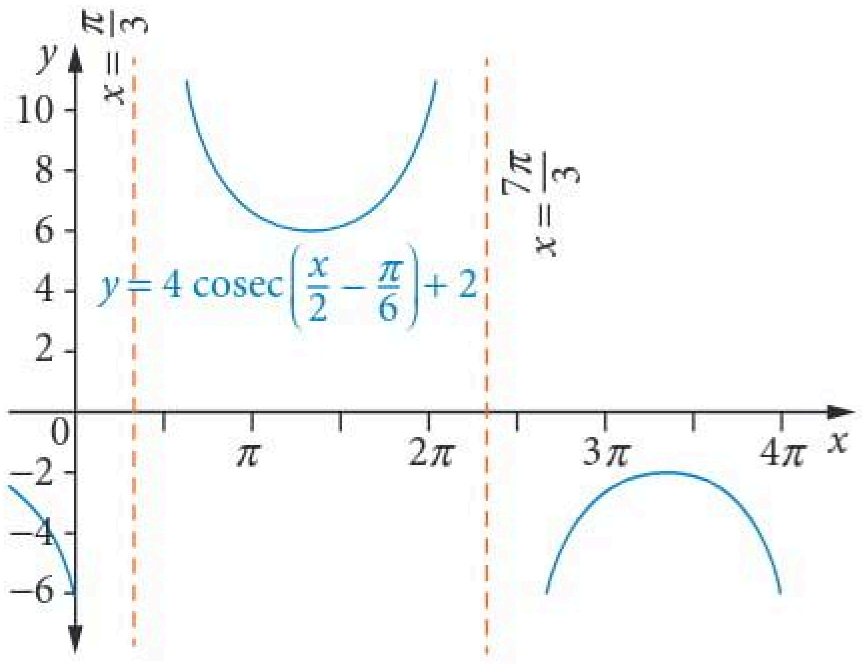
d



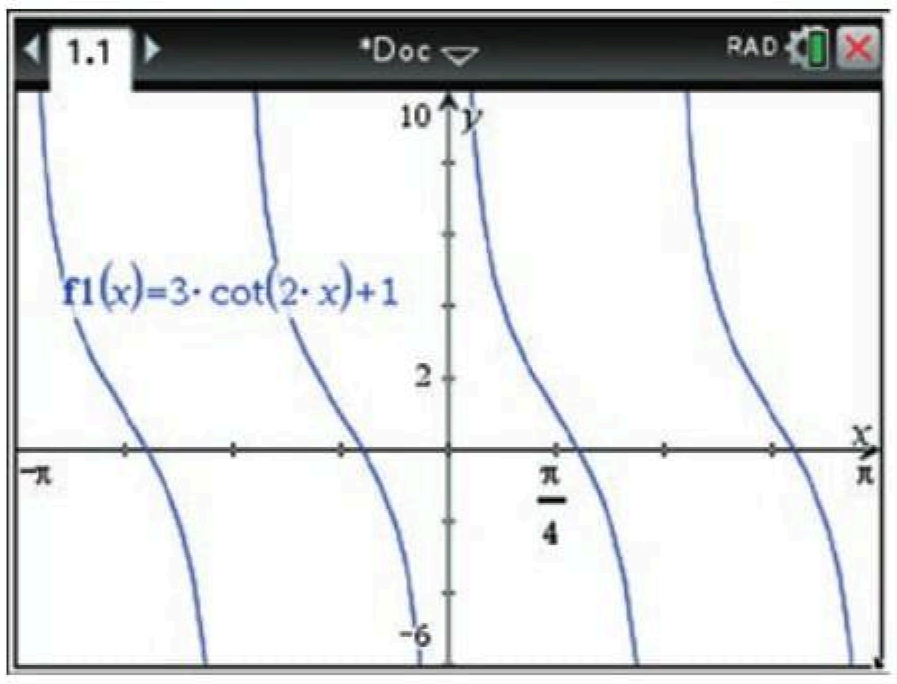
e



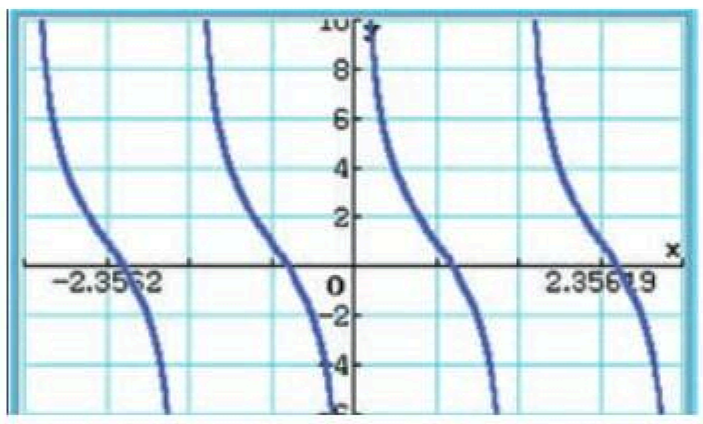
f



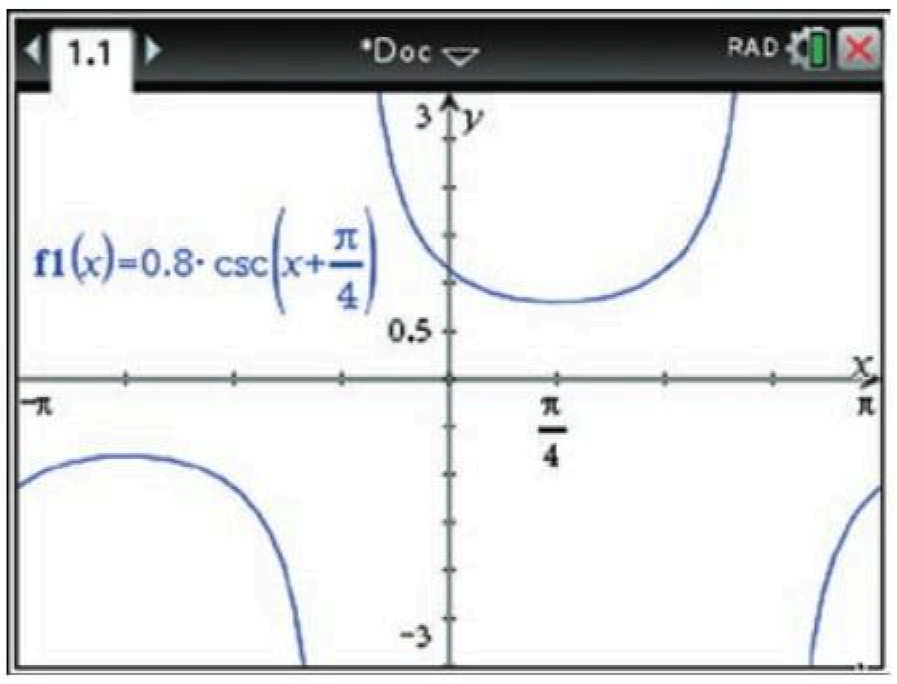
8 a



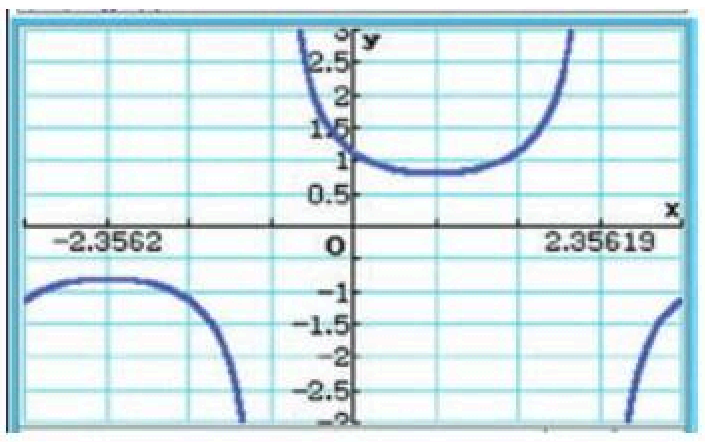
$f1(x)=3\cdot\cot(2\cdot x)+1$



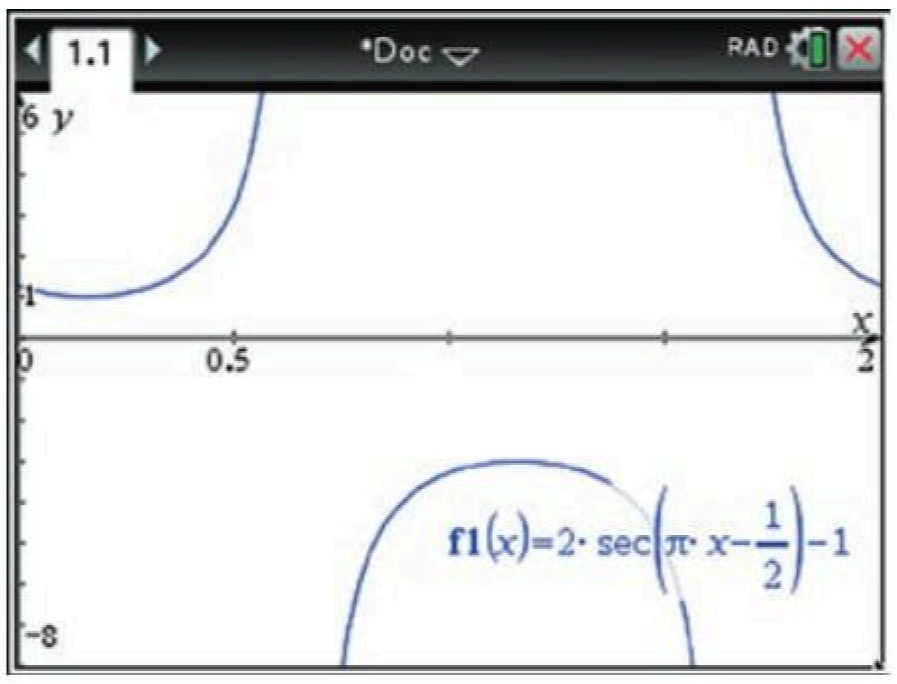
b



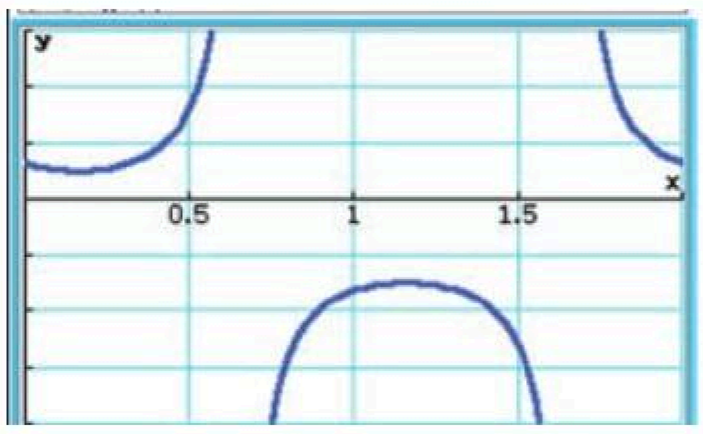
$f1(x)=0.8\cdot\csc\left(x+\frac{\pi}{4}\right)$



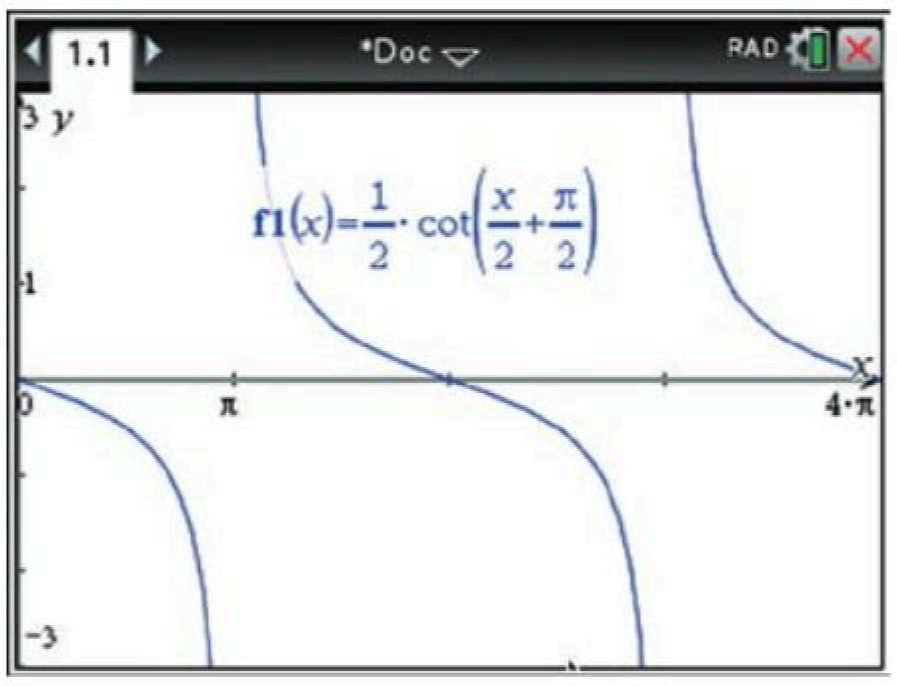
c



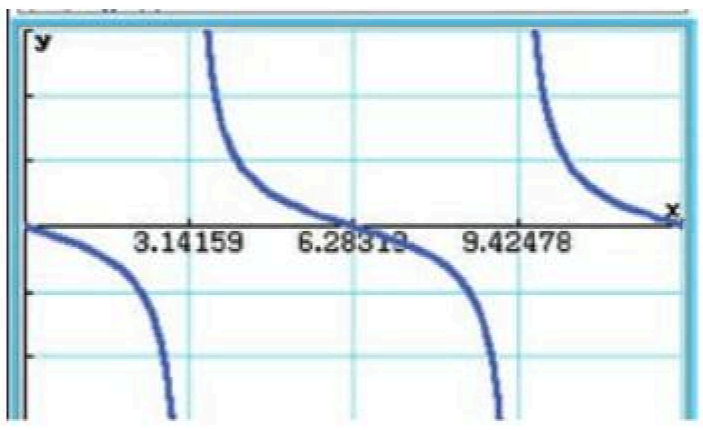
$f1(x)=2\cdot\sec\left(\pi\cdot x-\frac{1}{2}\right)-1$



d



$f1(x)=\frac{1}{2}\cdot\cot\left(\frac{x}{2}+\frac{\pi}{2}\right)$



9 a $f(x) = 2\sec(x) + 3$

b $f(x) = \text{cosec}(\pi x) - 2$

c $f(x) = 3\cot\left(\frac{x}{2}\right) + 1$

d $f(x) = 2\operatorname{cosec}\left(2x - \frac{\pi}{2}\right) - 3$

e $f(x) = \frac{1}{2}\sec(2\pi x) - 2$

f $f(x) = \frac{1}{4}\cot(2x) + 1$

10 a $x = (6n-1)\frac{\pi}{3}$ or $x = (3n-1)\frac{\pi}{3}$

b $x = (6n-1)\frac{\pi}{9}, \frac{2\pi}{9}(n+1)$

c $x = (4n-1)\pi$ or $x = (12n+5)\frac{\pi}{3}$

d $x = 2n\pi \pm \cos^{-1}\left(\frac{2}{3}\right)$ or $x = 2n\pi$

11 a $\frac{\pi}{8} \le x < \frac{\pi}{4}, \frac{3\pi}{4} < x \le \frac{7\pi}{8}, \frac{9\pi}{8} \le x < \frac{5\pi}{4}$ or $\frac{7\pi}{4} < x \le \frac{15\pi}{8}$

b $\pi < x \le \frac{7\pi}{6}, \frac{11\pi}{6} \le x < 2\pi$

c $0 \le x < \frac{3\pi}{8}, \frac{11\pi}{24} \le x < \frac{7\pi}{8}, \frac{23\pi}{24} \le x < \frac{11\pi}{8}, \frac{35\pi}{24} \le x < \frac{15\pi}{8}$

d $x = 0, \frac{5\pi}{18} \le x \le \frac{7\pi}{18}, x = \frac{2\pi}{3}, \frac{17\pi}{18} \le x \le \frac{19\pi}{18}, x = \frac{4\pi}{3}, \frac{29\pi}{18} \le x \le \frac{31\pi}{18}, x = 2\pi$

12 a $0 < x \le \frac{\pi}{9}, \frac{\pi}{3} < x \le \frac{4\pi}{9}$

$[0 < x \le 0.349, 1.047 < x \le 1.396]$

b $x = \pm 1.150, x = \pm 1.991$

c $x = n\pi \pm \frac{\pi}{6}$

d $\frac{\pi}{2} < x < \pi$

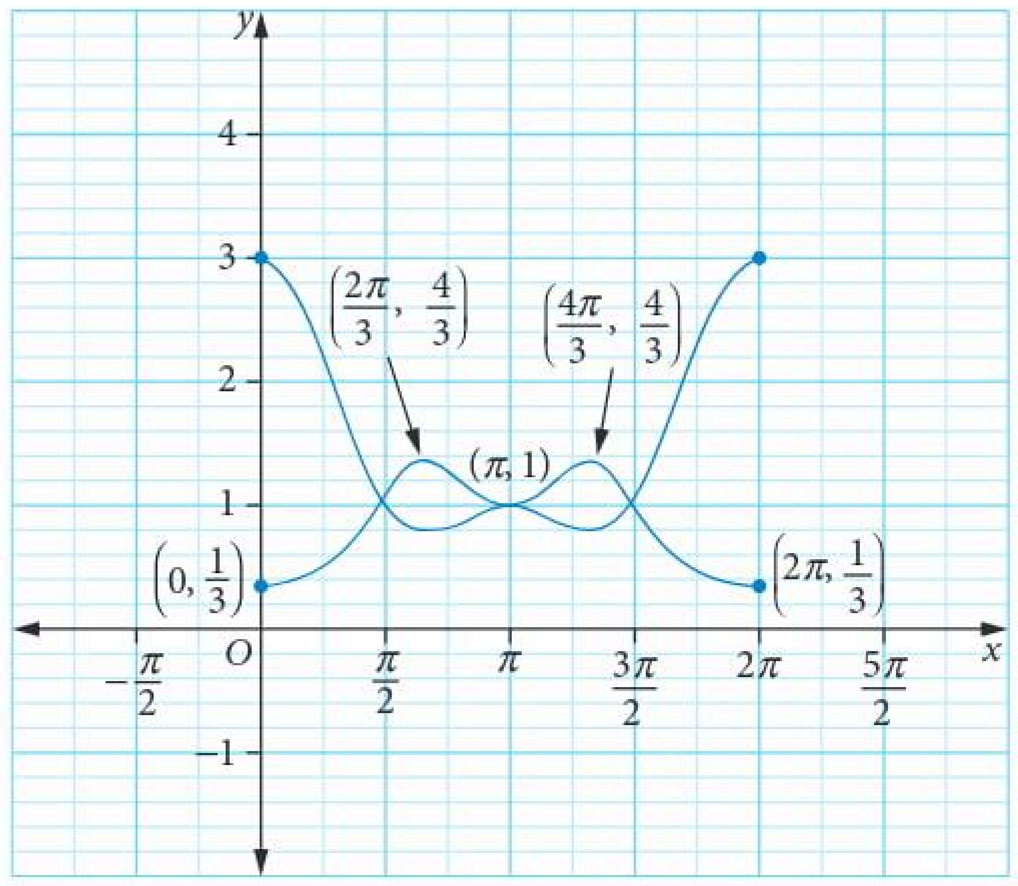
e $0.709 < x < 0.984$

f $0.120 \le x < 1.0472, 1.146 \le x < 2.094, 2.215 \le x < 3.142$

13 a i $-2\cos(x)\sin(x) - \sin(x)$ or $-\sin(2x) - \sin(x)$

ii $\left(\frac{2\pi}{3}, \frac{3}{4}\right), (\pi, 1), \left(\frac{4\pi}{3}, \frac{3}{4}\right)$

b



14 D **15** E **16** A **17** E

EXERCISE 2.7

1 D **2** B

3 a $\frac{\sqrt{2}+\sqrt{6}}{4}$ **b** $\sqrt{6}-\sqrt{2}$ **c** $2+\sqrt{3}$ **d** $1-\sqrt{2}$

4 First use the expansion of cos (2*x*) to find

a $\sin\left(-\frac{\pi}{12}\right)$ **b** $\cos\left(\frac{7\pi}{12}\right)$

c both $\sin\left(-\frac{7\pi}{12}\right)$ and $\cos\left(-\frac{7\pi}{12}\right)$

5 a $-\sqrt{2(2+\sqrt{2})}$ **b** $\frac{\sqrt{2}+\sqrt{6}}{4}$ **c** $1-\sqrt{2}$

6 First use the expansion of cos (2*x*) to find

a $\sin\left(\frac{3\pi}{8}\right)$ **b** $\cos\left(-\frac{7\pi}{8}\right)$,

c both $\sin\left(\frac{5\pi}{8}\right)$ and $\cos\left(\frac{5\pi}{8}\right)$

7 a $-\frac{\pi}{2}, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ **b** $-\pi, \pi$

8 a $-\pi \le x \le -\frac{7\pi}{8}, -\frac{5\pi}{8} \le x \le -\frac{3\pi}{8}, -\frac{\pi}{8} \le x \le \frac{\pi}{8}, \frac{3\pi}{8} \le x \le \frac{5\pi}{8}, \frac{7\pi}{8} \le x \le \pi$

b $x \in \left[\frac{\pi}{9}, \frac{2\pi}{9}\right] \cup \left[\frac{4\pi}{9}, \frac{5\pi}{9}\right] \cup \left[\frac{7\pi}{9}, \frac{8\pi}{9}\right]$

9 $\frac{a+b}{2}$ **10** $\frac{1}{4}$ **11** $-\frac{2\sqrt{6}}{3}$

12 a $2\sqrt{2}$ **b** $\sqrt{4+\sqrt{2}}$

c Change to sin (*x*) and cos (*x*) and a common denominator

d Use cos (2*x*) and $\sec^2(x) = 1 + \tan^2(x)$

13 $x = \sqrt{2}, -\sqrt{2}$ **14** $\frac{1}{2}$ **15** $-\frac{1}{5}$

16 B **17** A **18** E **19** C **20** D

21 Use the Pythagorean identity:

$\sin\left(\frac{\pi}{6}\right) = 2\sin\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{12}\right)$

EXERCISE 2.8

1 C **2** C

3 a $-\frac{\pi}{6}$ **b** $\frac{\pi}{4}$ **c** $\frac{5\pi}{6}$

d $-\frac{\pi^2}{4}$ **e** 0 **f** −2

4 a $-\frac{\pi}{4}$ **b** $-\frac{\pi}{6}$ **c** $\frac{\pi}{4}$

d −0.8481 **e** 0.3764 **f** 1.3734

5 a domain: *R*, range: $\left(-\frac{3\pi}{2} + 2, \frac{3\pi}{2} + 2\right)$

b domain: [−4, 0], range: $\left[-\frac{\pi}{6} - 4, \frac{\pi}{6} - 4\right]$

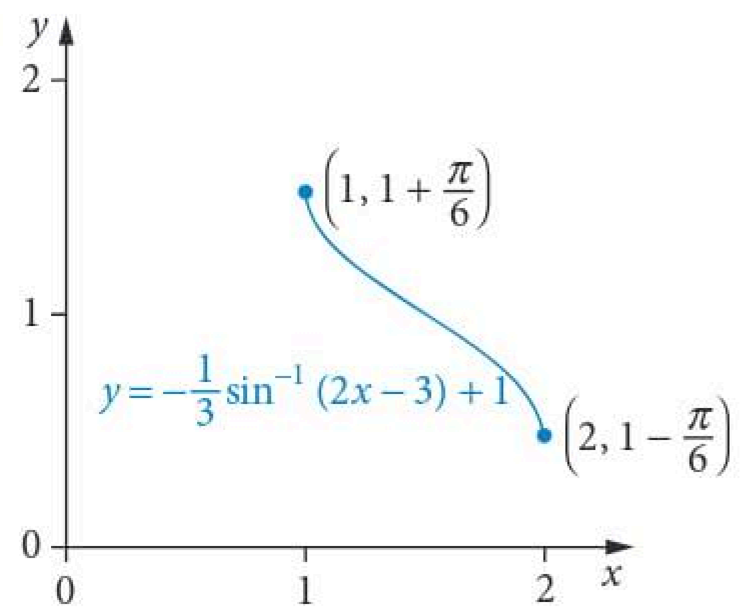
c domain: [3, 9], range: $[-\pi + 1, \pi + 1]$

d domain: $\left[-\frac{1}{2}, \frac{1}{2}\right]$, range: $[-\pi + 3, \pi + 3]$

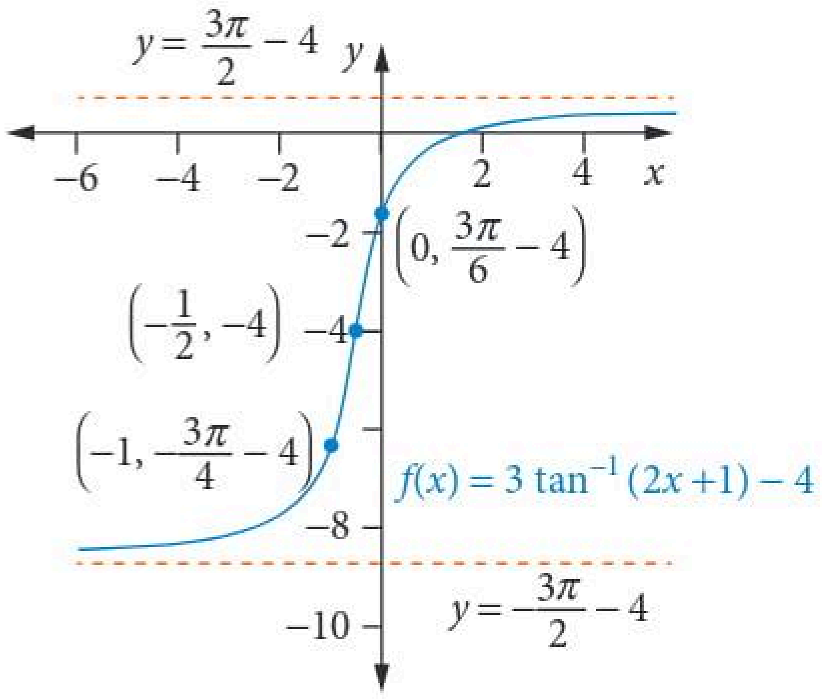
e domain: $[-\sqrt{3}, \sqrt{3}]$, range: $\left[1, 1 + \frac{2\pi}{3}\right]$

f domain: [0, 2], range: $\left[-2, \frac{\pi}{2} - 2\right]$

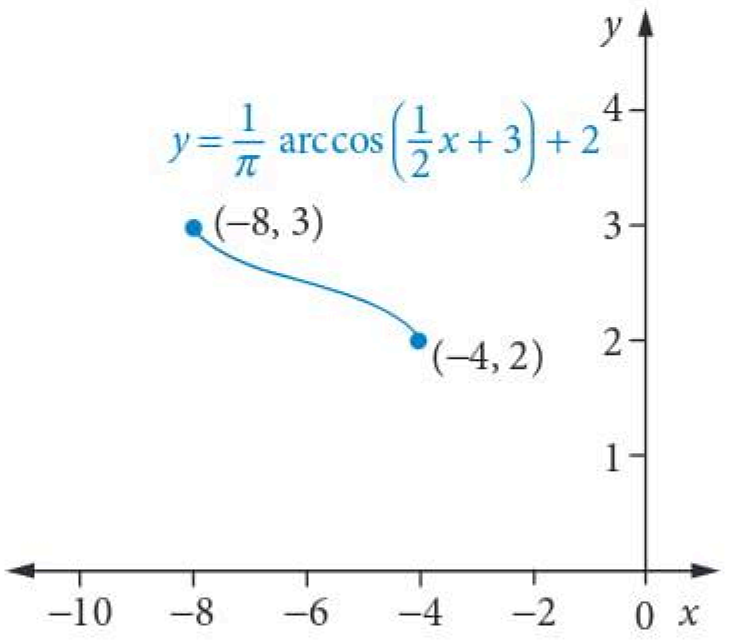
6 a



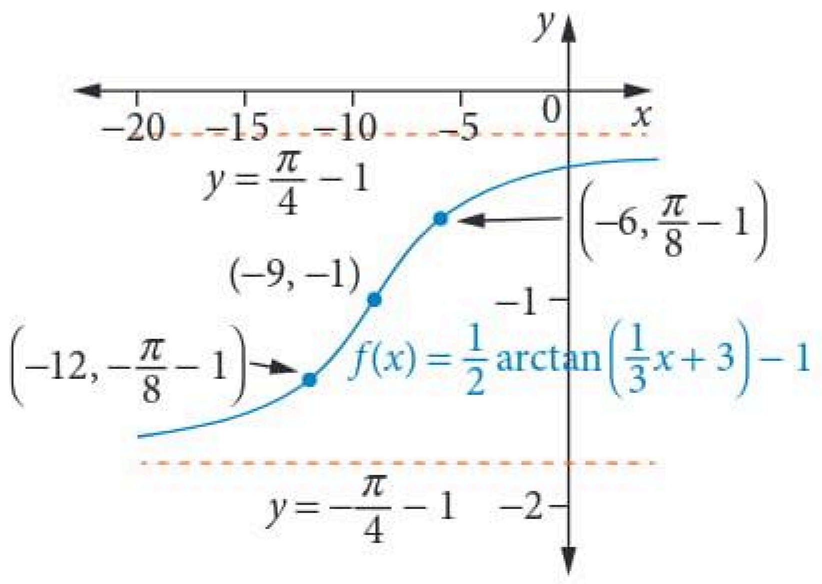
b



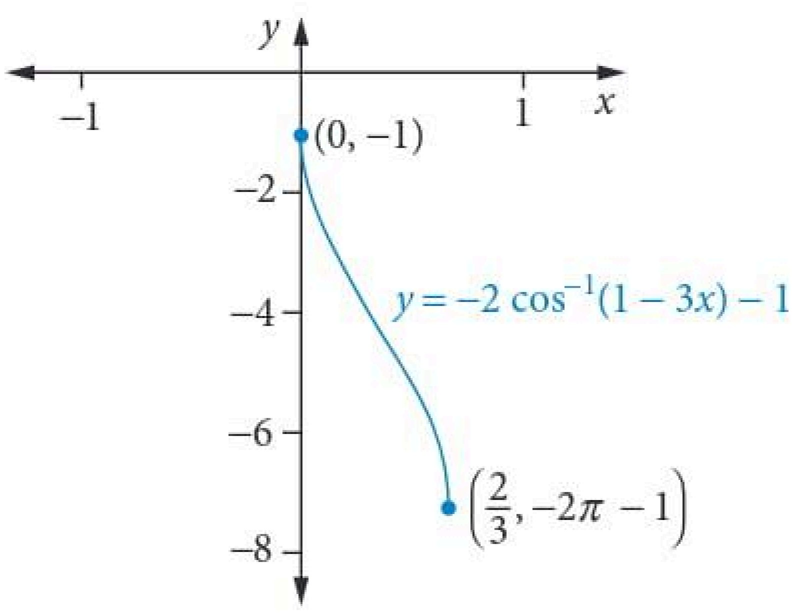
c



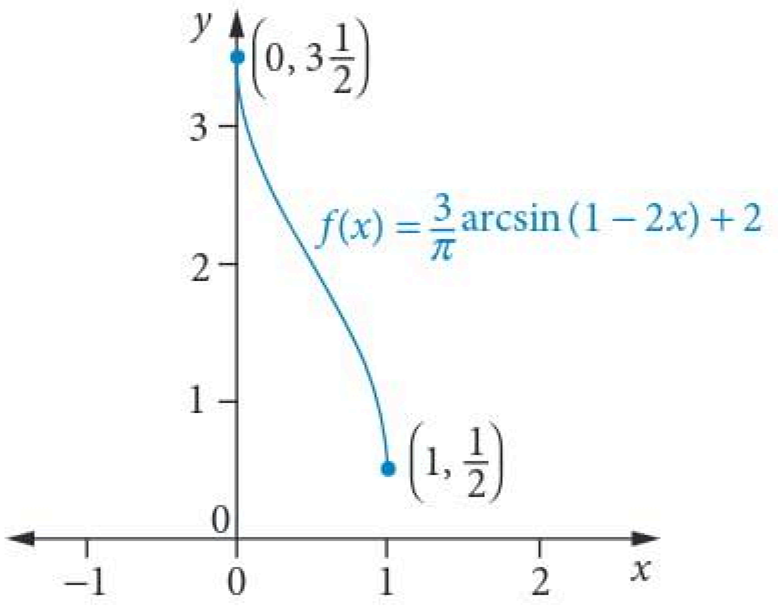
d



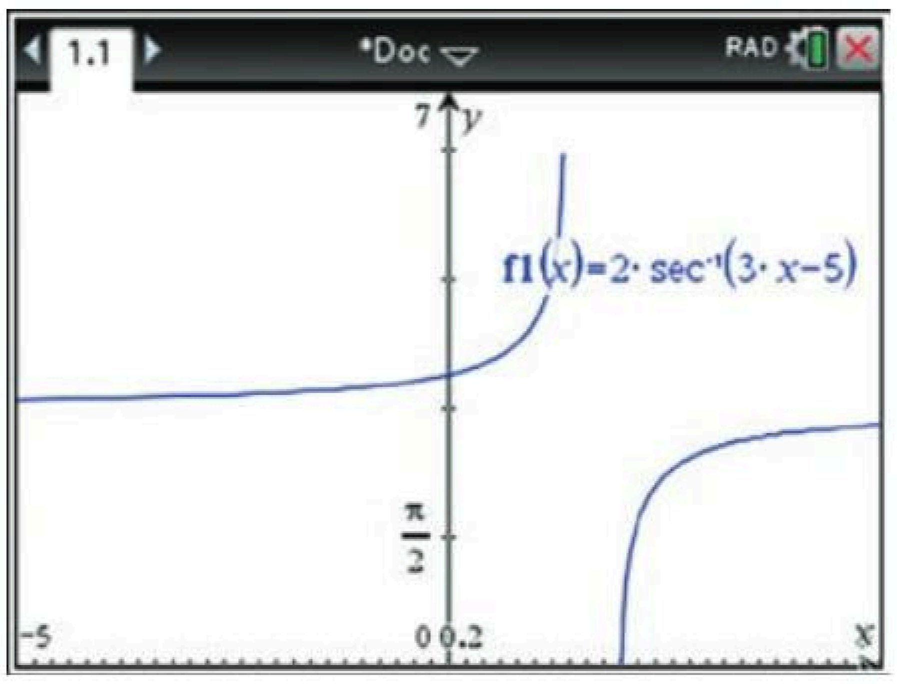
e

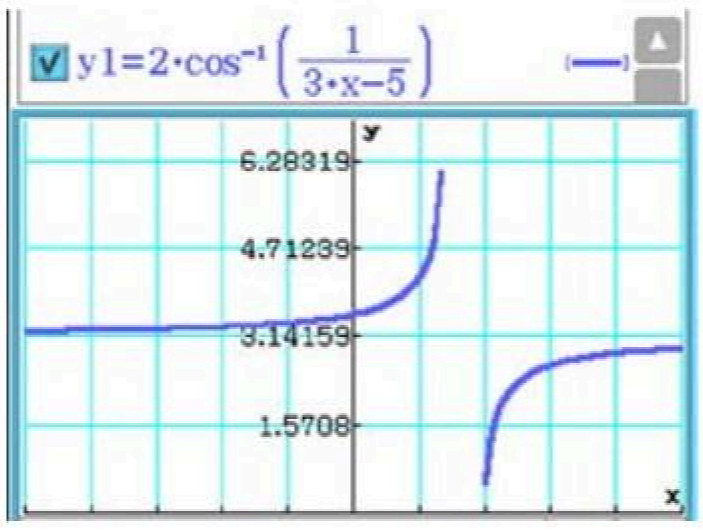


f

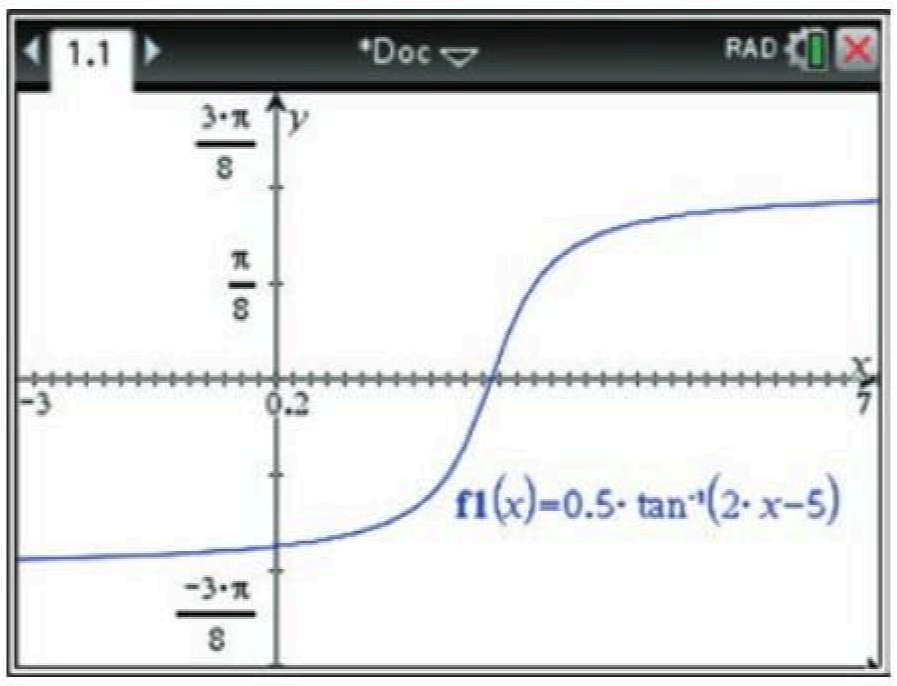


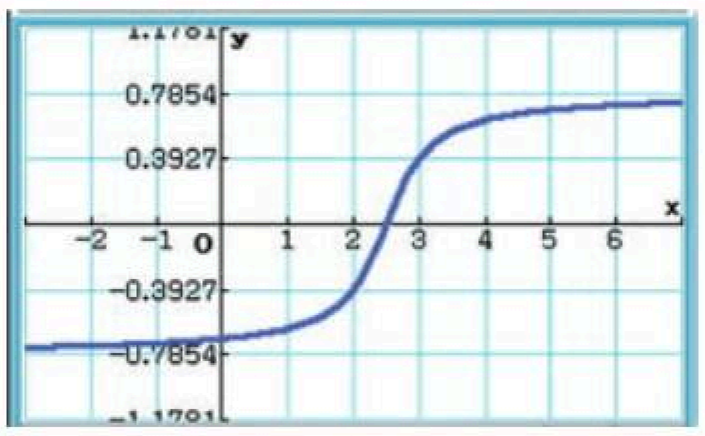
7 a



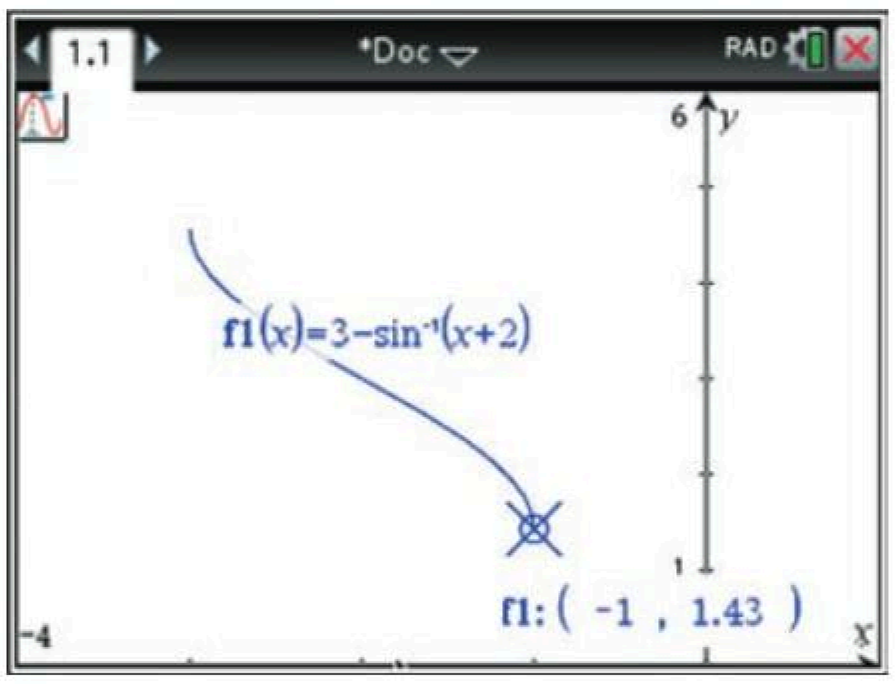


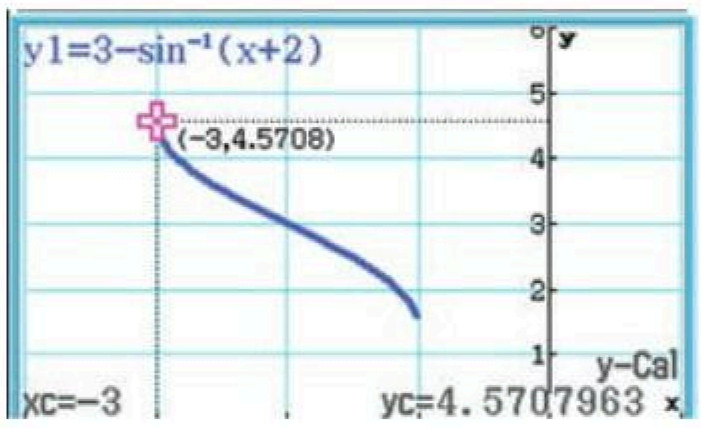
b





c





8 **a** domain: $\left[\frac{c-1}{b}, \frac{c+1}{b}\right]$, range from $-\frac{a\pi}{2}$ to $\frac{a\pi}{2}$
(or vice versa depending on the sign of a)

b For $a \geq 0$: domain: $\left[\frac{-c}{b}, \frac{1-c}{b}\right]$, range: $\left[0, \sqrt{\frac{a\pi}{2}}\right]$.

For a < 0 domain: $\left[\frac{-c-1}{b}, \frac{-c}{b}\right)$, range: $\left[-\sqrt{\frac{a\pi}{2}}, 0\right)$.

c domain: $R \setminus \left\{\frac{c}{b}\right\}$, range: $\left(-\infty, -\frac{2a}{\pi}\right) \cup \left(\frac{2a}{\pi}, \infty\right)$

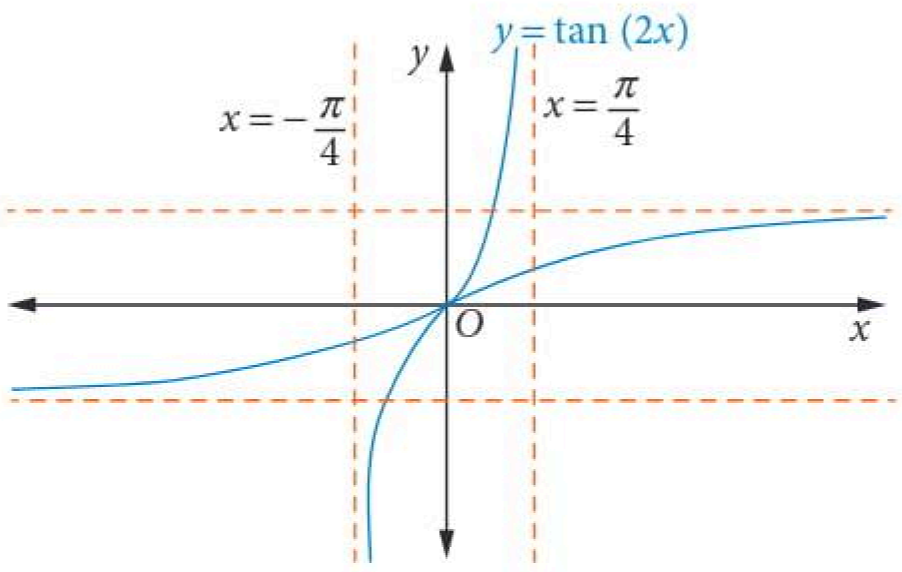
d domain: $\left[-\frac{1}{b}, \frac{1}{b}\right]$, range from $a \times e^{-0.5c\pi}$ to $a \times e^{0.5c\pi}$
(or vice versa depending on the sign of a)

e domain: $\left[\frac{c-1}{b}, \frac{c+1}{b}\right]$,
range: $\left[-\frac{(c-1)\pi}{2b} + a, \frac{(c+1)\pi}{2b} + a\right]$

f domain: R, range: $\left[-a, \frac{\pi^2}{4} - a\right)$

9 **a** **i** $y = \pm\frac{\pi}{4}$

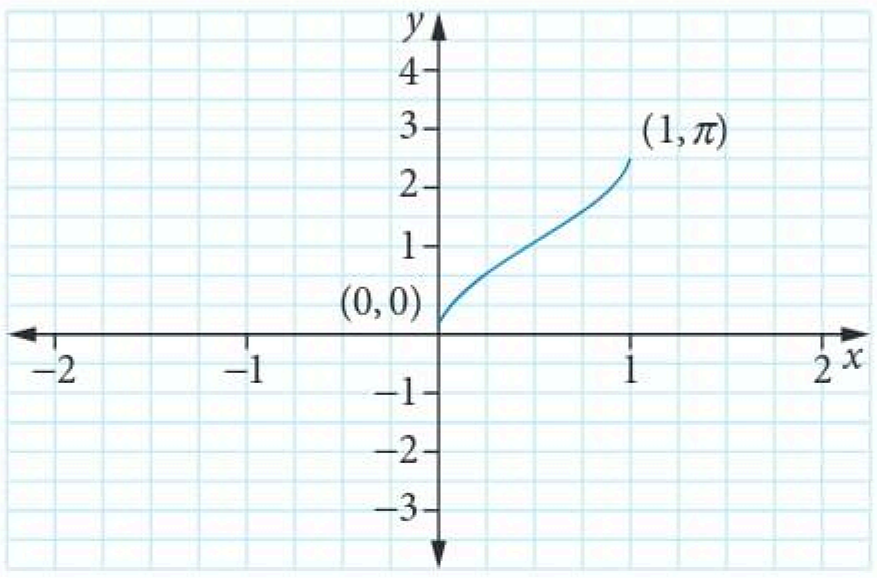
ii



b $\frac{\pi}{6}$

10 **a** domain: $[0, 1]$, range: $[0, \pi]$

b



11 domain: $x \in [-2, 2]$, range: $\left[0, \sqrt{\pi}\right]$

12 **a** **i** $[-2, 2]$

ii $R \setminus \left[\frac{1}{5}, \frac{1}{5}\right]$

iii $[-2, 2] \setminus \left[-\frac{1}{5}, \frac{1}{5}\right]$

b $\frac{5\sqrt{7}}{16}$

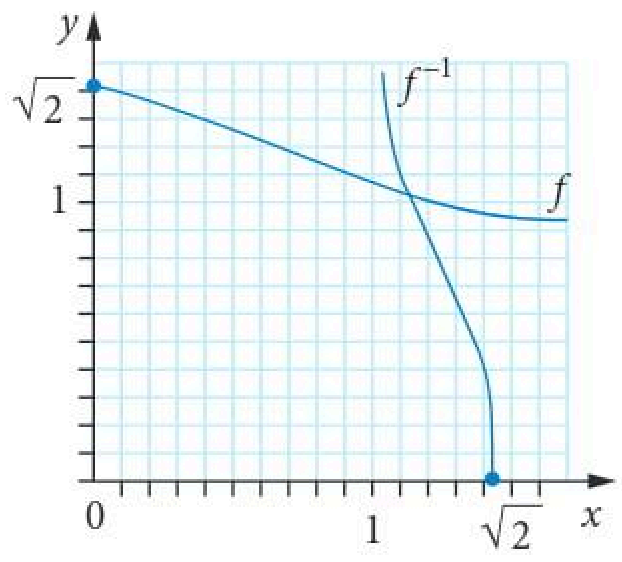
13 $[0, \infty)$ **14** C **15** D **16** D
17 E **18** B **19** E **20** A
21 A **22** B **23** B

24 **a** $\sqrt{2}$, 1

b $f^{-1}(x) = \arcsin\left(\sqrt{2 - x^2}\right)$, domain: $\left[1, \sqrt{2}\right]$,

range: $\left[0, \frac{\pi}{2}\right]$

c



d 1.099

25 **a** $\left(\sqrt{2}, \frac{3\pi}{4}\right)$

b $g(x) = \begin{cases} -3\arccos\left(-\frac{x}{2}\right), & -2 \leq x < -\sqrt{2} \\ -3\arcsin\left(-\frac{x}{2}\right), & -\sqrt{2} < x \leq 0 \end{cases}$

CUMULATIVE EXAMINATION 1

1 $d = 16$ 76% **2** $\overrightarrow{AB} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ 86%

3 $x = (2n+1)\frac{\pi}{2}, 2n\pi + \frac{\pi}{3}, 2n\pi + \frac{2\pi}{3}$ for $n \in Z$ 28%

4 $x = -2$ or 22 75%

CUMULATIVE EXAMINATION 2

Section A

1 A 77% **2** E 62% **3** D 85%
4 D 43% **5** A 69%

Section B

1 **a** $a = 6, b = 2, c = -3$ 45% **b** $\cos(\theta) = \frac{4}{9}$ 75%

c $2\sqrt{65}$ square units 27%

d $\overrightarrow{AB} = (2, -1, -2), \overrightarrow{AD} = (2, 4, -4)$ 68%

Let $\underset{\sim}{v} = 6\underset{\sim}{i} + 2\underset{\sim}{j} + 5\underset{\sim}{k}$

$\overrightarrow{AB} \cdot \underset{\sim}{v} = 12 - 2 - 10 = 0$, so $\overrightarrow{AB}$ is perpendicular to $\underset{\sim}{v}$.

