AMSI AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

ICE-EM MATHEMATICS 10 &10A

THIRD EDITION

Peter Brown
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David Hunt
Robert McLaren
Bill Pender
Brian Woolacott

INTERNATIONAL CENTRE OF EXCELLENCE FOR EDUCATION IN MATHEMATICS

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Contents

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Cambridge University Press
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Preface

ICE-EM Mathematics Third Edition is a series of textbooks for students in years 5 to 10 throughout Australia who study the Australian Curriculum and its state variations.

The program and textbooks were developed in recognition of the importance of mathematics in modern society and the need to enhance the mathematical capabilities of Australian students. Students who use the series will have a strong foundation for work or further study.

Background

The International Centre of Excellence for Education in Mathematics (ICE-EM) was established in 2004 with the assistance of the Australian Government and is managed by the Australian Mathematical Sciences Institute (AMSI). The Centre originally published the series as part of a program to improve mathematics teaching and learning in Australia. In 2012, AMSI and Cambridge University Press collaborated to publish the Second Edition of the series to coincide with the introduction of the Australian Curriculum, and we now bring you the Third Edition.

The series

ICE-EM Mathematics Third Edition provides a progressive development from upper primary to middle secondary school. The writers of the series are some of Australia's most outstanding mathematics teachers and subject experts. The textbooks are clearly and carefully written, and contain background information, examples and worked problems.

For the Third Edition, the series has been carefully edited to present the content in a more streamlined way without compromising quality. There is now one book per year level and the flow of topics from chapter to chapter and from one year level to the next has been improved.

The year 10 textbook incorporates all material for the 10A course, and selected topics in earlier books carefully prepare students for this. *ICE-EM Mathematics Third Edition* provides excellent preparation for all of the Australian Curriculum's year 11 and 12 mathematics courses.

For the Third Edition, *ICE-EM Mathematics* now comes with an Interactive Textbook: a cutting-edge digital resource where all textbook material can be answered online (with students' working-out), additional quizzes and features are included at no extra cost. See 'The Interactive Textbook and Online Teaching Suite' on page xiii for more information.

Author biographies

Peter Brown

Peter Brown studied Pure Mathematics and Ancient Greek at Newcastle University, and completed postgraduate degrees in each subject at the University of Sydney. He worked for nine years as a mathematics teacher in NSW State schools. Since 1990, he has taught Pure Mathematics at the School of Mathematics and Statistics at the University of New South Wales (UNSW). He was appointed Director of First Year Studies at UNSW from 2011 to 2015. He specialises in Number Theory and History of Mathematics and has published in both areas. Peter regularly speaks at teacher inservices, Talented Student days and Mathematics Olympiad Camps. In 2008 he received a UNSW Vice Chancellor's Teaching Award for educational leadership.

Michael Evans

Michael Evans has a PhD in Mathematics from Monash University and a Diploma of Education from La Trobe University. He currently holds the honorary position of Senior Consultant at the Australian Mathematical Sciences Institute at the University of Melbourne. He was Head of Mathematics at Scotch College, Melbourne, and has also taught in public schools and in recent years has returned to classroom teaching. He has been very involved with curriculum development at both state and national levels. In 1999, Michael was awarded an honorary Doctor of Laws by Monash University for his contribution to mathematics education, and in 2001 he received the Bernhard Neumann Award for contributions to mathematics enrichment in Australia.

Garth Gaudry

Garth Gaudry was Head of Mathematics at Flinders University before moving to UNSW, where he became Head of School. He was the inaugural Director of AMSI before he became the Director of AMSI's International Centre of Excellence for Education in Mathematics. Previous positions include membership of the South Australian Mathematics Subject Committee and the Eltis Committee appointed by the NSW Government to enquire into Outcomes and Profiles. He was a life member of the Australian Mathematical Society and Emeritus Professor of Mathematics, UNSW.

David Hunt

David Hunt graduated from the University of Sydney in 1967 with an Honours degree in Mathematics and Physics, then obtained a master's degree and a doctorate from the University of Warwick. He was appointed to a lectureship in Pure Mathematics at UNSW in early 1971, where he is currently an honorary Associate Professor. David has taught courses in Pure Mathematics from first year to master's level and was Director of First Year Studies in Mathematics for five years. Many of David's activities outside UNSW have centred on the Australian Mathematics Trust. These contributions as well as those to the International Mathematical Olympiad movement were recognised by the award of the Paul Erdos medal in 2016.

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Robert McLaren

Robert McLaren graduated from the University of Melbourne in 1978 with a Bachelor of Science (Hons) and a Diploma of Education. He commenced his teaching career in 1979 at The Geelong College and has taught at a number of Victorian Independent Schools throughout his career. He has been involved in textbook writing, curriculum development and VCE examination setting and marking during his teaching life. He has taught mathematics at all secondary levels and has a particular interest in problem solving. Robert is currently Vice Principal at Scotch College in Melbourne.

Bill Pender

Bill Pender has a PhD in Pure Mathematics from Sydney University and a BA (Hons) in Early English from Macquarie University. After a year at Bonn University, he taught at Sydney Grammar School from 1975 to 2008, where he was Subject Master for many years. He has been involved in the development of NSW Mathematics syllabuses since the early 1990s, and was a foundation member of the Education Advisory Committee of AMSI. He has also lectured and tutored at Sydney University and at UNSW, and given various inservice courses. Bill is the lead author of the NSW calculus series *Cambridge Mathematics.*

Brian Woolacott

Brian Woolacott graduated from the University of Melbourne in 1978 with a Bachelor of Science and a Diploma of Education. In 1979 he started his teaching career at Scotch College, Melbourne, and during his career he has taught at all secondary levels. For 13 years, Brian was the Co-ordinator of Mathematics for Years 9 and 10, and during this time he was involved in co-authoring a number of textbooks for the Year 9 and 10 levels. Brian is currently the Dean of Studies at Scotch College.

How to use this resource

The textbook

Each chapter in the textbook addresses a specific Australian Curriculum content strand and set of sub-strands. The exercises within chapters take an integrated approach to the concept of proficiency strands, rather than separating them out. Students are encouraged to develop and apply Understanding, Fluency, Problem-solving and Reasoning skills in every exercise.

The series places a strong emphasis on understanding basic ideas, along with mastering essential technical skills. Mental arithmetic and other mental processes are major focuses, as is the development of spatial intuition, logical reasoning and understanding of the concepts.

Problem-solving lies at the heart of mathematics, so *ICE-EM Mathematics* gives students a variety of different types of problems to work on, which help them develop their reasoning skills. Challenge exercises at the end of each chapter contain problems and investigations of varying difficulty that should catch the imagination and interest of students. Further, two 'Review and Problem-solving' chapters in each 7–10 textbook contain additional problems that cover new concepts for students who wish to explore the subject even further.

The Interactive Textbook and Online Teaching Suite

Included with the purchase of the textbook is the Interactive Textbook. This is the online version of the textbook and is accessed using the 16-character code on the inside cover of this book.

The Online Teaching Suite is the teacher version of the Interactive Textbook and contains all the support material for the series, including tests, worksheets, skillsheets, curriculum documentation and more.

For more information on the Interactive Textbook and Online Teaching Suite, see page xiii.

The Interactive Textbook and Online Teaching Suite are delivered on the *Cambridge HOTmaths* platform, providing access to a world-class Learning Management System for testing, task management and reporting. They do not provide access to the *Cambridge HOTmaths* stand-alone resource that you or your school may have used previously. For more information on this resource, contact Cambridge University Press.

AMSI's TIMES and SAM modules

The TIMES and SAM web resources were developed by the *ICE-EM Mathematics* author team at AMSI and are written around the structure of the Australian Curriculum. These resources have been mapped against your *ICE-EM Mathematics* book and are available to teachers and students via the AMSI icon on the Dashboard of the Interactive Textbook and Online Teaching Suite.

The Interactive Textbook and the Online Teaching Suite

Interactive Textbook

The Interactive Textbook is the online version of the print textbook and comes included with purchase of the print textbook. It is accessed by first activating the code on the inside cover. It is easy to navigate and is a valuable accompaniment to the print textbook.

Students can show their working

All textbook questions can be answered online within the Interactive Textbook. Students can show their working for each question using either the Draw tool for handwriting (if they are using a device with a touch-screen), the Type tool for using their keyboard in conjunction with the pop-up symbol palette, or by importing a file using the Import tool.

Once a student has completed an exercise they can save their work and submit it to the teacher, who can then view the student's working and give feedback to the student, as they see appropriate.

Auto-marked quizzes

The Interactive Textbook also contains material not included in the textbook, such as a short auto-marked quiz for each section. The quiz contains 10 questions which increase in difficulty from question 1 to 10 and cover all proficiency strands. There is also space for the student to do their working underneath each quiz question. The auto-marked quizzes are a great way for students to track their progress through the course.

Additional material for Year 5 and 6

For Years 5 and 6, the end-of-chapter Challenge activities as well as a set of Blackline Masters are now located in the Interactive Textbook. These can be found in the 'More resources' section, accessed via the Dashboard, and can then easily be downloaded and printed.

Online Teaching Suite

The Online Teaching Suite is the teacher's version of the Interactive Textbook. Much more than a 'Teacher Edition', the Online Teaching Suite features the following:

- The ability to view students' working and give feedback – When a student has submitted their work online for an exercise, the teacher can view the student's work and can give feedback on each question.
- For Years 5 and 6, access to Chapter tests, Blackline Masters, Challenge exercises, curriculum support material, and more.
- For Years 7 to 10, access to Pre-tests, Chapter tests, Skillsheets, Homework sheets, curriculum support material, and more.
- A Learning Management System that combines task-management tools, a powerful test generator, and comprehensive student and whole-class reporting tools.

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We gratefully acknowledge the major contributions made by those schools that participated in the Pilot Program during the development of the *ICE-EM Mathematics* program.

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CHAPTER

1

Number and Algebra

Consumer arithmetic

This chapter reviews some important practical financial topics, such as investing and borrowing money, income tax and GST, inflation, depreciation, profits and losses, discounts and commissions. Formulas for compound interest and depreciation are introduced.

Everything in this chapter requires calculations with percentages. We are assuming that you are using a calculator, so we have made little attempt to set questions where the numbers work out nicely.

Nevertheless, you should always look over your work and check that the answers to your calculations are reasonable and sensible.

When the calculator displays numbers with many decimal places, you will need to round the answer in some way that is appropriate in the context of the question. This is an important skill in everyday life.

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1A Review of percentages

We first review the calculation techniques involving percentages, which you have learned in previous years.

- To convert a percentage to a decimal, move the decimal point two places to the left. For example:

  $27\% = 0.27$

- To convert a percentage to a fraction, multiply by $\frac{1}{100}$. For example:

  $27\% = \frac{27}{100}$ and $2\frac{1}{2}\% = \frac{2\frac{1}{2}}{100} = \frac{5}{200} = \frac{1}{40}$

- To convert a decimal or a fraction to a percentage, multiply by 100%. For example:

  $$\begin{aligned} 0.35 &= 0.35 \times 100\% \\ &= 35\% \end{aligned} \quad \text{and} \quad \begin{aligned} \frac{3}{5} &= \frac{3}{5} \times 100\% \\ &= 60\% \end{aligned}$$

- To find a percentage of a quantity, convert the percentage to a decimal or a fraction, and then multiply the quantity by it. For example:

  $$\begin{aligned} 3.5\% \text{ of } 1250 &= 1250 \times 0.035 \\ &= 43.75 \end{aligned} \quad \text{or} \quad \begin{aligned} 3.5\% \text{ of } 1250 &= 1250 \times \frac{35}{1000} \\ &= 43\tfrac{3}{4} \end{aligned}$$

- To calculate the percentage that one quantity, a, is of another quantity, b:
  - first convert both quantities to the same unit of measurement
  - then form the fraction $\frac{a}{b}$ and multiply it by 100%.

  For example, to express 32 cm as a percentage of 2.4 m:

  First, write 2.4 m = 240 cm

  Then $\frac{32}{240} \times \frac{100}{1}\% = 13\frac{1}{3}\%$

  So 32 cm is $13\frac{1}{3}\%$ of 2.4 m

Finding the original amount

We now introduce another important method that will be used with percentages throughout this chapter.

- To find the original amount, given 12% of it, divide by 12%.

Example 1

Ken saves 12% of his after-tax salary every week. If he saves \$108 a week, what is his after-tax salary?

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Solution

Savings $=$ after-tax salary $\times$ 12%

Reversing this:

$$\begin{aligned}\text{After-tax salary} &= \text{savings} \div 12\% \\ &= \text{savings} \div 0.12 \qquad \text{(Replace 12\% by 0.12.)} \\ &= 108 \div 0.12 \\ &= \$900\end{aligned}$$

This technique of writing the percentage factor on the right and reversing the process using division is needed in many practical situations. It will be applied throughout this chapter to commissions, profit and loss, income tax and interest.

Commission

A **commission** is a fee that is charged by an agent who sells goods or services on behalf of someone else. The person who owns the goods or services is called the vendor, and the commission charged is usually determined as a percentage of the selling price.

Example 2

The Dandy Bay Gallery charges a commission of 8.6% on the selling price.

a An antique vase was sold recently for \$18 000. How much did the gallery receive, and how much was left for the vendor?

b The gallery received a commission of \$215 for selling a painting. What was the selling price of the painting, and what did the vendor actually receive?

Solution

a

$$\begin{aligned}\text{Commission} &= 18\,000 \times 8.6\% \\ &= 18\,000 \times 0.086 \\ &= \$1548\end{aligned}$$

$$\begin{aligned}\text{Amount received by vendor} &= 18\,000 - 1548 \\ &= \$16\,452\end{aligned}$$

b

Commission $=$ selling price $\times$ 8.6%

Reversing this:

$$\begin{aligned}\text{Selling price} &= \text{commission} \div 8.6\% \\ &= 215 \div 0.086 \\ &= \$2500\end{aligned}$$

$$\begin{aligned}\text{Amount received by vendor} &= 2500 - 215 \\ &= \$2285\end{aligned}$$

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Profit and loss as percentages

Is an annual profit of \$20 000 a great performance or a modest performance? For a business with annual sales of \$100 000, such a profit would be considered very large. For a business with annual expenditure of \$100 000 000, however, it would be considered a very poor performance.

For this reason, it is often convenient to express profit and loss as percentages of the total costs.

Example 3

The owners of Budget Shoe Shop spent \$6 600 000 last year buying shoes and paying salaries and other expenses. They made a 2% profit on these costs.

a What was their profit last year?

b What was the total of their sales?

c In the previous year, their costs were \$5 225 000 and their sales were only \$5 145 000. What percentage loss did they make on their costs?

d Two years ago their costs were \$5 230 000 and their sales were \$6 125 000. What percentage profit did they make on their costs?

Solution

a $$\begin{aligned}\text{Profit} &= 6\,600\,000 \times 2\% \\ &= 6\,600\,000 \times 0.02 \\ &= \$132\,000\end{aligned}$$

b $$\begin{aligned}\text{Total sales} &= \text{total costs} + \text{profit} \\ &= 6\,600\,000 + 132\,000 \\ &= \$6\,732\,000\end{aligned}$$

c $$\begin{aligned}\text{Last year, loss} &= \text{total costs} - \text{total sales} \\ &= 5\,225\,000 - 5\,145\,000 \\ &= \$80\,000\end{aligned}$$

$$\begin{aligned}\text{Percentage loss} &= \frac{80\,000}{5\,225\,000} \times \frac{100}{1}\% \\ &\approx 1.53\% \text{ (Correct to the nearest 0.01\%.)}\end{aligned}$$

$$\left[\begin{aligned}\text{Alternatively, profit} &= \text{total sales} - \text{total costs} \\ &= 5\,145\,000 - 5\,225\,000 \\ &= -\ \$80\,000 \\ \text{Percentage change} &\approx -1.53\% \\ &= 1.53\% \text{ loss}\end{aligned}\right]$$

d $$\begin{aligned}\text{Profit} &= \text{total sales} - \text{total costs} \\ &= 6\,125\,000 - 5\,230\,000 \\ &= \$895\,000\end{aligned}$$

$$\begin{aligned}\text{Percentage profit} &= \frac{895\,000}{5\,230\,000} \times \frac{100}{1}\% \\ &\approx 17.11\% \text{ (Correct the nearest 0.01\%.)}\end{aligned}$$

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Example 4

Andrew's paint shop made a profit of 6.4% on total costs last year. If the actual profit was \$87 000, what were the total costs, and what were the total sales?

Solution

$\text{Profit} = \text{costs} \times 6.4\%$

$$\begin{aligned}\text{Reversing this, costs} &= \text{profit} \div 6.4\% \\ &= 87\,000 \div 0.064 \\ &= \$1\,359\,375\end{aligned}$$

$$\begin{aligned}\text{Hence, total sales} &= \text{profit} + \text{costs} \\ &= 87\,000 + 1\,359\,375 \\ &= \$1\,446\,375\end{aligned}$$

Income tax

Income tax rates are often **progressive**. This means that the more you earn, the higher the rate of tax you pay on each extra dollar earned.

Australian income tax rates are progressive, but they often change, so here is an example using the rates of the fictional nation of Plusionta, where taxation rates have not changed for many years.

Example 5

Income tax in the fictional nation of Plusionta is calculated as follows.

- There is no tax on the first \$12 000 that a person earns in any one year.
- From \$12 001 to \$30 000, the tax rate is 15c for each dollar over \$12 000.
- From \$30 001 to \$75 000, the tax rate is 25c for each dollar over \$30 000.
- For incomes exceeding \$75 000, the tax rate is 35c for each dollar over \$75 000.

Find the income tax payable by a person whose taxable income for the year is:

a \$10 500 **b** \$26 734 **c** \$72 000 **d** \$455 000

Solution

a There is no tax.

b Tax on first \$12 000 = \$0

Tax on remaining \$14 734

$= 14\,734 \times 0.15$

$= \$2210.10$

This is the total tax payable.

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c Tax on first \$12 000 = \$0

Tax on next \$18 000
$= 18\,000 \times 0.15$
$= \$2700$

Tax on remaining \$42 000
$= 42\,000 \times 0.25$
$= \$10\,500$

$$\begin{aligned}\text{Total tax} &= 2700 + 10\,500\\ &= \$13\,200\end{aligned}$$

d Tax on first \$12 000 = \$0

Tax on next \$18 000 = \$2700

Tax on next \$45 000
$= 45\,000 \times 0.25$
$= \$11\,250$

Tax on remaining \$380 000
$= 380\,000 \times 0.35$
$= \$133\,000$

$$\begin{aligned}\text{Total tax} &= 2700 + 11\,250 + 133\,000\\ &= \$146\,950\end{aligned}$$

Simple interest

When money is lent by a bank or other lender, whoever borrows the money normally makes a payment, called **interest**, for the use of the money.

The amount of interest paid depends on:

- the **principal**, which is the amount of money borrowed
- the **rate** at which interest is charged
- the **time** for which the money is borrowed.

This section will deal only with **simple interest**. In simple interest transactions, interest is paid on only the original amount borrowed.

Conversely, if a person invests money in a bank or elsewhere, the bank pays the person interest because the bank uses the money to finance its own investments.

Formula for simple interest

Suppose that I borrow \$$P$ for T years at an interest rate of R per annum.

$$\begin{aligned}\text{Interest paid at the end of each year} &= P \times R\\ \text{Total interest, \$}I\text{, paid over } T \text{ years} &= P \times R \times T\\ &= PRT\end{aligned}$$

This gives us the well-known **simple interest formula**.

$$I = PRT \qquad (\text{Interest} = \text{principal} \times \text{rate} \times \text{time})$$

Note: The interest rate is normally given per year, so the time must also be written in years. In some books, R is written as $r\%$.

'Per annum' means 'per year'. It will sometimes be abbreviated to 'p.a.'.

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Example 6

Find the simple interest on $16 000 for eight years at 7.5% p.a.

Solution

$$\begin{aligned} I &= PRT \\ &= 16\,000 \times 7.5\% \times 8 \\ &= 16\,000 \times 0.075 \times 8 \\ &= 9600 \end{aligned}$$

Thus, the simple interest is $9600.

Reverse use of the simple interest formula

There are four pronumerals in the formula $I = PRT$. If any three are known, then substituting them into the simple interest equation allows the fourth to be found.

Example 7

John borrows $120 000 from his parents to put towards an apartment. His parents agree that John should only pay simple interest on what he borrows. Ten years later, John repays his parents $216 000, which includes simple interest on the loan. What was the interest rate?

Solution

$P = 120\,000$ and $T = 10$.

The total interest paid was $\$216\,000 - \$120\,000 = \$96\,000$, so $I = 96\,000$

$$\begin{aligned} I &= PRT \\ 96\,000 &= 120\,000 \times R \times 10 \\ R &= \frac{96\,000}{1\,200\,000} \times \frac{100}{1}\% \qquad \text{(Interest rates are normally written as percentages.)} \\ &= 8\% \end{aligned}$$

The interest rate was 8%.

Simple interest formula

- Suppose that a principal $\$P$ is invested for T years at an interest rate R p.a. Then the total interest $\$I$ is given by:

  $I = PRT$
- If the interest rate R is given per year, the time T must be given in years.
- The formula has four pronumerals. If any three are known, the fourth can be found by substitution and solving the resulting equation.

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Exercise 1A

1 Express each percentage as a decimal.

a 56% **b** 8.2% **c** 12% **d** 3.75%

e 215% **f** 0.8% **g** $88\frac{1}{4}\%$ **h** $\frac{7}{8}\%$

2 Express each percentage as a fraction in lowest terms.

a 45% **b** 64% **c** $67\frac{1}{2}\%$ **d** $66\frac{2}{3}\%$

e 8.25% **f** 5.6% **g** 120% **h** 150%

i 7.25% **j** $12\frac{3}{4}\%$ **k** $\frac{1}{2}\%$ **l** 7.8%

3 Express each fraction or decimal as a percentage.

a $\frac{4}{5}$ **b** $\frac{7}{8}$ **c** $\frac{7}{16}$ **d** $1\frac{1}{2}$

e $\frac{9}{20}$ **f** $\frac{5}{3}$ **g** 0.46 **h** 0.025

i 1.4 **j** 1.125 **k** 0.000 75 **l** $2\frac{1}{4}$

4 Copy and complete this table.

	Percentage	Fraction	Decimal
a	64%		
b		$\frac{3}{5}$	
c			0.16
d	20.5%		
e			1.4
f		$\frac{5}{8}$	

5 Evaluate each amount, correct to two decimal places.

a 15% of 60 **b** 36% of 524

c 120% of 436 **d** 140.5% of 720

e 3.8% of 73 **f** 0.5% of 220

6 Evaluate each amount, correct to the nearest cent where necessary.

a 52% of \$50 **b** 24.2% of \$1050

c 110% of \$1590 **d** 0.30% of \$900

e $8\frac{1}{4}\%$ of \$2000 **f** $\frac{3}{4}\%$ of \$1060

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7 Find what percentage the first quantity is of the second quantity, correct to one decimal place.

a 9 km, 150 km

b \$5, \$400

c 28 kg, 600 kg

d 80 m, 50 m

8 Find what percentage the first quantity is of the second quantity, correct to two decimal places. You will first need to express both quantities in the same unit.

a 48 cents, \$10.00

b 3.4 cm, 2 m

c 28 hours, 4 weeks

d 250 m, 8 km

e 40 km, 1250 m

f 1 day, 2 years

9 There are 640 students at a primary school, 7% of whom have red hair. Calculate the number of students in the school who have red hair.

10 A sample of a certain alloy weighs 2.6 g.

a Aluminium makes up 58% of the alloy. What is the weight of the aluminium in the sample?

b The percentage of lead in the alloy is 0.28%. What is the weight of the lead in the sample?

11 A soccer match lasted 94 minutes (including injury time). If Team A was in possession for 65% of the match, for how many minutes and seconds was Team A in possession?

12 A football club with 15 000 members undertook a membership drive, and the membership increased by 110%.

a How many new members joined the club?

b What is the size of the club's membership now?

Example 1

13 Find the original quantity, given that:

a 5% of it is \$24

b 30% of it is 72 minutes

c 90% of it is 216 cm

d 7% of it is \$15.26

e 0.5% of it is 4 mm

f 15% of it is 56 mm

14 Sometimes customers pay a deposit on an item and then later pay the rest of the full price. Find the full price when a deposit of \$570 is 30% of the full price.

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Example 2 **15** Find the selling price if the commission and the commission rate are as given.

a Commission \$46, rate 8%

b Commission \$724, rate 5.6%

Example 3 **16** Find the percentage profit or loss on costs in these situations.

a Costs \$26 000 and sales \$52 000

b Costs \$182 000 and sales \$150 000

Example 4 **17** **a** A company made a profit of \$28 000, which was a 5.4% profit on its costs. Find the costs and the total sales.

b A company made a loss of \$750 000, which was a 6.5% loss on its costs. Find the costs and the total sales.

Example 5 **18** This question uses the income tax rates in the fictional nation of Plusionta. They are:

- There is no tax on the first \$12 000 that a person earns in any one year.
- From \$12 001 to \$30 000, the tax rate is 15c for each dollar over \$12 000.
- From \$30 001 to \$75 000, the tax rate is 25c for each dollar over \$30 000.
- For incomes exceeding \$75 000, the tax rate is 35c for each dollar over \$75 000.

a Find the income tax payable on:

i \$9000 **ii** \$15 000 **iii** \$38 000 **iv** \$400 000

b What percentage of each person's income was paid in income tax in parts **i–iv** of part **a**?

c Find the income if the income tax on it was:

i \$1580 **ii** \$3860 **iii** \$15 200 **iv** \$15 000

Example 6 **19** \$20 000 is invested at 8% p.a. simple interest for five years.

a How much interest will be earned each year?

b Use the formula $I = PRT$ to find how much interest will be earned over the five-year period.

20 Find the total simple interest earned in each investment.

a \$4000 for three years at 6% p.a.

b \$7500 for six years at 4.5% p.a.

Example 7 **21** Find the rate R in each simple interest investment.

a Interest of \$7200 on \$8000 for 12 years

b Interest of \$3 400 000 on \$12 500 000 for four years

22 Find the time T involved in each simple interest investment.

a Interest of \$2500 on \$1000 at 5% p.a. **b** Interest of \$91 200 on \$30 000 at 8% p.a.

23 Find the principal P in each simple interest investment.

a Interest of \$4320 at 4.8% p.a. for six years

b Interest of \$5020 at 6.75% p.a. for three years

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1B Percentage increase and decrease

When a quantity is increased or decreased, the change is often expressed as a percentage of the original amount.

This section reviews a concise method of dealing with percentage increase and decrease. The method will be applied in various ways throughout the remaining sections of the chapter.

Percentage increase

The Shining Path Cleaning Company made a profit of \$421 000 last year, and increased its profit this year by 23%.

We can find the new profit in one step by using the fact that the new profit is $100\% + 23\% = 123\%$ of the old profit.

$$\begin{aligned}\text{New profit} &= 421\,000 \times 123\% \\ &= 421\,000 \times 1.23 \\ &= \$517\,830\end{aligned}$$

When using a calculator, this is a simpler method than calculating the profit separately and adding it on. It will also allow us to handle repeated increases more easily and will make it simpler to reverse the process.

Percentage decrease

The same method can be used to calculate percentage decreases. For example, Grey Gully Station recently sold 41% of its 2288 head of cattle to the meatworks.

We can calculate how many head of cattle the station now has by using the fact that $100\% - 41\% = 59\%$ of its cattle remain.

$$\begin{aligned}\text{Number of remaining head of cattle} &= 2288 \times 59\% \\ &= 2288 \times 0.59 \\ &\approx 1350 \text{ (Correct to the nearest integer.)}\end{aligned}$$

Percentage increase and decrease

- To increase an amount by, say, 15%, multiply by $1 + 0.15 = 1.15$.
- To decrease an amount by, say, 15%, multiply by $1 - 0.15 = 0.85$.

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Finding the percentage increase or decrease

The method used in the following example is in keeping with the other methods covered in this chapter. It requires fewer calculations than finding the actual increase or decrease and then expressing that change as a percentage of the original amount. In all cases, subtracting the calculated percentage by 100% determines the percentage change.

Example 8

The water stored in the main Warrabimbie Dam has increased from 1677 gigalitres to 2043 gigalitres in three months. What percentage increase is this?

Solution

$$\begin{aligned}\frac{\text{New storage}}{\text{Old storage}} &= \frac{2043}{1677} \times \frac{100}{1}\% \\ &\approx 121.82\% \text{ (Correct to the nearest 0.01\%.)}\end{aligned}$$

Thus, the storage has increased by about $121.82\% - 100\% = 21.82\%$.

Reversing the process to find the original amount

Harry claims that his mathematics mark of 78 constitutes a 45% increase on his previous mathematics mark. What was his previous mark?

This mark is $100\% + 45\% = 145\%$ of the previous mark.

Hence, this mark $=$ previous mark $\times 1.45$

$$\begin{aligned}\text{Reversing this, previous mark} &= \text{this mark} \div 1.45 \\ &= 78 \div 1.45 \\ &\approx 54 \text{ (Correct to the nearest mark.)}\end{aligned}$$

Thus, to find the original amount, we divide by 1.45, because dividing by 1.45 is the reverse process of multiplying by 1.45.

Exactly the same principle applies when an amount has been decreased by a percentage, as shown in the following example.

Example 9

The price of bananas has decreased by 70% over the last year to \$3 per kilogram. What was the price a year ago?

Solution

The new price is $100\% - 70\% = 30\%$ of the old price.

Hence, new price $=$ old price $\times 0.30$

$$\begin{aligned}\text{Reversing this, old price} &= \text{new price} \div 0.30 \\ &= 3.00 \div 0.3 \\ &= \$10 \text{ per kilogram}\end{aligned}$$

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Example 10

Ria has had mixed results with the shares that she bought three years ago. Shares in White Manufacturing rose 37% to \$14.56, but shares in Black Tile Distributors fell 28% to \$8.76. Find the prices she originally paid for these two shares, correct to the nearest cent.

Solution

White Manufacturing shares are now $100\% + 37\% = 137\%$ of their previous value.

Thus, new value = original price × 1.37

$$\begin{aligned}\text{Reversing this, original price} &= 14.56 \div 1.37\\ &\approx \$10.63 \text{ (Correct to the nearest cent.)}\end{aligned}$$

Black Tile Distributors shares are now $100\% - 28\% = 72\%$ of their original value.

Thus, new value = original price × 0.72

$$\begin{aligned}\text{Reversing this, original price} &= 8.76 \div 0.72\\ &\approx \$12.17 \text{ (Correct to the nearest cent.)}\end{aligned}$$

Finding the original amount

- To find the original amount after an increase of, say, 15%, divide the new amount by $1 + 0.15 = 1.15$.
- To find the original amount after a decrease of, say, 15%, divide the new amount by $1 - 0.15 = 0.85$.

Discounts

It is common for a shop to **discount** the price of an item. This can be done to sell stock of a slow-moving item more quickly, or simply to attract customers into the shop.

Discounts are normally expressed as a percentage of the original price.

Example 11

The Tie Knot Shop is expecting new stock and needs to make room on its shelves. It has discounted all its prices by 45% to try to sell some of its existing stock.

a What is the discounted price of a tie with an original price of \$90?

b What was the original price of a tie with a discounted price of \$90?

Solution

The discounted price of each item is $100\% - 45\% = 55\%$ of the old price.

a
$$\begin{aligned}\text{Discounted price} &= \text{original price} \times 0.55\\ &= 90 \times 0.55\\ &= \$49.50\end{aligned}$$

b
$$\begin{aligned}\text{Original price} &= \text{discounted price} \div 0.55\\ &= 90 \div 0.55\\ &\approx \$163.64\end{aligned}$$
(Correct to the nearest cent.)

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The GST

In 1999 the Australian Government introduced a Goods and Services Tax, or GST for short. This tax applies to nearly all goods and services in Australia.

The current rate of GST is 10% of the pre-tax price of the good or service.

- When GST applies, GST is added to the pre-tax price. This is easily done by multiplying by 1.10.
- Conversely, if a quoted price already includes the GST, the pre-tax price is obtained by dividing by 1.10.

Example 12

The current GST rate is 10% of the pre-tax price.

a The pre-tax price of a large fridge is $2150. What will the fridge cost after GST is added, and how much will be paid to the government?

b I recently paid $495 to have a tree pruned. What was the price before adding GST, and how much GST was paid to the government?

Solution

The after-tax price is 110% of the pre-tax price.

a

$$\begin{aligned}\text{After-tax price} &= 2150 \times 1.10\\ &= \$2365\end{aligned}$$

$$\begin{aligned}\text{Tax} &= 2365 - 2150\\ &= \$215\end{aligned}$$

Alternatively,

$$\begin{aligned}\text{tax} &= 2150 \times 0.1\\ &= 215\end{aligned}$$

$$\begin{aligned}\text{After-tax price} &= \$2150 + \$215\\ &= \$2365\end{aligned}$$

b

$$\begin{aligned}\text{Pre-tax price} &= 495 \div 1.10\\ &= \$450\end{aligned}$$

(Divide by 1.10 to reverse the process.)

$$\begin{aligned}\text{Tax} &= 495 - 450\\ &= \$45\end{aligned}$$

Inflation

The prices of goods and services in Australia and other countries usually increase by a small amount every year. This gradual rise in prices is called **inflation**, and is measured by taking the average percentage increase in the prices of a large range of goods and services.

Other things, such as salaries and pensions, are often adjusted automatically every year to take account of inflation.

High rates of inflation are damaging to society, and governments generally try to keep inflation low.

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Example 13

The economy in Espirito Santo is booming as a result of its mineral exports, but unfortunately, with a change of government, inflation has also taken hold. Last year inflation was 28%, meaning that on average, prices have increased by 28% over the last year.

a If the average winter electricity bill was $460 last year, give an estimate of this year's bill, based on the inflation rate.

b If a new Hunter Flash station wagon now costs $\$38\,000$, give an estimate of its cost a year ago, based on the inflation rate, correct to the nearest $\$100$.

Solution

We estimate this year's prices as $100\% + 28\% = 128\%$ of last year's prices.

a $\begin{aligned}\text{Estimate of this year's bill} &= 460 \times 1.28\\ &\approx \$588.80\end{aligned}$

b $\begin{aligned}\text{Estimate of cost last year} &= 38\,000 \div 1.28\\ &\approx \$29\,700 \text{ (Correct to the nearest \$100.)}\end{aligned}$

Exercise 1B

1 Increase each amount by the given percentage.

a $570, 10% **b** $9320, 5% **c** $456, 6% **d** $3120, 8%

2 Decrease each amount by the given percentage.

a $9000, 10% **b** $4560, 5% **c** $826, 3% **d** $9520, 4%

3 Traffic on all roads has increased by an average of 12% during the past 12 months. By multiplying by $112\% = 1.12$, estimate the number of vehicles now on a road where the number of vehicles a year ago was:

a $32\,000$ per day **b** $153\,000$ per day **c** 248 per day

4 Rainfall across Victoria has decreased over the last 10 years by about 38%. By multiplying by $62\% = 0.62$, estimate, correct to the nearest mm, the annual rainfall this year at a place where the rainfall 10 years ago was:

a 700 mm **b** 142 mm **c** 1268 mm

Example 8

5 The number of shops in different shopping centres in Borrington changed from 2011 to 2012, but by quite different percentage amounts. Find the percentage increase or decrease in the number of shops where the numbers during 2011 and 2012, respectively, were:

a 200 and 212 **b** 85 and 160 **c** 156 and 122 **d** 198 and 110

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Example 9

6 **a** An amount is decreased by 10% and the new amount is \$567. What was the original amount?

b An amount is increased by 10% and the new amount is \$5676. What was the original amount?

7 Phoenix Finance Pty Ltd recently issued bonus shares that increased by 14% the number of shares held by each of the company's shareholders. By dividing by 114% = 1.14, find the original holding of a shareholder who now holds:

a 228 shares **b** 8321 shares **c** 77 682 shares

Example 10

8 A research institute is trying to find out how much water Lake William had in it 8000 years ago. The lake now contains 7600 megalitres, but there are various conflicting theories about the percentage change over the last 8000 years. Find how much water the lake had 8000 years ago, correct to the nearest 10 megalitres, if in the last 8000 years the volume has:

a risen by 60% **b** fallen by 33% **c** risen by 312% **d** fallen by 88%

Example 11

9 A clothing store is offering a 35% discount on all its summer stock. Find the discounted price of an item with a marked price of:

a \$80 **b** \$48 **c** \$680 **d** \$1.60

Example 11

10 A furniture shop is offering a 55% discount at its end-of-year sale. Find the original price of an item with a discounted price of:

a \$1400 **b** \$327 **c** \$24.50

11 Mr Brown bought parcels of shares in June last year. He has a spreadsheet showing the value at which he bought his shares, the value at 31 December last year, and the percentage increase or decrease in their value. (Decreases are shown with a negative sign.) Unfortunately, a virus has corrupted one entry in each row of his spreadsheet. Help him fix his spreadsheet by calculating the missing values, correct to two decimal places.

Company	Value at purchase	Value at 31 December	Percentage increase
a	\$20 000		40%
b	\$14 268		–58%
c	\$3128.72		341.27%
d		\$80 000	15%
e		\$114 262	258.3%
f		\$32 516.24	–92.29%
g	\$50 000	\$52 000	
h	\$21 625	\$34 648	
i	\$48 372.11	\$40 072.11	

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Example 12 **12** The GST is a tax on most goods and services, which is calculated at the rate of 10% of the pre-tax price.

a Find the after-tax price on goods or services with a pre-tax price of:

i \$2740 **ii** \$134 000 **iii** \$8.20

b Find the pre-tax price on goods or services with an after-tax price of:

i \$3927 **ii** \$426 877 **iii** \$3.19

c Find the after-tax price on goods or services on which the GST is:

i \$442 **ii** \$347 114 **iii** \$0.47

Example 13 **13** **a** Prices have increased with inflation by an average of 3.4% since the same time last year. Estimate today's price for an item that one year ago cost:

i \$3000 **ii** \$24.15 **iii** \$361

b Estimate the price a year ago of an item that now costs:

i \$4200 **ii** \$14.30 **iii** \$76

(When finding your estimate, assume that the average increase applies to all items.)

14 **a** A table originally priced at \$370 was increased in price by 100%. What percentage discount will restore it to its original price?

b The number of daily passengers on the Jarrabine ferry-bus was 156, and in one year it increased by 25%. What percentage decrease next year would restore the number of passengers to its original value?

c Thien had savings of \$15 000, but he spent 45% of this last year. By what percentage of the new amount must he increase his savings to restore them to their original value?

d The profit of the Audry Goldfish Guild last year was \$3650, but this year it decreased by 42%. By what percentage must the profit increase next year to restore it to its original value?

15 **a** Find, correct to two decimal places, the percentage decrease necessary to restore a quantity to its original value if it has been increased by:

i 10% **ii** 18%

iii 360% **iv** 4.1%

b Find, correct to two decimal places, the percentage increase necessary to restore a quantity to its original value if it has been decreased by:

i 10% **ii** 18%

iii 80% **iv** 4.1%

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1C Repeated increases and decreases

Repeated increases

The method used in the last section becomes very useful when two or more successive increases or decreases are applied, because the original amount can simply be multiplied successively by two or more factors. Here is a typical example.

Example 14

Internet sales of Ferret Virus Guard have been increasing dramatically. Three years ago the number of registered users was 20 874. In the three years since then, the number of users rose by 8.1% in the first year, a further 46.4% in the second year, and then a further 112.8% in the third year.

a How many users were there at the end of the first year?

b How many users were there at the end of the second year?

c How many users are there now, at the end of the third year?

d What has the percentage increase in users been over the three years?

Solution

a After one year, the number of users was 108.1% of the original number.

$$\begin{aligned}\text{Hence, number after one year} &= 20\,874 \times 1.081\\ &\approx 22\,565 \text{ (Correct to the nearest integer.)}\end{aligned}$$

b After two years, the number of users was 146.4% of the number after one year.

$$\begin{aligned}\text{Hence, number after two years} &= 20\,874 \times 1.081 \times 1.464\\ &\approx 33\,035 \text{ (Correct to the nearest integer.)}\end{aligned}$$

c After three years, the number of users is 212.8% of the number after two years.

$$\begin{aligned}\text{Hence, number after three years} &= 20\,874 \times 1.081 \times 1.464 \times 2.128\\ &\approx 70\,298 \text{ (Correct to the nearest integer.)}\end{aligned}$$

d
$$\begin{aligned}\text{Number after three years} &= \text{original number} \times 1.081 \times 1.464 \times 2.128\\ &\approx \text{original number} \times 3.37\end{aligned}$$

This is about 337% of the original amount, or 337% − 100% = 237% more than it.
Hence, the number of users has increased by about 237% over the three years.

Note: We could have solved part **d** by calculating $\dfrac{70\,298}{20\,874} \approx 3.37$. It is best to keep the answer to **c** in your calculator for this calculation.

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Repeated decreases

The same method also applies to percentage decreases, as in Example 15.

Example 15

The median value of houses in the city of Winchester was reasonably stable for several years at \$340 000. After the local lead mine closed, however, the value of houses fell disastrously. One year later, the median house value had dropped by 58% and, over the next year, the median house value dropped a further 46%.

a What was the median house value one year later?

b What was the median house value two years later?

c What percentage of the original median house value was lost over the two years?

Solution

a One year later, the median value was $100\% - 58\% = 42\%$ of the original.

$$\begin{aligned}\text{Hence, median value after one year} &= 340\,000 \times 0.42 \\ &= \$142\,800\end{aligned}$$

b Two years later, the value was $100\% - 46\% = 54\%$ of the value after one year.

$$\begin{aligned}\text{Hence, median value after two years} &= 340\,000 \times 0.42 \times 0.54 \\ &= \$77\,112\end{aligned}$$

c
$$\begin{aligned}\text{Median value after two years} &= \text{original value} \times 0.42 \times 0.54 \\ &\approx \text{original value} \times 0.23\end{aligned}$$

Hence, about 77% of the value was lost over the two years ($23\% - 100\% = -77\%$).

Combinations of increases and decreases

Some problems involve both increases and decreases. They can be solved in the same way.

Example 16

During July 2011, 34% more patients at St Michael's Hospital were treated than in June 2011. Monthly patient numbers then fell by 40% during August, rose by 30% during September, and finally fell by 24% during October 2011.

a What is the percentage increase or decrease over the four months, correct to the nearest 1%?

b If 5783 patients were treated during the month of June 2011, how many patients were treated during October 2011?

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Solution

a $$\begin{aligned}\text{Final monthly number} &= \text{original monthly number} \times 1.34 \times 0.60 \times 1.30 \times 0.76\\ &\approx \text{original monthly number} \times 0.79\end{aligned}$$

$79\% - 100\% = -21\%$, so the number has decreased by about 21% over the four months.

b $$\begin{aligned}\text{Final monthly number} &= \text{original monthly number} \times 1.34 \times 0.60 \times 1.30 \times 0.76\\ &= 5783 \times 1.34 \times 0.60 \times 1.30 \times 0.76\\ &\approx 4594 \text{ patients (Correct to the nearest integer.)}\end{aligned}$$

In the above example, you may notice that the sum of the percentages is $34\% - 40\% + 30\% - 24\% = 0\%$, but this is completely irrelevant to the problem.

Repeated increases and decreases

- To apply successive increases of, say, 15%, 24% and 38% to a quantity, multiply the quantity by $1.15 \times 1.24 \times 1.38$.
- To apply successive decreases of, say, 15%, 24% and 38% to a quantity, multiply the quantity by $0.85 \times 0.76 \times 0.62$.

Reversing the process to find the original amount

As we have already learned, division reverses the process to find the original amount, as in the following example.

Example 17

The cat population in Grahamsville grew by 145% over a decade, and then fell by 40% over the next decade.

a What was the percentage increase in the cat population over the 20 years?

b If the final population was 10 000, what was the original population 20 years earlier?

Solution

a After one decade, the population was $100\% + 145\% = 245\%$ of the original population.

After the second decade, the population was $100\% - 40\% = 60\%$ of the increased population.

$$\begin{aligned}\text{Thus, final population} &= \text{original population} \times (2.45 \times 0.60)\\ &= \text{original population} \times 1.47\end{aligned}$$

So the population increased by $147\% - 100\% = 47\%$.

b Reversing this:

$$\begin{aligned}\text{Original population} &= \text{final population} \div (2.45 \times 0.60)\\ &= 10\,000 \div (2.45 \times 0.60)\\ &= 6803 \text{ (Correct to the nearest integer.)}\end{aligned}$$

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Reversing repeated increases and decreases

- To find the original quantity after successive increases of, say, 15%, 24% and 38% to that quantity, divide the final quantity by $(1.15 \times 1.24 \times 1.38)$.
- To find the original quantity after successive decreases of, say, 15%, 24% and 38% to that quantity, divide the final quantity by $(0.85 \times 0.76 \times 0.62)$.

Successive equal percentage increases and decreases

When all the percentage changes are the same, we can use powers to reduce the number of calculations.

Example 18

The number of people visiting the Olympus Cinema each week has been decreasing by 4% per year for the last 10 years.

a What is the percentage decrease in weekly attendance over the 10 years?

b If there are 1250 visitors per week now, what was the weekly attendance 10 years ago?

Solution

Each year the weekly attendance is 96% of the previous year's weekly attendance.

a Final weekly attendance

$$= \text{original weekly attendance} \times 0.96 \times 0.96 \times \ldots \times 0.96$$

$$= \text{original weekly attendance} \times (0.96)^{10}$$

$$\approx \text{original weekly attendance} \times 0.66$$

Here the cinema's weekly attendance has decreased by about 34% over the 10 years.

b From part **a**:

Final weekly attendance $=$ original weekly attendance $\times (0.96)^{10}$

Reversing this:

$$\begin{aligned}\text{Original weekly attendance} &= \text{final weekly attendance} \div (0.96)^{10}\\ &= 1250 \div (0.96)^{10}\\ &\approx 1880 \text{ (Correct to the nearest integer.)}\end{aligned}$$

Exercise 1C

1 **a** Find the final value after \$10 000 is successively increased by 5%, 8% and 10%.

b Find the final value after \$10 000 is successively decreased by 8%, 7% and 6%.

c Find the final value after \$90 000 has been increased by 10% ten times. (Give your answer correct to the nearest cent.)

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Example 14

2 Three years ago, apples cost \$2.80 per kg, but the price has increased by 8%, 15% and 10% in the past three successive years. Multiply by $1.08 \times 1.15 \times 1.1$ to find the price of apples now.

3 The dividend per share in Knowledge Bank Company has risen over the last four years by 32%, 112%, 155% and 8%, respectively. Find the total dividend received by a shareholder whose dividend four years ago was:

a \$1000 **b** \$12 472 **c** \$16.64 **d** \$512.21

4 Land rates in Crookwell Shire have risen by 6% every year for the last seven years.

a By what percentage have the land rates risen over the seven-year period?

b Find the rates now payable by a landowner whose rates seven years ago were:

i \$1000 **ii** \$17 268.24 **iii** \$216.04

Example 15

5 Since a new 'Fitness for Freedom' program was introduced in a community, the number of people classified as overweight in that community has been falling. In four successive years, the number of overweight people fell by 4.8%, 7.1%, 10.5% and 6.2%, respectively. Find, correct to two decimal places, the percentage decrease over the four-year period.

Example 16

6 Calculate the total increase or decrease in a quantity when:

a it is increased by 20% and then decreased by 20%

b it is increased by 80% and then decreased by 80%

Example 17

7 The price of beans has been rising. The price has risen by 10%, 15% and 35% in three successive years, and they now cost \$3.40 per kg. By dividing successively by 1.35, then by 1.15, and then by 1.10, find the:

a price one year ago **b** price two years ago **c** original price three years ago

Example 18

8 Shares in Value Radios have been falling by 18% per year for the last five years.

a Find the present worth of a parcel of shares with an original worth five years ago of:

i \$1000 **ii** \$24 000 **iii** \$11 328 512

b By what percentage has the value fallen over the five-year period?

9 A particular strain of bacteria increases its population on a certain prepared Petri dish by 18% every hour. Calculate the size of the original population four hours ago if there are now:

a 10 000 bacteria **b** 1 000 000 bacteria **c** 120 000 bacteria

10 A potato is taken from boiling water at 100°C and placed in a fridge at 0°C. Every minute after this, the temperature of the potato drops by 16%.

a Find the temperature of the potato after:

i 4 minutes **ii** 8 minutes **iii** 20 minutes

b François measures the temperature of the potato and finds it to be 12°C. Find its temperature:

i 3 minutes ago **ii** 6 minutes ago **iii** 10 minutes ago

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11 Here is a table of the annual inflation rate in Australia for the years ending 30 June 2005 to 30 June 2010 (from the Reserve Bank of Australia website).

Year	2005	2006	2007	2008	2009	2010
Inflation rate	2.3%	2.7%	3.8%	2.3%	4.4%	1.8%

Calculate the percentage increase in prices, correct to one decimal place:

a over the whole six-year period **b** during the first four-year period

12 The radioactivity of any sample of the element iodine-131 decreases by 55% every seven days. Find the percentage reduction in radioactivity over each of the periods given below. (Give percentages correct to one decimal place.)

a 3 weeks **b** 10 weeks **c** 26 weeks

13 **a** A coat is discounted by 50%, and the resulting price is then increased by 50%. By what percentage is the price increased or decreased from its original value?

b The price of a coat is increased by 50%, and the resulting price is then decreased by 50%. By what percentage is the price increased or decreased from its original value?

c Can you explain the relationship between your answers to parts **a** and **b**?

1D Compound interest

With simple interest, the interest is always calculated on the original amount: the principal.

With **compound interest**, the interest is applied periodically to the balance of an account (whether it is an amount borrowed or an amount invested). For example, when interest is compounded annually (per annum), the interest is calculated on the balance at the end of each year. The interest earned in one year plus the previous balance becomes subject to interest calculations in the next year, and so on.

The first example below is done using the method of percentage increase developed in the previous two sections. After that, we will develop a general formula for compound interest.

Example 19

Siri has invested $100 000 for five years with the St Michael Bank. The bank pays her interest at the rate of 6% p.a., compounded annually.

a How much will the investment be worth at the end of:

i one year? **ii** two years? **iii** five years?

b What is the percentage increase of her original investment at the end of five years?

c What is the total interest earned over the five years?

d What would the simple interest on the five-year investment have been, assuming the same interest rate of 6% p.a.?

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Solution

Each year the investment is worth 106% of its value the previous year.

a **i** Balance at the end of one year $= 100\,000 \times 1.06$
$= \$106\,000$

ii Balance at the end of two years $= 100\,000 \times 1.06 \times 1.06$
$= 100\,000 \times (1.06)^2$
$= \$112\,360$

iii Balance at the end of five years $= 100\,000 \times 1.06 \times 1.06 \times 1.06 \times 1.06 \times 1.06$
$= 100\,000 \times (1.06)^5$
$\approx \$133\,822.56$ (Correct to the nearest cent.)

b Final amount $=$ original amount $\times (1.06)^5$
$\approx$ original amount $\times 1.3382$

So the total increase over five years is about 33.82%.

c Total interest $\approx 133\,822.56 - 100\,000$
$\approx \$33\,822.56$

d Simple interest $= PRT$
$= 100\,000 \times 0.06 \times 5$
$= \$30\,000$

We will often use the word ‘amount’ for the ‘balance’ of the account.

A formula for compound interest

It is not difficult to develop a formula for compound interest, provided that the interest rate is constant throughout the loan.

Suppose that a principal $\$P$ is invested for n units of time at an interest rate R per unit time, and that compound interest is paid. (The unit of time may be years, months, days or any other length of time.)

Let $\$A_n$ be the amount that the investment is worth after n units of time. That is, A_n is the balance of the account.

At the end of each unit of time, the amount increases by a factor of $1 + R$.

Thus, $A_1 = P(1 + R)$

and $A_2 = A_1(1 + R)$

$= P(1 + R) \times (1 + R) = P(1 + R)^2$

and $A_3 = A_2(1 + R)$

$= P(1 + R)^2 \times (1 + R) = P(1 + R)^3$

Continuing this process for n units of time gives:

$$A_n = P(1 + R)^n$$

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Compound interest

Suppose that a principal $\$P$ is invested at an interest rate R per unit time, and that compound interest is paid. The amount $\$A_n$ that the investment is worth after n units of time is:

$$A_n = P(1 + R)^n$$

Thus, in the previous example, the amount at the end of five years would be calculated as:

Substituting $P = 100\,000$, $R = 0.06$ and $n = 5$

$$\begin{aligned} A_5 &= 100\,000 \times (1.06)^5 \\ &\approx \$133\,822.56 \end{aligned}$$

Example 20

Wesley has retired, and he has invested \$200 000.

a How much will his investment grow to after four years if he has invested the money at 0.5% per month compound interest?

b How much would it have grown to had Wesley invested the money at 6% p.a. compound interest for four years?

c How much would the investment grow to if he had invested it at 3% per six months compound interest for four years?

Solution

Each amount is calculated to the nearest cent.

a The money is invested for 48 months, and $R = 0.5\% = 0.005$ per annum.

$$\begin{aligned} A_n &= P(1 + R)^n \\ A_{48} &= 200\,000 \times (1.005)^{48} \\ &\approx \$254\,097.83 \end{aligned}$$

b The money is invested for four years, and $R = 6\% = 0.06$ per annum.

$$\begin{aligned} A_n &= P(1 + R)^n \\ A_4 &= 200\,000 \times (1.06)^4 \\ &\approx \$252\,495.39 \end{aligned}$$

c The money is invested for 8 periods of six months, and $R = 3\% = 0.03$ per six-month period.

$$\begin{aligned} A_n &= P(1 + R)^n \\ A_8 &= 200\,000 \times (1.03)^8 \\ &\approx \$253\,354.02 \end{aligned}$$

Note: The calculations of **a** and **b** in the above example show that 0.5% per month, compounded monthly, earns slightly more interest than 6% p.a., compounded annually. With compound interest, the more frequent the compounding, the greater the amount of interest.

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Compound interest on a loan

Exactly the same principles apply when someone borrows money from a bank and the bank charges compound interest on the loan. If no repayments are made, the amount owing compounds in the same way, and can grow quite rapidly.

Example 21

Assad is setting up a home renovation business and needs to borrow \$360 000 from a bank. The bank will charge him interest of 1% per month. Assad will pay the whole loan off all at once eight years later.

a How much will Assad owe the bank at the end of one year?

b How much will Assad owe the bank at the end of two years?

c How much will Assad owe the bank at the end of eight years?

d What is the percentage increase in the money owed at the end of eight years?

e What is the total interest that Assad will pay on the loan?

f What would the simple interest on the loan have been, assuming the same interest rate of 1% per month?

Solution

Each amount is calculated to the nearest cent.

The units of time are months, and $R = 1\% = 0.01$ per month.

a
$$\begin{aligned}\text{Amount owing at the end of one year} &= P(1+R)^n\\ &= 360\,000 \times (1.01)^{12}\\ &\approx \$405\,657.01\end{aligned}$$

b
$$\begin{aligned}\text{Amount owing at the end of two years} &= 360\,000 \times (1.01)^{24}\\ &\approx \$457\,104.47\end{aligned}$$

c
$$\begin{aligned}\text{Amount owing at the end of eight years} &= 360\,000 \times (1.01)^{96}\\ &\approx \$935\,738.25\end{aligned}$$

d
$$\begin{aligned}\text{Final amount} &= \text{original amount} \times (1.01)^{96}\\ &\approx \text{original amount} \times 2.60\end{aligned}$$

So the percentage increase over eight years is approximately 160%.

e
$$\begin{aligned}\text{Total interest} &\approx 935\,738.25 - 360\,000\\ &\approx \$575\,738.25\end{aligned}$$

f
$$\begin{aligned}\text{Simple interest} &= PRT\\ &= 360\,000 \times 0.01 \times 96\\ &= \$345\,600\end{aligned}$$

Note: Making no repayments on a loan that is accruing compound interest can be a risky business practice because, as this example makes clear, the amount owing grows with increasing rapidity as time goes on. Similarly, not making regular payments on a credit card can be disastrous.

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Reversing the process to find the original amount

If we are given the final amount A_n, the interest rate R and the number n of units of time, we can substitute into the compound interest formula and solve the resulting equation to find the principal P.

Example 22

Carla wants to borrow money for six years to start a business, and then pay all the money back, with interest, at the end of that time. The bank will charge compound interest at a rate of 0.8% per month, and will limit her final debt, including interest, to \$1 000 000. What is the maximum amount that Carla can borrow?

Solution

The units of time are months, so $n = 72$ and $R = 0.8\% = 0.008$ per month.

Hence, $A_{72} = P \times (1.008)^{72}$

Substituting $A_{72} = 1\,000\,000$, the maximum amount that Carla can owe at the end of the loan:

$$\begin{aligned} 1\,000\,000 &= P \times (1.008)^{72} \\ P &= 1\,000\,000 \div (1.008)^{72} \\ &\approx \$563\,432.23 \end{aligned}$$

Carla can borrow a maximum of \$563 432.23.

Exercise 1D

Note: This exercise is based on the compound interest formula $A_n = P(1 + R)^n$. Remember that if an interest rate is given ‘per annum’ then it is assumed that the compounding occurs annually, and if it is given per month then the compounding occurs every month, and so on.

Example 19

1 Ming invested \$100 000 for five years at 7% p.a. interest, compounded annually.

a Find the amount invested after one year.

b Find the amount invested after two years.

c Find the amount invested after five years.

d Find the percentage increase in the investment over the five-year period, correct to two decimal places.

e Find the total interest earned over the five years.

f Find the simple interest on the principal of \$100 000 over the five years at the same annual interest rate.

2 The population of a town increases at a rate of 5.8% p.a. for 10 years, compounded annually. Initially, the population was 34 000.

a What was the population at the end of the 10-year period?

b What was the total percentage increase, correct to the nearest 1%?

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Example 20

3 A couple takes out a housing loan of \$380 000 over a period of 25 years. They make no repayments during the 25-year period.

a **i** How much money would they owe if compound interest were payable at 6% p.a.?

ii What would the percentage increase in the debt be, correct to the nearest 1%?

b **i** How much money would they owe if compound interest were payable at 0.5% per month?

ii What would the percentage increase in the debt be, correct to the nearest 1%?

Example 21

4 Emmanuel has borrowed \$300 000 for seven years at 9% p.a. interest, compounded annually, in order to start his carpentry business. He intends to pay the whole amount back, plus interest, at the end of the seven years.

a Find the amount owing after one year.

b Find the amount owing after seven years.

c Find the percentage increase in the debt over the seven-year period, correct to two decimal places.

d Find the total interest charged over the seven years.

e Find the simple interest on the principal of \$300 000 over the seven-year period at the same annual interest rate.

5 **a** Find the compound interest on \$1000 at 12% p.a. for 100 years.

b Find the compound interest on \$1000 at 1% per month for 100 years.

c Find the simple interest on \$1000 at 12% p.a. for 100 years.

6 A student borrows \$20 000 from a bank for six years. Compound interest at 9% p.a. must be paid.

a How much money is owed to the bank at the end of the six-year period?

b How much of this amount is interest?

Example 22

7 Money borrowed at an interest rate of 8% p.a. grew to \$100 000 in seven years. Find:

a the original amount invested

b the total percentage increase in the investment, correct to the nearest 1%

8 Emily wants to invest some money now so that it will grow to \$250 000 in eight years' time. The compound interest rate is 0.5% per month.

a How much should she invest now?

b What will the total percentage increase be, correct to the nearest 1%?

9 A bank offers 0.7% per month compound interest. How much needs to be invested if the investment is to be worth \$100 000 in:

a 10 years?

b 25 years?

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10 The population of the mountain town of Granite Peak has been growing at 7.4% every year and has now reached 80 000. Find the population:

a one year ago

b two years ago

c five years ago

d 10 years ago

11 Mr Brown has had further difficulties with the virus that attacked his spreadsheet entries. The spreadsheet calculated interest compounded annually on various amounts, at various interest rates, for various periods of time. Help him reconstruct the missing entries.

	Principal	Rate	Number of years	Final amount	Total interest
a	\$3000	7%	25		
b	\$3 000 000	5.2%	12		
c		7%	25	\$3000	
d		5.2%	12	\$3 000 000	

12 **a** Mr Yang invested \$90 000 at a compound interest rate of 6% p.a. for three years. The tax office wants to know exactly how much interest he earned each year. Calculate these figures for Mr Yang.

b Repeat these calculations with the rate of interest of 0.5% per month.

13 WestPlaza Holdings sold one of its shopping centres for \$20 000 000 and invested the money at a daily compound interest rate of 0.016%. How much interest did the company earn in the first year?

14 Find the percentage increase in each situation (correct to the nearest 0.01%).

a \$100 000 is borrowed at a compound interest rate of 0.01% per day for one year.

b \$1 000 000 is borrowed at a compound interest rate of 0.02% per day for one year.

15 Find the total percentage growth, correct to the nearest 0.1%, in a compound interest investment:

a at 15% p.a. for two years

b at 10% p.a. for three years

c at 6% p.a. for five years

d at 5% p.a. for six years

e at 3% p.a. for 10 years

f at 2% p.a. for 15 years

g What do you observe about these results?

16 A doctor took out a six-year loan to start a medical practice. For the first three years, he was charged compound interest at a rate of 9% p.a. For the second three-year period, he was charged compound interest at a rate of 13% p.a. Find the total percentage increase in the money owing, correct to the nearest 1%.

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1E Compound depreciation

Depreciation occurs when the value of an asset reduces as time passes. For example, a person may buy a car for \$50 000, but after six years the car will be worth a lot less, because the motor will be worn, the car will be out of date, the body and interior may have a few scratches, and so on.

Accountants usually make the assumption that an asset, such as a car, depreciates at the same rate every year. This rate is called the **depreciation rate**. In the following example, the depreciation rate is taken to be 20%.

In many situations, simple depreciation is used, but in other situations the depreciation is compounded. We will deal only with compound depreciation.

This first example is done using the methods of percentage decrease developed in Sections 1C and 1D. After that, we will develop a general formula for depreciation, as we did for compound interest.

Example 23

A person bought a car six years ago for \$50 000, and assumed that the value of the car would depreciate at 20% p.a.

a What value did the car have at the end of two years?

b What value does the car have now, after six years?

c What is the percentage decrease in value over the six-year period?

d What is the average reduction in value, in dollars p.a., on the car over the six-year period due to depreciation?

Solution

The value each year is taken to be $100\% - 20\% = 80\%$ of the value in the previous year.

a Value at the end of two years $= 50\,000 \times 0.80 \times 0.80$

$$= 50\,000 \times (0.80)^2$$
$$= \$32\,000$$

b Value at the end of six years $= 50\,000 \times 0.80 \times 0.80 \times 0.80 \times 0.80 \times 0.80 \times 0.80$

$$= 50\,000 \times (0.80)^6$$
$$= \$13\,107.20$$

c Final value $=$ original value $\times (0.80)^6$

$\approx$ original value $\times\ 0.26$

Hence, the percentage decrease over six years is about $100\% - 26\% = 74\%$.

d Depreciation over six years $= 50\,000 - 13\,107.20$

$$= \$36\,892.80$$

Average reduction in value per year $= \$36\,892.80 \div 6$

$$= \$6148.80 \text{ per year}$$

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A formula for depreciation

We can develop a formula for depreciation, as we did for compound interest. The two formulas are very similar.

Suppose that an asset originally worth P depreciates at a rate, R, per unit time.

Let A_n be the value of the asset after n units of time.

At the end of each unit of time, the value decreases by a factor of $1 - R$.

$$\begin{aligned} \text{Thus,}\quad A_1 &= P(1-R) \\ \text{and}\quad A_2 &= A_1(1-R) \\ &= P(1-R)\times(1-R) = P(1-R)^2 \\ \text{and}\quad A_3 &= A_2(1-R) \\ &= P(1-R)^2\times(1-R) = P(1-R)^3 \end{aligned}$$

Continuing this process for n units of time gives:

$$A_n = P(1-R)^n$$

Compound depreciation

Suppose that an asset with an original value P depreciates at a rate R per unit time. The value A_n of that asset after n units of time is:

$$A_n = P(1-R)^n$$

Thus, in the previous example, the value at the end of six years would be calculated as:

Substituting $P = 50\,000$, $R = 0.8$ and $n = 6$

$$\begin{aligned} A_6 &= 50\,000\times(0.8)^6 \\ &= \$13\,107.20 \end{aligned}$$

Example 24

A piece of machinery that cost $560 000 depreciates at 30% p.a.

a What is its value after six years?

b What is the average reduction in value per year?

c If it had depreciated at 15% p.a., what would its value have been after six years?

Solution

a Here, $R = 0.3$, so $1 - R = 0.7$

$$\begin{aligned} A_n &= P(1-R)^n \\ A_6 &= 560\,000\times(0.7)^6 \\ &= \$65\,883.44 \end{aligned}$$

(continued over page)

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b Depreciation $= 560\,000 - 65\,883.44$

$= \$494\,116.56$

Average reduction per year $\approx 494\,117.56 \div 6$

$\approx \$82\,352.76$

c Here, $R = 0.15$, so $1 - R = 0.85$

$A_n = P(1 - R)^n$

$A_6 = 560\,000 \times (0.85)^6$

$\approx \$211\,203.73$

Reversing the process to find the original amount

Given the final depreciated value A_n and the depreciation rate R we can substitute into the depreciation formula and solve the resulting equation to find the original value of an item P.

Example 25

A company buys its staff new cars every four years. At the end of the four years, it offers to sell the cars to the staff on the assumption that they have depreciated at 22.5% p.a. The company is presently offering cars for sale at $9000 each.

a What did each car cost the company originally?

b What is the average reduction in value, in dollars p.a., on each car?

Solution

a Here, $R = 0.225$, so $1 - R = 0.775$

$A_n = P(1 - R)^n$

$9000 = P \times (0.775)^4$

$P = 9000 \div (0.775)^4$

$\approx \$24\,948$ (Correct to the nearest dollar.)

Each car originally cost the company about $24 948.

b Loss of value $\approx 24\,948 - 9000$

$= \$15\,948$

Average loss per year $\approx 15\,948 \div 4$

$= \$3987$

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Exercise 1E

Example 23, 24

1 The landlord of a large block of home units purchased washing machines for its units six years ago for \$600 000, and is assuming a depreciation rate of 30% p.a.

a Find the estimated value after one year.

b Find the estimated value after two years.

c Find the estimated value after six years.

d What is the percentage decrease in value over the six-year period?

e What is the average reduction in value, in dollars p.a., on the washing machines over the six-year period due to depreciation?

2 **a** A computer shop spent \$320 000 installing alarms at its premises. If it depreciated them at 20% p.a., find the estimated value after six years, and the percentage reduction of value over that period.

b The business borrowed the money to install the alarms, paying 8% p.a. compound interest for the six years. How much did it owe at the end of the six years?

3 A school bought a bus for \$90 000, depreciated it at 30% p.a., and sold it again five years later for \$20 000. Was the price that they obtained better or worse than the depreciated value, and by how much?

4 The Online Grocery spent \$4 540 000 buying computers for its offices, and depreciated them for taxation purposes at 40% p.a. Find the value of the computers at the end of each of the first four years, and the amount of the loss that the company could claim against its taxable income for each of those four years.

5 Sandra and Kevin each received \$80 000 from their parents. Sandra invested the money at 6.5% p.a. compounded annually, whereas Kevin bought a sports car that depreciated at a rate of 20% p.a. What were the values of their investments at the end of six years?

6 Taxis depreciate at 50% p.a., and other cars depreciate at 22.5% p.a.

a What is the total percentage reduction in value on each type of vehicle after six years?

b What is the difference in value after six years of a fleet of taxis and a fleet of other cars, if both fleets originally cost \$10 000 000?

Example 25

7 Mr Startit's 10-year-old car is worth \$6500, and has been depreciating at 22.5% p.a.

a By substituting into $A_n = P(1 - R)^n$, find how much (to the nearest dollar) it was estimated to be worth a year ago.

b How much, correct to the nearest dollar, was it estimated to be worth two years ago?

c How much, correct to the nearest dollar, was it estimated to be worth 10 years ago?

d What is the total percentage reduction in value on the car over the 10-year period?

e What was the average reduction in value in dollars per year over the 10-year period?

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8 Ms Rinoldis' seven-year-old car is worth $5600, and has been depreciating at 22.5% p.a.

a How much, correct to the nearest dollar, was it worth four years ago?

b How much, correct to the nearest dollar, was it worth seven years ago?

c What is the total percentage reduction in value on the car over the seven-year period?

d What was the average reduction in value in dollars per year over the seven-year period?

e Ms Rinoldis, however, only bought the car four years ago, at its depreciated value at that time.

i What has been Ms Rinoldis' average loss in dollars over the four years she has owned the car?

ii What was the average loss in dollars over the first three years of the car's life?

9 I take a sealed glass container and remove 60% of the air. Then I remove 60% of the remaining air. I do this six times altogether. What percentage of the original air is left in the container?

10 The number of trees on Green Plateau fell by 5% every year for 10 years. Then the numbers rose by 5% every year for 20 years. What was the total percentage gain or loss of trees over the 30-year period?

11 **a** Find the total percentage decrease in an investment with a value that decreased at:

i 15% p.a. for two years

ii 10% p.a. for three years

iii 6% p.a. for five years

iv 5% p.a. for six years

v 3% p.a. for 10 years

vi 2% p.a. for 15 years

b What do you observe about these results?

12 A special depreciation ruling was obtained from the Taxation Office on a particular piece of scientific apparatus. For the first four years, it depreciates at 32% p.a., and for the second four years, it depreciates at 22% p.a. Find the total percentage decrease in value.

13 I take 500 mL of a liquid and dilute it with 100 mL of water. Then I take 500 mL of the mixture and again dilute it with 100 mL of water. I repeat this process 20 times in all. What percentage of the original liquid remains in the mixture at the end?

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Review exercise

1 Find the simple interest payable in each case.

a $10 000 borrowed for six years at 9% p.a.

b $3000 borrowed for 15 years at 6% p.a.

c $1500 borrowed for 40 years at 2.5% p.a.

2 What principal will earn $1000 simple interest at:

a 4% p.a. over five years?

b 10% p.a. over three years?

c 6% p.a. over eight years?

d 2.5% p.a. over 10 years?

3 At what rate of simple interest will:

a $10 000 grow to $14 000 over a five-year period?

b $8000 grow to $10 000 over a two-year period?

c $1500 grow to $2000 over a three-year period?

4 If a woman borrows $750 to buy a television and agrees to pay back $870 in one year's time, what annual rate of simple interest is she being charged?

5 An investor bought an antique table for $6000. He paid 5% as a deposit and borrowed the remainder from a bank for two years at 18% p.a. simple interest, payable monthly. How much interest does he have to pay each month?

6 Find the new value if:

a 60 is increased by 10%

b 50 is increased by 150%

c 80 is decreased by 20%

d 200 is increased by $12\frac{1}{2}\%$

e 400 is decreased by 2.5%

f 312 is decreased by $5\frac{1}{4}\%$

7 A clothing store offers a 15% discount on all its summer stock. How much will I need to pay in total if I buy a shirt with a marked price of $35, a pair of shorts with a marked price of $25, and a cotton sweater with a marked price of $50?

8 A general store in a country town adds 8% to the recommended retail price of all its stock due to transport costs. What will be the total charge if I buy a torch with a retail price of $9.50, a tin of coffee with a retail price of $11.30, and a hat with a retail price of $42.00?

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9 **a** What is the final value if:

i 90 is increased by 10%?

ii 120 is decreased by 20%?

iii 96 is increased by 4%?

iv 108 is decreased by 8%?

b Look carefully at the results of part **a i–iv**. Do they surprise you?

10 A computer store states that it will reduce the price of a computer by 10% each day until it is sold. The original price of the computer is $2500.

a What is the sale price after the first reduction?

b What is the sale price after the second reduction?

c What is the sale price after the third reduction?

d What single percentage decrease has the same effect as the three 10% reductions?

11 A manufacturer of suits can produce a suit at a cost of $250. When he sells it to a clothing store owner, he makes a 20% profit on the suit. To cover costs, the store owner increases the cost of the suit a further 30%.

a For what price does the store owner sell the suit?

b What is the total percentage increase in the cost of the suit?

12 The population of a country increased by 3%, 2.6% and 1.8% in three successive years. What was the total percentage increase in the country's population over the three-year period?

13 Find the balance if:

a $2000 is invested for 10 years at 8% p.a. compounded annually

b $5000 is invested for six years at 1% per month compound interest

c $500 is invested for 40 years at 3% per six-month period compound interest

14 A piece of machinery has an initial value of $25 000. Due to usage and age, its value depreciated by 8% each year. Find the value of the piece of machinery after:

a three years **b** five years **c** 10 years **d** n years

15 A new car is valued at $26 000. It is estimated to depreciate by 12% each year.

a Find its depreciated value after five years.

b Find its depreciated value after 10 years.

c Find how many years it takes for its depreciated value to fall below $11 000.

16 Jo bought a four-year old car for $20 880.25 at its correct depreciated value. If the car has a depreciation rate of 15% p.a., then find the value of the car when it was brand new.

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Challenge exercise

1 Waleed owns a portfolio of shares that he purchased for \$38 000. For the first four years, the portfolio appreciated in value at an average of 8% each year, but for the next four years, it depreciated in value at an average of 8% each year. Calculate:

a the value of the portfolio, correct to the nearest dollar, at the end of these eight years

b the equivalent simple interest rate of change in value per year, correct to two decimal places, over these eight years.

2 Two banks offer the following investment packages.

Bank A: 6.5% p.a. compounded annually, fixed for six years

Bank B: 5.3% p.a. compounded annually, fixed for eight years

a Which bank's package will yield the greater interest?

b If a customer invests \$10 000 in Bank A, how much would she have to invest with Bank B to produce the same amount produced by Bank A at the end of the investment period?

3 The Happy Pumpkin fruit shop sells grapes at a price 10% cheaper than the Akrivo Stafili fruit shop and 10% more expensive than the Costa fruit shop.

A customer buys \$50 worth of grapes from the Happy Pumpkin fruit shop. He obtains n kilograms of grapes for his \$50.

a What is the cost, in terms of n, of 1 kg of grapes from the Happy Pumpkin fruit shop?

b If he buys $\frac{n}{2}$ kilograms of grapes from the Akrivo Stafili fruit shop and $\frac{n}{2}$ kilograms of grapes from the Costa fruit shop, how much does he pay?

c If he buys $\frac{n}{4}$ kilograms of grapes from the Akrivo Stafili fruit shop and $\frac{3n}{4}$ kilograms of grapes from the Costa fruit shop, how much does he pay?

d If he buys $\frac{3n}{4}$ kilograms of grapes from the Akrivo Stafili fruit shop and $\frac{n}{4}$ kilograms of grapes from the Costa fruit shop, how much does he pay?

4 The population of a town decreases by 11% during 2012. What percentage increase is necessary during 2013 for the population to be restored to its population immediately before 2012?

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5 The length of a rectangle is increased by 12% and the width is decreased by 10%. What is the percentage change in the area?

6 A man earns a salary of \$2440 for working a 44-hour week. His weekly salary is increased by 12.5% and his hours are reduced by 10%. Find the percentage increase in his new hourly salary.

7 In a particular country in 2011, 12% of the population was unemployed and 88% was employed. In 2012, 10% of the unemployed people became employed and 10% of those employed became unemployed. What percentage of the population was employed at the end of 2012?

8 Andrea buys a house and she spends an extra 10% of what she paid for the house on repairs. She takes out a loan and pays 5% p.a. compound interest on the total amount spent (including repairs). Three years later she sells the house for \$565 100 and she gains 20% on the whole investment purchase price. How much did she pay for the house?

9 Anthony invests $\$P$ for 2 years at $r\%$ p.a. compound interest. At the end of the 2 years, Anthony receives his original $\$P$ and $\dfrac{\$P}{10}$ in interest. Find r, correct to two decimal places.

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CHAPTER

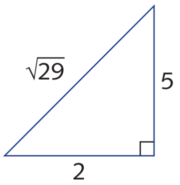
2

Number and Algebra

Review of surds

In this chapter, we revise our work on **surds**, which are a special class of irrational numbers that you studied in *ICE-EM Mathematics Year 9*.

Surds, such as $\sqrt{29}$, arise when we use Pythagoras' theorem.



$$5^2 + 2^2 = 29$$

The values of the trigonometric ratios of some common angles are surds.

For example, $\cos 30^\circ = \dfrac{\sqrt{3}}{2}$.

Surds also arise when solving quadratic equations and can often be treated as if they are pronumerals. Skills in manipulating surds strengthen algebraic skills.

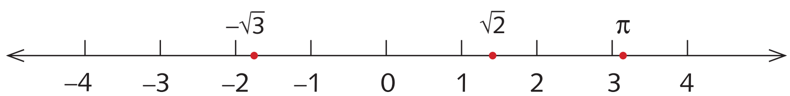
2A Irrational numbers and surds

Irrational numbers

When we apply Pythagoras' theorem, we often obtain numbers such as $\sqrt{2}$. The numbers $\sqrt{2}$ and $\sqrt{3}$ are examples of **irrational** numbers and so is π, the number that arises from circles. Note that $\sqrt{2}$ means the positive square root of 2.

Recall that a **rational** number is a number that can be written as $\frac{p}{q}$, where p is an integer and q is a non-zero integer.

A real number is a point on the number line. Every rational number is real but, as we have seen, not every real number is rational. A real number that is not rational is called **irrational**.



As we have mentioned, every real number is a point on the number line and, conversely, every point on the number line is real.

Surds can always be approximated by decimals, but working with exact values enables us to see important relationships and gives insights that would be lost if we approximated everything.

Surds

We can take the n^{th} root of any positive number a. The n^{th} root of a, written as $\sqrt[n]{a}$, is the positive number whose n^{th} power is a. Thus, $\sqrt[n]{a} = b$ is equivalent to the statement $b^n = a$.

If $\sqrt[n]{a}$ is irrational, then it is called a **surd**. If $\sqrt[n]{a}$ is rational, then a is the n^{th} power of a rational number. Hence, $\sqrt{3}$, $\sqrt[3]{5}$ and $\sqrt[5]{7}$ are surds. On the other hand, $\sqrt[3]{8} = 2$ and $\sqrt[4]{81} = 3$, so they are not surds.

Approximations to surds can be found using a calculator.

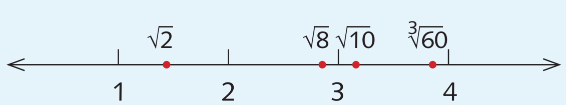
Example 1

Use your calculator to arrange the surds $\sqrt{8}$, $\sqrt{10}$, $\sqrt{2}$ and $\sqrt[3]{60}$ in order of size on the number line.

Solution

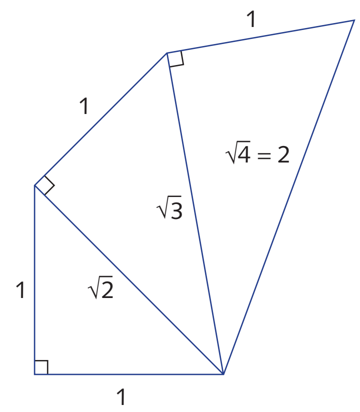
We use a calculator to find an approximation to each number, correct to two decimal places.

$\sqrt{8} \approx 2.83$ $\quad$ $\sqrt{10} \approx 3.16$ $\quad$ $\sqrt{2} \approx 1.41$ $\quad$ $\sqrt[3]{60} \approx 3.91$



Constructing some surds geometrically

We can use Pythagoras' theorem to construct lengths of $\sqrt{2}$, $\sqrt{3}$ and so on.



Using a ruler, draw a length of 1 unit.

Using your ruler and compasses or set square, draw a right angle at the end of your interval and mark off 1 unit.

Joining the endpoints, we have a length $\sqrt{2}$ units by Pythagoras' theorem.

If we now draw an interval of length 1 unit perpendicular to the hypotenuse, as shown in the diagram, and form another right-angled triangle, then the new hypotenuse is $\sqrt{3}$ units in length. We can continue this process, as shown, to construct the numbers $\sqrt{5}$, $\sqrt{6}$ and so on.

Irrational numbers and surds

- Every **real** number is a point on the number line and, conversely, every point on the number line is a real number.
- Every **rational** number is a real number. A real number that is not rational is called an **irrational** number.
- If a is a positive rational number and $\sqrt[n]{a}$ is irrational, then $\sqrt[n]{a}$ is called a **surd**.

Arithmetic with surds

We will review the basic rules for working with square roots.

When we write $2\sqrt{3}$, we mean $2 \times \sqrt{3}$. As in algebra, we can omit the multiplication sign.

If a and b are positive numbers, then:

$$\left(\sqrt{a}\right)^2 = a$$
$$\sqrt{a^2} = a$$
$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$
$$\sqrt{a} \div \sqrt{b} = \sqrt{\frac{a}{b}}$$

For example:

$$\left(\sqrt{11}\right)^2 = 11$$
$$\sqrt{3^2} = 3$$
$$\sqrt{3} \times \sqrt{7} = \sqrt{3 \times 7} = \sqrt{21}$$
$$\sqrt{35} \div \sqrt{5} = \sqrt{\frac{35}{5}} = \sqrt{7}$$

The first two of these rules remind us that, for positive numbers, squaring and taking a square root are **inverse processes**. For example:

$$\left(\sqrt{7}\right)^2 = 7 \text{ and } \sqrt{7^2} = 7$$

Also, $\sqrt{\pi^2} = \pi$ and $\left(\sqrt{\pi}\right)^2 = \pi$. Note that $\sqrt{\pi}$ is not a surd.

Take the surd $\sqrt{12}$. We can factor out the perfect square 4 from 12, and write:

$$\begin{aligned} \sqrt{12} &= \sqrt{4 \times 3} \\ &= \sqrt{4} \times \sqrt{3} \qquad \left(\sqrt{ab} = \sqrt{a} \times \sqrt{b}\right) \\ &= 2\sqrt{3} \end{aligned}$$

Hence, $\sqrt{12}$ and $2\sqrt{3}$ are equal. We will regard $2\sqrt{3}$ as a **simpler form** than $\sqrt{12}$, since the number under the square root sign is smaller.

To **simplify** a surd (or a multiple of a surd), we write it so that the number under the square root sign has no factors that are perfect squares (apart from 1). For example:

$$\sqrt{12} = 2\sqrt{3}$$

We shall also refer to any rational multiple of a surd as a surd. For example, $4\sqrt{7}$ is a surd.

In mathematics, we are often instructed to leave our answers in **surd form**. This means that we should not approximate the answer using a calculator, but leave the answer – in simplest form – expressed using square roots, cube roots etc. This is also called **giving the exact value** of the answer.

We can use our knowledge of factorising whole numbers to simplify surds.

Example 2

Simplify:

a $\sqrt{50}$

b $\sqrt{27}$

Solution

a $\sqrt{50} = \sqrt{25} \times \sqrt{2}$
$= 5\sqrt{2}$

b $\sqrt{27} = \sqrt{9} \times \sqrt{3}$
$= 3\sqrt{3}$

We look for factors of the number under the square root sign that are perfect squares. Sometimes we may need to do this in stages.

Example 3

Simplify:

a $\sqrt{588}$

b $7\sqrt{243}$

c $6\sqrt{162}$

Solution

a
$$\begin{aligned}\sqrt{588} &= \sqrt{4 \times 147}\\ &= 2\sqrt{147}\\ &= 2\sqrt{49 \times 3}\\ &= 2 \times 7\sqrt{3}\\ &= 14\sqrt{3}\end{aligned}$$

b
$$\begin{aligned}7\sqrt{243} &= 7\sqrt{3^5}\\ &= 7\sqrt{3^4 \times 3}\\ &= 7 \times 9\sqrt{3}\\ &= 63\sqrt{3}\end{aligned}$$

c
$$\begin{aligned}6\sqrt{162} &= 6\sqrt{81 \times 2}\\ &= 6\sqrt{9^2 \times 2}\\ &= 54\sqrt{2}\end{aligned}$$

In some problems, we need to reverse this process.

Example 4

Express each as the square root of a whole number.

a $5\sqrt{7}$

b $7\sqrt{6}$

Solution

a $5\sqrt{7} = \sqrt{5^2} \times \sqrt{7}$
$= \sqrt{25 \times 7}$
$= \sqrt{175}$

b $7\sqrt{6} = \sqrt{49 \times 6}$
$= \sqrt{294}$

Example 5

Simplify each expression.

a $\sqrt{5} \times \sqrt{7}$

b $\sqrt{3} \times \sqrt{11}$

c $\sqrt{5} \times \sqrt{30}$

d $\sqrt{3} \times \sqrt{15}$

Solution

a $\sqrt{5} \times \sqrt{7} = \sqrt{35}$

b $\sqrt{3} \times \sqrt{11} = \sqrt{33}$

c $\sqrt{5} \times \sqrt{30} = \sqrt{150}$
$= \sqrt{25 \times 6}$
$= 5\sqrt{6}$

d $\sqrt{3} \times \sqrt{15} = \sqrt{45}$
$= \sqrt{9 \times 5}$
$= 3\sqrt{5}$

Example 6

Simplify each expression.

a $\sqrt{15} \div \sqrt{3}$

b $\dfrac{\sqrt{70}}{\sqrt{14}}$

Solution

a $\sqrt{15} \div \sqrt{3} = \sqrt{\dfrac{15}{3}}$
$= \sqrt{5}$

b $\dfrac{\sqrt{70}}{\sqrt{14}} = \sqrt{\dfrac{70}{14}}$
$= \sqrt{5}$

Algebra of surds

- If a and b are positive numbers, then:

$$\left(\sqrt{a}\right)^2 = a$$
$$\sqrt{a^2} = a$$
$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$
$$\sqrt{a} \div \sqrt{b} = \sqrt{\frac{a}{b}}$$

- A surd is in its **simplest form** if the number under the square root sign has no factors that are perfect squares (apart from 1).
- To simplify a surd, take out any square factors.

  For example, $\sqrt{50} = \sqrt{25 \times 2} = 5\sqrt{2}$

Exercise 2A

Example 1

1 Arrange the irrational numbers $\sqrt{3}, \sqrt{6}, \sqrt[3]{30}$ and $\sqrt[5]{60}$ in order of size on the number line.

Example 2, 3

2 Simplify:

a $\sqrt{8}$ **b** $\sqrt{12}$ **c** $\sqrt{32}$ **d** $\sqrt{50}$ **e** $\sqrt{54}$

f $\sqrt{108}$ **g** $\sqrt{98}$ **h** $\sqrt{200}$ **i** $\sqrt{288}$ **j** $\sqrt{147}$

k $\sqrt{112}$ **l** $\sqrt{175}$ **m** $\sqrt{245}$ **n** $\sqrt{294}$ **o** $\sqrt{225}$

p $\sqrt{900}$ **q** $\sqrt{450}$ **r** $\sqrt{800}$ **s** $\sqrt{1000}$ **t** $\sqrt{1728}$

Example 3

3 Simplify:

a $2\sqrt{75}$ **b** $4\sqrt{125}$ **c** $6\sqrt{99}$ **d** $3\sqrt{150}$ **e** $2\sqrt{720}$

f $5\sqrt{245}$ **g** $6\sqrt{32}$ **h** $7\sqrt{50}$ **i** $11\sqrt{108}$ **j** $56\sqrt{100}$

k $7\sqrt{75}$ **l** $3\sqrt{176}$ **m** $2\sqrt{208}$ **n** $5\sqrt{275}$ **o** $4\sqrt{300}$

Example 4

4 Express each as the square root of a whole number.

a $2\sqrt{2}$ **b** $3\sqrt{5}$ **c** $7\sqrt{3}$ **d** $6\sqrt{6}$ **e** $10\sqrt{3}$

f $4\sqrt{10}$ **g** $11\sqrt{5}$ **h** $7\sqrt{50}$ **i** $6\sqrt{3}$ **j** $3\sqrt{20}$

Example 5

5 Simplify:

a $\sqrt{2} \times \sqrt{3}$ **b** $\sqrt{7} \times \sqrt{11}$ **c** $\sqrt{8} \times \sqrt{5}$ **d** $\sqrt{3} \times \sqrt{13}$ **e** $\sqrt{6} \times \sqrt{8}$

Example 6

6 Simplify:

a $\dfrac{\sqrt{10}}{\sqrt{2}}$ **b** $\dfrac{\sqrt{10}}{\sqrt{5}}$ **c** $\dfrac{\sqrt{30}}{\sqrt{5}}$ **d** $\dfrac{\sqrt{50}}{\sqrt{10}}$ **e** $\dfrac{\sqrt{18}}{\sqrt{6}}$ **f** $\dfrac{\sqrt{24}}{\sqrt{3}}$

7 Simplify:

a $\sqrt{2} \times \sqrt{7}$ **b** $\sqrt{2} \times \sqrt{8}$ **c** $\sqrt{3} \times \sqrt{6}$ **d** $\left(\sqrt{2}\right)^3 \times \left(\sqrt{3}\right)^2$

e $\sqrt{12} \div \sqrt{2}$ **f** $\left(\sqrt{5}\right)^2 - \sqrt{5^2}$ **g** $\sqrt{5^3} \times \sqrt{5}$ **h** $\sqrt{18} \div \sqrt{2}$

8 Complete:

a $\sqrt{5} \times \ldots = \sqrt{30}$ **b** $\sqrt{12} \times \ldots = \sqrt{36}$ **c** $\sqrt{15} \times \ldots = \sqrt{45}$

d $\dfrac{\sqrt{100}}{\ldots} = \sqrt{20}$ **e** $\dfrac{\ldots}{\sqrt{11}} = \sqrt{3}$ **f** $\dfrac{\sqrt{20}}{\ldots} = 2$

9 **a** Find the area of a rectangle with height $\sqrt{7}$ cm and width $\sqrt{3}$ cm.

b Find the area of a triangle with base $\sqrt{6}$ cm and height $\sqrt{5}$ cm.

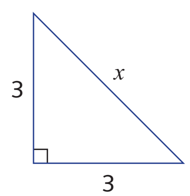
c Find the area of a square with side length $\sqrt{17}$ cm.

d A square has area 11 cm^2. What is the length of each side?

e A square has area 63 cm^2. What is the length of each side?

10 Use Pythagoras' theorem to find the value of x in exact form.

a



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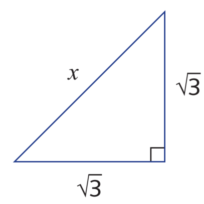

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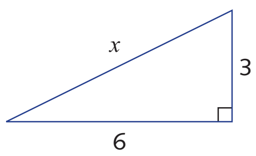

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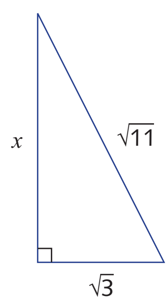

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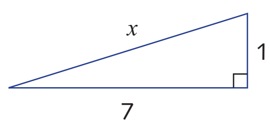

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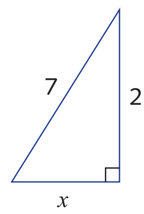


11 A square has side length $4\sqrt{7}$ cm. Find:

a the area of the square **b** the length of a diagonal

12 A rectangle has length 7 cm and width $\sqrt{3}$ cm. Find:

a the area of the rectangle **b** the length of a diagonal

2B Addition and subtraction of surds

Consider the calculation $4\sqrt{7} + 5\sqrt{7} = 9\sqrt{7}$. We can think of this as 4 lots of $\sqrt{7}$ plus 5 lots of $\sqrt{7}$ equals 9 lots of $\sqrt{7}$. This is very similar to algebra, where we write $4x + 5x = 9x$. We regard the numbers $4\sqrt{7}$ and $5\sqrt{7}$ as **like surds** since they are both multiples of $\sqrt{7}$.

On the other hand, in algebra we cannot simplify $4x + 7y$, because $4x$ and $7y$ are not like terms. Similarly, it is not possible to write $4\sqrt{2} + 7\sqrt{3}$ in a simpler way. The surds $4\sqrt{2}$ and $7\sqrt{3}$ are **unlike surds**, since one is a multiple of $\sqrt{2}$ while the other is a multiple of $\sqrt{3}$.

We can only simplify the sum or difference of like surds.

Example 7

Simplify:

a $2\sqrt{2} + 7\sqrt{2} - 4\sqrt{2}$

b $4\sqrt{7} + 3\sqrt{5} - 2\sqrt{5} + 8\sqrt{7}$

Solution

a $2\sqrt{2} + 7\sqrt{2} - 4\sqrt{2} = 5\sqrt{2}$

b $4\sqrt{7} + 3\sqrt{5} - 2\sqrt{5} + 8\sqrt{7} = 12\sqrt{7} + \sqrt{5}$

When dealing with expressions involving surds, we should simplify the surds first and then look for like terms.

Example 8

Simplify:

a $\sqrt{8} + 7\sqrt{2} - \sqrt{32}$

b $\sqrt{27} + 3\sqrt{5} + \sqrt{45} - 4\sqrt{3}$

Solution

a
$$\begin{aligned}\sqrt{8} + 7\sqrt{2} - \sqrt{32} &= 2\sqrt{2} + 7\sqrt{2} - 4\sqrt{2}\\ &= 5\sqrt{2}\end{aligned}$$

b
$$\begin{aligned}\sqrt{27} + 3\sqrt{5} + \sqrt{45} - 4\sqrt{3} &= 3\sqrt{3} + 3\sqrt{5} + 3\sqrt{5} - 4\sqrt{3}\\ &= 6\sqrt{5} - \sqrt{3}\end{aligned}$$

Addition and subtraction of surds

- Simplify each surd first, then look for like surds.
- We can add and subtract like surds.

Exercise 2B

Example 7

1 Simplify:

a $6\sqrt{2} + 7\sqrt{2}$

b $12\sqrt{3} + 13\sqrt{3}$

c $-6\sqrt{2} + 9\sqrt{2}$

d $-13\sqrt{5} - 14\sqrt{5}$

e $-19\sqrt{3} + 21\sqrt{3} - 4\sqrt{3}$

f $13\sqrt{5} - 16\sqrt{5} + 25\sqrt{5}$

2 Simplify:

a $5\sqrt{2} + 6\sqrt{3} + 7\sqrt{2} - 4\sqrt{3}$

b $7\sqrt{7} - 4\sqrt{5} + 3\sqrt{7} - 6\sqrt{5}$

c $8\sqrt{11} - 7\sqrt{10} + 5\sqrt{11} + 4\sqrt{10}$

d $\sqrt{3} + 4\sqrt{2} - 5\sqrt{3} + 6\sqrt{2}$

e $9\sqrt{15} - 4\sqrt{7} - 3\sqrt{15}$

f $8\sqrt{5} + 5\sqrt{8} + 3\sqrt{5} - 6\sqrt{8}$

3 Complete:

a $5\sqrt{2} + \ldots = 11\sqrt{2}$

b $9\sqrt{3} + \ldots = 14\sqrt{3}$

c $6\sqrt{5} - \ldots = \sqrt{5}$

d $11\sqrt{2} - \ldots = -4\sqrt{2}$

e $7\sqrt{3} + \ldots = 2\sqrt{3}$

f $4\sqrt{5} - \ldots = 6\sqrt{5}$

g $2\sqrt{3} + 4\sqrt{5} + \ldots = 5\sqrt{3} + 8\sqrt{5}$

h $7\sqrt{11} - 6\sqrt{5} + \ldots = 8\sqrt{11} + 2\sqrt{5}$

i $9\sqrt{10} - 4\sqrt{3} + \ldots = \sqrt{10} - \sqrt{3}$

j $6\sqrt{5} + 3\sqrt{2} + \ldots = 2\sqrt{5} - 5\sqrt{2}$

Example 8

4 Simplify:

a $\sqrt{12} + \sqrt{27}$

b $\sqrt{8} + \sqrt{18}$

c $3\sqrt{8} - 4\sqrt{2}$

d $\sqrt{45} - 3\sqrt{20}$

e $3\sqrt{32} - 4\sqrt{27} + 5\sqrt{18}$

f $5\sqrt{147} + 3\sqrt{48} - \sqrt{12}$

g $3\sqrt{45} + \sqrt{20} + 7\sqrt{5}$

h $4\sqrt{63} + 5\sqrt{7} - 8\sqrt{28}$

i $\sqrt{44} + 5\sqrt{176} + 2\sqrt{99}$

j $2\sqrt{363} - 5\sqrt{243} + \sqrt{192}$

5 Simplify:

a $\sqrt{72} - \sqrt{50}$

b $\sqrt{48} + \sqrt{12}$

c $\sqrt{8} + \sqrt{2} + \sqrt{18}$

d $\sqrt{12} + 4\sqrt{3} - \sqrt{75}$

e $\sqrt{32} - \sqrt{200} + 3\sqrt{50}$

f $4\sqrt{5} - 4\sqrt{20} - \sqrt{45}$

g $\sqrt{54} + \sqrt{24}$

h $\sqrt{27} - \sqrt{48} + \sqrt{75}$

i $\sqrt{45} + \sqrt{80} - \sqrt{125}$

j $\sqrt{2} + \sqrt{32} + \sqrt{72}$

k $3\sqrt{20} - 4\sqrt{5} + \frac{1}{2}\sqrt{5}$

l $5\sqrt{18} - 3\sqrt{20} - 4\sqrt{5}$

6 Simplify:

a $\sqrt{12} + 3\sqrt{8} - 2\sqrt{27} + \sqrt{32}$

b $4\sqrt{18} - 2\sqrt{20} + 3\sqrt{5} + 6\sqrt{8}$

c $6\sqrt{12} + 9\sqrt{40} - 2\sqrt{27} - \sqrt{90}$

d $4\sqrt{27} - 3\sqrt{18} + 2\sqrt{108} - \sqrt{200}$

7 Find the value of x if:

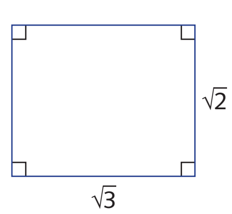
a $\sqrt{63} - \sqrt{28} = \sqrt{x}$

b $\sqrt{80} - \sqrt{45} = \sqrt{x}$

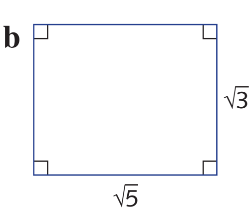
c $\sqrt{54} - 2\sqrt{24} = -\sqrt{x}$

8 For each rectangle, find (in exact form) the perimeter and area, and the length of the diagonal.

a



b



2C Multiplication and division of surds

When multiplying two surds, we multiply the numbers outside the square root sign together and, similarly, multiply the numbers under the square root sign. A similar procedure applies for division. These procedures are captured by the following general rules:

$$a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd},$$

where b and d are positive numbers.

$$a\sqrt{b} \div c\sqrt{d} = \frac{a}{c}\sqrt{\frac{b}{d}},$$

where b and d are positive numbers and $c \neq 0$.

As usual, we should always give the answer in simplest form.

Example 9

Find:

a $5\sqrt{7} \times 3\sqrt{2}$

b $15\sqrt{77} \div 3\sqrt{7}$

Solution

a $5\sqrt{7} \times 3\sqrt{2} = 15\sqrt{14}$

$\left(\text{This is } 5 \times \sqrt{7} \times 3 \times \sqrt{2}\right)$

b $15\sqrt{77} \div 3\sqrt{7} = \dfrac{15 \times \sqrt{77}}{3 \times \sqrt{7}}$

$= 5\sqrt{11}$

$\left(15 \div 3 = 5 \text{ and } \sqrt{77} \div \sqrt{7} = \sqrt{11}\right)$

Example 10

Find:

a $5\sqrt{6} \times 7\sqrt{10}$

b $\dfrac{18\sqrt{10}}{3\sqrt{5}}$

c $\left(2\sqrt{7}\right)^2$

d $\left(2\sqrt{3}\right)^3$

Solution

a $5\sqrt{6} \times 7\sqrt{10} = 35\sqrt{60}$

$= 35\sqrt{4 \times 15}$

$= 70\sqrt{15}$

b $\dfrac{18\sqrt{10}}{3\sqrt{5}} = 6\sqrt{2}$

c $\left(2\sqrt{7}\right)^2 = 2\sqrt{7} \times 2\sqrt{7}$

$= 4 \times 7$

$= 28$

d $\left(2\sqrt{3}\right)^3 = 2\sqrt{3} \times 2\sqrt{3} \times 2\sqrt{3}$

$= 4 \times 3 \times 2 \times \sqrt{3}$

$= 24\sqrt{3}$

The distributive law

We can apply the distributive law to expressions involving surds, just as we do in algebra.

Example 11

Expand and simplify:

a $2\sqrt{5}\left(6+3\sqrt{5}\right)$

b $-4\sqrt{3}\left(\sqrt{6}-2\sqrt{3}\right)$

Solution

a $$\begin{aligned}2\sqrt{5}\left(6+3\sqrt{5}\right) &= 12\sqrt{5}+6\sqrt{25}\\ &= 12\sqrt{5}+30\end{aligned}$$

b $$\begin{aligned}-4\sqrt{3}\left(\sqrt{6}-2\sqrt{3}\right) &= -4\sqrt{18}+8\times 3\\ &= -12\sqrt{2}+24\end{aligned}$$

In algebra, you learned how to expand brackets such as $(a+b)(c+d)$. These are known as **binomial products**. You multiply each term in the second bracket by each term in the first, then add. This means you expand out $a(c+d)+b(c+d)$ to obtain $ac+ad+bc+bd$. We use this idea again when multiplying out binomial products involving surds. (Remember to be very careful with the signs.)

Example 12

Expand and simplify:

a $\left(2\sqrt{3}-1\right)\left(4\sqrt{3}+2\right)$

b $\left(3\sqrt{2}-4\sqrt{3}\right)\left(5\sqrt{3}-\sqrt{2}\right)$

c $\left(\sqrt{2}+\sqrt{3}\right)\left(\sqrt{5}-\sqrt{7}\right)$

d $\left(\sqrt{2}+\sqrt{3}\right)\left(\sqrt{2}-\sqrt{3}\right)$

Solution

a $$\begin{aligned}\left(2\sqrt{3}-1\right)\left(4\sqrt{3}+2\right) &= 2\sqrt{3}\left(4\sqrt{3}+2\right)-1\left(4\sqrt{3}+2\right)\\ &= 8\sqrt{9}+4\sqrt{3}-4\sqrt{3}-2\\ &= 24-2\\ &= 22\end{aligned}$$

b $$\begin{aligned}\left(3\sqrt{2}-4\sqrt{3}\right)\left(5\sqrt{3}-\sqrt{2}\right) &= 3\sqrt{2}\left(5\sqrt{3}-\sqrt{2}\right)-4\sqrt{3}\left(5\sqrt{3}-\sqrt{2}\right)\\ &= 15\sqrt{6}-3\sqrt{4}-20\sqrt{9}+4\sqrt{6}\\ &= 15\sqrt{6}-6-60+4\sqrt{6}\\ &= 19\sqrt{6}-66\end{aligned}$$

c $$\begin{aligned}\left(\sqrt{2}+\sqrt{3}\right)\left(\sqrt{5}-\sqrt{7}\right) &= \sqrt{2}\left(\sqrt{5}-\sqrt{7}\right)+\sqrt{3}\left(\sqrt{5}-\sqrt{7}\right)\\ &= \sqrt{10}-\sqrt{14}+\sqrt{15}-\sqrt{21}\end{aligned}$$

d $$\begin{aligned}\left(\sqrt{2}+\sqrt{3}\right)\left(\sqrt{2}-\sqrt{3}\right) &= \sqrt{2}\left(\sqrt{2}-\sqrt{3}\right)+\sqrt{3}\left(\sqrt{2}-\sqrt{3}\right)\\ &= 2-\sqrt{6}+\sqrt{6}-3\\ &= -1\end{aligned}$$

Multiplication and division of surds

- For positive numbers b and d, $a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd}$.
- For positive numbers b and d, $a\sqrt{b} \div c\sqrt{d} = \frac{a}{c}\sqrt{\frac{b}{d}}$, where $c \neq 0$ and $d \neq 0$.
- We can apply the distributive law to expressions involving surds.
- We can expand binomial products involving surds just as we do in algebra:
  $(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd$.
- Always give the answer in simplest form.

Exercise 2C

1 Simplify:

a $\sqrt{5} \times \sqrt{3}$ **b** $\sqrt{5} \times \sqrt{11}$ **c** $\sqrt{8} \times \sqrt{14}$

d $\sqrt{5} \times \sqrt{13}$ **e** $\sqrt{6} \times \sqrt{2}$ **f** $\sqrt{18} \times \sqrt{3}$

2 Simplify:

a $\dfrac{\sqrt{30}}{\sqrt{6}}$ **b** $\dfrac{\sqrt{6}}{\sqrt{3}}$ **c** $\dfrac{\sqrt{42}}{\sqrt{7}}$

d $\dfrac{\sqrt{35}}{\sqrt{7}}$ **e** $\dfrac{\sqrt{33}}{\sqrt{11}}$ **f** $\dfrac{\sqrt{40}}{\sqrt{5}}$

Example 9

3 Simplify:

a $4\sqrt{2} \times 3\sqrt{5}$ **b** $7\sqrt{3} \times 11\sqrt{5}$ **c** $9\sqrt{5} \times 6\sqrt{7}$

d $8\sqrt{2} \times 4\sqrt{3}$ **e** $-6\sqrt{3} \times 5\sqrt{2}$ **f** $7\sqrt{2} \times \left(-4\sqrt{11}\right)$

g $-3\sqrt{7} \times \left(-4\sqrt{11}\right)$ **h** $-6\sqrt{2} \times \left(-3\sqrt{7}\right)$ **i** $11\sqrt{7} \times \left(-2\sqrt{6}\right)$

Example 9

4 Simplify:

a $\dfrac{12\sqrt{6}}{6\sqrt{2}}$ **b** $\dfrac{25\sqrt{15}}{10\sqrt{3}}$

c $\dfrac{8\sqrt{48}}{12\sqrt{8}}$ **d** $\dfrac{-20\sqrt{6}}{8\sqrt{3}}$

e $\dfrac{-32\sqrt{45}}{16\sqrt{15}}$ **f** $\dfrac{8\sqrt{3}}{24\sqrt{6}}$

5 Simplify:

a $\sqrt{2} \times \sqrt{2}$ **b** $\sqrt{11} \times \sqrt{11}$ **c** $3\sqrt{3} \times \sqrt{3}$

d $8\sqrt{5} \times \sqrt{5}$ **e** $3\sqrt{2} \times 5\sqrt{2}$ **f** $6\sqrt{3} \times 5\sqrt{3}$

g $4\sqrt{7} \times \left(-2\sqrt{7}\right)$ **h** $-2\sqrt{3} \times \left(-4\sqrt{3}\right)$ **i** $7\sqrt{6} \times \left(-3\sqrt{6}\right)$

6 Complete:

a $2\sqrt{3} \times \ldots = 6$
b $4\sqrt{2} \times \ldots = 8$
c $5\sqrt{2} \times \ldots = 20$
d $2\sqrt{3} \times \ldots = 18$
e $2\sqrt{7} \times \ldots = 42$
f $2\sqrt{5} \times \ldots = 60$
g $8\sqrt{2} \times \ldots = 96$
h $3\sqrt{3} \times \ldots = 108$
i $2\sqrt{5} \times \ldots = 100$
j $3\sqrt{8} \times \ldots = 96$
k $\sqrt{2} \times \ldots = 64$
l $4\sqrt{5} \times \ldots = 1000$

7 Complete:

a $9\sqrt{5} \times \ldots = -27\sqrt{15}$
b $6\sqrt{2} \times \ldots = -18\sqrt{10}$
c $\dfrac{15\sqrt{6}}{\ldots} = 5\sqrt{2}$
d $\dfrac{28\sqrt{22}}{\ldots} = 4\sqrt{11}$
e $\dfrac{\ldots}{5\sqrt{7}} = 5\sqrt{5}$
f $\dfrac{\ldots}{3\sqrt{7}} = 8\sqrt{6}$
g $4\sqrt{2} \times (\ldots) + 8\sqrt{6} = 20\sqrt{6}$
h $3\sqrt{5} \times (\ldots) - 2\sqrt{10} = 16\sqrt{10}$
i $\dfrac{12\sqrt{6}}{\ldots} + 3\sqrt{2} = 7\sqrt{2}$
j $\dfrac{\ldots}{8\sqrt{3}} - 4\sqrt{5} = -2\sqrt{5}$

Example 10a, b

8 Simplify:

a $2\sqrt{3} \times 4\sqrt{2} + 8\sqrt{6}$
b $7\sqrt{3} \times 5\sqrt{5} + 8\sqrt{15}$
c $16\sqrt{6} - 2\sqrt{3} \times 5\sqrt{2}$
d $18\sqrt{10} - 3\sqrt{5} \times 4\sqrt{2}$
e $3\sqrt{6} \times 5\sqrt{5} - 8\sqrt{15} \times 4\sqrt{2}$
f $8\sqrt{20} \times 3\sqrt{2} - 5\sqrt{5} \times 5\sqrt{8}$
g $\dfrac{12\sqrt{6}}{4\sqrt{3}} + 5\sqrt{2}$
h $\dfrac{16\sqrt{15}}{4\sqrt{3}} - 8\sqrt{5}$
i $\dfrac{5\sqrt{20}}{10\sqrt{10}} + \dfrac{3\sqrt{2}}{2}$
j $\dfrac{6\sqrt{10}}{9\sqrt{5}} + \dfrac{\sqrt{2}}{3}$

Example 10c

9 Simplify:

a $(2\sqrt{2})^2$
b $(2\sqrt{3})^2$
c $(3\sqrt{5})^2$
d $(5\sqrt{6})^2$
e $(3\sqrt{7})^2$
f $(2\sqrt{11})^2$
g $(5\sqrt{10})^2$
h $(a\sqrt{b})^2$

Example 10d

10 Simplify:

a $(\sqrt{2})^1$
b $(\sqrt{2})^2$
c $(\sqrt{2})^3$
d $(\sqrt{2})^4$
e $(\sqrt{2})^5$
f $(\sqrt{2})^6$
g $(\sqrt{3})^3$
h $(\sqrt{5})^3$
i $(2\sqrt{2})^3$
j $(3\sqrt{2})^3$
k $(4\sqrt{3})^3$
l $(2\sqrt{5})^3$
m $(5\sqrt{5})^3$
n $(2\sqrt{7})^3$
o $(2\sqrt{2})^6$
p $(3\sqrt{3})^5$

Example 11 **11** Expand and simplify:

a $\sqrt{2}(\sqrt{3}+\sqrt{5})$ **b** $\sqrt{7}(\sqrt{5}+\sqrt{6})$ **c** $\sqrt{5}(\sqrt{7}-\sqrt{2})$

d $3\sqrt{2}(2\sqrt{5}-3\sqrt{3})$ **e** $4\sqrt{3}(5\sqrt{2}+6\sqrt{5})$ **f** $4\sqrt{3}(\sqrt{2}-1)$

g $3\sqrt{5}(2\sqrt{3}+\sqrt{5})$ **h** $2\sqrt{6}(3\sqrt{3}+2\sqrt{2})$ **i** $4\sqrt{10}(3\sqrt{5}-4\sqrt{2})$

j $2\sqrt{3}(4\sqrt{3}-\sqrt{6})$ **k** $3\sqrt{2}(5\sqrt{2}+4\sqrt{10})$ **l** $3\sqrt{6}(4\sqrt{3}-\sqrt{6})$

m $4\sqrt{5}(2\sqrt{20}-3\sqrt{8})$ **n** $3\sqrt{7}(5\sqrt{35}-2\sqrt{21})$ **o** $3\sqrt{11}(2\sqrt{22}-4\sqrt{33})$

Example 12 **12** Expand and simplify:

a $(\sqrt{3}+\sqrt{2})(\sqrt{5}+\sqrt{7})$ **b** $(\sqrt{5}+\sqrt{7})(\sqrt{11}+\sqrt{6})$

c $(\sqrt{5}+\sqrt{2})(\sqrt{3}-\sqrt{7})$ **d** $(\sqrt{3}+\sqrt{6})(\sqrt{5}-\sqrt{7})$

e $(3\sqrt{2}+4\sqrt{3})(2\sqrt{2}-\sqrt{3})$ **f** $(5\sqrt{3}-\sqrt{5})(2\sqrt{3}-3\sqrt{5})$

g $(4\sqrt{5}+1)(2\sqrt{5}-3)$ **h** $(2\sqrt{3}+3\sqrt{6})(5\sqrt{2}-\sqrt{6})$

i $(3\sqrt{2}+\sqrt{7})(4\sqrt{2}-5\sqrt{7})$ **j** $(2\sqrt{7}+\sqrt{5})(\sqrt{3}-2\sqrt{5})$

k $(2\sqrt{2}-\sqrt{7})(5\sqrt{2}-2\sqrt{7})$ **l** $(3\sqrt{7}-8)(\sqrt{3}-3\sqrt{5})$

13 If $x = 2\sqrt{3}$ and $y = -3\sqrt{6}$, find:

a xy **b** $\frac{y}{x}$ **c** $x^2 + y^2$ **d** $\frac{1}{x^2} + \frac{1}{y^2}$

e x^3 **f** x^3y^2 **g** $\frac{y^2}{x^3}$ **h** $x^2 - y^2$

14 If $x = 2 + \sqrt{3}$ and $y = 2 - \sqrt{3}$, find:

a $x + y$ **b** $x + 2y$ **c** $3x + 2y$ **d** $x - y$

e $x - 2y$ **f** xy **g** $\sqrt{3xy}$ **h** $\sqrt{xy}$

15 Find the area and perimeter of a rectangle with:

a length $2\sqrt{3}$ and width $4\sqrt{2}$

b length $2\sqrt{3}$ and width $4\sqrt{3}$

c length $7 + 2\sqrt{5}$ and width $7 - 2\sqrt{5}$

d length $1 + \sqrt{5}$ and width $2 + \sqrt{5}$

16 The hypotenuse of a right-angled triangle has length $8 + \sqrt{3}$. Another side has length $4\sqrt{3} + 2$.

Find:

a the length of the third side

b the perimeter of the triangle

c the area of the triangle

17 A square has side length $2 + 5\sqrt{3}$.
Find:

a the perimeter of the square

b the area of the square

2D Special products

In algebra, you learned the following special identities. Recognising and applying these identities is important.

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$
$$(a - b)(a + b) = a^2 - b^2$$

These identities are also useful when dealing with surds, and the following examples demonstrate this use. The last identity listed above is known as the **difference of squares** identity, and we will pay particular attention to this.

Example 13

Expand and simplify:

a $\left(\sqrt{7} + \sqrt{3}\right)^2$ **b** $\left(5\sqrt{2} - \sqrt{3}\right)^2$ **c** $\left(\sqrt{3} + 5\sqrt{6}\right)^2$

Solution

a
$$\begin{aligned}\left(\sqrt{7} + \sqrt{3}\right)^2 &= \left(\sqrt{7}\right)^2 + 2\left(\sqrt{7}\right)\left(\sqrt{3}\right) + \left(\sqrt{3}\right)^2\\ &= 7 + 2\sqrt{21} + 3\\ &= 10 + 2\sqrt{21}\end{aligned}$$

b
$$\begin{aligned}\left(5\sqrt{2} - \sqrt{3}\right)^2 &= \left(5\sqrt{2}\right)^2 - 2\left(5\sqrt{2}\right)\left(\sqrt{3}\right) + \left(\sqrt{3}\right)^2\\ &= 50 - 10\sqrt{6} + 3\\ &= 53 - 10\sqrt{6}\end{aligned}$$

c
$$\begin{aligned}\left(\sqrt{3} + 5\sqrt{6}\right)^2 &= \left(\sqrt{3}\right)^2 + 2\left(\sqrt{3}\right)\left(5\sqrt{6}\right) + \left(5\sqrt{6}\right)^2\\ &= 3 + 10\sqrt{18} + 150\\ &= 153 + 30\sqrt{2}\end{aligned}$$

You should always express your answer in simplest form.

Notice what happens when we use the difference of squares identity.

Example 14

Expand and simplify:

a $\left(\sqrt{11} - \sqrt{5}\right)\left(\sqrt{11} + \sqrt{5}\right)$ **b** $\left(2\sqrt{3} + \sqrt{4}\right)\left(2\sqrt{3} - \sqrt{4}\right)$

Solution

a
$$\begin{aligned}\left(\sqrt{11} - \sqrt{5}\right)\left(\sqrt{11} + \sqrt{5}\right) &= \left(\sqrt{11}\right)^2 - \left(\sqrt{5}\right)^2\\ &= 11 - 5\\ &= 6\end{aligned}$$

b
$$\begin{aligned}\left(2\sqrt{3} + 4\right)\left(2\sqrt{3} - 4\right) &= \left(2\sqrt{3}\right)^2 - (4)^2\\ &= 12 - 16\\ &= -4\end{aligned}$$

When we apply the difference of squares identity, the answer is an integer.

Special products

We can apply the identities

$(a + b)^2 = a^2 + 2ab + b^2$

$(a - b)^2 = a^2 - 2ab + b^2$

$(a - b)(a + b) = a^2 - b^2$

to calculations involving surds.

Exercise 2D

Example 13

1 Simplify:

a $\left(\sqrt{5} + \sqrt{2}\right)^2$ **b** $\left(\sqrt{3} + \sqrt{7}\right)^2$

c $\left(\sqrt{2} - \sqrt{3}\right)^2$ **d** $\left(\sqrt{7} - \sqrt{6}\right)^2$

e $\left(2\sqrt{3} + 1\right)^2$ **f** $\left(3\sqrt{2} - 2\right)^2$

g $\left(2\sqrt{3} + \sqrt{2}\right)^2$ **h** $\left(4\sqrt{2} - 3\sqrt{3}\right)^2$

i $\left(2\sqrt{5} + 3\sqrt{7}\right)^2$ **j** $\left(3\sqrt{2} - 4\sqrt{5}\right)^2$

k $\left(4 - 3\sqrt{2}\right)^2$ **l** $\left(2 - 5\sqrt{3}\right)^2$

m $\left(\frac{1}{2} - \sqrt{3}\right)^2$ **n** $\left(5 - \frac{\sqrt{3}}{2}\right)^2$

Example 14

2 Simplify:

a $\left(4 - \sqrt{3}\right)\left(4 + \sqrt{3}\right)$ **b** $\left(\sqrt{7} + 2\right)\left(\sqrt{7} - 2\right)$ **c** $\left(\sqrt{5} + \sqrt{3}\right)\left(\sqrt{5} - \sqrt{3}\right)$

d $\left(\sqrt{6} - \sqrt{5}\right)\left(\sqrt{6} + \sqrt{5}\right)$ **e** $\left(2\sqrt{3} + 1\right)\left(2\sqrt{3} - 1\right)$ **f** $\left(3\sqrt{2} + 4\right)\left(3\sqrt{2} - 4\right)$

g $\left(3\sqrt{2} + 2\sqrt{3}\right)\left(3\sqrt{2} - 2\sqrt{3}\right)$ **h** $\left(3\sqrt{6} + 2\sqrt{5}\right)\left(3\sqrt{6} - 2\sqrt{5}\right)$ **i** $\left(2\sqrt{2} + \sqrt{7}\right)\left(2\sqrt{2} - \sqrt{7}\right)$

j $\left(3\sqrt{5} - 4\sqrt{3}\right)\left(3\sqrt{5} + 4\sqrt{3}\right)$ **k** $\left(\frac{1}{2} - \frac{1}{2}\sqrt{3}\right)\left(\frac{1}{2} + \frac{1}{2}\sqrt{3}\right)$ **l** $\left(3 - \frac{\sqrt{3}}{2}\right)\left(3 + \frac{\sqrt{3}}{2}\right)$

3 Find the area of a square with side length:

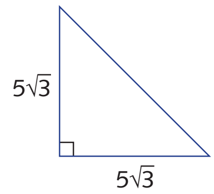
a $2 + \sqrt{3}$ **b** $2 - \sqrt{3}$ **c** $5 + 2\sqrt{3}$ **d** $5 - 2\sqrt{3}$

4 If $x = 2 + \sqrt{5}$ and $y = \sqrt{5} - 2$, find:

a xy **b** $x + y$ **c** $x^2 + y^2$ **d** $x^2 - y^2$

5 **a** Find the area of the triangle.

b Find the length of the hypotenuse.



6 If $x = 5 + 2\sqrt{3}$ and $y = 2 + 5\sqrt{3}$, find:

a $x + y$ **b** $x - y$ **c** xy **d** $x^2 + y^2$ **e** $x^2 - y^2$

7 The two shorter sides of a right-angled triangle have length $7 + 2\sqrt{3}$ and $7 - 2\sqrt{3}$.

Find:

a the length of the hypotenuse of the triangle

b the perimeter of the triangle

c the area of the triangle

2E Rationalising denominators

In the expression $\frac{2}{\sqrt{3}} + \frac{\sqrt{3}}{2}$, the first term has a square root in the denominator. This makes it difficult to tell if the surds are like surds or not.

Fractions involving surds are usually easier to deal with when the surd is in the numerator and there is a whole number in the denominator.

To express a fraction in such a way is called **rationalising the denominator**.

When we multiply the numerator and denominator of a fraction by the same number, we form an equivalent fraction. The same happens with a quotient involving surds.

Example 15

Rationalise the denominator of:

a $\frac{1}{\sqrt{3}}$ **b** $\frac{4}{2\sqrt{5}}$ **c** $\frac{4\sqrt{3} - 1}{4\sqrt{6}}$

Solution

a $$\begin{aligned}\frac{1}{\sqrt{3}} &= \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}\\ &= \frac{\sqrt{3}}{3}\end{aligned}$$

b $$\begin{aligned}\frac{4}{2\sqrt{5}} &= \frac{4}{2\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}\\ &= \frac{4\sqrt{5}}{10}\\ &= \frac{2\sqrt{5}}{5}\end{aligned}$$

c $$\begin{aligned}\frac{4\sqrt{3} - 1}{4\sqrt{6}} &= \frac{4\sqrt{3} - 1}{4\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}\\ &= \frac{4\sqrt{18} - \sqrt{6}}{24}\\ &= \frac{4\sqrt{9 \times 2} - \sqrt{6}}{24}\\ &= \frac{12\sqrt{2} - \sqrt{6}}{24}\end{aligned}$$

Example 16

Simplify $\dfrac{2}{\sqrt{3}} + \dfrac{\sqrt{3}}{2}$.

Solution

$$\begin{aligned}\frac{2}{\sqrt{3}} + \frac{\sqrt{3}}{2} &= \frac{2\sqrt{3}}{3} + \frac{\sqrt{3}}{2} \\ &= \frac{4\sqrt{3} + 3\sqrt{3}}{6} \\ &= \frac{7\sqrt{3}}{6}\end{aligned} \qquad \left(\frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}\right)$$

Binomial denominators

In the expression $\dfrac{1}{\sqrt{7} - \sqrt{5}}$, it is more difficult to remove the surds from the denominator.

In the following example, we explain how this can be done by using the difference of squares identity.

In the section on special products, we saw that:

$$\begin{aligned}\left(\sqrt{7} - \sqrt{5}\right)\left(\sqrt{7} + \sqrt{5}\right) &= \left(\sqrt{7}\right)^2 - \left(\sqrt{5}\right)^2 \\ &= 7 - 5 \\ &= 2, \qquad \text{which is rational.}\end{aligned}$$

So:

$$\begin{aligned}\frac{1}{\sqrt{7} - \sqrt{5}} &= \frac{1}{\sqrt{7} - \sqrt{5}} \times \frac{\sqrt{7} + \sqrt{5}}{\sqrt{7} + \sqrt{5}} \\ &= \frac{\sqrt{7} + \sqrt{5}}{7 - 5} \\ &= \frac{\sqrt{7} + \sqrt{5}}{2}\end{aligned}$$

Using the difference of two squares identity in this way is an important and initially surprising technique.

Example 17

Rationalise the denominators of the following, simplifying where possible.

a $\dfrac{2\sqrt{5}}{2\sqrt{5} - 2}$

b $\dfrac{\sqrt{3} + \sqrt{2}}{3\sqrt{2} + 2\sqrt{3}}$

Solution

a
$$\frac{2\sqrt{5}}{2\sqrt{5}-2} = \frac{2\sqrt{5}}{2\sqrt{5}-2} \times \frac{2\sqrt{5}+2}{2\sqrt{5}+2}$$
$$= \frac{20+4\sqrt{5}}{20-4}$$
$$= \frac{20+4\sqrt{5}}{16}$$
$$= \frac{4\left(5+\sqrt{5}\right)}{16}$$
$$= \frac{5+\sqrt{5}}{4}$$

b
$$\frac{\sqrt{3}+\sqrt{2}}{3\sqrt{2}+2\sqrt{3}} = \frac{\sqrt{3}+\sqrt{2}}{3\sqrt{2}+2\sqrt{3}} \times \frac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}$$
$$= \frac{\left(\sqrt{3}+\sqrt{2}\right)\left(3\sqrt{2}-2\sqrt{3}\right)}{18-12}$$
$$= \frac{3\sqrt{6}-6+6-2\sqrt{6}}{6}$$
$$= \frac{\sqrt{6}}{6}$$

Rationalising denominators

- To rationalise the denominator of $\frac{2}{\sqrt{3}}$, multiply top and bottom by $\sqrt{3}$.
- To rationalise a denominator with two terms, we use the difference of squares identity.
  - In an expression such as $\frac{3}{5+\sqrt{3}}$, multiply top and bottom by $5-\sqrt{3}$.
  - In an expression such as $\frac{\sqrt{2}}{7-3\sqrt{2}}$, multiply top and bottom by $7+3\sqrt{2}$.
- Rationalising a denominator allows us to identify like and unlike surds.

Exercise 2E

Example 15

1 Rationalise the denominator and simplify:

a $\frac{5}{\sqrt{3}}$ **b** $\frac{6}{\sqrt{2}}$ **c** $\frac{7}{\sqrt{7}}$

d $\frac{3}{\sqrt{5}}$ **e** $\frac{3}{\sqrt{3}}$ **f** $\frac{\sqrt{5}}{\sqrt{2}}$

g $\frac{\sqrt{6}}{\sqrt{3}}$ **h** $\frac{2}{3\sqrt{2}}$ **i** $\frac{4}{3\sqrt{6}}$

j $\frac{2\sqrt{3}}{3\sqrt{5}}$ **k** $\frac{3\sqrt{5}}{4\sqrt{3}}$ **l** $\frac{8+\sqrt{3}}{2\sqrt{3}}$

m $\frac{3-2\sqrt{2}}{5\sqrt{2}}$ **n** $\frac{3\sqrt{2}+4\sqrt{3}}{3\sqrt{2}}$ **o** $\frac{\sqrt{5}-2\sqrt{3}}{4\sqrt{2}}$

Example 16

2 Rationalise the denominator and simplify:

a $\frac{3}{\sqrt{2}} + \frac{4}{\sqrt{3}}$ **b** $\frac{3}{\sqrt{7}} - \frac{2}{\sqrt{5}}$ **c** $\frac{3\sqrt{2}}{4\sqrt{3}} + \frac{1}{2\sqrt{5}}$ **d** $\frac{3\sqrt{2}}{\sqrt{7}} - \frac{3}{2\sqrt{2}}$

e $\frac{4}{\sqrt{7}} - \frac{2}{\sqrt{5}}$ **f** $\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{11}}$ **g** $\frac{4}{2\sqrt{3}} - \frac{5}{\sqrt{7}}$ **h** $\frac{2}{\sqrt{3}} - \frac{4}{\sqrt{11}}$

Example 17

3 Rationalise the denominator and simplify:

a $\frac{1}{\sqrt{5}-2}$ **b** $\frac{1}{7+\sqrt{6}}$ **c** $\frac{3}{\sqrt{6}+2}$ **d** $\frac{6}{2-\sqrt{3}}$ **e** $\frac{2}{\sqrt{3}-\sqrt{2}}$

f $\frac{4}{\sqrt{5}-\sqrt{2}}$ **g** $\frac{\sqrt{3}}{\sqrt{5}+\sqrt{2}}$ **h** $\frac{\sqrt{7}}{\sqrt{3}+\sqrt{2}}$ **i** $\frac{2}{\sqrt{5}-\sqrt{3}}$ **j** $\frac{4}{\sqrt{7}-\sqrt{3}}$

k $\frac{4\sqrt{2}}{2\sqrt{2}-\sqrt{3}}$ **l** $\frac{2\sqrt{3}}{\sqrt{3}+2\sqrt{2}}$ **m** $\frac{3\sqrt{5}}{3\sqrt{2}-\sqrt{10}}$ **n** $\frac{2\sqrt{3}}{3\sqrt{2}-1}$ **o** $\frac{4\sqrt{2}}{2\sqrt{3}+3}$

p $\frac{4\sqrt{2}+\sqrt{3}}{3\sqrt{2}-\sqrt{3}}$ **q** $\frac{2\sqrt{3}+\sqrt{5}}{2\sqrt{5}-3\sqrt{3}}$ **r** $\frac{2\sqrt{5}+1}{2\sqrt{5}-1}$ **s** $\frac{3\sqrt{2}+\sqrt{5}}{4\sqrt{2}-\sqrt{5}}$ **t** $\frac{3\sqrt{2}-\sqrt{5}}{3\sqrt{2}+\sqrt{5}}$

4 Rationalise the denominator and simplify:

a $\frac{1}{\sqrt{3}+\sqrt{2}} + \frac{2\sqrt{3}-1}{2\sqrt{3}-2}$ **b** $\frac{3}{2\sqrt{5}+1} + \frac{1}{\sqrt{5}-\sqrt{2}}$

c $\frac{3\sqrt{2}}{\sqrt{2}+\sqrt{5}} + \frac{4\sqrt{2}}{3\sqrt{2}-1}$ **d** $\frac{2}{2\sqrt{5}+\sqrt{3}} - \frac{4}{2\sqrt{5}-\sqrt{3}}$

e $\frac{4}{2\sqrt{3}+\sqrt{2}} - \frac{1}{2\sqrt{3}-\sqrt{2}}$ **f** $\frac{\sqrt{2}+\sqrt{3}}{\sqrt{2}-\sqrt{3}} + \frac{\sqrt{2}-\sqrt{3}}{\sqrt{2}+\sqrt{3}}$

5 If $x = 2\sqrt{3}$ and $y = \frac{7}{\sqrt{3}}$, find, in simplest form:

a $x + y$ **b** $x - y$ **c** xy **d** $\frac{x}{y}$ **e** $x^3 + y^3$

6 A rectangle has area 10. The length of the rectangle is $\sqrt{2} - 1$. Find the width of the rectangle in simplest form.

7 If $x = \sqrt{3} + 1$ and $y = \sqrt{3} + 2$, find, in simplest form:

a $\frac{1}{x} + \frac{1}{y}$ **b** $\frac{1}{x^2} + \frac{1}{y^2}$ **c** x^3 **d** y^3

8 Given that $x = \frac{2}{3\sqrt{2}-1}$, find the value of each expression, giving answers in simplest form with a rational denominator.

a x^2 **b** $\frac{1}{x}$ **c** $\frac{1}{x^2}$ **d** $x + \frac{1}{x}$ **e** $x^2 + \frac{1}{x}$ **f** $x^2 + \frac{1}{x^2}$

9 Repeat question **8** for:

i $x = \frac{\sqrt{5}}{\sqrt{5}+1}$ **ii** $x = \frac{2\sqrt{3}}{5-2\sqrt{6}}$

Review exercise

1 Simplify:

a $2\sqrt{2}+3\sqrt{2}-\sqrt{2}$ **b** $-4\sqrt{6}-3\sqrt{6}+8\sqrt{6}$

c $\sqrt{3}-2\sqrt{2}+2\sqrt{3}+\sqrt{2}$ **d** $\sqrt{5}-3\sqrt{2}-4\sqrt{5}+7\sqrt{2}$

2 Simplify:

a $3\sqrt{2}+2\sqrt{2}$ **b** $\sqrt{32}-\sqrt{18}$ **c** $\sqrt{28}-6\sqrt{7}$ **d** $\sqrt{75}+6\sqrt{3}$

3 Simplify:

a $2\sqrt{3}\times5\sqrt{6}$ **b** $3\sqrt{5}\times2\sqrt{10}$ **c** $4\sqrt{2}\times3\sqrt{5}$ **d** $7\sqrt{6}\times4\sqrt{7}$

4 Simplify:

a $\sqrt{72}$ **b** $\sqrt{45}$ **c** $\sqrt{24}$ **d** $\sqrt{27}$ **e** $\sqrt{80}$ **f** $\sqrt{44}$

g $3\sqrt{8}$ **h** $4\sqrt{12}$ **i** $9\sqrt{50}$ **j** $3\sqrt{108}$ **k** $10\sqrt{32}$

5 Write each as a single surd.

a $5\sqrt{3}$ **b** $4\sqrt{7}$ **c** $11\sqrt{2}$ **d** $5\sqrt{5}$

e $8\sqrt{6}$ **f** $9\sqrt{11}$ **g** $4\sqrt{13}$ **h** $4\sqrt{11}$

6 Simplify:

a $\sqrt{32}+\sqrt{50}$ **b** $\sqrt{20}+\sqrt{75}$

c $9\sqrt{3}-2\sqrt{27}$ **d** $3\sqrt{63}+5\sqrt{28}$

e $4\sqrt{20}+3\sqrt{80}-3\sqrt{45}$ **f** $7\sqrt{54}+5\sqrt{216}+2\sqrt{24}$

7 Simplify:

a $\sqrt{32}+4\sqrt{8}+2\sqrt{50}-3\sqrt{2}$ **b** $5\sqrt{32}-3\sqrt{50}+4\sqrt{8}-3\sqrt{18}$

c $7\sqrt{2}+4\sqrt{8}-3\sqrt{54}+5\sqrt{24}$ **d** $5\sqrt{28}-\sqrt{147}+2\sqrt{63}-5\sqrt{48}$

8 Simplify:

a $2\sqrt{3}\left(3+\sqrt{3}\right)$ **b** $5\sqrt{2}\left(3\sqrt{2}-2\right)$

c $4\sqrt{3}\left(2\sqrt{3}-4\sqrt{7}\right)$ **d** $5\sqrt{5}\left(6-2\sqrt{5}\right)$

e $3\sqrt{7}\left(4-\sqrt{7}\right)$ **f** $3\sqrt{3}\left(5\sqrt{3}-4\sqrt{2}\right)$

9 Expand and simplify:

a $\left(2\sqrt{2}+1\right)\left(3\sqrt{2}-2\right)$ **b** $\left(5\sqrt{3}-2\right)\left(2\sqrt{3}-1\right)$ **c** $\left(\sqrt{7}-\sqrt{5}\right)\left(\sqrt{7}+\sqrt{5}\right)$

d $\left(2\sqrt{5}-\sqrt{3}\right)\left(2\sqrt{5}+\sqrt{3}\right)$ **e** $\left(7\sqrt{2}+4\sqrt{3}\right)\left(7\sqrt{2}-4\sqrt{3}\right)$ **f** $\left(\sqrt{5}+\sqrt{3}\right)^2$

g $\left(2\sqrt{3}+\sqrt{2}\right)^2$ **h** $\left(2\sqrt{3}-\sqrt{2}\right)^2$ **i** $\left(2\sqrt{3}-\sqrt{2}\right)\left(2\sqrt{3}+\sqrt{2}\right)$

j $\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)$ **k** $\left(\sqrt{7}-2\right)^2$ **l** $\left(5+\sqrt{3}\right)^2$

10 Rationalise each denominator and simplify where possible.

a $\dfrac{5}{\sqrt{3}}$ **b** $\dfrac{7}{2\sqrt{3}}$ **c** $\dfrac{4}{3\sqrt{2}}$ **d** $\dfrac{6}{7\sqrt{2}}$

e $\dfrac{5\sqrt{2}}{3\sqrt{6}}$ **f** $\dfrac{2\sqrt{3}}{4\sqrt{2}}$ **g** $\dfrac{6\sqrt{7}}{7\sqrt{6}}$ **h** $\dfrac{42\sqrt{7}}{12\sqrt{6}}$

11 Rationalise each denominator and simplify where possible.

a $\dfrac{1}{\sqrt{5}-\sqrt{7}}$ **b** $\dfrac{\sqrt{5}}{2\sqrt{5}-3}$

c $\dfrac{2}{3+\sqrt{5}}$ **d** $\dfrac{2}{3-\sqrt{5}}$

12 Rationalise each denominator.

a $\dfrac{1}{\sqrt{2}-1}$ **b** $\dfrac{1}{\sqrt{3}+2}$

c $\dfrac{1}{\sqrt{3}+\sqrt{2}}$ **d** $\dfrac{1}{\sqrt{3}-\sqrt{2}}$

13 Find integers p and q such that $\dfrac{\sqrt{5}}{\sqrt{5}-2} = p + q\sqrt{5}$.

14 Simplify:

a $\dfrac{3}{\sqrt{5}-2} + \dfrac{2}{\sqrt{5}+2}$

b $\dfrac{2}{6-3\sqrt{3}} - \dfrac{1}{2\sqrt{3}+3}$

15 If $x = \dfrac{1}{2-\sqrt{3}}$ and $y = 2+\sqrt{3}$, find, in simplest form:

a $x + y$ **b** $x - y$ **c** xy **d** $\dfrac{x}{y}$

16 If $x = 2+\sqrt{3}$ and $y = 4-\sqrt{3}$, find, in simplest form:

a $x + y$ **b** $x - y$ **c** $x^2 + y^2$ **d** $x^2 - y^2$

e $\dfrac{1}{x}$ **f** $\dfrac{1}{x} + \dfrac{1}{y}$ **g** $\dfrac{1}{x} - \dfrac{1}{y}$ **h** xy

17 A square has sides of length $2+\sqrt{3}$. Find:

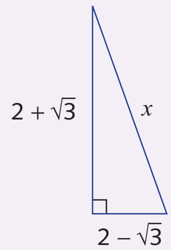
a the perimeter

b the area

18 A rectangle has area 20. The width is $2+\sqrt{3}$. Find the length of the rectangle in simplest form.

19 Find the value of x in each diagram.

a



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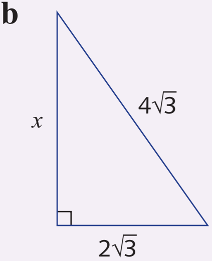

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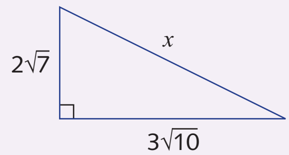

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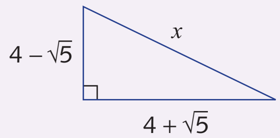

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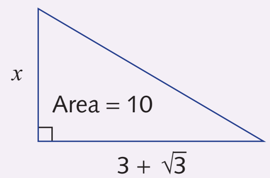

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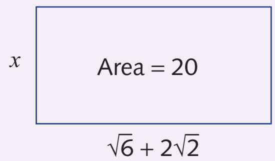


20 The number $\phi = \dfrac{\sqrt{5}+1}{2}$ is known as the **golden ratio**.

a Find:

i ϕ^2 **ii** $\dfrac{1}{\phi}$ **iii** $1+\dfrac{1}{\phi}$ **iv** ϕ^3 **v** $\phi+\dfrac{1}{\phi^2}$

b Show that:

i $\phi^2 = \phi + 1$ **ii** $\phi^3 = \phi^2 + \phi$ **iii** $\phi = 1 + \dfrac{1}{\phi}$

21 A square has area 50. Find its perimeter.

22 Simplify:

a $\dfrac{2}{\sqrt{3}-2}+\dfrac{2}{\sqrt{3}+2}$

b $\dfrac{2}{\sqrt{3}-2}-\dfrac{2}{\sqrt{3}+2}$

c $\dfrac{2}{\sqrt{3}-2}\times\dfrac{2}{\sqrt{3}+2}$

d $\dfrac{2}{\sqrt{3}-2}\div\dfrac{2}{\sqrt{3}+2}$

23 A rectangle has area 30 cm^2 and length $\sqrt{5}$ cm. Find its perimeter.

24 For $x = 3 + 2\sqrt{5}$ and $y = 3 - 2\sqrt{5}$, find:

a $x + y$

b xy

c $\dfrac{1}{x}+\dfrac{1}{y}$

Challenge exercise

1 **a** Show that $\left(\sqrt{x}+\sqrt{y}\right)^2 = x + y + 2\sqrt{xy}$.

b Use this result to find:

i $\sqrt{16+2\sqrt{55}}$

ii $\sqrt{16-2\sqrt{55}}$

iii $\sqrt{11+2\sqrt{30}}$

2 Simplify:

a $\left(\sqrt{a+b}+a\right)\left(\sqrt{a+b}-a\right)$

b $\left(2\sqrt{1+x^2}+1\right)\left(2\sqrt{1+x^2}-1\right)$

c $\left(\sqrt{1+x}+\sqrt{1-x}\right)\left(\sqrt{1+x}-\sqrt{1-x}\right)$

d $\dfrac{a-b}{\sqrt{a}-\sqrt{b}}$

e $\left(\sqrt{a+b}+\sqrt{a-b}\right)^2$

3 Solve these equations for x. (Make sure you check your solutions.)

a $6x - \sqrt{x} = 12$

b $\sqrt{x} - \sqrt{x-5} = 1$

c $\sqrt{7x-5} - \sqrt{2x} = \sqrt{15-7x}$

d $2\sqrt{x} - \sqrt{4x-11} = 1$

e $\dfrac{6\sqrt{x}-11}{3\sqrt{x}} = \dfrac{2\sqrt{x}+1}{\sqrt{x}+6}$

f $\sqrt{9+2x} - \sqrt{2x} = \dfrac{5}{\sqrt{9+2x}}$

4 Expand $\left(\sqrt{x}-1+\dfrac{1}{\sqrt{x}}\right)^2$.

5 Express $\dfrac{1}{\sqrt[3]{5}}$ with a rational denominator.

6 Express $\dfrac{1}{1+\sqrt{3}+\sqrt{5}+\sqrt{15}}$ with a rational denominator.

Hint: Factorise the denominator.

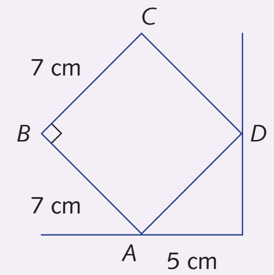
7 Express $\dfrac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$ with a rational denominator.

8 Simplify $\dfrac{a^2+ab+b^2}{a+\sqrt{ab}+b}$.

9 **a** Show that $\sqrt{9\frac{9}{80}} = 9\sqrt{\dfrac{9}{80}}$ and $\sqrt{4\frac{4}{15}} = 4\sqrt{\dfrac{4}{15}}$.

b Describe all mixed numbers that have this property.

10 A square box of side 7 cm is leaning against a vertical wall, as shown below. Find the height of point C above the floor.



11 Simplify $\dfrac{1}{1+\sqrt{3}} + \dfrac{1}{\sqrt{3}+\sqrt{5}} + \dfrac{1}{\sqrt{5}+\sqrt{7}} + \dfrac{1}{\sqrt{7}+3}$.

12 Simplify $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{2}+\sqrt{2+\sqrt{3}}} - \dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{2}+\sqrt{2-\sqrt{3}}}$.

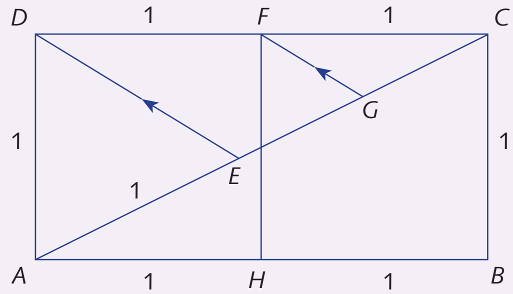
13 $[x]$ is defined as the largest integer, n, such that $n \le x$.
For example, $[1.78] = 1$ and $[2.31] = 2$.

Calculate:

a $1 + [\sqrt{2}] + [\sqrt{3}] + 2 + \ldots + [\sqrt{99}] + 10$

b $1 + [\sqrt{2}] + [\sqrt{3}] + 2 + \ldots + [\sqrt{200}]$

14 In the diagram below, squares $AHFD$ and $HBCF$ are drawn with common side FH. Diagonal AC is drawn and E is a point on AC such that $AE = 1$. G is a point on AC so that FG is parallel to DE.



Find:

a i AC

ii EC

iii EG

iv GC

b Show that $EG^2 + EG = 1$.

CHAPTER

3

Number and Algebra

Algebra review

Algebra is used in almost all areas of mathematics. It is also used by engineers, architects, applied scientists and economists to solve practical problems. In this chapter, we review the techniques of linear equations, linear inequalities, quadratic equations and algebraic fractions introduced in previous years. These techniques will be needed in the following chapters of *ICE-EM Mathematics Year 10*. In particular, algebra will play a central role in the study of coordinate geometry, and of exponentials and logarithms.

3A Expanding brackets and collecting like terms

Like terms

We recall that terms such as $2a^2b$ and $6a^2b$ are called **like terms** and terms such as $2a^2b$ and $6ab^2$ are called **unlike terms**. Only like terms can be added or subtracted to give a single term. This enables algebraic expressions to be simplified by collecting like terms.

Example 1

Simplify:

a $9a + 4b - 3a + 5b$ **b** $2ab + 3b^2 - 5ab - 4b^2$ **c** $2m^2n - mn^2 + 3m^2n + 2mn$

Solution

a $9a + 4b - 3a + 5b = 6a + 9b$

b $2ab + 3b^2 - 5ab - 4b^2 = -3ab - b^2$

c $2m^2n - mn^2 + 3m^2n + 2mn = 5m^2n - mn^2 + 2mn$

Expanding brackets

Recall that the distributive law states:

$$a(b + c) = ab + ac$$

Going from the left to the right of this identity is called ‘expanding brackets’. For example:

$$2x(x - 5) = 2x^2 - 10x$$

Example 2

Expand the brackets.

a $4(2x + 3)$ **b** $-2(x - 4)$ **c** $p(p + n)$ **d** $-3a(2a + 4)$

Solution

a $4(2x + 3) = 8x + 12$

b $-2(x - 4) = -2x + 8$

c $p(p + n) = p^2 + pn$

d $-3a(2a + 4) = -6a^2 - 12a$

Example 3

Expand and simplify each expression.

a $5a(a+1)+6a$ **b** $6d(d+5)-3d$ **c** $4b(2b-1)+7$

Solution

a $5a(a+1)+6a = 5a^2+5a+6a$
$= 5a^2+11a$

b $6d(d+5)-3d = 6d^2+30d-3d$
$= 6d^2+27d$

c $4b(2b-1)+7 = 8b^2-4b+7$

Example 4

Expand and simplify each expression.

a $2(b+5)+3(b+2)$ **b** $3(x-2)-2(x+1)$

c $5(a+1)-2(a-4)$ **d** $3(2a+3b)+2(3a-2b)$

Solution

a $2(b+5)+3(b+2) = 2b+10+3b+6$
$= 5b+16$

b $3(x-2)-2(x+1) = 3x-6-2x-2$
$= x-8$

c $5(a+1)-2(a-4) = 5a+5-2a+8$
$= 3a+13$

d $3(2a+3b)+2(3a-2b) = 6a+9b+6a-4b$
$= 12a+5b$

Binomial products

In general:

$$\begin{aligned}(a+b)(c+d) &= a(c+d)+b(c+d)\\ &= ac+ad+bc+bd\end{aligned}$$

Example 5

Expand and collect like terms.

a $(x+3)(x+2)$ **b** $(x-2)(x+5)$ **c** $(x-4)(x-3)$ **d** $(2y+1)(3y-4)$

Solution

a $(x+3)(x+2) = x(x+2)+3(x+2)$
$= x^2+2x+3x+6$
$= x^2+5x+6$

b $(x-2)(x+5) = x(x+5)-2(x+5)$
$= x^2+5x-2x-10$
$= x^2+3x-10$

(continued over page)

c $\begin{aligned}(x-4)(x-3) &= x(x-3)-4(x-3)\\ &= x^2-3x-4x+12\\ &= x^2-7x+12\end{aligned}$

d $\begin{aligned}(2y+1)(3y-4) &= 2y(3y-4)+1(3y-4)\\ &= 6y^2-8y+3y-4\\ &= 6y^2-5y-4\end{aligned}$

Note: With practice, you should be able to expand and collect like terms mentally. This will enable you to write just the final answer.

Expanding squares

The expansion of an expression such as $(x+3)^2$, $(x-5)^2$ or $(2x-7)^2$ has a special form. For example:

$$\begin{aligned}(x+3)^2 &= (x+3)(x+3)\\ &= x^2+3x+3x+9\\ &= x^2+6x+9\end{aligned}$$

$$\begin{aligned}(2x-7)^2 &= (2x-7)(2x-7)\\ &= 4x^2-14x-14x+49\\ &= 4x^2-28x+49\end{aligned}$$

In general:

$$\begin{aligned}(a+b)^2 &= a(a+b)+b(a+b)\\ &= a^2+ab+ba+b^2\\ &= a^2+2ab+b^2\end{aligned}$$

Similarly:

$$\begin{aligned}(a-b)^2 &= a(a-b)-b(a-b)\\ &= a^2-ab-ba+b^2\\ &= a^2-2ab+b^2\end{aligned}$$

In words:

To expand a square of the form $(a+b)^2$, take the sum of the squares of the terms and add twice the product of the terms.

Example 6

Expand:

a $(x+5)^2$ **b** $(x-7)^2$ **c** $(2x-5)^2$ **d** $(5x-3y)^2$

Solution

a $\begin{aligned}(x+5)^2 &= x^2+2\times 5x+5^2\\ &= x^2+10x+25\end{aligned}$

b $\begin{aligned}(x-7)^2 &= x^2-2\times 7x+7^2\\ &= x^2-14x+49\end{aligned}$

c $\begin{aligned}(2x-5)^2 &= (2x)^2-2\times 10x+5^2\\ &= 4x^2-20x+25\end{aligned}$

d $\begin{aligned}(5x-3y)^2 &= (5x)^2-2\times 15xy+(3y)^2\\ &= 25x^2-30xy+9y^2\end{aligned}$

Once again, with a little practice, the middle step can be left out.

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Expanding the difference of two squares

We now look at another special type of expansion: one that produces the difference of two squares.

$$\begin{aligned}(2x+3)(2x-3) &= 2x(2x-3)+3(2x-3)\\ &= 4x^2-6x+6x-9\\ &= 4x^2-9, \text{ which we can write as}\\ &= (2x)^2-3^2\end{aligned}$$

Notice these points:

- The product $(2x + 3)(2x - 3)$ is of the form 'sum of two terms × difference of those terms'.
- The answer is of the form 'first term squared − second term squared'.

In general:

$$\begin{aligned}(a+b)(a-b) &= a(a-b)+b(a-b)\\ &= a^2-ab+ba-b^2\\ &= a^2-b^2\end{aligned}$$

This identity is called the **difference of two squares**.

Example 7

Expand:

a $(5x - 7)(5x + 7)$

b $(4 + 3x)(4 - 3x)$

Solution

a $$\begin{aligned}(5x-7)(5x+7) &= (5x)^2-7^2\\ &= 25x^2-49\end{aligned}$$

b $$\begin{aligned}(4+3x)(4-3x) &= 4^2-(3x)^2\\ &= 16-9x^2\end{aligned}$$

Quadratic identities review

- $(a + b)(c + d) = ac + ad + bc + bd$
- $(a + b)^2 = a^2 + 2ab + b^2$
- $(a - b)^2 = a^2 - 2ab + b^2$
- $(a + b)(a - b) = a^2 - b^2$

The last identity listed is known as the difference of two squares.

Exercise 3A

Example 1a

1 Simplify each expression by collecting like terms.

a $3x + 5x$ **b** $6x - 4x$ **c** $2x - 8x$

d $6a + 3b - 2a + 4b$ **e** $5s - 7t - 6s + 9t$ **f** $2x - 4y + 6y - 5x$

g $6a - 3b + 4b - 2a$ **h** $2a - 6a + 7b - 11b$ **i** $5x + 7y - 2x - 11y$

Example 1b, c

2 Simplify each expression by collecting like terms.

a $5ab - 2ab$ **b** $7x^2y + 3x^2y - 4x^2y$

c $6cd^3 - 2cd^3 + 8cd^3$ **d** $3st^2 - 4s^2t + 5s^2t + 6st^2$

e $9mn - 8m^2n - 7mn + 12m^2n$ **f** $5x^2y - 11xy + 7x^2y + 11xy$

Example 2a, b

3 Expand each expression.

a $3(a + 2)$ **b** $7(b + 3)$ **c** $2(2a - 5)$

d $12(2p + 9)$ **e** $-3(4b + 9)$ **f** $-5(4m + 5)$

g $-2(3a - 1)$ **h** $-6(4b - 7)$ **i** $-5(2b - 7)$

j $\frac{1}{2}(2x + 6)$ **k** $\frac{1}{4}(16a + 8)$ **l** $\frac{1}{3}(12t - 9)$

m $-\frac{1}{2}(10\ell - 6)$ **n** $\frac{2}{3}\left(6x + \frac{3}{4}\right)$ **o** $-\frac{3}{5}\left(\frac{x}{6} - \frac{1}{3}\right)$

Example 2c, d

4 Expand:

a $a(a + 4)$ **b** $c(c - 5)$ **c** $2g(3g - 5)$ **d** $4h(5h - 7)$

e $3j(4j + 7)$ **f** $-k(5k - 4)$ **g** $-\ell(3\ell - 1)$ **h** $3c(2a + b)$

i $5d(2d - 4e)$ **j** $5m(2m - 4n)$ **k** $5x(2xy + 3)$ **l** $3p(2 - 5pq)$

5 A student expanded brackets and obtained the following answers. In each question, identify and correct the student's mistake and write the correct expansion.

a $4(a + b) = 4a + b$ **b** $5(a + 1) = 5a + 6$

c $8(p - 7) = 8p + 56$ **d** $-3(p - 5) = -3p - 15$

e $a(a + b) = 2a + ab$ **f** $2m(3m + 5) = 6m + 10m$

g $-6(x - 5) = 6x + 30$ **h** $3a(4a - 7) = 12a^2 - 7$

i $4a(3a + 5) = 7a^2 + 20a$ **j** $3x(2x - 7y) = 6x^2 - 21y$

Example 3

6 Expand and collect like terms.

a $8(a + 2) + 7$ **b** $3(d + 5) - 12$ **c** $2(e - 5) + 15$

d $5(g + 2) + 8g$ **e** $6(i - 5) - 3i$ **f** $2a(4a + 3) + 7a$

g $5b(2b - 3) + 6b$ **h** $3b(3b - 5) - 7b^2$ **i** $5(2 - 4p) + 20p$

j $\frac{2}{3}(x + 3) + \frac{x}{6}$ **k** $-\frac{3}{5}(5x - 3) + 3x$ **l** $-\frac{3}{4}(3 - 5x) - 3x$

Example 4

7 Expand and collect like terms.

a $2(y+1)+3(y+4)$
b $5(a-2)+2(a-1)$
c $2(3b-2)+5(2b-1)$
d $3(a+5)-2(a+7)$
e $2(3y-4)-3(2y-1)$
f $x(x-2)+3(x-2)$
g $2p(p+1)+5(p+1)$
h $3z(z+4)-z(3z+2)$
i $3y(y-4)+y(y-4)$
j $4z(4z-2)-z(z+2)$

Example 5

8 Expand and collect like terms.

a $(x+7)(x+3)$
b $(a+5)(a+8)$
c $(a+3)(a+9)$
d $(x-5)(x-1)$
e $(x-3)(x-2)$
f $(a-7)(a-4)$
g $(x-7)(x+6)$
h $(x+11)(x-3)$
i $(x+3)(x-8)$
j $(2x+3)(3x+4)$
k $(3x+2)(5x+4)$
l $(2x-1)(3x+5)$
m $(4x-3)(5x+1)$
n $(2a-5)(a-3)$
o $(2p+5)(3p-2)$
p $(3x-2)(4x+7)$
q $(5x-2)(3x-8)$
r $(2x-5)(3x-1)$

9 Expand and collect like terms.

a $(2a+b)(a+3b)$
b $(3m+n)(2m-3n)$
c $(6p-5q)(p+q)$
d $(3x-2a)(x+5a)$
e $(3x-y)(2x+5y)$
f $(6c-d)(c+6d)$
g $(2p-q)(3q+4p)$
h $(2a+b)(3b-a)$
i $(2p+5q)(3q-2p)$

Example 6

10 Expand:

a $(x+1)^2$
b $(3x+2)^2$
c $(3a+4b)^2$
d $(2x+3y)^2$
e $(x-7)^2$
f $(x-y)^2$
g $(2a-3b)^2$
h $(3a-4b)^2$
i $(2x-3y)^2$

Example 7

11 Expand and simplify:

a $(x+3)(x-3)$
b $(a-7)(a+7)$
c $(2a-3)(2a+3)$
d $(5-x)(5+x)$
e $(7-3y)(7+3y)$
f $(2m+p)(2m-p)$
g $(3x-2y)(3x+2y)$
h $(8m-5n)(8m+5n)$
i $(7r-2t)(7r+2t)$

12 Expand and simplify:

a $(x-8)(x+8)$
b $(2a+1)^2$
c $(2a+1)(2a-1)$
d $(3-2x)(3+2x)$
e $(3+2a)^2$
f $(3-2x)^2$
g $(a+3b)^2$
h $(a+3b)(a-3b)$
i $(a-3b)^2$

3B Solving linear equations and inequalities

The following are examples of **linear equations**. In each case, the highest power of x is 1.

$$2x + 3 = 7, \quad 4(x + 1) = 2(x - 3) \quad \text{and} \quad \frac{x}{2} - 1 = 7$$

The equation $x^2 + x = 6$ is *not* a linear equation because it contains a term in x^2.

Consider these examples.

Example 8

Solve each equation.

a $2x - 3 = 0$ **b** $4(2x - 3) = 6$ **c** $\frac{x}{12} + \frac{1}{3} = \frac{5}{6}$ **d** $2(x - 2) = 3(x + 4)$

Solution

a
$$\begin{aligned} 2x - 3 &= 0 \\ 2x &= 3 \\ x &= \frac{3}{2} \end{aligned}$$

b
$$\begin{aligned} 4(2x - 3) &= 6 \\ 8x - 12 &= 6 \\ 8x &= 18 \\ x &= \frac{18}{8} \\ &= \frac{9}{4} \end{aligned}$$

c
$$\begin{aligned} \frac{x}{12} + \frac{1}{3} &= \frac{5}{6} \\ x + 4 &= 10 \\ x &= 6 \end{aligned}$$

(Multiply both sides by 12, which is the lowest common denominator of the fractions.)

d
$$\begin{aligned} 2(x - 2) &= 3(x + 4) \\ 2x - 4 &= 3x + 12 \\ -4 &= x + 12 \\ -16 &= x \end{aligned}$$

Therefore, $x = -16$.

Linear inequalities

The symbols < and >

- The symbol < means 'is less than'.
- The symbol > means 'is greater than'.

The symbols ≤ and ≥

- The symbol ≤ means 'is less than or equal to'.
- The symbol ≥ means 'is greater than or equal to'.

The statement '$4 \leq 6$' means that either $4 < 6$ or $4 = 6$ is true. It is correct because $4 < 6$. It does not matter that the second part of the statement, '$4 = 6$', is false.

It is also correct to say that '$3 \geq 3$', because $3 = 3$. Once again, it does not matter that the part of the statement that says '$3 > 3$' is false.

Solving inequalities

We will make use of the following properties to solve linear inequalities.

- If we add or subtract the same number to both sides of an inequality, then the resulting inequality is true.

- If we multiply or divide both sides of an inequality by a positive number, then the resulting inequality is true.
- If we multiply or divide both sides of an inequality by a negative number, then we must reverse the inequality sign to make the resulting inequality true.

Example 9

Solve each inequality and graph each solution on a number line.

a $3x - 5 < 0$

b $3 - 8x \geq 15$

c $7x - 2 \leq 3x + 4$

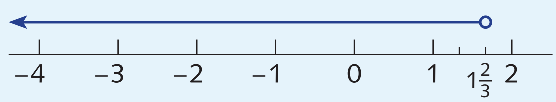
d $2(3x - 4) - 5(2x + 3) \geq 0$

Solution

a $3x - 5 < 0$

$3x < 5$ (Add 5 to both sides of the inequality.)

$x < \frac{5}{3}$ (Divide both sides by 3.)

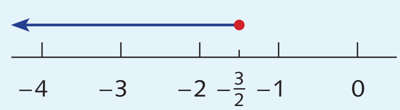


b $3 - 8x \geq 15$

$-8x \geq 12$ (Subtract 3 from both sides of the inequality.)

$x \leq \frac{12}{-8}$ (Divide both sides by −8 and reverse the inequality.)

$x \leq -\frac{3}{2}$

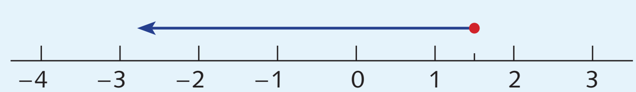


c $7x - 2 \leq 3x + 4$

$4x - 2 \leq 4$ (Subtract $3x$ from both sides.)

$4x \leq 6$ (Add 2 to both sides.)

$x \leq \frac{3}{2}$ (Divide both sides by 4.)



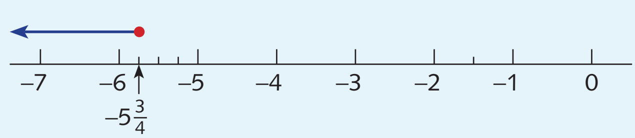
d $2(3x - 4) - 5(2x + 3) \geq 0$

$6x - 8 - 10x - 15 \geq 0$ (Expand the brackets.)

$-4x - 23 \geq 0$ (Collect like terms.)

$-4x \geq 23$ (Add 23 to both sides.)

$x \leq -5\frac{3}{4}$ (Divide both sides by −4 and reverse the inequality sign.)



Note: Infinitely many numbers satisfy a linear inequality. These sets of solutions are indicated on the above number lines.

Word problems

The techniques covered in this chapter can be used to solve problems involving practical situations. The key step is the choice of pronumeral.

Example 10

Fran has earned marks of 76, 84, 92 and 96 on maths tests so far. What must her next mark be for her average to be 86?

Solution

Let x be Fran's mark in the fifth test.

$$\begin{aligned}\text{Then } \frac{x + 76 + 84 + 92 + 96}{5} &= 86\\ \frac{x + 348}{5} &= 86\\ x + 348 &= 430\\ x &= 82\end{aligned}$$

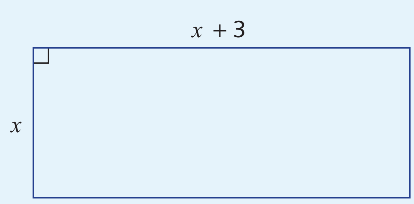
Hence, Fran's mark in the fifth test must be 82.

Example 11

The length of a room is 3 metres more than its width. The perimeter of the room is 30 metres. Find the length and width of the room.

Solution

Let x m be the width of the room. Then the length of the room is $(x + 3)$ m.



$$\begin{aligned}\text{The perimeter} &= x + x + (x + 3) + (x + 3)\\ &= 4x + 6\end{aligned}$$

$$\begin{aligned}\text{Thus, } 4x + 6 &= 30\\ 4x &= 24\\ x &= 6\end{aligned}$$

Hence, the width is 6 m and the length is 9 m.

Exercise 3B

Example 8a, b, c

1 Solve:

a $3a + 2 = 8$

b $5b - 2 = 13$

c $4x - 7 = 11$

d $2y + 3 = -7$

e $3(x - 4) = 6$

f $5(y - 2) = 12$

g $\frac{x}{2} + 3 = 5$

h $5 - \frac{y}{3} = 3$

i $\frac{3a}{2} - 4 = \frac{1}{2}$

j $\frac{2}{3} - \frac{5b}{6} = \frac{3}{2}$

Example 8d

2 Solve:

a $2x - 3 = 4x + 3$

b $5x + 1 = 2x + 7$

c $7x + 4 = 4x - 5$

d $3x - 7 = 5x - 3$

e $3(x - 5) = 2(4 - x)$

f $7(3 - a) = 5(2a - 1)$

g $5(2y + 1) = 3(y - 3)$

h $6(5 - 3x) = 5(2x - 5)$

Example 9a, b

3 Solve each inequality and graph each solution on a number line.

a $2x > 6$

b $3x \le 6$

c $-5x \ge 20$

d $-\frac{x}{2} < 4$

e $3x + 4 \ge 7$

f $2x - 7 < 9$

g $6 - 2x < 5$

h $7 - 3x \ge 5$

Example 9c, d

4 Solve:

a $3x + 4 > x - 7$

b $5x - 6 \le 3x + 8$

c $6x + 7 \le 7x - 6$

d $3x + 5 > 4x - 6$

e $5(x - 2) - 2(3x + 1) \ge 0$

f $5(2x - 3) < 4(x + 3)$

g $3(x + 4) - 4(x + 2) > 0$

h $2(3x - 7) - 5(2x + 3) \le 0$

i $2(x - 4) \ge 3(x + 4)$

j $6(3 - x) + 2(1 + x) > 0$

Example 10

5 Katsu has earned marks of 88, 94, 92 and 98 on maths tests so far. What must his next mark be in order for his average mark to be 80?

Example 11

6 The length of a large rectangular room is to be 3 m more than twice its width. If the perimeter is to be 36 m, find the length and width of the room.

7 One number is 9 less than twice another, and their sum is 42. Find the two numbers.

8 \$540 is divided among three people A, B and C, so that A gets three times as much as C, and B gets twice as much as C. How much does C get?

9 A student runs three times as fast as she walks. She walked for 3 hours and ran for 2 hours and altogether travelled 36 km. What was her speed of walking?

10 A man bought a house and had some renovations done. The renovations cost $\frac{1}{6}$ of the price of the house. He paid \$980 000 for the house and the renovations. What was the price of the house?

3C More difficult linear equations and inequalities

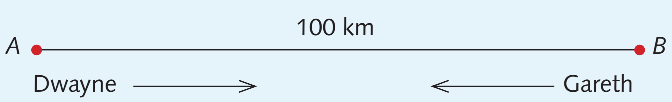
Problem-solving often involves introducing algebra, translating the problem into an equation and then solving the equation. An important first step is to introduce an appropriate pronumeral for one of the unknown (and often directly required) quantities.

Example 12

Gareth and Dwayne drive at constant speeds towards each other on a straight road from two towns, A and B, 100 km apart. Gareth drives $10\ \text{km/h}$ faster than Dwayne. They meet after 1 hour. At what speeds are Gareth and Dwayne driving?

Solution

Let V km/h be the speed of Dwayne. Then $(V + 10)\ \text{km/h}$ is the speed of Gareth.



After one hour, Dwayne has gone $V\ \text{km}$ from A and Gareth has gone $(V + 10)$ km from B.

$$\begin{aligned} V + V + 10 &= 100 \\ 2V &= 90 \\ V &= 45 \end{aligned}$$

Hence, Dwayne is driving at $45\ \text{km/h}$ and Gareth is driving at $55\ \text{km/h}$.

Example 13

A chemist has 119 millilitres of 10% acid solution by volume. How much pure acid must be added to obtain a 15% acid solution? Check your answer.

Solution

Let x millilitres be the volume of acid to be added to obtain a 15% acid solution.

$$\begin{aligned} \text{Original volume of pure acid} &= 10\% \text{ of } 119 \\ &= 11.9 \text{ millilitres} \end{aligned}$$

In the new solution, the volume of acid = $(x + 11.9)$ millilitres.

The new volume of solution = $(119 + x)$ millilitres.

In the new solution, the volume of acid = $0.15(119 + x)$ millilitres.

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So, $x + 11.9 = 0.15(119 + x)$

$$\begin{aligned} x + 11.9 &= 0.15 \times 119 + 0.15x \\ 0.85x &= 0.15 \times 119 - 11.9 \\ 0.85x &= 5.95 \\ x &= 7 \end{aligned}$$

Thus, 7 millilitres of pure acid must be added to obtain the 15% solution.

Check: Volume of new solution $119 + 7 = 126$ millilitres

Volume of acid $= 7 + 11.9 = 18.9$ millilitres

Proportion of acid $= \dfrac{18.9}{126} = 0.15$

Example 14

Two companies, Ornate Wares and Extravagant Products, employ sales staff. Ornate pays its employees a 15% commission on all sales while Extravagant pays its employees \$200 a week and a 10% commission. For what amount of sales in one week will Ornate pay more than Extravagant?

Solution

Let $\$x$ be the amount of sales in one week.

Ornate pays 15% of $\$x = \$0.15x$.

Extravagant pays 10% of $\$x + \$200 = \$0.1x + 200$.

If Ornate pays more than Extravagant:

$$\begin{aligned} 15\% \text{ of } x &> 10\% \text{ of } x + 200 \\ 0.15x &> 0.1x + 200 \\ 0.05x &> 200 \\ x &> \frac{200}{0.05} \\ x &> 4000 \end{aligned}$$

Hence, Ornate will pay more than Extravagant when sales for one week exceed \$4000.

Exercise 3C

Example 12

1 Giorgio and Fred ride their bicycles towards each other from points 12 km apart. Giorgio rides 2 km/h faster than Fred, and they meet in 1 hour. How fast did each of them ride?

Example 13

2 How many litres of pure (100%) acid must be added to 4 L of a solution that is 88% acid by volume in order to obtain a 92% solution?

3 A collection of 27 coins consists of 20-cent and \$2 coins, and has a value of \$23.40. Find the number of coins of each denomination.

4 Anna sells home-delivery products for two different companies. The first company pays her 10% commission, and the second company pays her 6% commission. Next month, she wants to earn \$600. If she plans on monthly sales of \$7600 in total from both companies, what amount must she sell for the company that pays 10% commission?

5 A factory has two smokestacks. The rate of particle pollution produced is 12 kg/h for the first smokestack and 32 kg/h for the second. The manufacturing process requires that the first smokestack operates 2 hours longer than $1\frac{1}{2}$ times as long as the second. A local pollution control board has ordered that the factory produce no more than 374 kg of particle pollution per day. What is the maximum number of hours per day that each smokestack can operate?

6 A chemist has 500 mL of a salt solution that contains 2% salt, in units of g/cm^3. How much water must be added to reduce the concentration to 0.5%?

7 A driver travels from town A to town B at a constant speed, and then from town B to town C at a different constant speed. He takes 4 hours to go from A to B, and the same time to go from B to C, even though the distance from B to C is 30 km greater. If his overall average speed is 80 km/h, find the distances between the three towns.

8 The owner of a 20-room motel must take \$660 000 a year to cover costs and earn a reasonable profit.

a If the average occupancy level is 70% and the same daily rate is charged for all occupied rooms, what should this rate be (correct to the nearest dollar)?

b On average, two or more people are in 75% of the occupied rooms. If the owner decides to charge \$30 more than the single occupancy rate if two or more people occupy a room, what should the two rates be? Give your answer correct to the nearest dollar.

9 Suppose that you work for \$25.50 per hour. Of the total you earn, you save 25%. How many hours must you work in a week to save at least \$163?

Example 14

10 Two companies, A and B, each offer you a sales position. Both jobs are essentially the same, but A pays a straight 8% commission while B pays \$51 per week plus 5% commission. For what amount of weekly sales will A pay more money?

11 At a certain school, the mark out of 100 for the Term 1 exam and twice the mark out of 100 for the Term 3 exam are added together. The students must obtain at least 150 marks to achieve a satisfactory grade. A student obtains x marks in the Term 1 exam.

a Write an appropriate inequality to show the mark, y, that the student must obtain in the Term 3 exam in order to pass.

b Solve this inequality for:

i $x = 35$

ii $x = 49$

3D Formulas

A **formula** relates different quantities. For instance, the formula $V = \pi r^2 h$ relates the radius r and the height h of a cylinder with the volume V of the cylinder.

The pronumeral on the left-hand side is called the **subject** of the formula. In the formula above, the subject is V. We say that the subject has been expressed in terms of the other pronumerals r and h. π is a number, not a pronumeral.

Pronumerals in a formula are often called **variables**.

When the variables other than the subject are given particular values, the value of the subject can be determined by substitution.

Example 15

The volume of metal in a tube is given by the formula $V = \pi\ell[r^2 - (r - t)^2]$, where ℓ is the length of the tube, r is the radius of the outside surface and t is the thickness of the material.

Find V when:

a $\ell = 40,\ r = 5$ and $t = 0.5$

b $\ell = 100,\ r = 2$ and $t = 0.2$

Solution

a $V = \ell[r^2 - (r - t)^2]$

When $\ell = 40,\ r = 5$ and $t = 0.5$,

$$\begin{aligned} V &= \pi \times 40[5^2 - (5 - 0.5)^2] \\ &= 40\pi(25 - 20.25) \\ &= 40\pi \times 4.75 \\ &= 190\pi \end{aligned}$$

b When $\ell = 100,\ r = 2$ and $t = 0.2$

$$\begin{aligned} V &= \pi \times 100[4 - (2 - 0.2)^2] \\ &= \pi \times 100(4 - 3.24) \\ &= \pi \times 100 \times 0.76 \\ &= 76\pi \end{aligned}$$

Sometimes we need to calculate the value of a variable that is not the subject of the formula. In this situation, we have a choice of methods.

Method 1

Substitute the values for the known variables, then solve the resulting equation for the unknown variable.

Method 2

Rearrange the formula to make the required variable the subject, then substitute.

In each case, an equation has to be solved. In Method 1, the equation involves numbers, while in Method 2, the equation involves pronumerals.

Example 16

In the formula $a = \sqrt{\dfrac{3V}{h}}$, find V when $h = 4$ and $a = 5$.

Solution

Method 1

Substitute $h = 4$ and $a = 5$ into the formula.

$$5 = \sqrt{\frac{3V}{4}}$$

$$5^2 = \left(\sqrt{\frac{3V}{4}}\right)^2 \qquad \text{(Square both sides of the equation.)}$$

$$25 = \frac{3V}{4}$$

$$100 = 3V$$

$$V = \frac{100}{3}$$

Method 2

Rearrange the formula first, then substitute.

$$a = \sqrt{\frac{3V}{h}}$$

$$a^2 = \frac{3V}{h}$$

$$a^2h = 3V$$

$$V = \frac{a^2h}{3}$$

When $a = 5$ and $h = 4$,

$$V = \frac{5^2 \times 4}{3}$$

$$= \frac{100}{3}$$

Method 2 is preferable, especially if substitutions of several different sets of values are to be carried out.

Example 17

In the formula $ut = 2s - vt$, find t when $u = 12$, $v = 18$ and $s = 25$.

Solution

Rearrange the formula first.

$$\begin{aligned} ut &= 2s - vt \\ ut + vt &= 2s \\ t(u + v) &= 2s \qquad \text{(Take out the common factor } t\text{.)} \\ t &= \frac{2s}{u + v} \end{aligned}$$

When $u = 12$, $v = 18$ and $s = 25$,

$$\begin{aligned} t &= \frac{2 \times 25}{12 + 18} \\ &= \frac{5}{3} \end{aligned}$$

Example 18

Make t the subject of the formula $\frac{1}{r} = \frac{1}{s} - \frac{1}{t}$.

Solution

$$\begin{aligned} \frac{1}{r} &= \frac{1}{s} - \frac{1}{t} \\ \frac{1}{t} &= \frac{1}{s} - \frac{1}{r} \\ \frac{1}{t} &= \frac{r - s}{sr} \qquad \text{(Use a common denominator.)} \\ t &= \frac{sr}{r - s} \qquad \text{(Take the reciprocal of both sides.)} \end{aligned}$$

Exercise 3D

1 Find the value of U if $U = QJ + W$, and $Q = 8$, $J = 15$ and $W = 7$.

2 Find the value of s if $s = \left(\frac{u + v}{2}\right)t$, and $u = 3.5$, $v = 6.1$ and $t = 9$.

3 Find the value of h, correct to one decimal place, if $h = \frac{9gRs}{2v}$ and $g = 9.8$, $R = 2.5$, $s = 3$ and $v = 7.4$.

4 Given that $v = u + at$, find t, given that $v = 16$, $u = 10$ and $a = 5$.

5 Given that $t = 2\pi\sqrt{\dfrac{L}{g}}$, find L, given that $t = \pi$ and $g = 9.8$.

6 The perimeter P of a semicircle of radius r is given by the formula $P = \pi r + 2r$. Find r, correct to two decimal places, if $P = 15$.

Example 15

7 The area A of the annulus formed by two concentric circles of radii R and r is given by the formula $A = \pi(R^2 - r^2)$, where $R > r$. Use this formula to find the area, correct to the nearest cm^2, of the annulus formed by two concentric circles of radii:

a 12 cm and 6 cm **b** 8.2 cm and 5.7 cm **c** 49 cm and 35 cm

8 The sum S of the consecutive whole numbers from 1 to n inclusive is given by the formula $S = \dfrac{n(n+1)}{2}$. Find the sum of the integers from:

a 1 to 30 **b** 1 to 50 **c** 1 to 120 **d** 51 to 120

9 Find the value of w if $\dfrac{w-u}{rg} = v$, and $v = 7.25$, $u = 3.1$, $r = 1.2$ and $g = 9.8$.

Example 16

10 The time T (in seconds) taken for one complete swing of a pendulum is given by $T = 2\pi\sqrt{\dfrac{L}{g}}$, where $g = 9.8$ and L is the length of the pendulum in metres. Find, correct to the nearest centimetre, the length of a pendulum that takes 2 seconds to complete one swing.

Example 16

11 The volume, V cm^3, of a cylinder is given by $V = \pi r^2 h$, where r cm is the radius and h cm is the height. Find the radius, correct to the nearest millimetre, of a cylinder with a volume of 100 cm^3 and a height of 8 cm.

Example 18

12 Make the variable in the brackets the subject of the formula. In this question all variables only take positive values.

a $x + y = z$ (y) **b** $A = \pi rs$ (r)

c $ax = y^2 t$ (y) **d** $w = \dfrac{1}{2}mv^2$ (v)

e $2\sqrt{x} = m$ (x) **f** $k = \sqrt{4mt}$ (m)

g $ap = b^2 - bp$ (p) **h** $x = \dfrac{1+t}{1-t}$ (t)

i $c^2 = a^2 + b^2$ (a) **j** $v = \sqrt{\dfrac{n-q}{r}}$ (r)

k $v = \sqrt[3]{\dfrac{n-q}{r}}$ (q) **l** $t = 2\pi\sqrt{\dfrac{L}{g}}$ (L)

m $t = 2\pi\sqrt{\dfrac{h-k}{g}}$ (k) **n** $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = \dfrac{1}{m}$ (a)

o $(A-B)^2 + C^2 = A^2$ (A) **p** $v = c\left(\dfrac{1}{r} - \dfrac{1}{s}\right)$ (s)

3E Factorising a difference of two squares

You will recall the important identity $(a + b)(a - b) = a^2 - b^2$. This is called the **difference of two squares**. We can now use this identity the other way around to factorise an expression that is the difference of two squares.
That is:

$$a^2 - b^2 = (a - b)(a + b)$$

Recall that in any factorisation the first step is to take out any common factor.

Example 19

Factorise:

a $x^2 - 9$ **b** $25 - y^2$ **c** $3a^2 - 27$ **d** $-16 + 9x^2$

Solution

a $x^2 - 9 = x^2 - 3^2$
$= (x + 3)(x - 3)$

b $25 - y^2 = 5^2 - y^2$
$= (5 + y)(5 - y)$

c There is a common factor of 3.
$3a^2 - 27 = 3(a^2 - 9)$
$= 3(a^2 - 3^2)$
$= 3(a + 3)(a - 3)$

d $-16 + 9x^2 = (3x)^2 - 4^2$
$= (3x - 4)(3x + 4)$

In some circumstances it is useful to factorise using surds.

In general, expressions of the form $x^2 - b$ (where $b > 0$) can be factorised as:

$$\begin{aligned} x^2 - b &= x^2 - \left(\sqrt{b}\right)^2 \\ &= \left(x + \sqrt{b}\right)\left(x - \sqrt{b}\right) \end{aligned}$$

Example 20

Factorise:

a $x^2 - 2$ **b** $x^2 - 12$

Solution

a $x^2 - 2 = x^2 - \left(\sqrt{2}\right)^2$
$= \left(x + \sqrt{2}\right)\left(x - \sqrt{2}\right)$

b $x^2 - 12 = x^2 - \left(\sqrt{12}\right)^2$
$= x^2 - \left(2\sqrt{3}\right)^2$
$= \left(x + 2\sqrt{3}\right)\left(x - 2\sqrt{3}\right)$

Having considered some standard examples of factorisation using surds, we will now look at some slightly harder examples.

Example 21

Factorise:

a $(x+1)^2 - 3$

b $(x-2)^2 - 5$

c $18 - 3x^2$

Solution

a $(x+1)^2 - 3 = (x+1)^2 - \left(\sqrt{3}\right)^2$
$= \left(x + 1 + \sqrt{3}\right)\left(x + 1 - \sqrt{3}\right)$

b $(x-2)^2 - 5 = (x-2)^2 - \left(\sqrt{5}\right)^2$
$= \left(x - 2 + \sqrt{5}\right)\left(x - 2 - \sqrt{5}\right)$

c $18 - 3x^2 = 3(6 - x^2)$ (Take out a common factor.)
$= 3\left(\left(\sqrt{6}\right)^2 - x^2\right)$
$= 3\left(\sqrt{6} + x\right)\left(\sqrt{6} - x\right)$

Exercise 3E

Example 19

1 Factorise:

a $b^2 - 4$ **b** $9 - a^2$ **c** $9x^2 - 16$ **d** $25a^2 - 1$

e $16y^2 - 64$ **f** $a^2 - 4b^2$ **g** $9x^2 - y^2$ **h** $8x^2 - 2$

i $12 - 3y^2$ **j** $5m^2 - 20n^2$ **k** $27r^2 - 3t^2$ **l** $1 - 9a^2b^2$

Example 20a

2 Factorise:

a $x^2 - 3$ **b** $x^2 - 7$ **c** $x^2 - 13$ **d** $x^2 - 6$

Example 20b

3 Factorise each expression, writing all surds in simplest form.

a $x^2 - 20$ **b** $x^2 - 18$ **c** $x^2 - 27$ **d** $x^2 - 24$

e $x^2 - 40$ **f** $x^2 - 28$ **g** $x^2 - 125$ **h** $x^2 - 200$

i $x^2 - \frac{7}{25}$ **j** $x^2 - \frac{3}{4}$ **k** $x^2 - \frac{5}{9}$ **l** $x^2 - \frac{11}{16}$

Example 21

4 Factorise each expression using surds.

a $(x+2)^2-2$ b $(x-1)^2-3$ c $(x-5)^2-5$ d $(x-4)^2-\frac{1}{2}$

e $(x-1)^2-12$ f $(x+3)^2-8$ g $(x+3)^2-\frac{5}{9}$ h $(x-2)^2-\frac{7}{9}$

i $(2x+1)^2-27$ j $(2x-3)^2-\frac{2}{3}$ k $(4x-1)^2-\frac{3}{5}$ l $(6z+2)^2-4$

5 Factorise each expression using surds.

a $2x^2-14$ b $3x^2-6$ c $5x^2-15$ d $4x^2-28$

e $3x^2-36$ f $2x^2-36$ g $2(x+1)^2-10$ h $3(x+2)^2-6$

i $3(x-2)^2-24$ j $2(x+1)^2-24$ k $-2(x+3)^2+4$ l $-4(x-2)^2+20$

6 a Factorise $a-b$ using surds. b Hence, simplify $\dfrac{a-b}{\sqrt{a}-\sqrt{b}}$.

3F Monic quadratics and grouping

Monic quadratics

When we expand $(x+p)(x+q)$, we obtain:

$$\begin{aligned}(x+p)(x+q) &= x(x+q)+p(x+q)\\ &= x^2+xq+px+pq\\ &= x^2+(p+q)x+pq\end{aligned}$$

The coefficient of x is the sum of p and q, and the constant term is the product of p and q.

A **monic quadratic** is a quadratic in which x^2 has coefficient 1. To factorise a monic quadratic, look for two numbers that add to give the coefficient of x, and that multiply together to give the constant term.

Example 22

Factorise:

a $x^2-3x-18$ b $x^2-11x+30$ c $3x^2-3x-36$

Solution

a We are looking for two numbers with product -18 and sum -3. The numbers -6 and 3 satisfy both conditions, so:

$x^2-3x-18=(x-6)(x+3)$

b The numbers -6 and -5 have a sum of -11 and a product of 30, so:

$x^2-11x+30=(x-6)(x-5)$

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c $3x^2 - 3x - 36$ is not a monic quadratic expression, but the quadratic obtained by taking out a factor of 3 is monic.

First, take out the common factor.

$$\begin{aligned} 3x^2 - 3x - 36 &= 3(x^2 - x - 12) \\ &= 3(x-4)(x+3) \end{aligned}$$

because -4 and 3 have sum -1 and product -12.

Grouping

We now discuss how to factorise certain expressions involving four terms. The four terms are grouped into two pairs of two, and each pair is factorised separately. The resulting expression is then factorised.

Example 23

a Factorise $3ax - a + 12x - 4$.

b Factorise $ab - 6 + 2a - 3b$.

Solution

a $\underbrace{3ax - a} + \underbrace{12x - 4} = a(3x-1) + 4(3x-1)$

$= (3x-1)(a+4)$ (Take out the common factor $(3x - 1)$.)

b $ab - 6 + 2a - 3b = ab + 2a - (3b + 6)$ (Change the order to obtain two pairs, each with a common factor.)

$= a(b+2) - 3(b+2)$

$= (b+2)(a-3)$

[Alternatively, $ab - 6 + 2a - 3b = ab - 3b + 2a - 6$

$= b(a-3) + 2(a-3)$

$= (a-3)(b+2)$]

Exercise 3F

Example 22

1 Factorise:

a $x^2 + 9x + 8$
b $x^2 + 8x + 12$
c $x^2 + 6x + 8$
d $x^2 + 12x + 36$
e $x^2 + 6x + 9$
f $x^2 + 11x + 24$
g $x^2 + x - 6$
h $x^2 + x - 30$
i $x^2 + 3x - 40$
j $x^2 + 4x - 60$
k $x^2 - 7x + 12$
l $x^2 - 10x + 25$
m $x^2 - 18x + 32$
n $x^2 - 2x - 35$
o $x^2 - 4x - 21$
p $3x^2 - 18x + 15$
q $2x^2 + 10x - 48$
r $4x^2 - 8x - 96$
s $5x^2 + 50x - 120$
t $3x^2 - 6x - 9$
u $2x^2 + 6x - 56$

Example 23

2 Factorise:

a $mx + 2m + 3x + 6$

b $x^2 - 3x - 4x + 12$

c $ab - 7a - 9b + 63$

d $2x^2 - 2x + 5x - 5$

e $3ab + 6a + 5b + 10$

f $6x^2 - 4xy + 3xy - 2y^2$

g $2ab - 12a - 5b + 30$

h $x^3 - 4x + 2x^2 - 8$

i $12x - 8xy - 15 + 10y$

j $10x^2 + 15xy - 4xy - 6y^2$

3 Factorise:

a $x^2 - 4x - 77$

b $z^2 - 25$

c $2x^2 - 6x$

d $20a^2 - 5a^3$

e $xy + 2x + 3y + 6$

f $x^2 - xy + xy - y^2$

g $x^2 + 14x + 40$

h $x^2 - 14x + 49$

i $x^2 - 11$

j $(x - 6)^2 - 18$

k $81 - (a - 2)^2$

l $x^2 - 15x + 26$

m $9 + z^2 + 6z$

n $15 - 2x - x^2$

o $2x^2 - 14x + 24$

p $3x^2 - 27x + 24$

q $100 - 25x^2$

r $64p^2 - 81q^2$

s $xy + xz + y^2 + yz$

t $6a^2 - 9a - 2ab + 3b$

3G Non-monic quadratics

Factorising the general quadratic $ax^2 + bx + c$

There are many methods for factorising non-monic quadratics. We present only one method here. Consider the quadratic expression $4x^2 + 5x - 6$. To factorise this quadratic:

Step 1 Multiply the coefficient of x^2 by the constant term.

$4 \times (-6) = -24$

Step 2 Find two numbers with product -24 and that sum to give the coefficient of x, which is 5. The numbers are 8 and -3, because:

$8 \times (-3) = -24$ and $8 + (-3) = 5$

Step 3 Split the middle term. $5x$ is the sum of $8x$ and $-3x$.

$5x = 8x - 3x$

Step 4 Use grouping to complete the factorisation.

$$\begin{aligned}4x^2 + 5x - 6 &= 4x^2 + 8x - 3x - 6\\ &= 4x(x + 2) - 3(x + 2)\\ &= (x + 2)(4x - 3)\end{aligned}$$

Quadratics whose factors involve only integers can be factorised using this method.

Factorising non-monic quadratics

To factorise a quadratic of the form $ax^2 + bx + c$:

- find two numbers, α and β, whose sum is b and whose product is ac
- write the middle term as $\alpha x + \beta x$
- complete the factorisation using grouping.

Example 24

Factorise $2x^2 + 9x + 4$.

Solution

Multiply 2 by 4 to obtain 8. Find two numbers with product 8 and sum 9.
The numbers are 8 and 1.

$$\begin{aligned} 2x^2 + 9x + 4 &= 2x^2 + 8x + x + 4 \qquad \text{(Split } 9x = 8x + x\text{)} \\ &= 2x(x + 4) + 1(x + 4) \\ &= (2x + 1)(x + 4) \end{aligned}$$

[Alternatively, if we write $9x = x + 8x$, then:

$$\begin{aligned} 2x^2 + 9x + 4 &= 2x^2 + x + 8x + 4 \\ &= x(2x + 1) + 4(2x + 1) \\ &= (2x + 1)(x + 4) \end{aligned}$$

]

Example 25

Factorise:

a $6x^2 - 19x + 10$

b $6x^2 - 7x - 3$

Solution

a $6 \times 10 = 60$, $(-15) \times (-4) = 60$ and $(-15) + (-4) = -19$

$$\begin{aligned} 6x^2 - 19x + 10 &= 6x^2 - 15x - 4x + 10 \\ &= 3x(2x - 5) - 2(2x - 5) \\ &= (3x - 2)(2x - 5) \end{aligned}$$

b $6 \times (-3) = -18$, $(-9) \times 2 = -18$ and $(-9) + 2 = -7$

$$\begin{aligned} 6x^2 - 7x - 3 &= 6x^2 - 9x + 2x - 3 \\ &= 3x(2x - 3) + 1(2x - 3) \\ &= (3x + 1)(2x - 3) \end{aligned}$$

Exercise 3G

1 Factorise:

a $2x^2 + 2x + x + 1$
b $5x^2 - 5x - 2x + 2$
c $6x^2 - 3x - 4x + 2$
d $8x^2 - 4x + 6x - 3$
e $20x^2 - 15x - 16x + 12$
f $12x^2 + 24x - 36x - 72$

Example 24, 25

2 Factorise:

a $2x^2 + 7x + 6$
b $2x^2 + 11x + 5$
c $2x^2 + 9x + 9$
d $3x^2 + 11x + 6$
e $3x^2 + 10x + 3$
f $3x^2 + 14x + 8$
g $2x^2 + x - 10$
h $2x^2 - 13x - 7$
i $3x^2 - x - 10$
j $5x^2 + 7x - 6$
k $3x^2 + 26x - 9$
l $7x^2 + 3x - 10$
m $5x^2 - 4x - 12$
n $3x^2 + 4x - 4$
o $4x^2 + 11x - 3$

3 Factorise:

a $6x^2 - x - 2$
b $8x^2 - 2x - 3$
c $6x^2 + 5x - 6$
d $15x^2 - 16x + 4$
e $9x^2 - 12x + 4$
f $4x^2 + 12x + 9$
g $10x^2 - 9x - 9$
h $12x^2 + 19x - 18$
i $10x^2 - 19x + 6$
j $12x^2 - 17x + 6$
k $12x^2 - 13x + 3$
l $15x^2 - 31x + 10$
m $20x^2 - 58x + 20$
n $24x^2 + 6x - 9$
o $12x^2 - 26x + 12$
p $12x^2 + 34x + 24$
q $40x^2 - 10x - 15$
r $16x^2 - 68x + 42$

4 Factorise:

a $6x^2 + 7x + 2$
b $6x^2 - 11x + 3$
c $6x^2 + 19x + 15$
d $9x^2 - 18x + 8$
e $a^2 - a - 56$
f $4 - 5x - 6x^2$
g $9 + 8x - x^2$
h $24x^2 - 4x - 8$
i $3a^2 + 56 - 31a$
j $100 - 5x^2$
k $(2x - 1)^2 - 15$
l $90 - 2(a - 3)^2$

5 Factorise:

a $x^2 - y^2 - 3(x - y)$
b $x^2 - y^2 - 5(x + y)$
c $x^2 - 2x + 1 - 3(x - 1)$
d $x^2 + 2x + 1 + 5(x + 1)$
e $a^2 - 4b^2 - a - 2b$
f $9x^2 - 4y^2 - 3x - 2y$
g $a^2 - 4d^2 - a - 2d$
h $x^2 + 3x - 9(x + 3)$
i $4x^2 + 4x + 1 + 3(2x + 1)$
j $7(x - 11) + x^2 - 22x + 121$

3H An introduction to algebraic fractions

Algebraic fractions occur in many parts of mathematics. For example, you have seen them arise in rate problems. In this section we simplify, add and subtract algebraic fractions and solve equations involving algebraic fractions.

Simplifying algebraic fractions

There are two steps to simplifying an algebraic fraction.

- First, factor the numerator and denominator.
- Second, cancel any fractions common to the numerator and denominator.

In some examples, you need to use the fact that $\dfrac{a-b}{b-a} = -1$, because $\dfrac{a-b}{b-a} = \dfrac{-(b-a)}{b-a}$.

Example 26

Simplify:

a $\dfrac{24x}{36}$ **b** $\dfrac{2x+4}{x+2}$ **c** $\dfrac{x^2-4}{x+2}$ **d** $\dfrac{(x-3)(x+4)}{(x+4)(3-x)}$

Solution

a $\dfrac{24x}{36} = \dfrac{2x}{3}$

b $\dfrac{2x+4}{x+2} = \dfrac{2(x+2)}{x+2}$
$= 2$

c $\dfrac{x^2-4}{x+2} = \dfrac{(x-2)(x+2)}{x+2}$
$= x-2$

d $\dfrac{(x-3)(x+4)}{(x+4)(3-x)} = \dfrac{(x-3)(x+4)}{-(x+4)(x-3)}$
$= -1$

Adding and subtracting algebraic fractions

To add and subtract algebraic fractions, we use a common denominator, just as we do in arithmetic. In this section we only add and subtract algebraic fractions with numerical denominators.

Example 27

Express each with a common denominator.

a $\dfrac{x}{3} + \dfrac{x}{2}$ **b** $\dfrac{x}{3} - \dfrac{x}{4}$ **c** $\dfrac{x-3}{2} + \dfrac{2x-4}{6}$ **d** $\dfrac{3x-7}{2} - \dfrac{2x-4}{3}$

Solution

a $\dfrac{x}{3}+\dfrac{x}{2}=\dfrac{2x}{6}+\dfrac{3x}{6}$
$=\dfrac{5x}{6}$

b $\dfrac{x}{3}-\dfrac{x}{4}=\dfrac{4x}{12}-\dfrac{3x}{12}$
$=\dfrac{x}{12}$

c $\dfrac{x-3}{2}+\dfrac{2x-4}{6}=\dfrac{3(x-3)}{6}+\dfrac{2x-4}{6}$
$=\dfrac{3x-9+2x-4}{6}$
$=\dfrac{5x-13}{6}$

d $\dfrac{3x-7}{2}-\dfrac{2x-4}{3}=\dfrac{3(3x-7)-2(2x-4)}{6}$
$=\dfrac{9x-21-4x+8}{6}$
$=\dfrac{5x-13}{6}$

Solving equations involving algebraic fractions

To solve an equation involving algebraic fractions, multiply through by the lowest common denominator of all the fractions. This removes all the fractions in one step.

Example 28

Solve:

a $\dfrac{x}{4}+\dfrac{2x}{5}=52$

b $\dfrac{3x+1}{7}-\dfrac{2x-1}{6}=6$

Solution

a $\dfrac{x}{4}+\dfrac{2x}{5}=52$

Multiply through by 20

$$\begin{aligned}\frac{20x}{4}+\frac{20\times 2x}{5}&=20\times 52\\ 5x+8x&=1040\\ 13x&=1040\\ x&=80\end{aligned}$$

b $\dfrac{3x+1}{7}-\dfrac{2x-1}{6}=6$

Multiply through by 42

$$\begin{aligned}\frac{42(3x+1)}{7}-\frac{42(2x-1)}{6}&=42\times 6\\ 6(3x+1)-7(2x-1)&=42\times 6\\ 18x+6-14x+7&=42\times 6\\ 4x+13&=252\\ 4x&=239\\ x&=\frac{239}{4}\end{aligned}$$

Note: With practice, the first step above can be performed mentally.

Exercise 3H

Example 26

1 Simplify:

a $\frac{8x}{10}$ **b** $\frac{11a}{44}$ **c** $\frac{9x+3}{3x+1}$

d $\frac{x^2-36}{x+6}$ **e** $\frac{4x^2-81}{2x-9}$ **f** $\frac{(x-5)(x+4)}{(x+4)(5-x)}$

g $\frac{(2x-5)(x+4)}{(x-4)(5-2x)}$ **h** $\frac{(2x-5)(x+4)}{x^2-16}$ **i** $\frac{(2x-5)(2x+4)}{2(x+4)(2x-5)}$

Example 27

2 Express each with a common denominator.

a $\frac{x}{5}+\frac{x}{4}$ **b** $\frac{2x}{5}-\frac{x}{3}$ **c** $\frac{3x}{10}+\frac{x}{4}$

d $\frac{2a}{5}+\frac{3a}{4}$ **e** $\frac{x+2}{5}+\frac{x-1}{4}$ **f** $\frac{x+3}{4}-\frac{x+1}{3}$

g $\frac{z+3}{7}-\frac{z+2}{5}$ **h** $\frac{2c-1}{3}+\frac{c+2}{5}$ **i** $\frac{2b+3}{4}-\frac{b-2}{6}$

j $\frac{3k+1}{14}-\frac{2k+3}{21}$ **k** $\frac{3-x}{2}+\frac{1-2x}{6}$ **l** $\frac{5x+2}{5}-\frac{3-4x}{6}$

Example 28

3 Solve:

a $\frac{x}{6}+\frac{x}{3}=3$ **b** $\frac{x}{4}-\frac{x}{8}=1$ **c** $\frac{2x}{3}+\frac{x}{5}=\frac{13}{5}$

d $\frac{3x}{2}-\frac{4x}{7}=3$ **e** $\frac{5x}{3}-\frac{3x}{4}=\frac{11}{6}$ **f** $\frac{x}{5}-\frac{x}{10}=2$

g $\frac{x-2}{5}-\frac{x-5}{3}=\frac{1}{15}$ **h** $\frac{x-2}{2}+\frac{x}{3}=\frac{3}{2}$ **i** $\frac{x-1}{9}+\frac{3x-7}{4}=\frac{13}{18}$

j $\frac{2x-3}{6}-\frac{x-3}{2}=\frac{1}{3}$ **k** $\frac{1}{2}-\frac{x-2}{5}=\frac{2x-3}{10}$ **l** $1-\frac{3x+7}{2}=\frac{x+4}{4}$

3I Further algebraic fractions

The methods used in this section were introduced in the previous section, but here they are applied to algebraic fractions whose numerators and denominators are quadratics.

Simplifying, multiplying and dividing algebraic fractions

As before, there are two steps.

- First, factor each numerator and denominator completely.
- Second, complete the calculation by cancelling fractions.

Example 29

Simplify:

a $\dfrac{x^2 + 4x - 21}{x^2 - 49}$

b $\dfrac{2x^2 - x - 6}{x^2 - x - 2}$

Solution

a $$\begin{aligned}\frac{x^2 + 4x - 21}{x^2 - 49} &= \frac{\cancel{(x+7)}(x-3)}{\cancel{(x+7)}(x-7)}\\ &= \frac{x-3}{x-7}\end{aligned}$$

b $$\begin{aligned}\frac{2x^2 - x - 6}{x^2 - x - 2} &= \frac{(2x+3)\cancel{(x-2)}}{\cancel{(x-2)}(x+1)}\\ &= \frac{2x+3}{x+1}\end{aligned}$$

Example 30

Simplify:

a $\dfrac{2x^2 + x - 21}{2x^2 + 7x} \times \dfrac{x^2 + 3x}{x^2 - 9}$

b $\dfrac{2x^2 + 9x - 5}{x^2 - 25} \div \dfrac{x^2 + 9x + 20}{x^2 + 5x}$

Solution

a $$\begin{aligned}\frac{2x^2 + x - 21}{2x^2 + 7x} \times \frac{x^2 + 3x}{x^2 - 9} &= \frac{\cancel{(2x+7)}\,\cancel{(x-3)}}{\cancel{x}\,\cancel{(2x+7)}} \times \frac{\cancel{x}\,\cancel{(x+3)}}{\cancel{(x-3)}\,\cancel{(x+3)}}\\ &= 1\end{aligned}$$

b $$\begin{aligned}\frac{2x^2 + 9x - 5}{x^2 - 25} \div \frac{x^2 + 9x + 20}{x^2 + 5x} &= \frac{2x^2 + 9x - 5}{x^2 - 25} \times \frac{x^2 + 5x}{x^2 + 9x + 20}\\ &= \frac{(2x-1)\cancel{(x+5)}}{(x-5)\cancel{(x+5)}} \times \frac{x\cancel{(x+5)}}{(x+4)\cancel{(x+5)}}\\ &= \frac{x(2x-1)}{(x-5)(x+4)}\end{aligned}$$

Note: In **b** the calculation is only valid when $x \neq -5$, $x \neq 5$ and $x \neq -4$.

Adding and subtracting algebraic fractions

To add and subtract algebraic fractions, first find a common denominator. You may need to simplify the answer you obtain.

When the denominators are algebraic expressions, factor them completely before finding the common denominator.

Example 31

Simplify:

a $\dfrac{2}{x+3} - \dfrac{1}{x+4}$

b $\dfrac{x^2 + 5xy - 4y^2}{x^2 - 16y^2} - \dfrac{2xy}{2x^2 + 8xy}$

Solution

a
$$\begin{aligned}\frac{2}{x+3} - \frac{1}{x+4} &= \frac{2(x+4) - (x+3)}{(x+3)(x+4)} \\ &= \frac{2x + 8 - x - 3}{(x+3)(x+4)} \\ &= \frac{x+5}{(x+3)(x+4)}\end{aligned}$$

b
$$\begin{aligned}\frac{x^2 + 5xy - 4y^2}{x^2 - 16y^2} - \frac{2xy}{2x^2 + 8xy} &= \frac{x^2 + 5xy - 4y^2}{x^2 - 16y^2} - \frac{y}{x+4y} \\ &= \frac{x^2 + 5xy - 4y^2}{(x-4y)(x+4y)} - \frac{y}{x+4y} \\ &= \frac{x^2 + 5xy - 4y^2 - y(x-4y)}{(x-4y)(x+4y)} \\ &= \frac{x^2 + 5xy - 4y^2 - xy + 4y^2}{(x-4y)(x+4y)} \\ &= \frac{x^2 + 4xy}{(x-4y)(x+4y)} \\ &= \frac{x(x+4y)}{(x-4y)(x+4y)} \\ &= \frac{x}{x-4y}\end{aligned}$$

(The lowest common multiple of denominators is $(x - 4y)(x + 4y)$.)

Solving equations involving algebraic fractions

Example 32

Solve:

a $\dfrac{x+4}{x-4} + \dfrac{x-4}{x+4} = \dfrac{10}{3}$

b $\dfrac{x-5}{2x} = \dfrac{x-4}{3}$

Solution

a

$$\frac{x+4}{x-4} + \frac{x-4}{x+4} = \frac{10}{3}$$

$$\frac{3\cancel{(x-4)}(x+4)(x+4)}{\cancel{x-4}} + \frac{3(x-4)\cancel{(x+4)}(x-4)}{\cancel{x+4}} = 10(x+4)(x-4)$$

(Multiply through by $3(x-4)(x+4)$.)

$$\begin{aligned}
3(x^2+8x+16) + 3(x^2-8x+16) &= 10(x^2-16) \\
3x^2+24x+48+3x^2-24x+48 &= 10x^2-160 \\
6x^2+96 &= 10x^2-160 \\
256 &= 4x^2 \\
64 &= x^2 \\
x^2 &= 64 \\
x &= 8 \text{ or } x = -8
\end{aligned}$$

b

$$\begin{aligned}
\frac{x-5}{2x} &= \frac{x-4}{3} \\
\frac{6x(x-5)}{2x} &= \frac{6x(x-4)}{3} \qquad \text{(Multiply through by } 6x.) \\
3(x-5) &= 2x(x-4) \\
3x-15 &= 2x^2-8x \\
0 &= 2x^2-11x+15 \\
0 &= 2x^2-6x-5x+15 \\
0 &= 2x(x-3)-5(x-3) \\
0 &= (x-3)(2x-5) \\
x-3 = 0 &\text{ or } 2x-5=0 \\
x = 3 &\text{ or } x = \frac{5}{2}
\end{aligned}$$

Exercise 3I

Example 29

1 Express each fraction in its simplest form.

a $\dfrac{a^2 + ab}{ab}$

b $\dfrac{x^2 + xy}{x^2 - xy}$

c $\dfrac{a^2 - b^2}{(a + b)^2}$

d $\dfrac{b^2 - 1}{b^2 - b}$

e $\dfrac{x^2 - y^2}{(x - y)^2}$

f $\dfrac{3x^2 - 3xy^2}{x^2 + xy}$

g $\dfrac{x^2 + 6x + 8}{x^2 - 4}$

h $\dfrac{a^2 + a - 6}{2a + 6}$

i $\dfrac{x^2 - 8x + 15}{x^2 + 2x - 35}$

j $\dfrac{x^2 + x - 2}{x^2 - 1}$

k $\dfrac{2x^2 - 7x + 3}{(2x - 6)^2}$

l $\dfrac{3x^2 + 5xy - 2y^2}{4x^2 + 7xy - 2y^2}$

Example 30

2 Simplify:

a $\dfrac{2x}{2x + y} \times \dfrac{2xy + x^2}{6x^2}$

b $\dfrac{a^2 - 5a + 6}{a} \times \dfrac{a + 2}{a^2 - 2a}$

c $\dfrac{x^2 - 3x}{2x^2 + 7x + 3} \times \dfrac{2x^2 - 3x - 2}{x^2 - 5x + 6}$

d $\dfrac{x^2 - 4x + 4}{6x - 2} \times \dfrac{9x^2 - 6x + 1}{3x^2 - 12}$

e $\dfrac{a^2 - 1}{a} \times \dfrac{a^2}{a^2 + 2a + 1}$

f $\dfrac{4x - 12}{3x} \times \dfrac{9x^2}{6x - 18}$

g $\dfrac{x^2 - x}{3x - 3} \div \dfrac{3x}{3x - 2}$

h $\dfrac{4x - 12}{3x} \div \dfrac{6x - 18}{9x^2}$

i $\dfrac{2x}{3x - 3} \div \dfrac{4y}{x^2 - x}$

j $\dfrac{x^2 - 2xy + y^2}{x^2 - y^2} \div \dfrac{x^2 + xy}{x^2 + 2xy + y^2}$

Example 31

3 Express each with a common denominator.

a $\dfrac{x}{x - 2} - \dfrac{3}{x + 2}$

b $\dfrac{2x - 3}{3x - 9} - \dfrac{x - 2}{2x - 6}$

c $\dfrac{4}{x + 3} + \dfrac{12}{x^2 - 9}$

d $\dfrac{x + 3}{x + 2} - \dfrac{x + 2}{x + 3}$

e $\dfrac{1}{x^2 - 4} - \dfrac{1}{x^2 + 2x - 8}$

f $\dfrac{x}{x^2 - 4x + 3} - \dfrac{2}{x^2 - x - 6}$

g $\dfrac{1}{x - 1} + \dfrac{2}{x^2 + 2x - 3}$

h $\dfrac{1}{x - 2} - \dfrac{7}{2x^2 - x - 6}$

i $\dfrac{a - 2}{a^2 - 3a + 4} - \dfrac{a - 1}{a^2 - a - 2}$

4 Simplify:

a $\dfrac{1}{x+3} - \dfrac{2(2-x)}{3-x} - \dfrac{x-15}{x^2-9}$

b $\dfrac{2}{x+7} + \dfrac{3}{7-x} - \dfrac{2(7+2x)}{x^2-49}$

c $\dfrac{6}{a^2-1} - \dfrac{3}{a+1} + \dfrac{3}{a-1}$

d $\dfrac{a-x}{x} + \dfrac{a+x}{a} - \dfrac{a^2-x^2}{2ax}$

e $\dfrac{1}{x^2-9x+20} + \dfrac{1}{x^2-11x+30}$

f $\dfrac{1}{2x^2-x-1} - \dfrac{3}{6x^2-x-2}$

g $\dfrac{4}{4-7a-2a^2} - \dfrac{4}{3-a-10a^2}$

h $\dfrac{5}{a^2-9} + \dfrac{6}{a-3} + \dfrac{7}{a+3}$

5 Simplify:

a $\dfrac{2x^2+17x+21}{3x^2+26x+35}$

b $\dfrac{3x^2+23x+14}{3x^2+41x+26}$

c $\dfrac{x^2-5x}{x^2-4x-5}$

d $\dfrac{a^2-121}{a^2-4} \div \dfrac{a+11}{a+2}$

e $\dfrac{16x^2-9a^2}{x^2-4} \times \dfrac{x-2}{4x-3a}$

f $\dfrac{x^2-14x-15}{x^2-4x-45} \div \dfrac{x^2-12x-45}{x^2-6x-27}$

Example 32

6 Solve the equations.

a $\dfrac{x}{6} + \dfrac{3x+5}{3} = 4$

b $\dfrac{2x-3}{4} - \dfrac{x+1}{3} = 0$

c $\dfrac{2x+3}{5} - \dfrac{6+3x}{4} = \dfrac{101}{20}$

d $\dfrac{x-8}{7} + \dfrac{x-3}{3} + \dfrac{5}{21} = 0$

e $\dfrac{x+5}{6} - \dfrac{x+1}{9} = \dfrac{x+3}{4}$

f $\dfrac{4(x+2)}{3} - \dfrac{6(x-7)}{7} = 12$

7 **a** Express $\dfrac{4}{x} + \dfrac{8}{5x}$ as a single fraction.

b Solve the equation $\dfrac{4}{x} + \dfrac{8}{5x} = 1$.

8 Solve the equation $\dfrac{1}{x} + \dfrac{1}{2x} + \dfrac{3}{4x} + \dfrac{5}{12} = \dfrac{7}{24}$ by first multiplying both sides of the equation by $24x$.

9 Solve the equation $\dfrac{3x}{x-1} - \dfrac{2x}{x+1} = \dfrac{x^2+10}{x^2-1}$ by first multiplying both sides of the equation by x^2-1.

10 Solve each equation.

a $\dfrac{x-3}{x-4} = \dfrac{x+12}{x+8}$

b $\dfrac{x}{x-3} + \dfrac{2}{x-5} = 1$

c $\dfrac{x+2}{x-3} + \dfrac{x+2}{x-6} = 2$

d $\dfrac{3x+2}{x-1} + \dfrac{2x-4}{x+2} = 5$

e $\dfrac{1}{x+1} - \dfrac{2}{x+2} + \dfrac{1}{x+4} = 0$

f $\dfrac{3x}{x-1} - \dfrac{2x}{2x-1} = 2$

Review exercise

1 Simplify each expression by collecting like terms.

a $4x + 9x$

b $5p^3q - 9p^3q + 11p^3q$

c $ab - 2ab^2 - 6ab + ab^2$

d $8x^3y - 2x^2y + 12x^3y - 8x^2y$

2 Expand each expression.

a $-2(a + 4)$ **b** $-3(b + 6)$ **c** $-4(3b - 5)$

d $-5(4b - 7)$ **e** $\frac{2}{3}(12p + 6)$ **f** $-\frac{1}{4}(16p - 20)$

3 Expand:

a $d(d - 9)$ **b** $e(3e + 1)$ **c** $f(5f + 6)$

d $-2m(5m - 4)$ **e** $-3n(5n + 7)$ **f** $2p(3q - 5r)$

g $-3x(2x + 5y)$ **h** $-2z(3z - 4y)$ **i** $2a(3 + 4ab)$

4 Expand and collect like terms.

a $2(c + 7) - 9$ **b** $4(h + 1) + 3h$ **c** $4(1 - 3q) + 15q$

d $2a(3a + 2b) - 6a^2$ **e** $\frac{5}{6}(x - 4) + \frac{3x}{4}$ **f** $5(b - 2) - 4(b + 3)$

g $4y(3y - 5) + 3(3y - 5)$ **h** $2p(3p + 1) - 4(2p + 1)$ **i** $(m + 4)(m - 5)$

j $(6x - 5)(x + 2)$ **k** $(7x - 1)(x - 5)$ **l** $(m + 3n)(2m + n)$

5 Solve each equation.

a $7x - 4 = 17$ **b** $3x = 25 - 2x$ **c** $7x = 18 - 2x$

d $\frac{x}{3} + 2 = 7$ **e** $\frac{3x}{2} + 2 = 8$ **f** $3x - 7 = x + 1$

g $\frac{7x}{2} + 2x = 12 - x$ **h** $\frac{x}{2} + \frac{x}{5} = 14$ **i** $3(2x - 1) + 3x = 15$

6 Solve each inequality. Graph each solution on a number line.

a $x + 1 > 5$ **b** $2x - 1 < 6$ **c** $\frac{x + 1}{2} \geq -4$

d $-2x + 1 \leq 6$ **e** $4 - 7x \geq 6$ **f** $\frac{-3x + 2}{6} \leq 1$

g $\frac{2x - 1}{3} \leq \frac{x + 1}{4}$ **h** $\frac{5x + 6}{3} \leq 2x + 7$ **i** $4 - 6x \leq 3 + 2x$

7 The total resistance R of two resistors with resistances r_1 and r_2, joined in parallel, is given by the formula $\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$.

a Find R if $r_1 = 4$ and $r_2 = 3$.

b Find r_1 if $R = 2$ and $r_2 = 6$.

8 Suppose that $P = 4(a + b)$.

a Find P when $a = 6$ and $b = 7$.

b Make a the subject of the formula.

c Find a when $P = 20$ and $b = 6$.

9 Make the variable in brackets the subject of the formula. All variables take positive values.

a $A = \pi r^2 + \pi rh \quad (h)$

b $A = \pi r\sqrt{h^2 + r^2} \quad (h)$

c $E = \frac{1}{2}m(v^2 - u^2) \quad (u)$

d $\frac{ax + by}{c} = x - b \quad (x)$

10 Factorise:

a $49 - z^2$ **b** $9 - 4a^2$ **c** $25 - 64x^2$ **d** $a^2b^2 - 4c^2d^2$

e $9x^2 - 1$ **f** $36p^2 - 49q^2$ **g** $1 - 100b^2$ **h** $121a^2 - 81b^2$

11 Factorise using surds:

a $x^2 - 11$ **b** $x^2 - 15$ **c** $x^2 - 28$ **d** $x^2 - 32$

12 Factorise using surds:

a $(x + 4)^2 - 2$ **b** $(x - 3)^2 - 5$ **c** $(2x + 1)^2 - 20$ **d** $(3 - 2x)^2 - 10$

13 Factorise:

a $(a + b)^2 - c^2$ **b** $(a - b)^2 - c^2$ **c** $(x + y)^2 - 4z^2$

14 Factorise:

a $2ax - a + 6x - 3$ **b** $ab + 5b - 2a - 10$

c $2xy + 3y + 10x + 15$ **d** $3mx - 21x + 2m - 14$

15 Factorise:

a $x^2 - 3x + 2$ **b** $c^2 + 12c + 11$ **c** $x^2 - 4x - 5$

d $y^2 + 9y - 10$ **e** $z^2 - 6z - 16$ **f** $c^2 + 8c - 20$

g $p^2 - 11p - 26$ **h** $k^2 - 11k - 42$ **i** $x^2 + 4x - 60$

16 Factorise:

a $2x^2 + 10x - 132$ **b** $3x^2 + 12x - 231$ **c** $3x^2 - 18x - 21$

d $3x^2 - 9x - 30$ **e** $3x^2 - 33x + 90$ **f** $6x^2 + 6x - 12$

17 Factorise:

a $2x^2 + 11x + 12$ **b** $6s^2 - 11st - 10t^2$ **c** $12x^2 - 7x - 12$

d $3x^2 + 8x + 4$ **e** $2x^2 + x - 3$ **f** $4x^2 + 4x - 3$

g $3x^2 + 8x - 3$ **h** $6x^2 + 11x - 10$ **i** $3x^2 + x - 2$

18 Solve:

a $\dfrac{2x-1}{3} = \dfrac{x+2}{2}$ **b** $\dfrac{3x-2}{5} = \dfrac{2x-5}{3}$

c $\dfrac{5x-3}{2} = 2x + 4$ **d** $\dfrac{x}{4} + \dfrac{3x+5}{2} = 4$

e $\dfrac{x}{2} + \dfrac{x}{3} + \dfrac{x}{4} + \dfrac{x}{5} = \dfrac{77}{6}$ **f** $\dfrac{x+4}{14} + \dfrac{x-4}{8} = 1$

g $\dfrac{x-2}{6} - \dfrac{x+1}{7} = \dfrac{x+3}{5}$ **h** $4 - \dfrac{x-9}{8} = \dfrac{x}{22} - \dfrac{1}{2}$

19 Simplify:

a $\dfrac{1}{x-2} - \dfrac{2}{x-1} + \dfrac{1}{x-3}$

b $\dfrac{3x}{x^2 - 3x + 2} - \dfrac{4}{x-1} + \dfrac{1}{x-2}$

c $\dfrac{8}{x^2 - 5x + 6} - \dfrac{5}{x^2 - 3x + 2} - \dfrac{3}{x^2 - 4x + 3}$

20 **a** Find a number such that if 5, 15 and 35 are added separately to it, the product of the first and third results is equal to the square of the second.

b A cellar contains only bottles of port, claret, sherry and brandy. Of these, $\frac{1}{5}$ of the bottles are port and $\frac{1}{5}$ are claret. The cellar also contains 15 dozen bottles of sherry and 30 bottles of brandy. How many bottles of port and claret does it contain?

Challenge exercise

1 Factorise:

a $(3x + 7y)^2 - (2x - 3y)^2$

b $(5x + 2y)^2 - (3x - y)^2$

2 Factorise:

a $a^2 - 2ax + x^2 - 4b^2$

b $9a^2 - c^2 + 2cx - x^2$

c $2bd - a^2 - c^2 + b^2 + d^2 + 2ac$

d $y^2 + 2by + b^2 - a^2 - 6ax - 9x^2$

3 Expand:

a $(a + b + c)^2$

b $(7a^2 - 3x)(49a^4 + 21a^2x + 9x^2)$

4 Factorise:

a $28x^4y + 64x^3y - 60x^2y$

b $x^2p^2 - 8y^2p^2 - 4x^2q^2 + 32y^2q^2$

5 Solve the equation $\frac{3}{16}(x - 1) - \frac{5}{12}(x - 4) = \frac{2}{5}(x - 6) + \frac{5}{48}$.

6 **a** The width of a room is two-thirds its length. If the width were 3 m more and the length 3 m less, the room would be square. Find the length and width of the room.

b The length of a room exceeds its width by 3 m. If the length increased by 3 m and the width diminished by 2 m, the area would be unaltered. Find the length and width of the room.

7 A train that travels a km at a constant speed in b hours is p times as fast as a bus. If the bus takes m hours to cover the distance between two places, how many kilometres apart are the two places?

8 Jonathan is c km due west of Anna when they start simultaneously walking in an easterly direction. Jonathan walks at a constant rate of p km/h and Anna walks at a constant rate of q km/h. How far will Jonathan have walked when he overtakes Anna? (Assume that $p > q$.)

CHAPTER

4

Number and Algebra

Lines and linear equations

Coordinate geometry takes the surprising approach of using algebra to solve geometric problems.

This chapter continues the development of coordinate geometry begun in *ICE-EM Mathematics Year 9*.

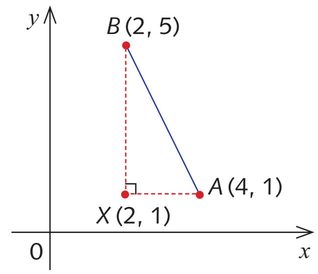
Each point in the plane is represented by an ordered pair (x, y) and each line is the set of points that satisfies a linear equation $ax + by + c = 0$.

The gradient of a line allows us to answer most questions about parallelism and perpendicularity. In principle, every geometric problem can be solved using coordinate geometry.

4A Distance between two points and midpoint of an interval

Distance formula

Consider the points $A(4, 1)$ and $B(2, 5)$ in the number plane. The length of the interval AB can be found using Pythagoras' theorem. This is called the **distance** between the points A and B.



Form the right-angled triangle ABX, as shown, where X is the point $(2, 1)$. Then:

$$BX = 5 - 1 = 4 \quad \text{and} \quad AX = 4 - 2 = 2$$

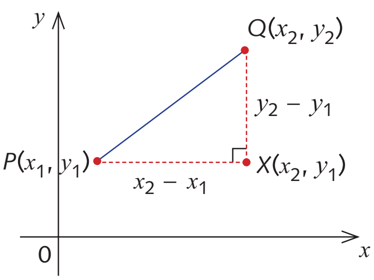
By Pythagoras' theorem:

$$\begin{aligned} AB^2 &= AX^2 + BX^2 \\ &= 2^2 + 4^2 \\ &= 20 \\ AB &= \sqrt{20} \\ &= 2\sqrt{5} \end{aligned}$$

The length of interval AB is $2\sqrt{5}$.

The general case

We can use the above idea to obtain a formula for the distance between any two points. Suppose that $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points, as shown opposite.



Form the right-angled triangle PQX, where X is the point (x_2, y_1). Then:

$$PX = x_2 - x_1 \quad \text{and} \quad QX = y_2 - y_1$$

By Pythagoras' theorem:

$$\begin{aligned} PQ &= PX^2 + QX^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \end{aligned}$$

You will notice that our diagram assumes that $x_2 - x_1$ and $y_2 - y_1$ are positive. If either or both are negative, it is not necessary to change the formula, as we are squaring. In other words:

$$PQ^2 = (\text{square of the difference of } x\text{-values}) + (\text{square of the difference of } y\text{-values})$$

Therefore:

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In practice, we sometimes work out the square of PQ and then take the square root.

Example 1

Use the distance formula to find the distance between each pair of points.

a $A(2,\ 3)$ and $B(5,\ 7)$

b $A(1, 2)$ and $B(-1, -3)$

c $A(2,\ 4)$ and $B(5,\ 4)$

d $A(-2, 3)$ and $B(-4, -3)$

Solution

a
$$\begin{aligned} AB^2 &= (5-2)^2 + (7-3)^2 \\ &= 3^2 + 4^2 \\ &= 25 \\ AB &= 5 \end{aligned}$$

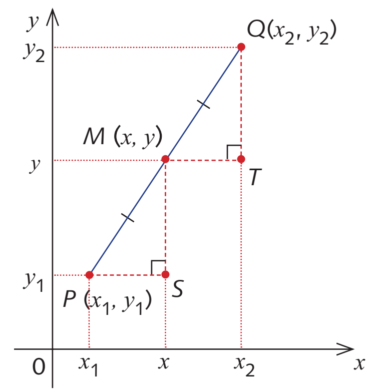
b
$$\begin{aligned} AB^2 &= (-1-1)^2 + (-3-2)^2 \\ &= (-2)^2 + (-5)^2 \\ &= 4 + 25 \\ &= 29 \\ AB &= \sqrt{29} \end{aligned}$$

c
$$\begin{aligned} AB^2 &= (5-2)^2 + (4-4)^2 \\ &= 3^2 + 0^2 \\ &= 9 \\ AB &= 3 \end{aligned}$$

d
$$\begin{aligned} AB^2 &= [-4-(-2)]^2 + (-3-3)^2 \\ &= (-2)^2 + (-6)^2 \\ &= 4 + 36 \\ &= 40 \\ AB &= \sqrt{40} \\ &= 2\sqrt{10} \end{aligned}$$

Midpoint formula

We can find a formula for the midpoint of any interval. Let $P(x_1,\ y_1)$ and $Q(x_2,\ y_2)$ be two points and let $M(x,\ y)$ be the midpoint of the interval PQ.



The triangles PMS and MQT are congruent triangles (AAS), so $PS = MT$ and $MS = QT$.

Hence, the x-coordinate of M is the average of x_1 and x_2,

Therefore, $x = \dfrac{x_1 + x_2}{2}$.

The y-coordinate of M is the average of y_1 and y_2.

Therefore, $y = \dfrac{y_1 + y_2}{2}$.

The coordinates of M are $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

Example 2

Find the coordinates of the midpoint, M, of interval AB, where A and B have coordinates $(-2, 6)$ and $(3, -7)$, respectively.

Solution

x-coordinate of $M = \dfrac{-2+3}{2} = \dfrac{1}{2}$ y-coordinate of $M = \dfrac{6+(-7)}{2} = -\dfrac{1}{2}$

The coordinates of M are $\left(\dfrac{1}{2}, -\dfrac{1}{2}\right)$.

Distance between two points and midpoint of an interval

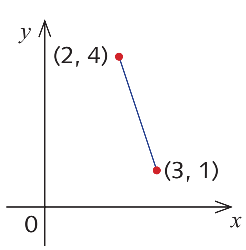
Consider two points, $P(x_1, y_1)$ and $Q(x_2, y_2)$.

- The distance between the points P and Q is given by the expression $PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.
  That is, $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- The midpoint M of the interval PQ has coordinates $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

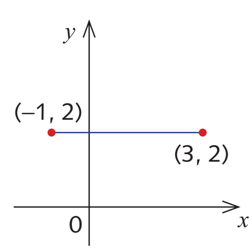
Exercise 4A

1 Find the distance between the labelled points in each diagram.

a



b



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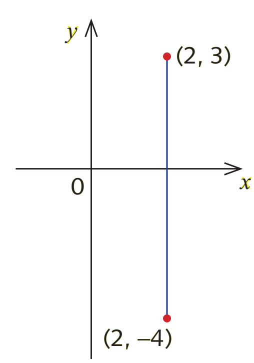


Example 1

2 Find the distance between points A and B.

a $A(1, 2)$, $B(0, 0)$ **b** $A(-1, 6)$, $B(4, 8)$

c $A(-2, 8)$, $B(6, 4)$ **d** $A(-2, -6)$, $B(3, 2)$

e $A(-3, 4)$, $B(4, 3)$ **f** $A(-3, -4)$, $B(3, 4)$

Example 2

3 Find the midpoint of the interval AB.

a $A(-1, 2),\ B(3, 6)$

b $A(2, 8),\ B(-1, 2)$

c $A(2, 4),\ B(6, 8)$

d $A(-2, -6),\ B(-4, -8)$

e $A(-1, 5),\ B(2, 7)$

f $A(-12, 16),\ B(2, 8)$

4 **a** Find the midpoint M of the interval AB where the coordinates of A are $(6, 2)$ and the coordinates of B are $(6, 8)$.

b Let C be the point $(10, 5)$. Find the distance between:

i A and C

ii A and B

c Describe triangle ABC.

5 **a** The distance between two points $A(2, u)$ and $B(-3, 4)$ is 5. Find the value of u.

b The distance between two points $P(4, -2)$ and $Q(v, -5)$ is $\sqrt{34}$. Find the possible values for v. Draw a diagram to illustrate the result.

c The distance between two points $A(3, -2)$ and $B(w, 4)$ is 10. Find the possible values for w. Draw a diagram to illustrate the result.

6 The triangle ABC has vertices $A(0, 0)$, $B(3, 0)$ and $C(3, 4)$.

a Find the distance between A and C.

b Find the midpoint M of AC.

c Find the length of:

i AM

ii BM

iii CM

7 **a** $M(4, 2)$ is the midpoint of the interval AC, where C has coordinates $(12, 3)$. Find the coordinates of A.

b $M(10, -2)$ is the midpoint of the interval AC, where A has coordinates $(-2, 6)$. Find the coordinates of C.

8 Show that ΔPQR is a right-angled triangle where the coordinates of P, Q and R are $(3, 3)$, $(3, -1)$ and $(6, 3)$, respectively.

9 Show that the triangle with vertices $X(-3, 1)$, $Y(0, 2)$ and $Z(-2, 4)$ is isosceles.

10 Show that the points $A(-1, -3)$, $B(4, 0)$, $C(5, 7)$ and $D(0, 2)$ are the vertices of a rhombus.

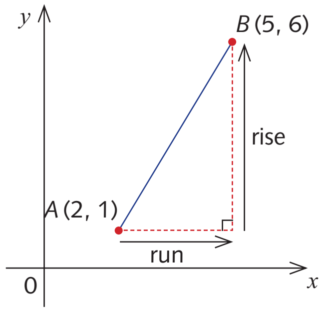
11 A, B, C and D are the points $(0, -5)$, $(-4, -1)$, $(4, 3)$ and $(-8, -9)$, respectively. Show that AB and CD bisect each other.

12 The points $A(-5, 0)$, $B(-3, -4)$, $C(2, 1)$ and $D(0, 5)$ are the vertices of a quadrilateral. Show that $ABCD$ is a parallelogram.

4B Gradient

Gradient of an interval

The **gradient** of an interval AB is defined as $\frac{\text{rise}}{\text{run}}$, where the **rise** is the change in the y-values as you move from A to B and the **run** is the change in the x-values as you move from A to B.

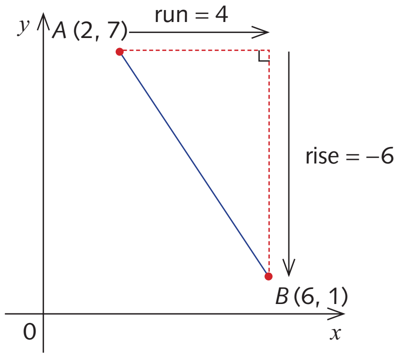


For the points $A(2, 1)$ and $B(5, 6)$:

$$\text{gradient of interval } AB = \frac{\text{rise}}{\text{run}} = \frac{6-1}{5-2} = \frac{5}{3}$$

Notice that as you move from A to B along the interval, the y-value increases as the value increases. This means the gradient is **positive**.

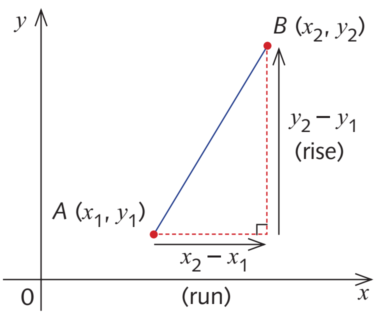
In the diagram to the right:



$$\text{gradient of interval } AB = \frac{\text{rise}}{\text{run}} = \frac{1-7}{6-2} = \frac{-6}{4} = -\frac{3}{2}$$

The rise from A to B is negative and the run from A to B is positive, so the gradient is **negative**.

In general, provided $x_2 \neq x_1$:

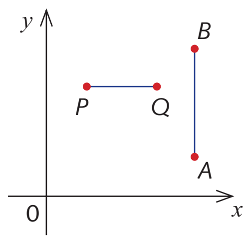


$$\text{gradient of interval } AB = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

Since $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$, it does not matter which point we take as the first and which point we take as the second.

If the rise is zero, the interval is horizontal, as shown by the interval PQ at the right. The gradient of the interval is zero.

If the run is zero, the interval is vertical, as shown by the interval AB at the right. The interval does not have a gradient.



Gradient of PQ is zero.
Gradient of AB is not defined.

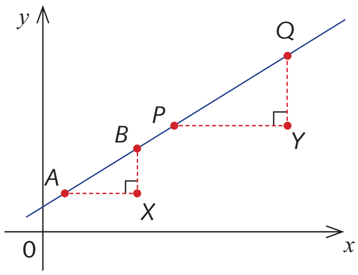
Gradient of a line

The **gradient of a line** is defined to be the gradient of any interval within the line.

Any two intervals on a line have the same gradient. We can prove this as follows.

Triangle ABX is similar to triangle PQY, as the corresponding angles are equal by parallel lines. Therefore:

$$\frac{QY}{PY} = \frac{BX}{AX} \quad \text{(Ratio of sides in similar triangles.)}$$



That is, the intervals have the same gradient. Therefore, the definition of the gradient of a line makes sense.

Example 3

Find the gradient AB.

a $A(3, -2)$, $B(2, -6)$

b $A(-1, -3)$, $B(-2, 6)$.

Solution

a
$$\begin{aligned}\text{Gradient} &= \frac{-6 - (-2)}{2 - 3} \\ &= \frac{-6 + 2}{-1} \\ &= 4\end{aligned}$$

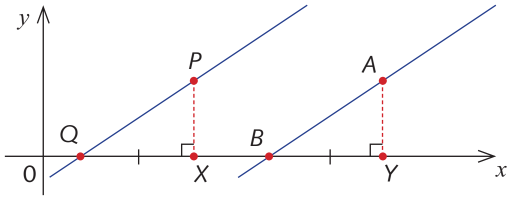
b
$$\begin{aligned}\text{Gradient} &= \frac{-3 - 6}{-1 - (-2)} \\ &= \frac{-9}{-1 + 2} \\ &= -9\end{aligned}$$

Parallel lines

If two non-vertical lines are **parallel**, then they have the same gradient. Conversely, if two lines have the same gradient, then they are parallel.

We can prove this as follows.

In the diagram on the right, two lines are drawn and the right-angled triangles PQX and ABY are drawn, with $QX = BY$.



If the lines are parallel, then $\angle PQX = \angle ABY$ (corresponding angles).

The two triangles are congruent by the AAS test. Hence, $PX = AY$, so $\frac{AY}{BY} = \frac{PX}{QX}$.

The gradients are equal.

Conversely, if the gradients are equal, then $PX = AY$.

The triangles are congruent by the SAS test.

Hence, the corresponding angles PQX and ABY are equal and the lines are parallel.

Note: This proof does not work for lines that are parallel to one of the axes.

Example 4

Show that the line passing through the points $A(6, 4)$ and $B(7, 11)$ is parallel to the line passing through $P(0, 0)$ and $Q(1, 7)$.

Solution

$$\begin{aligned}\text{Gradient of } AB &= \frac{11-4}{7-6}\\ &= \frac{7}{1}\\ &= 7\end{aligned}$$

$$\begin{aligned}\text{Gradient of } PQ &= \frac{7-0}{1-0}\\ &= \frac{7}{1}\\ &= 7\end{aligned}$$

The two lines have the same gradient, so they are parallel.

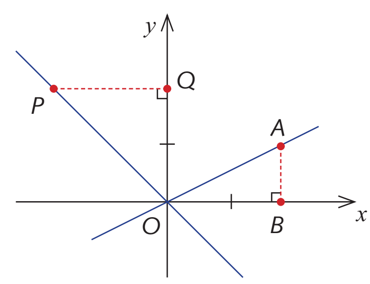
Perpendicular lines

Two lines are **perpendicular** if the product of their gradients is −1 (or if one is vertical and the other horizontal). Conversely, if two lines are perpendicular (but are not parallel with the axes), then the product of their gradients is −1.

Here is a proof of this remarkable result.

Draw two lines passing through the origin, with one of the lines having positive gradient and the other negative gradient.

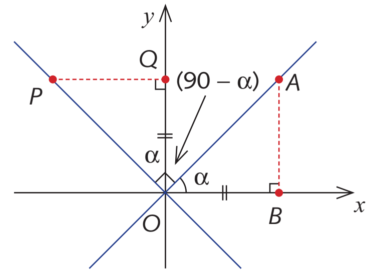
Form right-angled triangles OPQ and OAB, with $OQ = OB$.



$$\text{Gradient of the line } OA = \frac{AB}{BO}$$

$$\text{Gradient of the line } OP = -\frac{OQ}{PQ}$$

$$\begin{aligned}\text{Product of gradients} &= -\frac{OQ}{PQ} \times \frac{AB}{BO}\\ &= -\frac{OQ}{PQ} \times \frac{AB}{OQ} \qquad (\text{since } OB = OQ)\\ &= -\frac{AB}{PQ}\end{aligned}$$



If the lines are perpendicular, then $\angle POQ = \angle AOB$.

Therefore, triangles OPQ and OAB are congruent (AAS), so $PQ = AB$ and the product of the gradients is -1.

Conversely, if the product of the gradients is -1, then $AB = PQ$ since, by the above, the product of the gradients $= -\dfrac{AB}{PQ}$.

This implies that the triangles OBA and OQP are congruent (SAS). Therefore, $\angle POQ = \angle AOB$ and so $\angle AOP = 90° - \alpha + \alpha = 90°$.

We have now proved the result for lines through the origin. However, this will suffice for any pair of lines in the plane (not parallel with the axes).

Example 5

Show that the line through the points $A(6, 0)$ and $B(0, 12)$ is perpendicular to the line through $P(8, 10)$ and $Q(4, 8)$.

Solution

$$\text{Gradient of } AB = \frac{12-0}{0-6} = -2$$

$$\text{Gradient of } PQ = \frac{10-8}{8-4} = \frac{2}{4} = \frac{1}{2}$$

$$(\text{Gradient of } AB) \times (\text{Gradient of } PQ) = -2 \times \frac{1}{2} = -1$$

Hence, the lines are perpendicular.

Gradient of non-vertical lines

- The **gradient of an interval**, AB, connecting the two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\dfrac{y_2 - y_1}{x_2 - x_1}$.
- The **gradient of a line** is defined as the gradient of any interval within the line.
- Two lines are **parallel** if they have the same gradient. Conversely, if two lines are parallel, then they have the same gradient.
- Two lines are **perpendicular** if the product of their gradients is -1 (or if one is vertical and the other horizontal). Conversely, if two lines are perpendicular, then the product of their gradients is -1.

Exercise 4B

Example 3

1 Find the gradient of each interval.

a $A(6,3),\ B(2,0)$ **b** $A(-2,6),\ B(0,10)$ **c** $A(-1,10),\ B(6,-4)$

d $A(2,3),\ B(-4,5)$ **e** $A(6,7),\ B(-2,-3)$ **f** $A(10,0),\ B(0,10)$

g $A(10,0),\ B(0,-10)$ **h** $A(4,3),\ B(6,3)$ **i** $A(4,-3),\ B(-5,10)$

Example 4

2 Show that the line passing through $A(1,\ 6)$ and $B(2,\ 7)$ is parallel to the line passing through $X(-1,6)$ and $Y(2,9)$.

3 The line passing through the points $(1,4)$ and $(3,a)$ has gradient 2. Find the value of a.

4 The line passing through the points $(-4,6)$ and $(b,2)$ has gradient $\frac{1}{2}$. Find the value of b.

5 Complete:

	Coordinates of A	Coordinates of B	Gradient of AB
a	(2, 1)	(5, 13)	…
b	(–1, 3)	(0, –1)	…
c	(–1, 2)	(2, …)	2
d	(–4, 10)	(2, …)	$-\frac{1}{2}$
e	(…, 5)	(7, 9)	$\frac{2}{3}$
f	(…, –4)	(1, –13)	–3

Example 5

6 Show that the line passing through the points $A(5,60)$ and $B(-1,12)$ is perpendicular to the line passing through $P(7,10)$ and $Q(23,8)$.

7 Find the gradient of a line perpendicular to a line with gradient:

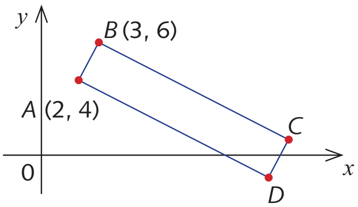
a 6 **b** $-\frac{1}{2}$ **c** $\frac{3}{2}$ **d** $-\frac{4}{5}$ **e** -1

8 $ABCD$ is a rectangle.

a Find the gradient of interval AB.

b Find the gradient of interval CD.

c Find the gradient of interval AD.



9 **a** Plot the four points $A(0,0)$, $B(3,0)$, $C(5,2)$ and $D(2,2)$.

b Use gradients to show that $ABCD$ is a parallelogram.

c Find the midpoint of DB and explain why it is the same as the midpoint of AC.

