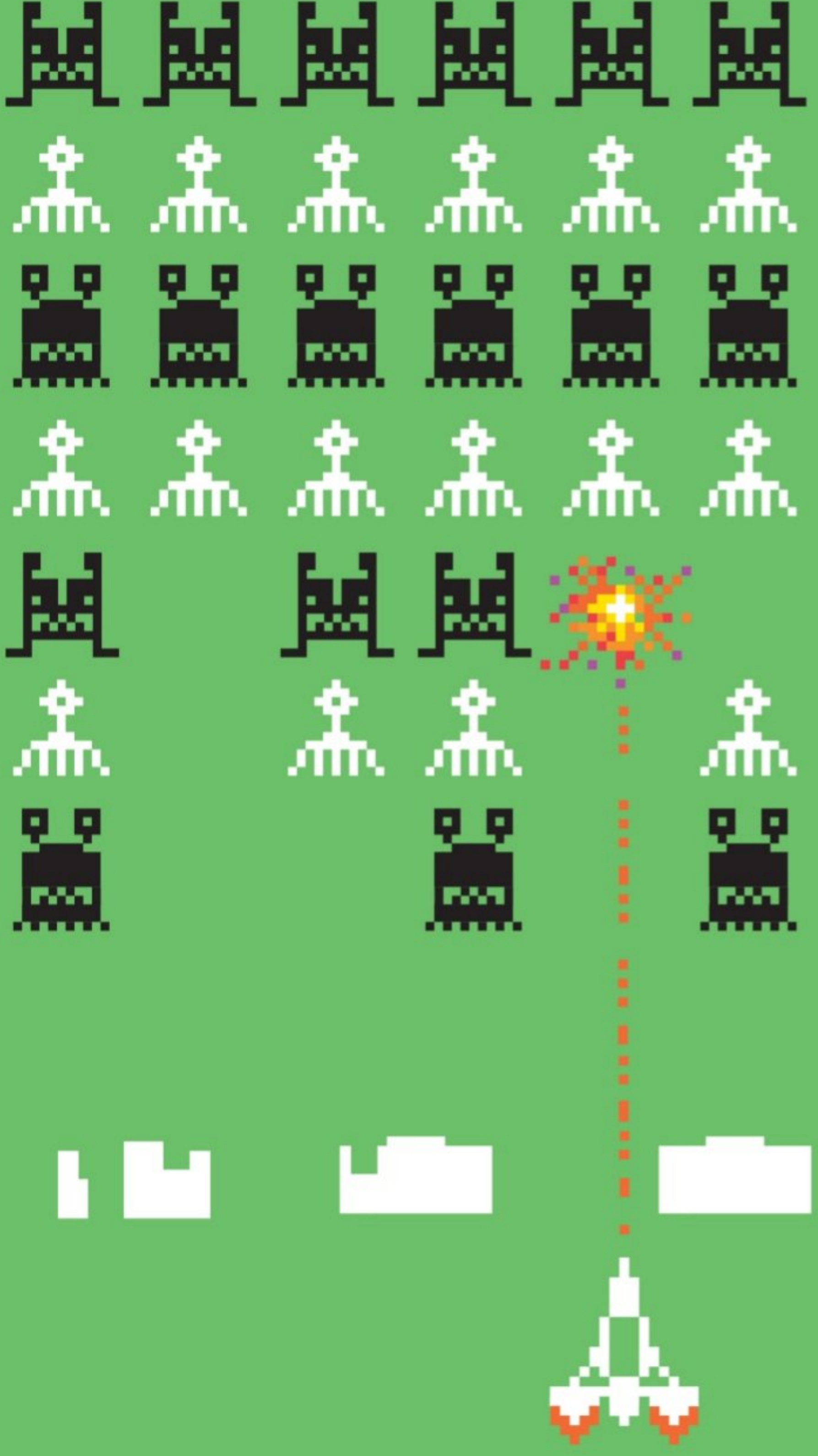
MATHEMATICS

OXFORD MATHS 10/10A



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OXFORD

VICTORIAN CURRICULUM

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Published in Australia by
Oxford University Press
Level 8, 737 Bourke Street, Docklands, Victoria 3008, Australia.

© Thomas Christiansen, Alexander Blanksby, Morgan Levick, Eugene Roizman,
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First published 2022

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ISBN 9780190329310

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Printed in Singapore by Markono Print Media Pte Ltd

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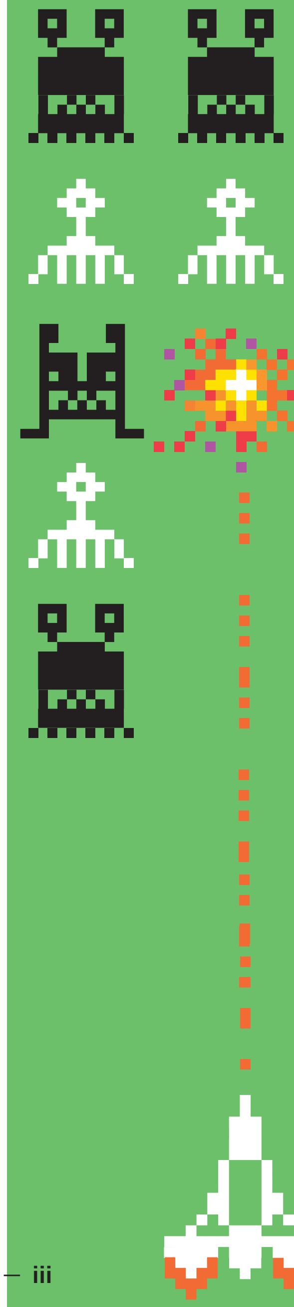
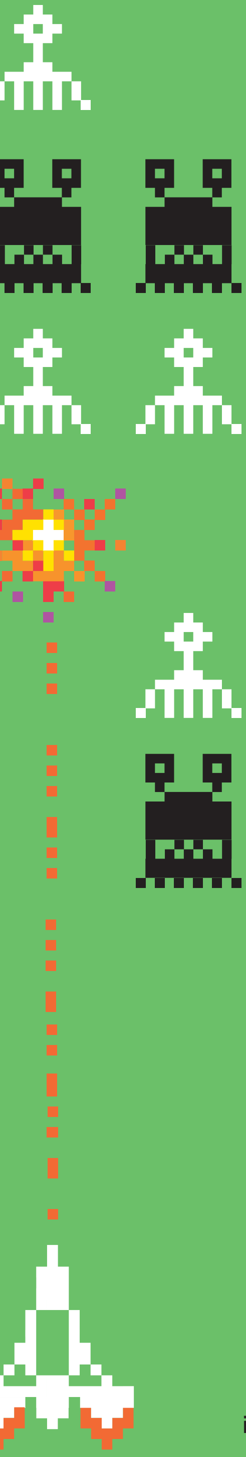
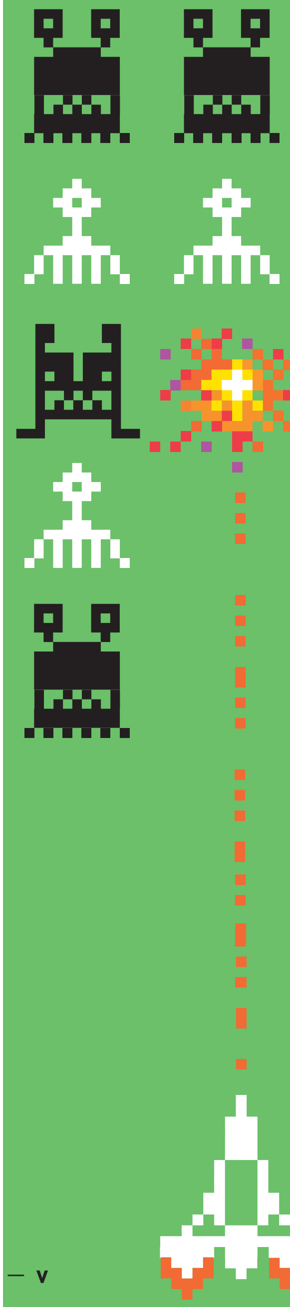






Introducing Oxford Maths 7–10

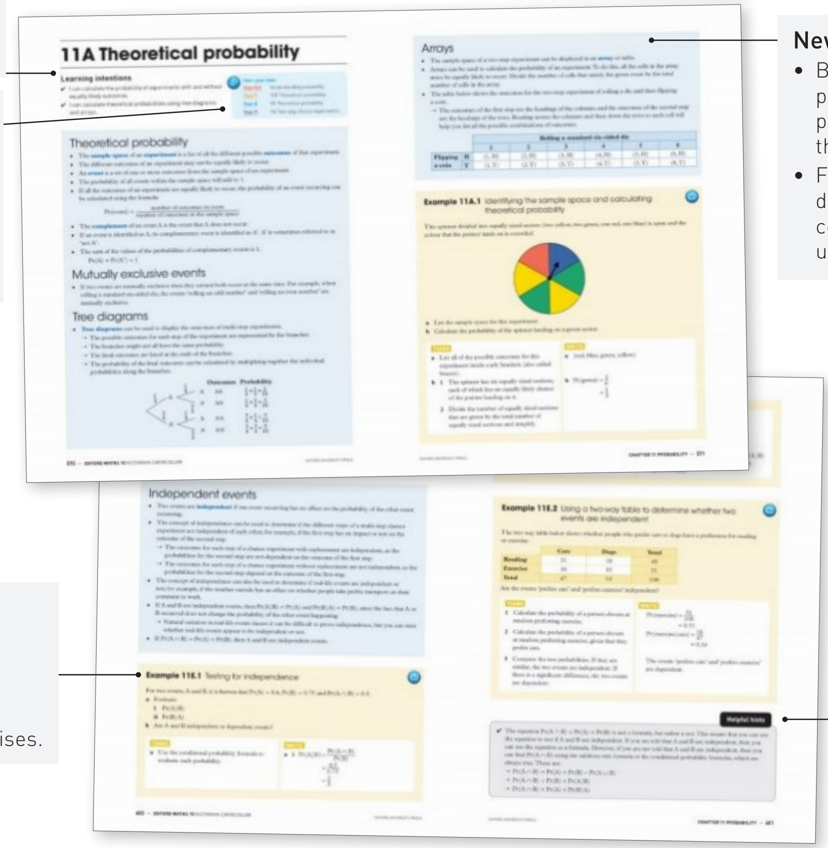
Oxford Maths 7-10 Victorian Curriculum utilises an innovative suite of print and digital resources to guide students on a focused mathematics journey. The series makes maths accessible to students with differing levels of understanding, increasing engagement by giving learners the opportunity to achieve success at their own skill level while also providing comprehensive syllabus coverage.

Key features of Student Books

- Complete access to all digital resources available on Student obook pro.
- Australian Maths Trust (AMT) spreads offer unique questions designed to challenge students and build engagement.
- STEAM projects encourage inter-disciplinary thinking.
- Semester reviews provide an opportunity to revise key concepts from each semester.
- An illustrated glossary of newly introduced mathematical terms.

Each chapter opens with:

- **Prerequisite skills** with reference to an online diagnostic pre-test and interactive skillsheets.
- **Curriculum links** to all relevant content descriptions in the VCAA mathematics syllabus.
- **Materials** used to complete the exercises.



Learning intentions

- Signpost the foundational skills being developed in each section.

Inter-year links

- Provide easy access to support and extension material from each of the 7–10 Student Books as students build knowledge year on year.

New theory

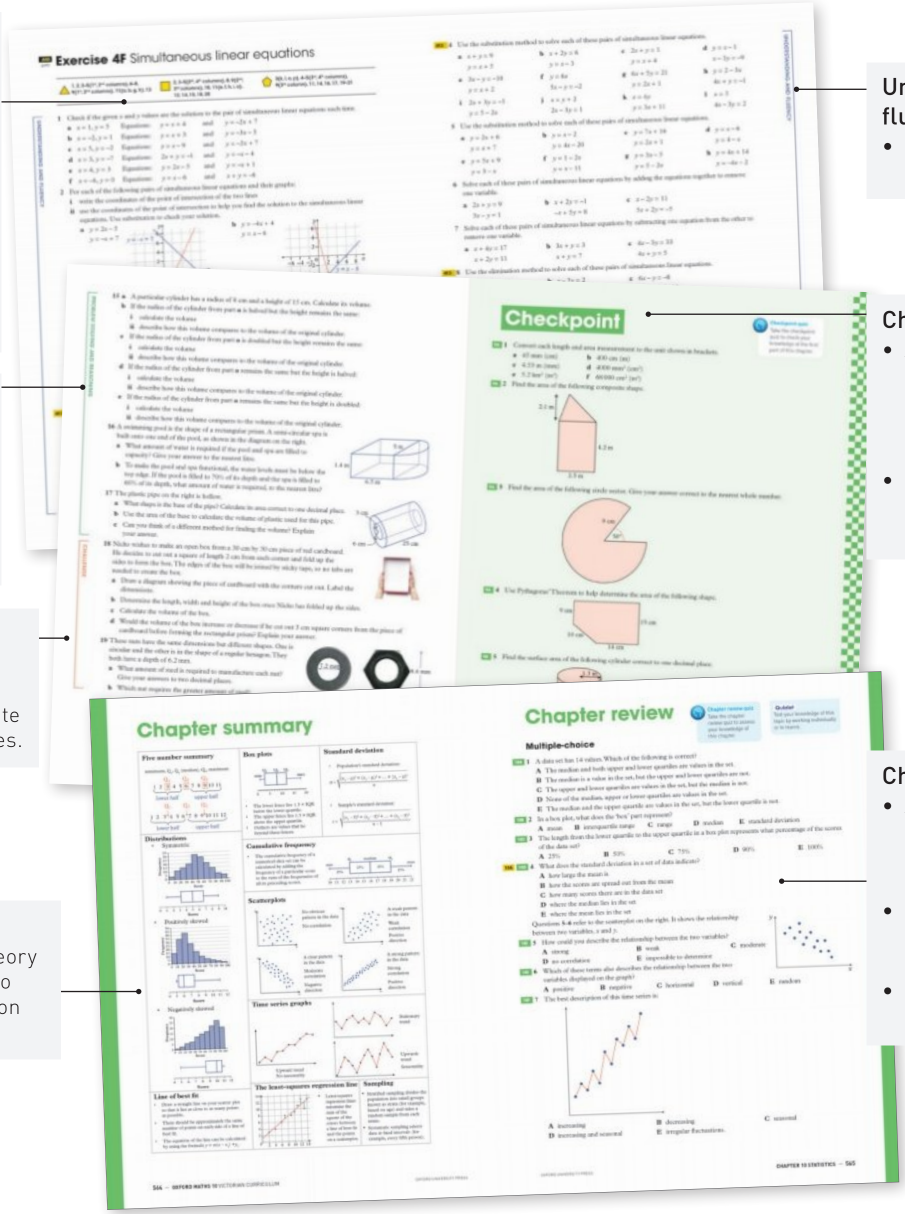
- Backed by the latest pedagogical research to promote engagement with the material.
- Filled with precise diagrams that bring key concepts to life, and aid understanding.

Worked examples

- Outline a step-by-step thought process for solving essential questions with direct reference to the exercises.

Helpful hints

- Provide additional strategies for tackling problems.
- Highlight important elements of the theory.
- Point out common misconceptions.



Differentiated learning pathways

- Each exercise is separated into three pathways, tailoring for students of all skill levels.
- Each pathway can be assigned based on results of the diagnostic pre-tests that are recommended at the beginning of every chapter.

Understanding and fluency

- Basic exercises dedicated to practising key concepts.

Problem solving and reasoning

- Comprehensive exercises bring together new ideas and provide engaging contexts from real-world problems.

Checkpoint

- A section in the middle of each chapter dedicated to summarising key skills and encouraging memory retention.
- Reference to an online checkpoint quiz to gauge student progress.

Challenge

- Advanced exercises designed to build engagment and anticipate future learning outcomes.

Chapter reviews

- Additional practice questions to further consolidate understanding at the end of each chapter.
- Reference to an online chapter review quiz to track results.
- Reference to Quizlet test to revise new terminology.

Chapter summary

- Condenses all the theory from each section into one accessible revision page.

Problem solving through design thinking

- Each STEAM project investigates a real-world problem that students are encouraged to problem-solve using design thinking.

Integrated STEAM projects

- Take the hard work out of cross-curricular learning with engaging STEAM projects. Two fully integrated projects are included at the end of each book in the series, and are scaffolded and mapped to the Science, Maths and Humanities curricula. The same projects also feature in the corresponding Oxford Humanities and Oxford Science series to assist cross-curricular learning.

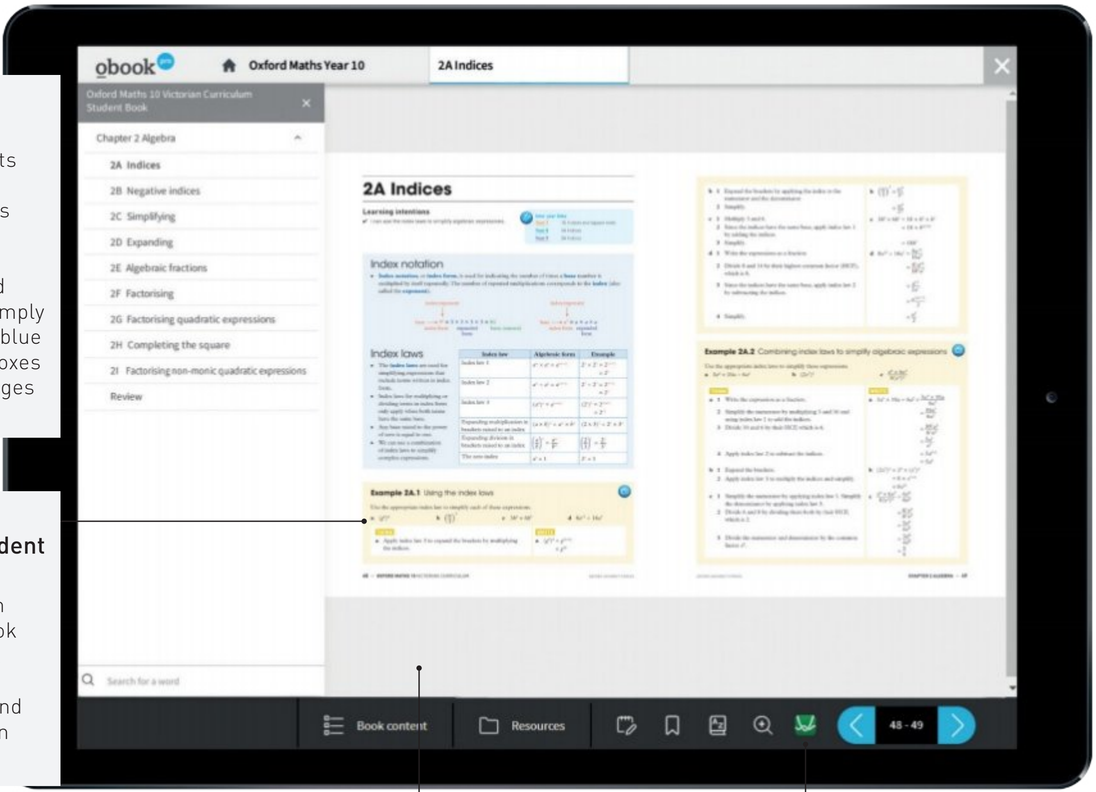
Full digital support

- Each STEAM project is supported by a wealth of digital resources, including student booklets (to scaffold students through the design-thinking process of each project), videos to support key concepts and skills, and implementation and assessment advice for teachers.

Key features of Student obook pro

obook pro

- > Student obook pro is a completely digital product delivered via Oxford's online learning platform, **Oxford Digital**.
- > It offers a complete digital version of the Student Book with interactive note-taking, highlighting and bookmarking functionality, allowing students to revisit points of learning.
- > A complete ePDF of the Student Book is also available for download for offline use and read-aloud functionality.



Integrated digital resources

- Integrated hotspots allow students to access diagnostics tests, quizzes, interactive skill sheets, videos and inter-year links simply by clicking on the blue digital resource boxes throughout the pages of the book.

Complete digital version of the Student Book

- The digital version of the Student Book is true to the print version, making it easy to navigate and transition between print and digital.

Toolbar features

- Notes can be added and saved to the text by simply selecting and highlighting.
- Bookmarks can be saved to any page.
- *Australian Concise Oxford Dictionary* can provide immediate definitions to any word within the text.

Desmos integration

- Our partnership with Desmos allows students to access a suite of calculator tools as they read through the text, providing convenient graphical support as well as the opportunity to investigate plane geometry and Cartesian coordinates.

Benefits for students

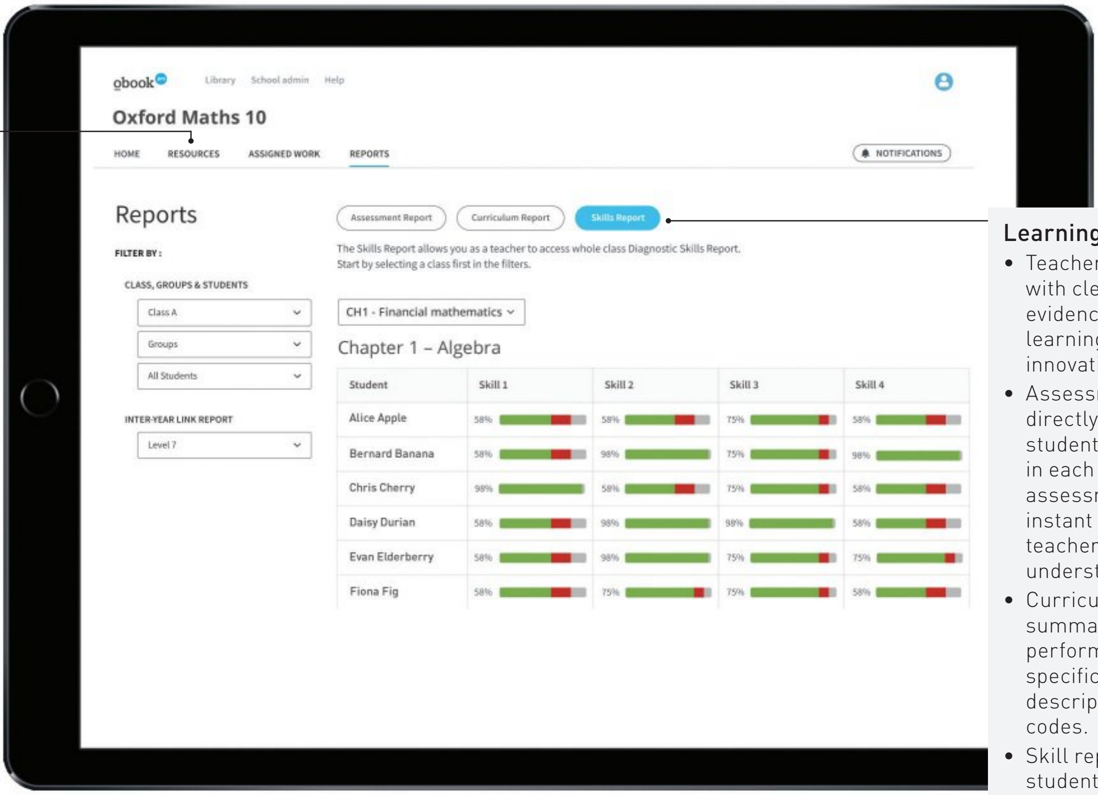
- > Integrated *Australian Concise Oxford Dictionary* look-up feature
- > Targeted instructional videos for every worked example question
- > Groundwork resources to support assumed knowledge
- > Interactive assessments to consolidate understanding
- > Auto-marked practice exam question sets
- > Integrated Quizlet sets, including real-time online quizzes with live leaderboards
- > Access to online assessment results to track progress.

Key features of Teacher obook pro

> Teacher obook pro is a completely digital product delivered via Oxford's online learning platform, **Oxford Digital**.

> Each chapter and topic of the Student Book is accompanied by full teaching support, including assessment reporting, worked solutions, chapter tests, detailed teacher notes and lesson plans.

> Teachers can use their Teacher obook pro to share notes and easily assign resources or assessments to students, including due dates and email notifications.



Oxford Maths 10

Reports

The Skills Report allows you as a teacher to access whole class Diagnostic Skills Report. Start by selecting a class first in the filters.

Chapter 1 – Algebra

Student	Skill 1	Skill 2	Skill 3	Skill 4
Alice Apple	58%	58%	75%	58%
Bernard Banana	58%	98%	75%	98%
Chris Cherry	98%	58%	75%	58%
Daisy Durian	58%	98%	98%	58%
Evan Elderberry	58%	98%	75%	75%
Fiona Fig	58%	75%	75%	58%

Learning pathway reports

- Teachers are provided with clear and tangible evidence of student learning progress through innovative reports.
- Assessment reports directly show how students are performing in each online interactive assessment, providing instant feedback for teachers about areas of understanding.
- Curriculum reports summarise student performance against specific curriculum content descriptors and curriculum codes.
- Skill reports indicate the students' understanding of a specific skill in mathematics.

Additional resources

- Each chapter of the Student Book is accompanied by additional interactive skillsheets, worksheets, investigations and topic quizzes to help students progress.

Benefits for teachers

> Diagnostic pre-tests and chapter tests that track students' progress against Study Design key knowledge, providing detailed learning pathway reports that differentiate each student's ability in each skill

> Assign reading and assessments to students either individually, or in groups – administration is taken care of!

> Ability to set-up classes, monitor student progress and graph results

> Worked solutions for every Student Book question

> Detailed teacher notes, teaching programs and lesson plans.

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Financial mathematics

Index

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

✔ Fractions, decimals and percentages
✔ Percentages
✔ Multiplying and dividing decimals

Curriculum links

- Connect the compound interest formula to repeated applications of simple interest using appropriate digital technologies (VCMNA328)

Materials

✔ Calculator

1A Calculating percentages

Learning intentions

- ✔ I can convert between fractions, decimals and percentages.
- ✔ I can calculate the percentage of a quantity.
- ✔ I can express one quantity as a percentage of another quantity.

Inter-year links

Years 5/6	Calculating percentages
Year 7	4I Calculating percentages
Year 8	3B Calculating percentages
Year 9	1A Calculator skills

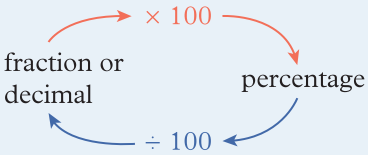
Fractions, decimals and percentages

- Fractions, decimals and **percentages** are closely related. They are different forms of numbers that can be used to represent the same quantities.

  For example, consider this table in which each row displays the same quantity in three different ways (as a percentage, as a fraction, and as a decimal).

Percentage	Fraction	Decimal
0.19%	$\frac{19}{1000}$	0.0019
10%	$\frac{1}{10}$	0.10
180%	$\frac{9}{5}$	1.80

- Percentages are often easier to read than fractions or decimals, because they allow some fractions and decimals to be expressed using whole numbers.
- In everyday life, it is common to express amounts as percentages, especially when they are amounts of money.
- Since per cent is literally 'per centum' or 'per hundred', we convert to and from percentages by adjusting the figures by a factor of 100, using the procedure shown below.



	Fraction	Decimal
Converting from a percentage to a …	Write the percentage as a fraction with a **denominator** of 100 and simplify if possible. $25\% = \frac{25}{100} = \frac{1}{4}$	Divide the percentage by 100 by moving the decimal point two places to the left. $33.3\% = 0.333$
Converting to a percentage from a …	Multiply the **numerator** by 100 and express the fraction in simplified or decimal form. $\frac{1}{5} = \frac{100}{5}\% = 20\%$	Multiply the decimal by 100 by moving the decimal point two places to the right. $0.75 = 75\%$

Percentage of a quantity

- To calculate a **percentage of a quantity**, write the percentage as a decimal and multiply it by the quantity.
  For example,

$$\begin{aligned}25\% \text{ of } \$160 &= 0.25 \times \$160\\ &= \$40\end{aligned}$$

A quantity as a percentage

- To write one quantity as a percentage of another quantity, divide the first quantity by the second quantity and multiply by 100.
  For example, to express \$12 as a percentage of \$60,

$$\begin{aligned}\frac{\$12}{\$60} \times 100\% &= 0.2 \times 100\%\\ &= 20\%\end{aligned}$$

Example 1A.1 Converting from percentages

Write 16% as a fraction and as a decimal.

THINK	**WRITE**
1 To convert a percentage to a fraction, write the percentage as the numerator of a fraction that has 100 as the denominator, then simplify the fraction by cancelling the common factor of 4.	$16\% = \frac{16}{100}$ $= \frac{4}{25}$
2 To convert a percentage to a decimal, divide the percentage by 100, by moving the decimal point of the percentage two places to the left.	$16\% = 16.0 \div 100.0$ $= 0.16$

Example 1A.2 Converting to percentages

Write each of these values as a percentage.

a $\frac{7}{25}$ **b** 0.835

THINK	**WRITE**
a To convert a fraction to a percentage, multiply the fraction by 100. Then simplify the fraction.	**a** $\frac{7}{25} = \frac{7}{25} \times 100\%$ $= \frac{7 \times \cancel{100}^{4}}{\cancel{25}^{1}}\%$ $= 28\%$
b To convert a decimal to a percentage, multiply the decimal by 100 by moving the decimal point two places to the right.	**b** $0.835 = 0.835 \times 100\%$ $= 83.5\%$

Example 1A.3 Calculating a percentage of a quantity

Calculate 12.5% of \$79.92.

THINK	**WRITE**
1 Convert the percentage to a decimal.	$12.5\% \text{ of } \$79.92 = 0.125 \times \$79.92$
2 Multiply by the quantity.	$= \$9.99$

Example 1A.4 Calculating a quantity as a percentage

Write \$345.75 as a percentage of \$400, correct to two decimal places.

THINK	**WRITE**
1 Divide the first quantity by the second quantity.	$\frac{\$345.75}{\$400} \times 100\% = 0.864\,375 \times 100\%$
2 Multiply by 100.	$= 86.4375\%$
3 Round to two decimal places.	$\approx 86.44\%$

Helpful hints

✔ When converting to and from percentages, it can be helpful to remember that 'per cent' means 'per hundred'. So the symbol '%' following an amount means we need to divide that amount by 100:

$$x\% = \frac{x}{100}$$

✔ Be careful to maintain the correct order of operations when typing expressions into calculators. Sometimes, parts of a calculation need to be entered separately before combining them to find the result. In other cases, brackets can be typed into the calculator to ensure the correct order of operations.

✔ The formula for writing one quantity as a percentage of another specified quantity can be thought of as a rearrangement of the formula for finding a percentage of a quantity.

$$\text{Quantity 1 as a percentage of quantity 2: } x\% = \frac{\text{quantity 1}}{\text{quantity 2}} \times 100\%$$

and:

$$\text{A percentage of quantity 2: quantity 1} = \frac{x\%}{100\%} \times \text{quantity 2}$$

ANS p648

Exercise 1A Calculating percentages

△ 1–8, 10–13, 14(a–c)

■ 4–7, 9, 11, 13, 15–17, 20

⬟ 4–7(b, d, f, h), 11, 13, 15, 16(g–i), 17–19, 21

1A.1 **1** Write each of these percentages as a decimal.

a 15% **b** 90% **c** 112% **d** $75\frac{1}{2}\%$

e 18.8% **f** 0.25% **g** 38.4% **h** 203%

UNDERSTANDING AND FLUENCY

2 Write each of these percentages as a fraction in simplest form.

a 21% **b** 85% **c** 225% **d** 64%

e 38% **f** 80.2% **g** 110% **h** $5\frac{1}{4}\%$

i $15\frac{3}{4}\%$ **j** 0.55% **k** $8\frac{2}{5}\%$ **l** 0.004%

1A.2 3 Write each of these fractions as a percentage.

a $\frac{3}{5}$ **b** $\frac{3}{8}$ **c** $\frac{7}{10}$ **d** $\frac{13}{20}$

e $\frac{11}{16}$ **f** $\frac{15}{24}$ **g** $\frac{23}{20}$ **h** $\frac{5}{2}$

4 Complete this table by filling in the missing values. Write all decimals correct to two decimal places and all fractions and percentages in their simplest exact form.

	Percentage	Fraction	Decimal
a		$\frac{1}{6}$	$0.1\dot{6}$
b		$\frac{2}{3}$	$0.\dot{6}$
c		$\frac{7}{6}$	
d	$85\frac{1}{6}\%$		

	Percentage	Fraction	Decimal
e	$46\frac{2}{3}\%$		
f	$12\frac{1}{3}\%$		
g		$\frac{11}{12}$	
h		$\frac{35}{9}$	

1A.3 5 Calculate each of these.

a 12% of \$480 **b** 20% of \$3000 **c** 35% of \$567 **d** 150% of \$888

e 65% of \$1920 **f** 5% of \$14 000 **g** 40% of \$10 500 **h** 225% of \$4880

6 Calculate each of these, correct to two decimal places.

a 11% of \$145.95 **b** $42\frac{1}{2}\%$ of \$180.99 **c** $12\frac{1}{2}\%$ of \$95.55 **d** 45.5% of \$852

e 17.5% of \$1550.25 **f** $7\frac{1}{4}\%$ of \$39.95 **g** 2.75% of \$29.45 **h** $10\frac{1}{8}\%$ of \$455.95

1A.4 7 Write these percentages, rounding your answers to two decimal places where appropriate.

a \$90 as a percentage of \$450 **b** \$80 as a percentage of \$130

c \$29 as a percentage of \$150 **d** \$36.50 as a percentage of \$145

e \$29.95 as a percentage of \$58 **f** \$152.50 as a percentage of \$65

g \$12.85 as a percentage of \$94.50 **h** \$140.20 as a percentage of \$255.95

i \$345.60 as a percentage of \$180.75

8 Shayden bought this scarf and, at the end of winter, sold it for \$6.55.

a Write Shayden's selling price as a fraction of the \$19.95 purchase price, in simplest form.

b Write the fraction you wrote for part **a** as a percentage and as a decimal, both rounded to two decimal places.

9 A store manager recorded the payment methods used by customers, and found that 35% of payments were in cash and the rest were electronic. Of the electronic payments, five-eighths were payments by credit card, three-tenths were electronically withdrawn from savings accounts and the rest were debit card purchases.

a What fraction of the total payments were in electronic form?

b Of all the electronic payments, what fraction describes the payments made with a debit card?

c What percentage of electronic payments were made with a debit card?

d Show that purchases made by withdrawals from savings accounts represent 19.5% of the store's total sales.

e What percentage of the store's total sales does each of the other forms of electronic payments represent?

10 Bruno buys raw coffee beans and roasts them to sell to customers. A 60 kg sack of raw beans has a wholesale price of $600. After roasting them, Bruno sells the beans for $14 per 250 g bag.

a What is the wholesale price of the raw beans per 250 g?

b Write Bruno's selling price as a fraction of the wholesale price using the price for 250 g of beans calculated in part **a**.

c Convert the fraction you wrote for part **c** to a percentage and briefly explain what this percentage means to Bruno's business.

11 A camping store offers a 20% discount to general customers and a 26.5% discount to VIP customers.

a How much discount would a non-VIP customer receive on the tent shown on the right?

b How much discount would a VIP customer receive on the tent?

c What fraction of the original price does each discount represent?

12 a Calculate $33\frac{1}{3}\%$ of $\$400$.   **b** Calculate $\frac{1}{3}$ of $\$400$.

c What do you notice about your answers for parts **a** and **b**? Why do you think this is so?

13 Calculations with percentages, fractions and decimals can be simplified using some basic number facts.

a Calculate each of these.

i 10% of $356.00   **ii** 5% of $356.00   **iii** 1% of $356.00

b Consider part **ai**. Describe a shortcut for calculating 10% of any amount without using a calculator.

c How does your answer for part **aii** compare with your answer for part **ai**? Describe a shortcut for calculating 5% of any amount without using a calculator.

d How does your answer for part **aiii** compare with your answer for part **ai**? Describe a shortcut for calculating 1% of any amount without using a calculator.

14 Test your methods from question **13** with each of these calculations. Use a calculator to check that each answer is correct.

a 10% of $438.50   **b** 5% of $438.50   **c** 1% of $438.50

d 1% of $229.85   **e** 5% of $1458.99   **f** 10% of $148.35

g 1% of $295.10   **h** 10% of $95.66   **i** 5% of $2458.45

15 a Explore how you can extend your findings from question **13**. Calculate each of the following values.

i 15% of $356.00   **ii** 20% of $356.00   **iii** $\frac{1}{2}\%$ of $\$356.00$

b Consider each percentage value in part **a**. Briefly describe a shortcut for calculating those percentages of an amount without using a calculator. (Hint: How does each percentage relate to 10%?)

16 Test your methods from question **15** with each of these calculations. Use a calculator to check that each answer is correct.

a 15% of $845.80   **b** 20% of $845.80   **c** $\frac{1}{2}\%$ of $\$845.80$

d 15% of $336.90   **e** $\frac{1}{2}\%$ of $\$1254.00$ **f** 20% of $265.00

g 20% of $4780.95   **h** $\frac{1}{2}\%$ of $\$45.50$ **i** 15% of $384.40

17 Calculations involving a percentage of an amount when the original amount is not known can be solved using the unitary method. Consider a laptop that is discounted by 12.5%, which represents a discount of $56.25.

a Let x represent the original (pre-discount) price of the laptop. The discount calculation is:

12.5% of $x = \$56.25$

For the unitary method, find how much 1% represents (one unit).
Calculate 1% of the original price. (Hint: Divide the amount we have found for 12.5% by 12.5 to find 1%.)

b The original price of the laptop represents the full amount, or 100%. Use your answer from part **a** to calculate 100% of the original price. (Hint: Multiply the amount you found for 1% by 100.)

c What is the original pre-discount price of the laptop?

18 You can use an alternative method to the one described in question **17** to obtain the same result.

PROBLEM SOLVING AND REASONING

a Reconsider the initial calculation 12.5% of $x = \$56.25$. Write the percentage as a decimal and show that the equation simplifies to $0.125x = \$56.25$.

b Look at the equation formed in part **a**. What operation can you perform to solve the equation for x?

c Solve the equation to determine the value for x.

d How does the value you obtained in part **c** compare with the value obtained in question **17b**?

e Consider the methods used in this question and in question **17**. Briefly explain why they produce the same result.

19 A shoe store has a 20% off everything sale.

a Anton buys a pair of shoes that normally cost \$80. How much does he pay for them?

b Tian buys a pair of shoes that normally cost \$108. How much does she pay for them?

c Pavel buys a pair of shoes and his receipt states that he saved \$13. What is the normal price of the pair of shoes he bought, and how much did he pay for them?

CHALLENGE

20 To find the discount price of an item, it is best to consider the percentage of the full price that someone must pay. For example, if a \$40 pair of shoes is selling for 30% off then the customer will pay 70% of the original price. Therefore, the selling price is $0.7 \times \$40 = \$28$.

a Find the selling price of a \$60 pair of jeans that is discounted 20%.

b Find the selling price of a \$90 jacket that is discounted 35%.

c Develop a formula to calculate the selling price, S, from the original price, P, after a discount, D%.

d Rearrange your formula to make the original price, P, the subject.

e Using your formula from part **d**, find the original price of an item that was purchased for \$56 during a 20% off everything sale.

f Using your formula from part **d**, find the original price of an item that was purchased for \$45 during a 75% off everything sale.

21 Ride-share companies like Uber use something called ‘surge pricing’ when many people are trying to book rides at the same time in a similar location. Surge pricing has two effects, both of which aim to fix the temporary imbalance between supply and demand. First, surge pricing reduces the number of customers, since not all of them will be willing to pay the higher fee. Second, surge pricing increases supply, since drivers who aren’t working will be alerted on their phone that they will be paid more than usual if they start driving.

Surge pricing is communicated with a multiplier. For example, ‘× 1.3’ would indicate that the price for the ride will be 1.3 times what it would cost during normal operations.

a Avi books a ride home that usually costs him \$23. If he takes this ride during a surge pricing period of ‘× 1.7’, how much will his ride home cost?

b Paolo books a ride during a surge pricing period and the app requires him to confirm that he agrees to the price multiplier of ‘× 1.4’. He pays \$38.50 for the ride. How much would this ride have cost Paolo during normal operations?

c In the early hours of the morning on New Year’s Day, after the fireworks, surge pricing can be very high in big cities. Shiloh books a ride when the multiplier is ‘× 3.2’ and pays \$53.76. How much would this ride have cost during normal operations?

d Shiloh’s driver Nadira keeps 75% of the fare for a ride she provides (the ride-share company keeps the other 25%). How much extra money did Nadira earn from driving Shiloh than she would have earned from the same ride during normal operations?

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Calculating percentages

CAS instructions
CAS basics

Topic quiz
1A

1B Financial calculations

Learning intentions

- ✔ I can calculate a percentage increase and a percentage decrease.
- ✔ I can determine percentage profit and percentage loss.

Inter-year links

Years 5/6	Calculating percentages
Year 7	4I Calculating percentages
Year 8	3C Financial calculations
Year 9	1C Mark-ups and discounts

Mark-ups and discounts

- A **mark-up** is the amount of money added to the original price of a product or service to determine its new selling price.

  selling price = original price + mark-up

- A percentage mark-up is an example of a **percentage increase** for which the mark-up is expressed as a percentage of the original price. The selling price after a mark-up can be calculated using the rule:

  selling price = (100% + percentage mark-up) × original price

  For example, if a $250 dress is marked up by 10%, then:

  selling price of dress = (100% + 10%) × $250
  = 110% × $250
  = $275

- A **discount** is the amount of money subtracted from the original price of a product or service to determine its new selling price.

  selling price = original price – discount

- A percentage discount is an example of a **percentage decrease** for which the discount is expressed as a percentage of the original price. The selling price after a discount can be calculated using the rule:

  selling price = (100% – percentage discount) × original price

  For example, if an $80 beauty treatment is discounted by 20%, then:

  selling price of treatment = (100% – 20%) × $80
  = 80% × $80
  = $64

Profit and loss

- A **profit** occurs when the selling price is higher than the original cost.

  profit = selling price – cost price

- A **loss** occurs when the selling price is lower than the original cost.

  loss = cost price – selling price

profit ⟶ selling price > cost price

loss ⟶ selling price < cost price

- **Percentage profit** and **percentage loss** are measures of the profit or loss expressed as a percentage of the cost price.

$$\text{percentage profit} = \frac{\text{profit}}{\text{cost price}} \times 100\%$$

$$\text{percentage loss} = \frac{\text{loss}}{\text{cost price}} \times 100\%$$

For example, if a greengrocer buys pineapples from a farmer for \$1.20 each and then sells them for \$3.00 each, the greengrocer makes a profit of \$1.80 on the sale of each pineapple. To express this as a percentage profit:

$$\begin{aligned}\text{percentage profit} &= \frac{\$3.00 - \$1.20}{\$1.20} \times 100\% \\ &= \frac{\$1.80}{\$1.20} \times 100\% \\ &= 150\%\end{aligned}$$

Example 1B.1 Working with percentage mark-ups

Calculate the selling price after a $55\frac{1}{2}\%$ mark-up on the original price of \$600.

THINK	**WRITE**
1 Write the formula for finding the selling price after a mark-up and substitute the relevant values into it.	selling price = (100% + percentage mark-up) × original price $= (100\% + 55\frac{1}{2}\%) \times \$600$
2 Add the percentages and convert the percentage fraction to a decimal.	$= 155.5\% \times \$600$ $= 1.555 \times \$600$
3 Multiply the decimal by the original price.	= \$933

Example 1B.2 Working with percentage discounts

Calculate the selling price after a $17\frac{1}{2}\%$ discount on the original price of \$229.95.

THINK	**WRITE**
1 Write the formula for finding the selling price after a discount and substitute the relevant values into it.	selling price = (100% – percentage discount) × original price $= (100\% - 17\frac{1}{2}\%) \times \$229.95$
2 Subtract the percentages and convert the percentage fraction to a decimal.	$= 82.5\% \times \$229.95$ $= 0.825 \times \$229.95$
3 Multiply by the original price and round the price to the nearest cent (two decimal places).	= \$189.71

Example 1B.3 Calculating percentage profit and percentage loss

For each situation below, write the profit or loss amount as a percentage of the original cost, correct to two decimal places where appropriate.

a A couch is manufactured for $455 and is later sold for $799.99.

b A computer is bought for $2150 and then sold for $400.

THINK	WRITE
a **1** Identify that a profit has been made because the selling price is greater than the cost price. Calculate the profit by subtracting the cost price from the selling price.	**a** profit = selling price – cost price = $799.99 – $455 = $344.99
2 Write the percentage profit as a fraction, with the profit as the numerator and the cost price as the denominator.	profit as a fraction $= \frac{\$344.99}{\$455}$
3 Express the fraction in decimal form, correct to four decimal places.	≈ 0.7582
4 Convert to a percentage by multiplying by 100%.	percentage profit $\approx 0.7582 \times 100\%$ $= 75.82\%$
b **1** Identify that a loss has been made by observing that the selling price is less than the cost price. Calculate the loss by subtracting the selling price from the cost price.	**b** loss = cost price – selling price = $2150 – $400 = $1750
2 Write the percentage loss as a fraction, with the loss as the numerator and the cost price as the denominator.	loss as a fraction $= \frac{\$1750}{\$2150}$
3 Express the fraction in decimal form, correct to four decimal places.	≈ 0.8140
4 Convert to a percentage by multiplying by 100%.	percentage loss $\approx 0.8140 \times 100\%$ $= 81.40\%$

Helpful hints

✔ You can also think of loss as negative profit.

For example, if a bicycle is bought for $350 and sold for $200, then you can determine the loss using the 'profit' formula:

profit = selling price – cost price
= $200 – $350
= –$150

So, there was a $150 loss made on the bicycle.

✔ In some cases, you might prefer to calculate the discount/mark-up amount first, then add it to or subtract it from the original price. This method can be represented by expanding the brackets in the formulas given earlier.

marked up selling price = original price + percentage mark-up × original price
discounted selling price = original price – percentage discount × original price

For example, a 25% discount on the price of a $36 house plant can be calculated like this:

$$\begin{aligned}\text{discount} &= 25\% \times \$36\\ &= 0.25 \times \$36\\ &= \$9\\ \text{selling price} &= \$36 - \$9\\ &= \$27\end{aligned}$$

Can you see how this method gives the same result as the method used earlier would have?

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Exercise 1B Financial calculations

△ 1–7, 9, 11, 13–15, 16(a–d) ■ 2–3(b, d, f), 5, 6, 8–10, 12, 17, 18, 20 ⬟ 5, 6, 9, 14, 16(e–h), 18–21

UNDERSTANDING AND FLUENCY

1 Calculate each of these, rounding to two decimal places where appropriate.

a 15% of $150 **b** 35% of $3200 **c** 8.5% of $957.65

d $122\frac{1}{2}$% of $400 **e** 150% of $30 **f** 230% of $75

1B.1 **2** Calculate the selling price for each of these, rounding to the nearest cent.

a 35% mark-up on a price of $340 **b** $15\frac{1}{2}$% mark-up on a price of $18 600

c 40.5% mark-up on a price of $844.20 **d** 125% mark-up on a price of $350.50

e 265% mark-up on a price of $1420.90 **f** $165\frac{1}{2}$% mark-up on a price of $680.95

1B.2 **3** Calculate the discounted selling price for each of these.

a 25% discount on a price of $185

b $17\frac{1}{2}$% discount on a price of $9200

c 31.5% discount on a price of $650.95

d 12.5% discount on a price of $165.50

e 28.5% discount on a price of $325.99

f $45\frac{1}{2}$% discount on a price of $499.45

4 For each of these, determine:

i the selling price **ii** the mark-up or discount amount.

a A smart phone bought for $89.90 is sold with a mark-up of 155%.

b A piece of jewellery originally marked at $1495.80 is offered for sale with a discount of 12.5%.

c Sport shorts marked at $49.90 are offered for sale with a 35% discount.

d Network cabling purchased for $2.85 per metre is marked up by 480%.

1B.3 **5** For each of the situations below:

i state whether a profit or loss has been made and determine the amount

ii write the amount of the profit or loss as a percentage of the original cost, correct to two decimal places where appropriate.

a A jacket is bought for $84.95 and later sold for $53.60.

b A market stallholder buys fruit for $1.70 per kilogram and sells it for $5.95 per kilogram.

c A new car is purchased for $37 935 and sold later for $17 390.

d A laptop is bought for $548 and sold for $652.

e A bookstore buys a book for $32.95 and sells it for $47.80.

f Computer games are bought at a market for $42.95 and sold later for $16.

6 The following are the original prices of some goods and their percentage mark-up or discount when they are sold. In each case, calculate:

i the selling price after the discount or mark-up

ii the amount of the discount or mark-up.

Where appropriate, round your answers to the nearest cent.

a \$1285 discounted by 20%
b \$1800 discounted by $15\frac{1}{2}\%$
c \$540 marked up 185%
d \$450 marked up 125.5%
e \$346.95 discounted by 17.5%
f \$850.40 discounted by 62.5%
g \$3458.99 marked up $87\frac{1}{2}\%$
h \$6240.55 marked up 255%

7 A school fête raised \$2915.65 for charity. Compare this with the \$2584.80 raised the previous year.

a How much more money was raised than the previous year?

b Write the increase as a percentage of the previous year's amount, correct to two decimal places.

8 Diana buys an MP3 player for \$141.45 and sells it later for \$99.

a What is the amount of Diana's loss?

b Write the amount of Diana's loss as a fraction of the price she originally paid.

c State the loss as a percentage of the original price, rounded to two decimal places.

9 For each of these:

i state the amount of the profit or loss

ii write the profit or loss amount as a fraction of the original price

iii write the profit or loss as a percentage of the original price (rounded to the nearest 1%).

a original price \$68, selling price \$92
b original price \$30 000, selling price \$12 850
c original price \$145.50, selling price \$99
d original price \$37.95, selling price \$12.50
e original price \$699.45, selling price \$1225.50
f original price \$294.58, selling price \$346.99

10 The coach of a soccer club decided to purchase new match balls. The balls were originally priced at \$84.95 each but were being offered at an 18% discount.

a Based on the advertised discount, what percentage of the original price will the new price be?

b Calculate the selling price of each soccer ball.

c What is the amount saved per ball as a result of the discount?

d Ordering more than 20 balls makes the soccer club eligible for an additional $5\frac{1}{2}\%$ discount. The coach decides to order 25 balls. How much will the club pay for the 25 balls?

e As a result of the bulk purchase of 25 balls, what is the price of each soccer ball?

11 Bonnie is planning to hire a dress for her school formal. When she first enquired, the hire charge was quoted as a booking deposit of \$50 plus \$85 on collection of the dress. When she next contacted the store, she learned that the collection fee had increased by 12.5%.

a What percentage of the original collection fee will the new collection fee be?

b What is the total amount Bonnie will pay to hire the dress, including the booking deposit?

c By how much has the hire charge increased, to the nearest cent?

12 Ricardo needs to purchase a new camera for his photography course. His local camera store is offering cameras for sale at a discount of 18%. Ricardo is interested in purchasing the camera shown on the right, priced at $1248.

- **a** What is the selling price of the camera following the discount?
- **b** As a valued customer of the store, Ricardo receives an additional 4.5% discount. Apply this discount to your answer for part **a** to determine the amount Ricardo will pay for the camera.
- **c** Ricardo received an 18% discount and a 4.5% discount. Is your answer for part **b** the same as it would have been if Ricardo had received a 22.5% discount (18% + 4.5%) on the original price?
- **d** What was the total discount amount Ricardo received on this purchase?
- **e** Write the amount of the discount as a percentage of the original price.
- **f** The percentage you wrote in part **e** represents the successive discounts as a single percentage amount. How does this single amount compare with the two successive percentage discounts of 18% and 4.5%?

13 Calculate the original price in each of these situations. Where necessary, round each answer to the nearest cent. (Hint: Substitute the given values into the formula for percentage discount or percentage mark-up. Let the original price be represented by the pronumeral x, then rearrange the formula to solve it for x.)

- **a** A smart phone sells for $297.50 following a discount of 15%.
- **b** A sport skirt sells for $37.50 after a discount of 17.5%.
- **c** Jewellery sells for $229.95 after a mark-up of 245%.
- **d** An artist sells her printed T-shirt for $20.95 following a 135.5% mark-up.

14 Dennis plans to buy a new computer game that is for sale at a 15% discount. He is about to pay when he learns that a further 4.25% has been taken off the new price, after the 15% discount has been applied.

- **a** Dennis expected to pay $74.46 for the game. What is the selling price after the additional discount?
- **b** What was the original price of the game?
- **c** What single calculation can be applied to the original price to determine the selling price after the two successive discounts of 15% and 4.25%?

15 James works in telephone sales and, for his gross weekly income, receives 2.75% commission on the total value of all his sales plus a retainer of $450. In a week when James' sales total $12 452, what was his gross weekly income?

16 Laura earns a weekly retainer of $925.50 and 4.125% commission on the total value of her weekly sales. Calculate her gross income for the weeks in which the total value of her sales were:

a $0
b $15 000
c $175
d $21 587
e $1648.90
f $6381.45
g $489
h $8468.20

PROBLEM SOLVING AND REASONING

17 Elijah is a car salesperson who currently earns $6000 per month. The manager of the car dealership decides to offer a different payment structure to employees in an effort to encourage them to sell more cars. Elijah is asked if he wants to continue earning $6000 per month or if he wants to choose one of the two new payment structures.

Option 1: Earn a fixed salary of $6000 per month.

Option 2: Earn $4000 per month plus an extra $350 per car sold.

Option 3: Earn $3000 per month plus 2% of the sale price of each car sold.

Last month Elijah sold 6 cars, which sold for a total of $152 000.

Last year Elijah sold 70 cars, which sold for a total of 1.98 million dollars.

a Based on his last month of sales, which payment option should Elijah choose?

b Based on his last year of sales, which payment option should Elijah choose?

c Is there any logical reason for the choice if Elijah chooses option 1?

18 Francesca decides to sell her old laptop online for $800. In the listing she declares that she will drop the price by 10% every week until it is sold. Emilia is interested in the laptop and sends Francesca the following message: 'Hi Francesca. Can you please clarify how the price will drop? Will it drop by 10% of the original price each week, or by 10% of the current price each week?'

a Calculate the price of the laptop after one week for each type of price drop described.

b Calculate the price of the laptop after two weeks for each type of price drop described.

c Francesca replies that the price will drop by 10% of the new price each week. If Emilia is willing to pay $500 for the laptop, how many weeks after the listing was posted must she wait? Justify your answer with calculations.

CHALLENGE

19 A department store holds a sale in which all items are 30% off. Sarah is an employee of the store and also receives a 20% staff discount. However, this 20% comes off the sale price of an item, not the original price. Sarah plans to buy a pair of jeans, a shirt, and a pair of shoes.

a If the jeans normally cost $80, at what price can Sarah purchase them?

b If the shirt normally costs $30, at what price can Sarah purchase it?

c After both the discounts are applied, what is the total percentage discount Sarah receives during the sale?

d Use your result from part **c** to find the normal price of the pair of shoes, if Sarah paid $64.40 for them.

e If Sarah's 20% staff discount came off the price before the 30% discount came off due to the sale, would the total percentage discount Sarah receives be more, less, or the same? Explain with calculations.

20 Jacob has $1000 invested in shares. One day the stock price for his shares drops by 10% and the next day the stock price increases by 10%.

a Jacob assumes his shares will be worth $1000 again, but he is mistaken. How much are Jacob's shares worth now?

b Explain why a decrease of 10% followed by an increase of 10% does not give back the original amount.

21 Alex has $3000 worth of shares that she believes will increase in value. Over the course of one month, the value of the shares increased by 50%. The next month the share price also increased by 50%.

a Explain, without calculations, why Alex's shares are now worth more than twice her initial investment.

b How much are Alex's shares worth?

c What is the percentage increase in the value of Alex's shares, from her initial $3000 investment to their current value?

Check your Student obook pro for these digital resources and more:

Interactive skillsheet Mark-ups, discounts and GST	**Interactive skillsheet** Percentage profit and percentage loss	**Topic quiz** 1B

1C Simple interest

Learning intentions

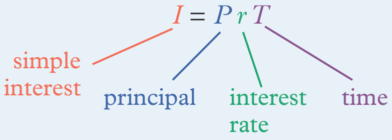
- ✔ I can calculate simple interest.
- ✔ I can calculate the interest rate, principal and time of an investment or loan.

Inter-year links

Year 7	6H Solving equations using inverse operations
Year 8	6B Solving equations using inverse operations
Year 9	1E Simple interest

Calculating simple interest

- **Interest** is a sum of money added to an initial amount that was either borrowed or invested.
  - → If the initial sum of money was borrowed then it is a **loan** and the borrower is charged interest. The borrower must repay the interest as well as the initial amount borrowed.
  - → If the initial money was invested then it is an **investment** and the investor is paid the interest.
- **Simple interest** can be calculated using the formula:



  - → I = interest, the amount a borrower is charged by a lender, or the amount that an investor is paid for his or her investment
  - → P = **principal**, which is the initial amount of money borrowed or invested
  - → r = **interest rate**, the fraction/decimal/percentage that the value of the investment/loan increases by over a given interval of time, usually given in units per annum (p.a.), meaning 'per year'
  - → T = **time**, the period of time that has passed since the principal was borrowed or invested, usually expressed in years.

Solving for other quantities using the simple interest formula

- The simple interest formula is not only used to calculate simple interest. The equation can also be used to find any one of the values (principal, interest rate or time) when the other three quantities are known.

$I = P r T$ Solve for principal

$I = P r T$ Solve for interest rate

$I = P r T$ Solve for time

- To find P, r or T, if the other three values are known, use inverse operations to solve the equation by dividing both sides of the formula by the two known quantities on the right-hand side.

Example 1C.1 Calculating simple interest

For a loan of $\$18\,000$, borrowed at an interest rate of 6.8% p.a. for 21 months, calculate:

a the amount of simple interest charged

b the total amount repaid at the end of the loan

c the amount of each repayment if equal repayments are made monthly.

THINK	**WRITE**
a **1** Identify the known values. Write the rate as a decimal and the time in years.	**a** $P = \$18\,000$ $r = 6.8\%$ $= 0.068$ $T = 21$ months $= \frac{21}{12}$ years $= 1.75$ years
2 Substitute the values for the pronumerals in the formula and calculate *I*.	$I = PrT$ $I = \$18\,000 \times 0.068 \times 1.75$ $= \$2142$
b The total repaid at the end of the loan is the interest added to the principal.	**b** Total amount repaid $= \$18\,000 + \$2142$ $= \$20\,142$
c The amount of each repayment is found by dividing the total amount to be paid by the number of instalments.	**c** Number of instalments $= 21$ Amount of each repayment $= \frac{\$20\,142}{21}$ $= \$959.14$

Example 1C.2 Solving for other quantities in the simple interest formula

Use the simple interest formula, $I = PrT$, to help you find each unknown value below.

a $I = \$800$, $P = \$4000$, $r = 1\%$, $T = ?$

b $I = \$3279.60$, $P = ?$, $r = 4.5\%$, $T = 4$ years

c $I = \$20\,150$, $P = \$124\,000$, $r = ?$, $T = 15$ months

THINK	**WRITE**
a **1** Identify the known values. Write the rate as a decimal.	**a** $I = \$800$ $P = \$4000$ $r = 1\%$ $= 0.01$
2 Substitute the known values into the formula and simplify.	$I = PrT$ $\$800 = \$4000 \times 0.01 \times T$ $= \$40 \times T$
3 Use inverse operations to solve the equation for *T*.	$\frac{\$800}{\$40} = \frac{\$40 \times T}{\$40}$ $T = 20$
4 Write the answer in years.	The time is 20 years.

THINK	WRITE
b **1** Identify the known values. Write the rate as a decimal and the time in years.	**b** $I = \$3279.60$ $r = 4.5\%$ $= 0.045$ $T = 4$ years
2 Substitute the known values into the formula and simplify.	$I = PrT$ $\$3279.60 = P \times 0.045 \times 4$ $= P \times 0.18$
3 Use inverse operations to solve the equation for P.	$\frac{\$3279.60}{0.18} = \frac{P \times 0.18}{0.18}$ $P = 18\,220$
4 Write the answer in dollars.	The principal is $18 220.
c **1** Identify the known values. Write the time in years.	**c** $I = \$20\,150$ $P = \$124\,000$ $T = 15$ months $= \frac{15}{12}$ years $= 1.25$ years
2 Substitute the values in the formula and simplify.	$I = PrT$ $\$20\,150 = \$124\,000 \times r \times 1.25$ $= \$155\,000 \times r$
3 Use inverse operations to solve the equation for r.	$\frac{\$20\,150}{\$155\,000} = \frac{\$155\,000 \times r}{\$155\,000}$ $r = 0.13$
4 Write the answer as a percentage per annum.	The interest rate is 13% p.a.

Helpful hints

✔ Instead of substituting the known values into the formula and using inverse operations to find the unknown value, try rearranging the simple interest formula to create a new formula with the unknown value on the left-hand side, and the known values on the right-hand side. This is called changing the subject of the formula. If you do this, you will be able to substitute the known values into your new formula and calculate the unknown value immediately!

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Exercise 1C Simple interest

△ 1–7, 8(a–d), 10, 11, 14, 15

□ 3–7, 8(e–h), 9, 12, 15, 16, 19

⬠ 3, 4, 8(e–h), 13, 15, 17, 18, 20, 21

1 Convert each of these times to years. (Hint: 1 year = 12 months = 52 weeks = 365 days)

a 24 months
b 48 months
c 27 months
d 45 months
e 7 months
f 52 weeks
g 65 weeks
h 4 weeks
i 312 weeks
j 78 weeks
k 730 days
l 73 days

UNDERSTANDING AND FLUENCY

2 Calculate the simple interest in each of these cases.

a $P = \$5000$, $r = 6\%$, $T = 4$ years

b $P = \$850$, $r = 12\%$, $T = 2$ years

c $P = \$125\,000$, $r = 8.4\%$, $T = 3$ years

d $P = \$64\,000$, $r = 3.6\%$, $T = 5$ years

e $P = \$9550$, $r = 10\%$, $T = 2.5$ years

f $P = \$22\,000$, $r = 12.5\%$, $T = 3$ years

1C.1 3 For each loan below, calculate the:

i amount of simple interest charged

ii total amount repaid at the end of the loan

iii amount of each repayment if equal repayments are made monthly.

- **a** $8000 borrowed at an interest rate of 4% p.a. over 2 years
- **b** $15 000 borrowed at an interest rate of 10% p.a. over 4 years
- **c** $1250 borrowed at an interest rate of 5.4% p.a. over 3 years
- **d** $14 000 borrowed at an interest rate of 8.2% p.a. over 30 months
- **e** $25 000 borrowed at an interest rate of 6.8% p.a. over 18 months
- **f** $6550 borrowed at an interest rate of 4.5% p.a. over 104 weeks

4 For each investment below, calculate the:

i amount of simple interest earned

ii value of the investment at the end of the term.

- **a** $12 000 invested at an interest rate of 3% p.a. for 3 years
- **b** $66 000 invested at an interest rate of 5% p.a. for 2 years
- **c** $125 000 invested at an interest rate of 6% p.a. for 2.5 years
- **d** $5500 invested at an interest rate of 4.2% p.a. for 12 months
- **e** $8250 invested at an interest rate of 8.5% p.a. for 15 months
- **f** $4580 invested at an interest rate of 3.6% p.a. for 36 months

5 Find the value of T to determine how long these earnings on investments would take.

a How long will it take an investment of $10 000 with a simple interest rate of 3.5% p.a. to earn $1400?

b How long does $1600 invested at an interest rate of 8% p.a. take to earn $320 in simple interest?

c How long will it take an investment of $8500 with a simple interest rate of 5.6% p.a. to earn $1428?

6 Find the value of P to determine the principal in each of these situations.

a How much needs to be invested for 4 years at an interest rate of 7% p.a. to earn $4200 in simple interest?

b How much was borrowed at an interest rate of 8.8% p.a. over 2.5 years if it cost the borrower $1430 in simple interest?

c How much was borrowed at an interest rate of 4.75% p.a. over 3 years if it cost the borrower $2850 in simple interest?

7 Find the value of r to determine the interest rate in each of these situations.

a What annual interest rate is applied to a loan of $8000 if the total simple interest charged over 2 years is $704?

b What annual interest rate is applied to a loan of $16 000 if the total simple interest charged over 4.5 years is $7560?

c What annual interest rate is applied to a $25 000 investment if the total simple interest earned over 30 months is $2812.50?

1C.2 8 Find each unknown value below.

a $I = ?$, $P = \$8000$, $r = 3\%$, $T = 9$ years

b $I = \$1000$, $P = \$5000$, $r = 5\%$, $T = ?$

c $I = \$4680$, $P = ?$, $r = 6.5\%$, $T = 4$ years

d $I = \$8160$, $P = \$64\,000$, $r = ?$, $T = 18$ months

e $I = ?$, $P = \$12\,000$, $r = 7.2\%$, $T = 3$ years

f $I = \$3864$, $P = \$9200$, $r = 8.4\%$, $T = ?$

g $I = \$4320$, $P = ?$, $r = 4.8\%$, $T = 4.5$ years

h $I = \$675$, $P = \$22\,500$, $r = ?$, $T = 15$ months

9 If the P values in question **8** represent amounts borrowed, what is the total amount to be repaid in each case?

10 For the smart television shown, an electronics store offers a purchase plan in which simple interest is charged at 12.75% p.a. for 20 months.

a What is the total amount of interest charged for the term of the purchase plan?

b What is the total cost of the television if the purchase is made with this plan?

c If the total amount to be repaid is divided into equal monthly instalments for the term of the plan, how much is each repayment?

d The store offers another option for customers who pay cash. This option involves a discount of 2.5% off the advertised price. What is the price for cash customers?

e Calculate the difference in the price of the same television purchased using each of the two options.

11 A bank offers an interest rate of 1.4% p.a. on its everyday accounts, with interest calculated on the daily balances. Consider the account balances shown below.

Date	Transaction	Amount	Balance
01/05	Opening balance		$985.90
09/05	Withdrawal at ATM	$60.00	$925.90
11/05	Deposit (Pay)	$875.55	$1801.45
20/05	EFTPOS Purchase	$185.95	
22/05	Gym membership	$69.50	
25/05	Deposit (Pay)	$875.55	
31/05	Interest for May		

a The account balances following three of the transactions are not shown. State the missing account balances.

b The opening balance of $985.90 applies for the first eight days of the month because each new balance applies on the date the transaction is made and until the next transaction. How many days does each balance on this account apply for?

c For each new balance on the account, calculate the simple interest based on the number of days to which each balance applies. Remember that 8 days is equivalent to $\frac{8}{365}$ years.

d Add all the amounts from part **c** to calculate the total interest for the month.

e What is the account balance at the end of May, if the total interest is added to the account at the end of the last day of the month?

12 A bank is offering these interest rates to attract investment in term deposits. Interest is calculated at the end of the investment term.

Bank interest rates per annum for different investment amounts and terms				
Terms	**Amounts invested**			
	\$5000 to <\$10 000	**\$10 000 to <\$20 000**	**\$20 000 to <\$75 000**	**\$75 000 to <\$250 000**
1 to <2 months	2.5%	2.5%	2.5%	2.5%
2 to <6 months	6.25%	6.25%	6.25%	6.25%
6 to <8 months	3.25%	3.25%	3.25%	3.25%
8 to <12 months	5.35%	5.35%	5.35%	5.35%
12 to <24 months	5.4%	5.4%	5.4%	5.4%

- **a** Answer these questions using the information in the table.
  - **i** What is the minimum amount that can be invested in a term deposit?
  - **ii** Do the interest rates change as the amount invested increases?
  - **iii** The bank appears to have a special rate for one period of time. What investment term receives this rate?
- **b** What interest rate will be offered for a 12-month investment of \$45 000?
- **c** Calculate the amount of simple interest earned on \$45 000 invested for 12 months.
- **d** Compare your answer to part **c** with the option of investing the same amount for three 4-month terms. (Assume that the interest rate will stay the same for the second and third term.)
- **e** Reconsider the option of investing \$45 000 for three 4-month terms, as calculated in part **d**. This time, at the end of the each term, the interest earned is added to the investment amount and the interest for the second term is calculated using that new total. How does the amount of interest for the second term change?
- **f** Compare all the investment options described. Which earned the greatest amount of interest?

13 Some savings accounts offered by banks include bonus interest rates if certain conditions are met. These bonus rates are seen as a reward to encourage people to save and earn more interest.

Consider Saul's account balance for November, shown on the right. His account offers interest at 1.2% p.a. (calculated daily and added into the account at the end of the month), plus a bonus 4.55% p.a. (calculated daily, without considering the 1.2% interest, and only added once at the end of the month) if no withdrawals are made in the month and the balance increases by at least \$250 each month.

Date	**Transaction**	**Amount**	**Balance**
01/11	Opening balance		\$8400.90
15/11	Deposit – Pay	\$655.80	
19/11	Deposit – at branch	\$240.00	
29/11	Deposit – Pay	\$655.80	
30/11	Interest		

- **a** Will this month's account receive the bonus interest rate? Provide a reason to support your answer.
- **b** Calculate the total interest earned for the month. (Hint: First determine the account balances following each transaction.)
- **c** State the final account balance for the month.
- **d** On the final day of the month, Saul made an EFTPOS purchase at a clothing store for \$100 and the transaction did not appear on his statement. How would the balances and the interest calculations have differed if the withdrawal had appeared on the statement?

14 A term deposit advertised on radio offers a fixed interest rate of 6.8% p.a. for 3 months.

- **a** What do you think the expression 'fixed interest rate' means for the 3 months?
- **b** How much interest is earned on an investment of \$12 000 in this term deposit?
- **c** What amount of money must be invested to earn \$500 in interest?
- **d** At the end of the term, the rate of interest decreases to a new fixed interest amount of 6.15% p.a. for 100 days. What amount of money must be invested to earn the same amount of interest as you calculated would be earned in part **b**?

PROBLEM SOLVING AND REASONING

15 Jacqui is saving money to purchase a racing bike like the one shown. She has saved $1980 and hopes to invest it to earn the remaining amount from interest on that investment. The best simple interest rate she can find is 7.2% p.a.

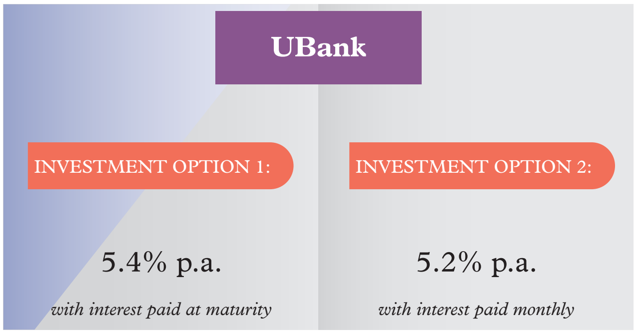
- **a** For how long must Jacqui invest the money in order to earn the extra amount she needs? Give your answer in years, correct to one decimal place.
- **b** Jacqui wants to investigate ways to earn the money quicker with a shorter investment term. What interest rate does she need if she wishes to have the money in 18 months? Correct the interest rate to one decimal place.
- **c** What will the interest rate need to be if Jacqui wishes to have the money in 8 months? Correct the interest to the nearest per cent.
- **d** Consider the rates you calculated for parts **b** and **c**. What problems do you think Jacqui may find in achieving the return she needs on her investment?

16 The statement shown on the right is for a credit card. Interest on purchases made with this card is charged from the day of purchase. To avoid extra charges, the total amount spent plus interest must be paid each month.

Date	Description	Amount
05/09	Clothing store purchase	$120.00
09/09	BPAY to Electricity provider	$356.00
12/09	ABC Bookstore	$39.90
19/09	Ticketmaster	$185.00
25/09	Party hire	$150.00
30/09	Petrol	$65.85
	Interest charge for the month of September	$8.94

- **a** Determine the amounts owing throughout the month and the total number of days that interest is charged on those amounts. Remember that interest is charged from the day of purchase.
- **b** How much needs to be paid at the end of the month to avoid any extra charges?
- **c** What is the annual interest rate (% p.a.) that is charged on this credit card account? (Hint: Use the total interest shown and remember that 4 days is equivalent to $\frac{4}{365}$ years.)

17 Banks offer various interest rates depending on the length of the investment and the frequency of the interest payments.



UBank

INVESTMENT OPTION 1:

5.4% p.a.

with interest paid at maturity

INVESTMENT OPTION 2:

5.2% p.a.

with interest paid monthly

- **a** How much interest is earned on $10 000 invested for 1 month under each of the investment options on the right?
- **b** Consider how much interest is earned on $10 000 invested for 3 months under each of the simple interest investment options described.
- **c** Explore what happens to the value of a $10 000 investment in option 2 if the monthly interest amounts are added to the invested amount at the end of each month.

  What is the value of the investment after the first month's interest calculation? This total value becomes the principal for the second month.
- **d** Using your answer from part **c**, how much interest is earned in the second month of the investment and what is the new value of the investment when that interest is added to the total?
- **e** Using the new value from part **d**, how much interest is earned in the third month and what is the final value of the investment when that interest is added to the total?
- **f** What is the total amount of interest earned on the $10 000 invested for 3 months using the methods explored in parts **c–e**?
- **g** Compare the amount of interest calculated for part **f** with the total amount of interest you calculated for option 2 in part **b**. Are the amounts the same or different? Provide a brief explanation as to why this is the case.

PROBLEM SOLVING AND REASONING

18 In question **17**, the interest rates for the two investment options were different. Reconsider the calculations in parts **c–f** using only the interest rate offered in option 1 (5.4% p.a.).

a If the monthly interest amounts are added to the investment at the end of each month, what will the final value of the investment be after 3 months? (Hint: You may wish to use the parts of question **17** to assist you with the steps of this calculation.)

b How does the total amount of interest found using this method compare to the amount of interest calculated on the investment at maturity? (Compare your answer with your answer for option 1 in question **17b**.)

c Consider the final value of the investment you calculated in part **a** for an investment of $10 000 with the interest added to the principal every month. If you are planning to have the 5.4% p.a. interest paid after 3 months without adding to the principal, and you want the final value of your investment to be equal to the final value calculated in part **a**, how much does the initial principal need to be?

d If the total amount of interest from part **a** is to be earned on an investment of $10 000 for 3 months, with interest paid at maturity, what interest rate should be applied?

CHALLENGE

19 A concept related to simple interest is flat rate depreciation, where an item loses value by a constant amount. For example, if a gaming console that was purchased for $600 loses $120 of value each year this would be a flat rate depreciation of 20%. The formula for calculating the amount of flat rate depreciation is almost the same as the formula for calculating simple interest. That is $D = PrT$, where D is the amount depreciated and r is the flat rate of depreciation, as a decimal. P and T represent principal and time, as they do in the simple interest formula.

a Consider a printer purchased for $240 that depreciates at a flat rate of 15% per year. Calculate the value of the printer after:

i 1 year

ii 2 years

iii 4 years.

b Explain why the flat rate depreciation model for this printer no longer makes sense at 7 years.

c Suggest a more realistic calculation model for the value of the printer when it is 7 or more years old.

20 For each short-term investment below, calculate the total amount of interest earned if interest is calculated:

i at maturity

ii monthly, with interest added to the investment at the end of each month.

a $10 000 invested at 3.6% p.a. for 2 months

b $15 000 invested at 8.4% p.a. for 3 months

c $35 000 invested at 12.75% p.a. for 4 months

d $8400 invested at 21.5% p.a. for 3 months

21 Reconsider the amount of interest earned from each investment in question **20** when the monthly interest amounts are added to the investment (part **ii**). If, instead, the interest is calculated at maturity, what interest rate should be applied to the principal to earn that same amount of interest over the term of the investment? Correct the interest rate to two decimal places and correct the interest to the nearest cent.

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Interactive skillsheet Calculating simple interest

Interactive skillsheet Calculating simple interest rate

Investigation Deferred payments

Topic quiz 1C

Interactive skillsheet Calculating the principal by simple interest formula

Interactive skillsheet Calculating the time period for an investment

Checkpoint

Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this chapter.

1A **1** Calculate each of the following.

a 50% of $180 **b** 20% of $60 **c** 75% of $200 **d** 130% of $150

1A **2** Write the first amount as a percentage of the second amount each time.

a $15 as a percentage of $50 **b** $60 as a percentage of $150

c $22 as a percentage of $80 **d** $140 as a percentage of $80

1A **3** Pierre opened a packet of 40 jellybeans and ate 12 of them.

a Find the fraction of the packet that Pierre ate, in simplest form.

b What percentage of the jellybeans are left?

1B **4** Calculate the selling price for each of these.

a 20% mark-up on $150 **b** 60% mark-up on $85

c 42.5% mark-up on $20 **d** 125% mark-up on $40

1B **5** Calculate the selling price for each of these.

a 25% discount on $60 **b** 30% discount on $2000

c 15% discount on $11 **d** 9% discount on $64

1B **6** A small business owner buys a pack of 24 soft drink cans for $18 and sells the cans for $2.50 each.

a How much does she pay for each individual can from her supplier?

b What is the total profit made from selling 24 cans at $2.50 each?

c Write the profit as a percentage of the original cost, rounded to one decimal place.

1B **7** Ankit buys a video game for $120. Once he is finished playing the game he sells it for $75.

a What is the amount of the loss Ankit has made?

b Write the loss amount as a fraction of the price Ankit originally paid. Write your fraction in simplest form.

c State the percentage loss on the original price.

1C **8** Calculate the simple interest in each case.

a $P = \$5000$, $r = 4\%$, $T = 3$ years **b** $P = \$12\,000$, $r = 2.5\%$, $T = 7$ years

c $P = \$8\,500$, $r = 3.8\%$, $T = 5$ years

1C **9** For each investment below, calculate:

i the amount of simple interest earned **ii** the value of the investment at the end of the term.

- **a** $5000 is invested at an interest rate of 4% p.a. for 3 years
- **b** $12 000 is invested at an interest rate of 3.5% p.a. for 2 years
- **c** $132 800 is invested at an interest rate of 2.75% p.a. for 8 years

1C **10** William takes out a loan of $4500 at a simple interest rate of 5% p.a. He plans to pay the loan back over 4 years.

a Find the total amount of simple interest changed over 4 years.

b Calculate the total amount of money William will have to repay.

c If William pays back an equal amount each month for the four years, then how much will he pay per month?

1C **11** Rahul invests $8000 in a bank account that earns 4% p.a. simple interest.

a How much simple interest will the account earn each year?

b How much money will be in the account in 5 years?

c If Rahul wants to have $10 000 in his account in 5 years, then what rate of simple interest does he require per year?

1D Compound interest

Learning intentions

✔ I can understand how compound interest works.

✔ I can use the compound interest formula to calculate compound interest.

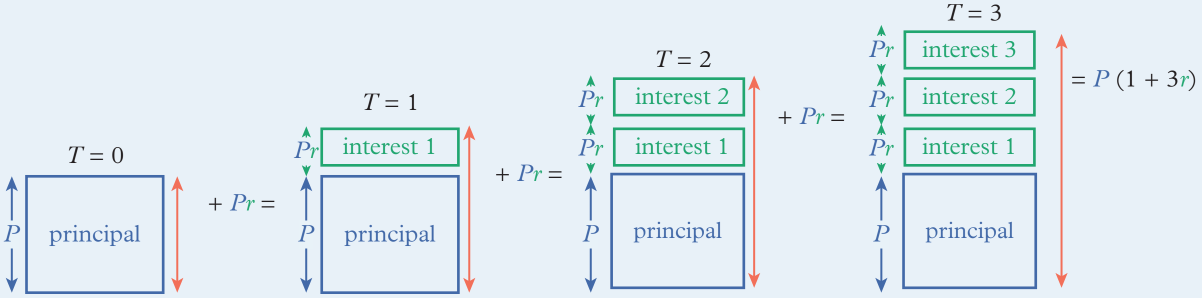
✔ I can calculate the interest rate for a compounding period as well as the number of compounding periods over time.

Inter-year links

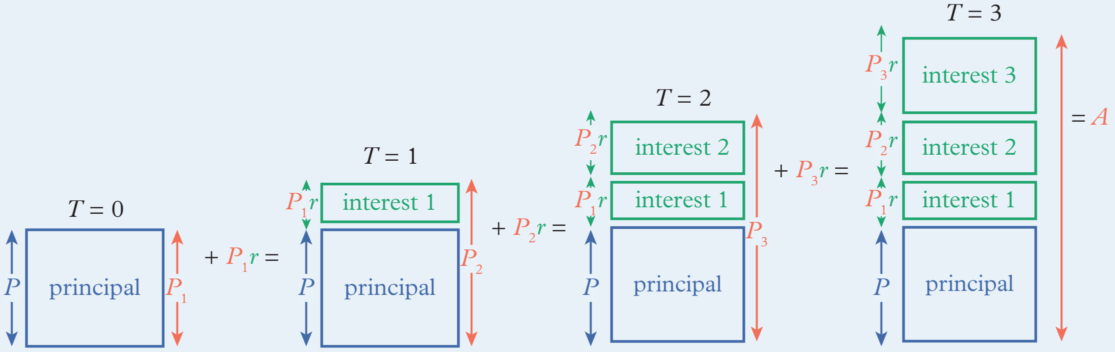
Year 7	6E Substitution
Year 8	5B Substitution
Year 9	1E Simple interest

Compound interest

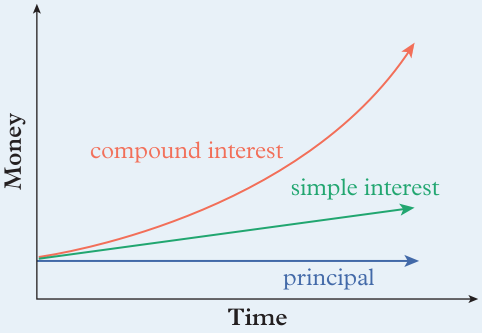
- Using simple interest, the total value of an investment or loan increases by the same amount of interest each time period. Consider the diagram below to see how an amount of money increases over 3 years as a result of simple interest.



- **Compound interest** is generated when the simple interest is added to the principal at regular intervals, so that the amount of money used for calculating the interest continues to increase over time.
- The **compounded value** (A) of an investment or loan is the final amount after a given number of compounding periods have elapsed: $A = P + I$
- A **compounding period** is the regular interval of time at which interest is calculated and added to the principal. Consider the diagram below to see how the compounded value increases over 3 years with a compounding period of 1 year.



- An amount of money that is subject to compound interest generally increases much faster than an amount subject to simple interest. As time passes, the difference between the amount generated by compound interest and that generated by simple interest will increase.



Compound interest formula

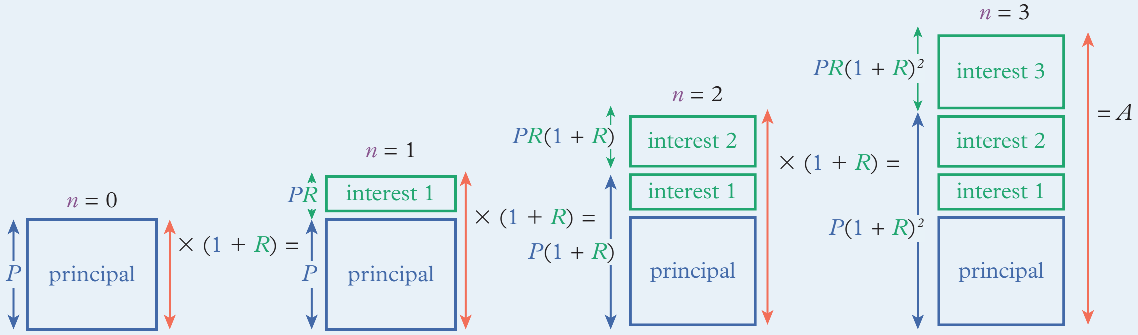
- The compounded value of an amount invested or borrowed can be calculated using the formula:

$$A = P(1 + R)^n$$

compounded value | principal | interest rate of compounding period | number of compounding periods

  → A = the compounded value of the investment or loan (the total amount after n periods)
  → P = the principal, the original amount of money borrowed or invested
  → R = the interest rate per compounding period, written as a decimal
  → n = the number of compounding periods.

- If r is the interest rate per annum, then $R = r \times \dfrac{\text{compounding period}}{\text{1 year}}$
- Consider the diagram below to see how the total value of a borrowed or invested amount increases due to interest over 3 compounding periods.



- The compound interest is the difference between the compounded value and the principal (that is, the difference between the final amount and the initial amount): $I = A - P$

  For example, if interest of 6% p.a. is compounded every 2 months on an investment of \$500 that lasts for half a year, then the value of R and n are calculated as shown below.

$$\begin{aligned} R &= r \times \frac{\text{compounding period}}{\text{1 year}} \\ &= 6\% \times \frac{\text{2 months}}{\text{12 months}} \\ &= 6\% \times \frac{1}{6} \\ &= 1\% \\ &= 0.01 \end{aligned}$$

$$\begin{aligned} n &= \frac{\text{time of investment}}{\text{compounding period}} \\ &= \frac{\text{half a year}}{\text{2 months}} \\ &= \frac{\text{6 months}}{\text{2 months}} \\ &= 3 \end{aligned}$$

  The compounded value will be:

$$\begin{aligned} A &= \$500 \times (1 + 0.01)^3 \\ &= \$500 \times 1.01^3 \\ &= \$500 \times 1.030\,301 \\ &\approx \$515.15 \end{aligned}$$

  and the amount of compound interest will be:

$$\begin{aligned} I &= A - P \\ &= \$515.15 - \$500 \\ &= \$15.15 \end{aligned}$$

Example 1D.1 Understanding compound interest

An amount of \$20 000 is invested at 4% p.a. with interest calculated at the end of each year and added to the principal amount. Use the simple interest formula to calculate the following amounts:

a the compounded value after 1 year
b the compounded value after 2 years
c the compound interest after 3 years

THINK	WRITE
a **1** Identify the known values for use in the simple interest formula.	**a** $P_1 = \$20\,000$ $r = 4\%$ $= 0.04$ $T = 1$ year
2 Calculate the simple interest and add that amount to the principal to find the compounded value of the investment after 1 year.	$I_1 = P_1 rT$ $I_1 = \$20\,000 \times 0.04 \times 1$ $= \$800$ $A_{1\text{ year}} = \$20\,000 + \$800$ $= \$20\,800$
b **1** Use the compounded value after 1 year as the new principal for use in the simple interest formula.	**b** $P_2 = \$20\,800$ $I_2 = P_2 rT$
2 Calculate the interest again and add that amount to the new principal to find the compounded value after 2 years.	$I_2 = \$20\,800 \times 0.04 \times 1$ $= \$832$ $A_{2\text{ years}} = \$20\,800 + \$832$ $= \$21\,632$
c **1** Use the compounded value after 2 years as the new principal for use in the simple interest formula.	**c** $P_3 = \$21\,632$ $I_3 = P_3 rT$
2 Calculate the interest again and add this to the new principal to find the compounded value after 3 years.	$I_3 = \$21\,632 \times 0.04 \times 1$ $= \$865.28$ $A_{3\text{ years}} = \$21\,632 + \$865.28$ $= \$22\,497.28$
3 Subtract the initial principal from the compounded value after 3 years to find the amount of compound interest earned after 3 years.	$I = A_{3\text{ years}} - P$ $= \$22\,497.28 - \$20\,000$ $= \$2497.28$

Example 1D.2 Using the compound interest formula

For an amount of \$14 000 invested at a compound interest rate of 3.5% p.a. for 5 years, calculate:

a the value of the investment at the end of the 5 years
b the total amount of compound interest earned.

THINK	WRITE
a 1 Identify the variables for the compound interest formula. Give *R* as a decimal.	**a** $P = \$14\,000$ $R = 3.5\%$ $= 0.035$ $n = 5$
2 Substitute the known values into the formula and calculate the result. Give your answer to the nearest cent by rounding the answer to two decimal places.	$A = P(1 + R)^n$ $A = \$14\,000 \times (1 + 0.035)^5$ $= \$14\,000 \times 1.035^5$ $\approx \$16\,627.61$
b The amount of compound interest earned is the difference between the final amount and the initial amount.	**b** $I = A - P$ $= \$16\,627.61 - \$14\,000$ $= \$2627.61$

Example 1D.3 Finding *R* and *n*

For an amount of $11 000 invested at 6.9% p.a. for 8 years, with interest compounded every 4 months:

a write the given interest rate per annum in a form that matches the compounding period

b state the values for *R* and *n* that are to be used in the compound interest formula.

THINK	WRITE
a 1 Identify the interest rate per annum and the compounding period.	**a** $r = 6.9\%$ compounding period = 4 months
2 To find the interest rate, *R*, per compounding period, first express the compounding period as a fraction of a year by writing the compounding period as the numerator and 1 year as the denominator. Then multiply this fraction by the interest rate per annum.	$R = \frac{\text{4 months}}{\text{1 year}} \times 6.9\%$
3 Simplify the fraction by converting 1 year to the appropriate unit of time. Then calculate to find the interest rate per compounding period.	$= \frac{\text{4 months}}{\text{12 months}} \times 6.9\%$ $= \frac{1}{3} \times 6.9\%$ $= 2.3\%$
b 1 Convert the interest rate per compounding period to a decimal value.	**b** $R = 2.3\%$ $= 0.023$
2 Divide the time of the investment by the compounding period to find the number of compounding periods.	$n = \frac{\text{8 years}}{\text{4 months}}$ $= \frac{8 \times \text{12 months}}{\text{4 months}}$ $= \frac{96}{4}$ $= 24$

Helpful hints

✔ You can also calculate n using the unitary method. To do this, you need to figure out how many compounding periods occur in 1 year and then multiply that number by the number of years the money has been invested or borrowed.

For instance, in Example 1D.3 the number of compounding periods in a year is 3, so:

$$\begin{aligned} n &= 3 \times 8 \\ &= 24 \end{aligned}$$

ANS p650

Exercise 1D Compound interest

△ 1–4, 5(a, c, e), 6, 7, 10, 11, 13, 16 ■ 3, 4, 5(b, d), 6, 7, 10, 11, 14, 15, 19, 21 ⬠ 4, 8, 9, 11, 12, 17, 18, 20–22

UNDERSTANDING AND FLUENCY

1 Calculate the values missing from the table below for an investment of $100 at 10% p.a over the course of 6 years.

	Investment of $100 at 10% p.a.				
	Years	**Simple interest**	**Value of principal + simple interest**	**Compound interest**	**Value of principal + compound interest**
	1	$10	$110	$10	$110
	2	$10	$120	$I = \$110 \times 10\%$ $= \$11$	$A = \$110 + \$11$ $= \$121$
a	3			$I = \$121 \times 10\%$ $= \$12.10$	$A = \$121 + \$12.10$ $= \$133.10$
b	4			$13.31	
c	5				
d	6				

1D.1 **2** Use the simple interest formula to calculate the following amounts for each investment below and find the compound interest after 3 years:

i the compounded value after 1 year

ii the compounded value after 2 years

iii the compound interest after 3 years.

a $40 000 is invested at 5% p.a.

b $6000 is invested at 2% p.a.

c $1000 is invested at 4% p.a.

d $20 000 is invested at 3.5% p.a.

e $6500 is invested at 7% p.a.

f $15 800 is invested at 2.4% p.a.

3 For each investment below, state the:

i number of compounding periods

ii time length of each compounding period.

a $5000 is invested at 4.5% p.a. for 2 years with interest calculated at the end of each year.

b $12 000 is invested at 9% p.a. for 3 years with interest calculated at the end of each year.

c $8500 is invested at 5% p.a. for 4 years with interest calculated at the end of each 6 months.

d $25 000 is invested at 8.4% p.a. for 2 years with interest calculated at the end of each month.

4 Mac is investigating ways to earn more interest on an investment. His bank offers him an interest rate of 5.2% p.a. with interest compounded annually. Mac has $8500 to invest for 3 years.

a From the given information:

i how many compounding periods will there be in Mac's investment?

ii how long is each compounding period?

b How much interest will Mac earn in the first year?

c Add to the principal the interest for the first year that you calculated for part **b**. This will be the new amount you will use for the principal for the second year.

d How much interest will Mac earn in the second year?

e Add the interest from part **d** to the principal you used for the second year. This will be the new amount for the principal for the third (final) year.

f How much interest will Mac earn in the final year of his investment?

g What will be the final value of Mac's investment?

h Compare the result you found in part **g** with the final value that would be obtained after 3 years from earning simple interest on the $8500 investment at a rate of 5.2% p.a.

5 For each of the investments shown in the table on the right, calculate the amount of interest earned if interest is compounded each year using the same method that was outlined in question **4**, parts **a–g**.

	Principal	Interest rate (p.a.)	Term
a	$8 400	12%	3 years
b	$12 000	5.4%	4 years
c	$25 000	3.75%	3 years
d	$4 590	9.5%	4 years
e	$150 000	6.25%	2 years
f	$35 500	15.8%	5 years

6 Use the compound interest formula to find the value of A each time.

a $A = ?, P = \$10\,000, R = 0.05, n = 4$

b $A = ?, P = \$5000, R = 0.06, n = 3$

c $A = ?, P = \$600, R = 0.08, n = 4$

d $A = ?, P = \$44\,000, R = 0.038, n = 5$

e $A = ?, P = \$100\,000, R = 0.015, n = 8$

f $A = ?, P = \$19\,500, R = 0.0025, n = 10$

1D.2 **7** For each investment below, calculate:

i the value of the investment at the end of the given period

ii the total amount of compound interest earned.

a $7500 is invested at 5% p.a. for 3 years, with interest compounded annually

b $9000 is invested at 4.5% p.a. for 3 years, with interest compounded annually

c $22 500 is invested at 6.8% p.a. for 5 years, with interest compounded annually

d $40 000 invested at 7.3% p.a. for 4 years, with interest compounded annually

8 For each investment listed in the table on the right, use the compound interest formula to calculate the final value of the investment if interest is compounded annually.

9 Calculate the total amount of interest earned by each investment listed in the table in question **8**.

	Principal	Interest rate (p.a.)	Term
a	$3 200	6.4%	4 years
b	$4 590	9.8%	5 years
c	$16 000	2.85%	2 years
d	$25 900	10.5%	3 years
e	$100 000	4.99%	5 years
f	$255 000	12.15%	6 years

UNDERSTANDING AND FLUENCY

1D.3 **10** For each investment below:

i write the given interest rate so that it matches the compounding period

ii state the values for R and n that are to be used in the compound interest formula.

- **a** $15 000 is invested at 4.8% p.a. for 3 years with interest calculated quarterly. (Hint: 'Quarterly' means every quarter of a year, or every three months.)
- **b** $22 000 is invested at 5.9% p.a. for 4 years compounded every 6 months.
- **c** $3500 is invested at 8.4% p.a. for 2 years with interest calculated monthly.
- **d** $14 500 is invested at 9.6% p.a. for 5 years with interest calculated at the end of each month.

11 Use the compound interest formula to determine the value of A for each set of information provided in the table on the right.

	Principal	Interest rate (p.a.)	Term	Compounding period
a	$4800	8.15%	3 years	annually
b	$9500	8.8%	4 years	every 6 months
c	$18 000	4.84%	5 years	quarterly
d	$35 000	16.8%	4 years	monthly
e	$50 000	6.95%	3 years	every 6 months
f	$150 000	8.16%	1 year	quarterly
g	$500 000	4.98%	2 years	every 4 months
h	$750 000	10.26%	3 years	every 2 months

PROBLEM SOLVING AND REASONING

12 The compound interest formula can be found by continuously increasing an amount of money by a percentage of the current amount, rather than by a percentage of the initial amount. In this question you will derive the compound interest formula by generalising from an example.