$\overrightarrow{AD} \cdot \underset{\sim}{v} = 12 + 8 - 20 = 0$, so $\overrightarrow{AD}$ is perpendicular to $\underset{\sim}{v}$.

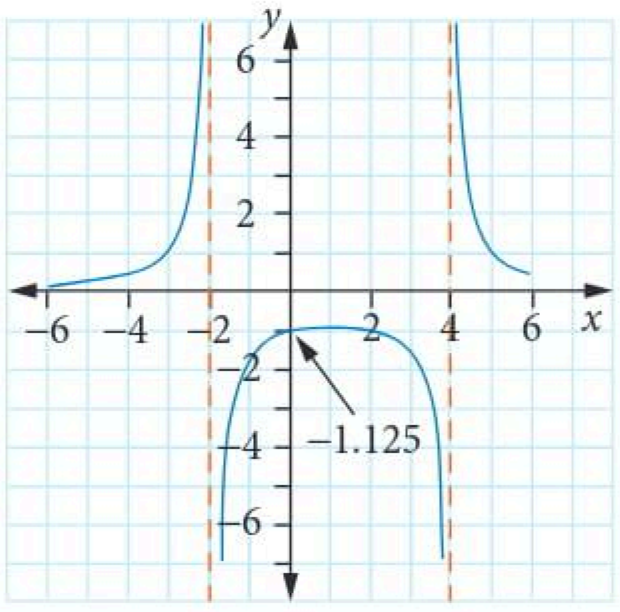
$\underset{\sim}{v} = \sqrt{36 + 4 + 25} = \sqrt{65}$

Thus $\hat{\underset{\sim}{v}} = \frac{1}{\sqrt{65}}(6\underset{\sim}{i} + 2\underset{\sim}{j} + 5\underset{\sim}{k})$ is a unit vector perpendicular to the base of the pyramid.

2 **a** $(1, -1)$ 93%

b $x = 2, x = 4, y = 0$ 88%

c 83%



CHAPTER 3

EXERCISE 3.1

1 **a** iv **b** i, iii **c** ii, iii

d i, iii **e** ii, iii **f** iv

2 **a** deductive reasoning **b** neither

c inductive

3 **a** Not all cockatoos are white.

b It cannot be verified that all left-handed people use left-handed scissors.

c The conclusion is based on a sample of homes. It doesn't include all display homes.

d Finding a parking spot depends on variables such as time of day, which shops are visited and luck.

4 E

5 **a** 'Most triangles' does not mean all triangles. Some triangles have an obtuse interior angle.

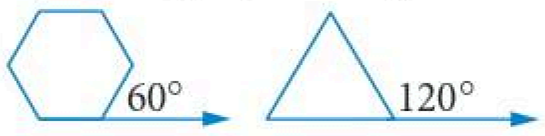
b A polygon consists of straight sides. A circle does not.

c There are two-dimensional shapes such as a circle that are not polygons.

6 **a** For $n = 3$, $6n - 1 = 17$, which is prime, so it is true. For $n = 11$, $6n - 1 = 65$, which is not prime.

b A square has opposite sides of equal length and it has the properties of a rectangle. However, a parallelogram and rhombus have opposite sides of equal length, but they are not rectangles.

c



d For $x = 3$, $3^2 > 3$, so true. For $x = \frac{1}{2}$, $\left(\frac{1}{2}\right)^2 < \frac{1}{2}$, so false.

e For $n = 4$, $4^2 = 16$, so true. For $n = 6$, $6^2 = 36$, so false.

7 A **8** D **9** D

10 **a** $1 + (-1 + 1) + (-1 + 1) - \ldots = 1$ or

$(1 - 1) + (1 - 1) + \ldots = 0$

b
$$\begin{aligned} S &= 1 - 1 + 1 - 1 + 1 - \ldots \\ &= 1 - (1 - 1 + 1 - \ldots) \\ &= 1 - S \end{aligned}$$

$2S = 1 \Rightarrow S = \frac{1}{2}$

11 **a** $t(n) = \frac{6n + 1}{3}, n = 1, 2, 3, \ldots$

b $t(4) = \frac{25}{3} = 8\frac{1}{3}, t(5) = \frac{31}{3} = 10\frac{1}{3}, t(6) = \frac{37}{3} = 12\frac{1}{3}$

c $t(n) = 2n + \frac{1}{3}$. The whole number part is even, starting with 8. The fraction is always $\frac{1}{3}$.

12 **a** $f(1) = 3, f(2) = 9, f(3) = 18, f(4) = 30$

b $f(n) = f(n - 1) + 3n$ **c** $f(7) = 108$

13 C **14** C

15 **a** $1^2 = 1, 2^2 = 4, 3^2 = 9, 4^2 = 16$

$1^3 = 1, 2^3 = 8, 3^3 = 27, 4^3 = 64$

$1^4 = 1, 2^4 = 16, 3^4 = 81, 4^4 = 256$

b $3^2 - 2^3 = 1$

16 **a** $x_3 = 12 + 4 \times 3 = 24$

$x_4 = 24 + 4 \times 4 = 40$

$x_5 = 40 + 4 \times 5 = 60$

b $x_n = x_{n-1} + 4n$

$x_5 = x_4 + 2 \times 5 = 60$

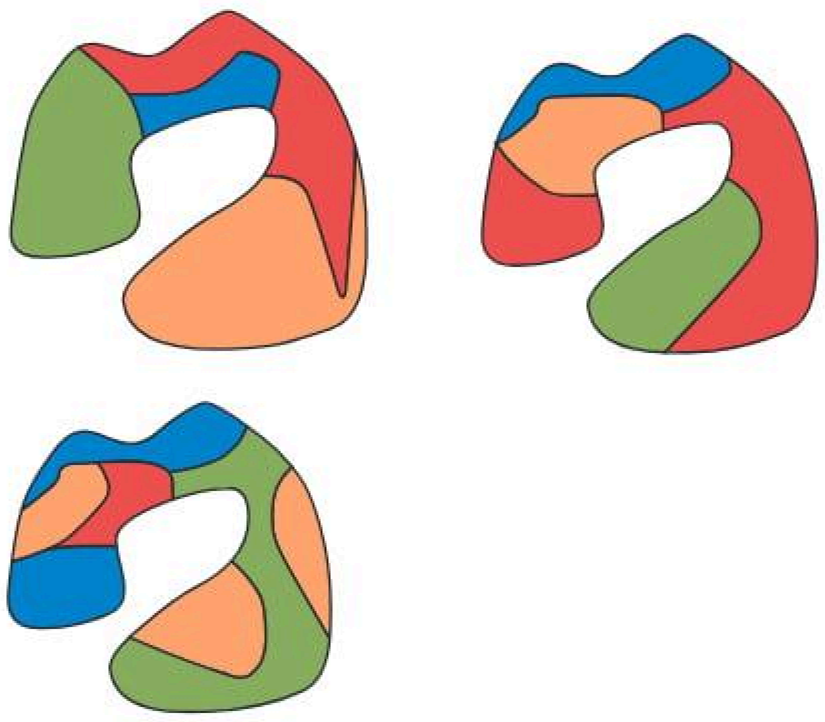
c $x_n = 12n^2$

17 **a** $20 \div 2 = 10$, $10 \div 2 = 5$, $3 \times 5 + 1 = 16$, $16 \div 2 = 8$, $8 \div 2 = 4$, $4 \div 2 = 2$, $2 \div 2 = 1$

b $3 \times 25 + 1 = 76$, $76 \div 2 = 38$, $38 \div 2 = 19$, $3 \times 19 + 1 = 58$, $58 \div 2 = 29$, $3 \times 29 + 1 = 88$, $88 \div 2 = 44$, $44 \div 2 = 22$, $22 \div 2 = 11$, $3 \times 11 + 1 = 34$, $34 \div 2 = 17$, $3 \times 17 + 1 = 52$, $52 \div 2 = 26$, $26 \div 2 = 13$, $3 \times 13 + 1 = 40$, $40 \div 2 = 20$, $20 \div 2 = 10$, $10 \div 2 = 5$, $3 \times 5 + 1 = 16$, $16 \div 2 = 8$, $8 \div 2 = 4$, $4 \div 2 = 2$, $2 \div 2 = 1$

c Start with any whole number. If the number is even, divide by 2, and if the number is odd, multiply by 3 and add 1. Apply the method to each result until the answer is 1.

18 **a** No more than four colours needed.



b A maximum of four colours are needed to colour any region.

19 **a**

Point	Number of intersections
1	5
2	10
3	7
4	9
5	8
6	6
7	5
8	4
9	3
10	2

b Odd number of intersections; inside even number of intersections: outside.

20 **a** The number of squares crossed is the sum of rows (R) and columns (C) minus the highest common factor of R and C.

b $198 + 187 - 11 = 374$

21 **a** $\text{LHS} = |-5 + 3| = |-2| = 2$

$\text{RHS} = |-5| + |3| = 5 + 3 = 8, \ 2 \le 8$

b $x \le |x|, y \le |y|$

$x + y \le |x| + |y|$

$|x + y| \le ||x| + |y||$

But $||x| + |y|| = |x| + |y|$, so $|x + y| \le |x| + |y|$.

EXERCISE 3.2

1 A

2 A

3 **a** If it's winter, it's cold and people ski.

b If it's summer or it's not cold, I swim.

c If people are skiing or it's cold, then it's not summer and I am not swimming.

4 **a** A: A number is rational. B: A number can be expressed as the ratio of two integers. $A \leftrightarrow B$.

b A: I pass tests if I study hard B: When I study hard I pass tests. $A \leftrightarrow B$.

c A: It's raining B: It's cold C: I stay indoors $(A \vee B) \rightarrow C$.

5 A: A quadratic equation has two x-intercepts.

B: The discriminant of a quadratic equation is positive.

6 Show $\neg(A \wedge B) = \neg A \vee \neg B$

$A = \{2,3,4,5,6,7,8,9\}, B = \{1,3,5,7\}$
$A \wedge B = \{3,5,7\}, \neg(A \wedge B) = \{1,2,4,6,8,9,10\}$
$\neg A = \{1,10\}, \neg B = \{2,4,6,8,9,10\},$
$\neg A \vee \neg B = \{10\}$

Show $\neg(A \vee B) = \neg A \wedge \neg B$
$A \vee B = \{1,2,3,4,5,6,7,8,9\}, \neg(A \vee B) = \{10\}$
$\neg A \wedge \neg B = \{10\}$

7 **a** $A \rightarrow (B \vee C)$ **b** $A \wedge B \wedge C \rightarrow \neg D$

c $\neg A \rightarrow \neg B \vee C$

8 A rational number need not be the sum of a recurring and a terminating decimal.

9 *Converse*: If the points are collinear, then they lie on the same line.

Biconditional: Points lie on the same line if and only if they are collinear.

10 E

11 **a** 'If it is a hot day, then it is sunny'. Not always true. Could be a humid overcast day.

b 'When I go to the supermarket to buy food, I am hungry'. Not always true.

c 'If this shape has exactly three sides, then it is a triangle'. Always true.

12 **a** A pair of lines intersect at one point if and only if they are not parallel.

b The square of a natural number is odd if and only if the number is odd.

13 **a** If two angles are supplementary, then their sum is 180°.

b The sum of two angles is 180° if and only if the two angles are supplementary.

14 *Converse*: 'If a figure has four right angles, then it is a square'. Not a good definition because a rectangle also has four right angles.

15 n is divisible by both 3 and 4 means $n = 12m, m \in N$, which is divisible by 12.

n divisible 12 means $n = 12m = 3 \times 4 \times m,\ m \in N$.
n is divisible by both 3 and 4.

16 If n is odd. $n = 2m + 1,\ m \in N$,
$n^2 = 4(m^2 + m) + 1 = 4k + 1,\ k \in N$, which is odd.

17 Show $\neg(A \wedge B) = \neg A \vee \neg B$

$A \wedge B = \{5,7\}, \neg(A \wedge B) = \{1,2,3,4,6,8,9,10\}$

$\neg A = \{1,2,3,10\}, \neg B = \{2,4,6,8,9,10\},$

$\neg A \vee \neg B = \{2, 10\}$

Show $\neg(A \vee B) = \neg A \wedge \neg B$

$A \vee B = \{1,3,4,5,6,7,8,9\}, \neg(A \vee B) = \{2,10\}$
$\neg A \wedge \neg B = \{2,10\}$

18 Continuity is a necessary but not sufficient condition for differentiability. For example, $y = |x|$ is continuous at $x = 0$ but not differentiable at that point.

So $P \rightarrow Q$ is true. The converse, $Q \rightarrow P$ means that if the function is continuous at a point, then it is also differentiable there. This is not the case for $y = |x|$ at $x = 0$. Thus, continuity is a necessary but not a sufficient condition for differentiability.

19
$$\begin{aligned}\neg(A_1 \wedge A_2 \wedge \ldots \wedge A_N) &= \neg(A_1 \wedge (A_2 \wedge \ldots \wedge A_N)) \\ &= \neg A_1 \vee \neg(A_2 \wedge \ldots \wedge A_N)) \\ &= \neg A_1 \vee \neg(A_2 \wedge (A_3 \wedge \ldots \wedge A_N)) \\ &= \neg A_1 \vee \neg A_2 \vee \neg(A_3 \wedge \ldots \wedge A_N))\end{aligned}$$

Continue in this way until
$\neg A_1 \vee \neg A_2 \vee \ldots \vee \neg(A_{N-1} \vee A_N)) =$
$\neg A_1 \vee \neg A_2 \vee \ldots \wedge \neg A_{N-1} \vee \neg A_N$

20 **a**
$$\begin{aligned}\neg((A \vee \neg B) \vee C) &= \neg(A \vee \neg B) \wedge \neg C \\ &= \neg A \wedge \neg(\neg B) \wedge \neg C \\ &\neg A \wedge \neg B \wedge \neg C\end{aligned}$$

b $\neg(\neg(A \vee B) \wedge (\neg C \wedge \neg D))$
$= (A \vee B) \vee \neg(\neg C \wedge \neg D)$
$\equiv A \vee B \vee C \vee D$

c $\neg(A \vee B \vee C)$
$= \neg((A \vee B) \vee C)$
$= \neg(A \vee B) \wedge \neg C$
$\equiv \neg A \wedge \neg B \wedge \neg C$

or

$\neg(A \vee B \vee C)$
$= \neg(A \vee (B \vee C))$
$= \neg A \wedge \neg(B \vee C)$
$\equiv \neg A \wedge \neg B \wedge \neg C$

21 $A \rightarrow (B \vee C) \vee (B \wedge C)$

22 Proofs: see worked solutions

EXERCISE 3.3

1 D

2 B

3 $n = 2k + 1,\ k \in N$

$$\begin{aligned}(2k+1)^4 &= 2(8k^4 + 16k^3 + 12k^2 + 4k) + 1 \\ &= 2m + 1,\ m \in N\end{aligned}$$

4 Proof. See worked solutions

5 $m = 2a + 1,\ a \in N$
$n = 2b + 1,\ b \in N$
$mn = 2(2ab + a + b) + 1 = 2p + 1, p \in N$

6 $n = 2k,\ k \in N$

$7n + 4 = 2(7k + 2) = 2p,\ p \in N$

7 $n = 2m$

$2n^2 + 4n + 13 = 2(4m^2 + 4m + 6) + 1$

$= 2k + 1,\ (k = 4m^2 + 4m + 6)$

8 $a = kb, k \in N$ so $a \le b$.

9 $3n + 2$ odd $\rightarrow 3n$ is odd because odd + even = odd.

$3n$ odd means n odd, because odd × odd = odd.

10 a $\forall x \in R^+ (x^2 > 0)$

b $\forall x \in R\, \exists y \in R\, (y = x + 1)$

c $\forall x \in R^+\, \forall y \in R^+\, \exists A \in R^+ (A = xy)$

d $\forall x \in R\, \forall y \in R\, \exists z = x + yi \in C\, (x^2 + y^2 \ge 0)$

e $\forall a \in R \setminus \{0\} \forall b \in R\, \forall c \in R\, \exists z \in C$

$(b^2 - 4ac < 0)$

11 a true

b false; $m = 2, n = \frac{2}{3}$

c true

d false; $m = 7, n = 5$ but $7 \ne \sqrt{5}$

12 a $m = 2a\ ,\ n = 2b,\ a, b$ integers

$m - n\ = 2(a - b) = 2k,\ k$ integer

b $(m + n)^2 = m^2 + n^2 - 2mn = M - N$ is even, where $M = m^2 + n^2$ is even and $N = 2mn$ is even.

13 $m = \frac{a}{b}, n = \frac{c}{d},\ a, b, c, d \in Z$

$mn = \frac{ac}{bd} = \frac{e}{f}, e, f \in N \Rightarrow mn$ is rational

14 Hint: Show the expression is divisible by 2 and by 3.

One of n or $n + 1$ is even and the other odd, so the product is even, hence divisible by 2.

15 Hint:

a Use integers a and b with $m = 2a + 1, n = 2b + 1$ to express $m^2 + n^2$ in the form $2k$, where k is an expression involving a and b.

b Modify your answer to part **a** so that it is in the form $4p + q$, where p and q are integers.

16 a $3 \times (-2) - 9 = -15$

b $(4xy + 2z + 1)^2 = 2[2(2xy + z)^2 + 2(2xy + z)] + 1 = 2n + 1,\ n = 2(2xy + z)^2 + 2(2xy + z)$

17 The question is asking to show that if s is odd, then $s^2 + 3s + 5$ is odd.

$s = 2k + 1,\ s^2 + 3s + 5 = 2(2k^2 + 5k + 4) + 1 = 2q + 1$, which describes an odd number.

18 b^2 is odd, a^2 is even, $b^2 + a^2$ is odd $b^2 + a^2 + 1$ is even, so $b^2 + a^2 + 1 = 2k$.

19 $m = 2k + 1, k \in N$

$2k + 1 = (k + 1)^2 - k^2$

$= a^2 - b^2,\ a = k + 1,\ b = k$

20 a The product of two consecutive whole numbers is even.

Let n be the first number, so $n + 1$ is the second number.

$n(n + 1)$; the product of odd/even or even/odd is even.

b The sum of two complex numbers is a complex number.

$z = z_1 + z_2 i,\ w = w_1 + w_2 i$

$z + w = (z_1 + w_1) + (z_2 + w_2)i = a + bi \in C)$, with $a = z_1 + w_1,\ b = z_2 + w_2$

21 a $3^n = 3^1 \times 3^2 \times 3^3 \times 3^4 \times \ldots \times 3^n$

$3^1 \times 3^2$ is odd $(3^1 \times 3^2) \times 3^3$ odd × odd = odd

$(3^1 \times 3^2 \times 3^3) \times 3^4$ odd × odd = odd and so on up to 3^n.

b 3^n is odd, so $3^n + 1$ even and $(3^n + 1)n$ is the product of n even numbers, which is even.

22 Let $x = 2m + 1$, for integer m.

$(2m + 1)^n = k_0 (2m)^n 1^0 + k_1 (2m)^{n-1} 1^1 +$

$k_2 (2m)^{n-2} 1^2 + \ldots + k_n$

$= 2\left(k_0 2^{n-2} m^{n-1} + k_1 2^{n-3} m^{n-2} + k_2 2^{n-4} m^{n-3} + \ldots + 1\right) + k_n$

$= 2K + k_n$, K a constant

$2K$ is even so need $k_n = 1$ to have $= 2K + 1$, odd.

EXERCISE 3.4

1 C

2 A

3 Prove that n even will lead to $5n - 7$ is odd.

Let $n = 2m$, for integer m.

$5n - 7 = 10m - 7 = 2(5m - 4) + 1$, which is odd.

4 D

5 Show if n is not divisible by 3, then n^3 is not divisible by 3.

Not divisible by 3 means remainder after division is 1 or 2

$n = 3k + 1$ or $3k + 2$ for $k \in N$.

If $n = 3k + 1,\ n^3 = 3(9k^3 + 9k^2 + 3k) + 1$, which is not divisible by 3.

If $n = 3k + 2,\ n^3 = 3(9k^3 + 18k^2 + 12k + 3) - 1$, which is not divisible by 3.

6 B

7 Assume $\sqrt[3]{2} = \frac{p}{q}$, p, q integers with no common divisor greater than 1.

$2q^3 = p^3$, p^3 has a factor of 2, so p has factor 2.
Let $p = 2k$, integer k.

$q^3 = 4k^3$, so q^3 has factor 2. Both p and q have factor 2, which is a contradiction.

8 Prove that $\neg c \rightarrow \neg(ab + c)$. c is even, so ab is even.

$\neg(ab + c)$ means $ab + c$ is even.

9 a Assume that there exist integers m, n such that $15m + 25n = 1$.

b 5 divides $15m + 25n$ but 5 does not divide 1. This is not possible.

10 Let $2n + 1, 2n + 3$ be the consecutive odd numbers.
Then $(2n + 3)^2 - (2n + 3)^2 = 8(n + 1)$.

So $8(n + 1) = 4k + 1$ or $8(n + 1) = 4k + 2$ or $8(n + 1) = 4k + 3$

$4(2n + 2 - k) = 1$ or $4(2n + 2 - k) = 2$ or $4(2n + 2 - k) = 3$, each of which is not possible.

11 Show that if n is odd then $n^3 + 5$ is even.

Let $n = 2k + 1$. Then $n^3 + 5 = (2k + 1)^3 + 1$
$= 2(4k^3 + 6k^2 + 3k + 1) = 2m$, integer m.

12 Show that $n^3 + 5$ odd and n odd leads to a contradiction.

$n^3 + 5 = 2k + 1$, for integer k.

$n^3 = 2(k - 2) = 2p$, integer p.

This implies that n is even because it is divisible by 2, which is a contradiction to the assumption that n is even.

13 Show that if 3 divides m and 3 divides n then 3 divides mn.

$m = 3k_1$, integer k_1. $mn = (3k_1)n = 3(k_1n) = 3k_2$, integer k_2

Hence 3 divides mn.

14 **a** Write statements P: n^2 is even Q: n is even.

b If n is odd then n^2 is odd.

c n is odd, so $n = 2k + 1$, integer k.

$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2p + 1$, integer p.

15 Assume $\forall a, b \in Z$, $a^2 - 4b = 2$.

$a^2 = 2 + 4b$, so a^2 is even. Hence a is even, so $a = 2k$, integer k.

$a^2 - 4b = 4k^2 - 4b = 2$ or $2(k^2 - b) = 1$

This is not possible because $k^2 - b$ is an integer.

16 The contrapositive is to prove 'If both real numbers are rational, then their sum is rational'.

$\frac{a}{b}, \frac{c}{d}$, integers a, b, c, d.

Sum is $\frac{ad + bc}{bd} = \frac{m}{n}$, integers m, n.

17 Show that if both m and n are even, then mn is even.

$m = 2x$, $n = 2y$, $mn = 2(2xy) = 2p$, p integer.

18 Assume that there is a smallest positive rational number of the form $\frac{a}{b}$.

Then we have $\frac{a}{2b} < \frac{a}{b}$, and $\frac{a}{2b}$ is rational because both a and $2b$ are integers.

Hence there is no smallest *positive rational number*.

19 **a** P_N is the product of all the primes and contains all the primes, so $P_N + 1$ can't be prime.

b If $P_N + 1$ is composite, it is divisible by at least one of *the known primes*.

But when $P_N + 1$ is divided by the prime(s), there is a remainder of 1. However, $P_N + 1$ can't be both divisible by a number and have remainder 1.

20 Assume the sum of the squares of two odd numbers is odd.

Let $a = 2m + 1$, $b = 2n + 1$, (m, n integers).

$a^2 + b^2$ is odd. This means $4m^2 + 4n^2 + 2m + 2n + 2 = 2k + 1$, for integer k.

$2(2m^2 + 2n^2 + m + n) + 1 = 2k$, $2p + 1 = 2k$, for integer p.

LHS is odd RHS is even hence a contradiction.

21 **a** For all natural numbers, if $3n + 2$ is odd, then n is odd.

b Show that if n is even, then $3n + 2$ is even.

$n = 2m$, integer m

$3n + 2 = 2(3m) + 2 = 2p + 2$, integer p.

EXERCISE 3.5

Only partial solutions shown for this exercise: see worked solutions.

1 E **2** B

3–8 Proof: see worked solutions

9 True for $n = 1$

Assume true for $n = k$ and prove true for $n = k + 1$

$9k - 1 = 8p$, for integer p.

$9k = 8p + 1$

$$9k + 1 - 1 = 9 \times 9k - 1$$
$$= 9(8p + 1) - 1$$
$$= 8(9p + 1), \text{ which is divisible by 8.}$$

10–15 Proof: see worked solutions

16 True for $n = 1$, LHS = $\cos(x)$

$$\text{RHS} = \frac{\sin(2x)}{2\sin(x)} = \frac{2\sin(x)\cos(x)}{2\sin(x)} = \cos(x)$$

Assume true for $n = k$, so let $P(k) = \frac{\sin(2^k x)}{2^k \sin(x)}$

$$P(k+1) = P(k) \times \frac{\sin(2^k x)}{2^{k+1}\sin(x)} = \cos\left(2^k x\right) \times \frac{\sin(2^k x)}{2^{k+1}\sin(x)}$$

$$= \frac{\sin(2(2^k x))}{2^{k+1}\sin(x)}$$

$$P(k+1) = \frac{\sin(2^{k+1} x)}{2^{k+1}\sin(x)}$$

17 True for $n = 1$. Assume true for $n = k$, so $uk = 2^k - 1$

$$u_{k+1} = 2_{uk} + 1 = 2(2^k - 1) + 1$$
$$= 2^{k+1} + 1, \text{ so true for } n = k + 1$$

18 Use the method of induction to prove $u_1 + u_2 + u_3 + \ldots + u_n = 3^n - 1$ given that the nth term is $u_n = 2 \times 3^{n-1}$.

True for $n = 1$, $2 \times 3^{1-1} = 2$, $3^1 - 1 = 2$.

Assume true for $n = k$.

$$\text{LHS} = (u_1 + u_2 + u_3 + \ldots + u_\text{k}) + 2 \times 3^{k+1-1}$$
$$= 3^k - 1 + 2 \times 3^k$$
$$= 3 \times 3^k - 1$$
$$= 3^{k+1} - 1$$
$$= P(k + 1)$$

19 Apply the method of induction to prove that for all natural numbers, n,

$$\frac{1}{2 \times 5} + \frac{1}{5 \times 8} + \frac{1}{8 \times 11} + \ldots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{(6n+4)}.$$

$$P(n) = \frac{n}{(6n+4)}$$

True for $n = 1$, LHS $= \frac{1}{10}$, RHS $= P(1) = \frac{1}{10}$

Assume true for $n = k$.

Then

$$\frac{1}{2 \times 5} + \frac{1}{5 \times 8} + \frac{1}{8 \times 11} + \ldots + \frac{1}{(3k-1)(3k+2)} = \frac{k}{(6k+4)}$$

$n = k + 1$

$$\left(\frac{1}{2 \times 5} + \frac{1}{5 \times 8} + \frac{1}{8 \times 11} + \ldots + \frac{1}{(3k-1)(3k+2)}\right) + \frac{1}{(3k+2)(3k+5)}$$

$$= \frac{k}{6k+4} + \frac{1}{(3k+2)(3k+5)} = \frac{k}{2(3k+2)} + \frac{1}{(3k+2)(3k+5)}$$

$$= \frac{k(3k+5)+2}{2(3k+2)(3k+5)} = \frac{k+1}{6(k+1)+4} = P(k+1)$$

20 True for $n = 2$

$$\prod_{i=2}^{2}\left(1 - \frac{1}{2^2}\right) = 1 - \frac{1}{2^2} = \frac{3}{4} \text{ and } \frac{n+1}{2n} = \frac{3}{4}$$

Assume true for $n = k$.

That is, $P(k) = \prod_{i=2}^{k}\left(1 - \frac{1}{i^2}\right) = \frac{k+1}{2k}$

$n = k + 1$

$$\prod_{i=2}^{k}\left(1 - \frac{1}{i^2}\right) \times \left(1 - \frac{1}{(k+1)^2}\right)$$

$$= \frac{k+1}{2k} \times \left(1 - \frac{1}{(k+1)^2}\right)$$

$$= \frac{(k+1)+1}{2(k+1)}$$

$$= P(k+1)$$

CUMULATIVE EXAMINATION 1

1 **a** If i is not a real number, then it is an imaginary number.

b i is not a real number if and only if it is an imaginary number.

c If i is a real number, then it is not an imaginary number.

2 odd number, $t_n = t_{n-1} + t_{n-2}$

3 $\underset{\sim}{u} + \underset{\sim}{v} = (a+2)\underset{\sim}{i} + (a-1)\underset{\sim}{j} - \underset{\sim}{k}$

$$\underset{\sim}{u} \cdot (\underset{\sim}{u} + \underset{\sim}{v}) = a(a+2) - (a-1) - 1$$
$$= a^2 + a$$
$$= a(a+1)$$

$a(a+1) = 0 \Rightarrow a = 0,\ a = -1$

Take $a = -1$

4 **a** $\frac{(x-2)^2}{1^2} + \frac{(y-1)^2}{b^2} = 1$ **b** $a = 1, b = 1$

CUMULATIVE EXAMINATION 2

Section A

1 C **2** B **3** B **4** C **5** D

Section B

1 $P(n) = 9^n + 3 = 4m$

True for $n = 1$, $P(1) = 12$ is divisible by 4.

Assume true for $n = k$

$P(k+1) = 9^{k+1} + 3 = 9(9^k) + 3$

$= 9(4m - 3) + 3 = 36m - 24$

$= 4(9m - 6)$, which is divisible by 4.

2 If a divides b, then $b = ka$, for integer k. This means a divides b and a is the greatest common divisor. If the greatest common divisor is a, then $b = ka$, for integer k, so a divides b.

3 **a** P: $c \geq a + b$, c is hypotenuse, a, b two shorter sides.

b $c^2 \geq (a+b)^2$

$c^2 \geq a^2 + b^2 + 2ab$

$c^2 \geq c^2 + 2ab$, since $a^2 + b^2 = c^2$

$2ab \leq 0$. This implies a or b or a and b are zero or either a or b is negative. This is a contradiction.

4 **a** $x^2 - 2x + 4 - \frac{19}{2x+5}$

b $y = x^2 - 2x + 4$, $x = -2.5$

5 **a** $a = 1$, $b = 3$ **b** 2

CHAPTER 4

EXERCISE 4.1

1 **a** i **b** i **c** $-i$

d -1 **e** $-i$

2 **a** $\text{Re}(z) = -4$, $\text{Im}(z) = 2$

b $\text{Re}(w) = \frac{3}{5}$, $\text{Im}(w) = -\frac{7}{5}$

c $\text{Re}(z) = \sqrt{3} + 4$, $\text{Im}(z) = 0$

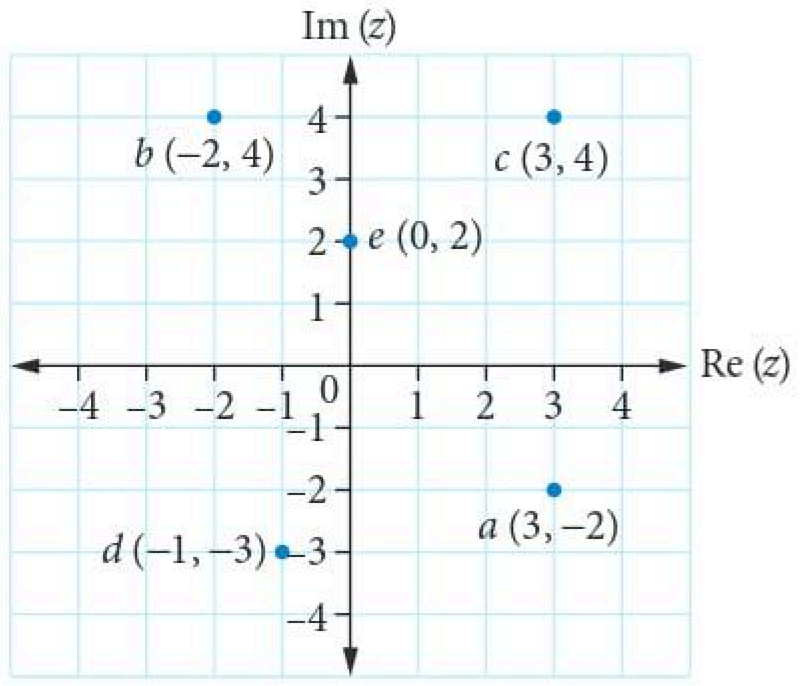
d $\text{Re}(w) = \frac{x}{x+y}$, $\text{Im}(w) = -\frac{y}{x+y}$

e $\text{Re}(z) = a + c$, $\text{Im}(z) = b + d$

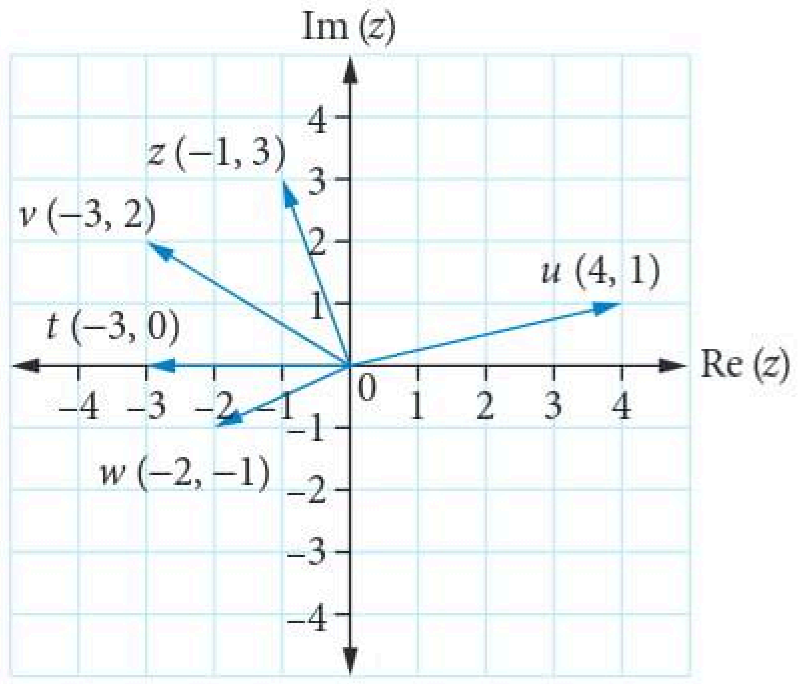
3 **a** $-5 - 6i$ **b** $\frac{5-2i}{3}$ **c** $1 - i\sqrt{3}$

d $a + bi - ci$ **e** $\frac{ac - b^2}{c^2 + b^2} + \frac{b(a+c)}{c^2 + b^2}i$

4



5



6 **a** $2 - 7i$ **b** $2 - 12i$ **c** $-6 - 5i$

d $8 - 3i$ **e** $-4 + 2i$

7 **a** $10 + 12i$ **b** $4 + 11i$ **c** $9 + 5i$

d $5i$ **e** $-10 - 9i$

8 **a** $-8 - 31i$ **b** $52 + 26i$ **c** $98 + 14i$

d $-2 - 36i$ **e** $-53 - 31i$

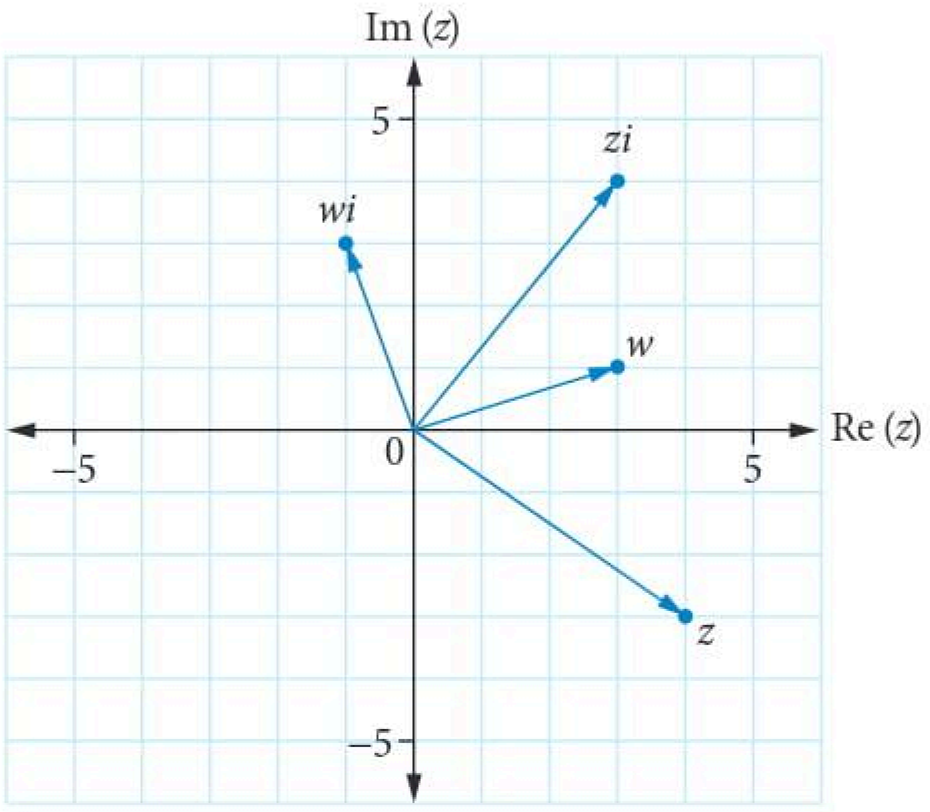
9 **a** $1 - 3i$ **b** $-\frac{13}{25} - \frac{9}{25}i$ **c** $\frac{8}{25} - \frac{13}{50}i$

d $-\frac{2}{41} - \frac{23}{41}i$ **e** $-\frac{11}{20} + \frac{3}{5}i$

10 **a** $6 + 7i$ **b** $-\frac{27}{74} - \frac{23}{74}i$ **c** $48 - 4\ \sqrt{(3)i}$

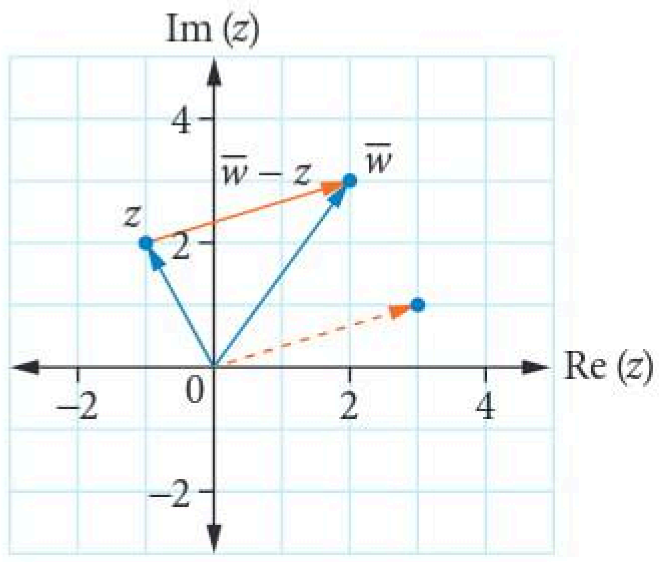
d $1 - 4i$ **e** $-\frac{21}{17} - \frac{16}{17}i$

11 a



b It rotates the complex number $\frac{\pi}{2}$ anticlockwise around the origin.

12



13 $4 + 4\sqrt{3}i$

14 E

15 A

16 C

17 C

18 A

EXERCISE 4.2

1 B

2 C

3 a $3\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

b $2\,\text{cis}\left(\frac{\pi}{6}\right)$

c $4\,\text{cis}\left(-\frac{\pi}{6}\right)$

d $3\,\text{cis}\left(\frac{\pi}{2}\right)$

e $5\text{ cis }(-\theta)$, where $\theta = \arctan\left(\frac{4}{3}\right)$

4 a $-\frac{5\sqrt{2}}{2} - \frac{5\sqrt{2}}{2}i$

b $-\sqrt{2} - \sqrt{2}i$

c -4

d $3\sqrt{3} - 3i$

e $-\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}i$

5 a $1.5\,\text{cis}\left(-\frac{5\pi}{12}\right)$

b $2\,\text{cis}\left(-\frac{7\pi}{12}\right)$

c $2.5\text{ cis}\left(\frac{\pi}{4}\right)$

d $8\text{ cis }(-\pi)$

e $\frac{5}{3}\text{cis}\left(-\frac{\pi}{12}\right)$

6 a $5\text{ cis }(-\theta)$, where $\theta = \tan^{-1}\left(\frac{3}{4}\right)$

b $\sqrt{5}\,\text{cis}\,\theta$, where $\theta = \tan^{-1}\left(\frac{1}{2}\right)$

c $2\,\text{cis}\left(-\frac{\pi}{6}\right)$

d $6\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$

e $13\,\text{cis}\,(\theta)$, where $\theta = \pi - \tan^{-1}\left(\frac{5}{12}\right)$

f $2\sqrt{3} - 3i$

g $-3i$

h $-\frac{5}{2} - \frac{5\sqrt{3}}{2}i$

i $-\sqrt{2} - \sqrt{2}i$

j $2.5\left(\sqrt{6} - \sqrt{2} - \sqrt{2}\right) + 2.5\left(\sqrt{6} + \sqrt{2}\right)i$

7 a $z - 2 + i = k(1 + 3i),\ k \in R,\ 3x - y - 7 = 0$

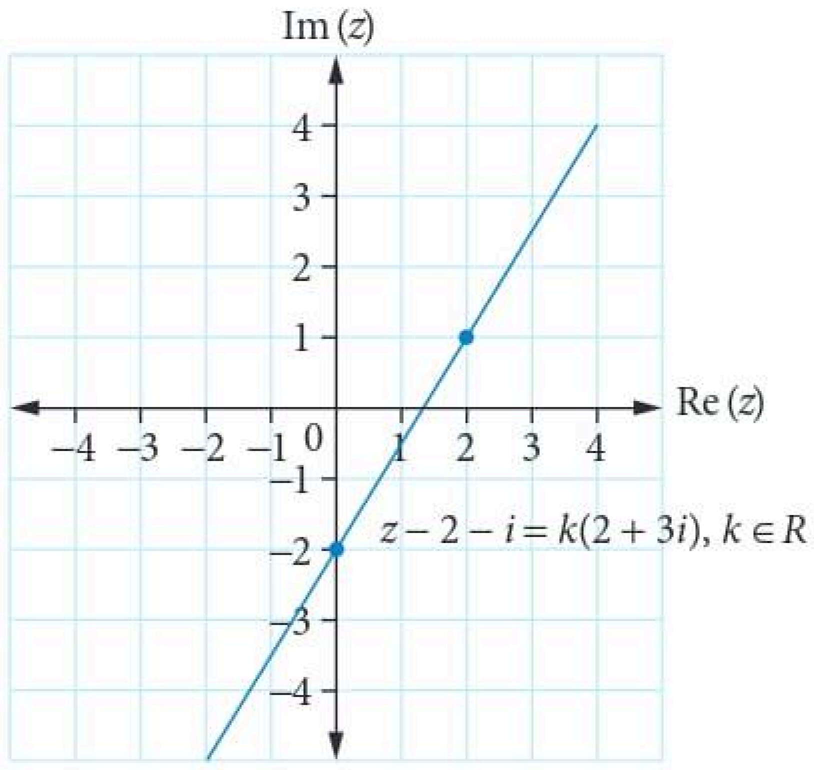
b $|z - 4 - 2i| = |z + 3 - i|,\ x + 7y + 10 = 0$

c $z + 1 - i = k(1 + 2i),\ k \in R,\ 2x - y + 3 = 0$

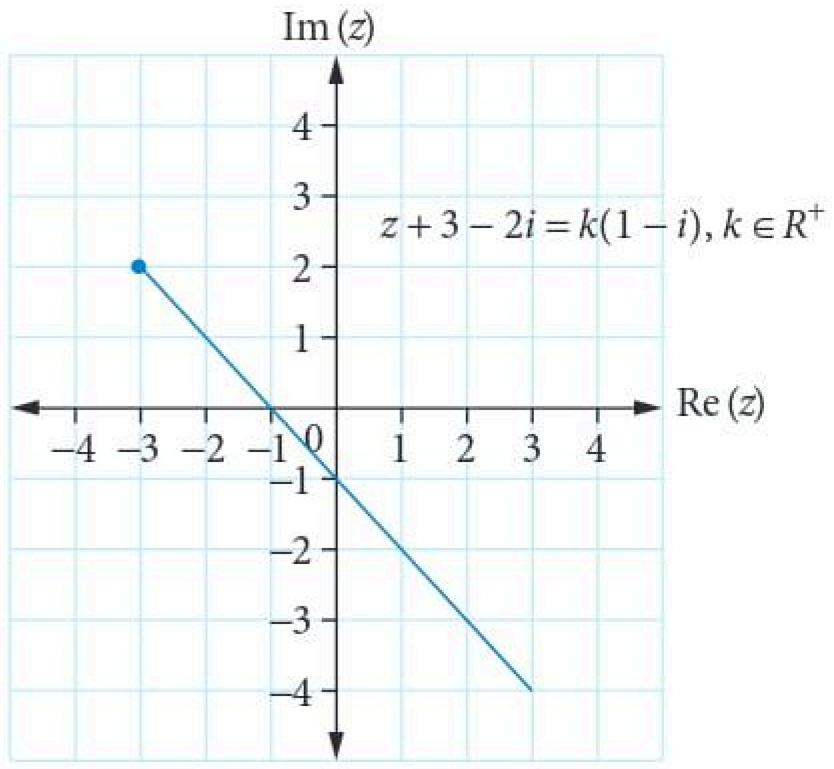
d $\text{Re}\,(z) = 1$

e $z + 3 + 2i = k\,\text{cis}\left(-\frac{5\pi}{6}\right)$ or $z + 3 + 2i = k\left(\sqrt{3} + i\right)$, $k \in R,\ \sqrt{3}x - 3y + 3\sqrt{3} - 6 = 0$

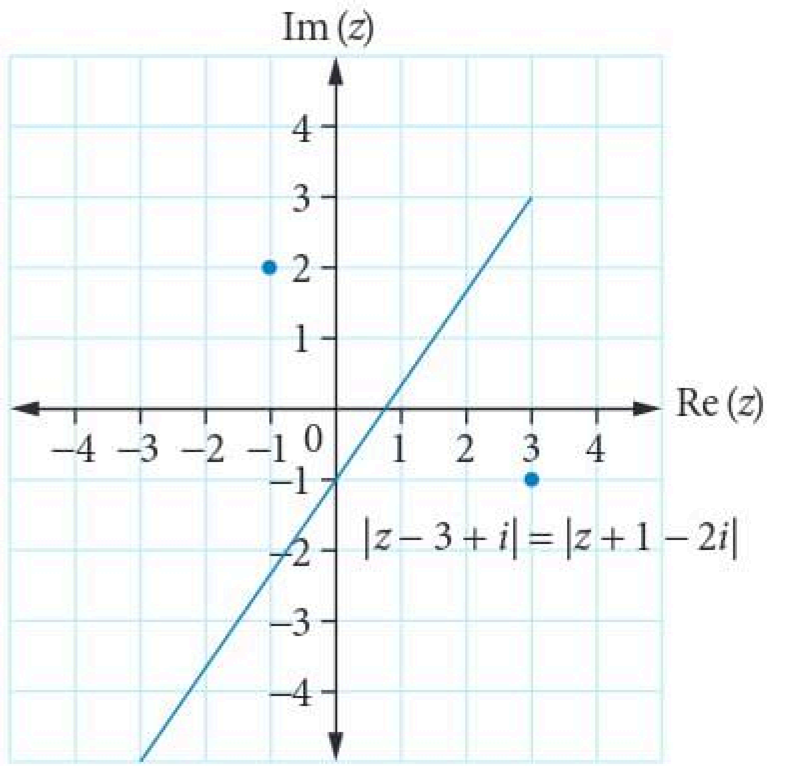
8 a The line through $2 + i$ with slope $\frac{3}{2}$.



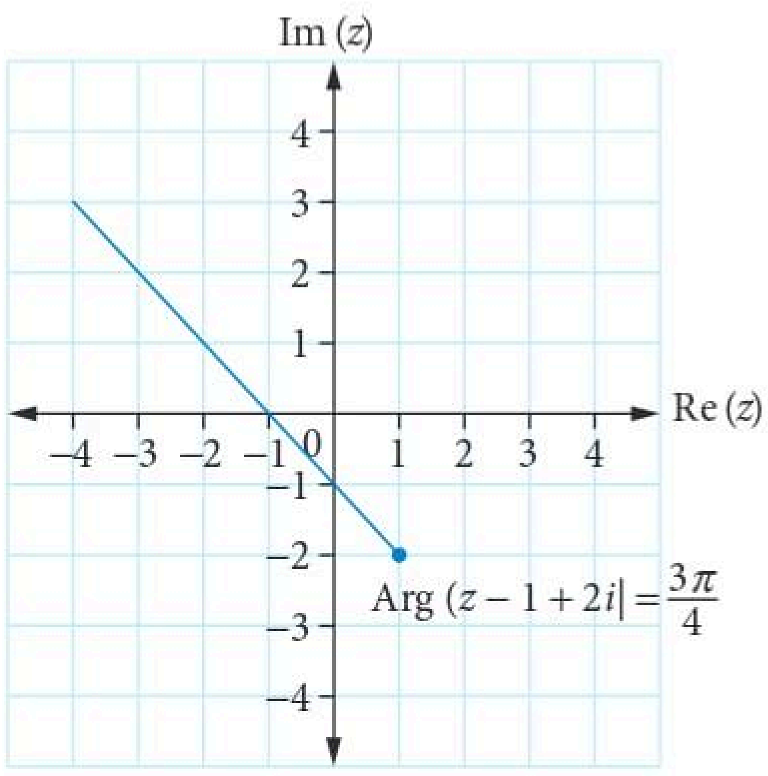
b The ray from $-3 + 2i$ with inclination $-\frac{\pi}{4}$.