10 The vertices of a quadrilateral $ABCD$ are the points $(-4,-2)$, $(3,9)$, $(8,1)$ and $(2,-3)$, respectively. E, F, G and H are the midpoints of AB, BC, CD and DA, respectively. Show that $EFGH$ is a parallelogram.

11 $A(3,6)$ and $B(4,7)$ are two adjacent vertices of a square $ABCD$.

a Find the length of each side of the square.

b Find the gradient of AB.

c Find the gradient of CD.

12 In each part, find the gradients of intervals AB and BC, and state whether A, B and C lie on the same line (are collinear) or not.

a $A(3,6)$, $B(-1,4)$, $C(4,11)$

b $A(3,8)$, $B(2,5)$, $C(1,2)$

c $A(4,11)$, $B(-1,-4)$, $C(2,5)$

d $A(4,5)$, $B(-1,-6)$, $C(3,7)$

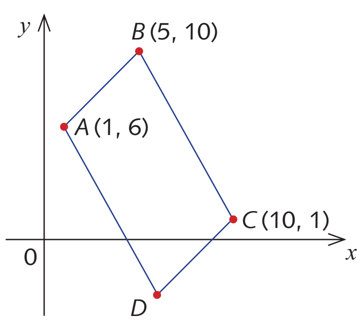
13 $ABCD$ is a parallelogram.

a Find the gradient of interval AB.

b Find the gradient of interval CD.

c Find the coordinates of D.

d Find the coordinates of the midpoints of AC and BD.



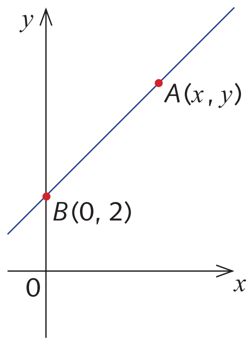
4C Gradient–intercept form and the general form of the equation of a line

Gradient–intercept form

In earlier work, we have seen that the equation $y = mx + b$ represents a line with gradient m and y-intercept b. This is called the **gradient–intercept form** of the equation of a line. Conversely, every non-vertical line has an equation of the form $y = mx + b$.

To illustrate this, consider the line with gradient 3 and y-intercept 2. That is, $m = 3$ and $b = 2$.

Let $A(x, y)$ be any point on this line.



$$\begin{aligned}\text{Gradient of interval } AB &= \frac{\text{rise}}{\text{run}}\\ &= \frac{y-2}{x-0}\\ &= \frac{y-2}{x}\end{aligned}$$

We know the gradient of the line is 3. Therefore:

$$\frac{y-2}{x} = 3$$
$$y - 2 = 3x$$
$$y = 3x + 2$$

Hence, the equation of the line is $y = 3x + 2$.

The equation relates the x- and y-coordinates of any point on the line.

In general lines with gradient m and y-intercept b have equation $y = mx + b$. Conversely the points whose coordinates satisfy th equation $y = mx + b$ always lie on a line with gradient m and y-intercept b.

Example 6

a The gradient of a line is -6 and the y-intercept is 2. Find the equation of the line.

b The equation of a line is $y = -7x + 3$. State the gradient and y-intercept.

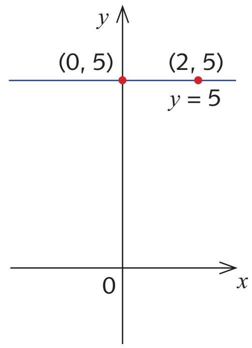
Solution

a The equation of the line is $y = -6x + 2$.

b The gradient is -7 and the y-intercept is 3.

Horizontal lines

All points on a horizontal line have the same y-coordinate, but the x-coordinate can take any value. Thus, the equation of the horizontal line through the point $(0, 5)$ is $y = 5$. The equation of the horizontal line through the point $(2, 5)$ is also $y = 5$.

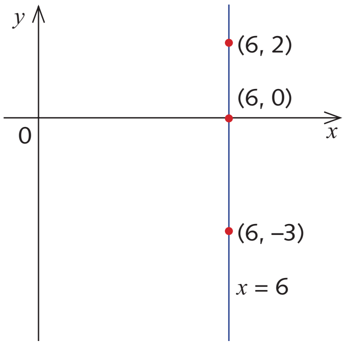


In general, the equation of the horizontal line through $P(a, b)$ is $y = b$.

A horizontal line has gradient 0 because all y-values are the same.

Vertical lines

All points on a vertical line have the same x-coordinate, but the y-coordinate can take any value. Thus, the equation of the vertical line through the point $(6, 0)$ is $x = 6$.



In general, the equation of the vertical line through $P(a, b)$ is $x = a$ or $x - a = 0$. Note that because this line does not have a gradient, it cannot be written in the form $y = mx + b$.

The form of the equation for a vertical or a horizontal line sometimes seems strange. It becomes clearer if we realise that the equation $x = a$ is shorthand for the statement $\{(x, y) : x = a\}$. This is read as the 'set of points (x, y) such that $x = a$'. Similarly, the equation $y = b$ is shorthand for the statement $\{(x, y) : y = b\}$. This is read as the 'set of points (x, y) such that $y = b$'.

The general form of an equation of a line

The equation $y = 2x - 3$ can be written as $-2x + y + 3 = 0$.

The equation $2x - 3y = 6$ can be written as $2x - 3y - 6$ and the equation $x = 6$ can be written as $x - 6 = 0$.

The **general form** for the equation of a line is $ax + by + c = 0$, where a, b and c are constants, and either $a \neq 0$ or $b \neq 0$. The equation of *every line* can be written in general form.

Example 7

Write each equation in general form.

a $y = -\frac{2}{3}x + 4$

b $y = -\frac{4}{5}x + \frac{2}{3}$

Solution

a

$$\begin{aligned} y &= -\frac{2}{3}x + 4 \\ 3y &= -2x + 12 \\ 2x + 3y - 12 &= 0 \end{aligned}$$

b

$$\begin{aligned} y &= -\frac{4}{5}x + \frac{2}{3} \\ 15y &= -12x + 10 \\ 12x + 15y - 10 &= 0 \end{aligned}$$

The general form is not unique. For example, the line $2x + 3y - 12 = 0$ is the same as the line $20x + 30y - 120 = 0$.

Example 8

Write the equation of each line in gradient–intercept form, and state its gradient and y-intercept.

a $2x + y + 6 = 0$

b $3x - 2y + 7 = 0$

Solution

a

$$\begin{aligned} 2x + y + 6 &= 0 \\ y &= -2x - 6 \\ \text{gradient} &= -2 \\ y\text{-intercept is } & -6 \end{aligned}$$

b

$$\begin{aligned} 3x - 2y + 7 &= 0 \\ 3x + 7 &= 2y \\ y &= \frac{3}{2}x + \frac{7}{2} \\ \text{gradient} &= \frac{3}{2} \\ y\text{-intercept is } & \frac{7}{2} \end{aligned}$$

Sketching a line given its equation

A line can be sketched if the coordinates of two points are known.

For lines that are not parallel to one of the axes and do not pass through the origin, a useful procedure to sketch the line is to find the intercepts with the axes. Find the x-intercept by substituting $y = 0$, and find the y-intercept by substituting $x = 0$.

A non-vertical line passing through the origin has an equation of the form $y = mx$. A second point on the line can be determined from the equation by substituting a non-zero value of x into the equation. This is recommended because it identifies the steepness of the line.

Example 9

Sketch the graph of:

a $y = 2x + 4$

b $y = -3x + 8$

c $2x + 3y + 12 = 0$

d $3x + 2y = 10$

e $x = 4$

f $y = -3x$

Solution

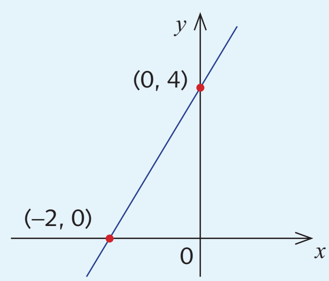
a $y = 2x + 4$

When $x = 0$, $y = 4$

When $y = 0$, $2x + 4 = 0$

$$x = \frac{-4}{2}$$

$$= -2$$



b $y = -3x + 8$

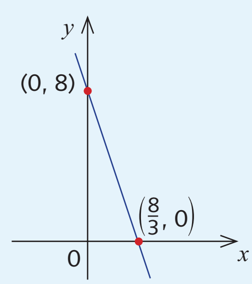
When $x = 0$, $y = 8$

When $y = 0$, $-3x + 8 = 0$

$$-3x = -8$$

$$x = \frac{8}{3}$$

$$x = 2\tfrac{2}{3}$$



c $2x + 3y + 12 = 0$

When $x = 0$, $3y + 12 = 0$

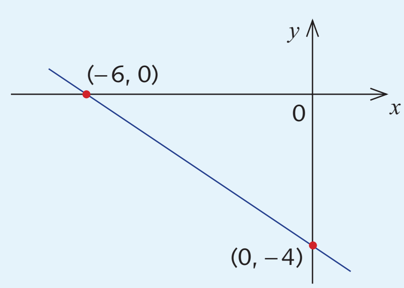
$$3y = -12$$

$$y = -4$$

When $y = 0$, $2x + 12 = 0$

$$2x = -12$$

$$x = -6$$



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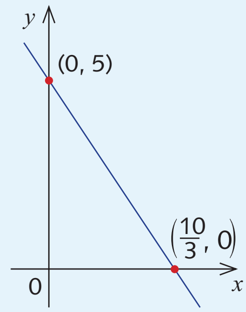
d $3x + 2y = 10$

When $x = 0$, $2y = 10$

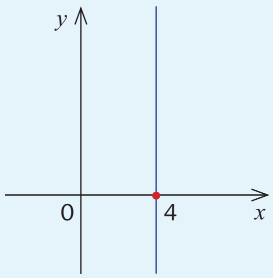
$$y = 5$$

When $y = 0$, $3x = 10$

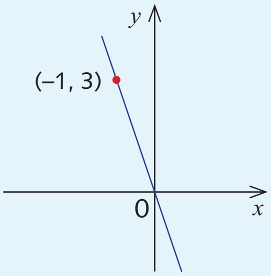
$$x = \frac{10}{3}$$



e $x = 4$



f $y = -3x$



Given one coordinate of a point on a line, the equation of a line can be used to find the other coordinate.

Example 10

The following points lie on the line with equation $5x - 4y = 20$. Find the value of each pronumeral.

a $(0, a)$ **b** $(b, 0)$ **c** $(d, -6)$ **d** $(f, 10)$

Solution

a $5 \times 0 - 4a = 20$

$$a = -5$$

b $5b - 4 \times 0 = 20$

$$5b = 20$$

$$b = 4$$

c $5d - 4 \times (-6) = 20$

$$5d + 24 = 20$$

$$5d = -4$$

$$d = -\frac{4}{5}$$

d $5f - 4 \times 10 = 20$

$$5f = 60$$

$$f = 12$$

Gradient–intercept form and the general form of the equation of a line

- The **gradient–intercept form** of the equation of a line is $y = mx + b$, where m is the gradient and b is the y-intercept.
- **The general form** of the equation of a line is $ax + by + c = 0$, where a, b and c are constants, and $a \neq 0$ or $b \neq 0$.

Exercise 4C

Example 6

1 Write the gradient and y-intercept of each line.

a $y = 4x + 2$ **b** $y = -\frac{2}{3}x + 5$ **c** $y = -7x + 10$ **d** $y = -\frac{4}{11}x + \frac{2}{3}$

2 Write the equation of the line with the given gradient and y-intercept.

a gradient $= 8$, y-intercept is 3 **b** gradient $= 11$, y-intercept is 5

c gradient $= -6$, y-intercept is -7 **d** gradient $= -\frac{3}{4}$, y-intercept is $\frac{2}{5}$

3 Sketch the graph of each line.

a $y = 1$ **b** $y = 2$ **c** $y = -2$ **d** $y = -3$

e $x = 2$ **f** $x = 4$ **g** $x = -1$ **h** $x = -3$

i $y + 3 = 0$ **j** $y - 1 = 0$ **k** $x + 3 = 0$ **l** $x - 1 = 0$

m Which of these lines have a gradient and what is it?

Example 7

4 Express each equation in general form.

a $y = -2x + 6$ **b** $y = -\frac{2}{3}x + 11$ **c** $y = -\frac{3}{5}x - \frac{2}{3}$ **d** $y = \frac{4}{7}x + \frac{1}{6}$

e $y = -\frac{2}{5}x - \frac{4}{5}$ **f** $y = \frac{2}{5}x - \frac{3}{10}$ **g** $x = \frac{1}{3}y + 4$ **h** $\frac{3}{4}x = \frac{4}{3}y - 3$

Example 8

5 Express each equation in gradient–intercept form.

a $3x - 2y = 6$ **b** $5x + 2y + 10 = 0$ **c** $3y - 2x + 12 = 0$

d $6y - x + 18 = 0$ **e** $15y - 2x + 18 = 0$ **f** $2x - 3y + 12 = 0$

g $5x + 4y + 20 = 10$ **h** $6x - 4y - 24 = 0$ **i** $3x - 5y + 15 = 0$

6 Find the gradient and y-intercept in each case.

a $2y - 3x = 12$ **b** $4x + y + 24 = 0$ **c** $3x + 8y + 48 = 0$

d $4x - 7y + 56 = 0$ **e** $11x + 4y = 44$ **f** $10x - 5y = -20$

g $3x - 7y + 42 = 0$ **h** $2x - 7y = 14$ **i** $-10x - 2y = -40$

Example 9a, b, c, d

7 Find the x- and y-intercepts in each case.

a $y = 2x - 10$ **b** $y = 3x + 11$ **c** $3x + 8y = 48$

d $5x - 4y + 80 = 0$ **e** $3x - 7y - 42 = 0$ **f** $5x - 2y + 11 = 0$

8 Sketch the graph of each line by first finding the x- and y-intercepts.

a $y = -2x + 12$ **b** $3y = -2x + 24$ **c** $6x - 3y = 18$

d $y = x + 18$ **e** $y = 2x - 11$ **f** $3x - 7y = 20$

g $4x - 7y = 28$ **h** $7x - 2y = 11$ **i** $8x - 4y + 20 = 0$

Example 9e, f

9 **a** Give the equation of the line parallel to the y-axis and passing through the point $(1, 5)$.

b Give the equation of the line parallel to the x-axis and passing through the point $(-2, 5)$.

c Give the equation of the line parallel to the y-axis and passing through the point $(-4, -7)$.

Example 9e, f

10 Sketch the graph of:

a $x = 3$ **b** $y = 2x$ **c** $y = -2x$ **d** $y = 4$ **e** $y = -4x$ **f** $y = \frac{1}{4}x$

Example 10

11 The following points lie on the line with equation $3x - 12y = 30$. Find the value of each pronumeral.

a $(0, a)$ **b** $(b, 0)$ **c** $(1, c)$ **d** $(d, -6)$ **e** $(4, e)$ **f** $(f, 10)$

12 The following points lie on the line with equation $y = -\frac{1}{2}x - 4$. Find the value of each pronumeral.

a $(0, a)$ **b** $(b, 0)$ **c** $(1, c)$ **d** $(d, -6)$ **e** $(4, e)$ **f** $(f, 10)$

13 A line has equation $y = -4x + c$. The point $(6, 10)$ is on the line. Find the value of c.

14 A line has equation $2x - by + 7 = 0$. The point $(6, -5)$ is on the line. Find the value of b.

15 A line has equation $ax - 3y + 15 = 0$ and gradient 4. Find the value of a.

16 A line has equation $3x - by + 10 = 0$ and gradient $-\frac{1}{2}$. Find the value of b.

4D Point–gradient form of an equation of a line

Equation of a line given the gradient and a point on the line

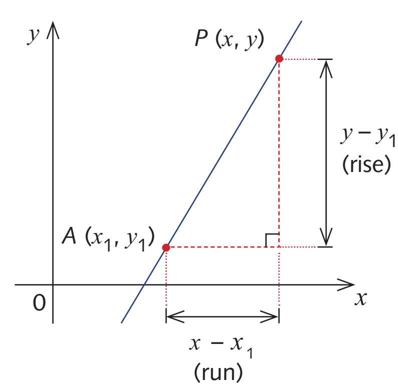
Suppose that we know the gradient m of a line and a point $A(x_1, y_1)$ on the line. Let $P(x, y)$ be any point on the line. Then:

$$m = \frac{y - y_1}{x - x_1}$$

and so:

$$y - y_1 = m(x - x_1)$$

This equation is called the **point–gradient** form of the line.



Example 11

Find the equation of the line that is parallel to the line with equation $y = -2x + 6$ and:

a passes through the point $A(1, 10)$

b passes through the point $B(-1, 0)$

Solution

The gradient of the line $y = -2x + 6$ is -2.

a Therefore the line through the point $A(1, 10)$ parallel to $y = -2x + 6$ has equation:

$$
\begin{aligned}
y - 10 &= -2(x - 1)\\
y - 10 &= -2x + 2\\
y &= -2x + 12 \text{ or } 2x + y - 12 = 0
\end{aligned}
$$

b The line through the point $B(-1, 0)$ parallel to $y = -2x + 6$ has equation:

$$
\begin{aligned}
y - 0 &= -2(x + 1)\\
y &= -2x - 2 \text{ or } 2x + y + 2 = 0
\end{aligned}
$$

Give the equation of the line in gradient–intercept form or general form – whichever you prefer.

Example 12

Find the equation of the line ℓ that is perpendicular to the line with equation $y = \frac{2}{3}x - 3$ and passes through the point $P(1, 6)$.

Solution

The gradient of $y = \frac{2}{3}x - 3$ is $\frac{2}{3}$.

$\frac{2}{3} \times \left(-\frac{3}{2}\right) = -1$, so the gradient of ℓ is $-\frac{3}{2}$

Since it passes through the point $(1, 6)$, the equation of the line is:

$$
\begin{aligned}
y - 6 &= -\frac{3}{2}(x - 1)\\
2y - 12 &= -3(x - 1)\\
2y - 12 &= -3x + 3\\
3x + 2y - 15 &= 0 \text{ or } y = -\frac{3}{2}x + \frac{15}{2}
\end{aligned}
$$

Equation of a line given two points

In Section 4B, we saw that the gradient m of a line passing through two points, $A(x_1, y_1)$ and $B(x_2, y_2)$, is given by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

We can now find the equation of a line, given the coordinates of two points on the line, as follows:

- Find the gradient of the line.
- Use the point–gradient form with either one of the points.

Example 13

Find the equation of the line passing through (2, 6) and (–3, 7).

Solution

$$\text{Gradient of line} = \frac{7-6}{-3-2} = -\frac{1}{5}$$

The point–gradient form with $m = -\dfrac{1}{5}$ and $(x_1, y_1) = (2, 6)$ gives:

$$y - 6 = -\frac{1}{5}(x - 2)$$

$5y - 30 = -(x - 2)$ (Multiply both sides by 5.)

$5y - 30 = -x + 2$

Hence, $x + 5y - 32 = 0$ is the general form of the line.

Check that both points lie on the line.

Note: the same equation can be established using $(x_1, y_1) = (-3, 7)$

The point–gradient form

- The equation of a line, **given the gradient m and one point $A(x_1, y_1)$** on the line, is: $y - y_1 = m(x - x_1)$
- To find the equation of a line, **given two points $A(x_1, y_1)$ and $B(x_2, y_2)$** use the point–gradient formula:

  $y - y_1 = m(x - x_1)$, where the gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Exercise 4D

Example 11

1 **a** Find the equation of the line with gradient 6 that passes through the point $(5, 6)$.

b Find the equation of the line with gradient -4 that passes through the point $(2, 6)$.

c Find the equation of the line with gradient $\frac{1}{2}$ that passes through the point $(-1, 8)$.

d Find the equation of the line parallel to the line $y = -3x + 8$ and passing through the point $(1, 8)$.

e Find the equation of the line with gradient 0 that passes through the point $(-3, 6)$.

f Find the equation of the line parallel to the line $x = -4$ and passing through the point $(-7, 11)$.

Example 12

2 **a** Find the equation of the line perpendicular to the line $y = -3x + 6$ and passing through the point $(-2, 8)$.

b Find the equation of the line perpendicular to the line $x + 2y = 6$ and passing through the point $(1, 2)$.

c Find the equation of the line perpendicular to the line $2x - y = 6$ and passing through the point $(6, -3)$.

3 **a** Find the midpoint of the interval AB, where the coordinates of A and B are $(2, -1)$ and $(3, 6)$, respectively.

b Find the gradient of the line that passes through points A and B.

c Find the equation of the line AB.

d Find the equation of the perpendicular bisector of the interval AB.

4 **a** Find the equation of the line with gradient -4 that passes through the point $(0, -6)$.

b Find the equation of the line with gradient -4 that passes through the point $(3, 8)$.

c Find the equation of the line that passes through the points $(-4, 8)$ and $(-6, -2)$.

d Find the equation of the line that is parallel to the line $y = -2x + 3$ and passes through the point with coordinates $(-1, -10)$.

Example 13

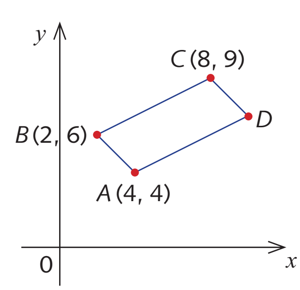
5 Find the equation of the line that passes through the two given points in each case.

a $(0, -4)$ and $(4, 0)$ **b** $(-3, 0)$ and $(0, -9)$ **c** $(2, 4)$ and $(-6, 12)$

d $(6, 3)$ and $(7, 3)$ **e** $(1, 4)$ and $(1, 8)$ **f** $(0, -3)$ and $(4, 6)$

6 Show that the points $A(1, 1)$, $B(3, 11)$ and $C(-2, -14)$ all lie on the same line (are collinear) and find the equation of this line. Do this by finding the equation of AB and checking that point C lies on the line.

7 $ABCD$ is a parallelogram with vertices $A(4, 4)$, $B(2, 6)$ and $C(8, 9)$. Find:



a the equation of the line BC **b** the equation of the line AB

c the equation of the line AD **d** the gradient of the line CD

e the distance AB **f** the distance CD

8 $A(1, 1)$, $B(1, 6)$, $C(6, 6)$ and $D(6, 1)$ are the vertices of a square $ABCD$.

a Find the midpoint of:

i AC **ii** BD

b Find the gradient of AC and BD, and hence show that AC is perpendicular to BD.

4E Review of simultaneous linear equations

In this section, we revise the standard methods for solving simultaneous linear equations. The solutions are the coordinates of the point of intersection of the two lines given by the linear equation.

Solving a pair of simultaneous equations means finding the values of x and y that satisfy both equations.

Example 14

Find the coordinates of the point of intersection of the lines $y = x - 1$ and $y = 2x - 3$ and sketch the lines on the one set of axes.

Solution

At the point (x, y) of intersection of the graphs, the y-coordinates of both graphs are the same.

Therefore, $x - 1 = 2x - 3$

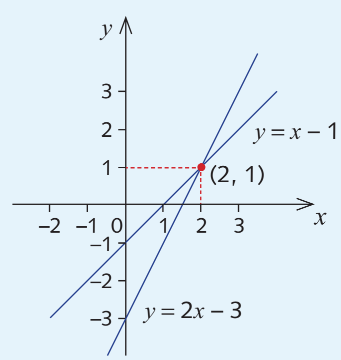
$$x + 2 = 2x$$

$$x = 2$$

Substituting into either equation gives $y = 1$.

The coordinates of the point of intersection are $(2, 1)$.

The solution of the simultaneous equations $y = x - 1$ and $y = 2x - 3$ is $x = 2$ and $y = 1$.



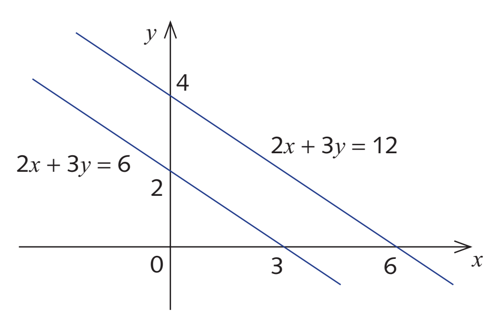
Note that the process outlined above can be done without the graph.

Lines that are parallel and lines that coincide

Simultaneous linear equations do not always have a unique solution. There are two geometric situations in which lines do not intersect at a single point.

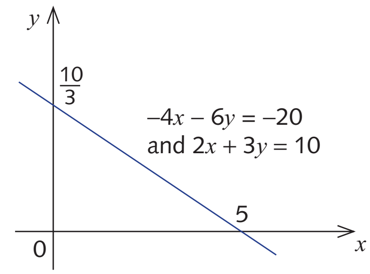
Parallel lines

The equations $2x + 3y = 12$ and $2x + 3y = 6$ represent parallel lines. There are no solutions to this pair of simultaneous equations since the lines do not meet.



Lines that coincide

Sometimes we have two equations that represent the same line. The equations $-4x - 6y = -20$ and $2x + 3y = 10$ represent the same line. This can be checked by showing that the lines have the same intercepts. We say that there are infinitely many solutions to this pair of equations since every point on the line satisfies both equations.



Solution by substitution

We recall that to solve a pair of simultaneous equations that involve pronumerals x and y, we can make either x or y the subject of one of the equations and substitute into the other equation. This method of solving a pair of simultaneous equations is called the **substitution method**. This method was used in Example 14.

Example 15

Solve this pair of equations for x and y.

$$x = 2y - 3 \qquad (1)$$
$$2x - 3y = 7 \qquad (2)$$

Solution

Substitute for x into equation (2), using equation (1):

$$\begin{aligned} 2(2y - 3) - 3y &= 7 \\ 4y - 6 - 3y &= 7 \\ y - 6 &= 7 \\ y &= 13 \end{aligned}$$

Using equation (1) gives:

$$\begin{aligned} x &= 2 \times 13 - 3 \\ &= 23 \end{aligned}$$

Thus the solution is $x = 23$, $y = 13$.

That is, the corresponding lines meet at $(23, 13)$.

Note: You should always check your answers by substituting into both of the original equations.

Solution by elimination

The other standard method for solving simultaneous equations is called the **elimination method**. This method relies on combining the two equations so that one of the pronumerals is eliminated; that is, we add or subtract multiples of the two equations to eliminate one pronumeral.

Example 16

Solve this pair of equations for x and y.

$$3x + y = 13 \quad (1)$$
$$x - y = 3 \quad (2)$$

Solution

Adding equations (1) and (2) gives:

$$4x = 16 \quad (3)$$
$$x = 4$$

Substituting into equation (1) gives:

$$12 + y = 13$$
$$y = 1$$

Therefore, the solution is $x = 4$, $y = 1$.

That is, the corresponding lines meet at (4, 1).

Example 17

Solve this pair of equations for x and y.

$$2x + 3y = 14 \quad (1)$$
$$2x - y = 6 \quad (2)$$

Solution

Subtracting the two equations will eliminate x and produce a single equation involving only y.

$$(1) - (2): \quad 4y = 8$$
$$y = 2$$

Substituting $y = 2$ into equation (1) gives:

$$2x + 6 = 14$$
$$2x = 8$$
$$x = 4$$

Hence, the solution is $x = 4$, $y = 2$.

That is, the corresponding lines meet at (4, 2).

Sometimes it is necessary to multiply both sides of an equation by a factor to enable a pronumeral to be eliminated. Remember that the new equation formed is equivalent to the first. This is shown in the following example.

Example 18

Solve this pair of equations for x and y.

$x - 3y = 2 \quad (1)$

$4x + y = 21 \quad (2)$

Solution

We make the coefficients of y the same by multiplying equation (2) by 3. Then we have:

$$x - 3y = 2 \quad (1)$$

$$(2) \times 3: \quad 12x + 3y = 63 \quad (3)$$

Now y can be eliminated by adding the two equations.

$$(1) + (3): \quad 13x = 65$$
$$x = 5$$

Substituting into equation (1) gives:

$$5 - 3y = 2$$
$$-3y = -3$$
$$y = 1$$

Hence, the solution is $x = 5$, $y = 1$ and the corresponding lines meet at $(5, 1)$.

In the next example, it is necessary to find two new equivalent equations in order to eliminate a pronumeral.

Example 19

Solve this pair of equations for x and y.

$3x + 5y = 1 \quad (1)$

$5x + 3y = 7 \quad (2)$

Solution

We choose to eliminate x, so we proceed as follows.

$$(1) \times 5: \quad 15x + 25y = 5 \quad (3)$$
$$(2) \times 3: \quad 15x + 9y = 21 \quad (4)$$
$$(3) - (4): \quad 16y = -16$$
$$y = -1$$

Substituting into equation (1) gives:

$$3x - 5 = 1$$
$$3x = 6$$
$$x = 2$$

The solution is $x = 2$, $y = -1$ and the corresponding lines meet at $(2, -1)$.

Check that $(2, -1)$ satisfies both equations (1) and (2).

Review of simultaneous linear equations

- A pair of simultaneous equations has either one, zero or infinitely many solutions. These cases occur, respectively, when the two lines meet at a point, are parallel or coincide.
- A pair of simultaneous equations can be solved using either the **substitution method** or the **elimination method**.
  - In the **substitution method**, make x or y the subject of one equation and substitute into the other equation.
  - In the **elimination method**, add or subtract suitable multiples of the two equations to eliminate one pronumeral.

Exercise 4E

Example 14

1 For each pair of equations, sketch the graphs and find the coordinates of the point of intersection.

a $y = 3x + 1$
$y = 2x + 2$

b $y = 3 - 2x$
$y = x - 3$

c $y = 2x + 1$
$y = 5x + 3$

Example 15

2 For each pair of equations, solve using the substitution method.

a $y = 3x$
$2x - 3y = 9$

b $x = 2y$
$3x + 2y = 6$

c $y = 2x + 1$
$x - 3y = 4$

d $x = 1 - 3y$
$4x - 3y = 12$

e $y = 1 - 2x$
$y = 5x + 2$

f $x = \dfrac{y}{3} + 2$
$7x - 5y = 10$

Example 16, 17, 18, 19

3 For each pair of equations, solve using the elimination method.

a $x - y = 3$
$2x + y = 9$

b $3x + y = 5$
$5x - y = 3$

c $x + y = 1$
$2x + y = 4$

d $2x + 3y = 4$
$5x + 3y = 1$

e $2x - 3y = 4$
$2x + y = 12$

f $3x + 2y = 5$
$3x + 5y = 26$

g $2x + y = 4$
$3x + 2y = 7$

h $4x - y = 5$
$3x + 4y = -1$

i $x + 2y = 2$
$3x + 5y = 3$

j $2x - 3y = 7$
$3x + 2y = 4$

k $5x + 4y = 20$
$2x + 5y = 10$

l $7x - 5y = 15$
$3x - 4y = 13$

m $2x - 3y = -9$
$3x - 2y = -1$

n $2x + 3y = 2$
$3x + 7y = -7$

o $2x + 5y = -35$
$3x - 2y = 8$

4 Solve each pair of equations.

a $y = 5x - 1$
$2x - 7y = 35$

b $7x - 9y = 63$
$5x + 8y = 40$

c $4x + 7y - 24 = 0$
$6x + 9y - 17 = 0$

d $x = 3 - 4y$
$7y - 3x = 21$

e $7x - 11y = 48$
$5x - 6y = 27$

f $\frac{1}{2}x - \frac{2}{3}y = 4$
$\frac{2}{3}x + \frac{3}{4}y = 7$

g $y = \frac{2}{5}x - 8$
$y = -\frac{2}{7}x + 3$

h $y = 3x - 2$
$2x + 3y = 4$

i $x + 7y = 0$
$3x - 4y = 24$

j $3x - 7y - 42 = 0$
$2x - 3y - 18 = 0$

k $x = 3y - 5$
$2x - 3y = 21$

l $y = \frac{1}{4}x + 7$
$y = -\frac{3}{5}x - 4$

5 The line $y = 2x$ intersects the line $y = x + 6$ at the point A. Find the equation of the line that passes through A and has gradient 3.

6 The line $y = 2x - 4$ intersects the line $y = -3x + 6$ at the point B. Find the equation of the line that passes through B and is:

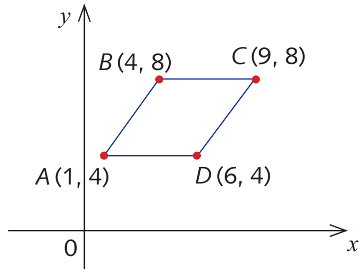
a parallel to the x-axis

b parallel to the y-axis

7 The line $y = x$ intersects the line $y = 2x + 1$ at the point A, and intersects the line $y = -3x + 12$ at the point B. The line $y = 2x + 1$ meets the line $y = -3x + 12$ at the point C. Find the coordinates of the vertices of triangle ABC.

8 The line that passes through the points $A(0, 2)$ and $B(1, 4)$ meets the line that passes through the points $C(1, 8)$ and $D(-1, 10)$ at the point E. Find the equations of the lines AB and CD and hence find the coordinates of E.

9 $ABCD$ is a rhombus.



a Find the equation of:

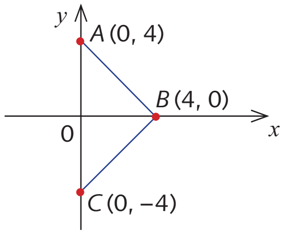
i AC

ii BD

b Use the results of part **a** to find the coordinates of the point of intersection of AC and BD.

c Show that the point of intersection is the midpoint of both AC and BD and that AC is perpendicular to BD.

10 $A(0, 4)$, $B(4, 0)$ and $C(0, -4)$ are the vertices of triangle ABC.



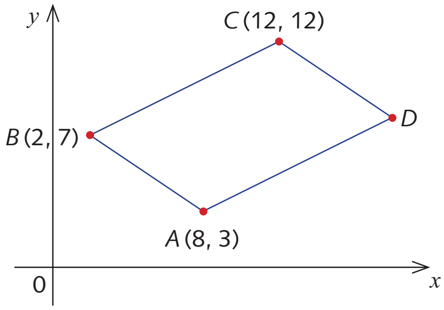
a Find the equation of the perpendicular bisector of interval:

i AB **ii** BC

b Find the coordinates of the intersection of the two perpendicular bisectors found in part **a**.

11 The line with equation $y = mx + 3$ intersects the line with equation $3x + 4y + 12 = 0$ at the point $\left(1, -\dfrac{15}{4}\right)$ Find the value of m.

12 The diagram opposite shows a parallelogram $ABCD$ in which A is the point with coordinates $(8, 3)$, B is the point with coordinates $(2, 7)$ and C is the point with coordinates $(12, 12)$. X is a point on BC such that AX is perpendicular to BC. Find:



a the equation of the line AD

b the equation of the line AX

c the coordinates of X

d the distance AX

e the distance BC

f the area of the parallelogram

13 Find a and b if $ax - 10y = 8$ and $6x + by = 12$ represent the same line.

14 Show that the straight lines $2x - 3y = 7$, $3x - 4y = 13$ and $8x - 11y = 33$ meet at a point.

15 Find the equation of the straight line that passes through the origin and the point of intersection of the lines:

a $x - y - 4 = 0$ and $7x + y + 20 = 0$ **b** $\dfrac{x}{a} + \dfrac{y}{b} = 1$ and $\dfrac{x}{b} + \dfrac{y}{a} = 1$

16 The line $ax - by + 3 = 0$ is parallel to the line $3x + 2y - 4 = 0$ and passes through the point $(1, -2)$. Find a and b.

17 Given three points, $A(0, 5)$, $B(8, 7)$ and $C(4, 1)$, calculate the coordinates of the point of intersection of the perpendicular bisectors of the lines AB and BC.

18 In the quadrilateral $ABCD$, the points A, B and D are at $(3, 3)$, $(0, 1)$ and $(6, 2)$, respectively. The line BD bisects the line AC at right-angles at the point M.

a Find the equation of BD and of AC.

b Calculate the coordinates of M.

c Calculate the length AM.

d Find the area of quadrilateral $ABCD$.

4F Solving word problems using simultaneous equations

In this section, we look at how simultaneous equations can be used to solve problems expressed in words.

When solving a problem expressed in words:

- introduce pronumerals
- translate all the relevant facts into equations
- solve the equations and check your solutions
- write a conclusion in words.

Example 20

The attendance at an evening performance of a local theatre production was 420 people and the box office receipts were $3840. Admission costs were $13 for each adult and $4 for each child. How many of each type of ticket were sold?

Solution

Let c be the number of child tickets sold, and let a be the number of adult tickets sold.

$$\begin{aligned} c + a &= 420 & (1)\\ 4c + 13a &= 3840 & (2)\\ (1)\times 4:\quad 4c + 4a &= 1680 & (3)\\ (2)-(3):\quad 9a &= 2160 &\\ a &= 240 & \end{aligned}$$

Substituting in equation (1) gives:

$$\begin{aligned} c + 240 &= 420\\ c &= 180 \end{aligned}$$

Hence, 240 adult tickets and 180 child tickets were sold.

The above question could also be solved by using one variable. For example, if we let x be the number of children, then the number of adults is $420 - x$.

Exercise 4F

Solve each of these problems by introducing two pronumerals and forming a pair of simultaneous equations.

1 The sum of two numbers is 112 and their difference is 22. Find the two numbers.

2 In a game of netball, the winning team won by 9 goals. In total, 83 goals were scored in the game. How many goals did each team score?

3 A father is 28 years older than his daughter. In six years' time, he will be three times her age. Find their present ages.

Example 20

4 A stallholder at a local market sells articles at either \$2 or \$5 each. On a particular market day, he sold 101 articles and took \$331 in revenue. How many articles were sold at each price?

5 Four times Brian's age exceeds Andrew's age by 20 years and one-third of Andrew's age is less than Brian's age by two years. Find their ages.

6 A ball of string of length 150 m is cut into 8 pieces of one length and 5 pieces of another length. The total length of three of the first 8 pieces exceeds that of two of the second 5 pieces by 2 m. Find the length of the pieces.

7 A manufacturer of lawn fertiliser produces bags of fertiliser in two sizes, standard and jumbo. To transport bags to retail outlets, he uses a van with a carrying capacity of one tonne. He discovers that he can transport either 110 standard bags and 60 jumbo bags or 50 standard bags and 100 jumbo bags at any one time. Find the weight of each type of bag.

8 Ten thousand tickets were sold for a concert. Some tickets sold for \$80 each and the remainder sold for \$60 each. If the total receipts were \$640 000, how many tickets of each price were sold?

9 The cooling system of Ennio's car contains 7.5 L of coolant, which is $33\frac{1}{3}\%$ antifreeze. How much of this solution must be drained from the system and replaced with 100% antifreeze so that the solution in the cooling system will contain 50% antifreeze?

10 A motorist travelled a total distance of 432 km and had an average speed of 80 km/h on highways and an average speed of 32 km/h while passing through towns. If the journey took 6 hours, find how long the motorist spent travelling on highways.

11 A car leaves Melbourne at 8 a.m., travelling at a constant speed of 80 km/h. It is followed at 10 a.m. by another car travelling on the same road at a constant speed of 110 km/h. At what time will the second car overtake the first?

12 One alloy of iron contains 52% iron and another contains 36% iron. How many tonnes of each alloy should be used to make 200 tonnes of 40% iron alloy?

13 Two aeroplanes pass each other in flight while travelling in opposite directions. Each aeroplane continues on its flight for 45 minutes, after which time the aeroplanes are 840 km apart. The speed of the first aeroplane is $\frac{3}{4}$ of the speed of the other aeroplane. Calculate the average speed of each aeroplane.

14 Six model horses and 7 model cows can be bought for \$250. Thirteen model cows and 11 model horses can be bought for \$460. What is the cost of each model animal?

15 If 1 is added to the numerator of a fraction $\frac{a}{b}$, it simplifies to $\frac{1}{5}$. If 1 is subtracted from the denominator, it simplifies to $\frac{1}{7}$. Find the fraction $\frac{a}{b}$.

16 A hiker walks a certain distance. If he had gone 1 km/h faster, he would have walked the distance in $\frac{4}{5}$ of the time. If he had walked 1 km/h slower, he would have taken $2\frac{1}{2}$ hours longer to travel the distance. Find the distance.

Review exercise

1 Find the distance between the points A and B.

a $A(1, 6)$, $B(3, -2)$ **b** $A(1, 5)$, $B(7, 5)$ **c** $A(-1, 6)$, $B(-1, -6)$

d $A(-2, -8)$, $B(-1, -3)$ **e** $A(2, 7)$, $B(-3, 10)$ **f** $A(-1, 6)$, $B(7, 10)$

2 Find the midpoint of the interval AB.

a $A(1, 6)$, $B(2, -4)$ **b** $A(2, 3)$, $B(-4, 6)$ **c** $A(1, -10)$, $B(-2, 10)$

d $A(-1, -3)$, $B(10, 13)$ **e** $A(-2, 6)$, $B(-1, 7)$ **f** $A(3, -4)$, $B(6, -2)$

3 Find the gradient of the line that passes through each pair of points.

a $(1, 2)$ and $(5, 18)$ **b** $(2, 3)$ and $(4, 9)$ **c** $(-2, 1)$ and $(1, 10)$

d $(1, -2)$ and $(3, 0)$ **e** $(-3, -4)$ and $(0, -2)$ **f** $(-1, -2)$ and $(1, -7)$

g $(0, -6)$ and $(-2, 0)$ **h** $(3, 5)$ and $(7, 5)$ **i** $(6, -3)$ and $(2, -3)$

4 The line passing through the points $(-1, 6)$ and $(4, b)$ has gradient -2. Find the value of b.

5 Write down the gradient and y-intercept for each equation.

a $y = 2x + 4$ **b** $y = x - 4$ **c** $y = \frac{1}{2}x + 1$

d $y = -2x + 5$ **e** $y = -x + 6$ **f** $y = 2 - x$

g $y = 4 - 3x$ **h** $3x + y = 4$ **i** $2x - 3y = 6$

j $3x + 4y = 12$ **k** $y = 2x$ **l** $y = 3x$

m $y = -4x$ **n** $-3x + 2y = 0$ **o** $2y = -3x + 6$

6 Sketch the graph of each equation by finding the x- and y-intercepts.

a $x + y = 4$ **b** $2x + y = 2$ **c** $3x + 4y = 12$

d $2x - y = 4$ **e** $3x - 2y = 6$ **f** $\frac{3}{2}x - y = 12$

7 Sketch the graph of each equation.

a $y = 2x - 3$ **b** $y = 3x - 2$ **c** $2x - y = 1$

d $2x - 5y = 10$ **e** $x = 2y + 1$ **f** $x = 3y - 2$

g $y = 4 - x$ **h** $y = 1 - 3x$ **i** $x = 2$

j $x = -1$ **k** $y - 3 = 0$ **l** $y = -2$

m $y = 2(x + 1)$ **n** $x = \frac{y + 1}{3}$ **o** $\frac{x}{2} + \frac{y}{3} = 1$

p $\frac{x}{4} - y = 1$ **q** $\frac{2x}{3} - \frac{3y}{2} = 1$ **r** $\frac{1}{2}x - 2y = 3$

8 **a** Find the equation of the line with gradient -6 that passes through the point $(1, 5)$.

b Find the equation of the line that is perpendicular to the line with equation $3x + 2y = 8$, and that passes through the point $(-1, 4)$.

9 Find the equation of the line that passes through the points:

a (5, 6) and (–4, 10) **b** (3, 4) and (–2, 8) **c** (–2, 6) and (1, 10)

10 Solve each pair of simultaneous equations.

a $5x + 3y = 15$
$x - y = 6$

b $\frac{x}{3} + \frac{y}{5} = 1$
$3x + 5y = 15$

11 Solve each pair of simultaneous equations.

a $y = 3x + 2$
$y = x - 4$

b $y = 2 - 3x$
$y = 10 + x$

c $y = 5 - x$
$y = 10 - 2x$

d $3x - y = 2$
$y + 3x = 4$

e $\frac{x}{3} = 6 - \frac{y}{3}$
$\frac{3x}{4} - 3 = 2y$

f $2y - \frac{5x}{3} = -4$
$5x + \frac{y}{2} = \frac{21}{2}$

12 The vertices of ΔABC are $A(3, 4)$, $B(8, 10)$, $C(5, -1)$.

a Find the equation of the perpendicular bisector of:

i AB **ii** BC

b Find the coordinates of the point of intersection of the two perpendicular bisectors.

13 The equation of the perpendicular bisector of AB is $3y = 2x - 1$. The coordinates of A are (1, 4). Find the coordinates of B.

14 Show that the points (2, 0), (5, 3), (3, 6) and (0, 3) are the vertices of a parallelogram. Find the equation of each of its sides.

15 Show that the points (1, 4), (–4, –1) and (2, 3) are the vertices of a right-angled triangle.

16 **a** Prove that the points (3, –2), (7, 6), (–1, 2) and (–5, –6) are the vertices of a rhombus.

b Find the length of each of the diagonals of the rhombus.

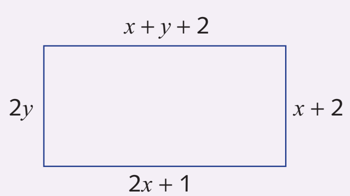
17 Find the two numbers whose sum is 138 and whose difference is 88.

18 Six stools and four chairs cost \$580 but five stools and two chairs cost \$350. Find the cost of each chair and each stool.

19 Three points have coordinates $A(1, 2)$, $B(3, 10)$ and $C(p, 8)$. Find the values of p if:

a A, B and C are collinear **b** AC is perpendicular to AB

20 Find the perimeter of the rectangle shown below.



21 Prove that the lines $2y - x = 2$, $y + x = 7$ and $y = 2x - 5$ are concurrent. (That is, they intersect at only one point.)

Challenge exercise

1 Tom begins in Mildura and travels a distance of 300 km to Broken Hill at a constant speed of 80 km/h. Steve, beginning at the same time, travels at a constant speed of 100 km/h from Mildura to Broken Hill with a 30-minute rest after travelling 150 km.

a Let d be the distance (in km) from Mildura, and t the time (in hours) after Tom and Steve leave Mildura. On a single set of axes, draw graphs to illustrate the journeys of Tom and Steve (d against t).

b From the graphs, find:

i when and where Tom overtakes Steve

ii when and where Steve overtakes Tom

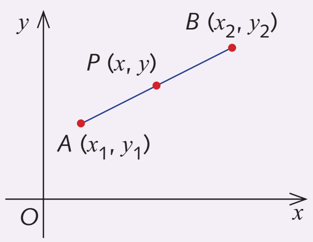
iii the distance Tom still has to travel to Broken Hill at the time Steve arrives at Broken Hill

2 For the interval AB, the coordinates of A and B are (x_1, y_1) and (x_2, y_2), respectively.

a If M is a point on AB such that $AM : MB = 3 : 1$, find the coordinates of M.

b If N is a point on AB such that $AN : NB = 3 : 2$, find the coordinates of N.

3 The point P divides the interval AB in the ratio $m : n$. That is $AP : PB = m : n$. Find the coordinates of P.

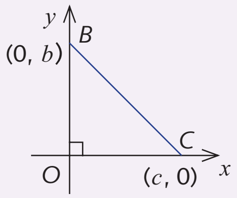


4 $O(0, 0)$, $B(0, b)$ and $C(c, 0)$ are the vertices of a right-angled triangle, with the right angle at O.

a Find the coordinates of the midpoint M of BC.

b Find the distances:

i OM **ii** MB **iii** MC



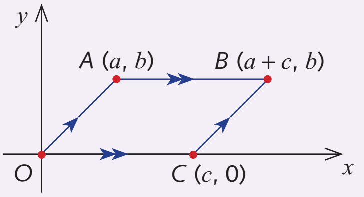
5 $OABC$ is a parallelogram.

a Find the equations of:

i OB **ii** AC

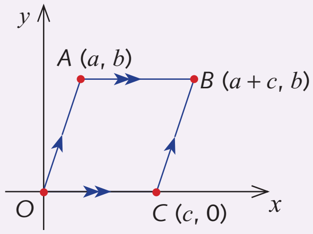
b Find the coordinates of the midpoints of:

i OB **ii** AC



Note that the diagonals of the parallelogram bisect each other.

6 $OABC$ is a rhombus, with vertices $O(0, 0)$, $A(a, b)$, $B(a + c, b)$ and $C(c, 0)$.



a Find the gradients of the lines:

i OB **ii** AC

b Show that (gradient of OB) × (gradient of AC) $= \dfrac{b^2}{a^2 - c^2}$.

c Find the length of OA.

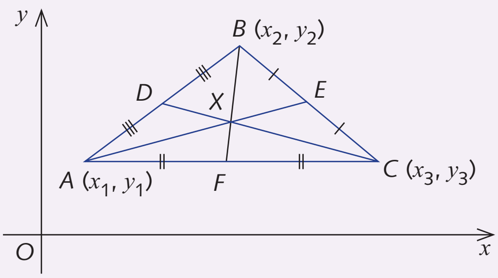
d Use the fact that $OA = OC$ to show that $c^2 = a^2 + b^2$, and hence that OB is perpendicular to AC.

7 AE, BF and CD are the medians of ΔABC. They are concurrent at the point X. We also have:

$BX = 2XF$

$CX = 2XD$

$AX = 2XE$

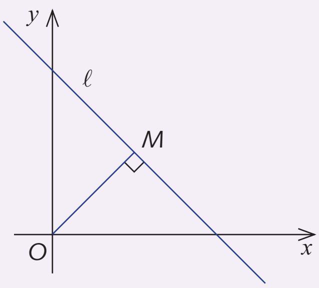


Show that X has coordinates:

$$\left(\frac{1}{3}(x_1 + x_2 + x_3), \frac{1}{3}(y_1 + y_2 + y_3)\right)$$

8 The line ℓ has equation $ax + by + c = 0$.

Show that $OM = \dfrac{c}{\sqrt{a^2 + b^2}}$.



9 $A(2, 6)$, $B(8, 11)$ and $C(4, 4)$ are the vertices of ΔABC.

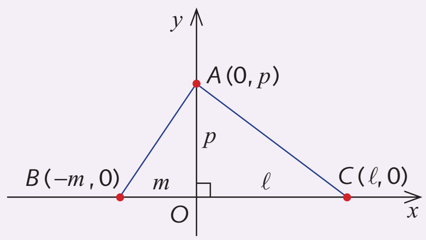
Line BC intersects the x-axis at P.

Line CA intersects the x-axis at Q.

Line AB intersects the x-axis at R.

Show that $\dfrac{BP}{PC} \times \dfrac{CQ}{QA} \times \dfrac{AR}{RB} = 1$.

10 We shall prove that the altitudes of a triangle are concurrent.



For triangle ABC, we choose a set of axes with the origin O on BC so that BOA is a right-angle.

Let $OA = p$, $OB = m$ and $OC = \ell$, so that the coordinates of A, B and C are $(0, p)$, $(-m, 0)$ and $(\ell, 0)$.

a Find the gradient of lines AB and CA.

b Find the equation of the line that is perpendicular to AB and passes through C (the altitude from C to AB).

c Find the equation of the line that is perpendicular to AC and passes through B (the altitude from B to AC).

d Show that the three altitudes of the triangle intersect at $\left(0, \dfrac{m\ell}{p}\right)$. That is, the altitudes are concurrent.

11 **a** Show that the area of a triangle ABC with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is $\pm\frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3)$.

b Show that the area of a quadrilateral whose vertices taken in order are $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$ is

$\pm\frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1 - x_2y_1 - x_3y_2 - x_4y_3 - x_1y_4)$,

where the sign is chosen to provide a positive answer.

12 Two lines have equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.

a Show that the lines are parallel if $a_1b_2 = a_2b_1$.

b Show that the lines are perpendicular if $a_1a_2 + b_1b_2 = 0$.

13 **a** Show that the line passing through the point (x_1, y_1) and parallel to the line $ax + by + c = 0$ is $ax + by = ax_1 + by_1$.

b Show that the line passing through the point (x_1, y_1) and perpendicular to the line $ax + by + c = 0$ is $bx - ay = bx_1 - ay_1$.

14 Show that the three lines:

$a_1x + b_1y + c_1 = 0$

$a_2x + b_2y + c_2 = 0$

$a_3x + b_3y + c_3 = 0$

are concurrent if $a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) = 0$.
The converse is always true.

CHAPTER

5

Number and Algebra

Quadratic equations

Quadratic equations turn up frequently in mathematics, and being able to solve them is a fundamental skill.

The ancient Babylonians were solving quadratic equations more than 5000 years ago!

In this chapter, we will revise and extend the basic methods of solving equations based on factorising, and then explore how to solve quadratic equations when the factorising method does not work. Techniques include completing the square and determining a general formula for the solution of quadratic equations.

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5A Solution of quadratic equations

Equations that can be written in the form $ax^2 + bx + c = 0$, where $a \neq 0$, are **quadratic equations**.

In *ICE-EM Mathematics Year 9*, we learned some of the methods of solving quadratic equations.

The method you learned used the following idea. If the product of two numbers is zero, then at least one of the numbers is zero.

In symbols, if $mn = 0$, then either $m = 0$ or $n = 0$ (or both).

To solve a quadratic equation $ax^2 + bx + c = 0$, you should first attempt to factorise the quadratic expression on the left to express it as a product of two factors and then use the above idea.

Example 1

Solve each equation.

a $x^2 - 6x = 0$ **b** $x^2 - 5x + 6 = 0$

Solution

a $x^2 - 6x = 0$

$x(x - 6) = 0$

Hence, $x = 0$ or $x - 6 = 0$

so $x = 0$ or $x = 6$

b $x^2 - 5x + 6 = 0$ (Look for two numbers with a product that is 6 and that sum to -5.)

$(x - 2)(x - 3) = 0$

Hence, $x - 2 = 0$ or $x - 3 = 0$

So $x = 2$ or $x = 3$

We can check by substitution that the two numbers obtained are solutions to the original equation. For example, in Example **1b**:

If $x = 2$:

$$\begin{aligned} \text{LHS} &= x^2 - 5x + 6 \\ &= 2^2 - 5 \times 2 + 6 \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

If $x = 3$:

$$\begin{aligned} \text{LHS} &= x^2 - 5x + 6 \\ &= 3^2 - 5 \times 3 + 6 \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

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Example 2

Solve:

a $x^2 - 16 = 0$ **b** $x^2 + 16 = 0$

Solution

a There are two ways we could do this. The simpler way is to write:

$$x^2 = 16$$

$$x = 4 \quad \text{or} \quad x = -4$$

[Alternatively, we could factorise using the difference of two squares identity.

$$\begin{aligned} x^2 - 16 &= 0 \\ (x-4)(x+4) &= 0 \\ x - 4 = 0 \quad &\text{or} \quad x + 4 = 0 \\ x = 4 \quad &\text{or} \quad x = -4 \end{aligned}$$
]

b We write the equation as $x^2 = -16$. There is no solution, since the square of any real number is positive or zero. This equation has no solution.

Note: $x^2 + 16$ cannot be factorised.

Always remember to move all of the terms onto the left-hand side of the equation when you want to solve a quadratic equation by factorising.

Example 3

Solve:

a $x^2 = 17x$ **b** $x^2 = 7x - 6$

Solution

a

$$\begin{aligned} x^2 &= 17x \\ x^2 - 17x &= 0 \\ x(x - 17) &= 0 \\ x = 0 \quad &\text{or} \quad x - 17 = 0 \\ x = 0 \quad &\text{or} \quad x = 17 \end{aligned}$$

b

$$\begin{aligned} x^2 &= 7x - 6 \\ x^2 - 7x + 6 &= 0 \\ (x - 6)(x - 1) &= 0 \\ x - 6 = 0 \quad &\text{or} \quad x - 1 = 0 \\ x = 6 \quad &\text{or} \quad x = 1 \end{aligned}$$

Note: A very common mistake is to 'cancel out the x' in the first line of part **a** above and obtain $x = 17$. You should never do this – you must *always* factorise. Otherwise, you will lose the solution $x = 0$.

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Quadratic equations

- If $mn = 0$, then $m = 0$ or $n = 0$ (or both).
- To solve a quadratic equation using the factorising method, move all terms to the left-hand side, factorise and use the result stated above.
- If a pronumeral is a common factor, never divide by it – instead, always factorise.

Factorising general quadratic expressions

We will review a method of factorising general quadratic expressions when the coefficients do not have a common factor. There are a number of such methods, but we will only give one method here.

To factorise, for example, $3x^2 + 11x + 6$, we go through the following steps.

First, we multiply the coefficient of x^2 by the constant term.

$$3 \times 6 = 18$$

Next, we find two numbers with product 18 and sum 11, the coefficient of x. The numbers are 9 and 2. Using these numbers:

$$\begin{aligned} 3x^2 + 11x + 6 &= 3x^2 + 9x + 2x + 6 && \text{(Split the } 11x \text{ term into } 9x + 2x.) \\ &= 3x(x + 3) + 2(x + 3) && \text{(Factorise in pairs.)} \\ &= (x + 3)(3x + 2) && \text{(Take out the common factor, } (x + 3).) \end{aligned}$$

It doesn't matter if we split $11x$ as $2x + 9x$ instead.

Thus, $3x^2 + 11x + 6 = (x + 3)(3x + 2)$.

This is the method presented in Section 3G of this book.

Example 4

Solve:

a $3x^2 + 11x + 6 = 0$ **b** $6x^2 + 7x + 2 = 0$

Solution

a

$$\begin{aligned} 3x^2 + 11x + 6 &= 0 \\ (x + 3)(3x + 2) &= 0 && \text{(Using the factorisation shown above.)} \\ x + 3 = 0 \quad &\text{or} \quad 3x + 2 = 0 \\ x = -3 \quad &\text{or} \quad x = -\frac{2}{3} \end{aligned}$$

b

$$\begin{aligned} 6x^2 + 7x + 2 &= 0 && \text{(Find two numbers that multiply to give } 6 \times 2 = 12 \\ & && \text{and add to give 7. They are 4 and 3.)} \\ 6x^2 + 4x + 3x + 2 &= 0 && \text{(Split the middle term.)} \\ 2x(3x + 2) + 1(3x + 2) &= 0 && \text{(Factorise in pairs.)} \\ (3x + 2)(2x + 1) &= 0 && \text{(Take out the common factor.)} \\ 3x + 2 = 0 \quad &\text{or} \quad 2x + 1 = 0 \\ x = -\frac{2}{3} \quad &\text{or} \quad x = -\frac{1}{2} \end{aligned}$$

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Note: It does not matter in which order we split the middle term. In the example on the previous page, we could write $4x + 3x$ or $3x + 4x$, and factorise in pairs. Try it for yourself!

Common factor

If there is a factor common to all of the coefficients in the equation, we can divide both sides by this common factor.

Example 5

Solve:

a $10x^2 - 40x - 210 = 0$ **b** $24x^2 = 46x - 10$

Solution

a
$$\begin{aligned} 10x^2 - 40x - 210 &= 0 \\ x^2 - 4x - 21 &= 0 \quad \text{(Divide both sides of the equation by 10.)} \\ (x-7)(x+3) &= 0 \\ x - 7 = 0 \quad &\text{or} \quad x + 3 = 0 \\ x = 7 \quad &\text{or} \quad x = -3 \end{aligned}$$

b
$$\begin{aligned} 24x^2 &= 46x - 10 \quad \text{(Divide both sides of the equation by 2} \\ 12x^2 - 23x + 5 &= 0 \quad \text{and rearrange.)} \\ 12x^2 - 20x - 3x + 5 &= 0 \quad (12 \times 5 = 60\text{. Find two numbers with a} \\ 4x(3x-5) - 1(3x-5) &= 0 \quad \text{product that is 60 and sum that is } -23. \\ (3x-5)(4x-1) &= 0 \quad \text{The numbers are } -20 \text{ and } -3.) \\ 3x - 5 = 0 \quad &\text{or} \quad 4x - 1 = 0 \\ x = \frac{5}{3} \quad &\text{or} \quad x = \frac{1}{4} \end{aligned}$$

Quadratic equations of the form $ax^2 + bx + c = 0$, when $a \neq 0$

- If the coefficients have a common factor, divide through by that factor.
- To factorise a quadratic expression such as $ax^2 + bx + c$, find two numbers, α and β, whose product is ac and whose sum is b. Write the middle term as $\alpha x + \beta x$ and then factorise.
- To solve a quadratic equation using the factorising method, write the equation in the form $ax^2 + bx + c = 0$, then factorise the quadratic and write down the solutions.

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Exercise 5A

1 Solve each equation.

a $x(x+3)=0$ **b** $x(x-7)=0$ **c** $3x(x+5)=0$

d $(x-3)(x+6)=0$ **e** $(x+7)(x+9)=0$ **f** $(x-10)(x-7)=0$

g $4x(5x+4)=0$ **h** $(4x+3)(3x-2)=0$ **i** $(2x+7)(x+4)=0$

j $(2x-3)(3x+4)=0$ **k** $3(2x-5)(x+4)=0$ **l** $7(2-3x)(4-3x)=0$

Example 1a

2 Solve each quadratic equation by factorising.

a $x^2-5x=0$ **b** $x^2+7x=0$ **c** $x^2+8x=0$

d $x^2-25x=0$ **e** $x^2=-18x$ **f** $x^2=\frac{1}{2}x$

Example 1b

3 Solve each quadratic equation by factorising.

a $x^2+9x+8=0$ **b** $x^2+8x+12=0$ **c** $x^2+12x+27=0$

d $x^2+12x+36=0$ **e** $x^2-6x+8=0$ **f** $x^2+x-6=0$

g $x^2+x-30=0$ **h** $x^2+3x-40=0$ **i** $x^2+4x-60=0$

j $x^2-7x+6=0$ **k** $x^2-7x+12=0$ **l** $x^2-10x+25=0$

m $x^2-18x+32=0$ **n** $x^2-4x-21=0$ **o** $x^2-20x+100=0$

Example 2, 3

4 Solve, if possible:

a $x^2=8x$ **b** $x^2=17x-16$ **c** $3x-x^2-2=0$

d $x^2+4=0$ **e** $15=8x-x^2$ **f** $-100-x^2=0$

g $x^2=-3x$ **h** $h^2=20-h$ **i** $x^2+9=0$

j $9a-10=-a^2$ **k** $8y=y^2+7$ **l** $a^2-1=0$

Example 5a

5 Solve each equation by first dividing both sides by a common factor.

a $2x^2+6x+4=0$ **b** $3a^2-15a+18=0$ **c** $4x^2+8x-140=0$

Example 4

6 Solve:

a $2x^2+11x+12=0$ **b** $3x^2+13x+4=0$ **c** $2x^2+7x+6=0$

d $2x^2-3x-2=0$ **e** $2x^2-9x+9=0$ **f** $3x^2-10x+8=0$

g $10x^2+23x+12=0$ **h** $6x^2-17x+12=0$ **i** $8x^2=6x+5$

j $12x^2=x+6$ **k** $12x^2=5x+2$ **l** $6x^2+11x=10$

m $3x^2=19x+14$ **n** $5x^2+17x+6=0$ **o** $12x^2-31x-15=0$

p $15x^2+224x=15$ **q** $72x^2-145x+72=0$ **r** $6+5x-6x^2=0$

Example 5b

7 Solve each equation, remembering first to divide both sides by any common factor.

a $12x^2-22x+8=0$ **b** $72x^2-78x-15=0$ **c** $12x^2-21x+9=0$

d $10x^2+5x-30=0$ **e** $72x^2-228x+120=0$ **f** $90x^2=75x+60$

g $100x^2-290x+100=0$ **h** $160x^2+136x+24=0$ **i** $10x^2-25x+10=0$

j $28x^2-49x-105=0$ **k** $42m^2-2m-4=0$ **l** $8x^2+46x-70=0$

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5B Rearranging to standard form

In many mathematical problems and applications, equations arise that do not initially appear to be quadratic equations. We often need to rearrange these equations to the standard form for a quadratic equation.

Some equations involve fractions in which the pronumeral may appear in the denominator. You will need to take care when solving these. We always assume that the pronumeral cannot take a value that makes the denominator equal to zero. It is a wise idea to check that your answers are the correct solutions to the initially given equation.

Example 6

Solve:

a $1 + \dfrac{5}{x} = \dfrac{6}{x^2}$ **b** $x = \dfrac{5x - 4}{x}$

Solution

a

$$\begin{aligned} 1 + \frac{5}{x} &= \frac{6}{x^2} \\ x^2 + 5x &= 6 \qquad \text{(Multiply both sides of the equation by } x^2\text{.)} \\ x^2 + 5x - 6 &= 0 \\ (x + 6)(x - 1) &= 0 \\ x &= -6 \quad \text{or} \quad x = 1 \end{aligned}$$

We can check that these are the correct solutions by substitution.

If $x = -6$:

$\text{LHS} = 1 - \dfrac{5}{6}$ and $\text{RHS} = \dfrac{6}{36}$

$= \dfrac{1}{6}$ $= \dfrac{1}{6}$

so LHS = RHS

If $x = 1$:

$\text{LHS} = 1 + \dfrac{5}{1}$ and $\text{RHS} = \dfrac{6}{1}$

$= 6$ $= 6$

so LHS = RHS

b

$$\begin{aligned} x &= \frac{5x - 4}{x} \\ x^2 &= 5x - 4 \qquad \text{(Multiply both sides by } x\text{.)} \\ x^2 - 5x + 4 &= 0 \qquad \text{(Rearrange.)} \\ (x - 1)(x - 4) &= 0 \\ x - 1 &= 0 \quad \text{or} \quad x - 4 = 0 \\ x &= 1 \quad \text{or} \quad x = 4 \end{aligned}$$

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Another standard technique in algebra that is useful in the solution of equations is **cross-multiplication**. This is using the result:

$$\frac{a}{b} = \frac{c}{d} \text{ is equivalent to } ad = bc$$

Example 7

Solve $\frac{x-2}{3} = \frac{5}{x}$.

Solution

$$\begin{aligned}
\frac{x-2}{3} &= \frac{5}{x} \\
x(x-2) &= 3 \times 5 && \text{(Multiply by } 3x.\text{)} \\
x^2 - 2x &= 15 \\
x^2 - 2x - 15 &= 0 && \text{(Rearrange.)} \\
(x+3)(x-5) &= 0 \\
x + 3 = 0 \quad &\text{or} \quad x - 5 = 0 \\
x = -3 \quad &\text{or} \quad x = 5
\end{aligned}$$

Example 8

Solve $\frac{x+1}{x-1} - \frac{3}{x+2} = 1$.

Solution

$$\begin{aligned}
\frac{x+1}{x-1} - \frac{3}{x+2} &= 1 \\
(x-1)(x+2)\left[\frac{x+1}{x-1} - \frac{3}{x+2}\right] &= (x-1)(x+2) && \text{(Multiply by } (x-1)(x+2).\text{)} \\
(x+1)(x+2) - 3(x-1) &= (x-1)(x+2) \\
x^2 + 3x + 2 - 3x + 3 &= x^2 + x - 2 \\
-x + 5 &= -2 \\
-x &= -7 \\
x &= 7
\end{aligned}$$

Check solution:

When $x = 7$, LHS $= \frac{8}{6} - \frac{3}{9} = \frac{4}{3} - \frac{1}{3} = 1$ and RHS $= 1$.

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Exercise 5B

1 Solve:

a $x(x-7) = 18$

b $x^2 = 4(x+8)$

c $x^2 = \frac{1}{2}(5x+12)$

d $5(x^2+5) = 6x^2$

e $3x^2 = 4(x^2+4)$

f $(x+1)(x-1) = 2(x+1)^2$

g $x(x-3) = 2x(x+1)$

h $(x-4)(x-2) = 3$

i $(9+x)(9-x) = 17$

j $(2x-1)(3x+1) = 11$

k $5x(2x-3) + 7(2x-3) = 0$

l $3x - 8 = \frac{x^2}{4}$

Example 6, 7, 8

2 Solve:

a $x + 5 = \frac{14}{x}$

b $\frac{15}{x} = x - 2$

c $\frac{6}{x} - x = 1$

d $x - 1 = \frac{2}{x}$

e $x + \frac{6}{x} = 7$

f $x + \frac{32}{x} = 18$

g $\frac{x+1}{3} = \frac{10}{x}$

h $\frac{x+1}{4} = \frac{5}{x}$

i $x + \frac{2}{x} = -\frac{9}{2}$

j $\frac{2}{2x-3} = \frac{x}{4x-6}$

k $\frac{1}{x-1} - \frac{1}{x+3} = \frac{1}{35}$

l $\frac{4}{x-1} - \frac{5}{x+2} = \frac{3}{x}$

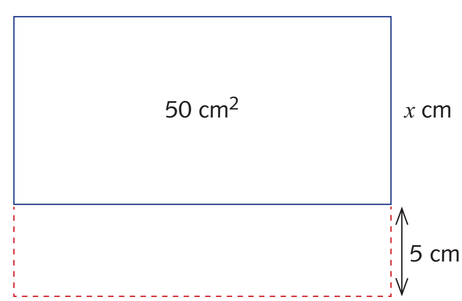
m $6(4x+5) + \frac{7}{x}(4x+5) = 0$

n $\frac{2}{x+3} + \frac{x+3}{2} = \frac{10}{3}$

3 The rectangle on the right has area 50 cm^2.

a The width is x cm. Find the length of the rectangle in terms of x.

b The rectangle is extended by 5 cm to form a square. Form a quadratic equation and find x.



5C Applications of quadratic equations

When we apply mathematics to real-world problems, we often obtain equations to solve. In many cases, these equations are quadratic equations. It is extremely important to keep in mind that some of the solutions we obtain to the equations may *not* be solutions to the real-world problem. For example, a quadratic equation may yield negative or fractional solutions, which may not make sense as answers to the original problem.

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Example 9

The formula for the number of diagonals of a polygon with n sides is $D = \frac{n}{2}(n - 3)$.

How many sides are there in a polygon with 35 diagonals?

Solution

$$D = 35, \text{ so } \frac{n}{2}(n-3) = 35$$

$$\begin{aligned} n(n-3) &= 70 \qquad \text{(Multiply both sides by 2.)} \\ n^2 - 3n - 70 &= 0 \\ (n-10)(n+7) &= 0 \\ n - 10 = 0 \quad &\text{or} \quad n + 7 = 0 \\ n = 10 \quad &\text{or} \quad n = -7 \end{aligned}$$

The value $n = -7$ does not make sense in this problem.

Hence, the polygon has 10 sides.

Example 10

A rectangle has one side 3 cm longer than the other. The rectangle has area 54 cm^2. What is the length of the shorter side?

Solution

Let x cm be the length of the shorter side. The other side has length $(x + 3)$ cm.

$$\text{Area} = x(x+3) = 54 \text{ cm}^2$$

$$\begin{aligned} x^2 + 3x - 54 &= 0 \\ (x-6)(x+9) &= 0 \\ x - 6 = 0 \quad &\text{or} \quad x + 9 = 0 \\ x = 6 \quad &\text{or} \quad x = -9 \end{aligned}$$

Since length must be positive, the solution to the problem is $x = 6$.

Hence, the shorter side has length 6 cm.

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Exercise 5C

Use quadratic equations to solve each problem. Clearly define any pronumerals introduced into your solution.

Example 9

1 The formula for the number of diagonals of a polygon with n sides is $D = \frac{n}{2}(n - 3)$. How many sides does a polygon with 44 diagonals have?

Example 10

2 The length of a rectangle is 4 cm greater than its width. If its area is 96 cm^2, find the length of the rectangle.

3 The sum S of the first n positive integers (that is, $1 + 2 + 3 + \dots + n$) is given by $S = \frac{n}{2}(n + 1)$. What value of n gives a sum of 136 ?

4 A number is squared and then doubled. The result is 45 more than the original number. What is the original number?

5 The difference of two numbers is 16 and the sum of their squares is 130. Find the two numbers.

6 A triangle has base length 4 cm greater than its height. If the area of the triangle is 48 cm^2, find the height of the triangle.

7 A piece of sheet metal measuring 50 cm × 40 cm has squares cut out of each corner so that it can be bent and formed into an open box (with no lid) with a base area of 1344 cm^2. Find the dimensions of the box.

8 Find two numbers such that the sum of their squares is 74 and their sum is 12.

9 A man travels 108 km at a constant speed and finds that the journey would have taken $4\frac{1}{2}$ hours less if he had travelled at a speed 2 km/h faster. What was his speed?

10 The perimeter of a rectangle is 40 cm and its area is 84 cm^2.

a If the width of the rectangle is x cm, express the length of the rectangle in terms of x.

b Find the length and width of the rectangle.

11 A rectangular swimming pool 12 m by 8 m is surrounded by a concrete path of uniform width. If the area of the path is 224 m^2, find the path's width.

12 In a right-angled triangle, one of the sides adjacent to the right angle is 4 cm longer than the other side. The area of the triangle is 48 cm^2. Find the length of each of the three sides.

13 A train travels 300 km at a constant speed. If the speed had been 5 km/h faster, the journey would have taken 2 hours less. Find the speed of the train.

14 One of the parallel sides of a trapezium is 5 cm longer than the other, and its height is half the length of the shorter parallel side. If the area is 225 cm^2, find the lengths of the parallel sides.

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5D Perfect squares and completing the square

Perfect squares

In most of the examples we have looked at so far, the quadratic equations had two solutions.

If the quadratic expression is a perfect square, then there is only one solution to the equation.

Example 11

Solve:

a $x^2 - 6x + 9 = 0$ **b** $9x^2 - 12x + 4 = 0$

Solution

a

$$\begin{aligned} x^2 - 6x + 9 &= 0 \\ (x - 3)(x - 3) &= 0 \\ (x - 3)^2 &= 0 \\ x &= 3 \end{aligned}$$

b

$$\begin{aligned} 9x^2 - 12x + 4 &= 0 \\ 9x^2 - 6x - 6x + 4 &= 0 \\ 3x(3x - 2) - 2(3x - 2) &= 0 \\ (3x - 2)(3x - 2) &= 0 \\ (3x - 2)^2 &= 0 \\ x &= \frac{2}{3} \end{aligned}$$

($9 \times 4 = 36$. Factors of 36 that sum to -12 are -6 and -6.)

Note: perfect squares can also be factored 'on inspection' using the identities $a^2 + 2ab + b^2 = (a + b)^2$ and $a^2 - 2ab + b^2 = (a - b)^2$.

Completing the square

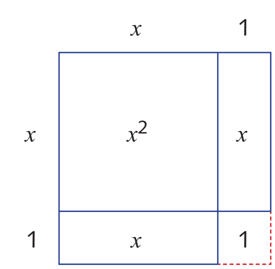
What number must be added to $x^2 + 6x$ to make a perfect square?

It is 9, which is the square of half of the coefficient of x, because $x^2 + 6x + 9 = (x + 3)^2$.

The process of completing the square is an important technique that has many important applications. This section is a basic introduction to this technique.

The key step in a monic expression is to take half the coefficient of x and square it.

Now consider the quadratic expression $x^2 + 2x - 6$. Focus on $x^2 + 2x$.
(In the diagram, a 1×1 square must be added to 'complete the square'.)



We say that the related perfect square is $x^2 + 2x + 1$.

$$\begin{aligned} x^2 + 2x - 6 &= x^2 + 2x + 1 - 1 - 6 \qquad \text{(Add and subtract 1.)} \\ &= (x^2 + 2x + 1) - 7 \\ &= (x + 1)^2 - 7 \end{aligned}$$

This process is called **completing the square**.

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Example 12

a What number must we add to $x^2 - 12x$ to produce a perfect square?

b What number must we add to $x^2 + 3x$ to produce a perfect square?

Solution

a Half the coefficient of x is -6. Its square is 36, so $x^2 - 12x + 36 = (x - 6)^2$.

Hence, 36 must be added to produce a perfect square.

b Half the coefficient of x is $\frac{3}{2}$. Its square is $\frac{9}{4}$.

So $x^2 + 3x + \frac{9}{4} = \left(x + \frac{3}{2}\right)^2$.

Hence, $\frac{9}{4}$ must be added to produce a perfect square.

Perfect squares and completing the square

- If the quadratic expression is a **perfect square**, the corresponding quadratic equation has only one solution.
- To **complete the square** for the expression $x^2 + bx$, take half the coefficient of x (that is, $\frac{b}{2}$) and add and subtract its square, $\left(\frac{b}{2}\right)^2$.

Example 13

Complete the square.

a $x^2 + 6x + 8$

b $x^2 + 3x - 5$

Solution

a
$$\begin{aligned} x^2 + 6x + 8 &= (x^2 + 6x + 9) - 9 + 8 \\ &= (x + 3)^2 - 1 \end{aligned}$$

b
$$\begin{aligned} x^2 + 3x - 5 &= \left(x^2 + 3x + \frac{9}{4}\right) - \frac{9}{4} - 5 \\ &= \left(x + \frac{3}{2}\right)^2 - \frac{29}{4} \end{aligned}$$

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Completing the square for non-monic expressions

When the co-efficient of x^2 is not 1, then this value needs to be factored out of the quadratic expression before the process of completing the square can be applied. The final step is to multiply both the perfect square and the constant term by this factored out value.

Example 14

Complete the square.

a $-x^2 + 2x + 5$ **b** $3x^2 + 12x - 1$ **c** $2x^2 - 5x + 1$

Solution

a $-x^2 + 2x + 5$

$$= -(x^2 - 2x - 5)$$
$$= -[(x^2 - 2x + 1) - 1 - 5]$$
$$= -[(x - 1)^2 - 6]$$
$$= -(x - 1)^2 + 6$$

b $3x^2 + 12x - 1$

$$= 3(x^2 + 4x - \frac{1}{3})$$
$$= 3\left[\left(x^2 + 4x + 4\right) - 4 - \frac{1}{3}\right]$$
$$= 3\left[(x + 2)^2 - \frac{13}{3}\right]$$
$$= 3(x + 2)^2 - 13$$

c $2x^2 + 5x + 1$

$$= 2\left(x^2 + \frac{5x}{2} + \frac{1}{2}\right)$$
$$= 2\left[\left(x^2 + \frac{5x}{2} + \frac{25}{16}\right) - \frac{25}{16} + \frac{1}{2}\right]$$
$$= 2\left[\left(x + \frac{5}{4}\right)^2 - \frac{17}{16}\right]$$
$$= 2\left(x + \frac{5}{4}\right)^2 - \frac{17}{8}$$

Exercise 5D

Example 11

1 Solve:

a $x^2 + 2x + 1 = 0$ **b** $x^2 + 4x + 4 = 0$ **c** $x^2 + 8x + 16 = 0$

d $x^2 - 10x + 25 = 0$ **e** $x^2 + x + \frac{1}{4} = 0$ **f** $x^2 - 3x + \frac{9}{4} = 0$

g $x^2 - \frac{4}{5}x + \frac{4}{25} = 0$ **h** $x^2 + \frac{3}{2}x + \frac{9}{16} = 0$ **i** $9x^2 - 6x + 1 = 0$

j $25x^2 + 10x + 1 = 0$ **k** $25x^2 - 20x + 4 = 0$ **l** $49x^2 + 28x + 4 = 0$

m $49x^2 - 70x + 25 = 0$ **n** $9x^2 + 30x + 25 = 0$

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2 Which of these expressions is:

a the result of a perfect square expansion?

b a difference of squares expansion?

i	$x^2 + 8x + 16$	ii	$x^2 - 16$	iii	$2x^2 + 3x + 1$
iv	$9 - y^2$	v	$25 - 10x + x^2$	vi	$x^2 + 4x + 1$
vii	$4x^2 - 25$	viii	$x^2 + 9$	ix	$4x^2 + 12x + 9$
x	$64 - 49a^2$	xi	$b^2 - 6b + 8$	xii	$36a^2 - 49b^2$

Example 12

3 What must be added to each expression to make it a perfect square?

a $x^2 + 4x$
b $x^2 + 8x$
c $x^2 - 10x$
d $x^2 - 12x$
e $x^2 + 20x$
f $x^2 + 3x$
g $x^2 + x$
h $x^2 - 7x$
i $x^2 - 11x$

Example 13

4 Complete the square:

a $x^2 + 6x + 10$
b $x^2 + 8x - 5$
c $x^2 + 12x - 10$
d $x^2 - 10x + 6$
e $x^2 - 6x - 8$
f $x^2 - 20x + 5$
g $x^2 + 3x - 2$
h $x^2 + x + 1$
i $x^2 - 5x + 6$
j $x^2 - x - 10$
k $x^2 + 3x + 7$
l $x^2 - 11x + 1$

Example 14

5 Complete the square:

a $3x^2 + 6x + 12$
b $5x^2 + 30x + 10$
c $3x^2 - 12x + 15$
d $-x^2 - 2x + 4$
e $-x^2 + 8x - 10$
f $4 - 6x - x^2$
g $3x^2 - 6x - 1$
h $2x^2 - 12x + 33$
i $4x^2 - 48x + 99$
j $2x^2 + 3x + 2$
k $4x^2 - x - 4$
l $3x^2 - 8x + 9$
m $5x^2 - x + 1$
n $2x^2 - 5x - 7$
o $4 - x - 3x^2$

5E Solving quadratic equations by completing the square

In all our examples so far, the quadratic expression factorised nicely and gave us integer or rational solutions. This is not always the case. For example, $x^2 - 7 = 0$ has solutions $x = \sqrt{7}$ and $x = -\sqrt{7}$.

Quadratic equations with integer coefficients

Quadratic equations with integer coefficients can have:

- integer or rational solutions, for example, $x^2 - 1 = 0$
- solutions involving surds, for example, $x^2 - 7 = 0$
- no solution, for example, $x^2 + 1 = 0$.

The method of completing the square enables us to deal with all quadratic equations.

Historically, quadratic equations were solved by completing the square. This method always works, even when we cannot easily factorise the quadratic expression. A typical example is $x^2 + 2x - 9 = 0$.

Here are the steps for solving the quadratic $x^2 + 2x - 9 = 0$.

We first complete the square on the left-hand side. Half the coefficient of x is 1; its square is 1.

$$(x^2 + 2x + 1) - 1 - 9 = 0 \qquad \text{(Add and subtract the square of half the coefficient of } x.)$$

$$(x + 1)^2 - 10 = 0$$

$$(x + 1)^2 = 10$$

$$x + 1 = \sqrt{10} \quad \text{or} \quad x + 1 = -\sqrt{10}$$

$$\text{Finally, } x = -1 + \sqrt{10} \quad \text{or} \quad x = -1 - \sqrt{10}$$

These two numbers are the solutions to the original equation. Note that checking by substitution is hard. It is more efficient to check each step in the calculation.

Example 15

Solve $x^2 - 6x - 2 = 0$.

Solution

Method 1

$$x^2 - 6x - 2 = 0$$

$$(x^2 - 6x + 9) - 9 - 2 = 0 \qquad \text{(Complete the square.)}$$

$$(x - 3)^2 = 11$$

$$x - 3 = \sqrt{11} \quad \text{or} \quad x - 3 = -\sqrt{11}$$

$$\text{Hence, } x = 3 + \sqrt{11} \quad \text{or} \quad x = 3 - \sqrt{11}.$$

Method 2

$$x^2 - 6x - 2 = 0$$

$$x^2 - 6x = 2$$

$$x^2 - 6x + 9 = 2 + 9$$

$$(x - 3)^2 = 11$$

$$x - 3 = \sqrt{11} \quad \text{or} \quad x - 3 = -\sqrt{11}$$

$$\text{Hence, } x = 3 + \sqrt{11} \quad \text{or} \quad x = 3 - \sqrt{11}.$$

Method 1 and Method 2 are essentially the same. Adding and subtracting a number on one side of an equation has the same effect as adding that number to both sides of the equation. When solving quadratic equations, we will generally use Method 2.

Example 16

Solve:

a $x^2 + 8x + 6 = 0$ **b** $x^2 - 7x - 3 = 0$

Solution

a
$$\begin{aligned} x^2 + 8x + 6 &= 0 \\ x^2 + 8x &= -6 \\ x^2 + 8x + 16 &= -6 + 16 \\ (x + 4)^2 &= 10 \\ x + 4 &= \sqrt{10} \quad \text{or} \quad x + 4 = -\sqrt{10} \end{aligned}$$

Hence, $x = -4 + \sqrt{10}$ or $x = -4 - \sqrt{10}$.

b
$$\begin{aligned} x^2 - 7x - 3 &= 0 \\ x^2 - 7x &= 3 \\ x^2 - 7x + \frac{49}{4} &= 3 + \frac{49}{4} && \text{(Complete the square.)} \\ \left(x - \frac{7}{2}\right)^2 &= \frac{61}{4} \\ x - \frac{7}{2} &= \frac{\sqrt{61}}{2} \quad \text{or} \quad x - \frac{7}{2} = -\frac{\sqrt{61}}{2} \\ x &= \frac{7 + \sqrt{61}}{2} \quad \text{or} \quad x = \frac{7 - \sqrt{61}}{2} \end{aligned}$$

Example 17

Solve $3x^2 + 5x - 1 = 0$.

Solution

$$\begin{aligned} 3x^2 + 5x - 1 &= 0 \\ 3x^2 + 5x &= 1 \\ x^2 + \frac{5x}{3} &= \frac{1}{3} && \text{(Divide all terms by the coefficient of } x^2\text{.)} \\ x^2 + \frac{5x}{3} + \frac{25}{36} &= \frac{1}{3} + \frac{25}{36} \\ \left(x + \frac{5}{6}\right)^2 &= \frac{37}{36} \\ x + \frac{5}{6} &= \frac{\sqrt{37}}{6} \quad \text{or} \quad x + \frac{5}{6} = -\frac{\sqrt{37}}{6} \quad \left(\sqrt{\frac{37}{36}} = \frac{\sqrt{37}}{6}\right) \\ x &= \frac{-5 + \sqrt{37}}{6} \quad \text{or} \quad x = \frac{-5 - \sqrt{37}}{6} \end{aligned}$$

There are quadratic equations that cannot be solved. Consider, for example, $x^2 - 6x + 12 = 0$.

$$\begin{aligned} x^2 - 6x + 12 &= 0 \\ x^2 - 6x + 9 - 9 + 12 &= 0 \\ (x-3)^2 + 3 &= 0 \\ (x-3)^2 &= -3 \end{aligned}$$

Since $(x-3)^2 \geq 0$ for all values of x, there is no solution to the equation $(x-3)^2 = -3$.

Solution of quadratic equations by completing the square

- To solve a quadratic equation of the form $ax^2 + bx + c = 0$ by completing the square, we:
  - move the constant, c, to the right-hand side
  - divide all terms by the coefficient of x^2, a
  - add the square of half the coefficient of x, $\left(\frac{b}{2a}\right)^2$, to both sides of the equation
  - solve for x.
- Fractions and square roots often occur in this procedure.
- We can also show that a quadratic equation has no solution using this procedure.

Exercise 5E

1 Solve:

a $x^2 - 5 = 0$ **b** $x^2 - 11 = 0$ **c** $2x^2 - 6 = 0$

d $4x^2 - 8 = 0$ **e** $50 - 5x^2 = 0$ **f** $40 - 8x^2 = 0$

Example 15, 16a

2 Solve each equation by completing the square.

a $x^2 + 2x - 1 = 0$ **b** $x^2 + 4x + 1 = 0$ **c** $x^2 - 12x + 23 = 0$

d $x^2 + 6x + 7 = 0$ **e** $x^2 - 8x - 1 = 0$ **f** $x^2 + 8x - 4 = 0$

g $x^2 + 10x + 1 = 0$ **h** $x^2 + 12x - 5 = 0$ **i** $x^2 - 10x - 50 = 0$

j $x^2 + 20x + 5 = 0$ **k** $x^2 - 100x - 80 = 0$ **l** $x^2 - 50x + 10 = 0$

Example 16b

3 Solve each equation by completing the square.

a $x^2 + x - 1 = 0$ **b** $x^2 - 3x + 1 = 0$ **c** $x^2 - 5x - 1 = 0$

d $x^2 + 3x - 2 = 0$ **e** $x^2 + 5x - 4 = 0$ **f** $x^2 - 3x - 5 = 0$

g $x^2 - 7x - 100 = 0$ **h** $x^2 - 3x - 6 = 0$ **i** $x^2 - 9x - 5 = 0$

j $x^2 - x - 5 = 0$ **k** $x^2 - 3x + 1 = 0$ **l** $x^2 - 5x + 3 = 0$

Example 17

4 Solve each equation by completing the square.

a $3x^2 - 12x + 3 = 0$ **b** $3x^2 + 6x - 12 = 0$ **c** $-x^2 - 2x + 4 = 0$

d $-x^2 + 8x - 10 = 0$ **e** $-x^2 - 6x + 12 = 0$ **f** $3x^2 + 24x - 12 = 0$

g $2x^2 - 3x - 2 = 0$ **h** $3x^2 - 8x - 6 = 0$ **i** $4x^2 - x - 4 = 0$

j $5x^2 + x - 1 = 0$ **k** $2x^2 - 5x - 3 = 0$ **l** $3x^2 + 10x - 15 = 0$

m $\frac{1}{6}x^2 + \frac{1}{3}x - 1 = 0$ **n** $\frac{3}{4}x^2 - \frac{3}{2}x - 3 = 0$ **o** $\sqrt{2}x^2 - 4x - 2\sqrt{2} = 0$

5 Solve the equations. Some of them will factorise by trial and error, for some you will have to complete the square, and some will have no solution.

a $x^2 + 6x - 8 = 0$ **b** $x^2 - 3x - 10 = 0$ **c** $x^2 + 6x - 7 = 0$

d $x^2 - 4x - 3 = 0$ **e** $x^2 + x - 6 = 0$ **f** $x^2 - x - 3 = 0$

g $2x^2 + 5x + 2 = 0$ **h** $3x^2 - 2x - 1 = 0$ **i** $x^2 + 2x - 5 = 0$

j $x^2 + 6x - 5 = 0$ **k** $x^2 + 4x + 6 = 0$ **l** $x^2 - 6x + 10 = 0$

m $4x^2 - 25 = 0$ **n** $9x^2 - 1 = 0$ **o** $2x^2 + 4x - 70 = 0$

p $3x^2 - 3x - 36 = 0$ **q** $4x^2 - 5 = 0$ **r** $9x^2 + 7 = 0$

s $6x^2 + x - 12 = 0$ **t** $12x^2 + 23x + 5 = 0$ **u** $x^2 + 6x + 9 = 0$

v $3x^2 + 6x + 2 = 0$ **w** $2x^2 - 8x + 5 = 0$ **x** $5x^2 + 2x - 5 = 0$

y $12x^2 + 5x - 2 = 0$ **z** $4x^2 - x + 4 = 0$

6 Solve:

a $x(x + 2) = 5$ **b** $x(x - 2) = 1$ **c** $x = \frac{7}{x} - 4$

d $x + \frac{4}{x} = -6$ **e** $\frac{x + 1}{x} = x$ **f** $\frac{x + 3}{x} = \frac{2x}{3}$

5F The quadratic formula

The method of completing the square always works. From this it is possible to develop a general formula for the solutions, if they exist, of a quadratic equation in terms of the coefficients in the given equation. This formula is known as the **quadratic formula**. If you are interested in computer programming, you may like to write a program that inputs the coefficients of a quadratic and uses the formula to find the solutions.

To derive the formula, we start with a general quadratic equation of the form:

$$ax^2 + bx + c = 0, \quad \text{where } a \neq 0$$

And begin to solve for x.

$$\begin{aligned}
ax^2 + bx &= -c \\
x^2 + \frac{b}{a}x &= \frac{-c}{a} \\
x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 &= \left(\frac{b}{2a}\right)^2 - \frac{c}{a} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2} - \frac{c}{a} \\
&= \frac{b^2 - 4ac}{4a^2}
\end{aligned}$$

If $b^2 - 4ac$ is negative, the equation has no solution.

If $b^2 - 4ac$ is positive or zero, then we can solve for x and obtain:

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$= \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad -\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\therefore x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Summarising the result:

When solving $ax^2 + bx + c = 0$, first calculate $b^2 - 4ac$.

- If $b^2 - 4ac$ is negative, then there is no solution.
- If $b^2 - 4ac$ is positive, then $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ or $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.
- If $b^2 - 4ac = 0$, then there is one solution: $x = \dfrac{-b}{2a}$.

You do not need to remember the details of the derivation of this formula, but you should memorise the formula.

Example 18

Use the quadratic formula to solve:

a $x^2 - 7x + 12 = 0$ **b** $x^2 + 3x - 1 = 0$ **c** $x^2 - 10x - 3 = 0$

Solution

a Here $a = 1$, $b = -7$, $c = 12$,

so $b^2 - 4ac = (-7)^2 - 4(1)(12)$

$= 49 - 48 = 1$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{7 + \sqrt{1}}{2} \quad \text{or} \quad x = \frac{7 - \sqrt{1}}{2}$$

$$x = 4 \quad \text{or} \quad x = 3$$

Note that this equation is much easier to solve by factorising.

(continued over page)

b Here $a = 1,\ b = 3,\ c = -1,$

so $b^2 - 4ac = (3)^2 - 4(1)(-1)$

$= 9 + 4 = 13$

$x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ or $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

$x = \dfrac{-3 + \sqrt{13}}{2}$ or $x = \dfrac{-3 + \sqrt{13}}{2}$

c Here $a = 1,\ b = -10,\ c = -3,$

so $b^2 - 4ac = (-10)^2 - 4(1)(-3)$

$= 100 + 12 = 112$

$x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ or $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

$x = \dfrac{10 + \sqrt{112}}{2}$ or $x = \dfrac{10 - \sqrt{112}}{2}$

$x = \dfrac{10 + 4\sqrt{7}}{2}$ or $x = \dfrac{10 - 4\sqrt{7}}{2}$ (Simplify the surd.)

$x = \dfrac{2(5 + 2\sqrt{7})}{2}$ or $x = \dfrac{2(5 - 2\sqrt{7})}{2}$

$x = 5 + 2\sqrt{7}$ or $x = 5 - 2\sqrt{7}$ (Cancel common factors.)

Example 19

Use the quadratic formula to solve $x^2 - 3x - 5 = 0$, giving your answers correct to two decimal places.

Solution

Here $a = 1, b = -3, c = -5,$

so $b^2 - 4ac = (-3)^2 - 4(1)(-5)$

$= 9 + 20 = 29$

$x = \dfrac{3 + \sqrt{29}}{2}$ or $x = \dfrac{3 - \sqrt{29}}{2}$

$x \approx 4.19$ or $x \approx -1.19$ (Correct to two decimal places.)

Solving quadratic equations – a summary

We now have three methods for solving a quadratic equation:

- completing the square
- factorisation
- the quadratic formula.

- It is a good idea to calculate $b^2 - 4ac$ first to check that it is positive or zero, otherwise there will be no solution.
- Only use the quadratic formula or complete the square if you cannot see how to factorise the quadratic expression.
- When you use the quadratic formula, take care to simplify the surd and cancel any common factors.
- A quadratic equation for which the coefficient of x^2 is 1 and in which the coefficient of x is even can be solved more quickly and efficiently by completing the square than by using the quadratic formula. You should be in the habit of using both methods and, for a given situation, choosing the one you think will be the faster.

Example 20

Solve each quadratic equation, using any method.

a $x^2 - 9x + 14 = 0$ **b** $x^2 - 8x - 1 = 0$ **c** $3x^2 - 7x + 1 = 0$

Solution

a This quadratic equation factorises easily.

$$x^2 - 9x + 14 = 0$$
$$(x - 2)(x - 7) = 0$$
$$x = 2 \quad \text{or} \quad x = 7$$

b This quadratic equation does not factorise easily, but the coefficient of x is even.

$$x^2 - 8x - 1 = 0$$
$$x^2 - 8x = 1$$
$$x^2 - 8x + 16 = 1 + 16$$
$$(x - 4)^2 = 17$$
$$x = 4 + \sqrt{17} \quad \text{or} \quad x = 4 - \sqrt{17}$$

c The quadratic formula is best here.

$$3x^2 - 7x + 1 = 0$$

Now $a = 3$, $b = -7$, $c = 1$,

so $b^2 - 4ac = 49 - 12$
$= 37$

$$x = \frac{7 + \sqrt{37}}{6} \quad \text{or} \quad x = \frac{7 - \sqrt{37}}{6}$$

The Discriminant

The **discriminant** is the name given to $b^2 - 4ac$, the part of the quadratic formula under the square root sign. It is often denoted by the capital Greek letter delta, Δ (i.e. $\Delta = b^2 - 4ac$).

As noted, the discriminant can be used to determine the number of solutions in the quadratic equation. However, when a, b and c are rational numbers, and $\Delta > 0$, it also determines the *nature* of the solution. If the discriminant is the square of a rational number (for example, 64, 1, 25, $\frac{1}{4}$ or $\frac{49}{36}$), then the solutions are rational numbers. Otherwise, the solutions contain a surd.

Example 21

Determine the number of solutions in the following quadratic equations. Where solutions exist, state their nature.

a $3x^2 + 8x + 1 = 0$

b $4x^2 + 7x + 5 = 0$

c $9x^2 - 60x + 100 = 0$

d $8x^2 - 2x - 3 = 0$

Solution

a Here $a = 3, b = 8, c = 1,$

so $\begin{aligned} b^2 - 4ac &= 64 - 12 \\ &= 52 \end{aligned}$

$\Delta = 52$, so $\Delta > 0$ and not the square of a rational number.

Therefore, the equation has two irrational solutions.

b Here $a = 4, b = 7, c = 5,$

so $\begin{aligned} b^2 - 4ac &= 49 - 80 \\ &= -31 \end{aligned}$

$\Delta = -31$, so $\Delta < 0$.

Therefore, the equation has no solutions.

c Here $a = 9, b = -60, c = 100,$

so $\begin{aligned} b^2 - 4ac &= 3600 - 3600 \\ &= 0 \end{aligned}$

$\Delta = 0$

Therefore, the equation has one rational solution.

(*Note*: The value of the single solution is easily determined using the formula, $x = -\frac{b}{2a} = \frac{60}{18} = \frac{10}{3}$.)

d Here $a = 8, b = -2, c = -3,$

so $\begin{aligned} b^2 - 4ac &= 4 + 96 \\ &= 100 \end{aligned}$

$\Delta = 100$, so $\Delta > 0$ and the square of a rational number.

Therefore, the equation has two rational solutions.

Note: If the discriminant is positive and the square of a rational number, then the expression can usually be factorised easily.

The quadratic formula for $ax^2 + bx + c = 0$

- First calculate $\Delta = b^2 - 4ac$.
- If Δ is negative, then the equation $ax^2 + bx + c = 0$ has no solution.
- The solution of $ax^2 + bx + c = 0$, with $a \neq 0$, is given by:
- $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ or $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

  provided that Δ is positive or zero.

Exercise 5F

Example 18

1 Use the quadratic formula to solve each quadratic equation. Give your answers in simplest surd form.

a $x^2 - 8x + 1 = 0$ **b** $x^2 - 2x - 8 = 0$ **c** $x^2 - 3x - 1 = 0$

d $x^2 - 4x - 12 = 0$ **e** $x^2 + 5x + 2 = 0$ **f** $x^2 + 9x + 3 = 0$

g $x^2 - 8x + 2 = 0$ **h** $x^2 + 2x - 4 = 0$ **i** $x^2 + 12x + 3 = 0$

2 Use the quadratic formula to solve each quadratic equation. Give your answers in simplest surd form.

a $3x^2 + 2x - 7 = 0$ **b** $5x^2 + 3x - 1 = 0$ **c** $4x^2 - 6x + 1 = 0$

d $7x^2 - 9x + 2 = 0$ **e** $5x^2 + 3x - 2 = 0$ **f** $7x^2 - x - 1 = 0$

g $2x^2 + 12x - 1 = 0$ **h** $3x^2 - 20x - 2 = 0$ **i** $3x^2 - 4x - 5 = 0$

Example 19

3 Use the quadratic formula to solve each quadratic equation, giving your answers to two decimal places where appropriate.

a $5x^2 - 7x - 1 = 0$ **b** $x^2 - 8x + 1 = 0$ **c** $x^2 - 3x - 10 = 0$

d $x^2 + 15x + 3 = 0$ **e** $2x^2 - 10x + 12 = 0$ **f** $5x^2 - 15x - 7 = 0$

g $2x^2 - 5x - 2 = 0$ **h** $5x^2 - 3x - 1 = 0$ **i** $2x^2 - 7x + 1 = 0$

Example 21

4 The quadratic formula states that the solutions of a quadratic equation are given by

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

a What can you conclude about the number of solutions of a quadratic equation if:

i $b^2 - 4ac < 0$? **ii** $b^2 - 4ac = 0$? **iii** $b^2 - 4ac > 0$?

b Determine the number of solutions of each quadratic equation, and where they exist, state their nature. You do not need to find the solutions.

i $x^2 + 8x - 5 = 0$ **ii** $3x^2 - 7x + 2 = 0$ **iii** $x^2 + 6x + 9 = 0$

iv $4x^2 - 4x + 1 = 0$ **v** $x^2 + 7x + 13 = 0$ **vi** $2x^2 + 11x + 17 = 0$

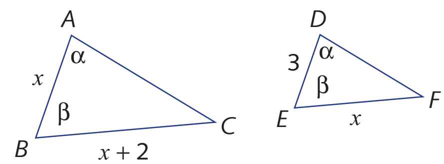
vii $3x^2 - 4x - 2 = 0$ **viii** $-2x^2 + 3x + 7 = 0$

5 In each problem, introduce one pronumeral, and then construct and solve a quadratic equation. Remember to check that your solutions make sense.

a What positive real number is one more than its reciprocal?

b A rectangle has length 5 cm greater than its width. If the area of the rectangle is 30 cm^2, find the width of the rectangle (correct to two decimal places).

c Consider the triangles ABC and DEF, which have side lengths and angles as marked.



Use similar triangles to find the value of x in surd form.

6 The interval AB is extended to point P so that $AB \times AP = BP^2$.

If $AB = 8$ cm, find the lengths of AP and BP.

7 A farmer sells sheep at \$75 a head. The sheep cost \$$x$ each. The farmer finds she has made $x\%$ profit on the sale of the sheep. Find x.

8 Find two numbers whose difference is 5 and the sum of whose squares is 100.

9 An investor invests \$10 000 at $x\%$ p.a. compound interest for 2 years. He finds that he receives \$20 more in interest than if he had invested it at a simple interest rate of $x\%$ p.a. Find x.

10 A car travels 500 km at a constant speed. If it had travelled at a speed 10 km/h less, it would have taken 1 hour more to travel the distance. Find the speed of the car.

11 A rectangular field is 405 m^2 in area, and its perimeter is 200 m. Find the length of its sides.

Example 20

12 Solve each quadratic equation, using any method.

a $x^2 - 11x + 28 = 0$ **b** $x^2 - 12x - 4 = 0$ **c** $3x^2 + 2x - 6 = 0$

Review exercise

1 Solve each quadratic equation by factorising.

a $x^2 - 3x - 18 = 0$ **b** $3x^2 + 5x - 2 = 0$ **c** $2x^2 + x - 1 = 0$

d $6x^2 + 7x = 3$ **e** $x^2 = \frac{1}{2}(5x + 12)$ **f** $x + \frac{2}{x} = -\frac{9}{2}$

g $x(x - 2) = 8$ **h** $(x - 1)(x + 1) = 2(x + 1)^2$ **i** $2x^2 - 3x = 5$

j $2 - 7x + \frac{5}{x} = 0$ **k** $2x^2 = 11x - 5$ **l** $2x^2 + 11x + 5 = 0$

m $4x^2 - 10x - 6 = 0$ **n** $6x^2 = 20x - 6$ **o** $18x^2 - 12x + 2 = 0$

p $9x^2 - 42x + 49 = 0$ **q** $6x - \frac{49}{x} + 7 = 0$

2 Solve each quadratic equation by completing the square.

a $x^2 - 8x + 15 = 0$

b $t^2 - 11t + 30 = 0$

c $x^2 - 4x + 1 = 0$

d $x^2 + 2x - 1 = 0$

e $y^2 + y = 3$

f $v^2 - 20v = 7$

g $x^2 - 3x = 7$

h $z^2 - 2z = 3$

i $2z^2 + 4z = 64$

j $2x^2 - 5x + 2 = 0$

k $3x^2 + x - 3 = 0$

l $4x^2 - 3x - 2 = 0$

3 Use the quadratic formula to find exact solutions to each quadratic equation.

a $x^2 - 2x - 24 = 0$

b $2x^2 + 3x - 2 = 0$

c $x^2 + x - 1 = 0$

d $2x^2 + 5x + 1 = 0$

e $2x^2 + 2x - 3 = 0$

f $3x^2 - x - 1 = 0$

4 Solve:

a $x^2 - 2x - 1 = 0$

b $2x^2 + 5x = 4$

c $4x^2 - x - 1 = 0$

d $\dfrac{1}{x} = \dfrac{x-1}{4}$

e $\dfrac{2x-1}{5} = \dfrac{1}{3x+2}$

f $\dfrac{2x+1}{5} = \dfrac{-x}{3x-2}$

5 Solve:

a $x^2 - 7x + 9 = 0$

b $2x^2 + 7x = 3$

c $10x^2 = 2x + 5$

d $5x^2 - 8x + 2 = 0$

e $4x^2 - 6x = 3$

f $2x^2 - 9x = 4$

g $2x^2 - 5x = 1$

h $3x^2 + 4x = 2$

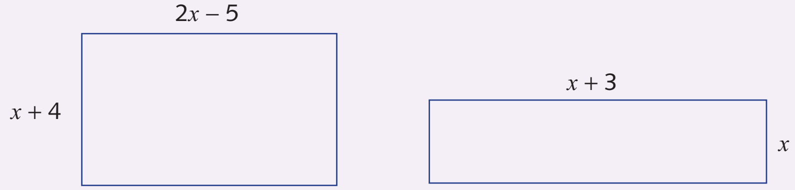
6 For each problem, introduce a pronumeral and construct a quadratic equation to solve it.

a The difference of two numbers is 16 and the sum of their squares is 130. What are the numbers?

b The perimeter of a rectangle is 50 cm and its area is 144 cm^2. Find its length and width.

c Two numbers differ by 2, but the difference of their reciprocals is $\frac{2}{15}$. What are the numbers?

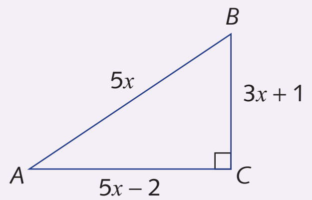
7 The rectangles shown have equal area. Find the value of x.



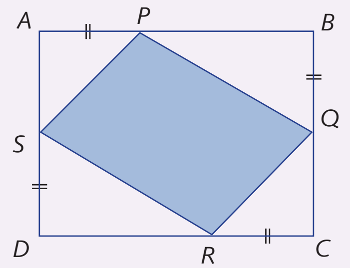
8 A man travels 196 km by train and returns in a car that travels 21 km/h faster. If the total journey takes 11 hours, find the speeds of the train and the car.

9 A wire 80 cm in length is cut into two parts and each part is bent to form a square. If the sum of the areas of the squares is 300 cm^2, find the lengths of the sides of the two squares.

10 The lengths of the sides of a right-angled triangle are $(3x + 1)$ cm, $5x$ cm and $(5x - 2)$ cm. Find the area of the triangle.



11 In the diagram below, $ABCD$ is a rectangle in which $AB = 16$ cm and $BC = 12$ cm and $AP = BQ = CR = DS$. The area of the shaded figure $PQRS$ is 112 cm^2. Find the length of AP.



Challenge exercise

1 Solve the equation $x^4 - 13x^2 + 36 = 0$ by treating it as a quadratic in x^2.

2 Solve the equation $(x^2 - 2x)^2 - 11(x^2 - 2x) + 24 = 0$.

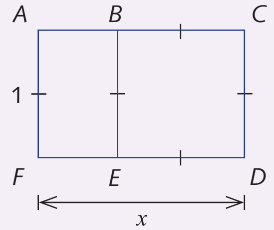
3 A **golden rectangle** is a rectangle such as $ACDF$ below, with sides of length 1 and x, and with the property that if a 1×1 square ($BCDE$) is removed, the resulting rectangle ($ABEF$) is similar to the original one. (That is, $ACDF$ is an enlargement of $ABEF$.)

a Show that $\dfrac{x-1}{1} = \dfrac{1}{x}$.

b Solve this equation to show that $x = \dfrac{1+\sqrt{5}}{2}$.

(This number is known as the **golden ratio**.)

c Check that your answer satisfies the equation in part **a**.



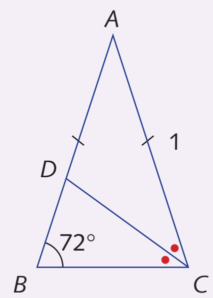
4 Find a monic quadratic equation that has roots equal to $2 - \sqrt{3}$ and $2 + \sqrt{3}$.

5 What is the average of the solutions and what is the product of the solutions of $2x^2 + 14x + 17 = 0$?

6 If x is a solution of $x + \frac{1}{x} = 3$, find $x^2 + \frac{1}{x^2}$.

7 Solve $\sqrt{7x} - \sqrt{3x} = 4$.

8 Take an isosceles triangle ABC with a base angle of 72° and $AB = AC = 1$. Bisect one of the base angles, for example, C (as shown in the diagram) and join CD.



a Prove that triangle ABC is similar to triangle CDB.

b Let $BC = x$. Prove that $x^2 + x - 1 = 0$ and hence solve for x.

c Drop a perpendicular from A to BC and show that $\cos 72° = \frac{\sqrt{5} - 1}{4}$.

d Find the cosine of 18° in simplest surd form.

9 A number, x, is defined by:

$$x = \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{3 + \dots}}}}$$

where the dots indicate that the pattern continues forever.

a As the pattern repeats indefinitely, we can write $x = \cfrac{1}{2 + \cfrac{1}{3 + x}}$. Simplify this to give a quadratic equation for x.

b Solve the quadratic equation.

c Find the number $y = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}}$

10 Solve the equation $\frac{x - a}{x + a} = \frac{x + a}{2x - a}$ for x, where a is a constant.

CHAPTER

6

Measurement and Geometry

Surface area and volume

In this chapter, we will review and extend the ideas of the surface area and the volume of a solid. Problems involving calculating volume and surface area are very practical and important. Most of the chapter concerns prisms and pyramids, which are the most common polyhedra. We will also see how to find the volume and surface area of solids such as cones, cylinders and spheres.

This chapter contains a number of formulas for volumes and areas.

6A Review of prisms and cylinders

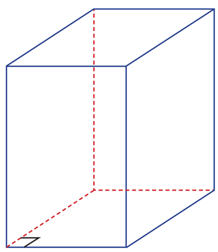
A **polyhedron** is a solid bounded by polygons called **faces**.

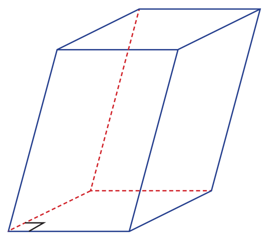
Two adjacent faces meet at an **edge** and two adjacent edges meet at a **vertex**.

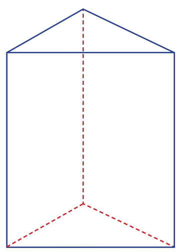
A **prism** is a polyhedron that has two congruent and parallel faces and all its remaining faces are parallelograms.

A **right prism** is a prism in which the top and bottom polygons are vertically above each other, and the vertical polygons connecting their faces are rectangles. A prism that is not a right prism is often called an **oblique prism**.

Some examples of prisms are shown below.

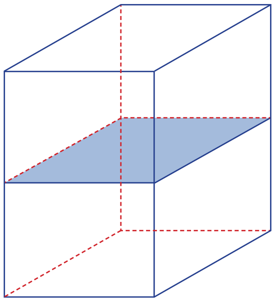

Right rectangular prism


Oblique rectangular prism


Right triangular prism

When we refer to a prism we generally mean a right prism.

A prism with a rectangular base is called a **rectangular prism**, while a **triangular prism** has a triangular base.

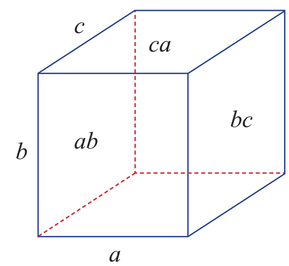


You will notice that if we slice a prism by a plane parallel to its base, then the cross-section is congruent to the base and so has the same area as the base.

In this chapter, we will use the pronumeral S for the surface area of a solid and the pronumeral V for the volume of a solid.

Surface areas

Surface area of a prism



The surface area of a prism is the sum of the areas of its faces.

A rectangular prism with dimensions a, b and c has six faces. These occur in opposite pairs; the faces with areas ab, bc and ca each occur twice.

Thus:

$$\text{Surface area of a rectangular prism} = 2\,(ab + bc + ca)$$

We do not need to learn this formula. We can simply find the area of each face and take the sum of the areas. The same idea applies to all other types of prisms.

In the diagrams in this chapter, assume that all quadrilaterals are rectangles unless the context indicates otherwise.

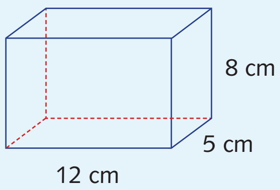
Surface area of a prism

To find the surface area of a prism, find the area of each face and calculate the sum of the areas.

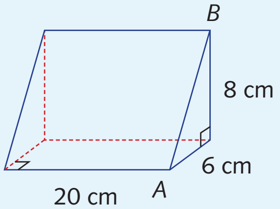
Example 1

Find the surface area of each prism.

a



b



Solution

a The prism has six faces.

$$\begin{aligned}\text{Area of top rectangle} &= 12 \times 5\\ &= 60\text{ cm}^2\end{aligned}$$

$$\begin{aligned}\text{Area of front rectangle} &= 12 \times 8\\ &= 96\text{ cm}^2\end{aligned}$$

$$\begin{aligned}\text{Area of side rectangle} &= 8 \times 5\\ &= 40\text{ cm}^2\end{aligned}$$

$$\begin{aligned}\text{Thus, } S &= 2 \times 60 + 2 \times 96 + 2 \times 40\\ &= 392\text{ cm}^2\end{aligned}$$

b We need to find the length AB in the diagram. We can find this using Pythagoras' theorem.

$$\begin{aligned}AB^2 &= 6^2 + 8^2\\ &= 100\end{aligned}$$

so $AB = 10$ cm

$$\begin{aligned}\text{Area of the sloping rectangle} &= 10 \times 20\\ &= 200\text{ cm}^2\end{aligned}$$

$$\begin{aligned}\text{Area of each triangle} &= \frac{1}{2} \times 6 \times 8\\ &= 24\text{ cm}^2\end{aligned}$$

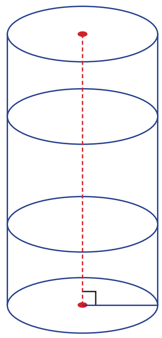
$$\begin{aligned}\text{Area of the base rectangle} &= 6 \times 20\\ &= 120\text{ cm}^2\end{aligned}$$

$$\begin{aligned}\text{Area of the back rectangle} &= 8 \times 20\\ &= 160\text{ cm}^2\end{aligned}$$

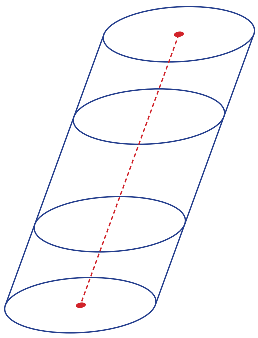
$$\begin{aligned}\text{Thus, } S &= 24 + 24 + 120 + 160 + 200\\ &= 528\text{ cm}^2\end{aligned}$$

Surface area of a cylinder

A **cylinder** is a solid that has parallel circular discs of equal radius at the top and the bottom. Each cross-section parallel to the base is a circle, and the centres of these circular cross-sections lie on a straight line. If that line is perpendicular to the base, the cylinder is called a **right cylinder**. When we use the word 'cylinder' in this book, we will generally mean a right cylinder.



Right cylinder



Oblique cylinder

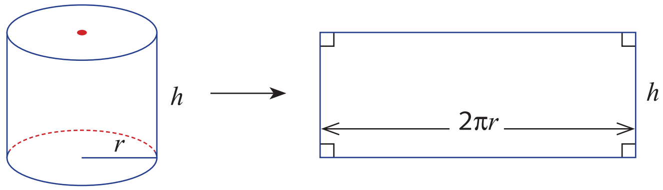
We will use a dot (•) to indicate the centre of the circular base or top.

As we did with prisms, we find the surface area of a cylinder by adding up the area of the curved section of the cylinder, and the area of the two circles.

Surface area of the curved surface

Suppose we have a cylinder with base radius r and height h. If we roll the cylinder along a flat surface through one revolution, as shown in the diagram, the curved surface traces out a rectangle. The width of the rectangle is the height of the cylinder, while the length of the rectangle is the circumference of the circle, which is $2\pi r$, so the area of the curved part is $2\pi rh$. Thus:

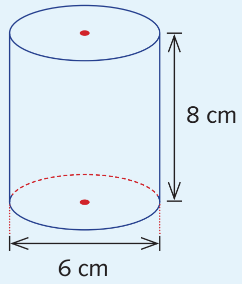
Curved surface area of cylinder $= 2\pi rh$



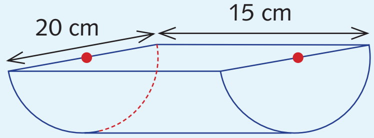
Example 2

Calculate the surface area of each solid, correct to two decimal places.

a



b



Solution

a This is a cylinder with radius 3 cm and height 8 cm.

$$\begin{aligned}\text{Area of curved surface} &= 2\pi rh\\ &= 2\times\pi\times 3\times 8\\ &= 48\pi\ \text{cm}^2\end{aligned}$$

$$\begin{aligned}\text{Area of a circular end} &= \pi r^2\\ &= \pi\times 3^2\\ &= 9\pi\ \text{cm}^2\end{aligned}$$

$$\begin{aligned}\text{Hence, } S &= 48\pi + 9\pi + 9\pi\\ &= 66\pi\ \text{cm}^2\\ &= 207.35\ \text{cm}^2 \qquad \text{(Correct to two decimal places.)}\end{aligned}$$

b

$$\begin{aligned}\text{Area of curved section} &= \frac{1}{2}\times 2\pi rh\\ &= \pi\times 10\times 15\\ &= 150\pi\ \text{cm}^2\end{aligned}$$

$$\begin{aligned}\text{Area of a semicircle} &= \frac{1}{2}\times\pi r^2\\ &= \frac{1}{2}\pi\times 10^2\\ &= 50\pi\ \text{cm}^2\end{aligned}$$

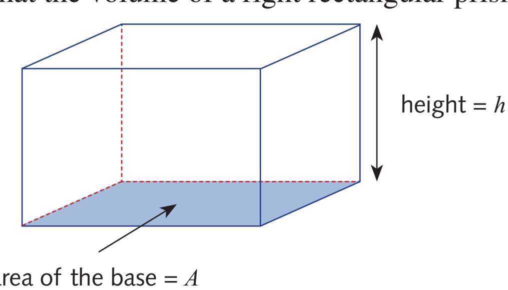
$$\begin{aligned}\text{Area of top rectangle} &= 20\times 15\\ &= 300\ \text{cm}^2\end{aligned}$$

$$\begin{aligned}S &= 150\pi + 50\pi + 50\pi + 300\\ &= (250\pi + 300)\ \text{cm}^2\\ &\approx 1085.40\ \text{cm}^2 \qquad \text{(Correct to two decimal places.)}\end{aligned}$$

Volume

Volume of a rectangular prism

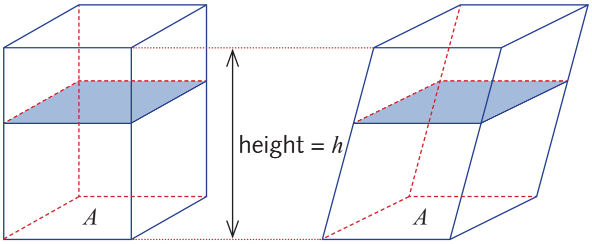
We have seen in earlier work that the volume of a right rectangular prism is given by:



$$\begin{aligned}\text{Volume of a right rectangular prism} &= \text{area of base}\times\text{height}\\ &= A\times h\end{aligned}$$

That is, $V = A\times h$

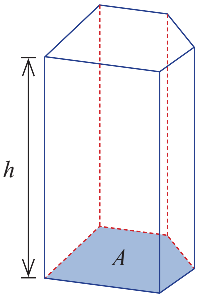
Suppose that we have two solids of the same height. If the cross-sections of the two solids, taken at the same distance above the base, have the same area, it can be shown that the solids have the same volume. This is known as **Cavalieri's principle**.



Cross-sectional areas are the same.

Cavalieri's principle allows us to say that the volume V of *any* rectangular prism, right or oblique, is given by $V = A \times h$.

Volume of a prism



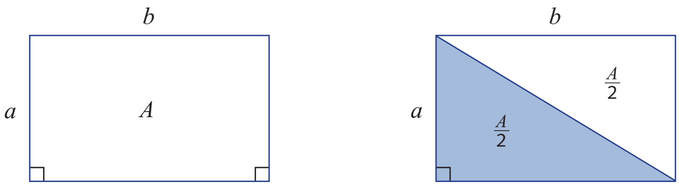
The volume V of a prism, right or oblique, is given by the formula:

$$V = Ah$$

where A is the area of the base and h is the height, as discussed previously.

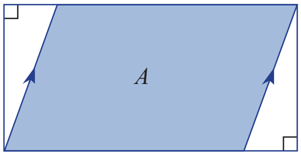
The proof is in five steps. In all of the following, the height is h.

Step 1: The base of the prism is a right-angled triangle



The right rectangular prism has volume $V = Ah$. If it is cut in half, we obtain two prisms of the same volume $\frac{Ah}{2}$ base area $\frac{A}{2}$ and height h.

Step 2: The base of the prism is a parallelogram of area A



$A = R - 2T$, where R is the area of the rectangle and T is the area of each right angled triangle.

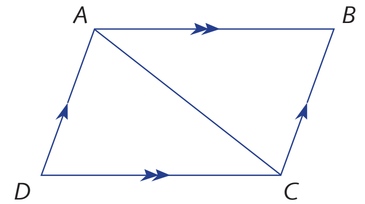
The volume of the prism:

$$\begin{aligned} V &= hR - 2hT \\ &= h(R - 2T) \\ &= hA \end{aligned}$$

So the formula holds if the base of the prism is a parallelogram.

Step 3: The base is a triangle *ABC*

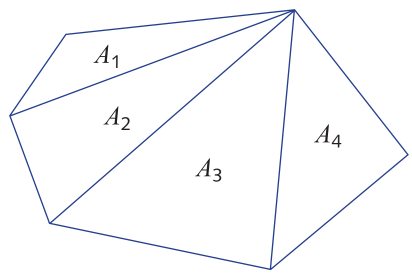
The area of the triangle ABC is half the area of the parallelogram $ABCD$.



The prism with triangular base ABC is half the volume of the prism with the parallelogram $ABCD$ as the base. So the formula holds if the base of the prism is a triangle.

Step 4: The base of the prism is a polygon

As an example, the convex hexagon can be cut up into four triangles.



Clearly the volume of the prism is:

$$\begin{aligned} V &= A_1h + A_2h + A_3h + A_4h \\ &= (A_1 + A_2 + A_3 + A_4)h \\ &= Ah, \end{aligned}$$

where A is the area of the hexagon.

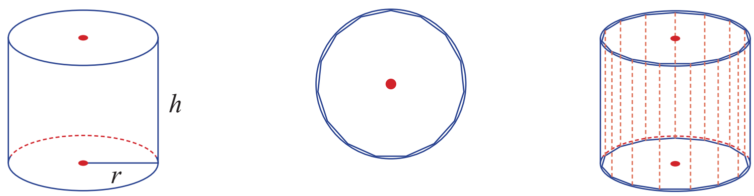
This argument applies to any right prism.

Step 5: Oblique prisms

Use Cavalieri's principle exactly as above.

Volume of a cylinder

The area of a regular polygon inscribed in a circle approximates the area of that circle. The greater the number of sides, the better the approximation. A cylinder has a circular base. Since $V = Ah$ holds for any polygon-based prism, it seems reasonable that the volume of the cylinder should be the area of its circular base multiplied by the height.



Thus, the volume of a cylinder with radius r and height h is equal to the area of the circular cross-section, πr^2, multiplied by the height, h.

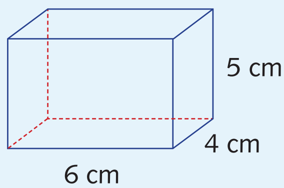
$$\text{Volume of a cylinder} = \pi r^2 h$$

Cavalieri's principle shows that this also applies to an oblique cylinder.

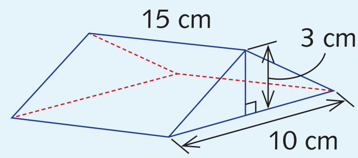
Example 3

Calculate the volume of each solid.

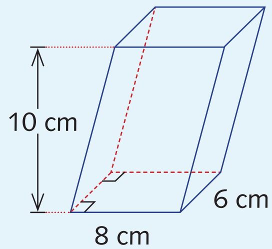
a



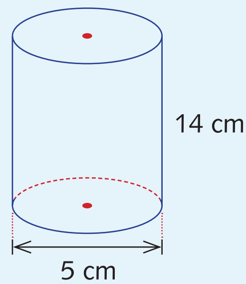
b



c



d



Solution

a $V = Ah$
$= 6 \times 4 \times 5$
$= 120\text{ cm}^3$

b The solid is a triangular prism.

$V = Ah$
$= (\text{area of triangular base}) \times (\text{height of prism})$
$= \left(\frac{1}{2} \times 10 \times 3\right) \times 15$
$= 225\text{ cm}^3$

c This is a rectangular prism of height 10 cm.

$V = Ah$
$= 8 \times 6 \times 10$
$= 480\text{ cm}^3$

d This is a cylinder of radius 2.5 cm and height 14 cm.

$V = \pi r^2 h$
$= \pi \times 2.5^2 \times 14$
$= 87.5\pi\text{ cm}^3$

Surface area and volume

- A **prism** is a polyhedron with two parallel congruent polygonal faces and all other faces parallelograms.
- The **surface area** of a prism is the sum of the areas of its faces.
- The surface area of the curved part of a cylinder with radius r and height h is given by:

  Curved surface area $= 2\pi rh$
- The volume of a prism is given by the product of the area of the base A and the height h:

  $V = Ah$
- The volume of a cylinder with radius r and height h is given by:

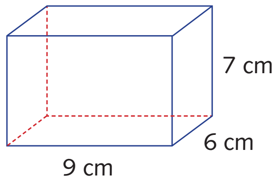
  $V = \pi r^2 h$

Exercise 6A

Example 1, 2

1 Calculate the surface area of each solid.

a



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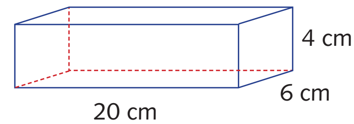

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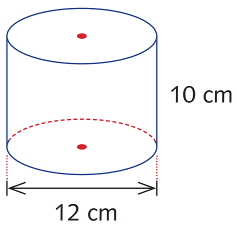

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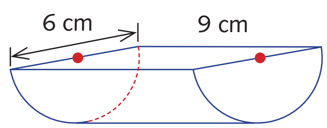

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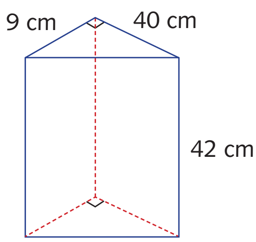

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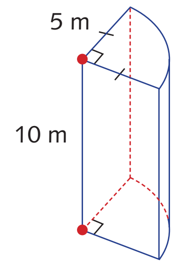

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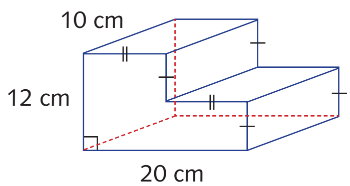

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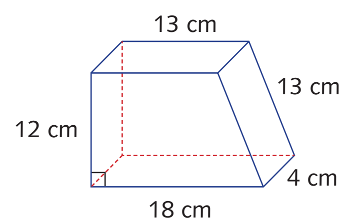

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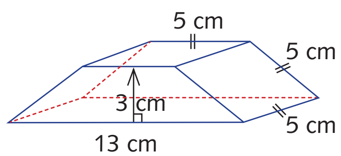

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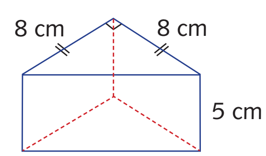


2 The base of a right prism of height 12 cm is an equilateral triangle.

a If the side length of the triangle is 6 cm, use Pythagoras' theorem to calculate the height of the triangle.

b Calculate the surface area of the prism.

3 A cube has surface area 486 m^2. Find its side length.

4 A cylinder of base radius 8 cm has curved surface area 72π cm^2. Find its height.

5 A rubbish bin is in the shape of an open cylinder.

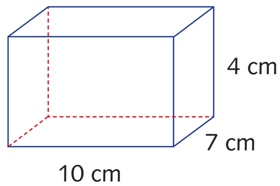
a If the bin has radius 15 cm and height 40 cm, find its total surface area (do not include the lid).

b If the bin has radius 15 cm and curved surface area 3500 cm^2, find the height of the bin, correct to one decimal place.

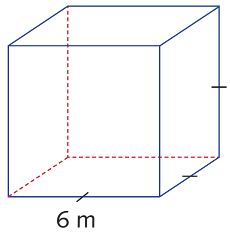
Example 3

6 Calculate the volume of each solid.

a



b



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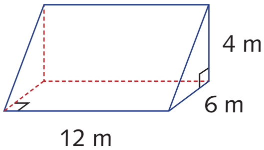

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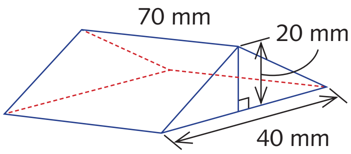

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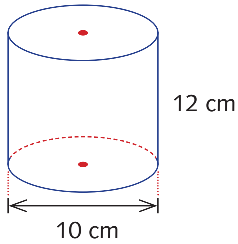

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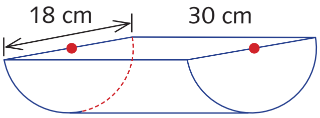

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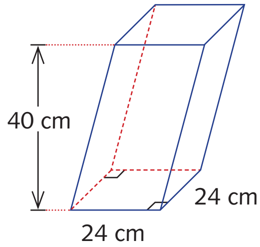

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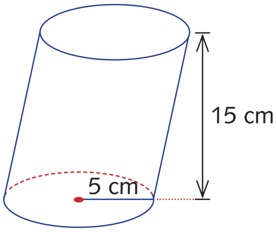

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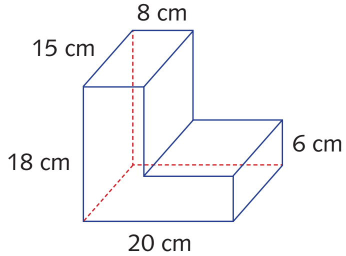

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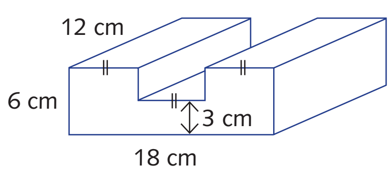

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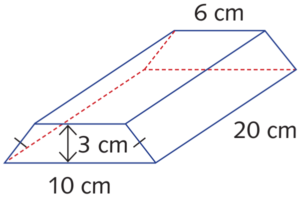

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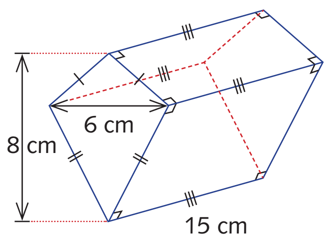
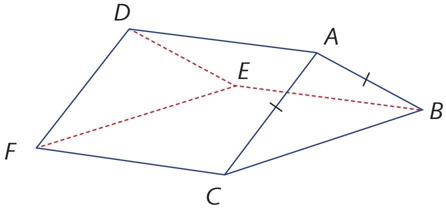


7 In the diagram below, $ABCFED$ is a right triangular prism, with $AB = AC$.



a If $AB = 8$ cm and $BC = 12$ cm, find the height of ΔABC, correct to two decimal places.

b If $AD = 24$ cm, find the volume of the prism, correct to two decimal places.

8 A manufacturer of drink cans produces cylindrical cans with a volume of 1000 cm^3.

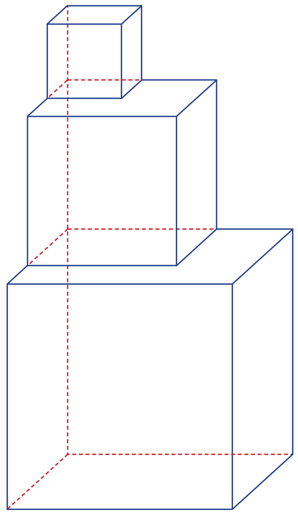
a If the radius of the can is 4 cm, find the height of the can, correct to one decimal place.

b If the height of the can is 8 cm, find the radius of the can, correct to two decimal places.

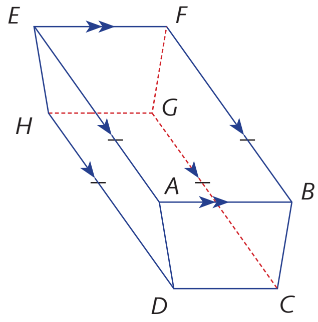
9 **a** If a cube has volume 64 cm^3, find its side length.

b If a cube has volume 400 cm^3, find its side length, correct to one decimal place.

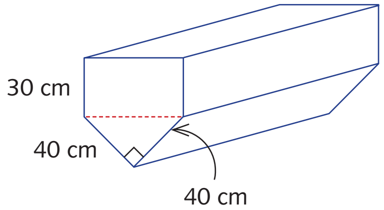
10 A modern sculpture consists of three cubes, with side lengths 0.5 m, 1 m and 1.5 m, respectively, placed on top of each other as shown in the diagram opposite. Calculate the surface area of the sculpture. Do not include the base of the large cube in your calculations.



11 The figure shows a trapezoidal prism. Its ends $ABCD$ and $EFGH$ are congruent trapezia, with $DC = 4$ cm and $AB = 6$ cm. If $AE = 15$ cm and the volume is 300 cm^3, find the height of the trapezium $ABCD$.



12 A farmer is making a trough that needs to be filled with water once each day. The trough is in the shape of a prism with pentagonal ends, as shown opposite. The farmer has 60 horses that each drink about 10 L of water per day. Using the fact that 1 cm^3 = 1 mL, what is the smallest length the trough needs to be to water the horses each day?

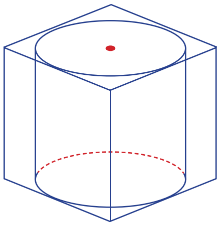


13 A cylindrical vase of radius 5 cm and height 14 cm just fits into a box. Find:

a the volume of the cylinder, correct to two decimal places

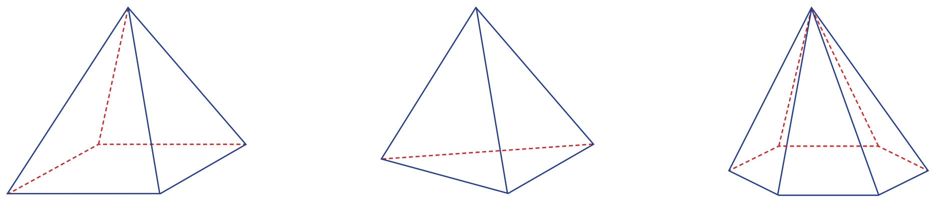
b the volume of the box

c the volume of unused space inside the box, correct to two decimal places



6B Pyramids

A **pyramid** is a polyhedron with a polygonal base and triangular sides that meet at a point called the **vertex**. The pyramid is named according to the shape of its base.

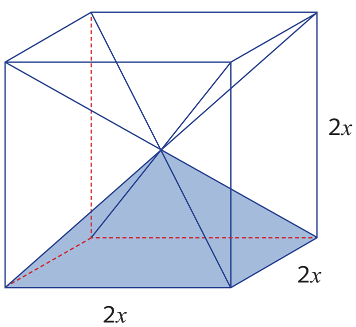


If we drop a perpendicular from the vertex of the pyramid to the base, then the length of the perpendicular is called the **height** of the pyramid.

A **right** pyramid is one whose vertex is directly above the centre of its base.

Volume of a pyramid

Here is a method for determining the formula for the volume of a right, square-based pyramid.



Consider a cube of side length $2x$. If we draw the long diagonals as shown, then we obtain 6 square pyramids, one of which is shaded in the diagram. Each of these pyramids has base area $2x \times 2x$ and height x. Let V be the volume of each pyramid.

The volume of the cube is $(2x)^3 = 8x^3$. Hence:

$$6 \times V = 8x^3$$

so $$V = \frac{4}{3} \times x^3$$

Now the area of the base of each pyramid is $(2x)^2 = 4x^2$ and the height of each pyramid is x, so in this case we can write:

$$\begin{aligned} V &= \frac{4}{3} \times x^3 \\ &= \frac{1}{3} \times 4x^2 \times x \end{aligned}$$

or $$V = \frac{1}{3} \times \text{area of base} \times \text{height}$$

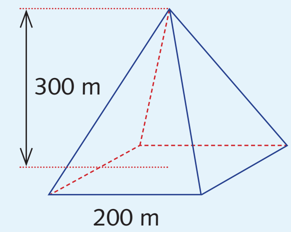
It is possible to extend this result to any pyramid by using geometric arguments and Cavalieri's principle. We then have the following important result.

$$\text{Volume of a pyramid} = \frac{1}{3} \times \text{base area} \times \text{height}$$

That is, $V = \frac{1}{3}Ah$

Example 4

Calculate the volume of a square pyramid with base of side length 200 m and height 300 m.



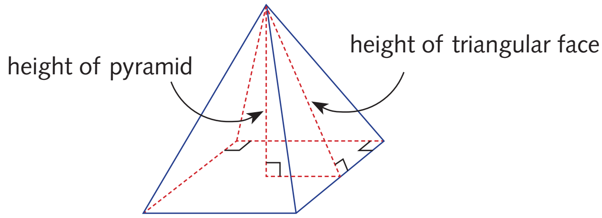
Solution

$$V = \frac{1}{3} \times \text{base area} \times \text{height}$$
$$= \frac{1}{3} \times 200 \times 200 \times 300$$
$$= 4\,000\,000 \text{ m}^3$$

The volume of the pyramid is $4\,000\,000 \text{ m}^3$.

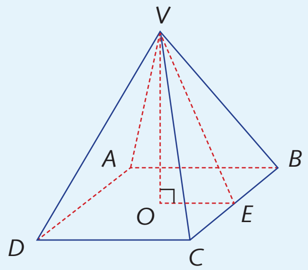
Surface area of a pyramid

To find the surface area of a pyramid, we need to find the areas of the surfaces bounding the pyramid, which will consist of a number of triangles together with the base. You will often have to use Pythagoras' theorem in calculating the surface area of a pyramid.



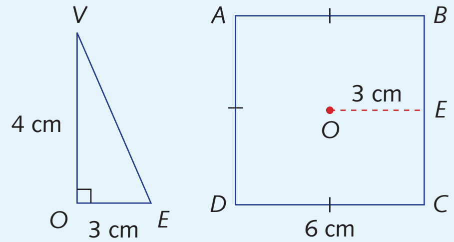
Example 5

VABCD is a right, square-based pyramid with vertex *V* and base *ABCD*, with *V* vertically above the centre of the square base. The height of the pyramid is 4 cm and the side length of the base is 6 cm. Find the surface area of the pyramid.



Solution

We need to find the height VE of triangle VBC, using Pythagoras' theorem.



$$\begin{aligned} VE^2 &= VO^2 + OE^2 \\ &= 4^2 + 3^2 \\ &= 25 \end{aligned}$$

Hence, $VE = 5$ cm.

$$\begin{aligned} \text{Area of } \Delta VCB &= \frac{1}{2} \times CB \times VE \\ &= \frac{1}{2} \times 6 \times 5 \\ &= 15\text{ cm}^2 \end{aligned}$$

$$\begin{aligned} \text{Area of base} &= 6 \times 6 \\ &= 36\text{ cm}^2 \end{aligned}$$

$$\begin{aligned} \text{Surface area} &= 4 \times 15 + 36 \\ &= 96\text{ cm}^2 \end{aligned}$$

Hence, the surface area of the pyramid is 96 cm^2.

Pyramids

- A **pyramid** is a polyhedron with a polygonal base and triangular faces that meet at a point called the **vertex**.
- The pyramid is named according to the shape of its base.
- The volume of a pyramid with base area A and height h is given by:

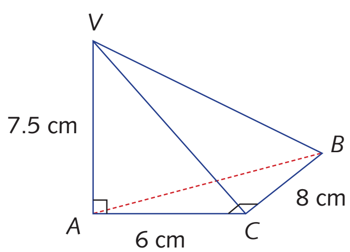
$$V = \frac{1}{3}Ah$$

Exercise 6B

In this exercise, assume all pyramids are right pyramids unless otherwise stated.

Example 4

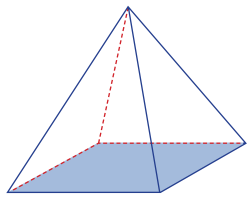
1 In the diagram opposite, $VABC$ is a triangular pyramid. The base ΔABC is right-angled, with $AC = 6$ cm and $BC = 8$ cm. The vertex V of the pyramid is vertically above A, and $VA = 7.5$ cm. Calculate the volume of the pyramid.



Example 4

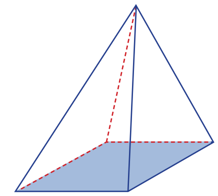
2 Calculate the volume of each of the following pyramids. The area, A cm^2, of the shaded face and the height, h cm, are given in each case.

a



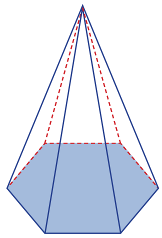
$A = 24$, $h = 10$

b



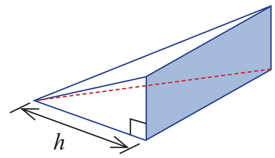
$A = 20$, $h = 15$

c



$A = 72$, $h = 9$

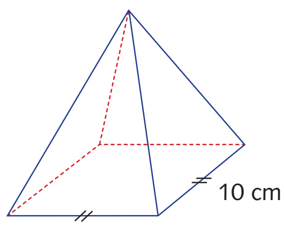
d



$A = 45$, $h = 12$

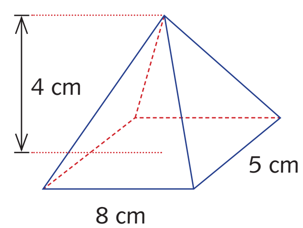
3 Calculate the volume of each pyramid.

a

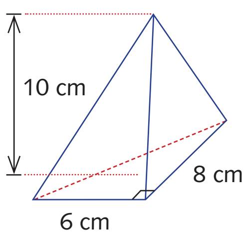


height = 12 cm

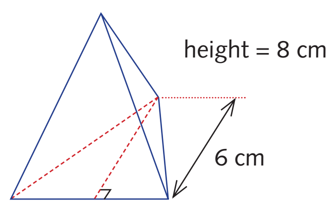
b



c



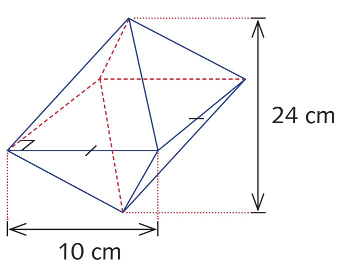
d



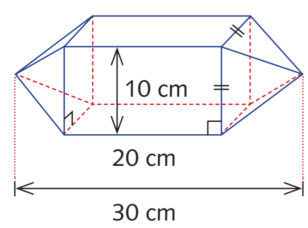
4 When it was built, the Great Pyramid of Cheops in Egypt had a height of 145.75 m and its base was a square of side length 229 m. Find its volume in cubic metres, correct to one decimal place.

5 Calculate the volume of each solid, giving the answer correct to two decimal places.

a

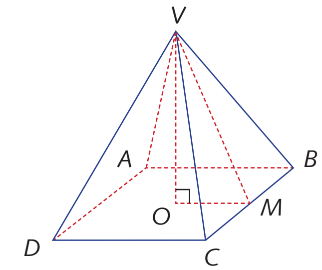


b



Example 5

6 In the diagram opposite, $VABCD$ is a square pyramid and O is the centre of square $ABCD$. If the height of the pyramid $VO = 12$ cm and $AB = 10$ cm, find:



a the length OM, where M is the midpoint of BC

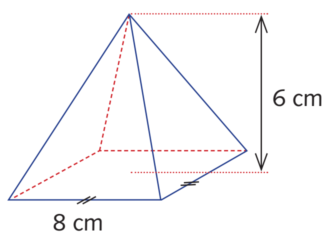
b the length VM

c the area of ΔVCB

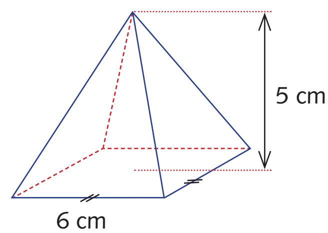
d the surface area of the pyramid

7 Use a technique similar to that in question **6** to find the surface areas of these square pyramids, assuming the vertex is above the centre of the square.

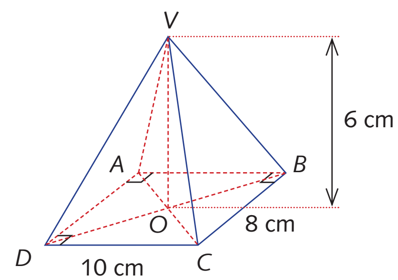
a



b



8 In the diagram opposite, $VABCD$ is a rectangular pyramid. If $VO = 6$ cm is the height of the pyramid, $CD = 10$ cm and $BC = 8$ cm, find:

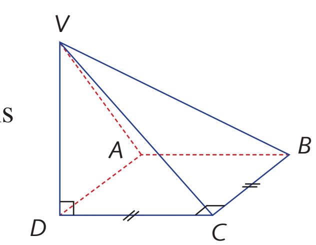


a the height of ΔVBC

b the height of ΔVDC

c the surface area of the pyramid

9 In the diagram opposite, $VABCD$ is a square pyramid, with vertex V directly above D. If $AB = 8$ cm and $VD = 6$ cm:



a name each triangle in the diagram and state what type of triangle it is

b find the length of:

i VC **ii** VB **iii** VA

c find the surface area of the pyramid

6C Cones

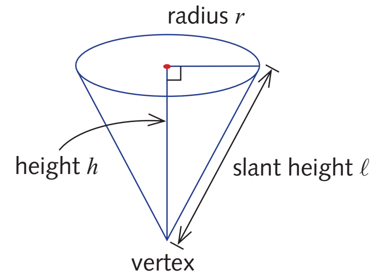
A **cone** is a solid that is formed by taking a circle and a point, called the **vertex**, which lies above or below the circle. We then join the vertex to each point on the circle. Of course, the cone can be in any orientation.

If the vertex is directly above or below the centre of the circular base, we call the cone a **right cone**. In this section, the only cones we consider are right cones, which we will simply call cones.

The distance from the vertex of the cone to the centre of the circular base is called the **height**, h, of the cone.

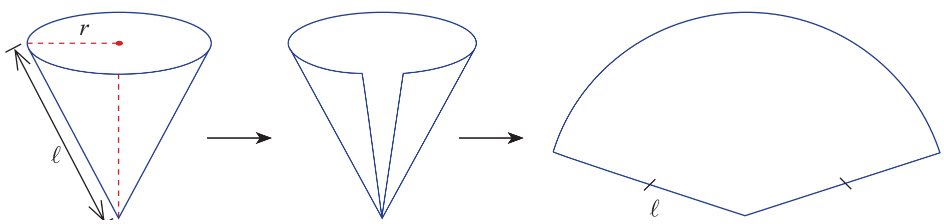
The distance from the vertex of the cone to the circumference of the circular base is called the **slant height**, ℓ, of the cone.

We will use a dot (•) to indicate the centre of the base.

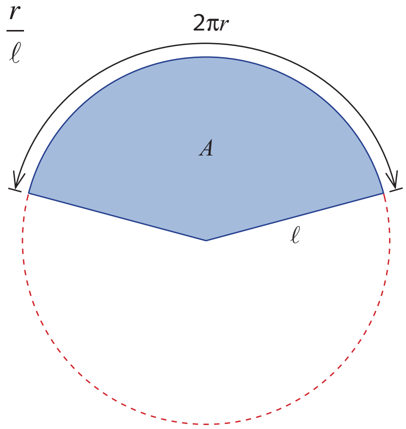


Surface area of a cone

To calculate the surface area of a cone, we need to find the area of each surface. To find the area of the curved surface, we cut and open up the curved surface to form a sector, as shown below.



- The arc length of the sector $=$ circumference of the circular base of the cone $= 2\pi r$
- The proportion of a whole circle $= \dfrac{\text{arc length}}{\text{whole circumference}} = \dfrac{2\pi r}{2\pi\ell} = \dfrac{r}{\ell}$
- Area of sector $A = \left(\dfrac{r}{\ell}\right) \times \pi\ell^2 = \pi r\ell$



Thus, the surface area of the curved part of a cone is given by:

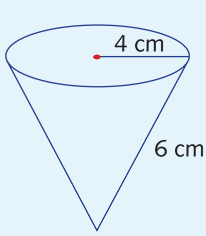
Curved surface area $= \pi r\ell$,

where r is the base radius and ℓ is the slant height.

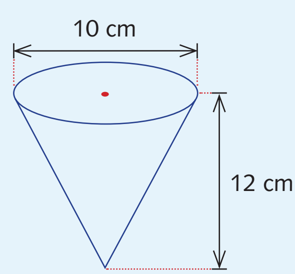
Example 6

Find the surface area of each cone. Give your answer correct to two decimal places.

a



b



Solution

a We have $r = 4$ and $\ell = 6$.

$$\begin{aligned}\text{Area of curved surface} &= \pi r \ell \\ &= \pi \times 4 \times 6 \\ &= 24\pi \text{ cm}^2\end{aligned}$$

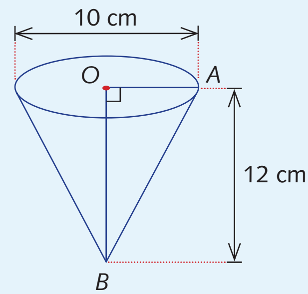
$$\begin{aligned}\text{Area of base} &= \pi r^2 \\ &= 16\pi \text{ cm}^2\end{aligned}$$

$$\begin{aligned}\text{Thus, } S &= 24\pi + 16\pi \\ &= 40\pi \text{ cm}^2 \\ &\approx 125.66 \text{ cm}^2 \qquad \text{(Correct to two decimal places.)}\end{aligned}$$

Hence, the surface area of the cone is approximately 125.66 cm^2.

b We first find the slant height AB.

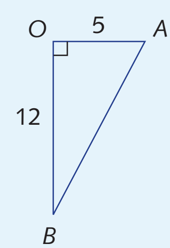
By Pythagoras' theorem:



$$\begin{aligned}AB^2 &= 5^2 + 12^2 \\ &= 169 \\ AB &= 169 \text{ cm}\end{aligned}$$

Hence, $r = 5$ and $\ell = 13$.

$$\begin{aligned}\text{Area of curved surface} &= \pi r \ell \\ &= \pi \times 5 \times 13 \\ &= 65\pi \text{ cm}^2\end{aligned}$$



$$\begin{aligned}\text{Area of base} &= \pi r^2 \\ &= 25\pi \text{ cm}^2\end{aligned}$$

$$\begin{aligned}\text{Then } S &= 65\pi + 25\pi \\ &= 90\pi \text{ cm}^2 \\ &\approx 282.74 \text{ cm}^2 \qquad \text{(Correct to two decimal places.)}\end{aligned}$$

Hence, the surface area of the cone is approximately 282.74 cm^2.

Example 7

The curved surface area of a cone is 44 cm^2 and the base radius is 2 cm. Find, correct to two decimal places:

a the slant height of the cone

b the height of the cone

Solution

a Curved surface area of a cone $= \pi r \ell$

Here $r = 2$, so $2\pi\ell = 44 \text{ cm}^2$.

$$\begin{aligned}\ell &= \frac{22}{\pi} \text{ cm} \\ &\approx 7.00 \text{ cm} \qquad \text{(Correct to two decimal places.)}\end{aligned}$$

Hence, the slant height is approximately 7.00 cm.

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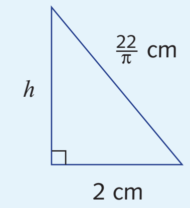
b Using Pythagoras' theorem:



$$h^2 + 2^2 = \left(\frac{22}{\pi}\right)^2$$

$$\text{so} \quad h = \sqrt{\left(\frac{22}{\pi}\right)^2 - 2^2}$$

$$\approx 6.71 \text{ cm} \qquad \text{(Correct to two decimal places.)}$$

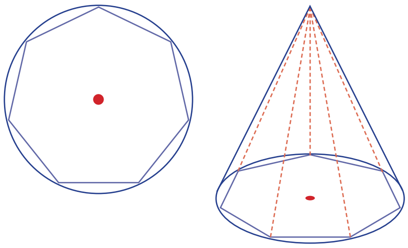
Hence, the height of the cone is approximately 6.71 cm.

Volume of a cone

The formula for the volume of a cone is the same as the formula for the volume of a pyramid, which is $\frac{1}{3} \times$ area of the base $\times$ height. For a cone, the base area is πr^2, so:

$$\text{Volume of a cone} = \frac{1}{3}\pi r^2 h,$$

where r is the radius of the base and h is the height.



To illustrate this informally, imagine constructing a polygon inside the circular base of the cone and joining the vertex of the cone to each of the vertices of the polygon.

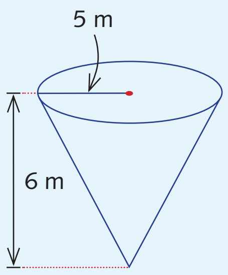
This would give us a polygonal pyramid with volume equal to:

$$\frac{1}{3} \times \text{area of the base} \times \text{height}$$

The more sides we take in the polygon, the area of the base gets closer and closer to πr^2, so the volume of the cone equals $\frac{1}{3}\pi r^2 h$.

Example 8

Calculate the volume of a cone with base radius 5 m and height 6 m.



Solution

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3} \times \pi \times 5^2 \times 6$$
$$= 50\pi \text{ m}^3$$

The volume of the cone is 50π m^3.

Right cones

- In a **cone**, the distance between the vertex and the centre of the base is called the **height**, h, of the cone.
- The length of a straight line joining the vertex to a point on the circumference of the circle is called the **slant height**, ℓ, of the cone.
- The surface area of the curved part of a cone is given by $\pi r\ell$, where r is the base radius and ℓ is the slant height.
- The volume of a cone is given by:

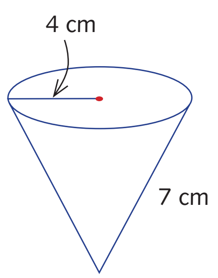
  $V = \frac{1}{3}\pi r^2 h$, where r is the base radius and h is the height.

Exercise 6C

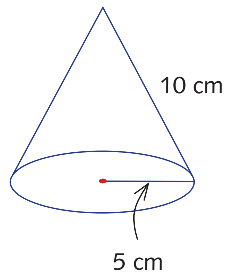
Example 6

1 Calculate the total surface area of each cone, including the base. Give your answers correct to two decimal places.

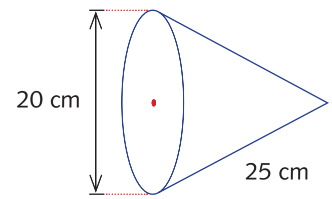
a



b

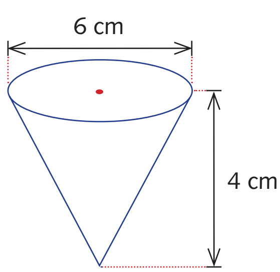


c



2 Calculate the total surface area of each cone.

a



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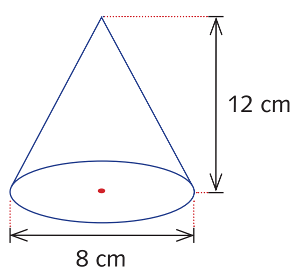

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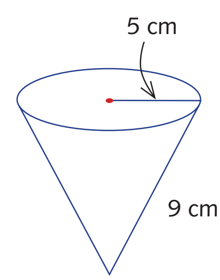


Example 7

3 The curved surface of a cone of base radius 4 cm has surface area 40π cm^2. Find:

a the slant height of the cone

b the height of the cone, correct to four decimal places

4 A cone has base radius 10 cm and total surface area 1000 cm^2. Find, correct to two decimal places:

a the surface area of the curved part of the cone

b the slant height of the cone

c the height of the cone

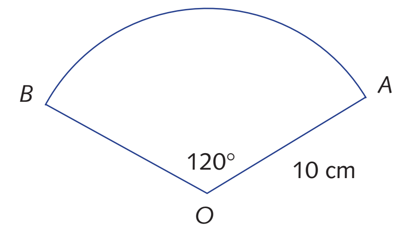
5 **a** Calculate the area of the sector shown opposite. Give your answer correct to two decimal places.

b If the radii *OA* and *OB* are joined together to form the curved surface of a cone, find, correct to two decimal places:

i the slant height of the cone

ii the base radius of the cone

iii the height of the cone



6 A cone has radius r cm and height h cm.

a If $r = 5$ and $h = 10$, find:

i the slant height of the cone

ii the surface area of the curved part of the cone

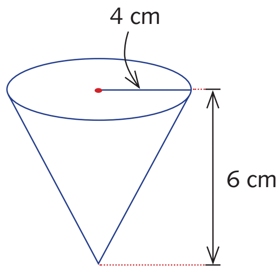
iii the angle of the sector we get if we cut the curved part of the cone

b If $r = h$, find the angle of the sector that produces the curved part of the cone.

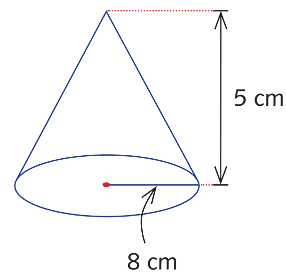
Example 8

7 Calculate the volume of each solid. Give your answers correct to two decimal places.

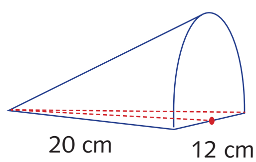
a



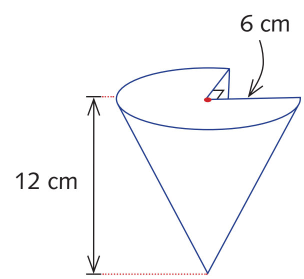
b



c



d

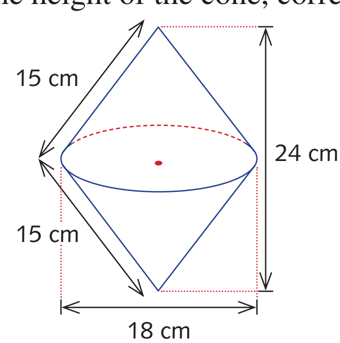


8 A cone with diameter 6 cm has a volume of 120 cm^3. Find the height of the cone, correct to the nearest millimetre.

9 For the solid shown, find:

a the volume

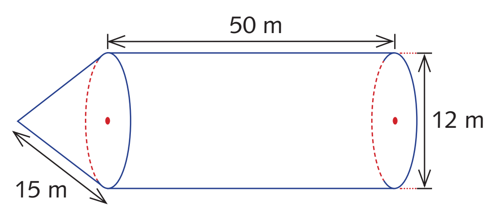
b the total surface area



10 For the solid shown, find:

a the volume

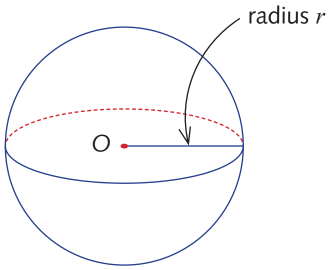
b the total surface area



11 A mound of earth is shaped like a cone. It is 6 metres high with a radius of 25 metres. Find the cost, to the nearest dollar, of moving the mound if it costs $5 to move one cubic metre.

6D Spheres

The word **sphere** comes from the Greek word *sphaira*, meaning 'ball'. Every point on the surface of a sphere lies at a distance r, called the **radius** of the sphere, from a fixed point O, called the **centre** of the sphere. It is difficult to derive the formulas for the surface area and volume of a sphere. These formulas are discussed in the Challenge exercises.



We will use a dot (•) to indicate the centre of a sphere.

Surface area of a sphere

The surface area of a sphere is given by:

$$S = 4\pi r^2,$$

where r is the radius of the sphere.

Example 9

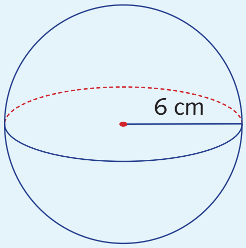
Calculate, correct to two decimal places, the surface area of a sphere:

a with radius 6 cm

b with diameter 10 cm

Solution

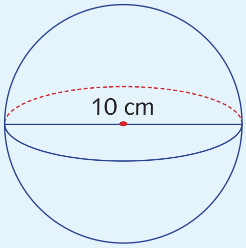
a We have



$$\begin{aligned} S &= 4\pi r^2 \\ &= 4 \times \pi \times 6^2 \\ &= 144\pi \text{ cm}^2 \\ &\approx 452.39 \text{ cm}^2 \quad \text{(Correct to two decimal places.)} \end{aligned}$$

The surface area is approximately 452.39 cm^2.

b The diameter = 10 cm, so r = 5 cm.



$$\begin{aligned} \text{Then } S &= 4\pi r^2 \\ &= 4 \times \pi \times 5^2 \\ &= 100\pi \text{ cm}^2 \\ &\approx 314.16 \text{ cm}^2 \quad \text{(Correct to two decimal places.)} \end{aligned}$$

The surface area is approximately 314.16 cm^2.

Volume of a sphere

The formula for the volume of a sphere is:

$$V = \frac{4}{3}\pi r^3,$$

where r is the radius of the sphere.

Example 10

Calculate the volume of a sphere with a diameter of 30 m.

Solution

The radius is 15 m.

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3} \times \pi \times 15^3 \\ &= 4500\pi \text{ m}^3 \end{aligned}$$

Example 11

A sphere has volume 2800 cm^3. Find the radius of the sphere, correct to the nearest millimetre.

Solution

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ 2800 &= \frac{4}{3}\pi r^3 \\ \text{so} \quad r^3 &= \frac{2800 \times 3}{4\pi} \\ r &= \sqrt[3]{\frac{2800 \times 3}{4\pi}} \\ &\approx 8.7 \text{ cm} \qquad \text{(Correct to one decimal place.)} \end{aligned}$$

The radius is approximately 87 mm.

Spheres

- The surface area of a sphere of radius r is given by:
  $S = 4\pi r^2$, where r is the radius of the sphere.
- The volume of a sphere of radius r is given by:
  $V = \frac{4}{3}\pi r^3$, where r is the radius of the sphere.

Exercise 6D

Example 9

1 Calculate the surface area, correct to two decimal places, of:

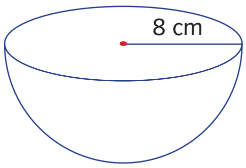
a a sphere of radius 8 cm

b a sphere of radius 15 cm

c a sphere of diameter 14 cm

d a sphere of diameter 21 cm

2 For the solid hemisphere shown opposite, find:



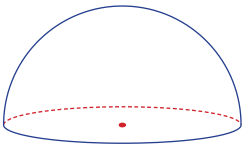
a the area of the flat surface

b the surface area of the curved part of the hemisphere

c the total surface area of the solid hemisphere

3 A sphere has a surface area of $500\ \text{cm}^2$. Find its radius, correct to four decimal places.

4 A hemispherical tent is made using $28\ \text{m}^2$ of material. Find, correct to two decimal places, the radius of the tent if:

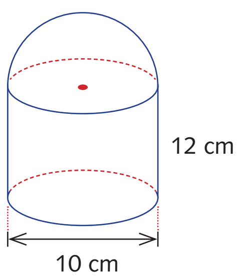


a the tent does not have a material floor

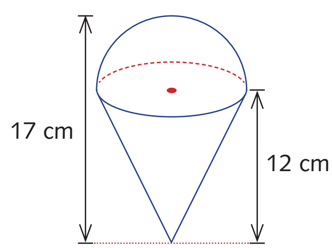
b the tent does have a material floor

5 Calculate the surface area of each object.

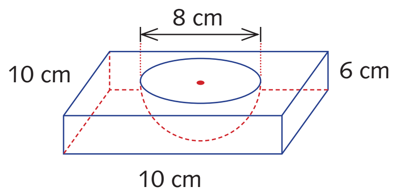
a



b



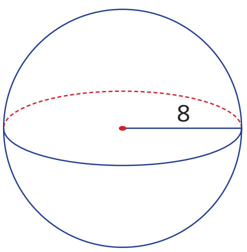
c



Example 10

6 Calculate the volume of each solid. Give your answers correct to two decimal places.

a



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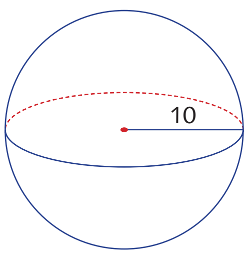

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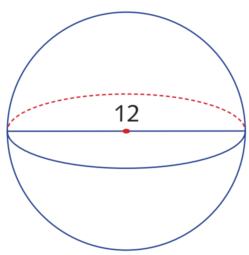

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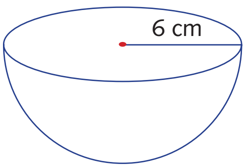

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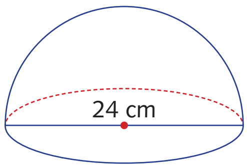
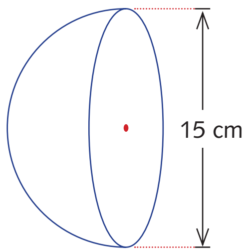


f



Example 11

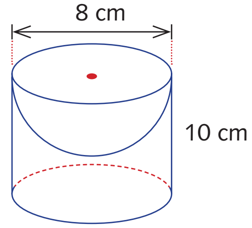
7 **a** Calculate, correct to two decimal places, the radius of a sphere with a volume of $1000\ \text{cm}^3$.

b Calculate, correct to the nearest millimetre, the diameter of a sphere with a volume of $3000\ \text{cm}^3$.

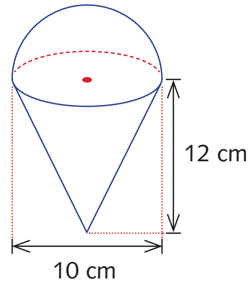
c Calculate, correct to one decimal place, the diameter of a hemispherical bowl with a volume of $250\ \text{cm}^3$.

8 Calculate the volume of each solid. Where appropriate, give your answers correct to two decimal places.

a



b



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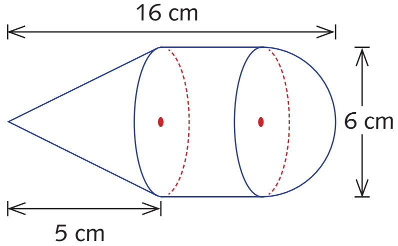


9 A spherical soccer ball of diameter 22 cm is packaged in a box that is in the shape of a cube with edges of length 24 cm. Find, correct to the nearest cm^3, the volume of unused space inside the box.

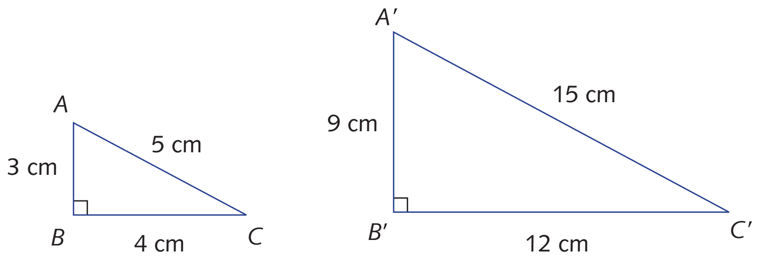
10 Tennis balls are packaged in cylindrical canisters. A tennis ball can be considered to be a sphere of diameter 70 mm. If the canister has base diameter 75 mm and height 286 mm, and each canister holds four balls, find the volume of unused space inside the canister, correct to two decimal places.

6E Enlargement

In this section, we will investigate what happens to the area and volume of figures under enlargement.

In the diagram below, $\Delta A'B'C'$ is an enlargement of ΔABC. The sides of the larger triangle are three times the lengths of those of the smaller one.

We say that the **enlargement factor** is 3.



Plane figures and enlargements

Notice what happens when we compare the areas of the two triangles. The area of ΔABC is 6 cm^2, and the area of $\Delta A'B'C'$ is 54 cm^2. In this case, the area of the larger triangle is 9 times the area of the smaller one. Since $9 = 3^2$, we see that the area of the smaller triangle is multiplied by the **square** of the enlargement factor.

Example 12

By what factor does the area of a circle with radius 2 cm change when we enlarge the radius by a factor of 5?

Solution

A circle with radius 2 cm has area $\pi \times 2^2 = 4\pi\text{ cm}^2$.

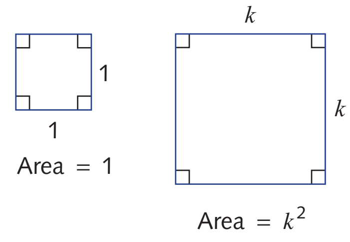
Enlarging by a factor of 5 gives a radius of 10 cm.

A circle with radius 10 cm has area $\pi \times 10^2 = 100\pi\text{ cm}^2$.

The area has been multiplied by a factor of $\dfrac{100\pi}{4\pi} = 25$.

(Notice that this is the square of the enlargement factor.)

In general, if each of the dimensions of a plane figure is enlarged by a factor of k, then the area of the figure is multiplied by a factor of k^2.



Example 13

A regular pentagon has area 45 cm^2. If the sides of the pentagon are enlarged by a factor of 8, what is the area of the resulting pentagon?

Solution

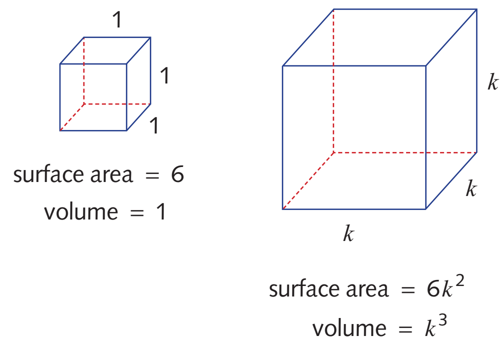
The enlargement factor is 8, so the area is multiplied by a factor of $8^2 = 64$.

Hence, the area of the resulting pentagon is $64 \times 45 = 2880\text{ cm}^2$.

Solids and enlargements

The diagram to the right enables us to draw the following general conclusions. If the dimensions of a solid are enlarged by a factor of k, then the surface area of the figure is multiplied by a factor of k^2.

If the dimensions of a solid are enlarged by a factor of k, then the volume of the figure is multiplied by a factor of k^3.



Example 14

A spherical balloon has radius 10 cm. It is inflated so that the radius becomes 15 cm. By what factor has the following changed?

a surface area **b** volume

Solution

a Method 1 – Find the two areas

For a sphere, $A = 4\pi r^2$

Hence, surface area of the balloon $= 4 \times \pi \times 10^2 = 400\pi \text{ cm}^2$

If the radius becomes 15 cm, then the surface area becomes $4 \times \pi \times 15^2 = 900\pi \text{ cm}^2$,

so the surface area has been multiplied by a factor of $\dfrac{900\pi}{400\pi} = \dfrac{9}{4}$

Method 2 – Enlargement factor method

Enlargement factor for the radius is $\dfrac{15}{10} = \dfrac{3}{2}$.

Hence, enlargement for the surface area is $\left(\dfrac{3}{2}\right)^2 = \dfrac{9}{4}$.

b Method 1 – Find the two volumes

For a sphere, $V = \dfrac{4}{3}\pi r^3$.

Hence, volume of the balloon $= \dfrac{4}{3} \times \pi \times 10^3 = \dfrac{4000}{3}\pi \text{ cm}^3$.

If the radius becomes 15 cm, then the volume becomes $\dfrac{4}{3} \times \pi \times 15^3 = 4500\pi \text{ cm}^3$, so the volume has been multiplied by a factor of $4500\pi \div \dfrac{4000\pi}{3} = \dfrac{27}{8}$.

Method 2 – Enlargement factor method

The enlargement factor for the radius is $\dfrac{3}{2}$.

Hence, the enlargement factor for the volume is $\left(\dfrac{3}{2}\right)^3 = \dfrac{27}{8}$.

Scale drawings and scale models

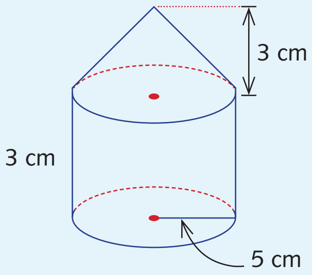
A designer, architect or engineer will often build a **scale model** of the object being designed. The real object is an enlargement of the model. A scale of, for example, 1 : 40 means that the dimensions of the real object are 40 times the dimensions of the model.

Example 15

A farmer builds a scale model of a silo, as shown opposite. The scale of the model is 1 : 100.

a Find the volume of the model.

b What is the volume of the silo, in cubic metres?



Solution

a Volume of cone $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3} \times \pi \times 5^2 \times 3$

$= 25\pi \text{ cm}^3$

Volume of cylinder $= \pi r^2 h$

$= \pi \times 5^2 \times 3$

$= 75\pi \text{ cm}^3$

Volume of model $= 25\pi + 75\pi$

$= 100\pi \text{ cm}^3$

b The enlargement factor is 100, so the volume of the silo is $100^3 \times$ volume of the model

Hence, the volume of the silo is:

$V = (100\pi \times 100^3) \text{ cm}^3$

$= 100\pi \text{ m}^3 \qquad (100^3 \text{ cm}^3 = 1 \text{ m}^3)$

Enlargement

Let $k > 0$.

- If a plane figure is enlarged by a factor of k, then the area of the figure is multiplied by a factor of k^2.
- If a solid is enlarged by a factor of k, then the surface area of the solid is multiplied by a factor of k^2.
- If a solid is enlarged by a factor of k, then the volume of the solid is multiplied by a factor of k^3.

Exercise 6E

Example 12, 13

1 Find what factor the area of the smaller figure must be multiplied by to find the area of the larger figure. Work this out by:

i calculating the areas
ii using the enlargement factor method

a

b

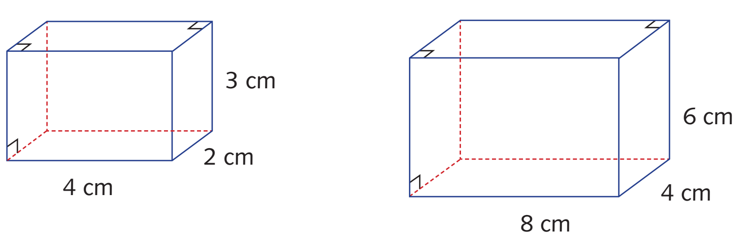
c

Example 14

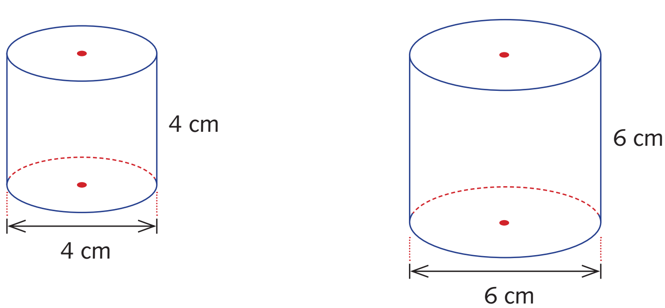
2 i Find what factor the total surface area of the smaller solid must be multiplied by in order to obtain the surface area of the larger solid. In parts **a**, **c** and **e**, do this by finding the surface areas; in parts **b** and **d**, use the enlargement factor method.

ii Find what factor the volume of the smaller solid must be multiplied by in order to obtain the volume of the larger solid. In parts **b** and **d**, do this by finding the volumes; in parts **a**, **c** and **e**, use the enlargement factor method.

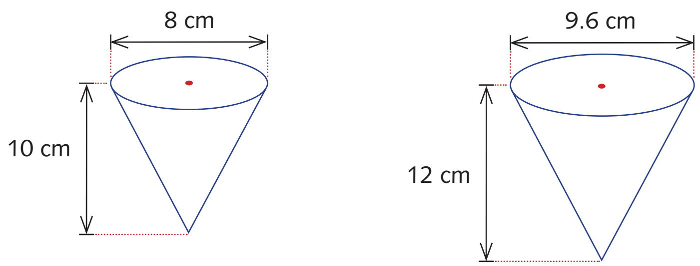
a



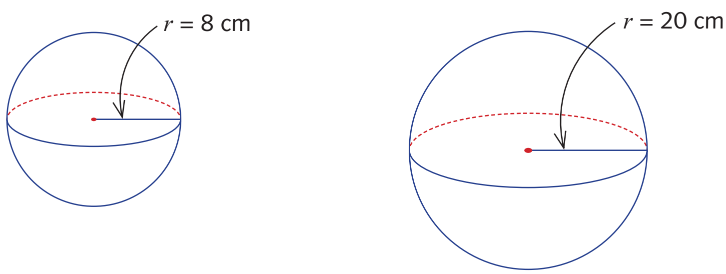
b



c



d



3 A cube has a volume of 343 cm^3. The cube is enlarged by a factor of 5.

a What is the volume of the resulting cube?

b Find the surface area of the resulting cube.

4 By what factor must the radius of a spherical balloon be multiplied if the volume is to be increased from 760 cm^3 to $389\,120\text{ cm}^3$?

5 A cylindrical container holds 125 cm^3 of liquid when full, and requires 240 cm^2 of material to manufacture. The company wants to increase the height and radius by the same enlargement factor to produce a cylindrical container that can hold 1000 cm^3 of liquid. How much material will be required to produce the new container?

6 A solid, A, is enlarged to form a new solid, B. If the surface area of B is twice the surface area of A, by what factor is the volume of A multiplied to give the volume of B?

Example 15

7 A model car has scale 1 : 24; that is, 1 cm on the model represents 24 cm on the actual car.

a If the model car requires 300 cm^2 of material to be made, how much material is required to make the actual car?

b If the volume of the actual car's interior is 4.5 m^3, find the volume of the model's interior, in cm^3 and correct to one decimal place.

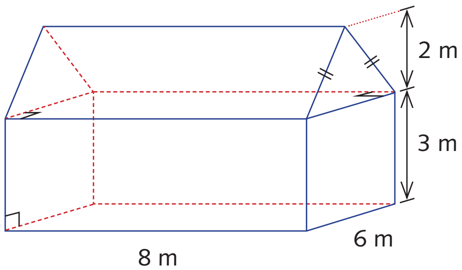
8 A barn is in the shape of a triangular prism on top of a rectangular prism, as shown below.

a Find the volume of the barn.

b If a model of scale 1 : 50 is made of the barn, find the volume of the model, in cm^3.

c Find the outside surface area of the barn (not including the floor).

d How much material, in cm^2 and correct to one decimal place, is needed to make the model?



9 **a** The base of a triangle is increased by a factor of 7, while the height is kept the same. What happens to the area of the triangle?

b The sides of the square base of a cube are each increased in length by a factor of 6, but the height is kept the same. This produces a square prism. By what factor has the volume changed?

c The radius of a cylinder is trebled, but the height is kept the same. By what factor has the volume changed?

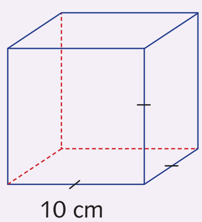
d The radius of a cone is multiplied by 9, but the height is kept the same. By what factor has the volume changed?

Review exercise

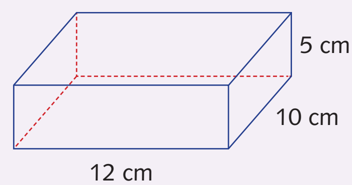
1 For each solid, calculate:

i the volume
ii the surface area

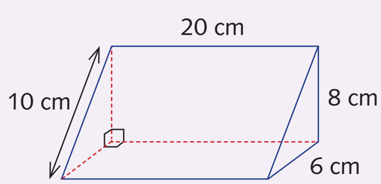
a



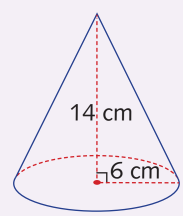
b



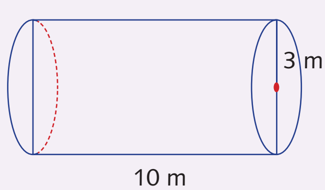
c



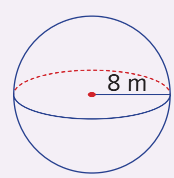
d



e



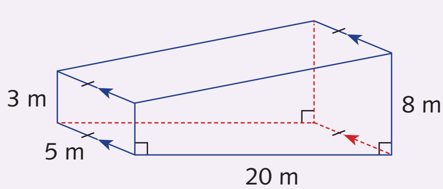
f



2 For each solid, calculate:

i the volume
ii the surface area

a



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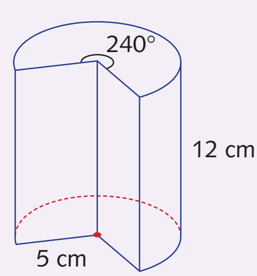
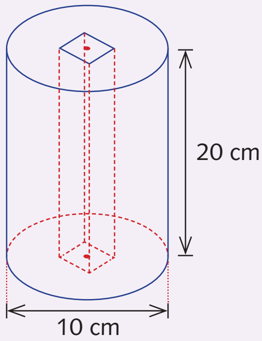
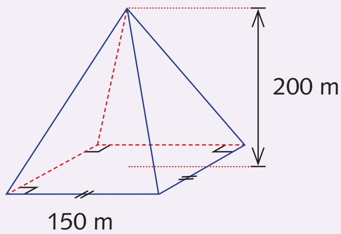


3 A solid cylinder has a shaft with a square cross-section through it, as shown in the diagram below. The volume of the solid is $(500\pi - 80)$ cm^3. Calculate the length of a side of the square cross-section.



4 Calculate the volume of each solid.

a



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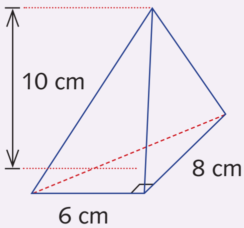

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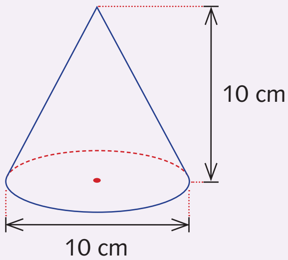

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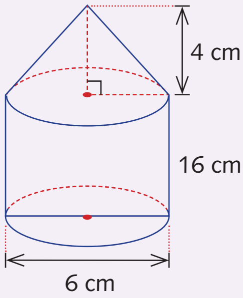
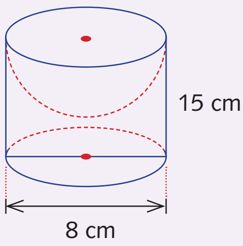


5 A triangular pyramid has all its edges equal in length. This is called a **regular tetrahedron**. Calculate the length of each edge, given that the surface area is $64\sqrt{3}$ cm^2.

6 For each solid shown, calculate:

i the volume
ii the surface area

a



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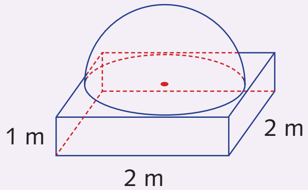
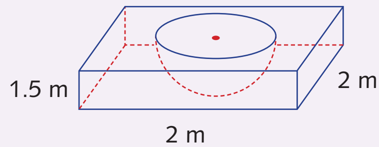


7 A hemispherical bowl is carved out of a solid block of marble, as shown.



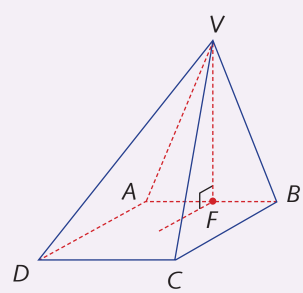
After the bowl is carved, the volume of marble remaining is $\left(6 - \frac{144}{125}\pi\right)$ m^3. Calculate the radius of the hemisphere.

8 As part of a major development, an architect designed a building and had a model made to a scale of 1 : 400; that is, 1 cm on the model is 4 m on the building.

a The external surface area of the building is 10 600 m^2. Calculate the external surface area of the model, in cm^2.

b The volume of the building is 60 000 m^3. Calculate the volume of the model, in cm^3.

9 In the pyramid below, F is the midpoint of AB. VF is vertical and is 5 m in length. $ABCD$ is horizontal and rectangular, with $AB = 6$ m and $BC = 8$ m.



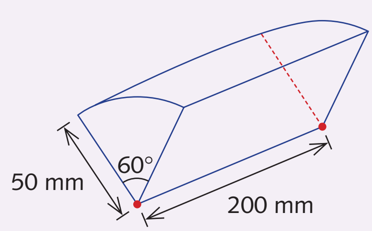
Find:

a the volume of this solid

b the surface area of this solid

10 Find the volume and surface area of the solids.

a



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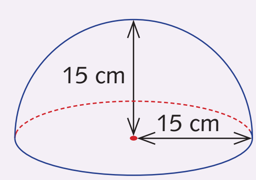

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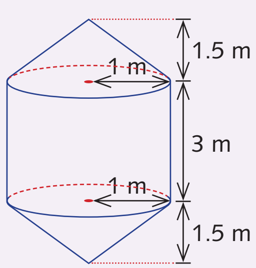

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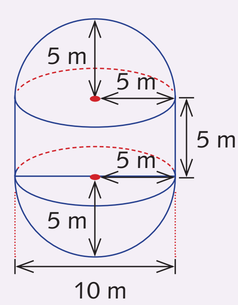
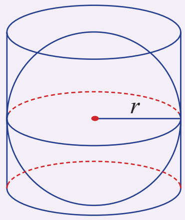


Challenge exercise

1 A **parallelepiped** is a six-faced polyhedron, each face of which is a parallelogram. A certain parallelepiped with a square base has volume 192 cm^3. Each side of its base is one-third of its height. Find the length of each side of the base.

2 The figure at the right shows a sphere of radius r fitting exactly into a cylinder. The sphere touches the cylinder at the top, bottom and curved surface. Show that the surface area of the sphere is equal to the area of the curved surface of the cylinder.

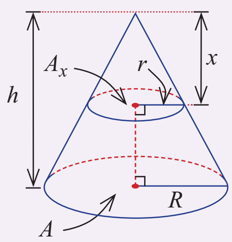


3 The surface area of a cube is x cm^2 and its volume is y cm^3. If $x = y$, find the length of the side of the cube.

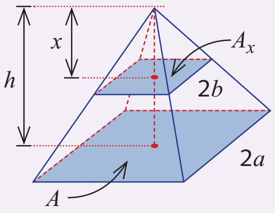
4 A cone has radius R, base area A and height h. A horizontal slice is taken at a distance x units from the vertex, as shown. Let A_x be the area of the circular slice and r be the radius.

a Use similar triangles to show that $\dfrac{r}{R} = \dfrac{x}{h}$.

b Show that the ratio $A_x : A = x^2 : h^2$.

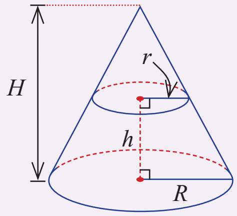


5 A square pyramid has base length $2a$, base area A and height h. A horizontal slice is taken at a distance x units from the vertex, as shown. Let A_x be the area of the square slice and $2b$ be the side length of the square slice. Use the method of question **4** to show that $A_x : A = x^2 : h^2$.



6 The portion of a right cone remaining after a smaller cone is cut off it is called a **frustum**. Suppose that the top and bottom are circles of radius r and R, respectively. Also, suppose that the height of the frustum is h and the height of the original cone is H.

a Show that the volume V of the frustum is $\dfrac{1}{3}\pi\left[H(R^2 - r^2) + r^2h\right]$.



b Use similar triangles to show that $H = \dfrac{hR}{R - r}$.

c Deduce that $V = \dfrac{1}{3}\pi h(R^2 + r^2 + rR)$.

7 **a** Use the method of question **6** to show that the volume of a truncated square pyramid with height h and square base and square top of side lengths x and y, respectively, is given by $\dfrac{1}{3}h(x^2 + y^2 + xy)$. (A truncated pyramid is formed in a similar way to a frustum.)

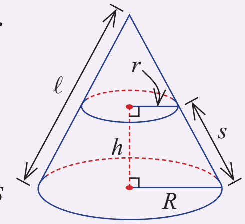
b If the top square has a side length that is half that of the bottom square, what is the ratio of the volume of the truncated pyramid to that of the whole pyramid?

8 A cone has height h and base radius r.

a Show that the surface area of the cone is $\pi r(r + \sqrt{r^2 + h^2})$.

b Suppose that the height and radius are equal. Show that the surface area is $\pi r^2(1 + \sqrt{2})$.

9 The frustum of a cone has base radii r and R, and the slant height is s.



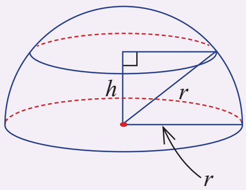
a Let the slant height of the full cone be ℓ.

Show that $\ell = \dfrac{sR}{R - r}$.

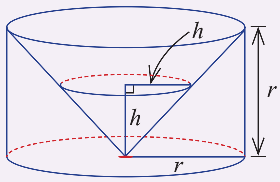
b Hence, show that the surface area of the curved section is $\pi(r + R)s$ and that the total surface area is $\pi(r^2 + R^2) + \pi(r + R)s$.

c Show that the latter can be written as $\pi(r + R)\sqrt{(R - r)^2 + h^2} + \pi(r^2 + R^2)$, where h is the height of the frustum.

10 Cavalieri's principle states that if we have two solids of the same height and the cross-sections of each solid taken at the same distance from the base have the same area, then the solids have the same volume. Take a hemisphere of radius r and look at the area of a typical cross-section at height h above the base.



Also, consider a cylinder of height r and radius r, with a cone cut out of it, also of height r and base radius r. We also take a cross-section at height h.



a Show that the radius of the circular cross-section of the sphere at height h is $\sqrt{r^2 - h^2}$.

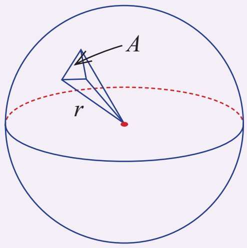
b Deduce that the area of the cross-section is $\pi(r^2 - h^2)$.

c Draw a diagram of the cross-section of the cylinder with the cone removed, and show that the area of the cross-section at height h is also $\pi(r^2 - h^2)$.

d We now conclude that the two solids have the same volume, by Cavalieri's principle. Find the volume of the cylinder minus the cone.

e Deduce the formula for the volume of the sphere.

11 Consider a sphere of radius r split up into very small pyramids, as shown. The volume of each pyramid $= \dfrac{1}{3}Ar$, where A is the area of the base of the pyramid on the surface of the sphere. The base of each pyramid is considered to be a plane surface.



Consider the sum of the volumes of these small pyramids to show that the surface area of the sphere is $4\pi r^2$.

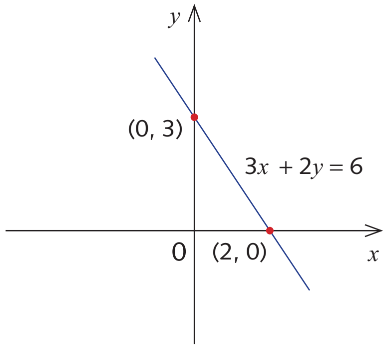
CHAPTER

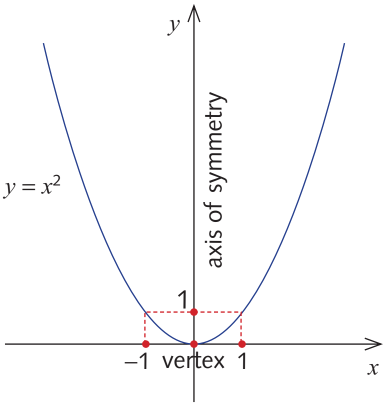
7

Number and Algebra

The parabola

In earlier years you learned how to draw curves by plotting some points from a table of values and joining them up. In particular, you should be familiar with examples such as the straight line and the parabola $y = x^2$.

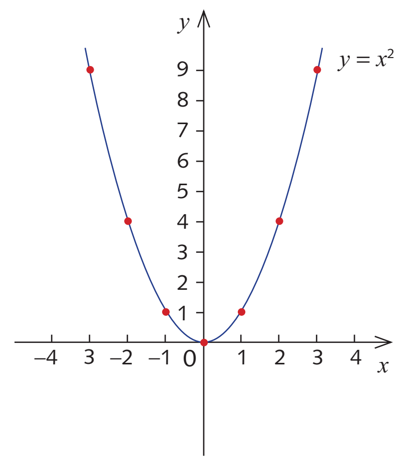




In this chapter, we will learn techniques for sketching the graphs of **quadratics** such as $y = x^2 + 1$, $y = 2x^2 - 3x$ and $y = -x^2 + 4x + 6$. These graphs are also called **parabolas**.

7A Parabolas congruent to $y = x^2$

We are all familiar with the graph of $y = x^2$, which was studied in both *ICE-EM Mathematics Year 8* and *Year 9*. In this chapter we will call it the **basic parabola**.

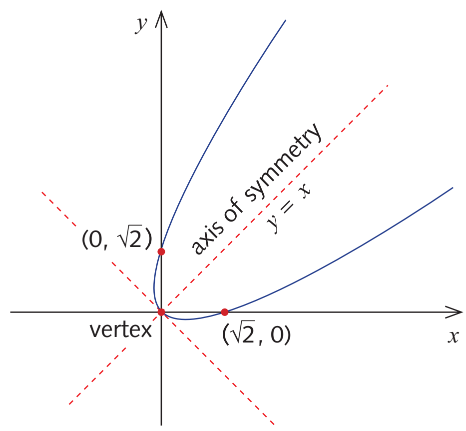


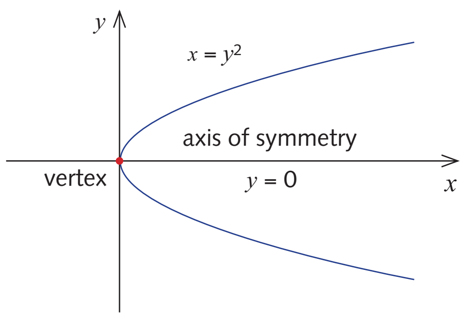
It has the following properties:

- The graph is symmetrical about the y-axis, $x = 0$. For example, the y-value at $x = 3$ is the same as the y-value at $x = -3$. In general, the y-value at $x = p$ is the same as the y-value at $x = -p$. We can visualise this as follows: if the graph were to be folded along the y-axis, the part on the left of the y-axis would land directly on top of the part on the right. The y-axis is called **the axis of symmetry** of the basic parabola.
- The minimum value of y occurs at the origin. It is called a **minimum turning point** since the y-values at points to both the left and the right of the origin are greater than the y-value at the origin. This turning point is called the **vertex** of the parabola.
- The **arms** of the parabola continue indefinitely, becoming steeper the higher they go.

Recall that two geometrical figures are said to be **congruent** if one can be transformed to the other by a sequence of translations, rotations and reflections. In this section we will look at parabolas congruent to $y = x^2$.

Imagine rotating the figure $y = x^2$ about the origin through 45° clockwise. The image you get is still called a parabola. Its vertex is $(0, 0)$ and its axis of symmetry is the line $y = x$, as shown in the left-hand figure below.

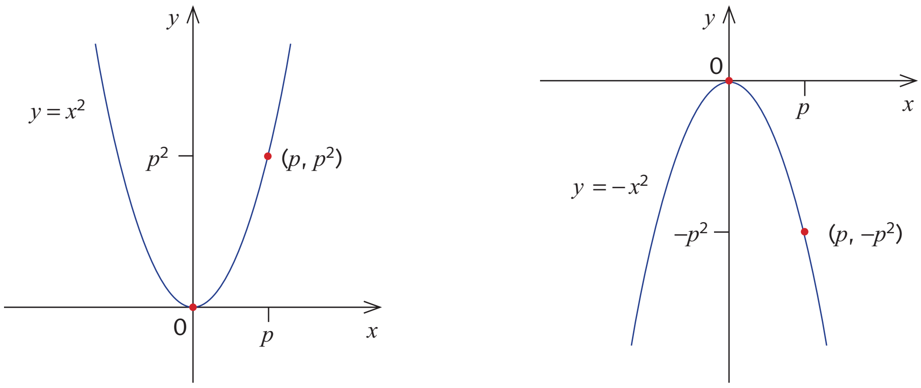




Similarly, rotating the basic parabola clockwise through 90° yields another parabola, $x = y^2$, as shown in the right-hand figure above.

However, for the rest of this chapter we shall only consider parabolas whose axis of symmetry is parallel to the y-axis; consequently, rotations will not be further discussed.

Reflection in the x-axis



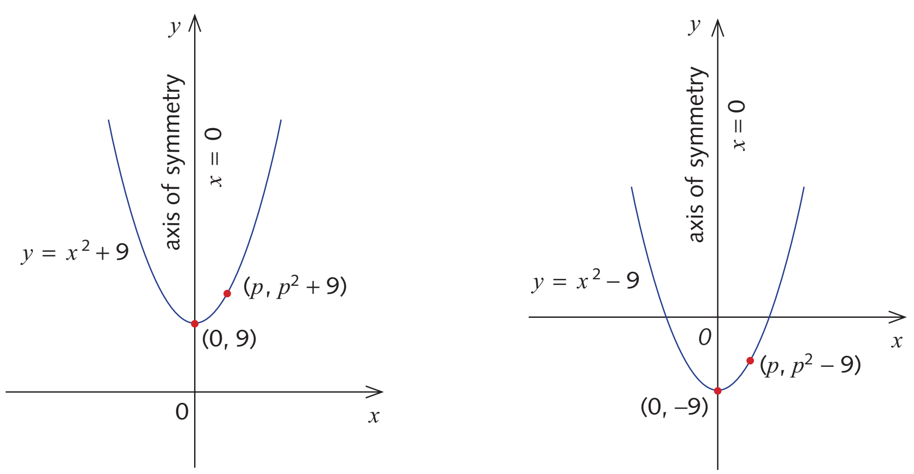
When the parabola $y = x^2$ is reflected in the x-axis, the point (3, 9) is reflected to the point (3, −9). In general, the point $(p,\ p^2)$ on the parabola $y = x^2$ is reflected to the point $(p, -p^2)$. Hence, the equation satisfied by the points of the image is $y = -x^2$.

The vertex remains at the origin under the transformation. However, the y-value of the vertex now represents a **maximum turning point** since the y-values at points to the left and right of the origin are less than the y-value at the origin.

Translations of $y = x^2$

Vertical translations

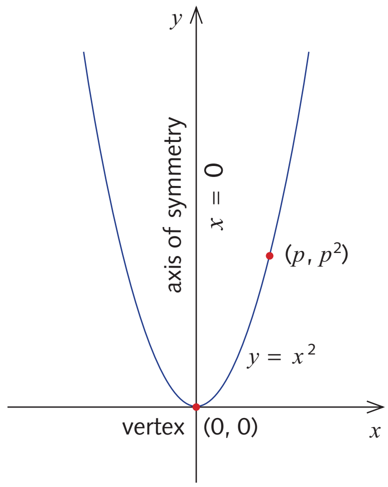
We now look at what happens when we translate $y = x^2$ up or down nine units.

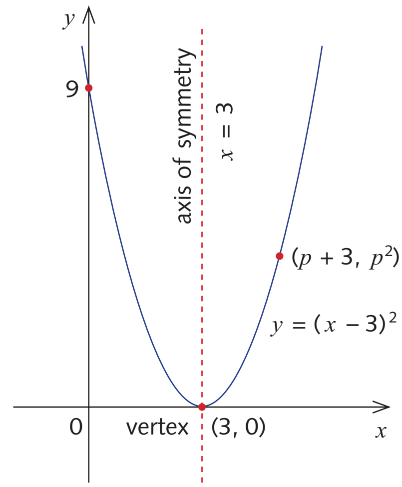


Translating $y = x^2$ nine units up shifts $(p,\ p^2)$ to $(p,\ p^2 + 9)$ so the equation becomes $y = x^2 + 9$. This is a parabola with vertex (0, 9). Similarly, $y = x^2$ becomes $y = x^2 - 9$ when translated nine units down.

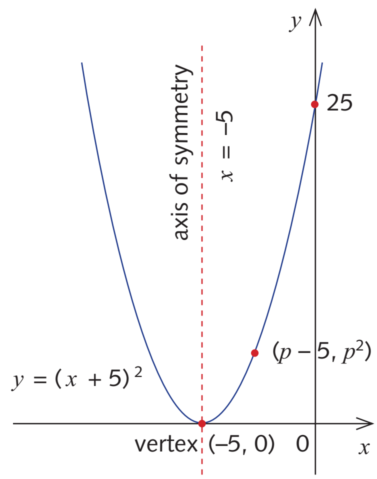
Horizontal translations

What happens when we make horizontal translations; that is, a translation to the left or to the right? This is a little trickier.





Every point on the basic parabola $y = x^2$ has coordinates $(p,\ p^2)$,as in the first diagram above. If we translate the parabola three units to the right, then the vertex (0, 0) goes to (3, 0), and the axis of symmetry, $x = 0$ goes to $x = 3$. The general point $(p,\ p^2)$ goes to the point $(p + 3,\ p^2)$. That is, each point on the image has coordinates $x = p + 3$, $y = p^2$. Eliminating p, $y = p^2 = (x - 3)^2$.

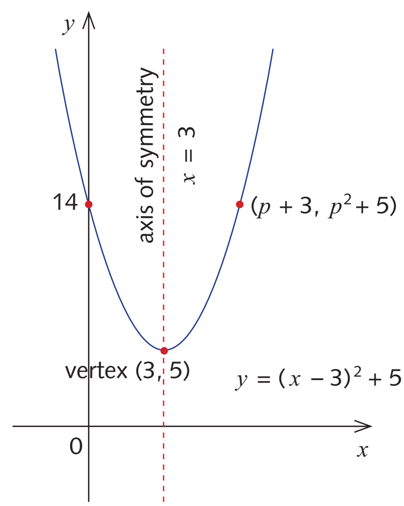


Thus, $y = (x - 3)^2$ is the equation of the parabola formed by translating $y = x^2$ three units to the right, as in the figure above on the right. The y-intercept is 9.

Similarly, translating $y = x^2$ five units to the left shifts the point $(p,\ p^2)$ to the point $(p - 5,\ p^2)$. In the same way, we see that the image parabola has equation $y = (x + 5)^2$, as in the diagram opposite. Once again, as a check, the vertex is (–5, 0), which is the image of (0, 0) under this translation. The y-intercept is 25.

In summary, translating $y = x^2$ five units to the left gives the parabola $y = (x + 5)^2$, as in the figure to the right.

General translations



Finally, if we translate $y = x^2$ three units to the right and five units up, then $y = x^2$ becomes $y = (x - 3)^2 + 5$ or $y - 5 = (x - 3)^2$. The y-intercept is 14. Its axis of symmetry is $x = 3$ and its vertex is (3, 5), as in the diagram opposite.

Translating it three units to the right moves $y = x^2$ to $y = (x - 3)^2$ and translating this five units up takes it to $y = (x - 3)^2 + 5$.

In summary, translating $y = x^2$ three units to the right and five units up gives the parabola $y = (x - 3)^2 + 5$, as in the figure to the right.

Example 1

Sketch each parabola and give the y-intercept, axis of symmetry and vertex.

a $y = (x - 3)^2 - 4$ **b** $y = (x + 2)^2 + 6$

Solution

a $y = (x - 3)^2 - 4$

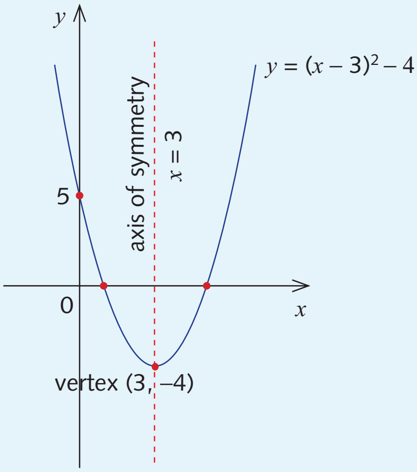
The graph of $y = (x - 3)^2 - 4$ is obtained by translating the graph of $y = x^2$ three units to the right and four units down.

Therefore, the axis of symmetry is $x = 3$.

The vertex is at $(3, -4)$.

When $x = 0$, $y = (-3)^2 - 4 = 5$

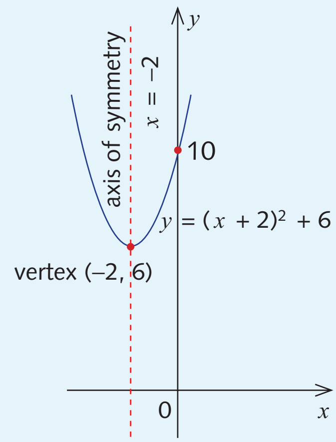
So the y-intercept is 5.



b The graph of $y = (x + 2)^2 + 6$ is obtained by translating the graph of $y = x^2$ two units to the left and six units up.
The axis of symmetry is $x = -2$ and the vertex $(-2, 6)$.

When $x = 0$, $y = (0 + 2)^2 + 6$
$= 10$

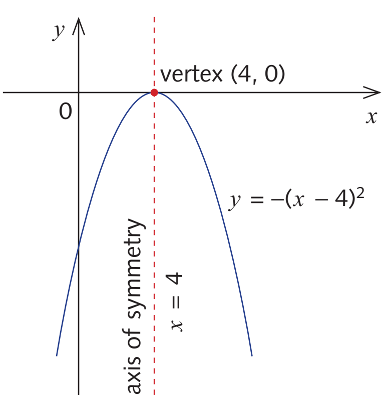
The y-intercept is 10.



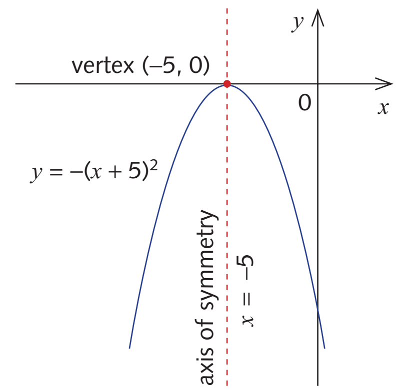
Translations of $y = -x^2$

Starting with $y = -x^2$ and translating left or right and up or down, we can construct many examples.

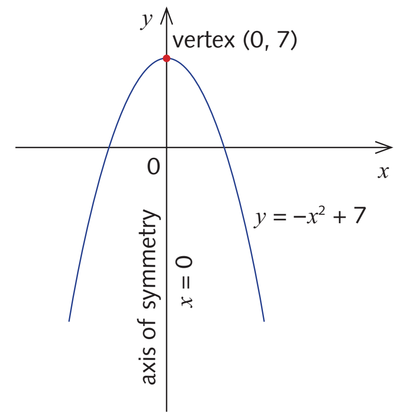
For example, if we translate $y = -x^2$ four units to the right, we obtain the parabola $y = -(x - 4)^2$ with axis of symmetry $x = 4$ and vertex $(4, 0)$. (As a check, $y \le 0$ and only equals 0 when $x = 4$; hence, the maximum value of y occurs when $x = 4$).



Similarly, translating $y = -x^2$ five units to the left yields $y = -(x + 5)^2$.



Next, if we translate $y = -x^2$ seven units upwards we obtain the parabola $y = -x^2 + 7$, which has axis of symmetry $x = 0$ and vertex (0, 7).



Example 2

Sketch each parabola and give the y-intercept, axis of symmetry and vertex.

a $y = -x^2 - 8$ **b** $y = -(x + 3)^2 - 4$ **c** $y = -(x - 2)^2 + 6$

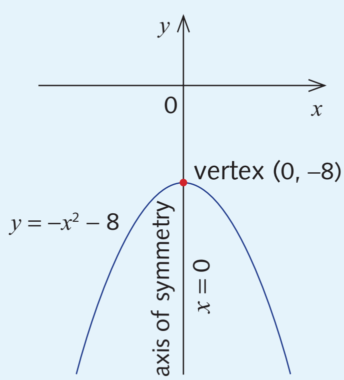
Solution

a $y = -x^2 - 8$

The graph $y = -x^2 - 8$ is obtained from the graph $y = -x^2$ by translating 8 units down.

The axis of symmetry is $x = 0$ and the vertex is (0, –8).

The graph has no x-intercepts.



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b $y = -(x+3)^2 - 4$

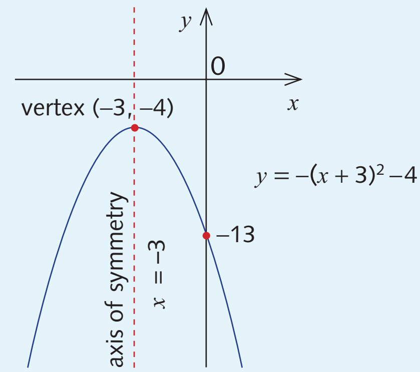
When $x = 0$, $y = -9 - 4$
$= -13$

The y-intercept is -13.

The graph $y = -(x+3)^2 - 4$ is obtained from the graph $y = -x^2$ by translating 3 units to the left and 4 units down.

The axis of symmetry is $x = -3$ and the vertex is $(-3, -4)$.

The graph has no x-intercepts.



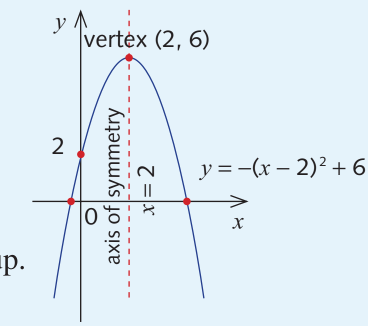
c $y = -(x-2)^2 + 6$

When $x = 0$, $y = -(0-2)^2 + 6$
$= -4 + 6$
$= 2$

The y-intercept is 2.

The graph $y = -(x-2)^2 + 6$ is obtained from the graph $y = -x^2$ by translating 2 units to the right and 6 units up.
The axis of symmetry is $x = 2$ and the vertex is $(2, 6)$.

The graph has two x-intercepts: $2 + \sqrt{6}$ and $2 - \sqrt{6}$.



The parabola

Properties of the parabola $y = x^2$

- The y-axis is the axis of symmetry, called simply the **axis** of the parabola.
- The minimum y-value occurs when $x = 0$.
- The origin is the vertex of the parabola.

Properties of the parabola $y = a(x-h)^2 + k$, $a = 1$ or $a = -1$

- The axis of symmetry is $x = h$, and the vertex is (h, k).
- When $a = 1$, the parabola is 'upright' and k is the minimum y-value ($y \geq k$ for all x)
- When $a = -1$, the parabola is 'upside down' and k is the maximum y-value ($y \leq k$ for all x)

Exercise 7A

1 Find the axis of symmetry and the vertex for:

a $y = (x-4)^2$　**b** $y = x^2 - 4$　**c** $y = (x-2)^2 + 6$

d $y = (x+3)^2 + 7$　**e** $y = (x+2)^2 + 3$　**f** $y = -x^2 + 9$

g $y = (x-3)^2 - 4$　**h** $y = (x+2)^2 - 3$　**i** $y = (x-6)^2 + 6$

j $y = -(x+1)^2$　**k** $y = -(x-2)^2 + 1$　**l** $y = -(x+3)^2 + 5$

2 Find the y-intercept of:

a $y = (x - 2)^2 - 7$ **b** $y = (x - 7)^2 - 3$ **c** $y = (x + 1)^2 + 4$

d $y = -(x - 3)^2$ **e** $y = -(x + 2)^2 - 4$ **f** $y = -(x - 2)^2 + 6$

Example 1

3 Sketch the graphs of the following, labelling the y-intercept and vertex.

a $y = (x - 5)^2$ **b** $y = (x - 1)^2 - 3$ **c** $y = (x + 2)^2 + 3$

d $y = (x - 4)^2 - 3$ **e** $y = (x - 1)^2 + 6$ **f** $y = (x - 4)^2 - 4$

Example 2

4 Sketch the graphs of the following, labelling the y-intercept and vertex.

a $y = -x^2 - 7$ **b** $y = -x^2 + 7$ **c** $y = -(x - 3)^2 + 5$

d $y = -(x - 3)^2 - 7$ **e** $y = -(x + 4)^2$ **f** $y = -(x - 6)^2$

g $y = -(x + 4)^2 - 3$ **h** $y = -(x + 3)^2 + 11$ **i** $y = -(x - 1)^2 + 6$

5 Write the equation of the parabola obtained when the basic parabola, $y = -x^2$, is:

a translated 3 units to the right

b translated b units to the left

c translated 6 units down

d translated c units up

6 Write the equation of the parabola obtained when the basic parabola, $y = x^2$, is:

a translated 3 units up and 4 units to the left

b translated 5 units down and 6 units to the right

c translated a units to the right and b units up

d translated c units down and d units to the left

7 Consider the parabola $y = (x + 3)^2 - 8$. Sketch this parabola. What is the equation of the image if it is:

a translated 8 units up and 3 units to the right?

b translated 2 units to the left and 3 units down?

c translated a units to the right and b units up?

8 Consider the parabola $y = (x - 1)^2 + a$. Find the value of a if the y-intercept is:

a 1 **b** 3 **c** 0 **d** -7

9 Consider the parabola $y = -(x - 2)^2 + b$. Find the value of b if the y-intercept is:

a 1 **b** 3 **c** -4 **d** -7

10 Consider the basic parabola $y = x^2$. Draw a sketch of the parabola if it is:

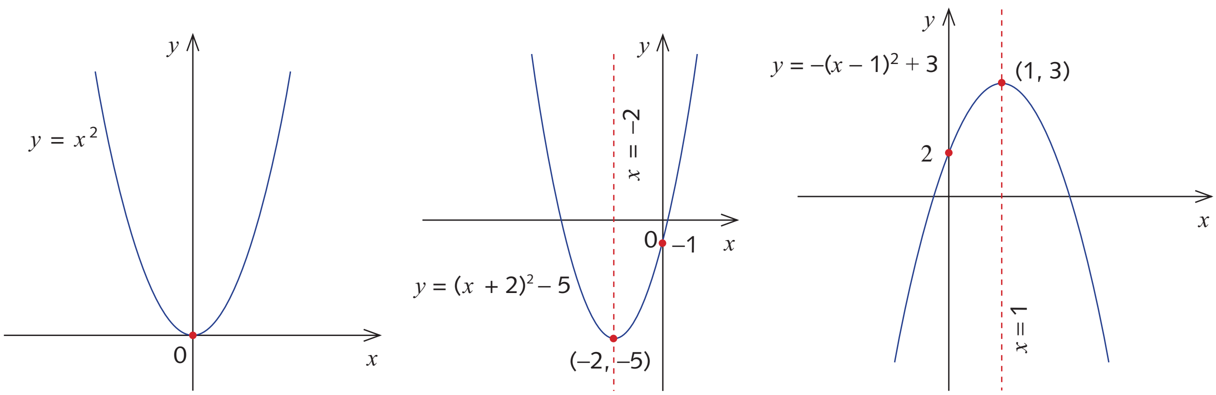
a rotated about the origin through 135° clockwise

b rotated about the origin through 90° anticlockwise

7B Sketching the graph of the quadratic $y = ax^2 + bx + c$, where $a = \pm 1$

In Section 7A we saw that $y = x^2$ becomes:

- $y = (x + 2)^2 - 5$ when translated two units to the left and five units down.
- $y = -(x - 1)^2 + 3$ when reflected in the x-axis and then translated one unit to the right and 3 units up.



By expanding brackets in each of the equations above we obtain an equation of the form $y = ax^2 + bx + c$. Any equation of this form represents a parabola. How do we find its axis of symmetry and its vertex? We do this by putting it into one of the forms above using the method of **completing the square**, which you studied in Chapter 5.

Recall that to complete the square when the coefficient of x^2 is one ($a = 1$), we add and subtract the square of half the coefficient of x. When the coefficient of x^2 is not one, that is, $a \neq 1$ or 0, we first factor a out of the expression, and then multiply through by a at the final step.

Example 3

Complete the square and hence sketch the graph of:

a $y = x^2 + 6x + 13$ **b** $y = -x^2 - 3x - 5$

Solution

a $y = x^2 + 6x + 13$ (Add and subtract the square of half the coefficient of x.)

$= (x^2 + 6x + 3^2) + 13 - 3^2$

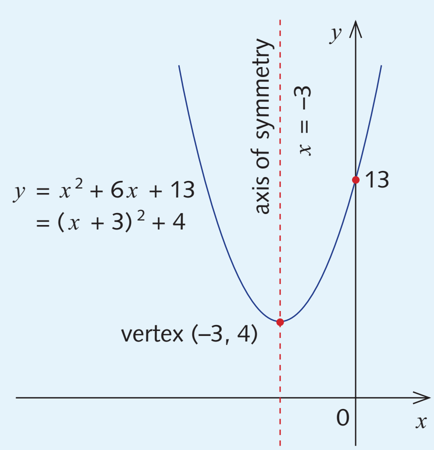
$= (x + 3)^2 + 4$

The axis of symmetry is $x = -3$.

The vertex is $(-3, 4)$.

When $x = 0$, $y = 13$, so the y-intercept is 13.

Note that this parabola has no x-intercepts since the minimum value of y is 4, which is positive. This is the case since $(x + 3)^2 \geq 0$.



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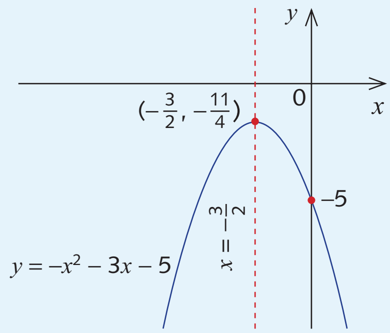
b $y = -x^2 - 3x - 5$

When $x = 0$, $y = -5$, so the y-intercept is -5.

Completing the square:

$$\begin{aligned} y &= -[x^2 + 3x + 5] \\ &= -\left[\left(x^2 + 3x + \left(\frac{3}{2}\right)^2\right) + 5 - \left(\frac{3}{2}\right)^2\right] \\ &= -\left[\left(x + \frac{3}{2}\right)^2 + \frac{11}{4}\right] \\ &= -\left(x + \frac{3}{2}\right)^2 - \frac{11}{4} \end{aligned}$$



So the axis of symmetry is $x = -\dfrac{3}{2}$

and the vertex is $\left(-\dfrac{3}{2}, -\dfrac{11}{4}\right)$.

Note that there are no x-intercepts, since $y \le -\dfrac{11}{4}$ for all x.

When sketching parabolas, the special features are:

- the axis of symmetry
- the vertex
- the y-intercept and the x-intercepts.

Example 4

Sketch the following parabolas. First find the y-intercept, then complete the square to find the axis of symmetry and the vertex of the parabola, then find the x-intercepts if they exist.

a $y = x^2 - 6x$ **b** $y = -x^2 - 10x$

Solution

a If $x = 0$, then $y = 0$.

Hence, the y-intercept is 0.

Also, $$\begin{aligned} y &= x^2 - 6x \\ &= (x^2 - 6x + 9) - 9 \\ &= (x - 3)^2 - 9 \end{aligned}$$

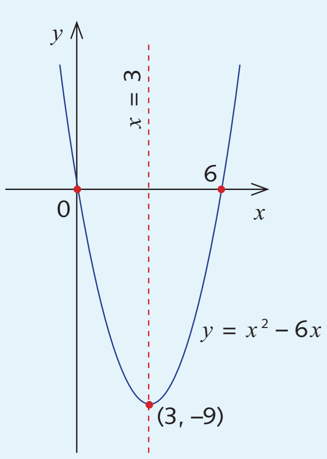
The axis of symmetry is $x = 3$ and the vertex is $(3, -9)$.

When $y = 0$, $x^2 - 6x = 0$

So $x(x - 6) = 0$,

$x = 0$ or $x = 6$

Hence, the x-intercepts are 0 and 6.



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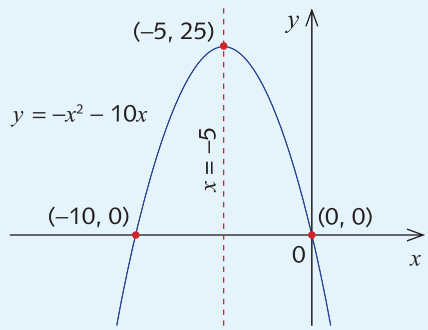
b Consider the equation $y = -x^2 - 10x$.

When $x = 0$, $y = 0$, so the y-intercept is 0.

Completing the square:

$$\begin{aligned} y &= -x^2 - 10x \\ &= -[x^2 + 10x] \\ &= -[(x^2 + 10x + 25) - 25] \\ &= -[(x + 5)^2 - 25] \\ &= -(x + 5)^2 + 25 \end{aligned}$$



Hence, the axis of symmetry is $x = -5$ and the vertex is $(-5, 25)$.

When $y = 0$, $-x^2 - 10x = 0$

so $-x(x + 10) = 0$

$x = 0$ or $x = -10$

Thus, the x-intercepts are $x = 0$ and $x = -10$.

x-intercepts

Some parabolas have x-intercepts and some do not. After completing the square to sketch a parabola, you will know whether or not it has x-intercepts. To find these, substitute $y = 0$ into the quadratic equation and solve the resulting equation for x. There are three ways to do this:

- factorise the quadratic (as in Example 4)
- complete the square
- use the quadratic formula.

These methods were discussed in detail in Chapter 5. Since the sketching technique discussed so far has included completing the square, the second method is usually used.

Example 5

Sketch the following parabolas. First find the y-intercept, then complete the square to find the axis of symmetry and the vertex of the parabola, then find the x-intercepts if they exist.

a $y = x^2 + 6x - 7$ **b** $y = -x^2 + 8x + 13$ **c** $y = x^2 + x + 1$

Solution

a $y = x^2 + 6x - 7$

When $x = 0$, $y = -7$.

The y-intercept is -7.

$$\begin{aligned} y &= x^2 + 6x - 7 \\ &= (x^2 + 6x + 9) - 9 - 7 \\ &= (x + 3)^2 - 16 \end{aligned}$$

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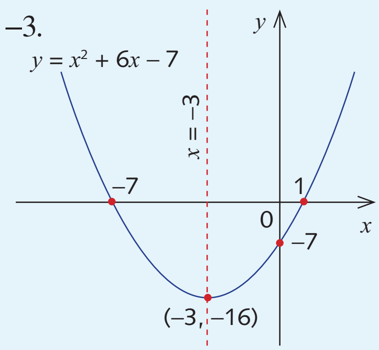
The vertex is at $(-3, -16)$ and the axis of symmetry is $x = -3$.



Substitute $y = 0$, then

$$\begin{aligned} (x+3)^2 - 16 &= 0 \\ (x+3)^2 &= 16 \\ x + 3 &= 4 \quad \text{or} \quad x + 3 = -4 \\ x &= 1 \quad \text{or} \quad x = -7 \end{aligned}$$

b The parabola is $y = -x^2 + 8x + 13$.

When $x = 0$, $y = 13$, so the y-intercept is 13.

Completing the square:

$$\begin{aligned} y &= -[x^2 - 8x - 13] \\ &= -[(x^2 - 8x + 16) - 13 - 16] \qquad \text{(Complete the square inside the brackets.)} \\ &= -[(x-4)^2 - 29] \\ &= -(x-4)^2 + 29 \end{aligned}$$

So the axis of symmetry is $x = 4$, and the vertex is $(4, 29)$.

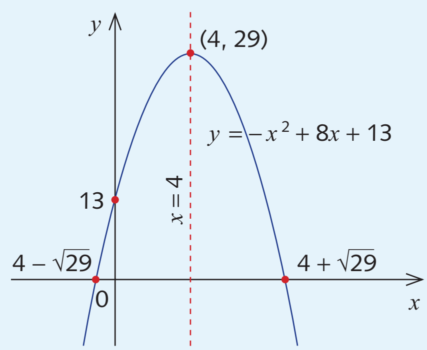
When $y = 0$,

$$\begin{aligned} -(x-4)^2 + 29 &= 0 \\ (x-4)^2 &= 29 \\ x - 4 &= \sqrt{29} \quad \text{or} \quad x - 4 = -\sqrt{29} \end{aligned}$$

so $x = 4 + \sqrt{29}$, which is positive

or $x = 4 - \sqrt{29}$, which is negative

Thus, the x-intercepts are $4 + \sqrt{29}$ and $4 - \sqrt{29}$.

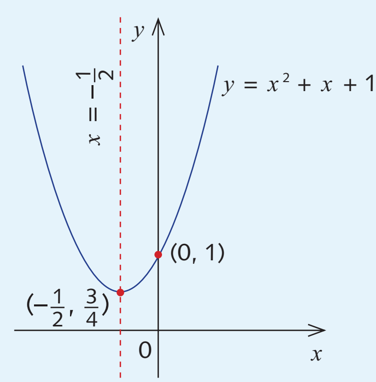


c If $x = 0$, then $y = 1$.

Thus, the y-intercept is 1.

$$\begin{aligned} \text{Also, } y &= x^2 + x + 1 \\ &= \left(x^2 + x + \frac{1}{4}\right) + 1 - \frac{1}{4} \\ &= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} \end{aligned}$$

The axis of symmetry is $x = -\frac{1}{2}$ and the vertex is $\left(-\frac{1}{2}, \frac{3}{4}\right)$. This graph has no x-intercepts since the minimum value of y is $\frac{3}{4}$, which is positive.



Graphing $y = ax^2 + bx + c$, where $a = 1$ or $a = -1$

- When $a = 1$, complete the square by adding and subtracting $\left(\frac{b}{2}\right)^2$, to write the quadratic in the form $y = (x - h)^2 + k$.
- When $a = -1$, first factor out -1 before completing the square. Then remove the outer brackets and multiply both terms by -1 to write the quadratic equation in the form $y = -(x - h)^2 + k$.
- The axis of symmetry is $x = h$.
- The coordinates of the vertex can now be read off. They are (h, k). The graph is a translation of $y = x^2$.
- When $a = 1$, k is the minimum y-value ($y \geq k$ for all x) and when $a = -1$, k is the maximum y-value ($y \leq k$ for all x)
- Find the y-intercept by substituting $x = 0$ in $y = ax^2 + bx + c$. Note: The y-intercept will always be $(0, c)$.
- Find the x-intercepts, if they exist, by substituting $y = 0$ and solving the resulting quadratic equation.

Exercise 7B

1 For each parabola:

i determine the y-intercept

ii write down the axis of symmetry and the vertex

iii sketch the parabola

iv determine from the sketch whether the parabola has no x-intercepts, one x-intercept or two x-intercepts

a $y = x^2 + 3$ **b** $y = x^2 - 7$ **c** $y = (x - 2)^2 + 4$

d $y = (x - 3)^2 - 7$ **e** $y = (x + 5)^2$ **f** $y = (x - 7)^2 - 1$

g $y = -(x - 3)^2$ **h** $y = -(x + 1)^2 + 5$ **i** $y = -(x - 2)^2 - 1$

Example 3,4

2 For each parabola:

i determine the y-intercept

ii complete the square

iii write down the axis of symmetry and the vertex

iv sketch the parabola

v determine whether the parabola has no x-intercepts, one x-intercept or two x-intercepts (Do not find the x-intercepts.)

a $y = x^2 - 6x$ **b** $y = x^2 + 6x$ **c** $y = x^2 + 2x - 4$

d $y = x^2 + 4x - 1$ **e** $y = x^2 + 6x - 3$ **f** $y = x^2 + 12x - 4$

g $y = x^2 - 3x + 5$ **h** $y = x^2 - 5x + 2$ **i** $y = x^2 + 7x$

j $y = -x^2 - 2x$ **k** $y = -x^2 + 8x + 7$ **l** $y = -x^2 + 5x - 7$

3 Find the x-intercepts of the parabolas.

a $y = (x + 1)^2 - 4$ **b** $y = (x + 2)^2 - 9$ **c** $y = -(x - 3)^2 + 25$

d $y = (x - 4)^2 - 7$ **e** $y = (x + 3)^2 - 11$ **f** $y = -(x + 1)^2 + 4$

g $y = (x - 2)^2 - 8$ **h** $y = -(x - 5)^2 + 18$ **i** $y = (x - 3)^2 - 50$

j $y = -(x + 3)^2 + 20$ **k** $y = (x + 2)^2 - 32$ **l** $y = 16 - (x + 1)^2$

Example 5

4 Find the x-intercepts of the parabolas by completing the square.

a $y = x^2 + 2x - 4$ **b** $y = x^2 - 6x + 7$ **c** $y = x^2 - 8x + 13$

d $y = x^2 + 4x - 4$ **e** $y = x^2 + 10x - 11$ **f** $y = x^2 - 20x - 50$

g $y = -x^2 + 12x + 13$ **h** $y = -x^2 + 6x - 4$ **i** $y = -x^2 + 4x + 8$

Example 5

5 For each parabola:

- **i** determine the y-intercept
- **ii** complete the square
- **iii** find the axis of symmetry and the vertex
- **iv** determine the x-intercepts, if any, using the completed square expression
- **v** sketch the parabola, marking all of the above features

a $y = x^2 + 4x - 5$ **b** $y = x^2 + 4x + 5$ **c** $y = x^2 + 6x + 9$

d $y = x^2 - 6x - 7$ **e** $y = x^2 + 8x - 3$ **f** $y = x^2 - x - 2$

g $y = x^2 + 5x + 10$ **h** $y = x^2 + 7x - 3$ **i** $y = x^2 - 2x + 4$

j $y = -x^2 + 4x + 3$ **k** $y = -x^2 + 12x + 4$ **l** $y = -x^2 + 2x - 2$

m $y = -x^2 + x + 1$ **n** $y = -x^2 - 5x - 1$ **o** $y = -x^2 + 11x + 20$

7C The general quadratic $y = ax^2 + bx + c$

Graphing the parabola $y = 3x^2$

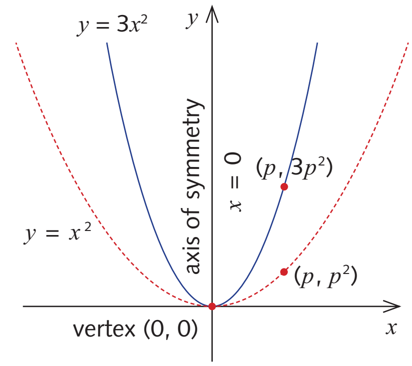
Consider the parabola $y = 3x^2$, sketched overpage. Its axis of symmetry is $x = 0$ and its vertex is $(0, 0)$. The graph of $y = x^2$ is sketched on the same set of axes.

Each point on the parabola $y = x^2$ has coordinates (p, p^2) and the matching point on the parabola $y = 3x^2$ is $(p, 3p^2)$. The parabola $y = 3x^2$ is obtained from the parabola $y = x^2$ by stretching by a factor 3 from the x-axis.

Given our investigations in the earlier sections of this chapter, we would expect that:

- translations of $y = 3x^2$ yield congruent parabolas of the form $y = 3x^2 + bx + c$
- every parabola of the form $y = 3x^2 + bx + c$ can be obtained by translating $y = 3x^2$.

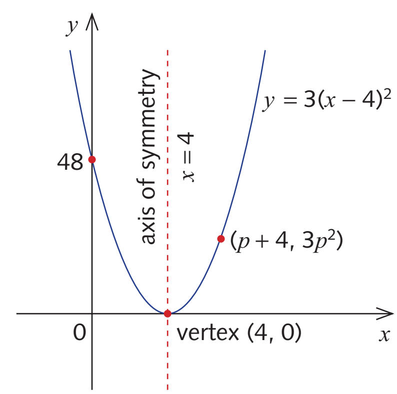
Both these statements are true.



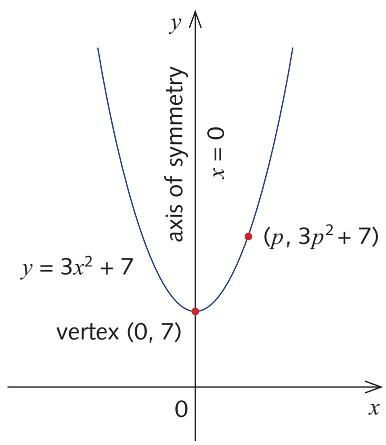
Translations of $y = 3x^2$

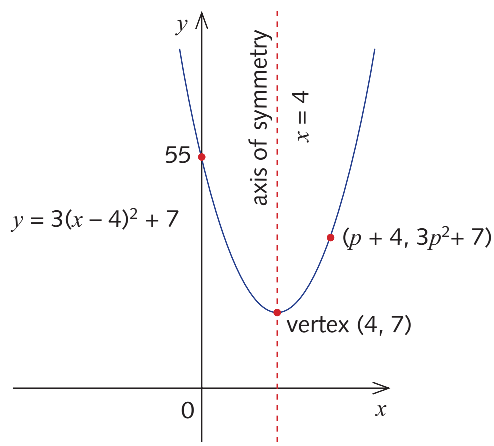
Suppose we translate $y = 3x^2$ four units to the right. The point $(p,\ 3p^2)$ goes to $(p + 4,\ 3p^2)$. Hence, we obtain the parabola $y = 3(x - 4)^2$.

The y-intercept is 48. The axis of symmetry is $x = 4$ and the vertex is $(4, 0)$.



Similarly, if we translate $y = 3x^2$ seven units up, then the image is the parabola $y = 3x^2 + 7$ (see left-hand figure below).





If we perform both translations; that is, four units to the right and then seven units up, then the parabola $y = 3x^2$ becomes the parabola $y = 3(x - 4)^2 + 7$ (see the right-hand figure above). The axis of symmetry is $x = 4$ and the vertex is $(4, 7)$.

In summary, translating $y = 3x^2$ four units to the right and seven units up gives the parabola $y = 3(x - 4)^2 + 7$, as in the figure above.

Translations of $y = ax^2$, $a \neq 0$

Clearly the above discussion holds just as well for translations of, for example, $y = 2x^2$, $y = 10x^2$ or $y = \frac{1}{2}x^2$. However, it is equally applicable to $y = -3x^2$, $y = -2x^2$ or $y = -\frac{1}{4}x^2$, where the basic parabola, $y = x^2$, has been not only stretched by a certain factor from the x-axis (3, 2 and $\frac{1}{4}$ respectively), but also reflected in the x-axis. This is shown in Example 6 overpage.

To complete the square in $y = ax^2 + bx + c$, write:

$$y = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$$

and then complete the square inside the square brackets.

Example 6

Find the y-intercept, the axis of symmetry and the vertex of each parabola by completing the square. Sketch their graphs.

a $y = 2x^2 + 4x + 9$

b $y = 3x^2 + 6x - 13$

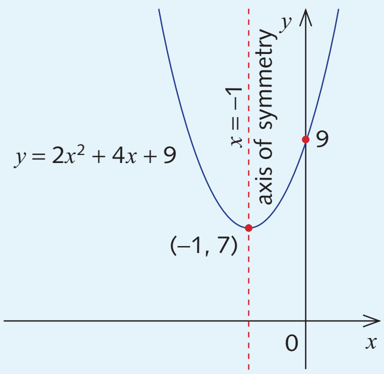
c $y = -2x^2 - 4x - 6$

d $y = -2x^2 - x + 21$

Solution

a Consider the equation $y = 2x^2 + 4x + 9$.

When $x = 0$, $y = 9$, so the y-intercept is 9.



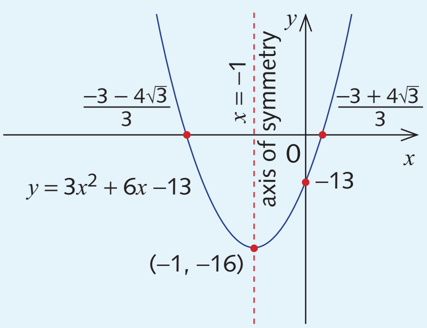
$$\begin{aligned} y &= 2\left[x^2 + 2x + \frac{9}{2}\right] \\ &= 2\left[(x^2 + 2x + 1) + \frac{9}{2} - 1\right] \\ &= 2\left[(x + 1)^2 + \frac{7}{2}\right] \\ &= 2(x + 1)^2 + 7 \end{aligned}$$

The axis of symmetry is $x = -1$ and the vertex is $(-1, 7)$.

Note: This parabola has no x-intercepts since $y \geq 7$ for all x.

b Consider the equation $y = 3x^2 + 6x - 13$.

The y-intercept is -13.



$$\begin{aligned} y &= 3\left[x^2 + 2x - \frac{13}{3}\right] \\ &= 3\left[(x^2 + 2x + 1) - \frac{13}{3} - 1\right] \\ &= 3\left[(x + 1)^2 - \frac{16}{3}\right] \\ &= 3(x + 1)^2 - 16 \end{aligned}$$

The axis of symmetry is $x = -1$ and the vertex is $(-1, -16)$.

We see from the graph that there are x-intercepts. To find them we substitute $y = 0$ and obtain:

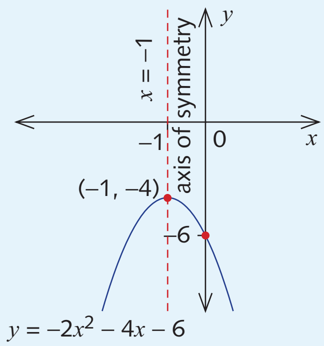
$$\begin{aligned} 3(x + 1)^2 &= 16 \\ (x + 1)^2 &= \frac{16}{3} \\ x + 1 &= \frac{4}{\sqrt{3}} \quad \text{or} \quad x + 1 = -\frac{4}{\sqrt{3}} \end{aligned}$$

$x = \dfrac{-3 + 4\sqrt{3}}{3}$, which is positive, or $x = \dfrac{-3 - 4\sqrt{3}}{3}$, which is negative.

(continued over page)

c Consider the equation $y = -2x^2 - 4x - 6$.
The y-intercept is -6.

$$\begin{aligned} y &= -2[x^2 + 2x + 3] \\ &= -2[x^2 + 2x + 1 - 1 + 3] \\ &= -2[(x+1)^2 + 2] \\ &= -2(x+1)^2 - 4 \end{aligned}$$

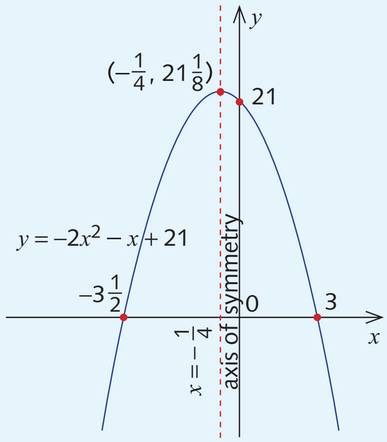


The axis of symmetry is $x = -1$ and the vertex is $(-1, -4)$.

The parabola has no x-intercepts since $y \le -4$ for all x.

d Consider the equation $y = -2x^2 - x + 21$.
The y-intercept is 21.

$$\begin{aligned} y &= -2\left[x^2 + \frac{x}{2} - \frac{21}{2}\right] \\ &= -2\left[\left(x^2 + \frac{x}{2} + \frac{1}{16}\right) - \frac{21}{2} - \frac{1}{16}\right] \\ &= -2\left[\left(x + \frac{1}{4}\right)^2 - 10\tfrac{9}{16}\right] \\ &= -2\left(x + \frac{1}{4}\right)^2 + 21\tfrac{1}{8} \end{aligned}$$



The axis of symmetry is $x = -\dfrac{1}{4}$ and the vertex is $\left(-\dfrac{1}{4}, 21\tfrac{1}{8}\right)$.

$$\begin{aligned} y &= -2x^2 - x + 21 \\ &= -(2x + 7)(x - 3) \end{aligned}$$

When $y = 0$, $x = 3$ or $x = -3\tfrac{1}{2}$. The axis of symmetry is their average, $x = -\dfrac{1}{4}$.

(See next section.)

A formula for the axis of symmetry

One thing is clear from some of the previous examples. In practice, completing the square can be technically difficult to carry out and therefore prone to error.

Fortunately, there is a formula for the axis of symmetry of the parabola $y = ax^2 + bx + c$. To derive the formula, we begin by completing the square in the general case:

$$\begin{aligned} y &= ax^2 + bx + c \\ &= a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] \\ &= a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right] \qquad \text{(Add and subtract the square of half the coefficient of } x.) \\ &= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right] \\ &= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \end{aligned}$$

This expression shows that the minimum or maximum of the quadratic occurs when $x = -\frac{b}{2a}$.