Consider the table on the right, which shows information about a $10 000 investment that receives 5% p.a. interest compounded annually.

Years since investment	Calculation using previous year's value	Calculation using initial investment value	Value of investment
0			10 000
1	$10\,000 \times (1.05)$	$10\,000 \times (1.05)$	10 500
2	$10\,500 \times (1.05)$	$10\,000 \times (1.05)^2$	11 025
3	$11\,025 \times (1.05)$	$10\,000 \times (1.05)^3$	11 576.25
4	$11\,576.25 \times (1.05)$	$10\,000 \times (1.05)^4$	12 155.06

- **a** Explain why multiplying a value by 1.05 increases that value by 5%.
- **b** The table shows the value of the investment for the first four years. Write an expression for the value of the investment after n years.
- **c** Write an expression for the value after n years if the interest rate is R (as a decimal), rather than 5%.
- **d** Write an expression for the value of the investment after n years if the interest rate is R (as a decimal), rather than 5%, and the principal value is P, rather than $10 000.

13 Maria and her family have begun long-term planning for an overseas holiday. They have invested their savings of $16 000 for 3 years in a bank account, which offers them 5.75% p.a. interest compounded annually.

- **a** How many compounding periods will there be in the family's investment?
- **b** What is the length of each compounding period?
- **c** Identify the values for P, R and n.
- **d** Use the compound interest formula to calculate the final value of the investment after the 3-year investment period.
- **e** How much interest will the family have earned on this investment?
- **f** Maria estimates that she will need $20 000 to fund the entire overseas holiday. If she invests the money for a fourth year, will she achieve this goal?

14 Carolyn borrows \$45 000 to set up a photography studio for her daughter. The loan is for 4 years with interest charged at 8.95% p.a. compounding annually. The terms of the loan require Carolyn to repay the total amount at the end of the 4-year term.

a Use the compound interest formula to calculate the total amount that Carolyn must repay at the end of the 4-year term.

b How much of the amount in part **a** represents the interest charged?

15 While investigating different investment options offered by his bank, Bailey decided he wanted to explore the following two options:

Investment Option 1: Interest rate of 5.35% p.a. with interest paid at maturity

Investment Option 2: Interest rate of 4.95% p.a. with interest compounded annually

He has \$9000 to invest and, initially, he plans to invest the money for 3 years.

a For each option, what would be the value of Bailey's investment at the end of the 3-year term?

b Which option would Bailey be advised to take for his 3-year investment? Provide a reason for your answer.

c How do the answers to parts **a** and **b** change if Bailey invests the money for 5 years?

16 Sheetal has saved \$8400 and is exploring the benefits of investing with an online bank. She finds that the bank will pay 8.6% p.a. interest on amounts invested for at least 36 months, with interest compounded quarterly. She decides to invest her \$8400 for 5 years.

a Write the annual interest rate as a rate per quarter.

b Use your answer from part **a** to state the value for R in the compound interest formula.

c How many compounding periods, n, will there be in Sheetal's investment?

d Use the compound interest formula to determine the final value of the investment.

e What is the total amount of interest Sheetal will expect to earn on her investment?

17 An investment of \$10 000 is made for 4 years with a bank that offers an annual interest rate of 8.4%. Answer the following questions to explore how the frequency of the compounding periods affects the final value of the investment.

a Calculate the final value of the investment if the interest is calculated at the end of the term; that is, at maturity.

b Calculate the final value of the investment if the interest is compounded:

i every 6 months **ii** quarterly **iii** every 2 months **iv** monthly.

c Use your results from part **b** to briefly explain what happens to the final value of an investment as the number of compounding periods increases. Will this always be the case?

18 Through a store purchase plan, Troy is informed that interest is calculated as compound interest based on the amount borrowed. In this situation, would Troy be better off with a plan involving a higher frequency of compounding periods or fewer compounding periods?

19 Erica knows she can earn 6.8% p.a. from her bank for her investments. She plans to invest some money for 2 years with interest compounded quarterly.

a For Erica's investment, state:

i the number of compounding periods, n

ii the value for R to be used in the compound interest formula.

b At the end of her 2-year investment term, Erica hopes her investment will have grown to \$6500. Which variable of the compound interest formula does this \$6500 represent?

c Substitute all the known values into the compound interest formula and solve the equation for the unknown variable.

d To the nearest dollar, how much should Erica invest at the start of her investment if she is aiming at having an amount of \$6500 after 2 years?

PROBLEM SOLVING AND REASONING

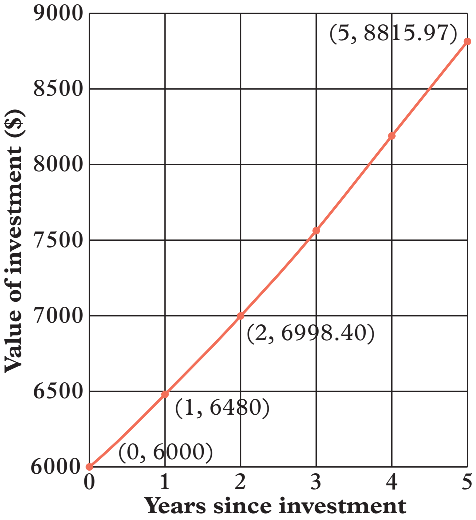
20 Consider the graph on the right, which shows the value of Raj's investment in a high interest savings account.

a How much money did Raj invest?

b What is the interest rate per annum for the savings account with interest compounded annually?

c Find the coordinates of the two points on the graph that are not already labelled.

d What will be the balance in Raj's account 8 years after investment?



CHALLENGE

21 If the value of something depreciates by the same percentage at regular intervals, this is called 'reducing balance depreciation'.

The formula for reducing balance depreciation is similar to the compound interest formula, $V = P(1 - R)^n$, where V is the value of the item after n time periods. P is the principal (in this case, the purchase price of the item) and R is the rate of depreciation (as a decimal).

a Identify the important difference between the formula for reducing balance depreciation and the formula for compound interest. Explain what you think the reason is for the difference.

b Consider a computer purchased for $2000 which depreciates by 20% each year.

Copy the table on the right and fill in the missing values to show the computer's depreciation.

c After how many years will the computer first be worth less than $100?

d In the formula, $(1 - R)$ evaluates to 0.8, since R is 0.2. Explain what the number 0.8 represents, in terms of the value of the computer.

e Explain, in words, why the value of the computer will never reach $0, according to this model of depreciation.

f Do you think this model of depreciation is more realistic than the flat rate depreciation model described in question **19** of exercise **1C**? Explain your answer.

Years since purchase	Value
0	$2000
1	$1600
2	$1280
3	
4	

22 Consider an amount of $5000 invested at 5% p.a. for 5 years, with interest compounded annually.

a What is the value of the investment at the end of each year during this term?

b How much interest is earned in each year of the investment?

c Display the growth of this investment as a line graph, with the years on the horizontal axis and the value of the investment on the vertical axis. You may wish to use digital technology to present your graph.

d Repeat parts **a–c** with the same values, but for these compounding periods:

- **i** quarterly
- **ii** every 6 months
- **iii** monthly.

e If the interest for this investment was calculated using simple interest, briefly explain how the value of the investment could be shown on a graph.

f How would a graph displaying the investment's growth using simple interest compare with the compound interest graphs? You may wish to compare the graphs by displaying them all on the one set of axes.

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Interactive skillsheet Calculating compound interest

CAS instructions Spreadsheets and compound interest

Topic quiz 1D

1E Compound interest calculations

Learning intentions

- ✔ I can use the compound interest formula to calculate the principal.
- ✔ I can use the compound interest formula to calculate the interest rate.

Inter-year links

Year 7	6H Solving equations using inverse operations
Year 8	6B Solving equations using inverse operations
Year 9	1F Simple interest calculations

Using the compound interest formula to find other unknown quantities

- The compound interest formula is not only used to calculate the compounded value of a loan or investment. The equation can also be used to find the principal or the interest rate, as long as the other values are known.

$A = P(1 + R)^n$ Solve for principal

$A = P(1 + R)^n$ Solve for interest rate per compounding period

Calculating the principal using the compound interest formula

- To find P using $A = P(1 + R)^n$, use inverse operations to solve the equation by dividing both sides of the formula by the value of $(1 + R)^n$.

  For example, if an 8% p.a. investment is compounded annually and grows to the value of \$2459 after 3 years, the principal can be calculated as follows:

$$\begin{aligned} \$2459 &= P(1 + 0.08)^3 \\ \$2459 &= P(1.08)^3 \\ \$2459 &= P \times 1.259\,712 \\ \frac{\$2459}{1.259\,712} &= P \\ P &= \$1952.03 \end{aligned}$$

Calculating the interest rate using the compound interest formula

- To find R using $A = P(1 + R)^n$, we use inverse operations.
  1. Divide both sides of the equation by the value of P.
  2. Find $\sqrt[n]{}$ of both sides of the equation.
  3. Subtract 1 from both sides of the equation.
- To find the annual interest rate, multiply R by the number of compounding periods in a year.

For example, if an investment of \$500 is compounded every 6 months and grows to have a value of \$653.48 after 2 years, then the annual interest rate, correct to one decimal place, can be calculated using the method below.

$$\begin{aligned} n &= \frac{2 \text{ years}}{6 \text{ months}} \\ &= \frac{24 \text{ months}}{6 \text{ months}} \\ &= 4 \end{aligned}$$

$$\begin{aligned} A &= P(1 + R)^n \\ \$653.48 &= \$500 \times (1 + R)^4 \\ \frac{\$653.48}{\$500} &= (1 + R)^4 \\ 1.306\,96 &= (1 + R)^4 \\ \sqrt[4]{1.306\,96} &= 1 + R \\ 1.069\,216\,31 &= 1 + R \\ R &= 0.069\,216\,31 \\ R &\approx 6.9\% \text{ (every 6 months)} \end{aligned}$$

compounding periods in a year = 2

$$\begin{aligned} \text{interest rate} &= 2 \times 6.9\% \text{ p.a.} \\ &= 13.8\% \text{ p.a.} \end{aligned}$$

Example 1E.1 Calculating the principal for compound interest

How much must be invested at an interest rate of 8.4% p.a. for the value to increase to \$9000 over 2 years with interest compounded quarterly?

THINK	**WRITE**
1 Identify the known values for use in the compound interest formula.	$A = \$9000$ $R = \frac{8.4\%}{4}$ $= 2.1\%$ per quarter $= 0.021$ $n = \frac{2 \text{ years}}{\text{one-quarter of 1 year}}$ $= \frac{24 \text{ months}}{3 \text{ months}}$ $= 8$
2 Substitute the values into the formula and simplify where possible.	$\$9000 = P(1 + 0.021)^8$ $\$9000 = P \times 1.021^8$
3 Solve for P by dividing both sides of the equation by $(1 + R)^n$.	$\frac{\$9000}{1.021^8} = \frac{P \times 1.021^8}{1.021^8}$ $P = \$7621.43$

Example 1E.2 Calculating the interest rate for compound interest

For an investment of \$6000 that increases in value to \$6900 after 5 months, with interest compounded monthly:

a find the value of R as a decimal, correct to four decimal places

b state the annual interest rate, correct to one decimal place.

THINK	**WRITE**
a 1 Identify the known values for the compound interest formula.	**a** $A = \$6900$ $P = \$6000$ $n = \frac{5 \text{ months}}{1 \text{ month}}$ $= 5$
2 Substitute the values into the formula.	$A = P(1 + R)^n$ $\$6900 = \$6000 \times (1 + R)^5$
3 Solve for R by first dividing both sides of the equation by P. If preferable, swap both sides of the equation.	$\frac{\$6900}{\$6000} = \frac{\$6000 \times (1 + R)^5}{\$6000}$ $(1 + R)^5 = 1.15$
4 Find $\sqrt[n]{\ }$ of both sides of the equation. Round to four decimal places.	$\sqrt[5]{(1 + R)^5} = \sqrt[5]{1.15}$ $1 + R = 1.028\,346\ldots$
5 Subtract 1 from both sides to find R correct to four decimal places.	$R = 0.0283$ (to four decimal places)
b 1 Find the number of compounding periods per year.	compounding periods per year $= \frac{1\text{year}}{1\text{month}}$ $= 12$
2 Multiply R by this number.	interest rate $= 12 \times 0.0283$ $= 0.3396$
3 Convert to a percentage correct to one decimal place.	interest rate = 34.0% p.a. (to one decimal place)

Helpful hints

- ✔ In most instances you will need to use a calculator to find square roots, cube roots and roots of a higher order. However, sometimes you can use the index laws and your knowledge of square roots to simplify the root yourself.

  For example,
  $$\begin{aligned}\sqrt[4]{16} &= \sqrt[4]{4^2}\\ &= \sqrt[4]{(2^2)^2}\\ &= \sqrt[4]{2^{2\times 2}}\\ &= \sqrt[4]{2^4}\\ &= 2\end{aligned}$$

- ✔ Remember to avoid rounding errors when calculating interest rates. Only round your calculations to the appropriate number of decimal places at the very last step.

  For example, in Example 1E.2 if we rounded R to two decimal places at the end of part a, we would get $R = 0.03$ instead of 0.0283.

  The answer to part b would be different from 34.0%: annual interest rate $= 12 \times R$
  $$\begin{aligned}&= 0.36\\ &= 36.0\% \text{ (to one decimal place)}\end{aligned}$$

ANS p651

Exercise 1E Compound interest calculations

△ 1–6, 8(a, c, d), 9–11 □ 2, 3, 5, 6, 8(b, e, f), 10, 12–14 ⬠ 2, 6, 8(b, e, f), 10, 12, 13, 15–17

UNDERSTANDING AND FLUENCY

1E.1 **1** Find the value of the principal (P) for each of these investments.

- **a** How much must be invested at an interest rate of 10.5% p.a. for the value to increase to \$30 000 over 4 years with interest compounded annually?
- **b** How much must be invested at an interest rate of 5.4% p.a. for the value to increase to \$10 000 over 3 years with interest compounded every 6 months?
- **c** How much must be invested at an interest rate of 6.4% p.a. for the value to increase to \$5500 over 5 years with interest compounded quarterly?
- **d** How much must be invested at an interest rate of 7.2% p.a. for the value to increase to \$8000 over 2 years with interest compounded every month?

2 Use the compound interest formula to determine the value of P, to the nearest dollar, for each set of investment information (**a**–**e**) provided in the table on the right.

	Final value (compounded value)	Interest rate (p.a.)	Term	Compounding period
a	\$5 200	6.75%	3 years	annually
b	\$10 000	4.2%	6 years	monthly
c	\$22 000	12.5%	4 years	every 6 months
d	\$16 650	15.5%	3 years	quarterly
e	\$225 000	9.95%	$4\frac{1}{2}$ years	every 6 months

3 How much is the total interest earned by each investment in question **2**?

1E.2 **4** For each investment below:

- **i** find the value of R as a decimal, correct to four decimal places
- **ii** state the annual interest rate, correct to one decimal place.
  - **a** After 1 year an investment of \$1000 increased in value to \$1170, with interest compounded every 6 months.
  - **b** After 2 years an investment of \$2500 increased in value to \$2800, with interest compounded annually.
  - **c** After 6 months an investment of \$3750 increased in value to \$4050, with interest compounded quarterly.

5 Consider an investment of \$12 000 that increased in value to \$15 000 over a 3-year period, with interest compounded annually.

- **a** From this information, which variable from the compound interest formula don't you know the value of?
- **b** Substitute the known values into the compound interest formula and show that it simplifies to $\frac{5}{4} = (1 + R)^3$.
- **c** Copy and complete the calculation shown on the right to solve the equation for R by performing inverse operations to both sides of the equation. (Note that the inverse operation to cubing a number is finding the cube root of a number.)

$$\begin{aligned} \frac{5}{4} &= (1 + R)^3 \\ 1.25 &= (1 + R)^3 \\ (1 + R)^3 &= 1.25 \\ \sqrt[3]{(1 + R)^3} &= \sqrt[3]{\underline{\qquad\qquad}} \\ 1 + R &= \sqrt[3]{\underline{\qquad\qquad}} \\ 1 + R &= \underline{\qquad\qquad} \\ R &= 0.077\,217\ldots \end{aligned}$$

- **d** The value for R in part **c** is the rate per compounding period expressed as a decimal. Write this as an annual interest rate, correct to one decimal place.

6 For each investment below:

i find the value of R as a decimal, correct to four decimal places

ii state the annual interest rate, correct to one decimal place.

a After 3 years an investment of \$1600 increased in value to \$2180, with interest compounded annually.

b After 2 years an investment of \$7500 increased in value to \$8200, with interest compounded every 6 months.

c After 18 months an investment of \$10 000 increased in value to \$11 000, with interest compounded every 6 months.

7 Jade's parents invested \$5500 for her in a savings account, where it increased in value to \$8000 in 4 years, with interest compounding annually.

a Write the values for A, P and n.

b Use the compound interest formula to find the value of R. Give your answer correct to three decimal places.

c What annual interest rate was applied to Jade's savings account?

8 Use the compound interest formula with the information in this table to determine the following for each set of investment values (**a–f**):

i the value for R (as a decimal)

ii the annual interest rate.

	Principal	Final value	Term	Compounding period
a	\$1 000	\$1 500	3 years	annually
b	\$8 000	\$12 000	4 years	annually
c	\$16 000	\$19 000	2 years	every 6 months
d	\$3 500	\$4 000	1 year	quarterly
e	\$12 000	\$16 500	3 years	every 6 months
f	\$25 000	\$29 500	2 years	quarterly

9 Riley aims to have \$15 000 by the time he turns 18 so that he can buy a car. He opens a bank savings account that pays interest at a rate of 7.2% p.a. compounded monthly. Riley plans to keep the money invested under these terms for 3 years, but he needs to know how much he should invest to be sure he has \$15 000 at the end of those 3 years.

a How many compounding periods, n, will there be in Riley's 3-year investment?

b What is the interest rate, R, per compounding period?

c From the given information, which variable in the compound interest formula don't you know the value of?

d Use the compound interest formula to calculate the amount of money Riley must invest for it to increase to \$15 000 over the 3-year term of the investment.

10 An investment of \$25 000 increased in value to over \$30 000 over a period of time in a savings account that earns 5.0% p.a. with interest compounded annually.

a From the given information, which variable in the compound interest formula don't you know the value of?

b Substitute the known values into the compound interest formula and show that it simplifies to $1.2 = (1.05)^n$.

c One strategy for solving the equation for n is the 'guess, check and improve' method. Substitute different values for n into the equation and evaluate the right-hand side until the value on that side is greater than the value on the left-hand side. That value for n represents the solution to the equation.

Value for n	Left-hand side	Right-hand side	Comment
1	1.2	$1.05^1 = 1.05$	Value on the right-hand side is too small.
2	1.2	$1.05^2 = 1.1025$	
3	1.2	$1.05^3 =$	
4	1.2	$1.05^4 =$	

d The value for n obtained in part **c** is the number of compounding periods for the investment. Briefly explain how this relates to this investment.

UNDERSTANDING AND FLUENCY

PROBLEM SOLVING AND REASONING

11 Yousef watched his savings grow from $850 to over $1100 in a savings account that earned 4.2% p.a. with interest compounded annually.

a Use the given information to determine the values for A, P and R.

b Substitute the given values into the compound interest formula and simplify.

c Use the 'guess, check and improve' method to solve the equation for n. You may wish to use a similar table to the one used in question **10**.

d How many compounding periods did it take for Yousef's savings to grow from $850 to over $1100?

12 For each set of investment information (**a–f**) given in the table below, state the number of compounding periods, n, needed for the initial principal to exceed the final value given.

	Principal	Final value	Interest rate (p.a.)	Investment period
a	$900	$1 200	5.0%	annually
b	$12 000	$14 000	3.5%	annually
c	$15 000	$18 500	4.8%	every 6 months
d	$650	$900	10.5%	quarterly
e	$35 000	$42 000	6.8%	every 6 months
f	$50 000	$55 590	7.2%	quarterly

13 For each investment in question **12** (**a–f**), write the investment period in years.

14 Shannon is saving money to buy a car. She wants to have $10 000 in 4 years.

a If her savings account pays 4% p.a. compounded yearly, how much does she need to have in her account now in order to have $10 000 in four years?

b If her savings account pays 4% p.a. compounded monthly, how much does she need to have in her account now in order to have $10 000 in four years?

c If she only has $8000 in her account, what annual rate of interest does Shannon need, as a percentage rounded to two decimal places, to have $10 000 in 4 years if interest is compounded:

- **i** yearly?
- **ii** quarterly?
- **iii** monthly?
- **iv** daily?

15 Consider the table on the right, which shows the balance of an account earning compound interest.

Time period	Amount ($)
0	5000
1	5045
2	5090.41
3	5136.22
4	5182.44
5	5229.09
6	5276.15

a If the time periods are quarters, and interest is compounded quarterly, find the interest rate per annum. Give your answer as a percentage, rounded to one decimal place.

b If the time periods are months, and interest is compounded monthly, find the interest rate per annum. Give your answer as a percentage, rounded to one decimal place.

c If the time periods are weeks, and interest is compounded weekly, find the interest rate per annum. Give your answer as a percentage, rounded to one decimal place.

d If the table represents the value in a regular savings account, do you think the time periods are quarters, months or weeks? Explain your reasoning.

e If the table represents a credit card debt, do you the think the time periods are quarters, months or weeks? Explain your reasoning.

CHALLENGE

16 **a** If an investment has a principal, P, an interest rate of r % p.a., a compounding period of C years, and an investment term of T years, write the compound interest formula in terms of P, r, C and T. (Hint: Determine equations for R and n first.)

Compare your formula to $A = P(1 + R)^n$.

b Explain what happens to the value for R as C decreases.

c Explain what happens to the value for n as C decreases.

d Use your calculator to investigate what happens to $(1 + R)^n$ as the compounding period decreases for a fixed investment term, T, and a fixed annual interest rate, r. Use your observations to guess what happens to A in your formula as C decreases. Explain your answer.

e What happens when the compounding period is equal to the investment term? What kind of interest does your formula now represent? Explain your answer.

17 Samantha deposits \$30 000 in a bank account that receives 4.8% p.a. compounded quarterly.

a Use the 'guess, check and improve' method to find the number of compounding periods required for Samantha's account to grow to at least:

i \$32 000

ii \$40 000.

b It is possible to solve for n in the compound interest formula, but this requires a special kind of function called a logarithm. In algebra, a logarithm is written as $\log_a$, where a is a number referred to as the 'base' of the logarithm.

Scientific calculators usually only include logarithms with bases that are either 10 or Euler's number ($e \approx 2.72$), so the following calculation uses a base 10 logarithm to determine a formula for n.

$$\begin{aligned}
A &= P(1 + R)^n \\
\frac{A}{P} &= (1 + R)^n \\
\log_{10}\left(\frac{A}{P}\right) &= \log_{10}[(1 + R)^n] \\
\log_{10}\left(\frac{A}{P}\right) &= n\log_{10}(1 + R) \\
n &= \frac{\log_{10}\left(\frac{A}{P}\right)}{\log_{10}(1 + R)}
\end{aligned}$$

Use a scientific calculator and the formula provided to solve the questions in part **a**.

c Explain why your answers for **b** must be rounded up to the nearest whole number, rather than rounded down.

d Use the formula provided to find the number of compounding periods required for Samantha's account balance to grow to at least:

i \$35 000

ii \$45 000

iii \$50 000.

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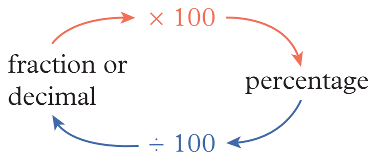
Interactive skillsheet Calculating compound interest rate	**Interactive skillsheet** Calculating the principal by compound interest formula	**Investigation** Rain, rain and more rain!	**Topic quiz** 1E

Chapter summary

Fractions, decimals and percentages



Percentage of a quantity

- To calculate a percentage of a quantity, write the percentage as a decimal and multiply it by the quantity.

  For example,

  $$\begin{aligned} 25\% \text{ of } \$160 &= 0.25 \times \$160 \\ &= \$40 \end{aligned}$$

A quantity as a percentage

- To write one quantity as the percentage of another quantity, divide the first quantity by the second quantity and multiply by 100%.

  For example,

  $$\begin{aligned} \frac{\$12}{\$60} \times 100\% &= 0.2 \times 100\% \\ &= 20\% \end{aligned}$$

Profit and loss

profit ⟶ selling price > cost price

loss ⟶ selling price < cost price

profit = selling price – cost price

loss = cost price – selling price

$$\text{percentage profit} = \frac{\text{profit}}{\text{cost price}} \times 100\%$$

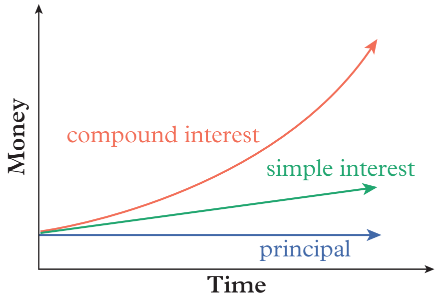
$$\text{percentage loss} = \frac{\text{loss}}{\text{cost price}} \times 100\%$$

Mark-ups and discounts

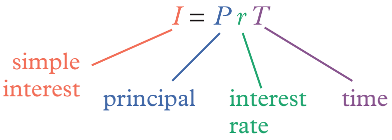
- A mark-up is the amount added to the original price. It is calculated using a percentage increase.
- A discount is the amount subtracted from the original price. It is calculated using a percentage decrease.

Interest

- Interest is a sum of money added to a principal amount that was either invested or borrowed.

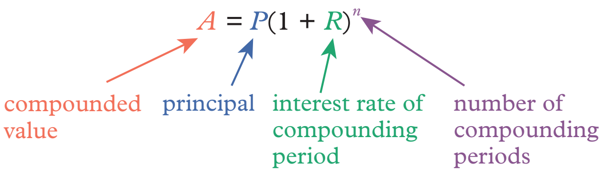


Simple interest



$$I = PrT$$

Compound interest



$$A = P(1 + R)^n$$

Mark-ups and discounts

selling price = (100% + percentage mark-up) × original price

selling price = (100% – percentage discount) × original price

Chapter review

Chapter review quiz
Take the chapter review quiz to assess your knowledge of this chapter.

Quizlet
Test your knowledge of this topic by working individually or in teams.

Multiple-choice

1A **1** What is $22\frac{1}{2}\%$ of \$284.95, rounded to the nearest cent?

A \$63.26 **B** \$64.10 **C** \$64.11 **D** \$62.69 **E** \$262.45

1A **2** \$35 as a percentage of \$80 is closest to which of the following?

A 35% **B** 50% **C** 23% **D** 230% **E** 44%

1B **3** What is the selling price after a discount of 10.5% on the original price of \$120?

A \$12.60 **B** \$107.40 **C** \$109.50 **D** \$132.60 **E** \$130.50

1B **4** Pedro buys a used computer for \$250 and resets it to factory conditions. He then sells it for \$420. What is his profit, as a percentage of the purchase price?

A 60% **B** 160% **C** 100% **D** 168% **E** 68%

1C **5** An amount of \$1500 is borrowed at 6.75% p.a. simple interest for 24 months. Which set of values should be substituted into the simple interest formula for this loan?

A $P = 1500, r = 6.75, T = 24$
B $P = 1500, r = 6.75, T = 2$
C $P = 1500, r = 0.0675, T = 2$
D $P = 1500, r = 0.0675, T = 24$
E $P = 1500, r = 0.675, T = 24$

1C **6** What is the annual interest rate applied to a loan of \$20 000 if the amount of simple interest charged on the loan over 3 years is \$2550?

A 0.0425% **B** 0.354% **C** 4.25% **D** 14.875% **E** 3.54%

1D **7** An amount of \$10 000 invested for 4 years earns interest at 8.4% p.a. compounded annually. What is the value of the investment after 2 years?

A \$11 750.56 **B** \$11 680 **C** \$12 737.61 **D** \$13 807.57 **E** \$11 822.44

1D **8** If an amount of \$4800 is invested for 3 years at 3.6% p.a. with interest compounded monthly, what is the number of compounding periods?

A 3 **B** 6 **C** 12 **D** 36 **E** 152

1E **9** Over a 4-year period, the value of an investment grew from \$18 500 to \$27 000 with interest compounded quarterly. Given this information, which variable in the compound interest formula don't you know the value of?

A A **B** P **C** R **D** n **E** None of them

1E **10** The annual interest rate on an investment compounded quarterly is 8.25% p.a. What is the interest rate per compounding period?

A 8.25% **B** 2.0625% **C** 0.0825% **D** 0.020 625% **E** 4.125%

Short answer

1A **1** Write each of these percentages as a decimal.

a 24% **b** 35.8% **c** $16\frac{1}{4}\%$ **d** 0.272%

1A **2** Write each of these percentages as a fraction in simplest form.

a 86% **b** 235% **c** $15\frac{1}{5}\%$
d 8.65% **e** $13\frac{2}{3}\%$ **f** 0.0512%

1A **3** Calculate each of these.

a 15% of \$685 **b** 170% of \$9500 **c** $17\frac{1}{2}\%$ of \$98.50

1A **4** **a** Write \$85 as a percentage of \$680.
b Write \$39.95 as a percentage of \$72.50, rounding your answer to two decimal places.

1A **5** A CD priced at \$23.95 is sold at a sale for \$12.45. Write the selling price as an exact percentage of the original price.

1B **6** Parts **a–d** below list the original prices and the percentage discount or percentage mark-up on some goods. In each case, calculate:

i the selling price after the discount or mark-up

ii the amount of the discount or mark-up, rounded to the nearest cent.

- **a** original price \$2550; discount $15\frac{1}{2}$%
- **b** original price \$899.50; mark-up 125%
- **c** original price \$18 045; mark-up 38.4%
- **d** original price \$1945.90; discount 37.5%

1B **7** For each of the following:

i state the value of the profit or loss

ii write the profit or loss as a percentage of the original price (rounded to the nearest 1%).

- **a** original price \$139.85, selling price \$171.30
- **b** original price \$27 490, selling price \$9813.95

1C **8** Calculate the simple interest if:

- **a** P = \$2500, r = 6% p.a. and T = 3 years
- **b** P = \$28 000, r = 4.25% p.a. and T = 5 years

1C **9** For each investment below, calculate:

i the amount of simple interest earned

ii the value of the investment at the end.

- **a** \$150 000 invested at an interest rate of 4.8% p.a. for $3\frac{1}{2}$ years.
- **b** \$9450 invested at an interest rate of 2.95% p.a. for 5 months.

1C **10** Find the unknown value for each simple interest situation below. Remember to write the value for r as a percentage and to write time in years.

- **a** I = \$308, P = \$3850, r = 6.4% p.a., T = ?
- **b** I = \$765, P = ?, r = 4.25% p.a., T = 12 months
- **c** I = \$9843.75, P = 35 000, r = ?, T = 2.5 years

1D **11** For each investment below, interest is calculated at the end of each year and added to the principal amount. Calculate the total amount of interest compounded each year during the term of the investment. Show the value of the investment at the end of each compounding period.

- **a** \$15 000 is invested at 4.5% p.a. for 3 years.
- **b** \$6850 is invested at 6.4% p.a. for 4 years.

1D **12** Given the information provided in the table below, use the compound interest formula to determine the value of A for each investment (**a–c**).

	Principal	Interest rate (p.a.)	Term	Compounding period
a	\$2 250	7.5%	2 years	annually
b	\$18 750	9.2%	4 years	quarterly
c	\$150 000	6.24%	4 years	every 6 months

1E **13** With reference to the compound interest formula, find each unknown value, correct to two decimal places.

- **a** A = ?, P = \$5650, R = 0.058, n = 3
- **b** A = \$20 000, P = ?, R = 0.08, n = 4
- **c** A = \$10 000, P = \$8000, R = ?, n = 6

1E **14** Tahlia deposited \$6000 in a savings account and the amount in the account grew to \$6054.16 after three compounding periods.

- **a** Find the value of R (as a decimal), rounded to three decimal places.
- **b** Find the interest rate per year (as a percentage) if interest is compounded:
  - **i** yearly **ii** quarterly **iii** monthly **iv** weekly.
- **c** Which of the interest rates per year from part **b** do you think are realistic for a regular savings account at a bank?

Analysis

1 Susan invested \$12 000 with her bank for 4 years with interest compounded quarterly throughout the term of the investment. The interest rate earned on the investment varied at different stages.

In the first year, the interest rate was $5\frac{1}{2}$% p.a.; for the next 2 years the interest rate remained fixed at 5.7% p.a.; and the rate increased to 6.2% p.a. for the entire final year.

- **a** How many compounding periods were there for the term of the investment?
- **b** Consider the annual interest rates stated above. What value for R would be used in the compound interest formula in the:
  - **i** first year?
  - **ii** middle two years?
  - **iii** final year?
- **c** Determine the value of the investment at the end of the first year.
- **d** State the values for P, R and n you would use to calculate the growth in the investment from the start of the second year to the end of the third year.
- **e** What was the value of the investment at the end of the third year?
- **f** What was the value of the investment at the end of the whole 4-year term?
- **g** What was the total amount of interest Susan earned on her investment?
- **h** Write the interest from part **g** as a percentage of the original amount invested, correct to two decimal places.
- **i** Susan had hoped that her investment would grow to \$15 500 by the end of the 4-year term. Keeping the compounding periods the same, what constant annual interest rate would Susan need to have received to achieve her desired final value? Correct the interest rate to two decimal places.
- **j** If the number of compounding periods was increased and the interest was compounded monthly, how does the answer to part **i** change?

2 Jiwoo has \$4000 to put into a savings account. His local bank offers him two account options.

Option 1: Simple interest at 4% p.a.

Option 2: Compound interest at 3.5% p.a. compounded annually

- **a** How much money will be in Jiwoo's account after 1 year if he chooses:
  - **i** option 1?
  - **ii** option 2?
- **b** How much money will be in Jiwoo's account after 10 years if he chooses:
  - **i** option 1?
  - **ii** option 2?
- **c** After how many years will option 2 first become better than option 1?
- **d** Jiwoo chooses option 2 but asks if the interest can be compounded monthly, instead of yearly. The bank manager replies that it can be compounded monthly if the rate drops to 3.4% p.a.

  How much money would be in the account after 1 year if interest was compounded monthly at a rate of 3.4% p.a.?
- **e** Should Jiwoo agree to drop the rate to 3.4% p.a. in order to have interest compounded monthly? Explain your answer.

2

Algebra

Index

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Indices
- ✔ Roots
- ✔ Highest common factor
- ✔ Like terms
- ✔ Equivalent fractions

Curriculum links

- Factorise algebraic expressions by taking out a common algebraic factor (VCMNA329)
- Simplify algebraic products and quotients using index laws (VCMNA330)
- Apply the four operations to simple algebraic fractions with numerical denominators (VCMNA331)
- Expand binomial products and factorise monic quadratic expressions using a variety of strategies (VCMNA332)
- Factorise monic and non-monic quadratic expressions and solve a wide range of quadratic equations derived from a variety of contexts (VCMNA362) [10A]

2A Indices

Learning intentions

✔ I can use the index laws to simplify algebraic expressions.

Inter-year links

Year 7	1G Indices and square roots
Year 8	4A Indices
Year 9	2A Indices

Index notation

- **Index notation**, or **index form**, is used for indicating the number of times a **base** number is multiplied by itself repeatedly. The number of repeated multiplications corresponds to the **index** (also called the **exponent**).

$$\text{base} \longrightarrow \underset{\text{index form}}{\overset{\overset{\text{index/exponent}}{\downarrow}}{3^4}} = \underset{\text{expanded form}}{3 \times 3 \times 3 \times 3} = \underset{\text{basic numeral}}{81}$$

$$\text{base} \longrightarrow \underset{\text{index form}}{\overset{\overset{\text{index/exponent}}{\downarrow}}{a^3}} = \underset{\text{expanded form}}{a \times a \times a}$$

Index laws

- The **index laws** are used for simplifying expressions that include terms written in index form.
- Index laws for multiplying or dividing terms in index form only apply when both terms have the same base.
- Any base raised to the power of zero is equal to one.
- We can use a combination of index laws to simplify complex expressions.

Index law	Algebraic form	Example
Index law 1	$a^m \times a^n = a^{(m+n)}$	$2^3 \times 2^5 = 2^{(3+5)}$ $= 2^8$
Index law 2	$a^m \div a^n = a^{(m-n)}$	$2^5 \div 2^3 = 2^{(5-3)}$ $= 2^2$
Index law 3	$(a^m)^n = a^{(m \times n)}$	$(2^3)^5 = 2^{(3\times5)}$ $= 2^{15}$
Expanding multiplication in brackets raised to an index	$(a \times b)^m = a^m \times b^m$	$(2 \times 3)^5 = 2^5 \times 3^5$
Expanding division in brackets raised to an index	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\left(\frac{2}{3}\right)^5 = \frac{2^5}{3^5}$
The zero index	$a^0 = 1$	$2^0 = 1$

Example 2A.1 Using the index laws

Use the appropriate index law to simplify each of these expressions.

a $(g^3)^4$ **b** $\left(\frac{w}{3}\right)^3$ **c** $3k^2 \times 6k^5$ **d** $8a^{12} \div 16a^7$

THINK	WRITE
a Apply index law 3 to expand the brackets by multiplying the indices.	**a** $(g^3)^4 = g^{(3\times4)}$ $= g^{12}$

b 1 Expand the brackets by applying the index to the numerator and the denominator.	**b** $\left(\frac{w}{3}\right)^3 = \frac{w^3}{3^3}$
2 Simplify.	$= \frac{w^3}{27}$
c 1 Multiply 3 and 6.	**c** $3k^2 \times 6k^5 = 18 \times k^2 \times k^5$
2 Since the indices have the same base, apply index law 1 by adding the indices.	$= 18 \times k^{(2+5)}$
3 Simplify.	$= 18k^7$
d 1 Write the expressions as a fraction	**d** $8a^{12} \div 16a^7 = \frac{8a^{12}}{16a^7}$
2 Divide 8 and 16 by their highest common factor (HCF), which is 8.	$= \frac{\cancel{8}^{1}a^{12}}{\cancel{16}^{2}a^7}$
3 Since the indices have the same base, apply index law 2 by subtracting the indices.	$= \frac{a^{12}}{2a^7}$
	$= \frac{a^{(12-7)}}{2}$
4 Simplify.	$= \frac{a^5}{2}$

Example 2A.2 Combining index laws to simplify algebraic expressions

Use the appropriate index laws to simplify these expressions.

a $3a^4 \times 10a \div 6a^2$ **b** $(2z^7)^3$ **c** $\frac{x^6 \times 6x^2}{8(x^4)^2}$

THINK	WRITE
a 1 Write the expression as a fraction.	**a** $3a^4 \times 10a \div 6a^2 = \frac{3a^4 \times 10a}{6a^2}$
2 Simplify the numerator by multiplying 3 and 10 and using index law 1 to add the indices.	$= \frac{30a^5}{6a^2}$
3 Divide 30 and 6 by their HCF, which is 6.	$= \frac{\cancel{30}^{5}a^5}{\cancel{6}^{1}a^2}$
	$= \frac{5a^5}{a^2}$
4 Apply index law 2 to subtract the indices.	$= 5a^{5-2}$
	$= 5a^3$
b 1 Expand the brackets.	**b** $(2z^7)^3 = 2^3 \times (z^7)^3$
2 Apply index law 3 to multiply the indices and simplify.	$= 8 \times z^{7\times3}$
	$= 8z^{21}$
c 1 Simplify the numerator by applying index law 1. Simplify the denominator by applying index law 3.	**c** $\frac{x^6 \times 6x^2}{8(x^4)^2} = \frac{6x^8}{8x^8}$
2 Divide 6 and 8 by dividing them both by their HCF, which is 2.	$= \frac{\cancel{6}^{3}x^8}{\cancel{8}^{4}x^8}$
	$= \frac{3x^8}{4x^8}$
3 Divide the numerator and denominator by the common factor x^8.	$= \frac{3\cancel{x^8}}{4\cancel{x^8}}$
	$= \frac{3}{4}$

Helpful hints

✔ Remember, index laws 1 and 2 only apply if the expressions have the same base.

✔ Expanding brackets raised to an index only works if the brackets contain division and multiplication. You cannot apply the same method to expand brackets containing addition and subtraction!

$(a + b)^m \neq a^m + b^m$ and $(a - b)^m \neq a^m - b^m$

In order to expand these expressions, you will need to use the distributive law covered in section **2D**.

✔ Avoid errors in your working out by applying the index laws one at a time.

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Exercise 2A Indices

△ 1–7, 8(a, b), 10(a–e)

□ 1(e–h), 3(d, g–i), 4–6(2nd, 3rd columns), 7, 8(b, d), 9–10(f–i), 12, 17

⬠ 6–7(f–i), 8, 9–10(f–i), 13–17

UNDERSTANDING AND FLUENCY

1 Use appropriate index laws to simplify these expressions.

a $a^7 \div a^3$ **b** $(b^4)^5$ **c** $c^9 \times c^6$ **d** $d^3 \div d^2$

e $y^5 \times y^5$ **f** $(m^7)^2$ **g** $n^{15} \div n$ **h** $h \times h^6$

i $\frac{x^{11}}{x^5}$ **j** $(p^6)^3$ **k** $\frac{w^4}{w}$ **l** $a^2 \times a \times a^7$

2 a What is the value of a term expressed in index form with a power of zero?

b Write the value of each of these.

i 3^0 **ii** 100^0 **iii** m^0 **iv** $5^9 \div 5^9$

3 Use the property $a^0 = 1$ to simplify each of these expressions.

a $3a^0$ **b** $(3a)^0$ **c** $8m^0$ **d** $(8m)^0$

e $(-5x)^0$ **f** $a^0 + b^0$ **g** $j^0 - k^0$ **h** $p^0 + p^0$

i $2c^0 + 4g^0$ **j** $w^0 + x^0 - y^0$ **k** $(m + n)^0$ **l** x^0

4 Use the index law $(a \times b)^m = a^m \times b^m$ to write each of these expressions without brackets.

a $(a \times b)^5$ **b** $(k \times p)^9$ **c** $(4 \times c)^3$

d $(7 \times y)^{13}$ **e** $(xy)^4$ **f** $(cd)^7$

g $(3w)^2$ **h** $(6g)^6$ **i** $(2p)^8$

5 Use the index law $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ to write each of these expressions without brackets.

a $\left(\frac{a}{b}\right)^4$ **b** $\left(\frac{f}{g}\right)^7$ **c** $\left(\frac{k}{2}\right)^6$

d $\left(\frac{5}{x}\right)^6$ **e** $\left(\frac{d}{c}\right)^{-2}$ **f** $\left(\frac{m}{3}\right)^{-5}$

2A.1 **6** Use the appropriate index laws to simplify these expressions.

a $5x^2 \times x^3$ **b** $2a^2 \times 3a^5$ **c** $(3x)^2$

d $(2x^4)^3$ **e** $\frac{4a^5}{a^2}$ **f** $\frac{12x^8}{6x^3}$

g $\frac{4x^3}{2x^3}$ **h** $3x^6 \times 2x^2 \times 5x^4$ **i** $\frac{(6a^4)^2}{12a^5}$

2A.2 **7** Use the appropriate index laws to simplify these expressions.

a $\frac{4y^2 \times 12y^5}{6y^3}$ **b** $3(2x^3)^3$ **c** $(3k)^2 \times (2k)^3$

d $(3b^5)^2 \times 5b^2$ **e** $3a^3 \times (2a^4)^2 \times 5a^9$ **f** $15c^4 \times 6c^3 \div (3c)^2$

g $\frac{(2n^2)^2 \times 5n^3}{10n^5}$ **h** $\frac{3v^2}{4} \times \frac{8v^4}{6}$ **i** $\left(\frac{5m^5}{2}\right)^2 \times 4m^3$

PROBLEM SOLVING AND REASONING

8 Use the index laws to simplify each expression. Write your answers in index form.

a $\dfrac{p^4 \times (p^5)^3}{p^{10}}$ **b** $\dfrac{2(t^2)^8 \times (3t^5)^3}{3t^{11}}$

c $\dfrac{(5m^{11}n^{10})^8 \times (5mn^6)^2}{(5m^9n^7)^5 \times (5m^2n^3)^3}$ **d** $\dfrac{(8j^5p)^0 \times (j^0p^4)^3}{(jp^2)^6}$

9 Use the index laws to decide whether each of these statements is true or false. Explain your reasoning. For each false statement, change the right-hand side to make the statement true.

a $x^7 \times x \times x^7 = x^{14}$ **b** $(5a)^3 = 5 \times a^3$ **c** $-k^0 = -1$

d $a^2b^3 \times a^3b^4 = a^{12}b^{12}$ **e** $x^7y^4 \div y^6 = \dfrac{x^7}{y^2}$ **f** $\left(\dfrac{m}{n}\right)^5 = \dfrac{m^5}{n}$

g $\dfrac{w^6 \times w^6}{w^{12}} = 0$ **h** $\dfrac{(b^5)^4 \times b^2}{(b^3)^7} = b$ **i** $\dfrac{a^8b^5}{a^4b^3} \times \dfrac{a^2b^2}{a^9b^4} = a^3$

10 Find the value of x that will make the statement true.

a $a^x \times a^3 = a^{12}$ **b** $b^x = 1$ **c** $c^x \div c^6 = c^4$

d $(d^3)^x = d^{12}$ **e** $\left(\dfrac{m}{n}\right)^x = \left(\dfrac{m^7}{n^7}\right)$ **f** $\dfrac{w^x \times w^4}{w^7} = w^6$

g $(2a^x)^5 = 32a^{30}$ **h** $\dfrac{(n^3)^x \times n^4}{(n^2)^6} = n^7$ **i** $\dfrac{a^3b^6}{a^4b^x} \times \dfrac{a^7(b^2)^3}{a^5b^4} = a$

11 Write $\dfrac{8^5 \times 2^4}{4^7}$ in simplest index form with a base of 2.

12 Write $\dfrac{3^{4x} \times 9^{2x}}{27^x \times 3^{3x}}$ in simplest index form with a base of 3.

13 Use the appropriate index laws to simplify $\dfrac{25^{3x} \times 625^x}{125^{2x} \times 3^{3x}}$.

14 Find the value of x and y to make each of these statements true.

a $a^xb \times a^y(b^4)^y = a^{12}b^{33}$ **b** $\dfrac{a^xb^y}{a^2b^3} \times \dfrac{a^{x+y}b^y}{a^3b^7} = a^3b^{14}$

CHALLENGE

15 A square number is a number that can be written in the form n^2, where n is a whole number.

a Show that $\dfrac{2^{13} \times 3^{11} \times 5^9}{30^7}$ is a square number.

b Find the possible values of a if $\dfrac{5^7 \times 7^5}{a}$ is a square number.

16 Evaluate $\dfrac{6^5 \times 5^7}{2^4 \times 15^5}$ without a calculator.

17 It is possible to apply indices to negative numbers, but it is necessary to include a bracket. For example, -5^2 is negative 5^2, meaning $-5^2 = -25$. However, $(-5)^2$ is -5×-5 and therefore $(-5)^2 = 25$.

a Evaluate each the following. It may be helpful to write the expression in expanded form each time.

i 2^2 **ii** $(-2)^2$ **iii** 2^3

iv $(-2)^3$ **v** 10^5 **vi** $(-10)^5$

b Evaluate each of the following without a calculator.

i $(-2)^5$ **ii** $(-3)^4$ **iii** $(-5)^3$

c Decide if each of the following will evaluate to a positive number or a negative number. Explain your answer.

i $(-4)^{10}$ **ii** $(-7)^7$ **iii** $(-123)^{456}$

Check your Student obook pro for these digital resources and more:

pro

Interactive skillsheet
Index laws

CAS instructions
Simplifying indices

Topic quiz
2A

2B Negative indices

Learning intentions

✔ I can write algebraic terms with only positive indices.

✔ I can use the index laws to simplify expressions with negative indices.

Inter-year links

Year 8	2C Multiplying and dividing fractions
Year 9	2D Negative indices

Negative indices

- A **negative index** indicates the number of times that the base is divided repeatedly. This can be shown by extending the same pattern that is applied to positive indices in the reverse direction.

multiply by the base →

$$\ldots, 3^{-3}=\frac{1}{27}, 3^{-2}=\frac{1}{9}, 3^{-1}=\frac{1}{3}, 3^{0}=1, 3^{1}=3, 3^{2}=9, 3^{3}=27, \ldots$$

← divide by the base

Algebraic property	Example
$a^{-1}=\frac{1}{a}$	$2^{-1}=\frac{1}{2}$
$\frac{1}{a^{-1}}=a$	$\frac{1}{3^{-1}}=3$
$\left(\frac{a}{b}\right)^{-1}=\frac{b}{a}$	$\left(\frac{2}{3}\right)^{-1}=\frac{3}{2}$

- The **reciprocal** of a number can be expressed using an index of −1. The algebraic properties on the right apply for all non-zero values of *a* and *b*.
- The index laws apply to negative indices in the same way they do for positive indices. For example, we can apply index law 3 to generalise the properties above for any index.
- These properties allow for expressions containing negative indices to be expressed using only positive indices.
  For example,
  $$3hr^{-2}=\frac{3h}{r^2}$$
- They can also be used to write fractions in index form.
  For example,
  $$\frac{2b^3}{a}=2a^{-1}b^3$$

Algebraic property	Example
$a^{-m}=\frac{1}{a^m}$	$3^{-2}=\frac{1}{3^2}$ $=\frac{1}{9}$
$\frac{1}{a^{-m}}=a^m$	$\frac{1}{4^{-3}}=4^3$ $=64$
$\left(\frac{a}{b}\right)^{-m}=\left(\frac{b}{a}\right)^m$	$\left(\frac{2}{3}\right)^{-4}=\left(\frac{3}{2}\right)^4$ $=\frac{81}{16}$

Example 2B.1 Writing terms with positive indices

Write an equivalent term containing positive indices for $\left(\frac{z}{3}\right)^{-2}$

THINK	WRITE
1 Apply $\left(\frac{a}{b}\right)^{-m}=\left(\frac{b}{a}\right)^m$ to write the expression with a positive index.	$\left(\frac{z}{3}\right)^{-2}=\left(\frac{3}{z}\right)^2$
2 Expand the brackets by applying the index to the numerator and the denominator.	$=\frac{3^2}{z^2}$
3 Simplify.	$=\frac{9}{z^2}$

Example 2B.2 Combining index laws to simplify negative indices

Use appropriate index laws to simplify these expressions. Write your answers using only positive indices.

a $\left(\frac{x^2}{x^{-1}}\right)^{-3}$ **b** $6g^{-4} \div (3g^{-2}) \times 2g$ **c** $\frac{3}{m^{-4}} \times (m^2)^{-2}$

THINK	**WRITE**
a **1** Apply index law 2 to subtract the indices of the fraction inside the bracket. Remember that subtracting a negative number is the same as adding a positive number.	**a** $\left(\frac{x^2}{x^{-1}}\right)^{-3} = (x^{(2-[-1])})^{-3}$ $= (x^{(2+1)})^{-3}$ $= (x^3)^{-3}$
2 Apply index law 3 to multiply the indices.	$= x^{(3\times[-3])}$ $= x^{-9}$
3 Apply $a^{-m} = \frac{1}{a^m}$ to write the expression with a positive index.	$= \frac{1}{x^9}$
b **1** Write the expression as a fraction.	**b** $6g^{-4} \div (3g^{-2}) \times 2g = \frac{6g^{-4} \times 2g}{3g^{-2}}$
2 Simplify the numerator by multiplying 6 by 2 in the numerator and applying index law 1 to add the indices.	$= \frac{12g^{(-4+1)}}{3g^{-2}}$
3 Divide 12 and 3 by their HCF of 3.	$= \frac{\cancel{12}^{4}g^{-3}}{\cancel{3}^{1}g^{-2}}$ $= \frac{4g^{-3}}{g^{-2}}$
4 Apply index law 2 to subtract the indices.	$= 4g^{(-3-[-2])}$ $= 4g^{(-3+2)}$ $= 4g^{-1}$
5 Apply $a^{-1} = \frac{1}{a}$ to write the number as a fraction with a positive index.	$= 4 \times \frac{1}{g}$ $= \frac{4}{g}$
c **1** Separate the denominator from the numerator and write each factor with a positive index by applying: $\frac{1}{a^{-m}} = a^m$ and $a^{-m} = \frac{1}{a^m}$	**c** $\frac{3}{m^{-4}} \times (m^2)^{-2} = 3 \times \frac{1}{m^{-4}} \times (m^2)^{-2}$ $= 3 \times m^4 \times (m^2)^{-2}$
2 Apply index law 3 to multiply the indices, then index law 2 to subtract the indices.	$= 3 \times m^4 \times m^{-4}$ $= 3 \times m^{(4+[-4])}$ $= 3m^0$
3 Simplify. Remember that any base raised to a zero index is equal to one.	$= 3$

Helpful hints

✔ Don't confuse negative indices with negative numbers.
For example,

$$2^{-3} = \frac{1}{2^3} \quad \text{and} \quad 2^{-3} \neq -(2^3)$$

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Exercise 2B Negative indices

△ 1–7(1st, 2nd columns), 8(a–c), 11(a, b),

□ 1–4(4th column), 5–7(3rd, 4th columns), 8(c, e, f), 9(b, d, f), 11, 13(a)

⬠ 4–7(4th column), 8(d, g, h), 9(d–f), 10, 12–16

UNDERSTANDING AND FLUENCY

2B.1 **1** Write each of these terms using only positive indices.

a x^{-5} **b** $c^{-9}d^4$ **c** j^5k^{-3} **d** $x^{-1}y^8$

e $2a^3b^{-6}$ **f** $3p^{-4}q^{-5}$ **g** $w^4x^{-5}y^7$ **h** $k^{-3}m^5n^{-8}$

i $6d^2e^{-7}f$ **j** $11a^{-3}b^{-5}c^{-9}$ **k** $\frac{1}{x^{-3}}$ **l** $\frac{b^4}{c^{-6}}$

m $\frac{4m^3}{p^{-7}}$ **n** $\frac{a^{-4}}{b^{-2}}$ **o** $\frac{8x^{-2}}{2y^{-5}}$ **p** $\frac{c^{-1}d^8}{e^{-4}}$

2 You have seen that the index laws also apply to terms with negative indices. Simplify each of these expressions.

a $x^5 \times x^{-2}$ **b** $m^4 \div m^{-3}$ **c** $(k^{-6})^{-2}$ **d** $y^{-6} \times y^{10}$

e $a^{-4} \div a^{-9}$ **f** $(d^{-5})^3$ **g** $y^3 \times y^{-4} \times y^2$ **h** $n^{-7} \times n^{-3} \times n^{12}$

3 Write each of these expressions without brackets and using only positive indices.

a $(x \times y)^{-4}$ **b** $(4 \times a)^{-6}$ **c** $(9 \times n)^{-1}$ **d** $(5p)^{-3}$

e $(3p)^{-2}$ **f** $(km)^{-5}$ **g** $\left(\frac{x}{4}\right)^{-3}$ **h** $\left(\frac{7}{d}\right)^{-1}$

4 Use the appropriate index laws to simplify these expressions.

a $2x^6 \times 7x^3$ **b** $x^7 \times 3x^{-5}$ **c** $x^3 \times x^8 \times x^4$ **d** $5x^9 \times x \times 4x^{-2}$

e $8x^6 \div 2x^3$ **f** $30x^9 \div 5x^3$ **g** $6x^5 \div 14x^4$ **h** $12x^{10} \div 8x^5$

i $(x^3)^5 \times x^2$ **j** $(x^6)^4 \times (x^5)^2$ **k** $x^7 \times 2x^8 \times (x^2)^4$ **l** $(x^4)^3 \times (x^5)^6 \times 5x^{-7}$

5 Simplify each of these expressions. Write your answers using only positive indices.

a $x^3 \times x^{-5}$ **b** $2x^{-4} \times x^{-1}$ **c** $7x^{-3} \times 3x^6 \times 2x^{-2}$ **d** $4x^{-8} \times 5x^3 \times x$

e $x^{-11} \div x^{-9}$ **f** $6x^4 \div (3x^{-5})$ **g** $3x^{-7} \div (12x^2)$ **h** $9x^6 \div (15x^{10})$

i $(x^{-3})^2 \times x^{-5}$ **j** $(x^{-4})^6 \times 10x^{18}$ **k** $3x^{-6} \times (x^{-1})^7 \times (x^3)^2$ **l** $(x^{-2})^7 \times (x^{-5})^{-1} \times (x^{-3})^4$

2B.2 **6** Use the appropriate index laws to simplify each of these expressions. Write your answers using only positive indices.

a $(x^3)^{-2}$ **b** $(2y^{-3})^2$ **c** $(w^{-3})^{-4}$ **d** $\left(\frac{2x^4}{x^{-3}}\right)^2$

e $\left(\frac{a^4}{3}\right)^{-2}$ **f** $(2m)^{-2} \times (2m)^3$ **g** $\left(\frac{h^{-4}}{2}\right)^{-3}$ **h** $3c^{-4} \times 5c^{-8} \times 2c^3$

i $(2d)^{-2} \times 8d^5$ **j** $(6t^2)^{-2} \times 72(t^4)^{-3}$ **k** $(4x^{-2})^2 \times (2x^{-3})^{-3} \div (2x^4)$ **l** $\left(\frac{a^2}{3}\right)^{-3} \times \left(\frac{2a^{-1}}{3}\right)^2$

7 Use the index laws to help you simplify each of these expressions. Write your answers using only positive indices.

a $(ab)^4 \times a^7b^3$ **b** $(4p)^3 \times (6p)^2$ **c** $(2y^3)^5$ **d** $3(m^2n)^6$

e $\frac{d^5}{h^4} \times \left(\frac{d}{h}\right)^7$ **f** $\left(\frac{5w}{y}\right)^3$ **g** $\left(\frac{a^5}{b^{-2}}\right)^6$ **h** $\left(\frac{a^{-4}b^2}{c^7}\right)^4$

i $\left(\frac{wx^5}{u^3}\right)^4 \times \left(\frac{u^5}{wx}\right)^3$ **j** $\left(\frac{c^6}{a^6n^2}\right)^2 \times \frac{(5a^4)^3}{c^8n^{-5}}$ **k** $\frac{(m^5n^4)^3 \times (m^6n)^4}{(mn^3)^5}$ **l** $\frac{(2x^2)^6(y^4)^2}{(4x^4y^5)^3}$

PROBLEM SOLVING AND REASONING

8 Use the index laws to decide whether each of the following statements is true or false. Explain your reasoning. For each false statement, change the right-hand side to make the statement true.

a $\frac{a^5}{a^9} = a^4$ **b** $\frac{x^3}{x^6} = x^{\frac{1}{2}}$ **c** $\frac{y^3}{y^9} = \frac{1}{y^6}$

d $\left(\frac{3a^2b^{-1}}{c^3}\right)^2 = \frac{6a^4}{b^2c^6}$ **e** $(5m^4n^{-3})^{-2} = \frac{n^6}{25m^8}$ **f** $\frac{2a^{-4}b^3}{5^{-1}ab^2} = \frac{10b}{a^5}$

g $\frac{3(x^{-2}y^1)^3}{x^3y^{-2}z} = \frac{3y^5}{x^5z}$ **h** $(6abc)^{-2} \times (3a^{-1}b)^2 \div (4a^4c^2)^{-1} = 1$

9 Find the value of x that will make each of these statements true.

a $\frac{a^x}{a^4} = \frac{1}{a}$ **b** $\frac{3}{a^x} = 3a^4$ **c** $\left(\frac{2a^{-2}b}{c^3}\right)^x = \frac{a^4c^6}{4b^2}$

d $\frac{(ab)^x}{a^2b^4} = \frac{a}{b}$ **e** $\frac{a^3b^x}{a^{3x}b^2} = \frac{1}{b}$ **f** $\frac{y^7 \times y^x}{y^{2x}} = y^2$

10 Simplify each of these expressions.

a $x^{2a}y^{b+3} \times x^5y^{3b-2}$ **b** $\frac{m^{3x+4}n^{x-2}}{m^xn^{x-5}}$

11 Evaluate each of the following.

a 5^{-2} **b** 3^{-3} **c** $\frac{5^{-2}}{10^{-3}}$

d $\left(\frac{3^{-1}}{2^2}\right)^{-2}$ **e** $4^{-2} + 2^{-4}$ **f** $3^{-2} + 3^{-1} + 3^0$

12 Simplify each of these expressions.

a $x^{2a+1} \times x^{a-5}$ **b** $(y^m)^2 \times y^{m+5}$ **c** $\frac{a^{3x+3}}{a^{2x}}$ **d** $\frac{b^{2x}}{b^y} \times b^3 \times \frac{b^{y+1}}{b^x}$

13 Use index laws to simplify each of the following expressions. Write your answers with positive indices only.

a $\frac{3a^2b^4c}{8a^5b^3} \times \frac{4a^5b^4}{b^3c^2}$ **b** $\frac{10ac^5}{5a^5b^9c^2} \times \frac{3a^4b^3c}{(2bc^3)^2}$ **c** $\frac{4(x^2y^{-1}z)^3}{10x^4y^2} \div \frac{2x^5z^2}{5x^3y^4z^{-1}}$

14 **a** Express $\left(\frac{3a^2b^3}{2c^4}\right)^{-3}$ in expanded form, using positive indices only, by:

- **i** raising each numeral and pronumeral to the power of –3 and then simplifying
- **ii** flipping the fraction and then raising each numeral and pronumeral to the power of +3.

b Which method did you prefer and why?

15 Simplify $\left(\frac{c^7d^3}{4a^3b^{-6}}\right)^{-2} \times \left(\frac{2a^2b^{-4}}{c^5d^2}\right)^{-3}$ as far as possible.

CHALLENGE

16 **a** Starting with the simple statement that $a^m = 1 \times a^m$ for all values of a and m, use inverse operations and the index laws to prove that $a^0 = 1$.

b Starting with $a^0 = 1$, use inverse operations and the index laws to prove that $a^{-1} = \frac{1}{a}$. (Hint: Use index law 1 and the fact that $0 = 1 + [-1]$.)

c Explain why the index law $a^{-1} = \frac{1}{a}$ only works for non-zero values of a.

d Starting with $a^{-1} = \frac{1}{a}$, use index laws to prove that $a^{-m} = \frac{1}{a^m}$ for any value of m. (Hint: Start by raising both sides of $a^{-1} = \frac{1}{a}$ to the index m.)

e Starting with $a^{-m} = \frac{1}{a^m}$ use inverse operations to prove that $a^m = \frac{1}{a^{-m}}$.

f Starting with $a^{-m} = \frac{1}{a^m}$ use the corresponding property $a^m = \frac{1}{a^{-m}}$ and index laws to prove that:

$\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$ (Hint: Start by multiplying both sides of $a^{-m} = \frac{1}{a^m}$ by b^m.)

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Interactive skillsheet
Negative indices

Topic quiz
2B

2C Simplifying

Learning intentions

✔ I can evaluate algebraic expressions using substitution.

✔ I can identify like terms to simplify algebraic expressions.

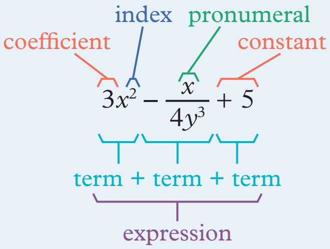
✔ I can multiply and divide algebraic terms.

Inter-year links

Year 7	6F Simplifying
Year 8	5C Adding and subtracting algebraic terms
Year 9	3A Simplifying

Evaluating algebraic expressions

- The terms in an algebraic expression are separated by a plus or minus sign.
- An algebraic term can be written as the product of a **coefficient** (number) multiplied by any combination of **pronumerals** raised to any index. If there are no pronumerals in the term, then it is a **constant term**.
- A pronumeral can be referred to as either a **variable** or an unknown.



- When pronumerals are given numerical values, an algebraic expression can be evaluated by **substitution**.

  For example, if $x = 3$ and $y = 2$:

$$\begin{aligned} x^2 + 5y - 2\frac{x}{y} &= 3^2 + 5(2) - 2\left(\frac{3}{2}\right) \\ &= 9 + 10 - 3 \\ &= 16 \end{aligned}$$

Adding and subtracting terms

- **Like terms** are the terms of an algebraic expression that contain the same pronumerals raised to the same power. The order of the pronumerals does not matter.

  For example, the following are sets of like terms:
  - → x^2 and $3x^2$
  - → 3, 16 and 100
  - → $2y^2z$ and $8zy^2$
  - → $4w^3h^4t$, $2w^3th^4$ and $10th^4w^3$
- Like terms can be added or subtracted to simplify an algebraic expression.

  For example,

$$7x^2y + 2 + 3yx^2 - 8 = 10x^2y - 6$$

Multiplying and dividing terms

- To multiply algebraic terms, multiply the coefficients and combine the pronumerals, using index laws if necessary.
  For example,
  $$\begin{aligned} 3a^3b \times -2a^2b^2 &= 3 \times -2 \times a^{(3+2)} \times b^{(1+2)} \\ &= -6a^5b^3 \end{aligned}$$
- To divide algebraic terms, divide the coefficients and combine the pronumerals, using index laws if necessary.
  For example,
  $$\begin{aligned} \frac{6a^5b^3}{3a^3b} &= \frac{{}^{2}\cancel{6} \times a^5 \times b^3}{{}^{1}\cancel{3} \times a^3 \times b} \\ &= 2 \times a^{(5-3)} \times b^{(3-1)} \\ &= 2a^2b^2 \end{aligned}$$
- If the terms being divided cannot be simplified using only positive indices, leave them in fraction form.
  For example,
  $$3t \div 6r^2 = \frac{t}{2r^2}$$

Example 2C.1 Using substitution to evaluate algebraic expressions

If $a = -2$, $b = 5$ and $c = -1$, evaluate:

a a^3bc^2 **b** $2b - ab + ac$

THINK	**WRITE**
a Substitute the known values and calculate the value.	**a** $\begin{aligned} a^3bc^2 &= (-2)^3(5)(-1)^2 \\ &= -8 \times 5 \times 1 \\ &= -40 \end{aligned}$
b Substitute the known values and calculate the value.	**b** $\begin{aligned} 2b - ab + ac &= 2(5) - (-2)(5) + (-2)(-1) \\ &= 10 + 10 + 2 \\ &= 22 \end{aligned}$

Example 2C.2 Adding and subtracting algebraic terms

Simplify each of these expressions by collecting like terms.

a $9a + 12b - 3 + 3a + 2a - 1$ **b** $2m^2n - 8n - 10n + n^{-2} + 3m + 4nm^2$

THINK	**WRITE**
a 1 Rearrange the expression, grouping like terms together.	**a** $9a + 12b - 3 + 3a + 2a - 1$ $= (9a + 3a + 2a) + (12b) + (-3 - 1)$
2 Simplify by combining like terms.	$= 14a + 12b - 4$
b 1 Rearrange the expression by grouping like terms together. Write each term with its pronumerals in alphabetical order so that the like terms can be seen clearly.	**b** $2m^2n - 8n - 10n + n^{-2} + 3m + 4nm^2$ $= (2m^2n + 4m^2n) + (n^{-2}) + (3m) + (-8n - 10n)$
2 Simplify by combining like terms.	$= 6m^2n + n^{-2} + 3m - 18n$

Example 2C.3 Using index laws to multiply and divide algebraic terms

Simplify these expressions by writing each one as a single term with positive indices.

a $2ab^2 \times 6a^{-3}c$ **b** $12pqr^2 \div (4qr)$

THINK	WRITE
a **1** Multiply the coefficients and apply index law 1 to multiply the terms with the same base by adding their indices.	**a** $2ab^2 \times 6a^{-3}c = 12 \times a^{(1+[-3])}b^2c$ $= 12a^{-2}b^2c$
2 Write the answer using positive indices.	$= \dfrac{12\,b^2\,c}{a^2}$
b **1** Write the expression as a fraction.	**b** $12pqr^2 \div (4qr) = \dfrac{12pqr^2}{4qr}$
2 Simplify the coefficients by dividing them both by their HCF of 4.	$= \dfrac{\cancel{12}^{3}pqr^2}{\cancel{4}^{1}qr}$ $= \dfrac{3pqr^2}{qr}$
3 Apply index law 2 to divide the terms with the same base by subtracting their indices.	$= 3pq^{(1-1)}r^{(2-1)}$
4 Simplify. Remember that any base raised to a zero index is equal to 1.	$= 3pq^0r^1$ $= 3pr$

Helpful hints

✔ The order in which you write the coefficients and pronumerals in an algebraic term doesn't matter. But it is easier to identify like terms if you write the pronumerals in alphabetical order. For example,

$$acb + cb - ba + 12bca + 8ab - 6bac = abc + 12abc - 6abc + bc - ab + 8ab$$
$$= 7abc + 7ab + bc$$

✔ Recall the following rules for algebraic notation.

→ To simplify products, leave out the multiplication sign and place the number first. For example,

$$7 \times x = 7x$$

and:

$$7 \times (x + 2) = 7(x + 2)$$

→ When a pronumeral is multiplied by 1, the 1 does not need to be included. For example,

$$1 \times x = x$$

→ Division is represented using fractions. For example,

$$x \div 7 = \frac{x}{7}$$

and:

$$(x + 2) \div 7 = \frac{(x + 2)}{7}$$

→ Terms with coefficients that are fractions can be written in two ways. For example,

$\frac{1}{7}x$ is the same as $\frac{x}{7}$

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Exercise 2C Simplifying

△ 1, 2, 3–4(1st column), 5–7, 8(a, c, e), 10–12 □ 2, 4, 7, 8, 10, 13, 14, 17(a–d) ⬠ 4, 7–9, 15–18

UNDERSTANDING AND FLUENCY

1 For each of the following terms:

i identify the coefficient **ii** write an example of a like term.

a $3abcd$ **b** $-4mn$ **c** xy^2 **d** $9k^2m^4p$

2A.1 **2** If $a = 1$, $b = 3$ and $c = -2$, evaluate:

a abc **b** $ab + bc$ **c** $ac - 2bc$ **d** a^4b^2

e $3ab^2c - ac$ **f** $7ac + 3bc - 4ac$ **g** $a^3b^2 + bc^3$ **h** $2a^2b + 3ac^2 - b^3$

2A.2 **3** Simplify each of these expressions by collecting like terms.

a $8x - 5x + 6x$ **b** $3ab - 4ab - 9ab$ **c** $4y^2 + y^2 + 5y^2$

d $7m^2n + 2m^2n - m^2n$ **e** $4a + 6d + 8a + 3d$ **f** $k - 3m + 5m + 7k$

g $4 + 8y - 2 - 11y$ **h** $a^2 + 6a - 5a^2 + a$ **i** $3xy + 2x^2 - xy + x^2$

4 Simplify each of these expressions by collecting like terms.

a $5a + 2b - a + 3b + 7a - 5b$ **b** $6mn - 3n - n + 2n^2 + m - 7mn$

c $x + 4x^2 - 8x + 1 + 5x - x^2$ **d** $2x^2y - 5x^2y - 4xy + 3 + 7xy - 8$

e $9p^2 - 3p + 2p^3 + 6p^2 - 9 + 3p$ **f** $a^2b + 4a^2 + 8ba^2 - b^2 + 3ab^2 - 5a^2$

5 Simplify each of the following expressions by writing each one as a single term.

a $5ab \times 2cd$ **b** $-7xy \times 3mn$ **c** $4kp \times k$

6 Simplify each of the following expressions by writing each one as a single term with positive indices.

a $abc \div b$ **b** $6mnp \div mp$ **c** $8wx \div 4wy$

2A.3 **7** Simplify each of the following expressions by writing each one as a single term with positive indices.

a $4x^2y^2 \times x^3y$ **b** $3ab^2 \times 2a^5b$ **c** $4m^5n^{-3} \times 6m^2n$

d $2x^4y^{-3}z^{-2} \times 5x^3y^5z^{-1}$ **e** $(10xy^2) \div (5x^3)$ **f** $(6x^2)^2 \div (9x^3)$

g $12x^2y^{-4} \times 3x^3z^7$ **h** $(3x^{-2}y^4)^2 \times 2x^3y^3$ **i** $(5a^{-2}b^3)^{-2} \times (10b)^2$

8 If $a = 3$, $b = -2$ and $c = 4$, evaluate each of the following expressions. (Hint: Simplify each expression first.)

a $4a + 3b + 2c - 2a - c + b$ **b** $9ab + 7a - 10a + ab$ **c** $a^2b + ab^2 + a^2c - 2a^2b + 5a^2c$

d $3abc \times b^2c \times 6a$ **e** $12a^2b^2c \div (3bc^2)$ **f** $ac^2 \times 2ab^3 \div (8a^2bc)$

9 The surface area of this box can be calculated using the expression $2lw + 2lh + 2wh$. Calculate the surface area of a box with each of the following sets of dimensions.

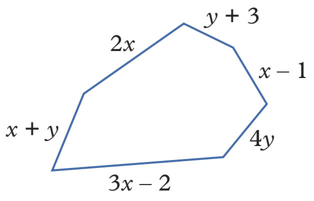
a $l = 25$ cm, $w = 15$ cm, $h = 10$ cm

b $l = 0.5$ m, $w = 0.3$ m, $h = 0.2$ m

10 The formula for the area, A, of a circle with radius r is $A = \pi r^2$. Calculate the area when $r = 5$ cm. Leave your answer as an exact value containing π.

11 Write the perimeter of each of these shapes as an algebraic expression in simplest form.

a



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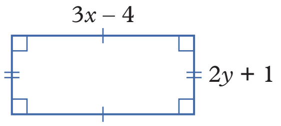

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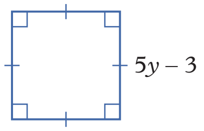
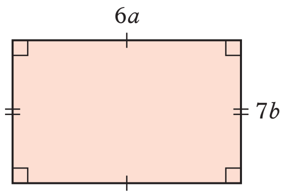


PROBLEM SOLVING AND REASONING

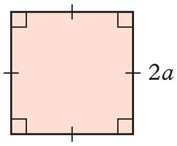
12 Calculate the perimeter of each shape in question **11** if $x = 7$ cm and $y = 2$ cm.

13 Write the area of each shaded shape below as an algebraic expression in simplest form.

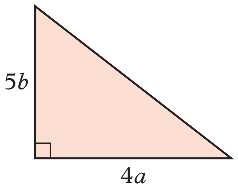
a



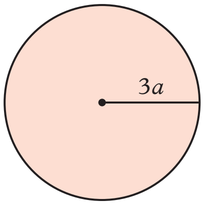
b



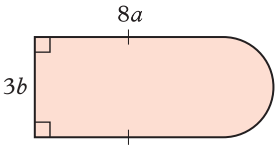
c



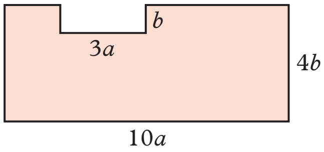
d Leave your answer as an exact value containing π.



e Leave your answer as an exact value containing π.

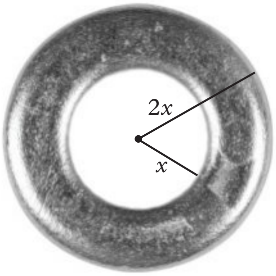


f



14 Calculate the area of each shape in question **13** if $a = 4$ m and $b = 3$ m. Leave your answer as an exact value containing π.

15 Consider the top surface of a metal washer shaped like an annulus as shown on the right.



- **a** Write an algebraic expression for the area of the top surface of one washer.
- **b** List three possible values for x that will result in an area between 400 mm^2 and 500 mm^2 for the surface area of the top of the washer.
- **c** Write an algebraic expression for the total length of the outer and inner edges of one washer.
- **d** Use the values you wrote for part **b** to calculate the total length of the outer and inner edges of one washer, correct to one decimal place.

CHALLENGE

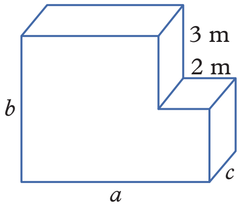
16 Use the order of operations and index laws to simplify each of the following expressions.

- **a** $2 \times (3a + 5a) - 4a$
- **b** $4b \times 3a + 5 \times (ab + 3ab)$
- **c** $2a \div 2 + 5a \times 3b - 4a$
- **d** $19xy^2 + 8x \div 4 + 5x^2y^2 \div x - 3x$
- **e** $3 \times (2a)^2 + (5ab^3)^2 - 2 \times (3a)^2$
- **f** $(-6xy)^2 + 4x^2y^2 \div x^2 + (3y)^2 - 5xy^2 \times 4x$

17 Simplify each of the following expressions by collecting like terms.

- **a** $5ab + 2ba$
- **b** $3abc + 4bac$
- **c** $6x^2y - 2xy + 5yx - yx^2$
- **d** $14map + 2amp + 3pam - 19mpa$
- **e** $2a^2b \times 4b + 3ab^2 \times 5a$
- **f** $\frac{6xy^3z^2}{2yx} + \frac{5y^3x^2z}{y^3} - \frac{14z^3x^5}{7x^3z^2} + \frac{(4zy)^2}{2}$

18 Consider the object on the right. (Hint: All angles are right angles, meaning all opposite sides are parallel and of equal length.)



- **a** Write an expression for the volume of the shape in cubic metres (m^3) in terms of a, b, and c in simplest form.
- **b** Write an expression for the surface area of the shape in square metres (m^2) in terms of a, b, and c in simplest form.

Check your Student obook pro for these digital resources and more:

pro

Interactive skillsheet
Multiplying terms

Interactive skillsheet
Dividing terms

Topic quiz
2C

2D Expanding

Learning intentions

- ✔ I can expand algebraic expressions with one pair of brackets.
- ✔ I can expand binomial products.

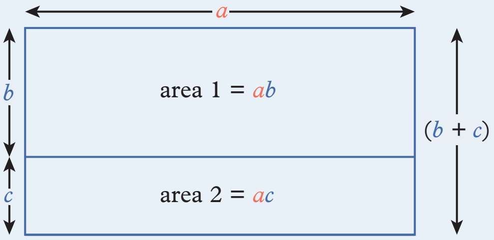
Inter-year links

Year 7	6D Order of operations in algebra
Year 8	5F Expanding
Year 9	3B Expanding

Expanding one pair of brackets

- The **distributive law** is used to distribute or expand products over addition and subtraction.

$$a(b + c) = ab + ac$$



total area = $a(b + c) = ab + ac$

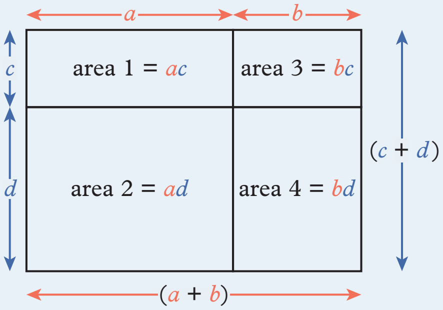
Expanding binomial products

- A **binomial** is an algebraic expression that contains two terms. Examples are:

$x + 1$ $\quad a^3 - 3a^4$ $\quad 21m^2 + n$

- The distributive law can be used to expand the product of two binomials.

$$(a + b)(c + d) = ac + ad + bc + bd$$



total area = $(a + b)(c + d) = ac + ad + bc + bd$

- Expanding binomial products often requires simplification using index laws and the identification of like terms.
- The **difference of two squares** is a specific form of expansion with the simplified rule:

$$(a + b)(a - b) = a^2 - b^2$$

- The **expansion of a perfect square** is a specific form of expansion with the simplified rule:

$$(a + b)^2 = a^2 + 2ab + b^2$$

Example 2D.1 Expanding one pair of brackets

Expand each of these algebraic expressions to remove the brackets.

a $4(a + 11)$ **b** $5k(k - 6)$ **c** $6g^3(g^4 + 3g)$ **d** $cd(3d - c^2)$

THINK	WRITE
a 1 Multiply each term inside the brackets by the term outside the brackets.	**a** $4(a + 11) = 4 \times a + 4 \times 11$
2 Simplify by performing the multiplications.	$= 4a + 44$
b 1 Multiply each term inside the brackets by the term outside the brackets.	**b** $5k(k - 6) = 5k \times k + 5k \times (-6)$
2 Simplify by performing the multiplications. Remember to take care with the + and – signs when simplifying.	$= 5k^2 - 30k$
c 1 Multiply each term inside the brackets by the term outside the brackets.	**c** $6g^3(g^4 + 3g) = 6g^3 \times g^4 + 6g^3 \times 3g$
2 Simplify by performing the multiplications. Apply index law 1, multiplying the terms with the same base by adding their indices.	$= 6g^7 + 18g^4$
d 1 Multiply each term inside the brackets by the term outside the brackets.	**d** $cd(3d - c^2) = cd \times 3d + cd \times (-c^2)$
2 Simplify by performing the multiplications. Apply index law 2, multiplying the terms with the same base by adding their indices.	$= 3cd^2 - c^3d$

Example 2D.2 Expanding a binomial product

Expand each of these algebraic expressions to remove the brackets.

a $(x + 4)(y + 9)$ **b** $(-f + 3)(4f - 6)$ **c** $(2z^2 + z)(z + 8)$ **d** $(h^2g - 2)(h + 4g^3)$

THINK	WRITE
a 1 Multiply each term inside the second pair of brackets, *y* and 9, by the first term in the first pair of brackets, *x*. Then multiply them by the second term in the first pair of brackets, 4.	**a** $(x + 4)(y + 9)$ $= x \times y + x \times 9 + 4 \times y + 4 \times 9$
2 Simplify each term.	$= xy + 9x + 4y + 36$
b 1 Multiply each term inside the second pair of brackets, 4*f* and –6, by the first term in the first pair of brackets, –*f*. Then multiply them by the second term in the first pair of brackets, 3.	**b** $(-f + 3)(4f - 6)$ $= (-f) \times 4f + (-f) \times (-6) + 3 \times 4f + 3 \times (-6)$
2 Simplify each term.	$= -4f^2 + 6f + 12f - 18$
3 Simplify any like terms.	$= -4f^2 + 18f - 18$

c **1** Multiply each term inside the second pair of brackets, z and 8, by the first term in the first pair of brackets, $2z^2$. Then multiply them by the second term in the first pair of brackets, z.	**c** $(2z^2 + z)(z + 8)$ $= 2z^2 \times z + 2z^2 \times 8 + z \times z + z \times 8$
2 Use index laws to simplify each term.	$= 2z^3 + 16z^2 + z^2 + 8z$
3 Simplify any like terms.	$= 2z^3 + 17z^2 + 8z$
d **1** Multiply each term inside the second pair of brackets, h and $4g^3$, by the first term in the first pair of brackets, h^2g. Then multiply them by the second term in the first pair of brackets, –2.	**d** $(h^2g - 2)(h + 4g^3)$ $= h^2g \times h + h^2g \times 4g^3 + (-2) \times h + (-2) \times 4g^3$
2 Use index laws to simplify each term.	$= gh^3 + 4g^4h^2 - 2h - 8g^3$

Helpful hints

- ✔ It's important to be able to recognise when an expression is the difference of two squares, or a perfect square. These are both very useful in algebra!
- ✔ Remember to pay close attention to minus signs when you are expanding an expression! Using brackets around negative terms can make it clearer.

ANS p655

Exercise 2D Expanding

▲ 1–2(1st, 2nd columns), 3, 5–6(1st column), 7, 8 ■ 3, 4, 5–6(2nd column), 7, 8, 10, 12 ⬟ 4, 5–6(3rd column), 7–9, 11, 13

1 Expand each of these algebraic expressions to remove the brackets.

a $3(a + 7)$ **b** $5(b - 4)$ **c** $2(6 - c)$ **d** $4(1 + d)$

e $-6(e + 3)$ **f** $-7(f - 2)$ **g** $-3(g - 1)$ **h** $-9(5 - h)$

i $x(x + 2)$ **j** $2y(y - 4)$ **k** $-k(k + 5)$ **l** $-3p(p - 1)$

m $m(3n + k)$ **n** $4a(3b + 6c)$ **o** $-w(w - 8x)$ **p** $-2p(2 + 5p)$

2D.1 **2** Expand each of these algebraic expressions to remove the brackets.

a $3(a - 4)$ **b** $-5(b - 3)$ **c** $4(2x + 3)$ **d** $-7(2y + 4)$

e $5a(2 - a)$ **f** $2m(m^2 + 3)$ **g** $8ab(3a + 1)$ **h** $4w(2x - w)$

i $5k^2(3 - 2k^2)$ **j** $6x^2y(2x - y^3)$ **k** $5cd(3c - 2d)$ **l** $-2a^2c(4a - 3b)$

3 Expand and simplify each of these algebraic expressions.

a $2(x + 5) + 7x$ **b** $4(y - 7) + 15$ **c** $m(n + 6) - 3m$

d $-3x(2 - x) + 5x^2 + x$ **e** $7p^3 - 8p + 5p(1 - p^2)$ **f** $x^4(x^3 - 2x) - x^5$

4 Expand and simplify each of these algebraic expressions.

a $4(x - 2) + 2(x + 5)$ **b** $m(m + 8) + 6(m - 3)$ **c** $2a(3a + 1) - 3(2 - 4a)$

d $k^2(k^3 + 2) + 5(k^2 + 1)$ **e** $3p^4(p^2 - 2) + 2p(p^3 + 6)$ **f** $5y^3(2y^5 + 3) - 2y(7y^2 - 4)$

UNDERSTANDING AND FLUENCY

2D.2 **5** Expand each of these algebraic expressions to remove the brackets.

a $(a + 4)(b + 2)$
b $(c + 3)(d + 6)$
c $(m + 7)(n + 5)$
d $(x + 6)(y + 8)$
e $(k + 5)(p - 1)$
f $(f - 2)(g + 3)$
g $(x - 9)(y - 4)$
h $(2m + 3)(n + 1)$
i $(4x + 5)(3y - 7)$
j $(3a - 2b)(5c - 4d)$
k $(x + 2)(x + 5)$
l $(y + 9)(y - 1)$
m $(m - 3)(m - 11)$
n $(p - 4)(p - 2)$
o $(5 - k)(6 - k)$
p $(4x + 7)(x + 3)$
q $(3a + 2)(a - 5)$
r $(8w - 7)(w - 2)$
s $(6p + 5)(2p + 7)$
t $(2x + 3)(4x - 1)$
u $(5y - 2)(3y - 4)$

6 Expand each of these algebraic expressions to remove the brackets.

a $(2a - b)(a + 3)$
b $(4t + u)(t - u)$
c $(2x + 3y)(x + 4y)$
d $(5 - x)(x + 4)$
e $(6x^2 + y)(x + 3y)$
f $(ab^2 + 1)(b^2 - 2a)$
g $(a + b)(b + c)$
h $(b + 3c)(b + d)$
i $(a + 2b)(3 - c)$
j $(a^3 + b^2)(3ab + 2)$
k $(x^4 - 3z)(5x - 2y)$
l $(c^2 + 4a^3b)(c^2 - 4a^3b)$

7 **a** Expand each of these binomial products.

i $(x + 2)(x - 2)$
ii $(k + 7)(k - 7)$
iii $(m + 8)(m - 8)$
iv $(w - 6)(w + 6)$
v $(y - 1)(y + 1)$
vi $(a + b)(a - b)$

b Describe any pattern or shortcut you can see in part **a**. What is special about the two binomial factors that were multiplied together in part **a**?

c The binomial products in this question are all examples of the 'difference of two squares'. Why do you think the rule is called the 'difference of two squares'?

d Does it matter whether the product is $(a + b)(a - b)$ or $(a - b)(a + b)$? Explain.

e Use the 'difference of two squares' rule to expand each of these binomial products.

i $(a - 3)(a + 3)$
ii $(x + 10)(x - 10)$
iii $(m + n)(m - n)$
iv $(3 + x)(3 - x)$
v $(1 + d)(1 - d)$
vi $(2x - 5)(2x + 5)$
vii $(2 - 3k)(2 + 3k)$
viii $(4g + h)(4g - h)$
ix $(5y + 2w)(5y - 2w)$

8 **a** Expand each of these binomial products.

i $(x + 3)(x + 3)$
ii $(y + 8)(y + 8)$
iii $(a + b)(a + b)$
iv $(k - 5)(k - 5)$
v $(p - 6)(p - 6)$
vi $(x - y)(x - y)$

b Describe any pattern or shortcut you can see in part **a**. What is special about the two binomial factors that were multiplied together in part **a**?

c These are all examples of 'perfect squares'. Why do you think the rule is called the 'perfect square' rule?

d Use the 'perfect square' rule to expand each of these algebraic expressions.

i $(a + 2)^2$
ii $(x + 5)^2$
iii $(p + 4)^2$
iv $(y + 10)^2$
v $(k + 9)^2$
vi $(m + n)^2$
vii $(5 + x)^2$
viii $(1 + 2d)^2$
ix $(3w + 7)^2$

e Why is $(a - b)^2 = a^2 - 2ab + b^2$ also a perfect square?

f Use the 'perfect square' rule to expand each of these algebraic expressions.

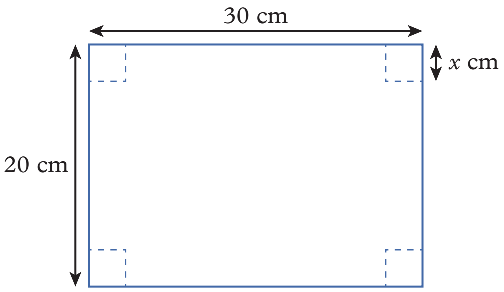
i $(a - 3)^2$
ii $(b - 4)^2$
iii $(x - 9)^2$
iv $(n - 11)^2$
v $(2 - x)^2$
vi $(g - h)^2$
vii $(4w - 1)^2$
viii $(5 - 2y)^2$
ix $(3m - 2p)^2$

9 Use the 'difference of two squares' and 'perfect square' rules to expand and simplify each of these expressions.

a $(x^2 + 2)(x^2 - 2)$
b $(x^2 - 5)(x^2 + 5)$
c $(x^2 + y^2)(x^2 - y^2)$
d $(x^2 + 4)^2$
e $(x^2 - 3)^2$
f $(x^3 - 1)^2$

PROBLEM SOLVING AND REASONING

10 A rectangular piece of cardboard is 30 cm long and 20 cm wide. A square of side length x cm is cut from each corner so that the cardboard can be folded to form an open box.



a Write an expression containing x for:

i the length of the box **ii** the width of the box **iii** the height of the box.

b Draw a labelled diagram of the open box.

c Write an expression for:

i the volume of space contained in the box **ii** the inner surface area of the box.

Simplify each of your expressions by expanding them to remove any brackets.

d If $x = 3$ cm, calculate:

i the volume of space contained in the box **ii** the inner surface area of the box.

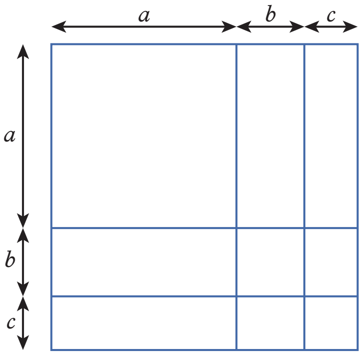
CHALLENGE

11 It is possible to expand brackets with three terms in each bracket.

a Use the diagram on the right to help you expand $(a + b + c)^2$.

b Expand $(x + 2y + 3)^2$.

c Expand $(2a - 3b + c)(2a + 3b - c)$.



12 It is possible to expand products of more than two brackets.
For example,

$$\begin{aligned}(a + b)^3 &= (a + b)(a + b)(a + b)\\ &= (a^2 + 2ab + b^2)(a + b)\\ &= a^3 + a^2b + 2a^2b + 2ab^2 + ab^2 + b^3\\ &= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

Expand each of the following:

a $(x + 1)^3$ **b** $(x - 3)^3$ **c** $(x + 2)^4$ **d** $(x + y)^2(x - y)^2$

13 In the diagram on the right, the area of the shaded rectangle is $(x - 2)(x - 5)$, which can be expanded to be expressed as $x^2 - 7x + 10$.

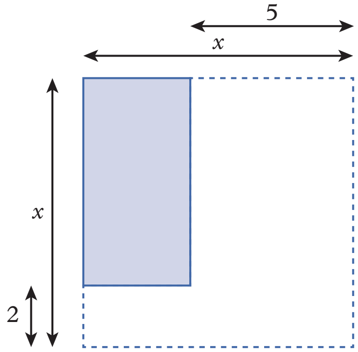
a Copy the diagram into your workbook and draw:

i the outline of a rectangle with area $2 \times x$ square units

ii the outline of a rectangle with area $5 \times x$ square units

iii the outline of a rectangle with area 10 square units.

b Explain why +10 appears in the expansion of $(x - 2)(x - 5)$, even though the rectangle with area 10 square units is not shaded.



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Interactive skillsheet
Expanding over one pair of brackets

Interactive skillsheet
Expanding binomial products

Investigation
Binomial expansion and Pascal's Triangle

CAS instructions
Expanding

Topic quiz
2D

2E Algebraic fractions

Learning intentions

- ✔ I can simplify algebraic fractions by identifying common factors.
- ✔ I can multiply and divide algebraic fractions.
- ✔ I can add and subtract algebraic fractions.

Inter-year links

Year 7	3B Equivalent fractions
Year 8	5E Dividing algebraic terms
Year 9	3A Simplifying

Simplifying algebraic fractions

- **Algebraic fractions** are fractions that contain at least one pronumeral. They can be simplified just like any other fractions, by identifying common factors of the numerator and the denominator.
- The fastest way to simplify a fraction is to divide the numerator and the denominator by their highest common factor (HCF).

  For example, the HCF of $2x^2$ and $4x$ is $2x$, so to simplify $\frac{2x^2}{4x}$, we divide the numerator and denominator by $2x$, as shown below.

  $$\underset{\div 2x}{\overset{\div 2x}{\frac{2x^2}{4x} = \frac{x}{2}}}$$

  This gives us $\frac{x}{2}$.

Multiplying and dividing algebraic fractions

- The product of two algebraic fractions is the product of the numerators over the product of the denominators. For example,

  $$\begin{aligned}\frac{3x}{2} \times \frac{x}{x+1} &= \frac{3x \times x}{2 \times (x+1)} \\ &= \frac{3x^2}{2(x+1)}\end{aligned}$$

- The **quotient** of two algebraic fractions is found by multiplying the fraction before the division sign by the reciprocal of the fraction after the division sign. For example,

  $$\begin{aligned}\frac{2x}{y} \div \frac{5}{x} &= \frac{2x}{y} \times \frac{x}{5} \\ &= \frac{2x^2}{5y}\end{aligned}$$

Adding and subtracting algebraic fractions

- Adding and subtracting algebraic fractions is no different from adding and subtracting any other fractions.
  1. Identify the lowest common denominator (LCD) by identifying the lowest common multiple (LCM) of the denominators.
  2. Write each fraction as an equivalent fraction with the LCD.