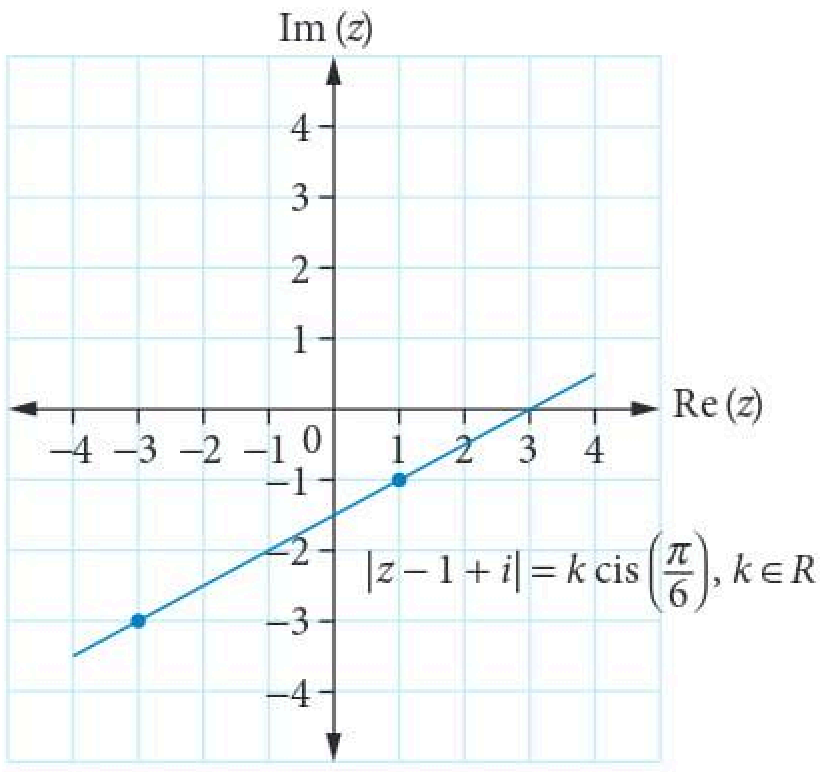
c The perpendicular bisector of $3 - i$ and $2i - 1$.



d The ray from $1 - 2i$ with inclination $\frac{3\pi}{4}$.

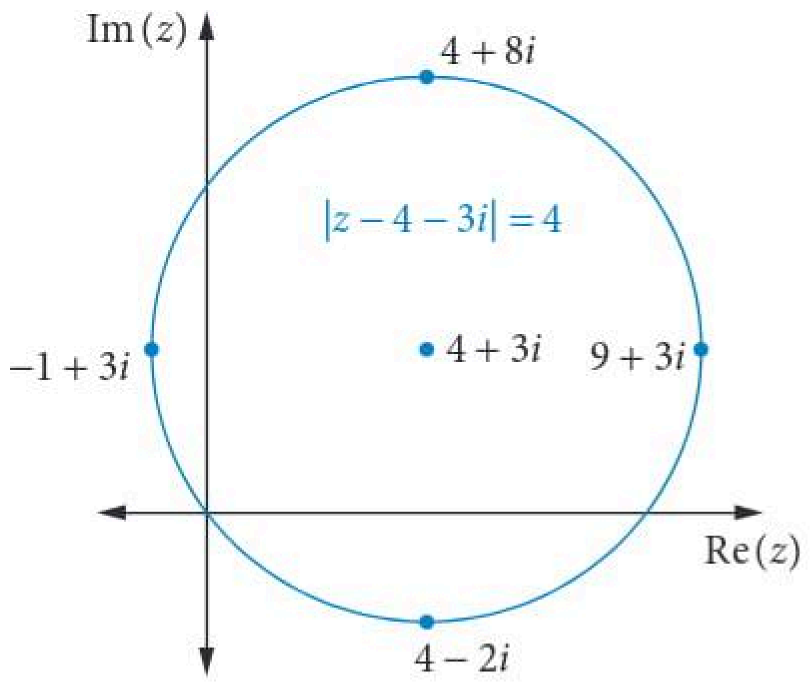


e The line through $1 - i$ with inclination $\frac{\pi}{6}$.

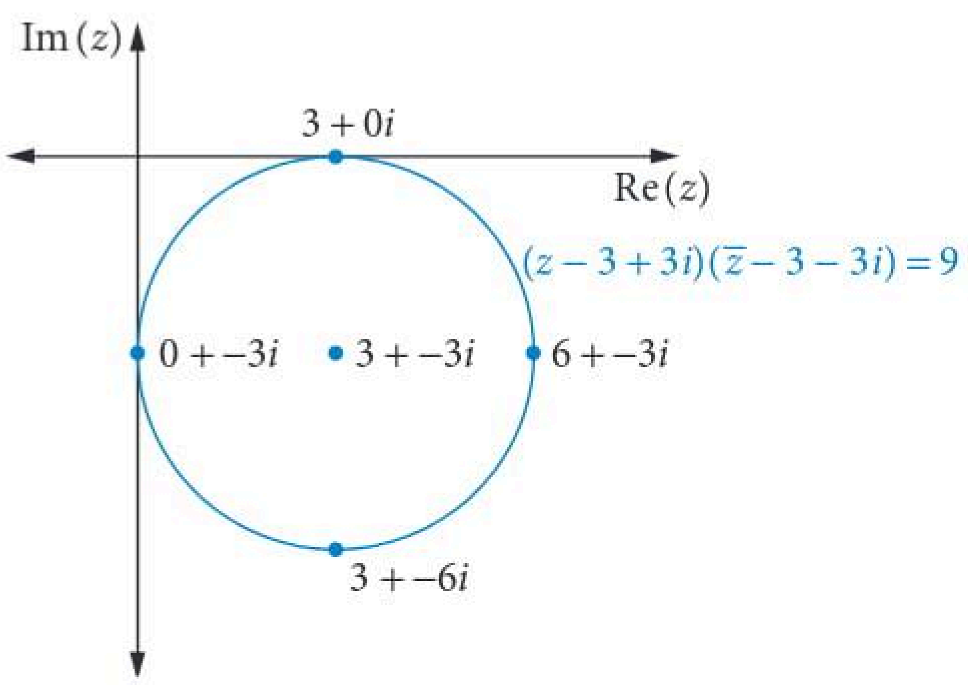


9 **a** $|z - 3 - 2i| = 5$ **b** $|z + 1 + i| \leq 4$ **c** $|z + 5 - 3i| > 2$

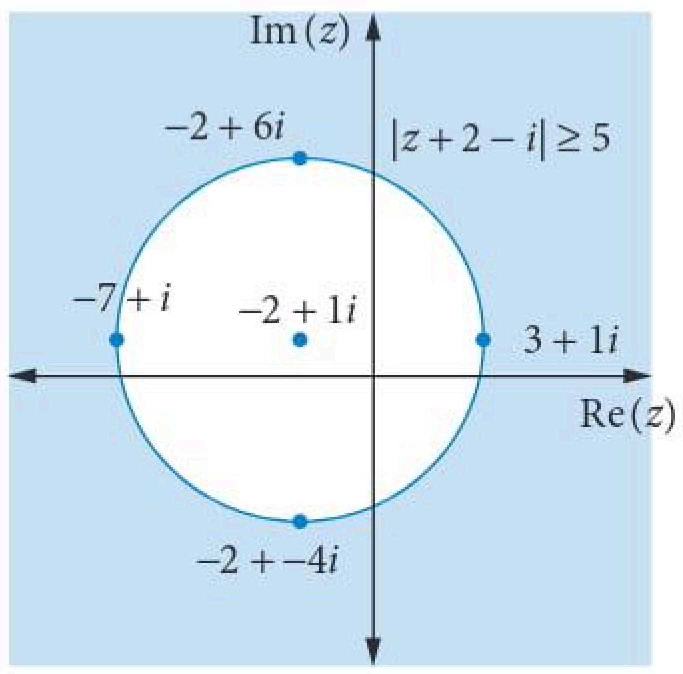
10 **a** The circle of radius 4 with centre $4 + 3i$.



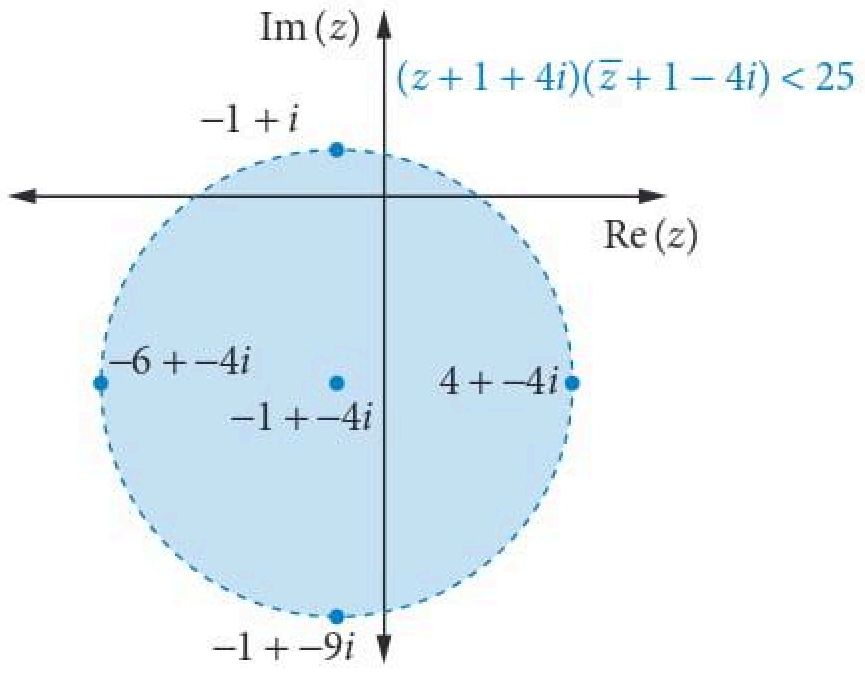
b The circle of radius 3 with centre $3 - 3i$.



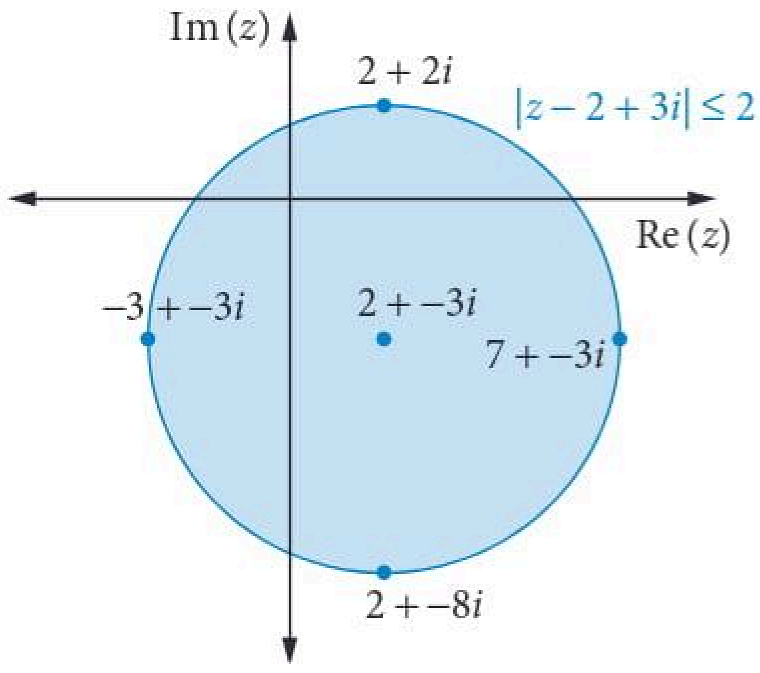
c The outside and circumference of the circle of radius 5 with centre $-2 + i$.



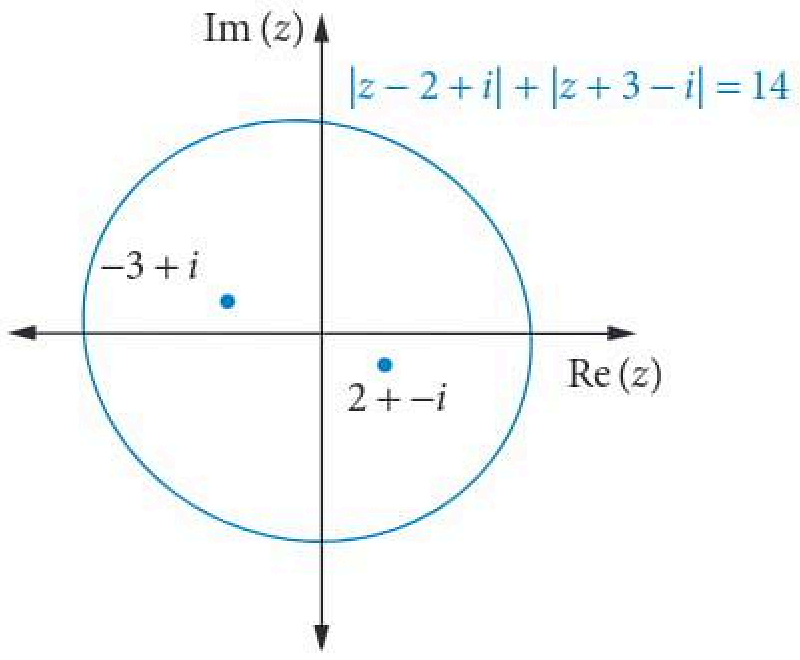
d The inside of the circle of radius 5 with centre $-1 - 4i$.



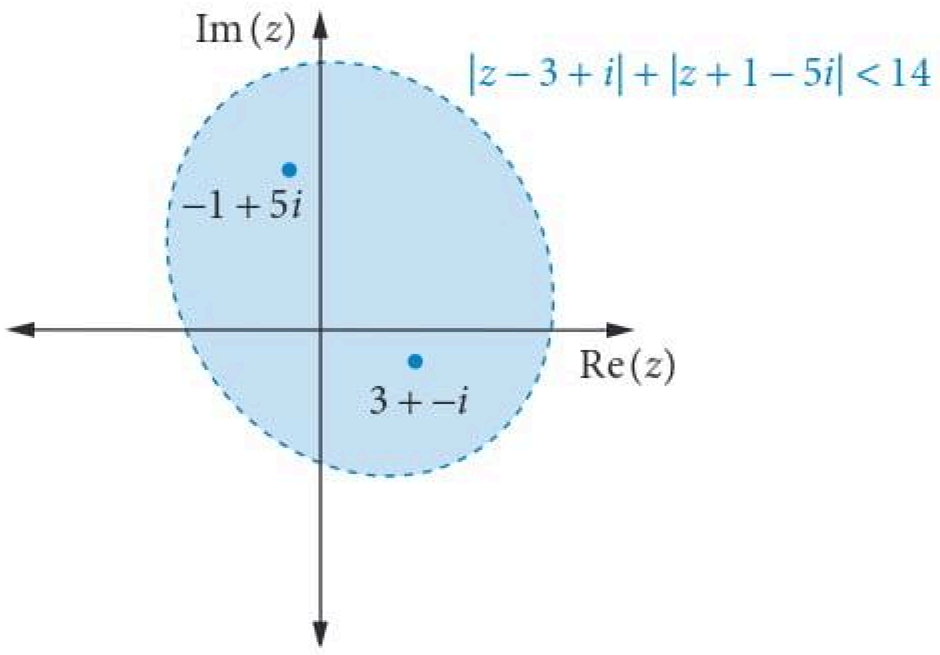
e The inside and circumference of the circle of radius 2 with centre $2 - 3i$.



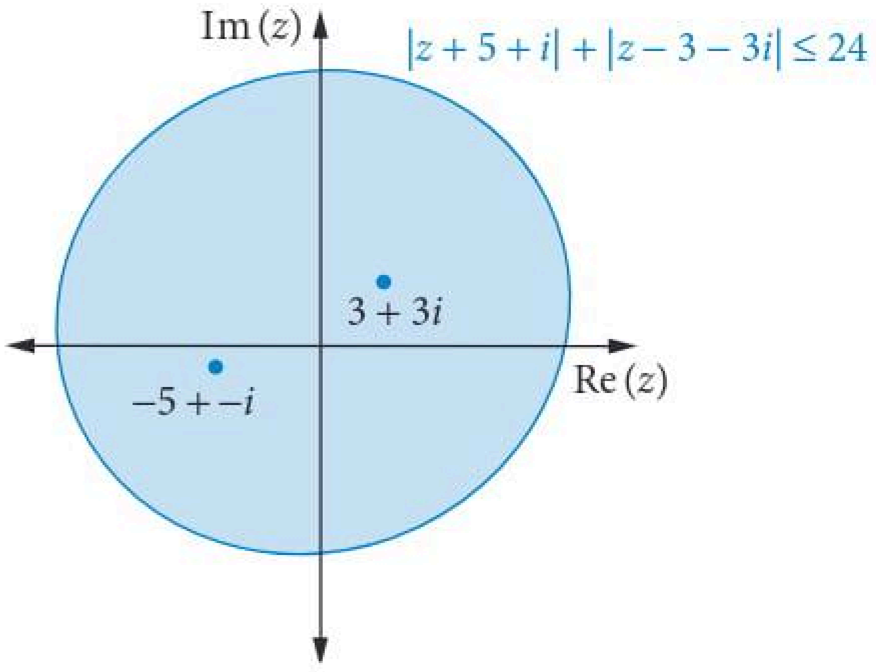
11 **a** The ellipse with foci $2 - i$ and $-3 + i$, major semi-axis 7 and minor semi-axis $\frac{\sqrt{167}}{2}$.



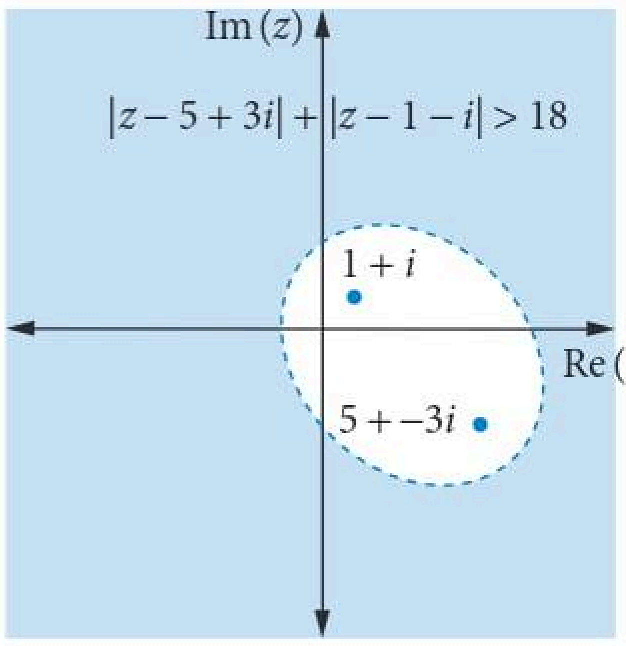
b The inside of the ellipse with foci $3 - i$ and $-1 + 5i$, major semi-axis 7 and minor semi-axis 6.



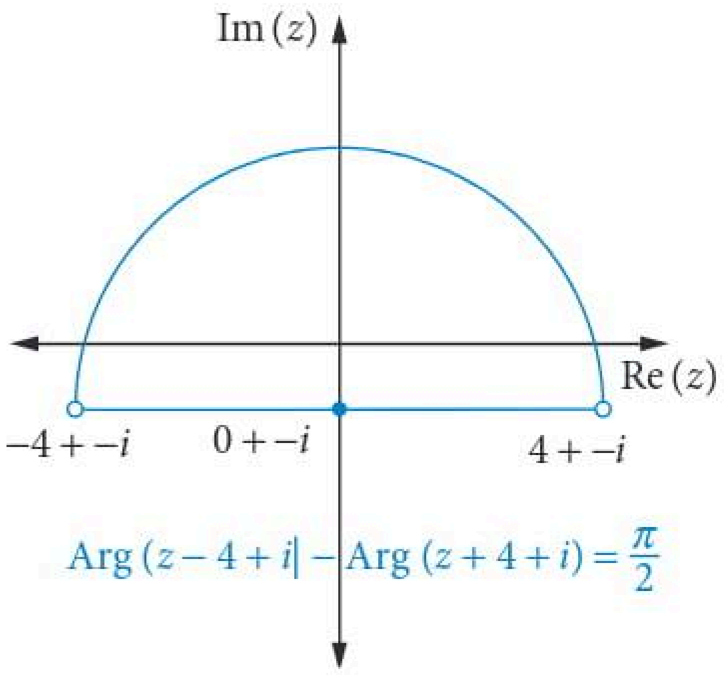
c The inside and circumference of the ellipse with foci $-5 - i$ and $3 + 3i$, major semi-axis 12 and minor semi-axis $\sqrt{124}$.



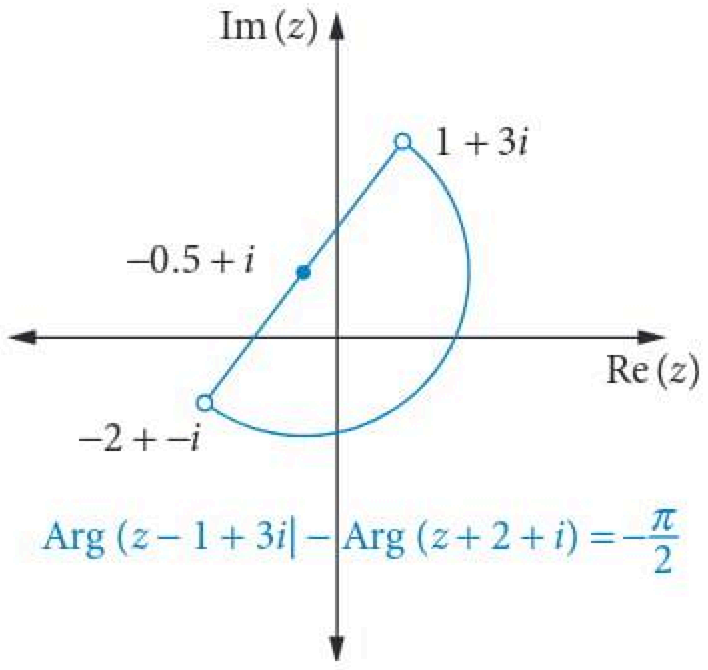
d The outside of the ellipse with foci $5 - 3i$ and $1 + i$, major semi-axis 9 and minor semi-axis $\sqrt{61}$.



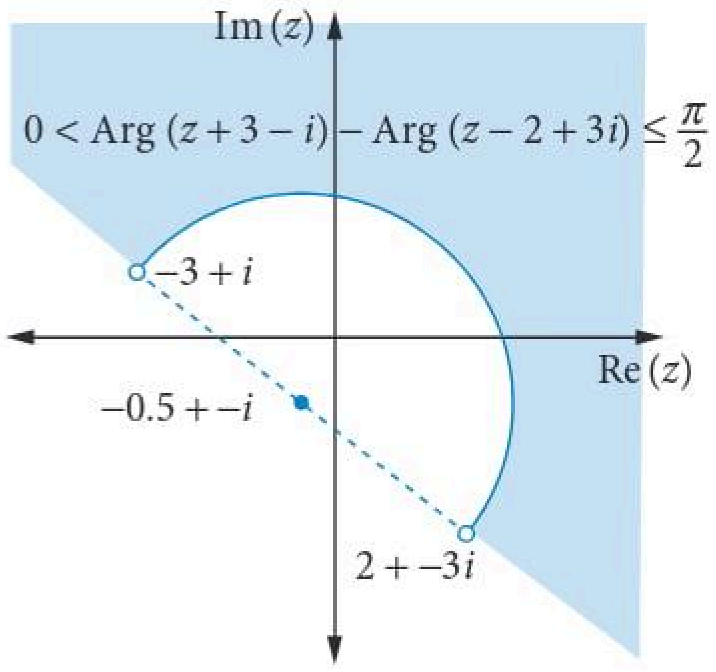
12 a The semicircle on the right-hand side of the diameter from $4 - i$ to $-4 - i$.



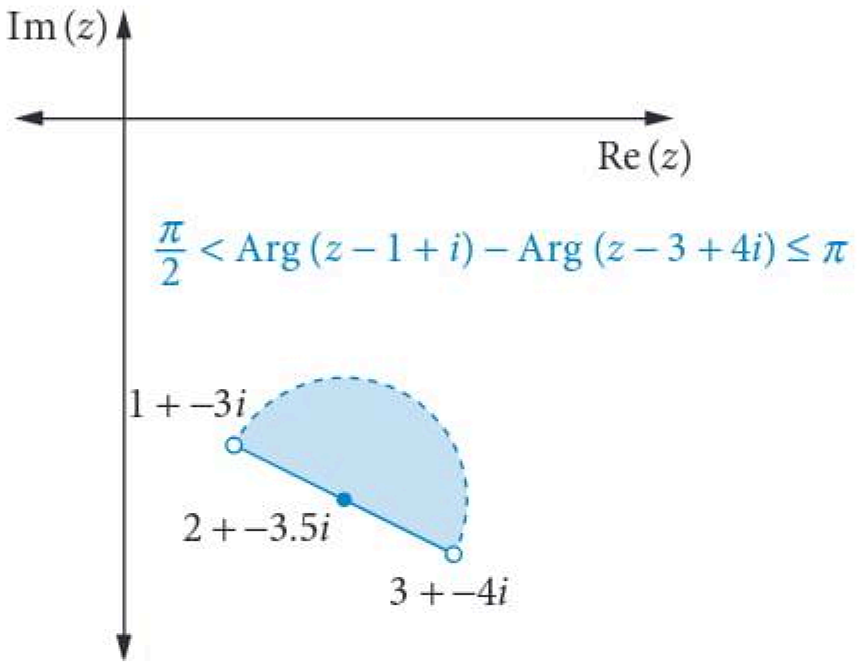
b The semicircle on the left-hand side of the diameter from $1 + 3i$ to $-2 - i$.



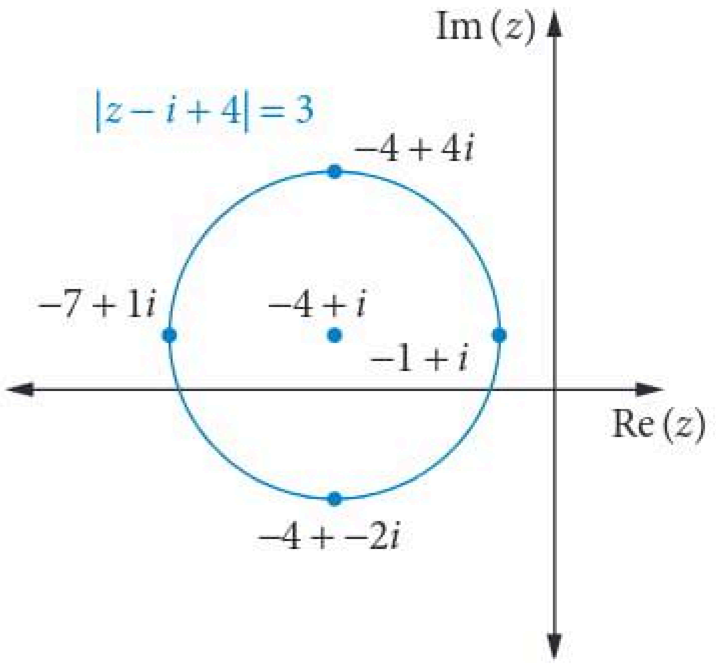
c The outside and circumference of the semicircle on the right-hand side of the diameter from $-3 + i$ to $2 - 3i$.



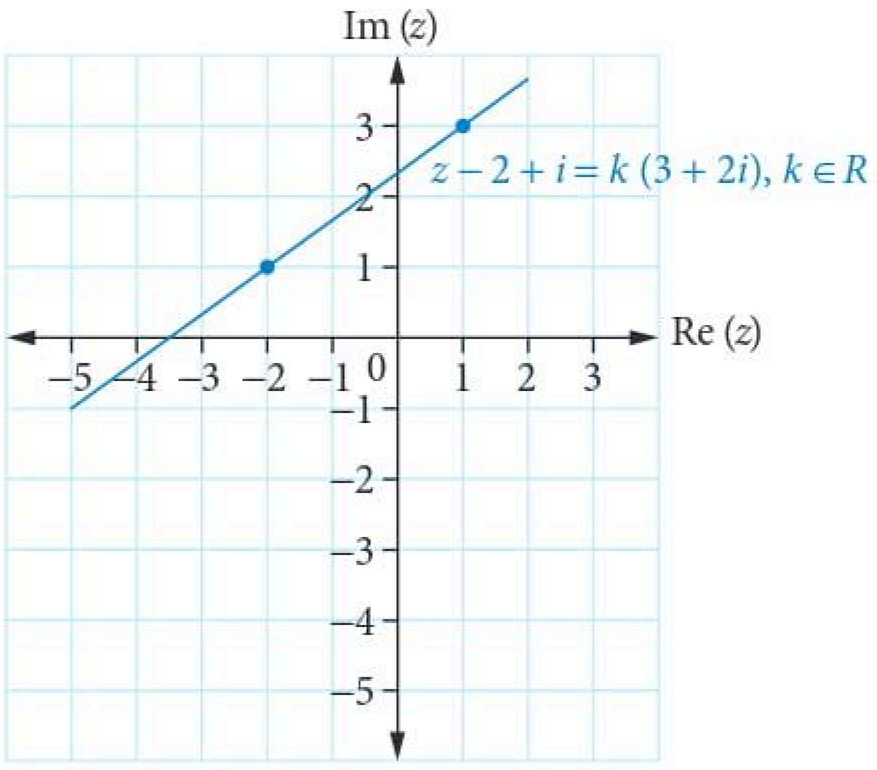
d The inside and diameter of the semicircle on the right-hand side of the diameter from $1 - i$ to $3 - 4i$.



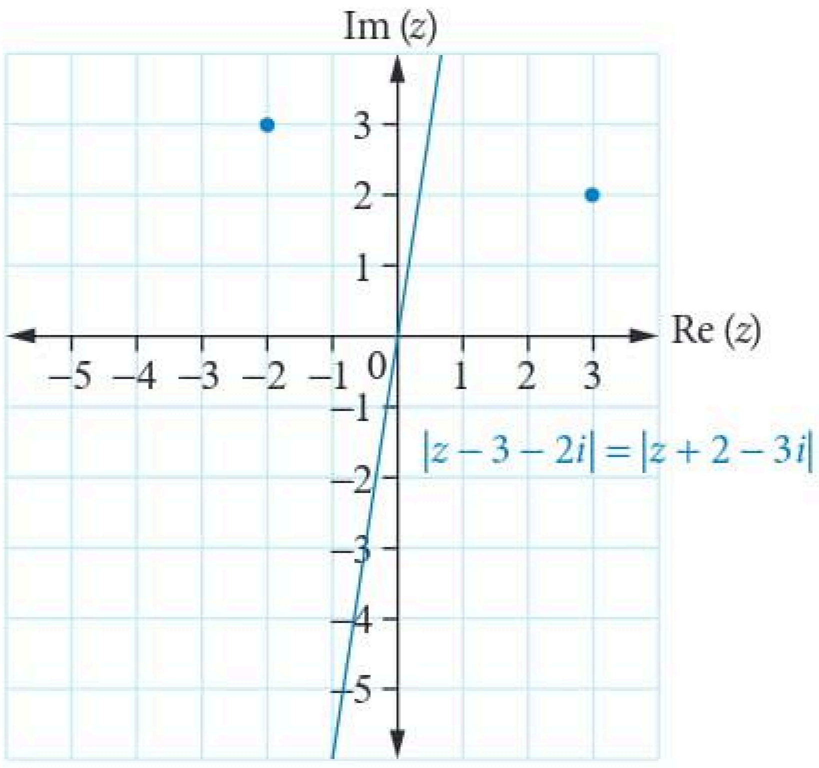
13 a The circle of radius 3 with centre $i - 4$.



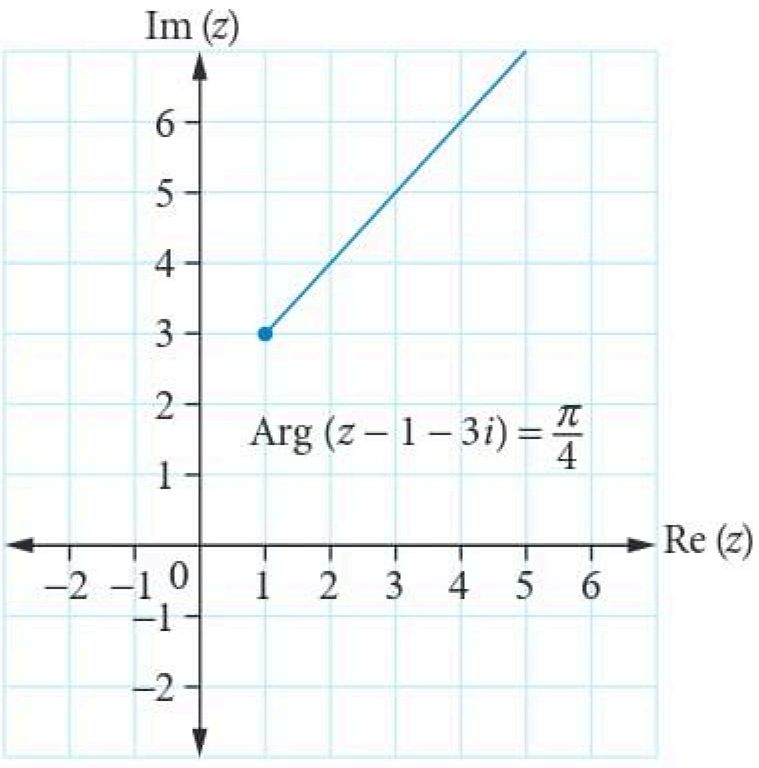
b The straight line through $2 - i$ and slope $\frac{2}{3}$.



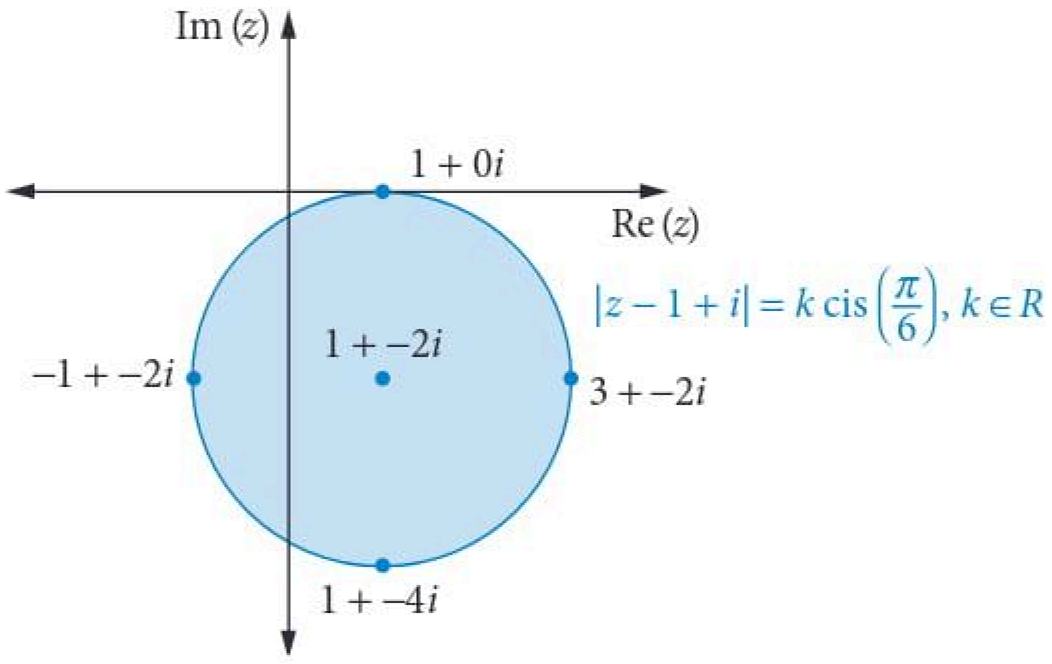
c The perpendicular bisector of $3 + 2i$ and $3i - 2$.



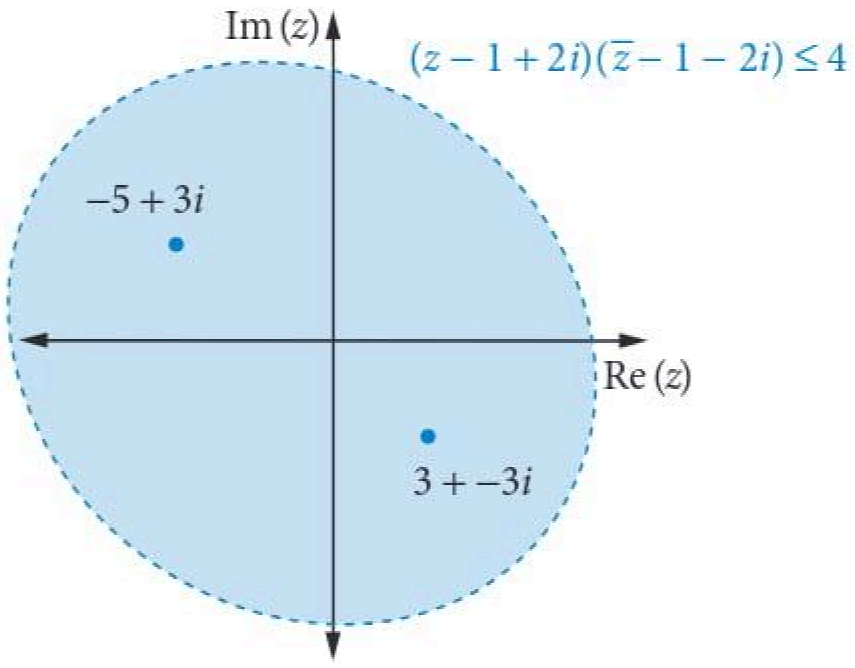
d The ray from $1 + 3i$ with inclination $\frac{\pi}{4}$.



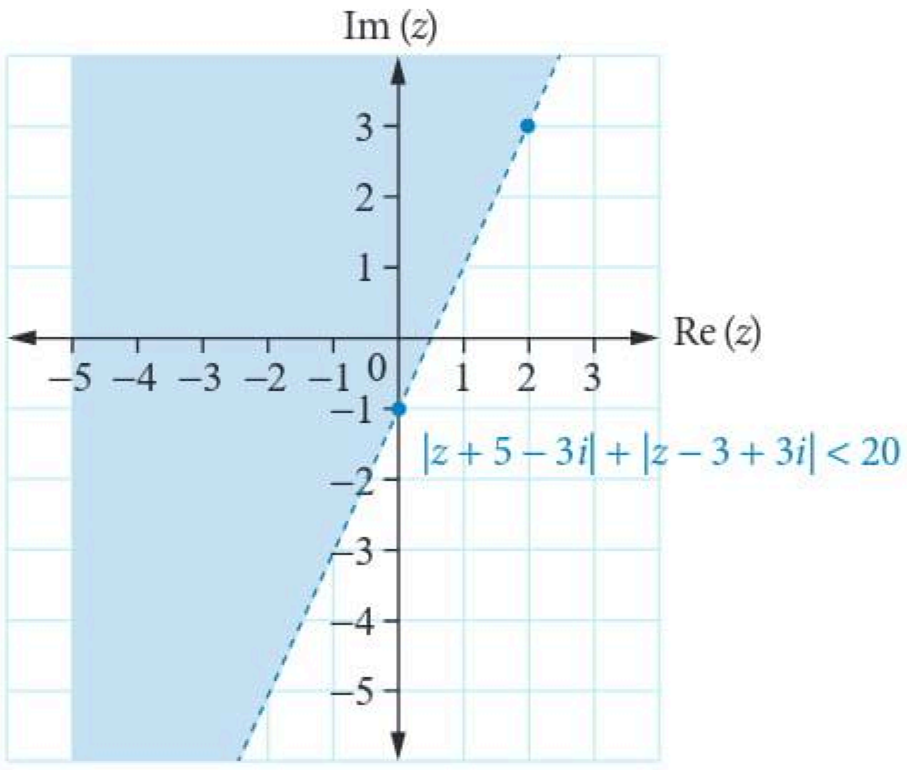
e The inside and circumference of the circle of radius 2 and centre $1 - 2i$.



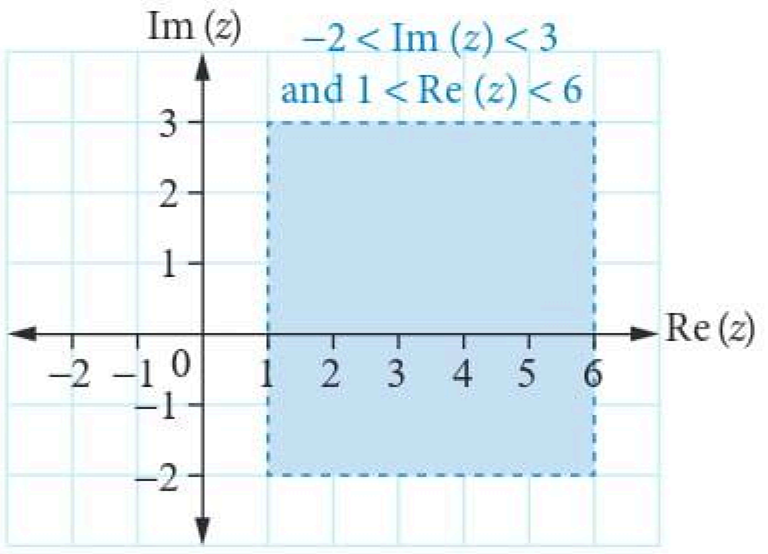
f The inside of the ellipse with foci $3i - 5$ and $3 - 3i$, major semi-axis 10 and minor semi-axis $\sqrt{75}$.



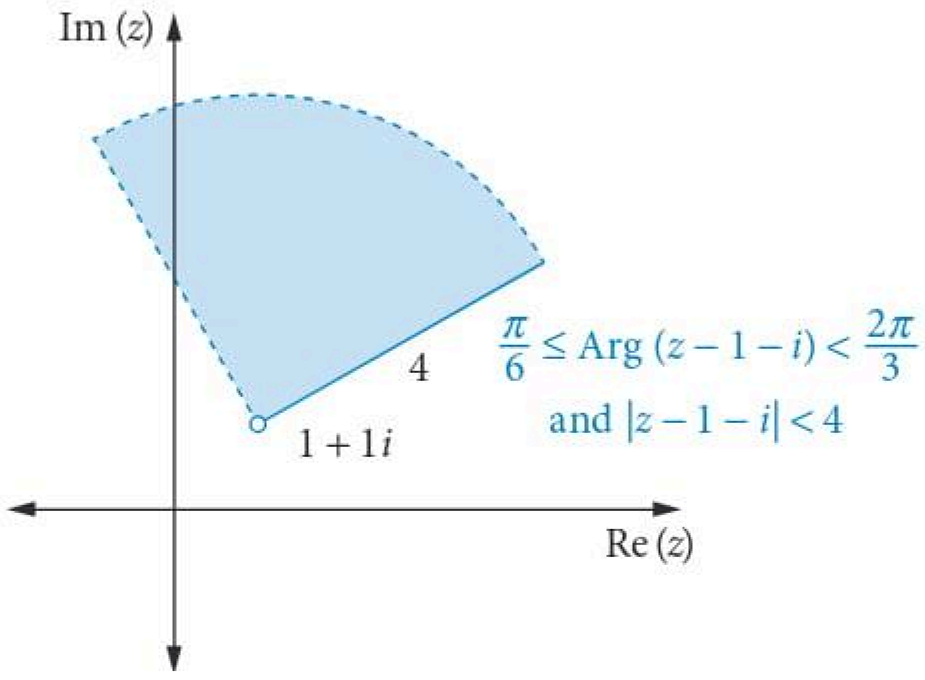
g The area above the line through $-i$ with slope 2.



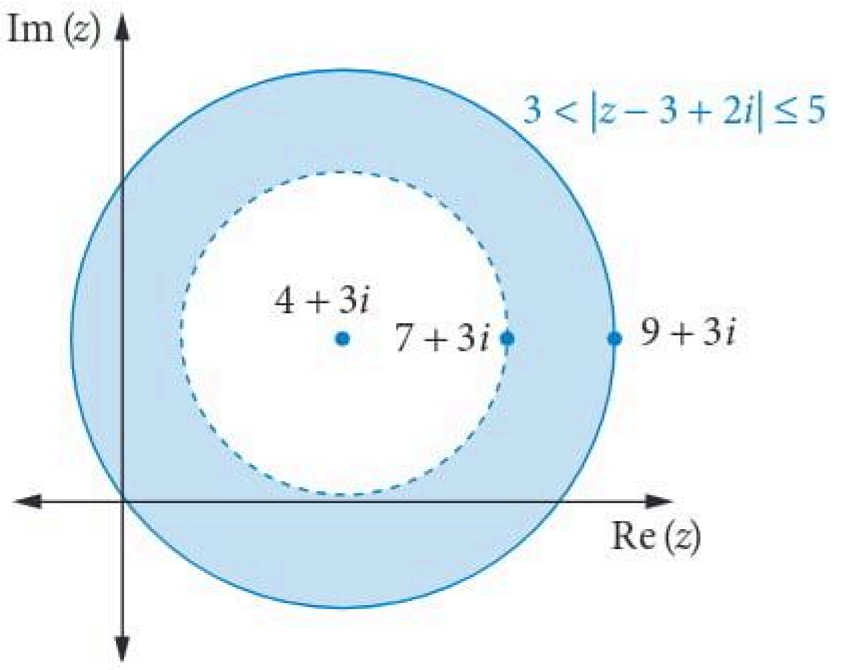
14 **a** The area enclosed by the lines $z = -2i$, $z = 3i$, $z = 1$ and $z = 6$, not including the lines.



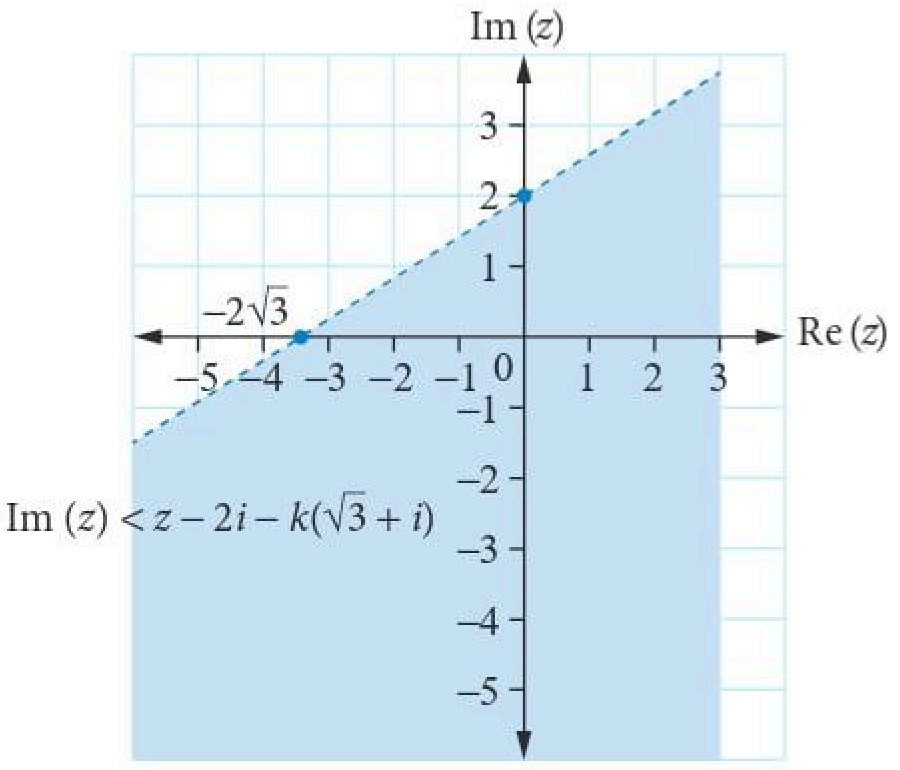
b The sector of the circle with centre $1 + i$ and radius 4 between the rays with inclinations $\frac{\pi}{6}$ and $\frac{2\pi}{3}$, including the ray with inclination $\frac{\pi}{6}$.



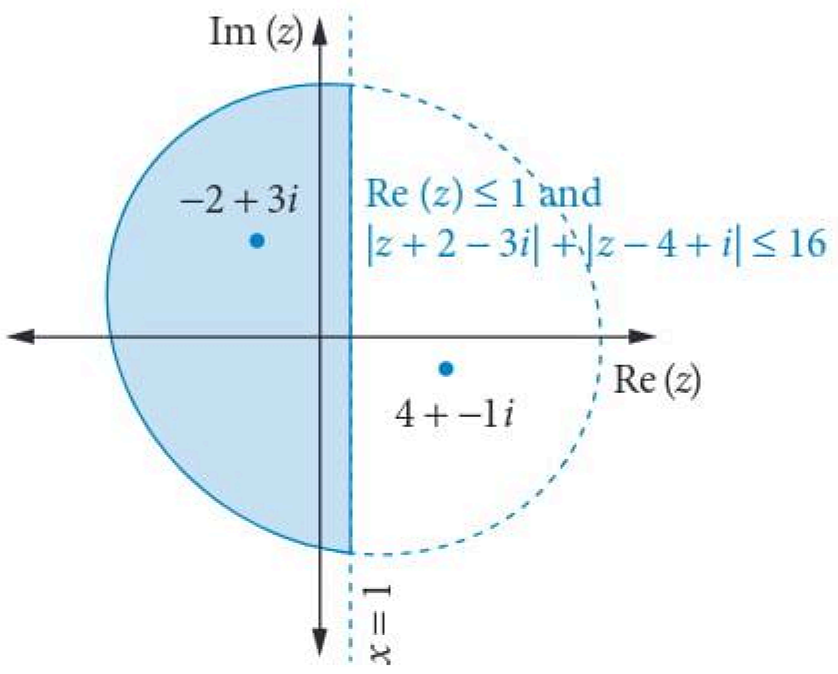
c The annulus with centre $3 - 2i$ and radii 3 and 5, including the outer circumference.



d The area under the line through $2i$ with inclination $\frac{\pi}{6}$.



e The area to the left of $z = 1$ and inside the ellipse with foci $-2 + 3i$ and $4 - i$, major semi-axis 8 and minor semi-axis $\sqrt{51}$, including all the boundaries.