Hence, we have shown that the axis of symmetry of the parabola $y = ax^2 + bx + c$ is $x = -\frac{b}{2a}$.

Once the axis of symmetry is known, then the y-coordinate of the vertex can be determined by substituting $x = -\frac{b}{2a}$ into the quadratic. To complete the sketch, find the y-intercept and the x-intercepts if they exist.

We have also shown that when we write the equation of the parabola in the form $y = a(x - h)^2 + k$, then (h, k) is the vertex and $x = h$ is the axis of symmetry.

The general parabola $y = ax^2 + bx + c$

- The parabola $y = ax^2 + bx + c$ is a translation of, and congruent to, the parabola $y = ax^2$.
- To complete the square, first take out a as a factor, $y = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$, and then complete the square inside the brackets.
- The equation for the parabola $y = ax^2 + bx + c$ can also be written in the form $y = a(x - h)^2 + k$, where (h, k) is the vertex and $x = h$ is the axis of symmetry.
- The axis of symmetry of the parabola $y = ax^2 + bx + c$ has equation $x = -\frac{b}{2a}$. The y-coordinate of the vertex can be found by substitution.

Example 7

Sketch $y = 2x^2 + 8x + 19$ using the formula for the axis of symmetry.

Solution

When $x = 0$, $y = 19$.

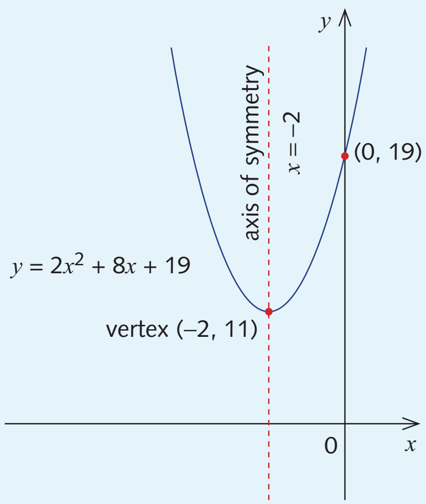
The y-intercept is 19.

The axis of symmetry is:

$$x = -\frac{b}{2a} = -\frac{8}{4} = -2$$

To find the vertex, we calculate the y-value when $x = -2$, which gives $y = 8 - 16 + 19 = 11$, so the vertex is $(-2, 11)$.

The graph has no x-intercepts.



Finding the equation of a parabola given the vertex and one other point

Given the vertex and one other point on a parabola, we can find the equation of the parabola. Since the vertex is (h, k), the equation is $y = a(x - h)^2 + k$. The value of a can be found by substituting in the values of the coordinates of the other point.

Example 8

A parabola has vertex at $(1, 3)$ and passes through the point $(3, 11)$. Find its equation.

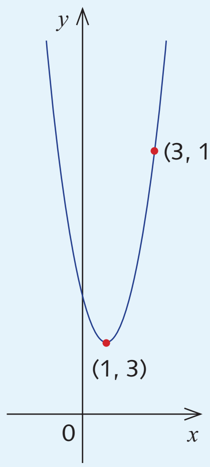
Solution

The sketch shows the information that has been given. Since the vertex is at $(1, 3)$, the equation must be of the form $y = a(x - 1)^2 + 3$ for some $a \neq 0$.

Since $(3, 11)$ is on the parabola,

$$\begin{aligned} 11 &= a(3-1)^2 + 3 \\ 11 &= a \times 4 + 3 \\ 4a + 3 &= 11 \\ 4a &= 8 \\ a &= 2 \end{aligned}$$

Hence, the equation of the parabola is $y = 2(x - 1)^2 + 3$.



Exercise 7C

1 **a** Sketch the graphs of these parabolas on the one set of axes.

i $y = x^2$ **ii** $y = 3x^2$ **iii** $y = \frac{1}{3}x^2$

b Sketch the graphs of these parabolas on the one set of axes.

i $y = -x^2$ **ii** $y = -3x^2$ **iii** $y = \frac{1}{3}x^2$

c Sketch the graphs of these parabolas on the one set of axes.

i $y = x^2$ **ii** $y = 2x^2$ **iii** $y = 3x^2$

2 Sketch the parabolas. In each case, determine the x- and y-intercepts and the vertex of the parabola.

a $y = 2x^2 + 1$ **b** $y = 4x^2 - 1$ **c** $y = 6x^2 - 1$

d $y = -2x^2 + 8$ **e** $y = -2x^2 + 9$

3 Consider the following parabolas. Determine those that are congruent to each other using translations and reflections in the x-axis.

a $y = 2x^2$ **b** $y = 3x^2$ **c** $y = -5x^2$

d $y = 2x^2 + 7x + 9$ **e** $y = -2x^2 + 5x$ **f** $y = -3x^2 - 7x - 11$

g $y = 2x^2 - 6x$ **h** $y = 5x^2 + 6x + 13$ **i** $y = 7x^2 + 6x + 13$

j $y = x^2$ **k** $y = -x^2 - x + 1$ **l** $y = 5 - 3x^2$

Example 6

4 Write each parabola in the form $y = a(x - h)^2 + k$.

a $y = 3x^2 + 6x + 3$ **b** $y = 3x^2 - 9x + 17$

c $y = 2x^2 + 10x - 13$ **d** $y = 2x^2 + 7x + 3$

e $y = 3x^2 + 5x + 7$ **f** $y = -5x^2 + 20x + 37$

Example 6

5 For each parabola:

i determine the y-intercept

ii complete the square

iii find the axis of symmetry and the vertex

iv determine any x-intercepts

v sketch the parabola

a $y = 2x^2 + 4x + 3$ **b** $y = -2x^2 - 4x + 7$

c $y = 3x^2 + 12x - 19$ **d** $y = 3x^2 + 8x - 2$

e $y = 2x^2 + x - 15$ **f** $y = 2x^2 - x + 5$

Example 7

6 Use the formula $x = \dfrac{-b}{2a}$ to find the axis of symmetry and vertex of the parabolas. Which of them have x-intercepts?

a $y = x^2 - x + 3$ **b** $y = 3x^2 + 5x - 13$

c $y = 2x^2 + x + 1$ **d** $y = 5x^2 + 3x + 7$

e $y = -3x^2 + 5x - 7$ **f** $y = -x^2 + 3x + 2$

Example 8

7 Consider the parabola $y = 2(x - 1)^2 + k$. Find the value of k if the y-intercept is:

a 8 **b** 10 **c** -6 **d** 0

8 Consider the parabola $y = a(x - 2)^2 - 6$. Find the value of a if the y-intercept is:

a 6 **b** -2 **c** -4 **d** -18

9 Consider the parabola $y = 2(x - h)^2 + 3$. Find the value of h if the y-intercept is:

a 5 **b** 21 **c** 4 **d** 9

Example 8

10 A parabola has vertex $(1, -2)$ and passes through the point $(3, 2)$. Find its equation.

11 A parabola has vertex $(-2, -1)$ and passes through the point $(1, 26)$. Find its equation.

12 A parabola has y-intercept 4 and vertex at $(1, 6)$. Find its equation.

13 Sketch the parabolas. In each case, determine the x- and y-intercepts, the vertex and the axis of symmetry.

a $y = 2(x - 1)^2 + 3$ **b** $y = -2(x - 1)^2 + 8$

c $y = -4(x - 2)^2 + 12$ **d** $y = -4(x + 3)^2 + 12$

e $y = 4(x + 2)^2 - 16$ **f** $y = 2(x + 2)^2 - 12$

g $y = 4 - 2(x - 3)^2$ **h** $y = 3(x + 1)^2 - 15$

i $y = 5(x + 4)^2 + 1$

7D Symmetry and factorisation

In this section, we will deal with another method for sketching parabolas. It is based on the fact that if we can locate two points on the parabola that are symmetric with respect to the axis of symmetry, then the axis of symmetry and hence the vertex can be found.

Sketching parabolas in factorised form

Sometimes we are given a quadratic in factorised form. For example, $y = (x - 6)(x - 4)$ or $y = 5(x - 1)(x - 3)$. These parabolas are easy to sketch since the axis of symmetry is simply given by the *average* of the two x-intercepts.

A second easy case is when the parabola is given with the square already completed, for example, $y = -7x^2 + 1$.

Example 9

Factor if necessary, and sketch, marking the intercepts, axis of symmetry and vertex.

a $y = (x - 6)(x - 4)$ **b** $y = 6 + x - x^2$ **c** $y = 5x^2 - 20x + 15$

d $y = -3(x + 5)(x + 7)$ **e** $y = -7x^2 + 1$

Solution

a $y = (x - 6)(x - 4)$

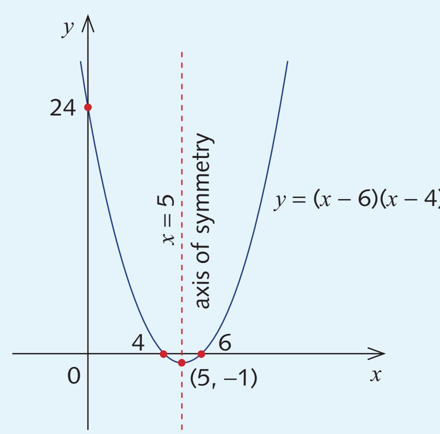
When $x = 0$, $y = 24$, so the y-intercept is 24.

When $y = 0$, $x = 4$ or $x = 6$, so the x-intercepts are 4 and 6.

Taking the average of the x-intercepts, $\dfrac{4 + 6}{2}$.

Therefore, $x = 5$ is the axis of symmetry.

When $x = 5$, $y = (5 - 6)(5 - 4) = -1$, so the vertex is $(5, -1)$.



b $y = -x^2 + x + 6$

This is an upside-down parabola.

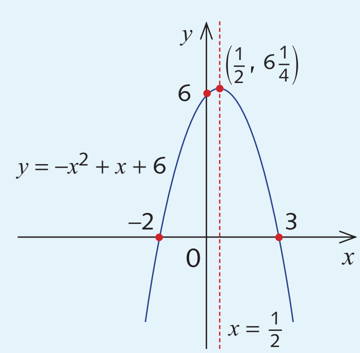
When $x = 0$, $y = 6$, so this is the y-intercept.

$$\begin{aligned} y &= -x^2 + x + 6 \\ &= -(x^2 - x - 6) \\ &= -(x - 3)(x + 2) \end{aligned}$$

So the x-intercepts are 3 and -2.

$\dfrac{3 + (-2)}{2} = \dfrac{1}{2}$, therefore the axis of symmetry is $x = \dfrac{1}{2}$.

When $x = \dfrac{1}{2}$, $y = -\dfrac{1}{4} + \dfrac{1}{2} + 6 = 6\tfrac{1}{4}$, so the vertex is $\left(\dfrac{1}{2}, 6\tfrac{1}{4}\right)$.



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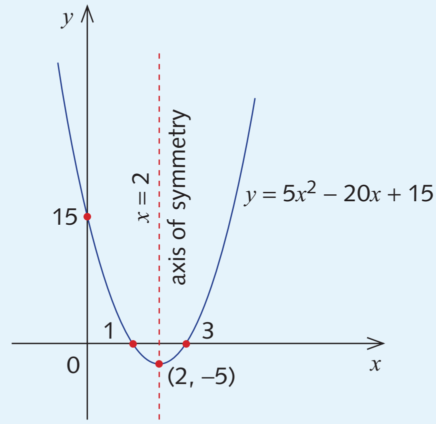
c $y = 5x^2 - 20x + 15$

When $x = 0$, $y = 15$. So the y-intercept is 15.

$$\begin{aligned} y &= 5x^2 - 20x + 15 \\ &= 5(x^2 - 4x + 3) \\ &= 5(x - 3)(x - 1) \end{aligned}$$

The two x-intercepts are $x = 1$ and $x = 3$.

Hence, the axis of symmetry is $x = 2$ and the vertex is $(2, -5)$.

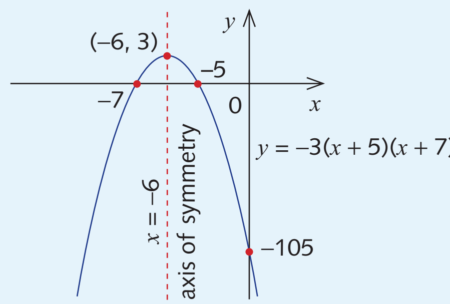


d $y = -3(x + 5)(x + 7)$

The parabola is upside-down with y-intercept -105.

The two x-intercepts are $x = -5$ and $x = -7$.

Hence, the axis of symmetry is $x = -6$ and the vertex is $(-6, 3)$.



e $y = -7x^2 + 1$

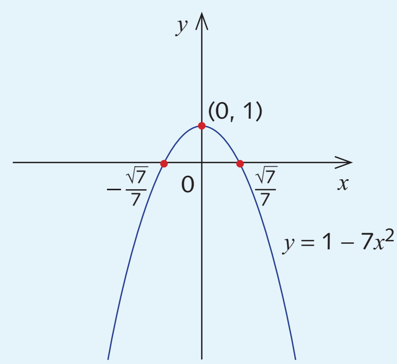
The y-intercept is 1, the axis of symmetry is $x = 0$ and the vertex is $(0, 1)$.

The parabola is upside-down.

When $y = 0$, $1 - 7x^2 = 0$

$$\begin{aligned} x^2 &= \frac{1}{7} \\ x &= \frac{1}{\sqrt{7}} \quad \text{or} \quad x = -\frac{1}{\sqrt{7}} \\ x &= \frac{\sqrt{7}}{7} \quad \text{or} \quad x = -\frac{\sqrt{7}}{7} \end{aligned}$$

So the x-intercepts are $x = -\dfrac{\sqrt{7}}{7}$ and $\dfrac{\sqrt{7}}{7}$.



Finding the equation of a parabola from the x-intercepts

If a parabola has two known x-intercepts at $x = u$ and $x = v$, or exactly one x-intercept at $x = t$, then we know the parabola has the form $y = a(x - u)(x - v)$ or $y = a(x - t)^2$, respectively. The value of a can be determined by substitution if we know the coordinates of some other point on the parabola.

Example 10

A parabola has x-intercepts $x = 5$ and $x = -5$, and y-intercept at 10. Find its equation.

Solution

$(x - 5)$ and $(x + 5)$ are factors. Therefore, $y = a(x - 5)(x + 5)$ for some $a \neq 0$.

When $x = 0$, $y = 10$.

$$\begin{aligned} \text{So } 10 &= a(-5)(5) \\ a &= -\frac{10}{25} = -\frac{2}{5} \\ \text{So } \quad y &= -\frac{2}{5}(x - 5)(x + 5). \end{aligned}$$

Exercise 7D

1 A parabola has x-intercepts of 1 and 7. What is the x-coordinate of the vertex?

2 A parabola has x-intercepts of -2 and 4. What is the x-coordinate of the vertex?

Example 9

3 Find the x-intercepts, the y-intercept and the vertex of each parabola and sketch it.

a $y = x(x - 4)$ **b** $y = (x - 3)(x - 2)$ **c** $y = 5(x + 1)(x - 3)$

d $y = 2(x + 1)(5 - x)$ **e** $y = 2(x - 5)(x - 6)$ **f** $y = 3(x - 1)(x + 2)$

g $y = 5(x - 4)(x + 2)$ **h** $y = -6(x - 4)(x + 3)$ **i** $y = 7(2x - 1)(x + 1)$

Example 9

4 Factor each quadratic and hence find the x-intercepts.

a $y = x^2 + 6x + 5$ **b** $y = x^2 + 7x + 12$ **c** $y = x^2 - 3x - 18$

d $y = x^2 + 2x - 15$ **e** $y = 2x^2 - 19x - 10$ **f** $y = 16 - x^2$

g $y = 1 - 4x^2$ **h** $y = x^2 - 3x$ **i** $y = 2x^2 + 8x$

Example 9

5 Sketch each parabola, clearly labelling the x- and y-intercepts, the axis of symmetry and the vertex.

a $y = x^2 - 6x + 8$ **b** $y = x^2 - 4x + 3$ **c** $y = x^2 + 4x - 12$

d $y = x^2 - 2x$ **e** $y = x^2 + 3x$ **f** $y = 2x^2 - 3x + 1$

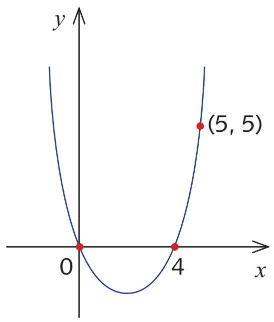
g $y = 2x^2 + 7x + 6$ **h** $y = 6x^2 - 7x + 2$ **i** $y = 8x^2 + 6x + 1$

j $y = 9 - x^2$ **k** $y = 3x - 2x^2$ **l** $y = 3 + x - 2x^2$

Example 10

6 Find the equation of each parabola.

a



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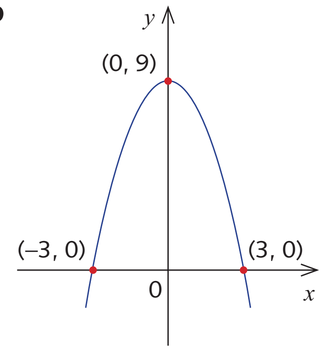

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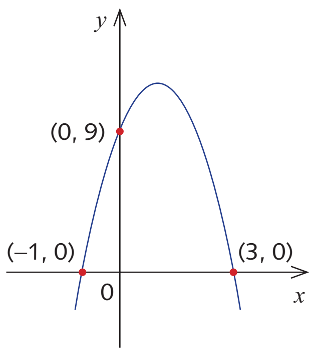

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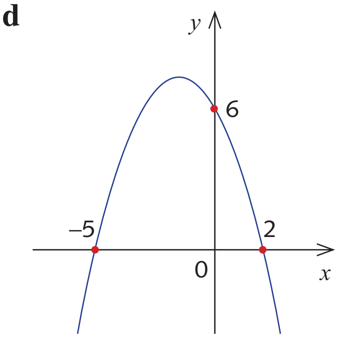


7 A parabola has x-intercepts of -1 and 2 and passes through the point $(4, 10)$. Find its equation.

8 A parabola has x-intercepts of -2 and 3 and y-intercept -3. Find its equation.

9 A parabola has x-intercepts of $\frac{1}{2}$ and 2 and passes through the point $(1, -3)$. Find its equation.

7E Sketching via the discriminant

In this section we will discuss how determining the value of the discriminant can assist with parabola sketching.

The discriminant $\Delta = b^2 - 4ac$ and sketching $y = ax^2 + bx + c$

We now have a few approaches to sketching parabolas when given in general form, $y = ax^2 + bx + c$.

- Complete the square to transpose the equation into the form $y = a(x - h)^2 + k$.
- Find the x-intercepts via factorisation and use the symmetry property to find the x-value of the vertex.
- Use the rule $x = -\dfrac{b}{2a}$ to find the x-value of the vertex.

In all three cases we need to label the y-intercept $(0, c)$, and label x-intercepts, if they exist.

In Chapter 5 we discussed the following techniques for finding x-intercepts (solving equations of the form, $ax^2 + bx + c = 0$).

- Completing the square
- Factorisation
- The quadratic formula

If you are asked to sketch $y = ax^2 + bx + c$, which approach should you take? Knowing the value of the discriminant, $\Delta = b^2 - 4ac$, can assist you in approaching the sketch efficiently. See the table below.

$\Delta = b^2 - 4ac$	Number of x-intercepts
$\Delta < 0$	0
$\Delta = 0$	1
$\Delta > 0$	2

The use of the discriminant is demonstrated in the following example.

Example 11

Sketch the following by whatever means, labelling all key features.

a $y = -3x^2 + 8x - 6$

b $y = 4x^2 - 4x - 3$

c $y = 5x^2 + 3x - 12$

Solution

a $y = -3x^2 + 8x - 6$

$$\begin{aligned}\Delta = b^2 - 4ac &= 8^2 - 4(-3)(-6)\\ &= 64 - 72 = -8\end{aligned}$$

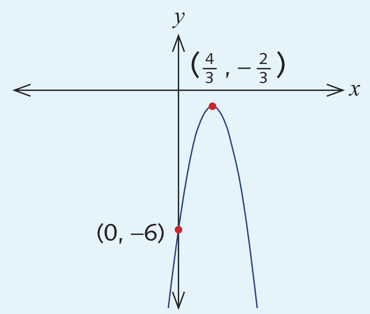
$\Delta < 0$, therefore there are no x-intercepts.

The axis of symmetry is:

$$x = -\frac{b}{2a} = -\frac{8}{2(-3)} = \frac{4}{3}$$

The y-value of the vertex is:

$$\begin{aligned}-3\left(\frac{4}{3}\right)^2 + 8\left(\frac{4}{3}\right) - 6 &= -\frac{48}{9} + \frac{32}{3} - 6\\ &= \frac{-48 + 96 - 54}{9}\\ &= -\frac{2}{3}\end{aligned}$$



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b $y = 4x^2 - 4x - 3$

$$\begin{aligned}\Delta = b^2 - 4ac &= (-4)^2 - 4(4)(-3)\\ &= 16 + 48 = 64\end{aligned}$$

$\Delta > 0$ and is the square of a rational number, therefore there are two *rational* x-intercepts.

Find the x-intercepts via factorising:

$$\begin{aligned}4x^2 + 2x - 6x - 3 &= 0\\ 2x(2x+1) - 3(2x+1) &= 0\\ (2x+1)(2x-3) &= 0\end{aligned}$$

$$2x + 1 = 0 \quad \text{or} \quad 2x - 3 = 0$$

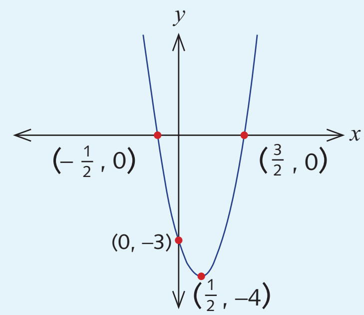
$$x = -\frac{1}{2} \text{ or } \qquad x = \frac{3}{2}$$

Axis of symmetry:

$$x = \left(-\frac{1}{2} + \frac{3}{2}\right) \div 2 = \frac{1}{2}$$

y-value of vertex is:

$$4\left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right) - 3 = 1 - 2 - 3 = -4$$



c $y = 5x^2 + 3x - 12$

$$\begin{aligned}\Delta = b^2 - 4ac &= (3)^2 - 4(5)(-12)\\ &= 9 + 240 = 249\end{aligned}$$

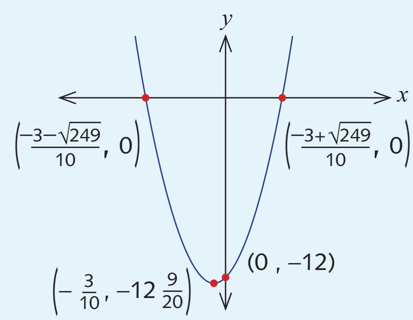
$\Delta > 0$ but is not the square of a rational number, therefore there are two *irrational* x-intercepts.

The axis of symmetry is:

$$x = -\frac{b}{2a} = -\frac{3}{2(5)} = -\frac{3}{10}$$

The y-value of the vertex is:

$$\begin{aligned}5\left(-\frac{3}{10}\right)^2 + 3\left(-\frac{3}{10}\right) - 12 &= \frac{45}{100} - \frac{9}{10} - 12\\ &= -12\tfrac{9}{20}\end{aligned}$$



Find the x-intercepts via the quadratic formula: $a = 5,\ b = 3,\ c = -12$

$$x = \frac{-3 - \sqrt{249}}{10} \quad \text{or} \quad x = \frac{-3 + \sqrt{249}}{10}$$

Exercise 7E

Example 11

For each of the following, determine the value of the discriminant and hence sketch the parabola using the most efficient approach. Label all intercepts and the vertex.

a $y = 4x^2 - 2x$ **b** $y = 3x^2 - 12x + 12$ **c** $y = x^2 + 4x + 2$

d $y = 2x^2 - 4x + 5$ **e** $y = -2x^2 + 6x + 3$ **f** $y = -x^2 - 4x + 12$

g $y = 3x - 4x^2$ **h** $y = 4x^2 + 12x + 15$ **i** $y = 25x^2 - 30x + 9$

j $y = -5x^2 - 4x + 10$ **k** $y = -4 + x - 2x^2$ **l** $y = 4x^2 - 4x - 15$

7F Applications involving quadratics

Many practical problems can be solved using quadratics.

For example:

$$s = 30t - 4.9t^2$$

is a formula used to calculate the height, s metres, of a cricket ball t seconds after it has been thrown in the air vertically with an initial speed of 30 m/s.

Example 12 shows a problem about a right-angled triangle that leads to a quadratic equation.

Example 12

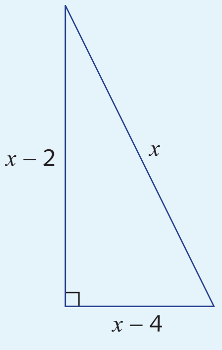
The two sides of a right-angled triangle are, respectively, 2 cm and 4 cm shorter than the hypotenuse. Find the side lengths of the triangle.

Solution

Let x cm be the length of the hypotenuse. Then the two other sides have lengths $(x - 2)$ cm and $(x - 4)$ cm.

By Pythagoras' theorem:

$$\begin{aligned} x^2 &= (x-2)^2 + (x-4)^2 \\ x^2 &= x^2 - 4x + 4 + x^2 - 8x + 16 \\ 0 &= x^2 - 12x + 20 \\ (x-10)(x-2) &= 0 \\ x &= 10 \text{ or } 2 \end{aligned}$$



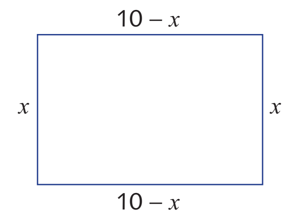
But the solution $x = 2$ is impossible, since it leads to a triangle with a negative side length.

Hence, $x = 10$ and the side lengths are 6 cm, 8 cm and 10 cm.

Minimum-maximum problems

Suppose we have 20 centimetres of wire, which is to be bent to form a rectangle. The area of the rectangle will change as its dimensions change, as you can see in the table below.

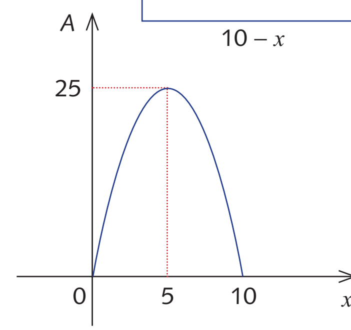
Length	2	2.5	4	5	6	7
Width	8	7.5	6	5	4	3
Area	16	18.75	24	25	24	21



Suppose that we have a rectangle with perimeter 20 cm. Let x cm be the length. Then the width is $(10 - x)$ cm.

Suppose that the area is $A\,\text{cm}^2$, then $A = x(10 - x)$
$= 10x - x^2$

The graph of A against x is an upside-down parabola, but note that the x-values are restricted to between 0 and 10.



The x-intercepts are 0 and 10, so the parabola has $x = 5$ as its axis of symmetry. Hence, it has a maximum value of 25 at $x = 5$. This value of x makes the rectangle into a square.

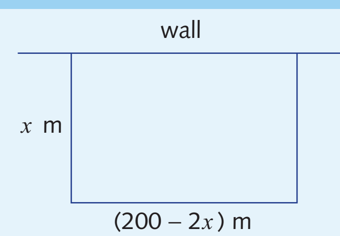
Thus, of all the rectangles with fixed perimeter, the square is the one with greatest area, as you may have noticed from the table of values.

The idea of maximising (or minimising) a quantity using parabolas has many uses.

Example 13

A farmer needs to construct a small rectangular paddock using a long wall for one side of the paddock. He has enough posts and wire to erect 200 m of fence. What are the dimensions of the paddock if the fences are to enclose the largest possible area?

Solution

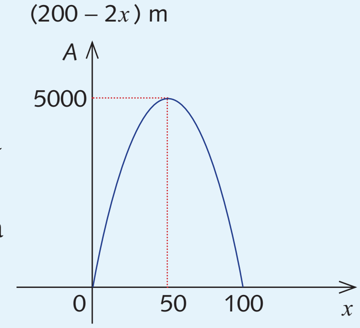


Let x m be the length of the side perpendicular to the wall.

Then the length of the side parallel to the wall is $(200 - 2x)$ m.

Let A m^2 be the area of the paddock.

Then $A = x(200 - 2x)$
$= 200x - 2x^2$



Hence, $A = 0$ when $x = 0$ or $x = 100$ (as in the sketch). Taking the average, the axis of symmetry is $x = 50$ and the maximum area occurs when $x = 50$.

Thus, the dimensions of the paddock are 50 m by 100 m, and the area is 5000 square metres.

As we saw in Example 12, the algebra can lead to a number of possible answers. Some of these may not be admissible as they do not satisfy the requirements of the question.

Exercise 7F

The solutions to most of the practical problems should begin with a diagram.

1 A rectangular paddock is 50 m longer than it is wide. If the area of the paddock is 10 400 m^2, find its dimensions.

2 A large triangular road sign with base length equal to its height has an area of 1800 cm^2. Find the length of the base.

3 The product of two consecutive positive integers is 650. Find the numbers.

4 The product of two consecutive even numbers is 224. Find the numbers.

5 The product of two consecutive odd numbers is 195. Find the numbers.

6 One more than a certain positive number is five less than the number squared. Find the number.

7 If Tom's age is squared, it will be equal to his age in 56 years' time. How old is he now?

8 If the amount of Karlima's savings is squared and then doubled, the amount would be \$66 more than her savings now. How much has she saved?

Example 12

9 A right-angled triangle has one side 7 cm longer than the side perpendicular to it. If the hypotenuse is 17 cm, find the side lengths of the triangle.

10 A right-angled triangle has hypotenuse 9 cm longer than its shortest side. Given that the third side is 21 cm long, find the side lengths of the triangle.

11 A piece of sheet metal 50 cm by 40 cm has squares cut out of each corner so that it can be bent and formed into a lidless box with a base area of 1200 cm^2. Find the dimensions of the box.

12 The formula for finding the number of diagonals of a convex polygon with n sides is $\frac{n}{2}(n-3)$. How many sides does a polygon with 902 diagonals have? (As an interesting counting argument, prove the formula for the number of diagonals of a convex polygon.)

13 Show that the sum of the first n positive integers is $\frac{n(n+1)}{2}$. How many integers are needed to produce a sum of 136?

14 The height (h metres) of an arrow above the ground, t seconds after release from the bow, is given by $h = 23.7t - 4.9t^2$. Find the time taken for the arrow to reach a height of 27 metres, correct to two decimal places.

15 What is the minimum value of $x^2 - 6x + 2$?

16 What is the maximum value of $-x^2 + 3x - 1$?

17 What is the maximum and minimum value of $3x^2 + 7x - 2$ if:

a $-3 \le x \le 0$?

b $0 \le x \le 3$?

18 A piece of wire is 100 cm long. Find the dimensions of the rectangle formed by bending this wire when the area is a maximum.

Example 13

19 A farmer has a straight fence along the boundary of his property. He wishes to fence an enclosure for a bull and has enough materials to erect 300 m of fence. What would be the dimensions of the largest possible rectangular paddock, assuming that he uses the existing boundary fence as one of its sides?

20 The height, h metres, reached by a ball after t seconds when thrown vertically upwards is given by $h = 25t - 4.9t^2$. Find, correct to three decimal places, the maximum height reached and the time the ball is in the air.

21 A rectangular piece of land of area 5000 m^2 is to be enclosed by a wall, and then divided into three equal regions by partition walls parallel to one of its sides. If the total length of the walls is 445 m, calculate the possible dimensions of the land.

22 A rectangle is constructed so that one vertex is at the origin, and another vertex is on the graph of $y = 3 - \dfrac{2x}{3}$, where $x > 0$ and $y > 0$ and adjacent sides are on the axes. What is the maximum possible area of the rectangle?

7G Quadratic inequalities

In this section, we answer such questions as:

For which values of x is $x^2 - 1 < 0$?

When is $x^2 + 8x + 7 \geq 0$?

Recall the method for solving a linear inequality.

For example:

$$\begin{aligned} 3x + 7 &< 5x + 11 \\ -2x &< 4 \\ x &> -2 \end{aligned}$$

Note: When dividing through by a negative number, the inequality is reversed.

When solving quadratic inequalities, a graphical technique is used. The following examples explore this technique.

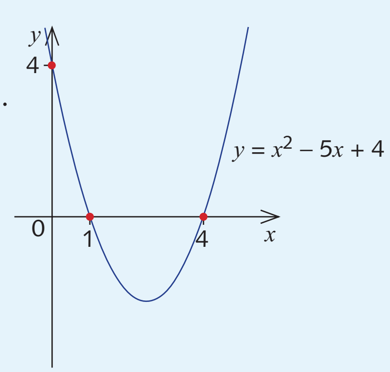
Example 14

a Solve the inequality $x^2 - 5x + 4 < 0$.

b Solve the inequality $x^2 - 5x + 4 \geq 0$.

Solution

The graph of $y = x^2 - 5x + 4 = (x - 4)(x - 1)$ is drawn.



a The y-values are negative when the graph is below the x-axis.
Thus, $x^2 - 5x + 4 < 0$ if $1 < x < 4$.

b The y-values are positive when the graph is above the x-axis.
Thus, $x^2 - 5x + 4 > 0$ if $x > 4$ or $x < 1$.

The y-values are equal to 0 when $x = 4$ or $x = 1$.

Thus, $x^2 - 5x + 4 \geq 0$ if $x \geq 4$ or $x \leq 1$.

Example 15

a Solve the inequality $-x^2 + 5x - 6 < 0$. **b** Solve the inequality $-x^2 + 5x - 6 \geq 0$.

Solution

The graph of $y = -x^2 + 5x - 6 = -(x - 2)(x - 3)$ is drawn.

a The y-values are negative when the graph is below the x-axis.

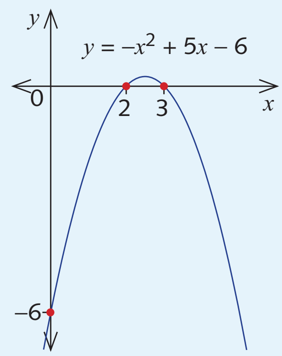
Thus, $-x^2 + 5x - 6 < 0$ if $x < 2$ or $x > 3$.

b The y-values are positive when the graph is above the x-axis.

Thus, $-x^2 + 5x - 6 > 0$ if $2 < x < 3$.

The y-values are equal to 0 when $x = 2$ or $x = 3$.

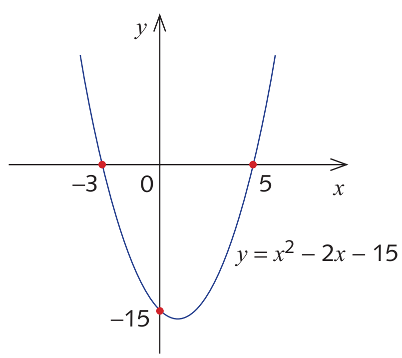
Thus, $x^2 + 5x - 6 \geq 0$ if $2 \leq x \leq 3$.



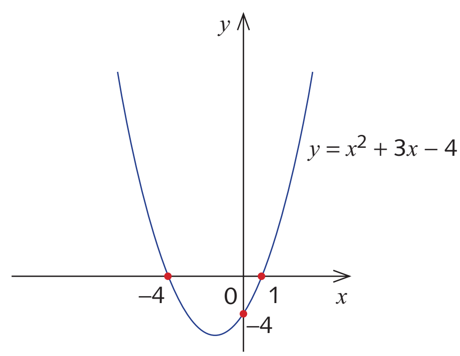
Exercise 7G

1 Use the graphs given to find the set of x-values described by each inequality.

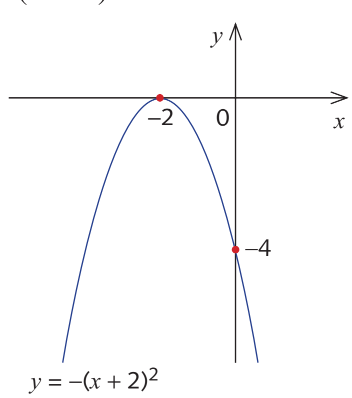
a $x^2 - 2x - 15 < 0$



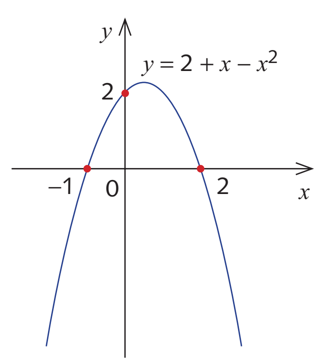
b $x^2 + 3x - 4 > 0$



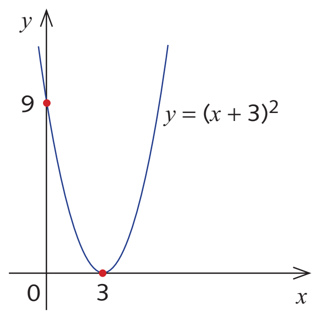
c $-(x + 2)^2 \geq 0$



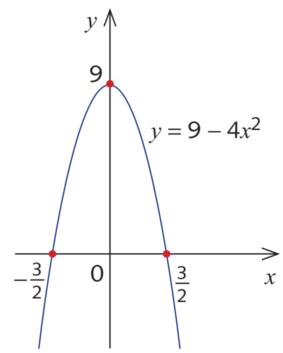
d $2 + x - x^2 \leq 0$



e $(x+3)^2 \geq 0$



f $9 - 4x^2 < 0$



2 Sketch a graph and find all values of x such that:

a $(x-3)(x+2) > 0$ **b** $(x+1)(x+4) \leq 0$

c $(x-5)(x-2) \geq 0$ **d** $x(x+3) < 0$

Example 14

3 Solve the quadratic inequalities.

a $x^2 + 3x - 70 > 0$ **b** $x^2 - 5x - 24 < 0$

c $x^2 + 9x + 20 \geq 0$ **d** $x^2 - 7x + 12 \leq 0$

Example 15

4 Solve the quadratic inequalities.

a $-x^2 - 3x + 40 > 0$ **b** $-x^2 + 5x + 24 \geq 0$

c $-x^2 + 12x - 35 \leq 0$ **d** $-x^2 + 11x < 0$

Review exercise

1 Find the y-intercept of:

a $y = x^2 + 5x + 2$ **b** $y = 5(x-3)^2 - 21$ **c** $y = 3x^2 + 2x$

d $y = 2 - 2(x+1)^2$ **e** $y = 5 - (x-1)^2$ **f** $y = -3(x+2)^2 - 4$

2 Consider the parabola $y = (x-h)^2 + 5$. Find the value of h if the y-intercept is:

a 5 **b** 21 **c** 14 **d** 9

3 Find the x-intercepts of each parabola.

a $y = x^2 + 3x - 4$ **b** $y = 2x^2 + 13x + 6$ **c** $y = 8x^2 - 6x - 9$

d $y = 8x^2 - 16x - 10$ **e** $y = x^2 - 49$ **f** $y = 2x^2 - 10x$

4 Find the exact values of the x-intercepts of each parabola.

a $y = (x+2)^2 - 5$ **b** $y = (x-3)^2 - 2$ **c** $y = 2(x+1)^2 - 10$

d $y = 3(x-2)^2 - 15$ **e** $y = 5(x-3)^2 - 7$ **f** $y = 6 - 3(x-2)^2$

5 Find the exact values of the x-intercepts by completing the square.

a $y = x^2 + 4x - 2$

b $y = x^2 - 6x + 1$

c $y = 2x^2 + 10x + 3$

d $y = -2x^2 - 8x + 5$

6 State whether the graph of each quadratic has a maximum or minimum turning point (vertex).

a $y = x^2 + 6x - 5$

b $y = -x^2 + 2x + 1$

c $y = 7 - 2x - 3x^2$

d $y = 3x^2 - 2x + 1$

7 Determine which pairs of parabolas are congruent.

$y = x^2$, $y = -2x^2$, $y = 3x^2$, $y = 3x^2 + 1$,

$y = 2 + 3x - 4x^2$, $y = 3 - 2x^2$, $y = x^2 - x$, $y = 1 + 4x^2$

8 State the transformations that need to be applied to the graph of $y = x^2$ to obtain the graph of:

a $y = x^2 - 1$

b $y = x^2 + 2$

c $y = 4 - x^2$

d $y = 1 - x^2$

Note: There are many possible answers to this question.

9 State the transformations that need to be applied to the graph of $y = x^2$ to obtain the graph of:

a $y = (x + 2)^2$

b $y = (x - 1)^2$

c $y = -(x + 1)^2$

d $y = (x + 1)^2 - 3$

e $y = (x - 2)^2 - 3$

f $y = 1 - (x - 3)^2$

10 Write the equation of the parabola obtained when the graph of $y = x^2$ is:

a translated 2 units to the left

b translated 3 units to the right and 1 unit up

c translated 2 units down and 5 units to the right

d translated 3 units to the left and 2 units down

11 Write the equation of the parabola obtained when the graph of $y = 3x^2$ is:

a translated 3 units to the left and 2 units up

b translated 3 units to the right and 2 units down

12 Write the equation of the parabola obtained when the graph of $y = x^2$ is:

a reflected in the x-axis and translated 1 unit to the right

b reflected in the x-axis and translated 2 units to the left

c reflected in the x-axis, then translated 1 unit to the left and 2 units down

13 For each parabola, state the coordinates of the vertex.

a $y = (x - 1)^2 + 2$ **b** $y = (x + 2)^2 + 3$ **c** $y = (x + 4)^2 - 2$

d $y = (x - 5)^2 + 11$ **e** $y = -3(x + 2)^2 - 1$ **f** $y = 4 - 2(x - 3)^2$

14 A parabola has vertex $(1, -2)$ and passes through the point $(3, 2)$. Find its equation.

15 A parabola has x-intercepts of -5 and 3 and passes through the point $(1, -12)$. Find its equation.

16 A parabola has x-intercepts of -2 and -4 and a y-intercept of -8. Find its equation.

17 Sketch the graph of each quadratic, clearly labelling the x- and y-intercepts, the axis of symmetry and the vertex.

a $y = x^2 - 6x + 5$ **b** $y = x^2 - 4x - 12$ **c** $y = x^2 - 3x$

d $y = x^2 + 5x$ **e** $y = 16 - x^2$ **f** $y = 3x - 9x^2$

18 Sketch the graph of each quadratic, clearly labelling the x- and y-intercepts, the axis of symmetry and the vertex.

a $y = (x - 3)^2 + 4$

b $y = 3(x + 1)^2 - 6$

c $y = 5 - (x + 3)^2$

d $y = 6 - 3(x - 5)^2$

19 A parabola has vertex $(2, -4)$ and passes through the point $(1, 7)$. Find its equation.

20 A parabola has equation $y = 3(x + h)^2 + 4$ and y-intercept 7. Find the value of h.

21 Sketch the graph of each quadratic, clearly labelling the x- and y-intercepts, the axis of symmetry and the vertex.

a $y = x^2 + 2x - 5$ **b** $y = x^2 - 6x + 2$ **c** $y = -x^2 - 4x - 7$

d $y = -x^2 + 8x - 13$ **e** $y = 2x^2 + 4x + 5$ **f** $y = 7 + 6x - 2x^2$

22 In a right-angled triangle, one side is 7 cm longer than its shortest side and the hypotenuse is 8 cm longer than its shortest side. Find the side lengths of the triangle.

23 A piece of sheet metal 50 cm × 60 cm has squares cut out of each corner so that it can be bent and formed into a lidless box with a base area of $2184\ \text{cm}^2$. Find the length, width and height of the box.

24 A farmer has a straight fence along the boundary of his property. He wishes to fence an enclosure for a bull and has enough materials to erect 500 m of fence. What would be the dimensions of the largest possible paddock, assuming that he uses the existing boundary fence as one of its sides?

25 By considering a graph, solve:

a $(x - 5)(x + 3) < 0$

b $(x + 2)(x + 5) \le 0$

c $x(x - 2) > 0$

Challenge exercise

1 **a** What is the maximum value of $2x^2 + 9x - 5$ if $-2 \le x \le 0$?

b What is the minimum value of $2x^2 + 9x - 5$ if $0 \le x \le 2$?

2 Consider the quadratic inequality $x^2 + 4x + c \le 0$. For each of the following sets of values of x, find the values of c for which the given set satisfies the inequality:

a $-7 \le x \le 3$

b $x = -2$

c no x values

3 The distance between two towns is 120 km by road and 150 km by rail. A train takes 10 minutes longer than a car, whose average speed is 10 km/h less than the train's average speed.

The purpose of this problem is to find the average speed of the car.

a Let the average speed of the car be x km/h and let the time taken by the car be t hours. Show that the information in the question gives:

$$xt = 120 \qquad (1)$$

$$(x+10)\left(t+\frac{1}{6}\right) = 150 \qquad (2)$$

b Subtract (1) from (2) to obtain a linear equation linking x and t.

c Make t the subject of this linear equation, substitute it into (1) and solve for x, obtaining $x = 80$ or $x = 90$.

d Calculate the corresponding values of t and check that both pairs of solutions make sense.

The next six questions are similar to the previous question. That is, it is best to introduce two variables, eliminate one and then solve the resulting quadratic equation. Do not forget to check that the solutions are feasible; that is, that they make sense and satisfy the original problem.

4 A train could save 1 hour on a journey of 200 km by increasing its average speed by 10 km/h. What is the original speed of the train?

5 A farmer purchased a number of cattle for \$3600. Five of them died, but he sold the remainder at \$20 per head more than he paid for them, making a profit of \$400. How many did he buy?

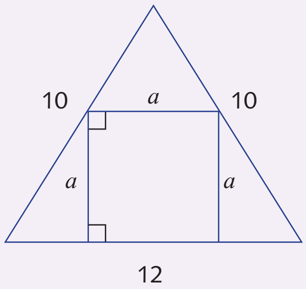
6 The distance between two towns is 80 km by road and 90 km by rail. A car takes 15 minutes longer than a train, whose average speed is 8 km/h greater than the car's average speed. Find the average speed of the car and of the train.

7 In cricket, batting average = $\dfrac{\text{total number of runs scored}}{\text{number of times out}}$. In a season, a cricketer scored 1800 runs. If he had been out on one more occasion, his average would have been three runs less. What is his average?

8 Two boys, one of whom can run 1 m/s faster than the other, compete in a 400 m race. The slower competitor is given a 20 m start and loses by 10 seconds. What was the average speed of each runner (correct to three decimal places)?

9 A and B are two towns, 120 km apart. A car starts from A to travel to B at the same time as a second car, whose speed is 20 km/h faster than the first, starts from B to travel to A. The slower car reaches B 1 hour and 48 minutes after it passes the other car. Find their speeds.

10 The diagram shows a square inscribed in an isosceles triangle with side lengths 10, 10 and 12. Find a.



11 To solve $x^2 - gx + h = 0$ graphically, let A be the point $(0, 1)$ and B the point (g, h). Draw a circle with AB as its diameter. Then the points (if any) where the circle cuts the x-axis are the roots of $x^2 - gx + h = 0$.

a Illustrate the method by graphically solving $x^2 - 5x + 6 = 0$.

b Prove that the method works.

Note: This construction is called Carlyle's method.

12 Take a piece of string of length 100 cm. Cut it into two pieces, x cm and $(100 - x)$ cm, and form the first piece into a circle and the other into a square.

a Write down a quadratic expression for the combined area enclosed by the separate pieces.

b Find the minimum possible sum of the two areas and the value of x for which it occurs.

13 Recall that two geometric figures are by definition *congruent* if there is a sequence of translations, rotations and reflections taking one figure to the other. Also recall that two geometric figures are *similar* if we can enlarge one figure so that its enlargement is congruent to the other figure.

In this question we will show that all parabolas are similar. It is not, however, true that all parabolas are congruent.

a Explain why the ideas in Section 7B show that every parabola $y = x^2 + ax + b$ is congruent to the basic parabola $y = x^2$.

b Explain why the ideas in Section 7B show that every parabola $y = -x^2 + ax + b$ is congruent to the basic parabola $y = x^2$.

c Let $a > 0$. Explain why the ideas in Section 7C show that every parabola $y = ax^2 + bx + c$ and every parabola $y = -ax^2 + bx + c$ is congruent to the parabola $y = ax^2$.

d Every point on $y = x^2$ has coordinates (p, p^2) for some p. Find a similar expression for the points on $y = 5x^2$. Show that the transformation taking (x, y) to $(x, 5y)$ maps $y = x^2$ to $y = 5x^2$. Show that this transformation is not a similarity transformation.

e Show that there is an enlargement that takes $y = x^2$ to $y = 5x^2$.

f Show that all parabolas are similar.

14 In Section 7D we discussed methods for sketching parabolas using symmetry about the axis of symmetry. Here is another method.

For the parabola $y = ax^2 + bx + c$:

First find the two points where $y = c$ meets the parabola. These are $(0, c)$ and $\left(-\frac{b}{a}, c\right)$.

Then find the vertex, knowing that the x-coordinate of the vertex is the average of the x-coordinates 0 and $-\frac{b}{a}$. Sketch the parabola using these three points. Use this method to sketch:

a $y = x^2 + 8x + 17$

b $y = 2x^2 + 5x - 3$

c $y = dx^2 + ex - f$

CHAPTER

8

Measurement and Geometry

Review of congruence and similarity

This chapter reviews our knowledge of geometry. In particular, we review congruence tests and similarity tests for triangles.

Congruence and similarity are extremely useful tools in geometrical arguments. Both congruence and similarity have many applications and you will meet some of these in this chapter.

8A Review of triangles

An initial discussion of the properties of triangles appeared in *ICE-EM Mathematics Year 8* and *ICE-EM Mathematics Year 9*. We briefly review them here.

Triangles

We recall:

- The sum of the interior angles of a triangle is 180°.
- An exterior angle of a triangle equals the sum of the opposite interior angles.

Isosceles and equilateral triangles

- The base angles of an isosceles triangle are equal.
- Conversely, if two angles of a triangle are equal, then the sides opposite those angles are equal.
- Each interior angle of an equilateral triangle is 60°.
- Conversely, if the three angles of a triangle are equal, then the triangle is equilateral.

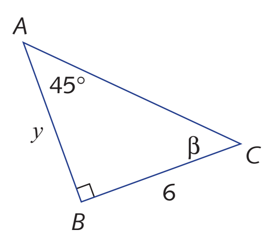
Polygons

- The angle sum of a quadrilateral is 360°.
- The sum of the interior angles of a convex polygon is $(n - 2)\,180°$.
- The sum of the exterior angles of a convex polygon is 360°.

Exercise 8A

1 Find the values of x, y, α, β and γ.

a



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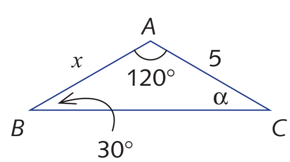

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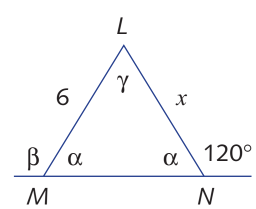

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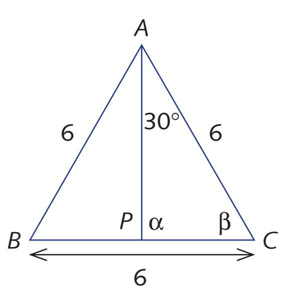

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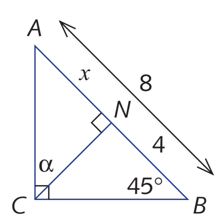

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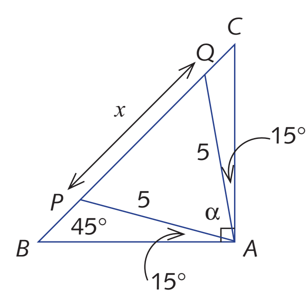

2 Find the values of x, y, α, β, γ and θ. Give reasons in your solutions. Points marked O are the centres of circles.

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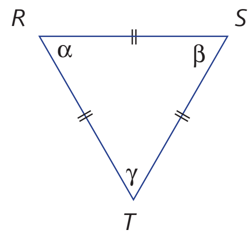

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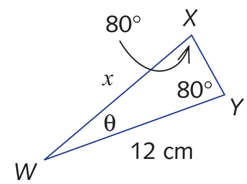

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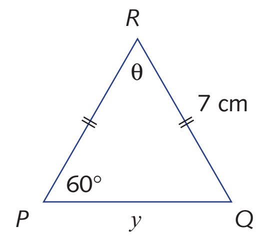

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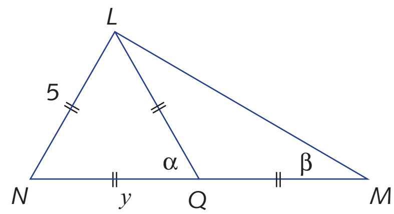

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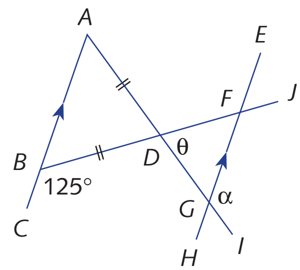

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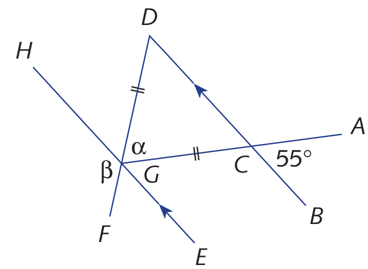

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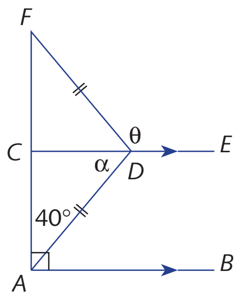

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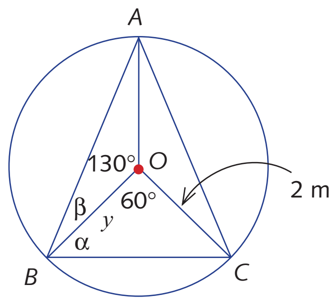

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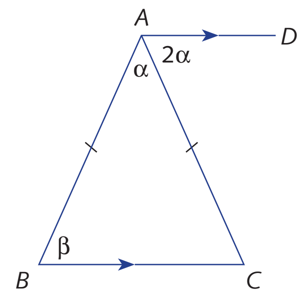

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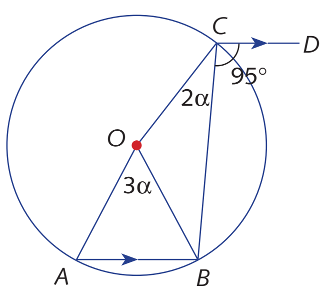

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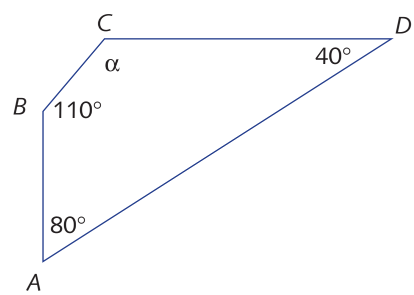

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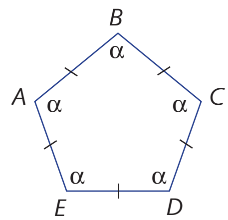

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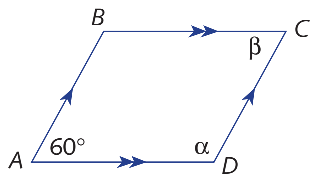


3 The exterior angles of a regular polygon are each 60°. How many sides does the polygon have?

4 Three angles of a pentagon are each 156° and the remaining angles are equal. Find the size of the two remaining angles.

8B Congruence

In *ICE-EM Mathematics Year 8* and *Year 9* we introduced the idea of **congruent figures**.

Congruent figures

- Two plane figures are called **congruent** if one figure can be moved on top of the other figure, by a sequence of translations, rotations and reflections, so that they coincide exactly.
- Congruent figures have exactly the same shape and size.
- When two figures are congruent, we can match up every part of one figure with the corresponding part of the other, so that:
  – matching angles have the same size
  – matching intervals have the same length
  – matching regions have the same area.

The congruence arguments used in this chapter involve only congruent triangles. In *ICE-EM Mathematics Year 8* and *Year 9* we developed four tests for two triangles to be congruent, as follows.

The four standard congruence tests for triangles

Two triangles are congruent if:

SSS: the three sides of one triangle are respectively equal to the three sides of the other triangle, **or**

AAS: two angles and one side of one triangle are respectively equal to two angles and the matching side of the other triangle, **or**

SAS: two sides and the included angle of one triangle are respectively equal to two sides and the included angle of the other triangle, **or**

RHS: the hypotenuse and one side of one right-angled triangle are respectively equal to the hypotenuse and one side of the other right-angled triangle.

The statement 'Triangle ABC is congruent to triangle PQR' is written as:

$$\Delta ABC \equiv \Delta PQR,$$

where the vertices are written in matching order.

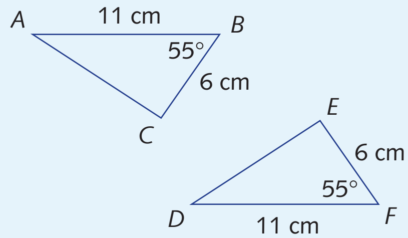
When a congruence test is used to justify the congruence, the test's initials are placed in brackets after the congruence statement, as in Example 1 on the next page.

Example 1

If the two triangles are congruent, write down a congruence statement and the congruence test used to justify it.

If they are not, explain why not.

a



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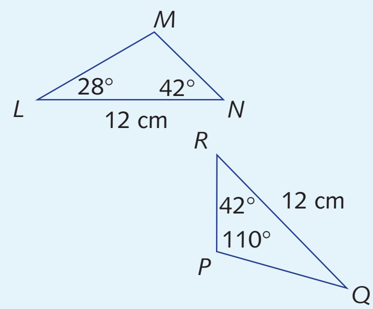

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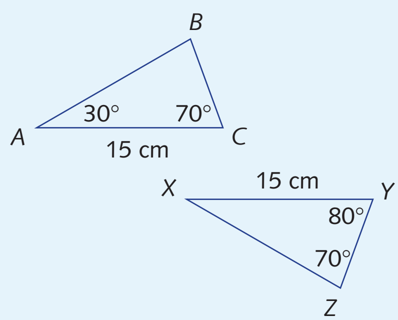

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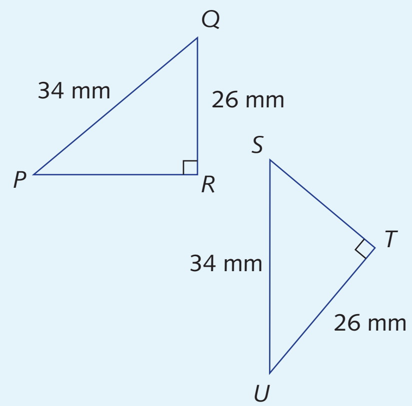


Solution

a $\Delta ABC \equiv \Delta DFE$ (SAS)

b In $\Delta PQR, \angle RQP = 180° - (42 + 110)°$

$= 28°$

So $\Delta LMN \equiv \Delta QPR$ (ASA)

c In ΔABC, $\angle ABC = 80°$. In ΔXYZ, $\angle ZXY = 30°$. $XZ \neq 15$ cm, since ΔXYZ is not isosceles. Hence, ΔABC is not congruent to ΔXYZ, because $AC \neq XZ$.

d $\Delta PQR \equiv \Delta UST$ (RHS)

Quadrilaterals

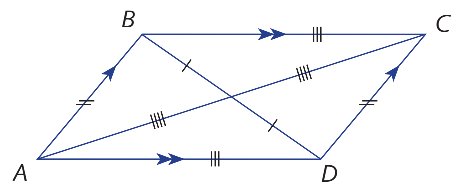
The sum of the interior angles of a quadrilateral is 360°.

Congruence of triangles is used to establish properties of special quadrilaterals. A proof for each of the properties listed overpage was given in *ICE-EM Mathematics Year 9*.

Parallelograms

A **parallelogram** is a quadrilateral whose opposite sides are parallel.

A parallelogram has the following properties:



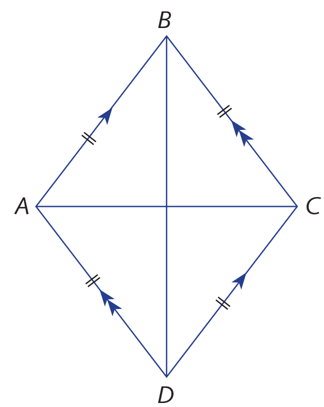
- The opposite angles of a parallelogram are equal.
- The opposites sides of a parallelogram are equal.
- The diagonals of a parallelogram bisect each other.

Here are four well known tests for a parallelogram:

- If the opposite angles of a quadrilateral are equal, then the quadrilateral is a parallelogram.
- If the opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram.
- If one pair of opposite sides of a quadrilateral are equal and parallel, then the quadrilateral is a parallelogram.
- If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

Rhombuses

A **rhombus** is a quadrilateral with four equal sides. A rhombus is a parallelogram. (We also note that because the opposite sides of a parallelogram are equal, it is always sufficient to establish that just two adjacent sides are equal.)



The following are properties of a rhombus:

- The diagonals of a rhombus bisect each other at right angles.
- The diagonals of a rhombus bisect the vertex angles through which they pass.

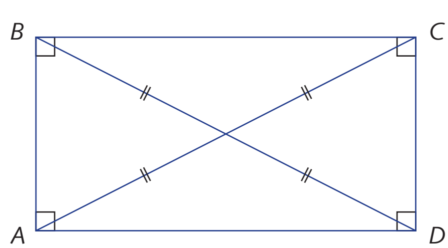
Here are two tests for whether a quadrilateral is a rhombus.

- If a quadrilateral is a parallelogram with two adjacent sides equal, then the parallelogram is a rhombus.
- If the diagonals of a quadrilateral bisect each other at right angles, then the quadrilateral is a rhombus.

Rectangles

A **rectangle** is a quadrilateral in which all angles are right angles.

The following are properties of a rectangle:



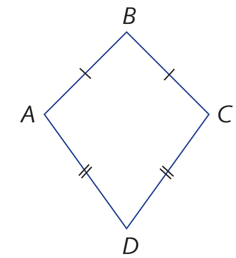
- A rectangle is a parallelogram.
  - Its opposite sides are equal and parallel.
  - Its diagonals bisect each other.
- The diagonals of a rectangle are equal.

Here are three tests for a rectangle:

- A parallelogram with one right angle is a rectangle.
- If all angles of a quadrilateral are equal, then the quadrilateral is a rectangle.
- If the diagonals of a quadrilateral are equal and bisect each other, then the quadrilateral is a rectangle.

Kites

A **kite** is a quadrilateral with two pairs of adjacent equal sides.



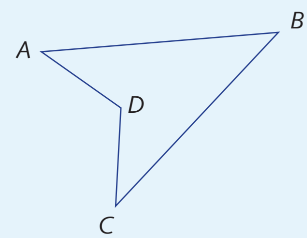
Example 2

Congruence is used to prove many results with quadrilaterals and triangles.

In the kite $ABCD$,

$DA = DC$ and $BA = BC$.

Prove that $\angle BAD = \angle BCD$.



Solution

Join D to B.

In the triangles ABD and CBD,

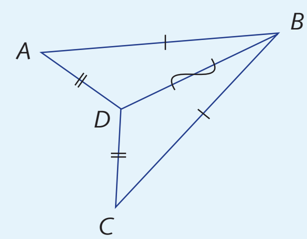
$BA = BC$ (given)

$DA = DC$ (given)

DB is common,

so $\Delta BDA \equiv \Delta BDC$ (SSS)

Hence, $\angle BAD = \angle BCD$ (matching angles of congruent triangles)

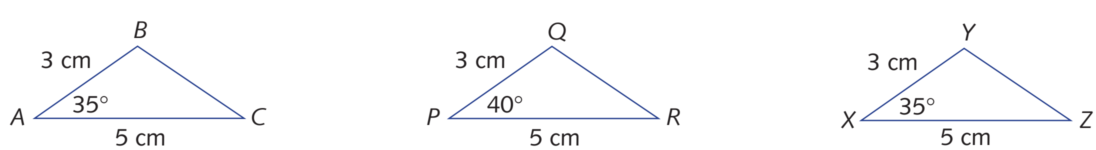


Exercise 8B

Example 1

1 In each part, find a pair of congruent triangles. State the congruence test used.

a



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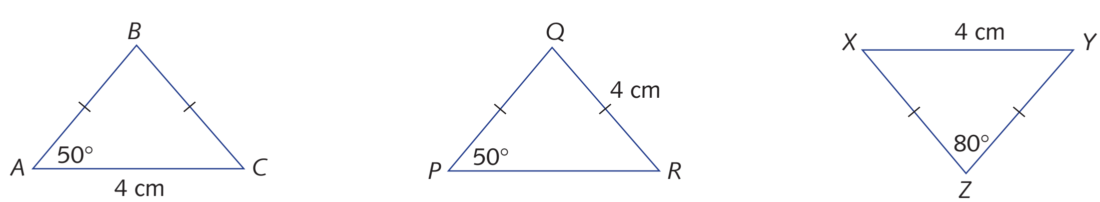

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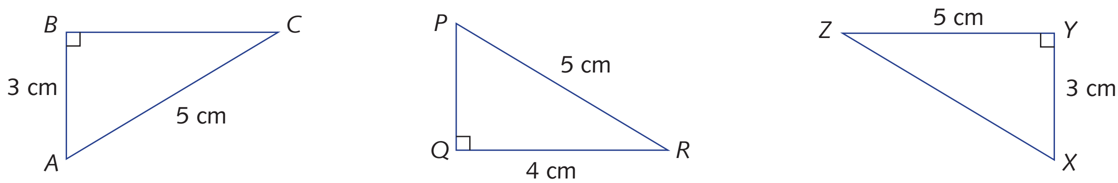

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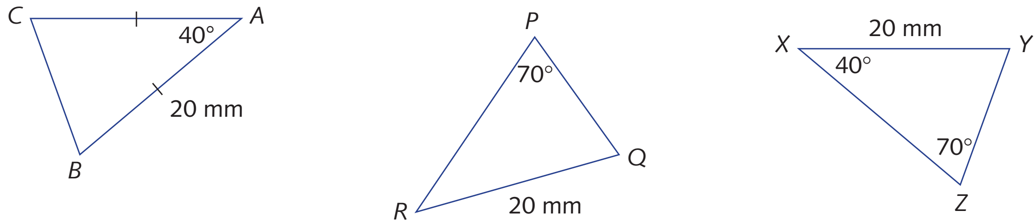

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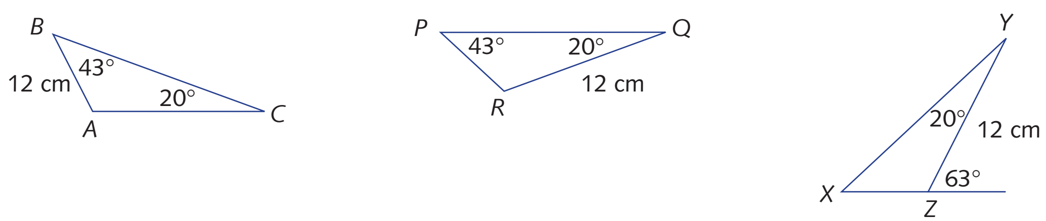
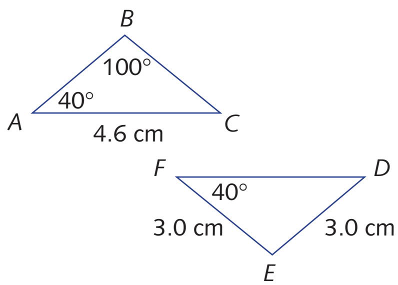


2 In each part, it is known that $\Delta ABC \equiv \Delta DEF$. Determine the unknown angles and side lengths. (Side lengths are given correct to one decimal place.)

a



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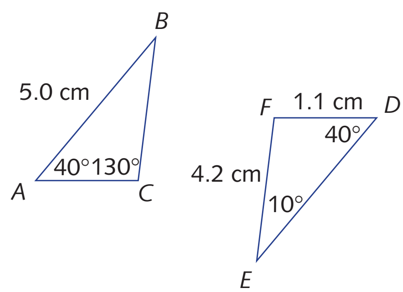

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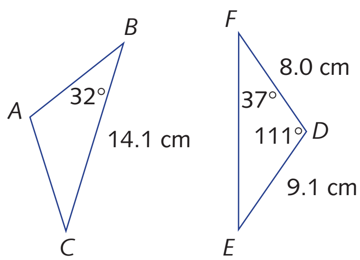

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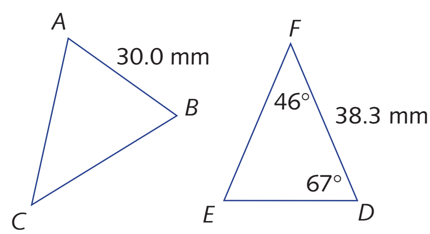
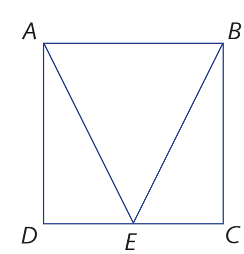


Example 2

3 In the diagram at the right, $ABCD$ is a square and $DE = EC$.

a Draw a diagram and prove that $\Delta ADE \equiv \Delta BCE$.

b Prove that $AE = BE$.

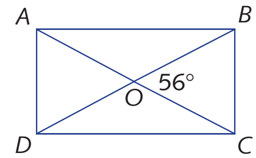


4 $PRSV$ is a square. The midpoint of PV is X and T is the midpoint of SV.

a Draw a diagram and prove that $RX = RT$.

b Join RV and prove that $\angle TRV = \angle XRV$.

5 The diagonals of a rectangle $ABCD$ meet at O and $\angle BOC = 56°$.

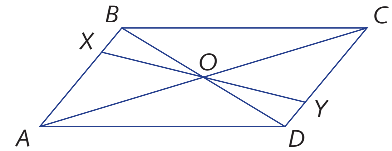


a Give reasons why $OB = OC$.

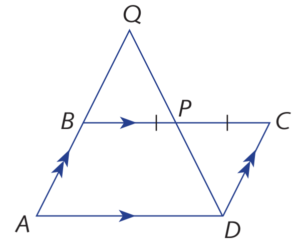
b Use this to find:

i $\angle AOD$ **ii** $\angle AOB$ **iii** $\angle OBC$ **iv** $\angle ABO$

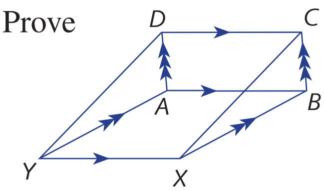
6 The diagonals of the parallelogram $ABCD$ intersect at O. A line through O meets the sides AB and CD at X and Y, respectively. Prove that $OX = OY$.



7 In a parallelogram $ABCD$, P is the midpoint of BC. Both DP and AB are produced to meet at Q. Prove that $AQ = 2AB$.



8 Two parallelograms, $ABCD$ and $ABXY$, are on the same base, AB. Prove that $DCXY$ is a parallelogram.



9 The diagonals of a square $ABCD$ meet at O. The point K lies on AB such that $AK = AO$. Prove that $\angle AOK = 3\angle BOK$.

10 Recall that a kite is a quadrilateral with two pairs of adjacent equal sides. Prove the following properties of a kite. You will need to draw a separate diagram for each point.

a If one diagonal of a quadrilateral bisects the two vertex angles through which it passes, then the quadrilateral is a kite.

b If one diagonal of a quadrilateral is the perpendicular bisector of the other diagonal, then the quadrilateral is a kite.

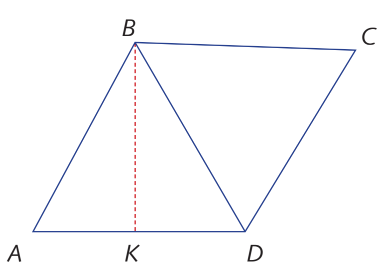
11 Draw a diagram and prove that, in a parallelogram, opposite sides are equal and opposite angles are equal.

12 Draw a diagram and prove that the diagonals of a parallelogram bisect each other.

13 Draw a diagram and prove that the diagonals of a rhombus are perpendicular.

14 Draw a diagram and prove that diagonals of a rectangle are equal.

15 $ABCD$ is a rhombus.
The bisector of $\angle ABD$ meets AD at K.
Prove that $\angle AKB = 3\angle ABK$.



16 $ABCD$ is a rectangle. Equilateral triangles ABX and DAY are drawn outside $ABCD$. Draw a diagram and prove that triangle CXY is equilateral.

17 Draw a diagram and prove that if each angle of a quadrilateral is equal to the opposite angle then the quadrilateral is a parallelogram.

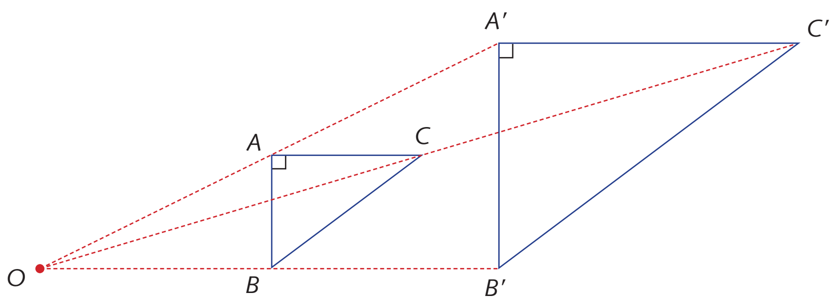
18 Draw a diagram and prove that if each side of a quadrilateral is equal to the opposite side then the quadrilateral is a parallelogram.

8C Enlargements and similarity

Enlargements

- An **enlargement** stretches a figure by the same factor in all directions.
- An enlargement transformation is specified by its **centre of enlargement** and its **enlargement factor**.
- When a figure is enlarged:
  - matching lengths are in the same ratio and
  - matching angles are equal.

The image is thus a **scale drawing** of the original figure.



In the diagram at the right, O is the centre of enlargement. $\Delta A'B'C'$ is an enlargement by factor 2 of ΔABC.

Since the enlargement factor is 2, $\dfrac{A'B'}{AB} = \dfrac{B'C'}{BC} = \dfrac{C'A'}{CA} = 2$.

Similarity

Two figures are called **similar** if we can enlarge one figure so that its enlargement is congruent to the other figure. In simple terms, this means that by enlarging or shrinking one of two items, we get the other item, perhaps translated, rotated or reflected.

Thus, similar figures have the same shape, but not necessarily the same size, just as a scale drawing has the same shape as the original, but has a different size.

Similar figures

- Two figures are called **similar** if there is an enlargement of one figure that is congruent to the other figure.
- Matching lengths in similar figures are in the same ratio, called the **similarity ratio**.
- Matching angles in similar figures are equal.

Similarity tests for triangles

As with congruence, most problems involving similarity come down to establishing that two triangles are similar. In this section we review the four similarity tests for triangles. For each congruence test there is a corresponding similarity test.

The AAA similarity test

- If the angles of one triangle are respectively equal to the angles of another triangle, then the two triangles are similar.

Note: When using this test, it is sufficient to prove that just two pairs of angles are equal – the third pair must then also be equal since the angle sum of any triangle is 180°. Thus, the test is often called 'the AA similarity test'.

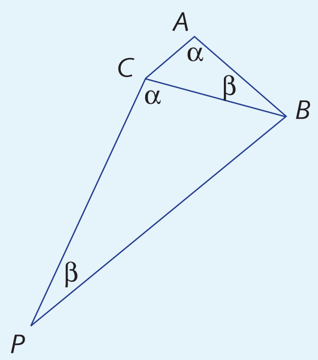
This similarity test corresponds to the AAS congruence test.

Example 3

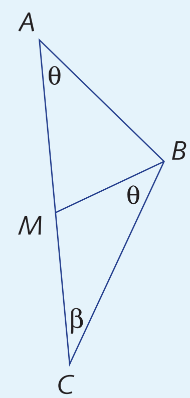
For each diagram, write a similarity statement beginning with 'ΔABC is similar to …' and state the test you used.

Be careful to name the vertices in matching order.

a



b



Solution

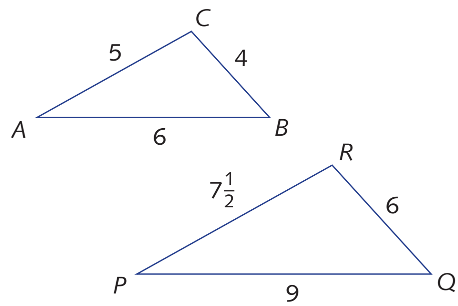
a ΔABC is similar to ΔCPB (AAA)

b ΔABC is similar to ΔBMC (AAA)

Note: A similarity statement should not only appeal to the test used, but also list the vertices of the triangles in matching order.

The SSS similarity test

If we can match up the sides of one triangle with the sides of the other so that the ratio of matching lengths is constant, then the triangles are similar.



The statement that the two triangles shown in the box above are similar is thus written as:

ΔABC is similar to ΔPQR (SSS)

The similarity factor is $\frac{3}{2}$.

The SSS similarity test corresponds to the SSS congruence test.

Ratios between triangles and ratios within triangles

When two triangles are similar, we can read off the ratios of the matching lengths between the triangles in the box above. That is:

$$\frac{PQ}{AB} = \frac{RQ}{CB} = \frac{PR}{AC} = \frac{3}{2}$$

Alternatively, we can read off the ratios within the triangles. Thus, for the triangles above:

$$\frac{PQ}{PR} = \frac{AB}{AC} = \frac{6}{5}$$

and $\frac{PQ}{RQ} = \frac{AB}{CB} = \frac{3}{2}$

and $\frac{RQ}{PR} = \frac{CB}{AC} = \frac{4}{5}$

Note that $\frac{a}{b} = \frac{x}{y}$ is equivalent to $\frac{a}{x} = \frac{b}{y}$ because both statements are equivalent to $ay = bx$.

That is, equal ratios between the triangles is equivalent to equal ratios within triangles.

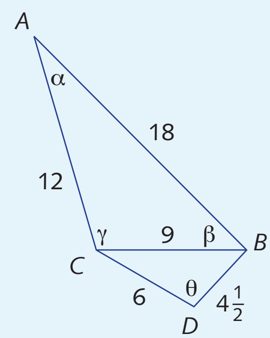
It does not matter whether you use ratios between triangles or ratios within triangles.

Example 4

a Prove that the two triangles in the diagram are similar.

b Which of the two marked angles are equal?

Solution



a In the triangles ΔABC and ΔCBD:

$$\frac{AB}{CB} = 2 \text{ (given)}$$

$$\frac{BC}{BD} = 2 \text{ (given)}$$

$$\frac{CA}{DC} = 2 \text{ (given)}$$

so ΔABC is similar to ΔCBD (SSS).

b Hence, $\gamma = \theta$. (matching angles of similar triangles)

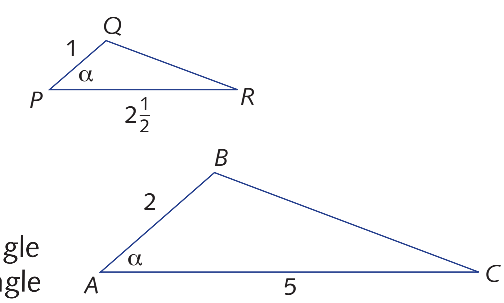
The SAS similarity test

There are two ways of stating the test:

- If the ratios of two pairs of matching sides are equal and the included angles are equal, then the two triangles are similar.

  OR

- If the ratio of the lengths of two sides of one triangle is equal to the ratio of the lengths of another triangle and the included angles are equal, then the two triangles are similar.



The statement that the two triangles in the box above are similar is thus written as:

ΔABC is similar to ΔPQR (SAS)

Consider ΔPQR and ΔABC, as shown in the box above. The *ratios of matching lengths* are:

$$\frac{AC}{PR} = \frac{AB}{PQ} = 2$$

The ratios *within* the triangles are:

$$\frac{PR}{PQ} = \frac{AC}{AB} = \frac{5}{2}$$

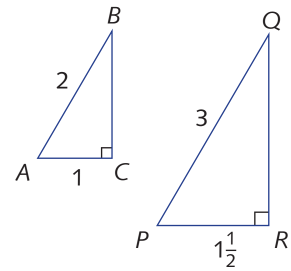
The RHS similarity test

There are two ways of stating the test:

- If the ratio of the hypotenuses and the ratio of one pair of sides of a right-angled triangle are equal, then the triangles are similar.

  OR

- If the ratio of the hypotenuse and one side of one right-angled triangle is equal to the ratio of the hypotenuse and one side of another right-angled triangle, then the two triangles are similar.



The statement that the two triangles in the box above are similar is thus written as:

ΔABC is similar to ΔPQR (RHS)

Consider ΔABC and ΔPQR as shown in the box on the previous page. The *ratios of matching lengths* are:

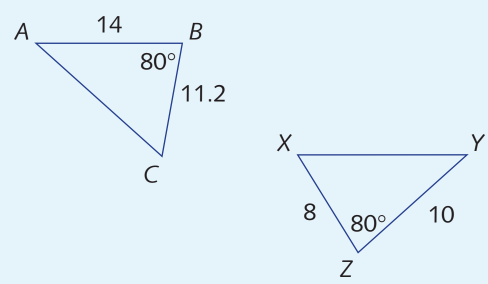
$$\frac{PR}{AC} = \frac{QP}{AB} = \frac{3}{2}$$

The ratios *within* the triangles are:

$$\frac{BA}{AC} = \frac{QP}{PR} = 2$$

Example 5

Determine whether the triangles shown are similar and, if they are, state the appropriate similarity test.



Solution

In ΔABC and ΔYZX,

$\angle ABC = \angle YZX = 80°$.

$\frac{AB}{YZ} = \frac{14}{10} = \frac{7}{5}$ and

$\frac{BC}{ZX} = \frac{11.2}{8} = \frac{7}{5}$

ΔABC is similar to ΔYZX (SAS).

Alternatively,

$\frac{BA}{BC} = \frac{14}{11.2} = 1.25$

$\frac{ZY}{ZX} = \frac{10}{8} = 1.25$ and

$\angle ABC = \angle YZX = 80°$

so ΔABC is similar to ΔYZX (SAS).

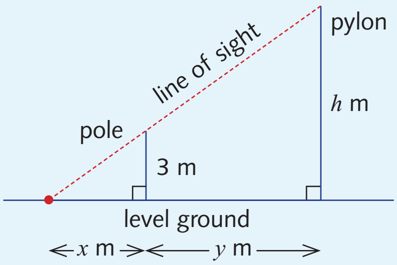
Similar figures can be used to calculate magnitudes of angles and lengths in practical situations, as in the following examples.

Example 6

Some students estimate the height of an electricity pylon using the following method. One student holds a 3-metre pole vertical while another student sights from ground level. The pole is moved until the top of the pole lines up with the top of the pylon, as shown in the diagram.

The distances x metres and y metres are measured and it is found that $x = 4.2$ and $y = 75.6$.

Using similar triangles, calculate the approximate height of the pylon.

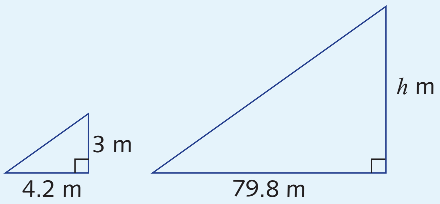


Solution

The two triangles are similar (AAA).

$$\text{Thus, } \frac{h}{3} = \frac{x + y}{x}$$
$$= \frac{79.8}{4.2}$$
$$h = 3 \times \frac{79.8}{4.2} = 57$$

Hence, the height of the pylon is 57 metres.

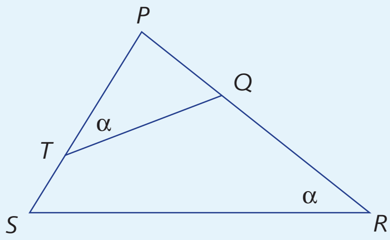


Example 7

In the figure shown,

$\angle PTQ = \angle PRS = \alpha$.

Prove that $PQ \times PR = PT \times PS$.



Solution

In ΔPTQ and ΔPSR,

$\angle PTQ = \angle PRS$ (given)

$\angle TPQ = \angle SPR$ (same angle)

so ΔPTQ is similar to ΔPRS (AAA).

Hence, $\dfrac{PQ}{PS} = \dfrac{PT}{PR}$ (matching sides of similar triangles)

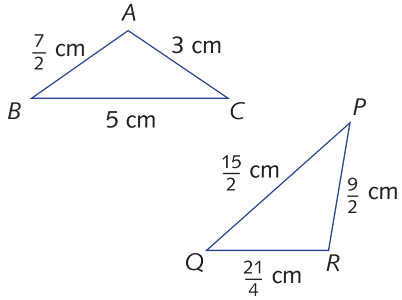
$PQ \times PR = PT \times PS$

Exercise 8C

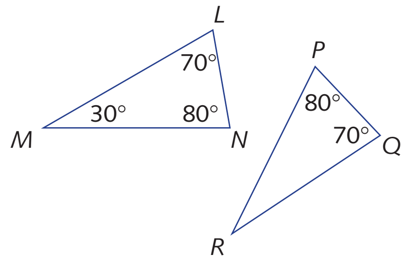
Example 3, 4, 5

1 Determine whether the triangles in each pair are similar. If they are similar, state the appropriate similarity test.

a



b



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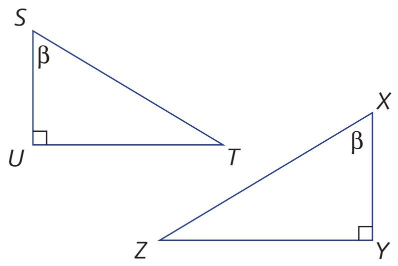
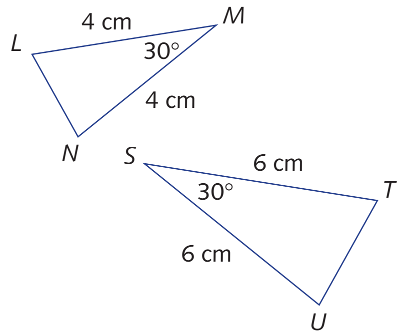


d



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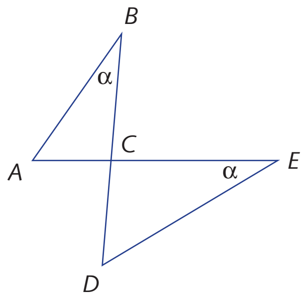
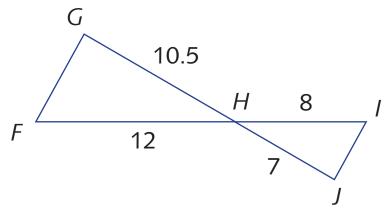


f



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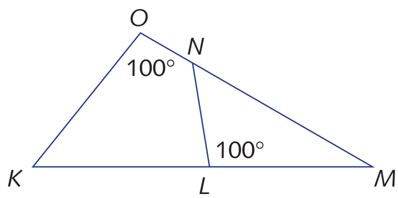

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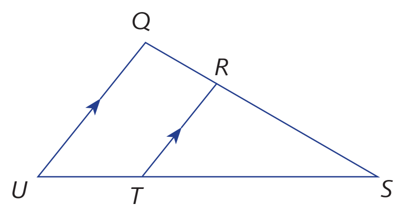
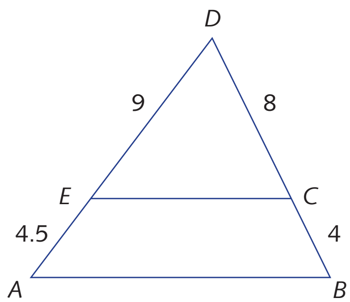
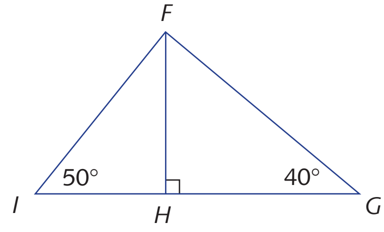


i

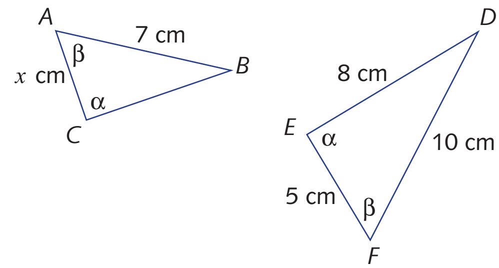


j (Compare all three triangles)



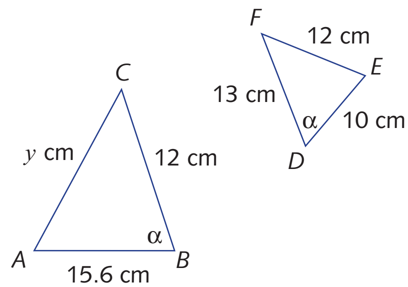
2 **a** State why these two triangles are similar.

b Calculate x.



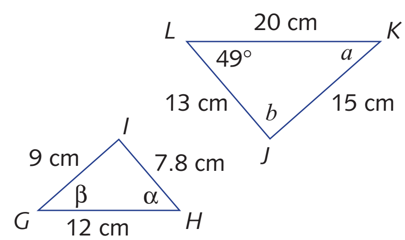
3 **a** State why these two triangles are similar.

b Calculate y.



4 **a** State why these two triangles are similar.

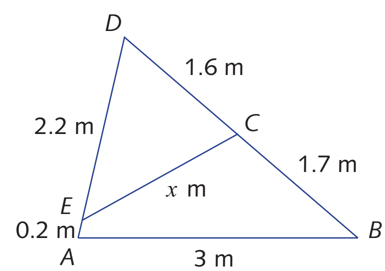
b Find α and find a, then b in terms of β.



5 **a** In the diagram shown, name two triangles that are similar and state why they are similar.

b Write down the three equal ratios.

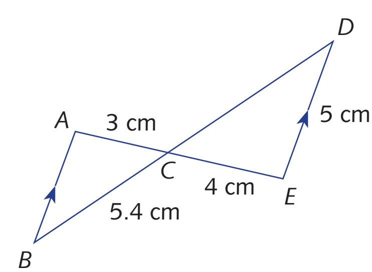
c Calculate the value of x.



6 **a** List the three pairs of equal angles in the figure.

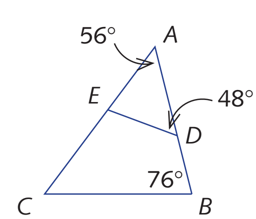
b Find the length of AB.

c Find the length of DC.



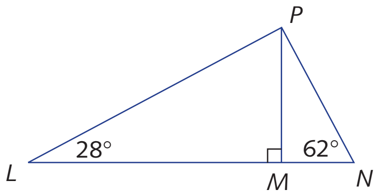
7 **a** In the diagram shown, are the two triangles similar? If so, why?

b If $AD = 6$ cm, $DB = 4$ cm and $AE = 7$ cm, calculate AC.



8 **a** State why ΔLMP is similar to ΔPMN.

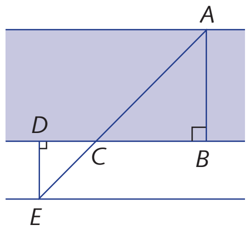
b If $PM = 10$ cm and $MN = 6$ cm, calculate LM.



Example 6

9 At a certain time of day, a flagpole casts a shadow 15 m long, and at the same time a stick 30 cm high casts a shadow 24 cm long. Assuming that both the stick and the flagpole are perpendicular to the horizontal ground, find the height of the flagpole.

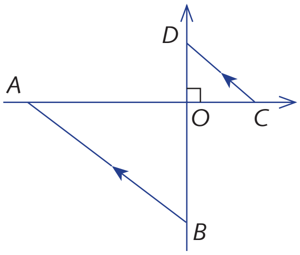
10 This diagram represents a river (shaded) with a tree on the bank at point A. A man stands directly opposite A, on the opposite bank, at point B. He then walks 100 m along the bank, to point C, where he places a peg. He then walks a further 50 m to point D, turns 90° and walks 65 m to point E, where he finds that E, C and A are in a straight line. Find the width of the river.



11 A line from the top of a church steeple to the ground just passes over the top of a pole 3 m high, and meets the ground at a point A, 2 m from the base of the pole. If the distance of A from a point directly below the church steeple is 22 m, find the height of the steeple.

12 In the figure opposite, the line OC is perpendicular to line OD and $OA = 2OC$. B is a point on OD and $AB \parallel CD$.

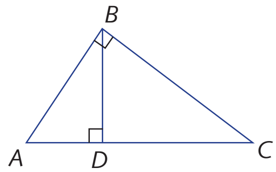
Prove that ΔAOB is similar to ΔCOD and hence prove that $OB = 2OD$.



13 In the figure shown, ABC is a triangle, right-angled at B, and $BD \perp AC$. Prove that:

a ΔABD is similar to ΔACB

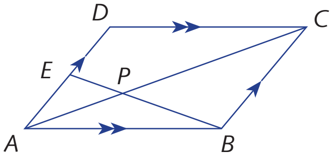
b ΔBCD is similar to ΔACB



14 $ABCD$ is a parallelogram and E is the midpoint of AD. The intervals BE and AC intersect at P. Prove that:

a ΔAPE is similar to ΔCPB

b $AC = 3AP$

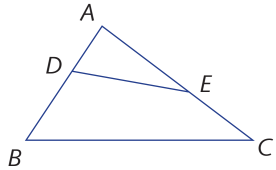


Example 7

15 In ΔABC, D lies on AB and E lies on the interval AC such that $\angle EDB$ and $\angle ACB$ are supplementary angles.

a Prove that ΔADE is similar to ΔACB.

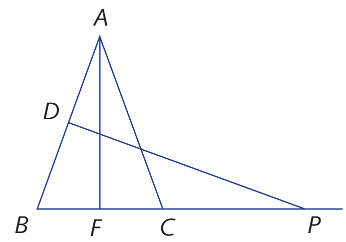
b Prove that $\dfrac{AE}{AB} = \dfrac{AD}{AC}$.



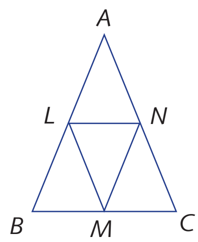
16 In the figure shown, ΔABC is isosceles with $AB = AC$. The point F lies on BC such that $AF \perp BC$. The point P lies on BC and the point D lies on AB such that $DP \perp AB$.

a Prove that ΔPBD is similar to ΔACF.

b Prove that $\dfrac{FC}{DB} = \dfrac{AC}{PB}$.



17 In the diagram shown, $AB = AC$ and L, M and N are midpoints of AB, BC and CA, respectively. Prove that $LM = NM$.



18 Complete the following proof of Pythagoras' theorem.

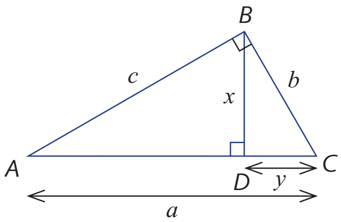
a Show that ΔABC is similar to ΔADB.

b Show that $a^2 - ay = c^2$.

c Show that ΔABC is similar to ΔBDC.

d Show that $ay = b^2$.

e Deduce that $a^2 = b^2 + c^2$.



8D Similarity and intervals within a triangle

Similarity is a useful tool to analyse an interval joining points on two sides of a triangle, as in the example below.

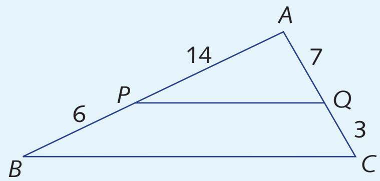
Example 8

In triangle ABC,
$AP = 14$, $PB = 6$, $AQ = 7$ and $QC = 3$.

a Prove that ΔABC is similar to ΔAPQ.

b Prove that PQ is parallel to BC.

c Find the ratio $PQ : BC$.



Solution

a In triangles APQ and ABC,

$$\frac{AB}{AP} = \frac{AC}{AQ} = \frac{10}{7}$$

$\angle BAC = \angle PAQ$

ΔABC is similar to ΔAPQ (SAS).

b Since the triangles are similar, $\angle ABC = \angle APQ$.

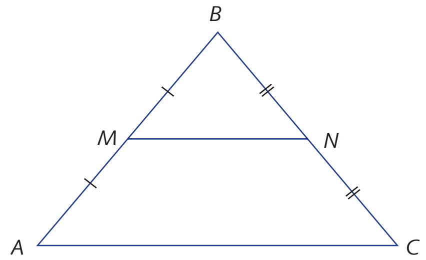
Therefore, corresponding angles are equal.

Thus, $BC \parallel PQ$.

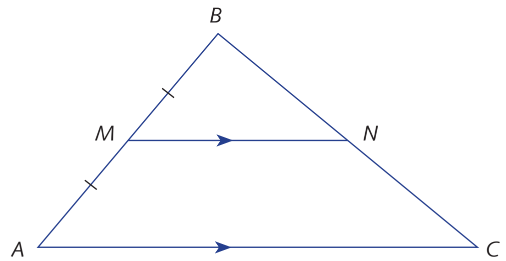
c $PQ : BC = 7 : 10$ (matching sides of similar triangles)

Exercise 8D

1 Prove that the interval joining the midpoints of two sides of a triangle is parallel to the third side and half its length.



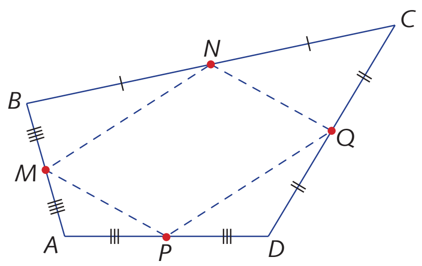
2 Prove that the line through the midpoint of one side of a triangle parallel to another side meets the third side of the triangle at its midpoint.



3 Prove that the intervals joining the midpoints of the sides of a triangle dissect the triangle into four congruent triangles, each similar to the original triangle.

4 Prove that the midpoints of the sides of a quadrilateral form the vertices of a parallelogram.

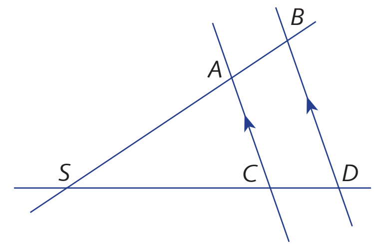
- Point N is the *midpoint* of BC
- Point Q is the *midpoint* of CD
- Point P is the *midpoint* of AD
- Point M is the *midpoint* of AB



5 Draw a diagram of a triangle ABC with a point P on AB and a point Q on AC such that $AP = 12$, $PB = 9$, $AQ = 4$ and $QC = 3$.

Prove that $PQ \parallel BC$.

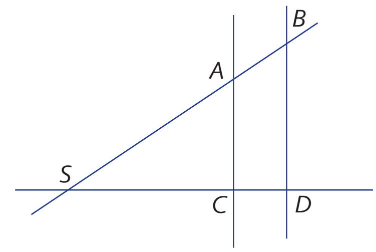
6 The point S is the intersection point of two lines. Points A and B are the points of intersection of the first line with two parallel lines, such that B is further away from S than A. Similarly, points C and D are the intersections of the second line with the two parallel lines, such that D is further away from S than C.



Prove that:

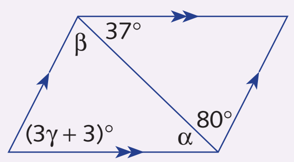
a $SA : AB = SC : CD$ **b** $SB : AB = SD : CD$ **c** $SA : SB = SC : SD$

7 The point S is the intersection point of two lines. Points A and B are the points of intersection of the first line with two other lines, such that B is further away from S than A. Similarly, points C and D are the intersection points of the second line with the two other lines, such that D is further away from S than C. Prove that if $SA : AB = SC : CD$, then the two intercepting lines AC and BD are parallel.



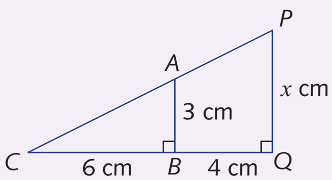
Review exercise

1 Find the value of α, β and γ in the diagram at the right.



2 **a** Name the similar triangles in the diagram at the right.

b Find x.

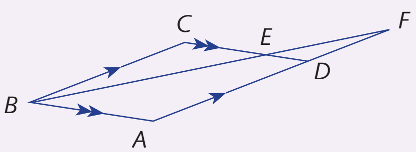


3 In the parallelogram $ABCD$, E is a point on CD, and BE and AD are produced to meet at F.

a Prove that triangle BEC is similar to triangle FED.

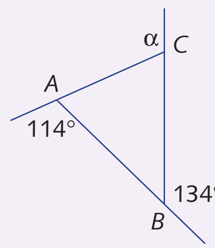
b Given that $CD = 3ED$, $AB = 6$ and $BC = 8$:

i find ED **ii** find DF

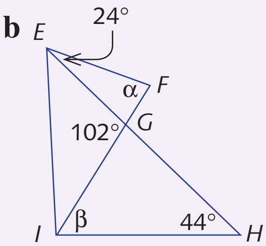


4 Find the value of the Greek letters in each diagram.

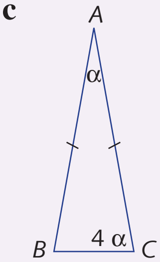
a



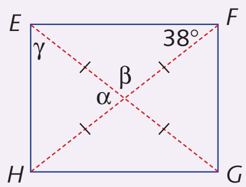
b



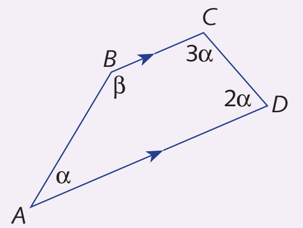
c



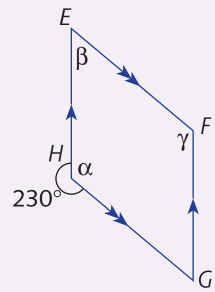
d



e



f



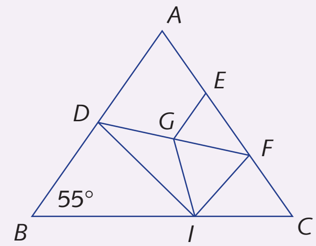
5 In the diagram at the right, ΔABC is isosceles with $AB = AC$, $\angle ABC = 55°$, $AD \parallel EG \parallel FI$ and $DI \parallel AC$.

a Find $\angle BAC$ and $\angle ACB$.

b Find $\angle FIC$ and prove that ΔFIC is isosceles.

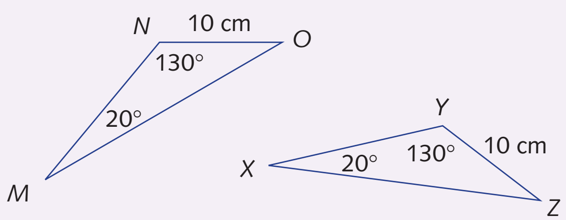
c Prove that $DB = DI$.

d If $GD = GI$ and $\angle FGI = 34°$, find $\angle GIF$ and $\angle EFG$.

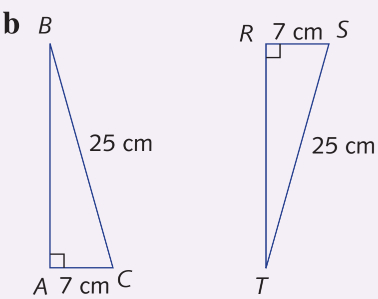


6 For each pair of triangles below, write a congruence statement, including the appropriate congruence test.

a



b

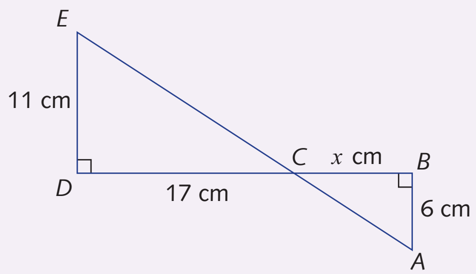


7 AB and DC are parallel sides of a trapezium $ABCD$. The diagonals of the trapezium intersect at O. Prove that $\dfrac{BO}{OD} = \dfrac{AO}{OC}$.

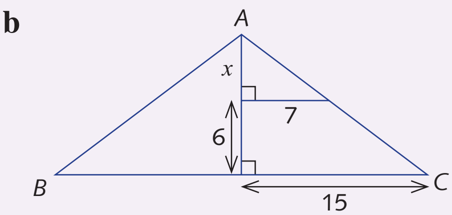
8 BE and CF are altitudes of a triangle ABC. Prove that $\dfrac{BE}{CF} = \dfrac{AB}{AC}$.

9 Find the value of x in each diagram.

a



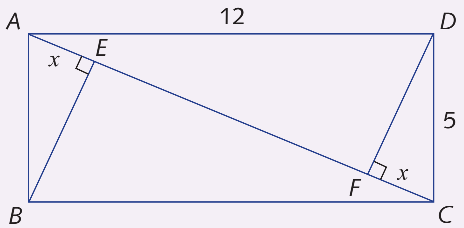
b



10 $ABCD$ is a rectangle with $AD = 12$ and $DC = 5$, and BE and DF are perpendicular to AC.

a Find AC.

b Find EF.



11 In triangle ABC, $AB = 8$ cm, $BC = 5$ cm and $CA = 6$ cm. The side AB is produced to D so that $BD = 16$ cm, and AC is extended to E so that $CE = 26$ cm. Find DE.

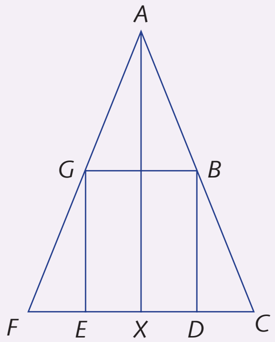
Challenge exercise

1 Attic space in a particular house has the shape of a triangular prism.

Triangle AFC is isosceles with $AF = AC = 4$ m and $FC = 3$ m.

A box in the form of a rectangular prism is placed in the attic, touching both sides.

A cross section is shown in the diagram below, where the face $EGBD$ of the box is shown.



The box is 204 cm wide; that is, $ED = GB = 204$ cm.

a Find the length of AB.

b Find the length of AX, where X is the midpoint of FC.

c Find the height, EG, of the box.

d A box in the shape of a cube is to be placed on top of this box. Find the length, correct to the nearest cm, of an edge of the largest cube that could fit.

2 The diagonals of a square $ABCD$ are AC and BD, which intersect at O. The bisector of $\angle BAC$ cuts BO at X and BC at Y. Prove that $CY = 2OX$. (*Hint*: Let AY meet CD at Z and consider ΔACZ.)

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3 A, B, C and D are points on a straight line so that $AB = BC = CD$. Also, $BPQC$ is a parallelogram. If $BP = 2BC$, prove that PD is perpendicular to AQ.

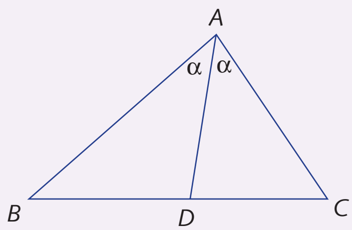
4 In triangle ABC, $AB = AC$ and $\angle ABC = 2\angle BAC$. The segment BC is produced to D so that $2\angle CAD = \angle BAC$. The point F lies on AB so that CF is perpendicular to AB. Prove that $AD = 2CF$. (*Hint*: Find an extra isosceles triangle and draw an altitude.)

5 X and Y are the midpoints of the sides PS and SR of a parallelogram $PQRS$. Prove that the area of triangle SXY is one eighth the area of the parallelogram.

6 **a** In ΔABC, AD bisects $\angle BAC$.

Prove that $\dfrac{BD}{DC} = \dfrac{BA}{AC}$.

Hint: Construct CE parallel to DA to meet BA extended at E.



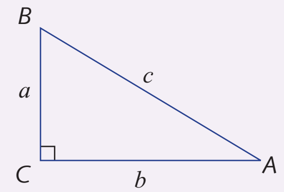
b The bisectors of the angle A and the angle C of a quadrilateral $ABCD$ meet at point E on the diagonal BD.

Prove that $\dfrac{AD}{AB} = \dfrac{CD}{CB}$.

c The bisectors of the angles A, B and C of ΔABC meet the opposite sides at D, E and F.

Prove that $\dfrac{BD}{DC} \times \dfrac{CE}{EA} \times \dfrac{AF}{FB} = 1$.

7 Triangle ABC is right-angled at C. This question leads you through another proof of Pythagoras' theorem using enlargements.



a Enlarge triangle ABC by a factor of b to form triangle $A'B'C'$ and mark the side lengths of each side on a diagram of triangle $A'B'C'$.

b Enlarge triangle ABC by a factor of a to form triangle $A''B''C''$, and show the side lengths of each side on a diagram of triangle $A''B''C''$.

c Join triangle $A''B''C''$ and $A'B'C'$ along sides $B'C'$ and $C''A''$ with C'' and C' coinciding.

d Show that the new triangle formed is similar to triangle ABC.

e What is the enlargement factor that transforms triangle ABC to this triangle?

f Deduce Pythagoras' theorem.

8 Prove that the lines joining the midpoints of opposite sides of any quadrilateral bisect each other.

CHAPTER

9

Number and Algebra

Indices, exponentials and logarithms – part 1

You often hear people talk about 'exponential growth' or 'exponential decay', generally in connection with business, investment, ecology and science. This chapter will explain what these terms mean.

In *ICE-EM Mathematics Year 9*, you learned how to graph parabolas such as $y = x^2$ and $y = 3x^2 - 4$. In this chapter, you will learn what the exponential and logarithm functions are, and how to draw their graphs.

9A Review of powers and integer indices

In *ICE-EM Mathematics Year 9*, you learned that a number such as 2 could be raised to any integer power, so that:

$$2^0 = 1, \qquad 2^1 = 2, \qquad 2^2 = 4, \qquad 2^3 = 8, \qquad \ldots$$

and

$$2^{-1} = \frac{1}{2}, \qquad 2^{-2} = \frac{1}{4}, \qquad 2^{-3} = \frac{1}{8}, \qquad \ldots$$

In the statement $2^5 = 32$, we call 2^5 a **power**, we call 2 the **base** and we call 5 the **index** or the **exponent**.

In general, if a is any number and n is a positive integer, we define a^n to be the product of n factors of a, and we define:

$$a^{-n} \text{ to be } \frac{1}{a^n},$$

provided a is non-zero.

Also, we define:

$$a^0 = 1$$

All of the index laws follow directly from these definitions. It is important to be able to recall and use these laws. In this chapter, we will use the index laws repeatedly.

Index laws

Recall that if m and n are integers and a and b are any non-zero numbers:

Index law 1	$a^m a^n = a^{m+n}$	Index law 4	$(ab)^n = a^n b^n$
Index law 2	$\dfrac{a^m}{a^n} = a^{m-n}$	Index law 5	$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$
Index law 3	$(a^m)^n = a^{mn}$		

Example 1

a Evaluate:

i 7^2 **ii** 2^8

b Write each number in index form with a prime-number base.

i 128 **ii** 343 **iii** $\dfrac{1}{25}$

Solution

a **i** $7^2 = 49$ **ii** $2^8 = 256$

b **i** $128 = 2^7$ **ii** $343 = 7^3$ **iii** $\dfrac{1}{25} = 5^{-2}$

Example 2

a Simplify each expression.

i $x^7 \times x^2 \times x^3$ **ii** $x^2z^3 \times x^7z^2$ **iii** $2a^2b \times 7a^3b^2$

b Simplify each quotient.

i $\dfrac{a^3b^7}{ab^2}$ **ii** $\dfrac{60a^3b^2}{5a^2b}$

c Simplify $(a^2)^3 \times a^4$.

Solution

a i $x^7 \times x^2 \times x^3 = x^{12}$ **ii** $x^2z^3 \times x^7z^2 = x^9z^5$ **iii** $2a^2b \times 7a^3b^2 = 14a^5b^3$

b i $\dfrac{a^3b^7}{ab^2} = a^2b^5$ **ii** $\dfrac{60a^3b^2}{5a^2b} = 12ab$

c
$$\begin{aligned}(a^2)^3 \times a^4 &= a^6 \times a^4\\ &= a^{10}\end{aligned}$$

Example 3

Simplify these expressions.

a $(x^2y^3)^4$ **b** $(2m^2)^3 \times (3m)^3$ **c** $\dfrac{(a^2b^3)^4}{(ab^2)^3}$

Solution

a $(x^2y^3)^4 = x^8y^{12}$

b
$$\begin{aligned}(2m^2)^3 \times (3m)^3 &= 8m^6 \times 27m^3\\ &= 216m^9\end{aligned}$$

c
$$\begin{aligned}\frac{(a^2b^3)^4}{(ab^2)^3} &= \frac{a^8b^{12}}{a^3b^6}\\ &= a^5b^6\end{aligned}$$

Here are two useful facts:

- $\left(\dfrac{a}{b}\right)^{-1} = \dfrac{b}{a}$, since $\dfrac{a}{b} \times \dfrac{b}{a} = 1$
- Similarly, $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$

Example 4

Evaluate:

a $\left(\frac{4}{7}\right)^{-1}$ **b** 4^{-3} **c** 10^{-3} **d** $5a^0$ **e** $\left(\frac{2}{3}\right)^{-3}$

Solution

a $\left(\frac{4}{7}\right)^{-1} = \frac{7}{4}$

b $4^{-3} = \frac{1}{4^3} = \frac{1}{64}$

c $10^{-3} = \frac{1}{10^3} = \frac{1}{1000}$

d $5a^0 = 5 \times 1 = 5$

e $\left(\frac{2}{3}\right)^{-3} = \left(\frac{3}{2}\right)^{3} = \frac{27}{8}$

Example 5

Simplify these products, expressing each pronumeral in the answer with a positive index.

a $a^{-4} \times a^{-6}$ **b** $2a^4 \times 5a^{-6}$ **c** $(m^{-3}n^{-5})^4 \times (m^{-7}n^3)^{-5}$

Solution

a $a^{-4} \times a^{-6} = a^{-10} = \frac{1}{a^{10}}$

b $2a^4 \times 5a^{-6} = 10a^{-2} = \frac{10}{a^2}$

c $(m^{-3}n^{-5})^4 \times (m^{-7}n^3)^{-5} = m^{-12}n^{-20} \times m^{35}n^{-15} = \frac{m^{23}}{n^{35}}$

Exercise 9A

Example 1a

1 Evaluate:

a 4^2 **b** 5^3 **c** 2^6 **d** 3^3 **e** 10^4 **f** 6^3

Example 1bi, ii

2 Write each number in index form with a prime number base.

a 8 **b** 64 **c** 81 **d** 32 **e** 625 **f** 243

Example 1biii

3 Write each number in index form with a prime number base.

a $\frac{1}{13}$ **b** $\frac{1}{49}$ **c** $\frac{1}{13^3}$ **d** $\frac{1}{1024}$ **e** $\frac{9}{729}$ **f** $(121)^{-5}$

Example 2a

4 Simplify each expression.

a $a^4 \times a^6 \times a^5$ **b** $a^7 \times a^3 \times a$ **c** $m^4 \times m^3 \times m^8$

d $p^4 \times p^5 \times p^2$ **e** $a^2b \times a^4b^6$ **f** $m^4n^2 \times m^5n^4$

g $2a^4b^3 \times 4ab^2$ **h** $3x^3y \times 5x^2y^3$ **i** $3x^3y^7 \times 5x^5y^2$

Example 2b

5 Simplify each quotient, expressing each pronumeral in the answer with a positive index.

a $\dfrac{x^2y^3}{xy^4}$ **b** $\dfrac{30x^2y^4}{20xy^2}$ **c** $\dfrac{a^6m^4}{a^2m}$

d $\dfrac{25x^4y^6}{20x^3y^5}$ **e** $\dfrac{8ab^3c^4}{12ab^5c^2}$ **f** $\dfrac{16x^3y^6z^8}{4y^7z^8}$

Example 3a

6 Simplify:

a $(a^2b^4)^3$ **b** $(x^3y^5)^7$ **c** $(ab^2c^3d^4)^5$

d $(2a^3b)^2$ **e** $(3a^2b^4)^2$ **f** $(4a^3b^2)^3$

Example 3b, c

7 Simplify each expression, writing each pronumeral in the answer with a positive index.

a $(3m^3)^2 \times 2m^6$ **b** $(2p^5)^2 \div (4p^6)$ **c** $\dfrac{(a^2b)^3}{ab^4} \times \dfrac{a^2b^5}{ab}$

d $\dfrac{m^4n^2}{mn^3} \div \dfrac{(mn^2)^3}{m^5n^8}$ **e** $\dfrac{a^4b^6}{(ab^2)^2} \times \dfrac{a^5b}{a^2b}$ **f** $\dfrac{(p^4q)^2}{pq^3} \div \dfrac{pq}{(p^2q)^3}$

Example 4

8 Evaluate:

a 2^{-1} **b** 2^{-2} **c** 3^{-1} **d** 3^{-2} **e** 10^{-3}

f $\left(\dfrac{7}{8}\right)^{-1}$ **g** $\left(\dfrac{15}{14}\right)^{-1}$ **h** $\left(\dfrac{3}{5}\right)^{-2}$ **i** $\left(\dfrac{2}{3}\right)^{-3}$ **j** $\left(\dfrac{5}{11}\right)^{-3}$

k 3^{-4} **l** $5 + a^0$ **m** $\dfrac{4a^0}{(5b)^0}$ **n** $(2 + a)^0$ **o** $(4^3)^0$

9 Simplify each expression, writing each pronumeral in the answer with a positive index.

a $(2x^2y)^{-1}$ **b** $(3x^2y^2)^{-2}$ **c** $(4xy^{-1})^{-3}$

d $(2x^2y^{-2})^{-3}$ **e** $(3x^{-2}y^{-2})^{-3}$ **f** $(2x^5y^5)^2$

Example 5

10 Simplify each expression, writing each pronumeral in the answer with a positive index.

a $m^6 \times m^{-2} \times m^{-5}$ **b** $2a^{-1}b^3 \times 4a^{-3}b^{-6}$ **c** $5p^2q^{-1} \times 3pq^{-4}$

d $\dfrac{15p^4q^{-2}}{10p^{-7}q^4}$ **e** $\dfrac{5x^{-2}y^{-3}}{10x^4y^{-4}}$ **f** $(2x^{-1})^{-4}$

g $(2a^{-1}b^3)^{-2} \times 4(a^2b)^{-3}$ **h** $(m^{-2}n^3)^4 \times (m^{-5}n^2)^{-3}$ **i** $\left(\dfrac{m^2n^{-1}}{p^4}\right)^{-2}$

j $\left(\dfrac{a^{-1}b^4}{c^{-1}}\right)^{-2}$ **k** $\dfrac{2a^{-1}b^2}{a^3b^{-2}} \times \dfrac{4a^6b^{-1}}{6ab^{-2}}$ **l** $\dfrac{m^2n^{-3}}{m^4n^2} \times \dfrac{(mn^2)^{-3}}{m^4n^6}$

m $\dfrac{x^4y^{-1}}{(x^2y)^{-3}} \div \dfrac{xy^2}{(xy)^{-2}}$

n $\dfrac{a^{-6}b^4}{(a^2b)^{-3}} \div \dfrac{(a^2b)^{-1}}{ab^3}$

o $\dfrac{(2a^4b^{-2})^3}{c^2} \times \dfrac{(2^2a^{-3}b^2)^{-1}}{c}$

p $\dfrac{(m^2n^3)^2}{p^{-3}} \times (mnp^{-2})^{-3}$

q $\dfrac{(a^2)^3}{b^3} \div \left(\dfrac{a}{b^2}\right)^{-2}$

r $\dfrac{(2a^4)^2}{b^7} \div \dfrac{(a^2)^{-3}}{2b}$

11 Calculate:

a $\dfrac{3^{-1} + 3^{-2}}{3 + 3^2}$

b $\dfrac{2^{-2} + 2^{-4}}{2^2 + 2^4}$

c $\dfrac{2^{-2} - 2^{-4}}{2^2 - 2^4}$

12 Calculate $\dfrac{2^{-1} + 2^{-2} + 2^{-3}}{2 + 2^3 + 2^4}$.

13 If $x = 1$, find the value of $3^x + 3^{1-x} + 3^{x-2}$.

14 Simplify each expression, writing each pronumeral in the answer with a positive index.

a $\dfrac{x - y^{-1}}{x^{-1} - y}$

b $(x^{-1} + y^{-1})(x^{-1} - y^{-1})$

c $\dfrac{3xy}{x^{-1} + y^{-1}}$

d $\dfrac{x^{-1} + y^{-1}}{x^{-2} + y^{-2}}$

e $(x^{-2} + y)^{-2}$

f $(x^{-2} + y^{-1})^{-1}$

9B Scientific notation and significant figures

Many mathematical problems have exact answers, such as $\dfrac{13}{7}$, $\sqrt{2} + \sqrt{3}$ or 400π. However, in the real world, very large numbers and very small numbers are common and, nearly always, these can only be determined approximately. To express large and small numbers conveniently, we use **scientific notation**, also known as **standard form**.

In science, whenever we measure something it is an approximation. Scientific notation and **significant figures** are useful in expressing these numbers. To deal with approximations we use significant figures.

Scientific notation or standard form

By definition, a positive number is in **scientific notation** if it is written as:

$a \times 10^b$, where $1 \leq a < 10$ and b is an integer

This notation is also called the **standard form** for a number. In contrast, for example, 2345.6789, is called the **decimal notation** for that number.

Example 6

Write each number in scientific notation.

a 2100 **b** 0.0062 **c** 764 000 000 **d** 0.000 000 2345

Solution

a $2100 = 2.1 \times 10^3$

b $0.0062 = 6.2 \times 10^{-3}$

c $764\,000\,000 = 7.64 \times 10^8$

d $0.000\,000\,2345 = 2.345 \times 10^{-7}$

Note: If the number is greater than 1, then the exponent of 10 is positive or zero when the number is written in scientific notation. If the number is positive and less than 1, then the exponent is negative.

When the number is written in scientific notation, the exponent records how many places the decimal point has to be moved to the left or right to produce the decimal notation.

Example 7

Write each number in decimal notation.

a 7.2×10^3 **b** 5.832×10^{-2} **c** 3.61×10^5

Solution

a $7.2 \times 10^3 = 7200$ **b** $5.832 \times 10^{-2} = 0.058\,32$ **c** $3.61 \times 10^5 = 361\,000$

Example 8

Evaluate each expression without using a calculator. Give your answers in scientific notation.

a $(4 \times 10^4) \times (2.1 \times 10^3)$ **b** $\dfrac{6.3 \times 10^5}{7 \times 10^6}$

c $(1.5 \times 10^5)^2 \times (9.0 \times 10^{-12})$

Solution

a
$$\begin{aligned}(4 \times 10^4) \times (2.1 \times 10^3) &= 4 \times 2.1 \times 10^4 \times 10^3\\ &= 8.4 \times 10^7\end{aligned}$$

b
$$\begin{aligned}\frac{6.3 \times 10^5}{7 \times 10^6} &= 6.3 \div 7 \times 10^5 \div 10^6\\ &= 0.9 \times 10^{-1}\\ &= 9.0 \times 10^{-2}\end{aligned}$$

c
$$\begin{aligned}(1.5 \times 10^5)^2 \times (9.0 \times 10^{-12}) &= 1.5^2 \times 9.0 \times 10^{10} \times 10^{-12}\\ &= 2.25 \times 9.0 \times 10^{-2}\\ &= 20.25 \times 10^{-2}\\ &= 2.025 \times 10^{-1}\end{aligned}$$

Significant figures

Every time we record a physical measurement, we write down an approximation to the 'true value'. For example, we may say that a standard A4 sheet of paper is 30 cm by 21 cm. This has a conventional meaning and says that the actual length is between 29.5 cm and 30.5 cm. If we measure the sheet of paper more accurately, we could say that it is $29.7\text{ cm} \times 21.0\text{ cm}$. This means that we believe that the actual length is between 29.65 cm and 29.75 cm. Similarly, if we say a girl's height is 156 cm to the nearest centimetre, this means that her actual height is between 155.5 and 156.5 cm.

In this situation, we say that a measurement recorded as 156 cm is **correct to three significant figures**. Similarly, when we say that the width of the paper is 21 cm, this is correct to two significant figures.

Using approximations of π as another example, we say that 3.14 is π correct to three significant figures and $3.141\,59$ is π correct to six significant figures. When we **round** a number, we record it correct to a certain number of significant figures.

The rules for rounding require you to first identify the last significant digit. Then:

- if the next digit is $0, 1, 2, 3$ or 4, round down
- if the next digit is $5, 6, 7, 8$ or 9, round up.

So $\pi = 3.141\,592\,654\ldots$ is rounded to $3, 3.1, 3.14, 3.142, 3.1416, 3.141\,59, 3.141\,593$ and so on, depending on the number of significant figures required.

We use the symbol $\approx$ to mean that two numbers are approximately equal to each other.

Significant figures and scientific notation

Recording a number in scientific notation makes it clear how many significant figures have been recorded. For example, it is unclear whether 800 is written to $1, 2$ or 3 significant figures. However, when written in scientific notation as 8.00×10^2, 8.0×10^2 or 8×10^2, it is clear how many significant figures are recorded.

Example 9

State the number of significant figures to which each of these numbers is recorded.

a 7.321×10^8 **b** 7.200×10^9 **c** 2.0×10^{-5}

d -5.6789×10^{-9} **e** $213\,205$ **f** $-0.001\,240$

Solution

a 7.321×10^8 has 4 significant figures.
b 7.200×10^9 has 4 significant figures.
c 2.0×10^{-5} has 2 significant figures.
d -5.6789×10^{-9} has 5 significant figures.
e $213\,205 = 2.132\,05 \times 10^5$ has 6 significant figures.
f $-0.001\,240 = -1.240 \times 10^{-3}$ has 4 significant figures.

Example 10

Write each of the following numbers correct to the number of significant figures specified in the brackets.

a 214 (2)	**b** 0.000 6786 (3)	**c** 13.999 99 (6)
d −137.4895 (5)	**e** 0.000 532 (2)	**f** 132.007 31 (6)

Solution

a
$$\begin{aligned} 214 &= 2.14 \times 10^2 \\ &= 2.1 \times 10^2 \\ &\approx 210 \end{aligned}$$
(Correct to 2 significant figures.)

b
$$\begin{aligned} 0.000\,6786 &= 6.786 \times 10^{-4} \\ &\approx 0.000\,679 \end{aligned}$$
(Correct to 3 significant figures.)

c
$$\begin{aligned} 13.999\,99 &= 1.399\,999 \times 10 \\ &\approx 14.000 \end{aligned}$$
(Correct to 6 significant figures.)

d
$$\begin{aligned} -137.4895 &= -1.374\,895 \times 10^2 \\ &\approx -137.49 \end{aligned}$$
(Correct to 5 significant figures.)

e
$$\begin{aligned} 0.000\,532 &= 5.32 \times 10^{-4} \\ &\approx 0.000\,53 \end{aligned}$$
(Correct to 2 significant figures.)

f
$$\begin{aligned} 132.007\,31 &= 1.320\,0731 \times 10^2 \\ &\approx 132.007 \end{aligned}$$
(Correct to 6 significant figures.)

Scientific notation and significant figures

- **Scientific notation**, or **standard form**, is a convenient way to represent very large and very small numbers.
- To represent a number in scientific notation, insert a decimal point after the first non-zero digit and multiply by an appropriate power of 10. For example: $75\,684\,000\,000\,000 = 7.5684 \times 10^{13}$ and $0.000\,000\,000\,38 = 3.8 \times 10^{-10}$
- The term for a number expressed without a multiple of a power of 10 is **decimal notation** or decimal form.
- A number may be expressed with different numbers of **significant figures**. For example:

  3.1 has 2 significant figures, 3.14 has 3 significant figures,
  3.141 has 4 significant figures
- To write a number to a specified number of significant figures, first write the number in scientific notation and then round it correct to the required number of significant figures.
- To **round** a number to a required number of significant figures, first write the number in scientific notation and identify the last significant digit. Then:
  - if the next digit is 0, 1, 2, 3 or 4, round down
  - if the next digit is 5, 6, 7, 8 or 9, round up.

Exercise 9B

Scientific notation

Example 6

1 Write each number in scientific notation.

a 63
b 0.4
c 0.62
d 7400
e 21 000 000
f 0.000 26
g -0.086
h 2 000 000 000 000
i 0.000 091 345
j 57 320
k 0.003 012
l 0.100 0510

2 a At the beginning of 2011, the population of Australia was estimated to be approximately 22.5 million. Write this number in scientific notation.

b The wavelength of red light is 6700Å, where $1\text{Å} = 10^{-10}$ m. Write this wavelength of red light in metres, using scientific notation.

c The Sun is approximately 150 billion metres from the Earth. Using scientific notation, write this distance in metres.

Example 7

3 Write each number in decimal notation.

a 6.4×10^3
b 9.2×10^4
c 4.8×10^{-2}
d 8.7×10^{-3}
e 7.412×10^6
f -4.02×10^2
g -4.657×10^{-3}
h 47.26×10^0

Example 8

4 Simplify each number, writing your answer in scientific notation.

a $(2 \times 10^3) \times (4 \times 10^2)$
b $(5 \times 10^3) \times (2 \times 10^2)$
c $(6 \times 10^4) \times (2.1 \times 10^3)$
d $(4 \times 10^3) \times (5.1 \times 10^2)$
e $(4 \times 10^{-3}) \times (5 \times 10^{-2})$
f $(2 \times 10^{-3})^2$
g $(1.1 \times 10^{-8})^2$
h $\dfrac{(2 \times 10^{-8})^3}{4 \times 10^{-3}}$
i $(5 \times 10^4) \div (2 \times 10^3)$
j $(1.2 \times 10^6) \div (4 \times 10^7)$
k $\dfrac{(2 \times 10^5)(4 \times 10^4)}{1.6 \times 10^3}$
l $\dfrac{(2 \times 10^{-1})^5}{(4 \times 10^{-2})^3}$

5 Using your calculator where necessary, write each number in scientific notation.

a $(2.7 \times 10^6) \times (3.8 \times 10^2)$
b $(5.3 \times 10^4) \times (1.1 \times 10^{-3})$
c $\dfrac{9.6 \times 10^{14}}{1.6 \times 10^{21}}$
d $\sqrt{9.61 \times 10^{12}} \times 1.4 \times 10^3$
e $\dfrac{8.4 \times 10^4}{\sqrt{4.9 \times 10^5}}$
f $\sqrt[3]{64 \times 10^9} \times \sqrt[5]{1024 \times 10^{-10}}$

6 At the beginning of 2011, the population of Australia was estimated to be approximately 22.5 million. If the population stayed the same for the next year, and each person in Australia produced an average of 0.712 kg of waste each day, how many tonnes of waste would be produced by Australians in the following year? (1 tonne = 1000 kg, 1 year = 365 days.) Express your answer in scientific notation.

7 A light year is the distance light travels in a year. Light travels at approximately 3×10^5 km/s.

a How wide is our galaxy (in kilometres) if it is approximately 230 000 light years across?

b How far from us (in kilometres) is the farthest galaxy detected by optical telescopes if it is approximately 13×10^9 light years from us?

c How long does it take light to travel from the Sun to the Earth if the distance between the Sun and the Earth is 1.4951×10^8 km?

8 The mass of a hydrogen atom is approximately 1.674×10^{-27} kg and the mass of an electron is approximately 9.1×10^{-31} kg. How many electrons, correct to the nearest whole number, will it take to equal the mass of a single hydrogen atom?

Significant figures

Example 9, 10

9 Write each of these numbers in scientific notation, correct to the number of significant figures indicated in the brackets.

a 576.63	(4)	**b** 472.61	(3)	**c** 472.61	(2)
d 472.61	(1)	**e** 0.051 237	(4)	**f** 0.051 237	(3)
g 0.051 237	(2)	**h** 0.051 237	(1)	**i** 1603.29	(4)
j 1603.29	(3)	**k** 1603.29	(2)	**l** 1603.29	(1)
m 2.9935×10^{27}	(4)	**n** 2.9935×10^{27}	(3)	**o** 2.9935×10^{27}	(2)
p 2.9935×10^{27}	(1)	**q** 573 007	(3)	**r** 0.006 534	(1)

Example 10

10 Write each of these numbers in decimal notation, correct to three significant figures.

a 5.6023	**b** 537.97	**c** 9673.47	**d** 732 412
e 0.003 511	**f** 0.014 187	**g** 372.2	**h** 478 000

11 A cylindrical wire in an electrical circuit has radius 3.41×10^{-4} m and length 8.02×10^{-2} m. Calculate its volume in m^3, correct to three significant figures, giving the answer in scientific notation.

12 The formula for kinetic energy is $E = \frac{1}{2}mv^2$.

a Find the value of E correct to three significant figures, when $m = 9.21 \times 10^{-11}$ and $v = 3.00 \times 10^7$.

b Find the value of v correct to four significant figures, when $E = 2.834 \times 10^{-10}$ and $m = 6.418 \times 10^{-29}$.

13 For each measurement, identify the range within which the true value lies.

a 15 cm	**b** 2.00×10^3 kg	**c** 18.67 m	**d** 4.8745×10^7 mL

9C Powers with rational indices

We begin by considering what we mean by powers such as $3^{\frac{1}{2}}$, $2^{\frac{1}{3}}$ and $\pi^{\frac{1}{10}}$, in which the exponent is the reciprocal of a positive integer.

Recall that if a is positive, $\sqrt{a}$ is the positive number whose square is a. That is:

$$\left(\sqrt{a}\right)^2 = a = a^1$$

For this reason, we introduce an alternative notation for $\sqrt{a}$: we write it as $a^{\frac{1}{2}}$. We do this because then we preserve the third index law:

$$\left(a^{\frac{1}{2}}\right)^2 = a^{2\times\frac{1}{2}} = a^1$$

Keep in mind that $a^{\frac{1}{2}}$ is nothing more than an alternative notation for $\sqrt{a}$.

Similarly, every positive number a has a cube root, $\sqrt[3]{a}$. It is the positive number whose cube is a; that is, $(\sqrt[3]{a})^3 = a = a^1$.

We define $a^{\frac{1}{3}}$ to be $\sqrt[3]{a}$. The third index law continues to hold.

$$\left(a^{\frac{1}{3}}\right)^3 = a^{3\times\frac{1}{3}} = a^1$$

The same can be done for $\sqrt[4]{a}$, $\sqrt[5]{a}$ and so on. The alternative notations are:

$\sqrt[4]{a} = a^{\frac{1}{4}}$, $\sqrt[5]{a} = a^{\frac{1}{5}}$ and so on.

n^{th} root

Let a be positive or zero and let n be a positive integer. Define $a^{\frac{1}{n}}$ to be the n^{th} root of a.

That is, $a^{\frac{1}{n}}$ is the positive number whose n^{th} power is a.

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

For example, $a^{\frac{1}{2}} = \sqrt{a}$ and $a^{\frac{1}{3}} = \sqrt[3]{a}$.

Using a calculator, it is easy to obtain approximations for square roots, cube roots or any higher-order root.

$$\sqrt{10} \approx 3.1623,\ \sqrt[3]{10} \approx 2.1544,\ \sqrt[4]{10} \approx 1.7783,\ \sqrt[5]{10} \approx 1.5849, \ldots$$

Using our new notation, here are some other numerical approximations, all recorded correct to five significant figures.

$$2^{\frac{1}{5}} \approx 1.1487,\ 10^{\frac{1}{8}} \approx 1.3335,\ 0.2^{\frac{1}{4}} \approx 0.668\,74,\ 3.2^{\frac{1}{6}} \approx 1.2139$$

Use your calculator to check these calculations.

Example 11

Without using your calculator, evaluate:

a $8^{\frac{1}{3}}$ **b** $1024^{\frac{1}{2}}$ **c** $1024^{\frac{1}{5}}$ **d** $\left(\frac{1}{729}\right)^{\frac{1}{2}}$ **e** $\left(\frac{1}{729}\right)^{\frac{1}{6}}$

Solution

a $2^3 = 8$, so $8^{\frac{1}{3}} = 2$

b $1024 = 2^{10}$, so $1024^{\frac{1}{2}} = (2^{10})^{\frac{1}{2}} = 2^5 = 32$

c Similarly, $1024^{\frac{1}{5}} = 2^2 = 4$

d $729 = 3^6$, so $\left(\frac{1}{729}\right)^{\frac{1}{2}} = \left(\frac{1}{3^6}\right)^{\frac{1}{2}} = \frac{1}{3^3} = \frac{1}{27}$

e Similarly, $\left(\frac{1}{729}\right)^{\frac{1}{6}} = \frac{1}{3}$

We now come to the main definition. If a is a positive number, p is an integer and q is a positive integer, then we define:

$a^{\frac{p}{q}} = \left(a^{\frac{1}{q}}\right)^p$ which means $\left(\sqrt[q]{a}\right)^p$. This is the p^{th} power of the q^{th} root of a.

For example:

$$8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = 2^2 = 4$$

Throughout the rest of this chapter, we will avoid using the radical symbol $\sqrt{\ }$ wherever possible. We begin with some simple calculations and then investigate how the index laws behave when we have rational powers of numbers.

Example 12

Without using your calculator, find:

a $8^{\frac{4}{3}}$ **b** $81^{\frac{5}{4}}$ **c** $100\,000^{\frac{3}{5}}$ **d** $0.01^{\frac{3}{2}}$

Solution

a Using the definition,

$$8^{\frac{4}{3}} = \left(8^{\frac{1}{3}}\right)^4 \qquad (\text{since } 2^3 = 8)$$
$$= 2^4$$
$$= 16$$

b Since $81 = 3^4$, we have

$$81^{\frac{5}{4}} = \left(81^{\frac{1}{4}}\right)^5$$
$$= 3^5$$
$$= 243$$

c Since $100\,000 = 10^5$,

$$100\,000^{\frac{3}{5}} = 10^3$$
$$= 1000$$

d $0.01^{\frac{3}{2}} = (10^{-2})^{\frac{3}{2}}$

$$= 10^{-3}$$
$$= 0.001$$

Example 13

a Write each number in the form $\sqrt[n]{a}$.

i $7^{\frac{1}{3}}$ **ii** $11^{\frac{1}{5}}$

b Write each number in index form.

i $\sqrt[3]{17}$ **ii** $\left(\sqrt[5]{13}\right)^2$ **iii** $7\sqrt{7}$ **iv** $6^2 \times \sqrt[5]{6}$

Solution

a i $7^{\frac{1}{3}} = \sqrt[3]{7}$

ii $11^{\frac{1}{5}} = \sqrt[5]{11}$

b i $\sqrt[3]{17} = 17^{\frac{1}{3}}$

ii $\left(\sqrt[5]{13}\right)^2 = \left(13^{\frac{1}{5}}\right)^2 = 13^{\frac{2}{5}}$

iii $7\sqrt{7} = 7 \times 7^{\frac{1}{2}}$

$$= 7^{\frac{3}{2}}$$

iv $6^2 \times \sqrt[5]{6} = 6^2 \times 6^{\frac{1}{5}}$

$$= 6^{\frac{11}{5}}$$

Example 14

Calculate the exact value of each number.

a $16^{-\frac{1}{2}}$ **b** $125^{-\frac{1}{3}}$ **c** $32^{-\frac{1}{5}}$

Solution

a $16^{-\frac{1}{2}} = \left(\frac{1}{16}\right)^{\frac{1}{2}}$

$$= \frac{1}{4}$$

b $125^{-\frac{1}{3}} = \left(\frac{1}{125}\right)^{\frac{1}{3}}$

$$= \frac{1}{5}$$

c $32^{-\frac{1}{5}} = \left(\frac{1}{32}\right)^{\frac{1}{5}}$

$$= \frac{1}{2}$$

Index laws for rational indices

The five index laws introduced previously for integer indices are equally valid for rational indices. This follows from the definition of a^x, where x is rational.

We will leave a discussion of the proofs of these laws to the Challenge exercises.

Using index law 3, we note that, for rational x and y:

$$(a^x)^y = a^{xy} = a^{yx} = (a^y)^x$$

Hence:

$$a^{\frac{p}{q}} = \left(a^{\frac{1}{q}}\right)^p = (a^p)^{\frac{1}{q}}$$

This means that when we evaluate $a^{\frac{p}{q}}$, it does not matter if we take the q^{th} root first and the p^{th} power second, or the p^{th} power first and the q^{th} power second. For example:

$$4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3 = 2^3 = 8$$

and also:

$$4^{\frac{3}{2}} = (4^3)^{\frac{1}{2}} = 64^{\frac{1}{2}} = 8$$

Example 15

Simplify:

a $3^{\frac{1}{3}} \times 3^{\frac{1}{2}}$ **b** $5^{\frac{1}{2}} \div 5^{\frac{2}{3}}$ **c** $27^{\frac{4}{3}}$ **d** $16^{\frac{3}{4}}$ **e** $\left(\frac{16}{25}\right)^{-\frac{3}{2}}$

Solution

a
$$\begin{aligned}3^{\frac{1}{3}} \times 3^{\frac{1}{2}} &= 3^{\frac{1}{3}+\frac{1}{2}}\\ &= 3^{\frac{5}{6}}\end{aligned}$$

b
$$\begin{aligned}5^{\frac{1}{2}} \div 5^{\frac{2}{3}} &= 5^{\frac{1}{2}-\frac{2}{3}}\\ &= 5^{-\frac{1}{6}}\end{aligned}$$

c
$$\begin{aligned}27^{\frac{4}{3}} &= \left(27^{\frac{1}{3}}\right)^4\\ &= 3^4\\ &= 81\end{aligned}$$

d
$$\begin{aligned}16^{\frac{3}{4}} &= \left(16^{\frac{1}{4}}\right)^3\\ &= 2^3\\ &= 8\end{aligned}$$

e
$$\begin{aligned}\left(\frac{16}{25}\right)^{-\frac{3}{2}} &= \left(\frac{25}{16}\right)^{\frac{3}{2}}\\ &= \left(\left(\frac{25}{16}\right)^{\frac{1}{2}}\right)^3\\ &= \left(\frac{5}{4}\right)^3\\ &= \frac{125}{64}\end{aligned}$$

Example 16

Simplify each of these expressions, writing your answers with positive indices.

a $a^{\frac{2}{3}} \times a^{\frac{1}{2}}$ **b** $m^{\frac{1}{2}} \div m^{\frac{3}{5}}$ **c** $(32m^{\frac{3}{4}})^{\frac{2}{5}}$

Solution

a
$$a^{\frac{2}{3}} \times a^{\frac{1}{2}} = a^{\frac{2}{3}+\frac{1}{2}}$$
$$= a^{\frac{7}{6}}$$

b
$$m^{\frac{1}{2}} \div m^{\frac{3}{5}} = m^{\frac{1}{2}-\frac{3}{5}}$$
$$= m^{-\frac{1}{10}}$$
$$= \frac{1}{m^{\frac{1}{10}}}$$

c
$$\left(32m^{\frac{3}{4}}\right)^{\frac{2}{5}} = 32^{\frac{2}{5}} m^{\frac{3}{4}\times\frac{2}{5}}$$
$$= \left(2^5\right)^{\frac{2}{5}} m^{\frac{3}{10}}$$
$$= 4m^{\frac{3}{10}}$$

Index laws for rational indices

If a and b are positive numbers and x and y are rational numbers, then:

Index law 1 $a^x a^y = a^{x+y}$

Index law 2 $\dfrac{a^x}{a^y} = a^{x-y}$

Index law 3 $(a^x)^y = a^{xy}$

Index law 4 $(ab)^x = a^x b^x$

Index law 5 $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

Exercise 9C

Example 11a, b, c

1 Calculate the exact value of each number.

a $4^{\frac{1}{2}}$ **b** $49^{\frac{1}{2}}$ **c** $27^{\frac{1}{3}}$ **d** $32^{\frac{1}{5}}$ **e** $1000^{\frac{1}{3}}$ **f** $625^{\frac{1}{4}}$

Example 11d, e

2 Calculate the exact value of each number.

a $\left(\dfrac{1}{125}\right)^{\frac{1}{3}}$ **b** $\left(\dfrac{1}{64}\right)^{\frac{1}{2}}$ **c** $\left(\dfrac{1}{64}\right)^{\frac{1}{6}}$ **d** $\left(\dfrac{1}{10\,000}\right)^{\frac{1}{2}}$ **e** $\left(\dfrac{1}{10\,000}\right)^{\frac{1}{4}}$

Example 12

3 Calculate the exact value of each number.

a $27^{\frac{2}{3}}$ **b** $64^{\frac{2}{3}}$ **c** $81^{\frac{3}{4}}$ **d** $32^{\frac{3}{5}}$ **e** $9^{\frac{5}{2}}$

f $100^{\frac{5}{2}}$ **g** $\left(\frac{1}{8}\right)^{\frac{2}{3}}$ **h** $\left(\frac{8}{27}\right)^{\frac{4}{3}}$ **i** $121^{\frac{3}{2}}$ **j** $343^{\frac{2}{3}}$

k $\left(\frac{4}{25}\right)^{\frac{3}{2}}$ **l** $\left(\frac{25}{36}\right)^{\frac{3}{2}}$ **m** $\left(\frac{1}{10\,000}\right)^{\frac{3}{4}}$ **n** $\left(\frac{49}{100}\right)^{\frac{1}{2}}$ **o** $\left(\frac{32}{243}\right)^{\frac{4}{5}}$

Example 13a

4 Write each number in the form $\sqrt[n]{a}$.

a $2^{\frac{1}{3}}$ **b** $3^{\frac{1}{4}}$ **c** $20^{\frac{1}{5}}$ **d** $10^{\frac{1}{4}}$ **e** $9^{\frac{1}{5}}$

Example 13b

5 Write each number in index form.

a $\sqrt[3]{4}$ **b** $\sqrt[7]{13}$ **c** $\sqrt[4]{5}$ **d** $\sqrt[3]{11}$

e $\left(\sqrt[3]{5}\right)^2$ **f** $\left(\sqrt{7}\right)^3$ **g** $\left(\sqrt[5]{11}\right)^3$ **h** $\left(\sqrt[7]{10}\right)^2$

Example 13b

6 Write each number in index form.

a $5\sqrt{5}$ **b** $5 \times \sqrt[3]{5}$ **c** $6 \times \sqrt[4]{6}$

d $7 \times \sqrt[5]{7}$ **e** $5^2 \times \sqrt[3]{5}$ **f** $11^2 \times \sqrt{11}$

Example 14

7 Calculate the exact value of each number.

a $9^{-\frac{1}{2}}$ **b** $16^{-\frac{1}{4}}$ **c** $121^{-\frac{1}{2}}$

d $100^{-\frac{1}{2}}$ **e** $1\,000\,000^{-\frac{1}{2}}$ **f** $1331^{-\frac{1}{3}}$

Example 15c, d, e

8 Calculate the exact value of each number.

a $16^{-\frac{3}{4}}$ **b** $100^{-\frac{3}{2}}$ **c** $\left(\frac{1}{8}\right)^{-\frac{2}{3}}$ **d** $125^{-\frac{4}{3}}$ **e** $1000^{-\frac{2}{3}}$ **f** $32^{-\frac{3}{5}}$

Example 15a

9 Simplify, expressing each answer with a positive index.

a $2^{\frac{2}{3}} \times 2^{\frac{1}{3}}$ **b** $3^{\frac{4}{5}} \times 3^{\frac{1}{3}}$ **c** $7^{\frac{1}{5}} \times 7^{\frac{2}{5}}$

d $3^{\frac{1}{4}} \times 3^{\frac{1}{3}}$ **e** $10^{\frac{1}{2}} \times 10$ **f** $10^{\frac{2}{3}} \times 10^{\frac{1}{4}}$

g $3^{\frac{2}{3}} \times 3^{-\frac{1}{5}}$ **h** $2^{\frac{1}{5}} \times 2^{-\frac{1}{4}}$ **i** $5^{\frac{2}{5}} \times 5^{-\frac{7}{10}}$

Example 15b

10 Simplify, expressing each answer with a positive index.

a $2^{\frac{3}{5}} \div 2^{\frac{1}{5}}$ **b** $7^{\frac{11}{3}} \div 7^{\frac{8}{3}}$ **c** $8^{\frac{1}{4}} \div 8^{\frac{1}{7}}$

d $7^{\frac{2}{3}} \div 7^{\frac{1}{2}}$ **e** $8^{\frac{8}{9}} \div 8^{\frac{5}{9}}$ **f** $10^{\frac{3}{7}} \div 10^{\frac{5}{7}}$

11 Use your calculator to find the value of each of these numbers, correct to five significant figures.

a $10^{\frac{3}{5}}$ **b** $24^{\frac{2}{3}}$ **c** $86^{\frac{4}{7}}$ **d** $127^{\frac{3}{11}}$

e $19.6^{\frac{3}{4}}$ **f** $1.8^{\frac{3}{2}}$ **g** $\left(\pi + \pi^2\right)^3$ **h** $\left(\sqrt{3} + \pi\right)^{\frac{4}{7}}$

Example 16

12 Simplify each expression. In your answers, use only positive indices.

a $m^{\frac{2}{3}} \times m^{\frac{1}{4}}$ **b** $a^{\frac{2}{5}} \times a^{\frac{1}{3}}$ **c** $x^{\frac{1}{2}}y^{\frac{1}{3}} \times x^{\frac{1}{4}}y^{\frac{2}{5}}$

d $a^{\frac{4}{5}}b^{\frac{1}{3}} \times a^{\frac{3}{10}}b^{\frac{1}{2}}$ **e** $a^{\frac{4}{5}} \div a^{\frac{3}{10}}$ **f** $m^{\frac{2}{3}} \div m^{\frac{5}{6}}$

g $b^{\frac{4}{7}} \div b^{\frac{1}{3}}$ **h** $\left(2m^{\frac{4}{5}}\right)^2$ **i** $\left(3m^{\frac{1}{2}}\right)^{-3}$

j $\left(5a^{-\frac{1}{2}}\right)^{-4}$ **k** $\left(4m^{-\frac{2}{3}}\right)^2 \times 5m^{\frac{3}{4}}$ **l** $\left(2m^{-\frac{3}{4}}\right)^{-2} \times 4m^{\frac{2}{3}}$

m $\left(8m^6\right)^{\frac{1}{3}} \times \left(16m^2\right)^{\frac{1}{4}}$ **n** $\left(27m^{-6}\right)^{\frac{1}{3}} \times \left(64m^2\right)^{-\frac{1}{2}}$

13 Evaluate each number, giving the answers correct to four significant figures.

a $6^{1.2}$ **b** $18.5^{2.1}$ **c** $0.84^{-0.7}$ **d** $1.59^{-0.1}$ **e** $12.6^{-1.8}$ **f** $5.9^{-3.7}$

14 Simplify each expression, giving your answers with positive indices.

a $a^{1.6} \times a^{3.2}$ **b** $m^{4.7} \times m^{1.3}$

c $p^{8.2} \div p^{4.6}$ **d** $b^{4.1} \div b^{2.85}$

e $(2p^{1.3})^2$ **f** $(4p^{2.1})^3$

g $4a^{1.3}b^{0.6} \div (8a^2b^{-1})$ **h** $12m^{-1.2}n^{3.5} \div \left(18(mn^{-1.5})^3\right)$

i $\dfrac{a^{1.2}b^{4.3}}{(ab^{-1})^{1.2}} \times \dfrac{ab^{0.6}}{a^{1.8}b}$ **j** $\dfrac{m^{0.9}n}{(mn^{1.5})^2} \times \dfrac{1}{mn^{3.8}}$

9D Graphs of exponential functions

In the previous section, we saw how to define 2^x for all rational numbers x. There are a number of ways of defining 2^x for all real numbers x, but it is not possible to deal with them in this book. The calculator gives approximations to 2^x and we will use these values. Consider the following list of approximate values of powers of 2.

$2^1 = 2$ $2^{1.1} \approx 2.1435$ $2^{1.2} \approx 2.2974$

$2^{1.3} \approx 2.4623$ $2^{1.4} \approx 2.6390$ $2^{1.5} \approx 2.8284$ $(2^{1.5} = 2\sqrt{2})$

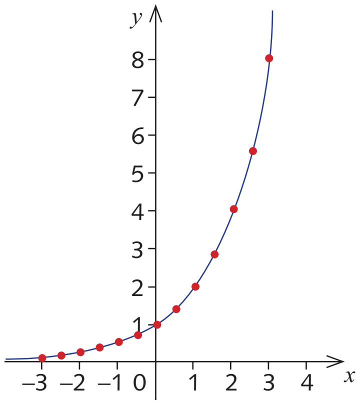
This list of values suggests that 2^x increases as x increases. This is in fact the case.

Throughout the rest of this chapter, you will often need to use your calculator to calculate values of exponential functions.

Consider the function $y = 2^x$. A table of approximate values, correct to three decimal places, follows.

x	−3	−2.5	−2	−1.5	−1	−0.5	0	0.5	1	1.5	2	2.5	3
y	0.125	0.177	0.25	0.354	0.5	0.707	1	1.414	2	2.828	4	5.657	8

By plotting these values and connecting them up with a smooth curve, we obtain the graph of $y = 2^x$.



Key features:

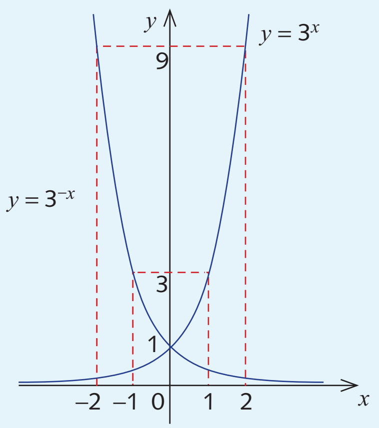
- $y = 2^x$ is an **increasing** function; that is, 2^x increases as x increases.
- A y-intercept occurs at $(0, 1)$ but there is no x-intercept.
- As x moves away from 0 in the negative direction (to the left), the value of 2^x gets close to 0, but it never equals 0. Why? We say that the x-axis is an **asymptote** for the graph of $y = 2^x$.

Example 17

Produce a table of values for the functions $y = 3^x$ and $y = 3^{-x}$. Draw the graphs on the same set of axes.

Solution

x	−3	−2	−1	0	1	2	3
3^x	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27
3^{-x}	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$



Note: $y = 3^x$ is an **increasing** function; that is, 3^x increases as x increases; and that $y = 3^{-x}$ is a **decreasing** function; that is, 3^{-x} decreases as x increases.

The two graphs in Example 17 are reflections of each other in the y-axis.

Note: $\frac{1}{3} = 3^{-1}$; hence, $\left(\frac{1}{3}\right)^x = (3^{-1})^x = 3^{-x}$.

Multiplication by a constant

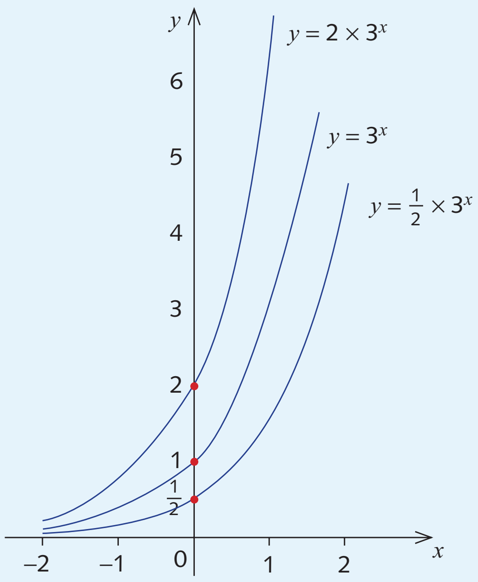
In applications, exponential functions often occur multiplied by a constant.

Example 18

Draw the graphs of $y = 3^x$, $y = \frac{1}{2} \times 3^x$ and $y = 2 \times 3^x$ on the same set of axes. (Produce a table of values first.)

Solution

x	–2	–1	0	1	2
3^x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$\frac{1}{2} \times 3^x$	$\frac{1}{18}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{9}{2}$
2×3^x	$\frac{2}{9}$	$\frac{2}{3}$	2	6	18



The different graphs in Example 18 are roughly the same shape and the y-intercept of the curve is the constant that multiplies the exponential function.

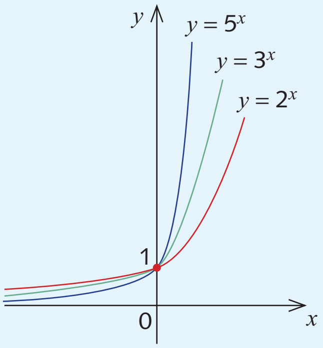
Next, we will investigate how exponential functions change for different values of the base.

Example 19

Draw the graphs of $y = 2^x$, $y = 3^x$ and $y = 5^x$ on the same set of axes.

Solution

x	–3	–2	–1	0	1	2	3
2^x	0.125	0.25	0.5	1	2	4	8
3^x	0.037	0.111	0.333	1	3	9	27
5^x	0.008	0.04	0.2	1	5	25	125



All three graphs in Example 19 pass through the point $(0, 1)$ but they have different 'gradients'. That is, 5^x increases more quickly than 3^x, which increases more quickly than 2^x. So, for example, $2^x < 5^x$ if $x > 0$, but $2^x > 5^x$ if $x < 0$.

Graphs of exponential functions

- To graph an exponential function, first create a table of values, then plot the points on a set of axes.
- If the exponential is multiplied by a constant, the y-intercept is that constant.
- The graph of $y = a^{-x}$, where $a > 0$, is the reflection of the graph of $y = a^x$ in the y-axis.
- The x-axis is an asymptote of the graph of $y = a^x$ and of $y = a^{-x}$, where $a > 0$ and $a \neq 1$.

Exercise 9D

Example 17 **1** For each function, produce a table of values for $x = -2, -1, 0, 1, 2$, and use it to draw a graph.

a $y = 2^x$ **b** $y = 2^{-x}$ **c** $y = 4^x$ **d** $y = 5^{-x}$

Example 18 **2** Sketch the graphs of $y = 4^x$, $y = 2 \times 4^x$ and $y = 3 \times 4^x$ on a single set of axes.

3 Sketch the graphs of $y = 2^x$, $y = 2 \times 2^x$ and $y = \frac{1}{2} \times 2^x$ on a single set of axes.

Example 19 **4** Sketch the graph of $y = 2^{-x}$, $y = 3^{-x}$ and $y = 5^{-x}$ on the one set of axes.

9E Exponential equations

From the previous section, we have seen that the graph of $y = 2^x$ is increasing and the graph of

$y = 2^{-x} = \left(\frac{1}{2}\right)^x$ is decreasing.

In general, suppose that a is a positive number different from 1. Since the graphs of $y = a^x$ are either increasing or decreasing (unless $a = 1$), there is only one value of x for each value of y. Hence, we know that if $a^c = a^d$, then $c = d$.

In the following examples, this fact is used to solve **exponential equations**. From the above discussion it can be seen that there is only one solution for x to the equation $a^x = y$, provided that y is positive.

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Example 20

Solve each equation for x.

a $2^x = 32$ **b** $10^x = 10\,000$ **c** $5^x = 625$

Solution

a $2^x = 32$

Since $32 = 2^5$

$2^x = 2^5$

$x = 5$

b $10^x = 10\,000$

Since $10\,000 = 10^4$

$10^x = 10^4$

$x = 4$

c $5^x = 625$

Since $625 = 5^4$

$5^x = 5^4$

$x = 4$

Example 21

Solve each equation for x.

a $2^x = \frac{1}{8}$ **b** $7^x = \frac{1}{343}$ **c** $7^x = 1$

Solution

a $2^x = \frac{1}{8}$

Since $\frac{1}{8} = 2^{-3}$

$2^x = 2^{-3}$

$x = -3$

b $7^x = \frac{1}{343}$

Since $\frac{1}{343} = 7^{-3}$

$7^x = 7^{-3}$

$x = -3$

c $7^x = 1$

Since $7^0 = 1$

$x = 0$

In Example 22, we first write each side of the equation as a power with the same base.

Example 22

Solve each equation for x.

a $16^x = 32$ **b** $81^x = 243$ **c** $256^x = 32$

Solution

a
$$\begin{aligned} 16^x &= 32 \\ (2^4)^x &= 32 \\ 2^{4x} &= 2^5 \\ 4x &= 5 \\ x &= \frac{5}{4} \end{aligned}$$

b
$$\begin{aligned} 81^x &= 243 \\ (3^4)^x &= 3^5 \\ 4x &= 5 \\ x &= \frac{5}{4} \end{aligned}$$

c
$$\begin{aligned} 256^x &= 32 \\ (2^8)^x &= 2^5 \\ 8x &= 5 \\ x &= \frac{5}{8} \end{aligned}$$

Example 23

Solve:

a $3^{2x-1} = 81$

b $6^{x-1} = 36\sqrt{6}$

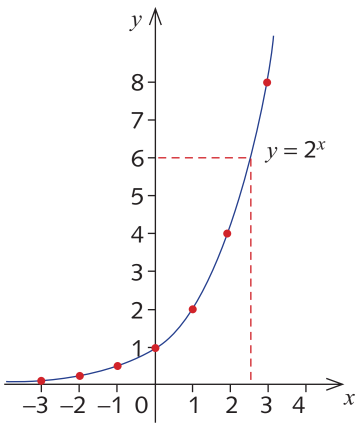
Solution

a
$$\begin{aligned} 3^{2x-1} &= 81 \\ 3^{2x-1} &= 3^4 \\ 2x - 1 &= 4 \\ 2x &= 5 \\ x &= \frac{5}{2} \end{aligned}$$

b
$$\begin{aligned} 6^{x-1} &= 36\sqrt{6} \\ 6^{x-1} &= 6^2 \times 6^{\frac{1}{2}} \\ 6^{x-1} &= 6^{\frac{5}{2}} \\ x - 1 &= \frac{5}{2} \\ x &= \frac{7}{2} \end{aligned}$$

Consider the exponential equation $2^x = 6$.
Since $2^2 = 4$ and $2^3 = 8$, x must be between 2 and 3.
From the graph opposite, we can estimate x to be about 2.5. From a calculator, one obtains 2.58 as a better approximation. The value of x is called $\log_2 6$, which is $\approx 2.584\,962$.

We will discuss logarithms in a later section of this chapter.



Example 24

Between which two integers does x lie if:

a $2^x = 70$?

b $2^x = 200$?

Solution

The graph of $y = 2^x$ is increasing.

a $2^6 = 64$ and $2^7 = 128$

Therefore, x lies between 6 and 7.

b $2^7 = 128$ and $2^8 = 256$

Therefore, x lies between 7 and 8.

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Solving exponential equations

For any positive value a, if $a^c = a^d$, then $c = d$.

For $a > 0$ and $a \neq 1$, the equation $a^x = y$, where $y > 0$, can be solved, and there is only one solution for x.

Exercise 9E

Example 20

1 Solve:

a $2^x = 8$ **b** $2^x = 512$ **c** $3^x = 243$ **d** $10^x = 100$

e $11^x = 1331$ **f** $20^x = 400$ **g** $6^x = 216$ **h** $10^x = 100\,000$

i $5^x = 125$ **j** $3^x = 729$ **k** $4^x = 256$ **l** $4^x = 1024$

Example 21

2 Solve:

a $2^x = \frac{1}{16}$ **b** $4^x = \frac{1}{256}$ **c** $5^x = 1$ **d** $10^x = 0.001$

e $10^x = \frac{1}{100\,000}$ **f** $7^x = \frac{1}{343}$ **g** $3^x = \frac{1}{243}$ **h** $2^x = \frac{1}{1024}$

Example 22

3 Solve:

a $121^x = 11$ **b** $121^x = 1331$ **c** $9^x = 27$ **d** $64^x = 16$

e $25^x = 125$ **f** $125^x = 25$ **g** $1000^x = 100$ **h** $10\,000^x = 1000$

4 Solve:

a $27^a = 243$ **b** $4^b = 128$ **c** $128^c = 32$ **d** $625^d = 125$

e $1000^e = 10$ **f** $\left(\frac{1}{8}\right)^f = 4$ **g** $27^x = \frac{1}{3}$ **h** $(0.01)^x = 1000$

Example 23

5 Solve:

a $3^{x-2} = 27$ **b** $5^{1-x} = 125$ **c** $4^{3x-1} = 64$ **d** $32^{3x+1} = 128$

e $2^{3-x} = \sqrt{8}$ **f** $(\sqrt{7})^x = 343$ **g** $4^{x-1} = \frac{1}{16\sqrt{2}}$ **h** $3^{3-x} = 27^{x-1}$

Example 24

6 Identify which two integers x lies between if:

a $2^x = 19$ **b** $5^x = 30$ **c** $2^x = 40$ **d** $10^x = 500$

e $3^x = 90$ **f** $7^x = 50$ **g** $11^x = 100$ **h** $13^x = 200$

i $2^{-x} = 0.1$ **j** $5^{-x} = 2$ **k** $5^{-x} = 0.3$ **l** $10^{-x} = 0.045$

9F Exponential growth and decay

We begin by looking at two examples.

Exponential growth

The first example is a mathematical model of the number of bacteria in a culture.

Initially, there are 1000 bacteria in a culture. The number of bacteria is doubling every hour.

Therefore:

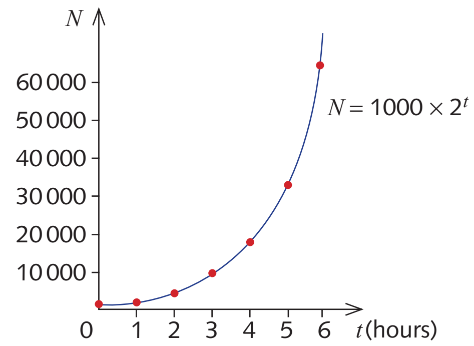
- after 1 hour there are 1000×2 bacteria
- after 2 hours there are $1000 \times 2 \times 2 = 1000 \times 2^2$ bacteria
- after 3 hours there are $1000 \times 2^2 \times 2 = 1000 \times 2^3$ bacteria.

Following this pattern, there are 1000×2^t bacteria after t hours. This can be written as a formula. Let N be the number of bacteria after t hours. Then:

$$N = 1000 \times 2^t$$

A graph can be plotted by first producing a table of values.

t	0	1	2	3	4	5	6
N	1000	2000	4000	8000	16 000	32 000	64 000



This is an example of **exponential growth**.

Exponential decay

Radioactivity is a natural phenomenon in which atoms of one element 'decay' to form atoms of another element by emitting a particle such as an alpha particle.

A sample of a radioactive substance that is widely used in medical radiology initially has a mass of 100 g. The substance decays over time, its quantity halving every hour. Let M grams be the mass present after t hours.

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Therefore:

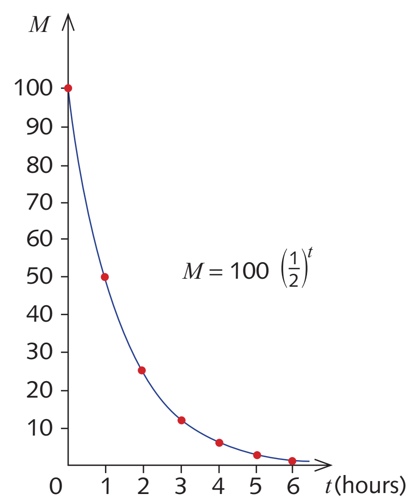
- after 1 hour the mass is $100 \times \frac{1}{2}$ g
- after 2 hours the mass is $100 \times \frac{1}{2} \times \frac{1}{2} = 100 \times \left(\frac{1}{2}\right)^2$ g
- after 3 hours the mass is $100 \times \left(\frac{1}{2}\right)^2 \times \frac{1}{2} = 100 \times \left(\frac{1}{2}\right)^3$ g.

Following this pattern, there are $100 \times \left(\frac{1}{2}\right)^t$ grams of the radioactive substance after t hours. So:

$$M = 100\left(\frac{1}{2}\right)^t$$

A table is constructed and the graph is plotted.

t	0	1	2	3	4	5	6
M	100	50	25	12.5	6.25	3.13	1.56



This is an example of **exponential decay**.

Formulas for exponential growth and decay

The two previous examples concern populations or quantities that can be described by a formula of the form:

$$P = A \times B^t$$

In this formula, A and B are positive constants and t is a variable that is usually time measured in seconds, hours or years, depending on the application.

If $t = 0$, then $P = A$, so A is the initial amount.

If $B = 1$, then $P = A$ for all values of t.

If $B > 1$, we say that P grows exponentially.

If $B < 1$, we say that P decays exponentially.

It is possible to estimate both future and past sizes of the population by substituting positive and negative values for t.

Example 25

For the rule $y = 20 \times 3^t$:

a Complete the table of values.

t	0	1	2	3
y				

b Plot the graph of y against t.

c Find the value y, correct to two decimal places, when:

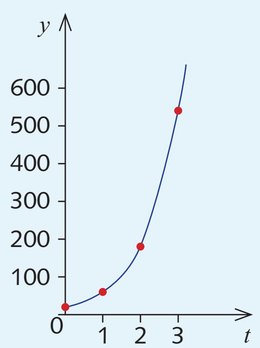
i $t = 0.5$ **ii** $t = 2.5$ **iii** $t = 2.8$

Solution

a Complete the table of values.

t	0	1	2	3
y	20	60	180	540

b



c Using a calculator:

i When $t = 0.5$, $y = 34.64$ **ii** When $t = 2.5$, $y = 311.77$

iii When $t = 2.8$, $y = 433.48$

Exercise 9F

Example 25

1 For the formula $y = 200 \times 2^t$:

a Complete the table of values.

t	0	1	2	3	4	5
y						

b Plot the graph of y against t.

c Using your calculator, find the value of y, correct to two decimal places, when:

i $t = 0.6$ **ii** $t = 2.2$ **iii** $t = 3.5$

2 For the formula $y = 200 \times \left(\frac{1}{2}\right)^t$:

a Complete the table of values.

t	0	1	2	3	4	5
y						

b Plot the graph of y against t.

c Using your calculator, find the value of y, correct to two decimal places, when:

i $t = 0.6$ **ii** $t = 3.2$ **iii** $t = 4.6$

3 **a** For $y = 60 \times 8^t$, find the value of y when:

i $t = 0$ **ii** $t = 2$ **iii** $t = 2.5$

b For $y = 1000 \times (0.1)^t$, find the value of y when:

i $t = 0$ **ii** $t = 1$ **iii** $t = 3$ **iv** $t = 4$

4 On 1 January 2011, the population of the world was estimated to be $7\,074\,000\,000 = 7.074 \times 10^9 = A$. Assume that the population of the world is increasing at the rate of 3% per year, so that $N = A(1.03)^t$ after t years.

a Estimate what the population of the world will be on 1 January 2016.

b Estimate the population on 1 January 2111.

5 A liquid cools from its original temperature of $95°C$ to a temperature $T°C$ in t minutes. Given that $T = 95(0.96)^t$, find:

a the value of T when $t = 10$

b the value of T when $t = 20$

6 The number of finches on an island, N, at time t years after 1 January 2010 is approximately described by the rule $N = 80\,000 \times (1.008)^t$.

a Identify (from the rule) the annual percentage increase in finches on the island after 1 January 2010.

b How many finches were there on the island on 1 January 2010?

c How many finches will there be on the island on 1 January 2020?

7 The number of bacteria, N, in a certain culture is halving every hour, so $N = A \times \left(\frac{1}{2}\right)^t$, where t is the time in hours after 2 p.m. on a particular day. Assume that there are initially 1000 bacteria.

a State the value of A.

b Estimate the number of bacteria in the culture when:

i $t = 2$ **ii** $t = 3$ **iii** $t = 5$

9G Logarithms

In Section 9D, we saw how to sketch the graph of $y = 2^x$. When we wish to determine the value of y for particular values of x, for example, $6 = 2^x$, the concept of a logarithm is used.

Consider the number fact $2^3 = 8$. When making the exponent the subject of this relationship, we express it as $\log_2 8 = 3$.

This is read as either 'log to the base 2 of 8 is (equal to) 3' or 'the log of 8 to the base 2 is (equal to) 3'.

For example:

- $3^5 = 243$ is equivalent to $\log_3 243 = 5$
- $5^{-2} = \frac{1}{25}$ is equivalent to $\log_5 \frac{1}{25} = -2$
- $10^2 = 100$ is equivalent to $\log_{10} 100 = 2$
- $8^{\frac{2}{3}} = 4$ is equivalent to $\log_8 4 = \frac{2}{3}$

The logarithm of a number to base a is the index to which a is raised to give that number.

In general, the logarithm can be defined as follows.

If $a > 0$ and $a \neq 1$ and $a^x = y$, then $\log_a y = x$.

Logarithms were invented in the seventeenth century to assist in astronomical calculations. They have a number of important properties, which will be discussed in detail in Chapter 14.

Example 26

Evaluate these logarithms.

a $\log_2 32$ **b** $\log_3 81$ **c** $\log_{10} 1000$ **d** $\log_2 1024$

Solution

a $2^5 = 32$, so $\log_2 32 = 5$

b $3^4 = 81$, so $\log_3 81 = 4$

c $10^3 = 1000$, so $\log_{10} 1000 = 3$

d $2^{10} = 1024$, so $\log_2 1024 = 10$

Example 27

Evaluate these logarithms.

a $\log_4 \frac{1}{16}$ **b** $\log_{10} 0.001$ **c** $\log_3 \frac{1}{27}$ **d** $\log_2 \frac{1}{1024}$

Solution

a $4^{-2} = \frac{1}{16}$, so $\log_4 \frac{1}{16} = -2$

b $10^{-3} = 0.001$, so $\log_{10} 0.001 = -3$

c $3^{-3} = \frac{1}{27}$, so $\log_3 \frac{1}{27} = -3$

d $2^{-10} = \frac{1}{1024}$, so $\log_2 \frac{1}{1024} = -10$

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On most calculators, the button labelled 'log' calculates $\log_{10} x$, for any positive number x.

Example 28

Calculate these logarithms correct to four decimal places.

a $\log_{10} 3$

b $\log_{10} 842$

c $\log_{10} 2$

d $\log_{10} 0.0005$

Solution

a $\log_{10} 3 \approx 0.4771$

b $\log_{10} 842 \approx 2.9253$

c $\log_{10} 2 \approx 0.3010$

d $\log_{10} 0.0005 \approx -3.3010$

In general, simple logarithmic equations are best solved by first converting them into their equivalent exponential form.

Example 29

Find the value of x.

a $\log_2 32 = x$

b $\log_8 \frac{1}{64} = x$

c $\log_2 x = 5$

d $\log_x 16 = 2$

e $\log_{36} x = -\frac{1}{2}$

f $\log_7 x = 2$

Solution

a $\log_2 32 = x$, is equivalent to $2^x = 32$, so $x = 5$

b $\log_8 \frac{1}{64} = x$, is equivalent to $8^x = \frac{1}{64}$, so $x = -2$.

c $\log_2 x = 5$, is equivalent to $2^5 = x$, so $x = 32$.

d $\log_x 16 = 2$, is equivalent to $x^2 = 16$, so $x = 4$ (since $x > 0$).

e $\log_{36} x = -\frac{1}{2}$, is equivalent to $36^{-\frac{1}{2}} = x$, so $x = \frac{1}{6}$.

f $\log_7 x = 2$, is equivalent to $7^2 = x$, so $x = 49$.

Logarithms

The logarithm of a number to base a is the index to which a is raised to give this number.

If $a > 0$ and $a \neq 1$ and $a^x = y$, then $\log_a y = x$.

Exercise 9G

1 Copy and complete:

a $2^3 = 8$ is equivalent to $\log_2 8 = \ldots$
b $10^2 = 100$ is equivalent to $\log_{10} 100 = \ldots$
c $7^2 = 49$ is equivalent to $\log_7 \ldots = \ldots$
d $3^4 = \ldots$ is equivalent to $\log_3 \ldots = \ldots$
e $5^3 = \ldots$ is equivalent to $\log_5 \ldots = \ldots$
f $7^3 = \ldots$ is equivalent to $\log_7 \ldots = \ldots$
g $2^5 = \ldots$ is equivalent to $\log_2 \ldots = \ldots$
h $10^4 = \ldots$ is equivalent to $\log_{10} \ldots = \ldots$
i $10^{-3} = \ldots$ is equivalent to $\log_{10} \ldots = \ldots$
j $2^{-1} = \ldots$ is equivalent to $\log_2 \ldots = \ldots$

Example 26

2 Evaluate each logarithm.

a $\log_2 4$ **b** $\log_2 64$ **c** $\log_2 128$ **d** $\log_2 4096$
e $\log_2 1$ **f** $\log_2 256$ **g** $\log_{10} 1000$ **h** $\log_5 25$

3 Evaluate:

a $\log_3 27$ **b** $\log_5 625$ **c** $\log_4 64$ **d** $\log_8 64$
e $\log_6 216$ **f** $\log_7 1$ **g** $\log_6 1296$ **h** $\log_9 729$

Example 27

4 Evaluate:

a $\log_2 \frac{1}{4}$ **b** $\log_5 \frac{1}{5}$ **c** $\log_3 \frac{1}{9}$ **d** $\log_{11} \frac{1}{121}$
e $\log_5 \frac{1}{125}$ **f** $\log_4 \frac{1}{1024}$ **g** $\log_3 \frac{1}{81}$ **h** $\log_7 \frac{1}{343}$

5 Evaluate:

a $\log_{10} 10$ **b** $\log_{10} 1$ **c** $\log_{10} 1000$ **d** $\log_{10} 100\,000$
e $\log_{10} 10^{100}$ **f** $\log_{10} \frac{1}{100}$ **g** $\log_{10} 0.000\,0001$ **h** $\log_{10} 10^{-13}$

6 If $a > 0$, what is $\log_a a$?

7 If $a > 0$, what is $\log_a 1$?

Example 28

8 Use your calculator to evaluate each logarithm correct to four decimal places.

a $\log_{10} 789$ **b** $\log_{10} 0.0003$ **c** $\log_{10} 72\,000\,000$
d $\log_{10}(5.3950 \times 10^{-3})$ **e** $\log_{10}(635 \times 10^{54})$ **f** $\log_{10} 0.000\,123\,45$

Example 29

9 Find the value of x.

a $\log_2 64 = x$ **b** $\log_3 243 = x$ **c** $\log_4 \frac{1}{256} = x$ **d** $\log_{10} \frac{1}{1000} = x$
e $\log_3 x = 3$ **f** $\log_5 x = 2$ **g** $\log_2 x = -3$ **h** $\log_{25} x = -\frac{1}{2}$
i $\log_x 16 = 4$ **j** $\log_x 16 = 2$ **k** $\log_x 125 = 3$ **l** $\log_x \frac{1}{8} = -3$

Review exercise

1 Simplify:

a $(a^3)^4 \times a^5$ **b** $(2m^3)^4 \times (3m)^4$ **c** $\dfrac{(a^3b^2)^4}{(a^2b^2)^3}$

2 Evaluate:

a 4^{-2} **b** $6a^0$ **c** 10^{-4} **d** $\left(\dfrac{2}{3}\right)^{-4}$

3 Simplify each expression, writing each pronumeral with a positive index.

a $a^{-3} \times a^{-5}$ **b** $2a^3 \times 7a^{-6}$ **c** $\dfrac{12a^4}{3a^6}$ **d** $(2a^{-1})^2 \times (4a^2)^{-2}$

4 Write each term with positive indices only.

a b^{-3} **b** $2x^{-4}$ **c** $5x^{-3}$

d $\dfrac{x^{-3}}{2}$ **e** $\dfrac{a^{-4}}{5}$ **f** $\dfrac{2}{x^{-2}}$

g $\dfrac{4}{x^{-3}}$ **h** $\dfrac{4a^{-2}}{b^{-3}}$ **i** $\dfrac{5m^{-1}}{6m^{-4}}$

5 Express each power as a fraction.

a 6^{-2} **b** 4^{-3} **c** 2^{-4} **d** 5^{-1} **e** 10^{-2}

6 Simplify each expression.

a 5^0 **b** $5a^0$ **c** $(5a)^0$

d $6 + a^0$ **e** $(4 + a)^0$ **f** $2 + 3b^0$

g $\left(\dfrac{2}{3}\right)^0$ **h** $\dfrac{2^0}{3}$ **i** $\dfrac{4a^0}{(7b)^0}$

7 Simplify each expression, giving your answers with positive indices.

a $\dfrac{2a^2(2b)^3}{2ab^2}$ **b** $\dfrac{a^2b^3}{ab} \times \dfrac{a^2b^5}{a^2b^2}$ **c** $\dfrac{(2a)^2 \times 8b^3}{16a^2b^2}$

d $\dfrac{2a^2b^3}{8a^2b^2} \div \dfrac{16(ab)^2}{2ab}$ **e** $\dfrac{8a^6}{6a^3} \div \dfrac{4(a^2)^4}{(3a)^3}$ **f** $\dfrac{3a^3}{6a^{-1}}$

8 Write $\dfrac{2^n \times 8^n}{2^{2n} \times 16}$ in the form 2^{an+b}.

9 Write $2^{-x} \times 3^{-x} \times 6^{2x} \times 3^{2x} \times 2^{2x}$ as a power of 6.

10 Simplify each product.

a $2^{\frac{1}{3}} \times 2^{\frac{1}{6}} \times 2^{-\frac{2}{3}}$ **b** $a^{\frac{1}{4}} \times a^{\frac{2}{5}} \times a^{-\frac{1}{10}}$

c $2^{\frac{1}{3}} \times (2^{\frac{2}{5}})^5$ **d** $(2^{\frac{1}{3}})^2 \times 2^{\frac{1}{3}} \times 2^{-\frac{2}{5}}$

11 Write each number in scientific notation.

a 4200 **b** 0.0062 **c** 740 000 000 **d** 0.000 0002

12 Write each number in decimal notation.

a 5.4×10^3 **b** 11.2×10^4 **c** 6.8×10^{-2} **d** 9.7×10^{-3}

e 1.8×10^{-1} **f** 6.4×10^{-5} **g** 7.41×10^6 **h** 4.02×10^2

13 Write each number correct to the number of significant figures specified in the brackets.

a 18 (1) **b** 495 (1) **c** 416 (2)

d 34 200 (2) **e** 0.006 81 (2) **f** 0.049 21 (3)

g 475.2 (2) **h** 598.7 (2) **i** 0.006 842 (1)

14 Evaluate:

a $\log_2 8$ **b** $\log_2 16$ **c** $\log_2 \frac{1}{4}$ **d** $\log_3 1$

e $\log_5 \frac{1}{25}$ **f** $\log_4 \frac{1}{64}$ **g** $\log_3 \frac{1}{81}$ **h** $\log_7 \frac{1}{343}$

15 Evaluate:

a $\log_{10} 10$ **b** $\log_{10} 100\,000$ **c** $\log_{10} 10^{15}$

d $\log_{10} \frac{1}{10}$ **e** $\log_{10} \frac{1}{100}$ **f** $\log_{10} 10^{-6}$

16 Solve each equation for x.

a $4^x = 32^{x+1}$ **b** $(3^{x+2})^3 = \frac{1}{3}$ **c** $3^{x+1} = \frac{1}{81^x}$

d $5^{3x} \div 5^{2(x-1)} = 1$ **e** $2^7 \times 4^x = \frac{1}{8}$ **f** $9x = 27^4$

17 Evaluate:

a $2^{\frac{1}{3}} \times 12^{\frac{2}{3}} \times 6^{\frac{1}{3}}$ **b** $2^{-3} \times 4^{\frac{1}{2}} \times 8^{\frac{1}{3}}$ **c** $8^{-\frac{4}{3}} \times 32^{\frac{6}{5}}$

d $8^{-\frac{2}{3}} \times 16^{-\frac{5}{4}}$ **e** $16^{-\frac{3}{4}} \times 4$ **f** $7^{\frac{3}{2}} \times 7^{-1}$

18 Simplify, expressing your answers with positive indices.

a $\left(\frac{3^{-2}a}{b^2}\right)^{-2} \times \frac{27}{a^2b^2}$ **b** $\left(\frac{x^2}{y^{-2}}\right)^{-3} \times \left(\frac{x^2}{y^3}\right)$

c $\frac{(3a^2)^3}{(2ab^2)^2} \times \frac{(2b)^{-5}}{(3a)^{-4}}$ **d** $\frac{(a^2b)^3 \times (ab^3)^{-1}}{(a^{-1}b)^{-4}}$

19 Solve for x.

a $\log_2 x = 5$ b $\log_3 x = 7$ c $\log_5 x = 0$

d $\log_7 x = 2$ e $\log_{10} x = -1$ f $\log_5 x = -\frac{1}{2}$

g $\log_x 25 = 2$ h $\log_x 81 = 4$ i $\log_x 10\,000 = 4$

20 The population of a town is initially 8000. Every year the population increases by 5%. What is the population of the town after:

a 1 year? b 3 years? c n years?

Challenge exercise

1 a If $2^y = x$, what is $15 \times 2^{y+3}$, in terms of x?

b If $3^x = 2$, find 3^{7x}.

c If $4^y = x$, what is 4^{y-2}, in terms of x?

2 Find the value of x if $a^x = \dfrac{\sqrt{a^8}}{\left(\sqrt{a^6}\right)^4}$.

3 Find the value of x if $t^x = \sqrt[3]{\dfrac{t}{\sqrt{t}}}$.

4 Find the value of x if $\dfrac{4^{\frac{1}{2}}}{\sqrt[3]{8^2}} = 2^x$.

5 a Evaluate $2^8 + 2^{11} + 2^n$ for n between 1 and 8.

b Find the value of $n > 8$ such that $2^8 + 2^{11} + 2^n$ is a perfect square.

6 a Prove that the index laws hold for negative integer exponents. (Use the laws for positive integer exponents.)

For example, the product-of-powers result can be proved in the following way for negative integer exponents.

Consider $a^{-p}a^{-q}$ where p and q are positive integers.

$$\begin{aligned}
a^{-p}a^{-q} &= \frac{1}{a^p} \times \frac{1}{a^q} \\
&= \frac{1}{a^p a^q} \\
&= \frac{1}{a^{p+q}} && \text{(Index law 1 for positive integers)} \\
&= a^{-(p+q)} \\
&= a^{-p+(-q)}
\end{aligned}$$

b Prove that the index laws hold for fractional exponents.

For example, $a^{\frac{1}{n}} \times a^{\frac{1}{m}} = a^{\frac{1}{n}+\frac{1}{m}}$ can be proved in the following way.

$$\begin{aligned} a^{\frac{1}{n}} \times a^{\frac{1}{m}} &= a^{\frac{m}{nm}} \times a^{\frac{n}{nm}} \\ &= \sqrt[nm]{a^m} \times \sqrt[nm]{a^n} \\ &= \sqrt[nm]{a^m \times a^n} && \text{(Index law 1)} \\ &= \sqrt[nm]{a^{m+n}} \\ &= a^{\frac{m+n}{nm}} \\ &= a^{\frac{1}{n}+\frac{1}{m}} \end{aligned}$$

The result can easily be extended to $a^{\frac{p}{n}} \times a^{\frac{q}{m}} = a^{\frac{p}{n}+\frac{q}{m}}$.

7 Solve each pair of equations for x and y.

a $25^x = 125^y$, $16^x \div 8 = 2 \times 4^{2y}$

b $3^y - 1 = 9^x$, $4^y \times 64^x = 128$

c $10^{5y} = 10^5 \times 100^x$, $49^y = 7 \times 7^x$

d $a^{2x} = a^{y-1}$, $b^{2+y} = b^{3x}$

8 Simplify:

a $\dfrac{a^{\frac{2}{3}} + 2a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}} - c^{\frac{2}{3}}}{a^{\frac{1}{3}} + b^{\frac{1}{3}} - c^{\frac{1}{3}}}$

b $\dfrac{x^2 + x^{-2} - 1}{x + x^{-1} - 3^{\frac{1}{2}}}$

9 Expand:

a $\left(3a^{\frac{2}{3}} - 2a^{-\frac{2}{3}}b^{\frac{1}{2}} - b^{-\frac{1}{2}}\right)\left(a^{\frac{1}{3}} - 2b\right)$

b $\left(a^{\frac{3}{4}} + a^{\frac{1}{2}}b^{\frac{1}{2}} + a^{\frac{1}{4}}b + b^{\frac{3}{2}}\right)\left(a^{\frac{1}{4}} - b^{\frac{1}{2}}\right)$

10 Without using a calculator, list the numbers $2^{\frac{1}{2}}$, $3^{\frac{1}{3}}$ and $5^{\frac{1}{5}}$ in order from greatest to least.

11 The areas of the side, front and bottom faces of a rectangular prism are $2x$, $\dfrac{y}{2}$ and xy. Find the volume of the prism in terms of x and y.

12 Simplify $\dfrac{5^{3x+1} - 5^{3x-1} + 24}{24 \times 5^{3x} + 120}$.

13 Find the sum of the digits of $10^{2008} - 2008$.

CHAPTER

10

Review and problem-solving

10A Review

Chapter 1: Consumer arithmetic

1 The sum of \$10 000 is borrowed for 5 years at 9% p.a. simple interest. How much interest is paid?

2 The sum of \$7500 is borrowed at 7.5% p.a. simple interest and \$1687.50 is paid in interest. For how many years has the money been borrowed?

3 The sum of \$5600 is borrowed for 4 years and \$1948.80 is paid in interest. Calculate the (per annum) rate of simple interest charged.

4 A department store is offering a 40% discount on all items in the store. Calculate the discounted price on the following items:

a a jacket with a marked price of \$399

b a dress with a marked price of \$120

5 A pair of shoes marked at \$220 is sold for \$176. What percentage discount has been allowed?

6 A music store is offering a 45% discount during a sale.

Calculate the original marked price of:

a a DVD that has a sale price of \$13.20

b a boxed set of DVDs that has a sale price of \$66

7 Calculate the missing entries.

	Original value	New value	Percentage change
a	120		10% decrease
b	90		15% increase
c	60		40% decrease
d		26	30% increase
e	500	375	
f	140	350	
g		203	20% decrease

8 Find the single percentage change that is equivalent to:

a a 20% increase followed by a 20% decrease

b a 10% increase followed by a 5% increase

c an 8% decrease followed by a 4% increase

d a 10% decrease followed by a 10% decrease

9 Due to market demands, the cost of petrol increases by 2%, 5% and 4% in three successive months. By what percentage has the cost of petrol increased over the three-month period?

10 A quantity is increased by 10%. What further percentage change, applied to the increased value, is required to produce these changes?

a Increase of 32% **b** Increase of 15.5%

c Decrease of 12% **d** Decrease of 6.5%

11 Calculate the amount that an investment of $30 000 will be worth if it is invested at 6.5% p.a. for 10 years compounded annually.

12 Calculate the amount that an investment of $12 000 will be worth if it is invested at 8% p.a. for 6 years compounded:

a annually **b** quarterly (assume 2% per quarter) **c** monthly (assume $\frac{2}{3}$% per month)

Chapter 2: Review of surds

1 Simplify by collecting like surds:

a $2\sqrt{2}+3\sqrt{3}-\sqrt{3}+3\sqrt{2}$ **b** $5\sqrt{5}-3+2\sqrt{5}+7$

2 Simplify:

a $\sqrt{18}$ **b** $\sqrt{128}$ **c** $4\sqrt{72}$ **d** $3\sqrt{27}$

3 Simplify:

a $\sqrt{50}-3\sqrt{8}$ **b** $3\sqrt{12}+4\sqrt{75}$ **c** $\sqrt{147}+\sqrt{243}$

d $4\sqrt{63}-2\sqrt{28}$ **e** $3\sqrt{45}+\sqrt{72}+6\sqrt{8}-\sqrt{20}$ **f** $\dfrac{a}{\sqrt{a}}+\sqrt{a}$

4 Expand and simplify:

a $\sqrt{8}\left(\sqrt{6}-\sqrt{2}\right)$ **b** $\left(5+\sqrt{2}\right)\left(\sqrt{2}-3\right)$

c $\left(3\sqrt{2}-\sqrt{5}\right)\left(3\sqrt{2}+\sqrt{5}\right)$ **d** $\left(\sqrt{3}-\sqrt{2}\right)^2$

5 Simplify:

a $\sqrt{8}\times\sqrt{8}$ **b** $\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{a}+\sqrt{b}\right)$ **c** $\left(\sqrt{a}+\sqrt{b}\right)^2$

6 Rationalise the denominator and simplify:

a $\dfrac{2\sqrt{27}}{\sqrt{18}}$ **b** $\dfrac{2\sqrt{3}+3\sqrt{2}}{\sqrt{6}}$ **c** $\dfrac{\sqrt{3}+5}{\sqrt{3}}$

d $\dfrac{3\sqrt{5}-1}{\sqrt{5}}$ **e** $\dfrac{5\sqrt{12}+2\sqrt{10}}{\sqrt{5}}$ **f** $\dfrac{2\sqrt{6}-\sqrt{3}}{3\sqrt{3}}$

7 Express with a rational denominator in simplest form:

a $\dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$ **b** $\dfrac{\sqrt{3}+\sqrt{2}}{3\sqrt{3}-2\sqrt{2}}$ **c** $\dfrac{2\sqrt{2}+1}{\sqrt{6}-2}$ **d** $\dfrac{3\sqrt{10}+2}{2\sqrt{5}+\sqrt{2}}$

Chapter 3: Algebra review

1 Simplify:

a $6mn^2-7m+3mn^2+4m$ **b** $2x\times 3y-6x^2-6xy+3x\times 2x$

c $\dfrac{4a^2b^3}{2ab}+3a^2-\dfrac{6a^3b^2}{3ab^2}+5ab^2$ **d** $\dfrac{6p^3q^3}{pq}+\dfrac{2p^3q^2}{q^2}-\dfrac{14p^5q}{2p^2q}+\dfrac{12p^4q^2}{3p^2}$

2 Expand and collect like terms for each expression.

a $3(a+2)+2(a-1)$

b $5(b+3)-3(b-2)$

c $2x(x+5)+4x(x-3)$

d $3y(y-1)-4y(2y-5)$

e $(2x+1)(x+5)$

f $(2a+7)(3a-2)$

g $(2y+3)(y+2)-(y-1)(y+3)$

h $(b+5)(3b+1)-(2b-3)(b-2)$

3 Expand and collect like terms for each expression.

a $3(x+3)(2x+5)$

b $2(2a+1)(3a-4)$

c $2(2y+1)(y+2)+3(y-2)(2y+3)$

d $5(b+2)(2b+1)-3(b-1)(b-3)$

4 Expand and collect like terms for each expression.

a $\frac{3}{4}(a+2)+\frac{1}{2}(a-1)$

b $5\left(\frac{b}{3}+\frac{1}{6}\right)-3\left(\frac{b}{6}-\frac{1}{2}\right)$

c $\frac{2}{3}x(x+5)+\frac{1}{4}x(x-3)$

d $\frac{2}{5}y\left(\frac{3}{4}y-1\right)-\frac{1}{2}y\left(2y-\frac{3}{5}\right)$

e $\left(\frac{2}{3}x+1\right)\left(x+\frac{1}{4}\right)$

f $\left(\frac{2}{3}y+3\right)\left(\frac{5}{6}y+2\right)-\left(y-\frac{1}{4}\right)\left(y+\frac{1}{3}\right)$

5 Solve:

a $2x-7=10$

b $5-3y=15$

c $5x+3=2x-8$

d $7y+5=5y-3$

e $\frac{3x+2}{5}=8$

f $\frac{5y-2}{4}+2=\frac{3y+2}{3}$

g $4(x-3)=3x+4$

h $\frac{2x-5}{3}-\frac{x-2}{5}=4$

i $3(2x-5)=2\left(4x+\frac{3}{2}\right)$

j $\frac{4(x+5)}{5}=1+\frac{2x-7}{2}$

6 A gardener has 60 m of garden edging, which she uses to set out a rectangular garden with width 5 m less than the length. Let x metres be the length of the garden.

a Find, in terms of x, the width of the garden.

b Hence, form an equation and solve it to find the length and width of the garden.

7 In an effort to catch a bus, I walked for 10 minutes and ran for 5 minutes. I know I can run 4 times as fast as I can walk. What was my running speed, in km/h, if I travelled a total of 3 km to catch the bus?

8 A completely filled car radiator with capacity 8 L contains a mixture of 40% antifreeze (by volume). If the radiator is partly drained and refilled with pure antifreeze, how many litres should be drained from the radiator so as to have a mixture of 70% antifreeze?

9 Solve each inequality.

a $3x-7>5$

b $4(2x-3)\geq 2x-1$

c $\frac{x-1}{4}-\frac{x-2}{5}<2$

d $\frac{x+5}{3}-\frac{3x+1}{2}\geq 4$

e $\frac{4(2-x)}{3}-2\geq\frac{22}{3}$

f $4(3-x)<3-3(4-x)$

10 The power used by a furnace, P watts, is related to the resistance of the wiring, R ohms, and the current, I amps, according to the formula $P=RI^2$. Find the power used by a furnace with wire resistance of 2.5×10^{-1} ohms that draws a current of 6.2×10^{3} amps.

11 Given the relationship $s = ut + \frac{1}{2}at^2$:

a calculate s when $u = 20.8$, $t = 1.5$ and $a = 9.8$

b rearrange the formula to make a the subject

12 The formula for the time of swing, T seconds, of a pendulum is $T = 2\pi\sqrt{\frac{p}{g}}$, where p metres is the length of the pendulum and g is a constant related to gravity.

a Make g the subject of this formula.

b The time of swing is found to be 3 seconds when the length of the pendulum is 2.24 m. What is the value of g (correct to one decimal place)?

13 Make x the subject of each formula.

a $ax + b = c$ **b** $a(x + b) = c$ **c** $\frac{ax + b}{c} = d$

d $rx + b = tx + c$ **e** $\sqrt{\frac{x}{y}} = a$ **f** $\frac{1}{x} + \frac{1}{y} = \frac{1}{c}$

g $m = \sqrt{\frac{n - p}{x}}$ **h** $\frac{ax + b}{a^2} - \frac{x + 1}{b} = 0$ **i** $\frac{y - 3}{2} + 1 = \frac{x - 2}{3}$

14 Expand:

a $(x + 5)(x - 5)$ **b** $(x + 2)(x - 2)$ **c** $(3a + 1)(3a - 1)$

d $(5x + 2y)(5x - 2y)$ **e** $\left(\frac{1}{2}a + 1\right)\left(\frac{1}{2}a - 1\right)$ **f** $\left(\frac{1}{4}x + \frac{2}{3}y\right)\left(\frac{1}{4}x - \frac{2}{3}y\right)$

15 Factorise:

a $x^2 - 36$ **b** $a^2 - 64$ **c** $81b^2 - 1$ **d** $9x^2 - 4y^2$

16 Factorise:

a $x^2 - 18x$ **b** $3x^2 - 18x$ **c** $18b^2 - 50$ **d** $12b^2 - 27$

e $28x^2 - 63y^2$ **f** $54a^2 - 24b^2$ **g** $\frac{1}{4}x^2 - y^2$ **h** $\frac{3}{4}x^2 - \frac{12}{25}y^2$

17 Factorise:

a $x^2 + 5x + 6$ **b** $x^2 + 8x + 12$ **c** $x^2 - 3x + 2$

d $x^2 - 6x + 5$ **e** $x^2 - 9x + 18$ **f** $x^2 - 5x - 6$

g $x^2 - 3x - 10$ **h** $x^2 - 2x - 8$ **i** $x^2 - 4x - 21$

18 Factorise:

a $2x^2 + 7x + 6$ **b** $3x^2 + 19x + 6$ **c** $5x^2 + 19x + 12$

d $2x^2 - 5x + 2$ **e** $3x^2 - 13x + 10$ **f** $7x^2 - 23x + 18$

g $3x^2 - 7x - 6$ **h** $5x^2 - 6x - 8$ **i** $2x^2 - 11x - 21$

19 Factorise:

a $4x^2 + 8x + 3$ **b** $6x^2 + 13x + 6$ **c** $4x^2 + 19x + 12$

d $4x^2 - 16x + 15$ **e** $6x^2 - 19x + 10$ **f** $10x^2 - 27x + 18$

g $4x^2 - 4x - 15$ **h** $6x^2 - 11x - 10$ **i** $8x^2 - 2x - 15$

20 Factorise:

a $2x^2 - 8x - 42$ **b** $5x^2 + 25x + 30$ **c** $4x^2 - 12x - 16$

d $3x^2 + 12x - 15$ **e** $6x^2 + 9x - 15$ **f** $8x^2 + 20x + 8$

g $10x^2 - 55x - 30$ **h** $-x^2 + 10x - 24$ **i** $-6x^2 - 14x - 8$

21 Express with a common denominator:

a $\frac{7x-1}{5} + \frac{3x-4}{7}$ **b** $\frac{2}{x+1} + \frac{3}{x+2}$ **c** $\frac{1}{x+2} - \frac{4}{x-3}$

d $\frac{5}{2x^2+3x} - \frac{1}{2x^2+5x+3}$ **e** $\frac{4}{x^2+x} - \frac{3}{x^2-1}$ **f** $\frac{3}{x^2-9} - \frac{2}{3+2x-x^2}$

22 Simplify:

a $\frac{x^2+4x+3}{x^2+x-6} \times \frac{x^2-4}{x^2+5x+4}$ **b** $\frac{x^2+7x+6}{x^2+x-2} \times \frac{x^2-x-6}{2x^2-5x-3}$

c $\frac{x^2+3x-4}{x^2+4x} \div \frac{x^2+x-2}{x+2}$ **d** $\frac{3x^2-3x}{2x^2+3x+1} \div \frac{9-9x}{2x^2+7x+3}$

Chapter 4: Lines and linear equations

1 Find the distance between each pair of points.

a (6, 4), (0, 0) **b** (3, 2), (5, 4) **c** (–3, 2), (2, 5) **d** (–4, –3), (–1, 2)

2 Find the midpoint of the interval AB, where:

a $A = (6, 4)$ and $B = (0, 0)$ **b** $A = (3, 2)$ and $B = (5, 4)$

c $A = (-3, 2)$ and $B = (2, 5)$ **d** $A = (-4, -3)$ and $B = (-1, 2)$

3 Find the gradient of the line that passes through each pair of points.

a (6, 4), (0, 0) **b** (3, 2), (5, 4) **c** (–3, 2), (2, 5) **d** (–4, –3), (–1, 2)

4 Write the gradient and y-intercept of each line.

a $y = 2x - 1$ **b** $y = x + 3$ **c** $y = -x + 7$

d $x + y = 4$ **e** $2x + 3y = 1$ **f** $3x - 4y = 2$

5 Find the gradient of a line that is:

i parallel **ii** perpendicular

to the line with equation:

a $y = 3x + 2$ **b** $y = 1 - 2x$ **c** $y = \frac{1}{2}x + 2$ **d** $y = -\frac{2}{3}x + 2$

6 Sketch the graph of each equation, and mark the intercepts.

a $y = 2x + 1$ **b** $y = 3 - 2x$ **c** $2x + 3y = 6$ **d** $3x - 4y = 12$

e $y = -2x$ **f** $y = 4x$ **g** $y = -4$ **h** $x = 3$

7 Find the equation of the line with:

a a gradient of 3 passing through (0, 2) **b** a gradient of 2 passing through (0, 1)

c a gradient of -1 passing through (0, -3) **d** a gradient of $-\frac{1}{2}$ passing through (0, 4)

8 Find the equation of the line passing through the points:

a (2, 0) and (0, 3) **b** (2, 2) and (0, 1) **c** (1, 3) and (2, 3)

d (5, 3) and (5, –2) **e** (1, 1) and (2, 3) **f** (–2, 3) and (2, –1)

9 Solve each pair of simultaneous equations for x and y.

a $y = x + 1$, $x + 2y = 8$

b $y = 2x - 1$, $2x + 5y = 7$

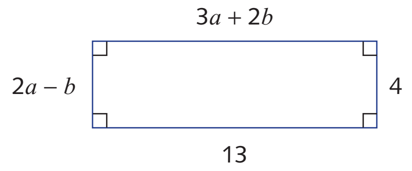
c $x + y = 1$, $3x + 2y = 8$

d $2x - 3y = -1$, $6x + 6y = 7$

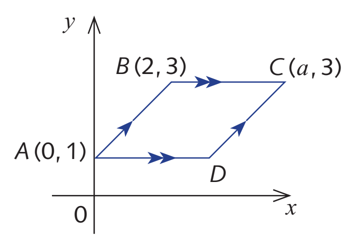
e $5x + 3y = 15$, $3x + 2y = 8$

f $2x - 3y = -10$, $3x + 2y = 7$

10 Find the values of a and b in the diagram shown.



11 $ABCD$ is a parallelogram, as shown opposite, where $a > 2$.



a If $a = 5$, find the length of BC.

b Find, in terms of a:

i the length BC

ii the coordinates of the point D

c i Find the gradient of the line AC in terms of a.

ii Find the gradient of the line BD in terms of a.

iii Show that when $a = 5$, the gradient of the line BD is –2.

d For $a = 5$, find:

i the gradient of the line AC

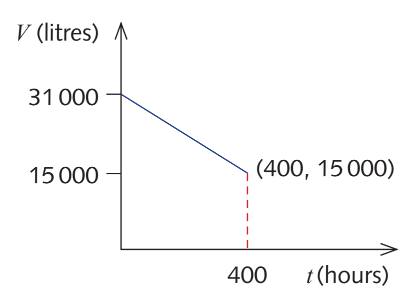
ii the equation of the line AC

iii algebraically, the coordinates of the intersection point of the line AC with the line BD, given that the equation of the line BD is $y = -2x + 7$

e i Find the length of AC in terms of a.

ii Find the exact value of a (as a surd in simplest form) so that $AC = 7$.

12 Water was leaking from a tank at a constant rate. The graph shows the volume of water (V litres) remaining in the tank after t hours.



a How many litres of water were initially in the tank?

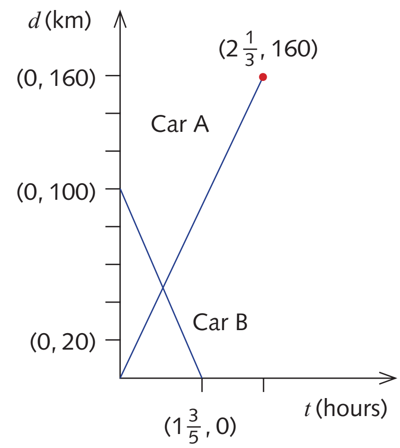
b How many litres were leaking from the tank per hour?

c Write a rule for finding the number of litres remaining (V) after t hours.

d When would the tank be empty if the leaking continued at this rate?

13 The graph shown represents the trips of two cars, A and B, along the Hume Highway.

The vertical axis is the d axis, where d km is the distance from Melbourne along the Hume Highway. The horizontal axis is the t axis, where t hours is the time of travel. Assume that both cars started their trip at 9 a.m.



a Describe these aspects of each car's trip.

i Where did it start?

ii Where did it finish?

iii What was the time taken?

iv What was the average speed?

b Find the equation of the graph of each car's trip (in terms of d and t).

c Find the time at which they passed each other, giving your answer to the nearest minute.

Chapter 5: Quadratic equations

1 Solve:

a $x^2 = 16$ **b** $7x^2 = 28$ **c** $2x^2 - 98 = 0$

d $4x^2 - 25 = 0$ **e** $4x^2 - 1 = 0$ **f** $12x^2 - 75 = 0$

2 Solve:

a $4x^2 - 6x = 0$ **b** $27x^2 + 9x = 0$ **c** $5x^2 - 3x = 0$

d $18x^2 = 9x$ **e** $8x = 28x^2$ **f** $-3x^2 - 15x = 0$

g $14x - 2x^2 = 0$ **h** $\frac{1}{2}x^2 - 6x = 0$ **i** $24x^2 = -6x$

3 Solve:

a $a^2 - a - 12 = 0$ **b** $t^2 + 8t + 15 = 0$ **c** $m^2 + 4m - 21 = 0$

d $n^2 - 3n - 4 = 0$ **e** $x^2 - 8x + 16 = 0$ **f** $b^2 - 6b = 27$

4 Solve:

a $2x^2 - 19x + 35 = 0$ **b** $9f^2 - 36f + 11 = 0$ **c** $-3x^2 - 23x + 8 = 0$

d $12y^2 + 21 = -32y$ **e** $3x^2 - 2x - 1 = 0$ **f** $12x^2 + 8x = 15$

g $-2x^2 - 5x + 12 = 0$ **h** $3x^2 = 18x - 27$

5 Solve:

a $b^2 - 6b + 9 = 0$ **b** $x^2 + 10x + 25 = 0$ **c** $2x^2 + 4x + 2 = 0$

d $3b^2 - 24b + 48 = 0$ **e** $4x^2 + 12x + 9 = 0$ **f** $3y^2 - 30y + 75 = 0$

6 Factorise, using surds:

a $x^2 - 5$

b $(x + 2)^2 - 8$

c $2(x - 3)^2 - 10$

7 Solve each equation by completing the square.

a $y^2 + 2y - 4 = 0$ **b** $a^2 - 4a - 2 = 0$

c $x^2 - 2x = \frac{5}{2}$ **d** $x^2 - 7x + 2 = 0$

e $y^2 + \frac{1}{2}y = \frac{1}{16}$ **f** $2x^2 - x = 4$

g $n^2 = 5n + 4$ **h** $16x^2 + 8x = 1$

8 Solve:

a $5d^2 - 10 = 0$ **b** $\frac{2y^2}{3} - 5 = 0$ **c** $\frac{3(x-10)^2}{5} - 12 = 0$

d $y^2 - 8y + 3 = 0$ **e** $1 = m^2 - m$ **f** $3n + 3 = n^2$

9 In a right-angled triangle, the hypotenuse is 8 cm longer than the shortest side, and the third side of the triangle is 7 cm longer than the short side. Let x cm be the shortest side length.

a Express the other two side lengths in terms of x.

b Hence, form an equation and solve it to find the side lengths of the triangle.

10 The height, h metres above sea level, to which a rocket has risen t seconds after launching from sea level is given by $h = ut - 4.9t^2$, where u metres per second is the launch velocity.

a Calculate the height above sea level 4 seconds after the launch of a rocket with a launch velocity of 115 m/s.

b If the launch velocity can be a maximum of 500 m/s, calculate the longest possible time of flight, to the nearest second.

(*Hint*: At the end of a flight, the height above sea level is 0 m.)

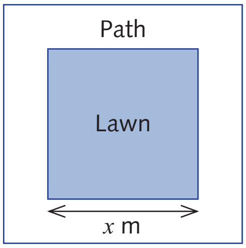
11 A sheet of cardboard 24 cm long and 17 cm wide has squares of side length x cm cut from each corner so that it can be folded to form an open box with base area of 228 cm^2.

a Express the length and width of the base in terms of x.

b Write an expression involving x and solve it for x.

c Find the dimensions of the box.

12 A square lawn is surrounded by a concrete path 2 m wide. If the lawn has sides of length x metres, find, in terms of x:



a the area of the lawn

b the area of the concrete path

The area of the concrete path is $1\frac{1}{4}$ times that of the lawn.

c Write an equation that can be used to find x.

d Solve this equation to find the dimensions of the lawn.

13 For the quadratic equation $x^2 + bx + 4 = 0$, find the values of b for which the equation has:

a one solution **b** two solutions **c** no solutions

14 For the quadratic equation $ax^2 - 4x + 3 = 0$, find the values of a for which the equation has:

a one solution **b** two solutions **c** no solutions

Chapter 6: Surface area and volume

1 For a rectangular prism measuring 30 cm × 20 cm × 10 cm, calculate:

a the surface area

b the volume

2 A rectangular prism has a surface area of 550 cm^2. If its length is 15 cm and its width is 10 cm, calculate the height of the rectangular prism.

3 A rectangular prism has a volume of 660 cm^3. If its length is 12 cm and its width is 11 cm, calculate the height of the rectangular prism.

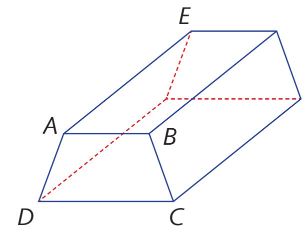
4 The cross-section $ABCD$ of the prism shown is an isosceles trapezium with $AB = 8$ cm, $DC = 14$ cm, $AD = BC = 5$ cm and $AE = 20$ cm.

Calculate:

a the area of $ABCD$

b the surface area of the prism

c the volume of the prism



5 A cylindrical water tank stands on its circular base. It has a diameter of 2 m and a height of 1.5 m.

a Calculate the volume of the tank, to the nearest litre.

b Calculate the depth of water in the tank, to the nearest centimetre, when it contains 2000 litres of water.

6 Find answers to these questions in cm^2 and cm^3.

a A square-based pyramid has base side length 10 cm and perpendicular height 12 cm.

Calculate:

i the surface area **ii** the volume

b A cone has a radius of 6 cm and a slant height of 10 cm.

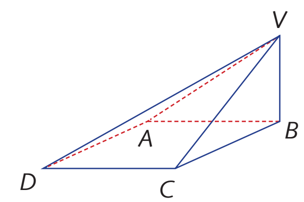
Calculate:

i the surface area **ii** the volume

7 In the pyramid $VABCD$ shown, VB is perpendicular to rectangle $ABCD$, $AB = 12$ m, $BC = 8$ m and $VB = 5$ m.

a Calculate the surface area of the pyramid in m^2, correct to one decimal place.

b Calculate the volume of the pyramid.



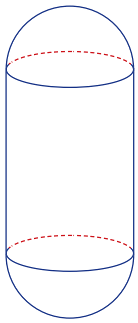
8 The curved surface area of a cone is 80π cm^2 and the area of the circular base is 16π cm^2.

a Calculate the radius of the cone.

b Calculate the exact perpendicular height of the cone.

c Calculate the volume of the cone, correct to the nearest cm^3.

9 A storage tank is constructed as a cylinder with a hemisphere at each end of the cylinder. The radius of the cylinder is 1.5 m and the overall length of the tank is 6 m.



Calculate:

a the surface area of the tank in m^2

b the volume of the tank

10 Fill in the missing entries in the table below.

	Length scale factor	Area scale factor	Volume scale factor
a	3		
b	1.5		
c		4	
d		36	
e			125
f			729

Chapter 7: The parabola

1 Find the x-intercepts of the graph for each equation.

a $y = x^2 + 4x + 3$

b $y = 2x^2 - 11x - 6$

c $y = (x + 4)^2 - 3$

d $y = 3(x - 2)^2 - 6$

2 Express each equation in the form $y = a(x - h)^2 + k$, and hence state the coordinates of the vertex of each graph.

a $y = x^2 + 6x + 3$

b $y = x^2 - 4x + 2$

c $y = 2x^2 + 6x + 1$

d $y = 3x^2 + 8x + 2$

3 **a** A parabola has x-intercepts -1 and 4, and y-intercept 8. Find the equation of the parabola.

b A parabola has x-intercepts 3 and 5, and passes through the point (1, 8). Find the equation of the parabola.

4 **a** A parabola has vertex (3, –2) and y-intercept 16. Find the equation of the parabola.

b A parabola has vertex (2, 5) and passes through the point (1, 2). Find the equation of the parabola.

5 Write the equation of the parabola obtained when the graph of $y = x^2$ is:

a stretched by a factor of 3 from the x-axis and translated 2 units to the right

b reflected in the x-axis and then translated 1 unit to the left and 3 units up

c translated 5 units to the right and 4 units down

6 Sketch the graph, labelling the vertex, axes of symmetry and the intercepts.

a $y = x^2 - 6x + 5$ **b** $y = -x^2 - x + 6$ **c** $y = 4 - x^2$

d $y = (x + 3)^2$ **e** $y = (x - 1)^2 - 4$ **f** $y = x^2 - 2$

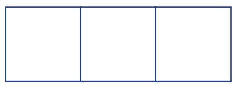
g $y = (x - 3)^2 + 2$ **h** $y = 2 - (x + 1)^2$ **i** $y = x^2 + 5x - 3$

j $y = 10 - 6x^2 - 11x$ **k** $y = 4x^2 + 7x + 6$ **l** $y = 2x^2 - x - 7$

7 For the graph with equation $y = 3x^2 - 2x - 1$, find the coordinates of the:

a vertex **b** x-intercepts

8 A gardener is planning to establish a vegetable garden. The garden will have a wooden border and two wooden dividers to form three partitions, as shown in the diagram. Twenty-four metres of timber is used for the border and the dividers. Let x m be the length of the dividers and two of the sides of the garden, as indicated in the diagram.



a Express the other side length of the garden in terms of x.

b Let A m^2 be the area of the garden. Write an equation for the area of the garden in terms of x.

c Find the length and width of the garden in order for the area to be a maximum.

9 **a** By expressing the quadratic equation $y = x^2 + 2x - 7$ in the form $y = a(x - h)^2 + k$, find the coordinates of the turning point.

b Find the points of intersection with the axes of the graph of $y = x^2 + 2x - 7$.

c Sketch the graph of $y = x^2 + 2x - 7$, marking on your sketch the points found in **a** and **b**.

d Solve $x^2 + 2x - 7 \le 0$ for x.

10 **a** Sketch the graph of $y = 4x^2 - 8x + 1$, labelling clearly the coordinates of the turning point and the points of intersection with the axes.

b Solve $4x^2 - 8x + 1 < 0$ for x.

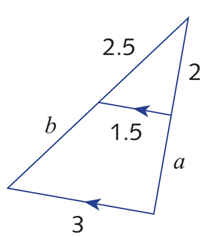
11 Solve for x:

a $x^2 + x < 30$ **b** $x^2 + 5x \ge -6$ **c** $-x^2 + 4x + 60 \le 0$

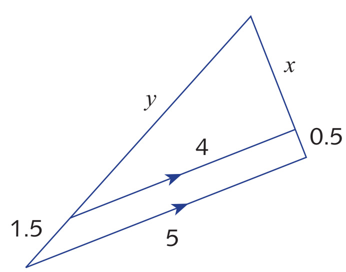
Chapter 8: Review of congruence and similarity

1 Find the value of the pronumerals.

a



b

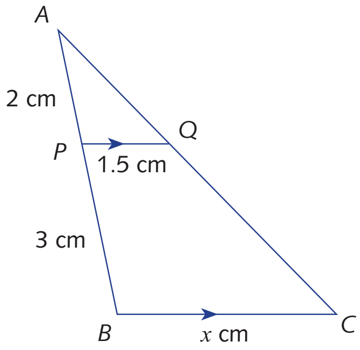


2 A vertical stick of length 30 cm casts a shadow of length 5 cm. Find the length of the shadow cast by a 1 metre ruler placed in the same position at the same time of day.

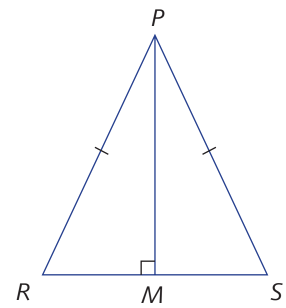
3 **a** In the figure shown, $PQ \parallel BC$.

i Prove that ΔAPQ is similar to ΔABC.

ii Find the value of x.

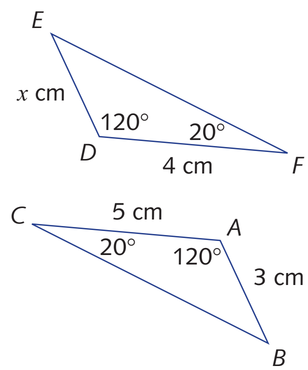


b In the figure shown, $PM \perp RS$ and $PR = PS$. Prove that $\Delta PMR \equiv \Delta PMS$.

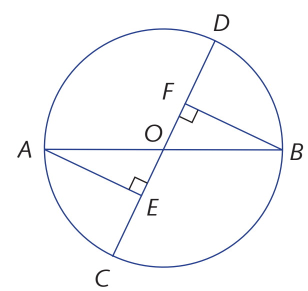


4 **a** **i** State, in abbreviated form, why ΔABC is similar to ΔDEF.

ii Calculate x.



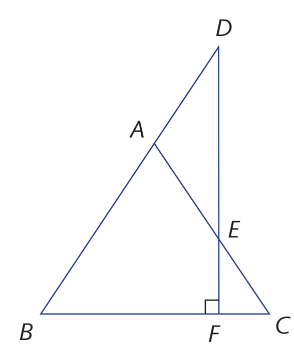
b In the diagram, AB and CD are diameters of the circle with centre O, and AE and BF are perpendicular to CD. State, in abbreviated form, why $\Delta AEO \equiv \Delta BFO$.



5 Complete the proof that, in the figure shown, ΔDAE is isosceles.

Given: In ΔABC, $AB = AC$, D is on the ray from B through A, $DF \perp BC$ and DF intersects AC at E.

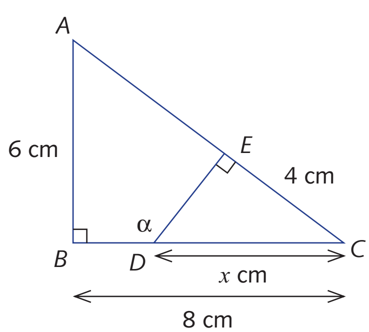
Prove: ΔDAE is isosceles.



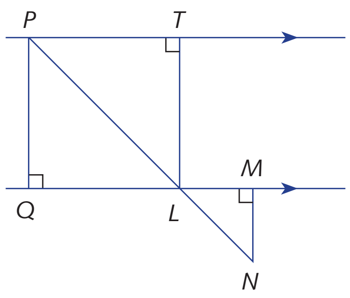
6 **a** Prove that ΔDEC is similar to ΔABC.

b Calculate x.

c Use trigonometry to calculate α, correct to two decimal places.



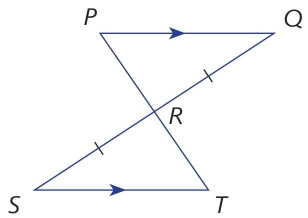
7 In order to calculate the distance across a straight canal, some scouts place markers Q, L, M and N in the positions shown. P is a pumping station and T is a large tree.



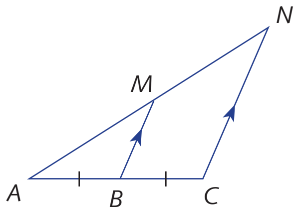
a Name all pairs of similar triangles in the diagram and give the abbreviated reason why they are similar.

b The scouts measure QL to be 60 m, LM to be 40 m and MN to be 50 m. Calculate the distance across the canal.

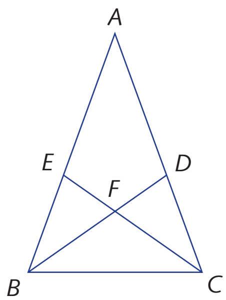
8 In the diagram, $PQ \parallel ST$ and $QR = SR$. Prove that triangles PQR and TSR are congruent.



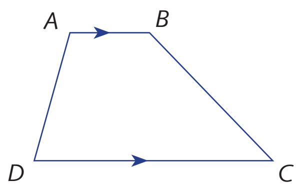
9 In the diagram, $AB = BC$ and $BM \parallel CN$. Prove that $CN = 2BM$.



10 In this diagram, ΔABC is isosceles. $AB = AC$ and $BE = CD$. Prove that $EC = DB$.



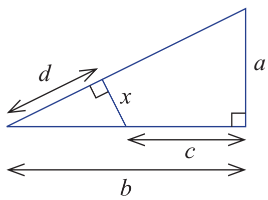
11 $ABCD$ is a trapezium with $AB \parallel DC$ and $AB = \frac{1}{3}DC$. The diagonals of this trapezium intersect at O.



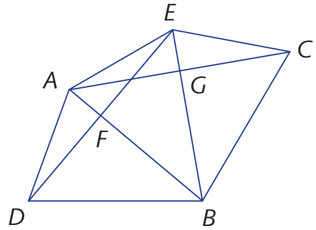
a Prove that ΔABO is similar to ΔCDO.

b Hence, prove that $3AC = 4OC$.

12 Find the formula, with x as its subject, that can be used to calculate the value of x if a, b, c and d are known.



13 In the diagram, $AB = BC$, $BE = BD$, BA intersects DE at right angles and BE intersects AC at right angles.



a Prove that $\Delta DFB \equiv \Delta EFB$.

b Prove that $\Delta ABD \equiv \Delta CBE$.

14 $ABCD$ is a parallelogram with the size of $\angle BAD < 90°$. E is on the ray CB such that ΔABE is isosceles with $AB = AE$. F is on the ray CD such that ΔADF is isosceles with $AD = AF$.

a Prove that ΔABE is similar to ΔADF.

b Prove that $DE = BF$.

15 ABC is a triangle, M is a point in the interval AB such that $AB = 3AM$, and N is a point in the interval AC such that $AC = 3AN$.

a Prove that $BC \parallel MN$.

b If BN and CM intersect at P, prove that $BP = 3PN$.

Chapter 9: Indices, exponentials and logarithms – part 1

1 Simplify each expression, writing your answers with positive powers.

a $(a^3)^2 \times a^{-2}$ **b** $(2x^2y)^3 \times 3xy^2$ **c** $\left(\frac{a}{b}\right)^2 \times b^3$

d $\frac{a^{-2}b^3}{a^3b^4} \times \frac{a^2b^3}{ab^2}$ **e** $\frac{8x^4y^2}{4x^3y}$ **f** $\frac{3x^2y^3}{9x^3y} \times \frac{6xy}{y^2}$

g $\frac{12xy^2}{x^2y} \div \frac{6x^3y}{y^3}$ **h** $\frac{ab^2}{a^3b^{-2}} \div \frac{a^2b^{-1}}{a^3b^3}$ **i** $\frac{x^2y^3}{x^{-1}y^2} \times \frac{x^{-3}y^2}{x^2y^{-1}}$

2 Express each number in scientific notation.

a 3200 **b** 576 000 **c** 0.000 267 **d** 0.025

3 Evaluate each expression, giving your answers to four significant figures and in scientific notation.

a $3.267 \times 10^6 \times 2.76 \times 10^{-2}$ **b** $\frac{5.567 \times 10^2 \times 2.78 \times 10^{-2}}{3.4 \times 10^4}$

c $\frac{2.34 \times 10^{-6} \times 1.76 \times 10^{-4}}{6.32 \times 10^{-5}}$ **d** $\frac{1.267 \times 10^{-10} \times 2.543 \times 10^{-12}}{1.27 \times 10^{-4} + 3.276 \times 10^{-3}}$

4 Evaluate:

a $\sqrt[3]{27}$ **b** $\sqrt[4]{81}$ **c** $\sqrt[4]{16}$ **d** $\sqrt[5]{32}$ **e** $\sqrt[5]{243}$ **f** $\sqrt[3]{64}$

5 Evaluate:

a $8^{\frac{2}{3}}$ **b** $16^{\frac{3}{4}}$ **c** $27^{\frac{2}{3}}$ **d** $4^{\frac{3}{2}}$ **e** $9^{-\frac{3}{2}}$ **f** $125^{-\frac{2}{3}}$

6 Simplify:

a $\left(b^{\frac{2}{3}}\right)^3 \times b^2$ **b** $\sqrt{\frac{a^4}{b^2}}$ **c** $\sqrt[3]{\frac{a^6}{b^3}}$ **d** $\frac{a^{\frac{1}{3}}b^{\frac{2}{3}}}{ab^2} \div \frac{ab^{\frac{4}{3}}}{a^2b^3}$

e $\sqrt[5]{\frac{a^5}{b^{10}}}$ **f** $\sqrt[3]{\frac{27a^2}{b^3}}$ **g** $\sqrt{\frac{32a^4b}{2ab^3}}$ **h** $\left(a^{\frac{1}{4}}\right)^2 \times \left(a^{\frac{1}{4}}\right)^3$

7 Sketch the graph of each equation.

a $y = 2^x$ **b** $y = 3^x$ **c** $y = 5^{-x}$ **d** $y = -5^x$

8 Solve for x.

a $243^x = 3$ **b** $625^x = 25$ **c** $\left(\frac{1}{9}\right)^x = 81$

d $10\,000^x = 1000$ **e** $(0.0001)^x = 1000$ **f** $(0.001)^x = 0.000\,01$

9 Solve for x.

a $7^{x-3} = 49$ **b** $5^{5-x} = 625$ **c** $4^{2x-3} = 32$

d $16^{2x-1} = 32^{3-2x}$ **e** $5^{-5-7x} = 625^{3+2x}$ **f** $10^{4-3x} = 100^{5-2x}$

10 A biologist discovers that the number of organisms present in a Petri dish increases by 8% each minute. If there are initially 5000 organisms present in the dish, find the number of organisms in the dish:

a after 1 minute **b** after 2 minutes **c** after x minutes **d** after 20 minutes

11 The population of a town is initially 4200, and each year the population decreases by 2%.

a What is the population of the town after:

i 1 year? **ii** 2 years? **iii** x years?

b On a single set of axes, sketch the graphs of:

i $y = 4200 \times 0.98^x$ **ii** $y = 3200$

c Use your calculator and your answer to part **b** to find the minimum number of years it will take for the population of the town to drop below 3200.

12 Evaluate:

a $\log_2 16$ **b** $\log_7 49$ **c** $\log_{25} 5$ **d** $\log_{25} 125$

10B Miscellaneous questions

1 Two trains travel between towns A and B. They leave at the same time, with one train travelling from A to B and the other from B to A. They arrive at their destination one hour and four hours, respectively, after passing one another. The slower train travels at 35 km/h.

a How far does the slower train travel after they pass?

b If the faster train travels at x km/h, how far, in terms of x, does the faster train travel after they pass?

c Hence, find the number of hours, in terms of x, each train has travelled before they pass.

d Hence, find the speed of the faster train.

2 **a** A car left town A and travelled at a constant speed towards town B, 150 km away. Half an hour later, an express train left A travelling at a constant speed towards B, and overtook the car 90 km from A. The speed of the car was 80 km/h. Find:

i the time for which the car had been travelling before it was overtaken by the train

ii the time for which the train had been travelling before it overtook the car

iii the speed of the train

b A car left town A and travelled at a constant speed to town B, which is d km away. At a time n hours later, an express train left A travelling at a constant speed to B and overtook the car m km from B. The speed of the car was v km/h. Find formulas for:

i the distance from town A to the point where the train passes the car

ii the time, T hours, for which the car was travelling before it was overtaken by the train in terms of d, m and v

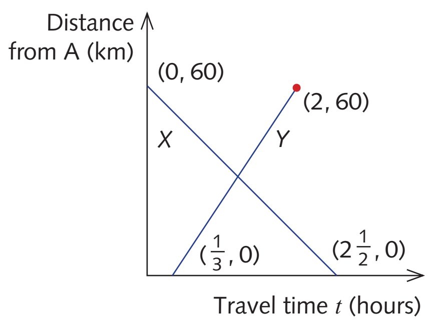
iii the time, t hours, for which the train was travelling before it overtook the car in terms of d, m, v and n

iv the speed of the train, w km/h, in terms of d, m, v and n

v the speed of the car, v km/h, in terms of d, m, n and w

c Given that $d = 150$, $m = 90$, $n = 0.5$ and $w = 108$, find the speed of the car.

3 Two cyclists are riding on the same road between two points, A and B, which are 60 km apart. Cyclist X starts first and is riding from B to A. Cyclist Y starts 20 minutes later and is riding from A to B. The distance–time graph opposite shows all the information.



Find:

a how long it takes each cyclist to ride between A and B

b the average speed of each cyclist on the ride

c how far from A they pass each other

4 In a triathlon event, two competitors, Alan and Shen, are keen rivals. The event consists of an 800 m swim, a 50 km bicycle ride and a 20 km run. Alan can swim at 2 km/h, cycle at 35 km/h and run at 10 km/h (all average speeds). Shen can swim at 2.4 km/h, cycle at 30 km/h and run at 12 km/h (all average speeds). Assume no time is lost when transitioning between legs.

a Find the distance between Shen and Alan when Alan has completed the swim.

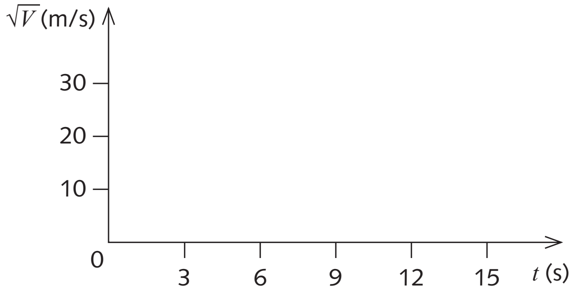
b Find which of the two competitors finishes first, and the difference between their times, to the nearest minute.

5 Lindy is speeding in her car along a straight road at a constant speed of 20 m/s (72 km/h). She passes a stationary police motorcyclist, John. Three seconds later, John starts in pursuit. He accelerates for 6 seconds until he reaches his maximum speed, which he maintains until he overtakes Lindy. Let t seconds be the time elapsed since Lindy passed John.

John's speed, v m/s, at any time until he reaches his maximum speed at $t = 9$, is given by $v = 5(t - 3)$ for $3 \le t \le 9$.

a Find John's maximum speed.

b Copy this set of axes.



i Sketch the speed–time graph for Lindy.

ii On the same set of axes, sketch John's speed–time graph for $3 \le t \le 9$.

iii On the same set of axes, sketch John's speed–time graph for $t \ge 9$.

c Find the value of t when John and Lindy are travelling at equal speeds.

d What is John's acceleration (rate of change of speed) for $3 \le t \le 9$?

e Given that the distance travelled by an object is equal to the area under its speed–time graph (above the t-axis), find:

i the distance travelled by Lindy in the first 9 seconds

ii an expression for the distance travelled by Lindy after t seconds

f Find:

i the distance travelled by John by the time he reaches his maximum speed

ii the total distance travelled by John when $t = 12$

iii an expression for the total distance travelled by John, t seconds after Lindy passed him, for $t \ge 9$

g **i** Use your answers to parts **e** and **f** to find the value of t when John draws level with Lindy.

ii How far has John travelled by the time he draws level with Lindy?

6 Consider the lines shown in the diagram.

a Find the gradient of:

i AB **ii** CD

b Find the equation of:

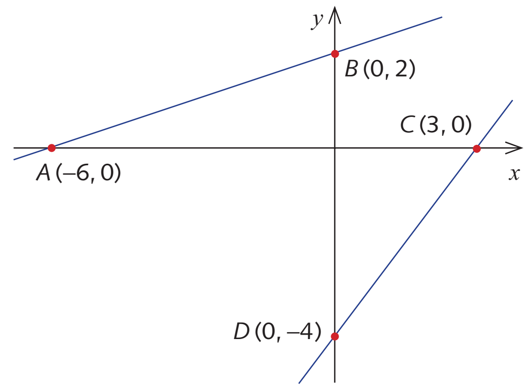
i AB **ii** CD

c Find the coordinates of E, the point of intersection of the line AB and the line CD.

d Find the area of quadrilateral $ABCD$.

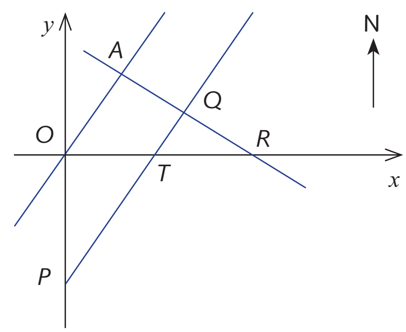
e Find the area of ΔDBE.

f If A and D remain fixed but $B = (0, 2b)$ and $C = (3b, 0)$, find the coordinates of E, the point of intersection of the line AB and the line CD,
commenting on the special cases when $b = 0$, $b = 2$ and $b = -2$.



7 A surveyor has drawn lines on a map to represent straight roads between towns positioned at O, A, P, Q, T and R, as shown. Cartesian axes have been drawn so that equations can be assigned to roads. Distances are measured in kilometres.

The road through towns O and A has equation $y = \sqrt{2}x$, while the road through towns P and Q has equation $y = \sqrt{2}x - 20$. The road through towns A, Q and R has equation $y = \dfrac{-1}{\sqrt{2}}x + 25$. The direction due north is shown on the diagram.



Express all answers in parts **a** to **d** as exact values in surd form.

a Find the distance from town O to town P (OP).

b Find the distances:

i OT **ii** OR

c Find the coordinates of town Q at the intersection of the road from A to R with the road from P to Q.

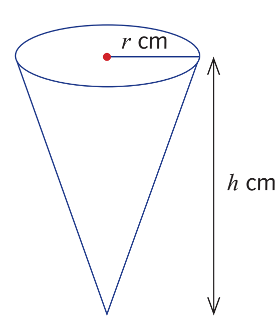
d A new road is to be built through towns positioned at P and R.

i Find the coordinates of P and R.

ii Find the gradient of the line from P to R.

iii Find the equation of the line that runs through P and R.

8 The cone shown in this diagram has an open circular top of radius r cm and depth h cm. The radius of the cone is equal to one-third of the height; that is, $r = \frac{h}{3}$.



a Express, in terms of r:

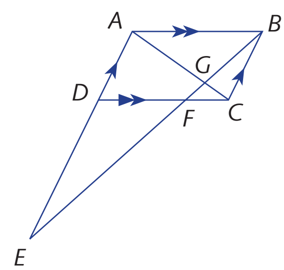
i h **ii** V **iii** A

b If the cone holds 50 cm^3 of water, find:

i the depth of water in the cone, correct to three significant figures

ii the curved surface area of the cone covered by water, correct to three significant figures

9 Using the diagram shown:



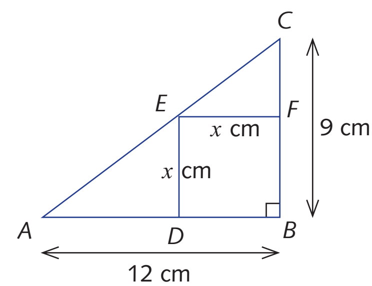
a prove that ΔAGB is similar to ΔCGF

b name two triangles similar to ΔEFD

c given that $DF : FC = 2 : 1$, and using your answers to parts **a** and **b**, find:

i $AB : DF$ **ii** $EF : EB$

10 a In the right-angled triangle ABC, there is a square $BDEF$, as shown.

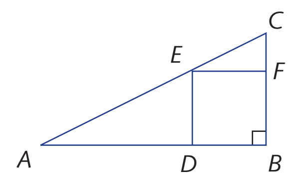


i What is the abbreviated reason for ΔEFC to be similar to ΔABC?

ii Hence, find x.

iii Hence, find the area of the square $BDEF$ as a fraction of the area of ΔABC.

b In this diagram, square $BDEF$ is inside ΔABC, as shown. If $BC = x$ cm, $EF = y$ cm and $AB = 2BC$:

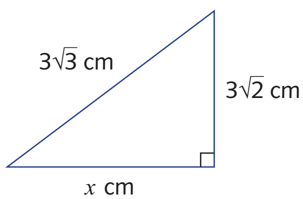


i find the relationship between x and y

ii hence, find the area of the square $BDEF$ as a fraction of the area of ΔABC

11 a Find the exact value of x in the following diagrams.

i



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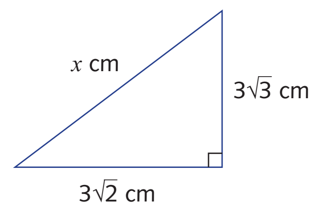

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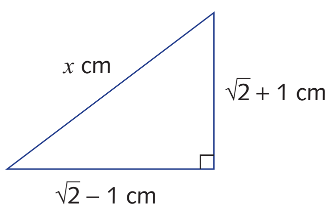

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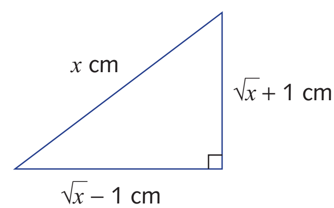
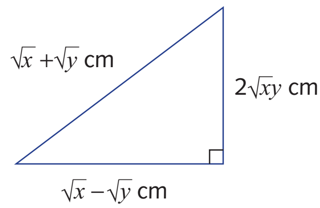
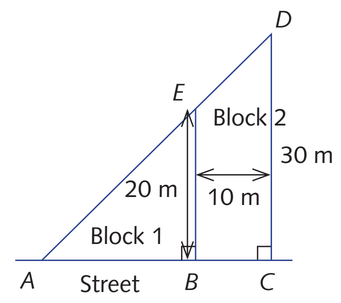


b Find the relationship between x and y, with y as the subject of the formula.



12 A triangular region of land (ΔACD) is divided up into two areas, Block 1 and Block 2, to make way for residential development. Details regarding the plan are provided in the given diagram.



a Prove that ΔABE is similar to ΔACD.

b By letting $AB = x$ metres, write, in terms of x:

i the length AC

ii an equation linking x with the lengths EB, AC and CD

c Solve the equation in part **b ii** to find the length AB.