3 The sum of the algebraic fractions with a common denominator is then found by adding the numerators.

For example, $\frac{x+1}{4} + \frac{x}{6}$

The LCD of $\frac{x+1}{4}$ and $\frac{x}{6}$ is 12, so:

$$\begin{aligned}\frac{x+1}{4} + \frac{x}{6} &= \frac{x+1}{4} \times \frac{3}{3} + \frac{x}{6} \times \frac{2}{2}\\ &= \frac{3(x+1)}{12} + \frac{2x}{12}\\ &= \frac{3x+3+2x}{12}\\ &= \frac{5x+3}{12}\end{aligned}$$

- The difference of algebraic fractions with a common denominator is found by subtracting the numerators. So, using the same algebraic fractions from the previous example, we have:

$$\begin{aligned}\frac{x+1}{4} - \frac{x}{6} &= \frac{3(x+1)}{12} - \frac{2x}{12}\\ &= \frac{3x+3-2x}{12}\\ &= \frac{x+3}{12}\end{aligned}$$

Example 2E.1 Simplifying algebraic fractions

Simplify each of these algebraic fractions.

a $\frac{7a}{7ab}$ **b** $\frac{8(2x+1)}{16}$ **c** $\frac{(x+9)(3x+1)}{(x+2)(x+9)}$

THINK	WRITE
a 1 Look for factors common to both the numerator and the denominator to determine the highest common factor (HCF). The HCF is 7*a*.	**a** $\frac{7a}{7ab} = \frac{\cancel{7a}^{1}}{\cancel{7a}^{1}\,b}$
2 Divide both the numerator and the denominator by 7*a*.	$= \frac{1}{b}$
b 1 Look for factors common to both the numerator and the denominator to determine the HCF. The HCF is 8.	**b** $\frac{8(2x+1)}{16} = \frac{\cancel{8}^{1}(2x+1)}{\cancel{16}^{2}}$
2 Divide both the numerator and the denominator by 8.	$= \frac{2x+1}{2}$
c 1 Look for factors common to both the numerator and the denominator to determine the HCF. The HCF is $(x + 9)$.	**c** $\frac{(x+9)(3x+1)}{(x+2)(x+9)} = \frac{\cancel{(x+9)}^{1}(3x+1)}{(x+2)\,\cancel{(x+9)}^{1}}$
2 Divide both the numerator and the denominator by $(x + 9)$.	$= \frac{3x+1}{x+2}$

Example 2E.2 Multiplying and dividing algebraic fractions

Write each of these expressions as one fraction in simplest form.

a $\frac{x(x-2)}{(x+3)(2x+1)} \times \frac{(x+1)(2x+1)}{(x-2)(x+3)}$ **b** $\frac{x+4}{x^2} \div \frac{3(x+4)}{x-9}$

THINK	WRITE
a **1** Write the expression as a single fraction by multiplying the numerators together and the denominators together.	**a** $\frac{x(x-2)}{(x+3)(2x+1)} \times \frac{(x+1)(2x+1)}{(x-2)(x+3)}$ $= \frac{x(x-2) \times (x+1)(2x+1)}{(x+3)(2x+1) \times (x-2)(x+3)}$
2 Look for common factors. Divide both the numerator and the denominator by $(x-2)$ and $(2x+1)$.	$= \frac{x\cancel{(x-2)}^1(x+1)\cancel{(2x+1)}^1}{(x+3)\cancel{(2x+1)}^1\cancel{(x-2)}^1(x+3)}$ $= \frac{x(x+1)}{(x+3)(x+3)}$
3 Simplify.	$= \frac{x(x+1)}{(x+3)^2}$
b **1** Write the expression as the product of two fractions.	**b** $\frac{x+4}{x^2} \div \frac{3(x+4)}{x-9} = \frac{x+4}{x^2} \times \frac{x-9}{3(x+4)}$
2 Write the expression as a single fraction by multiplying the numerators together and the denominators together.	$= \frac{(x+4) \times (x-9)}{x^2 \times 3(x+4)}$
3 Look for common factors. Divide both the numerator and the denominator by $(x+4)$.	$= \frac{\cancel{(x+4)}^1(x-9)}{3x^2\cancel{(x+4)}^1}$ $= \frac{x-9}{3x^2}$

Example 2E.3 Adding and subtracting algebraic fractions with numerical denominators

Write each of these expressions as one fraction in simplest form.

a $\frac{2x+3}{7} + \frac{x-1}{3}$ **b** $\frac{7x}{10} - \frac{x}{6}$

THINK	WRITE
a **1** The LCM of 7 and 3 is 21. So write each fraction as an equivalent fraction with an LCD of 21.	**a** $\frac{2x+3}{7} + \frac{x-1}{3} = \frac{2x+3}{7} \times \frac{3}{3} + \frac{x-1}{3} \times \frac{7}{7}$ $= \frac{3(2x+3)}{21} + \frac{7(x-1)}{21}$
2 Write the expression as a single fraction by adding the numerators.	$= \frac{3(2x+3)+7(x-1)}{21}$
3 Expand the brackets.	$= \frac{6x+9+7x-7}{21}$
4 Simplify the numerator by adding and subtracting like terms.	$= \frac{13x+2}{21}$

b 1 The LCM of 10 and 6 is 30. So write each fraction as an equivalent fraction with an LCD of 30.	**b** $\frac{7x}{10}-\frac{x}{6}=\frac{7x}{10}\times\frac{3}{3}-\frac{x}{6}\times\frac{5}{5}$ $=\frac{21x}{30}-\frac{5x}{30}$
2 Write the expression as a single fraction by subtracting the numerators.	$=\frac{21x-5x}{30}$ $=\frac{16x}{30}$
3 Simplify by dividing the numerator and the denominator by their HCF of 2.	$=\frac{8x}{15}$

Example 2E.4 Adding and subtracting algebraic fractions with algebraic denominators

Write each of these expressions as one fraction in simplest form.

a $\frac{1}{3x}+\frac{4}{x}$

b $\frac{7x}{x+1}-\frac{x-3}{x}$

THINK	WRITE
a 1 The LCM of $3x$ and x is $3x$. So write each fraction as an equivalent fraction with an LCD of $3x$.	**a** $\frac{1}{3x}+\frac{4}{x}=\frac{1}{3x}+\frac{4}{x}\times\frac{3}{3}$ $=\frac{1}{3x}+\frac{12}{3x}$
2 Write the expression as a single fraction by adding the numerators.	$=\frac{1+12}{3x}$ $=\frac{13}{3x}$
b 1 The LCM of $(x + 1)$ and x is $x(x + 1)$. So write each fraction as an equivalent fraction with an LCD of $x(x + 1)$.	**b** $\frac{7x}{x+1}-\frac{x-3}{x}=\frac{7x}{x+1}\times\frac{x}{x}-\frac{x-3}{x}\times\frac{x+1}{x+1}$ $=\frac{7x\times x}{x(x+1)}-\frac{(x-3)(x+1)}{x(x+1)}$
2 Write the expression as a single fraction by subtracting the numerators.	$=\frac{7x^2-(x-3)(x+1)}{x(x+1)}$
3 Expand the binomial product in the numerator.	$=\frac{7x^2-(x^2-3x+x-3)}{x(x+1)}$ $=\frac{7x^2-(x^2-2x-3)}{x(x+1)}$
4 Simplify the numerator by adding and subtracting like terms.	$=\frac{7x^2-x^2+2x+3}{x(x+1)}$ $=\frac{6x^2+2x+3}{x(x+1)}$

Helpful hints

- ✔ You don't need to keep the brackets around a binomial factor if the other factors have been removed. For example,

$$\frac{2(x+3)}{4} = \frac{x+3}{2}$$

- ✔ Remember, if you want to turn an algebraic fraction into a sum of algebraic terms, you must divide *every* term in the numerator by the denominator.

$$\frac{x+9}{3} \neq x+3$$

- ✔ You don't need to find the LCD to add and subtract fractions! While the LCD is often the simplest to work with, any common denominator will do. The easiest way to find equivalent fractions with common denominators is to multiply each fraction by the denominator of the other fraction.

$$\begin{aligned}\frac{a}{c}+\frac{b}{d} &= \frac{a}{c}\times d+\frac{b}{d}\times c\\ &= \frac{ad}{cd}+\frac{bc}{dc}\\ &= \frac{ad+bc}{cd}\end{aligned}$$

ANS p656

Exercise 2E Algebraic fractions

△ 1(b, d, f, h), 2–4, 5(a, c, e, g, i, j), 6(a, c), 7–9(1st, 2nd columns), 12(a–c)

■ 3–4(2nd column), 5(b, d, f, h, j, k), 6(b, d), 7–9(3rd, 4th columns), 10, 12(d, e), 14, 15

⬠ 3(3rd column), 5(h–l), 6(b, d), 7–9(3rd column), 11, 13, 16–18

UNDERSTANDING AND FLUENCY

1 Simplify each of these algebraic fractions.

a $\frac{3x}{6}$ **b** $\frac{9x}{9xy}$ **c** $\frac{8x}{2x}$ **d** $\frac{4xy}{6y}$

e $\frac{2(x+3)}{10}$ **f** $\frac{8(x-5)}{4}$ **g** $\frac{x(x-4)}{x}$ **h** $\frac{3x(x+1)}{3x}$

2 Simplify each of these algebraic fractions.

a $\frac{(x-2)(x+5)}{(x+1)(x-2)}$ **b** $\frac{(x-6)(x+4)}{(x-2)(x+4)}$ **c** $\frac{(x-1)(x+3)}{(x+3)(x+1)}$

d $\frac{(x-7)(x+2)}{(x-7)(x-8)}$ **e** $\frac{(x+4)(x-5)}{(x+4)(x-5)}$ **f** $\frac{(x-1)(x+1)}{(x+1)(x-1)}$

2E.1 **3** Simplify each of these algebraic fractions.

a $\frac{8x}{4xy}$ **b** $\frac{14abc}{7ac}$ **c** $\frac{5(a+1)}{a(a+1)}$

d $\frac{x(x+3)}{2(x+3)}$ **e** $\frac{5(m-2)}{15m(m-2)}$ **f** $\frac{(n+3)(n+7)}{2(n+3)}$

g $\frac{3y^2(y+x)}{y(x+y)}$ **h** $\frac{x^2(x-4)}{x(x+1)(x-4)}$ **i** $\frac{5(y-3)(y+4)}{(y+4)(y-4)}$

j $\frac{(t+2)(r+3)}{(r+3)(r+2)}$ **k** $\frac{(x+2)(x-1)(2x+7)}{(2x+7)(x-4)(x+2)}$ **l** $\frac{(p^2+2)(p+6)(p-2)}{3(p+6)(p+2)(p^2+2)}$

4 Simplify each of these expressions.

a $\frac{3x}{x+2}\times\frac{x+2}{6x}$ **b** $\frac{x-3}{x}\times\frac{5x}{x-3}$ **c** $\frac{x+4}{x+1}\times\frac{x+1}{x+4}$

d $\frac{6x}{x-5}\div\frac{2x}{x-5}$ **e** $\frac{x+7}{4x}\div\frac{x+7}{20x}$ **f** $\frac{8x}{2x+1}\div\frac{8}{2x+1}$

2E.2 **5** Write each of these expressions as a single fraction in simplest form.

a $\frac{a+2}{a-4}\times\frac{2a+1}{a+2}$

b $\frac{x(x+1)}{x+5}\times\frac{x+5}{x(x-3)}$

c $\frac{t+3}{4(t+5)}\div\frac{t(t+3)}{8(t+5)}$

d $\frac{m(m+2)}{(m-3)(3m+1)}\times\frac{3m+1}{m(m+7)}$

e $\frac{(x+3)(x-2)}{x(x-2)}\times\frac{x(x+5)}{(x-3)}$

f $\frac{x+4}{(x-2)(x+4)}\div\frac{x+5}{x-2}$

g $\frac{2y+3}{y-4}\div\frac{y+3}{3(y-4)}$

h $\frac{(m+1)(m-2)}{(m+4)(m-3)}\times\frac{m(m+4)}{(m-1)(m-2)}$

i $\frac{(c^2+6)(c-4)}{(c+2)(c^2+3)}\times\frac{(2c+1)(c^2+3)}{(c-4)(2c+1)}$

j $\frac{n^2(n+p)}{n+2p}\times\frac{(n-p)(n+2p)}{n^3(n+p)}$

k $\frac{x^2y^3(x+2y)}{y(x-2y)(x+y)}\div\frac{x^4(x+2y)(x+1)}{(x+y)(x-2y)}$

l $\frac{5(2a+b)}{(4-3a)(b+2)}\times\frac{(a+b)(b+2)}{(a-b)(a+b)}\div\frac{b+2a}{2(4-3a)(a^2+b^2)}$

6 Write each of these expressions as a single fraction in simplest form.

a $\frac{6(x+5)(x-5)}{(x-2)(x+2)}\times\frac{(x-1)(x+2)}{3(x+5)(x-1)}$

b $\frac{(x+3)(x+8)}{4x(x+2)(x-3)}\times\frac{4x(x+2)(x-6)}{(x+3)(x+6)}$

c $\frac{(2x-1)(3x+2)}{(3x-2)(x+1)}\div\frac{(x+2)(2x-1)}{(5x+1)(3x-2)}$

d $\frac{5x(3x-5)}{(2x+5)(7x-4)}\div\frac{10(3x-5)(x+8)}{(7x-4)(2x+5)}$

2E.3 **7** Write each of these expressions as a single fraction in simplest form.

a $\frac{x}{4}+\frac{x}{5}$

b $\frac{x}{2}-\frac{x}{7}$

c $\frac{x}{6}+\frac{x}{3}$

d $\frac{x}{8}-\frac{x}{10}$

e $\frac{2x}{3}+\frac{x}{2}$

f $\frac{3x}{4}-\frac{x}{8}$

g $\frac{4x}{5}+\frac{3x}{7}$

h $\frac{x^2}{2}-\frac{x^2}{9}$

i $\frac{5a}{3}+\frac{3a}{5}$

j $\frac{b^2}{2}-\frac{b}{4}$

k $\frac{c+1}{4}+\frac{c}{6}$

l $\frac{m-2}{3}+\frac{m+1}{4}$

m $\frac{x-2}{2}-\frac{x}{5}$

n $\frac{4k+1}{4}-\frac{2k}{5}$

o $\frac{5x+1}{8}+\frac{x-3}{10}$

p $\frac{y}{2}+\frac{y}{4}+\frac{y}{8}$

q $\frac{x}{6}-\frac{x-1}{3}$

r $\frac{b+4}{8}-\frac{b+1}{12}$

s $\frac{2a-1}{4}-\frac{3a+1}{6}$

t $\frac{x+3}{2}+\frac{2x-3}{5}+\frac{x}{4}$

8 Write each of these expressions as a single fraction in simplest form.

a $\frac{x+1}{2}+\frac{x+5}{3}$

b $\frac{x+2}{5}+\frac{x-4}{6}$

c $\frac{x-3}{4}+\frac{x-1}{8}$

d $\frac{x+4}{3}-\frac{x+2}{4}$

e $\frac{x-6}{8}-\frac{x-3}{12}$

f $\frac{x+1}{2}-\frac{x+4}{10}$

g $\frac{2x+3}{3}+\frac{x-7}{5}$

h $\frac{x+5}{6}+\frac{2x+1}{9}$

i $\frac{4x+7}{3}+\frac{5x-3}{2}$

j $\frac{3x-1}{4}-\frac{2x-5}{5}$

k $\frac{4-x}{7}-\frac{x-4}{3}$

l $\frac{x^2+2}{5}-\frac{x^2+3}{8}$

2E.4 **9** Write each of these expressions as a single fraction in simplest form.

a $\frac{1}{x}+\frac{1}{x}$

b $\frac{1}{3a}+\frac{1}{a}$

c $\frac{1}{b}-\frac{5}{b^2}$

d $\frac{1}{x}-\frac{x}{3}$

e $\frac{2}{m+1}+\frac{3}{m}$

f $\frac{k}{k^2+1}+\frac{3}{k}$

g $\frac{3}{y+2}+\frac{1}{y-3}$

h $\frac{x}{x+3}+\frac{2}{x}$

i $\frac{t}{t+5}+\frac{2}{t-1}$

j $\frac{5}{x}-\frac{2}{2x+1}$

k $\frac{5}{x+2}-\frac{3}{x+1}$

l $\frac{4}{2h-1}-\frac{3}{h+2}$

m $\frac{m}{m-3}-\frac{1}{2m+1}$

n $\frac{a^2}{3a+1}+\frac{4}{a}$

o $\frac{x}{2}-\frac{2x+3}{x+1}$

p $\frac{x+1}{x}-\frac{x}{x+1}$

PROBLEM SOLVING AND REASONING

10 Find the algebraic expression that is equal to a if $\frac{a}{2} + \frac{x+1}{4} = \frac{3x-5}{4}$.

11 Find the algebraic expression that is equal to b if $\frac{b}{5} - \frac{x-2}{2} = \frac{x+12}{10}$.

12 Simplify each of these expressions using the most appropriate method.

a $\frac{x}{3} + \frac{2x}{5} + \frac{x}{2}$ **b** $\frac{5x}{4} + \frac{x}{8} - \frac{x}{2}$ **c** $\frac{4x}{3} - \frac{5x}{6} - \frac{3x}{8}$

d $\frac{2x^2}{7} \times \frac{3}{10x} \times \frac{5}{x}$ **e** $\frac{4}{x} \div \frac{12}{x^2} - \frac{5x}{7}$ **f** $\frac{3x}{4} \times \frac{2x(x-7)}{6x^2} + \frac{x+2}{3}$

13 For each of the following, expand the brackets and write the final answer as a single fraction in simplest form. For some, it may be helpful to simplify before expanding.

a $\left(\frac{1}{x} + x\right)\left(\frac{1}{x} - x\right)$ **b** $\left(\frac{1}{x} + x\right)^2$

c $\frac{x}{x+1}\left(\frac{2}{x-1} - \frac{1}{x}\right)$ **d** $\left(\frac{2}{x+1} - \frac{1}{x}\right)\left(\frac{1}{x-1} - \frac{1}{2x}\right)$

14 Consider the expression $\frac{x}{x+1} - \frac{x-1}{x}$.

a Write this expression as a single fraction in simplest form.

b Evaluate:

i $\frac{6}{7} - \frac{5}{6}$ **ii** $\frac{9}{10} - \frac{8}{9}$

15 Consider the expression $\frac{x}{y} - \frac{x}{y+1}$.

a Write this expression as a single fraction in simplest form.

b Evaluate:

i $\frac{5}{6} - \frac{5}{7}$ **ii** $\frac{7}{10} - \frac{7}{11}$

CHALLENGE

16 Consider the following algebraic property: $x - y = -(y - x)$.

Use this property to simplify the following algebraic fractions.

a $\frac{a-b}{b-a}$ **b** $\frac{4(m-n)}{n-m}$ **c** $\frac{(x-1)(5-x)(x-3)}{(x+4)(1-x)(x-5)}$

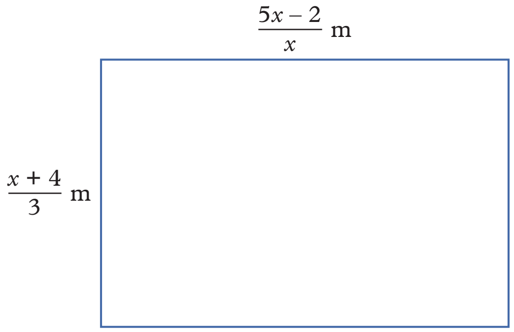
17 Write each of these expressions as a single fraction.

a $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ **b** $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$

18 Consider the rectangle on the right.

a Write an expression for the area of the rectangle.

b Write an expression for the perimeter of the rectangle, expressed as just a single fraction.



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2E

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Checkpoint

Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this chapter.

2A **1** Use the appropriate index law to simplify each of these expressions.

a $x^2 \times x^7$ **b** $y^6 \div y^2$ **c** $\dfrac{2m^4}{m^3}$

d $(a^2)^5$ **e** $(3xy^2)^2$ **f** $\left(\dfrac{2a}{b^2}\right)^3$

2A **2** Use the property $a^0 = 1$ to simplify each of the following.

a x^0 **b** $2y^0$ **c** $(3a^2)^0$ **d** $x^0 + y^0 + z^0$

2B **3** Rewrite each of these terms with positive indices.

a y^{-3} **b** $4x^{-2}$ **c** $\dfrac{a^{-7}}{6}$

d $\dfrac{2}{b^{-3}}$ **e** $\dfrac{a^{-2}}{2b^5}$ **f** $5c^{-3}d^4$

2B **4** Simplify the following expression, writing your answer with positive indices only.

$$\frac{3a^2b^3}{a^3} \times \left(\frac{2a^{-3}}{b^2}\right)^2$$

2C **5** Simplify each of these expressions.

a $4x + 3x - x$ **b** $5m^2 + 3m^2$ **c** $3a + 5b + 4a$

d $c + 3d - 4c + 2d$ **e** $3a^2 - 4a + 2a^2 + 2a$ **f** $2x^2y + 3xy - 4x^2y + xy^2$

2C **6** For each of the following, find the missing term(s) that will make the equation true.

a $4ab + ? = 7ab$ **b** $5x^2y + 3xy + ? = 11xy + 5x^2y$

c $a^2b - b^2a + ? = 2ab^2 + a^2b$ **d** $2x^2 + 3x - 6 + ? = 2x^2 + 6$

2D **7** Expand and simplify each of these algebraic expressions.

a $3(a + 7)$ **b** $3(2y + 1) + 4$

c $2(x - 5) + x + 2$ **d** $-3(x + 2) - 4(x - 4)$

2D **8** Expand and simplify each of the following.

a $(a + 1)(a + 2)$ **b** $(x - 4)(x + 6)$

c $(k - 5)(k + 5)$ **d** $(2x - 3)(x + 2)$

2D **9** Expand this expression:

$$(x^2 - y)(x + 2y)$$

2E **10** Write each of these expressions as a single fraction in simplest form.

a $\dfrac{x}{2} + \dfrac{x}{3}$ **b** $\dfrac{y+1}{3} + \dfrac{5y}{2}$ **c** $\dfrac{m^2+1}{4} - \dfrac{m}{5}$ **d** $\dfrac{2x+1}{3} - \dfrac{x+2}{7}$

2E **11** Simplify each of these algebraic fractions.

a $\dfrac{a(a+4)}{3(a+4)}$ **b** $\dfrac{(x+4)(x-3)}{5(x-3)}$

c $\dfrac{(x+4)(x-3)}{(x+7)(2x-3)} \div \dfrac{(x-3)(x+4)}{2x-3}$ **d** $\dfrac{m^2(m-1)(m+3)}{(m+3)(m+1)} \times \dfrac{(m+1)(m-4)}{m^3(m-1)}$

2E **12** Simplify this expression:

$$\frac{x}{x-2} - \frac{3}{x+1}$$

2F Factorising

Learning intentions

- ✔ I can factorise algebraic expressions by identifying the highest common factor.
- ✔ I can factorise algebraic expressions by grouping terms.
- ✔ I can factorise algebraic expressions using the difference of two squares rule.
- ✔ I can factorise algebraic expressions using the perfect square rule.

Inter-year links

Years 5/6	Factors and multiples
Year 7	2D Factors and the highest common factor
Year 8	5G Factorising
Year 9	3C Factorising using the HCF

Factorising using the highest common factor

- The distributive law can be used to factorise algebraic expressions by identifying the HCF of the terms in the expression.

$$ab + ac = a(b + c)$$

For example, the HCF of $2x^2$ and $-6x$ is $2x$, so:

$$\underset{\text{expanded form}}{2x^2 - 6x} = \underset{\text{factorised form}}{2x(x - 3)}$$

Factorising by grouping terms

- It is possible to find binomial factors of large algebraic expressions by grouping pairs of terms and factorising each pair using their HCF.

$$\begin{aligned} ac + ad + bc + bd &= a(c + d) + b(c + d) \\ &= (a + b)(c + d) \end{aligned}$$

For example, $x^2 + 2x - 3x + 6$ can be factorised into a binomial product by grouping the terms $x^2 + 2x$ and $-3x - 6$, and then factorising each of them with a common factor of $x + 2$:

$$\begin{aligned} x^2 + 2x - 3x - 6 &= x(x + 2) - 3(x + 2) \\ &= (x + 2)(x - 3) \end{aligned}$$

Factorising using the 'difference of two squares' rule

- Binomials that have the same terms but a different sign, such as $a + b$ and $a - b$, are **conjugates** of each other.
- If the expression is the difference of two squares, you can factorise it into the product of conjugate pairs using the 'difference of two squares' rule:

$$a^2 - b^2 = (a + b)(a - b)$$

- If the terms of a binomial expression have opposite signs, but both terms are not perfect squares, you can still factorise the expression using surds. For example,

$$x^2 - 3 = (x + \sqrt{3})(x - \sqrt{3})$$

Factorising using the ‘perfect square’ rule

- If the expression is of the same form as a perfect square, then you can factorise it using the ‘perfect square’ rule.

$$a^2 + 2ab + b^2 = (a + b)^2$$

For example,

$$\begin{aligned} x^2 + 6x + 9 &= x^2 + 2(x)(3) + 3^2 \\ &= (x + 3)^2 \end{aligned}$$

Example 2F.1 Factorising using the highest common factor

Factorise each of these expressions.

a $4n + 36$ **b** $t^2 - 7t$ **c** $5x^5y - 13x^2y^3$ **d** $3klm + 9km - 6k$

THINK	WRITE
a **1** Identify that the HCF is 4 and write each term as a product of the HCF and its other factor.	**a** $4n + 36 = 4 \times n + 4 \times 9$
2 Use the distributive law. Write the HCF before a pair of brackets containing the sum of the remaining factors.	$= 4(n + 9)$
b **1** Identify that the HCF is t and write each term as a product of the HCF and its other factor.	**b** $t^2 - 7t = t \times t + t \times (-7)$
2 Use the distributive law to factorise the expression.	$= t(t - 7)$
c **1** Identify that the HCF is x^2y and write each term as a product of the HCF and its other factor.	**c** $5x^5y - 13x^2y^3 = x^2y \times 5x^3 + x^2y \times (-13y^2)$
2 Use the distributive law to factorise the expression.	$= x^2y(5x^3 - 13y^2)$
d **1** Identify that the HCF is $3k$ and write each term as a product of the HCF and its other factor.	**d** $3klm + 9km - 6k = 3k \times lm + 3k \times 3m + 3k \times (-2)$
2 Use the distributive law. Write the HCF before a pair of brackets containing the sum of the remaining factors.	$= 3k(lm + 3m - 2)$

Example 2F.2 Factorising by grouping terms

Factorise $a^2 + 3a - 15b - 5ab$ by grouping terms.

THINK	**WRITE**
1 Check for the HCF of all four terms. In this case there isn't one. Pair the terms in groups of two so that each pair has a common factor.	$a^2 + 3a - 15b - 5ab = (a^2 + 3a) + (-15b - 5ab)$
2 Use the distributive law to factorise each pair of terms.	$= a(a + 3) - 5b(a + 3)$
3 Factorise using the common binomial factor, which is $(a + 3)$. Put the common binomial factor in front of a new pair of brackets containing the sum of the other factors.	$= (a + 3)(a - 5b)$

Example 2F.3 Factorising using the 'difference of two squares' rule

Factorise each of these expressions.

a $16h^2 - 49$

b $12a^4 - 27b^2$

c $(r + 11)^2 - 1$

THINK	**WRITE**
a **1** Check for the HCF of all the terms. In this case there isn't one. Write each term as a square in index form.	**a** $16h^2 - 49 = (4h)^2 - 7^2$
2 Factorise using the 'difference of two squares' rule.	$= (4h + 7)(4h - 7)$
b **1** Check for the HCF of all the terms. In this case, 3 is the HCF. Using the distributive law, factorise using the HCF.	**b** $12a^4 - 27b^2 = 3(4a^4 - 9b^2)$
2 Write each term in the binomial factor as a square in index form.	$= 3([2a^2]^2 - [3b]^2)$
3 Factorise the binomial factor using the 'difference of two squares' rule.	$= 3(2a^2 + 3b)(2a^2 - 3b)$
c **1** Check for the HCF of all the terms. In this case, there isn't one. Write each term as a square in index form.	**c** $(r + 11)^2 - 1 = (r + 11)^2 - 1^2$
2 Factorise using the 'difference of two squares' rule.	$= (r + 11 + 1)(r + 11 - 1)$
3 Simplify each of the bracketed expressions.	$= (r + 12)(r + 10)$

Example 2F.4 Factorising using the 'difference of two squares' rule and surds

Factorise each of these expressions.

a $25x^2 - 7$ **b** $(y + 5)^2 - 12$

THINK	WRITE
a **1** Check for the HCF of the two terms. In this case there isn't one. Write each term as a square in index form.	**a** $25x^2 - 7 = (5x)^2 - (\sqrt{7})^2$
2 Factorise using the 'difference of two squares' rule.	$= (5x + \sqrt{7})(5x - \sqrt{7})$
b **1** Check for the HCF of the two terms. In this case there isn't one. Write each term as a square in index form.	**b** $(y + 5)^2 - 12 = (y + 5)^2 - (\sqrt{12})^2$
2 Simplify the surd by factorising 12 so it is a product of 3 and 4, where 4 is a perfect square.	$= (y + 5)^2 - (\sqrt{4 \times 3})^2$ $= (y + 5)^2 - (\sqrt{4} \times \sqrt{3})^2$ $= (y + 5)^2 - (2\sqrt{3})^2$
3 Factorise using the 'difference of two squares' rule.	$= (y + 5 + 2\sqrt{3})(y + 5 - 2\sqrt{3})$

Example 2F.5 Factorising using the 'perfect square' rule

Factorise:

a $t^2 - 18t + 81$ **b** $16x^2 - 56x + 49$

THINK	WRITE
a **1** Check for the HCF of all three terms. In this case there isn't one. Identify that the square root of t^2 is t, the negative square root of 81 is –9, and $-18t$ is twice the product of both these roots: $2 \times (-9) \times t$.	**a** $t^2 - 18t + 81 = t^2 + 2(-9)t + (-9)^2$
2 Factorise using the 'perfect square' rule.	$= (t - 9)^2$
b **1** Check for the HCF of all three terms. In this case there isn't one. Identify that the square root of $16x^2$ is $4x$, the negative square root of 49 is –7, and $-56x$ is twice the product of both these roots: $2 \times (-7) \times (4x)$.	**b** $16x^2 - 56x + 49 = (4x)^2 + 2 \times (-7) \times (4x) + (-7)^2$
2 Factorise using the 'perfect square' rule.	$= (4x - 7)^2$

Helpful hints

✔ It's important that you are able to recognise when an expression can be factorised by grouping terms, or by identifying it as the difference of two squares, or as a perfect square.

✔ Factorising an expression is the reverse process of expanding an expression. You can check if your factorised solution is correct by expanding it to see if you get the same expression that you started with.

$$\xrightarrow{\text{expanding}}$$
$$7(a + 2) = 7a + 14$$
$$\xleftarrow[\text{factorising}]{}$$

✔ Always check to see if the terms of an expression have any common factors. It is always best to start with common factors before you move on with any other factorising methods.

For example, it is easier to identify that $3x^2 + 18x + 27$ can be solved using the 'perfect square' rule once you take out 3 as a common factor:

$$3(x^2 + 6x + 9) = 3(x + 3)^2$$

✔ Remember that the difference of two squares requires one term to be positive, and the other to be negative. For example, the following are NOT the difference of two squares:

$$x^2 + y^2 \quad \text{and} \quad -a^2 - 4$$

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Exercise 2F Factorising

△ 1–2(1st, 2nd columns), 3, 4(a, e, g), 5, 6(a, d, g, j), 7(a–f), 8, 9(1st, 2nd columns), 10(a, d, g), 11, 12, 13(a, e, g, h, j), 14, 15(a, c)

■ 1–2(4th column), 4(b, e, h, i), 6–7(2nd column), 9(2nd column), 10, 13(3rd, 4th columns), 15(d, g, i), 17, 18

⬟ 2(4th column), 4(g–i), 6–7(3rd column), 9(j–r), 10, 13(4th column), 15(i–l), 16, 17, 19–21

UNDERSTANDING AND FLUENCY

2F.1 **1** Factorise each of these expressions.

a $5a + 35$ **b** $30b + 6$ **c** $24 + 33c$ **d** $12d + 8$

e $8e - 18$ **f** $35f - 21$ **g** $24 + 16g$ **h** $10 - 40h$

i $9j + 9k$ **j** $16w - 28x$ **k** $m^2 + 2m$ **l** $n^2 - 6n$

m $5a + a^2$ **n** $9p - p^2$ **o** $3q^2 + 3q$ **p** $2r^2 - 8r$

2 Factorise each of these expressions.

a $3x - 12$ **b** $6y - 15$ **c** $8x^2 - 16x$ **d** $4k^2 + 6k$

e $5c + 11c^2$ **f** $8u - 4u^2$ **g** $6m^2 - 3m$ **h** $y^2 + 5y^3$

i $14t - 8t^2$ **j** $9p^3 + 3p^2$ **k** $4ab + 3a$ **l** $5t^2u - 3tu$

m $4m^2n + 2mn^2$ **n** $6x^2y + 3xy$ **o** $15c^2 + 21cd$ **p** $18a^2b^3c - 12a^4bc^2$

3 Complete the following factorisations so each expression is factorised using a negative HCF.

a $-2ab - 4a^2 = (-2a) \times \_\_ + (-2a) \times 2a$

$= -2a(\_\_ + 2a)$

b $-10x^3 + 35x = \_\_ \times 2x^2 + \_\_ \times (-7)$

$= \_\_(2x^2 - 7)$

4 Factorise each of these expressions using a negative HCF.

a $-6mn - 18$ **b** $-3ab - 3bc$ **c** $-y^2 - 10y$

d $-18x^2 - 9x$ **e** $-4k + 8k^2$ **f** $-20a^3 - 30a$

g $-12x^2y + 14xy$ **h** $-w^5 + w^7$ **i** $-4b^3 + 2b^2 - 6b^4$

5 **a** Using the binomial factor $(x - 5)$ as the HCF, show that $y(x - 5) + 2(x - 5)$ is written in factorised form as $(x - 5)(y + 2)$.

b In the same way, show that $4a(3 + k^2) - 9(3 + k^2) = (3 + k^2)(4a - 9)$.

UNDERSTANDING AND FLUENCY

6 Factorise each of these expressions using a HCF that is a binomial factor.

a $a(x + 3) + 5(x + 3)$ **b** $m(k - 2) + 4(k - 2)$ **c** $y(y + 5) + 2(y + 5)$

d $x(x - 1) + 9(x - 1)$ **e** $k(k + 6) - 3(k + 6)$ **f** $p(p - 9) - 6(p - 9)$

g $7(4 - a) + a(4 - a)$ **h** $3n(2n - 5) + 4(2n - 5)$ **i** $x(y^2 + 2) + 8(y^2 + 2)$

j $a^2(a - 4) + 3(a - 4)$ **k** $2x(y^2 + 1) - 5(y^2 + 1)$ **l** $5x(7 - x^3) + 2(7 - x^3)$

2F.2 **7** Factorise each of these expressions by grouping terms.

a $ab + 4b + 3a + 12$ **b** $x^2 - 7x + 2xy - 14y$ **c** $mn + 5m - 6n - 30$

d $a^2 + 2a + 5a + 10$ **e** $x^2 - x + 4x - 4$ **f** $c^2 + 8c - 3c - 24$

g $6y - 3y^2 + 8 - 4y$ **h** $p^3 + 2p^2 + 5p + 10$ **i** $x^4 + x^2 + 3x^2 + 3$

j $2m^3 + 4m - 3m^2 - 6$ **k** $4 + 4k^2 + k^5 + k^3$ **l** $xy - 15 + 5y - 3x$

8 **a** Expand $(x + 5)(x - 5)$ using the 'difference of two squares' rule.

b Explain how you can factorise $x^2 - 25$.

2F.3 **9** Factorise each of these expressions.

a $x^2 - 4$ **b** $a^2 - 36$ **c** $100 - y^2$

d $64b^2 - 9$ **e** $25 - 49p^2$ **f** $4a^2 - 81b^2$

g $3m^2 - 3$ **h** $8k^2 - 18$ **i** $(xy)^2 - 16w^2$

j $m^6 - n^2$ **k** $4a^{10} - 36b^6$ **l** $(x^2y)^2 - w^4$

m $(h + 3)^2 - 25$ **n** $(c - 4)^2 - 9$ **o** $(2 - x)^2 - x^2$

p $1 - (a + b)^2$ **q** $4 - (y - 5)^2$ **r** $(x + 3)^2 - (x - 6)^2$

2F.4 **10** Factorise each of these expressions. (Hint: Where needed, use a surd to write a term as a square.)

a $x^2 - 7$ **b** $a^2 - 13$ **c** $19 - y^2$

d $16k^2 - 5$ **e** $(p + 8)^2 - 2$ **f** $(m - 4)^2 - 6$

g $(x + 7)^2 - 3$ **h** $(y - 1)^2 - 19$ **i** $(d - 17)^2 - 17$

11 **a** Expand $(x + 5)^2$ using the 'perfect square' rule.

b Explain how you can factorise $x^2 + 10x + 25$.

12 **a** Expand $(y - 8)^2$ using the 'perfect square' rule.

b Explain how you can factorise $y^2 - 16y + 64$.

2F.5 **13** Factorise each of these expressions using what you know from the 'perfect square' rule.

a $x^2 + 12x + 36$ **b** $m^2 + 8m + 16$ **c** $p^2 + 20p + 100$ **d** $y^2 - 6y + 9$

e $x^2 - 14x + 49$ **f** $g^2 - 4g + 4$ **g** $1 + 2a + a^2$ **h** $25 - 10x + x^2$

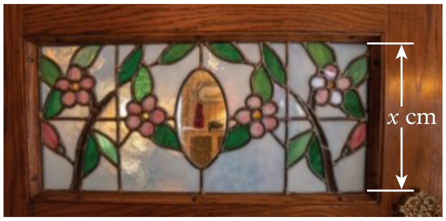
i $81 - 18b + b^2$ **j** $w^2 + 2wx + x^2$ **k** $k^2 - 2km + m^2$ **l** $9x^2 + 12x + 4$

PROBLEM SOLVING AND REASONING

14 **a** Write an expression for the missing side length of each rectangular item below.

i The area of the welcome mat is $(x^2 - 100)$ cm^2.

ii The area of the stained glass panel is $(2x^2 + 3x)$ cm^2.



b Write an expression in factorised form for the perimeter of each of the items above.

PROBLEM SOLVING AND REASONING

15 Simplify each of these algebraic fractions by factorising the numerator and denominator when possible.

a $\dfrac{a^2 + 2a}{ab + 2b}$ **b** $\dfrac{xy + 2x}{3y^2 + 6y}$ **c** $\dfrac{abc^2 - 4ab^2}{ab}$ **d** $\dfrac{2mn + m^2n^2}{mn + 2}$

e $\dfrac{x^2 - y^2}{2x + 2y}$ **f** $\dfrac{3x + 15}{x^2 + 5x}$ **g** $\dfrac{9x^2 - 15x}{12x - 20}$ **h** $\dfrac{m^2 - 49}{5m^2 - 35m}$

i $\dfrac{a^2 + 2ab + b^2}{a^2 - b^2}$ **j** $\dfrac{x^2 + 6x + 9}{x^2 - 9}$ **k** $\dfrac{16a^2 - 25}{8a^2 - 10a}$ **l** $\dfrac{4x^2 + 28x + 49}{49 - 4x^2}$

16 Simplify each of the following.

a $\dfrac{3x + 21}{2x^2 - 10x} \times \dfrac{4x^2 - 20x}{x^2 - 49}$ **b** $\dfrac{x^2 - 1}{10x^2 + 30x} \times \dfrac{15x + 45}{x^2 + 2x + 1} \times \dfrac{2x^2 + 2x}{3x - 3}$

17 Consider an odd number represented by the pronumeral n.

a Write expressions for the next two odd numbers.

b Write an expression for the sum of n and the next two odd numbers.

c Factorise your expression from part **b**. Explain how this relates to one of the three odd numbers.

d Use the shortcut you found in part **c** to find the sum of 423, 425 and 427 without adding.

e Investigate a shortcut for adding five consecutive odd numbers. Show your working.

f Use the shortcut you found in part **e** to find the sum of 2671, 2673, 2675, 2677 and 2679 without adding.

CHALLENGE

18 In question **17**, you found expressions for the sum of three consecutive odd numbers and five consecutive odd numbers.

a Investigate shortcuts for finding the sum of an even number of consecutive odd terms. Start with two consecutive odd numbers, then investigate for four consecutive odd numbers, then six consecutive odd numbers, where the first term is n each time.

b Without adding, find the sum of the numbers listed below.

73, 75, 77, 79, 81, 83, 85, 87, 89, 91

19 The difference between two numbers is 7 and the difference between the squares of the two numbers is 105. What is the sum of the two numbers?

20 Factorising using the 'difference of two squares' method can be useful for simplifying certain multiplications. Evaluate each of the following without a calculator by using the 'difference of two squares' factorisation method.

a 19×21 **b** 28×32 **c** 83×77 **d** 54×46

21 Legend has it that the famous mathematician Friedrich Gauss was once punished for misbehaving in primary school by being told he had to find the sum of all the numbers from 1 to 100 before leaving class. Much to his teacher's surprise, he did so in seconds.

The shortcut Gauss probably used was to start by pairing the highest and lowest numbers together, like so:

$(1 + 100) + (2 + 99) + (3 + 98) + (4 + 97) + \ldots$

a What is the sum of each binomial pair in Gauss's method?

b How many binomial pairs will there be once all of the numbers from 1 to 100 are grouped together?

c Use your answers from parts **b** and **c** to write the sum of all the numbers from 1 to 100 as the product of two numbers. Evaluate this product to find Gauss's solution.

d This method can be generalised to write the sum of all the numbers from 1 to n, for any positive integer n. Write the sum of all the numbers from 1 to n as a factorised expression in terms of n.

Check your Student obook pro for these digital resources and more: pro

Interactive skillsheet
Factorising using the HCF

Interactive skillsheet
Factorising the difference of two squares

CAS instructions
Factorising

Topic quiz
2F

Interactive skillsheet
Factorising by grouping terms

2G Factorising quadratic expressions

Learning intentions

✔ I can factorise quadratic expressions by considering each term.

Inter-year links

Years 5/6	Factors and multiples
Year 7	2D Factors and the highest common factor
Year 8	5G Factorising
Year 9	3E Factorising quadratic expressions

Quadratic trinomials

- A quadratic expression is an algebraic expression that contains a squared pronumeral, with no indices greater than 2 in the expression. The following are all quadratic expressions:

$$6t^2 \qquad x^2 + 5 \qquad k^2 + 14k + 30 \qquad 4b^2 - a^2$$

- A **trinomial** is an algebraic expression that contains three terms.
- A **quadratic trinomial** is an algebraic expression of the form $ax^2 + bx + c$, where a, b and c are non-zero constants.

$$ax^2 + bx + c$$

a is the **leading coefficient**, b is the coefficient of the linear term, and c is the constant term.

- A **monic quadratic** is a quadratic in which the leading coefficient is equal to 1.

$$x^2 + bx + c$$

Factorising monic quadratic trinomials

- Expanding a binomial product of the form $(x + m)(x + n)$ results in a monic quadratic trinomial:

$$\begin{aligned}(x + m)(x + n) &= x^2 + mx + nx + mn\\ &= x^2 + (m + n)x + mn\\ &= x^2 + bx + c \qquad \text{where } b = m + n \text{ and } c = m \times n\end{aligned}$$

- The process can be reversed to factorise a monic quadratic trinomial, $x^2 + bx + c$, by finding two numbers (m and n) that add to give b and multiply to give c.

For example, in the quadratic trinomial $x^2 + 4x + 3$, the linear coefficient is 4 (that is, $b = 4$) and the constant is 3 (that is, $c = 3$). So the values $m = 3$ and $n = 1$ satisfy the conditions $b = m + n$ and $c = m \times n$.

$$\begin{aligned}x^2 + 4x + 3 &= x^2 + (3 + 1)x + (3 \times 1)\\ &= (x + 3)(x + 1)\end{aligned}$$

$$\xleftarrow{\text{expanding}}$$

$$\underset{\substack{\text{quadratic trinomial}\\ \text{(expanded form)}}}{x^2 + 4x + 3} = \underset{\substack{\text{binomial product}\\ \text{(factorised form)}}}{(x + 1)(x + 3)}$$

$$\xrightarrow{\text{factorising}}$$

Example 2G.1 Factorising simple quadratic trinomials

Factorise each of these quadratic trinomials.

a $x^2 + 6x + 5$ **b** $x^2 + 9x + 18$

THINK	**WRITE**
a **1** For $x^2 + 6x + 5$, list the factor pairs of the constant term, 5. Remember that, if the positive/negative sign is changed for both factors, the product will be the same. So there are two combinations of factor pairs with the same numerals.	**a** $1 \times 5 = 5 \rightarrow 1 + 5 = 6$ $-1 \times -5 = 5 \rightarrow -1 - 5 = -6$
2 Add the factor pairs together and identify which pair add to the linear coefficient, 6.	
3 Write the expression in factorised form.	$x^2 + 6x + 5 = (x + 1)(x + 5)$
b **1** For $x^2 + 9x + 18$, list the factor pairs of the constant term, 18.	**b** $1 \times 18 = 18 \rightarrow 1 + 18 = 19$ $-1 \times -18 = 18 \rightarrow -1 - 18 = -19$ $2 \times 9 = 18 \rightarrow 2 + 9 = 11$ $-2 \times -9 = 18 \rightarrow -2 - 9 = -11$ $3 \times 6 = 18 \rightarrow 3 + 6 = 9$ $-3 \times -6 = 18 \rightarrow -3 - 6 = -9$
2 Add the factor pairs together and identify which pair add to the linear coefficient, 9.	
3 Write the expression in factorised form.	$x^2 + 9x + 18 = (x + 3)(x + 6)$

Example 2G.2 Factorising more complex quadratic trinomials

Factorise each of these quadratic trinomials.

a $x^2 + 2x - 3$ **b** $x^2 - 7x - 8$ **c** $x^2 - 5x + 6$

THINK	**WRITE**
1 List the factor pairs of the constant term. Remember that, if the positive/negative sign is changed for both factors, the product will be the same. So there are always two combinations of factor pairs with the same numerals. **2** Add the factor pairs together and identify which pair add to the linear coefficient. **3** Write the expression in factorised form.	**a** $1 \times -3 = -3 \rightarrow 1 - 3 = -2$ $-1 \times 3 = -3 \rightarrow -1 + 3 = +2$ $x^2 + 2x - 3 = (x + 3)(x - 1)$ **b** $1 \times -8 = -8 \rightarrow 1 - 8 = -7$ $-1 \times 8 = -8 \rightarrow -1 + 8 = 7$ $2 \times -4 = -8 \rightarrow 2 - 4 = -2$ $-2 \times 4 = -8 \rightarrow -2 + 4 = 2$ $x^2 - 7x - 8 = (x - 8)(x + 1)$ **c** $1 \times 6 = 6 \rightarrow 1 + 6 = 7$ $-1 \times -6 = 6 \rightarrow -1 - 6 = -7$ $2 \times 3 = 6 \rightarrow 2 + 3 = 5$ $-2 \times -3 = 6 \rightarrow -2 - 3 = -5$ $x^2 - 5x + 6 = (x - 3)(x - 2)$

Example 2G.3 Factorising quadratic trinomials by first taking out a common factor

Factorise these by first taking out a common factor.

a $2x^2 - 14x + 12$ **b** $-x^2 + 4x - 3$

THINK

1 If the coefficient of x^2 is not 1, look for a common factor of the three terms (–1 can be a common factor). Write the HCF before the brackets. Inside the brackets, write the results of dividing each term by the HCF.

2 List the factor pairs of the constant term each time. Remember that, if the positive/negative sign is changed for both factors, the product will be the same. So there are always two combinations of factor pairs with the same numerals.

3 Add the factor pairs together and identify which pair add to the linear coefficient.

4 Write the expression in factorised form.

WRITE

a $2x^2 - 14x + 12 = 2(x^2 - 7x + 6)$

$1 \times 6 = 6 \rightarrow 1 + 6 = 7$

$-1 \times -6 = 6 \rightarrow -1 - 6 = -7$

$2 \times 3 = 6 \rightarrow 2 + 3 = 5$

$-2 \times -3 = 6 \rightarrow -2 - 3 = -5$

$$\begin{aligned} 2x^2 - 14x + 12 &= 2(x^2 - 7x + 6) \\ &= 2(x - 6)(x - 1) \end{aligned}$$

b $-x^2 + 4x - 3 = -(x^2 - 4x + 3)$

$1 \times 3 = 3 \rightarrow 1 + 3 = 4$

$-1 \times -3 = 3 \rightarrow -1 - 3 = -4$

$$\begin{aligned} -x^2 + 4x - 3 &= -(x^2 - 4x + 3) \\ &= -(x - 3)(x - 1) \end{aligned}$$

Helpful hints

✔ You can always check your factorised quadratic by expanding your solution to see if you get the same expression that you started with.

✔ You don't need to list all the factor pairs of the constant term. Just write each new pair as you go and stop when you find the pair that add to give the linear term.

✔ You don't always need to consider all the factor pairs of the constant term, c, to factorise a quadratic of the form $x^2 + bx + c$.

If the constant term, c, is positive, then both factors will have the same sign. So:

→ if b is also positive, both factors will be positive

→ if b is negative, both factors will be negative.

✔ Remember, always look for common factors! You should start by checking to see if the leading coefficient is a common factor of all the terms.

ANS
p658

Exercise 2G Factorising quadratic expressions

▲ 1, 2, 3(1st column), 4, 5, 6(1st column), 7(a, d, g, h, k), 8(a, b, e, f), 9(a, c), 10

■ 3(2nd, 3rd columns), 6–7(2nd, 3rd columns), 8(e–h), 9(b, d), 12, 15

⬟ 3(3rd column), 6–7(2nd, 3rd columns), 8(e–h), 9(b, d), 11, 13–14, 16

1 For each of these pairs of numbers, identify the two numbers that multiply together to give the first number and add together to give the second number.

a 7, 8 **b** 16, 10 **c** 45, 14 **d** 60, 16 **e** 36, 13 **f** 24, 11

UNDERSTANDING AND FLUENCY

2 Use your results from question **1** to factorise each of these quadratic trinomials.

a $x^2 + 8x + 7$ **b** $x^2 + 10x + 16$ **c** $x^2 + 14x + 45$

d $x^2 + 16x + 60$ **e** $x^2 + 13x + 36$ **f** $x^2 + 11x + 24$

2G.1 **3** Factorise each of these quadratic trinomials.

a $x^2 + 3x + 2$ **b** $x^2 + 5x + 6$ **c** $x^2 + 11x + 10$

d $x^2 + 12x + 27$ **e** $x^2 + 7x + 10$ **f** $x^2 + 9x + 14$

g $x^2 + 6x + 9$ **h** $x^2 + 12x + 11$ **i** $x^2 + 11x + 18$

j $x^2 + 12x + 32$ **k** $x^2 + 16x + 63$ **l** $x^2 + 14x + 40$

m $x^2 + 11x + 30$ **n** $x^2 + 13x + 22$ **o** $x^2 + 15x + 36$

4 For each of these pairs of numbers, identify the two numbers that multiply together to give the first number and add together to give the second number.

a –8, 7 **b** 6, –7 **c** –35, –2

d –15, 2 **e** –8, –2 **f** 30, –11

5 Use your results from question **4** to factorise each of these quadratic trinomials.

a $x^2 + 7x - 8$ **b** $x^2 - 7x + 6$ **c** $x^2 - 2x - 35$

d $x^2 + 2x - 15$ **e** $x^2 - 2x - 8$ **f** $x^2 - 11x + 30$

2G.2 **6** Factorise each of these quadratic trinomials.

a $x^2 + 4x - 12$ **b** $x^2 - 8x - 20$ **c** $x^2 - 5x + 6$

d $x^2 + 6x - 27$ **e** $x^2 - 13x + 40$ **f** $x^2 - x - 20$

g $x^2 + 4x - 21$ **h** $x^2 - 14x + 33$ **i** $x^2 - 3x - 28$

j $x^2 + x - 12$ **k** $x^2 - 7x - 18$ **l** $x^2 - 15x + 56$

2G.3 **7** Factorise each of these quadratic trinomial by first taking out a common factor.

a $2x^2 + 18x + 16$ **b** $3x^2 + 15x + 18$ **c** $7x^2 + 14x - 56$

d $-5x^2 - 35x - 60$ **e** $-4x^2 + 28x - 40$ **f** $6 - 5x - x^2$

g $2x^2 + 14x - 36$ **h** $19x^2 - 38x - 57$ **i** $-3x^2 + 33x + 78$

j $6x - 2x^2 - 4$ **k** $\frac{1}{2}x^2 + \frac{3}{2}x - 2$ **l** $\frac{1}{3}x^2 + x - 6$

8 Simplify each of the following algebraic fractions.

a $\frac{x^2 + 6x + 8}{x + 2}$ **b** $\frac{x + 2}{x^2 - 4}$ **c** $\frac{x^2 + 6x}{x^2 + 4x - 12}$

d $\frac{x^2 - 8x + 15}{x^2 - 9}$ **e** $\frac{2x^2 - 50}{x^2 + 7x + 10}$ **f** $\frac{11x^2 - 22x}{11x^2 - 44}$

g $\frac{x^2 + 2x + 1}{1 - x^2}$ **h** $\frac{4x^2 + 8x - 60}{2x^2 + 18x + 40}$

9 Simplify each of these expressions using the most appropriate method.

a $\frac{x^2 + x - 20}{x^2 - 6x + 8} \times \frac{x^2 - 4}{x^2 - 25} \times \frac{8x^2 - 40x}{4x^2 + 8x}$ **b** $\frac{x}{x + 7} \times \frac{x^2 - 49}{3x} \div \frac{x^2 - 5x - 14}{x^2 - 4x - 12}$

c $\frac{3x^2 + 3x}{x - 3} \times \frac{x^2 - 7x + 12}{6x^2 + 6x} + \frac{x + 5}{3}$ **d** $\frac{x^2 + 9x + 18}{2x + 6} \div \frac{3x + 18}{5x - 1} + \frac{3x - 5}{8}$

10 Choose the most appropriate method to factorise each of these expressions. Then factorise them.

a $x^2 + 15x + 44$ **b** $x^2 - 2x - 15$ **c** $x^2 + 4x + 4$

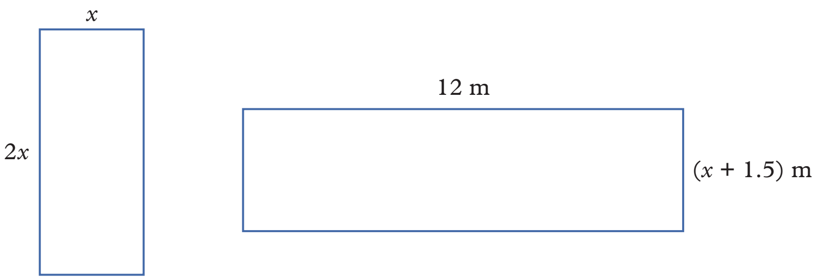
d $3x^2 - 18x + 24$ **e** $5x^2 - 45$ **f** $2x^2 - 8x$

11 Explain why each of these expressions cannot be factorised using the methods covered so far.

a $x^2 + 9$ **b** $(x - 3)^2 + 16$ **c** $(x + 1)^2 + 4$

d $x^2 - 4x + 5$ **e** $(x + 2)^2 - 4x$ **f** $x^2 + 12x + 40$

12 Consider the following two rectangles:

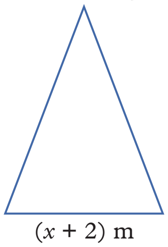


a Write an expression for the area of each rectangle in terms of x.

b Write an expression for the total area of the two rectangles, in terms of x.

c The total area of the two rectangles can be combined and reshaped to form two equally sized squares. What is the side length of these squares, in terms of x?

13 The triangle below has a base of $(x + 2)$ metres and an area of $\left(\frac{1}{2}x^2 + 4x + 6\right)$ square metres. Find an expression, in terms of x, for the height of the triangle.



14 It is possible for an expression to be a quadratic with two variables.

a Expand each of the following:

i $(x + y)(x + y)$ **ii** $(x + 2y)(x + 3y)$

b Factorise each of the following:

i $x^2 + 10x + 16$ **ii** $x^2 + 10xy + 16y^2$

c Factorise each of the following:

i $x^2 + 7x + 10$ **ii** $x^2 + 7xy + 10y^2$

d Factorise each of the following:

i $x^2 - 6xy + 8y^2$ **ii** $y^2 - 9xy + 14x^2$ **iii** $4x^2 + 5xy + y^2$

15 If possible, factorise each of the following quadratic expressions, using a method already covered in this chapter.

a $x^2 + 6x$ **b** $x^2 + 6x + 1$ **c** $x^2 + 6x + 2$ **d** $x^2 + 6x + 3$

e $x^2 + 6x + 4$ **f** $x^2 + 6x + 5$ **g** $x^2 + 6x + 6$ **h** $x^2 + 6x + 7$

i $x^2 + 6x + 8$ **j** $x^2 + 6x + 9$

16 What values can k have for each quadratic trinomial below to be able to be factorised using the methods already covered in this chapter?

a $x^2 - 6x + k$ **b** $x^2 + 20x + k$

PROBLEM SOLVING AND REASONING

CHALLENGE

Check your Student obook pro for these digital resources and more: pro

Interactive skillsheet
Factorising quadratic expressions

Topic quiz
2G

2H Completing the square

Learning intentions

- ✔ I can add a term to a quadratic binomial to make a perfect square.
- ✔ I can factorise quadratic expressions by completing the square.

Inter-year links

Year 7	1G Indices and square roots
Year 8	5G Factorising
Year 9	3D Factorising the difference of perfect squares

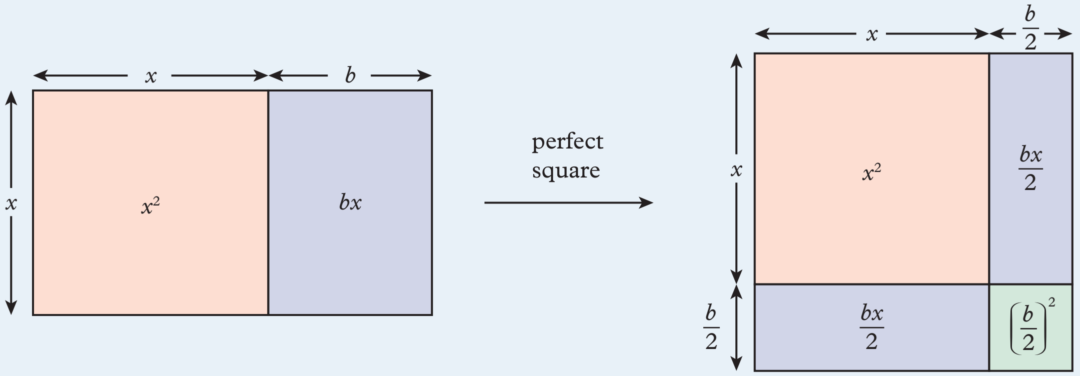
Adding a term to make a perfect square

- For any quadratic of the form $x^2 + bx$, a constant term of the form $\left(\frac{b}{2}\right)^2$ can be added to make it a perfect square.

$$x^2 + 2x\left(\frac{b}{2}\right) + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

‘perfect square’ rule

$$a^2 + 2ab + b^2 = (a + b)^2$$

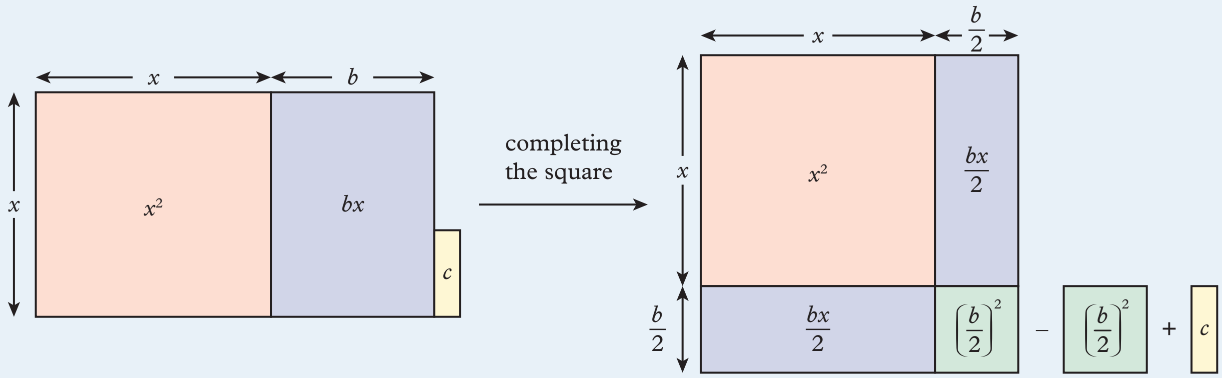


For example, consider the expression $x^2 + 6x$. The coefficient of the linear term is 6, so we can add $\left(\frac{6}{2}\right)^2 = 9$ to the binomial to make it a perfect square:

$$\begin{aligned} x^2 + 6x + 9 &= x^2 + 2(x)(3) + 3^2 \\ &= (x + 3)^2 \end{aligned}$$

Completing the square

- **Completing the square** is the process of changing a quadratic expression from the general form, $a + bx + c$, to the form $a(x - h)^2 - k$ by applying the ‘perfect square’ rule.
- To complete the square for a monic quadratic, $x^2 + bx + c$, the term $\left(\frac{b}{2}\right)^2$ must be *added and subtracted* from the expression to change it to the form $(x - h)^2 - k$.



- Just as zero can be added to any number without changing its value, an expression equivalent to zero can be added to an expression without changing its value. In this case the expression is:

$$+\left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 = 0$$

$$\begin{aligned} x^2 + bx + c &= x^2 + bx + 0 + c \\ &= x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \\ &= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \\ &= (x - h)^2 - k \end{aligned}$$

when $h = -\frac{b}{2}$ and $k = \left(\frac{b}{2}\right)^2 - c$

For example, to complete the square for $x^2 + 8x + 1$, first find the constant term needed to make this a perfect square: $\left(\frac{b}{2}\right)^2 = \left(\frac{8}{2}\right)^2 = 16$

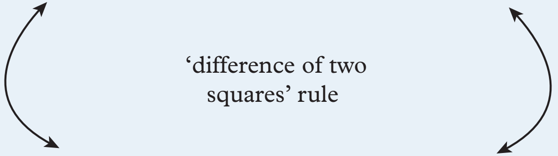
We then add and subtract this value to complete the square:

$$\begin{aligned} x^2 + 8x + 1 &= x^2 + 8x + 0 + 1 \\ &= x^2 + 8x + 16 - 16 + 1 \\ &= (x^2 + 8x + 16) - 16 + 1 \\ &= (x + 4)^2 - 15 \end{aligned}$$

Factorising using the 'difference of two squares' rule

- Once the quadratic is of the form $(x - h)^2 - k$, where $k = \left(\frac{b}{2}\right)^2 - c$ is a positive constant, it can be factorised to be a binomial product using the 'difference of two squares' rule.

$$(x - h)^2 - k = \left(x - h + \sqrt{k}\right)\left(x - h - \sqrt{k}\right)$$



$$a^2 - b^2 = (a + b)(a - b)$$

For example, to factorise the expression $x^2 + 8x + 1$, from the previous example:

$$\begin{aligned} x^2 + 8x + 1 &= (x + 4)^2 - 15 \\ &= (x + 4 + \sqrt{15})(x + 4 - \sqrt{15}) \end{aligned}$$

Example 2H.1 Adding a term to make a perfect square

Add a term to each of these expressions to make a new expression that is a perfect square in expanded form.

a $x^2 + 10x$ **b** $x^2 - 8x$

THINK

a **1** For $x^2 + 10x$, substitute the coefficient of the linear term, 10, into the expression $\left(\frac{b}{2}\right)^2$ to find the correct term.

2 Add the term to make a perfect square.

b **1** For $x^2 - 8x$, substitute the coefficient of the linear term, −8, into the expression $\left(\frac{b}{2}\right)^2$ to find the correct term.

2 Add the term to make a perfect square.

WRITE

a
$$\begin{aligned}\left(\frac{b}{2}\right)^2 &= \left(\frac{10}{2}\right)^2 \\ &= 5^2 \\ &= 25\end{aligned}$$

$x^2 + 10x + 25$

b
$$\begin{aligned}\left(\frac{b}{2}\right)^2 &= \left(\frac{-8}{2}\right)^2 \\ &= (-4)^2 \\ &= 16\end{aligned}$$

$x^2 - 8x + 16$

Example 2H.2 Factorising by completing the square

Factorise each of these quadratic trinomials by completing the square.

a $x^2 + 12x + 15$ **b** $x^2 - 2x - 7$ **c** $x^2 + x - 3$

THINK

1 Substitute the coefficient of the linear term into the expression $\left(\frac{b}{2}\right)^2$ each time to find the term to make a perfect square.

2 Add and subtract the term from the quadratic so that the overall expression remains unchanged.

3 Factorise the perfect square expression using the 'perfect square' rule and simplify the constant terms at the end of the expression.

4 Write as the difference of two squares. Simplify the square root if possible.

5 Factorise using the 'difference of two squares' rule.

WRITE

a The coefficient of the linear term in $x^2 + 12x + 15$ is 12.

$$\begin{aligned}\left(\frac{12}{2}\right)^2 &= 6^2 \\ &= 36\end{aligned}$$

$$\begin{aligned}x^2 + 12x + 15 &= x^2 + 12x + 36 - 36 + 15 \\ &= (x^2 + 12x + 36) - 36 + 15 \\ &= (x^2 + 2[6]x + 6^2) - 21 \\ &= (x + 6)^2 - 21 \\ &= (x + 6)^2 - (\sqrt{21})^2 \\ &= (x + 6 + \sqrt{21})(x + 6 - \sqrt{21})\end{aligned}$$

b The coefficient of the linear term in $x^2 - 2x - 7$ is -2.

$$\begin{aligned}\left(\frac{-2}{2}\right)^2 &= (-1)^2\\ &= 1\end{aligned}$$

$$\begin{aligned}x^2 - 2x - 7 &= x^2 - 2x + 1 - 1 - 7\\ &= (x^2 - 2x + 1) - 1 - 7\\ &= (x^2 + 2[-1]x + [-1]^2) - 8\\ &= (x - 1)^2 - 8\\ &= (x - 1)^2 - (\sqrt{8})^2\\ &= (x - 1)^2 - (2\sqrt{2})^2\\ &= (x - 1 + 2\sqrt{2})(x - 1 - 2\sqrt{2})\end{aligned}$$

c The coefficient of the linear term in $x^2 + x - 3$ is 1.

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$\begin{aligned}x^2 + x - 3 &= x^2 + x + \frac{1}{4} - \frac{1}{4} - 3\\ &= (x^2 + x + \tfrac{1}{4}) - \frac{1}{4} - 3\\ &= \left(x^2 + 2\left[\frac{1}{2}\right]x + \left[\frac{1}{2}\right]^2\right) - \frac{1}{4} - \frac{12}{4}\\ &= \left(x + \frac{1}{2}\right)^2 - \frac{13}{4}\\ &= \left(x + \frac{1}{2}\right)^2 - \left(\frac{\sqrt{13}}{2}\right)^2\\ &= \left(x + \frac{1 + \sqrt{13}}{2}\right)\left(x + \frac{1 - \sqrt{13}}{2}\right)\end{aligned}$$

Helpful hints

✔ Remember to always simplify surds by factorising any square numbers.

✔ Remember that you must always add and also subtract the $\left(\frac{b}{2}\right)^2$ term for the expression to remain unchanged.

ANS
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Exercise 2H Completing the square

△ 1–5, 6(1st column), 7, 8, 11

■ 3–5, 6(2nd column), 7, 9, 10, 12, 13(a, b, e)

⬟ 3, 4, 6(2nd column), 7, 10, 12, 14–16

1 **a** **i** Expand $(x + 3)^2$ using the 'perfect square' rule.

ii Use your answer for part **i** to factorise $x^2 + 6x + 9$.

b **i** Expand $(x - 9)^2$ using the 'perfect square' rule.

ii Use your answer for part **i** to factorise $x^2 - 18x + 81$.

c Factorise each of these quadratic trinomials.

i $x^2 + 8x + 16$
ii $x^2 + 4x + 4$
iii $x^2 - 12x + 36$
iv $x^2 + 16x + 64$
v $x^2 - 10x + 25$
vi $x^2 - 20x + 100$

UNDERSTANDING AND FLUENCY

2 Copy and complete each of these expressions to make a perfect square.

a $x^2 + 6x + \_\_ = (x + 3)^2$ **b** $x^2 + 10x + \_\_ = (x + 5)^2$

c $x^2 - 8x + \_\_ = (x - 4)^2$ **d** $x^2 - 4x + \_\_ = (x - 2)^2$

e $x^2 + 2x + \_\_ = (x + 1)^2$ **f** $x^2 - 16x + \_\_ = (x - 8)^2$

2H.1 **3** Add a term to each of these expressions to make a new perfect square in expanded form.

a $x^2 + 14x$ **b** $x^2 - 6x$

c $x^2 + 18x$ **d** $x^2 + 20x$

e $x^2 - 12x$ **f** $x^2 + 22x$

4 Factorise each perfect square expression you wrote for question **3**.

5 If $x^2 + 6x + 7$ is written as $(x^2 + 6x + 9) - 2$, explain how knowledge of the 'perfect square' rule and the 'difference of two squares' rule can be used to factorise it.

2H.2 **6** Factorise each of these quadratic trinomials by completing the square.

a $x^2 + 8x + 9$ **b** $x^2 - 10x + 14$

c $x^2 + 4x - 6$ **d** $x^2 + 2x - 5$

e $x^2 - 6x - 4$ **f** $x^2 - 12x + 13$

g $x^2 + 10x - 4$ **h** $x^2 - 14x + 46$

i $x^2 + 8x + 7$ **j** $x^2 + 3x + 1$

k $x^2 - 5x + 3$ **l** $x^2 + 11x + 9$

7 Factorise each of these quadratic trinomials by first taking out a common factor.

a $2x^2 + 8x + 2$ **b** $-x^2 - 6x + 2$

c $-5x^2 + 10x + 5$ **d** $3x^2 - 15x + 6$

PROBLEM SOLVING AND REASONING

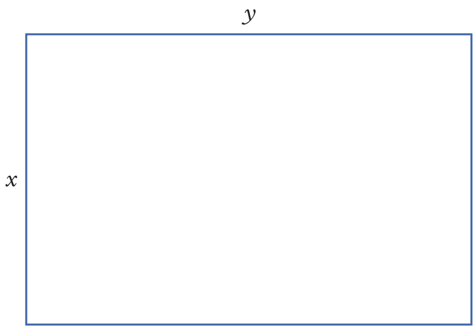
8 To create a square from the rectangle on the right, other rectangles can be cut out or attached.

a What are the dimensions of the rectangle needing to be cut out to turn the rectangle into a square?

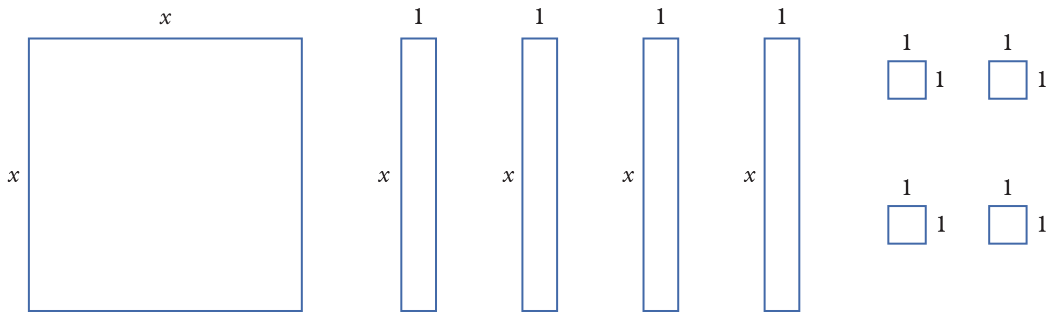
b If a rectangle is cut out to turn the rectangle into a square, write an expression for the area of the resulting square.

c What are the dimensions of the rectangle needing to be attached to turn the rectangle into a square?

d If a rectangle is attached to turn the rectangle into a square, write an expression for the area of the resulting square.



9 **a** Show how the following shapes can be arranged to form a square.



b Express the area of the square you arranged in part **a** as an expanded quadratic.

c Express the area of the square you arranged in part **a** as a factorised quadratic.

10 In the challenge section of Exercise **2G**, you were asked to factorise as many of the following quadratic expressions as possible using the methods that had been covered. Now that you know how to complete the square, factorise each of the following quadratics using the most appropriate method.

a $x^2 + 6x$ **b** $x^2 + 6x + 1$ **c** $x^2 + 6x + 2$ **d** $x^2 + 6x + 3$

e $x^2 + 6x + 4$ **f** $x^2 + 6x + 5$ **g** $x^2 + 6x + 6$ **h** $x^2 + 6x + 7$

i $x^2 + 6x + 8$ **j** $x^2 + 6x + 9$

11 a Factorise $x^2 + 7x + 6$ by completing the square.

b Look at your answer to part **a**. Is there an easier way of obtaining it? Explain.

12 Not all quadratic trinomials can be factorised.

If a quadratic cannot be factorised by completing the square, then it cannot be factorised at all. For example, a quadratic like $x^2 + 2x + 5$ becomes $(x + 1)^2 + 3$ after completing the square, and this expression cannot be factorised because it cannot be expressed as the difference of two squares. Complete the square for each of the following expressions to find the specified values.

a Find the values of c for which $x^2 + 8x + c$ cannot be factorised.

b Find the values of c for which $x^2 - 10x + c$ cannot be factorised.

c Find the values of b for which $x^2 + bx + 1$ cannot be factorised.

13 Explain why each of these expressions cannot be factorised using the methods covered in this chapter so far.

a $x^2 + 6x + 10$ **b** $x^2 - 8x + 21$ **c** $3x^2 + 7x + 2$

d $12x^2 + 64x + 45$ **e** $2x^2 + 4x + 4$

CHALLENGE

14 Factorise each of the following expressions, if possible.

a $2x^2 + 6x - 56$ **b** $x^2 - 3x + 5$ **c** $x^3 + 3x^2 - 18x$

d $12 + x - x^2$ **e** $98 - 2x^2$ **f** $4x^2 + 25$

15 Non-monic quadratics can also be factorised by completing the square. For example:

$$\begin{aligned} 2x^2 + 4x - 9 &= 2\left(x^2 + 2x - \frac{9}{2}\right) \\ &= 2\left[(x^2 + 2x + 1) - 1 - \frac{9}{2}\right] \\ &= 2\left[\left(x + 1\right)^2 - \frac{11}{2}\right] \\ &= 2\left(x + 1 - \sqrt{\frac{11}{2}}\right)\left(x + 1 + \sqrt{\frac{11}{2}}\right) \\ &= 2\left(x + 1 - \frac{\sqrt{22}}{2}\right)\left(x + 1 + \frac{\sqrt{22}}{2}\right) \end{aligned}$$

Factorise each of the following quadratics by completing the square:

a $2x^2 + 12x + 5$ **b** $2x^2 + 7x + 1$ **c** $3x^2 + 11x - 1$

16 a Complete the square to factorise the general monic quadratic trinomial $x^2 + bx + c$.

b Considering the fact that you cannot take the square root of a negative number, write an algebraic fraction in terms of b and c that must be *greater than or equal to zero* for $x^2 + bx + c$ to be factorised.

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Interactive skillsheet
Factorising by completing the square

CAS instructions
Completing the square

Topic quiz
2H

2I Factorising non-monic quadratic expressions

Learning intentions

- ✔ I can factorise non-monic quadratic expressions by splitting the middle term.
- ✔ I can factorise non-monic quadratic expressions by completing the square.

Inter-year links

Year 8	5G Factorising
Year 9	3E Factorising quadratic expressions

Non-monic quadratic trinomials

- A **non-monic quadratic** trinomial is a quadratic trinomial in which the leading coefficient is not equal to one. These trinomials are often more difficult to factorise.

$$ax^2 + bx + c \quad \text{when } a \neq 1$$

- When a non-monic quadratic trinomial $ax^2 + bx + c$ can be factorised, it has the general form $(mx + n)(px + q)$ where m, n, p and q are non-zero constants.

$$ax^2 + bx + c = (mx + n)(px + q)$$

Factorising by splitting the middle term

- Expanding the general form of the factorised non-monic quadratic trinomial gives:

$$\begin{aligned}(mx + n)(px + q) &= mpx^2 + mqx + npx + nq\\ &= mpx^2 + (mq + np)x + nq\\ &= ax^2 + bx + c\end{aligned}$$

So: $a = m \times p$ $\quad b = mq + np$ $\quad c = n \times q$

This means there are two numbers, mq and np, which add to give b and multiply to give $a \times c$.

$$b = mq + np \quad \text{and} \quad a \times c = (mq)(np)$$

- If the middle term, bx, of a quadratic trinomial, $ax^2 + bx + c$, is split into the sum of two terms whose coefficients multiply to give $a \times c$, the quadratic can be factorised by grouping terms. This method of factorisation is called splitting the middle term.

For example, $3x^2 + 10x + 8$ is in the form $ax^2 + bx + c$. We have $a = 3$, $b = 10$ and $c = 8$. To split the middle term $10x$ we must find two numbers that add to give b (which is 10) and multiply to give $a \times c$ (which is $3 \times 8 = 24$).

The numbers 4 and 6 satisfy these conditions:

$$4 + 6 = 10 \quad \text{and} \quad 4 \times 6 = 24$$

So:

$$\begin{aligned}3x^2 + 10x + 8 &= 3x^2 + 4x + 6x + 8\\ &= x(3x + 4) + 2(3x + 4)\\ &= (3x + 4)(x + 2)\end{aligned}$$

Factorising by completing the square

- A non-monic quadratic trinomial can also be factorised by completing the square. This is done by taking out the leading coefficient as a factor.

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a} + \frac{c}{a}\right)$$

Then we can complete the square using the normal method.

Example 2I.1 Factorising by splitting the middle term

Factorise each of these quadratic trinomials by first splitting the middle term.

a $3x^2 + 8x + 4$ **b** $6x^2 + 7x - 3$ **c** $10x^2 - x - 2$

THINK			**WRITE**
a	**1**	Identify the values of $a \times c$ and b in the general form of the trinomial, $ax^2 + bx + c$.	**a** $3x^2 + 8x + 4$ $b = 8$ and $a \times c = 3 \times 4 = 12$
	2	List the factor pairs of the product $a \times c = 12$. Remember to consider both positive and negative factors.	$1 \times 12 = 12 \rightarrow 1 + 12 = 13$ $-1 \times -12 = 12 \rightarrow -1 - 12 = -13$ $2 \times 6 = 12 \rightarrow 2 + 6 = 8$ $-2 \times -6 = 12 \rightarrow -2 - 6 = -8$ $3 \times 4 = 12 \rightarrow 3 + 4 = 7$ $-3 \times -4 = 12 \rightarrow -3 - 4 = -7$
	3	Add the factor pairs together and identify which pair add to the linear coefficient ($b = 8$).	
	4	Split the middle term of the trinomial into two terms using the pair of numbers from the previous step as the coefficients.	$3x^2 + 8x + 4 = 3x^2 + 2x + 6x + 4$
	5	Factorise by grouping terms.	$= x(3x + 2) + 2(3x + 2)$ $= (3x + 2)(x + 2)$
b	**1**	Identify the values of $a \times c$ and b in the general form of the trinomial, $ax^2 + bx + c$.	**b** $6x^2 + 7x - 3$ $b = 7$ and $a \times c = 6 \times (-3) = -18$
	2	List the factor pairs of the product $a \times c = -18$. Remember to consider both positive and negative factors.	$1 \times -18 = -18 \rightarrow 1 + -18 = -17$ $-1 \times 18 = -18 \rightarrow -1 + 18 = 17$ $2 \times -9 = -18 \rightarrow 2 - 9 = -7$ $-2 \times 9 = -18 \rightarrow -2 + 9 = 7$ $3 \times -6 = -18 \rightarrow 3 - 6 = -3$ $-3 \times 6 = -18 \rightarrow -3 + 6 = 3$
	3	Add the factor pairs together and identify which pair add to the linear coefficient ($b = 7$).	
	4	Split the middle term of the trinomial into two terms using the pair of numbers from the previous step as the coefficients.	$6x^2 + 7x - 3 = 6x^2 - 2x + 9x - 3$
	5	Factorise by grouping terms.	$= 2x(3x - 1) + 3(3x - 1)$ $= (3x - 1)(2x + 3)$
c	**1**	Identify the values of $a \times c$ and b in the general form of the trinomial, $ax^2 + bx + c$.	**c** $10x^2 - x - 2$ $b = -1$ and $a \times c = 10 \times (-2) = -20$
	2	List the factor pairs of the product $a \times c = -20$. Remember to consider both positive and negative factors.	$1 \times -20 = -20 \rightarrow 1 - 20 = -19$ $-1 \times 20 = -20 \rightarrow -1 + 20 = 19$ $2 \times -10 = -20 \rightarrow 2 - 10 = -8$ $-2 \times 10 = -20 \rightarrow -2 + 10 = 8$ $4 \times -5 = -20 \rightarrow 4 - 5 = -1$ $-4 \times 5 = -20 \rightarrow -4 + 5 = 1$
	3	Add the factor pairs together and identify which pair add to the linear coefficient ($b = -1$).	
	4	Split the middle term of the trinomial into two terms using the pair of numbers from the previous step as the coefficients.	$10x^2 - x - 2 = 10x^2 + 4x - 5x - 2$
	5	Factorise by grouping terms.	$= 2x(5x + 2) - 1(5x + 2)$ $= (5x + 2)(2x - 1)$

Example 21.2 Factorising by completing the square

Factorise $5x^2 - 10x + 1$ by completing the square.

THINK	WRITE
1 Factorise the trinomial by writing the expression as a product of the leading coefficient, which is 5, and a monic quadratic.	$5x^2 - 10x + 1 = 5\left(x^2 - 2x + \frac{1}{5}\right)$
2 The linear term of the monic quadratic is −2, so add and subtract the term $\left(\frac{-2}{2}\right)^2 = 1$ from the quadratic so that the overall expression remains unchanged.	$= 5\left(x^2 - 2x + 1 - 1 + \frac{1}{5}\right)$ $= 5\left([x^2 - 2x + 1] - \frac{5}{5} + \frac{1}{5}\right)$
3 Factorise the perfect square expression using the 'perfect square' rule and simplify the constant terms at the end of the expression.	$= 5\left(\left[x - 1\right]^2 - \frac{4}{5}\right)$
4 Write as the difference of two squares. Simplify the square root by identifying any perfect square factors and rationalising the denominator.	$= 5\left(\left[x - 1\right]^2 - \left[\sqrt{\frac{4}{5}}\right]^2\right)$ $= 5\left(\left[x - 1\right]^2 - \left[\frac{2}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}\right]^2\right)$ $= 5\left(\left[x - 1\right]^2 - \left[\frac{2\sqrt{5}}{5}\right]^2\right)$
5 Factorise using the 'difference of two squares' rule.	$= 5\left(x - 1 + \frac{2\sqrt{5}}{5}\right)\left(x - 1 - \frac{2\sqrt{5}}{5}\right)$

Helpful hints

✔ Factorising non-monic quadratic trinomials by completing the square often results in an expression containing fractions with surds. In these cases, always remember to rationalise the denominator!

ANS p660

Exercise 21 Factorising non-monic quadratic expressions

▲ 1, 2, 3–5(1st, 2nd columns), 6(a, c, e), 7(a, c), 8(a, c, e, g), 12

■ 3–6(2nd, 3rd columns), 7(e, f), 8(d, f, g, i), 9, 10, 13

⬟ 4–6(2nd, 3rd columns), 7(e, f), 8(f–i), 11, 14–16

1 Each quadratic trinomial below is in the form $ax^2 + bx + c$. For each one:

i identify ac and b

ii find two numbers that multiply together to give ac and add together to give b

iii use the two numbers you found in part **ii** to split the middle term into two terms.

a $2x^2 + 5x + 2$ **b** $3x^2 + 11x + 6$ **c** $4x^2 + 17x + 4$

d $3x^2 + 10x - 8$ **e** $6x^2 - 7x - 3$ **f** $5x^2 - 13x + 6$

2 Factorise each of these expressions by grouping terms.

a $3x^2 + 2x + 9x + 6$ **b** $4x^2 + x + 16x + 4$ **c** $2x^2 + x + 4x + 2$

d $6x^2 - 9x + 2x - 3$ **e** $3x^2 - 2x + 12x - 8$ **f** $5x^2 - 3x - 10x + 6$

2I.1 **3** Factorise each of these quadratic trinomials by first splitting the middle term.

a $2x^2 + 7x + 6$ **b** $3x^2 + 5x + 2$ **c** $4x^2 + 12x + 5$

d $7x^2 + 9x + 2$ **e** $5x^2 + 14x + 8$ **f** $2x^2 + 9x + 9$

g $6x^2 + 7x + 2$ **h** $10x^2 + 17x + 3$ **i** $8x^2 + 14x + 5$

4 Factorise each of these quadratic trinomials by first splitting the middle term.

a $3x^2 - 8x - 3$ **b** $4x^2 + 5x - 6$ **c** $2x^2 - 9x + 10$

d $5x^2 - x - 4$ **e** $8x^2 + 10x - 3$ **f** $4x^2 + 4x - 15$

g $7x^2 - 18x + 8$ **h** $12x^2 + x - 1$ **i** $3x^2 - 11x - 20$

5 Factorise each of these quadratic trinomials. Where appropriate, take out any common factors first. (Hint: The common factor can be negative.)

a $4x^2 + 10x + 6$ **b** $12x^2 + 27x + 6$ **c** $10x^2 + 38x - 8$

d $8x^2 + 22x + 15$ **e** $6x^2 + 3x - 30$ **f** $12x^2 - 8x - 4$

g $-4x^2 - 6x - 2$ **h** $-12x^2 - 21x + 6$ **i** $-6x^2 - 17x - 12$

6 Simplify each of these expressions. Remember to factorise first.

a $\dfrac{2x^2 + 15x + 18}{x + 6}$ **b** $\dfrac{3x + 4}{3x^2 + 13x + 12}$ **c** $\dfrac{4x^2 + 18x + 14}{2x + 2}$

d $\dfrac{6x^2 - 5x + 1}{3x^2 + 14x - 5}$ **e** $\dfrac{5x^2 - 28x - 12}{2x^2 - 15x + 18}$ **f** $\dfrac{-8x^2 + 4x + 40}{2x^2 - x - 10}$

7 Simplify each of these expressions.

a $\dfrac{5x^2 - 8x + 3}{x^2 - 1} \times \dfrac{2x^2 + x - 1}{2x^2 - 7x + 3}$ **b** $\dfrac{2x^2 + 13x + 21}{6x^2 - x - 2} \times \dfrac{3x^2 - 14x + 8}{x^2 - x - 12}$

c $\dfrac{4x^2 + 13x - 12}{2x^2 + 13x + 20} \div \dfrac{12x^2 - 5x - 3}{6x^2 + 17x + 5}$ **d** $\dfrac{4x^2 + 28x + 49}{4x^2 - 49} \div \dfrac{2x^2 - x - 28}{2x^2 + 15x + 28}$

e $\dfrac{3x^2 + 6x - 24}{3x^2 - 7x + 2} \times \dfrac{15x^2 + x - 2}{6x + 24}$ **f** $\dfrac{2x^3 - x^2 - 3x}{2x^2 + 9x - 18} \div \dfrac{x^3 - 9x}{x^2 + 9x + 18}$

2I.2 **8** Factorise each of these quadratic trinomials by completing the square.

a $3x^2 + 18x + 21$ **b** $2x^2 - 20x + 40$ **c** $5x^2 + 10x - 30$

d $3x^2 + 12x + 3$ **e** $2x^2 - 12x - 3$ **f** $4x^2 + 8x - 1$

g $6x^2 - 24x - 5$ **h** $3x^2 + 24x + 1$ **i** $9x^2 - 18x + 2$

9 Simplify the following expression by first factorising each fraction.

$$\frac{2x^2 - 3x - 20}{x^3 - 16x} \times \frac{3x^2 + 16x + 16}{4x^2 - 25} \div \frac{9x^2 + 24x + 16}{70x - 28x^2}$$

10 a Factorise $4x^2 + 7x - 2$.

b Show that $-4x^2 - 7x + 2$ factorises to $(1 - 4x)(x + 2)$.

c Factorise each of these quadratic trinomials.

i $-2x^2 + x + 3$ **ii** $-3x^2 + 16x - 5$

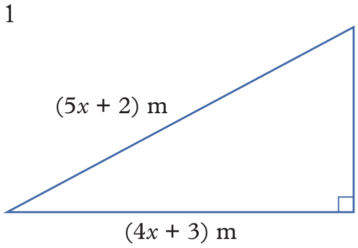
iii $-5x^2 - 3x + 14$ **iv** $-4x^2 + 5x + 6$

v $-6x^2 - x + 2$ **vi** $-25x^2 + 10x - 1$

11 Consider the right-angled triangle shown on the right.

a Find an expression for the length of the missing side, in terms of x.

b The expression you wrote for part **a** should include a square root symbol. Factorise the expression appearing under that square root symbol.



12 Factorise each of the following by first completing the square.

a $x^2 + 8x + 5$ **b** $2x^2 + 8x + 5$ **c** $3x^2 + 8x + 5$

13 Decide whether each of these expressions can be factorised using the methods covered in this chapter so far. Write those that can be factorised in factorised form.

a $2x^2 + 4x + 6$ **b** $15x^2 - 2x - 8$

c $3x^2 - 12x + 3$ **d** $4x^2 - 8x + 16$

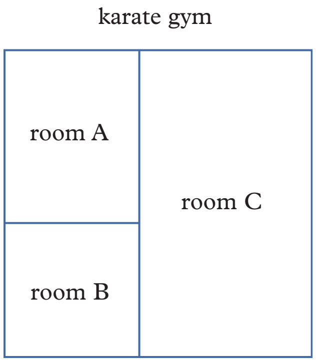
e $-2x^2 + 12x - 22$ **f** $-6x^2 + 69x - 189$

CHALLENGE

14 A karate instructor bought a gym with three training rooms: A, B and C. To make each room a safe place for training, the instructor plans to set down some large square mats of length x metres.

The dimensions of all the rooms are a number of mats plus or minus an integer number of metres. Training room A has an area of $(8x^2 + 2x - 15)$ m². The length of the square floor of training room B is the same as the smaller floor dimension of training room A. Assume $x < 4$ m.

Answer the questions about the karate gym.



a Write expressions for the dimensions of the floors of training rooms A and B in terms of the length of the mats.

b Write an expression for the total floor area of training room A and B combined, in terms of the length of the mats.

c If the shape of the karate gym is a square, what are the dimensions of the gym?

d Write an expression for the total area of training rooms A and C combined, as the difference of two squares.

e Factorise your answer for part **d**.

f Divide the total area of the gym by the area of one mat to determine an expression for the maximum number of mats that the instructor will need to cover the floor of the gym. Give your answer as a trinomial.

g If $x = 3$ m, what is the maximum number of mats that can fit on the floor of the gym?

15 A quartic is an expression of the form $ax^4 + bx^3 + cx^2 + dx + e$ where a, b, c, d and e are constants. If a quartic does not include any indices that are odd numbers (so the b and d coefficients are zero), it can be factorised using the same methods for factorising quadratics.

For example, $x^4 + 5x^2 + 6$ can be factorised to $(x^2 + 3)(x^2 + 2)$.

a Factorise:

i $x^4 + 7x^2 + 10$ **ii** $x^4 + 14x^2 + 24$ **iii** $x^4 + 4x^2 + 3$

b Write each of the following quartics as the product of one quadratic factor and two linear factors, by using the 'difference of two squares' method of factorisation.

i $x^4 - 1$ **ii** $x^4 - 16$ **iii** $x^4 - 81$

c Write each of the following quartics as the product of four linear factors.

i $x^4 - 13x^2 + 36$ **ii** $x^4 - 26x^2 + 25$ **iii** $x^4 - 11x^2 + 10$

16 a Complete the square for the general quadratic trinomial $ax^2 + bx + c$.

b Use your answer for part **a** to factorise the general quadratic trinomial $ax^2 + bx + c$.

c Considering the fact that you cannot find the square root of a negative number, write a simplified expression in terms of a, b and c that *must be non-negative* for the quadratic to be factorised.

Check your Student obook pro for these digital resources and more: pro

Interactive skillsheet
Factorising non-monic quadractic expressions

Topic quiz
2I

Chapter summary

Index laws and properties

$a^m \times a^n = a^{(m+n)}$	$a^m \div a^n = a^{(m-n)}$
$(a^m)^n = a^{(m \times n)}$	$(a \times b)^m = a^m \times b^m$
$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$
$a^0 = 1$	$a^{-1} = \frac{1}{a}$

Expanding

- Use the distributive law to expand brackets.

$$a(b + c) = ab + ac$$

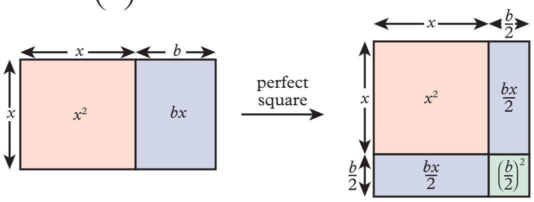
$$(a + b)(c + d) = ac + ad + bc + bd$$

Factorising by grouping terms

$$x^2 + 2x - 3x - 6 = x(x + 2) - 3(x + 2)$$
$$= (x + 2)(x - 3)$$

Adding a term to make a perfect square

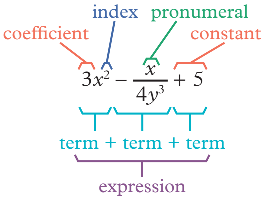
- Add $\left(\frac{b}{2}\right)^2$ to $x^2 + bx$ to make it a perfect square.



$$x^2 + 2x\left(\frac{b}{2}\right) + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

Simplifying terms and expressions

- Like terms contain the same pronumerals raised to the same indices.



- Like terms can be added and subtracted by adding and subtracting their cooefficients.

$$7x^2y + 2 + 3yx^2 - 8 = 10x^2y - 6$$

Factorising using the HCF

$$\xleftarrow{\text{expanding}}$$

$$2x^2 - 6x = 2x(x - 3)$$

expanded form factorised form

$$\xrightarrow{\text{factorising}}$$

Factorising monic quadratics

$$x^2 + bx + c$$

- Find numbers, m and n, that add to give b and multiply to give c.

$$x^2 + (m + n)x + mn = (x + m)(x + n)$$

- Substitute those two numbers into the factorised form (the binomial product).

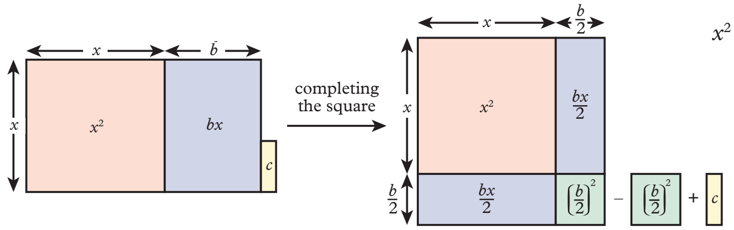
Factorising by splitting the middle term

$$ax^2 + bx + c = mpx^2 + mqx + npx + nq$$

- Find the unknowns that add to give b and multiply to give $a \times c$.
- Use these numbers to split the linear term.
- Factorise by grouping terms.

$$mpx^2 + mqx + npx + nq = (mx + n)(px + q)$$

Completing the square



$$x^2 + bx + c = x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$
$$= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

Using the difference of two squares to factorise

$$(x - h)^2 - k = \left(x - h + \sqrt{k}\right)\left(x - h - \sqrt{k}\right)$$

Chapter review

Chapter review quiz
Take the chapter review quiz to assess your knowledge of this chapter.