15	$\frac{11\pi}{12}$	**16**	B	**17**	A	**18**	A
19	C	**20**	C	**21**	D	**22**	C
23	C	**24**	B	**25**	C	**26**	E
27	D	**28**	D	**29**	A		

30 **a** $w = l \text{ cis}\left(\theta + \frac{\pi}{3}\right)$

b **i** $p = \frac{\sqrt{3}}{2} l \text{ cis}\left(\theta + \frac{\pi}{6}\right)$

ii $u = l \text{ cis}(\theta)$

$$p^{12} = \frac{729}{4096} u^{12}$$

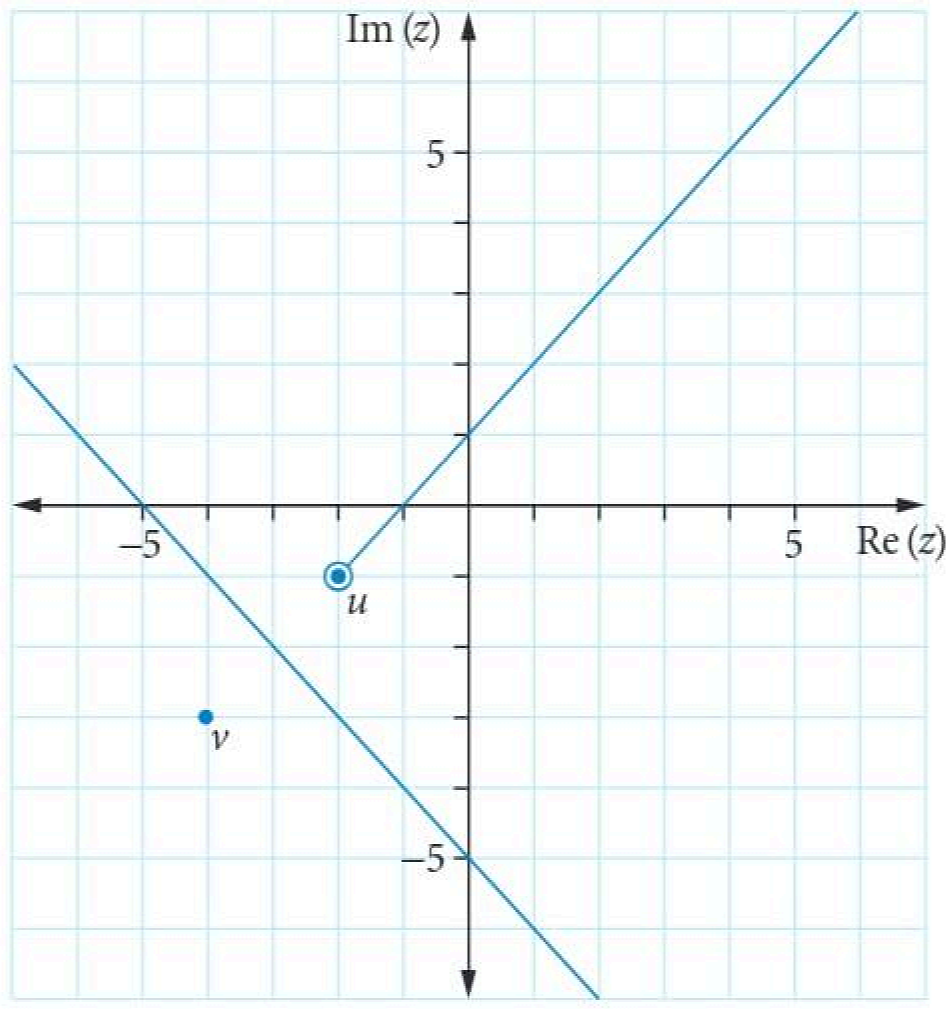
$n = 12, a = 729, b = 4096$

c $\frac{w}{u} = \frac{l \text{cis}\left(\theta + \frac{\pi}{3}\right)}{l \text{cis}\,\theta} = \text{cis}\left(\frac{\pi}{3}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

d $a = \frac{65\sqrt{3}}{3}, b = \frac{7\sqrt{3}}{3}$

31 **a** $y = -x - 5$

b



c The perpendicular bisector of the line segment joining the points represented by u and v.

d **i** See diagram in part **b** above.

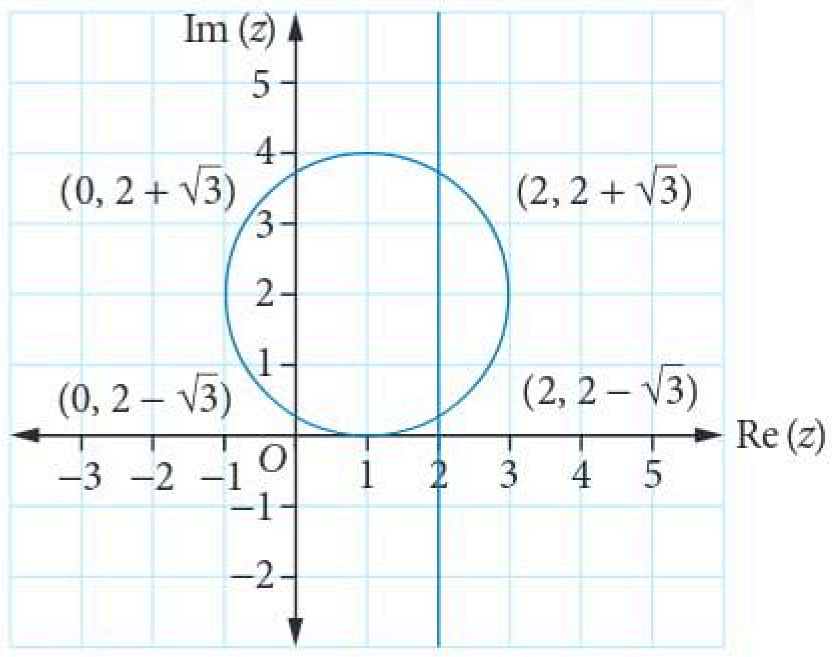
ii $y = x + 1, x > -2$

e $z_c = -\frac{5}{3} - \frac{10}{3}i$, radius $= \frac{5\sqrt{3}}{3}$

32 **a** centre $(1, 2)$, radius 2

b Use $|z|^2 = x^2 + y^2$ on both sides of the equation.

c, d

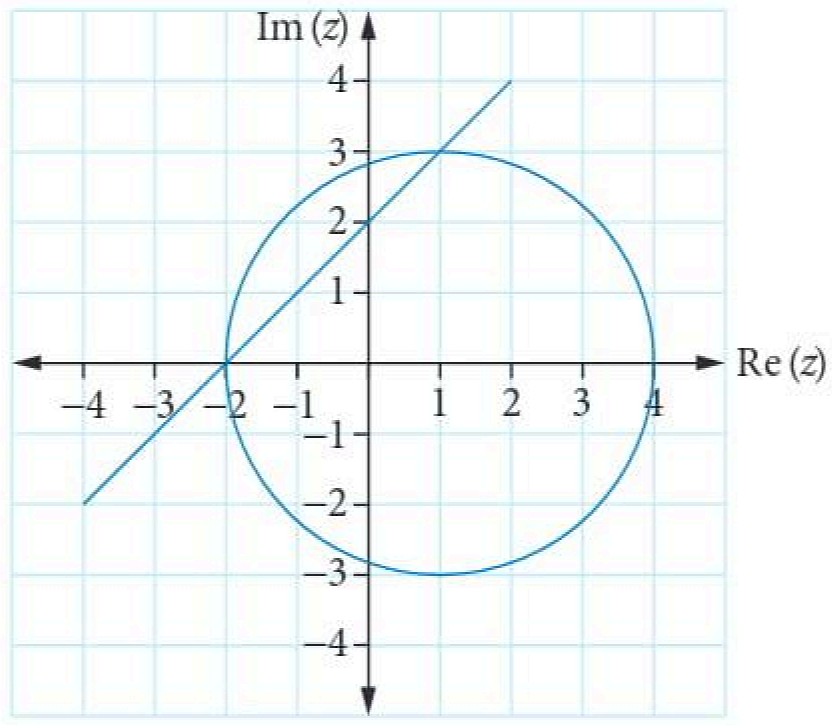


e $A = \frac{4\pi}{3} - \sqrt{3}$

33 **a** $y = x + 2$

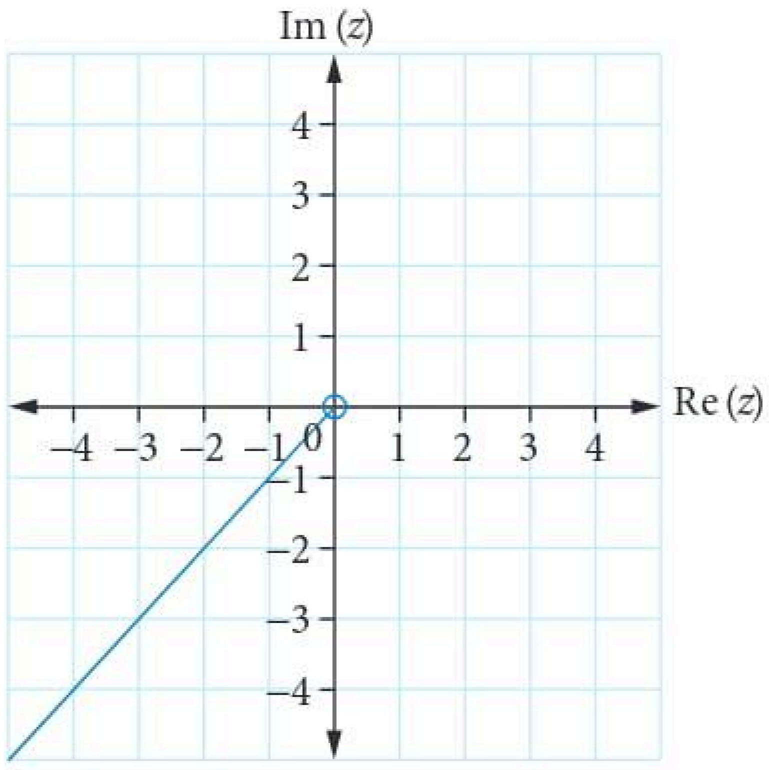
b $(-2, 0), (1, 3)$

c



d $A = \frac{27\pi}{4} + \frac{9}{2}$

e



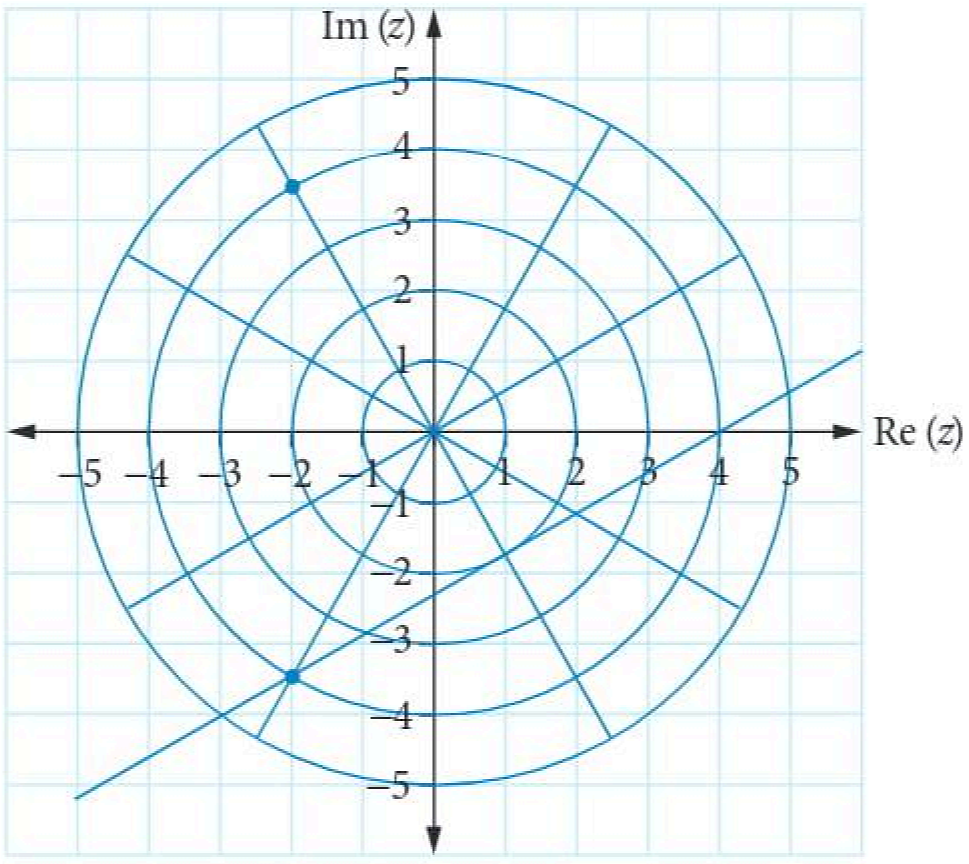
f $\left(-1,-\frac{3}{4}\right)\cup\left(\frac{1}{4},1\right)$

34 a $4\,\text{cis}\left(-\frac{2\pi}{3}\right)$ **b** $-2\pm 2\sqrt{3}i$

c $-\left(2-2\sqrt{3}i\right), -\left(\overline{2-2\sqrt{3}i}\right)$

d Use $|z|^2 = x^2 + y^2$ on both sides of the equation

e



f $b = -4 - \overline{a}$

g $A = 4\sqrt{3} + \frac{32\pi}{3}$

EXERCISE 4.3

1 C, a semicircle **2** D

3 a $4\sqrt{2}, -\frac{3\pi}{4}, -4-4i$

b $32, -\frac{\pi}{6}, 16\sqrt{3}-16i$

c $1024, \frac{3\pi}{4}, -512\sqrt{2}+512\sqrt{2}i$

d 64, π, −64

e $16, \frac{2\pi}{3}, -8+8\sqrt{3}i$

4 a $-7+24i$ **b** $-46-9i$ **c** $597+122i$

d $0.5-0.5\sqrt{3}i$ **e** $128+128\sqrt{3}i$

5 a 8 **b** 6 **c** 5

6 a $32\sqrt{3}-32-\left(32\sqrt{3}+32\right)i$ **b** $128\sqrt{3}+128i$

c $-2-2\sqrt{3}+\left(2-2\sqrt{3}\right)i$ **d** $8+8\sqrt{3}i$

e $\frac{\sqrt{3}+1+(\sqrt{3}-1)i}{2}$

7 $z(1, 1), z^2(0, 2), z^3(-4, 0)$

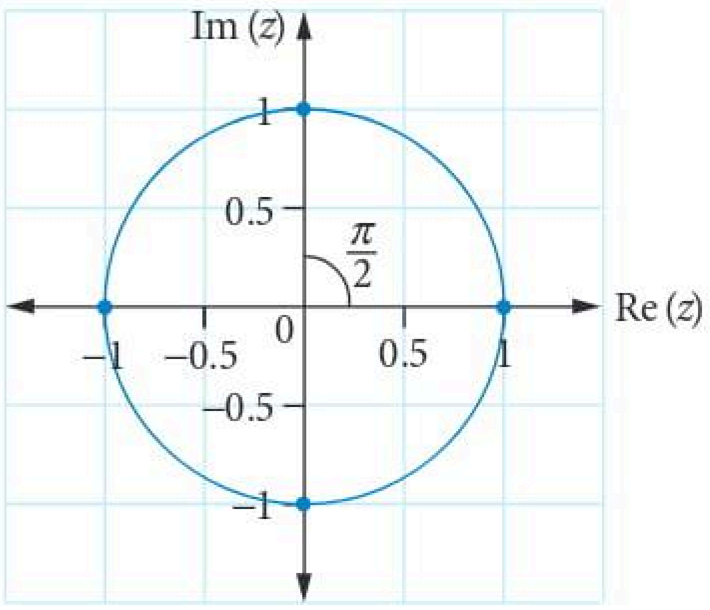
8 a Find modulus and argument **b** $-8-8\sqrt{3}i$

9 A **10** B **11** E **12** E **13** A

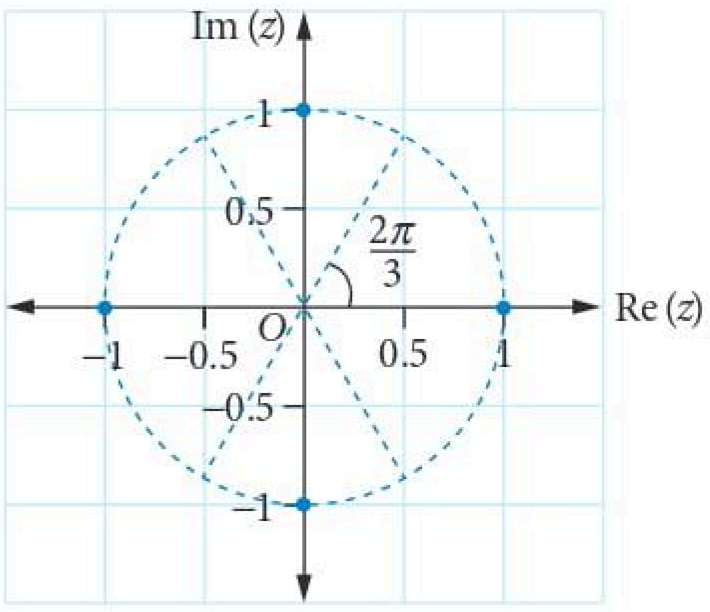
EXERCISE 4.4

1 B **2** C

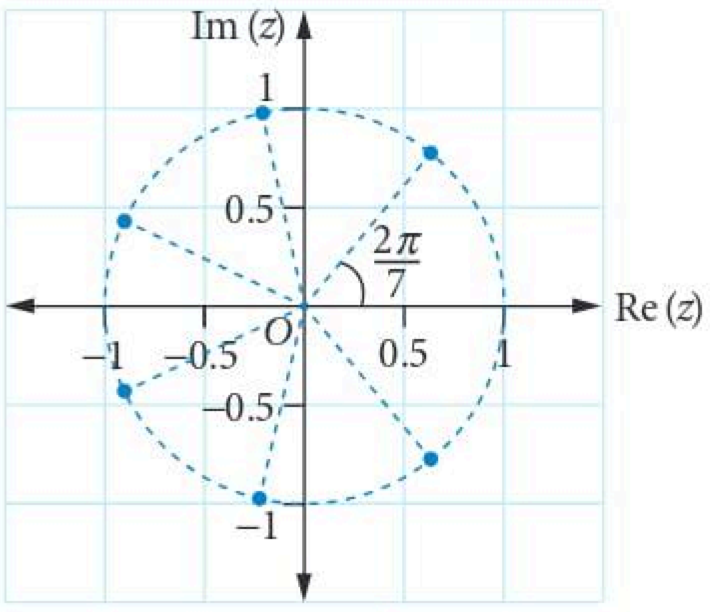
3 a $z = 1, -1, i, -i$



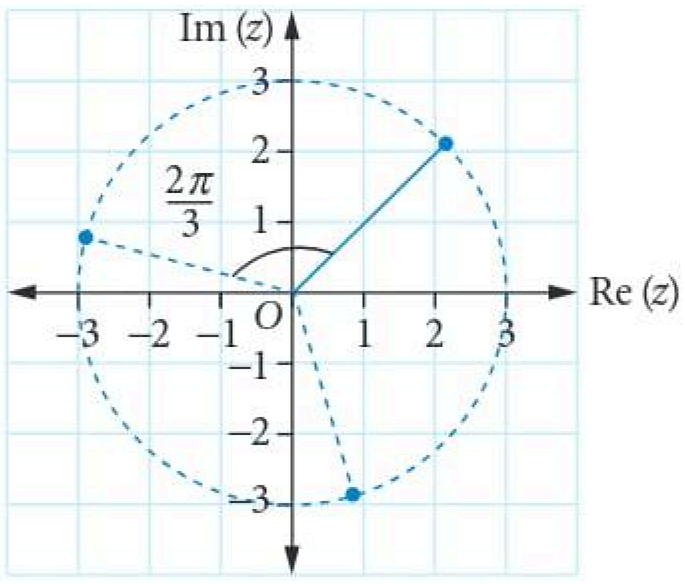
b $1, \frac{1}{2}+\frac{\sqrt{3}}{2}i, -\frac{1}{2}+\frac{\sqrt{3}}{2}i, -1, -\frac{1}{2}-\frac{\sqrt{3}}{2}i, \frac{1}{2}-\frac{\sqrt{3}}{2}i$



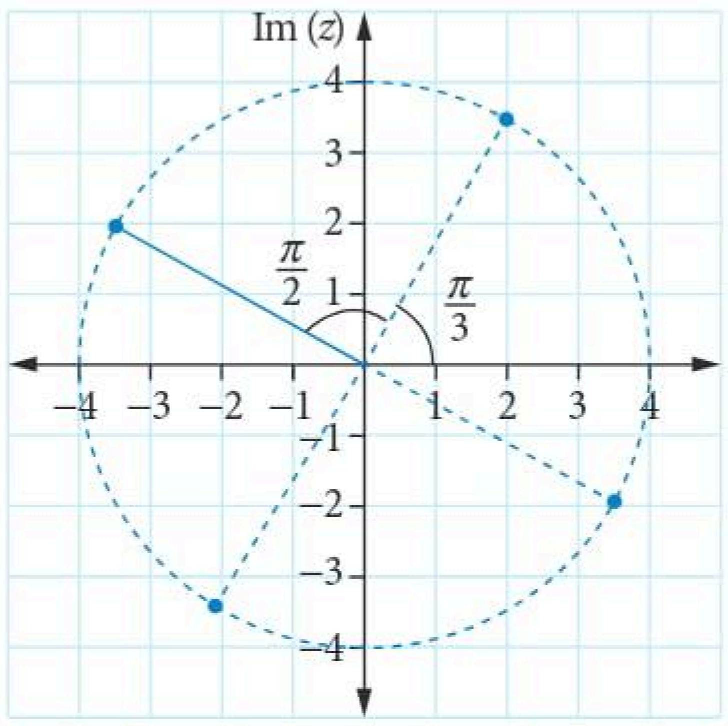
c $\text{cis}\left(\frac{2k\pi}{7}\right)$ for $k = -3, -2, -1, 0, 1, 2, 3$



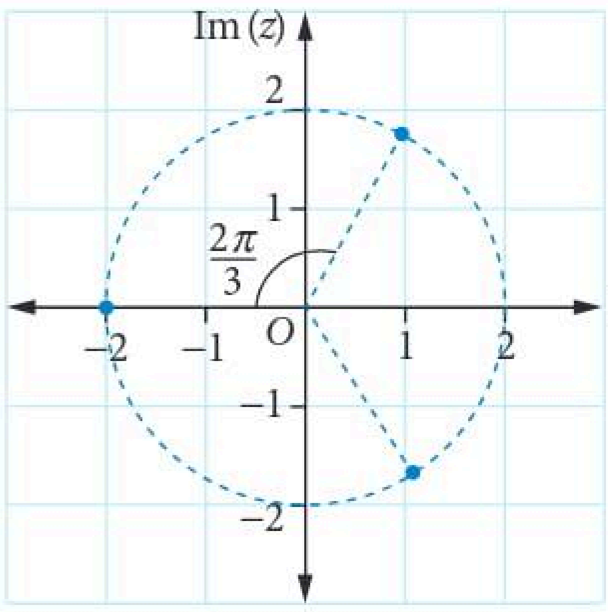
4 a $\frac{3\sqrt{2}}{2}+\frac{3\sqrt{2}}{2}i, \frac{-3(\sqrt{6}+\sqrt{2})}{4}+\frac{3(\sqrt{6}-\sqrt{2})}{4}i,$
$\frac{3(\sqrt{6}-\sqrt{2})}{4}-\frac{3(\sqrt{6}+\sqrt{2})}{4}i$



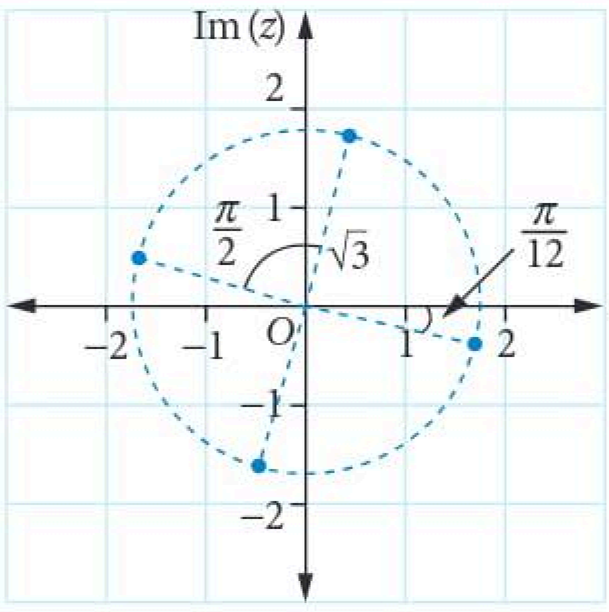
b $2 + 2\sqrt{3}i, -2\sqrt{3} + 2i, -2 - 2\sqrt{3}i, 2\sqrt{3} - 2i$



c $-2, 1 + \sqrt{3}i, 1 - \sqrt{3}i$



d $\sqrt{3}\,\text{cis}\left[\frac{(6k-1)\pi}{12}\right]$ for $k = -1, 0, 1$ and 2



5 a $-\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, 1$

b $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i -$

$\frac{\sqrt{2}}{2}i,\ i, -i, -1, 1$

c $1 + i, 1 - i, -1 + i, -1 - i$

d $\frac{\sqrt{3}}{2} + \frac{3}{2}i, \frac{\sqrt{3}}{2} - \frac{3}{2}i, \frac{\sqrt{3}}{2} + \frac{3}{2}i, -\frac{\sqrt{3}}{2} - \frac{3}{2}i, -\sqrt{3}, \sqrt{3}$

e $-\sqrt{2} - \sqrt{2}i, -\sqrt{2} + \sqrt{2}i, \sqrt{2} - \sqrt{2}i, \sqrt{2} + \sqrt{2}i$

6 $\text{cis}\left(-\frac{3\pi}{4}\right), \text{cis}\left(-\frac{\pi}{12}\right), \text{cis}\left(\frac{7\pi}{12}\right)$

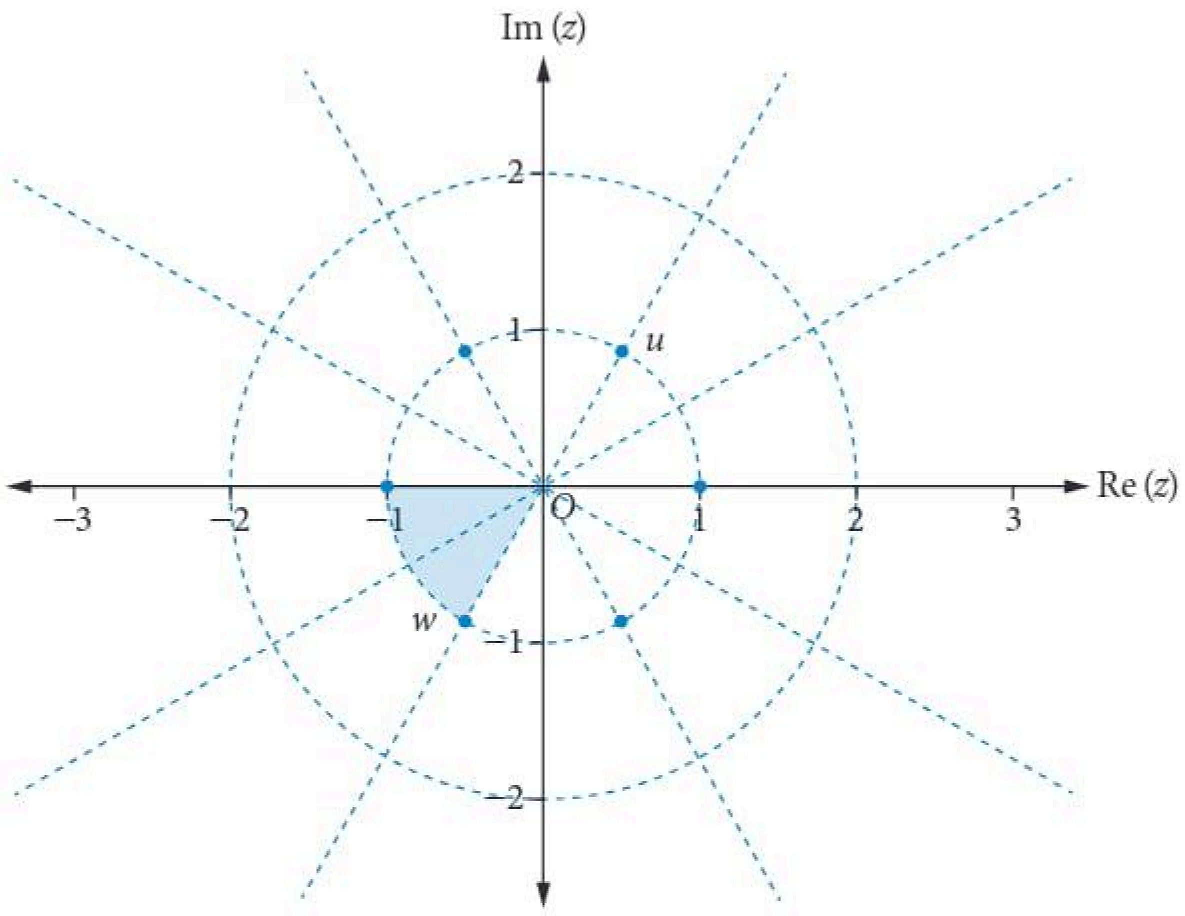
7 B

8 E

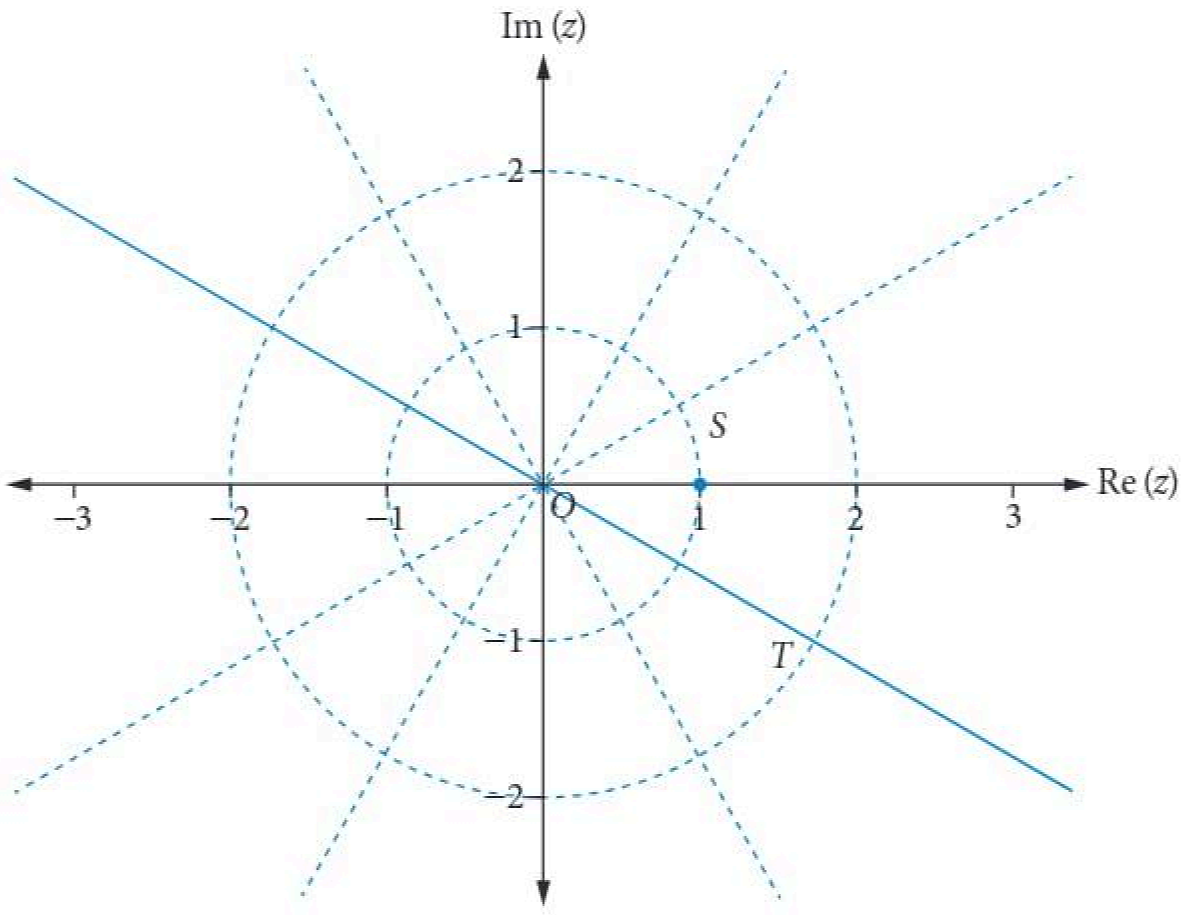
9 a **i** $\text{cis}\left(\frac{\pi}{3}\right)$

ii Apply de Moivre's theorem.

iii



b **i, ii**



iii $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

EXERCISE 4.5

1 D

2 B

3 a $(z + 2 + \sqrt{2}i)(z + 2 - \sqrt{2}i)$

b $\left(z - \frac{5}{2} + \frac{\sqrt{3}}{2}i\right)\left(z - \frac{5}{2} - \frac{\sqrt{3}}{2}i\right)$

c $3\left(z - \frac{1}{3} + \frac{\sqrt{2}}{3}i\right)\left(z - \frac{1}{3} - \frac{\sqrt{2}}{3}i\right)$

d $2\left(z - \frac{5}{2} + \frac{\sqrt{3}}{2}i\right)\left(z - \frac{5}{2} - \frac{\sqrt{3}}{2}i\right)$

e $(2z - 5 + \sqrt{3}i)(2z - 5 - \sqrt{3}i)$

4 a $3 - 6i$ **b** $-1 + 2i$ **c** $7 + 64i$

d 0 **e** $-102 + 53i$

5 a $3 - 3i$ **b** $7i$ **c** 9

d $6 - i$ **e** $-2 - 20i$

6 a $z - 2$ **b** $z + 1 - 2i$ **c** $z - 3 - 2i$

d $z - i, 3z + i$ **e** $z + 2, z - i$

7 **a** $\left(z+\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)\left(z-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)$

b $(z+i)(z+i)\left(z+\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\left(z-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$
$\left(z-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\left(z-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)$

c $(z+1)(z-1)(z+i)(z-i)$

d $\left(z+\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)\left(z-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$

e $(z+i)\left(z-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)\left(z+\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)$

8 **a** $(z+1)(z-1+2i)$ **b** $(z-2)(z-2-i)$

c $(z+i)(z+3-3i)$ **d** $(z-2i)(z+1+3i)$

e $(z-3)(z-2-2i)$

9 **a** $(3z-2+i)(z-1)(z+1)$

b $2(z+1-2i)\left(z-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)\left(z+\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)$

c $(z-i)(z+2)(z-1-2i)$

d $(z+i)(z-2i)(z+3-i)$

e $(z-2)(z+i)(z-2i)(z+2+2i)$

10 $(z+1)(z+4+2i)(z+4-2i)$

11 C **12** C **13** D **14** B

EXERCISE 4.6

1 A **2** B

3 **a** $\frac{3}{2}+\frac{\sqrt{11}}{2}i, \frac{3}{2}-\frac{\sqrt{11}}{2}i$

b $-2-\sqrt{2}i, -2+\sqrt{2}i$

c $\frac{\sqrt{2}}{3}+\frac{4}{3}i, \frac{\sqrt{2}}{3}-\frac{4}{3}i$

d $-\frac{\sqrt{3}}{2}+\frac{\sqrt{17}}{2}i, \frac{\sqrt{3}}{2}-\frac{\sqrt{17}}{2}i$

e $\frac{5}{4}+\frac{\sqrt{7}}{4}i, \frac{5}{4}-\frac{\sqrt{7}}{4}i$

f $\frac{1}{2}+\frac{\sqrt{15}}{6}i, \frac{1}{2}-\frac{\sqrt{15}}{6}i$

4 **a** $z^2-4z+13=0$ **b** $z^2+2z+5=0$

c $z^2-4z+7=0$ **d** $9z^2-6z+3=0$

e $16z^2+16z+7=0$

5 **a** $-i, \frac{5}{2}i$

b $\sqrt{2}-\left(2-\sqrt{6}\right)i, -\sqrt{2}-\left(2+\sqrt{6}\right)i$

c $-\sqrt{2}-\left(\sqrt{2}+3\right)i, \sqrt{2}+\left(\sqrt{2}-3\right)i$

d $\frac{\sqrt{6}}{2}+\frac{\sqrt{2}-4}{2}i, -\frac{\sqrt{6}}{2}+\frac{\sqrt{2}+4}{2}i$

e $\frac{1}{2}i, -2i$

f $2i, \frac{2}{3}i$

6 **a** $3-4i$ **b** $-3+5i$ and $-3-5i$

7 **a** $3, i, -i$ **b** $1, 1+2i, 1-2i$

c $-\frac{2}{3}, 2+i, 2-i$ **d** $-1, -1+i, -1-i$

e $2, -\frac{1}{2}+\frac{3}{2}i, -\frac{1}{2}+\frac{3}{2}i$ **f** $-1, i, \frac{4}{3}i$

g $-1, 2+i, \frac{3}{2}-i$ **h** $i, 4-2i, -\frac{1}{2}+\frac{3}{2}i$

i $-2, 3, 2-3i$ **j** $1+i, 2i, -\frac{1}{3}+\frac{1}{3}i$

k $1, -2, 3+i, 3+i$

8 **a** $2i, -2i, 5+i$ **b** $1+i, 2+i, -i$

c $2i, 2+i, 1+2i$ **d** $1+2i, 1-i, 1+i, 2+i$

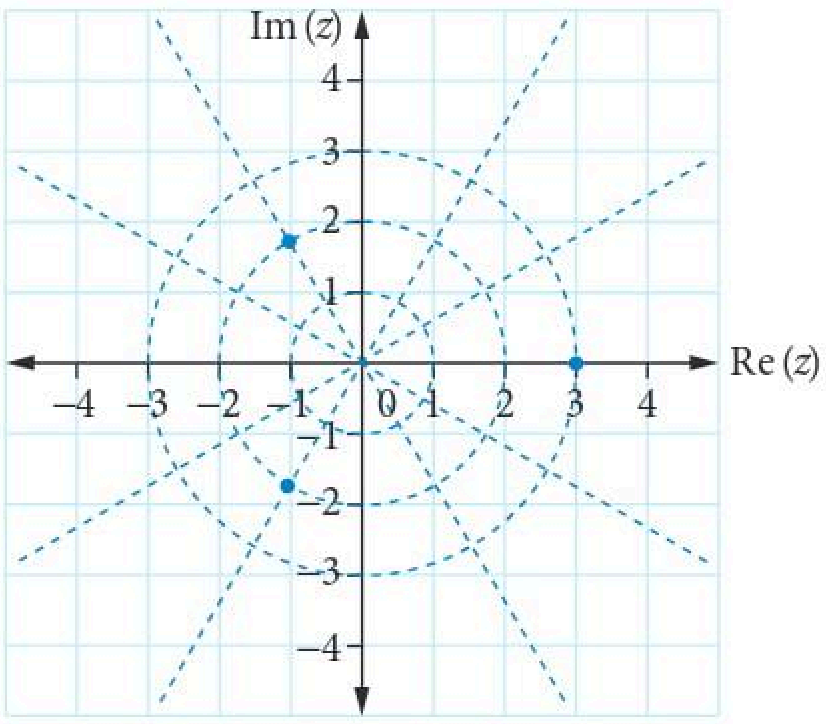
e $2i, -3i, 3+4i, -2+3i$

9 $1+i, 2$

10 **a** Show $p\left(\sqrt{5}-i\right)=0$ **b** $-2i, 2i$

11 **a** $-1-\sqrt{3}i, 3$

b



12 $\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}i, -\frac{\sqrt{6}}{2}-\frac{\sqrt{2}}{2}i, \frac{\sqrt{6}}{2}-\frac{\sqrt{2}}{2}i, -\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}i$

13 E **14** B **15** C **16** D

17 **a** **i** $z_1 = 2\sqrt{3}\,\text{cis}\left(-\frac{\pi}{3}\right)$

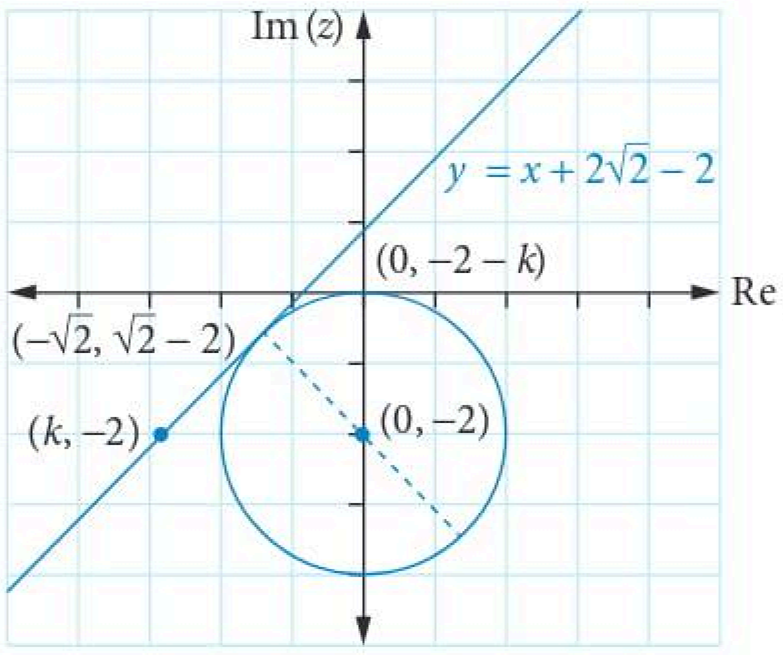
ii $\frac{2\pi}{3}$

iii $-2\sqrt{3}$ and $\sqrt{3}+3i$

b **i** 4

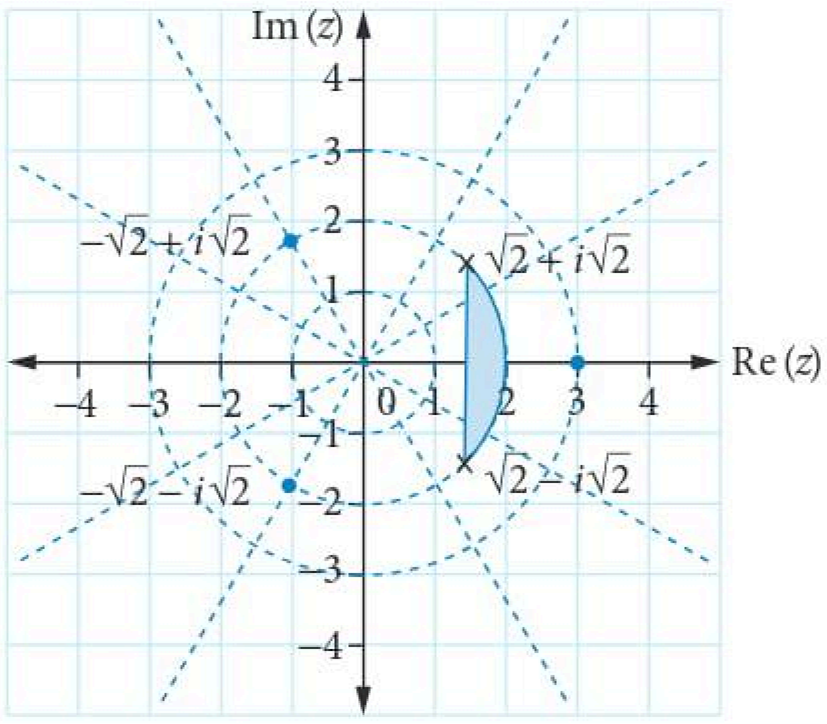
ii $[x+(y+2)i][x-(y+2)i]=4 \Rightarrow x^2+(y+2)^2=4$

iii



c $-2\sqrt{2}$

18 **a**



b Solve $y = \pm x$ and $x^2 + y^2 = 4$. $z = \pm\sqrt{2} \pm i\sqrt{2}$

c $\sqrt{2} - \sqrt{2}i, -\sqrt{2} + \sqrt{2}i, -\sqrt{2} - \sqrt{2}i$; all roots are shown in part **a** above.

d $\left(z - \sqrt{2} - \sqrt{2}i\right)\left(z + \sqrt{2} - \sqrt{2}i\right)\left(z + \sqrt{2} + \sqrt{2}i\right)\left(z - \sqrt{2} + \sqrt{2}i\right)$

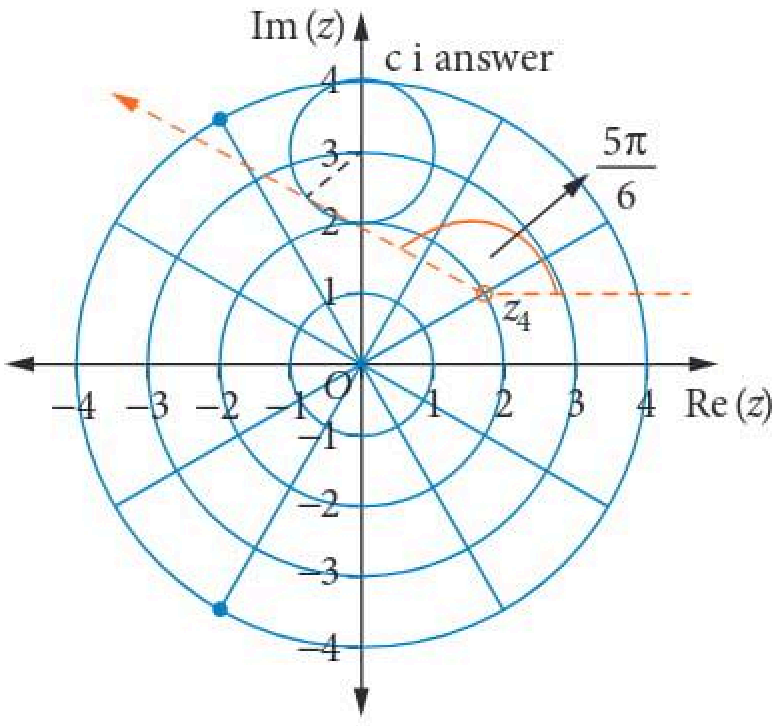
e Segment shaded in part *a*

f $\pi - 2 \left(\text{segment has angle } \frac{\pi}{2}\right)$

19 a i $z_2 = \overline{z}_3$, they are complex conjugates

ii $\alpha = -3, \beta = 9$ and $\gamma = -27$

b



c i See diagram above.

ii $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$

CUMULATIVE EXAMINATION 1

1 $p \in R \setminus \{-\sqrt{5}, \sqrt{5}\}$ 63%

2 'If n is not divisible by 2, then n^3 is not divisible by 8'.