The subdivider now considers moving the position of the fence BE with the given information: $AC = 30$ m, $BE = AB$ and $BE \perp AC$. (The length DC remains at 30 m and $AC \perp CD$.)

d Given that $AB = x$ metres, find, in terms of x:

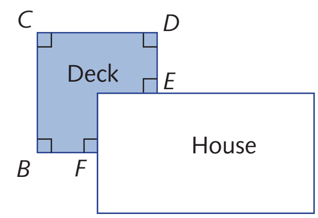
i the area of Block 1

ii the area of Block 2

e If the subdivider requires that the area of Block 1 is to be the same as the area of Block 2:

i write an equation in x to represent this situation

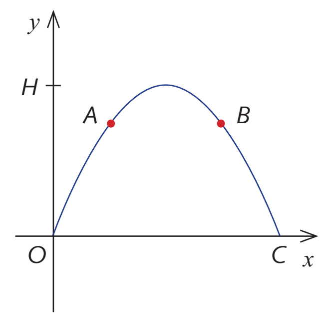
ii find the length AB (x metres) as a surd in simplest form



13 A builder has been contracted to construct a deck for a family on the corner of their house, as shown. The contract requirements are that $BF = DE$ and $BC = CD$, the total length of railing is $BF + BC + CD + DE = 30$ metres and that the deck has the maximum possible area. If $BF = x$ m and A m^2 = area of the deck:

a construct a formula relating A and x with A the subject

b hence, find the maximum possible area of the deck



14 A weather rocket is fired so that it follows a parabolic path, just over weather balloons A and B, as shown. It has been fired to follow the path with equation $y = \frac{1}{5}x - \frac{1}{50}x^2$, where x and y are measured in kilometres.

a Find how far the rocket travels horizontally from O to point C.

b Find H kilometres, the maximum height reached by the rocket.

Given that A and B are both at a height of 200 metres:

c find the coordinates of A and B, expressing your answer correct to two decimal places

d hence, find the horizontal distance, AB, between the balloons, correct to two decimal places

15 Gipps Road and Bells Road are two non-intersecting roads in the country. The government wishes to build a road running North–South that connects these two roads. They employ you to work out where to build this connecting road in order to minimise its length. The path of Gipps Road is given by the equation $y = x^2 - 4x + 9$, while the path of Bells Road is given by the equation $y = 2x - 2$. In this model the positive direction of the x-axis runs due East while the positive direction of the y-axis runs due North. All lengths are in kilometres.

a Sketch a graph of Bells Road. Clearly mark in the x- and y-intercepts.

b **i** By completing the square, write the equation for Gipps Road in the form $y = (x - h)^2 + k$.

ii What will be the y-value for Gipps Road when $x = 0$?

iii Hence, on the same set of axes used to sketch Bells Road, sketch the graph for the path of Gipps Road. Clearly label the turning point and y-intercept.

c Find the distance between the two points on the roads where $x = 0$.

d When $x = a$, find, in terms of a, the y-value of:

i Bells Road

ii Gipps Road

e Hence, show that the North–South distance, d km, between the two roads when $x = a$ is given by $d = a^2 - 6a + 11$.

f On a new set of axes, sketch a graph of d against a. Clearly label the turning point.

g Hence, report back to the government on how long and how far East of the origin the North–South connection road should be built in order to minimise its length.

16 **a** Show, by completing the square, that $y = 3x^2 + 6x - 7$ can be written in the form $y = 3(x + 1)^2 - 10$.

A two-dimensional *Space Invaders*-type game involves a coordinate system whereby the x- and y-axes are centrally located on the screen and a space station is located at $P(-1, -12)$. An enemy spacecraft approaches and attacks the space station while flying on the path described by $y = 3x^2 + 6x - 7$.

b Use the result from part **a** to complete the following.

i Write the coordinates of the turning point of the path of the spacecraft.

ii Find the exact coordinates of where the path of the spacecraft cuts the x-axis, leaving your answer in surd form.

c Sketch a graph showing the path of the spacecraft and the position of the space station, P. Label the turning point and the x- and y-intercepts for the path of the spacecraft.

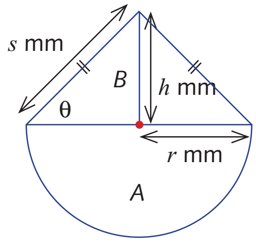
d If one unit represents 100 km, find the distance between the spacecraft and the space station when they are closest to each other.

A second spacecraft flies on the path $y = 5x + 3$.

e Show that the x-coordinates of the intersection points of the paths of the two spacecraft can be found by solving $3x^2 + x - 10 = 0$.

f Solve the equation in part **e** and hence state the coordinates of the intersection points of the paths of the two spacecraft.

17 The cross-section through the centre of a diamond cut at The Perfect Diamond Company is of the shape shown in the diagram. Region A is semicircular and region B is an isosceles triangle. The semicircle has radius r mm and the isosceles triangle has height h mm, slant height s mm and slant angle θ, as shown.



a Use trigonometric ratios to find a formula for:

i h in terms of r and θ **ii** s in terms of r and θ

b Find a formula for the area of:

i region A in terms of r and π **ii** region B in terms of r and θ

The Perfect Diamond Company's secret is to make sure that the cross-sectional areas of regions A and B are equal.

c Show that this leads to an equation that can be simplified to $\tan\theta = \dfrac{\pi}{2}$.

d Solve the equation in part **c** to find the value of θ for diamonds cut at The Perfect Diamond Company. Round off your answer to the nearest tenth of a degree.

e Find, to two decimal places, the total area of the cross-section through the centre of a diamond if the radius, r, is 2 mm.

10C Problem-solving

1 A line with gradient -2 passes through the point $(r, -3)$. A second line, perpendicular to the first, meets this line at the point (a, b). The second line passes through the point $(6, r)$. Find a and b in terms of r.

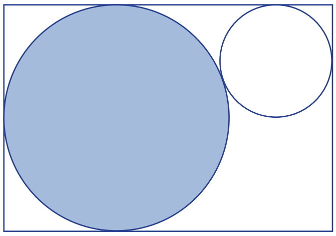
2 Find all the ordered pairs of integers such that $x^2 - y^2 = 140$.

3 **a** The sum of the lengths of the shorter sides of a right-angled triangle is 34. Find the length of the hypotenuse of the triangle if the area is:

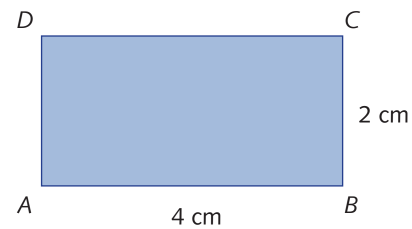
i 30 cm^2 **ii** 32 cm^2

b The area of a rectangle is 12 cm^2 and its perimeter is 14 cm. What is the length of the diagonal of the rectangle?

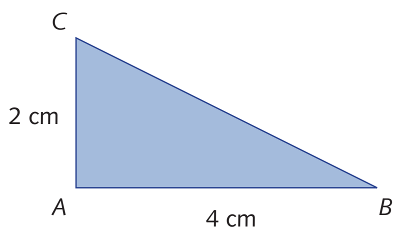
4 A circle (shown shaded) just fits inside a 2 m × 3 m rectangle. What is the radius, in metres, of the largest circle that will also fit inside the rectangle but will not intersect with the shaded circle?



5 **a** The rectangle $ABCD$ is rotated about the side AB. Find the volume of the solid defined by this rotation.

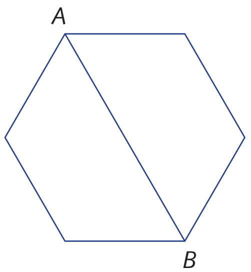


b Triangle ABC is rotated about the side AB. Find the volume of the solid defined by this rotation.



c A circle of radius 2 cm is rotated about a diameter. Find the volume of the solid defined by this rotation.

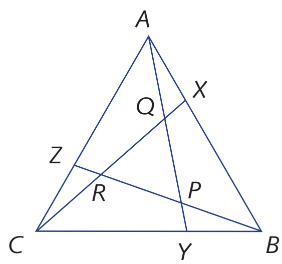
d A regular hexagon with side length 2 cm is rotated about the diagonal AB. Find the volume of the solid produced.



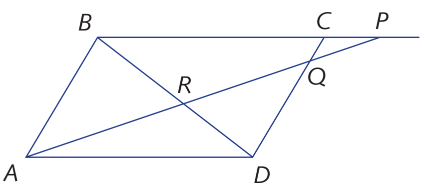
6 Prove, using coordinates, that the line intervals joining the midpoints of successive sides of any quadrilateral form a parallelogram.

7 If P is any point in the plane of a rectangle $ABCD$, prove that $(PA)^2 + (PC)^2 = (PB)^2 + (PD)^2$.

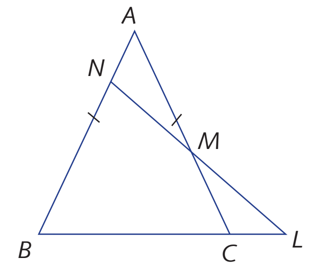
8 ΔABC is equilateral, X is on AB and $AX : XB = 1 : 2$. Y is on BC and $BY : YC = 1 : 2$. Z is on CA and $CZ : ZA = 1 : 2$. AY, BZ and CX intersect at P, Q and R. Prove that the area of ΔPQR is one-seventh of the area of ΔABC.



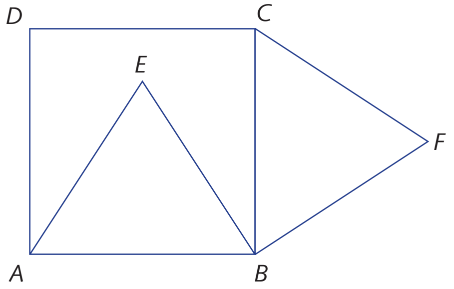
9 $ABCD$ is a parallelogram and P is any point on BC produced. Prove that: $AR^2 = RQ \times RP$.



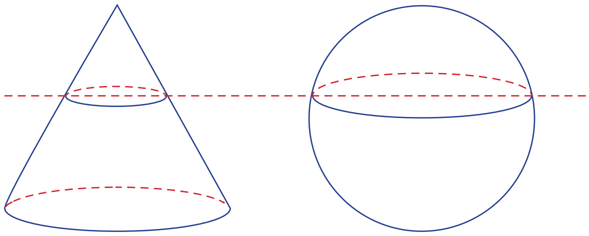
10 ΔABC is an isosceles triangle. L is a point on BC produced so that there are points N and M on AB and AC, respectively, so that $NM = ML$. Find the ratio $BN : CM$.



11 On square $ABCD$, an equilateral triangle ABE is constructed internally and an equilateral triangle BCF is constructed externally. Prove that the points D, E and F are collinear.



12 A sphere has radius 5 cm. A cone has height 10 cm and its base has radius 5 cm.

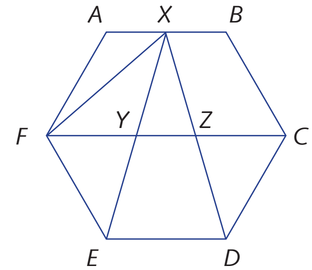


The sphere and the cone sit on a horizontal surface. Find the height of the horizontal plane above the surface that gives circular cross-sections of the sphere and the cone of equal area.

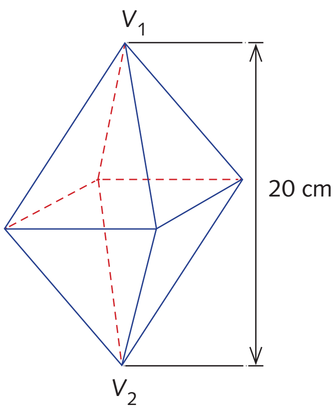
13 $ABCDEF$ is a regular hexagon. X is the midpoint of AB. XE and XD are drawn to meet FC at Y and Z, respectively.

Find the ratio:

Area of quadrilateral $YZDE$: Area of ΔFYX



14 The solid shown is a regular octahedron. The distance between the vertices V_1 and V_2 is 20 cm. Find the sum of the lengths of the edges of the octahedron.



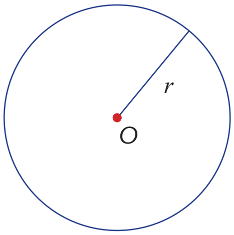
CHAPTER

11

Number and Algebra

Circles, hyperbolas and simultaneous equations

A circle with centre O and radius r is the set of all points whose distance from the centre O is equal to r.



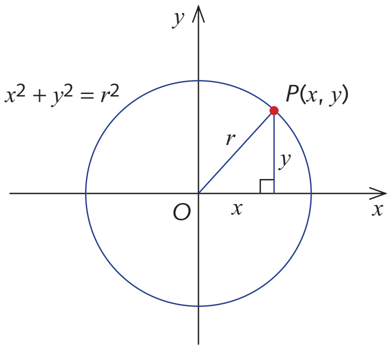
In this chapter, we study circles using the techniques of coordinate geometry.

We also introduce rectangular hyperbolas, and describe methods for finding the coordinates of the points of intersection of hyperbolas, parabolas and circles with straight lines.

11A Cartesian equation of a circle

Circles with centre the origin

Consider a circle in the coordinate plane with centre the origin and radius r. Throughout this chapter, we will always assume that $r > 0$.



If $P(x, y)$ is a point on the circle, then its distance from the origin is r. By Pythagoras' theorem, this gives $x^2 + y^2 = r^2$.

Conversely, if a point $P(x, y)$ satisfies the equation $x^2 + y^2 = r^2$, then its distance from $O(0, 0)$ is $\sqrt{x^2 + y^2} = r$, so it lies on the circle with centre the origin and radius r.

Example 1

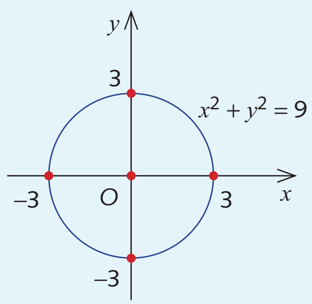
Sketch the graphs of the circles with the following equations.

a $x^2 + y^2 = 9$

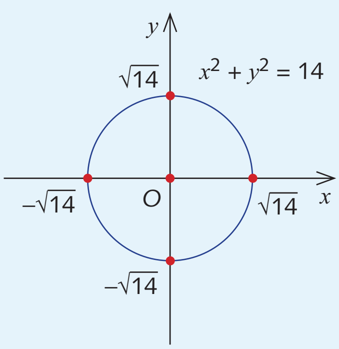
b $x^2 + y^2 = 14$

Solution

a Centre is (0,0) and radius is 3.



b Centre is (0,0) and radius is $\sqrt{14}$.



Example 2

Sketch the graph of the circle $x^2 + y^2 = 25$ and verify that the points $(3, 4)$, $(-3, 4)$, $(-3, -4)$ and $(4, -3)$ lie on the circle.

Solution

The circle has centre the origin and radius 5.

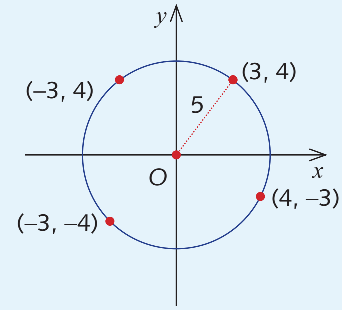
To verify that a point lies on the circle, we substitute the coordinates into $x^2 + y^2 = 25$.

The point $(3, 4)$ lies on the circle, since $3^2 + 4^2 = 25$.

The point $(-3, 4)$ lies on the circle, since $(-3)^2 + 4^2 = 25$.

The point $(-3, -4)$ lies on the circle, since $(-3)^2 + (-4)^2 = 25$.

The point $(4, -3)$ lies on the circle, since $(4)^2 + (-3)^2 = 25$.

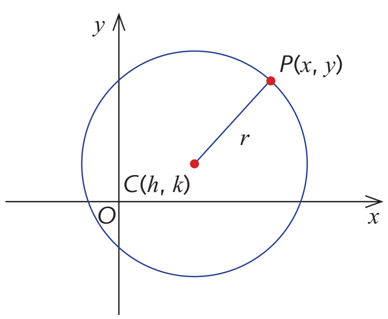


Circles with centre not the origin

Now take a circle in the coordinate plane with centre at the point $C(h, k)$ and radius r.

If $P(x, y)$ is a point on the circle, then by the distance formula:

$$(x - h)^2 + (y - k)^2 = r^2$$



Conversely, if a point $P(x, y)$ satisfies the equation $(x - h)^2 + (y - k)^2 = r^2$, then its distance from (h, k) is r, so it lies on a circle with centre $C(h, k)$ and radius r.

We call $(x - h)^2 + (y - k)^2 = r^2$ the **standard form for the equation of a circle**.

Circles

- The circle with centre $O(0,0)$ and radius r has equation:

  $x^2 + y^2 = r^2$

- The standard form for the equation of the circle with centre (h, k) and radius r is:

  $(x - h)^2 + (y - k)^2 = r^2$

Example 3

Sketch the graph of each circle, showing any intercepts.

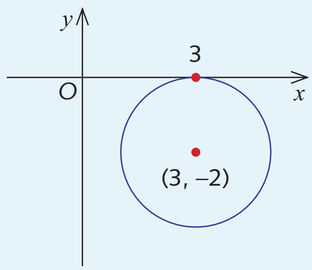
a $(x-3)^2 + (y+2)^2 = 4$

b $(x+1)^2 + (y-3)^2 = 25$

Solution

a The circle has centre $(3, -2)$ and radius 2, and hence the circle touches the x-axis. That is, it meets the x-axis but does not cross it.

The circle does not meet the y-axis.



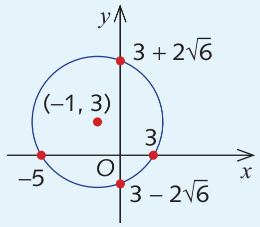
b The circle has centre $(-1,3)$ and radius 5.

Put $y = 0$ into the equation to find where the circle cuts the x-axis.

$$(x+1)^2 + (0-3)^2 = 25$$
$$(x+1)^2 + 9 = 25$$
$$(x+1)^2 = 16$$
$$x + 1 = 4 \quad \text{or } x + 1 = -4$$
$$x = 3 \quad \text{or} \quad x = -5$$

Put $x = 0$ into the equation to find where the circle cuts the y-axis.

$$(0+1)^2 + (y-3)^2 = 25$$
$$1 + (y-3)^2 = 25$$
$$(y-3)^2 = 24$$
$$y - 3 = 2\sqrt{6} \quad \text{or } y - 3 = -2\sqrt{6}$$
$$y = 3 + 2\sqrt{6} \quad \text{or} \quad y = 3 - 2\sqrt{6}$$



Note: The circle $(x-3)^2 + (y+2)^2 = 4$ is a translation of the circle $x^2 + y^2 = 4$, three units to the right and two units down.

The circle $(x+1)^2 + (y-3)^2 = 25$ is the image of the circle $x^2 + y^2 = 25$ under a translation of 1 unit to the left and 3 units up.

Finding the centre and radius of a circle by completing the square

In Example 3a, we sketched the graph of $(x-3)^2 + (y+2)^2 = 4$.
Expanding the brackets, we obtain:

$$x^2 - 6x + 9 + y^2 + 4y + 4 = 4$$

which simplifies to:

$$x^2 + y^2 - 6x + 4y + 9 = 0$$

This is still the equation of the circle with centre $(3, -2)$ and radius 2, but in this form it is not clear what the centre and radius are.

Completing the square enables us to reverse the process and to express the equation in standard form. We can then read off the centre and radius.

Converting to standard form

To find the centre and radius of a circle, complete the square in both x and y to write the equation in standard form. Then read off the centre and the radius.

Example 4

Express each equation in the standard form $(x-h)^2 + (y-k)^2 = r^2$ and hence write down the centre and the radius of the circle.

a $x^2 + 4x + y^2 + 6y + 4 = 0$

b $x^2 - 4x + y^2 + 8y - 5 = 0$

Solution

a

$$\begin{aligned} (x^2 + 4x) + (y^2 + 6y) &= -4 \\ (x^2 + 4x + 4) + (y^2 + 6y + 9) &= -4 + 4 + 9 \\ (x+2)^2 + (y+3)^2 &= 9 \\ &= 3^2 \end{aligned}$$

(Group together the x-terms and y-terms on one side of the equation.)

(Complete the square for the quadratic in x and the quadratic in y.)

Hence, the centre of the circle is $(-2, -3)$ and the radius is 3.

b

$$\begin{aligned} (x^2 - 4x) + (y^2 + 8y) &= 5 \\ (x^2 - 4x + 4) + (y^2 + 8y + 16) &= 5 + 4 + 16 \\ (x-2)^2 + (y+4)^2 &= 25 \\ &= 5^2 \end{aligned}$$

(Complete the square.)

Hence, the centre of the circle is $(2, -4)$ and the radius is 5.

Exercise 11A

Example 1

1 Sketch the graph of each circle, marking any intercepts.

a $x^2 + y^2 = 25$ **b** $x^2 + y^2 = 1$ **c** $x^2 + y^2 = 2$ **d** $x^2 + y^2 = 3$

2 Sketch the graphs, marking any intercepts.

a $y^2 = 4 - x^2$ **b** $y^2 = -x^2 + 10$ **c** $x^2 = 5 - y^2$ **d** $x^2 = -y^2 + 8$

Example 2

3 Check whether or not each point lies on the circle $x^2 + y^2 = 100$.

a $(6,8)$ **b** $(10,10)$ **c** $(20,80)$

d $(-6,8)$ **e** $(5\sqrt{2},5\sqrt{2})$ **f** $(10,0)$

4 Check whether or not each point lies on the circle $x^2 + y^2 = 169$.

a $(5,12)$ **b** $(100,69)$ **c** $(-5,-12)$

d $(-5,12)$ **e** $(-13\sqrt{2},13\sqrt{2})$ **f** $(0,13)$

Example 3

5 Sketch the graphs, showing the x- and y-intercepts.

a $(x-1)^2 + (y-2)^2 = 4$ **b** $(x-3)^2 + (y-4)^2 = 25$ **c** $(x-2)^2 + (y-3)^2 = 9$

d $(x-3)^2 + (y-1)^2 = 16$ **e** $(x-1)^2 + y^2 = 4$ **f** $x^2 + (y-4)^2 = 16$

Example 4

6 Complete the square in x and y to find the coordinates of the centre and the radius of each circle.

a $x^2 + 4x + y^2 + 6y + 9 = 0$ **b** $x^2 - 2x + y^2 + 8y + 4 = 0$

c $x^2 - 6x + y^2 - 8y = 39$ **d** $x^2 - 14x + y^2 - 8y + 40 = 0$

e $x^2 - 8x + y^2 - 6y + 15 = 0$ **f** $x^2 - 8x + y^2 - 4y + 10 = 0$

7 Write down the equation of the circle with:

a centre $(1,3)$ and radius 3 **b** centre $(-2,1)$ and radius 4

c centre $(4,-1)$ and radius 1 **d** centre $(2,0)$ and radius 2

8 Show that the point $(17,17)$ lies on the circle with centre $(5,12)$ and radius 13. Find the equation of the circle.

9 Find the equation of the circle with centre $(3,-4)$ passing through the origin.

10 a Find the equation of the circle with centre $(6,7)$ that touches the y-axis.

b Find the equation of the circle with centre $(6,7)$ that touches the x-axis.

11 The interval AB joins the points $A(2,6)$ and $B(8,6)$. Find:

a the distance AB **b** the midpoint of AB

c the equation of the circle with diameter AB

12 The interval AB joins the points $A(1,6)$ and $B(3,-8)$. Find:

a the distance AB **b** the midpoint of AB

c the equation of the circle with diameter AB

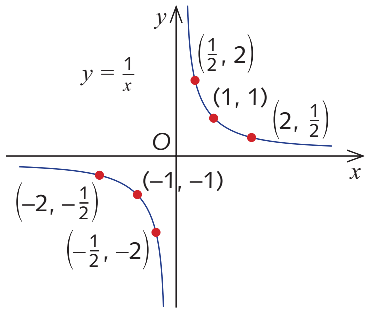
11B The rectangular hyperbola

The basic rectangular hyperbola

In Chapter 7, we called $y = x^2$ the basic parabola and then we showed how to obtain other parabolas from the basic parabola by using transformations.

Similarly, we shall call the hyperbola $y = \frac{1}{x}$ the **basic rectangular hyperbola**.

To see what the graph of $y = \frac{1}{x}$ looks like, begin by considering the table of values below.



x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y	$-\frac{1}{2}$	-1	-2	–	2	1	$\frac{1}{2}$

Since division by zero is not allowed, there is no y-value when $x = 0$.

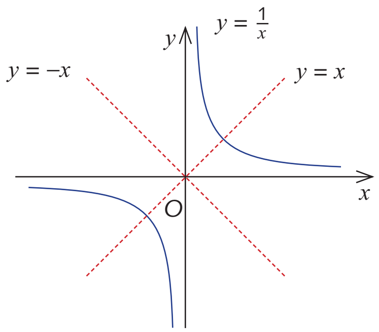
To see more clearly what is happening to the curve close to zero, we produce the table of values for $y = \frac{1}{x}$ below.

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
y	-10	-100	-1000	–	1000	100	10

Can you see how these values are reflected in the graph close to the y-axis?

And as x gets larger and larger in the positive and negative directions (moves further away from zero), y gets smaller and smaller. For example, when $x = 100$, $y = 0.01$ and when $x = 10\,000$, $y = 0.0001$. Similarly, when $x = -100$, $y = -0.01$ and when $x = -10\,000$, $y = -0.0001$.

Features of $y = \frac{1}{x}$



- There are no x-intercepts and no y-intercepts.
- When x is a large positive number, y is a small positive number.
- When x is a small positive number, y is a large positive number.
- Similar results hold for large and small negative values of x.
- The x-axis and the y-axis are called **asymptotes** to the graph. The graph gets very close to each of these lines, but never meets them.
- The lines $y = x$ and $y = -x$ are axes of symmetry for the graph of $y = \frac{1}{x}$.

The types of transformations applied to parabolas in Chapter 7 will now be applied to a **rectangular hyperbola**. The word 'rectangular' means that the asymptotes are perpendicular.

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Reflection in the x-axis

In Chapter 7, we saw that $y = -x^2$ is the reflection of $y = x^2$ in the x-axis. Similarly, $y = -\frac{1}{x}$ is the reflection of $y = \frac{1}{x}$ in the x-axis.

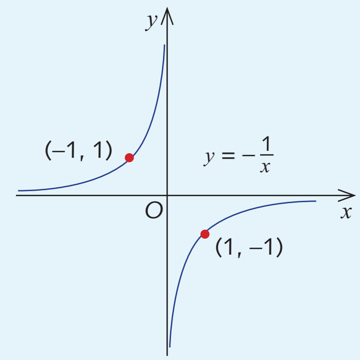
Example 5

Sketch the graph of $y = -\frac{1}{x}$.

Solution

The graph of $y = -\frac{1}{x}$ is the reflection of $y = \frac{1}{x}$ in the x-axis.

The graph of $y = -\frac{1}{x}$ has been drawn.



Horizontal translations

In Chapter 7, we saw that the graph of $y = x^2$ becomes:

- the graph of $y = (x - 5)^2$ when translated 5 units to the right
- the graph of $y = (x + 4)^2$ when translated 4 units to the left.

In a similar way, the graph of $y = \frac{1}{x}$ becomes:

- the graph of $y = \frac{1}{x - 5}$ when translated 5 units to the right
- the graph of $y = \frac{1}{x + 4}$ when translated 4 units to the left.

Example 6

Sketch the graph of $y = \frac{1}{x - 3}$.

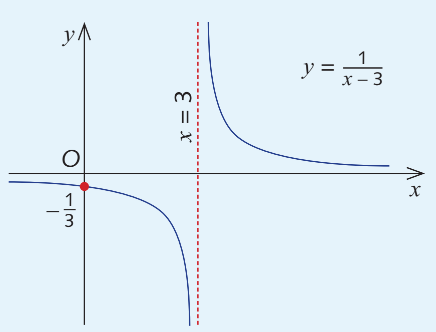
Solution

The graph is obtained by translating the graph of $y = \frac{1}{x}$ three units to the right.

The vertical asymptote has equation $x = 3$.

The horizontal asymptote remains $y = 0$.

The y-intercept is found by putting $x = 0$ into the equation, and so it is $-\frac{1}{3}$. There are no x-intercepts.



Vertical translations

In Chapter 7, we saw that the graph of $y = x^2$ becomes:

- the graph of $y = x^2 + 5$ when translated 5 units up
- the graph of $y = x^2 - 4$ when translated 4 units down.

In a similar way, the graph of $y = \frac{1}{x}$ becomes:

- the graph of $y = \frac{1}{x} + 5$ when translated 5 units up
- the graph of $y = \frac{1}{x} - 4$ when translated 4 units down.

Example 7

Sketch the graph of $y = \frac{1}{x} + 2$.

Solution

The graph of $y = \frac{1}{x} + 2$ is obtained by translating the graph of $y = \frac{1}{x}$ two units up.

The horizontal asymptote has equation $y = 2$. The vertical asymptote remains $x = 0$.

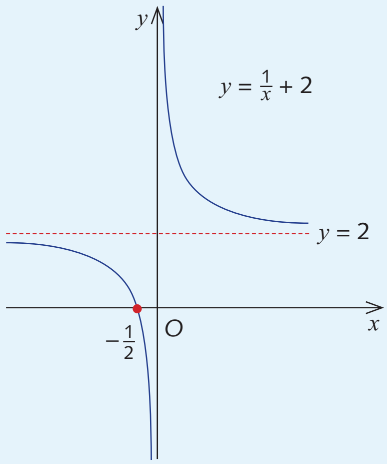
To find the x-intercept, put $y = 0$ into the equation.

$$0 = \frac{1}{x} + 2$$

$$\frac{1}{x} = -2$$

$$x = -\frac{1}{2}$$

There is no y-intercept.



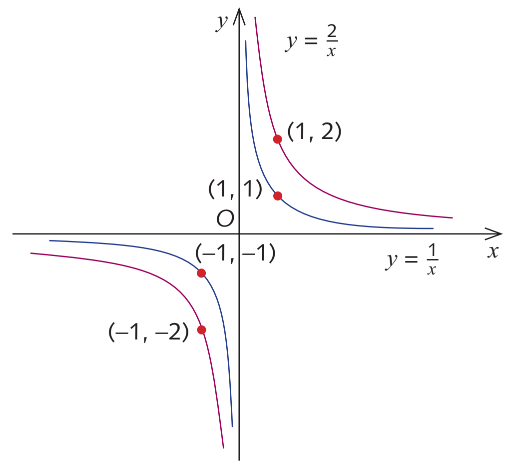
Translations of the basic rectangular hyperbola

- The graph of $y = \frac{1}{x-h}$, where h is a positive number, can be obtained by translating the graph of $y = \frac{1}{x}$ by h units to the right. The equation of the vertical asymptote is $x = h$.
- The graph of $y = \frac{1}{x} + k$, where k is a positive number, can be obtained by translating the graph of $y = \frac{1}{x}$ by k units up. The equation of the vertical asymptote is $y = k$.
- Similar statements apply for translations to the left and translations down.

The rectangular hyperbola $y = \dfrac{a}{x}$

The graph of $y = \dfrac{2}{x}$ is obtained from the graph of $y = \dfrac{1}{x}$ by transforming each point $\left(p, \dfrac{1}{p}\right)$, where $p \neq 0$, to $\left(p, \dfrac{2}{p}\right)$. The y-coordinate is multiplied by 2.

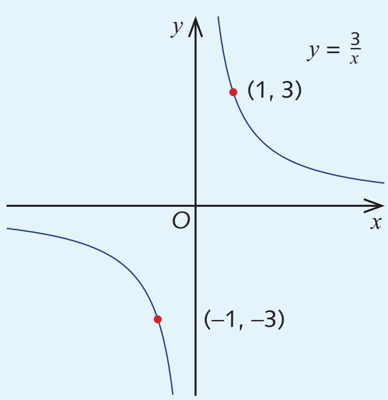
The rectangular hyperbola $y = \dfrac{2}{x}$ is obtained by stretching the basic hyperbola $y = \dfrac{1}{x}$ by a factor of 2 from the x-axis.

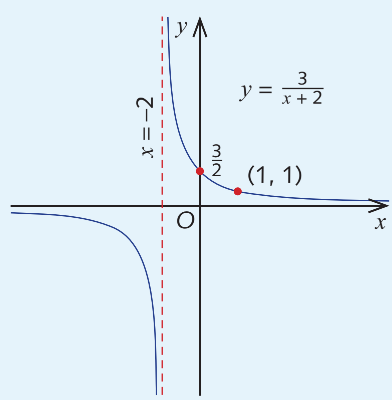


Example 8

Sketch the graph of $y = \dfrac{3}{x + 2}$ by first sketching the graph of $y = \dfrac{3}{x}$.

Solution





The graph of $y = \dfrac{3}{x + 2}$ is obtained by translating the graph of $y = \dfrac{3}{x}$ two units to the left.

Exercise 11B

1 Given that $y = \dfrac{1}{x}$, find y when:

a $x = 2$ **b** $x = -2$ **c** $x = \dfrac{1}{2}$ **d** $x = \dfrac{3}{2}$ **e** $x = -\dfrac{2}{3}$

2 Given that $y = \dfrac{12}{x}$, find y when:

a $x = 3$ **b** $x = 4$ **c** $x = -\dfrac{1}{2}$ **d** $x = -\dfrac{3}{2}$ **e** $x = \dfrac{3}{4}$

3 Given that $y = -\frac{1}{x}$, find y when:

a $x = -1$ **b** $x = -2$ **c** $x = -\frac{1}{2}$ **d** $x = \frac{3}{2}$ **e** $x = -\frac{3}{2}$

4 Given that $y = \frac{1}{x-3}$, find y when:

a $x = 4$ **b** $x = 2$ **c** $x = 5$ **d** $x = 3\frac{1}{2}$ **e** $x = 2\frac{3}{4}$

5 **a** Sketch the graph of $y = \frac{4}{x}$.

b Find the values of y when $x = -4, -2, -1, 1, 2$ and 4, and plot the corresponding points on the graph.

6 **a** Sketch the graph of $y = \frac{1}{x+2}$.

b Find the values of y when $x = -4, -3, 2\frac{1}{2}, 1\frac{1}{2}, -1$ and 0, and plot the corresponding points on the graph.

7 On the hyperbola $y = \frac{12}{x}$, find the value of y when x equals:

a -0.001 **b** -0.2 **c** 3 **d** 24 **e** 144

8 On the hyperbola $y = \frac{6}{x-3}$, find the value of y when x equals:

a 0 **b** 1 **c** 2.99 **d** 3.01 **e** 1000

9 On the hyperbola $y = \frac{12}{x+3}$, find the value of y when x equals:

a 0 **b** -2.9 **c** -2.99 **d** -3.01 **e** -3.001

Example 5 **10** Sketch each graph, and indicate two points on each graph.

a $y = \frac{3}{x}$ **b** $y = \frac{3}{2x}$ **c** $y = -\frac{1}{x}$ **d** $y = -\frac{3}{x}$

Example 6 **11** Sketch each graph, showing the asymptotes and any intercepts.

a $y = \frac{1}{x-4}$ **b** $y = \frac{1}{x-2}$ **c** $y = \frac{1}{x+3}$ **d** $y = \frac{-1}{x+1}$

Example 7 **12** Sketch each graph, showing the asymptotes and any intercepts.

a $y = \frac{1}{x} + 1$ **b** $y = \frac{1}{x} - 3$ **c** $y = -\frac{1}{x} + 4$ **d** $y = \frac{1}{x} - 1$

Example 8 **13** Sketch each graph, showing the asymptotes and any intercepts.

a **i** $y = \frac{6}{x}$ **ii** $y = \frac{6}{x-3}$

b **i** $y = \frac{10}{x}$ **ii** $y = \frac{10}{x-5}$

c **i** $y = \frac{4}{x}$ **ii** $y = \frac{4}{x+2}$

d **i** $y = \frac{-3}{x}$ **ii** $y = \frac{-3}{x+1}$

14 Sketch each graph, showing the asymptotes and any intercepts.

a $y = \frac{2}{x} + 1$ **b** $y = \frac{4}{x} - 3$ **c** $y = -\frac{12}{x} + 4$ **d** $y = \frac{2}{x} - 1$

11C Intersections of graphs

In this section we will look at the intersections of:

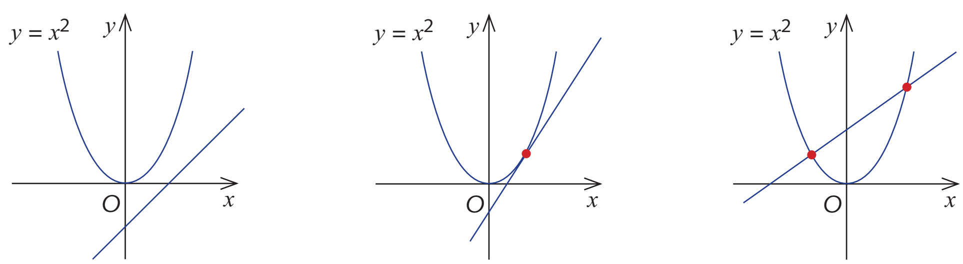
- lines and parabolas
- lines and circles
- lines and rectangular hyperbolas.

In Chapter 4, we looked at the intersections of lines.

Two distinct lines meet at zero points or 1 point. In the situations listed above, there are always 0, 1 or 2 points of intersection. We find these points of intersection by solving simultaneous equations.

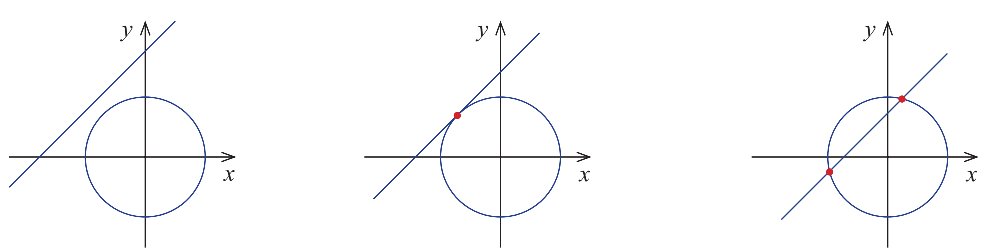
That is, we shall be using algebra to solve problems in geometry.

A straight line and a parabola



0 points of intersection 1 point of intersection 2 points of intersection

A straight line and a circle

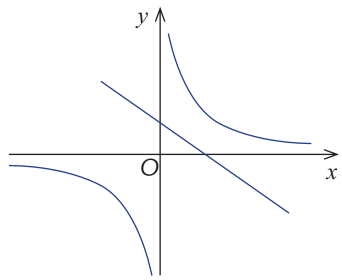


0 points of intersection 1 point of intersection 2 points of intersection

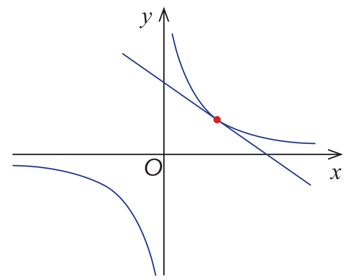
When there is just one point of intersection between a circle and a line, the line is called a **tangent** to the circle (see Chapter 13).

A straight line and a rectangular hyperbola

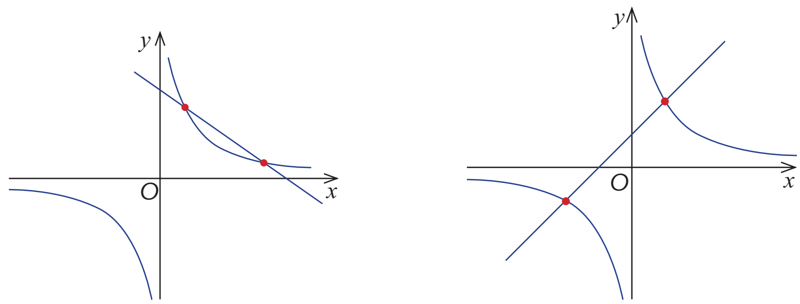
The following diagrams show that when a straight line and a rectangular hyperbola are drawn there may be 0, 1 or 2 points of intersection.



0 points of intersection



1 point of intersection



2 points of intersection

Example 9

Find the coordinates of the points of intersection of the graphs of $y = 4 - x^2$ and $y = 4 - x$, and illustrate your answer graphically.

Solution

To find the points of intersection, solve the equations simultaneously.

$y = 4 - x^2 \qquad (1)$

$y = 4 - x \qquad (2)$

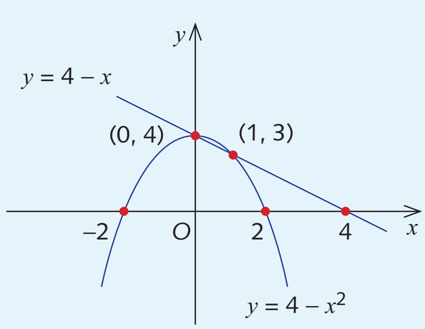
At the points of intersection, the y-values are the same, so:

$$4 - x^2 = 4 - x$$
$$x^2 - x = 0$$
$$x(x - 1) = 0$$
$$x = 0 \text{ or } x = 1$$

When $x = 0, y = 4$.

When $x = 1, y = 3$.

So the two points of intersection are $(0, 4)$ and $(1, 3)$.



Equating the two expressions for y is called **eliminating *y***. We have used this previously for simultaneous linear equations. For all three curves, parabolas, circles and rectangular hyperbolas, a quadratic equation results from eliminating y. This equation will have 0, 1 or 2 solutions, each situation graphically related to 0, 1 or 2 points of intersection, respectively.

Example 10

Find the points of intersection of the circle $x^2 + y^2 = 5$ and the line $y = x + 1$. Illustrate this graphically.

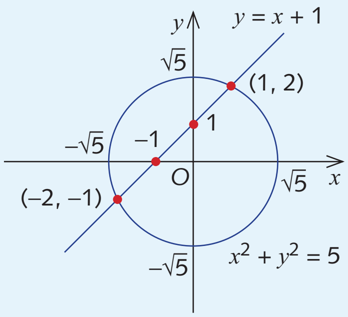
Solution

We have $x^2 + y^2 = 5$ (1)

$y = x + 1$ (2)

Substituting the right-hand side of (2) into (1):

$$\begin{aligned} x^2 + (x+1)^2 &= 5 \\ x^2 + x^2 + 2x + 1 &= 5 \\ 2x^2 + 2x - 4 &= 0 \\ x^2 + x - 2 &= 0 \\ (x+2)(x-1) &= 0 \\ x = -2 \text{ or } x &= 1 \end{aligned}$$



To find the y-values, substitute the values of x into equation (2).

When $x = -2$, $y = -1$.

When $x = 1$, $y = 2$.

So the line cuts the circle at the points $(-2, -1)$ and $(1, 2)$.

Substituting the x-values into equation (1) does not determine the y-values. Check what happens yourself.

Example 11

Find the point of intersection of the line $y = 2x + 5$ and the circle $x^2 + y^2 = 5$. Illustrate this graphically.

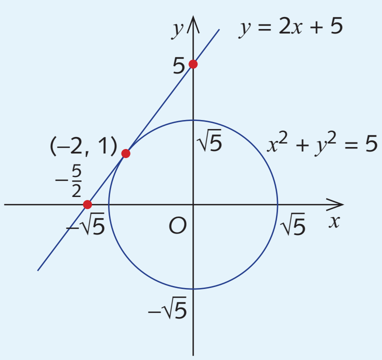
Solution

$y = 2x + 5$ (1)

$x^2 + y^2 = 5$ (2)

Substituting from (1) into (2):

$$\begin{aligned} x^2 + (2x+5)^2 &= 5 \\ x^2 + 4x^2 + 20x + 25 &= 5 \\ 5x^2 + 20x + 20 &= 0 \\ x^2 + 4x + 4 &= 0 \\ (x+2)^2 &= 0 \\ x &= -2 \\ \text{and} \quad y &= 2 \times (-2) + 5 = 1 \end{aligned}$$



Thus the line meets the circle at one point $(-2, 1)$.

Example 12

Show that the line $y = x + 4$ does not meet the circle $x^2 + y^2 = 1$. Illustrate this graphically.

Solution

$$y = x + 4 \quad (1)$$
$$x^2 + y^2 = 1 \quad (2)$$

Substituting from (1) into (2):

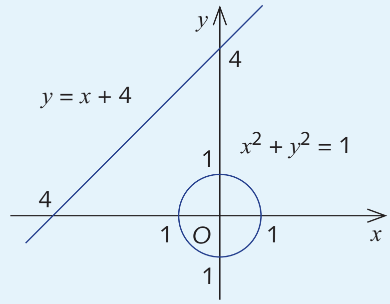
$$x^2 + (x + 4)^2 = 1$$
$$x^2 + x^2 + 8x + 16 = 1$$
$$2x^2 + 8x + 15 = 0$$

For this quadratic equation:

$$\Delta = b^2 - 4ac = 64 - 4 \times 2 \times 15$$
$$= 64 - 120$$
$$= -56 < 0$$

so $b^2 - 4ac < 0$ and there are no solutions to the quadratic equation.

Hence, the line does not meet the circle.



Example 13

Find where the hyperbola $y = \dfrac{2}{x}$ meets the line $y = x + 1$ and illustrate this graphically.

Solution

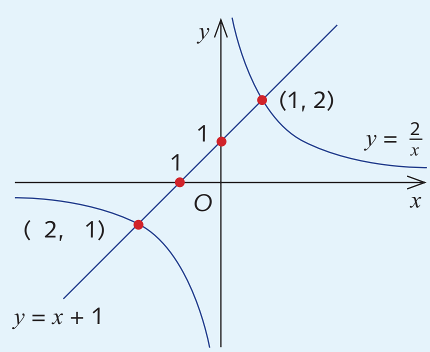
$$y = \frac{2}{x} \quad (1)$$
$$y = x + 1 \quad (2)$$

Eliminating y from equations (1) and (2):

$$x + 1 = \frac{2}{x}$$
$$x^2 + x = 2$$
$$x^2 + x - 2 = 0$$
$$(x + 2)(x - 1) = 0$$
$$x = -2 \text{ or } x = 1$$

Thus, $\quad y = -1$ or $y = 2 \quad$ (either from equation (1) or (2))

Hence, the hyperbola meets the line at $(-2, -1)$ and $(1, 2)$.



Intersection of graphs

- To find the points of intersection of graphs, solve the equations simultaneously.
- A line meets a parabola, a rectangular hyperbola or a circle at 0, 1 or 2 points.

Exercise 11C

Example 9

1 Find the coordinates of the points of intersection of:

a $y = x^2$ and $y = 4$

b $y = x^2$ and $y = 1$

c $y = (x - 1)^2$ and $y = 2x - 3$

d $y = x^2$ and $y = 7x - 12$

2 Find the coordinates of the points of intersection of:

a $y = x^2 + 3x + 3$ and $y = x + 2$

b $y = x^2 + 5x + 2$ and $y = x + 7$

c $y = x^2 + 2x + 4$ and $y = x + 6$

d $y = 2x^2 + 3x + 1$ and $y = 2x + 1$

e $y = 3x^2 + x + 2$ and $y = 3x + 3$

f $y = 6x^2 + 9x + 5$ and $y = 2x + 3$

Example 10

3 Find the coordinates of the points of intersection of:

a $x^2 + y^2 = 4$ and $x = 2$

b $x^2 + y^2 = 9$ and $y = 0$

c $x^2 + y^2 = 32$ and $y = x$

d $x^2 + y^2 = 81$ and $y = 2\sqrt{2}x$

Example 10, 11, 12

4 For each pair of curves, find the points of intersection and illustrate with a graph.

a $x^2 + y^2 = 10$ and $y = x + 2$

b $x^2 + y^2 = 17$ and $y = 3 - x$

c $x^2 + y^2 = 26$ and $x + y = 4$

d $x^2 + y^2 = 20$ and $y = 2x$

e $x^2 + y^2 = 5$ and $y = 2x - 3$

f $x^2 + y^2 = 8$ and $y = x + 4$

g $x^2 + y^2 = 18$ and $x + y = 6$

h $x^2 + y^2 = 25$ and $3x + 4y = 25$

i $x^2 + y^2 = 4$ and $x + y = 6$

j $x^2 + y^2 = 9$ and $y = 2x + 8$

Example 13

5 For each pair of curves, find the coordinates of the points of intersection and illustrate with a graph.

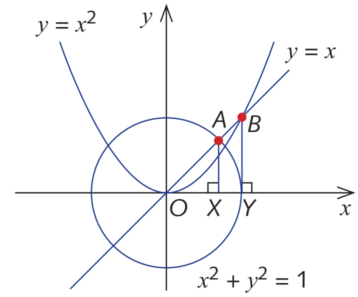
a $y = x - 2$ and $y = \dfrac{3}{x}$

b $y = 2x - 1$ and $y = \dfrac{1}{x}$

c $y = 3x - 1$ and $y = \dfrac{2}{x}$

d $y = -\dfrac{1}{x}$ and $y = -x$

6 The circle $x^2 + y^2 = 1$, the parabola $y = x^2$ and the line $y = x$ are drawn on the same axes. Let A and B be the points of intersection of $y = x$ with the circle and parabola, respectively. XA and YB are drawn perpendicular to the x-axis.



   a Find the coordinates of A and B.

   b Find the area of triangles OAX and OBY.

7 Where does the line $3y - x = 7$ meet the circle $(x - 3)^2 + y^2 = 10$?

8 Show that the line $y = 2x$ does not meet the circle $(x - 5)^2 + y^2 = 4$.

9 Find the values of a for which the graphs of $y = x + a$ and $x^2 + y^2 = 9$ intersect at:

   a one point   **b** two points   **c** no points.

10 Find the points of intersection of the circles $x^2 + y^2 = 9$ and $(x - 2)^2 + y^2 = 9$.

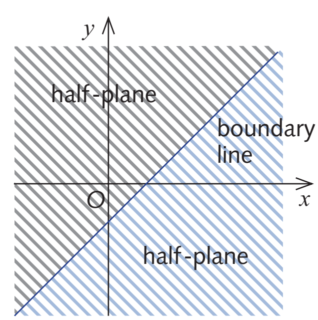
11D Regions of the plane

When we plot a set of points satisfying an inequality, we generally obtain a region of the plane, not a curve or a line.

Half-planes

A straight line divides the plane into three non-overlapping regions:

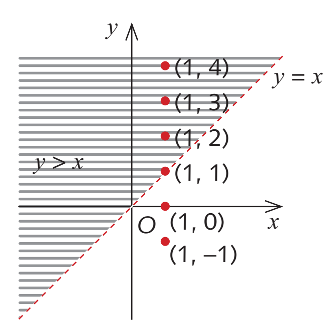
- the points that lie on the line
- the points that lie on one side of the line
- the points that lie on the other side of the line.



Regions consisting of all the points on one side of a line are called **half-planes**.

The region may or may not include the points on the line. The line is often called the **boundary line** of the half-plane.

The region of the plane defined by the inequality $y > x$ consists of all the points (x, y) whose y-coordinate is greater than the x-coordinate. The points $(1, 2)$, $(1, 3)$ and $(1, 4)$ are all in this region, whereas $(1, 0)$ and $(1, -1)$ are not in the region.



The region $y > x$ contains all the points above the line $y = x$. This is because if you choose any point on the line $y = x$ (for example, $(1, 1)$), then all the points (x, y) above the point $(1, 1)$ have $y > x$, and those below have $y < x$.

The region $y > x$ is shown above. The line $y = x$ is dashed to show that it is not included in the region $y > x$.

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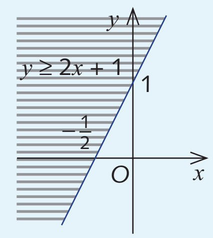
Example 14

Sketch the region defined by the inequality $y \geq 2x + 1$.

Solution



We first sketch the boundary line $y = 2x + 1$.

The boundary line has been drawn as a solid line since it is included in the required region.

Method 1 – Using the inequality with *y* the subject

If you choose any point on the line $y = 2x + 1$, for example, $(0, 1)$, then all the points above $(0,\ 1)$ have $y \geq 2x + 1$ and those below have $y < 2x + 1$. Hence, we shade the region above the line $y = 2x + 1$.

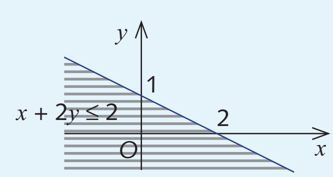
Method 2 – Using a test point *not* on the boundary line

Test the point $(0, 0)$. Since $0 \leq 2 \times 0 + 1$, the point $(0, 0)$ does not belong to the region. Hence, the required region is above the line.

Example 15

Sketch the region defined by the inequality $x + 2y \leq 2$.

Solution



First sketch the boundary line $x + 2y = 2$.

Method 1

The inequality can be rearranged to make y the subject:

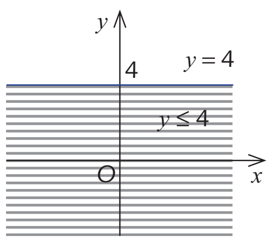
$$y \leq -\frac{1}{2}x + 1$$

Hence, we shade the region below the line.
We include the line, since points on the line satisfy $y = -\frac{1}{2}x + 1$.

Method 2

Test the point $(0, 0)$. Since $0 + 2 \times 0 \leq 2$, the point $(0, 0)$ does belong to the region. Hence, the required region is below the line.

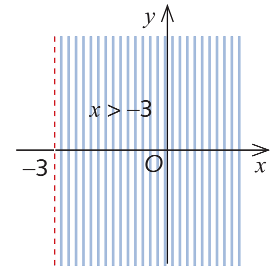
Boundaries parallel to the *x*-axis



The inequality $y \leq 4$ describes the half-plane with boundary line $y = 4$. All of the points below $y = 4$ and the points on $y = 4$ are included in the region.

Boundaries parallel to the y-axis

The inequality $x > -3$ describes the half-plane with the boundary line $x = -3$. All of the points to the right of $x = -3$ are in the half-plane. The points on $x = -3$ are not included, so the line is dashed.



Intersection of regions

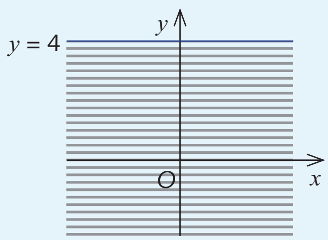
To sketch the intersection of two regions, sketch the regions and see which points they have in common. **Corner points** are those points where the boundary lines of the half-planes meet. They should always be labelled in the sketch.

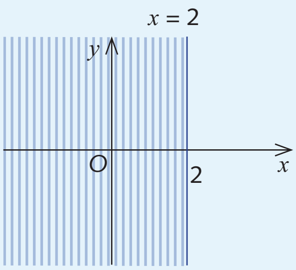
Example 16

Sketch the region of the plane defined by $y \le 4$ and $x \le 2$.

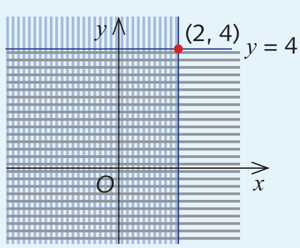
Solution

Sketch the region $y \le 4$ and the region $x \le 2$.



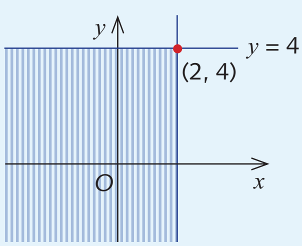


The boundary lines $x = 2$ and $y = 4$ intersect at the corner point $(2, 4)$.



The region is $y \le 4$ and $x \le 2$.

Alternatively, it can be shown as:



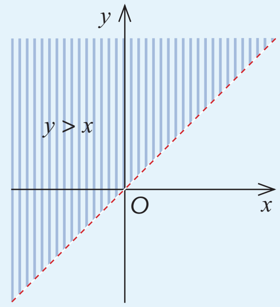
Example 17

Sketch the region defined by the inequalities $y > x$ and $x + y \leq 4$.

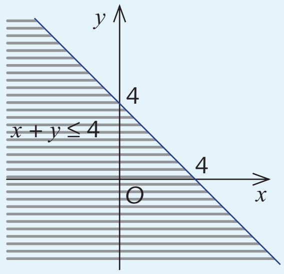
Solution

First sketch the region $y > x$.

Draw the line $y = x$ and shade the region above the line.



Draw $x + y = 4$ and test the origin, $0 + 0 \leq 4$.



To find the corner point, solve the simultaneous equations.

$$y = x \qquad (1)$$

$$x + y = 4 \qquad (2)$$

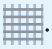
Substitute (1) into (2):

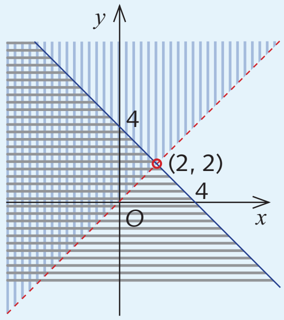
$$x + x = 4$$

$$x = 2$$

From equation (1):

$$y = 2$$

The corner point is $(2, 2)$ but it does not lie in the required region, since it is not a member of $y > x$. It is therefore indicated by an open circle. The region $y > x$ and $x + y \leq 4$ is .



Discs

A circle divides the plane into three regions. The points in the plane are either on the circle, inside the circle or outside the circle. The set of points inside and on a circle is called a **disc**.

Example 18

Sketch the regions.

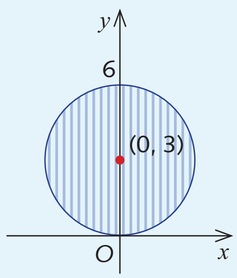
a $x^2 + (y - 3)^2 \le 9$

b $x^2 + (y - 3)^2 > 9$

Solution

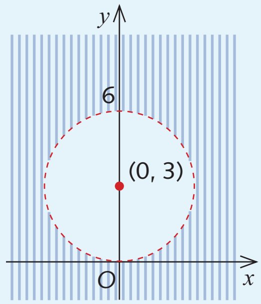
a First draw the circle $x^2 + (y - 3)^2 = 9$. This circle has centre $(0, 3)$ and radius 3. The region is the set of points whose distance from $(0, 3)$ is less than or equal to 3 units.

The region is the shaded disc.



b The region $x^2 + (y - 3)^2 > 9$ is the set of points whose distance from $(0, 3)$ is greater than to 3.

The region is the shaded area outside the disc.



Exercise 11D

Example 14, 15

1 Sketch each region.

a $y > x + 1$

b $y < 2x + 3$

c $y \le 2x - 1$

d $y > 1 - x$

e $2x + y \le 4$

f $3x - 2y > 6$

g $3x + y > 1$

h $x - 2y < 1$

i $y \le 2x$

j $y \ge 3x$

k $x \ge 3$

l $x < 1$

m $y < 2$

n $y \le -2$

o $4x + 3y \le 12$

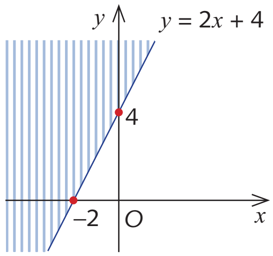
p $2x - y \le 8$

q $2x - y \ge 4$

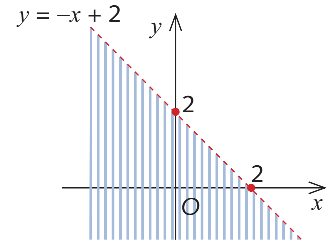
r $y < 3 - 2x$

2 Write down the inequalities that describe each given region.

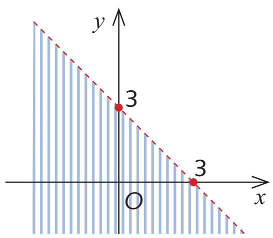
a



b



c



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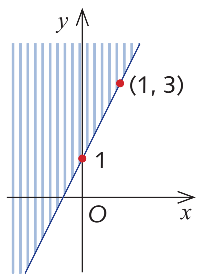


Example 16, 17

3 Sketch the regions satisfying the given inequalities. Find the coordinates of the corner points.

a $y > x$ and $x + y \le 6$

b $y \le 2x$ and $2x + y > 4$

c $x + y \le 4$ and $2x + y \le 6$

d $x + 2y \le 8$ and $3x + y \le 9$

e $y \ge x + 1$ and $y > 3x - 5$

f $y \le 1 - 2x$ and $y \ge \frac{1}{2}x$

g $x \le 2$ and $y \le 1$

h $x \le -2$ and $y \ge 2$

i $x \ge 1$, $x \le 3$, $y \ge 0$ and $y \le 4$

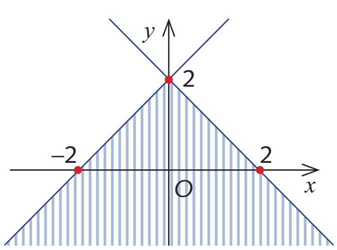
j $x \ge 0$, $y \ge 0$ and $x + y \le 4$

k $x \ge 0$, $y \ge 0$, $x + y \le 6$ and $x + 2y \le 8$

l $y \ge x$, $y \ge 0$ and $y \le \frac{1}{2}x + 2$

4 Write down the inequalities whose intersections are the shaded region.

a



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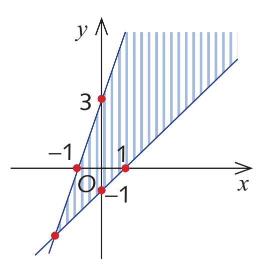

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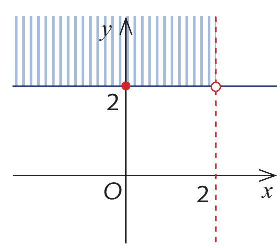

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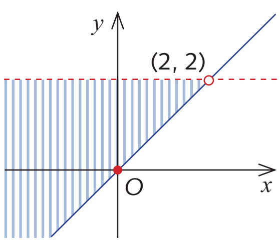

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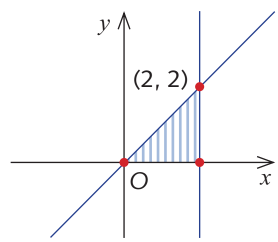

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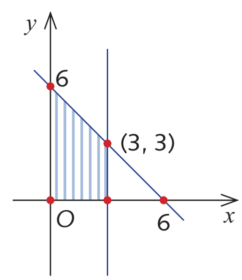

Example 18

5 Sketch each region.

a $x^2 + y^2 < 4$

b $(x - 2)^2 + y^2 \ge 9$

c $(x + 3)^2 + (y - 1)^2 > 16$

d $(x + 1)^2 + (y + 2)^2 \le 1$

6 Sketch $y > \frac{1}{x}$.

Review exercise

1 Sketch each graph.

a $x^2 + y^2 = 9$ **b** $2x^2 + 2y^2 = 8$ **c** $x^2 + y^2 = 5$ **d** $x^2 + y^2 = \frac{9}{4}$

2 Sketch each graph.

a $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = 1$ **b** $(x + 1)^2 + (y + 1)^2 = 4$

c $(x + 3)^2 + (y + 4)^2 = 25$ **d** $x^2 + (y + 4)^2 = 16$

e $(x - 3)^2 + (y + 5)^2 = 4$ **f** $(x - 1)^2 + (y - 1)^2 = 25$

3 Complete the square to find the centre and the radius of each circle.

a $x^2 + 4x + y^2 + 8y = 0$ **b** $x^2 + y^2 + 4x + 2y - 5 = 0$

c $2x^2 + 2y^2 - 8x + 5y + 3 = 0$ **d** $x^2 + y^2 - 4x + 6y - 37 = 0$

4 Write down the equation of the circle with:

a centre $(0, 0)$ and radius $= 3$ **b** centre $(-1, 4)$ and radius $= 6$

c centre $(2, 5)$ and radius $= 1$ **d** centre $(-2, -6)$ and radius $= 4$

5 Sketch the graph of each rectangular hyperbola.

a $y = \frac{4}{x}$ **b** $y = \frac{5}{2x}$ **c** $y = -\frac{4}{x}$

6 Sketch the graph of each rectangular hyperbola. Include the asymptotes.

a $y = \frac{1}{x + 2}$ **b** $y = \frac{1}{x - 4}$ **c** $y = \frac{1}{x + 5}$ **d** $y = \frac{-1}{x - 1}$

7 Find the points where each parabola meets the line.

a $y = 2x^2 - 3x + 4$ and $y = 12 - 3x$ **b** $y = 2 - x - 3x^2$ and $y = -7x + 2$

8 Find the points of intersection of:

a $x^2 + y^2 = 9$ and $x = 3$ **b** $x^2 + y^2 = 16$ and $y = 0$

c $x^2 + y^2 = 16$ and $y = \sqrt{3}x$

9 Find the points of intersection of:

a $3y + 4x = 25$ and $x^2 + y^2 = 25$ **b** $x^2 + y^2 = 29$ and $y = 3x - 1$

10 Sketch each region.

a $y > x + 2$ **b** $y \geq 2x - 4$ **c** $y > 2 - x$ **d** $2x + y \leq 6$

e $3x + 2y > 6$ **f** $y \leq -1$ **g** $x < -1$ **h** $y \leq 3x$

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11 Sketch each region and find the coordinates of the corner points.

a $x > 4$ and $y \le -3$

b $y \le 2x$ and $x \le 6$

c $x + y \le 4$ and $y \le 2x$

d $y \le 1 - 2x$ and $y > x + 2$

e $2x + y \le 6$ and $x + y \ge 4$

f $x + y \le 6$ and $y \ge -2x + 3$

12 Sketch the regions.

a $(x-1)^2 + y^2 \le 1$

b $(x-3)^2 + (y-4)^2 \le 25$

c $x^2 + y^2 > 36$

d $(x-2)^2 + y^2 > 9$

13 Find the points where the hyperbola meets the line.

a $y = x - 1,\ y = \dfrac{12}{x}$

b $y = 2x - 7,\ y = -\dfrac{3}{x}$

c $y = \dfrac{6}{x},\ x = 3$

d $y = \dfrac{9}{x},\ y = 4 - x$

Challenge exercise

1 By considering suitable translations, sketch the graph of:

a $y = 1 + \dfrac{1}{x+4}$

b $y = 2 + \dfrac{1}{x-3}$

2 By considering suitable transformations, sketch the graph of:

a $y = 2 + \dfrac{3}{x-4}$

b $y = 4 + \dfrac{2}{x-5}$

3 By considering suitable transformations, sketch the graph of:

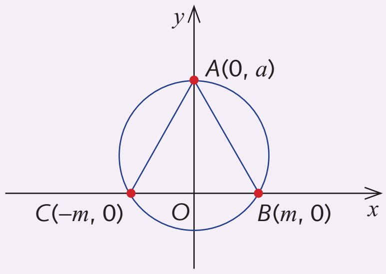
a $y = 1 - \dfrac{1}{x+4}$

b $y = 3 - \dfrac{1}{x+2}$

c $y = 2 + \dfrac{3}{x-2}$

d $y = 4 - \dfrac{5}{x+2}$

4 Triangle ABC is equilateral with vertices $A(0, a)$, $B(m, 0)$ and $C(-m, 0)$. First show $a = \sqrt{3}m$.



a Find, in terms of a, the equation of the perpendicular bisector of:

i AC

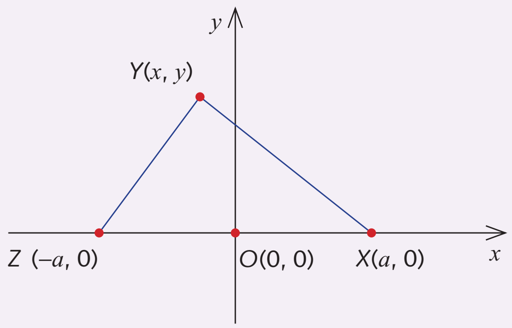
ii AB

b Show that the two perpendicular bisectors meet at $X\left(0, \dfrac{a}{3}\right)$.

c Find the distance AX in terms of a.

d Find the equation of the circle with centre $X\left(0, \dfrac{a}{3}\right)$ and radius AX.

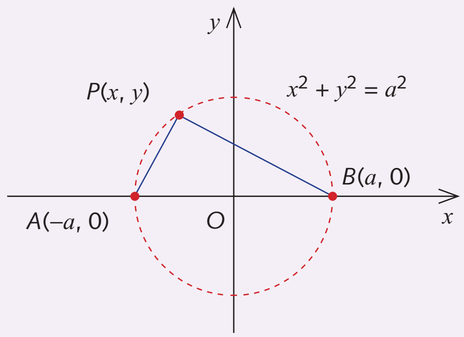
5 **a** XYZ is a right-angled triangle with the right angle at Y. $O(0,0)$ is the midpoint of XZ. The coordinates of X and Z are $(a,0)$ and $(-a,0)$, respectively.



i Use the fact that XY is perpendicular to ZY to show that $x^2 + y^2 = a^2$.

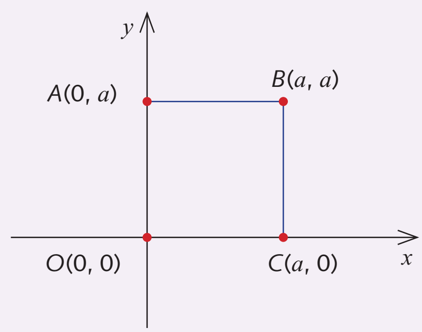
ii Hence, show that $OX = OY = OZ$.

b $P(x, y)$ is a point on the circle $x^2 + y^2 = a^2$. Show that PA is perpendicular to PB.



Note: This proves the important result that the diameter of a circle subtends a right-angle at the circumference. You will encounter this result ag ain in Chapter 13.

6 $ABCO$ is a square of side length a. Show that the equation of the circle passing through all four vertices is $x^2 + y^2 - ax - ay = 0$.



7 The points $O(0,0)$, $A(a,0)$ and $B(0, b)$ lie on a circle.

a Find the equation of the perpendicular bisector of:

i OA **ii** OB

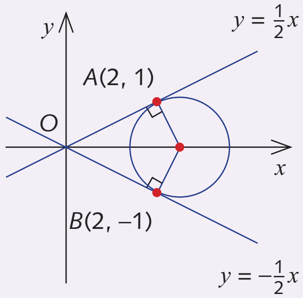
b Find the coordinates of the point of intersection of the perpendicular bisectors of OA and OB.

c Show that the perpendicular bisector of AB also passes through this point.

d Find the equation of the circle passing through O, A and B.

8 Find the equation of the circle that passes through the points (a, b), $(a, -b)$ and $(a + b,\ a - b)$.

9 The lines $y = \frac{1}{2}x$ and $y = -\frac{1}{2}x$ meet the circle at $(2,1)$ and $(2, -1)$, as shown in the diagram. Find the equation of the circle.



10 Sketch each graph.

a $(x - 4)(y - 3) = 2$

b $(x - 2)(y - 3) = 2$

11 Sketch each graph.

a $(x - y)(x + y) = 0$

b $(y - x^2)(y + x^2) = 0$

c $(x^2 - y^2)(x + y^2) = 0$

d $(y^2 - x)(y^2 + x) = 0$

12 Show that the circles $x^2 + y^2 - 2x - 3y = 0$ and $x^2 + y^2 + x - y = 6$ intersect on the x-axis and y-axis.

13 Find the points of intersection of the circles $x^2 + y^2 + x - 3y = 0$ and $2x^2 + 2y^2 - x - 2y - 15 = 0$.

14 The general equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$. Find the equation of the circle passing through the points $(-1,3)$, $(2,2)$ and $(1,4)$.

15 Show that $y = ax + b$, where $a > 0$, always meets $y = \frac{1}{x}$.

CHAPTER

12

Measurement and Geometry

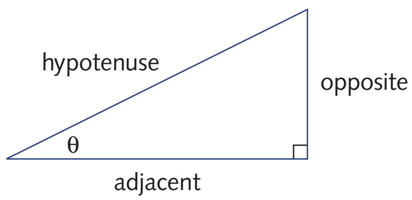
Further trigonometry

Trigonometry begins with the study of relationships between sides and angles in a right-angled triangle.

In this chapter, we will review the basics of the trigonometry of right-angled triangles, look at applications to three-dimensional problems, and extend our study of trigonometry to triangles that are not right-angled.

12A Review of the basic trigonometric ratios

By similarity, the ratio of any two sides in a right-angled triangle is always the same, once we have fixed the angles.



We choose one of the two acute angles and call it the **reference angle**.

The side opposite the reference angle is called the **opposite**, the side opposite the right angle is called the **hypotenuse** and the remaining side, which is between the reference angle and the right-angle, is called the **adjacent**.

The three basic trigonometric ratios are the **sine**, **cosine** and **tangent** ratios.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

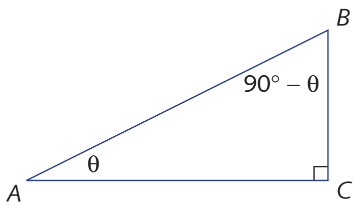
You should learn the three ratios for sine, cosine and tangent by heart and remember them. A simple mnemonic is:

SOHCAHTOA

for **S**ine: **O**pposite/**H**ypotenuse, **C**osine: **A**djacent/**H**ypotenuse, **T**angent: **O**pposite/**A**djacent

Complementary angles

In the diagram, the angles at A and B are complementary; that is, they add to 90°.



The side opposite A is the side adjacent to B and vice versa. Hence, the sine of θ is the cosine of (90° – θ) and vice versa.

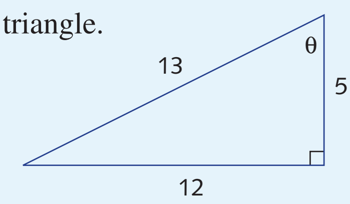
$$\sin\theta = \cos(90^\circ - \theta)$$

$$\cos\theta = \sin(90^\circ - \theta)$$

For example, $\sin 60^\circ = \cos 30^\circ$ and $\cos 10^\circ = \sin 80^\circ$.

Example 1

Write down the sine, cosine and tangent ratios for the angle θ in this triangle.



Solution

$$\sin\theta = \frac{12}{13} \qquad \cos\theta = \frac{5}{13} \qquad \tan\theta = \frac{12}{5}$$

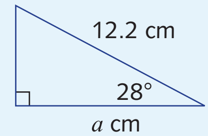
Once a reference angle is given, the approximate numerical value of each of the three ratios can be obtained from a calculator. We can use this idea to find unknown sides in a right-angled triangle.

Example 2

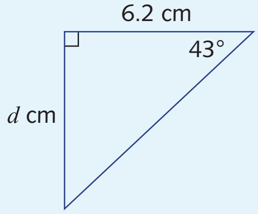
Find, correct to two decimal places, the value of the pronumeral in each triangle.

a

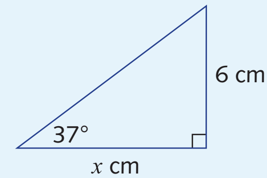
b



c



d



Solution

a
$$\begin{aligned}\sin 15^\circ &= \frac{\text{opposite}}{\text{hypotenuse}}\\ \sin 15^\circ &= \frac{x}{8}\\ x &= 8 \times \sin 15^\circ\\ &\approx 2.07\end{aligned}$$

b
$$\begin{aligned}\cos 28^\circ &= \frac{\text{adjacent}}{\text{hypotenuse}}\\ \cos 28^\circ &= \frac{a}{12.2}\\ a &= 12.2 \times \cos 28^\circ\\ &\approx 10.77\end{aligned}$$

c
$$\begin{aligned}\tan 43^\circ &= \frac{\text{opposite}}{\text{adjacent}}\\ \tan 43^\circ &= \frac{d}{6.2}\\ d &= 6.2 \tan 43^\circ\\ &\approx 5.78\end{aligned}$$

d
$$\begin{aligned}\tan 37^\circ &= \frac{\text{opposite}}{\text{adjacent}}\\ \tan 37^\circ &= \frac{6}{x}\\ x \tan 37^\circ &= 6\\ x &= \frac{6}{\tan 37^\circ}\\ &\approx 7.96\end{aligned}$$

Finding angles

In order to apply trigonometry to finding angles rather than side lengths in right-angled triangles, we need to be able to go from the value of sine, cosine or tangent back to the angle.

What is the acute angle whose sine is 0.5?

The calculator gives $\sin 30^\circ = 0.5$, so we write $\sin^{-1} 0.5 = 30^\circ$.

The opposite process of finding the sine of an angle is to find the **inverse sine** of a number.

When θ is an acute angle, the statement $\sin^{-1} x = \theta$ means $\sin \theta = x$.

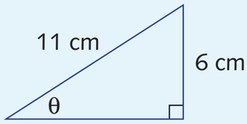
This notation is standard, but is rather misleading. The index -1 does NOT mean *one over,* as it normally does in algebra. To help you avoid confusion, you should always read $\sin^{-1} x$ as *inverse sine of* x and $\tan^{-1} x$ as *inverse tan of* x, and so on.

For example, the calculator gives $\cos^{-1} 0.8192 \approx 35^\circ$ (read this as *inverse cosine of 0.8192 is approximately 35°*).

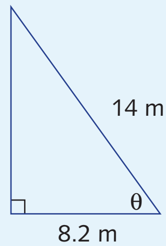
Example 3

Calculate the value of θ, correct to one decimal place.

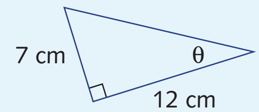
a



b



c



Solution

a $\sin\theta = \dfrac{6}{11}$

$\theta = \sin^{-1}\left(\dfrac{6}{11}\right)$

$\approx 33.1°$

b $\cos\theta = \dfrac{8.2}{14}$

$\theta = \cos^{-1}\left(\dfrac{8.2}{14}\right)$

$\approx 54.1°$

c $\tan\theta = \dfrac{7}{12}$

$\theta = \tan^{-1}\left(\dfrac{7}{12}\right)$

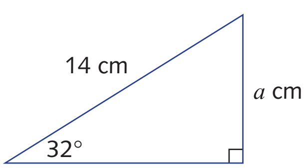
$\approx 30.3°$

Exercise 12A

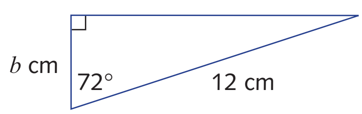
Example 2

1 Calculate the value of each pronumeral, correct to two decimal places.

a



b



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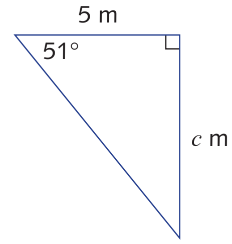

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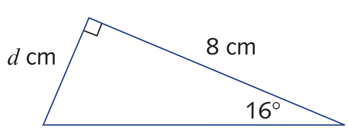
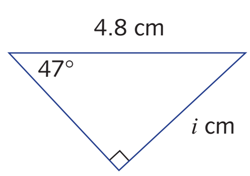
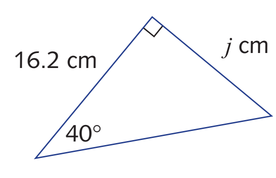


e

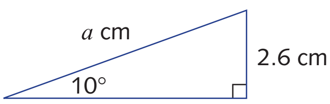


f

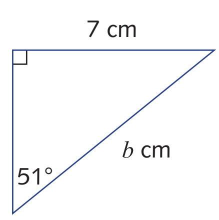


2 Calculate the value of the pronumeral, correct to two decimal places.

a



b



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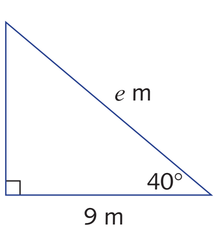
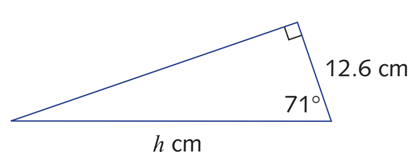


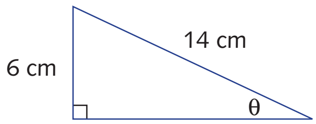
d



Example 3

3 Calculate the value of θ, correct to one decimal place.

a



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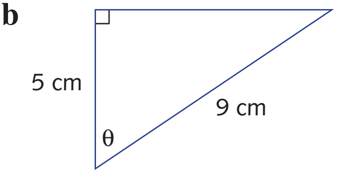

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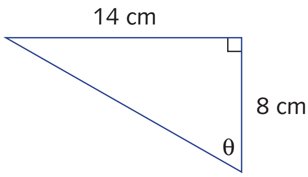

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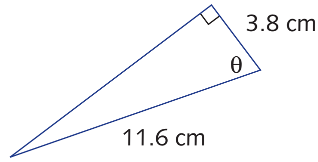

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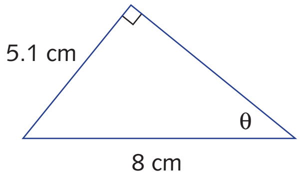

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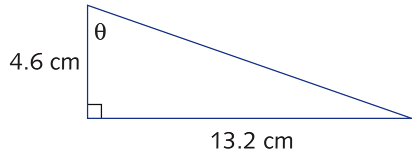
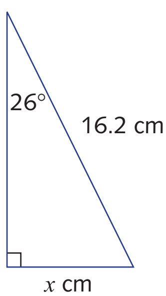
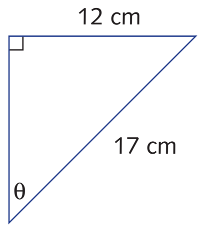


4 Calculate the value of each pronumeral. Give side lengths correct to two decimal places and angles correct to one decimal place.

a



b



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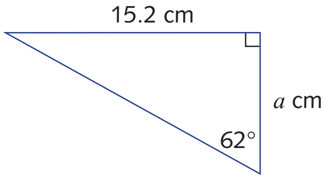

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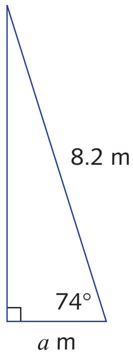

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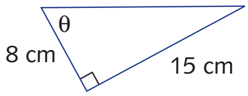

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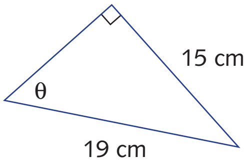

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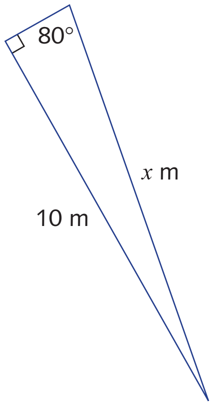

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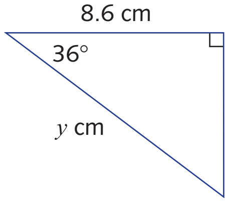

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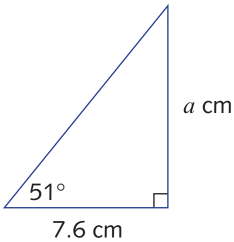
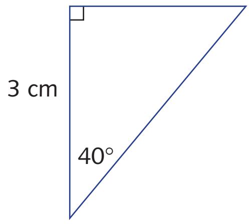
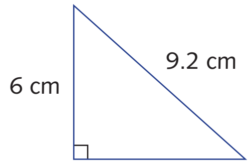


5 Find all sides, correct to two decimal places, and all angles, correct to one decimal place.

a



b



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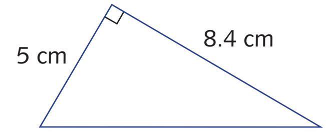
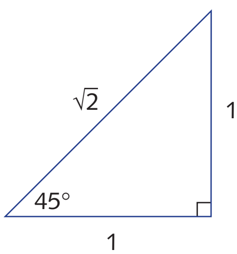


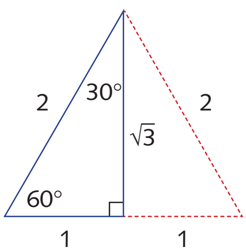
12B Exact values

The trigonometric ratios for the angles $30°$, $45°$ and $60°$ occur very frequently and can be expressed using surds.



The value of the trigonometric ratios for $45°$ can be found from the diagram opposite. It is an isosceles triangle with shorter sides 1.

The values of the trigonometric ratios for $30°$ and $60°$ can be found by drawing an altitude in an equilateral triangle.



The values are given in the table.

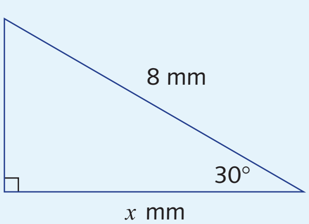
θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45°$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Check the details in the triangles and the entries in the table. You can either learn the table or remember the diagrams to construct the table.

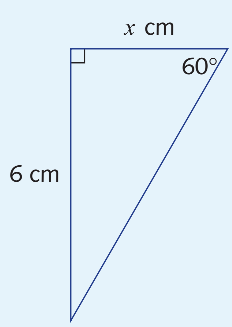
Example 4

Find the exact value of x.

a



b



Solution

a We have $\cos 30° = \dfrac{x}{8}$

$$\begin{aligned} x &= 8\cos 30° \\ &= 8 \times \frac{\sqrt{3}}{2} \\ &= 4\sqrt{3} \end{aligned}$$

b We have $\tan 60° = \dfrac{6}{x}$

so $\dfrac{6}{x} = \sqrt{3}$

Hence
$$\begin{aligned} x &= \frac{6}{\sqrt{3}} \\ &= \frac{6}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \quad \text{(Rationalise the denominator.)} \\ &= 2\sqrt{3} \end{aligned}$$

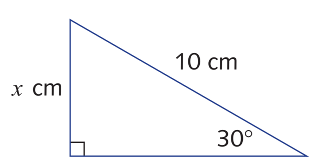
Alternatively, we could have worked with the complementary angle, $\tan 30° = \dfrac{x}{6}$.

Exercise 12B

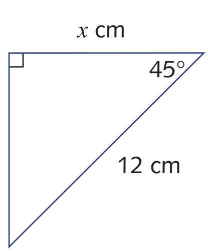
Example 4

1 Find the exact value of x.

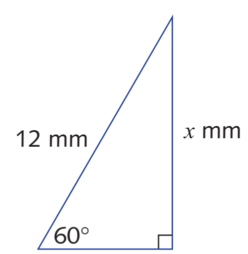
a



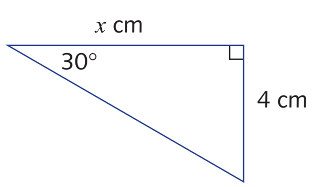
b



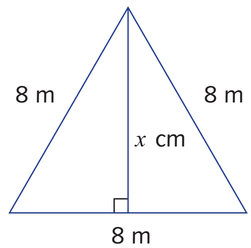
c



d



e



2 Find the exact values, rationalising the denominator where appropriate.

a $(\sin 60°)^2 + (\cos 60°)^2$

b $(\tan 30°)^2 - \dfrac{1}{(\cos 30°)^2}$

c $\dfrac{\tan 60° - \tan 45°}{1 + \tan 60° \times \tan 45°}$

d $\sin 45° \times \cos 60° + \cos 45° \times \sin 60°$

e $2 \sin 30° \times \cos 30°$

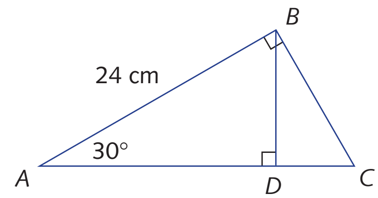
f $2(\cos 45°)^2 - 1$

3 $ABCD$ is a rhombus with $\angle ABD = 30°$. Find the exact length of each diagonal if the side lengths are 10 cm.

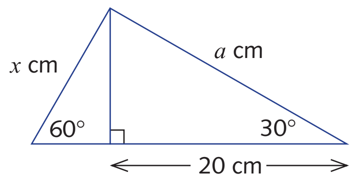
4 Find exact values of:

a AC **b** AD **c** BC

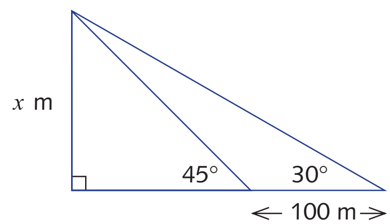
d DC **e** BD



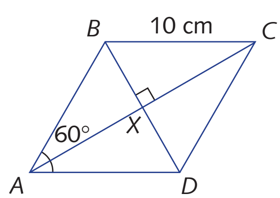
5 Find the exact values of a and x.



6 Find the exact value of x.



7 $ABCD$ is a rhombus with sides 10 cm. Find AX.



12C Three-dimensional trigonometry

We can apply our knowledge of trigonometry to solve problems in three dimensions. To do this you will need to draw careful diagrams and look for right-angled triangles. Sometimes it is helpful to draw a separate diagram showing the right-angled triangle.

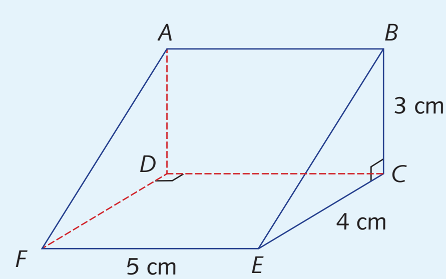
Example 5

In the triangular prism shown, find:

a the length CF

b the length BF

c the angle BFC, correct to one decimal place.

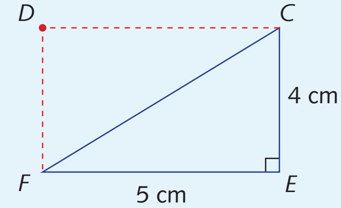


Solution

a Applying Pythagoras' theorem to ΔCEF:

$$\begin{aligned} CF^2 &= 4^2 + 5^2 \\ &= 41 \end{aligned}$$

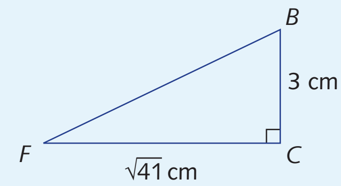
Hence, $CF = \sqrt{41}$ cm



b Applying Pythagoras' theorem to ΔBCF:

$$\begin{aligned} BF^2 &= 3^2 + \left(\sqrt{41}\right)^2 \\ &= 50 \end{aligned}$$

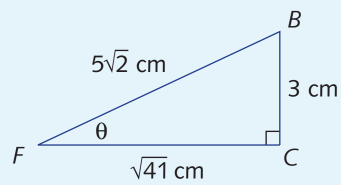
Hence, $BF = 5\sqrt{2}$ cm



c To find the angle BFC, draw ΔBCF and let $\angle BFC = \theta$.

Now $\tan\theta = \dfrac{3}{\sqrt{41}}$

so $\theta \approx 25.1°$ (Correct to one decimal place.)

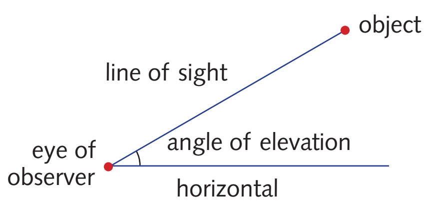


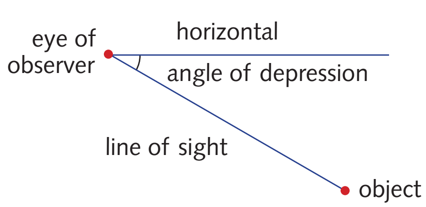
Angles of elevation and depression

When a person looks at an object that is higher than the person's eye, the angle between the line of sight and the horizontal is called the **angle of elevation**.

On the other hand, when the object is lower than the person's eye, the angle between the horizontal and the line of sight is called the **angle of depression**.

In practice, 'eye of observer' is replaced by a point on the ground.

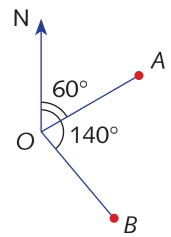




Bearings

Bearings are used to indicate the direction of an object from a fixed reference point, O. **True bearings** give the angle $\theta°$ from north, measured clockwise. We write a true bearing of $\theta°$ as $\theta°$T, where $\theta°$ is an angle between 0° and 360°. It is customary to write the angle using three digits, so 0°T is written 000°T, 15°T is written 015°T, and so on.

For example, in the diagram opposite, the true bearing of A from O is 060°T, and the true bearing of B from O is 140°T.



Example 6

A tower is situated due north of a point A and due west of a point B. From A, the angle of elevation of the top of the tower is 18°. In addition, B (which is on the same level as A) is 52 metres from A and has a bearing of 064°T from A. Find, correct to one decimal place:

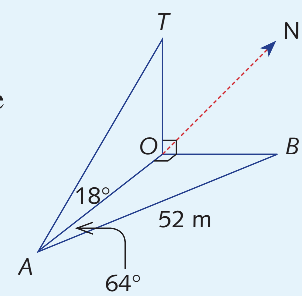
a the distance from A to the base of the tower

b the height of the tower

c the angle of elevation of the top of the tower from B.

Solution

Draw the tower OT and mark the point A level with the base of the tower. The line AO then points north. We can then mark all the given information on the diagram. The triangle AOT is vertical and triangle AOB is horizontal.



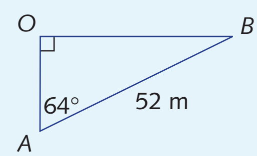
a In ΔAOB, $\cos 64° = \dfrac{OA}{52}$

so $OA = 52 \cos 64°$

$= 22.795 \ldots$ (Keep this in your calculator for part **b**.)

≈ 22.8 (Correct to one decimal place.)

A is approximately 22.8 metres from the base of the tower.



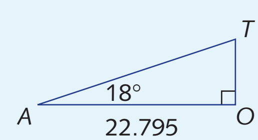
b In ΔAOT, $\tan 18° = \dfrac{OT}{OA}$

that is, $OT = OA \times \tan 18°$

$= 7.406 \ldots$ (Keep this in your calculator for part **c**.)

≈ 7.4 (Correct to one decimal place.)

The tower is approximately 7.4 metres high.



c In ΔTOB, $\tan\theta = \dfrac{OT}{OB}$

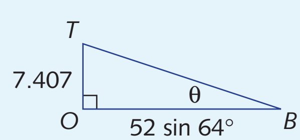
Now from ΔAOB, $OB = 52 \sin 64°$

Hence, $\tan\theta = \dfrac{OT}{52 \sin 64°}$

≈ 0.1585

so $\theta \approx 9.0°$ (Correct to one decimal place.)

The angle of elevation of the top of the tower from B is approximately 9.0°.

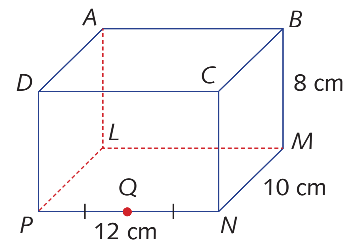


Do not re-enter a rounded result into your calculator; it is much more accurate to store the un-rounded number and use it in subsequent steps.

Exercise 12C

Example 5

1 In the rectangular prism shown opposite, find:



a BN

b $\angle BNM$ (correct to one decimal place)

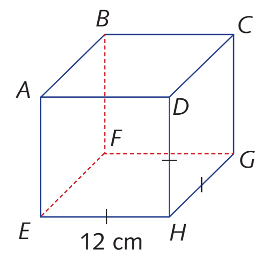
c BP

d the angle BPM (correct to one decimal place)

e MQ, where Q is the midpoint of PN

f the angle BQM (correct to one decimal place)

2 In the cube shown opposite, find:



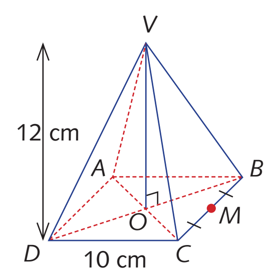
a CE

b $\angle CEG$ (correct to one decimal place)

c $\angle CBE$

d $\angle CEB$ (correct to one decimal place)

3 In the square pyramid shown opposite, find:



a AC **b** OC **c** VC

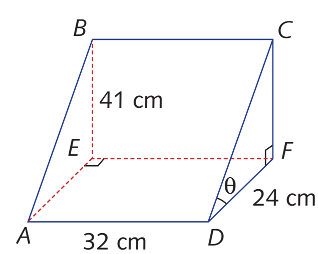
d $\angle VCO$ (correct to one decimal place)

e OM, where M is the midpoint of BC

f $\angle VMO$ (correct to one decimal place)

g $\angle VBM$ (correct to one decimal place)

4 $AEFD$ is a horizontal rectangle. $ABCD$ is a rectangle inclined at an angle θ to the horizontal. $AD = 32$ cm, $AE = 24$ cm and $BE = 41$ cm. Find, correct to one decimal place where necessary:



a DC

b AF

c $\angle CAF$

Example 6

5 The base of a tree is situated 50 metres due north of a point P. The angle of elevation of the top of the tree from P is 32°.

a Find the height of the tree, correct to one decimal place.

b Q is a point 100 metres due east of P. Find:

i the distance of Q from the base of the tree

ii the angle of elevation of the top of the tree from Q, correct to one decimal place

iii the bearing of the tree from Q, correct to one decimal place

6 Dillon and Eugene are both looking at a tower of height 35 metres. Dillon is standing due south of the tower and he measures the angle of elevation from the ground to the top of the tower to be 15°. Eugene is standing due east of the tower and he measures the angle of elevation from the ground to the top of the tower to be 20°. Find, correct to one decimal place:

a the distance Dillon is from the foot of the tower

b the distance Eugene is from the foot of the tower

c the distance between Dillon and Eugene

d the bearing of Dillon from Eugene

7 From a point A, a lighthouse is on a bearing of 026°T and the top of the lighthouse is at angle of elevation of 20.25°. From a point B, the lighthouse is on a bearing of 296°T and the top of the lighthouse is at angle of elevation of 10.20°. If A and B are 500 metres apart, find the height of the lighthouse, correct to the nearest metre.

8 From the top of a cliff that runs north–south, the angle of depression of a yacht, 200 metres out to sea and due east of the observer, is 20°. When the observer next looks at the yacht, he notices that it has sailed 150 metres parallel to the cliff.

a Find the height of the cliff, correct to the nearest metre.

b Find the distance the yacht is from the observer after it has sailed 150 metres parallel to the cliff, correct to the nearest metre.

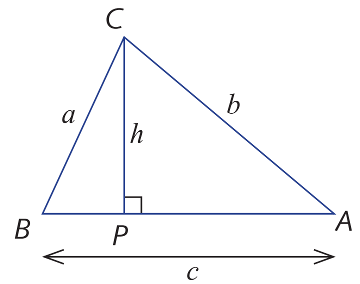
c Find the angle of depression of the yacht from the top of the cliff when it is in its new position, correct to the nearest degree.

9 A mast is held in position by means of two taut ropes running from the ground to the top of the mast. One rope is of length 40 metres and makes an angle of 58° with the ground. Its anchor point with the ground is due south of the mast. The other rope is 50 metres long and its anchor point is due east of the mast. Find the distance, correct to the nearest metre, between the two anchor points.

12D The sine rule

In many situations we encounter triangles that are not right-angled. We can use trigonometry to deal with these triangles as well. One of the two key formulas for doing this is known as the sine rule.

We begin with an acute-angled triangle, ABC, with side lengths a, b and c, as shown below. (It is standard to write a lower case letter on a side and the corresponding upper case letter on the angle opposite that side.) Drop a perpendicular, CP, of length h, from C to AB.



In ΔAPC we have $\sin A = \dfrac{h}{b}$, so $h = b\sin A$.

Similarly, in ΔCPB we have $\sin B = \dfrac{h}{a}$, so $h = a\sin B$.

Equating these expressions for h, we have:

$$b\sin A = a\sin B$$

which we can write as:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

The same result holds for the side c and angle C, so we can write:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

This is known as the **sine rule**.

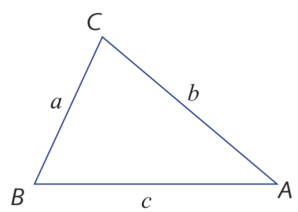
In words, this says 'any side of a triangle over the sine of the opposite angle equals any other side of the triangle over the sine of its opposite angle'.

This result also holds in an obtuse-angled triangle. We will look at that case later.

The sine rule

In any triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



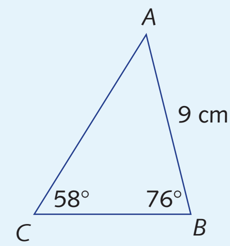
For example, the sine rule can be used to find an unknown length of a side of a triangle when a side length and the angles are known. This is closely related to the AAS congruence test.

Example 7

In ΔABC = AB = 9 cm, $\angle ABC = 76°$ and $\angle ACB = 58°$.

Find, correct to two decimal places:

a AC **b** BC



Solution

a Apply the sine rule:

$$\frac{AC}{\sin 76°} = \frac{9}{\sin 58°}$$

$$AC = \frac{9\sin 76°}{\sin 58°}$$

$$\approx 10.30 \text{ cm}$$

b To find BC, we need the angle $\angle CAB$ opposite it.

$$\angle CAB = 180° - 58° - 76°$$

$$= 46°$$

Then by the sine rule:

$$\frac{BC}{\sin 46°} = \frac{9}{\sin 58°}$$

$$BC = \frac{9\sin 46°}{\sin 58°}$$

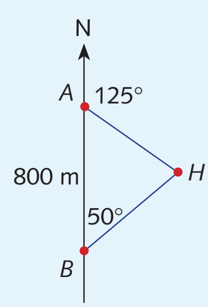
$$\approx 7.63 \text{ cm}$$

Example 8

From two points A and B, which are 800 metres apart on a straight north–south road, the bearings of a house are 125°T and 050°T, respectively. Find how far each point is from the house, correct to the nearest metre.

Solution

We draw a diagram to represent the information.



We can find the angles in ΔAHB.

$$\angle HAB = 180° - 125°$$

$$= 55°$$

and $\angle AHB = 180° - 50° - 55°$

$$= 75°$$

Apply the sine rule to ΔABH:

$$\frac{BH}{\sin 55°} = \frac{800}{\sin 75°}$$

$$BH = \frac{800\sin 55°}{\sin 75°}$$

$$\approx 678.44 \text{ m}$$

Similarly, $\dfrac{AH}{\sin 50°} = \dfrac{800}{\sin 75°}$

and so $AH = \dfrac{800\sin 50°}{\sin 75°}$

$$\approx 634.45 \text{ m}$$

Thus, A and B are approximately 634 metres and 678 metres from the house, respectively.

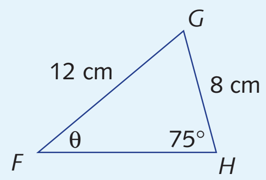
Finding angles

The sine rule can also be used to find angles in a triangle, provided that one of the known sides is opposite a known angle.

At this stage we can only deal with acute angled triangles.

Example 9

Find the angle θ in the triangle FGH, correct to the nearest degree.



Solution

Apply the sine rule to ΔFGH:

$$\frac{8}{\sin\theta} = \frac{12}{\sin 75°}$$

To make the algebra easier, take the reciprocal of both sides:

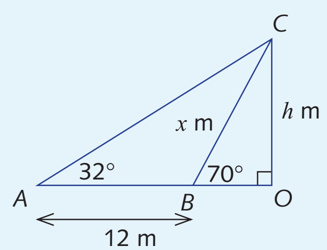
$$\frac{\sin\theta}{8} = \frac{\sin 75°}{12}$$

$$\begin{aligned}\text{Hence, } \sin\theta &= \frac{8\sin 75°}{12}\\ &= 0.6440\ldots\end{aligned}$$

Hence, $\theta \approx 40°$ (Correct to the nearest degree.)

Example 10

Find the length of OC in the diagram, correct to one decimal place.



Solution

$OC = h$ m and $BC = x$ m.

$\angle ACB + 32° = 70°$ (exterior angle of ΔABC)

The angle $\angle ACB = 38°$

Applying the sine rule:

$$\frac{x}{\sin 32°} = \frac{12}{\sin 38°}$$

$$x = \frac{12\sin 32°}{\sin 38°}$$

$$= 10.3287\ldots \quad \text{(Keep this in your calculator.)}$$

In triangle BCO:

$$\sin 70° = \frac{h}{x}$$

so

$$h = x \times \sin 70°$$

$$\approx 9.7 \quad \text{(Correct to one decimal place.)}$$

The length OC is approximately 9.7 m.

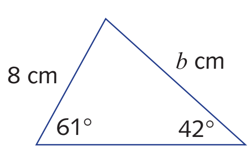
Note: Alternatively, h can be calculated directly as $\dfrac{12\sin 32°}{\sin 38°} \times \sin 70°$.

Exercise 12D

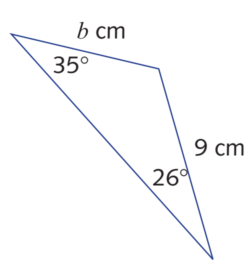
Example 7

1 Find the value of b, correct to two decimal places.

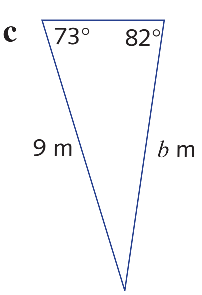
a



b

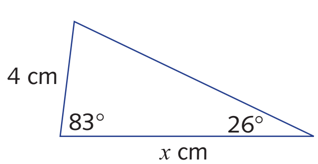


c

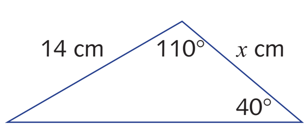


2 Find the value of x, correct to two decimal places.

a



b



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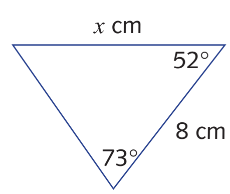


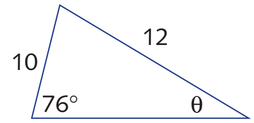
3 **a** In ΔABC, $A = 62°$, $B = 54°$ and $a = 8$. Find b, correct to two decimal places.

b In ΔABC, $B = 47°$, $C = 82°$ and $b = 10$. Find a, correct to two decimal places.

Example 9

4 Find the value of θ, correct to the nearest degree.

a



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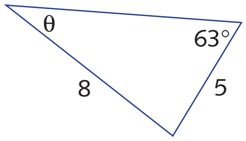

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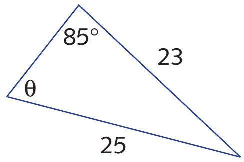
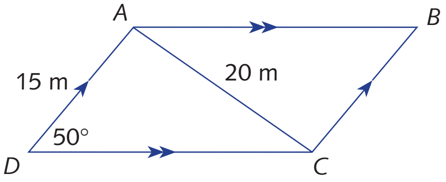


5 In ΔABC, $A = 71°$, $a = 18$ cm and $b = 14$ cm. Find, correct to two decimal places:

a B **b** C **c** c

6 $ABCD$ is a parallelogram with $\angle ADC = 50°$. The shorter diagonal, AC, is 20 m, and $AD = 15$ m. Find $\angle ACD$ and hence the length of the side DC, correct to two decimal places.



Example 8

7 Two hikers, Paul and Sayo, are both looking at a distant landmark. From Paul, the bearing of the landmark is 222°T and, from Sayo, the bearing of the landmark is 300°T. If Sayo is standing 800 m due south of Paul, find, correct to the nearest metre:

a the distance from Paul to the landmark

b the distance from Sayo to the landmark

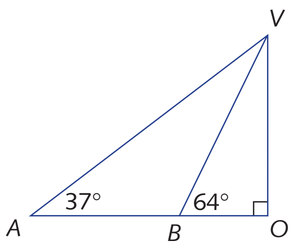
8 An archaeologist wishes to determine the height of an ancient temple. From a point A at ground level, she measures the angle of elevation of V, the top of the temple, to be 37°.

She then walks 100 m towards the temple to a point B. From here, the angle of elevation of V from ground level is 64°. Find:

a $\angle AVB$

b VB, correct to two decimal places

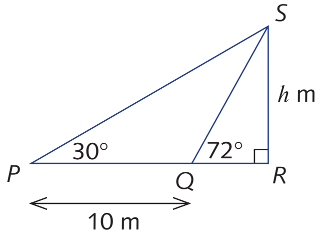
c OV, the height of the temple, to the nearest metre



9 A hillside is inclined at 26° to the horizontal. From the bottom of the hill, Alex observes a vertical tree whose base is 40 m up the hill from the point where Alex is standing. If the angle of elevation of the top of the tree is 43° from the point where Alex is standing, find the height of the tree, correct to the nearest metre.

Example 10

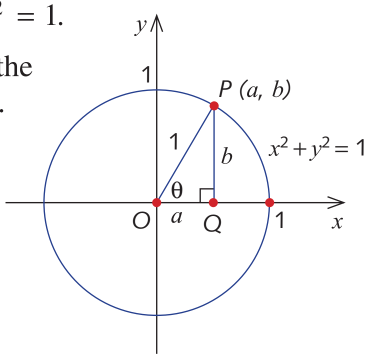
10 Find h, correct to the nearest centimetre.



12E Trigonometric ratios of obtuse angles

We have seen that we can use the sine rule to find sides and angles in acute-angled triangles. What happens when one of the angles is obtuse? We can extend our definition of the basic trigonometric functions to obtuse angles by using coordinate geometry.

We begin by drawing a circle of radius 1 in the Cartesian plane, with its centre at the origin. The equation of the circle is $x^2 + y^2 = 1$.



Take a point $P(a, b)$ on the circle in the first quadrant and form the right-angled triangle POQ with O at the origin. Let $\angle POQ$ be θ.

$$\cos\theta = \frac{OQ}{OP} = \frac{a}{1} = a, \text{ and}$$

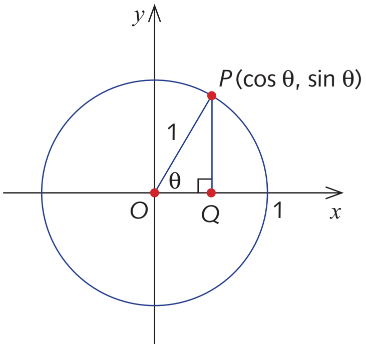
$$\sin\theta = \frac{QP}{OP} = \frac{b}{1} = b$$

But a is the x-coordinate of P and b is the y-coordinate of P.

Hence, the coordinates of the point P are $(\cos\theta, \sin\theta)$.

We can now turn this idea around and say that if θ is the angle between OP and the positive x-axis, then:

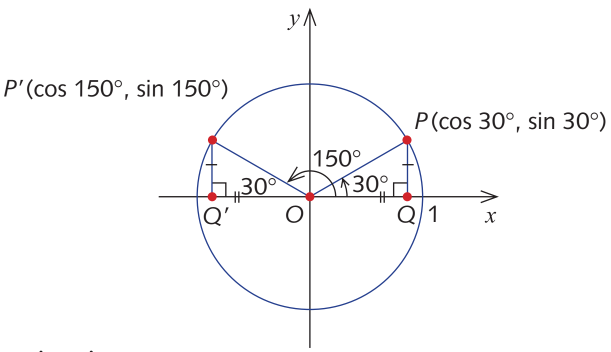
- the cosine of θ is defined to be the x-coordinate of the point P on the unit circle
- the sine of θ is defined to be the y-coordinate of the point P on the unit circle.



This definition can be applied to all angles θ, but in this chapter we will restrict the angle θ to $0° \le \theta \le 180°$.

Now take θ to be 30°, so P has coordinates $(\cos 30°, \sin 30°)$.

Suppose that we move the point P around the circle to P' so that P' makes an angle of 150° with the positive x-axis. (Recall that 30° and 150° are supplementary angles.)



The coordinates of P' are $(\cos 150°, \sin 150°)$. But we can see that triangles OPQ and $OP'Q'$ are congruent, so the y-coordinates of P and P' are the same. That is:

$$\sin 150° = \sin 30°$$

The x-coordinates have the same magnitude but opposite sign, so:

$$\cos 150° = -\cos 30°$$

From this example, we can see the following rules.

Supplementary angles

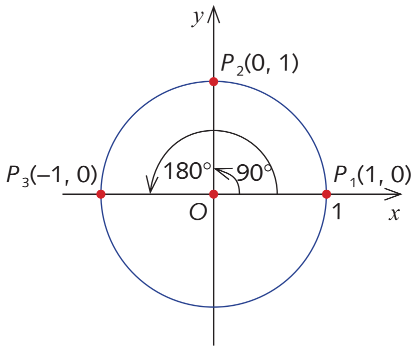
- The sines of two supplementary angles are the same.
- The cosines of two supplementary angles are opposite in sign.
- In symbols:

  $\sin\theta = \sin(180^\circ - \theta)$ and $\cos\theta = -\cos(180^\circ - \theta)$

We can extend the definition of sine and cosine to angles beyond 180°. This will be done later in this book.

The angles 0°, 90° and 180°

We have defined $\cos\theta$ and $\sin\theta$ as the x- and y-coordinates of the point P on the unit circle. Taking the axis intercepts P_1, P_2 and P_3 from the unit circle diagram to the right, we obtain the following table of values. These values should be memorised.



θ	0°	90°	180°
sin θ	0	1	0
cos θ	1	0	–1

Example 11

Find the exact value of:

a sin 150° **b** cos 150° **c** sin 120° **d** cos 120°

Solution

a
$$\begin{aligned}\sin 150^\circ &= \sin(180-150)^\circ\\ &= \sin 30^\circ\\ &= \frac{1}{2}\end{aligned}$$

b
$$\begin{aligned}\cos 150^\circ &= -\cos(180-150)^\circ\\ &= -\cos 30^\circ\\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

c
$$\begin{aligned}\sin 120^\circ &= \sin(180-120)^\circ\\ &= \sin 60^\circ\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

d
$$\begin{aligned}\cos 120^\circ &= -\cos(180-120)^\circ\\ &= -\cos 60^\circ\\ &= -\frac{1}{2}\end{aligned}$$

Note: You can verify these results using your calculator.

Example 12

Find, correct to the nearest degree, the acute and obtuse angle whose sine is:

a approximately 0.7431 **b** $\dfrac{1}{\sqrt{2}}$ **c** $\dfrac{\sqrt{3}}{2}$

Solution

a If $\sin\theta = 0.7431$ and θ is acute, then the calculator gives $\theta = \sin^{-1} 0.7431 \approx 48°$. Hence, the solutions are 48° and 132°, correct to the nearest degree, because 132° is the supplement of 48°.

b If $\sin\theta = \dfrac{1}{\sqrt{2}}$

$\theta = 45°$ or $\theta = 180° - 45°$

That is, $\theta = 45°$ or $\theta = 135°$

c If $\sin\theta = \dfrac{\sqrt{3}}{2}$

$\theta = 60°$ or $\theta = 180° - 60°$

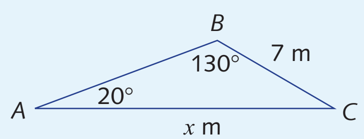
That is, $\theta = 60°$ or $\theta = 120°$

More on the sine rule

The sine rule also holds in obtuse-angled triangles. A proof is given in question **7** of Exercise 12E. We now see how to apply the sine rule in obtuse-angled triangles.

Example 13

Find the value of x, correct to one decimal place.



Solution

Apply the sine rule to ΔABC:

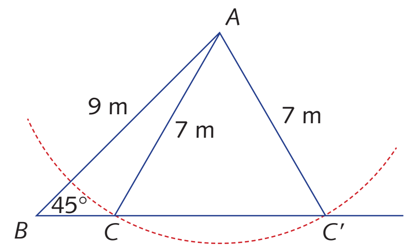
$$\frac{x}{\sin 130°} = \frac{7}{\sin 20°}$$

$$x = \frac{7\sin 130°}{\sin 20°} \quad (\sin 130° = \sin 50°)$$

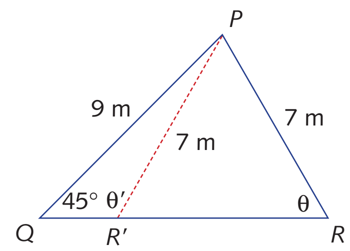
$$\approx 15.7 \quad \text{(Correct to one decimal place.)}$$

The ambiguous case

You are given the following information about a triangle.
A triangle has side lengths 9 m and 7 m and an angle of 45° between the 9 m side and the unknown side.
How many triangles satisfy these properties?



In the diagram, the triangles ABC and ABC' both have sides of length 9 m and 7 m, and both contain an angle of 45° opposite the side of length 7 m. Despite this, the triangles are different. (Recall that the included angle was required in the SAS congruence test.)



Hence, given the data that a triangle $\angle PQR$ has $PQ = 9$ m, $\angle PQR = 45°$ and $PR = 7$ m, the angle opposite PQ is not determined. There are two non-congruent triangles that satisfy the given data.

Let $PR' = 7$ m so that $\theta = \angle PRQ$ is acute and $\theta' = \angle PR'Q$ is obtuse.

Applying the sine rule to the ΔPRQ, we have:

$$\frac{9}{\sin\theta} = \frac{7}{\sin 45°}$$

$$\sin\theta = \frac{9\sin 45°}{7}$$

$$= 0.9091\ldots$$

The calculator tells us that $\sin^{-1}(0.9091\ldots)$ is approximately 65°. Hence $\theta \approx 65°$.

The triangle $PR'R$ is isosceles, so $\angle PR'R$ is 65° and $\theta' = 180° - 65° = 115°$. Since $\sin 65° = \sin 115°$, the triangle $PR'Q$ also satisfies the given data.

Exercise 12E

1 Copy and complete:

a $\sin 115° = \sin$ ___ **b** $\cos 123° = -\cos$ ___

c $\sin 138° = \sin$ ___ **d** $\cos 95° = -\cos$ ___

Example 11

2 Find the exact value of:

a $\sin 135°$ **b** $\cos 135°$

Example 12

3 a Find the acute and the obtuse angle whose sine is $\frac{1}{2}$.

b Find, correct to the nearest degree, two angles whose sine is approximately 0.5738.

c Find, correct to the nearest degree, an angle whose cosine is approximately -0.8746.

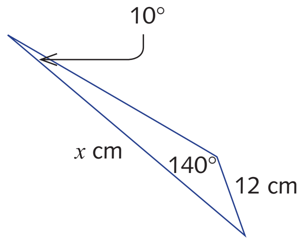
4 Copy and complete:

θ	30°	120°	150°	90°	135°
$\sin\theta$		$\frac{\sqrt{3}}{2}$			$\frac{1}{\sqrt{2}}$
$\cos\theta$			$-\frac{\sqrt{3}}{2}$	0	

Example 13

5 Use the sine rule to find the value of x, correct to two decimal places.

a



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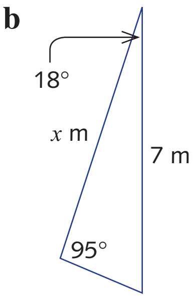
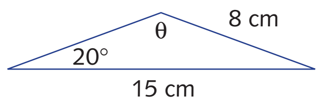


6 Given that θ is an obtuse angle, find its value, correct to the nearest degree.

a



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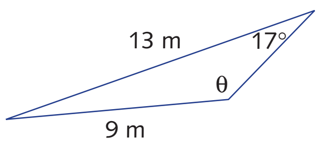
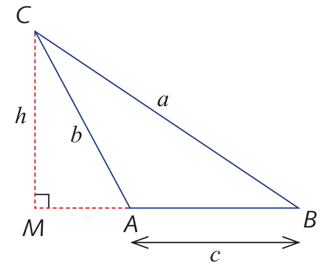


7 Suppose that $\angle A$ in triangle ABC is obtuse.



a Explain why $\sin\angle A = \sin\angle CAM$.

b Use triangle ACM to find a formula for h in terms of A and b.

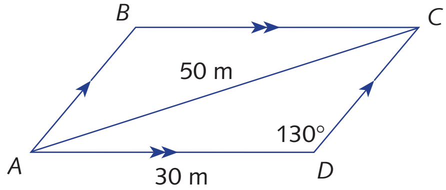
c Use triangle BCM to find a formula for h in terms of B and a.

d Deduce that $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$.

That is, we have proved the sine rule holds in obtuse-angled triangles.

8 The angle between the two sides of a parallelogram is 93°. If the longer side has length 12 cm and the longer diagonal has length 14 cm, find the angle between the long diagonal and the short side of the parallelogram, correct to the nearest degree.

9 $ABCD$ is a parallelogram. $\angle CDA = 130°$, the long diagonal AC is 50 m and $AD = 30$ m. Find the length of the side DC, correct to one decimal place.



10 Sonia starts at O and walks 600 metres due east to point A. She then walks on a bearing of 250°T to point B, 750 metres from O. Find:

a the bearing of B from O, correct to the nearest degree

b the distance from A to B, correct to the nearest metre

11 A point M is one kilometre due east of a point C. A hill is on a bearing of 028°T from C and is 1.2 km from M. Find:

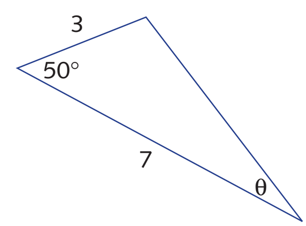
a the bearing of the hill from M, correct to the nearest degree

b the distance, correct to the nearest metre, between C and the hill

12F The cosine rule

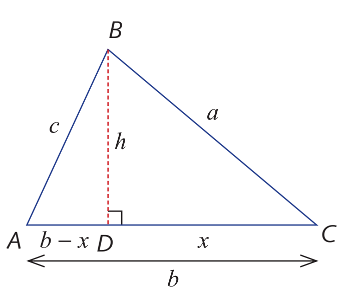
We know, from the SAS congruence test, that a triangle is completely determined if we are given two of its sides and the included angle. If we want to know the third side and the two other angles, the sine rule does not help us.

You can see from the diagram that there is not enough information to apply the sine rule. This is because the known angle is not opposite one of the known sides.



Fortunately there is another rule called the **cosine rule** which we can use in this situation.

Suppose that ABC is a triangle and that the angles A and C are acute. Drop a perpendicular from B to AC and mark the side lengths as shown in the diagram.



In ΔBDA, Pythagoras' theorem tells us that:

$$c^2 = h^2 + (b - x)^2$$

Also in ΔCBD, by Pythagoras' theorem we have:

$$h^2 = a^2 - x^2$$

Substituting this expression for h^2 into the first equation and expanding:

$$\begin{aligned} c^2 &= a^2 - x^2(b - x)^2 \\ &= a^2 - x^2 + b^2 - 2bx + x^2 \\ &= a^2 + b^2 - 2bx \end{aligned}$$

Finally, from ΔCBD, we have $\dfrac{x}{a} = \cos C$. That is, $x = a \cos C$ and so:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

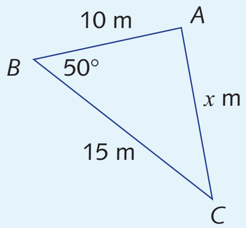
Notes:

- By relabelling the sides and angle, we could also write $a^2 = b^2 + c^2 - 2bc \cos A$ and $b^2 = a^2 + c^2 - 2ac \cos B$.
- If $C = 90°$, then, since $\cos 90° = 0$, we obtain Pythagoras' theorem. Thus the cosine rule can be thought of as 'Pythagoras' theorem with a correction term'.
- The cosine rule is also true if C is obtuse. This is proven in the exercises.

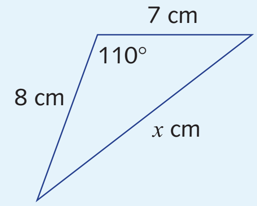
Example 14

Find the value of x, correct to one decimal place.

a



b



Solution

a Applying the cosine rule to ΔABC:

$$x^2 = 10^2 + 15^2 - 2 \times 10 \times 15 \times \cos 50°$$
$$= 132.16 \ldots$$

so $x \approx 11.5$

b Applying the cosine rule:

$$x^2 = 7^2 + 8^2 - 2 \times 7 \times 8 \times \cos 110°$$
$$= 151.30 \ldots$$

so $x \approx 12.3$ (Correct to one decimal place.)

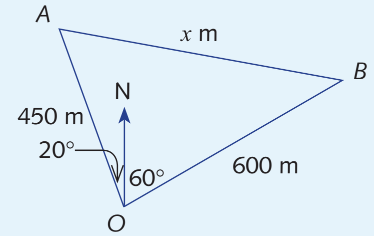
Note that in Example 14a, $x^2 < 10^2 + 15^2$ since $\cos 50°$ is positive. In Example 14b, $x^2 > 7^2 + 8^2$ since $\cos 110°$ is negative.

Example 15

A tower at A is 450 metres from O on a bearing of 340°T and a tower at B is 600 metres from O on a bearing of 060°T. Find, correct to the nearest metre, the distance between the two towers.

Solution

We draw a diagram to represent the information.



Now $\angle AOB = 80°$. Let $AB = x$ m.

Applying the cosine rule:

$$x^2 = 450^2 + 600^2 - 2 \times 450 \times 600 \times \cos 80°$$
$$= 468\,729.98 \ldots$$

that is, $x \approx 684.63 \ldots$

Hence, the towers are 685 metres apart, correct to the nearest metre.

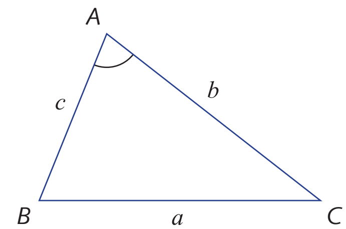
The cosine rule

In any triangle ABC:

$a^2 = b^2 + c^2 - 2bc\cos A$,

where A is the angle opposite a.

The cosine rule can be used to find the length of the third side of a triangle when the lengths of two sides and the size of the included angle are known.

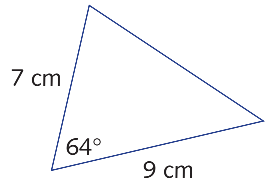


Exercise 12F

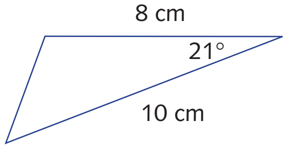
Example 14

1 In each triangle, calculate the unknown side length, giving your answer correct to two decimal places.

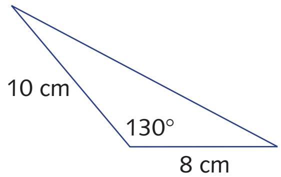
a



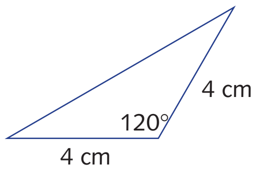
b



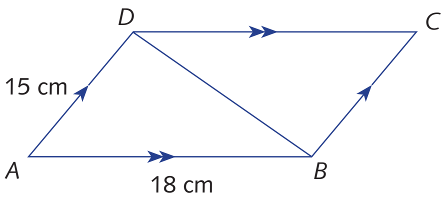
c



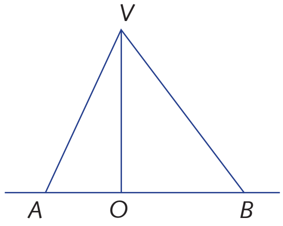
d



2 $ABCD$ is a parallelogram with sides 15 cm and 18 cm. The angle at A is 65°. Find the length of the shorter diagonal, correct to two decimal places.



3 A vertical pole OV is being held in position by two ropes, VA and VB. If $VA = 6$ m, $VB = 6.5$ m, $\angle OVB = 32°$ and $\angle OVA = 27°$ find, correct to one decimal place, the distance AB.



Example 15

4 A ship is 300 km from port on a bearing of 070°T. A second ship is 400 km from the same port and on a bearing of 140°T. How far apart, correct to the nearest kilometre, are the two ships?

5 A pilot flies a plane on course for an airport 600 km away. Unfortunately, due to an error, his bearing is out by 2°. After travelling 700 km he realises he is off course. How far from the airport is he, correct to the nearest kilometre?

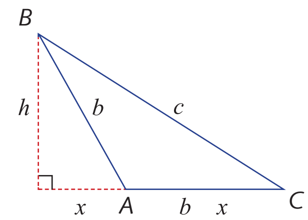
6 A rhombus $PQRS$ has side lengths 8 m, and contains an angle of 128°.

a Find the length of the longer diagonal, correct to two decimal places.

b Find the length of the shorter diagonal, correct to two decimal places.

c Find the area of the rhombus, correct to two decimal places.

7 Prove the cosine rule when the included angle, A is obtuse.



12G Finding angles using the cosine rule

The SSS congruence test tells us that once three sides of a triangle are known, the angles are uniquely determined. The question is, how do we find them?

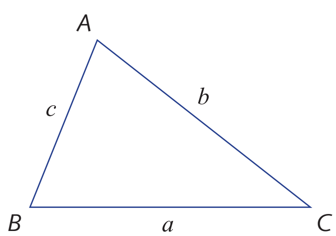
Given three sides of a triangle, we can substitute the information into the cosine rule and rearrange to find the cosine of one of the angles and hence the angle.

If you prefer, you can learn or derive another form of the cosine rule, with $\cos C$ as the subject.

Rearranging $c^2 = a^2 + b^2 - 2ab\cos C$ we have:

$$2ab\cos C = a^2 + b^2 - c^2$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

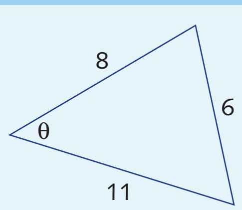


Example 16

A triangle has side lengths 6 cm, 8 cm and 11 cm. Find the smallest angle in the triangle.

Solution

The smallest angle in the triangle is opposite the smallest side.



Applying the cosine rule:

$$6^2 = 8^2 + 11^2 - 2 \times 8 \times 11 \times \cos\theta$$

$$\cos\theta = \frac{8^2 + 11^2 - 6^2}{2 \times 8 \times 11}$$

$$= \frac{149}{172}$$

and so $\theta \approx 32.2°$ (Correct to one decimal place.)

There is no ambiguous case when we use the cosine rule to find an angle. In the following example, the unknown angle is obtuse.

Example 17

In ABC, $a = 6$, $b = 20$ and $c = 17$. Find the size of $\angle ABC$, correct to one decimal place.

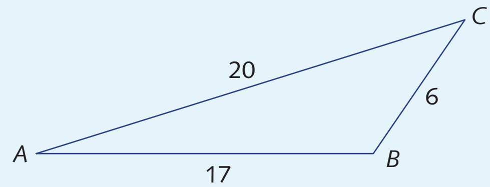
Solution

Applying the cosine rule:

$$20^2 = 6^2 + 17^2 - 2 \times 6 \times 17 \cos B$$

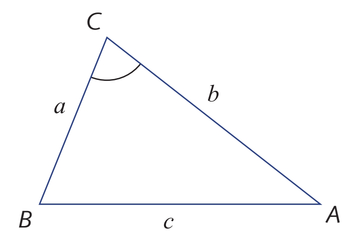
$$\cos B = \frac{6^2 + 17^2 - 20^2}{2 \times 6 \times 17} = \frac{-75}{204}$$

and so $B = 111.6°$ (Correct to one decimal place.)



Using the cosine rule to find an angle

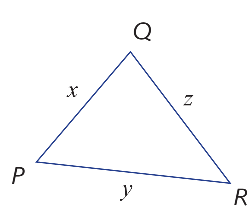
- The cosine rule can be used to determine the size of any angle in a triangle where the three side lengths are known.
- In any triangle $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$, where C is the opposite angle C.



Exercise 12G

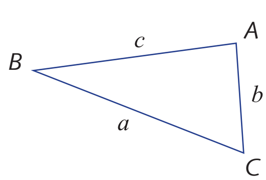
1 Copy and complete the statement of the cosine rule.

a



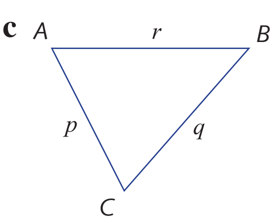
$x^2 =$

b



$b^2 =$

c

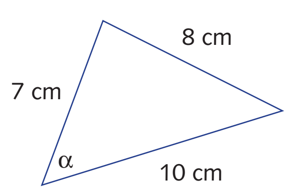


$p^2 =$

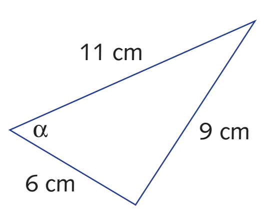
Example 16

2 Calculate α, giving the answer correct to one decimal place.

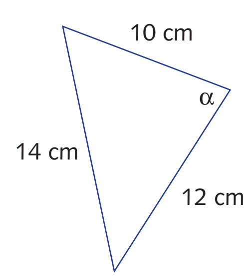
a



b



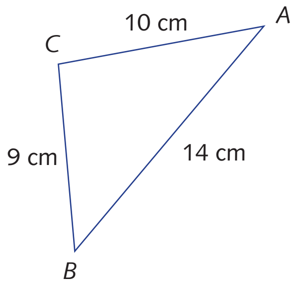
c



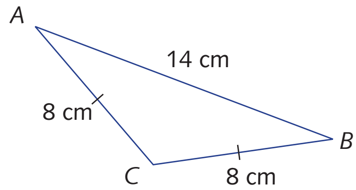
Example 17

3 Find all angles. Give answers correct to one decimal place.

a



b



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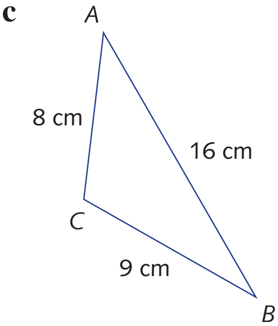


4 Calculate the size of the smallest angle of the triangle whose side lengths are 30 mm, 70 mm and 85 mm. Give your answer correct to one decimal place.

5 A triangle has sides of length 9 cm, 13 cm and 18 cm. Calculate the size of the largest angle, correct to one decimal place.

6 Find all the angles of a triangle whose sides are in the ratio 4 : 8 : 11, to the nearest degree.

7 A parallelogram has sides of length 12 cm and 18 cm. The longer diagonal has length 22 cm. Find, correct to one decimal place, the size of the obtuse angle between the two sides.

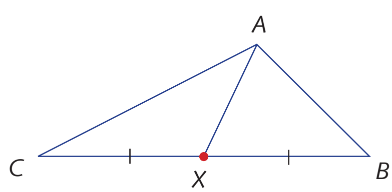
8 In ΔABC, $AB = 6$ cm, $AC = 10$ cm, $BC = 14$ cm and X is the midpoint of side BC.

a Find, correct to one decimal place:

i $\angle ACB$ **ii** the length AX

AX is called a **median** of the triangle. A median is the line segment from a vertex to the midpoint of the opposite side.

b Find the length of the other two medians, correct to one decimal place.



12H Area of a triangle

If we know two sides and an included angle of a triangle, then by the SAS congruence test the area is determined. We will now find a formula for the area.

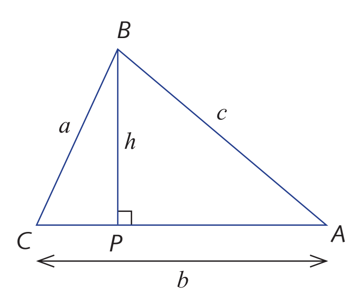
In ΔABC on the right, drop a perpendicular from A to BC. Then in ΔAPC:

$$\frac{h}{a} = \sin C$$

That is:

$$h = a\sin C$$

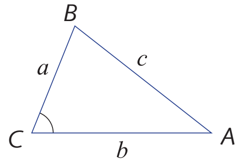
Hence, the area of $\Delta ABC = \frac{1}{2}bh = \frac{1}{2}ab\sin C$



Thus, the area of a triangle is half the product of any two sides times the sine of the included angle.

Area of a triangle

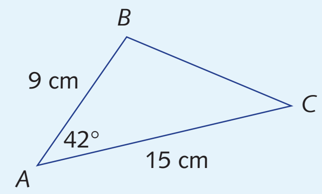
Area $= \frac{1}{2}ab\sin C$, where C is included angle.



Note that if $C = 90°$, then since $\sin 90° = 1$, the area formula becomes $\frac{1}{2}ab$. The formula also applies when the angle is obtuse. This is proved in the exercises.

Example 18

Calculate the area of the triangle ABC, correct to one decimal place.



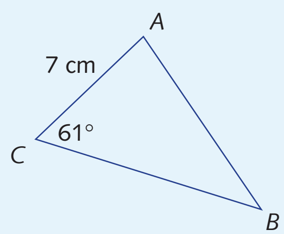
Solution

$$\text{Area of } \Delta ABC = \frac{1}{2} \times 9 \times 15 \times \sin 42°$$
$$\approx 45.2 \quad \text{(Correct to one decimal place.)}$$

So the area of the triangle is 45.2 cm^2.

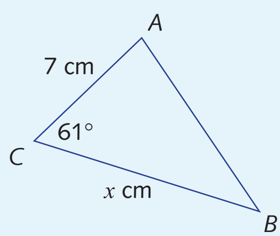
Example 19

The triangle shown has area 34 cm^2. Find the length of BC, correct to two decimal places.



Solution

Let $BC = x$ cm



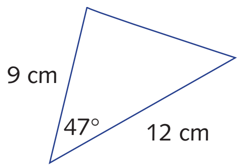
$$34 = \frac{1}{2} \times 7 \times x \times \sin 61°$$
$$x = \frac{68}{7\sin 61°}$$
$$\approx 11.11 \qquad \text{(Correct to two decimal places.)}$$
$$BC \approx 11.11 \text{ cm}$$

Exercise 12H

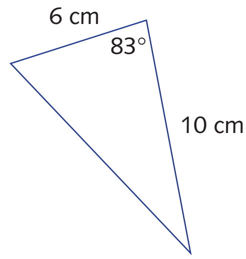
Example 18

1 Calculate each area, correct to one decimal place.

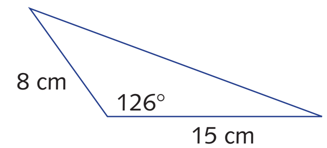
a



b

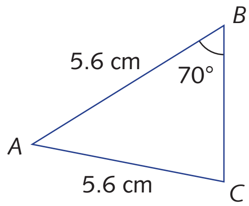


c



2 Calculate each area, correct to two decimal places.

a



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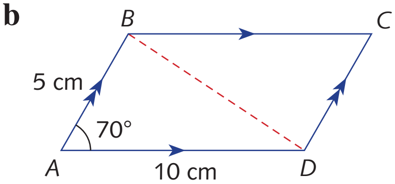
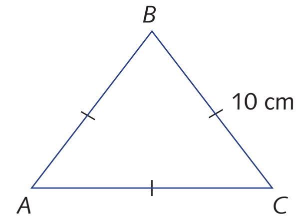


c

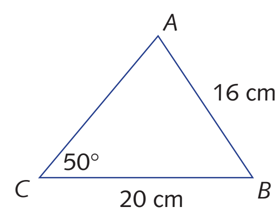


3 In ΔABC shown opposite, $\angle CAB$ is an acute angle.

a Use the sine rule to find $\angle CAB$, correct to one decimal place.

b Find $\angle ABC$, correct to one decimal place.

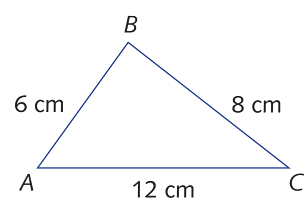
c Find the area of the triangle, correct to the nearest square centimetre.



4 In ΔABC shown opposite:

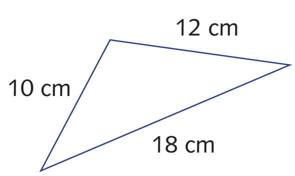
a use the cosine rule to find $\angle BAC$

b find the area of the triangle, correct to the nearest square centimetre

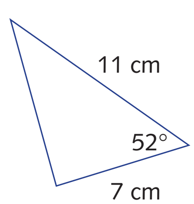


5 Calculate the area of each triangle, correct to the nearest square centimetre.

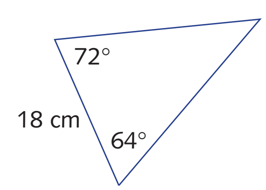
a



b



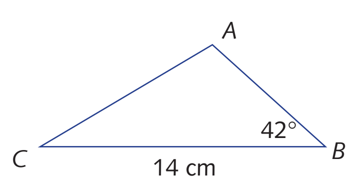
c



Example 19

6 In ΔABC shown opposite, the area of the triangle is 40 cm^2. Find, correct to one decimal place:

a AB **b** AC **c** $\angle ACB$



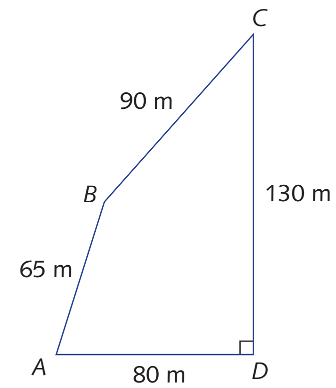
7 An acute-angled triangle of area 60 cm^2 has side lengths of 16 cm and 20 cm. What is the magnitude of the included angle, correct to the nearest degree?

8 An irregular block of land, $ABCD$, has dimensions shown opposite. Calculate, correct to one decimal place:

a the length AC

b $\angle ABC$

c the area of the block



9 $ABCDE$ is a regular pentagon with side lengths 10 cm. Diagonals AD and AC are drawn. Find:

a $\angle AED$

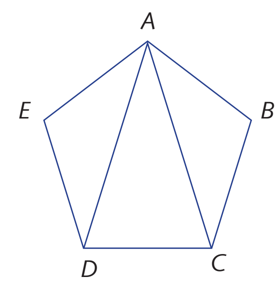
b the area of ΔADE, correct to two decimal places

c AD, correct to two decimal places

d $\angle ADE$ **e** $\angle ADC$ **f** $\angle DAC$

g the area of ΔADC, correct to two decimal places

h the area of the pentagon, correct to two decimal places



10 A quadrilateral has diagonals of length 12 cm and 18 cm. If the angle between the diagonals is 65°, find the area of the quadrilateral, correct to the nearest square centimetre.

11 The sides of a triangle ABC are enlarged by a factor, k. Use the area formula to show that the area is enlarged by the factor, k^2.

12 Prove that the formula Area $= \frac{1}{2} ab \sin C$ gives the area of a triangle when C is obtuse.

13 A triangle has sides of length 8 cm, 11 cm and 15 cm.

a Find the size of the smallest angle in the triangle, correct to two decimal places.

b Calculate, correct to two decimal places, the area of the triangle.

c Calculate the perimeter of the triangle.

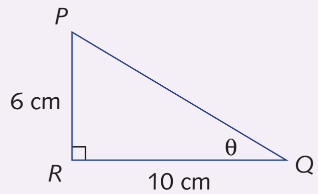
d Let s = half the perimeter of the triangle. The area of the triangle can be found using Heron's formula: Area $= \sqrt{s(s-a)(s-b)(s-c)}$, where a, b and c are the lengths of the three sides. Use this formula to calculate the area of the triangle, correct to two decimal places.

e Check that your answers to parts **b** and **d** are the same.

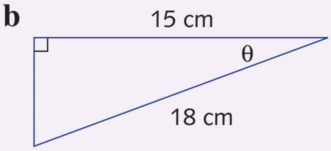
Review exercise

1 Calculate the value of the pronumeral in each triangle. Give all side lengths correct to two decimal places and all angles correct to one decimal place.

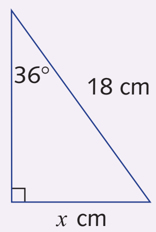
a



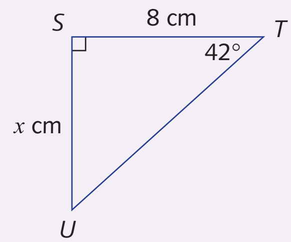
b



c

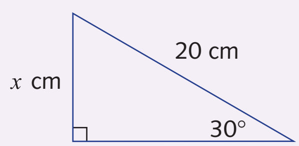


d

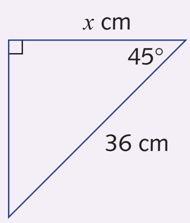


2 Find the exact value of the pronumeral in each triangle.

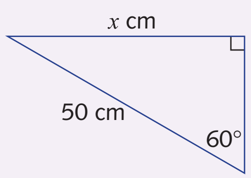
a



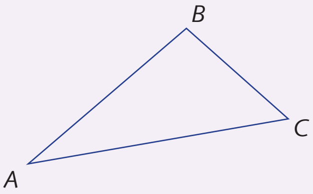
b



c



3 $AB = 8$ cm, $BC = 6$ cm and $AC = 12$ cm. Find the magnitude of each of the angles of triangle ABC correct to one decimal place.



4 A triangular region is enclosed by straight fences of lengths 42.8 metres, 56.6 metres and 72.1 metres.

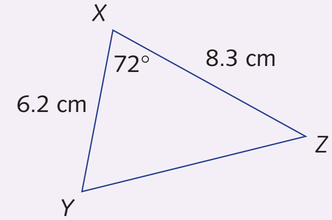
a Find the angle between the 42.8 m and the 56.6 m fences, correct to the nearest degree.

b Find the area of the region, correct to the nearest square metre.

5 In a triangle ABC, $\sin A = \frac{1}{8}$, $\sin B = \frac{3}{4}$ and $a = 8$. Find, using the sine rule, the value of b.

6 In a triangle, ABC, $a = 5$, $b = 6$ and $\cos C = \frac{1}{5}$. Find c.

7 Find the area of triangle XYZ, correct to two decimal places.



8 For a triangle ABC, $AC = 16.2$ cm, $AB = 18.6$ cm and $\angle ACB = 60°$. Find, correct to one decimal place:

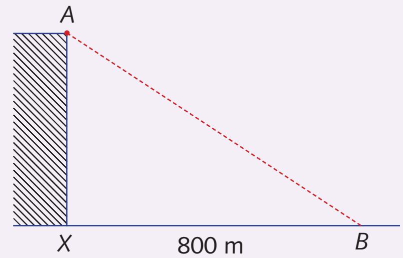
a $\angle ABC$

b $\angle BAC$

c the length of CB

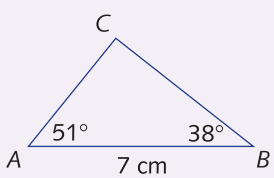
d the area of the triangle

9 The angle of depression from a point A to a ship at point B is 10°. If the distance BX from B to the foot of the cliff at X is 800 m, find the height of the cliff, correct to the nearest metre.

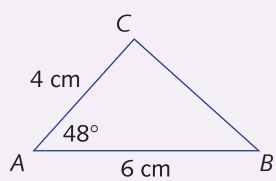


10 Calculate the lengths of the unknown sides and the sizes of the unknown angles, correct to two decimal places.

a



b



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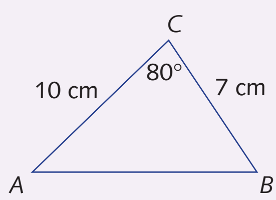
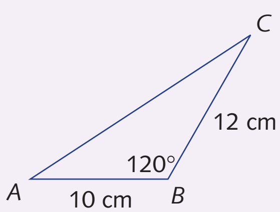
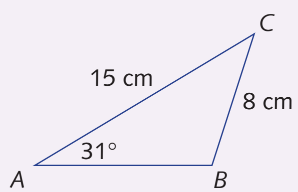


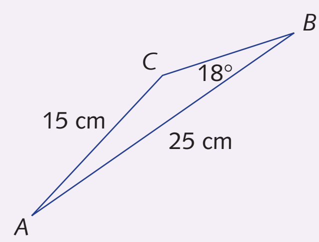
d



e

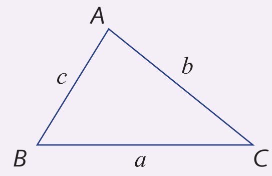


f



Challenge exercise

1 Write down two formulas for the area of triangle ABC and deduce the sine rule from those two formulas.



2 Simi is standing 200 metres due east of Ricardo. From Ricardo, the angle of elevation from the ground to the top of a building due north of Ricardo is 12°. From Simi, the angle of elevation from the ground to the top of the building is 9°.

a Let the height of the building be h metres. Express the following in terms of h:

i the distance from Ricardo to the foot of the building

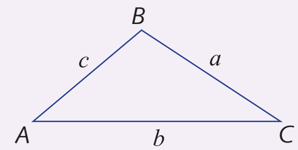
ii the distance from Simi to the foot of the building

b Use your answers to part **a** and Pythagoras' theorem to find the height of the building correct to one decimal place.

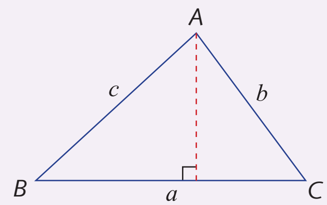
c On what bearing is the building from Simi?

3 For triangle ABC, show that

$$\text{Area} = \frac{a^2 \sin B \sin C}{2 \sin A}$$



4 Here is an alternative proof of the cosine rule. Assume ΔABC is acute-angled.



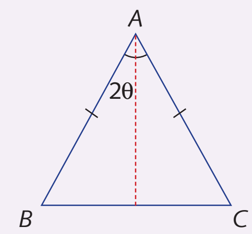
a Prove that $a = b\cos C + c\cos B$.

b Write down corresponding results for b and c.

c Show that $a^2 = a(b\cos C + c\cos B)$ and, using corresponding results for b^2 and c^2, prove the cosine rule.

d Check that a similar proof works for an obtuse-angled triangle.

5 ABC is an isosceles triangle, with $AB = AC = 1$. Suppose $\angle BAC = 2\theta$.



a Show that $BC^2 = 2(1 - \cos 2\theta)$.

b Show that $BC = 2 \sin\theta$.

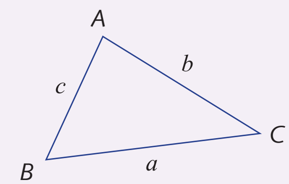
c Deduce that $1 - \cos 2\theta = 2(\sin\theta)^2$.

d Deduce that $\cos 2\theta = \cos^2\theta - \sin^2\theta$.

6 **a** Use the cosine rule to show that

$$1 + \cos A = \frac{(b+c)^2 - a^2}{2bc} \text{ and}$$

$$1 - \cos A = \frac{a^2 - (b-c)^2}{2bc}$$



b Let $s = \dfrac{a+b+c}{2}$.

Show that $1 + \cos A = \dfrac{2s(s-a)}{bc}$

and $1 - \cos A = \dfrac{2(s-b)(s-c)}{bc}$

c Use the fact that $(\sin A)^2 = 1 - (\cos A)^2$ to show that the square of the area of ΔABC is $s(s-a)(s-b)(s-c)$ and deduce Heron's formula for the area of a triangle:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

7 Given two sides and a non-included angle, describe the conditions for 0, 1, or 2 triangles to exist.

CHAPTER

13

Measurement and Geometry

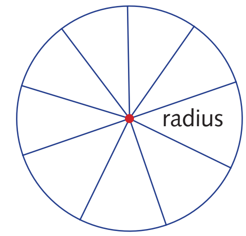
Circle geometry

You have already seen how powerful Euclidean geometry is when working with triangles. For example, Pythagoras' theorem and all of trigonometry arise from Euclidean geometry.

When applied to circles, geometry also produces beautiful and surprising results. In this chapter, you will see how useful congruence and similarity are in the context of circle geometry.

13A Angles at the centre and the circumference

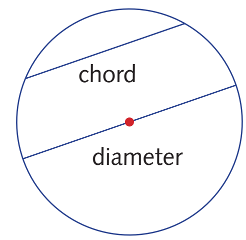
A **circle** is the set of all points that lie a fixed distance (called the **radius**) from a fixed point (called the **centre**).



While we use the word 'radius' to mean this fixed distance, we also use 'radius' to mean any interval joining a point on the circle to the centre. The radii (plural of radius) of a circle radiate out from the centre, like the spokes of a bicycle wheel. (The word *radius* is a Latin word meaning 'spoke' or 'ray'.)

A **chord** of a circle is the interval joining any two points on the circle. The word *chord* is a Greek word meaning 'cord' or 'string'. A tightly stretched string that is plucked gives out a musical note, which is the origin of the word *chord* in music.

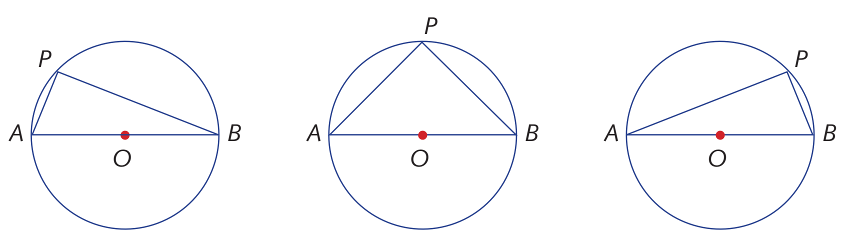
A chord that passes through the centre of the circle is called a **diameter**.



Angles in a semicircle

We will start this chapter with an important result about circles. The discovery of this result was attributed to Thales (~ 600 BC) by later Greek mathematicians, who claimed that it was the first theorem ever consciously stated and proved in mathematics.

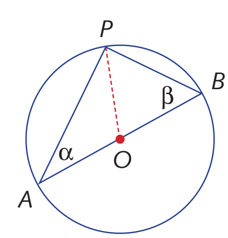
In each diagram below, the angle $\angle P$ is called an **angle in a semicircle**. It is formed by taking a diameter AOB, choosing any other point P on the circle, and joining the chords PA and PB.



These diagrams lead us to ask the question, 'What happens to $\angle P$ as P takes different positions around the semicircle?' Thales discovered a marvellous fact: $\angle P$ is always a right angle.

Theorem: An angle in a semicircle is a right angle.

Proof: Draw the radius OP.



Because the radii are equal, we have two isosceles triangles ΔAOP and ΔBOP.

Let $\angle BAP = \alpha$ and $\angle ABP = \beta$.

Then $\angle OPA = \alpha$ (base angles of isosceles ΔOPA)

and $\angle OPB = \beta$ (base angles of isosceles ΔOPB).

Adding up the interior angles of the triangle ΔABP,

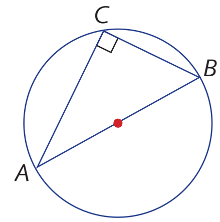
$$\alpha + \alpha + \beta + \beta = 180^\circ$$

$$\alpha + \beta = 90^\circ$$

So $\angle APB = 90^\circ$, which is the required result.

Angles in a semicircle (Thales' theorem)

An angle in a semicircle is a right angle.



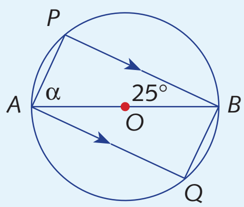
Two slightly different proofs of this result are given in Exercise 13A.

Example 1

In the diagram shown, O is the centre of the circle.

a Find α.

b Prove that $APBQ$ is a rectangle.



Solution

a First, $\angle P = 90°$ (angle in a semicircle)

so $\alpha = 65°$ (angle sum of ΔAPB)

b Also, $\angle Q = 90°$ (angle in a semicircle)

so $\angle PAQ = 90°$ and $\angle PBQ = 90°$ (co-interior angles, $AQ \parallel BP$)

so $APBQ$ is a rectangle, being a quadrilateral with interior angles that are all 90°.

Arcs and segments

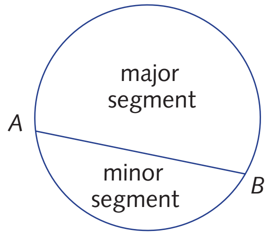
Our next result needs some additional words.

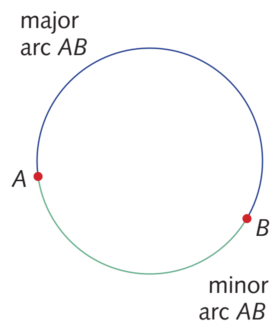
A chord AB divides the circle into two regions called **segments**.

If AB is not a diameter, the regions are unequal. The larger region is called the **major segment** and the smaller region is called the **minor segment**.

Similarly, the points A and B divide the circumference into two pieces called **arcs**.

If AB is not a diameter, the arcs are unequal. The larger piece is called the **major arc** and the smaller piece is called the **minor arc**. Notice that the phrase 'the arc AB' could refer to either arc, and we often need to clarify which arc we mean.





Angles at the centre and the circumference

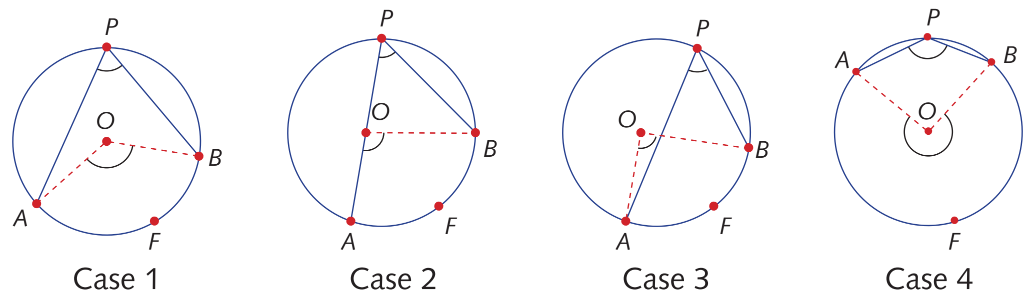
Thales' theorem about an angle in a semicircle is a special case of a more general result.

Consider a fixed arc AFB. Consider a point P on the other arc. Join the chords AP and BP to form the angle $\angle APB$.

We call this angle $\angle APB$ **an angle at the circumference subtended by the arc** AFB. The word *subtends* comes from Latin and literally means 'stretches under' or 'holds under'. We also say $\angle APB$ **stands on the arc** AFB.

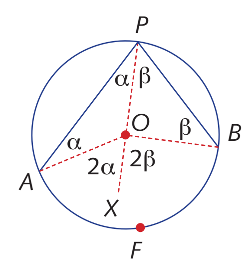
As with angles in a semicircle, we ask, 'What happens to $\angle APB$ as P takes different positions around the arc?'

We will begin our study by focusing on $\angle APB$ in relation to $\angle AOB$, the angle subtended at the centre of the circle by the arc AFB. There are four cases to consider.



In the first three cases, AFB is a minor arc and $\angle APB$ is acute. However, in the fourth case, AFB is a major arc and $\angle APB$ is obtuse.

Case 1



Suppose that AFB is a minor arc, and A, B and P are located as in the diagram. We draw all three radii, AO, BO and PO, and produce PO to X.

Since AO and PO are radii, ΔAOP is isosceles. Let the equal angles be α.

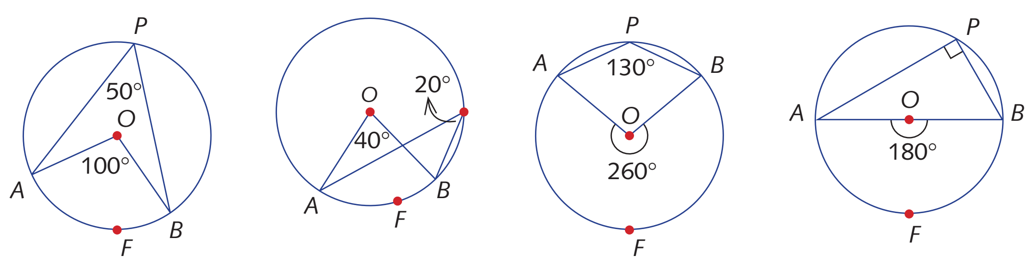
Similarly, ΔBOP is isosceles. Let the equal angles be β.

Next, using the fact that an exterior angle of a triangle is equal to the sum of the two opposite interior angles, $\angle AOX = 2\alpha$ and $\angle BOX = 2\beta$.

Hence, $\angle AOB = 2\alpha + 2\beta$ and $\angle APB = \alpha + \beta$. The following result has now been proved for case 1.

Theorem: The angle at the centre subtended by an arc of a circle is twice an angle at the circumference subtended by the same arc.

The proof for Case 4 is the same as the above proof for Case 1. It relates the obtuse angle $\angle APB$ to the reflex angle $\angle AOB$. The other two cases will be dealt with in question **7** of Exercise 13A. This will complete the proof of the theorem. Some examples of the relationship between angles at the circumference and at the centre, when subtended by a common arc, are illustrated in the following diagrams.



Semicircles

As we can see from the fourth example from the previous page, when the arc is a semicircle, the angle at the centre is 180° and the angle at the circumference is 90°. You will recognise that this is precisely the situation covered by Thales' theorem, which is thus a **special case** of our new theorem.

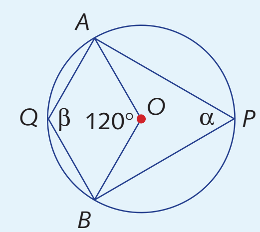
Thus, the two theorems are an excellent example of a **theorem and its generalisation**. This situation occurs routinely throughout mathematics. For example, the cosine rule can be thought of as a generalisation of Pythagoras' theorem.

Angles at the centre and the circumference

The angle at the centre subtended by an arc of a circle is twice an angle at the circumference subtended by the same arc.

Example 2

Find α and β in the diagram shown, where O is the centre of the circle.



Solution

$\alpha = 60°$ (angle at the centre is half the angle at the circumference on the same arc AQB)

Next, reflex $\angle AOB = 240°$ (angles in a revolution at O)

so $\beta = 120°$ (angle at the centre is half the angle at the circumference on the same arc APB)

Exercise 13A

Note: Points labelled O in this exercise are always centres of circles.

1 **a** **i** Use compasses to draw a large circle with centre O, and draw a diameter AOB.

ii Draw an angle $\angle APB$ in one of the semicircles. What is its size?

b **i** Draw another large circle, and draw a chord AB that is not a diameter.

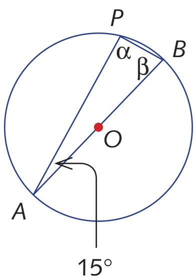
ii Draw the angle at the centre and an angle at the circumference subtended by the minor arc AB. How are these two angles related?

iii Mark the angle at the centre on the major arc AB and draw an angle at the circumference subtended by this major arc. How are these two angles related?

Example 1

2 Find the values of α, β, γ and θ, giving reasons.

a



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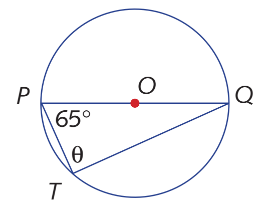

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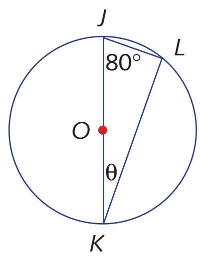

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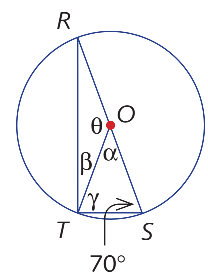

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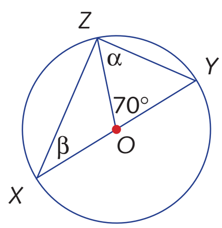

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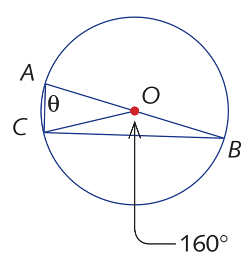
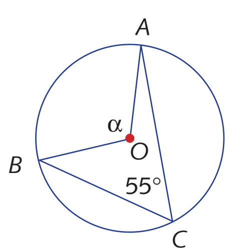


Example 2

3 Find the values of α, β, γ and θ, giving reasons.

a



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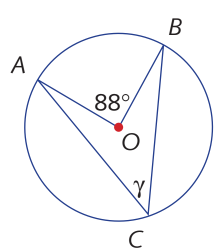

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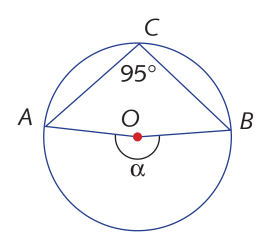

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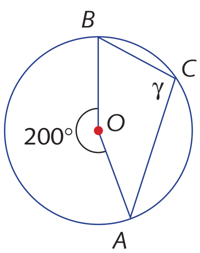

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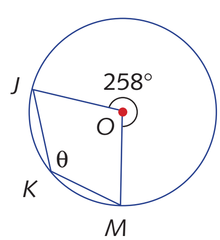

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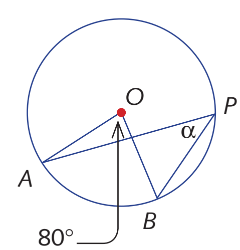

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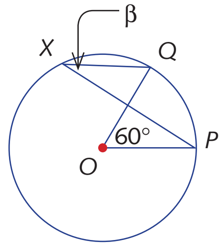

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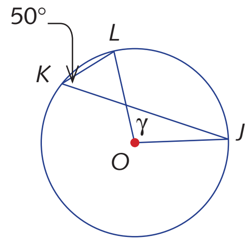

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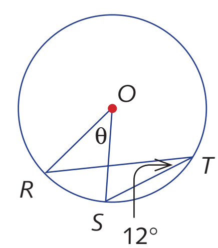

Example 2

4 Find the values of α, β, γ and θ, giving reasons.

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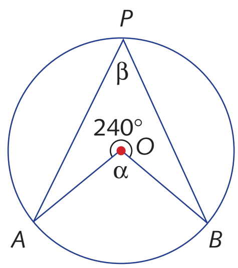

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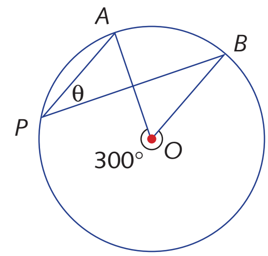

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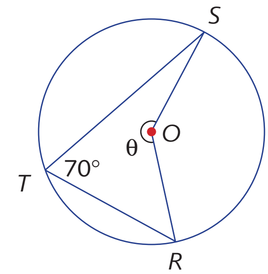

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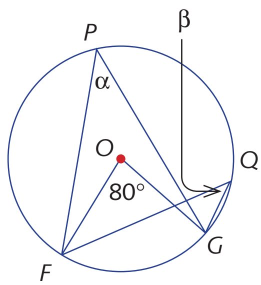

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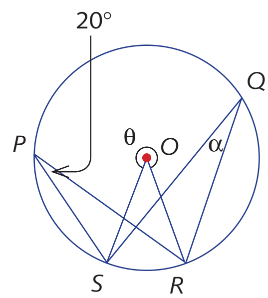

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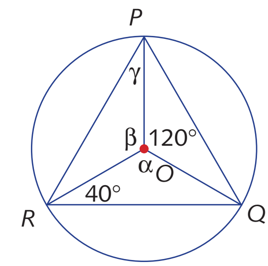

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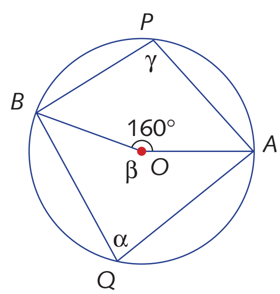

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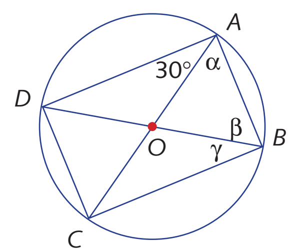

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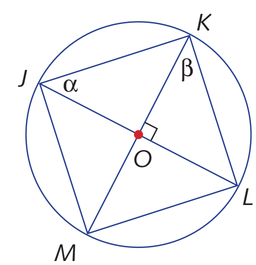
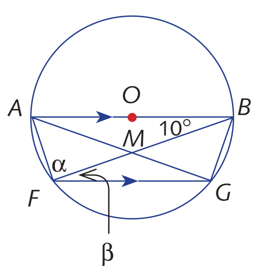


5 Find the values of α, β and γ, giving reasons.

a



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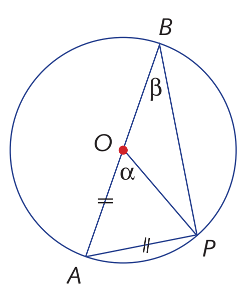

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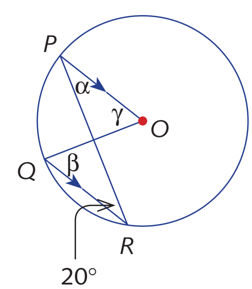

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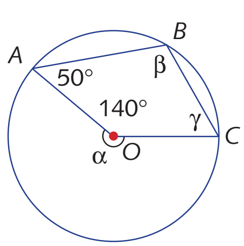

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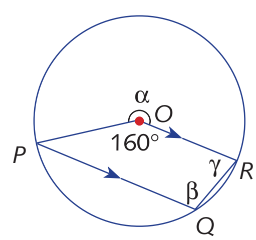

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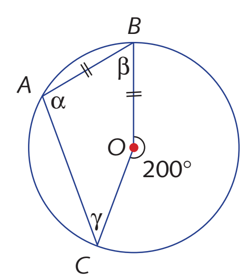
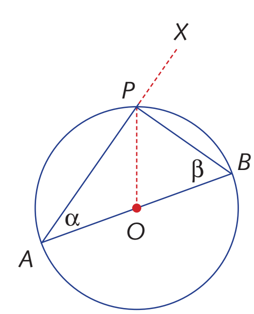


6 Thales' theorem states that: *An angle in a semicircle is a right angle*. This question develops two other proofs of Thales' theorem. We must prove, in each part, that $\angle APB = 90°$.



a (*Euclid's proof*) Join PO, and produce AP to X.

Let $\angle A = \alpha$ and $\angle B = \beta$.

i Prove that $\angle APB = \alpha + \beta$, and that $\angle XPB = \alpha + \beta$.

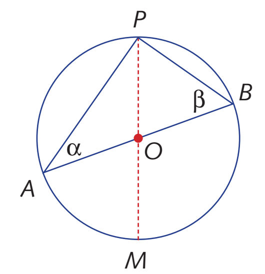
ii Hence, prove that $\alpha + \beta = 90°$.

b Join PO and produce it to M.

i Prove that $\angle AOM = 2\alpha$ and $\angle BOM = 2\beta$.

ii Hence, prove that $2\alpha + 2\beta = 180°$.

iii Deduce that $\alpha + \beta = 90°$.



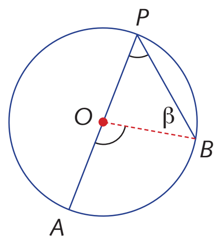
7 Prove that: *An angle at the centre subtended by an arc is twice an angle at the circumference subtended by the same arc.*

We proved Case 1 of this and noted Case 4 follows in the same manner. We pointed out that there are two other cases to consider.

a Using the diagram to the right:

i prove that $\angle APB = \beta$

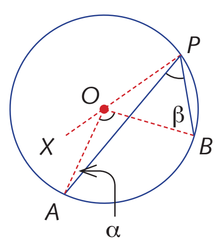
ii prove that $\angle AOB = 2\beta$



b Using the diagram to the right:

i prove that $\angle APB = \beta - \alpha$

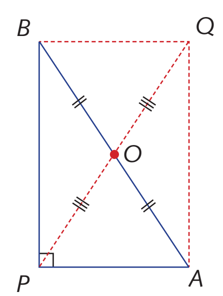
ii prove that $\angle AOB = 2(\beta - \alpha)$



8 The converse of Thales' theorem is established by proving the following result:

The midpoint of the hypotenuse of a right-angled triangle is equidistant from the three vertices of the triangle.

Let ΔABP be right-angled at P, and let O be the midpoint of the hypotenuse AB. Draw PO and produce it to Q so that $PO = OQ$. Draw AQ and BQ.



a Explain why $APBQ$ is a parallelogram.

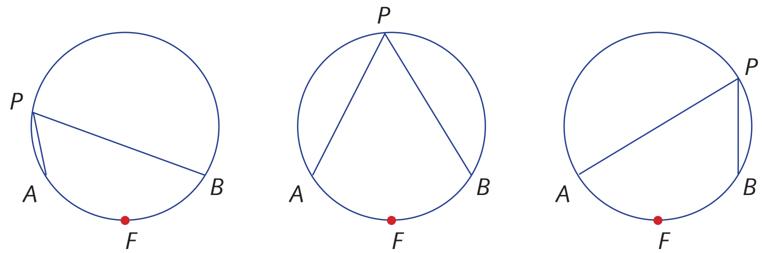
b Hence, explain why $APBQ$ is a rectangle.

c Hence, explain why $AO = BO = PO$ and why the circle with diameter AB passes through P.

9 (*An application of the angle at the centre and circumference theorem*) A horse is travelling around a circular track at a constant speed. A punter standing at the very edge of the track is following him with binoculars. Explain why the punter's binoculars are rotating at a constant rate.

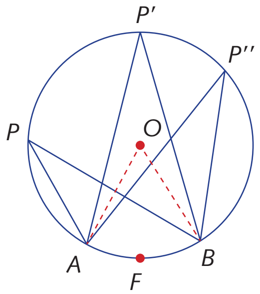
13B Angles at the circumference and cyclic quadrilaterals

Let us look at three angles at the circumference, all subtended by the same arc AFB.



We know already that all three angles are half the angle $\angle AOB$ at the centre subtended by this same arc.

It follows immediately that all three angles are equal. This result is important enough to state as a separate theorem, in the box below.



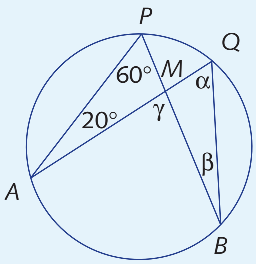
Angles at the circumference

Angles at the circumference of a circle subtended by the same arc are equal.

This is often stated as 'Angles in the same segment are equal'.

Example 3

Find α, β and γ in the diagram to the right.



Solution

First, $\alpha = 60°$ (angles on the same arc AB)

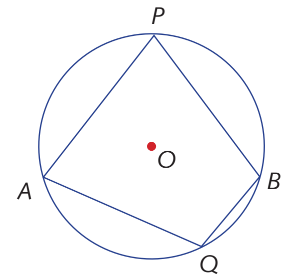
Second, $\beta = 20°$ (angles on the same arc PQ)

Third, $\gamma = 80°$ (exterior angle of ΔAPM)

Cyclic quadrilaterals

A **cyclic quadrilateral** is a quadrilateral whose vertices all lie on a circle. We also say that the points A, B, P and Q are **concyclic**.

The opposite angles $\angle P$ and $\angle Q$ of the cyclic quadrilateral $APBQ$ are closely related.



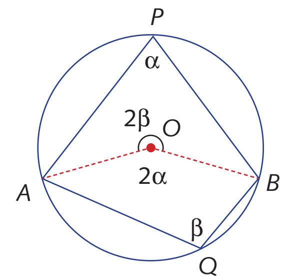
The key to finding the relationship is to draw the radii AO and BO.

First, $\angle P$ is half the angle $\angle AOB$ at the centre on the same arc AQB; we have marked these angles α and 2α.

Second, $\angle Q$ is half the reflex angle $\angle AOB$ at the centre on the same arc APB; we have marked these angles β and 2β.

$2\alpha + 2\beta = 360°$ (angles in a revolution at O)

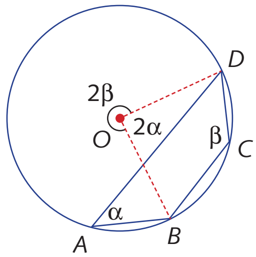
so $\alpha + \beta = 180°$



Hence, the opposite angles $\angle P$ and $\angle Q$ are supplementary.

The diagram could also have been drawn as shown, but the proof is unchanged.

We usually state this as a result about cyclic quadrilaterals.



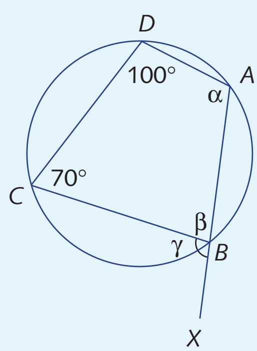
Cyclic quadrilaterals

The opposite angles of a cyclic quadrilateral are supplementary.

An interesting alternative proof is given as question **8** in Exercise 13B. Every cyclic quadrilateral is convex because none of its angles are reflex, but not every convex quadrilateral is cyclic.

Example 4

Find α, β and γ in the diagram shown.



Solution

$\alpha + 70° = 180°$ (opposite angles of cyclic quadrilateral $ABCD$)

$\alpha = 110°$

$\beta + 100° = 180°$ (opposite angles of cyclic quadrilateral $ABCD$)

$\beta = 80°$

$\gamma + \beta = 180°$ (straight angle at B)

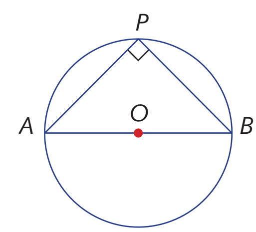
$\gamma = 100°$

The converse of Thales' theorem

We began this chapter by proving Thales' theorem:

An angle in a semicircle is a right angle.

Thales' theorem has an important converse, which was proved in question **8** of exercise 13A. It elegantly uses the diagonal properties of rectangles.

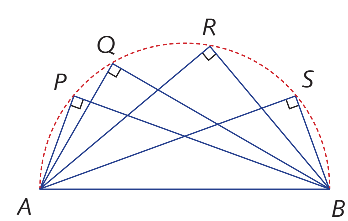


Converse of Thales' theorem

If an interval AB subtends a right angle at a point P, then P lies on the circle with diameter AB.

Here is a diagram that illustrates the converse of Thales' theorem very nicely. Suppose that AB is a line interval. A person walks from A to B in a curved path $APQRSB$ so that AB always subtends a right angle at his position.

The converse of Thales' theorem tells us that his path is a semicircle.



Exercise 13B

Note: Points labelled O in this exercise are always centres of circles.

1 **a** Draw a large circle, and draw a chord AB that is not a diameter.

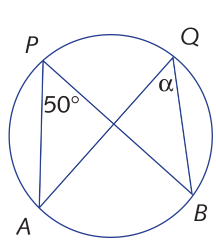
b Draw two angles at the circumference standing on the minor arc AB. How are these two angles related?

c Draw two angles at the circumference standing on the major arc AB. How are these two angles related?

Example 3

2 Find the values of α, β and θ, giving reasons.

a



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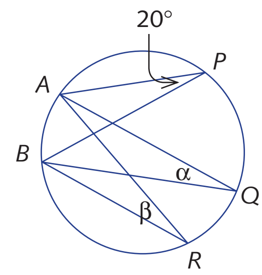

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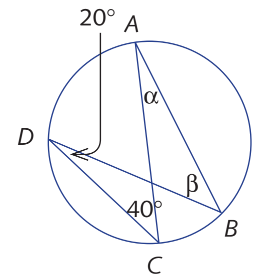

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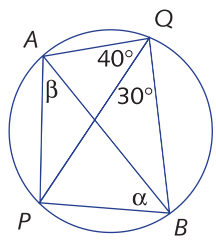

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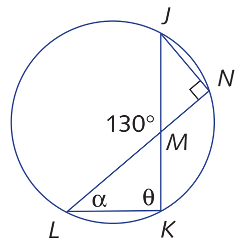

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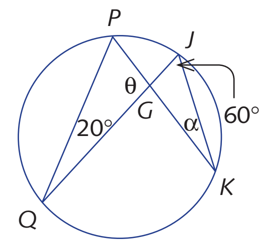
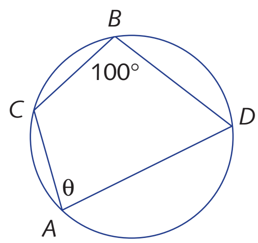


Example 4

3 Find the values of α, β, γ and θ, giving reasons.

a



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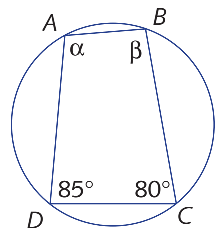

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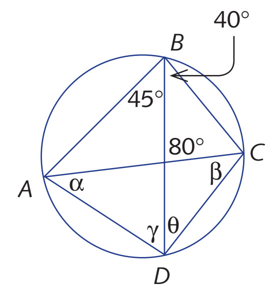

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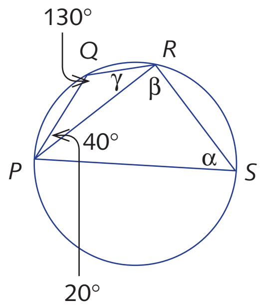

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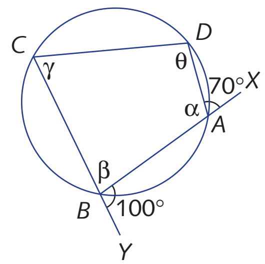

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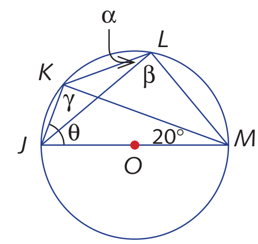
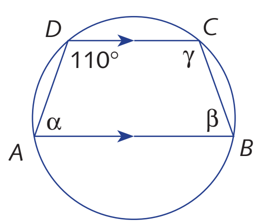


4 Find the values of α, β, γ and θ, giving reasons.

a



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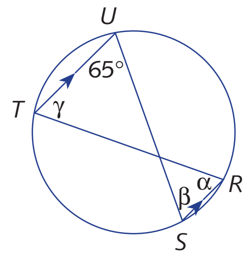

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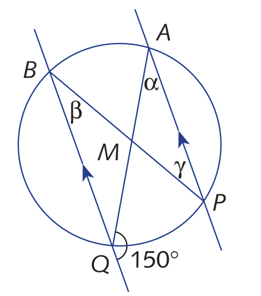

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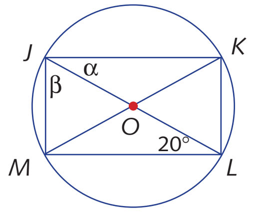

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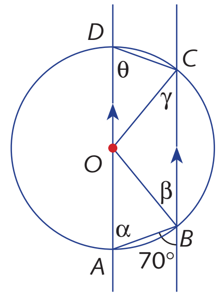

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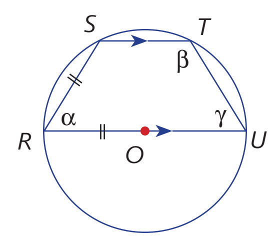

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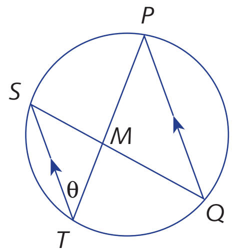
5 **a**

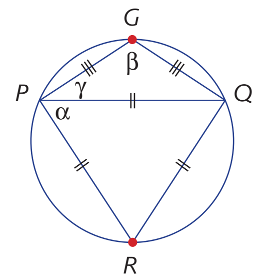


i Prove that

$\angle P = \angle Q = \angle S = \angle T$.

ii Prove that $PT = SQ$.

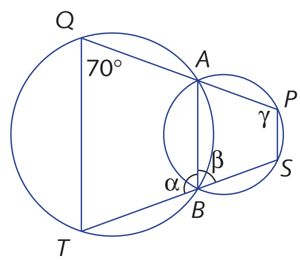
b



i Find α, β and γ.

ii Prove that $PQ \perp GR$.

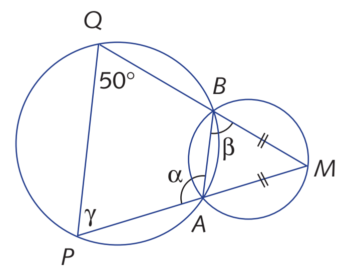
6 **a**



i Find α, β and γ.

ii Prove that $PS \parallel QT$.

b

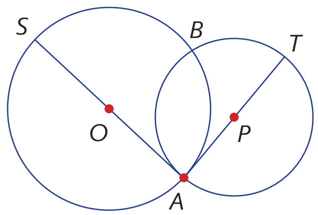


i Find α, β and γ.

ii Prove that $AB \parallel PQ$.

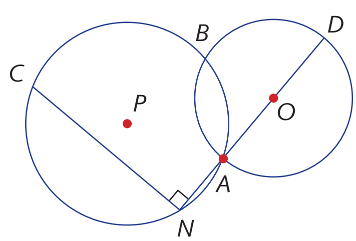
iii Prove that $AP = BQ$.

7 The centres of the circles are O and P.



a **i** Find $\angle ABS$ and $\angle ABT$.

ii Hence, prove that S, B and T are collinear.



b **i** Find $\angle ABC$ and $\angle ABD$.

ii Hence, prove that C, B and D are collinear.

iii Why is AC a diameter?

8 Here is an alternative proof that: *The opposite angles of a cyclic quadrilateral are supplementary*.

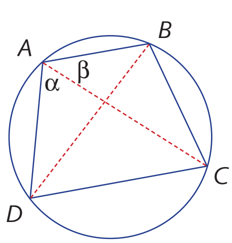
In the cyclic quadrilateral $ABCD$, draw the diagonals AC and BD.

Let $\alpha = \angle DAC$ and $\beta = \angle BAC$.

a Prove that $\angle DBC = \alpha$ and $\angle BDC = \beta$.

b Hence, prove that $\angle DCB = 180° - (\alpha + \beta)$.

c Deduce that $\angle DAB + \angle DCB = 180°$.



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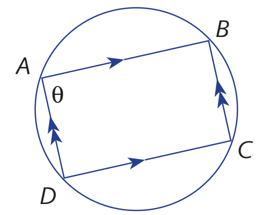
9 **a** Prove that: *A cyclic parallelogram is a rectangle*.

In the cyclic parallelogram $ABCD$, let $\angle A = \theta$.

i Give reasons why $\angle C = 180° - \theta$ and why $\angle C = \theta$.

ii Hence, prove that $ABCD$ is a rectangle.

b Use part **a** to prove that: *A cyclic rhombus is a square*.

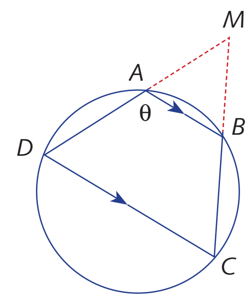


10 Prove that: *In a cyclic trapezium that is not a parallelogram, the non-parallel sides have equal length.*

Let $ABCD$ be a cyclic trapezium with $AB \parallel DC$ and $AD \nparallel BC$. Suppose DA meets CB at M and let $\angle DAB = \theta$.

a Prove that ΔABM and ΔDCM are isosceles.

b Hence, prove that $AD = BC$.

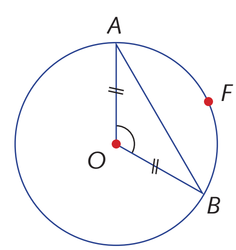


13C Chords and angles at the centre

Take a minor arc AFB of a circle and join the radii AO and BO. The angle $\angle AOB$ at the centre is called the **angle subtended at the centre** by the arc AFB. It is also called the angle at the centre subtended by the chord AB.

$OA = OB$ (radii of a circle)

Therefore, ΔAOB is isosceles. This is the key idea in the results of this section.



Equal chords and equal angles at the centre

Suppose now that two chords each subtend an angle at the centre of the circle. Two results about this situation can be proved.

Theorem: Chords of equal length subtend equal angles at the centre of the circle.

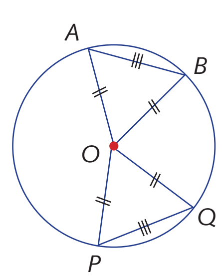
Proof: In the diagram, AB and PQ are chords of equal length.

$OA = OB = OP = OQ$ (radii of a circle)

From the diagram,

$\Delta AOB \equiv \Delta POQ$ (SSS)

Hence, $\angle AOB = \angle POQ$ (matching angles of congruent triangles)



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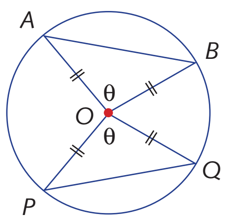
Theorem: Conversely, chords subtending equal angles at the centre have equal length.

Proof: In the diagram, AB and PQ subtend equal angles at O,

so $\Delta AOB \equiv \Delta POQ$ (SAS)

Hence, $AB = PQ$ (matching sides of congruent triangles)



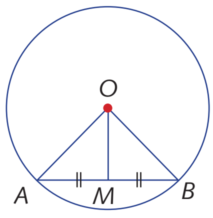
Chords and angles at the centre

- Chords of equal length subtend equal angles at the centre of a circle.
- Conversely, chords subtending equal angles at the centre of a circle have equal length.

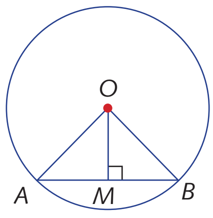
The midpoint of a chord

Three theorems about the midpoint of a chord are stated below. The proofs are dealt with in question **6** of Exercise 13C.

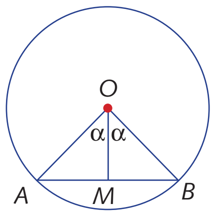
Theorem: The interval joining the midpoint of a chord to the centre of a circle is perpendicular to the chord, and bisects the angle at the centre subtended by the chord.



Theorem: The perpendicular from the centre of a circle to a chord bisects the chord, and bisects the angle at the centre subtended by the chord.



Theorem: The bisector of the angle at the centre of a circle subtended by a chord bisects the chord, and is perpendicular to it.



Chords and calculations

The circle theorems stated above can be used in conjuction with Pythagoras' theorem and trigonometry to calculate lengths and angles.

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Example 5

A chord of length 12 cm is drawn in a circle of radius 8 cm.

a How far is the chord from the centre (that is, the perpendicular distance from the centre to the chord)?

b What angle, correct to one decimal place, does the chord subtend at the centre?

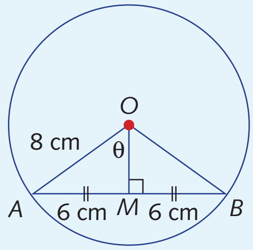
Solution

a Draw the perpendicular OM from O to the chord. By a midpoint of the chord theorem, M is the midpoint of AB, so $AM = 6$.

Hence, $OM^2 = 8^2 - 6^2$ (Pythagoras' theorem)

$$OM = \sqrt{28}$$

$$= 2\sqrt{7} \text{ cm}$$



b Let $\theta = \angle AOM$

Then, $\sin\theta = \dfrac{6}{8}$

$$\theta \approx 48.59°$$

So, $\angle AOB = 2\theta \approx 97.2°$

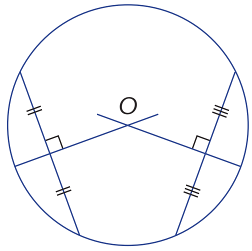
The midpoint of a chord

- The interval joining the midpoint of a chord to the centre of a circle is perpendicular to the chord, and bisects the angle at the centre subtended by the chord.
- The perpendicular from the centre of a circle to a chord bisects the chord, and bisects the angle at the centre subtended by the chord.
- The bisector of the angle at the centre of a circle subtended by a chord bisects the chord, and is perpendicular to it.

Finding the centre of a circle

Suppose that we have a circle. How do we find its centre?

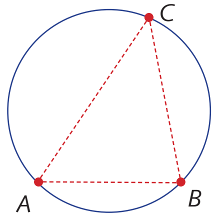
The first midpoint-of-a-chord theorem above tells us that the centre lies on the perpendicular bisector of every chord. Thus, if we draw two chords that are not parallel, and construct their perpendicular bisectors, the point of intersection of the bisectors will be the centre of the circle.



The circumcircle of a triangle

Here is an important fact about circles.

Suppose that three points A, B and C form a triangle, meaning that they are not collinear. Then there is a circle passing through all three points. The circle is called the **circumcircle** of ΔABC, and its centre is called the **circumcentre** of the triangle.



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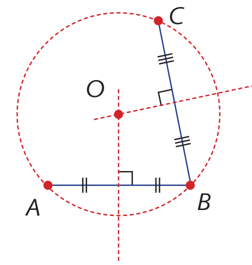
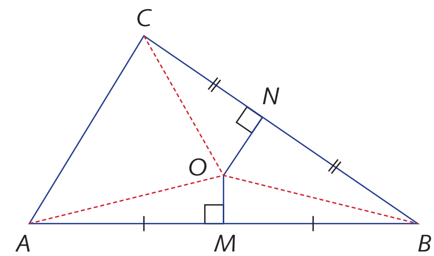
Here is a simple construction of the circumcentre and circumcircle.



Construct the perpendicular bisectors of two sides AB and BC, and let them meet at O. Then O is the circumcentre of ΔABC, and we can use it to draw the circumcircle through A, B and C.

Here is the theorem that justifies all the previous remarks.

Theorem: The intersection of the perpendicular bisectors of two sides of a triangle is the centre of a circle passing through all three verticles.



Proof: Let ABC be a triangle.

Let M be the midpoint of AB, and let N be the midpoint of BC.

Let the perpendicular bisectors of AB and BC meet at O, and join AO, BO and CO.

Then $\Delta AOM \equiv \Delta BOM$ (SAS)

so $AO = BO$ (matching sides of congruent triangles),

and $\Delta CON \equiv \Delta BON$ (SAS)

so $CO = BO$ (matching sides of congruent triangles),

so $AO = BO = CO$

Hence, O is equidistant from A, B and C, so the circle with centre O and radius AO passes through A, B and C.

The centre of a circle and circumcentre of a triangle

- To find the centre of a given circle, construct the perpendicular bisectors of two non-parallel chords, and take their point of intersection.
- To find the circumcentre of a given triangle, construct the perpendicular bisectors of two sides, and take their point of intersection.

Exercise 13C

Note: Points labelled O in this exercise are always centres of circles.

1 **a** Draw a large circle (and ignore the fact that you may be able to see the mark that the compasses made at the centre). Draw two non-parallel chords AB and PQ, then construct their perpendicular bisectors. The point where the bisectors intersect is the centre of the circle.

b Draw a large triangle ABC.

i Construct the perpendicular bisectors of two sides, and let them intersect at O. Construct the circle with circumcentre O passing through all three vertices of the triangle.

ii Construct the perpendicular bisector of the third side. It should also pass through O.

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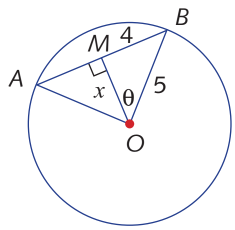
Example 5

2 Find the exact value of x, as a surd if necessary. Then use trigonometry to find the value of θ, correct to one decimal place.

a



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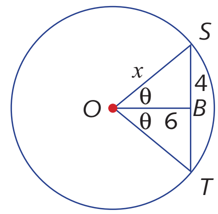

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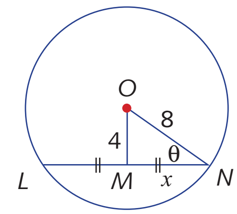


3 **a** A chord subtends an angle of $90°$ at the centre of a circle of radius 12 cm.

- **i** How long is the chord?
- **ii** How far is the midpoint of the chord from the centre?

b In a circle of radius 20 cm, the midpoint of a chord is 16 cm from the centre.

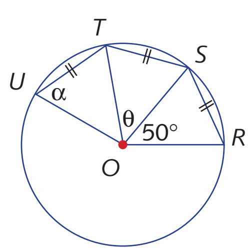
- **i** How long is the chord?
- **ii** What angle does the chord subtend at the centre, correct to one decimal place?

c In a circle of radius 10 cm, a chord has length 16 cm.

- **i** What is the perpendicular distance from the chord to the centre?
- **ii** What angle does the chord subtend at the centre, correct to one decimal place?

4 Find the values of α, β, γ and θ, giving reasons.

a



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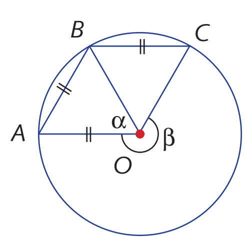

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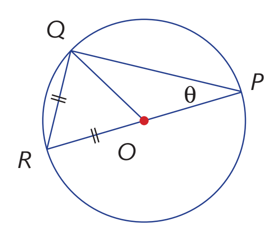

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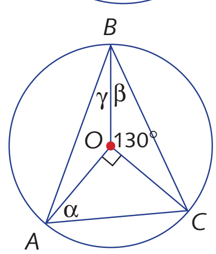

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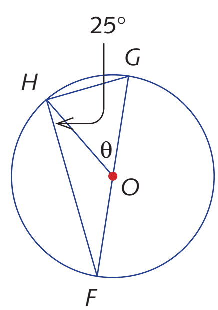

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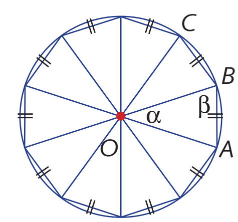
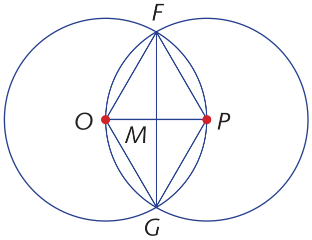


5 Let two circles of radius 1 and centres O and P each pass through the centre of the other, and intersect at F and G.

Let FG meet OP at M.

a Find $\angle FPO$, $\angle FGO$ and $\angle FMO$.

b Find the length of the common chord FG.



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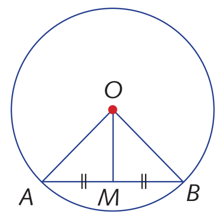
6 This question leads you through the proofs of the three theorems in the text about the midpoint of a chord.

a Prove that: *The line joining the midpoint of a chord to the centre is perpendicular to the chord, and bisects the angle at the centre subtended by the chord.*

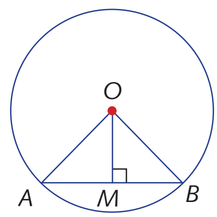


Let M be the midpoint of the chord AB.

i Prove that $\Delta AOM \equiv \Delta BOM$.

ii Hence, prove that $OM \perp AB$ and that OM bisects $\angle AOB$.

b Prove that: *The perpendicular from the centre to a chord bisects the chord, and bisects the angle at the centre subtended by the chord.*

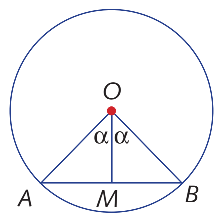


Let M be the foot of the perpendicular from O to chord AB.

i Prove that $\Delta AOM \equiv \Delta BOM$.

ii Hence, prove that M is the midpoint of AB and that OM bisects $\angle AOB$.

c Prove that: *The bisector of the angle at the centre subtended by a chord bisects the chord, and is perpendicular to it.*

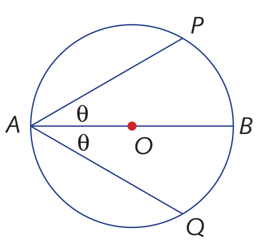


Let the bisector of $\angle AOB$ meet the chord AB at M.

i Prove that $\Delta AOM \equiv \Delta BOM$.

ii Hence, prove that M is the midpoint of AB and that $OM \perp AB$.

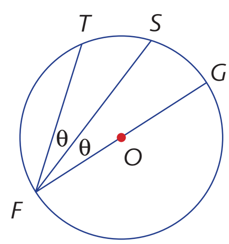
7 a



i Prove that $\Delta AOP \equiv \Delta AOQ$.

ii Prove that $AP = AQ$.

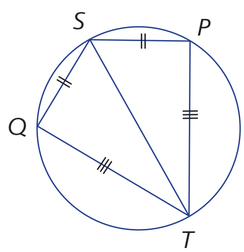
b



i Prove that $\angle FSO = \angle TFS$.

ii Prove that $FT \parallel OS$.

c



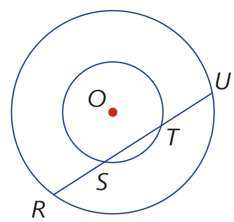
i Prove that $\Delta PST \equiv \Delta QST$.

ii Prove that $\angle P = \angle Q$.

iii Prove that $\angle P + \angle Q = 180°$.

iv Prove that ST is a diameter.

d



i Prove that $\angle OST = \angle OTS$.

ii Prove that $\angle ORU = \angle OUR$.

iii Prove that $\Delta ORT \equiv \Delta OUS$.

iv Prove that $RS = TU$.

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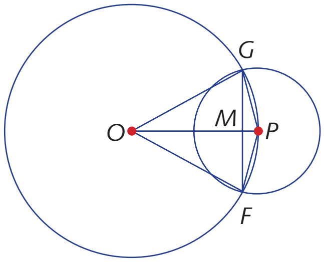
8 Prove that: *When two circles intersect, the line joining their centres is the perpendicular bisector of their common chord.*



Let two circles with centres O and P intersect at F and G. Let OP meet the common chord FG at M.

a Prove that $\Delta FOP \equiv \Delta GOP$.

b Prove that $\Delta FOM \equiv \Delta GOM$.

c Hence, prove that $FM = GM$ and $OP \perp FG$.

9 Let AB be an interval with midpoint M, and let P be a point in the plane not on AB.

a Prove that if P is equidistant from A and B, then P lies on the perpendicular bisector of AB.

b Conversely, prove that if P lies on the perpendicular bisector of AB, then P is equidistant from A and B.

c Use parts **a** and **b** to prove that a circle has only one centre.

d For these questions you will need to think in three dimensions.

i A circle is drawn on a piece of paper that lies flat on the table. Is there any other point in three-dimensional space, other than the centre of the circle, that is equidistant from all the points on the circle?

ii What geometrical object is formed by taking, in three dimensions, the set of all points that are a fixed distance from a given point?

iii What geometrical object is formed by taking, in three dimensions, the set of all points that are a fixed distance from a given line?

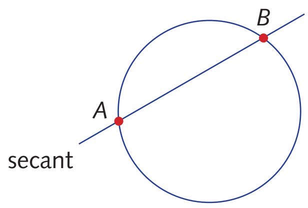
iv What geometrical object is formed by taking, in three dimensions, the set of all points that are a fixed distance from a given interval?

v What geometrical object is formed by taking, in three dimensions, the set of all points that are equidistant from the endpoints of an interval?

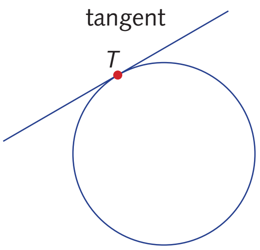
vi How could you find the centre of a sphere?

13D Tangents and radii

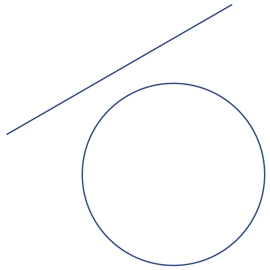
The diagrams show that a line can intersect a circle at two points, one point or no points.



two points



one point



no points

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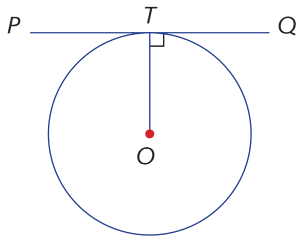
- A line that intersects a circle at two points is called a **secant**, because it *cuts* the circle into two pieces. (The word *secant* comes from Latin and means 'cutting'.)
- A line that intersects a circle at just one point is called a **tangent**. It *touches* the circle at that *point of contact*, but does not pass inside it. (The word *tangent* also comes from Latin and means 'touching'.)

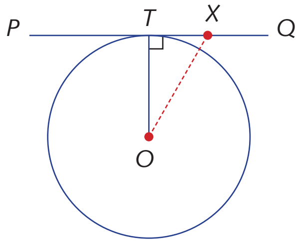
Constructing a tangent



In the diagram, OT is a radius of a circle. The line PTQ is perpendicular to the radius OT.

Could PQ intersect the circle at a second point (other than T)? The symmetry of the diagram about the line OT suggests that it cannot, and here is the proof.

Theorem: The line through a point on a circle perpendicular to the radius at that point is the tangent at that point.



Proof: Let OT be a radius, and let PTQ be perpendicular to OT. Let X be a point other than T on PTQ.

Then $OX^2 = OT^2 + TX^2$ (Pythagoras' theorem)

$OX^2 > OT^2$ since TX is non-zero,

so $OX > OT$, and OT is the radius of the circle

and thus X lies outside the circle.

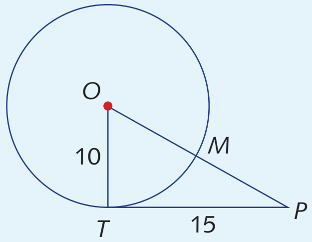
Hence, PTQ intersects the circle at only the one point, T, and so PTQ is a tangent to the circle.

Example 6

In the diagram, TP is a tangent to the circle with centre O.

a Find OP and MP.

b Find $\angle TOP$, correct to one decimal place.



Solution

a We know that $OT \perp TP$ (radius and tangent),

so $OP^2 = 10^2 + 15^2$ (Pythagoras' theorem)

$= 325$

$OP = 5\sqrt{13}$

Hence, $MP = 5\sqrt{13} - 10$

b Let $\theta = \angle TOP$

Then, $\tan\theta = \dfrac{15}{10}$

$\theta \approx 56.3°$

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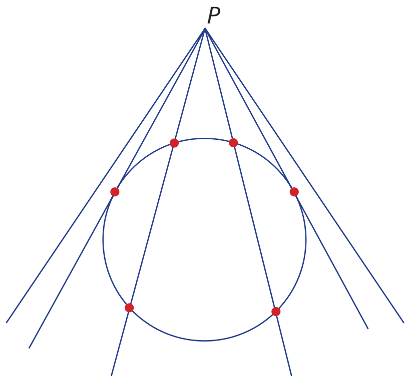
Tangents from an external point

Let P be a point outside a circle. The diagram shows how different lines through P intersect the circle at two, one or no points. You can see that there are clearly exactly two tangents to the circle from P.

We now prove that these two tangents from the point P to the circle have equal length.



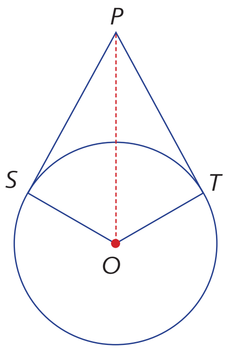
Theorem: The tangents to a circle from a point outside have equal length.

Proof: Let P be a point outside the circle with centre O.
Let the tangents from P touch the circle at S and T.

In the triangles OPS and OPT,

$OP = OP$ (common)

$OS = OT$ (radii)



$\angle PSO = \angle PTO = 90°$ (radius and tangent)

so $\Delta OPS \equiv \Delta OPT$ (RHS)

Hence, $PS = PT$ (matching sides of congruent triangles)

Alternative proof: ΔPST and ΔPTO are right-angles

Therefore, $PS^2 = PO^2 - OS^2$ (Pythagoras' theorem)

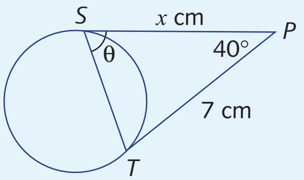
$= PO^2 - OT^2$ (radii of a circle)

$= PT^2$ (Pythagoras' theorem)

Example 7

The intervals PS and PT are tangents.

Find θ and x.



Solution

First, $x = 7$ (tangents from an external point)

Hence, $\angle T = \theta$ (base angles of isosceles ΔPST)

so $\theta = 70°$ (angle sum of ΔPST)

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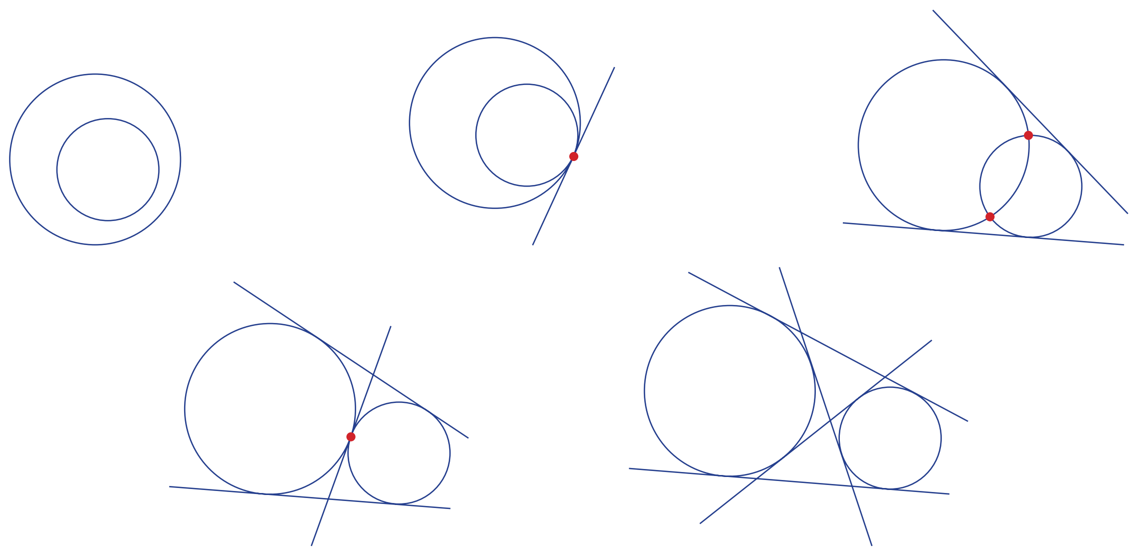
Common tangents and touching circles

The five diagrams below show all the ways in which two circles of different radii can intersect. Start with the smaller circle inside the larger, and move it slowly to the right. Notice that the two circles can intersect at two, one or no points.



The various lines are all the **common tangents** to the two circles. There are 0, 1, 2, 3 and 4 common tangents in the five successive diagrams.

In the second and fourth diagrams, the two circles **touch** each other, and they have a **common tangent at the point of contact**.

Tangents to a circle

Tangent and radius

- The line through a point on a circle perpendicular to the radius at that point is the tangent at that point.

Tangents from outside the circle

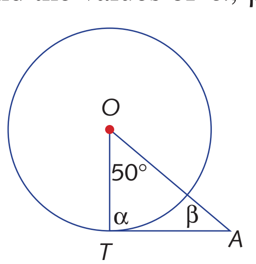
- The tangents to a circle from a point outside have equal length.

Exercise 13D

Note: Points labelled O in this exercise are always centres of circles.

1 Find the values of α, β, γ and θ, giving reasons. In each diagram, a tangent is drawn at T.

a



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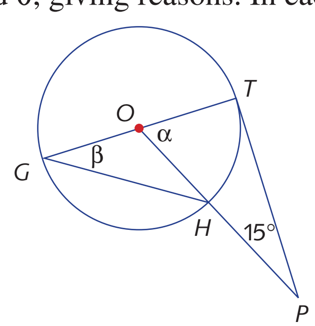

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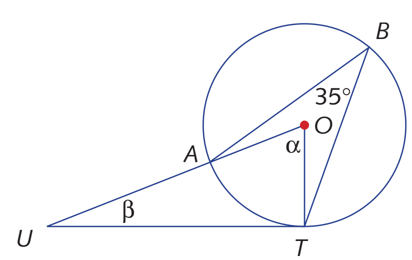

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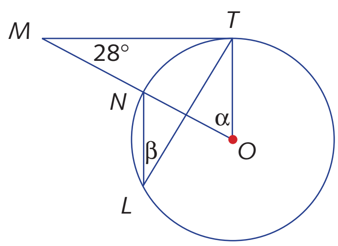

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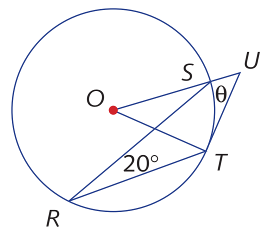

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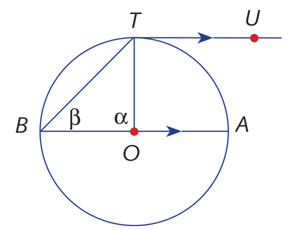

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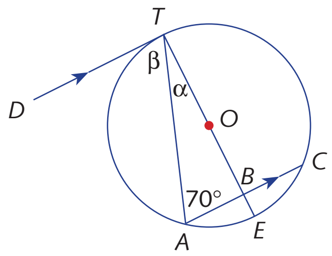

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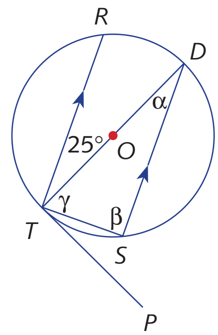

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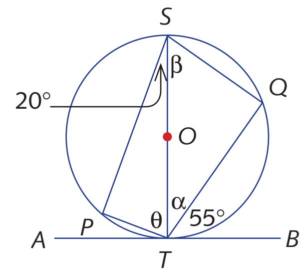

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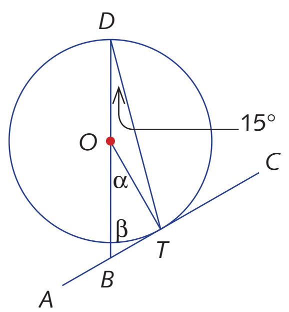

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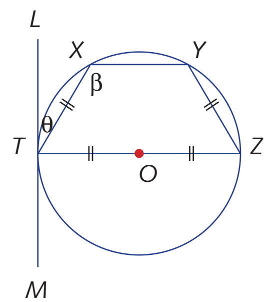

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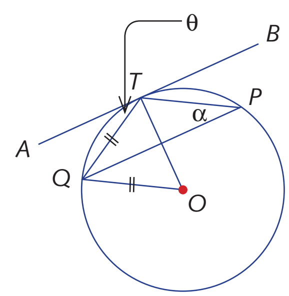
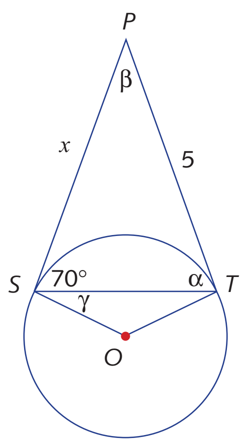


Example 7

2 Find the values of α, β, γ and θ, and the values of x, y and z. In the diagrams, tangents are drawn to the circle at S, T and U.

a



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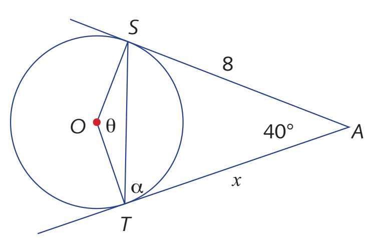

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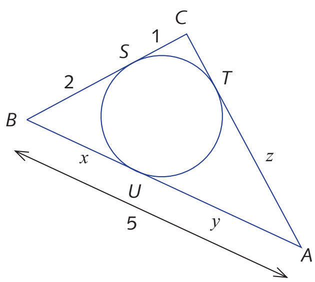

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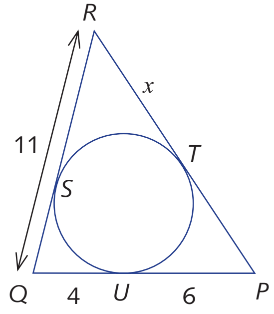

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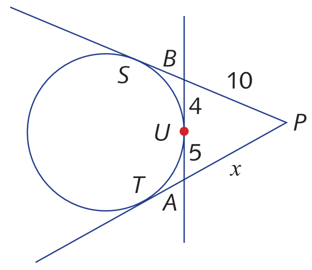

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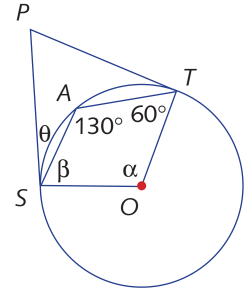

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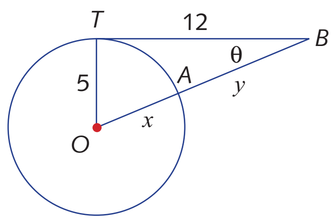
Example 6

3 Find the values of x, y and θ correct to two decimal places, where tangents are drawn at S and T.

a



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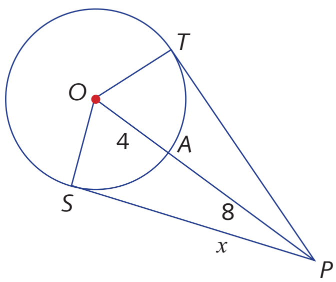

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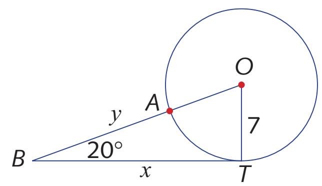
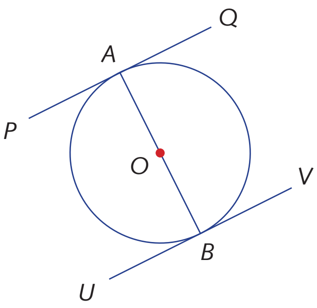
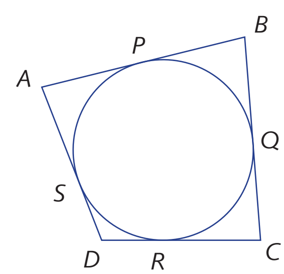


4 Prove that: *The tangents at the endpoints of a diameter are parallel.*

Let PAQ and UBV be the tangents at the endpoints of a diameter AOB. Prove that $PQ \parallel UV$.



5 The tangents at the four points P, Q, R and S on a circle form a quadrilateral $ABCD$. Prove that $AB + CD = AD + BC$.

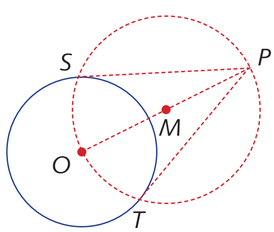


6 This question describes the method of construction of tangents to a circle from an external point P.

a Draw a circle with centre O and choose a point, P, outside the circle. Let M be the midpoint of OP, and hence draw the circle with diameter OP. Let the circles intersect at S and T, and join PS and PT.

b Prove that PS and PT are tangents to the original circle.

c Deduce that $PS = PT$.



7 Let PS and PT be the two tangents to a circle with centre O from a point, P, outside the circle.

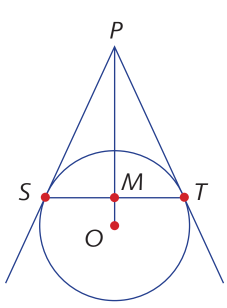
a **i** Prove that $\Delta PSO \equiv \Delta PTO$.

ii Hence, prove that the tangents have equal length, and that OP bisects the angle between the tangents and bisects the angle between the radii at OS and OT.

b Join the chord ST and let it meet PO at M.

i Prove that $\Delta SPM \equiv \Delta TPM$.

ii Hence, prove that OP is the perpendicular bisector of ST.



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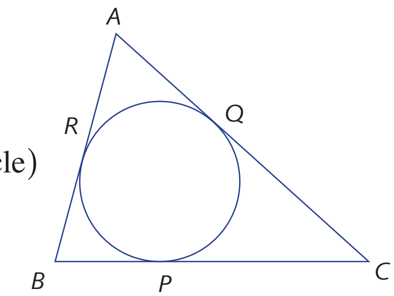
8 The circle in the diagram is called the **incircle** of triangle ABC. It touches the three sides of ΔABC at P, Q and R.

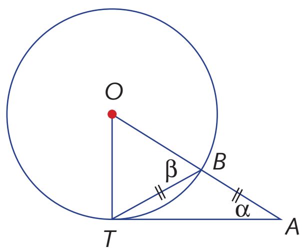


Prove that:

$$\text{Area of triangle} = \frac{1}{2} \times (\text{perimeter of triangle}) \times (\text{radius of circle})$$

You will need to join the radii OP, OQ, and OR and the intervals OA, OB and OC, where O is the centre of the incircle.

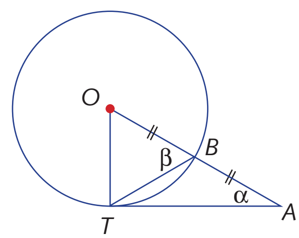
9 **a**



i Prove that $\alpha = 30°$.

ii Find β.

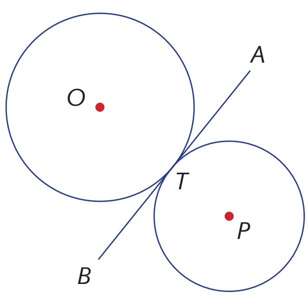
b



i Prove that $\sin\alpha = \frac{1}{2}$.

ii Prove that $\beta = 60°$.

10 Prove that: *When two circles touch, their centres and their point of contact are collinear.*



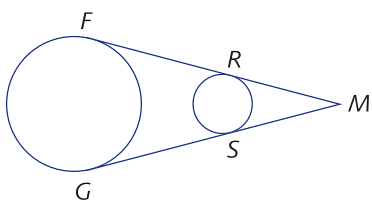
a Let two circles with centres O and P touch externally at T. Let ATB be the common tangent at T.

i Find $\angle ATO$ and $\angle ATP$.

ii Hence, prove that O, T and P are collinear.

b Draw a diagram of two circles touching internally, and prove the theorem in this case.

11 **a** Each tangent, FR and GS, in the diagram is called a **direct common tangent** because the two circles lie on the same side of the tangent. Produce the two tangents to meet at M.

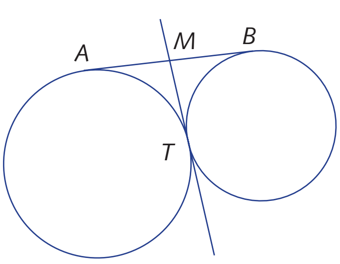


i Prove that $MF = MG$ and $MR = MS$.

ii Hence, prove that $FR = GS$.

b Draw a diagram showing **indirect common tangents**, and prove that they also have equal length. (*Note*: Indirect common tangents cross over, and intersect between the two circles.)

12 Let AB be a direct common tangent of two circles touching externally. Let the common tangent at the point of contact, T, meet AB at M.



a Prove that $MA = MB = MT$.

b Hence, deduce that $\angle ATB$ is a right angle.

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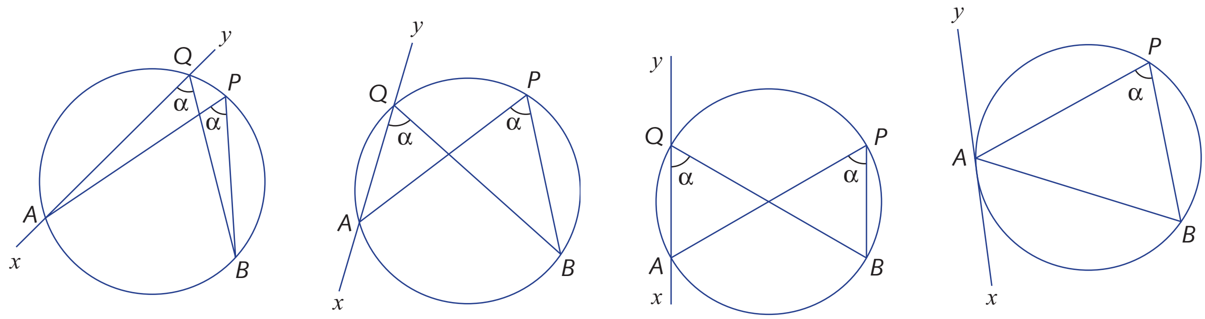
13E The alternate segment theorem

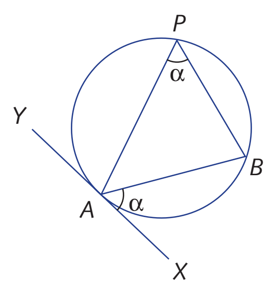
Consider the secant XY that intersects a circle at points A and Q. Consider also points P and B and the angles subtended at P and Q by the arc AB.



As you can see from the images above, as Q approaches A along the circumference, $\angle XQB$ and $\angle P$ remain equal. It therefore seems reasonable to suppose that as Q coincides with A, and XY becomes tangent to the circle at A, $\angle XAB$ and $\angle P$ are equal.

This is in fact the case, and is the **alternate segment theorem**.

The **alternate segment** ('alternate' here simply means 'other') is the segment of the circle on the other side of the chord AB from $\angle XAB$. The angle $\angle P$ is an *angle in the alternate segment*.



Theorem: The angle between a tangent and a chord is equal to any angle in the alternate segment.

Proof: Let AB be a chord of a circle and let XAY be a tangent at A. Let P be a point on the circle on the other side of the chord AB from $\angle XAB$. Let $\angle P = \theta$.

We must prove that $\angle XAB = \angle P$.

Draw the diameter AON, and join BN.

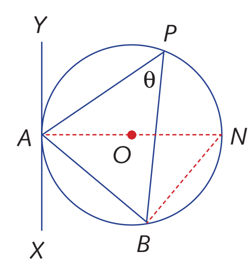
Then $\angle N = \theta$ (angles on the same arc AB)

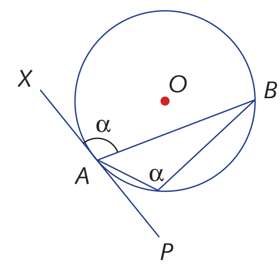
and $\angle NBA = 90^\circ$ (angle in the semicircle NBA)

and $\angle NAX = 90^\circ$ (radius and tangent)

Hence, $\angle NAB = 9\ ^\circ - \theta$ (angle sum of $\Delta\, NAB$)

so $\angle XAB = \theta$ (adjacent angles in a right angle)





Note: This proof is only valid when $\angle XAB$ is acute. In the exercises we will prove the result when $\angle XAB$ is obtuse.

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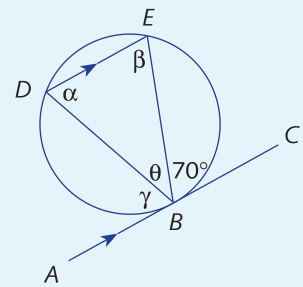
Example 8

Find α, β, γ and θ in the figure shown to the right.



Solution

$\alpha = 70°$ (alternate segment theorem)

$\beta = 70°$ (alternate angles, $DE \parallel AC$)

$\gamma = 70°$ (alternate angles, $DE \parallel AC$)

$\theta = 40°$ (angles in a straight angle at B)

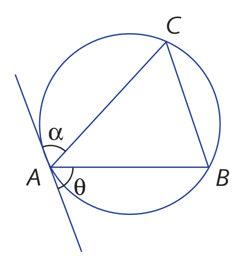
The alternate segment theorem

The angle between a tangent and a chord is equal to any angle in the alternate segment.

Exercise 13E

Note: Points labelled O in this exercise are always centres of circles.

1 Draw a large circle and a chord AB. At one end of the chord, draw the tangent to the circle.



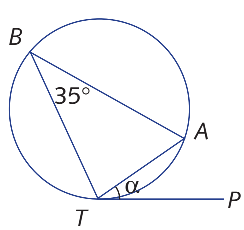
   a Mark one of the angles θ between the tangent and the chord AB, then draw any angle in the alternate segment. How are these two angles related?

   b Mark the angle α between the tangent and the chord AC. Which angle is equal to α?

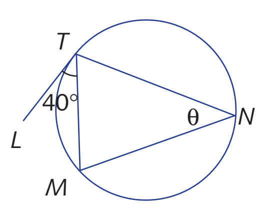
Example 8

2 Find the values of α, β, γ and θ, giving reasons. In each diagram, a tangent is drawn at T.

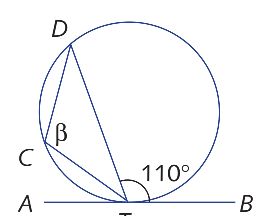
a



b



c



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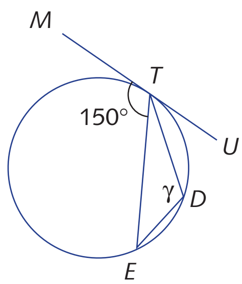
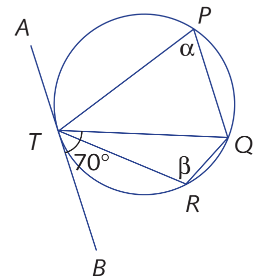
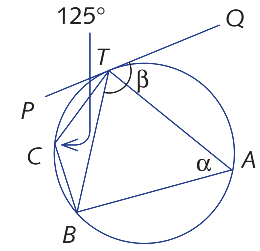
d



e

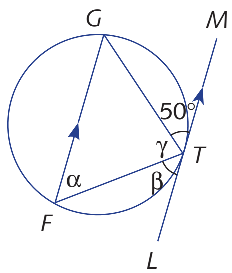


f



3 Find the values of α, β, γ and θ, giving reasons. In the diagrams, tangents are drawn at S, T and U.

a



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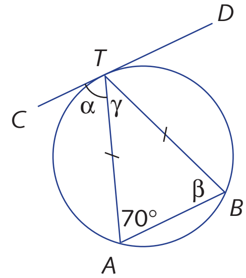

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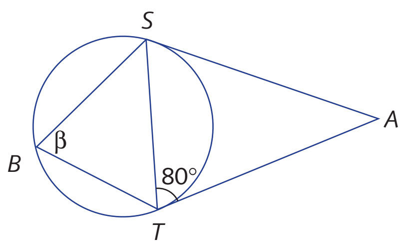

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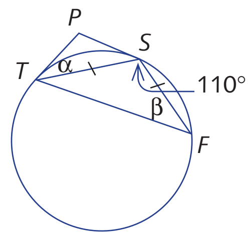

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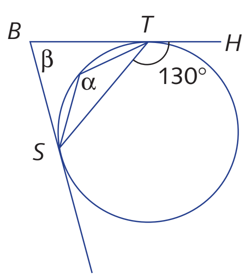

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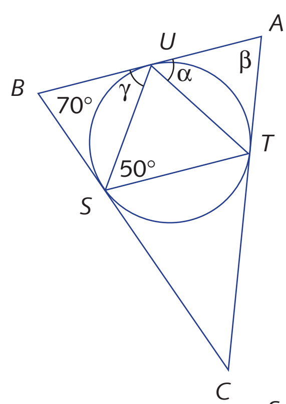
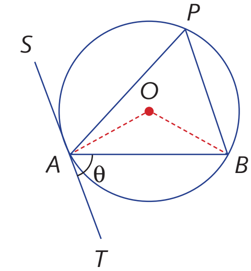


4 Here is a different proof of the alternate segment theorem.



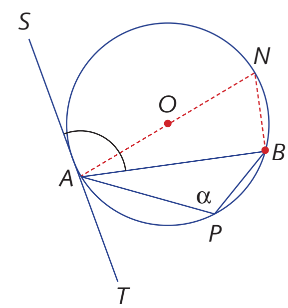
Let AB be a chord of a circle, and let SAT be the tangent at A. Let $\theta = \angle BAT$ be an acute angle. We must prove that $\angle APB = \theta$.

a Join the radii OA and OB. What is the size of $\angle BAO$?

b What is the size of $\angle AOB$?

c Hence, prove that $\angle APB = \theta$.

5 Show that the alternate segment theorem holds when the angle between the tangent and the chord is obtuse.



Let $\angle P = \alpha$. We must prove $\angle SAB = \alpha$.

a Construct diameter AN and chord NB. Show that $\angle ANB = 180° - \alpha$.

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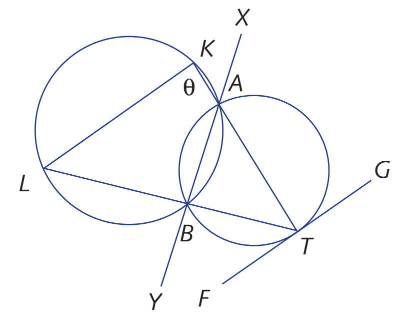
b Show that $\angle NAB = \alpha - 90°$.

c Using $\angle SAB = \angle SAN + \angle NAB$, show that $\angle SAB = \alpha$.

d Use the technique used in question **4** to prove the alternate segment theorem for an obtuse angle.

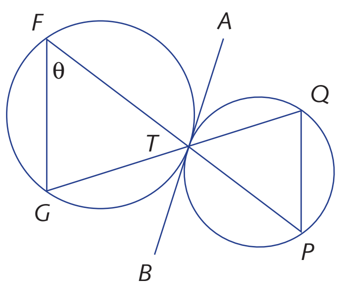
6 Choose T on the smaller circle. Let FTG be the tangent at T and construct lines KAT and TBL as shown in the diagram. Let $\angle LKT = \theta$.



a Prove that $\angle GTA = \theta$.

b Hence, prove that $LK \parallel FG$.

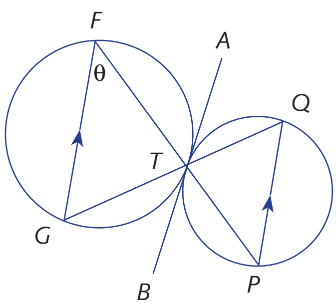
7 The two circles in the diagram touch externally at T, with common tangent ATB at the point of contact, and FTP and GTQ are straight lines.



a Let $\angle F = \theta$. Prove that $\angle GTB = \theta$ and $\angle QTA = \theta$.

b Hence, prove that $FG \parallel QP$.

8 The two circles in the diagram touch externally at T, with common tangent ATB at the point of contact. Suppose P, T and F are collinear and $GF \parallel QP$.



a Let $\theta = \angle F$. Prove that $\angle P = \theta$.

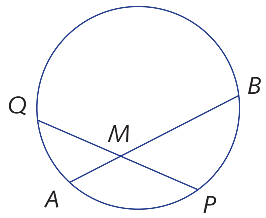
b Hence, prove that the points G, T and Q are collinear.

13F Similarity and circles

Intersecting chords

Take a point, M, inside a circle, and draw two chords, AMB and QMP, through M. Each chord is thus divided into two subintervals called **intercepts**. In the following, we shall prove that:

$$AM \times BM = PM \times QM$$



This is a very interesting and useful result called the *intersecting chord theorem*.

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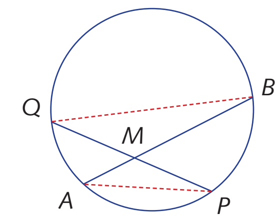
Theorem: When two chords of a circle intersect, the product of the intercepts on one chord equals the product of the intercepts on the other chord.

Proof: Draw the intervals AP and BQ to make two triangles AMP and QMB.



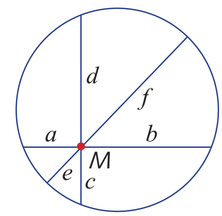
ΔAMP is similar to ΔQMB (AAA).

Hence,

$$\frac{AM}{QM} = \frac{PM}{BM} \text{ (matching sides of similar triangles)}$$

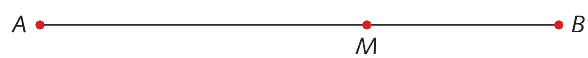
so, $AM \times BM = PM \times QM$.

Note: If we have a family of chords passing through a point, we can apply the theorem to see that $ab = cd = ef$.

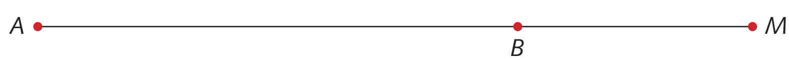


Intercepts

A point M on an interval AB divides that interval into two subintervals AM and MB, called intercepts.



For the next two theorems, we will need to apply this definition to the situation where the dividing point M is still on the line AB, but is outside the interval AB.

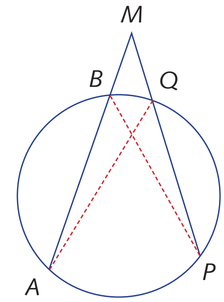


The intercepts are still AM and BM. Everything works in exactly the same way provided that both intercepts are measured from M.

Secants from an external point

Now take a point, M, outside a circle, and draw two secants, MBA and MQP, to the circle from M. Provided that we continue to take our lengths from M to the circle, the statement of the result is the same. That is:

$$AM \times BM = PM \times QM$$



Theorem: When two secants intersect outside a circle, the product of the intercepts on one secant equals the product of the intercepts on the other secant.

The proof by similarity is practically the same as when M is inside the circle, and we will address this in question **2** of Exercise 13F.

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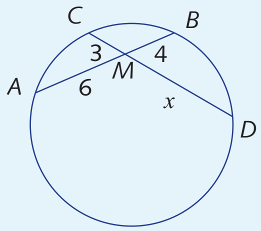
Example 9

Find x in each diagram.

a



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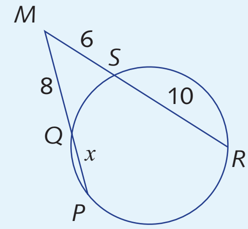


Solution

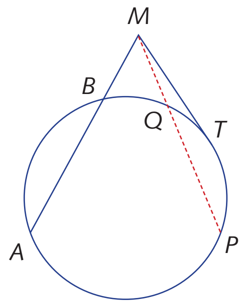
a Using intersecting chords:

$$3 \times x = 6 \times 4$$
$$x = 8$$

b Using secants from an external point:

$$8 \times (8 + x) = 6 \times (6 + 10)$$
$$8(8 + x) = 96$$
$$8 + x = 12$$
$$x = 4$$

Tangent and secant from an external point



As the point Q moves towards T, the line MQP becomes the tangent at T. Thus, the previous product $PM \times QM$ has become the square TM^2. That is:

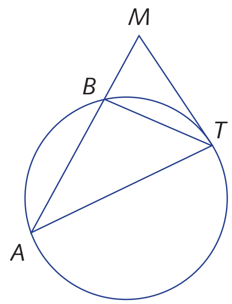
$$AM \times BM = TM^2$$

We therefore have a new theorem. For completeness, we give another proof.

Theorem: When a secant and a tangent to a circle intersect, the product of the intercepts on the secant equals the square of the tangent.

That is, $AM \times BM = TM^2$

Proof: Let M be a point external to a circle.
Let TM be a tangent from M. Suppose a secant from M cuts the circle at A and B.



Draw the intervals AT and BT, and look at the two triangles AMT and TMB.

$\angle AMT = \angle TMB$ (common angle)

$\angle MAT = \angle MTB$ (alternate segment theorem)

so ΔAMT is similar to ΔTMB (AAA).

Hence, $\dfrac{AM}{TM} = \dfrac{TM}{BM}$ (matching sides of similar triangles)

so $AM \times BM = TM^2$

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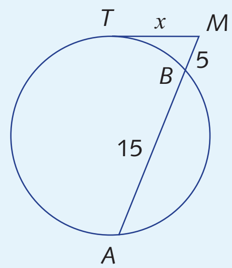
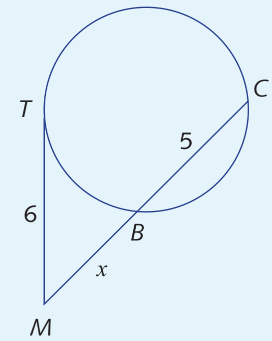
Example 10

Find x in each diagram, given that MT is a tangent to the circle.

a



b



Solution

We use the tangent and secant theorem in each part.

a $x^2 = 5 \times (5 + 15)$

$x^2 = 100$

$x = 10$ (since x is positive)

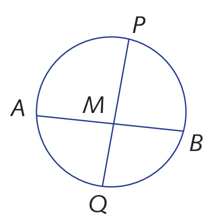
b $x \times (x + 5) = 6^2$

$x^2 \times 5x - 36 = 0$

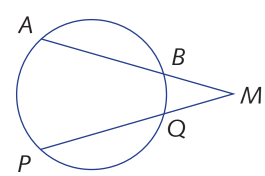
$(x + 9)(x - 4) = 0$

$x = 4$ (since x is positive)

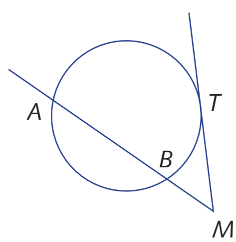
Chords, secants and tangents



$AM \times BM = PM \times QM$



$AM \times BM = PM \times QM$



$AM \times BM = TM^2$

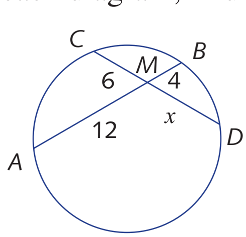
Exercise 13F

Note: Points labelled O in this exercise are always centres of circles.

Example 9, 10

1 In each diagram, find the value of x, giving reasons. Tangents are drawn at the point T.

a



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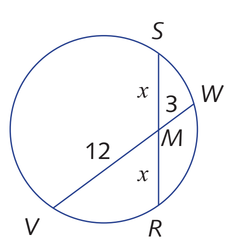

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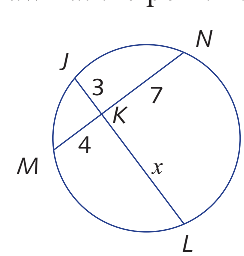

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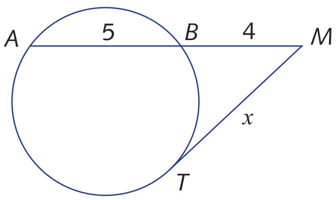

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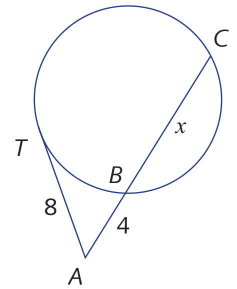

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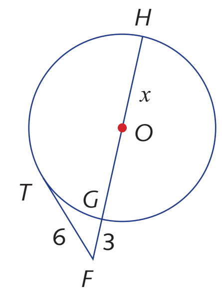

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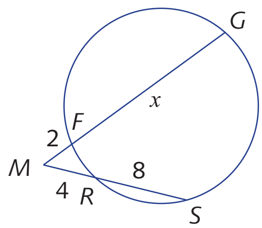

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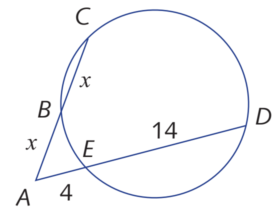

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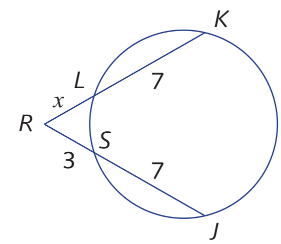

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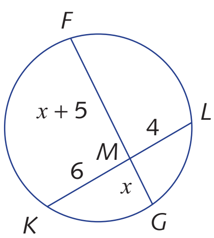

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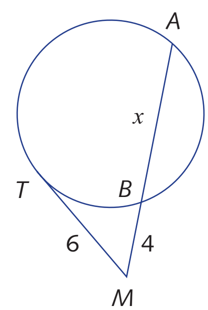

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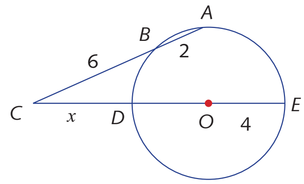
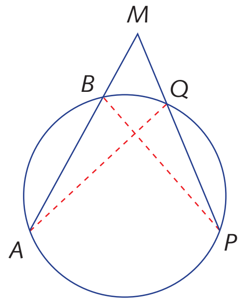


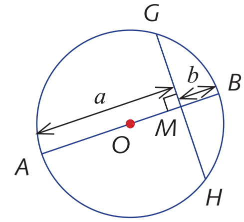
2 Let secants from a point, M, external to the circle cut the circle at points A and B and P and Q, as shown in the diagram. Prove that $AM \times BM = PM \times QM$.