Quizlet
Test your knowledge of this topic by working individually or in teams.

Multiple-choice

2A **1** What is the simplified form of $x^2 \times (x^3)^4$?

A x^9 **B** x^{24} **C** x^{20} **D** x^{14} **E** x^5

2B **2** What is $\frac{36x^2y}{4x^3y^{-2}}$ when it is simplified?

A $\frac{9}{xy^3}$ **B** $\frac{9y^3}{x}$ **C** $9x^{-1}y^{-3}$ **D** $\frac{9}{xy}$ **E** $9xy^3$

2C **3** Which of the following expressions is equivalent to $a^2b - 2ab - ab^2 + 5ab$?

A $7ab$ **B** $a^2b + 3ab + ab^2$ **C** $3ab$

D $3a^2b^2$ **E** $a^2b + 3ab - ab^2$

2D **4** What is the expanded form of $6t^2w(4w^3 + 3t^4)$?

A $24t^2w^3 + 18t^6w$ **B** $24t^2w^4 + 18t^8w$ **C** $24t^2w^4 + 18t^6w$

D $24t^2w^3 + 18t^8w$ **E** $10t^2w^4 + 9t^6w$

2D **5** What is the expanded form of $(x - 4)(x + 3)$?

A $x^2 + x + 12$ **B** $x^2 + x - 12$ **C** $x^2 - x + 12$ **D** $x^2 - x - 12$ **E** $x^2 + 7x - 12$

2E **6** What does $\frac{2x}{3} + \frac{x}{4}$ equal?

A $\frac{3x}{7}$ **B** $\frac{11x}{12}$ **C** $\frac{3x}{12}$ **D** $\frac{11x}{7}$ **E** $\frac{x^2}{6}$

2F **7** What is the fully factorised form of $8x^2y^3 + 12x^3y$?

A $8x^2y(y^2 + 4x)$ **B** $4x^2y(2y^2 + 3x)$ **C** $4xy(2xy^2 + 3x^2)$

D $4xy(2y^2 + 3x)$ **E** $8x^3y^3(x + y^2)$

2F **8** What is the factorised form of $3(a - 2) - b(a - 2)$?

A $(a - 2)(3 - b)$ **B** $(a - 2)(b - 3)$ **C** $(a - 2)(3 + b)$

D $(a + 2)(3 + b)$ **E** $-3b(a - 2)$

2G **9** The quadratic trinomial $x^2 + 7x + c$ *cannot* be factorised for which of the following values of c?

A -18 **B** -8 **C** 10 **D** 12 **E** 18

2G **10** Which of these is a quadratic trinomial?

A $5x^2 + 4$ **B** $2x^3 + 4x^2 - 6$ **C** $x + 5x - 3$ **D** $4x - 2 + 3x^2$ **E** $x^2 - 3$

2H **11** Which of the following expressions is equivalent to $2x^2 + 6x + 7$?

A $2(x + 3)^2 - 2$ **B** $2\left(x + \frac{3}{2}\right)^2 - 2$ **C** $2(x + 3)^2 + 5$

D $2\left(x + \frac{3}{2}\right)^2 - \frac{5}{2}$ **E** $2\left(x + \frac{3}{2}\right)^2 + \frac{5}{2}$

10A **2I** **12** What is the factorised form of $9x^2 + 21x + 10$?

A $(9x + 5)(x + 2)$ **B** $(9x + 10)(x + 1)$ **C** $(3x + 5)(3x + 2)$

D $(3x + 10)(3x + 1)$ **E** $(9x + 2)(x + 5)$

Short answer

2A **1** Simplify each of the following.

a $a^2 \times a^5$ **b** $(2x^2y)^3 \times 3xy^2$ **c** $3(x^2y)^0$ **d** $\frac{12ab^2c^5}{(2bc)^2}$

2B **2** Write each of these terms in simplified form with positive indices.

a $\frac{xy^{-3}}{x^{-2}y}$ **b** $\frac{4a^2b^{-1}}{12a^{-2}b^4}$ **c** $\frac{18p^3q^{-1}r^5}{15p^{-1}q^4r^{-2}}$

2B **3** Simplify each of these expressions, writing the answers with positive indices.

a $(x^2)^3 \times x^{-2} \div x^{-4}$ **b** $(p^2q^{-3})^2 \div p^4q^{-3} \times p^{-1}q$ **c** $(2a^2)^{-3} \times (3ab^{-1})^{-2} \div 12b^5$

2C **4** Simplify each of these expressions.

a $4xy - 10xy + x^2 - 4x^2 + 2x^2y - 5xy^2$ **b** $ab^2 + (ab)^2 - 3a^2b^2 + 2a^2b - 4b^2a$

c $pq - 5pq + p^2 - q^2 - (3p)^2 + (2q)^2$

2C **5** Name the like terms in each of these lists.

a $3abc^2,\ a^2b^2c^2,\ -2a^2b^2c,\ (2abc)^2$ **b** $5xyz^2,\ -2z^2xy,\ 3yz^2x,\ 4x^2y^2y^2$

c $2p^2q^2,\ -(pq)^2,\ p^2 \times q \times -q,\ (p \div q)^2$

2C **6** If $a = -2$, $b = -1$ and $c = 3$, evaluate each of these expressions.

a $a^2 + b^2 + c^2$ **b** $abc - ab^2c - c^3 \div b$ **c** $(a + b)^2 - (c - a)^2$

2D **7** Expand and simplify each of these.

a $(x - 3)(x + 7)$ **b** $(x + 2)^2$ **c** $(x - 4)^2$

d $(2x + 1)(x - 4)$ **e** $(a - b)(a + b)$ **f** $(a^2 + b)(a + b^2)$

2D **8** Expand each of these expressions, giving your answers in simplest form.

a $(2x^2 - 4)(3x^2 + 5)$ **b** $(2x + 3y)(2x - 3y)$ **c** $(3x^2 - 4)^2$

2D **9** Expand and simplify each of these expressions.

a $3(x^2 - 2) - x(x + 5)$ **b** $-2x(3 - x) + 5x(2 - 3x)$ **c** $x(x + 1) - x(x - 1) + x(x - 3)$

2E **10** Simplify each of these expressions.

a $\frac{3x}{4} + \frac{4x}{3}$ **b** $\frac{x-2}{4} - \frac{3-x}{5}$ **c** $\frac{4x-1}{12} - \frac{3-2x}{8}$ **d** $\frac{4}{3x} + \frac{5}{4x}$

2F **11** Factorise each of these expressions by grouping terms.

a $x^2 - 3x - 4x + 12$ **b** $x^3 + x^2 - x - 1$ **c** $16x + 6x^2 - 32 - 12x$

2F **12** Use the 'difference of two squares' rule to factorise each of these expressions. Write your answers in simplest form.

a $18x^2 - 8y^2$ **b** $(x - 3)^2 - x^2$ **c** $(x + 2)^2 - (x - 2)^2$

2F **13** Factorise each of these expressions, including surds in your answer if necessary.

a $x^2y^2 - 13$ **b** $(x + 1)^2 - 3$ **c** $7 - (a - 1)^2$

2G **14** Factorise each of these expressions.

a $x^2 + 8x + 12$ **b** $x^2 - 2x - 63$ **c** $-x^2 + 6x - 8$ **d** $-6x^2 - 6x + 36$

2G **15** Simplify each of these expressions.

a $\frac{x^2 - 4}{x^2 - x - 6} \times \frac{x^2 + x - 12}{x^2 + x - 6}$ **b** $\frac{x^2 - 5x + 6}{x^2 - 16} \div \frac{x^2 + x - 6}{x^2 - 3x - 4}$

2H **16** Factorise each of these expressions by first completing the square.

a $x^2 + 10x + 20$ **b** $x^2 - 8x - 1$

10A **2I** **17** Factorise each of these expressions.

a $3x^2 + 29x + 18$ **b** $8x^2 + 10x - 3$ **c** $2x^2 - 7x - 22$ **d** $12x^2 - 17x + 6$

10A **2I** **18** Factorise each of these expressions, if possible.

a $2x^2 + 12x - 4$ **b** $5x^2 - 20x + 30$ **c** $6x^2 + 21x - 90$ **d** $-4x^2 + 12x - 2$

Analysis

Solve the following questions with methods covered in this chapter.

a Expand each of the following: **i** $(2x + 3)^2$ **ii** $(x + 4)^2$

b Find the quadratic, in expanded form, that results from $(2x + 3)^2 - (x + 4)^2$.

10A **c** Factorise the quadratic found in part **b**.

d Use the difference of two squares to find a quicker way to factorise $(2x + 3)^2 - (x + 4)^2$.

e Factorise each of the following.

i $(3x + 4)^2 - (x + 5)^2$ **ii** $(2x + 1)^2 - (x - 3)^2$

iii $(x + 4)^2 - (x + 2)^2$ **iv** $(2x - 5)^2 - (x - 4)^2$

3

Real numbers

Index

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

✓ Indices
✓ Roots
✓ Multiplying terms
✓ Dividing terms
✓ Equivalent fractions

Curriculum links

- Define rational and irrational numbers and perform operations with surds and fractional indices (VCMNA355) [10A]
- Use the definition of a logarithm to establish and apply the laws of logarithms and investigate logarithmic scales in measurement (VCMNA356) [10A]
- Solve simple exponential equations (VCMNA360) [10A]

Materials

✓ Calculator

3A Rational and irrational numbers

Learning intentions

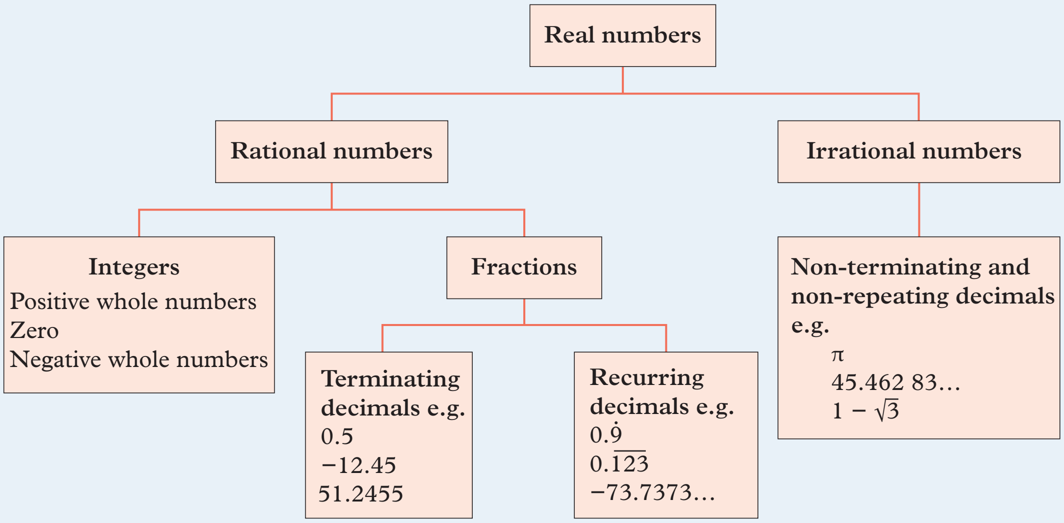
✔ I can determine whether a number is rational or irrational.
✔ I can determine whether a square root is a surd.

Inter-year links

Year 8	4E Roots
Year 9	2F Surds

Rational and irrational numbers

- The real number system is made up of **rational numbers** and **irrational numbers**.
- Rational numbers are numbers that can be written as a fraction in which both the numerator and denominator are integers. They are whole numbers (integers), fractions, terminating decimals and recurring decimals.
- Irrational numbers are numbers that cannot be written exactly as fractions. They include **surds** and other non-terminating decimals that are not recurring.



Surds

- A surd is an irrational root. Surds cannot be simplified to remove the root.
  - → $\sqrt{1} = 1$
    So $\sqrt{1}$ is not a surd.
  - → $\sqrt{2} = 1.414\,213\,5\ldots$
    $\sqrt{2}$ cannot be written as a fraction, an integer, or a recurring or terminating decimal, so it is a surd.
  - → $\sqrt{3} = 1.732\,050\,8\ldots$
    $\sqrt{3}$ cannot be written as a fraction, an integer, or a recurring or terminating decimal, so it is a surd.
  - → $\sqrt{4}$ can be simplified to 2.
    So $\sqrt{4}$ is not a surd.
- Surds can be other types of roots such as cube roots ($\sqrt[3]{2}$) or fourth roots ($\sqrt[4]{2}$).

Basic rules for square roots

- The inverse operation of finding the square root of a number is raising a number to the power of 2.

  $(\sqrt{a})^2 = a$

- The inverse operation of raising a number to the power of 2 is finding the square root of a number.

  $\sqrt{a^2} = a$

- A number multiplied by a surd can be written in the form $a\sqrt{b}$, where a and b are both numbers.

  $a \times \sqrt{b} = a\sqrt{b}$

Example 3A.1 Identifying rational and irrational numbers

Determine whether each of these numbers is rational or irrational.

a 0.26 **b** $-\frac{2}{3}$ **c** $-\sqrt{11}$ **d** $\sqrt{16}$

e $2.3\overline{54}$ **f** 6.584 381 257… **g** $\sqrt[3]{9}$

THINK	**WRITE**
a 0.26 is a terminating decimal. A terminating decimal can be written exactly as a fraction: $0.26 = \frac{26}{100}$ So 0.26 is rational.	**a** 0.26 is a rational number.
b Fractions are rational numbers, so $-\frac{2}{3}$ is rational.	**b** $-\frac{2}{3}$ is a rational number.
c $-\sqrt{11}$ is approximately equal to –3.316 624 79… It is a non-terminating and non-recurring decimal which cannot be written as a fraction, so $-\sqrt{11}$ is irrational.	**c** $-\sqrt{11}$ is an irrational number.
d $\sqrt{16}$ simplifies to 4. Integers are rational numbers, so $\sqrt{16}$ is rational.	**d** $\sqrt{16}$ is a rational number.
e $2.3\overline{54}$ represents the recurring decimal 2.354 545 454… Recurring decimals are rational numbers, so $2.3\overline{54}$ is rational.	**e** $2.3\overline{54}$ is a rational number.
f 6.584 381 257… is a non-terminating decimal number. Its decimal place values do not have a pattern of repeating digits, so 6.584 381 257… is irrational.	**f** 6.584 381 257… is an irrational number.
g $\sqrt[3]{9}$ is approximately equal to 2.080 083 823… It is a non-terminating and non-recurring decimal which cannot be written as a fraction, so $\sqrt[3]{9}$ is irrational.	**g** $\sqrt[3]{9}$ is irrational.

Example 3A.2 Identifying surds

Which of these numbers are surds?

a $\sqrt{9}$ **b** $2\sqrt{6}$ **c** $4\sqrt{25}$

THINK	**WRITE**
a $\sqrt{9}$ can be simplified to 3, so $\sqrt{9}$ is not a surd.	**a** $\sqrt{9}$ is not a surd.
b $2\sqrt{6} = 2 \times \sqrt{6}$ $\sqrt{6} = 2.449\,489\,743\ldots$ so $2\sqrt{6}$ is a surd.	**b** $2\sqrt{6}$ is a surd.
c $4\sqrt{25} = 4 \times \sqrt{25}$ $\sqrt{25}$ can be simplified to 5. $4\sqrt{25} = 4 \times 5 = 20$, which is rational. So $4\sqrt{25}$ is not a surd.	**c** $4\sqrt{25}$ is not a surd.

Helpful hints

✔ Some people think that decimals are more ‘exact’ than fractions. However, mathematicians prefer fractions because it is never possible to tell if a decimal has been rounded, truncated, or has only been measured to a certain number of decimal places. This uncertainty is not an issue with a fraction, which has an integer for its numerator and an integer for its denominator.

ANS p661

Exercise 3A Rational and irrational numbers

△ 1–12, 16, 17, 20 ■ 2, 3, 5–9, 11(a, c, f, i, k), 13, 14, 17–19, 21 ⬟ 2, 3, 6, 7, 9, 13–15, 21–25

UNDERSTANDING AND FLUENCY

1 Match each number below with the correct term from the list **A–F**.

a $\frac{3}{4}$
b 45.710 326 58…
c –9
d $\sqrt{23}$
e 0.154 715 471 547…
f 6.213 764 29

A recurring decimal
B integer
C terminating decimal
D fraction
E surd
F non-terminating and non-recurring decimal

3A.1 **2** Determine whether each of these numbers is rational or irrational.

a $\frac{4}{5}$ **b** –5 **c** $\sqrt{3}$ **d** $\sqrt{64}$ **e** 2.548 247 001…
f 5.684 **g** $-\frac{4}{9}$ **h** $-3\sqrt{5}$ **i** $5\frac{1}{6}$ **j** –1.5
k $\sqrt{9}$ **l** $\sqrt{8}$ **m** $\sqrt[3]{125}$ **n** 0.001 002 142… **o** $\sqrt[4]{32}$

3A.2 **3** Which of these numbers are surds?

a $\sqrt{21}$ **b** $-\sqrt{16}$ **c** $\sqrt[4]{64}$ **d** $-\sqrt{6}$
e $\frac{1}{2}\sqrt{9}$ **f** $5\sqrt{5}$ **g** $-3\sqrt{27}$ **h** $-\sqrt[3]{27}$
i $\sqrt[3]{3}$ **j** $3\sqrt{3}$ **k** $3 + \sqrt{3}$ **l** $\sqrt{5} - 1$
m $\frac{\sqrt{100}}{5}$ **n** $\frac{\sqrt{6}}{5}$ **o** $\frac{\sqrt{6}}{\sqrt{36}}$ **p** $\frac{2\sqrt{2}}{4} - \sqrt{2}$

UNDERSTANDING AND FLUENCY

4 Use your calculator to help you write a decimal approximation, rounded to four decimal places, for each of the surds and irrational numbers in question **3**.

5 Which irrational numbers in question **2** are surds?

6 Calculate the value of each of these, without using a calculator:

a $\sqrt{121}$
b $5\sqrt{36}$
c $-4\sqrt{64}$
d $-\frac{\sqrt{16}}{4}$
e $4\sqrt{16} + \sqrt{64}$
f $\sqrt{36} + \sqrt{64}$
g $\sqrt{36 + 64}$
h $\sqrt{9 + 16}$
i $\sqrt{25 + 144}$
j $-\sqrt{9} - \frac{\sqrt{36}}{2}$
k $2\sqrt{100} - \frac{3\sqrt{144}}{4}$
l $\sqrt{9^2}$

7 Which of these are rational numbers and which are irrational?

a $-\sqrt{81}$
b $\frac{2}{3}$
c $\sqrt{0.64}$
d $\sqrt{0.036}$
e 3.358 294 721…
f 8.4
g $-\frac{1}{7}$
h $-3\sqrt{0.004}$
i $\sqrt{1.369}$
j 4.435 353 535…
k $\sqrt{20.25}$
l $\frac{3}{4}\sqrt{100}$

8 Which of the irrational numbers in question **7** are surds?

9 State whether each of these numbers is rational or irrational:

a $\sqrt{\frac{4}{16}}$
b $\sqrt{\frac{1}{12}}$
c $\frac{\sqrt{16}}{\sqrt{5}}$
d $\sqrt{\frac{1}{16}}$
e $\frac{2\sqrt{6}}{\sqrt{4}}$
f $\sqrt{\frac{1}{2}}$
g $\frac{\sqrt{6}}{\sqrt{25}}$
h $\sqrt{\frac{7}{49}}$

10 The Greek letter π (pi) is used as the symbol for a number commonly used in mathematics (equal to 3.141 592…). Is π a rational or irrational number? Provide a reason for your answer.

11 Use your calculator to help you find the value of each of the following expressions, correct to four decimal places.

a $\sqrt{5} + \sqrt{6}$
b $2\sqrt{3} + \sqrt{11}$
c $\sqrt{30} - \sqrt{15}$
d $4\sqrt{8} - 2\sqrt{10}$
e $\sqrt{7} \times \sqrt{5}$
f $4\sqrt{3} \times 3\sqrt{7}$
g $2\sqrt{24} \div \sqrt{8}$
h $(2\sqrt{7} - 4\sqrt{3})^2$
i $\frac{5\sqrt{6}}{\sqrt{7}}$
j $\frac{5}{4\sqrt{11}}$
k $\frac{4\sqrt{2}}{2 + \sqrt{3}}$
l $\frac{3 - \sqrt{5}}{\sqrt{3} + 5}$

PROBLEM SOLVING AND REASONING

12 Consider the two fractions $\sqrt{\frac{5}{8}}$ and $\frac{\sqrt{5}}{\sqrt{8}}$.

a Use your calculator to help you determine the approximate value for each of $\sqrt{\frac{5}{8}}$ and $\frac{\sqrt{5}}{\sqrt{8}}$, to six decimal places.

b Are the two fractions equivalent? Explain.

c Use your results to comment on the relationship between fractions written in the forms $\frac{\sqrt{a}}{\sqrt{b}}$ and $\sqrt{\frac{a}{b}}$.

d Is $\sqrt{\frac{5}{8}}$ a rational number or an irrational number? Provide a reason for your answer.

e Write each of these fractions in the form $\sqrt{\frac{a}{b}}$ to help you decide if it is a rational or irrational number.

i $\frac{\sqrt{32}}{\sqrt{2}}$
ii $\frac{\sqrt{15}}{\sqrt{3}}$
iii $\frac{\sqrt{7}}{\sqrt{28}}$

13 a Calculate:

i $\sqrt{4}$
ii $\sqrt{9}$

b Between which two consecutive integers will $\sqrt{5}$ lie?

c How does a knowledge of perfect squares help you decide which two consecutive integers a surd will lie between?

d Between which two consecutive integer values will each of these surds lie?

i $\sqrt{11}$
ii $-\sqrt{20}$
iii $\sqrt{131}$
iv $-\sqrt{48}$
v $\sqrt{180}$
vi $\sqrt{300}$

14 Knowing which two integers a surd lies between can help us approximate values when more calculations are involved. We can apply the same operations to the two consecutive integers as are applied to the surd.

a Use the fact that $\sqrt{26}$ lies between 5 and 6 to determine which two integers the results of the following calculations lie between.

i $\sqrt{26} + 3$ **ii** $\sqrt{26} - 10$ **iii** $10 - \sqrt{26}$
iv $3\sqrt{26}$ **v** $3\sqrt{26} - 10$ **vi** $3(\sqrt{26} - 10)$

b Determine which two rational numbers each of the following lies between.

i $\dfrac{\sqrt{14}}{5}$ **ii** $\dfrac{5}{\sqrt{14}}$ **iii** $\dfrac{6 + \sqrt{51}}{2}$
iv $\dfrac{6 - \sqrt{51}}{2}$ **v** $-5 + \dfrac{\sqrt{91}}{6}$ **vi** $-5 + \dfrac{6}{\sqrt{91}}$

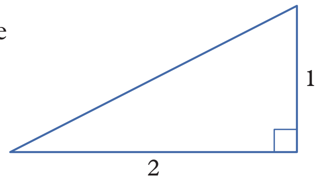
15 Determine which two rational numbers the answer to each of the following calculations lies between.

a $\sqrt[3]{11}$ **b** $\sqrt[4]{11}$ **c** $\sqrt[3]{36}$ **d** $\sqrt[5]{400\,000}$
e $7\sqrt[3]{25}$ **f** $7 - \sqrt[3]{70}$ **g** $3 - 5\sqrt[3]{144}$ **h** $\dfrac{6 + 2\sqrt[3]{300}}{3}$

16 You will have seen by now that the decimal value for a surd that is obtained using a calculator is only an approximation. But the surd represents an exact value.

a Use Pythagoras' Theorem to show that the exact length of the hypotenuse of the triangle in this diagram is $\sqrt{5}$.

b Write a decimal approximation for the length of the hypotenuse, to two decimal places.



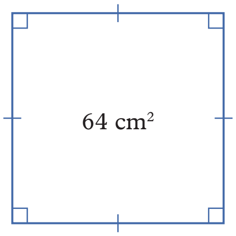
17 A square has an area of 64 cm².

a What is the length of each side of the square?

b Calculate the length of the diagonal of the square:

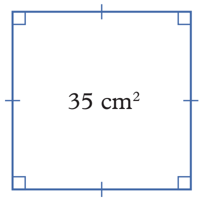
i as an exact value

ii rounded to two decimal places.

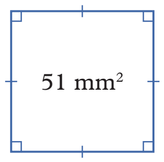


18 Repeat question **17** for squares with these areas:

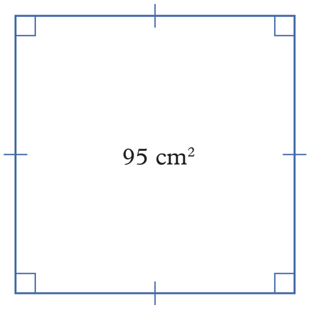
a



b

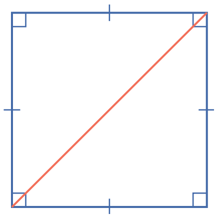


c



19 a Multiply the side length of each square from questions **17** and **18** by $\sqrt{2}$. Compare the results for each square with the diagonal length you calculated for it. What do you notice?

b Use Pythagoras' Theorem to show that the length of the diagonal of any square is always its side length multiplied by $\sqrt{2}$. (Hint: 'Show that' means to show all the necessary calculations.)



20 Show that each of these decimals is a rational number by writing it as a fraction in simplest form.

a 0.4 **b** 0.65 **c** 0.49 **d** 1.6 **e** 2.57 **f** 0.851

21 You will have seen by now that recurring decimals are rational numbers. For example: $0.\dot{6}$ is equivalent to $\frac{2}{3}$.

You can verify that $\frac{2}{3} = 0.666\,666\ldots$ by using a calculator to find $2 \div 3$.

To write a recurring decimal as a fraction (and to show it is rational), there is a series of steps to follow.

a In the calculation below, match each step (**1–6**) with its corresponding line of working (**A–F**). Then write the lines of working in the correct order.

Steps:

1 Let x represent the recurring decimal.

2 Multiply the recurring decimal by 10 to write an equation for $10x$.

3 Form a new subtraction equation from the two equations for x and $10x$. Then carry out the subtraction on each side of that equation.

4 Write the simplified equation that results from the subtraction.

5 Divide both sides of the equation by 9 to obtain x on the left-hand side of the equation and a fraction on the right-hand side.

6 Write the recurring decimal as a fraction, in simplest form.

Lines of working:

A $0.\dot{6} = \frac{2}{3}$

B $x = \frac{6}{9}$

C $10x = 6.666\,666\ldots$

D $x = 0.666\,666\ldots$

E $10x - x = 6.666\,666\ldots - 0.666\,666\ldots$

F $9x = 6$

b Why did you multiply x by 10 in the calculation above? How did this help you obtain a whole number on the right-hand side of the equation in step **4**?

22 Use the method shown in question **21a** to write each of these recurring decimals as a fraction. (Use a calculator to check your answers.)

a $0.\dot{2}$ **b** $0.\dot{7}$ **c** $2.\dot{3}$ **d** $5.\dot{6}$

23 a Complete the working below to write $0.\overline{36}$ as a fraction.

$$\begin{aligned}\text{Let: } x &= 0.363\,636\ldots\\ 100x &= \underline{\qquad\qquad}\\ 100x - x &= \underline{\qquad\qquad} - 0.363\,636\ldots\\ 99x &= \underline{\qquad\qquad}\\ x &= \frac{\square}{99}\\ \text{So: } \quad 0.\overline{36} &= \frac{\square}{11}\end{aligned}$$

b Explain why multiplying x by 100 was helpful in this case.

24 Write each of these recurring decimals as a fraction.

a $0.\overline{14}$ **b** $0.\dot{2}\dot{1}$ **c** $0.\overline{48}$ **d** $1.\dot{7}\dot{3}$

25 Write each of these recurring decimals as a fraction. Show all your working.

a $0.333\,333\,333\ldots$ **b** $0.\overline{24}$ **c** $0.\overline{785}$ **d** $0.\dot{4}\dot{1}$

e $0.5\dot{6}$ **f** $3.\overline{18}$ **g** $4.2\dot{5}$ **h** $3.5\overline{126}$

Check your Student obook pro for these digital resources and more: pro

Interactive skillsheet
Terminating, non-terminating and recurring decimals

Topic quiz
3A

3B Simplifying surds

Learning intentions

✔ I can simplify surds in $\sqrt{a}$ form.

✔ I can simplify surds in $a\sqrt{b}$ form.

Inter-year links

Year 8	4E Roots
Year 9	2F Surds

Basic rules for square roots

- The inverse operation of 'finding the square root' of a number is 'squaring' a number. When a number is squared, it is 'raised to the power of 2' (or the number has 'an index of 2'). Another name for an index is 'exponent'.

$$(\sqrt{a})^2 = a$$

- The inverse operation of 'squaring' a number is 'finding the square root' of the number.

$$\sqrt{a^2} = a$$

- A number multiplied by a surd can be written in the form $a\sqrt{b}$, where a and b are both numbers.

$$a \times \sqrt{b} = a\sqrt{b}$$

- The square root of a product can be written as the product of the square roots of its positive factors.

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b}$$

Simplifying square roots to surds

- A square root is simplified to a surd by taking out any square factors.
- To simplify a square root (in this case, $\sqrt{12}$):

1 Rewrite the number under the root sign as a product of its factors, with the first factor being the highest possible perfect square.	$\sqrt{12} = \sqrt{4 \times 3}$
2 Write the square root as the product of its square root factors, using $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$.	$= \sqrt{4} \times \sqrt{3}$
3 Simplify the square roots of any perfect squares.	$= 2 \times \sqrt{3}$
4 Multiply what is left and simplify the expression.	$= 2\sqrt{3}$

Example 3B.1 Simplifying a surd in $\sqrt{a}$ form

Write $\sqrt{54}$ in its simplest form.

THINK	**WRITE**
1 Rewrite the number under the root sign as a product of its factors, with the first factor being the highest possible perfect square.	$\sqrt{54} = \sqrt{9 \times 6}$
2 Write the square root as the product of its square root factors, using $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$.	$= \sqrt{9} \times \sqrt{6}$
3 Simplify the square roots of any perfect squares.	$= 3 \times \sqrt{6}$
4 Multiply what is left and simplify the expression.	$= 3\sqrt{6}$

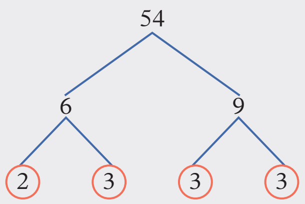
Example 3B.2 Simplifying a surd in $a\sqrt{b}$ form

Write $4\sqrt{72}$ in its simplest form.

THINK	WRITE
1 Rewrite the number under the root sign as a product of its factors, with the first factor being the highest possible perfect square.	$4\sqrt{72} = 4\sqrt{36 \times 2}$ $= 4 \times \sqrt{36} \times \sqrt{2}$
2 Simplify the square roots.	$= 4 \times 6 \times \sqrt{2}$
3 Multiply the coefficients and simplify the surd.	$= 24 \times \sqrt{2}$ $= 24\sqrt{2}$

Helpful hints

- ✔ Remember that factor trees are helpful for determining factors that are square numbers because they make it easy to see a number multiplied by itself to give a square number. For example, in the factor tree shown, $3 \times 3 = 3^2 = 9$ is a square number factor of 54.
- ✔ If there are no repeated prime factors of a root to be found, then the root is a surd and cannot be simplified.
- ✔ Simplifying surds is a very important maths skill for Years 11 and 12.



ANS p662

Exercise 3B Simplifying surds

△ 1, 2, 3–4(1st–3rd columns), 5, 6(1st–3rd columns), 8, 9, 13, 14

■ 3–4(3rd–5th columns), 6(3rd–5th columns), 8, 9(b, d, j, l), 10, 11, 13, 14, 15(b, d, h, j, m)

⬟ 3–4(4th, 5th columns), 6(4th, 5th columns), 8, 9(i–l), 10(a), 11, 12, 16, 17, 18(c, d)

UNDERSTANDING AND FLUENCY

1 Calculate the value of each of these.

a $\sqrt{16}$ **b** $\sqrt{81}$ **c** $\sqrt{49}$ **d** $\sqrt{4}$ **e** $\sqrt{100}$

f $\sqrt{25}$ **g** $3\sqrt{36}$ **h** $4\sqrt{121}$ **i** $\sqrt{1}$ **j** $\frac{1}{2}\sqrt{16}$

2 Simplify each of these expressions.

a $\sqrt{81} \times \sqrt{7}$ **b** $\sqrt{16} \times \sqrt{3}$ **c** $\sqrt{4} \times \sqrt{6}$

d $\sqrt{25} \times \sqrt{10}$ **e** $\sqrt{9} \times \sqrt{2}$ **f** $\sqrt{5} \times \sqrt{36}$

g $2 \times \sqrt{16} \times \sqrt{11}$ **h** $3 \times \sqrt{100} \times \sqrt{5}$ **i** $\frac{1}{2}\sqrt{144} \times \sqrt{10}$

3B.1 3 Simplify the following roots.

a $\sqrt{28}$ **b** $\sqrt{50}$ **c** $\sqrt{24}$ **d** $\sqrt{27}$ **e** $\sqrt{48}$

f $\sqrt{75}$ **g** $\sqrt{63}$ **h** $\sqrt{60}$ **i** $\sqrt{56}$ **j** $\sqrt{45}$

k $\sqrt{128}$ **l** $\sqrt{108}$ **m** $\sqrt{180}$ **n** $\sqrt{147}$ **o** $\sqrt{162}$

p $\sqrt{125}$ **q** $\sqrt{450}$ **r** $\sqrt{288}$ **s** $\sqrt{270}$ **t** $\sqrt{567}$

3B.2 4 Write each of these in simplest form.

a $5\sqrt{20}$ **b** $3\sqrt{8}$ **c** $6\sqrt{12}$ **d** $9\sqrt{18}$ **e** $4\sqrt{80}$

f $3\sqrt{90}$ **g** $12\sqrt{150}$ **h** $-3\sqrt{48}$ **i** $7\sqrt{96}$ **j** $-2\sqrt{245}$

k $8\sqrt{192}$ **l** $10\sqrt{252}$ **m** $-5\sqrt{112}$ **n** $7\sqrt{99}$ **o** $-12\sqrt{68}$

p $100\sqrt{300}$ **q** $\frac{1}{2}\sqrt{320}$ **r** $\frac{1}{2}\sqrt{216}$ **s** $8\sqrt{248}$ **t** $\frac{3}{4}\sqrt{176}$

5 Show that the surd $5\sqrt{27}$ simplifies to $15\sqrt{3}$. (Hint: 'Show that' means to show all the necessary calculations.)

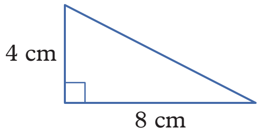
6 Write each of these in simplest form.

a $\sqrt{44}$
b $\sqrt{125}$
c $\sqrt{120}$
d $\sqrt{72}$
e $\sqrt{150}$
f $-\sqrt{216}$
g $\sqrt{126}$
h $\sqrt{240}$
i $-3\sqrt{18}$
j $\sqrt{245}$
k $\frac{1}{2}\sqrt{192}$
l $\frac{\sqrt{108}}{3}$
m $-2\sqrt{48}$
n $15\sqrt{80}$
o $3\sqrt{84}$
p $\frac{3}{5}\sqrt{450}$
q $-8\sqrt{800}$
r $-\frac{2}{3}\sqrt{900}$
s $-\sqrt{1000}$
t $\frac{\sqrt{288}}{4}$
u $\frac{5\sqrt{1372}}{14}$
v $\frac{-\sqrt{384}}{2}$
w $\frac{7\sqrt{1280}}{32}$
x $\frac{3}{4}\sqrt{648}$

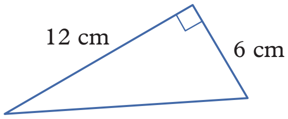
PROBLEM SOLVING AND REASONING

7 Use Pythagoras' Theorem to calculate the exact length of the unknown side in each triangle below. Where appropriate, write the answer in simplest surd form.

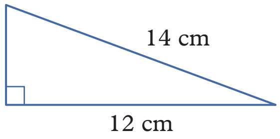
a



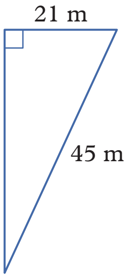
b



c



d



8 A surd in which the entire number is written beneath the root sign can be called an entire surd. For example, $\sqrt{6}$ can be classified as an entire surd, but $3\sqrt{6}$ cannot.

a Copy and complete the following to write $3\sqrt{6}$ as an entire surd.

$$\begin{aligned}3\sqrt{6} &= \sqrt{3^2}\times\sqrt{6}\\ &= \sqrt{\_\_}\times\sqrt{6}\\ &= \sqrt{\_\_\times 6}\\ &= \sqrt{\_\_}\end{aligned}$$

b Consider the first line of working in part **a**. Why do you think the coefficient has been written as a perfect square beneath the root sign?

c How do you think the surd we started with ($3\sqrt{6}$) and the answer (the entire surd) are related to each other?

d Briefly explain how you could check that your answer is correct when you write a surd as an entire surd.

e Follow the working shown in part **a** to write $4\sqrt{3}$ as an entire surd.

9 Write each of these as an entire surd.

a $3\sqrt{5}$
b $2\sqrt{3}$
c $4\sqrt{2}$
d $6\sqrt{7}$
e $4\sqrt{10}$
f $5\sqrt{11}$
g $8\sqrt{6}$
h $5\sqrt{5}$
i $4\sqrt{15}$
j $-8\sqrt{2}$
k $12\sqrt{7}$
l $-5\sqrt{3}$

10 a Write $a\sqrt{b}$ as an entire surd.

b Use the general form of the entire surd you wrote in part **a** and check that it produces the same answers you got for the surds in question **9**.

c In your own words, explain how a surd is expressed as an entire surd.

11 Write each of these as an entire surd.

a $m\sqrt{n}$
b $7\sqrt{a}$
c $x\sqrt{5}$
d $10\sqrt{xy}$
e $d\sqrt{2c}$
f $3\sqrt{4abc}$
g $a\sqrt{ab^2}$
h $xy\sqrt{x^2yz^2}$

12 Write each of these in simplest form.

a $\sqrt{x^2y}$
b $\sqrt{x^2y^2}$
c $\sqrt{ab^2}$
d $\sqrt{x^2yz^2}$
e $\sqrt{x^4}$
f $\sqrt{x^4y}$
g $\sqrt{x^4y^2}$
h $\sqrt{a^4b^2c}$

13 An alternative method for simplifying a surd involves writing the number appearing under the root sign as a product of its prime factors. Consider the surd $\sqrt{12}$.

a Write 12 as a product of its prime factors in index form.

b Write the product of the prime factors of 12 under the root sign and show that it simplifies to $2\sqrt{3}$.

14 When using the method described in question **13**, you will notice that not all numbers under the root sign result in perfect squares. But, often, they can be written differently so that they do contain perfect squares. Consider $\sqrt{54}$.

a Write 54 as a product of its prime factors in index form.

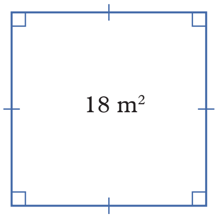
b Use your answer to part **a** to show that $\sqrt{3^2 \times 3 \times 2}$ is an equivalent form of $\sqrt{54}$.

c Simplify your answer and use your calculator to check that it is correct.

15 Use the methods from questions **13** and **14** to simplify each of the following.

a $\sqrt{18}$ **b** $\sqrt{28}$ **c** $\sqrt{32}$ **d** $\sqrt{27}$ **e** $\sqrt{24}$ **f** $\sqrt{50}$
g $3\sqrt{20}$ **h** $2\sqrt{128}$ **i** $5\sqrt{63}$ **j** $8\sqrt{40}$ **k** $7\sqrt{48}$ **l** $6\sqrt{45}$
m $\sqrt{180}$ **n** $5\sqrt{147}$ **o** $6\sqrt{162}$ **p** $\sqrt{125}$ **q** $2\sqrt{450}$ **r** $\sqrt{288}$

16 Consider this square with an area of 18 m².



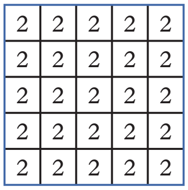
a Write the exact side length of the square as an entire surd and a simplified surd.

b The 18 m² square is divided into nine 2 m² squares.
Write the exact side length of each of the 2 m² squares.

c Describe the relationship between the side lengths of the two different sized squares from parts **a** and **b** and the number of squares the 18 m² square is divided into.

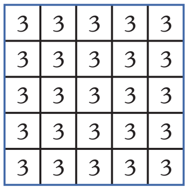
d Determine the side length of each of the following blue squares. (Hint: In part **i**, the area of each small square is 2 m²; in part **ii**, the area of each small square is 3 m².) Write your answer as an entire surd and a simplified surd each time.

i



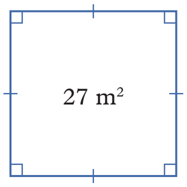
2	2	2	2	2
2	2	2	2	2
2	2	2	2	2
2	2	2	2	2
2	2	2	2	2

ii

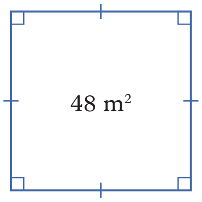


3	3	3	3	3
3	3	3	3	3
3	3	3	3	3
3	3	3	3	3
3	3	3	3	3

iii



iv



17 a Simplify each of the following surds.

i $\sqrt{8}$ **ii** $\sqrt{27}$ **iii** $\sqrt{125}$ **iv** $\sqrt{a^3}$, where $a > 0$

b Simplify each of the following surds.

i $\sqrt{32}$ **ii** $\sqrt{243}$ **iii** $\sqrt{3125}$ **iv** $\sqrt{a^5}$, where $a > 0$

c Simplify each of the following surds.

i $\sqrt[3]{16}$ **ii** $\sqrt[3]{81}$ **iii** $\sqrt[3]{625}$ **iv** $\sqrt[3]{a^4}$

d Simplify each of the following surds.

i $\sqrt[4]{32}$ **ii** $\sqrt[4]{243}$ **iii** $\sqrt[4]{3125}$ **iv** $\sqrt[4]{a^5}$, where $a > 0$

18 Arrange the surds in each of the following lists in ascending order.

a $\sqrt{46}$, $\sqrt{22}$, $\sqrt{42}$, $\sqrt{10}$, $\sqrt{27}$, $\sqrt{5}$, $\sqrt{21}$, $\sqrt{30}$
b $2\sqrt{3}$, 3, 6, $\sqrt{6}$, $\sqrt{39}$, $2\sqrt{6}$, $\sqrt{11}$, $4\sqrt{3}$
c $3\sqrt{3}$, $2\sqrt{5}$, $3\sqrt{6}$, $\sqrt{83}$, $\sqrt{22}$, $\sqrt{43}$, 4, $2\sqrt{11}$
d $7\sqrt{10}$, $12\sqrt{2}$, $3\sqrt{43}$, $2\sqrt{57}$, $3\sqrt{14}$, 20, $4\sqrt{5}$, $2\sqrt{6}$

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Simplifying surds

CAS instructions
Simplifying surds

Topic quiz
3B

3C Multiplying and dividing surds

Learning intentions

- ✔ I can multiply two or more surds together.
- ✔ I can divide a surd by a surd.

Inter-year links

Year 8	5D Multiplying algebraic terms
Year 9	3A Simplifying

Multiplying surds

- A square root of a number can be written as the product of the square roots of that number's positive factors.

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b}$$

- When multiplying surds that are in the form $a\sqrt{b}$, multiply the surd terms and multiply the coefficients.

$$a\sqrt{b} \times c\sqrt{d} = a \times c \times \sqrt{b \times d}$$

Dividing surds

- A square root divided by a square root can be written as the square root of the quotient.

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

- When dividing surds that are in the form $a\sqrt{b}$, divide the surd terms and divide the coefficients.

$$a\sqrt{b} \div c\sqrt{d} = \frac{a\sqrt{b}}{c\sqrt{d}} = \frac{a}{c} \times \sqrt{\frac{b}{d}}$$

Example 3C.1 Multiplying surds

Simplify the following multiplication expressions that contain roots.

a $\sqrt{6} \times \sqrt{5}$ **b** $\sqrt{15} \times \sqrt{6}$ **c** $5\sqrt{3} \times 8\sqrt{11}$

THINK	WRITE
a Multiply the square roots together. Remember: $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$.	**a** $\sqrt{6} \times \sqrt{5} = \sqrt{6 \times 5}$ $= \sqrt{30}$
b 1 Multiply the square roots together. Remember: $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$. 2 Simplify the square root.	**b** $\sqrt{15} \times \sqrt{6} = \sqrt{15 \times 6}$ $= \sqrt{90}$ $= \sqrt{9 \times 10}$ $= \sqrt{9} \times \sqrt{10}$ $= 3 \times \sqrt{10}$ $= 3\sqrt{10}$

c **1** Multiply the square roots together and multiply the coefficients together.	**c** $5\sqrt{3} \times 8\sqrt{11} = 5 \times 8 \times \sqrt{3} \times \sqrt{11}$
2 Write the product of the coefficient first. Then write the product of the surd terms under the root sign.	$= 40 \times \sqrt{3 \times 11}$ $= 40 \times \sqrt{33}$ $= 40\sqrt{33}$

Example 3C.2 Dividing surds

Simplify the following division expressions that contain roots.

a $\sqrt{27} \div \sqrt{3}$ **b** $\dfrac{\sqrt{35}}{\sqrt{15}}$ **c** $\dfrac{36\sqrt{60}}{9\sqrt{12}}$

THINK	**WRITE**
a **1** Divide one square root by the other. Remember: $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$.	**a** $\sqrt{27} \div \sqrt{3} = \dfrac{\sqrt{27}}{\sqrt{3}}$
2 Simplify the fraction.	$= \sqrt{\dfrac{27}{3}}$ $= \sqrt{9}$
3 Simplify the square root.	$= 3$
b **1** Divide one square root by the other. Remember: $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$.	**b** $\dfrac{\sqrt{35}}{\sqrt{15}} = \sqrt{\dfrac{35}{15}}$
2 Simplify the fraction.	$= \sqrt{\dfrac{7}{3}}$
3 Simplify the square root. $\sqrt{7}$ and $\sqrt{3}$ are surds. Write in the form $\dfrac{\sqrt{a}}{\sqrt{b}}$.	$= \dfrac{\sqrt{7}}{\sqrt{3}}$
c **1** Divide the surd terms and divide the coefficients.	**c** $\dfrac{36\sqrt{60}}{9\sqrt{12}} = \dfrac{36 \times \sqrt{60}}{9 \times \sqrt{12}}$ $= \dfrac{36}{9} \times \dfrac{\sqrt{60}}{\sqrt{12}}$
2 Divide one square root by the other.	$= \dfrac{36}{9} \times \sqrt{\dfrac{60}{12}}$
3 Simplify the fractions. $\sqrt{5}$ is already in surd form.	$= 4 \times \sqrt{5}$ $= 4\sqrt{5}$

Helpful hints

- ✔ Make sure that the horizontal line that is part of the root symbol covers all relevant numbers or pronumerals.
- ✔ Your skills with fractions will be very useful to you in this chapter. Simplifying when dividing roots is very similar to multiplying and dividing fractions.

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Exercise 3C Multiplying and dividing surds

△ 1(e–l), 2, 3, 4(e–l), 5(e–l), 6–9, 10(a, c, e, g, k), 11, 12(a, d, e, f), 13, 14, 16, 21(a, b)

◻ 1(e–l), 2(a, b, e, f), 3, 4(g–l), 5(i–l), 7(e–h), 8, 10(c, d, g, h), 11(f, h, i, k), 12(d, f, g, h), 17, 18(c–f), 21(a, d, e), 25

⬠ 4(k–p), 5(i–l), 10(h–l), 11(h–k), 12(g–i), 15, 19, 20, 21, 23, 24, 26

UNDERSTANDING AND FLUENCY

Do not use a calculator for the questions in this exercise.

3C.1 **1** Evaluate the following multiplication expressions that contain roots.

a $\sqrt{7} \times \sqrt{5}$
b $\sqrt{3} \times \sqrt{11}$
c $\sqrt{2} \times -\sqrt{7}$
d $-\sqrt{15} \times \sqrt{7}$
e $\sqrt{13} \times \sqrt{5}$
f $-\sqrt{2} \times \sqrt{11}$
g $\sqrt{7} \times \sqrt{3}$
h $\sqrt{19} \times \sqrt{5}$
i $\sqrt{7} \times 3\sqrt{11}$
j $5\sqrt{3} \times 4\sqrt{2}$
k $-4\sqrt{2} \times 7\sqrt{11}$
l $12\sqrt{6} \times -3\sqrt{5}$
m $11\sqrt{2} \times 6\sqrt{17}$
n $8\sqrt{6} \times 8\sqrt{7}$
o $4\sqrt{3} \times -6\sqrt{11}$
p $12\sqrt{3} \times 11\sqrt{5}$

2 Calculate the value of each of these.

a $\sqrt{7} \times \sqrt{7}$
b $\sqrt{3} \times \sqrt{3}$
c $\sqrt{2} \times \sqrt{2}$
d $\sqrt{6} \times \sqrt{6}$
e $\sqrt{5} \times \sqrt{5}$
f $-\sqrt{11} \times -\sqrt{11}$
g $\sqrt{10} \times \sqrt{10}$
h $\sqrt{9} \times \sqrt{9}$

3 **a** What did you notice about each result in question **2**?

b Simplify $\sqrt{a} \times \sqrt{a}$, where a is a positive real number.

4 Calculate the value of each of these, writing your answer in surd form where appropriate.

a $3\sqrt{3} \times \sqrt{3}$
b $-\sqrt{11} \times 7\sqrt{11}$
c $4\sqrt{2} \times 5\sqrt{2}$
d $6\sqrt{5} \times 2\sqrt{5}$
e $\sqrt{5} \times 3\sqrt{2}$
f $15\sqrt{6} \times 2\sqrt{5}$
g $-4\sqrt{2} \times 8\sqrt{13}$
h $12\sqrt{3} \times \frac{1}{4}\sqrt{11}$
i $2\sqrt{12} \times 7\sqrt{3}$
j $\frac{1}{2}\sqrt{5} \times 3\sqrt{20}$
k $\frac{3}{4}\sqrt{13} \times -8\sqrt{2}$
l $\frac{3}{5}\sqrt{7} \times \frac{5}{3}\sqrt{15}$
m $-12\sqrt{6} \times -6\sqrt{7}$
n $\frac{1}{2}\sqrt{15} \times \frac{3}{4}\sqrt{13}$
o $15\sqrt{10} \times -\frac{1}{3}\sqrt{7}$
p $-10\sqrt{17} \times 1.1\sqrt{5}$

3C.2 **5** Simplify the following expressions that contain roots.

a $\sqrt{34} \div \sqrt{2}$
b $\sqrt{24} \div \sqrt{8}$
c $\sqrt{15} \div \sqrt{15}$
d $\sqrt{15} \div -\sqrt{3}$
e $\dfrac{\sqrt{42}}{\sqrt{7}}$
f $\dfrac{\sqrt{56}}{\sqrt{8}}$
g $4\sqrt{18} \div \sqrt{3}$
h $22\sqrt{40} \div 11\sqrt{8}$
i $\dfrac{-24\sqrt{35}}{8\sqrt{7}}$
j $\dfrac{16\sqrt{50}}{8\sqrt{10}}$
k $-12\sqrt{60} \div 3\sqrt{6}$
l $8\sqrt{55} \div 16\sqrt{11}$

6 Calculate the value of each of these.

a $\sqrt{32} \div \sqrt{2}$ **b** $\sqrt{24} \div \sqrt{6}$ **c** $\sqrt{54} \div \sqrt{6}$ **d** $\sqrt{40} \div \sqrt{10}$

7 Calculate the value of each of these, writing your answers in surd form.

a $12\sqrt{15} \div -\sqrt{3}$ **b** $35\sqrt{24} \div 7\sqrt{8}$ **c** $-12\sqrt{48} \div 4\sqrt{24}$

d $\frac{18\sqrt{35}}{9\sqrt{7}}$ **e** $\frac{-32\sqrt{42}}{-8\sqrt{6}}$ **f** $\frac{35\sqrt{5}}{-15}$

g $\frac{-24}{-8\sqrt{3}}$ **h** $\sqrt{240} \div 5\sqrt{48}$ **i** $\frac{12\sqrt{27}}{\sqrt{432}}$

8 Show that the result of the calculation $6\sqrt{3} \times 3\sqrt{2}$ is $18\sqrt{6}$. (Hint: 'Show that' means to show all the necessary calculations.)

9 Consider $\frac{24\sqrt{40}}{8\sqrt{20}}$. By dividing the coefficients and then dividing the surd terms, show that the result is $3\sqrt{2}$.

10 Calculate the value of each of these expressions. Write your answer in fractional surd form where appropriate.

a $6\sqrt{15} \div \sqrt{15}$ (Remember: $6\sqrt{15} \div \sqrt{15}$ is equivalent to $\frac{6\sqrt{15}}{\sqrt{15}}$.)

b $\frac{12\sqrt{30}}{16\sqrt{6}}$ **c** $44\sqrt{34} \div 28\sqrt{42}$ **d** $\frac{-20\sqrt{75}}{25\sqrt{45}}$

e $\frac{36\sqrt{35}}{16\sqrt{7}}$ **f** $\frac{22\sqrt{5}}{-33\sqrt{5}}$ **g** $\frac{8\sqrt{3} \times 6}{-\sqrt{3}}$

h $\frac{1}{2}\sqrt{15} \div 4\sqrt{8}$ **i** $9\sqrt{32} \div 18\sqrt{8}$ **j** $\frac{4\sqrt{3} \times 6}{3\sqrt{6}}$

k $\frac{9\sqrt{2} \times 5}{\sqrt{18}}$ **l** $\frac{8\sqrt{21}}{12\sqrt{14}}$

11 Simplify each of these expressions.

a $\frac{\sqrt{54}}{\sqrt{6}}$ **b** $\sqrt{\frac{81}{100}}$ **c** $\sqrt{32} \times 3\sqrt{2}$

d $\frac{\sqrt{3}}{\sqrt{27}}$ **e** $5\sqrt{12} \times -2\sqrt{3}$ **f** $\sqrt{\frac{36}{9}}$

g $7\sqrt{8} \times \sqrt{18}$ **h** $4\sqrt{6} \times \sqrt{8} \times 6\sqrt{12}$ **i** $8\sqrt{45} \div 5\sqrt{5}$

j $\frac{\sqrt{32}}{16\sqrt{8}}$ **k** $\frac{-4\sqrt{75}}{15\sqrt{3}}$ **l** $\frac{-5\sqrt{343}}{15\sqrt{7}}$

12 Simplify each of these fractions, writing your answer in fraction form where appropriate.

a $\frac{3\sqrt{5} \times 4\sqrt{6}}{6\sqrt{3}}$ **b** $\frac{5\sqrt{7} \times \sqrt{6}}{25\sqrt{14}}$ **c** $\frac{12\sqrt{48}}{3\sqrt{2} \times 5\sqrt{6}}$

d $\frac{11\sqrt{132}}{4\sqrt{11} \times -2\sqrt{6}}$ **e** $\frac{8\sqrt{3} \times 2\sqrt{8}}{6\sqrt{6} \times \sqrt{2}}$ **f** $\frac{4\sqrt{5} \times -3\sqrt{24}}{5\sqrt{10} \times 6\sqrt{2}}$

g $\frac{\sqrt{3} \times \sqrt{15} \times \sqrt{8}}{\sqrt{6} \times 9\sqrt{12}}$ **h** $\frac{4\sqrt{12} \times 3\sqrt{10} \times -6\sqrt{7}}{5\sqrt{8} \times 8\sqrt{15}}$ **i** $\frac{-5\sqrt{6} \times 2\sqrt{28} \times -\sqrt{44}}{3\sqrt{21} \times -4\sqrt{18}}$

13 To square a number, we multiply the number by itself. Find the value of each of these expressions.

a $(\sqrt{5})^2$ **b** $(\sqrt{8})^2$ **c** $(\sqrt{2})^2$

d $(\sqrt{7})^2$ **e** $(\sqrt{9})^2$ **f** $(4\sqrt{5})^2$

g $(3\sqrt{6})^2$ **h** $(8\sqrt{3})^2$ **i** $(9\sqrt{11})^2$

14 Study your answer for each part of question **13**.

a Does the square of a surd expressed in the form $\sqrt{a}$ or $a\sqrt{b}$ result in a rational answer or an irrational answer?

b Show that your answer to part **a** will always be true.

c Write the surds in parts **f–i** as entire surds.

d Describe the connection between the number beneath the square root of an entire surd and the square of that surd.

15 Simplify each of these expressions.

a $(\sqrt{x})^2$ **b** $(\sqrt{ab})^2$

c $\sqrt{6y} \times \sqrt{6y}$ **d** $\sqrt{x+y} \times \sqrt{x+y}$

16 Find the value of each of these expressions.

a $\left(\frac{5}{\sqrt{7}}\right)^2$ **b** $\left(\frac{1}{\sqrt{5}}\right)^2$

c $\left(-\frac{3}{\sqrt{6}}\right)^2$ **d** $\left(\frac{\sqrt{10}}{\sqrt{13}}\right)^2$

17 Consider $3\sqrt{392} \times 5\sqrt{192}$.

a Calculate the value of the expression using each of these methods:

Method 1: Multiply the coefficients, then multiply the surd terms and write the surd in simplest form.

Method 2: Simplify each surd first and then perform the multiplication.

b Do both methods from part **a** produce the same answer?

c When do you think it is best to use:

i Method 1? **ii** Method 2?

18 Calculate the value of each of the following, writing each answer as a fraction in simplest surd form when necessary.

a $3\sqrt{48} \div \sqrt{6}$ **b** $\frac{2\sqrt{54}}{6\sqrt{3}}$ **c** $15\sqrt{104} \div 25\sqrt{8}$

d $\frac{54\sqrt{75}}{63\sqrt{15}}$ **e** $\frac{-21\sqrt{84}}{6\sqrt{7}}$ **f** $\frac{4\sqrt{300}}{3\sqrt{25}}$

g $\frac{8\sqrt{960}}{-\sqrt{48}}$ **h** $\sqrt{384} \div 5\sqrt{64}$ **i** $\frac{-\sqrt{504}}{\sqrt{54}}$

19 Calculate the value of each of these, writing your answer in simplest surd form.

a $\frac{22\sqrt{48}}{8\sqrt{2} \times 4\sqrt{3}}$ **b** $\frac{4\sqrt{10} \times 6\sqrt{24}}{2\sqrt{5} \times 3\sqrt{2}}$

c $\frac{3\sqrt{8} \times 7\sqrt{12}}{12\sqrt{6} \times -8\sqrt{2}}$ **d** $\frac{2\sqrt{15} \times 4\sqrt{12} \times \sqrt{10}}{\sqrt{2} \times 5\sqrt{12}}$

e $\frac{3\sqrt{14} \times -5\sqrt{12}}{4\sqrt{2} \times \sqrt{7}}$ **f** $\frac{2\sqrt{35} \times 3\sqrt{2} \times 6\sqrt{8}}{5\sqrt{5} \times 12\sqrt{14}}$

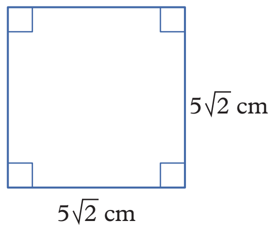
g $\frac{-4\sqrt{8} \times 3\sqrt{7} \times 12\sqrt{45}}{2\sqrt{14} \times 6\sqrt{15} \times 3\sqrt{3}}$ **h** $\frac{-6\sqrt{26} \times 12\sqrt{2} \times 8\sqrt{12} \times 16\sqrt{33}}{2\sqrt{11} \times 4\sqrt{13} \times 18\sqrt{3} \times 22\sqrt{8}}$

20 Simplify $\frac{a\sqrt{bc^2} \times b^2\sqrt{ac}}{c\sqrt{ab^2}}$, where a, b, and c are positive real numbers.

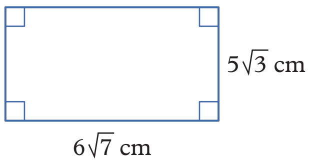
21 For each of these shapes, calculate:

i the area, correct to two decimal places **ii** the exact area.

a



b



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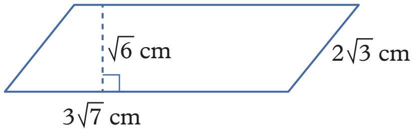

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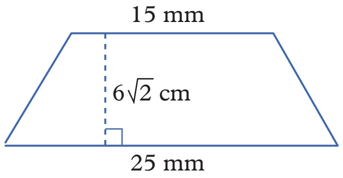

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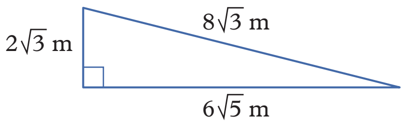

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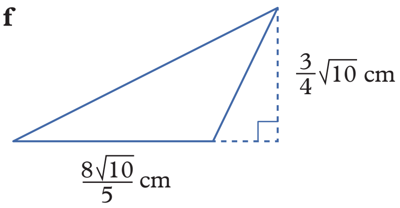
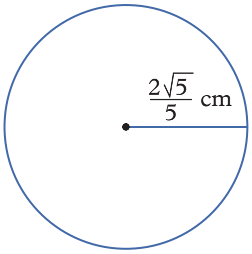


22 For each shape below, calculate:

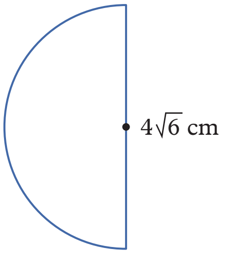
i the area, correct to two decimal places

ii the exact area.

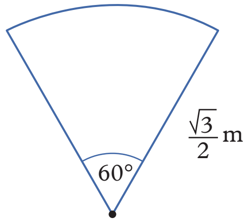
a



b

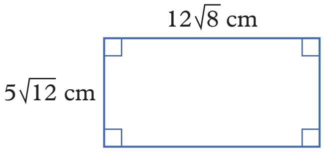


c

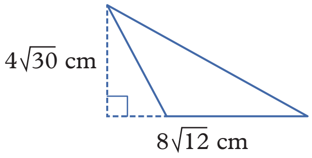


23 Calculate the exact area of each of these shapes. Write your answers in simplest surd form.

a



b



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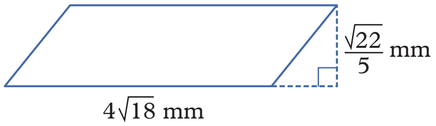

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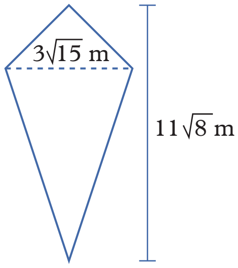

24 Consider a square with dimensions expressed as a surd of the type $\sqrt{a}$. Is it possible for the value representing the perimeter to be the same as the value representing the area of this square? Explain your answer.

25 Determine the exact simplified perimeter of a square with an area of:

a 5 m^2

b 17 m^2

c 18 m^2

d 20 m^2

e $\frac{3}{4}$ m^2

f $\frac{175}{36}$ m^2

26 For each of the following, determine the smallest surd, x, that will make the product an integer.

a $x \times \sqrt{180}$

b $x \times \sqrt{5400}$

c $x \times \sqrt{2835}$

d $x \times \sqrt{8064}$

e $x \times \sqrt{1911}$

f $x \times \sqrt{11\,154}$

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Multiplying and dividing surds

Topic quiz
3C

3D Adding and subtracting surds

Learning intentions

- ✔ I can identify like surds.
- ✔ I can add and subtract like surds.
- ✔ I can add and subtract surds after first simplifying.

Inter-year links

Year 8	5C Adding and subtracting algebraic terms
Year 9	3A Simplifying

Adding and subtracting surds

- **Like surds** contain exactly the same surd term, regardless of the coefficient.
  For example, $2\sqrt{2}$ and $3\sqrt{2}$ are like surds but $2\sqrt{5}$and $2\sqrt{3}$ are unlike surds.
- To add or subtract like surds, add or subtract the coefficients.
  $$a\sqrt{b} + c\sqrt{b} = (a + c)\sqrt{b}$$
- Sometimes seemingly unlike surds can be simplified to produce like surds. Simplify each surd in an expression by rewriting the number under each root sign as a product of its factors, with the first factor being the highest possible perfect square. Write surds in their simplest form before attempting to add or subtract.
  For example,
  $$\begin{aligned} 2\sqrt{27} - \sqrt{12} &= 2\sqrt{9 \times 3} - \sqrt{4 \times 3} \\ &= 2 \times \sqrt{9} \times \sqrt{3} - \sqrt{4} \times \sqrt{3} \\ &= 2 \times 3\sqrt{3} - 2\sqrt{3} \\ &= 6\sqrt{3} - 2\sqrt{3} \\ &= 4\sqrt{3} \end{aligned}$$

Example 3D.1 Identifying like surds

State whether the surds in each of these pairs are like surds or not.

a $2\sqrt{6}$ and $5\sqrt{6}$

b $7\sqrt{6}$ and $7\sqrt{3}$

c $5\sqrt{2}$ and $\sqrt{8}$

THINK	**WRITE**
a The surds are like surds because they have exactly the same surd term.	**a** Yes, they are like surds.
b The surds are not like surds because they do not have exactly the same surd term.	**b** No, they are not like surds.
c $\sqrt{8}$ needs to be simplified first. $\sqrt{8} = 2\sqrt{2}$ $\sqrt{8}$ and $2\sqrt{2}$ are like surds because $2\sqrt{2}$ and $5\sqrt{2}$ have exactly the same surd term.	**c** Yes, they are like surds.

Example 3D.2 Adding and subtracting surds

Simplify each of these expressions.

a $7\sqrt{3} + 4\sqrt{3} - \sqrt{3}$ **b** $6\sqrt{5} - 3\sqrt{2} - 8\sqrt{2} + 5\sqrt{5}$

THINK	**WRITE**
a 1 Group the like surds together. All the surds in this expression are like surds.	**a** $\enclose{circle}{7\sqrt{3}}\enclose{circle}{+ 4\sqrt{3}}\enclose{circle}{- \sqrt{3}} = 7\sqrt{3} + 4\sqrt{3} - 1\sqrt{3}$
2 Simplify the expression by adding and subtracting the coefficients.	$= 10\sqrt{3}$
b 1 Group the like surds together. There are two groups of like surds in this expression: $\sqrt{5}$ and $\sqrt{2}$	**b** $\enclose{circle}{6\sqrt{5}}\enclose{circle}{- 3\sqrt{2}}\enclose{circle}{- 8\sqrt{2}}\enclose{circle}{+ 5\sqrt{5}} = 6\sqrt{5} + 5\sqrt{5} - 3\sqrt{2} - 8\sqrt{2}$
2 Simplify the like terms in the expression by adding and subtracting the coefficients.	$= 11\sqrt{5} - 11\sqrt{2}$

Example 3D.3 Adding and subtracting surds after first simplifying

Simplify the following expressions.

a $3\sqrt{2} + 4\sqrt{8}$ **b** $4\sqrt{48} - 2\sqrt{75}$

THINK	**WRITE**
a 1 Simplify each surd in the expression. The first term, $3\sqrt{2}$, is already in its simplest form. Rewrite the number under the other root sign as a product of its factors, with the first factor being the highest possible perfect square.	**a** $3\sqrt{2} + 4\sqrt{8} = 3\sqrt{2} + 4\sqrt{4 \times 2}$ $= 3\sqrt{2} + 4 \times \sqrt{4} \times \sqrt{2}$ $= 3\sqrt{2} + 4 \times 2 \times \sqrt{2}$
2 Simplify the like terms in the expression by adding the coefficients.	$= 3\sqrt{2} + 8\sqrt{2}$ $= 11\sqrt{2}$
b 1 Simplify each surd in the expression, rewriting the number under each root sign as a product of its factors, with the first factor being the highest possible perfect square.	**b** $4\sqrt{48} - 2\sqrt{75} = 4\sqrt{16 \times 3} - 2\sqrt{25 \times 3}$ $= 4 \times \sqrt{16} \times \sqrt{3} - 2 \times \sqrt{25} \times \sqrt{3}$ $= 4 \times 4 \times \sqrt{3} - 2 \times 5 \times \sqrt{3}$
2 Simplify the like terms in the expression by subtracting the coefficients.	$= 16\sqrt{3} - 10\sqrt{3}$ $= 6\sqrt{3}$

Helpful hints

- ✔ Pay attention to your surds! Do not attempt to simplify the sum or difference of unlike surds!
- ✔ Starting every question by asking yourself ‘Can I simplify these surds?’ is a great way to avoid common mistakes.

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Exercise 3D Adding and subtracting surds

△ 1, 2(1st column), 3–5, 6, 7(1st column), 8(a, c, e, g), 9(a–f), 10(a–c), 11, 12 □ 2(2nd column), 4, 5, 6–10(2nd column), 13, 14, 15(a, b), 17 ⬠ 6–10(2nd column), 16, 18–21

UNDERSTANDING AND FLUENCY

3D.1 **1** State whether the terms in each of these pairs are like surds or not.

a $3\sqrt{5}$ and $\sqrt{5}$
b $4\sqrt{2}$ and $2\sqrt{4}$
c 3 and $\sqrt{3}$
d $-3\sqrt{6}$ and $6\sqrt{6}$
e $5\sqrt{5}$ and $2\sqrt{2}$
f $3\sqrt{7}$ and $3\sqrt{10}$
g $2\sqrt{2}$ and $-\sqrt{2}$
h $2\sqrt{2}$ and $\sqrt{8}$

3D.2 **2** Simplify each of these expressions by adding (or subtracting) the like terms, if possible. (Hint: Look for like surds.)

a $2\sqrt{3} + \sqrt{2} + 5\sqrt{3}$
b $4\sqrt{3} - 2\sqrt{2} + \sqrt{2}$
c $-3\sqrt{10} - 4\sqrt{7} - \sqrt{6} + 2\sqrt{3}$
d $6\sqrt{13} + 4\sqrt{39} - \sqrt{3} + 2\sqrt{13}$
e $9\sqrt{2} - \sqrt{2} - 6\sqrt{5} + 5\sqrt{2}$
f $3\sqrt{6} + 4\sqrt{5} - \sqrt{6} + 5 + 2\sqrt{6} - 4$
g $3\sqrt{11} + 4\sqrt{55} - \sqrt{5}$
h $7\sqrt{21} - 6\sqrt{3} + 5\sqrt{3} - 6\sqrt{21}$
i $6\sqrt{6} + 3\sqrt{5} - \sqrt{6} + 5 + 2\sqrt{6}$
j $-3\sqrt{15} - 2\sqrt{7} - \sqrt{7} - 4\sqrt{15}$

3 Write each surd in simplest form and then identify any like surds in each list.

a $2\sqrt{3}$ $\sqrt{2}$ $4\sqrt{2}$ $2\sqrt{5}$ $3\sqrt{2}$
b $3\sqrt{7}$ $14\sqrt{2}$ $7\sqrt{3}$ $2\sqrt{14}$
c $-3\sqrt{6}$ $3\sqrt{12}$ $\sqrt{3}$ $\sqrt{18}$ $2\sqrt{27}$
d $9\sqrt{32}$ $4\sqrt{8}$ $\sqrt{128}$ $5\sqrt{200}$

3D.3 **4** Simplify each of these expressions.

a $5\sqrt{2} + 4\sqrt{8}$
b $4\sqrt{36} - 2\sqrt{12}$
c $3\sqrt{24} + \sqrt{45}$
d $2\sqrt{28} + 8\sqrt{63}$
e $4\sqrt{72} + 5\sqrt{98}$
f $-2\sqrt{8} + 7\sqrt{16}$

5 Simplify each of these expressions.

a $5\sqrt{27} + \sqrt{12} - 8\sqrt{48}$
b $2\sqrt{20} - 4\sqrt{80} + 2\sqrt{45}$
c $\sqrt{28} - 3\sqrt{112} + 2\sqrt{63}$
d $5\sqrt{12} + \sqrt{27} - 8\sqrt{3}$
e $2\sqrt{8} + 4\sqrt{32} + 3\sqrt{2}$
f $\sqrt{7} - 3\sqrt{28} + 5\sqrt{7}$

6 Simplify each of these expressions.

a $3\sqrt{45} - \sqrt{5} - 2\sqrt{80} + 2\sqrt{20}$
b $\sqrt{108} - 4\sqrt{27} - 2\sqrt{12} + 2\sqrt{75}$
c $3\sqrt{24} - 4\sqrt{54} + 5\sqrt{24}$
d $3\sqrt{18} - 4\sqrt{50} - \sqrt{8} + 2\sqrt{98}$
e $3\sqrt{24} - 7\sqrt{54} - \sqrt{96} + 4\sqrt{20}$
f $4\sqrt{108} - 5\sqrt{75} - 6\sqrt{48} + \sqrt{8}$
g $\sqrt{128} - 5\sqrt{27} + 2\sqrt{32} - 3\sqrt{75}$
h $2\sqrt{75} - 6\sqrt{3} + 2\sqrt{48} - 6\sqrt{12}$
i $3\sqrt{12} - \sqrt{28} + 5\sqrt{8} - 2\sqrt{20}$
j $6\sqrt{54} + \sqrt{45} - 5\sqrt{96} - 2\sqrt{27}$

7 Simplify each of these expressions.

a $\sqrt{180} + 4\sqrt{125} - 3\sqrt{245}$
b $4\sqrt{243} + 3\sqrt{300} - 2\sqrt{432}$
c $3\sqrt{338} - 12\sqrt{147} - 2\sqrt{50} + \sqrt{48}$
d $3\sqrt{216} - 4\sqrt{125} - 2\sqrt{147} + 5\sqrt{45}$
e $\sqrt{63} + 7\sqrt{567} - 2\sqrt{96} + 8\sqrt{588}$
f $\sqrt{200} - 4\sqrt{180} + 5\sqrt{864} - 3\sqrt{160}$
g $-4\sqrt{320} - 25 + \sqrt{245} - 36 + \sqrt{27}$
h $12\sqrt{72} - 6\sqrt{192} + 2\sqrt{128} - 5\sqrt{507}$

8 Simplify each of these surd expressions containing pronumerals.

For example: $\sqrt{27x} + \sqrt{12x} = \sqrt{9 \times 3 \times x} + \sqrt{4 \times 3 \times x}$

$$= 3\sqrt{3x} + 2\sqrt{3x}$$
$$= 5\sqrt{3x}$$

a $\sqrt{x} + \sqrt{16x} - \sqrt{25x}$
b $3\sqrt{4x} + 2\sqrt{x} - 5\sqrt{36x}$
c $\sqrt{9x} + \sqrt{12x} + \sqrt{16x} - 2\sqrt{48x}$
d $\sqrt{8xy} - 2\sqrt{32xy} + \frac{1}{3}\sqrt{18xy}$
e $\sqrt{150xy} + 2\sqrt{384xy} - \sqrt{6xy}$
f $\sqrt{24x^2y} + 3x\sqrt{96y} - 5\sqrt{54x^2y}$
g $\sqrt{3xy^2} - 5y\sqrt{12x} + 7\sqrt{27xy^2}$
h $3\sqrt{100x^2y} - \sqrt{147x^2y} + \frac{1}{2}\sqrt{48x^2y} + \sqrt{16x^2y}$
i $2x\sqrt{80xy} + 5\sqrt{12xy} - \sqrt{125x^2y} - 2\sqrt{432x^2y}$

9 Remember the distributive law? It is used to expand expressions, as shown below.

$$a(b + c) = ab + ac$$

The distributive law can be applied to expressions involving surds.

Expand each of the expressions below. Where necessary, write all surds in simplest form.

a $6(2 + \sqrt{3})$
b $8(\sqrt{5} - 4)$
c $5(4 - 6\sqrt{2})$
d $4(3\sqrt{2} + 2\sqrt{3})$
e $2(5\sqrt{6} - 3\sqrt{5})$
f $-5\sqrt{7}(2\sqrt{6} - 3\sqrt{10} + 2)$
g $3\sqrt{6}(5\sqrt{3} + 2\sqrt{10})$
h $5\sqrt{7}(2\sqrt{7} - 12)$
i $6\sqrt{11}(2\sqrt{3} - 5\sqrt{7} + \sqrt{11})$
j $(3 - 2\sqrt{5} + 5\sqrt{15})\sqrt{5}$
k $-3\sqrt{3}(4\sqrt{6} - 2\sqrt{10} + 9\sqrt{15})$

10 Remember how to expand binomial products? An example is shown below.

$$(a + b)(c + d) = ac + ad + bc + bd$$

Binomial products can also contain surds. (Expressions such as '$\sqrt{5} + \sqrt{3}$' are referred to as binomial surds. Can you see why?)

Expand and simplify each of these expressions.

a $(5 + 3\sqrt{3})(2 + 4\sqrt{6})$
b $(3\sqrt{2} + 6)(5\sqrt{5} - 8)$
c $(7\sqrt{5} - 3\sqrt{2})(4\sqrt{2} + 8\sqrt{6})$
d $(\sqrt{6} + 2\sqrt{3})(6\sqrt{2} + 5\sqrt{3})$
e $(4\sqrt{8} - 5\sqrt{5})(3\sqrt{3} - 9\sqrt{8})$
f $(4\sqrt{10} + 3\sqrt{5})(7\sqrt{2} - 6\sqrt{12})$

11 Expand and simplify each of these perfect squares.

a $(5 + \sqrt{6})^2$
b $(\sqrt{7} - 3)^2$
c $(3\sqrt{5} + 2\sqrt{3})^2$
d $(4\sqrt{6} - 3\sqrt{7})^2$

12 You might have noticed that the results of the expansions in question **11** followed a pattern that you've seen before; that is, the 'perfect square' rule.

a Show that the expansion of $(a + b)^2$ will always produce the expression $a^2 + 2ab + b^2$.
b Show that the expansion of $(a - b)^2$ will always produce the expression $a^2 - 2ab + b^2$.
c Use these rules to help you check your answers to question **11**.

13 Expand and simplify each of these expressions.

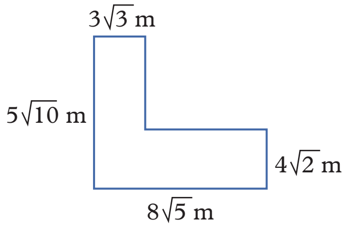
a $(4 + \sqrt{6})(4 - \sqrt{6})$
b $(5\sqrt{5} - 3)(5\sqrt{5} + 3)$
c $(2\sqrt{3} + \sqrt{2})(2\sqrt{3} - \sqrt{2})$
d $(4\sqrt{5} + 7\sqrt{3})(4\sqrt{5} - 7\sqrt{3})$

14 Consider each expansion in question **13**.

a What do you notice about the terms inside each pair of brackets?
b Is the result of the expansion a rational number or an irrational number? Why do you think this is the case?
c You may have noticed that the results of the expansions in question **13** followed a pattern that you've seen before, the 'difference of two squares' rule. Show that the expansion of binomial products in the form $(a + b)(a - b)$ or $(a - b)(a + b)$ will always be $a^2 - b^2$.
d Use this rule to help you check your answers for question **13**.

15 Use the information provided to calculate the exact area of each of the following composite shapes.

a



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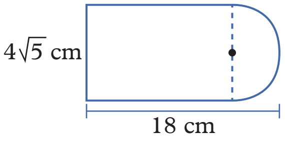

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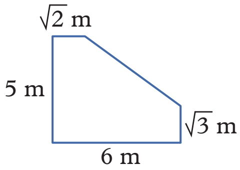
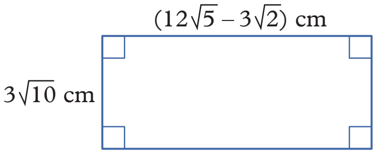


PROBLEM SOLVING AND REASONING

16 Calculate the exact area of each of these shapes. Write your answers in simplest surd form.

a



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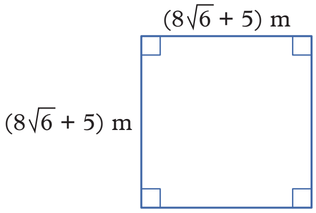

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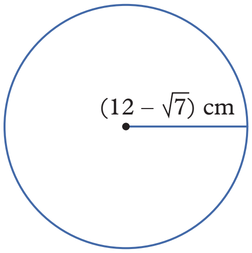
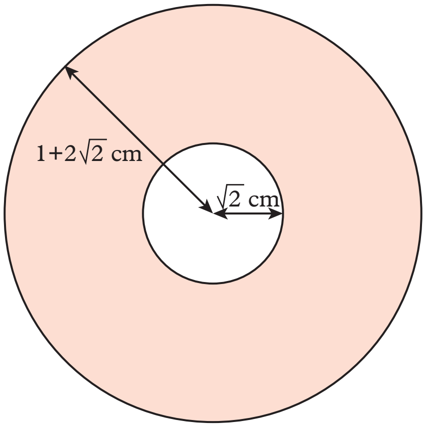


d



17 Calculate the exact perimeter of each shape in parts **a** and **b** of question **15**.

18 Evaluate each of the following.

a $\sqrt{3}(\sqrt{8} + \sqrt{72})$

b $-7\sqrt{15} + 5\sqrt{3} \times 2\sqrt{5}$

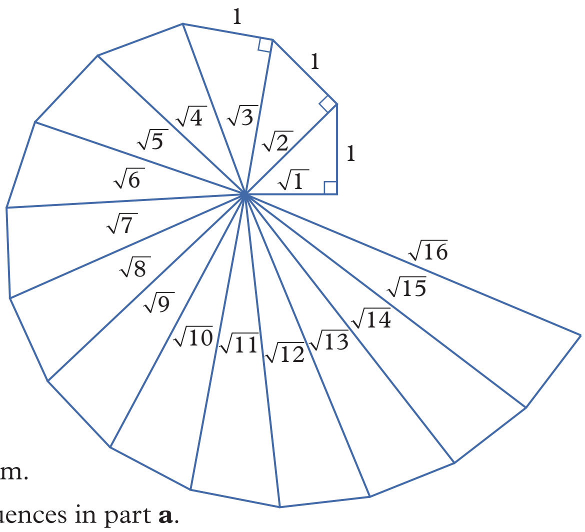
c $\dfrac{8\sqrt{30}}{3\sqrt{10}} + \dfrac{5\sqrt{42}}{6\sqrt{14}} + \dfrac{5\sqrt{105}}{9\sqrt{35}}$

d $\sqrt{50}\left(\dfrac{1}{\sqrt{10}} + \dfrac{1}{\sqrt{5}}\right) + \dfrac{\sqrt{10}}{2}(\sqrt{8} + \sqrt{18} + \sqrt{4})$

e $\dfrac{3\sqrt{55} - 5\sqrt{35}}{\sqrt{5}} + \dfrac{2\sqrt{77} - 2\sqrt{121}}{\sqrt{11}}$

f $\dfrac{6}{\sqrt{2}} \times \dfrac{6}{\sqrt{3}} + \left(\dfrac{4}{\sqrt{2}} + 2\sqrt{3}\right)\left(3\sqrt{2} - \dfrac{9}{\sqrt{3}}\right)$

19 The spiral of Theodorus is formed by consecutively forming a right-angled triangle on the hypotenuse of the previous right-angled triangle, so that the length of one of the shorter sides is always 1. Determine the total length of line segments drawn for this section of the spiral of Theodorus. Write your answer as a simplified expression.



CHALLENGE

20 **a** Describe the rule for generating the next term in each of the following sequences.

- **i** $3\sqrt{5},\ 8\sqrt{5},\ 13\sqrt{5},\ 18\sqrt{5}, \ldots$
- **ii** $2\sqrt{3},\ 3\sqrt{3} + 7,\ 4\sqrt{3} + 14,\ 5\sqrt{3} + 21$
- **iii** $3\sqrt{2},\ 6,\ 6\sqrt{2},\ 12, \ldots$
- **iv** $\sqrt{5},\ 10,\ 20\sqrt{5},\ 200, \ldots$

b Continue each of the sequences in part **a** to the tenth term.

c Evaluate the sum of the first 10 terms in each of the sequences in part **a**.

21 For each of the following lists, calculate:

i the mean

ii the median.

a $\sqrt{2},\ \sqrt{4},\ \sqrt{8},\ \sqrt{16},\ \sqrt{32},\ \sqrt{64},\ \sqrt{128},\ \sqrt{256}$

b $\sqrt{5},\ \sqrt{3},\ 5\sqrt{2},\ 2\sqrt{5},\ 3\sqrt{3},\ \sqrt{2}$

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Investigation
Tangrams and surds

Topic quiz
3D

3E Rationalising the denominator

Learning intentions

- ✔ I can rationalise the denominator of simple fractions.
- ✔ I can identify the conjugate of a surd expression.
- ✔ I can rationalise the denominator of a fraction using the conjugate surd.

Inter-year links

Year 8	5D Multiplying algebraic terms
Year 9	3B Expanding

Rationalising the denominator of a simple fraction

- Fractions that contain surds are conventionally written with a rational denominator. In most cases, it is an integer denominator.
- To write a surd fraction with a rational denominator:

  Step 1: Identify the surd in the denominator.

  Step 2: Multiply the numerator and the denominator of the fraction by that surd.

  Step 3: Simplify the resulting fraction.

$$\frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$$

Conjugate of a binomial expression

- The conjugate of a binomial expression with one surd has the same two terms but the surd term has a different sign. The conjugate of $a + \sqrt{b}$ is $a - \sqrt{b}$ and the conjugate of $a - \sqrt{b}$ is $a + \sqrt{b}$.

  For example, the conjugate of $2 - \sqrt{3}$ is $2 + \sqrt{3}$ and the conjugate of $\sqrt{3} - 2$ is $-\sqrt{3} - 2$.
- The product of conjugate expressions is the difference of the two squares of the two terms.

$$(a + b)(a - b) = a^2 - b^2$$
$$\underbrace{(a + \sqrt{b})(a - \sqrt{b})}_{\text{conjugates}} = \underbrace{a^2 - b}_{\text{rational number}}$$

For example, the conjugate of $\sqrt{2} + \sqrt{3}$ is $\sqrt{2} - \sqrt{3}$ because:

$$\begin{aligned}(\sqrt{2} + \sqrt{3})(\sqrt{2} - \sqrt{3}) &= \sqrt{2}^2 + \sqrt{3} \times \sqrt{2} - \sqrt{3} \times \sqrt{2} - \sqrt{3}^2 \\ &= \sqrt{2}^2 - \sqrt{3}^2 \\ &= 2 - 3 \\ &= -1 \qquad \text{which is a rational number.}\end{aligned}$$

Similar working can be used to show that the reverse is true and that the conjugate of $\sqrt{2} - \sqrt{3}$ is $\sqrt{2} + \sqrt{3}$.

Rationalising the denominator using the conjugate surd

- When the denominator of a fraction is a binomial including surds, the denominator can be rationalised by multiplying the numerator and denominator of the fraction by the conjugate of the denominator.

$$\frac{1}{a + \sqrt{b}} \times \frac{a - \sqrt{b}}{a - \sqrt{b}} = \frac{a - \sqrt{b}}{a^2 - b}$$

Example 3E.1 Rationalising the denominator of a fraction

Rewrite each of these fractions with a rational denominator.

a $\frac{7}{2\sqrt{5}}$ **b** $\frac{2\sqrt{3}}{\sqrt{2}}$ **c** $\frac{1-\sqrt{3}}{\sqrt{2}}$

THINK	**WRITE**
a 1 Multiply the numerator and the denominator by the surd term of the denominator.	**a** $\frac{7}{2\sqrt{5}} = \frac{7}{2\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$ $= \frac{7\sqrt{5}}{2\sqrt{25}}$
2 Simplify the fraction.	$= \frac{7\sqrt{5}}{2 \times 5}$ $= \frac{7\sqrt{5}}{10}$
b 1 Multiply the numerator and the denominator by the surd term of the denominator.	**b** $\frac{2\sqrt{3}}{\sqrt{2}} = \frac{2\sqrt{3}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$ $= \frac{2\sqrt{6}}{\sqrt{4}}$
2 Simplify the fraction.	$= \frac{\cancel{2}^{1}\sqrt{6}}{\cancel{2}^{1}}$ $= \sqrt{6}$
c 1 Multiply the numerator and the denominator by the surd term of the denominator.	**c** $\frac{1-\sqrt{3}}{\sqrt{2}} = \frac{1-\sqrt{3}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$
2 Expand the expression using the distributive law.	$= \frac{(1-\sqrt{3})\sqrt{2}}{\sqrt{4}}$ $= \frac{\sqrt{2}-\sqrt{6}}{\sqrt{4}}$
3 Simplify the fraction.	$= \frac{\sqrt{2}-\sqrt{6}}{2}$

Example 3E.2 Rationalising the denominator of surd fractions and simplifying

Write $\frac{\sqrt{8}}{\sqrt{5}}$ with a rational denominator, in simplest surd form.

THINK	**WRITE**
1 Multiply the numerator and the denominator by the surd term of the denominator.	$\frac{\sqrt{8}}{\sqrt{5}} = \frac{\sqrt{8}}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$ $= \frac{\sqrt{40}}{\sqrt{25}}$
2 Simplify the denominator.	$= \frac{\sqrt{40}}{5}$
3 Simplify the surd in the numerator by writing the number under the root sign as a product of two factors, one of which is a perfect square.	$= \frac{\sqrt{4 \times 10}}{5}$ $= \frac{\sqrt{4} \times \sqrt{10}}{5}$ $= \frac{2\sqrt{10}}{5}$

Example 3E.3 Identifying the conjugate of an expression

State the conjugate for each of these expressions.

a $7 + \sqrt{2}$ **b** $1 - \sqrt{5}$

THINK	WRITE
a Use the 'difference of two squares' rule to identify the conjugate. Multiplying $7 + \sqrt{2}$ by $7 - \sqrt{2}$ gives $49 - 2 = 47$, which is a rational number. So $7 - \sqrt{2}$ is the conjugate of $7 + \sqrt{2}$.	**a** $7 - \sqrt{2}$
b Use the 'difference of two squares' rule to identify the conjugate. Multiplying $1 - \sqrt{5}$ by $1 + \sqrt{5}$ gives $1 - 5 = -4$, which is a rational number. So $1 + \sqrt{5}$ is the conjugate of $1 - \sqrt{5}$.	**b** $1 + \sqrt{5}$

Example 3E.4 Rationalising the denominator using the conjugate

Write $\frac{2}{3 + \sqrt{6}}$ with a rational denominator.

THINK	WRITE
1 Multiply the numerator and the denominator by the conjugate of the denominator. The conjugate of $3 + \sqrt{6}$ is $3 - \sqrt{6}$.	$\frac{2}{3+\sqrt{6}} = \frac{2}{3+\sqrt{6}} \times \frac{3-\sqrt{6}}{3-\sqrt{6}}$ $= \frac{2(3-\sqrt{6})}{(3+\sqrt{6})(3-\sqrt{6})}$
2 Simplify the fraction by first expanding the expression in the numerator and then the expression in the denominator using the 'difference of two squares' rule: $(a + b)(a - b) = a^2 - b^2$.	$= \frac{6 - 2\sqrt{6}}{9 - 6}$
3 Check that your answer has a rational denominator.	$= \frac{6 - 2\sqrt{6}}{3}$

Helpful hints

✔ It can be helpful to work out the binomial multiplication before you multiply the denominator. Then you can easily see if your multiplication will rationalise the denominator.
For example, $(4 + \sqrt{3})(4 - \sqrt{3}) = 16 - 3 = 13$

So then: $\frac{1}{4 + \sqrt{3}} \times \frac{4 - \sqrt{3}}{4 - \sqrt{3}} = \frac{4 - \sqrt{3}}{13}$

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Exercise 3E Rationalising the denominator

△ **1–4, 5–7(1st, 2nd columns), 8, 9(a–c), 10, 11(a–c), 14**

□ **3–7(3rd, 4th columns), 8–10, 12, 13(a, b, e), 14(e, f), 15**

⬠ **5–7(3rd, 4th columns), 8–10, 11(e, f), 12, 13(e, f), 16**

UNDERSTANDING AND FLUENCY

1 **a** Find the answer for each of these multiplication calculations. Write your answers in simplest form.

i $\frac{1}{2} \times 1$ **ii** $\frac{3}{4} \times \frac{1}{1}$ **iii** $\frac{3}{4} \times \frac{4}{4}$ **iv** $\frac{2}{5} \times \frac{5}{5}$

b What did you notice about each of your results in part **a**?

c What is special about the second value in each of the multiplications in this question?

2 **a** Calculate results for these by multiplying the numerators together and multiplying the denominators together.

i $\frac{2}{\sqrt{3}} \times \frac{1}{1}$ **ii** $\frac{2}{\sqrt{3}} \times \frac{3}{3}$ **iii** $\frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$ **iv** $\frac{2}{\sqrt{2}} \times \frac{2}{2}$

b Which result in part **a** is a fraction with a rational denominator?

c What operation was performed so that a rational denominator was obtained?