3 a Find modulus and argument 81%

b $-12\sqrt{12}\,i = -24\sqrt{3}\,i$ 84%

c $n = 6k, k \in Z$ 39%

d $n = 6k + 3, k \in Z$ 26%

CUMULATIVE EXAMINATION 2

Section A

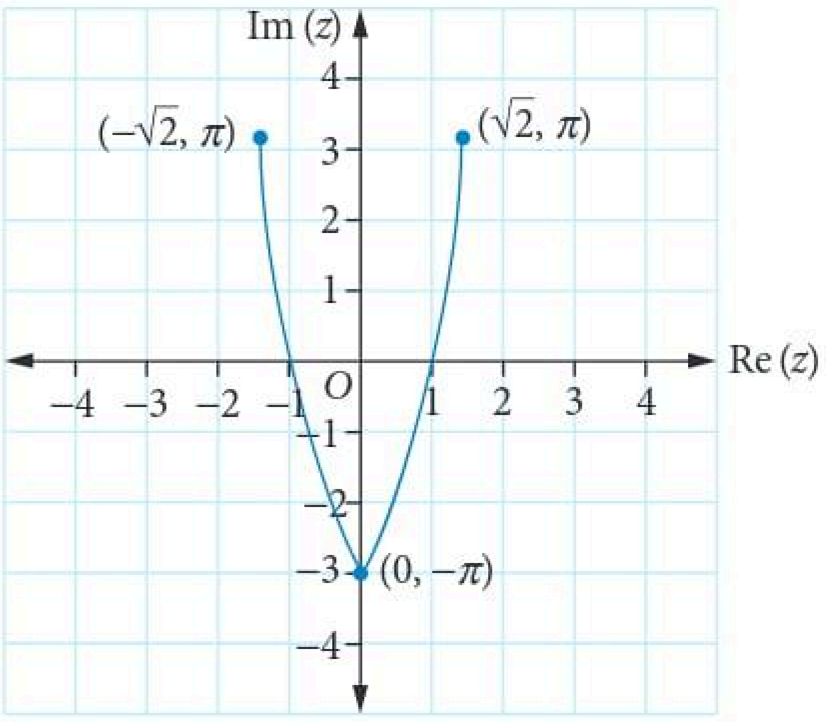
1 D 60% **2** C 75% **3** E 75%

4 A 56% **5** D 38%

Section B

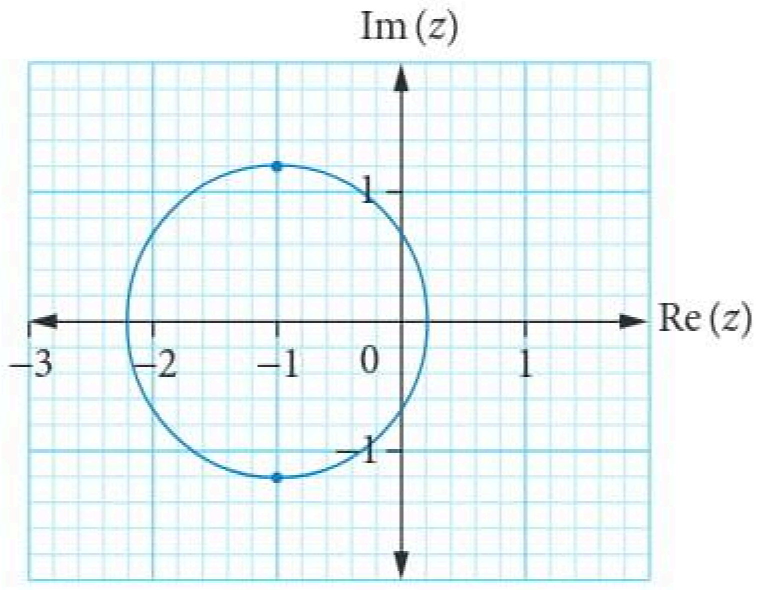
1 a domain: $\left[-\sqrt{2}, \sqrt{2}\right]$, range: $[-\pi, \pi]$ 76%

b 76%



2 a i Use the quadratic formula or complete the square. 68%

ii 83%



b i $m = 1, n = \frac{\sqrt{6}}{2}$ 64%

ii $(x + 1)^2 + y^2 = 1.5$ 59%

iii 68%

c $-1 \le d \le 5$ 19%

d $p = \frac{b}{2a},\ q = \frac{\sqrt{b^2 - 4ac}}{2|a|}$ 15%

CHAPTER 5

EXERCISE 5.1

1 $\frac{dy}{dx} = 6x^2(1 - 3x^3)$

2 $\frac{dy}{dx} = \frac{5}{(x + 2)^2}$

3 $f'(x) = -16x(3 - 4x^2)$

4 $\frac{5}{4}$

5 D **6** E **7** C

8 a $35x^6$ **b** $-\frac{3}{5x^{\frac{5}{3}}}$ **c** $\frac{3x}{\sqrt{3x^2 + 1}}$

9 a 35 **b** $-\frac{3}{5}$ **c** $\frac{3}{2}$

10 $\frac{1}{8}$

11 A **12** C **13** D **14** E

15 D **16** A **17** B **18** D

19 E **20** E **21** A **22** D

EXERCISE 5.2

1 $f'(1) = 12$ **2** B

3 $8\sin^3(2x)\cos(2x)$

4 1 **5** E **6** D

7 $\cos^2(x) - \sin^2(x) = \cos(2x)$

8 $-2\operatorname{cosec}^2(2x)$

9 $-(12x^2 - 1)\sin(4x^3 - x)$

10 $-12x^4\sin(x^3) + 8x\cos(x^3)$

11 C **12** C **13** A

14 A **15** B **16** B

EXERCISE 5.3

1 $\frac{2}{\pi}$ **2** B

3 $\frac{dy}{dx} = \frac{10}{4x^2 + 25}$; domain: R; range: $\left(0, \frac{2}{5}\right]$

4 −1 **5** E **6** D

7 Proof: see worked solutions.

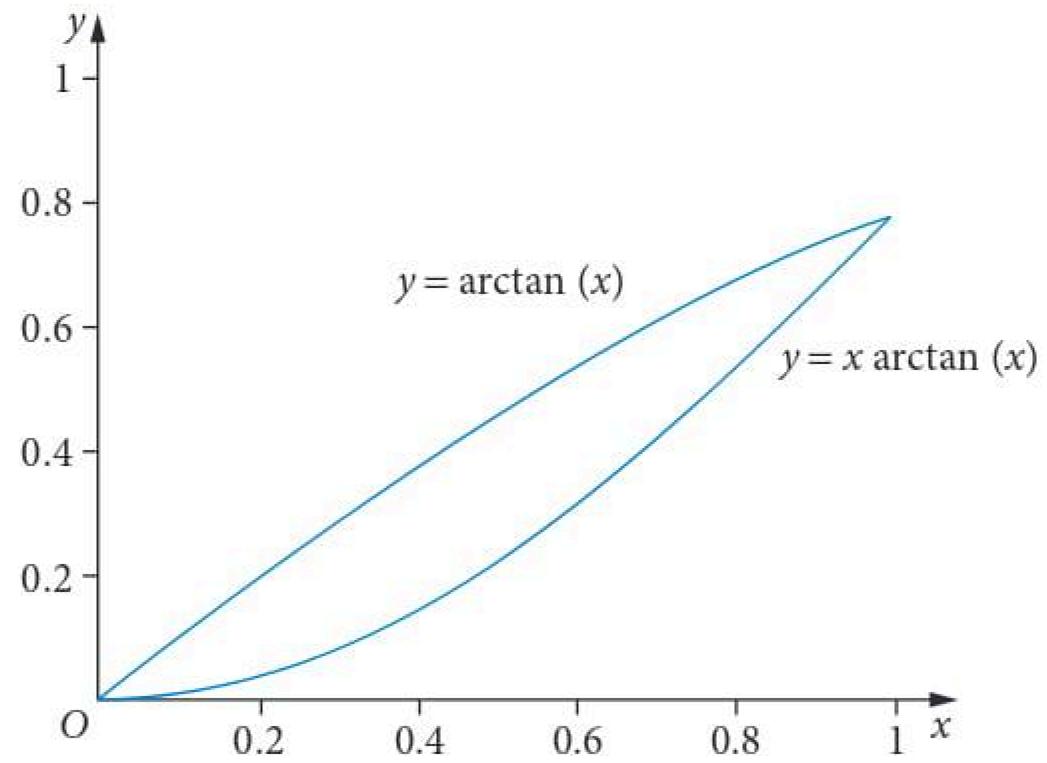
8 **a** domain: $x \in [-4, 0]$ range: $[-4, -2]$

b $f'(x) = \frac{2}{\pi\sqrt{-x(x+4)}}$

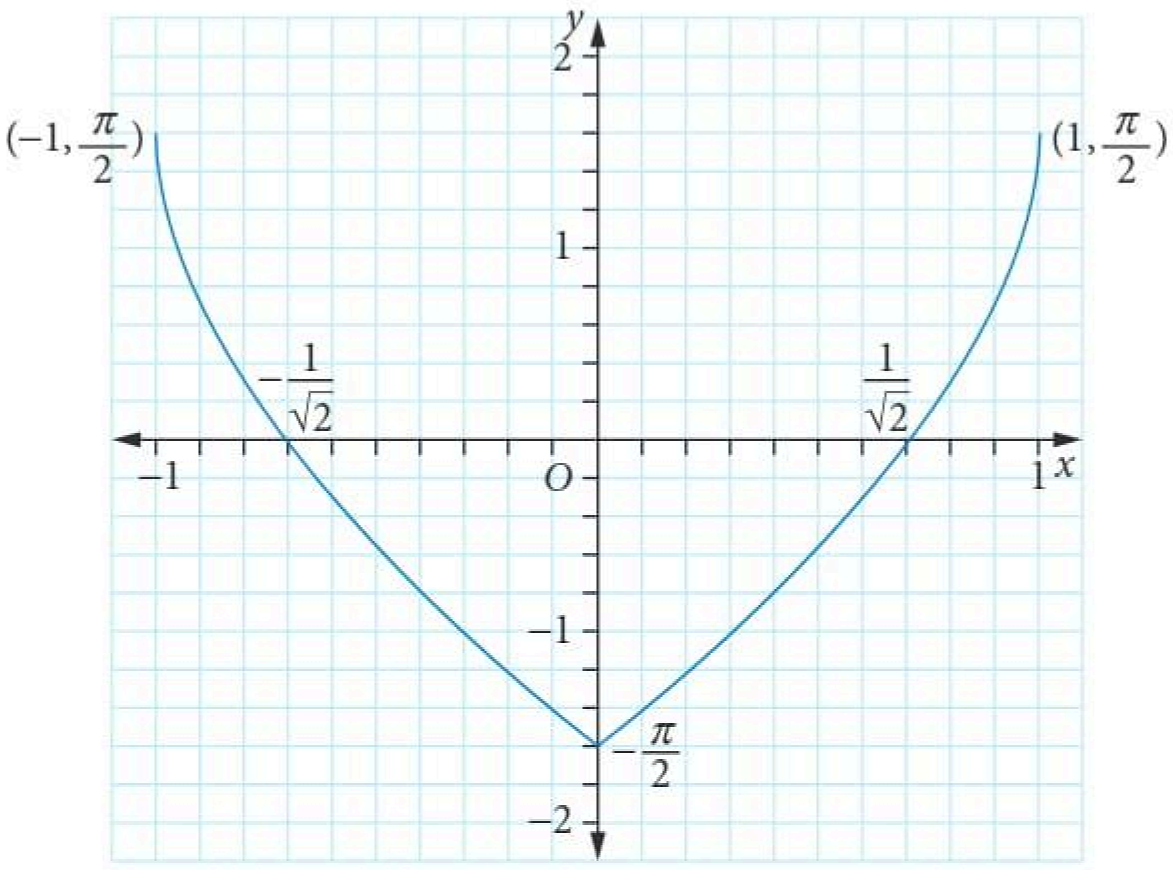
9 A **10** D **11** A **12** E

13 **a** $f'(x) = \frac{x}{1+x^2} + \arctan(x), f'(0) = 0$

b



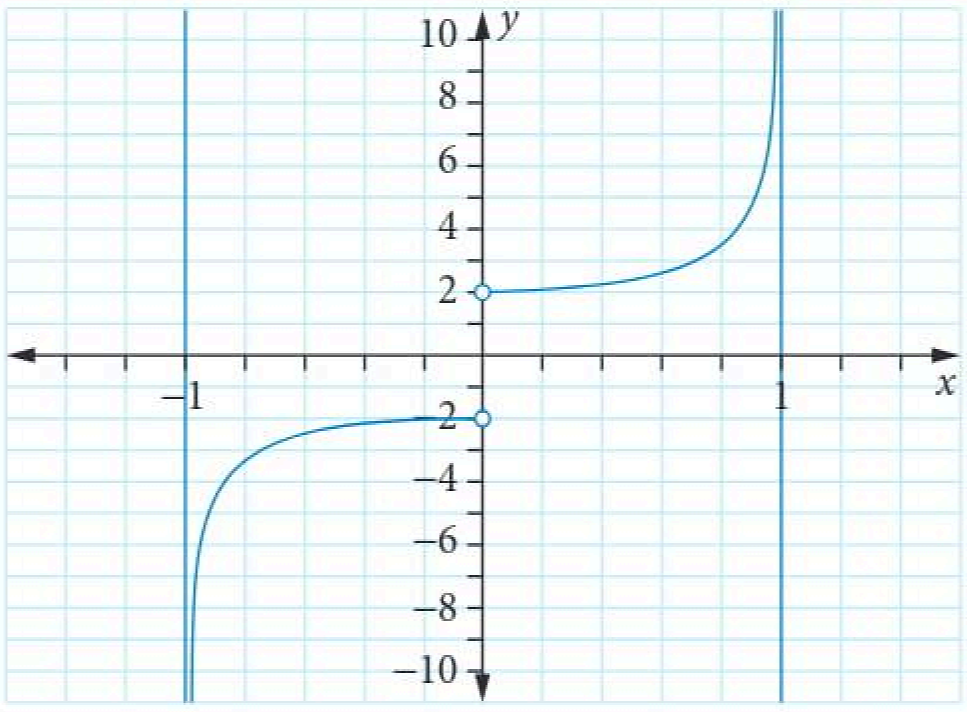
14 **a**



b Proof: see worked solutions, $a = 1$

c $f'(x) = \begin{cases} \frac{2}{\sqrt{1-x^2}} & \text{for } x \in (0, 1) \\ -\frac{2}{\sqrt{1-x^2}} & \text{for } x \in (-1, 0) \end{cases}$

d



EXERCISE 5.4

1 $\frac{2x}{x^4 + 1}$

2 $\frac{-1}{(\arcsin(x))^2\sqrt{1-x^2}}$, defined for $(-1, 1)\backslash\{0\}$ OR $(-1, 0) \cup (0, 1)$

3 **a** $4e^3$ **b** 1 **c** $2\log_e(2) + 1$

4 **a** $\frac{x^2 - 1}{x(x^2 + 1)}$ **b** $\frac{2x}{x^2 + 1}$ **c** $2x\log_e(3x) + x$

5 D

6 A

7 **a** $\frac{e^x(x-2)}{x^3}$ **b** $\frac{e^{6x}(6x-1)}{3x^2}$

c $\frac{2e^{5x}(5x-3)}{5x^4}$

8 **a** $e^{-4x}(2x - 4x^2)$ **b** $\frac{e^{\sqrt{x}}(x - 2\sqrt{x})}{2x^{\frac{5}{2}}}$

c $15e^{5x}\cos(3e^{5x})$

9 **a** $3e^2$ **b** $9e^2$ **c** $100e^2$

10 **a** $9e$ **b** $-e$ **c** $e + 2$

d $12e(2e-3)^5$

11 **a** 1 **b** $\frac{3}{2}$ **c** 1 **d** $2\cot(1)$

12 B **13** A **14** E **15** B

EXERCISE 5.5

1 $y = 3\ln(2(x-1))$ **2** E

3 $\frac{d^2v}{dt^2} = 24t + 16$ **4** $\frac{d^2y}{dx^2} = -\frac{1}{x^2}$

5 $\frac{d^2y}{dx^2} = -2$ **6** $\frac{d^2y}{dx^2} = 0$

7 $\frac{d^2y}{dx^2} = 42x^5 - 40x^3 + 12x^2$

8 **a** $\frac{d^2y}{dx^2} = 42x^5 - 40x^3 + 48x^2$

b $\frac{d^2y}{dx^2} = -20\cos(2x)$ **c** $\frac{d^2y}{dx^2} = 4$

d $\frac{d^2y}{dx^2} = 20x^{-6}$

9 **a** $f''(x) = \frac{-1}{4(2-x)^{\frac{3}{2}}}$ **b** $f''(x) = \frac{96}{(3x-1)^3}$

10 B

11 $f''\left(\frac{\pi}{2}\right) = -\frac{8\pi}{\left(1+\pi^2\right)^2}$.

12 $x = \frac{7}{18}$

13 $f''(x) = 72x^7$

14 $\frac{d^2y}{dx^2} = 40x^3 - 6x$

15 $f'(1) = 11$ and $f''(-2) = 168$

16 $f'(-1) = -16$ and $f''(2) = 40$

17 $g'(4) = -\frac{1}{32}$

18 $\frac{d^2h}{dt^2} = 26$

19 E **20** D **21** B

EXERCISE 5.6

1 $f'(x) = 4x - \frac{1}{4}$ and $f''(x) = 4$

2 A

3 Non-stationary point of inflection at $x = -\frac{1}{2}$.

4 Stationary point of inflection at $x = -1$ where the graph changes from concave up to concave down.

And at $x = \frac{3}{2}$, the graph changes from concave down to concave up.

5 Maximum turning point at $x = 1$ and stationary point of inflection at $x = 4$.

6 A **7** E **8** C

9 $f''(x) = 12x^2 = 0$ has solution $x = 0$. But concavity does not change on both sides of $x = 0$, because $f''(-1) = 12 > 0$ and $f''(1) = 12 > 0$. (0, 0) is not a point of inflection. It is a minimum point.

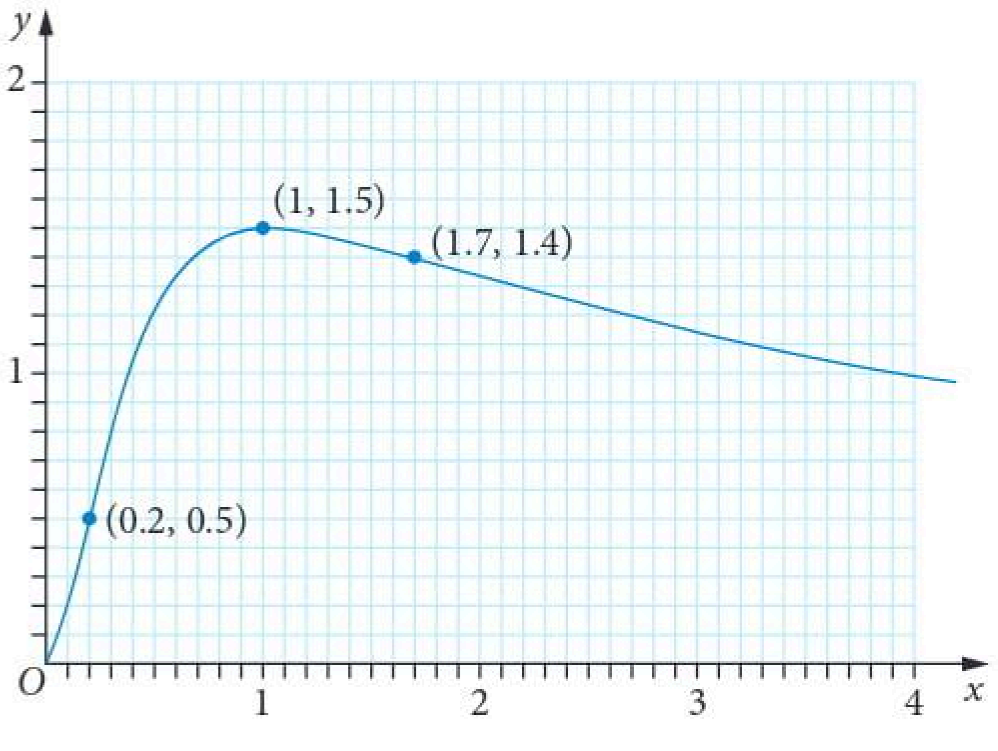
10 $\frac{-x\sin(x) - \cos(x)}{x^2}$

11 $90\pi^2$ **12** C **13** B

14 **a** (1, 1.5), $f''(1) = -\frac{9}{8} < 0$. Thus (1, 1.5) is the maximum turning point.

b **i** $9x^4 - 26x^2 + 1 = 0$ **ii** (0.2, 0.5) and (1.7, 1.4)

c



EXERCISE 5.7

1 $x = \pm 1$ **2** E

3 $\frac{dV}{dt} = 8\pi r^2$

4 $\frac{dr}{dt} = -\frac{1}{5\pi}$ mm/s; decreasing at rate of $\frac{1}{5\pi}$ mm/s.

5 $\frac{40\pi}{3}$ cm^3/s

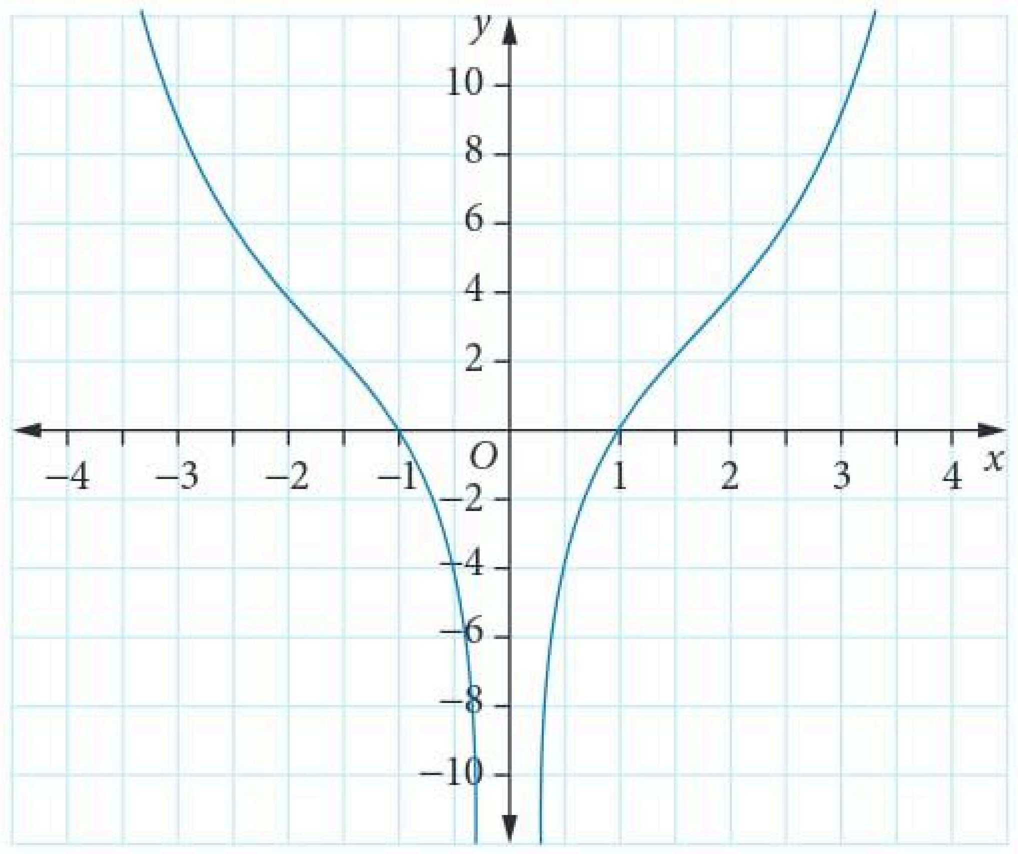
6 D **7** C **8** C **9** A

10 **a** **i** 2000π L **ii** 1.34 m

b $\frac{dh}{dt} = \frac{2000}{8000\tan^{-1}(h) + \frac{8000h}{1+h^2}}$

11 **a** $a = 3.2$, $b = 0.3$

b



c 0.11 cm/s

12 **a** $\frac{dy}{dt} = \frac{6-3x^3}{(1+x^3)^{\frac{3}{2}}} \times 2$ **b** $\frac{dA}{dt} = \frac{24\pi x(6-3x^3)}{(1+x^3)^2}$

c $\sqrt[3]{2}$

13 **a** Proof: see worked solutions

b domain: $(1, \infty)$, range: $(1, \infty)$

c Proof: see worked solutions

EXERCISE 5.8

1 $\frac{56}{65}$ **2** A **3** $-\frac{\sin(y)}{x\cos(y)}$

4 $-\frac{49}{6}$ **5** -3 **6** D

7 A

8 **a** (0, 1) **b** $\frac{dy}{dx} = \frac{2y - 9x^2 - k}{5 - 2y - 2x}$

c $\frac{dy}{dx} = -6$

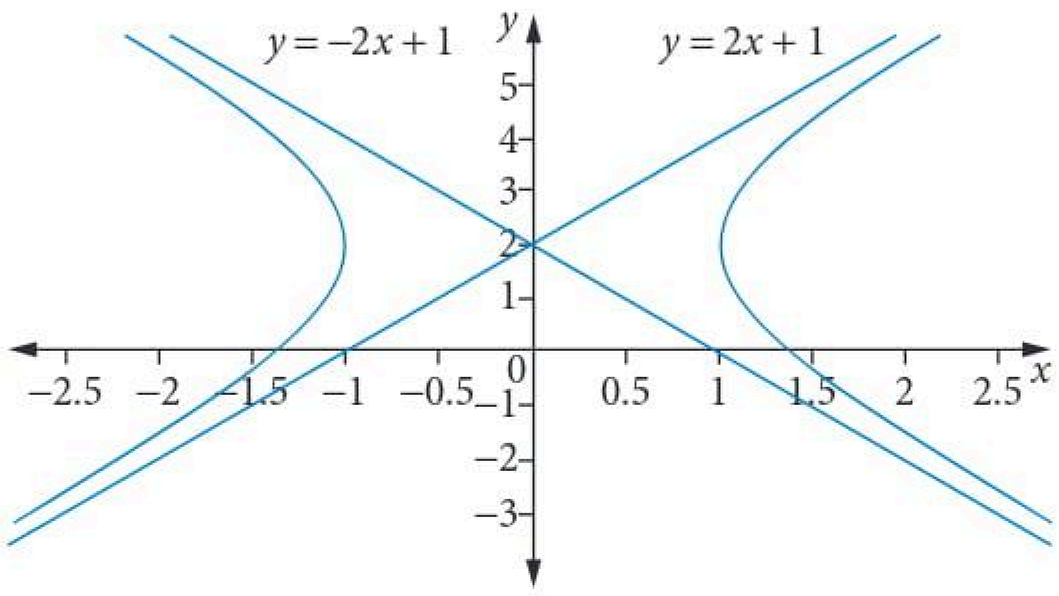
9 $\frac{dy}{dx} = \frac{3}{5}$

10 **a** $\frac{dy}{dx} = \frac{y}{9-x}$ **b** $\frac{dy}{dx} = \frac{1}{9}$

11 $\frac{dy}{dx} = \frac{-4}{13}$

12 **a** Asymptotes at $y = 2x + 2$ and $y = -2x + 2$

Intercepts at $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$



b $-\frac{8}{\sqrt{12}}$

13 $\frac{dy}{dx} = \frac{-18}{\pi\sqrt{3}+6}$ **14** A

CUMULATIVE EXAMINATION 1

1 $\frac{dy}{dx} - 2\frac{d^2y}{dx^2} = -\frac{3}{2}$

2 a i $x \in [-2, 2]$ 67% **ii** $R \setminus \left[-\frac{1}{5}, \frac{1}{5}\right]$ 33%

iii $x \in [-2, 2] \setminus \left[-\frac{1}{5}, \frac{1}{5}\right]$ 26%

b $\frac{5\sqrt{7}}{16}$ 20%

CUMULATIVE EXAMINATION 2

Section A

1 C 58% **2** B 60% **3** B 46%

4 E 6% **5** D

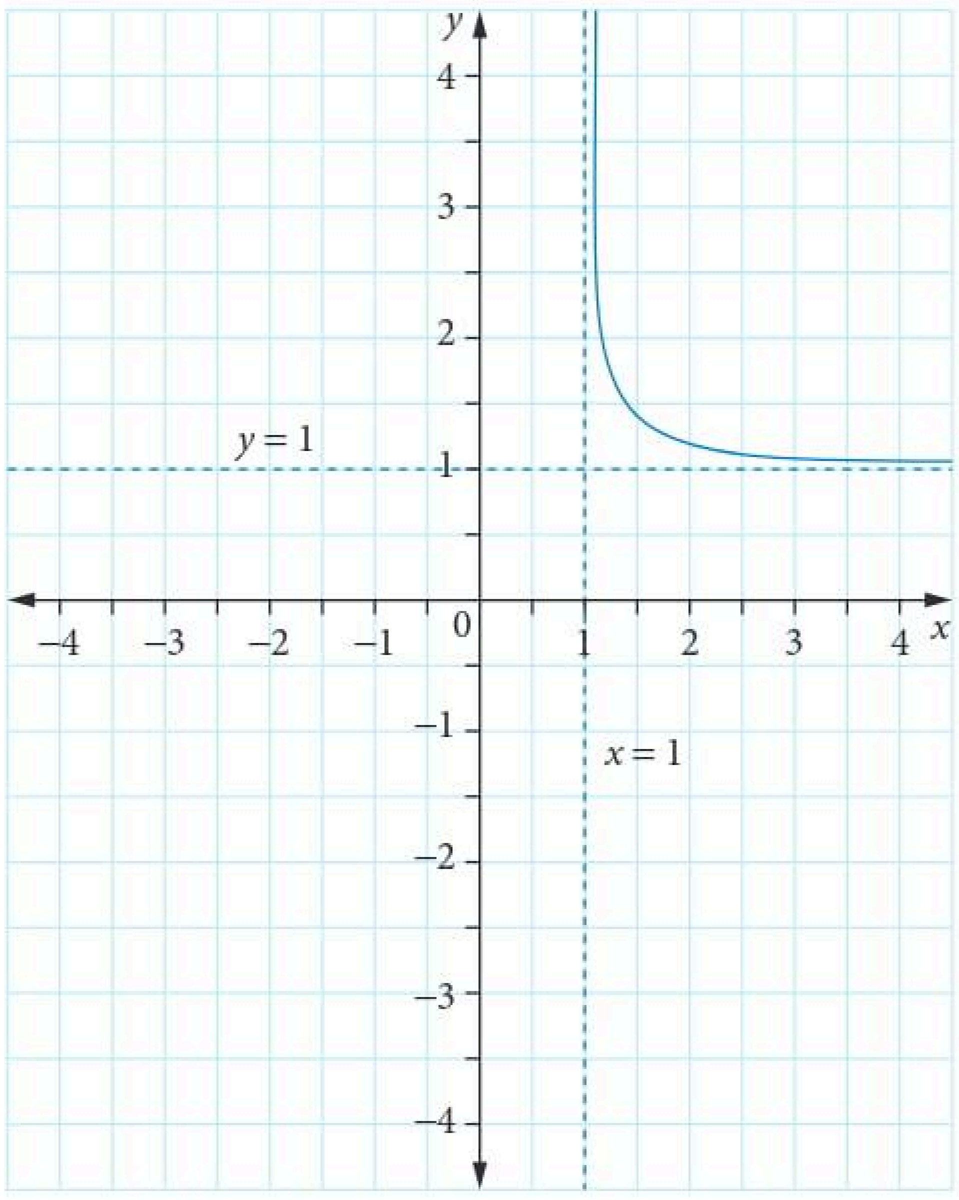
Section B

1 a Proof: see worked solutions

b $x \in (1, \infty),\ y \in (1, \infty)$

c Proof: see worked solutions

d

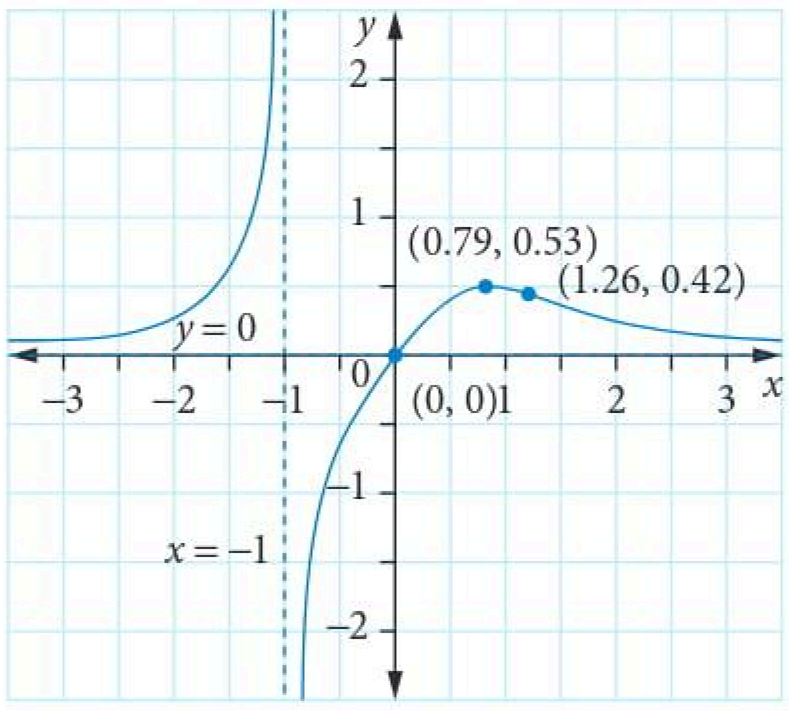


2 a i $x = -1, y = 0$ 36%

ii $f'(x) = \frac{1-2x^3}{\left(1+x^3\right)^2}$, $(0.79,\ 0.53)$ 90%

iii $(1.26, 0.42)$ 52%

b 83%



CHAPTER 6

EXERCISE 6.1

1 a $2x - 5y - 14 = 0$, straight line

b $\frac{(x-1)^2}{9} - \frac{(y-2)^2}{4} = 1$, horizontal hyperbola with centre $(1, 2)$

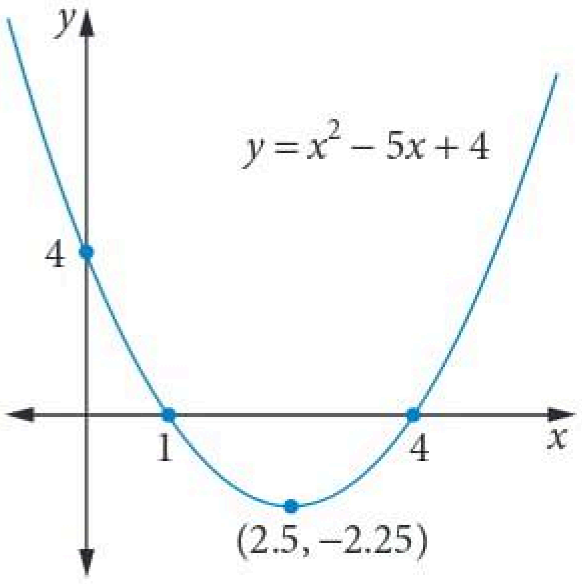
c $y = (x+1)^3$, cubic with point of inflection $(-1, 0)$

d $(x+3)^2 + (y-4)^2 = 25$, circle of radius 5 with centre $(-3, 4)$

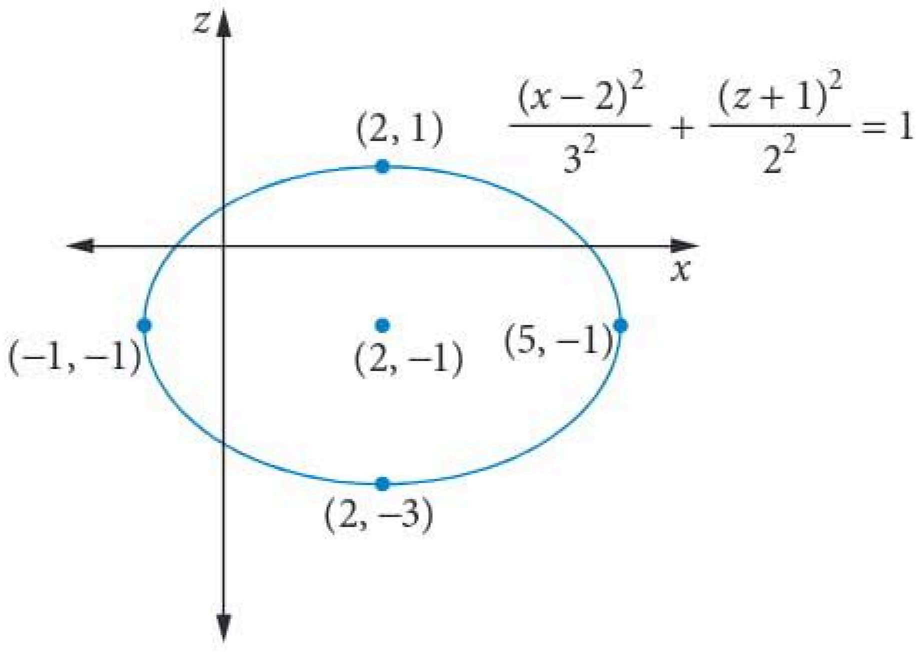
2 A cosine function of magnitude 3 along the line $z = 2y$ and perpendicular to the y-z plane.

3 A straight line through the origin with projections $x + 4y = 0$, $x - 2z = 0$ and $2y + z = 0$ on the x-y, x-z and y-z planes respectively.

4



5



6 $y^2 + z^2 = 5$, a circle of radius $\sqrt{5}$ in the y-z plane with centre at the origin.

7 A parabola on the plane perpendicular to the x-axis through $x = 1$ with equation $z = 4y^2$.

8 $\underset{\sim}{r}(t) = (t-1)\underset{\sim}{i} + [4(t-4)^2 + 2]\underset{\sim}{j}$

9 D **10** A

EXERCISE 6.2

1 B **2** E

3 a $\underset{\sim}{r}(t) = (3 + 2t)\underset{\sim}{i} + (8 - 10t)\underset{\sim}{j}$

b $\underset{\sim}{r}(t) = (-5 + 7t)\underset{\sim}{i} + (3 + 4t)\underset{\sim}{j} + (8 - 8t)\underset{\sim}{k}$

c $\underset{\sim}{r}(t) = (6 + 2t)\underset{\sim}{i} + (-1 + 4t)\underset{\sim}{j} + (-4 + 10t)\underset{\sim}{k}$

d $\underset{\sim}{r}(t) = (5 - 6t)\underset{\sim}{i} + (-8 + 2t)\underset{\sim}{j} + t\underset{\sim}{k}$

e $\underset{\sim}{r}(t) = (-8 + 8t)\underset{\sim}{i} + (8 - t)\underset{\sim}{j} + (-8 + 7t)\underset{\sim}{k}$

f $\underset{\sim}{r}(t) = (-3 + 8t)\underset{\sim}{i} + (-6 + 10t)\underset{\sim}{j} + (-8 + 6t)\underset{\sim}{k}$

4 a $x = 9 - 8t, y = -8 + 10t, z = -7t$

b $x = -4 + 7t, y = -1 + 9t, z = 5 - 10t$

c $x = -8 + 6t, y = -4 + 13t, z = -4 + 9t$

d $x = 2 + 5t, y = 9 - 15t, z = -9 + 6t$

e $x = -5 + 10t, y = -7 + 13t, z = 2 - 10t$

f $x = 7 - 4t, y = -1 - 7t, z = -7 - 2t$

5 $\utilde{i} + 6\utilde{j} - 2\utilde{k} + t\,(3\utilde{i} + 5\utilde{j} - 2\utilde{k})$

6 **a** $\frac{x-6}{-3} = \frac{y-1}{2}$ or $2x + 3y - 15 = 0$

b $\frac{x+2}{4} = \frac{y-4}{5}$ or $5x - 4y + 26 = 0$

c $\frac{x+3}{5} = \frac{y}{2} = \frac{z+8}{9}$

d $-x - 3 = \frac{y+2}{7} = \frac{z-8}{-4}$

e $\frac{x-2}{5} = \frac{y+5}{10} = \frac{z+9}{16}$

f $\frac{x-4}{3} = \frac{y-4}{3} = \frac{z-6}{4}$

7 **a** $\utilde{r}\,(t) = 5\,\utilde{i} - \utilde{j} - 4\utilde{k} + t\,(-11\utilde{i} + 4\utilde{j} + 12\utilde{k}), 0 \le t \le 1$

b $\utilde{r}\,(t) = -6\utilde{i} + 8\utilde{j} - 2\utilde{k} + \mathrm{t}\,(11\utilde{i} - 7\utilde{j} - 5\utilde{k}), 0 \le t \le 1$

c $\utilde{r}\,(t) = -\utilde{i} - 8\utilde{j} + 7\utilde{k} + t\,(3\utilde{i} + 3\utilde{j} - 16\utilde{k}), 0 \le t \le 1$

d $\utilde{r}\,(t) = -\utilde{j} - 5\utilde{k} + t\,(-7\utilde{i} + 4\utilde{j} + 6\utilde{k}), 0 \le \mathrm{t} \le 1$

e $\utilde{r}\,(t) = 3\utilde{i} + 6\utilde{j} + 3\utilde{k} + t\,(-4\utilde{i} - 10\utilde{j} - 3\utilde{k}), 0 \le t \le 1$

f $\utilde{r}\,(t) = 5\utilde{i} - 6\utilde{j} + t\,(4\,\utilde{i} - 2\utilde{j} + \utilde{k}), 0 \le t \le 1$

8 **a** Y **b** N **c** Y

d Y **e** Y **f** N

9 **a** Y **b** N **c** N

d N **e** Y

10 **a** Y **b** Y **c** N

d N **e** Y

11 $10\utilde{i} + 30\utilde{j} + 20\utilde{k} + t\,(30\utilde{i} + 50\utilde{j} + 20\utilde{k})$

12 $-2\utilde{i} + 4\utilde{j} - 3\utilde{k} + t\,(\utilde{i} - \sqrt{2}\utilde{j} + \sqrt{3}\utilde{k})$

13 $\utilde{a} + \utilde{u} + t\,(\utilde{v} + \utilde{w} - \utilde{u})$

14 $2x + 3y + 5 = 0$

15 B **16** D **17** A

EXERCISE 6.3

1 D **2** B

3 **a** Y **b** Y **c** N

d Y **e** N

4 **a** $-53\utilde{i} + 5\utilde{j} - 35\utilde{k}$ **b** $-12\utilde{i} - 9\utilde{j} - 25\utilde{k}$

c $41\utilde{i} - 20\utilde{j} + 17\utilde{k}$ **d** $-32\utilde{i} + 17\utilde{j} + 18\utilde{k}$

5 **a** $\frac{1}{\sqrt{91}}(3\utilde{i} + 9\utilde{j} - \utilde{k})$ **b** $\frac{1}{3}(\utilde{i} + 2\utilde{j} - 2\utilde{k})$

c $\frac{1}{7}2\,(\utilde{i} - 6\utilde{j} + 3\utilde{k})$ **d** $\frac{1}{\sqrt{57}}(2\utilde{i} + 2\utilde{j} - 7\utilde{k})$

6 **a** $4\utilde{i} - 9\utilde{j} + 7\utilde{k}$ **b** $28\utilde{i} - 21\utilde{j} - 15\utilde{k}$

c $41\utilde{i} - 17\utilde{j} - 20\utilde{k}$ **d** $35\utilde{i} - 17\utilde{j} - 37\utilde{k}$

e $18\utilde{i} - 26\utilde{j} - \utilde{k}$

7 **a** $\frac{1}{15}(2\utilde{i} - 11\utilde{j} + 10\utilde{k})$ **b** $\frac{1}{43}(7\utilde{i} - 42\utilde{j} - 6\utilde{k})$

c $\frac{1}{\sqrt{110}}(6\utilde{i} - 5\utilde{j} + 7\utilde{k})$ **d** $\frac{1}{9}(7\utilde{i} - 4\utilde{j} - 4\utilde{k})$

e $\frac{1}{\sqrt{69}}(7\utilde{i} + 4\utilde{j} - 2\utilde{k})$

8 **a** $-52\utilde{i} - 5\utilde{j} + 25\utilde{k}$ **b** $11\utilde{i} + 21\utilde{j} + 14\utilde{k}$

c $53\utilde{i} - 59\utilde{j} - 14\utilde{k}$ **d** $14\utilde{i} + 6\utilde{j} - 25\utilde{k}$

e $17\utilde{j} + 51\utilde{k}$

9 $3\utilde{i} + 4\utilde{j} - \utilde{k}$

10 C

EXERCISE 6.4

1 E **2** B

3 **a** $5x - y + 3z = -13$ **b** $9x + y - 4z = 12$

c $4x + 4y - 7z = 44$ **d** $7x + 4y + 8z = -64$

e $3y - 7x = -80$

4 **a** $\mathrm{p} \cdot (3\utilde{i} - 3\utilde{j} + 5\utilde{k}) = 40$

b $\mathrm{p} \cdot (9\utilde{k} - 4\utilde{i} - 2\utilde{j}) = 84$

c $\mathrm{p} \cdot (9\utilde{j} - 8\utilde{i} + 2\utilde{k}) = -15$

d $\mathrm{p} \cdot (9\utilde{i} - 2\utilde{j} + 2\utilde{k}) = 23$

e $\mathrm{p} \cdot (9\utilde{j} - 9\utilde{i} - \utilde{k}) = 55$

5 **a** $x - y - 3z = 13$ **b** $3x + 2y - 22z = -3$

c $10x - 71y + 13z = -43$ **d** $x + 3y - z = 13$

e $2y - 14x + 69z = 1$

6 **a** $-59x + 37y - 6z = 289$ **b** $6x - 5y - 6z = 34$

c $5x - 2y - 3z = 0$ **d** $-4x + 15y + 14z = 5$

e $28x + 33y - 45z = 10$ **f** $25x - y + 16z = -19$

g $x + 3y - z = 6$ **h** $-x - 51y + 8z = -33$

7 **a** Y, N, N **b** Y, N, N

c Y, N, N **d** Y, N, N

e N, N, Y **f** Y, N, N

8 **a** $-\utilde{i} + (2 + 3t)\,\utilde{j} + (-3 + 4t)\,\utilde{k}$

b $(2 + 5t)\,\utilde{i} + (-2 + 5t)\,\utilde{j} + \left(-\frac{5}{2} + t\right)\utilde{k}$

c $(5 + t)\,\utilde{i} - 4\utilde{j} + (-3 - 3t)\,\utilde{k}$

d $(-6 - 3t)\,\utilde{i} + (2 + 4t)\,\utilde{j} + (2 - 4t)\,\utilde{k}$

e $\left(\frac{2}{5} - 2t\right)\utilde{i} + (-4 + 5t)\,\utilde{j} + 5\utilde{k}$

9 no

10 $4x - y + 5z = -4$

11 $x - y + 3z = 20$

12 $\arccos\left(\frac{8}{45}\right)$

13 A **14** B

15 **a** $5 - t = 3 \Rightarrow t = 2$, but $3 + 4 \times 2 \ne 7$

b $x + 3y - 4z = -40$ **c** $\left(\frac{47}{13}, -\frac{41}{13}, \frac{111}{13}\right)$

d $\frac{8}{13}\utilde{i} - \frac{24}{13}\utilde{j} - \frac{20}{13}\utilde{k}$ and $\overrightarrow{BC}$ is in the direction of the line, $\utilde{n} = -\utilde{i} - 3\utilde{j} + 4\utilde{k}$

$\overrightarrow{AB} \cdot \utilde{n} = \frac{8}{13} \times (-1) + -\frac{24}{13} \times (-3) + -\frac{20}{13} \times 4 = \frac{8}{13} +$

$\frac{72}{13} - \frac{80}{13} = 0$

Hence $\overrightarrow{AB}$ is perpendicular to $\utilde{n}$, so $\angle ABC = \frac{\pi}{2}$.

e Suppose the line is given by $\underset{\sim}{r}(t) = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k} + t(n_1\underset{\sim}{i} + n_2\underset{\sim}{j} + n_3\underset{\sim}{k})$.

Then the plane has normal vector $n_1\underset{\sim}{i} + n_2\underset{\sim}{j} + n_3\underset{\sim}{k}$ and has equation $n_1x + n_2y + n_3z = d$ for some $d \in R$. Since B is on the plane, it satisfies the equation, so $n_1b1 + n_2b_2 + n_3b_3 = d$.

The equation of the plane is $n_1x + n_2y + n_3z = n_1b_1 + n_2b_2 + n_3b_3$.

CUMULATIVE EXAMINATION 1

1 a $x = \frac{\pi}{2} + 2n\pi, n \in Z$

b $f''(x) = 1 - \sin(x) \geq 0$, no sign change, so there are no points of inflection.

2 $-(2nx^{n-1}) - n^2x^{n-2} + nx^{n-2} - x^n)e^{-x}$ or $x^{n-2}(x^2 - 2nx + n(n-1))e^{-x}$ 91%

3 $\underset{\sim}{r}(t) = (-2 + 5t)\underset{\sim}{i} + (3 + 2t)\underset{\sim}{j} + (-1 - 4t)\underset{\sim}{k}$

4 a $4\underset{\sim}{i} + \underset{\sim}{j} + 5\underset{\sim}{k}$ **b** $4x + y + 5z = 13$

CUMULATIVE EXAMINATION 2

Section A

1 D **2** B **3** D

4 A **5** C

Section B

1 a $\frac{dy}{dx} = -\frac{9x}{16y}$

b $z = a + \frac{16b}{9a}$

c $\frac{dz}{dt} = \left(1 - \frac{9x^2 + 16y^2}{9x^2y}\right)\frac{dx}{dt}$

d $\frac{dz}{dt} = \frac{3\sqrt{15} - 32}{2\sqrt{15}} \approx -2.6312$

e As x increases from 0 to 8, z moves from $+\infty$ back towards the ellipse.