This is the proof of the theorem stated on page 410.



3 Prove the result of Question **2** by drawing a tangent MT and using the 'tangent and secant' theorem.

4 Let AOB be a diameter of a circle, and let GH be a chord perpendicular to AB, meeting AB at M.



a Why is M the midpoint of GH?

b Let $g = GM$, $a = AM$ and $b = BM$. Prove that $g^2 = ab$.

c Explain why the radius of the circle is $\dfrac{a+b}{2}$.

d Prove that $\sqrt{ab} \le \dfrac{a+b}{2}$.

This is the well known Arithmetic mean–Geometric mean inequality.

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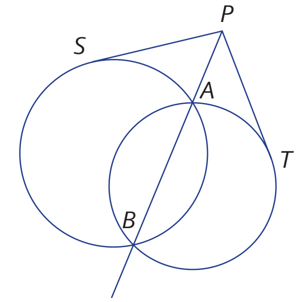
5 Let P be a point on the common secant AB of two intersecting circles. Let PS and PT be tangents from P, one to each circle. Prove that $PS = PT$.



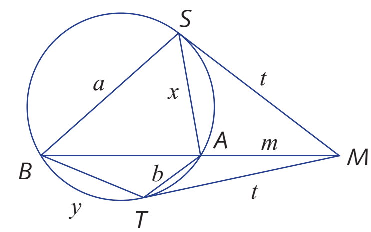
6 In the diagram, MS and MT are tangents from an external point, M.

a Prove that ΔMSA is similar to ΔMBS.

b Hence, prove that $\frac{a}{x} = \frac{t}{m}$.

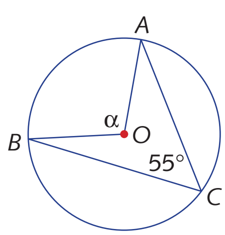
c Similarly, prove that $\frac{y}{b} = \frac{t}{m}$.

d Hence, prove that $ab = xy$.



Review exercise

1 Find the values of the pronumerals.

a


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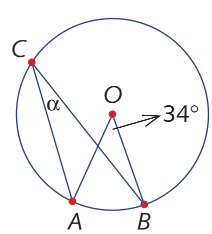

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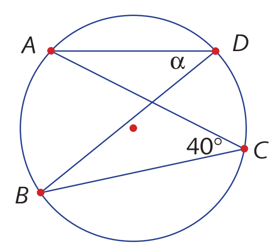

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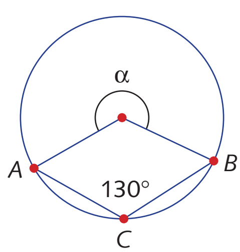

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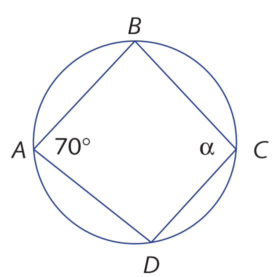

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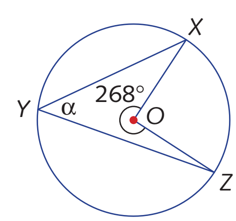

2 Find the values of the pronumerals.

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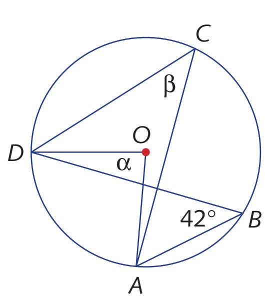

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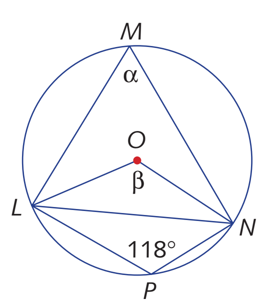

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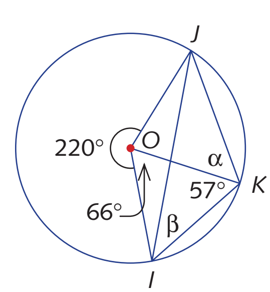
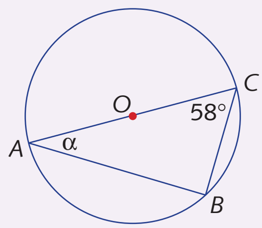


3 Find the values of the pronumerals.

a



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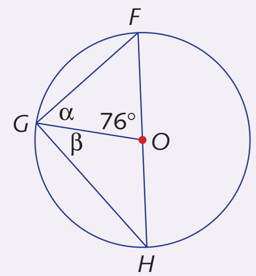

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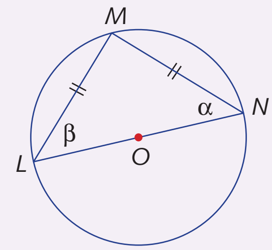


4 $ABCD$ is a cyclic quadrilateral. Its diagonals AC and BD intersect at P. Prove that ΔAPD is similar to ΔBPC.

5 The quadrilateral $ABCD$ has its vertices on a circle with centre O. The diagonal AB is a diameter of the circle and $AC = BD$. Prove that $AD = BC$.

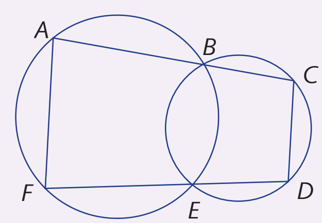
6 $ABCD$ is a cyclic quadrilateral with AD parallel to BC. The diagonals AC and BD intersect at P. Prove that $\angle APB = 2\angle ACB$.

7 $ABCD$ is a cyclic quadrilateral. Chord AB is produced and a point E is marked on the line AB so that B is between A and E. Prove that $\angle EBC = \angle ADC$.

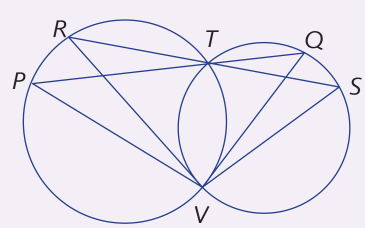
8 $PQRS$ is a cyclic quadrilateral. The diagonal PR bisects both $\angle SPQ$ and $\angle SRQ$. Prove that $\angle PQR$ is a right angle.

9 In the diagram opposite, the two circles intersect at B and E.

Prove that AF is parallel to CD.



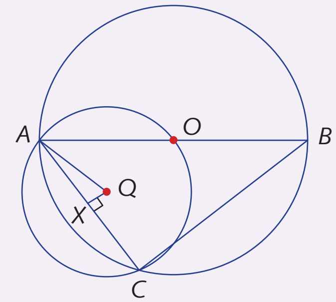
10 Two circles intersect at T and V. The intervals PTQ and RTS are drawn as shown. Prove that $\angle PVR = \angle QVS$.



11 In the figure, AOB is the diameter of the circle ABC with centre O. The point Q is the centre of another circle that passes through the points A, O and C, and $QX \perp AC$.

a Prove that $\angle AQX = 2\angle ABC$.

b Show that $AB^2 = BC^2 + 4\left(AQ^2 - XQ^2\right)$.



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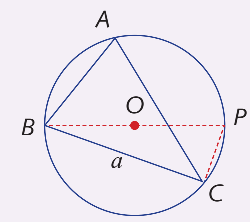
Challenge exercise

1 Here is a form of the sine rule that shows its connection with the circumcircle of a triangle.

In any triangle ABC, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R,$

where R is the radius of the circumcircle of triangle ABC. Assume that A is acute.

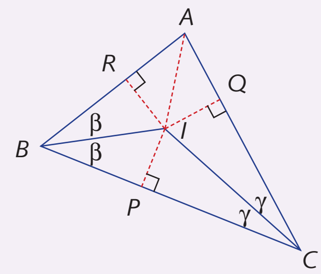


Draw the circumcircle of ΔABC, and let O be the circumcentre.

a Hence, prove that $\dfrac{a}{\sin A} = 2R$.

b Prove the result when $\angle A$ is obtuse.

2 Prove that: *In any triangle ABC, the bisectors of the vertex angles are concurrent, and the resulting* **incentre** *is the centre of a circle that touches all sides of the triangle.*



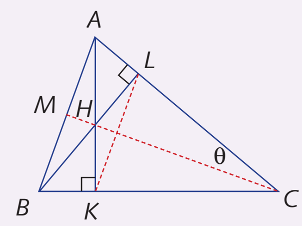
Let the angle bisectors of $\angle B$ and $\angle C$ meet at I, and draw IA. Draw the perpendiculars IP, IQ and IR to the sides BC, CA and AB, respectively.

a Use congruence to prove that $IR = IP$, and that $IP = IQ$.

b Use congruence to prove that IA bisects $\angle A$.

c Why does the circle with centre I and radius IR touch all three sides of the triangle?

3 Prove that: *In any triangle ABC, the altitudes are concurrent (their intersection is called the* **orthocentre** *of the triangle).*



Let the altitudes AK and BL meet at H. Draw CH and produce it to meet AB at M.

a Prove that C, K, H and L are concyclic.

b Let $\angle ACM = \theta$. Prove that $\angle AKL = \theta$.

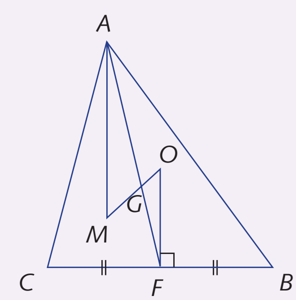
c Prove that B, K, L and A are concyclic, and hence prove that $\angle ABL = \theta$.

d Hence, prove that CM is an altitude of the triangle. That is, we have proved that the three altitudes are concurrent.

4 Euler discovered a wonderful theorem that the Greeks had missed:

The orthocentre, the centroid and the circumcentre of a triangle are collinear, with the centroid dividing the interval joining the orthocentre and circumcentre in the ratio $2:1$.

(This line is called the **Euler line**.)



Note: The **centroid** is the point of intersection of the **medians** of a triangle, which are the lines drawn from any vertex of a triangle to the midpoint of the opposite side.

Let O and G be the circumcentre and centroid, respectively, of ΔABC. Draw OG and produce it to a point, M, such that $OG : GM = 1 : 2$.

a Let F be the midpoint of BC. Use the fact that the centroid, G, divides the median AF in the ratio 2 : 1 to prove that ΔGOF is similar to ΔGMA.

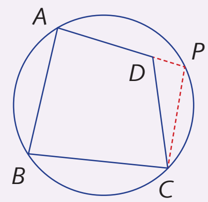
b Hence, prove that M lies on the altitude from A.

c Show that point M is the point H constructed in Question **3**.

5 Prove the following converse of the cyclic quadilateral theorem:

If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

Let the opposite angles of the quadrilateral $ABCD$ be supplementary. Draw the circle through the points A, B and C. Let AD, produced if necessary, meet the circle at P, and draw PC.



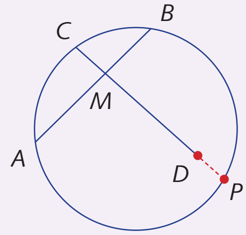
a Prove that $\angle P = \angle D$.

b Hence, prove that the points P and D coincide.

6 Prove the following converse of the intersecting chords theorem:

Suppose that two intervals AB and CD intersect at M, and that $AM \times BM = CM \times DM$. Then the points A, B, C and D are concyclic.

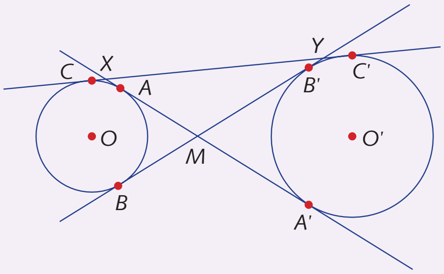
Draw the circle through the points A, B and C. Let CD, produced if necessary, meet the circle at P.



a Prove that $PM \times CM = AM \times BM$.

b Hence, prove that the points P and D coincide.

7 Take two non-intersecting circles in the plane with centres O and O'. Draw two indirect common tangents AA' and BB', and one direct tangent CC', where A, B and C lie on the first circle, and A', B' and C' lie on the second circle. Produce AA' and BB' to meet CC' at X and Y.



a Prove that $AA' = BB'$.

b Prove that $AA' = XY$.

c Describe what happens when the two circles are touching each other externally.

8 Two circles intersect at A and B. A straight line passing through A meets the two circles respectively at C and D.

a Show that any two triangles CBD formed in this way are similar.

b Which of these triangles has the larger area?

9 Two circles touch externally at P, and a common tangent touches them at A and B. Let the common tangent at P meet AB at C.

a Show that C is the midpoint of AB.

b A line passing through P meets the two circles at D and E. Draw the tangents to each circle at D and at E. Show that the tangents are parallel.

10 If ΔABC has side lengths a, b and c, prove that:

$$\frac{2 \times (\text{Area of } \Delta ABC)}{abc} = \frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b}$$

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CHAPTER

14

Number and Algebra

Indices, exponentials and logarithms – part 2

In Chapter 9, starting with integer powers of numbers, we developed the ideas of the exponential function and the logarithmic function. We learned basic properties, such as:

$$2^x 2^y = 2^{x+y} \qquad \text{and} \qquad \log_2(xy) = \log_2 x + \log_2 y$$

In this chapter, we will investigate the change of base formula and meet a range of new applications, especially applications to science.

14A Logarithm rules

In Section 9G, we introduced logarithms. Logarithms are closely related to indices. Recall that the logarithm of a number to base a is the **index** to which a is raised to give this number. For example:

$3^4 = 81$ is equivalent to $\log_3 81 = 4$

$10^6 = 1\,000\,000$ is equivalent to $\log_{10} 1\,000\,000 = 6$

$5^{-3} = \frac{1}{125}$ is equivalent to $\log_5 \frac{1}{125} = -3$

$16^{\frac{3}{4}} = 8$ is equivalent to $\log_{16} 8 = \frac{3}{4}$

In general, the **logarithmic function** is defined as follows:

If $a > 0$, $a \neq 1$ and $y = a^x$, then $\log_a y = x$

Logarithms obey a number of important laws. Each one comes from a property of indices.

Index laws

If a and b are positive numbers and x and y are rational numbers, then:

Index law 1 $a^x a^y = a^{x+y}$ Index law 2 $\frac{a^x}{a^y} = a^{x-y}$

Index law 3 $(a^x)^y = a^{xy}$ Index law 4 $(ab)^x = a^x b^x$

Index law 5 $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

The first three index laws have a direct correspondence to the first three logarithmic laws, which are developed below.

Suppose $a > 0$ and $a \neq 1$ for the rest of this section.

Logarithmic Law 1 If x and y are positive numbers, then $\log_a xy = \log_a x + \log_a y$.
That is, the logarithm of a product is the sum of the logarithms.

Suppose that $\log_a x = c$ and $\log_a y = d$

That is, $x = a^c$ and $y = a^d$

Then $xy = a^c \times a^d$

$= a^{c+d}$ (by Index law 1)

So $\log_a xy = \log_a a^{c+d}$

$= c + d$

$= \log_a x + \log_a y$

Logarithmic Law 2 If x and y are positive numbers, then $\log_a \frac{x}{y} = \log_a x - \log_a y$.

That is, the logarithm of a quotient is the difference of their logarithms.

Suppose that $\log_a x = c$ and $\log_a y = d$

That is, $x = a^c$ and $y = a^d$

$$
\begin{aligned}
\text{Then} \qquad \frac{x}{y} &= \frac{a^c}{a^d} \\
&= a^{c-d} && \text{(by Index law 2)} \\
\text{So} \qquad \log_a \frac{x}{y} &= \log_a a^{c-d} \\
&= c - d \\
&= \log_a x - \log_a y
\end{aligned}
$$

Logarithmic Law 3 If x is a positive number and n is any rational number, then $\log_a(x^n) = n\log_a x$.

This follows from index law 3. Suppose that $\log_a x = c$. That is, $x = a^c$.

$$
\begin{aligned}
\text{Then} \qquad x^n = \left(a^c\right)^n &= a^{cn} && \text{(by Index law 3)} \\
\text{So} \qquad \log_a(x^n) &= \log_a(a^{cn}) \\
\text{Hence,} \qquad \log_a(x^n) &= cn \\
&= n\log_a x, \text{ as required}
\end{aligned}
$$

Logarithmic Law 4 If x is a positive number, then $\log_a \frac{1}{x} = -\log_a x$.

This follows from logarithm law 3.

$$
\begin{aligned}
\log_a \frac{1}{x} &= \log_a x^{-1} && \text{(definition)} \\
&= -\log_a x && \text{(logarithm law 3)}
\end{aligned}
$$

Logarithmic Law 5 $\log_a 1 = 0$ and $\log_a a = 1$

Let the base a be a positive number, with $a \neq 1$.

Since $a^0 = 1$, we have $\log_a 1 = 0$.

Similarly, since $a^1 = a$, we have $\log_a a = 1$.

Example 1

Write each statement in logarithmic form.

a $2^4 = 16$ **b** $5^3 = 125$ **c** $10^{-3} = 0.001$ **d** $2^{-4} = \frac{1}{16}$

Solution

a $2^4 = 16$ so $\log_2 16 = 4$

b $5^3 = 125$ so $\log_5 125 = 3$

c $10^{-3} = 0.001$ so $\log_{10} 0.001 = -3$

d $2^{-4} = \dfrac{1}{16}$ so $\log_2 \dfrac{1}{16} = -4$

Example 2

Evaluate each logarithm.

a $\log_2 256$ **b** $\log_2 \sqrt[3]{2}$ **c** $\log_3 81$

d $\log_9 81$ **e** $\log_5 \dfrac{1}{5}$ **f** $\log_7 \dfrac{1}{49}$

Solution

Method 1

a $256 = 2^8$, so $\log_2 256 = 8$

b $\sqrt[3]{2} = 2^{\frac{1}{3}}$, so $\log_2 \sqrt[3]{2} = \dfrac{1}{3}$

c $81 = 3^4$, so $\log_3 81 = 4$

d $81 = 9^2$, so $\log_9 81 = 2$

e
$$\begin{aligned}\log_5 \frac{1}{5} &= \log_5 5^{-1}\\ &= -1\end{aligned}$$

f
$$\begin{aligned}\log_7 \frac{1}{49} &= \log_7 7^{-2}\\ &= -2\end{aligned}$$

Method 2

The following method introduces a pronumeral x.

a
$$\begin{aligned}\text{Let } x &= \log_2 256\\ \text{so } 2^x &= 256 = 2^8\\ x &= 8\end{aligned}$$

b
$$\begin{aligned}\text{Let } x &= \log_2 \sqrt[3]{2}\\ \text{so } 2^x &= \sqrt[3]{2} = 2^{\frac{1}{3}}\\ x &= \frac{1}{3}\end{aligned}$$

c
$$\begin{aligned}\text{Let } x &= \log_3 81\\ \text{so } 3^x &= 81 = 3^4\\ x &= 4\end{aligned}$$

d
$$\begin{aligned}\text{Let } x &= \log_9 81\\ \text{so } 9^x &= 81 = 9^2\\ x &= 2\end{aligned}$$

Example 3

Solve each logarithmic equation.

a $\log_2 x = 5$ **b** $\log_7(x - 1) = 2$ **c** $\log_x 64 = 6$ **d** $\log_x \dfrac{1}{25} = -2$

Solution

a $\log_2 x = 5$

so $x = 2^5$

$= 32$

b $\log_7 (x - 1) = 2$

so $x - 1 = 7^2$

$= 49$

$x = 50$

c $\log_x 64 = 6$

so $x^6 = 64$

$x^6 = 2^6$

$x = 2$, since $x > 0$

d $\log_x \dfrac{1}{25} = -2$

so $x^{-2} = \dfrac{1}{25}$

$x^2 = 25$

$x = 5$, since $x > 0$

Example 4

Write each statement in logarithmic form.

a $y = b^x$ **b** $a^x = N$ **c** $7^0 = 1$ **d** $3\sqrt{3} = 3^{\frac{3}{2}}$

Solution

a $y = b^x$ becomes $x = \log_b y$

b $a^x = N$ becomes $x = \log_a N$

c $7^0 = 1$ becomes $\log_7 1 = 0$

d $3\sqrt{3} = 3^{\frac{3}{2}}$ becomes $\log_3 3\sqrt{3} = \dfrac{3}{2}$

Example 5

Given $\log_7 2 = \alpha$, $\log_7 3 = \beta$ and $\log_7 5 = \gamma$, express each in terms of α, β and γ.

a $\log_7 6$ **b** $\log_7 75$ **c** $\log_7 \dfrac{15}{2}$

Solution

a $\log_7 6 = \log_7 (2 \times 3)$

$= \log_7 2 + \log_7 3$

$= \alpha + \beta$

b $\log_7 75 = \log_7 (3 \times 25)$

$= \log_7 3 + \log_7 5^2$

$= \log_7 3 + 2\log_7 5$

$= \beta + 2\gamma$

c $\log_7 \dfrac{15}{2} = \log_7 15 - \log_7 2$

$= \log_7 (3 \times 5) - \log_7 2$

$= \log_7 3 + \log_7 5 - \log_7 2$

$= \beta + \gamma - \alpha$

Exercise 14A

Example 2

1 Calculate each logarithm.

a $\log_2 8$ **b** $\log_3 27$ **c** $\log_2 2048$ **d** $\log_7 1$

e $\log_5 625$ **f** $\log_7 343$ **g** $\log_{10} 10\,000$ **h** $\log_{10} 1\,000\,000$

2 Calculate:

a $\log_2 \dfrac{1}{16}$ **b** $\log_3 \dfrac{1}{27}$ **c** $\log_{10} \dfrac{1}{10}$ **d** $\log_{10} 0.01$

e $\log_5 \dfrac{1}{125}$ **f** $\log_6 \dfrac{1}{36}$ **g** $\log_2 \dfrac{1}{1024}$ **h** $\log_{10} 0.0001$

3 Evaluate:

a $\log_2 2\sqrt{2}$ **b** $\log_3 9\sqrt{3}$ **c** $\log_6 36\sqrt{6}$ **d** $\log_2 4\sqrt{2}$

e $\log_3(27\sqrt{3})$ **f** $\log_{10}\left(\dfrac{1}{100\sqrt{10}}\right)$ **g** $\log_5(5^2 \times \sqrt[3]{5})$ **h** $\log_8 \sqrt{2}$

Example 3a, b

4 Solve each equation for x.

a $\log_2 x = 5$ **b** $\log_3 x = 6$ **c** $\log_{10} x = 3$

d $\log_{10} x = -3$ **e** $\log_{10} x = -4$ **f** $\log_5 x = 4$

g $\log_2(x - 3) = 1$ **h** $\log_2(x + 4) = 6$ **i** $\log_2(x - 5) = 3$

Example 3c, d

5 Solve each equation.

a $\log_x 81 = 2$ **b** $\log_x 8 = 6$ **c** $\log_x 1024 = 5$

d $\log_x 1024 = 10$ **e** $\log_x 9 = 2$ **f** $\log_x 1000 = 3$

Example 1, 4

6 Write each statement in logarithmic form.

a $2 = \left(\sqrt{2}\right)^2$ **b** $0.001 = 10^{-3}$ **c** $\left(\dfrac{1}{2}\right)^{-1} = 2$ **d** $1024 = 32^2$

e $10^x = N$ **f** $5\sqrt{2} = 5^{\frac{3}{2}}$ **g** $5^0 = 1$ **h** $13^1 = 13$

7 Write each statement in exponential form.

a $\log_2 32 = 5$ **b** $\log_3 81 = 4$ **c** $\log_{10} 0.001 = -3$

d $\log_3 27\sqrt{3} = \dfrac{7}{2}$ **e** $\log_b y = x$ **f** $\log_a N = x$

8 Simplify:

a $\log_3 7 + \log_3 5$ **b** $\log_2 3 + \log_2 5$ **c** $\log_2 9 + \log_2 7$

d $\log_{10} 5 + \log_{10} 20$ **e** $\log_6 4 + \log_6 9$ **f** $\log_3 7 + \log_3 \dfrac{1}{7}$

9 Simplify:

a $\log_3 100 - \log_3 10$ **b** $\log_7 20 - \log_7 10$ **c** $\log_7 21 - \log_7 3$

d $\log_3 17 - \log_3 51$ **e** $\log_5 100 - \log_5 10$ **f** $\log_5 10 - \log_5 2$

10 Simplify:

a $\log_2 3 + \log_2 5 + \log_2 7$

b $\log_3 100 - \log_3 10 - \log_3 2$

c $\log_5 7 + \log_5 343 - 2\log_5 49$

d $\log_7 25 + \log_7 3 - \log_7 75$

Example 5

11 Given that $\log_{10} 2 = \alpha$, $\log_{10} 3 = \beta$, $\log_{10} 5 = \gamma$ and $\log_{10} 7 = \delta$, express in terms of α, β, γ and δ:

a $\log_{10} 12$

b $\log_{10} 75$

c $\log_{10} 210$

d $\log_{10} 6\,000\,000$

e $\log_{10} 1875$

f $\log_{10} 1050$

g $\log_{10}(2^a 3^b 5^c 7^d)$

h What does $\alpha + \gamma$ equal?

12 Find a relation between x and y that does not involve logarithms.

a $\log_3 x + \log_3 y = \log_3(x + y)$

b $2\log_{10} x - 3\log_{10} y = -1$

c $\log_5 y = 3 + 2\log_5 x$

d $\log_7(1 + y) - \log_7(1 - y) = x$

13 $V = \frac{4}{3}\pi r^3$ is the volume of a sphere of radius r. Express $\log_2 V$ in terms of $\log_2 r$.

14 If $y = a \times 10^{bx}$, express x in terms of the other pronumerals.

15 Solve $\log_{10} A = bt + \log_{10} P$ for A.

14B Change of base

In Section 14A we studied logarithms to one base (which was a positive number other than 1) and their relationships, such as:

$$\log_a x + \log_a y = \log_a xy$$

Often we need to work with different bases and, in particular, calculate quantities such as $\log_5 8$, which is clearly between 1 and 2. It is of immediate concern that some calculators do not have the capacity to calculate $\log_5 8$ directly, but they can calculate $\log_{10} 8$ and $\log_{10} 5$.

We will show that $\log_5 8 = \dfrac{\log_{10} 8}{\log_{10} 5} \approx 1.2920$.

This is a special case of the **change of base formula**:

$$\log_b c = \frac{\log_a c}{\log_a b}$$

where a, b and c are positive numbers, $a \neq 1$ and $b \neq 1$.

The change of base formula is very important in later mathematics.

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Proof 1

Let $x = \log_b c$ so, $b^x = c$

Taking logarithms to base a of both sides:

$$\log_a b^x = \log_a c$$

$$x \log_a b = \log_a c \qquad \text{(Logarithm law 3)}$$

$$x = \frac{\log_a c}{\log_a b}$$

That is, $\log_b c = \dfrac{\log_a c}{\log_a b}$

Proof 2

If $\log_a b = e$, then $a^e = b$

Similarly, if $\log_b c = f$, then $b^f = c$

Hence, $c = b^f = (a^e)^f = a^{ef}$

So $\log_a c = ef = \log_a b \times \log_b c$

$$\text{and } \log_b c = \frac{\log_a c}{\log_a b}$$

Change of base formula

- If a, b and c are positive numbers, $a \neq 1$ and $b \neq 1$ then:

  $$\log_b c = \frac{\log_a c}{\log_a b}$$

- This formula can also be written as:

  $$\log_a c = \log_a b \times \log_b c$$

These formulas are called 'change of base' formulas, since they allow the calculation of logarithms to the base b from knowledge of logarithms to the base a.

Example 6

By changing to base 2, calculate $\log_{16} 8$.

Solution

$\log_2 8 = 3$ and $\log_2 16 = 4$,

$$\text{hence, } \log_{16} 8 = \frac{\log_2 8}{\log_2 16} = \frac{3}{4}$$

So $\log_{16} 8 = \dfrac{3}{4}$

As a check, $16^{\frac{3}{4}} = (2^4)^{\frac{3}{4}} = 2^3 = 8$

Example 7

Calculate $\log_7 8$, correct to four decimal places, using base 10 logarithms

Solution

Changing from base 7 to base 10:

$$\begin{aligned}\log_7 8 &= \frac{\log_{10} 8}{\log_{10} 7}\\ &\approx 1.0686\end{aligned}$$

As a check, $7^{1.0686} \approx 7.9997$ with a calculator.

Example 8

If $3^x = 7$, calculate x, correct to four decimal places.

Solution

$$\begin{aligned}x = \log_3 7 &= \frac{\log_{10} 7}{\log_{10} 3}\\ &\approx 1.7712\end{aligned}$$

Example 9

Suppose that $a > 0$. Find the exact value of $\log_{a^2} a^3$.

Solution

$$\begin{aligned}\log_{a^2} a^3 &= \frac{\log_a a^3}{\log_a a^2}\\ &= \frac{3\log_a a}{2\log_a a}\\ &= \frac{3}{2}\end{aligned}$$

As a check, $(a^2)^{\frac{3}{2}} = a^3$

Exercise 14B

In this exercise, a, b and c are positive and not equal to 1.

Example 6 **1** **a** By changing to base 3, calculate $\log_9 243$.

b By changing to base 2, calculate $\log_8 32$.

Example 7 **2** Use the change of base formula to convert to base 10 and calculate these logorithms, correct to four decimal places.

a $\log_7 9$ **b** $\log_5 3$ **c** $\log_3 5$

d $\log_3 13$ **e** $\log_{19} 17$ **f** $\log_7 \frac{1}{4}$

Example 8 **3** Solve for x, correct to four decimal places.

a $2^x = 5$ **b** $3^x = 18$ **c** $5^x = 2$

d $5^x = 17$ **e** $2^{-x} = 7$ **f** $3^{-x} = 5$

4 Solve for x, correct to four decimal places.

a $(0.01)^x = 7$ **b** $5^{1-2x} = 3$ **c** $4^{2x-1} = 7^{x-3}$ **d** $3^{3x-3} = 5^{5x-5}$

5 Simplify:

a $(\log_a b)(\log_b a)$ **b** $(\log_a b)(\log_b c)(\log_c a)$

Example 9 **6** Change to base a and simplify.

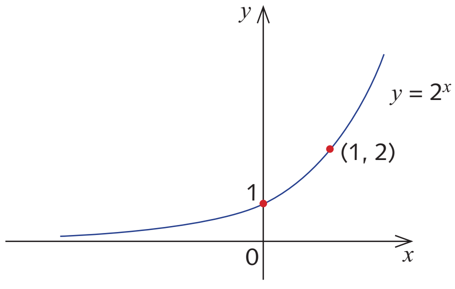
a $\log_{a^2} a^3$ **b** $\log_{a^2} a^7$ **c** $\log_{a^3} a^5$ **d** $\log_{\sqrt[3]{a}} \sqrt[11]{a}$

e $\log_a a^8 - \log_a a^7 + \log_a a^{11}$ **f** $\log_{\sqrt{a}} \sqrt[3]{a} + \log_{\sqrt[3]{a}} \sqrt[4]{a} + \log_{\sqrt[4]{a}} \sqrt[5]{a}$

14C Graphs of exponential and logarithm functions

We saw the basic shape of the graph of an exponential function in Chapter 9.

For example, $y = 2^x$ is graphed to the right.

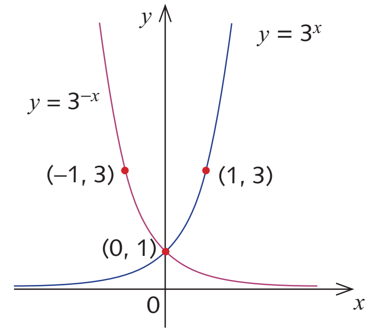


The graph has the following features:

- The y-intercept is 1.
- There is no x-intercept.
- The y-values are always positive.
- As x takes large positive values, 2^x becomes very large.
- As x takes large negative values, 2^x becomes very small.
- The x-axis is an asymptote to the graph.

Here are the graphs of $y = 3^x$ and $y = 3^{-x}$ drawn on the same axes.

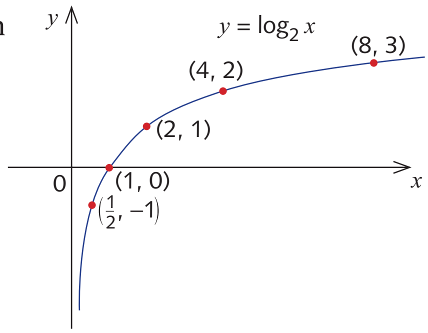
Notice that $y = 3^x$ is the reflection of $y = 3^{-x}$ in the y-axis.



Simple logarithm graphs

We can also draw the graph of $y = \log_2 x$. As usual, we begin with a table of values.

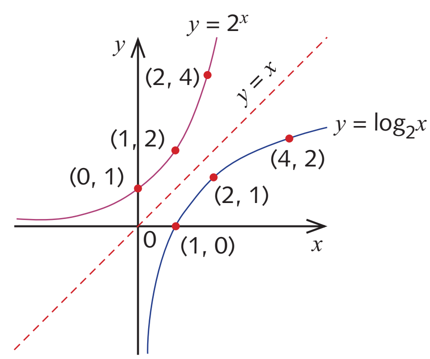
x	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16
$y = \log_2 x$	−4	−3	−2	−1	0	1	2	3	4



How are the graphs of $y = \log_2 x$ and $y = 2^x$ related?

Here is a table of values of $y = 2^x$. The graphs of $y = 2^x$ and $y = \log_2 x$ are shown on the one set of axes.

x	−4	−3	−2	−1	0	1	2	3	4
$y = 2^x$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16



If the point (a, b) lies on $y = 2^x$, then $b = 2^a$.

Hence, we can write $a = \log_2 b$, so (b, a) lies on the graph of $y = \log_2 x$.

Thus, each point on $y = \log_2 x$ can be obtained by taking a point on $y = 2^x$ and interchanging the x and y values.

The midpoint of (a, b) and (b, a) is $\left(\frac{a+b}{2}, \frac{b+a}{2}\right)$, and thus always lies on the line $y = x$.

Graphically this means (a, b) is the reflection of (b, a) in the line $y = x$ and vice versa. This is evident in the above pair of graphs.

From this we can list some of the features of the graph of $y = \log_2 x$.

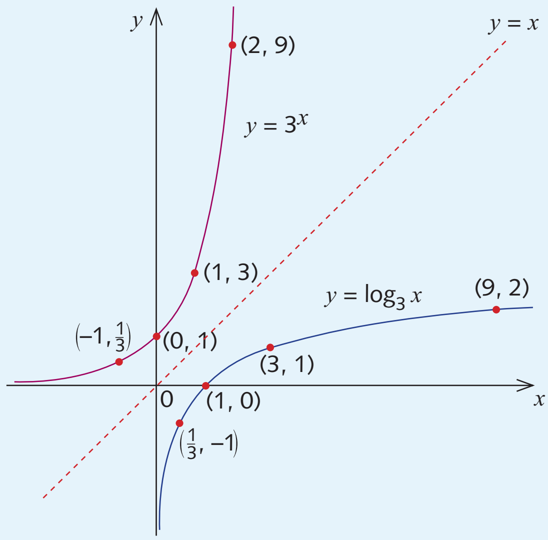
- The graph is to the right of the y-axis. (This is because the function is only defined for $x > 0$.)
- The y-axis is a vertical asymptote to the graph.
- The x-intercept is $(1, 0)$, corresponding to $\log_2 1 = 0$.
- The graph does not have a y-intercept.
- As x takes very large positive values, $\log_2 x$ becomes large positive.
- As x takes very small positive values, $\log_2 x$ becomes large negative.
- The graph is a reflection of $y = 2^x$ in the line $y = x$.

Example 10

Use the graph of $y = 3^x$ to assist in sketching $y = \log_3 x$.

Solution

First draw the graph of $y = 3^x$.



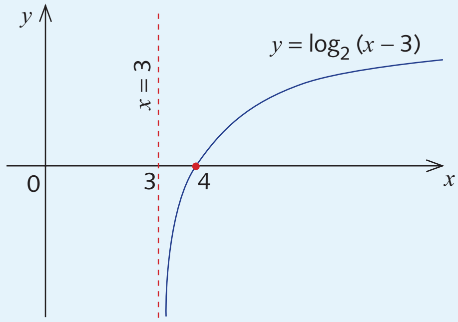
The two graphs are reflections of each other in the line $y = x$.

Example 11

Sketch the graph of $y = \log_2(x - 3)$.

Solution

Translate the graph of $y = \log_2 x$ three units to the right.



Note that the line $x = 3$ is an asymptote to the graph.

Example 12

Sketch the graphs of $y = \log_3 x$ and $y = \log_5 x$ on the same set of axes.

Solution

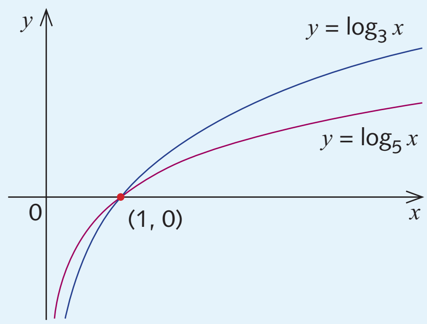
x	$\frac{1}{25}$	$\frac{1}{5}$	1	5	25
$y = \log_3 x$	–2.93	–1.46	0	1.46	2.93
$y = \log_5 x$	–2	–1	0	1	2

$\log_3 5 = \dfrac{\log_{10} 5}{\log_{10} 3} \approx 1.46$

$\log_3 25 = \log_3 5^2 = 2\log_3 5 \approx 2.93$

$\log_3 \dfrac{1}{5} = -\log_3 5 \approx -1.46$

$\log_3 \dfrac{1}{25} = \log_3 5^{-2} = -2\log_3 5 \approx -2.93$



The table of values shows that:

$\log_3 x > \log_5 x$ if $x > 1$ and $\log_5 x > \log_3 x$ if $0 < x < 1$

Exercise 14C

Example 10

1 a Use the graph of $y = 4^x$ to draw the graph of $y = \log_4 x$.

b Use the graph of $y = 5^x$ to draw the graph of $y = \log_5 x$.

2 For each of these logarithm functions, produce a table of values for (x, y), using the following y-values: –2, –1, 0, 1, 2. Use the table to draw the graph of the function.

a $y = \log_{10} x$ **b** $y = \log_6 x$

3 Draw each set of graphs on the same axes.

a $y = 3^x$, $y = 3^x + 1$, $y = 3^x - 2$ **b** $y = 5^x$, $y = 2 \times 5^x$, $y = \frac{1}{2} \times 5^x$

c $y = 2^x$, $y = 2^{-x}$ **d** $y = \left(\frac{1}{2}\right)^x$, $y = \left(\frac{1}{2}\right)^{-x}$

Example 11

4 a Sketch the graphs of $y = \log_2 x$ and $y = \log_3 x$ on the same set of axes, for y values between –3 and 3.

b In what ways are the graphs similar?

c How do the graphs differ?

d Without using a table of values, sketch the graph of $y = \log_4 x$ on the same set of axes used in part **a**.

Example 12

5 Sketch the following graphs.

a $y = \log_3 x,\ x > 0$

b $y = \log_3(x - 1),\ x > 1$

c $y = \log_3(x + 5),\ x > -5$

d $y = 2\log_3 x,\ x > 0$

e $y = \log_3(x) + 2,\ x > 0$

6 Sketch $y = 2^x$, $y = 3^x$, $y = \log_2 x$ and $y = \log_3 x$ on the one set of axes.

14D Applications to science, population growth and finance

In Section 9F of you saw that in a given experiment, the growth in bacteria could be described using an **exponential function**, such as $N = 1000 \times 2^t$.

Here, N is the number of bacteria at time t, measured in hours.

Equations of this type arise in many practical situations in which we know the value of N, but want to solve for t.

Logarithms are needed for such calculations.

Example 13

Initially there are 1000 bacteria in a given culture. The number of bacteria, N, is doubling every hour, so $N = 1000 \times 2^t$, where t is measured in hours.

a How many bacteria are present after 24 hours? Give your answer correct to three significant figures.

b How long is it until there are one million bacteria? Give your answer correct to three significant figures.

Solution

a After 24 hours, $N = 1000 \times 2^{24}$

$$\approx 1.68 \times 10^{10}$$

b If $N = 10^6$, then $10^6 = 1000 \times 2^t$

$$\begin{aligned} 2^t &= 1000 \\ \log_{10} 2^t &= \log_{10} 1000 \\ t \log_{10} 2 &= 3 \\ t &= \frac{3}{\log_{10} 2} \\ &\approx 9.97 \text{ hours} \end{aligned}$$

There are one million bacteria after approximately 9.97 hours.

The following example illustrates the use of logarithms in estimating the age of fossils.

Example 14

The carbon isotope carbon-14, C^{14}, occurs naturally but decays with time. Measurements of carbon-14 in fossils are used to estimate the age of samples.

If M is the mass of carbon-14 at time t years and M_0 is the mass at time $t = 0$, then $M = M_0 10^{-kt}$ where $k = 5.404\,488\,252 \times 10^{-5}$.

All 10 digits are needed to achieve reasonable accuracy in these calculations.

a Calculate the fraction left after 100 years as a percentage.

b Calculate the fraction left after 10 000 years as a percentage.

c Calculate the half-life of C^{14}. That is, after how long does $M = \frac{1}{2}M_0$?

Solution

a When $t = 100$,
$$\begin{aligned} M &= M_0 10^{-100k} \\ \frac{M}{M_0} &= 10^{-100k} \\ &\approx 0.987\,63 \\ &\approx 98.76\% \end{aligned}$$

That is, the fraction left after 100 years is 98.76%.

b When $t = 10\,000$,
$$\begin{aligned} \frac{M}{M_0} &= 10^{-10\,000k} \\ &\approx 0.288\,11 \\ &\approx 28.81\% \end{aligned}$$

That is, the fraction left after 10 000 years is 28.81%.

c $M = \frac{1}{2}M_0$ when $\frac{1}{2} = 10^{-kt}$
$$\begin{aligned} \log_{10}\frac{1}{2} &= -kt \\ kt &= \log_{10} 2 \\ t &= \frac{\log_{10} 2}{k} \\ &\approx 5570.000\,001 \\ &\approx 5570 \text{ years} \end{aligned}$$

That is, the half-life of C^{14} is about 5570 years.

Compound interest

In Section 1D, we introduced the compound interest formula:

$$A_n = P(1+R)^n$$

where A_n is the amount that the investment is worth after n units of time, P is the principal and R is the interest rate.

Logarithms can be used to find the value of n in this formula given R, P and A_n.

Example 15

\$50 000 is invested on 1 Jan at 8% per annum. Interest is only paid on 1 Jan of each year. At the end of how many years will the investment be worth

a \$75 000?

b \$100 000?

Solution

a $A_n = P(1+R)^n$

$A_n = 75\,000,\ P = 50\,000$ and $R = 0.08,$ so

$$75\,000 = 50\,000(1.08)^n$$

$$\frac{3}{2} = (1.08)^n$$

$$\log_{10}\frac{3}{2} = n\log_{10}(1.08) \quad \text{(Take logarithms of both sides.)}$$

$$n = \frac{\log_{10}\left(\frac{3}{2}\right)}{\log_{10}(1.08)}$$

$$= 5.268\,44\ldots$$

At the end of the sixth year, the investment will be worth $\$50\,000(1.08)^6 = \$79\,343.72$.

At the end of the fifth year, the investment will be worth $\$50\,000(1.08)^5 = \$73\,466.40$.

The investment will be worth more than \$75 000 at the end of the sixth year.

b $A_n = P(1+R)^n$

$A_n = 100\,000,\ P = 50\,000$ and $R = 0.08,$ so

$$100\,000 = 50\,000(1.08)^n$$

$$2 = (1.08)^n$$

$$\log_{10}(2) = n\log_{10}(1.08) \quad \text{(Take logarithms of both sides.)}$$

$$n = \frac{\log_{10}(2)}{\log_{10}(1.08)}$$

$$= 9.006\,46\ldots$$

At the end of the tenth year, the investment will be worth $\$50\,000(1.08)^{10} = \$107\,946.25$.

At the end of the ninth year, the investment will be worth $\$50\,000(1.08)^9 = \$99\,950.23$.

The investment will be worth more than \$100 000 at the end of the tenth year.

Exercise 14D

Example 13

1 A culture of bacteria initially has a mass of 3 grams and its mass doubles in size every hour. How long will it take to reach a mass of 60 grams?

2 A culture of bacteria initially weighs 0.72 grams and is multiplying in size by a factor of five every day.

a Write down a formula for M, the weight of bacteria in grams after t days.

b What is the weight after two days?

c How long will the culture take to double its weight?

d The mass of the Earth is about 5.972×10^{24} kg. After how many days will the culture weigh the same as the Earth?

e Discuss your answer to part **d**.

3 The population of the Earth at the beginning of 1976 was four billion. Assume that the rate of growth is 2% per year.

a Write a formula for P, the population of the Earth in year t, $t \geq 1976$.

b What will be the population in 2076?

c When will the population reach 10 billion?

4 The population of the People's Republic of China in 1970 was 750 million. Assume that its rate of growth is 4% per annum.

a Write down a formula for C, the population of China in year t, $t \geq 1970$.

b When would the population of China reach two billion?

c With the assumptions of question **3**, when would the population of China be equal to half the population of the Earth?

d When would everyone in the world be Chinese? (Discuss your answer.)

Example 14

5 The mass M of a radioactive substance is initially 10 g and 20 years later its mass is 9.6 g. If the relationship between M grams and t years is of the from $M = M_o 10^{-kt}$, find:

a M_o and k

b the half-life of the radioactive substance

Example 15

6 An amount of \$80 000 is invested on 1 Jan at a compound interest rate of 7% per annum. Interest is only paid on 1 Jan of each year. At the end of how many years will the investment be worth:

a \$110 000 ?

b \$200 000?

7 A man now owes the bank \$47 000, after taking out a loan n years ago with an interest rate of 10% per annum. He borrowed \$26 530. Find n.

8 The formula for the calculation of compound interest is $A_n = P(1 + R)^n$. Find, correct to one decimal place:

a A_n if $P = \$50\,000$, $R = 8\%$ and $n = 3$

b P if $A_n = \$80\,000$, $R = 5\%$ and $n = 4$

c n if $A_n = \$60\,000$, $R = 2\%$ and $P = \$20\,000$

d n if $A_n = \$90\,000$, $R = 4\%$ and $P = \$20\,000$

Review exercise

1 Calculate each logarithm.

a $\log_2 16$ **b** $\log_5 125$ **c** $\log_2 512$ **d** $\log_7 1$

e $\log_3 \frac{1}{27}$ **f** $\log_2 \frac{1}{64}$ **g** $\log_{10} 10\,000$ **h** $\log_{10}(0.001)$

2 Solve each logarithmic equation.

a $\log_x 16 = 2$ **b** $\log_x 64 = 6$

c $\log_x 2048 = 11$ **d** $\log_x 512 = 3$

e $\log_x 25 = 2$ **f** $\log_x 125 = 3$

3 Write each statement in logarithmic form.

a $1024 = 2^{10}$ **b** $10^x = a$

c $6^0 = 1$ **d** $11^1 = 11$

e $3^x = b$ **f** $5^4 = 625$

4 Write each statement in exponential form.

a $\log_3 81 = 4$ **b** $\log_2 64 = 6$

c $\log_{10} 0.01 = -2$ **d** $\log_b c = a$

e $\log_a b = c$

5 Simplify:

a $\log_2 11 + \log_2 5$ **b** $\log_2 7 + \log_2 5$

c $\log_6 11 + \log_6 7$ **d** $\log_3 8 - \log_3 32$

e $\log_5 200 - \log_5 40$ **f** $\log_5 30 - \log_5 6$

6 Simplify:

a $\log_2 5 + \log_2 4 + \log_2 7$ **b** $\log_5 1000 - \log_5 100 - \log_5 10$

c $\log_7 7 + \log_7 343 - 3\log_7 49$ **d** $\log_3 25 + 2\log_3 5 - 2\log_3 75$

7 Use the change of base formula to convert to base 10 and calculate each to four decimal places.

a $\log_7 11$ **b** $\log_5 7$ **c** $\log_3 24$

d $\log_3 35$ **e** $\log_{16} 8$ **f** $\log_3 \frac{1}{4}$

8 Solve for x, correct to four decimal places.

a $2^x = 7$ **b** $3^x = 78$

c $5^x = 28$ **d** $5^x = 132$

e $2^{-x} = 5$ **f** $3^{-x} = 15$

9 Solve for x.

a $\log_2(2x - 3) = 4$ **b** $\log_3 3x = 4$

c $\log_2(3 - x) = 2$ **d** $\log_{10} x = 4$

e $\log_4(5 - 2x) = 3$ **f** $\log_2(x - 6) = 2$

10 Sketch each graph.

a $y = \log_5 x,\ x > 0$

b $y = \log_3(x - 2),\ x > 2$

c $y = \log_2(x + 4),\ x > -4$

d $y = \log_2(x) + 5,\ x > 0$

11 Express y in terms of x when:

a $\log_{10} y = 1 + \log_{10} x$

b $\log_{10}(y + 1) = 2 + \log_{10} x$

12 Simplify $\log_2\left(\frac{8}{75}\right) - 2\log_2\left(\frac{3}{5}\right) - 4\log_2\left(\frac{3}{2}\right)$.

13 If $\log_{10} x = 0.6$ and $\log_{10} y = 0.2$, evaluate $\log_{10}\left(\frac{x^2}{\sqrt{y}}\right)$.

14 **a** Express $3 + \log_2 5$ as a single logarithm.

b Express $5 - \log_2 5$ as a single logarithm.

15 An amount of \$120 000 is invested on 1 Jan at a compound interest rate of 8% per annum. Interest only paid on 1 Jan of each year. At the end of how many years will the investment be worth:

a \$160 000? **b** \$200 000?

Challenge exercise

Throughout this exercise, the bases a and b are positive and not equal to 1.

1 Consider a right-angled triangle with side lengths a, b and c, with c the hypotenuse. Prove that $\log_{10} a = \frac{1}{2}\log_{10}(c+b) + \frac{1}{2}\log_{10}(c-b)$.

2 Simplify $\log_a(a^2 + a) - \log_a(a + 1)$.

3 Show that $3\log_{10} x + 2\log_{10} y - \frac{1}{2}\log_{10} z = \log_{10}\left(\frac{x^3y^2}{\sqrt{z}}\right)$.

4 Solve for x:

a $\log_2(x+1) - \log_2(x-1) = 3$

b $(\log_{10} x)(\log_{10} x^2) + \log_{10} x^3 - 5 = 0$

c $(\log_2 x^2)^2 - \log_2 x^3 - 10 = 0$

d $(\log_3 x)^2 = \log_3 x^5 - 6$

5 Solve each set of simultaneous equations.

a $9^x = 27^{y-3}$, $16^{x+1} = 8^y \times 2$

b $8^x = 32^{y+1}$, $5^{x-1} = 25^y$

c $49^{x+3} = 343^{y-1}$, $2^{x+y} = 8^{x-2y}$

d $8^x = 4^y$, $7^{3x+3} = 343^y$

6 Solve the equation $(\log_a x)(\log_b x) = \log_a b$ for x where a and b are positive numbers different from 1.

7 If $a = \log_8 225$ and $b = \log_2 15$, find a in terms of b.

8 **a** Show that $\log_{10} 3$ cannot be a rational number.

b Show that $\log_{10} n$ cannot be a rational number if n is any positive integer that is not a whole number power of 10.

9 Prove that $\log_a\left(\frac{xy}{z}\right) + \log_a\left(\frac{yz}{x}\right) + \log_a\left(\frac{zx}{y}\right) = \log_a x + \log_a y + \log_a z$.

10 If x and y are distinct positive numbers, $a > 0$ and $\frac{\log_a x}{y-z} = \frac{\log_a y}{z-x} = \frac{\log_a z}{x-y}$, show $xyz = 1$ and $x^x y^y z^z = 1$.

11 If $2\log_a x = 1 + \log_a(7x - 10a)$, find x in terms of a, where a is a positive constant and x is positive.

CHAPTER

15

Statistics and Probability

Probability

In this chapter, we continue our study of probability. In particular, we introduce the important ideas of sampling with and without replacement. The other important new ideas in this chapter are the concepts of conditional probability and independence.

15A Review of probability

We first review the basic ideas of probability that we introduced in Chapter 12 of *ICE-EM Mathematics Year 9*.

Sample spaces with equally likely outcomes

In *ICE-EM Mathematics Year 9*, we looked at the experiment of throwing two dice and recording the values on the uppermost faces. The results can be displayed in an array, as shown here.

Die 1 \ Die 2	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

The **sample space**, ξ, for this experiment is the set of ordered pairs displayed in the array.

That is, $\xi = \{(1, 1), (1, 2), ..., (6, 6)\}$. The 36 **outcomes** of this experiment are equally likely and each outcome has probability $\frac{1}{36}$.

Example 1

Two dice are thrown and the value on each die is recorded. Find the probability that:

a the sum of the two values is 5

b the sum of the two values is less than or equal to 3

Solution

The sample space ξ is as described as above. The size of ξ is 36.

a Let A be the event that the sum is 5.

$A = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$

$P(A) = \frac{4}{36} = \frac{1}{9}$

b Let B be the event that the sum is less than or equal to 3.

$B = \{(1, 1), (1, 2), (2, 1)\}$

$P(B) = \frac{3}{36} = \frac{1}{12}$

Sample spaces with non-equally likely outcomes

We can change the experiment to: Two dice are thrown and the *sum* of the values on the uppermost faces is recorded.

This leads to a different sample space:

$$\xi = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

The outcomes are no longer equally likely, since, for example, we can only obtain a total of 2 by throwing a 1 and a 1, but there are 5 ways to obtain a sum of 6.

We can determine the probability of each of these outcomes from the array on the previous page.

The probabilities are listed in the table below.

Outcome	2	3	4	5	6	7	8	9	10	11	12
P(outcome)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

The sum of the probabilities of the outcomes is 1.

Events

An **event** is a subset of the sample space. For example, in the experiment of throwing two dice and recording the sum of the uppermost faces, an event is a subset of:

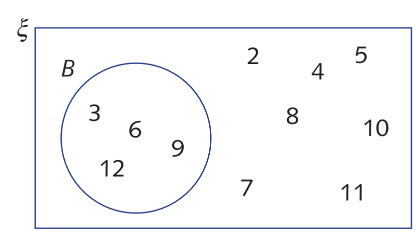
$$\xi = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

For example, the event B {outcomes whose sum is divisible by 3} is the subset:

$$B = \{3, 6, 9, 12\}$$

We will often use a more colloquial description of such events. For example, we will say B is the event 'the sum is divisible by 3'.

An outcome is **favourable** to an event if it is a member of that event. For example, $6 \in B$ and $5 \notin B$. The event B can be illustrated with a Venn diagram.



Probability of an event

The probability p of an outcome is a number between 0 and 1 inclusive.

Probabilities are assigned to outcomes in such a way that the sum of the probabilities of all the outcomes in the sample space ξ is 1.

The probability of the event A is written as $P(A)$. Thus, $P(A)$ is the sum of the probabilities of the outcomes that are favourable to the event A.

Hence, $0 \le P(A) \le 1$, for each event A. That is, the probability of an event is a number between 0 and 1 inclusive. In particular, $P(\xi) = 1$.

For the event B {outcomes whose sum is divisible by 3} in the previous example:

$$\begin{aligned} P(B) &= P(3) + P(6) + P(9) + P(12) \\ &= \frac{2}{36} + \frac{5}{36} + \frac{4}{36} + \frac{1}{36} \\ &= \frac{1}{3} \end{aligned}$$

For an experiment in which all of the outcomes are equally likely:

$$\text{Probability of an event} = \frac{\text{number of outcomes favourable to that event}}{\text{total number of outcomes}}$$

This is *not* the case for the experiment of throwing two dice and recording the sum, as we learned that such an event had non-equally likely outcomes.

Example 2

If a die is rolled, what is the probability that a number greater than 4 is obtained?

Solution

When a die is rolled once, there are six equally likely outcomes
$\xi = \{1, 2, 3, 4, 5, 6\}$

Let A be the event 'a number greater than four is obtained'.
Then $A = \{5, 6\}$

Hence, $P(A) = \frac{2}{6} = \frac{1}{3}$

Example 3

A standard pack of playing cards consists of four suits: Hearts, Diamonds, Clubs and Spades. Each suit has 13 cards consisting of an Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen and King.

The pack is shuffled and a card is drawn at random.

For this experiment the size of ξ is 52.

a What is the probability that it is a King?

b What is the probability that it is a Heart?

Solution

a Let K be the event 'drawing a King'.

There are four Kings in the pack of 52 cards.

$P(K) = \frac{4}{52} = \frac{1}{13}$

b Let H be the event 'drawing a Heart'.

There are 13 Hearts in the pack of 52 cards.

$P(H) = \frac{13}{52} = \frac{1}{4}$

Example 4

One box contains 4 discs labelled as shown.

1 3 6 8

A second box contains 5 discs labelled as as shown.

2 4 5 7 9

A disc is taken from each of the boxes and the larger of the two numbers is recorded.

a What is a sample space for the experiment?

b Find the probability of each outcome.

c Find the probability that the number obtained is less than 5.

Solution

There are 20 different pairs that can be drawn from the two boxes. Each of these pairs is equally likely to occur. The larger of the two numbers is recorded in the array.

Box 1 \ Box 2	2	4	5	7	9
1	2	4	5	7	9
3	3	4	5	7	9
6	6	6	6	7	9
8	8	8	8	8	9

a The sample space is $\xi = \{2, 3, 4, 5, 6, 7, 8, 9\}$

b From the array:

$$P(2) = \frac{1}{20},\ P(3) = \frac{1}{20},\ P(4) = \frac{2}{20},\ P(5) = \frac{2}{20},$$

$$P(6) = \frac{3}{20},\ P(7) = \frac{3}{20},\ P(8) = \frac{4}{20},\ P(9) = \frac{4}{20}$$

c $$\begin{aligned} P(\{2, 3, 4\}) &= P(2) + P(3) + P(4) \\ &= \frac{1}{20} + \frac{1}{20} + \frac{2}{20} \\ &= \frac{1}{5} \end{aligned}$$

Review of probability

- A **sample space**, ξ, consists of all possible outcomes of an experiment.
- Each outcome has a probability p between 0 and 1. That is, $0 \le p \le 1$.
- The sum of the probabilities of all outcomes is 1.
- An **event**, A, is a subset of ξ. A member of A is called an outcome **favourable** to A.
- $P(A)$ is the sum of the probabilities of all outcomes favourable to A.
- For an experiment in which all the outcomes are equally likely:

$$\text{Probability of an event} = \frac{\text{number of outcomes favourable to that event}}{\text{total number of outcomes}}$$

Exercise 15A

Example 1, 2

1 David has 13 marbles. Five of them are pink, three are blue, three are green and two are black. If he chooses a marble at random, what is the probability that it is green?

Example 3

2 A debating team consists of five boys and seven girls. If one of the team is chosen at random to be the leader, what is the probability that the leader is a girl?

3 A basketball team consists of five players: Adams, Brown, Cattogio, O'Leary and Nguyen. If a player is chosen at random, what is the probability that his name starts with a consonant?

4 Slips of paper numbered $1, 2, 3, ..., 10$ are placed in a hat and one is drawn at random. What is the probability that the number on the slip of paper is not a multiple of four?

5 A bag contains 11 balls. Three of these are black and eight are blue. A ball is taken from the bag at random. What is the probability that it is blue?

Example 4

6 One box contains 4 discs labelled as shown.

(2) (3) (7) (8)

A second box contains 5 discs labelled as shown.

(5) (6) (7) (8) (9)

A disc is taken from each of the boxes and the larger of the two numbers is recorded.

a List the sample space for the experiment.

b Find the probability of each outcome.

c Find the probability that the number obtained is greater than 6.

7 A box contains 3 discs labelled as shown.

(1) (2) (3)

A second box contains 3 discs labelled as shown.

(2) (4) (7)

A disc is taken randomly from each box and the result is recorded as an ordered pair, for example, $(1, 7)$.

a List the sample space for the experiment.

b Find the probability of each outcome.

c Find the probability that there is an even number on both of the selected discs.

8 A box contains 4 discs labelled as shown.

1 2 3 4

A second box contains 3 discs labelled as shown.

1 2 3

A disc is taken randomly from each box and the sum of the numbers on the two discs is recorded.

a List the sample space for the experiment.

b Find the probability of each outcome.

c Find the probability that the sum is less than 5.

9 Two dice are thrown and the values on the uppermost faces recorded. What is the probability of:

a obtaining an even number on both dice?

b obtaining exactly one 6?

c obtaining a 3 on one die and an even number on the other?

10 Two dice are thrown and the difference of the values on the uppermost faces is recorded: outcome = value on die 1 – value on die 2

a List the sample space for this experiment.

b What is the probability of obtaining a negative number?

c What is the probability of obtaining a difference of 0?

d What is the probability of obtaining a difference of -1?

e What is the probability of obtaining a difference that is exactly divisible by 3?

11 A bag contains six balls: three red balls numbered 1 to 3, two white balls numbered 1 and 2, and one yellow ball. Two balls are selected one after the other, at random, and the first is replaced before the second is withdrawn.

a List the sample space.

b Find the probability that:

i both balls are the same colour

ii the two balls selected are different colours

12 The surnames of 800 male students on a school roll vary in length from 3 letters to 11 letters as follows:

Number of letters	3	4	5	6	7	8	9	10	11
Number of boys	16	100	171	206	144	97	51	13	2

If a boy is selected at random from those in this school, what is the probability that his surname contains:

a four letters? **b** more than eight letters? **c** less than five letters?

15B The complement, union and intersection

The complement of A

In some problems, the outcomes in the event A can be difficult to count; whereas the event 'not A' may be easier to deal with.

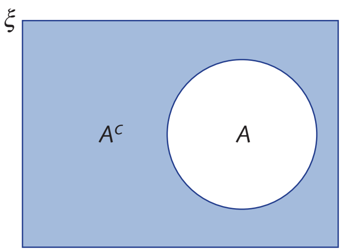
The event 'not A' consists of every possible outcome in the sample space ξ that it is not in A. The set 'not A' is called the **complement** of A and is denoted by A^c.

Every outcome in the sample space ξ is contained in exactly one of A or A^c.

Therefore:

$$P(A) + P(A^c) = 1 \text{ and so } P(A^c) = 1 - P(A)$$

This can be illustrated with a Venn diagram.



Example 5

A card is drawn from a standard pack. What is the probability that it is not the King of Hearts?

Solution

Let A be the event 'the King of Hearts is drawn'.

Then A^c is the event 'the King of Hearts is not drawn'.

$$P(A) = \frac{1}{52}$$

$$\begin{aligned} P(A^c) &= 1 - P(A) \\ &= 1 - \frac{1}{52} \\ &= \frac{51}{52} \end{aligned}$$

The probability that the card drawn is not the King of Hearts is $\frac{51}{52}$.

Union and intersection

Sometimes, rather than just considering a single event, we want to look at two or more events.

We return to our example of throwing two dice and taking the sum of the numbers on the uppermost faces.

Recall that $\xi = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

Let A be the event 'a number divisible by 3 is obtained'.

Let B be the event 'a number greater than 5 is obtained'.

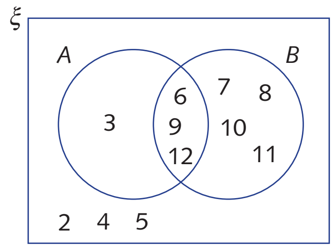
The events A and B are:

$A = \{3, 6, 9, 12\}$

$B = \{6, 7, 8, 9, 10, 11, 12\}$

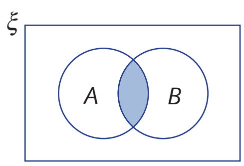
and $A \cap B = \{6, 9, 12\}$

Here is the Venn diagram illustrating these events.

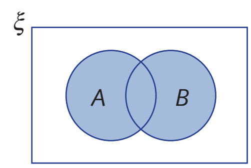


The outcomes favourable to the event 'the number is divisible by 3 *and* greater than 5' is the **intersection** of the sets A and B; that is, $A \cap B$. The event $A \cap B$ is often called 'A *and* B'.

The outcomes favourable to the event 'the number is divisible by 3 *or* greater than 5' is the **union** of the sets A and B; that is, $A \cup B$. The event $A \cup B$ is often called 'A *or* B'.



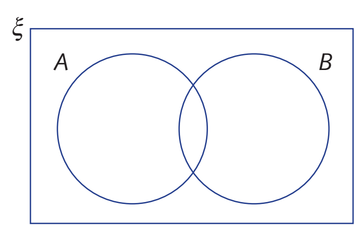
$A \cap B$ is shaded



$A \cup B$ is shaded

For an outcome to be in the event $A \cup B$, it must be in *either* the set of outcomes for A or the set of outcomes for B. Of course, it could be in both sets.

For an outcome to be in the event $A \cap B$, it must be in *both* the set of outcomes for A and the set of outcomes for B.



We recall the **addition rule** for probability.

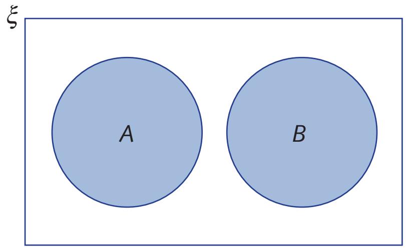
For any two events, A and B:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

This is clear from the Venn diagram.

Two events are **mutually exclusive** if they have no outcomes in common. That is:

$A \cap B = \varnothing$, where $\varnothing$ is the empty set



In this case, when A and B are mutually exclusive, the addition rule becomes:

$$P(A \cup B) = P(A) + P(B)$$

Here are some examples using these ideas.

Example 6

Two dice are thrown and the sum of the numbers on the uppermost faces is recorded. What is the probability that the sum is:

a even?

b greater than 7?

c less than 5?

d greater than 7 or less than 5?

e even and greater than 7?

f even or greater than 7?

Solution

Recall that $\xi = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

Outcome	2	3	4	5	6	7	8	9	10	11	12
P(outcome)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Let A be the event 'the sum is even'

B be the event 'the sum is greater than 7'

C be the event 'the sum is less than 5'

Then $A = \{2, 4, 6, 8, 10, 12\}$

$B = \{8, 9, 10, 11, 12\}$

$C = \{2, 3, 4\}$

a Using the table:

$$P(A) = P(2) + P(4) + P(6) + P(8) + P(10) + P(12)$$
$$= \frac{1}{36} + \frac{3}{36} + \frac{5}{36} + \frac{5}{36} + \frac{3}{36} + \frac{1}{36}$$
$$= \frac{1}{2}$$

That is, the probability that the sum is even is $\frac{1}{2}$.

b $$P(B) = P(8) + P(9) + P(10) + P(11) + P(12)$$
$$= \frac{5}{36} + \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36}$$
$$= \frac{5}{12}$$

That is, the probability that the sum is greater than 7 is $\frac{5}{12}$.

c $$P(C) = P(2) + P(3) + P(4)$$
$$= \frac{1}{36} + \frac{2}{36} + \frac{3}{36}$$
$$= \frac{1}{6}$$

That is, the probability that the sum is less than 5 is $\frac{1}{6}$.

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d $P(\text{the sum is greater than 7 or less than 5}) = P(B \cup C)$

Now $B \cap C = \varnothing$, so B and C are mutually exclusive events

$$P(B \cup C) = P(B) + P(C) = \frac{7}{12}$$

e $P(\text{the sum is even and greater than 7}) = P(A \cap B)$

$$P(A \cap B) = P(8) + P(10) + P(12) = \frac{5}{36} + \frac{3}{36} + \frac{1}{36} = \frac{1}{4}$$

f $P(\text{the sum is even or greater than 7}) = P(A \cup B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{5}{12} - \frac{1}{4} = \frac{2}{3}$$

Example 7

The eye colour and gender of 150 people were recorded. The results are shown in the table below.

Gender \ Eye colour	Blue	Brown	Green	Grey
Male	20	25	5	10
Female	40	35	5	10

What is the probability that a person chosen at random from the sample:

a has blue eyes?

b is male?

c is male and has green eyes?

d is female and does not have blue eyes?

e has blue eyes or is female?

f is male or does not have green eyes?

Solution

Let A be the event 'has blue eyes'

B be the event 'has brown eyes'

G be the event 'has green eyes'

M be the event 'is male'

F be the event 'is female'

(continued over page)

a $P(A) = \dfrac{60}{150} = \dfrac{2}{5}$

b $P(M) = \dfrac{60}{150} = \dfrac{2}{5}$

c $P(M \cap G) = \dfrac{5}{150} = \dfrac{1}{30}$

d $P(F \cup A^c) = \dfrac{35 + 5 + 10}{150}$

$= \dfrac{50}{150}$

$= \dfrac{1}{3}$

e $P(A \cup F) = \dfrac{20 + 40 + 35 + 5 + 10}{150}$

$= \dfrac{110}{150}$

$= \dfrac{11}{15}$

[Alternatively, using the addition rule,

$P(A \cup F) = P(A) + P(F) - P(A \cap F)$

$= \dfrac{2}{5} + \dfrac{3}{5} - \dfrac{40}{150}$ *Note*: $P(F) = 1 - P(M)$

$= \dfrac{11}{15}$]

f $P(M \cup G^c) = \dfrac{20 + 25 + 5 + 10 + 40 + 35 + 10}{150}$

$= \dfrac{145}{150}$

$= \dfrac{29}{30}$

Note: This can also be calculated using the addition rule or by noting that this is the complement of the event 'The person has green eyes and is female',

$1 - P(F \cap G) = 1 - \dfrac{5}{150} = \dfrac{29}{30}$.

Complement, or, and

- The event 'not A' includes every outcome of the sample space ξ that is not in A. The event 'not A' is called the complement of A and is denoted by A^c.

  $P(A^c) = 1 - P(A)$
- An outcome in the event $A \cup B$, is either in A or B, or both.
- An outcome in the event $A \cap B$, is in both A and B.
- For any two events A and B:

  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- Two events A and B are mutually exclusive if $A \cap B = \varnothing$ and in this case $P(A \cup B) = P(A) + P(B)$.

Exercise 15B

Example 5

1 A number is chosen at random from the first 15 positive whole numbers. What is the probability that it is not a prime number?

2 A card is drawn at random from an ordinary pack of 52 playing cards. What is the probability that it is not a King?

3 A number is chosen at random from the first 30 positive whole numbers. What is the probability that it is not divisible by 7?

4 In a raffle, 1000 tickets are sold. If you buy 50 tickets, what is the probability that you will not win first prize?

5 A letter is chosen at random from the 10 letters of the word COMMISSION. What is the probability that the letter is:

a N? **b** S? **c** a vowel? **d** not S?

Example 6

6 A card is drawn at random from a pack of playing cards. Find the probability that the card chosen:

a is a Club

b is a court card (i.e. an Ace, King, Queen or Jack)

c is a Club and a court card **d** is a Club or a court card

e has a face value between 2 and 5 inclusive and is a court card

f has a face value between 2 and 5 inclusive or is a court card

7 A standard die is thrown and the uppermost number is noted. Find the probability that the number is:

a even and a six **b** even or a six

c less than or equal to four and a six **d** less than or equal to three or a six

e even and less than or equal to four **f** odd or less than or equal to three

Example 7

8 A survey of 200 people was carried out to determine hair and eye colour. The results are shown in the table below.

Eye colour \ Hair colour	Fair	Brown	Red	Black
Blue	25	9	6	18
Brown	16	16	18	22
Green	15	17	22	16

What is the probability that a person chosen at random from this group has:

a blue eyes? **b** red hair?

c fair or brown hair? **d** blue or brown eyes?

e red hair and green eyes? **f** eyes that are not green?

g hair that is not red? **h** fair hair and blue eyes?

i eyes that are not blue or hair that is not fair?

In the following questions, use an appropriate Venn diagram.

9 In a group of 100 students, 60 study mathematics, 70 study physics and 30 study both mathematics and physics.

a Represent this information on a Venn diagram.

b One student is selected at random from the group. What is the probability that the student studies:

i mathematics but not physics?

ii physics but not mathematics?

iii neither physics nor mathematics?

10 In a group of 40 students, 26 play tennis and 19 play soccer. Assuming that each of the 40 students plays at least one of these sports, find the probability that a student chosen at random from this group:

a plays both tennis and soccer

b plays only tennis

c plays only one sport

d plays only soccer

11 In a group of 65 students, 30 students study geography, 42 study history and 20 study both history and geography. If a student is chosen at random from the group of 65 students, find the probability that the student studies:

a history or geography

b neither history nor geography

c history but not geography

d exactly one of history or geography

12 A number is selected at random from the integers 1 to 1000 inclusive. Find the probability the number is:

a divisible by 5

b divisible by 9

c divisible by 11

d divisible by 5 and 9

e divisible by 5 and 11

f divisible by 9 and 11

g divisible by 5, 9 and 11

13 In a group of 85 people, 33 own a microwave, 28 own a DVD player and 38 own a computer. In addition, 6 people own both a microwave and a DVD player, 9 own both a DVD player and a computer, 7 own both a computer and a microwave and 2 people own all three items. Draw a Venn diagram representing this information. If a person is chosen at random from the group, what is the probability that the person:

a does not own a microwave, a computer or a DVD player?

b owns exactly one of the three items?

c owns exactly two of the three items?

14 If a card is drawn at random from a pack of 52 playing cards, what is the probability that it will be:

a a Heart or the Ace of Clubs?

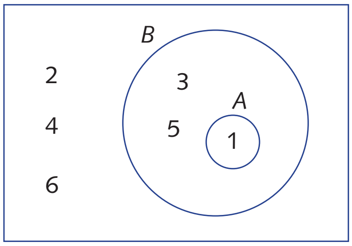
b a Heart or an Ace?

c a Heart or a Diamond?

15 From a set of 15 cards whose faces are numbered 1 to 15, one card is drawn at random. What is the probability that it is a multiple of 3 or 5?

15C Conditional probability

The probability of an event, A, occurring when it is known that some event, B, has occurred is called the probability of ***A* given *B*** and is written $P(A \mid B)$. This is the idea of **conditional probability**.



Suppose we roll a fair die and define event A as 'rolling a one' and event B as 'rolling an odd number'.

The events A and B are shown on the Venn diagram to the right.

What is the probability that a one was rolled given the information that an odd number was rolled?

We are being asked to find $P(A \mid B)$.

The knowledge that event B has occurred restricts the sample space for this calculation to $B = \{1, 3, 5\}$. Since the outcomes of a fair die are equally likely to occur, we can calculate $P(A \mid B)$:

$$P(A \mid B) = \frac{1}{3}$$

The understanding that an event has occurred requires us to adjust our probability calculations in the light of this information.

Example 8

In a group of 200 students, 42 study French only, 25 study German only and 8 study both. Find the probability that a student studies French given that they study German.

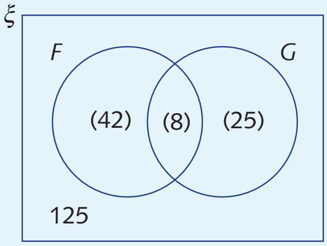
Solution

The sample space ξ is the set of 200 students.

Let F be the event 'a student studies French'.

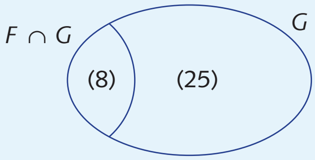
Let G be the event 'a student studies German'.

This information can be represented in a Venn diagram.



P (a student studies French given that they study German) is written as $P(F \mid G)$.

To find this, we consider G as a new sample space. The corresponding Venn diagram is as shown.



$\mid G \mid = 33$ and $\mid F \cap G \mid = 8$. Hence, $P(F \mid G) = \dfrac{8}{33}$.

In this problem, we are regarding G as a sample space in its own right and calculating the probablity of $F \cap G$ as an event in the sample space G. Thus, we have:

$$P(F \mid G) = \frac{\mid F \cap G \mid}{\mid G \mid}$$

Example 9

A bowl contains blue and black marbles. Some of the marbles have A marked on them and others have B marked on them. The number of each type is given in the table below.

	Black marble	Blue marble
Marked A	50	27
Marked B	22	13

A marble is randomly taken out of the bowl. Find the probability that:

a it is a marble marked A

b it is a marble marked A given that it is blue

c it is a blue marble

d it is a blue marble given that it is marked B

Solution

There are 112 marbles.

a $P(\text{a marble marked } A) = \dfrac{77}{112} = \dfrac{11}{16}$

b $P(\text{a marble marked } A \mid \text{it is blue}) = \dfrac{27}{40}$

c $P(\text{a blue marble}) = \dfrac{40}{112} = \dfrac{5}{14}$

d $P(\text{a blue marble} \mid \text{it is marked } B) = \dfrac{13}{35}$

Note that in Example 9, $P(\text{Is blue} \cap \text{Marked } B) = \dfrac{13}{112}$ and $P(\text{Marked } B) = \dfrac{35}{112}$.

Hence $Pr(\text{Is blue} \mid \text{Marked } B) = \dfrac{P(\text{Is blue} \cap \text{Marked } B)}{P(\text{Marked } B)} = \dfrac{13}{112} \div \dfrac{35}{112} = \dfrac{13}{35}$.

In general:

Conditional probability

Suppose that A and B are two subsets of a sample space ξ. Then for the events A and B

$$P(A \text{ given } B) = P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Example 10

Given that for two events, A and B, $P(A) = 0.6$, $P(B) = 0.4$ and $P(A \cup B) = 0.8$, find:

a $P(A \mid B)$

b $P(B \mid A)$

Solution

a $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

We know $P(B)$, but $P(A \cap B)$ is required.

The addition rule states, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Therefore, $P(A \cap B) = P(A) + P(B) - P(A \cup B)$
$= 0.6 + 0.4 - 0.8$
$= 0.2$

Hence, $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$
$= \dfrac{0.2}{0.4} = \dfrac{1}{2}$

b $P(B \mid A) = \dfrac{P(B \cap A)}{P(A)}$ $\quad (B \cap A = A \cap B)$
$= \dfrac{0.2}{0.6} = \dfrac{1}{3}$

Exercise 15C

Example 8, 9

1 A bowl contains green and red normal jelly beans and green and red double-flavoured jelly beans. The number of each type is given in the following table.

	Green	Red
Normal jelly bean	13	18
Double-flavoured jelly bean	9	8

A jelly bean is randomly taken out of the bowl. Find the probability that:

a it is a double-flavoured jelly bean

b it is a green jelly bean

c it is a green normal jelly bean

d it is a green jelly bean given that it is a normal jelly bean

e it is a double-flavoured jelly bean given that it is a red jelly bean

f it is a double-flavoured jelly bean given that it is a green jelly bean

2 A die is tossed. What is the probability that an outcome greater than 4 is obtained, given that:

a an even number is obtained?

b a number greater than 2 is obtained?

3 Two coins are tossed. What is the probability of obtaining two heads given that at least one head is obtained?

4 A card is drawn from a standard pack of cards. What is the probability that:

a a court card is drawn given that it is known that the card is a Heart?

b the 8 or 9 of Clubs is drawn given that it is known that a black card is drawn?

5 In a traffic survey during a 30-minute period, the number of people in each passing car was noted, and the results tabulated as follows.

Number of people in a car	1	2	3	4	5
Number of cars	60	50	40	10	5

Total: 165

a What is the probability that there was 1 person in a car during this period?

b What is the probability that there was more than 1 person in a car during this period?

c What is the probability that there were less than 2 people in a car during this period given that there were less than 4 people in the car?

d What is the probability that there were 5 people in a car during this period given that there were more than 3 people in the car?

6 A group of 2000 people, eligible to vote, were asked their age and candidate preference in an upcoming election, with the following results.

	18–25 years	26–40 years	Over 40 years	Total
Candidate A	400	200	170	770
Candidate B	500	460	100	1060
No preference	100	40	30	170
Total	1000	700	300	2000

What is the probability that a person chosen at random from this group:

a is from the 26–40 age group?

b prefers Candidate B?

c is from 26–40 age group given that they prefer Candidate A?

d prefers Candidate B given that they are in the 18–25 years age group?

7 A prize is going to be awarded at the end of a concert. It is announced that the winner will be chosen randomly.

The number of people at the concert in different age groups is given in the following table.

Age group	0–5	6–11	12–18	19–29	30–40	Older than 40
Number of people In age group	10	150	350	420	125	85

What is the probability that the prize winner is:

a 40 or less?

b between 12 and 29?

c older than 11?

d older than 18 given they are older than 11?

e 29 or less given that they are 40 or less?

8 A game is devised by two friends, Aalia and Rachael. They roll two dice and take the smaller number from the larger, or they write 0 if the numbers are the same. Aalia wins if the difference is less than 3.

a Complete the table of differences.

Die 1 \ Die 2	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2					
4	3					
5	4					
6	5					

b Draw up a table giving the outcomes of the experiment and their probabilities.

c Find the probability that Aalia wins.

d Find the probability that Rachael wins.

e Find the probability that Aalia wins given that the difference is less than 4.

f Find the probability of Rachael winning given that the difference is less than 4.

9 An urn contains 25 marbles numbered from 1 to 25 . A marble is drawn from the urn. What is the probability that:

a the marble numbered 3 is drawn given that it is odd?

b a marble with a number less than 10 is drawn given that it is less than 20?

c a marble with a number greater than 10 is drawn given that it is greater than 5?

d a marble with a number greater than 10 is drawn given that it is less than 20?

e a marble with a number divisible by 10 is drawn given that it is divisible by 5?

10 In a group of 85 people, 33 own a microwave, 28 own a DVD player and 38 own a computer. In addition, 6 people own both a microwave and a DVD player, 9 own both a DVD player and a computer, 7 own both a computer and a microwave and 2 people own all three items. If a person is chosen at random from the group, what is the probability that the person:

a owns a microwave given that they own a DVD player?

b owns a computer given that they own a DVD player?

c owns a computer given that they own a DVD player and a microwave?

Example 10

11 Given that for two events A and B, $P(A) = 0.2$, $P(B) = 0.6$ and $P(A \cup B) = 0.7$, Find:

a $P(A \mid B)$

b $P(B \mid A)$

12 Given that for two events A and B, $P(B) = 0.5$, $P(A \mid B) = 0.2$ and $P(A \cup B) = 0.7$. Find $P(A)$.

15D Independent events

Consider the situation where a coin is tossed twice.

Toss 1 \ Toss 2	H	T
H	(H, H)	(H, T)
T	(T, H)	(T, T)

If we define A as the event 'the second toss is a head' and B as the event 'the first toss is a head', then $A = \{(T,H),(H, H)\}$ and $B = \{(H, T),(H, H)\}$. What is $P(A \mid B)$?

By definition, $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

$$= \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

Hence, $P(A \mid B) = P(A)$. This is unsurprising since there are two separate coin tosses driving events A and B and one does not impact upon the other. That is, the probability of event A occurring is unaffected by event B having occurred. This is an example of independent events.

Two events A and B are **independent** if the occurrence of one event does not affect the probability of the occurrence of the other. That is, if:

$P(A \mid B) = P(A)$ or $P(B \mid A) = P(B)$

Thus when two events A and B are independent:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = P(A), \quad \text{if } P(B) \neq 0$$

Therefore:

$$P(A \cap B) = P(A) \times P(B)$$

This equation provides a convenient alternative to testing whether two events A and B are independent.

In the special case that one event or the other is impossible to occur, i.e. $P(A) = 0$ or $P(B) = 0$, this rule is also satisfied since both sides of the equation are zero. In this special case, we say that A and B are also independent.

Independent events

Events A and B are independent if and only if:

$P(A \cap B) = P(A) \times P(B)$

Example 11

200 teenagers and young adults less than 23 years old were interviewed about their use of the social medium, Snapchatter. The results, as well as their ages are given in the following table.

Use Snapchatter	Age (years), A		Total
	$13 \leq A < 18$	$18 \leq A < 23$	
Yes	77	63	140
No	28	32	60
Total	105	95	200

Is the use of Snapchatter independent of age among teenagers and young adults?

Solution

From the table:

$$P(13 \leq A < 18 \cap Yes) = \frac{77}{200} = 0.335$$

$$P(13 \leq A < 18) \times P(Yes) = \frac{105}{200} \times \frac{140}{200} = \frac{147}{400} = 0.3675$$

Hence,

$$P(13 \leq A < 18 \cap Yes) \neq P(13 \leq A < 18) \times P(Yes)$$

Therefore, these events are not independent.

That is, Snapchatter use is not independent of age among teenagers and young adults.

Example 12

In a certain rural town, the probability that a randomly selected person has more than one sibling (S) is 0.6 and the probability that they live 'in the east of town' (between the north east and south east of the centre) (E) is 0.3. If these events are independent, then find the following probabilities.

a A person from the east of town has more than one sibling.

b A person has no more than one sibling and does not live in the east of town.

Solution

a A person from the east of town who has more than one sibling is represented by $E \cap S$,

$$P(E \cap S) = P(E) \times P(S) \qquad (E \text{ and } S \text{ are independent})$$
$$= 0.6 \times 0.3 = 0.18$$

b A person not from the east of town who has no more than one sibling is represented by $E^c \cap S^c$,

$$P(E^c \cap S^c) = P(E^c) \times P(S^c) \quad (E \text{ and } S \text{ are independent, therefore } E^c \text{ and } S^c \text{ are independent.})$$
$$= 0.4 \times 0.7 = 0.28$$

Confusion often arises between independent and mutually exclusive events. As discussed previously, two events A and B are mutually exclusive means that $A \cap B = \varnothing$ and hence that $P(A \cap B) = 0$. Therefore, two events will only be mutually exclusive **and** independent if the probability of at least one of them is zero.

Example 13

Consider rolling a die. Define event A as 'rolling a number divisible by 3' and event B as 'rolling an even number'.

a Are events A and B independent?

b Are events A and B mutually exclusive?

Solution

a Favourable outcomes for the these events are $A = \{3, 6\}$, $B = \{2, 4, 6\}$.

Therefore, $A \cap B = \{6\}$.

Since all outcomes $\{1, 2, 3, 4, 5, 6\}$ are equally likely,

$$P(A) = \frac{2}{6} = \frac{1}{3}, \quad P(B) = \frac{3}{6} = \frac{1}{2} \quad \text{and } P(A \cap B) = \frac{1}{6}.$$

This means, $P(A \cap \mathrm{B}) = P(A) \times P(B) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$.

Hence, events A and B are independent.

b $P(A \cap B) \neq 0$. Therefore, events A and B are not mutually exclusive.

Exercise 15D

Example 11

1 100 people were surveyed about their attitudes to the use of headgear in professional boxing and classified according to sex. The results are shown in the table below.

Should the use of headgear be mandatory in professional boxing?

	Male	Female	Total
Yes	25	30	55
No	35	10	45
Total	60	40	100

Is attitude to the use of headgear in professional boxing independent of sex?

2 80 adults were surveyed about whether they play computer games more than once a week and their age category was recorded ('30 years or older' and 'less than 30'). The results are shown below.

Play computer games >1 per week	Age (years)		Total
	< 30	≥ 30	
Yes	35	20	55
No	10	15	25
Total	45	35	80

Is playing computer games multiple times a week independent of age category?

3 200 road accidents recorded by the Traffic Authority were studied in terms of vehicle speed at the time of collision, relative to the local speed limit, and the accident severity. The results are shown below.

Accident severity	Speed of vehicle at collision			Total
	Not speeding	$\leq$ 10 km/h over limit	$>$ 10 km/h over limit	
Minor	41	88	17	146
Major	2	31	21	54
Total	43	109	38	200

a Find the probability that a randomly selected accident in the study is classed as 'major'.

b Find the probability that an accident was classed as major given that it collided at a speed greater than 10 km/h over the local speed limit.

c Hence, explain why accidents in this study classed as 'major' are not independent of the event that they were travelling at a speed greater than 10 km/h over the limit.

d By focussing on the events, 'Minor' accident and 'Not speeding', conduct an alternative test to that used in part **c** to show that they are also not independent events.

Example 12

4 The probability that a person does their grocery shopping at Colesworth is $\frac{3}{5}$, and the probability that a person is left-handed is $\frac{1}{7}$. If these events are independent, find the following probabilities.

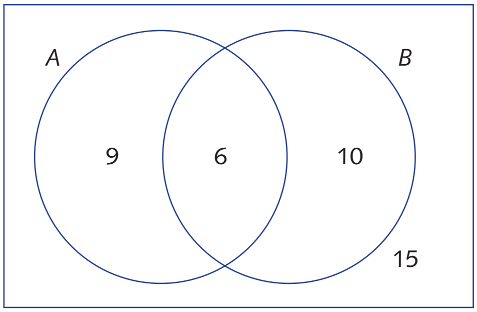
a A person does their grocery shopping at Colesworth and is left-handed.

b A person is not left-handed but does their grocery shopping at Colesworth.

c A person is not left-handed and does not do their grocery shopping at Colesworth.

d A person does their grocery shopping at Colesworth or is left-handed.

5 Events A and B are shown in the Venn diagram. Show that A and B are independent.



Example 13

6 Consider rolling a die on two separate occasions. Define event A as 'rolling a 4 on the first throw' and event B as 'rolling at least 10 as the sum of the two numbers shown'.

a Create a table (array) showing the sample space of a die rolled twice.

b Determine whether events A and B are independent.

c Determine whether events A and B are mutually exclusive.

15E Sampling with replacement and without replacement

A random experiment is any repeatable procedure with clear but unpredictable outcomes like, for example, tossing a coin or rolling a die. **Sampling** is a type of experiment that concerns making a number of random selections from a set of things. This set from which random selections are drawn is known as the **population**. We will be considering sampling with replacement and without replacement.

As we know, the conditional probability of an event A given that event B has already occurred is given by:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \text{ if } P(B) \neq 0$$

The formula can be re-arranged to give the **multiplication rule of probability**:

$$P(A \cap B) = P(A \mid B) \times P(B)$$

This rule can be used when calculating probabilities in multi-stage experiments. Sampling with replacement involves making a selection, observing the outcome and then returning the item to the population before another selection is made. Since replacement occurs, the outcome at one stage is not affected by the outcome at any other stage.

Multi-stage sampling with replacement

A bag contains three red balls, R_1, R_2 and R_3, and two black balls, B_1 and B_2. A ball is drawn at random and its colour recorded. It is then put back in the bag, the balls are mixed thoroughly and a second ball is drawn. Its colour is also noted. The sample space is shown in the array below.

First ball \ Second ball	R_1	R_2	R_3	B_1	B_2
R_1	(R_1, R_1)	(R_1, R_2)	(R_1, R_3)	(R_1, B_1)	(R_1, B_2)
R_2	(R_2, R_1)	(R_2, R_2)	(R_2, R_3)	(R_2, B_1)	(R_2, B_2)
R_3	(R_3, R_1)	(R_3, R_2)	(R_3, R_3)	(R_3, B_1)	(R_3, B_2)
B_1	(B_1, R_1)	(B_1, R_2)	(B_1, R_3)	(B_1, B_1)	(B_1, B_2)
B_2	(B_2, R_1)	(B_2, R_2)	(B_2, R_3)	(B_2, B_1)	(B_2, B_2)

The sample space ξ contains the 25 pairs listed above. They are equally likely and each outcome has probability $\frac{1}{25}$ of occurring.

Let A be the event 'both balls are red', B be the event 'the first ball is red' and C be the event 'the second ball is red'.

From the array:

$P(A) = \frac{9}{25}$, $P(B) = \frac{15}{25} = \frac{3}{5}$ and $P(C) = \frac{15}{25} = \frac{3}{5}$

The event $B \cap C$ is 'the first and second balls are red', which is the same as event A. That is, $A = B \cap C$.

We note that $P(A) = \frac{9}{25}$ and $P(B) \times P(C) = \frac{9}{25}$.

$P(A) = P(B \cap C) = P(B) \times P(C)$ is true.

Multi-stage sampling without replacement

We start with the same bag of coloured balls as previously described. A ball is drawn at random and its colour recorded. The ball is *not* put back in the bag. A second ball is drawn at random from the remaining balls and its colour recorded.

The sample space ξ is listed in the array on the next page. There are the $5 \times 4 = 20$ outcomes in the sample space ξ.

First ball \ Second ball	R_1	R_2	R_3	B_1	B_2
R_1	–	(R_1, R_2)	(R_1, R_3)	(R_1, B_1)	(R_1, B_2)
R_2	(R_2, R_1)	–	(R_2, R_3)	(R_2, B_1)	(R_2, B_2)
R_3	(R_3, R_1)	(R_3, R_2)	–	(R_3, B_1)	(R_3, B_2)
B_1	(B_1, R_1)	(B_1, R_2)	(B_1, R_3)	–	(B_1, B_2)
B_2	(B_2, R_1)	(B_2, R_2)	(B_2, R_3)	(B_2, B_1)	–

The – indicates that the pair cannot occur.

Again, let A be the event 'both balls are red', B be the event' 'the first ball is red' and C be the event 'the second ball is red'. It is also the case that event $B \cap C$ is the same as event A. That is, $A = B \cap C$.

From the array, $P(A) = \frac{6}{20} = \frac{3}{10}$, $P(B) = \frac{12}{20} = \frac{3}{5}$ and $P(C) = \frac{12}{20} = \frac{3}{5}$.

We note that $P(A) = \frac{3}{10}$ and $P(B) \times P(C) = \frac{9}{25}$.

So in this case $P(B \cap C) \neq P(B) \times P(C)$.

The events B and C are not independent. The result of the second draw is not independent of the result of the first. This should not be a surprise since, if the first ball drawn is red, there are two reds and two blacks left. On the other hand, if the first ball drawn is black, there are three reds and one black left.

To apply the multiplication principle in this case, we need to calculate $P(C \mid B)$. We note that if B has occurred then there are four balls left; two of them are red and two black.

Therefore, $P(C \mid B) = \frac{2}{4} = \frac{1}{2}$

$$\begin{aligned} P(B \cap C) &= P(B) \times P(C \mid B) \\ &= \frac{3}{5} \times \frac{1}{2} \\ &= \frac{3}{10} \\ &= P(A), \text{ as expected} \end{aligned}$$

Tree diagrams and probability

Drawing an array containing all possibilities is only sensible for small cases such as those dealt with earlier in this section. For example, if one draws two cards from a pack of cards without replacement then there are 52×51 possibilities.

Another useful method for calculating probabilities is a **tree diagram**.

Tree diagrams were introduced in *ICE-EM Mathematics Year 9* as a means of methodically listing the sample space of a multi-stage experiment involving equally likely outcomes. However, they can also be used to visually support the multiplication rule of probability in any multi-stage experiment via branches of the tree.

Consider again the experiment of drawing two balls from a bag containing 3 red and 2 black balls without replacement.

On the first draw, the probability of choosing a red is $\frac{3}{5}$ and the probability of choosing a black $\frac{2}{5}$.

If a red ball is chosen first, then there are 2 red and 2 black balls to choose from on the second draw. This means we can determine the following conditional probabilities:

$$P(\textit{red second} \mid \textit{red first}) = \frac{2}{4} = \frac{1}{2}$$

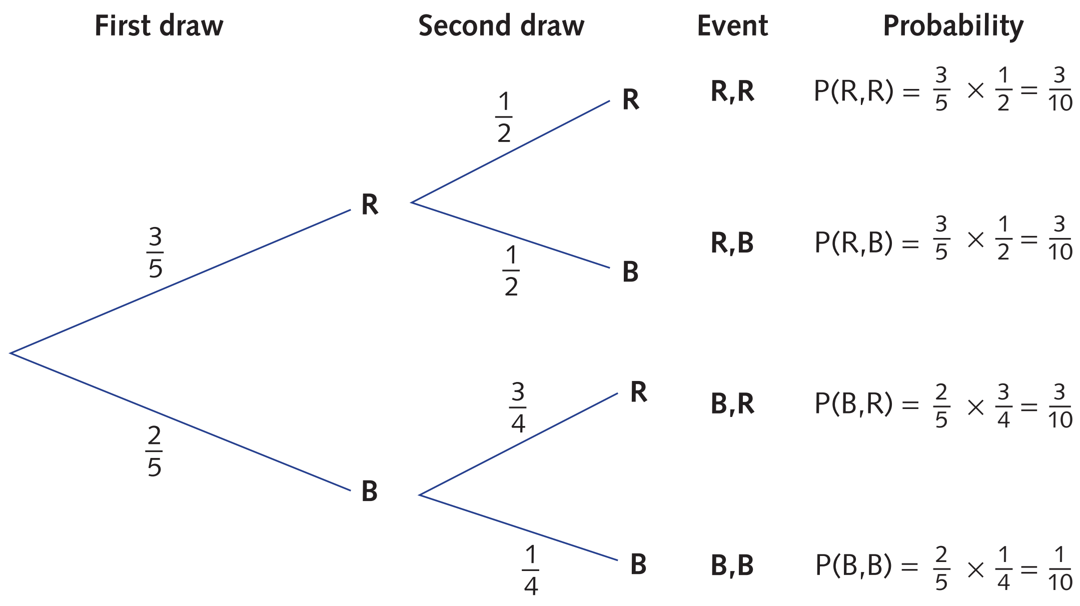
$$P(\textit{black second} \mid \textit{red first}) = \frac{2}{4} = \frac{1}{2}$$

If a black ball is chosen first, then there are 3 red and 1 black ball to choose from on the second draw. Therefore:

$$P(\textit{red second} \mid \textit{black first}) = \frac{3}{4}$$

$$P(\textit{black second} \mid \textit{black first}) = \frac{1}{4}$$

Overall, there are four possible events involving colour in this two-stage event; red first – red second (R, R), red first – black second (R, B), black first – red second (B, R) and black first – black second (B, B). These events are represented in the four branch tree diagram below. Respective probabilities are placed along each arm, with conditional probabilities placed along the second arms, as shown. Event probability is then determined by multiplying probabilities along the respective branch. This is justified by the multiplication rule of probability.



A 20 branch tree diagram (5×4) could have been used to model the situation in a similar manner to the array on page 464, in terms of equally likely outcomes $(R_1, R_2, R_3, B_1, B_2)$. This however, would have been far less efficient.

Consider the following example that does *not* concern sampling.

Example 14

A coin is tossed three times and the uppermost face is recorded each time.

a List the sample space.

b Find the probability of obtaining two heads.

Solution

a $\xi = \{\text{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}\}$

b **Method 1**

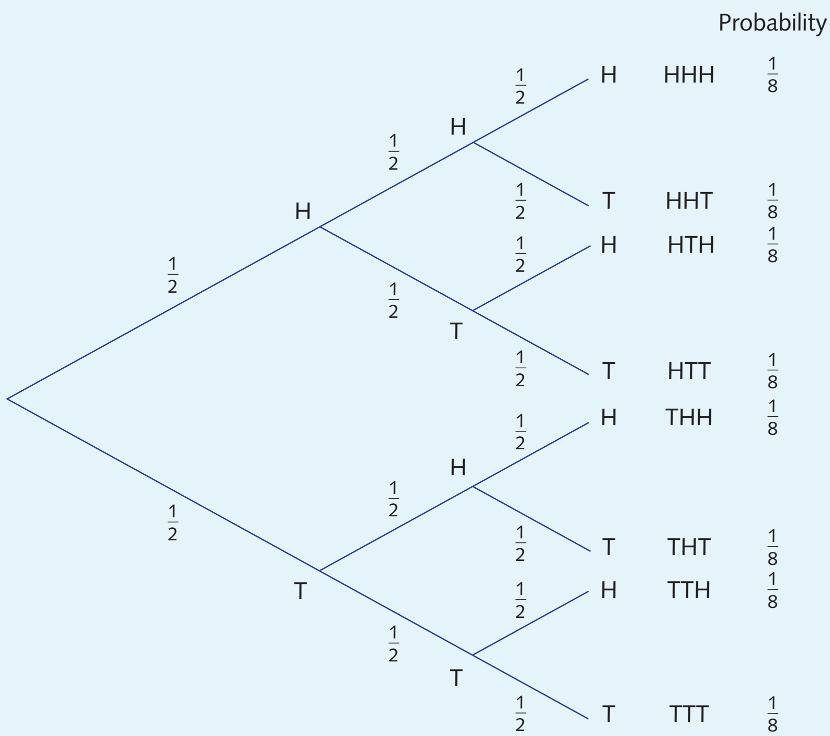
Let A be the event two heads are obtained.

$A = \{\text{HHT, HTH, THH}\}$

$P(A) = \frac{3}{8}$

Method 2

We draw a tree diagram. It is clear in this case that each throw is independent of each of the others, so probabilities along successive branches remain fixed.



The probability of a head or a tail at each stage is $\frac{1}{2}$.

The three required arms of the tree are HHT, HTH and THH with two heads.

$$\begin{aligned} P\text{ (two heads)} &= \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} \\ &= \frac{3}{8} \end{aligned}$$

Note: The above example involves equally likely outcomes, so a tree diagram may be only useful for methodically listing the sample space.

Cards

A deck of cards consists of 52 cards –13 Hearts, 13 Diamonds, 13 Spades and 13 Clubs. Each suit consists of a 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K and A.

If one card is drawn, then the probabilities are easy to calculate. For example:

$$P(\text{King of Hearts is drawn}) = \frac{1}{52},\ P(\text{a Heart is drawn}) = \frac{13}{52} = \frac{1}{4}$$

$$P(\text{a King is drawn}) = \frac{4}{52} = \frac{1}{13}$$

If two cards are drawn *with replacement* then, for example:

$$P(\text{two Hearts are drawn}) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$

On the other hand, if two cards are drawn *without replacement*, then the conditional probabilities vary depending on the first drawn card. For example, if the King of Hearts is drawn, then on the second draw:

$$P(\text{Heart}) = \frac{12}{51},\ P(\text{Club}) = \frac{13}{51},\ P(\text{Spade}) = \frac{13}{51} \text{ and } P(\text{Diamond}) = \frac{13}{51}$$

In this type of situation, a tree diagram is useful for assisting with the application of the multiplication principle.

Example 15

A card is taken at random from a pack and not replaced. A second card is then taken from the pack and the result noted.

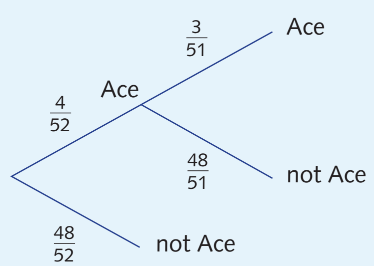
a What is the probability that the two cards are Aces?

b What is the probability that the two cards are Hearts?

c What is the probability of obtaining one Heart and one Club?

Solution

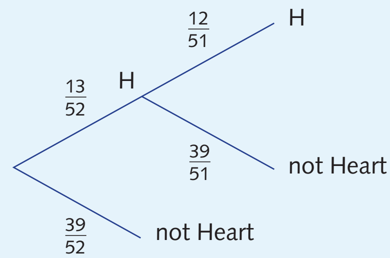
a



$$P(\text{two Aces}) = \frac{4}{52} \times \frac{3}{51} = \frac{1}{221}$$

(continued over page)

b $P(\text{two Hearts}) = \dfrac{13}{52} \times \dfrac{12}{51} = \dfrac{1}{17}$



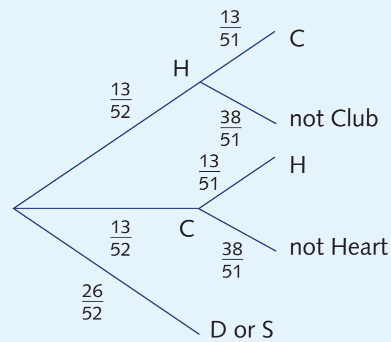
c $P(\text{one Heart and one Club})$

$= P(\text{the first card is a Heart and the second card a Club})$

$\quad + P(\text{the first card is a Club and the second card a Heart})$

$= \dfrac{13}{52} \times \dfrac{13}{51} + \dfrac{13}{52} \times \dfrac{13}{51}$

$= \dfrac{13}{102}$



Exercise 15E

Example 14

1 A card is drawn at random from a pack of 52 playing cards. It is replaced and the pack is shuffled. A second card is then drawn. What is the probability of the event:

a both cards are Diamonds?
b neither card is a Diamond?
c only one of the cards is a Diamond?
d only the first card is a Diamond?
e only the second card is a Diamond?
f at least one of the cards is a Diamond?

Example 15

2 One card is drawn at random from a pack of 52 playing cards. It is not replaced. A second card is then drawn. What is the probability of the event:

a both cards are Diamonds?
b neither card is a Diamond?
c only one of the cards is a Diamond?
d only the first card is a Diamond?
e only the second card is a Diamond?
f at least one of the cards is a Diamond?

3 A bag contains 8 red balls and 5 black balls. A ball is taken and its colour noted. It is not replaced. A second ball is taken and its colour noted. Find the probability of obtaining:

a a red ball followed by a black ball
b a red and a black ball
c two red balls
d two black balls

4 Giorgia has 5 red ribbons, 3 blue ribbons and 6 green ribbons in a drawer. Giorgia randomly takes one ribbon out and then a second (no replacement). What is the probability that she obtains:

a 2 red ribbons?
b a red and a blue ribbon?
c a green and a red ribbon?
d 2 blue ribbons?

5 Cube A has 5 red faces and 1 white face, cube B has 3 red faces and 3 white faces and cube C has 2 red faces and 4 white faces. The 3 cubes are tossed. What is the probability of:

a 3 red faces uppermost?

b 3 white faces uppermost?

c red with A and B and white with C?

d red with A and white with B and C?

e at least 1 red face uppermost?

6 A bag of confectionary has 27 chocolates and 35 toffees in it.

Sanjesh takes out one item from the bag and then a second without replacing the first. What is the probability of obtaining:

a two chocolates?

b two toffees?

c a chocolate and a toffee?

7 A bag contains 15 blue balls and 10 green balls. A ball is taken out and its colour noted. It is replaced. A second ball is taken out and its colour noted. Find the probability of obtaining:

a a green ball followed by a blue ball

b a green and a blue ball

c two green balls

d two balls of the same colour

8 A coin is tossed four times. What is the probability of:

a four heads?

b four tails?

c head, tail, head, tail, in that order?

d heads in the first three tosses but not in the fourth?

e a head in at least one of the four tosses?

9 A die is tossed three times. What is the probability of obtaining:

a three 6s?

b no 6s?

c three odd numbers?

d three even numbers?

e a 6 in the first two tosses only?

f a 6, not a 6, and a 6 in that order?

10 A box contains chocolates and toffees with green and red wrapping. The number of each type of confectionary and its wrapping colour is given below. One item is removed from the box.

	Green wrapping	Red wrapping
Chocolate	48	60
Toffee	20	25

Find the probability of obtaining an item with green wrapping and the probability of obtaining an item with green wrapping given that it is a chocolate.

Review exercise

1 **a** A bag contains 2 red marbles and 3 black marbles. Two marbles are drawn from the bag. Each marble is replaced after it is drawn and the bag is shaken. Find the probability of selecting:

i two black marbles

ii two red marbles

iii a red and a black marble

iv at least one red marble

b From the same bag of marbles as in part **a**, two marbles are selected without replacing the first marble. Find the probability that the selection contains:

i two red marbles

ii a black and a red marble

iii two marbles of the same colour

2 Discs with the digits 0 to 9 are placed in a box. A disc is drawn at random, its digit is recorded, then it is replaced in the box. A second disc is then drawn and its digit is recorded. Find the probability:

a that the two digits are the same

b of drawing an even digit and an odd digit

c that the first digit is a 6 and the second digit is odd

3 A number is chosen by throwing a die in the shape of a regular tetrahedron with the numbers 2, 4, 6, 8 on the faces, and noting the number that is face down. A second number is obtained by throwing a fair six-sided die and noting the number on its uppermost face. These two numbers are then added together.

a Complete the table, showing all possible outcomes.

		Roll of the six-sided die					
		1	2	3	4	5	6
Roll of the four-sided die	2	3					
	4						
	6			9			
	8					13	

b Find the probability of:

i A: the event in which the total score exceeds 8

ii B: the event in which the total score is 10

c If C is the event in which the total score is less than 13, find $A \cap C$ and $P(A \cap C)$.

d Are the events A and C independent? Justify your answer.

4 In a group of 100 students, 60 study mathematics, 50 study physics and 20 study both mathematics and physics. One of the mathematics students is selected at random. What is the probability that he also studies physics?

5 An odd digit is selected at random and then a second odd digit is chosen at random (they may be equal). What is the probability that the sum of the two digits is greater than 10?

6 An urn contains 8 red marbles, 7 white marbles and 5 black marbles. One marble is drawn at random from the urn. What is the probability that it is:

a red or black?

b not white?

c neither black nor white?

7 A cube has 4 red faces and 2 white faces; another has 3 red and 3 white; another 2 red and 4 white. The 3 cubes are tossed. What is the probability that there are at least 2 red faces uppermost?

8 A number is selected at random from the integers 1 to 100 inclusive. What is the probability it is:

a divisible by 3?

b divisible by 7?

c divisible by both 3 and 7?

d divisible by 3 but not by 7?

e divisible by 7 but not by 3?

Challenge exercise

1 A tennis team consists of 4 players who must be chosen from a group of 6 boys and 5 girls.

a Find the number of ways the team can be picked:

i without restriction

ii with 2 boys and 2 girls in the team

iii if at least 2 girls must be in the team

iv if no more than 2 boys are to be in the team

b If the team consists of 2 girls (Joanne and Freda) and two boys (Peter and Stuart) from which two pairs of mixed doubles must be selected, how many ways can the mixed doubles pairs be selected?

c During a particular tournament, the probability of the first mixed doubles pair winning each match it plays is 0.4 and the probability of the second pair winning each match it plays is 0.7.

i Find the probability that both pairs win their first match.

ii Find the probability that the first pair wins 2 and loses 1 of their first 3 matches.

iii Find the probability that the second pair wins their second and third match, given that they won their first match.

2 A box contains 35 apples, of which 25 are red and 10 are green. Of the red apples, five contain an insect and of the green apples, one contains an insect. Two apples are chosen at random from the box. Find the probability that:

a both apples are red and at least one contains an insect

b at least one apple contains an insect given that both apples are red

c both apples are red given that at least one is red

3 Four-digit numbers are to be formed from the digits $4, 5, 6, 7, 8, 9$.

a For each of the cases below, find how many four-digit numbers can be formed if:

i any digit may appear up to four times in the number

ii no digit may appear more than once in the number

iii there is at least one repeated digit, but no digit appears more than twice in a number

b Find the probability that a four-digit number chosen at random from the set of numbers in part **a i** contains at least one six.

4 Each of three boxes has two drawers. One box contains a diamond in each drawer, another contains a pearl in each drawer, and the third contains a diamond in one drawer and a pearl in the other. A box is chosen, a drawer is opened and found to contain a diamond. What is the probability that there is a diamond in the other drawer of that box?

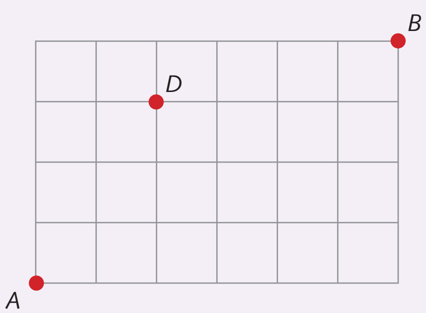
5 If you hold two tickets in a lottery for which n tickets were sold and 5 prizes are to be given, what is the probability that you will win at least one prize?

6 If A and B are mutually exclusive events, show that

$$P(A \mid A \cup B) = \frac{P(A)}{P(A) + P(B)}.$$

7 **a** In the diagram shown, in how many different ways can you get from A to B if you are only allowed to move to the right and upwards?

b What is the probability that a random journey from A to B passes through the point D? (Only moves to the right and up are allowed.)



CHAPTER

16

Number and Algebra

Direct and inverse proportion

People working in science, economics and many other areas look for relationships between various quantities of interest. These relationships often turn out to be linear, quadratic or hyperbolic. That is, the graph relating these quantities is a straight line, a parabola or a rectangular hyperbola.

In Chapter 3, we revised the use of formulas. In this chapter we are mainly concerned with formulas for which the associated graphs are either straight lines or rectangular hyperbolas. In the first case we have direct proportion, and in the second we have inverse proportion. We have met direct proportion in Chapter 18 of *ICE-EM Mathematics Year 9.*

To take a very simple example, the formula $V = IR$ is called Ohm's law and relates voltage V, current I, and resistance R. The law is fundamental in the study of electricity. If R is a constant, V is directly proportional to I. If V is a constant, I is inversely proportional to R.

In part, because our examples are drawn from physical problems, in this chapter variables will take mostly positive values.

16A Direct proportion

Andrew drives from his home at a constant speed of 100 km/h.

The formula for the distance, d km, travelled in t hours is:

$$d = 100t$$

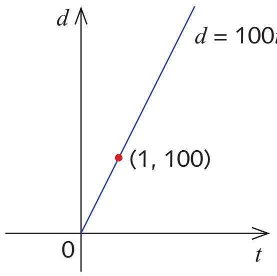
Andrew will go twice as far in twice the time, three times as far in three times the time and so on.

We say that d is **directly proportional** to t. The number 100 is called the **constant of proportionality**.

The statement 'd is directly proportional to t' is written as:

$$d \propto t$$

The graph of d against t is a straight line passing through the origin. The gradient of the line is 100.



By considering the gradient of the line, we see that for values t_1 and t_2 with corresponding values d_1 and d_2:

$$\frac{d_1}{t_1} = \frac{d_2}{t_2} = 100$$

That is, the constant of proportionality is the gradient of the straight line graph, $d = 100t$, which, in this example, is the speed of the car.

Quantities proportional to the square or cube

A metal ball is dropped from the top of a tall building and the distance it falls is recorded each second.

From physics, the formula for the distance, d metres, the ball has fallen in t seconds, is given by:

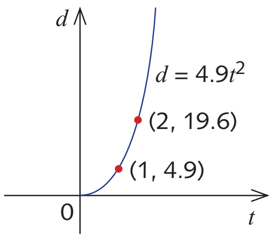
$$d = 4.9t^2$$

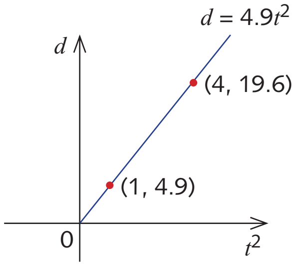
In this case, we say that d is **directly proportional** to the square of t.

The first diagram to the right is a graph of d against t.
Since t is positive, the graph is half a parabola.

The second diagram to the right is a graph of d against t^2.

t	0	1	2	3
t^2	0	1	4	9
d	0	4.9	19.6	44.1





This graph is now a straight line passing through the origin. The gradient of this line is 4.9.

The statement 'd is directly proportional to t^2' is written as:

$$d \propto t^2$$

This means that for any two values, t_1 and t_2, with corresponding values d_1 and d_2:

$$\frac{d_1}{t_1^2} = \frac{d_2}{t_2^2} = 4.9$$

So once again the gradient of the line is the constant of proportionality.

Finding the constant of proportionality

If we can relate two variables so that the graph is a straight line through the origin, then the constant of proportionality is the gradient of that line. Thus, to find the constant of proportionality, just one pair of non-zero values is needed.

Example 1

From physics, the kinetic energy, E mJ (mJ is the abbreviation for microjoules), of a body in motion is directly proportional to the square of its speed, v m/s. If a body travelling at a speed of 10 m/s has energy 400 mJ, find:

a the constant of proportionality

b the formula for E in terms of v

c the energy of the body when it travels at a speed of 15 m/s

d the speed if the moving body has energy 500 mJ

Solution

a Kinetic energy is directly proportional to the square of the speed.

$$E \propto v^2$$

so $E = kv^2$, for some constant k

We know that $E = 400$ when $v = 10$

so $400 = 100k$

$$k = 4$$

b From part **a**, $E = 4v^2$.

c When $v = 15$,

$$\begin{aligned} E &= 4 \times 15^2 \\ &= 900 \end{aligned}$$

Therefore, the body travelling at speed 15 m/s has energy of 900 mJ.

d When $E = 500$,

$$\begin{aligned} 500 &= 4 \times v^2 \\ v^2 &= 125 \\ v &= \sqrt{125} \quad (\text{since } v > 0) \\ &= 5\sqrt{5} \\ &\approx 11.18 \text{ m/s} \quad (\text{Correct to two decimal places.}) \end{aligned}$$

Therefore, the body has energy 500 mJ when travelling at $5\sqrt{5}$ m/s.

The procedure for solving the previous example was as follows.

- Write down the statement of proportionality.
- Write this statement as an equation involving a constant, k.
- Substitute the given information to obtain the value of k.
- Rewrite the formula with the determined value of k.

Example 2

The mass, w grams, of a plastic material required to mould a solid ball is directly proportional to the cube of the radius, r cm, of the ball. If 40 grams of plastic is needed to make a ball of radius 2.5 cm, what size ball can be made from 200 grams of the same type of plastic?

Solution

$w \propto r^3$ so $w = kr^3$ for some constant k.

We know that $w = 40$ when $r = 2.5$

$$\begin{aligned} \text{so, } 40 &= k \times (2.5)^3 \\ k &= 2.56 \end{aligned}$$

Thus the formula is $w = 2.56r^3$

$$\begin{aligned} \text{When } w = 200,\ 200 &= 2.56r^3 \\ r^3 &= 78.125 \\ r &= \sqrt[3]{78.125} \\ r &\approx 4.27 \end{aligned}$$

Thus, a ball with a radius of approximately 4.3 cm can be made from 200 grams of plastic.

Note: It is a fact that the mass of a ball of constant density is given by density × volume. The volume is $\frac{4}{3}\pi r^3$ and so the mass of a ball is proportional to r^3.

Increase and decrease

If one quantity is proportional to another, we can investigate what happens to one of the quantities when the other is changed.

Suppose that $a \propto b$, then $a = kb$ for a positive constant, k.

If the value of b is doubled, then the value of a is doubled. For example, if $b = 1$, then $a = k$. So $b = 2$ gives $a = 2k$.

Similarly, if the value of b is tripled, then the value of a is tripled.

These ideas can be used in a variety of situations.

Example 3

Given that $y \propto \sqrt{x}$, what is the percentage change in:

a y when x is increased by 20%?

b x when y is decreased by 30%?

Solution

Since $y \propto \sqrt{x}$, $y = k\sqrt{x}$

a When $x = 1$, $y = k$

If x is increased by 20%, then $x = 1.2$,

$$\text{so } y = k\sqrt{1.2} \approx 1.095k$$

and y is approximately 109.5% of its previous value.

Thus, y has increased by approximately 9.5%.

b Making x the subject in $y = k\sqrt{x}$:

$$y^2 = k^2x$$
$$x = \frac{y^2}{k^2}$$

When $y = 1$, $x = \dfrac{1}{k^2}$

If y is decreased by 30%, then $y = 0.7$ and $x = \dfrac{0.49}{k^2}$, so x is 49% of its previous value.

Thus, x has decreased by 51%.

Direct proportion

- y is directly **proportional** to x^n if there is a positive constant k such that $y = kx^n$.
- The symbol $\propto$ is used for 'is proportional to'. We write $y \propto x^n$.
- The constant k is called the **constant of proportionality**.
- If y is directly proportional to x, then the graph of y against x^n is a straight line through the origin. The gradient of the line is the constant of proportionality.

Exercise 16A

Throughout this exercise, all variables take only positive values.

Example 1

1 **a** Given that $a \propto b$, and that $b = 3$ when $a = 1$, find the formula for a in terms of b.

b Given that $m \propto n$, and that $m = 15$ when $n = 3$, find the formula for m in terms of n.

2 Consider the following table of values.

p	0	1	4	9	16
q	0	4	8	12	16
$\sqrt{p}$					

a Plot the graph of q against p.

b Complete the table of values and calculate $\dfrac{q}{\sqrt{p}}$ for each pair $(q, \sqrt{p})$.

c Assuming that there is a simple relationship between the two variables, find a formula for q in terms of p.

Example 2

3 **a** Given that $m \propto n^2$ and that $m = 12$ when $n = 2$, find the formula for m in terms of n and the exact value of:

i m when $n = 5$ **ii** n when $m = 27$

b Given that $a \propto \sqrt{b}$ and that $a = 30$ when $b = 9$, find the formula for a in terms of b and the exact value of:

i a when $b = 16$ **ii** b when $a = 25$

4 In each part, find the formula connecting the pronumerals.

a $R \propto s$ and $s = 7$ when $R = 28$

b $P \propto T$ and $P = 12$ when $T = 100$

c a is directly proportional to the square root of b and $a = 12$ when $b = 9$

d V is directly proportional to r^3 and $V = 216$ when $r = 3$

5 In each of the following tables, $y \propto x$. Find the constant of proportionality and complete the tables.

a

x	2	8	12	18
y	$\frac{1}{2}$			

b

x		3	6	15
y	16		48	

6 On a particular road map, a distance of 0.5 cm on the map represents an actual distance of 10 km. What actual distance would a distance of 6.5 cm on the map represent?

7 The estimated cost $\$C$ of building a brick veneer house on a concrete slab is directly proportional to the area A of floor space in square metres. If it costs \$90 000 for 150 m^2, how much floor space would you expect for \$126 300?

8 The power p kW needed to run a boat varies as the cube of its speed, s m/s. If 400 kW will run a boat at 3 m/s, what power, correct to the nearest kW, is needed to run the same boat at 5 m/s?

9 If air resistance is neglected, the distance d metres that an object falls from rest is directly proportional to the square of the time t seconds of the fall. An object falls to 9.6 metres in 1.4 seconds. How far will the object fall in 4.2 seconds?

Example 3

10 The surface area of a sphere, A cm^2, is directly proportional to the square of the radius, r cm. What is the effect on:

a the surface area when the radius is tripled?

b the radius when the surface area is tripled?

11 Given that $m \propto n^5$, what is the effect on:

a m when n is doubled? **b** m when n is halved?

c n when m is multiplied by 243? **d** n when m is divided by 1024?

12 Given that $a \propto \sqrt{b}$, what is the effect, correct to two decimal places, on a when b is:

a increased by 25%? **b** decreased by 8%?

13 Given that $p \propto \sqrt[3]{q}$, what is the effect on:

a p when q is increased by 20%?

b p when q is decreased by 5%?

c q when p is increased by 10%?

d q when p is decreased by 10%?

16B Inverse proportion

We know that:

$$\text{distance} = \text{speed} \times \text{time} \quad (d = vt)$$

Rearranging gives:

$$\text{time} = \frac{\text{distance}}{\text{speed}} \quad \left(t = \frac{d}{v}\right)$$

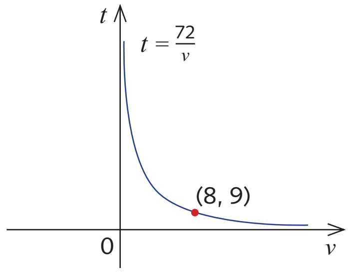
The distance between two towns is 72 km. The time t hours taken to cover this distance at v km/h is given by the formula:

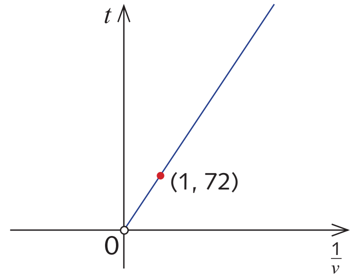
$$t = \frac{72}{v}$$

As v increases, t decreases, and as v decreases, t increases.

This is an example of **inverse proportion**. We write $t \propto \frac{1}{v}$ and say t is **inversely proportional** to v. The number 72 is **the constant of proportionality**.

The graph of t against v is a branch of the rectangular hyperbola $t = \frac{72}{v}$, and the graph of t against $\frac{1}{v}$ is a straight line with gradient 72.





Example 4

Suppose that two towns, A and B, are 144 km apart.

a Write down the formula for the time taken, t hours, to travel from A to B at a speed of v km/h.

b Draw a graph of t against v.

c If the car is driven at 24 km/h, how long does it take to complete the journey?

d If the trip takes 90 minutes, at what speed is the car driven?

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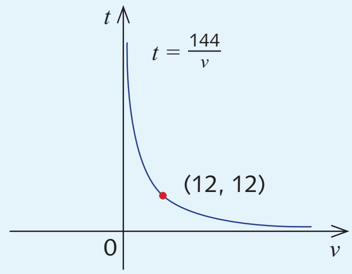
Solution

a

$$t = \frac{144}{v}$$

b



c When $v = 24$,

$$t = \frac{144}{24}$$
$$= 6$$

It takes 6 hours.

d When $t = \frac{90}{60} = \frac{3}{2}$ h,

$$\frac{3}{2} = \frac{144}{v}$$

so $v = 144 \times \frac{2}{3} = 96$

The speed is 96 km/h.

Note: If $a \propto \frac{1}{b}$ then $ab = k$, where k is a positive constant. Conversely, if $ab = k$ for all values of a and b, then $a \propto \frac{1}{b}$.

Example 5

The volume, V cm^3, of a quantity of gas kept at a constant temperature is inversely proportional to the pressure, P kPa. If the volume is 500 cm^3 when the pressure is 80 kPa, find the volume when the pressure is 25 kPa.

Solution

V is inversely proportional to $P\left(V \propto \frac{1}{P}\right)$

so $V = \frac{k}{P}$ or $VP = k$ for some constant k.

We know that $V = 500$ when $P = 80$

$$k = 500 \times 80$$
$$= 40\,000$$

so $V = \frac{40\,000}{P}$

When $P = 25$, $V = \frac{40\,000}{25}$

$$= 1600$$

Thus, the volume of the gas at 25 kPa is 1600 cm^3.

Example 6

If a is inversely proportional to the cube of b and $a = 2$ when $b = 3$, find the formula relating a and b. Then find:

a a when $b = 2$

b b when $a = \frac{27}{32}$

Solution

$a \propto \frac{1}{b^3}$

That is, $a = \frac{k}{b^3}$ or $ab^3 = k$ for some positive constant k.

We know that $a = 2$ when $b = 3$,

so $k = 54$

Hence, $a = \frac{54}{b^3}$

a When $b = 2$, $a = \frac{54}{8}$

$= 6.75$

b When $a = \frac{27}{32}$, $b^3 = 54 \div \frac{27}{32}$

$= 64$

Hence, $b = 4$

As with direct proportion, we are sometimes interested in the effect on one variable when the other one is changed. As before, we can take a particular value of one variable to work out the change in the other variable.

Example 7

Given that $y \propto \frac{1}{x^2}$, find, correct to the nearest 0.1%:

a the percentage change in y when x is decreased by 10%

b the percentage change in x when y is increased by 10%

Solution

$y \propto \frac{1}{x^2}$

that is, $y = \frac{k}{x^2}$, for some positive constant k.

a When $x = 1$, $y = k$.

When x is decreased by 10%, the new value of $x = 0.9$.

Then, $y = \frac{k}{0.9^2}$ so new value of $y \approx 1.235k$.

Thus, y is approximately 123.5% of its previous value.

That is, y has increased by approximately 23.5%.

(continued over page)

b $y = \dfrac{k}{x^2}$ so when $y = 1$,

$x^2 = k$

$x = \sqrt{k}$

When y is increased by 10%,

the new value of $y = 1.1$

so the new value of x is given by:

$1.1 = \dfrac{k}{x^2}$

$x^2 = \dfrac{k}{1.1}$

$x \approx 0.953\sqrt{k}$

That is, x is approximately 95.3% of its previous value.

Thus, x has decreased by approximately 4.7%.

[Alternatively, make x as the subject of the relationship.

$x^2 = \dfrac{k}{y}$

$x = \sqrt{\dfrac{k}{y}}$

When $y = 1$, $x = \sqrt{k}$

When $y = 1.1$, $x = \sqrt{\dfrac{k}{1.1}} = \dfrac{\sqrt{k}}{\sqrt{1.1}}$]

Inverse proportion

- y is **inversely proportional** to x^n when y is directly proportional to $\dfrac{1}{x}$.
- We write $y \propto \dfrac{1}{x}$ when $y = \dfrac{k}{x}$ or $xy = k$, where k is a positive constant.
- If y is inversely proportional to x, then the graph of y against $\dfrac{1}{x^n}$ is a straight line and the gradient of the line is equal to the **constant of proportionality**.
- If $y \propto \dfrac{1}{x}$, then for any pair of values x_1 and y_1, $x_1y_1 = k$.

Exercise 16B

1 Consider the following table of values.

a	1	2	3	4	5
b	15	7.5	5	3.75	3

a Plot the graph of b against $\dfrac{1}{a}$.

b Assuming that there is a simple relationship between the two variables, find a formula for b in terms of a.

2 Consider the following table of values.

x	1	2	5	10
y	100	25	4	1
x^2y	100			

a Complete the table of values for x^2y.

b Assuming that there is a simple relationship between the two variables, find a formula for y in terms of x.

3 Write each statement in symbols.

a The speed v km/h of a car over a given distance is inversely proportional to the time t hours of travel.

b m is inversely proportional to the square root of n.

c s is inversely proportional to the cube of t.

Example 5 **4** y is inversely proportional to x. If $x = 2$ when $y = 3$, find a formula relating x and y, and calculate:

a y when $x = \frac{3}{2}$

b x when $y = \frac{2}{3}$

Example 6 **5** Given that a is inversely proportional to b^2 and that $a = 6$ when $b = 2$, find a formula for a in terms of b, and calculate:

a a when $b = 3$

b b when $a = 3$

6 Given that $p \propto \frac{1}{\sqrt{q}}$ and that $p = 5$ when $q = 4$, find a formula for p in terms of q, and calculate:

a p when $q = 9$

b q when $p = 4$

7 For the data below, we assume that $y \propto \frac{1}{x}$. Find the constant of proportionality and complete the table.

x	1	2		4	
y	12		4		24

8 For the data below, we assume that $y \propto \frac{1}{x^2}$. Find the constant of proportionality and complete the table.

x	2		8		16
y	8	2		0.32	

Example 4 **9** If a car travels at an average speed of 60 km/h, it takes 77 minutes to complete a certain trip. To complete the same trip in 84 minutes, what average speed is required?

10 Timber dowelling comes in fixed lengths. If 48 pieces, each 3.6 cm long, can be cut from a fixed length, how many pieces 3.2 cm long can be cut from the same fixed length?

11 The illumination from a light is inversely proportional to the square of the distance from the light source. If the illumination is 3 units when seen from 4 metres away, find:

a the illumination when seen from 6 metres

b the distance from the light source when the illumination is 12 units

12 Given that $m \propto \frac{1}{n^2}$, what is the effect on:

a m when n is doubled?

b m when n is halved?

c n when m is multiplied by 16?

d n when m is divided by 9?

Example 7 **13** Given that $a \propto \frac{1}{b}$, what is the effect, correct to the nearest 0.1%, on a when b is:

a increased by 15%?

b decreased by 12%?

14 Given that $p \propto \frac{1}{q^3}$, what is the effect, correct to two decimal places, on:

a p when q is increased by 10%?

b p when q is decreased by 10%?

c q when p is increased by 20%?

d q when p is decreased by 20%?

15 We know that a cone of height h and radius r has volume $V = \frac{1}{3}\pi r^2 h$.

For cones of the same volume, height is inversely proportional to the radius squared.

a What is the effect on:

i the height when the radius is doubled?

ii the radius when the height is multiplied by 9?

b For cones of volume 12π cm^3, state the constant of proportionality in the relationship $h \propto \frac{1}{r^2}$.

16C Proportionality in several variables

Often a particular physical quantity is dependent on several other variables. For example, the distance d a motorist travels depends on both the speed v at which he travels and the time t taken for the trip. These variables are related by the formula $d = vt$. We say that d is directly proportional to v and t.

If $y = kxz$ for a positive constant k, we say that y is directly proportional to x and z. Similarly, if $a = \frac{kb^3}{c^2}$, where k is a positive constant, we say that a is directly proportional to b^3 and inversely proportional to c^2.

Example 8

Suppose that a is directly proportional to b and to the square of c.

If $a = 36$ when $b = 3$ and $c = 2$, find:

a the formula connecting a, b and c

b the value of a when $b = 4$ and $c = 1$

c the value of b when $a = 48$ and $c = 3$

d the value of c when $a = 64$ and $b = 6$

Solution

a $a \propto bc^2$, so $a = kbc^2$ for some constant k.

Substitute $a = 36$, $b = 3$ and $c = 2$ to find k.

$36 = k \times 3 \times 2^2$; hence, $k = 3$.

Thus, $a = 3bc^2$.

b When $b = 4$ and $c = 1$:

$$\begin{aligned} a &= 3 \times 4 \times 1^2 \\ &= 12 \end{aligned}$$

c When $a = 48$ and $c = 3$:

$$\begin{aligned} 48 &= 3 \times b \times 3^2 \\ b &= \frac{16}{9} \end{aligned}$$

d When $a = 64$ and $b = 6$:

$$\begin{aligned} 64 &= 3 \times 6 \times c^2 \\ c^2 &= \frac{32}{9} \\ c &= \frac{4\sqrt{2}}{3} \end{aligned}$$

Example 9

Suppose that y is directly proportional to x and inversely proportional to z.

If $y = \frac{1}{5}$ when $x = \frac{2}{5}$ and $z = \frac{3}{5}$, find:

a the formula for y in terms of x and z

b the value of y when $x = 1$ and $z = \frac{3}{8}$

c the value of z when $x = 2$ and $y = \frac{1}{6}$

Solution

a $y = \frac{kx}{z}$, for some positive constant k.

We know that $y = \frac{1}{5}$ when $x = \frac{2}{5}$ and $z = \frac{3}{5}$

so $\frac{1}{5} = k \times \frac{2}{5} \div \frac{3}{5}$

and $k = \frac{1}{5} \times \frac{5}{2} \times \frac{3}{5}$

$= \frac{3}{10}$

Hence, $k = \frac{3}{10}$ and $y = \frac{3x}{10z}$.

b $y = \frac{3x}{10z}$ so when $x = 1$ and $z = \frac{3}{8}$

so $y = \frac{3}{10} \div \frac{3}{8} = \frac{4}{5}$

c $y = \frac{3x}{10z}$ so when $x = 2$ and $y = \frac{1}{6}$

$$\begin{aligned} \frac{1}{6} &= \frac{3 \times 2}{10z} \\ \text{so } 10z &= 36 \\ z &= 3.6 \end{aligned}$$

Example 10

Suppose that a is directly proportional to the square of b and inversely proportional to c. Find the effect on a when:

a b is halved and c is doubled

b b is increased by 10% and c is increased by 20%

Solution

$a = \dfrac{kb^2}{c}$ for some positive constant k.

a When $b = 1$ and $c = 1$, $a = k$

When $b = \dfrac{1}{2}$ and $c = 2$, $a = \dfrac{1}{4}k \div 2$

$= \dfrac{k}{8}$

Thus, the value of a is divided by 8.

b When $b = 1$ and $c = 1$, $a = k$

When b is increased by 10% and c is increased by 20%.

So, $b = 1.1$ and $c = 1.2$

$$a = \frac{1.1^2}{1.2}k$$

$$= \frac{121}{120}k$$

Thus, a is increased by approximately 0.83%.

Exercise 16C

Example 8

1 If $a \propto bc$, write down the formula relating the variables and complete the following table.

a	12	24		48	72
b	1		2		2
c	1	2	1	2	

2 If $r \propto \dfrac{s}{t}$, write down the formula relating the variables and complete the following table.

r	24	12		48	4
s	1		2		2
t	1	2	2	1	

Example 9

3 Suppose that y is directly proportional to x and inversely proportional to w. If $y = 2$ when $x = 7$ and $w = 14$, find y when $x = 10$ and $w = 8$.

4 Assume that y is directly proportional to the square of x and inversely proportional to the square root of z.

a Write a formula for y in terms of x and z.

b If $y = 6$ when $x = 2$ and $z = 4$, find y when $x = 3$ and $z = 16$.

5 Suppose that a is directly proportional to b and the cube of c.

a Write a formula for a in terms of b and c.

b If $a = 96$ when $b = 3$ and $c = 2$, find b when $a = 16$ and $c = \frac{1}{2}$.

6 The amount of heat, H units, produced by an electric heater element is directly proportional to the square of the current, i amperes, flowing through the element, to the electrical resistance, R ohms, and to the time, t seconds, for which the current has been flowing.

a Write down the formula for H in terms of i, R and t.

b If 256 units of heat are produced by a current of 2 amp through a resistance of 40 ohms for 10 seconds, how much heat is produced by a current of 4.5 amp through a resistance of 60 ohms for 15 seconds?

7 A model aeroplane attached to one end of a string moves in a horizontal circle. The tension, T N (or newtons), in the string is directly proportional to the square of the speed, v m/s, and inversely proportional to the radius, r m, of the circle. If the radius is 10 m and the speed is 20 m/s, the tension is 60 N. Find the tension if the radius is 15 m and the speed is 30 m/s.

8 The frequency n (the number of vibrations per second) of a piano string varies directly as the square root of the tension, T N, in the string and inversely as the length, ℓ cm, of the string. A string 30 cm long under a tension 25 N has a frequency of 256 vibrations per second (this is the pitch called 'middle C'). If the tension is changed to 30 N, to what must the length be changed, correct to two decimal places, for the string to emit the same note?

9 The quantity t is directly proportional to m and n, and is inversely proportional to the square of r. If $t = \frac{45}{4}$ when $m = 3$, $n = 5$ and $r = 4$, find:

a r when $t = 6$, $m = 9$ and $n = 8$

b n when $t = 8$, $r = 12$ and $m = 4$

Example 10

10 If y is directly proportional to the cube of x and inversely proportional to the square of z, what is the effect on y if:

a both x and z are doubled?

b x is increased in the ratio $3 : 2$ and z is decreased in the ratio $1 : 2$?

11 If y is directly proportional to the square of x and inversely proportional to the square root of z, what is the effect on y if:

a x and z are increased by 10%?

b x is increased by 20% and z is decreased by 15%?

12 The force of attraction F between two particles of masses m_1 and m_2 that are distance d apart varies directly as the product of the masses, and inversely as the square of the distance between them.

a What is the effect on F if the distance between the two masses is doubled?

b What is the effect on the force if the distance between the two particles is halved and the mass of one particle is trebled?

13 The value of g, the acceleration due to gravity on the surface of a planet or moon, varies directly as the planet or moon's mass and inversely as the square of the radius of the planet. The mass of the Moon is $\frac{1}{80}$ of the mass of the Earth, and the radius of the moon is $\frac{3}{11}$ the radius of the earth. Given that the value of g on the surface of the earth is $9.8\ \text{m/s}^2$, find the value of g, correct to two decimal places, on the surface of the moon.

Review exercise

1 Write each of the following in words.

a $x \propto y$ **b** $p \propto n^2$ **c** $a \propto \sqrt{b}$ **d** $p \propto q^3$

2 **a** Given that $p \propto q$ and $p = 12$ when $q = 1.5$, find the exact value of:

i p when $q = 6$ **ii** q when $p = 81$

b Given that $a \propto b^2$ and $a = 20$ when $b = 4$, find the formula for a in terms of b and:

i a when $b = 5$ **ii** a when $b = 12$

3 In each of the following tables, $y \propto x$. Find the constant of proportionality in each case and complete the tables.

a

x	0	1	2	3
y	0	12		

b

x	2	8	12	18
y	3			

4 Given that $y \propto x^3$, what is the effect on y when x is:

a doubled? **b** multiplied by 3? **c** divided by 4?

5 Given that $m \propto \sqrt{n}$, what is the effect on:

a m when n is doubled? **b** m when n is divided by 4?

6 Given that $a \propto b^2$, what is the effect on a when b is:

a increased by 5%? **b** decreased by 8%?

7 y is inversely proportional to x. If $x = 5$ when $y = 8$, find:

a y when $x = \frac{3}{2}$

b x when $y = \frac{2}{3}$

8 a is inversely proportional to b^2. If $a = 8$ when $b = 2$, find:

a a when $b = 4$

b b when $a = 9$

9 z is directly proportional to the square of x and inversely proportional to the square root of y. If $z = 12$ when $x = 2$ and $y = 4$, find z when $x = 6$ and $y = 32$.

10 The quantity y is directly proportional to x and the cube of z. If $y = 108$ when $x = 3$ and $z = 2$, find x when $y = 24$ and $z = \frac{1}{2}$.

Challenge exercise

1 The electrical resistance, R ohms, in a wire is directly proportional to its length, L m, and inversely proportional to the square of its diameter, D mm. A certain wire 100 m long with a diameter 0.4 mm has a resistance 1.4 ohms.

a Find the equation connecting R, L and D.

b Find the resistance (correct to one decimal place) of a wire of the same material if it is 150 m in length and has a diameter of 0.25 mm.

c If the length and diameter are doubled, what is the effect on the resistance?

d If the length is increased by 10% and the diameter is decreased by 5%, what is the percentage change on the resistance? (Give your answer correct to one decimal place.)

2 If $a \propto c$ and $b \propto c$, prove that $a + b$, $a - b$ and $\sqrt{ab}$ are directly proportional to c.

3 It is known that $a \propto x$, $b \propto \frac{1}{x^2}$ and $y = a + b$. If $y = 30$ when $x = 2$ or $x = 3$, find the expression for y in terms of x.

4 If $x^2 + y^2$ is directly proportional to $x + y$ and $y = 2$ when $x = 2$, find the value of y when $x = \frac{4}{5}$.

5 For stones of the same quality, the value of a diamond is proportional to the square of its weight. Find the loss incurred by cutting a diamond worth \$$C$ into two pieces whose weights are in the ratio $a : b$.

6 If $a + b \propto a - b$, prove that $a^2 + b^2 \propto ab$.

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CHAPTER

17

Number and Algebra

Polynomials

You have spent some time over the last two years studying quadratics, learning to factorise them and learning to sketch their graphs. In this chapter, we take the next step and study polynomials such as $x^3 - x$ and $x^4 + x^2 + x - 14$ that contain higher powers of x. Just as we factorised, solved and graphed quadratics, we shall do the same for polynomials.

Some modern electronic devices, such as mobile phones and Blu-ray discs, use error-correcting codes that are based on calculations using polynomials.

17A The language of polynomials

A polynomial is an expression, such as:

$$x^5 - 5x^2 + 7x\,, \qquad 3x^7 + 2 \qquad \text{and} \qquad \frac{1}{5}x^2 + 2x - 5$$

A polynomial may have any number of terms (the word 'polynomial' means 'many terms'), but each term must be a multiple of a whole-number power of x. (Recall that a whole number is any number in the sequence 0, 1, 2, 3, ...)

The term of *highest index* among the non-zero terms is called the **leading term**. Its coefficient is called the **leading coefficient**, and its index is called the **degree** of the polynomial. Thus:

- $x^5 - 5x^2 + 7x$ has leading term x^5, leading coefficient 1 and degree 5
- $3x^7 + 2$ has leading term $3x^7$, leading coefficient 3 and degree 7
- $\frac{1}{5}x^2 + 2x - 5$ has leading term $\frac{1}{5}x^2$, leading coefficient $\frac{1}{5}$ and degree 2.

A **monic** polynomial has leading coefficient 1; thus, $x^5 - 5x^2 + 7x$ is a monic polynomial. The other two examples above are **non-monic** because neither of the leading coefficients is 1.

The second term in the polynomial $3x^7 + 2$ is called the **constant term**, because it does not involve x. The constant term in $x^5 - 5x^2 + 7x$ is zero.

Some names for polynomials

You are already familiar with some simple polynomials.

- Polynomials of degree 2, such as $x^2 + 6x + 2$, are called **quadratic**.
- Polynomials of degree 1, such as $7x - 3$, are called **linear**.
- A non-zero number such as 8 is regarded as a polynomial of degree 0, because we can write $8 = 8x^0$, and is called a **constant polynomial**.

The number zero is regarded as a polynomial called the **zero polynomial**. It has no terms, so the leading term and the degree of the zero polynomial are not defined.

In this chapter you will begin to study polynomials of degree higher than 2.

- Polynomials of degree 3 are called **cubic polynomials**.
- Polynomials of degree 4 are called **quartic polynomials**.
- Polynomials of degree 5 are called **quintic polynomials**.

Beyond these, we drop the Latin name and refer to a polynomial by its degree.

For example, $x^6 + 2x^3 + x + 2$ is a polynomial of degree 6.

When we write down an expression for a general polynomial, we need to use dots:

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

where n is a whole number, the coefficients $a_0, a_1, a_2, \ldots, a_n$ are real numbers, and $a_n \neq 0$.

If a polynomial is written in this form, we call it the **standard form** of the polynomial.

A new notation for polynomials and substitution

We need a simple way of naming a polynomial such as $x^3 + 2x^2 - 4x - 5$, so that we can talk about it easily. The new notation is:

$$P(x) = x^3 + 2x^2 - 4x - 5$$

The x in brackets indicates that x is the variable in the polynomial.

This notation is called **function notation**.

When we substitute the number 5 for x, the number we obtain is written $P(5)$ and:

$$\begin{aligned} P(5) &= 5^3 + 2 \times 5^2 - 4 \times 5 - 5 \\ &= 125 + 50 - 20 - 5 \\ &= 150 \end{aligned}$$

Similarly, $P(a) = a^3 + 2a^2 - 4a - 5$.

Polynomials

- A **polynomial** is an expression that can be written in the form:
  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

  where n is a whole number, and the coefficients $a_0, a_1, a_2, \dots, a_n$ are real numbers, $a_n \neq 0$.
- The number 0 is called **zero polynomial**. It has no terms, so the leading term and the degree of the zero polynomial are not defined.
- The **leading term** of the polynomial is the term of highest index, $a_n x^n$, among those with a non-zero coefficient.
- The **degree** of the polynomial is the index of the leading term, and the **leading coefficient** is the coefficient of the leading term.
- A **monic polynomial** is a polynomial whose leading coefficient is 1.
- The **constant term** is the term of index 0 (this is the term not involving x).

Example 1

State whether each of the following functions is a polynomial. If it is a polynomial, arrange the terms in descending order by degree. Then state the leading term, the leading coefficient, the degree and the constant term, and say whether or not the polynomial is monic.

a $P(x) = x^3 - 5x^2 - x^6$

b $Q(x) = x^2 + x^{-2}$

c $R(x) = \dfrac{9}{2} + x$

d $S(x) = 5$

Solution

a $P(x) = -x^6 + x^3 - 5x^2$

This is a non-monic polynomial.

The leading term is $-x^6$, the leading coefficient is -1, the degree is 6 and the constant term is 0.

b $Q(x)$ is not a polynomial, because the index of the term x^{-2} is not a whole number.

(continued over page)

c $R(x) = x + \frac{9}{2}$

This is a monic polynomial.

The leading term is x, the leading coefficient is 1, the degree is 1 and the constant term is $\frac{9}{2}$.

d $S(x) = 5$ is a non-monic polynomial.

The leading term is 5, the leading coefficient is 5, the degree is 0 and the constant term is 5.

Example 2

Expand each expression, and then state the degree, the leading coefficient and the constant term.

a $P(x) = (3x + 2)^2$

b $Q(x) = 3x(5 - x^2)(5 + x^2)$

Solution

a $P(x) = 9x^2 + 12x + 4$

The degree is 2, the leading coefficient is 9, and the constant term is 4.

b $$\begin{aligned} Q(x) &= 3x(25 - x^4) \\ &= -3x^5 + 75x \end{aligned}$$

The degree is 5, the leading coefficient is −3, and the constant term is 0.

Example 3

If $P(x) = x^4 - 2x^2 + 10x + 11$, then find $P(3)$, $P(0)$, $P(-1)$ and $P(2a)$.

Solution

$$\begin{aligned} P(3) &= 3^4 - 2 \times 3^2 + 10 \times 3 + 11 \\ &= 81 - 18 + 30 + 11 \\ &= 104 \end{aligned}$$

$$\begin{aligned} P(-1) &= 1 - 2 - 10 + 11 \\ &= 0 \end{aligned}$$

$$\begin{aligned} P(0) &= 0 - 0 + 0 + 11 \\ &= 11 \end{aligned}$$

$$\begin{aligned} P(2a) &= (2a)^4 - 2(2a)^2 + 10(2a) + 11 \\ &= 16a^4 - 8a^2 + 20a + 11 \end{aligned}$$

Example 4

If $P(x) = x^4 - 3x^3 + ax + 2$, and $P(-2) = 0$, find a.

Solution

$$\begin{aligned} P(-2) &= 0 \\ 16 + 24 - 2a + 2 &= 0 \\ 2a &= 42 \\ a &= 21 \end{aligned}$$

Exercise 17A

Example 1

1 State whether or not each expression is a polynomial.

a $5x^2 + 6x + 3$ **b** $\frac{1}{x^7} + x$ **c** $1 - 5x^5 + 10x^{10}$

d $3x + 4$ **e** $\sqrt{5}x^3 - \sqrt{5}$ **f** $2x - 1$

g $(x - 5)^2$ **h** $\frac{1}{4}x^{21} + 7x^{24}$ **i** $x + \sqrt{x}$

j $\frac{3}{5}x^4 - \pi x^2 + \pi$ **k** -6 **l** $\frac{2 - 3x}{2 + 3x}$

Example 2

2 State the degree, the leading coefficient, and the constant term of each polynomial. Rearrange the terms first.

a $x^3 + 5x - 6$ **b** $5x^4 - 5x^2 - 7x$ **c** $7 - 4x$

d 15 **e** $5 - 2x + 7x^3$ **f** $8 - 4x + 3x^2$

g $\frac{1}{2}x^2 - 14x^3$ **h** $\frac{x^5}{5} + \frac{x^3}{3}$ **i** $-x^5 - 3x^6 - x^3 + \frac{1}{3} + x^4$

Example 1

3 State whether each polynomial is monic or non-monic.

a $x^4 + x$ **b** $-2x^3 + 5x - 2$ **c** $\frac{1}{2}x^3 + x^4$

d 1 **e** $5x + 2x^2 - x^3$ **f** $\frac{4x + 6x^3}{6}$

Example 3

4 Let $P(x) = x^3 - x - 6$. Find:

a $P(1)$ **b** $P(-1)$ **c** $P(2)$ **d** $P(-2)$ **e** $P(0)$

f $P(-3)$ **g** $P(a)$ **h** $P(2a)$ **i** $P(-a)$

5 Find $Q(-1)$, $Q(2)$, $Q(-10)$ and $Q(0)$ for each polynomial.

a $Q(x) = x^5 + x^4 + x^3 + x^2$ **b** $Q(x) = x^4 - 2x^3 - 4x^2 - 8x + 32$

c $Q(x) = 5x^3 - 3x^5 + 1$ **d** $Q(x) = x^2 + 8x - 20$

6 Expand and simplify each polynomial, and then state its leading term, its degree and its constant term.

a $A(x) = (x - 5)^2$ **b** $B(x) = (x - 5)(x + 10)$

c $C(x) = x^2(x - 3x^5)$ **d** $D(x) = x(x + 6)^2$

e $E(x) = 3x^3(x^2 + 1)^2$ **f** $F(x) = x^2 + 9 - (x + 3)^2$

g $G(x) = (x + 2)(x + 3)(x + 4)$ **h** $H(x) = (x + 1)^2 + (x + 2)^2 + (x + 3)^2$

Example 4

7 **a** Find a if $P(x) = x^5 - 3x^3 - 5x + a$ and $P(2) = 4$.

b Find b if $Q(x) = 2x^3 - 3x^2 + bx + 4$ and $Q(-1) = 0$.

c Find a and d if $S(x) = x^4 - 4x^3 + ax^2 - 2x + d$ and $S(-1) = S(-2) = 0$.

8 **a** Find the monic polynomial $P(x)$ of degree 2, constant term 1 and with $P(-1) = 7$.

b Write down an example of a quartic polynomial $Q(x)$ with leading coefficient 2, constant term 0 and with $Q(-2) = 12$.

c Find the quartic polynomial $R(x)$ with equal coefficients such that $R(-1) = 7$.

17B Adding, subtracting and multiplying polynomials

To add or subtract two polynomials, simply collect like terms.

Example 5

For the polynomials $P(x) = 3x^4 - 2x^2 + x - 1$, $Q(x) = x^2(7x^3 + 2)$ and $R(x) = 3x^4 - x - 3$, find:

a $P(x) + Q(x)$

b $P(x) - R(x)$

c $P(x) - Q(x) + R(x)$

d $2P(x) + 3Q(x)$

Solution

First, we need to expand $Q(x)$ and obtain $Q(x) = 7x^5 + 2x^2$.

a
$$\begin{aligned} P(x) + Q(x) &= (3x^4 - 2x^2 + x - 1) + (7x^5 + 2x^2) \\ &= 7x^5 + 3x^4 + x - 1 \end{aligned}$$

b
$$\begin{aligned} P(x) - R(x) &= (3x^4 - 2x^2 + x - 1) - (3x^4 - x - 3) \\ &= 3x^4 - 2x^2 + x - 1 - 3x^4 + x + 3 \\ &= -2x^2 + 2x + 2 \end{aligned}$$

c
$$\begin{aligned} P(x) - Q(x) + R(x) &= (3x^4 - 2x^2 + x - 1) - (7x^5 + 2x^2) + (3x^4 - x - 3) \\ &= -7x^5 + 6x^4 - 4x^2 - 4 \end{aligned}$$

d
$$\begin{aligned} 2P(x) + 3Q(x) &= 2(3x^4 - 2x^2 + x - 1) + 3(7x^5 + 2x^2) \\ &= 6x^4 - 4x^2 + 2x - 2 + 21x^5 + 6x^2 \\ &= 21x^5 + 6x^4 + 2x^2 + 2x - 2 \end{aligned}$$

Multiplying polynomials

To multiply two polynomials, we use the distributive law a number of times. We multiply each term in the first polynomial by the second polynomial and add these expressions together. We then expand the brackets, collect like terms and write the polynomial in standard form.

Example 6

The polynomials $P(x)$, $Q(x)$ and $R(x)$ are given by $P(x) = x^3 - x^2 + x - 1$, $Q(x) = 3x^3 - 2x^2$ and $R(x) = -x^4 + 2x^3 - 3x^2$. Find:

a $P(x)Q(x)$

b $Q(x)R(x)$

Solution

a
$$\begin{aligned} P(x)Q(x) &= (x^3 - x^2 + x - 1)(3x^3 - 2x^2) \\ &= x^3(3x^3 - 2x^2) - x^2(3x^3 - 2x^2) + x(3x^3 - 2x^2) - (3x^3 - 2x^2) \\ &= 3x^6 - 2x^5 - 3x^5 + 2x^4 + 3x^4 - 2x^3 - 3x^3 + 2x^2 \\ &= 3x^6 - 5x^5 + 5x^4 - 5x^3 + 2x^2 \end{aligned}$$

b
$$\begin{aligned} Q(x)R(x) &= (3x^3 - 2x^2)(-x^4 + 2x^3 - 3x^2) \\ &= 3x^3(-x^4 + 2x^3 - 3x^2) - 2x^2(-x^4 + 2x^3 - 3x^2) \\ &= -3x^7 + 6x^6 - 9x^5 + 2x^6 - 4x^5 + 6x^4 \\ &= -3x^7 + 8x^6 - 13x^5 + 6x^4 \end{aligned}$$

Addition, subtraction and multiplication of polynomials

- Polynomials can be added, subtracted and multiplied using the usual rules of algebra.
- The sum, difference and product of two polynomials is always another polynomial.

Exercise 17B

Example 5

1 Find the sum $P(x) + Q(x)$ and the difference $P(x) - Q(x)$, given that:

a $P(x) = x^3 + 3x + 5$ and $Q(x) = 2x^3 - 3x^2 - 4x$

b $P(x) = 2x^3 - 3x^2 - 4x + 5$ and $Q(x) = -2x^3 + 3x^2 + 5x - 2$

c $P(x) = 4x^2 - 3x + 6$ and $Q(x) = 4x^2 - 3x - 6$

d $P(x) = x^4 - x^2 + x - 1$ and $Q(x) = x^3 - x^2 + x - 1$

e $P(x) = 5x^3 + 2x^2 - x - 5$ and $Q(x) = 5 - 5x^3 - 2x^2 + x$

2 For $P(x) = 2x^3 - 3x^2 + 7$, $Q(x) = 4x^5 - 2x^2 + 2$ and $R(x) = 3x^5 - x^3 - 2$, find:

a $2P(x) + 3Q(x)$

b $5P(x) - 4Q(x)$

c $3P(x) - 2Q(x) + R(x)$

d $2P(x) - 3Q(x) + 4R(x)$

Example 6

3 Find the product $P(x)Q(x)$, given that:

a $P(x) = x^3$ and $Q(x) = 5x^3 - 2x^2 + 7x$

b $P(x) = x^3 + 1$ and $Q(x) = x^3 - 1$

c $P(x) = x^4 + x^2 + 1$ and $Q(x) = x^2 - 1$

d $P(x) = -x^3 + x$ and $Q(x) = -x^2 - 3x$

e $P(x) = x^2 + x + 1$ and $Q(x) = x^2 - x + 1$

4 Look at the examples in question **3**. Then copy and complete:

a 'When two non-zero polynomials $P(x)$ and $Q(x)$ are multiplied, the degree of the product …'

b 'The constant term of $P(x)Q(x)$ is …'

5 The square of a polynomial $P(x)$ is $(P(x))^2 = P(x)P(x)$. Find the square of $P(x)$ given that:

a $P(x) = x - 7$ **b** $P(x) = -x^2 + 3$ **c** $P(x) = x^3 - 7x$

d $P(x) = 3x^5 + 5x^3$ **e** $P(x) = x^2 + x + 1$ **f** $P(x) = x^4 + x^2 + 1$

6 Look at the examples in question **5**. Then copy and complete:

a 'When a non-zero polynomial $P(x)$ is squared, the degree of the square …'

b 'The constant term of $(P(x))^2$ is …'

c 'If … then the square $(P(x))^2$ is monic.'

7 Expand and simplify $D(x)Q(x) + R(x)$ given that:

a $D(x) = x - 1$, $Q(x) = -5x^3 - 7$ and $R(x) = -10$

b $D(x) = 2x + 3$, $Q(x) = 3x^3 - 5x^2 - 1$ and $R(x) = 6$

c $D(x) = x^2 - 7$, $Q(x) = -4x^2 - 3x - 2$ and $R(x) = 7x - 12$

8 Expand and simplify:

a $(x - 2)(x + 1)(x + 2)(x - 1)$ **b** $(x + 1)(x + 2)(x + 3)(x + 4)$

17C Dividing polynomials

Whenever we add, subtract or multiply two polynomials, the result is another polynomial. Division of polynomials, however, does not usually result in a polynomial. For example:

$$\begin{aligned}\frac{3x^4 - 5x^2 + 7}{x^2} &= \frac{3x^4}{x^2} - \frac{5x^2}{x^2} + \frac{7}{x^2}\\ &= 3x^2 - 5 + \frac{7}{x^2}\end{aligned}$$

This is not a polynomial.

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You are already familiar with this situation from arithmetic with integers. Adding, subtracting and multiplying integers always results in an integer, but division with integers may or may not result in an integer.

For example:

$$\begin{aligned}\frac{32}{5} &= \frac{30}{5} + \frac{2}{5}\\ &= 6 + \frac{2}{5}\end{aligned}$$

This can also be written without fractions in terms of the quotient and remainder:

$$32 \div 5 = 6 \text{ remainder } 2$$

From this we can also write 32 as the subject in this relationship as follows:

$$32 = 5 \times \underbrace{6}_{\text{quotient}} + \overbrace{2}^{\text{remainder}}$$

We can see that we have three different ways of writing the same division statement. Similarly, we can write:

$$(3x^4 - 5x^2 + 7) \div x^2 = 3x^2 - 5 \text{ remainder } 7$$

From this we can write $3x^4 - 5x^2 + 7 = x^2(3x^2 - 5) + 7$.

Division of whole numbers

Before we try to divide polynomials, let us review long division of whole numbers by converting 283 months to years and months. That is, we must perform the division $283 \div 12$.

$$\begin{array}{r l} 23 & \leftarrow \text{quotient} \\ 12\overline{)283} & \\ \underline{24} & \\ 43 & \\ \underline{36} & \\ 7 & \leftarrow \text{remainder} \end{array}$$

Thus $283 \div 12 = 23$ remainder 7, which we write as:

$$283 = 12 \times 23 + 7$$

The final remainder 7 had to be less than the divisor 12.

Hence, 283 months is 23 years and 7 months.

- The number 12 that we are dividing by is called the **divisor**.
- The number 283 that we are dividing into is called the **dividend**.
- The number of years is 23, which is called the **quotient**.
- The number of months left over is 7, which is called the **remainder**.

In general, we can write the dividend as the subject of a division statement in the following way.

Let p (the **dividend**) and d (the **divisor**) be whole numbers, with $d > 0$.

- If $\frac{p}{d} = q$ **remainder** r (q is the **quotient**), then $p = dq + r$, where $0 \le r < d$.
- When the remainder, r, is zero, then d is a **factor** of p because $p = dq$.

Division of polynomials

These ideas also apply to polynomials.

The key idea when dividing one polynomial by another is to keep working with the leading terms.

The following example shows how to divide $P(x) = 5x^4 - 7x^3 + 2x - 4$ by $D(x) = x - 2$ and how to write this result as $P(x) = D(x)Q(x) + R(x)$.

There must be a column for each successive power of x. Thus we leave a gap for the missing term in x^2 in $P(x)$. Alternatively, '$+ 0x^2$' could have been written in its place.

We begin by dividing the leading term of $P(x)$ by the leading term of $Q(x)$.

$$\begin{array}{rrrrr}
 & 5x^3 & & & \\ \cline{2-5}
x-2 \,\big) & 5x^4 & -7x^3 & +2x & -4 \\
 & 5x^4 & -10x^3 & & \\ \cline{2-5}
 & & 3x^3 & +2x & -4
\end{array}$$

(Divide x into $5x^4$, giving the $5x^3$ which is written directly above the leading term of the dividend.)
(Multiply $x - 2$ by $5x^3$.)
(Subtract line 2 from line 1.)

The process will now be repeated with $3x^3 + 2x - 4$ as the new dividend. The whole process is shown next.

Keep dividing by the leading term of the divisor, adding terms progressively to the quotient.

$$\begin{array}{rrrrrr}
 & 5x^3 & +3x^2 & +6x & +14 & \\ \cline{2-6}
x-2 \,\big) & 5x^4 & -7x^3 & & +2x & -4 \\
 & 5x^4 & -10x^3 & & & \\ \cline{2-6}
 & & 3x^3 & & +2x & -4 \\
 & & 3x^2 & -6x^2 & & \\ \cline{3-6}
 & & & 6x^2 & +2x & -4 \\
 & & & 6x^2 & -12x & \\ \cline{4-6}
 & & & & 14x & -4 \\
 & & & & 14x & -28 \\ \cline{5-6}
 & & & & & 24
\end{array}$$

(Divide x into $5x^4$, giving $5x^3$.)
(Multiply $x - 2$ by $5x^3$, then subtract.)
(Divide x into $3x^3$, giving $3x^2$.)
(Multiply $x - 2$ by $3x^2$, then subtract.)
(Divide x into $6x^2$, giving $6x$.)
(Multiply $x - 2$ by $6x$, then subtract.)
(Divide x into $14x$, giving 14.)
(Multiply $x - 2$ by 14, then subtract.)
(This is the final remainder.)

Hence, $5x^4 - 7x^3 + 2x - 4 = (x - 2)(5x^3 + 3x^2 + 6x + 14) + 24.$ (1)

We recommend the above method, although other layouts are possible.

- The final remainder must either be zero, or have degree less than the degree of the divisor $x - 2$.
- We use the same names as for integer division:
  - The polynomial $x - 2$ that we are dividing by is called the **divisor**.
  - The polynomial $5x^4 - 7x^3 + 2x - 4$ that we are dividing into is called the **dividend**.
  - The **quotient** is the polynomial $5x^3 + 3x^2 + 6x + 14$.
  - The **remainder** is the polynomial 24.

This process is called the **division algorithm** for polynomials.

The final statement, (1), is an identity that must be true for all values of x.

We can perform a partial check that the division has been done correctly by substituting some small values of x into the final statement marked (1):

When $x = 1$, LHS $= 5 - 7 + 2 - 4$
$= -4$

RHS $= (-1) \times (5 + 3 + 6 + 14) + 24$
$= -28 + 24$
$= -4$

When $x = 2$, LHS $= 80 - 56 + 4 - 4$
$= 24$

RHS $= 0 \times (\ldots) + 24$
$= 24$

Notice that $x = 2$ was particularly easy to substitute into the RHS, because $x - 2 = 0$.

Example 7

Divide $P(x) = 5x^4 - 7x^3 + 2x - 4$ by $D(x) = x^2 - 2$. Express the result in the form $P(x) = D(x)Q(x) + R(x)$, where the degree of $R(x)$ is less than the degree of $D(x)$.

Solution

This time the divisor $x^2 - 2$ has degree 2, so the remainder will either be zero, or have degree 0 or 1.

$$
\begin{array}{r|rrrrr l}
 & 5x^2 & -7x & +10 & & & \\
\hline
x^2 - 2 & 5x^4 & -7x^3 & & +2x & -4 & \text{(Divide } x^2 \text{ into } 5x^4, \text{ giving } 5x^2.) \\
 & 5x^4 & & -10x^2 & & & \text{(Multiply } x^2 - 2 \text{ by } 5x^2, \text{ then subtract.)} \\
\hline
 & & -7x^3 & +10x^2 & +2x & -4 & \text{(Divide } x^2 \text{ into } -7x^3, \text{ giving } -7x.) \\
 & & -7x^3 & & +14x & & \text{(Multiply } x^2 - 2 \text{ by } -7x, \text{ then subtract.)} \\
\hline
 & & & 10x^2 & -12x & -4 & \text{(Divide } x^2 \text{ into } 10x^2, \text{ giving } 10.) \\
 & & & 10x^2 & & -20 & \text{(Multiply } x^2 - 2 \text{ by } 10, \text{ then subtract.)} \\
\hline
 & & & & -12x & +16 & \text{(This is the final remainder.)}
\end{array}
$$

Hence, $5x^4 - 7x^3 + 2x - 4 = (x^2 - 2)(5x^2 - 7x + 10) + (-12x + 16)$ (2)

The remainder $-12x + 16$ has degree 1, which is less than the degree of the divisor $x^2 - 2$, which is 2.

Again, we can perform a partial check by substituting some small values of x into the final statement marked (2):

When $x = 0$, LHS $= -4$

RHS $= -20 + 16$
$= -4$

When $x = 1$, LHS $= 5 - 7 + 2 - 4$
$= -4$

RHS $= (1 - 2) \times (5 - 7 + 10) + (-12 + 16)$
$= -8 + 4 = -4$

A full check may be made by expanding the right-hand side of (2).

Factors of polynomials

When one polynomial is a factor of another, then the remainder after division is zero. You have already seen this with whole numbers. For example, 7 is a factor of 42, and when we divide 42 by 7, we obtain $42 = 7 \times 6 + 0$.

We can then go on to factorise 42 completely into primes as $42 = 7 \times 3 \times 2$.

Here is an example of dividing a polynomial by one of its factors.

Example 8

a Divide $x^3 + 5x^2 - 4x - 20$ by $x + 5$.

b Hence, factorise $x^3 + 5x^2 - 4x - 20$ into linear factors.

Solution

a

$$\begin{array}{r|l} & x^2 - 4 \\ \cline{2-2} x + 5 & x^3 + 5x^2 - 4x - 20 \\ & x^3 + 5x^2 \\ \cline{2-2} & - 4x - 20 \\ & - 4x - 20 \\ \cline{2-2} & 0 \end{array}$$

Since the remainder is zero, $x + 5$ is a factor of $x^3 + 5x^2 - 4x - 20$ and $x^3 + 5x^2 - 4x - 20 = (x + 5)(x^2 - 4)$.

b Since $x^2 - 4 = (x - 2)(x + 2)$, the complete factorisation is $x^3 + 5x^2 - 4x - 20 = (x + 5)(x + 2)(x - 2)$.

A formal statement of the division algorithm

In the next few sections, we will need a formal algebraic statement of the division algorithm for polynomials. It is very similar to the statement for whole numbers.

Dividing polynomials

Let $P(x)$ (the **dividend**) and $D(x)$ (the **divisor**) be polynomials, with $D(x) \neq 0$.

- When we divide $P(x)$ by $D(x)$, we obtain two more polynomials, $Q(x)$ (the **quotient**) and $R(x)$ (the **remainder**), such that:

  1 $P(x) = D(x)Q(x) + R(x)$, and

  2 either $R(x) = 0$ or $R(x)$ has degree less than $D(x)$.

- When the remainder $R(x)$ is zero, then $D(x)$ is a **factor** of $P(x)$.

  The polynomial $P(x)$ then factorises as the product $P(x) = D(x)Q(x)$.

Exercise 17C

1 Carry out each whole-number division, using long division when necessary. Write the result of the division in the form $p = dq + r$, where $0 \le r < d$.
For example, $47 \div 10 = 4$ remainder 7, which we write as $47 = 10 \times 4 + 7$.

a $68 \div 11$ **b** $1454 \div 12$ **c** $2765 \div 21$

Example 7

2 Use the division algorithm to divide $P(x)$ by $D(x)$. Express each result in the form $P(x) = D(x)Q(x) + R(x)$, where either $R(x) = 0$ or the degree of $R(x)$ is less than the degree of $D(x)$.

a $P(x) = x^2 + 6x + 1$, $D(x) = x + 2$

b $P(x) = x^3 - 5x^2 - 12x + 30$, $D(x) = x + 5$

c $P(x) = 5x^3 - 7x^2 - 6$, $D(x) = x - 3$

d $P(x) = x^4 + 3x^2 - 3x$, $D(x) = x + 2$

e $P(x) = 4x^3 - 4x^2 + 1$, $D(x) = 2x + 1$

f $P(x) = x^4 + 3x^3 - 3x^2 - 4x + 1$, $D(x) = x + 1$

3 **a** Find the quotient and remainder when $x^4 + x^3 + x^2 + x + 1$ is divided by $x^2 + 2x$.

b Find the quotient and remainder when $x^4 - 2x^3 + 3x^2 - 4x + 5$ is divided by $x^2 - 2$.

4 Divide $P(x)$ by $D(x)$ in each case. Express each result in the form $P(x) = D(x)Q(x) + R(x)$.

a $P(x) = x^3 + 5x^2 - x + 2$, $D(x) = x^2 + x + 1$

b $P(x) = x^3 - 4x^2 - 3x + 7$, $D(x) = x^2 - 2x + 3$

c $P(x) = x^4 + 5x^2 + 3$, $D(x) = x^2 - 3x - 3$

d $P(x) = x^5 - 3x^4 - 9x^2 + 9$, $D(x) = x^3 - x^2 + x - 1$

5 **a** If a polynomial is divided by a polynomial of degree 1 and the remainder is non-zero, what are the possible degrees of the remainder?

b If a polynomial is divided by a polynomial of degree 2 and the remainder is not zero, what are the possible degrees of the remainder?

c A polynomial has remainder $R(x)$ of degree 2 after division by $D(x)$. What are the possible degrees of $D(x)$?

d A polynomial of degree 6 is divided by a polynomial of degree 2. What is the degree of the quotient?

Example 8

6 **a** Use long division to prove that $P(x) = x^3 + x^2 - 41x - 105$ is divisible by $x + 5$. Hence, factorise $P(x)$ completely.

b Use long division to prove that $P(x) = x^4 + 10x^3 + 37x^2 + 60x + 36$ is divisible by $x^2 + 4x + 4$. Hence, factorise $P(x)$ completely.

7 **a** **i** Divide $x^4 - 3x^3 - 5x^2 + x - 7$ by $x + 5$.

ii Hence, find a if $x^4 - 3x^3 - 5x^2 + x + a$ is divisible by $x + 5$.

b **i** Divide $x^4 - 3x^3 - 5x^2 + x - 7$ by $x^2 + 5$.

ii Hence, find a and b if $x^4 - 3x^3 - 5x^2 + ax + b$ is divisible by $x^2 + 5$.

17D The remainder theorem and factor theorem

Long division of a polynomial $P(x)$ by another polynomial is a cumbersome process. Sometimes we are only interested in the remainder, and unfortunately this only appears at the very end of the division algorithm.

Proof of the remainder theorem

The remainder theorem enables us to find the remainder. When we divide $P(x)$ by a factor of the form $x - \alpha$, the remainder is a constant, which we will call r. That is:

$$P(x) = (x - \alpha)Q(x) + r$$

When we substitute $x = \alpha$ into this identity, we get:

$$\begin{aligned} P(\alpha) &= 0 \times Q(\alpha) + r \\ &= r \end{aligned}$$

So we have an interesting result – the remainder is simply $P(\alpha)$. This result is called the **remainder theorem**, and it allows us to find the remainder easily without performing the division algorithm. It also allows us to find linear factors, as we will see next.

The remainder theorem

Let $P(x)$ be a polynomial and let α be a constant. When $P(x)$ is divided by $x - \alpha$, the remainder is $P(\alpha)$.

Keep in mind two things about this theorem.

- It tells us nothing at all about the quotient.
- It only applies when the divisor has the form $x - \alpha$ (that is, when the divisor is a monic linear polynomial).

Example 9

Find the remainder when $2x^3 + 4x^2 - 5x - 7$ is divided by $x - 3$:

a by long division

b by the remainder theorem

Solution

a

$$\begin{array}{r l}
 & 2x^2 + 10x + 25 \\ \cline{2-2}
x - 3 \big) & 2x^3 + 4x^2 - 5x - 7 \\
 & 2x^3 - 6x^2 \\ \cline{2-2}
 & 10x^2 - 5x - 7 \\
 & 10x^2 - 30x \\ \cline{2-2}
 & 25x - 7 \\
 & 25x - 75 \\ \cline{2-2}
 & 68
\end{array}$$

Thus, the remainder is 68.

b Using the remainder theorem, the remainder is:

$$\begin{aligned} P(3) &= 2 \times 27 + 4 \times 9 - 5 \times 3 - 7 \\ &= 54 + 36 - 15 - 7 \\ &= 68 \end{aligned}$$

The above example shows how much easier it is to find the remainder using the remainder theorem.

Example 10

Find the remainder when $P(x) = x^4 - 3x^2 - 10x - 24$ is divided by:

a $x - 3$ **b** $x + 2$ **c** x **d** $x + 5$

Solution

a We are dividing by $x - 3$, so the remainder is:

$$\begin{aligned} P(3) &= 81 - 27 - 30 - 24 \\ &= 0 \end{aligned}$$

Thus, $x - 3$ is a factor of $P(x)$.

b We are dividing by $x + 2 = x - (-2)$, so the remainder is:

$$\begin{aligned} P(-2) &= 16 - 12 + 20 - 24 \\ &= 0 \end{aligned}$$

Thus, $x + 2$ is a factor of $P(x)$.

c We are dividing by $x = x - 0$, so the remainder is:

$$\begin{aligned} P(0) &= 0 - 0 - 0 - 24 \\ &= -24 \end{aligned}$$

d We are dividing by $x + 5 = x - (-5)$, so the remainder is:

$$\begin{aligned} P(-5) &= 625 - 75 + 50 - 24 \\ &= 576 \end{aligned}$$

Using remainders to find coefficients

In some situations, we do not know all the coefficients of a polynomial, but we do know the remainder after division by one or more linear polynomials. This may give us enough information to work out the unknown coefficients.

Example 11

The polynomial $P(x) = x^5 - 7x^3 + ax + 1$ has remainder 13 after division by $x - 1$. Find the value of the coefficient a.

Solution

The remainder theorem tells us that, after dividing $P(x)$ by $x - 1$, the remainder is $P(1)$.

$$P(1) = 13$$

$$\begin{aligned} \text{Thus,}\quad 1 - 7 + a + 1 &= 13 \\ a - 5 &= 13 \\ a &= 18 \end{aligned}$$

The factor theorem

Suppose we want to know whether $x + 3$ is a factor of the polynomial $P(x) = x^3 + 2x^2 - 5x - 6$. All we need to do is to find the remainder after division by $x + 3$.

- If the remainder is 0, then we know that $x + 3$ is a factor.
- If the remainder is not 0, then we know that $x + 3$ is not a factor.

Using the remainder theorem, the remainder is $P(-3)$.

$$\begin{aligned} P(-3) &= -27 + 18 + 15 - 6 \\ &= 0 \end{aligned}$$

so $x + 3$ is a factor of $P(x)$.

Is $x + 2$ a factor of $P(x)$? After dividing by $x + 2$, the remainder is $P(-2)$.

$$\begin{aligned} P(-2) &= -8 + 8 + 10 - 6 \\ &= 4 \neq 0 \end{aligned}$$

so $x + 2$ is not a factor of $P(x)$.

Indeed, by performing polynomial division we can show that:

$$\begin{aligned} P(x) &= (x + 3)(x^2 - x - 2) \\ &= (x + 3)(x - 2)(x + 1) \end{aligned}$$

This suggests $P(2) = 0$ and $P(-1) = 0$, which is easily verified.

The factor theorem

Let $P(x)$ be a polynomial and let a be α constant.

- If $P(\alpha) = 0$, then $(x - \alpha)$ is a factor of $P(x)$.
- If $P(\alpha) \neq 0$, then $(x - \alpha)$ is not a factor of $P(x)$.

Example 12

The polynomial $P(x) = 3x^6 - 5x^3 + ax^2 + bx + 10$ is divisible by $x + 1$ and $x - 2$.
Find the values of the coefficients a and b.

Solution

Since $x + 1$ is a factor, $P(-1) = 0$

$$\begin{aligned} 3 + 5 + a - b + 10 &= 0 \\ a - b &= -18 \qquad (1) \end{aligned}$$

Since $x - 2$ is a factor, $P(2) = 0$

$$\begin{aligned} 192 - 40 + 4a + 2b + 10 &= 0 \\ 4a + 2b &= -162 \\ 2a + b &= -81 \qquad (2) \end{aligned}$$

Adding (1) and (2), $\quad 3a = -99$

$\quad a = -33$

Substituting into (1), $\quad b = -15$

Thus, $a = -33$ and $b = -15$, and $P(x) = 3x^6 - 5x^3 - 33x^2 - 15x + 10$

Exercise 17D

Example 10

1 Use the remainder theorem to find the remainder, and state whether or not $D(x)$ is a factor of $P(x)$.

a $P(x) = x^2 - 5x + 2$, $D(x) = x + 4$

b $P(x) = 3x^2 - 16x + 21$, $D(x) = x - 3$

c $P(x) = x^3 - 6x^2 + 1$, $D(x) = x + 5$

d $P(x) = x^3 - 11x^2 + 8x + 20$, $D(x) = x - 10$

2 Use the remainder theorem to find the remainder when $P(x) = x^4 - 6x^2 + 3x + 2$ is divided by each linear polynomial $D(x)$. Then state whether or not $D(x)$ is a factor of $P(x)$.

a $D(x) = x - 1$ **b** $D(x) = x + 1$ **c** $D(x) = x - 3$ **d** $D(x) = x + 3$

3 Use the remainder theorem to find the remainder when each polynomial $P(x)$ is divided by $D(x) = x + 1$. Then state whether or not $x + 1$ is a factor of $P(x)$.

a $P(x) = 5x^2 - 7x - 12$

b $P(x) = 5x^2 + 7x - 12$

c $P(x) = x^6 - 4x^4 + 6x^2 - 2$

d $P(x) = 7x^5 - 3x^3 - 2x - 2$

e $P(x) = 4x^5 + 5x^4 - 3x + 2$

f $P(x) = x^{100} - x^{99} + x - 1$

4 Check systematically which, if any, of $x + 1$, $x - 1$, $x + 2$, $x - 2$, $x + 4$ and $x - 4$ are factors of each polynomial $P(x)$.

a $P(x) = x^3 + x^2 - 4x - 4$ **b** $P(x) = x^4 + 5x^3 + 3x^2 - 5x - 4$

Example 11

5 Use the factor theorem to answer these questions.

a Find k if $x - 1$ is a factor of $P(x) = 5x^3 - 2x^2 + kx - 7$.

b Find m if $x + 2$ is a factor of $P(x) = 5x^3 + mx^2 - 7x + 10$.

6 Use the remainder theorem to answer these questions.

a When the polynomial $P(x) = 2x^4 - x^2 + x - p$ is divided by $x - 2$, the remainder is 2. Find p.

b When the polynomial $P(x) = x^3 - bx^2 + 6x - 24$ is divided by $x + 2$, the remainder is 48. Find b.

Example 12

7 **a** When the polynomial $P(x) = x^4 - 5x^3 + 6x^2 - ax + b$ is divided by $x - 3$, the remainder is 20, and when $P(x)$ is divided by $x + 2$, the remainder is 30. Find a and b.

b Find a and b, given that the polynomial $P(x) = x^4 + x^3 - ax^2 + bx + 2$ is divisible by $x - 2$ and $x - 1$.

c Find a, b and c, given that $P(x) = 5x^9 - ax^6 + 17x^4 + bx^3 - 26x + c$ is divisible by x, by $x + 1$ and by $x - 1$.

17E Factorising polynomials

The factor theorem often allows us to find a linear factor of $P(x)$ of the form $x - \alpha$. Then by long division, $P(x) = (x - \alpha)Q(x)$, where the degree of $Q(x)$ is one less than the degree of $P(x)$. We may be able to repeat this process to obtain the complete factorisation of $P(x)$.

In this section, for simplicity, we will only look for factors with integer coefficients.

For example, let us examine this polynomial:

$$P(x) = x^3 + 4x^2 - 7x - 10$$

- First, we search systematically for a factor. In question **8** of Exercise 17E, you will prove that if $x - \alpha$ is a factor of a polynomial with integer coefficients, then the only integer possibilities for α are the factors of the constant term -10. Thus, we only need to try substituting $1, -1, 2, -2, 5, -5, 10$ and -10 into $P(x)$.

  $P(1) = 1 + 4 - 7 - 10 = -12 \neq 0$

  $P(-1) = -1 + 4 + 7 - 10 = 0$, so $x + 1$ is a factor.

- Next, we use long division to divide $P(x)$ by $x + 1$, and obtain:

  $P(x) = (x + 1)(x^2 + 3x - 10)$

$$\begin{array}{r|l}
 & x^2 + 3x - 10 \\ \hline
x + 1 & x^3 + 4x^2 - 7x - 10 \\
 & x^3 + x^2 \\ \hline
 & 3x^2 - 7x - 10 \\
 & 3x^2 + 3x \\ \hline
 & -10x - 10 \\
 & -10x - 10 \\ \hline
 & 0
\end{array}$$

- Now we factorise the quadratic $x^2 + 3x - 10$.

  By inspection, $x^2 + 3x - 10 = (x + 5)(x - 2)$, so we have factorised the cubic into 3 linear factors. That is:

  $P(x) = x^3 + 4x^2 - 7x - 10 = (x + 1)(x + 5)(x - 2)$

Example 13

Factorise the polynomial $P(x) = x^4 - 2x^3 - 8x + 16$.

Solution

- We only need to test the positive and negative factors of 16.

  $P(1) = 1 - 2 - 8 + 16 \neq 0$, so $x - 1$ is not a factor of $P(x)$.

  $P(-1) = 1 + 2 + 8 + 16 \neq 0$, so $x + 1$ is not a factor of $P(x)$.

  $P(2) = 16 - 16 - 16 + 16 = 0$, so $x - 2$ is a factor.

  After long division of $P(x)$ by $x - 2$, $P(x) = (x - 2)(x^3 - 8)$.

- Let $Q(x) = x^3 - 8$.

  $x - 1$ and $x + 1$ are not factors of $Q(x)$ since $P(x) = (x - 2)Q(x)$, and they are not factors of $P(x)$.

  However, $Q(2) = 8 - 8 = 0$, so $x - 2$ is a factor of $Q(x)$ as well.

  After long division of $Q(x)$ by $x - 2$, $Q(x) = (x - 2)(x^2 + 2x + 4)$.

(continued over page)

- The quadratic $x^2 + 2x + 4$ cannot be factorised, because:

  $$\begin{aligned} x^2 + 2x + 4 &= (x^2 + 2x + 1) + 3 \\ &= (x + 1)^2 + 3 \qquad (\text{Alternatively, } \Delta = (2)^2 - 4(1)(4) = -12 < 0) \end{aligned}$$

  Hence, $P(x) = (x - 2)^2(x^2 + 2x + 4)$ is the complete factorisation of $P(x)$.

Note:

- The first step in factorising this particular polynomial can also be done by grouping:

  $$\begin{aligned} P(x) &= x^3(x - 2) - 8(x - 2) \\ &= (x - 2)(x^3 - 8) \end{aligned}$$

  There can be many ways to solve a mathematical problem!
- It is easy to miss repeated factors of a polynomial.

Taking out a common factor

As with all methods of factorising, you should first do a quick check for common factors and deal with these before doing anything else. The following example demonstrates this.

Example 14

Factorise the polynomial $P(x) = 2x^5 - 22x^4 + 78x^3 - 90x^2$.

Solution

- $2x^2$ is a common factor of all the terms, and we take this out first; thus, $P(x) = 2x^2(x^3 - 11x^2 + 39x - 45)$.
- Let $Q(x) = x^3 - 11x^2 + 39x - 45$. We now try to factorise $Q(x)$.
- We need only test the positive and negative factors of 45.

  $Q(1) = 1 - 11 + 39 - 45 = -16 \neq 0$

  $Q(-1) = -1 - 11 - 39 - 45 = -96 \neq 0$

  $Q(3) = 27 - 99 + 117 - 45 = 0$, so $x - 3$ is a factor
- After long division, we obtain $Q(x) = (x - 3)(x^2 - 8x + 15)$.
- The quadratic factors as $x^2 - 8x + 15 = (x - 3)(x - 5)$, and so $P(x) = 2x^2(x - 3)^2(x - 5)$.

Hence, we have a complete factorisation of the quintic (degree 5 polynomial) into a constant times 5 linear factors.

Factorising a polynomial

Suppose that $P(x)$ is a polynomial with integer coefficients.

- Take out any common factor, including powers of x.
- Try to find a factor $x - \alpha$ of $P(x)$ by testing whether $P(\alpha) = 0$. The only integer possibilities for α are the positive and negative factors of the constant term.
- Having found a factor, use long division to factorise the polynomial as $(x - \alpha)Q(x)$, where $Q(x)$ has degree 1 less than the degree of $P(x)$.
- Repeat this process on $Q(x)$ to try to complete the factorisation of $P(x)$.

We should admit at this point that most polynomials are extremely difficult to factorise. Nevertheless, polynomials that can be factorised occur in many important situations and, in any case, all mathematics begins by first dealing with the simplest cases.

For example, the polynomial $x^4 - 3x^3 + 4x^2 - 14x + 48$ factorises as $(x^2 + 2x + 6)(x^2 - 5x + 8)$ and has no linear factors at all. So the techniques described in this section will not provide a pathway to factorisation in this case.

Exercise 17E

1 **a** Write down, in factored form, the monic quadratic polynomial $P(x)$ with factors $x - 12$ and $x + 9$.

b Expand $P(x)$, then show that $P(12)$ and $P(-9)$ are both zero.

2 Write down, in factored form, the monic quartic polynomial $P(x)$ with factors $x - 1$, $x + 1$, $x - 2$ and $x + 2$.

3 **a** For the cubic polynomial $P(x) = x^3 - 6x^2 + 11x - 6$, show that $P(1) = 0$.

b Divide $P(x)$ by $x - 1$.

c Hence, factor $P(x)$ into linear factors.

Example 13

4 Use the method given in this section to factorise these cubic polynomials.

a $P(x) = x^3 + 6x^2 + 11x + 6$

b $P(x) = x^3 - 7x^2 - x + 7$

c $P(x) = x^3 + 3x^2 - 13x - 15$

d $P(x) = x^3 + x^2 - 21x - 45$

e $P(x) = x^3 + x^2 - 5x + 3$

f $P(x) = x^3 + 3x^2 - 4$

5 Factorise these polynomials into linear factors.

a $P(x) = x^4 - 5x^3 + 5x^2 + 5x - 6$

b $P(x) = x^4 + 12x^3 + 46x^2 + 60x + 25$

Example 14

6 By first taking out a common factor, write each polynomial as a constant times a product of linear factors.

a $P(x) = 3x^3 + 6x^2 - 39x + 30$

b $P(x) = 5x^3 - 5x^2 - 20x + 20$

c $P(x) = x^4 + x^3 - 4x^2 - 4x$

d $P(x) = x^5 + 4x^4 - 2x^3 - 12x^2 + 9x$

7 Factorise each polynomial as a product of linear factors and one quadratic factor.

a $P(x) = x^3 + 2x^2 + 2x - 5$

b $P(x) = x^3 + 4x^2 + 4x + 3$

c $P(x) = x^5 + 4x^4 - 15x^3 + 6x^2$

d $P(x) = x^4 + 4x^3 - 2x^2 - 17x - 6$

8 Suppose that $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial with integer coefficients, and suppose that $P(\alpha) = 0$, where α is an integer. Show that α is a factor of the constant term a_0. This justifies the second dot-point on page 507.

17F Polynomial equations

If a polynomial $P(x)$ can be completely factorised, we can then easily find the solutions of the **polynomial equation** $P(x) = 0$.

Example 15

Solve $x^3 + 4x^2 - 7x - 10 = 0$.

Solution

At the beginning of the last section, we found the factorisation:

$x^3 + 4x^2 - 7x - 10 = (x + 1)(x - 2)(x + 5)$

Hence, the equation becomes:

$(x + 1)(x - 2)(x + 5) = 0$

so $x + 1 = 0$ or $x - 2 = 0$ or $x + 5 = 0$

Thus, the solutions are $x = -1$, $x = 2$ and $x = -5$.

Example 16

Solve $2x^5 - 22x^4 + 78x^3 - 90x^2 = 0$

Solution

In Example 14 of the last section, we found the factorisation:

$2x^5 - 22x^4 + 78x^3 - 90x^2 = 2x^2(x - 3)^2(x - 5)$

Hence, the equation becomes:

$2x^2(x - 3)^2(x - 5) = 0$

so $x^2 = 0$ or $(x - 3)^2 = 0$ or $x - 5 = 0$

Thus, the solutions are $x = 0$, 3 and 5.

This quintic equation in Example 16 has only three solutions. The polynomial has repeated factors x and $x - 3$. We can think of the solutions $x = 0$ and $x = 3$ as occurring twice because they arise from the square factors x^2 and $(x - 3)^2$. We therefore say that the solutions 0 and 3 have **multiplicity** 2. There are now five solutions to the quintic equation, counted by multiplicity.

Example 17

Solve $x^4 + 16 = 2x^3 + 8x$.

Solution

Moving all terms to the left: $x^4 - 2x^3 - 8x + 16 = 0$.

In Example 13 of the last section, we found the factorisation:

$x^4 - 2x^3 - 8x + 16 = (x - 2)^2(x^2 + 2x + 4)$

Hence, the equation becomes $(x - 2)^2(x^2 + 2x + 4) = 0$

so $x - 2 = 0$ or $x^2 + 2x + 4 = 0$

The quadratic equation has no solution, as $x^2 + 2x + 4 = (x + 1)^2 + 3$ (or $\Delta < 0$)

Thus, the only solution is $x = 2$.

In previous examples the solutions were integers. In some cases, the solutions may be surds.

We recall the formula for solving a quadratic equation. If $ax^2 + bx + c = 0$, then

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Example 18

Solve $x^4 + 7x^3 - 2x^2 - 7x + 1 = 0$.

Solution

The polynomial $x^4 + 7x^3 - 2x^2 - 7x + 1$ has factorisation $(x - 1)(x + 1)(x^2 + 7x - 1)$

Hence, the equation becomes $(x - 1)(x + 1)(x^2 + 7x - 1) = 0$

Thus, the solutions are $x = 1$, $x = -1$ and the solutions to $x^2 + 7x - 1 = 0$

Using the quadratic formula, $\Delta = b^2 - 4ac = 49 + 4$
$= 53$,

so the quadratic has solutions $x = \dfrac{-7 + \sqrt{53}}{2}$ and $x = \dfrac{-7 - \sqrt{53}}{2}$

Hence, the quartic equation has four solutions: $x = 1, -1, \dfrac{-7 + \sqrt{53}}{2}$ and $\dfrac{-7 - \sqrt{53}}{2}$

Solving polynomial equations

- Move all terms to the left-hand side, with zero on the right.
- Factorise the polynomial on the left as far as possible.
- Hence, write down all solutions, using the quadratic formula if necessary.

It should be noted that a polynomial equation of degree n cannot have more than n solutions. For example, a quartic has at most four solutions. This follows from the factor theorem.

Exercise 17F

1 Solve these polynomial equations.

a $(x+7)(x-5)(x+6)=0$

b $(x-3)^2(x+1)=0$

c $5(x-2)(x-4)(x-6)(x-8)=0$

d $4x(x-7)^2(x+8)^2=0$

2 Solve these polynomial equations.

a $(x-3)(x^2+6x-8)=0$

b $(x+5)^2(3x^2-2x-2)=0$

c $5x^3(x-7)(x+6)(x^2+2x+5)=0$

d $-2(x-2)^2(x-5)^4(x^2-10)=0$

Example 15, 16

3 Use the factor theorem to factorise the left-hand side of each equation, then solve it.

a $x^3-2x^2-13x-10=0$

b $x^3-3x^2-4x+12=0$

c $x^5+3x^4-25x^3+21x^2=0$

d $x^4-5x^3-15x^2+5x+14=0$

Example 17, 18

4 Solve:

a $x^3-7x^2+11x-5=0$

b $x^3-x^2-8x+12=0$

c $x^4-12x^3+46x^2-60x+25=0$

d $x^4+x^3-2x^2+4x-24=0$

e $x^5+9x^4+21x^3+19x^2+6x=0$

f $x^5-4x^3-2x^2+3x+2=0$

5 Solve:

a $x^3-7x^2+11x+3=0$

b $x^3+4x^2+10x+7=0$

c $x^5-2x^4-10x^3+23x^2-6x=0$

d $x^5-3x^3-4x^2+2x+4=0$

17G Sketching polynomials

In this section, we will sketch the graphs of polynomial functions given in factorised form. We begin by looking at the graphs of polynomials that do not have any repeated factors.

Consider the polynomial function $y = x(x - 2)(x + 3)$. When we substitute $x = 0$, $x = 2$ or $x = -3$ into this polynomial, we get zero.

These values are called the **zeros of the polynomial**. No other value of x will make the polynomial zero.

We saw earlier how to sketch the graph of a quadratic function. The graph of a quadratic function is a smooth curve. Among the key features we looked for were the points at which the curve cuts the coordinate axes.

In the example above, the graph of $y = x(x - 2)(x + 3)$ cuts the x-axis at the zeros; that is, at $x = 0$, $x = 2$, and $x = -3$.

The graph cuts the y-axis when $x = 0$, so the y-intercept is 0.

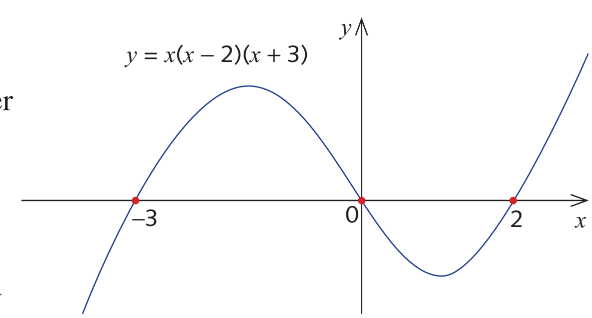
To get a picture of the overall shape of the curve, we can substitute some test points.

x	–4	–3	–1	0	1	2	3
y	–24	0	6	0	–4	0	18
Sign of y	–	0	+	0	–	0	+

We can represent the sign of y using a **sign diagram**:

Sign of y	–	0	+	0	–	0	+
x values		–3		0		2	

With this information, we can begin to give a sketch of the graph of $y = x(x - 2)(x + 3)$.



The sign diagram tells us that the graph cuts the x-axis at the points $x = -3$, 0 and 2, and also whether the graph is above or below the x-axis on each side of these points. It does not tell us the maximum and minimum values of y between the zeros.

It is important to note that unlike a parabola, the x-value of a turning point will not always lie midway between successive zeros.

Moreover, notice that if x is a large positive number, then $P(x)$ is also large and positive. For example, if $x = 10$, then $y = 1040$.

If x is a large negative number, then $P(x)$ is also a large negative number. For example, if $x = -10$, then $y = -840$.

Example 19

Sketch the graph of $y = (x + 2)(x + 1)(x - 1)(x - 2)$.

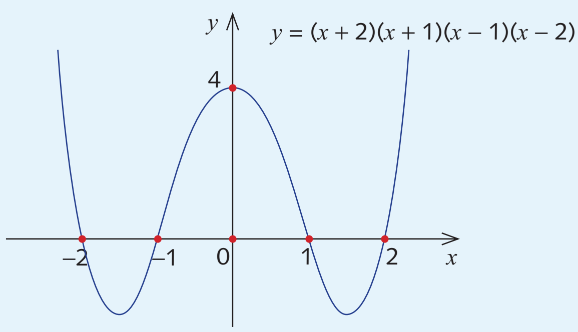
Solution

The zeros are at $x = -2, -1, 1$ and 2. These are the x-intercepts of the polynomial.
When $x = 0$, the y-intercept is 4.

We make up a sign diagram (use your own test points):

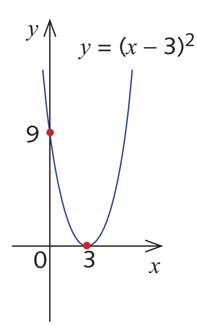
Sign of y	+	0	–	0	+	0	–	0	+
x values		–2		–1		1		2	

The graph is:



Graphs of polynomials with repeated factors

We know from Chapter 7 in *ICE-EM Mathematics Year 10* that the graph of the parabola $y = (x - 3)^2$ is as shown to the right.



So what does the graph of $y = (x - 3)^3$ look like?

In this section, we will examine the graphs of polynomials such as $y = (x - 2)^3$ and $y = (x + 3)^4$, which have repeated factors.

Odd powers

Let us begin with $y = x^3$.

At $x = 0$, $y = 0$, so the graph cuts the axes at $(0, 0)$.

We look at the sign of y near $x = 0$. Since the cube of a negative number is negative, the y-values are negative for $x < 0$ and positive for $x > 0$.

We can represent the signs using the following diagram.

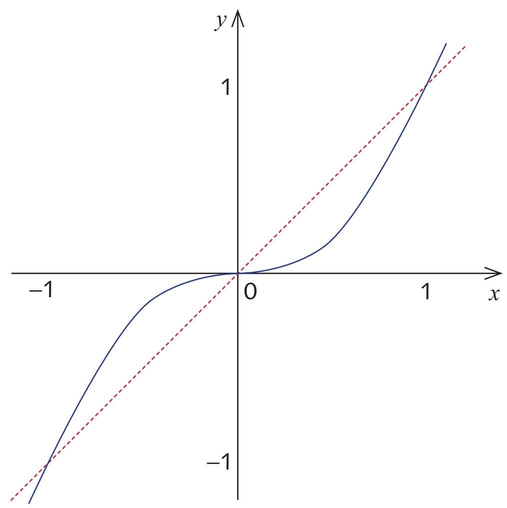
Sign of y	–	0	+
x values	–1	0	1

The change in sign near 0 tells us that the curve *cuts* the x-axis there. It moves from below the x-axis to above the x-axis.

But, what happens near the origin? The point (1, 1) lies on the curve $y = x^3$. Cubing a number between 0 and 1 makes it *smaller*.

So for an x-value between 0 and 1, $x^3 < x$ and the point on $y = x^3$ is below the corresponding point on the line $y = x$.

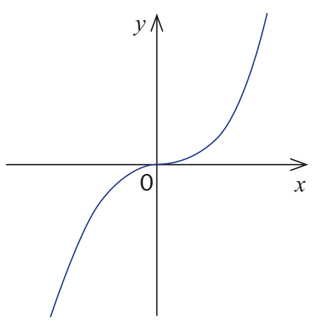
Similarly, if $x > 1$ then $x^3 > x$, so the point on $y = x^3$ is above the corresponding point on the line $y = x$.



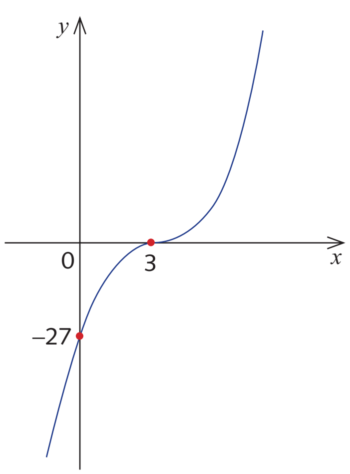
Similarly, if $-1 < x < 0$ then $x^3 > x$ and if $x < -1$ then $x^3 < x$.

Thus, near zero, the graph is quite 'flat' and then starts to increase sharply for $x > 1$, and similarly on the other side.

Whenever we are dealing with polynomials that have **repeated factors**, the graph will be 'flat' near zero, which comes from the repeated factor.



To sketch the graph of $y = (x - 3)^3$, we observe that it is obtained by translating the graph of $y = x^3$ three units to the right. The curve cuts the x-axis at 3. It cuts the y-axis at -27 when $x = 0$, and is flat near $x = 3$.



Even powers

The next example shows how to deal with even powers.

Example 20

Sketch the graphs of $y = x^4$ and $y = (x + 3)^4$.

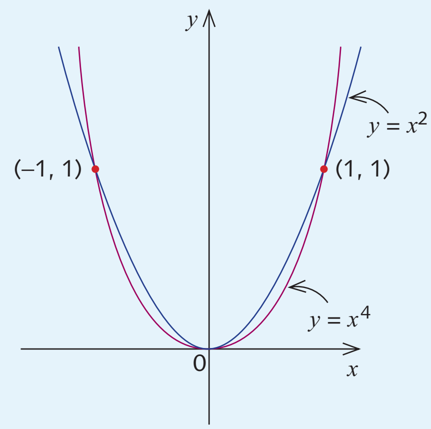
Solution

The function $y = x^4$ has a repeated factor, x. It cuts the coordinate axes at $(0, 0)$. Since the fourth power of any number is always positive, the sign diagram is:

Sign of y	$+$	0	$+$
x value		0	

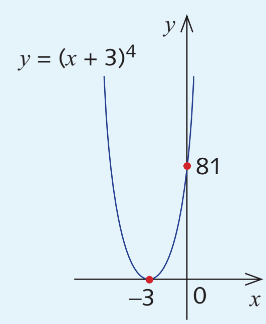
Since the sign of y is the same either side of 0, the graph *touches* the x-axis at 0.

The diagram shows the graphs of $y = x^2$ and $y = x^4$ for comparison. Notice that $y = x^4$ is below $y = x^2$ for x-values between -1 and 1 but above it for $x > 1$ and $x < -1$.



Since $y = x^2$ is flat near the origin, so is $y = x^4$.

To draw the graph of $y = (x + 3)^4$, we simply translate the graph of $y = x^4$ three units to the left, so the graph touches the x-axis at $x = -3$.



Graphs of polynomials with repeated factors

- The graph of $y = (x - a)^n$, where n is a whole number greater than 1
  - touches the x-axis if n is even
  - cuts the x-axis if n is odd.
- The graph of a polynomial with a repeated factor $x - a$ is flat near $x = a$.

It is a good idea to draw a sign diagram each time. We will see in the next section that the sign diagram is very helpful in sketching more complicated polynomials.

Exercise 17G

Example 19

1 Identify the zeros of each polynomial. Draw sign diagrams and sketch the curves.

a $y = (x - 2)(x - 4)$

b $y = x(x - 2)(x - 4)$

c $y = (x + 3)(x - 1)(x - 3)$

d $y = (x + 2)(x + 1)(x - 3)$

Example 20

2 Identify the zeros of each polynomial. Draw sign diagrams and sketch the curves.

a $y = (x - 1)^2$

b $y = (x - 1)^3$

c $y = (x - 1)^4$

3 Sketch:

a $y = (x + 2)^2$

b $y = (x + 2)^3$

c $y = (x + 2)^4$

4 Consider the polynomial $y = (x + 2)(x - 1)(x + 4)$.

a Sketch the graph.

b For what values of x is the graph above the x-axis?

c For what values of x is the graph below the x-axis?

5 The factorisation of each polynomial is not complete. Complete the factorisation, find the zeros of the polynomials and sketch the graphs.

a $y = 3x(x^2 - 16)$

b $y = (x^2 - 36)(x^2 - 4)$

6 **a** A monic cubic polynomial, $P(x)$, has zeros at $x = 2$, $x = 4$ and $x = 6$. Write down the equation of the polynomial. Draw the graph of $y = P(x)$.

b A monic cubic polynomial, $P(x)$, has one zero of multiplicity 3 at $x = -3$. Write down the equation of the polynomial. Draw the graph of $y = P(x)$.

c A monic cubic polynomial, $P(x)$, has one zero of multiplicity 3 at $x = 2$. Write down the equation of the polynomial. Draw the graph of $y = P(x)$.

7 **a** Draw the graph of $y = x(x - 1)(x + 1)$.

b Draw the graph of $y = -x(x - 1)(x + 1)$.

8 **a** Draw the graph of $y = (x - 1)^3$.

b Draw the graph of $y = -(x - 1)^3$.

9 **a** Draw the graph of $y = (x + 3)^4$.

b Draw the graph of $y = -(x + 3)^4$.

17H Further sketching of polynomials

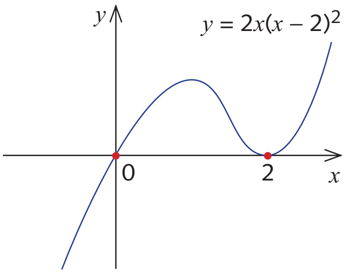
We shall now sketch polynomials in factored form.

Let us begin with $y = 2x(x - 2)^2$, which has one repeated factor. The graph cuts the x-axis at $x = 0$ and $x = 2$. The y-intercept is 0.

We now draw a sign diagram for this function.

Sign of y	$-$		$+$		$+$
x values		0		2	

We obtain the signs by substituting x-values less than zero, between 0 and 2, and greater than 2, into the equation and noting the sign of the answer.



There is a change of sign at $x = 0$, so the graph cuts the x-axis at 0.

There is no change of sign at $x = 2$, so the graph touches the x-axis at $x = 2$.

As we saw in the previous section, the graph is flat near $x = 2$, since $(x - 2)$ is a repeated factor.

Example 21

Sketch $y = (x + 3)^3(x - 1)^3$.

Solution

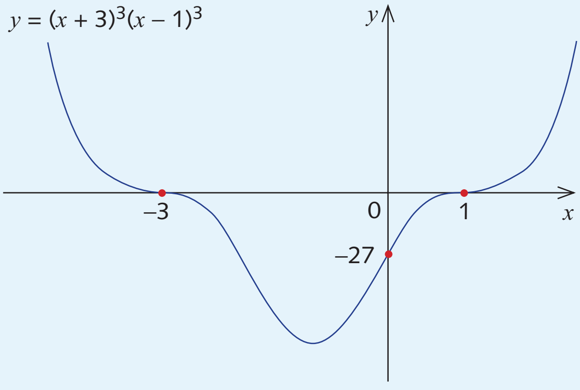
The zeros are at $x = -3$ and $x = 1$. The y-intercept is -27.

The sign diagram is:

Sign of y	$+$		$-$		$+$
x values		-3		1	

The changes in sign tell us that the graph cuts the x-axis at the two zeros. The curve is flat near both zeros.

The graph is:



Note: As in the previous examples, we do not know the minimum value of y for the x-values between -3 and 1. To find this, we need techniques from a branch of mathematics known as calculus or we can use the symmetry of the graph about $x = -1$.

Exercise 17H

Example 21

1 Identify the zeros of each polynomial. Draw sign diagrams and sketch the curves.

a $y = x(x - 2)^2$ **b** $y = (x - 2)^2(x - 4)^2$

c $y = x^2(x + 3)$ **d** $y = (x + 2)^2(x + 1)^3$

2 Identify the zeros of each polynomial. Draw sign diagrams and sketch the curves.

a $y = (x - 4)^2(x + 4)^2$ **b** $y = (x - 4)^3(x + 1)^3$

c $y = x^3(x - 4)^4$ **d** $y = x^4(x + 2)^4$

3 Sketch:

a $y = (x + 2)^2(x - 1)^2$ **b** $y = (x + 2)^3(x - 2)^3$

4 Consider the polynomial $y = (x + 3)^3(x - 1)^2$.

a Sketch the graph.

b For what values of x is the graph above the x-axis?

c For what values of x is the graph below the x-axis?

5 The factorisation of each polynomial is not complete. Complete the factorisation, find the zeros of the polynomials and sketch the graphs.

a $y = (3x^2 - 3)(x^2 - 9)^4$ **b** $y = x^2(20 - 5x^2)$

6 **a** A monic polynomial, $P(x)$, of degree 6 has triple zeros at $x = 2$ and $x = 4$. Write down the equation of the polynomial. Draw the graph of $y = P(x)$.

b A monic polynomial, $P(x)$, of degree 5 has a triple zero at $x = -3$ and a double zero at $x = 1$. Write down the equation of the polynomial. Draw the graph of $y = P(x)$.

7 **a** Draw the graph of $y = x^2(x - 1)^2$.

b Draw the graph of $y = -x^2(x - 1)^2$.

8 **a** Draw the graph of $y = (x - 1)^3(x + 1)^3$.

b Draw the graph of $y = -(x - 1)^3(x + 1)^3$.

Review exercise

1 State whether or not each expression is a polynomial.

a $5x^2 + 3x - 4$ **b** $3 - 2x$ **c** $\dfrac{x - 3}{8}$

d $\dfrac{x^2}{x^2 - 2}$ **e** $\dfrac{x + 2}{\sqrt{x}}$ **f** $2(x - 1)^3 - 2x + 1$

2 State the degree of each polynomial.

a $x^2 + 3x$ **b** $x^3 - 5x + 7$

c $2x^4 - 5x^2 + 7$ **d** $3 - 5x - 6x^2$

e $9 - x - x^3$

3 Let $P(x) = x^3 + 2x - 1$. Find:

a $P(1)$ **b** $P(-1)$ **c** $P(2)$

d $P(-2)$ **e** $P(a)$ **f** $P(2a)$

4 Find a if $P(x) = x^3 + 2x - a$ and $P(1) = 6$.

5 Find a if $P(x) = x^3 + 2ax - a$ and $P(1) = 0$.

6 Find the sum $P(x) + Q(x)$, the difference $P(x) - Q(x)$ and the product $P(x)Q(x)$.

a $P(x) = x + 3,\ Q(x) = x^2 + 2x + 3$

b $P(x) = x^2 + 1,\ Q(x) = x^2 + 3$

c $P(x) = 2x + 1,\ Q(x) = x^2 - 2x + 1$

7 Use the division algorithm to divide $P(x)$ by $D(x)$. Find the quotient and the remainder.

a $P(x) = 6x^3 + 7x^2 - 15x + 4,\ D(x) = x - 1$

b $P(x) = 2x^3 - 3x^2 + 5x + 3,\ D(x) = x + 1$

c $P(x) = x^3 - 7x^2 + 6x + 1,\ D(x) = x - 3$

d $P(x) = x^3 - 2x^2 + 3x + 1,\ D(x) = x - 2$

8 Factorise each polynomial.

a $P(x) = x^3 - 2x^2 - 5x + 6$

b $P(x) = 2x^3 + 7x^2 - 7x - 12$

c $P(x) = 2x^3 + 3x^2 - 17x + 12$

d $P(x) = 6x^3 - 5x^2 - 17x + 6$

9 If $x^3 + ax^2 + bx - 4$ is exactly divisible by $x + 4$ and $x - 1$, find the values of a and b.

10 When the polynomial $P(x) = x^3 + 2x^2 - 5x + d$ is divided by $x - 2$, the remainder is 10. Find the value of $\boldsymbol{a}$.

11 If $3x^3 + ax^2 + bx - 6$ is exactly divisible by $x + 2$ and $x - 3$, find the values of a and b.

12 Consider the polynomial $P(x) = x^3 + ax^2 + b$.

a Find $P(w) - P(-w)$ in terms of w.

b Find the values of a and b if the graph of $y = P(x)$ passes through the point with coordinates $(1,\ 3)$ and $(2,\ 4)$.

13 Find the x-intercepts and y-intercepts of the graphs of each of the following.

a $y = x^3 - x^2 - 2x$

b $y = x^3 - 2x^2 - 5x + 6$

c $y = x^3 + 2x^2 - x - 2$

d $y = 3x^3 - 4x^2 - 13x - 6$

e $y = 5x^3 + 12x^2 - 36x - 16$

f $y = 6x^3 - 5x^2 - 2x + 1$

14 Sketch the graphs of:

a $y = 2x(x^2 - 4)$

b $y = (x + 2)^3$

c $y = (x - 2)^4$

d $y = x^2(x + 3)^2$

e $y = x(x + 2)^2$

f $y = (x - 3)^2(x + 1)^2$

Challenge exercise

1 Find the value of a, given that $x^2 + 1$ is a factor of $x^4 - 3x^3 + 3x^2 + ax + 2$.

2 Express $x^4 + 4$ as the product of two quadratic polynomials with integer coefficients.

3 The remainder when $x^5 - 3x^2 + ax + b$ is divided by $(x - 1)(x - 2)$ is $11x - 10$. Find a and b.

4 **a** If $(x - a_1)(x - a_2)(x - a_3) = x^3 + bx^2 + cx + d$ then show $a_1 + a_2 + a_3 = -b$, $a_1a_2 + a_2a_3 + a_1a_3 = c$ and $a_1a_2a_3 = -d$.

b Hence, find the monic cubic equation with roots, $x = 1$, $x = 2$ and $x = 3$.

5 $P(x)$ is a polynomial of degree 5 such that $P(x) - 1$ is divisible by $(x - 1)^3$ and $P(x)$ itself is divisible by x^3. Find $P(x)$.

6 $x^5 + 2x^3 + ax^2 + b$ is divisible by $x^3 + 1$. Find the values of a and b.

7 Without long division, find the remainder when $x^{49} + x^{25} + x^9 + x$ is divided by $x^3 - x$.

8 **a** Show that $(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$.

b Show that $(x^2 + 1)(x^2 + 4)(x^2 - 2x + 2)(x^2 + 2x + 2) = ((x^2 + 2)^2 + x^2)(x^4 + 4)$.

c Hence, express $(x^2 + 1)(x^2 + 4)(x^2 - 2x + 2)(x^2 + 2x + 2)$ as the sum of the squares of two polynomials having integer coefficients.

9 Let $P(x)$ be a polynomial leaving remainder A when divided by $(x - a)$, and remainder B when divided by $(x - b)$, where $a \neq b$. Find the remainder when $P(x)$ is divided by $(x - a)(x - b)$.

10 Find all ordered pairs such that $x + y^2 = 2$ and $y + x^2 = 2$.

CHAPTER

18

Statistics and Probability

Statistics

In previous books in this series, we have looked at the **measures of central tendency**, such as the mean and the median.

In this chapter, we discuss two measurements of spread – the **interquartile range** and **standard deviation**. The representation of numerical data by **boxplots** is also introduced.

In our study of statistics up to now, we have often associated one measurement with an item. For example, the height of each person in a class, the number of possessions obtained by a player in a football match or the number of marks obtained by a student in a test.

In the last two sections of this chapter, we look at associating a pair of numbers with an item, for example, the height and weight of a person or the age and salary of an employee. This is called **bivariate data**.

When a measurement is collected or recorded at successive intervals of time, it is referred to as **time-series** data. This type of bivariate data is also introduced in this chapter.

18A The median and the interquartile range

The median has been introduced and discussed in earlier books in this series. We review it here, because it is the measure of central tendency used when working with the interquartile range as a measurement of spread.

Median

We often see the median value being used to describe the housing market in a city. The median is the 'middle value' when all values are arranged in numerical order.

Here are 13 numbers in numerical order:

$2, 2, 3, 3, 3, 4, \boxed{5}, 11, 13, 18, 18, 19, 21$

This data set has an odd number of values. The middle value is 5, since it has the same number of values on either side of it. Hence, the median of this data set is 5.

Here is a set of 12 numbers, arranged in numerical order:

$1, 3, 4, 4, 5, \boxed{7}, \boxed{9}, 11, 13, 13, 19, 21$

This data set has an even number of values. The middle values are 7 and 9. We take the average of 7 and 9 to calculate the median.

$$\begin{aligned}\text{Median} &= \frac{7+9}{2}\\ &= 8\end{aligned}$$

Hence, the median of this data set is 8, even though this value does not occur in the data set.

Median

- When a data set has an odd number of values and they are arranged in numerical order, the median is the middle value.
- When a data set has an even number of values and they are arranged in numerical order, the median is the average of the two middle values.
- When a data set with n items is arranged in numerical order, the median lies in the $\left(\frac{n+1}{2}\right)^{\text{th}}$ position.

Example 1

Calculate the median of the data sets.

a 33 35 43 29 53 39 45

b 5 7 9 5 12 10

Solution

a To locate the median, first put the values in numerical order. This gives:

29 33 35 39 43 45 53

Median = 39

(continued over page)

b Again, the values are placed in numerical order.

5 5 7 9 10 12

$$\begin{aligned}\text{Median} &= \frac{7+9}{2}\\ &= 8\end{aligned}$$

Quartiles and the interquartile range

The **interquartile range(IQR)** measures the spread of the middle 50% of the data in an ordered data set.

We use the interquartile range to see how closely the data are grouped around the median. When we calculate the interquartile range, we organise the data into **quartiles**, each containing 25% of the data. The word ‘quartile’ is related to ‘quarter’.

Olivia has been playing Sudoku on the internet. Her last 11 games were all rated ‘diabolical’, and her times, correct to the nearest minute and arranged in ascending order, were:

8, 12, 14, 14, 16, 18, 19, 19, 25, 78, 523

The range of these times is 523 – 8 = 515.

Clearly the range does not give a clear picture of Olivia’s considerable skills, because the last two times, 78 and 523, are outliers. An **outlier** is a single data value far away from the rest of the data. That is, it is much larger or much smaller than all of the other values. Outliers have a huge influence on the value of both the mean and the range. (In fact, the time of 78 minutes occurred when Olivia left the game running over dinner, and the time of 523 minutes occurred when Olivia left the game running overnight.)

Because of situations like this, the interquartile range is often a better measure of the spread of the data than the range. Here is the procedure for finding it.

Step 1: Find the median. Divide the data into two equal groups. Omit the median (middle value) if there is an odd number of values. In Olivia’s case, there are 11 values so, omitting the median 18, the two groups of 5 are:

8, 12, 14, 14, 16 and 19, 19, 25, 78, 523

Step 2: The **lower quartile** is the median of the lower set of values. In Olivia’s case, the lower quartile is 14.

Step 3: The **upper quartile** is the median of the upper set of values. In Olivia’s case, the upper quartile is 25.

Step 4: The **interquartile range** is the difference between the two quartiles. In Olivia’s case:

$$\begin{aligned}\text{Interquartile range} &= 25 - 14\\ &= 11\end{aligned}$$

Thus, the middle 50% of Olivia’s times have a spread of 11 minutes.

Notice that the interquartile range is unaffected by the lower quarter and the upper quarter of the values. Hence, the large sizes of two of Olivia's times, when she left the game running to eat dinner and to sleep, do not affect the interquartile range.

The calculations begin slightly differently when there is an even number of results. For example, suppose that Olivia played one more game, which she solved in 22 minutes.

There are now 12 results to arrange in ascending order:

8, 12, 14, 14, 16, 18, 19, 19, 22, 25, 78, 523

Step 1: Since there is an even number of results, we divide them into two equal groups of 6. (The median lies 'between' the 6th and 7th member of the ordered data set. That is, in the $\frac{12+1}{2} = 6.5$th position.)

8, 12, 14, 14, 16, 18 and 19, 19, 22, 25, 78, 523

Step 2: The lower quartile is now $\frac{14+14}{2} = 14$.

Step 3: The upper quartile is now $\frac{22+25}{2} = 23\frac{1}{2}$.

Step 4: The interquartile range is now $23\frac{1}{2} - 14 = 9\frac{1}{2}$.

In this case, the middle 50% of Olivia's times have a spread of $9\frac{1}{2}$ minutes.

The minimum, maximum, median and the two quartiles are sometimes called the **five-number summary**. Sometimes the lower quartile is called the first quartile, because it marks the first quarter of the ordered data. The median is then the second quartile, although this term is seldom used. The upper quartile is called the third quartile.

We denote the lower quartile by Q_1 and the upper quartile by Q_3. We sometimes use the abbreviation IQR for the interquartile range.

Example 2

Find the interquartile range of the data set:

26 19 25 13 24 23 23 25 20 28 23

Solution

First arrange in order and locate the median.

13 19 20 23 23 $\boxed{23}$ 24 25 25 26 28

$\uparrow$

median = 23

There are 11 data values. The 6th value is 23, $\left(\frac{11+1}{2} = 6\text{th value}\right)$, so the median is 23.

The lower group contains 5 values. The 3rd value is 20. So the lower quartile is 20.
Similarly, the upper quartile is 25.

$$\begin{aligned}\text{Thus, interquartile range} &= 25 - 20\\ &= 5\end{aligned}$$

That is, the middle 50% of data values have a spread of 5.

Example 3

For the stem-and-leaf plot opposite, find the median and the quartiles.

3|4 means 34.

2	4 6 7 8 9
3	$\boxed{0}$ 1 1 3 4 6 7
4	1 4 5 5 $\boxed{7}$ 8 9
5	0 1 2

Solution

There are 22 data values. First locate the median to divide the data into two equal groups.

The median lies in the $\dfrac{22+1}{2} = 11.5$th position of the ordered set. The 11th value is 36 and the 12th value is 37, so the median is 36.5.

The lower group contains 11 values. The 6th value is 30. So the lower quartile is 30. Similarly, the upper quartile is 47.

Measures of spread

- The **range** is the difference between the highest and lowest values in a data set.
- The **interquartile range** measures the spread of the middle 50% of the data in an ordered data set.
- To calculate the interquartile range, find the difference between the upper quartile Q_3 and the lower quartile Q_1.

Exercise 18A

Example 2 **1** Find the range and interquartile range of each data set.

a 7 5 15 10 13 3 20 7 15

b 8 5 1 7 5 7 8 10 5 7

c 4 0 6 4 6 7 9 4

d 3 13 8 11 1 18 5 13

Example 3 **2** Locate the median and the quartiles for each of the following stem-and-leaf plots. State the interquartile range for each data set.

a

2	0 1 2 4 4 7 7 9
3	1 1 1 2 2 4 6 6 7 8 9
4	0 1 2 2 4

3|2 means 32

b

5	4 4 6 7 7 9
6	1 4 4 4 6 7 8
7	1 5 7 8 9 9
8	0 1 1 2 3 4 6
9	1 3 4 5

6|1 means 61

3 Find the mean, the mode, the median and the interquartile range of this data set.

Value	0	1	2	3	4	5	6	7	8	9	10
Frequency	5	2	0	7	1	8	4	6	0	2	11

4 Complete the following table for the positions of the median and the quartiles for data sets of 100 and 101 items. (*Note*: A position of 8.5 means it is between the eighth and ninth data values).

	Number of data items	Lower quartile position	Median position	Upper quartile position
a	100			
b	101			

5 The stem-and-leaf plot opposite gives the height in centimetres of 20 students in a class.

a What is the range of the height of students in the class?

b What is the median height of students in the class?

c What is the interquartile range?

14	4 5 6
15	0 1 2 8
16	0 0 1 2 4 5 7
17	2 6 7 8
18	0 2

15 | 1 means 151

6 The stem-and-leaf plot opposite gives the lengths in centimetres of 15 leaves that have fallen from a tree. The values are given correct to one decimal place. Find the interquartile range of the leaf lengths.

4	4
5	5 1 8 44
6	3 1 2 4
7	7 2 7
8	
9	4 3

9 | 4 means 9.4

7 The following figures are the amounts a family spent on food each week for 13 weeks.

\$148 \$143 \$152 \$149 \$158

\$155 \$147 \$152 \$158 \$139

\$143 \$150 \$141

a Find the median, upper quartile and lower quartile.

b Find the interquartile range of the amounts spent.

8 Write down two sets of seven whole numbers with minimum data value 3, lower quartile 5, median 10, upper quartile 12 and maximum data value 13.

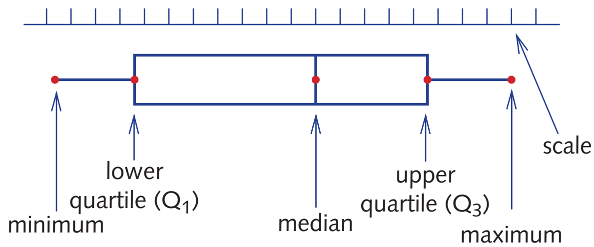
9 The median is always between the two quartiles. Is the mean always between the two quartiles? If not, give an example of seven whole numbers where the mean is above the upper quartile and an example where the mean is below the lower quartile.

10 **a** For a data set, the minimum value is 8 and the range is 27. Find the maximum value.

b For a particular data set, the upper quartile is 25.6, and the interquartile range is 11.9. Find the lower quartile.

18B Boxplots

A useful way of displaying the maximum value and the minimum value, the upper and lower quartiles and the median of a data set (the five-number summary) is a **boxplot**.



The rectangle is called the **box**.

The horizontal lines from the lower and upper quartiles to the minimum and maximum are called the **whiskers**. In a boxplot, the box itself indicates the location of the middle 50% of the data.

Boxplots are especially useful for large data sets. A boxplot is a visual summary of some of the main features of the data set. Boxplots are also useful for comparing related data sets – see Questions 9, 10 and 11 in Exercise 18B.

Example 4

The weights of 20 students are recorded here. The weights are given to the nearest kilogram.

48 52 54 54 55 58 58 61 62 63 63 64 65 66 66 67 69 70 72 79

a Find the median, upper quartile, lower quartile and interquartile range.

b Draw a boxplot for this data.

Solution

a There are 20 data values. Therefore, the median $= \dfrac{63 + 63}{2} = 63$ kg

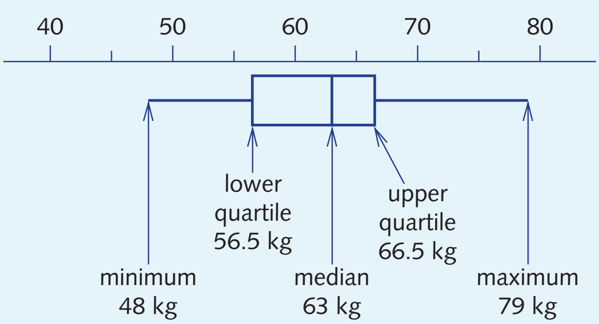
Divide the data into two equal groups of 10.

48 52 54 54 55 58 58 61 62 63   63 64 65 66 66 67 69 70 72 79

The lower quartile $= \dfrac{55 + 58}{2} = 56.5$ kg   The upper quartile $= \dfrac{66 + 67}{2} = 66.5$ kg

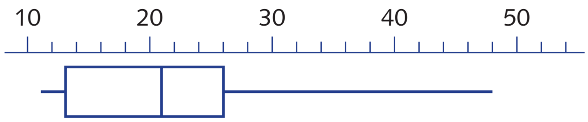
The interquartile range $= 66.5 - 56.5$
$= 10$ kg

b



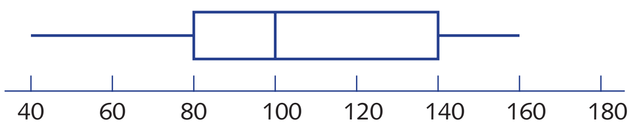
Exercise 18B

1 The boxplot below shows the price (in \$) of 20 different brands of sports shirts.



What is the cost of the most expensive and least expensive sports shirt?

2 The boxplot below gives information regarding the annual salaries (in thousands of dollars) of employees in a large company.



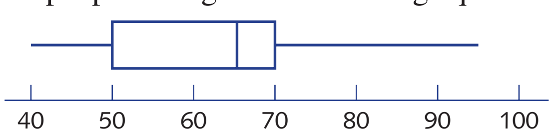
a What is the lowest salary?

b What is the range of the salaries?

c What is the median salary?

d What is the interquartile range?

3 The boxplot below gives information about the marks out of 100 obtained by a group of 40 people on a general knowledge quiz.



a What was the lowest mark obtained on the quiz?

b What was the median mark obtained on the quiz?

c What was the range of marks?

d What was the interquartile range?

4 Construct a boxplot for the data set given in Exercise 18A, question **2b**.

Example 4

5 The pulse rates of 21 adult females are recorded.

60 61 67 68 69 70 70 70 73 74 75 75 76 77 77 78 79 80 81 89 90

a Find the median, upper quartile, lower quartile and interquartile range.

b Draw a boxplot for this data.

6 In a boxplot for a large data set, approximately what percentage of the data set is:

a below the median?

b below the lower quartile?

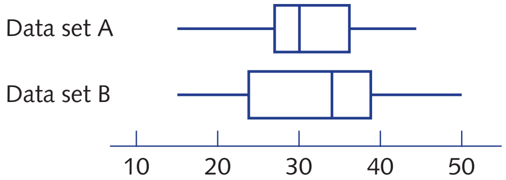
c in the box?

d in each whisker?

7 In a boxplot, is one whisker always longer than the other?

8 In a boxplot, why is the median not always in the centre of the box?

9 Here are two boxplots drawn on the one scale.



Which data set has:

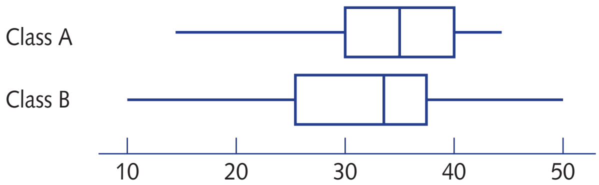
a the greater median?

b the greater range?

c the greater interquartile range?

d the greater largest data value?

10 Students in two classes sat the same mathematics test. Their results are shown in the two boxplots below.



a Which class had the higher median mark?

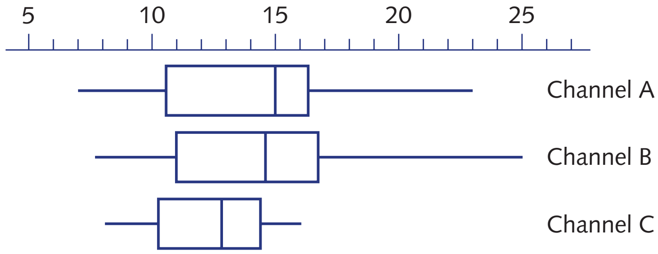
b Which class had the higher interquartile range?

c In which class was the highest mark for the test obtained?

d In which class was the lowest mark for the test obtained?

e Which class did better on the test? Give reasons for your choice. (Class discussion)

11 The ratings for a number of television programs on Channel A, Channel B and Channel C were collated. The information is shown in the boxplots below. (If a program has a rating of 14, it means that 14% of the viewing audience watched that particular program.)



a Write down the approximate values of the median, quartiles and maximum and minimum values for each channel.

b Which channel has the largest interquartile range?

c If the winning channel is the one with the highest rated program, which channel is the winner? Which is second? Which is third?

d If the winning channel is the one with the largest median, rank the channels.

e Can you find a criterion that makes Channel C the winning channel?

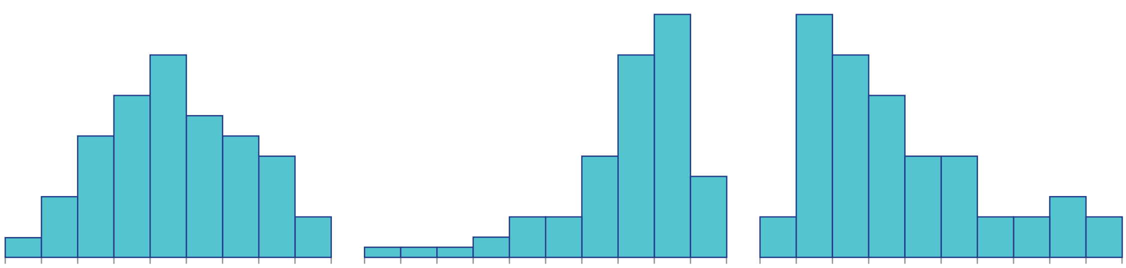
18C Boxplots, histograms and outliers

It is common to use a form of the boxplot that is designed to illustrate any possible outliers in the data. Outliers are unusual, or 'freak', values that differ greatly in magnitude from the majority of data values.

- Any point that is more than 1.5 IQRs away from the end of the box is classified as an outlier. That is, if a data value is greater than $Q_3 + 1.5 \times \text{IQR}$ or less than it $Q_1 - 1.5 \times \text{IQR}$ is considered to be an outlier. An outlier is indicated by a marker, as shown in the diagram above.
- The whiskers end at the highest and lowest data values that lie within 1.5 IQRs from the ends of the box.

Comparing a boxplot to the histogram of the same data

In *ICE-EM Mathematics Year 9* we looked at different shapes of histograms and the distributions of data, and in particular we used the terms **symmetric**, **positively skewed** and **negatively skewed** to describe the shapes.



Symmetric distribution   Negatively skewed distribution   Positively skewed distribution

The following examples look at representing data with histograms and boxplots.

Example 5

The house prices of 50 houses sold in a town over a period of two years are recorded. The prices are in thousands of dollars.

110, 110, 120, 130, 140, 150, 150, 170, 170, 170, 180, 190, 200, 210, 210, 230, 270, 270, 290, 310, 340, 340, 340, 340, 350, 360, 360, 365, 365, 400, 400, 400, 400, 410, 430, 440, 450, 460, 460, 460, 460, 564, 678, 678, 750, 760, 904, 1320, 2350, 2350

a Find the quartiles, the median and the interquartile range.
b Calculate $1.5 \times \text{IQR}$.
c Name the outliers.
d Draw a histogram and boxplot of this information. The boxplot should show outliers.
e **i** Calculate the mean, including the outliers.
  ii Calculate the mean, not including the outliers.

Solution

a The data has been given in ascending order. There are 50 data values. The median is the mean of the 25th and 26th values.

Median $= \$355\,000$

Q_1 is the median of the lower set of 25 values. This is the 13th value.

$Q_1 = \$200\,000$

Q_3 is the median of the upper set of 25 values.

$Q_3 = \$460\,000$

$\text{IQR} = \$260\,000$

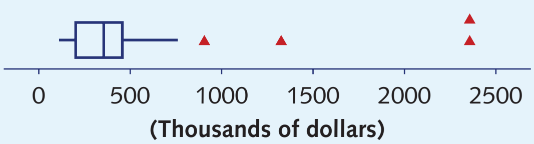
b $1.5 \times \text{IQR} = 1.5 \times (Q_3 - Q_1) = \$390\,000$

Hence, a value is an outlier if it is greater than $460\,000 + 390\,000 = \$850\,000$

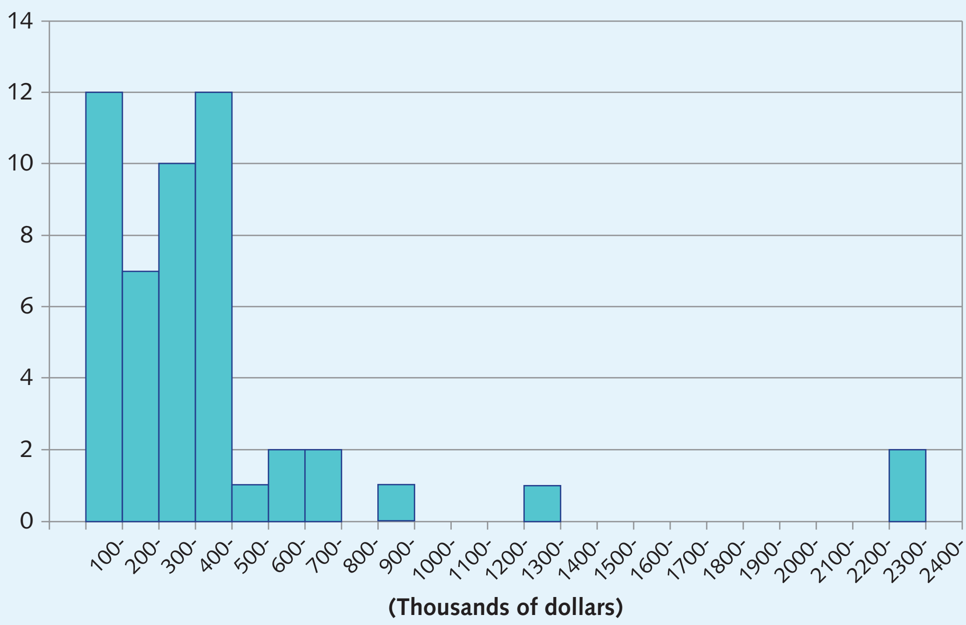
or less than $200\,000 - 1.5 \times 260\,000$

c The outliers are $904 000, $1 320 000, $2 350 000 and $2 350 000.

d



(Note: The right-hand whisker ends with the value $760 000)



The classes are $100 000 to $199 000, $200 000 to $299 000 etc.

e **i** Mean with outliers = $449 300, to the nearest $100.

ii Mean without outliers = $337 800, to the nearest $100.

It could be said that the distribution has a positive skew. The left-hand whisker is short. Most of the values lie in the interval from $100 000 to $500 000.

Example 6

The waiting times in seconds at a ticket counter were as follows:

0, 0, 3, 5, 5, 5, 9, 10, 12, 13, 16, 17, 18, 18, 21, 22, 23, 23, 24, 24, 24, 24, 24, 25, 25, 25, 26, 26, 27, 28, 29, 28, 29, 29, 28, 30, 31, 31, 31, 32, 34, 34, 33, 33, 33, 34, 34, 33, 34, 35, 35, 35, 36, 36, 37, 38, 39, 38, 39, 39, 38, 40, 41, 41, 52

a Find Q_1, the median, Q_3 and the IQR.

b Draw a boxplot, showing outliers.

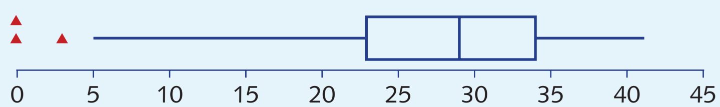
c Draw a histogram.

d Comment on the shape of the histogram and the boxplot.

Solution

a $Q_1 = 22.5$, median $= 29$, $Q_3 = 34.5$, $\text{IQR} = Q_3 - Q_1 = 12$

b

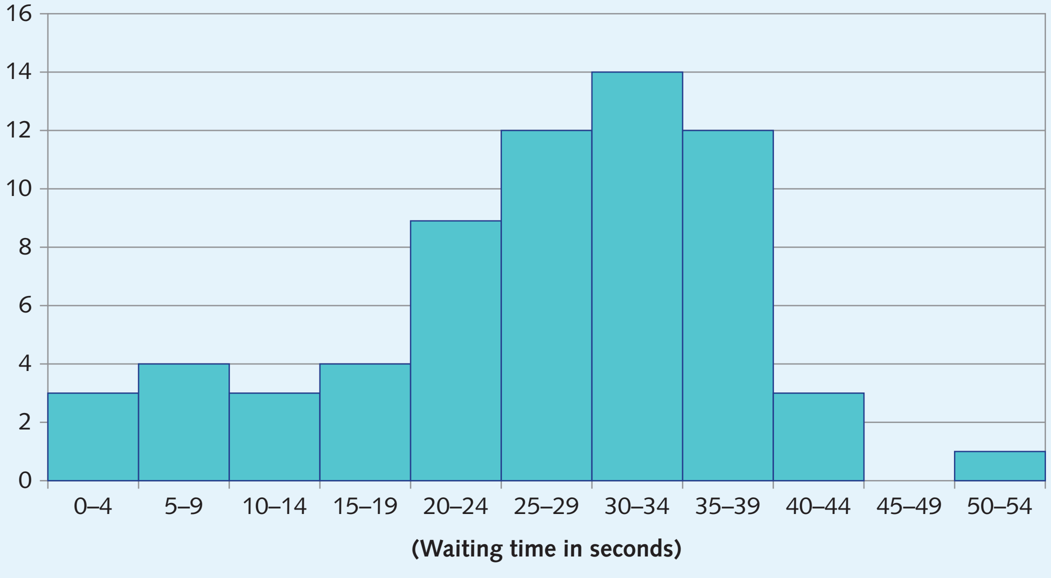


$Q_3 + 1.5 \times \text{IQR} = 34.5 + 1.5 \times 12 = 52.5$

$Q_1 - 1.5 \times \text{IQR} = 22.5 - 1.5 \times 12 = 4.5$

Therefore, the values 0, 0 and 3 are considered to be outliers.

c



d There is a negative skew. The right-hand whisker is short. The left-hand whisker is longer, indicating a tailing off of the data values. The values 0, 0 and 3 are outliers.