3E.1 **3** Rewrite each of these fractions with a rational denominator.

a $\frac{3}{\sqrt{6}}$ **b** $\frac{2}{\sqrt{7}}$ **c** $\frac{3}{\sqrt{3}}$ **d** $\frac{4\sqrt{5}}{\sqrt{7}}$

e $\frac{5}{3\sqrt{2}}$ **f** $\frac{7}{4\sqrt{5}}$ **g** $\frac{2\sqrt{3}}{\sqrt{5}}$ **h** $-\frac{5}{2\sqrt{5}}$

i $\frac{1}{7\sqrt{7}}$ **j** $\frac{4\sqrt{11}}{\sqrt{10}}$ **k** $\frac{8\sqrt{5}}{3\sqrt{2}}$ **l** $\frac{9\sqrt{6}}{4\sqrt{5}}$

3E.2 **4** Rewrite each of these fractions with a rational denominator. Where appropriate, write the numerator in simplest surd form.

a $\frac{\sqrt{6}}{\sqrt{3}}$ **b** $\frac{\sqrt{10}}{\sqrt{5}}$ **c** $\frac{2\sqrt{10}}{\sqrt{6}}$ **d** $\frac{3\sqrt{8}}{\sqrt{3}}$

e $\frac{\sqrt{15}}{3\sqrt{5}}$ **f** $\frac{5\sqrt{10}}{7\sqrt{2}}$ **g** $\frac{2\sqrt{12}}{\sqrt{5}}$ **h** $\frac{\sqrt{14}}{4\sqrt{7}}$

i $\frac{\sqrt{20}}{2\sqrt{2}}$ **j** $\frac{7\sqrt{15}}{6\sqrt{3}}$ **k** $\frac{\sqrt{2}}{\sqrt{18}}$ **l** $\frac{6\sqrt{12}}{5\sqrt{8}}$

5 Rewrite each of these fractions with a rational denominator. Where appropriate, write the numerator in simplest surd form.

a $\frac{-\sqrt{6}}{\sqrt{2}}$ **b** $\frac{-\sqrt{15}}{\sqrt{6}}$ **c** $\frac{2\sqrt{8}}{3\sqrt{7}}$ **d** $\frac{2\sqrt{6}}{\sqrt{8}}$

e $\frac{6\sqrt{15}}{7\sqrt{3}}$ **f** $\frac{3\sqrt{20}}{\sqrt{5}}$ **g** $\frac{5}{\sqrt{x}}$ **h** $\frac{x}{x\sqrt{y}}$

i $\frac{4x}{\sqrt{2y}}$ **j** $\frac{\sqrt{x}}{x\sqrt{y}}$ **k** $\frac{\sqrt{2y}}{2y\sqrt{x}}$ **l** $\frac{-3\sqrt{x}}{6x\sqrt{yz}}$

3E.3 **6** State the conjugate for each of these expressions.

a $3 + \sqrt{2}$ **b** $2 - \sqrt{3}$ **c** $-\sqrt{6} + 5$ **d** $-4 + 3\sqrt{5}$

e $3\sqrt{5} - 2$ **f** $-4\sqrt{7} - 1$ **g** $1 + 2\sqrt{3}$ **h** $\sqrt{2} + \sqrt{3}$

i $5 - \sqrt{3}$ **j** $-5\sqrt{6} + 2$ **k** $3\sqrt{7} - 2$ **l** $-5 - 2\sqrt{5}$

3E.4 **7** Rewrite each of these fractions with a rational denominator.

a $\frac{1}{2 + \sqrt{3}}$ **b** $\frac{3}{3 - \sqrt{2}}$ **c** $\frac{5}{6 + \sqrt{5}}$ **d** $\frac{3}{3 - 4\sqrt{5}}$

e $\frac{\sqrt{2}}{5 + 3\sqrt{2}}$ **f** $\frac{2\sqrt{3}}{1 + 2\sqrt{3}}$ **g** $\frac{3}{\sqrt{5} - 1}$ **h** $\frac{3\sqrt{7}}{2\sqrt{6} + 5}$

i $\frac{-4\sqrt{2}}{\sqrt{3} - 2}$ **j** $\frac{\sqrt{3}}{\sqrt{5} + 2}$ **k** $\frac{2\sqrt{5}}{3\sqrt{6} + 10}$ **l** $\frac{6\sqrt{10}}{5\sqrt{2} - 5}$

8 Rewrite each of these fractions with a rational denominator.

a $\frac{\sqrt{2}+1}{2\sqrt{3}+3}$ **b** $\frac{5+2\sqrt{3}}{4+\sqrt{2}}$

c $\frac{1+\sqrt{3}}{3-\sqrt{5}}$ **d** $\frac{3+\sqrt{5}}{\sqrt{6}+1}$

e $\frac{2\sqrt{3}-2}{3\sqrt{5}+2}$ **f** $\frac{5\sqrt{2}-1}{3+2\sqrt{5}}$

9 Write each of these fractions in simplest form, with a rational denominator.

a $\frac{3\sqrt{2}-4}{4+3\sqrt{2}}$ **b** $\frac{7-\sqrt{5}}{\sqrt{5}-7}$ **c** $\frac{x}{x-\sqrt{y}}$

d $\frac{\sqrt{x}}{\sqrt{x}-y}$ **e** $\frac{x+y}{x\sqrt{x}+y}$ **f** $\frac{x-\sqrt{y}}{x+\sqrt{y}}$

10 You can perform addition and subtraction with surd fractions in a way that is similar to how you do it with common fractions.

Consider $\frac{3}{\sqrt{2}}+\frac{1}{2}$.

a First write the fractions with a common denominator and add the two fractions. Then write your answer with a rational denominator.

b Repeat the addition, but this time rationalise the denominator of the first fraction *before* performing the addition.

11 Calculate the value of each of these expressions.

a $\frac{3}{\sqrt{5}}+\frac{1}{5}$ **b** $\frac{4}{\sqrt{3}}-\frac{1}{3}$

c $\frac{3\sqrt{2}}{2}+\frac{2}{\sqrt{3}}$ **d** $\frac{1}{4\sqrt{3}}+\frac{3}{2\sqrt{2}}$

e $\frac{5}{\sqrt{5}}-\frac{2}{3\sqrt{6}}$ **f** $\frac{2\sqrt{3}}{5\sqrt{2}}+\frac{\sqrt{5}}{4\sqrt{3}}$

12 Now consider the expression $\frac{1}{\sqrt{5}-1}+\frac{3}{\sqrt{5}+1}$.

a Write each fraction in the expression as an equivalent fraction with a rational denominator.

b Add your two fraction answers from part **a**, writing your answer in simplest form.

13 Evaluate each of these expressions after first expressing each fraction with a rational denominator.

a $\frac{2}{\sqrt{6}-2}+\frac{2}{\sqrt{6}+2}$ **b** $\frac{2}{\sqrt{5}+2}-\frac{2}{\sqrt{5}-2}$

c $\frac{2}{\sqrt{6}-1}+\frac{2}{\sqrt{3}+1}$ **d** $\frac{5}{\sqrt{2}+1}-\frac{4}{\sqrt{5}-2}$

e $\frac{2\sqrt{3}}{2\sqrt{2}+5}+\frac{\sqrt{3}}{3\sqrt{2}+2}$ **f** $\frac{\sqrt{3}+1}{\sqrt{3}+4}-\frac{\sqrt{3}-1}{\sqrt{3}+1}$

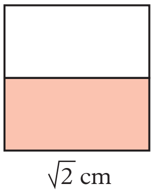
14 Write the reciprocal of each of the following. Write your answers with rational denominators and simplified surds.

a $\sqrt{2}$ **b** $5\sqrt{3}$ **c** $\frac{\sqrt{28}}{7}$

d $\frac{\sqrt{27}}{4}$ **e** $\sqrt{6}+2$ **f** $\frac{3+\sqrt{5}}{3-\sqrt{5}}$

PROBLEM SOLVING AND REASONING

15 Consider the square on the right, which has been divided into congruent rectangles.



a State the area of the whole square.

b State the area of the shaded rectangle.

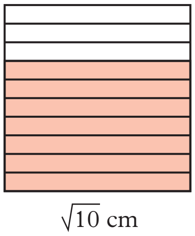
c The width of a rectangle can be determined by dividing its area by its length. Determine the width of the shaded rectangle. Leave your answer with an irrational denominator.

d The width of the shaded rectangle can also be determined by considering the fraction that the width of the shaded rectangle is of the square's side length. State what fraction the shaded rectangle's width is of the square's side length.

e State the width of the shaded rectangle in the form $\frac{a}{b}\sqrt{c}$, where a, b and c are integers.

f Describe the connection between the answers from parts **c** and **e** referring to 'rationalising the denominator'.

g Use a similar method to determine the width of the shaded region in the square on the right, with both a rational and irrational denominator.



CHALLENGE

16 When an expression contains more than one surd, the expression has two conjugates resulting from changing the sign on one of the surd terms. For example, the conjugates of $\sqrt{a} + \sqrt{b}$ are $\sqrt{a} - \sqrt{b}$ and $-\sqrt{a} + \sqrt{b}$. The conjugates of $-\sqrt{6} - \sqrt{5}$ are $\sqrt{6} - \sqrt{5}$ and $-\sqrt{6} + \sqrt{5}$.

a Write all of the conjugates of the following.

i $\sqrt{2} + \sqrt{3}$ **ii** $\sqrt{5} - \sqrt{3}$ **iii** $-2\sqrt{5} - 5\sqrt{6}$ **iv** $-2\sqrt{5} + 3\sqrt{7}$

b Evaluate the following products.

i $(\sqrt{2} + \sqrt{3})(\sqrt{2} - \sqrt{3})$ **ii** $(\sqrt{2} + \sqrt{3})(-\sqrt{2} + \sqrt{3})$ **iii** $(\sqrt{2} + \sqrt{3})(-\sqrt{2} - \sqrt{3})$

iv $(-\sqrt{2} + \sqrt{3})(\sqrt{2} - \sqrt{3})$ **v** $(\sqrt{2} - \sqrt{3})(-\sqrt{2} - \sqrt{3})$

c We can rationalise a denominator involving multiple surd terms by making use of the 'difference of two squares' rule. However, it is important that suitable conjugates are multiplied to result in a rational expression.

Write the two conjugates which could be used to rationalise the denominator of each of the following.

i $\dfrac{1}{\sqrt{5} - \sqrt{3}}$ **ii** $\dfrac{1}{2\sqrt{2} + 5\sqrt{5}}$ **iii** $\dfrac{1}{-\sqrt{11} + 3\sqrt{2}}$ **iv** $\dfrac{1}{-5\sqrt{13} - 4\sqrt{3}}$

d Use the conjugates from part **c** to rationalise the denominator for each fraction in part **c**.

e Lucy was rationalising the denominator of $\dfrac{4}{-\sqrt{2} + \sqrt{3}}$. Her working is shown.

$$\frac{4}{-\sqrt{2} + \sqrt{3}} \times \frac{\sqrt{2} - \sqrt{3}}{\sqrt{2} - \sqrt{3}} = \frac{4(\sqrt{2} - \sqrt{3})}{-2 + \sqrt{6} + \sqrt{6} - 3}$$
$$= \frac{4\sqrt{2} - 4\sqrt{3}}{2\sqrt{6} - 5}$$

i Explain why Lucy did not get a rational denominator.

ii Explain what she could do next to her last line to rationalise the denominator.

f Rationalise the denominator of each of the following fractions.

i $\dfrac{2}{\sqrt{3} + \sqrt{5}}$ **ii** $\dfrac{-4\sqrt{2}}{\sqrt{3} - \sqrt{2}}$ **iii** $\dfrac{\sqrt{3}}{\sqrt{5} + \sqrt{2}}$ **iv** $\dfrac{2\sqrt{5}}{3\sqrt{6} + 2\sqrt{10}}$

v $\dfrac{6\sqrt{10}}{5\sqrt{2} - 2\sqrt{5}}$ **vi** $\dfrac{5\sqrt{2} - 1}{3\sqrt{3} + 2\sqrt{5}}$ **vii** $\dfrac{\sqrt{7} - \sqrt{5}}{\sqrt{5} - \sqrt{7}}$ **viii** $\dfrac{x + y}{x\sqrt{x} + \sqrt{y}}$

g Evaluate $(\sqrt{2} + \sqrt{3} + \sqrt{5})(\sqrt{2} + \sqrt{3} - \sqrt{5})$.

h Write $\dfrac{4}{\sqrt{2} + \sqrt{3} + \sqrt{5}}$ with a rational denominator.

Check your Student obook pro for these digital resources and more:

pro

Interactive skillsheet
Rationalising the denominator

Interactive skillsheet
Conjugate of a binomial expression

Interactive skillsheet
Rationalising the denominator using the conjugate surd

Topic quiz
3E

Checkpoint

Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this chapter.

10A **3A** **1** State if the following numbers are rational or irrational.

a $\frac{4}{3}$ **b** -12.01 **c** $19.279\,102\,865\,725\ldots$ **d** $-2\frac{5}{7}$ **e** π **f** $\sqrt{2}$

10A **3A** **2** State if the following roots are surds or are not surds.

a $\sqrt{15}$ **b** $\sqrt{16}$ **c** $\sqrt{100}$ **d** $\sqrt[3]{8}$ **e** $\sqrt{160}$ **f** $\sqrt{256}$

10A **3B** **3** Simplify the following surds.

a $\sqrt{8}$ **b** $\sqrt{45}$ **c** $\sqrt{175}$ **d** $\sqrt{600}$

10A **3B** **4** Simplify the following surds.

a $5\sqrt{12}$ **b** $7\sqrt{54}$ **c** $25\sqrt{72}$ **d** $5\sqrt{1280}$

10A **3C** **5** Evaluate the following products.

a $-\sqrt{5} \times -\sqrt{7}$ **b** $\sqrt{14} \times -\sqrt{22}$ **c** $-6\sqrt{3} \times 5\sqrt{6}$ **d** $\sqrt{18} \times \sqrt{112}$

10A **3C** **6** Evaluate the following quotients.

a $\dfrac{-\sqrt{60}}{-\sqrt{10}}$ **b** $\dfrac{-8\sqrt{34}}{12\sqrt{2}}$

c $\dfrac{\sqrt{864}}{-\sqrt{384}}$ **d** $\dfrac{9\sqrt{525}}{3\sqrt{252}}$

e $\dfrac{-5\sqrt{72}}{6\sqrt{15}}$ **f** $\dfrac{12\sqrt{338}}{13\sqrt{148}}$

10A **3C** **7** Evaluate the following.

a $\dfrac{\sqrt{54} \times \sqrt{12}}{\sqrt{18}}$ **b** $\dfrac{8\sqrt{5} \times 5\sqrt{6}}{10\sqrt{3} \times 6\sqrt{15}}$

10A **3D** **8** Evaluate the following sums and differences.

a $3\sqrt{2} + 9\sqrt{2}$ **b** $3\sqrt{2} - 9\sqrt{2}$

c $5\sqrt{3} + 3\sqrt{5} + 15\sqrt{5} + 7\sqrt{3}$ **d** $21\sqrt{7} - 15\sqrt{11} - 23\sqrt{7} + 32\sqrt{11}$

10A **3D** **9** Evaluate the following sums and differences.

a $\sqrt{8} + \sqrt{128} + \sqrt{18}$ **b** $\sqrt{75} - \sqrt{27}$

c $\sqrt{150} + \sqrt{108} + \sqrt{54} + \sqrt{192}$ **d** $\sqrt{4000} - \sqrt{242} - \sqrt{90} - \sqrt{72}$

10A **3D** **10** Evaluate the following.

a $\dfrac{5\sqrt{6} + 13\sqrt{6}}{3\sqrt{2}}$ **b** $3\sqrt{3}(6\sqrt{2} - 2\sqrt{6})$

c $(\sqrt{5} - 2)(3 - \sqrt{7})$ **d** $(\sqrt{8} + \sqrt{27})(\sqrt{125} + \sqrt{48})$

e $(7\sqrt{12} + 6\sqrt{5})(5\sqrt{6} - 3\sqrt{40})$ **f** $(4\sqrt{32} + 2\sqrt{7})(5\sqrt{22} - \sqrt{72})$

10A **3E** **11** Rationalise the denominators of the following fractions.

a $\dfrac{5}{\sqrt{2}}$ **b** $\dfrac{12}{\sqrt{3}}$ **c** $\dfrac{\sqrt{20}}{\sqrt{7}}$ **d** $\dfrac{\sqrt{55}}{\sqrt{30}}$

10A **3E** **12** Rationalise the denominators of the following fractions.

a $\dfrac{5}{1 + \sqrt{2}}$ **b** $\dfrac{\sqrt{3}}{3 - \sqrt{3}}$

c $\dfrac{2\sqrt{5}}{3\sqrt{5} - 2}$ **d** $\dfrac{2 + \sqrt{7}}{-\sqrt{63} + 7}$

e $\dfrac{4 - \sqrt{5}}{-13 + \sqrt{5}}$ **f** $\dfrac{\sqrt{10} + 2\sqrt{3}}{2 - \sqrt{128}}$

3F Fractional indices

Learning intentions

✔ I can apply the index laws to fractional indices.

Inter-year links

Year 8	4B Multiplying and dividing numbers with the same base
Year 9	2B Index laws 1 and 2

Fractional indices

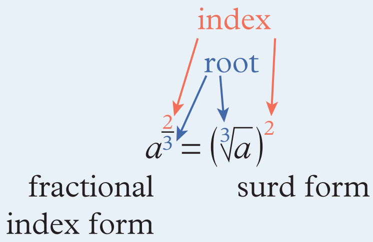
- Fractional indices are an alternate notation for expressing roots and indices together.

$$\text{base} \longrightarrow a^{\frac{1}{3}} \longleftarrow \text{index/exponent/power}$$

index form

→ The numerator of the fractional index is the number the base is raised to.

→ The denominator of the fractional index refers to the type of root applied to the base. Numbers raised to a fractional index also have an equivalent surd form:



$$a^{\frac{2}{3}} = \left(\sqrt[3]{a}\right)^2$$

- Expressions that contain fractional indices obey the usual index laws.

	Integer indices	Fractional indices
Zero index law	$a^0 = 1$	$a^0 = 1$
Index law 1	$a^5 \times a^3 = a^{5+3}$	$a^{\frac{1}{5}} \times a^{\frac{1}{3}} = a^{\frac{1}{5}+\frac{1}{3}}$
Index law 2	$a^5 \div a^3 = a^{5-3}$	$a^{\frac{1}{5}} \div a^{\frac{1}{3}} = a^{\frac{1}{5}-\frac{1}{3}}$
Index law 3	$\left(a^5\right)^3 = a^{5\times3}$ $(ab)^3 = a^3b^3$ $\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$	$\left(a^{\frac{1}{5}}\right)^{\frac{1}{3}} = a^{\frac{1}{5}\times\frac{1}{3}}$ $(ab)^{\frac{1}{3}} = a^{\frac{1}{3}}b^{\frac{1}{3}}$ $\left(\frac{a}{b}\right)^{\frac{1}{3}} = \frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}}$

- **Zero index law:** Excluding 0, any number or variable with an index of 0 is equal to 1.
- **Index law 1:** When multiplying numbers or variables in index form that have the same base, add the indices.
- **Index law 2:** When dividing numbers or variables in index form that have the same base, subtract the second index from the first index.
- **Index law 3:** When raising a number or variable to two indices, multiply the indices. Every base number or variable inside brackets should have its index multiplied by the index outside the brackets.

Example 3F.1 Applying the index laws to fractional indices

Simplify each of these expressions.

a $x^{\frac{1}{4}} \times x^{\frac{2}{3}}$

b $\dfrac{x^{\frac{3}{5}}}{x^{\frac{1}{3}}}$

c $\left(x^{\frac{1}{3}}\right)^{\frac{1}{2}}$

THINK	WRITE
a Add the indices of the common base, *x*. To add the fraction indices, convert them to equivalent fractions with a common denominator.	**a** $x^{\frac{1}{4}} \times x^{\frac{2}{3}} = x^{\frac{1}{4}+\frac{2}{3}}$ $= x^{\frac{3}{12}+\frac{8}{12}}$ $= x^{\frac{11}{12}}$
b Subtract the indices of the common base, *x*. To subtract the fraction indices, convert them to equivalent fractions with a common denominator.	**b** $\dfrac{x^{\frac{3}{5}}}{x^{\frac{1}{3}}} = x^{\frac{3}{5}-\frac{1}{3}}$ $= x^{\frac{9}{15}-\frac{5}{15}}$ $= x^{\frac{4}{15}}$
c Multiply the index of every base inside the brackets by the index outside the brackets.	**c** $\left(x^{\frac{1}{3}}\right)^{\frac{1}{2}} = x^{\frac{1}{3}\times\frac{1}{2}}$ $= x^{\frac{1}{6}}$

Example 3F.2 Raising a term to a fractional index

Simplify $4\left(x^{\frac{2}{3}}y^{3}\right)^{\frac{5}{6}}$

THINK	WRITE
1 Raise each term inside the brackets to the index outside the brackets.	$4\left(x^{\frac{2}{3}}y^{3}\right)^{\frac{5}{6}} = 4\left(x^{\frac{2}{3}}\right)^{\frac{5}{6}}\left(y^{3}\right)^{\frac{5}{6}}$
2 Multiply the index of every base inside the brackets by the index outside the brackets.	$= 4 \times x^{\frac{2}{3}\times\frac{5}{6}} \times y^{3\times\frac{5}{6}}$
3 Simplify the fractional indices.	$= 4 \times x^{\frac{10}{18}} \times y^{\frac{15}{6}}$ $= 4x^{\frac{5}{9}}y^{\frac{5}{2}}$

Example 3F.3 Multiplying terms with fractional indices

Simplify $5x^{\frac{1}{2}}y^{\frac{1}{3}} \times 2x^{\frac{1}{5}}y^{\frac{3}{5}}$.

THINK	WRITE
1 Multiply the coefficients and multiply the terms with the same bases. Add the indices of the common bases, *x* and *y*.	$5x^{\frac{1}{2}}y^{\frac{1}{3}} \times 2x^{\frac{1}{5}}y^{\frac{3}{5}} = 5 \times 2 \times x^{\frac{1}{2}} \times x^{\frac{1}{5}} \times y^{\frac{1}{3}} \times y^{\frac{3}{5}}$ $= 10 \times x^{\frac{1}{2}+\frac{1}{5}} \times y^{\frac{1}{3}+\frac{3}{5}}$
2 Simplify the index terms with the same base.	$= 10 \times x^{\frac{7}{10}} \times y^{\frac{14}{15}}$ $= 10x^{\frac{7}{10}}y^{\frac{14}{15}}$

Helpful hints

- ✔ Take care when writing fractional indices. They should be smaller than the base and sit high up on the shoulder of the base to avoid confusion between the base and the index. For example, $x^{\frac{1}{3}}$ not $x\frac{1}{3}$.
- ✔ Take care not to mix up the index laws.
  - → When multiplying numbers with the same base, add the indices.
  - → When dividing numbers with the same base, subtract the indices.
  - → If a number raised to a power is enclosed in brackets and then raised to another power, multiply the indices.
- ✔ Keep the index laws and the rules for negative indices written somewhere close until they become second nature to you.

$$a^{-1} = \frac{1}{a} \qquad a^{-m} = \frac{1}{a^m} \qquad \frac{1}{a^{-m}} = a^m$$

ANS p666

Exercise 3F Fractional indices

△ 1–9, 11, 12, 13(a–e), 14 ☐ 1–4(e–h), 5–9, 11, 12, 13(g–l), 15, 18, 19, 21 ⬠ 5, 6, 10, 11, 15–17, 19–22

UNDERSTANDING AND FLUENCY

3F.1 **1** Use the appropriate index law to simplify each of these expressions.

a $x^{\frac{1}{5}} \times x^{\frac{2}{5}}$ **b** $x^{\frac{3}{10}} \times x^{\frac{1}{5}}$ **c** $x^{\frac{3}{4}} \times x^{\frac{1}{8}}$ **d** $x^{\frac{2}{5}} \times x^{\frac{1}{6}}$

e $x^{\frac{5}{6}} \times x^{\frac{1}{6}}$ **f** $x^{\frac{2}{3}} \times x^{\frac{1}{4}}$ **g** $x^{\frac{1}{4}} \times x^{\frac{1}{4}}$ **h** $x^{\frac{1}{3}} \times x^{\frac{1}{6}}$

2 Use the appropriate index law to simplify each of these expressions.

a $x^{\frac{13}{11}} \div x^{\frac{9}{11}}$ **b** $x^{\frac{3}{8}} \div x^{\frac{1}{8}}$ **c** $\dfrac{x^{\frac{1}{2}}}{x^{\frac{3}{7}}}$ **d** $\dfrac{x^{\frac{4}{5}}}{x^{\frac{3}{4}}}$

e $\dfrac{x^{\frac{1}{3}}}{x^{\frac{1}{6}}}$ **f** $x^{\frac{2}{5}} \div x^{\frac{3}{10}}$ **g** $\dfrac{x^{\frac{9}{4}}}{x^{\frac{1}{2}}}$ **h** $x^{\frac{6}{7}} \div x^{\frac{6}{7}}$

3 Use the appropriate index law to simplify each of these expressions.

a $\left(x^{\frac{3}{7}}\right)^{\frac{5}{4}}$ **b** $\left(x^{\frac{1}{3}}\right)^{\frac{3}{5}}$ **c** $\left(x^{\frac{3}{7}}\right)^{0}$ **d** $\left(x^{\frac{2}{5}}\right)^{\frac{3}{4}}$

e $\left(x^{\frac{1}{8}}\right)^{\frac{4}{9}}$ **f** $\left(x^{\frac{2}{3}}\right)^{\frac{3}{4}}$ **g** $\left(x^{\frac{27}{2}}\right)^{\frac{8}{9}}$ **h** $\left(x^{\frac{8}{45}}\right)^{\frac{9}{56}}$

4 Simplify each of these expressions.

a $5x^{\frac{1}{3}} \times 2x^{\frac{1}{4}}$ **b** $6x^{\frac{2}{5}} \times 5x^{\frac{3}{8}}$ **c** $9x^{\frac{3}{7}} \times -4x^{\frac{2}{9}}$ **d** $4x^{\frac{1}{4}} \div x^{\frac{1}{6}}$

e $12x^{\frac{3}{5}} \div 6x^{\frac{3}{8}}$ **f** $\dfrac{32x^{\frac{5}{6}}}{8x^{\frac{4}{5}}}$ **g** $3x^{\frac{1}{2}} \times 2x^{\frac{1}{3}}$ **h** $\dfrac{x^{\frac{3}{5}}}{4x^{\frac{1}{2}}}$

3F.2 **5** Simplify each of these expressions.

a $\left(5x^{\frac{1}{3}}\right)^{\frac{3}{5}}$ **b** $\left(16x^{\frac{2}{5}}\right)^{\frac{1}{4}}$ **c** $\left(27x^{\frac{3}{8}}\right)^{\frac{2}{3}}$ **d** $2\left(x^{\frac{2}{5}}y^{\frac{3}{4}}\right)^{\frac{1}{2}}$

e $4\left(x^{\frac{1}{5}}y^{\frac{3}{7}}\right)^{\frac{2}{3}}$ **f** $5\left(x^{\frac{2}{3}}y^{\frac{4}{5}}\right)^{\frac{3}{8}}$ **g** $5\left(x^{2}y^{\frac{3}{5}}\right)^{\frac{1}{2}}$ **h** $7\left(x^{\frac{1}{5}}y^{\frac{4}{11}}\right)^{\frac{5}{3}}$

3F.3 **6** Simplify each of these expressions.

a $4x^{\frac{1}{4}}y^{\frac{1}{5}} \times 7x^{\frac{1}{3}}y^{\frac{3}{4}}$ **b** $-6x^{\frac{2}{3}}y^{\frac{4}{7}} \times 12x^{\frac{5}{6}}y^{\frac{3}{7}}$ **c** $9x^{\frac{1}{6}}y^{\frac{2}{5}} \times 5x^{\frac{5}{8}}y$

d $25x^{\frac{3}{5}}y^{\frac{1}{4}} \div 5x^{\frac{1}{5}}y^{\frac{1}{12}}$ **e** $12x^{\frac{3}{8}}y^{\frac{5}{11}} \div 6x^{\frac{1}{5}}y^{\frac{1}{3}}$ **f** $\dfrac{7x^{\frac{1}{2}}y^{\frac{3}{7}}}{7x^{\frac{3}{7}}y^{\frac{2}{5}}}$

7 Consider the numerical expression $3^{\frac{1}{2}} \times 3^{\frac{1}{2}}$.

a Which one of the index laws would you use to simplify it?

b Use the index law you chose to simplify the expression. Then check your answer with a calculator.

c Now consider the surd expression $\sqrt{3} \times \sqrt{3}$. Multiply the two surds and write your answer in simplest form.

d Compare your answers for parts **b** and **c**. What do you notice?

e Repeat parts **b–d** for each of these pairs of expressions.

i $2^{\frac{1}{2}} \times 2^{\frac{1}{2}}$ and $\sqrt{2} \times \sqrt{2}$
ii $5^{\frac{1}{2}} \times 5^{\frac{1}{2}}$ and $\sqrt{5} \times \sqrt{5}$
iii $10^{\frac{1}{2}} \times 10^{\frac{1}{2}}$ and $\sqrt{10} \times \sqrt{10}$
iv $x^{\frac{1}{2}} \times x^{\frac{1}{2}}$ and $\sqrt{x} \times \sqrt{x}$

f Using your results from part **e**, explain what you notice about the relationship between the fractional index of $\frac{1}{2}$ and the square root sign.

8 Now consider the numerical expression $3^{\frac{1}{3}} \times 3^{\frac{1}{3}} \times 3^{\frac{1}{3}}$.

a Use the appropriate index law to simplify the expression. Then check your answer with a calculator.

b Calculate $\sqrt[3]{3} \times \sqrt[3]{3} \times \sqrt[3]{3}$.

c Compare your answers for parts **a** and **b**. What do you notice?

d Repeat parts **a–c** for each of these pairs of expressions.

i $2^{\frac{1}{3}} \times 2^{\frac{1}{3}} \times 2^{\frac{1}{3}}$ and $\sqrt[3]{2} \times \sqrt[3]{2} \times \sqrt[3]{2}$
ii $5^{\frac{1}{3}} \times 5^{\frac{1}{3}} \times 5^{\frac{1}{3}}$ and $\sqrt[3]{5} \times \sqrt[3]{5} \times \sqrt[3]{5}$
iii $10^{\frac{1}{3}} \times 10^{\frac{1}{3}} \times 10^{\frac{1}{3}}$ and $\sqrt[3]{10} \times \sqrt[3]{10} \times \sqrt[3]{10}$
iv $x^{\frac{1}{3}} \times x^{\frac{1}{3}} \times x^{\frac{1}{3}}$ and $\sqrt[3]{x} \times \sqrt[3]{x} \times \sqrt[3]{x}$

e What are the differences between the surd expression in question **8b** and the surd expression in question **7c**?

f Use your results to explain what you notice about the relationship between the fractional index of $\frac{1}{3}$ and the cube root sign.

9 The fractional indices you have seen so far and their corresponding surd forms have involved fractions with a numerator of 1. Now consider the term $x^{\frac{2}{3}}$.

a Show that this term has the equivalent forms of $\left(x^2\right)^{\frac{1}{3}}$ and $\left(x^{\frac{1}{3}}\right)^2$.

b Show that both expressions in part **a** are equivalent to $\sqrt[3]{x^2}$ and $(\sqrt[3]{x})^2$.

c Use a calculator to substitute each of the values below for x in $x^{\frac{2}{3}}$. Check that it produces the same results as substituting those values into each of the expressions from parts **a** and **b**.

i $x = 5$
ii $x = 2$
iii $x = 3$
iv $x = 10$
v $x = 12$
vi $x = 27$

10 Show that any fractional index of the form $\frac{m}{n}$ is equivalent to $\sqrt[n]{(\ )^m}$ or $\left(\sqrt[n]{\ }\right)^m$.

That is, show that $x^{\frac{m}{n}}$ is equivalent to $\sqrt[n]{x^m}$ or $(\sqrt[n]{x})^m$.

11 Calculate each of these values, without using a calculator.

a $25^{\frac{1}{2}}$
b $36^{\frac{1}{2}}$
c $1^{\frac{1}{2}}$
d $144^{\frac{1}{2}}$
e $100^{\frac{1}{2}}$
f $196^{\frac{1}{2}}$
g $8^{\frac{1}{3}}$
h $27^{\frac{1}{3}}$
i $125^{\frac{1}{3}}$
j $1^{\frac{1}{3}}$
k $1^{\frac{1}{5}}$
l $32^{\frac{1}{5}}$

12 Look at each part of question **11**.

i If each fractional index was negative, briefly explain how the answers would change.

ii What would the answer for each part of question 11 be if the indices were negative?

iii Which parts can have a negative base? Explain why.

iv What would the value of each of the parts you identified in part **iii** be if the bases were negative?

13 Write each of these values in simplest surd form.

a $5^{\frac{1}{2}}$
b $8^{\frac{1}{2}}$
c $24^{\frac{1}{2}}$
d $108^{\frac{1}{2}}$
e $152^{\frac{1}{2}}$
f $54^{\frac{1}{3}}$
g $128^{\frac{1}{3}}$
h $320^{\frac{1}{2}}$
i $243^{\frac{1}{2}}$
j $432^{\frac{1}{2}}$
k $5^{\frac{3}{2}}$
l $6^{\frac{3}{2}}$

14 Write each of these expressions using fractional indices.

a $\sqrt{x^3}$
b $\sqrt{6}$
c $\sqrt{x^3 y^5}$
d $\sqrt{3xy^3}$
e $\sqrt{16x^{-1}y}$
f $\sqrt[3]{x^4 y^2}$

PROBLEM SOLVING AND REASONING

15 Simplify each of these expressions, writing your answers with fractional indices.

a $12x^{\frac{1}{4}}y^{\frac{1}{5}} \times 9x^{\frac{1}{5}}y^{\frac{2}{3}}$ **b** $\dfrac{x^{\frac{1}{3}}y^{\frac{3}{4}}}{5x^{\frac{1}{9}}y^{\frac{2}{5}}}$ **c** $\left(4x^{\frac{2}{5}}y^{\frac{1}{2}}\right)^{\frac{1}{2}}$ **d** $\sqrt{4x^3y} \times 3\sqrt{x^5y^3}$

e $3\sqrt{x^4y^3} \div \sqrt{9xy^2}$ **f** $(\sqrt{x^3y^5})^{\frac{1}{3}}$ **g** $\sqrt{8x^3yz^5} \times 2\sqrt{3xz^3}$ **h** $\left(\sqrt[3]{x^6y^{\frac{1}{3}}z}\right)^2$

16 Write each of these expressions in simplest form.

a $\sqrt{16x^2y}$ **b** $\sqrt{12x^{-3}y^7}$ **c** $\sqrt{\dfrac{72x^4y^3}{24xy}}$

d $\sqrt[3]{27xy^2}$ **e** $\sqrt[4]{16x^3y^2z}$ **f** $\sqrt{343x^{-1}y^5}$

17 Simplify each of these expressions, writing your answers with positive indices where appropriate.

a $3x^{-\frac{1}{3}}y^{\frac{2}{3}} \times 6x^{\frac{3}{4}}y^{-\frac{1}{5}}$ **b** $12x^{\frac{1}{4}}y^{\frac{1}{3}} \times 6x^{-\frac{2}{3}}y^{-\frac{1}{4}}$ **c** $\dfrac{xy^2}{x^{\frac{1}{3}}y^{\frac{5}{2}}}$ **d** $\dfrac{3x^{\frac{1}{3}}y^{\frac{2}{5}}}{12x^{\frac{2}{5}}y^{\frac{3}{4}}}$

e $\left(\dfrac{x^{-\frac{1}{3}}}{y^3}\right)^{\frac{1}{4}}$ **f** $\left(\dfrac{2x^{-\frac{2}{5}}y^{\frac{3}{4}}}{y^2}\right)^4$ **g** $\dfrac{5x^{-\frac{1}{2}} \times 4x^{\frac{1}{3}}}{2x^{-\frac{2}{5}}}$ **h** $\sqrt{\dfrac{x^{-\frac{1}{3}}y^{\frac{2}{3}}}{3x^{-\frac{1}{4}}y^{-\frac{1}{5}}}}$

18 Consider the expression $\dfrac{3^{\frac{2}{3}} \times 9^{-\frac{1}{2}}}{81^{\frac{1}{4}}}$.

a Use an appropriate combination of the index laws to simplify the expression.

b Use your working and your answer from part **a** to help you solve this equation for x:

$$9^x = \frac{3^{\frac{2}{3}} \times 9^{-\frac{1}{2}}}{81^{\frac{1}{4}}}$$

19 Solve each of these equations for x.

a $25^{x+2} = \dfrac{1}{5}$ **b** $27^{x-3} = 9^{x+1}$ **c** $8^x = \dfrac{16^{-\frac{1}{4}} \times 2^{\frac{1}{2}}}{64^{\frac{1}{6}}}$

CHALLENGE

20 Consider the diagram on the right.

a Refer to the diagram to compare multiplying by a fraction and raising a number to a fractional index. Describe what is similar and what is different.

b Evaluate each of the following.

i $81^{\frac{1}{4}}$ **ii** $81^{\frac{3}{4}}$ **iii** $81^{\frac{5}{4}}$

iv $343^{\frac{7}{3}}$ **v** $81^{\frac{5}{2}}$ **vi** $256^{\frac{3}{8}}$

c Write the values appearing in the questions in part **b** in surd form.

d Describe how the denominator of an index and the root of a surd are related, as well as how the numerator of an index and the index of a surd are related. Your descriptions refer to the number of repeated factors the base has.

$$6 = \boxed{2 + 2 + 2}$$
$$\frac{1}{3} \times 6 = \boxed{2}$$
$$\frac{2}{3} \times 6 = \boxed{2 + 2}$$
$$\frac{4}{3} \times 6 = \boxed{2 + 2 + 2} + \boxed{2}$$

$$8 = \boxed{2 \times 2 \times 2}$$
$$8^{\frac{1}{3}} = \boxed{2}$$
$$8^{\frac{2}{3}} = \boxed{2 \times 2}$$
$$8^{\frac{4}{3}} = \boxed{2 \times 2 \times 2} \times \boxed{2}$$

21 a Use indices to show that $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$, where a and b are positive integers.

b Use indices to show that $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$, where a and b are positive integers.

22 Write each of the following in the form $\sqrt[a]{x^b}$, where a and b are positive integers.

a $10^{2.5}$ **b** $2^{4.25}$ **c** $3^{0.16}$ **d** $5^{5.555}$

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Fractional indices

Topic quiz
3F

3G Logarithms

Learning intentions

- ✔ I can write equivalent expressions for logarithms.
- ✔ I can evaluate logarithms and understand them as a numerical value.
- ✔ I can solve simple logarithmic equations.

Inter-year links

Year 8	4A Indices
Year 9	2A Indices

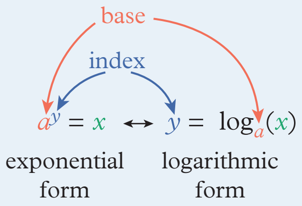
Logarithms and exponentials

- Any equation in which the pronumeral appears as an index is called an exponential function.

  For example, $2^y = 8$ is an exponential equation with the solution $y = 3$.
- A **logarithm** is a function that asks, 'What index applied to a gives x?' It is the number to which the base is raised to produce a specific number.

  For example, in the equation above, the logarithm is $y = \log_2(8) = 3$. It is read as 'the logarithm, base 2, of 8 is equal to 3'.

  The diagram below shows the general relationship between logarithms and exponentials.



- To evaluate a logarithm, ask yourself, 'What index applied to a gives x?' In the diagram above, the answer is y.
- Logarithms and exponentials are the inverse operations of each other.
- Logarithmic identities include:

$$\log_a(a) = 1$$

$$\log_a(1) = 0$$

Example 3G.1 Converting between exponential form and logarithmic form

For each of these, write an equivalent equation in the form specified.

a $3^4 = 81$ (in logarithmic form) **b** $\log_2(16) = 4$ (in exponential form)

THINK	WRITE
a Remember that $a^y = x$ is the same as $\log_a(x) = y$. Identify the value of each variable and rewrite the equation.	**a** $\log_3(81) = 4$
b Recall that $\log_a(x) = y$ is the same as $a^y = x$. Identify the value of each variable and restate the equation.	**b** $2^4 = 16$

Example 3G.2 Calculating the value of logarithms

Find the value of each of these logarithms.

a $\log_3(9)$ **b** $\log_2(64)$

THINK	WRITE
a Determine the value of the logarithm. The term $\log_3(9)$ is asking, ‘What index applied to 3 gives 9?’ Since $3^2 = 9$, the value of the logarithm is 2.	**a** $\log_3(9) = 2$
b Determine the value of the logarithm. The term $\log_2(64)$ is asking, ‘What index applied to 2 gives 64?’ Since $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6 = 64$, the value of the logarithm is 6.	**b** $\log_2(64) = 6$

Example 3G.3 Converting from exponential form with a negative index to logarithmic form

For each part:

i find the value of the term

ii write an equation in logarithmic form.

a 5^{-1} **b** $\left(\frac{1}{4}\right)^{-2}$ **c** 3^{-3}

THINK	WRITE
a **1** Write an equivalent value with a positive index and simplify.	**a** **i** $5^{-1} = \frac{1}{5^1}$ $= \frac{1}{5}$
2 Write it in log form. (Remember that $a^y = x$ is the same as $\log_a(x) = y$.)	**ii** $\log_5\left(\frac{1}{5}\right) = -1$
b **1** Write an equivalent value with a positive index and simplify.	**b** **i** $\left(\frac{1}{4}\right)^{-2} = \left(\frac{4}{1}\right)^2$ $= 16$
2 Write it in log form. (Remember that $a^y = x$ is the same as $\log_a(x) = y$.)	**ii** $\log_{\frac{1}{4}}(16) = -2$
c **1** Write an equivalent value with a positive index and simplify.	**c** **i** $3^{-3} = \frac{1}{3^3}$ $= \frac{1}{27}$
2 Write it in log form. (Remember that $a^y = x$ is the same as $\log_a(x) = y$.)	**ii** $\log_3\left(\frac{1}{27}\right) = 3$

Helpful hints

✔ When the base of the logarithm and the number in the brackets are the same, the result is always 1. It doesn't matter how big or small the values are! For example, $\log_{238\,042}(238\,042) = 1$.

✔ Be careful when writing the base of the logarithms. The base needs to be written in a smaller size than the number in the brackets. For example, write $\log_2(3)$ rather than $\log_{2}(3)$.

ANS p667

Exercise 3G Logarithms

△ 1–9, 14(a, b)

□ 1–5(3rd, 4th columns), 6, 7, 10–13, 15, 18(a)

⬠ 1–5(3rd, 4th columns), 12, 13, 16–18

UNDERSTANDING AND FLUENCY

3G.1 1 Write each of these equations in logarithmic form.

a $3^3 = 27$ **b** $4^1 = 4$ **c** $5^4 = 625$ **d** $10^3 = 1000$

e $6^2 = 36$ **f** $9^3 = 729$ **g** $3^4 = 81$ **h** $2^7 = 128$

2 Write each of these equations in exponential form.

a $\log_2(32) = 5$ **b** $\log_3(9) = 2$ **c** $\log_5(25) = 2$ **d** $\log_7(49) = 2$

e $\log_4(16) = 2$ **f** $\log_2(1024) = 10$ **g** $\log_6(216) = 3$ **h** $\log_{10}(100) = 2$

3G.2 3 Calculate each of these values.

a $\log_2(16)$ **b** $\log_4(256)$ **c** $\log_5(125)$ **d** $\log_{10}(1000)$

e $\log_8(64)$ **f** $\log_3(27)$ **g** $\log_9(81)$ **h** $\log_2(32)$

i $\log_{10}(10\,000)$ **j** $\log_5(125)$ **k** $\log_6(36)$ **l** $\log_3(81)$

m $\log_2(8)$ **n** $\log_4(64)$ **o** $\log_7(49)$ **p** $\log_3(243)$

3G.3 4 **i** Find the value of each of the terms below.

ii Write an equation in logarithmic form for each term.

a 6^{-1} **b** 4^{-1} **c** 2^{-2} **d** 2^{-5}

e 5^{-2} **f** 10^{-1} **g** 10^{-3} **h** 3^{-3}

i $\left(\frac{1}{2}\right)^{-1}$ **j** $\left(\frac{1}{3}\right)^{-2}$ **k** $\left(\frac{1}{5}\right)^{-3}$ **l** $(0.01)^{-1}$

5 Calculate the value of each of these.

a $\log_2\left(\frac{1}{8}\right)$ **b** $\log_3\left(\frac{1}{27}\right)$ **c** $\log_{10}(0.01)$ **d** $\log_4\left(\frac{1}{4}\right)$

e $\log_3\left(\frac{1}{81}\right)$ **f** $\log_2\left(\frac{1}{64}\right)$ **g** $\log_4\left(\frac{1}{64}\right)$ **h** $\log_8\left(\frac{1}{64}\right)$

6 **a** Evaluate the following, rounding correct to four decimal places where required.

i 2^4 **ii** $2^{\frac{1}{4}}$ **iii** 2^{-4} **iv** $2^{-\frac{1}{4}}$

b Is there a real number such that 2 to the power of that real number is −16? (Find the value of $\log_2(-16)$.)

c Repeat part **b** for each of the following values.

i $\log_5(-25)$ **ii** $\log_3(-27)$ **iii** $\log_{10}(-1000)$

iv $\log_6(-36)$ **v** $\log_4(-64)$ **vi** $\log_8(-8)$

d What can you conclude about the logarithm of any negative number to any base?

7 **a** Now consider $\log_1(5)$. Is it possible for this to be written in an equivalent unique index form?

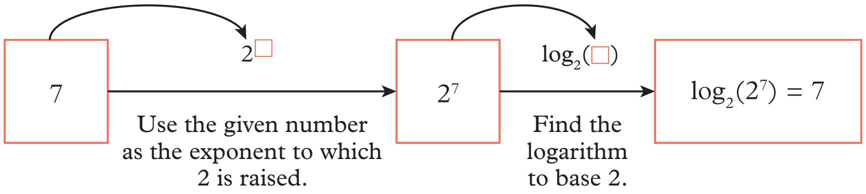
b Is it possible for any of the following to be written in an equivalent index form?

i $\log_1(3)$ **ii** $\log_1(10)$ **iii** $\log_1(6)$ **iv** $\log_1(4)$ **v** $\log_1(1)$

c What can you conclude about the logarithm of any number to the base 1?

8 Since the exponential and logarithmic operations are inverse operations, they ‘undo’ each other. If you raise a base to a power or exponent (which is called ‘exponentiating’), then you find the logarithm to the same base of that result, you will get the original number you raised the base to.

a Describe the process shown below.



b Evaluate each of the following.

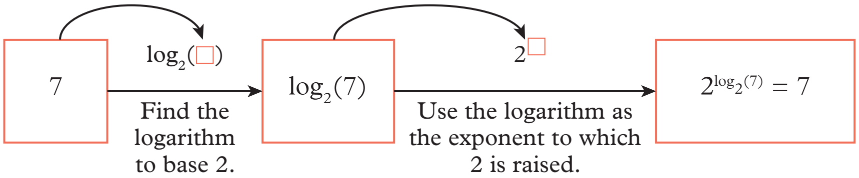
i $\log_5(5^7)$ **ii** $\log_7(7^5)$ **iii** $\log_{12}(12^9)$ **iv** $\log_2(2^{140})$

v $\log_{19}(19^{-6})$ **vi** $\log_{456}(456^{123})$ **vii** $\log_{\frac{3}{4}}\left(\left[\frac{3}{4}\right]^{\frac{5}{7}}\right)$ **viii** $\log_{1089}\left(1089^{-\frac{37}{19}}\right)$

c Describe, in terms of how we evaluate logarithms, why $\log_a(a^x) = x$ for $a > 0$.

9 If we find the logarithm of a number to a given base, then exponentiate that with the same base, we get the original number back. (See question **9a** if you don’t know what ‘exponentiate’ means.)

a Describe the process shown below.



b Evaluate each of the following.

i 5^x, where $x = \log_5(7)$ **ii** 7^x, where $x = \log_7(5)$

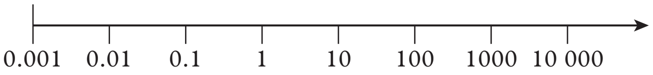
iii 21^x, where $x = \log_{21}(11)$ **iv** 2^x, where $x = \log_2(531)$

v 101^x, where $x = \log_{101}(8)$ **vi** 963^x, where $x = \log_{963}(147)$

vii $\left(\frac{9}{7}\right)^x$, where $x = \log_{\left(\frac{9}{7}\right)}\left(\frac{6}{17}\right)$ **viii** 63^x, where $x = \log_{63}\left(\frac{514}{21}\right)$

c Describe, in terms of how we evaluate exponential expressions, why $a^{\log_a(x)} = x$, when $a > 0$ and $x > 0$.

10 When we are measuring length across several powers of 10, we often use logarithms of the values so that the values are easier to work with. Most commonly, a base 10 logarithmic scale is used, in which an increase of 1 unit corresponds to a value 10 times greater.



A knowledge of the inverse relationship between logarithms and exponential operations is useful when solving logarithmic equations.

Consider the equation $\log_x(64) = 2$, when $x > 0$.

a Copy and complete the following, using both sides of the equation as powers to which x is raised.

$\log_x(64) = 2$

$x^{\square} = x^{\square}$

$\_\_\_\_ = x^{\square}$

$x = \_\_\_\_$

b What is the value of x? Explain why there is only one solution.

c Write the exponential equation in the last line of your answer for part **a** in the form $x^a = b^a$.

d Notice that, in the equation shown in part **c**, the two indices are the same. If those indices are the same, equating the bases will produce the solution. How does the answer found using this method compare with your answer to part **b**?

11 Use the methods discussed in question **10** to solve each of these logarithmic equations for x.

a $\log_x(64) = 6$ **b** $\log_x(125) = 3$ **c** $\log_x(16) = 2$ **d** $\log_x(8) = 3$

e $\log_x(128) = 7$ **f** $\log_x(81) = 4$ **g** $\log_x\left(\frac{1}{4}\right) = -1$ **h** $\log_x\left(\frac{1}{8}\right) = -3$

12 Consider the equation $\log_4(x) = 3$.

a Copy and complete the following, using both sides of the equation as powers to which 4 is raised.

$$\begin{aligned} \log_4(x) &= 3 \\ 4^{\square} &= 4^{\square} \\ \_\_\_ &= 4^{\square} \end{aligned}$$

b Calculate the index term to find the value of x.

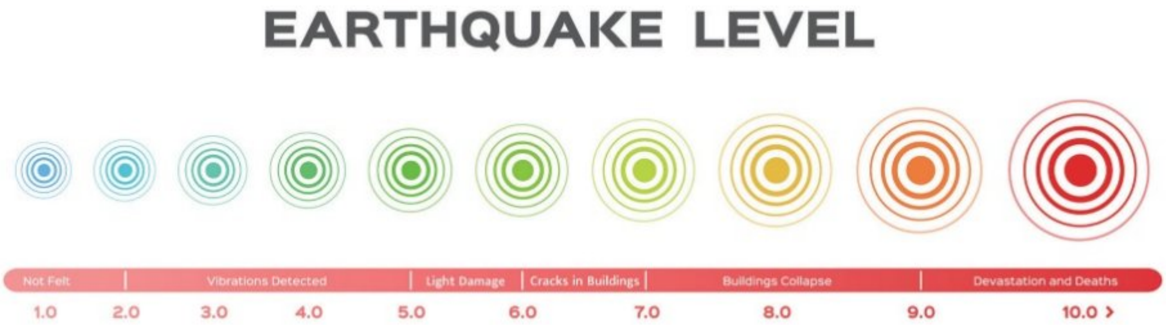
c Briefly explain how using both sides of the equation as powers to which the base, 4, is raised expresses the logarithmic equation in index form.

13 Use the method from question **12** to solve each of these logarithmic equations for x.

a $\log_3(x) = 3$ **b** $\log_5(x) = 2$ **c** $\log_6(x) = 1$ **d** $\log_{10}(x) = 4$

e $\log_2(x) = 5$ **f** $\log_4(x) = 3$ **g** $\log_3(x) = -3$ **h** $\log_2(x) = -2$

14 The Richter scale is used to measure the magnitude (size) of an earthquake by taking the (base 10) logarithm of the amplitude (height) of the largest seismic wave, measured in micrometres (μm).



a Determine the magnitude of earthquakes with their largest seismic waves having the following amplitudes, correct to one decimal place. (Hint: 1000 μm = 1 mm)

i 320 000 μm **ii** 1200 μm **iii** 75 600 μm

iv 600 μm **v** 21 460 mm **vi** 146 m

b Determine the amplitudes, in micrometres, of seismic waves of the following magnitudes correct to the nearest micrometre.

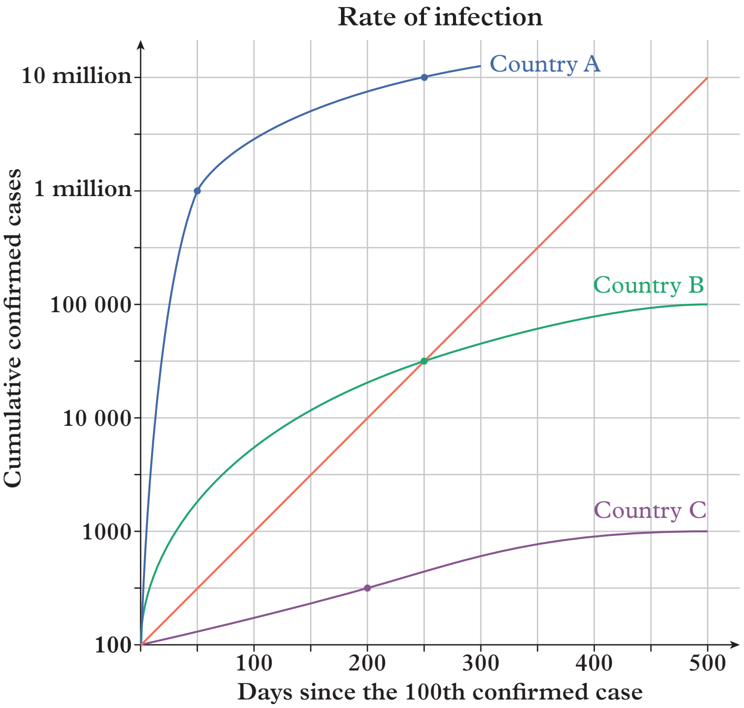
i 5.0 **ii** 6.0 **iii** 6.5 **iv** 1.5 **v** 7.8 **vi** 4.3

c Each increase of one unit on the Richter scale corresponds to the measure of an earthquake's magnitude that is 10 times greater. Determine how many times greater the larger magnitude is in each of these pairs, correct to four significant figures.

i 4.2 and 5.2 **ii** 4.2 and 4.7 **iii** 4.2 and 5.7

iv 6.1 and 7.9 **v** 1.2 and 7.3 **vi** 5.1 and 5.2

15 When growth or change occurs at an exponential rate, we often graph the change on a logarithmic scale rather than a linear scale. The number of people infected by a virus in three countries has been plotted against the number of days since the 100th infection below, on a base 10 logarithmic scale.



a An orange line has been drawn at a 45° angle on the graph so that it passes through the points (100, 1000), (200, 10 000), (300, 100 000). Describe what this line represents.

b Use the graph to determine:

i the number of days it took for Country A to go from its 100th confirmed case to its one millionth

ii the number of days it took for Country A to go from its one millionth confirmed case to its ten millionth

iii the number of days it took for Country B to go from its 100th confirmed case to its hundred thousandth

iv correct to the nearest integer, the number of cumulative confirmed cases in Country B after 250 days

v the number of cumulative confirmed cases in Country C after 500 days.

c Describe the importance of the point where Country B crosses the orange line described in part **a**.

16 Logarithmic scales are also useful for plotting data on a histogram that involves very small and very large values so that the smaller values do not become excessively grouped on the scale. The average weights of 47 breeds of animals, in kilograms, are listed below.

1390	1100	725	475	386	190	180	154.25	117.5	100	90	64.475
60	44.984	28.25	26.625	25.5	20	16.5	13.25	9.3	8.6	7.875	6.81
6	5.765	4.25	3.1	2.25	2.06	1.25	0.425	0.397	0.336	0.15	0.29
0.2	0.186	0.173	0.115	0.1	0.07	0.06	0.055	0.05	0.028	0.018	

a Complete the grouped data tables using the average weights listed above.

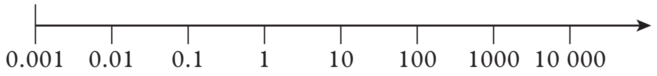
i

Class interval	Frequency
0–<200	
200–<400	
400–<600	
600–<800	
800–<1000	
1000–<1200	
1200–<1400	

ii

Class interval	Frequency
0.01–<0.1	
0.1–<1	
1–<10	
10–<100	
100–<1000	
1000–<10 000	

b Draw a histogram for each of the grouped data tables in part **a**. Use a horizontal, base 10 logarithmic scale for the table in part **ii** as shown here.



c Describe the shape of the two histograms using 'symmetric' or 'skewed'.

d Explain why using the logarithmic scale helps us to understand the distribution of average weights better than the linear scale does.

17 Evaluate the following.

a $\log_2(\sqrt{2})$ **b** $\log_{\sqrt{2}}(2)$ **c** $\log_2\left(\frac{1}{\sqrt{8}}\right)$ **d** $\log_2\left(\sqrt[3]{128}\right)$

e $\log_3(3\sqrt{3})$ **f** $\log_5\left(\sqrt[8]{125}\right)$ **g** $\sqrt{\log_3(81)}$ **h** $\log_{\sqrt{125}}\left(\frac{\sqrt{5}}{5}\right)$

18 Bels (B) are a unit of measurement for loud noise, named in honour of Alexander Graham Bell (the inventor of the telephone). Bels are measured on a logarithmic scale, on which an increase in 1 B is an increase of 10 times in the loudness. More frequently, decibels (dB) are used: 10 dB = 1 B.

a Determine how many times louder the larger sounds in each of these pairs is, correct to four significant figures.

i 15 dB and 25 dB **ii** 60 dB and 70 dB

iii 50 dB and 90 dB **iv** 95 dB and 145 dB

v 10 dB and 100 dB **vi** 99 dB and 100 dB

b The gain, G, in decibels, is 10 times the base 10 logarithm of the ratio, $\frac{P_{\text{out}}}{P_{\text{in}}}$, of the power in watts (W) of the output power level, P_{out}, to the input power level, P_{in}. That is, $G = 10\log_{10}\left(\frac{P_{\text{out}}}{P_{\text{in}}}\right)$.

Determine the gain, correct to the nearest decibel, given the input and output power levels provided.

i output = 5 W and input = 50 W

ii output = 50 W and input = 5 W

iii output = 50 W and input = 5000 W

iv output = 2 W and input = 8 W

v output = 240 W and input = 240 W

vi output = 100 W and input = 1 W

c Determine the output power level, in watts correct to one decimal place, given the input power level and gain provided.

i input = 5 W and gain = 30 dB

ii input = 5 W and gain = 3 dB

iii input = 50 W and gain = 3 dB

iv input = 500 W and gain = 30 dB

v input = 240 W and gain = 60 dB

vi input = 240 W and gain = 20 dB

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Interactive skillsheet
Converting between exponential form and logarithmic form

Interactive skillsheet
Calculating the value of logarithms

Topic quiz
3G

3H Logarithm laws

Learning intentions

✔ I can evaluate the logarithm of 1.

✔ I can simplify logarithmic expressions involving addition and subtraction.

✔ I can simplify logarithmic expressions involving indices.

✔ I can simplify expressions using the logarithm laws.

Inter-year links

Year 8	4A Indices
Year 9	2C Index law 3 and the zero index

Logarithm laws

- The logarithm laws can be applied to expressions in logarithmic form.
- When $a > 0$, the logarithm laws include:
  - → The **logarithm of a product** is equal to the sum of the logarithms of the factors.

    $$\log_a(xy) = \log_a(x) + \log_a(y)$$

  - → The **logarithm of a quotient** is equal to the difference of the logarithm of the dividend and the logarithm of the divisor.

    $$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$$

  - → The **logarithm of a power** is equal to the product of the index of x and the logarithm of x.

    $$\log_a(x^n) = n \times \log_a(x)$$

- Recall that logarithmic identities include:

$$\log_a(a) = 1$$

$$\log_a(1) = 0$$

Example 3H.1 Calculating logarithmic expressions

Calculate each of these values.

a $\log_3(1)$ **b** $\log_4(4)$

THINK	WRITE
a The logarithm of 1 is always equal to 0, no matter what the base is. $1 = 3^0$, so $\log_3(1) = 0$.	**a** $\log_3(1) = 0$
b The logarithm of a number to a base equal to the number itself is always 1. $4^1 = 4$, so $\log_4(4) = 1$.	**b** $\log_4(4) = 1$

Example 3H.2 Adding and subtracting logarithms

Use the appropriate logarithm law to calculate:

a $\log_4(32) + \log_4(2)$ **b** $\log_5(50) - \log_5(2)$

THINK	WRITE
a 1 Both logarithms in the sum contain the same base, so use the 'logarithm of a product' law: $\log_a(x) + \log_a(y) = \log_a(xy)$	**a** $\log_4(32) + \log_4(2) = \log_4(32 \times 2)$
2 Which logarithm is the index of base 4 that will give an answer of 64? The logarithm is 3.	$= \log_4(64)$ $= \log_4(4^3)$ $= 3$
b 1 Both logarithms in the expression contain the same base, so use the 'logarithm of a quotient' law: $\log_a(x) - \log_a(y) = \log_a\left(\frac{x}{y}\right)$	**b** $\log_5(50) - \log_5(2) = \log_5\left(\frac{50}{2}\right)$
2 Which logarithm is the index of base 5 that will give an answer of 25? The logarithm is 2.	$= \log_5(25)$ $= \log_5(5^2)$ $= 2$

Example 3H.3 Determining the logarithm of a number in index form

Use the appropriate logarithm law to calculate:

a $\log_2(4^3)$ **b** $\log_5\left(\frac{1}{5}\right)$

THINK	WRITE
a 1 Use the 'logarithm of a power' law to simplify the logarithm: $\log_a(x^n) = n \times \log_a(x)$	**a** $\log_2(4^3) = 3 \times \log_2(4)$
2 Calculate the logarithm and simplify.	$= 3 \times \log_2(2^2)$ $= 3 \times 2$ $= 6$
b 1 Write the fraction as an integer with a negative index.	**b** $\log_5\left(\frac{1}{5}\right) = \log_5(5^{-1})$
2 Use the 'logarithm of a power' law to simplify the logarithm: $\log_a(x^n) = n \times \log_a(x)$	$= -1 \times \log_5(5)$
3 Use the logarithm identity $\log_a(a) = 1$ to simplify the expression.	$= -1 \times 1$ $= -1$

Example 3H.4 Simplifying a logarithmic expression

Calculate $\log_6(48) + \log_6(3) - 2\log_6(2)$.

THINK	WRITE
1 Use the 'logarithm of a power' law to rewrite the third term.	$\log_6(48) + \log_6(3) - 2\log_6(2)$ $= \log_6(48) + \log_6(3) - \log_6(2^2)$
2 Use the 'logarithm of a product' law to simplify the first two terms.	$= \log_6(48 \times 3) - \log_6(2^2)$
3 Simplify the product.	$= \log_6(144) - \log_6(4)$
4 Use the 'logarithm of a quotient' law to simplify the expression to one term.	$= \log_6\left(\frac{144}{4}\right)$ $= \log_6(36)$
5 Work out the value of the logarithm. (Remember: $6^2 = 36$)	$= \log_6(6^2)$ $= 2$

Helpful hints

- ✔ Order of operations is important when solving logarithms. Recall your knowledge of BIDMAS. The I stands for indices, which can also be called exponents. Logarithms and exponential equations are in the same category as indices. After brackets, you'll need to simplify or apply the logarithms and exponents.
- ✔ Remember that for all real numbers, a logarithm cannot be applied to a negative number. This is important when solving logarithmic equations for which there might be two solutions.

  For example, $\log_2(x^2) = \log_2(9)$

  $$x^2 = 9$$
  $$x = \pm 3$$

  Except, x cannot equal negative 3, so $x = 3$ only.

ANS p668

Exercise 3H Logarithm laws

△ 1–5, 6(a–f), 7–11, 12–13(1st column), 14(1st, 2nd columns), 15, 16

■ 3, 5, 6(g–l), 9–11, 12–13(2nd column), 14(3rd, 4th columns), 15, 17(a, c, e), 18, 19(a, c, e), 21

⬠ 6(g–l), 9, 11, 13(2nd column), 14(3rd, 4th columns), 17, 18, 19(d–f), 20, 22

3H.1 **1** Evaluate each of the following.

a $\log_5(5)$ **b** $\log_3(3)$ **c** $\log_{10}(10)$ **d** $4\log_5(5)$

e $3\log_3(3)$ **f** $\log_3(1)$ **g** $\log_2(1)$ **h** $4\log_5(1)$

2 Use the 'logarithm of a product' law, $\log_a(xy) = \log_a(x) + \log_a(y)$, to write each of the following expressions as the sum of two logarithms.

a $\log_{10}(3 \times 5)$ **b** $\log_2(6 \times 3)$ **c** $\log_5(2.5 \times 10)$ **d** $\log_2(xy)$

3H.2 **3** Write each of these expressions as the logarithm of a product.

a $\log_3(6) + \log_3(2)$ **b** $\log_{10}(5) + \log_{10}(3)$ **c** $\log_2(10) + \log_2(5)$

d $\log_4(6) + \log_4(12)$ **e** $\log_a(5) + \log_a(3)$ **f** $\log_2(x) + \log_2(y)$

4 Use the 'logarithm of a quotient' law, $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$, to write each of the following expressions as the difference of two logarithms.

a $\log_{10}\left(\frac{5}{7}\right)$
b $\log_2\left(\frac{5}{3}\right)$
c $\log_5\left(\frac{11}{20}\right)$
d $\log_2\left(\frac{a}{b}\right)$

5 Write each of these expressions as the logarithm of a quotient.

a $\log_3(12) - \log_3(5)$
b $\log_{10}(50) - \log_{10}(15)$
c $\log_2(25) - \log_2(10)$
d $\log_4(36) - \log_4(9)$
e $\log_a(3) - \log_a(15)$
f $\log_2(x) - \log_2(y)$

6 Use the appropriate logarithm law to evaluate each of the following.

a $\log_3(9) + \log_3(9)$
b $\log_6(108) - \log_6(3)$
c $\log_2(8) + \log_2(4)$
d $\log_{10}(500) + \log_{10}(20)$
e $\log_4(256) - \log_4(16)$
f $\log_3(243) - \log_3(9)$
g $\log_6(9) + \log_6(4)$
h $\log_5(175) - \log_5(7)$
i $\log_2(32) + \log_2(16)$
j $\log_{10}(25) + \log_{10}(4)$
k $\log_3(54) - \log_3(6)$
l $\log_4(768) - \log_4(3)$

7 Use the appropriate logarithm laws to calculate these values.

a $\log_6(27) + \log_6(4) - \log_6(3)$
b $\log_{10}(60) - \log_{10}\left(\frac{3}{5}\right)$
c $\log_3\left(\frac{3}{2}\right) + \log_3(18)$

8 Use the 'logarithm of a power' law, $\log_a(x^n) = n \times \log_a(x)$, to write each of the following expressions as the product of an index and a logarithm.

a $\log_3(7^2)$
b $\log_5(2^9)$
c $\log_4(8^3)$
d $\log_9(11^4)$

3H.3 **9** Use the appropriate logarithm law to evaluate each of the following.

a $\log_3(3^5)$
b $\log_4(4^2)$
c $\log_6(6^5)$
d $\log_3(9^4)$
e $3\log_3(27)$
f $4\log_4(64)$
g $5\log_5(5^2)$
h $\frac{1}{2}\log_4(4^4)$

10 Use the appropriate logarithm law to evaluate each of the following.

a $\log_{15}\left(\frac{1}{15^2}\right)$
b $\log_{12}\left(\frac{1}{12}\right)$
c $\log_{19}\left(\left[\frac{1}{19}\right]^2\right)$
d $\log_{103}\left(\frac{1}{103^{14}}\right)$
e $5\log_2\left(\frac{1}{64}\right)$
f $7\log_6\left(\frac{1}{216}\right)$
g $\frac{1}{9}\log_3\left(\frac{4}{108}\right)$
h $\frac{5}{2}\log_9\left(\frac{3}{243}\right)$

11 Write each of these in a simpler form and evaluate them.

a $2\log_9(3)$
b $8\log_4(2)$
c $2\log_3(9)$
d $\log_a(\sqrt{a})$
e $\log_3(5) - \log_3\left(\frac{5}{3}\right)$
f $2\log_x(\sqrt{xy}) - \log_x(y)$

3H.4 **12** Use the appropriate logarithm laws to evaluate each of the following.

a $\log_6(12) + \log_6(9) + \log_6(2)$
b $\log_2(24) + \log_2(6) - \log_2(36)$
c $\log_5(40) + \log_5(10) - \log_5(16)$
d $\log_3(15) - \log_3(45)$
e $\log_4(2) - \log_4(32)$
f $\log_{10}(20) + \log_{10}(45) + \log_{10}\left(\frac{10}{9}\right)$

13 Evaluate each of the following.

a $3\log_4(2) + \log_4(8)$
b $\log_5(225) - 2\log_5(3)$
c $2\log_3(9) + 2\log_3(3)$
d $4\log_{10}(5) + 4\log_{10}(2)$
e $\log_2(72) - 2\log_2(6)$
f $5\log_2(4) - \log_2(2^3)$
g $2\log_2(4) + \log_2(14) - \log_2(7)$
h $4\log_3(2) - \log_3(32) + \log_3(54)$
i $\log_5(15) + 2\log_5(5) + 3\log_5(2) - \log_5(24)$
j $3\log_{10}(5) + 2\log_{10}(3) + \log_{10}\left(\frac{8}{9}\right)$

14 Solve each of these logarithmic equations for x.

a $\log_x(27) = 3$
b $\log_4(64) = x$
c $\log_3(x) = 2$
d $\log_7(2401) = x$
e $\log_6(x) = 3$
f $\log_2(x) = 6$
g $\log_5\left(\frac{1}{125}\right) = x$
h $\log_x(6) = \frac{1}{2}$
i $\log_3(x) = 0$
j $\log_x\left(\frac{1}{16}\right) = 4$
k $\log_2(\sqrt{2}) = x$
l $\log_x\left(\frac{1}{16}\right) = -2$

PROBLEM SOLVING AND REASONING

15 Fractions containing logarithmic expressions in the numerator and the denominator can be simplified in a similar way to common fractions; that is, by cancelling any common factors in the numerator and denominator.

Consider $\frac{\log_{10}(3)}{\log_{10}(9)}$.

a Use a calculator to find the value of this fraction.

b Show that an equivalent logarithmic expression for the denominator of the fraction is $2\log_{10}(3)$.

c Show that the fraction simplifies to the same answer you obtained in part **a**.

16 Three students attempt to simplify the fraction $\frac{\log_6(8)}{\log_6(2)}$.

Hailey says the correct answer is 3, Danielle thinks the answer is 4 and Joyce is certain that it must be $\log_6\left(\frac{8}{2}\right)$ or $\log_6(4)$. Which student do you think is correct and why?

17 Simplify each of these expressions.

a $\frac{\log_{10}(16)}{\log_{10}(2)}$ **b** $\frac{\log_6(216)}{\log_6(36)}$ **c** $\frac{\log_3(8)}{\log_3(2)}$

d $\frac{\log_4(64)}{\log_4(256)}$ **e** $\frac{\log_{10}(16)}{\log_{10}(64)}$ **f** $\frac{\log_5(125)}{\log_5(5)}$

18 You can use the logarithm laws to solve more logarithmic equations.

Consider the equation $\log_5(2x) + \log_5(3) = \log_5(12)$. The following parts of this question highlight two techniques that can be used to solve equations like this. Work through both parts to solve the equation in two different ways.

a **i** Subtract $\log_5(12)$ from both sides of the equation and show that the use of the logarithm laws can help you to simplify the equation to $\log_5\left(\frac{x}{2}\right) = 0$.

ii Write the simplified equation from part **i** in index form and solve it for x.

b **i** Look at the original equation. Apply the appropriate law to the left-hand side of the equation to show that it simplifies to $\log_5(6x) = \log_5(12)$.

ii If the same base appears on both sides, the number parts of each logarithm can be equated; that is, $6x = 12$. Show that this method produces the same value for x as you found in part **a**.

19 Use either method from question **18** to solve each of these logarithmic equations for x.

a $\log_3(x) + \log_3(5) = \log_3(15)$ **b** $\log_4(2x) + \log_4(8) = 3$

c $\log_5(5x) - \log_5(4) = 2$ **d** $\log_5(10) - \log_5(x) = \log_5(15)$

e $\log_2(2x + 1) - \log_2(x) = 4$ **f** $\log_{10}(2x) + \log_{10}(3) = 4 + \log_{10}(5)$

20 Solve this logarithmic equation for x: $\log_5(x^2 - 9x + 21) = 0$

CHALLENGE

21 Show that the solution to this equation is $\frac{1}{2}$: $\log_{10}(2^{2x+1}) = \log_{10}(4)$

22 If $2\log_x\left(\frac{7}{5}\right) + \log_x\left(\frac{704}{35}\right) - 3\log_x\left(\frac{4}{5}\right) = \log_x(y)$, find the value of y.

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Logarithm of a power

Interactive skillsheet
Simplifying a logarithmic expression

3I Exponential equations

Learning intentions

✔ I can solve simple exponential equations.
✔ I can solve simple exponential equations describing population growth.

Inter-year links

Year 8	4A Indices
Year 9	2C Index law 3 and the zero index

Solving exponential equations

- To solve exponential equations of the form $a^x = y$ for x, use one of the two methods below.

  Method 1: If y can be written in index form as a term with the base a, you can equate the exponents. That is, if $a^x = a^m$ then $x = m$.

  For example, $2^x = 8$ can be written as $2^x = 2^3$, so $x = 3$.

  In some cases, you might need to rewrite both sides of the equation so they have the same base.

  For example, $8^x = 4$ can be written as $2^{3x} = 2^2$.

  To calculate x, you need to divide both indices by 3: $2^{\frac{3x}{3}} = 2^{\frac{2}{3}}$.

  Simplifying shows that $x = \frac{2}{3}$.

  Method 2: Take the logarithm with the same base of both sides and simplify.

  For example, $2^x = 8$ can be written as:

$$\begin{aligned} 2^x &= 8 \\ \log_2(2^x) &= \log_2(8) \\ \log_2(2^x) &= \log_2(2^3) \\ x &= 3 \end{aligned}$$

- Exponential relationships can be used to model many real situations involving growth and decay (for populations or investments, for example).

Example 3I.1 Solving exponential equations by writing with the same base

Solve each of these equations.

a $2^x = 32$ **b** $9^x = 27$

THINK	WRITE
a **1** Write the equation so that both sides have the same base (in this case, 2).	**a** $2^x = 32$ $2^x = 2^5$
2 Solve the equation by equating the indices. The bases are the same and it is an equation, so the indices are equal.	$x = 5$
b **1** Write the equation so that both sides have the same base (in this case, 3).	**b** $9^x = 27$ $(3^2)^x = 3^3$
2 Equate the indices because it is an equation and the bases are the same.	$3^{2x} = 3^3$ $2x = 3$
3 Solve the linear equation for x.	$x = \frac{3}{2}$

Example 3I.2 Solving exponential equations using logarithmic form

Solve each of these equations. Give your answer as an exact value each time.

a $2^x = 5$ **b** $4 \times 3^x = 2$ **c** $5^{x-4} = 125$ **d** $2^{x+3} = 48$

THINK	WRITE
a Write the equation in logarithmic form. The logarithm cannot be simplified so x is already in exact form.	**a** $2^x = 5$ $x = \log_2(5)$
b **1** Simplify by dividing both sides of the equation by 4.	**b** $4 \times 3^x = 2$ $3^x = \frac{1}{2}$
2 Write the equation in logarithmic form.	$x = \log_3\left(\frac{1}{2}\right)$
3 Simplify the logarithm. The fraction can be simplified using negative indices.	$= \log_3(2^{-1})$ $= -\log_3(2)$
c **1** Write the equation in logarithmic form.	**c** $5^{x-4} = 125$ $x - 4 = \log_5(125)$
2 Solve for x by adding 4 to both sides.	$x = \log_5(125) + 4$
3 Which index of base 5 will give an answer of 125? The answer is 3, then simplify.	$x = 3 + 4$ $= 7$
d **1** Write the equation in logarithmic form.	**d** $2^{x+3} = 48$ $x + 3 = \log_2(48)$
2 Solve for x by subtracting 3 from both sides.	$x = \log_2(48) - 3$
3 Use $\log_a(a) = 1$ to write 3 as a logarithm with the base 2. Then use the 'logarithm of a power' law to simplify.	$x = \log_2(48) - 3\log_2(2)$ $= \log_2(48) - \log_2(8)$
4 Use the 'logarithm of a quotient' law to simplify. Leave your solution in exact form.	$= \log_2\left(\frac{48}{8}\right)$ $= \log_2(6)$

Helpful hints

- ✔ Remember that logarithms 'undo' exponentials, and exponentials 'undo' logarithms. These are inverse operations.
- ✔ You'll need expert skills in negative indices, factors, surds and logarithm laws to effectively solve exponential equations!

ANS p669

Exercise 3I Exponential equations

△ **1–8, 9(a, c, e, g), 11, 12, 15** ■ **3–9, 10, 14, 16(a–c), 18** ⬟ **3–6, 8(c, e, f), 9(e–h), 13, 14, 16(d–f), 17, 19**

3I.1 **1** Find x by equating the indices to solve each of these equations.

a $2^x = 2^9$ **b** $3^x = 3^6$ **c** $4^x = 4^{-3}$ **d** $8^x = 8$

e $5^{x-2} = 5^6$ **f** $2^{x+4} = 2^8$ **g** $10^{2-x} = 10^{-5}$ **h** $2^{-x-7} = 2^3$

i $9^{2x} = 9^6$ **j** $6^{3x} = 6^{15}$ **k** $7^{2x-1} = 7^{11}$ **l** $3^{4x+5} = 3^5$

2 Solve each of these equations.

a $2^x = 8$ **b** $3^x = 9$ **c** $4^x = 1024$ **d** $10^x = 10\,000$

e $6^{x-5} = 36$ **f** $2^{x+3} = 64$ **g** $9^{2x-1} = 729$ **h** $3^{-x+4} = 243$

i $8^x = 128$ **j** $25^x = 125$ **k** $81^x = 27$ **l** $49^{2x} = 16\,807$

3I.2 **3** Solve each of these equations. Give your answer as an exact value each time.

a $2^x = 11$ **b** $5^x = 92$ **c** $92^x = 5$ **d** $14^x = \frac{1}{4}$

e $6 \times 3^x = 3$ **f** $2 \times 7^x = 53$ **g** $5^{2x} = 17$ **h** $10^{3x} = 75$

i $8^x + 3 = 36$ **j** $11^{6x} - 7 = 48$ **k** $3 \times 17^x + 5 = 41$ **l** $9 \times 12^{-4x} - 10 = 36$

4 Solve each of these equations. Give your answer as an exact value in the form $a \log_b (c) + d$, where a, b, c and d are rational numbers.

a $8^{x-1} = 10$ **b** $3^{x+1} = 12$ **c** $5^{x+3} = 2$ **d** $2^{x-5} = 6$

e $2^{3-x} = 48$ **f** $9^{-x+2} = 20$ **g** $6^{3-x} = 4$ **h** $2^{6-x} = 30$

i $4^{2x+3} = 24$ **j** $3^{4x+1} = 87$ **k** $8 \times 7^{5x+2} = 24$ **l** $6 \times 10^{4x+3} + 11 = 20$

5 Write your answers from question **4** in the form $a \log_b (c)$, where a, b and c are rational numbers.

6 Solve each of these equations. Give your answers correct to three significant figures.

a $9^x = 10$ **b** $5^x = 30$ **c** $\frac{1}{4} \times 2^x = 2000$ **d** $3^{6x} = 120$

e $10^{\frac{1}{8}x+5} = 40\,000$ **f** $7 \times 16^x + 9 = 130$ **g** $25^{3x} - 10 = 28$ **h** $6 \times 7^{2x-7} + 1 = 1258$

7 Solve the following equations.

a $2^{x^3} = 256$ **b** $2^{x^2} = 512$ **c** $3^{x^2} = 81$

8 Solve the following equations.

a $(3^x - 9)(2^x - 8) = 0$ **b** $(5^x - 125)(10^x - 10\,000) = 0$

c $(2^x - 64)(2^x - 16) = 0$ **d** $(3^x - 1)(3^x - 3)(3^x - 9) = 0$

e $\left(2^x - \frac{1}{32}\right)\left(4^x - \frac{1}{64}\right) = 0$ **f** $(4 \times 7^x - 28)\left(2^{3x-5} - 1\right) = 0$

9 The value of x in each of the exponential equations below is not a whole number.
Between which two whole numbers will the value of x be for each of the equations?
For example, $2^x = 10$ has a solution for x that is between 3 and 4, because $2^3 = 8$ and $2^4 = 16$.

a $2^x = 5$ **b** $3^x = 290$ **c** $5^x = 17$ **d** $10^x = 6850$

e $3^{2x} = 14$ **f** $7^{2x} = 3000$ **g** $4^{x+1} = 38$ **h** $6^{x-2} = 210$

10 The human population can be modelled using different exponential functions. One model uses $P = 2.6 \times (1.017)^t$, where P is the population in billions and t is the number of years since 1950.

a Use this model to find a population figure for:

i 2020 **ii** 2040

b In what year does this model predict that the population will reach 15 billion?

c The population in 2010 was estimated to be 6.9 billion. How did this compare to the population calculated using this model? Give reasons why you think the population numbers might be different.

11 After the water boils in an electric kettle, the temperature of the water T (in °C) after m minutes follows the relationship $T = 20 + 80 \times 0.8^m$ as it cools.

a What was the temperature of the boiled water at the start?

b What is the temperature of the water, to the nearest degree, after 3 minutes?

c How many minutes, correct to one decimal place, does it take for the temperature to be half the temperature it was at the start?

d What is the lowest temperature that this kettle of water can reach?

e Is the relationship discussed here showing growth or decay?

12 A population of rabbits can be modelled by the relationship $n = 9^t + 1$, where n is the number of rabbits after t months.

a How many rabbits were there to start with?

b How many rabbits are there after 2 months?

c How many months, correct to two decimal places, does it take for there to be 1000 rabbits?

d Is the model discussed here showing population growth or population decay?

13 Consider the exponential equation $9 \times 2^{6x-5} = 27$.

a Use index laws to write the equation in the form $b \times a^x = c$, where a, b and c are rational numbers.

b Solve the equation. Give your answer as an exact value.

14 Tristan fills a glass with tap water at room temperature, which is 25°C. He wants to cool it by placing it in the refrigerator which has a constant internal temperature of 4°C. The temperature, T°C, of the water in the glass t minutes after the glass is placed in the refrigerator is given by the equation $T = 21 \times 2.718^{-kt} + 4$, where k is a positive number.

a Tristan checks the temperature of the glass of water after 5 minutes to find it is 15°C. Determine the value of k, correct to two decimal places.

b Determine how much longer the water will take to cool to 9°C, correct to the nearest minute.

c Unhappy with how long the water will take to cool to 9°C, Tristan puts the glass in the freezer which has a constant internal temperature of −18°C. The temperature, T°C, of the water in the glass t minutes after the glass is placed in the freezer is given by the equation $T = 33 \times 2.718^{-mt} - 18$, where m is a positive number.

Tristan assumes that $m = k$ while the glass is in the freezer. Determine how long, to the nearest 30 seconds, Tristan expects the glass to take to cool to 9°C.

d Tristan checks the temperature of the glass of water after 1 minute to find it is 7°C. Determine the actual value of m, correct to two decimal places.

15 Explain why the following equations have no real solutions.

a $3^x = -1$ **b** $3^x = 0$

16 Solve the following equations by first factorising the expression.

a $(2^x)^2 - 8 \times 2^x = 0$ **b** $(5^2)^x - 5 \times 5^x = 0$ **c** $(2^x)^2 + 8 \times 2^x = 0$

d $10^{2x} - 15 \times 10^x = 0$ **e** $4^x - 6 \times 2^x + 8 = 0$ **f** $9^x - 5 \times 3^x - 14 = 0$

PROBLEM SOLVING AND REASONING

17 Bacteria grows by doubling every hour such that the number, N, of bacteria on a petri dish after t hours is given by $N = 45 \times 2^t$.

a State the number of bacteria on the petri dish initially.

b Determine the number of hours it would take for the number of bacteria to grow to 45 000:

i as an exact value

ii correct to the nearest hour.

c Initially, the petri dish was left for a day. Determine the number of bacteria on the petri dish after that day, correct to two significant figures.

d A virus is introduced to the petri dish that kills the bacteria. The number, N, of bacteria alive t hours after the virus is introduced is given by the equation $N = \dfrac{100}{1 - 0.99^{t+1}}$.

Determine the number of bacteria still alive one hour after the virus is introduced, correct to the nearest bacterium.

e Determine the number of hours it will take for the number of bacteria to reduce to 200:

i as an exact value

ii correct to the nearest hour.

CHALLENGE

18 Solve $\left(x^2 + x - 11\right)^{\left(x^2-5x-14\right)} = 1$ for x.

19 Scientific calculators often have buttons for only two types of logarithms: the common (base 10) logarithm, usually labelled 'log'; and the natural logarithm with a special base, e, that is approximately 2.71, labelled 'ln'. To evaluate logarithms with other bases on these calculators, we need to change the base of the logarithm we are evaluating to 10 (or to the special base, e).

For example, to change $\log_2(54)$ to a base 10 logarithm, we change the base to 10, then divide by the base 10 logarithm of 2. That is, $\log_2(54) = \dfrac{\log_{10}(54)}{\log_{10}(2)}$.

a Convert the following logarithms to base 10.

i $\log_3(7)$ **ii** $\log_7(3)$ **iii** $\log_{11}(45)$

iv $\log_{101}(12)$ **v** $\log_{19}(24)$ **vi** $\log_a(b)$

b Write the following quotients as a single logarithm, for which the base is the value appearing inside the brackets of the logarithm in the denominator.

i $\dfrac{\log_{10}(8)}{\log_{10}(9)}$ **ii** $\dfrac{\log_{10}(9)}{\log_{10}(8)}$ **iii** $\dfrac{\log_2(18)}{\log_2(5)}$

iv $\dfrac{\log_e(6)}{\log_e(15)}$ **v** $\dfrac{\log_e(75)}{\log_e(2)}$ **vi** $\dfrac{\log_x(v)}{\log_x(t)}$

c Write $x = \log_2(54)$ in exponential form.

d Solve the exponential equation from part **c** using a base 10 logarithm to show that $x = \dfrac{\log_{10}(54)}{\log_{10}(2)}$.

e Use a similar method to show that $\log_b(a) = \dfrac{\log_x(a)}{\log_x(b)}$.

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Solving exponential equations

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3I

Chapter summary

Surds

- $(\sqrt{a})^2 = a$
- $\sqrt{a^2} = a$

Multiplying surds

- $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
- $a\sqrt{b} \times c\sqrt{d} = a \times c \times \sqrt{b \times d}$

Dividing surds

- $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$
- $a\sqrt{b} \div c\sqrt{d} = \dfrac{a\sqrt{b}}{c\sqrt{d}} = \dfrac{a}{c} \times \sqrt{\dfrac{b}{d}}$

Adding and subtracting surds

- $a\sqrt{b} + c\sqrt{b} = (a + c)\sqrt{b}$
- $6\sqrt{5} - 3\sqrt{2} - 8\sqrt{2} + 5\sqrt{5}$

Negative indices

- $a^{-1} = \dfrac{1}{a}$
- $\dfrac{1}{a^{-m}} = a^m$
- $a^{-m} = \dfrac{1}{a^m}$

Rationalising the denominator

$$(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$$

conjugates rational number

- $\dfrac{a}{\sqrt{b}} \times \dfrac{\sqrt{b}}{\sqrt{b}} = \dfrac{a\sqrt{b}}{b}$
- $\dfrac{1}{a + \sqrt{b}} \times \dfrac{a - \sqrt{b}}{a - \sqrt{b}} = \dfrac{a - \sqrt{b}}{a^2 - b}$

Fractional indices

base $\rightarrow a^{\frac{1}{3}} \leftarrow$ index/exponent/power

index form

index

root

$$a^{\frac{2}{3}} = \left(\sqrt[3]{a}\right)^2$$

fractional index form surd form

	Integer indices	Fractional indices
Zero index law	$a^0 = 1$	$a^0 = 1$
Index law 1	$a^5 \times a^3 = a^{5+3}$	$a^{\frac{1}{5}} \times a^{\frac{1}{3}} = a^{\frac{1}{5}+\frac{1}{3}}$
Index law 2	$a^5 \div a^3 = a^{5-3}$	$a^{\frac{1}{5}} \div a^{\frac{1}{3}} = a^{\frac{1}{5}-\frac{1}{3}}$
Index law 3	$(a^5)^3 = a^{5\times 3}$ $(ab)^3 = a^3 b^3$ $\left(\dfrac{a}{b}\right)^3 = \dfrac{a^3}{b^3}$	$\left(a^{\frac{1}{5}}\right)^{\frac{1}{3}} = a^{\frac{1}{5}\times\frac{1}{3}}$ $(ab)^{\frac{1}{3}} = a^{\frac{1}{3}} b^{\frac{1}{3}}$ $\left(\dfrac{a}{b}\right)^{\frac{1}{3}} = \dfrac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}}$

Logarithms and exponentials

- $\log_a(1) = 0$
- $\log_a(a) = 1$
- $\log_a(x^n) = n \times \log_a(x)$
- $\log_a\left(\dfrac{x}{y}\right) = \log_a(x) - \log_a(y)$

base

index

$$a^y = x \leftrightarrow y = \log_a(x)$$

exponential form logarithmic form

Chapter review

Chapter review quiz
Take the chapter review quiz to assess your knowledge of this chapter.

Quizlet
Test your knowledge of this topic by working individually or in teams.

Multiple-choice

10A **3A** **1** Which of these numbers is irrational?

A $\sqrt{120}$ **B** $-\sqrt{16}$ **C** $3\sqrt{9}$ **D** $\sqrt[3]{8}$ **E** $\log_2(8)$

10A **3B** **2** Which of these numbers is the simplest form of $4\sqrt{32}$?

A $8\sqrt{8}$ **B** $4\sqrt{2}$ **C** $8\sqrt{2}$ **D** $16\sqrt{2}$ **E** $\sqrt{512}$

10A **3C** **3** Which of these surd expressions is equivalent to $\sqrt{35}$?

A $\sqrt{30} \times \sqrt{5}$ **B** $\sqrt{7} \times 5$ **C** $\sqrt{7} \times \sqrt{5}$ **D** $\sqrt{5} \times 5\sqrt{5}$ **E** $\sqrt{30} + \sqrt{5}$

10A **3D** **4** Which of these surd expressions does $2\sqrt{50} - 6\sqrt{2} + 4\sqrt{18}$ simplify to?

A $8\sqrt{2}$ **B** $2\sqrt{50} - 6\sqrt{2} + 4\sqrt{18}$ **C** $80\sqrt{2}$
D $16\sqrt{2}$ **E** $28\sqrt{2}$

10A **3D** **5** After expanding, what does $2\sqrt{6}(4\sqrt{3} + 5\sqrt{10})$ simplify to?

A $8\sqrt{18} + 10\sqrt{60}$ **B** $24\sqrt{2} + 20\sqrt{15}$ **C** $24\sqrt{2} + 5\sqrt{10}$
D $72\sqrt{2} + 40\sqrt{15}$ **E** $24\sqrt{2} + 5\sqrt{10}$

10A **3E** **6** Which of these is equivalent to $\frac{3\sqrt{5}}{2\sqrt{2}}$ written with a rational denominator?

A $\frac{3\sqrt{10}}{2}$ **B** $\frac{3\sqrt{10}}{4}$ **C** $\frac{3\sqrt{7}}{4}$ **D** $\frac{3\sqrt{10}}{8}$ **E** $\frac{15}{2\sqrt{10}}$

10A **3F** **7** Which of these expressions is *not* equivalent to $x^{\frac{3}{4}}$?

A $\sqrt[4]{x^3}$ **B** $(\sqrt[4]{x})^3$ **C** $(x^6)^{\frac{1}{8}}$ **D** $(x^3)^{\frac{1}{4}}$ **E** $(\sqrt[3]{x})^4$

10A **3G** **8** What is $x^y = z$ equivalent to in logarithmic form?

A $\log_x(z) = y$ **B** $\log_x(y) = z$ **C** $\log_z(x) = y$ **D** $\log_z(y) = x$ **E** $\log_y(z) = x$

10A **3H** **9** Which of these is an equivalent logarithmic expression for $\log_{10}(42)$?

A $\log_{10}(20) + \log_{10}(22)$ **B** $\log_{10}(6) + \log_{10}(7)$ **C** $\log_{10}(6) \times \log_{10}(7)$
D $\frac{\log_{10}(7)}{\log_{10}(6)}$ **E** $\frac{\log_{10}(84)}{\log_{10}(2)}$

10A **3I** **10** If $8^x = 32$, which of these is the solution to the equation?

A $x = 24$ **B** $x = \frac{1}{4}$ **C** $x = 4$ **D** $x = \frac{3}{5}$ **E** $x = \frac{5}{3}$

Short answer

10A **3A** **1** Determine whether each of these numbers is rational or irrational.

a $\frac{3}{7}$ **b** 0.256 371… **c** $-\sqrt{25}$
d $\frac{3\sqrt{5}}{2}$ **e** $4\sqrt{2} + 16$ **f** $0.\overline{215}$

10A **3A** **2** Determine which of these numbers are surds.

a $3\sqrt{8}$ **b** $\frac{1}{\sqrt{5}}$ **c** $6\sqrt{4}$
d $\sqrt{6} - 1$ **e** $\sqrt{\frac{9}{25}}$ **f** $\frac{\sqrt{6}}{4}$

10A **3A** **3** Use a calculator to find the value of each of these, correct to three decimal places.

a $2\sqrt{6}$ **b** $\frac{\sqrt{5} - 2}{4}$

10A 3A **4** Show that each of these numbers is rational.

a 0.65 **b** 0.2175 **c** $0.5\dot{2}$ **d** $1.\overline{23}$

10A 3B **5** Write each of these terms in simplest form.

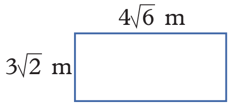
a $2\sqrt{24}$ **b** $3\sqrt{125}$ **c** $\frac{1}{2}\sqrt{108}$

10A 3C **6** Calculate the value of each of these, writing your answers in simplest form.

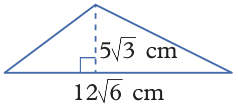
a $3\sqrt{12} \times 4\sqrt{6}$ **b** $-\sqrt{20} \times 5\sqrt{45}$

10A 3C **7** Calculate the exact area of each of these shapes, in simplest surd form.

a



b



10A 3C **8** Calculate the value of each of these.

a $\sqrt{11} \times \sqrt{7}$ **b** $3\sqrt{6} \times 9\sqrt{5}$

10A 3C **9** Calculate the value of each of these.

a $\sqrt{24} \div \sqrt{8}$ **b** $\frac{4\sqrt{45}}{\sqrt{15}}$

10A 3D **10** Simplify each of these expressions.

a $3\sqrt{2} - 6\sqrt{2} + \sqrt{2}$ **b** $3\sqrt{7} + 2\sqrt{5} - \sqrt{7} + 3\sqrt{5}$

c $2\sqrt{10} - \sqrt{2} + 4\sqrt{5} + \sqrt{2}$ **d** $5\sqrt{6} - 2\sqrt{6} + 5 + 3\sqrt{5} - 3$

10A 3D **11** Simplify each of these expressions.

a $2\sqrt{24} - 2\sqrt{54} - 5\sqrt{96} + \sqrt{20}$ **b** $\sqrt{108} - 2\sqrt{75} + 2\sqrt{48} + 3\sqrt{8}$

c $4\sqrt{180} - 6\sqrt{125} + 3\sqrt{245}$ **d** $3\sqrt{200} + 4\sqrt{180} + \sqrt{160} - 3\sqrt{128}$

10A 3D **12** Expand and simplify each of these expressions.

a $(2 + 4\sqrt{3})(3 - 2\sqrt{8})$ **b** $(2\sqrt{3} + 4\sqrt{5})^2$

10A 3E **13** Write each of these fractions with a rational denominator, writing the numerator in simplest surd form where appropriate.

a $\frac{5}{2\sqrt{6}}$ **b** $\frac{3\sqrt{15}}{2\sqrt{5}}$ **c** $\frac{4\sqrt{5}}{\sqrt{3} + 2\sqrt{6}}$ **d** $\frac{2 + \sqrt{6}}{3\sqrt{7} - 5\sqrt{3}}$

10A 3F **14** Use the appropriate index law to simplify each of these expressions. Write each answer:

i in fractional index form **ii** in an appropriate surd form.

a $\left(x^{\frac{3}{5}}\right)^{\frac{5}{2}}$ **b** $x^{\frac{1}{2}} \times x^{\frac{1}{3}}$ **c** $\frac{x^{\frac{2}{3}}}{9x^{\frac{1}{4}}}$

d $\sqrt{8x^3y^2} \times \sqrt{3xy}$ **e** $\left(9x^{\frac{1}{4}}y^{\frac{2}{5}}\right)^{\frac{1}{2}}$ **f** $\frac{\sqrt{3x^5y^2z}}{5\sqrt{8x^2yz^2}}$

10A 3G **15** Write each of these equations in logarithmic form.

a $3^4 = 81$ **b** $6^3 = 216$ **c** $2^5 = 32$ **d** $x^y = w$

10A 3G **16** Write each of these equations in exponential form.

a $\log_2(8) = 3$ **b** $\log_4(64) = 3$ **c** $\log_{10}\left(\frac{1}{10\,000}\right) = -4$ **d** $\log_x(z) = b$

10A 3H **17** Use the appropriate logarithm law to calculate the value of each of these.

a $\log_5(5^3)$ **b** $\log_4(96) - \log_4(6)$

c $3\log_9(3) + \log_9(27)$ **d** $2\log_3(4) - \log_3(32) + \log_3(54)$

10A 3H **18** Solve each of these logarithmic equations for x.

a $\log_x(32) = 5$ **b** $\log_5(x) = -3$ **c** $\log_4(256) = x$

d $\log_x\left(\frac{1}{8}\right) = 3$ **e** $\log_2(x) + \log_2(5) = \log_2(12)$ **f** $\log_5(4x) - \log_5(5) = 3$

10A 3I **19** Solve each of these equations.

a $4^{x+2} = 16$ **b** $81^{x+1} = 27$ **c** $5^{3x} = 15$ **d** $6^{2x+1} = 42$

Analysis

10A **1** A storage chest is the shape of a cube with edge lengths each measuring $\left(7\sqrt{10} + 11\sqrt{7}\right)$ cm.