2 a $x + y + 2z = 0$ **b** $x + y + 2z = 20$

c $\frac{15\sqrt{6}}{4}$

CHAPTER 7

EXERCISE 7.1

1 a $2x^5 + 2x^3 - 4x^2 + 5x + c$

b $-\frac{1}{x^2} + \frac{1}{x^3} + c$

c $\frac{2x^{\frac{3}{2}}}{3} + 3x^{\frac{1}{3}} - \frac{4x^{\frac{9}{4}}}{9} + c$

2 a $4e^{2x} - 2e^{-2x}$ **b** $2\log_e|x|$ **c** $\frac{1}{3}\log_e|x|$

3 a $\frac{1}{5}\log_e|5x - 2| + c$

b $\frac{3}{2}\log_e|4x + 3| + c$

c $-2\log_e|1 - 4x| + c$

4 a $\frac{(2x + 9)^4}{8} + c$ **b** $\frac{(4x - 1)^3}{6} + c$

c $\frac{-1}{10(5x + 6)^2} + c$

5 a $\frac{1}{10}\sin(10x) + c$ **b** $-\frac{1}{2}\cos(6x) + c$

c $\frac{2}{3}\sin(3x + 4) + c$

6 a $4\log_e(2)$ **b** $\frac{\sqrt{3}}{4}$ **c** $\frac{133}{3}$

7 a $x\log_e(x) + x\log_e(3) - x + c$

b $\frac{2x\sin(2x) + \cos(2x)}{4} + c$

8 a 8.389 **b** 0.048

9 $a = -1$

10 2 **11** A **12** A **13** D

14 E **15** D **16** C **17** E

EXERCISE 7.2

1 D **2** A

3 a $\sin^{-1}\left(\frac{x}{3}\right) + c$ **b** $4\cos^{-1}\left(\frac{x}{9}\right) + c$

4 a $\frac{1}{2}\sin^{-1}(2x) + c$ **b** $\frac{1}{3}\cos^{-1}\left(\frac{3x}{2}\right) + c$

5 a $\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) + c$ **b** $\frac{2}{3}\tan^{-1}\left(\frac{x}{3}\right) + c$

6 a $\frac{1}{10}\tan^{-1}\left(\frac{5x}{2}\right) + c$ **b** $\frac{1}{9}\tan^{-1}\left(\frac{9x}{4}\right) + c$

7 a $\frac{1}{4}\tan^{-1}\left(\frac{x - 1}{4}\right) + c$ **b** $\frac{1}{6}\tan^{-1}\left(\frac{x + 3}{6}\right) + c$

8 a $\frac{5\pi}{3}$ **b** $\frac{\pi}{12}$

9 a $\frac{\pi}{3}$ **b** $\frac{\pi}{9}$

10 a $(x + 5)^2 + 9$ **b** $\frac{1}{3}\tan^{-1}\left(\frac{x + 5}{3}\right) + c$

11 $\frac{\pi}{4a}$ **12** A **13** D **14** A

15 B **16** E **17** E **18** D

EXERCISE 7.3

1 D **2** E

3 a $\frac{2}{3}\left(3x^2 - 2x\right)^{\frac{3}{2}} + c$ **b** $\sin\left(e^x\right) + c$

4 a $(x^3 + 2)^4 + c$ **b** $\frac{-2}{\left(4x - 5x^2\right)} + c$

5 a $-\frac{1}{5}\cos^5(x) + c$ **b** $\frac{(\log_e(x))^2}{6} + c$

6 a $\frac{2(x + 5)^{\frac{5}{2}}}{5} - \frac{10(x + 5)^{\frac{3}{2}}}{3} + c$

b $-\frac{1}{x - 1} - \frac{1}{2(x - 1)^2} + c$

7 a $\frac{\sqrt{3}}{8}$ **b** $\frac{195}{4}$

8 a $\frac{76}{3}$ **b** $\frac{349}{20}$

9 $\sin(e) - \sin(1)$

10 $\int \frac{\sin(2x)}{\cos(2x)}\,dx$

Let $u = \cos(2x)$ $\frac{du}{dx} = -2\sin(2x)$

$-\frac{1}{2}\frac{du}{dx} = \sin(2x)$

$-\frac{1}{2}\int \frac{1}{u}\,du = -\frac{1}{2}\log_e|u| + c = \frac{1}{2}\log_e\left|u^{-1}\right| + c$

$\int \frac{\sin(2x)}{\cos(2x)}\,dx = \frac{1}{2}\log_e\left|\frac{1}{\cos(2x)}\right| + c = \frac{1}{2}\log_e|\sec(2x)| + c$

11 $-\frac{1}{6}$ **12** $\log_e\left(\sqrt{\frac{3}{2}}\right)$ **13** B

14 A **15** C **16** D

17 D **18** E **19** C

20 E **21** D **22** B

23 B **24** B **25** B

EXERCISE 7.4

1 C **2** D

3 a $\sin(x) - \frac{\sin^3(x)}{3} + c$

b $-\frac{1}{2}\cos(2x) + \frac{1}{6}\cos^3(2x) + c$

c $\sin(4x) - \frac{\sin^3(4x)}{3} + c$

d $\sin(x) - \frac{2}{3}\sin^3(x) + \frac{\sin^5(x)}{5} + c$

4 $\frac{1}{2}x - \frac{1}{4}\sin(2x) + c$

5 a $\frac{1}{2}x - \frac{1}{16}\sin(8x) + c$ **b** $\frac{1}{2}x + \frac{1}{8}\sin(4x) + c$

c $\frac{3}{2}x + \frac{3}{20}\sin(10x) + c$

6 $\tan(x) + c$

7 $\frac{x}{2} - \frac{1}{16}\sin(8x) + c$

8 $\frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5} + c;\ \frac{17}{480}$

9 a 48 **b** $-8\int_{\frac{\sqrt{3}}{2}}^{0} u^2\,du$ **c** $\sqrt{3}$

10 a $a = \frac{1}{2}, b = 1$ **b** $\frac{5}{48}$

11 C **12** B **13** B **14** C

15 E **16** E **17** A

EXERCISE 7.5

1 B **2** A

3 a $2\log_e|x-4| + 6\log_e|x+3| + c$

b $3\log_e|x-2| - \log_e|x+2| + c$

c $\log_e|2x+1| + 3\log_e|x+1| + c$

4 $\frac{-3}{10(3x+1)} + \frac{1}{10(x-3)}$

5 $\log_e\left(\frac{864}{343}\right)$

6 a $3\log_e|x-1| - \frac{1}{x-1} + c$

b $-2\log_e|x+4| - \frac{9}{x+4} + c$

7 $\log_e\left(\frac{9}{32}\right)$

8 $-\frac{2}{3}\log_e|3-x| - \frac{1}{3}\log_e|3+x|$

9 C **10** B **11** D **12** D

13 C **14** C **15** B **16** B

EXERCISE 7.6

1 E **2** B

3 a $\frac{1}{2\sqrt{x-x^2}}$ **b** $2\sin^{-1}\left(\sqrt{x}\right) + c$

4 a $\tan^{-1}\left(\frac{x}{3}\right) + \log_e\left|9+x^2\right| + c$

b $2\sin^{-1}\left(\frac{x}{3}\right) - \sqrt{9-x^2} + c$

5 a $u = \sqrt{2x},\ \frac{d}{dx}\left(\sin^{-1}\left(\sqrt{2x}\right)\right) = \frac{1}{\sqrt{2x}} \times \frac{1}{\sqrt{(1-2x)}}$

b $\frac{\pi}{12}$

6 $3\tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{2}\log_e\left(x^2+4\right) + c$

7 $2\sin^{-1}\left(\frac{x}{2}\right) - 6\sqrt{4-x^2} + c$

8 a $u = \sqrt{x},\ \frac{d\left(\tan^{-1}\left(\sqrt{x}\right)\right)}{dx} = \frac{1}{2\sqrt{x}} \times \frac{1}{(1+x)}$

b $2\tan^{-1}\left(\sqrt{x}\right)$

9 $\frac{1}{10}\log_e\left(4+25x^2\right) + \frac{1}{10}\tan^{-1}\left(\frac{5x}{2}\right) + c$

10 $\log_e(2) + \frac{\pi}{4}$

EXERCISE 7.7

1 B **2** D

3 a $\sin(x) - x\cos(x) + c$

b $\frac{1}{9}\cos(3x) + \frac{1}{3}x\sin(3x) + c$ **c** $xe^x - e^x + c$

d $\log_e|\cos(x)| + x\tan(x) + c$

4 a $\frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + c$

b $2\cos(x) - x^2\cos(x) + 2x\sin(x) + c$

c $2x\cos(x) + x^2\sin(x) - 2\sin(x) + c$

5 a $x\arcsin(x) + \sqrt{1-x^2} + c$

b $x\arccos(x) - \sqrt{1-x^2} + c$

c $x\log_e(x) - x + c$

6 **a** $\frac{1}{5}e^x\cos(2x)+\frac{2}{5}e^x\sin(2x)+c$

b $\frac{1}{2}e^x\sin(x)-\frac{1}{2}e^x\cos(x)+c$

c $\frac{2}{5}e^{2x}\sin(x)-\frac{1}{5}e^{2x}\cos(x)+c$

8 $\frac{25}{32}e^8-\frac{1}{32}$

9 $\frac{\pi}{24}+\frac{\sqrt{3}}{4}-\frac{1}{2}$

10 $\frac{2}{5}-\frac{\sqrt{2}}{10}e^{\frac{\pi}{8}}$

11 C **12** B

CUMULATIVE EXAMINATION 1

1 $2\log_e(2)-\log_e(3)$

2 $\frac{1}{4}\tan^{-1}(4x)+\frac{1}{4}\log_e(1+16x^2)+c$

3 $\frac{1}{2}\text{cis}\left(-\frac{\pi}{2}\right)$

CUMULATIVE EXAMINATION 2

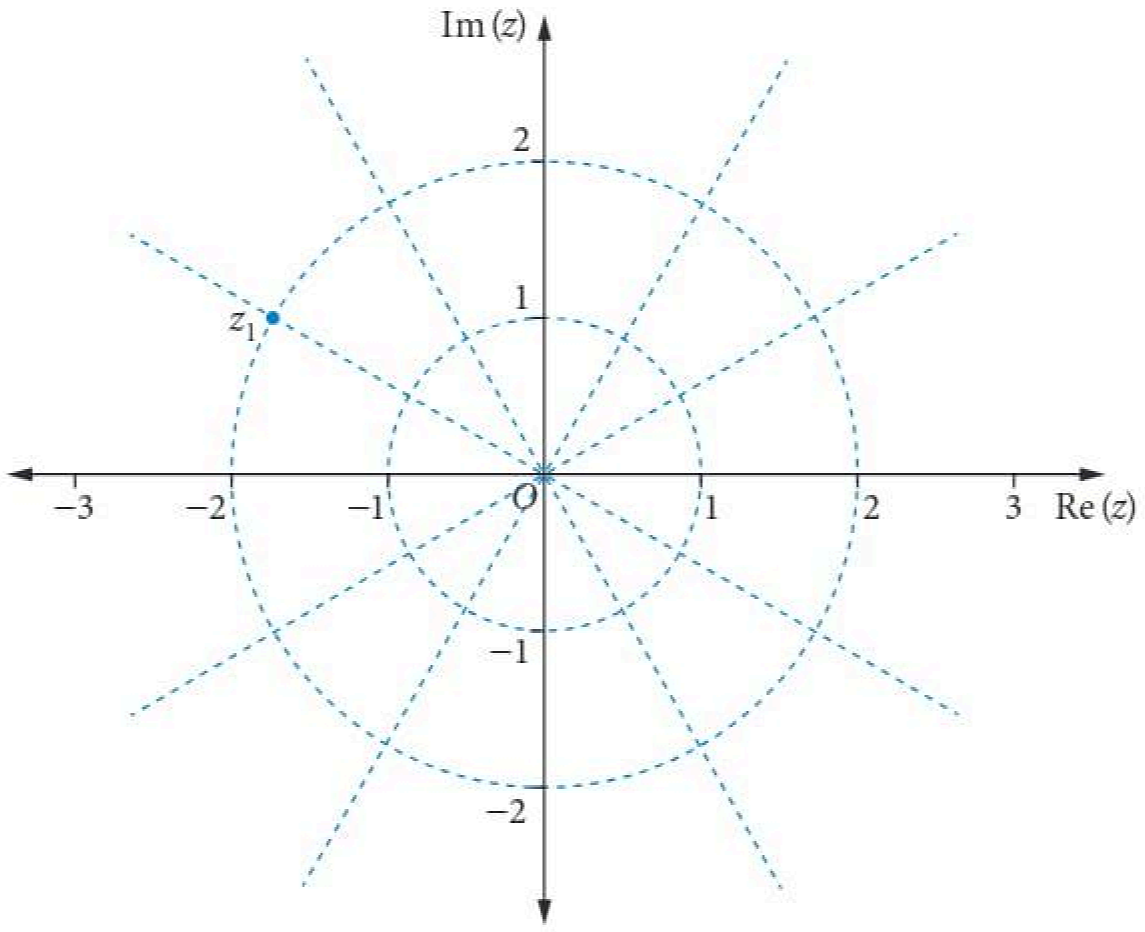
Section A

1 D 46% **2** E 71% **3** E 74%

4 B **5** A

Section B

1 **a** $z_1=2\text{ cis}\left(\frac{5\pi}{6}\right)$ 81% **b** 74%



c Use the quadratic formula or complete the square. 73%

2 **a** $x=\frac{1}{\cos(t)}, y=\frac{1}{\sin(t)}, \frac{x}{y}=\tan(t)$

$1+\tan^2(t)=\sec^2(t)$

$1+\frac{x^2}{y^2}=x^2$

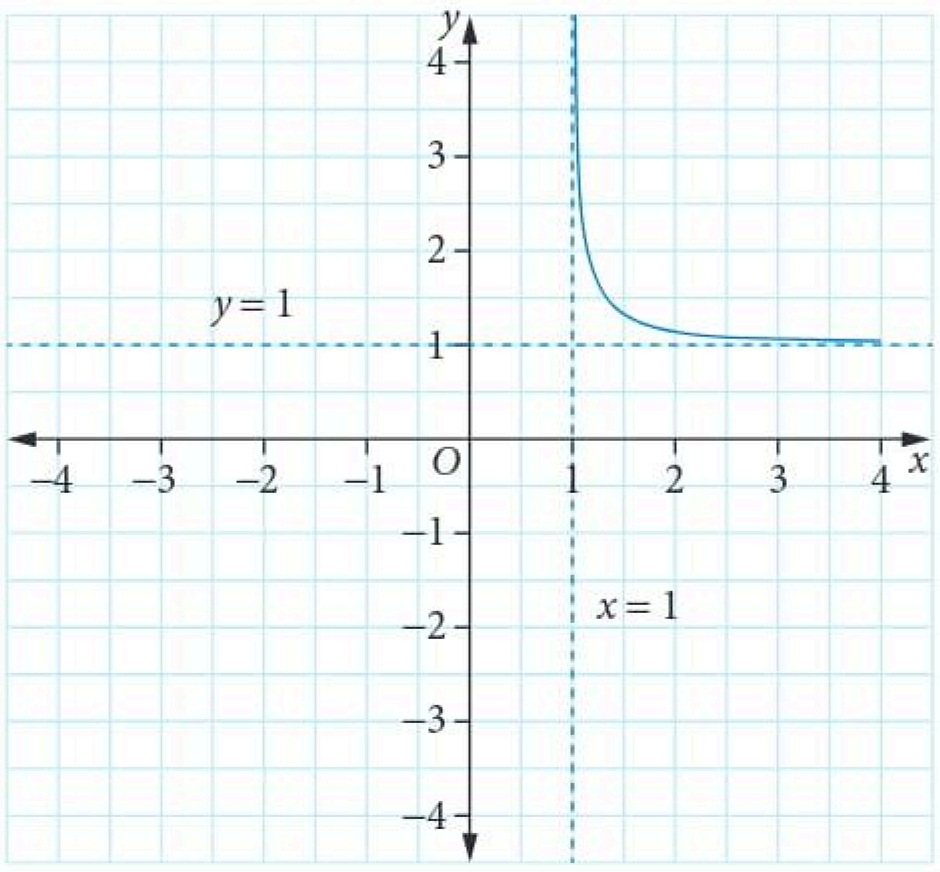
$y^2=\frac{x^2}{x^2-1}, y=\frac{x}{\sqrt{x^2-1}}$

b $x\in(1,\infty), y\in(1,\infty)$

c $\frac{dy}{dx}=\frac{-1}{(x^2-1)^{\frac{3}{2}}}$ is $\frac{\text{negative}}{\text{positive}}$.

Gradient is always negative.

d



3 **a** $\frac{1}{2}\int u^{-\frac{1}{2}}\,du$ **b** $\sqrt{x^2-1}$

CHAPTER 8

EXERCISE 8.1

1 **a** $\frac{\pi}{3}$ square units **b** 1 square unit

2 **a** $\frac{4}{3}$ square units **b** 2 square units

3 **a** 22.18 **b** 4.13

4 **a** $4\sqrt{2}$ square units **b** $\frac{10}{3}$ square units

5 **a** 2.13

b 0.111

6 **a** $\frac{\pi}{4}-\frac{1}{2}$ square units **b** 1 square unit

7 $4+\frac{\pi}{2}$

8 **a** Proof: see worked solutions **b** $-2+3\log_e(3)$

9 $\frac{4}{15}$ **10** $6\log_e(5)$

11 **a** $[0,\infty)$ **b** Proof: see worked solutions

c $\frac{\sqrt{3}\pi}{6}-\frac{\pi}{8}-\frac{1}{4}\log_e(2)$

12 $\frac{32\sqrt{2}}{35}$

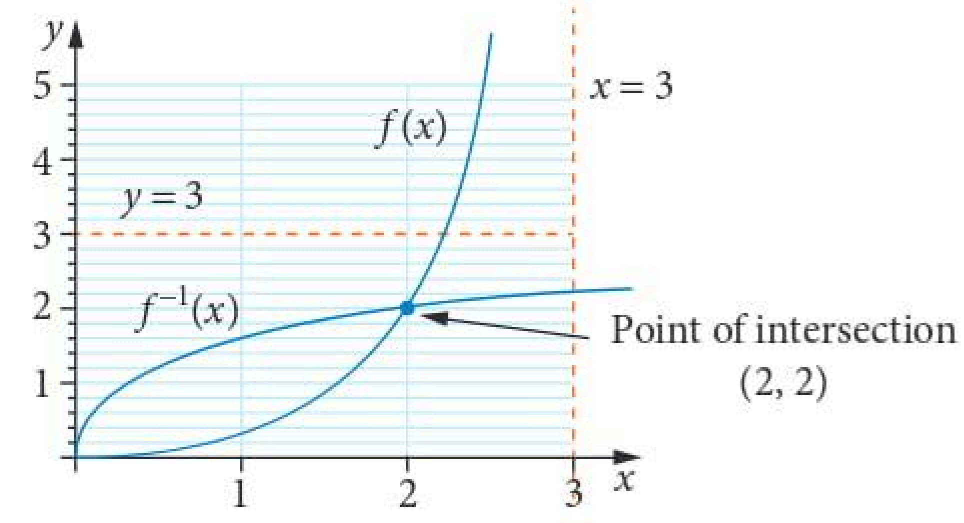
13 E

14 **a** 1.9875 m^2 **b** $A=3, B=-3$

c 2.18 **d** 52

15 **a** 2

b



c $\frac{6}{\pi}$ **d** 1.939

e **i** Proof: see worked solutions

ii $12-\left(\frac{24}{\pi}\right)\log_e\left(2+\sqrt{3}\right)$

16 **a** $(-2, 2)$

b local max at $(0, \log_e(4))$; asymptotes at $x = \pm 2$;

x intercepts $(0, \pm\sqrt{3})$, symmetric about the x-axis

c area of lower rectangle $< A <$ area of upper rectangle

d **i** $\log_e(4 - x^2) - \frac{2x^2}{4 - x^2}$

ii $-x + \log_e\left(\frac{2 + x}{2 - x}\right) + c$

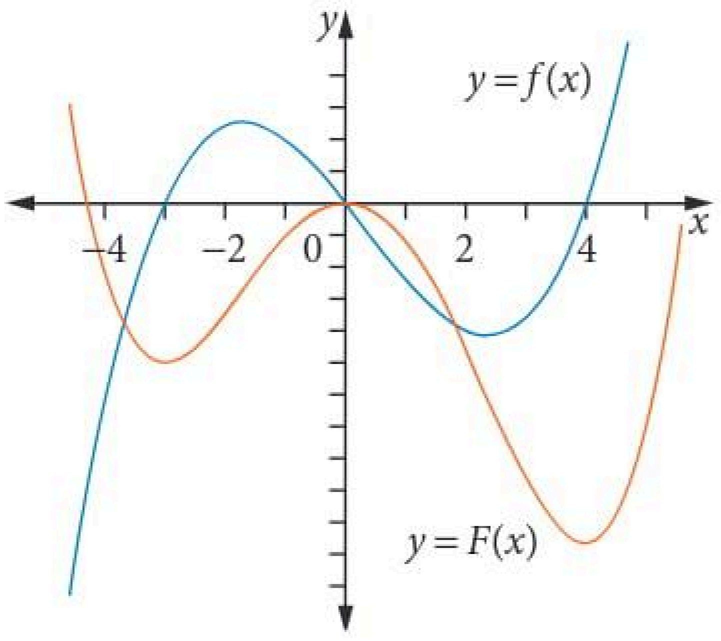
iii $-2 + 3\log_e(3)$

EXERCISE 8.2

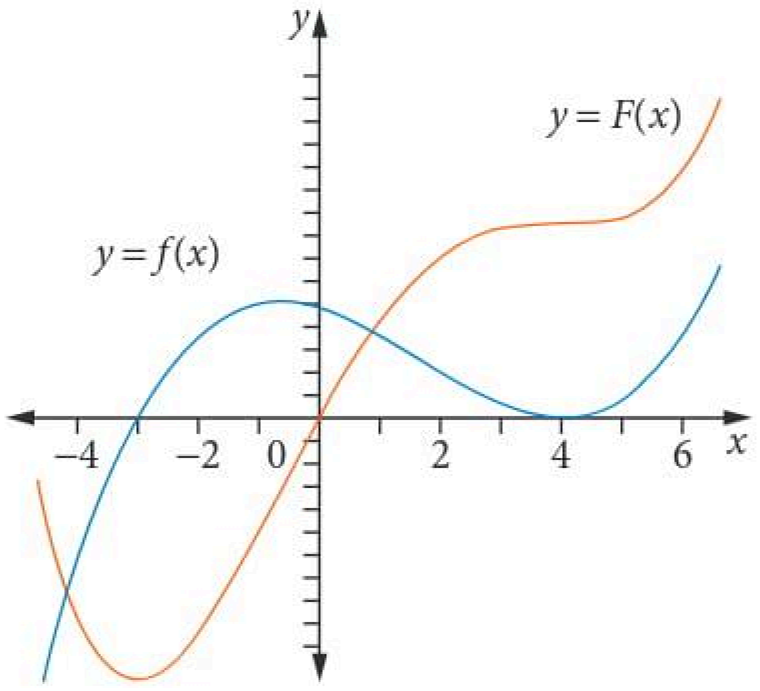
1 D **2** A

3 Any graph of the form $F(x) + c$ is correct

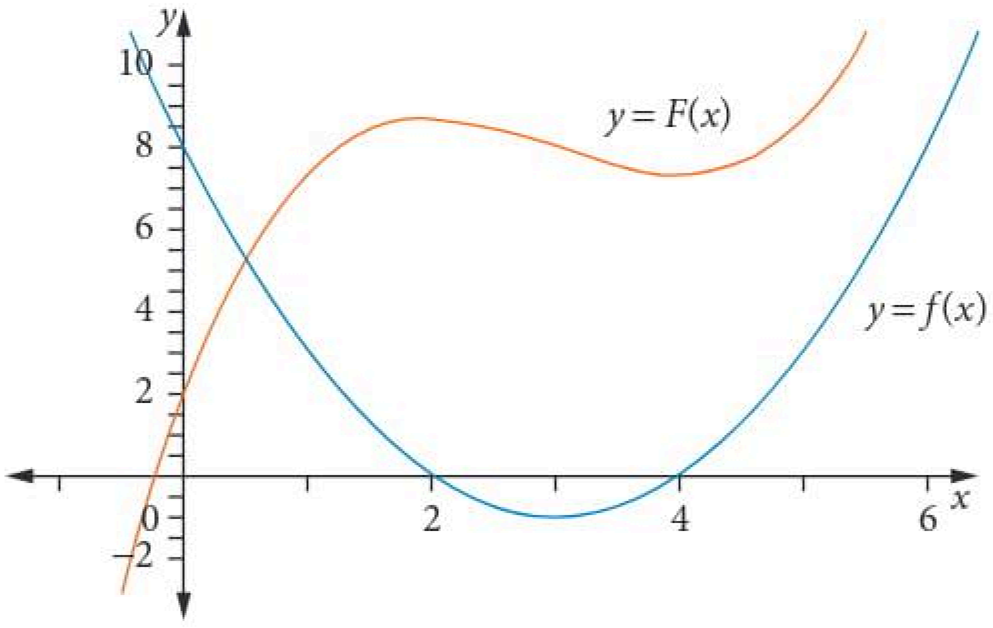
a



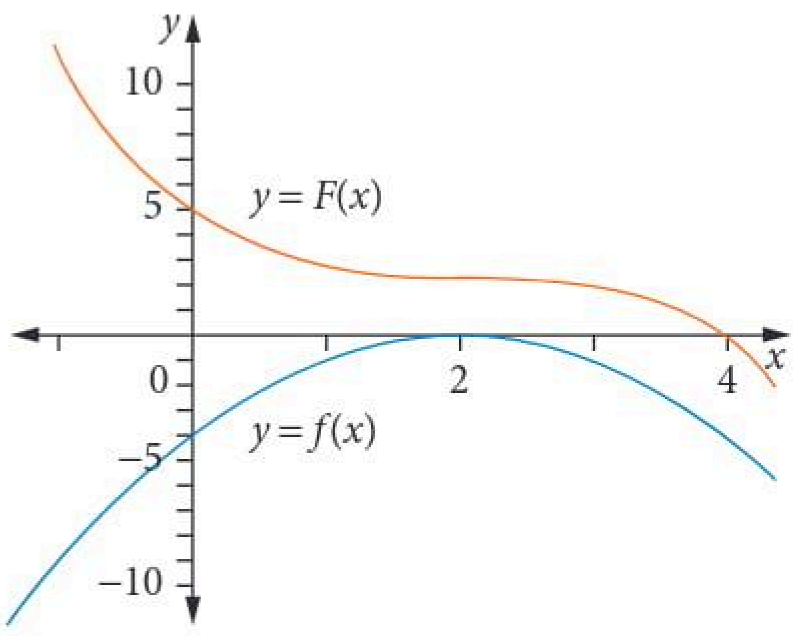
b



4 **a**



b



5 B **6** A

EXERCISE 8.3

1 A **2** C

3 **a** $\frac{\pi \log_e(3)}{4}$ cubic units

b 36π cubic units

4 **a** $\frac{\pi^2}{6} - \frac{\pi\sqrt{3}}{8}$ cubic units **b** $\frac{192\pi}{5}$ cubic units

5 $\frac{64\pi}{15}$ cubic units

6 **a** $V = \pi\int_0^a \left(e^{-x}\right)^2 dx$

b $V = \frac{\pi}{2}\left(1 - e^{-2a}\right)$

c $\log_e\left(\frac{3}{2}\right)$

7 $\pi\left(\frac{\pi}{4} + \log_e(2)\right)$

8 $\frac{\pi^2}{12} - \frac{\pi\sqrt{3}}{8}$

9 **a** Proof: see worked solutions

b $V = \pi\left(1 + \log_e\left(\frac{5}{3}\right)\right)$

10 **a** $\frac{\pi^2}{2}$ units^3 **b** $\frac{1}{k}V_s$ units^3

11 $\frac{\pi^4}{9} - \frac{\pi^2}{6} + \frac{\pi\sqrt{3}}{8}$

12 16π

13 **a** Proof: see worked solutions

b domain: $[-2, 2]$, range: $\left[0, \sqrt{\pi}\right]$

c $2\pi^2$

14 **a** π **b** $\frac{\pi^2}{2}$

15 D **16** C **17** A **18** C

19 **a** $V = \int_0^3 \frac{81\pi}{(x+2)^2(x-4)^2} dx$

b 12.85

20 **a** $V = \int_1^3 \pi\left(4 - \frac{4}{9}(x-1)^2\right)dx$ **b** $\frac{184\pi}{27}$

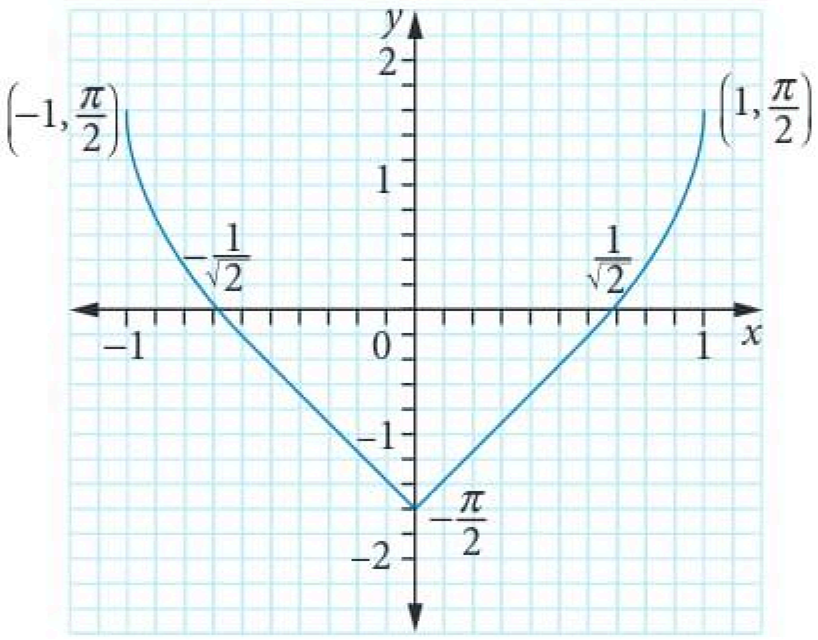
21 **a** $\int_0^1 \pi\left(2 - \sin^2(x)\right) dx$ **b** 5.4

22 **a** $\int_0^5 \pi\left(\frac{6x}{\sqrt{1 + x^3}}\right)^2 dx$ **b** $V = \pi\int_1^{126} \frac{12}{u} du$

c 182

23 $\frac{16\pi}{3}$

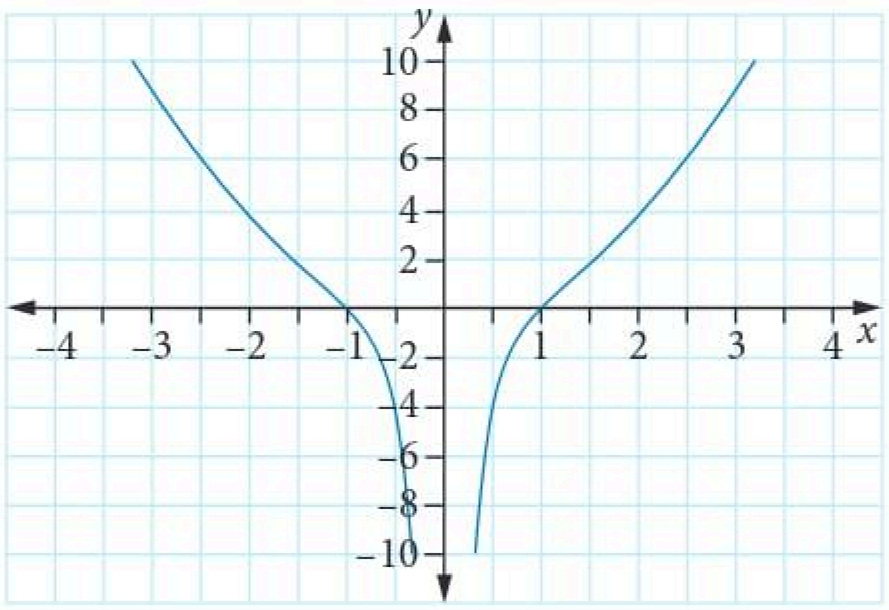
24 **a**



b **i** $V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi \times \frac{1}{2}\sin(y+1)\,dy$ **ii** $\frac{\pi^2}{2}$

25 **a** $a = 3.2, b = 0.3$

b



c Proof: see worked solutions

d **i** $\int_{-10}^{10} \pi\left(\frac{y+\sqrt{y^2+4}}{2}\right)dy$ **ii** 174.7

26 **a** Proof: see worked solutions **b** $V = 2\pi\int_1^2 \frac{1}{u} - \frac{1}{u^2}\,du$

c $V = 2\pi\log_e(2) - \pi$

EXERCISE 8.4

1 E **2** A

3 **a** $\frac{1625\sqrt{13}}{54} - \frac{37\sqrt{37}}{54}$ **b** $\frac{13\sqrt{13}-8}{27}$

4 **a** 215.74 **b** 7.06

5 **a** π **b** $4\sqrt{2}-2$

6 $\frac{14}{3}$ **7** $\frac{123}{32}$

8 **a** $\frac{-x}{\sqrt{1-x^2}}$ **b** $\frac{\pi}{6}$

9 $6a$ **10** B **11** E **12** C

13 A **14** C **15** B **16** B

17 $\frac{14}{3}$

18 **a** **i** $\int_{-3}^{-0.5}\sqrt{1+\left(\frac{2x^3+x^2-4}{x^2}\right)^2}\,dx$

ii 13.18

b $a = \pi, b = \frac{14}{3}, c = -\frac{33}{4}$

EXERCISE 8.5

1 C **2** B

3 **a** $3\sqrt{10}\pi$ units^2 **b** 24π

c $\frac{62\pi}{3}$

4 **a** $\frac{21\pi\sqrt{26}}{25}$ **b** $\frac{\pi\left(10\sqrt{10}-1\right)}{27}$

c $\frac{\pi}{6}\left[\left(13\sqrt{3}\right)-5\sqrt{5}\right]$

5 **a** $4\pi\sqrt{10}$ **b** $44\pi\sqrt{29}$ **c** 32π

6 $\frac{8\pi}{3}\left(5\sqrt{5}-3\sqrt{3}\right)$ **7** $\frac{\pi}{6}\left(5\sqrt{5}-1\right)$

8 $\frac{\pi}{96}\left(17\sqrt{17}-1\right)$ **9** 165π

10 E **11** A

12 **a** $a = \frac{2}{9}$ **b** $\frac{3\pi}{2}\int_0^2\sqrt{8y+9}\,dy$

c $\frac{49\pi}{4}$

13 **a** $\frac{x^2}{2} + y^2 = 1$ **b** $2\pi\int_0^{\pi}\sin(t)\sqrt{2\sin^2(t)+\cos^2(t)}\,dt$

c $\pi(\pi+2)$

CUMULATIVE EXAMINATION 1

1 **a** Proof: see worked solutions **80%**

b $x = 1, x = 3$ **70%**

c $\frac{2\pi}{3}$ **30%**

2 $\log_e(2)$

3 $\frac{\pi}{6}$

CUMULATIVE EXAMINATION 2

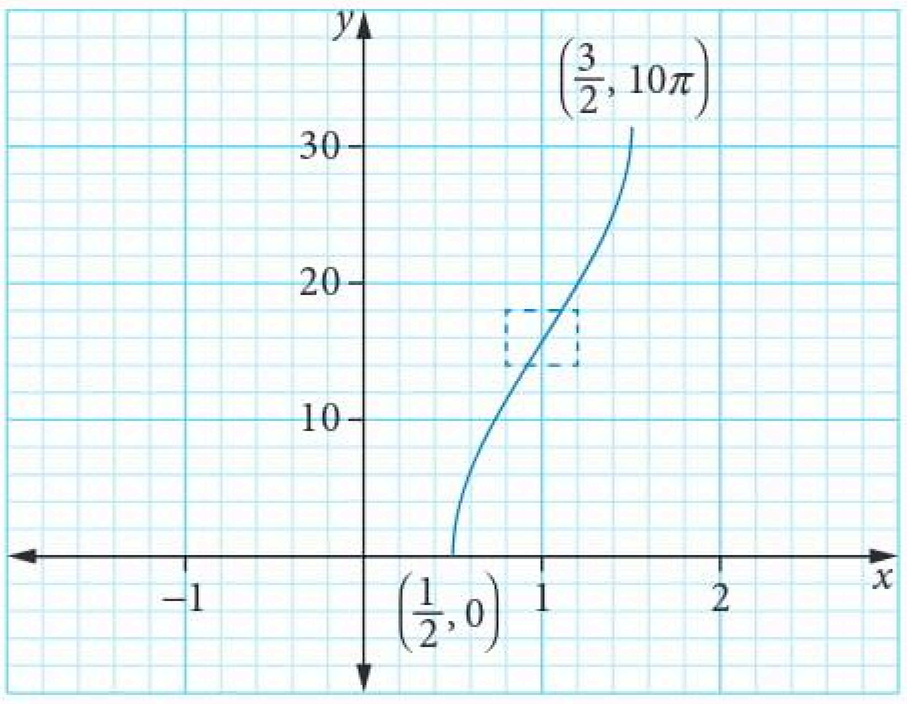
Section A

1 C **78%** **2** C **3** D

4 D **5** B **84%**

Section B

1 **a**



b **i** $\int_0^{10\pi}\pi\left(1-0.5\cos\left(\frac{y}{10}\right)\right)^2 dy$

ii $\frac{45\pi^2}{4}$

c $\frac{20}{\pi}$ **d** 31.14

2 a $-\dfrac{\sin(x)\cos(x)}{y}$ or $-\dfrac{\sin(2x)}{2y}$ 75%

b i $\sqrt{2}, 1$ 91%

ii 0, 0 85%

c $f^{-1}(x) = \arcsin\left(\sqrt{2-x^2}\right)$, domain: $[1, \sqrt{2}]$, range: $\left[0, \dfrac{\pi}{2}\right]$ 79%

CHAPTER 9

EXERCISE 9.1

1 a order 2, degree 1

b order 2, degree 2

c order 1, degree 2

2 a $\dfrac{dK}{dS} = kS$, k is the proportionality constant.

b $\dfrac{dB}{dr} = \dfrac{k}{r^3}$, k is the proportionality constant.

c $\dfrac{dH}{dt} = \dfrac{kA}{T}$, k is the proportionality constant.

d $\dfrac{d^2x}{dt^2} = k$, k is the proportionality constant.

3 $\dfrac{dy}{dx} = -2\sin(2x)$, $\dfrac{d^2y}{dx^2} = -4\cos(2x)$

$$\frac{d^2y}{dx^2} + 2y = -4\cos(2x) + 2 \times 2\cos(2x) = 0$$

4 $y = x^k$

$$\frac{dy}{dx} = kx^{k-1}$$

$$\frac{d^2y}{dx^2} = k(k-1)x^{k-2}$$

$$x\frac{d^2y}{dx^2} - (k-1)\frac{dy}{dx}$$

$$= x \times k(k-1)x^{k-2} - (k-1) \times kx^{k-1}$$

$$= k(k-1)x^{k-1} - k(k-1) \times x^{k-1}$$

$$= 0$$

5 a $\dfrac{dG}{dR} = \dfrac{k}{R^3}$

b $\dfrac{d^2G}{dR^2} = -\dfrac{3k}{R^4}$

$$= -\frac{3}{R} \times \frac{k}{R^3}$$

$$= -\frac{3}{R}\frac{dG}{dR}$$

$\dfrac{d^2G}{dR^2} = -\dfrac{3}{R}\dfrac{dG}{dR}$, hence $R\dfrac{d^2G}{dR^2} + 3\dfrac{dG}{dR} = 0$.

6 $m = -6, m = 2$

7 $a = 2, b = -1, c = 3, d = 5$

8 $k = 2$

9 c can be any positive real number.

10 $m = -2, n = 9$

11 $k = -2$ **12** $k = 3$

13 $m = -6, m = 2$

14 C **15** E **16** D **17** E

18 C **19** E **20** A

EXERCISE 9.2

1 C **2** E

3 a $y = \sin(2x) + \cos(x) + c$

b $x = \dfrac{1}{3}\tan^{-1}\dfrac{t}{3} + c$ **c** $P = e^{2f} - f + c$

4 a $y = \log_e\left|\dfrac{x^3 + x}{2}\right|$

b $x = \log_e|2(3s+2)(4s+1)|$

c $f = \dfrac{1}{2} - \dfrac{1}{4}\sin(4x)$

5 0.05 cm

6 $y = x + \log_e\left(\dfrac{1+x}{1-x}\right) + 4$

7 $y = -e^{1-\frac{1}{x}}$

8 $x = \dfrac{1}{3}\sin^{-1}\left(\dfrac{3y}{2}\right) + 1$

9 $y = \dfrac{3\sqrt{2}}{2}\tan^{-1}\left(\dfrac{\sqrt{2}(x+2)}{2}\right) - \dfrac{3\sqrt{2}}{2}\tan^{-1}(\sqrt{2}) + 2$

10 $y = \displaystyle\int \frac{e^x}{1+e^x}\,dx = \log_e\left|(1+e^x)\right| + c$

$y(0) = \log_e(2e) \Rightarrow \log_e(2) + c = \log_e(2e)$

$c = \log_e(2e) - \log_e(2) = 1$

11 $y = \dfrac{1}{3}(x^2 - 16)^{\frac{3}{2}} + 4$

12 $y = \dfrac{2}{3}\sqrt{\left(9 + x^2\right)^3} + 2$

13 a $h = 25t - 0.8t^2$ **b** 87.2 m

c Proof: see worked solutions

14 a $\dfrac{dx}{dt} = kt$ **b** 154 cm

15 D **16** C **17** E **18** D **19** B

20 $y = -\sqrt{16 - x^2}$

21 9 039 212

22 a $\dfrac{dy}{dx} = 0.03x^2 - 0.2x + 0.5$

b $\left(3\dfrac{1}{3}, 1\dfrac{25}{27}\right)$

c $y = 0.01x^3 - 0.1x^2 + 0.5x + 1$

d $B^2 = 3AC$

EXERCISE 9.3

1 B **2** B

3 $y = e^x + \sin(x) + c_1 x + c_2$

4 72 m

5 $y = \log_e x + 2x + 1$

6 a $x = 2600\left(1 - e^{-\frac{2t}{65}}\right)$ **b** 7 seconds

c 2600 metres

7 a $P = 5000e^{-0.1t} + t$ **b** 5000 people

c $t = 0$ years

8 $y = x^3 - x^2 + 2x - 1$

9 $\dfrac{dy}{dx} = e^x + c_1,\ y = e^x + c_1x + c_2,\ y = \dfrac{d^2y}{dx^2} + c_1x + c_2$

since $e^x = \dfrac{d^2y}{dx^2}$, so $\dfrac{d^2y}{dx^2} = y - c_1x + c_2$

10 $y = \log_e|(x+1)(x+2)|$

11 $k = 24$

12 D **13** B **14** B **15** C **16** C

17 **a** $\dfrac{dv}{dt} = 10$ **b** $v = 2 + 10t$ **c** $x = 2t + 5t^2$

d 1.8 s

18 **a** $80\dfrac{dy}{dx} = \dfrac{3}{2}x^2 - 4x + c,\ 0 = \dfrac{3}{2} \times 2^2 - 4 \times 2 + c,\ c = 2$

$$80y = \frac{1}{2}x^3 - 2x^2 + 2x + d,$$

$$0 = \frac{1}{2} \times 2^3 - 2 \times 2^2 + 2 \times 2 + d,\ d = 0$$

$$80y = 2x^2 - \frac{1}{2}x^3 - 2x$$

b 1.4°

c $x = \dfrac{2}{3}$; maximum deflection is $\dfrac{1}{135}$.

d 0.5°

19 **a** $V = \dfrac{5\pi}{3}R^3$

b $\dfrac{2\pi}{3}R^3(\text{hemisphere}) + \dfrac{1}{3}\pi R^2 H(\text{cone}) = \dfrac{5\pi}{3}R^3$

$2R + H = 5R$, hence $H = 3R$

c $R = 2$

EXERCISE 9.4

1 D **2** D

3 **a** $y = \pm\sqrt{\dfrac{2x + c}{3}}$ **b** $x = 3\sin(t + c)$

c $N = e^{-2t}$

4 $T = 25 + 75e^{-0.155t}$ °C, (t in min), another 5 min 5 s

5 $Q \approx 100e^{-0.000121t}$ %, (t in years), about 13 300 years

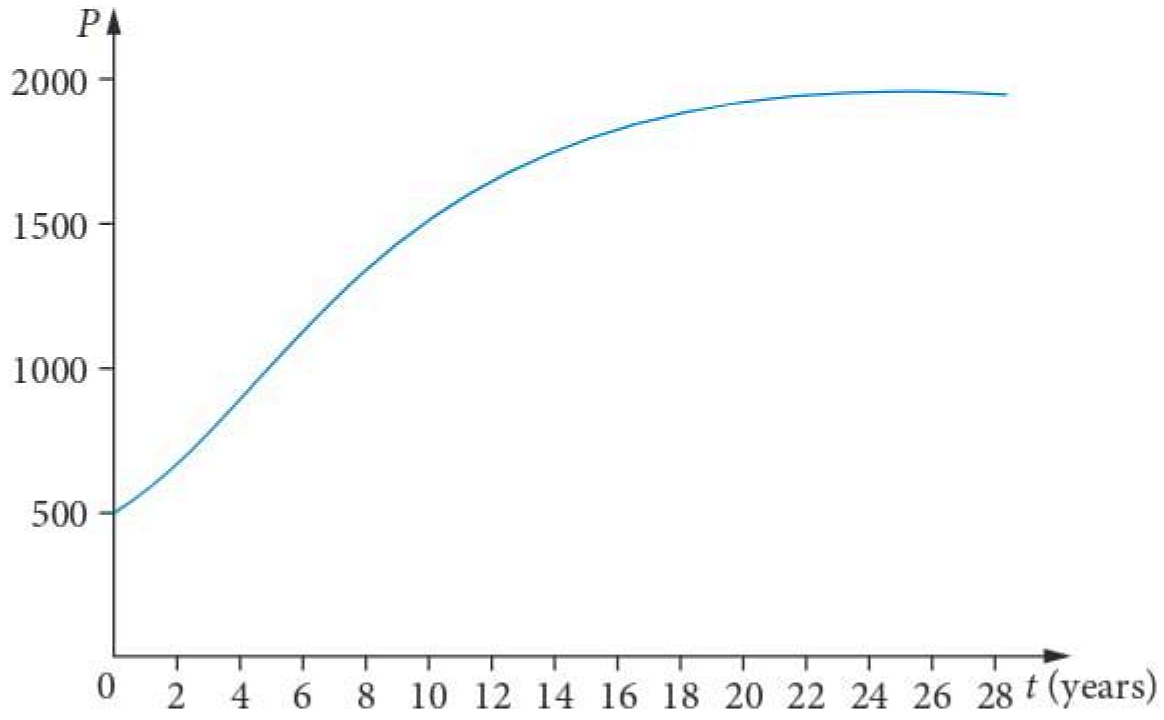
6 $P = 100e^{0.2t},\ t = 8,\ P = 506$ rabbits.

7 **a** $M = 300 - 290e^{-\frac{1}{20}t}$, 63 g

b 10 g

8 **a** 9.51 years

b



9 $y = 4 - \dfrac{1}{x+1}$

10 **a** Proof: see worked solutions

b $T = 65$

11 C **12** C **13** D **14** B **15** B

16 C **17** A **18** A **19** A **20** D

21 $T = 1100 - 1080e^{-0.3646t}$ °C, 12.31 pm, 1100°C

22 315°C

23 **a** about 6 years (5.973)

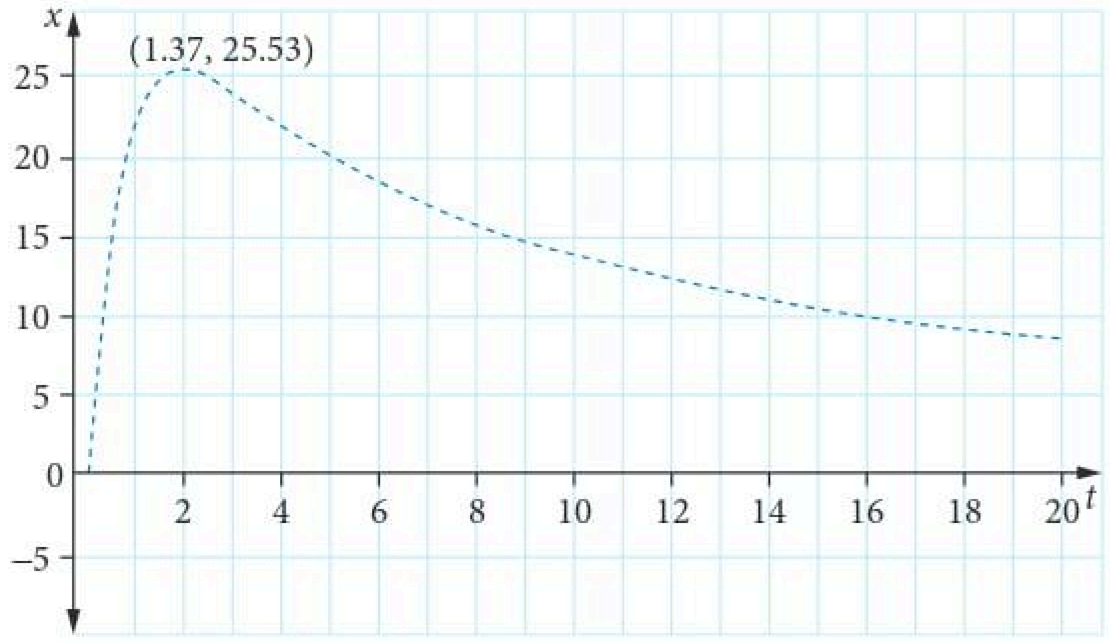
b about 83 each subsequent year

24 50.45 minutes

25 **a** $\dfrac{x}{10 + 10t}$ **b** Proof: see worked solutions

c **i-ii** Proofs: see worked solutions

d



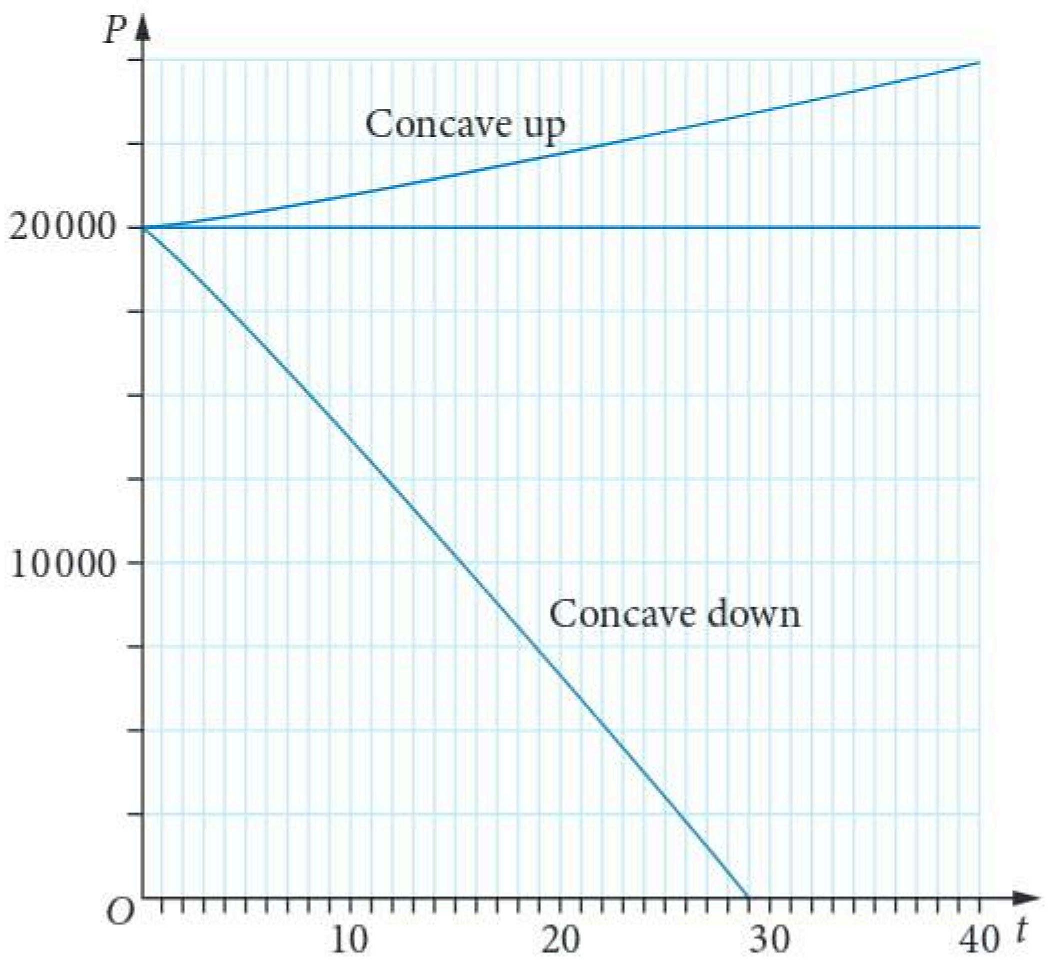
e **i** 0.485 **ii** 8.17

26 **a** student to verify

b 29 years

c student to show working

d **i**, **ii** and **iii**



e **i** $k = -0.0049$

ii Arrivals exceeded departures.