Example 7

Fifty-four lengths of wire are cut off by a machine. The resulting lengths measured in cm are as shown:

103, 104, 105, 106, 106, 106, 107, 107, 107, 107, 107, 108, 108, 108, 108, 108, 108, 108, 108, 109, 109, 109, 109, 109, 109, 109, 109, 110, 110, 110, 110, 110, 110, 110, 110, 110, 111, 111, 111, 111, 111, 111, 111, 112, 112, 112, 112, 113, 113, 113, 113, 114, 115, 116

a Find Q_1, the median, Q_3 and the IQR.

b Draw a boxplot, showing outliers.

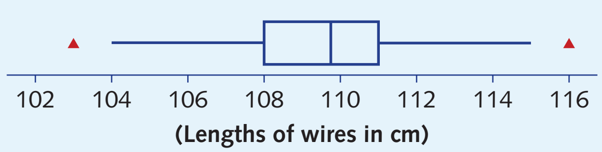
c Draw a histogram.

d Comment on the shape of the histogram and the boxplot.

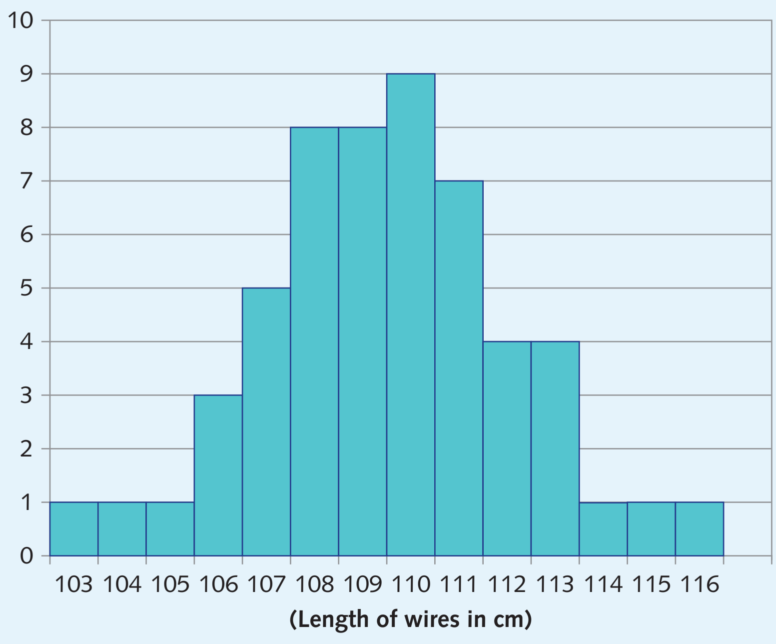
Solution

a $Q_1 = 108$ cm, median $= 109.5$ cm, $Q_3 = 111$ cm and IQR $= 3$ cm

b



c



d The histogram is symmetric. The whiskers on the boxplot are of equal length. The values 103 cm and 116 cm are outliers.

Exercise 18C

Example 5, 6

1 The heights, measured in centimetres, of 25 students in a class are:

170	175	133	153	164	189	143	133	167	145	150	164	169
159	177	186	173	164	177	168	142	155	153	167	166	

a Find Q_1, the median and Q_3. **b** Find the interquartile range.

c Draw a boxplot, showing any outliers.

Example 7

2 The annual incomes of 30 people, given correct to the nearest \$1000, are:

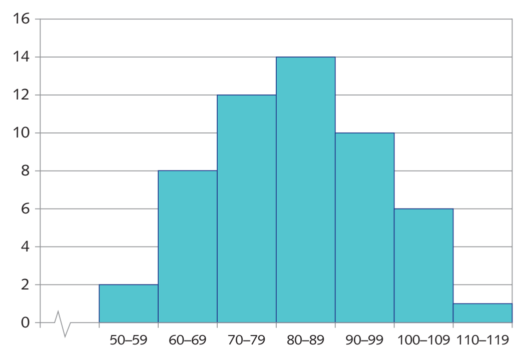
54 000	67 000	92 000	78 000	54 000	87 000	102 000	112 000
132 000	45 000	256 000	89 000	78 000	98 000	34 000	75 000
65 000	100 000	34 000	68 000	79 000	81 000	82 000	103 000
21 000	345 000	98 000	67 000	105 000	98 000		

a Find Q_1, the median and Q_3. **b** Find the interquartile range.

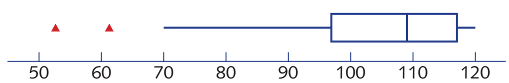
c Draw a boxplot, showing any outliers.

3 Match each histogram **a – c** with its box plot **i – iii** and describe the shape of the data distribution.

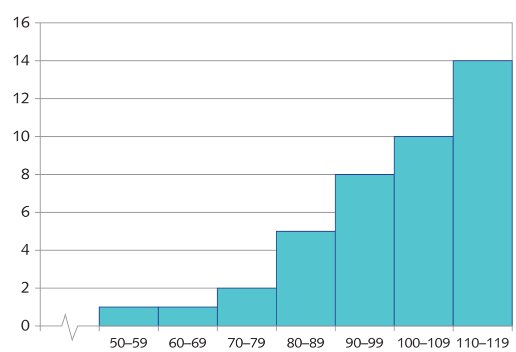
a



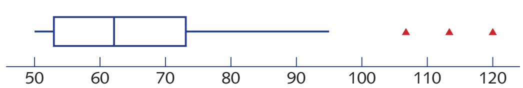
i



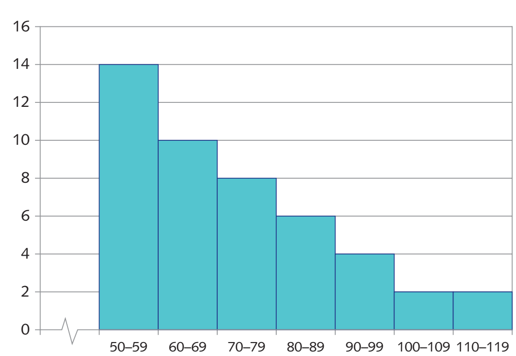
b



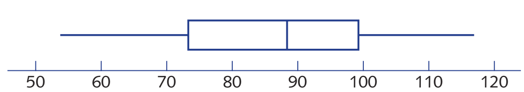
ii



c



iii



4 Consider the data shown in the stem-and-leaf plot.

```
15|6 8
16|9 9
17|0 1 3 3 4 5 8 8 9 9
18|0 0 1 3 3 4 7 7 8 8
19|1 2 3
```

a Draw a histogram.

b Find Q_1, the median, Q_3 and the IQR.

c Draw the boxplot.

d Comment on the shape of the histogram and the boxplot.

5 The lower and upper quartiles for a data set are 116 and 134. Which of the following data values would be classified as an outlier?

a 190 **b** 60 **c** 150

6 The speeds of 20 cars measured on a city street were recorded.

40	14	3	26	20	31	42	36	17	24
28	33	27	29	24	51	11	35	5	24

a Construct a stem-and-leaf diagram.

b Construct a boxplot.

c Comment on the shape of the distribution of data.

7 The reaction times (in milliseconds) of 20 people are listed here.

38	31	36	39	35	25	35	44	43	44
46	34	62	22	42	48	31	30	45	40

a Find the median, Q_1, Q_3 and the interquartile range.

b Construct a boxplot.

c Identify any outliers.

8 The weight loss (in kilograms) of 20 randomly selected people undertaking a special diet over three weeks is:

8	5	10	6	6	12	4	5	5	6
8	13	7	7	7	6	6	4	5	5

a Construct a dotplot of the data.

b Construct a boxplot of the data.

c Comment on the shape.

18D The mean and the standard deviation

Mean

The **mean** of a data set is a measure of its centre. The mean is calculated by adding together all the data values and then dividing the resulting sum by the number of data values.

$$\text{Mean} = \frac{\text{sum of values}}{\text{number of values}}$$

A more common name for the mean is 'average'. We use the symbol $\bar{x}$ to denote the mean.

For a set of data $x_1, x_2, x_3, \ldots, x_n$,

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$$

Example 8

A student obtained the following marks in seven tests:

43, 35, 41, 29, 33, 39 and 42

Calculate the mean mark correct to two decimal places.

Solution

$$\begin{aligned}\bar{x} &= \frac{43 + 35 + 41 + 29 + 33 + 39 + 42}{7}\\ &\approx 37.43 \quad \text{(Correct to two decimal places.)}\end{aligned}$$

For larger sets of data, a frequency table can be prepared. Let f_1 be the frequency of the data item x_1, let f_2 be the frequency of the data item x_2 and so on. In this case we can write:

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + \ldots + f_sx_s}{f_1 + f_2 + \ldots + f_s}$$

The numerator is the sum of the data items and the denominator is the number of data items.

Example 9

The following information gives the number of children in each of 20 families. Calculate the mean number of children per family.

Number of children x_i	Frequency f_i
0	4
1	5
2	7
3	4

Solution

Add in a column for $f_i\, x_i$.

Number of children x_i	Frequency f_i	$f_i\, x_i$
0	4	0
1	5	5
2	7	14
3	4	12
	Total = 20	Total = 31

$$\bar{x} = \frac{31}{20} = 1.55$$

It is obviously impossible for a family to have 1.55 children. The mean is not necessarily a member of the data set.

Standard deviation

The **standard deviation** of a set of data is a measure of how far the data values are spread out from the mean. The difference between each data item and the mean is called the **deviation** of the data value. The sum of the deviations is zero, which will be proved in question **10** of Exercise 18D.

The standard deviation is calculated from the squares of the deviations.

Here are the steps in finding the standard deviation:

- Calculate the mean.
- Square each of the deviations.
- Sum these squares.
- Divide the sum of the squares by the number of data values.
- Take the square root of the value obtained.

This is given by the formula:

$$\sigma = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2}{n}}$$

where the x_i are the data values, $\bar{x}$ is the mean and n is the number of data values.

We will use the Greek letter σ (sigma) to denote the standard deviation of a data set.

Example 10

Find the standard deviation, correct to two decimal places, for the data set.

5, 7, 11, 13, 14

Solution

$$\begin{aligned}\bar{x} &= \frac{5 + 7 + 11 + 13 + 14}{5}\\ &= 10\end{aligned}$$

$$\begin{aligned}\sigma^2 &= \frac{(5-10)^2 + (7-10)^2 + (11-10)^2 + (13-10)^2 + (14-10)^2}{5}\\ &= \frac{25 + 9 + 1 + 9 + 16}{5}\\ &= \frac{60}{5}\\ &= 12\end{aligned}$$

Hence, $\sigma = \sqrt{12} \approx 3.46$ (Correct to two decimal places.)

When calculating the standard deviation from a frequency table, we can use the following formula:

$$\sigma = \sqrt{\frac{f_1(x_1 - \bar{x})^2 + f_2(x_2 - \bar{x})^2 + f_3(x_3 - \bar{x})^2 + \ldots + f_s(x_s - \bar{x})^2}{f_1 + f_2 + \ldots + f_s}}$$

When frequencies are taken into account, we can see that this is the same formula as above.

We can calculate the standard deviation with an extended frequency table with five columns. Fill in the first three columns, then calculate $\bar{x}$. Fill in the other two columns and then calculate σ.

Example 11

Calculate the mean and standard deviation of the set of values, correct to two decimal places.

1, 3, 4, 5, 7, 3, 6, 9, 9, 4, 5, 2, 5, 7

Solution

x_i	f_i	$f_i\ x_i$	$(x_i - \bar{x})$	$f_i(x_i - \bar{x})^2$
1	1	1	–4	16
2	1	2	–3	9
3	2	6	–2	8
4	2	8	–1	2
5	3	15	0	0
6	1	6	1	1
7	2	14	2	8
9	2	18	4	32
	Total = 14	Total = 70		Total = 76

$\bar{x} = \frac{70}{14} = 5$

$$\begin{aligned}\sigma &= \sqrt{\frac{76}{14}}\\ &\approx 2.33 \quad \text{(Correct to two decimal places.)}\end{aligned}$$

Note: The sum of the deviations $x_i - \bar{x}$ is zero. Hence, the average of the deviations is not useful.

Mean and standard deviation

- The **mean** of a set of data is denoted by $\bar{x}$.
- The **standard deviation** of a data set is a measure of spread and is denoted by the Greek letter σ.
- There are two formulas for the standard deviation.

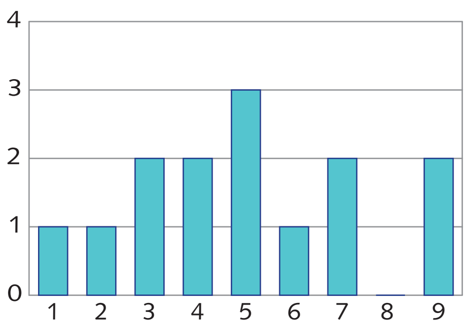
$$\sigma = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2}{n}}, \text{ when the data is in a list.}$$

$$\sigma = \sqrt{\frac{f_1(x_1 - \bar{x})^2 + f_2(x_2 - \bar{x})^2 + f_3(x_3 - \bar{x})^2 + \ldots + f_s(x_s - \bar{x})^2}{f_1 + f_2 + \ldots + f_s}}, \text{ when the data is in a}$$

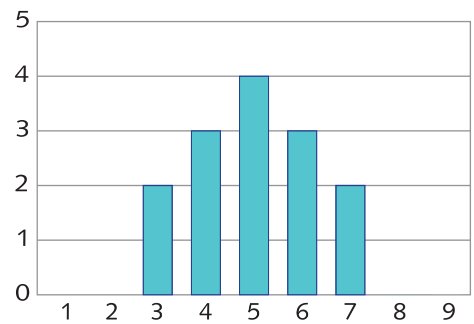
frequency table.

It is clear that the larger the standard deviation, the more spread out the data are about the mean.

For example, here is a bar chart of the data in Example 11, and also another set of 14 data items where the data are not as spread out but have the same mean.



$\bar{x} = 5$ and $\sigma \approx 2.33$



$\bar{x} = 5$ and $\sigma \approx 1.25$

In the following section we will see how the standard deviation may be used to make comparisons between data sets.

Use of calculators

Many calculators and spreadsheets have a built-in facility for calculating the standard deviation of a set of data.

To save time, we recommend using this facility for all but the simplest data sets. In particular, if $\bar{x}$ is not an integer, then calculating σ is very tedious.

It should be noted that in this book we calculate the standard deviation by dividing the sum of the squares of the deviations by n, the number of data items, and taking the square root. There is also another type of standard deviation that is obtained by dividing the sum of the squares of the deviations by $n - 1$, and taking the square root. Many calculators offer both versions. Sometimes they are denoted by symbols such as σ_n and σ_{n-1}. In this book, we only use σ_n.

Exercise 18D

Give all answers correct to two decimal places unless otherwise specified.

Example 8

1 During a 13-week football season, the number of kicks obtained by a particular player each week is:

18, 18, 20, 26, 10, 8, 21, 14, 16, 14, 12 and 16

Calculate the mean number of kicks obtained by the player.

2 The daily maximum temperature was recorded in two different cities for a week. The results are shown below.

City A: 28, 31, 34, 32, 31, 29, 28

City B: 26, 32, 36, 38, 37, 29, 25

Which city had the greater mean daily maximum temperature?

3 The average of 5 masses is 67 kg. If a mass of 25 kg is added, what is the average of the 6 masses?

4 During a term, a student has an average of 46 marks after the first four tests and his average for the next six tests is 38 marks. What is his average for the ten tests?

Example 10

5 **a** Calculate, correct to two decimal places, the mean and standard deviation for the data sets.

i 2, 4, 8, 10, 2, 9, 3, 8, 2, 2 **ii** 3, 6, 4, 5, 6, 7, 3, 4, 6, 6

b Comment on the results from part **a**.

Example 11

6 Complete the following extended frequency table to calculate the mean and standard deviation of the given data set.

x_i	f_i	$f_i\ x_i$	$(x_i - \bar{x})$	$f_i(x_i - \bar{x})^2$
1	2			
2	7			
3	6			
4	1			
5	2			
6	2			
	Total =	Total =		Total =

7 Use a calculator to find, correct to two decimal places, the mean and standard deviation for each data set.

a 3, 6, 7, 5, 8, 5, 10, 12, 13, 12, 6, 9, 12, 14, 15

b 8, 10, 12, 14, 16, 17, 19, 12, 11, 10, 14, 16, 18, 19

8 Twenty students sat a test and their results are given in the stem-and-leaf plot below.

1	2 2 8 9
2	2 4 5 6 8
3	0 2 6 8 8 9
4	0 1 2 3 6

1|2 means 12

a Calculate their mean mark.

b How many students obtained a mark higher than the mean mark?

c Find the standard deviation of their marks.

9 Twenty people completed a test worth 10 marks. Their scores are shown in the frequency table below.

Score	0	1	2	3	4	5	6	7	8	9	10
Number of people	0	2	0	1	1	2	4	6	0	2	2

a Calculate the mean mark.

b How many students obtained a mark lower than the mean mark?

c Find the standard deviation of their marks.

10 a Prove that the sum of the deviations for the data set a, b, c is zero.

b Prove that the sum of the deviations of any data set is zero.

18E Interpreting the standard deviation

Consider the data sets 4, 5, 6, 7, 8 and 2, 4, 6, 8, 10.

Both the data sets have a mean and median of 6. However, when we apply the formula for σ, it can be observed that the standard deviation for the second data set is $2\sqrt{2}$, which is twice the standard deviation of the first data set, $\sqrt{2}$. This reflects the difference in spread between the two data sets. That is, even though both have evenly distributed values, the spread of data from the mean is twice as great in the second data set as compared to the first.

Intervals about the mean

In the following we will look at a 'symmetric' set of data which 'tails off' as you move away from the mean in either direction.

The stem-and-leaf plot on the next page gives the incomes, in thousands of dollars, of 134 people.

0	8 8 9
1	0 0 2 2 3
2	4 4 4 4 4 4 8 8 8 8 8 8 8
3	1 1 1 2 2 4 4 4 4 6 6 6 6 7 7 7 7 7 8 8 8 8 8 9 9 9
4	1 1 1 1 1 2 2 2 3 3 3 4 4 4 4 4 5 5 5 5 6 6 7 7 7 7 7 7 8 8 9 9 9 9 9 9 9 9 9
5	0 0 0 0 0 1 1 1 1 1 1 1 1 2 2 2 3 3 3 4 4 4 4 4 5 7 7
6	3 3 3 3 3 6 6 6 6 9 9 9 9
7	7 7 8 9 9
8	6 6 6

7 | 7 means \$77 000

The mean is 45.1, the median is 45.5, and the standard deviation is 16.1.

We next consider intervals centred on the mean.

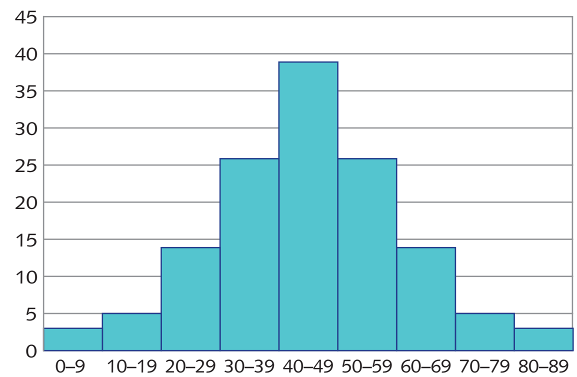
$\bar{x} + \sigma = 45.1 + 16.1 = 61.2$ and $\bar{x} - \sigma = 45.1 - 16.1 = 29.0$

We can observe from the plot above that there are 92 values between 29 and 61; hence, the percentage of values **within one standard deviation of the mean** is 68.7%. Also,

$\bar{x} + 2\sigma = 45.1 + 2 \times 16.1 = 77.3$ and
$\bar{x} - 2\sigma = 45.1 - 2 \times 16.1 = 12.9$

There are 121 values between 13 and 77.

Thus, the percentage of values **within two standard deviations of the mean** is 90.3%.



$\bar{x} - \sigma$ to $\bar{x} + \sigma$

$\bar{x} - 2\sigma$ to $\bar{x} + 2\sigma$

We have seen that about 69% of the data is within one standard deviation of the mean and about 90% of the data is within two standard deviations of the mean.

Histograms similar to this one occur frequently. In most cases like these the median and the mean are very close.

Example 12

David plays golf every Friday. He has recorded his score each Friday for five years, and has found that his mean score for all his games is 85 and the standard deviation of his scores is 5.2.

Find the range of scores that lie within:

a one standard deviation of the mean **b** two standard deviations of the mean

Solution

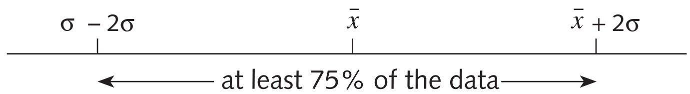
a $\bar{x} + \sigma = 85 + 5.2 = 90.2$ and $\bar{x} - \sigma = 85 - 5.2 = 79.8$
So the range of scores within one standard deviation of the mean is 80 to 90.

b $\bar{x} + 2\sigma = 85 + 10.4 = 95.4$ and $\bar{x} - 2\sigma = 85 - 10.4 = 74.6$
So the range of scores within the two standard deviations of the mean is 75 to 95.

A remarkable result known as Chebyshev's inequality states that, for *any set of data,* if we take an interval between $\bar{x} - k\sigma$ and $\bar{x} + k\sigma$, then all values can lie outside this interval for $0 < k \leq 1$, but for $k > 1$, at most $\frac{1}{k^2}$ of the data can lie outside this interval.

So, for example, taking $k = 2$, not more than $\frac{1}{4}$ of the data can be outside this interval.

So at least 75% of the data must lie inside this interval.



Using the standard deviation to compare data

To compare values from different data sets with approximately the same shape, it is useful to consider where they are positioned relative to their respective means. This can be achieved by using their respective standard deviations, and calculating where these values lie in terms of the number of standards above or below the mean.

Example 13

Gus scored 14 in a maths test and 14 in an English test. The scores of each student in the maths and English classes are listed below. In which test did Gus perform better, relative to the class results?

Maths test: 10, 13, 18, 17, 12, 16, 9, 8, 7, 11, 10, 12

English test: 15, 17, 18, 19, 18, 17, 19, 16, 14, 15, 14, 12

Solution

Maths test $\quad \bar{x} = \frac{143}{12} \approx 11.92, \quad \sigma \approx 3.38$

English test $\quad \bar{x} \approx 16.17, \quad \sigma \approx 2.11$

It can be seen that in the maths test Gus scored about 0.6 of a standard deviation above the mean $\left(\frac{14 - 11.92}{3.38} \approx 0.6\right)$ and in the English test Gus scored about 1 standard deviation below the mean $\left(\frac{14 - 16.17}{2.11} \approx -1\right)$. So Gus has done better relative to the class in the maths test.

Exercise 18E

1 Find the mean and standard deviation of each set of data.

a 5, 6, 6, 7, 8, 9, 22

b 11, 7, 8, 9, 8, 10, 10

c 1, 3, 7, 9, 11, 15, 17

Compare the sets of data using their means and standard deviations.

Example 12

2 The mean and standard deviation of each set of data is given. Find the range of values that is within:

i one standard deviation of the mean **ii** two standard deviations of the mean

a $\bar{x} = 35, \sigma = 2.5$

b $\bar{x} = 40, \sigma = 5$

c $\bar{x} = 35, \sigma = 8$

Example 13

3 The mathematics and English marks for a class of 15 students are given below.

Mathematics: 12, 16, 14, 19, 17, 18, 15, 15, 19, 20, 14, 18, 19, 15, 11

English: 10, 13, 16, 19, 20, 19, 18, 16, 15, 14, 17, 11, 15, 18, 17

a Calculate, correct to two decimal places, the mean and standard deviation for each set of marks.

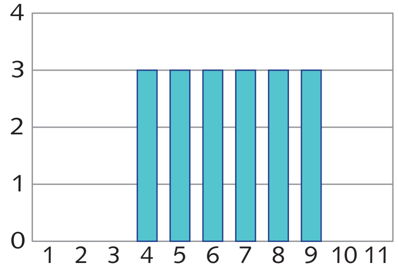
b If a student scored 16 for the mathematics test and 14 for the English test, which is the better mark relative to the class results?

4 The following table lists the marks of several students on different tests in English and mathematics. Compare the English and mathematics marks of each student.

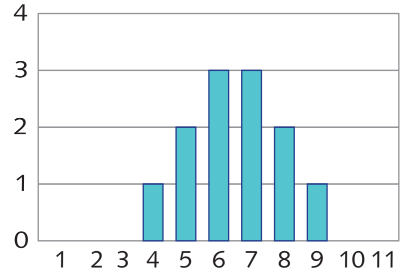
		Mark	Mean	Standard deviation
a	David			
	English	15	17	2
	Mathematics	13	17	3
b	Akira			
	English	42	30	6
	Mathematics	39	25	8
c	Katherine			
	English	70	75	5
	Mathematics	65	70	10
d	Daniel			
	English	70	55	9
	Mathematics	69	62	7

5 The bar charts of three sets of data are shown.

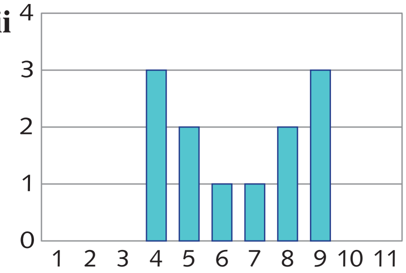
i



ii



iii



a For each set of data, calculate the mean and the standard deviation.

b Add 5 onto each data item in each of **i**, **ii** and **iii** and state the mean and standard deviation of each new set of data.

c Multiply each data item in each of **i**, **ii** and **iii** by 2 and state the mean and standard deviation of each new set of data.

6 (There is no arithmetic required in the following.)

Make up a list of 10 numbers so that the standard deviation is as large as possible and:

a every number is either 1 or 5

b every number is either 1 or 9

c every number is either 1 or 5 or 9, and at least two of them are 5

7 Repeat question **6**, but this time so the standard deviation is as small as possible.

8 An employer has 29 employees whose weekly salaries have $\bar{x} = \$429$ and $\sigma = \$1.53$. The employer decides to give a flat \$100 raise to every employee.

a What would be the change to the average annual salary paid by the employer?

b Would there be a change in the standard deviation?

c What would be the change in total weekly payments to employees?

18F Time-series data

A **time series** is a set of data that has been obtained by taking repeated measurements over time.

Maximum daily temperatures, average weekly wages, quarterly sales figures of a company and annual population of a city are all examples of a time series.

To represent the information obtained in a time series pictorially, a graph is drawn in which:

- the horizontal axis represents time
- the vertical axis represents the quantity that is being measured at regular intervals
- adjacent plotted points are joined by line intervals.

Example 14

The mean daily maximum temperature was measured each month in a particular city.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Mean daily max. temp (°C)	29.2	28.9	28.1	26.4	23.5	21.2	20.6	21.7	23.8	25.7	27.4	28.7

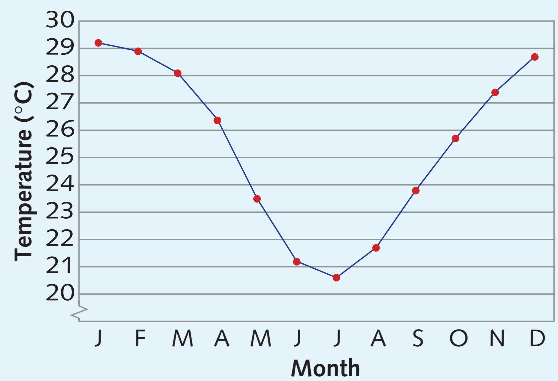
a Represent this information on a time-series plot.

b Briefly comment on the annual variation in daily maximum temperature.

Solution

a To construct a time-series plot, the months are placed on the horizontal axis and the vertical axis will represent the mean daily maximum temperature. The points are plotted and joined by lines.

The following time-series plot is obtained.



b There is a gradual decrease in the mean daily maximum temperature over the months January, February and March. During April, May and June, the mean daily maximum temperature falls quite quickly to a minimum during July. For the remainder of the year, there is a steady increase in the mean daily maximum temperature each month.

Exercise 18F

Example 14

1 **a** Construct a time-series plot for the average rainfall (in cm) in a particular city, which is given in the table below.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Rainfall (in cm)	16.2	17.5	14.2	9.1	9.6	7.1	6.2	4.1	3.3	9.3	9.6	12.6

b Use the time-series plot to write a brief description as to how the rainfall varies in this particular city.

2 The table below gives the annual profit (in $ million) of a particular company over a 10-year period. Construct a time-series plot of the information.

Year	1989	1990	1991	1992	1993	1994	1995	1996	1997	1998
Profit ($ million)	1.2	1.8	2.4	2.2	2.6	3.1	3.2	3.4	3.6	4.0

3 The table below gives the number of births that occurred in a hospital each month for a year.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Number of births	52	46	43	40	31	32	26	27	24	20	26	26

a Represent this information on a time-series plot.

b Briefly describe how the number of births recorded each month changed over the year.

4 The table below gives the position of a particular football team in a competition of 12 teams at the completion of each round throughout the season.

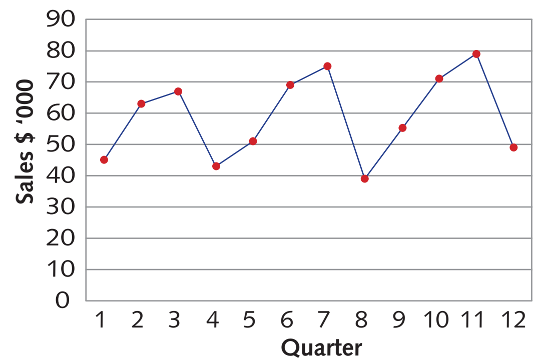
Round	1	2	3	4	5	6	7	8	9	10	11
Position	10	12	11	9	8	6	5	5	4	5	5
Round	12	13	14	15	16	17	18	19	20	21	22
Position	6	4	4	3	4	3	5	7	6	9	8

a Represent this information on a time-series plot.

b Briefly describe the progress of the team throughout the season.

5 The data below shows the quarterly sales of a department store over a period of three years. The quarters are labelled 1 to 12 in the corresponding time-series graph.

Sales quarter	Sales $'000
2009–1	45
2009–2	63
2009–3	67
2009–4	43
2010–1	51
2010–2	69
2010–3	75
2010–4	39
2011–1	55
2011–2	71
2011–3	79
2011–4	49



a In which quarter of each year are the sales figures the worst?

b In which quarter of each year are the sales figures the best?

c Are the sales figures improving? Compare the sales figures for the first quarter of each year and do the same for the other quarters.

6 The table below gives the quarterly sales figures for a car dealer for the period 2009–2011.

Number of sales	Q1	Q2	Q3	Q4
2009	72	62	90	98
2010	87	78	112	111
2011	90	84	132	117

a Represent this information on a time-series plot.

b Briefly describe how the car sales have altered over the given time period.

c Does it appear that the car dealer is able to sell more cars in a particular period each year?

18G Bivariate data

We often want to know if there is a relationship between the items in two different data sets.

- Is there a relationship between children's ages and their heights?
- Is there a relationship between people's heights and weights?
- Is there a relationship between students' marks in English and their marks in mathematics?

In each of the above, two pieces of information are to be collected from each person in the investigation and then the two data sets are to be compared. When two pieces of information are collected from each subject in an investigation, we are then concerned with **bivariate data**.

A **scatter graph** or **scatter plot** is a type of display that uses coordinates to display values for two variables for a set of data. The data is displayed as a collection of points, each having the value of one variable determining the position of the horizontal coordinate and the value of the other variable determining the position of the vertical coordinate.

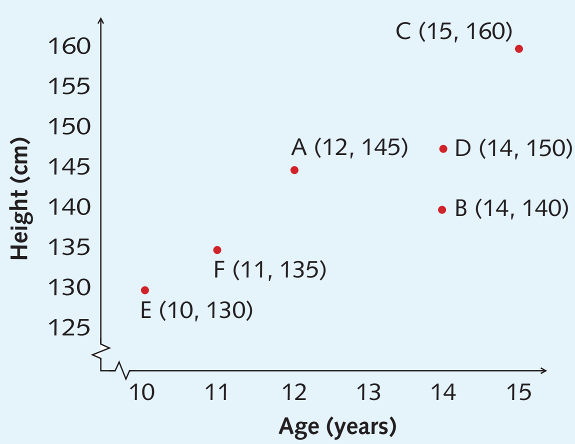
Example 15

The age (in years) and height (in cm) of a group of people was recorded. The data obtained is shown in the table on the right. Present the information in the table on a scatter plot.

Person	Age (years)	Height (cm)
Alan	12	145
Brianna	14	140
Chiyo	15	160
Danielle	14	150
Ezra	10	130
Frankie	11	135

Solution

The variables under consideration are age and height. The horizontal axis represents the age and the vertical axis represents height. The axes are broken (using the symbol ─⁄\/─) to allow us to focus on the data points.



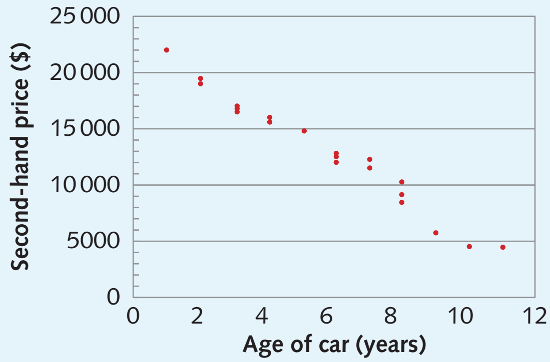
In this scatter plot, it is noted that points towards the top-right of the plot represent individuals who are older and taller. Points in the bottom-right represent individuals who are older but shorter than the rest of the group. The bottom-left of the plot represents people who are younger and shorter, while the top-left portion of the graph represents individuals who are younger but taller than the rest of the group.

We can see from the general trend of the points, which is upward as we move to the right, that the height of a child increases as the child grows older (for children in this data set).

Example 16

The second-hand price and age of a particular model of car are recorded in the table below, and the points plotted on a scatter plot.

Age of car (year)	Second-hand price($)
1	22 000
2	19 500
2	18 700
3	16 400
3	17 000
3	16 800
4	15 800
4	15 950
5	14 800
6	12 500
6	12 000
6	12 800
7	12 200
7	11 580
8	10 500
8	9200
8	8600
9	5700
10	4850
11	4500



a Describe the points in the top-left of the plot.
b Describe the points in the bottom-right of the plot.
c Describe the trend.

Solution

a The top-left of the scatter plot has points corresponding to relatively new second-hand cars with higher prices.
b The bottom-right of the scatter plot has points corresponding to older second-hand cars with lower prices.
c As the age of the car increases the value decreases.

Exercise 18G

Example 15

1 The table below gives the marks obtained by 10 students in a mathematics examination and an English examination.

Mathematics mark	72	50	96	58	86	94	78	66	85	78
English mark	78	64	70	46	88	72	70	62	72	74

Represent this information on a scatter plot, using the horizontal axis to represent the mathematics marks and the vertical axis to represent the English marks.

Break the axes so that the vertical axis starts near 40 and the horizontal axis starts near 50.

2 The table below gives the average monthly rainfall, in mm, and the average number of rainy days per month for twelve different cities in Australia.

Average rainfall (in mm)	161	175	142	90	96	71	62	41	33	93	96	126
Average number of rainy days	13	14	14	11	10	7	7	6	7	10	10	12

a Represent this information on a scatter plot. Use the horizontal axis to represent average monthly rainfall and the vertical axis to represent the average number of rainy days per month.

b Give a brief description of the relationship between rainy days and average rainfall.

3 The table below gives the amount of carbohydrates, in grams, and the amount of fat, in grams, in 100 g of a number of breakfast cereals.

Carbohydrates (in g)	88.7	67.0	77.5	61.7	86.8	32.4	72.4	77.1	86.5
Fat (in g)	0.3	1.3	2.8	7.6	1.2	5.7	9.4	10.0	0.7

a Represent this information on a scatter plot. Use the x-axis to represent the amount of carbohydrates and the y-axis to represent the amount of fat.

b Does there appear to be any relationship between the carbohydrate content and the fat content?

Example 16

4 The table below gives the IQ of a number of adults and the time, in seconds, for them to complete a simple puzzle.

IQ	115	118	110	103	120	104	124	116	110
Time (in seconds)	14	15	21	27	11	25	9	16	18

a Represent this information on a scatter plot. Use the x-axis to represent IQ and the y-axis to represent the time taken to complete the puzzle.

b Is there any trend in the data?

5 The table below gives the number of kicks and the number of handballs obtained by each player in an AFL team in a particular match.

Player	1	2	3	4	5	6	7	8	9	10	11
Number of kicks	3	20	7	19	7	6	2	9	7	26	3
Number of handballs	8	11	11	6	4	6	3	1	3	3	8
Player	12	13	14	15	16	17	18	19	20	21	22
Number of kicks	12	17	6	11	14	5	1	21	6	13	4
Number of handballs	4	5	0	3	8	3	0	11	0	17	11

a Represent this information on a scatter plot. Use the x-axis to represent the number of kicks and the y-axis to represent the number of handballs.

b Does your scatter plot support the claim, 'the more kicks a player obtains, the more handballs he gives'? Explain your answer.

6 The table below gives the number of 'goals for' (scored by the team) and the number of 'goals against' (scored by the opposing team) for each team in a soccer competition.

Team	A	B	C	D	E	F	G	H	I	J	K	L
Goals for	36	45	22	26	20	59	24	41	23	43	32	41
Goals against	31	16	33	26	64	16	53	42	47	21	49	14

a Represent this information on a scatter plot. Use the x-axis to represent 'goals for' and the y-axis to represent 'goals against'.

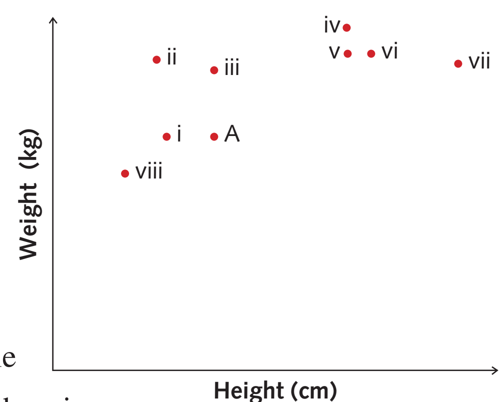
b Use your scatter plot to answer the following questions.

i Which team is the best team in the competition? Why?

ii Which team is the worst team in the competition? Why?

iii Which of team J and team H is better? Why?

7 The scatter plot at the right gives information about the height and weight of a number of people. Annabelle's height and weight is represented by the point A.



Write down the point that represents each of the following people.

a Barry, who is heavier and taller than Annabelle

b Chandra, who is shorter but heavier than Annabelle

c Dario, who is the same height as Barry but a little heavier

d Edwina, who is shorter and lighter than Chandra

e Frederick, who is the same weight as Barry but a bit taller

f George, who is the same height as Annabelle but heavier

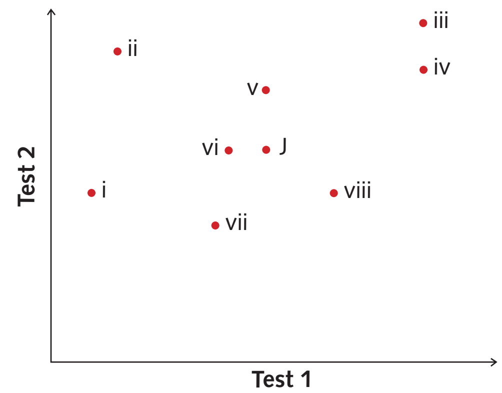
g Harriet, who is the same weight as Annabelle but shorter

h Ivan, who is the tallest person in the group

8 The scatter plot at the right gives the marks obtained by students in two tests.

John's marks on the tests are represented by the point J.

Which point represents each of the following students?



a Alex, who got the top mark in both tests

b Bao, who got the top mark in Test 1 but not in Test 2

c Charlene, who did better in Test 1 than John, but not as well on Test 2

d Drago, who did not do as well as Charlene on either test

e Eddie, who got the same mark as John for Test 2, but did not do as well as John on Test 1

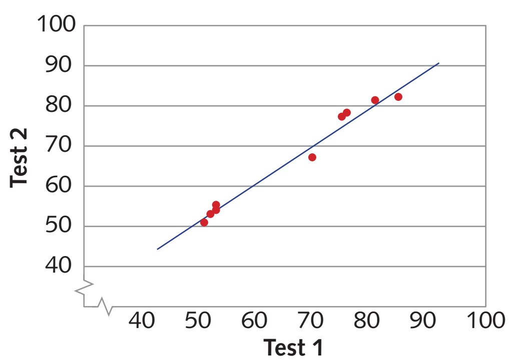
f Francis, who got the same mark as John for Test 1, but did better than John on Test 2

g Georgina, who got the lowest mark for Test 1

h Harvir, who had the greatest discrepancy between his two marks

9 The test results of a group of 9 students is recorded in the table and plotted on a scatter plot. A line has been drawn through the 'middle of the points'.

Test 1	Test 2
53	54
70	67
53	55
81	81
85	82
51	51
52	53
76	78
75	77



The equation for this line is Test 2 = 0.95 × Test 1 + 3.85.

a Use this equation to predict the Test 2 mark of a student if their mark on Test 1 was:

i 53 **ii** 54 **iii** 34 **iv** 84 **v** 67

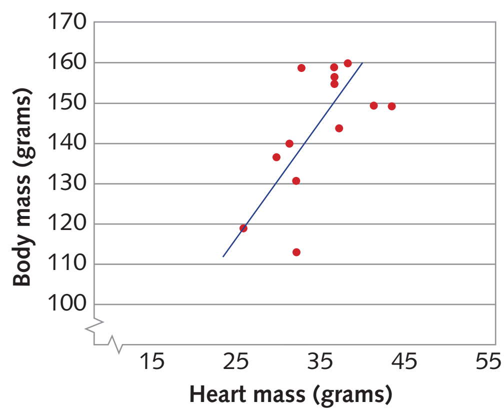
b Use this equation to predict the Test 1 mark of a student if their mark on Test 2 was:

i 53 **ii** 54 **iii** 34 **iv** 84 **v** 67

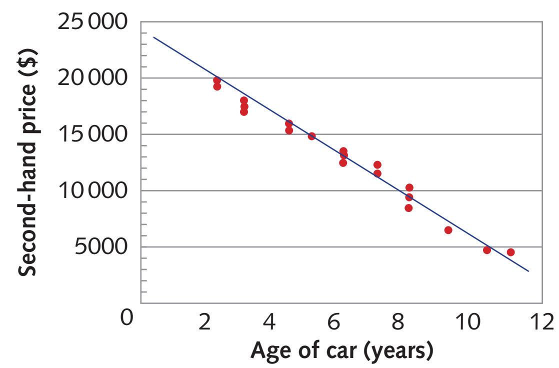
18H Line of best fit

Consider the four scatter plots below. A trend line or 'line of best fit' has been fitted to each 'by eye'. It is constructed by first noting the general trend, increasing or decreasing. A line (or curve) is then drawn through the middle of the scatter plot following that upwards or downwards trend, with roughly equal number of points above and below the line. The distance points lie from the line must also be taken into account.

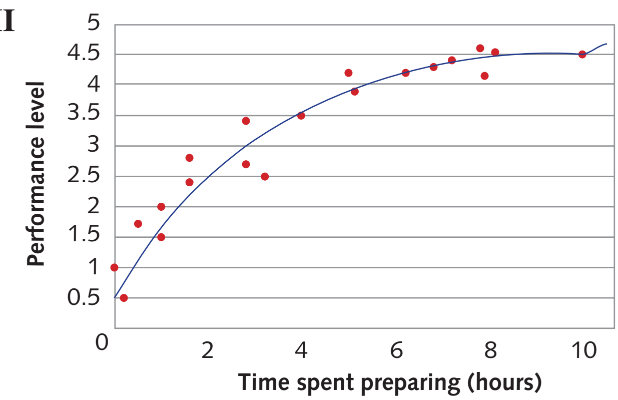
I



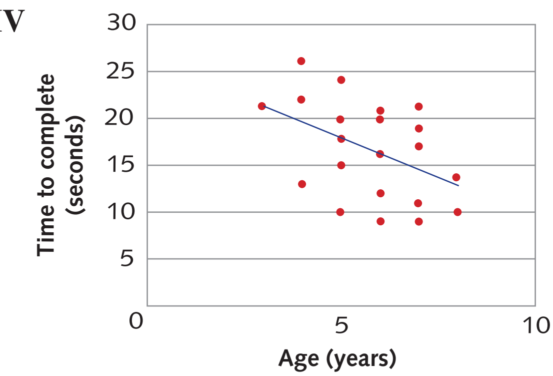
II



III



IV



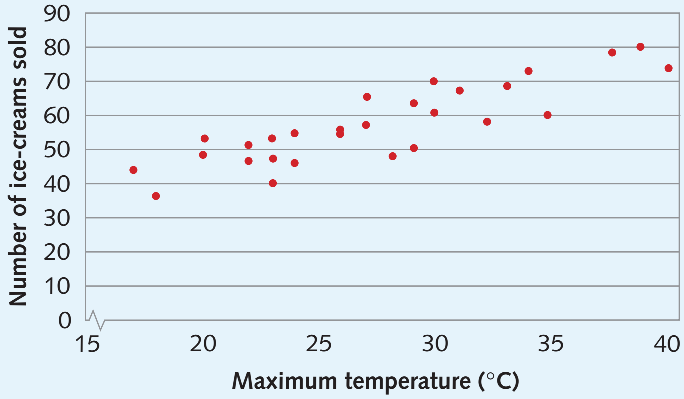
Observations

- Graphs I and III show an increasing trend whilst graphs II and IV show a decreasing trend.
- Graph II shows a strong linear relationship between the variables and all points are in close proximity to the line of best fit. However, graphs I and IV show moderately strong linear relationships between the variables.
- Graph III shows a non-linear relationship between variables and a 'curve' of best fit is suggested. The other graphs display a linear relationship.

In this section, only linear relationships will be studied. To determine the equation of the line of best fit we draw on skills that were introduced in Chapter 4.

Example 17

Consider the scatter plot below showing the relationship between ice-creams sold by vendor during the month of February and maximum temperature for the day.



a Draw a line of best fit by eye.

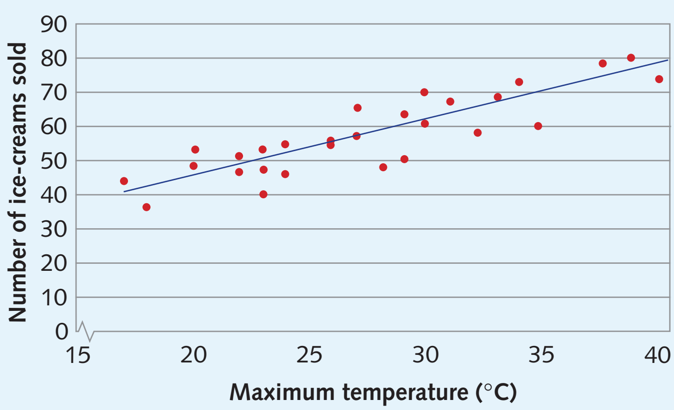
b Determine the equation of the line.

c Use the equation to predict the number of ice-creams the vendor will sell on a 35°C day.

d Use the equation to predict the maximum temperature of the day if the vendor sells 58 ice-creams.

Solution

a



Note: Small variations in the placement of the line of best fit is expected using this technique.

(continued over page)

b Use the point–gradient form, $y - y_1 = m(x - x_1)$, to find the equation of the line.
Note: The grid lines can assist you to find two points on the line. For improved accuracy, ensure they are not too close together.

Choose (34, 70) and (22, 50). (Other selections are possible.)

$$m = \frac{70-50}{34-22} = \frac{20}{12} = \frac{5}{3}$$
$$y - 50 = \frac{5}{3}(x - 22)$$
$$y = \frac{5}{3}x + 50 - \frac{110}{3}$$
$$y = \frac{5}{3}x + \frac{40}{3}$$

Interpreting this equation in the given context, we get;

$$\textit{Number of ice-creams sold} = \frac{5}{3} \times \left(\textit{maximum temperature } ^\circ\mathrm{C}\right) + \frac{40}{3}$$

c $$\textit{Number of ice-creams sold} = \frac{5}{3} \times 35 + \frac{40}{3}$$
$$= 71\tfrac{2}{3}$$
$$\approx 72 \text{ (Round up to the nearest integer.)}$$

d $$58 = \frac{5}{3} \times \left(\textit{maximum temperature } ^\circ\mathrm{C}\right) + \frac{40}{3}$$
$$174 = 5 \times \left(\textit{maximum temperature } ^\circ\mathrm{C}\right) + 40 \qquad \text{(multiplying all terms by 3)}$$
$$\therefore \textit{maximum temperature} = \frac{174-40}{5} = 26.8^\circ\mathrm{C}$$

Interpolation versus extrapolation

When we use the line of best fit to make predictions of values within the range of data already obtained it is called **interpolation**. In the example above, the predicted number of ice-creams sold, based on a maximum temperature of 35°C was interpolation. This is because 35°C lies between the minimum (17°C) and maximum (38°C) recorded temperatures. The same can be said for predicting the maximum temperature based on a sale of 58 ice-creams.

Extrapolation is the term used for making predictions outside the range of values already obtained. Extrapolation should be performed with a degree of caution, since there is no guarantee the noted relationship between variables will continue beyond the observed range.

For example, using the equation, *number of ice-creams sold* $= \frac{5}{3} \times$ (*maximum temperature* °C) $+ \frac{40}{3}$, to predict the maximum temperature when 20 ice-creams are sold is an act of extrapolation. The predicted maximum temperature of 4°C may not be feasible.

Lines of best fit by other techniques

You may have noticed that creating a line of best fit by eye is prone to variation and discrepancy. This is not desirable if we need to be consistent and accurate with fitting a line to data. Fortunately, there are several alternative approaches to drawing a line of best fit. The approach commonly used is called the least squares method.

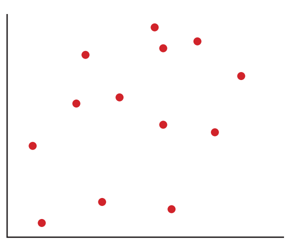
Line of best fit

- Drawing a line of best fit by eye consists of tracing the trend of the scatter plot with a straight line, ensuring that there are roughly equal numbers of points above and below the line, with distance of points from the line taken into account.
- Once two points have been identified on the straight line, the equation of the line can be determined using the point–gradient form.
- Interpolation is making predictions using data that lies within the range of observed values.
- Extrapolation is making predictions using data that lies outside the range of observed values. Caution must be used when predicting values based on extrapolation.

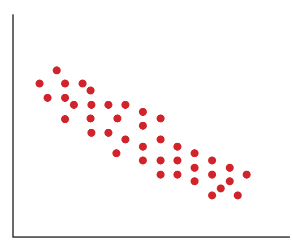
Exercise 18H

1 Copy these scatter plots and draw a line of best fit by eye though each.

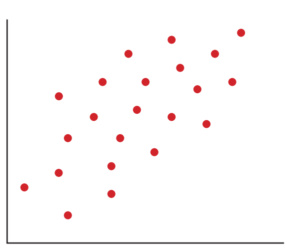
i



ii



iii



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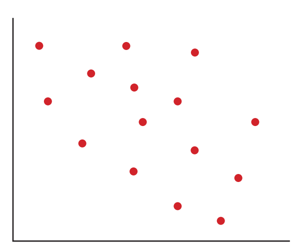


2 In the scatter plots in question **1**, comment on the following.

a Do the scatter plots display an increasing or decreasing trend?

b What is the strength of the relationships between y and x?

3 Data was collected on 100 adults comparing shoe size and height. Shoe sizes ranged from 6 to 13. An equation relating height (in cm) to shoe size was determined to be:

$$\textit{height} = 127.18 + 4.84 \times \textit{shoe size}$$

Use this equation to predict (to the nearest cm) the height of a person whose shoe size is as follows. Are you interpolating or extrapolating?

a size 7 **b** size 12 **c** size 14

4 A line of best fit for a scatter plot, relating the weight of a pumpkin (kg) to the number of seeds it contains, was found to pass through the points $(1, 300)$ and $(7, 540)$. Assume weight is on the x-axis.

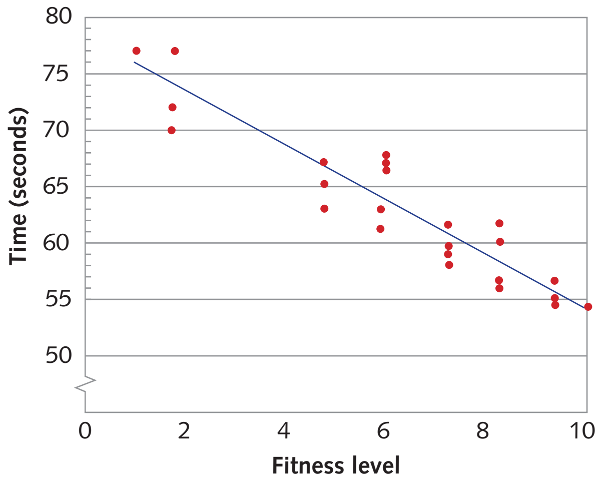
a Find the equation of the line of best fit.

b Use your equation to estimate the number of seeds a pumpkin contains that weighs 5.2 kg.

c Use your equation to estimate the weight of a pumpkin containing 600 seeds.

Example 17

5 A class of Year 10 PE students were asked to run a lap of the school's oval. Their times were recorded and compared against their fitness levels, which had been previously analysed and placed on a scale of 1 to 10. The teacher then drew a line of best fit over the scatter plot as shown.



a Determine the equation of the line of best fit.

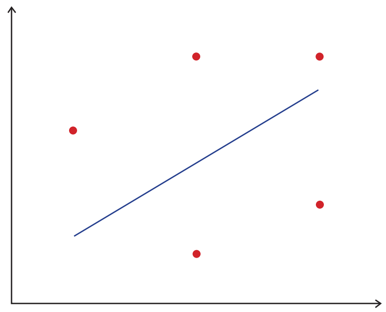
b Use the equation to predict the time it would take a Year 10 PE student to run a lap of the oval if that student has a fitness level of 3. Leave your answer correct to one decimal place.

c Use the equation to predict the fitness level of a Year 10 PE student if a lap of the oval is run in 62 seconds.

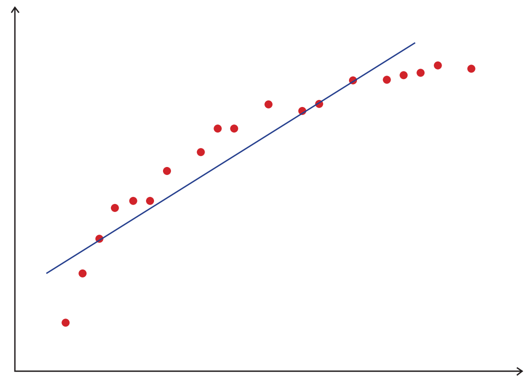
d Are these predictions examples of interpolation or extrapolation? Explain your answer.

6 State the problems with making predictions using the lines of best in the following scatter plots.

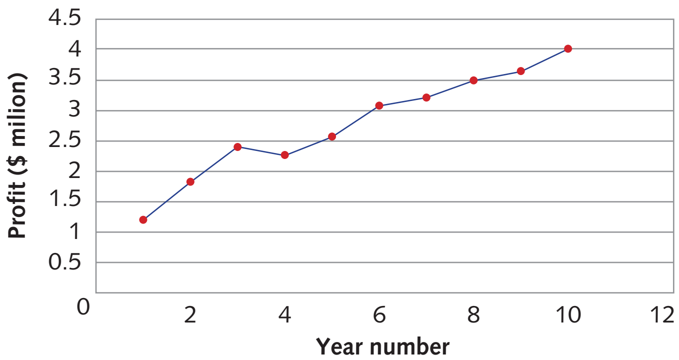
a



b



7 Consider the time series below, showing a company's profit for consecutive financial years over a 10 year period. 'Year 1' marks the financial year 1988–1989, 'Year 2' marks the financial year 1989–1990, and so on. 'Year 10' marks the financial year 1997–1998.



Create a line of best fit on the time series and use it to predict the company's profits, to the nearest $100 000, in the financial year 1998–1999. (Predicting future values in a time series based on previously observed values is called **forecasting**.) Is your answer an example of interpolation or extrapolation?

Review exercise

1 The stem-and-leaf plot on the right gives the times for which a class of 26 Year 10 students ran 100 m.

a What is the range of times to run 100 m in the class?

b What is the median time to run 100 m in the class?

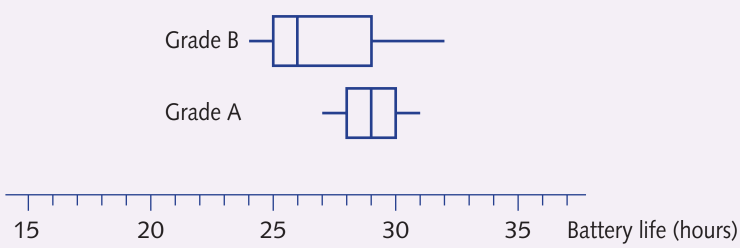
c What is the interquartile range?

d Would the median time change if the fastest and slowest times were removed?

11	5
12	3 4 6 9
13	0 0 2 6 8
14	0 1 2 4 7 9 9
15	1 2 4 5 5 5
16	3 4
17	
18	2

15|1 means 15.1 seconds

2 The 'life' of alkaline batteries is compared through continuous use in a standard product. 40 Grade A and 40 Grade B batteries are tested in this way. Their results are shown in the two boxplots below.



a State the median battery life for the Grade A and Grade B batteries.

b State the range in battery life for the Grade A and Grade B batteries.

c State the interquartile range for the Grade A and Grade B batteries.

d Determine the number of Grade A and Grade B batteries lasting longer than 29 hours.

e Describe the shape of data distributions for the Grade A and Grade B battery life.

f Under what criterion is the Grade B battery 'better' than the Grade A battery in this test?

3 The following data are the speeds of 45 semi-trailers passing a given point on an interstate highway. The speeds are measured in km/h.

88	90	93	94	95	96	98	100	100	100	100	100
101	102	102	102	103	103	103	104	105	106	106	107
109	109	110	110	110	112	113	114	116	117	118	120
120	121	128	130	130	139	141	144	150			

a Construct a dotplot of the data.

b Construct a boxplot of the data.

c Comment on the shape.

4 The number of times 35 randomly chosen Year 10 students go online in the course of a school day was recorded. The results are shown in the frequency table below.

Times online	0	1	2	3	4	5	6	7
Number of students	8	3	5	6	7	5	0	1

a Calculate the mean number of times students in this random sample go online.

b Find the standard deviation of the number of times students go online, correct to two decimal places.

c Find the range of times online that lie within one standard deviation of the mean.

d If every student in this sample went online one more time than what was recorded, determine the effect on the mean and standard deviation.

5 Kathryn scored 78% on both her history and mathematics tests. Both tests had a class mean of 70%, but history had a standard deviation of 8% and mathematics had a standard deviation of 12%. In which test did Kathryn perform better relative to the rest of the class?

6 The table below gives the quarterly sales figures for a Melbourne swimwear shop in the period 2014–2016.

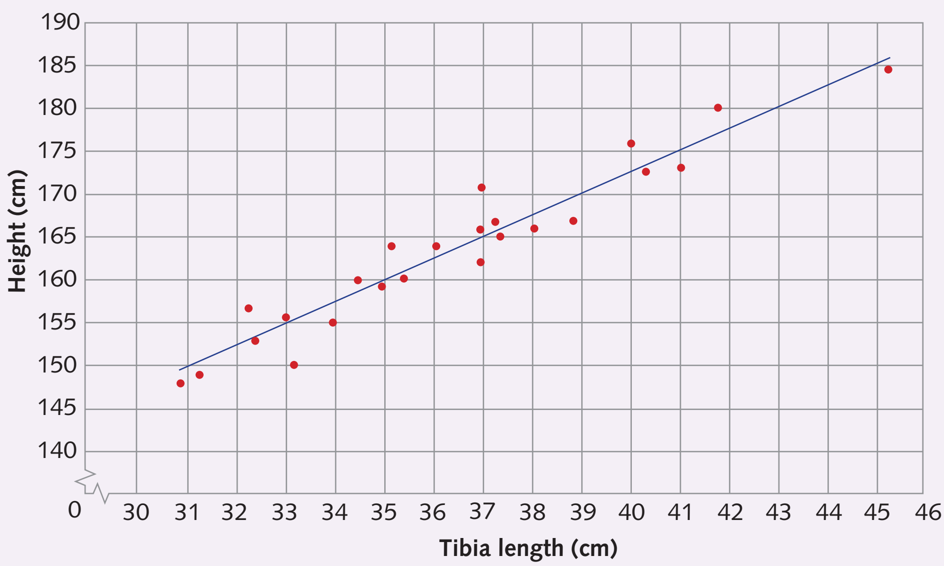
Sales $'000	January–March	April–June	July–September	October–December
2014	33	16	5	21
2015	35	19	8	26
2016	44	22	10	30

a Represent this information on a time-series plot. (Use numbers 1 to 12 to mark the quarters.)

b In which quarter of each year are the sales figures the best?

c Describe briefly how the quarterly sales figures change over time. Are the sales figures improving?

7 In an all-female class of Year 10 students, the length of each student's tibia (shin bone) and height (in centimetres) was recorded and graphed below. A line of best fit was drawn.



a Determine the equation of the line of best fit.

b Use the equation to predict the height of a Year 10 female with tibia length of 44 cm.

c Use the equation to predict the tibia length of a 145 cm tall Year 10 female.

d Are these predictions examples of interpolation or extrapolation? Explain your answer.

CHAPTER

19

Measurement and Geometry

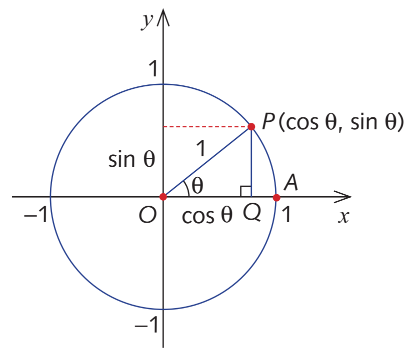
Trigonometric functions

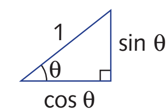
In Chapter 12, we saw how to extend the definition of the **trigonometric functions** to the second quadrant so that we could deal with obtuse-angled triangles. You probably realised that the ideas could be further extended so that we could give meaning to the trigonometric ratios of angles that were greater than 180°. We will do that in this chapter, and we will also draw the graphs of the trigonometric functions for all positive and negative angle sizes.

The graphs of sine and cosine functions are used to model wave motion and are therefore central to the applications of mathematics to any problem in which periodic motion is involved – from the motion of the tides and ocean waves to sound waves and modern telecommunications.

19A Angles in the four quadrants

We take a circle of radius 1, centre the origin, in the Cartesian plane.





From point P on the circle in the first quadrant, we construct the right-angled triangle POQ with O at the origin. Let $\angle POQ$ be θ.

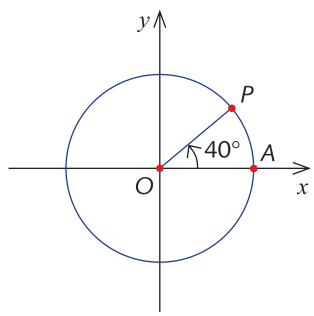
The length OQ is the x-coordinate of P, and since $\dfrac{OQ}{1} = \cos\theta$, the x-coordinate of P is $\cos\theta$.

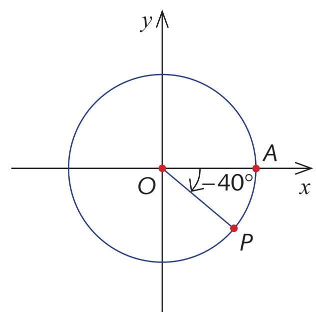
Similarly, the y-coordinate of P is the length PQ, which equals $\sin\theta$.

Hence, the coordinates of the point P are $(\cos\theta, \sin\theta)$.

Positive and negative angles

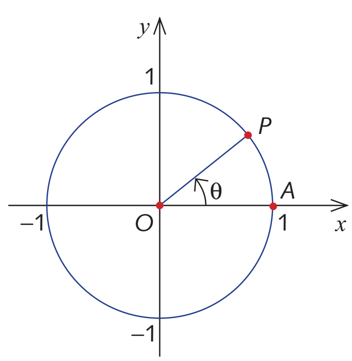
In this chapter, angles measured anticlockwise from OA will be called **positive angles**. Similarly, angles measured clockwise from OA will be called **negative angles**.



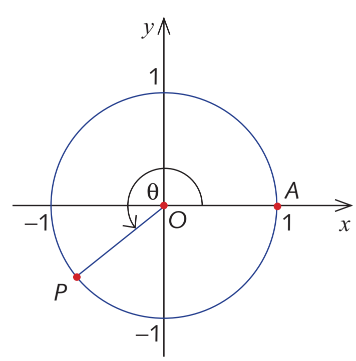


The definition of sine and cosine

Notice that each angle, positive or negative, determines a point, P, on the unit circle. For the moment we will only deal with positive angles between 0° and 360°.



$0° < \theta < 90°$



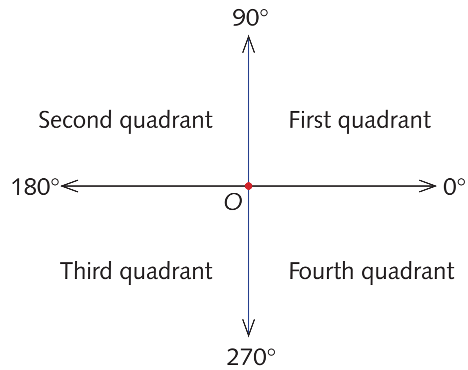
$180° < \theta < 270°$

For point P, determined by the angle θ, we define:

- the **cosine** of θ to be the x-coordinate of the point P
- the **sine** of θ to be the y-coordinate of the point P.

The four quadrants

The coordinate axes cut the plane into four quadrants. These are labelled anticlockwise around the origin, as the first, second, third and fourth quadrants.

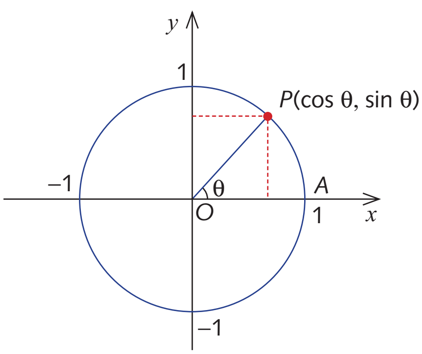


The signs of sin θ and cos θ

First quadrant

For θ in the first quadrant ($0° < \theta < 90°$):

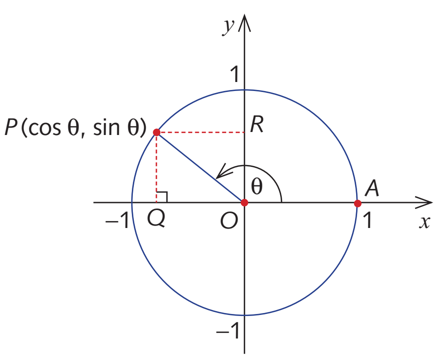
- the x-value is positive, so cos θ is positive
- the y-value is positive, so sin θ is positive.



Second quadrant

For θ in the second quadrant ($90° < \theta < 180°$):

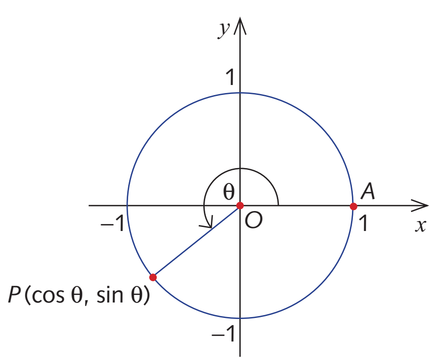
- the x-value is negative, so cos θ is negative
- the y-value is positive, so sin θ is positive.



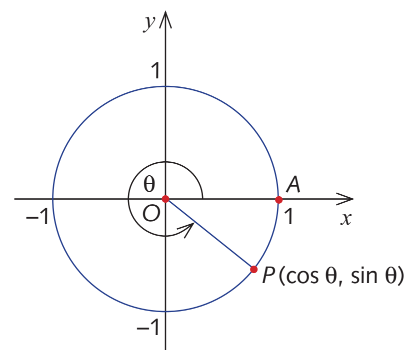
Third quadrant

For θ in the third quadrant ($180° < \theta < 270°$):

- the x-value is negative, so cos θ is negative
- the y-value is negative, so sin θ is negative.



Fourth quadrant

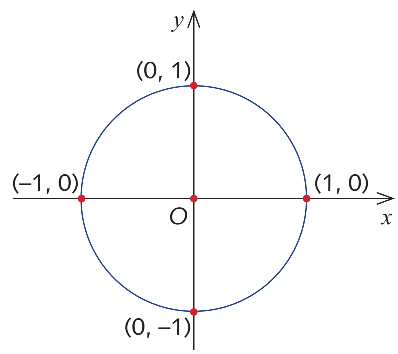


For θ in the fourth quadrant ($270° < \theta < 360°$):

- the x-value is positive, so cos θ is positive
- the y-value is negative, so sin θ is negative.

The angles $\theta = 0°, 90°, 180°$ and $270°$ correspond to the points (1, 0), (0, 1), (–1, 0) and (0, –1).
Using the coordinates of these points and the definition of sin θ and cos θ, we construct the following table.

P	θ	cos θ	sin θ
(1, 0)	0°	1	0
(0, 1)	90°	0	1
(–1, 0)	180°	–1	0
(0, –1)	270°	0	–1



Notes:

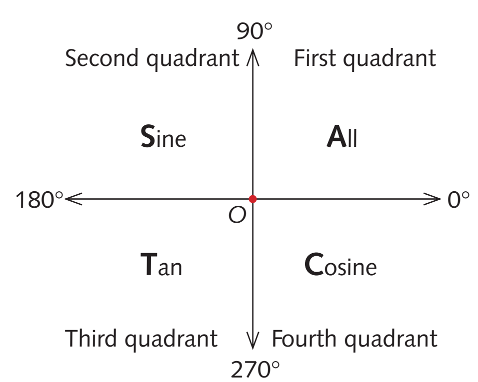
- $-1 \le \sin\theta \le 1$ for all θ
- $-1 \le \cos\theta \le 1$ for all θ
- In the first quadrant ($0° < \theta < 90°$):
  - As θ increases, sin θ increases and cos θ decreases
  - For any value a such that $0 < a < 1$, there is a *unique value* of θ such that $\sin\theta = a$. A similar statement holds for cosine.

The tangent ratio

For acute angles, we know that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}\left(= \dfrac{\text{opposite}}{\text{adjacent}}\right)$. For angles that are greater than 90°, we can define the tangent of θ by $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, where $\theta \ne 90°, 270°$.

This means that tan θ will be positive in quadrants where sin θ and cos θ are both positive or both negative. Hence, tan θ is positive in the first and third quadrants, and negative in the second and fourth quadrants.

To assist in remembering the signs of the three trigonometric functions in the various quadrants, notice that only one ratio is *positive* in the second, third and fourth quadrants. Hence, we can remember the signs by the diagram:



In the diagram, the bold letters tell you which ratio is positive in the given quadrant. The letters can be remembered by the mnemonic:

All **S**tations **T**o **C**entral

You can also remember the signs by just thinking about the coordinates of the point P on the unit circle corresponding to the given angle, since $\sin\theta$ is the y-coordinate and $\cos\theta$ is the x-coordinate.

It is often useful to draw a diagram showing the angle when calculating values of sine, cosine and tangent.

Example 1

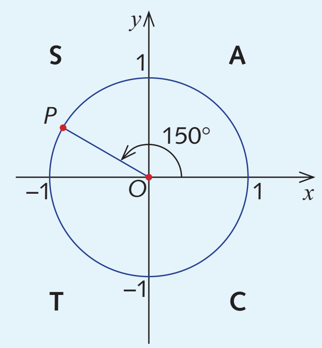
Draw a diagram and state the sign of the given ratio.

a $\sin 150°$ **b** $\tan 300°$ **c** $\cos 210°$

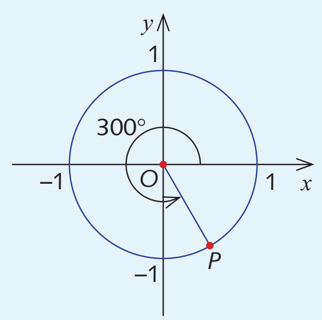
Solution

a The angle 150° lies in the second quadrant, hence $\sin 150°$ is positive.

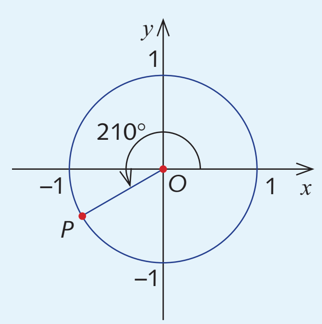
(Alternatively, P is above the x-axis, so $\sin\theta$, which is the y-coordinate of P, is positive.)



b The angle 300° lies in the fourth quadrant, hence $\tan 300°$ is negative.



c The angle 210° lies in the third quadrant, hence $\cos 210°$ is negative.



We can use a calculator to find the approximate numerical value of the trigonometric function of a given angle. Make sure that your calculator is in degree mode.

Example 2

Use the calculator to find, to four decimal places:

a $\sin 100°$ **b** $\tan 320°$ **c** $\cos 200°$

Solution

From the calculator:

a $\sin 100° \approx 0.9848$

(The angle 100° is in the second quadrant so sin 100° is positive.)

b $\tan 320° \approx -0.8391$

(The angle 320° is in the fourth quadrant so tan 320° is negative.)

c $\cos 200° \approx -0.9397$

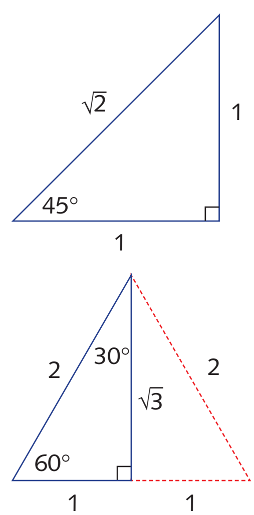
(The angle 200° is in the third quadrant so cos 200° is negative.)

Exact values

You should recall the following two triangles. From these you can read off the exact values of sine, cosine and tangent of 30°, 45° and 60°. These were derived in Section 12B of this book.

Alternatively, knowing, for example, that $\cos 60° = \frac{1}{2}$ and $\tan 45° = 1$, you can easily reconstruct the table.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$



These results can be used to determine the exact trigonometric functions for certain angles greater than 90°.

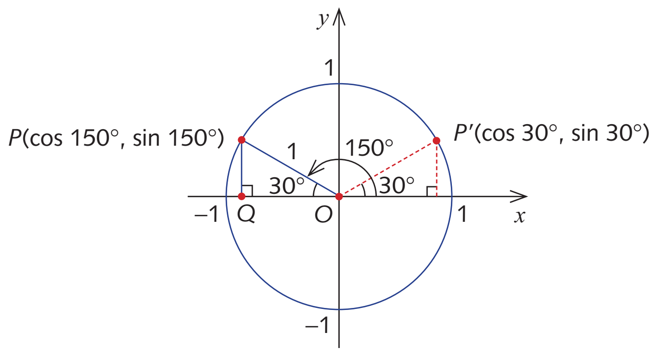
To find the value of sine and cosine for any θ, we introduce the concept of the **related angle**, which is always acute.

The related angle

When θ drives the point P on the unit circle into the second, third or fourth quadrant, the acute $\angle POQ$ makes an angle with the x-axis called the **related angle**.

The second quadrant

We begin by finding the exact value of $\cos 150°$ and $\sin 150°$.



The angle 150° corresponds to the point P in the second quadrant, as shown.

The coordinates of P are $(\cos 150°, \sin 150°)$.

The angle POQ is 30° and is called the **related angle** for 150°.

When we reflect the point P in the y-axis, we get the point $P'(\cos 30°, \sin 30°)$.

From ΔPOQ, we can see that $OQ = \cos 30°$ and $PQ = \sin 30°$, so the coordinates of P are $(-\cos 30°, \sin 30°)$.

Hence, $\cos 150° = -\cos 30° = -\dfrac{\sqrt{3}}{2}$ and $\sin 150° = \sin 30° = \dfrac{1}{2}$

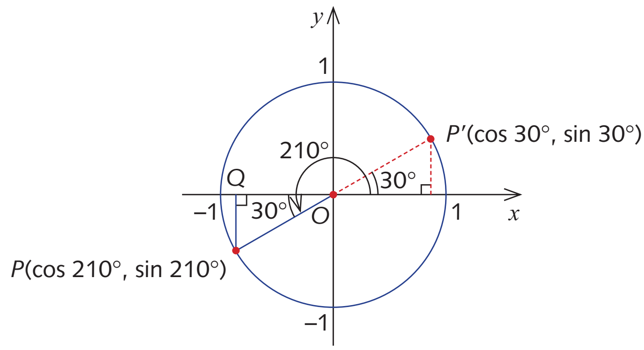
In general, if θ lies in the second quadrant, $180° - \theta$ is the **related angle** for θ.

The third quadrant

Next, we find the exact value of $\cos 210°$ and $\sin 210°$. The corresponding point P lies in the third quadrant. The coordinates of P are $(\cos 210°, \sin 210°)$. The angle POQ is 30° and is called the **related angle** for 210°.

So, $\cos 210° = -\cos 30° = -\dfrac{\sqrt{3}}{2}$

and $\sin 210° = -\sin 30° = -\dfrac{1}{2}$



When we rotate point P around O by 180°, we get the point $P'(\cos 30°, \sin 30°)$.

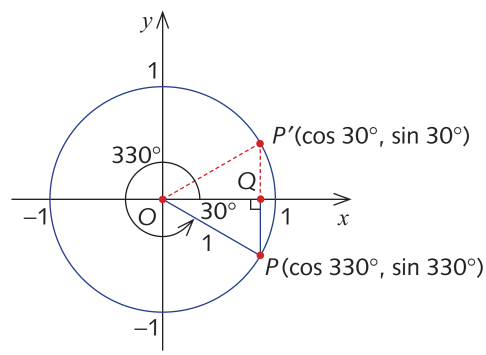
In general, if θ lies in the third quadrant, $\theta - 180°$ is the **related angle** for θ.

The fourth quadrant

Next, we find the exact value of $\cos 330°$ and $\sin 330°$. The corresponding point P lies in the fourth quadrant. The related angle is $360° - 330° = 30°$.

So $\cos 330° = \cos 30° = \dfrac{\sqrt{3}}{2}$

$\sin 330° = -\sin 30° = -\dfrac{1}{2}$



When we reflect point P in the x-axis, we get the point $P'(\cos 30°, \sin 30°)$.

In general, if θ lies in the fourth quadrant, $360° - \theta$ is the **related angle** of θ.

Trigonometric functions of angles

To find the trigonometric function of an angle, θ, between 0° and 360°:

- Find the **related angle** for θ, the acute angle between OP and the x-axis.
- Obtain the **sign** of the trigonometric function using, for example, the ASTC picture.
- Evaluate the trigonometric function of the related angle, and attach the appropriate sign.

In the next two examples, the sign of the trigonometric function will be determined using different approaches.

Example 3

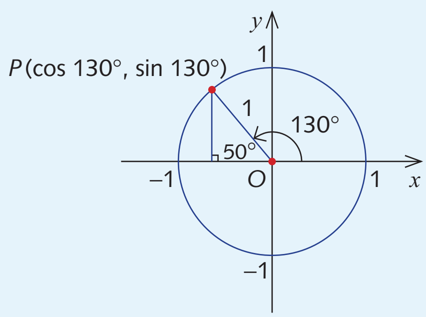
Without evaluating, express each number as the trigonometric function of an acute angle.

a sin 130° **b** cos 200° **c** tan 325° **d** sin 235°

Solution

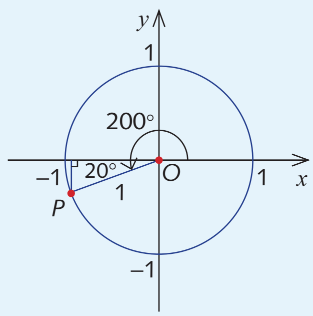
a The related angle is:
180° − 130° = 50°

The angle 130° is in the second quadrant,
so sin 130° = sin 50°



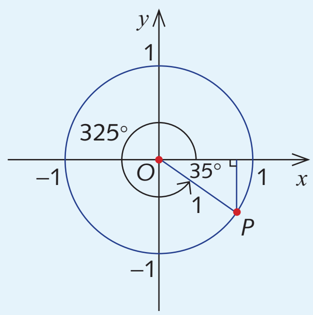
b The related angle is:
200° − 180° = 20°

The angle 200° is in the third quadrant,
so cos 200° = −cos 20°



c The related angle is:
360° − 325° = 35°

The angle 325° is in the fourth quadrant,
so tan 325° = −tan 35°



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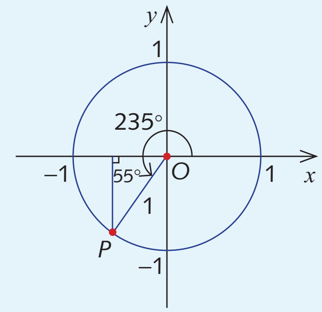
d The related angle is:

$235° - 180° = 55°$

The angle $235°$ is in the third quadrant, so $\sin 235° = -\sin 55°$



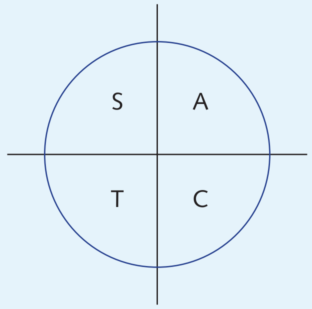
Example 4

Use the related angle to find the exact value of:

a $\sin 120°$ **b** $\cos 150°$ **c** $\tan 300°$ **d** $\cos 240°$

Solution

Recall:



a The related angle is $180° - 120° = 60°$.

Sine is positive in the second quadrant.

So, $\sin 120° = \sin 60°$

$= \frac{\sqrt{3}}{2}$

b The related angle is $180° - 150° = 30°$.

Only sine is positive in the second quadrant.

So, $\cos 150° = -\cos 30°$

$= \frac{-\sqrt{3}}{2}$

c The related angle is $360° - 300° = 60°$.

Only cosine is positive in the fourth quadrant.

So, $\tan 300° = -\tan 60°$

$= -\sqrt{3}$

d The related angle is $240° - 180° = 60°$.

Only tangent is positive in the third quadrant.

So, $\cos 240° = -\cos 60°$

$= -\frac{1}{2}$

Exercise 19A

Example 1

1 State which quadrant each angle is in.

a $120°$ **b** $225°$ **c** $240°$ **d** $300°$

e $135°$ **f** $263°$ **g** $172°$ **h** $310°$

Example 3

2 Without evaluating, express each number as the trigonometric function of an acute angle.

a $\sin 170°$ **b** $\cos 170°$ **c** $\tan 170°$

d $\sin 190°$ **e** $\cos 190°$ **f** $\tan 190°$

g $\sin 350°$ **h** $\cos 350°$ **i** $\tan 350°$

Example 4

3 Find the exact value of:

a $\sin 135°$ **b** $\cos 225°$ **c** $\tan 120°$

d $\tan 135°$ **e** $\sin 300°$ **f** $\cos 330°$

g $\tan 300°$ **h** $\sin 150°$ **i** $\cos 135°$

4 Which quadrant does θ lie in if:

a $\cos\theta > 0$ and $\sin\theta < 0$?

b $\cos\theta < 0$ and $\sin\theta > 0$?

c $\cos\theta < 0$ and $\sin\theta < 0$?

d $\cos\theta < 0$ and $\tan\theta > 0$?

e $\cos\theta < 0$ and $\tan\theta < 0$?

f $\sin\theta < 0$ and $\tan\theta > 0$?

5 a Draw the unit circle and mark the point P at $(1, 0)$. Use your diagram to complete the following.

$\cos 0° = ...$ $\sin 0° = ...$ $\tan 0° = ...$

b Repeat with P at $(0, 1)$ to complete the following.

$\cos 90° = ...$ $\sin 90° = ...$

c Repeat with P at $(-1, 0)$ to complete the following.

$\cos 180° = ...$ $\sin 180° = ...$ $\tan 180° = ...$

d Repeat with P at $(0, -1)$ to complete the following.

$\cos 270° = ...$ $\sin 270° = ...$

e What are the values of $\cos 360°$, $\sin 360°$ and $\tan 360°$?

f Why are $\tan 90°$ and $\tan 270°$ not defined?

6 Without using a calculator, find the exact value of:

a $\sin 90° \times \tan 135° \times \cos 135°$ **b** $\sin 330° \times \cos 360°$

c $\sin 360° \times \cos 330°$ **d** $2 \times \sin 135° \times \cos 135°$

e $\cos 225° \times \tan 180° + \sin 225° \times \sin 90°$ **f** $3\sin 240° - 2\cos 300°$

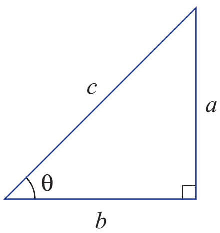
7 We use the notation $\sin^2\theta$ to mean $(\sin\theta)^2$, $\cos^2\theta$ to mean $(\cos\theta)^2$ and $\tan^2\theta$ to mean $(\tan\theta)^2$. This is the standard notation. Find the exact values of:

a $\sin^2 30°$ **b** $\cos^2 30°$ **c** $\tan^2 30°$

d $\sin^2 300°$ **e** $\tan^2 240°$ **f** $\cos^2 210°$

g $\sin^2 225° + \cos^2 225°$ **h** $\sin^2 330° + \cos^2 330°$

8 **a** Suppose that θ is an acute angle.
Use the diagram to show that $\cos^2\theta + \sin^2\theta = 1$.



b Explain why this result remains true when θ lies in the second quadrant.

c What happens in the other quadrants?

d Check that this result holds for $0°, 90°, 180°$ and $270°$.

9 The reciprocals of the sine, cosine and tangent functions are also important and are given the following names.

$\dfrac{1}{\sin\theta}$ is called the **cosecant** of θ and written as $\operatorname{cosec}\theta$.

$\dfrac{1}{\cos\theta}$ is called the **secant** of θ and written as $\sec\theta$.

$\dfrac{1}{\tan\theta}$ is called the **cotangent** of θ and written as $\cot\theta$.

Find the exact value of:

a $\sec 30°$ **b** $\cot 45°$ **c** $\operatorname{cosec} 60°$ **d** $\sec 120°$

e $\operatorname{cosec} 210°$ **f** $\cot 240°$ **g** $\cot 300°$ **h** $\sec 330°$

10 **a** Show $\tan^2\theta + 1 = \sec^2\theta$ for an acute angle.

b What happens for all angles between $0°$ and $360°$?

19B Finding angles

Since $\cos 120° = -\frac{1}{2}$ and $\cos 240° = -\frac{1}{2}$, there are two angles between $0°$ and $360°$ whose cosine is $-\frac{1}{2}$; they are $120°$ and $240°$.

In this section, we will learn how to find all angles in the range $0°$ to $360°$ that have a given trigonometric function. While the calculator is useful here, it will only give you one value of θ, when in general there are two. For example, a calculator gives $\cos^{-1}\left(-\frac{1}{2}\right)$ is $120°$, whereas the two solutions to $\cos\theta = -\frac{1}{2}$, for the range $0°$ to $360°$, are $\theta = 120°$ and $\theta = 240°$.

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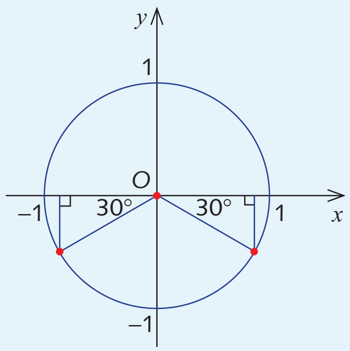
Example 5

Find all angles θ, in the range 0° to 360°, such that:

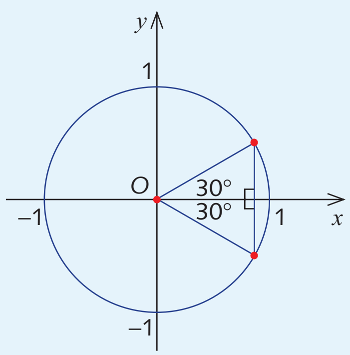
a $\sin\theta = -\dfrac{1}{2}$ **b** $\cos\theta = \dfrac{\sqrt{3}}{2}$ **c** $\tan\theta = -0.3640$

Solution

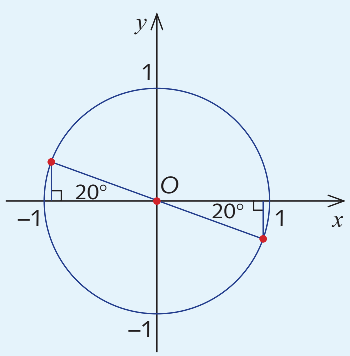
a The given value of sine is negative, so θ lies in the third or fourth quadrant. The related angle whose sine is $\frac{1}{2}$ is 30°. Hence, $\theta = 180° + 30°$ or $\theta = 360° - 30°$.
That is, $\theta = 210°$ or $330°$.
Note: Entering $\sin^{-1}\left(-\frac{1}{2}\right)$ into a calculator gives –30°. This is not in the range 0° to 360°.



b The given value of cosine is positive, so θ lies in the first or fourth quadrant. The acute angle whose cosine is $\frac{\sqrt{3}}{2}$ is 30°. Hence, $\theta = 30°$ or $\theta = 360° - 30°$.
That is, $\theta = 30°$ or $330°$.



c The given value of tangent is negative, so θ lies in the second or fourth quadrant. To find the acute angle whose tangent is 0.3640, enter $\tan^{-1} 0.3640$ into your calculator to obtain, approximately, 20°. Hence, to the nearest degree, $\theta \approx 180° - 20°$ or $\theta \approx 360° - 20°$. That is, $\theta \approx 160°$ or $340°$.



Note: In part **c** we find $\tan^{-1} 0.3640$ on the calculator and *not* $\tan^{-1}(-0.3640)$. Work from the related angle and then shift to the correct quadrant.

Finding angles

To find all angles from 0° to 360° that have a given value of a trigonometric function:

- use a circle diagram or the ASTC picture to work out which quadrant the angles are in
- find the related angle using a calculator when exact values are not given
- find all angles.

Exercise 19B

Example 5

1 Without using your calculator, find the angles θ, between 0° and 360° inclusive, for which (draw a diagram in each case):

a $\sin\theta = \frac{1}{2}$ **b** $\tan\theta = \sqrt{3}$ **c** $\cos\theta = \frac{1}{\sqrt{2}}$

d $\cos\theta = -\frac{1}{2}$ **e** $\sin\theta = -\frac{\sqrt{3}}{2}$ **f** $\tan\theta = 1$

2 Without using your calculator, find the angles θ, between 0° and 360° inclusive, for which:

a $\sin\theta = -\frac{1}{\sqrt{2}}$ **b** $\tan\theta = -\frac{1}{\sqrt{3}}$ **c** $\cos\theta = -\frac{\sqrt{3}}{2}$

d $\sin\theta = 1$ **e** $\cos\theta = 0$ **f** $\tan\theta = 0$

3 Draw a diagram first, and then, using a calculator, find to the nearest degree the angles θ, between 0° and 360° inclusive, such that:

a $\sin\theta = 0.1736$ **b** $\cos\theta = -0.9063$ **c** $\tan\theta = 2.1445$

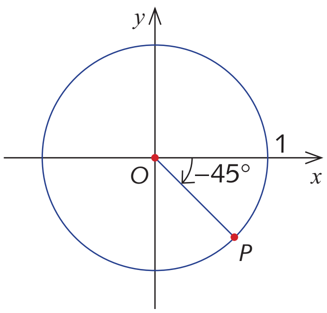
d $\sin\theta = -0.7986$ **e** $\cos\theta = 0.8090$ **f** $\tan\theta = -3.4874$

g $\cos\theta = -0.9455$ **h** $\tan\theta = 0.4245$ **i** $\sin\theta = -0.9781$

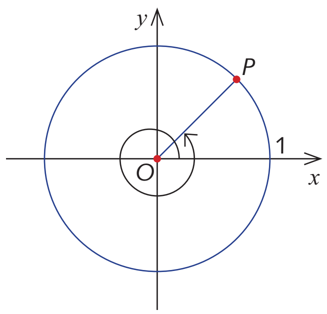
19C Angles of any magnitude

Angles greater than 360° and less than 0° arise naturally. If you turn three times in an anticlockwise direction, then you have turned through an angle of 1080°.

If you make a quarter turn in a clockwise direction, then we can think of this as an angle of –90°. The diagram shows an angle of –45°.



Since the sine and cosine of an angle are the y- and x-coordinates of the corresponding point P, $\sin(\theta + 360°) = \sin\theta$, $\cos(\theta + 360°) = \cos\theta$, $\sin(\theta - 360°) = \sin\theta$ and $\cos(\theta - 360°) = \cos\theta$.



It is clear that for any angle greater than 360° or less than 0°, the corresponding point P on the unit circle can be equivalently described by an angle between 0° and 360°.

Hence, to find the trigonometric function of an angle greater than 360°, we subtract a multiple of 360° to arrive at an angle between 0° and 360°.

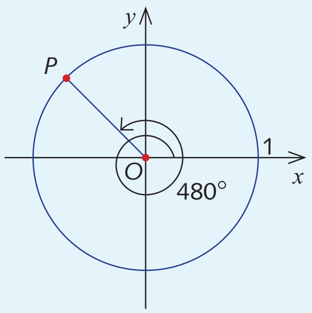
Similarly, to find the trigonometric function of a negative angle, we add a multiple of 360° to arrive at an angle between 0° and 360°.

Example 6

Find sin 480° in surd form.

Solution

$$
\begin{aligned}
\sin 480° &= \sin(480° - 360°) \\
&= \sin 120° && \text{(120° lies in the second quadrant.)} \\
&= \sin 60° && \text{(The related angle is 60°.)} \\
&= \frac{\sqrt{3}}{2}
\end{aligned}
$$

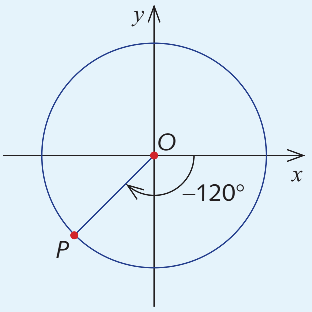


Example 7

Find cos (−120°) in surd form.

Solution

$$
\begin{aligned}
\cos(-120°) &= \cos(-120° + 360°) \\
&= \cos 240° && \text{(240° lies in the third quadrant.)} \\
&= -\cos 60° && \text{(The related angle is 60°.)} \\
&= -\frac{1}{2}
\end{aligned}
$$



Note: We are careful to distinguish clearly between an angle and its trigonometric function. For example, the angles 480° and 120° are *different* but their trigonometric functions are the *same*.

Exercise 19C

1 Draw a diagram representing each angle.

a 390° b 540° c 720° d 940°

2 Draw a diagram representing each angle.

a –150° b –330° c –720° d –540°

3 State the related angle for each angle in question **1**.

4 State the related angle for each angle in question **2**.

Example 6

5 Find, in surd form:

a sin 540° b cos 540° c tan 540° d sin 390°

e cos 840° f tan 480° g cos 660° h sin 405°

Example 7

6 Find the exact value of:

a sin (–60°) b cos (–135°) c tan (–225°)

d cos (–240°) e sin (–330°) f sin (–390°)

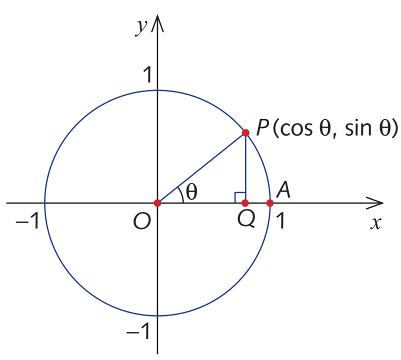
7 Find the exact value of:

a sin 720° b cos 720° c cos 450° d tan (–360°)

e sin (–270°) f cos (–90°) g tan (–180°)

19D The trigonometric functions and their symmetries

As usual, P is a point on the unit circle where PO makes an angle θ with OA.



As the angle θ varies from 0° to 90°, the length PQ, which equals $\sin\theta$, varies from 0 to 1. As θ varies from 0° to 360°, we can summarise the change in $\sin\theta$ by the following table.

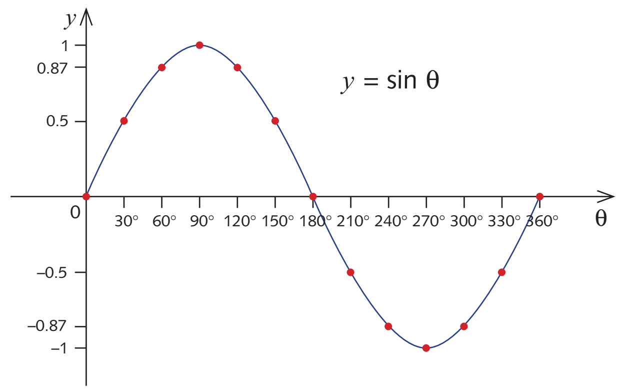
As θ increases from:	sin θ:
0° to 90°	increases from 0 to 1
90° to 180°	decreases from 1 to 0
180° to 270°	decreases from 0 to –1
270° to 360°	increases from –1 to 0

The way $\sin\theta$ increases and decreases can be represented graphically.

Using the values $\sin 30^\circ = \frac{1}{2} = 0.5$ and $\sin 60^\circ = \frac{\sqrt{3}}{2} \approx 0.87$, we can draw up the following table of values and then plot them.

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
sin θ	0	0.5	0.87	1	0.87	0.5	0	–0.5	–0.87	–1	–0.87	–0.5	0

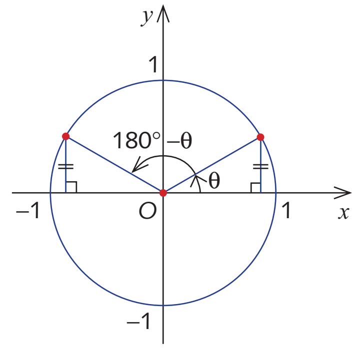
More points can be used to show that the shape is as shown in the following graph.

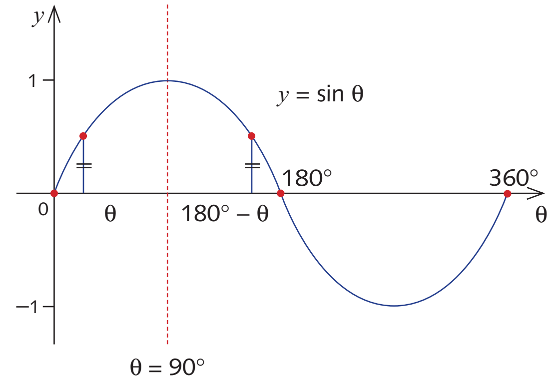


Electrical engineers and physicists call this a wave.

Symmetries

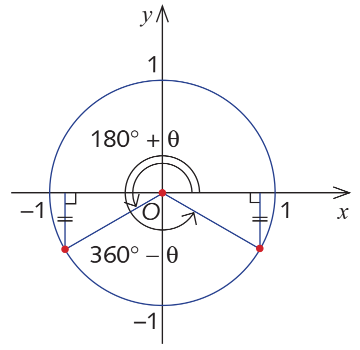
We have seen that if θ is between 0° and 90°, then $\sin\theta = \sin(180^\circ - \theta)$. The identity is shown by the equal intervals in the unit circle and related graph of $y = \sin\theta$ below.

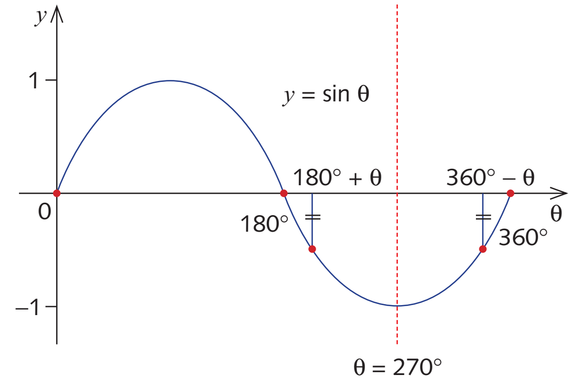




Hence, between 0° and 180°, the graph is symmetric about $\theta = 90^\circ$.

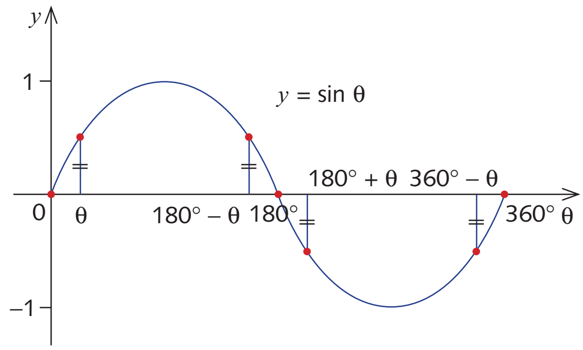
Similarly, for θ between 0° and 90°, $\sin(180^\circ + \theta) = \sin(360^\circ - \theta)$.





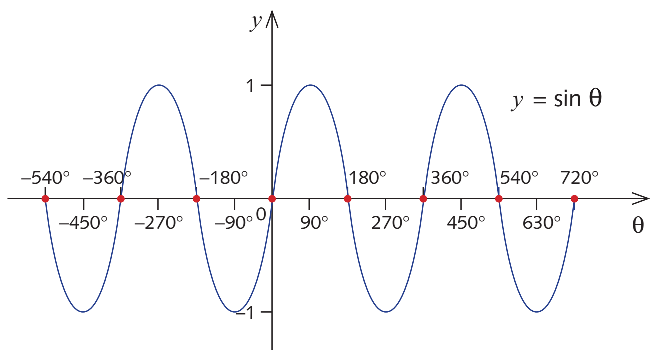
Hence, between 180° and 360°, the graph is symmetric about $\theta = 270^\circ$.

All intervals on the previous page are equal in magnitude since $\sin(360 - \theta) = -\sin\theta$. Therefore, we can summarise these observations in one diagram.



Extending the graph

We saw in Section 19C that the values of $\sin\theta$ remain the same when θ is increased or decreased by 360°. That is, $\sin\theta = \sin(\theta + 360°)$. Hence, the graph of $\sin\theta$ can be drawn for angles greater than 360° and less than 0°, as shown.

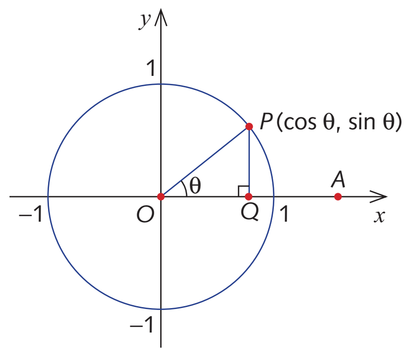


$\sin\theta$ is **periodic** and we call 360° the **period**.

The cosine graph

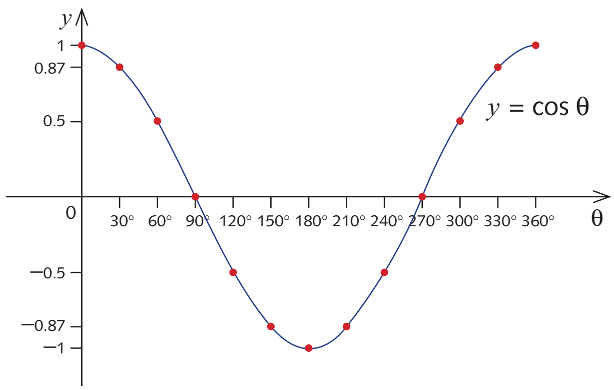
We can repeat for $\cos\theta$ the analysis we carried out for $\sin\theta$. In this case we look at the way OQ changes as θ varies from 0° to 360°.

As θ increases from:	cos θ:
0° to 90°	decreases from 1 to 0
90° to 180°	decreases from 0 to –1
180° to 270°	increases from –1 to 0
270° to 360°	increases from 0 to 1



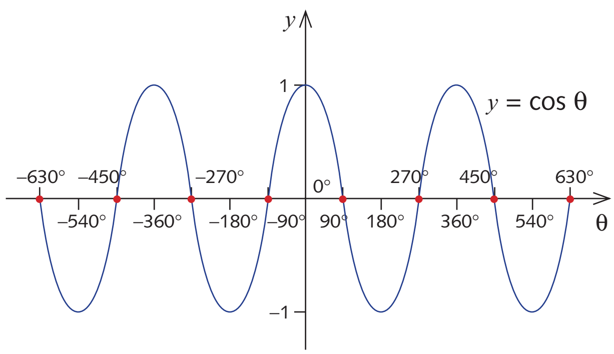
Using $\cos 60° = \frac{1}{2}$ and $\cos 30° \approx 0.87$, we can draw up the following table of values and then plot the points.

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\cos\theta$	1	0.87	0.5	0	–0.5	–0.87	–1	–0.87	–0.5	0	0.5	0.87	1



We will examine the symmetries of the graph of $y = \cos\theta$ in the exercises.

The values of $\cos\theta$ remain the same when θ is increased or decreased by 360°; that is, $\cos\theta = \cos(\theta + 360°)$. $\cos\theta$ is periodic with period 360°.



Hence, the graph of $\cos\theta$ can be drawn for angles greater than 360° and less than 0°, as shown.

Note that the graph of cosine is symmetric about the y-axis.

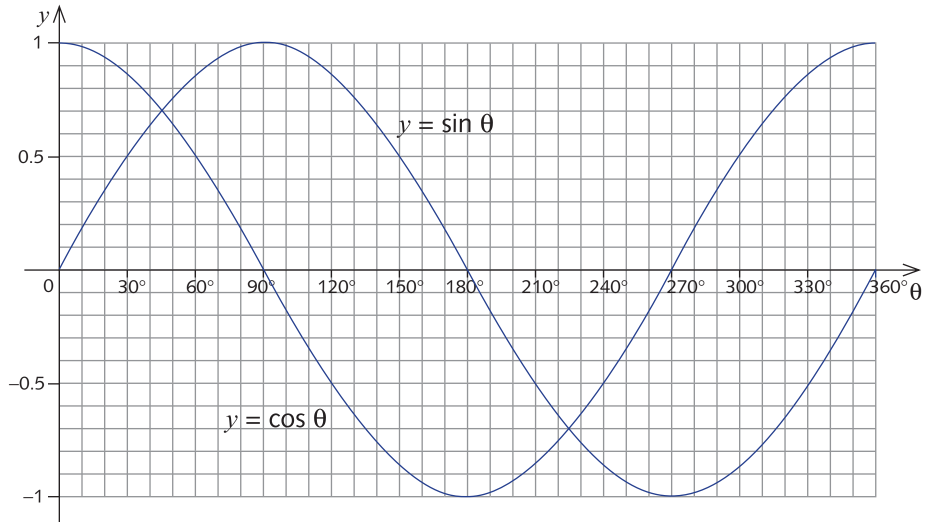
You should also notice that the graph of $y = \cos\theta$ is the same as the graph of $y = \sin\theta$ translated to the left by 90°. That is, $\cos\theta = \sin(90° + \theta)$.

Exercise 19D

1 Draw up a table of values of $y = \sin\theta$ for $0° \le \theta \le 90°$, correct to two decimal places, using increments of 10°. Using your table of values, plus symmetry, plot the graph of $y = \sin\theta$ for $0° \le \theta \le 360°$.

2 Draw up a table of values of $y = \cos\theta$ for $0° \le \theta \le 90°$, correct to two decimal places, using increments of 10°. Using your table of values, plus symmetry, plot the graph of $y = \cos\theta$ for $0° \le \theta \le 360°$.

3 Here are the graphs of $y = \sin\theta$ and $y = \cos\theta$ drawn on the same axes.



a From the graphs, read off the approximate value of:

i	$\cos 60°$	**ii**	$\sin 210°$	**iii**	$\sin 75°$	**iv**	$\cos 20°$
v	$\cos 150°$	**vi**	$\sin 25°$	**vii**	$\sin 235°$	**viii**	$\cos 305°$

b Find, from the graphs, two approximate values of θ between $0°$ and $360°$ for which:

i	$\sin = 0.5$	**ii**	$\cos\theta = -0.5$	**iii**	$\sin\theta = 0.9$	**iv**	$\cos\theta = 0.6$
v	$\sin\theta = 0.8$	**vi**	$\cos\theta = -0.8$	**vii**	$\sin\theta = -0.4$	**viii**	$\cos\theta = -0.3$

c Read from the graph the two values of θ, between $0°$ and $360°$, for which $\sin\theta = \cos\theta$.

4 **a** What are the maximum and minimum values of $\sin\theta$?

b Where do they occur?

5 **a** What are the maximum and minimum values of $\cos\theta$?

b Where do they occur?

6 Draw diagrams to illustrate:

a $\cos(180° - \theta) = -\cos\theta$

b $\cos(180° + \theta) = -\cos\theta$

c $\cos(360° - \theta) = \cos\theta$

7 Draw diagrams to illustrate:

a $\cos(-\theta) = \cos\theta$

b $\sin(-\theta) = -\sin\theta$

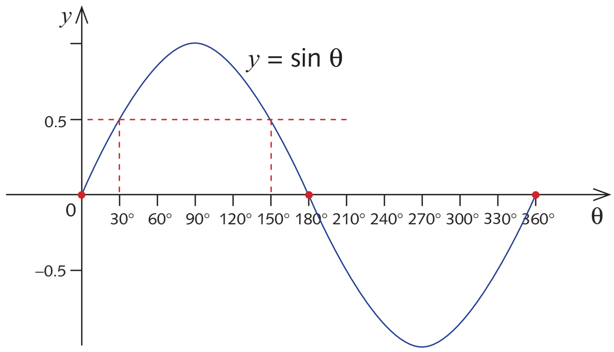
8 Draw up a table of values of $y = 3\sin 2\theta$ for $0° \le \theta \le 360°$. Use increments of $15°$ and work to one decimal place. Sketch the graph of $y = 3\sin 2\theta$ for $0° \le \theta \le 360°$.

9 Draw up a table of values of $y = 4\cos 2\theta$ for $0° \le \theta \le 360°$. Use increments of $15°$ and work to one decimal place. Sketch the graph of $y = 4\cos 2\theta$ for $0° \le \theta \le 360°$.

19E Trigonometric equations

Equations such as $\sin\theta = \frac{1}{2}$ and $\cos\theta = -\frac{1}{3}$ are examples of **trigonometric equations**.

Suppose we are asked to find *all* the angles θ such that $\sin\theta = \frac{1}{2}$. There are infinitely many solutions since, as we saw above, adding $360°$ to any solution will provide a new one. In this section we will restrict the range of the answers to be between $0°$ and $360°$. Hence, the equation $\sin\theta = \frac{1}{2}$ has solutions $\theta = 30°$ and $\theta = 150°$ in the range $0° \le \theta \le 360°$, since $\sin 30° = \sin 150° = \frac{1}{2}$. They are the only solutions in the given range, as shown in the diagram.



Linear trigonometric equations

When solving linear equations such as $3x - 2 = 2x + 3$, our approach was to isolate x on one side of the equation and the numbers on the other, to obtain $x = 5$.

When solving equations involving just one trigonometric ratio, treat the trigonometric function as a pronumeral and isolate it on one side of the equation using the usual rules of algebra.

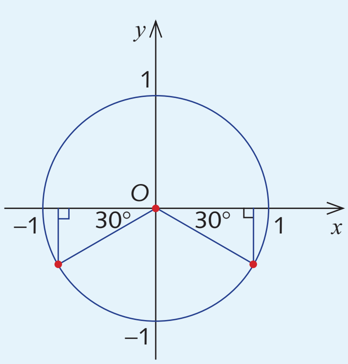
Example 8

Solve $2\sin\theta + 1 = 0$ for $0° \le \theta \le 360°$.

Solution

$$2\sin\theta + 1 = 0$$
$$\sin\theta = -\frac{1}{2}$$

The related angle is $30°$ because $\sin 30° = \frac{1}{2}$.



Here, the sine is negative, so θ lies in the third or fourth quadrant.
Hence, $\theta = 180° + 30° = 210$ or $\theta = 360° - 30° = 330°$.

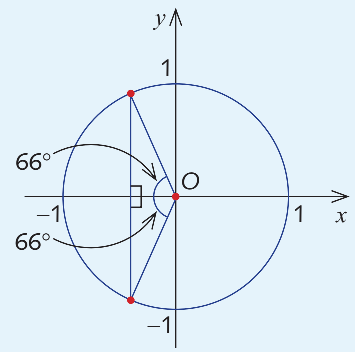
The solutions in the given range are $\theta = 210°$ and $\theta = 330°$.

Example 9

Solve, correct to the nearest degree, $5\cos\theta + 4 = 2$ for $0° \le \theta < 360°$.

Solution

$$
\begin{aligned}
5\cos\theta + 4 &= 2 \\
\cos\theta &= -\frac{2}{5} \\
&= -0.4
\end{aligned}
$$



From a calculator, the related angle is $\cos^{-1} 0.4 \approx 66°$, correct to the nearest degree. Since the cosine is negative, θ lies in the second or third quadrant.

Hence, $\theta \approx 180° - 66° = 114°$ or $\theta \approx 180° + 66° = 246°$.

Note: Remember to work with the related angle first and then shift to the correct quadrants.

Example 10

Solve $4\sin^2\theta = 1$ for $0° \le \theta \le 360°$.

Solution

$$
\begin{aligned}
4\sin^2\theta &= 1 \\
\sin^2\theta &= \frac{1}{4} \\
\sin\theta &= \frac{1}{2} \text{ or } \sin\theta = -\frac{1}{2}
\end{aligned}
$$

The related angle is $30°$ since $\sin 30° = \frac{1}{2}$.

Since $\sin\theta$ is either positive or negative, the angle can be in any one of the four quadrants, so $\theta = 30°, 150°, 210°$ or $330°$.

Exercise 19E

Example 8

1 Without using a calculator, solve each equation for $0° \le \theta \le 360°$.

a $2\sin\theta = 1$

b $2\sin\theta = \sqrt{3}$

c $2\sin\theta = -\sqrt{3}$

d $4\cos\theta - 2 = 0$

e $9\tan\theta = 9$

f $\sqrt{3}\tan\theta = -1$

g $2\cos\theta + \sqrt{3} = 0$

h $\frac{1}{\sqrt{3}}\tan\theta = 1$

Example 9

2 Solve each equation for $0° \le \theta \le 360°$, correct to the nearest degree.

a $\sin\theta = 0.58778$

b $3\cos\theta = 1.6776$

c $5\sin\theta = 4.455$

d $2\sin\theta = -1.4863$

e $7\cos\theta + 3 = 9.729$

f $9\sin\theta - 2 = -10.733$

Example 10

3 Solve each equation for $0° \le \theta \le 360°$.

a $\sin^2\theta = \frac{3}{4}$

b $\tan^2\theta = 1$

c $\cos^2 = \frac{1}{4}$

d $\sin^2\theta = \frac{1}{2}$

e $2\cos^2\theta = \frac{3}{2}$

f $3\tan^2\theta = 1$

4 Recall that $\frac{\sin\theta}{\cos\theta} = \tan\theta$. Use this to solve each equation for $0° \le \theta \le 360°$.

a $\sin\theta = \cos\theta$

b $\sqrt{3}\cos\theta - \sin\theta = 0$

Review exercise

1 Write down the related angle for:

a 35°

b 150°

c 310°

d 200°

e 430°

f 600°

g –60°

h –300°

2 Find the exact value of the sine, cosine and tangent function of:

a 150°

b 120°

c 135°

d 300°

e 210°

f 330°

g 240°

h 315°

3 If $A = 30°$, $B = 60°$ and $C = 45°$, find the value of:

a $\sin 2A$ b $2\sin A$

c $\cos 2B$ d $2\cos B$

e $\cos^2 B - \sin^2 B$ f $\tan 3C$

4 Draw up a table of values and draw the graph of $y = \sin 2\theta$ for $0° \le \theta \le 360°$.

5 Draw up a table of values and draw the graph of $y = \cos 2\theta$ for $0° \le \theta \le 360°$.

6 Solve for $0° \le \theta \le 360°$.

a $\cos\theta = -\frac{1}{2}$ b $\sin\theta = \frac{1}{\sqrt{2}}$

c $\tan\theta = 1$ d $\tan\theta = -\sqrt{3}$

e $\sin\theta = \frac{\sqrt{3}}{2}$ f $\sin^2\theta = \frac{1}{4}$

7 Find, in surd form, the value of:

a $\sin 675°$ b $\cos 480°$

c $\tan 510°$ d $\sin(-330°)$

e $\cos(-240°)$ f $\tan(-210°)$

8 Find, correct to the nearest degree, all values of θ between $0°$ and $360°$ such that:

a $\sin\theta = 0.5735$ b $\cos\theta = -0.587\,78$

c $\tan\theta = 2.1445$ d $\sin\theta = -0.8191$

9 Solve each equation for $0° \le \theta \le 360°$.

a $2\cos\theta - 1 = 0$ b $2\cos\theta + 1 = 0$

c $2\sin\theta - \sqrt{3} = 0$ d $2\sin\theta + \sqrt{3} = 0$

e $5\sin\theta + 5 = 0$ f $4\cos\theta - 4 = 0$

Challenge exercise

1 Find the exact value of
$\sin^2 120° \operatorname{cosec} 270° - \cos^2 315° \sec 180° - \tan^2 225° \cot 315°$.

2 Show that $\sin 420° \cos 405° + \cos 420° \sin 405° = \frac{\sqrt{3}+1}{2\sqrt{2}}$.

3 On the same set of axes, sketch the graphs of $y = \sin 3\theta$ for $0° < \theta$

$y = \sin 3\theta$ for $0° < \theta < 360°$ ·

4 Solve, for $0° \le \theta \le 360°$, $\sin^2\theta \sec\theta = 2 \tan\theta$.

5 On the same set of axes, sketch the graphs of $y = \cos\theta$ and $y = \sec\theta$ for $0° \le \theta \le 360°$, $\theta \ne 90°,\ 270°$.

6 On the same set of axes, sketch the graphs of $y = \sin\theta$ and $y = \operatorname{cosec}\theta$ for $0° \le \theta \le 360°$, $\theta \ne 0°, 180°, 360°$.

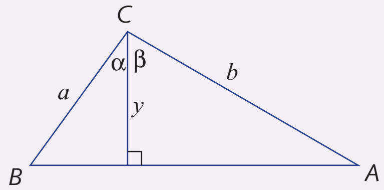
7 Solve each equation for θ, where $0° \le \theta \le 360°$.

a $2\cos^2\theta + 3\cos\theta - 2 = 0$

b $2\sin^2\theta + 5\sin\theta - 3 = 0$

c $-2\cos^2\theta + \sin\theta + 1 = 0$.

8 **a** In the diagram, show that $y = a\cos\alpha$ and $y = b\cos\beta$.



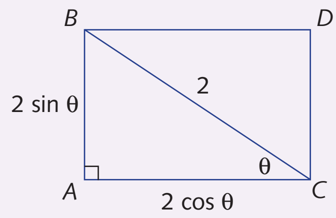
b Using the formula for the area of a triangle, $\frac{1}{2}ab\sin C$, prove that $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

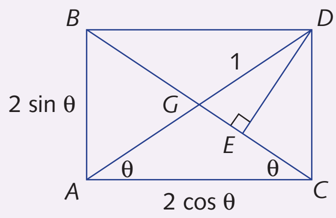
c Find the exact value of sin 75°.

9 **a** In the first diagram, state the area of triangle ABC.

b In the second diagram, show that $\angle DGC$ is 2θ and $DE = \sin 2\theta$.

c By comparing areas, show that $2\sin\theta\cos\theta = \sin 2\theta$.





CHAPTER

20

Number and Algebra

Functions and inverse functions

In earlier chapters, we have met a number of types of functions – polynomial functions including quadratics and cubics, exponential functions, logarithmic functions and trigonometric functions.

In this chapter we discuss two questions:

- What is a function?
- What is the inverse of a function, and which functions have inverses?

We shall meet the **vertical line test** and the **horizontal line test**, and develop a method for constructing the inverse of a function when it exists.

We will concentrate as much as possible on concrete examples rather than general theory.

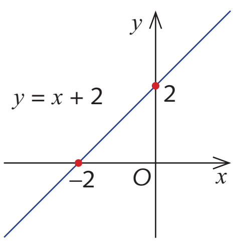
20A Functions and domains

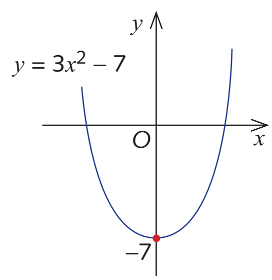
When a quantity y is **uniquely** determined by another quantity x as a result of some rule or formula, then we say ***y* is a function of *x***.

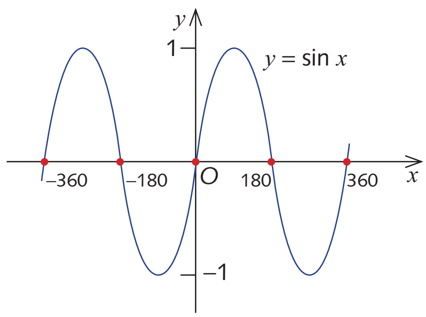
Here are some examples of functions:

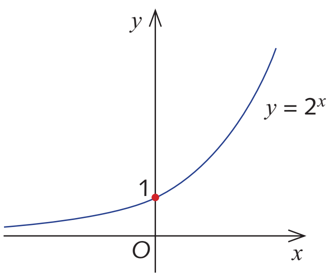
$$y = x + 2,\ y = 3x^2 - 7,\ y = \sin x,\ y = 2^x,\ y = \frac{1}{x} \text{ and } y = \log_2$$

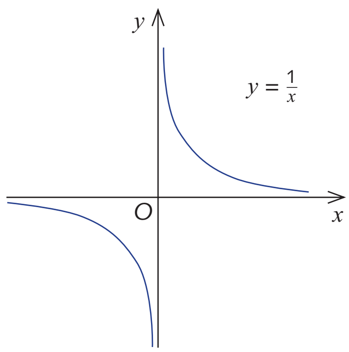
These are all examples of functions that we have met in earlier chapters of this book. We know how to draw their graphs.

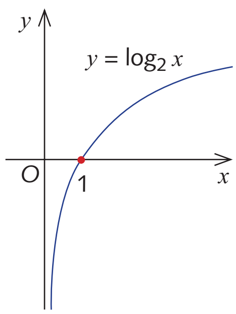












Domains

For the first four graphs above, there is a point on the graph corresponding to every x-value. That is, you can substitute any x-value into the formula to obtain a unique y-value.

We therefore say that the **natural domain** of the functions $y = x + 2$, $y = 3x^2 - 7$, $y = \sin x$ and $y = 2x$ is 'the set of all real numbers'.

For the graph of $y = \log_2 x$, there is a point on the graph corresponding to every positive x-value. That is, you can substitute any positive x-value into the formula to obtain a unique y-value.

For the graph of $y = \frac{1}{x}$, there is a point on the graph corresponding to every non-zero x-value. That is, you can substitute any non-zero x-value into the formula to obtain a unique y-value.

Definition

The set of allowable values of x is called the **natural domain** of the function.

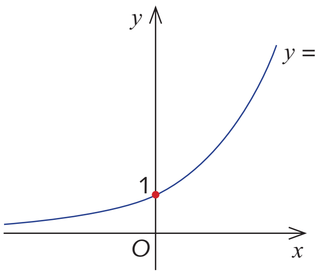
The natural domain of a function is often simply called the **domain** of the function. We will refer to it as the domain in this chapter.

The domain of the function $y = \log_2 x$ is the set of positive real numbers, $\{x : x > 0\}$, for which we will use the shorthand $x > 0$. We write, '$y = \log_2 x$, where $x > 0$'.

The domains of some functions that you have met previously are presented below.

Function	Domain
$y = 4x^3 + 2x^2 + 5x - 4$	all real numbers
$y = \cos 3x$	all real numbers
$y = \sqrt{x}$	$x \geq 0$
$y = \sqrt[3]{x}$	all real numbers
$y = \frac{1}{x}$	$x \neq 0$

To be a little more precise we say $y = 2^x$ for all real x is the **function**, whereas



is the **graph** of the function.

Example 1

What is the domain of each function?

a $y = \dfrac{6}{x - 1}$

b $y = \sqrt{x - 5}$

c $y = \dfrac{1}{x^2 - 4}$

d $y = \dfrac{x^2 - 6x + 3}{x^2 + 4}$

Solution

a The domain is $x \neq 1$, since the denominator must not be zero.

b $\sqrt{x}$ is only defined for $x \geq 0$. Hence, the domain of $y = \sqrt{x - 5}$ is $x \geq 5$.

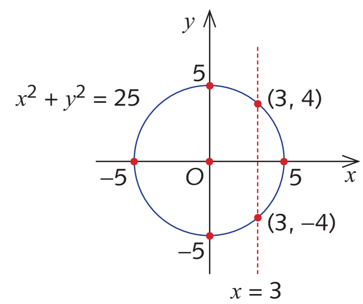
c The domain is all real numbers except 2 and –2, since the denominator is zero when $x = 2$ or $x = -2$.

d $x^2 + 4$ is never zero, so the domain is all real numbers.

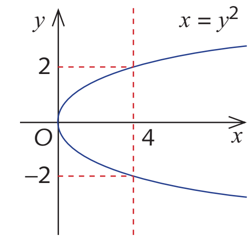
Note: You can often determine the domain of a function even though you may not be able to easily sketch its graph.

Graphs and the vertical line test

Not all graphs are the graphs of functions. For example, the graph of $x^2 + y^2 = 25$ is a circle with centre the origin and radius 5. When we substitute $x = 3$, we get two y-values, $y = -4$ and $y = 4$, because the line $x = 3$ cuts the circle at two points.



Hence, for some x-values, for example $x = 3$, there is not a unique y-value. Thus, this graph is not the graph of a function. Each vertical line, $x = c$, must meet the graph at, *at most*, one point for the graph to be the graph of a function.



In general, *if we can draw a vertical line that cuts a graph more than once, the graph is not the graph of a function.*

This is called the **vertical line test**.

The graph of the parabola to the right is not a graph of a function.

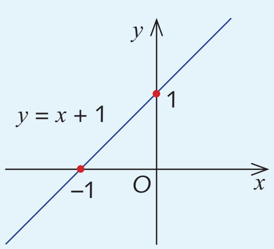
A vertical line has been drawn that crosses the graph at two places.

The y-values are not uniquely determined by the x-values.

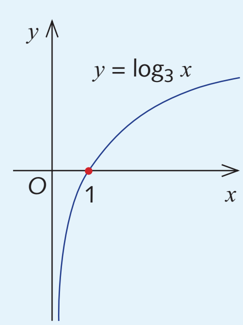
Example 2

State whether or not each graph is the graph of a function, and illustrate using the vertical line test.

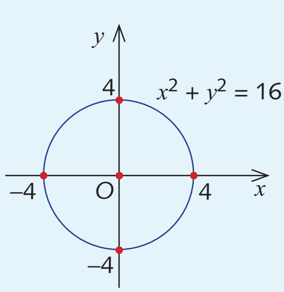
a



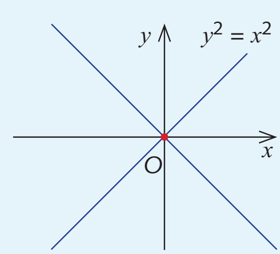
b



c

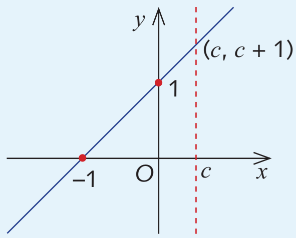


d



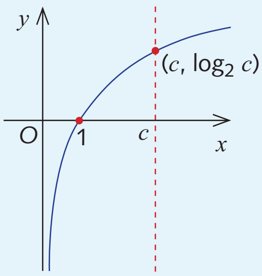
Solution

a



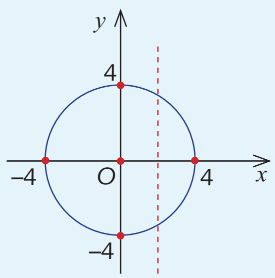
It is the graph of a function.

b



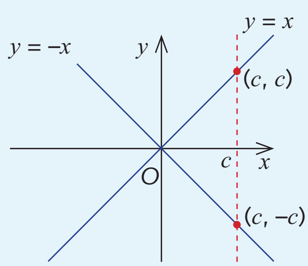
It is the graph of a function.

c



It is not the graph of a function.

d

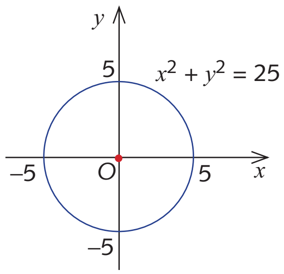


If $y^2 = x^2$, then $y = x$ or $y = -x$, so the graph consists of two straight lines. It is not the graph of a function.

Relations

- An equation such as $x^2 + y^2 = 25$ is called a **relation**. Indeed, the word 'relation' is very general, and any set of points in the Cartesian plane is a relation.

  The **vertical line test** determines whether or not a relation is a function.



- In a natural way, the circle $x^2 + y^2 = 25$ leads to two functions.

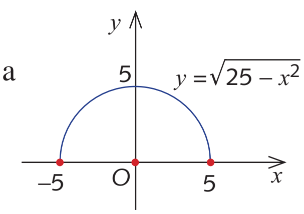
  Solving $x^2 + y^2 = 25$ for y:

  $y^2 = 25 - x^2$

  $y = \sqrt{25 - x^2}$ or $y = -\sqrt{25 - x^2}$

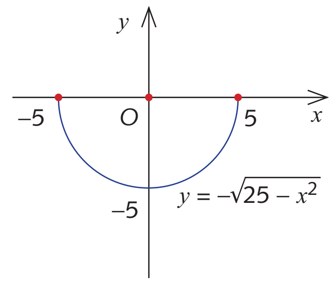
  The graph of the first of these is the top half of the circle.

  This graph satisfies the vertical line test. So $y = \sqrt{25 - x^2}$, $-5 \le x \le 5$ is a function.



Domain: $-5 \le x \le 5$

The graph of the second of these is the bottom half of the circle and the graph satisfies the vertical line test, so $y = -\sqrt{25 - x^2}$, $-5 \le x \le 5$ is a function.

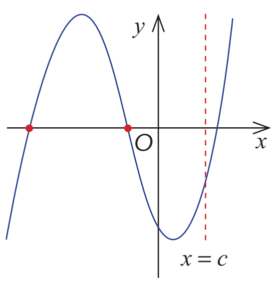


Domain: $-5 \le x \le 5$

Functions, domains and the vertical line test

- When a quantity, y, is uniquely determined by some other quantity, x, as a result of some rule or formula, then we say **y is a function of x**.
- The set of allowable values of x is called the **natural domain**, or **domain**, of the function.
- **Vertical line test**. Each vertical line, $x = c$, must meet a graph at, at most, one point for the graph to be the graph of a function.

  Notice that each vertical line meets the graphs of the functions on page 590 of this chapter at either zero or one point.



Exercise 20A

Example 1

1 What is the domain of each function?

a $y = 2x - 5$ **b** $y = x^2 + 5$ **c** $y = \dfrac{5}{x}$ **d** $y = \dfrac{1}{x - 2}$

e $y = \dfrac{3}{x + 4}$ **f** $y = \dfrac{4}{3x - 6}$ **g** $y = \dfrac{7}{x^2 - 4}$ **h** $y = \dfrac{3x + 2}{x^2 - 9}$

2 What is the domain of each function?

a $y = \sqrt{7x}$ **b** $y = \sqrt{7 + x}$ **c** $y = \sqrt{7 - x}$

d $y = \sqrt{7x - 1}$ **e** $y = \dfrac{1}{\sqrt{7x}}$ **f** $y = \dfrac{1}{\sqrt{x - 7}}$

3 What is the domain of each function?

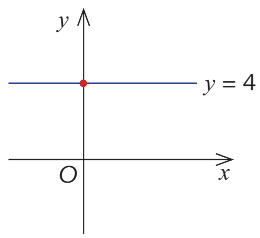
a $y = 2^x$ **b** $y = 7^{3x} + 5$ **c** $y = \log_5 x$

d $y = \log_3(x - 2)$ **e** $y = \log_2(-x)$ **f** $y = 2\sin x$

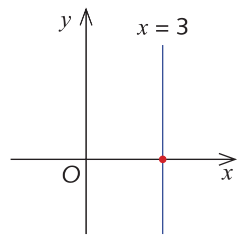
Example 2

4 Use the vertical line test to determine whether each graph is the graph of a function.

a



b



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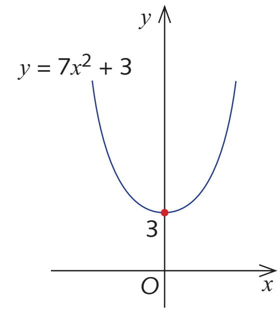

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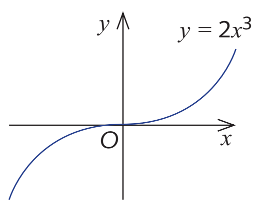
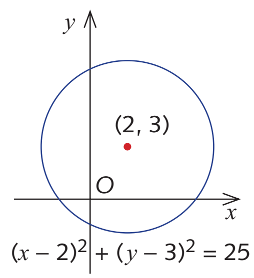
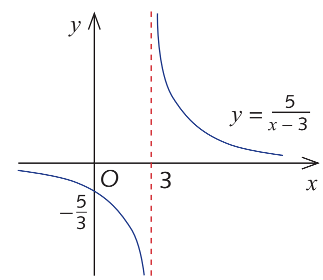


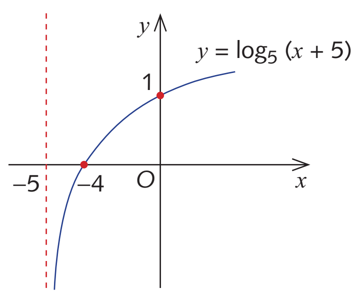
e



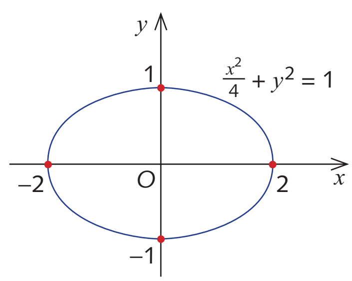
f



g



h



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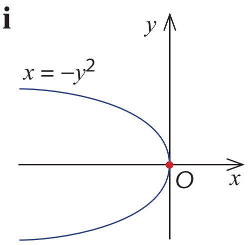


5 **a** Solve the equation $y^2 = 4x^2$ for y.

b Draw the graph of $y^2 = 4x^2$.

c Does the graph satisfy the vertical line test?

d Is the graph of $y^2 = 4x^2$ the graph of a function?

6 In a natural way the graph of $y^2 = x$ leads to two functions $y = \sqrt{x}$ and $y = -\sqrt{x}$.

a Draw the graph of $y^2 = x$.

b Draw the graphs of $y = \sqrt{x}$ and $y = -\sqrt{x}$.

20B Inverse functions

We start with a very simple example.

If we add three to a number and then subtract three, we get back to the original number.

The function $y = x + 3$ corresponds to adding three to a number and similarly the function $y = x - 3$ corresponds to subtracting three from a number.

The function $y = x + 3$ takes 2 to 5 and the function $y = x - 3$ takes 5 to 2.

$y = x + 3$

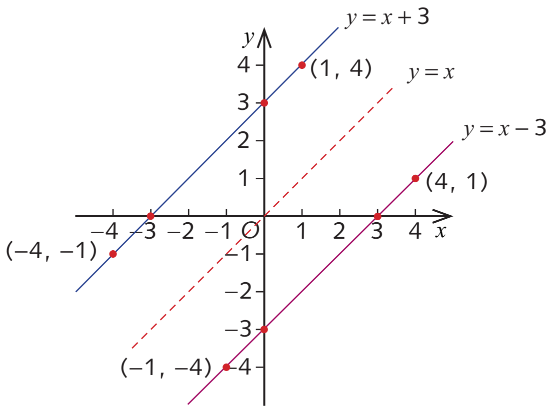
x	−4	−3	−2	−1	0	1	2	3	4
y	−1	0	1	2	3	4	5	6	7

$y = x - 3$

x	−1	0	1	2	3	4	5	6	7
y	−4	−3	−2	−1	0	1	2	3	4

The tables show the values of all x and y pairs on one function swap places on the other.

The graphs of the two functions are shown below. It is clear from the diagram that one of the functions is the reflection of the other in the line $y = x$.

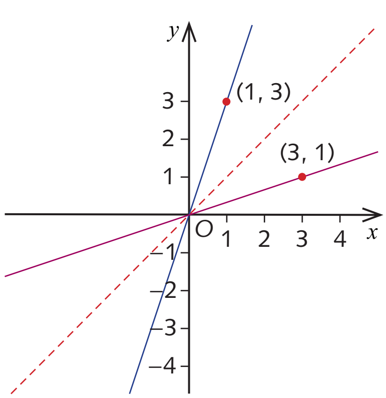


$y = x + 3$ and $y = x - 3$ are said to be *inverses* of each other. This will be defined formally in Section 20E.

A second very simple example.

If we multiply a number by 3 *and then divide by* 3*, we get back to the original number.*

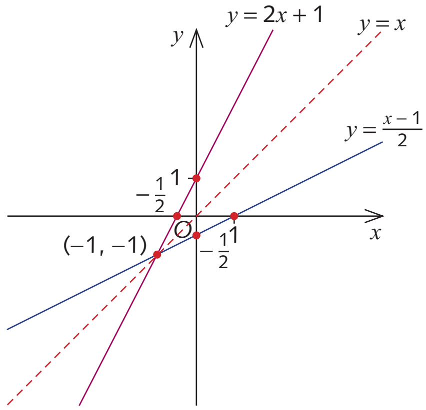
The function $y = 3x$ corresponds to multiplying a number by 3 and similary the function $y = \dfrac{x}{3}$ corresponds to dividing a number by 3.



What is the inverse of $y = 2x + 1$? If we double a number and add one we must first subtract one and then halve it to get back to the original number.

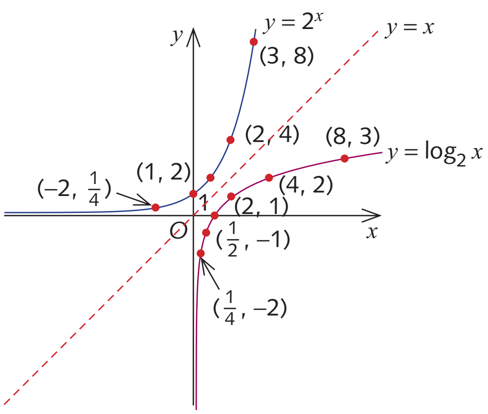
Thus $y = \dfrac{x - 1}{2}$ is the inverse of $y = 2x + 1$.

The graphs of the two functions are shown below. Each is the reflection of the other in the line $y = x$. The function $y = \frac{x-1}{2}$ is the inverse of the function $y = 2x + 1$ and $y = 2x + 1$ is the inverse of $y = \frac{x-1}{2}$.



Now consider the two functions $y = \log_2 x$ and $y = 2^x$.

The graphs of the two functions are shown below. Each graph is the reflection of the other in the line $y = x$. Moreover, for each point (a, b) on one function, the point (b, a) lies on the other. The function $y = \log_2 x$ is the inverse of the function $y = 2^x$ and $y = 2^x$ is the inverse of $y = \log_2 x$. This has been discussed in Chapter 14.



Constructing inverses

As we saw from the above examples, there is a simple method for finding the formula for the inverse of a function.

We interchange x and y and then make y the subject.

For example, if $y = x + 3$

Then $\quad x = y + 3 \quad$ (Interchanging x and y.)

$\quad\quad\quad y = x - 3$

$y = x - 3$ is the inverse function of $y = x + 3$ as we saw above.

Example 3

Find the inverse function of:

a $y = 2x + 1$ **b** $y = x^3$

Solution

a
$$y = 2x + 1$$
$$x = 2y + 1 \quad \text{(Interchanging } x \text{ and } y.\text{)}$$
so $y = \dfrac{x-1}{2}$ is the inverse function of $y = 2x + 1$

b
$$y = x^3$$
$$x = y^3 \quad \text{(Interchanging } x \text{ and } y.\text{)}$$
so $y = \sqrt[3]{x}$ is the inverse function of $y = x^3$

We return to our study of inverse functions in Section 20E of this chapter.

Exercise 20B

Example 3a

1 Find the inverse of each function. Sketch the graph of each function and its inverse on the one set of axes and also include the line $y = x$.

a $y = x + 4$

b $y = 2x + 2$

c $y = 2x - 1$

d $y = \dfrac{x-2}{3}$

e $y = 3x + 2$

f $y = \dfrac{2x-4}{3}$

g $y = 5x$

h $y = \dfrac{x}{3}$

i $y = 6 - 2x$

j $y = 5 - x$

k $y = 6 - 3x$

l $y = 2 - \dfrac{x}{2}$

Example 3b

2 Find the inverse of each function. Sketch the graph of each function and its inverse on the one set of axes and also include the line $y = x$.

a $y = x^3 + 1$

b $y = -x^3$

c $y = x^3 + 8$

d $y = \dfrac{1}{x} + 3$

e $y = 2x^3 - 4$

f $y = \dfrac{1}{x} - 3$

g $y = \dfrac{2}{x} - 3$

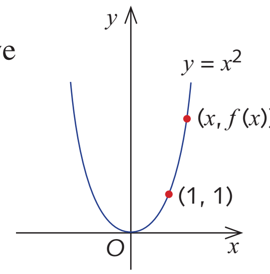
h $y = \dfrac{4}{x} - 1$

20C Function notation and the range of a function

In Section 20A we said that 'y is a function of x' if the value of y is uniquely determined by the value of x.

There is a standard and very convenient notation for functions. For example, we can write the function $y = x^2$ as:

$$f(x) = x^2$$



This is read as 'f of x is equal to x^2'.

To calculate the value of a function, we substitute the value of x.
So in this case:

$f(3) = 3^2 = 9$ $\qquad$ $f(0) = 0$ $\qquad$ $f(-2) = 4$ $\qquad$ $f(a) = a^2$

We say that the **graph of the function** $f(x) = x^2$ is the graph of $y = x^2$. So $f(x)$ is the y-value.

This new way of writing functions is called **function notation** and was introduced to mathematics by Leonhard Euler in 1735. We have previously used this in the chapter on polynomials but from now on we shall use it for all functions.

So, for example, the statement $P(x) = x^3 + 2x^2 - 5$ can be thought of as defining the polynomial $P(x)$ or the function $P(x)$.

$P(1) = -2$ is a value of the function P. It is also the value of the polynomial at $x = 1$.

Example 4

Let $f(x) = 3 - x^2$. Calculate:

a $f(0)$ $\qquad$ **b** $f(1)$ $\qquad$ **c** $f(-1)$

d $f(t)$ $\qquad$ **e** $f(2a)$ $\qquad$ **f** $f(a - 2)$

Solution

a $f(0) = 3 - 0^2 = 3$

b $f(1) = 3 - 1^2 = 2$

c $f(-1) = 3 - (-1)^2 = 2$

d $f(t) = 3 - t^2$

e $f(2a) = 3 - (2a)^2$
$\qquad = 3 - 4a^2$

f $f(a + 2) = 3 - (a - 2)^2$
$\qquad = 3 - (a^2 - 4a + 4)$
$\qquad = -a^2 + 4a - 1$

The natural domain and range of a function

Natural domain of a function

Recall from the previous section that the natural domain of a function is the set of all allowable x-values and can be known simply as the 'domain'. For example, the function $f(x) = \log_5 x$ has domain 'the positive real numbers', or simply $x > 0$.

Definition of the range of a function

The set of all values of $f(x)$ (or, if you like, the set of all y-values) is called the **range** of the function. To determine the range, it is best to first graph the function.

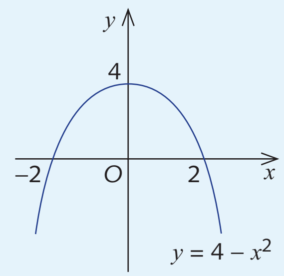
Example 5

What is the domain and the range of $f(x) = 4 - x^2$?

Solution

$f(x)$ is defined for all real numbers and so the domain is 'all real numbers'.

From the graph, the range of $f(x) = 4 - x^2$ is $y \le 4$.



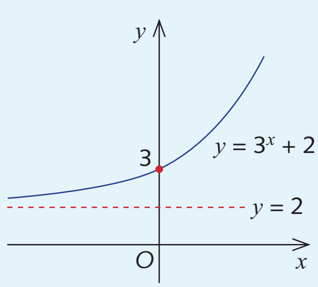
Example 6

What is the domain and the range of $f(x) = 3^x + 2$?

Solution

The domain of the function is all real numbers.

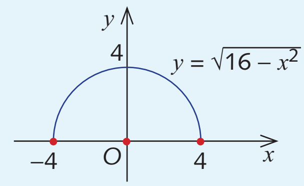
The range of the function is all real numbers greater than 2, or $y > 2$.



Example 7

What is the domain and the range of $f(x) = \sqrt{16 - x^2}$?

Solution



Suppose that $y = \sqrt{16 - x^2}$

Then $\quad y^2 = 16 - x^2$

$x^2 + y^2 = 16$

So the graph of $f(x) = \sqrt{16 - x^2}$ is the top half of the circle with centre the origin and radius 4.

From the graph: The domain of $f(x)$ is $-4 \le x \le 4$. The range of $f(x)$ is $0 \le y \le 4$.

Exercise 20C

Example 4

1 If $f(x) = 3 - 5x$, find:

a $f(0)$ **b** $f(4)$ **c** $f\left(\frac{3}{5}\right)$

d $f(1) + f(2)$ **e** $f(4)f(3)$ **f** $3f(10) - 4f(5)$

2 If $f(x) = x^2 + 2$, find:

a $f(2)$ **b** $f(0)$ **c** $f(-3)$

d $f\left(\frac{1}{2}\right)$ **e** $f(\sqrt{2})$ **f** $f(10) + f(20)$

3 If $g(x) = \frac{5 + x}{5 - x}$, find:

a $g(0)$ **b** $g(-5)$ **c** $g(7)$ **d** $g(1)$ **e** $g\left(\frac{5}{2}\right)$ **f** $g\left(-\frac{5}{2}\right)$

4 Let $f(x) = \frac{1}{x}$. Find x if:

a $f(x) = 6$ **b** $f(x) = \frac{5}{2}$ **c** $f(x) = f(-2)$

5 Let $k(x) = x^2 - 4x$. Find x if:

a $k(x) = 0$ **b** $k(x) = -4$ **c** $k(x) = 5$

d $k(x) = -5$ **e** $k(x) = 1$ **f** $k(x) = k(3)$

6 If $h(x) = x^2 - 4$, find and simplify:

a $h(a)$ **b** $h(y + 2)$ **c** $h(2b)$ **d** $h(-3c - 1)$ **e** $h(x^2)$ **f** $h(x^3)$

7 If $f(x) = x^2$, state whether each statement is true or false.

a $f(5) = f(3) + f(4)$ **b** $f(4) = 2f(3) - f(1)$

c $f(x + y) = f(x) + f(y)$ **d** $f(xy) = f(x)f(y)$

e $f(ax) = a^2 f(x)$ **f** $f(a + b) - f(a) - f(b) = 2ab$

8 If $g(x) = 3x$, state whether each statement is true or false.

a $g(3) = 2g(2) + 3g(1)$ **b** $g(2) = g(1) + 2g(0)$ **c** $g(x + y) = g(x) + g(y)$

d $g(x + y) = g(x)g(y)$ **e** $g(xy) = g(x)g(y)$ **f** $g(2a) = 2g(a)$

Example 5, 6, 7

9 Find the domain and the range of:

a $f(x) = 3 - x^2$ **b** $f(x) = \frac{2}{x}$ **c** $f(x) = 2^x + 4$

d $f(x) = \sqrt{9 - x^2}$ **e** $f(x) = 6 - 5x^2$ **f** $f(x) = x^2 + 4$

g $f(x) = 5^x - 3$ **h** $f(x) = 2^x + 7$ **i** $f(x) = -\sqrt{25 - x^2}$

j $f(x) = x^3 - 7$ **k** $f(x) = \frac{-3}{x}$ **l** $f(x) = \log_2(7 - x)$

m $f(x) = \sin x$ **n** $f(x) = \tan x \left(= \frac{\sin x}{\cos x} \right)$

20D Transformations of graphs of functions

In Chapter 7 of this book, we saw how to draw the graphs of quadratic functions starting with the basic parabola $y = x^2$ by:

- translating up and down
- translating to the left and to the right
- reflecting in the x-axis
- stretching from the x-axis.

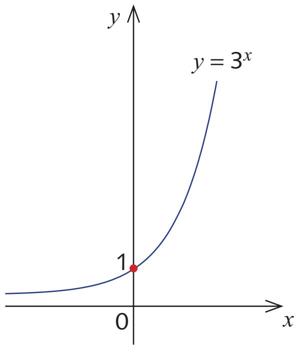
In Chapter 11 of this book, these transformations were applied to the graph of $y = \frac{1}{x}$.

These same transformations can be applied to any function and its graph. We will also see the effect of reflecting a graph in the y-axis.

Translations

The graph of $y = f(x) + a$ (where a is a constant) is the graph of $y = f(x)$ with a translation of a units in the vertical direction.

For example:



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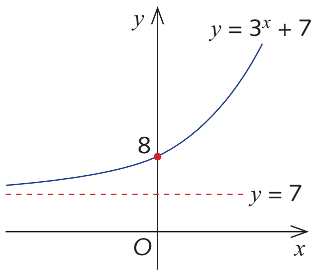
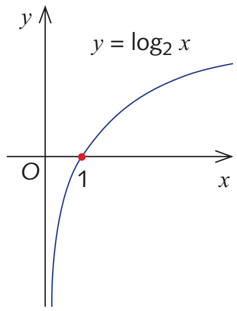


The graph of $y = f(x - h)$ is the graph of $y = f(x)$ with a translation of h units in the horizontal direction.

For example:



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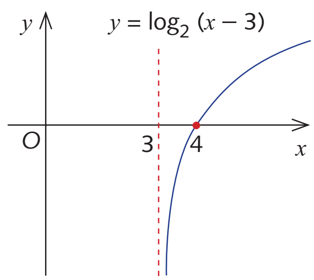
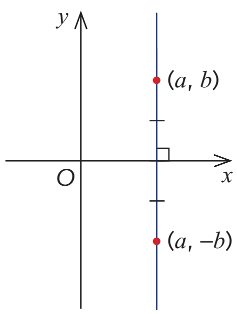


Reflection in the x-axis

The graph of $y = -f(x)$ is the reflection of the graph of $y = f(x)$ in the x-axis.

Combinations of translations and reflections in the x-axis

Reflection in the x-axis sends (2, 3) to (2, –3) and in general (a, b) to $(a, -b)$.

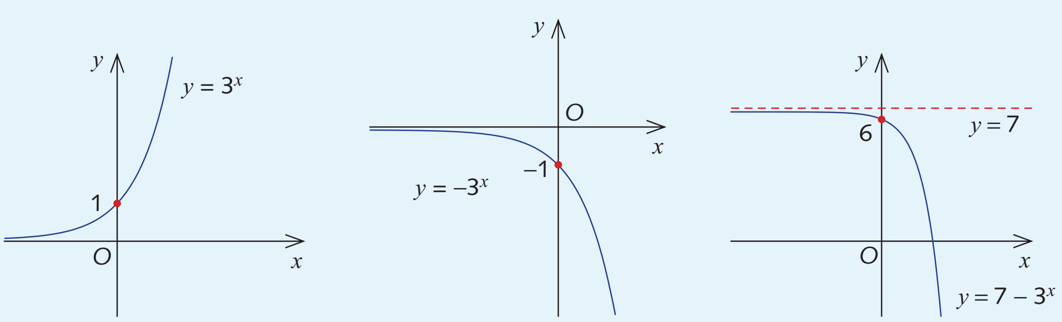


Example 8

Sketch the graph of $f(x) = 7 - 3^x$ and find its domain and range.

Solution

We start with the graph of $y = 3^x$ and reflect in the x-axis to obtain the graph of $y = -3^x$. Next, we translate the graph upwards 7 units to obtain the graph of $y = 7 - 3^x$.



The domain is the set of all real numbers and the range is $y < 7$.

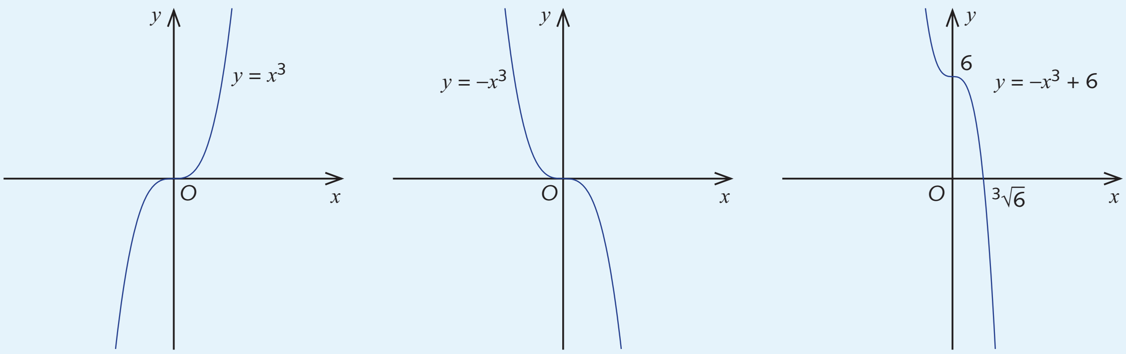
Example 9

a Sketch the graph of $f(x) = -x^3 + 6$ and find its domain and range.

b Sketch the graph of $f(x) = -(x^2 + 2)$ and find its domain and range.

Solution

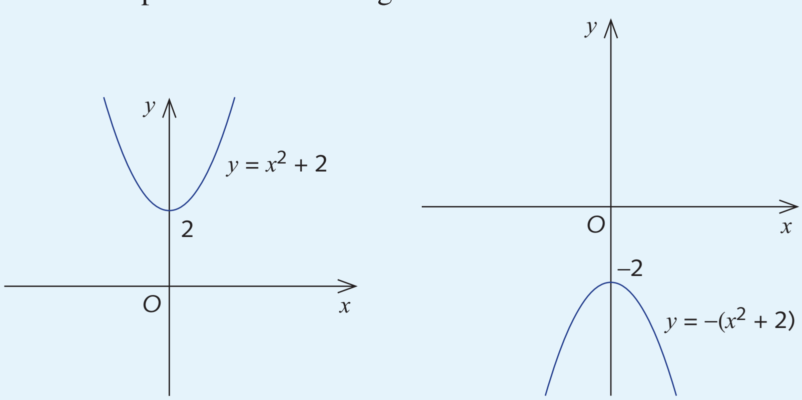
a The graph of $f(x) = -x^3 + 6$ can be drawn by first reflecting the graph of $g(x) = x^3$ in the x-axis and then translating 6 units up.



We can write $f(x) = -g(x) + 6$.

The range of $f(x)$ is ‘all real numbers’.

b The graph of $f(x) = -(x^2 + 2)$ can be drawn by first translating the graph of $g(x) = x^2$ two units up and then reflecting in the x-axis.

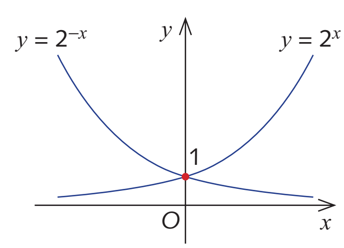


We can write $f(x) = -(g(x) + 2)$.

The range of $f(x)$ is $y < -2$.

Reflection in the y-axis

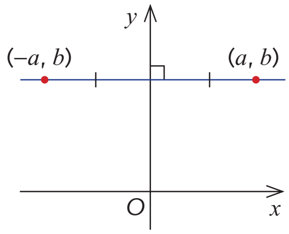
The graph of $y = 2^{-x}$ is the reflection of the graph of $y = 2^x$ in the y-axis.



Reflection in the y-axis sends $(2, 3)$ to $(-2, 3)$ and in general (a, b) to $(-a, b)$.

The point $(x, f(x))$ on the graph of $y = f(x)$ reflects to the point $(-x, f(x))$ and, similarly, $(-x, f(-x))$ goes to $(x, f(-x))$.

That is, the reflection of the graph of $y = f(x)$ in the y-axis is the graph of $y = f(-x)$.



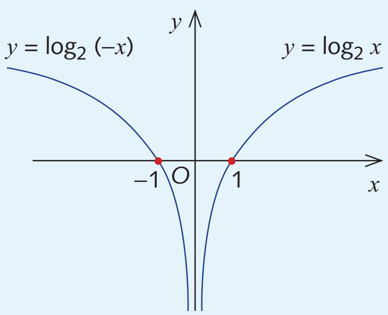
Example 10

Sketch the graph of $f(x) = \log_2(-x)$.

Solution

The graph of $f(x) = \log_2(-x)$ is the reflection of the graph of $y = \log_2 x$ in the y-axis.

Note: If $g(x) = \log_2(x)$ then $f(x) = g(-x)$.



Stretches from the x-axis

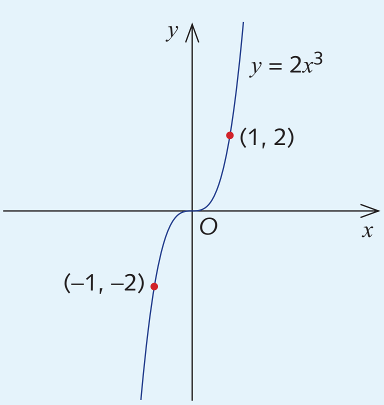
The graph of $y = af(x)$ is a stretch (or dilation) of the graph $y = f(x)$ from the x-axis by a factor of a. The point $(x, f(x))$ on the original graph becomes the point $(x, af(x))$ on the transformed graph.

Example 11

Sketch the graph of $f(x) = 2x^3$.

Solution

The graph of $f(x) = 2x^3$ is the stretch of the graph of $g(x) = x^3$ by a factor of 2.



Exercise 20D

1 Let $f(x) = 2x + 3$. Sketch the graphs of:

a $y = f(x)$ **b** $y = f(x) + 4$ **c** $y = -f(x)$ **d** $y = -f(x) + 2$

2 Let $f(x) = 3x$. Sketch the graphs of:

a $y = f(x)$ **b** $y = f(x) + 4$ **c** $y = -f(x)$ **d** $y = -f(-x)$

Example 8, 10

3 Use transformations to sketch the graphs of each function and find its domain and range.

a $f(x) = x^2 + 5$ **b** $f(x) = (x - 5)^2$ **c** $f(x) = (x + 4)^2$

d $f(x) = 3^{-x}$ **e** $f(x) = 5^x + 1$ **f** $f(x) = 5^x - 4$

g $f(x) = 2 + \log_3 x$ **h** $f(x) = \log_3(x - 4)$ **i** $f(x) = -\log_3(-x)$

Example 9

4 Sketch the graph of each function and find its domain and range.

a $f(x) = x^2 + 2$ **b** $f(x) = x^2 - 6x + 13$ **c** $f(x) = \sqrt{x}$

d $f(x) = \sqrt{2x} + 2$ **e** $f(x) = -\sqrt{x - 2}$ **f** $f(x) = 2 - \sqrt{x + 2}$

5 Let $f(x) = \sqrt{25 - x^2}$.

Sketch the graphs of $y = f(x)$, $y = f(x) + 5$ and $y = -f(x)$ on the one set of axes.

Example 11

6 Let $f(x) = x^3 - 3x^2 + 2x$.

Sketch the graphs of $y = f(x)$, $y = -f(x)$ and $y = -2f(x)$ on the one set of axes.

7 Let $f(x) = \dfrac{1}{x}$.

Sketch the graphs of $y = f(x)$, $y = 2f(x)$, $y = -f(x)$ and $y = -2f(x)$ on the one set of axes.

20E Composites and inverses

Composites of functions

Let $f(x) = x^2$ and $g(x) = 2x + 3$. We can combine these two functions to obtain a **composite function**. Since $f(x)$ is a number we can calculate $g(f(x))$.

For example, $f(3) = 3^2$ and $g(3^2) = 21$

Thus $g(f(3)) = g(3^2) = 21$

and for any x, $g(f(x)) = g(x^2) = 2x^2 + 3$

This procedure is called taking the **composite** of the two functions $f(x)$ and $g(x)$. This composite is a function, since there is a rule that uniquely determines $g(f(x))$.

Note:

- $f(g(3)) = f(9) = 81$ and in general

  $f(g(x)) = f(2x + 3) = (2x + 3)^2$, so $f(g(x)) \neq g(f(x))$
- The composite $g(f(a))$ is defined when $f(a)$ lies in the domain of g. For example, if $f(x) = \dfrac{1}{x}$ and $g(x) = x - 3$, the composite $f(g(3))$ is not defined, since $g(3) = 0$, which is not in the domain of f.

Example 12

Let $f(x) = \dfrac{1}{x - 3}$ and $g(x) = 2x + 5$.

a Find $g(f(4))$, $f(g(4))$, $g(f(x))$ and $f(g(x))$.

b Explain why $f(g(-1))$ does not exist.

c What are the domains of the functions $g(f(x))$ and $f(g(x))$?

Solution

a $g(f(4)) = g(1) = 7$, $f(g(4)) = f(13) = \dfrac{1}{10}$

$g(f(x)) = g\left(\dfrac{1}{x - 3}\right) = \dfrac{2}{x - 3} + 5$ $\qquad$ $f(g(x)) = f(2x + 5) = \dfrac{1}{2x + 2}$

b $g(-1) = 3$, which does not belong to the domain of $f(x)$. Hence, $f(g(-1))$ does not exist.

c $g(f(x))$ has domain $x \neq 3$ and $f(g(x))$ has domain $x \neq -1$.

Inverses of functions

In Section 20B, we introduced the idea of the inverse of a function. We now consider what happens when we compose a function with its inverse.

If we add 2 to a number and then subtract 2, we get back to the original number. We can express this as the composition of the functions $f(x) = x + 2$ and $g(x) = x - 2$.

$$f(g(x)) = f(x - 2) = x - 2 + 2 = x$$

and $g(f(x)) = g(x + 2) = x + 2 - 2 = x$

Applying $f(x)$ and then $g(x)$, or vice versa, returns the original value of x.

The functions $f(x) = x + 2$ and $g(x) = x - 2$ are said to be **inverses** of each other.

Two functions, $f(x)$ and $g(x)$, are **inverses** of each other if $f(g(x)) = x$ and $g(f(x)) = x$. The first equation must hold for all x in the domain of g and the second must hold for all x in the domain of f.

Of course, this is consistent with the idea of inverses introduced in Section 20B. Cubing a number and then finding the cube root returns the original number. Hence, we would expect $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ to be inverse functions. The following example demonstrates this.

Example 13

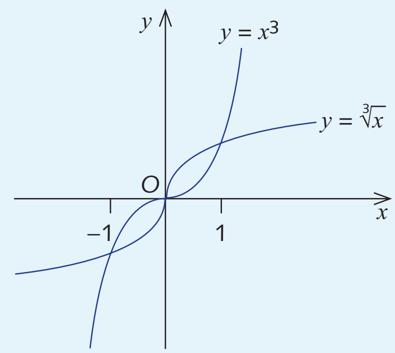
Show that $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverses and sketch their graphs.

Solution

$f(g(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x$

$g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x$

Hence, $f(x)$ and $g(x)$ are inverses of each other for all x.



Geometrically, reflecting the graph of a function in the line $y = x$ corresponds algebraically to interchanging x and y in the equation. This can be seen through the discussion in Section 20B. It can easily be proved that the point (a, b) is the reflection of the point (b, a) in the line $y = x$.

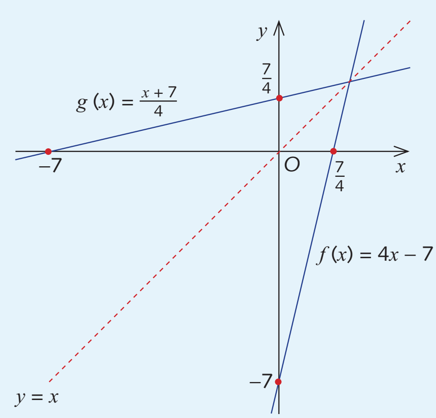
Example 14

Find the inverse function $g(x)$ of the function $f(x) = 4x - 7$. Sketch the graphs of $y = f(x)$, $y = g(x)$ and $y = x$ on the one set of axes.

Solution

$$\begin{aligned}
& & f(x) &= 4x - 7 \\
&\text{then} & y &= 4x - 7 \\
&\text{The inverse is} & x &= 4y - 7 \quad \text{(Interchange } x \text{ and } y.\text{)} \\
& & y &= \frac{x+7}{4} \\
&\text{so} & g(x) &= \frac{x+7}{4}
\end{aligned}$$

Geometrically, the graphs of $f(x)$ and $g(x)$ are reflections in the line $y = x$.

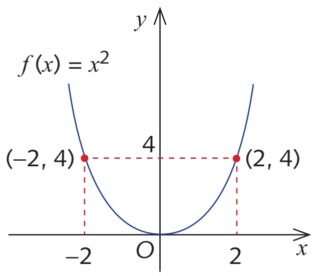


Note:

$$f(g(x)) = 4\left(\frac{x+7}{4}\right) - 7 = x + 7 - 7 = x \qquad g(f(x)) = \frac{(4x-7)+7}{4} = \frac{4x}{4} = x$$

The horizontal line test

Not all functions have inverse functions. For example, the function $f(x) = x^2$ does not have an inverse. We can see this by noting $f(2) = 4$ and $f(-2) = 4$.



So if the inverse $g(x)$ existed, we would have $g(4) = 2$ and $g(4) = -2$, which is impossible, because a function cannot have two y-values for the same x-value.

In general, a function, $f(x)$, has an inverse function when no *horizontal* line crosses the graph of $y = f(x)$ more than once.

This is called the **horizontal line test**.

Example 15

a Show that $f(x) = x^3 - 1$ satisfies the horizontal line test, and find its inverse function.

b Show that $f(x) = x(x-1)(x+1)$ does not satisfy the horizontal line test and hence does not have an inverse.

Solution

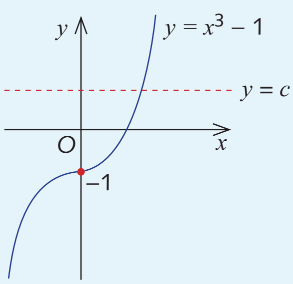
a Each horizontal line, $y = c$, meets the graph of $y = f(x)$ exactly once.

The function is $y = x^3 - 1$

The inverse is $x = y^3 - 1$ (Interchanging x and y.)

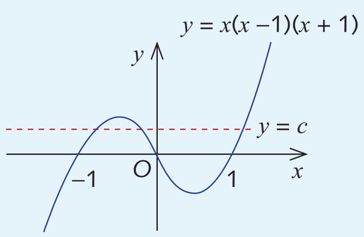
$$y = (x+1)^{\frac{1}{3}}$$

The inverse function of $f(x) = x^3 - 1$ is $g(x)$ $g(x) = (x+1)^{\frac{1}{3}}$



b The graph does not satisfy the horizontal line test, as shown in the diagram.

Hence, the function $f(x) = x(x-1)(x+1)$ does not have an inverse function.



Example 16

Find the domain and range of $f(x) = \dfrac{1}{x+3}$.

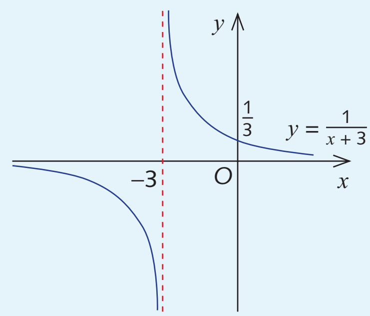
Show that $f(x)$ has an inverse function $g(x)$ and find $g(x)$.

Solution

If $f(x) = \dfrac{1}{x+3}$, then the domain of $f(x)$ is $x \neq 3$.

The range of $f(x)$ is $y \neq 0$.

Since the graph satisfies the horizontal line test, $f(x)$ has an inverse function.



$$\begin{aligned}
\text{Write } y &= \frac{1}{x+3}\\
\text{The inverse is } x &= \frac{1}{y+3} \qquad \text{(Interchanging } x \text{ and } y.)\\
y + 3 &= \frac{1}{x}\\
y &= \frac{1}{x} - 3
\end{aligned}$$

So the inverse function is $g(x) = \dfrac{1}{x} - 3$

The domain of $g(x)$ is $x \neq 0$ and the range of $g(x)$ is $y \neq 3$.

$$\begin{aligned}
\text{Check: } g(f(x)) &= g\left(\frac{1}{x+3}\right)\\
&= \frac{1}{\frac{1}{x+3}} - 3\\
&= x + 3 - 3\\
&= x
\end{aligned}
\qquad\qquad
\begin{aligned}
f(g(x)) &= f\left(\frac{1}{x} - 3\right)\\
&= \frac{1}{\frac{1}{x} - 3 + 3}\\
&= x \qquad \text{as required}
\end{aligned}$$

Note: When we reflect in the line $y = x$, every vertical line becomes a horizontal line. Thus, the horizontal line test for $f(x)$ becomes a vertical line test for its reflection. So they are really the same test, one for the function and the other for the inverse.

Composite and inverse

- If $f(x) = x + 2$ and $g(x) = x^3$, then
  $f(g(x)) = f(x^3) = x^3 + 2$ and $g(f(x)) = g(x+2) = (x+2)^3$
- Two functions, $f(x)$ and $g(x)$, are inverses of each other if $f(g(x)) = x$ and $g(f(x)) = x$. The first equation must hold for all x in the domain of g and the second for all x in the domain of f.
- If $f(x)$ and $g(x)$ are inverses of each other, then the domain of f is the range of g and vice-versa.

Exercise 20E

Example 12

1 Suppose that $f(x) = x - 2$ and $g(x) = x + 5$. Calculate:

a $g(f(0))$ **b** $g(f(2))$ **c** $g(f(7))$ **d** $g(f(a))$ **e** $g(f(x))$

Interpret these calculations in terms of translations along a line.

2 If $f(x) = x - 2$ and $g(x) = x^2 - 4$, find:

a $g(f(0))$ **b** $f(g(0))$ **c** $g(f(2))$ **d** $f(g(2))$ **e** $f(f(7))$

f $g(g(2))$ **g** $f(g(x))$ **h** $g(f(x))$ **i** $f(f(x))$ **j** $g(g(x))$

k Does $f(g(x)) = g(f(x))$?

3 If $f(x) = 3x - 2$ and $g(x) = \frac{1}{3}(x + 2)$, find:

a $g(f(2))$ **b** $f(g(2))$ **c** $g(f(4))$

d $f(g(4))$ **e** $f(g(x))$ **f** $g(f(x))$

g Describe the relationship between $f(x)$ and $g(x)$.

4 If $f(x) = \frac{1}{x - 1}$ and $g(x) = \frac{x + 1}{x}$, find:

a $g(f(2))$ **b** $f(g(2))$ **c** $g(f(4))$

d $f(g(4))$ **e** $f(g(x))$ **f** $g(f(x))$

g Describe the relationship between $f(x)$ and $g(x)$.

Example 14

5 Find the inverse function $g(x)$ of each function $f(x)$. Sketch the graph of each function and its inverse function on the one set of axes and also sketch the line $y = x$.

a $f(x) = x + 5$ **b** $f(x) = 3x - 2$ **c** $f(x) = 3x + 2$

d $f(x) = 4 - 3x$ **e** $f(x) = 3 - \frac{1}{2}x$

Example 13, 15

6 For each function $f(x)$, find the inverse function $g(x)$.

a $f(x) = x^3 - 2$ **b** $f(x) = 2 - x^3$ **c** $f(x) = 32x^5$

Example 16

7 For each function $f(x)$, find the domain. Then find the inverse function $g(x)$ and its domain.

a $f(x) = \frac{1}{x} + 1$ **b** $f(x) = \frac{1}{x + 1}$ **c** $f(x) = \frac{x + 2}{x - 2}$ **d** $f(x) = \frac{3x}{x + 2}$

8 Show that each function is its own inverse.

a $f(x) = 5 - x$ **b** $f(x) = -x$ **c** $f(x) = -\frac{1}{x}$

d $f(x) = \frac{6}{x}$ **e** $f(x) = \frac{2x - 2}{x - 2}$ **f** $f(x) = \frac{-3x - 5}{x + 3}$

9 For each function $f(x)$, find its domain. Then find the inverse function $g(x)$ and its domain.

a $f(x) = 3x$ **b** $f(x) = 2^{3x}$ **c** $f(x) = 5 \times 7^x$

d $f(x) = \log_5 x$ **e** $f(x) = 2\log_4 3x$ **f** $f(x) = \log_2(x - 3)$

g $f(x) = 5^{x-1}$ **h** $f(x) = 4 + \log_4 x$ **i** $f(x) = 5^{3x} + 5$

10 Consider the graph of the circle $x^2 + y^2 = 49$. Show that it is possible, in a natural way, to divide the circle into four pieces, each of which is the graph of a function that has an inverse.

Review exercise

1 Find the domain of each function.

a $y = 4x + 3$ **b** $y = \frac{7}{x}$ **c** $y = \frac{1}{x-5}$ **d** $y = \frac{3}{x+8}$

e $y = \sqrt{x-2}$ **f** $y = 2x^2 + 3$ **g** $y = \frac{2}{x+5}$ **h** $y = \sqrt{x+6}$

2 Let $h(x) = x^2 - 4$. Calculate:

a $h(0)$ **b** $h(1)$ **c** $h(-1)$ **d** $h(-4)$

e $h(a)$ **f** $h(-a)$ **g** $h(2a)$ **h** $h(a-2)$

3 Let $h(x) = 3 - 2x$. Calculate:

a $h(0)$ **b** $h(1)$ **c** $h(-1)$ **d** $h(-4)$

e $h(a)$ **f** $h(-a)$ **g** $h(2a)$ **h** $h(a-2)$

4 State the domain and range of:

a $f(x) = 5 - 2x$

b $f(x) = 4 - x^2$

c $f(x) = \frac{2}{x+6}$

5 Let $h(x) = 4x + 2$. Sketch the graphs of:

a $y = -h(x)$

b $y = h(x) + 5$

c $y = h(x) - 2$

d $y = 2h(x)$

6 Let $f(x) = x^2 - 2$.

Sketch the graphs of $y = f(x)$, $y = -f(x)$ and $y = f(x) + 3$ on the one set of axes.

7 If $f(x) = 2x + 1$ and $g(x) = 5 - x^2$, find:

a $g(f(0))$ **b** $f(g(0))$ **c** $g(f(2))$ **d** $f(g(2))$ **e** $f(f(7))$

f $g(g(2))$ **g** $f(g(x))$ **h** $g(f(x))$ **i** $f(f(x))$ **j** $g(g(x))$

k Is it true that $f(g(x)) = g(f(x))$?

8 Find the inverse function of each function.

a $f(x) = 3x - 4$

b $f(x) = 2 - 3x$

c $y = x^3 + 2$

d $y = \frac{1}{x+2}$

Challenge exercise

1 **a** Let $f(x) = 2x$. Show that $f(a + b) = f(a) + f(b)$ and $f(ka) = kf(a)$ for all real numbers a, b and k.

b Let $f(x) = x + 2$. Show that $f(a + b) \neq f(a) + f(b)$ for any real numbers a and b. Also show that $f(ka) = kf(a)$ for all real numbers a and b unless $k = 1$.

2 **a** Let $f(x) = 2^x$. Show that $f(x + y) = f(x)f(y)$ for all real numbers x and y.

b Let $f(x) = x$. Which whole numbers x and y satisfy $f(x + y) = f(x)f(y)$?

3 Assume that the domain is the real numbers for the functions being considered in the following.

A function $f(x)$ is said to be **even** if $f(x) = f(-x)$ for all x.

A function $f(x)$ is said to be **odd** if $f(-x) = -f(x)$.

a Give an example of an even function and an odd function.

b Prove that the sum of two even functions is an even function.

c Prove that the product of two even functions is an even function.

d Prove that the product of two odd functions is an even function.

e Prove that the composition of two odd functions is an odd function.

CHAPTER

21

Review and problem-solving

21A Review

Chapter 11: Circles, hyperbolas and simultaneous equations

1 Sketch the graph of:

a $x^2 + y^2 = 49$ **b** $x^2 + y^2 = 7$

c $(x - 2)^2 + y^2 = 4$ **d** $(x + 1)^2 + (y - 2)^2 = 16$

2 Write the equation of the circle with:

a centre (3, 0) and radius 4 **b** centre (–1, 2) and radius $\sqrt{3}$

3 Express each equation in the form $(x - h)^2 + (y - k)^2 = r^2$ and hence state the coordinates of the centre and the radius of the circle.

a $x^2 - 4x + y^2 + 6y + 9 = 0$ **b** $x^2 + 2x + y^2 + 8y + 1 = 0$

4 Sketch the graph of:

a $y = \dfrac{2}{x}$ **b** $y = 2 - \dfrac{1}{x}$ **c** $y = \dfrac{3}{x - 2}$ **d** $y = \dfrac{1}{x + 3} - 2$

5 Find the intersection points of:

a $y = x^2 + 2x - 3$, $y = 3x + 3$ **b** $y = 2x^2 + 3x - 3$, $y = 2x + 3$ **c** $y = 2x + 1$, $y = \dfrac{3}{x}$ **d** $y = 3x + 7$, $y = \dfrac{6}{x}$

6 Find the intersection points of:

a $x^2 + y^2 = 9$, $y = 2$ **b** $x^2 + y^2 = 4$, $x = 1$

c $x^2 + y^2 = 4$, $y = x + 2$ **d** $x^2 + y^2 = 16$, $y = 4 - \sqrt{2x}$

7 Find the coordinates of the points of intersection of $y + 2x = 1$ and $x^2 + y^2 = 13$.

8 Find the coordinates of the points of intersection of $4y = x^2 - 4$ and $2y - x = 10$.

9 Sketch each inequality.

a $y < 2x + 3$ **b** $x + 2y \le 6$ **c** $(x - 2)^2 + y^2 \le 1$

d $x^2 + (y - 2)^2 \le 4$ **e** $x^2 + y^2 > 9$ **f** $y > \dfrac{1}{x + 1}$

10 Sketch each region.

a $y \ge x$ and $x \ge 0$ and $x + y \le 6$

b $y \ge x$ and $y \le 2x$ and $y \le 6$

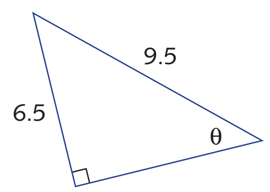
c $x \ge 0$ and $y \ge 0$ and $y \le 2x + 1$ and $x + y \le 8$

Chapter 12: Further trigonometry

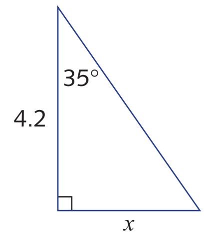
Unless otherwise stated, values should be calculated to one decimal place.

1 Find the value of each pronumeral.

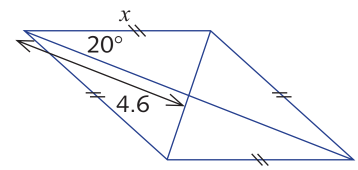
a



b



c



2 A 3 m ladder leans against a wall so that it makes an angle of 40° with the vertical.

a How far up the wall does it reach?

b How far is the foot of the ladder from the wall?

3 Find the angle of elevation of the sun when a tree 1.5 m high casts a shadow of 75 cm.

4 A hiker walks due south for 6 km then on a bearing of 270°T for 10 km and finally due north for 15 km.

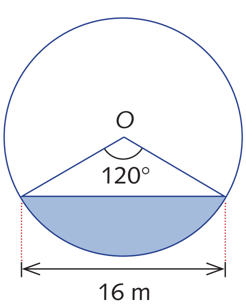
a Calculate the distance between the starting point and the finishing point.

b Calculate, to the nearest degree, the bearing of the starting point from the final position.

5 An aeroplane flies on a bearing of 060°T for 80 km and then on a bearing of 150°T for 70 km. What is the bearing of the starting point from the final position of the aeroplane?

6 An observer is 350 m from the shoreline, where a man is standing. Between the observer and the man is a sand dune 15 m high and 100 m from the sea. What is the minimum height above sea level that the observer's eye must be in order for him to see the man's feet?

7 The surface of the water in a horizontal pipe is 16 m wide and subtends an angle of 120° at the centre of the pipe, as shown. Find, correct to three decimal places:



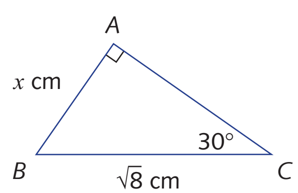
a the distance from the centre of the pipe to the water surface

b the diameter of the pipe

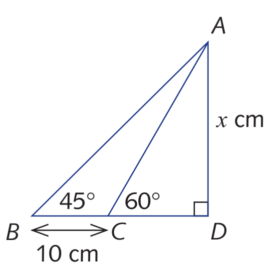
c the maximum depth of the water

8 Find the exact value of x.

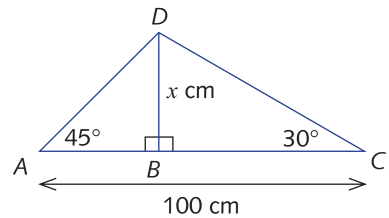
a



b



9 For the diagram shown, find the exact value of x.



10 A piece of wire 20 cm long is bent in the shape of a triangle with interior angles 30°, 60° and 90°. Find the length of the hypotenuse, giving your answer in surd form with a rational denominator.

11 A boat was sailing off the coast of Wilson's Promontory on a bearing of 350°T. At 1400 hours (2 p.m.), the bearing from the boat to South-East Point Lighthouse was 020°T and, at 1600 hours (4 p.m.), the bearing from the boat to the same lighthouse was 050°T. If the boat was travelling at 6 km/h, how far from the lighthouse was the boat at 1600 hours?

12 Find the missing side-lengths and angles for triangle ABC, given that:

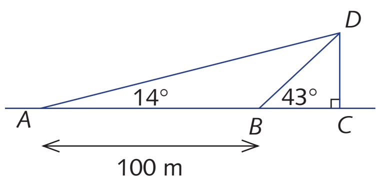
a $AB = 3$, $BC = 5$ and $\angle BAC = 50°$ **b** $AB = 6$, $AC = 4$ and $\angle ACB = 70°$

c $BC = 2$, $\angle BAC = 65°$ and $\angle ABC = 80°$

13 A hiker walks 5 km on a bearing of 143°T and then turns on a bearing of 121°T and walks a further 10 km. How far is the hiker from his starting position?

14 A scout measures the magnitudes of the angles of elevation to the top of a flagpole, CD, from two points (A and B) at ground level. A is 100 metres further away from the flagpole than B.

A, B, C and D are in the one vertical plane. If the angles are 43° and 14°, calculate the height of the flagpole, giving your answer correct to four significant figures.

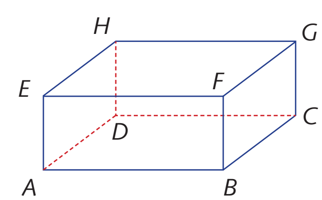


15 The bearing of a boat is taken from two points, A and B, which are on a jetty. The bearing of B from A is 090°T and $AB = 100$ m. The bearing of the boat from A is 045°T and from B is 030°. Find the distance of the boat from B, giving your answer as an exact value.

16 In the prism $ABCDEFGH$, $AB = 12$ cm, $BC = 5$ cm and $CG = 6$ cm. Find:

a the inclination of AG to the plane $ABCD$

b the inclination of HB to the plane $BCGF$



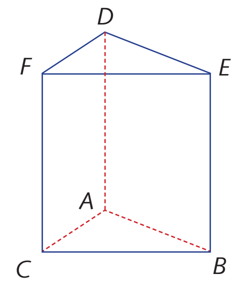
17 A right pyramid $VABCD$ stands on a square base $ABCD$ of side length 42 cm. If each sloping face makes an angle of 60° with the base, find:

a the height of the pyramid (correct to four significant figures)

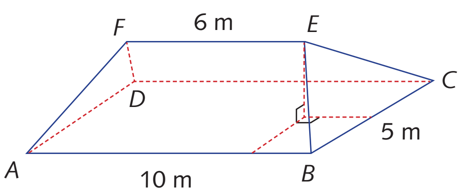
b the angle a sloping edge makes with the base (correct to one decimal place)

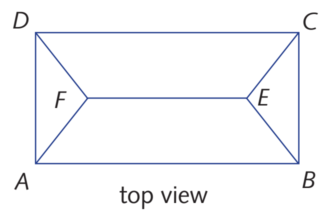
c the length of a sloping edge (correct to four significant figures)

18 *ABCDEF* is a right prism where $\angle BAC$ is a right angle. Given that $AB = 8$ cm, $AC = 3$ cm and $AD = 15$ cm, find the inclination of the interval *CE* to the face *ADEB*.



19 In the gable roof shown below, the ceiling *ABCD* lies in a horizontal plane and the slope of the opposite faces is the same. The ridge beam *FE* is parallel to the ceiling plane and 2 m above it.





top view

Calculate:

a the inclination of the face *EBC* to the ceiling

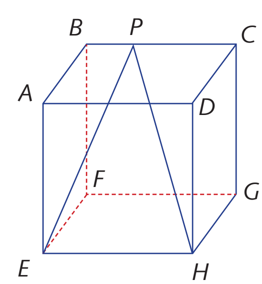
b the inclination of the rafter *EB* to the ceiling

20 *ABCDEFGH* is a cube with sides of length 5 cm.

P is a point on *BC*. Describe the location(s) of *P* so that $\angle EPH$ is:

a least **b** greatest

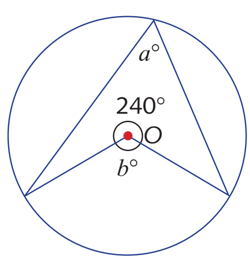
and state the size of $\angle EPH$ in each case, to two decimal places



Chapter 13: Circle geometry

1 Find the value of the pronumerals.

a



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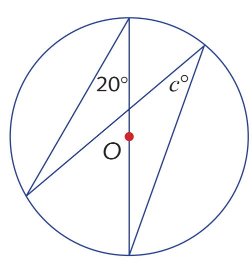

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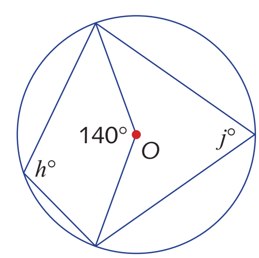

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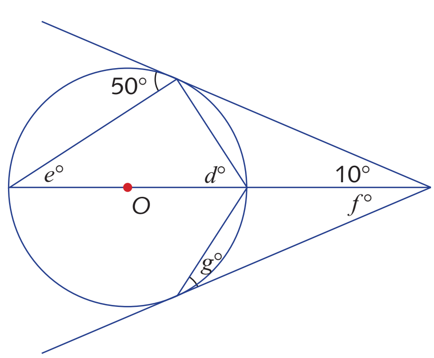
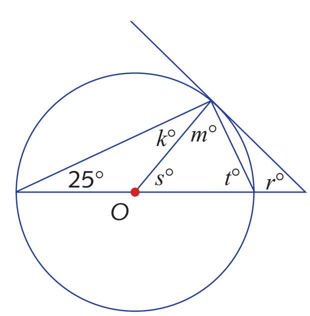
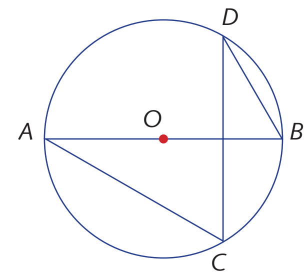


e



2 In the circle with centre O, AB is a diameter and $BC = OB$.



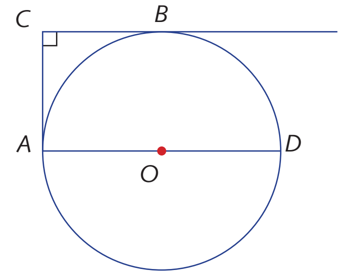
a Find the size of:

i $\angle ACB$ **ii** $\angle BOC$

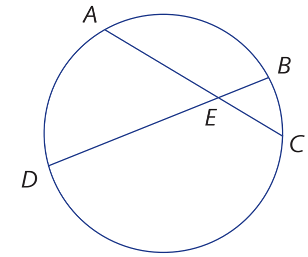
iii $\angle CAB$ **iv** $\angle CDB$

b If the radius of the circle is 6 cm, find AC.

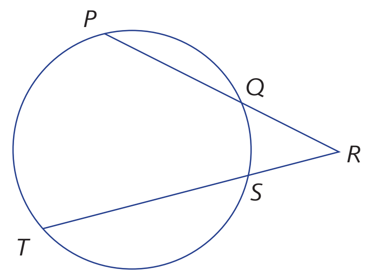
3 AD is the diameter of a circle ADB, with centre O. BC is the tangent to the circle at B, $AC \perp BC$ and AC is tangent to the circle at A. Prove that BA bisects $\angle CAD$.



4 AC and BD are two chords of a circle intersecting internally at E. Given that $AE = 6$ cm, $EC = 3$ cm and $DE = 9$ cm, find the length of BE.



5 PQ and TS are two secants of a circle intersecting externally at R. Given that $PQ = 5$ cm, $QR = 7$ cm and $SR = 4$ cm, find the length of TS.



6 $ABCD$ is a cyclic quadrilateral with BA and CD extended to meet at E. If $AD = 2$ cm, $BC = 5$ cm, $EA = 4$ cm and $AB = 11$ cm, find EC and ED.

7 P is a point inside triangle ABC. BP is extended to cut AC at Q and CP is extended to cut AB at R. If $BP \times PQ = CP \times PR$, prove that $AR \times AB = AQ \times AC$.

8 PT is a tangent to a circle where T is the point of tangency, and PXY is a secant.

a If $PT = 6$ cm and $PX = 4$ cm, find XY and PY.

b If $XY = 24$ cm and $PX = 3$ cm, find PT.

c If $XY = 21$ cm and $PT = 10$ cm, find PX.

9 AB is a chord of a circle ABC with centre O and TC is a tangent at C. If $\angle BCT = 75°$, find the size of $\angle BOC$.

10 AB is a chord of a circle and XAY is the tangent at A. AK and AL are chords bisecting $\angle XAB$ and $\angle YAB$, respectively. Prove that:

a $AL = BL$ **b** KL is the diameter of the circle

Chapter 14: Indices, exponentials and logarithms – part 2

1 Calculate each logarithm.

a $\log_2 16$ **b** $\log_3 81$ **c** $\log_2 1024$

d $\log_7 1$ **e** $\log_{10} 100\,000$

2 Calculate each logarithm.

a $\log_2 \frac{1}{32}$ **b** $\log_3 \frac{1}{243}$ **c** $\log_{10} \frac{1}{10\,000}$ **d** $\log_{10} 0.01$

e $\log_5 \frac{1}{625}$ **f** $\log_6 \frac{1}{216}$ **g** $\log_2 \frac{1}{2048}$ **h** $\log_{10} 0.000\,01$

3 Simplify:

a $\log_2 15 + \log_2 5$ **b** $\log_2 7 + \log_2 9$ **c** $\log_2 11 + \log_2 3$

d $\log_3 1000 - \log_3 10$ **e** $\log_7 200 - \log_7 5$ **f** $\log_7 42 - \log_7 6$

g $\log_3 15 - \log_3 45$ **h** $\log_5 1000 - \log_5 200$ **i** $\log_5 30 - \log_5 6$

4 Simplify:

a $\log_2 7 - \log_2 11 + \log_2 22$ **b** $\log_3 1000 - \log_3 10 - \log_3 5$

c $\log_5 7 + \log_5 49 - 2\log_5 343$ **d** $\log_{11} 25 + \log_{11} 3 - \log_{11} 125$

5 Solve each logarithmic equation for x.

a $\log_5 x = 3$ **b** $\log_2 x = 8$ **c** $\log_5(x + 5) = 4$

d $\log_2(6x - 3) = 10$ **e** $\log_2(5 - x) = 6$ **f** $\log_{10}(2x - 1) = 4$

6 Solve each logarithmic equation for x.

a $\log_x 27 = 3$ **b** $\log_x 16 = 6$ **c** $\log_x 2048 = 6$ **d** $\log_x 1000 = 3$

7 Sketch each graph.

a $y = \log_5 x, \quad x > 0$ **b** $y = \log_3(x - 2), \quad x > 2$

c $y = \log_3(x + 5), \quad x > -5$ **d** $y = 3\log_2 x, \quad x > 0$

e $y = \log_3(x) - 2, \quad x > 0$

Chapter 15: Probability

1 A fair die is rolled once. Find the probability that the number showing on the die is:

a divisible by 3 **b** an even number

2 A card is drawn at random from a standard deck of playing cards. Find the probability that the card is:

a a Heart **b** a Jack **c** the Ace of Hearts

d a court card (i.e. a Jack, King or Queen)

3 Two thousand tickets are sold in a raffle. If you buy 10 tickets, what is the probability that you will win first prize?

4 A fair coin is tossed 5 times. What is the probability of getting 3 heads from the 5 tosses?

5 Two dice are rolled and the sum of the values on the uppermost faces is noted. Find the probability that the sum is:

a 10

b 12

c less than 9

6 From a box containing 6 red and 4 blue spheres, 2 spheres are taken at random:

i with replacement

ii without replacement

In each case, find the probability that:

a both spheres are blue

b one is red and one is blue

7 A group of 1000 people, eligible to vote, were asked their age and their preferred candidate in an upcoming election, with the following results.

	18–25 years	26–40 years	Over 40 years	Total
Candidate A	200	100	85	385
Candidate B	250	230	50	530
Candidate C	50	20	15	85
Total	500	350	150	1000

What is the probability that a person chosen at random from this group:

a is between 18 and 25 years old?

b prefers Candidate A?

c is between 18 and 25 years old, given that they prefer Candidate A?

d prefers Candidate A, given that they are between 18 and 25 years old?

8 $P(A) = p$, $P(B) = \frac{3p}{2}$ and $P(A \cup B) = \frac{2}{3}$. Find p if:

a A and B are mutually exclusive

b A and B are independent

9 Of the patients reporting to a clinic, 35% have a headache, 50% have a fever, and 10% have both.

a What is the probability that a patient selected at random has either a headache, a fever or both?

b Are the events 'headache' and 'fever' independent? Explain your answer.

10 Records indicate that 60% of secondary students participate in sport, and 50% of secondary students regularly read books for leisure. They also show that 20% of students participate in sport and also read books for leisure. Use this information to find:

a the probability that a person selected at random does not read books for leisure

b the probability that a person selected at random does not read books for leisure, given that they do not participate in sport

Chapter 16: Direct and inverse proportion

1 In each of the following:

i find the constant of proportion and the formula for y in terms of x

ii find the missing numbers in the tables

a

x	1	4	8	
y		2	4	10

$y \propto x$

b

x	1	3	5	
y	2	18		14

$y \propto x^2$

c

x	2	5	7	11
y	$\frac{5}{2}$	1		

$y \propto \frac{1}{x}$

d

x	2	3		7
y	$\frac{1}{4}$		8	$\frac{1}{49}$

$y \propto \frac{1}{x^2}$

2 Given that $y \propto \sqrt{x}$, and if $y = 27$ when $x = 9$, find the formula for y in terms of x, and find:

a y when $x = 4$

b x when $y = 75$

3 The surface area of a sphere is directly proportional to the square of the radius. If the surface area of a spherical ball of radius 7 cm is 616 cm^2, find the surface area of a sphere of radius 3.5 cm.

4 Given that y is inversely proportional to x^2 and $y = 10$ when $x = 2$, find the formula for y in terms of x, and find:

a y when $x = 9$

b x when $y = 9$

5 Given that $c \propto ab^2$, find:

a the constant of proportionality and the formula for c in terms of a and b

b the missing numbers in the table

a	5		6	
b	1	2		3
c	10	24	48	54

6 a is proportional to x and inversely proportional to y. If $a = 8$ when $x = 7$ and $y = 14$, find a when $x = 14$ and $y = 7$.

7 z is proportional to the square of x and proportional to the square root of y. If $z = 72$ when $x = 2$ and $y = 4$, find z when $x = 3$ and $y = 9$.

8 The energy of a moving body is proportional to its mass and the square of its velocity. A mass of 3 kg has a velocity of 10 m/sec and its kinetic energy is 150 joule.

a Find the kinetic energy of a mass of 5 kg, moving with a velocity of 30 m/sec.

b What is the effect on the kinetic energy of doubling the mass and doubling the velocity?

Chapter 17: Polynomials

1 Let $P(x) = x^3 - 2x + 4$. Find:

a $P(1)$ **b** $P(-1)$ **c** $P(2)$

d $P(-2)$ **e** $P(0)$ **f** $P(a)$

2 a Find a, if $P(x) = x^4 - 3x^2 - 5x + a$ and $P(2) = 1$.

b Find b, if $Q(x) = x^3 - 3x^2 + bx + 6$ and $Q(-1) = 0$.

3 Find the sum $P(x) + Q(x)$ and the difference $P(x) - Q(x)$, given that:

a $P(x) = x^3 + 4x + 7$ and $Q(x) = -2x^3 + 3x^2 - 4x$

b $P(x) = -3x^5 - 3x + 7$ and $Q(x) = 3x^5 + x^2 - 7$

c $P(x) = 4x^3 - 5x^2 - 6x + 6$ and $Q(x) = -4x^3 + 5x^2 + 5x - 4$

4 Use the division algorithm to divide $P(x)$ by $D(x)$. Express each result in the form $P(x) = D(x)Q(x) + R(x)$, where either $R(x) = 0$ or the degree of $R(x)$ is less than the degree of $D(x)$.

a $P(x) = x^2 + 8x + 6$, $D(x) = x + 2$

b $P(x) = x^3 - 6x^2 - 12x + 30$, $D(x) = x + 6$

c $P(x) = 5x^3 - 7x^2 - 1$, $D(x) = x - 1$

5 Use the remainder theorem to find the remainder when the polynomial $P(x) = x^3 + 2x^2 - x + 3$ is divided by:

a $x - 3$ **b** $x - \frac{1}{2}$ **c** $x + \frac{1}{2}$

6 Find the value of a in the polynomial $ax^3 + 2x^2 + 3$ if the remainder is 3 when the polynomial is divided by $x - 2$.

7 Factorise each polynomial.

a $2x^3 + 5x^2 - x - 6$ **b** $2x^3 + x^2 - 7x - 6$ **c** $2x^4 - x^3 - 8x^2 + x + 6$

8 Solve each equation for x.

a $2x^3 + 5x^2 - x - 6 = 0$ **b** $2x^4 - x^3 - 8x^2 + x + 6 = 0$

9 Let $P(x) = x^3 - kx^2 + 2kx - k - 1$.

a Show that $P(x)$ is divisible by $x - 1$ for all k.

b If $P(x)$ is divisible by $x - 2$, find the value of k.

c Assuming that $x - 2$ divides $P(x)$, solve the equation $P(x) = 0$.

10 a Write $\dfrac{2x + 3}{x - 1}$ in the form $a + \dfrac{b}{x - 1}$.

b Write $\dfrac{4x^2 + 3x + 2}{x^2 + 2x}$ in the form $a + \dfrac{bx + c}{x^2 + 2x}$.

Chapter 18: Statistics

1 Calculate, correct to two decimal places, the mean and standard deviation for each data set.

a 3, 5, 6, 10, 12, 14, 11, 12, 11, 15, 5

b 7, 9, 11, 13, 15, 16, 18, 12, 11, 10, 14, 16, 18, 19

2 The body mass and heart mass of 14 ten-month old male mice are given in the table below.

Body mass (grams)	27	30	37	38	32	36	32	32	38	42	36	44	33	38
Heart mass (milligrams)	118	136	156	150	140	155	157	114	144	149	159	149	131	160

a Draw a scatter plot of the heart mass against the body mass.

b Draw a line of best fit and describe the main features of the scatter plot.

3 The following table represents the results of two different tests for a group of students.

Student	Test 1	Test 2
1	214	216
2	281	270
3	212	221
4	324	326
5	340	330
6	205	207
7	208	213
8	304	312
9	303	311

Draw the scatter plot of Test 2 against Test 1 and comment on the result.

4 A woman keeps a record of how long it takes her to get to work each day for a month. The times in minutes are as follows.

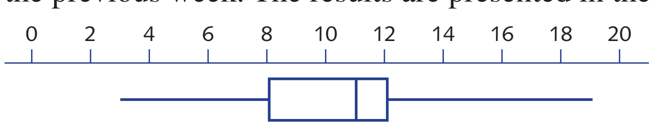
42	31	38	29	47	41	46	28	32	37	38
46	41	27	35	38	42	48	27	29	32	

a Find the median.

b Find the interquartile range.

c Use the information to construct a boxplot.

5 In a market survey, 200 people were asked how many hours of television they watched in the previous week. The results are presented in the boxplot below.



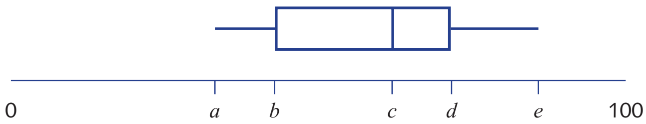
a What is the maximum number of hours anyone watched television?

b How many people watched more than 8 hours of television?

c What is the interquartile range?

d How many people watched between 8 hours and 11 hours of television?

6 **a** The boxplot shows the distribution of test scores in a class (Class A) of 20 students.



The lowest score in the class was 38, the range of the scores was 50 and the median was 61.

i Write down the values of a, c and e.

ii When all the test scores were added up the total was 1240.

What was the mean of the test scores?

b The stem-and-leaf plot shows the distribution of test scores in Class B for the same test.

4	4 7
5	2 3 3 6 9
6	2 3 7 8
7	1 5 6
8	3 6
9	0

4 | 7 is 47

i Assuming all students sat for the test, write down the number of students in Class B.

ii Find the median of the scores for Class B.

7 A community group is claiming that traffic volume on a suburban street has risen to 500 vehicles for the hour between 8 and 9 a.m. on weekdays. George lives on this street and decides to conduct his own test. The following data represents George's count of vehicles between 8 and 9 a.m. on Monday to Friday for 2 weeks.

	Monday	Tuesday	Wednesday	Thursday	Friday
Week 1	383	295	378	317	346
Week 2	15	339	311	341	357

a How might you explain the value of the outlier; that is, the value obtained for Monday of week 2?

For the remaining parts, ignore this outlier.

b Find the:

i mean, correct to one decimal place

ii median

iii interquartile range

c Represent the data as a boxplot.

d Give reasons which might explain the discrepancy between the community group's claim and the data gathered by George.

Chapter 19: Trigonometric functions

1 State which quadrant each angle is in.

a 160° **b** 245° **c** 240° **d** 300°

e 135° **f** 272° **g** 192° **h** 337°

2 Without evaluating, express each number as the trigonometric function of an acute angle.

a $\sin 175°$ **b** $\cos 150°$ **c** $\tan 160°$ **d** $\sin 200°$

e $\cos 200°$ **f** $\tan 185°$ **g** $\sin 355°$ **h** $\cos 350°$

3 Find the exact value of:

a $\cos 135°$ **b** $\sin 225°$ **c** $\sin 120°$ **d** $\tan 120°$

e $\sin 330°$ **f** $\cos 315°$ **g** $\tan 315°$ **h** $\sin 240°$

4 Without using a calculator, find the exact value of:

a $\sin 90° \times \sin 225° \times \cos 135°$ **b** $\sin 330° \times \cos 240°$

c $\sin 360° \times \cos 275°$ **d** $2 \times \sin 120° \times \cos 120°$

5 Using exact values, find the angles θ between 0° and 360° inclusive, with the given trigonometric function.

a $\cos\theta = \frac{1}{2}$ **b** $\tan\theta = -\sqrt{3}$ **c** $\sin\theta = \frac{1}{\sqrt{2}}$

d $\sin\theta = -\frac{1}{2}$ **e** $\cos\theta = -\frac{\sqrt{3}}{2}$ **f** $\tan\theta = -1$

6 Using a calculator, find, correct to two decimal places, the angles θ between 0° and 360° inclusive, such that:

a $\sin\theta = 0.2745$ **b** $\cos\theta = -0.9165$ **c** $\tan\theta = 2.2465$

d $\sin\theta = -0.8976$ **e** $\cos\theta = 0.7010$ **f** $\tan\theta = -2.5884$

7 Find, in surd form, each of the following.

a $\cos(-60°)$ **b** $\sin(-225°)$ **c** $\tan(-135°)$

d $\cos(-210°)$ **e** $\cos(-330°)$ **f** $\sin(-405°)$

8 Solve each equation for $0° \le \theta < 360°$.

a $2\cos\theta = 1$ **b** $2\cos\theta = -\sqrt{3}$ **c** $2\sin\theta + \sqrt{3} = 0$

d $6\cos\theta + 3 = 0$ **e** $8\tan\theta = 8$ **f** $\sqrt{3}\tan\theta = 1$

Chapter 20: Functions and inverse functions

1 Given that $f(x) = 2x - 1$, find:

a $f(0)$ **b** $f(4)$ **c** $f(-1)$ **d** $f(-5)$

2 The function f is defined by $f(x) = \frac{4}{x}$, $x \neq 0$. Find:

a $f\left(\frac{1}{2}\right)$ **b** $f(2)$ **c** $f(8)$ **d** $f(-2)$

3 If $f(x) = 3 - x$, find:

a $f(1)$ **b** $f(-1)$ **c** $f(5)$ **d** $f(-3)$

4 Find the value of a if:

a $f(x) = 5x - 4$ and $f(a) = 2$

b $f(x) = \frac{1}{x}\ (x \neq 0)$ and $f(a) = 5$

5 Write down the domain for each function.

a $f(x) = \frac{1}{x+2}$

b $f(x) = \frac{1}{3x-6}$

c $f(x) = \sqrt{5-x}$

d $g(x) = \sqrt{2x-4}$

e $g(x) = \frac{1}{x^2-9}$

f $f(x) = \log_2(x+7)$

g $f(x) = 2^x + 6$

h $h(x) = \log_2(2x-1)$

i $h(x) = \log_2(6-x)$

6 Sketch each function and write down its domain and its range.

a $f(x) = x^2 - 3$

b $g(x) = 6 - x^2$

c $f(x) = \log_2(x+3)$

d $g(x) = 3^x + 6$

e $h(x) = 6 - 2^x$

f $f(x) = \sqrt{16 - x^2}$

7 Let $f(x) = x^3$. Sketch the graph of $y = f(x)$, $y = f(-x)$ and $y = 2f(x)$ on the one set of axes.

8 Suppose that $f(x) = x^2$ and $g(x) = 2x - 3$. Calculate:

a $f(g(1))$

b $g(f(1))$

c $g(f(x))$

d $f(g(x))$

9 For each function $f(x)$, find the inverse function $g(x)$ and state its domain.

a $f(x) = 2x - 3$

b $f(x) = \frac{x-1}{2}$

c $f(x) = 2^x - 3$

d $f(x) = \log_3(x+1)$

e $f(x) = 8 - x^3$

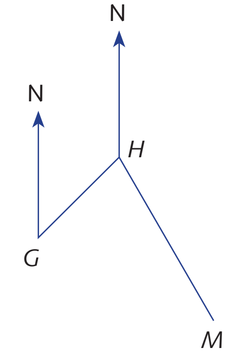
f $f(x) = x^3 - 8$

21B Problem-solving

1 A man starts from a point G and walks for 6 km on a bearing of 045° to a point H, then he walks 10 km on a bearing of 150° to a point M. From his position at M:

a how far is he from G, correct to one decimal place?

b what is the bearing of G from M, correct to one decimal place?

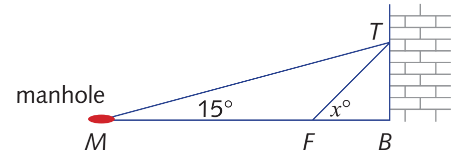


2 **a** A ladder $5\sqrt{3}$ m long leaning against a vertical wall makes an angle of $x°$ with the ground. If the foot of the ladder is a distance of $3\sqrt{3}$ m from the wall, then:

i find how far the ladder reaches up the wall

ii find x, correct to the nearest degree

b A manhole is at a point (M) where the angle of elevation to the top of the ladder (T) is 15°, as shown in the diagram.



Find the exact distance from the manhole to the foot of the ladder, given that $\tan 15° = 2 - \sqrt{3}$.

3 A ship is sailing on a bearing of 350°T. At 2 p.m., the bearing from the ship to North Cape Light is 080°T and the bearing from the ship to South Light is 105°T. It is clear from a map that the bearing from South Light to North Cape Light is 355°T, and they are 1.5 km apart.

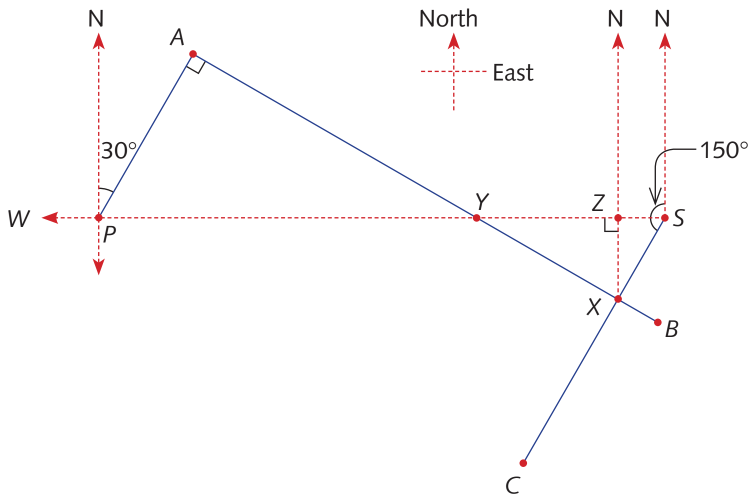
a Draw a diagram using A for the point that the bearings were taken from the ship, N for North Cape Light and S for South Light. Clearly label all bearings and true north directions.

b Draw ΔANS, indicating the angles and side lengths that are known.

c Find the distance from the 2 p.m. position of the ship to the North Cape Light, to the nearest metre.

d If the ship has maintained a constant course, find, to the nearest metre, the closest it came to South Light.

4 Pedro and Sam are both camping in the bush. Sam's campsite is 15 km due east of Pedro's campsite. At 9 a.m., they both walk out from their campsites. Initially Pedro walks 5 km to checkpoint A. From there, he turns right 90° and walks 15 km to checkpoint B. Sam just walks 10 km to checkpoint C. The paths Pedro and Sam follow from their campsites are indicated on the diagram below. The angles are given from due north. Let P and S represent Pedro and Sam's campsites, respectively.



Let X be the point where their paths cross and Y be the point of intersection of the lines PS and AX.

a Find each angle.

i $\angle APS$ **ii** $\angle AYP$ **iii** $\angle SYX$ **iv** $\angle SXY$

b **i** Find the distance PY.
ii Hence, calculate the distance YS.

c Prove that ΔPAY and ΔSXY are similar.

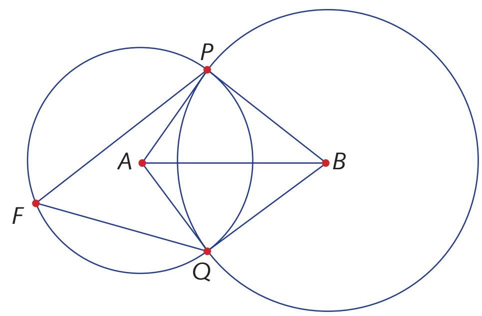
d Hence, find the distance SX.

e Find the exact values of:

i AX **ii** XB

f Hence, find how far apart Pedro and Sam finish up. Give your answer, correct to the nearest metre.

5 Two sprinklers, A and B, are set up to spray the circular areas shown in the diagram. Sprinkler A has a spray radius of 3 m and sprinkler B has a spray radius of 4 m. Points P and Q show the intersection of the circles. The sprinklers are 5 m apart.



a Explain why ΔPAB is a right-angled triangle.

b Which angle in ΔPAB is the right angle?

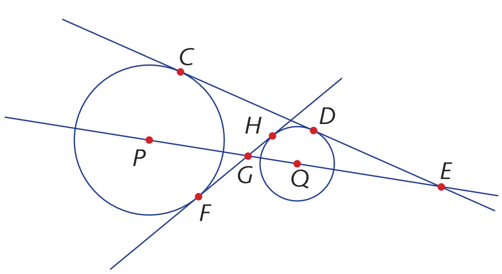
c Prove that $\Delta APB \equiv \Delta AQB$.

d Find, to the nearest degree, the size of:

i $\angle PAB$ **ii** $\angle QAP$

e F is a point on the circle with centre A.
Find $\angle PFQ$ and give a reason for your answer.

6 In the diagram to the right, the line CE and the line FH are tangents to both circles with centres P and Q. The points of tangency for CE are C and D, and the points of tangency for FH are F and H.

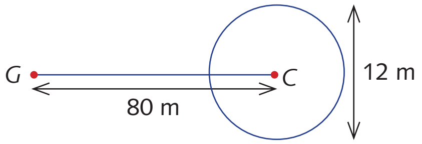


a Prove that ΔCPE is similar to ΔDQE.

b Prove that ΔGFP is similar to ΔGHQ.

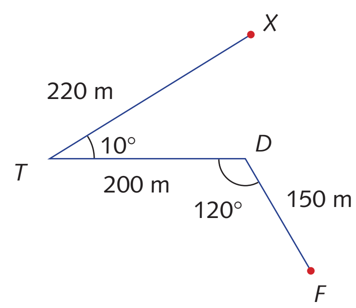
c Prove that $\dfrac{CE}{FG} = \dfrac{DE}{GH}$.

7 **a** This diagram represents a golfer at G, 80 m from the centre of a green, C, which can be represented by a circle of diameter 12 metres.

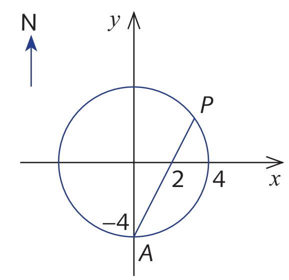


Calculate, to two decimal places, the greatest angle that the golfer can deviate either side of the direct line GC so that the golfer's ball can land on the green.

b This diagram below represents a 350 m golf hole. TD and DF represent the centre line of the fairway, with $\angle TDF = 120°$, $TD = 200$ m and $DF = 150$ m. The golfer hits 220 m, 10° left of the line TD, to a point, X. Find the distance, correct to two decimal places, from X to F.



8 Bob the gardener is planning a circular garden, as shown, that is divided into two sections by the string line AP. Dimensions are in metres. The direction north is indicated.

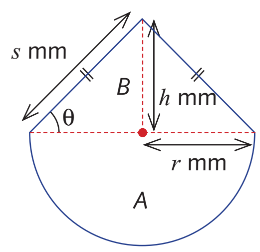


a Write down the equation of the circle.

b Find the equation for the straight line AP.

c A peg is placed at point P. By using your answers to parts **a** and **b**, find the coordinates of P, and hence state where the peg is relative to the centre of the garden.

9 The cross-section through the centre of a diamond cut at the Perfect Diamond Company is of the shape shown in the diagram. Region A is semicircular and region B is an isosceles triangle. The semicircle has radius r mm and the isosceles triangle has height h mm, slant height s mm and slant angle θ, as shown.



a Find a formula for:

i h in terms of r and θ

ii s in terms of r and θ

b Find a formula for the area of:

i region A in terms of r and π

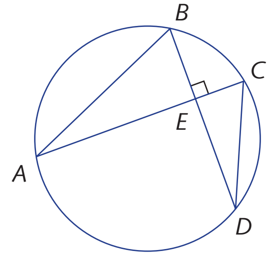
ii region B in terms of r and θ

c The Perfect Diamond Company's secret is to make sure that the cross-sectional areas of regions A and B are equal. Show that this leads to an equation that can be simplified to $\tan\theta = \dfrac{\pi}{2}$.

d Solve the equation in part **c** to find the value of θ, correct to one decimal place, for diamonds cut at the Perfect Diamond Company.

e Find, to two decimal places, the total area of the cross-section through the centre of a diamond if the radius r is 2 mm.

10 Suppose that the points A, B, C and D lie on a circle, with AC meeting BD at right-angles at E.



a If $\angle BAE = 30°$, find the size of:

i $\angle ABE$

ii $\angle CDE$

b If $\angle BAE = a°$ and $\angle ECD = b°$, find an equation relating a and b.

Next, suppose that $AC = 10$ cm, $BD = 10$ cm, $AE = x$ cm and $BE = y$ cm.

c Find:

i CE in terms of x

ii DE in terms of y

d Show that $x = y$ or $x + y = 10$.

e Find, in terms of x and y:

i area ΔABE

ii area ΔCED

Next, suppose that area (ΔABE) = area (ΔCED).

f Show that $x + y = 10$.

g Find the area of ΔABE in terms of x.

h find x if that the area of ΔABE is 12 cm^2.

Answers to exercises

Chapter 1 answers

Exercise 1A

1 a 0.56 b 0.082 c 0.12
d 0.0375 e 2.15 f 0.008
g 0.8825 h 0.008 75

2 a $\frac{9}{20}$ b $\frac{16}{25}$ c $\frac{27}{40}$
d $\frac{2}{3}$ e $\frac{33}{400}$ f $\frac{7}{125}$
g $\frac{6}{5}$ h $\frac{3}{2}$ i $\frac{29}{400}$
j $\frac{51}{400}$ k $\frac{1}{200}$ l $\frac{39}{500}$

3 a 80% b 87.5% c 43.75%
d 150% e 45% f $166\frac{2}{3}$%
g 46% h 2.5% i 140%
j 112.5% k 0.075% l 225%

4 a $\frac{16}{25}$, 0.64 b 60%, 0.6 c 16%, $\frac{4}{25}$
d $\frac{41}{200}$, 0.205 e 140%, $1\frac{2}{5}$ f 62.5%, 0.625

5 a 9 b 188.64 c 523.2
d 1011.6 e 2.77 f 1.1

6 a $26 b $254.10 c $1749
d $2.70 e $165 f $7.95

7 a 6% b 1.3%
c 4.7% d 160%

8 a 4.80% b 1.70% c 4.17%
d 3.13% e 3200% f 0.14%

9 45

10 a 1.508 g b 0.007 28 g

11 61 minutes and 6 seconds

12 a 16 500 b 31 500

13 a $480 b 240 mins c 240 cm
d $218 e 800 mm f $373\frac{1}{3}$ mm

14 $1900

15 a $575 b $12 929

16 a 100% profit b 17.6% loss

17 a costs $518 519, total sales $546 519
b costs $11 538 462, total sales $10 788 462

18 a i $0 ii $450
iii $4700 iv $127 700
b i 0% ii 3%
iii 12.37% iv 31.93%
c i $22 533.33 ii $34 640.00
iii $78 571.43 iv $78 000.00

19 a $1600 b $8000

20 a $720 b $2025

21 a 7.5% p.a. b 6.8% p.a.

22 a 50 years b 38 years

23 a $15 000 b $24 790.12

Exercise 1B

1 a $627 b $9786
c $483.36 d $3369.60

2 a $8100 b $4332
c $801.22 d $9139.20

3 a 35 840 b 171 360 c 278

4 a 434 mm b 88 mm c 786 mm

5 a 6% increase b 88% increase
c 22% decrease d 44% decrease

6 a $630 b $5160

7 a 200 b 7299 c 68 142

8 a 4750 megalitres b 11 340 megalitres
c 1840 megalitres d 63 330 megalitres

9 a $52 b $31.20
c $442 d $1.04

10 a $3111.11 b $726.77 c $54.44

11 a $28 000 b $5992.56 c $13 806.10
d $69 565.22 e $31 890.04 f $421 741.12
g 4% h 60.22% i −17.16%

12 a i $3014 ii $147 400 iii $9.02
b i $3570 ii $388 070 iii $2.90
c i $4862 ii $3 818 254 iii $5.17

13 a i $3102 ii $24.97 iii $373.27
b i $4061.90 ii $13.83 iii $73.50

14 a 50% b 20% c 81.8% d 72.4%

15 **a** **i** 9.09% **ii** 15.25% **iii** 78.26% **iv** 3.94%
b **i** 11.11% **ii** 21.95% **iii** 400% **iv** 4.28%

Exercise 1C

1 **a** $12 474 **b** $8042.64 **c** $233 436.82

2 $3.83

3 **a** $7706.79 **b** $96 119.13 **c** $128.24 **d** $3947.50

4 **a** 50.4%
b **i** $1503.63 **ii** $25 965.05 **iii** $324.84

5 25.75%

6 **a** decreased by 4% **b** decreased by 64%

7 **a** $2.52 **b** $2.19 **c** $1.99

8 **a** **i** $370.74 **ii** $8897.76 **iii** $4 199 930.76
b 62.9%

9 **a** 5158 **b** 51 589 **c** 61 895

10 **a** **i** 49.8°C **ii** 24.8°C **iii** 3.1°C
b **i** 20.2°C **ii** 34.2°C **iii** 68.6°C

11 **a** 18.6% **b** 11.6%

12 **a** 90.9% **b** 100.0% **c** 100.0%

13 **a** decrease of 25%
b decrease of 25%
c As multiplication is commutative, if the percentage increases and decreases are the same, but in different orders, the overall effect is the same.

Exercise 1D

1 **a** $107 000 **b** $114 490 **c** $140 255.17 **d** 40.26% **e** $40 255.17 **f** $35 000

2 **a** 59 750 **b** 76%

3 **a** **i** $1 630 910.87 **ii** 329%
b **i** $1 696 688.53 **ii** 346%

4 **a** $327 000 **b** $548 411.74 **c** 82.80% **d** $248 411.74 **e** $189 000

5 **a** $83 521 265.73 **b** $153 336 556.81 **c** $12 000

6 **a** $33 542.00 **b** $13 542.00

7 **a** $58 349.04 **b** 71%

8 **a** $154 880.98 **b** 61%

9 **a** $43 297.57 **b** $12 335.56

10 **a** 74 488 **b** 69 356 **c** 55 985 **d** 39 178

11 **a** $16 282.30, $13 282.30
b $5 512 011.72, $2 512 011.72
c $552.75, $2447.25
d $1 632 797.69, $1 367 202.31

12 **a** $5400, $5724, $6067.44
b $5551.00, $5893.38, $6256.87

13 $1 202 680.28

14 **a** 3.72% **b** 7.57%

15 **a** 32.3% **b** 33.1% **c** 33.8% **d** 34.0% **e** 34.4% **f** 34.6%
g The total percentage growth is approximately the same, but increases as the number of years increases.

16 87%

Exercise 1E

1 **a** $420 000 **b** $294 000 **c** $70 589.40 **d** 88.2% **e** $88 235.10

2 **a** $83 886.08, 73.8%
b $507 799.78

3 Depreciated value is $15 126. Price obtained is better by $4874, to the nearest dollar.

4 $2 724 000, $1 634 400, $980 640, $588 384;
$1 816 000, $1 089 600, $653 760, $392 256

5 Sandra $116 731.38, Kevin $20 971.52

6 **a** taxis 98.4%, other cars 78.3%
b taxis $156 250, other cars $2 166 757.03, difference = $2 010 507.03

7 **a** $8387 **b** $10 822 **c** $83 157 **d** 92.2% **e** $7666

8 **a** $15 523.20 **b** $33 348.49 **c** $83.2% **d** $3964.07
e **i** $2480.80 **ii** $5941.76

9 0.4096%

10 gain of 58.86%

11 **a** **i** 27.75% **ii** 27.10% **iii** 26.61% **iv** 26.49% **v** 26.26% **vi** 26.14%
b The total depreciation is approximately the same, but decreases as the number of years increases.

12 92.09% 13 2.6%

Review exercise

1 **a** \$5400 **b** \$2700 **c** \$1500

2 **a** \$5000 **b** \$3333.33 **c** \$2083.33 **d** \$4000

3 **a** 8% **b** $12\frac{1}{2}$% **c** $11\frac{1}{9}$%

4 16% 5 \$85.50

6 **a** 66 **b** 125 **c** 64 **d** 225 **e** 390 **f** 294.84

7 \$93.50 8 \$67.82

9 **a** **i** 99 **ii** 96 **iii** 99.84 **iv** 99.36
b No, they all end up close to 100.

10 **a** 2250 **b** \$2025 **c** \$1822.50 **d** 27.1%

11 **a** \$390 **b** 56%

12 7.58%

13 **a** \$4317.85 **b** \$10 235.50 **c** \$5320.45

14 **a** \$19 467.20 **b** \$16 477.04 **c** \$10 859.71 **d** $25\,000(0.92)^n$

15 **a** \$13 721.03 **b** \$7241.03 **c** 7 years

16 \$40 000

Challenge exercise

1 **a** \$37 036 **b** 0.32% decrease

2 **a** Bank B **b** \$9653.19

3 **a** $\$\frac{50}{n}$ **b** \$50.51 **c** \$47.98 **d** \$53.03

4 12.36% 5 0.8% increase

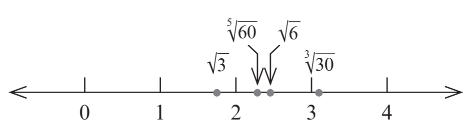
6 25% 7 80.4%

8 \$369 814.11 9 $r \approx 4.88$

Chapter 2 answers

Exercise 2A

1



2 **a** $2\sqrt{2}$ **b** $2\sqrt{3}$ **c** $4\sqrt{2}$ **d** $5\sqrt{2}$ **e** $3\sqrt{6}$ **f** $6\sqrt{3}$
g $7\sqrt{2}$ **h** $10\sqrt{2}$ **i** $12\sqrt{2}$ **j** $7\sqrt{3}$ **k** $4\sqrt{7}$ **l** $5\sqrt{7}$ **m** $7\sqrt{5}$ **n** $7\sqrt{6}$ **o** 15 **p** 30 **q** $15\sqrt{2}$ **r** $20\sqrt{2}$ **s** $10\sqrt{10}$ **t** $24\sqrt{3}$

3 **a** $10\sqrt{3}$ **b** $20\sqrt{5}$ **c** $18\sqrt{11}$ **d** $15\sqrt{6}$ **e** $24\sqrt{5}$ **f** $35\sqrt{5}$ **g** $24\sqrt{2}$ **h** $35\sqrt{2}$ **i** $66\sqrt{3}$ **j** 560 **k** $35\sqrt{3}$ **l** $12\sqrt{11}$ **m** $8\sqrt{13}$ **n** $25\sqrt{11}$ **o** $40\sqrt{3}$

4 **a** $\sqrt{8}$ **b** $\sqrt{45}$ **c** $\sqrt{147}$ **d** $\sqrt{216}$ **e** $\sqrt{300}$ **f** $\sqrt{160}$ **g** $\sqrt{605}$ **h** $\sqrt{2450}$ **i** $\sqrt{108}$ **j** $\sqrt{180}$

5 **a** $\sqrt{6}$ **b** $\sqrt{77}$ **c** $\sqrt{40} = 2\sqrt{10}$ **d** $\sqrt{39}$ **e** $\sqrt{48} = 4\sqrt{3}$

6 **a** $\sqrt{5}$ **b** $\sqrt{2}$ **c** $\sqrt{6}$ **d** $\sqrt{5}$ **e** $\sqrt{3}$ **f** $\sqrt{8} = 2\sqrt{2}$

7 **a** $\sqrt{14}$ **b** 4 **c** $3\sqrt{2}$ **d** $6\sqrt{2}$ **e** $\sqrt{6}$ **f** 0 **g** 25 **h** 3

8 **a** $\sqrt{5} \times \sqrt{6} = \sqrt{30}$ **b** $\sqrt{12} \times \sqrt{3} = \sqrt{36}$
c $\sqrt{15} \times \sqrt{3} = \sqrt{45}$ **d** $\dfrac{\sqrt{100}}{\sqrt{5}} = \sqrt{20}$
e $\dfrac{\sqrt{33}}{\sqrt{11}} = \sqrt{3}$ **f** $\sqrt{5}$

9 **a** $\sqrt{21}$ cm^2 **b** $\frac{1}{2}\sqrt{30}$ cm^2 **c** 17 cm^2 **d** $\sqrt{11}$ cm **e** $3\sqrt{7}$ cm

10 **a** $3\sqrt{2}$ **b** $\sqrt{6}$ **c** $3\sqrt{5}$ **d** $2\sqrt{2}$ **e** $5\sqrt{2}$ **f** $3\sqrt{5}$

11 **a** 112 cm^2 **b** $4\sqrt{14}$ cm

12 **a** $7\sqrt{3}$ cm^2 **b** $\sqrt{52} = 2\sqrt{13}$ cm

Exercise 2B

1 **a** $13\sqrt{2}$ **b** $25\sqrt{3}$ **c** $3\sqrt{2}$ **d** $-27\sqrt{5}$ **e** $-2\sqrt{3}$ **f** $22\sqrt{5}$

2 **a** $12\sqrt{2} + 2\sqrt{3}$ **b** $10\sqrt{7} - 10\sqrt{5}$
c $13\sqrt{11} - 3\sqrt{10}$ **d** $10\sqrt{2} - 4\sqrt{3}$
e $6\sqrt{15} - 4\sqrt{7}$ **f** $11\sqrt{5} - 2\sqrt{2}$

3 a $6\sqrt{2}$ b $5\sqrt{3}$ c $5\sqrt{5}$
d $15\sqrt{2}$ e $-5\sqrt{3}$ f $-2\sqrt{5}$
g $3\sqrt{3}+4\sqrt{5}$ h $\sqrt{11}+8\sqrt{5}$ i $3\sqrt{3}-8\sqrt{10}$
j $-4\sqrt{5}-8\sqrt{2}$

4 a $5\sqrt{3}$ b $5\sqrt{2}$ c $2\sqrt{2}$
d $-3\sqrt{5}$ e $27\sqrt{2}-12\sqrt{3}$ f $45\sqrt{3}$
g $18\sqrt{5}$ h $\sqrt{7}$ i $28\sqrt{11}$
j $-15\sqrt{3}$

5 a $\sqrt{2}$ b $6\sqrt{3}$ c $6\sqrt{2}$
d $\sqrt{3}$ e $9\sqrt{2}$ f $-7\sqrt{5}$
g $5\sqrt{6}$ h $4\sqrt{3}$ i $2\sqrt{5}$
j $11\sqrt{2}$ k $\frac{5\sqrt{5}}{2}$ l $15\sqrt{2}-10\sqrt{5}$

6 a $10\sqrt{2}-4\sqrt{3}$ b $24\sqrt{2}-\sqrt{5}$
c $6\sqrt{3}+15\sqrt{10}$ d $24\sqrt{3}-19\sqrt{2}$

7 a $x=7$ b $x=5$ c $x=6$

8 a $2\sqrt{2}+2\sqrt{3}, \sqrt{6}, \sqrt{5}$ b $2\sqrt{3}+2\sqrt{5}, \sqrt{15}, 2\sqrt{2}$

Exercise 2C

1 a $\sqrt{15}$ b $\sqrt{55}$ c $\sqrt{112}=4\sqrt{7}$
d $\sqrt{65}$ e $\sqrt{12}=2\sqrt{3}$ f $\sqrt{54}=3\sqrt{6}$

2 a $\sqrt{5}$ b $\sqrt{2}$ c $\sqrt{6}$
d $\sqrt{5}$ e $\sqrt{3}$ f $\sqrt{8}=2\sqrt{2}$

3 a $12\sqrt{10}$ b $77\sqrt{15}$ c $54\sqrt{35}$
d $32\sqrt{6}$ e $-30\sqrt{6}$ f $-28\sqrt{22}$
g $12\sqrt{77}$ h $18\sqrt{14}$ i $-22\sqrt{42}$

4 a $2\sqrt{3}$ b $\frac{5\sqrt{5}}{2}$ c $\frac{2\sqrt{6}}{3}$
d $\frac{-5\sqrt{2}}{2}$ e $-2\sqrt{3}$ f $\frac{1}{3\sqrt{2}}$

5 a 2 b 11 c 9
d 40 e 30 f 90
g −56 h 24 i −126

6 a $\sqrt{3}$ b $\sqrt{2}$ c $2\sqrt{2}$
d $3\sqrt{3}$ e $3\sqrt{7}$ f $6\sqrt{5}$
g $6\sqrt{2}$ h $12\sqrt{3}$ i $10\sqrt{5}$
j $8\sqrt{2}$ k $32\sqrt{2}$ l $50\sqrt{5}$

7 a $-3\sqrt{3}$ b $-3\sqrt{5}$ c $3\sqrt{3}$
d $7\sqrt{2}$ e $25\sqrt{35}$ f $24\sqrt{42}$
g $3\sqrt{3}$ h $6\sqrt{2}$ i $3\sqrt{3}$
j $16\sqrt{15}$

8 a $16\sqrt{6}$ b $43\sqrt{15}$ c $6\sqrt{6}$
d $6\sqrt{10}$ e $-17\sqrt{30}$ f $-\sqrt{40}=-2\sqrt{10}$
g $8\sqrt{2}$ h $-4\sqrt{5}$ i $2\sqrt{2}$
j $\sqrt{2}$

9 a 8 b 12 c 45
d 150 e 63 f 44
g 250 h a^2b

10 a $\sqrt{2}$ b 2 c $2\sqrt{2}$
d 4 e $4\sqrt{2}$ f 8
g $3\sqrt{3}$ h $5\sqrt{5}$ i $16\sqrt{2}$
j $54\sqrt{2}$ k $192\sqrt{3}$ l $40\sqrt{5}$
m $625\sqrt{5}$ n $56\sqrt{7}$ o 512
p $2187\sqrt{3}$

11 a $\sqrt{6}+\sqrt{10}$ b $\sqrt{35}+\sqrt{42}$ c $\sqrt{35}-\sqrt{10}$
d $6\sqrt{10}-9\sqrt{6}$ e $20\sqrt{6}+24\sqrt{15}$ f $4\sqrt{6}-4\sqrt{3}$
g $6\sqrt{15}+15$ h $18\sqrt{2}+8\sqrt{3}$ i $60\sqrt{2}-32\sqrt{5}$
j $24-6\sqrt{2}$ k $30+24\sqrt{5}$ l $36\sqrt{2}-18$
m $80-24\sqrt{10}$ n $105\sqrt{5}-42\sqrt{3}$ o $66\sqrt{2}-132\sqrt{3}$

12 a $\sqrt{15}+\sqrt{21}+\sqrt{10}+\sqrt{14}$
b $\sqrt{55}+\sqrt{30}+\sqrt{77}+\sqrt{42}$
c $\sqrt{15}-\sqrt{35}+\sqrt{6}-\sqrt{14}$
d $\sqrt{15}-\sqrt{21}+\sqrt{30}-\sqrt{42}$
e $5\sqrt{6}$
f $45-17\sqrt{15}$
g $37-10\sqrt{5}$
h $10\sqrt{6}-6\sqrt{2}+30\sqrt{3}-18$
i $-11-11\sqrt{14}$
j $2\sqrt{21}-4\sqrt{35}+\sqrt{15}-10$
k $34-9\sqrt{14}$
l $3\sqrt{21}-9\sqrt{35}-8\sqrt{3}+24\sqrt{5}$

13 a $-18\sqrt{2}$ b $\frac{-3\sqrt{2}}{2}$ c 66
d $\frac{11}{108}$ e $24\sqrt{3}$ f $1296\sqrt{3}$
g $\frac{9}{4\sqrt{3}}=\frac{3\sqrt{3}}{4}$ h −42

14 a 4 b $6-\sqrt{3}$ c $\sqrt{3}+10$
d $2\sqrt{3}$ e $3\sqrt{3}-2$ f 1
g $\sqrt{3}$ h 1

15 a $8\sqrt{6}, 4\sqrt{3}+8\sqrt{2}$ b $24, 12\sqrt{3}$
c $29, 28$ d $7+3\sqrt{5}, 6+4\sqrt{5}$

16 a $\sqrt{15}$ b $10+5\sqrt{3}+\sqrt{15}$ c $6\sqrt{5}+\sqrt{15}$

17 a $8+20\sqrt{3}$ b $79+20\sqrt{3}$

Exercise 2D

1 a $7+2\sqrt{10}$ b $10+2\sqrt{21}$ c $5-2\sqrt{6}$
d $13-2\sqrt{42}$ e $13+4\sqrt{3}$ f $22-12\sqrt{2}$
g $14+4\sqrt{6}$ h $59-24\sqrt{6}$ i $83+12\sqrt{35}$

j $98 - 24\sqrt{10}$ k $34 - 24\sqrt{2}$ l $79 - 20\sqrt{3}$

m $\frac{13}{4} - \sqrt{3}$ n $\frac{103}{4} - 5\sqrt{3}$

2 a 13 b 3 c 2
d 1 e 11 f 2
g 6 h 34 i 1
j -3 k $-\frac{1}{2}$ l $\frac{33}{4}$

3 a $7 + 4\sqrt{3}$ b $7 - 4\sqrt{3}$
c $37 + 20\sqrt{3}$ d $37 - 20\sqrt{3}$

4 a 1 b $2\sqrt{5}$ c 18 d $8\sqrt{5}$

5 a $\frac{75}{2}$ b $5\sqrt{6}$

6 a $7\sqrt{3} + 7$ b $3 - 3\sqrt{3}$ c $29\sqrt{3} + 40$
d $40\sqrt{3} + 116$ e -42

7 a $\sqrt{122}$ b $14 + \sqrt{122}$ c $\frac{37}{2}$

Exercise 2E

1 a $\frac{5\sqrt{3}}{3}$ b $3\sqrt{2}$ c $\sqrt{7}$
d $\frac{3\sqrt{5}}{5}$ e $\sqrt{3}$ f $\frac{\sqrt{10}}{2}$
g $\sqrt{2}$ h $\frac{\sqrt{2}}{3}$ i $\frac{2\sqrt{6}}{9}$
j $\frac{2\sqrt{15}}{15}$ k $\frac{\sqrt{15}}{4}$ l $\frac{1}{6}\left(8\sqrt{3} + 3\right)$
m $\frac{1}{10}\left(3\sqrt{2} - 4\right)$ n $\frac{1}{3}\left(3 + 2\sqrt{6}\right)$
o $\frac{1}{8}\left(\sqrt{10} - 2\sqrt{6}\right)$

2 a $\frac{1}{6}\left(9\sqrt{2} + 8\sqrt{3}\right)$ b $\frac{1}{35}\left(15\sqrt{7} - 14\sqrt{5}\right)$
c $\frac{1}{20}\left(5\sqrt{6} + 2\sqrt{5}\right)$ d $\frac{1}{28}\left(12\sqrt{14} - 21\sqrt{2}\right)$
e $\frac{20\sqrt{7} - 14\sqrt{5}}{35}$ f $\frac{11\sqrt{3} - 3\sqrt{11}}{33}$
g $\frac{14\sqrt{3} - 15\sqrt{7}}{21}$ h $\frac{22\sqrt{3} - 12\sqrt{11}}{33}$

3 a $\sqrt{5} + 2$ b $\frac{7 - \sqrt{6}}{43}$ c $\frac{3\left(\sqrt{6} - 2\right)}{2}$
d $6\left(2 + \sqrt{3}\right)$ e $2\left(\sqrt{3} + \sqrt{2}\right)$ f $\frac{4\left(\sqrt{5} + \sqrt{2}\right)}{3}$
g $\frac{1}{3}\left(\sqrt{15} - \sqrt{6}\right)$ h $\sqrt{21} - \sqrt{14}$ i $\sqrt{5} + \sqrt{3}$
j $\sqrt{7} + \sqrt{3}$ k $\frac{4}{5}\left(4 + \sqrt{6}\right)$ l $\frac{2}{5}\left(2\sqrt{6} - 3\right)$
m $\frac{3}{8}\left(3\sqrt{10} + 5\sqrt{2}\right)$ n $\frac{2}{17}\left(3\sqrt{6} + \sqrt{3}\right)$ o $\frac{4}{3}\left(2\sqrt{6} - 3\sqrt{2}\right)$
p $\frac{1}{15}\left(27 + 7\sqrt{6}\right)$ q $-4 - \sqrt{15}$ r $\frac{1}{19}\left(21 + 4\sqrt{5}\right)$
s $\frac{1}{27}\left(29 + 7\sqrt{10}\right)$ t $\frac{1}{13}\left(23 - 6\sqrt{10}\right)$

4 a $\frac{1}{4}\left(5 + 5\sqrt{3} - 4\sqrt{2}\right)$ b $\frac{1}{57}\left(37\sqrt{5} + 19\sqrt{2} - 9\right)$
c $\frac{1}{17}\left(17\sqrt{10} + 4\sqrt{2} - 10\right)$ d $-\frac{2}{17}\left(3\sqrt{3} + 2\sqrt{5}\right)$
e $\frac{1}{10}\left(6\sqrt{3} - 5\sqrt{2}\right)$ f -10

5 a $\frac{13\sqrt{3}}{3}$ b $-\frac{\sqrt{3}}{3}$ c 14
d $\frac{6}{7}$ e $\frac{559\sqrt{3}}{9}$

6 $10\left(\sqrt{2} + 1\right)$

7 a $\frac{3 - \sqrt{3}}{2}$ b $\frac{16 - 9\sqrt{3}}{2}$
c $6\sqrt{3} + 10$ d $15\sqrt{3} + 26$

8 a $\frac{4}{289}\left(19 + 6\sqrt{2}\right)$ b $\frac{1}{2}\left(3\sqrt{2} - 1\right)$
c $\frac{1}{4}\left(19 - 6\sqrt{2}\right)$ d $\frac{1}{34}\left(63\sqrt{2} - 13\right)$
e $\frac{1}{578}\left(915\sqrt{2} - 137\right)$ f $\frac{1}{1156}\left(5795 - 1638\sqrt{2}\right)$

9 i a $\frac{1}{8}\left(15 - 5\sqrt{5}\right)$ b $\frac{1}{5}\left(5 + \sqrt{5}\right)$
c $\frac{1}{5}\left(6 + 2\sqrt{5}\right)$ d $\frac{1}{20}\left(45 - \sqrt{5}\right)$
e $\frac{1}{40}\left(115 - 17\sqrt{5}\right)$ f $\frac{1}{40}\left(123 - 9\sqrt{5}\right)$
ii a $588 + 240\sqrt{6}$ b $\frac{1}{6}\left(5\sqrt{3} - 6\sqrt{2}\right)$
c $\frac{1}{36}\left(147 - 60\sqrt{6}\right)$ d $\frac{1}{6}\left(65\sqrt{3} + 66\sqrt{2}\right)$
e $\frac{1}{6}\left(3528 + 1440\sqrt{6} + 5\sqrt{3} - 6\sqrt{2}\right)$
f $\frac{5}{12}\left(1421 + 572\sqrt{6}\right)$

Review exercise

1 a $4\sqrt{2}$ b $\sqrt{6}$
c $3\sqrt{3} - \sqrt{2}$ d $4\sqrt{2} - 3\sqrt{5}$

2 a $5\sqrt{2}$ b $\sqrt{2}$
c $-4\sqrt{7}$ d $11\sqrt{3}$

3 a $30\sqrt{2}$ b $30\sqrt{2}$ c $12\sqrt{10}$ d $28\sqrt{42}$

4 a $6\sqrt{2}$ b $3\sqrt{5}$ c $2\sqrt{6}$
d $3\sqrt{3}$ e $4\sqrt{5}$ f $2\sqrt{11}$
g $6\sqrt{2}$ h $8\sqrt{3}$ i $45\sqrt{2}$
j $18\sqrt{3}$ k $40\sqrt{2}$

5 a $\sqrt{75}$ b $\sqrt{112}$ c $\sqrt{242}$
d $\sqrt{125}$ e $\sqrt{384}$ f $\sqrt{891}$
g $\sqrt{208}$ h $\sqrt{176}$

6 a $9\sqrt{2}$ b $2\sqrt{5}+5\sqrt{3}$ c $3\sqrt{3}$
d $19\sqrt{7}$ e $11\sqrt{5}$ f $55\sqrt{6}$

7 a $19\sqrt{2}$ b $4\sqrt{2}$
c $15\sqrt{2}+\sqrt{6}$ d $16\sqrt{7}-27\sqrt{3}$

8 a $6\sqrt{3}+6$ b $30-10\sqrt{2}$ c $24-16\sqrt{21}$
d $30\sqrt{5}-50$ e $12\sqrt{7}-21$ f $45-12\sqrt{6}$

9 a $10-\sqrt{2}$ b $32-9\sqrt{3}$ c 2
d 17 e 50 f $8+2\sqrt{15}$
g $14+4\sqrt{6}$ h $14-4\sqrt{6}$ i 10
j $a-b$ k $11-4\sqrt{7}$ l $28+10\sqrt{3}$

10 a $\frac{5\sqrt{3}}{3}$ b $\frac{7\sqrt{3}}{6}$ c $\frac{2\sqrt{2}}{3}$
d $\frac{3\sqrt{2}}{7}$ e $\frac{5\sqrt{3}}{9}$ f $\frac{\sqrt{6}}{4}$
g $\frac{\sqrt{42}}{7}$ h $\frac{7\sqrt{42}}{12}$

11 a $-\frac{1}{2}\left(\sqrt{5}+\sqrt{7}\right)$ b $\frac{1}{11}\left(10+3\sqrt{5}\right)$
c $\frac{1}{2}\left(3-\sqrt{5}\right)$ d $\frac{1}{2}\left(3+\sqrt{5}\right)$

12 a $\sqrt{2}+1$ b $2-\sqrt{3}$
c $\sqrt{3}-\sqrt{2}$ d $\sqrt{3}+\sqrt{2}$

13 $p=5, q=2$

14 a $5\sqrt{5}+2$ b $\frac{7}{3}$

15 a $2\left(\sqrt{3}+2\right)$ b 0
c $4\sqrt{3}+7$ d 1

16 a 6 b $2\sqrt{3}-2$ c $26-4\sqrt{3}$
d $12\sqrt{3}-12$ e $2-\sqrt{3}$ f $\frac{30-12\sqrt{3}}{13}$
g $\frac{22-14\sqrt{3}}{13}$ h $2\sqrt{3}+5$

17 a $4\sqrt{3}+8$ b $4\sqrt{3}+7$

18 $40-20\sqrt{3}$

19 a $\sqrt{14}$ b 6 c $\sqrt{118}$
d $\sqrt{42}$ e $\frac{30-10\sqrt{3}}{3}$ f $20\sqrt{2}-10\sqrt{6}$

20 a i $\frac{3+\sqrt{5}}{2}$ ii $\frac{\sqrt{5}-1}{2}$ iii $\frac{1+\sqrt{5}}{2}$
iv $2+\sqrt{5}$ v 2

21 $20\sqrt{2}$

22 a $-4\sqrt{3}$ b -8 c -4 d $-7-4\sqrt{3}$

23 $14\sqrt{5}$

24 a 6 b -11 c $-\frac{6}{11}$

Challenge execrise

1 a $\left(\sqrt{x}\right)^2+2\sqrt{x}\sqrt{y}+\left(\sqrt{y}\right)^2=x+y+2\sqrt{xy}$
b i $\sqrt{11}+\sqrt{5}$ ii $\sqrt{11}-\sqrt{5}$ iii $\sqrt{6}+\sqrt{5}$

2 a $a+b-a^2$ b $4(1+x^2)-1=3+4x^2$
c $2x$ d $\sqrt{a}+\sqrt{b}$
e $2a+2\sqrt{a^2-b^2}$

3 a $x=\frac{9}{4}$ b $x=9$ c $x=2$
d $x=9$ e $x=9$ f $x=8$

4 $x-2\sqrt{x}-\frac{2}{\sqrt{x}}+\frac{1}{x}+3$

5 $\frac{5^{\frac{2}{3}}}{5}$

6 $\frac{\sqrt{15}-\sqrt{5}-\sqrt{3}+1}{8}$

7 $\frac{2\sqrt{3}+3\sqrt{2}-\sqrt{30}}{12}$

8 $a+b-\sqrt{ab}$

9 a
$$\sqrt{9\frac{9}{80}}=\sqrt{\frac{729}{80}}=9\sqrt{\frac{9}{80}}$$
$$\sqrt{4\frac{4}{15}}=\sqrt{\frac{64}{15}}=4\sqrt{\frac{4}{15}}$$
b
$$\sqrt{a+\frac{b}{c}}=a\sqrt{\frac{b}{c}}$$
$$a+\frac{b}{c}=a^2\frac{b}{c}$$
$$a=\frac{b}{c}(a^2-1)$$
$$\frac{a}{a^2-1}=\frac{b}{c}$$
Mixed numbers of the form $a\frac{a}{a^2-1}$.