- **a** Draw a labelled diagram to represent the chest.
- **b** Calculate the exact area of one of the faces of the chest.
- **c** Is the area you calculated in part **b** a rational number or an irrational number?
- **d** All the external faces of the chest need to be covered with a special coating to protect it against wear and tear. What is the total surface area that needs to be covered?
- **e** If the special coating is sold for $12.80 per square metre, how much will it cost to cover the chest?
- **f** An alternative protective paint offers the best protection when three coats are applied. It is advertised that 1 L of this protective paint will cover an area of 5 m^2. Show that a 1 L can of the protective paint is enough to provide best protection for the chest.
- **g** Calculate the percentage of the paint in the 1 L can from part **f** that will be used to cover the chest with three coats.
- **h** A narrow 80 cm long pole needs to fit inside the chest. Show that it is not possible for the pole to lie along the bottom of the chest.
- **i** Is there any way that the pole from part **h** can fit inside the chest? Use appropriate mathematics to show your working and to support your answer.
- **j** What assumptions did you need to consider for the calculations you have performed in this task?

10A **2** Toby has begun beekeeping. He started with 10 000 bees in his four-frame hive. Two months later, there are now 40 000 bees in the hive. The number of bees, B, in Toby's hive after t months can be modelled using the equation $B = a \times 10^{bt}$.

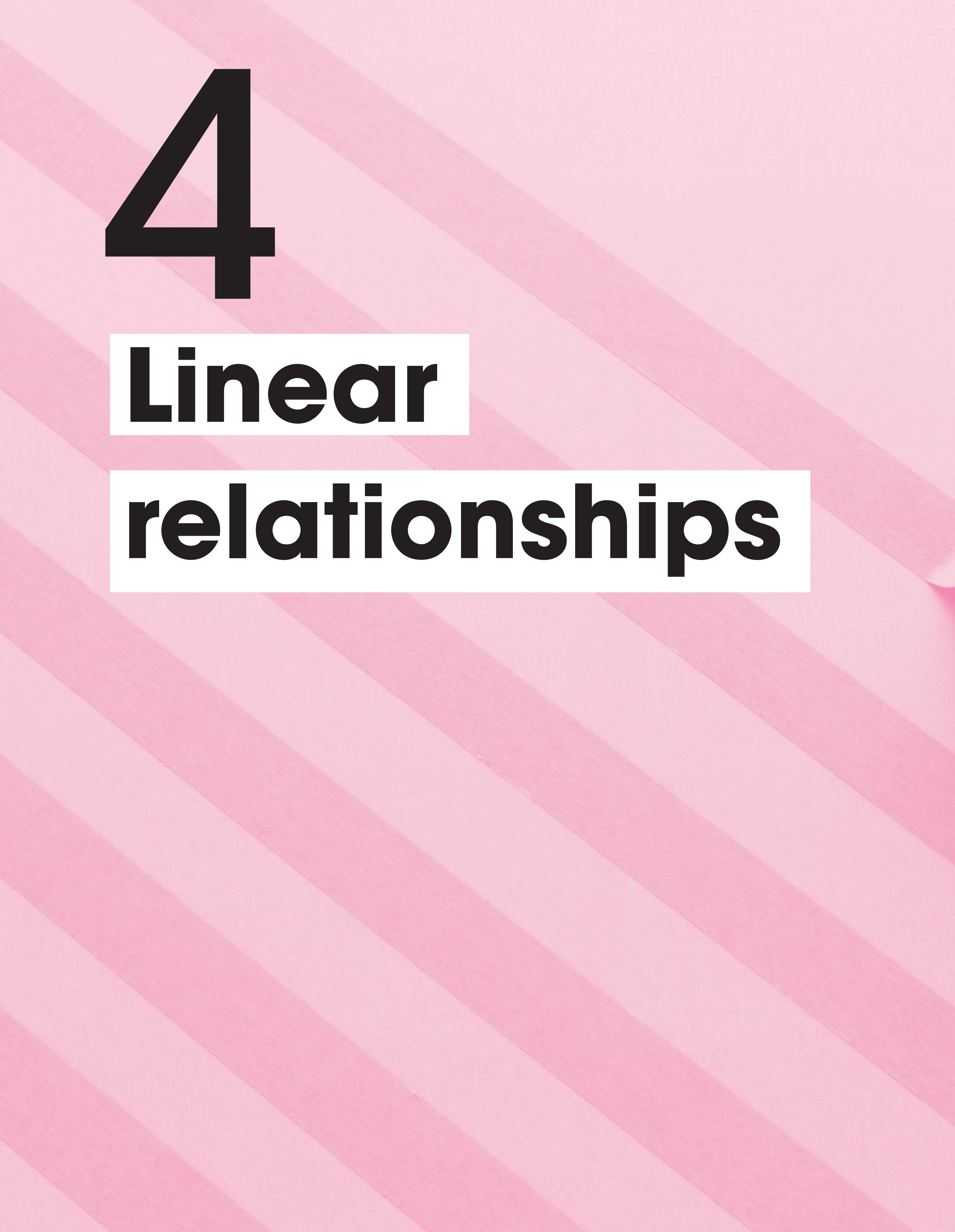
- **a** State the value of a.
- **b** Determine the value of b:
  - **i** as an exact value
  - **ii** as a decimal correct to two decimal places.
- **c** Write the equation modelling the number of bees in the form $B = a \times d^t$.
- **d** Determine the expected number of bees after 6 months.
- **e** Toby's four-frame hive can support up to 1 000 000 bees.

  After how many months does Toby need to expand his hive to allow for the population of bees to continue growing? Write your answer:
  - **i** as an exact value
  - **ii** correct to the nearest month.
- **f** After some years, Toby's population of bees became relatively constant at 10 million. Toby began to notice his population of bees declining in number. After a year, there was approximately 9.1 million bees.

  Determine the equation for the number of bees, B, n years after Toby noticed the population was declining, in the form $B = p \times q^n$.
- **g** If Toby does not make any changes, how many years will it take for the number of bees to drop below his original 10 000 bees?

4

Linear relationships



Index

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Number lines
- ✔ The Cartesian plane
- ✔ Plotting linear relationships
- ✔ Intercepts

Curriculum links

- Solve problems involving linear equations, including those derived from formulas (VCMNA335)
- Solve linear inequalities and graph their solutions on a number line (VCMNA336)
- Solve simultaneous linear equations, using algebraic and graphical techniques including using digital technology (VCMNA337)
- Solve problems involving gradients of parallel and perpendicular lines (VCMNA338)
- Solve linear equations involving simple algebraic fractions (VCMNA340)

Materials

- ✔ Ruler
- ✔ Graph paper

4A Solving linear equations

Learning intentions

- ✔ I can solve linear equations using inverse operations.
- ✔ I can solve linear equations with an unknown on both sides.
- ✔ I can rearrange linear equations.

Inter-year links

Year 7	6H Solving equations using inverse operations
Year 8	6C Solving equations with the unknown on both sides
Year 9	4A Solving linear equations

Linear equations

- A **linear equation** is an equation featuring pronumerals that are only raised to the power of 1. These pronumerals represent variable or unknown values.

Linear equations	Non-linear equations
$4x - 1 = 6$	$2x^2 = 32$
$y = x + 3$	$y = \frac{1}{x}$
$2a - \frac{3}{5}b = 10$	$a^3 + 6b = 0$

- If a linear equation has only one pronumeral, algebraic methods can be used to find a solution for the unknown value.
- If a linear equation has more than one pronumeral, then it will have more than one solution for each variable. It cannot be solved for numerical solutions by itself and can only be rearranged.

Solving linear equations using inverse operations

- If the same operation is performed on both sides of an equation, the result is an equation equivalent to the original equation and the value(s) of the unknown(s) will remain the same.
- Any operation can be undone by applying its inverse operation.
- Operations performed on an unknown in an equation can be undone by performing the appropriate inverse operation to both sides of the equation.

For example,
$$\begin{aligned} \frac{5x-4}{2} &= 8 && (\times 2)\\ 5x - 4 &= 16 && (+ 4)\\ 5x &= 20 && (\div 5)\\ x &= 4 \end{aligned}$$

Operation	Inverse operation
+ 3	− 3
− 3	+ 3
× 3	÷ 3
÷ 3	× 3

Rearranging linear equations

- A literal equation is an equation that mostly consists of pronumerals.
- A pronumeral is the **subject of an equation** if it is by itself on one side of the equation (usually the left-hand side) and does not appear on the other side of the equation.
- Inverse operations can be used to rearrange a literal linear equation to make any pronumeral the subject.

For example,
$$\begin{aligned} \frac{x}{a} - b &= c && (+ b)\\ \frac{x}{a} &= b + c && (\times a)\\ x &= a(b + c) \end{aligned}$$

Example 4A.1 Solving linear equations

Solve each of these equations by finding the value of x.

a $5x - 6 = -4$ **b** $2 = \frac{1 - x}{7}$

THINK	**WRITE**
a Identify the correct inverse operation and apply it to both sides of the equation until x is the subject of the equation.	**a** $5x - 6 = -4$ $(+\ 6)$ $5x = 2$ $(\div\ 5)$ $x = \frac{2}{5}$
b **1** Identify the correct inverse operation and apply it to both sides of the equation until x is the subject of the equation. The numerator of a fraction is always grouped as if it is enclosed in brackets, so the inverse operations in the numerator must be applied last.	**b** $2 = \frac{1 - x}{7}$ $(\times\ 7)$ $14 = 1 - x$ $(-\ 1)$
2 Change the sign when dividing by −1 on both sides of the equation. Write the solution so that x is on the left-hand side (LHS).	$13 = -x$ $(\div\ (-1))$ $x = -13$

Example 4A.2 Solving linear equations with an unknown in the denominator

Solve $\frac{24}{x + 2} - 1 = 5$ by finding the value of x.

THINK	**WRITE**
1 Use inverse operations to rearrange the equation so that the fraction is by itself on one side.	$\frac{24}{x + 2} - 1 = 5$ $(+\ 1)$
2 Multiply both sides of the equation by the denominator of the fraction.	$\frac{24}{x + 2} = 6$ $(\times\ (x + 2))$
3 Identify and apply the correct inverse operations to both sides of the equation.	$24 = 6(x + 2)$ $(\div\ 6)$ $4 = x + 2$ $(-\ 2)$ $2 = x$
4 Write the solution to the equation with x on the LHS.	$x = 2$

Example 4A.3 Solving linear equations with the unknown on both sides

Solve each of these equations by finding the value of x.

a $9x + 2 = 7x - 8$ **b** $5 - 2x = 6x - 1$

THINK	WRITE
a **1** Remove the pronumeral term from one side of the equation by subtracting the pronumeral that has the smaller coefficient ($7x$) from both sides of the equation.	**a** $9x + 2 = 7x - 8 \quad (-\,7x)$
2 Solve the equation using inverse operations.	$2x + 2 = -8 \quad (-\,2)$ $2x = -10 \quad (\div\,2)$ $x = -5$
b **1** Remove the negative pronumeral term by adding $2x$ to both sides of the equation.	**b** $5 - 2x = 6x - 1 \quad (+\,2x)$
2 Solve the equation using inverse operations.	$5 = 8x - 1 \quad (+\,1)$ $6 = 8x \quad (\div\,8)$ $\frac{6}{8} = x$
3 Simplify the solution and write x on the LHS.	$x = \frac{3}{4}$

Example 4A.4 Rearranging linear equations

Rearrange each equation to make x the subject on the left-hand side.

a $2x - m = 4n$ **b** $\frac{h+1}{x} = \frac{a}{b}$

THINK	WRITE
a Identify the correct inverse operation and apply it to both sides of the equation until x is the subject of the equation.	**a** $2x - m = 4n \quad (+\,m)$ $2x = 4n + m \quad (\div\,2)$ $x = \frac{4n + m}{2}$
b **1** Multiply both sides of the equation by the denominator containing x.	**b** $\frac{h+1}{x} = \frac{a}{b} \quad (\times\,x)$
2 Identify the inverse operation and apply it to both sides of the equation until x is the subject of the equation.	$h + 1 = \frac{ax}{b} \quad (\times\,b)$ $b(h + 1) = ax \quad (\div\,a)$ $\frac{b(h+1)}{a} = x$ $x = \frac{b(h+1)}{a}$ or, in expanded form: $x = \frac{bh + b}{a}$

Helpful hints

✔ Writing the inverse operation beside the appropriate line of working is a great way to keep track of your calculations.

$$\frac{x}{3} - 5 = 7 \qquad (+\ 5)$$

✔ Writing the pronumeral on the LHS of your solution is only a convention. Once you have isolated the unknown on one side of the equation, you have the solution.
For example, $3 = x$ is the same as $x = 3$.

✔ When given a problem in words, define your pronumerals before writing the equation to represent the problem.
For example, if you are told that a community garden grows twice as many tomatoes as zucchinis, you could define the pronumerals like this: t = the number of tomatoes, z = the number of zucchinis, and the formula can be written as: $2t = z$

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Exercise 4A Solving linear equations

△ **1(1st column), 2–4(1st, 2nd columns), 5–6(d, f, h, j, l), 7, 8(b, d, f, h), 9(a, c, e), 10(a, c, e, h, k, m), 11, 12**

■ **1–5(3rd column), 6–10(2nd, 3rd columns), 13, 14, 16, 17**

⬟ **1–5(3rd column), 6–10(2nd, 3rd columns), 13, 14, 16, 17**

4A.1 **1** Solve each of these equations.

a $3x - 4 = 15$ **b** $\frac{x}{5} + 6 = -2$ **c** $3 = 5 - 7x$
d $8 = \frac{5x}{3}$ **e** $\frac{x+3}{10} = -5$ **f** $6 = 3(x + 5)$
g $3(x - 2) = -11$ **h** $1 - \frac{x}{2} = 5$ **i** $-4(5 - 3x) = 9$
j $\frac{2x-5}{8} = -3$ **k** $\frac{7-6x}{5} = 10$ **l** $6 = \frac{5(x-4)}{7}$
m $-4 = \frac{9x}{4} + 4$ **n** $12 - \frac{8x}{15} = 8$ **o** $8 = \frac{4(11-3x)}{9}$

2 Solve each of these equations by finding the value of x. (Hint: You will need to perform operations that involve fractions and decimals.)

a $x - \frac{1}{2} = -9$ **b** $2x + 3.6 = 7$ **c** $8.5x = 34$
d $\frac{x}{0.28} + 3.2 = -5.6$ **e** $\frac{3-0.5x}{4} = 0.45$ **f** $\frac{11}{3}x = \frac{5}{7}$
g $\frac{4}{3}x + \frac{5}{6} = \frac{3}{2}$ **h** $\frac{3}{4}\left(x + \frac{9}{5}\right) = \frac{11}{10}$ **i** $\frac{8}{7}\left(\frac{21}{2}x + \frac{56}{5}\right) = \frac{48}{35}$

3 Solve each of these equations.

a $2(x + 4) - 5 = -7$ **b** $3(x - 1) + 2 = 8$ **c** $4(x + 3) + 2x = 0$
d $9(x - 5) - x = 3$ **e** $2(3x + 4) + 7 = 13$ **f** $7(2x - 11) - 4x = -7$
g $5(2 - x) - 1 = -6$ **h** $4(1 - 3x) + 3x = 3$ **i** $-3(4 - x) - 2x + 5 = -1$

UNDERSTANDING AND FLUENCY

4A.2 **4** Solve each of these equations.

a $\frac{8}{x} = 4$ **b** $\frac{45}{x} = -9$ **c** $\frac{6}{x} = -1$

d $\frac{-12}{x} = 3$ **e** $\frac{-20}{x} = -5$ **f** $\frac{15}{x} = 2$

g $\frac{10}{3x} = 1$ **h** $\frac{-2}{5x} = -7$ **i** $\frac{6}{x} + 5 = 8$

j $\frac{11}{x} + 7 = -4$ **k** $\frac{7}{2x} - 1 = 13$ **l** $\frac{-24}{5x} - 2 = 0$

m $\frac{5}{x+2} = 8$ **n** $2 = \frac{21}{x-7}$ **o** $\frac{10}{2x-5} = -3$

4A.3 **5** Solve each of these equations.

a $5x + 3 = 2x + 9$ **b** $6x - 7 = 4x + 3$

c $3x + 4 = x - 2$ **d** $4x - 5 = 2x - 3$

e $3x + 8 = 2x + 1$ **f** $5x + 6 = 2 + 3x$

g $4x - 5 = 13 - 2x$ **h** $2x - 1 = 11 - x$

i $2x + 3 = 6 - 5x$ **j** $3x - 5 = 7x + 11$

k $2 - x = 4x - 8$ **l** $8x + 5 = 11x + 20$

6 Solve each of these equations by first expanding to remove the brackets.

a $3(x + 2) = 2x + 11$ **b** $8x - 9 = 5(x - 3)$

c $7x + 3 = 6(x + 1)$ **d** $5(2x + 1) = 7x + 2$

e $4(x - 7) = 2 - x$ **f** $x - 4 = 2(3 - 2x)$

g $3(x + 6) = 2(x + 7)$ **h** $9(x - 3) = 5(x + 1)$

i $4(x - 1) = 2(x + 2)$ **j** $6(x + 5) = 4(5 - x)$

k $2(3x - 4) = 11(x - 3)$ **l** $7(x - 12) = -3(x - 2)$

7 Solve each of these equations. Start by multiplying both sides of the equation by the common denominator.

a $\frac{2x+1}{5} = \frac{x-7}{5}$ **b** $\frac{4x-3}{3} = \frac{2x+5}{3}$

c $\frac{7x-2}{4} = \frac{3x-10}{4}$ **d** $\frac{x+5}{6} = \frac{11-2x}{6}$

e $\frac{9-2x}{11} = \frac{4-7x}{11}$ **f** $\frac{3(x-4)}{2} = \frac{4-x}{2}$

8 Solve each of these equations. Start by writing each fraction as an equivalent fraction with the same denominator as the other. (Hint: Some solutions may not be integers.)

a $\frac{x-2}{5} = \frac{x+4}{2}$ **b** $\frac{x-4}{3} = \frac{x+5}{4}$ **c** $\frac{x+2}{2} = \frac{x+7}{3}$

d $\frac{x+3}{4} = \frac{x+6}{7}$ **e** $\frac{x-8}{3} = \frac{x+1}{6}$ **f** $\frac{3x+5}{4} = \frac{x+9}{2}$

g $\frac{2x+1}{5} = \frac{x-2}{4}$ **h** $\frac{x-7}{3} = \frac{5-x}{4}$ **i** $\frac{2x-1}{6} = \frac{3x+2}{5}$

9 Solve each of these equations.

a $\frac{5x}{6} + \frac{4x}{3} = \frac{13}{2}$ **b** $\frac{3x}{4} - \frac{4x}{3} = \frac{21}{12}$ **c** $\frac{7x}{8} = \frac{1}{2} + \frac{5x}{6}$

d $\frac{4x}{9} + \frac{7}{3} = \frac{7}{4} - \frac{11x}{12}$ **e** $\frac{2}{5}\left(\frac{3x}{4} + \frac{7}{2}\right) = \frac{9x}{5}$ **f** $\frac{55}{6}\left(\frac{14x}{3} - \frac{2}{5}\right) = \frac{22}{15}\left(\frac{5}{6} - \frac{15x}{2}\right)$

4A.4 **10** Rearrange each of these equations to make x the subject on the left-hand side.

a $x + a = b$ **b** $k = x - p$ **c** $cx = d$

d $g + 3x = h$ **e** $y = 4x + 5$ **f** $\frac{a+x}{m} = n$

g $7x - 2w = 5$ **h** $b - \frac{x}{a} = c$ **i** $k = \frac{h}{x}$

j $6x - e = 4x + f$ **k** $2(x + y) = v$ **l** $\frac{x}{p} + q = w$

m $\frac{a}{x} - b = c$ **n** $y = \frac{kx - m}{2n}$ **o** $x + 3d = 5c - 7x$

11 Liam is saving to buy a tennis racquet that costs \$219. He is able to save \$24 per week and currently has \$105. Follow the steps below to work out how many weeks it will take Liam to save enough for the racquet.

a Choose a pronumeral to represent the unknown quantity in the problem.

b Write an equation to represent the problem, using your chosen pronumeral for the unknown value.

c Solve the equation using inverse operations.

d Write your answer to the problem, in weeks.

12 For each of these problems, set up an equation and solve it using inverse operations.

a Edith bought four makeup kits online for a total cost of \$387, which included the delivery charge of \$29. What was the cost of each makeup kit?

b The perimeter of a rectangular car park is 88 m. If the length of the car park is 12 m more than its width, what are the car park's dimensions?

c Lisa, Sara and Bindi scored a combined total of 29 goals in a netball match. Lisa scored five more goals than Sara, and Bindi scored three goals less than Sara. How many goals did each girl score?

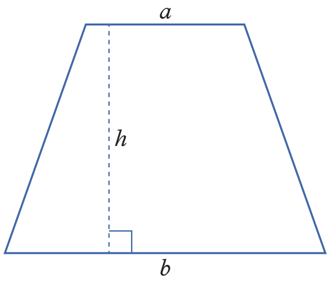
d Oliver has a budget of \$1800 for his birthday party. The cost of hiring a local hall is \$400 and catering is \$27 per person. What is the maximum number of people that Oliver can afford to attend his party?

13 Shana and Paulo have the same amount of money. Shana buys six dumplings and has \$3.80 left over. Paulo buys four dumplings and has \$9.20 left over.

a Write an equation to represent this situation.

b Solve the equation to find the cost of one dumpling.

14 The formula $A = \frac{h(a + b)}{2}$ links the area, A, of a trapezium with its three length measurements a, b and h, as shown in the diagram below.



a Calculate the area, A, of a trapezium for which $a = 4$ cm, $b = 6$ cm and $h = 10$ cm.

b Calculate the height, h, of a trapezium for which $a = 15$ mm, $b = 19$ mm and $A = 119$ mm^2. (Hint: Substitute the values for a, b and A into the formula and then solve the resulting equation to find h.)

c Calculate a when $b = 5$ m, $h = 3$ m and $A = 18$ m^2.

d Rearrange the formula to make a the subject.

e Calculate a when:

i $b = 37$ cm, $h = 24$ cm and $A = 1068$ cm^2

ii $b = 2.4$ m, $h = 6.1$ m and $A = 30.5$ m^2

PROBLEM SOLVING AND REASONING

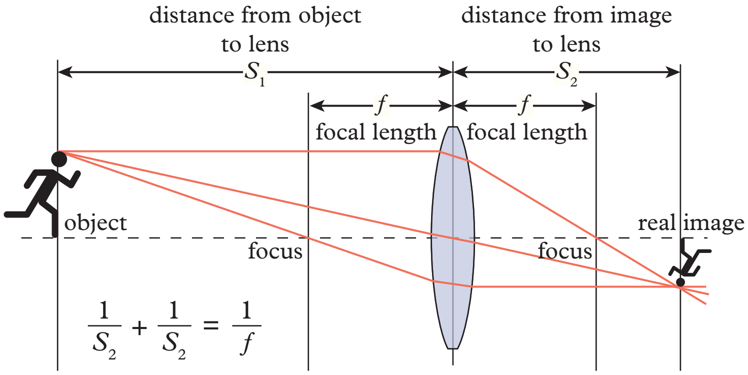
15 The optic equation, $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$, equates the sum of the reciprocals of two numbers with the reciprocal of a third. The optic equation is used for dealing with many systems, such as:

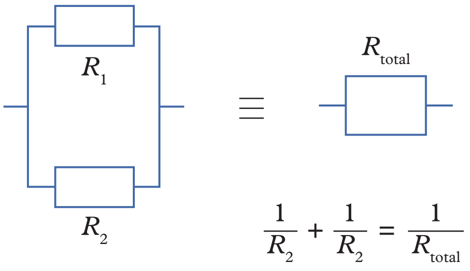
- the distances between an object and its image through a thin lens,

$$\frac{1}{\text{distance from object to lens}} + \frac{1}{\text{distance from image to lens}} = \frac{1}{\text{focal length}}$$

- the resistances in a parallel circuit,

$$\frac{1}{\text{resistance}_1} + \frac{1}{\text{resistance}_2} = \frac{1}{\text{total resistance}}$$





a Show that, for the optic equation, $c = \frac{ab}{a + b}$.

b Calculate:

 i the focal length when the distance from the object to the lens is 4 cm and the distance from the image to the lens is 12 cm

 ii the total resistance in a parallel circuit when the two resistances are 500 ohms and 2000 ohms.

c Rearrange the optic equation to make a the subject.

d Determine:

 i the distance to the object when the distance from the image is 10 cm and the focal length is 8 cm

 ii the resistance of a resistor that has been connected in parallel with a 16 ohm resistor, where the total resistance is 8 ohms.

CHALLENGE

16 The sum of five consecutive odd numbers is 50 more than triple the smallest number. Write an equation you can solve to find the five numbers. (Hint: Let the smallest number be x.)

17 Solve each of these equations.

a $\frac{2x + 5}{x - 4} = 7$ **b** $\frac{9 - 4x}{3x + 5} = 10$ **c** $\frac{5}{6} = \frac{7x - 3}{3 - 2x}$

18 Solve each of these equations. Use substitution to check your solutions.

a $\frac{3(2 - x)}{4} + 9 = 15$ **b** $\frac{4(2x - 1)}{3} - 5 = 4x - 17$ **c** $\frac{2x - 3}{5} + \frac{x + 7}{3} = 1$

d $\frac{x + 5}{2} - \frac{x - 7}{6} = 8$ **e** $\frac{3(2x - 3)}{4} + \frac{x + 23}{10} = 2x$ **f** $\frac{5x + 6}{3} = \frac{1 - 2x}{4} = 3x + 4$

19 Make x the subject of each of the following.

a $\frac{mx + p}{nx + q} = r$ **b** $\frac{a + by}{b + ax} = \frac{b - ay}{a - bx}$ **c** $\frac{a + by}{b - ax} = \frac{b + ay}{a - bx}$

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Solving equations with the unknown in the denominator

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Calendar capers

CAS instructions
Solving equations

Topic quiz
4A

4B Solving linear inequalities

Learning intentions

- ✔ I can represent linear inequalities using a number line.
- ✔ I can solve linear inequalities using inverse operations.
- ✔ I can solve linear inequalities with the unknown on both sides using inverse operations.

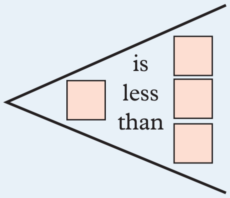
Inter-year links

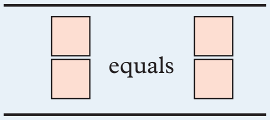
Year 7	6H Solving equations using inverse operations
Year 8	6C Solving equations with the unknown on both sides
Year 9	4A Solving linear equations

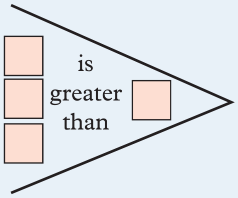
Linear inequalities

- An inequality is a mathematical statement that compares the values of two unequal expressions. Inequalities are written using inequality signs, like the ones shown in the table below.

Inequality sign	Meaning	Example	Meaning of example
$\neq$	'is **not** equal to'	$x \neq 2$	x is **not** equal to 2
$<$	'is less than'	$x < 2$	x is less than 2
$>$	'is greater than'	$x > 2$	x is greater than 2
$\leq$	'is less than or equal to'	$x \leq 2$	x is less than or equal to 2
$\geq$	'is greater than or equal to'	$x \geq 2$	x is greater than or equal to 2







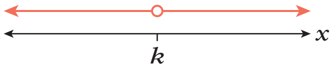
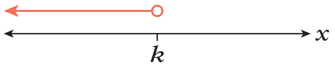
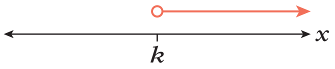
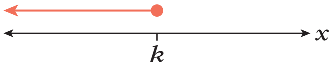
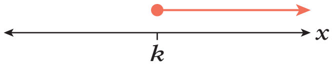
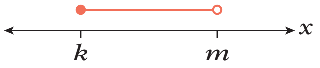
- A linear inequality is an inequality containing pronumerals only raised to the power of 1.
- Any variable can be limited to a set of possible values by writing an inequality that places it between two inequality signs, indicating that it must be greater than (or perhaps equal to) one number and less than (or perhaps equal to) another number.

  For example, $2 < x \leq 5$ means '2 is less than x, but x is less than or equal to 5', which is the same as 'x is greater than 2, but less than or equal to 5'.

Representing linear inequalities on a number line

- Inequalities involving a single variable can be represented on a number line, using rays and line segments to represent all of the values that satisfy the inequality (see the table on the next page).

  On a number line:
  - → a line or line segment indicates that all the values the line passes through are included in the possible range of values for the variable
  - → a ray (arrowed line) indicates that the possible values for the variable have no limit and continue to positive or negative infinity in the direction indicated
  - → a closed circle (solid dot ●) indicates that the endpoint value is included, meaning the value is either 'less than or equal to' (≤) or 'greater than or equal to' (≥) the value represented by that point on the number line
  - → an open circle (hollow dot ○) indicates that the value is not included, meaning the value is either 'less than' (<) or 'greater than' (>) the value represented by that point on the number line.

Inequality	Number line representation	Meaning
$x \neq k$		x is not equal to k
$x < k$		x is less than k
$x > k$		x is greater than k
$x \leq k$		x is less than or equal to k
$x \geq k$		x is greater than or equal to k
$k \leq x < m$		x is greater than or equal to k but less than m

Solving linear inequalities

- If the same operation is performed on both sides of an inequality sign, this results in an equivalent inequality of a different form. The possible values of the unknown will remain the same.
- Inverse operations can be used to solve linear inequalities. However, the inequality sign will be reversed when both sides of the inequality are multiplied or divided by a negative number.

  For example, $5 - 2x \leq 1 \quad (-\ 5)$

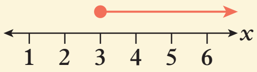
  $-2x \leq -4 \quad (\div\ (-\ 2))$

  $x \geq 2$

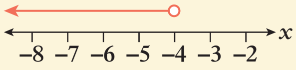
Example 4B.1 Representing linear inequalities on a number line

Write the inequality represented by each of these number lines.

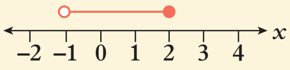
a



b



c



THINK	WRITE
a The line has a closed circle at 3, indicating that x can be equal to 3. The ray points to the right, indicating that x can be any value greater than 3. So x is greater than or equal to 3.	**a** $x \geq 3$
b The line has an open circle at –4, indicating that x cannot be equal to –4. The ray points to the left, indicating that x can be any value less than –4. So x is less than –4.	**b** $x < -4$
c The line has an open circle at –1, indicating that x cannot be equal to –1. The line segment extends to a closed circle at 2, indicating that x can be equal to 2. So x is greater than –1 but less than or equal to 2.	**c** $-1 < x \leq 2$

Example 4B.2 Solving linear inequalities

Solve each of these inequalities.

a $\frac{x}{4} - 3 \le 0$ **b** $-2 > 5x + 3$ **c** $3 > -1 - \frac{x}{7}$

THINK	**WRITE**
a **1** Apply the inverse operations to both sides in the correct order.	**a** $\frac{x}{4} - 3 \le 0$ (+ 3) $\frac{x}{4} \le 3$ (× 4) $x \le 12$
2 Check your solution. Substitute $x = 12$ into the LHS of the inequality. The result should equal the right-hand side (RHS). Then substitute any value for x that is less than 12 into the original inequality to check the inequality sign. The result should be less than or equal to the RHS.	Check by substituting $x = 12$ into $\frac{x}{4} - 3 \le 0$. LHS $= \frac{12}{4} - 3$ $= 0$ = RHS (For $x = 12$, LHS = RHS as required.) Check by substituting a value less than 12 (for example, $x = 8$). LHS $= \frac{8}{4} - 3$ $= -1$ $-1 < 0$ (For $x < 12$, LHS < RHS as required.)
b **1** Apply the inverse operations to both sides in the correct order.	**b** $-2 > 5x + 3$ ($-$ 3) $-5 > 5x$ (÷ 5) $-1 > x$
2 Write the solution with x on the LHS, reverse the inequality sign if needed.	$x < -1$
3 Check your solution. Substitute $x = -1$ into the RHS of the inequality. The result should equal the LHS. Then substitute any value for x that is less than -1 into the original inequality to check the inequality sign. The result should be less than the LHS.	Check by substituting $x = -1$ into $-2 > 5x + 3$. RHS $= 5(-1) + 3$ $= -2$ = LHS (For $x = -1$, LHS = RHS as required.) Check by substituting a value less than -1 (for example, $x = -2$). RHS $= 5(-2) + 3$ $= -7$ $-7 < -2$ (For $x < -1$, LHS > RHS as required.)
c **1** Apply the inverse operations to both sides in the correct order.	**c** $3 > -1 - \frac{x}{7}$ (+ 1) $4 > -\frac{x}{7}$ (× ($-$7)) $-28 < x$
2 Write the solution with x on the LHS, reverse the inequality sign if needed.	$x > -28$
3 Check your solution. Substitute $x = -28$ into the RHS of the inequality. The result should equal the LHS. Then substitute any value for x that is greater than -28 into the original inequality to check the inequality sign. The result should be less than the LHS.	Check by substituting $x = -28$ into $3 > -1 - \frac{x}{7}$. RHS $= -1 - \frac{-28}{7}$ $= -1 + 4$ $= 3$ = LHS (For $x = -28$, LHS = RHS as required.) Check by substituting a value greater than -28 (for example, $x = 0$). RHS $= -1 - \frac{0}{7}$ $= -1$ $-1 < 3$ (For $x > -28$, LHS > RHS as required.)

Example 4B.3 Solving inequalities with the unknown on both sides

Solve each of these inequalities.

a $7x - 2 \geq 3x - 10$ **b** $4 - 5x < 3 - 8x$

THINK

1 Use inverse operations to remove the pronumeral on the side where the coefficient is a number less than the coefficient on the other side.

2 Identify and apply the correct inverse operations, in order, to both sides of the inequality.

3 Check your solution by substituting appropriate values for x into the original inequality to see if true inequalities result.

WRITE

a $7x - 2 \geq 3x - 10 \quad (-\ 3x)$

$4x - 2 \geq -10 \quad (+\ 2)$

$4x \geq -8 \quad (\div\ 4)$

$x \geq -2$

Check by substituting $x = -2$ into $7x - 2 \geq 3x - 10$.

$\text{LHS} = 7(-2) - 2$

$= -16$

$\text{RHS} = 3(-2) - 10$

$= -16$ (For $x = -2$, LHS = RHS as required.)

Check by substituting a value greater than -2 (for example, $x = 0$).

$\text{LHS} = 7(0) - 2$

$= -2$

$\text{RHS} = 3(0) - 10$

$= -10$ (For $x > -2$, LHS > RHS as required.)

b $4 - 5x < 3 - 8x \quad (+\ 8x)$

$4 + 3x < 3 \quad (-\ 4)$

$3x < -1 \quad (\div\ 3)$

$x < -\frac{1}{3}$

Check by substituting $x = -\frac{1}{3}$ into $4 - 5x < 3 - 8x$.

$\text{LHS} = 4 - 5\left(-\frac{1}{3}\right)$

$= \frac{17}{3}$

$\text{RHS} = 3 - 8\left(-\frac{1}{3}\right)$

$= \frac{17}{3}$ (For $x = -\frac{1}{3}$, LHS = RHS as required.)

Check by substituting a value less than $-\frac{1}{3}$ (for example, $x = -1$).

$\text{LHS} = 4 - 5(-1)$

$= 9$

$\text{RHS} = 3 - 8(-1)$

$= 11$

$9 < 11$ (For $x < -\frac{1}{3}$, LHS < RHS as required.)

Helpful hints

- ✔ Be careful when multiplying and dividing inequalities by negative numbers.
- ✔ If you need to swap the LHS and RHS of an inequality, remember to reverse the inequality sign so that it is still pointing towards the lesser expression. For example, if $5 \leq x$, then $x \geq 5$.
- ✔ When solving linear inequalities with the unknown on both sides, always remove the pronumeral on the side where the coefficient is less. That way you will always have a positive value for the remaining coefficient. Remember, $4 > 2$ but $-4 < -2$.

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Exercise 4B Solving linear inequalities

△ 1, 2, 3(a, c, e, g, i), 4, 5(1st, 2nd columns), 6, 7, 8(a, d, f, g), 9, 11, 12

□ 2–3(2nd column), 5–7(3rd, 4th columns), 8(c, f, h, i), 9, 10, 13, 17(a, b)

⬠ 2–3(2nd column), 6–8(e–h), 9, 14–16, 17(d–f), 18

UNDERSTANDING AND FLUENCY

1 Write each of the following statements as an inequality.

a Two is less than three.

b Three is greater than two.

c Three is greater than or equal to two.

d Negative three is less than or equal to negative two.

e Three is not equal to negative three.

f Negative three is not equal to three.

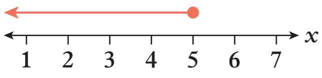
g Four is less than five which is less than six.

h Five is greater than one and less than or equal to ten.

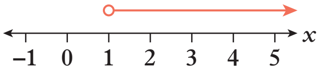
i Negative five is greater than or equal to negative ten and less than negative one.

4B.1 **2** Write the inequality represented on each of these number lines.

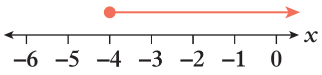
a



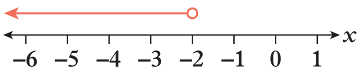
b



c



d



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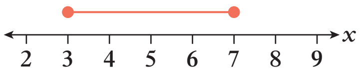

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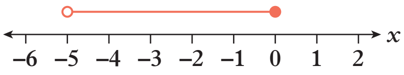

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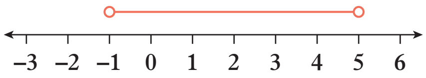

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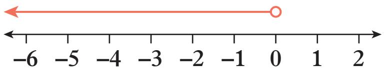

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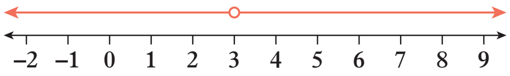
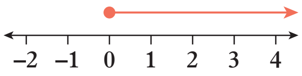
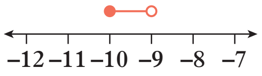


j



k



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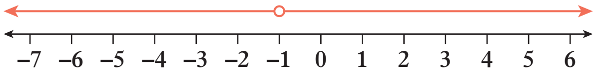

3 For each of these inequalities, show the values of x on a number line.

a $x > 4$

b $x \leq 3$

c $x \geq -2$

d $x < 0$

e $x \neq 2$

f $1 \leq x \leq 5$

g $-4 < x < 2$

h $0 < x \leq 6$

i $-2.5 \leq x < -0.5$

j $x \neq -2$

4 Apply the following operations to each side of the following inequalities. Ensure the correct inequality symbol is written between the two numbers.

a $-4 < 9$

b $x \geq 6$

c $-1 < x \leq 4$

i add five

ii subtract nine

iii multiply by three

iv multiply by one-half

v multiply by negative three

vi multiply by negative one-half

vii divide by three

viii divide by negative three

UNDERSTANDING AND FLUENCY

4B.2 **5** Solve each of these inequalities.

a $x + 5 > 9$ **b** $\frac{x}{10} \le 3$ **c** $-20 > 4x$ **d** $-3x \ge -15$

e $7 < -\frac{x}{10}$ **f** $\frac{x}{5} + 8 \le 11$ **g** $2x - 9 \ge -5$ **h** $26 > 14 - 4x$

i $-x + 3 \ge 12$ **j** $5(x + 3) \le 25$ **k** $\frac{x - 8}{12} > -3$ **l** $5 < 10x + 14$

m $12 - 14x > 14$ **n** $-17 \le -6x + 12$ **o** $-6(x - 4) < 13$ **p** $-11(3x + 5) \ge 40$

6 Solve each of these inequalities and show the solution on a number line each time.

a $3x + 2 \ge -13$ **b** $1 - 7x < -6$ **c** $5 - \frac{x}{6} \le 2$ **d** $\frac{x - 4}{2} < 5$

e $\frac{7 - x}{3} \ge -1$ **f** $\frac{2 + x}{5} > 3$ **g** $\frac{3x}{2} - 5 \le 4$ **h** $\frac{8 - x}{7} - 3 < -2$

4B.3 **7** Solve each of these inequalities. Use substitution to check your solutions.

a $4x - 3 > 2x + 5$ **b** $7x + 1 \le 3x - 7$ **c** $5x - 9 \ge 9 - x$ **d** $2x + 4 < 5x - 8$

e $3 - x > 4x - 2$ **f** $x + 11 \ge 7 - 3x$ **g** $8 - 3x \le 18 - 5x$ **h** $1 - 6x > -3x - 5$

8 Solve each of these inequalities by first expanding to remove brackets.

a $2(x - 3) > x + 5$ **b** $4x - 7 \le 3(x + 2)$ **c** $5(x + 1) \ge -3x - 11$

d $3(2x - 5) < 7x + 4$ **e** $9(x + 5) < 10(x + 6)$ **f** $3(3x - 1) \le -2(x - 4)$

g $4(x + 3) \ge 2(5x - 3)$ **h** $2(2 - x) > 3(1 - 2x)$ **i** $5(3 - 2x) \le -4(2x - 7)$

9 Solve each of these inequalities. Use substitution to check your solutions.

a $\frac{3x + 2}{7} \le \frac{2x - 5}{7}$ **b** $\frac{2 - x}{3} < \frac{6x - 5}{3}$ **c** $\frac{1 - 4x}{2} \ge \frac{11 - 5x}{4}$

d $\frac{x + 4}{5} > \frac{x + 2}{3}$ **e** $\frac{x - 4}{7} \ge \frac{4 - x}{2}$ **f** $\frac{4x - 1}{6} < \frac{3x + 1}{4}$

10 a How many solutions does a linear equation have?

b How many solutions can a linear inequality have? Explain.

PROBLEM SOLVING AND REASONING

11 Write an inequality statement to represent each of the following situations. Use a pronumeral for the unknown quantity each time.

a A house with a reserve price of \$650 000 is sold at auction. What price could the house have sold for?

b To go on a theme park ride, a person must be over 97 cm tall. What could be the height of a person on this ride?

c The speed limit along a road is 60 km/h. At what speeds could a person legally travel along this road?

d To ride the Jet Rescue ride at Sea World, your height must be in the range 125 cm to 196 cm. What could the height of a person on this ride be?

12 Todd is deciding how many packs of chewing gum he should buy. Each pack costs \$3 and he has \$20 in his wallet.

a Write an inequality to represent this situation and then solve it.

b List the number of packs of gum Todd could possibly buy.

13 Emily and Ethan are selling watermelons at a market. They start with 20 melons and agree to share any that are left at the end of the day.

a Write an expression for the number of melons they each will take home if they sell x watermelons.

b Emily and Ethan aim to take home no more than three watermelons each. Use your expression from part **a** to write an inequality for this situation and then solve it.

c How many watermelons can they sell to meet their goal?

14 To make calls from Australia to friends and family travelling overseas, Jada has a pre-paid phone card with a credit of \$50. For each call, she is charged a connection fee of 49c and a per minute rate depending on the country she is calling. The per minute charges are shown in the table on the right.

Country	Per minute rate
Egypt	\$1.47
India	\$1.35
Japan	\$0.80
Morocco	\$1.95

a For how many minutes could Jada talk to her friend travelling in Morocco and not go over her \$50 limit?

b Jada's friend is in Egypt and not Morocco. For how many minutes could Jada talk to her if she wants to save at least \$25 of her \$50 for another call?

c Jada decides to make two calls. The first call is for 12 minutes to her brother in Japan. How long is her second call to a friend in India if she uses between \$35 and \$50 for both calls combined?

15 a Square all parts of the following inequalities. Ensure the correct inequality symbols are written between the numbers.

i $2 < 3$ **ii** $-2 > -3$ **iii** $2 > -3$

iv $-2 < 3$ **v** $\frac{1}{2} > \frac{1}{3}$ **vi** $-\frac{1}{2} < \frac{1}{3}$

vii $-2 < 0 < 3$ **viii** $-3 < 0 < 2$ **ix** $-2 \neq 2$

b Explain why we need to be careful when squaring both sides of an inequality.

c Apply the reciprocal to all parts of the following inequalities. Ensure the correct inequality symbols are written between the numbers.

i $3 > \frac{1}{2}$ **ii** $-3 < \frac{1}{2}$ **iii** $-3 < -\frac{1}{2}$ **iv** $3 > -\frac{1}{2}$

v $\frac{1}{3} < \frac{1}{2}$ **vi** $-\frac{1}{3} < \frac{1}{2}$ **vii** $-\frac{1}{2} < 1 < 3$ **viii** $-3 < -1 < -\frac{1}{2}$

d Explain why we need to be careful when applying the reciprocal to both sides of an inequality.

16 A company makes and sells two products: X and Y. It costs the company \$4 for each of product X to be made and \$5 for each of product Y. The company needs the cost of producing the two products to be no more than \$100 per day. Let x be the number of product X the company makes and let y be the number of product Y the company makes.

a Write an inequality describing the cost of producing products X and Y for one day.

b State two other inequalities that must be true given that x and y represent the number of each product.

c Determine the greatest number of product X that could be produced in one day.

d Determine the greatest number of product Y that could be produced in one day.

e Determine the greatest number of product X that could be produced in one day if 8 of product Y are produced.

17 Solve each of these inequalities.

a $\frac{2(4-x)}{5} - 3 \leq 3$ **b** $\frac{3(7x-4)}{2} - 8 < 6x + 4$

c $\frac{5x-1}{3} + \frac{x-2}{4} \geq 3$ **d** $\frac{x-4}{8} - \frac{x+3}{2} > 1$

e $\frac{4(2x+1)}{5} + \frac{7-2x}{3} \geq 2x + 1$ **f** $\frac{3x+4}{2} - \frac{x+4}{6} \leq \frac{x-5}{3}$

18 Solve each of these inequalities.

a $-3 \leq 2x - 5 \leq 7$ **b** $15 < \frac{5(1-2x)}{3} < 25$

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Interactive skillsheet
Representing linear inequalities on a number line

Interactive skillsheet
Solving linear inequalties

Worksheet
Solving linear equations and inequalities

Topic quiz
4B

4C Sketching linear graphs

Learning intentions

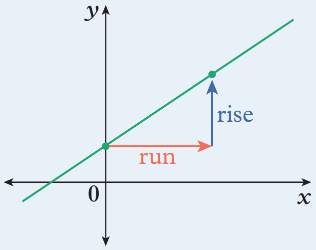
- ✔ I can sketch a linear graph using the gradient–intercept method.
- ✔ I can sketch a linear graph using the *x*-and-*y*-intercept method.
- ✔ I can sketch horizontal and vertical linear graphs.

Inter-year links

Years 5/6	The Cartesian plane
Year 7	5D The Cartesian plane
Year 8	6D Plotting linear and non-linear relationships
Year 9	4D Sketching linear graphs using intercepts

Linear graphs

- A linear equation with two pronumerals describes a **linear relationship** between two variables.
- A linear relationship can be represented on the Cartesian plane as a straight line called a **linear graph**.
- An intercept is the point where a graph crosses an axis.
- The **gradient** is a numerical measure of the slope or steepness of a graph. It can be calculated as the ratio of the change in the y values (the **rise**) to the change in the x values (the **run**).



$$\text{gradient} = \frac{\text{rise}}{\text{run}}$$

- If a linear graph is not vertical or horizontal, then two points must be plotted in order to sketch the graph.

The gradient–intercept method

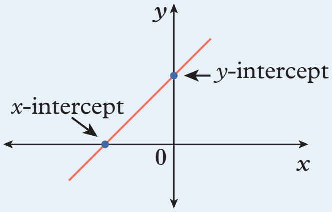
- The equations for all linear relationships can be expressed in the general **gradient–intercept form** shown on the right, where:
  - → m is the gradient of the line
  - → c is the y-coordinate of the y-intercept (the point where the line crosses the y-axis).

$$y = \underbrace{m}_{\text{gradient}}x + \underbrace{c}_{y\text{-intercept}}$$

- The gradient–intercept form can be used to identify the gradient and y-intercept, enabling us to sketch the graph.

The x-and-y-intercept method

- The x-intercept is the point at which the graph crosses the x-axis. That is, where $y = 0$.
- The y-intercept is the point at which the graph crosses the y-axis. That is, where $x = 0$.
- The values $x = 0$ and $y = 0$ can be substituted into a linear equation in order to find the line's x- and y-intercepts and sketch the graph.

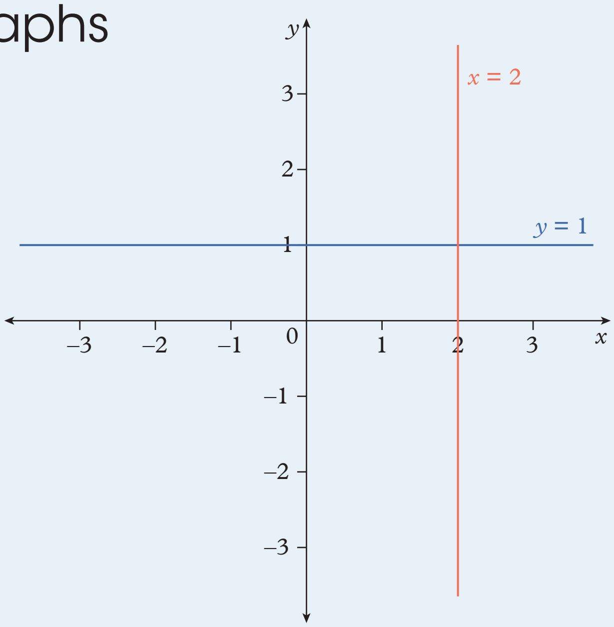


Horizontal and vertical graphs

- A horizontal graph has a gradient of zero, because the rise = 0. Horizontal graphs have equations of the form $y = c$, where c is any real number.

  For example, the blue graph line on the right shows the horizontal graph $y = 1$.
- A vertical graph has an undefined gradient because the run = 0. Vertical graphs have equations of the form $x = a$, where a is any real number.

  For example, the red graph line on the right shows the vertical graph $x = 2$.



Example 4C.1 The gradient–intercept method

Sketch the graph of $y = -\frac{1}{2}x + 3$.

THINK

1. Identify the gradient, m, and the y-intercept, c, by comparing this to the general gradient–intercept form of the equation: $y = mx + c$.
2. Write the gradient, m, and identify appropriate values for the rise and the run.
3. Plot a point at the y-intercept and label the coordinates (0, 3).
4. Add the rise to the y-coordinate and the run to the x-coordinate of the y-intercept to find a second point on the graph. Move two units to the right and one unit down to the point (2, 2).
5. Rule a straight line through the points. Label the graph with its equation.

WRITE

$y = -\frac{1}{2}x + 3$

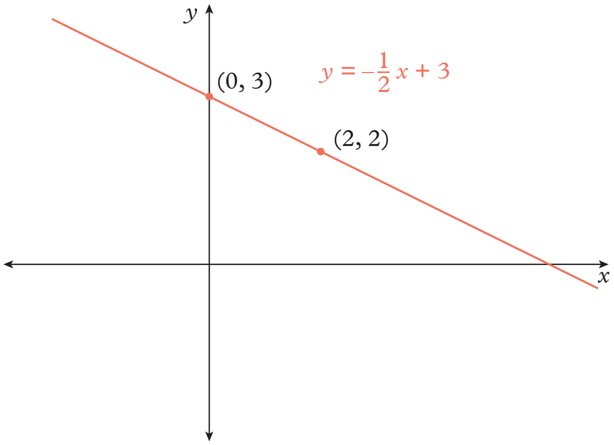
From $y = mx + c$:

$m = -\frac{1}{2}$

$c = 3$

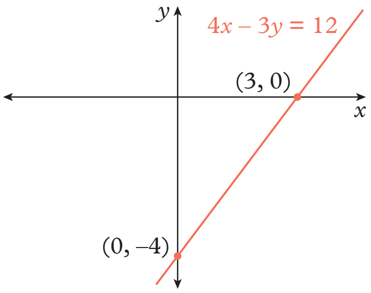
$\frac{\text{rise}}{\text{run}} = -\frac{1}{2}$

So, rise = -1, run = 2



Example 4C.2 The *x*- and *y*-intercept method

Sketch the graph of $4x - 3y = 12$ using the x- and y-intercept method.

THINK	WRITE
1 Determine the x-intercept by substituting $y = 0$ into the equation and solving for x.	For the x-intercept, $y = 0$: $4x - 3y = 12$ $4x - 3(0) = 12$ $4x = 12$ $x = 3$
2 Determine the y-intercept by substituting $x = 0$ into the equation and solving for y.	For the y-intercept, $x = 0$: $4x - 3y = 12$ $4(0) - 3y = 12$ $-3y = 12$ $y = -4$
3 Plot and label the x- and y-intercepts on the Cartesian plane. **4** Rule a straight line through the points. Label the graph with its equation.	

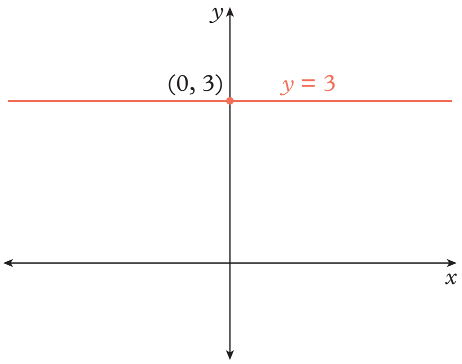
Example 4C.3 Horizontal and vertical graphs

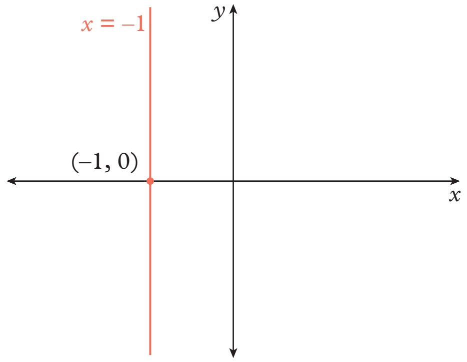
For each linear equation below:

i list the coordinates of any intercepts

ii sketch the graph.

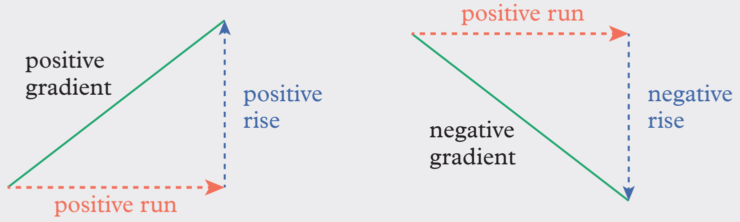
a $y = 3$ **b** $x = -1$

THINK	WRITE
a i The equation tells us that y is 3 for any value of x, so the graph will be a horizontal line passing through a y-intercept at $(0, 3)$.	**a i** y-intercept at $(0, 3)$
ii Rule a straight horizontal line through $(0, 3)$. Label the graph with its equation.	**ii** 

THINK	WRITE
b **i** The equation tells us that x is –1 for any value of y, so the graph will be a vertical line passing through an x-intercept at (–1, 0).	**b** **i** x-intercept at (–1, 0)
ii Rule a straight vertical line through (–1, 0). Label the graph with its equation.	**ii** 

Helpful hints

✔ Check that the slope of your graph matches the sign of the gradient. A positive gradient slopes upwards to the right and a negative gradient slopes downwards to the right.



✔ If you used the gradient–intercept method to sketch your graph, check the x- and y-intercepts of your graph by substituting $y = 0$ and $x = 0$ into the equation.

✔ If you used the x- and y-intercept method to sketch your graph, rearrange the equation to the form $y = mx + c$ and check to see if the gradient, m, matches the gradient of your graph.

ANS p674

Exercise 4C Sketching linear graphs

△ 1(a, c, e, g, i), 2(1st column), 3–6, 7(1st, 2nd column), 8–10, 11(a, b), 12(a–d), 13(a–c, g), 15, 16(b–g), 17(a–d), 19

□ 1(b, c, f, g), 2(b, e, h, k), 4, 6, 7(3rd, 4th columns), 9, 10, 13(f–i), 14, 17(e–h), 18, 20, 22

⬠ 2(j–l), 4, 6, 9, 10, 17(g–j), 18, 21, 23–25

4C.1 **1** Sketch the graph of each of these linear relationships using the gradient–intercept method.

a $y = 2x - 4$
b $y = \frac{2}{3}x + 1$
c $y = -3x + 5$
d $y = \frac{5}{4}x - 2$
e $y = -5x - 1$
f $y = -\frac{1}{2}x + 3$
g $y = 3x - 4$
h $y = x + 6$
i $y = -x - 3$

2 Rearrange each of these equations to be in the form $y = mx + c$ and then:

i identify the gradient
ii identify the y-intercept
iii sketch the graph.

a $y = 1 + 2x$
b $y = 4 - 3x$
c $y = -3 - \frac{3}{4}x$
d $y = 3(x - 2)$
e $y = -2(x - 3)$
f $y - 4 = -(5x + 6)$
g $3x + y = 2$
h $-4x + y = -5$
i $6x + 3y = 9$
j $-4x + 5y = 15$
k $3x - 2y = -4$
l $x + 3y + 6 = 0$

3 **a** If $y = 3x$ is compared to the gradient–intercept form, $y = mx + c$, then identify the value of:

i m

ii c

b Sketch the graph of $y = 3x$ using the gradient–intercept method.

4 Sketch each of the following linear relationships using the gradient–intercept method.

a $y = 2x$ **b** $y = -5x$ **c** $y = -x$ **d** $y = \frac{2}{5}x$

5 **a** If $y = 4$ is compared to the gradient–intercept form, $y = mx + c$, identify the value of:

i m

ii c

b Sketch the graph of $y = 4$ using the gradient–intercept method.

6 Sketch each of these linear relationships using the gradient–intercept method.

a $y = 6$ **b** $y = -2$ **c** $y = \frac{1}{3}$ **d** $y = -\frac{7}{2}$

4C.2 **7** Sketch the graph of each of these linear relationships using the x- and y-intercept method.

a $x + 2y = 2$ **b** $3x + y = -6$ **c** $2x + 4y = 8$ **d** $5x + 10y = 20$

e $4x + 3y = -12$ **f** $-6x + 2y = 6$ **g** $8x - y = -8$ **h** $-x + 4y = 2$

i $9x - 3y = -18$ **j** $-12x + 9y = 36$ **k** $7x + 4y = 21$ **l** $2x - y = 5$

8 For each of the linear relationships below:

i find the x-intercept **ii** find the y-intercept **iii** sketch the graph.

a $y = x + 5$ **b** $y = x - 4$ **c** $y = 3x + 6$

d $y = 2x - 8$ **e** $y = 2 - x$ **f** $y = -6x - 3$

9 Sketch the graph of each of these using the x- and y-intercept method.

a $2y = 3x + 6$ **b** $4y = 4x - 8$ **c** $x + 3y - 3 = 0$

d $3x + 6y + 18 = 0$ **e** $5x - 2y - 10 = 0$ **f** $2x - y - 7 = 0$

10 Consider the linear equation $y = 2x$.

a Find:

i the x-intercept

ii the y-intercept.

b Do you have enough information from part **a** to sketch the graph? Explain.

c What other information do you need? Discuss this with a classmate.

d Another way of identifying a second point to plot is to find the matching y value for a chosen x value. Any x value can be used, but we often choose $x = 1$.

Find the value of y in $y = 2x$ when x is 1. (Hint: Substitute $x = 1$ into the equation.)

e List the coordinates of the two points you can now use to sketch the graph of $y = 2x$.

f Plot the two points and rule a straight line through them to produce the graph of $y = 2x$. Label your graph with its equation.

11 Use the approach used in question **10** to sketch linear graphs with these equations.

a $y = 4x$ **b** $y = -3x$ **c** $y = x$ **d** $y = -\frac{1}{2}x$

4C.3 **12** For each of the linear equations below:

i list the coordinates of any intercepts **ii** sketch the graph.

a $y = 8$ **b** $x = 3$ **c** $y = -5$ **d** $x = -1$

e $x = 7$ **f** $y = 0$ **g** $x = 0$ **h** $x = -4.5$

13 How many axis intercepts does the graph of each of these linear equations have? List them.

a $y = 2x - 4$ **b** $y = -7$ **c** $4x + 6y = 12$

d $y = 9x$ **e** $y = 5(x - 10)$ **f** $3x - 5y + 15 = 0$

g $x = 28$ **h** $x = 3y + 6$ **i** $y = 5.6$

14 **a** Write $x = py + q$ in the form:

i $y = a(x - b)$

ii $y = mx + c$.

b What do the values of p and q correspond with on the graph of $x = py + q$?

c Write $x = 3$ in the form $x = py + q$.

d Identify the value of:

i p

ii q

e **i** Write your answer from part **c** in the forms $y = a(x - b)$ and $x = py + q$.

ii Use your answers from parts **b** and **e i** to explain why the gradient of the lines $x = 3$ and, more generally, $x = q$ are undefined.

15 Describe which method you think would be the best to use to sketch:

a linear graphs with equations in the general form $y = mx + c$ (for example, $y = 3x + 2$)

b linear graphs with equations in the general form $ax + by = d$ (for example, $4x + 2y = 8$)

c linear graphs that pass through the origin (for example, $y = 3x$)

d linear graphs that are horizontal lines (for example, $y = 7$)

e linear graphs that are vertical lines (for example, $x = -2$).

16 Use the most appropriate method to sketch the graph of each of these linear relationships.

a $5x - 2y = -10$ **b** $y = 2x + 4$ **c** $x + y = 5$ **d** $y = 2x$

e $y = 5$ **f** $y = 8 - 4x$ **g** $x = -9$ **h** $y = 3(x - 2)$

i $2x + 4y - 8 = 0$ **j** $y = -3x$ **k** $y = -2x + 7$ **l** $y = 3 - 6x$

17 Sketch each of these linear relationship for the given x or y values. Label the endpoints with their coordinates.

a $y = 5x$ for $x \geq 0$ **b** $y = 2x - 4$ for $x \leq 3$

c $y = 7$ for $-2 \leq x \leq 4$ **d** $x = -3$ for $-1 \leq y \leq 1$

e $2x + 3y = 12$ for $0 \leq x \leq 6$ **f** $y = x$ for $x > 2$

g $y = 3 - x$ for $x < 3$ **h** $y = -3$ for $-1 < x < 4$

i $x - 4y = 2$ for $-2 \leq x < 6$ **j** $x = 6$ for $y < 3$

18 For a line segment between the points (x_1, y_1) and (x_2, y_2), the length is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

a Sketch each of these linear relationships on the same Cartesian plane.

i $x = -6$ for $3 \leq y \leq 13$

ii $y = 3$ for $-6 \leq x \leq 2$

iii $y = -\frac{5}{4}x + \frac{11}{2}$ for $-6 \leq x \leq 2$

b Calculate the perimeter of the shape formed, correct to two decimal places.

c Calculate the area of the shape formed.

19 The amount of simple interest, I, earned on an investment, P, at $r = 5\%$ p.a. for $T = 10$ years is represented by the equation $I = PrT$, where I and P are dollar amounts.

a Substitute the values of r and T into the equation to determine the linear relationship between I and P. (Hint: Convert the percentage to a decimal.)

b Draw the graph of this relationship for P values from \$0 to \$5000.

c Use your graph from part **b** to find the amount of simple interest earned after 10 years for these investment amounts.

i \$1000 **ii** \$2500 **iii** \$4250

d Use your graph to find the amount that needs to be invested for 10 years to earn the following interest.

i \$750 **ii** \$1000 **iii** \$2250

e If the formula for simple interest is $I = PrT$, where r is the interest rate per annum written as a decimal and T is the number of years of investment, explain what the gradient of your graph represents.

PROBLEM SOLVING AND REASONING

20 A racing car accelerates from a constant speed of 20 m/s. Its speed, s, after t seconds of accelerating can be calculated using the formula $s = 20 + 5t$.

a Draw the graph of this linear relationship for $0 \leq t \leq 10$ (that is, t values from 0 to 10 seconds).

b What does the s-intercept represent on this graph?

c What does the gradient represent on this graph?

d From the graph, what is the speed of the racing car after:

i 3 seconds?

ii 5 seconds?

iii 7.5 seconds?

e Convert each speed in part **d** to kilometres per hour.

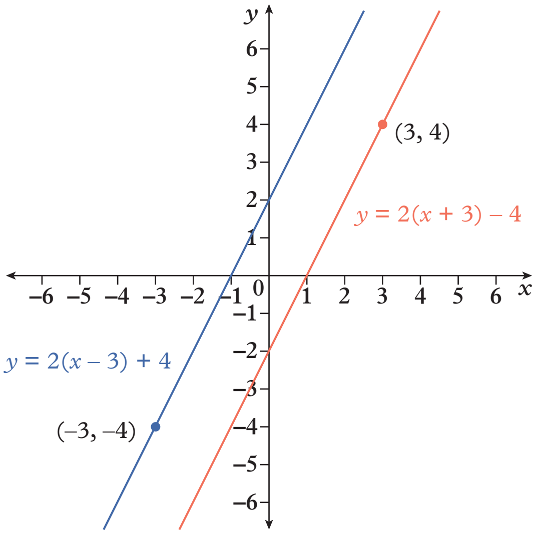
f From the graph, how long does it take the racing car to reach speeds of:

i 30 m/s? **ii** 55 m/s? **iii** 65 m/s?

g Using the graph, how long does it take the racing car to reach a speed of 216 km/h?

h Using the graph, what is the highest speed, in kilometres per hour, reached by the racing car? How long does it take to reach this speed?

21 The gradient–intercept form $y = mx + c$ is a specific case of the gradient–point form $y = m(x - h) + k$, where (h, k) is a point on the line. For example, $y = 2(x - 3) + 4$ passes through the point (3, 4) and $y = 2(x + 3) - 4$ passes through the point $(-3, -4)$.



a Substitute $(h, k) = (0, c)$ and show that the equation $y = m(x - h) + k$ becomes $y = mx + c$.

b Sketch the following lines by plotting the point (h, k) and then use the gradient to plot another point in the same way you would using the gradient–intercept method.

i $y = -3(x - 2) + 6$ **ii** $y = 3(x - 2) + 6$

iii $y = \frac{5}{2}(x + 3) - 4$ **iv** $y = -\frac{3}{4}(x + 7) + 10$

c Determine the coordinates for the x- and y-intercepts of $y = m(x - h) + k$ in terms of m, h and k.

d **i** Substitute $(h, k) = (b, 0)$ to determine a new general form for writing a linear relationship in terms of the x-intercept.

ii Explain what you can do to the equation $y = mx + c$ to write it in this form.

iii Explain how you can use this form to determine the x-intercept.

e Use your method from part **d** to write the following equations in the new general form and determine their x-intercepts.

i $y = 3x - 12$ **ii** $y = 2x - 5$ **iii** $y = -3x - 12$ **iv** $y = 2x + 5$

CHALLENGE

22 a Sketch these linear relationships on the same Cartesian plane:

- $2x + 4y = 8$, for $4 \leq x \leq 8$
- $y = -2$, for $-2 \leq x \leq 8$
- $x = -2$, for $-2 \leq y \leq 2$
- $y = -x$, for $-2 \leq x \leq 0$
- $y = 0$, for $0 \leq x \leq 4$.

b Use Pythagoras' Theorem to calculate the perimeter of the shape formed by the graph lines.

c Calculate the area of the shape formed.

CHALLENGE

23 a Sketch these linear relationships on the same Cartesian plane.

- $y = -4x + 10$, for $0 \le x \le 2$
- $y = 4x + 10$, for $-2 \le x \le 0$
- $x - 4y = -10$, for $-10 \le x \le -2$
- $4y = x - 10$, for $2 \le x \le 10$
- $y = 4x - 10$, for $0 \le x \le 2$
- $y = -4x - 10$, for $-2 \le x \le 0$
- $x + 4y = -10$, for $-10 \le x \le -2$
- $x + 4y = 10$, for $2 \le x \le 10$

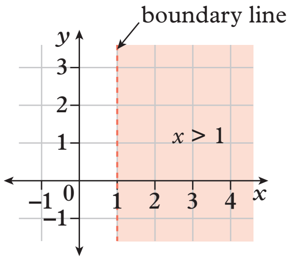
b Calculate the perimeter of the shape formed by the graph lines.

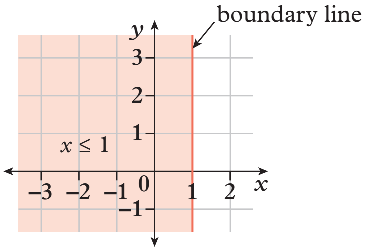
c Calculate the area of the shape formed.

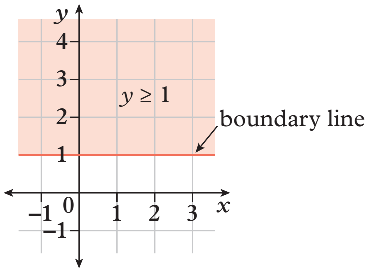
d Join the points (–2, –2) and (2, 2) to form a line segment. Write a rule to represent the line segment.

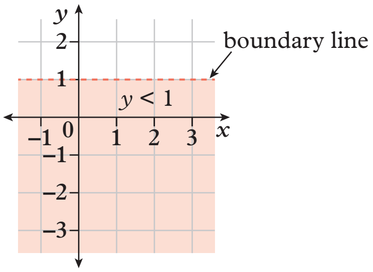
e Join the points (–2, 2) and (2, –2) to form a line segment. Write a rule to represent the line segment.

24 Linear inequalities can be sketched on a Cartesian plane as well as a number line. Where an inequality has an endpoint on the number line, a Cartesian plane has a boundary line. If we are including the values on the boundary line, we use a solid line. If we are not including the values on the boundary line, we use a dotted line. Instead of using a ray or line segment, we shade the region that satisfies the inequality.









a Sketch each of the following inequalities on a different Cartesian plane.

i $x \ge -3$ **ii** $x < 5$

iii $y > 2$ **iv** $y \le -1$

b Sketching inequalities on the Cartesian plane enables us to sketch linear inequalities involving two variables instead of just one. We can do this by first sketching the boundary line as a solid or dotted line, as required, then shading the region that satisfies the inequality.

Sketch each of the following inequalities on a different Cartesian plane. (Hint: Test points can be substituted into the inequality to determine whether they satisfy or do not satisfy the inequality.)

i $y > x$ **ii** $y \le x$ **iii** $y < 2x + 5$ **iv** $y \ge 2x + 5$

v $2x + 3y > 18$ **vi** $5x - 4y < 20$ **vii** $4x - 5y \le 20$ **viii** $x + y \ne 1$

25 When an equation is written in the form $ax + by = c$, we can determine both of the intercepts simultaneously.

a Determine the coordinates of the x- and y-intercepts of $ax + by = c$ in terms of a, b and c.

b Divide both sides of the equation $ax + by = c$ by c.

c Explain the relationship between the coefficients of x and y in your answer for part **b** and the coordinates in part **a**.

d Use the method in part **b** to help you determine the intercepts of the following lines.

i $2x + 3y = 5$ **ii** $4x - 7y = 11$ **iii** $2x - 3y = 12$

iv $9y - 2x = 36$ **v** $8x - 4y = -32$ **vi** $20x + 50y = 10$

Check your Student obook pro for these digital resources and more:

pro

Interactive skillsheet
Sketching linear relationships using intercepts

Interactive skillsheet
Sketching linear relationships using the gradient-intercept method

Worksheet
Linear graphs

CAS instructions
Graphing functions

Topic quiz
4C

Checkpoint

Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this chapter.

4A **1** Solve the following equations.

a $\frac{x}{3} - 5 = 2$ **b** $8(x + 6) = -24$

c $\frac{5 - 2x}{4} = 9$ **d** $4(3x - 7) - 5x = -31$

4A **2** Solve the following equations.

a $\frac{10}{x} = 4$ **b** $\frac{4}{x} = 12$

c $\frac{-18}{x + 5} = 9$ **d** $\frac{14}{7x - 9} = 3$

4A **3** Solve the following equations.

a $6x - 10 = 5x + 9$ **b** $5(3x - 2) = x$

c $9(4 - 7x) = 3(9x + 11)$ **d** $\frac{4x - 9}{3} = \frac{2x + 7}{4}$

4A **4** Rearrange each of the following equations to make x the subject.

a $ax + by = c$ **b** $\frac{f}{dx - e} = g$

c $j(x + k) = m(x - n)$ **d** $\frac{px - q}{s - tx} = r$

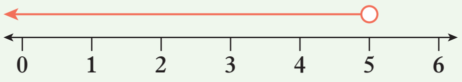
4B **5** Represent the values of x for each inequality on a number line.

a $x \leq 5$ **b** $x > -3$

c $x \neq 2$ **d** $x \geq 4$

4B **6** Write the inequality that is represented on each of these number lines.

a



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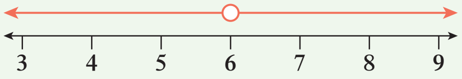
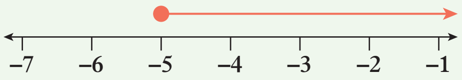
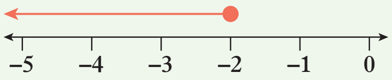


c



d



4B **7** Solve the following inequalities.

a $6x > 18$ **b** $-4x \geq 20$

c $-13 < \frac{x}{5} - 8$ **d** $7 - 3x < 4$

e $4x + 7 \neq 6x + 13$ **f** $4(3x - 4) \leq 3(3x + 2)$

4C **8** Sketch the graph of each of the following.

a $y = 3x - 1$ **b** $y = \frac{1}{4}x + 2$

c $y = -\frac{2}{5}x + 7$ **d** $y = \frac{7}{6}x$

4C **9** Sketch the graph of each of the following.

a $2x - 3y = 12$ **b** $4x + 5y = -100$

c $18x + 36y = 9$ **d** $14y - 15x = 24$

4C **10** Sketch the graph of each of the following.

a $x = 10$ **b** $y = -4$

c $x = -\frac{2}{3}$ **d** $y = \frac{3}{2}$

4D Determining linear equations

Learning intentions

- ✔ I can determine the gradient of a line given two points.
- ✔ I can determine linear equations given the gradient and a point or two points.
- ✔ I can determine the linear equation of a graph.

Inter-year links

Years 5/6	The Cartesian plane
Year 7	5D The Cartesian plane
Year 8	6F Finding linear equations
Year 9	4E Determining linear equations

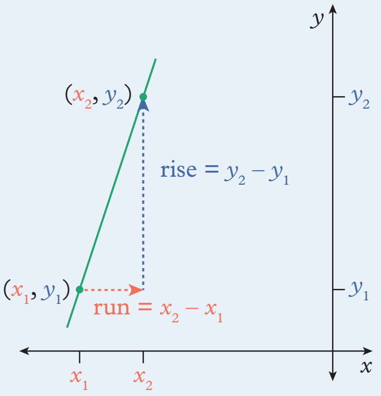
Determining the gradient

- The formula for the gradient, m, between any two points, (x_1, y_1) and (x_2, y_2) is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

For example, the gradient of a line that passes through (1, 3) and (2, –1) can be calculated as:

$$\begin{aligned} m &= \frac{3-(-1)}{1-2} \\ &= \frac{4}{-1} \\ &= -4 \end{aligned}$$



Determining linear equations

- To determine the equation of a line in the form $y = mx + c$:
  1. determine the gradient, m
  2. identify the coordinates of a point on the graph
  3. substitute the gradient and the coordinates into the formula $y = mx + c$ to determine c.
- If the y-coordinate of the y-intercept is easy to read from the graph, then it is quicker to substitute this value for c, rather than solve the equation algebraically.

Example 4D.1 Determining the gradient

Determine the gradient of a line that passes through (3, –8) and (–7, –10).

THINK	**WRITE**
1 Substitute the given x- and y-coordinates into the gradient formula.	Let $(x_1, y_1) = (3, -8)$ and $(x_2, y_2) = (-7, -10)$. $m = \frac{y_2 - y_1}{x_2 - x_1}$
2 Calculate and simplify the gradient.	$= \frac{(-10)-(-8)}{(-7)-3}$ $= \frac{-10+8}{-7-3}$ $= \frac{-2}{-10}$ $= \frac{1}{5}$

Example 4D.2 Determining the equation given the gradient

Determine the equation for a line with a gradient of 4 that passes through (–2, 7).

THINK	WRITE
1 Substitute the values for the gradient and the coordinates into the gradient–intercept equation: $y = mx + c$.	From the question: $m = 4 \quad x = -2 \quad y = 7$ $y = mx + c$ $7 = 4 \times (-2) + c$
2 Solve the equation for c and write the final equation in gradient–intercept form.	$7 = -8 + c \quad (+\ 8)$ $c = 15$ So, $y = 4x + 15$.

Example 4D.3 Determining the equation given two points

Determine the equations for the lines that pass through these given pairs of points.

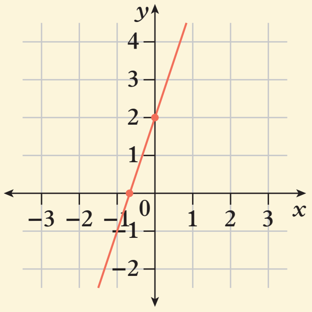
a (2, 6) and (–4, 3) **b** (0, 2) and (3, –7)

THINK	WRITE
a **1** Substitute the x- and y-coordinates into the gradient formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$.	**a** Let $(x_1, y_1) = (2, 6)$ and $(x_2, y_2) = (-4, 3)$. $m = \frac{y_2 - y_1}{x_2 - x_1}$ $= \frac{3 - 6}{-4 - 2}$ $= \frac{-3}{-6}$ $= \frac{1}{2}$
2 Substitute the values for the gradient and one of the sets of coordinates into the gradient–intercept equation: $y = mx + c$. **3** Solve for c and write the final equation in gradient–intercept form.	Use: $x = 2 \quad y = 6$ $y = mx + c$ $6 = \frac{1}{2} \times 2 + c$ $6 = 1 + c \quad (-\ 1)$ $c = 5$ So, $y = \frac{1}{2}x + 5$.
b **1** Substitute the x- and y-coordinates into the gradient formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$.	**b** Let $(x_1, y_1) = (0, 2)$ and $(x_2, y_2) = (3, -7)$. $m = \frac{y_2 - y_1}{x_2 - x_1}$ $= \frac{-7 - 2}{3 - 0}$ $= \frac{-9}{3}$ $= -3$
2 If one of the coordinates has an x value of 0, the point is a y-intercept, and the corresponding y value is the value of c in $y = mx + c$.	(0, 2) is the y-intercept. So, $c = 2$.
3 Write the final equation in gradient–intercept form.	$y = -3x + 2$.

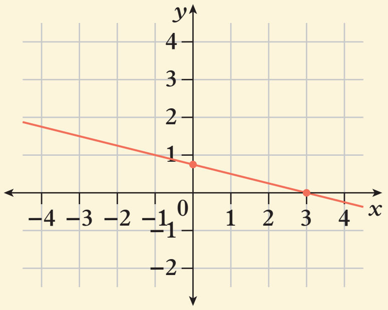
Example 4D.4 Determining the equation given the graph

Determine the equation of each of these linear graphs.

a



b



THINK	WRITE
a 1 Identify two coordinates on the graph.	**a** Let $(x_1, y_1) = (-1, -1)$ and $(x_2, y_2) = (0, 2)$.
2 Determine the value of the gradient, *m*, using the chosen coordinates.	$m = \frac{y_2 - y_1}{x_2 - x_1}$ $= \frac{2 - (-1)}{0 - (-1)}$ $= 3$
3 Determine the value of *c*, by identifying the *y*-coordinate of the *y*-intercept.	$c = 2$
4 Write the equation of the linear graph by substituting *m* and *c* into the equation.	$y = mx + c$ $y = 3x + 2$
b 1 Identify two integer (whole number) coordinates on the graph.	**b** Let $(x_1, y_1) = (-1, 1)$ and $(x_2, y_2) = (3, 0)$.
2 Determine the value of the gradient, *m*, using the chosen coordinates.	$m = \frac{y_2 - y_1}{x_2 - x_1}$ $= \frac{0 - 1}{3 - (-1)}$ $= -\frac{1}{4}$
3 The *y*-coordinate of the *y*-intercept is unclear on the graph, so choose a clearer coordinate to substitute into the gradient–intercept equation: $y = mx + c$.	Use: $x = 3 \quad y = 0$ $y = mx + c$ $0 = -\frac{1}{4} \times 3 + c$
4 Solve the equation for *c* and write the final equation in gradient–intercept form.	$0 = -\frac{3}{4} + c \qquad \left(+\frac{3}{4}\right)$ $c = \frac{3}{4}$ $y = -\frac{1}{4}x + \frac{3}{4}$

Helpful hints

✔ The order in which you substitute points into the formula for the gradient of a line won't affect the final value – you just need to make sure the *x*- and *y*-coordinates of a given point match up vertically when written as a fraction:

$$m = \frac{\overset{\text{point 2}}{y_2} - \overset{\text{point 1}}{y_1}}{x_2 - x_1}$$

✔ Remember, if you know the *y*-intercept, (0, *c*), all you need to do is determine the gradient, *m*, and substitute both values into the gradient–intercept form of the equation: $y = mx + c$.

ANS p685

Exercise 4D Determining linear equations

△ 1–3, 4(e–h), 5(a–e), 6, 7, 8(1st column), 9, 10, 15, 16

□ 1, 3, 4–5(e–h), 8(2nd column), 11, 12, 14, 17, 20

⬠ 3, 4–5(e–h), 8(2nd column), 13, 14, 18, 19, 21, 22

UNDERSTANDING AND FLUENCY

4D.1 **1** Determine the gradient of the line passing through each of these pairs of points.

- **a** (2, 3) and (3, 7)
- **b** (5, 4) and (10, –6)
- **c** (4, 2) and (6, 3)
- **d** (0, 5) and (4, 1)
- **e** (–3, –2) and (–1, 8)
- **f** (–1, –1) and (2, –10)

2 For each line passing through a pair of points given below:

- **i** determine the gradient
- **ii** determine the equation of the line by substituting the correct values into $y = mx + c$.
  - **a** (0, 2) and (2, 10)
  - **b** (0, –7) and (3, –4)
  - **c** (0, 5) and (4, 7)
  - **d** (–1, –4) and (0, 6)
  - **e** (–2, 1) and (0, –5)
  - **f** (–8, 6) and (0, 4)

3 Use the *x*- and *y*-intercepts given below to determine the equation of the linear graph on which they appear.

- **a** *x*-intercept: (2, 0), *y*-intercept: (0, 4)
- **b** *x*-intercept: (–3, 0), *y*-intercept: (0, 3)
- **c** *x*-intercept: (3, 0), *y*-intercept: (0, –6)
- **d** *x*-intercept: (1, 0), *y*-intercept: (0, 5)
- **e** *x*-intercept: (5, 0), no *y*-intercept
- **f** no *x*-intercept, *y*-intercept: (0, –2)

4D.2 **4** Use the given information to determine the equation for each of these lines.

- **a** gradient of 3; passes through (1, 4)
- **b** gradient of 2; passes through (3, 9)
- **c** gradient of –4; passes through (–1, 2)
- **d** gradient of –1; passes through (2, 7)
- **e** gradient of 5; passes through (–3, –6)
- **f** gradient of –3; passes through (–4, 11)
- **g** gradient of $\frac{6}{5}$; passes through (15, 4)
- **h** gradient of $-\frac{3}{4}$; passes through (8, −5)
- **i** gradient of $\frac{5}{2}$; passes through (−3, −6)
- **j** gradient of $-\frac{7}{9}$; passes through (−6, 4)

4D.3 **5** Determine the equations for the lines that pass through these given pairs of points.

a $(1, 3)$ and $(3, 7)$

b $(5, 2)$ and $(6, 8)$

c $(2, 4)$ and $(4, -2)$

d $(3, -2)$ and $(6, -8)$

e $(-2, -3)$ and $(2, 1)$

f $(-4, -2)$ and $(-1, -5)$

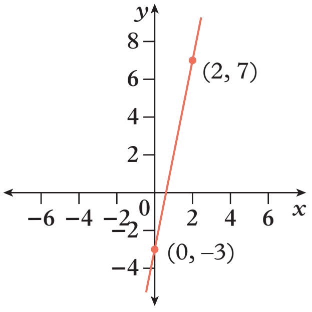
g $(1, -3)$ and $(4, 4)$

h $(-7, 5)$ and $(-2, -2)$

i $(-5, 3)$ and $(1, 4)$

6 Use $y = mx + c$ to help you determine the equation of the line passing through the points shown each time.

a



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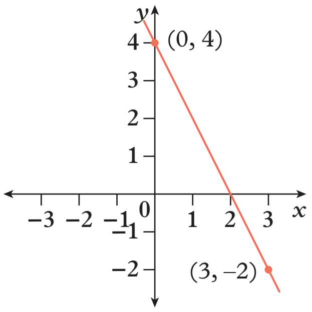

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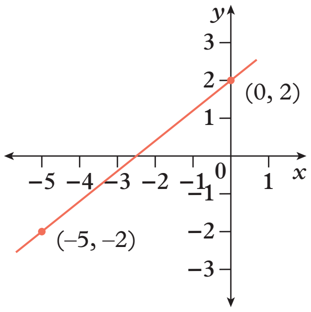
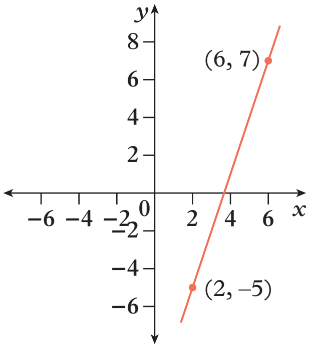


7 Determine the equation of the line passing through the points shown each time.

a



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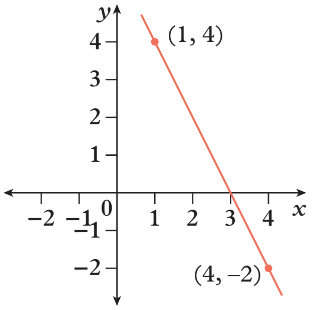

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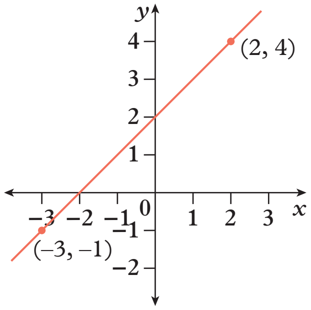
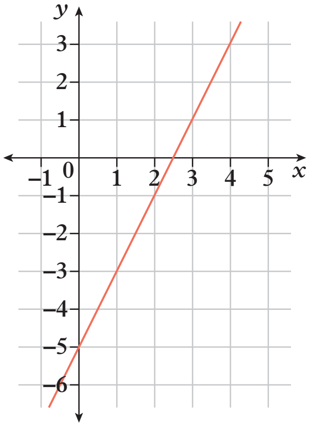


4D.4 **8** Determine the equation of each of these linear graphs.

a



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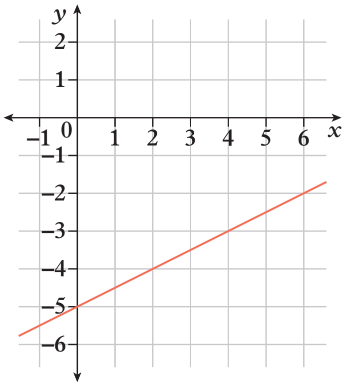

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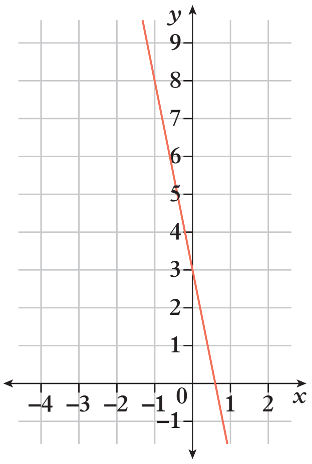
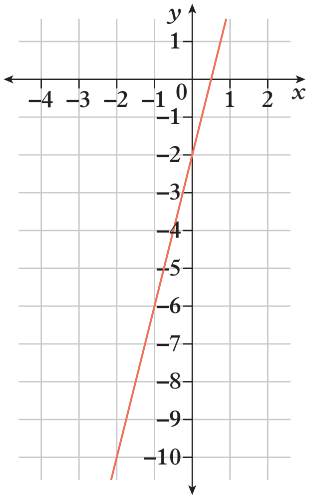


d

e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x
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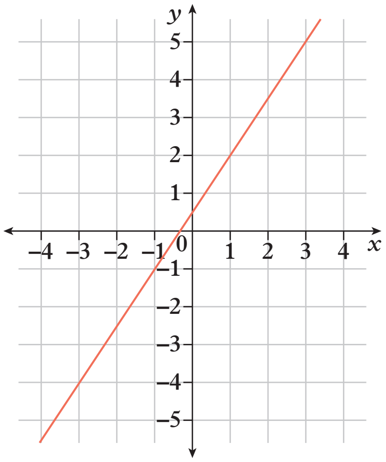
−2
−3
−4
−5
−6
−7
−8
−9
−10

f


y
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−2
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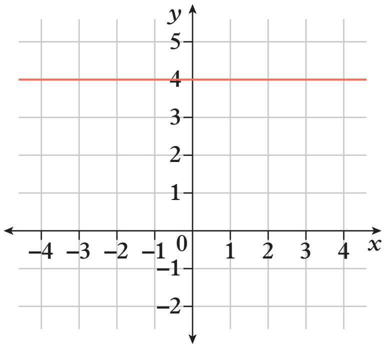
−4
−5

g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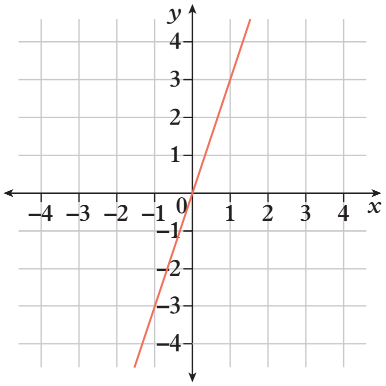
−1
−2

h


y
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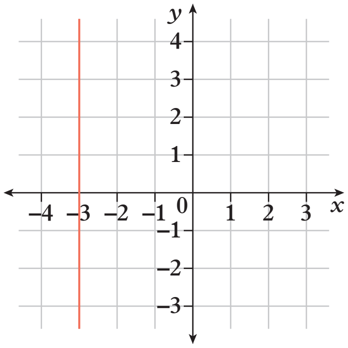
i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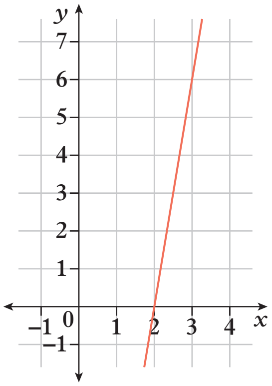
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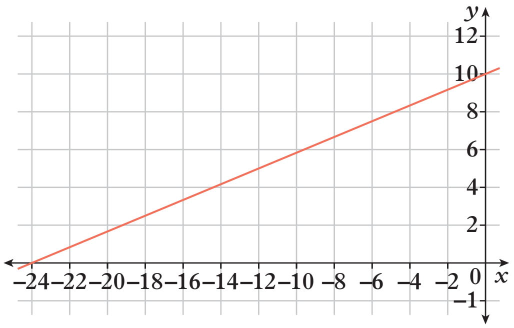
−1

k


y
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x
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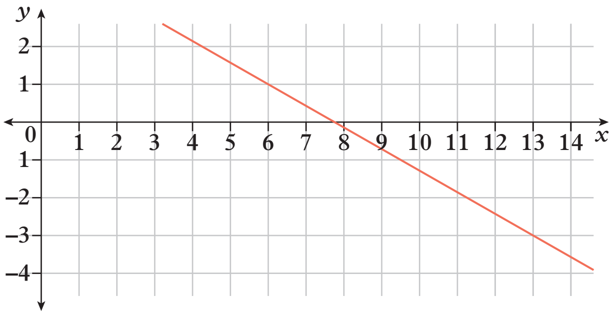
−1

l


y
2
1
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0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
1
−2
−3
−4

9 Liza and Tate are finding the gradient of a line that passes through (–2, 2) and (4, 5). Liza writes $m = \frac{5-2}{4-(-2)}$, while Tate writes $m = \frac{2-5}{-2-4}$.

a Show that Liza and Tate both find the gradient to be $m = \frac{1}{2}$.

b Juan attempts to find the gradient of the same line and writes $m = \frac{5-2}{-2-4}$. Explain what Juan has done wrong and how his answer will compare to the correct gradient found by Liza and Tate.

c Show that $m = \frac{y_1 - y_2}{x_1 - x_2}$ is equivalent to $m = \frac{y_2 - y_1}{x_2 - x_1}$.

10 Liza and Juan are now finding the equation of the line that passes through (–2, 2) and (4, 5). To find the y-intercept, c, Liza writes the equation $2 = \frac{1}{2}(-2) + c$ and Tate writes $5 = \frac{1}{2}(4) + c$.

a Show that Liza and Tate will both get the same value for c.

b Tate attempts to find the equation of the same line and says it is $y = 4x + 5$. Explain what Tate has done wrong.

11 Consider the equation $y = mx + c$.

a For a line passing through the point (h, k), the coordinates can be substituted into the equation $y = mx + c$ to produce $k = mh + c$. Rearrange this equation to make c the subject.

b Substitute the equation you wrote for c into $y = mx + c$.

c Rearrange your answer to part **b** and show that the equation can be expressed in gradient–point form: $y = m(x - h) + k$.