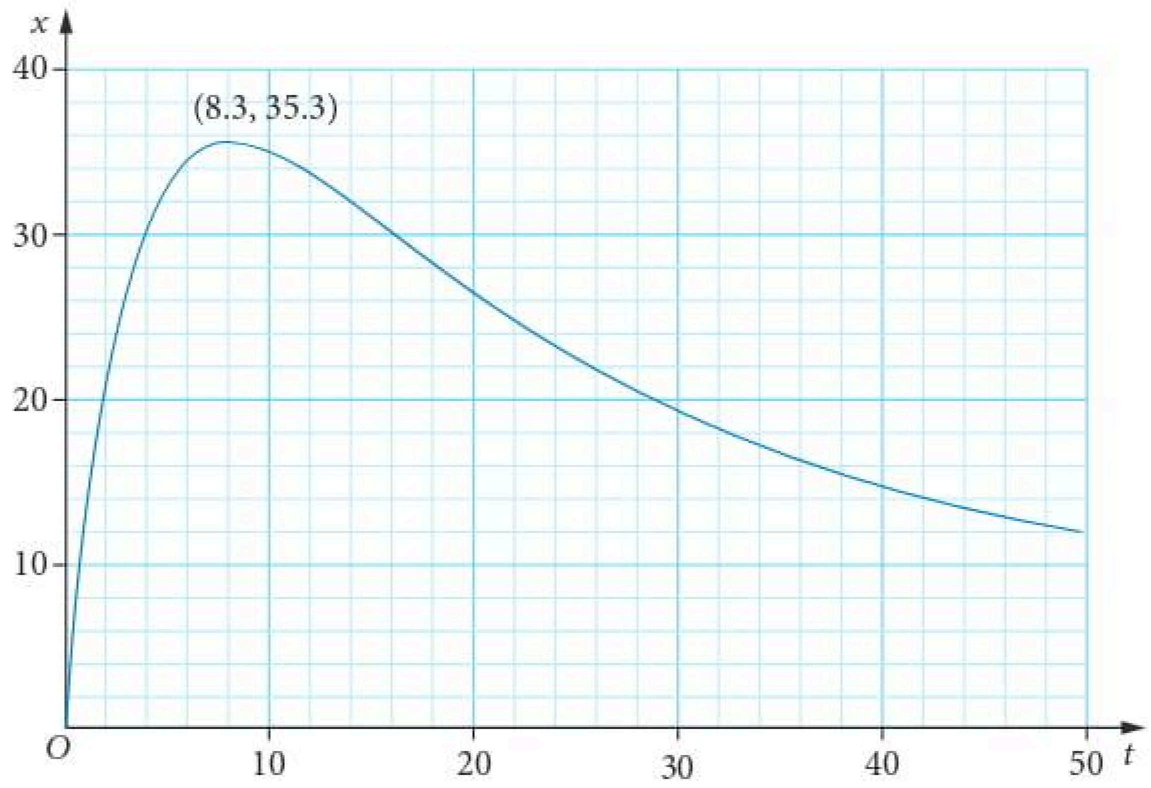
27 **a** concentration $= \dfrac{x}{10 + 10t}$

b $\dfrac{dx}{dt} = e^{-0.2t} \times 20 - \dfrac{x}{10 + 10t} \times 10$

c **i** $\dfrac{dx}{dt} = \dfrac{20e^{-0.2t}(t^2 + 7t + 31)}{(t+1)^2} - \dfrac{600}{(t+1)^2}$

ii Show that both sides of the equation simplify to $20e^{-0.2t}$. Initial condition: $x(0) = 0$.

d



e 51.6 g

28 a Proof: see worked solutions

b Answer as given, with $A = 38$.

c slightly curved, concave down from (0, 5) to (T, 48)
slightly curved, concave up from (T, 48) to ($4T$, 36)

d 4.76

29 a i $\dfrac{2}{P}+\dfrac{2}{1-P}$ **ii** Solve $\dfrac{dt}{dP}=\dfrac{2}{P}+\dfrac{2}{1-P}$ and rearrange to obtain $\dfrac{t-c}{2}=\log_e\left(\dfrac{P}{1-P}\right)$.

iii $P=\dfrac{e^{0.5t}}{1+e^{0.5t}}$

b 0.894

c $q = 0.62,\ r = 0.80,\ s = 1$

EXERCISE 9.5

1 B

2 B

3 $y = Ae^{\frac{15}{2}x^2}-\dfrac{8}{3}$

4 a $x = Ae^{\frac{1}{3}kt^3}$ **b** $k=\dfrac{1}{9}, c = 2$ **c** 5 seconds

5 $y=\dfrac{6e^{t^2}}{10-9e^{t^2}}$

6 $y = -0.5\log_e(e^{-12}-x)$

7 a $\theta = \cos^{-1}(1-\log_e(1+t))$

b Proof: see worked solutions

8 $y=\dfrac{1}{k}\left(\dfrac{kx+1}{e^{\frac{c}{k}}}-1\right)=\dfrac{kx+1}{ke^{\frac{c}{k}}}-\dfrac{1}{k}=A(kx+1)+B,$

$A=\dfrac{1}{ke^{\frac{c}{k}}}, B=-\dfrac{1}{k}$

9 $A = 9$

10 a $\dfrac{dQ}{dt}=0-\dfrac{3Q}{16+2t}=-\dfrac{3Q}{16+2t}$

b $Q=\dfrac{32}{(16+2t)^{\frac{3}{2}}}$

11 $|y| = e^{\left(1-\frac{1}{x}\right)}$ or $y=\pm e^{\left(1-\frac{1}{x}\right)}$

12 $y = 2-\sqrt{4+\dfrac{\pi}{2}-2\arcsin\left(\dfrac{x}{\sqrt{2}}\right)}$

13 C **14** A **15** E

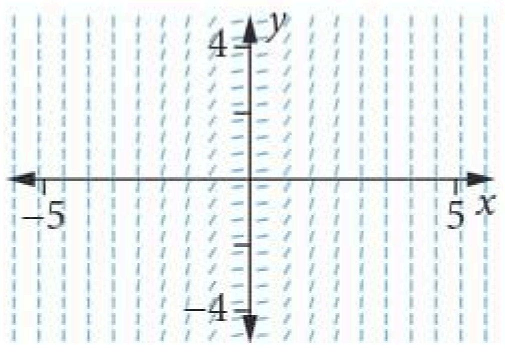
16 D **17** B **18** D

EXERCISE 9.6

1 A

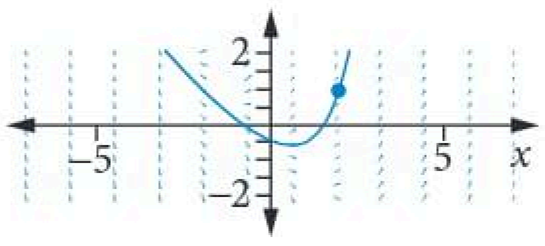
2 D

3 a

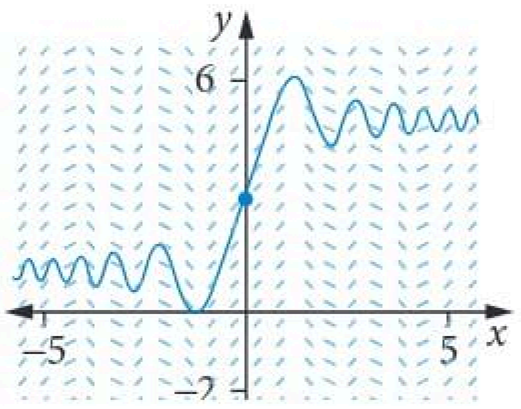


b cubic function of the form $y = x^3 + c$

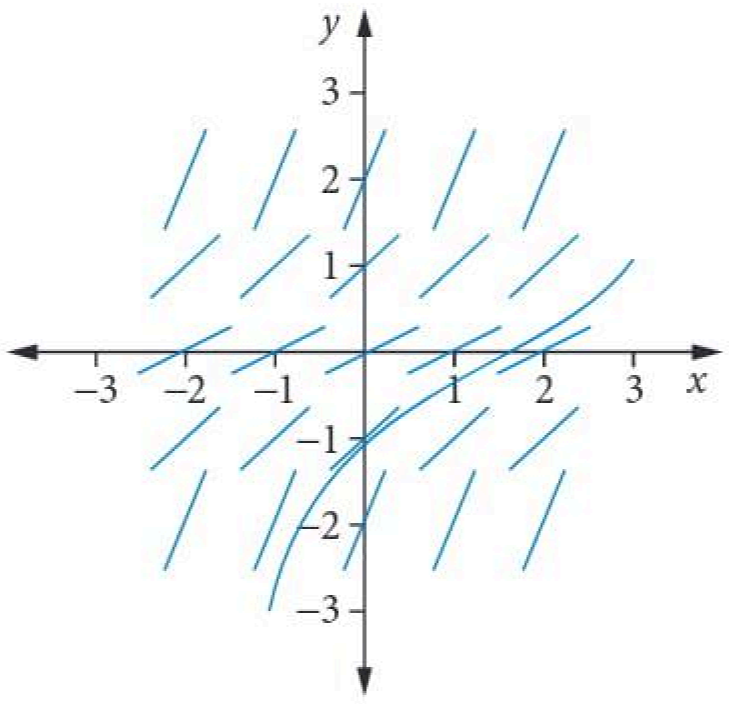
4



5



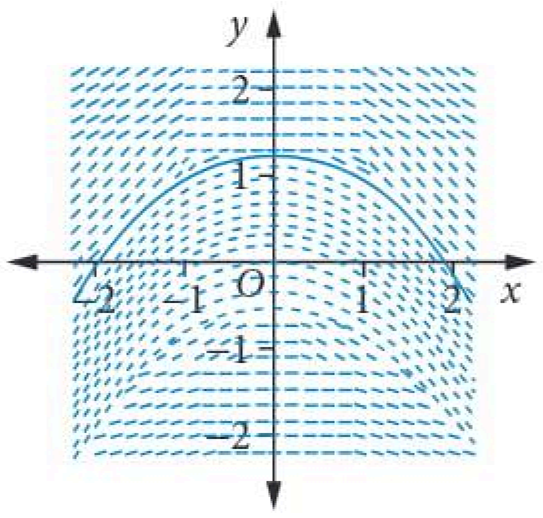
6 a



b $y = \tan\left(\dfrac{x}{2}-\dfrac{\pi}{4}\right)$

c See answer to part **a**.

7 a



$1.7 \le x \le 1.9$

b $2y^3 + 6y + 3x^2 - 11 = 0$

8 $y=-\dfrac{2}{3}x+10$

9 C

10 E **11** B **12** A **13** A

14 B **15** D **16** B **17** C

18 C **19** B **20** D **21** D

EXERCISE 9.7

1 B **2** C

3 (0.4, 5), (0.8, 5.192), (1.2, 5.96), (1.6, 7.688), (2, 10.76)

4 a $y = \sin(2x) + 3$

b $(0, 4), (\frac{3\pi}{4}, 5.570), (\frac{\pi}{2}, 5.570), (\frac{3\pi}{4}, 4.000), (\pi, 4)$

c 39%

5 20.9

6 a $\frac{131}{110}$ **b** $\log_e(1.2) + 1$

7 a $y = 2\tan\left(2x + \frac{\pi}{4}\right) - 2$ **b** $y_1 = 0.8$

8 D **9** D **10** E **11** D **12** C **13** D

14 C **15** C **16** B **17** B **18** B

19 a $y_{20} = y_{19} + 0.05\log_e(4 - 0.95^2)$ **b** 1.3029

c $y_{20} \approx \int_0^{x_{20}} \log_e(4 - x^2)dx = \int_0^1 \log_e(4 - x^2)dx = A$

CUMULATIVE EXAMINATION 1

1 $a = -4$ **60%**

2 $Q^2 - t^4 - 1 = 0$

3 a $x = 1, y = -1$

b $n = 4$

4 a $y = -(x - 1)^2 + 7$ **b** $2\frac{2}{3}$ units^2

CUMULATIVE EXAMINATION 2

Section A

1 A **65%** **2** B **51%** **3** D **66%**

4 C **5** B

Section B

1 a Differentiation of the solution with respect to *t* gives $\frac{1}{N}\frac{dN}{dt} = 1.2e^{-0.4t}$, and substituting into the left-hand side of the differential equation gives left-hand side $= 1.2e^{-0.4t} + 0.4 \times (6 - 3e^{-0.4t}) - 2.4 = 1.2e^{-0.4t} + 2.4 - 1.2e^{-0.4t} - 2.4 \Rightarrow$ left-hand side $= 0 =$ right-hand side. **42%**

b $N = 20$ **82%**

c 403 **61%**

d $\frac{d^2N}{dt^2} = (2 - 0.4\log_e(N))0.4N(6 - \log_e(N))$ **14%**

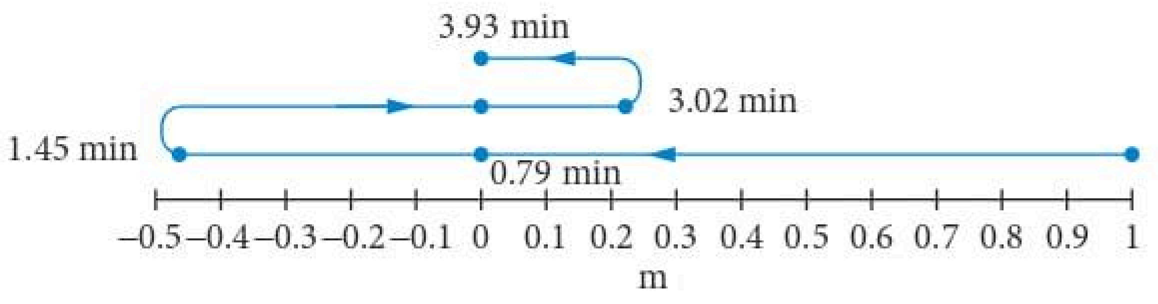
2 a (0, 0), (4, 2) **b** $\frac{4}{3}$

c $\frac{8}{3}$

CHAPTER 10

EXERCISE 10.1

1 a



b $\frac{2.36}{3.93} = 0.60$ m/min

c $-\frac{1.00}{3.93} = -0.25$ m/min

2 a $\frac{99}{9\frac{1}{3}} = 10.61$ km/h

b $\frac{95.78}{9\frac{1}{3}} = 10.26$ km/h

c 11.75°T

3 a $t = 0.6$ s, -3.06 cm

b 0.05 s, −8.525 cm/s **c** 32.79 cm

4 a $v = t^2 - 8t + 16$ **b** $22\frac{1}{3}$ m

5 $x = e^{\sqrt{3}t}$

6 a $k = 8$

b $x = \frac{1}{2\sqrt{2}}\sin(2\sqrt{2}t)$

7 a 7.92 m **b** 1 m/s **c** 0.8 s

d 0.4 s **e** 4.5 s

8

$$\begin{aligned}\frac{dx}{dt} &= -Ak\sin(kt) + Bk\cos(kt)\\ \frac{d^2x}{dt^2} &= -Ak^2\cos(kt) - Bk^2\sin(kt)\\ &= -k^2(A\cos(kt) + B\sin(kt))\\ &= -k^2x\end{aligned}$$

9 $a = -\frac{1}{k^2}, \quad b = \frac{2}{k}, \quad c = -\frac{\pi}{k^2}$

10 $a = 1, b = 3$

11 a $\frac{dv}{dt} = e^t + te^t = e^t(1 + t)$, using Product rule

b

$$\frac{dv}{dt} = te^t + e^t$$

$$\int \frac{dv}{dt}dt = \int te^t dt + \int e^t dt$$

$$v = x + e^t + c$$

$$t = 0, x = 0, v = 0 \Rightarrow c = -1$$

$$v - x = e^t - 1$$

$$x = v - e^t + 1$$

12 $\frac{4x}{(1 + x)^2}$

13 $v = -\sqrt{e^{2x-2} + 3}$

14 E **15** C **16** E **17** E

18 D **19** D **20** E **21** B

22 E **23** A **24** B **25** E

26 C **27** B **28** D **29** D

30 C **31** C

32 a $\frac{17\pi}{2}$ **b** 0.3

c $25 = 17\tan^{-1}\left(\frac{\pi T}{6}\right), T = 18.995, T = 19$

d i $v = -\frac{1}{100}(145t - t^2) + 25$

ii $0 = -\frac{1}{100}(145t - t^2) + 25, t = 20$

e **i** $\int_0^{19} 17\tan^{-1}\left(\frac{\pi T}{6}\right) dT, 25 \times 120$

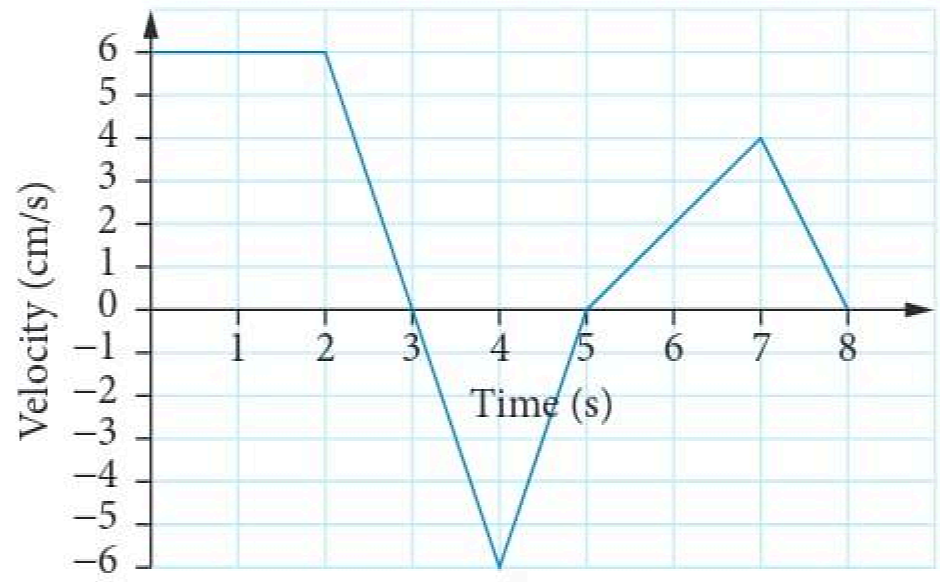
$\int_0^{20} -\frac{1}{100}(145t - t^2) + 25\, dt$

ii 3637

EXERCISE 10.2

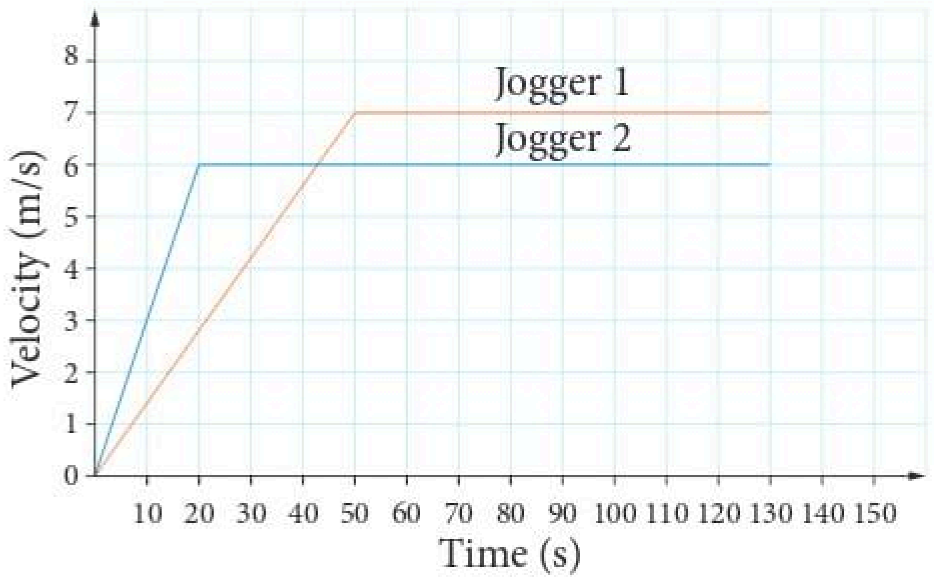
1 E **2** A

3 **a**



b 27 cm **c** 15 cm

4 **a**

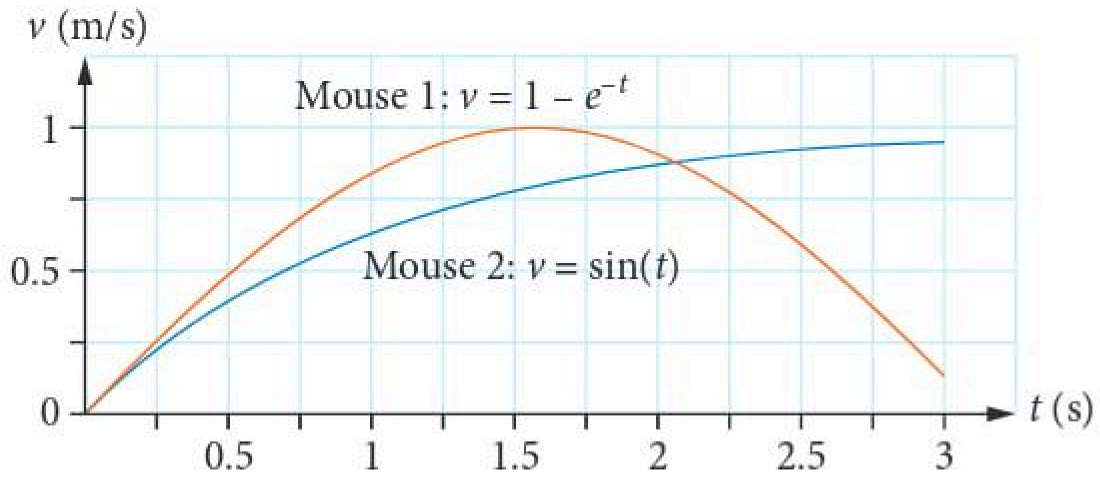


b 1 min 55 s **c** 630 m

5 **a** $v = \sqrt{\frac{g}{k}(1 - e^{-2kx})}$

b $k = 0.2$ **c** 7 m

6 **a**



b The second mouse travelled further by 28 cm.

7 A **8** B **9** E

10 A **11** D **12** C

13 C **14** B **15** B

16 E

17 **a** $v = \sqrt{g}\left(\frac{e^{2\sqrt{gt}} - 1}{e^{2\sqrt{gt}} + 1}\right)$ **b** 3.13 m s^{-1}

c 61.92 m

18 $T = 25$

19 **a** $t = 12, v = (12 - 4)\tan\left(\frac{\pi}{48} \times 12\right) = 8\tan\left(\frac{\pi}{4}\right) = 8$

b 36.6 s

20 $t = 3.7$

21 **a** $17 = 20 - \tan^{-1}(t), t = 14.1$, correct to one decimal place

b $\tan^{-1}(t) < \frac{\pi}{2}, V \rightarrow 20 - \pi \simeq 16.858$

c $\int_0^T (20 - 2\tan^{-1}(t))\, dt$

d $\int_0^8 (20 - 2\tan^{-1}(t))dt -$

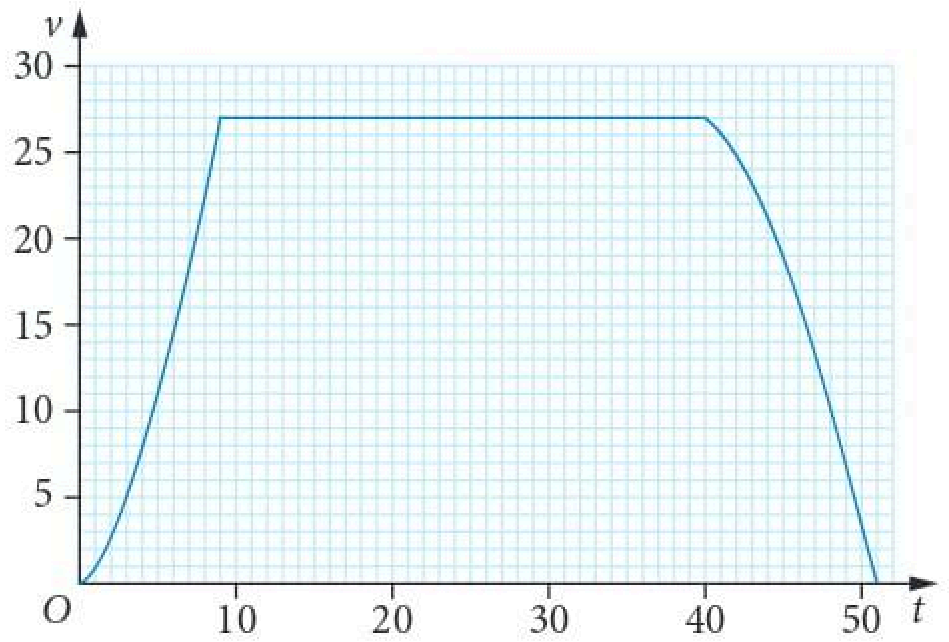
$\int_3^8 13\cos^{-1}\left(\frac{13 - 2t}{7}\right)dt = 60.7$ m, correct to one decimal place

e When $t = 8$, $V_{\text{police}} = 26.178$ (m/s), so

$60.7 + \int_8^{T_c} (20 - 2\tan^{-1}(t))\, dt = 26.178(T_c - 8)$ or equivalent.

f $T_c = 14.6 \simeq 15$ s, correct to the nearest second.

22 **a**



b 97.2 m **c** 206.3 m **d** 21.8 m s^{-1}

e $t_1 = 7.9, t_2 = 43.6$, correct to the nearest 0.1 s

f 417 m

EXERCISE 10.3

1 E **2** D

3 The van by 300 − 250 = 50 m.

4 97.5 s, 3.15 km

5 $u = 10$ m/s, $a = 10$ m/s^2

$s = 105$ m

$s = ut + \frac{1}{2}at^2$

$105 = 10t + 5t^2, t^2 + 2t - 21 = 0, t = t_2 = \sqrt{11}\sqrt{2} - 1$

$s = 35$ m, $35 = 10t + 5t^2, t^2 + 2t - 7 = 0,$

$t = t_1 = 2\sqrt{2} - 1$

$t_2 - t_1 = (\sqrt{11}\sqrt{2} - 1) - (2\sqrt{2} - 1)$

$= \sqrt{2}(\sqrt{11} - 2)$

6 Use $v^2 = u^2 + 2as$

$v = V_1, s = H, u = V$

$V_1^2 = V^2 + 2gH \quad \ldots[1]$

$v = V_2, s = \frac{1}{2}H, u = V$

$V_2^2 = V^2 + 2g\left(\frac{1}{2}H\right) = V^2 + gH \quad \ldots[2]$

$[2] - [1]: V_2^2 - V_1^2 = -gH$ so $V_2 = \sqrt{V_1^2 - gH}$

7 1.70 m s^{-1}

8 $t = 2.51$ s, $s = 19.28$ m

9 **a** $v\frac{dv}{dx} = 10 + kv^2$ or $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 10 + kv^2$

b $v = 100\sqrt{e^{0.002x} - 1}$

c 43 m

10 C **11** B **12** B **13** E

14 A **15** B **16** C

17 Use $v^2 = u^2 + 2as$ with $v = 0, a = -10$ to get

$-20s = -u^2$ or $s = \frac{u^2}{20}$.

Require $s \geq D$, $\frac{u^2}{20} \geq D$, $u \geq 2\sqrt{5D}$ m/s

18 $s = \frac{1}{2}at_1^2, \frac{1}{2}s = \frac{1}{2}at_2^2 \Rightarrow s = at_2^2$

$\frac{1}{2}at_1^2 = at_2^2 \Rightarrow t_1 = t_2\sqrt{2}$

19 9.6 s

20 **a** 12.5 m s^{-2}

b $v = u + at; v = 0 + 12.5 \times 8 = 100$

21 **a, b** Proof: see worked solutions

c **i** $t = \frac{v-u}{a}$ **ii** $s = ut + \frac{1}{2}at^2$

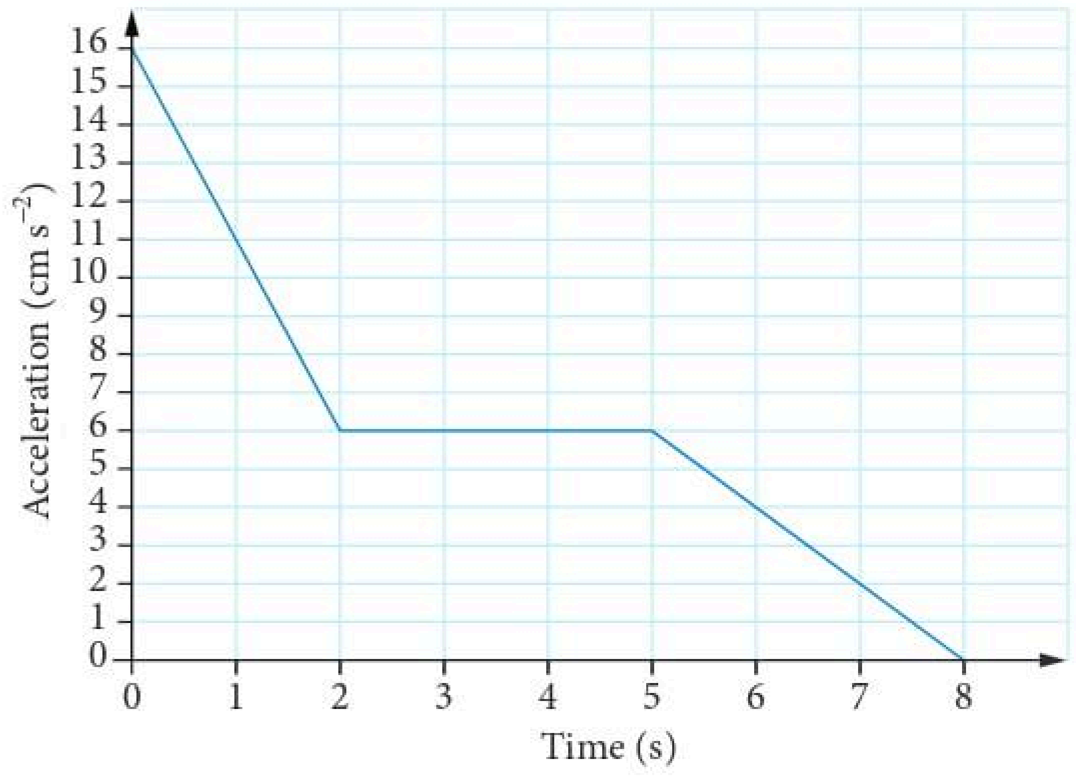
iii Proof: see worked solutions

EXERCISE 10.4

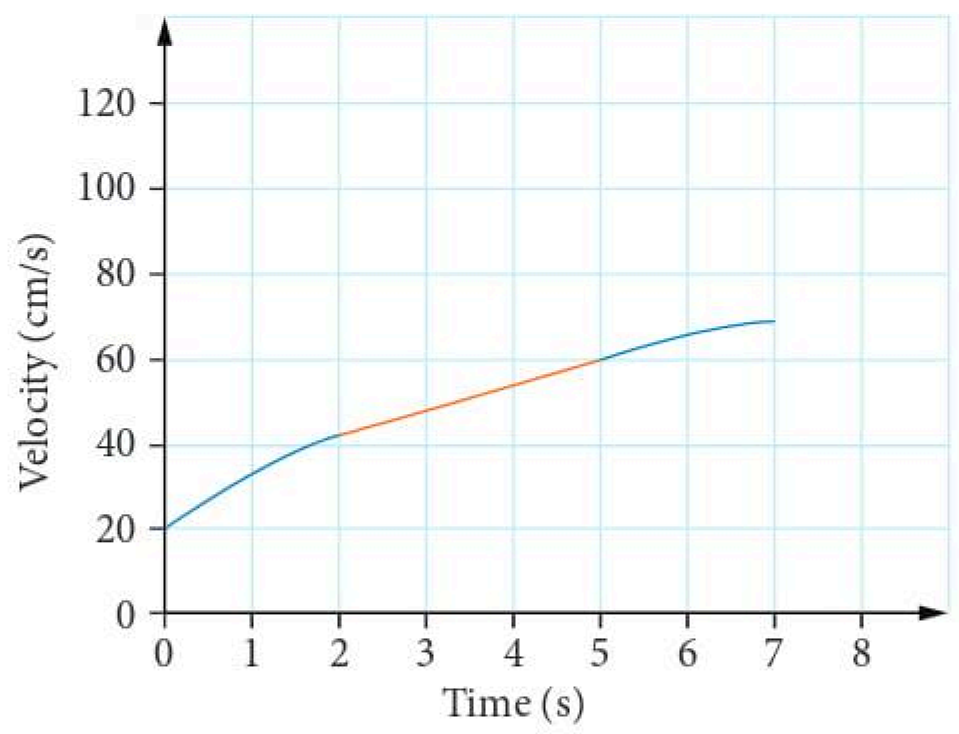
1 B **2** D

3 −394 m/s, −3760 m

4 **a**



b



c $412\frac{1}{3}$ cm

5 2.36 s, −0.07 m/s

6 20 s

7 $a(t) = -\frac{2}{3}t + \frac{1}{2}$

8 C **9** E **10** A **11** D

12 B **13** B **14** C **15** A

16 **a** $\frac{dv}{dt} = -10 - kt$

b $v = -10t - 0.5kt^2 + 20$ **c** 0.1

17 $\frac{d^2x}{dt^2} = -\frac{1}{(1-t)^2}$

18 $v_1(t) = e^t, v_2(t) = 5 - e^{t-2}$

Let $t = T$ satisfy $v_1(T) = v_2(T)$.

Then $e^T = \frac{5e^2}{1+e^2} \Rightarrow T = \log_e\left(\frac{5e^2}{1+e^2}\right)$.

$d_1 = \int_0^T e^t dt = e^T - 1, \; d_2 = \int_0^T (5 - e^{t-2})dt = 5T - e^{-2}(e^T - 1)$

$d_2 = 5T - e^{-2}(e^T - 1) = 5\log_e\left(\frac{5e^2}{1+e^2}\right) - e^{-2}d_1$

$d_1 + e^2 d_2 = 5e^2 \log_e\left(\frac{5e^2}{1+e^2}\right)$

19 **a** **i** $a = 6t - 3t^2$

ii $a_{\max} = 3 \text{ m/s}^2$

b $\frac{49}{4}$ s

20 **a** $\frac{10}{3}$ s **b** 2.5 s **c** 35.1 m/s

d **i** $v = 14\sin\left(\frac{\pi}{6} - \frac{t}{10}\right)$ or $v = 14\sin\left(\frac{\pi}{3} + \frac{t}{10}\right)$

ii $t = \frac{5\pi}{3}$ **iii** 18.8 m

CUMULATIVE EXAMINATION 1

1 $\frac{1}{2}(t_2 + t_3 - t_1)V$ **2** e^5 70% **3** $k = -2$ 67%

4 $k = 2$

CUMULATIVE EXAMINATION 2

Section A

1 E 59%   **2** E 48%   **3** C

4 E   **5** A

Section B

1 a $v = \sqrt{2 + \frac{2}{x}}$ **b** -1.25 m, 0.67 m

2 a 75 s, 1500 m   **b** 30 s, 600 m

3 a $|z - 4 - 4i| = 4,\ |z - 7 - 7i| = 7$

b $A(0.213, 5.287),\ B(5.287, 0.213)$

c $\text{Im}(z) = -\text{Re}(z) + 5.50$

4 a $a^2 - 4b > 0$ **b** $\frac{1}{x+2} - \frac{1}{x+3}$

c $y = x + 5$, oblique; $x = -2$, $x = -3$, vertical

CHAPTER 11

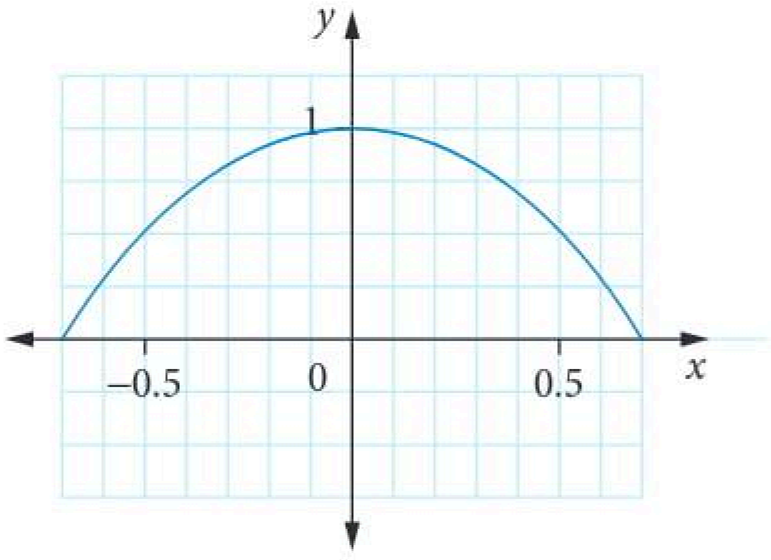
EXERCISE 11.1

1 $\underset{\sim}{r}(0) = \underset{\sim}{i}$ and $\underset{\sim}{r}(1) = 4\underset{\sim}{i} + 2\underset{\sim}{j}$

2 $x = 3y^2$

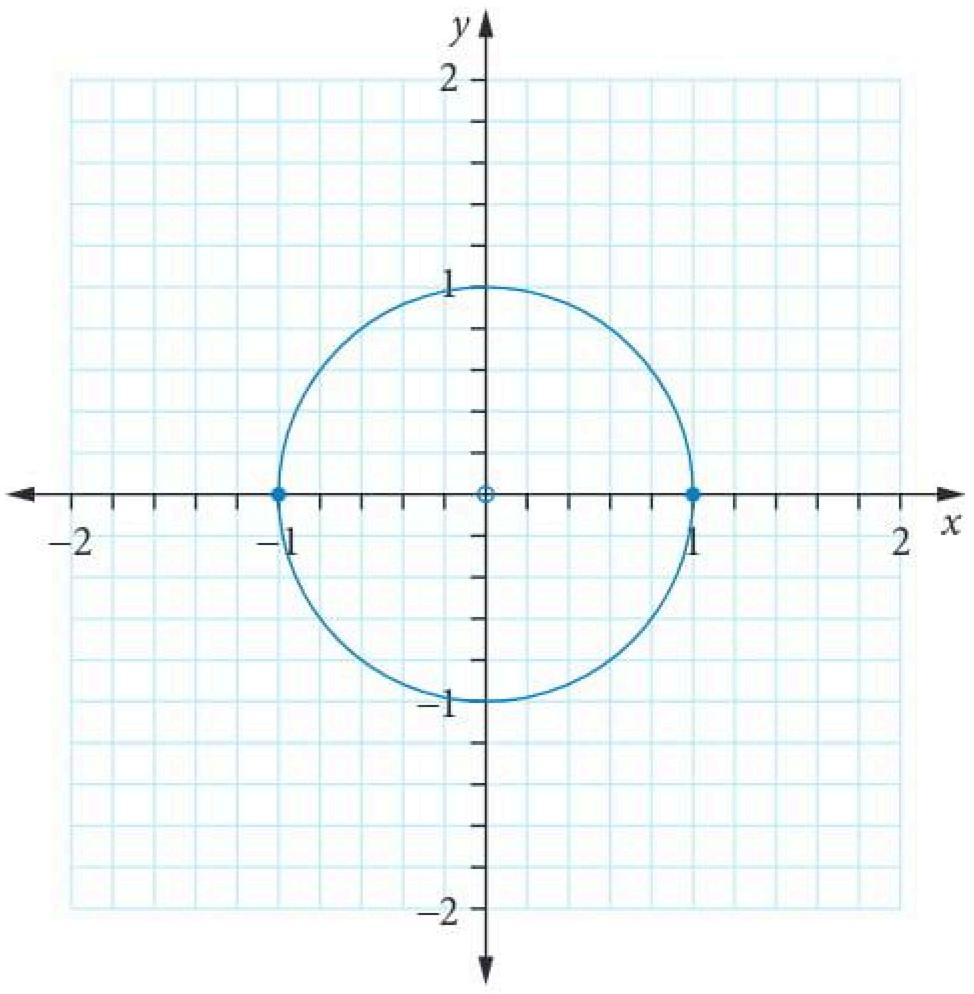
3 a $y = 1 - 2x^2$ for $x \in [-1, 1]$

b



4 a $x^2 + y^2 = 1$ for $x \in [-1, 1]$

b



5 D   **6** B   **7** D   **8** A

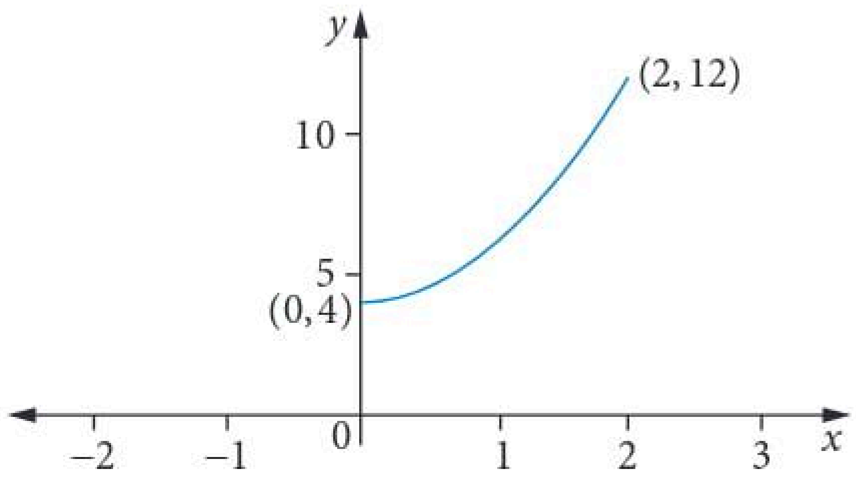
9 a Proof: see worked solutions

b $x = 1, x = 3$

c $\frac{2\pi}{3}$

10 a $y = 2x^2 + 4$

b



11 D   **12** E

EXERCISE 11.2

1 a $x^2 + y^2 = 9$

b domain: $[-3, 3]$, range: $[-3, 3]$

2 C

3 $\dot{\underset{\sim}{r}}(t) = 6t\underset{\sim}{i} - \underset{\sim}{j} + 2(t+1)\underset{\sim}{k}$ and $\ddot{\underset{\sim}{r}}(t) = 6\underset{\sim}{i} + 2\underset{\sim}{k}$

$\dot{\underset{\sim}{r}}(3) = 18\underset{\sim}{i} - \underset{\sim}{j} + 8\underset{\sim}{k}$ and $\ddot{\underset{\sim}{r}}(3) = 6\underset{\sim}{i} + 2\underset{\sim}{k}$

4 $\sqrt{14}$

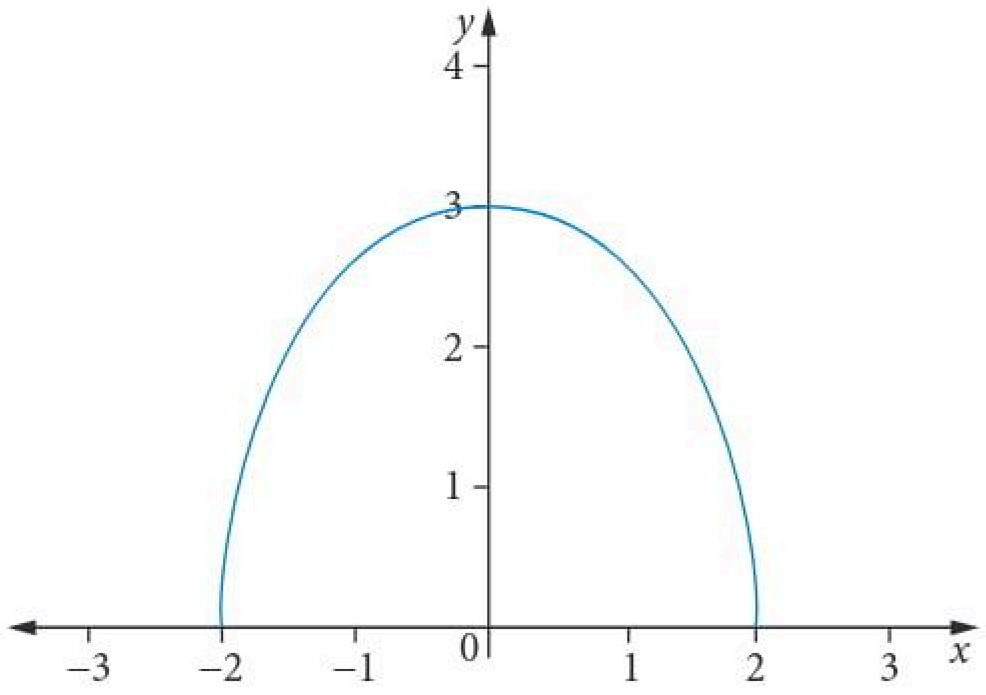
5 $\dot{\underset{\sim}{r}}(t) = \left(3 - t^2\right)\underset{\sim}{i} + \left(2 + \sin(t)\right)\underset{\sim}{j} + t^3\underset{\sim}{k}$

6 A   **7** A   **8** E

9 a $\underset{\sim}{r}(t) = 2\cos(2t)\underset{\sim}{i} + 3\sin(2t)\underset{\sim}{j}$

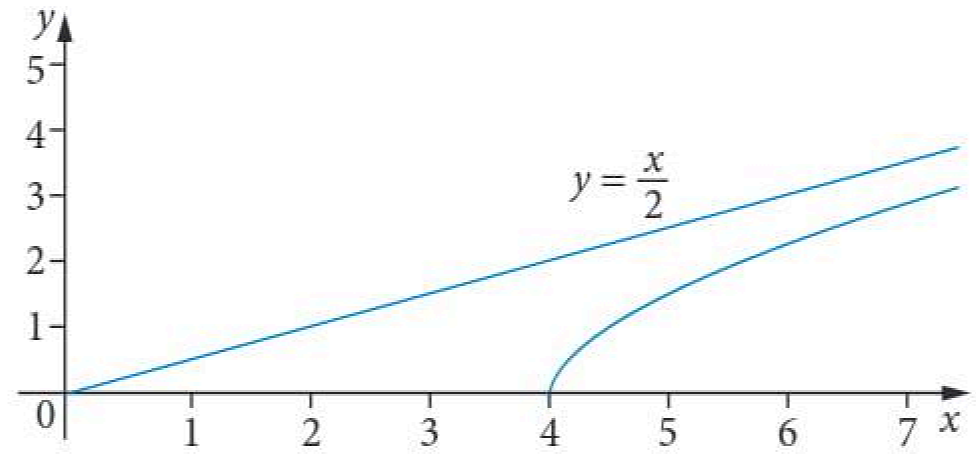
b $\frac{x^2}{4} + \frac{y^2}{9} = 1$

c



10 a Proof: see worked solutions

b



c $4\sqrt{3}$

11 a $\underset{\sim}{v}(t) = \left(\frac{2t}{\sqrt{t^2+2}} - 2t\right)\underset{\sim}{i} + \left(\frac{2t}{\sqrt{t^2+2}} + 2\right)\underset{\sim}{j}$

b $\frac{4\sqrt{6}}{3}$

c Proof: see worked solutions

d $\frac{7\pi}{12}$

12 a Proof: see worked solutions

b $\frac{1}{2}\sqrt{4 + 9\pi^2}$

c $\frac{3\pi}{2}$

13 $\underset{\sim}{a} = -\underset{\sim}{r}$

14 A **15** B **16** C **17** E

18 C **19** B **20** B **21** E

EXERCISE 11.3

1 $x^2 + y^2 = 1, \dot{\underset{\sim}{r}} = (3\sin(3t))\underset{\sim}{i} + (-3\cos(3t))\underset{\sim}{j}$

2 $\dot{\underset{\sim}{r}}(t) = (3t^2 + 1)\underset{\sim}{i} + (\sin(t) + 1)\underset{\sim}{j}$

3 $(-1, 4)$

4 $y = \dfrac{x-3}{3}$

5 $10\sqrt{5}$ m/s

6 D **7** C **8** E

9 $a = \dfrac{\pi - 1}{3}$

10 $a = -1, b = 0, c = 2$

11 E **12** C **13** E

14 **a, b** Proof: see worked solutions

c 30.625 m **d** 43 m/s **e** 35 m

15 **a** 24.5 m **b** 14 s **c** 12 s

d $\dot{\underset{\sim}{r}}(t) = \dfrac{5\pi}{6}\cos\left(\dfrac{\pi t}{6}\right)\underset{\sim}{i} - \dfrac{5\pi}{6}\sin\left(\dfrac{\pi t}{6}\right)\underset{\sim}{j} - \dfrac{t}{4}\underset{\sim}{k}$

e 4.4 m/s

f Proof: see worked solutions

16 **a** Proof: see worked solutions **b** 1118 m

c 4.6° **d** 56 s

e 3010 m

17 **a** Proof: see worked solutions **b** 25.4°

c 22.05 m **d** $320y = 168x - x^2$

e 0.75 m **f** 5 m

CUMULATIVE EXAMINATION 1

1 **a** Proof: see worked solutions **83%**

b $-8 - 8\sqrt{3}i$ **70%**

2 $\sqrt{\pi^2 + 1}$ **64%**

3 $\log_e\left(\dfrac{4}{3}\right)$

CUMULATIVE EXAMINATION 2

Section A

1 E **85%** **2** D **41%**

3 E **71%** **4** A **65%**

5 A **49%**

Section B

1 **a** 60° **b** $|\dot{\underset{\sim}{r}}(0)| = 12$

c Maximum height occurs when $t = 1.0639$.