10 $(2\sqrt{6}+5)$ cm

11 1

12 $\frac{2}{5}$

13 a 625 b 1799

14 a i $\sqrt{5}$ ii $\sqrt{5}-1$
iii $\frac{\sqrt{5}-1}{2}$ iv $\frac{\sqrt{5}-1}{2}$
b $\left(\frac{\sqrt{5}-1}{2}\right)^2+\left(\frac{\sqrt{5}-1}{2}\right)=1$

Chapter 3 answers

Exercise 3A

1 a $8x$ b $2x$ c $-6x$
d $4a+7b$ e $2t-s\ (=-s+2t)$ f $-3x+2y$
g $4a+b$ h $-4a-4b$ i $3x-4y$

2 a $3ab$ b $6x^2y$ c $12cd^3$
d $9st^2 + s^2t$ e $2mn + 4m^2n$ f $12x^2y$

3 a $3a + 6$ b $7b + 21$ c $4a - 10$
d $24p + 108$ e $-12b - 27$ f $-20m - 25$
g $2 - 6a$ h $42 - 24b$ i $35 - 10b$
j $x + 3$ k $4a + 2$ l $4t - 3$
m $3 - 5\ell$ n $4x + \frac{1}{2}$ o $\frac{1}{5} - \frac{x}{10}$

4 a $a^2 + 4a$ b $c^2 - 5c$ c $6g^2 - 10g$
d $20h^2 - 28h$ e $12j^2 + 21j$ f $4k - 5k^2$
g $\ell - 3\ell^2$ h $6ac + 3bc$ i $10d^2 - 20de$
j $10m^2 - 20mn$ k $10x^2y + 15x$ l $6p - 15p^2q$

5 a $4a + 4b$ b $5a + 5$
c $8p - 56$ d $15 - 3p$
e $a^2 + ab$ f $6m^2 + 10m$
g $30 - 6x$ h $12a^2 - 21a$
i $12a^2 + 20a$ j $6x^2 - 21xy$

6 a $8a + 23$ b $3d + 3$ c $2e + 5$
d $13g + 10$ e $3i - 30$ f $8a^2 + 13a$
g $10b^2 - 9b$ h $2b^2 - 15b$ i 10
j $\frac{5x}{6} + 2$ k $\frac{9}{5}$ l $\frac{3x}{4} - \frac{9}{4}$

7 a $5y + 14$ b $7a - 12$
c $16b - 9$ d $a + 1$
e -5 f $x^2 + x - 6$
g $2p^2 + 7p + 5$ h $10z$
i $4y^2 - 16y$ j $15z^2 - 10z$

8 a $x^2 + 10x + 21$ b $a^2 + 13a + 40$
c $a^2 + 12a + 27$ d $x^2 - 6x + 5$
e $x^2 - 5x + 6$ f $a^2 - 11a + 28$
g $x^2 - x - 42$ h $x^2 + 8x - 33$
i $x^2 - 5x - 24$ j $6x^2 + 17x + 12$
k $15x^2 + 22x + 8$ l $6x^2 + 7x - 5$
m $20x^2 - 11x - 3$ n $2a^2 - 11a + 15$
o $6p^2 + 11p - 10$ p $12x^2 + 13x - 14$
q $15x^2 - 46x + 16$ r $6x^2 - 17x + 5$

9 a $2a^2 + 7ab + 3b^2$ b $6m^2 - 7mn - 3n^2$
c $6p^2 + pq - 5q^2$ d $3x^2 + 13ax - 10a^2$
e $6x^2 + 13xy - 5y^2$ f $6c^2 + 35cd - 6d^2$
g $8p^2 + 2pq - 3q^2$ h $3b^2 + 5ab - 2a^2$
i $15q^2 - 4pq - 4p^2$

10 a $x^2 + 2x + 1$ b $9x^2 + 12x + 4$
c $9a^2 + 24ab + 16b^2$ d $4x^2 + 12xy + 9y^2$
e $x^2 - 14x + 49$ f $x^2 - 2xy + y^2$
g $4a^2 - 12ab + 9b^2$ h $9a^2 - 24ab + 16b^2$
i $4x^2 - 12xy + 9y^2$

11 a $x^2 - 9$ b $a^2 - 49$ c $4a^2 - 9$
d $25 - x^2$ e $49 - 9y^2$ f $4m^2 - p^2$
g $9x^2 - 4y^2$ h $64m^2 - 25n^2$ i $49r^2 - 4t^2$

12 a $x^2 - 64$ b $4a^2 + 4a + 1$
c $4a^2 - 1$ d $9 - 4x^2$
e $9 + 12a + 4a^2$ f $9 - 12x + 4x^2$
g $a^2 + 6ab + 9b^2$ h $a^2 - 9b^2$
i $a^2 - 6ab + 9b^2$

Exercise 3B

1 a $a = 2$ b $b = 3$ c $x = \frac{9}{2}$
d $y = -5$ e $x = 6$ f $y = \frac{22}{5}$
g $x = 4$ h $y = 6$ i $a = 3$
j $b = -1$

2 a $x = -3$ b $x = 2$ c $x = -3$
d $x = -2$ e $x = \frac{23}{5}$ f $a = \frac{26}{17}$
g $y = -2$ h $x = \frac{55}{28}$

3 a $x > 3$ b $x \le 2$ c $x \le -4$
d $x > -8$ e $x \ge 1$ f $x < 8$
g $x > \frac{1}{2}$ h $x \le \frac{2}{3}$

4 a $x > -\frac{11}{2}$ b $x \le 7$ c $x \ge 13$
d $x < 11$ e $x \le -12$ f $x < \frac{9}{2}$
g $x < 4$ h $x \ge -\frac{29}{4}$ i $x \le -20$
j $x < 5$

5 28

6 width 5 m, length 13 m

7 17 and 25

8 \$90

9 4 km/h

10 \$840 000

Exercise 3C

1 Giorgio 7 km/h, Fred 5 km/h

2 2 L

3 17 20-cent pieces, 10 \$2 coins

4 \$3600

5 $12\frac{1}{2}$ hours, 7 hours

6 1.5 L

7 A to B 305 km, B to C 335 km

8 a \$129 b \$107, \$137

9 26 hours

10 when sales exceed \$1700

11 a $y \geq \dfrac{150 - x}{2}$

b i $y \geq 57.5$ ii $y \geq 50.5$

Exercise 3D

1 127 2 43.2 3 44.7

4 $\dfrac{6}{5}$ 5 2.45 6 2.92

7 a 339 cm^2 b 109 cm^2 c 3695 cm^2

8 a 465 b 1275 c 7260 d 5985

9 88.36 10 99 cm 11 20 mm

12 a $y = z - x$ b $r = \dfrac{A}{\pi s}$ c $y = \sqrt{\dfrac{ax}{t}}$

d $v = \sqrt{\dfrac{2w}{m}}$ e $x = \dfrac{m^2}{4}$ f $m = \dfrac{k^2}{4t}$

g $p = \dfrac{b^2}{a+b}$ h $t = \dfrac{x-1}{x+1}$ i $a = \sqrt{c^2 - b^2}$

j $r = \dfrac{n-q}{v^2}$ k $q = n - v^3 r$ l $L = g\left(\dfrac{t}{2\pi}\right)^2$

m $k = h - g\left(\dfrac{t}{2\pi}\right)^2$ n $a = \dfrac{mbc}{bc - mc - mb}$

o $A = \dfrac{B^2 + C^2}{2B}$ p $s = \dfrac{rc}{c - rv}$

Exercise 3E

1 a $(b+2)(b-2)$ b $(3+a)(3-a)$

c $(3x+4)(3x-4)$ d $(5a+1)(5a-1)$

e $16(y+2)(y-2)$ f $(a+2b)(a-2b)$

g $(3x+y)(3x-y)$ h $2(2x+1)(2x-1)$

i $3(2+y)(2-y)$ j $5(m+2n)(m-2n)$

k $3(3r+t)(3r-t)$ l $(1+3ab)(1-3ab)$

2 a $\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)$ b $\left(x+\sqrt{7}\right)\left(x-\sqrt{7}\right)$

c $\left(x+\sqrt{13}\right)\left(x-\sqrt{13}\right)$ d $\left(x+\sqrt{6}\right)\left(x-\sqrt{6}\right)$

3 a $\left(x+2\sqrt{5}\right)\left(x-2\sqrt{5}\right)$ b $\left(x+3\sqrt{2}\right)\left(x-3\sqrt{2}\right)$

c $\left(x+3\sqrt{3}\right)\left(x-3\sqrt{3}\right)$ d $\left(x+2\sqrt{6}\right)\left(x-2\sqrt{6}\right)$

e $\left(x+2\sqrt{10}\right)\left(x-2\sqrt{10}\right)$ f $\left(x+2\sqrt{7}\right)\left(x-2\sqrt{7}\right)$

g $\left(x+5\sqrt{5}\right)\left(x-5\sqrt{5}\right)$ h $\left(x+10\sqrt{2}\right)\left(x-10\sqrt{2}\right)$

i $\left(x+\dfrac{\sqrt{7}}{5}\right)\left(x-\dfrac{\sqrt{7}}{5}\right)$ j $\left(x+\dfrac{\sqrt{3}}{2}\right)\left(x-\dfrac{\sqrt{3}}{2}\right)$

k $\left(x+\dfrac{\sqrt{5}}{3}\right)\left(x-\dfrac{\sqrt{5}}{3}\right)$ l $\left(x+\dfrac{\sqrt{11}}{4}\right)\left(x-\dfrac{\sqrt{11}}{4}\right)$

4 a $(x+2+\sqrt{2})(x+2-\sqrt{2})$

b $(x-1+\sqrt{3})(x-1-\sqrt{3})$

c $(x-5+\sqrt{5})(x-5-\sqrt{5})$

d $\left(x-4+\dfrac{\sqrt{2}}{2}\right)\left(x-4-\dfrac{\sqrt{2}}{2}\right)$

e $\left(x-1+2\sqrt{3}\right)\left(x-1-2\sqrt{3}\right)$

f $\left(x+3+2\sqrt{2}\right)\left(x+3-2\sqrt{2}\right)$

g $\left(x+3+\dfrac{\sqrt{5}}{3}\right)\left(x+3-\dfrac{\sqrt{5}}{3}\right)$

h $\left(x-2+\dfrac{\sqrt{7}}{3}\right)\left(x-2-\dfrac{\sqrt{7}}{3}\right)$

i $\left(2x+1+3\sqrt{3}\right)\left(2x+1-3\sqrt{3}\right)$

j $\left(2x-3+\dfrac{\sqrt{6}}{3}\right)\left(2x-3-\dfrac{\sqrt{6}}{3}\right)$

k $\left(4x-1+\dfrac{\sqrt{15}}{5}\right)\left(4x-1-\dfrac{\sqrt{15}}{5}\right)$

l $12z(3z+2)$

5 a $2\left(x+\sqrt{7}\right)\left(x-\sqrt{7}\right)$

b $3\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right)$

c $5\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)$

d $4\left(x+\sqrt{7}\right)\left(x-\sqrt{7}\right)$

e $3\left(x+2\sqrt{3}\right)\left(x-2\sqrt{3}\right)$

f $2\left(x+3\sqrt{2}\right)\left(x-3\sqrt{2}\right)$

g $2\left(x+1+\sqrt{5}\right)\left(x+1-\sqrt{5}\right)$

h $3\left(x+2+\sqrt{2}\right)\left(x+2-\sqrt{2}\right)$

i $3\left(x-2+2\sqrt{2}\right)\left(x-2-2\sqrt{2}\right)$

j $2\left(x+1+2\sqrt{3}\right)\left(x+1-2\sqrt{3}\right)$

k $-2\left(x+3+\sqrt{2}\right)\left(x+3-\sqrt{2}\right)$

l $-4\left(x-2+\sqrt{5}\right)\left(x-2-\sqrt{5}\right)$

6 a $\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{a}+\sqrt{b}\right)$ b $\sqrt{a}+\sqrt{b}$

Exercise 3F

1 a $(x+1)(x+8)$ b $(x+2)(x+6)$

c $(x+2)(x+4)$ d $(x+6)^2$

e $(x+3)^2$ f $(x+3)(x+8)$

g $(x+3)(x-2)$ h $(x+6)(x-5)$

i $(x+8)(x-5)$ j $(x+10)(x-6)$

k $(x-3)(x-4)$ l $(x-5)^2$

m $(x-2)(x-16)$ n $(x+5)(x-7)$

o $(x+3)(x-7)$ p $3(x-1)(x-5)$

q $2(x+8)(x-3)$ r $4(x+4)(x-6)$

s $5(x+12)(x-2)$ t $3(x+1)(x-3)$

u $2(x+7)(x-4)$

2 a $(m+3)(x+2)$ b $(x-4)(x-3)$

c $(a-9)(b-7)$ d $(2x+5)(x-1)$

e $(3a+5)(b+2)$ f $(2x+y)(3x-2y)$

g $(2a-5)(b-6)$ h $(x+2)^2(x-2)$

i $(4x-5)(3-2y)$ j $(5x-2y)(2x+3y)$

3 a $(x-11)(x+7)$ b $(z-5)(z+5)$
c $2x(x-3)$ d $5a^2(4-a)$
e $(x+3)(y+2)$ f $(x+y)(x-y)$
g $(x+10)(x+4)$ h $(x-7)^2$
i $(x-\sqrt{11})(x+\sqrt{11})$ j $(x-6-3\sqrt{2})(x-6+3\sqrt{2})$
k $(11-a)(7+a)$ l $(x-13)(x-2)$
m $(z+3)^2$ n $-(x+5)(x-3)$
o $2(x-3)(x-4)$ p $3(x-8)(x-1)$
q $25(2-x)(2+x)$ r $(8p-9q)(8p+9q)$
s $(x+y)(x+z)$ t $(3a-b)(2a-3)$

Exercise 3G

1 a $(2x+1)(x+1)$ b $(5x-2)(x-1)$
c $(3x-2)(2x-1)$ d $(4x+3)(2x-1)$
e $(5x-4)(4x-3)$ f $12(x-3)(x+2)$

2 a $(2x+3)(x+2)$ b $(2x+1)(x+5)$
c $(2x+3)(x+3)$ d $(3x+2)(x+3)$
e $(3x+1)(x+3)$ f $(3x+2)(x+4)$
g $(2x+5)(x-2)$ h $(2x+1)(x-7)$
i $(3x+5)(x-2)$ j $(5x-3)(x+2)$
k $(3x-1)(x+9)$ l $(7x+10)(x-1)$
m $(5x+6)(x-2)$ n $(3x-2)(x+2)$
o $(4x-1)(x+3)$

3 a $(3x-2)(2x+1)$ b $(4x-3)(2x+1)$
c $(3x-2)(2x+3)$ d $(5x-2)(3x-2)$
e $(3x-2)^2$ f $(2x+3)^2$
g $(5x+3)(2x-3)$ h $(4x+9)(3x-2)$
i $(5x-2)(2x-3)$ j $(4x-3)(3x-2)$
k $(4x-3)(3x-1)$ l $(5x-2)(3x-5)$
m $2(5x-2)(2x-5)$ n $3(4x+3)(2x-1)$
o $2(3x-2)(2x-3)$ p $2(3x+4)(2x+3)$
q $5(4x-3)(2x+1)$ r $2(4x-3)(2x-7)$

4 a $(2x+1)(3x+2)$ b $(2x-3)(3x-1)$
c $(2x+3)(3x+5)$ d $(3x-4)(3x-2)$
e $(a-8)(a+7)$ f $-(2x-1)(3x+4)$
g $-(x-9)(x+1)$ h $4(2x+1)(3x-2)$
i $(x-8)(3x-7)$ j $5\left(2\sqrt{5}-x\right)\left(2\sqrt{5}+x\right)$
k $\left(2x-1-\sqrt{15}\right)\left(2x-1+\sqrt{15}\right)$
l $\left(3\sqrt{10}-\sqrt{2}(a-3)\right)\left(3\sqrt{10}+\sqrt{2}(a-3)\right)$

5 a $(x+y-3)(x-y)$ b $(x-y-5)(x+y)$
c $(x-4)(x-1)$ d $(x+6)(x+1)$
e $(a+2b)(a-2b-1)$ f $(3x+2y)(3x-2y-1)$
g $(a+2d)(a-2d-1)$ h $(x+3)(x-9)$
i $(2x+1)(2x+4)$ j $(x-11)(x-4)$

Exercise 3H

1 a $\frac{4x}{5}$ b $\frac{a}{4}$ c 3
d $x-6$ e $2x+9$ f -1
g $-\frac{x+4}{x-4}$ h $\frac{2x-5}{x-4}$ i $\frac{x+2}{x+4}$

2 a $\frac{9x}{20}$ b $\frac{x}{15}$ c $\frac{11x}{20}$
d $\frac{23a}{20}$ e $\frac{9x+3}{20}$ f $\frac{5-x}{12}$
g $\frac{1-2z}{35}$ h $\frac{13c+1}{15}$ i $\frac{4b+13}{12}$
j $\frac{5k-3}{42}$ k $\frac{10-5x}{6}$ l $\frac{50x-3}{30}$

3 a $x=6$ b $x=8$ c $x=3$
d $x=\frac{42}{13}$ e $x=2$ f $x=20$
g $x=9$ h $x=3$ i $x=3$
j $x=4$ k $x=3$ l $x=-2$

Exercise 3I

1 a $\frac{a+b}{b}$ b $\frac{x+y}{x-y}$
c $\frac{a-b}{a+b}$ d $\frac{b+1}{b}$
e $\frac{x+y}{x-y}$ f $3(x-y)$
g $\frac{x+4}{x-2}$ h $\frac{a-2}{2}$
i $\frac{x-3}{x+7}$ j $\frac{x+2}{x+1}$
k $\frac{2x-1}{4(x-3)}$ l $\frac{3x-y}{4x-y}$

2 a $\frac{2y+x}{3(2x+y)}$ b $\frac{(a-3)(a+2)}{a^2}$
c $\frac{x}{(x+3)}$ d $\frac{(x-2)(3x-1)}{6(x+2)}$
e $\frac{a(a-1)}{a+1}$ f $2x$
g $\frac{3x-2}{9}$ h $2x$
i $\frac{x^2}{6y}$ j $\frac{x-y}{x}$

3 a $\frac{x^2-x+6}{(x-2)(x+2)}$ b $\frac{x}{6(x-3)}$
c $\frac{4x}{x^2-9}$ d $\frac{2x+5}{(x+2)(x+3)}$
e $\frac{2}{(x-2)(x+2)(x+4)}$ f $\frac{x^2+2}{(x-1)(x-3)(x+2)}$
g $\frac{x+5}{(x-1)(x+3)}$ h $\frac{2}{2x+3}$
i $\frac{a}{(a-4)(a+1)(a-2)}$

4 **a** $\dfrac{-2(x+4)}{x+3}$ **b** $\dfrac{5x+49}{49-x^2}$

c $\dfrac{12}{a^2-1}$ **d** $\dfrac{a^2+3x^2}{2ax}$

e $\dfrac{2}{(x-4)(x-6)}$ **f** $\dfrac{1}{(2x+1)(x-1)(3x-2)}$

g $\dfrac{-4(4a-1)}{(5a+3)(2a-1)(a+4)}$ **h** $\dfrac{13a+2}{a^2-9}$

5 **a** $\dfrac{2x+3}{3x+5}$ **b** $\dfrac{x+7}{x+13}$ **c** $\dfrac{x}{x+1}$

d $\dfrac{a-11}{a-2}$ **e** $\dfrac{4x+3a}{x+2}$ **f** $\dfrac{x+1}{x+5}$

6 **a** $x = 2$ **b** $x = \dfrac{13}{2}$ **c** $x = -17$

d $x = 4$ **e** $x = -\dfrac{1}{7}$ **f** $x = 7$

7 **a** $\dfrac{28}{5x}$ **b** $x = \dfrac{28}{5}$

8 $x = -18$ **9** $x = 2$

10 **a** $x = 8$ **b** $x = \dfrac{21}{5}$ **c** $x = \dfrac{54}{13}$

d $x = 6$ **e** $x = 2$ **f** $x = \dfrac{2}{5}$

Review exercise

1 **a** $13x$ **b** $7p^3q$

c $-5ab - ab^2$ **d** $20x^3y - 10x^2y$

2 **a** $-2a - 8$ **b** $-3b - 18$ **c** $-12b + 20$

d $-20b + 35$ **e** $8p + 4$ **f** $-4p + 5$

3 **a** $d^2 - 9d$ **b** $3e^2 + e$ **c** $5f^2 + 6f$

d $-10m^2 + 8m$ **e** $-15n^2 - 21n$ **f** $6pq - 10pr$

g $-6x^2 - 15xy$ **h** $-6z^2 + 8zy$ **i** $6a + 8a^2b$

4 **a** $2c + 5$ **b** $7h + 4$

c $4 + 3q$ **d** $4ab$

e $\dfrac{19x}{12} - \dfrac{10}{3}$ **f** $b - 22$

g $12y^2 - 11y - 15$ **h** $6p^2 - 6p - 4$

i $m^2 - m - 20$ **j** $6x^2 + 7x - 10$

k $7x^2 - 36x + 5$ **l** $2m^2 + 7mn + 3n^2$

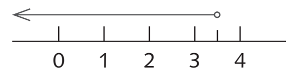
5 **a** $x = 3$ **b** $x = 5$ **c** $x = 2$

d $x = 15$ **e** $x = 4$ **f** $x = 4$

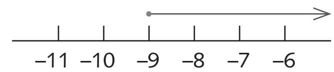
g $x = \dfrac{24}{13}$ **h** $x = 20$ **i** $x = 2$

6 **a** $x > 4$

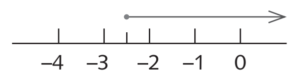
b $x < \dfrac{7}{2}$



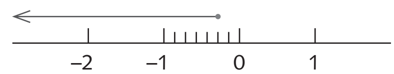
c $x \ge -9$



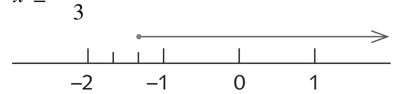
d $x \ge -\dfrac{5}{2}$



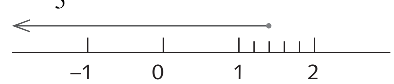
e $x \le -\dfrac{2}{7}$



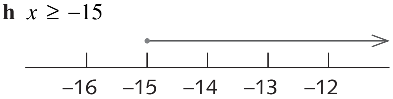
f $x \ge -\dfrac{4}{3}$



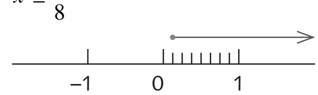
g $x \le \dfrac{7}{5}$



h $x \ge -15$



i $x \ge \dfrac{1}{8}$



7 **a** $\dfrac{12}{7}$ **b** 3

8 **a** 52 **b** $a = \dfrac{p}{4} - b$ **c** -1

9 **a** $h = \dfrac{A - \pi r^2}{\pi r}$ **b** $h = \sqrt{\left(\dfrac{A}{\pi r}\right)^2 - r^2}$

c $u = \sqrt{v^2 - \dfrac{2E}{m}}$ **d** $x = \dfrac{cb + by}{c - a} = \dfrac{b(c + y)}{c - a}$

10 **a** $(7 + z)(7 - z)$ **b** $(3 + 2a)(3 - 2a)$

c $(5 + 8x)(5 - 8x)$ **d** $(ab + 2cd)(ab - 2cd)$

e $(3x + 1)(3x - 1)$ **f** $(6p + 7q)(6p - 7q)$

g $(1 + 10b)(1 - 10b)$ **h** $(11a + 9b)(11a - 9b)$

11 **a** $\left(x - \sqrt{11}\right)\left(x + \sqrt{11}\right)$ **b** $\left(x - \sqrt{15}\right)\left(x + \sqrt{15}\right)$

c $\left(x - 2\sqrt{7}\right)\left(x + 2\sqrt{7}\right)$ **d** $\left(x - 4\sqrt{2}\right)\left(x + 4\sqrt{2}\right)$

12 **a** $\left(x+4-\sqrt{2}\right)\left(x+4+\sqrt{2}\right)$
b $\left(x-3-\sqrt{5}\right)\left(x-3+\sqrt{5}\right)$
c $\left(2x+1-2\sqrt{5}\right)\left(2x+1+2\sqrt{5}\right)$
d $\left(3-2x-\sqrt{10}\right)\left(3-2x+\sqrt{10}\right)$

13 **a** $(a+b+c)(a+b-c)$
b $(a-b+c)(a-b-c)$
c $(x+y+2z)(x+y-2z)$

14 **a** $(a+3)(2x-1)$ **b** $(a+5)(b-2)$
c $(2x+3)(y+5)$ **d** $(m-7)(3x+2)$

15 **a** $(x-1)(x-2)$ **b** $(c+1)(c+11)$
c $(x+1)(x-5)$ **d** $(y-1)(y+10)$
e $(z-8)(z+2)$ **f** $(c+10)(c-2)$
g $(p-13)(p+2)$ **h** $(k-14)(k+3)$
i $(x+10)(x-6)$

16 **a** $2(x-6)(x+11)$ **b** $3(x-7)(x+11)$
c $3(x-7)(x+1)$ **d** $3(x-5)(x+2)$
e $3(x-5)(x-6)$ **f** $6(x-1)(x+2)$

17 **a** $(2x+3)(x+4)$ **b** $(3s+2t)(2s-5t)$
c $(4x+3)(3x-4)$ **d** $(3x+2)(x+2)$
e $(x-1)(2x+3)$ **f** $(2x-1)(2x+3)$
g $(3x-1)(x+3)$ **h** $(3x-2)(2x+5)$
i $(3x-2)(x+1)$

18 **a** $x=4$ **b** $x=19$ **c** $x=11$
d $x=\dfrac{6}{7}$ **e** $x=10$ **f** $x=\dfrac{68}{11}$
g $x=-\dfrac{86}{37}$ **h** $x=33$

19 **a** $\dfrac{3x-7}{(x-2)(x-1)(x-3)}$
b $\dfrac{7}{(x-1)(x-2)}$
c $\dfrac{13}{(x-1)(x-2)(x-3)}$

20 **a** 5 **b** 70 bottles of each

Challenge exercise

1 **a** $(x+10y)(5x+4y)$ **b** $(8x+y)(2x+3y)$

2 **a** $(a-x+2b)(a-x-2b)$
b $(3a-x+c)(3a+x-c)$
c $(b+d+a-c)(b+d-a+c)$
d $(y+b+a+3x)(y+b-a-3x)$

3 **a** $a^2+b^2+c^2+2ab+2bc+2ac$
b $343a^6-27x^3$

4 **a** $4x^2y(7x-5)(x+3)$
b $(p+2q)(p-2q)\left(x+2\sqrt{2}y\right)\left(x-2\sqrt{2}y\right)$

5 $x=6$

6 **a** length 18 m, width 12 m
b length 15 m, width 12 m

7 $\dfrac{am}{bp}$ km

8 $\dfrac{cp}{p-q}$ km

Chapter 4 answers

Exercise 4A

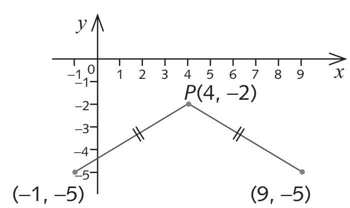
1 **a** $\sqrt{10}\approx 3.16$ **b** 4 **c** 7

2 **a** $\sqrt{5}\approx 2.24$ **b** $\sqrt{29}\approx 5.39$ **c** $4\sqrt{5}\approx 8.94$
d $\sqrt{89}\approx 9.43$ **e** $5\sqrt{2}\approx 7.07$ **f** 10

3 **a** $(1,4)$ **b** $\left(\dfrac{1}{2},5\right)$ **c** $(4,6)$
d $(-3,-7)$ **e** $\left(\dfrac{1}{2},6\right)$ **f** $(-5,12)$

4 **a** $(6,5)$
b **i** 5 **ii** 6
c AC and BC are equal, so ΔABC is isosceles.

5 **a** $u=4$ **b** $v=-1$ or 9 **c** $w=-5$ or 11



6 **a** 5 **b** $M\left(1\frac{1}{2},2\right)$
c **i** $2\frac{1}{2}$ **ii** $2\frac{1}{2}$ **iii** $2\frac{1}{2}$

7 **a** $(-4,1)$ **b** $(22,-10)$

8 PQ is perpendicular to PR.

9 $XY=XZ$

10 $AB=CD=DA=CA$

11 $(-2,-3)$

12 $AB=CD=\sqrt{20}$ and $BC=DA=\sqrt{50}$

Exercise 4B

1 **a** $\dfrac{3}{4}$ **b** 2 **c** -2
d $-\dfrac{1}{3}$ **e** $\dfrac{5}{4}$ **f** -1
g 1 **h** 0 **i** $-\dfrac{13}{9}$

2 both gradients = 1

3 $a = 8$ 4 $b = -12$

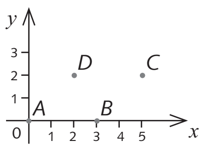
5 a 4 b −4 c 8
d 7 e 1 f −2

6 gradient of $AB = 8$, gradient of $PQ' = -\frac{1}{8}$

7 a $-\frac{1}{6}$ b 2 c $-\frac{2}{3}$
d $\frac{5}{4}$ e 1

8 a 2 b 2 c $-\frac{1}{2}$

9 a



b gradient of AB = gradient of $DC = 0$;
gradient of AD = gradient of $BC = 1$

c $\left(\frac{5}{2}, 1\right)$; Because the diagonals of a parallelogram bisect each other

10 $M_{EF} = M_{GH} = \frac{1}{4}$ and $M_{EH} = M_{FG} = 12$

11 a $\sqrt{2}$ b 1 c 1

12 a gradient of $AB = \frac{1}{2}$, gradient of $BC = \frac{7}{5}$, points not collinear
b both gradients = 3, points are collinear
c both gradients = 3, points are collinear
d gradient of $AB = \frac{11}{5}$, gradient of $BC = \frac{13}{4}$, points not collinear

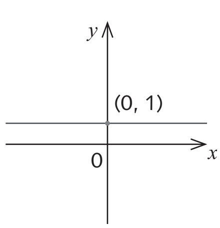
13 a 1 b 1 c (6, −3) d $\left(5\frac{1}{2}, 3\frac{1}{2}\right)$

Exercise 4C

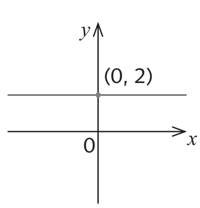
1 a gradient = 4, y-intercept is 2
b gradient = $-\frac{2}{3}$, y-intercept is 5
c gradient = −7, y-intercept is 10
d gradient = $-\frac{4}{11}$, y-intercept is $\frac{2}{3}$

2 a $y = 8x + 3$ b $y = 11x + 5$
c $y = -6x - 7$ d $y = -\frac{3}{4}x + \frac{2}{5}$

3 a



b



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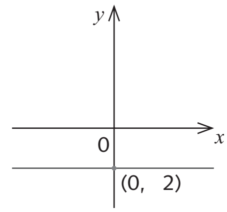
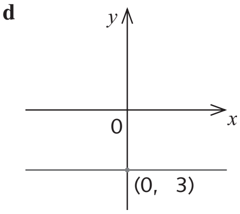
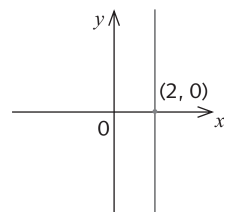


d



e



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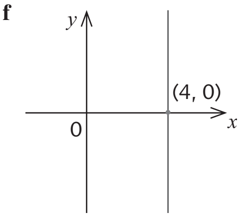

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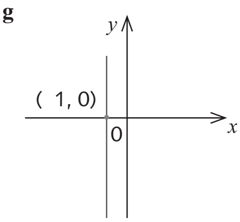

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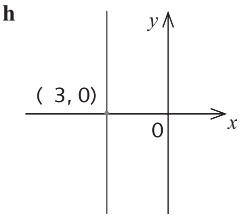

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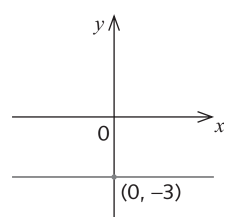

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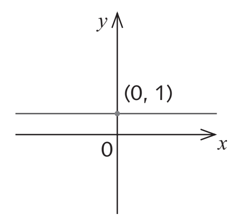
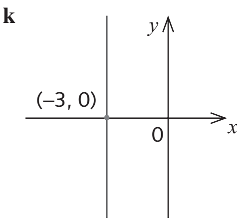


k



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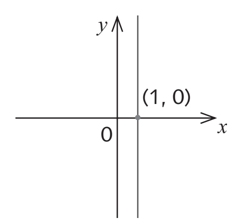


a, **b**, **c**, **d**, **i** and **j** have gradient = 0.
e, **f**, **g**, **h**, **k** and **l** do not have a gradient.

4 a $2x + y - 6 = 0$ b $2x + 3y - 33 = 0$
c $9x + 15y + 10 = 0$ d $-24x + 42y - 7 = 0$
e $2x + 5y + 4 = 0$ f $-4x + 10y + 3 = 0$
g $3x - y - 12 = 0$ h $9x - 16y + 36 = 0$

5 a $y = \frac{3}{2}x - 3$ b $y = -\frac{5}{2}x - 5$ c $y = \frac{2}{3}x - 4$
d $y = \frac{1}{6}x - 3$ e $y = \frac{2}{15}x - \frac{6}{5}$ f $y = \frac{2}{3}x + 4$
g $y = -\frac{5}{4}x - \frac{5}{2}$ h $y = \frac{3}{2}x - 6$ i $y = \frac{3}{5}x + 3$

6 a gradient = $\frac{3}{2}$, y-intercept is 6
b gradient = −4, y-intercept is −24
c gradient = $-\frac{3}{8}$, y-intercept is −6

d gradient $= \frac{4}{7}$, y-intercept is 8

e gradient $= -\frac{11}{4}$, y-intercept is 11

f gradient $= 2$, y-intercept is 4

g gradient $= \frac{3}{7}$, y-intercept is 6

h gradient $= \frac{2}{7}$, y-intercept is -2

i gradient $= -5$, y-intercept is 20

7 **a** x-intercept is 5, y-intercept is -10

b x-intercept is $-\frac{11}{3}$, y-intercept is 11

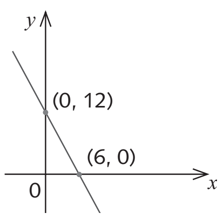
c x-intercept is 16, y-intercept is 6

d x-intercept is -16, y-intercept is 20

e x-intercept is 14, y-intercept is -6

f x-intercept is $-\frac{11}{5}$, y-intercept is $\frac{11}{2}$

8 **a**



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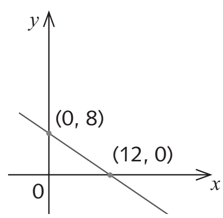

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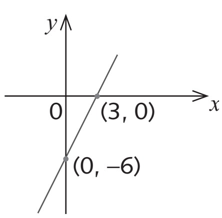

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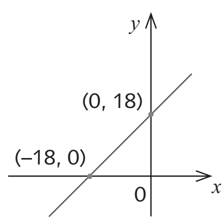

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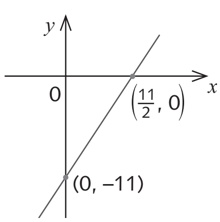

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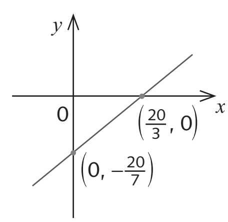

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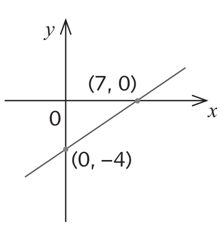

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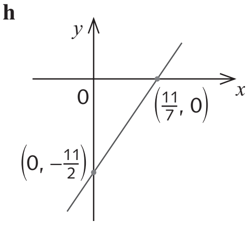
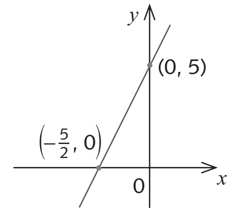
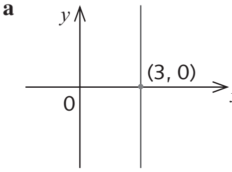


i

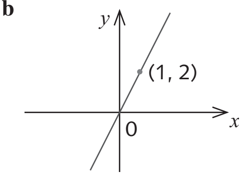


9 **a** $x = 1$ **b** $y = 5$ **c** $x = -4$

10 **a**



b



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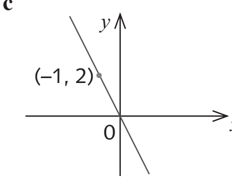
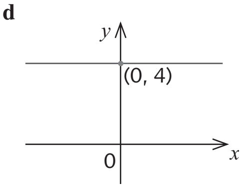


d



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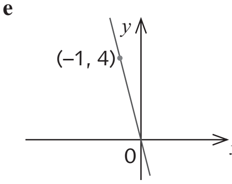

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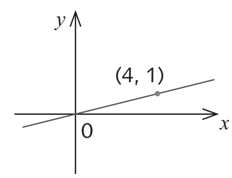


11 **a** $a = -\frac{5}{2}$ **b** $b = 10$ **c** $c = -\frac{9}{4}$

d $d = -14$ **e** $e = -\frac{3}{2}$ **f** $f = 50$

12 **a** $a = -4$ **b** $b = -8$ **c** $c = -\frac{9}{2}$

d $d = 4$ **e** $e = -6$ **f** $f = -28$

13 34 **14** $-\frac{19}{5}$

15 12 **16** -6

Exercise 4D

1 **a** $y = 6x - 24$ **b** $y = -4x + 14$ **c** $y = \frac{1}{2}x + 8\frac{1}{2}$

d $y = -3x + 11$ **e** $y = 6$ **f** $x = -7$

2 **a** $y = \frac{1}{3}x + 8\frac{2}{3}$ **b** $y = 2x$ **c** $y = -\frac{1}{2}x$

3 **a** $\left(\frac{5}{2}, \frac{5}{2}\right)$ **b** 7

c $y = 7x - 15$ **d** $y = -\frac{1}{7}x + 2\frac{6}{7}$

4 **a** $y = -4x - 6$ **b** $y = -4x + 20$

c $y = 5x + 28$ **d** $y = -2x - 12$

5 **a** $y = x - 4$ **b** $y = -3x - 9$ **c** $y = -x + 6$

d $y = 3$ **e** $x = 1$ **f** $y = \frac{9}{4}x - 3$

6 $y = 5x - 4$

7 **a** $y = \frac{1}{2}x + 5$ **b** $y = -x + 8$ **c** $y = \frac{1}{2}x + 2$

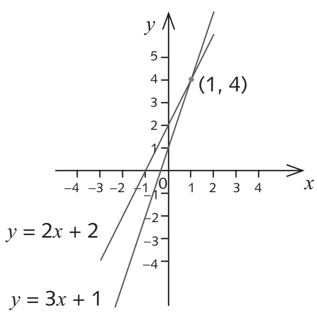
d -1 **e** $2\sqrt{2}$ **f** $2\sqrt{2}$

8 **a i** $(3\frac{1}{2}, 3\frac{1}{2})$ **ii** $(3\frac{1}{2}, 3\frac{1}{2})$

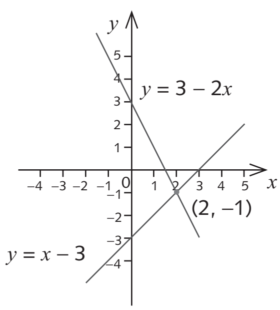
b $m_{AC} = 1, m_{BD} = -1$

Exercise 4E

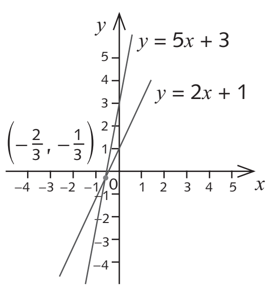
1 **a**



b



c



2 **a** $x = -\frac{9}{7}, y = -\frac{27}{7}$ **b** $x = \frac{3}{2}, y = \frac{3}{4}$

c $x = -\frac{7}{5}, y = -\frac{9}{5}$ **d** $x = \frac{13}{5}, y = -\frac{8}{15}$

e $x = -\frac{1}{7}, y = \frac{9}{7}$ **f** $x = \frac{5}{2}, y = \frac{3}{2}$

3 **a** $x = 4, y = 1$ **b** $x = 1, y = 2$

c $x = 3, y = -2$ **d** $x = -1, y = 2$

e $x = 5, y = 2$ **f** $x = -3, y = 7$

g $x = 1, y = 2$ **h** $x = 1, y = -1$

i $x = -4, y = 3$ **j** $x = 2, y = -1$

k $x = \frac{60}{17}, y = \frac{10}{17}$ **l** $x = -\frac{5}{13}, y = -\frac{46}{13}$

m $x = 3, y = 5$ **n** $x = -\frac{6}{7}, y = \frac{24}{7}$

o $x = -\frac{30}{19}, y = -\frac{121}{19}$

4 **a** $x = -\frac{28}{33}, y = -\frac{173}{33}$ **b** $x = \frac{864}{101}, y = -\frac{35}{101}$

c $x = -\frac{97}{6}, y = \frac{38}{3}$ **d** $x = -\frac{63}{19}, y = \frac{30}{19}$

e $x = \frac{9}{13}, y = -\frac{51}{13}$ **f** $x = \frac{552}{59}, y = \frac{60}{59}$

g $x = \frac{385}{24}, y = -\frac{19}{12}$ **h** $x = \frac{10}{11}, y = \frac{8}{11}$

i $x = \frac{168}{25}, y = -\frac{24}{25}$ **j** $x = 0, y = -6$

k $x = 26, y = \frac{31}{3}$ **l** $x = -\frac{220}{17}, y = \frac{64}{17}$

5 $y = 3x - 6$

6 **a** $y = 0$ **b** $x = 2$

7 $A(-1, -1), B(3, 3), C\left(\frac{11}{5}, \frac{27}{5}\right)$

8 $y = 2x + 2, x + y = 9, \left(\frac{7}{3}, \frac{20}{3}\right)$

9 **a i** $y = \frac{1}{2}x + \frac{7}{2}$ **ii** $y = -2x + 16$

b $(5, 6)$

10 **a i** $y = x$ **ii** $y = -x$

b $(0, 0)$

11 $m = -\frac{27}{4}$

12 **a** $y = \frac{1}{2}x - 1$ **b** $y = -2x + 19$ **c** $\left(\frac{26}{5}, \frac{43}{5}\right)$

d $\frac{14\sqrt{5}}{5} \approx 6.26$ **e** $5\sqrt{5} \approx 11.18$ **f** 70

13 $a = 4, b = -15$

14 The point is $(11, 5)$.

15 **a** $y = 3x$ **b** $y = x$

16 $a = 9, b = -6$

17 $\left(\frac{21}{5}, \frac{26}{5}\right)$

18 **a** $BD : 6y - x = 6;\ AC : y + 6x = 21$

b $\left(\frac{120}{37}, \frac{57}{37}\right)$ **c** $\frac{9\sqrt{37}}{37}$ **d** 9

Exercise 4F

1 45 and 67 **2** 46 and 37

3 The daughter is 8 and the father is 36.

4 58 \$2 items, 43 \$5 items

5 Andrew is 36 and Brian is 14

6 10 metres and 14 metres

7 standard: 5 kg, jumbo: 7.5 kg

8 8000 \$60 tickets sold and 2000 \$80 tickets sold

9 1.875 L **10** 5 hours **11** 3:20 p.m.

12 50 tonnes of the 52% iron alloy, 150 tonnes of the 36% iron alloy

13 480 km/h and 640 km/h

14 model horse \$30.00; model cow \$10.00

15 $\frac{2}{15}$ **16** 30 km

Review exercise

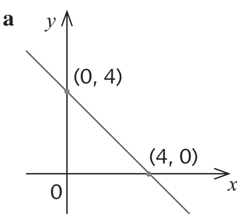
1 **a** $2\sqrt{17}$ **b** 6 **c** 12
d $\sqrt{26}$ **e** $\sqrt{34}$ **f** $4\sqrt{5}$

2 **a** $\left(\frac{3}{2}, 1\right)$ **b** $\left(-1, 4\frac{1}{2}\right)$ **c** $\left(-\frac{1}{2}, 0\right)$
d $\left(\frac{9}{2}, 5\right)$ **e** $\left(-\frac{3}{2}, \frac{13}{2}\right)$ **f** $\left(\frac{9}{2}, -3\right)$

3 **a** 4 **b** 3 **c** 3
d 1 **e** $\frac{2}{3}$ **f** $-\frac{5}{2}$
g –3 **h** 0 **i** 0

4 –4

5 **a** gradient $= 2$, y-intercept is 4
b gradient $= 1$, y-intercept is -4
c gradient $= \frac{1}{2}$, y-intercept is 1
d gradient $= -2$, y-intercept is 5
e gradient $= -1$, y-intercept is 6
f gradient $= -1$, y-intercept is 2
g gradient $= -3$, y-intercept is 4
h gradient $= -3$, y-intercept is 4
i gradient $= \frac{2}{3}$, y-intercept is -2
j gradient $= -\frac{3}{4}$, y-intercept is 3
k gradient $= 2$, y-intercept is 0
l gradient $= 3$, y-intercept is 0
m gradient $= -4$, y-intercept is 0
n gradient $= \frac{3}{2}$, y-intercept is 0
o gradient $= -\frac{3}{2}$, y-intercept is 3

6 **a**


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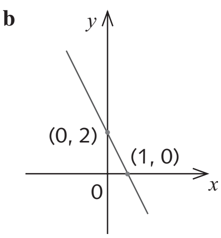

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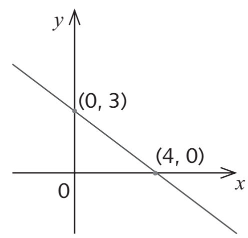

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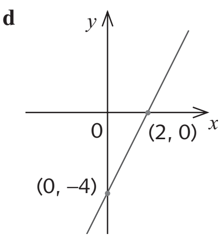

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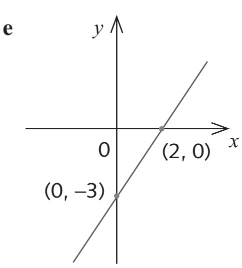

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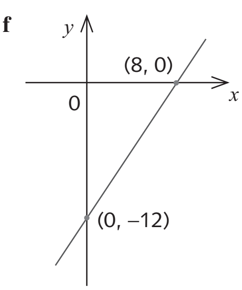
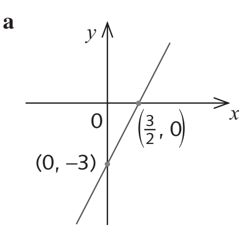


7 **a**


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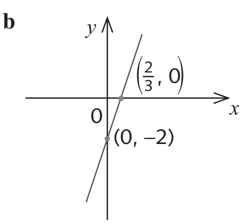

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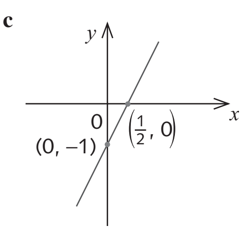

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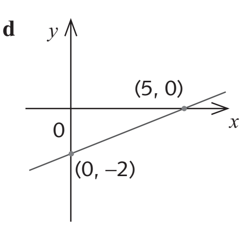

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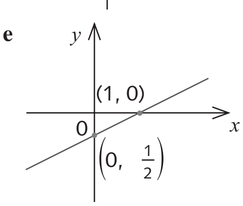

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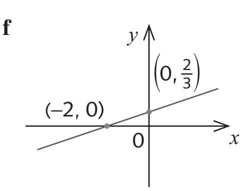

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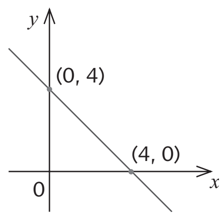

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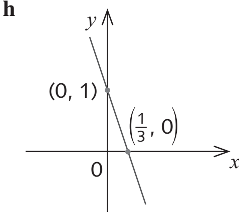

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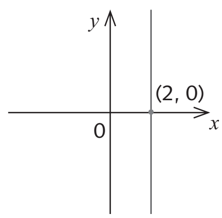

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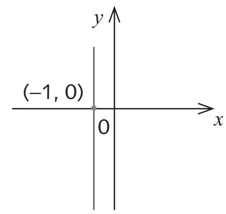

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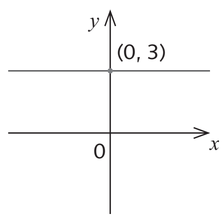

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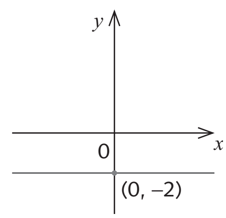
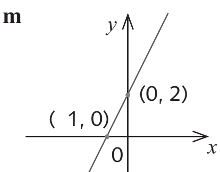


m



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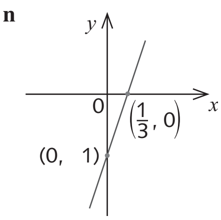

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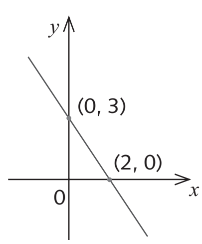

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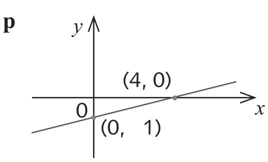
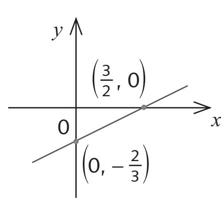
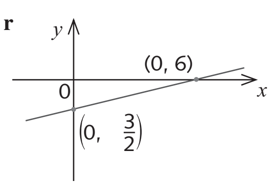


q



r



8 **a** $6x + y = 11$ **b** $2x - 3y + 14 = 0$

9 **a** $4x + 9y = 74$ **b** $4x + 5y = 32$
c $4x - 3y + 26 = 0$

10 **a** $x = 4\frac{1}{8}, y = -1\frac{7}{8}$ **b** $x = 1\frac{7}{8}, y = 1\frac{7}{8}$

11 **a** $x = -3, y = -7$ **b** $x = -2, y = 8$
c $x = 5, y = 0$ **d** $x = 1, y = 1$
e $x = \frac{156}{11}, y = \frac{42}{11}$ **f** $x = \frac{138}{65}, y = -\frac{3}{13}$

12 **a** **i** $y = -\frac{5}{6}x + \frac{139}{12}$ **ii** $y = -\frac{3}{11}x + \frac{69}{11}$
b $\left(\frac{701}{74}, \frac{273}{74}\right)$

13 $\left(\frac{57}{13}, -\frac{14}{13}\right)$

14 $y = x + 3; y = x - 2; 3x + 2y = 21; 3x + 2y = 6$

15 The gradient of $AC = -1$ and the gradient of $AB = 1$.

16 **b** $12\sqrt{2}, 4\sqrt{2}$

17 113 and 25

18 stool \$30; chair \$100

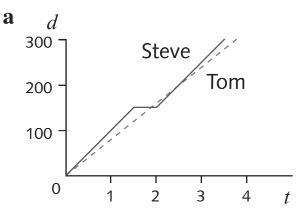
19 **a** $p = \frac{5}{2}$ **b** $p = -23$

20 $4x + 3y + 5 = 30$

21 The point is (4, 3).

Challenge exercise

1 **a**



b **i** After 1 hour $52\frac{1}{2}$ minutes, 150 km from Mildura
ii After $2\frac{1}{2}$ hours, 200 km from Mildura
iii 20 km

2 **a** $\left(\frac{1}{4}(x_1 + 3x_2), \frac{1}{4}(y_1 + 3y_2)\right)$
b $\left(\frac{1}{5}(2x_1 + 3x_2), \frac{1}{5}(2y_1 + 3y_2)\right)$

3 $\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}\right)$

4 **a** $\left(\frac{c}{2}, \frac{b}{2}\right)$
b **i** $\frac{\sqrt{c^2 + b^2}}{2}$ **ii** $\frac{\sqrt{b^2 + c^2}}{2}$ **iii** $\frac{\sqrt{c^2 + b^2}}{2}$

5 **a** **i** $y = \frac{b}{a + c}x$ **ii** $y = \frac{b}{a - c}(x - c)$
b **i** $\left(\frac{a + c}{2}\right), \frac{b}{2}$ **ii** $\left(\frac{a + c}{2}, \frac{b}{2}\right)$

6 **a** **i** $\frac{b}{a + c}$ **ii** $\frac{b}{a - c}$ **c** $\sqrt{a^2 + b^2}$

10 **a** $\frac{p}{m}; -\frac{p}{\ell}$ **b** $y = -\frac{m}{p}x + \frac{m\ell}{p}$

c $y = \frac{1}{p}x + \frac{\ell m}{p}$

Chapter 5 answers

Exercise 5A

1 **a** 0, −3 **b** 0, 7 **c** 0, −5
d 3, −6 **e** −7, −9 **f** 10, 7
g $0, -\frac{4}{5}$ **h** $-\frac{3}{4}, \frac{2}{3}$ **i** $-\frac{7}{2}, -4$
j $\frac{3}{2}, -\frac{4}{3}$ **k** $\frac{5}{2}, -4$ **l** $\frac{2}{3}, \frac{4}{3}$

2 **a** 0, 5 **b** 0, −7 **c** 0, −8
d 0, 25 **e** 0, −18 **f** $0, \frac{1}{2}$

3 **a** −1, −8 **b** −2, −6 **c** −3, −9
d −6 **e** 2, 4 **f** 2, −3
g 5, −6 **h** 5, −8 **i** 6, −10
j 1, 6 **k** 3, 4 **l** 5
m 2, 16 **n** −3, 7 **o** 10

4 **a** 0, 8 **b** 1, 16 **c** 1, 2
d no solution **e** 3, 5 **f** no solution
g 0, −3 **h** −5, 4 **i** no solution
j −10, 1 **k** 7, 1 **l** −1, 1

5 **a** −1, −2 **b** 3, 2 **c** −7, 5

6 **a** $-\frac{3}{2}, -4$ **b** $-\frac{1}{3}, -4$ **c** $-\frac{3}{2}, -2$
d $-\frac{1}{2}, 2$ **e** $\frac{3}{2}, 3$ **f** $\frac{4}{3}, 2$
g $-\frac{3}{2}, -\frac{4}{5}$ **h** $\frac{3}{2}, \frac{4}{3}$ **i** $-\frac{1}{2}, \frac{5}{4}$
j $-\frac{2}{3}, \frac{3}{4}$ **k** $-\frac{1}{4}, \frac{2}{3}$ **l** $\frac{2}{3}, -\frac{5}{2}$
m $-\frac{2}{3}, 7$ **n** $-\frac{2}{5}, -3$ **o** $-\frac{15}{12}, 3$
p $\frac{1}{15}, -15$ **q** $\frac{9}{8}, \frac{8}{9}$ **r** $-\frac{2}{3}, \frac{3}{2}$

7 **a** $\frac{1}{2}, \frac{4}{3}$ **b** $-\frac{1}{6}, \frac{5}{4}$ **c** $1, \frac{3}{4}$
d $-2, \frac{3}{2}$ **e** $\frac{2}{3}, \frac{5}{2}$ **f** $-\frac{1}{2}, \frac{4}{3}$
g $\frac{5}{2}, \frac{2}{5}$ **h** $-\frac{1}{4}, -\frac{3}{5}$ **i** $2, \frac{1}{2}$
j $3, -\frac{5}{4}$ **k** $-\frac{2}{7}, \frac{1}{3}$ **l** $\frac{5}{4}, -7$

Exercise 5B

1 **a** 9, −2 **b** 8, −4 **c** $-\frac{3}{2}, 4$
d −5, 5 **e** no solution **f** −3, −1
g 0, −5 **h** 5, 1 **i** 8, −8
j $-\frac{4}{3}, \frac{3}{2}$ **k** $\frac{3}{2}, -\frac{7}{5}$ **l** 4, 8

2 **a** −7, 2 **b** 5, −3 **c** −3, 2
d 2, −1 **e** 6, 1 **f** 16, 2
g −6, 5 **h** −5, 4 **i** $-\frac{1}{2}, -4$
j 4 **k** −13, 11 **l** $-\frac{1}{2}, 3$
m $-\frac{5}{4}, -\frac{7}{6}$ **n** $-\frac{7}{3}, 3$

3 **a** $\frac{50}{x}$ **b** $\frac{50}{x} = x + 5, x = 5$

Exercise 5C

1 11 sides **2** 12 cm **3** 16 **4** 5

5 9 and −7 or 7 and −9 **6** 8 cm

7 4 cm, 32 cm, 42 cm **8** 5 and 7

9 6 km/h

10 **a** $20 - x$ cm **b** 14 cm, 6 cm

11 4 m **12** 8 cm, 12 cm, $4\sqrt{3}$ cm

13 25 km/h **14** 20 cm and 25 cm

Exercise 5D

1 **a** −1 **b** −2 **c** −4
d 5 **e** $-\frac{1}{2}$ **f** $\frac{3}{2}$
g $\frac{2}{5}$ **h** $-\frac{3}{4}$ **i** $\frac{1}{3}$
j $-\frac{1}{5}$ **k** $\frac{2}{5}$ **l** $-\frac{2}{7}$
m $\frac{5}{7}$ **n** $-\frac{5}{3}$

2 **a** parts **i**, **v**, **ix** **b** parts **ii**, **iv**, **vii**, **x**, **xii**

3 **a** 4 **b** 16 **c** 25
d 36 **e** 100 **f** $\frac{9}{4}$
g $\frac{1}{4}$ **h** $\frac{49}{4}$ **i** $\frac{121}{4}$

4 **a** $(x+3)^2+1$ **b** $(x+4)^2-21$
c $(x+6)^2-46$ **d** $(x-5)^2-19$
e $(x-3)^2-17$ **f** $(x-10)^2-95$
g $\left(x+\frac{3}{2}\right)^2-\frac{17}{4}$ **h** $\left(x+\frac{1}{2}\right)^2+\frac{3}{4}$
i $\left(x-\frac{5}{2}\right)^2-\frac{1}{4}$ **j** $\left(x-\frac{1}{2}\right)^2-\frac{41}{4}$
k $\left(x+\frac{3}{2}\right)^2+\frac{19}{4}$ **l** $\left(x-\frac{11}{2}\right)^2-\frac{117}{4}$

5 **a** $3(x+1)^2+9$ **b** $5(x+3)^2-35$
c $3(x-2)^2+3$ **d** $-(x+1)^2+5$
e $-(x-4)^2+6$ **f** $-(x+3)^2+13$
g $3(x-1)^2-4$ **h** $2(x-3)^2+15$
i $4(x-6)^2-45$ **j** $2\left(x+\frac{3}{4}\right)^2+\frac{7}{8}$
k $4\left(x-\frac{1}{8}\right)^2-\frac{65}{16}$ **l** $3\left(x-\frac{4}{3}\right)^2+\frac{11}{3}$
m $5\left(x-\frac{1}{10}\right)^2+\frac{19}{20}$ **n** $2\left(x-\frac{5}{4}\right)^2-\frac{81}{8}$
o $-3\left(x+\frac{1}{6}\right)^2+\frac{49}{12}$

Exercise 5E

1 **a** $\pm\sqrt{5}$ **b** $\pm\sqrt{11}$ **c** $\pm\sqrt{3}$
d $\pm\sqrt{2}$ **e** $\pm\sqrt{10}$ **f** $\pm\sqrt{5}$

2 **a** $-1\pm\sqrt{2}$ **b** $-2\pm\sqrt{3}$ **c** $6\pm\sqrt{13}$
d $-3\pm\sqrt{2}$ **e** $4\pm\sqrt{17}$ **f** $-4\pm 2\sqrt{5}$
g $-5\pm 2\sqrt{6}$ **h** $-6\pm\sqrt{41}$ **i** $5\pm 5\sqrt{3}$
j $-10\pm\sqrt{95}$ **k** $50\pm 2\sqrt{645}$ **l** $25\pm\sqrt{615}$

3 **a** $\frac{-1\pm\sqrt{5}}{2}$ **b** $\frac{3\pm\sqrt{5}}{2}$ **c** $\frac{5\pm\sqrt{29}}{2}$
d $\frac{-3\pm\sqrt{17}}{2}$ **e** $\frac{-5\pm\sqrt{41}}{2}$ **f** $\frac{3\pm\sqrt{29}}{2}$
g $\frac{7\pm\sqrt{449}}{2}$ **h** $\frac{3\pm\sqrt{33}}{2}$ **i** $\frac{9\pm\sqrt{101}}{2}$
j $\frac{1\pm\sqrt{21}}{2}$ **k** $\frac{3\pm\sqrt{5}}{2}$ **l** $\frac{5\pm\sqrt{13}}{2}$

4 **a** $2\pm\sqrt{3}$ **b** $-1\pm\sqrt{5}$ **c** $-1\pm\sqrt{5}$
d $4\pm\sqrt{6}$ **e** $-3\pm\sqrt{21}$ **f** $-4\pm 2\sqrt{5}$
g $2, -\frac{1}{2}$ **h** $\frac{4\pm\sqrt{34}}{3}$ **i** $\frac{1\pm\sqrt{65}}{8}$
j $\frac{-1\pm\sqrt{21}}{10}$ **k** $3, -\frac{1}{2}$ **l** $\frac{-5\pm\sqrt{70}}{3}$
m $-1\pm\sqrt{7}$ **n** $1\pm\sqrt{5}$ **o** $\sqrt{2}\pm 2$

5 **a** $-3\pm\sqrt{17}$ **b** $5, -2$ **c** $1, -7$
d $2\pm\sqrt{7}$ **e** $2, -3$ **f** $\frac{1\pm\sqrt{13}}{2}$
g $-2, -\frac{1}{2}$ **h** $1, -\frac{1}{3}$ **i** $-1\pm\sqrt{6}$
j $-3\pm\sqrt{14}$ **k** no solution **l** no solution
m $\pm\frac{5}{2}$ **n** $\pm\frac{1}{3}$ **o** $-7, 5$
p $-3, 4$ **q** $\pm\frac{\sqrt{5}}{2}$ **r** no solution
s $-\frac{3}{2}, \frac{4}{3}$ **t** $-\frac{5}{3}, -\frac{1}{4}$ **u** -3
v $-1\pm\frac{\sqrt{3}}{3}$ **w** $2\pm\frac{\sqrt{6}}{2}$ **x** $-\frac{1\pm\sqrt{26}}{5}$
y $-\frac{2}{3}, \frac{1}{4}$ **z** no solution

6 **a** $-1\pm\sqrt{6}$ **b** $1\pm\sqrt{2}$ **c** $-2\pm\sqrt{11}$
d $-3\pm\sqrt{5}$ **e** $\frac{1\pm\sqrt{5}}{2}$ **f** $3, \frac{-3}{2}$

Exercise 5F

1 **a** $4\pm\sqrt{15}$ **b** $-2, 4$
c $\frac{3\pm\sqrt{13}}{2}$ **d** $-2, 6$
e $\frac{-5\pm\sqrt{17}}{2}$ **f** $\frac{-9\pm\sqrt{69}}{2}$
g $4\pm\sqrt{14}$ **h** $-1\pm\sqrt{5}$
i $-6\pm\sqrt{33}$

2 **a** $\frac{-1\pm\sqrt{22}}{3}$ **b** $\frac{-3\pm\sqrt{29}}{10}$
c $\frac{3\pm\sqrt{5}}{4}$ **d** $1, \frac{2}{7}$
e $-1, \frac{2}{5}$ **f** $\frac{1\pm\sqrt{29}}{14}$
g $\frac{-6\pm\sqrt{38}}{2}$ **h** $\frac{10\pm\sqrt{106}}{3}$
i $\frac{2\pm\sqrt{19}}{3}$

3 **a** 1.53, −0.13 **b** 7.87, 0.13
c −2, 5 **d** −0.20, −14.80
e 2, 3 **f** −0.41, 3.41
g −0.35, 2.85 **h** −0.24, 0.84
i 0.15, 3.35

4 **a** **i** no solutions **ii** one solution
iii two solutions

b **i** 2, irrational **ii** 2, rational

iii 1, rational **iv** 1, rational

v 0 **vi** 0

vii 2, irrational **viii** 2, irrational

5 **a** $\frac{1+\sqrt{5}}{2}$ **b** 3.52 cm **c** $\frac{3+\sqrt{33}}{2}$

6 $AP = 12 + 4\sqrt{5}$ cm, $BP = 4 + 4\sqrt{5}$ cm

7 50

8 $\frac{5+5\sqrt{7}}{2}$ and $\frac{-5+5\sqrt{7}}{2}$ or $\frac{5-5\sqrt{7}}{2}$ and $\frac{-5-5\sqrt{7}}{2}$

9 4.47

10 $5(\sqrt{201}+1)$ km/h $\approx$ 75.89 km/h

11 $50 - \sqrt{2095}$ and $50 + \sqrt{2095}$ $\approx$ 4.23 m and 95.77 m

12 **a** $x = 4$ or $x = 7$

b $x = 6 - 2\sqrt{10}$ or $x = 6 + 2\sqrt{10}$

c $x = \frac{-1-\sqrt{19}}{3}$ or $x = \frac{-1+\sqrt{19}}{3}$

Review exercise

1 **a** 6, −3 **b** $-2, \frac{1}{3}$

c $-1, \frac{1}{2}$ **d** $-\frac{3}{2}, \frac{1}{3}$

e $-\frac{3}{2}, 4$ **f** $-\frac{1}{2}, -4$

g −2, 4 **h** −1, −3

i $\frac{5}{2}, -1$ **j** $1, -\frac{5}{7}$

k $\frac{1}{2}, 5$ **l** $-\frac{1}{2}, -5$

m $-\frac{1}{2}, 3$ **n** $\frac{1}{3}, 3$

o $\frac{1}{3}$ **p** $\frac{7}{3}$

q $\frac{7}{3}, -\frac{7}{2}$

2 **a** 3, 5 **b** 5, 6

c $2 \pm \sqrt{3}$ **d** $-1 \pm \sqrt{2}$

e $\frac{-1 \pm \sqrt{13}}{2}$ **f** $10 \pm \sqrt{107}$

g $\frac{3 \pm \sqrt{37}}{2}$ **h** 3, −1

i $-1 \pm \sqrt{33}$ **j** $2, \frac{1}{2}$

k $\frac{-1 \pm \sqrt{37}}{6}$ **l** $\frac{3 \pm \sqrt{41}}{8}$

3 **a** 6, −4 **b** $-2, \frac{1}{2}$

c $\frac{-1 \pm \sqrt{5}}{2}$ **d** $\frac{-5 \pm \sqrt{17}}{4}$

e $\frac{-1 \pm \sqrt{7}}{2}$ **f** $\frac{1 \pm \sqrt{13}}{6}$

4 **a** $1 \pm \sqrt{2}$ **b** $\frac{-5 \pm \sqrt{57}}{4}$

c $\frac{1 \pm \sqrt{17}}{8}$ **d** $\frac{1 \pm \sqrt{17}}{2}$

e $1, -\frac{7}{6}$ **f** $\frac{1}{3}, -1$

5 **a** $\frac{7+\sqrt{13}}{2}, \frac{7-\sqrt{13}}{2}$ **b** $\frac{-7-\sqrt{73}}{4}, \frac{-7+\sqrt{73}}{4}$

c $\frac{1+\sqrt{51}}{10}, \frac{1-\sqrt{51}}{10}$ **d** $\frac{4+\sqrt{6}}{5}, \frac{4-\sqrt{6}}{5}$

e $\frac{3+\sqrt{21}}{4}, \frac{3-\sqrt{21}}{4}$ **f** $\frac{9+\sqrt{113}}{4}, \frac{9-\sqrt{113}}{4}$

g $\frac{5+\sqrt{33}}{4}, \frac{5-\sqrt{33}}{4}$ **h** $\frac{-2+\sqrt{10}}{3}, \frac{-2-\sqrt{10}}{3}$

6 **a** −7 and 9 or 7 and −9

b length 16 cm, width 9 cm

c 3, 5 or −3, −5

7 $x = 2\sqrt{5}$

8 train 28 km/h, car 49 km/h

9 $(10 + 5\sqrt{2})$ cm and $(10 - 5\sqrt{2})$ cm

10 6 cm^2 or $\frac{28}{27}$ cm^2

11 $AP = 4$ cm

Challenge exercise

1 $\pm 2, \pm 3$ **2** −1, −2, 3, 4

4 $x^2 - 4x + 1 = 0$ **5** $-\frac{7}{2}, \frac{17}{2}$

6 7 **7** $10 + 2\sqrt{21}$

8 **a** AA **b** $\frac{-1+\sqrt{5}}{2}$

d $\frac{\sqrt{2(5+\sqrt{5})}}{4}$

9 **a** $2x^2 + 6x - 3 = 0$ **b** $\frac{-3+\sqrt{15}}{2}$

c $\frac{-1+\sqrt{5}}{2}$

10 0, $5a$

Chapter 6 answers

Exercise 6A

1 **a** 318 cm^2 **b** 448 cm^2
c 192π cm^2 $\approx$ 603.2 cm^2 **d** $54 + 36\pi$ cm^2 $\approx$ 167.1 cm^2
e 4140 cm^2 **f** $100 + 37.5\pi$ m^2 $\approx$ 217.8 m^2
g 1000 cm^2 **h** 596 cm^2
i 194 cm^2 **j** $144 + 40\sqrt{2} \approx 200.6$ cm^2

2 **a** $3\sqrt{3} \approx 5.1962$ cm
b $216 + 18\sqrt{3} \approx 247.1769$ cm^2

3 9 m **4** 4.5 cm

5 **a** 1425π cm^2 $\approx$ 4476.8 cm^2
b $\frac{350}{3\pi}$ cm $\approx$ 37.1 cm

6 **a** 280 cm^3 **b** 216 m^3 **c** 144 m^3
d 28 000 mm^3 **e** 300π cm^3 $\approx$ 942.5 cm^3
f 1215π cm^3 $\approx$ 3817.0 cm^3 **g** 23 040 cm^3
h 375π cm^3 $\approx$ 1178.1 cm^3 **i** 3240 cm^3
j 1080 cm^3 **k** 480 cm^3 **l** 360 cm^3

7 **a** 5.29 cm **b** 761.98 cm^3

8 **a** 19.9 cm **b** 6.31 cm

9 **a** 4 cm **b** 7.4 cm^3

10 16.25 m^2

11 4 cm

12 $\frac{1500}{2 + 3\sqrt{2}}$ cm $\approx$ 240.28 cm

13 **a** 1099.56 cm^3 **b** 1400 cm^3 **c** 300.44 cm^3

Exercise 6B

1 60 cm^3

2 **a** 80 cm^3 **b** 100 cm^3 **c** 216 cm^3 **d** 180 cm^3

3 **a** 400 cm^3 **b** $53\frac{1}{3}$ cm^3 **c** 80 cm^3 **d** 96 cm^3

4 2 547 758.6 m^3

5 **a** 800 cm^3 **b** 2333.33 cm^3

6 **a** 5 cm **b** 13 cm **c** 65 cm^2 **d** 360 cm^2

7 **a** $64 + 32\sqrt{13} \approx 179.378$ cm^2
b $36 + 12\sqrt{34} \approx 105.971$ cm^2

8 **a** $\sqrt{61}$ cm $\approx$ 7.810 cm **b** $2\sqrt{13}$ cm $\approx$ 7.211 cm
c $(80 + 8\sqrt{61} + 20\sqrt{13})$ cm $\approx$ 214.593 cm^2

9 **a** ΔVDA, ΔVAB, ΔVCB, ΔVDC: all are right-angled triangles.
b **i** 10 cm **ii** $2\sqrt{41}$ cm $\approx$ 12.806 cm
iii 10 cm
c 192 cm^2

Exercise 6C

1 **a** 138.23 cm^2 **b** 235.62 cm^2 **c** 1099.56 cm^2

2 **a** 24π cm^2 **b** $(16\pi + 16\pi\sqrt{10})$ cm^2
c 70π cm^2

3 **a** 10 cm **b** 9.1652 cm

4 **a** 685.84 cm^2 **b** 21.83 cm **c** 19.41 cm

5 **a** 104.72 cm^2
b **i** 10 cm **ii** 3.33 cm **iii** 9.43 cm

6 **a** **i** $5\sqrt{5}$ cm $\approx$ 11.18 cm **ii** $25\pi\sqrt{5}$ cm^2 $\approx$ 175.62 cm^2
iii $\left(\frac{360}{\sqrt{5}}\right)^\circ \approx 161^\circ$
b $\left(\frac{360}{\sqrt{2}}\right)^\circ \approx 255^\circ$

7 **a** 100.53 cm^3 **b** 335.10 cm^3
c 376.99 cm^3 **d** 339.29 cm^3

8 127 mm

9 **a** 648π cm^3 **b** 270π cm^2

10 **a** $(1800 + 36\sqrt{2})\pi$ cm^3 **b** 726π cm^3

11 $19 635

Exercise 6D

1 **a** 804.25 cm^2 **b** 2827.43 cm^2
c 615.75 cm^2 **d** 1385.44 cm^2

2 **a** 64π cm^2 $\approx$ 201.06 cm^2 **b** 128π cm^2 $\approx$ 402.12 cm^2
c 192π cm^2 $\approx$ 603.19 cm^2

3 6.3078 cm

4 **a** 2.11 m **b** 1.72 m

5 **a** 195π cm^2 **b** 115π cm^2 **c** $(440 + 16\pi)$ cm^2

6 **a** 2144.66 **b** 4188.79 **c** 904.78
d 452.39 cm^3 **e** 3619.11 cm^3 **f** 883.57 cm^3

7 **a** 6.20 cm **b** 179 mm **c** 9.8 cm

8 **a** 368.61 cm^3 **b** 575.96 cm^3 **c** 329.87 cm^3

9 8249 cm^3

10 545 131.78 mm^3

Exercise 6E

1 **a** **i** 9 **ii** 9
b **i** $\frac{36}{25}$ **ii** $\frac{36}{25}$
c **i** $\frac{9}{4}$ **ii** $\frac{9}{4}$

2 **a i** 4 **ii** 8

b i $\frac{9}{4}$ **ii** $\frac{27}{8}$

c i $\frac{36}{25}$ **ii** $\frac{216}{125}$

d i $\frac{25}{4}$ **ii** $\frac{125}{8}$

3 **a** 42 875 cm^3 **b** 7350 cm^2

4 8 **5** 960 cm^2 **6** $2\sqrt{2}$

7 **a** 172 800 cm^2 **b** 325.5 cm^3

8 **a** 192 m^3 **b** 1536 cm^3

c $(96 + 16\sqrt{13})$ m^2 $\approx$ 153.7 m^2 **d** 614.8 cm^2

9 **a** The area of the triangle increases by a factor of 7.

b 36 **c** 9 **d** 81

Review exercise

1 **a i** 1000 cm^3 **ii** 600 cm^2

b i 600 cm^3 **ii** 460 cm^2

c i 480 cm^3 **ii** 528 cm^2

d i 168π cm^3 **ii** $(36\pi + 12\sqrt{58}\pi)$ cm^2

e i 90π m^3 **ii** 78π m^2

f i $\frac{2048}{3}\pi$ m^3 **ii** 256π m^2

2 **a i** 550 m^3 **ii** $(375 + 25\sqrt{17})$ cm^2 $\approx$ 478.1 cm^2

b i 200π cm^3 $\approx$ 628.3 cm^3

ii $\left(\frac{340\pi}{3} + 120\right)$ cm^2 $\approx$ 476.0 cm

3 2 cm

4 **a** 1 500 000 m^3 **b** 80 cm^3

c $\frac{250\pi}{3}$ cm^3 $\approx$ 261.8 cm^3 **d** 156π cm^3 $\approx$ 490.1 cm^3

5 8 cm

6 **a i** $\frac{592\pi}{3}$ cm^3 $\approx$ 619.94 cm^3

ii 168π cm^2 $\approx$ 527.8 cm^2

b i $\left(4 + \frac{2\pi}{3}\right)$ m^3 $\approx$ 6.09 m^3

ii $(16 + \pi)$ m^2 $\approx$ 19.1 m^2

7 1.2 m

8 **a** 662.5 cm^2 **b** 937.5 cm^3

9 **a** 80 m^3 **b** $\left(63 + 8\sqrt{34} + 3\sqrt{89}\right)$ m^3 $\approx$ 137.95 m^2

10 **a** $V = \frac{250\,000}{3}\pi$ mm^3, $S = \left(20\,000 + \frac{12\,500}{3}\pi\right)$ mm^2

b V = 2250 cm^3, $S = 675\pi$ m^2

c $V = 4\pi$ m^3, $S = (6 + \sqrt{13})\pi$ m^2

d $V = \frac{875}{3}\pi$ m^3, $S = 150\pi$ m^2

Challenge exercise

1 4 cm

3 6 cm

6 **c** Substitute the result of part **b** into that of part **a**.

7 **b** 7 : 8

9 **a** Use similar triangles.

c Show $s^2 = h^2 + (R - r)^2$

10 **a** Use Pythagoras' theorem

c cross-section area = area of cylinder cross-section – area of cone cross-section

d $\pi r^3 - \frac{1}{3}\pi r^3 = \frac{2}{3}\pi r^3$

e Volume of a sphere is twice the result in part **d**.

Chapter 7 answers

Exercise 7A

1 **a** x = 4, (4, 0) **b** x = 0, (0, – 4) **c** x = 2, (2, 6)

d x = –3, (–3, 7) **e** x = –2, (–2, 3) **f** x = 0, (0, 9)

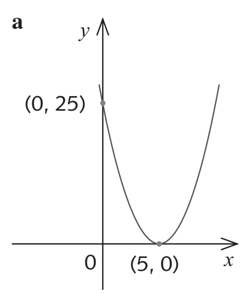
g x = 3, (3, –4) **h** x = –2, (–2, –3) **i** x = 6, (6, 6)

j x = –1, (–1, 0) **k** x = 2, (2, 1) **l** x = –3, (–3, 5)

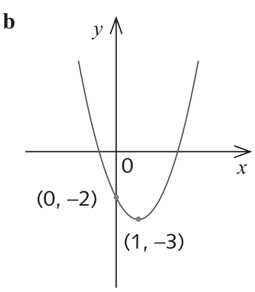
2 **a** –3 **b** 46 **c** 5

d –9 **e** –8 **f** 2

3 **a**



b



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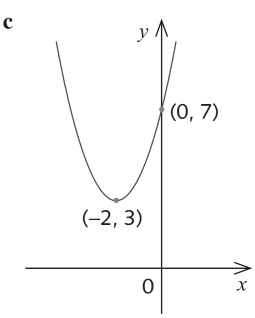

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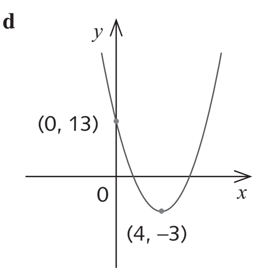

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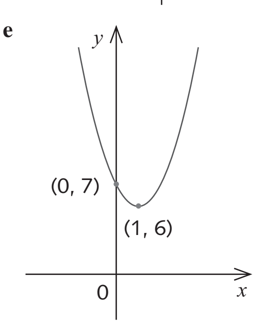

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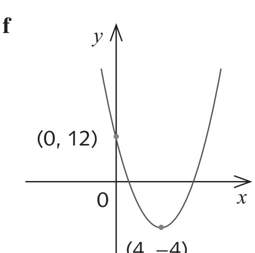
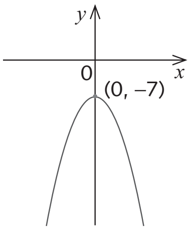


4 **a**



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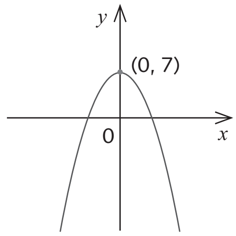

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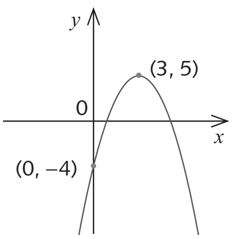

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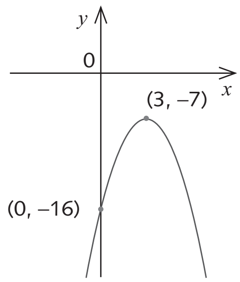

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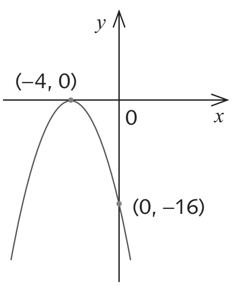

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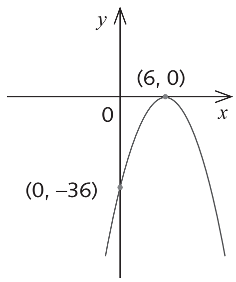

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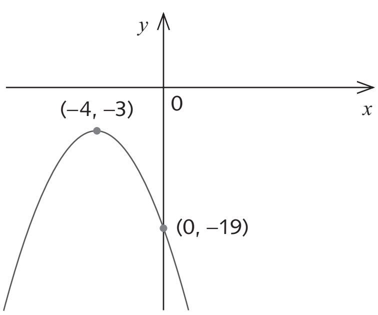

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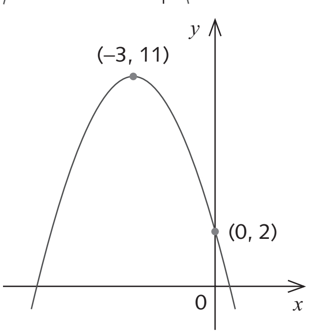
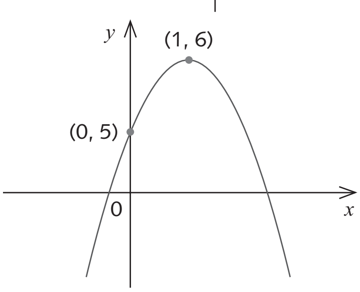


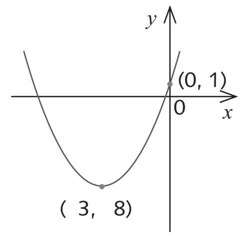
i



5 **a** $y = -(x-3)^2$ **b** $y = -(x+b)^2$
c $y = -x^2 - 6$ **d** $y = -x^2 + c$

6 **a** $y = (x+4)^2 + 3$ **b** $y = (x-6)^2 - 5$
c $y = (x-a)^2 + b$ **d** $y = (x+d)^2 - c$

7

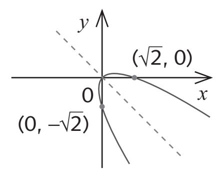


a $y = x^2$ **b** $y = (x+5)^2 - 11$
c $y = (x+3-a)^2 - 8 + b$

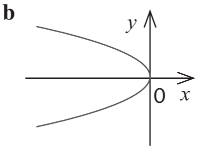
8 **a** $a = 0$ **b** $a = 2$ **c** $a = -1$ **d** $a = -8$

9 **a** $b = 5$ **b** $b = 7$ **c** $b = 0$ **d** $b = -3$

10 **a**

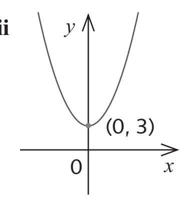


b



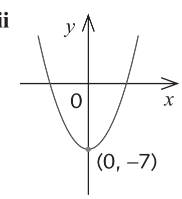
Exercise 7B

1 **a** **i** 3 **ii** $x = 0, (0, 3)$
iii



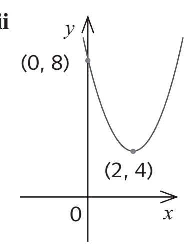
iv no x-intercepts

b **i** –7 **ii** $x = 0, (0, -7)$
iii



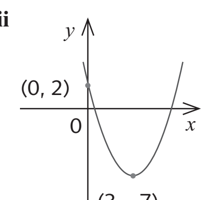
iv two x-intercepts

c **i** 8 **ii** $x = 2, (2, 4)$
iii



iv no x-intercepts

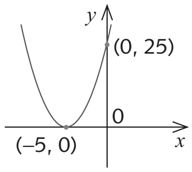
d **i** 2 **ii** $x = 3, (3, -7)$
iii



iv two x-intercepts

e **i** 25 **ii** $x = -5, (-5, 0)$

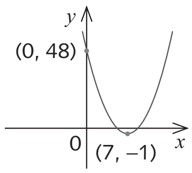
iii



iv one x-intercept

f **i** 48 **ii** $x = 7, (7, -1)$

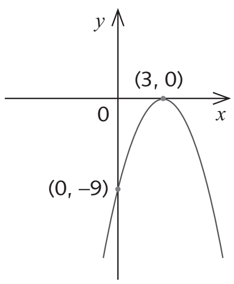
iii



iv two x-intercepts

g **i** -9 **ii** $x = 3, (3, 0)$

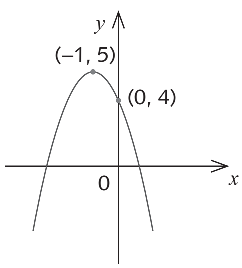
iii



iv one x-intercept

h **i** 4 **ii** $x = -1, (-1, 5)$

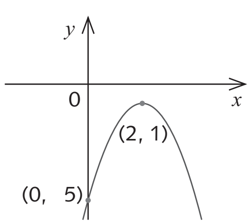
iii



iv two x-intercepts

i **i** -5 **ii** $x = 2, (2, -1)$

iii

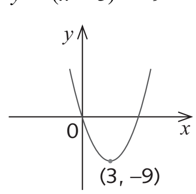


iv no x-intercepts

2 **a** **i** 0 **ii** $y = (x - 3)^2 - 9$

iii $x = 3, (3, -9)$

iv

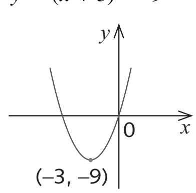


v two x-intercepts

b **i** 0 **ii** $y = (x + 3)^2 - 9$

iii $x = -3, (-3, -9)$

iv

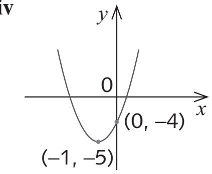


v two x-intercepts

c **i** -4 **ii** $y = (x + 1)^2 - 5$

iii $x = -1, (-1, -5)$

iv

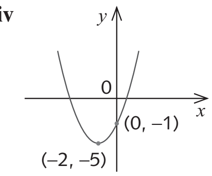


v two x-intercepts

d **i** -1 **ii** $y = (x + 2)^2 - 5$

iii $x = -2, (-2, -5)$

iv

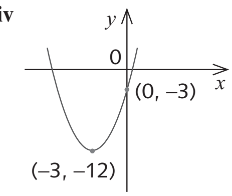


v two x-intercepts

e **i** -3 **ii** $y = (x + 3)^2 - 12$

iii $x = -3, (-3, -12)$

iv

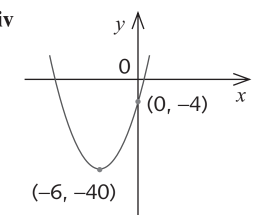


v two x-intercepts

f **i** -4 **ii** $y = (x + 6)^2 - 40$

iii $x = -6, (-6, -40)$

iv

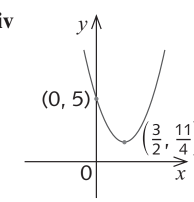


v two x-intercepts

g **i** 5 **ii** $y = \left(x - \frac{3}{2}\right)^2 + \frac{11}{4}$

iii $x = \frac{3}{2}, \left(\frac{3}{2}, \frac{11}{4}\right)$

iv

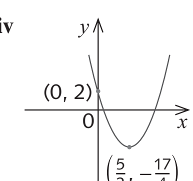


v two x-intercepts

h **i** 2 **ii** $y = \left(x - \frac{5}{2}\right)^2 - \frac{17}{4}$

iii $x = \frac{5}{2}, \left(\frac{5}{2}, -\frac{17}{4}\right)$

iv

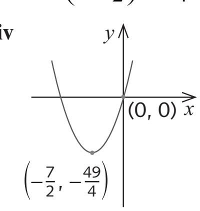


v two x-intercepts

i **i** 0 **ii** $y = \left(x + \frac{7}{2}\right)^2 - \frac{49}{4}$

iii $x = -\frac{7}{2}, \left(-\frac{7}{2}, -\frac{49}{4}\right)$

iv

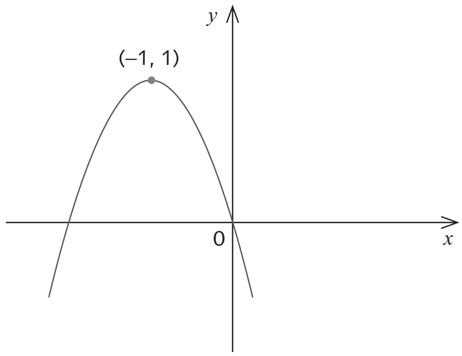


v two x-intercepts

j **i** 0 **ii** $y = -(x + 1)^2 + 1$

iii $x = -1, (-1, 1)$

iv

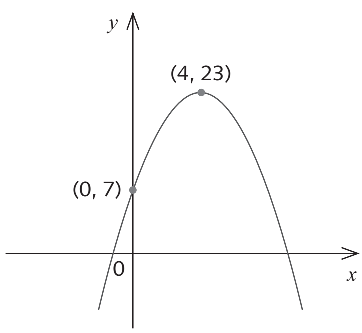


v two x-intercepts

k **i** 7 **ii** $y = -(x - 4)^2 + 23$

iii $x = 4, (4, 23)$

iv

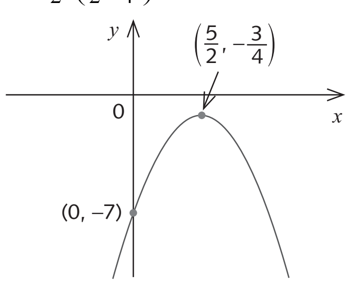


v two x-intercepts

l **i** -15 **ii** $y = -\left(x - \frac{5}{2}\right)^2 - \frac{3}{4}$

iii $x = \frac{5}{2}, \left(\frac{5}{2}, \frac{-3}{4}\right)$

iv



v no x-intercepts

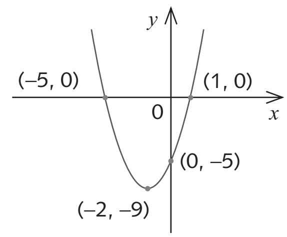
3 **a** $-3, 1$ **b** $-5, 1$
c $-2, 8$ **d** $4 - \sqrt{7},\ 4 + \sqrt{7}$
e $-3 - \sqrt{11},\ -3 + \sqrt{11}$ **f** $1, -3$
g $2 + 2\sqrt{2},\ 2 - 2\sqrt{2}$ **h** $5 + 3\sqrt{2},\ 5 - 3\sqrt{2}$
i $3 + 5\sqrt{2},\ 3 - 5\sqrt{2}$ **j** $-3 + 2\sqrt{5}, -3 - 2\sqrt{5}$
k $-2 + 4\sqrt{2},\ -2 - 4\sqrt{2}$ **l** $3, -5$

4 **a** $-1 - \sqrt{5},\ -1 + \sqrt{5}$ **b** $3 - \sqrt{2},\ 3 + \sqrt{2}$
c $4 - \sqrt{3},\ 4 + \sqrt{3}$ **d** $-2 - 2\sqrt{2}, -2 + 2\sqrt{2}$
e $-11, 1$ **f** $10 - 5\sqrt{6}, 10 + 5\sqrt{6}$
g $-1, 13$ **h** $3 + \sqrt{5},\ 3 - \sqrt{5}$
i $2 - 2\sqrt{3},\ 2 + 2\sqrt{3}$

5 **a** **i** -5 **ii** $y = (x + 2)^2 - 9$

iii $x = -2, (-2, -9)$ **iv** $-5, 1$

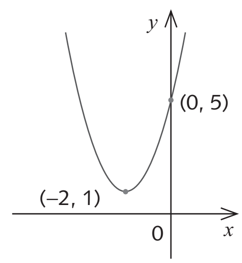
v



b **i** 5 **ii** $y = (x + 2)^2 + 1$

iii $x = -2,\ (-2,\ 1)$ **iv** no x-intercepts

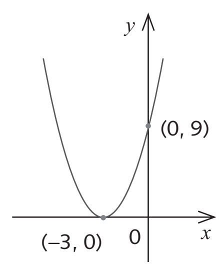
v



c **i** 9 **ii** $y = (x + 3)^2$

iii $x = -3, (-3, 0)$ **iv** -3

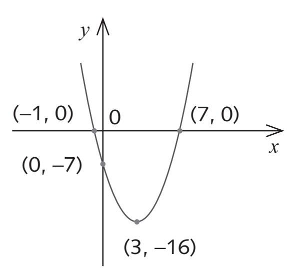
v



d **i** -7 **ii** $y = (x - 3)^2 - 16$

iii $x = 3, (3, -16)$ **iv** $-1, 7$

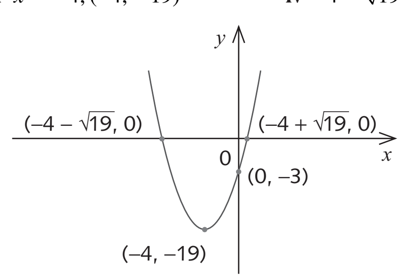
v



e **i** -3 **ii** $y = (x + 4)^2 - 19$

iii $x = -4, (-4, -19)$ **iv** $-4 - \sqrt{19},\ -4 + \sqrt{19}$

v



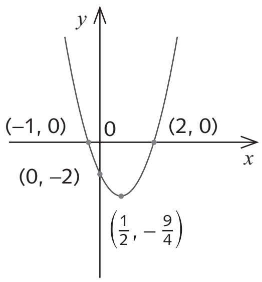
f **i** -2

ii $y = \left(x - \frac{1}{2}\right)^2 - \frac{9}{4}$

iii $x = \frac{1}{2}, \left(\frac{1}{2}, -\frac{9}{4}\right)$

iv $-1, 2$

v



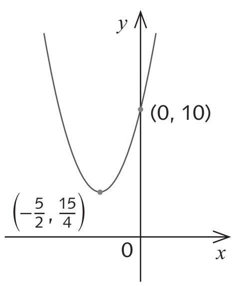
g **i** 10

ii $y = \left(x + \frac{5}{2}\right)^2 + \frac{15}{4}$

iii $x = -\frac{5}{2}, \left(-\frac{5}{2}, \frac{15}{4}\right)$

iv no x-intercepts

v



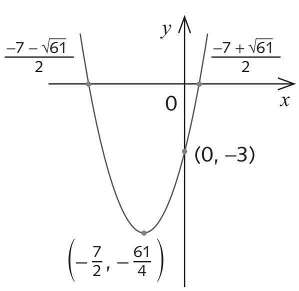
h **i** -3

ii $y = \left(x + \frac{7}{2}\right)^2 - \frac{61}{4}$

iii $x = -\frac{7}{2}, \left(-\frac{7}{2}, -\frac{61}{4}\right)$

iv $\frac{-7 + \sqrt{61}}{2}, \frac{-7 - \sqrt{61}}{2}$

v



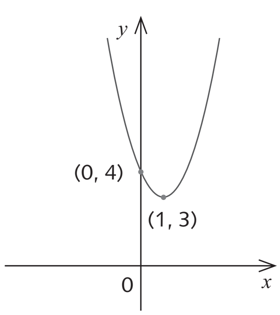
i **i** 4

ii $y = (x - 1)^2 + 3$

iii $x = 1, (1, 3)$

iv no x-intercepts

v



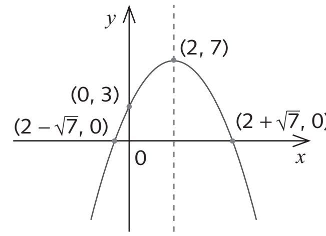
j **i** 3

ii $y = -(x - 2)^2 + 7$

iii $x = 2, (2, 7)$

iv $2 - \sqrt{7}, 2 + \sqrt{7}$

v



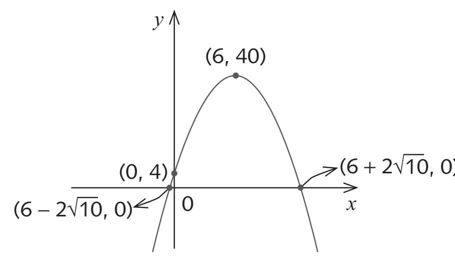
k **i** 6

ii $y = -(x - 6)^2 + 40$

iii $x = 6, (6, 40)$

iv $6 - 2\sqrt{10}, 6 + 2\sqrt{10}$

v



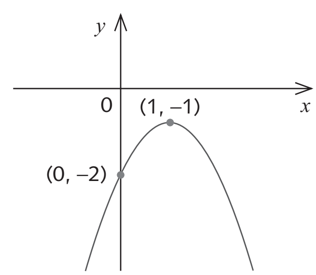
l **i** -2

ii $y = -(x - 1)^2 - 1$

iii $x = 1, (1, -1)$

iv no x-intercepts

v



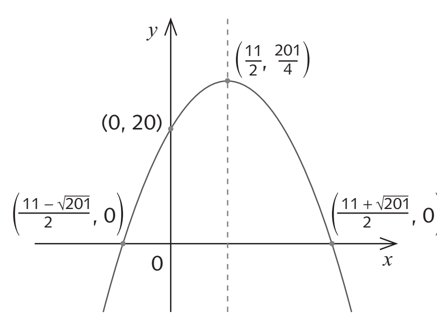
m **i** 1

ii $y = -\left(x - \frac{1}{2}\right)^2 + \frac{5}{4}$

iii $x = \frac{1}{2}, \left(\frac{1}{2}, \frac{5}{4}\right)$

iv $\frac{1 - \sqrt{5}}{2}, \frac{1 + \sqrt{5}}{2}$

v



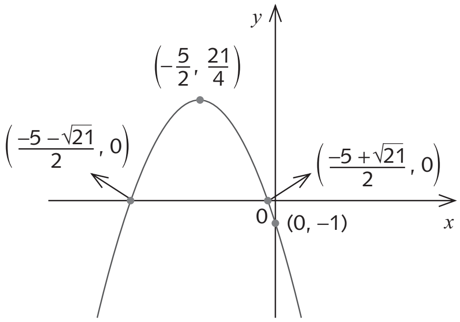
n i -1

ii $-\left(x+\frac{5}{2}\right)^2+\frac{21}{4}$

iii $x=\frac{-5}{2}$, $\left(\frac{-5}{2}, \frac{21}{4}\right)$

iv $\frac{-5-\sqrt{21}}{2}, \frac{-5+\sqrt{21}}{2}$

v



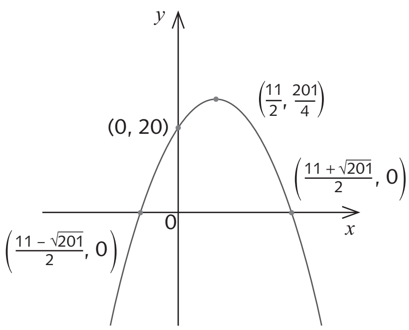
o i 20

ii $y=-\left(x-\frac{11}{2}\right)^2+\frac{201}{4}$

iii $x=\frac{11}{2}$, $\left(\frac{11}{2}, \frac{201}{4}\right)$

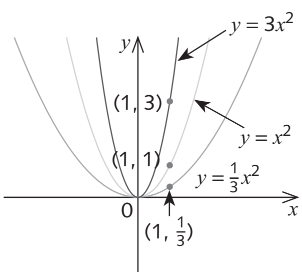
iv $\frac{11-\sqrt{201}}{2}, \frac{11+\sqrt{201}}{2}$

v

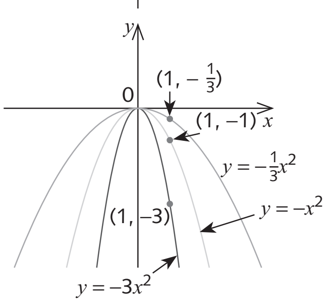


Exercise 7C

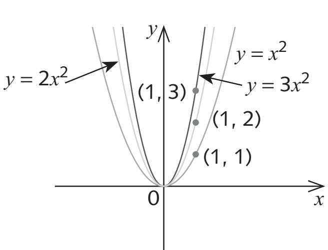
1 a



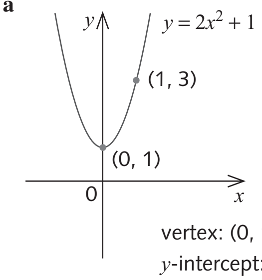
b



c

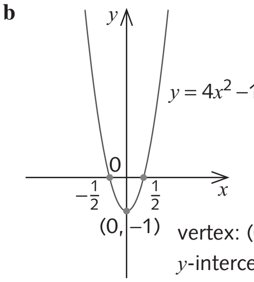


2 a



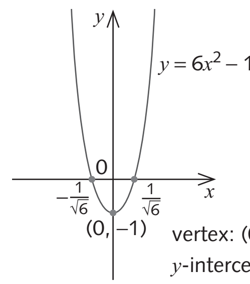
vertex: (0, 1)
y-intercept: 1

b



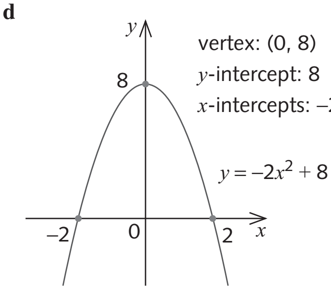
vertex: (0, –1)
y-intercept: –1
x-intercepts: $-\frac{1}{2}, \frac{1}{2}$

c



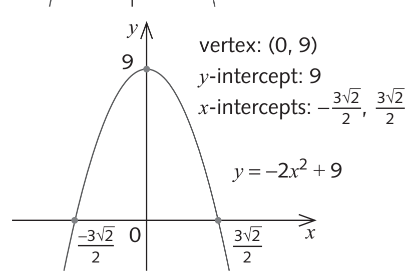
vertex: (0, –1)
y-intercept: –1
x-intercepts: $-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}$

d



vertex: (0, 8)
y-intercept: 8
x-intercepts: –2, 2

e



vertex: (0, 9)
y-intercept: 9
x-intercepts: $-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}$

3 **a, d, e, g** are congruent; **b, f, l** are congruent; **c, h** are congruent; **j and k** are congruent

4 **a** $y=3(x+1)^2$ **b** $y=3\left(x-\frac{3}{2}\right)^2+\frac{41}{4}$

c $y=2\left(x+\frac{5}{2}\right)^2-\frac{51}{2}$ **d** $y=2\left(x+\frac{7}{4}\right)^2-\frac{25}{8}$

e $y=3\left(x+\frac{5}{6}\right)^2+\frac{59}{12}$ **f** $y=-5(x-2)^2+57$

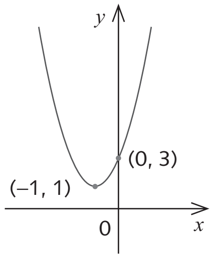
5 **a** **i** 3

ii $y = 2(x + 1)^2 + 1$

iii $x = -1, (-1, 1)$

iv no x-intercepts

v



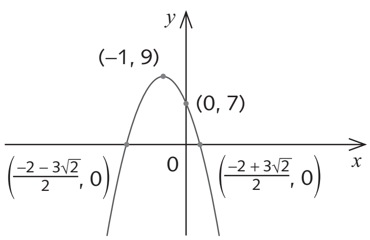
b **i** 7

ii $y = -2(x + 1)^2 + 9$

iii $x = -1, (-1, 9)$

iv $\dfrac{-2 + 3\sqrt{2}}{2}, \dfrac{-2 - 3\sqrt{2}}{2}$

v



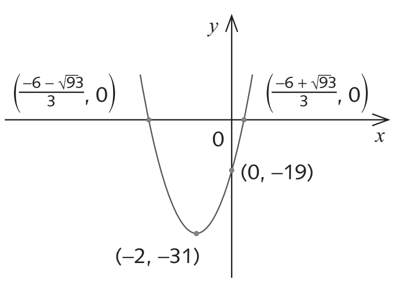
c **i** −19

ii $y = 3(x + 2)^2 - 31$

iii $x = -2, (-2, -31)$

iv $\dfrac{-6 + \sqrt{93}}{3}, \dfrac{-6 - \sqrt{93}}{3}$

v



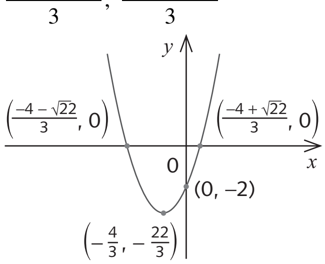
d **i** −2

ii $y = 3\left(x + \dfrac{4}{3}\right)^2 - \dfrac{22}{3}$

iii $x = -\dfrac{4}{3}, \left(-\dfrac{4}{3}, -\dfrac{22}{3}\right)$

iv $\dfrac{-4 + \sqrt{22}}{3}, \dfrac{-4 - \sqrt{22}}{3}$

v



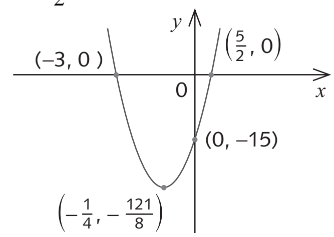
e **i** −15

ii $y = 2\left(x + \dfrac{1}{4}\right)^2 - \dfrac{121}{8}$

iii $x = -\dfrac{1}{4}, \left(-\dfrac{1}{4}, -\dfrac{121}{8}\right)$

iv $-3, \dfrac{5}{2}$

v



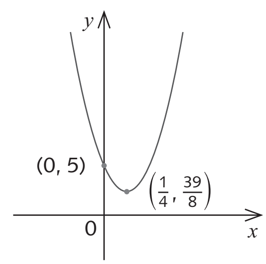
f **i** 5

ii $y = 2\left(x - \dfrac{1}{4}\right)^2 + \dfrac{39}{8}$

iii $x = \dfrac{1}{4}, \left(\dfrac{1}{4}, \dfrac{39}{8}\right)$

iv no x-intercepts

v



6 **a** $x = \dfrac{1}{2}, \left(\dfrac{1}{2}, \dfrac{11}{4}\right)$, no x-intercepts

b $x = -\dfrac{5}{6}, \left(-\dfrac{5}{6}, -\dfrac{181}{12}\right)$, has x-intercepts

c $x = -\dfrac{1}{4}, \left(-\dfrac{1}{4}, \dfrac{7}{8}\right)$, no x-intercepts

d $x = -\dfrac{3}{10}, \left(-\dfrac{3}{10}, \dfrac{131}{20}\right)$, no x-intercepts

e $x = \dfrac{5}{6}, \left(\dfrac{5}{6}, -\dfrac{59}{12}\right)$, no x-intercepts

f $x = \dfrac{3}{2}, \left(\dfrac{3}{2}, \dfrac{17}{4}\right)$, has x-intercepts

7 **a** 6 **b** 8 **c** −8 **d** −2

8 **a** 3 **b** 1 **c** $\dfrac{1}{2}$ **d** −3

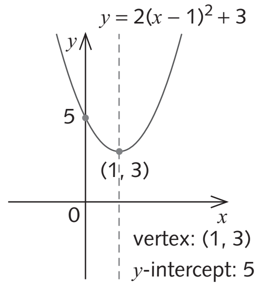
9 **a** ±1 **b** ±3 **c** $\pm\dfrac{\sqrt{2}}{2}$ **d** $\pm\sqrt{3}$

10 $y = (x - 1)^2 - 2$ or $y = x^2 - 2x - 1$

11 $y = 3(x + 2)^2 - 1$ or $y = 3x^2 + 12x + 11$

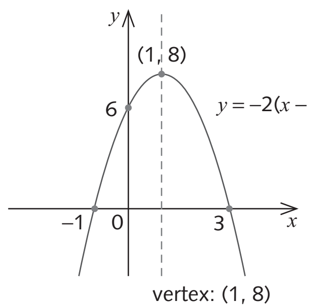
12 $y = -2(x - 1)^2 + 6$ or $y = -2x^2 + 4x + 4$

13 **a**



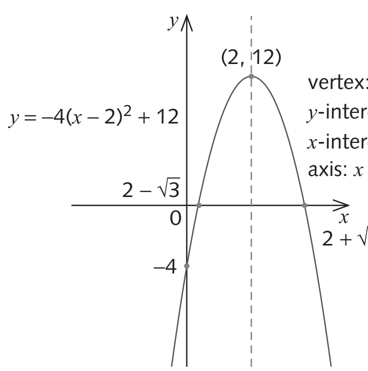
vertex: (1, 3)
y-intercept: 5
axis: $x = 1$

b



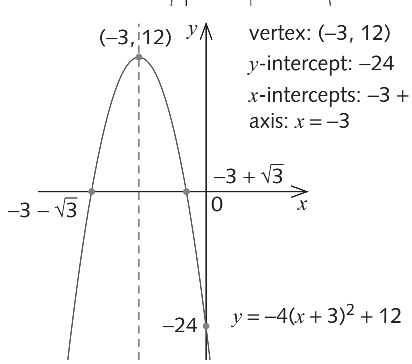
vertex: (1, 8)
y-intercept: 6
x-intercepts: –1, 3
axis: $x = 1$

c



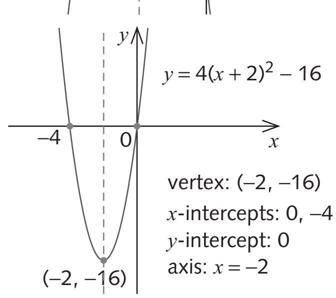
vertex: (2, 12)
y-intercept: –4
x-intercepts: $2 - \sqrt{3}$, $2 + \sqrt{3}$
axis: $x = 2$

d



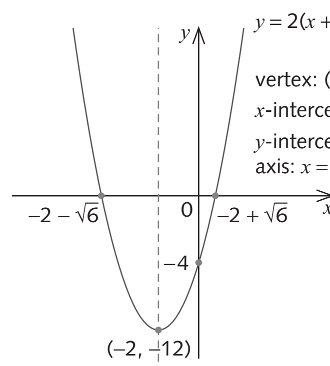
vertex: (–3, 12)
y-intercept: –24
x-intercepts: $-3 + \sqrt{3}$, $-3 - \sqrt{3}$
axis: $x = -3$

e



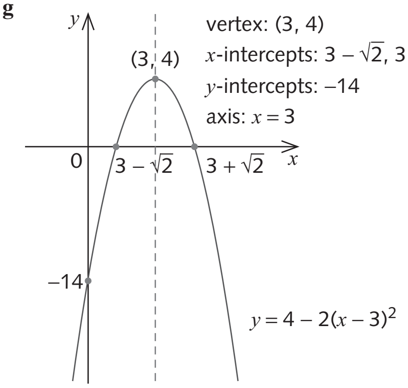
vertex: (–2, –16)
x-intercepts: 0, –4
y-intercept: 0
axis: $x = -2$

f



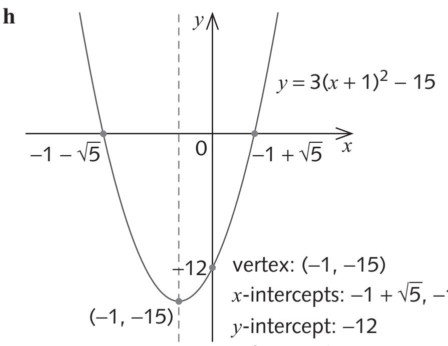
vertex: (–2, –12)
x-intercepts: $-2 - \sqrt{6}$, $-2 + \sqrt{6}$
y-intercept: –4
axis: $x = -2$

g



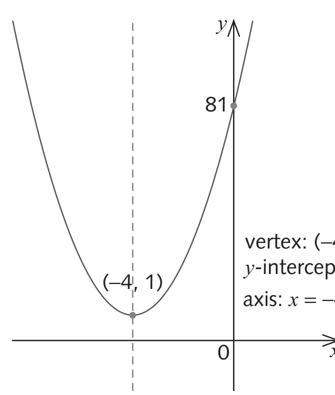
vertex: (3, 4)
x-intercepts: $3 - \sqrt{2}$, $3 + \sqrt{2}$
y-intercepts: –14
axis: $x = 3$

h



vertex: (–1, –15)
x-intercepts: $-1 + \sqrt{5}$, $-1 - \sqrt{5}$
y-intercept: –12
axis: $x = -1$

i

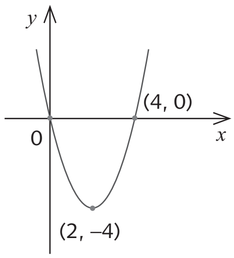


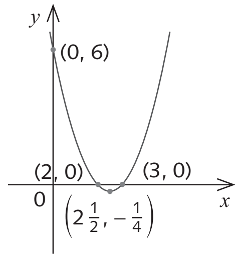
vertex: (–4, 1)
y-intercept: 81
axis: $x = -4$

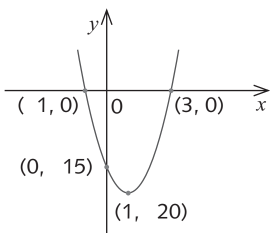
Exercise 7D

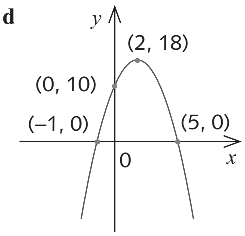
1 4

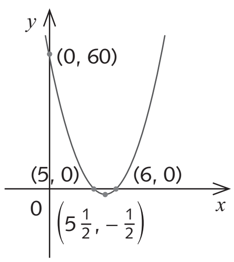
2 1

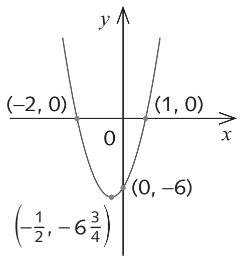
3 a


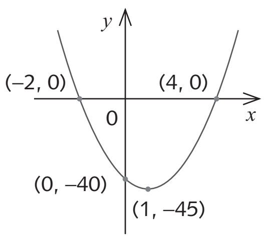
b


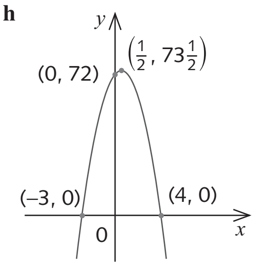
c


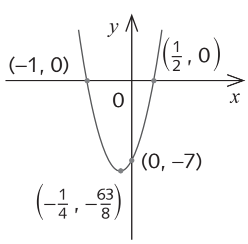
d


e


f


g


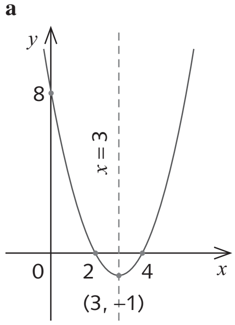
h


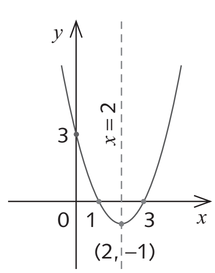
i


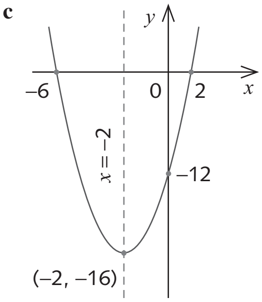
4 a –5, –1 b –4, –3 c –3, 6

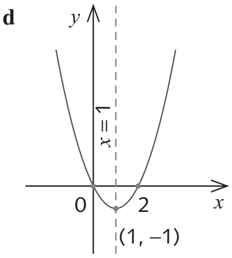
d –5, 3 e $-\frac{1}{2}$, 10 f –4, 4

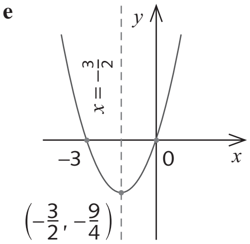
g $-\frac{1}{2}$, $\frac{1}{2}$ h 0, 3 i –4, 0

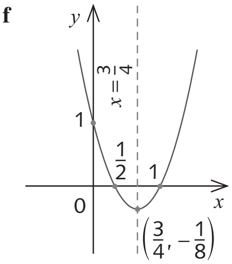
5 a


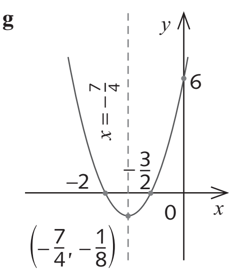
b


c


d


e


f


g


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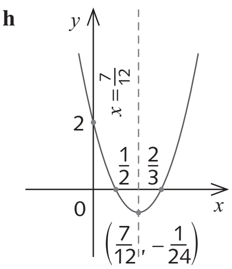

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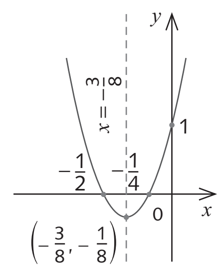

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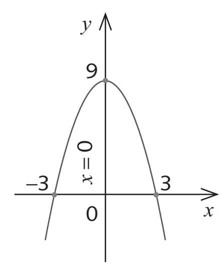

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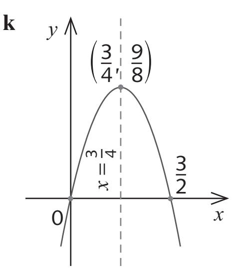
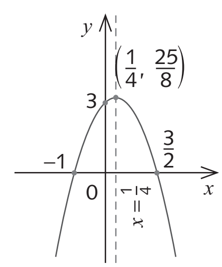


l


6 a $y = x^2 - 4x$ b $y = 9 - x^2$

c $y = -3x^2 + 6x + 9$ d $y = -\frac{3x^2}{5} - \frac{9x}{5} + 6$

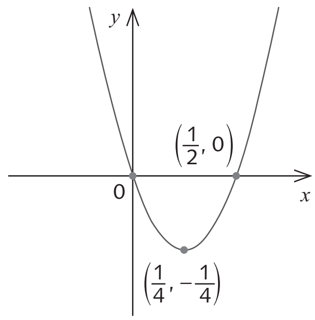
7 $y = x^2 - x - 2$

8 $y = \frac{1}{2}x^2 - \frac{x}{2} - 3$

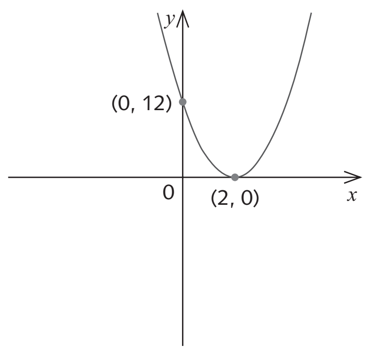
9 $y = 6x^2 - 15x + 6$

Exercise 7E

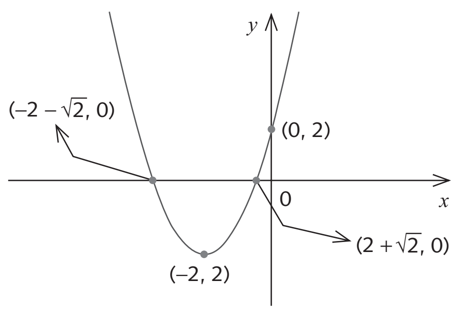
a



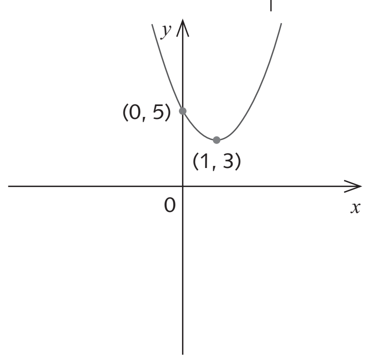
b



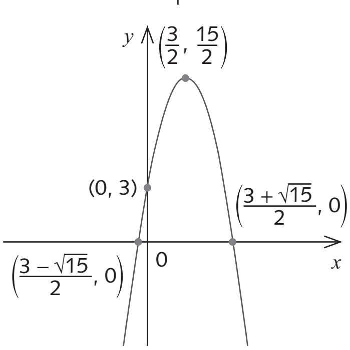
c



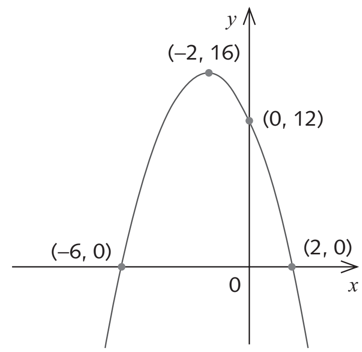
d



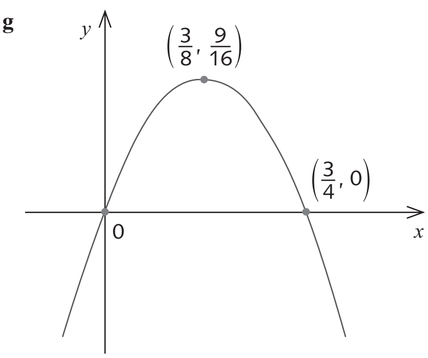
e



f



g



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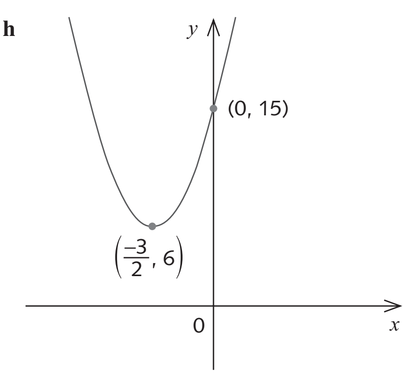

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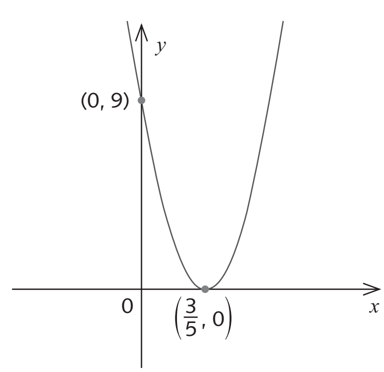

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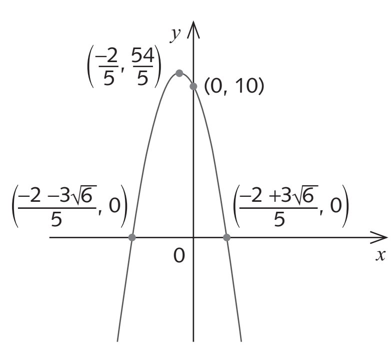

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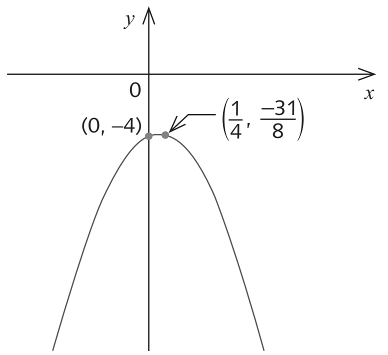

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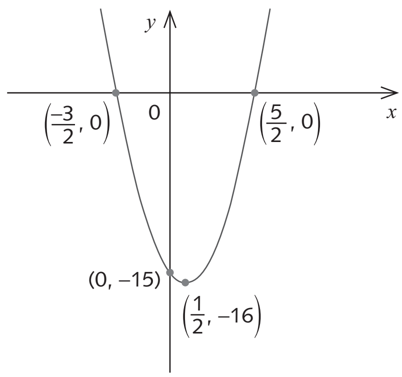


Exercise 7F

1 80 m, 130 m **2** 60 cm **3** 25, 26

4 14, 16 **5** 13, 15 **6** 3

7 8 years **8** \$6

9 8 cm, 15 cm, 17 cm

10 20 cm, 21 cm, 29 cm

11 5 cm, 30 cm, 40 cm

12 44 sides **13** 16 **14** 1.84 seconds

15 −7 **16** $\frac{5}{4}$

17 **a** max = 4, min = $\frac{-73}{12}$
b max = 46, min = −2

18 25 cm by 25 cm, 17 cm

19 75 m by 150 m

20 31.888 m, 5.102 seconds

21 62.5 m by 80 m, or 160 m by 31.25 m

22 $3\frac{3}{8}$ square units

Exercise 7G

1 **a** $-3 < x < 5$ **b** $x < -4$ or $x > 1$
c $x = -2$ **d** $x \le -1$ or $x \ge 2$
e all values of x **f** $x < -\frac{3}{2}$ or $x > \frac{3}{2}$

2 **a** $x < -2$ or $x > 3$ **b** $-4 \le x \le -1$
c $x \le 2$ or $x \ge 5$ **d** $-3 < x < 0$

3 **a** $x < -10$ or $x > 7$ **b** $-3 < x < 8$
c $x \le -5$ or $x \ge -4$ **d** $3 \le x \le 4$

4 **a** $-8 < x < 5$ **b** $-3 \le x \le 8$
c $x \le 5$ or $x \ge 7$ **d** $x < 0$ or $x > 11$

Review exercise

1 **a** 2 **b** 24 **c** 0
d 0 **e** 4 **f** −16

2 **a** 0 **b** ±4
c ±3 **d** ±2

3 **a** −4, 1 **b** $-6, -\frac{1}{2}$ **c** $-\frac{3}{4}, \frac{3}{2}$
d $-\frac{1}{2}, \frac{5}{2}$ **e** −7, 7 **f** 0, 5

4 **a** $-2-\sqrt{5}, -2+\sqrt{5}$ **b** $3-\sqrt{2}, 3+\sqrt{2}$
c $-1-\sqrt{5}, -1+\sqrt{5}$ **d** $2-\sqrt{5}, 2+\sqrt{5}$
e $3+\frac{\sqrt{35}}{5}, 3-\frac{\sqrt{35}}{5}$ **f** $2+\sqrt{2}, 2-\sqrt{2}$

5 **a** $-2-\sqrt{6}, -2+\sqrt{6}$
b $3-2\sqrt{2}, 3+2\sqrt{2}$
c $-\frac{5}{2}-\frac{\sqrt{19}}{2}, -\frac{5}{2}+\frac{\sqrt{19}}{2}$
d $-2-\frac{\sqrt{26}}{2}, -2+\frac{\sqrt{26}}{2}$

6 **a** minimum **b** maximum
c maximum **d** minimum

7 $y = x^2$ is congruent to $y = x^2 - x$,
$y = -2x^2$ is congruent to $y = 3 - 2x^2$,
$y = 3x^2$ is congruent to $y = 3x^2 + 1$,
$y = 2 + 3x - 4x^2$ is congruent to $y = 1 + 4x^2$

8 **a** translate 1 unit down
b translate 2 units up
c reflect in the x-axis then translate 4 units up
d reflect in the x-axis then translate 1 unit up
(There are other correct answers to this question.)

9 **a** translate 2 units left
b translate 1 unit right
c reflect in the x-axis then translate 1 unit left
d translate 1 unit left and 3 units down
e translate 2 units right and 3 units down
f reflect in the x-axis then translate 3 units right and 1 unit up
(There are other correct answers to this question.)

10 **a** $y = (x+2)^2$ **b** $y = (x-3)^2 + 1$
c $y = (x-5)^2 - 2$ **d** $y = (x+3)^2 - 2$

11 **a** $y = 3(x + 3)^2 + 2$ **b** $y = 3(x - 3)^2 - 2$

12 **a** $y = -(x - 1)^2$
b $y = -(x + 2)^2$
c $y = -(x + 1)^2 - 2$

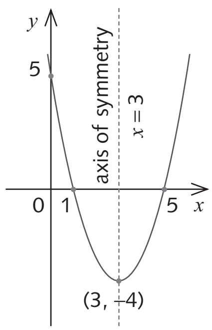
13 **a** $(1,\ 2)$ **b** $(-2,\ 3)$
c $(-4,\ -2)$ **d** $(5,\ 11)$
e $(-2,\ -1)$ **f** $(3, 4)$

14 $y = x^2 - 2x - 1$

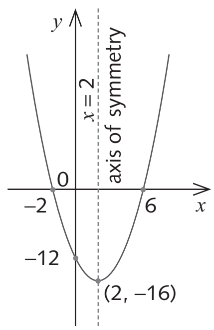
15 $y = x^2 + 2x - 15$

16 $y = -x^2 - 6x - 8$

17 **a**



b



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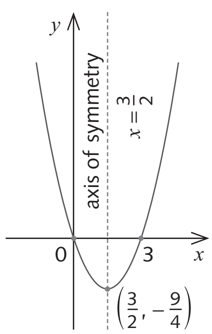
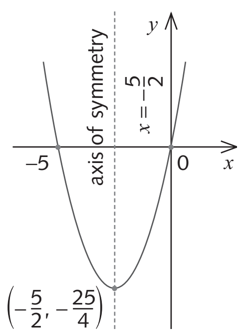


d



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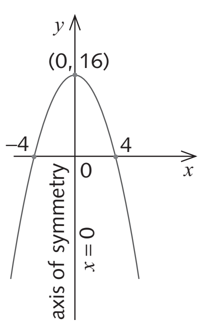
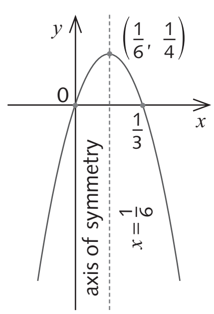
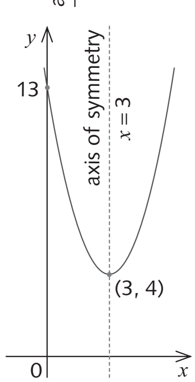


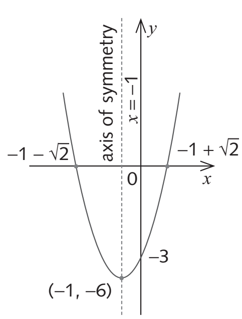
f



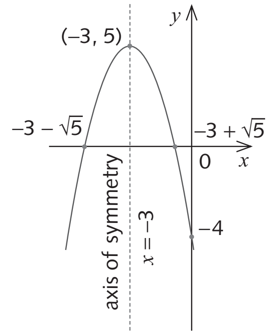
18 **a**



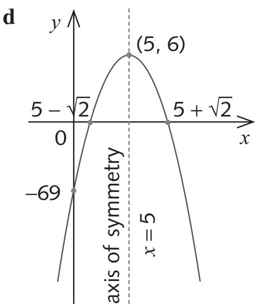
b



c

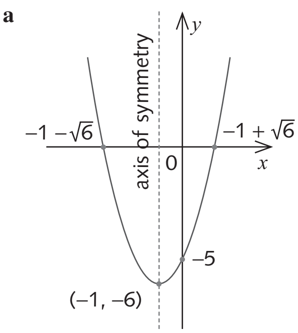


d

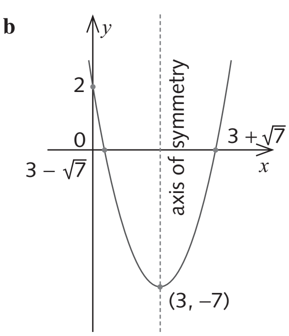


19 $y = 11(x - 2)^2 - 4$ **20** $h = 1$ or $h = -1$

21 **a**



b



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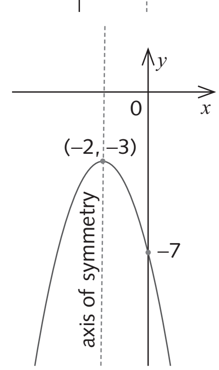

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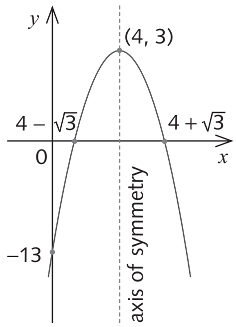

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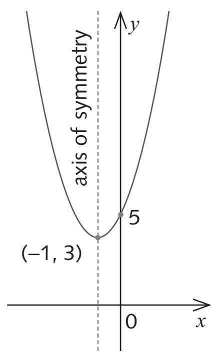

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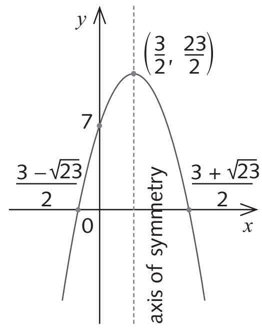


22 The side lengths are 5 cm, 12 cm and 13 cm.

23 52 cm, 42 cm and 4 cm

24 125 m by 250 m

25 **a** $-3 < x < 5$
b $-5 \le x \le -2$
c $x < 0$ or $x > 2$

Challenge exercise

1 **a** -5 **b** -5

2 **a** -21 **b** 4 **c** $c > 4$

3 **d** $x = 80,\ t = \frac{3}{2};\ x = 90,\ t = \frac{4}{3}$

4 40 km / h

5 45 cattle

6 32 km / h, 40 km / h

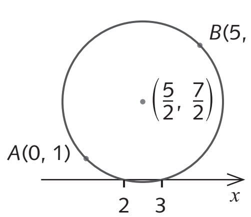
7 75 runs

8 4.844 m / s, 5.844 m / s

9 40 km / h, 60 km / h

10 $a = \frac{24}{5}$

11 **a** $\left(x - \frac{5}{2}\right)^2 + \left(y - \frac{7}{2}\right)^2 = \frac{25}{2}$



b Let $A(0,1)$ and $B(g,h)$ be the given points and $P(x,y)$ be any point. P is on the circle with diameter AB if $\angle APB = 90°$ (angle in semicircle). If gradient of $AP \times$ gradient of $BP = -1$, then

$\frac{y-1}{x} \times \frac{y-h}{x-g} = 1.$

So $(y-1)(y-h) + x(x-g) = 0$ is the equation of the circle. This meets the x-axis when $y = 0$. That is when $x^2 - gx + h = 0$.

12 **a** $\frac{x^2}{4\pi} + \frac{(100-x)^2}{16}$

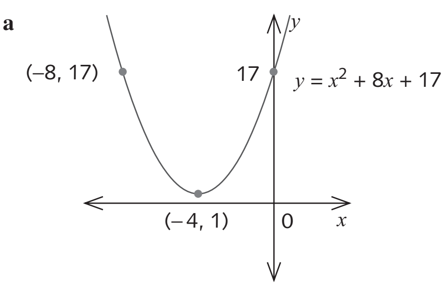
b The minimum area is $\frac{2500}{\pi + 4}$ cm^2 when $x = \frac{100\pi}{\pi + 4}$

13 **d** If $(x,\ y)$ goes to $(x,\ 5y)$, then $(a,\ a^2)$ goes to $(a,\ 5a^2)$. The graph of $y = x^2$ goes to the graph of $y = 5x^2$. $(0,\ 0)$ goes to $(0,\ 0)$, $(1,\ 0)$, goes to $(1,\ 0)$, and $(0,\ 1)$ goes to $(0,\ 5)$. It is not a similarity transformation since some distances are unchanged and others multiplied by 5. Note that 5 can be replaced by any $a > 0$.

e The enlargement with centre at $(0,\ 0)$ and enlargement factor $\frac{1}{5}$ maps $A(a,\ a^2)$ to $B\left(\frac{a}{5},\ \frac{a^2}{5}\right)$. Since $\frac{a^2}{5} = 5\left(\frac{a}{5}\right)^2$, B lies on $y = 5x^2$. That is $y = x^2$ is similar to $y = 5x^2$.

f By **c** $y = \pm ax^2 + bx + c$ is congruent to $y = ax^2$, $a > 0$. By **d** $y = ax^2$ is similar to $y = x^2$. That is, all parabolas are similar.

14 **a**



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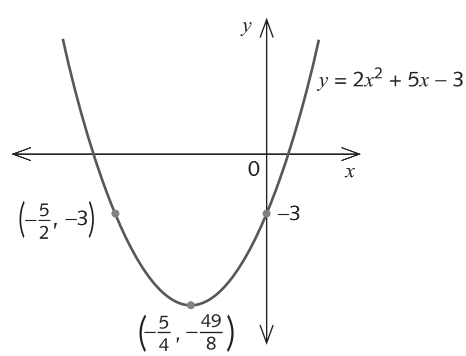

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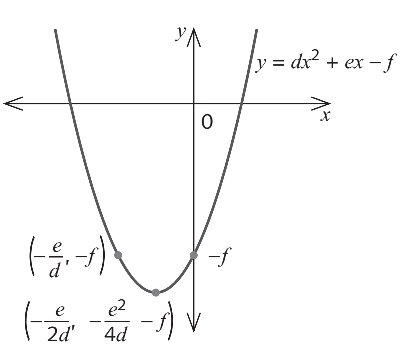


Chapter 8 answers

Exercise 8A

1 **a** $y = 6,\ \beta = 45°$ **b** $x = 5,\ \alpha = 30°$
c $\alpha = \gamma = 60°,\ \beta = 120°, x = 6$ **d** $\alpha = 90°,\ \beta = 60°$
e $x = 4,\ \alpha = 45°$ **f** $\alpha = 60°,\ x = 5$

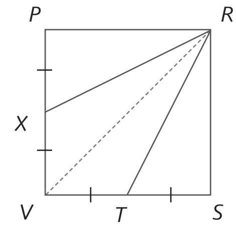
2 **a** $\alpha = \beta = \gamma = 60°$ (equilateral $\Delta\, RST$)
b $\theta = 20°$ (angle sum of $\Delta\, WXY$),
$x = 12$ cm (isosceles $\Delta\, WXY$)
c $\angle RQP = 60°$ (base angle of isosceles $\Delta\, PQR$),
$\theta = 60°$ (angle sum of $\Delta\, PQR$),
$y = 7$ cm (equilateral $\Delta\, RPQ$)
d $y = 5, \alpha = 60°$ (equilateral $\Delta\, LNQ$),
$\alpha = 2\beta$ (exterior angle of isosceles $\Delta\, LQM$), $\beta = 30°$
e $\angle ABD = 55°$ (straight angle at B),
$\angle BAD = 55°$ (base angle of isosceles $\Delta\, ABD$),
$\angle ADB = 70°$ (angle sum of ΔABD),
$\theta = 70°$ (vertically opposite angles at D),
$\angle AGE = 55°$ (alternate angles, $AC \parallel HE$),
$\alpha = 125°$ (straight angle at G)
f $\angle DCG = 55°$ (vertically opposite angles at C),
$\angle CDG = 55°$ (base angle of isosceles $\Delta\, CDG$),
$\alpha = 70°$ (angle sum of $\Delta\, CDG$),
$\angle AGE = 55°$ (corresponding angles, $BD \parallel HE$),
$\angle EGD = 55° + 70° = 125°$,
$\beta = 125°$ (vertically opposite angles at G)
g $\angle DAB = 50°$ (complementary angles),
$\alpha = 50°$ (alternate angles, $AB \parallel CE$),
$\angle AFD = 40°$ (base angle of isosceles $\Delta\, ADF$),
$\alpha + \angle FDC = 100°$ (angle sum of $\Delta\, ADF$),
$\angle FDC = 50°, \theta = 130°$ (straight angle at D)
h $y = 2$ m (radii of circle),
$\beta = 25°$ (base angle of isosceles $\Delta\, ABO$),
$\alpha = 60°$ (base angle of isosceles
$\Delta\, BCO$, angle sum of $\Delta\, BCO$)
i $3\alpha + \beta = 180°$ (co-interior angles, $AD \parallel BC$),
$\alpha + 2\beta = 180°$ (angle sum of isosceles triangle),
$\alpha = 36°, \beta = 72°$
j $\angle CBO = 2\alpha$ ($\Delta\, BOC$ isosceles),
$\angle OBA = 90 - \dfrac{3\alpha}{2}$ (angle sum of isosceles triangle),
$\angle OBC + \angle ABO = 95°$ (alternate angles, $AB \parallel CD$),
$\alpha = 10°$
k $\alpha = 130°$ (angle sum of quadrilateral)
l $\alpha = 108°$ (angle sum of regular pentagon)
m $\alpha = 120°$ (co-interior angles, $AB \parallel DC$),
$\beta = 60°$ (co-interior angles, $BC \parallel AD$)

3 6 sides **4** 36°

Exercise 8B

1 **a** $\Delta\, BAC \equiv \Delta YXZ$ (SAS) **b** $\Delta\, ABC \equiv \Delta\, XZY$ (ASA)
c $\Delta\, ABC \equiv \Delta\, PQR$ (SSS) **d** $\Delta\, ABC \equiv \Delta\, XYZ$ (ASA)
e $\Delta\, PQR \equiv \Delta\, XYZ$ (ASA)

2 **a** $\angle BCA = 40°, \angle FED = 100°, \angle FDE = 40°$,
$AB = BC = 3$ cm, $FD = 4.6$ cm
b $\angle ABC = 10°, \angle EFD = 130°, CB = 4.2$ cm,
$AC = 1.1$ cm, $ED = 5$ cm
c $\angle FED = 32°, \angle CAB = 111°, \angle BCA = 37°$,
$AB = 9.1$ cm, $\angle AC = 8$ cm, $FE = 14.1$ cm
d $\angle FED = 67°, \angle CAB = 67°, \angle ABC = 67°$,
$\angle BCA = 46°, CB = 38.3$ mm, $AC = 38.3$ mm,
$FE = 38.3$ mm, $ED = 30$ mm



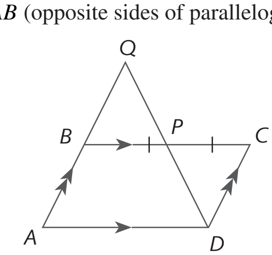
3 **a** $AD = BC$ (equal side lengths of square), $DE = CE$ (given),
$\angle ADE = \angle BCE$ (given),
$\Delta\, ADE \equiv \Delta\, BCE$ (SAS)
b $AE = BE$ (matching sides in congruent triangles)

4 **a** $PR = RS$ (equal side lengths of square), $PX = ST$
(half side lengths of square),
$\angle RPX = \angle RST = 90°$,
$\Delta RPX \equiv \Delta RST$ (SAS),
$RX = RT$ (matching sides in congruent triangles),
b $RX = RT$ (from **a**),
$VR = VR$ (common side),
$VT = VX$ (half side lengths of square)
$\Delta\, RXV \equiv \Delta\, RTV$ (SSS),
$\angle TRV = \angle XRV$ (matching angles of congruent triangles)

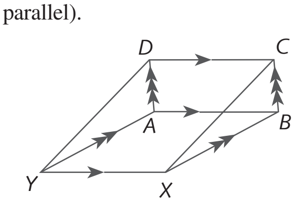
5 **a** Diagonals of a rectangle are equal and bisect each other.
b **i** $\angle AOD = 56°$ **ii** $\angle AOB = 124°$
iii $\angle OBC = 62°$ **iv** $\angle ABO = 28°$

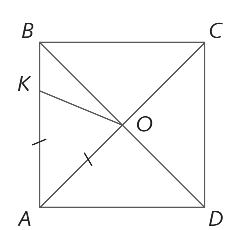
6 $\angle YDO = \angle OBX$ (alternate angles, $AB \parallel CD$),
$\angle BOX = \angle DOY$ (vertically opposite),
$OB = OD$ (diagonals of a parallelogram bisect each other),
$\Delta BOX \equiv \Delta DOY$ (AAS),
$OX = OY$ (matching sides of congruent triangles)

7 $BP = PC$ (P is the midpoint of BC),
$\angle QPB = \angle CPD$ (vertically opposite),
$\angle PQB = \angle CDP$ (alternate angles, $AQ \parallel DC$),
$\Delta BQP \equiv \Delta CDP$ (AAS),
$BQ = CD$ (matching sides of congruent triangles),
$CD = AB$ (opposite sides of parallelogram), $AQ = 2AB$



8 $DC = AB = YX$ (opposite sides of parallelogram),
$DC \parallel AB \parallel YX$ (opposite sides of parallelogram)
Therefore, $DCXY$ is a parallelogram (opposite sides equal and parallel).



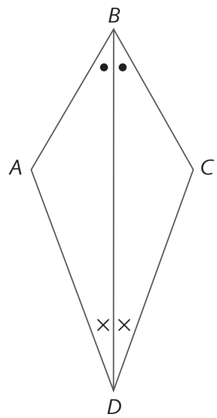


9 $\angle AKO = \angle AOK$ (isosceles triangle AKO),
$\angle KAO = 45°$ (diagonals of a square bisect the vertex angles),
$\angle AKO = \angle AOK = 67\tfrac{1}{2}°$ (angle sum of triangle),
$\angle BOA = 90°$ (diagonals of a square intersect at right angles),
$\angle KBO = 45°$ (diagonals of a square bisect the vertex angles),
$\angle BOK = 22\tfrac{1}{2}°$, $\angle AOK = 3\angle BOK$

10 **a** $\Delta DBC \equiv \Delta DBA$ (AAS)

$AB = CB$ (matching sides of congruent triangles)

$DA = DC$ (matching sides of congruent triangles)

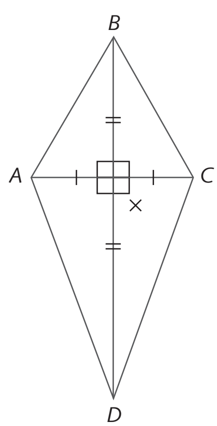


b $\Delta AXB \equiv \Delta CXB$ (SAS)

So $AB = CB$ (matching sides of congruent triangles)

$\Delta CDX \equiv \Delta ADX$ (SAS)

$AD = CD$ (matching sides of congruent triangles)



11 $\angle DAC = \angle BCA$ (alternate angles, $AD \parallel BC$)

$\angle ACD = \angle CAB$ (alternate angles, $AB \parallel CD$)

$\Delta ABC = \Delta CDA$ (SAS)

$CD = BA$ (matching sides of congruent triangles)

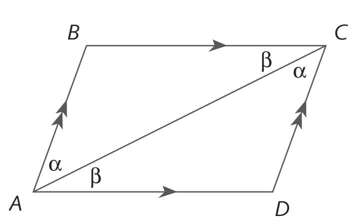
$AD = BC$ (matching sides of congruent triangles)

Thus, opposite sides are equal.

$\angle ABC = \angle CDA$ (matching angles of congruent triangles)

$\angle BAD = \alpha + \beta = \angle OCB$

Thus, opposite angles are equal.



12 Let X be the point of intersection of diagonals BD and AC.

We use the results of Question 11.

$AD = CB$

$\angle DAC = \angle BCA$ (alternate angles, $BC \parallel AD$)

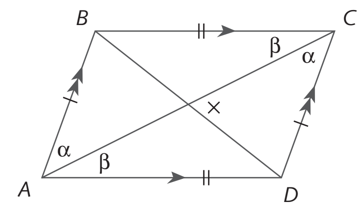
$\angle BXC = \angle DXA$ (vertically opposite)

$\angle BXC \equiv \angle DAX$ (AAS)

$CX = AX$ (matching sides of congruent triangles)

$DX = BX$ (matching sides of congruent triangles)

Thus, the diagonals of a parallelogram bisect each other.



13 A rhombus is a parallelogram.

$BC = CD$ (all sides of a rhombus are equal)

$BX = DX$ (diagonals of a parallelogram bisect each other)

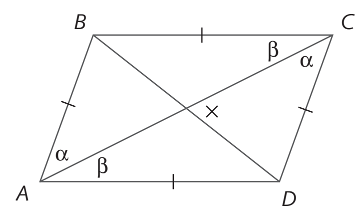
$\Delta BXC \equiv \Delta DXC$ (SSS)

$\angle BXC = \angle DXC$ (matching angles of congruent triangles)

$\angle BXC + \angle DXC = 180°$

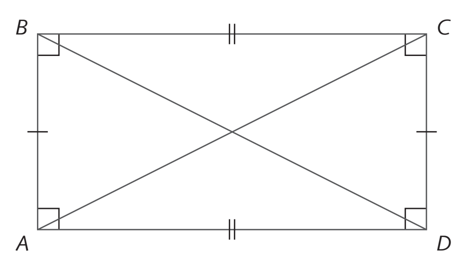
So $\angle BXC = \angle DXC = 90°$

Thus, diagonals of a rhombus are perpendicular.



14 $\Delta BAD \equiv \Delta DCB$ (SAS)

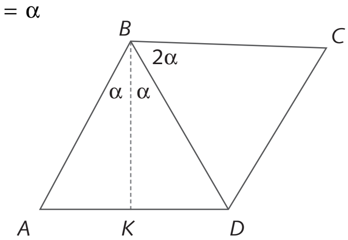
$BD = AC$ (matching sides of congruent triangles)



15 Let $\angle ABK = \angle DBK = \alpha$

Then $\angle CBD = 2\alpha$ (diagonals of rhombus bisect the vertex angles)

$\angle BAK = 180° - 4\alpha$ (co-interior angles $AD \parallel BC$)



$\angle BKA = 180° - (180° - 4\alpha) - \alpha$ (angle sum of triangle)

$= 3\alpha$

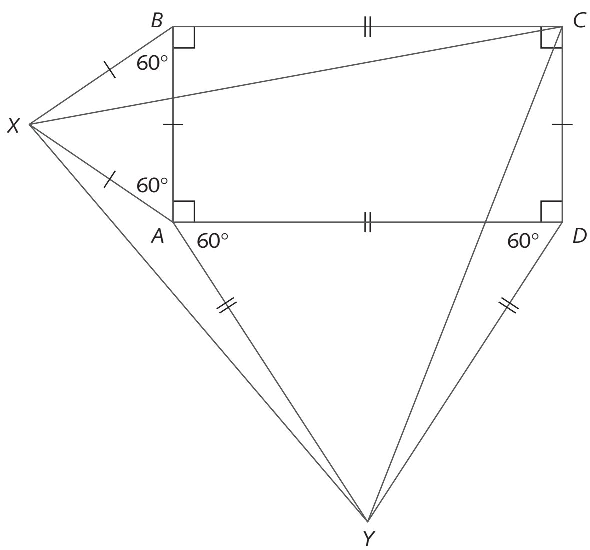
$= 3\angle \text{ABK}$

16 $\angle CBX = \angle CDY = 90° + 60° = 150°$

$\angle XAY = 360° - (90° + 60° + 60°) = 150°$

$\Delta XBC \equiv \Delta CDY \equiv \Delta XAY$ (SAS congruence test)

Hence, $XC = CY = YX$ (matching sides of congruent triangles)



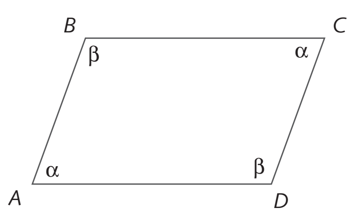
17 $2\alpha + 2\beta = 360°$ (angle sum of quadrilateral $ABCD$)

$\alpha + \beta = 180°$

so $\angle ABC + \angle BAD = 180°$

so $\angle BCD + \angle ADC = 180°$

Thus, $AB \parallel OC$ and $BC \parallel AD$ (co-interior angles are supplementary).



18 $BC = AD$ and $AB = DC$

$\Delta\ BCD \equiv \Delta\ DAB$ (SSS)

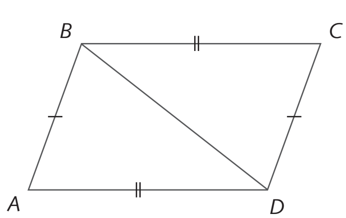
Hence,

$\angle ABD = \angle CDB$ (matching angles of congruent triangles)

$\angle ADB = \angle CBD$ (matching angles of congruent triangles)

Hence, $AB \parallel CD$ (alternate angles are equal)

and $BC \parallel AD$ (alternate angles are equal)



Exercise 8C

1 **a** ΔABC is similar to ΔRQP (SSS)

b ΔLMN is similar to ΔQRP (AAA)

c ΔSTU is similar to ΔXZY (AAA)

d ΔLMN is similar to ΔTSU (SAS)

e ΔBCA is similar to ΔECD (AAA)

f ΔGHF is similar to ΔJHI (SAS)

g ΔOMK is similar to ΔLMN (AAA)

h ΔSTR is similar to ΔSUQ (AAA)

i ΔEDC is similar to ΔADB (SAS)

j ΔIFG is similar to ΔIHF is similar to ΔFHG (AAA)

2 **a** AAA **b** $x = 3.5$

3 **a** SAS **b** $y = 14.4$

4 **a** ΔJKL is similar to ΔIGH (SSS)

b $\alpha = 49°\ a = \beta, b = 131 - \beta$

5 **a** ΔDEC is similar to ΔDBA (SAS)

b $\dfrac{EC}{BA} = \dfrac{ED}{DB} = \dfrac{DC}{DA}$

c $x = 2$

6 **a** $\angle ACB = \angle DCE, \angle CAB = \angle CED, \angle ABC = \angle EDC$

b 3.75 cm **c** 7.2 cm

7 **a** ΔADE is similar to ΔACB (AAA)

b $8\frac{4}{7}$ cm

8 **a** AAA **b** $LM = 16\frac{2}{3}$ cm

9 18.75 m

10 130 m

11 33 m

12 $\angle DOC = \angle AOB$ (vertically opposite),
$\angle ABO = \angle CDO$ (alternate angles, $AB \parallel CD$),
ΔAOB is similar to ΔCOD (AAA)

$\dfrac{OB}{OD} = \dfrac{OA}{OC} = 2, OB = 2OD$

13 **a** $\angle BAD = \angle BAC$ (common angle),
$\angle ADB = \angle ABC = 90°$,
ΔABD is similar to ΔACB (AAA)

b $\angle BCD = \angle BCA$ (common angle),
$\angle ABC = \angle BDC = 90°$, ΔBCD is similar to ΔACB (AAA)

14 **a** $\angle AEP = \angle CBP$ (alternate angles, $CB \parallel AD$),
$\angle EPA = \angle BPC$ (vertically opposite),
ΔAPE is similar to ΔCPB (AAA)

b $\dfrac{AP}{PC} = \dfrac{AE}{AD} = \dfrac{AE}{CB}$ (ratios of matching sides of similar triangles) and $AD = CB$ (opposite sides of a parallelogram) $\dfrac{AD}{PC} = \dfrac{AE}{CD} = \dfrac{AE}{CB} = \dfrac{1}{2}$,

$AP = 2PC$ and $AC = 3AP$

15 **a** $\angle EDB = 180° - \angle ACB$ (given),
$\angle ADE = \angle ACB$ (supplementary),
$\angle BAC = \angle DAE$ (common),
ΔADE is similar to ΔACB (AAA)

b $\dfrac{AE}{AB} = \dfrac{AD}{AC}$ (equal ratios of matching sides of similar triangles)

16 **a** $\angle PDB = 90°$ (given), $\angle AFB = 90°$ (given)
$\angle DBP = \angle ABF$ (common)
ΔPBD is similar to ΔABF (AAA)
$\Delta ABF \equiv \Delta ACF$ (RHS)
So ΔPDB is similar to ΔACF

b $\dfrac{FC}{DB} = \dfrac{AC}{PB}$ (matching sides of similar triangles)

17 $LM = \dfrac{1}{2}AC$ and $NM = \dfrac{1}{2}AB$. But $AC = AB$. Therefore, $LM = NM$.

18 **a** $\angle ABC = \angle ADB = 90°$, $\angle BAC = \angle BAD$ (common),
ΔABC is similar to ΔADB (AAA)

b $\dfrac{b}{x} = \dfrac{a}{c} = \dfrac{c}{a-y}$. Therefore, $a(a-y) = c^2$ (1)

c $\angle ABC = \angle BDC = 90°$, $\angle BCA = \angle BCD$ (common),
ΔABC is similar to ΔBDC (AAA)

d $\dfrac{c}{x} = \dfrac{b}{y} = \dfrac{a}{b}$, so $ay = b^2$ (2)

e From (1) $a(a-y) = c^2$ and $a^2 = c^2 + ay$
From (2) $ay = b^2$
$a^2 = b^2 + c^2$

Exercise 8D

1 Let M be the midpoint of AB.

Let N be the midpoint of BC.

$MB = \dfrac{1}{2}AB$ and $NB = \dfrac{1}{2}CB$

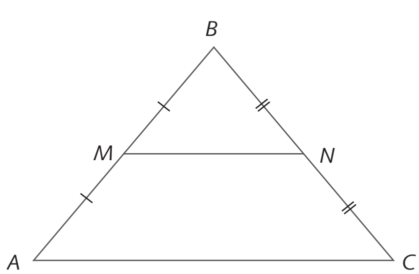
Hence, $\dfrac{MB}{AB} = \dfrac{1}{2}$ and $\dfrac{NB}{CB} = \dfrac{1}{2}$

ΔMBN is similar to ΔABC (SAS)

Hence, $\angle BAC = \angle BMN$ (matching angles of similar triangles)

$MN \parallel AC$ (corresponding angles equal)

Also, $MN = \dfrac{1}{2}AC$ (similarity factor of $\dfrac{1}{2}$)



2 Let M be the midpoint of AB and $MN \parallel AC$.

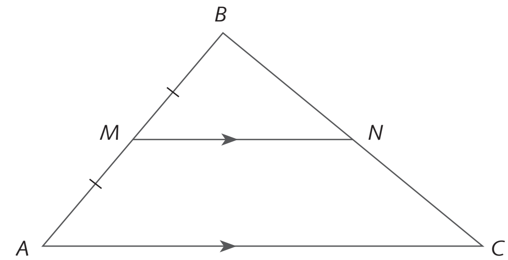
$\angle BMN = \angle BAC$ (corresponding angles, $MN \parallel AC$)

ΔBMN is similar to ΔBAC (AAA)

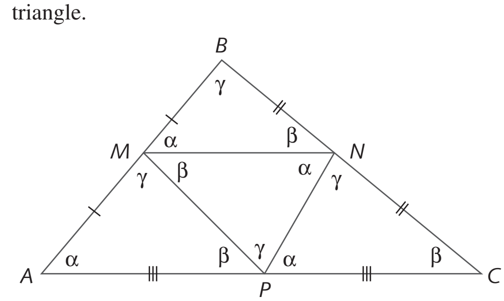
$BM = \dfrac{1}{2}BA$ (matching sides of similar angles)

$\dfrac{BM}{BA} = \dfrac{BN}{BC} = \dfrac{1}{2}$

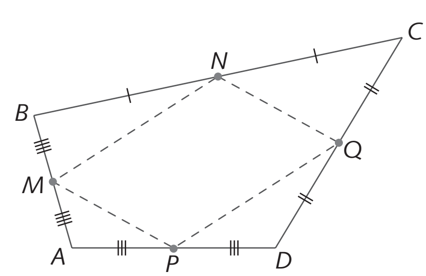
Thus, N is the midpoint of BC.



3 M is the midpoint of AB.
N is the midpoint of BC.
P is the midpoint of AC.
From question 1,
$MN = \dfrac{1}{2}AC$, $MP = \dfrac{1}{2}BC$, $PN = \dfrac{1}{2}AB$
Therefore,
$NP = MB = AM$
$MN = AP = PC$
$MP = BN = NC$
Hence, all the triangles are congruent by the SSS test.
All triangles have angles α, β and γ.
Therefore, each of the small triangles is similar to the large triangle.

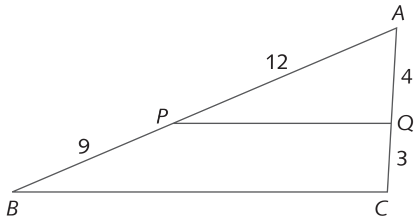


4 Let M, N, Q and P be the midpoints of AB, BC, CD and DA respectively.
Join vertices B and D.
In ΔCBD, $NQ \parallel BD$ (Question 1)
In ΔABD, $MP \parallel BD$ (Question 1)
Therefore, $NQ \parallel MP$.
In a similar way, $MN \parallel PQ$.
Thus, $MNQP$ is a parallelogram.



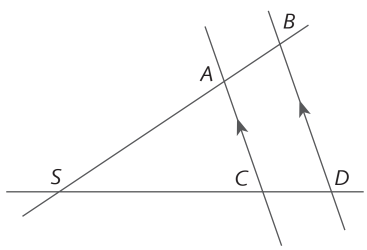
5 In ΔPAQ and ΔBAC
$\angle PAQ = \angle BAC$ (common)
$\dfrac{BA}{PA} = \dfrac{AC}{AQ} = \dfrac{7}{4}$
ΔPAQ is similar to ΔBAC (SAS)
$\angle APQ = \angle ABC$ (matching angles of similar triangles)

$PQ \parallel BC$ (corresponding angles are equal)



6 **a** ΔSAC is similar to ΔSBD

$$\frac{SA}{SA + AB} = \frac{SC}{SC + CD}$$
$$SA \times (SC + CD) = SC \times (SA + AB)$$
$$SA \times SC + SA \times CD = SC \times SA + SC \times AB$$
$$SA \times CD = SC \times AB$$
$$\frac{SA}{AB} = \frac{SC}{CD}$$
$$SA : AB = SC : CD$$



b ΔSAC is similar to ΔSBD

$$\frac{SB}{SB - AB} = \frac{SD}{SD - CD}$$
$$SB \times (SD - CD) = SD \times (SB - AB)$$
$$SB \times SD - SB \times CD = SD \times SB - SD \times AB$$
$$SB \times CD = SD \times AB$$
$$\frac{SB}{AB} = \frac{SD}{CD}$$

That is, $SB : AB = SD : CD$.

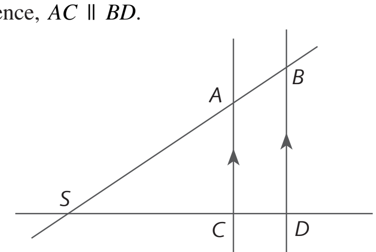
c $\frac{SA}{SB} = \frac{SC}{SD}$ or ΔSAC is similar to ΔSBD.

$SA : SB = SC : SD$

7
$$\frac{SA}{AB} = \frac{SC}{CD}$$
$$SA \times CD = AB \times SC$$
$$SA \times SC + SA \times CD = AB \times SC + SC \times SA$$
$$SA \times (SC + CD) = SC \times (AB + SA)$$
$$SA \times SD = SC \times SB$$
$$\frac{SA}{SB} = \frac{SC}{SD}$$

Therefore, ΔSAC is similar to ΔSBD (SAS).

Hence, $AC \parallel BD$.



Review exercise

1 $\alpha = 37°, \beta = 80°, \gamma = 20°$

2 **a** ΔPCQ is similar to ΔACB

b $x = 5$

3 **a** In ΔBEC and ΔFED, $\angle CBE = \angle DFE$ (alternate angles, $BC \parallel AD$)
$\angle CEB = \angle FED$ (vertically opposite),
ΔBEC is similar to ΔFED (AAA)

b **i** 2 **ii** 4

4 **a** $\alpha = 112°$

b $\alpha = 78°, \beta = 58°$

c $\alpha = 20°$

d $\beta = 104°, \alpha = 76°, \gamma = 52°$

e $\alpha = 36°, \beta = 144°$

f $\alpha = 130°, \beta = 50°, \gamma = 130°$

5 **a** $\angle BAC = 70°, \angle ACB = 55°$

b $\angle FIC = 55°$ (corresponding angles, $FI \parallel AB$).
$\angle FCI = 55°$. So ΔFIC is isosceles.

c $\angle DIB = 55°$ (corresponding angles, $AC \parallel DI$). ΔDBI is isosceles (equal base angles), $DB = DI$

d $\angle GIF = 53°, \angle EFG = 17°$

6 **a** $\Delta ONM \equiv \Delta ZYX$ (AAS)

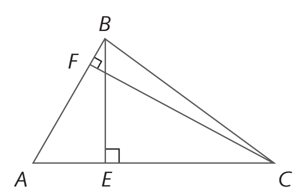
b $\Delta ABC \equiv \Delta RTS$ (RHS)

7 ΔAOB is similar to ΔCOD (AAA).

Ratio of matching sides gives: $\frac{BO}{OD} = \frac{AO}{OC}$

8 In ΔCFA and ΔBEA,
$\angle CFA = \angle BEA = 90°$ (altitudes),
$\angle FAC = \angle EAB$ (common),
ΔCFA is similar to ΔBEA (AAA)

$\frac{BE}{CF} = \frac{AB}{AC}$ (ratios of matching sides)



9 **a** $x = \frac{102}{11}$ **b** $x = \frac{21}{4}$

10 **a** $x = 13$ **b** $EF = \frac{119}{13}$

11 20 cm

Challenge exercise

Only outlines of proofs are given in the challenge exercise

1 **a** 272 cm **b** $50\sqrt{55}$ cm

c $16\sqrt{55}$ cm **d** approximately 67 cm

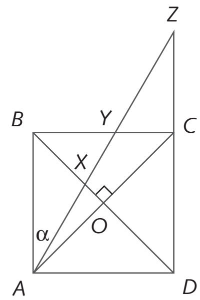
2 Produce AY to Z. Let $\angle BAY = \alpha$ $\angle AZC = \alpha$ (alternate angles, $BA \parallel ZC$)

Then $\angle YAC = \alpha$ (angle bisector), and ΔZCA is isosceles

So $ZC = CA = 2OA$

Next, ΔAOX is similar to ΔZCY (AAA)

$$\frac{CY}{OX} = \frac{ZC}{OA} = 2 \text{ or } CY = 2OX$$

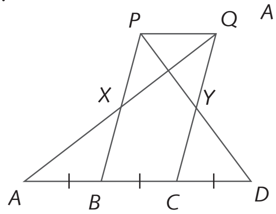


3 $PQ = BC = AB = CD = x$ and $BP = CQ = 2x$

$\Delta AXB \equiv QXP$ (AAS)

$\therefore PX = XB = x = QY = YC$

$PQYX$ is a rhombus and XQ and PY are diagonals of a rhombus. Hence, $AQ \perp DP$.

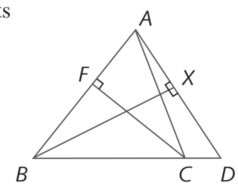


4 Show that $\angle BAD = \angle BDA = \angle ACF = 54°$.

Hence, ΔABD is isosceles with $AB = BD$.

Draw the perpendicular from B to AD. Let X be the point where the perpendicular meets AD. X is the midpoint of AD.

$\Delta AFC \equiv \Delta BXD$ (AAS). Hence, $FC = XD$ and $AD = 2FC$.



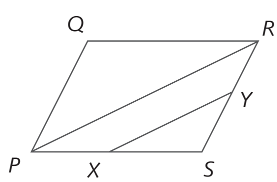
5 ΔSXY is similar to ΔSPR.

$SX = \frac{1}{2} PS$, $SY = \frac{1}{2} SR$

Therefore, area of $\Delta SXY = \frac{1}{4}$ area of ΔSPR

Therefore, area of $\Delta SPR = \frac{1}{2}$ area of parallelogram $PQRS$

Therefore, area of $\Delta XYS = \frac{1}{8}$ area of parallelogram $PQRS$



6 **a** $\angle BAD = \angle BEC$ (corresponding angles, $AD \parallel EC$)

$\angle DAC = \angle ACE$ (alternate angles, $AD \parallel EC$)

ΔBAD is similar to ΔBEC (AAA)

ΔAEC is isosceles; $AC = AE$ (base angles of ΔAEC)

$$\frac{BD}{BD + DC} = \frac{BA}{BA + AE}$$

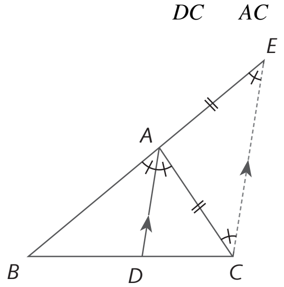
$$\frac{BD}{BD + DC} = \frac{BA}{BA + AC} \quad (AC = AE)$$

$$BD(BA + AC) = BA(BD + DC)$$

$$BD \times BA + BD \times AC = BA \times BD + BA \times DC$$

$$BD \times AC = BA \times DC$$

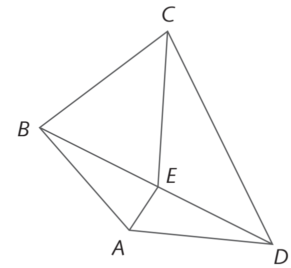
$$\frac{BD}{DC} = \frac{BA}{AC}$$



b From **a**, $\frac{BC}{CD} = \frac{BE}{ED}$ and $\frac{BA}{AD} = \frac{BE}{ED}$.

Therefore, $\frac{BC}{CD} = \frac{BA}{AD}$

Therefore, $\frac{AD}{AB} = \frac{CD}{CB}$

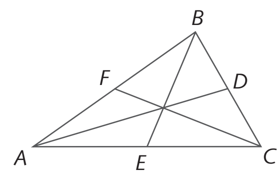


c From **a**, $\frac{AF}{FB} = \frac{AC}{BC}$

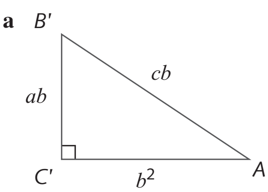
$$\frac{BD}{DC} = \frac{BA}{CA}$$

$$\frac{AE}{EC} = \frac{AB}{BC}$$

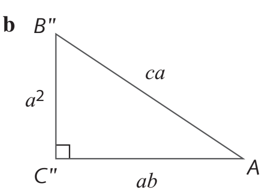
$$\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = \frac{\cancel{BA}}{\cancel{CA}} \times \frac{\cancel{BC}}{\cancel{AB}} \times \frac{\cancel{AC}}{\cancel{BC}} = 1$$



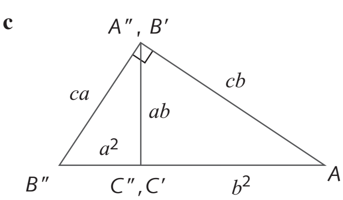
7 **a**



b



c



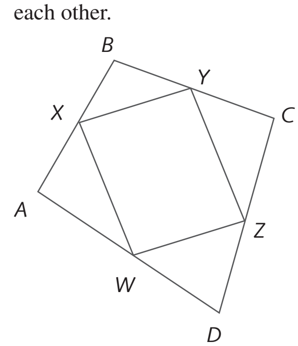
d $\angle B''A''A'$ is a right angle. The new triangle is similar to triangle ABC.

e c

f $a^2 + b^2 = c^2$

8 $XY = \frac{1}{2} AC = WZ$ and $YZ = \frac{1}{2} BD = XW$.

Therefore, $XYZW$ is a parallelogram. Diagonals bisect each other.



Chapter 9 answers

Exercise 9A

1 **a** 16 **b** 125 **c** 64
d 27 **e** 10 000 **f** 216

2 **a** 2^3 **b** 2^6 **c** 3^4
d 2^5 **e** 5^4 **f** 3^5

3 **a** 13^{-1} **b** 7^{-2} **c** 13^{-3}
d 2^{-10} **e** 3^{-4} **f** 11^{-10}

4 **a** a^{15} **b** a^{11} **c** m^{15}
d p^{11} **e** a^6b^7 **f** m^9n^6
g $8a^5b^5$ **h** $15x^5y^4$ **i** $15x^8y^9$

5 **a** $\frac{x}{y}$ **b** $\frac{3xy^2}{2}$ **c** a^4m^3
d $\frac{5xy}{4}$ **e** $\frac{2c^2}{3b^2}$ **f** $\frac{4x^3}{y}$

6 **a** a^6b^{12} **b** $x^{21}y^{35}$ **c** $a^5b^{10}c^{15}d^{20}$
d $4a^6b^2$ **e** $9a^4b^8$ **f** $64a^9b^6$

7 **a** $18m^{12}$ **b** p^4 **c** a^6b^3
d m^5n **e** a^5b^2 **f** $p^{12}q$

8 **a** $\frac{1}{2}$ **b** $\frac{1}{4}$ **c** $\frac{1}{3}$
d $\frac{1}{9}$ **e** $\frac{1}{1000}$ **f** $\frac{8}{7}$
g $\frac{14}{15}$ **h** $\frac{25}{9}$ **i** $\frac{27}{8}$
j $\frac{1331}{125}$ **k** $\frac{1}{81}$ **l** 6
m 4 **n** 1 **o** 1

9 **a** $\frac{1}{2x^2y}$ **b** $\frac{1}{9x^4y^4}$ **c** $\frac{y^3}{64x^3}$
d $\frac{y^6}{8x^6}$ **e** $\frac{x^6y^6}{27}$ **f** $4x^{10}y^{10}$

10 **a** $\frac{1}{m}$ **b** $\frac{8}{a^4b^3}$ **c** $\frac{15p^3}{q^5}$
d $\frac{3p^{11}}{2q^6}$ **e** $\frac{y}{2x^6}$ **f** $\frac{x^4}{16}$
g $\frac{1}{a^4b^9}$ **h** m^7n^6 **i** $\frac{p^8n^2}{m^4}$
j $\frac{a^2}{b^8c^2}$ **k** $\frac{4ab^5}{3}$ **l** $\frac{1}{m^9n^{17}}$
m $\frac{x^7}{y^7}$ **n** a^3b^{11} **o** $\frac{2a^{15}}{b^8c^3}$
p mn^3p^9 **q** $\frac{a^8}{b^7}$ **r** $\frac{8a^{14}}{b^6}$

11 **a** $\frac{1}{27}$ **b** $\frac{1}{64}$ **c** $-\frac{1}{64}$

12 $\frac{7}{208}$ **13** $\frac{13}{3}$

14 **a** $-\frac{x}{y}$ **b** $\frac{1}{x^2}-\frac{1}{y^2}$
c $\frac{3x^2y^2}{x+y}$ **d** $\frac{x^2y+xy^2}{x^2+y^2}$
e $\frac{x^4}{1+2x^2y+x^4y^2}$ **f** $\frac{x^2y}{x^2+y}$

Exercise 9B

1 **a** 6.3×10^1 **b** 4×10^{-1} **c** 6.2×10^{-1}
d 7.4×10^3 **e** 2.1×10^7 **f** 2.6×10^{-4}
g -8.6×10^{-2} **h** 2×10^{12} **i** 9.1345×10^{-5}
j 5.732×10^4 **k** 3.012×10^{-3} **l** $1.000\,51 \times 10^{-1}$

2 **a** 2.25×10^7 **b** 6.700×10^{-7} m **c** 1.5×10^{11} m

3 **a** 6400 **b** 92 000 **c** 0.048
d 0.0087 **e** 7 412 000 **f** −402
g −0.004 657 **h** 47.26

4 **a** 8×10^5 **b** 1×10^6 **c** 1.26×10^8
d 2.04×10^6 **e** 2×10^{-4} **f** 4×10^{-6}
g 1.21×10^{-16} **h** 2×10^{-21} **i** 2.5×10^1
j 3×10^{-2} **k** 5×10^6 **l** 5×10^0

5 **a** 1.026×10^9 **b** 5.83×10^1 **c** 6×10^{-7}
d 4.34×10^9 **e** 1.2×10^2 **f** 1.6×10^2

6 5.8473×10^9 kg $= 5.8473 \times 10^6$ tonnes

7 **a** 2.18×10^{18} km **b** 1.23×10^{23} km
c approximately 8 minutes 18 seconds

8 1840 electrons

9 **a** 5.766×10^2 **b** 4.73×10^2 **c** 4.7×10^2
d 5×10^2 **e** 5.124×10^{-2} **f** 5.12×10^{-2}
g 5.1×10^{-2} **h** 5×10^{-2} **i** 1.603×10^3
j 1.60×10^3 **k** 1.6×10^3 **l** 2×10^3
m 2.994×10^{27} **n** 2.99×10^{27} **o** 3.0×10^{27}
p 3×10^{27} **q** 5.73×10^5 **r** 7×10^{-3}

10 **a** 5.60 **b** 538 **c** 9670
d 732 000 **e** 0.003 51 **f** 0.0142
g 372 **h** 478 000

11 2.93×10^{-8} m^3

12 **a** 4.14×10^4 **b** 2.97×10^9

13 **a** between 14.5 cm and 15.5 cm
b between 1.995×10^3 kg and 2.005×10^3 kg
c between 18.665 m and 18.675 m
d between $4.874\,45 \times 10^7$ mL and $4.874\,55 \times 10^7$ mL

Exercise 9C

1 **a** 2 **b** 7 **c** 3
d 2 **e** 10 **f** 5

2 **a** $\frac{1}{5}$ **b** $\frac{1}{8}$ **c** $\frac{1}{2}$
d $\frac{1}{100}$ **e** $\frac{1}{10}$

3 **a** 9 **b** 16 **c** 27
d 8 **e** 243 **f** 100 000
g $\frac{1}{4}$ **h** $\frac{16}{81}$ **i** 1331
j 49 **k** $\frac{8}{125}$ **l** $\frac{125}{216}$
m $\frac{1}{1000}$ **n** $\frac{7}{10}$ **o** $\frac{16}{81}$

4 **a** $\sqrt[3]{2}$ **b** $\sqrt[4]{3}$ **c** $\sqrt[5]{20}$
d $\sqrt[4]{10}$ **e** $\sqrt[5]{9}$

5 **a** $4^{\frac{1}{3}}$ **b** $13^{\frac{1}{7}}$ **c** $5^{\frac{1}{4}}$
d $11^{\frac{2}{7}}$ **e** $5^{\frac{2}{3}}$ **f** $7^{\frac{3}{2}}$
g $11^{\frac{3}{5}}$ **h** $10^{\frac{2}{7}}$

6 **a** $5^{\frac{3}{2}}$ **b** $5^{\frac{4}{3}}$ **c** $6^{\frac{5}{4}}$
d $7^{\frac{6}{5}}$ **e** $5^{\frac{7}{3}}$ **f** $11^{\frac{5}{2}}$

7 **a** $\frac{1}{3}$ **b** $\frac{1}{2}$ **c** $\frac{1}{11}$
d $\frac{1}{10}$ **e** $\frac{1}{1000}$ **f** $\frac{1}{11}$

8 **a** $\frac{1}{8}$ **b** $\frac{1}{1000}$ **c** 4
d $\frac{1}{625}$ **e** $\frac{1}{100}$ **f** $\frac{1}{8}$

9 **a** 2 **b** $3^{\frac{17}{15}}$ **c** $7^{\frac{3}{5}}$
d $3^{\frac{7}{12}}$ **e** $10^{\frac{3}{2}}$ **f** $10^{\frac{11}{12}}$
g $3^{\frac{7}{15}}$ **h** $\frac{1}{2^{\frac{1}{20}}}$ **i** $\frac{1}{5^{\frac{3}{10}}}$

10 **a** $2^{\frac{2}{5}}$ **b** 7 **c** $2^{\frac{9}{28}}$
d $7^{\frac{1}{6}}$ **e** 2 **f** $\frac{1}{10^{\frac{2}{7}}}$

11 **a** 3.9811 **b** 8.3203 **c** 12.748
d 3.7477 **e** 9.3152 **f** 2.4150
g 2202.7 **h** 2.4721

12 **a** $m^{\frac{11}{12}}$ **b** $a^{\frac{11}{15}}$ **c** $x^{\frac{3}{4}}y^{\frac{11}{15}}$
d $a^{\frac{10}{11}}b^{\frac{5}{6}}$ **e** $a^{\frac{1}{2}}$ **f** $\frac{1}{m^{\frac{1}{6}}}$
g $b^{\frac{5}{21}}$ **h** $4m^{\frac{8}{5}}$ **i** $\frac{1}{27m^{\frac{3}{2}}}$
j $\frac{a^2}{625}$ **k** $\frac{80}{m^{\frac{7}{12}}}$ **l** $m^{\frac{13}{6}}$
m $4m^{\frac{5}{2}}$ **n** $\frac{3}{8m^3}$

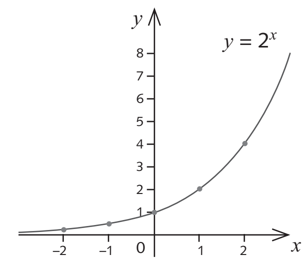
13 **a** 8.586 **b** 458.2 **c** 1.130
d 0.9547 **e** 0.010 46 **f** 0.001 406

14 **a** $a^{4.8}$ **b** m^6 **c** $p^{3.6}$
d $b^{1.25}$ **e** $4p^{2.6}$ **f** $64p^{6.3}$
g $\frac{b^{1.6}}{2a^{0.7}}$ **h** $\frac{2n^8}{3m^{4.2}}$ **i** $\frac{b^{5.1}}{a^{0.8}}$
j $\frac{1}{m^{2.1}n^{5.8}}$

Exercise 9D

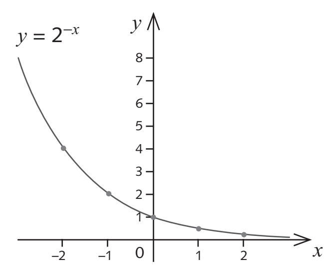
1 **a**

x	–2	–1	0	1	2
2^x	0.25	0.5	1	2	4



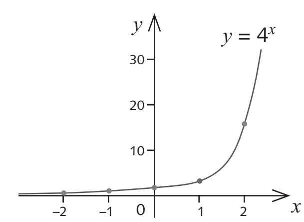
b

x	–2	–1	0	1	2
2^{-x}	4	2	1	0.5	0.25



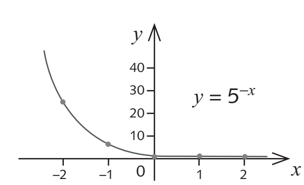
c

x	–2	–1	0	1	2
4^x	0.0625	0.25	1	4	16



d

x	–2	–1	0	1	2
5^{-x}	25	5	1	0.2	0.04



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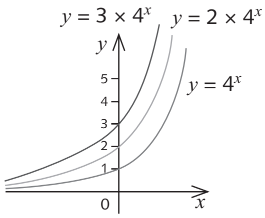

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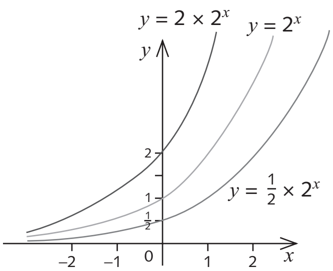

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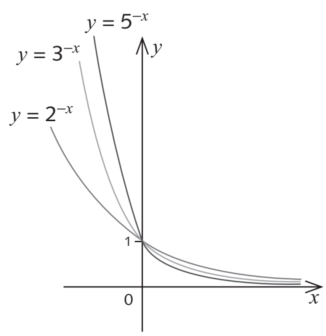


Exercise 9E

1 **a** 3 **b** 9 **c** 5 **d** 2
e 3 **f** 2 **g** 3 **h** 5
i 3 **j** 6 **k** 4 **l** 5

2 **a** –4 **b** –4 **c** 0 **d** –3
e –5 **f** –3 **g** –5 **h** –10

3 **a** $\frac{1}{2}$ **b** $\frac{3}{2}$ **c** $\frac{3}{2}$ **d** $\frac{2}{3}$
e $\frac{3}{2}$ **f** $\frac{2}{3}$ **g** $\frac{2}{3}$ **h** $\frac{3}{4}$

4 **a** $\frac{5}{3}$ **b** $\frac{7}{2}$ **c** $\frac{5}{7}$ **d** $\frac{3}{4}$
e $\frac{1}{3}$ **f** $-\frac{2}{3}$ **g** $-\frac{1}{3}$ **h** $-\frac{3}{2}$

5 **a** 5 **b** –2 **c** $\frac{4}{3}$ **d** $\frac{2}{15}$
e $\frac{3}{2}$ **f** 6 **g** $-\frac{5}{4}$ **h** $\frac{3}{2}$

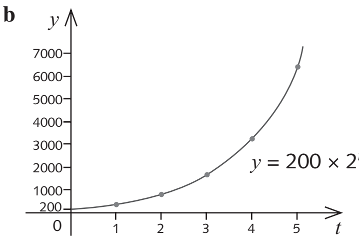
6 **a** 4 and 5 **b** 2 and 3 **c** 5 and 6
d 2 and 3 **e** 4 and 5 **f** 2 and 3
g 1 and 2 **h** 2 and 3 **i** 3 and 4
j –1 and 0 **k** 0 and 1 **l** 1 and 2

Exercise 9F

1 **a**

t	0	1	2	3	4	5
y	200	400	800	1600	3200	6400

b

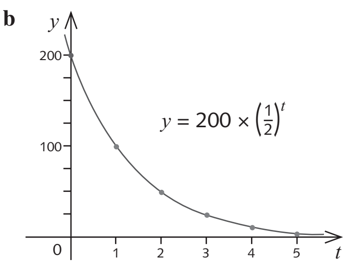


c **i** 303.14 **ii** 918.96 **iii** 2262.74

2 **a**

t	0	1	2	3	4	5
y	200	100	50	25	12.5	6.25

b



c **i** 131.95 **ii** 21.76 **iii** 8.25

3 **a** **i** 60 **ii** 3840 **iii** 10 861
b **i** 1000 **ii** 100
iii 1 **iv** 0.1

4 **a** 8.2007×10^9 **b** 1.360×10^{11}

5 **a** 63.2° **b** 42.0°

6 **a** 0.8% **b** 80 000 **c** 86 635

7 **a** 1000
b **i** 250 **ii** 125 **iii** 31

Exercise 9G

1 **a** $\log_2 8 = 3$ **b** $\log_{10} 100 = 2$ **c** $\log_7 49 = 2$
d $3^4 = 81$ is equivalent to $\log_3 81 = 4$
e $5^3 = 125$ is equivalent to $\log_5 125 = 3$
f $7^3 = 343$ is equivalent to $\log_7 343 = 3$
g $2^5 = 32$ is equivalent to $\log_2 32 = 5$
h $10^4 = 10\,000$ is equivalent to $\log_{10} 10\,000 = 4$
i $10^{-3} = 0.001$ is equivalent to $\log_{10} 0.001 = -3$
j $2^{-1} = 0.5$ is equivalent to $\log_2 0.5 = -1$

2 **a** 2 **b** 6 **c** 7 **d** 12
e 0 **f** 8 **g** 3 **h** 2

3 **a** 3 **b** 4 **c** 3 **d** 2
e 3 **f** 0 **g** 4 **h** 3

4 **a** -2 **b** -1 **c** -2 **d** -2
e -3 **f** -5 **g** -4 **h** -3

5 **a** 1 **b** 0 **c** 3 **d** 5
e 100 **f** -2 **g** -7 **h** -13

6 1

7 0

8 **a** 2.8971 **b** -3.5229 **c** 7.8573
d -2.2680 **e** 56.8028 **f** -3.9085

9 **a** 6 **b** 5 **c** -4
d -3 **e** 27 **f** 25
g $\frac{1}{8}$ **h** $\frac{1}{5}$ **i** 2
j 4 **k** 5 **l** 2

Review exercise

1 **a** a^{17} **b** $1296m^{16}$ **c** a^6b^2

2 **a** $\frac{1}{16}$ **b** 6
c $\frac{1}{10\,000}$ **d** $\frac{81}{16}$

3 **a** $\frac{1}{a^8}$ **b** $\frac{14}{a^3}$
c $\frac{4}{a^2}$ **d** $\frac{1}{4a^6}$

4 **a** $\frac{1}{b^3}$ **b** $\frac{2}{x^4}$ **c** $\frac{5}{x^3}$
d $\frac{1}{2x^3}$ **e** $\frac{1}{5a^4}$ **f** $2x^2$
g $4x^3$ **h** $\frac{4b^3}{a^2}$ **i** $\frac{5m^3}{6}$

5 **a** $\frac{1}{36}$ **b** $\frac{1}{64}$ **c** $\frac{1}{16}$
d $\frac{1}{5}$ **e** $\frac{1}{100}$

6 **a** 1 **b** 5 **c** 1
d 7 **e** 1 **f** 5
g 1 **h** $\frac{1}{3}$ **i** 4

7 **a** $8ab$ **b** ab^5 **c** $2b$
d $\frac{1}{32a}$ **e** $\frac{9}{a^2}$ **f** $\frac{a^4}{2}$

8 2^{2n-4}

9 6^{3x}

10 **a** $2^{-\frac{1}{6}}$ **b** $a^{\frac{11}{20}}$
c $2^{\frac{7}{3}}$ **d** $2^{\frac{3}{5}}$

11 **a** 4.2×10^3 **b** 6.2×10^{-3}
c 7.4×10^8 **d** 2×10^{-7}

12 **a** 5400 **b** 112 000
c 0.068 **d** 0.0097
e 0.18 **f** 0.000 064
g 7 410 000 **h** 402

13 **a** 20 **b** 500 **c** 420
d 34 000 **e** 0.0068 **f** 0.0492
g 480 **h** 600 **i** 0.007

14 **a** 3 **b** 4 **c** -2
d 0 **e** -2 **f** -3
g -4 **h** -3

15 **a** 1 **b** 5 **c** 15
d -1 **e** -2 **f** -6

16 **a** $-\frac{5}{3}$ **b** $-\frac{7}{3}$ **c** $-\frac{1}{5}$
d -2 **e** -5 **f** 6

17 **a** 12 **b** $\frac{1}{2}$ **c** 4
d $\frac{1}{128}$ **e** $\frac{1}{2}$ **f** $7^{\frac{1}{2}}$

18 **a** $\frac{3^7b^2}{3a^4}$ **b** $\frac{1}{x^{10}}$
c $\frac{3^7a^8}{2^7b^9}$ **d** ab^4

19 **a** 32 **b** $3^7 = 2187$
c 1 **d** $7^2 = 49$
e $\frac{1}{10}$ **f** $\frac{\sqrt{5}}{5}$
g 5 **h** 3 **i** 10

20 **a** 8400 **b** 9261 **c** $8000 \times (1.05)^n$

Challenge exercise

1 **a** $120x$ **b** 128 **c** $\frac{x}{16}$

2 -8 **3** $\frac{1}{6}$ **4** -1

5 **a** 2306, 2308, 2312, 2320, 2336, 2368, 2432, 2560
b $n = 12$

7 **a** $x = 3, y = 2$ **b** $x = \frac{1}{2}, y = 2$
c $x = 5, y = 3$ **d** $x = 3, y = 7$

8 **a** $a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}}$ **b** $x + x^{-1} + 3^{\frac{1}{2}}$

9 **a** $3a - 2a^{-\frac{1}{3}}b^{\frac{1}{2}} - a^{\frac{1}{3}}b^{-\frac{1}{2}} - 6a^{\frac{2}{3}}b + 4a^{-\frac{2}{3}}b^{\frac{3}{2}} + 2b^{\frac{1}{2}}$
b $a - b^2$

10 $3^{\frac{1}{3}} > 2^{\frac{1}{2}} > 5^{\frac{1}{5}}$ **11** xy

12 $\frac{1}{5}$ **13** 18 063

Chapter 10 answers

10A Review

Chapter 1: Consumer arithmetic

1 \$4500 **2** 3 years **3** 8.7% p.a.

4 **a** \$239.40 **b** \$72

5 20%

6 **a** \$24 **b** \$120

7 **a** 108 **b** 103.5
c 36 **d** 20
e 25% decrease **f** 150% increase
g 253.75

8 **a** 4% decrease **b** 15.5% increase
c 4.32% decrease **d** 19% decrease

9 11.384% increase

10 **a** 20% increase **b** 5% increase
c 20% decrease **d** 15% decrease

11 \$56 314.12

12 **a** \$19 042.49 **b** \$19 301.25 **c** \$19 362.03

Chapter 2: Review of surds

1 **a** $5\sqrt{2} + 2\sqrt{3}$ **b** $7\sqrt{5} + 4$

2 **a** $3\sqrt{2}$ **b** $8\sqrt{2}$
c $24\sqrt{2}$ **d** $9\sqrt{3}$

3 **a** $-\sqrt{2}$ **b** $26\sqrt{3}$
c $16\sqrt{3}$ **d** $8\sqrt{7}$
e $7\sqrt{5} + 18\sqrt{2}$ **f** $2\sqrt{a}$

4 **a** $4\sqrt{3} - 4$ **b** $2\sqrt{2} - 13$
c 13 **d** $5 - 2\sqrt{6}$

5 **a** 8 **b** $a - b$
c $a + 2\sqrt{ab} + b$

6 **a** $\sqrt{6}$ **b** $\sqrt{2} + \sqrt{3}$
c $\dfrac{3 + 5\sqrt{3}}{3}$ **d** $\dfrac{15 - \sqrt{5}}{5}$
e $2\sqrt{15} + 2\sqrt{2}$ **f** $\dfrac{2\sqrt{2} - 1}{3}$

7 **a** $\dfrac{7 + 2\sqrt{10}}{3}$ **b** $\dfrac{13 + 5\sqrt{6}}{19}$
c $\dfrac{4\sqrt{3} + 4\sqrt{2} + \sqrt{6} + 2}{2}$ **d** $\dfrac{14\sqrt{2} - \sqrt{5}}{9}$

Chapter 3: Algebra review

1 **a** $9mn^2 - 3m$ **b** 0
c $7ab^2 + a^2$ **d** $10p^2q^2 - 5p^3$

2 **a** $5a + 4$ **b** $2b + 21$
c $6x^2 - 2x$ **d** $17y - 5y^2$
e $2x^2 + 11x + 5$ **f** $6a^2 + 17a - 14$
g $y^2 + 5y + 9$ **h** $b^2 + 23b - 1$

3 **a** $6x^2 + 33x + 45$ **b** $12a^2 - 10a - 8$
c $10y^2 + 7y - 14$ **d** $7b^2 + 37b + 1$

4 **a** $\frac{5}{4}a + 1$ **b** $\frac{7}{6}b + \frac{7}{3}$
c $\frac{11x^2}{12} + \frac{31x^2}{12}$ **d** $-\frac{7y^2}{10} - \frac{y}{10}$
e $\frac{2x^2}{3} + \frac{7x}{6} + \frac{1}{4}$ **f** $-\frac{4y^2}{9} + \frac{15y}{4} + \frac{73}{12}$

5 **a** $x = \frac{17}{2}$ **b** $x = -\frac{10}{3}$ **c** $x = -\frac{11}{3}$
d $y = -4$ **e** $x = \frac{38}{3}$ **f** $y = -\frac{10}{3}$
g $x = 16$ **h** $x = \frac{79}{7}$ **i** $y = -9$
j $x = \frac{65}{2}$

6 **a** $(x - 5)$ metres
b length 17.5 m, width 12.5 m

7 24 km/h

8 4 litres

9 **a** $x > 4$ **b** $x \geq \frac{11}{6}$ **c** $x < 37$
d $x \leq -\frac{17}{7}$ **e** $x \leq -5$ **f** $x > 3$

10 9.61×10^6 watts

11 **a** $s = 42.225$ **b** $a = \dfrac{2(s - ut)}{t^2}$

12 **a** $g = \dfrac{4\pi^2 p}{T^2}$ **b** $g = 9.8$

13 **a** $x = \dfrac{c - b}{a}$ **b** $x = \dfrac{c - ab}{a}$ **c** $x = \dfrac{cd - b}{a}$
d $x = \dfrac{c - b}{r - t}$ **e** $x = ya^2$ **f** $x = \dfrac{yc}{y - c}$
g $x = \dfrac{n - p}{m^2}$ **h** $x = \dfrac{-(a + b)}{a}$ **i** $x = \dfrac{3y - 1}{2}$

14 **a** $x^2 - 25$ **b** $x^2 - 4$ **c** $9x^2 - 1$
d $25x^2 - 4y^2$ **e** $\frac{1}{4}a^2 - 1$ **f** $\frac{1}{16}x^2 - \frac{4}{9}y^2$

15 **a** $(x - 6)(x + 6)$ **b** $(a - 8)(a + 8)$
c $(9b - 1)(9b + 1)$ **d** $(3x - 2y)(3x + 2y)$

16 **a** $x(x-18)$ **b** $3x(x-6)$

c $2(3b-5)(3b+5)$ **d** $3(2b-3)(2b+3)$

e $7(2x-3y)(2x+3y)$ **f** $6(3a-2b)(3a+2b)$

g $\left(\frac{1}{2}x-y\right)\left(\frac{1}{2}x+y\right)$ **h** $3\left(\frac{1}{2}x-\frac{2}{5}y\right)\left(\frac{1}{2}x+\frac{2}{5}y\right)$

17 **a** $(x+2)(x+3)$ **b** $(x+2)(x+6)$

c $(x-1)(x-2)$ **d** $(x-1)(x-5)$

e $(x-3)(x-6)$ **f** $(x-6)(x+1)$

g $(x-5)(x+2)$ **h** $(x-4)(x+2)$

i $(x-7)(x+3)$

18 **a** $(x+2)(2x+3)$ **b** $(3x+1)(x+6)$

c $(5x+4)(x+3)$ **d** $(2x-1)(x-2)$

e $(3x-10)(x-1)$ **f** $(7x-9)(x-2)$

g $(3x+2)(x-3)$ **h** $(5x+4)(x-2)$

i $(2x+3)(x-7)$

19 **a** $(2x+1)(2x+3)$ **b** $(3x+2)(2x+3)$

c $(4x+3)(x+4)$ **d** $(2x-5)(2x-3)$

e $(2x-5)(3x-2)$ **f** $(5x-6)(2x-3)$

g $(2x-5)(2x-3)$ **h** $(2x-5)(3x+2)$

i $(4x+5)(2x-3)$

20 **a** $2(x+3)(x-7)$ **b** $5(x+2)(x+3)$

c $4(x+1)(x-4)$ **d** $3(x-1)(x+5)$

e $3(2x+5)(x-1)$ **f** $4(2x+1)(x+2)$

g $5(2x+1)(x-6)$ **h** $(4-x)(x-6)$

i $-2(3x+4)(x+1)$

21 **a** $\dfrac{34x+13}{35}$ **b** $\dfrac{5x+7}{(x+2)(x+1)}$

c $\dfrac{-3x-11}{(x+2)(x-3)}$ **d** $\dfrac{4x+5}{x(x+2)(2x+3)}$

e $\dfrac{x-4}{x(x-1)(x+1)}$ **f** $\dfrac{5x+9}{(x-3)(x+3)(x+1)}$

22 **a** $\dfrac{x+2}{x+4}$ **b** $\dfrac{(x+1)(x+6)}{(x-1)(x+1)}$

c $\dfrac{1}{x}$ **d** $\dfrac{-x(x+3)}{3(x+1)}$

Chapter 4: Lines and linear equations

1 **a** $2\sqrt{13}$ **b** $2\sqrt{2}$

c $\sqrt{34}$ **d** $\sqrt{34}$

2 **a** $(3,\ 2)$ **b** $(4,\ 3)$

c $\left(-\frac{1}{2}, \frac{7}{2}\right)$ **d** $\left(-\frac{5}{2}, -\frac{1}{2}\right)$

3 **a** $\frac{2}{3}$ **b** 1

c $\frac{3}{5}$ **d** $\frac{5}{3}$

4 **a** $m=2,\ c=-1$ **b** $m=1,\ c=3$

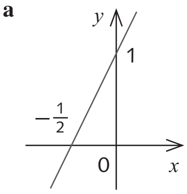
c $m=-1,\ c=7$ **d** $m=-1,\ c=4$

e $m=-\frac{2}{3},\ c=\frac{1}{3}$ **f** $m=\frac{3}{4},\ c=-\frac{1}{2}$

5 **a** **i** 3 **ii** $-\frac{1}{3}$ **b** **i** -2 **ii** $\frac{1}{2}$

c **i** $\frac{1}{2}$ **ii** -2 **d** **i** $-\frac{2}{3}$ **ii** $\frac{3}{2}$

6 **a**



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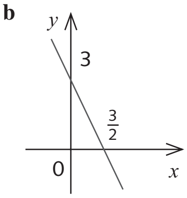

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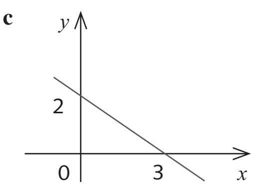

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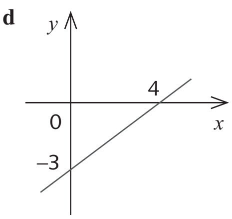

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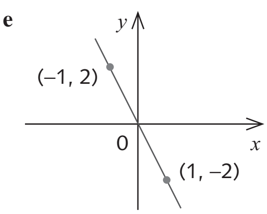

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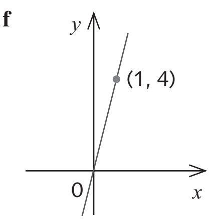

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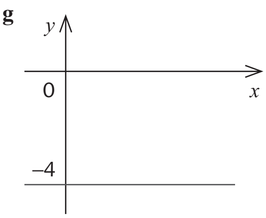

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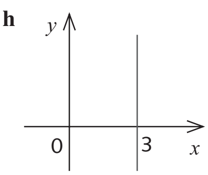


7 **a** $y=3x+2$ **b** $y=2x+1$

c $y=-x-3$ **d** $x+2y=8$

8 **a** $3x+2y=6$ **b** $2y-x=2$

c $y=3$ **d** $x=5$

e $y=2x-1$ **f** $y=1-x$

9 **a** $x=2, y=3$ **b** $x=1, y=1$

c $x=6, y=-5$ **d** $x=\frac{1}{2}, y=\frac{2}{3}$

e $x=6, y=-5$ **f** $x=\frac{1}{13}, y=\frac{44}{13}$

10 $a=3, b=2$

11 **a** 3

b **i** $a-2$ **ii** $(a-2,\ 1)$

c **i** $\dfrac{2}{a}$ **ii** $\dfrac{2}{4-a}$

d i $\frac{2}{5}$ ii $y = \frac{2}{5}x + 1$ iii $\left(\frac{5}{2}, 2\right)$

e i $\sqrt{a^2 + 4}$ ii $3\sqrt{5}$

12 a 31 000 L b 40 L/h

c $V = 31\,000 - 40t$ d after 775 hours

13 a i A started in Melbourne, B started 100 km from Melbourne.

ii A finished 160 km from Melbourne, B finished in Melbourne.

iii A took 2 hours 20 minutes, B took 1 hour 36 minutes.

iv A averaged $68\frac{4}{7}$ km/h, B averaged 62.5 km/h.

b A: $d = \frac{480}{7}t$, B: $d = -62.5t + 100$

c They passed after 46 minutes of travelling.

Chapter 5: Quadratic equations

1 a $-4, 4$ b $-2, 2$

c $-7, 7$ d $-\frac{5}{2}, \frac{5}{2}$

e $\frac{1}{2}, -\frac{1}{2}$ f $\frac{5}{2}, -\frac{5}{2}$

2 a $0, \frac{3}{2}$ b $0, -\frac{1}{3}$

c $0, \frac{3}{5}$ d $0, \frac{1}{2}$

e $0, \frac{2}{7}$ f $0, -5$

g $0, 7$ h $0, 12$

i $0, -\frac{1}{4}$

3 a $a = 4$ or -3 b $t = -3$ or -5

c $m = -7$ or 3 d $m = 4$ or -1

e $x = 4$ f $b = 9$ or -3

4 a $\frac{5}{2}, 7$ b $\frac{1}{3}, \frac{11}{3}$

c $\frac{1}{3}, -8$ d $-\frac{7}{6}, -\frac{3}{2}$

e $-\frac{1}{3}, 1$ f $-\frac{3}{2}, \frac{5}{6}$

g $\frac{3}{2}, -4$ h 3

5 a 3 b -5 c -1

d 4 e $-\frac{3}{2}$ f 5

6 a $(x + \sqrt{5})(x - \sqrt{5})$

b $(x + 2 - 2\sqrt{2})(x + 2 + 2\sqrt{2})$

c $2(x - 3 - \sqrt{5})(x - 3 + \sqrt{5})$

7 a $y = -1 \pm \sqrt{5}$ b $a = 2 \pm \sqrt{6}$

c $x = 1 \pm \frac{\sqrt{41}}{2}$ d $x = \frac{7 \pm \sqrt{41}}{2}$

e $y = -\frac{1}{4} \pm \frac{\sqrt{2}}{4}$ f $x = \frac{1 \pm \sqrt{33}}{4}$

g $n = \frac{5 \pm \sqrt{41}}{2}$ h $x = \frac{-1 + \sqrt{2}}{4}$

8 a $d = \pm\sqrt{2}$ b $y = \pm\frac{\sqrt{30}}{2}$

c $x = 10 \pm 2\sqrt{5}$ d $y = 4 \pm \sqrt{13}$

e $m = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$ f $n = \frac{3 \pm \sqrt{21}}{2}$

9 a $(x + 7)$ metres, $(x + 8)$ metres

b 5 cm, 12 cm, 13 cm

10 a 381.6 m b 102 seconds

11 a length $= 24 - 2x$ cm, width $= 17 - 2x$ cm

b $x = 2.5$

c 19 cm × 12 cm × 2.5 cm

12 a x^2 m^2 b $(8x + 16)$ m^2

c $8x + 16 = \frac{5x^2}{4}$ d 8 m × 8 m

13 a $b = \pm 4$ b $b < -4$ or $b > 4$

c $-4 < b < 4$

14 a $a = \frac{4}{3}$ b $a < \frac{4}{3}$ c $a > \frac{4}{3}$

Chapter 6: Surface area and volume

1 a 2200 cm^2 b 6000 cm^3

2 5 cm 3 5 cm

4 a 44 cm^2 b 728 cm^2 c 880 cm^3

5 a 4712 litres b 64 cm

6 a i 360 cm^2 ii 400 cm^3

b i 96π cm^2 ii 96π cm^3

7 a 254.6 m^2 b 160 m^3

8 a 4 cm b $8\sqrt{6}$ cm c 328 cm^3

9 a 18π m^2 b $\frac{45}{4}\pi$ m^3

10 a 9, 27 b 2.25, 3.375 c 2, 8

d 6, 216 e 5, 25 f 9, 81

Chapter 7: The parabola

1 a $-1, -3$ b $-\frac{1}{2}, 6$

c $-4 + \sqrt{3}, -4 - \sqrt{3}$ d $2 + \sqrt{2}, 2 - \sqrt{2}$

2 a $y = (x + 3)^2 - 6$, $(-3, -6)$

b $y = (x - 2)^2 - 2$, $(2, -2)$

c $y = 2\left(x + \frac{3}{2}\right)^2 - \frac{7}{2}$, $\left(-\frac{3}{2}, -\frac{7}{2}\right)$

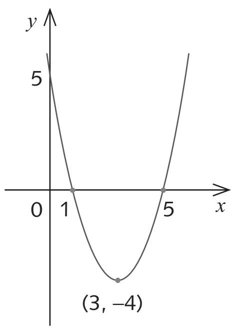
d $y = 3\left(x + \frac{4}{3}\right)^2 - \frac{10}{3}$, $\left(-\frac{4}{3}, -\frac{10}{3}\right)$

3 **a** $y = -2x^2 + 6x + 8$
b $y = x^2 - 8x + 15$

4 **a** $y = 2x^2 - 12x + 16$
b $y = -3x^2 + 12x - 7$

5 **a** $y = 3x^2 - 12x + 12$
b $y = -x^2 - 2x + 2$
c $y = x^2 - 10x + 21$

6 **a**



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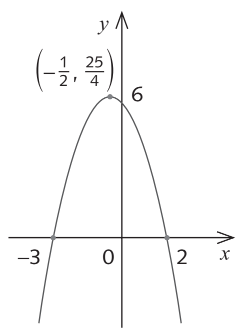

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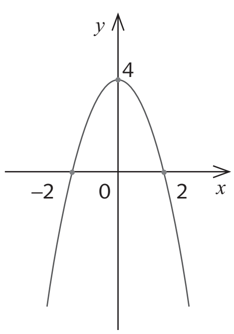
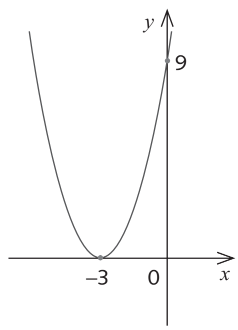


d



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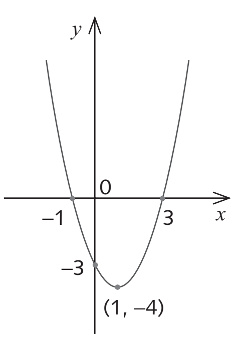

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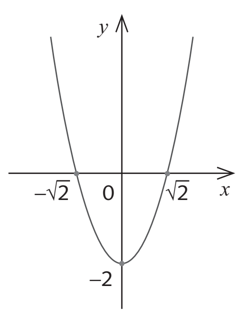

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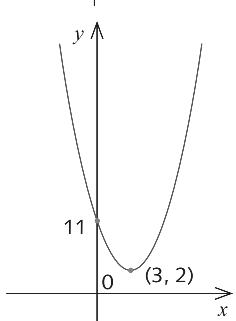

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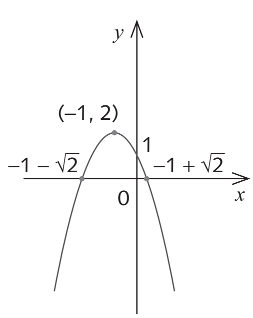

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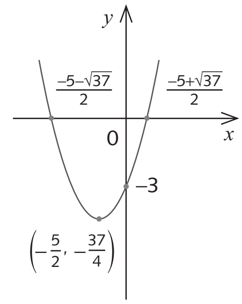

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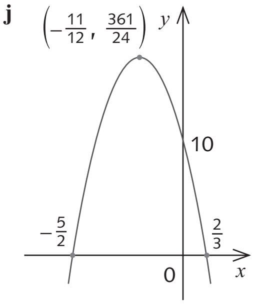

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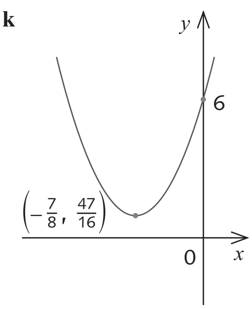

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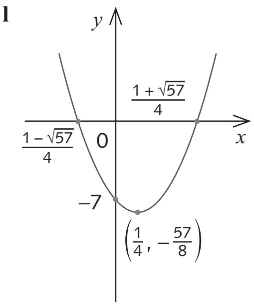
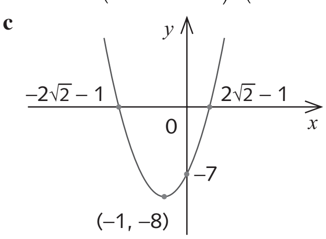


7 **a** $\left(\frac{1}{3}, -\frac{4}{3}\right)$ **b** $\left(-\frac{1}{3}, 0\right)$, $(1, 0)$

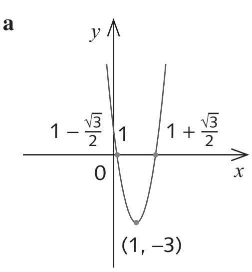
8 **a** $12 - 2x$ **b** $A = x(12 - 2x)$
c 3 m by 6 m

9 **a** $(-1, -8)$
b $(0, -7)$, $\left(-1 - 2\sqrt{2}, 0\right)$, $\left(-1 + 2\sqrt{2}, 0\right)$
c



d $-2\sqrt{2} - 1 \le x \le 2\sqrt{2} - 1$

10 **a**



b $1 - \frac{\sqrt{3}}{2} < x < 1 + \frac{\sqrt{3}}{2}$

11 **a** $-6 < x < 5$ **b** $x \le -3$ or $x \ge -2$
c $x \le -6$ or $x \ge 10$

Chapter 8: Review of congruence and similarity

Note: Proofs are not given in answers for this section.

1 **a** $a = 2, b = 2.5$ **b** $x = 2, y = 6$

2 $16\frac{2}{3}$ cm

3 **a** **i** AAA **ii** $x = 3.75$

b RHS

4 **a** **i** AAA **ii** $x = \frac{12}{5}$

b ASA

6 **a** $\angle DEC = \angle ABC$ (given both 90°), $\angle ECD = \angle BCA$ (common), ΔDEC is similar to ΔABC (AAA)

b $x = 5$ **c** $\alpha = 126.87°$

7 **a** ΔPQL is similar to ΔLTP is similar to ΔNML (AAA)

b $LT = 75$ m

12 $x = \frac{ad}{b}$

Chapter 9: Indices, exponentials and logarithms – part 1

1 **a** a^4 **b** $24x^7y^5$ **c** a^2b

d $\frac{1}{a^4}$ **e** $2xy$ **f** $2y$

g $\frac{2y^3}{x^4}$ **h** $\frac{b^8}{a}$ **i** $\frac{y^4}{x^2}$

2 **a** 3.2×10^3 **b** 5.76×10^5

c 2.67×10^{-4} **d** 2.5×10^{-2}

3 **a** 9.017×10^4 **b** 4.552×10^{-4}

c 6.516×10^{-6} **d** 9.468×10^{-20}

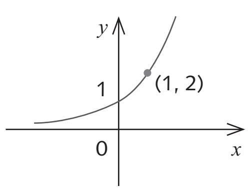
4 **a** 3 **b** 3 **c** 2 **d** 2 **e** 3 **f** 4

5 **a** 4 **b** 8 **c** 9 **d** 8 **e** $\frac{1}{27}$ **f** $\frac{1}{25}$

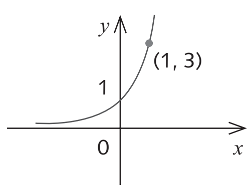
6 **a** b^4 **b** $\frac{a^2}{b}$ **c** $\frac{a^2}{b}$ **d** $a^{\frac{1}{3}}b^{\frac{1}{3}}$

e $\frac{a}{b^2}$ **f** $\frac{3a^{\frac{2}{3}}}{b}$ **g** $\frac{4a^{\frac{3}{2}}}{b}$ **h** a^2

7 **a**



b



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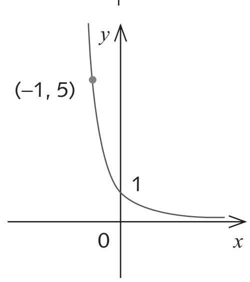
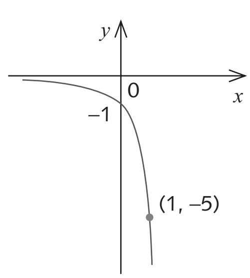


d



8 **a** $x = \frac{1}{5}$ **b** $x = \frac{1}{2}$ **c** $x = -2$

d $x = \frac{3}{4}$ **e** $x = -\frac{3}{4}$ **f** $x = \frac{5}{3}$

9 **a** $x = 5$ **b** $x = 1$ **c** $x = \frac{11}{4}$

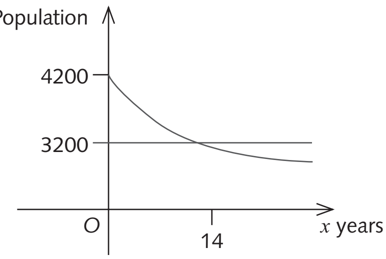
d $x = \frac{19}{18}$ **e** $x = -\frac{17}{15}$ **f** $x = 6$

10 **a** 5400 **b** 5832

c 5000×1.08^x **d** 23 305

11 **a** **i** 4116 **ii** 4037 **iii** 4200×0.98^x

b



c 14 years

12 **a** 4 **b** 2 **c** $\frac{1}{2}$ **d** $\frac{3}{2}$

10B Miscellaneous questions

1 **a** 140 km **b** x km

c $\frac{x}{35}$ hours, $\frac{140}{x}$ hours **d** 70 km/h

2 **a** **i** $1\frac{1}{8}$ hours **ii** $\frac{5}{8}$ hours **iii** 144 km/h

b **i** $(d - m)$ km **ii** $T = \frac{d - m}{v}$

iii $t = \frac{d - m}{v} - n$ **iv** $w = \frac{v(d - m)}{d - m - nv}$

v $v = \frac{w(d - m)}{wn + d - m}$

c $56\frac{16}{19}$ km/h

3 **a** X takes $2\frac{1}{2}$ hours, Y takes $1\frac{2}{3}$ hours

b X averages 24 km/h, Y averages 36 km/h

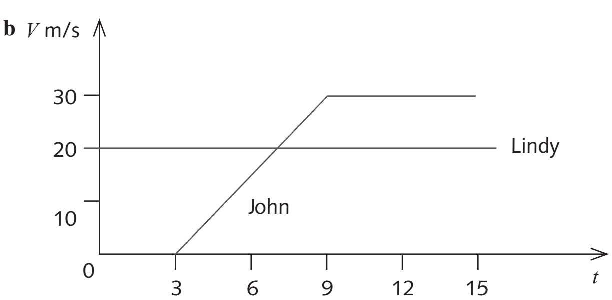
c They pass 31.2 km from A.

4 **a** 2 km

b Shen finishes first by approximately 10 minutes.

5 **a** 30 m/s

b



c 7 seconds **d** 5 m/s^2

e **i** 180 m **ii** $20t$ metres

f **i** 90 m **ii** 180 m

iii $(30t - 180)$ metres

g **i** 18 seconds **ii** 360 m

6 **a** **i** $\frac{1}{3}$ **ii** $\frac{4}{3}$

b **i** $y = \frac{1}{3}x + 2$ **ii** $y = \frac{4}{3}x - 4$

c (6, 4) **d** 27 **e** 18

f $\left(\frac{6b^2\ 12b}{4-b^2}, \frac{8b+4b^2}{4-b^2}\right)$; when $b = 0$, $E = (0, 0)$; when $b = 2$ the lines are parallel so E does not exist; when $b = -2$ the lines are the same.

7 **a** 20 km

b **i** $10\sqrt{2}$ km **ii** $25\sqrt{2}$ km

c $(15\sqrt{2}, 10)$

d **i** $P = (0, -20)$, $R = (25\sqrt{2},\ 0)$

ii $\frac{2\sqrt{5}}{5}$

iii $y = \frac{2\sqrt{2}}{5}x - 20$

8 **a** **i** $h = 3r$ **ii** $V = \pi r^3$ **iii** $A = \sqrt{10}\pi r^2$

b **i** 7.55 cm **ii** 62.9 cm^3

9 **a** $\angle AGB = \angle CGF$ (vertically opposite angles at G), $\angle GAB = \angle GCF$ (or $\angle ABG = \angle CFG$) (alternate angles in parallel lines), so ΔAGB is similar to ΔCGF (AAA)

b ΔEBA, ΔBFC

c **i** 3 : 2 **ii** 2 : 3

10 **a** **i** AAA **ii** $x = 5\frac{1}{7}$ **iii** $\frac{24}{49}$

b **i** $x = \frac{3y}{2}$ **ii** $\frac{4}{9}$

11 **a** **i** 3 **ii** $3\sqrt{5}$

iii $\sqrt{6}$ **iv** $1 + \sqrt{3}$

b $y = \frac{1}{x}$

12 **a** $\angle ABE = \angle ACD$ (given), $\angle BAE = \angle CAD$ (common), so ΔABE is similar to ΔACD (AAA)

b **i** $(x + 10)$ metres **ii** $\frac{x}{x+10} = \frac{2}{3}$

c 20 metres

d **i** $\frac{1}{2}x^2$ m^2 **ii** $\left(450 - \frac{1}{2}x^2\right)$ m^2

e **i** $\frac{1}{2}x^2 = 450 - \frac{1}{2}x^2$ **ii** $15\sqrt{2}$ metres

13 **a** $A = (15 - x)^2 - (15 - 2x)^2$

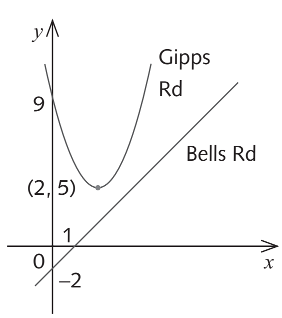
b 75 m^2

14 **a** 10 km **b** 0.5 km

c $A = (1.13, 0.2)$, $B = (8.87, 0.2)$

d 7.75 km

15 **a**



b **i** $y = (x - 2)^2 + 5$

ii 9

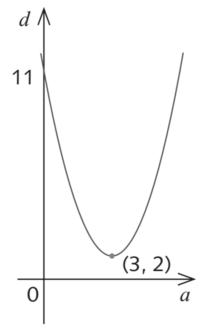
iii see graph

c 11 km

d **i** $y = 2a - 2$

ii $y = a^2 - 4a + 9$

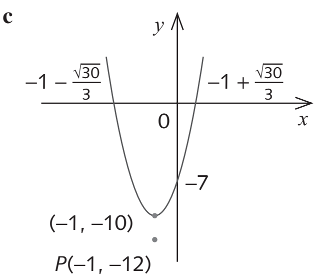
f



g 2 km, 3 km east of origin

16 **b** **i** $(-1, -10)$ **ii** $\left(-1+\frac{\sqrt{30}}{3}, 0\right)$, $\left(-1-\frac{\sqrt{30}}{3}, 0\right)$

c



d 200 km

e $3x^2 + 6x - 7 = 5x + 3$
$3x^2 + x - 10 = 0$

f $\left(\frac{5}{3}, \frac{34}{3}\right)$, $(-2, -7)$

17 **a** **i** $h = r\tan\theta$ **ii** $s = \frac{r}{\cos\theta}$

b **i** Area $(A) = \frac{1}{2}\pi r^2$

ii Area $(B) = r^2\tan\theta$

d 57.5° **e** $4\pi \approx 12.57$ mm^2

10C Problem-solving

1 $a = \frac{2r}{5}$, $b = \frac{6r}{5} - 3$

2 $x = 12$ and $y = 2$, or $x = 12$ and $y = -2$
or $x = -12$ and $y = 2$, or $x = -12$ and $x = -2$
or $x = 36$ and $y = 34$, or $x = 36$ and $y = -34$,
or $x = -36$ and $y = 34$, or $x = -36$ and $y = -34$

3 **a** **i** $2\sqrt{259}$ cm **ii** $2\sqrt{257}$ cm

b 5 cm

4 $\left(4 - 2\sqrt{3}\right)$ m

5 **a** 16π cm^3 **b** $\frac{16\pi}{3}$ cm^3

c $\frac{32\pi}{3}$ cm^3 **d** 8π cm^3

6 Midpoint of $AB = X\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

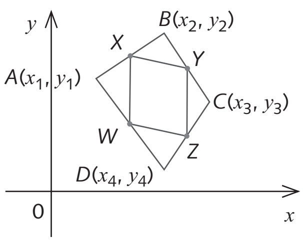
Midpoint of $BC = Y\left(\dfrac{x_2 + x_3}{2}, \dfrac{y_2 + y_3}{2}\right)$

Midpoint of $CD = Z\left(\dfrac{x_3 + x_4}{2}, \dfrac{y_3 + y_4}{2}\right)$

Midpoint of $DA = W\left(\dfrac{x_4 + x_1}{2}, \dfrac{y_4 + y_1}{2}\right)$

Gradient of $XY = \dfrac{y_3 - y_1}{x_3 - x_1}$

Gradient of $WZ = \dfrac{y_3 - y_1}{x_3 - x}$



Therefore, $XY \parallel WZ$

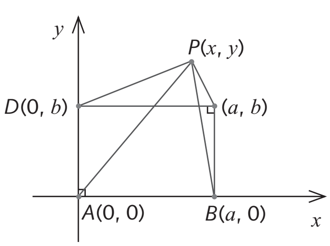
$XY = \dfrac{1}{2}\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}$

$WZ = \dfrac{1}{2}\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}$

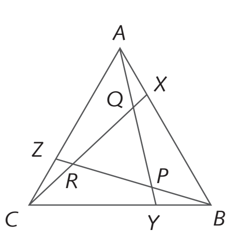
so $XY = WZ$

$XYZW$ is a parallelogram.

7 $(PA)^2 + (PC)^2 = x^2 + y^2 + (x - a)^2 + (y - b)^2$
$(PB)^2 + (PD)^2 = (x - a)^2 + y^2 + x^2 + (y - b)^2$
so $(PA)^2 + (PC)^2 = (PB)^2 + (PD)^2$



8 $\Delta AQX \equiv \Delta CRZ \equiv \Delta BPY$
Quad $AQRZ \equiv$ Quad $BPQX \equiv$ Quad $CRPY$
ΔPQR is equilateral.
ΔBPY is similar to ΔBCZ (AAA).
Assume $AB = BC = CA = 1$.
Then $CZ = AX = YB = \dfrac{1}{3}$.



Apply Pythagoras' theorem twice to find $ZB = \dfrac{\sqrt{7}}{3}$.

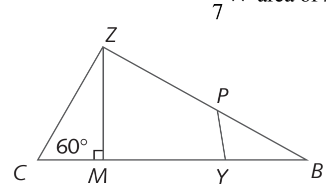
The enlargement factor from ΔBPY to ΔBCZ is $\sqrt{7}$.

$\therefore$ area of $\Delta BPY = \dfrac{1}{7}$ area of ΔBCZ

$= \dfrac{1}{21}$ area of ΔABC

area of $\Delta PQR = 3 \times$ area of ΔPBY

$= \dfrac{1}{7} \times$ area of ΔABC



9 ΔARD is similar to ΔPRB

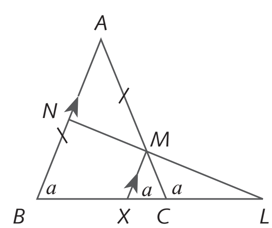
Therefore $\dfrac{AR}{RP} = \dfrac{RD}{RB}$

ΔBRA is similar to ΔDRQ

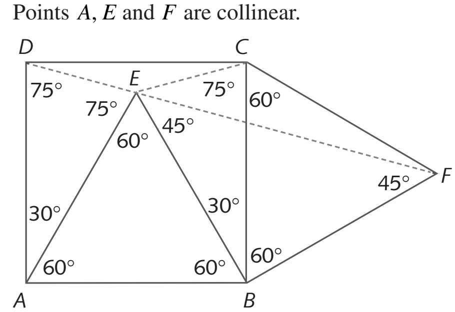
$\therefore \dfrac{RD}{RB} = \dfrac{RQ}{AR}$
$\therefore \dfrac{AR}{RP} = \dfrac{RQ}{AR}$
$\therefore AR^2 = RP \times RQ$

10 Draw $MX \parallel AB$.

$\angle ABC = \angle MXC$ (corresponding angles $AB \parallel MX$)
$\angle ABC = \angle MCX$ (ΔABC isosceles)
$\therefore \angle MXC = \angle MCX$ (ΔMXC isosceles)
$\therefore MX = MC$
ΔLNB is similar to ΔLMX
$NM = ML$ (given)
$\therefore LN = 2LM$ and $NB = 2MX = 2MC$



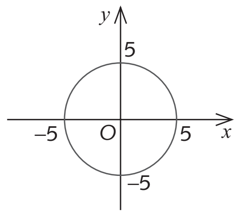
11 $\angle AEB = 60°$ (ΔAEB is isosceles)
$\angle DEA = 30°$ (complementary to $\angle EAB$)
$\angle CBE = 30°$ (complementary to $\angle EBC$)
$\Delta BEC = \Delta AED$ (SAS)
$\angle ADE = \angle AED = \angle BEC = \angle ECB = 75°$
(ΔBEC and ΔAED are isosceles)
$\angle EBF = 30° + 60°$ (square and equilateral triangle)
ΔBEF is isosceles
$\angle BEF = 45°$ (angle sum of ΔBEF)
Therefore,
$\angle AED + \angle AEB + \angle AEB + \angle BEF = 75° + 60° + 45° = 180°$
Points A, E and F are collinear.



12 2 cm **13** 2 : 1 **14** $120\sqrt{2}$

Chapter 11 answers

Exercise 11A

1 **a**


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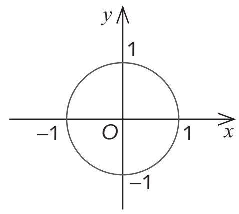

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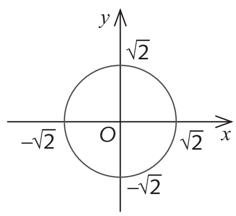

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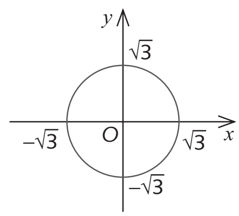

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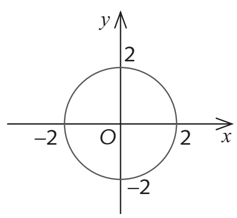

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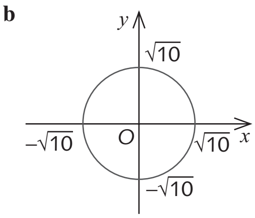

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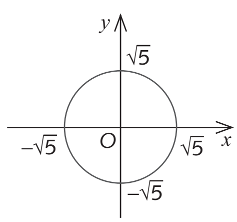

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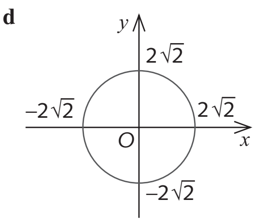
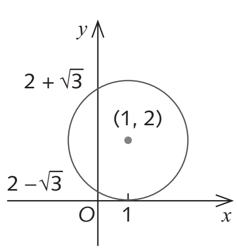


3 **a** yes **b** no **c** no
d yes **e** yes **f** yes

4 **a** yes **b** no **c** yes
d yes **e** no **f** yes

5 **a**


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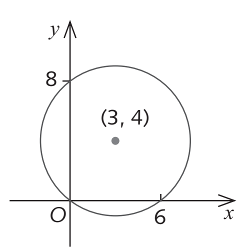

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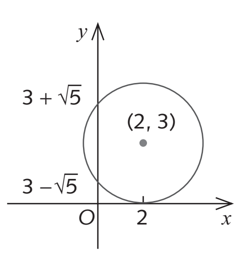

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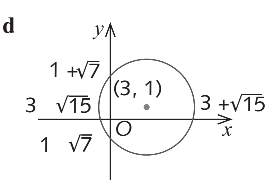

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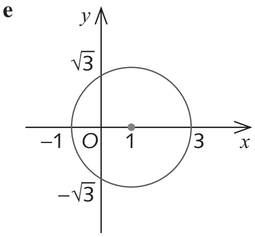
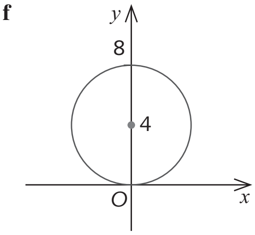


f


6 **a** centre $(-2, -3)$, radius 2
b centre $(1, -4)$, radius $\sqrt{13}$
c centre $(3, 4)$, radius 8
d centre $(7, 4)$, radius 5
e centre $(4, 3)$, radius $\sqrt{10}$
f centre $(4, 2)$, radius $\sqrt{10}$

7 **a** $(x-1)^2 + (y-3)^2 = 9$ **b** $(x+2)^2 + (y-1)^2 = 16$
c $(x-4)^2 + (y+1)^2 = 1$ **d** $(x-2)^2 + y^2 = 4$

8 $\sqrt{(17-5)^2 + (17-12)^2} = 13$, so the point $(17, 17)$ is 13 units from the centre at $(5, 12)$. The equation of the circle is $(x-5)^2 + (y-12)^2 = 169$.