12 Determine the equations of the lines with the features described in **a**–**h** below:

i expressed in the gradient–point form $y = m(x - h) + k$ derived in question **11**, where m is the gradient and (h, k) is a point on the line

ii expressed in the gradient–intercept form $y = mx + c$.

a gradient of $\frac{3}{4}$; passes through (8, 2)

b gradient of $-\frac{7}{3}$; passes through (15, −6)

c (−3, 5) and (1, 6) lie on the line

d (−6, 5) and (−1, −3) lie on the line

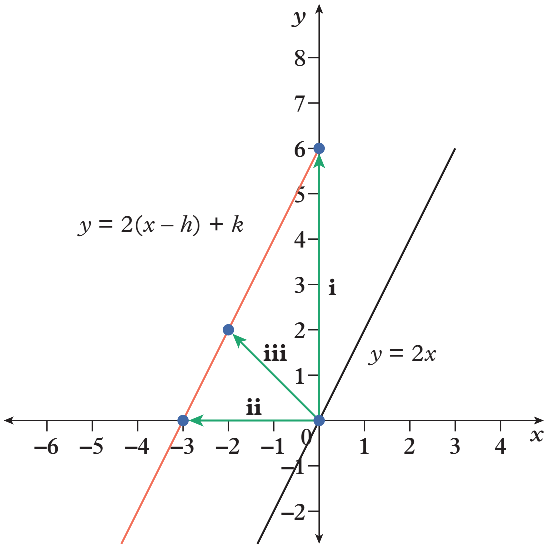
e (−4, 2) and (2, −1) lie on the line

f (−7, 3) and (4, 3) lie on the line

g x-intercept = 4, y-intercept = 3

h x-intercept = −5, y-intercept = 2

13 Consider the linear graphs $y = 2x$ and $y = 2(x - h) + k$ shown on the right. The line $y = 2(x - h) + k$ can be considered as a translation of the line $y = 2x$. Three possible translations are shown as arrows on the diagram.



a State the translation each arrow describes.

b Write the equation of the translated line in the form $y = 2(x - h) + k$ using the three points at the end of the arrows.

c Describe the translation required for the graph of the first equation to become the graph of the second equation each time:

i $y = 3x \rightarrow y = 3(x + 1) - 2$

ii $y = 3x \rightarrow y = 3(x - 1) + 2$

iii $y = -3x \rightarrow y = -3(x - 1) + 2$

iv $y = \frac{5}{2}x \rightarrow y = \frac{5}{2}(x + 4) + 6$

v $y = \frac{2}{5}x \rightarrow y = \frac{2}{5}\left(x - \frac{3}{2}\right) + \frac{4}{3}$

vi $y = 3x + 6 \rightarrow y = 3(x - 4) + 8$

14 Write an equation for a linear graph to represent each of these situations, where y represents the altitude and x represents the horizontal distance from the point of origin.

a From an altitude of 50 m, a road rises 1 m vertically over a horizontal distance of 200 m.

b From an altitude of 6 km, a plane descends 2 km vertically over a horizontal distance of 30 km.

c A road rises 20 m vertically over a horizontal distance of 100 m.

15 Jasper has \$80 in his bank account and decides to deposit \$20 more each week. If he doesn't withdraw any money:

a write an equation to represent the amount of money, a, in Jasper's account after n weeks

b calculate how long it would take for Jasper to build his account balance to \$500.

16 Inge owes \$1560 on her store credit card and has arranged to make regular payments without interest charges until the debt is paid. A monthly direct debit of \$195 from her bank account is paid to the store. If Inge doesn't buy any further items on her card:

a write an equation to represent the amount of money, w, Inge owes on her card after n months

b use your equation from part **a** to calculate the amount still owed after 3 months

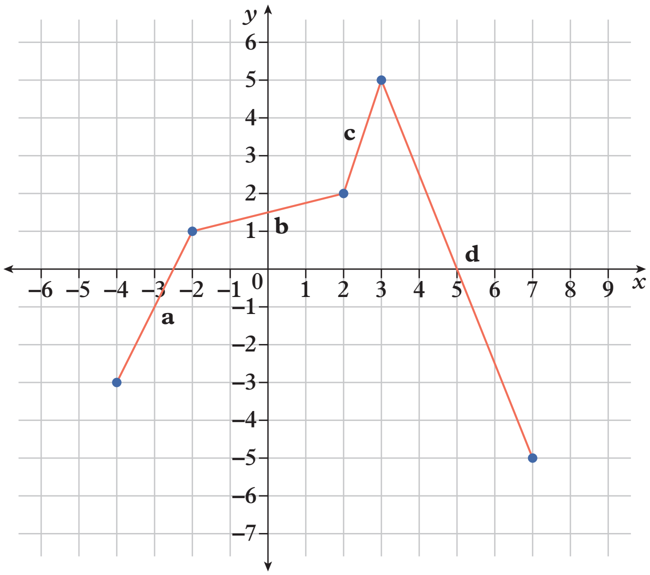
c calculate how long it will take Inge to clear the debt on her card.

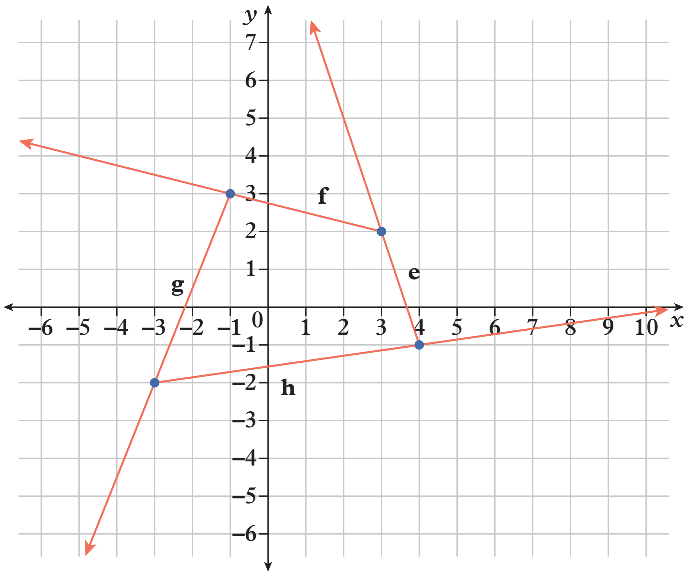
17 Each graph below is constructed from four line segments. For each of the line segments (**a**–**h**):

i determine the equation of the line segment

ii write an inequality describing the x values included in the line segment

iii write an inequality describing the y values included in the line segment.





18 A helicopter rises at a constant speed from a height of 20 m above the ground, covering a vertical distance of 30 m as it flies over a horizontal distance of 100 m. For the next 100 m it travels horizontally. It then descends at a constant speed over a further horizontal distance of 100 m before landing.

a Draw a graph to represent the changing height above the ground of the helicopter over the total 300 m horizontal distance it travelled.

b Identify the rule for each of the three linear sections of the graph you drew for part **a**.

c Use the appropriate rule to calculate the height of the helicopter for each of these horizontal distances on your graph.

i 30 m

ii 80 m

iii 150 m

iv 240 m

d What horizontal distance/s have been travelled when the helicopter is at a height of 40 m?

19 Substitute $m = \frac{y_2 - y_1}{x_2 - x_1}$ into $y = mx + c$ and rearrange to make c the subject, then show that the expression for c is the same when substituting either $(x, y) = (x_1, y_1)$ or $(x, y) = (x_2, y_2)$.

20 Determine the equation for each linear graph described by the information below. Write your answers without fractions.

a x-intercept of 4, passes through (20, 8)

b gradient of 0, passes through (–15, –23)

c passes through (–25, 10) and (11, –14)

d passes through the origin and (–17, –12)

21 Sketch the linear graph for each situation in question **14**. What values of x would be appropriate to use? Rewrite each equation you wrote in your answers to question **14** with the appropriate restriction of x values written as an inequality statement.

22 In the previous exercise we saw that, for the general form $ax + by = 1$, the x-intercept is given by the reciprocal of a and the y-intercept is given by the reciprocal of b. Use this fact to:

i write the equations of the following lines in the form $ax + by = 1$, where a and b are real numbers

ii write the equations in the form $px + qy = r$, where p, q and r are integers.

a x-intercept $= \frac{2}{3}$, y-intercept $= \frac{2}{7}$

b x-intercept $= -\frac{3}{5}$, y-intercept $= \frac{3}{2}$

c x-intercept $= \frac{3}{5}$, y-intercept $= -\frac{2}{5}$

d x-intercept $= -\frac{1}{2}$, y-intercept $= -\frac{1}{3}$

e x-intercept = 7, y-intercept = 11

f x-intercept = 1, y-intercept = –1

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Interactive skillsheet
Gradients

Interactive skillsheet
Determining linear equations

Topic quiz
4D

4E Parallel and perpendicular lines

Learning intentions

- ✔ I can determine if two lines are parallel or perpendicular from their equations.
- ✔ I can determine linear equations for parallel lines.
- ✔ I can determine linear equations for perpendicular lines.

Inter-year links

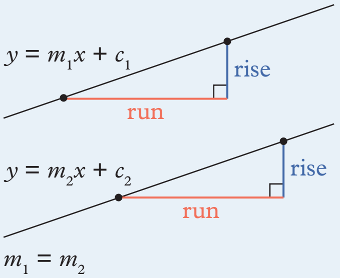
Year 8	6F Finding linear equations
Year 9	4C Gradients and intercepts

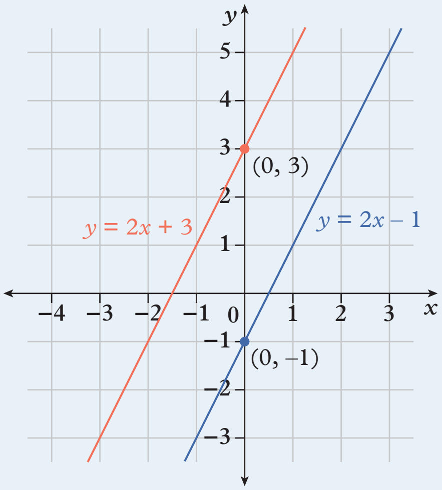
Parallel lines

- **Parallel** lines are two or more straight lines that never meet.
- Two lines are parallel if they have the same gradient.

  If the gradient–intercept equations for two lines are $y = m_1x + c_1$ and $y = m_2x + c_2$, the lines are parallel if $m_1 = m_2$.

  For example, the lines $y = 2x + 3$ and $y = 2x - 1$ shown on the right are parallel.



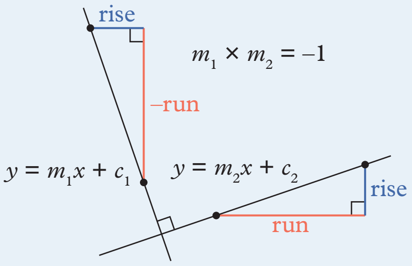


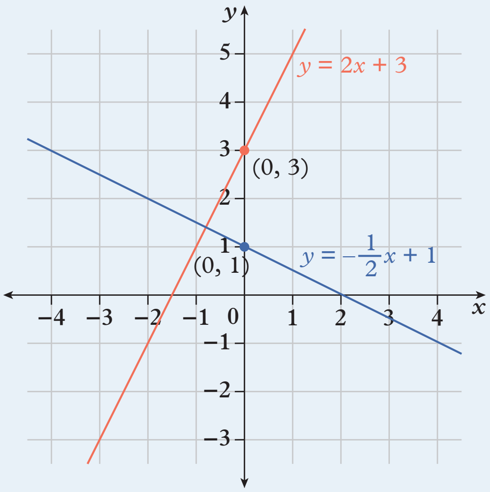
Perpendicular lines

- **Perpendicular** lines are straight lines that meet at right angles.
- Two lines are perpendicular if the product of their gradients is –1 or, expressed another way, if one gradient is the negative reciprocal of the other.

  If the equations of two lines written in gradient–intercept form are $y = m_1x + c_1$ and $y = m_2x + c_2$, the lines are perpendicular if $m_1 \times m_2 = -1$ or, expressed another way, if $m_1 = -\frac{1}{m_2}$.

  For example, the lines $y = 2x + 3$ and $y = -\frac{1}{2}x + 1$ are perpendicular because $2 \times \left(-\frac{1}{2}\right) = -1$.





Equations of parallel and perpendicular lines

- To determine the equation of a parallel or perpendicular line in the form $y = mx + c$:
  1. determine the gradient of the given line, m_2
  2. determine the gradient of the new line, m_1
     - → If lines are parallel, then their gradients are equal: $m_1 = m_2$.
     - → If lines are perpendicular, then their gradients are the negative reciprocal of each other: $m_1 = -\frac{1}{m_2}$.
  3. identify the x- and y-coordinates of a point on the new line
  4. substitute into the formula $y = mx + c$ to determine c.

Example 4E.1 Determining whether two lines are parallel or perpendicular

For each pair of equations, decide whether the lines are parallel, perpendicular or neither. Give reasons.

a $y - 2 = 5x$
$y - 5x = 9$

b $1 - 4y = 2x + 3$
$1 + 4x = 2y$

c $y = 3x - 2$
$3x + y = -2$

THINK	**WRITE**
a **1** Rearrange $y - 2 = 5x$ to make y the subject and identify the gradient.	**a** $y - 2 = 5x \quad (+ 2)$ $y = 5x + 2$ So the gradient of $y - 2 = 5x$ is 5.
2 Rearrange $y - 5x = 9$ to make y the subject and identify the gradient.	$y - 5x = 9 \quad (+ 5x)$ $y = 5x + 9$ So the gradient of $y - 5x = 9$ is 5.
3 State whether the lines are parallel, perpendicular or neither, by comparing the two gradients.	The lines are parallel because their gradients are the same.
b **1** Rearrange $1 - 4y = 2x + 3$ to make y the subject and identify the gradient.	**b** $1 - 4y = 2x + 3 \quad (- 1)$ $-4y = 2x + 2 \quad (\div -4)$ $y = -\frac{1}{2}x - \frac{1}{2}$ So the gradient of $1 - 4y = 2x + 3$ is $-\frac{1}{2}$.
2 Rearrange $1 + 4x = 2y$ to make y the subject and identify the gradient.	$1 + 4x = 2y$ $2y = 1 + 4x \quad (\div 2)$ $y = 2x + \frac{1}{2}$ So the gradient of $1 + 4x = 2y$ is 2.
3 State whether the lines are parallel, perpendicular or neither, by comparing the two gradients.	The lines are perpendicular because their gradients are the negative reciprocals of each other.
c **1** Identify the gradient of $y = 3x - 2$ by looking at the coefficient of x.	**c** The gradient of $y = 3x - 2$ is 3.
2 Rearrange $3x + y = -2$ to make y the subject and identify the gradient.	$3x + y = -2 \quad (- 3x)$ $y = -3x - 2$ So the gradient of $3x + y = -2$ is -3.
3 State whether the lines are parallel, perpendicular or neither, by comparing the two gradients.	The lines are neither parallel nor perpendicular. Their gradients are not equal, nor are they the negative reciprocals of each other.

Example 4E.2 Determining parallel lines

Determine the equation for a line that is parallel to the graph of $y = -7x - 5$ and passes through (2, 6).

THINK	WRITE
1 Identify the gradient of the graph of $y = -7x - 5$ and use this to define the gradient of the parallel line. Parallel lines have equal gradients.	The gradient of $y = -7x - 5$ is -7.
2 Substitute the values for the gradient and the coordinates into the gradient–intercept equation: $y = mx + c$.	For the parallel line, use $m = -7$ with $x = 2$ and $y = 6$. $y = mx + c$ $6 = -7 \times 2 + c$
3 Solve the equation for c and write the final equation in gradient–intercept form.	$6 = -14 + c$ (+ 14) $c = 20$ The equation is $y = -7x + 20$.

Example 4E.3 Determining perpendicular lines

Determine the equation for a line that is perpendicular to the graph of $y = \frac{2}{3}x + 11$ and passes through (10, –8).

THINK	WRITE
1 Identify the gradient for the graph of $y = \frac{2}{3}x + 11$ and use this to define a gradient for the perpendicular line. Perpendicular lines have gradients that are the negative reciprocal of each other, $m_1 = -\frac{1}{m_2}$.	The gradient of $y = \frac{2}{3}x + 11$ is $\frac{2}{3}$. For the perpendicular line, use $m = -\frac{3}{2}$. $\left(\frac{2}{3} \times \left[-\frac{3}{2}\right] = -1\right)$
2 Substitute the values for the gradient and the coordinates into the gradient–intercept equation: $y = mx + c$.	Use $x = 10$ and $y = -8$. $y = mx + c$ $-8 = -\frac{3}{2} \times 10 + c$
3 Solve the equation for c and write the final equation in gradient–intercept form.	$-8 = -15 + c$ (+ 15) $c = 7$ The equation is $y = -\frac{3}{2}x + 7$.

Helpful hints

✔ If y is already the subject of the equation, you do not need to rearrange the equation to the form $y = mx + c$ to identify the gradient, m. The coefficient of x will be the gradient.

For example, the gradient of $y = 1 - 4x$ is -4 and the gradient of $y = \frac{1 + 2x}{3}$ is $\frac{2}{3}$.

✔ Remember, if you know the y-intercept, $(0, c)$, all you need to do is determine the gradient, m, and substitute both values into the gradient–intercept equation $y = mx + c$.

ANS p688

Exercise 4E Parallel and perpendicular lines

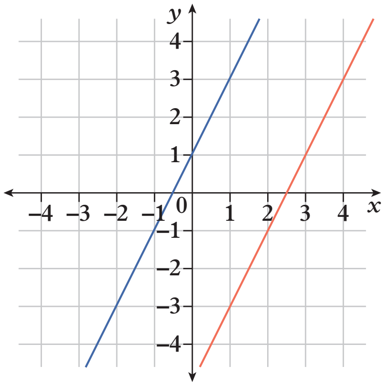
△ 1–3, 4(a, c, e, g), 5, 6(a, c, e, g), 7, 8(a, c, e, g), 9–11, 13, 16

□ 4, 5, 6(a, c, e, g), 7, 8(a, c, e, g), 9, 12, 17, 19, 21

⬠ 5, 6(g, h), 7, 8(g, h), 9, 14, 15, 18, 20, 22–26

1 **a** Calculate the gradients for the linear graphs on each of these Cartesian planes.

i



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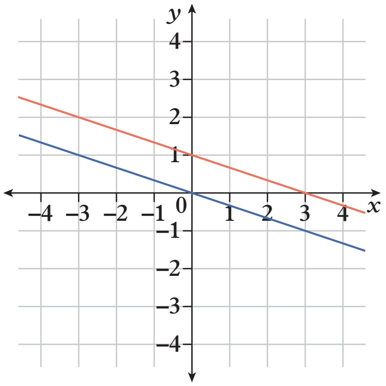


b What do you notice about both pairs of lines?

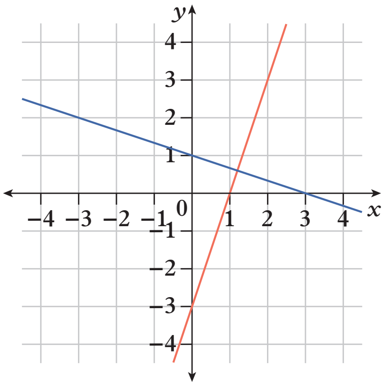
c What can you say about the gradients of parallel lines?

d Decide whether each equation below describes a linear graph parallel to those in part **a i**.

i $y = 2x - 9$ **ii** $y = 5 + 2x$ **iii** $y = 3x + 2$

2 **a** Calculate the gradients for the linear graphs on each of these Cartesian planes.

i



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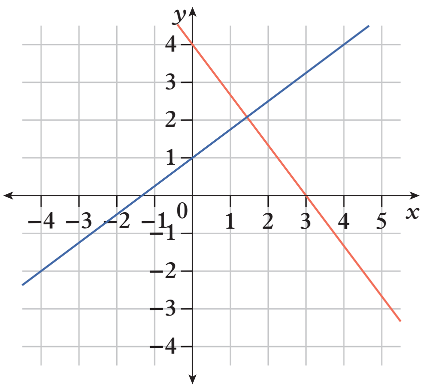


b Find the product of the gradients for each pair of lines in part **a**.

c What do you notice about both pairs of lines?

d What can you say about the product of the gradients of perpendicular lines?

e Write an equation for determining the gradient m_2 of a line that is perpendicular to another line with gradient m_1.

f Decide whether each of the following equations describes a linear graph that is perpendicular to those in part **a i**.

i $y = 3x - 4$

ii $y = \frac{1}{3}x + 4$

iii $y = 9 - \frac{1}{3}x$

3 If each of the numbers below is the gradient of a line, then find the gradient of the corresponding perpendicular lines.

a 5 **b** −8 **c** −1 **d** $\frac{1}{6}$ **e** $\frac{4}{5}$ **f** $-\frac{3}{2}$

4 For each linear equation below:

i rearrange the equation so that it is in gradient–intercept form: $y = mx + c$

ii write the gradient of a line parallel to the line described by the given equation

iii write the gradient of a line perpendicular to the line described by the given equation.

a $3x + y = 11$

b $-4x + y = 7$

c $-\frac{5}{3}x + y = \frac{2}{3}$

d $3x + 2y = 8$

e $5x - 2y = 4$

f $8x + 5y = -11$

g $9y - 3x = 15$

h $x = -5y - 3$

i $\frac{y-2}{x+5} = 1$

4E.1 **5** Decide whether each of these pairs of equations describes lines that are parallel, perpendicular or neither. Give a reason for your answer.

a $y = 4x - 3$ and $4x - y = 5$

b $3y - 2x = 12$ and $3x + 2y = 8$

c $y = \frac{3}{8}x$ and $3y = 4 - 8x$

d $2x + y = 3$ and $y = 2x + 6$

e $5x + y + 4 = 0$ and $x + 5y = 7$

f $2x - 7y = 11$ and $14y = 4x + 1$

g $8x - 12y = 7$ and $18y + 12x = -11$

h $6x + 3y = 7$ and $20y - 10x = 4$

4E.2 **6** Determine the equation for the line that is:

a parallel to the graph of $y = 4x + 3$ and passes through $(2, -5)$

b parallel to the graph of $y = -3x + 7$ and passes through $(-1, -3)$

c parallel to the graph of $y = -x - 6$ and passes through $(-4, 2)$

d parallel to the graph of $y = \frac{1}{2}x + 9$ and passes through $(6, 0)$

e parallel to the graph of $y = -\frac{1}{3}x - 8$ and passes through $(15, 0)$

f parallel to the graph of $y = \frac{2}{7}x + 5$ and passes through $(0, 11)$

g parallel to the graph of $y = \frac{3}{2} - \frac{5}{4}x$ and passes through $(0, -5)$

h parallel to the graph of $y = \frac{5}{6}x - \frac{9}{4}$ and passes through $(-9, 12)$.

7 Rearrange each of these equations to help you determine the equation for a line that is:

a parallel to the graph of $8x + 2y = 10$ and passes through $(5, -2)$

b parallel to the graph of $5x + 10y = 30$ and passes through $(-8, 1)$

c parallel to the graph of $7x - 3y = 5$ and passes through $(0, 6)$

d parallel to the graph of $40y - 24x = 6$ and passes through $(5, 0)$.

4E.3 **8** Determine the equation for the line that is:

a perpendicular to the graph of $y = 5x + 9$ and passes through $(5, -8)$

b perpendicular to the graph of $y = -7x - 3$ and passes through $(0, 3)$

c perpendicular to the graph of $y = x + 4$ and passes through $(-1, -2)$

d perpendicular to the graph of $y = -\frac{2}{3}x + 1$ and passes through $(-4, 4)$

e perpendicular to the graph of $y = \frac{4}{9}x - 7$ and passes through $(0, 8)$

f perpendicular to the graph of $y = -\frac{1}{3}x + 2$ and passes through $(-10, 0)$

g perpendicular to the graph of $y = \frac{14}{3} - \frac{5}{6}x$ and passes through $(9, 0)$

h perpendicular to the graph of $y = \frac{4}{7}x - \frac{8}{21}$ and passes through $(-28, 3)$.

9 Rearrange each of these equations to help you determine the equation for a line that is:

a perpendicular to the graph of $15x + 3y = 12$ and passes through $(-10, 7)$

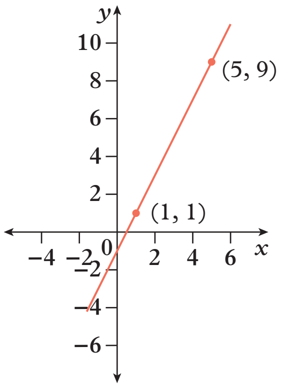
b perpendicular to the graph of $12x - 36y = 72$ and passes through $(-9, -6)$

c perpendicular to the graph of $9x + 7y = 13$ and passes through $(0, 47)$

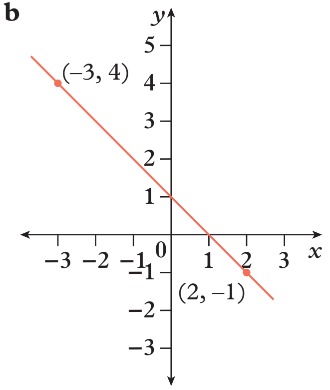
d perpendicular to the graph of $45y - 18x = 63$ and passes through $(-40, 0)$.

10 Write the equation for a line that is perpendicular to each line shown and has a y-intercept of 3.

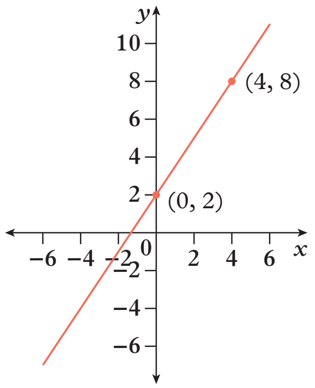
a



b



c



11 For each line shown in question **10**, write the equation for a line that has a y-intercept of –4 and is:

i parallel to the given line

ii perpendicular to the given line.

12 a Sketch each pair of graphs on the same Cartesian plane.

i $2x + 3y = 12$ and $2x - 3y = 12$

ii $2x + 3y = 12$ and $\frac{1}{2}x - \frac{1}{3}y = 12$

iii $2x + 3y = 12$ and $2x - 3y = \frac{1}{12}$

iv $5x + 2y = 20$ and $2x - 5y = 20$

v $5x + 2y = 20$ and $-2x + 5y = 20$

vi $5x + 2y = 20$ and $2x + 5y = -20$

b Determine which pairs of graphs are perpendicular.

c Write the equation of a line perpendicular to $ax + by = 1$ in the same form and in terms of a and b.

13 a Sketch the graphs of $x = 2$ and $y = 4$ on the same Cartesian plane.

b Explain why each of the following explanations is either correct or incorrect.

i Aamira says the graphs are perpendicular because they meet at a right angle.

ii Jack says the graphs are not perpendicular because $\frac{0}{1} \times \frac{1}{0} = 1 \neq -1$.

c Complete each of the following sentences.

i Vertical lines are perpendicular to horizontal lines because they meet at a ______ degree angle.

ii Vertical lines are perpendicular to horizontal lines, but we cannot use the formula ____ × ____ = ____ because the gradient of a vertical line is ____________.

iii Vertical lines are perpendicular to horizontal lines, so the equations ________ and ________ are perpendicular.

14 Show that the line joining (2, –3) and (4, 5) on the Cartesian plane is parallel to the graph of $y = 4x - 7$.

15 Show that the line joining (–11, –7) and (–1, –2) on the Cartesian plane is perpendicular to the graph of $y = -2x + 5$.

16 A line passes through two points, (–2, 3) and (7, 9).

a Find the gradient of the line.

b Write the equation for a linear graph that is:

i parallel to the line and passes through (6, –1) **ii** perpendicular to the line and passes through (–4, 8).

17 A line segment joins (–6, 7) and (0, –11). Write the equation for a linear graph that is:

a parallel to this line segment and passes through the origin

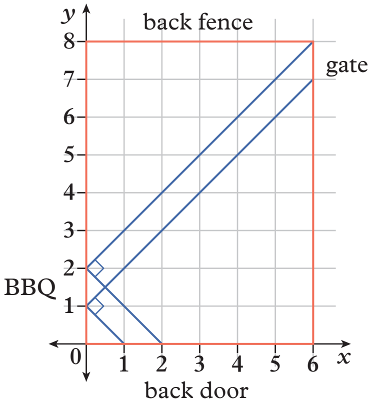
b perpendicular to this line segment and passes through (–6, 7)

c perpendicular to this line segment and passes through its midpoint.

(Hint: Use $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ to find the coordinates of the midpoint.)

PROBLEM SOLVING AND REASONING

18 In his backyard, Husayn is planning to add a concrete path that goes from the back door to the barbecue on the left fence, then continues to the gate in the back right corner. He wants the path to make a right turn at the barbecue. The path is planned to be 1 m wide and the back door is 1 m away from the left fence. Determine the equations of the four lines that mark the edge of the proposed path.



19 Four points are plotted on a Cartesian plane: $A(-2, 1)$, $B(5, 4)$, $C(-6, -4)$ and $D(8, 2)$.

a Is line segment AB parallel to line segment CD?

b Is line segment AC perpendicular to line segment BD?

20 The following three points are plotted on a Cartesian plane: $A(-2, 3)$, $B(1, 12)$ and $C(7, 10)$.

a Show that line segments drawn between these points form a right-angled triangle with the points A, B and C as vertices.

b Is the triangle ABC an isosceles right-angled triangle? Show your reasoning.

c Calculate the perimeter of the triangle, correct to one decimal place.

d Calculate the area of the triangle, correct to one decimal place.

(Hint: Use Pythagoras' Theorem to find the length between two points $\sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$.)

21 Use your knowledge of parallel lines and Pythagoras' Theorem to:

a show that quadrilateral $ABCD$, with vertices at $A(1, 4)$, $B(2, 6)$, $C(4, 7)$ and $D(3, 5)$, is a rhombus

b show that quadrilateral $EFGH$, with vertices at $E(1, 3)$, $F(8, 6)$, $G(4, -2)$ and $H(-3, -5)$, is a parallelogram.

CHALLENGE

22 Prove that quadrilateral $KLMN$, with vertices at $K(0, 0)$, $L(2, 4)$, $M(6, 2)$ and $N(4, -2)$, is a square. Then find its perimeter and area, correct to one decimal place.

23 Prove that quadrilateral $PQRS$, with vertices at $P(-7, 5)$, $Q(-3, 1)$, $R(0, 4)$ and $S(-4, 8)$, is a rectangle. Then find its perimeter and area, correct to one decimal place.

24 A perpendicular bisector is the line that passes through the midpoint of a line segment between two points at a right angle. Write the equation for the perpendicular bisector of the line segment joining $(-2, 3)$ and $(8, -7)$.

25 Prove that quadrilateral $WXYZ$, with vertices at $W(-9, -11)$, $X(-1, 8)$, $Y(6, 9)$ and $Z(7, 2)$, is a kite. (Hint: Also consider the diagonals of the quadrilateral.)

26 Without plotting points on a Cartesian plane, what type of quadrilateral is $ABCD$ if its vertices are $A(2, 1)$, $B(8, 4)$, $C(9, 1)$ and $D(-3, -5)$? Show your reasoning, using your knowledge of parallel and perpendicular lines.

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4E

4F Simultaneous linear equations

Learning intentions

- ✔ I can solve simultaneous linear equations graphically.
- ✔ I can solve simultaneous linear equations using substitution.
- ✔ I can solve simultaneous linear equations using elimination.

Inter-year links

Year 8	6E Solving linear equations using graphs
Year 9	4D Sketching linear graphs using intercepts

Simultaneous linear equations

- If linear equations contain some of the same variables, we call them **simultaneous linear equations**.
- The solutions to a pair of simultaneous linear equations are the values for the variables that are solutions to both equations.

  For example, to check whether $x = 2$ and $y = 7$ are solutions to $y = 5x - 3$ and $y = -x + 9$, substitute the value $x = 2$ into both equations.

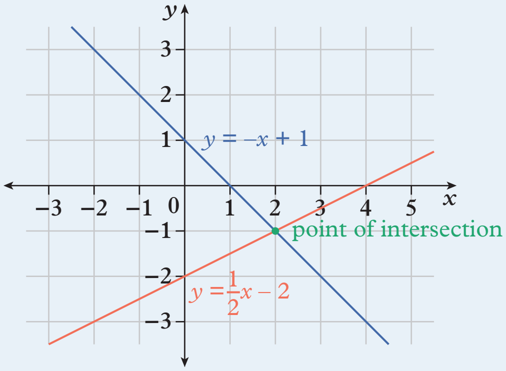
  $$\begin{aligned} y &= 5x - 3 \\ y &= 5(2) - 3 \\ y &= 10 - 3 \\ y &= 7 \end{aligned} \qquad \begin{aligned} y &= -x + 9 \\ y &= -1(2) + 9 \\ y &= -2 + 9 \\ y &= 7 \end{aligned}$$

  When $x = 2$, the result of both equations is $y = 7$, showing that the solution to the pair of simultaneous linear equations is $x = 2$ and $y = 7$.

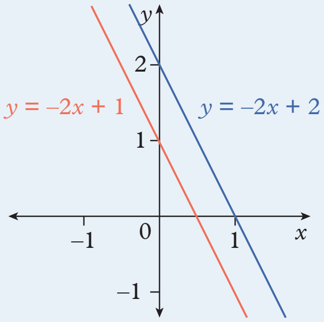
Solving simultaneous linear equations graphically

- If two linear graphs are not parallel, then they will have a **point of intersection**.

  For example, the pair of simultaneous linear equations $y = -x + 1$ and $y = \frac{1}{2}x - 2$ can be represented by the pair of linear graphs shown on the right.
- At the point of intersection, both lines will have the same x- and y-coordinates. So, the coordinates of the point of intersection will be the solution for the pair of simultaneous linear equations describing the pair of intersecting linear graphs.



- If the linear graphs are parallel, the simultaneous linear equations have no solution. The graphs do not intersect.



Solving simultaneous linear equations by substitution

- The **substitution method** involves substituting an expression from one equation into another equation.
- Using the substitution method is simplest when at least one of the equations already has one variable as the subject.

  For example, if $y = 4 - 2x$ and $y + 11 = 3x$ are being solved simultaneously, substituting the value for y from the first equation into the second equation will give us a value for x.

  $y = 4 - 2x$ and $y + 11 = 3x$

  $$\begin{array}{lrl} \text{So:} & (4 - 2x) + 11 = 3x & \\ & 4 - 2x + 11 = 3x & (+\ 2x) \\ & 4 + 11 = 5x & \\ & 15 = 5x & (\div\ 5) \\ & x = 3 & \end{array}$$

  Substituting the value $x = 3$ into either equation gives the result $y = -2$:

  $$\begin{array}{lll} y = 4 - 2x & y + 11 = 3x & \\ y = 4 - 2(3) & y + 11 = 3(3) & \\ y = 4 - 6 & y + 11 = 9 & (-\ 11) \\ y = -2 & y = -2 & \end{array}$$

  So, the solution to the pair of simultaneous equations is $x = 3$ and $y = -2$.

Solving simultaneous linear equations by elimination

- The LHS and RHS of two equations can be added or subtracted from each other to create a new equation. This idea can be expressed algebraically as follows.

  If: $a = b$ and $c = d$

  then: $a + c = b + d$

  An example is shown below.

  If: $6 = 1 + 5$ and $3 + 4 = 7$

  then: $6 + (3 + 4) = (1 + 5) + 7$
- The **elimination method** uses this idea to 'add' or 'subtract' simultaneous linear equations to create a new equation in which one of the variables has been eliminated.

  For example, the pair of simultaneous linear equations $2x + y = 4$ and $-3x + y = -11$ can be solved by elimination when the second equation is 'subtracted' from the first.

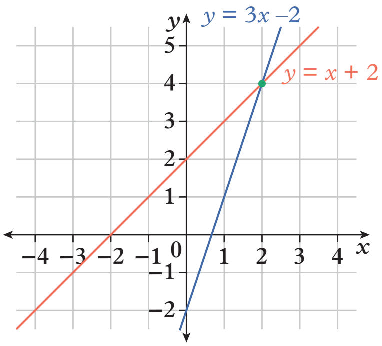
  $$\begin{array}{lrl} & 2x + y = 4 & ① \\ & -3x + y = -11 & ② \\ ① - ②: & (2x + y) - (-3x + y) = (4) - (-11) & \\ & 2x + 3x + y - y = (4 + 11) & \\ & 5x = 15 & (\div\ 5) \\ & x = 3 & \end{array}$$

  Substituting this value for x into either equation gives the result $y = -2$. The pair of simultaneous equations has been solved.
- One or both of the equations might need to be multiplied by a constant so that the coefficients become zero when they are added/subtracted.

Example 4F.1 Solving simultaneous linear equations graphically

Solve this pair of simultaneous linear equations using a graph.

$y = x + 2 \qquad\qquad y = 3x - 2$

THINK	WRITE
1 Sketch the graphs of $y = x + 2$ and $y = 3x - 2$ on the same Cartesian plane, using graph paper. Use a ruler to extend both lines until they reach their point of intersection.	
2 Use the coordinates of the point of intersection to determine a solution to the simultaneous linear equations.	The point of intersection is at (2, 4), so the solution to the simultaneous linear equations is $x = 2, y = 4$.
3 Check the solution by substituting the value for x into both equations. If both the calculated values for y match the solution, then the solution is correct.	Check: $y = x + 2$ $y = 3x - 20$ $y = (2) + 2$ $y = 3(2) - 2$ $y = 4$ $y = 6 - 2$ $y = 4$

Example 4F.2 Solving simultaneous linear equations by substitution

Solve this pair of simultaneous linear equations using the substitution method.

$5x + 2y = 2 \qquad\qquad y = 7x + 20$

THINK	WRITE
1 Number each equation for reference.	$5x + 2y = 2$ ① $y = 7x + 20$ ②
2 Equation ② has one variable (y) as its subject. Substitute the expression for y from equation ② into equation ①. Use brackets when substituting the expression.	Substituting $y = 7x + 20$ into ①: $5x + 2y = 2$ $5x + 2(7x + 20) = 2$ $5x + 14x + 40 = 2$
3 Solve for x.	$19x + 40 = 2$ $(-\ 40)$ $19x = -38$ $(\div\ 19)$ $x = -2$
4 Substitute the solution for x into either equation and solve it for y.	Substitute $x = -2$ into ②: $y = 7x + 20$ $y = 7(-2) + 20$ $y = -14 + 20$ $y = 6$ The solution is $x = -2, y = 6$.

Example 4F.3 Solving simultaneous linear equations by elimination

Solve this pair of simultaneous linear equations using the elimination method.

$y - 3x = -10 \qquad 3x + 2y = 16$

THINK	**WRITE**
1 Number each equation for reference.	$y - 3x = -10$ ① $3x + 2y = 16$ ②
2 The coefficient for x in equation ① is the negative value of the coefficient for x in equation ②. So, the equations can be added to remove x.	① + ②: $(y - 3x) + (3x + 2y) = (-10) + (16)$
3 Solve the resulting equation for y.	$y - 3x + 3x + 2y = -10 + 16$ $3y = 6 \quad (\div 3)$ $y = 2$
4 Substitute the solution for y into either equation to solve it for x.	Substitute $y = 2$ into ①: $y - 3x = -10$ $(2) - 3x = -10 \quad (-\ 2)$ $-3x = -12 \quad (\div -3)$ $x = 4$ The solution is $x = 4$, $y = 2$.

Example 4F.4 Solving simultaneous linear equations by elimination (multiplying by a constant)

Solve each of these pairs of simultaneous linear equations using the elimination method.

a $y - x = 11$ \
$3y - 2x = 27$

b $4x - 3y = -5$ \
$2y - 5x = 1$

THINK	**WRITE**
a **1** Number each equation for reference.	**a** $y - x = 11$ ① $3y - 2x = 27$ ②
2 The coefficient of x in equation ① is –1 and the coefficient of x in equation ② is –2. So equation ② can stay the same, but equation ① must be multiplied by 2.	2 × ①: $y - x = 11 \quad (\times 2)$ $2y - 2x = 22$
3 Equation '2 × ①' and equation ② have the same coefficients of x, so subtract to remove x.	(2 × ①) – ②: $(2y - 2x) - (3y - 2x) = (22) - (27)$
4 Solve the equation for y.	$2y - 2x - 3y + 2x = 22 - 27$ $-y = -5$ $y = 5$

5 Substitute the solution for y into either equation and solve it for x.	Substituting $y = 5$ into ①: $y - x = 11$ $(5) - x = 11 \quad (-\ 5)$ $-x = 6$ $x = -6$ The solution is $x = -6$, $y = 5$.
b **1** Number each equation for reference.	**b** $4x - 3y = -5$ ① $2y - 5x = 1$ ②
2 The coefficient of y in equation ① is –3 and the coefficient of y in equation ② is 2. So equation ① must be multiplied by 2 and equation ② must be multiplied by 3.	$2 \times$ ①: $4x - 3y = -5 \quad (\times\ 2)$ $8x - 6y = -10$ $3 \times$ ②: $2y - 5x = 1 \quad (\times\ 3)$ $6y - 15x = 3$
3 Equation ‘2 × ①’ and equation ‘3 × ②’ have opposite coefficients of y, so add them together to remove y.	$(2 \times$ ①$) + (3 \times$ ②$)$: $(8x - 6y) + (6y - 15x) = (-10) + (3)$ $8x - 6y + 6y - 15x = -10 + 3$
4 Solve for x.	$-7x = -7$ $x = 1$
5 Substitute the solution for x into either equation to solve it for y.	Substituting $x = 1$ into ①: $4x - 3y = -5$ $4(1) - 3y = -5$ $4 - 3y = -5 \quad (-\ 4)$ $-3y = -9 \quad (\div\ 3)$ $y = 3$ The solution is $x = 1$, $y = 3$.

Helpful hints

- ✔ Always number your equations for easy reference.
- ✔ Remember, you can always check your solution to a pair of simultaneous equations by substituting one of the values back into each equation to see if the result matches your solution for the other variable.
- ✔ When solving simultaneous linear equations graphically, make sure to choose a suitable size for the units of your x- and y-axes and use graph paper so that you can identify the intersection point with accuracy.
- ✔ When solving simultaneous linear equations using the substitution method, always make sure you substitute algebraic expressions enclosed in brackets, then remove the brackets by expanding in the next step.
- ✔ Remember, you can’t eliminate a variable by adding or subtracting equations unless the magnitude of the coefficients are the same.

ANS p690

Exercise 4F Simultaneous linear equations

△ 1, 2, 3–5(1st, 2nd columns), 6–8, 9(1st, 2nd columns), 11(a, b, g, h), 13

■ 2, 3–5(3rd, 4th columns), 8, 9(2nd, 3rd columns), 10, 11(e, f, h, i, o), 12, 14, 15, 18, 20

⬠ 3(k, l, o, p), 4–5(3rd, 4th columns), 9(3rd column), 11, 14, 16, 17, 19–21

UNDERSTANDING AND FLUENCY

1 Check if the given x and y values are the solution to the pair of simultaneous linear equations each time.

a $x = 1, y = 5$ Equations: $y = x + 4$ and $y = -2x + 7$

b $x = -2, y = 1$ Equations: $y = x + 3$ and $y = -3x - 5$

c $x = 5, y = -2$ Equations: $y = x - 9$ and $y = -2x + 7$

d $x = 3, y = -7$ Equations: $2x + y = -1$ and $y = -x - 4$

e $x = 4, y = 3$ Equations: $y = 2x - 5$ and $y = -x + 1$

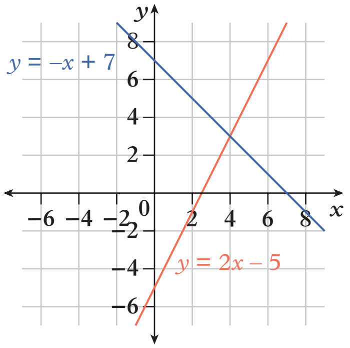
f $x = -6, y = 0$ Equations: $y = x - 6$ and $x + y = -6$

2 For each of the following pairs of simultaneous linear equations and their graphs:

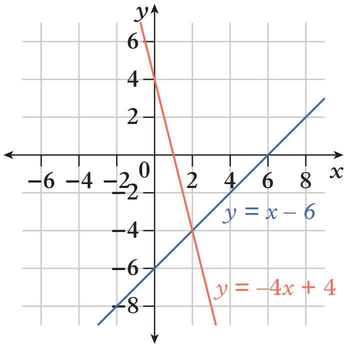
i write the coordinates of the point of intersection of the two lines

ii use the coordinates of the point of intersection to help you find the solution to the simultaneous linear equations. Use substitution to check your solution.

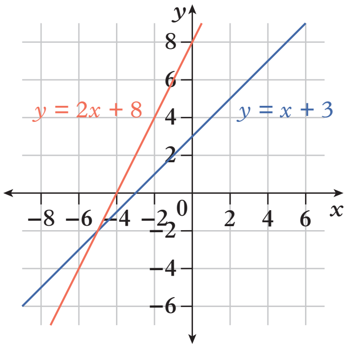
a $y = 2x - 5$
$y = -x + 7$



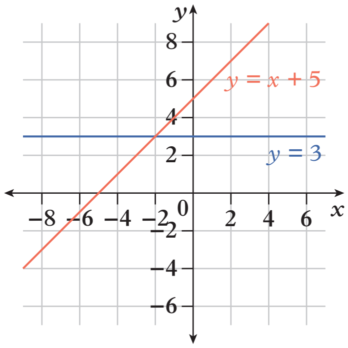
b $y = -4x + 4$
$y = x - 6$



c $y = 2x + 8$
$y = x + 3$



d $y = x + 5$
$y = 3$



4F.1 **3** Use graphs to help you solve these pairs of simultaneous linear equations (**a–p**). Use substitution to check your solutions.

a $y = 2x - 6$
$y = x - 1$

b $y = 3x$
$y = x + 4$

c $y = x - 2$
$y = 10 - x$

d $y = x + 7$
$y = -3x - 1$

e $y = 3x + 6$
$2x + y = 1$

f $y = -2x - 5$
$y = 2x + 7$

g $y = 2x - 4$
$y = x - 1$

h $2x + y = 6$
$y = x - 6$

i $y = 3x + 3$
$y = -x - 5$

j $2x + y = -4$
$x + y = 1$

k $x - y = 4$
$4x + y = 6$

l $3y = x + 12$
$y = x + 2$

m $y = \frac{7}{2}x + 13$
$-2x + 3y = 5$

n $y = -\frac{5}{4}x - \frac{11}{4}$
$3x - 2y = 11$

o $y = \frac{8}{5}x + \frac{23}{5}$
$9x + 16y = -134$

p $y = \frac{5}{7}x + \frac{19}{7}$
$7x + 4y = 1$

UNDERSTANDING AND FLUENCY

4F.2 **4** Use the substitution method to solve each of these pairs of simultaneous linear equations.

a $x + y = 9$
$y = x + 5$

b $x + 2y = 6$
$y = x - 3$

c $2x + y = 1$
$y = x + 4$

d $y = x - 1$
$x - 3y = -9$

e $3x - y = -10$
$y = x + 2$

f $y = 6x$
$5x - y = -2$

g $6x + 5y = 21$
$y = 2x + 1$

h $y = 2 - 3x$
$4x + y = -1$

i $2x + 3y = -5$
$y = 5 - 2x$

j $x = y + 2$
$2x - 3y = 1$

k $x = 4y$
$y = 3x + 11$

l $x = 5$
$4x - 3y = 2$

5 Use the substitution method to solve each of these pairs of simultaneous linear equations.

a $y = 2x + 6$
$y = x + 7$

b $y = x - 2$
$y = 4x - 20$

c $y = 7x + 16$
$y = 2x + 1$

d $y = x - 6$
$y = 4 - x$

e $y = 5x + 9$
$y = 3 - x$

f $y = 1 - 2x$
$y = x - 11$

g $y = 3x - 5$
$y = 5 - 2x$

h $y = 4x + 14$
$y = -4x - 2$

6 Solve each of these pairs of simultaneous linear equations by adding the equations together to remove one variable.

a $2x + y = 9$
$3x - y = 1$

b $x + 2y = -1$
$-x + 5y = 8$

c $x - 2y = 11$
$5x + 2y = -5$

7 Solve each of these pairs of simultaneous linear equations by subtracting one equation from the other to remove one variable.

a $x + 4y = 17$
$x + 2y = 11$

b $3x + y = 3$
$x + y = 7$

c $4x - 3y = 33$
$4x + y = 5$

4F.3 **8** Use the elimination method to solve each of these pairs of simultaneous linear equations.

a $-3x + y = 10$
$3x - 4y = -4$

b $x - 2y = 2$
$3x - 2y = 14$

c $6x - y = -8$
$-6x + 5y = 16$

d $6x + 5y = 19$
$4x + 5y = 21$

e $4x + 3y = 29$
$2x - 3y = 1$

f $-5x + y = 11$
$-5x - 2y = 23$

4F.4 **9** Use the elimination method to solve each of these pairs of simultaneous linear equations.

a $5x + 3y = 21$
$2x - y = 4$

b $x + 4y = -2$
$4x - 3y = 11$

c $3x + y = -5$
$7x + 2y = -14$

d $6x - 7y = 9$
$-2x + 5y = 5$

e $4x + 3y = -24$
$5x - 12y = 33$

f $2x + 5y = 26$
$10x - 3y = -38$

g $4x + 3y = 16$
$3x - 2y = -5$

h $3x + 7y = -2$
$5x + 11y = -4$

i $-8x + 5y = -21$
$5x + 2y = 8$

j $3x - 10y = -32$
$-7x + 3y = 34$

k $-9x + 2y = -35$
$8x - 9y = -5$

l $-4x - 5y = 22$
$-3x + 7y = -5$

10 Consider the three methods you have used for solving simultaneous linear equations:

i graphical
ii substitution
iii elimination

- **a** For each method, describe the situation for which its use is most appropriate.
- **b** For each method, describe the situation for which its use is least appropriate.
- **c** What could you do to make the least appropriate situation for each method more appropriate for its use?

PROBLEM SOLVING AND REASONING

11 Use the most appropriate method to solve each of these pairs of simultaneous equations. (Hint: Not all solutions are integers.)

a $y = 3x - 6$
$4x + 11y = 8$

b $y = 6x - 5$
$y = 3x - 4$

c $4x - 5y = 4$
$2x + 5y = 5$

d $y = -7$
$y = 4x + 29$

e $x = 5y - 6$
$x - y = 2$

f $2y = x - 4$
$3x - 2y = 0$

g $2x - y = -4$
$4x + y = 22$

h $x + 5y = -5$
$x - 2y = 9$

i $-x - 7y = 5$
$-2x - 7y = 3$

j $-3x + 4y = 14$
$3x - y = -8$

k $x + y = 11$
$x - y = 3$

l $5x + 2y = -18$
$-5x + 2y = 22$

m $y = 2.5x + 4.3$
$y = 7.5x + 8.3$

n $4x = y - 6$
$8x + 3y = 13$

o $6x - 7y = -119$
$8x - 5y = 19$

12 Explain why there is no solution to the simultaneous linear equations $y = 2x + 5$ and $2x - y = 1$.

13 Sophie has decided to make hand-painted cards. She invests \$60 in a set of paintbrushes and enough paint to make 40 cards. The cost of each plain white card is \$2 and she plans to sell the painted cards for \$6 each.

a Explain how the amount of money, a, to produce n cards (where $0 \le n \le 40$) can be written as $a = 2n + 60$.

b Explain how the amount of money, a, received for selling n cards can be written as $a = 6n$.

c Draw graphs of the two equations on the same Cartesian plane. Label the horizontal axis as n and the vertical axis as a.

d What are the coordinates of the point of intersection of the two graphs? What does this tell you?

e How many cards should Sophie produce and then sell to break even (recover her costs but nothing extra)?

f How many cards should Sophie produce and sell to make a profit?

14 The perimeter around the top of a rectangular billiard table is 810 cm.

a Use l and w to write a linear equation for the perimeter of the table.

b Use l and w to write a linear equation describing the relationship between the length and width of the top of the table if the length is twice the width.

c Solve the simultaneous linear equations you wrote for part **b** and find the dimensions of the top of the billiard table.

15 a Show that $x = 15$, $y = -5$ and $z = 5$ is a set of solutions to the simultaneous equations $x + y = 10$ and $x + z = 20$.

b Show that $x = 22$, $y = -12$ and $z = -2$ is a set of solutions to the simultaneous equations $x + y = 10$ and $x + z = 20$.

c Explain why we cannot solve the simultaneous equations $x + y = 10$ and $x + z = 20$ for one set of values of x, y and z.

16 a Show that the solution to the simultaneous equations $ax + by = c$ and $ax + qy = r$ is $x = \dfrac{br - cq}{ab - aq}$ and $y = \dfrac{c - r}{b - q}$.

b Solve the simultaneous equations $3x + 4y = 9$ and $3x + 6y = 7$ by substituting the values of a, b, c, q and r into the solution from part **a**.

c Show that the solution to the simultaneous equations $ax + by = c$ and $px + qy = r$ is $x = \dfrac{br - cq}{bp - aq}$ and $y = \dfrac{cp - ar}{bp - aq}$.

d Solve the simultaneous equations $3x + 4y = 9$ and $5x + 6y = 7$ by substituting the values of a, b, c, p, q and r into the solution from part **c**.

PROBLEM SOLVING AND REASONING

17 Chloe is planning a party and obtains prices from two catering companies. Each caterer has a fixed price for delivery and set-up, and a cost per person for the food.

Angie's Catering: fixed price \$200, \$28 per person

Cool Food Club: fixed price \$100, \$32 per person

Chloe is unsure how many people she will invite.

- **a** Write a linear equation to represent the cost of hiring Angie's Catering. Remember to define the two variables you are using.
- **b** Write a linear equation to represent the cost of hiring Cool Food Club.
- **c** Show the graphs of both linear equations on the same Cartesian plane.
- **d** Which company is cheaper if catering for 18 people? Explain how you can see this from your graph.
- **e** Which company is cheaper if catering for 36 people?
- **f** How many people can be catered for so that the cost of using either company is the same? What is this cost?
- **g** Write a summary advising Chloe on what her options are.

CHALLENGE

18 A linear graph has a y-intercept of –5. Another linear graph has a gradient of –3. The two linear graphs intersect at the point (4, –6).

- **a** Show the two graphs on the same Cartesian plane.
- **b** Find the equation for each linear graph.
- **c** Write the solution to the simultaneous linear equations.
- **d** Use substitution to check that the solution satisfies both equations.

19 Write a pair of equations whose graphs are perpendicular lines that both pass through the point (4, –3) such that:

- **a** the gradients for both lines are integer values
- **b** one of the lines has a gradient of zero
- **c** one of the lines has a gradient of $\frac{a}{b}$, where a and b are integers.

20 Explain why the pair of simultaneous linear equations $y = 4 - 3x$ and $6x + 2y = 8$ has an infinite number of solutions.

21 Consider this pair of simultaneous equations:

$7x - 6y = -14$ ①

$5x - 3y = -1$ ②

- **a** Demonstrate that the same solution is obtained using each of the strategies described below.

  Strategy A: Multiply equation ② by 2 to form equation ③ and then subtract equation ③ from equation ①.

  Strategy B: Multiply equation ② by –2 to form equation ④ and then add equations ① and ④.

  Strategy C: Multiply equation ① by 5 to form equation ⑤ and multiply equation ② by 7 to form equation ⑥. Subtract equation ⑥ from equation ⑤.
- **b** Are there other strategies you could use? Explain.

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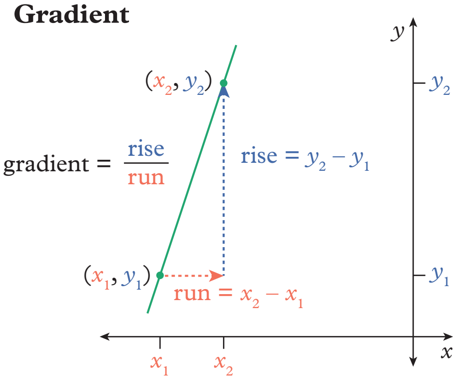
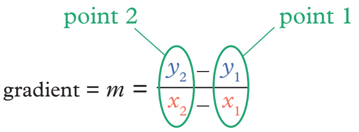
4F

Chapter summary

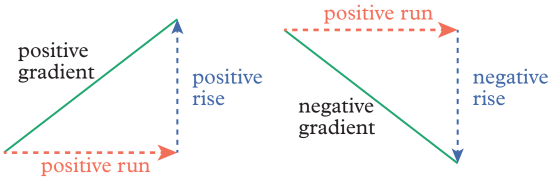
Gradient



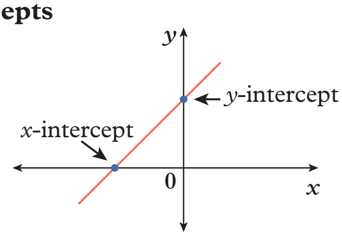
$$\text{gradient} = \frac{\text{rise}}{\text{run}}$$



$$\text{gradient} = m = \frac{y_2 - y_1}{x_2 - x_1}$$



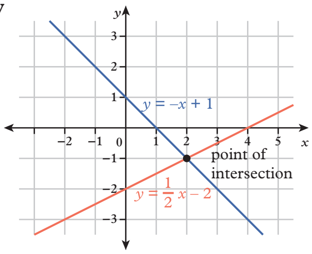
Intercepts



Simultaneous linear equations

1. Solving graphically

- The coordinates for the point of intersection of the two graphs give the solution to the simultaneous linear equations.



2. Solving by substitution

- Substitute an expression from one equation into the other equation to create a new equation containing only one variable.
  For example: $y = 4 - 2x$ and $y + 11 = 3x$
  So: $4 - 2x + 11 = 3x$
- Solve for the remaining variable, then substitute the solution into one of the original equations to solve for the other variable.

3. Solving by elimination

- Multiply each equation by a constant so that the coefficients of one of the variables is the same magnitude in both equations.
- Add or subtract the LHS and RHS of the equations to eliminate one of the variables.
  For example, $2x - y = 4$ ①
  $-3x - 2y = -6$ ②

2 × ① − ②: $2 \times (2x - y) - (-3x - 2y) = 2 \times (4) - (-6)$

$7x = 14$

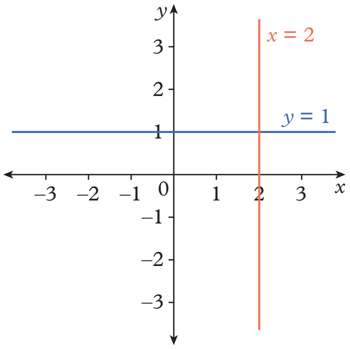
- Solve for the remaining variable, then substitute the solution into one of the original equations to solve for the other variable.

Gradient–intercept form

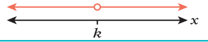
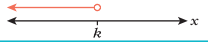
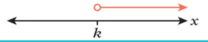
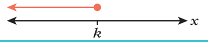
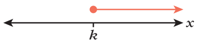
$$y = \underset{\text{gradient}}{m}x + \underset{y\text{-intercept}}{c}$$

Horizontal and vertical lines

- Horizontal: $y = c$, gradient = 0
- Vertical: $x = c$, gradient is undefined

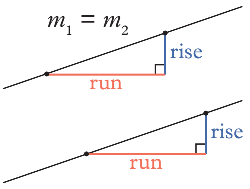


Linear inequalities

Inequality	Number line representation	Meaning
$x \neq k$		x is not equal to k
$x < k$		x is less than k
$x > k$		x is greater than k
$x \leq k$		x is less than or equal to k
$x \geq k$		x is greater than or equal to k
$k \leq x < m$		x is greater than or equal to k but less than m

Parallel lines

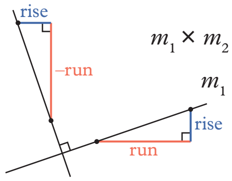
- Their gradients are equal.



$$m_1 = m_2$$

Perpendicular lines

- The product of their gradients is −1.



$$m_1 \times m_2 = -1$$

$$m_1 = -\frac{1}{m_2}$$

Chapter review

Chapter review quiz
Take the chapter review quiz to assess your knowledge of this chapter.

Quizlet
Test your knowledge of this topic by working individually or in teams.

Multiple-choice

4A **1** The solution to $\frac{x-4}{2} = -3$ is:

A $x = 2$ **B** $x = -2$ **C** $x = 10$ **D** $x = -10$ **E** $x = -14$

4A **2** Which of these equations has the solution $x = -5$?

A $\frac{12}{x} = -2.4$ **B** $\frac{2x+3}{13} = 1$ **C** $3x - 7 = 8$ **D** $\frac{3}{2-x} = 1$ **E** $3x + 1 = 8(x + 6)$

4B **3** Which of these statements is incorrect?

A $-7 < -4$ **B** $\frac{2}{3} < \frac{3}{4}$ **C** $-3.5 < -4.7$ **D** $\frac{8}{9} < \frac{9}{8}$ **E** $\frac{3}{4} \neq \frac{8}{16}$

4B **4** If $x \geq -6.2$, which of these could *not* be a value for x?

A 0 **B** -6.2 **C** 6.2 **D** -6.3 **E** -4.9

4C **5** The y-intercept of the graph of $12x + 4y = 8$ is:

A 8 **B** -8 **C** 2 **D** -2 **E** 2

4C **6** The gradient of the line $12x + 4y = 8$ is:

A 3 **B** -3 **C** 12 **D** -12 **E** 4

4D **7** The equation for the linear graph with a gradient of $\frac{3}{4}$ and a y-intercept of $\frac{5}{8}$ is:

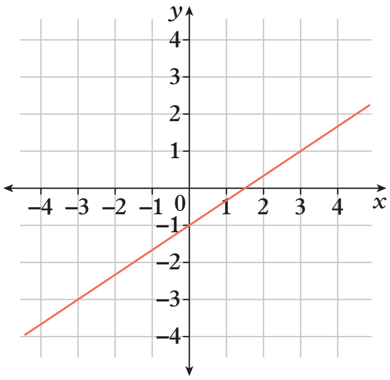
A $8x - 6y + 5 = 0$ **B** $8y = 6x - 5$ **C** $6x - 8y + 5 = 0$
D $6y = 8x + 5$ **E** $6x - 5y + 8 = 0$

4D **8** The gradient of the line passing through the points (1, 8) and (−3, −6) is:

A 1 **B** -1 **C** $\frac{2}{7}$ **D** $\frac{7}{2}$ **E** $-\frac{7}{2}$

9 The equation of the linear graph to the right is:

A $y = \frac{2}{3}x + 1$ **B** $y = \frac{3}{2}x - 1$ **C** $y = -\frac{2}{3}x - 1$
D $y = \frac{3}{2}x + 1$ **E** $y = \frac{2}{3}x - 1$



4E **10** A line parallel to the line $3x + 2y = 6$ would have a gradient of:

A $\frac{2}{3}$ **B** -3 **C** $-\frac{3}{2}$
D $\frac{3}{2}$ **E** $-\frac{2}{3}$

4E **11** A line has a gradient of $-\frac{3}{4}$. A line perpendicular to this line would have a gradient of:

A 3 **B** $-\frac{3}{4}$ **C** $\frac{3}{4}$ **D** $-\frac{4}{3}$ **E** $\frac{4}{3}$

4F **12** The pair of linear equations $y = x + 3$ and $y = -2x + 6$ intersect at which of these points?

A (−3, 0) **B** (3, 6) **C** (1, 8) **D** (1, 4) **E** (0, 6)

4F **13** The pair of linear equations $y = 3x + 1$ and $6x - 2y + 2 = 0$ have how many simultaneous solutions?

A none **B** one **C** two **D** three **E** many

4F **14** When $y = 3x - 4$ is substituted into the equation $x - 2y = 3$, the resulting equation is:

A $-5x - 8 = 3$ **B** $-5x + 8 = 3$ **C** $5x + 8 = 3$ **D** $5x - 8 = 3$ **E** $7x + 8 = 3$

4F **15** Consider the two equations:

$2x - y + 3 = 0$ ①
$4x - y + 5 = 0$ ②

If equation ② is subtracted from equation ①, the resulting equation will be:

A $-2x - 2 = 0$ **B** $-2x - 8 = 0$ **C** $2x + 2 = 0$ **D** $2x + 8 = 0$ **E** $2x - 2 = 0$

Short answer

4A **1** Solve each of these equations. Leave your answer as a fraction where appropriate.

a $3 - 2(4 - 5x) = -20$ **b** $4(3x + 5) = 3(5x - 3)$

4A **2** Solve each of these equations, leaving your answer as a fraction if appropriate.

a $\frac{4 + x}{3} = \frac{2x - 5}{3}$ **b** $\frac{5 - 3x}{2} = \frac{2x + 5}{3}$

4A **3** Rearrange each of these equations so that the coloured pronumeral in the brackets is the subject.

a $A = \frac{h(a + b)}{2}$ (b)

b $2(l + w)h$ (h)

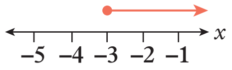
4B **4** For each inequality below, choose the x values from this list that make it a true statement:

-3.9 $1\frac{5}{8}$ 2.6 8.5 -8.5 $\frac{3}{4}$

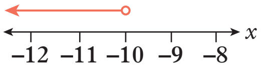
a $x > -1$ **b** $x \le 0.75$ **c** $x \ge 2.6$

4B **5** Write the inequality that is represented on each of these number lines.

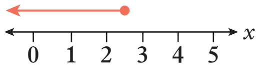
a



b



c



4B **6** Solve each of these inequalities.

a $5 - 2x < 7$ **b** $\frac{2 - 3x}{5} > 2$

4C **7** Sketch each of these linear equations using the gradient–intercept method.

a $y = -2x + 3$ **b** $x - 2y = 4$ **c** $5y - 10 = 15x$

4C **8** Sketch the graph of each of these equations, using the x- and y-intercept method.

a $2x - 4y - 12 = 0$ **b** $x + 3y + 9 = 0$ **c** $7x + 3y + 21 = 0$

4C **9** Find the equation for the line passing through each of these pairs of points.

a $(0, -4)$ and $(5, 6)$ **b** $(3, 6)$ and $(-2, 6)$ **c** $(-3, 5)$ and $(-3, -3)$

4D **10** Find the equation for the line that has:

a a gradient of 2 and passes through $(1, 5)$

b a gradient of -2 and passes through $(1, 5)$.

4D **11** Find the equation of the line that passes through both the given points in each of the following pairs. Write your answers in the form $y = mx + c$.

a $(1, 3)$ and $(2, 4)$ **b** $(-1, 5)$ and $(3\ -7)$ **c** $(-4, -4)$ and $(5, 5)$

4E **12** Write an equation for a line that is:

a parallel to the graph of $y = -4x + 5$; with y-intercept of -3

b perpendicular to the graph of $y = 2x - 7$; with y-intercept of -4.

4E **13** Write an equation for a line that is:

a perpendicular to the graph of $y = 3x + 2$ and passes through $(3, 2)$

b parallel to the graph of $y = 3x + 2$ and passes through $(-3, -2)$.

4F **14** Use a graph to help you solve each of these pairs of simultaneous equations.

a $y = 4x - 2$ **b** $2x - y + 5 = 0$

$y = -3x + 5$ $3x - y = 1$

4F **15** Use the substitution method to solve each of these pairs of simultaneous equations.

a $3x + 4y = -1$ **b** $y = 4x + 3$ **c** $7x - y = 4$

$y = x - 2$ $x - 2y = 8$ $y = 4x + 2$

4F **16** Use the elimination method to solve each of these pairs of simultaneous equations.

a $2x + y = 8$ **b** $x + 5y = 13$ **c** $2x - y = 7$

$4x - y = 4$ $x + 2y = 4$ $3x + 5y = 4$

Analysis

1 For a hot food stall, Sophie buys and then sells sausage rolls and party pies. The sausage rolls cost 71 cents each and the party pies cost 26 cents each. Sophie sells the sausage rolls for \$1.20 each and the party pies for 50 cents each. Sophie has a budget of \$60 per day. Let s be the number of sausage rolls Sophie buys and p be the number of party pies Sophie buys.

a Write an inequality that describes how Sophie can spend her budget each day.

b If Sophie buys 50 sausage rolls, write an inequality that describes the number of party pies she can buy.

c If Sophie buys 150 party pies, write an inequality that describes the number of sausage rolls she can buy.

d Assume Sophie is able to sell all the sausage rolls and party pies she buys on a particular day. Let P be the profit Sophie will make that day.

Write an equation that describes the profit Sophie will make.

e On another day, Sophie made \$50 profit.

Sketch the graph of the number of sausage rolls against the number of party pies Sophie could have bought and sold. Write the coordinates of the intercepts, correct to two decimal places. (Hint: Put sausage rolls on the vertical axis.)

f State the maximum number of party pies Sophie could have bought and sold on the day she made \$50 profit.

g On the day Sophie made \$50 profit, the price she paid for the sausage rolls and party pies was \$59.90. Determine the number of sausage rolls and party pies Sophie bought and sold that day.

2 **a** A triangle has vertices $X(4, -4)$, $Y(-2, 2)$ and $Z(1, 5)$. Draw this triangle on a Cartesian plane.

b Write an equation for each of the line segments XY, YZ and ZX and limit the values of x for each equation with an appropriate inequality statement.

c Show that triangle XYZ is right-angled.

d Let the length of the side opposite vertex X be represented by x, the side opposite vertex Y be represented by y, and the side opposite Z be represented by z. Calculate x, y and z, correct to two decimal places.

e Use the lengths from your answer to part **d** to demonstrate Pythagoras' Theorem.

f Determine the coordinates of the midpoint, M, of ZX. Join M to vertex Y.

g Calculate the length of MY.

h The line segment MY divides triangle XYZ into two smaller triangles. Considering the side lengths and angles, describe the shape of these triangles.

i Draw the perpendicular height of the triangle from vertex Y to the base XZ, meeting the base at H. Write the equation for the line segment YH.

j Solve a pair of simultaneous equations to find the coordinates of H.

k Calculate the length of YH.

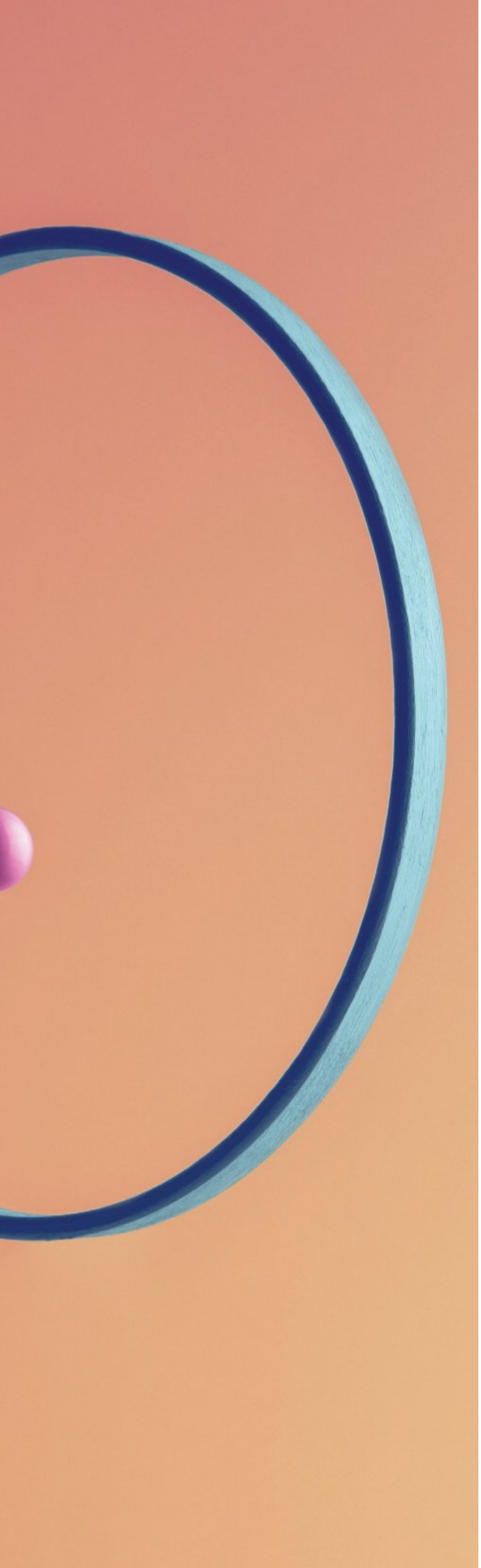
l Use two sets of different measurements to calculate the area of triangle XYZ. Show all your working and give your answer correct to one decimal place.

m Show that the length of $YH = \frac{xz}{y}$.

n Explain how a semi-circle can be drawn to enclose triangle XYZ. What does this show about the angle in a semi-circle?

5

Non-linear relationships



Index

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

✔ Solving equations using inverse operations
✔ Factorising quadratic expressions
✔ Factorising the difference of two squares
✔ The Cartesian plane
✔ Non-linear relationships

Curriculum links

- Solve simple problems involving inverse proportion (VCMNA327)
- Explore the connection between algebraic and graphical representations of relations such as simple quadratic, reciprocal, circle and exponential, using digital technology as appropriate (VCMNA339)
- Solve simple quadratic equations using a range of strategies (VCMNA341)
- Describe, interpret and sketch parabolas, hyperbolas, circles and exponential functions and their transformations (VCMNA359) [10A]
- Factorise monic and non-monic quadratic expressions and solve a wide range of quadratic equations derived from a variety of contexts (VCMNA362) [10A]

Materials

✔ Calculator

5A Solving quadratic equations

Learning intentions

- ✔ I can solve quadratic equations using the Null Factor Law.
- ✔ I can solve quadratic equations using surds.
- ✔ I can solve quadratic equations by completing the square.

Inter-year links

Year 7	6C Terms, expressions and equations
Year 8	6A Equations
Year 9	5A Solving quadratic equations

Quadratic equations

- The general form of a quadratic equation is:
  $ax^2 + bx + c = 0$ where a, b and c are constants.
- The factor form of a quadratic equation is: $(x - p)(x - q) = 0$ where p and q are constants.

The Null Factor Law

- The **Null Factor Law** states that, if the product of two factors is 0, one or both of the factors must be 0; that is:
  If $p \times q = 0$, then $p = 0$ or $q = 0$, or both $p = 0$ and $q = 0$.
- Some quadratic equations can be solved by making one side equal to zero and factorising the non-zero expression.
- When we apply the Null Factor Law to quadratic equations in factor form, $(x - p)(x - q) = 0$, we get:
  If $(x - p)(x - q) = 0$, then $x - p = 0$ or $x - q = 0$.
- Solutions can be written in a shorthand way using a ± ('plus or minus') sign. For example:
  If $x = -2$ or $x = 2$, that can be written as $x = \pm 2$.

Example 5A.1 Solving quadratic equations using the Null Factor Law

Solve each of these equations.

a $(x + 5)(x - 4) = 0$ **b** $x^2 - 4x - 12 = 0$ **c** $x^2 - 16 = 0$ **d** $5x^2 = -35x$

THINK	WRITE
a 1 Check that the quadratic expression is in factor form and that the other side of the equation is equal to 0.	**a** $(x + 5)(x - 4) = 0$
2 Apply the Null Factor Law by letting each factor equal 0.	$x + 5 = 0$ or $x - 4 = 0$
3 Solve each linear equation.	$x = -5$ or $x = 4$
b 1 Check that the quadratic expression is in factor form and that the other side of the equation is equal to 0.	**b** $x^2 - 4x - 12 = 0$
2 Factorise the quadratic expression.	$(x - 6)(x + 2) = 0$
3 Apply the Null Factor Law.	$x - 6 = 0$ or $x + 2 = 0$
4 Solve each linear equation.	$x = 6$ or $x = -2$

c **1** Check that the quadratic expression is in factor form and that the other side of the equation is equal to 0.	**c** $x^2 - 16 = 0$
2 Factorise the quadratic expression using the 'difference of two squares' rule.	$x^2 - 4^2 = 0$ $(x + 4)(x - 4) = 0$
3 Apply the Null Factor Law.	$x + 4 = 0$ or $x - 4 = 0$
4 Solve each linear equation. Alternatively, use ± to write the solution in a short way.	$x = -4$ or $x = 4$ $x = \pm 4$
d **1** Rearrange the equation so one side of the equation is equal to 0.	**d** $5x^2 = -35x$ $5x^2 + 35x = 0$
2 Factorise the quadratic expression by taking out a common factor.	$5x(x + 7) = 0$
3 Apply the Null Factor Law.	$5x = 0$ or $x + 7 = 0$
4 Solve each linear equation.	$x = 0$ or $x = -7$

Example 5A.2 Solving quadratic equations using surds

Solve each of these equations. Write the solutions as exact values.

a $x^2 - 5 = 0$ **b** $-2x^2 + 24 = 0$

THINK	WRITE
a **1** Check that the quadratic expression is in factor form and that the other side of the equation is equal to 0.	**a** $x^2 - 5 = 0$
2 Factorise the quadratic expression using the 'difference of two squares' rule.	$x^2 - (\sqrt{5})^2 = 0$ $(x + \sqrt{5})(x - \sqrt{5}) = 0$
3 Apply the Null Factor Law.	$x + \sqrt{5} = 0$ or $x - \sqrt{5} = 0$
4 Solve each linear equation.	$x = -\sqrt{5}$ or $x = \sqrt{5}$
Alternatively, write the solution using ±.	$x = \pm\sqrt{5}$
b **1** Check that the quadratic expression is in factor form and that the other side of the equation is equal to 0.	**b** $-2x^2 + 24 = 0$
2 Take out a common factor (in this case, –2).	$-2(x^2 - 12) = 0$
3 Divide both sides of the equation by –2.	$x^2 - 12 = 0$
4 Factorise the quadratic expression using the 'difference of two squares' rule.	$x^2 - (\sqrt{12})^2 = 0$ $(x + \sqrt{12})(x - \sqrt{12}) = 0$
5 Simplify the surd. $\sqrt{12} = \sqrt{4 \times 3} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3}$	$(x + \sqrt{4 \times 3})(x - \sqrt{4 \times 3}) = 0$ $(x + 2\sqrt{3})(x - 2\sqrt{3}) = 0$
6 Apply the Null Factor Law.	$x + 2\sqrt{3} = 0$ or $x - 2\sqrt{3} = 0$
7 Solve each linear equation. Alternatively, write the solution using ±.	$x = -2\sqrt{3}$ or $x = 2\sqrt{3}$ $x = \pm 2\sqrt{3}$

Example 5A.3 Factorising then solving quadratic equations

Solve $x^2 + 6x + 3 = 0$ by completing the square. Write the solutions as exact values.

THINK	**WRITE**
1 Check that the quadratic expression is in factor form and that the other side of the equation is equal to 0.	$x^2 + 6x + 3 = 0$
2 Rewrite the quadratic expression by completing the square.	$x^2 + 6x + 9 - 9 + 3 = 0$ $x^2 + 6x + 9 - 6 = 0$
3 Factorise the quadratic expression using the difference of two squares.	$(x + 3)^2 - (\sqrt{6})^2 = 0$ $(x + 3 + \sqrt{6})(x + 3 - \sqrt{6}) = 0$
4 Apply the Null Factor Law.	$x + 3 + \sqrt{6} = 0$ or $x + 3 - \sqrt{6} = 0$
5 Solve each linear equation. Alternatively, write the solution using ±.	$x = -3 - \sqrt{6}$ or $x = -3 + \sqrt{6} = 0$ $x = -3 \pm \sqrt{6}$

Helpful hints

- ✔ Watch for changes in sign when solving quadratic equations for x.

$$(x + 6)(x - 2) = 0$$
$$x + 6 = 0 \quad \text{or} \quad x - 2 = 0$$
$$x = -6 \quad \text{or} \quad x = +2$$

- ✔ Although it is possible to add, subtract and multiply by x, you must not divide both sides of an equation by x, in case $x = 0$.

$$\frac{1}{0} = \text{undefined}$$

- ✔ The Null Factor Law can only be used to solve equations if one side of the equation is equal to zero. You cannot use the rule if the equation is equal to any other real number.
  For example, if $x(x - 2) = 1$, it does not imply that $x = 1$ or $x - 2 = 1$.
  But, if $x(x - 2) = 0$, two possible solutions are $x = 0$ or $x - 2 = 0$.

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Exercise 5A Solving quadratic equations

△ 1(1st column), 2–6, 7(1st column), 8–12, 14

□ 1(2nd column), 2, 5, 6, 7(2nd column), 8, 10, 12, 13(a–f), 15, 16, 18(a–c)

⬠ 2(2nd column), 6, 7(i–l), 8, 10, 13(g–l), 15, 17–19

5A.1 **1** Solve each of the following equations. Show all steps of your working.

a $(x + 2)(x - 4) = 0$
b $(x + 7)(x + 1) = 0$
c $(x - 3)(x - 6) = 0$
d $(x + 9)(x - 9) = 0$
e $x(x - 5) = 0$
f $x(2x - 1) = 0$
g $2x(x + 1) = 0$
h $7x(2x + 4) = 0$
i $\frac{1}{2}x(4x - 3) = 0$
j $(x + \sqrt{2})(x - \sqrt{2}) = 0$
k $(x - \sqrt{5})(x + \sqrt{5}) = 0$
l $(x + 4\sqrt{3})(x - 4\sqrt{3}) = 0$
m $(2x + 3)(x - 4) = 0$
n $(3x - 1)(3x + 1) = 0$
o $(7x + 2)(4x - 5) = 0$

UNDERSTANDING AND FLUENCY

2 Solve each of these equations.

a $x^2 + 7x + 12 = 0$ **b** $x^2 + 3x - 10 = 0$

c $x^2 - 9x + 8 = 0$ **d** $x^2 - 25 = 0$

e $x^2 - 81 = 0$ **f** $4 - x^2 = 0$

g $x^2 - 3x = 0$ **h** $x^2 + x = 0$

i $2x^2 + 8x = 0$ **j** $10x^2 - 4x = 0$

k $3x^2 = 18x$ **l** $7x^2 = -70x$

3 Use substitution to check that your solutions to the first four quadratic equations in question **2** are correct.

4 Consider $3(x + 6)(x - 8) = 0$.

a Why is this equivalent to $(x + 6)(x - 8) = 0$?

b Solve $(x + 6)(x - 8) = 0$. Then write the solutions for $3(x + 6)(x - 8) = 0$.

c Use substitution to check that you have the correct solutions.

5 Solve each of these quadratic equations.

a $2(x + 5)(x - 6) = 0$ **b** $7(x - 2)(x - 4) = 0$ **c** $-5(x + 8)(x + 1) = 0$

d $-4(x + 9)(x - 3) = 0$ **e** $-11(x - 4)(x + 4) = 0$ **f** $-x(x + 7) = 0$

g $-6x(x - 2) = 0$ **h** $-2x(3x + 5) = 0$ **i** $-8(2x - 3)(x + 10) = 0$

6 Solve each of these quadratic equations. (Hint: Take out a common factor first.)

a $2x^2 + 6x + 4 = 0$ **b** $3x^2 + 6x - 45 = 0$ **c** $6x^2 - 30x + 24 = 0$

d $-5x^2 - 5x + 30 = 0$ **e** $-4x^2 + 16x + 48 = 0$ **f** $-x^2 - x + 56 = 0$

g $11x^2 - 44 = 0$ **h** $-7x^2 + 63 = 0$ **i** $-16x^2 + 8x = 0$

5A.2 **7** Solve each of these equations. Write the solutions as exact values.

a $x^2 - 3 = 0$ **b** $x^2 - 11 = 0$

c $2x^2 - 14 = 0$ **d** $x^2 - 8 = 0$

e $x^2 - 27 = 0$ **f** $3x^2 - 6 = 0$

g $-x^2 + 5 = 0$ **h** $-4x^2 + 24 = 0$

i $-2x^2 + 64 = 0$ **j** $5x^2 = 95$

k $8x^2 = 160$ **l** $156 - 3x^2 = 0$

5A.3 **8** Solve each of these equations by completing the square. Write the solutions as exact values.

a $x^2 + 2x - 6 = 0$ **b** $x^2 - 4x + 2 = 0$ **c** $x^2 + 8x + 11 = 0$

d $x^2 - 12x + 28 = 0$ **e** $x^2 + 20x + 25 = 0$ **f** $x^2 - x - 1 = 0$

9 Rewrite the solutions you found for each equation in question **8** correct to two decimal places.

10 Explain why you can divide both sides of $3(x - 2) = 0$ by 3 but you cannot divide both sides of $x(x - 2) = 0$ by x.

PROBLEM SOLVING AND REASONING

11 How many real solutions does each of these quadratic equations have?

a $(x - 3)(x - 5) = 0$ **b** $(x + 2)(x + 2) = 0$ **c** $(x - 4)(x + 4) = 0$

d $(x - 7)^2 = 0$ **e** $x(x - 6) = 0$ **f** $x^2 + 9 = 0$

12 Find solution/s for each of these equations, if real solutions exist. (Hint: Some of these equations cannot be solved.)

a $x^2 - 9 = 0$ **b** $x^2 + 4x - 77 = 0$ **c** $x^2 - 8x + 16 = 0$

d $x^2 + 4 = 0$ **e** $x^2 + 4x = 0$ **f** $x^2 - 6x - 1 = 0$

13 Solve each of these quadratic equations. (Hint: First rearrange the equation into the form $ax^2 + bx + c = 0$.)

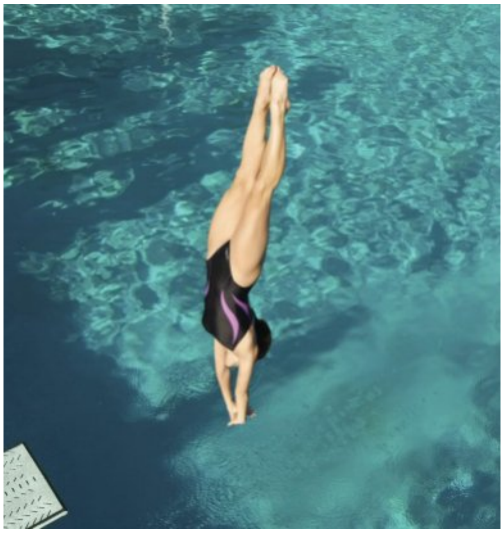
a $x^2 + 4x = 12$ **b** $x^2 + 4 = 5x$ **c** $6x^2 = 54$

d $x^2 - 3x - 20 = 20$ **e** $x^2 - 6x = 3x$ **f** $x^2 + 1 = 4x$

g $5x - x^2 = x$ **h** $x^2 + 21 = 11x + 3$ **i** $2x^2 - 3x = x^2 + 18$

j $x(x + 4) = 21$ **k** $(x - 5)^2 = 1$ **l** $x^2 + 3x + 10 = 1 - 3x$

PROBLEM SOLVING AND REASONING



14 Nutsa dives with her outstretched arms from a platform into a diving pool. The height of her fingertips above the pool surface can be represented by the quadratic relationship $h = -2t^2 + 4t + 6$, where h is the height above the water of her fingertips, in metres, after t seconds.

a What is the height of Nutsa's fingertips above the water after:

i 1 second?

ii 2 seconds?

b What is the height of Nutsa's fingertips when she starts the dive?

c How long does it take for the dive? (Hint: How long does it take for Nutsa's fingertips to hit the surface of the water?)

d Explain why there is only one time value for your answer to part **c** even though you have solved a quadratic equation that has two solutions.

15 Oliver buys rice to serve with stir-fried meals over the coming weeks. As time passes, the amount of rice Oliver has left can be represented by the quadratic relationship $y = x^2 - 10x + 25$, where y is the amount left, in kilograms, after x weeks.

a How much rice did Oliver buy?

b How much rice is left after 2 weeks?

c How long does it take for all of the rice to be used?

d Did Oliver use the same amount of rice each week? Explain.

16 The area of a rectangular vegetable plot is 70 m^2. The length is 3 m longer than the width.

a Write a quadratic equation to represent this. (Hint: Let x represent the width of the vegetable plot, in metres.)

b Solve the quadratic equation.

c Write the dimensions of the vegetable plot.

17 The width of a mobile phone screen is 5 cm less than its length. Write an equation to represent the situation if the area of the screen is 66 cm^2. Then solve your equation to find the dimensions of the screen.

CHALLENGE

18 Solve each of these quadratic equations by completing the square.

a $3x^2 + 18x + 21 = 0$ **b** $4x^2 + 8x - 16 = 0$

c $-x^2 + 4x - 1 = 0$ **d** $-6x^2 - 48x - 36 = 0$

e $2x^2 - 12x - 3 = 0$ **f** $5x^2 + 5x - 2 = 0$

19 Solve each of these quadratic equations.

a $2x^2 + 13x + 15 = 0$ **b** $5x^2 - 22x + 8 = 0$

c $6x^2 + x - 2 = 0$ **d** $4x^2 + 25x - 21 = 0$

e $3x^2 + 2x = 1$ **f** $7x^2 = 11x + 6$

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Interactive skillsheet
Solving quadratic equations

Investigation
Beware of irreversible operations

Topic quiz
5A

5B The quadratic formula

Learning intentions

✔ I can identify the number of solutions to a quadratic equation.

✔ I can solve quadratic equations using the quadratic formula.

Inter-year links

Year 7	6E Substitution
Year 8	5B Substitution

The quadratic formula

- The **quadratic formula** can be used to solve equations of the general form $ax^2 + bx + c = 0$, where a, b and c are constants and $a \neq 0$.
- The solution for $ax^2 + bx + c = 0$ is:

  $$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

  This can be written more simply as:

  $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The discriminant

- The discriminant is $\Delta = b^2 - 4ac$

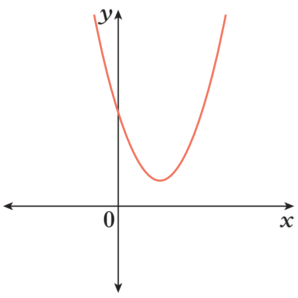
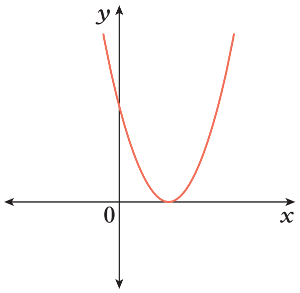
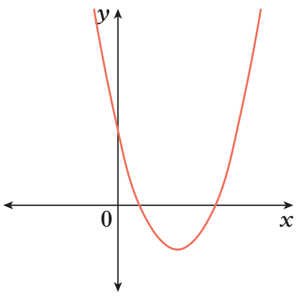
  Evaluating this expression after substituting values for a, b and c allows us to identify the number of real solutions for the quadratic equation.

  → If $\Delta < 0$, there are no real solutions.

  → If $\Delta = 0$, there is one real solution. It is $x = \frac{b}{2a}$ and the quadratic is a perfect square.

  → If $\Delta > 0$, there are two real solutions. They are:

  $$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Discriminant	$\Delta < 0$ (negative)	$\Delta = 0$ (zero)	$\Delta > 0$ (positive)
Number of solutions	No real solutions	One real solution	Two real solutions
Graph			

Example 5B.1 Identifying the number of solutions for a quadratic equation

Using the discriminant, identify the number of solutions for each of these quadratic equations.

a $3x^2 + 7x + 2 = 0$

b $x^2 + 16 = 8x$

c $-2x^2 + 3x - 5 = 0$

THINK

1. If necessary, rearrange each equation so that the right-hand side is 0.
2. Identify a, b and c.
3. Write the discriminant formula and substitute values for a, b and c into it. Simplify the result.
4. Identify the number of solutions.

WRITE

a $3x^2 + 7x + 2 = 0$

$a = 3 \qquad b = 7 \qquad c = 2$

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= 7^2 - 4 \times 3 \times 2\\ &= 49 - 24\\ &= 25\end{aligned}$$

Since $\Delta > 0$ (is positive), the equation has two solutions. You may also notice that since the discriminant is a perfect square, these solutions will be rational (no surds).

b $x^2 + 16 = 8x$

$x^2 - 8x + 16 = 0$

$a = 1 \qquad b = -8 \qquad c = 16$

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (-8)^2 - 4 \times 1 \times 16\\ &= 64 - 64\\ &= 0\end{aligned}$$

Since $\Delta = 0$, the equation has one solution.

c $-2x^2 + 3x - 5 = 0$

$a = -2 \qquad b = 3 \qquad c = -5$

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= 3^2 - 4 \times (-2) \times (-5)\\ &= 9 - 40\\ &= -31\end{aligned}$$

Since $\Delta < 0$, the equation has no solutions.

Example 5B.2 Using the quadratic formula to solve quadratic equations

Use the quadratic formula to solve each of these equations. If a solution exists, write the solutions as:

i exact values

ii approximate values, correct to two decimal places.

a $2x^2 - 5x + 1 = 0$ **b** $9x^2 + 12x + 4 = 0$ **c** $4x^2 = 3x - 1$

THINK

1. If necessary, rearrange the equation so that one side of the equation equals 0.
2. Identify *a*, *b* and *c*.
3. Determine the discriminant and identify the number of solutions.
4. Write the quadratic formula and substitute values for *a*, *b* and *c* into it. Simplify the result.
5. Write the solution as exact values.
6. Use a calculator to help you rewrite the values correct to two decimal places.

WRITE

a $2x^2 - 5x + 1 = 0$

$a = 2 \qquad b = -5 \qquad c = 1$

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (-5)^2 - 4 \times 2 \times 1\\ &= 25 - 8\\ &= 17\end{aligned}$$

The equation has two solutions.

$$\begin{aligned}x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\\ &= \frac{-(-5) \pm \sqrt{17}}{2(2)}\\ &= \frac{5 \pm \sqrt{17}}{4}\end{aligned}$$

i $x = \dfrac{5 - \sqrt{17}}{4}$ or $\dfrac{5 + \sqrt{17}}{4}$

ii $x \approx 0.22$ or 2.28

b $9x^2 + 12x + 4 = 0$

$a = 9 \qquad b = 12 \qquad c = 4$

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (12)^2 - 4 \times 9 \times 4\\ &= 144 - 144\\ &= 0\end{aligned}$$

The equation has one solution.

$$\begin{aligned}x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\\ &= \frac{-12 \pm \sqrt{0}}{2(9)}\\ &= \frac{-12}{18}\\ &= -\frac{2}{3}\end{aligned}$$

i $x = -\dfrac{2}{3}$

ii $x \approx -0.67$

c $4x^2 = 3x - 1$

$4x^2 - 3x + 1 = 0$

$a = 4 \qquad b = -3 \qquad c = 1$

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (-3)^2 - 4 \times 4 \times 1\\ &= 9 - 16\\ &= -7\end{aligned}$$

The equation has no solutions.

Helpful hints

- ✔ If the discriminant is positive, don't forget to evaluate both possible solutions using the quadratic formula.

  Remember that the quadratic formula is equal to $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, so:

  $$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

- ✔ The quadratic formula is very important and will be useful to know when studying VCE Maths. It might be helpful for you to memorise it now.
- ✔ Be careful when identifying the values of a, b and c. Make sure one side of the equation equals zero before identifying a, b and c.
- ✔ Be careful when substituting the values of a, b and c into the equation. You need to include the sign of each number.

  For example, if $-2x^2 - 6x - 4 = 0$, then $a = -2$, $b = -6$ and $c = -4$.

  So, the solutions for x are $x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(-2)(-4)}}{2(-2)}$.