Maximum height = 5.5 m

d Proof: see worked solutions

e 38.51 m

2 **a** **i** Proof: see worked solutions

ii $145\,000\pi$

b Proof: see worked solutions

c $\dfrac{dh}{dt} = \dfrac{-20\sqrt{h}}{\pi(45 + 4h)^2}$

d 9.9 hours

CHAPTER 12

EXERCISE 12.1

1 **a** 1.7 **b** 20

2 **a** 6 **b** 4

3 **a** $\dfrac{8}{9}$ **b** 288

4 mean = 70 min, variance = 89 min

5 mean = 210, variance = 410

6 **a** 84 **b** 60 **c** $2\sqrt{15}$

7 **a** 17 **b** $3\sqrt{41}$

8 mean = 78, variance 522

9 **a** $\dfrac{3\pi}{4}$ **b** $\dfrac{\pi^2}{1600}$ **c** $\dfrac{7\pi}{2}$

10 A **11** B **12** D **13** B

14 D **15** E **16** D **17** C

18 **a** $\dfrac{4}{3}$ **b** 10 **c** $\dfrac{2}{9}$ **d** 2

19 7.2

20 **a** mean = 30 min, variance = 6 min

b 600

EXERCISE 12.2

1 E **2** C

3 **a** mean = 20, standard deviation = $\dfrac{\sqrt{5}}{5}$

b 0.868

4 **a** $E(B - A) = 10$, $Var(B - A) = 1525$

b $Pr(B - A > 0) = 0.601$

5 497.570

6 **a** mean = 300, standard deviation = 10

b 0.841 **c** 310 mL

7 0.025 or 0.023

8 B **9** E **10** C **11** B

12 A **13** C **14** B **15** E

16 E **17** D **18** A

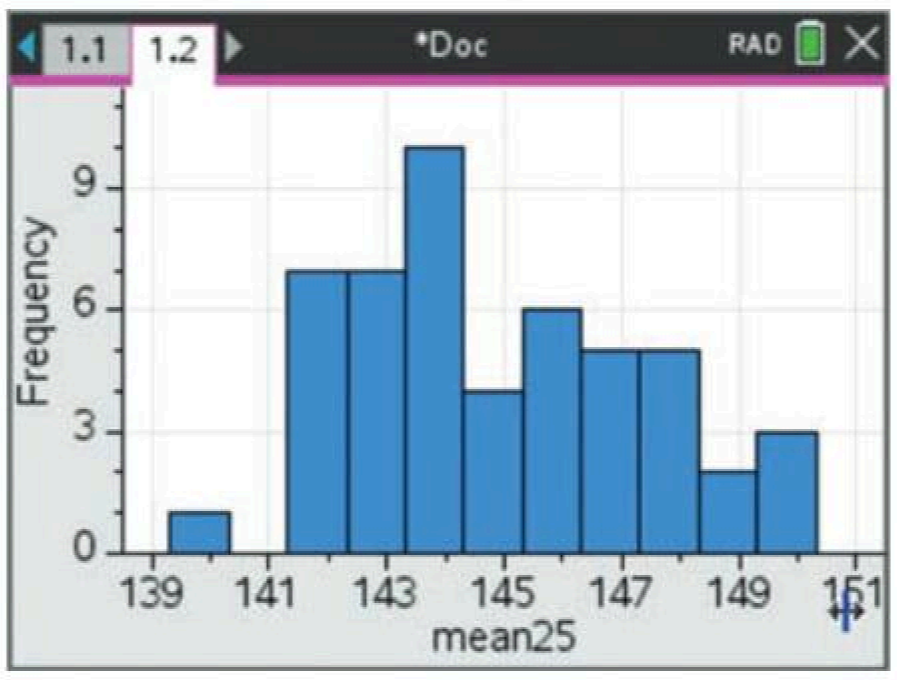
19 **a** mean = 35, standard deviation = $2\sqrt{5}$ **b** 0.673

20 0.945

EXERCISE 12.3

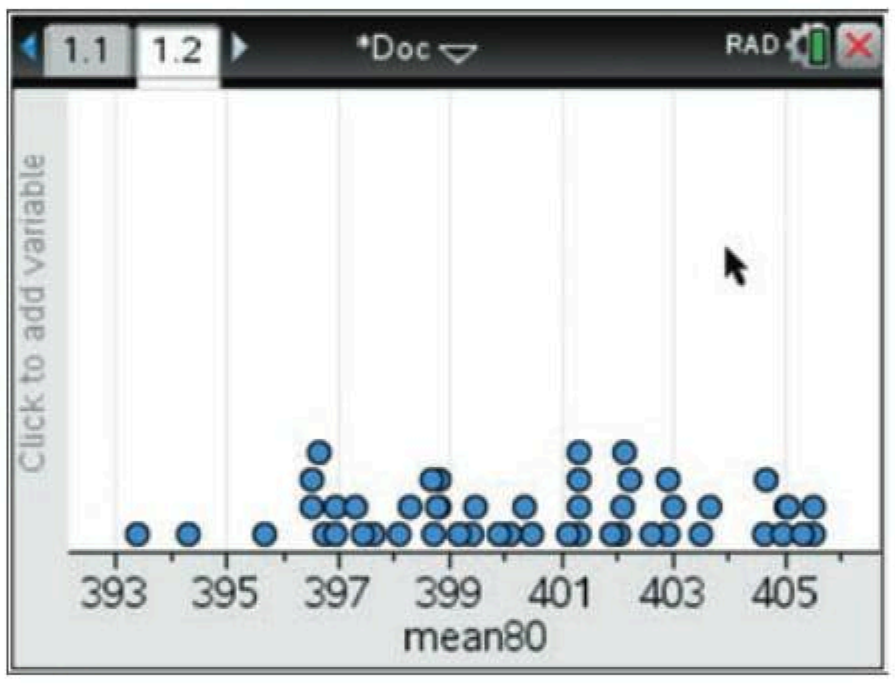
1 B **2** E

3



4 $\frac{7}{100}$

5 **a**



b average of the sample means = 400.41

6

Sample size	Mean of $\overline{x}$	Standard deviation of $\overline{x}$
20	249.474	11.376
200	249.793	3.904

Answers may vary slightly depending on the data generated.

7 0.15 g

8 A **9** C **10** B

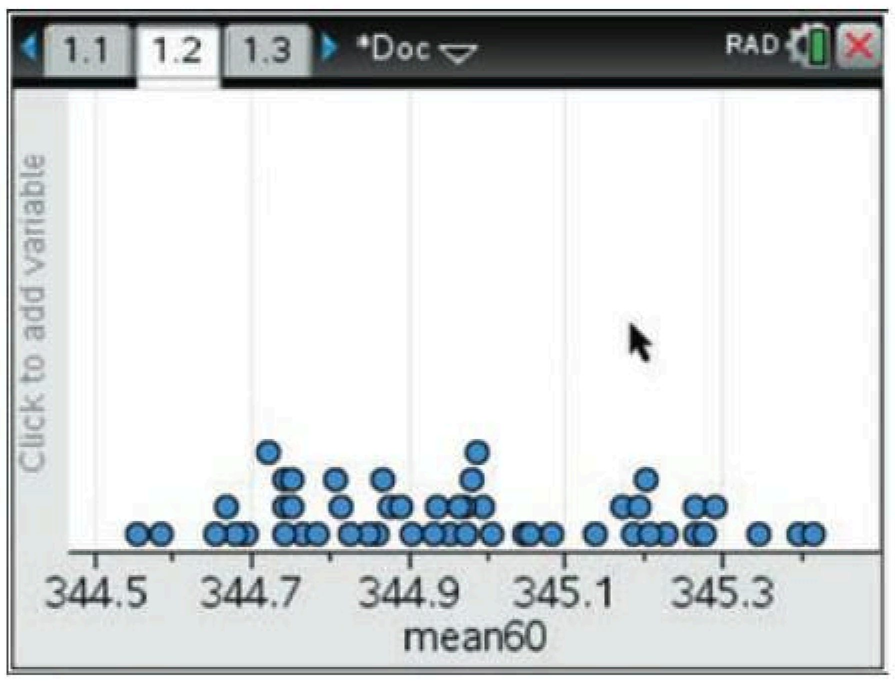
11 E **12** A **13** A

14 **a** population mean = 30

b population standard deviation = $0.566 \times \sqrt{50} \approx 4$

15 **a** 3.5 kg **b** 0.5 kg

16 **a**



b mean of sample means = 344.96, standard deviation of the sample means = 0.224

EXERCISE 12.4

1 B **2** C

3 **a** (30.6, 89.4) **b** (8.64, 101.36)

4 **a** 0.8849 **b** 0.9772

5 **a** (42.37, 47.63) **b** (25.51, 30.49)

6 (33.011, 33.789)

7 (83.3, 86.7)

8 **a** $n = 71$ **b** $n = 68$

9 mean = 250 mL, $n = 49$

10 (103.4, 106.6)

11 **a** mean = 48.2, $n = 144$ **b** 1764

12 **a** (51, 65) **b** 0.025

13 (5.4, 5.6)

14 B **15** E **16** E **17** D

18 A **19** B **20** A **21** C

22 C **23** A **24** A **25** D

26 D **27** (130, 136)

28 (115, 135) **29** 0.35 hours **30** 97

EXERCISE 12.5

1 C **2** E

3 **a** $H_0: \mu = 40, H_1: \mu < 40$ **b** $H_0: \mu = 75, H_1: \mu > 75$

4 **a** $H_0: \mu = 250, H_1: \mu \neq 250$ **b** $H_0: \mu = 5.5, H_1: \mu \neq 5.5$

5 **a** 0.24 **b** 0.89

6 **a** $H_0: \mu = 3000, H_1: \mu < 3000$

b $z = -4, p = 0.000\,032$

c The chance of getting a mean less than 3000 is 0.000 032 if H_0 is true. We reject the null hypothesis $\mu = 3000$ because p is less than 0.05 and therefore the result is significant.

The mean life of a light bulb is significantly less than 3000 hours.

$(z = -4, p = 0.000\,032, \overline{x} = 2800, \sigma = 500, n = 100)$

The retailers claim is supported.

7 **a** $H_0: \mu = 580$ h, $H_1: \mu \neq 580$ h

b $z = -1, p = 0.3173$

c The chance of getting a mean less than 577 or greater than 583 is 0.3173 if H_0 is true. We fail to reject the null hypothesis. The construction company's claim is not supported.

8 **a** −2.054 **b** −2.576 and 2.576

9 **a** $H_0: \mu = 175, H_1: \mu < 175$

b $z = -2.5$, critical z-value = −1.64

The test statistic $z = -2.5$ is less than the critical z-value = −1.645 and lies inside the rejection region. We reject the null hypothesis $\mu = 175$ g at the 0.05 significance level.

$(z = -2.5, \overline{x} = 170, \sigma = 10, n = 25)$

10 **a** $z = -2.326, z = 2.326$

b Reject the null hypothesis if the sample mean is less than 75.347 or greater than 84.653.

11 **a** $H_0: \mu = 200, H_1: \mu < 200$

b **i** 0.001 **ii** reject H_0 as $p < 0.001$

c (246.08, 253.92)

12 **a** $H_0: \mu = 4, H_1: \mu < 4$

b mean = 4; standard deviation = 0.75

13 C **14** D **15** B **16** E

17 A **18** D **19** D

20 **a** H_0: $\mu = 0.65$ kg/week, H_1: $\mu > 0.65$ kg/week

b $z = 2.683$ $p = 0.0036$

c The chance of getting a mean greater than 0.65 is 0.0036 if H_0 is true. We reject the null hypothesis.

d The mean life weight loss is significantly more than 0.65 kg/week.

$(z = 2.683, p = 0.0036, \overline{x} = 0.67, \sigma = 0.05, n = 45)$

The claim about the app is supported.

21 **a** H_0: $\mu = 65$ beats/min, H_1: $\mu \neq 65$ beats/min

b $z = 3$ $p = 0.026$

c The chance of getting a mean less than 62 or greater than 68 beats per minute is 0.0026 if H_0 is true. We reject the null hypothesis.

d The mean number of beats per minute is significantly more than 65 beats per minute.

($z = 3$, $p = 0.0026$, $\bar{x} = 68$, $\sigma = 10$, $n = 100$)

The medical association's claim is supported.

22 H_0: $\mu = 100$ H_1: $\mu > 100$

one-tail test: $z = 0.4216$, $p = 0.337$

The chance of getting a mean greater than 101 is 0.337 if H_0 is true. There is insufficient evidence to reject the null hypothesis, therefore we fail to reject the null hypothesis. ($z = 0.4216$, $p = 0.337$, $\bar{x} = 100$, $\sigma = 15$, $n = 40$)

The average IQ is 100 and the organisation's claim is not supported.

23 **a** 79.8

b Proof: see worked solutions **c** 99.6

d s = 5.1

e $p = 0.0569$, accept the dairy's claim.

24 **a** $E(\overline{X}) = 1.1$, $\mathrm{SD}(\overline{X}) = 0.032$

b H_0: $\mu = 1.1$, H_1: $\mu > 1.1$

c **i** $p = 0.0009$

ii $p < 0.05$, reject the null hypothesis at the 5% level of significance, supports the contention

d $\overline{x_c} = 1.153$

e 0.124

25 **a** 0.741 **b** 0.495

c H_0: $\mu = 375$, H_1: $\mu \neq 375$ **d** $p = 0.046$

e As $p < 0.05$ reject H_0. The sample suggests the machine is not working properly.

f $x_c = 372.1$

26 **a** H_0: $\mu = 128$, H_1: $\mu > 128$,

$p = 0.00621$, reject H_0 at the 5% significance level as $p < 0.05$, The evidence supports the researchers claim.

b (130, 136)

27 **a** mean = 3.55, standard deviation = 0.11

b H_0: $\mu = 3.55$, H_1: $\mu > 3.55$

c $p = Pr\left(\overline{X} > 3.85 \middle| \mu = 3.55\right) = 0.003$

d As $p < 0.01$ we reject H_0 at the 1% significance level.

e $Pr\left(\overline{X} > \overline{x}_{\text{critical}} \middle| \mu = 3.55\right) = 0.01$, $\overline{x}_{\text{critical}} = 3.806$

$\bar{x} \geq 3.806$

f $Pr\left(\overline{X} < 3.806 \middle| \mu = 3.83\right) = 0.41$

EXERCISE 12.6

1 E **2** C

3 **a** **i** H_0: $\mu = 12$, H_1: $\mu < 12$

ii The null hypothesis H_0: $\mu = 12$ is rejected when it is true. The potatoes have an average length of 12 cm however the inspector believes they do not. The potato-chip producer sends the truckload of acceptable potatoes away, which may result in lost revenue for the supplier.

iii The null hypothesis H_0: $\mu = 12$ is not rejected when it is false. The potatoes do not have an average length of 12 cm however the inspector believes they do. The producer uses the truckload of potatoes to make potato chips. The potatoes are undersize and this will upset the chip company and may affect the farmers contract

b **i** H_0: $\mu = 13.5$, H_1: $\mu < 13.5$

ii The null hypothesis H_0: $\mu = 13.5$ is rejected when it is true. The average response time is still 13.5 minutes however they believe the changes to the call centre have improved response times. This would result in changes being made to the call centre which will not have an impact on the response time of the ambulance as any improvement in response times is due to expected variation in the sample mean.

iii The null hypothesis H_0: $\mu = 13.5$ is not rejected when it is false. The average response time is no longer than 13.5 min but they believe it still is. This would result in changes appearing to have had no effect, and they may be cancelled, or not further improved upon.

4 D

5 A

6 **a** H_0: $\mu = 40\,000$ km, H_1: $\mu > 40\,000$ km

b $z = 2$, $p = 0.02275$

c The null hypothesis H_0: $\mu = 40\,000$ is rejected when it is true. The average life of tyres is 40 000 kilometres but it is believed reducing the maximum speed has improved it. This could result in maximum speeds being reduced in the false belief it will improve the life of the tyres.

d The null hypothesis H_0: $\mu = 40\,000$ is not rejected when it is false. The average life of tyres is not 40 000 km, but it is believed reducing the maximum speed has had no impact. This could result in drivers continuing to drive at maximum speeds of 110 km/h when they could improve the life of the tyres by reducing their maximum speeds.

e The chance of getting a mean greater than 40 000 km is 0.022 75 if H_0 is true. We reject the null hypothesis.

f The mean number of kilometres is significantly more than 40 000 km.

($z = 2$, $p = 0.02275$, $\bar{x} = 41\,000$, $\sigma = 5000$, $n = 100$)

The RACV's claim is supported.

g As the null hypothesis is rejected, there is a chance of making a type I error.

7 **a** H_0: $\mu = 7$ h, H_1: $\mu > 7$ h

b $z = 3.536$, $p = 0.000\,204$

c The null hypothesis H_0: $\mu = 7$ is rejected when it is true. The battery life of the phone is still 7 h but it is believed the software update has improved it. This could result in the update being installed in phones as they believe it will have an impact when it will in fact make no difference to the phone's battery life.

d The null hypothesis H_0: $\mu = 7$ is not rejected when it is false. The battery life of the phone not 7 h, but it is believed the software update has not improved the phone life. This could result in the update not being installed in phones as they believe it will not have an impact when it will in fact improve the battery life.

e The chance of getting a mean greater than 7 h is 0.000 204 if H_0 is true. We reject the null hypothesis.

The mean number of hours of battery life is significantly more than 7 h.

($z = 3.536$, $p = 0.000\,204$, $\overline{x} = 7.5$, $\sigma = 0.5$, $n = 50$)

The claim that the phone's software update improves battery life is supported.

CUMULATIVE EXAMINATION 1

1 $a = 2$, $b = 3$ 76%

2 **a** Proof: see worked solutions

b $128i$

3 $\frac{8\pi}{3}$ units^3

CUMULATIVE EXAMINATION 2

Section A

1 E 57%

2 B 56%

3 A

4 D

5 C

Section B

1 **a** mean = 190, standard deviation = 6.5

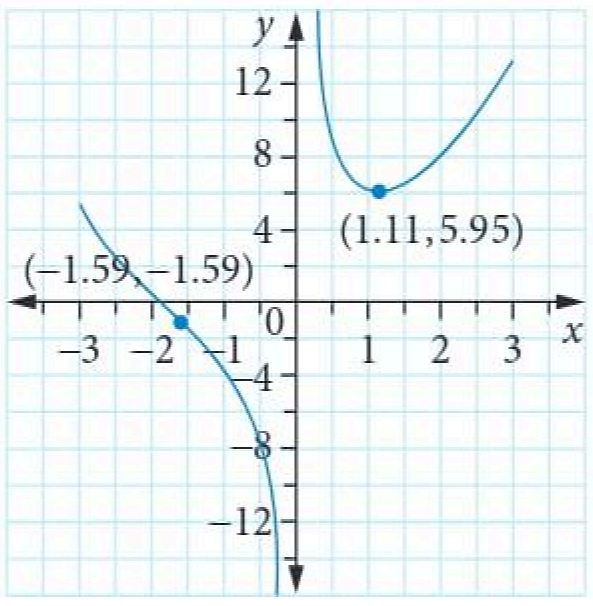
b 0.062

c 3.99

2 **a** (1.11, 5.95) 93%

b (−1.59, −1.59) 86%

c 72%



d **i** $\int_{-3}^{-0.5} \sqrt{\left(1 + \left(\frac{2x^3 + x^2 - 4}{x^2}\right)\right)^2}\, dx$ 73%

ii 13.18 74%

e $a = \pi$, $b = \frac{14}{3}$, $c = -\frac{33}{4} = -8.25$ 34%

Glossary and index

absolute value (or **modulus**) The unsigned magnitude of a real number, or the distance of the number from 0 on the number line. For example, the absolute value of −7, written as $|-7|$, is 7. Also, $|2| = 2$. Absolute value can be defined by the piecewise function:

$$x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

or by the formula $|x| = \sqrt{x^2}$. (p. 73)

acceleration The rate of change of velocity of a moving object, represented by the function $a = \frac{dv}{dt} = \dot{v} = \ddot{x}$, where v is the velocity ('signed speed') and x is the displacement. Also, $a = v\frac{dv}{dx}$ or $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$. Conversely, $v = \int a(t)\,dt$. (p. 423)

air resistance In addition to the gravitational acceleration, a retarding force acting in the opposite direction to the object's direction of motion. (p. 450)

alternative hypothesis (symbol: $\boldsymbol{H_1}$) A statement about a population parameter that is the opposite of and complementary to the null hypothesis, usually claiming some effect or difference. If the null hypothesis is rejected, then the alternative hypothesis is supported.
See also **null hypothesis.** (p. 529)

anti-derivative (or **integral** or **primitive**) The opposite of the derivative. The anti-derivative of $f(x)$ is a function $F(x)$ whose derivative is $f(x)$: $F'(x) = f(x)$. (p. 220)

arc length The length of the graph of a function over an interval from $x = x_1$ to $x = x_2$, given by the formulas $l = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$, $l = \int_{x_1}^{x_2} \sqrt{1 + \left[f'(x)\right]^2}\,dx$, or if the function has parametric equations in terms of t with the interval between $t = t_1$ and $t = t_2$, $l = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$ or $l = \int_{t_1}^{t_2} \sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2}\,dt$. (p. 340)

arccosine, arcsine, arctangent *See* **inverse sine**, **inverse cosine**, **inverse tangent** respectively.

Argand diagram (or **complex plane**) A number plane for representing complex numbers. The number $x + yi$ is represented by the point (x, y). (p. 149)

argument The angle that the vector of a complex number makes with the positive direction of the x-axis, written $\theta = \arg(z)$. The **principal argument** is the value of θ in the interval $(-\pi, \pi]$. (p. 154)

atomic sentence A statement that cannot be broken down into fewer sentences. (p. 113)

average speed The total distance divided by the total time taken. (p. 423)

average velocity The displacement divided by the total time taken. (p. 423)

Cartesian form (or **rectangular form**) The conventional form of the equation of a graph in which y is written as a function of x. (p. 44)
See also **polar form**.

carrying capacity In a logistic model for population growth, carrying capacity is the maximum number the environment can support. (p. 383)
See also **logistic model**.

central limit theorem A rule states that for sufficiently large ($n \geq 30$) random samples of a random variable X from a distribution with mean μ and standard deviation σ, the distribution of the means of the samples is approximately normal with mean μ and standard deviation $\frac{\sigma}{\sqrt{n}}$. (p. 521)

chain rule A formula for finding the derivative of a composite function. If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$. (p. 206)

complex conjugate The conjugate of the complex number $x + yi$ is $x - yi$, the same expression with the '+' changed to a '−' (or vice-versa). (p. 149)

complex numbers (set C) Numbers of the form $x + yi$, where x and y are real numbers and i is the imaginary number, where $i = \sqrt{-1}$. Includes all real and imaginary numbers. (p. 149)

complex plane *See* **Argand diagram**.

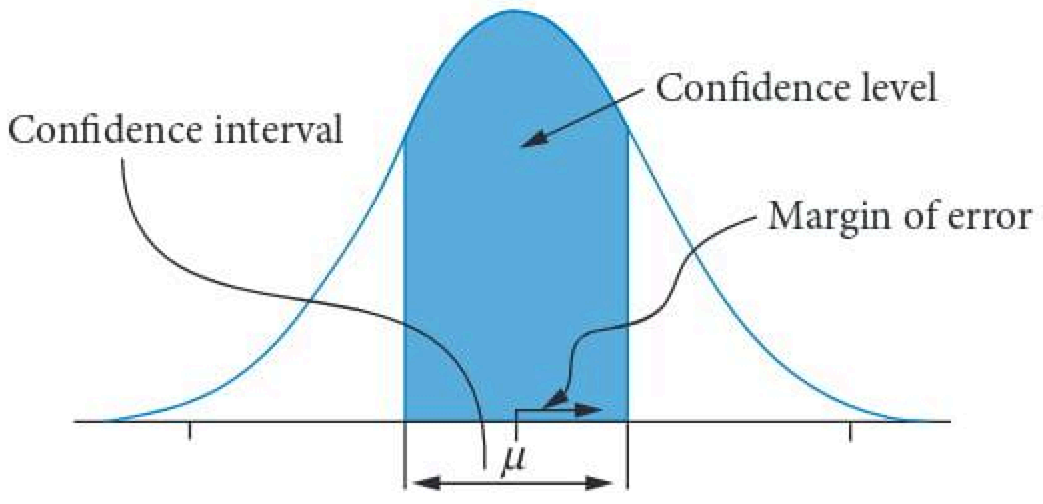
complex polynomial A polynomial whose coefficients and variable are complex numbers. (p. 176)

compound angle formulas *See* **sum and difference formulas**.

compound sentence A statement containing two or more atomic statements. (p. 113)

concavity *See* **second derivative**.

confidence interval An interval centred on the mean of a distribution that contains a specific proportion of the values of the distribution. For example, a 95% confidence interval contains 95% of the values of the distribution. (p. 519)



confidence level The proportion of values contained in a confidence interval, for example, a 95% confidence interval has a confidence level of 0.95. (p. 519)

conjecture A statement that is believed to be true but conclusive proof has not been found. (p. 116)

conjugate *See* **complex conjugate**.

conjugate root theorem If w is a complex root of a real polynomial, then its conjugate $\bar{w}$ is also a root. (p. 185)

constant of integration A constant added to an indefinite integral. (p. 372)

converse The converse to the logical statement $P \rightarrow Q$ (if P then Q) is $Q \rightarrow P$ (if Q then P). (p. 123)

cosecant Abbreviation: cosec. The reciprocal of sine; $\operatorname{cosec}(\theta) = \dfrac{1}{\sin(\theta)}$. (p. 76)

cotangent Abbreviation: cot. The reciprocal of tangent; $\cot(\theta) = \dfrac{1}{\tan(\theta)}$. (p. 76)

counterexample An example that is used to disprove a conjecture. (p. 116)

deceleration Negative acceleration. (p. 448)

deductive reasoning The process of reasoning from one or more statements to reach a conclusion using logic. (p. 115)

definite integral An integral of the form $\int_a^b f(x)\,dx$, whose value is the area under a curve and is read 'the integral of $f(x)$ between a and b with respect to x'. (p. 277)

degree (of a polynomial) The highest power of the variable in a polynomial. For example, $x^3 - 3x^2 + 6x$ has degree 3. (p. 363)

degree (of a differential equation) The power of the highest derivative of the equation, for example, $\left(\dfrac{dy}{dx}\right)^4 - \left(\dfrac{d^2y}{dx^2}\right)^3 + y = 0$ is of degree 3 (or is a third-degree differential equation). (p. 359)

de Moivre's theorem The identity for complex numbers states that $[\cos(\theta) + i\sin(\theta)]^n = \cos(n\theta) + i\sin(n\theta)$; useful for finding powers and roots of complex numbers. (p. 167)

De Morgan's laws The rules $\neg(P \vee Q) \equiv \neg P \wedge \neg\, Q$ and $\neg(P \wedge Q) \equiv \neg P \vee \neg\, Q$. (p. 125)

dependent variable A variable that is calculated from another variable according to a rule or equation. The variable on the vertical scale of a graph. For $y = f(x)$, y is the dependent variable. (p. 378)

differential equation An equation that contains one or more derivatives, for example, $\dfrac{d^2x}{dt^2} - \dfrac{dx}{dt} + 4 = 0$. (p. 359)

dihedral angle The angle between intersecting planes, same as the angle between their normals. (p. 265)

dilation Stretching or squashing (compressing) of a graph, either vertically or horizontally. (p. 59)

direct proof A series of statements given or implied that are used to prove the last statement. (p. 128)

direction field *See* **slope field**.

discrete mathematics A branch of mathematics that examines countable and distinct objects. (p. 137)

discriminant $\Delta = b^2 - 4ac$, the part of the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ that gives information about the number and type of roots of the quadratic equation, $ax^2 + bx + c = 0$. (p. 175)

displacement The change of position of an object from its start position to its end position (its 'signed distance'), represented by the function $x(t)$ or $\underset{\sim}{r}(t)$, where displacement is a function of time. (p. 423)

displacement vector The vector from one point to another. The change from position A to position B is written as $\overline{AB}$. (p. 254)

equation of a plane An equation that specifies the points in a plane, written in vector form or Cartesian form. (p. 262)

equations of kinematics Displacement (s) and velocity (v) equations for motion in a straight line with constant acceleration (a).

$v = u + at,\ s = ut + \dfrac{1}{2}at^2,\ v^2 = u^2 + 2as,\ s = \dfrac{1}{2}(u + v)t,$

where t is time, u is initial velocity. (p. 448)

Euler's method A numerical approach that uses an initial value to solve a first-order first-degree differential equation. (p. 406)

existential quantifier The symbol $\exists$, meaning 'there exists', which shows that the formula is true for at least one value of the quantity involved. (p. 129)

factor theorem If $z - a$ is a factor of the polynomial $P(z)$, then $P(a) = 0$. (p. 176)

fundamental theorem of algebra A (non-constant) polynomial equation has at least one complex root. (p. 177)

gradient field *See* **slope field**.

gravitational acceleration The constant acceleration of approximately 9.8 m s^{-2} due to the effects of Earth's gravitational force on a falling object. (p. 450)

growth and decay A first-order differential equation where the rate of change of the dependent variable is proportional to the dependent variable. If the proportionality constant is positive, the rate of change represents growth, and if the proportionality constant is negative, the rate of change represents decay or decline. (p. 385)

imaginary numbers Numbers of the form yi, where i is the imaginary number, $i = \sqrt{-1}$. (p. 149)

Im (*z*) The imaginary part of the complex number $x + yi$. $\operatorname{Im}(z) = y$. (p. 149)

implicit differentiation A method of differentiation for a function such as $3xy + y^2 = 2$, where the subject of the function (such as y in the above function) is not on its own. It involves using the chain rule. (p. 238)

indefinite integral An integral of the form $\int f(x)\,dx$, which is an anti-derivative function and is read 'the integral of $f(x)$ with respect to x'. (p. 278)

independent variable A variable that is used to calculate the value of another variable. The variable on the horizontal scale of a graph. For $y = f(x)$, x is the independent variable. (p. 360)

inductive reasoning A method to reach a general conclusion from premises containing specific evidence. (p. 115) *See also* **deductive reasoning**.

inference A conclusion that can be drawn on the basis of evidence and reasoning. (p. 113)

inflow rate/outflow rate The rate at which a product such as liquid or gas enters a contained system such as a tank and described by a first-order differential equation. The exit rate of the product is the outflow rate. (p. 382)

initial conditions A set of given coordinates that are used to determine one or more constants of integration or other constants. In Euler's method, it is the first (starting) approximation in the set of approximations for solving the differential equation. (p. 407)

integrand The expression that is being integrated. (p. 275)

integration by parts An integration technique such that for the functions $u(x)$ and $v(x)$: $\int u\frac{dv}{dx}\,dx = u.v - \int v\frac{du}{dx}\,dx$. (p. 303)

integration by substitution A method of integration for the product of functions where one of the functions is the derivative of a part of the other function, for example, for $\int (2x+1)\sqrt{x^2+x}\,dx$. (p. 286)

interval estimate of a parameter An interval that is likely to include the value of the parameter, such as a confidence interval. (p. 519)

inverse circular functions The functions that are the inverse of the sine, cosine and tangent functions, namely the **inverse sine**, **inverse cosine** and **inverse tangent** functions respectively. (p. 89) *See* below.

inverse cosine (or **arccosine**) Abbreviation: $\cos^{-1}$. The inverse function to the cosine function. $y = \arccos(x)$ if and only if $\cos(y) = x$ and $0 \le y \le \pi$. (p. 89)

inverse sine (or **arcsine**) Abbreviation: $\sin^{-1}$. The inverse function to the sine function. $y = \arcsin(x)$ if and only if $\sin(y) = x$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. (p. 89)

inverse tangent (or **arctangent**) Abbreviation: $\tan^{-1}$. The inverse function to the tangent function. $y = \arctan(x)$ if and only if $\tan(y) = x$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. (p. 89)

inversely proportional to If y is inversely proportional to x, then $y = \frac{k}{x}$, where k is a constant. (p. 360)

kinematics The study of the motion of an object without reference to any acting forces. (p. 423)
See also **equations of kinematics**.

limiting velocity *See* **terminal velocity**.

linear dependence of vectors If $\underset{\sim}{c}$ can be written as a linear combination of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ (i.e. $\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$, where m and n are not 0), then $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are linearly dependent. Otherwise they are linearly independent. (p. 8)

locus A set of points that satisfies a given condition, for example, equidistant from a point and a line. (p. 39)

logic A system of reasoning that allows conclusions to be drawn according to a set of rules. (p. 113)

logical argument A set of premises and inferences that allow a conclusion to be drawn. (p. 114)

logical connectives (logical operators) Symbols in logic that are used to connect or negate statements. (p. 122)

logically equivalent A relationship between two statements where they have the same truth value. (p. 124)

logistic model A model for population growth that takes into account the growth rate and carrying capacity of the population. (p. 383)
See also **carrying capacity**.

logistic population model A realistic model of population growth described by the equation $\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$, which takes into account the maximum possible population, K, that can be sustained. (p. 383)

margin of error In a confidence interval, the difference between the mean and the extremes. For the confidence interval $(\mu - E, \mu + E)$, the margin of error is E. (p. 520)

mathematical induction *See* **proof by mathematical induction**.

modulus The distance between a point on an Argand diagram representing a complex number $x + yi$, and the origin, written $|z|$. $|z| = \sqrt{x^2 + y^2}$. (p. 73)
See also **absolute value**.

modulus-argument form *See* **polar form**.

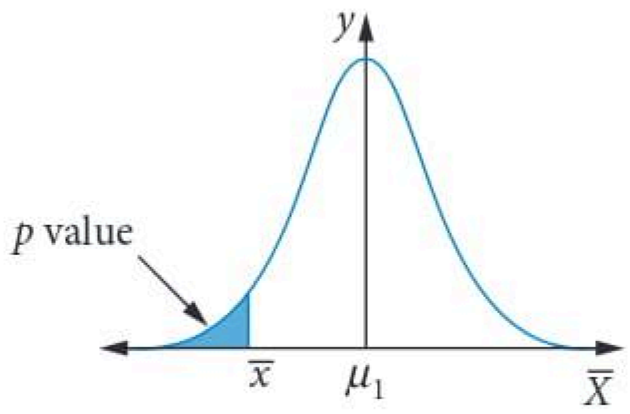
necessary and sufficient condition Conditions that show the biconditional statement $P \leftrightarrow Q$ to be true. A 'necessary condition' is required to show $P \rightarrow Q$, and a 'sufficient condition' is needed to show $Q \rightarrow P$. (p. 123)

Newton's law of cooling The rate at which an object loses heat is proportional to the temperature difference between the body and its surroundings. $\frac{dT}{dt} = -k(T - T_s)$ (p. 379)

normal (in vector geometry) A line that is perpendicular to a plane. (p. 258)

null hypothesis (symbol: H_0) A statement about a population parameter that is assumed to be true, usually claiming no effect or no difference. (p. 529)
See also **alternative hypothesis**.

one-tailed test A test used in hypothesis testing when the alternative hypothesis involves > or <. (p. 533)
See also **two-tailed test**.



open conjecture A conjecture that has been proposed but no formal proof has been provided. (p. 116)

order (of a differential equation) The highest derivative in the equation, for example, $\frac{d^2x}{dt^2} - \frac{dx}{dt} + 4 = 0$ is of order 2 (or a second-order differential equation). (p. 359)

***p*-value** The probability of obtaining a value in the sample or a more extreme value assuming the null hypothesis is true. (p. 530)

parameter (in statistics) A characteristic value of a particular population, such as the mean. (p. 519)

parameter (in algebra) An independent variable such as t or θ that is used in the parametric form of the equation of a graph, upon which the variables x and y are defined. (p. 251)

parametric equations Equations expressed in terms of a parameter. (p. 43)

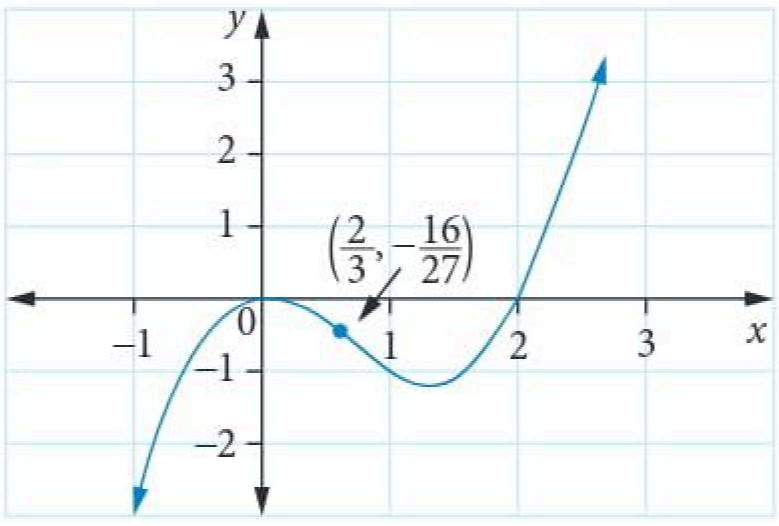
partial fraction One of the fractions when a rational function is written as the sum or difference of fractions, for example, in $\frac{5x+1}{(x-1)(x+2)} = \frac{2}{x-1} + \frac{3}{x+2}$ the partial fractions are $\frac{2}{x-1}$ and $\frac{3}{x+2}$. (p. 53)

particular solution The solution to a definite integral that includes the value of the constant of integration. (p. 373)

plane A flat surface that extends in all directions, such as a number plane. (p. 258)

point estimate of a parameter A statistical value obtained from a sample that estimates the parameter, such as a sample mean. (p. 519)

point of inflection A point on the graph where the concavity changes and where $\frac{d^2y}{dx^2} = 0$. (p. 227)



polar form (or **modulus-argument form** or **trigonometric form**) A form of a complex number expressed in terms of its modulus r, the length of its vector in the Argand diagram, and its principal argument θ, the angle the vector makes with the positive direction of the x-axis. (p. 45)

position–time line A diagrammatic way of depicting the position, direction and time taken of an object's motion. (p. 424)

predicate A statement which is taken to be true in the context of the logical argument. (p. 129)

premise A statement taken to be true which is used to form a conclusion. (p. 113)

principal argument *See* **argument**.

product rule A formula for finding the derivative of the product of two functions: $\frac{d}{dx}(uv) = u'v + uv'$. (p. 204)

projectile motion Motion in which a body is travelling under the influence of gravity. (p. 484)

proof by contradiction A method of proof that begins by assuming the opposite of what is required to be proven so as to lead to a contradiction. (p. 134)

proof by contrapositive A form of proof where to prove $P \rightarrow Q$ involves showing $\neg Q \rightarrow \neg P$. (p. 134)

proof by mathematical induction A form of mathematical proof that involves proving that if a conjecture is true for some integer $n = k$, then it will also be true for the next consecutive number, $n = k + 1$. (p. 137)

proportional to If y is proportional to x, then $y = kx$, where k is a constant. (p. 360)

proposition A statement proposing a concept that can be true or false. (p. 113)

quadratic formula The formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ that gives the solutions of the quadratic equation $ax^2 + bx + c = 0$. (p. 175)

quantifier The symbols $\forall$ and $\exists$ that are used to express the quantity in terms of 'there exists' and 'for every'. (p. 129)

quotient The answer to a division. (p. 51)

quotient rule A formula for finding the derivative of the ratio of two functions: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$. (p. 205)

rational function A function of the form $f(x) = \frac{p(x)}{q(x)}$, where p and q are polynomials and q is not the zero polynomial. (p. 50)

Re (*z*) The real part of the complex number $x + yi$. Re $(z) = x$. (p. 149)

realise the denominator To convert a complex expression of the form $\frac{z}{w}$ so that its denominator is real, by multiplying by $\frac{\bar{w}}{\bar{w}}$, where $\bar{w}$ is the conjugate of the denominator. (p. 150)

reciprocal functions The functions that are the reciprocals of the sine, cosine and tangent functions, namely the cosecant, secant and cotangent functions respectively. $\operatorname{cosec}(\theta) = \frac{1}{\sin(\theta)}$, $\sec(\theta) = \frac{1}{\cos(\theta)}$, $\cot(\theta) = \frac{1}{\tan(\theta)}$. (p. 51)

rectilinear motion Straight line motion of an object. (p. 423)

reflection Mirror-image or 'flipping' of a graph so that it is back-to-front or bottom-to-top. (p. 78)

resultant vector The vector that is the sum of 2 or more vectors. (p. 23)

retardation Negative acceleration. (p. 448)

roots of unity The solutions to the complex equation $z^n = 1$, the roots of 1. (p. 170)

scalar A quantity that has magnitude (size) but not direction. In comparison, a **vector** has both magnitude and direction. (p. 13)

scalar product (or **dot product**) The product of two vectors as a scalar (a real number), not a vector. $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta)$, where θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$. (p. 13)

scalar projection (or **scalar resolute**) The (scalar) magnitude of a vector projection (but not the direction). (p. 16)

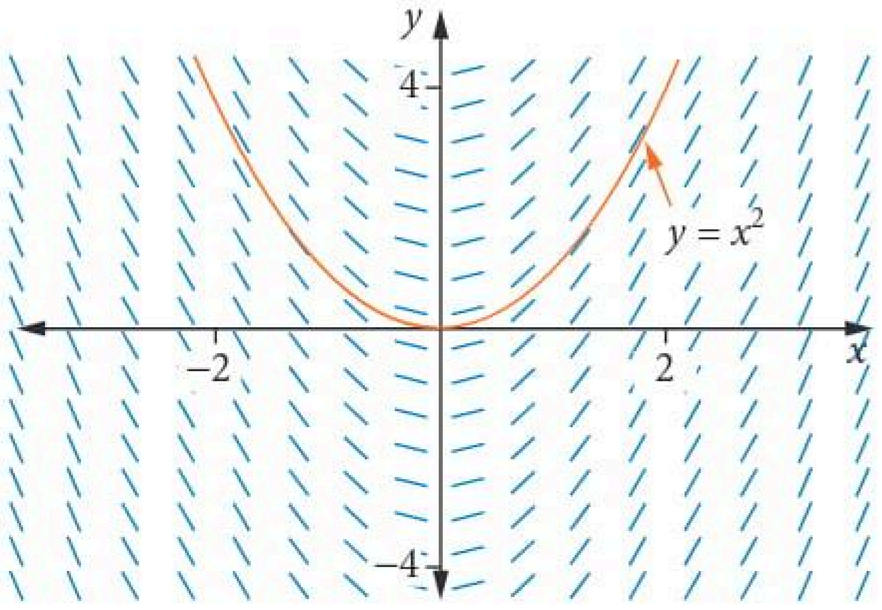
secant Abbreviation: sec. The reciprocal of cosine. $\sec(\theta) = \frac{1}{\cos(\theta)}$. (p. 76)

second derivative The derivative of the derivative of a function $y = f(x)$, written as $f''(x)$ or $\frac{d^2y}{dx^2}$, which indicates the concavity of the graph of the function. If $f''(x) > 0$, the graph is **concave up.** If $f''(x) < 0$, the graph is **concave down**. (p. 224)

separation of variables A method used to separate the product of two functions and to obtain two integrands, each of which can be integrated. (p. 392)
See also **integrand**.

significance level (symbol: α) The level at which a null hypothesis is rejected, for example, if $p < 0.05$, the observed difference is 'significant' and not due to chance. (p. 531)

slope field (or **gradient field** or **direction field**) A graph that displays a family of solutions to a differential equation for various values of the constant. (p. 397)



standard error The standard deviation of the sample means. (p. 515)

statement A sentence that is true or false. (p. 113)

statistic An estimate of a population parameter found from a sample. (p. 516)

step size In Euler's method, the difference between any two consecutive x values that are used to obtain the sequence of iterations that approximate the solution to the differential equation. (p. 406)

sum and difference identities (or **compound angle formulas**) The formulas for the trigonometric ratios of sums and differences of angles.

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

$$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

$$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$

$$\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$$

$$\tan(x-y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)} \text{ (p. 85)}$$

terminal velocity The predicted limiting velocity of a falling object when the time taken for its motion is considered to be infinite. (p. 440)

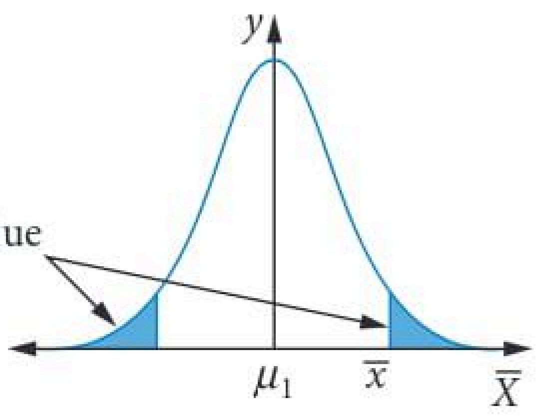
theorem A statement that is always true. (p. 166)

translation Shifting of a graph parallel to the x-axis (horizontally) or y-axis (vertically). (p. 78)

trigonometric identities (or **Pythagorean identities**) For any value x: $\sin^2(x) + \cos^2(x) = 1$, $\tan^2(x) + 1 = \sec^2(x)$, $\cot^2(x) + 1 = \text{cosec}^2(x)$. (p. 85)

truth table A table that summarises logical statements in terms of true/false. (p. 125)

two-tailed test A test used in hypothesis testing when the alternative hypothesis involves $\neq$ rather than $>$ or $<$.
See also **one-tailed test**. (p. 533)



type I error The error of rejecting a null hypothesis when it is true, sometimes called a false positive. (p. 544)

type II error The error of not rejecting a null hypothesis when it is false, sometimes called a false negative. (p. 544)

unit vector A vector that has a magnitude of 1. The unit vector heading in the direction of $\underset{\sim}{a}$ has the notation $\underset{\sim}{\hat{a}}$. (p. 4)

universal quantifier The symbol $\forall$, meaning 'every' or 'for all', indicating that the formula is true for all values of the quantity involved. (p. 129)

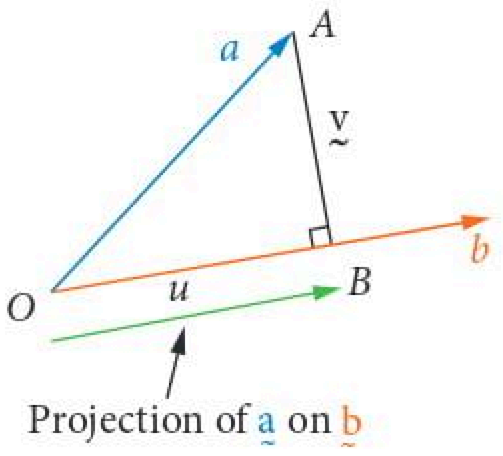
variable acceleration Acceleration that is not constant, such as the acceleration of a falling object experiencing air resistance. (p. 456)

vector A quantity that has both magnitude (size) and direction. In comparison, a **scalar** has magnitude only (and is a number). Examples of vector notation are $\underset{\sim}{a}$ or $\overrightarrow{AB}$. (p. 3)

vector equation An equation written as a vector or involving a vector or vectors, such as the vector equation of a line. (p. 251)

vector product (or **cross product**) The product of two vectors as a vector, not a scalar. $\underset{\sim}{a} \times \underset{\sim}{b}$ is perpendicular to both vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ and has magnitude $|\underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta)$, where θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$. (p. 19)

vector projection (or **vector resolute**) A vector that is the component of a main vector in the direction of another vector. For example, the vector projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ (or in the direction of $\underset{\sim}{b}$) is shown on the diagram. It is like the shadow of $\underset{\sim}{a}$ on $\underset{\sim}{b}$. (p. 16)



velocity The 'signed speed' of a moving object, represented by the function $v = \frac{dx}{dt} = \dot{x}$, the rate of change of the displacement, x. Conversely, $x = \int v(t)\,dt$. (p. 423)

velocity–time graph A graph of velocity against time that can be used to determine the distance/displacement by finding the area between the function and the horizontal axis. (p. 436)

verify a solution by substitution Substitute an expression into an equation and show that the left side has the same value as the right side. (p. 361)