9 $(x-3)^2 + (y+4)^2 = 25$

10 **a** $(x-6)^2 + (y-7)^2 = 36$ **b** $(x-6)^2 + (y-7)^2 = 49$

11 **a** 6 **b** $(5, 6)$
c $(x-5)^2 + (y-6)^2 = 9$

12 **a** $10\sqrt{2}$ **b** $(2, -1)$
c $(x-2)^2 + (y+1)^2 = 50$

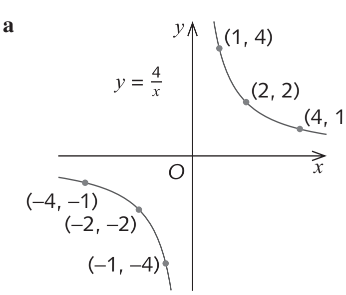
Exercise 11B

1 **a** $y = \frac{1}{2}$ **b** $y = -\frac{1}{2}$ **c** $y = 2$
d $y = \frac{2}{3}$ **e** $y = -\frac{3}{2}$

2 **a** $y = 4$ **b** $y = 3$ **c** $y = -24$
d $y = -8$ **e** $y = 16$

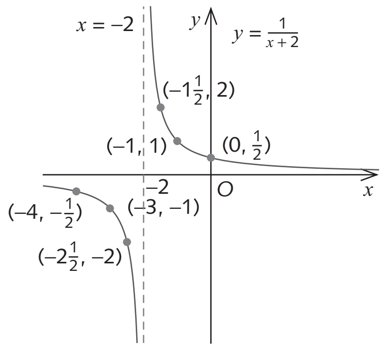
3 **a** $y = 1$ **b** $y = \frac{1}{2}$ **c** $y = 2$
d $y = -\frac{2}{3}$ **e** $y = \frac{2}{3}$

4 **a** $y = 1$ **b** $y = -1$ **c** $y = \frac{1}{2}$
d $y = 2$ **e** $y = -4$

5 **a**


b $-1, -2, -4, 4, 2, 1$

6 **a**



b $-\frac{1}{2}$, –1, –2, 2, 1, $\frac{1}{2}$

7 **a** –12 000 **b** –60 **c** 4

d $\frac{1}{2}$ **e** $\frac{1}{12}$

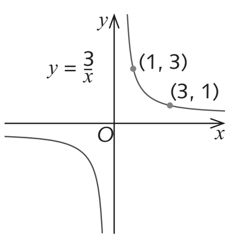
8 **a** –2 **b** –3 **c** –600

d 600 **e** $\frac{6}{997}$

9 **a** 4 **b** 120 **c** 1200

d –1200 **e** –12 000

10 **a**



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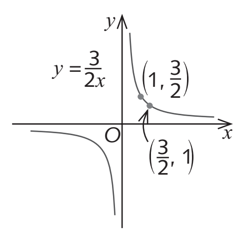
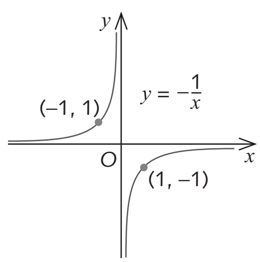
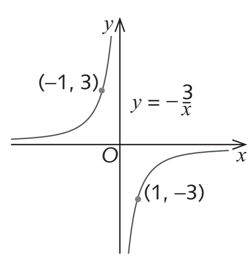


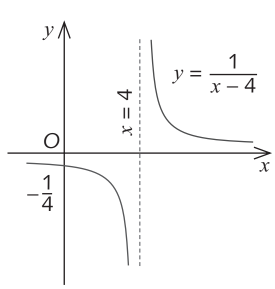
c



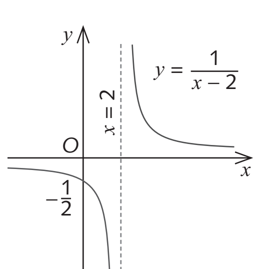
d



11 **a**



b



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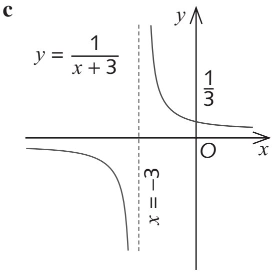

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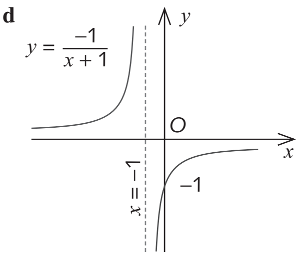
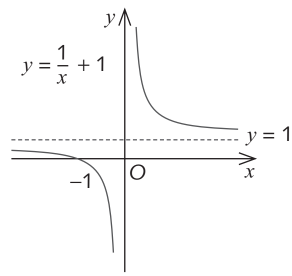
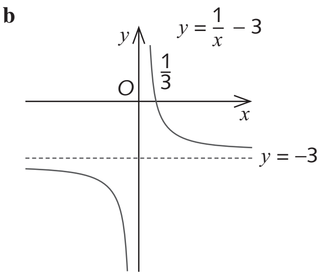


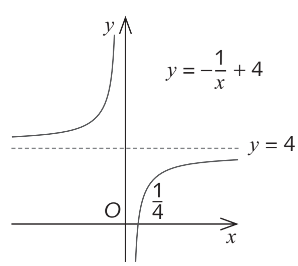
12 **a**



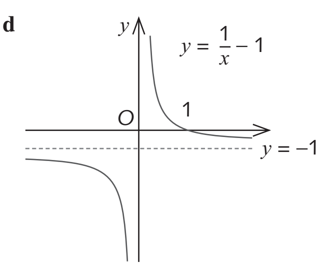
b



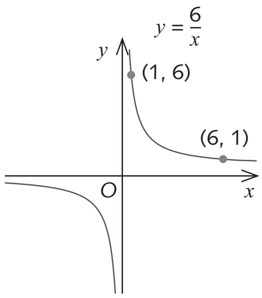
c



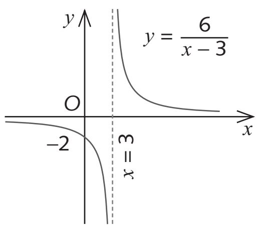
d



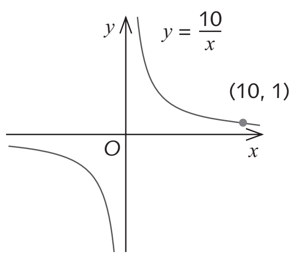
13 a i



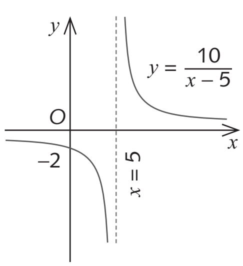
ii



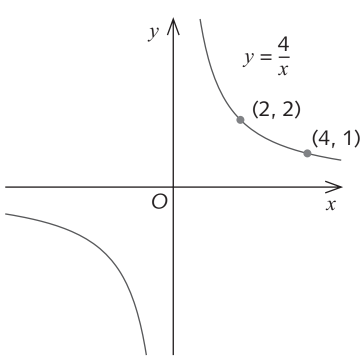
b i



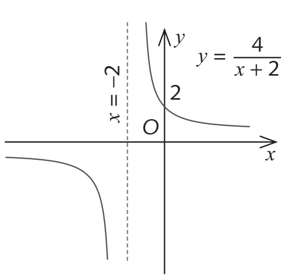
ii



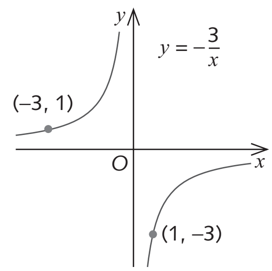
c i



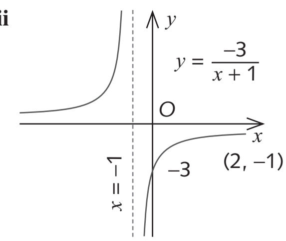
ii



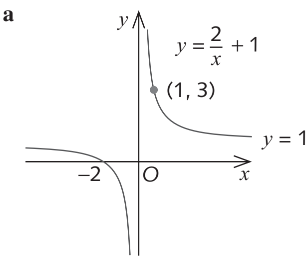
d i



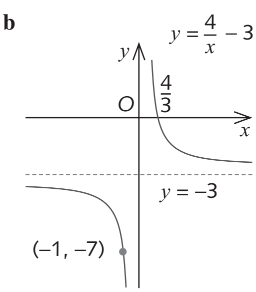
ii



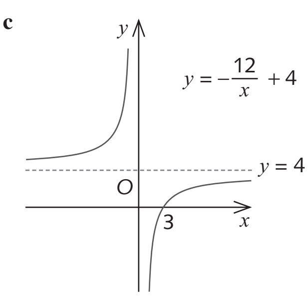
14 a



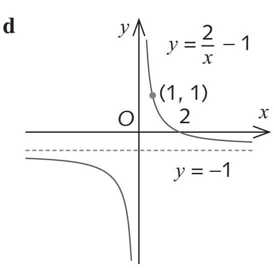
b



c



d



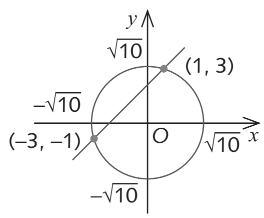
Exercise 11C

1 **a** (2, 4), (–2, 4) **b** (1, 1), (–1, 1)
c (2, 1) **d** (3, 9), (4, 16)

2 **a** (–1, 1) **b** (–5, 2), (1, 8) **c** (–2, 4), (1, 7)
d $\left(-\frac{1}{2}, 0\right)$, (0, 1) **e** $\left(-\frac{1}{3}, 2\right)$, (1, 6)
f $\left(-\frac{2}{3}, \frac{5}{3}\right), \left(-\frac{1}{2}, 2\right)$

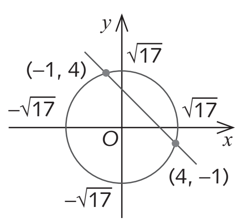
3 **a** (2, 0) **b** (–3, 0), (3, 0)
c (4, 4), (–4, –4) **d** $(-3, -6\sqrt{2}), (3, 6\sqrt{2})$

4 **a**



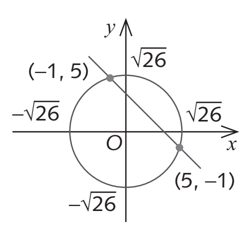
intersection points:
(1, 3), (–3, –1)

b



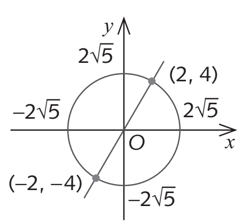
intersection points:
(4, –1), (–1, 4)

c



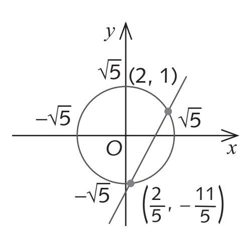
intersection points:
(–1, 5), (5, –1)

d



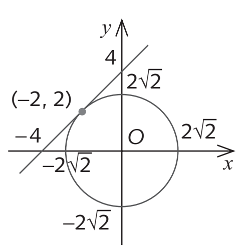
intersection points:
(–2, –4), (2, 4)

e



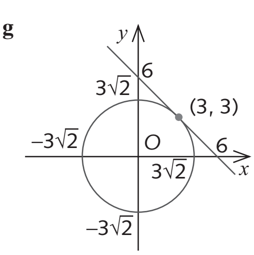
intersection points:
$\left(\frac{2}{5}, -\frac{11}{5}\right)$, (2, 1)

f



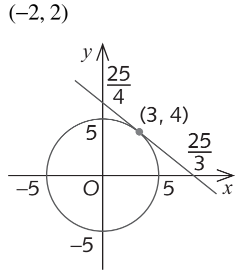
intersection point:
(–2, 2)

g



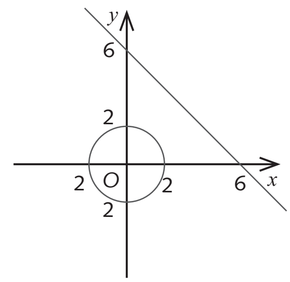
intersection point:
(3, 3)

h



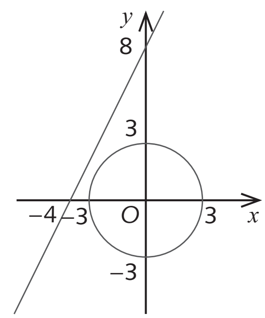
intersection point:
(3, 4)

i



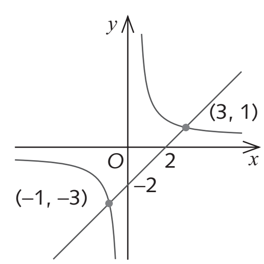
no intersection point

j

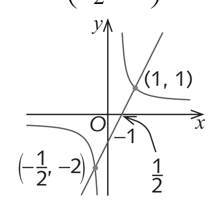


no intersection point

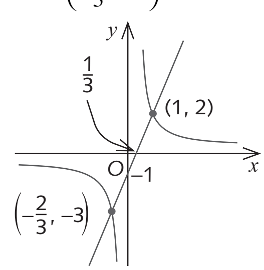
5 **a** (3, 1), (–1, –3)



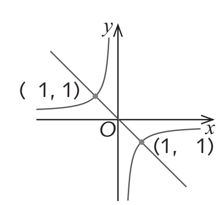
b $(1, 1), \left(-\frac{1}{2}, -2\right)$



c $(1, 2), \left(-\frac{2}{3}, -3\right)$



d (–1,1), (1,–1)



6 **a** $A = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $B = (1, 1)$
b area of ΔOAX is $\frac{1}{4}$, area of ΔOBY is $\frac{1}{2}$

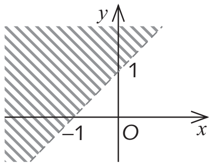
7 (2, 3)

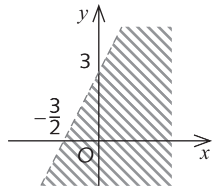
8 $(x - 5)^2 + 4x^2 = 4, (x - 1)^2 = -\frac{16}{5}$, so this quadratic equation has no solution and the line does not meet the circle.

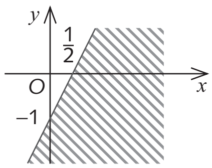
9 **a** $a = 3\sqrt{2}$ or $-3\sqrt{2}$ **b** $-3\sqrt{2} < a < 3\sqrt{2}$
c $a < -3\sqrt{2}$ or $a > 3\sqrt{2}$

10 $(1, 2\sqrt{2}), (1, -2\sqrt{2})$

Exercise 11D

1 **a**


b


c


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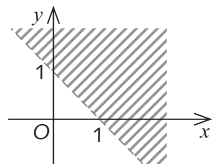

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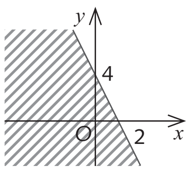

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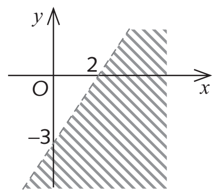

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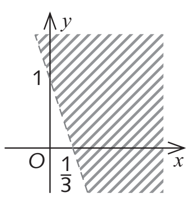

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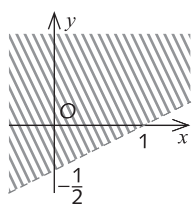

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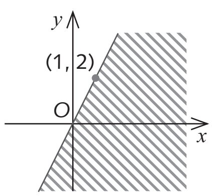

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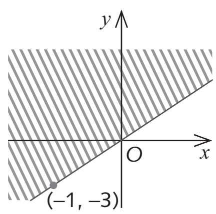

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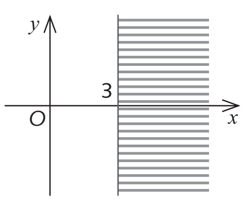

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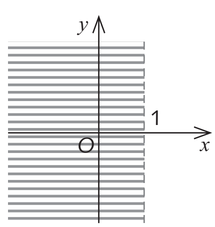
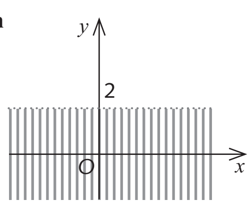


m


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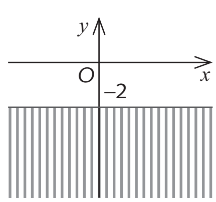

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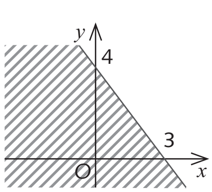

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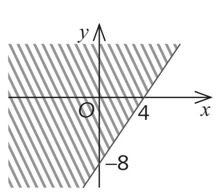

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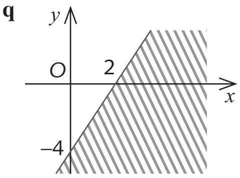

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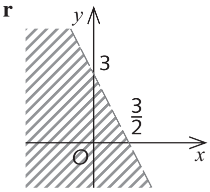
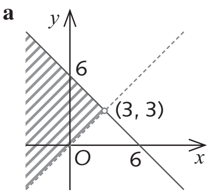
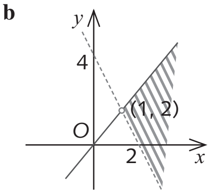
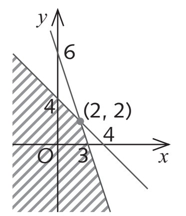


2 **a** $y \geq 2x + 4$ **b** $y < -x + 2$
c $x + y < 3$ **d** $y \geq 2x + 1$

3 **a**


b


c


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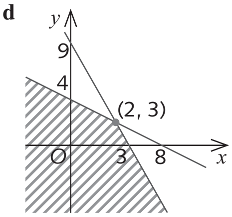

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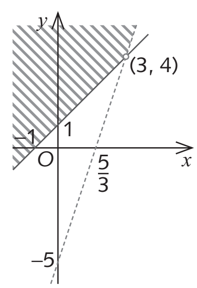

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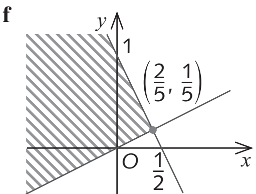

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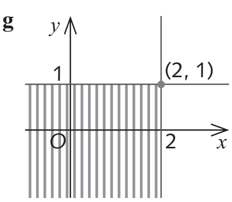

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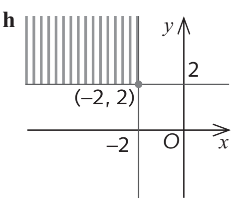

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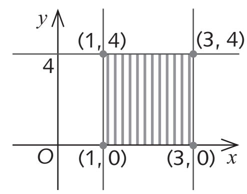

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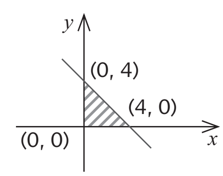

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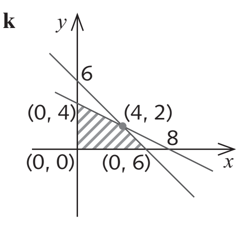

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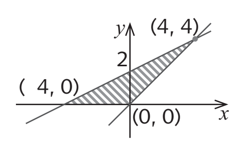


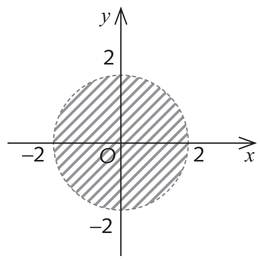
4 **a** $x + y \le 2, y - x \le 2$ **b** $y \ge x - 1, y \le 3x + 3$

c $y \ge 2, x < 2$ **d** $y \ge x, y < 2$

e $y \le x, x \le 2, y \ge 0$

f $x \ge 0 y \ge 0, x \le 3, x + y \le 6$

5 **a**



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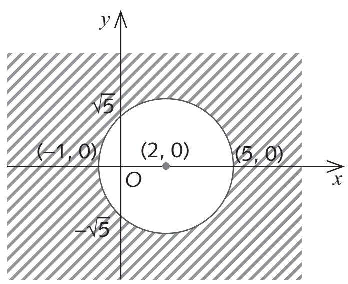

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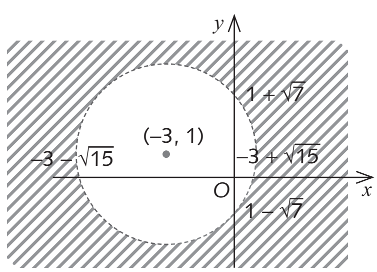

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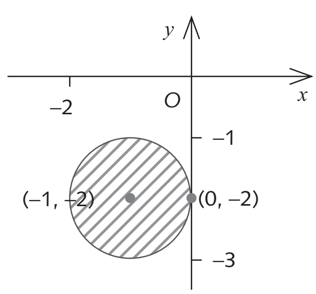

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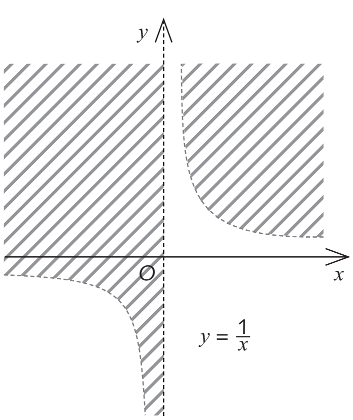
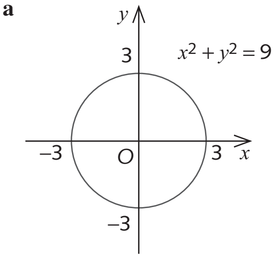


Review exercise

1 **a**



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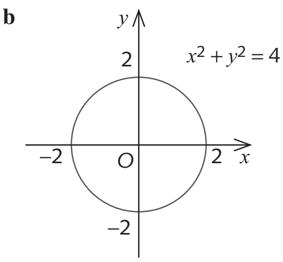

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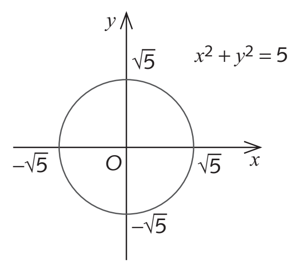

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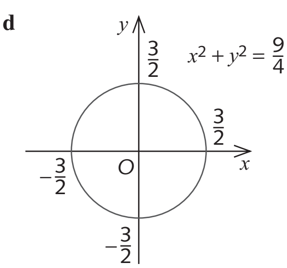
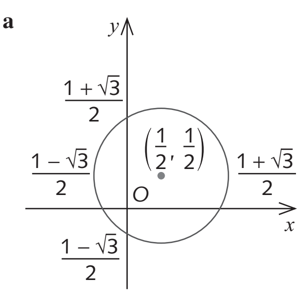


2 **a**



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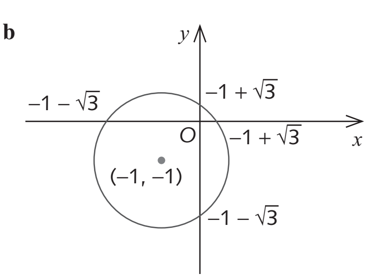

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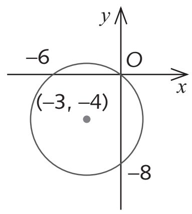

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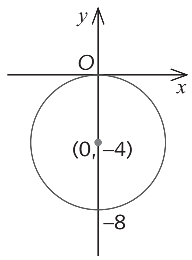

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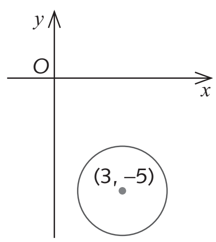

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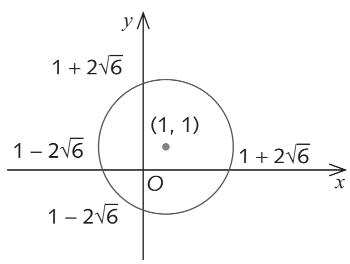


3 **a** centre $(-2, -4)$, radius $2\sqrt{5}$

b centre $(-2, -1)$, radius $\sqrt{10}$

c centre $\left(2, -\frac{5}{4}\right)$, radius $\frac{\sqrt{65}}{4}$

d centre $(2, -3)$, radius $5\sqrt{2}$

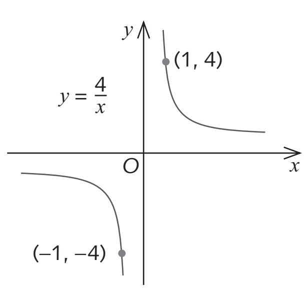
4 **a** $x^2 + y^2 = 9$

b $(x + 1)^2 + (y - 4)^2 = 36$

c $(x - 2)^2 + (y - 5)^2 = 1$

d $(x + 2)^2 + (y + 6)^2 = 16$

5 **a**



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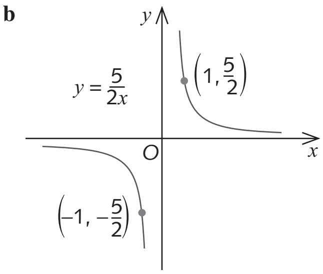

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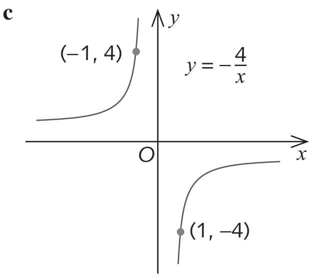
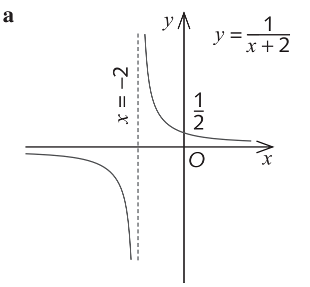


6 **a**



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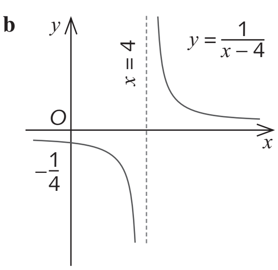

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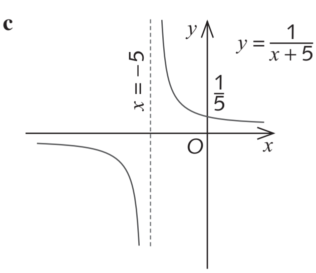

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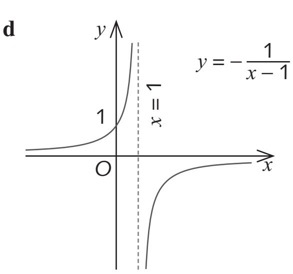
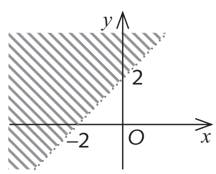


7 **a** $(-2, 18), (2, 6)$ **b** $(0, 2), (2, -12)$

8 **a** $(3, 0)$ **b** $(-4, 0), (4, 0)$
c $(-2, -2\sqrt{3}), (2, 2\sqrt{3})$

9 **a** $(4, 3)$ **b** $\left(-\frac{7}{5}, -\frac{26}{5}\right), (2, 5)$

10 **a**


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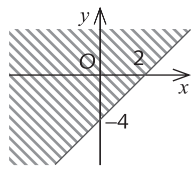

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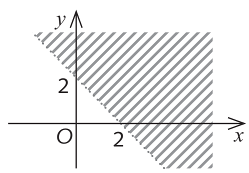

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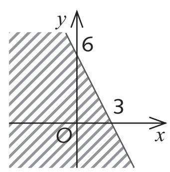

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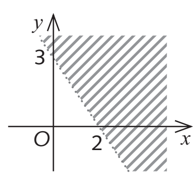

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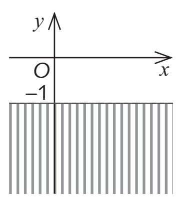

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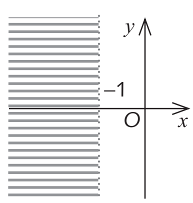

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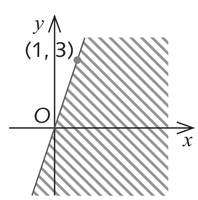
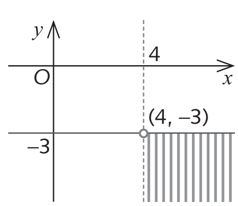


11 **a**


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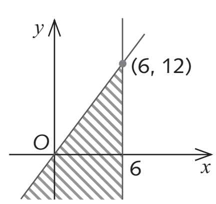

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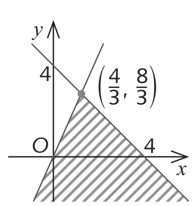

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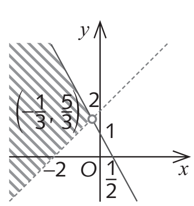

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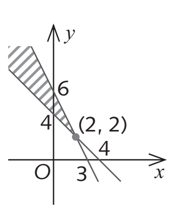

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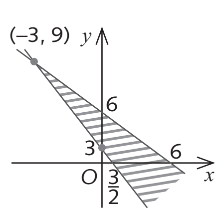
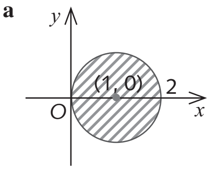


12 **a**


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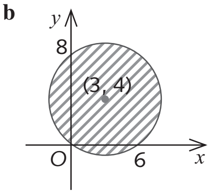

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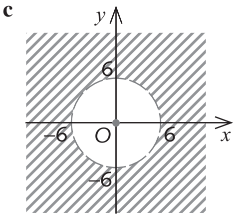

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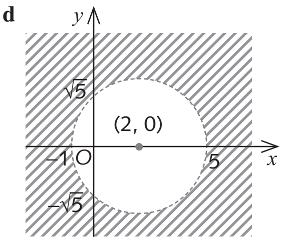
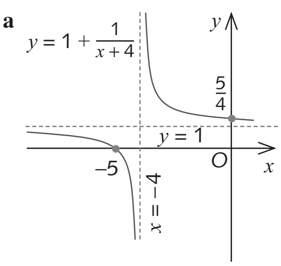


13 **a** $(4, 3), (-3, -4)$ **b** $\left(\frac{1}{2}, -6\right), (3, -1)$
c $(3, 2)$
d no points of intersection

Challenge exercise

1 **a**


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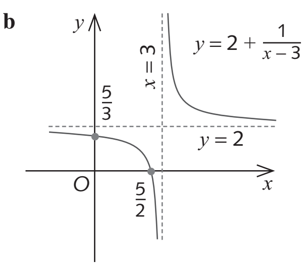
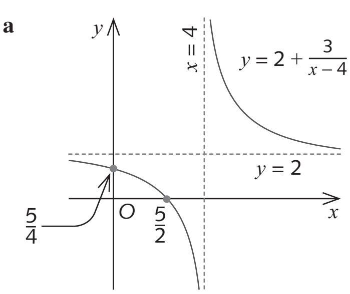


2 **a**


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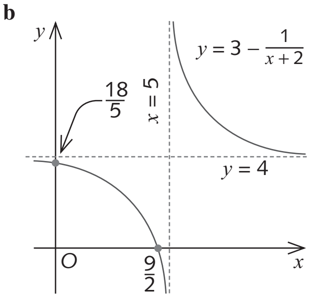
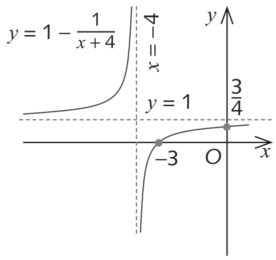
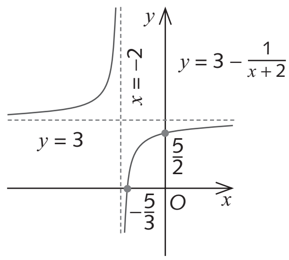


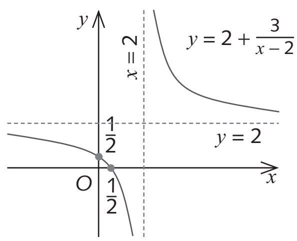
3 **a**



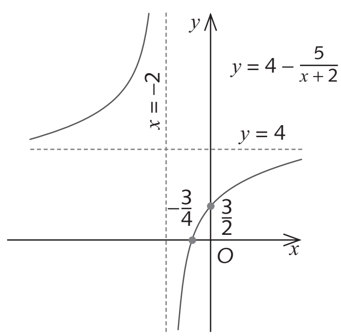
b



c



d



4 **a** **i** $y = -\frac{1}{\sqrt{3}}x + \frac{a}{3}$ **ii** $y = \frac{1}{\sqrt{3}}x + \frac{a}{3}$

c $\frac{2}{3}a$ **d** $x^2 + \left(y - \frac{a}{3}\right)^2 = \frac{4a^2}{9}$

6 Centre $\left(\frac{a}{2}, \frac{a}{2}\right)$, radius $\frac{\sqrt{2}}{2}a$, expand

$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{a}{2}\right)^2 = \left(\frac{\sqrt{2}}{2}a\right)^2$$

7 **a** **i** $x = \frac{a}{2}$ **ii** $y = \frac{b}{2}$ **b** $\left(\frac{a}{2}, \frac{b}{2}\right)$

d $\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 = \frac{a^2 + b^2}{4}$

8 $\left(x - \frac{a^2 + b^2}{2b}\right)^2 + y^2 = \frac{(a-b)^4 + 4b^4}{4b^2}$

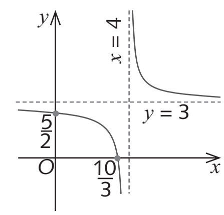
9 $\left(x - \frac{5}{2}\right)^2 + y^2 = \frac{5}{4}$

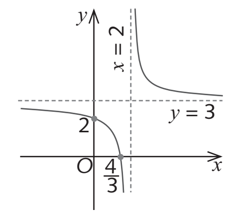
10 **a** $y - 3 = \frac{2}{x-4}$

$y = 3 + \frac{2}{x-4}$

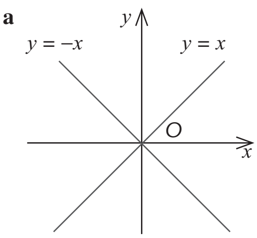
b $y - 3 = \frac{2}{x-2}$

$y = 3 + \frac{2}{x-2}$

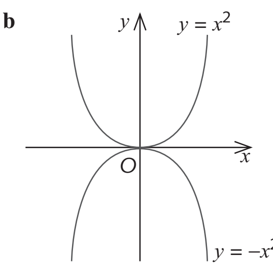




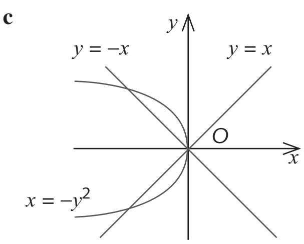
11 **a**



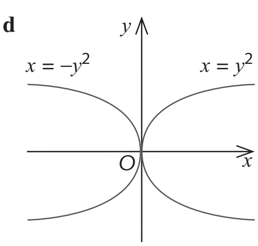
b



c



d



12 $(0, 3)$ and $(2, 0)$ **13** $\left(-\frac{9}{5}, \frac{12}{5}\right)$ and $(-1, 3)$

14 $x^2 + y^2 - x - 5y + 4 = 0$

15 $ax + b = \frac{1}{x}$, eliminating y.

$ax^2 + bx = 1$

$ax^2 + bx - 1 = 0$

$x = \frac{-b \pm \sqrt{b^2 + 4a}}{2a}$ and $b^2 + 4a > 0$, since $a > 0$

So there are always two points of intersection between the line and the hyperbola.

Chapter 12 answers

Exercise 12A

1 **a** $a = 7.42$ **b** $b = 3.71$ **c** $c = 6.17$
d $d = 2.29$ **e** $i = 3.51$ **f** $j = 13.59$

2 **a** $a = 14.97$ **b** $b = 9.01$ **c** $e = 11.75$
d $h = 38.70$

3 **a** 25.4° **b** 56.3° **c** 60.3°
d 70.9° **e** 39.6° **f** 70.8°

4 **a** $x = 7.10$ **b** $\theta = 44.9°$ **c** $a = 8.08$
d $a = 2.26$ **e** $\theta = 61.9°$ **f** $\theta = 52.1°$
g $x = 10.15$ **h** $y = 10.63$ **i** $a = 9.39$

5 **a** 3.92 cm, 2.52 cm, 50° **b** 6.97 cm, 40.7°, 49.3°
c 9.78 cm, 30.8°, 59.2°

Exercise 12B

1 **a** 5 **b** $6\sqrt{2}$ **c** $6\sqrt{3}$
d $4\sqrt{3}$ **e** $4\sqrt{3}$

2 **a** 1 **b** -1 **c** $2 - \sqrt{3}$
d $\dfrac{\sqrt{2} + \sqrt{6}}{4}$ **e** $\dfrac{\sqrt{3}}{2}$ **f** 0

3 10 cm and $10\sqrt{3}$ cm

4 **a** $16\sqrt{3}$ cm **b** $12\sqrt{3}$ cm **c** $8\sqrt{3}$ cm
d $4\sqrt{3}$ cm **e** 12 cm

5 $a = \dfrac{40\sqrt{3}}{3}$ and $x = \dfrac{40}{3}$ **6** $50\left(\sqrt{3} + 1\right)$

7 $5\sqrt{3}$ cm

Exercise 12C

1 **a** $2\sqrt{41}$ cm **b** 38.7° **c** $2\sqrt{77}$ cm
d 27.1° **e** $2\sqrt{34}$ cm **f** 34.4°

2 **a** $12\sqrt{3}$ cm **b** 35.3° **c** 90°
d 35.3°

3 **a** $10\sqrt{2}$ cm **b** $5\sqrt{2}$ cm **c** $\sqrt{194}$ cm
d 59.5° **e** 5 cm **f** 67.4°
g 69.0°

4 **a** 47.5 cm **b** 40 cm **c** 45.7°

5 **a** 31.2 m
b **i** $50\sqrt{5}$ m $\approx$ 111.8 m
ii 15.6°
iii 296.6°

6 **a** 130.6 m **b** 96.2 m **c** 162.2 m
d 216.4°T

7 81 m

8 **a** 73 m **b** 260 m **c** 16°

9 42 m

Exercise 12D

1 **a** 10.46 **b** 6.88 **c** 8.69

2 **a** 8.63 **b** 10.89 **c** 9.34

3 **a** 7.33 **b** 10.63

4 **a** 54° **b** 34° **c** 66°

5 **a** 47.34° **b** 61.66° **c** 16.76 cm

6 $\angle ACD \approx 35.07$, 26.01 m

7 **a** 708 m **b** 547 m

8 **a** 27° **b** 132.56 m **c** 119 m

9 16 m **10** 7.11 m

Exercise 12E

1 **a** 65° **b** 57° **c** 42° **d** 85°

2 **a** $\dfrac{1}{\sqrt{2}}$ **b** $-\dfrac{1}{\sqrt{2}}$

3 **a** 30°, 150° **b** 35°, 145° **c** 151°

4

θ	30°	120°	150°	90°	135°
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	1	$\frac{1}{\sqrt{2}}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	0	$-\frac{1}{\sqrt{2}}$

5 **a** 44.42 **b** 6.47

6 **a** 140° **b** 155°

7 **a** The two angles are supplementary, and the sines of two supplementary angles are the same.
b $h = b \sin A$ **c** $h = a \sin B$

8 59° **9** 25.1 m

10 **a** 234°T **b** 1285 m

11 **a** 341°T **b** 1282 m

Exercise 12F

1 **a** 8.65 cm **b** 3.82 cm
c 16.34 cm **d** 6.93 cm

2 17.91 cm **3** 6.2 m

4 410 km **5** 103 km

6 **a** 14.38 m **b** 7.01 m **c** 50.43 m^2

7 From ΔBXA

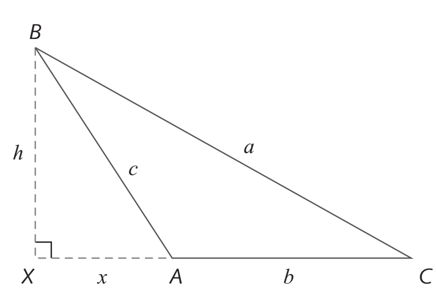
$c^2 = h^2 + x^2$ and $x = c\cos(180 - A) = -\cos A$

From ΔBXC,

$a^2 = (b + x)^2 + h^2$

Hence,

$$\begin{aligned} a^2 &= b^2 + 2bx + x^2 + c^2 - x^2 \\ &= b^2 + c^2 - 2bc\cos A \end{aligned}$$



Exercise 12G

1 a $x^2 = y^2 + z^2 - 2yz \cos R$
 b $b^2 = a^2 + c^2 - 2ac \cos B$
 c $p^2 = q^2 + r^2 - 2qr \cos B$

2 a 52.6° b 54.8° c 78.5°

3 a $A = 39.8°, B = 45.4°, C = 94.8°$
 b $A = B = 29.0°, C = 122.1°$
 c $A = 21.0°, B = 18.6°, C = 140.4°$

4 19.4°

5 108.4°

6 16°, 34°, 130°

7 92.1°

8 a i 21.8° ii 4.4 cm b 9.5 cm, 11.8 cm

Exercise 12H

1 a 39.5 cm^2 b 29.8 cm^2 c 48.5 cm^2

2 a 10.08 cm^2 b 46.98 cm^2 c 43.30 cm^2

3 a 73.2° b 56.8° c 134 cm^2

4 a 36.3° b 21 cm^2

5 a 57 cm^2 b 30 cm^2 c 199 cm^2

6 a 8.5 cm b 9.6 cm c 36.7°

7 22°

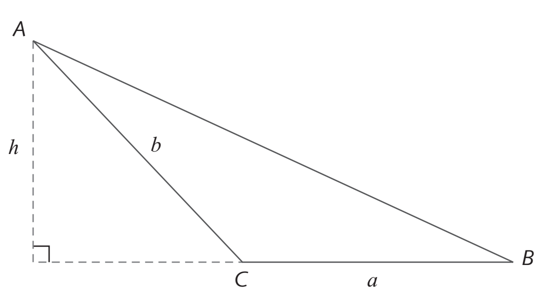
8 a 152.6 m b 159.7° c 6213.6 m^2

9 a 108° b 47.55 cm^2 c 16.18 cm
 d 36° e 72° f 36°
 g 76.94 cm^2 h 172.05 cm^2

10 98 cm^2

11 $\text{Area} = \frac{1}{2} \times (ka) \times (kb) \sin C = \frac{1}{2} k^2 \times ab \sin C$
$= k^2 \times \text{area of triangle } ABC$

12
$$\begin{aligned}\text{Area} &= \frac{1}{2} \times a \times h \\ &= \frac{1}{2} ab \sin (180 - C) \\ &= \frac{1}{2} ab \sin C\end{aligned}$$



13 a 31.29° b 42.85 cm^2 c 34 cm
 d 42.85 cm^2

Review exercise

1 a 31.0° b 33.6° c 10.58
 d 7.20

2 a 10 b $18\sqrt{2}$ c $25\sqrt{3}$

3 $A = 26.4°, B = 117.3°, C = 36.3°$

4 a 92° b 1211 m^2

5 48 6 7 7 24.47 cm^2

8 a 49.0° b 71.0° c 20.3 cm
 d 142.5 cm^2

9 141 m

10 a $\angle C = 91°, BC \approx 5.44$ cm, $AC \approx 4.31$ cm
 b $BC \approx 4.46$ cm, $\angle C \approx 90.19°, \angle B \approx 41.81°$
 c $AB \approx 11.17$ cm, $\angle B \approx 61.88°, \angle A \approx 38.12°$
 d $AC \approx 19.08$ cm, $\angle A \approx 33.00°, \angle C \approx 27.00°$
 e $\angle B \approx 105.05°, \angle C \approx 43.95°, AB \approx 10.78$ cm
 f $\angle C \approx 149.00°, \angle A \approx 13.00°, BC \approx 10.92$ cm

Challenge exercise

1 $\frac{1}{2} ab \sin C = \frac{1}{2} ac \sin B, \ \frac{b}{\sin B} = \frac{c}{\sin C}$

2 a i $\frac{h}{\tan 12°}$ ii $\frac{h}{\tan 9°}$
 b 47.5 m c 318°T

4 b $b = a \cos C + c \cos A, \ c = a \cos B + b \cos A$

7 If given angle A and sides b and a:
0 triangles if $a < b \sin A$ or A is obtuse and $a < b$
1 triangle if A is obtuse and $a > b$ or A is acute and $a = b \sin A$
2 triangles if A is acute and $b \sin A < a < b$

Chapter 13 answers

Exercise 13A

1 a ii 90°
 b ii The angle at the centre is twice any angle at the circumference standing on the same minor arc.
 iii The reflex angle at the centre is twice any angle at the circumference standing on the same major arc.

2 a $\alpha = 90°$ (Thales' theorem),
 $\beta = 75°$ (sum of angles in a triangle is 180°)
 b $\theta = 90°$ (Thales' theorem)
 c $\theta = 10°$ ($\angle JLK = 90°$, Thales' theorem, sum of angles in a triangle is 180°)
 d $\gamma = 70°$ ($OS = OT$ radii of circle and base angles of an isosceles triangle are equal),
 $\alpha = 40°$ (sum of angles in a triangle is 180°),
 $\theta = 140°$ (straight angle),
 $\beta = 20°$ (base angles of an isosceles triangle)
 e $\theta = 55°$ ($OZ = OY$ radii of circle, base angles of an isosceles triangle are equal and the sum of angles in a triangle is 180°),
 $\beta = 35°$ (external angle of isosceles triangle is 70°)
 f $\theta = 80°$ ($\angle AOC = 20°$ and ΔAOC is isosceles)

3 **a** $\theta = 110°$ (angle at the centre is twice angle at the circumference standing on the same arc)

b $\gamma = 44°$ (angle at the circumference is half angle at the centre standing on the same arc)

c $\alpha = 190°$ (angle at the centre is twice angle at the circumference standing on the same arc)

d $\gamma = 100°$ (angle at the circumference is half angle at the centre standing on the same arc)

e $\theta = 129°$ (angle at the circumference is half angle at the centre standing on the same arc)

f $\alpha = 40°$ (angle at the circumference is half angle at the centre standing on the same arc)

g $\beta = 30°$ (angle at the circumference is half angle at the centre standing on the same arc)

h $\gamma = 100°$ (angle at the centre is twice angle at the circumference standing on the same arc)

i $\theta = 24°$ (angle at the centre is twice angle at the circumference standing on the same arc)

4 **a** $\alpha = 120°$ (sum of angles about a point is 360°), $\beta = 60°$ (angle at the circumference is half angle at the centre standing on the same arc)

b $\theta = 30°$ ($\angle AOB = 60°$, sum of angles about a point is 360° and angle at the circumference is half angle at the centre standing on the same arc so θ is half of $\angle AOB$)

c $\theta = 220°$ ($\angle SOR = 140°$, angle at the centre is twice angle at the circumference standing on the same arc and $\theta + \angle SOR = 360°$, sum of angles about a point is 360°)

d $\alpha = \beta = 40°$ (any angle at the circumference is half angle at the centre standing on the same arc)

e $\theta = 320°$ ($\angle SOR = 40°$, angle at the centre is twice angle at the circumference standing on the same arc and $\theta + \angle SOR = 360°$, sum of angles about a point is 360°)

$\alpha = 20°$ (angle at the circumference is half angle at the centre standing on the same arc, $\angle SOR = 40°$)

f $\alpha = 100°$ ($OR = OQ$ radii so base angles of an isosceles triangle are both 40°, sum of angles in a triangle is 180°), $\beta = 140°$ (sum of angles about a point is 360°), $\gamma = 20°$ ($OR = OP$ radii, γ is a base angle of an isosceles triangle)

g $\alpha = 80°$ (angle at the circumference is half angle at the centre standing on the same arc), $\beta = 200°$ (sum of angles about a point is 360°), $\gamma = 100°$ (angles at the circumference is half angle at the centre standing on the same arc)

h $\alpha = 60°$ ($\angle DAB = 90°$, Thales' theorem, so $\alpha + 30° = 90°$),

$\beta = 60°$ ($OA = OB$ radii, so $\alpha = \beta$, base angles of isosceles triangle),

$\gamma = 30°$ ($\angle ABC = 90°$, Thales' theorem, so $\gamma + \beta = 90°$)

i $\alpha = \beta = 45°$ (both are base angles in isosceles triangles with the third angle 90°)

5 **a** $\alpha = 90°$ (Thales' theorem), $\beta = 10°$ (alternate angles, $AB \parallel FG$)

b $\alpha = 60°$ ($OP = OA = AP$ so the triangle is equilateral), $\beta = 30°$ (angle at the circumference is half angle at the centre standing on the same arc)

c $\alpha = 20°$ (alternate angles, $PO \parallel QR$), $\gamma = 40°$ (angle at the centre is twice angle at the circumference standing on the same arc), $\beta = 40°$ (alternate angles, $PO \parallel QR$)

d $\alpha = 220°$ (sum of angles about a point), $\beta = 110°$ (angle at the circumference is half angle at the centre standing on the same arc), $\gamma = 60°$ (sum of angles in a quadrilateral is 360°)

e $\alpha = 200°$ (sum of angles about a point), $\beta = 100°$ (angle at the circumference is half angle at the centre standing on the same arc), $\gamma = 80°$ (co-interior angles, $PQ \parallel OR$)

f $\alpha = 100°$ (angle at the circumference is half angle at the centre standing on the same arc), $\beta = 60°$ (construct OA, as $OB = AB = OA$ the triangle is equilateral), $\gamma = 40°$ (sum of angles in a quadrilateral is 360°)

6 **a** **i** $OA = OP$ so ΔAPO is isosceles with $\angle PAO = \angle APO = \alpha$,

$OB = OP$ so ΔPBO is isosceles with $\angle PBO = \angle BPO = \beta$,

so $\angle APB = \alpha + \beta$; $\angle XPB = \alpha + \beta$, external angle of ΔAPB

ii $\angle XPB + \angle APB = 180°$, $\alpha + \beta + \alpha + \beta = 180°$,

$2(\alpha + \beta) = 180°$, $\alpha + \beta = 90°$

b **i** $OA = OP$ so ΔAPO is isosceles with $\angle PAO = \angle APO = \alpha$, $OB = OP$ so ΔPBO is isosceles with $\angle PBO = \angle BPO = \beta$, $\angle AOM = 2\alpha$ and $\angle BOM = 2\beta$, external angles of ΔAPO and ΔPBO

ii $\angle AOM + \angle BOM = 180°$ so $2\alpha + 2\beta = 180°$

iii Thus $\alpha + \beta = 90°$

7 **a** **i** $OB = OP$ so ΔPBO is isosceles with $\angle PBO = \angle BPO = \beta$, so $\angle APB = \beta$

ii $\angle AOB = 2\beta$, (exterior angle of triangle)

b **i** $OA = OP$ so ΔAPO is isosceles with $\angle PAO = \angle APO = \alpha$,

$OB = OP$ so ΔPBO is isosceles with $\angle PBO = \angle BPO = \beta$,

so $\angle APB = \angle BPO - \angle APO = \beta - \alpha$

ii $\angle XOB = 2\beta$ (exterior angle of triangle), $\angle XOA = 2\alpha$ (exterior angle of triangle),

$\angle AOB = 2\beta - 2\alpha = 2(\beta - \alpha) = 2\angle APB$

8 **a** The diagonals bisect each other.

b A parallelogram with a right angle is a rectangle.

c The diagonals of a rectangle are equal and bisect each other, so $OA = OB = OP$. A circle with diameter AB has centre O and passes through P.

9 The angle at the centre is twice the angle at the circumference. As the horse moves from position 1 to position 2, angles are subtended both at the binoculars and the centre.

Exercise 13B

1 **b** They are equal.

c They are equal.

2 **a** $\alpha = 50°$ (angles at the circumference standing on the same arc)

b $\alpha = \beta = 20°$ (angles at the circumference standing on the same arc)

c $\alpha = 20°$ (angles at the circumference standing on the same arc)

$\beta = 40°$ (angles at the circumference standing on the same arc)

d $\alpha = 40°$ (angles at the circumference standing on the same arc)

$\beta = 30°$ (angles at the circumference standing on the same arc)

e $\theta = 90°$ (angles at the circumference standing on the same arc)

$\alpha = 40°$ ($\alpha + \theta = 130°$, exterior angle of triangle)

f $\alpha = 20°$ (angles at the circumference standing on the same arc)

$\theta = 100°$ ($\angle QPK = 60°$, angles at the circumference standing on the same arc, sum of angles in triangle is 180°)

3 **a** $\theta = 80°$ (opposite angles in a cyclic quadrilateral are supplementary)

b $\alpha = 100°$ (opposite angles in a cyclic quadrilateral are supplementary)

$\beta = 95°$ (opposite angles in a cyclic quadrilateral are supplementary)

c $\alpha = 40°$ (angles at the circumference standing on the same arc)

$\beta = 45°$ (angles at the circumference standing on the same arc)

$\theta = 35°$ ($80 = \beta + \theta$ exterior angle),
$\gamma = 60°$ (sum of angles in triangle ACD is 180°)

d $\alpha = 50°$ (opposite angles in a cyclic quadrilateral are supplementary)

$\beta = 90°$ (sum of angles in triangle is 180°),
$\gamma = 30°$ (sum of angles in triangle is 180°)

e $\alpha = 110°$ (straight angle),

$\gamma = 70°$ (opposite angles in a cyclic quadrilateral are supplementary)

$\beta = 80°$ (straight angle),

$\theta = 100°$ (opposite angles in a cyclic quadrilateral are supplementary)

f $\alpha = 20°$ (angles at the circumference standing on the same arc),

$\beta = 90°$ (Thales' theorem),

$\gamma = 90°$ (Thales' theorem),

$\theta = 70°$ (opposite angles in a cyclic quadrilateral are supplementary)

4 **a** $\alpha = 70°$ (co-interior angles, $DC \parallel AB$),

$\gamma = 110°$ (opposite angles in a cyclic quadrilateral are supplementary),

$\beta = 70°$ (opposite angles in a cyclic quadrilateral are supplementary)

b $\alpha = 65°$ (angles at the circumference standing on the same arc),

$\gamma = 65°$ (alternate, $TU \parallel SR$)

$\beta = 65°$ (alternate, $TU \parallel SR$)

c $\alpha = 30°$ ($\angle BQM = 30°$, alternate, $BQ \parallel AP$),

$\beta = 30°$ (angles at the circumference standing on the same arc),

$\gamma = 30°$ (angles at the circumference standing on the same arc)

d $\beta = 70°$ ($\angle JML = 90°$, Thales' theorem, sum of angles in triangle is 180°)

$\alpha = 20°$ ($\angle MJK = 90°$, Thales' theorem, $\alpha + \beta = 90°$)

e $\alpha = 70°$ (alternate, $AD \parallel BC$),

$\beta = 40°$ (ΔAOB is isosceles with base angles α, so $\angle AOB = 40°$, alternate, $AD \parallel BC$),

$\gamma = 40°$ (ΔBOC is isosceles with base angles γ and β),

$\theta = 70°$ ($\angle DOC = 40°$, alternate, $AD \parallel BC$ and ΔDOC is isosceles with base angles θ)

f $\alpha = 60°$ (construct SO, ΔRSO is equilateral),

$\beta = 120°$ (opposite angles in a cyclic quadrilateral are supplementary),

$\gamma = 60°$ (co-interior, $ST \parallel RU$)

5 **a** **i** $\angle Q = \angle T = \theta$ (angles at the circumference standing on the same arc),

$\angle P = \angle T = \theta$ (alternate, $PQ \parallel ST$), $\angle P = \angle S = \theta$ (angles at the circumference standing on the same arc),

so $\angle P = \angle Q = \angle S = \angle T = \theta$

ii $\angle P = \angle Q = \angle S = \angle T = \theta$, ΔSMT and ΔPMQ are isosceles, so $SM = TM$ and $PM = QM$, so $SM + MQ = TM + MP$ giving $SQ = TP$

b **i** $\alpha = 60°$ (ΔPQR is equilateral),

$\beta = 120°$ (opposite angles in a cyclic quadrilateral are supplementary),

$\gamma = 30°$ (base angle of isosceles triangle PGQ)

ii $\angle GRP = 30°$, $\angle GRQ = 30°$ (angles standing on the same arc),

$\angle PMR = 90°$ so $PQ \perp GR$

6 **a** **i** $\alpha = 110°$ (opposite angles in a cyclic quadrilateral are supplementary),

$\beta = 70°$ (straight angle),

$\gamma = 110°$ (opposite angles in a cyclic quadrilateral are supplementary)

ii As $\angle TQP$ and $\angle SPQ$ are co-interior and supplementary then $PS \parallel QT$.

b **i** $\alpha = 130°$ (opposite angles in a cyclic quadrilateral are supplementary),

$\beta = 50°$ (ΔABM is isosceles, straight angle at A),

$\gamma = 50°$ ($\angle QBA = 130°$, straight angle at B, opposite angles in a cyclic quadrilateral are supplementary)

ii $\angle QPM$ and $\angle BAP$ are co-interior and supplementary, so $PQ \parallel AB$

iii $\angle QPM = \angle PQM$, QMP is isosceles with $QM = PM$, also $BM = AM$, so $QB = PA$

7 **a** **i** $\angle ABS = 90°$ and $\angle ABT = 90°$ (Thales' theorem)

ii $\angle TBS = \angle ABS + \angle ABT = 180°$, so $\angle TBS$ is a straight angle $\therefore$ T, B and S are collinear

b **i** $\angle ABC = 90°$; (opposite angles in a cyclic quadrilateral are supplementary), $\angle ABD = 90°$ (Thales' theorem)

ii $\angle CBD = \angle ABC + \angle ABD = 180°$, so $\angle CBD$ is a straight angle $\therefore$ C, B and D are collinear

iii $\angle ANC = 90°$; by the converse of Thales' theorem $\angle ANC$ is in a semicircle with AC the diameter

8 a $\angle DBC = \angle DAC = \alpha$ (angles at the circumference standing on the same arc)

$\angle BDC = \angle BAC = \beta$ (angles at the circumference standing on the same arc)

b $\angle DBC + \angle BDC + \angle BCD = 180°$ (sum of angles in a triangle),

$\alpha + \beta + \angle BCD = 180°$, so $\angle BCD = 180° - \alpha - \beta$

c $\angle BAD = \alpha + \beta$; thus, $\angle BAD + \angle BCD = 180°$

9 a i $\angle C = 180° - \theta$ (opposite angles in a cyclic quadrilateral are supplementary),

$\angle C = \theta$ (opposite angles in a parallelogram are equal)

ii From **a** $180° - \theta = \theta$, so $\theta = 90°$, so $\angle C = \angle A = 90°$, similarly $\angle B = \angle D = 90°$, so $ABCD$ is a rectangle.

b A rhombus is a parallelogram with all sides equal and so if a cyclic parallelogram is a rectangle, a cyclic rhombus must be a square.

10 a $\angle MDC = 180° - \theta$ (co-interior, $AB \parallel DC$) and $\angle BCD = 180° - \theta$ (opposite angles in a cyclic quadrilateral are supplementary) $\therefore \Delta MDC$ is isosceles

$\angle MBA = 180° - \theta$ (corresponding angles, $AB \parallel DC$) and $\angle MAB = 180° - \theta$ (straight angle at A) $\therefore \Delta MAB$ is isosceles

b As ΔMDC is isosceles, $MD = MC$ and as ΔMAB is isosceles, $MA = MB$, so $AD = BC$

Exercise 13C

2 a $x = 3, \theta = 53.1°$

b $x = 2\sqrt{13}, \theta = 33.7°$

c $x = 4\sqrt{3}, \theta = 30°$

3 a i 17.0 cm   ii 8.5 cm

b i 24 cm   ii 73.7°

c i 6 cm   ii 106.3°

4 a $\theta = 50°$ and $\angle UOT = 50°$ (chords of equal length subtend equal angles at the centre of a circle)

$\alpha = 65°$ since ΔUOT is isosceles

b $\alpha = 60°$ (ΔABO is equilateral),

$\beta = 240°$ (ΔABO and ΔBCO are equilateral, sum of angles at a point)

c $\theta = 30°$ (ΔRQO is equilateral, so $\angle QRO = 60°$ and $\angle RQP = 90°$, Thales' theorem)

d $\alpha = 45°$ (base angle of isosceles triangle with third angle 90°),

$\beta = 25°$ (base angle of isosceles triangle with third angle 130°),

$\gamma = 20°$ (base angle of isosceles triangle with third angle 140°)

e $\theta = 50°$ ($\angle HOF = 130°$ as ΔHOF is isosceles, $\angle GOF$ is a straight angle)

f $\alpha = 36°$ (10 congruent triangles in a circle),

$\beta = 72°$ (base angle of isosceles triangle with third angle 36°)

5 a $\angle FPO = 60°$ (ΔFPO is equilateral), $\angle FGO = 30°$ (angle at the centre is twice angle at the circumference standing on the same arc) $\angle FMO = 90°$ (angle sum of triangle)

b $FG = \sqrt{3}$

6 a i $AO = BO$ (radii of the circle), $AM = BM$ (as M is midpoint of AB),

OM is common $\therefore \Delta AOM \equiv \Delta BOM$ (SSS)

ii $\angle AMO = \angle BMO$ (matching angles, $\Delta AOM \equiv \Delta BOM$) and

$\angle AMO + \angle BMO = 180°$ (straight angle at M), so $\angle AMO = \angle BMO = 90°$

$\angle AOM = \angle BOM$ (matching angles, $\Delta AOM \equiv \Delta BOM$)

b i $\angle AMO = \angle BMO = 90°$ (given), $AO = BO$ (radii of the circle),

OM is common $\therefore \Delta AOM \equiv \Delta BOM$ (RHS)

ii $\angle AOM = \angle BOM$ (matching angles, $\Delta AOM \equiv \Delta BOM$) and

$AM = BM$ (matching sides, $\Delta AOM \equiv \Delta BOM$) so OM bisects $\angle AOB$

c i $\angle AOM = \angle BOM = \alpha$ (given), $AO = BO$ (radii of the circle),

OM is common $\therefore \Delta AOM \equiv \Delta BOM$ (SAS)

ii $\angle AMO = \angle BMO$ (matching angles, $\Delta AOM \equiv \Delta BOM$) and

$\angle AMO + \angle BMO = 180°$ (straight angle at M), so $\angle AMO = \angle BMO = 90°$ $AM = BM$ (matching sides, $\Delta AOM \equiv \Delta BOM$)

7 a i Join OA and OP. $OA = OP = OQ$ (radii of the circle) so ΔAOP and ΔAOQ are isosceles with base angles θ making the third angle in both triangles $180° - 2\theta$, so $\Delta AOP \equiv \Delta AOQ$ (SAS)

ii $AP = AQ$ (matching sides, $\Delta AOP \equiv \Delta AOQ$)

b i Join OS $\angle TFS = \angle SFO = \theta$ (given) and ΔFSO is isosceles ($OF = OS$, radii),

so $\angle SFO = \angle FSO \therefore \angle TFS = \angle FSO$

ii Join OF. $\angle TFS$ and $\angle FSO$ are alternate and equal, so $FT \parallel OS$

c i $SQ = SP$ (given), $QT = PT$ (given), ST is common $\therefore \Delta SQT \equiv \Delta SPT$ (SSS)

ii $\angle P = \angle Q$ (matching angles, $\Delta SQT \equiv \Delta SPT$)

iii $\angle P + \angle Q = 180°$ (opposite angles in a cyclic quadrilateral are supplementary)

iv As $\angle P = \angle Q$ from **ii** and $\angle P + \angle Q = 180°$ from **iii** then $\angle P = \angle Q = 90°$, so ST is a diameter (converse of Thales' theorem)

d i Join OR, OS, OT and OU. ΔOST is isosceles with $OT = OS$ (radii of circle centre O through T), angles opposite equal sides are equal so $\angle OST = \angle OTS$

ii ΔOUR is isosceles with $OR = OU$ (radii of circle centre O through U), angles opposite equal sides are equal so $\angle OUR = \angle ORU$

iii $\angle OST = \angle OTS$ (from **i**), $\angle OUT = \angle ORS$ (from **ii**) and $OR = OU$ (radii), so $\Delta ORT \equiv \Delta OUS$ (AAS)

iv $RT = SU$ (matching sides, $\Delta ORT \equiv \Delta OUS$)

8 a $GO = FO$ (radii of the circle centre O), $PG = PF$ (radii of the circle centre P), OP is common $\therefore \Delta GOP \equiv \Delta FOP$ (SSS)

b $\angle FOM = \angle GOM$ (matching angles, $\Delta GOP \equiv \Delta FOP$)

$GO = FO$ (radii of the circle centre O), $\angle FOM = \angle GOM$ (from **b**), OM is common $\therefore \Delta GOM \equiv \Delta FOM$ (SAS)

c $\angle GMO = \angle FMO$ (matching angles, $\Delta FOM \equiv \Delta GOM$) and $\angle GMO + \angle FMO = 180°$ (straight angle at M), so $\angle GMO = \angle FMO = 90°$ and $OP \perp FG$, $FM = GM$ (matching sides, $\Delta GOM \equiv \Delta FOM$)

9 **a** $\Delta PAM \equiv \Delta PBM$ (SSS), so $\angle AMP = \angle BMP = 90°$

b $\Delta PAM \equiv \Delta PBM$ (SAS), so $AP = BP$

c Take three points A, B and C on the circle. Construct the perpendicular bisectors of AB and BC. By parts a and b, the centre lies on the perpendicular bisectors of AB and BC, so the intersection is the only centre of the circle.

d **i** Imagine the vertical line through the centre of the circle, and let P be any point on this line. Then P is equidistant from all the points on the circle.

ii A sphere

iii An infinite cylinder

iv A cylinder with hemispherical ends

v A plane perpendicular to the interval through the midpoint of the interval

vi One method is to take three chords, not in a plane. Take the plane perpendicular to each chord through its midpoint, then the intersection of the three planes is the centre of the sphere.

Exercise 13D

1 **a** $\alpha = 90°$ (tangent perpendicular to radius at common point on circle),

$\beta = 40°$ (sum of angles in a triangle)

b $\alpha = 75°$ ($\angle OFP = 90°$, sum of angles in a triangle),

$\beta = 37.5°$ (base angle of isosceles triangle with external angle 75°)

c $\alpha = 70°$ (angle at the centre is twice angle at the circumference standing on the same arc),

$\beta = 20°$ ($\angle OTU = 90°$, sum of angles in a triangle)

d $\alpha = 62°$ ($\angle MTO = 90°$, sum of angles in a triangle),

$\beta = 31°$ (angles at the circumference is half angle at the centre standing on the same arc)

e $\theta = 50°$ ($\angle UOT = 40°$, angle at the centre is twice angle at the circumference standing on the same arc and $\angle OTU = 90°$, sum of angles in a triangle)

f $\alpha = 90°$ ($\angle OTU = 90°$, alternate angles, $TU \parallel BA$),

$\beta = 45°$ (base angle of isosceles triangle with third angle 90°)

g $\beta = 70°$ (alternate angle, $DT \parallel AC$),
$\alpha = 20°$ ($\alpha + \beta = 90° = \angle DTO$)

h $\alpha = 25°$ (alternate angles, $TR \parallel SD$), $\beta = 90°$ (Thales' theorem),

$\gamma = 65°$ (sum of angles in a triangle)

i $\alpha = 35°$ ($\angle BTS = 90°$),

$\beta = 55°$ ($\angle SQT = 90°$, Thales' theorem, sum of angles in triangle)

$\theta = 70°$ ($\angle SPT = 90°$, Thales' theorem, sum of angles in triangle)

j $\alpha = 30°$ ($\angle DTO = 15°$, angle at circumference is half angle at centre standing on same arc),
$\beta = 60°$ ($\angle OTB = 90°$, sum of angles in a triangle)

k $\theta = 30°$ (ΔTXO is equilateral, $\angle XTO = 60°$, $\angle LTO = 90°$),

$\beta = 120°$ ($\angle YZO = 60°$, sum of opposite angles in a cyclic quadrilateral)

l $\alpha = 30°$ (ΔTQO is equilateral, $\angle TOQ = 60°$, angle at the circumference is half angle at the centre standing on the same arc),
$\theta = 30°$ (complementary with $\angle QTO = 60°$)

2 **a** $x = 5$ (tangents to a circle from an external point have equal length),

$\alpha = 70°$ (base angle of an isosceles triangle),
$\beta = 40°$ (sum of angles in a triangle),
$\gamma = 20°$ ($\angle OSP = 90°$)

b $x = 8$ (tangents to a circle from an external point have equal length),

$\alpha = 70°$ (base angle of an isosceles triangle),

$\theta = 140°$ ($\angle T = \angle S = 90°$, sum of angles in a quadrilateral)

c $x = 2$ (tangents to a circle from an external point have equal length),

$y = 3, z = 3$ (tangents to a circle from an external point have equal length)

d $x = 7$ ($SQ = 4$, tangents to a circle from an external point have equal length, so $RS = 7$)

e $x = 9$ ($SB = 4$ (equal tangents), $SP = 14$ and $TP = 14$ (equal tangents), $TA = 5$ (equal tangents), $x = 14 - 5$

f $\alpha = 100°$ (reflex $\angle SOT = 260°$, angle at the centre is twice angle at the circumference standing on the same arc, sum of angles at point O),

$\beta = 70°$ (sum of angles in a quadrilateral),
$\theta = 20°$ ($\angle PSO = 90°$)

3 **a** $x = 5$ (radius), $y = 8$ ($OB = 13$ by Pythagoras' theorem), $\theta \approx 22.62°$

b $x = \sqrt{133} \approx 11.53$

c $x \approx 19.23$, $y \approx 13.47$

4 $\angle PAB = 90°$ (tangent perpendicular to radius) and $\angle VBA = 90°$ (tangent perpendicular to radius), $\angle PAB$ and $\angle VBA$ are alternate and equal therefore $PQ \parallel UV$

5
$$\begin{aligned} AB + CD &= AP + PB + CR + DR \\ &= AS + BQ + QC + SD \quad \text{(tangents to a circle from an external point have equal length)} \\ &= AD + BC \end{aligned}$$

6 **b** Join OS and OT. $\angle OSP$ and $\angle OTP$ are angles in a semi-circle, centre M, so by Thales' theorem $\angle OSP = \angle OTP = 90°$. As OS and OT are radii of circle, centre O, $PS \perp OS$ and $PT \perp OT$, so PS and PT are tangents to circle centre O.

c $PS = PT$ (tangents to a circle from an external point have equal length)

7 **a** **i** $\angle PSO = \angle PTO$ (tangent perpendicular to radius), $SO = TO$ (radii of circle), PO is common. Thus $\Delta PSO \equiv \Delta PTO$ (RHS)

ii $PS = PT$ (matching sides, $\Delta PSO \equiv \Delta PTO$), $\angle SPO = \angle TPO$ (matching angles, $\Delta PSO \equiv \Delta PTO$), $\angle SOP = \angle TOP$ (matching angles, $\Delta PSO \equiv \Delta PTO$)

b **i** $\angle SPO = \angle TPO$ (from **aii**), $SP = TP$ (from **aii**), PM is common

Thus $\Delta PSM \equiv \Delta PTM$ (SAS)

ii $SM = TM$ (matching sides, $\Delta PSM \equiv \Delta PTM$),

$\angle SMP = \angle TMP$ (matching angles, $\Delta PSM \equiv \Delta PTM$) and

$\angle SMP + \angle TMP = 180°$ (straight angle at M), so $\angle SMP = \angle TMP = 90°$.

Thus OP is the perpendicular bisector of ST

8 $RO \perp AB, QO \perp AC, PO \perp BC$ (tangent perpendicular to radius)

Area of $\Delta ABC =$ area of $\Delta AOC +$ area of $\Delta BOC +$ area of ΔAOB

$$= \frac{1}{2} \times AC \times OQ + \frac{1}{2} \times BC \times OP + \frac{1}{2} \times AB \times OR$$
$$= \frac{1}{2} \times \text{radius} \times (AC + BC + AB)$$
$$= \frac{1}{2} \times (\text{radius of circle}) \times (\text{perimeter of triangle})$$

9 **a** **i** $\angle BTA = \alpha$ (base angle of isosceles triangle),
$\angle BTO = \beta$ (base angle of isosceles triangle),
$\alpha + \beta = 90°$ (tangent perpendicular to radius),
$\beta = 2\alpha$ (exterior angle of ΔABT).
So $3\alpha = 90°, \alpha = 30°$.

ii $\beta = 60°$

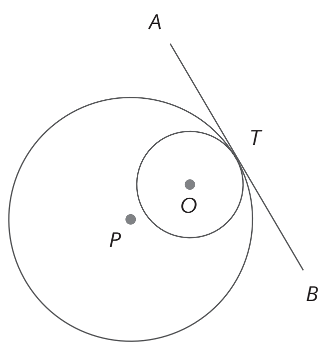
b **i** $OT = OB$ (radii) and $OB = BA$ (given), so
$OA = 2OT, \angle OTA = 90°, \sin\alpha = \frac{OT}{OA} = \frac{OT}{2OT} = \frac{1}{2}$

ii $\sin\alpha = \frac{1}{2}, \alpha = 30°$,
so $\angle TOA = 60°$ (sum of angles in a triangle), ΔOBT is equilateral and $\beta = 60°$

10 **a** **i** $\angle ATO = 90°$ (tangent perpendicular to radius),
$\angle ATP = 90°$ (tangent perpendicular to radius)

ii $\angle OTP = \angle ATO + \angle ATP = 90° + 90° = 180°$, so $\angle OTP$ is a straight angle so O, T and P are collinear

b

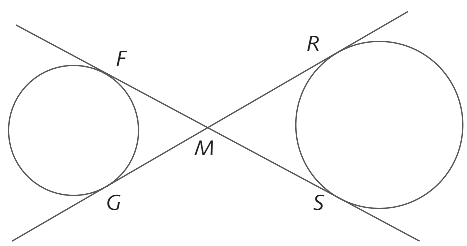


$\angle ATO = 90°$ (tangent perpendicular to radius),
$\angle ATP = 90°$ (tangent perpendicular to radius),
$\angle ATO = \angle ATP = 90°$ so O, T and P are collinear

11 **a** **i** $MF = MG$ (tangents to a circle from an external point have equal length)
$MR = MS$ (tangents to a circle from an external point have equal length)

ii $FR = MF - MR = MG - MS = GS$

b



$MF = MG$ (tangents to a circle from an external point have equal length)
$MR = MS$ (tangents to a circle from an external point have equal length)
$FS = FM + MS = GM + MR = GR$

12 **a** $MA = MT$ (tangents to a circle from an external point have equal length)
$MB = MT$ (tangents to a circle from an external point have equal length)
$\therefore MA = MT = MB$

b Circle centre M and passing through A, B and T has a diameter AB, so $\angle ATB$ is in a semicircle, hence by Thales' theorem $\angle ATB = 90° \therefore AT \perp BT$

Exercise 13E

2 **a** $\alpha = 35°$ (alternate segment theorem)
b $\theta = 40°$ (alternate segment theorem)
c $\beta = 110°$ (alternate segment theorem)
d $\gamma = 150°$ (alternate segment theorem)
e $\alpha = 70°$ (alternate segment theorem),
$\beta = 110°$ (opposite angles in a cyclic quadrilateral are supplementary)
f $\beta = 125°$ (alternate segment theorem),
$\alpha = 55°$ (opposite angles in a cyclic quadrilateral are supplementary)

3 **a** $\alpha = 50°$ (alternate segment theorem),
$\beta = 50°$ (alternate angles, $FG \parallel LM$ alternate segment theorem),
$\gamma = 80°$ (straight angle)
b $\beta = 70°$ (base angles of isosceles triangle are equal),
$\alpha = 70°$ (alternate segment theorem),
$\gamma = 40°$ (angles in a triangle)
c $\beta = 80°$ (alternate segment theorem)
d $\beta = 35°$ (ΔTSF is isosceles),
$\alpha = 35°$ (alternate segment theorem)
e $\alpha = 130°$ (alternate segment theorem),
$\beta = 80°$ ($\angle BTS = 50°$, straight angle at T and ΔBTS is isosceles, tangents to a circle from an external point have equal length)
f $\alpha = 50°$ (alternate segment theorem),
$\beta = 80°$ (ΔUAT is isosceles, tangents to a circle from an external point have equal length),
$\gamma = 55°$ (ΔBUS is isosceles, tangents to a circle from an external point have equal length)

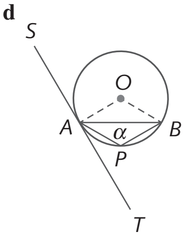
4 **a** $\angle BAO = 90° - \theta$ ($\angle OAT = 90°$, tangent perpendicular to radius)
b $\angle AOB = 180° - 2(90° - \theta)$, ($\Delta AOB$ is isosceles with base angles $\angle BAO$ and $\angle OBA$, sum of angles in a triangle), so $\angle AOB = 2\theta$
c $\angle APB = \theta$ (angle at the circumference is half angle at the centre standing on the same arc)

5 **a** $\angle ANB = 180 - \alpha$ (opposite angles in a cyclic quadrilateral are supplementary)
b $\angle ABN = 90°$ (Thales' theorem)

$$\begin{aligned}\angle NAB &= 180 - (\angle ABN + \angle ANB)\\ &= 180 - (180 - \alpha + 90)\\ &= \alpha - 90\end{aligned}$$

c
$$\begin{aligned}\angle SAB &= 90 + (\alpha - 90)\\ &= \alpha\end{aligned}$$

d



$\angle AOB = 2\alpha$ on major arc AB ($\angle APB$ on same arc)

$\angle AOB = 360 - 2\alpha$ on minor arc, AB

$$\angle OAB = \frac{180 - \angle AOB}{2} \quad (\Delta AOB \text{ is isosceles})$$
$$= \frac{180 - (360 - 2\alpha)}{2}$$
$$= \alpha - 90$$

$$\angle SAB = \angle SAO + \angle OAB$$
$$= 90 + (\alpha - 90)$$
$$= \alpha$$

6 **a** $\angle LBA = 180° - \theta$ (opposite angles in a cyclic quadrilateral are supplementary)

$\angle TBA = \theta$ (straight angle at B), $\angle GTA = \theta$ (alternate segment theorem)

b $\angle GTA$ and $\angle LKT$ are alternate and equal so $LK \parallel FG$

7 **a** $\angle GTB = \theta$ (alternate segment theorem), $\angle QTA = \theta$ (vertically opposite to $\angle GTB$)

b $\angle QPT = \theta$ (alternate segment theorem), $\angle QPT$ and $\angle GFT$ are alternate and equal so $FG \parallel QP$

8 **a** $\angle P = \theta$ (alternate, $FG \parallel QP$)

b $\angle GTB = \theta$ (alternate segment theorem), $\angle ATQ = \theta$ (alternate segment theorem), so $\angle GTB = \angle ATQ$, since ATB is a tangent then GTQ must be collinear for $\angle GTB$ and $\angle ATQ$ to be vertically opposite.

Exercise 13F

1 **a** $x = 8$ (products of the intervals on intersecting chords are equal)

b $x = 6$ (products of the intervals on intersecting chords are equal)

c $x = \frac{28}{3}$ (products of the intervals on intersecting chords are equal)

d $x = 6$ (tangent and secant from an external point)

e $x = 12$ (tangent and secant from an external point)

f $x = \frac{9}{2}$ (tangent and secant from an external point)

g $x = 22$ (secants from an external point)

h $x = 6$ (secants from an external point)

i $x = 3$ (secants from an external point)

j $x(x + 5) = 24$; $x = 3$ (products of the intervals on intersecting chords are equal)

k $6^2 = 4(x + 4)$; $x = 5$ (tangent and secant from an external point)

l $x(x + 8) = 48$; $x = 4$ (secants from an external point)

2 ΔMAQ is similar to ΔMPB (AAA)

$\frac{AM}{PM} = \frac{QM}{BM}$ (matching sides in similar triangles)

Therefore, $AM \times BM = QM \times PM$

3 $AM \times BM = TM^2$ and $PM \times QM = TM^2$

Therefore, $AM \times BM = PM \times QM$

4 **a** $\Delta OGM \equiv \Delta OHM$ (RHS as $OG = OH$, radii, $\angle OMG = \angle OMH$, given, and OM is common), so $MG = MH$ (corresponding sides)

b $GM \times HM = AM \times BM$, so $g \times g = a \times b$, $g^2 = ab$

c diameter is $a + b$, so the radius is $\frac{a+b}{2}$

d $0 \le GM \le$ radius, so $0 \le g \le$ radius and $g = \sqrt{ab}$, so

$0 \le \sqrt{ab} \le \text{radius}\left(= \frac{a+b}{2}\right)$

5 $PT^2 = PA \times PB$ (tangent and secant from an external point) and $PS^2 = PA \times PB$ (tangent and secant from an external point), so $PT = PS$

6 **a** $\angle MSA = \angle MBS$ (alternate segment theorem), $\angle SMA = \angle BMS$ (common), so ΔMSA is similar to ΔMBS (AA)

b $\frac{BS}{SA} = \frac{SM}{AM}$ (ratio of matching sides in similar triangles are equal), so $\frac{a}{x} = \frac{t}{m}$

c $\angle MTA = \angle MBT$ (alternate segment theorem), $\angle TMA = \angle BMT$ (common), so ΔMTA is similar to ΔMBT (AA) and $\frac{BT}{TA} = \frac{TM}{AM}$ (ratio of matching sides in similar triangles are equal),

so $\frac{y}{b} = \frac{t}{m}$

d From **b** and **c** $\frac{a}{x} = \frac{t}{m} = \frac{y}{b}$, so $ab = xy$

Review exercise

1 **a** $\alpha = 110°$ **b** $\alpha = 17°$ **c** $\alpha = 40°$

d $\alpha = 260°$ **e** $\alpha = 110°$ **f** $\alpha = 46°$

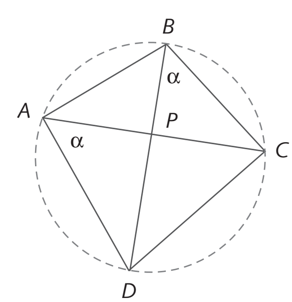
2 **a** $\beta = 42°, \alpha = 84°$

b $\alpha = 62°, \beta = 124°$

c $\alpha = 53°, \beta = 37°$

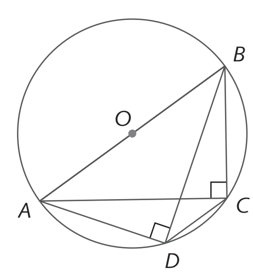
3 **a** $\alpha = 32°$ **b** $\alpha = 52°, \beta = 38°$ **c** $\alpha = 45°, \beta = 45°$

4



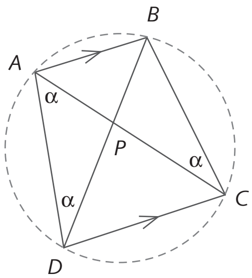
$\angle DBC = \angle CAD$ (angles subtended by the same arc)
$\angle BCA = \angle BDA$ (angles subtended by the same arc)
ΔAPD is similar to ΔBPC (AAA)

5



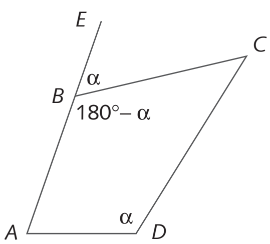
$\angle BDA = \angle BCA = 90°$ (Thales' theorem)
$AB = BA$ (common)
$\Delta ABC \equiv \Delta BAD$ (RHS)
Thus, $AD = BC$ (matching sides of congruent triangles)
Alternatively, use Pythagoras' theorem

6



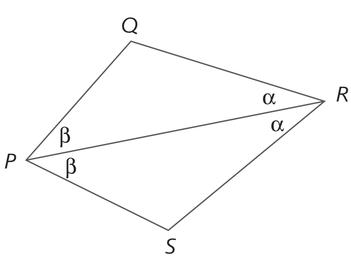
Let $\angle BCA = \alpha$
$\angle BDA = \alpha$ (angles subtended by the same arc)
$\angle DAC = \alpha$ (alternate angles $BC \parallel AD$)
$\angle BPA = 2\angle ACB$ (exterior angle of ΔAPD)

7



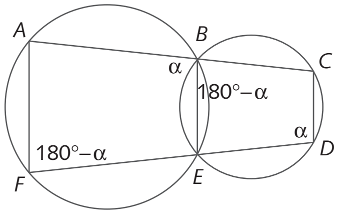
$\angle ABC = 180 - \angle CDA$ (cyclic quadrilateral)
$\angle EBC = 180 - \angle ABC$ (straight line)
$\angle EBC = \angle CDA$

8



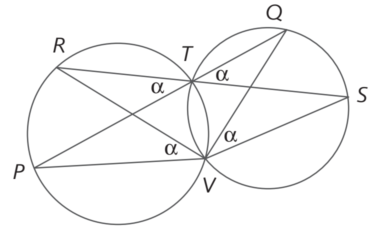
$\Delta PQR \equiv \Delta PSR$ (AAS)
Let $\angle QPR = \angle RPS = \beta$ and $\angle QRP = \angle SRP = \alpha$
$2\alpha + 2\beta = 180°$
$\alpha + \beta = 90°$
Thus $\angle PQR = \angle RSP = 90°$

9



Join B to E. Let $\angle EDC = \alpha$.
$\angle CBE = 180° - \alpha$ (opposite angles of cyclic quadrilateral)
$\angle ABE = \alpha$ (supplementary angles)
$\angle AFE = 180° - \alpha$ (opposite angles of a cyclic quadrilateral)
$\angle AFE$ and $\angle CDE$ are co-interior angles and
$\angle CDE + \angle AFD = 180°$
Thus, $AF \parallel CD$

10



Let $\angle RVP = \alpha$
$\angle RTP = \angle RTP = \alpha$ (angles subtended by the same arc)
$\angle STQ = \alpha$ (vertically opposite)
$\angle QTS = \angle SVQ$ (angles subtended by the same arc)
Thus, $\angle QVS = \angle PVR$

11 **a** $\angle ADC = \frac{1}{2}\angle AQC = \angle AQX$ (angle subtended at the centre is twice the angle subtended at the circumference in the small circle)

$\angle ABC = \frac{1}{2}\angle ADC$ (angle subtended at the centre is twice the angle subtended at the circumference in the large circle)

$\therefore \angle AQX = 2\angle ABC$

b $\angle ACB = 90°$ (Thales' theorem)

$AB^2 = BC^2 + AC^2$ (Pythagoras' theorem in ΔABC)
$= BC^2 + 4(AQ^2 - XQ^2)$ (Pythagoras' theorem in ΔAQX)

Challenge exercise

1 **a** $\angle BCP = 90°$ (Thales' theorem)

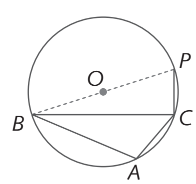
$\angle P = \angle A$ (angles on the circumference standing on the same arc)

In ΔBCP, $\sin P = \frac{a}{BP}$, so $BP = \frac{a}{\sin A}$

So $2R = \frac{a}{\sin A}$

b $\angle P = 180° - \angle A$ (opposite angles in a cyclic quadrilateral are supplementary)

In ΔBCP, $\sin(180° - A) = \frac{a}{BP}$, so $2R = \frac{a}{\sin A}$,
since $\sin(180° - \alpha) = \sin \alpha$



2 **a** $\Delta BRI \equiv \Delta BPI$ (AAS) as $\angle RBI = \angle PBI$ ($\angle B$ is bisected), $\angle BRI = \angle BPI$ (right angles), and BI is common $\therefore$
$IR = IP$ (corresponding sides of congruent triangles)
$\Delta CPI \equiv \Delta CQI$ (AAS) as $\angle PCI = \angle QCI$ ($\angle C$ is bisected), $\angle CPI = \angle CQI$ (right angles), and CI is common $\therefore$ $IP = IQ$ (corresponding sides of congruent triangles)

b From **a** $IP = IQ = IR$, $\angle ARI = \angle AQI$ (right angles), and AI is common
$\therefore$ $\Delta ARI \equiv \Delta AQI$ (RHS). So $\angle RAI = \angle QAI$ (corresponding angles of congruent triangles)
$\therefore$ IA bisects $\angle A$

c $IP = IQ = IR =$ radius, $IR \perp AB, IP \perp BC, IQ \perp AC$, so AB, BC and CA are tangents to the circle

3 **a** $\angle CLH = 90°$ so CH is a diameter of a circle that passes through L (converse of Thales' theorem) and $\angle CKH = 90°$ so CH is a diameter of a circle that passes through K (converse of Thales' theorem), so C, K, H and L are concyclic.

b C, K, H and L are concyclic, so $\angle HCL = \angle HKL$ (angles on the circumference standing on the same arc) and $\angle AKL = \angle HKL = \theta$

c $\angle AKB = 90°$ so AB is a diameter of a circle that passes through K (converse of Thales' theorem) and $\angle BLA = 90°$ so AB is a diameter of a circle that passes through L (converse of Thales' theorem), so B, K, L and A are concyclic. As B, K, L and A are concyclic $\angle ABL = \angle AKL$ (angles on the circumference standing on the same arc) and $\angle ABL = \angle AKL = \theta$

d AMC is similar to ALB (AA) as $\angle ABL = \angle ACM$ (both θ), $\angle BAL = \angle CAM$ (common)
$\therefore\ \angle BLA = \angle CMA = 90°$ (corresponding angles in similar triangles) meaning CM is an altitude of ΔABC

4 **a** $MG : GO = 2 : 1$ (given), $AG : GF = 2 : 1$ (centroid property),
$\angle AGM = \angle FGO$ (vertically opposite) $\therefore GMA$ is similar to ΔGOF (SAS)

b $\angle OFG = \angle MAG$ (corresponding angles in similar triangles), so $MA \parallel OF$ (alternate angles equal)
$\therefore$ line $AM \perp CB$ since $FO \perp CB$, so M lies on the altitude from A

c By the same argument, $BM \perp AC$ and $CM \perp AB$. Hence, $M = H$ from question 3.

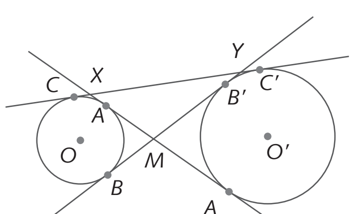
5 **a** $\angle ABC + \angle APC = 180°$ (as $APCB$ is a cyclic quadrilateral), $\angle ABC + \angle ADC = 180°$ (given), so $\angle APC = \angle ADC$

b D is on AP and $\angle APC = \angle ADC$, so D and P coincide.

6 **a** $PM \times CM = AM \times BM$ (The product of the intervals on intersecting chords are equal)

b $DM \times CM = AM \times BM$ (given) and
$PM \times CM = AM \times DM$ (from **a**), then $DM = PM$. D is on MP and $DM = PM$, so P and D coincide.

7



a Let the intersection of AA' and BB' be M.
$AM = BM$ and $A'M = B'M$ (tangents to a circle from an external point have equal length)
As $AA' = AM + A'M$ and $BB' = BM + B'M$, then $AA' = BB'$

b $YC' = YB'$, $XC = XA$, $XA' = XC'$, $YC = YB$, (tangents to a circle from an external point have equal length),

$XY = XA + AA' - YC'$ (**eqn 1**) and
$XY = YB' + BB' - CX$ (**eqn 2**)

eqn 1 + eqn 2 gives

$2XY = XA + AA' - YC' + YB' + BB' - CX$

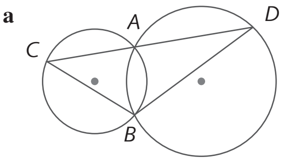
$2XY = XA - CX + AA' + BB' - YC' + YB'$

$2XY = AA' + BB'$

$2XY = 2AA'$ so $XY = AA'$

c The indirect common tangents (AA' and BB') become the same common tangent at the point of contact and $AA' = BB' = XY$.

8 **a**

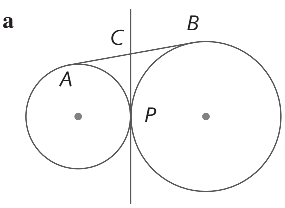


$\angle D$ is constant as angles on the circumference standing on the same arc are equal and $\angle C$ is constant as angles on the circumference standing on the same arc are equal. So any triangles drawn as described are similar (AAA).

b Area $= \dfrac{1}{2} BC \times BD \sin B$

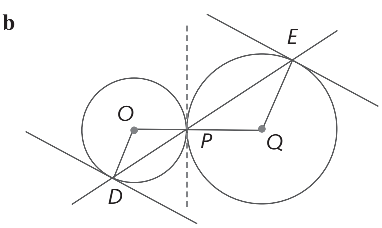
B is a constant. $\therefore$ largest area when BC and BD are diameters. (If BC is a diameter, $\angle CAB = 90°$. Therefore $\angle BAD = 90°$. Thus BD is a diameter by the converse of Thales' theorem.)

9 **a**



$CA = CP$ and $CB = CP$ (tangents to a circle from an external point have equal length), so $CA = CP = CB$
$\therefore\ C$ is the midpoint of AB.

b



Let the centres of the circles be O and Q.
As the circles touch at P, the radii OP and QP are $\perp$ to the common tangent at P, so OPQ is a line. ΔDOP is isosceles as $OD = OP$ (radii), so $\angle ODP = \angle OPD$ and ΔEQP is isosceles as $QE = QP$ (radii), so $\angle QPE = \angle QEP$.
$\angle OPD = \angle QPE$ (vertically opposite)
$\therefore\ \angle ODP = \angle QEP$, so $OD \parallel QE$ (alternate and equal angles). Finally the tangent at D is perpendicular to OD and the tangent at E is perpendicular to EQ, so the tangents must also be parallel.

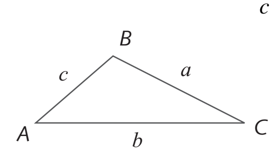
10 Area of $\Delta ABC = \dfrac{1}{2} ab \sin C$

$= \dfrac{1}{2} bc \sin A$

$= \dfrac{1}{2} ac \sin B$

$$\therefore \frac{2 \times \text{Area of } \Delta ABC}{abc} = \frac{ab \sin C}{abc} = \frac{bc \sin A}{abc} = \frac{ac \sin B}{abc}$$

$$= \frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b}$$



Chapter 14 answers

Exercise 14A

1 **a** 3 **b** 3 **c** 11
d 0 **e** 4 **f** 3
g 4 **h** 6

2 **a** −4 **b** −3 **c** −1
d −2 **e** −3 **f** −2
g −10 **h** −4

3 **a** $\frac{3}{2}$ **b** $\frac{5}{2}$ **c** $\frac{5}{2}$
d $\frac{5}{2}$ **e** $\frac{7}{2}$ **f** $-\frac{5}{2}$
g $\frac{7}{3}$ **h** $\frac{1}{6}$

4 **a** 32 **b** 729 **c** 1000
d $\frac{1}{1000}$ **e** $\frac{1}{10\,000}$ **f** 625
g 5 **h** 60 **i** 13

5 **a** 9 **b** $\sqrt{2}$ **c** 4
d 2 **e** 3 **f** 10

6 **a** $\log_{\sqrt{2}} 2 = 2$ **b** $\log_{10} 0.001 = -3$ **c** $\log_{\frac{1}{2}} 2 = -1$
d $\log_{32} 1024 = 2$ **e** $\log_{10} N = x$ **f** $\log_5 5\sqrt{2} = \frac{3}{2}$
g $\log_5 1 = 0$ **h** $\log_{13} 13 = 1$

7 **a** $2^5 = 32$ **b** $3^4 = 81$ **c** $10^{-3} = 0.001$
d $3^{\frac{7}{2}} = 27\sqrt{3}$ **e** $b^x = y$ **f** $a^x = N$

8 **a** $\log_3 35$ **b** $\log_2 15$ **c** $\log_2 63$
d 2 **e** 2 **f** 0

9 **a** $\log_3 10$ **b** $\log_7 2$ **c** 1
d −1 **e** $\log_5 10$ **f** 1

10 **a** $\log_2 105$ **b** $\log_3 5$ **c** 0
d 0

11 **a** $2\alpha + \beta$ **b** $\beta + 2\gamma$ **c** $\alpha + \beta + \gamma + \delta$
d $7\alpha + \beta + 6\gamma$ **e** $\beta + 4\gamma$ **f** $\alpha + \beta + 2\gamma + \delta$
g $a\alpha + b\beta + c\gamma + d\delta$ **h** 1

12 **a** $xy = x + y$ **b** $x^2 = \frac{1}{10}y^3$ **c** $y = 125x^2$
d $\frac{1+y}{1-y} = 7^x$

13 $\log_2 V = \log_2 \frac{4\pi}{3} + 3\log_2 r$,
or $\log_2 V = 2 + \log_2 \pi - \log_2 3 + 3\log_2 r$

14 $x = \frac{1}{b}\log_{10}\frac{y}{a}$

15 $A = P \times 10^{bt}$

Exercise 14B

1 **a** $\frac{5}{2}$ **b** $\frac{5}{3}$

2 **a** 1.1292 **b** 0.6826 **c** 1.4650
d 2.3347 **e** 0.9622 **f** −0.7124

3 **a** 2.3219 **b** 2.6309 **c** 0.4307
d 1.7604 **e** −2.8074 **f** −1.4650

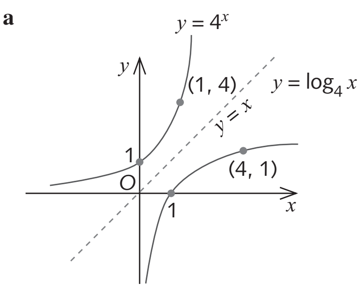
4 **a** −0.4225 **b** 0.1587 **c** −5.3847
d 1

5 **a** 1 **b** 1

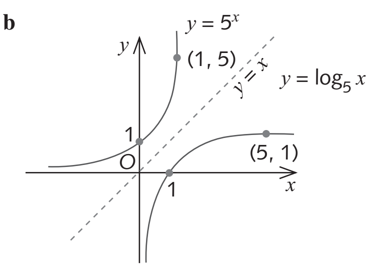
6 **a** $\frac{3}{2}$ **b** $\frac{7}{2}$ **c** $\frac{5}{3}$
d $\frac{3}{11}$ **e** 12 **f** $\frac{133}{60}$

Exercise 14C

1 **a**

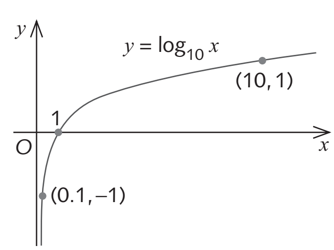


b



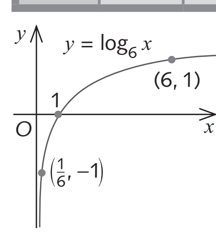
2 **a**

x	0.01	0.1	1	10	100
$\log_{10} x$	−2	−1	0	1	2

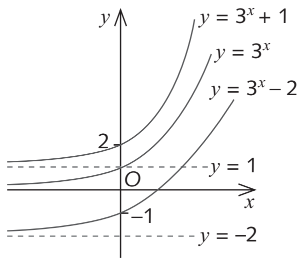


b

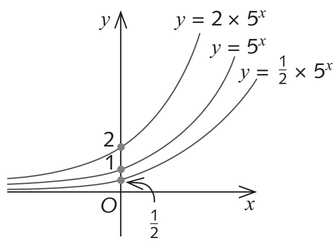
x	$\frac{1}{36}$	$\frac{1}{6}$	1	6	36
$\log_6 x$	−2	−1	0	1	2



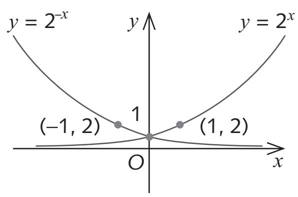
3 **a**



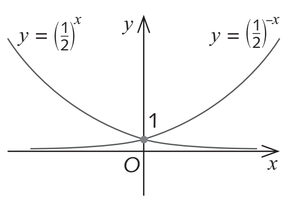
b



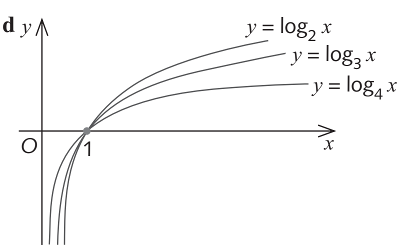
c



d



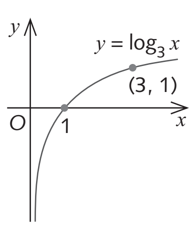
4 **a, d**



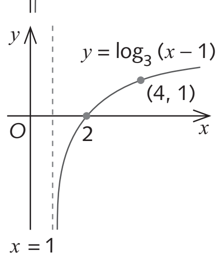
b They have the same x-intercept, no y-intercept and the same asymptote, the y-axis.

c $\log_2 x$ grows faster than $\log_3 x$ as x grows

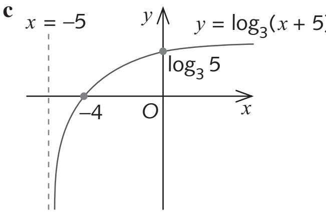
5 **a**



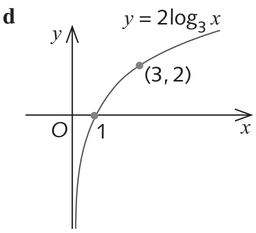
b



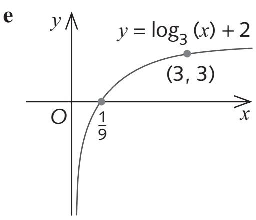
c



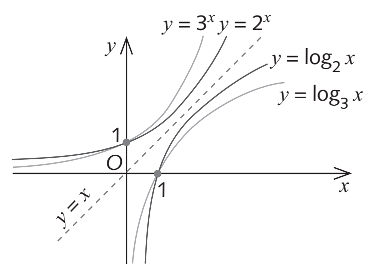
d



e



6



Exercise 14D

1 4 hours 19 minutes

2 **a** $M = 0.72 \times 5^t$ g

b 18 g

c 10 hours 20 minutes

d 39.94 days

e This is unrealistic, as there will not be enough food to keep the bacteria alive, so they will not multiply at the same rate for long.

3 **a** $4 \times 1.02^{t-1976}$ billion

b 28.98 billion **c** 2022

4 **a** $750 \times 1.04^{t-1970}$ million

b 1995 **c** 2014

d 2050 (this is clearly impossible, because if the population of China increases at the same rate, the world population will increase at a rate getting closer to 4%, not the assumed 2%)

5 **a** $M_0 = 10, k = 8.864 \times 10^{-4}$

b 339.59 years

6 **a** 5 **b** 14

7 6

8 **a** \$62 985.60 **b** \$65 816.20 **c** 55.5

d 38.3

Review exercise

1 **a** 4 **b** 3 **c** 9 **d** 0 **e** −3 **f** −6 **g** 4 **h** −3

2 **a** 4 **b** 2 **c** 2 **d** 8 **e** 5 **f** 5

3 **a** $\log_2 1024 = 10$ **b** $\log_{10} a = x$ **c** $\log_6 1 = 0$ **d** $\log_{11} 11 = 1$ **e** $\log_3 b = x$ **f** $\log_5 625 = 4$

4 **a** $3^4 = 81$ **b** $2^6 = 64$ **c** $10^{-2} = 0.01$ **d** $b^a = c$ **e** $a^c = b$

5 **a** $\log_2 55$ **b** $\log_2 35$ **c** $\log_6 77$ **d** $-\log_3 4$ **e** 1 **f** 1

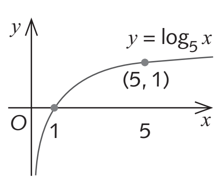
6 **a** $\log_2 140$ **b** 0 **c** −2 **d** −2

7 **a** 1.2323 **b** 1.2091 **c** 2.8928 **d** 3.2362 **e** 0.75 **f** −1.2619

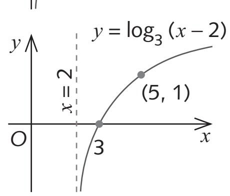
8 **a** 2.8074 **b** 3.9656 **c** 2.0704 **d** 3.0339 **e** −2.3219 **f** −2.4650

9 **a** $\frac{19}{2}$ **b** 27 **c** −1 **d** 10 000 **e** $-\frac{59}{2}$ **f** 10

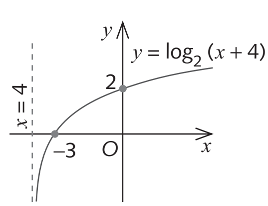
10 **a**



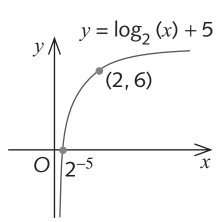
b



c



d



11 **a** $y = 10x$ **b** $y + 1 = 100x$

12 $\log_2 3 - 1$

13 1.1

14 **a** $\log_2 40$ **b** $\log_2\left(\frac{32}{5}\right)$

15 **a** 4 years **b** 7 years

Challenge exercise

1 From Pythagoras' theorem, $a^2 = c^2 - b^2 = (c-b)(c+b)$; now take log of both sides

2 1

4 **a** $\frac{9}{7}$ **b** 10 or $10^{\frac{-5}{2}}$ **c** 4 or $2^{-\frac{5}{4}}$ **d** 9 or 27

5 **a** $x = 3, y = 5$ **b** $x = 5, y = 2$ **c** $x = -\frac{63}{8}, y = -\frac{9}{4}$ **d** $x = 2, y = 3$

6 Change base: $\frac{\log_a x \log_a x}{\log_a b} = \log_a b$, so $\log_a x = \pm \log_a b$,

$x = b$ or $\frac{1}{b}$

7 $a = \frac{2b}{3}$

8 **b** Suppose $\log_{10} n = \frac{p}{q}$, then $10^{\frac{p}{q}} = n$, $10^p = n^q$, then for n to be an integer, it must contain an equal power of 2 and 5, and no other prime factors; that is, it is a power of 10.

10 Let $\alpha = \frac{\log_a x}{y-z} = \frac{\log_a y}{z-x} = \frac{\log_a z}{x-y}$

$\log_a x = \alpha y - \alpha z$

$\log_a y = \alpha z - \alpha x$

$\log_a z = \alpha x - \alpha y$

Add together

$\log_a x + \log_a y + \log_a z = 0$

$xyz = 1$

Now, $x = a^{\alpha y - \alpha z}$, $y = a^{\alpha z - \alpha x}$ and $z = a^{\alpha x - \alpha y}$

$$x^x y^y z^z = a^{\alpha yx - \alpha zx + \alpha zy - \alpha xy + \alpha xz - \alpha yz} = a^0 = 1$$

11 $x = 2a$ or $x = 5a$

Chapter 15 answers

Exercise 15A

1 $\frac{3}{13}$ **2** $\frac{7}{12}$ **3** $\frac{3}{5}$

4 $\frac{4}{5}$ **5** $\frac{8}{11}$

6 **a** $\xi = \{5, 6, 7, 8, 9\}$

b $P(5) = \frac{1}{10}, P(6) = \frac{1}{10}, P(7) = \frac{1}{4}, P(8) = \frac{7}{20}, P(9) = \frac{1}{5}$

c $\frac{4}{5}$

7 **a** $\xi = \left\{\begin{matrix}(1,\ 2),(1,\ 4),(1,\ 7),(2,\ 2),(2,\ 4),(2,\ 7),(3,\ 2),\\(3,\ 4),(3,\ 7)\end{matrix}\right\}$

b equally likely, $\frac{1}{9}$ **c** $\frac{2}{9}$

8 **a** $\xi = \{2, 3, 4, 5, 6, 7\}$

b $P(2) = \frac{1}{12}, P(3) = \frac{1}{6}, P(4) = \frac{1}{4}, P(5) = \frac{1}{4},$ $P(6) = \frac{1}{6}, P(7) = \frac{1}{12}$

c $P(\text{sum is less than } 5) = \frac{1}{2}$

9 **a** $\frac{1}{4}$ **b** $\frac{5}{18}$ **c** $\frac{1}{6}$

10 **a** $\xi = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$

b $\frac{5}{12}$ **c** $\frac{1}{6}$ **d** $\frac{5}{36}$

e $\frac{1}{3}$

11 **a** $\xi = \{(R_1, R_1), (R_1, R_2), (R_1, R_3), (R_1, W_1), (R_1, W_2),$ $(R_1, Y), (R_2, R_1), (R_2, R_2), (R_2, R_3), (R_2, W_1), (R_2, W_2),$ $(R_2, Y), (R_3, R_1), (R_3, R_2), (R_3, R_3), (R_3, W_1),$ $(R_3, W_2), (R_3, Y), (W_1, R_1), (W_1, R_2), (W_1, R_3),$ $(W_1, W_1), (W_1, W_2), (W_1, Y), (W_2, R_1), (W_2, R_2),$ $(W_2, R_3), (W_2, W_1), (W_2, W_2), (W_2, Y), (Y, R_1),$ $(Y, R_2), (Y, R_3), (Y, W_1), (Y, W_2), (Y, Y)\}$

b **i** $\frac{7}{18}$ **ii** $\frac{11}{18}$

12 **a** $\frac{1}{8}$ **b** $\frac{33}{400}$ **c** $\frac{29}{200}$

Exercise 15B

1 $\frac{3}{5}$ **2** $\frac{12}{13}$

3 $\frac{13}{15}$ **4** $\frac{19}{20}$

5 **a** $\frac{1}{10}$ **b** $\frac{1}{5}$ **c** $\frac{2}{5}$ **d** $\frac{4}{5}$

6 **a** $\frac{1}{4}$ **b** $\frac{4}{13}$ **c** $\frac{1}{13}$

d $\frac{25}{52}$ **e** 0 **f** $\frac{8}{13}$

7 **a** $\frac{1}{6}$ **b** $\frac{1}{2}$ **c** 0

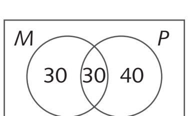
d $\frac{2}{3}$ **e** $\frac{1}{3}$ **f** $\frac{2}{3}$

8 **a** $\frac{29}{100}$ **b** $\frac{23}{100}$ **c** $\frac{49}{100}$

d $\frac{13}{20}$ **e** $\frac{11}{100}$ **f** $\frac{13}{20}$

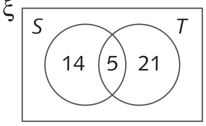
g $\frac{77}{100}$ **h** $\frac{1}{8}$ **i** $\frac{7}{8}$

9 **a** ξ



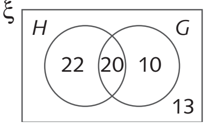
b **i** $\frac{3}{10}$ **ii** $\frac{2}{5}$ **iii** 0

10 ξ



a $\frac{1}{8}$ **b** $\frac{21}{40}$ **c** $\frac{7}{8}$ **d** $\frac{7}{20}$

11 ξ



a $\frac{4}{5}$ **b** $\frac{1}{5}$ **c** $\frac{22}{65}$ **d** $\frac{32}{65}$

12 **a** $\frac{1}{5}$ **b** $\frac{111}{1000}$ **c** $\frac{9}{100}$ **d** $\frac{11}{500}$

e $\frac{9}{500}$ **f** $\frac{1}{100}$ **g** $\frac{1}{500}$

13 **a** $\frac{6}{85}$ **b** $\frac{61}{85}$ **c** $\frac{16}{85}$

14 **a** $\frac{7}{26}$ **b** $\frac{4}{13}$ **c** $\frac{1}{2}$

15 $\frac{7}{15}$

Exercise 15C

1 **a** $\frac{17}{48}$ **b** $\frac{11}{24}$ **c** $\frac{13}{48}$

d $\frac{13}{31}$ **e** $\frac{4}{13}$ **f** $\frac{9}{22}$

2 **a** $\frac{1}{3}$ **b** $\frac{1}{2}$ **3** $\frac{1}{3}$

4 **a** $\frac{4}{13}$ **b** $\frac{1}{13}$

5 **a** $\frac{4}{11}$ **b** $\frac{7}{11}$ **c** $\frac{2}{5}$ **d** $\frac{1}{3}$

6 **a** $\frac{7}{20}$ **b** $\frac{53}{100}$ **c** $\frac{20}{77}$ **d** $\frac{1}{2}$

7 **a** $\frac{211}{228}$ **b** $\frac{77}{114}$ **c** $\frac{49}{57}$ **d** $\frac{9}{14}$ **e** $\frac{186}{211}$

8 **a**

	1	**2**	**3**	**4**	**5**	**6**
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

b

Outcome	0	1	2	3	4	5
Probability	$\frac{1}{6}$	$\frac{5}{18}$	$\frac{2}{9}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{18}$

c $\frac{2}{3}$ **d** $\frac{1}{3}$ **e** $\frac{4}{5}$ **f** $\frac{1}{5}$

9 a $\frac{1}{13}$ b $\frac{9}{19}$ c $\frac{3}{4}$ d $\frac{9}{19}$ e $\frac{2}{5}$

10 a $\frac{3}{14}$ b $\frac{9}{28}$ c $\frac{1}{3}$

11 a $\frac{1}{6}$ b $\frac{1}{2}$

12 0.3

Exercise 15D

1 No. Example: $P(M \cap Yes) \neq P(M) \times P(Yes)$ $[0.6 \times 0.55 = 0.33 \neq 0.25]$

2 No. Example: $P(< 30 \cap Yes) \neq P(< 30) \times P(Yes)$$[0.5625 \times 0.6875 \approx 0.387 \neq 0.4375]$

3 a 0.27 b $\frac{21}{38} \approx 0.55$

c $P(M) \neq P(M| > 10 \text{ km/h})$

d $P(Minor \cap Not\ speeding) \neq P(Minor) \times P(Not\ speeding)$ $\left[\frac{41}{200} \neq \frac{146}{200} \times \frac{43}{200}\right]$

4 a $\frac{3}{35}$ b $\frac{18}{35}$ c $\frac{12}{35}$ d $\frac{23}{35}$

5 $P(A) = P(A|B) = \frac{3}{8}$ or $P(A \cap B) = P(A) \times P(B) = \frac{3}{20}$ or $P(B) = P(B|A) = \frac{2}{5}$

6 a

		Die 2					
		1	2	3	4	5	6
Die 1	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

b $P(A) = \frac{6}{36} = \frac{1}{6}$, $P(B) = \frac{6}{36} = \frac{1}{6}$, $P(A \cap B) = \frac{1}{36}$. Events independent since $P(A \cap B) = P(A) \times P(B)$.

c $P(A \cap B) \neq 0$. Therefore, events A and B are not mutually exclusive.

Exercise 15E

1 a $\frac{1}{16}$ b $\frac{9}{16}$ c $\frac{3}{8}$

d $\frac{3}{16}$ e $\frac{3}{16}$ f $\frac{7}{16}$

2 a $\frac{1}{17}$ b $\frac{19}{34}$ c $\frac{13}{34}$

d $\frac{13}{68}$ e $\frac{13}{68}$ f $\frac{15}{34}$

3 a $\frac{10}{39}$ b $\frac{20}{39}$ c $\frac{14}{39}$ d $\frac{5}{39}$

4 a $\frac{10}{91}$ b $\frac{15}{91}$ c $\frac{30}{91}$ d $\frac{3}{91}$

5 a $\frac{5}{36}$ b $\frac{1}{18}$ c $\frac{5}{18}$ d $\frac{5}{18}$ e $\frac{17}{18}$

6 a $\frac{351}{1891}$ b $\frac{595}{1891}$ c $\frac{945}{1891}$

7 a $\frac{6}{25}$ b $\frac{12}{25}$ c $\frac{4}{25}$ d $\frac{13}{25}$

8 a $\frac{1}{16}$ b $\frac{1}{16}$ c $\frac{1}{16}$ d $\frac{1}{16}$ e $\frac{15}{16}$

9 a $\frac{1}{216}$ b $\frac{125}{216}$ c $\frac{1}{8}$

d $\frac{1}{8}$ e $\frac{5}{216}$ f $\frac{5}{216}$

10 $P(G) = P(G|C) = \frac{4}{9}$

Review exercise

1 a i $\frac{9}{25}$ ii $\frac{4}{25}$ iii $\frac{12}{25}$ iv $\frac{16}{25}$

b i $\frac{1}{10}$ ii $\frac{3}{5}$ iii $\frac{2}{5}$

2 a $\frac{1}{10}$ b $\frac{1}{2}$ c $\frac{1}{20}$

3 a

	1	2	3	4	5	6
2	3	4	5	6	7	8
4	5	6	7	8	9	10
6	7	8	9	10	11	12
8	9	10	11	12	13	14

b i $P(A) = \frac{1}{2}$ ii $P(B) = \frac{1}{8}$

c $A \cap C = \{9, 10, 11, 12\}$; $P(A \cap C) = \frac{5}{12}$

d No. $P(A) \times P(C) = \frac{1}{2} \times \frac{11}{12} = \frac{11}{24} \neq \frac{10}{24} = P(A \cap C)$

4 $\frac{1}{3}$ 5 $\frac{2}{5}$

6 a $\frac{13}{20}$ b $\frac{13}{20}$ c $\frac{2}{5}$

7 $\frac{1}{2}$

8 a $\frac{33}{100}$ b $\frac{7}{50}$ c $\frac{1}{25}$ d $\frac{29}{100}$ e $\frac{1}{10}$

Challenge exercise

1 a i 330 ii 150 iii 215 iv 215

b 2

c i 0.28 ii 0.288 iii 0.49

2 a $\frac{22}{119}$ b $\frac{11}{30}$ c $\frac{6}{11}$

3 a i 1296 ii 360 iii 810

b $\frac{671}{1296}$

4 $\frac{2}{3}$ 5 $\frac{10(n-3)}{n(n-1)}$ for $n > 4$

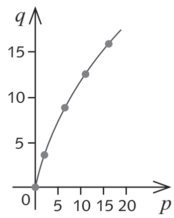
7 a 210 b $\frac{5}{21}$

Chapter 16 answers

Exercise 16A

1 **a** $a = \frac{b}{3}$ **b** $m = 5n$

2 **a**



b

$\sqrt{p}$	0	1	2	3	4
q	0	4	8	12	16

$\frac{q}{\sqrt{p}} = 4$, for each pair (p, q)

c $q = 4\sqrt{p}$

3 **a** $m = 3n^2$
i 75 **ii** 3

b $a = 10\sqrt{b}$
i 40 **ii** 6.25

4 **a** $R = 4s$ **b** $P = 0.12T$
c $a = 4\sqrt{b}$ **d** $V = 8r^3$

5 **a** $y = \frac{x}{4}$

x	2	8	12	18
y	$\frac{1}{2}$	2	3	$\frac{9}{2}$

b $y = 8x$

x	2	3	6	15
y	16	24	48	120

6 130 km **7** 210.5 m^2

8 1852 kW **9** 86.4 m

10 **a** The surface area is multiplied by 9.
b The radius is multiplied by $\sqrt{3}$.

11 **a** m is multiplied by 32
b m is multiplied by $\frac{1}{32}$
c n is tripled
d n is divided by 4

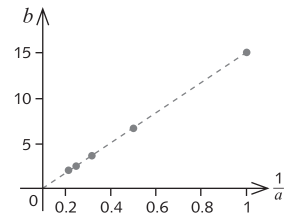
12 **a** a is increased by 11.8%
b a is decreased by approximately 4.08%

13 **a** p is increased by 6.27%
b p is decreased by 1.70%
c q is increased by 33.1%
d q is decreased by 27.1%

Exercise 16B

1 a

$\frac{1}{a}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
b	15	7.5	5	3.75	3



b $b = \frac{15}{a}$

2 a

x	1	2	5	10
y	100	25	4	1
x^2y	100	100	100	100

b $y = \frac{100}{x^2}$

3 **a** $v \propto \frac{1}{t}$ **b** $m \propto \frac{1}{\sqrt{n}}$ **c** $s \propto \frac{1}{t^3}$

4 $xy = 6$ or $y = \frac{6}{x}$

a 4 **b** 9

5 $ab^2 = 24$

a $\frac{8}{3}$ **b** $2\sqrt{2}$

6 $p^2q = 100$

a $\frac{10}{3}$ **b** $\frac{25}{4}$

7 $k = 12$ so $y = \frac{12}{x}$

x	1	2	3	4	$\frac{1}{2}$
y	12	6	4	3	24

8 $k = 32$ so $y = \frac{32}{x^2}$

x	2	4	8	10	16
y	8	2	0.5	0.32	0.125

9 55 km/h **10** 54

11 **a** $1\frac{1}{3}$ units **b** 2 m

12 **a** m is divided by 4 **b** m is multiplied by 4
c n is divided by 4 **d** n is tripled

13 **a** a is decreased by approximately 13.04%
b a is increased by approximately 13.64%

14 **a** p is decreased by approximately 24.87%
b p is increased by approximately 37.17%
c q is decreased by approximately 5.90%
d q is increased by approximately 7.72%

15 **a** **i** The height is divided by 4.
ii The radius is divided by 3.
b $k_1 = 36$

Exercise 16C

1 $a = 12bc$

a	12	24	24	48	72
b	1	1	2	2	2
c	1	2	1	2	3

2 $r = \dfrac{24s}{t}$

r	24	12	24	48	4
s	1	1	2	2	2
t	1	2	2	1	12

3 $y = 5$

4 **a** $y = \dfrac{kx^2}{\sqrt{z}}$ **b** $y = \dfrac{27}{4}$

5 **a** $a = kbc^3$ **b** $b = 32$

6 **a** $H = ki^2Rt$ **b** 2916 units

7 90 N

8 $6\sqrt{30}$ cm ≈ 32.86 cm

9 **a** $r = 12$ **b** $n = 24$

10 **a** y is doubled **b** y is multiplied by 13.5

11 **a** y is multiplied by approximately 1.154
b y is multiplied by approximately 1.562

12 **a** F is divided by 4 **b** F is multiplied by 12

13 1.65 m/s^2

Review exercise

1 **a** x is directly proportional to y
b p is directly proportional to the square of n
c a is directly proportional to the square root of b
d p is directly proportional to the cube of q

2 **a** **i** $p = 48$ **ii** $q = \dfrac{81}{8}$
b $a = \dfrac{5b^2}{4}$
i $a = \dfrac{125}{4}$ **ii** $a = 180$

3 **a** $k = 12$, $y = 12x$

x	0	1	2	3
y	0	12	24	36

b $k = 1.5$, $y = 1.5x$

x	2	8	12	18
y	3	12	18	27

4 **a** y is multiplied by 8
b y is multiplied by 27
c y is divided by 64

5 **a** m is multiplied by $\sqrt{2}$ (increase by 41% since $\sqrt{2} \approx 1.41$)
b m is halved

6 **a** a is increased by 10.25%
b a is decreased by 15.36%

7 **a** $y = \dfrac{80}{3}$ **b** $x = 60$

8 **a** $a = 2$ **b** $b = \dfrac{4\sqrt{2}}{3}$

9 $z = 27\sqrt{2}$ 10 $x = \dfrac{128}{3}$

Challenge exercise

1 **a** $R = \dfrac{0.002\,24\,L}{D^2}$
b $5.736 \approx 5.4$ ohm
c resistance is halved
d resistance is increased by 21.9%

2 $a = kc$ and $b = Kc$
$a + b = (k + K)c$ so $(a + b) \propto c$
$a - b = (k - K)c$ so $(a - b) \propto c$
$\sqrt{ab} = \sqrt{kc \times Kc} = \sqrt{kK}c$ so $\sqrt{ab} \propto c$

3 $y = \dfrac{150x}{19} + \dfrac{1080}{19x^2}$ 4 $y = \dfrac{12}{5}$ 5 $\dfrac{2abC}{(a+b)^2}$

6
$$\begin{aligned} a + b &= k(a - b) \\ a^2 + 2ab + b^2 &= k^2(a^2 - 2ab + b^2) \\ 2abk^2 + 2ab &= k^2a^2 - a^2 + k^2b^2 - b^2 \\ 2ab(k^2 + 1) &= (k^2 - 1)(a^2 + b^2) \\ a^2 + b^2 &= \frac{k^2 + 1}{k^2 - 1} \times 2ab \\ a^2 + b^2 &\propto ab \end{aligned}$$

Chapter 17 answers

Exercise 17A

1 **a** yes **b** no **c** yes
d yes **e** yes **f** yes
g yes **h** yes **i** no
j yes **k** yes **l** no

2 **a** 3, 1, −6 **b** 4, 5, 0 **c** 1, −4, 7
d 0, 15, 15 **e** 3, 7, 5 **f** 2, 3, 8
g 3, −14, 0 **h** 5, $\frac{1}{5}$, 0 **i** 6, $-3\frac{1}{3}$

3 **a** monic **b** non-monic **c** monic
d monic **e** non-monic **f** monic

4 **a** −6 **b** −6 **c** 0
d −12 **e** −6 **f** −30
g $a^3 - a - 6$ **h** $8a^3 - 2a - 6$ **i** $-a^3 + a - 6$

5 **a** 0, 60, −90 900, 0 **b** 39, 0, 11 712, 32
c −1, −55, 295 001, 1 **d** −27, 0, 0, −20

6 **a** $x^2 - 10x + 25,\ x^2, 2, 25$
b $x^2 + 5x - 50,\ x^2,\ 2, -50$
c $-3x^7 + x^3,\ -3x^7, 7, 0$
d $x^3 + 12x^2 + 36x,\ x^3, 3, 0$
e $3x^7 + 6x^5 + 3x^3,\ 3x^7, 7, 0$
f $-6x, -6x, 1, 0$
g $x^3 + 9x^2 + 26x + 24,\ x^3,\ 3,\ 24$
h $3x^2 + 12x + 14,\ 3x^2,\ 2,\ 14$

7 **a** $a = 6$
b $b = -1$
c $a = -15, d = 8$

8 **a** $P(x) = x^2 - 5x + 1$ **b** $Q(x) = 2x^4 + 10x$
c $R(x) = 7x^4 + 7x^3 + 7x^2 + 7x + 7$

Exercise 17B

1 **a** $3x^3 - 3x^2 - x + 5, -x^3 + 3x^2 + 7x + 5$
b $x + 3, 4x^3 - 6x^2 - 9x + 7$
c $8x^2 - 6x, 12$
d $x^4 + x^3 - 2x^2 + 2x - 2, x^4 - x^3$
e $0, 10x^3 + 4x^2 - 2x - 10$

2 **a** $12x^5 + 4x^3 - 12x^2 + 20$
b $-16x^5 + 10x^3 - 7x^2 + 27$
c $-5x^5 + 5x^3 - 5x^2 + 15$
d 0

3 **a** $5x^6 - 2x^5 + 7x^4$ **b** $x^6 - 1$
c $x^6 - 1$ **d** $x^5 + 3x^4 - x^3 - 3x^2$
e $x^4 + x^2 + 1$

4 **a** equals the sum of the degrees of $P(x)$ and $Q(x)$
b the product of the constant terms of $P(x)$ and $Q(x)$

5 **a** $x^2 - 14x + 49$ **b** $x^4 - 6x^2 + 9$
c $x^6 - 14x^4 + 49x^2$ **d** $9x^{10} + 30x^8 + 25x^6$
e $x^4 + 2x^3 + 3x^2 + 2x + 1$
f $x^8 + 2x^6 + 3x^4 + 2x^2 + 1$

6 **a** is twice the degree of $P(x)$
b equal to the square of the constant term of $P(x)$
c the leading coefficient of $P(x)$ is 1 or −1

7 **a** $-5x^4 + 5x^3 - 7x - 3$
b $6x^4 - x^3 - 15x^2 - 2x + 3$
c $-4x^4 - 3x^3 + 26x^2 + 28x + 2$

8 **a** $x^4 - 5x^2 + 4$
b $x^4 + 10x^3 + 35x^2 + 50x + 24$

Exercise 17C

1 **a** $68 = 11 \times 6 + 2$ **b** $1454 = 12 \times 121 + 2$
c $2765 = 21 \times 131 + 14$

2 **a** $x^2 + 6x + 1 = (x + 2)(x + 4) - 7$
b $x^3 - 5x^2 - 12x + 30 = (x + 5)(x^2 - 10x + 38) - 160$
c $5x^3 - 7x^2 - 6 = (x - 3)(5x^2 + 8x + 24) + 66$
d $x^4 + 3x^2 - 3x = (x + 2)(x^3 - 2x^2 + 7x - 17) + 34$
e $4x^3 - 4x^2 + 1 = (2x + 1)\left(2x^2 - 3x + \frac{3}{2}\right) - \frac{1}{2}$
f $x^4 + 3x^3 - 3x^2 - 4x + 1 = (x + 1)(x^3 + 2x^2 - 5x + 1)$

3 **a** $R(x) = -5x + 1, Q(x) = x^2 - x + 3$
b $Q(x) = x^2 - 2x + 5, R(x) = 15 - 8x$

4 **a** $x^3 + 5x^2 - x + 2 = (x^2 + x + 1)(x + 4) - 6x - 2$
b $x^3 - 4x^2 - 3x + 7 = (x^2 - 2x + 3)(x - 2) - 10x + 13$
c $x^4 + 5x^2 + 3 = (x^2 - 3x - 3)(x^2 + 3x + 17) + 60x + 54$
d $x^5 - 3x^4 - 9x^2 + 9 = (x^3 - x^2 + x - 1)(x^2 - 2x - 3) - 9x^2 + x + 6$

5 **a** 0 **b** 0 or 1
c 3 or higher **d** 4

6 **a** $P(x) = (x + 5)(x + 3)(x - 7)$
b $P(x) = (x + 2)^2(x + 3)^2$

7 **a** **i** $x^4 - 3x^3 - 5x^2 + x - 7$
$= (x + 5)(x^3 - 8x^2 + 35x - 174) + 863$
ii −870

b **i** $x^4 - 3x^3 - 5x^2 + x - 7$
$= (x^2 + 5)(x^2 - 3x - 10) + 16x + 43$
ii $a = -15, b = -50$

Exercise 17D

1 **a** 38, not a factor **b** 0, is a factor
c −274, not a factor **d** 0, is a factor

2 **a** 0, is a factor **b** −6, not a factor
c 38, not a factor **d** 20, not a factor

3 **a** 0, is a factor **b** −14, not a factor
c 1, not a factor **d** −4, not a factor
e 6, not a factor **f** 0, is a factor

4 **a** $x + 1, x + 2, x - 2$ **b** $x + 1, x - 1, x + 4$

5 **a** $k = 4$ **b** $m = 4$

6 **a** $p = 28$ **b** $b = -23$

7 **a** $a = -14, b = -22$ **b** $a = 9, b = 5$
c $a = 17, b = 21, c = 0$

Exercise 17E

1 **a** $P(x) = (x - 12)(x + 9)$ **b** $x^2 - 3x - 108$

2 $P(x) = (x - 1)(x + 1)(x - 2)(x + 2)$

3 **b** $P(x) = (x-1)(x^2-5x+6)$
c $P(x) = (x-1)(x-2)(x-3)$

4 **a** $(x+1)(x+2)(x+3)$ **b** $(x+1)(x-1)(x-7)$
c $(x+1)(x+5)(x-3)$ **d** $(x-5)(x+3)^2$
e $(x-1)^2(x+3)$ **f** $(x-1)(x+2)^2$

5 **a** $(x+1)(x-1)(x-2)(x-3)$ **b** $(x+1)^2(x+5)^2$

6 **a** $3(x-2)(x-1)(x+5)$ **b** $5(x-2)(x-1)(x+2)$
c $x(x-2)(x+1)(x+2)$ **d** $x(x-1)^2(x+3)^2$

7 **a** $(x-1)(x^2+3x+5)$ **b** $(x+3)(x^2+x+1)$
c $x^2(x-2)(x^2+6x-3)$ **d** $(x-2)(x+3)(x^2+3x+1)$

8 If $P(\alpha) = 0$, then, $a_n\alpha^n + a_{n-1}\alpha^{n-1} + a_{n-2}\alpha^{n-2} + \ldots + a_1\alpha + a_0 = 0$
Therefore, $a_0 = -\alpha(a_n\alpha^{n-1} + a_{n-1}\alpha^{n-2} + a_{n-2}\alpha^{n-3} + \ldots + a_1)$ and α divides a_0

Exercise 17F

1 **a** $-7, 5, -6$ **b** $3, -1$
c $2, 4, 6, 8$ **d** $0, 7, -8$

2 **a** $3, -3+\sqrt{17}, -3-\sqrt{17}$
b $-5, \frac{1}{3}\left(1+\sqrt{7}\right), \frac{1}{3}\left(1-\sqrt{7}\right)$
c $0, 7, -6$
d $2, 5, \sqrt{10}, -\sqrt{10}$

3 **a** $-2, -1, 5$ **b** $-2, 2, 3$
c $-7, 0, 1, 3$ **d** $-2, -1, 1, 7$

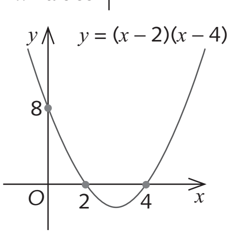
4 **a** $1, 5$ **b** $-3, 2$ **c** $1, 5$
d $-3, 2$ **e** $-6, -1, 0$ **f** $-1, 1, 2$

5 **a** $(x-3)(x^2-4x-1)$; $3, 2-\sqrt{5}, 2+\sqrt{5}$
b $(x+1)(x^2+3x+7)$; -1
c $x(x-2)(x-3)(x^2+3x-1)$; $0, 2, 3, \frac{1}{2}\left(-3-\sqrt{13}\right), \frac{1}{2}\left(-3+\sqrt{13}\right)$
d $(x+1)(x-1)(x-2)(x^2+2x+2)$; $-1, 1, 2$

Exercise 17G

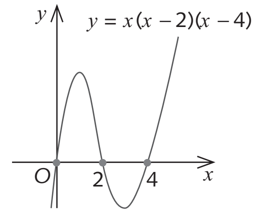
1 **a** $x = 2, x = 4$

Sign of y	+ 0 − 0 +
x values	2 4



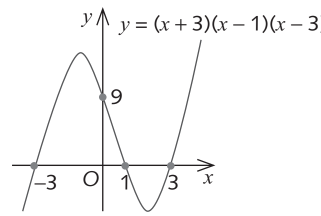
b $x = 0, x = 2, x = 4$

Sign of y	− + − +
x values	0 2 4



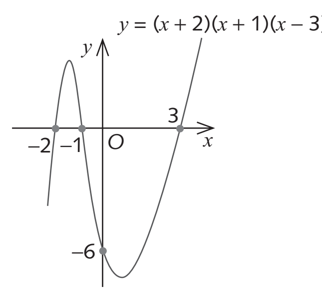
c $x = -3, x = 1, x = 3$

Sign of y	− + − +
x values	−3 1 3



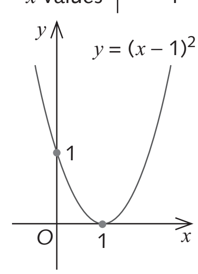
d $x = -2, x = -1, x = 3$

Sign of y	− + − +
x values	−2 −1 3



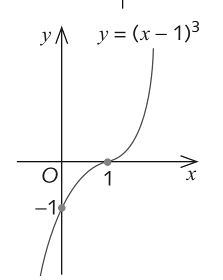
2 **a** $x = 1$

Sign of y	+ +
x values	1



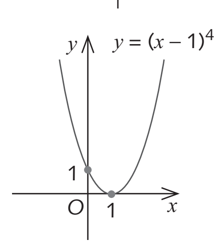
b $x = 1$

Sign of y	− +
x values	1

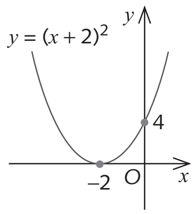


c $x = 1$

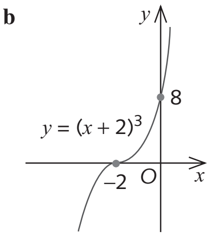
Sign of y	+ +
x values	1



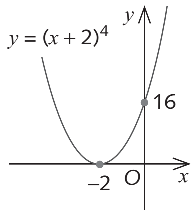
3 **a**



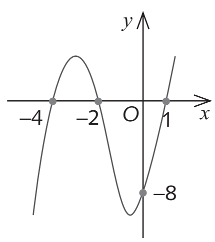
b



c



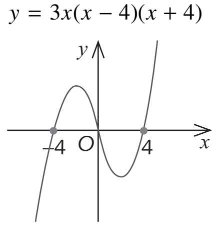
4 **a** $y = (x + 2)(x - 1)(x + 4)$



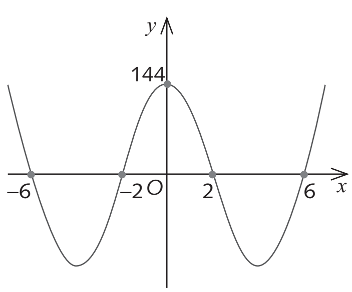
b $x > 1, -4 < x < -2$

c $x < -4, -2 < x < 1$

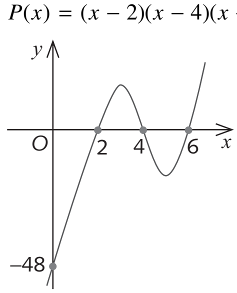
5 **a** $y = 3x(x - 4)(x + 4)$



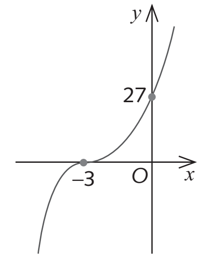
b $y = (x - 6)(x + 6)(x - 2)(x + 2)$



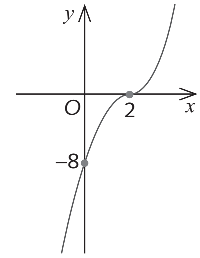
6 **a** $P(x) = (x - 2)(x - 4)(x - 6)$



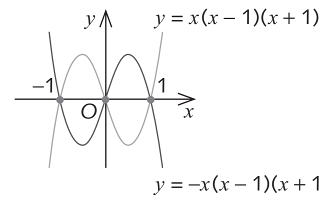
b $P(x) = (x + 3)^3$



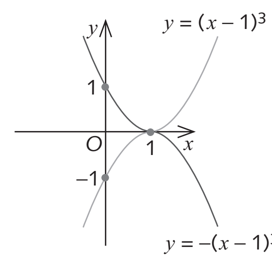
c $P(x) = (x - 2)^3$



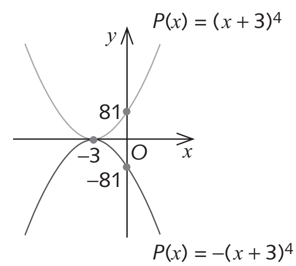
7 **a – b**



8 **a – b**



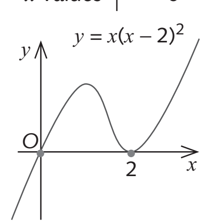
9 **a – b**



Exercise 17H

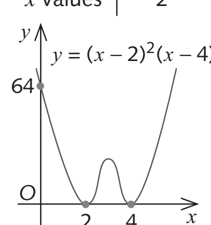
1 **a** $x = 0, x = 2$

Sign of y	–	0	+	0	+
x values		0		2	



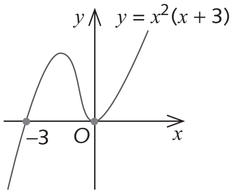
b $x = 2, x = 4$

Sign of y	+	0	+	0	+
x values		2		4	



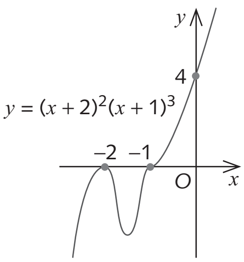
c $x = 0, x = -3$

Sign of y	$-$	0	$+$	0	$+$
x values		-3		0	



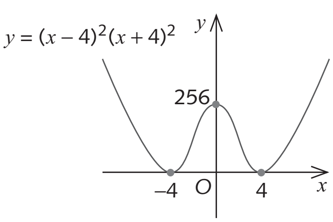
d $x = -2, x = -1$

Sign of y	$-$	0	$-$	0	$+$
x values		-2		-1	



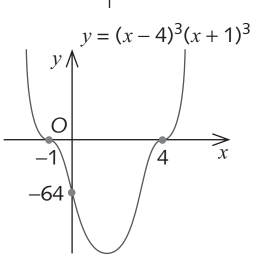
2 **a** $x = 4, x = -4$

Sign of y	$+$	0	$+$	0	$+$
x values		-4		4	



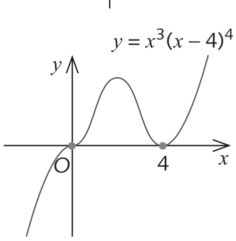
b $x = -1, x = 4$

Sign of y	$+$	0	$-$	0	$+$
x values		-1		4	



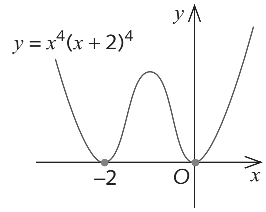
c $x = 0, x = 4$

Sign of y	$-$		$+$		$+$
x values		0		4	

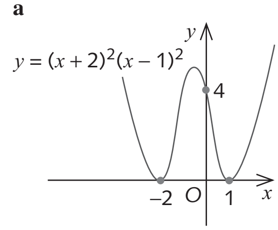


d $x = 0, x = -2$

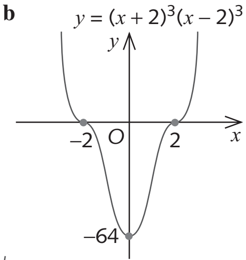
Sign of y	$+$		$+$		$+$
x values		-2		0	



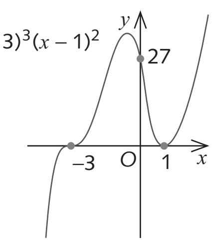
3 **a**



b

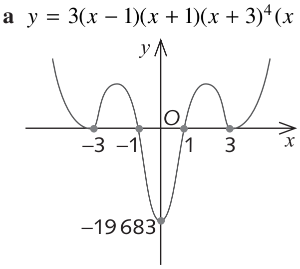


4 **a** $y = (x + 3)^3(x - 1)^2$

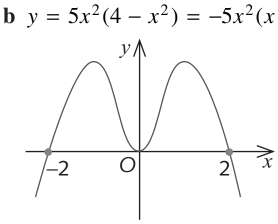


b $x > 1, -3 < x < 1$ **c** $x < -3$

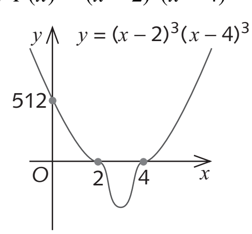
5 **a** $y = 3(x - 1)(x + 1)(x + 3)^4(x - 3)^4$



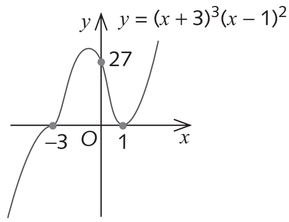
b $y = 5x^2(4 - x^2) = -5x^2(x - 2)(x + 2)$



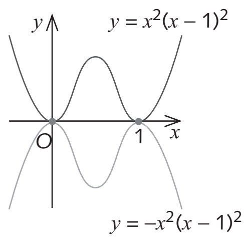
6 **a** $P(x) = (x - 2)^3(x - 4)^3$



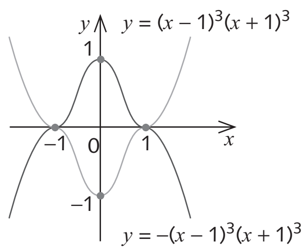
b $P(x) = (x + 3)^3(x - 1)^2$



7 **a – b**



8 **a – b**



Review exercise

1 **a** polynomial **b** polynomial
c polynomial **d** not a polynomial
e not a polynomial **f** polynomial

2 **a** 2 **b** 3 **c** 4 **d** 2 **e** 3

3 **a** 2 **b** –4 **c** 11
d –13 **e** $a^3 + 2a - 1$ **f** $8a^3 + 4a - 1$

4 $a = -3$ **5** $a = -1$

6 **a** $P(x) + Q(x) = x^2 + 3x + 6$; $P(x) - Q(x)$
$= -x^2 - x$; $P(x)Q(x)$
$= x^3 + 5x^2 + 9x + 9$
b $P(x) + Q(x) = 2x^2 + 4$; $P(x) - Q(x)$
$= -2$; $P(x)Q(x) = x^4 + 4x^2 + 3$
c $P(x) + Q(x) = x^2 + 2$; $P(x) - Q(x)$
$= 4x - x^2$; $P(x)Q(x) = 2x^3 - 3x^2 + 1$

7 **a** $6x^2 + 13x - 2$, remainder 2
b $2x^2 - 5x + 10$, remainder –7
c $x^2 - 4x - 6$, remainder –17
d $x^2 + 3$, remainder 7

8 **a** $(x - 3)(x - 1)(x + 2)$ **b** $(x + 4)(x + 1)(2x - 3)$
c $(x - 1)(x + 4)(2x - 3)$ **d** $(2x + 3)(x - 2)(3x - 1)$

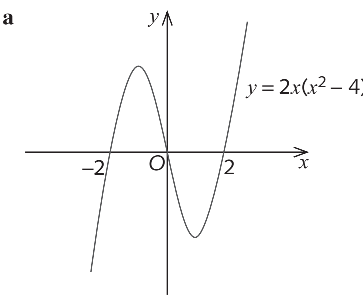
9 $a = 4, b = -1$ **10** $d = 4$

11 $a = -2, b = -19$

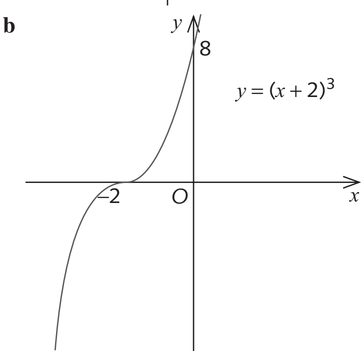
12 **a** $2w^3$ **b** $a = -2, b = 4$

13 **a** x-intercepts: –1, 0, 2 y-intercept: 0
b x-intercepts: –2, 1, 3 y-intercept: 6
c x-intercepts: –2, –1, 1 y-intercept: –2
d x-intercepts: $-1, -\frac{2}{3}, 3$ y-intercept: –6
e x-intercepts:, $-4, -\frac{2}{5}, 2$ y-intercept: –16
f x-intercepts: $-\frac{1}{2}, \frac{1}{3}, 1$ y-intercept: 1

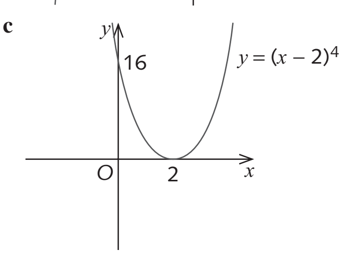
14 **a**



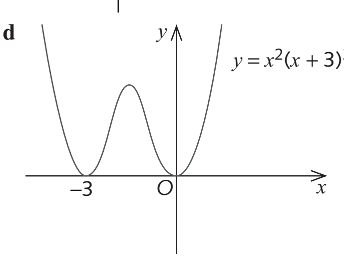
b



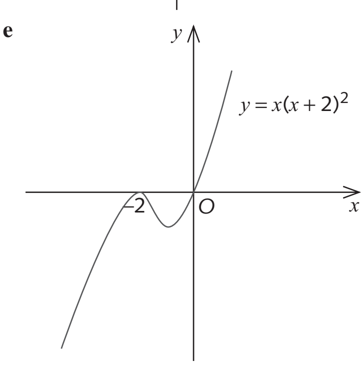
c



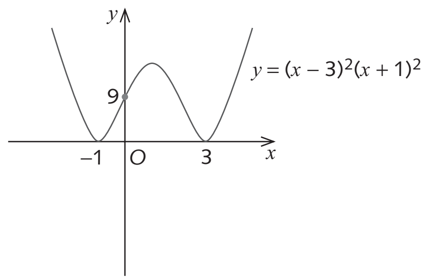
d



e



f



Challenge exercise

1 $a = -3$

2 $(x^2 - 2x + 2)(x^2 + 2x + 2)$

3 $b = 14,\ a = -11$

4 **b** $x^3 - 6x^2 + 11x - 6$

5 $P(x) = 6x^5 - 15x^4 + 10x^3$

6 $a = 1$ and $b = 2$

7 $4x$

8 **c** $(x^4 + 2x^2 + 2x)^2 + (x^3 - 2x^2 - 4)^2$

9 $\dfrac{(A - B)}{a - b}x - \dfrac{(Ba - Ab)}{a - b}$

10 $x = \dfrac{1 + \sqrt{5}}{2},\ y = \dfrac{1 - \sqrt{5}}{2};\ x = \dfrac{1 - \sqrt{5}}{2},\ y = \dfrac{1 + \sqrt{5}}{2};$
$x = 1,\ y = 1;\ x = -2,\ y = -2$

Chapter 18 answers

Exercise 18A

1 **a** range = 17, IQR = 9 **b** range = 9, IQR = 3
c range = 9, IQR = 2.5 **d** range = 17, IQR = 9

2 **a** lower quartile = 27, median = 32, upper quartile = 38.5, IQR = 11.5
b lower quartile = 64, median = 76, upper quartile = 82, IQR = 18

3 mean = 5.7, mode = 10, median = 5.5, interquartile range = 6

4

	number of data items	lower quartile position	median position	upper quartile position
a	100	25.5	50.5	75.5
b	101	25.5	51	76.5

5 **a** 38 cm **b** 161.5 cm **c** 22.5 cm

6 IQR = 1.4 cm

7 **a** median = \$149, lower quartile = \$143, upper quartile = \$153.50
b IQR = \$10.50

8 3 5 7 1 0 1 2 1 2 1 3 and 3 5 6 1 0 1 1 1 2 1 3 (others are possible)

9 No. 1 1 1 2 2 3 1 8, mean = 4, lower quartile = 1 and upper quartile = 3;
1 1 3 1 3 1 4 1 4 1 6, mean = 12, lower quartile = 13 and upper quartile = 14

10 **a** 35 **b** 13.7

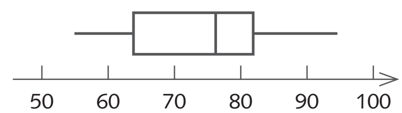
Exercise 18B

1 \$49 and \$10.50, respectively

2 **a** \$40 000 **b** \$120 000 **c** \$100 000 **d** \$60 000

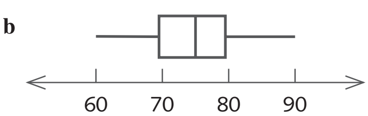
3 **a** 40 **b** 65 **c** 55 **d** 20

4



5 **a** median = 75, upper quartile = 78.5, lower quartile = 69.5, IQR = 9

b



6 **a** 50% **b** 25% **c** 50% **d** 25%

7 no

8 It depends on the spread of the values between the lower quartile and the median compared with those between the median and the upper quartile.

9 **a** B **b** B **c** B **d** B

10 **a** A **b** B **c** B **d** B
e Class A. Lowest mark is higher, and lower quartile, median and upper quartile are all higher. Only the maximum mark is lower.

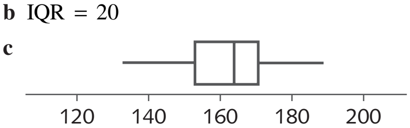
11 **a**

channel	minimum	lower quartile	median	upper quartile	maximum
A	7	10.5	15	16.3	23
B	7.8	11	14.6	16.7	25
C	8	10	13	14.5	16

b channel A **c** channel B, channel A, channel C
d channel A, channel B, channel C
e If the criterion is the highest rating for the lowest rating show, then Channel C is the winner.

Exercise 18C

1 **a** $Q_1 = 151.5,\ Q_3 = 171.5$, median = 164
b IQR = 20
c

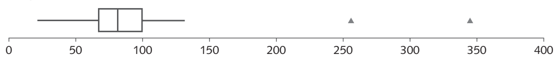


There are no outliers

2 **a** $Q_1 = 67\,000, Q_3 = 100\,000$, median $= 81500$

b IQR $= 33\,000$

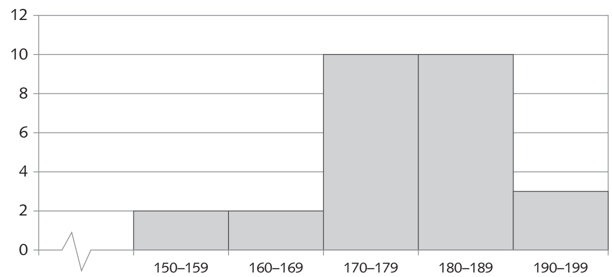
c



3 **a** **iii**, symmetric **b** **i**, negative skew

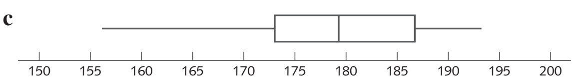
c **ii**, positive skew

4 **a**



b $Q_1 = 173$, median $= 179$, $Q_3 = 187$, IQR=14

c



d symmetric

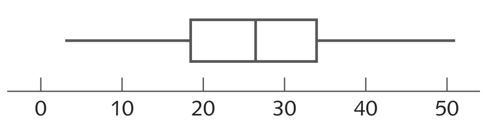
5 **a** and **b** are outliers; **c** is not an outlier

6 **a**

0	35
1	147
2	0 4 4 4 6 7 8 9
3	1 3 5 6
4	0 2
5	1

5 |1 mean 51

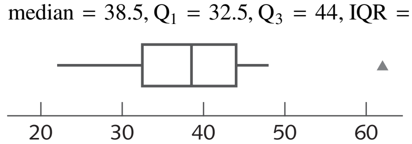
b median $= 26.5$, $Q_1 = 18.5$, $Q_3 = 34$, IQR $= 15.5$



c The distribution is symmetric. There are no outliers.

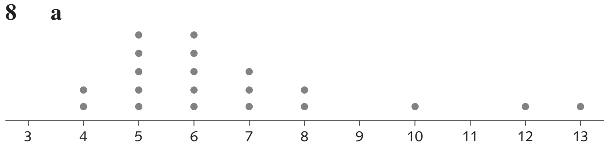
7 **a** median $= 38.5$, $Q_1 = 32.5$, $Q_3 = 44$, IQR $= 11.5$

b

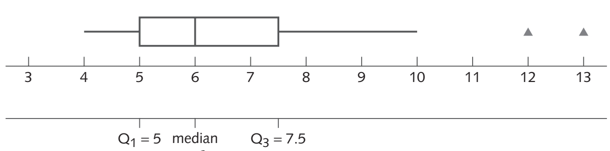


c 62 is an outlier

8 **a**



b



c The distribution has a slight positive skew.

minimum $= 4$, $Q_1 = 5$, median $= 6$, $Q_3 = 7.5$, maximum $= 13$, 12 kg and 13 kg are outliers

Exercise 18D

1 15.54

2 City A : 30.43, City B : 31.86; City B has the greater mean daily maximum temperature.

3 60 kg **4** 41.2

5 **a** **i** $\bar{x} = 5, \sigma = 3.16$ **ii** $\bar{x} = 5, \sigma = 1.34$

b Both data sets have the same mean but data set **i** is more spread out.

6

x_i	f_i	$f_i x_i$	$(x_i - \bar{x})$	$f_i(x_i - \bar{x})^2$
1	2	2	–2	8
2	7	14	–1	7
3	6	18	0	0
4	1	4	1	1
5	2	10	2	8
6	2	12	3	18
	Total = 60	Total = 60		Total = 60

$\sigma = 1.45, \bar{x} = 3$

7 **a** $\bar{x} = 9.13, \sigma = 3.61$ **b** $\bar{x} = 14, \sigma = 3.46$

8 **a** $\bar{x} = 30.55$ **b** 10 **c** 10.22

9 **a** 6.15 **b** 10 **c** 2.46

10 **a** $\bar{x} = \dfrac{a+b+c}{3}$ and the sum of the deviations

$$= a - \bar{x} + b - \bar{x} + c - \bar{x}$$
$$= (a+b+c) - 3\bar{x}$$
$$= (a+b+c) - 3 \times \frac{a+b+c}{3}$$
$$= 0$$

b $\bar{x} = \dfrac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$ and

the sum of the deviations $= (x_1 - \bar{x}) + (x_2 - \bar{x}) + (x_1 - \bar{x}) + \ldots + (x_n - \bar{x})$

$$= (x_1 + x_2 + x_3 + \ldots + x_n) - n\bar{x}$$
$$= (x_1 + x_2 + x_3 + \ldots + x_n) - n \times \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$$
$$= 0$$

Exercise 18E

1 **a** $\bar{x} = 9, \sigma = 5.45$ **b** $\bar{x} = 9, \sigma = 1.31$

c $\bar{x} = 9, \sigma = 5.45$

All three data sets have the same mean. Data set **b** is less spread out than the other two, with a standard deviation of 1.31. Data sets **a** and **c** have the same mean and standard deviation. (The size of σ for **a** would be much smaller if the outlier 22 is omitted. It drops from 5.45 to 1.34. This does not happen when any one value is omitted from **c**.)

2 **a** **i** between 32.5 and 37.5

ii between 30 and 40

b **i** between 35 and 45

ii between 30 and 50

c **i** between 27 and 43 **ii** between 19 and 51

3 **a** Mathematics : $\bar{x} = 16.13$ $\sigma = 2.63$;

English : $\bar{x} = 15.87$ $\sigma = 2.83$

b The mathematics result is about 0.05 standard deviations below the mean, whereas the English result is 0.66 standard deviations below the mean. The mathematics mark is better.

4 **a** David's mark for English is one standard deviation below the mean.

David's mathematics mark is more than one standard deviation below the mean.

His English mark is better.

b Akira's mark for English is two standard deviations above the mean.

Akira's mathematics mark is less than two standard deviations above the mean.

Her English mark is better.

c Katherine's mark for English is one standard deviation below the mean.

Katherine's mathematics mark is half a standard deviation below the mean.

Her mathematics mark is better.

d Daniel's mark for English is more than one standard deviation above the mean.

Daniel's mathematics mark is one standard deviation above the mean.

His English mark is better.

5 **a** **i** $\bar{x} = 6.5, \sigma = 1.71$ **ii** $\bar{x} = 6.5, \sigma = 1.38$

iii $\bar{x} = 6.5, \sigma = 1.98$

b **i** $\bar{x} = 11.5, \sigma = 1.71$ **ii** $\bar{x} = 11.5, \sigma = 1.38$

iii $\bar{x} = 11.5, \sigma = 1.98$

c **i** $\bar{x} = 13, \sigma = 3.42$ **ii** $\bar{x} = 13, \sigma = 2.77$

iii $\bar{x} = 13.0, \sigma = 3.96$

6 **a** 1 1 1 1 1 5 5 5 5 5 **b** 1 1 1 1 1 9 9 9 9 9

c 1 1 1 1 5 5 9 9 9 9

7 **a** 1 1 1 1 1 1 1 1 1 1 **b** 1 1 1 1 1 1 1 1 1 1

c 5 5 5 5 5 5 5 5 5 5

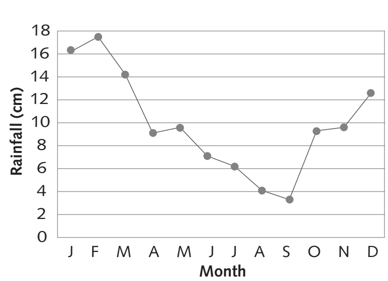
8 **a** average annual salary increases by about \$5200

b the standard deviation remains unchanged

c an increase of \$2900

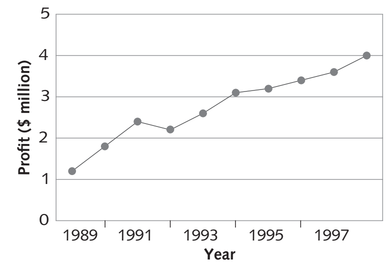
Exercise 18F

1 **a**

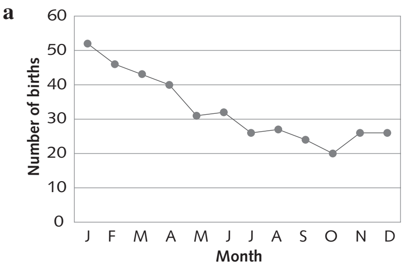


b The rainfall is high in January and February and decreases to reach a minimum in September. The rainfall from September to December increases significantly.

2

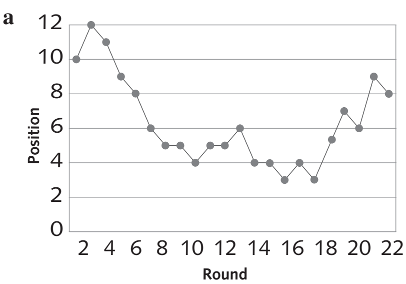


3 **a**



b The number of births fell over the first half of the year. From July until the end of the year, the birth rate was roughly constant. The maximum occurred in January and the minimum in October.

4 **a**



b The team improved its position quite markedly during the first half of the season and maintained a position in the top five teams between rounds 7 and 18 (except for round 12, when it was sixth). However, during the last five rounds, the team's position deteriorated again to eighth at the end of the season.

5 **a** fourth quarter **b** third quarter

c yes

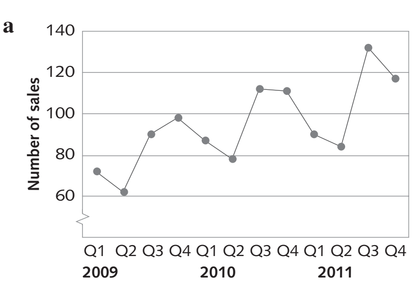
1st quarter sales: 45, 51, 55

2nd quarter sales: 63, 69, 71

3rd quarter sales: 67, 75, 79

4th quarter sales: 43, 39, 49

6 **a**

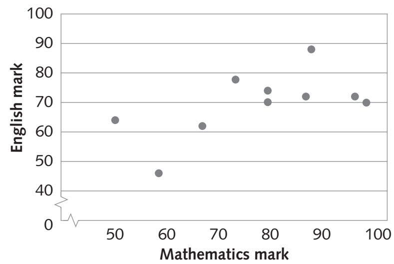


b Car sales per quarter have shown a general upward trend, with major fluctuations.

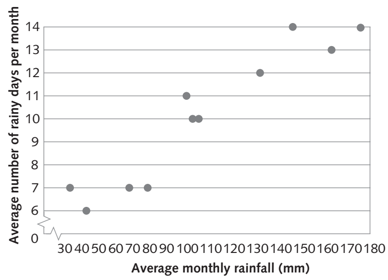
c It seems that the car dealer is able to sell more cars in the third and fourth quarters each year than in the first and second quarters.

Exercise 18G

1

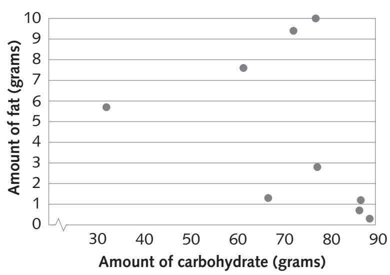


2 **a**



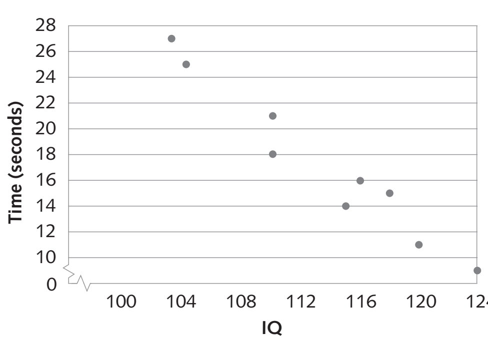
b The number of rainy days per month generally increases as the average monthly rainfall increases.

3 **a**



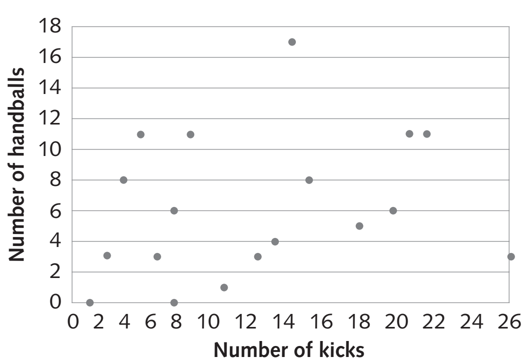
b There is no apparent relationship between carbohydrate content and fat content.

4 **a**



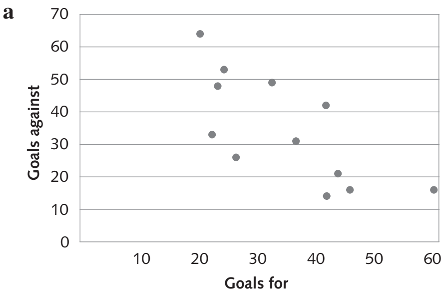
b The time taken to complete a puzzle tends to decrease as IQ increases.

5 **a**



b The scatter plot does not support the claim ‘the greater the kicks, the greater the handballs’. This would only be supported by an upwards trend from left to right.

6 **a**



b **i** The best team is F, with easily the greatest ‘goals for’ and very nearly the least ‘goals against’.

ii The worst team is E, with the least ‘goals for’ and the greatest ‘goals against’.

iii J is better than H – while the ‘goals for’ are about the same, the ‘goals against’ clearly favour J.

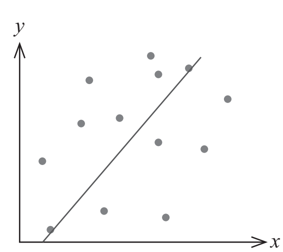
7 **a** B = v **b** C = ii **c** D = iv
d E = viii **e** F = vi **f** G = iii
g H = i **h** I = vii

8 **a** A = iii **b** B = iv **c** C = viii
d D = vii **e** E = vi **f** F = v
g G = i **h** H = ii

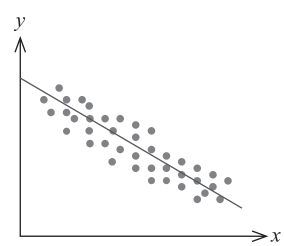
9 **a** **i** 54 **ii** 55 **iii** 36
iv 84 **v** 67 or 68
b **i** 52 **ii** 53 **iii** 32
iv 84 **v** 66

Exercise 18H

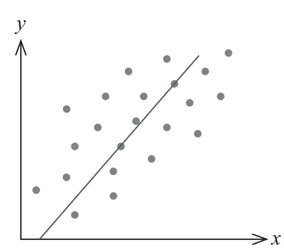
1 **i**



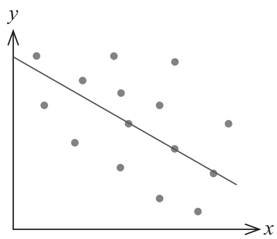
ii



iii



iv



2 a i increasing ii decreasing
iii increasing iv decreasing
b i weak ii strong
iii moderate iv weak

3 a 161 cm, interpolation
b 185 cm, interpolation
c 195 cm, extrapolation

4 a $number\ of\ seeds = 40 \times (weight\ kg) + 260$
b 468 seeds
c 8.5 kg

5 a Using points (4, 69) and (8, 59) we get the equation,
$time = -\frac{5}{2} \times (fitness\ level) + 79$
b 71.5 seconds
c 7 (rounded up from 6.8)
d Interpolation. A fitness level of 3 and time of 62 seconds is within the data range already obtained.

6 a Too few points to judge if there is a relationship between variables. If there is, it appears weak at best, so using the line of best fit to make predictions would be unreliable.
b The form of the scatterplot is not linear. Hence, using a linear equation to make predictions would be inappropriate.

7 $4 300 300, extrapolation

Review exercise

1 a 6.7 seconds b 14.3 seconds
c 2.4 seconds d no

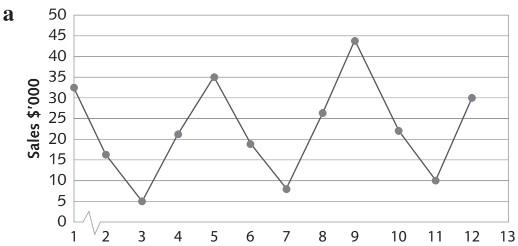
2 a Grade A; 29 hours, Grade B; 26 hours
b Grade A; 4 hours, Grade B; 8 hours
c Grade A; 2 hours, Grade B; 4 hours
d Grade A; 20(50% of 40), Grade B; 10(25% of 40)
e Grade A; symmetrical, Grade B; positively skewed
f In samples of 40 batteries, a Grade B battery lasted longer than any other (32 hours)

3 a ***See below on this page***
b ***See below on this page***

4 a 2.6 b 1.93 c 1 to 4
d The mean would increase by one (from 2.6 to 3.6), but the standard deviation would remain the same (1.93).

5 Kathryn performed better in history, relative to the rest of the class, since she achieved a score that was a full standard deviation above the mean, whereas in mathematics her score was $\frac{2}{3}$ of a standard deviation above the mean.

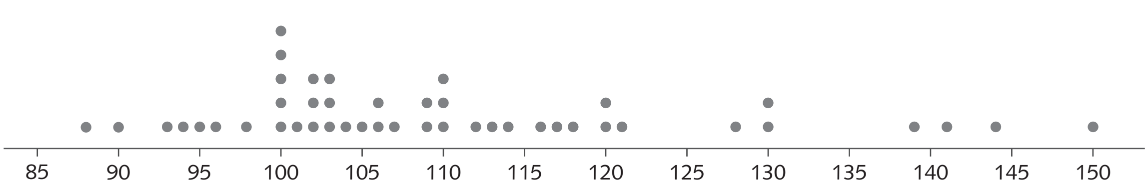
6 a



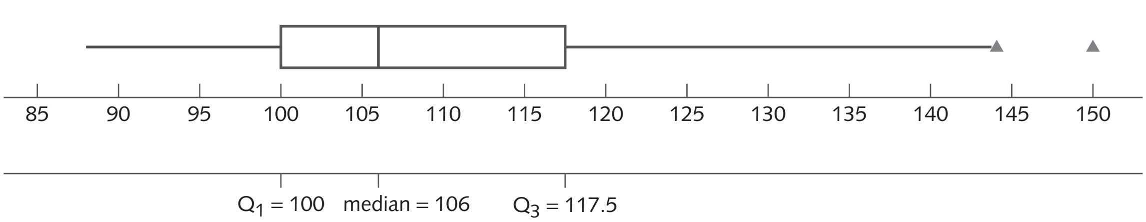
b In the first quarter (Jan–March)
c Sales figures are seasonally changing, with greatest sales in and around the summer period, and least sales in the winter to early spring period (July–September). Nevertheless, it can be observed that there is a general upwards trend in sales.

7 a $height(cm) = 2.5 \times tibia(cm) + 72.5$ (Determined using points (45, 185) and (31, 150).)
b 182.5 cm
c 29 cm
d The answer to (**b**) is interpolation since a tibia length of 44 cm is within the data range already collected. The answer to (**c**) is extrapolation since a height of 145 cm is outside the data range already collected.

3 a



b



Chapter 19 answers

Exercise 19A

1 **a** second **b** third **c** third **d** fourth
e second **f** third **g** second **h** fourth

2 **a** $\sin 10°$ **b** $-\cos 10°$ **c** $-\tan 10°$
d $-\sin 10°$ **e** $-\cos 10°$ **f** $\tan 10°$
g $-\sin 10°$ **h** $\cos 10°$ **i** $-\tan 10°$

3 **a** $\frac{1}{\sqrt{2}}$ **b** $-\frac{1}{\sqrt{2}}$ **c** $-\sqrt{3}$
d -1 **e** $-\frac{\sqrt{3}}{2}$ **f** $\frac{\sqrt{3}}{2}$
g $-\sqrt{3}$ **h** $\frac{1}{2}$ **i** $-\frac{1}{\sqrt{2}}$

4 **a** fourth **b** second **c** third
d third **e** second **f** third

5 **a** 1, 0, 0 **b** 0, 1
c −1, 0, 0 **d** 0, −1
e 1, 0, 0 **f** $\frac{1}{0}$ is not defined

6 **a** $\frac{1}{\sqrt{2}}$ **b** $-\frac{1}{2}$ **c** 0
d -1 **e** $-\frac{1}{\sqrt{2}}$ **f** $-\frac{3\sqrt{3}}{2}-1$

7 **a** $\frac{1}{4}$ **b** $\frac{3}{4}$ **c** $\frac{1}{3}$ **d** $\frac{3}{4}$
e 3 **f** $\frac{3}{4}$ **g** 1 **h** 1

8 **a** $\left(\frac{a}{c}\right)^2+\left(\frac{b}{c}\right)^2=\frac{a^2+b^2}{c^2}=1$, by Pythagoras' theorem
b $\cos^2\theta=\left(\frac{-b}{c}\right)^2=\frac{b^2}{c^2}$, the rest remains the same
c it is still true

9 **a** $\frac{2}{\sqrt{3}}$ **b** 1 **c** $\frac{2}{\sqrt{3}}$ **d** -2
e -2 **f** $\frac{1}{\sqrt{3}}$ **g** $-\frac{1}{\sqrt{3}}$ **h** $\frac{2}{\sqrt{3}}$

10 **a** From question **8**, $\sin^2\theta+\cos^2\theta=1$, now divide both sides of the equation by $\cos^2\theta$.
b The identity holds for all θ between 0° and 360° except when $\cos\theta=0$. That is, when $\theta=90°$ and 270°.

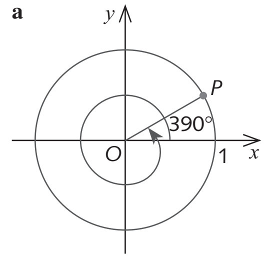
Exercise 19B

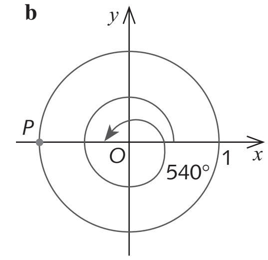
1 **a** 30°, 150° **b** 60°, 240° **c** 45°, 315°
d 120°, 240° **e** 240°, 300° **f** 45°, 225°

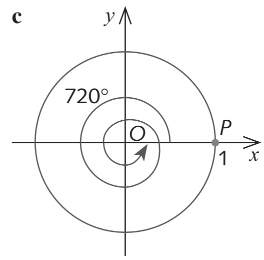
2 **a** 225°, 315° **b** 150°, 330°
c 150°, 210° **d** 90°
e 90°, 270° **f** 0°, 180°, 360°

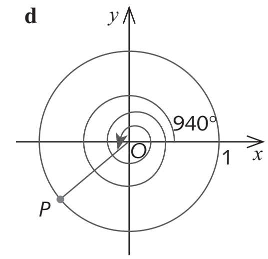
3 **a** 10°, 170° **b** 155°, 205° **c** 65°, 245°
d 233°, 307° **e** 36°, 324° **f** 106°, 286°
g 161°, 199° **h** 23°, 203° **i** 258°, 282°

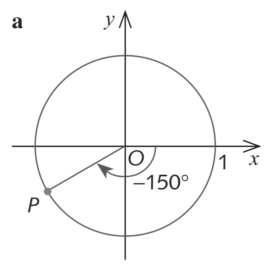
Exercise 19C

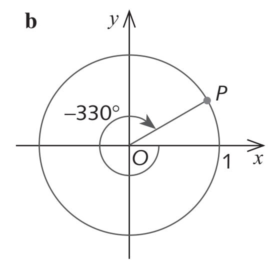
1 **a**


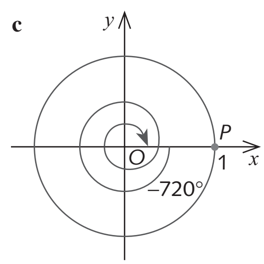
b


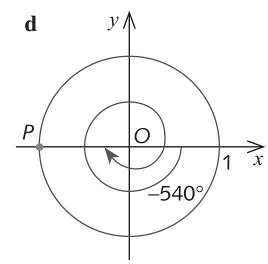
c


d


2 **a**


b


c


d


3 **a** 30° **b** 0° **c** 0° **d** 40°

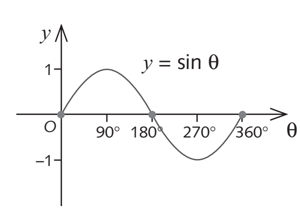
4 **a** 30° **b** 30° **c** 0° **d** 0°

5 **a** 0 **b** −1 **c** 0 **d** $\frac{1}{2}$
e $-\frac{1}{2}$ **f** $-\sqrt{3}$ **g** $\frac{1}{2}$ **h** $\frac{1}{\sqrt{2}}$

6 **a** $-\frac{\sqrt{3}}{2}$ **b** $-\frac{1}{\sqrt{2}}$ **c** -1
d $-\frac{1}{2}$ **e** $\frac{1}{2}$ **f** $-\frac{1}{2}$

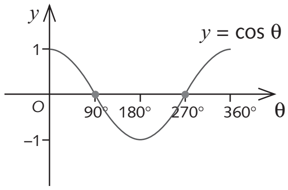
7 **a** 0 **b** 1 **c** 0 **d** 0
e 1 **f** 0 **g** 0

Exercise 19D

1


θ	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
sin θ	0.00	0.17	0.34	0.50	0.64	0.77	0.87	0.94	0.98	1.00

2



θ	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
cos θ	1.00	0.98	0.94	0.87	0.77	0.64	0.50	0.34	0.17	0.00

3 **a** **i** 0.5 **ii** –0.9 **iii** 0.9 **iv** 0.95
v –0.85 **vi** 0.40 **vii** –0.8 **viii** 0.55

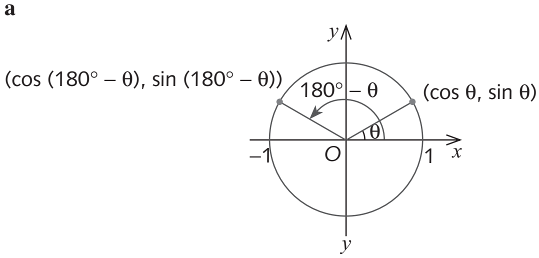
b **i** 30°, 150° **ii** 120°, 240° **iii** 65°, 115°
iv 55°, 305° **v** 55°, 125° **vi** 145°, 215°
vii 205°, 335° **viii** 105°, 255°

c 45° and 225°

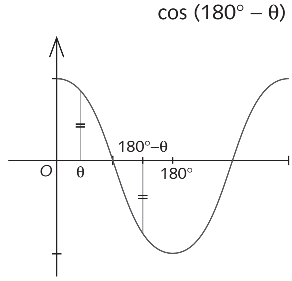
4 **a** 1, –1
b maximum: 90° (plus multiples of 360°); minimum: 270° (plus multiples of 360°)

5 **a** 1, –1
b maximum: 0° (plus multiples of 360°); minimum: 180° (plus multiples of 360°)

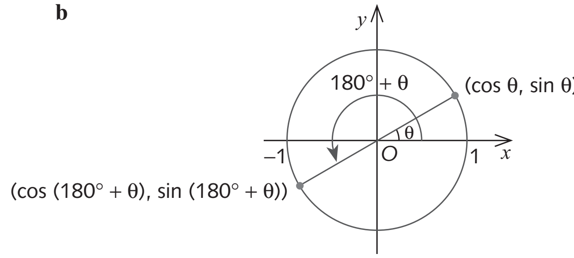
6 **a**



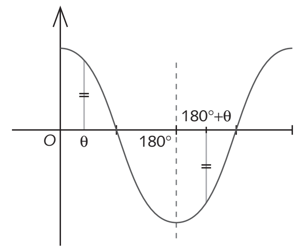
cos (180° – θ) = –cos θ



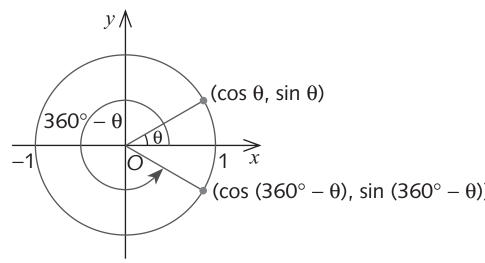
b



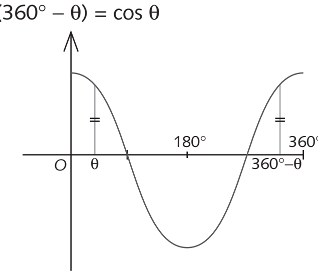
cos (180° + θ) = –cos θ



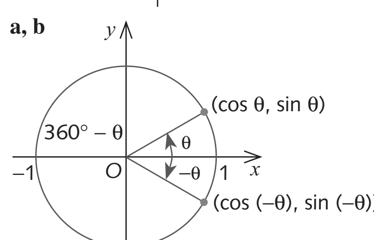
c



cos (360° – θ) = cos θ



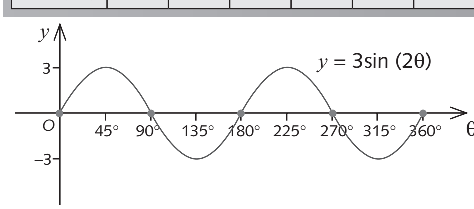
7 **a, b**



cos (–θ) = cos θ
sin (–θ) = –sin θ

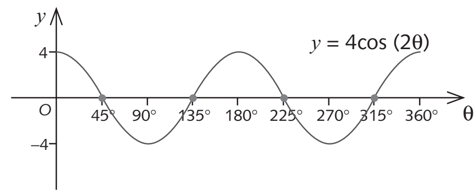
8

θ	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
3sin (2θ)	0	1.5	2.6	3	2.6	1.5	0	–1.5	–2.6	–3	–2.6	–1.5	0
θ		195°	210°	225°	240°	255°	270°	285°	300°	315°	330°	345°	360°
3sin (2θ)		1.5	2.6	3	2.6	1.5	0	–1.5	–2.6	–3	–2.6	–1.5	0



9

θ	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
4cos (2θ)	4	3.5	2	0	–2	–3.5	–4	–3.5	–2	0	2	3.5	4
θ		195°	210°	225°	240°	255°	270°	285°	300°	315°	330°	345°	360°
4cos (2θ)		3.5	2	0	–2	–3.5	–4	–3.5	–2	0	2	3.5	4



Exercise 19E

1 **a** 30°, 150° **b** 60°, 120° **c** 240°, 300° **d** 60°, 300° **e** 45°, 225° **f** 150°, 330° **g** 150°, 210° **h** 60°, 240°

2 **a** 36°, 144° **b** 56°, 304° **c** 63°, 117° **d** 228°, 312° **e** 16°, 344° **f** 256°, 284°

3 **a** 60°, 120°, 240°, 300° **b** 45°, 135°, 225°, 315° **c** 60°, 120°, 240°, 300° **d** 45°, 135°, 225°, 315° **e** 30°, 150°, 210°, 330° **f** 30°, 150°, 210°, 330°

4 **a** 45°, 225° **b** 60°, 240°

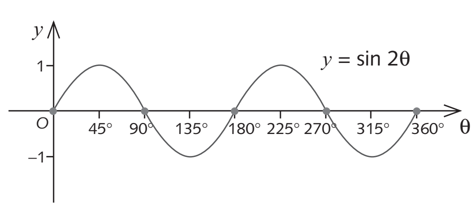
Review exercise

1 **a** 35° **b** 30° **c** 50° **d** 20° **e** 70° **f** 60° **g** 60° **h** 60°

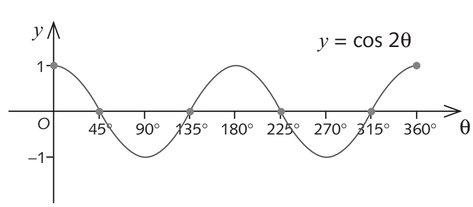
2 **a** $\frac{1}{2}, -\frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{3}}$ **b** $\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\sqrt{3}$ **c** $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -1$ **d** $-\frac{\sqrt{3}}{2}, \frac{1}{2}, -\sqrt{3}$ **e** $-\frac{1}{2}, -\frac{\sqrt{3}}{2}, \frac{1}{\sqrt{3}}$ **f** $-\frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{3}}$ **g** $-\frac{\sqrt{3}}{2}, -\frac{1}{2}, \sqrt{3}$ **h** $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1$

3 **a** $\frac{\sqrt{3}}{2}$ **b** 1 **c** $-\frac{1}{2}$ **d** 1 **e** $-\frac{1}{2}$ **f** –1

4



5



6 **a** 120°, 240° **b** 45°, 135° **c** 45°, 225° **d** 120°, 300° **e** 60°, 120° **f** 30°, 150°, 210°, 330°

7 **a** $-\frac{1}{\sqrt{2}}$ **b** $-\frac{1}{2}$ **c** $-\frac{1}{\sqrt{3}}$ **d** $\frac{1}{2}$ **e** $-\frac{1}{2}$ **f** $-\frac{1}{\sqrt{3}}$

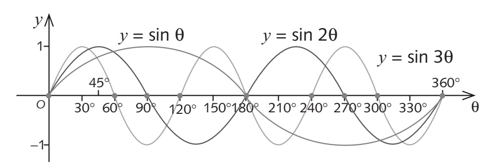
8 **a** 35°, 145° **b** 126°, 234° **c** 65°, 245° **d** 235°, 305°

9 **a** 60°, 300° **b** 120°, 240° **c** 60°, 120° **d** 240°, 300° **e** 270° **f** 0°, 360°

Challenge exercise

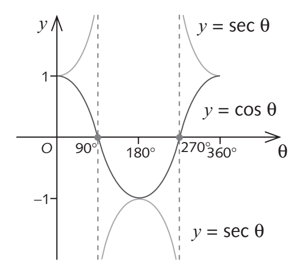
1 $\frac{3}{4}$

3

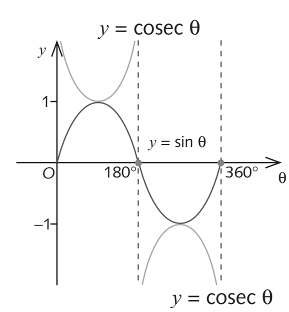


4 0°, 180°, 360°

5



6



7 **a** 60°, 300° **b** 30°, 150° **c** 30°, 150°, 270°

8 **b** Area $= \frac{1}{2}ab\sin(\alpha+\beta) = \frac{1}{2}ya\sin\alpha + \frac{1}{2}yb\sin\beta$

$= \frac{1}{2}(ab\sin\alpha\cos\beta + ab\sin\beta\cos\alpha).$

Thus $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$

c Use $\alpha = 45°$ and $\beta = 30°$: $\sin 75° = \frac{\sqrt{2}+\sqrt{6}}{4}$

9 **a** $2\sin\theta\cos\theta$ **b** $\angle CGD = \theta + \theta = 2\theta$

c $2\sin\theta\cos\theta = \text{area } ABC = \text{area } BCD = \frac{1}{2}BC \times DE = \sin 2\theta$

Chapter 20 answers

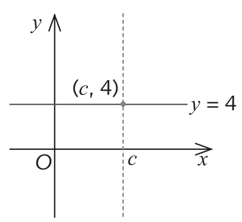
Exercise 20A

1 **a** all real numbers **b** all real numbers
c $x \neq 0$ **d** $x \neq 2$
e $x \neq -4$ **f** $x \neq 2$
g $x \neq 2$ and $x \neq -2$ **h** $x \neq 3$ and $x \neq -3$

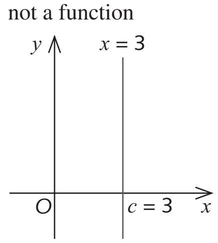
2 **a** $x \geq 0$ **b** $x \geq -7$ **c** $x \leq 7$
d $x \geq \frac{1}{7}$ **e** $x > 0$ **f** $x > 7$

3 **a** all real numbers **b** all real numbers
c $x > 0$ **d** $x > 2$
e $x < 0$ **f** all real numbers

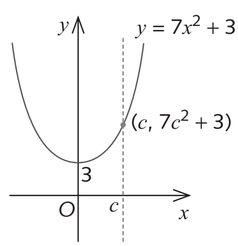
4 **a** function



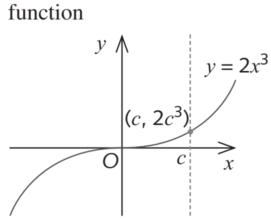
b not a function



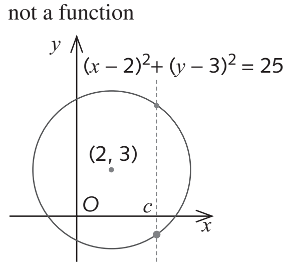
c function



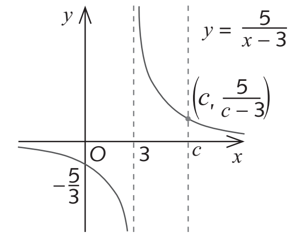
d function



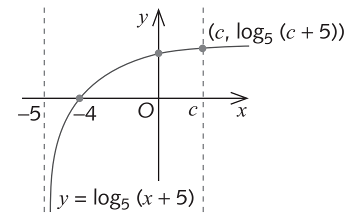
e not a function



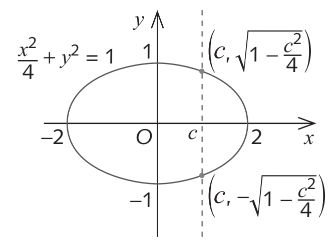
f function



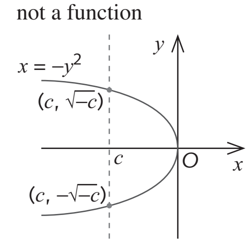
g function



h not a function

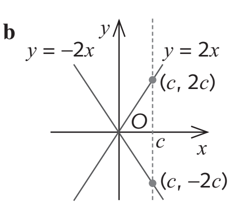


i not a function



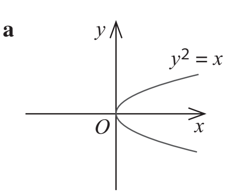
5 **a** $y = 2x$ or $y = -2x$

b

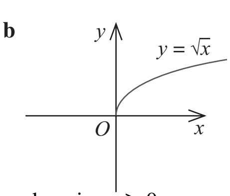


c no
d not a function

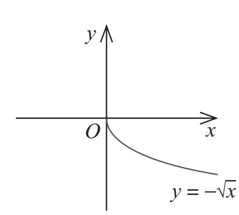
6 **a**



b



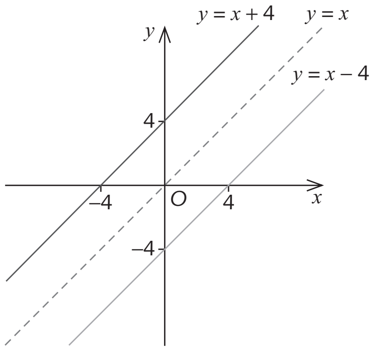
domain: $x \geq 0$



domain: $x \geq 0$

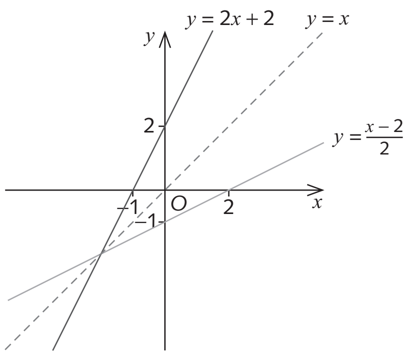
Exercise 20B

1 a



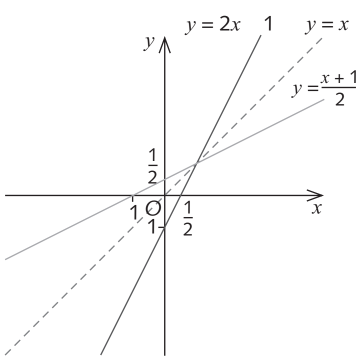
inverse: $y = x - 4$

b



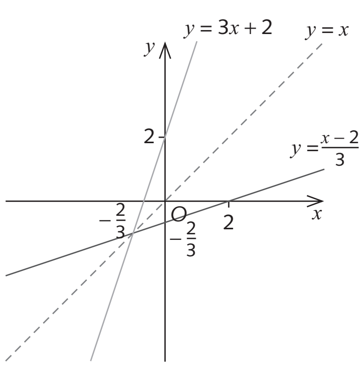
inverse: $y = \dfrac{x-2}{2}$

c



inverse: $y = \dfrac{x+1}{2}$

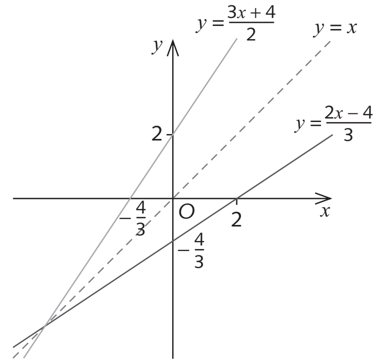
d, e



$y = \dfrac{x-2}{3}$ is the inverse of $y = 3x + 2$

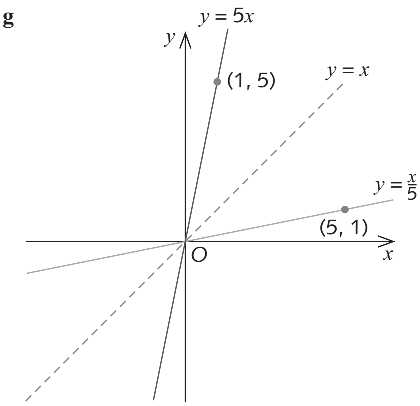
and $y = 3x + 2$ is the inverse of $y = \dfrac{x-2}{3}$

f



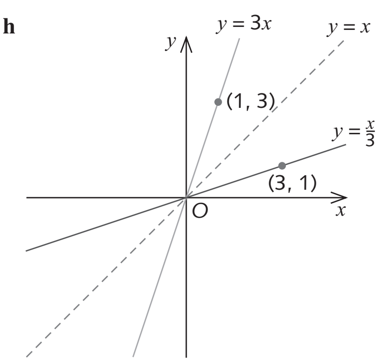
inverse: $y = \dfrac{3x+4}{2}$

g



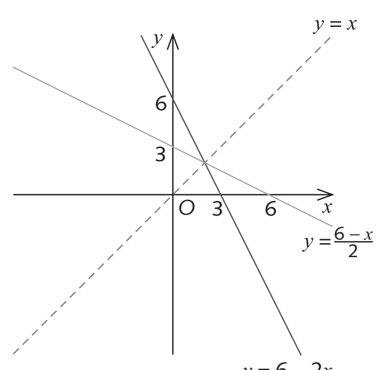
inverse: $y = \dfrac{x}{5}$

h



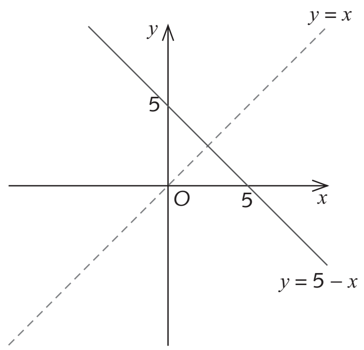
inverse: $y = 3x$

i



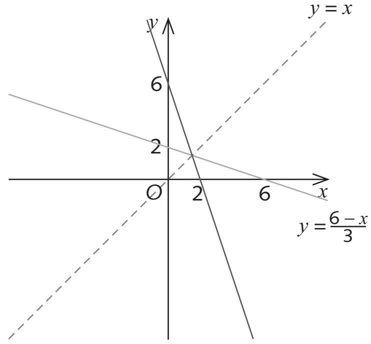
inverse: $y = \dfrac{6-x}{2}$

j



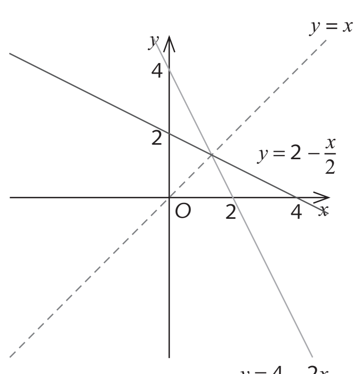
inverse: $y = 5 - x$

k



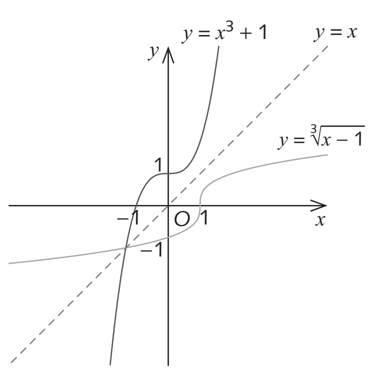
inverse: $y = \dfrac{6 - x}{3}$

l

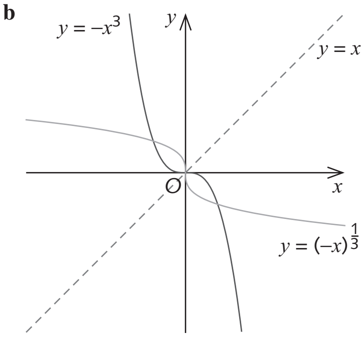


inverse: $y = 4 - 2x$

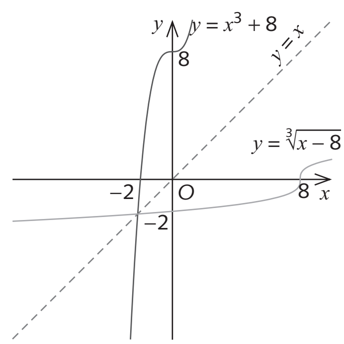
2 **a**



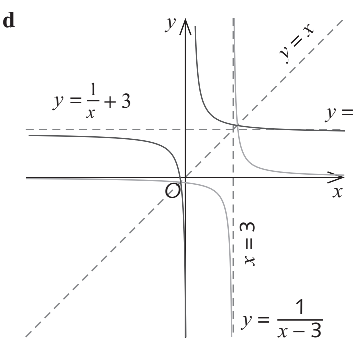
b



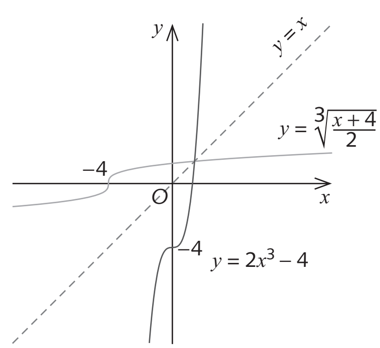
c



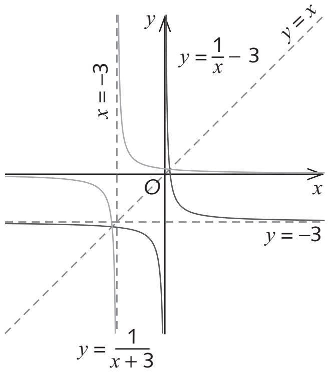
d



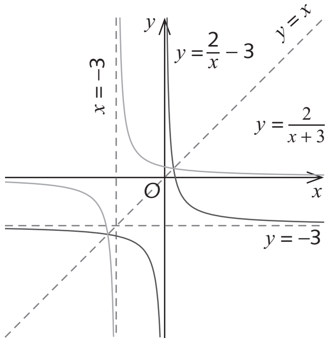
e



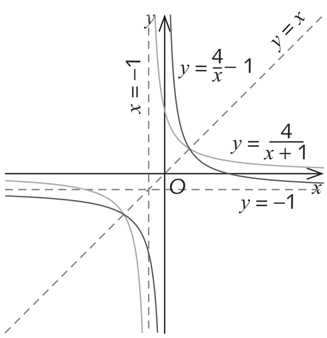
f



g



h



Exercise 20C

1	**a** 3	**b** −17	**c** 0
	d −9	**e** 204	**f** −53
2	**a** 6	**b** 2	**c** 11
	d $\frac{9}{4}$	**e** 4	**f** 504
3	**a** 1	**b** 0	**c** −6
	d $\frac{3}{2}$	**e** 3	**f** $\frac{1}{3}$
4	**a** $\frac{1}{6}$	**b** $\frac{2}{5}$	**c** −2

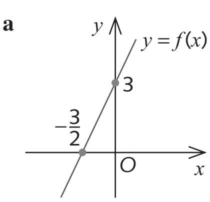
5 **a** $x = 0$ or $x = 4$ **b** $x = 2$
c $x = -1$ or $x = 5$ **d** no values of x
e $x = 2 - \sqrt{5}$ or $x = 2 + \sqrt{5}$ **f** $x = 1$ or $x = 3$

6	**a** $a^2 - 4$	**b** $y^2 + 4y$	**c** $4b^2 - 4$
	d $9c^2 + 6c - 3$	**e** $x^4 - 4$	**f** $x^6 - 4$
7	**a** true	**b** false	**c** false
	d true	**e** true	**f** true
8	**a** false	**b** false	**c** true
	d false	**e** false	**f** true

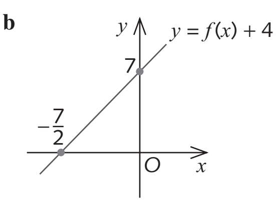
9 **a** all real numbers, $y \le 3$
b $x \ne 0, y \ne 0$
c all real numbers, $y > 4$
d $-3 \le x \le 3, 0 \le y \le 3$
e all real numbers, $y \le 6$
f all real numbers, $y \ge 4$
g all real numbers, $y > -3$
h all real numbers, $y > 7$
i $-5 \le x \le 5, -5 \le y \le 0$
j all real numbers, all real numbers
k $x \ne 0, y \ne 0$
l $x < 7$, all real numbers
m all real numbers, $-1 \le y \le 1$
n $x \ne \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$, all real numbers

Exercise 20D

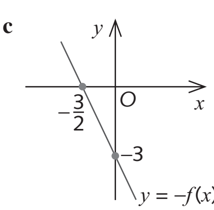
1 **a**



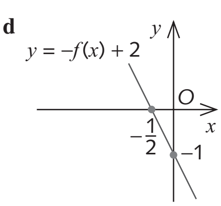
b



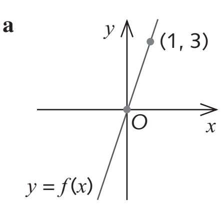
c



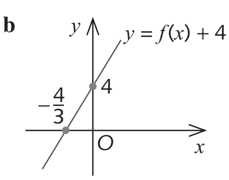
d



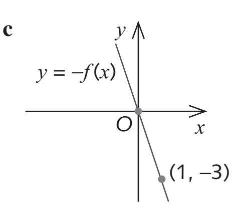
2 **a**



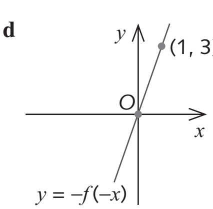
b



c



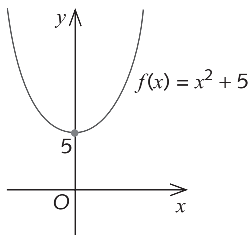
d



Indeed in this case, $-f(-x) = f(x)$

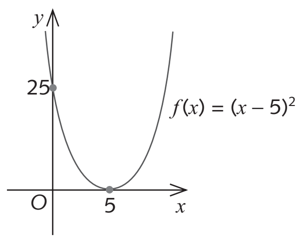
3 **a** domain: all real numbers

range: $y \geq 5$



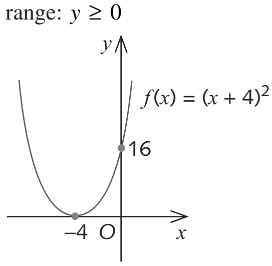
b domain: all real numbers

range: $y \geq 0$



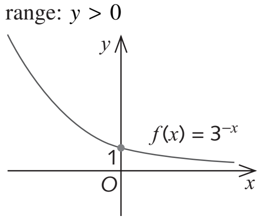
c domain: all real numbers

range: $y \geq 0$



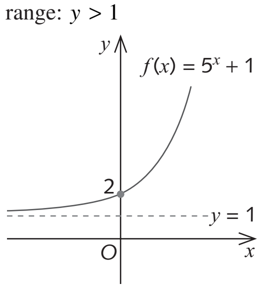
d domain: all real numbers

range: $y > 0$



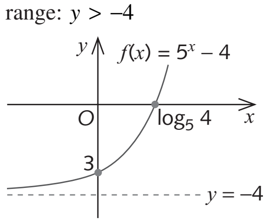
e domain: all real numbers

range: $y > 1$



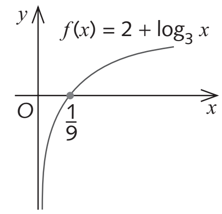
f domain: all real numbers

range: $y > -4$



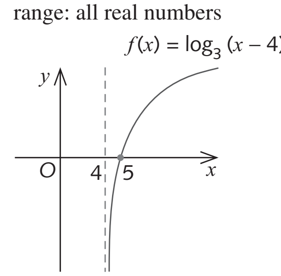
g domain: $x > 0$

range: all real numbers



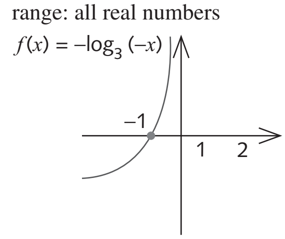
h domain: $x > 4$

range: all real numbers



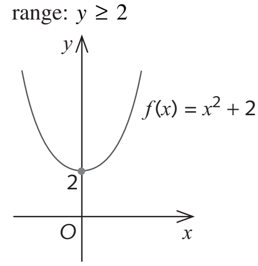
i domain: $x < 0$

range: all real numbers



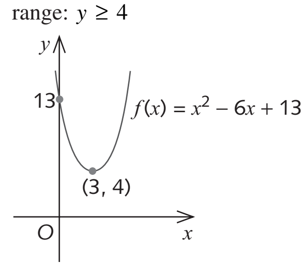
4 **a** domain: all real numbers

range: $y \geq 2$



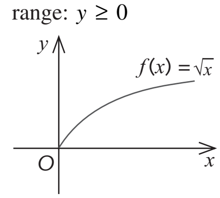
b domain: all real numbers

range: $y \geq 4$



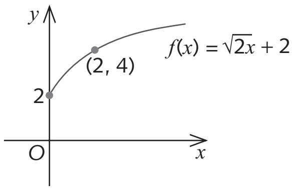
c domain: $x \geq 0$

range: $y \geq 0$



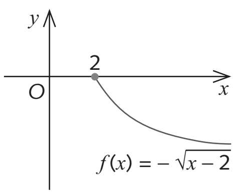
d domain: $x \geq 0$

range: $y \geq 2$



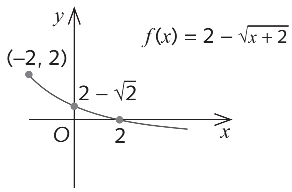
e domain: $x \geq 2$

range: $y \leq 0$

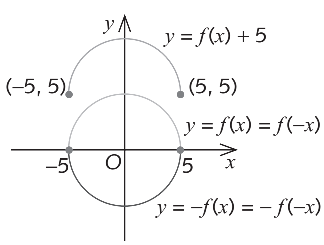


f domain: $x \geq -2$

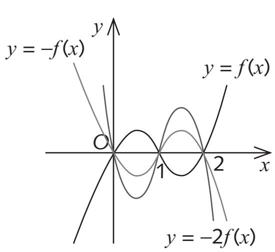
range: $y \leq 2$



5



6



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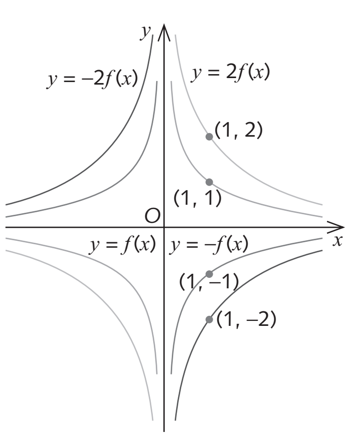


Exercise 20E

1 **a** 3 **b** 5 **c** 10

d $a + 3$ **e** $g(f(x)) = x + 3$

The result is a translation of 3 units to the right.

2 **a** 0 **b** −6 **c** −4

d −2 **e** 3 **f** −4

g $x^2 - 6$ **h** $x^2 - 4x$ **i** $x - 4$

j $x^4 - 8x^2 + 12$ **k** $f(g(x)) \neq g(f(x))$

3 **a** 2 **b** 2 **c** 4

d 4 **e** x **f** x

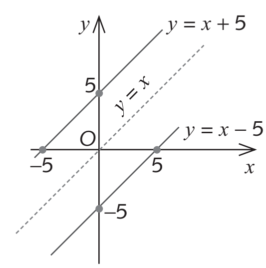
g $f(g(x)) = x$ and $g(f(x)) = x$, f and g are inverses of each other.

4 **a** 2 **b** 2 **c** 4

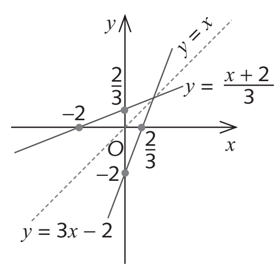
d 4 **e** x **f** x

g $f(g(x)) = x$ and $g(f(x)) = x$, f and g are inverses of each other.

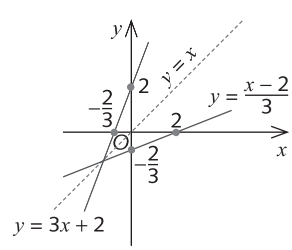
5 **a** $g(x) = x - 5$



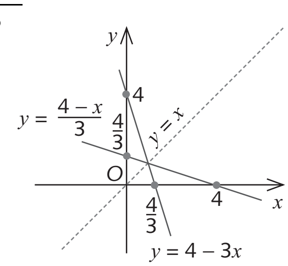
b $g(x) = \dfrac{x + 2}{3}$



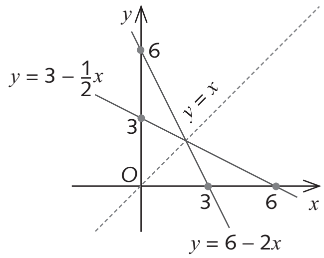
c $g(x) = \dfrac{x - 2}{3}$



d $g(x) = \dfrac{4 - x}{3}$



e $g(x) = 6 - 2x$



6 **a** $g(x) = \sqrt[3]{x + 2}$

b $g(x) = \sqrt[3]{2 - x}$

c $g(x) = \sqrt[5]{\dfrac{x}{32}}$

7 **a** domain: $x \neq 0$, $g(x) = \dfrac{1}{x - 1}$, $x \neq 1$

b domain: $x \neq -1$, $g(x) = \dfrac{1}{x} - 1$, $x \neq 0$

c domain: $x \neq 2$, $g(x) = \dfrac{2(x + 1)}{x - 1}$, $x \neq 1$

d domain: $x \neq -2$, $g(x) = \dfrac{2x}{3 - x}$, $x \neq 3$

8 **a** $f(f(x)) = f(5 - x) = 5 - (5 - x) = x$

b $f(f(x)) = f(-x) = x$

c $f(f(x)) = f\left(-\dfrac{1}{x}\right) = x$

d $f(f(x)) = f\left(\dfrac{6}{x}\right) = x$

e $f(f(x)) = f\left(\dfrac{2x - 2}{x - 2}\right) = x$

f $f(f(x)) = f\left(\dfrac{-3x - 5}{x + 3}\right) = x$

9 **a** domain: all real numbers; $g(x) = \dfrac{x}{3}$; domain of $g(x)$: all real numbers

b domain: all real numbers; $g(x) = \dfrac{1}{3}\log_2 x$; domain of $g(x)$: $x > 0$

c domain: all real numbers; $g(x) = \log_7 \dfrac{x}{5}$; domain of $g(x)$: $x > 0$

d domain: $x > 0$; $g(x) = 5^x$; domain of $g(x)$: all real numbers

e domain: $x > 0$; $g(x) = \dfrac{1}{3} \times 4^{\frac{x}{2}}$; domain of $g(x)$: all real numbers

f domain: $x > 3$; $g(x) = 2^x + 3$; domain of $g(x)$: all real numbers

g domain: all real numbers; $g(x) = \log_5(x) + 1$; domain of $g(x)$: $x > 0$

h domain: $x > 0$; $g(x) = 4^{x-4}$; domain of $g(x)$: all real numbers

i domain: all real numbers; $g(x) = \dfrac{1}{3}\log_5(x - 5)$; domain of $g(x)$: $x > 5$

10 $y = \sqrt{49 - x^2}$, $-7 \leq x \leq 0$; $y = \sqrt{49 - x^2}$, $0 \leq x \leq 7$;

$y = -\sqrt{49 - x^2}$, $-7 \leq x \leq 0$; $y = -\sqrt{49 - x^2}$, $0 \leq x \leq 7$

Review exercise

1 **a** all real numbers **b** $x \neq 0$

c $x \neq 5$ **d** $x \neq -8$

e $x \geq 2$ **f** all real numbers

g $x \neq -5$ **h** $x \geq -6$

2 **a** -4 **b** -3 **c** -3

d 12 **e** $a^2 - 4$ **f** $a^2 - 4$

g $4(a^2 - 1)$ **h** $a^2 - 4a$

3 **a** 3 **b** 1 **c** 5

d 11 **e** $3 - 2a$ **f** $3 + 2a$

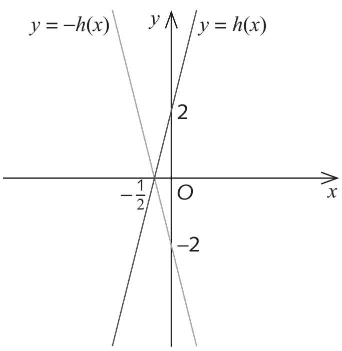
g $3 - 4a$ **h** $7 - 2a$

4 **a** domain: all real numbers; range: all real numbers

b domain: all real numbers; range: $y \leq 4$

c domain: $x \neq -6$; range: $y \neq 0$

5 **a**



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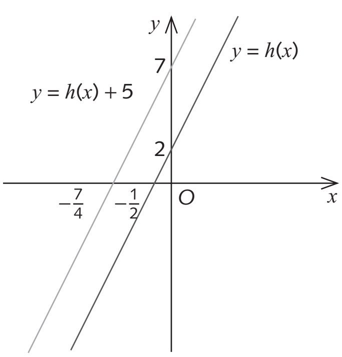

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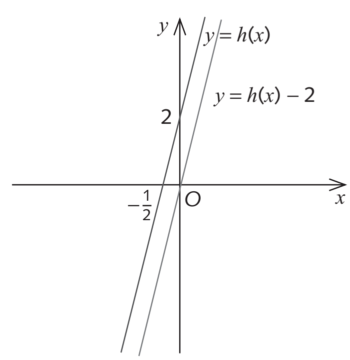

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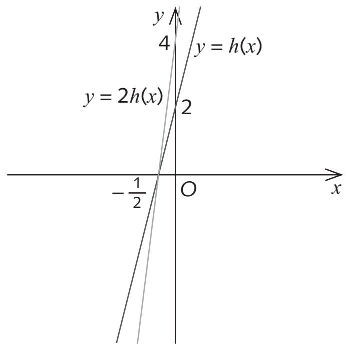

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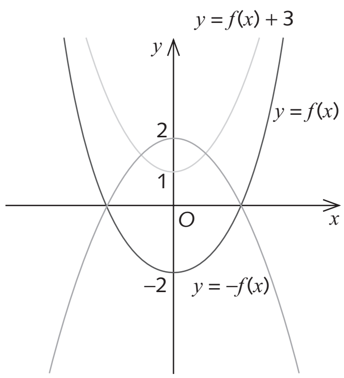


7 **a** 4 **b** 11 **c** −20

d 3 **e** 31 **f** 4

g $11 - 2x^2$ **h** $-4x^2 - 4x + 4$ **i** $4x + 3$

j $-x^4 + 10x^2 - 20$ **k** $f(g(x)) \neq g(f(x))$

8 **a** $y = \dfrac{x + 4}{3}$ **b** $y = \dfrac{2 - x}{3}$

c $y = \sqrt[3]{x - 2}$ **d** $y = \dfrac{1}{x} - 2$

Challenge exercise

1 **a** $f(a + b) = 2(a + b) = 2a + 2b = f(a) + f(b)$

$f(ka) = 2ka = kf(a)$

b Assume $f(a + b) = f(a) + f(b)$

Then $a + b + 2 = a + 2 + b + 2 = a + b + 4$, which is a contradiction.

Assume $f(ka) = kf(a)$.

Then, $ka + 2 = ka + 2k$.

Thus, $k = 1$.

2 **a** $f(x + y) = 2^{x+y} = 2^x \times 2^y = f(x)f(y)$

b $x + y = xy$. $x = 2$ and $y = 2$, and $x = y = 0$

3 **a** $f(x) = x^2$ is even and $f(x) = x^3$ is odd.

b Let $f(x)$ and $g(x)$ be even functions. The sum function

$(f + g)(x) = f(x) + g(x)$

$(f + g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f + g)(x)$.

c Let $f(x)$ and $g(x)$ be even functions. The product function $(fg)(x) = f(x)g(x)$

$(fg)(-x) = f(-x) \times g(-x) = f(x)g(x) = (fg)(x)$

d Let $f(x)$ and $g(x)$ be odd functions. The product function $(fg)(x) = f(x)g(x)$

$$(fg)(-x) = f(-x) \times g(-x) = -f(x) \times (-g(x))$$
$$= f(x)\,g(x) = (fg)(x)$$

e Let $f(x)$ and $g(x)$ be odd functions. Then

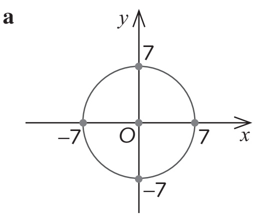
$f(g(-x)) = f(-g(x)) = -f(g(x))$

Chapter 21 answers

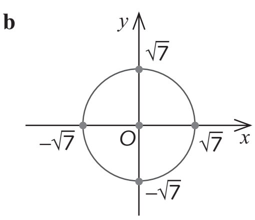
21A Review

Chapter 11: Circles, hyperbolas and simultaneous equations

1 **a**



b



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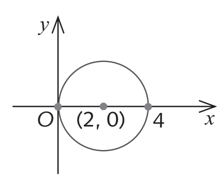

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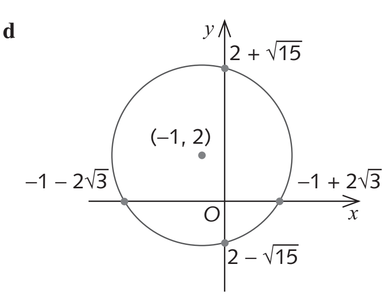


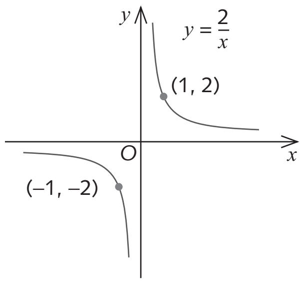
2 **a** $(x - 3)^2 + y^2 = 16$

b $(x + 1)^2 + (y - 2)^2 = 3$

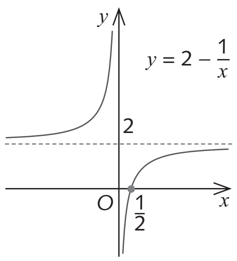
3 **a** $(x - 2)^2 + (y + 3)^2 = 4$, centre $(2, -3)$, radius 2

b $(x + 1)^2 + (y + 4)^2 = 16$, centre $(-1, -4)$, radius 4

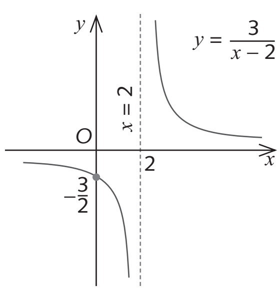
4 **a** asymptotes: $x = 0$ and $y = 0$



b asymptotes: $x = 0$, $y = 2$; x-intercept $x = \frac{1}{2}$

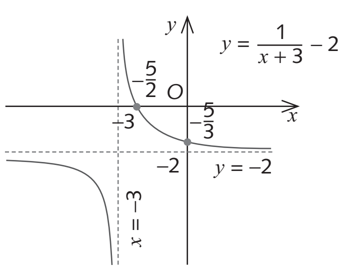


c asymptotes: $y = 0$, $x = 2$, $x = 2$; y-intercept $= -\frac{3}{2}$



d asymptotes: $x = -3$ and $y = -2$

x-intercept $= -\frac{5}{2}$, y-intercept $= -\frac{5}{3}$



5 **a** $(3, 12), (-2, -3)$ **b** $(-2, -1), \left(\frac{3}{2}, 6\right)$

c $\left(-\frac{3}{2}, -2\right), (1, 3)$ **d** $(-3, -2), \left(\frac{2}{3}, 9\right)$

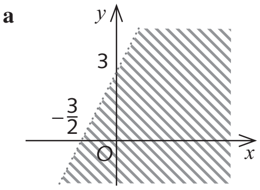
6 **a** $(\sqrt{5}, 2), (-\sqrt{5}, 2)$ **b** $(1, \sqrt{3}), (1, -\sqrt{3})$

c $(0, 2), (-2, 0)$ **d** $(0, 4), \left(\frac{8\sqrt{2}}{3}, -\frac{4}{3}\right)$

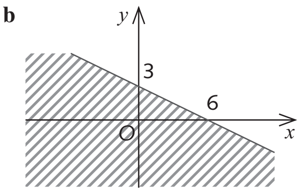
7 $\left(-\frac{6}{5}, \frac{17}{5}\right), (2, -3)$

8 $(-4, 3), (6, 8)$

9 **a**



b



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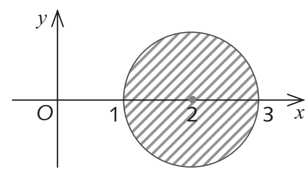

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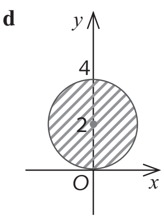

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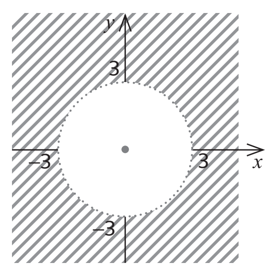

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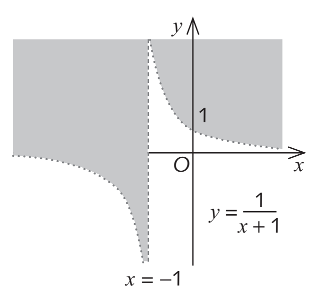
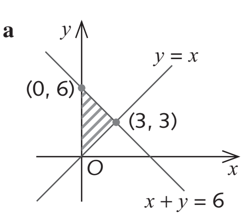
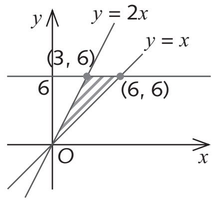
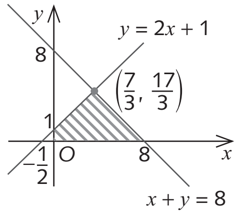


10 **a**



b 

c 

Chapter 12: Further trigonometry

1 **a** $\theta \approx 43.2°$ **b** $x \approx 2.9$ **c** $x \approx 4.9$

2 **a** 2.3 m **b** 1.9 m

3 63.4°

4 **a** 13.5 km **b** 132.0°T

5 281.2°T **6** 52.5 m

7 **a** 4.619 m **b** 18.475 m **c** 4.619 m

8 **a** $x = \sqrt{2}$ **b** $x = 5\left(3 + \sqrt{3}\right)$

9 $x = 50\left(\sqrt{3} - 1\right)$

10 $\dfrac{20\left(3 - \sqrt{3}\right)}{3}$ cm

11 12 km

12 **a** $B \approx 102.6°, C \approx 27.4°, AC \approx 6.4$
b $A \approx 71.2°, B \approx 38.8°, BC \approx 6.0$
c $C = 35°, AB \approx 1.3, AC \approx 2.17$

13 14.8 km **14** 34.03 m

15 $100\left(\sqrt{3} + 1\right)$ m

16 **a** 24.8° **b** 56.9°

17 **a** 36.37 cm **b** 50.8° **c** 46.96 cm

18 10.0°

19 **a** 45° **b** 32.0°

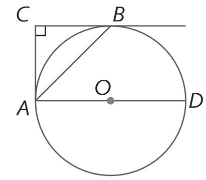
20 **a** P at B or C, $\angle EPH \approx 35.26°$
b P is midpoint of BC, $\angle EPH \approx 38.94°$

Chapter 13: Circle geometry

1 **a** $a = 60, b = 120$
b $c = 20$
c $h = 110, j = 70$
d $d = 50, e = 40, f = 10, g = 40$
e $k = 25, m = 65, r = 40, s = 50, t = 65$

2 **a** **i** 90° **ii** 60° **iii** 30° **iv** 30°
b $6\sqrt{3}$ cm

3 $CB = CA$ (equal tangents). Hence, $\angle CAB = \angle CBA = 45°$ (isosceles triangle), $\angle CAD = 90°$ (tangent to circle at A). Therefore, BA bisects $\angle CAD$.



4 $BE = 2$ cm **5** $TS = 17$ cm

6 $EC = 10, ED = 6$

7 If $BP \times PQ = CP \times PR$

then $\dfrac{BP}{CP} = \dfrac{PR}{PQ}$

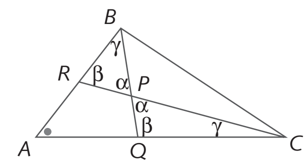
$\angle BPR = \angle CPQ$ (vertically opposite)

ΔRPB is similar to ΔQPC (SAS)

ΔABQ is similar to ΔACR (AAA)

so $\dfrac{AR}{AC} = \dfrac{AQ}{AB}$

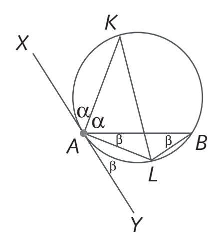
$AR \times AB = AC \times AQ$



8 **a** $PY = 9$ cm, $XY = 5$ cm **b** $PT = 9$ cm
c $PX = 4$ cm

9 $\angle BOC = 150°$

10 **a** Let $\angle YAL = \beta$
Then $\angle LAB = \beta$ (given)
$\angle ABL = \beta$ (alternate segment theorem)
So ΔALB is isosceles, $AL = BL$
b Let $\angle XAK = \alpha$. Then $\angle KAB = \alpha$ (given)
Therefore, $2\alpha + 2\beta = 180°$
$\alpha + \beta = 90°$
$\angle KAL$ is a right angle. Therefore, KL is a diameter (converse of Thale's theorem)



Chapter 14: Indices, exponentials and logarithms – part 2

1 **a** 4 **b** 4 **c** 10
d 0 **e** 5

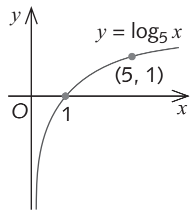
2 **a** −5 **b** −5 **c** −4
d −2 **e** −4 **f** −3
g −11 **h** −5

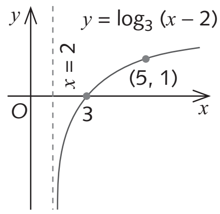
3 **a** $\log_2 75$ **b** $\log_2 63$ **c** $\log_2 33$
d $\log_3 100$ **e** $\log_7 40$ **f** 1
g −1 **h** 1 **i** 1

4 **a** $\log_2 14$ **b** $\log_3 20$
c $-\log_5 343 = -3\log_5 7$ **d** $\log_{11}\dfrac{3}{5}$

5 **a** 125 **b** 256 **c** 620
d $\dfrac{1027}{6}$ **e** −59 **f** $\dfrac{10\,001}{2}$

6 **a** 3 **b** $2^{\frac{2}{3}}$ **c** $2^{\frac{11}{6}}$ **d** 10

7 **a**


b


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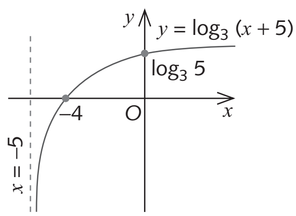

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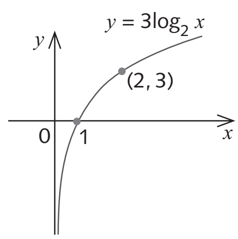
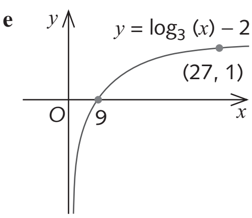


e


Chapter 15: Probability

1 **a** $\frac{1}{3}$ **b** $\frac{1}{2}$

2 **a** $\frac{1}{4}$ **b** $\frac{1}{13}$ **c** $\frac{1}{52}$ **d** $\frac{3}{13}$

3 $\frac{1}{200}$ 4 $\frac{5}{16}$

5 **a** $\frac{1}{12}$ **b** $\frac{1}{36}$ **c** $\frac{13}{18}$

6 **a i** $\frac{4}{25}$ **ii** $\frac{2}{15}$

b i $\frac{12}{25}$ **ii** $\frac{8}{15}$

7 **a** $\frac{1}{2}$ **b** $\frac{77}{200}$ **c** $\frac{40}{77}$ **d** $\frac{2}{5}$

8 **a** $p = \frac{4}{15}$ **b** $p = \frac{1}{3}$

9 **a** 0.75
b They are not independent as $P(H \cap F) \neq P(H)P(F)$

10 **a** 0.5 **b** 0.25

Chapter 16: Direct and inverse proportion

1 **a i** $\frac{1}{2}$, $y = \frac{1}{2}x$ **ii**

x	1	4	8	20
y	$\frac{1}{2}$	2	4	10

b i 2, $y = 2x^2$ **ii**

x	1	3	5	$\sqrt{7}$
y	2	18	50	14

c i 5, $y = \frac{5}{x}$ **ii**

x	2	5	7	11
y	$\frac{1}{2}$	1	$\frac{5}{7}$	$\frac{5}{11}$

d i 1, $y = \frac{1}{x^2}$ **ii**

x	2	3	$\frac{1}{2\sqrt{2}}$	7
y	$\frac{1}{4}$	$\frac{1}{9}$	8	$\frac{1}{49}$

2 $y = 9\sqrt{x}$ **a** $y = 18$ **b** $x = \frac{625}{9}$

3 154 cm^2

4 $y = \frac{40}{x^2}$ **a** $y = \frac{40}{81}$ **b** $x = \frac{2\sqrt{10}}{3}$

5 **a** 2, $c = 2ab^2$ **b**

a	5	3	6	3
b	1	2	2	3
c	10	24	48	54

6 $a = 32$ 7 $z = 243$

8 **a** 2250 joules
b 8 times the original energy

Chapter 17: Polynomials

1 **a** 3 **b** 5 **c** 8
d 0 **e** 4 **f** $a^3 - 2a + 4$

2 **a** $a = 7$ **b** $b = 2$

3 **a** $P(x) + Q(x) = -x^3 + 3x^2 + 7$,
$P(x) - Q(x) = 3x^3 - 3x^2 + 8x + 7$
b $P(x) + Q(x) = x^2 - 3x$,
$P(x) - Q(x) = -6x^5 - x^2 - 3x + 14$
c $P(x) + Q(x) = -x + 2$,
$P(x) - Q(x) = 8x^3 - 10x^2 - 11x + 10$

4 **a** $P(x) = (x + 2)(x + 6) - 6$
b $P(x) = (x + 6)(x^2 - 12x + 60) - 330$
c $P(x) = (x - 1)(5x^2 - 2x - 2) - 3$

5 **a** 45 **b** $\frac{25}{8}$ **c** $\frac{31}{8}$

6 $a = -1$

7 **a** $(x - 1)(x + 2)(2x + 3)$
b $(x - 2)(x + 1)(2x + 3)$
c $(x - 2)(x - 1)(x + 1)(2x + 3)$

8 **a** $x = 1$ or $x = -2$ or $x = -\frac{3}{2}$
b $x = 2$ or $x = 1$ or $x = -1$ or $x = -\frac{3}{2}$

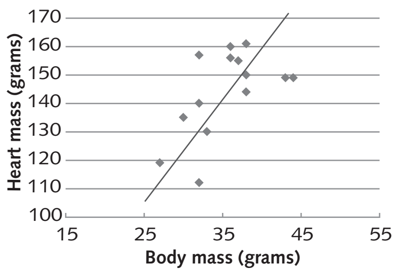
9 **a** $P(1) = 0$ for all x. **b** $k = 7$
c $x = 1$ or $x = 2$ or $x = 4$

10 **a** $2 + \dfrac{5}{x-1}$ **b** $4 + \dfrac{2-5x}{x^2+2x}$

Chapter 18: Statistics

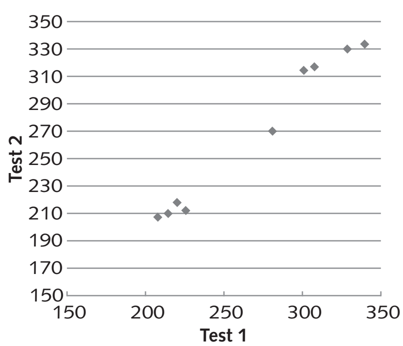
1 **a** $\bar{x} = 9.45, \sigma = 3.85$ **b** $\bar{x} = 13.50, \sigma = 3.54$

2 **a**



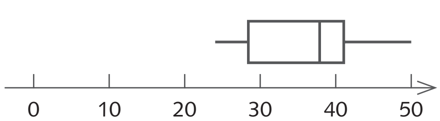
b There is some tendency for the heart mass to increase as the body mass increases. The relationship could be described as linear and of moderate strength.

3 As the scores on Test 1 increase, the scores on Test 2 increase.



4 **a** 38 **b** 11.5

c



5 **a** 19 **b** approximately 150

c 4 hours **d** approximately 50

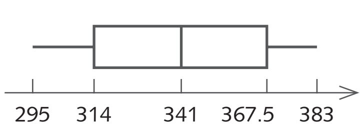
6 **a** **i** $a = 38, c = 61, e = 88$ **ii** 62

b **i** 17 **ii** 63

7 **a** public holiday, roadworks, accident

b **i** 340.8 **ii** 341 **iii** 53.5

c



d George may have sampled over school holiday period. Community group may have sampled for a small number of days and obtained high counts, or may be exaggerating.

Chapter 19: Trigonometric functions

1 **a** second **b** third **c** third

d fourth **e** second **f** fourth

g third **h** fourth

2 **a** sin 5° **b** −cos 30° **c** −tan 20°

d −sin 20° **e** −cos 20° **f** tan 5°

g −sin 5° **h** cos 10°

3 **a** $-\dfrac{\sqrt{2}}{2}$ **b** $-\dfrac{\sqrt{2}}{2}$ **c** $\dfrac{\sqrt{3}}{2}$

d $-\sqrt{3}$ **e** $-\dfrac{1}{2}$ **f** $\dfrac{\sqrt{2}}{2}$

g -1 **h** $-\dfrac{\sqrt{3}}{2}$

4 **a** $\dfrac{1}{2}$ **b** $\dfrac{1}{4}$ **c** 0 **d** $-\dfrac{\sqrt{3}}{2}$

5 **a** 60°, 300° **b** 120°, 300° **c** 45°, 135°

d 210°, 330° **e** 150°, 210° **f** 135°, 315°

6 **a** 15.93°, 164.07° **b** 156.42°, 203.58°

c 66.00°, 246.00° **d** 243.84°, 296.16°

e 45.49°, 314.51° **f** 111.12°, 291.12°

7 **a** $\dfrac{1}{2}$ **b** $\dfrac{\sqrt{2}}{2}$ **c** 1

d $-\dfrac{\sqrt{3}}{2}$ **e** $\dfrac{\sqrt{3}}{2}$ **f** $-\dfrac{\sqrt{2}}{2}$

8 **a** 60°, 300° **b** 150°, 210° **c** 240°, 300°

d 120°, 240° **e** 45°, 225° **f** 30°, 210°

Chapter 20: Functions and inverse functions

1 **a** −1 **b** 7 **c** −3 **d** −11

2 **a** 8 **b** 2 **c** $\dfrac{1}{2}$ **d** −2

3 **a** 2 **b** 4 **c** −2 **d** 6

4 **a** $\dfrac{6}{5}$ **b** $\dfrac{1}{5}$

5 **a** $x \neq -2$ **b** $x \neq 2$

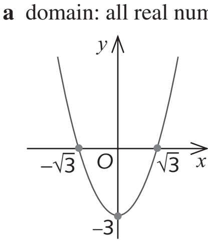
c $x \leq 5$ **d** $x \geq 2$

e $x \neq 3$ and $x \neq -3$ **f** $x > -7$

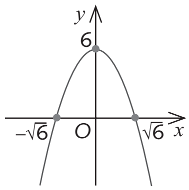
g all real numbers **h** $x > \dfrac{1}{2}$

i $x < 6$

6 **a** domain: all real numbers, space and range: $y \geq -3$

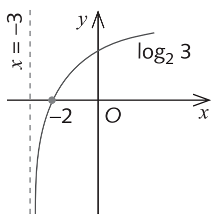


b domain: all real numbers, space and range: $y \le 6$



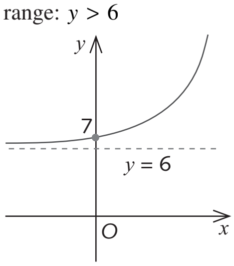
c domain: $x > -3$

range: all real numbers



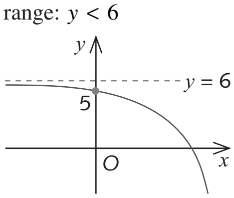
d domain: all real numbers

range: $y > 6$



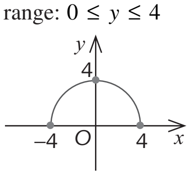
e domain: all real numbers

range: $y < 6$



f domain: $-4 \le x \le 4$

range: $0 \le y \le 4$



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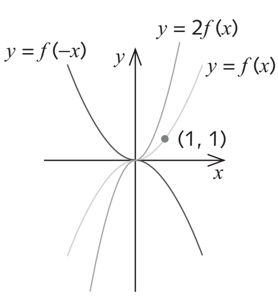


8 **a** 1 **b** −1

c $g(f(x)) = 2x^2 - 3$ **d** $f(g(x)) = (2x - 3)^2$

9 The domian for **a**, **b**, **d**, **e** and **f** is all real numbers. The domain for **c** is the real numbers > −3.

a $g(x) = \dfrac{x+3}{2}$ **b** $g(x) = 2x + 1$

c $g(x) = \log_2(x + 3)$ **d** $g(x) = 3^x - 1$

e $g(x) = \sqrt[3]{8 - x}$ **f** $g(x) = \sqrt[3]{8 + x}$

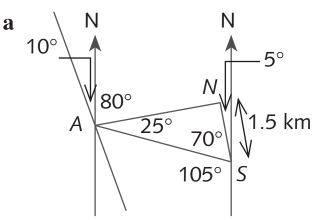
21B Problem-solving

1 **a** 10.2 km **b** 295.5°

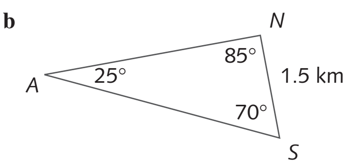
2 **a** **i** $4\sqrt{3}$ m **ii** 53°

b $(5\sqrt{3} + 12)$ m

3 **a**



b



c 3.335 km **d** 3.205 km

4 **a** **i** 60° **ii** 30° **iii** 30° **iv** 90°

b **i** 10 km **ii** 5 km

c $\angle PAY = \angle SXY = 90°$, $\angle APY = \angle XSY = 60°$, $\angle AYP = \angle SYX = 30°$, so ΔPAY is similar to ΔSXY (AAA).

d 2.5 km

e **i** $\dfrac{15\sqrt{3}}{2}$ km **ii** $\dfrac{30 - 15\sqrt{3}}{2}$ km

f 7.765 km

5 **a** 3, 4, 5 triangle **b** $\angle APB$

c $AP = AQ$ (radii of smaller circle), $BP = BQ$ (radii of larger circle), $AB = AB$ (common), so $\Delta APE = \Delta AQB$ (SSS).

d **i** 53° **ii** 106°

e 53° ($\angle PFQ$ is angle on circumference standing on the same arc as $\angle PAQ$, the angle at the centre of circle)

6 **a** $\angle CEP = \angle DEQ$ (common), $\angle ECP = \angle EDQ = 90°$ (tangent $\perp$ radius), so ΔCPE is similar to ΔDQE (AAA).

b $\angle GFP = \angle GHQ = 90°$ (tangent $\perp$ radius), $\angle FGP = \angle HGQ$ (vertically opposite angles at G), so ΔGFP is similar to ΔGHQ (AAA).

c From ΔCPE and ΔDQE, $\frac{CE}{DE} = \frac{CP}{DQ}$

From ΔGFP and ΔGHQ, $\frac{FG}{GH} = \frac{PF}{HQ} = \frac{CP}{DQ}$ (radii), so

$\frac{CE}{DE} = \frac{FG}{GH}$ so $\frac{CE}{FG} = \frac{DE}{GH}$

7 **a** deviation of 4.30°

b $\sqrt{(275 - 220\cos 10°)^2 + (220\sin 10° + 75\sqrt{3})^2}$

≈ 177.94 m

8 **a** $x^2 + y^2 = 16$ **b** $y = 2x - 4$

c $P = \left(\frac{16}{5}, \frac{12}{5}\right)$, peg is 3.2 m east and 2.4 m north of the centre of the garden

9 **a** **i** $h = r\tan\theta$ **ii** $s = \frac{r}{\cos\theta}$

b **i** Area $(A) = \frac{1}{2}\pi r^2$ **ii** Area $(B) = r^2\tan\theta$

c $\frac{1}{2}\pi r^2 = r^2\tan\theta$, $\frac{\pi}{2} = \tan\theta$

d 57.5° **e** $4\pi \approx 12.57$ mm^2

10 **a** **i** 60° **ii** 30°

b $a + b = 90$

c **i** $CE = (10 - x)$ cm **ii** $DE = (10 - y)$ cm

d $x(10 - x) = y(10 - y) \Rightarrow 0 = 10(y - x) - (y^2 - x^2)$

$\Rightarrow 0 = 10(y - x) - (y - x)(y + x)$

$0 = (y - x)(10 - y - x)$

e **i** $\frac{xy}{2}$ cm^2 **ii** $\frac{1}{2}(10 - x)(10 - y)$ cm^2

f $\frac{xy}{2} = \frac{1}{2}(10 - x)(10 - y)$

$xy = 100 - 10y - 10x + xy$

$10(x + y) = 100$

$x + y = 10$

g $\frac{x(10 - x)}{2}$ cm^2

h $x = 6$ or 4