- ✔ If you have already calculated the discriminant of a quadratic expression, you can substitute it into the quadratic formula.

  For example, $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{\Delta}}{2a}$.

ANS
p695

Exercise 5B The quadratic formula

△ 1, 2, 3–4(1st, 2nd columns), 5, 6, 8, 10, 12

■ 2, 3–4(2nd, 3rd columns), 5–9, 14, 15

⬟ 3–4(2nd, 3rd columns), 5–7, 11, 13, 14, 16

UNDERSTANDING AND FLUENCY

5B.1 **1** Using the discriminant, identify the number of solutions for each of these quadratic equations.

a $x^2 + 5x + 7 = 0$
b $3x^2 - 8x - 4 = 0$
c $x^2 - 9 = 0$
d $2x^2 + 3x + 4 = 0$
e $12x - x^2 = 36$
f $6x^2 - 5 = 7x$

2 Complete the following for each quadratic equation below.

i Calculate the discriminant.
ii Identify the number of solutions.
iii Solve, by first factorising and then using the Null Factor Law.
iv Check whether your answer for part **iii** confirms your result for part **ii** each time.

a $x^2 - 2x - 15 = 0$
b $x^2 + 4x + 4 = 0$
c $x^2 - 49 = 0$
d $3x^2 - 18x = 0$
e $x^2 - 6x + 7 = 0$
f $x^2 + 2x + 3 = 0$

5B.2 **3** Use the quadratic formula to solve each of these equations. If a solution exists, write the solutions as:

i exact values
ii approximate values, correct to two decimal places.

a $x^2 - 7x + 5 = 0$
b $3x^2 - x - 1 = 0$
c $2x^2 + x - 4 = 0$
d $-3x^2 - x + 9 = 0$
e $-6x^2 - 5x + 3 = 0$
f $x^2 - 2x - 1 = 0$
g $x^2 + 8x + 4 = 0$
h $4x^2 - 12x + 1 = 0$
i $-x^2 + 4x + 2 = 0$
j $5x^2 + 10x - 3 = 0$
k $2x^2 = 11 - 3x$
l $x^2 + 2x = 7$

4 Use the quadratic formula to find any solutions to each of the following quadratic equations. If solutions exist, write them as exact values.

- **a** $x^2 + 3x = 1$
- **b** $x^2 + 6x + 12 = 0$
- **c** $\frac{1}{2}x^2 + 3x + 2 = 0$
- **d** $x^2 + 3x + 5 = 0$
- **e** $2x^2 + 5x + 1 = 0$
- **f** $x^2 + 16 = 8x$
- **g** $x^2 + 2x = 4$
- **h** $x^2 = 3 - 7x$
- **i** $10 = x^2 + 8x$
- **j** $x^2 - 5 = x$
- **k** $x^2 + 5 = x$
- **l** $x^2 + \sqrt{5}\,x = 1$

5 Use the quadratic formula to find the solutions to each of the following quadratic equations.

- **a** $m^2 + 3m - 7 = 0$
- **b** $u^2 + 4u = 1$
- **c** $r^2 + 9r + 1 = 0$
- **d** $y^2 - 3 = 2y$
- **e** $3h^2 + 7h = 1$
- **f** $w^2 = 5 + 6w$

6 Solve $\frac{1}{x} = x - 3$ by first multiplying both sides by x to create a quadratic equation.

PROBLEM SOLVING AND REASONING

7 Copy and complete the lines of working below to derive the quadratic formula. You will complete the square to factorise the left-hand side and then use the Null Factor Law to find the solutions for $ax^2 + bx + c = 0$ (where $a \neq 0$).

THINK	**WRITE**
1 Write the quadratic equation.	$ax^2 + bx + c = 0$
2 Take out the coefficient of x^2 as a common factor.	$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0$
3 Divide both sides by a.	$x^2 + \frac{b}{a}x + \underline{\qquad} = 0$
4 Add a new third term to complete the square and compensate by subtracting the same term.	$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{\underline{\quad}} = 0$
5 Form a perfect square using the first three terms and simplify the fourth term.	$\left(x + \frac{b}{2a}\right)^2 - \frac{\underline{\quad}}{4a^2} + \frac{c}{a} = 0$
6 Group the last two terms together.	$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2}{4a^2} - \frac{c}{a}\right) = 0$
7 Combine the last two terms using a common denominator.	$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\underline{\quad} - 4ac}{4a^2}\right) = 0$
8 Write the left-hand side as the difference of two squares.	$\left(x + \frac{b}{2a}\right)^2 - \left(\sqrt{\frac{b^2 - 4ac}{4a^2}}\right)^2 = 0$
9 Simplify the second square.	$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2 = 0$
10 Factorise the left-hand side using the 'difference of two squares' rule.	$\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} - \underline{\qquad\qquad}\right)^2 = 0$
11 Use the Null Factor Law.	$x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} = 0$ or $x + \frac{b}{2a} - \underline{\qquad\qquad} = 0$
12 Solve each linear equation.	$x = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$ or $x = -\frac{b}{2a} + \underline{\qquad\qquad}$
13 Simplify each solution.	$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ or $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$

So, the quadratic formula for finding the solutions to $ax^2 + bx + c = 0$ is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

8 Consider the quadratic equation $x^2 - 5x + 6 = 0$.

- **a** Solve the equation by factorising the left-hand side and using the Null Factor Law.
- **b** Now solve the equation again, using the quadratic formula.
- **c** Which method was quicker or easier to use?
- **d** The solutions to any quadratic equation can be found using the quadratic formula. Why might you try using the method described in part **a** first?

PROBLEM SOLVING AND REASONING

9 The profit, p (in dollars), made from selling n soft toys is given by the relationship $p = -0.1n^2 + 17n + 200$.

a What is the profit when 30 toys are sold?

b What is the smallest number of toys that need to be sold to make a profit of at least \$850?

10 A rectangular paddock is 20 m longer than it is wide and has an area of 5000 m^2. Solve a quadratic equation to find the dimensions of the paddock, correct to one decimal place.

11 The perimeter of a framed painting is 100 cm. Within the 2 cm frame, the area of the painting that can be viewed is 300 cm^2. Find the dimensions of the framed painting, correct to the nearest millimetre.

12 When answering each of these, consider the result for $\sqrt{b^2 - 4ac}$ in the quadratic formula.

a Explain why there are no real solutions to a quadratic equation when the discriminant is negative.

b Explain why there is only one real solution when the discriminant is zero.

c Explain why there are two real solutions when the discriminant is positive.

13 The discriminant of a quadratic can be used to find the number of solutions, but it can also be used to find the nature of the solutions, if they exist. If a quadratic has integer coefficients and the discriminant is a square number, then the solutions to the quadratic will be rational (that is, whole numbers or fractions).

a Consider each of the following quadratics, which have integer coefficients. Find the discriminant and use it to determine whether the solutions to the quadratic will be rational or irrational.

i $y = x^2 + 5x - 2$

ii $y = 2x^2 + 3x - 2$

iii $y = x^2 - 7x + 10$

iv $y = 5x^2 + 11x + 4$

v $y = 3x^2 - x - 4$

b Explain with reference to the quadratic formula why, if the discriminant is a square number, the solution is guaranteed to be rational, assuming all coefficients are integers.

14 a Find the x-intercepts, if they exist, of the graphs of each of the following equations. You do not need to sketch the graph.

i $y = x^2 + 7x + 10$

ii $y = x^2 - 6x + 8$

iii $y = x^2 + 3x + 5$

iv $y = x^2 - 8x + 16$

v $y = -2x^2 + 6x - 7$

b Explain how the discriminant of a quadratic relates to the number of x-intercepts of its graph.

CHALLENGE

15 a Find the value of k so $kx^2 + 6x + 3 = 0$ has only one real solution.

b For what values of k does $kx^2 + 6x + 3 = 0$ have:

i two real solutions?

ii no real solutions?

16 For each of these equations, find the value of k that will give the number of real solutions shown in square brackets.

a $x^2 + 2x + k = 0$ [one real solution]

b $2x^2 - 4x + k = 0$ [two real solutions]

c $kx^2 + 6x - 1 = 0$ [no real solutions]

d $kx^2 - 3x + 1 = 0$ [one real solution]

e $3x^2 + kx + 3 = 0$ [two real solutions]

f $x^2 + kx + 25 = 0$ [no real solutions]

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Interactive skillsheet
The quadratic formula

Worksheet
Solving quadratic equations

Topic quiz
5B

5C Sketching parabolas using intercepts

Learning intentions

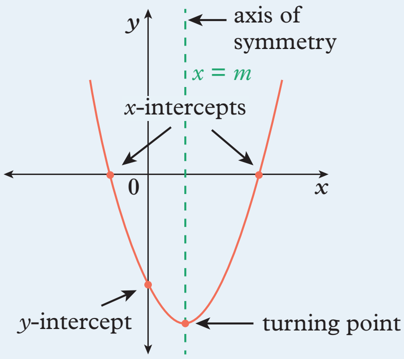
- ✔ I can find the x- and y-intercepts of a quadratic graph from its equation.
- ✔ I can find the coordinates of the turning point for a quadratic graph with two x-intercepts.
- ✔ I can sketch quadratic graphs using the intercepts and turning point.

Inter-year links

Year 7	6E Substitution
Year 8	5B Substitution
Year 9	5C Sketching parabolas using intercepts

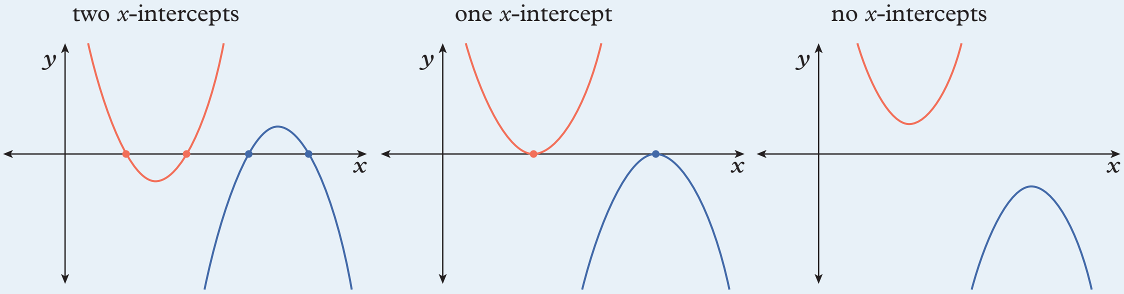
Parabolas

- The graph of a quadratic relationship is called a **parabola**.
- The key features of a parabola include:
  - → x- and y-intercepts (parabolas can have two, one or no x-intercepts)
  - → a **turning point** or vertex
  - → an **axis of symmetry**, $x = m$, where m is the x-coordinate of the turning point.

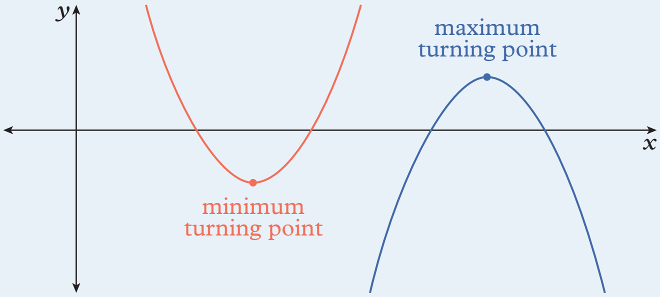


Sketching parabolas

- One way of sketching a parabola is to use the x- and y-intercepts. The coordinates of the turning point and the orientation of the parabola (upright or inverted) can also be identified.
  - → The y-intercept is found by substituting $x = 0$ into the equation and simplifying.
  - → The x-intercept(s) are found by substituting $y = 0$ into the equation and solving for x. A parabola can have two, one or no x-intercepts.

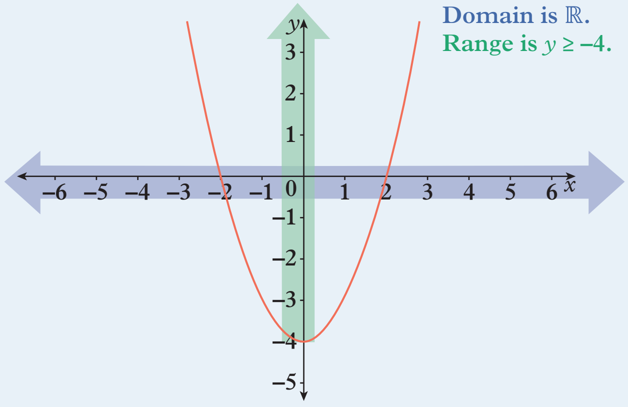


- A **minimum turning point** is the turning point of a section of a graph that opens upwards or is 'upright'.
- A **maximum turning point** is the turning point of a section of a graph that opens downwards or is 'inverted'.
- Turning points can be identified using the axis of symmetry or the halfway point between intercepts (or any two x values with the same y value).



Domain and range

- The **domain** of a graph is the complete set of possible x values that will output real y values.
  For example, the parabola $(x - 2)(x + 2) = 0$ has as its domain all real x values, which can be represented as $\mathbb{R}$.
- The **range** is the y values that would result from substituting the complete set of possible x values.
  For example, the parabola $(x - 2)(x + 2) = 0$ has as its range all real y values greater than or equal to –4, which can be represented as $y \geq -4$.



Domain is $\mathbb{R}$.
Range is $y \geq -4$.

Example 5C.1 Finding the coordinates of the x- and y-intercepts

For each quadratic relationship below, find the coordinates of the:

i x-intercepts **ii** y-intercept.

a $y = x^2 - 6x$ **b** $y = x^2 + 7x + 10$

THINK	WRITE
a i 1 To find the x-coordinate of the x-intercept, substitute $y = 0$ into the quadratic equation.	**a i** $y = x^2 - 6x$ For the x-intercepts, $y = 0$: $x^2 - 6x = 0$
2 Factorise the quadratic expression.	$x(x - 6) = 0$
3 Solve the equation using the Null Factor Law.	$x = 0$ or $(x - 6) = 0$ $x = 0$ or $x = 6$ The x-intercepts are $(0, 0)$ and $(6, 0)$.
ii To find the y-coordinate of the y-intercept, substitute $x = 0$ into the quadratic equation and solve for y.	**ii** For the y-intercept, $x = 0$: $y = x^2 - 6x$ $y = (0)^2 - 6(0)$ $= 0$ The y-intercept is $(0, 0)$.
b i 1 To find the x-coordinate of the x-intercept, substitute $y = 0$ into the quadratic equation.	**b i** $y = x^2 + 7x + 10$ For the x-intercepts, $y = 0$: $x^2 + 7x + 10 = 0$
2 Factorise the quadratic expression.	$(x + 2)(x + 5) = 0$
3 Solve the equation using the Null Factor Law.	$x + 2 = 0$ or $x + 5 = 0$ $x = -2$ or $x = -5$ The x-intercepts are $(-2, 0)$ and $(-5, 0)$.
ii To find the y-coordinate of the y-intercept, substitute $x = 0$ into the quadratic equation and solve for y.	**ii** For the y-intercept, $x = 0$: $y = x^2 + 7x + 10$ $y = (0)^2 + 7(0) + 10$ $= 10$ The y-intercept is $(0, 10)$.

Example 5C.2 Finding the coordinates of the turning point using *x*-intercepts

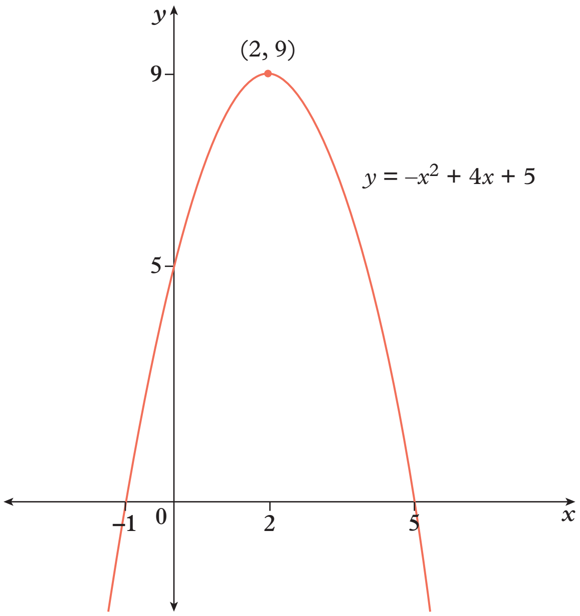
Find the coordinates of the turning point for $y = x^2 - 6x$.

THINK	**WRITE**
1 Calculate the *x*-coordinate of the turning point by finding the point midway between the *x*-intercepts.	$y = x^2 - 6x$ $= x(x - 6)$ $x = 0$ or $x = 6$ The *x*-intercepts are $(0, 0)$ and $(6, 0)$. $x = \frac{0 + 6}{2}$ $= \frac{6}{2}$ $= 3$
2 Calculate the *y*-coordinate of the turning point by substituting the *x*-coordinate into the quadratic equation.	If $x = 3$: $y = x^2 - 6x$ $y = (3)^2 - 6(3)$ $= 9 - 18$ $= -9$
3 Write the coordinates of the turning point.	The turning point is $(3, -9)$.

Example 5C.3 Sketching a parabola using the intercepts

Sketch the graph of $y = -x^2 + 4x + 5$ by first finding the *x*- and *y*-intercepts. Label the turning point with its coordinates. List the domain and range of the graph.

THINK	**WRITE**
1 Find the *x*-intercepts by substituting $y = 0$ into the equation and solving it for *x*. Factorise so that the Null Factor Law can be used.	For the *x*-intercepts, $y = 0$: $-x^2 + 4x + 5 = 0$ $-(x^2 - 4x - 5) = 0$ $x^2 - 4x - 5 = 0$ $(x + 1)(x - 5) = 0$ $x + 1 = 0$ or $x - 5 = 0$ $x = -1$ or $x = 5$ The *x*-intercepts are $(-1, 0)$ and $(5, 0)$.
2 Find the *y*-intercept by substituting $x = 0$ into the equation and simplifying.	For the *y*-intercept, $x = 0$: $y = -x^2 + 4x + 5$ $y = -(0)^2 + 4(0) + 5$ $= 5$ The *y*-intercept is $(0, 5)$.

3 Find the coordinates of the turning point. First, calculate the x value in the middle of the two x-intercepts, then substitute that value into the equation to find the corresponding y value. Use both values to write down the coordinates of the turning point.	For the turning point: $x = \frac{-1 + 5}{2}$ $= \frac{4}{2}$ $= 2$ If $x = 2$: $y = -x^2 + 4x + 5$ $y = -(2)^2 + 4(2) + 5$ $= -4 + 8 + 5$ $= 9$ The turning point is (2, 9).
4 Plot the intercept points and the turning point on the Cartesian plane.	
5 Draw an inverted parabola through the points and label the graph with the equation.	
6 List the domain by identifying the x values used to produce the graph. The set of real numbers is represented by $\mathbb{R}$.	Domain is $\mathbb{R}$.
7 List the range by identifying the y values used to produce the graph.	Range is $y \leq 9$.

Example 5C.4 Sketching a parabola using the quadratic formula

Sketch the graph of $y = x^2 + 6x + 7$, using the quadratic formula to find the x-intercepts. Label the turning point.

THINK

1 Find the x-intercepts by substituting $y = 0$ into the equation and solving it for x. The quadratic expression is not easy to factorise, so use the quadratic formula to find the solutions.

The surd $\sqrt{8}$ can be simplified to $2\sqrt{2}$.

$\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2}$

For the x-intercepts, $y = 0$:

$x^2 + 6x + 7 = 0$

$a = 1 \quad b = 6 \quad c = 7$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-6 \pm \sqrt{6^2 - 4(1)(7)}}{2(1)} \\ &= \frac{-6 \pm \sqrt{8}}{2} \\ &= -3 \pm \sqrt{2} \end{aligned}$$

The x-intercepts are $(-3 - \sqrt{2}, 0)$ and $(-3 + \sqrt{2}, 0)$.

2 Find the y-intercept by substituting $x = 0$ into the equation and simplifying.

For the y-intercept, $x = 0$:

$$\begin{aligned} y &= x^2 + 6x + 7 \\ y &= (0)^2 + 6(0) + 7 \\ &= 7 \end{aligned}$$

The y-intercept is $(0, 7)$.

3 Find the coordinates of the turning point. First, calculate the x value in the middle of the two x-intercepts, then substitute that value into the equation to find the corresponding y value. Use both values to write down the coordinates of the turning point.

For the turning point:

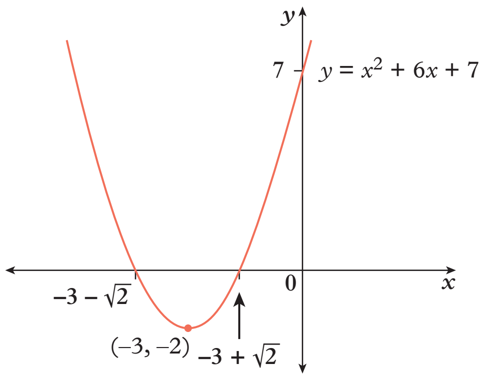
$$\begin{aligned} x &= \frac{(-3 - \sqrt{2}) + (-3 + \sqrt{2})}{2} \\ &= \frac{-6}{2} \\ &= -3 \end{aligned}$$

If $x = -3$:

$$\begin{aligned} y &= (-3)^2 + 6(-3) + 7 \\ &= 9 - 18 + 7 \\ &= -2 \end{aligned}$$

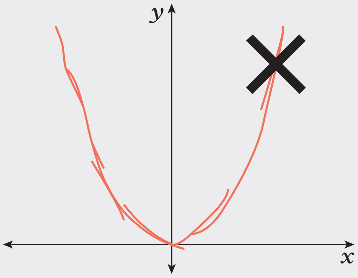
The turning point is $(-3, -2)$.

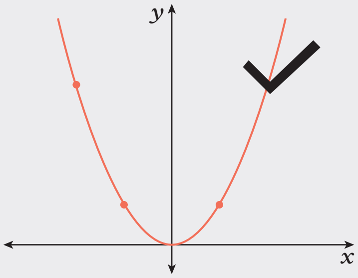
4 Draw an upright parabola through the x- and y-intercepts. Label the intercepts with exact values.

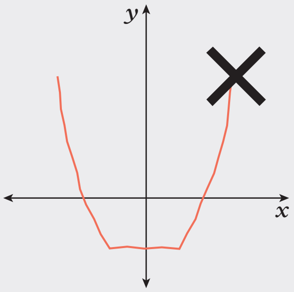


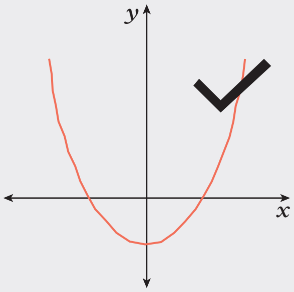
Helpful hints

- ✔ Rather than drawing messy inaccurate parabolas, try starting at the turning point and drawing a smooth line up or down through the other plotted points.
- ✔ Label your working out so that it is clear when you are calculating the x-intercepts, the y-intercept and the turning point. This can make it easier to check your calculations when you are doing a test.
- ✔ Make sure that your parabola has a clearly defined turning point. If your graph has a flat bottom or top where the turning point should be, you may lose marks when doing a test.









ANS
p696

Exercise 5C Sketching parabolas using intercepts

△ 1–5, 6(a–e), 7, 8(a, b, e), 9, 10 □ 1–3, 5, 6(c–g), 8(d–f), 10, 11, 14(a, b), 15(a, b) ⬠ 6(d–i), 8(d–f), 10, 12, 13, 14(b, c), 15

UNDERSTANDING AND FLUENCY

5C.1 **1** For each quadratic relationship below, find:

i the x-intercepts **ii** the y-intercept.

a $y = x^2 + 6x$ **b** $y = -x^2 + 10x$ **c** $y = x^2 - 6x + 5$

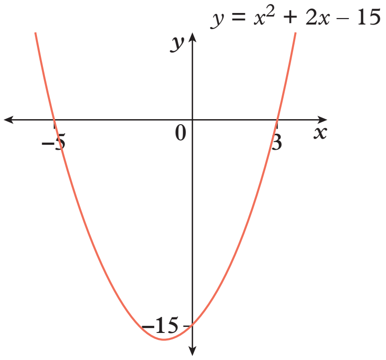
d $y = x^2 - 1$ **e** $y = -x^2 - 4x + 12$ **f** $y = 9 - x^2$

2 For each quadratic relationship in question **1**, state whether its graph is an upright parabola or an inverted parabola.

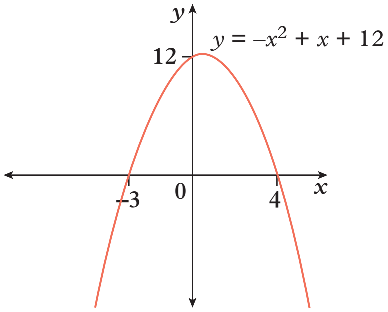
5C.2 **3** Find the coordinates of the turning point for each quadratic relationship in question **1**.

4 Find the coordinates of the turning point for each of these parabolas.

a



b



UNDERSTANDING AND FLUENCY

5 Sketch the graph of each quadratic relationship in question **1** using your answers for questions **1–3** to help you.

5C.3 **6** Use intercepts to help you sketch the graph of each quadratic relationship below. Label each turning point with its coordinates. List the domain and range of each graph.

a $y = x^2 - 6x + 8$
b $y = -x^2 + 10x - 16$
c $y = x^2 + 4x - 12$
d $y = 4 - x^2$
e $y = x^2 + 2x$
f $y = -x^2 + 10x - 25$
g $y = x^2 - 4x - 21$
h $y = x^2 - x - 6$
i $y = x^2 + 7x + 10$

7 Consider the graphs of $y = (x + 5)(x - 1)$, $y = 2(x + 5)(x - 1)$ and $y = -(x + 5)(x - 1)$.

a Find the x-intercepts for each graph.

b Explain why the graphs of these equations will be different even though each of the parabolas has the same x-intercepts.

c Find the y-intercept for each parabola.

d Find the coordinates of the turning point for each parabola.

e Sketch all three parabolas on the same Cartesian plane.

5C.4 **8** Sketch the graph of each of these quadratic relationships using the quadratic formula to find the x-intercepts. Label the turning point.

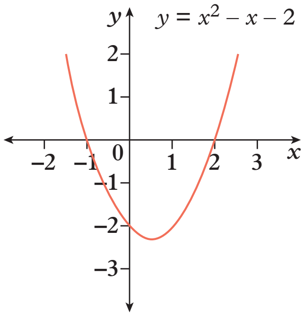
a $y = x^2 + 4x - 1$
b $y = x^2 - 8x + 3$
c $y = x^2 + 6x - 8$
d $y = x^2 - 2x - 10$
e $y = -x^2 + 2x + 5$
f $y = -x^2 - x + 4$

PROBLEM SOLVING AND REASONING

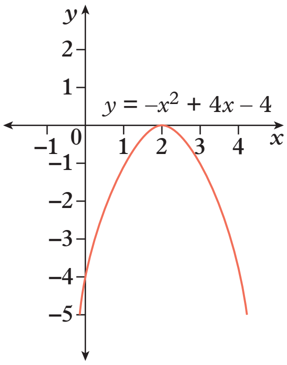
9 Consider the graphs of the three quadratic relationships shown below.

a How many x-intercepts does each parabola have?

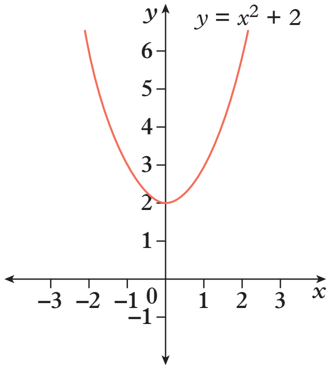
i



ii



iii



b Use the graphs to help you solve the following:

i $x^2 - x - 2 = 0$
ii $-x^2 + 4x - 4 = 0$
iii $x^2 + 2 = 0$

c Show how you can use the discriminant ($\Delta = b^2 - 4ac$) to verify the number of x-intercepts for each parabola.

d Show how you can use the quadratic formula to find the x-intercepts for each parabola.

10 Before sketching a parabola, you can work out the number of x-intercepts it will have.

i How many x-intercepts will each of the parabolas identified by the following rules have?

ii Find the coordinates of the x-intercepts.

a $y = x^2 - 4x + 4$
b $y = x^2 + 2x + 3$
c $y = x^2 - 6x + 7$

11 Thomas throws a discus. The position of the discus can be represented by $h = -0.01(d + 4)(d - 72)$, where h is the height of the discus above the ground and d is the horizontal distance from where the discus was thrown. Both h and d are in metres.

a Sketch the graph of this relationship by finding the intercepts.

b Thomas is standing on a platform. At what height off the ground was the discus thrown?

c What horizontal distance did the discus travel before hitting the ground?

d What is the greatest height that the discus reached?

12 A ball was kicked off the ground. Its path can be represented by $y = -0.05x^2 + 1.6x$, where x is the horizontal distance (in metres) and y is the vertical distance (in metres).

a Sketch the graph of this relationship.

b What was the maximum height the ball reached? What horizontal distance had it travelled when it reached its maximum height?

c What horizontal distance did the ball travel before hitting the ground?

d If the ball was caught at a height of 1.8 m above the ground as it was moving downwards, what horizontal distance did the ball travel?

CHALLENGE

13 Consider the graph of $y = ax^2 + bx + c$.

a Use the quadratic formula to write expressions for the x-intercepts.

b Use the x-intercepts to show that the x-coordinate of the turning point is $-\frac{b}{2a}$.

c Write an expression for the y-coordinate of the turning point.

14 Use the intercepts to help you sketch the graph of each of these quadratic relationships. Label the turning point with its coordinates.

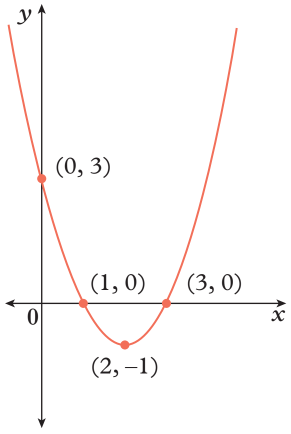
a $y = x^2 + 2x - 7$

b $y = -x^2 + 4x + 2$

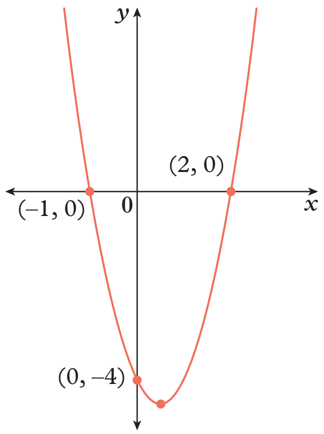
c $y = x^2 - 2x - 1$

15 The following parabolas all have equations in the form $y = a(x - m)(x - n)$. Find the equation for each graph.

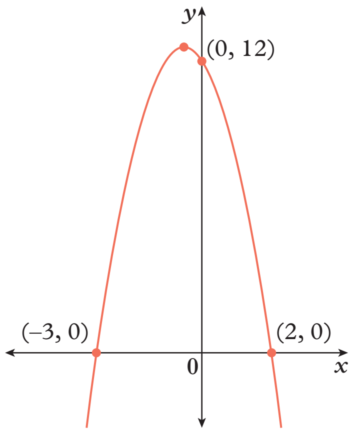
a



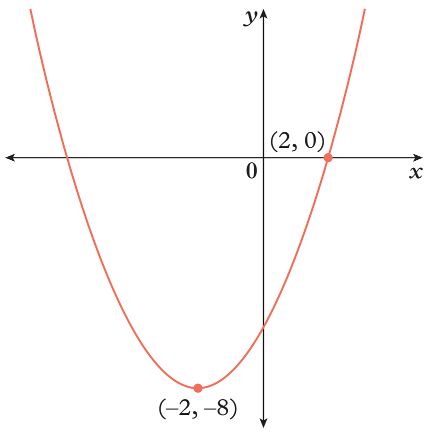
b



c



d



Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Sketching parabolas using intercepts

Investigation
The connection between linear and quadratic graphs

Topic quiz
5C

5D Sketching parabolas using transformations

Learning intentions

✔ I can sketch quadratic graphs of the form $y = a(x - h)^2 + k$ using reflections, stretches, vertical translations and horizontal translations.

Inter-year links

Years 5/6	Transformations
Year 7	8E Rotations and reflections
Year 8	7C Transformations
Year 9	5D Sketching parabolas using transformations

Turning point form

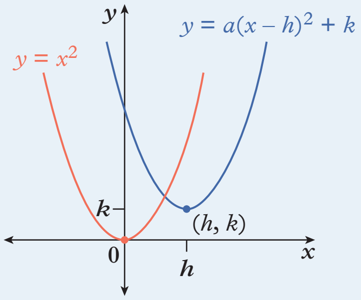
- Quadratic relationships can be written in **turning point form**, as shown below where the coordinates of the turning point are (h, k):

$$y = a(x - h)^2 + k$$

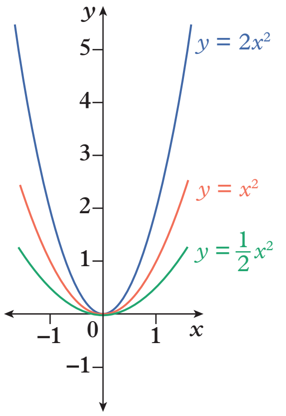
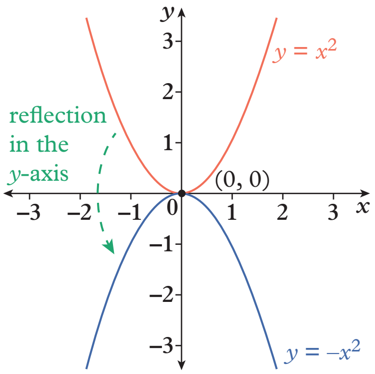
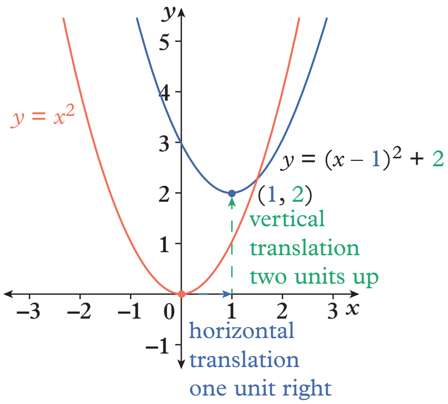
Vertical translation of k units
If $k > 0$, up
If $k < 0$, down

Stretch in the y-direction
If $a > 0$, vertical stretch up
If $0 < a < 1$, horizontal stretch outward
But if $a < 0$, invert

Horizontal translation of h units
If $h > 0$, right
If $h < 0$, left



- The following **transformations** can be performed on the graph of $y = x^2$ to produce the graph of $y = a(x - h)^2 + k$:
  - → a stretch of a factor of a in the y-direction. If a is greater than 1 or less than –1, the parabola becomes thinner.
  - → a **reflection** in the y-axis if $a < 0$
  - → a horizontal **translation** of h units right or left
  - → a vertical translation of k units up or down.

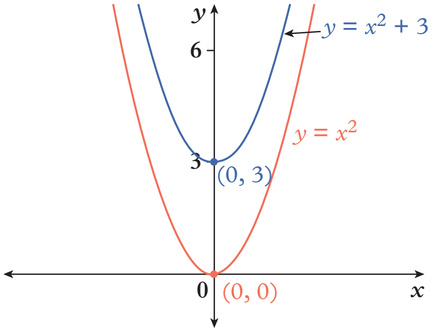
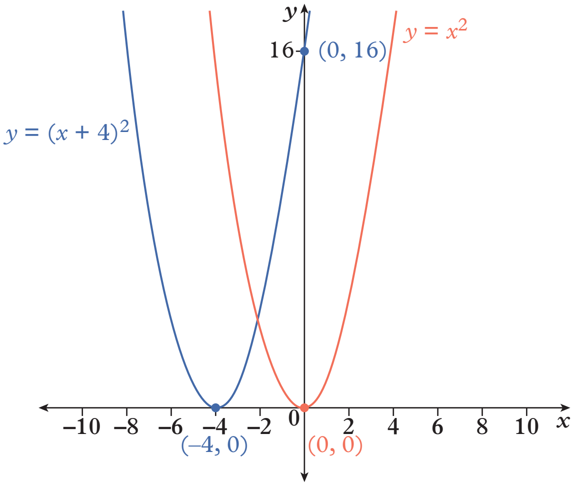
Stretches	Reflection	Translation
		

- A combination of transformations can be performed on the graph of $y = x^2$ to produce the graph of $y = a(x - h)^2 + k$.

Example 5D.1 Sketching a parabola using vertical and horizontal translations

Sketch the graph of $y = x^2$ on a Cartesian plane, then perform a translation to sketch the graph of each quadratic relationship below. Label the turning point and y-intercept on each parabola with their coordinates.

a $y = x^2 + 3$ **b** $y = (x + 4)^2$

	THINK	**WRITE**
a	**1** Identify the transformation. The equation has the form $x^2 + k$ where $k = 3$, so the graph of $y = x^2$ undergoes a vertical translation of 3 units up, resulting in a turning point of $(h, k) = (0, 3)$.	$y = x^2 \boxed{+ 3}$ Vertical translation of three units up
	2 Sketch the graphs of $y = x^2$ and $y = x^2 + 3$, and label their turning points.	The turning point is (0, 3). 
b	**1** Identify the transformation. The equation has the form $(x - h)^2$ where $h = -4$, so the graph of $y = x^2$ undergoes a horizontal translation of 4 units to the left, resulting in a turning point of $(h, k) = (-4, 0)$.	$y = (x \boxed{+ 4})^2$ Horizontal translation of four units to the left
	2 Find the y-intercept by substituting $x = 0$ into the equation and simplifying.	The turning point is (–4, 0). For the y-intercept, $x = 0$: $y = (x + 4)^2$ $= (0 + 4)^2$ $= 16$ The y-intercept is (0, 16).
	3 Sketch the graphs of $y = x^2$ and $y = (x + 4)^2$, and label their turning points and intercepts.	

Example 5D.2 Sketching a parabola using two transformations

Sketch the graph of each of the following parabolas by performing transformations on $y = x^2$. Label the y-intercept and the turning point each time.

a $y = -(x + 3)^2$ **b** $y = (x - 2)^2 - 1$

THINK

WRITE

a

1 Identify the transformations. The equation has the form $a(x - h)^2$ where $a = -1$, so the graph of $y = x^2$ is reflected in the x-axis. Because $h = -3$, the graph of $y = x^2$ also undergoes a horizontal translation 3 units to the left, resulting in a turning point of $(h, k) = (-3, 0)$.

$$y = \boxed{-}(x\boxed{+\,3})^2$$

Reflection in the x-axis

Horizontal translation of three units to the left

The turning point is $(-3, 0)$.

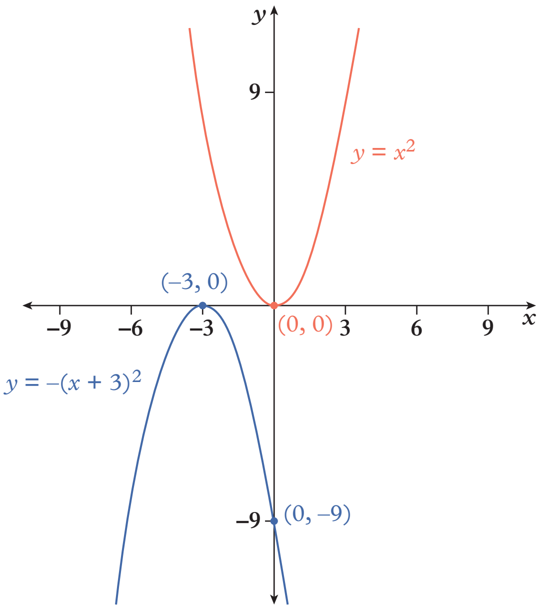
2 Find the y-intercept by substituting $x = 0$ into the equation and simplifying.

For the y-intercept, $x = 0$:

$$\begin{aligned} y &= -(x + 3)^2 \\ y &= -(0 + 3)^2 \\ &= -9 \end{aligned}$$

The y-intercept is $(0, -9)$.

3 Sketch the graphs of $y = x^2$ and $y = -(x + 3)^2$, and label their turning points and intercepts.



b

1 Identify the transformations. The equation has the form $(x - h)^2 + k$ where $h = 2$ and $k = -1$, so the graph of $y = x^2$ undergoes a horizontal translation 2 units to the right and a vertical translation 1 unit down, giving a turning point of $(h, k) = (2, -1)$.

$$y = (x\boxed{-\,2})^2\boxed{-\,1}$$

Horizontal translation of two units to the right

Vertical translation of one unit down

The turning point is $(2, -1)$.

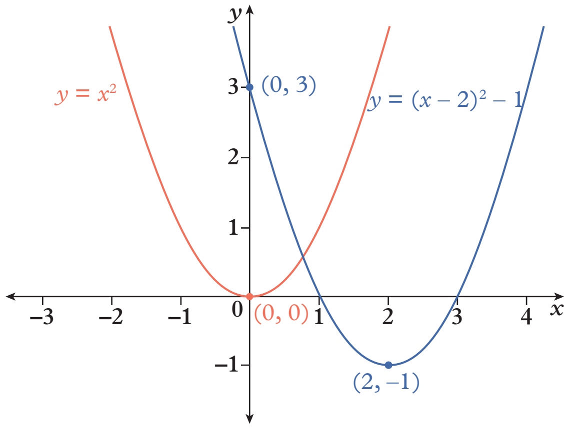
2 Find the y-intercept by substituting $x = 0$ into the equation and simplifying.

For the y-intercept, $x = 0$:

$$\begin{aligned} y &= (x - 2)^2 - 1 \\ y &= (0 - 2)^2 - 1 \\ &= 3 \end{aligned}$$

The y-intercept is $(0, 3)$.

3 Sketch the graphs of $y = x^2$ and $y = (x - 2)^2 - 1$, and label their turning points and intercepts.



Example 5D.3 Sketching a parabola using multiple transformations

Sketch the graph of $y = x^2$ on a Cartesian plane, then perform transformations on $y = x^2$ to sketch the graph of $y = -3(x - 1)^2 + 2$. For both graphs, label the y-intercept and the turning point.

THINK

1 Identify the transformations. The equation has the form $a(x - h)^2 + k$ where $a = -3$, $h = 1$ and $k = 2$, so the graph of $y = x^2$ is:

- reflected in the x-axis
- stretched in the y-direction by a factor of 3
- translated horizontally 1 unit to the right
- translated vertically 2 units up

resulting in a turning point of $(h, k) = (1, 2)$.

2 Find the y-intercept by substituting $x = 0$ into the equation and simplifying.

3 Sketch the graphs of $y = x^2$ and $y = -3(x - 1)^2 + 2$, and label their turning points and intercepts.

WRITE

$$y = \boxed{-3}(x \boxed{-1})^2 \boxed{+2}$$

Reflection in the x-axis and stretched in the y-direction by a factor of 3

Horizontal translation of one unit to the right

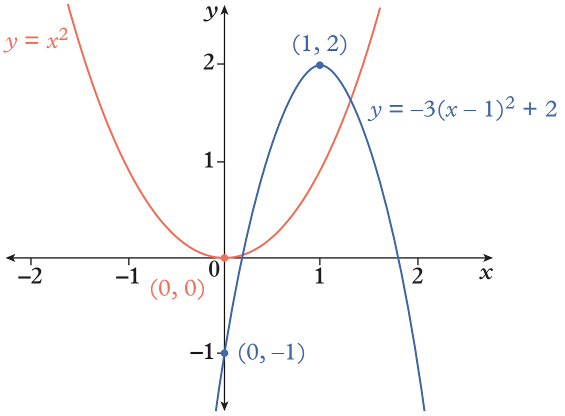
Vertical translation of two units up

The turning point is (1, 2).

For the y-intercept, $x = 0$:

$$y = -3(x - 1)^2 + 2$$
$$y = -3(0 - 1)^2 + 2$$
$$= -1$$

The y-intercept is (0, –1).



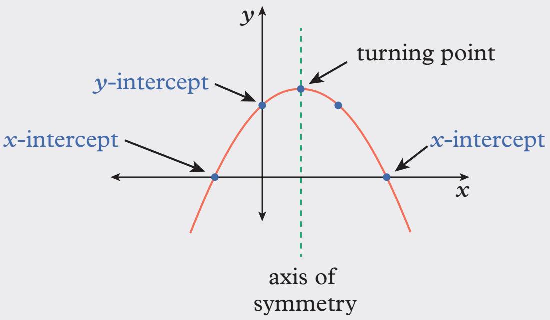
Helpful hints

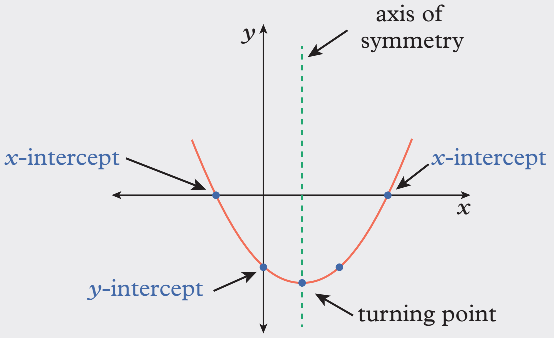
- ✔ In turning point form:
  - → the size and direction of vertical translations are determined by the constant outside the brackets
  - → the size and direction of horizontal translations are determined by the constant that is inside the brackets with x.

$$y = a(x - h)^2 + k$$

Horizontal translation of h units   Vertical translation of k units

- ✔ Always label key features when sketching a graph. These include the intercepts and the turning points.
- ✔ If you need help drawing a symmetrical curve, it can help to draw a dashed line of symmetry.





- ✔ For $y = a(x - h)^2 + k$, if $-1 < a < 1$, the graph of $y = x^2$ has undergone a 'stretch' in the direction of the y-axis which actually compresses the graph.

  For example, if $y = x^2$ becomes $y = \frac{1}{2}x^2$ then it has been 'stretched' by a factor of $\frac{1}{2}$ in the direction of the y-axis, which really means it has been compressed by a factor of 2 in the direction of the y-axis.

ANS
p699

Exercise 5D Sketching parabolas using transformations

▲ 1–7, 8(a, b, d, e), 9–11, 12(a, b), 13

■ 2–7, 8(a, c, d, f), 9, 12(c, d), 14, 15, 17(a, c), 18(a, b)

⬟ 2–3(b, d, f), 5, 6(d–f), 8, 9, 12, 14, 16–19

1 For each set of quadratic rules below:

i draw the graphs on the same Cartesian plane

ii compare each parabola with the graph of $y = x^2$

iii identify the transformation (or transformations) performed. Describe the effect of different coefficients for x^2 and any constants that are added or subtracted.

a $y = x^2$
$y = 2x^2$
$y = 4x^2$
$y = \frac{1}{2}x^2$
$y = \frac{1}{4}x^2$

b $y = x^2$
$y = x^2 + 1$
$y = x^2 + 2$
$y = x^2 - 1$
$y = x^2 - 2$

c $y = x^2$
$y = (x - 1)^2$
$y = (x - 2)^2$
$y = (x + 1)^2$
$y = (x + 2)^2$

d $y = x^2$
$y = -x^2$
$y = -x^2 + 2$
$y = -2x^2$
$y = \frac{1}{2}x^2$

UNDERSTANDING AND FLUENCY

5D.1 **2** Perform transformations on the graph of $y = x^2$ to help you sketch the graph of each quadratic equation below. Clearly show the coordinates of the turning point each time.

a $y = x^2 + 3$
b $y = x^2 + 2$
c $y = x^2 - 5$
d $y = x^2 + 6$
e $y = x^2 - 4$
f $y = x^2 - 7.5$

3 Perform a transformation on the graph of $y = x^2$ to help you sketch the graph of each quadratic equation below. Clearly show the y-intercept and the coordinates of the turning point each time.

a $y = (x - 4)^2$
b $y = (x + 3)^2$
c $y = (x - 5)^2$
d $y = (x + 2)^2$
e $y = (x - 8)^2$
f $y = (x + 1.5)^2$

4 Match each equation below with a description (**A–F**) of the transformation/s that must be performed on the graph of $y = x^2$ to produce its graph.

a $y = x^2 - 9$
b $y = (x + 9)^2$
c $y = 9x^2$
d $y = -9x^2$
e $y = -x^2 + 9$
f $y = -(x - 9)^2$

A stretched in the y-direction by a factor of 9
B translated 9 units down
C reflected in the x-axis and then translated 9 units right
D translated 9 units left
E reflected in the x-axis and then translated 9 units up
F reflected in the x-axis and stretched by a factor of 9 in the y-direction

5 Write the rule for the parabola that would be produced by performing each transformation (or combination of transformations) below on the graph of $y = x^2$.

a stretched in the y-direction by a factor of 6
b reflected in the x-axis
c translated 5 units right
d translated 2 units down
e stretched in the y-direction by a factor of $\frac{1}{4}$ and reflected in the x-axis
f reflected in the x-axis and translated 3 units up

5D.2 **6** Sketch the graph of each of the following quadratic relationships by performing transformations on $y = x^2$. Label the y-intercept and the turning point each time.

a $y = (x - 1)^2 + 2$
b $y = (x + 5)^2 - 3$
c $y = -(x - 2)^2 + 7$
d $y = -(x + 4)^2 - 6$
e $y = (x + 7)^2 + 4$
f $y = -(x - 6)^2 - 1$

7 Use a graphing calculator or other digital technology to check the graphs you sketched for question **6**.

5D.3 **8** For each quadratic relationship:

i identify whether the graph will be narrower or wider than the graph of $y = x^2$
ii identify whether the parabola will be upright or inverted
iii identify the coordinates of the turning point
iv sketch the graph.

a $y = 2(x - 4)^2 - 8$
b $y = -3(x + 1)^2 + 12$
c $y = 5(x - 3)^2$
d $y = -4x^2 + 1$
e $y = \frac{1}{3}(x + 6)^2 - 3$
f $y = -\frac{1}{2}(x - 2)^2 - 9$

9 Write the equation for the parabola produced after performing each set of transformations below on the graph of $y = x^2$.

a stretched in the y-direction by a factor of 2 then translated 5 units right

b reflected in the x-axis then translated 3 units down and 4 units left

c stretched in the y-direction by a factor of $\frac{1}{3}$ and reflected in the x-axis

d stretched in the y-direction by a factor of 5, reflected in the x-axis then translated 7 units up

10 A quadratic relationship has the equation $y = (x + 2)^2 + 4$. What are its smallest and largest possible y values? Explain.

11 A quadratic relationship has the equation $y = -(x - 6)^2 - 1$. What are its smallest and largest possible y values? Explain.

12 Sketch each of the following parabolas by first completing the square for the quadratic, labelling all relevant points as exact values.

a $y = x^2 + 6x + 9$ **b** $y = x^2 - 2x + 3$ **c** $y = x^2 - 4x + 1$ **d** $y = 3x^2 - 6x + 5$

13 Riley tries to kick a goal during a game of football, but accidentally hits the post, scoring a behind. The height of the ball above the ground can be represented by the relationship $h = -4(t - 1)^2 + 5$, where h is the height in metres after t seconds.

a What are the coordinates of the turning point of the graph of this relationship?

b Sketch the graph, from $t = 0$ to $t = 1.5$, and list the domain and range.

c Use the graph to find:

i the height at which the ball made contact with Riley's boot

ii the maximum height of the ball during Riley's kick

iii the height of the football when it hits the post, 1.5 seconds after it was kicked.

d If the ball had not hit the post, how long would it have taken for the ball to hit the ground after Riley kicked it? Give your answer correct to one decimal place.

14 Ella's bank account balance over the first few weeks of the year can be represented by the relationship $m = 20(t - 4)^2 + 50$, where m is the amount in dollars in her account after t weeks.

a Sketch a graph of the relationship, from $t = 0$ to $t = 10$. List the domain and range.

b How much did Ella have in her account at the start of the year?

c What is the lowest amount Ella had in the account in the first 10 weeks?

d After how many weeks did Ella have the same account balance as she did at the start of the year?

e What is the account balance after 10 weeks?

15 A soccer ball is kicked from the ground into the air. The path of the ball follows the relationship $y = -\frac{1}{40}(x - 20)^2 + 10$, where y is the height above the ground of the ball for a horizontal distance x from where the ball was kicked. Both x and y are in metres.

a Sketch the graph of this relationship from when the ball was kicked to when it landed. List its domain and range.

b What was the maximum height of the soccer ball?

c What horizontal distance did the ball travel before it hit the ground?

PROBLEM SOLVING AND REASONING

16 The equation, $H = -x^2 + 6x + 1$, models the path of a poorly hit baseball, where H represents the height above the ground of the ball (in metres) and x represents the horizontal distance travelled by the ball (in metres).

a Substitute $x = 0$ into the equation to find the height above the ground of the ball when it was hit by the batter.

b Complete the square for $-x^2 + 6x + 1$.

c Find the maximum height reached by the ball.

d Using the quadratic formula, or otherwise, find the horizontal distance the ball travelled before hitting the ground. Give your answer in metres rounded to two decimal places.

e Use the information found in parts **a–d** to help you sketch the graph of H versus x, labelling all relevant points.

CHALLENGE

17 From the given information, write an equation for each of these quadratic relationships.

a The graph is an upright parabola, with turning point at (2, 6), and y-intercept of 10.

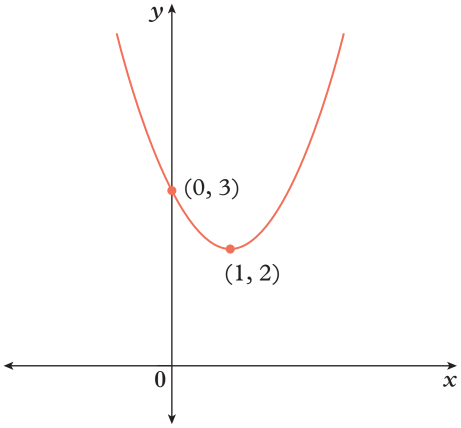
b The graph is an upright parabola, with turning point at (–4, –12), and y-intercept of 20.

c The graph is an inverted parabola, with turning point at (–1, 7), and y-intercept of 4.

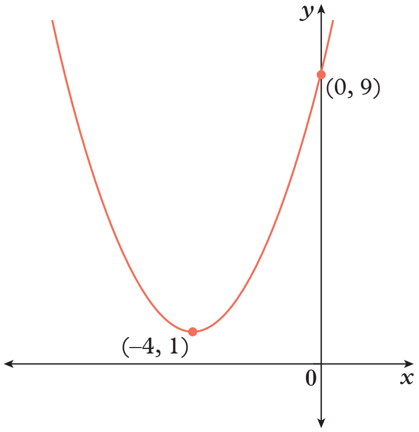
d The graph is an inverted parabola, with turning point at (3, –5), and y-intercept of –8.

18 The following parabolas all have equations in the form $y = a(x - h)^2 + k$. Find the equation for each graph.

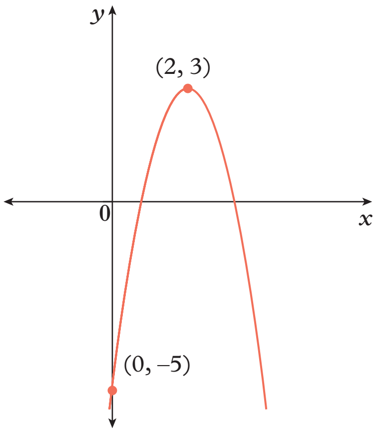
a



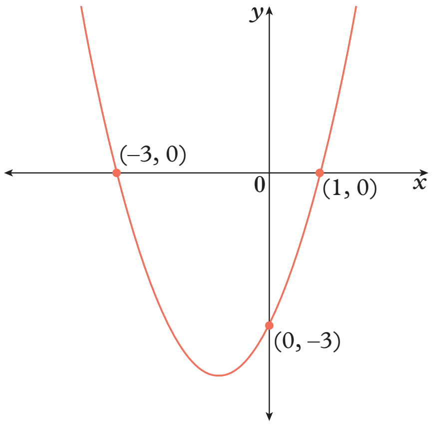
b



c



d



19 a Show that $(2 - x)^2$ is equivalent to $(x - 2)^2$ by expanding each expression.

b Sketch each of the following graphs, labelling all relevant points.

i $y = (2 - x)^2 + 1$

ii $y = (1 - x)^2 - 4$

iii $y = -\frac{1}{3}(3 - x)^2 - 1$

iv $y = -(-x - 2)^2 - 1$

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Sketching parabolas using transformations

Topic quiz
5D

Checkpoint

Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this chapter.

5A **1** Use the Null Factor Law to solve each of the following equations.

a $x(x-2)=0$ **b** $(x+3)(x-4)=0$

c $(x+1)(x-1)=0$ **d** $(2x+3)(x-5)=0$

5A **2** Solve each of the following quadratics by first factorising and then applying the Null Factor Law.

a $x^2+5x=0$ **b** $2x^2-12x=0$

c $x^2+6x+8=0$ **d** $x^2-22x+40=0$

e $x^2+16x-17=0$ **f** $x^2+x-6=0$

5A **3** Solve each of the following equations and give your answers as exact values.

a $x^2-4=0$ **b** $x^2-7=0$

c $x^2=22$ **d** $5x^2=45$

e $3x^2-15=0$ **f** $-2x^2+12=0$

5B **4** Find the value of the discriminant for each of the following quadratic equations.

a $x^2+5x+7=0$ **b** $x^2+4x=2$

c $2x^2+6x+5=0$ **d** $3x^2+7x=10$

5B **5** Use the discriminant to identify the number of solutions for each of the following quadratic equations.

a $x^2-5x+3=0$ **b** $x^2-6x=-9$

c $2x^2+7x=10$ **d** $2x^2+5x+4=0$

5B **6** Use the quadratic formula to solve the following quadratic equations.

a $x^2+3x+1=0$ **b** $x^2-2x+1=0$

c $3x+1=x^2$ **d** $5x^2+x=2$

e $4x^2+12x+9=0$ **f** $\frac{1}{2}x^2+5x=3$

5C **7** Find the x-intercepts and the y-intercept for the parabolas with the following equations. You do not need to sketch the graphs.

a $y=x(x-4)$ **b** $y=(x+3)(x-1)$

c $y=x^2-x-2$ **d** $y=x^2+6x-16$

5C **8** Sketch the graph of $y=(x-1)(x-5)$, labelling all relevant points.

5C **9** Sketch the graph of $y=-2(x+1)(x-2)$, labelling all relevant points.

5D **10** Sketch the graphs of each of the following, labelling all relevant points.

a $y=x^2+1$ **b** $y=x^2-4$

c $y=(x-2)^2$ **d** $y=(x+3)^2$

5D **11** Sketch the graphs of each of the following, labelling all relevant points with exact values.

a $y=(x-3)^2+1$ **b** $y=2(x+1)^2+2$

c $y=-x^2+1$ **d** $y=-(x+1)^2-1$

e $y=-(x-2)^2+1$ **f** $y=(x-3)^2-7$

5D **12** Write the rule for each parabola resulting from performing each of these sets of transformations on the graph of $y=x^2$.

a a translation 2 units right and 1 unit up

b a reflection in the x-axis, then a translation 3 units left and 1 unit up

c a stretch in the y-direction by a factor of 2, then a translation 3 units down

d a stretch in the y-direction by a factor of 3 and a reflection in the x-axis, then a translation 4 units right and 2 units down

5E Circles

Learning intentions

- ✔ I can sketch simple circles of the form $x^2 + y^2 = r^2$.
- ✔ I can sketch circles of the form $(x - h)^2 + (y - k)^2 = r^2$ using vertical translations and horizontal translations.
- ✔ I can determine the equation of a circle from a description or a graph.

Inter-year links

Year 7	8D Translations
Year 8	7C Transformations
Year 9	5E Circles and other non-linear relationships

Circles

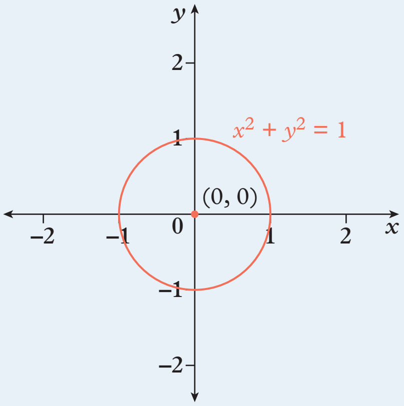
- The graph of $x^2 + y^2 = 1$ is a circle with centre at (0, 0) and a radius of $r = 1$ unit.
- The general equation of a circle graph is: $(x - h)^2 + (y - k)^2 = r^2$
  where:
  - → the circle has a radius of r units
  - → the coordinates of the centre of the circle are (h, k).

$$(x - h)^2 + (y - k)^2 = r^2$$

Horizontal translation of h units
If $h > 0$, right
If $h < 0$, left

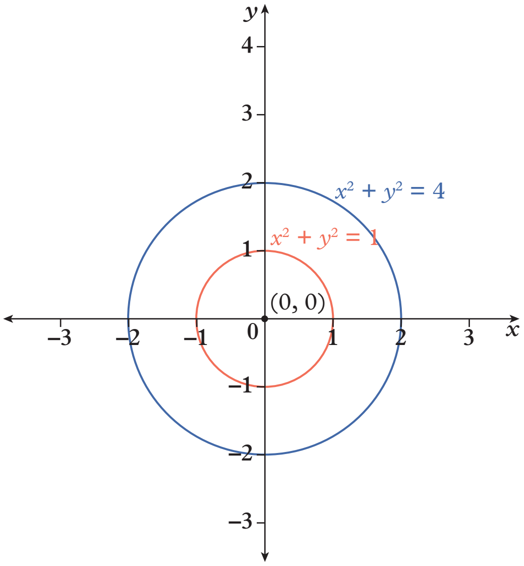
Vertical translation of k units
If $k > 0$, up
If $k < 0$, down

Radius of r units

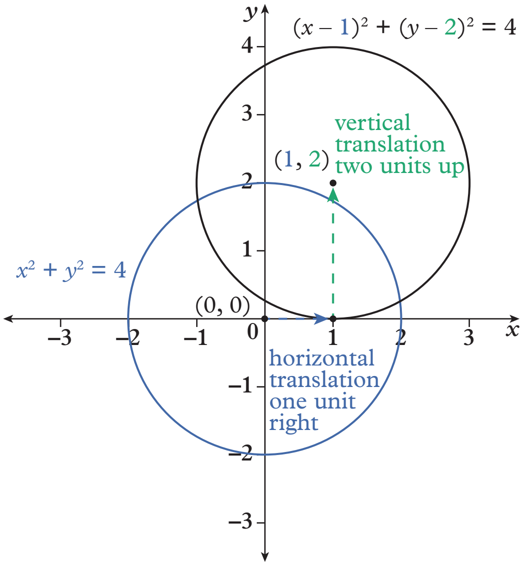


- The following transformations can be performed on the graph of $x^2 + y^2 = r^2$ to produce the graph of $(x - h)^2 + (y - k)^2 = r^2$:
  - → a stretch in all directions as r increases
  - → a horizontal translation of h units right or left
  - → a vertical translation of k units up or down.

Stretch

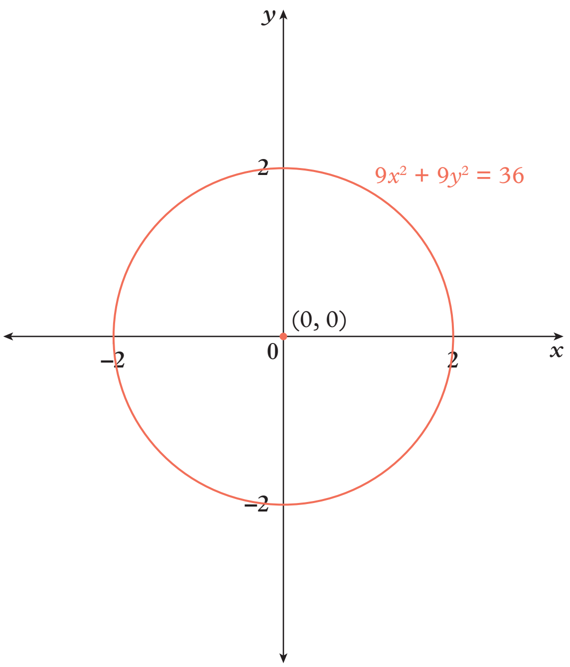


Translation



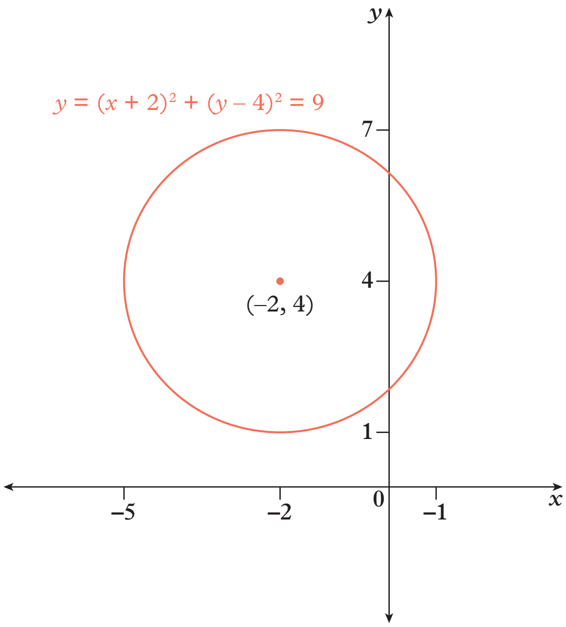
Example 5E.1 Sketching simple circles

Sketch the graph of $9x^2 + 9y^2 = 36$ by first finding the x- and y-intercepts. List the domain and range of the graph.

THINK	**WRITE**
1 Rearrange the equation to the form $x^2 + y^2 = r^2$.	$9x^2 + 9y^2 = 36$ $x^2 + y^2 = \frac{36}{9}$ $x^2 + y^2 = 4$
2 Identify the centre of the circle. The equation has the form $x^2 + y^2 = r^2$, or $(x - 0)^2 + (y - 0)^2 = r^2$, so there is no translation of the circle from the origin. Therefore, the centre of the circle is at (0, 0).	The centre is (0, 0).
3 Identify the radius.	$r^2 = 4$ $r = \sqrt{4}$ $= 2$
4 Identify the x-intercepts. The x-intercepts will be 2 units left and 2 units right of the origin.	The x-intercepts are (–2, 0) and (2, 0).
5 Identify the y-intercepts. The y-intercepts will be 2 units below and 2 units above the origin.	The y-intercepts are (0, –2) and (0, 2).
6 Sketch the graph. Label the graph with its equation.	
7 Identify the domain and range of the graph.	The domain is $-2 \leq x \leq 2$. The range is $-2 \leq y \leq 2$.

Example 5E.2 Sketching circles using vertical and horizontal translations

Use transformations to help you sketch the graph of $y = (x + 2)^2 + (y - 4)^2 = 9$. List the domain and range of the graph.

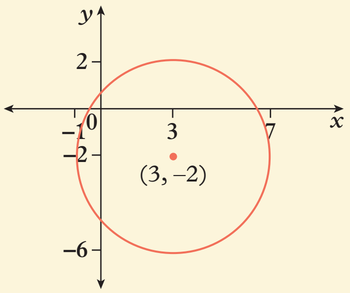
THINK	**WRITE**
1 Identify the translations used. The equation has the form $y = (x - h)^2 + (y - k)^2 = r^2$ where $h = -2$ and $k = 4$, so the graph of $x^2 + y^2 = 9$ undergoes a horizontal translation of 2 units to the left and a vertical translation of 4 units up, giving a centre of $(h, k) = (-2, 4)$.	$y = (x \underbrace{+ 2}_{\text{Horizontal translation of two units left}})^2 + (y \underbrace{- 4}_{\text{Vertical translation of four units up}})^2 = 9$ $y = (x + 2)^2 + (y - 4)^2 = 9$ The centre is (–2, 4).
2 Identify the radius.	$r^2 = 9$ $r = 3$
3 Sketch the graph by identifying the four points that are 3 units left, 3 units right, 3 units below and 3 units above the centre: (–5, 4), (1, 4), (–2, 1) and (–2, 7).	
4 Identify the domain and range of the graph. The x values extend from –5 to 1 and the y values extend from 1 to 7.	Domain is $-5 \le x \le 1$. Range is $1 \le y \le 7$.

Example 5E.3 Determining the equation of a circle

Determine the equation of the circle in each case below.

a a circle with a radius of 6 units and a centre at (–2, 5)

b



THINK

The general equation of a circle is $y = (x - h)^2 + (y - k)^2 = r^2$. Identify the values of h, k and r, substitute them into the equation and simplify.

WRITE

a The centre is $(-2, 5) = (h, k)$.

$h = -2$

$k = 5$

Radius: $r = 6$

$(x - h)^2 + (y - k)^2 = r^2$

$(x - [-2])^2 + (y - 5)^2 = 6^2$

$(x + 2)^2 + (y - 5)^2 = 36$

b The centre is $(3, -2) = (h, k)$

$h = 3$

$k = -2$

Radius: $r = 4$

$(x - h)^2 + (y - k)^2 = r^2$

$(x - 3)^2 + (y - [-2])^2 = 4^2$

$(x - 3)^2 + (y + 2)^2 = 16$

Helpful hints

- ✔ Remember that the values of h and k have the opposite sign to the one appearing with them in the equation.
  For example, in the equation $y = (x + 2)^2 + (y - 4)^2 = 9$, $h = -2$ and $k = +4$
- ✔ Make sure you sketch your circles on a set of axes drawn to the same scale, because scales that don't match will make your circle look like an oval or an ellipse.
- ✔ Remember that the value of r is the positive square root of the right-hand side of the equation.
  For example, the radius of $(x - 1)^2 + (y - 3)^2 = 4$ is 2, not 4.
- ✔ Make sure that you mark and label the centre of your circles!

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Exercise 5E Circles

△ 1–7, 10, 14	■ 1(e, f), 3–8, 10, 11, 15, 16, 18	⬠ 1(e, f), 3(b, d, f, h), 5(b, d, f, h), 6, 7, 9, 10, 12, 13, 15, 17, 19

UNDERSTANDING AND FLUENCY

5E.1 **1** Sketch the graph of each of these equations and list the domain and range each time.

a $x^2 + y^2 = 9$

b $x^2 + y^2 = 1$

c $x^2 + y^2 = 64$

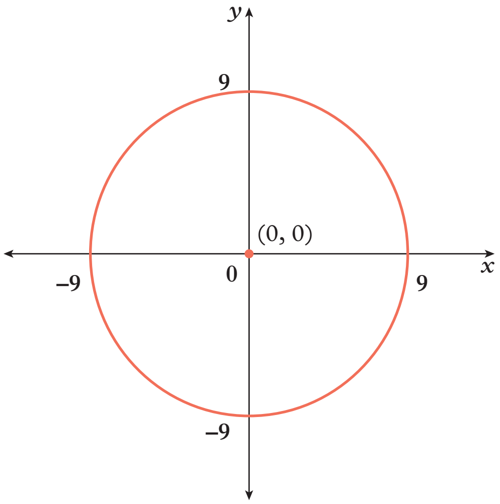
d $4x^2 + 4y^2 = 100$

e $15x^2 + 15y^2 = 60$

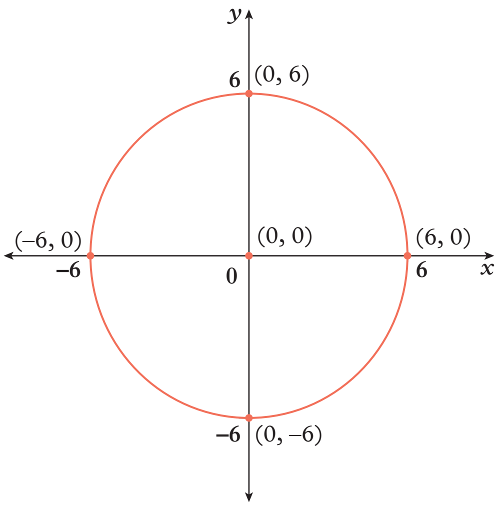
f $\frac{x^2}{16} + \frac{y^2}{16} = 1$

2 Write the equation for each of these circles.

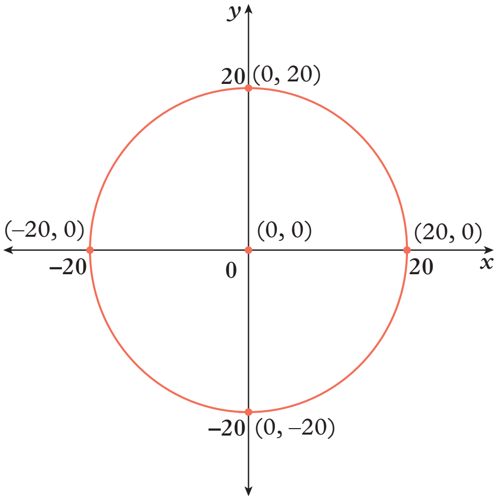
a



b



c



5E.2 **3** Sketch the graph of each of these equations. List the centre, radius, domain and range of each graph.

a $(x - 3)^2 + (y - 2)^2 = 9$

b $(x - 5)^2 + (y - 7)^2 = 25$

c $(x + 1)^2 + (y - 3)^2 = 64$

d $(x - 6)^2 + (y + 6)^2 = 36$

e $(x - 2)^2 + y^2 = 16$

f $x^2 + (y + 5)^2 = 4$

g $(x + 3)^2 + (y + 5)^2 = 49$

h $(x + 4)^2 + (y + 3)^2 = 81$

4 Use digital technology to reproduce the first four graphs from question **3**. Compare those graphs to the ones you sketched.

UNDERSTANDING AND FLUENCY

5E.3 **5** Use the information provided to write the equation for each of these circles.

a a circle with radius 12 units and centre at (0, 0)

b a circle with radius 15 units and centre at (0, 0)

c a circle with radius 1 unit and centre at (4, –2)

d a circle with radius 7 units and centre at (–5, –6)

e a circle with radius 3 units and centre at (4, 4)

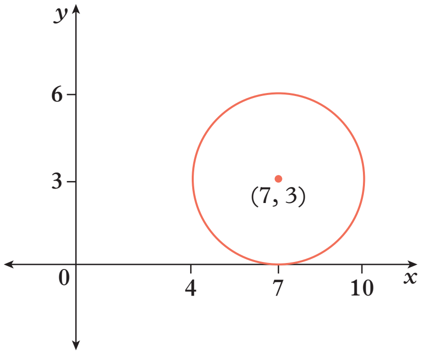
f a circle with radius 6 units and centre at (0, –5)

g a circle with radius 1.5 units and centre at (–1, 2.5)

h a circle with radius $\sqrt{5}$ units and centre at (2, –3)

6 Write the equation for each of these circles.

a



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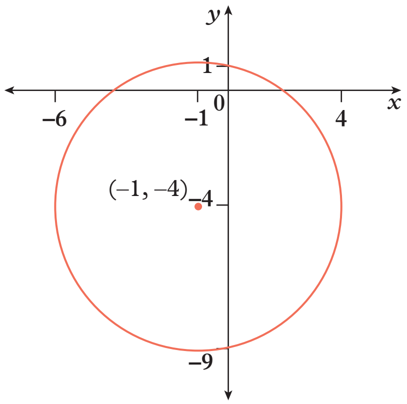
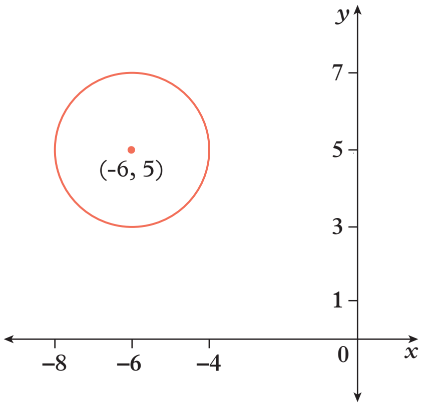


c



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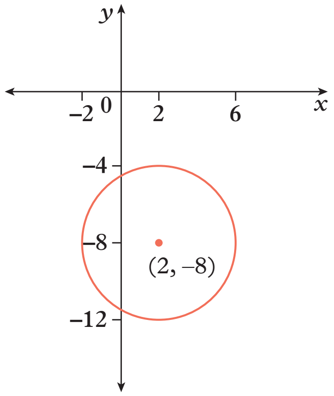


7 **a** The circle with equation $x^2 + y^2 = 1$ is stretched in all directions by a factor of 3, then translated 2 units right and 1 unit down. What is the equation of the transformed circle?

b The circle with equation $x^2 + y^2 = 1$ is transformed to become a circle with equation $(x + 7)^2 + (y - 4)^2 = 25$. Describe the transformations that have occurred. (Hint: You must describe the stretch before the translations.)

PROBLEM SOLVING AND REASONING

8 **a** Complete the square for each of these quadratics:

i $x^2 + 6x + 2$

ii $y^2 - 4y + 7$

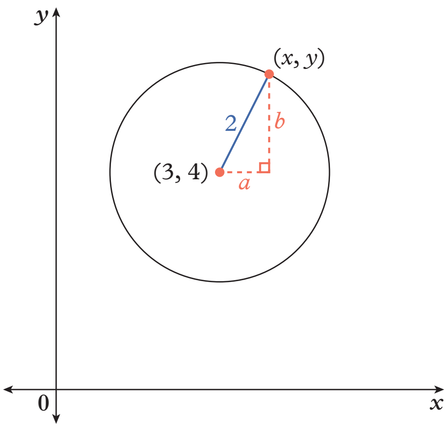
b Sketch the circle with equation $x^2 + 6x + 2 + y^2 - 4y + 7 = 0$.

9 A circle is a set of points that are equidistant from a single point. This geometric description of a circle, along with Pythagoras' Theorem, explains the Cartesian equation of a circle.

Consider the following diagram, where (x, y) is generalised to be any point on the circle.

a Explain why $a = x - 3$ and $b = y - 4$.

b Use Pythagoras' Theorem to explain why the Cartesian equation of this circle is $(x - 3)^2 + (y - 4)^2 = 2^2$.



PROBLEM SOLVING AND REASONING

10 Consider $x^2 + y^2 = 25$.

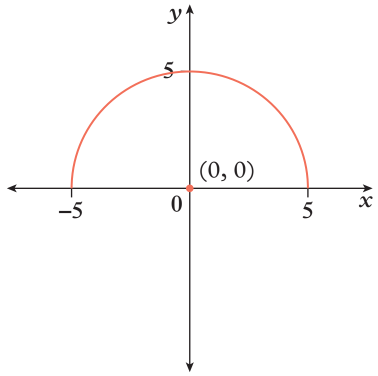
a Draw a graph of the equation.

b Rearrange the equation to make y the subject.

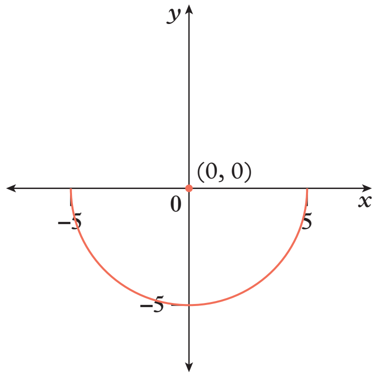
c Match each of these equations to one of the graphs shown below (**A** and **B**). Explain your reasoning.

i $y = -\sqrt{25 - x^2}$ **ii** $y = \sqrt{25 - x^2}$

A



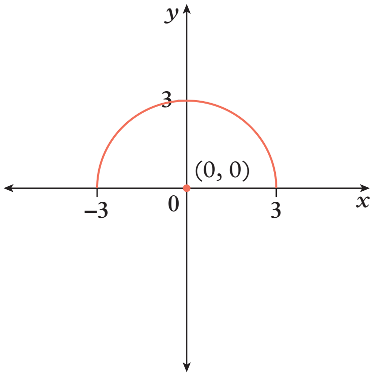
B



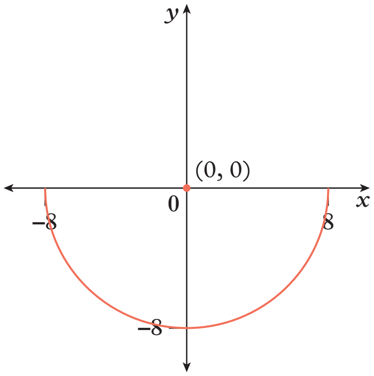
d Explain how the two half circles **A** and **B** are related to the graph of $x^2 + y^2 = 25$.

11 Write the equation for each of these half circles.

a



b



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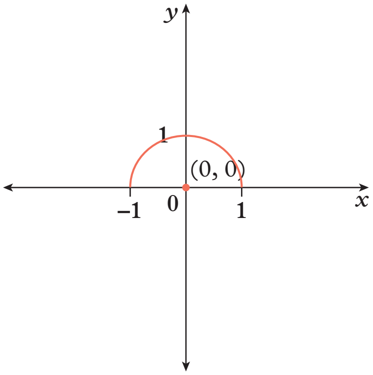


12 Sketch the graph of each of these half circles. List the domain and range of each graph.

a $y = -\sqrt{49 - x^2}$ **b** $y = \sqrt{16 - x^2}$ **c** $y = -\sqrt{100 - x^2}$

13 a Rearrange $x^2 + y^2 = 25$ to make x the subject of the equation.

b Your answer to part **a** produces two equations. What would the graph of each equation look like?

14 An arch is modelled using the relationship $(x - 3)^2 + y^2 = 9$, where x is the horizontal distance from the place the left-hand side of the arch meets the ground and y is the corresponding height at that distance. Both x and y are measured in metres, $x \geq 0$ and $y \geq 0$.

a Sketch the graph of this relationship.

b What is the horizontal distance along the ground from one side of the arch to the other?

c What is the maximum height of the arch?

d Describe the position on the ground below a point where the arch is at half its maximum height.

15 The outline of a spa is mapped onto a Cartesian plane representing the outdoor area of a seaside resort, using the relationship $(x - 6.2)^2 + (y - 4.8)^2 = 2.25$. All measurements are in metres. The outdoor area is 15 m long (shown on the positive x-axis) and 10 m wide (shown on the positive y-axis).

a Sketch the graph of this relationship.

b What is the minimum distance from the water to the edge of the outdoor area?

c Calculate the area of the spa to the nearest square metre.

d What percentage of the outdoor area is taken up by the spa?

16 Use a graphical method to find the points of intersection of the graphs with equations $x^2 + y^2 = 25$ and $y = x + 7$.

17 Use an algebraic method to find the points of intersection of the graphs with equations:

a $x^2 + y^2 = 9$ and $y = x - 3$

b $(x + 3)^2 + (y - 8)^2 = 4$ and $y = x + 9$.

18 Find the equation for a circle that passes through the four points (–10, 5), (–4, 11), (2, 5) and (–4, –1).

19 A tangent to a circle is a straight line which just touches the circle once, at one point on its circumference. The radius of the circle at the point of tangency is perpendicular to the tangent line.

Consider the circle with equation $(x - 1)^2 + (y - 2)^2 = 5$.

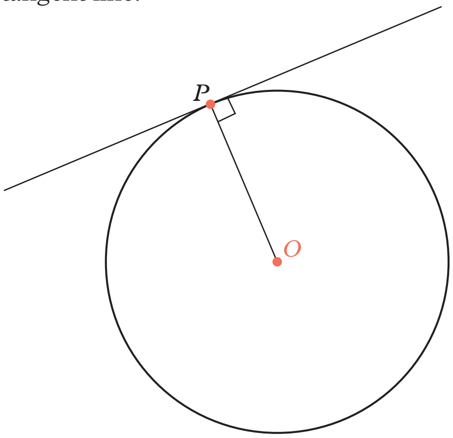
a What is the centre of this circle?

b Show that (3, 3) is a point on the circle.

c Find the gradient of the radius that connects the centre of the circle to the point (3, 3).

d Find the gradient of the line perpendicular to the radius referred to in part **c**.

e Use the gradient you found in part **d** to find the equation of the tangent to the circle at (3, 3). Sketch the graphs of the circle and your line on a graphing calculator to confirm that the straight line is a tangent to the circle.



Check your Student obook pro for these digital resources and more:

- **Interactive skillsheet** Circles
- **Worksheet** Circles on the Cartesian plane
- **CAS instructions** Graphing circles
- **Topic quiz** 5E

5F Exponential relationships

Learning intentions

- ✔ I can sketch simple exponential relationships of the form $y = a^x$.
- ✔ I can sketch exponential relationships using reflections and stretches.
- ✔ I can sketch exponential relationships using vertical translations and horizontal translations.

Inter-year links

Year 7	8D Translations
Year 8	7C Transformations
Year 9	5E Circles and other non-linear relationships

Exponential relationships

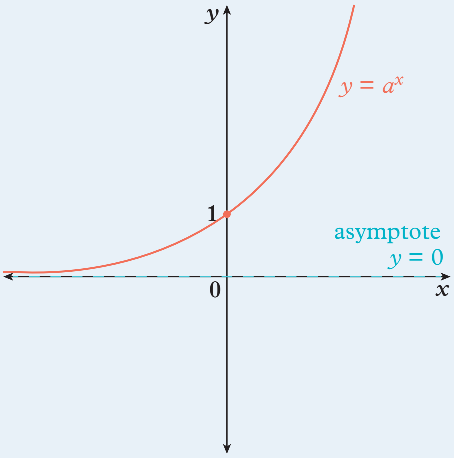
- Exponential relationships can be used to model many real situations involving growth and decay; for example, situations involving populations or investments.
- The basic form of an **exponential relationship** has the following equation:

index/exponent

$$y = a^x$$

base

where $a > 1$.



- The graph has:
  - → a y-intercept of 1
  - → a horizontal asymptote along the x-axis.
- An **asymptote** is a boundary or line that a curve approaches but never reaches. The asymptote for $y = a^x$ is the horizontal line with equation $y = 0$.
- The general equation of an exponential relationship is:

  $y = B \times a^{x-h} + k$

  where:
  - → the graph has an asymptote at $y = k$
  - → the graph has a y-intercept at $(0, Ba^{-h} + k)$.

$$y = B \times a^{x-h} + k$$

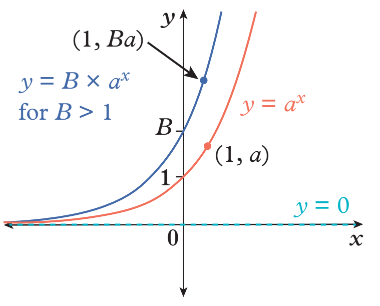
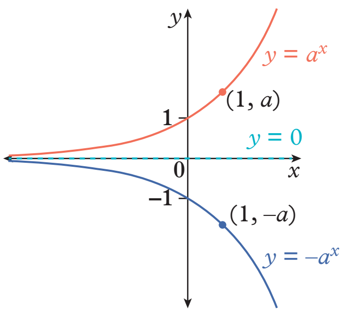
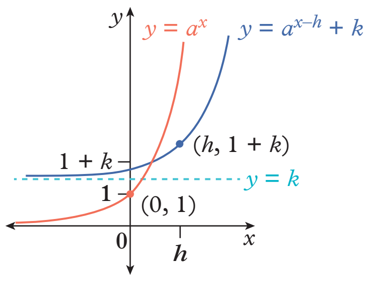
Vertical translation of k units
If $k > 0$, up
If $k < 0$, down

If $B > 0$, upright
If $B < 0$, inverted

If $a > 1$, upright

Horizontal translation of h units
If $h > 0$, right
If $h < 0$, left

- The following transformations can be performed on the graph of $y = a^x$ to produce a sketch of $y = B \times a^{x-h} + k$:
  - → a non-uniform stretch in the y-direction for $x > 0$ and a non-uniform compression for $x < 0$ as a increases
  - → a stretch in the y-direction as B increases
  - → a reflection in the x-axis if $B < 0$
  - → a horizontal translation of h units right or left
  - → a vertical translation of k units up or down.

Stretch	Reflection	Translation
		

Example 5F.1 Sketching the graph of an exponential relationship using stretching transformations

On the same Cartesian plane, sketch the graph of each of these exponential relationships. Label the y-intercept and asymptote.

a $y = 2^x$ **b** $y = 4^x$ **c** $y = 2 \times 2^x$

THINK

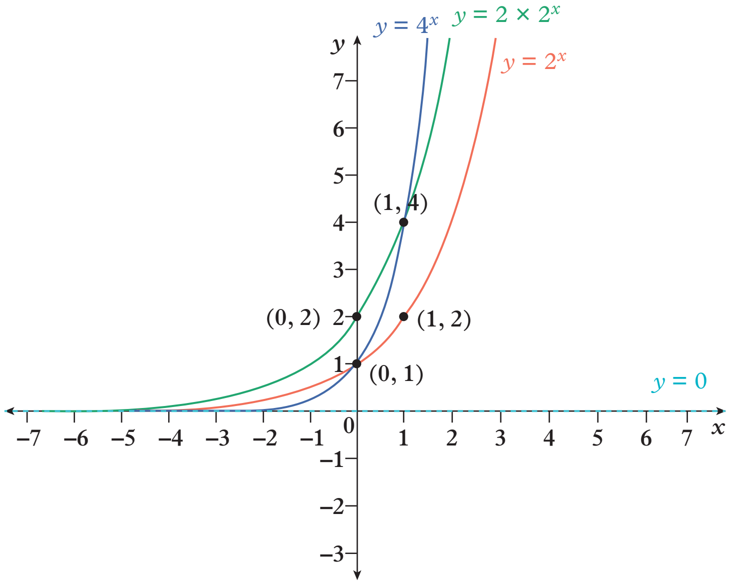
1. Identify the asymptote. The asymptote for $y = a^x$ is the horizontal line with equation $y = 0$.
2. Find the y-intercepts by substituting $x = 0$ into each equation.
3. Label another point on the curve with its coordinates, for scale. Find the value of y when $x = 1$.
4. Sketch the curve for an exponential relationship through the y-intercept and the labelled point, using the horizontal asymptote as a boundary. Label the asymptote and the y-intercept.

WRITE

a Horizontal asymptote: $y = 0$
When $x = 0$, $y = 2^0 = 1$.
y-intercept: (0, 1)
When $x = 1$, $y = 2^1 = 2$.
Point on curve is (1, 2).

b Horizontal asymptote: $y = 0$
When $x = 0$, $y = 4^0 = 1$.
y-intercept: (0, 1)
When $x = 1$, $y = 4^1 = 4$.
Point on curve is (1, 4).

c Horizontal asymptote: $y = 0$
When $x = 0$, $y = 2 \times 2^0 = 2$.
y-intercept: (0, 2)
When $x = 1$, $y = 2 \times 2^1 = 4$.
Point on curve is (1, 4).



Example 5F.2 Sketching the graph of an exponential relationship using vertical translations

On the same Cartesian plane, sketch the graphs of these exponential relationships. Label the y-intercept and asymptote each time.

a $y = 2^x$ **b** $y = 2^x + 1$ **c** $y = 2^x - 3$

THINK

1. Identify the asymptote. The asymptote for $y = a^x + b$ is the horizontal line with equation $y = b$.
2. Find the y-intercept.
3. Label another point on the curve with its coordinates. Find the value of y when $x = 1$.
4. Sketch the curve for an exponential relationship through the y-intercept and the labelled point, using the horizontal asymptote as a boundary. Label the y-intercept and asymptote for each graph.

WRITE

a Horizontal asymptote: $y = 0$

When $x = 0, y = 2^0 = 1$.

y-intercept: (0, 1)

When $x = 1, y = 2^1 = 2$.

Point on curve is (1, 2).

b Horizontal asymptote: $y = 1$

When $x = 0, y = 2^0 + 1 = 2$.

y-intercept: (0, 2)

When $x = 1, y = 2^1 + 1 = 3$.

Point on curve is (1, 3).

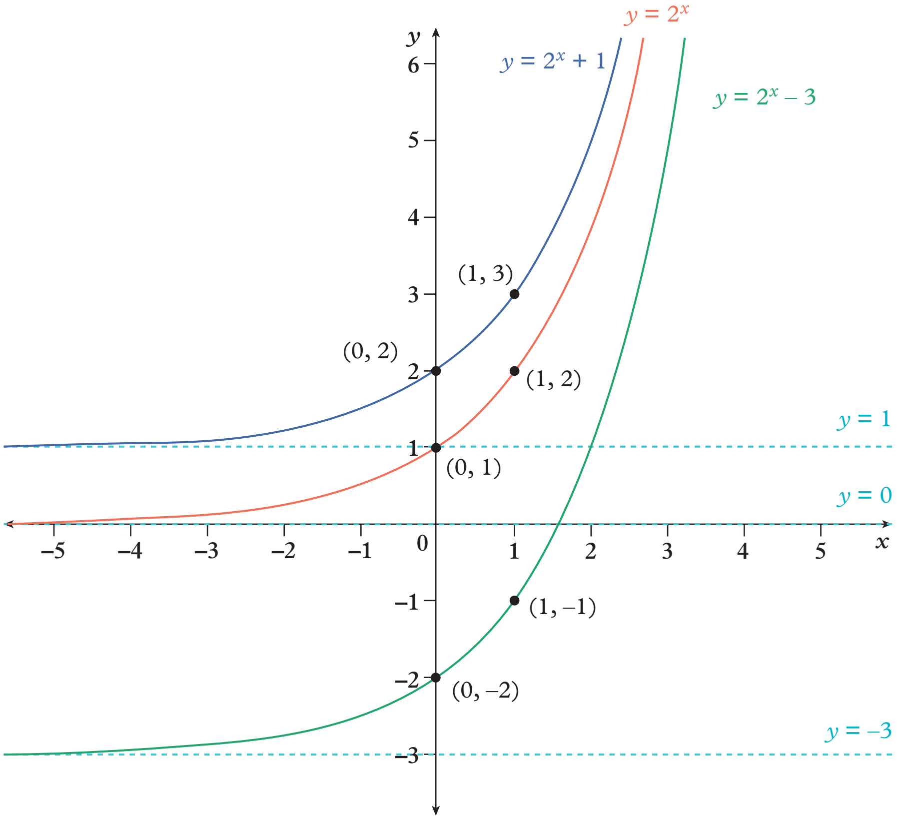
c Horizontal asymptote: $y = -3$

When $x = 0, y = 2^0 - 3 = -2$.

y-intercept: (0, –2)

When $x = 1, y = 2^1 - 3 = -1$.

Point on curve is (1, –1).



Example 5F.3 Sketching the graph of an exponential relationship using horizontal translations

On the same Cartesian plane, sketch the graphs of these exponential relationships. Label the y-intercept and asymptote for each graph.

a $y = 2^x$ **b** $y = 2^{x+1}$ **c** $y = 2^{x-3}$

THINK

1 Identify the asymptote. The asymptote for $y = a^x$ is the horizontal line with equation $y = 0$.

2 Find the y-intercept.

3 Label another point on the curve with its coordinates. Find the value of y when $x = 1$.

4 Sketch the curve for an exponential relationship through the y-intercept and the labelled point, using the horizontal asymptote as a boundary. Label the asymptote and the y-intercept for each graph.

WRITE

a Horizontal asymptote: $y = 0$

When $x = 0$, $y = 2^0 = 1$.

y-intercept: $(0, 1)$

When $x = 1$, $y = 2^1 = 2$.

Point on curve is $(1, 2)$.

b Horizontal asymptote: $y = 0$

When $x = 0$, $y = 2^{0+1} = 2$.

y-intercept: $(0, 2)$

When $x = 1$, $y = 2^{1+1} = 4$.

Point on curve is $(1, 4)$.

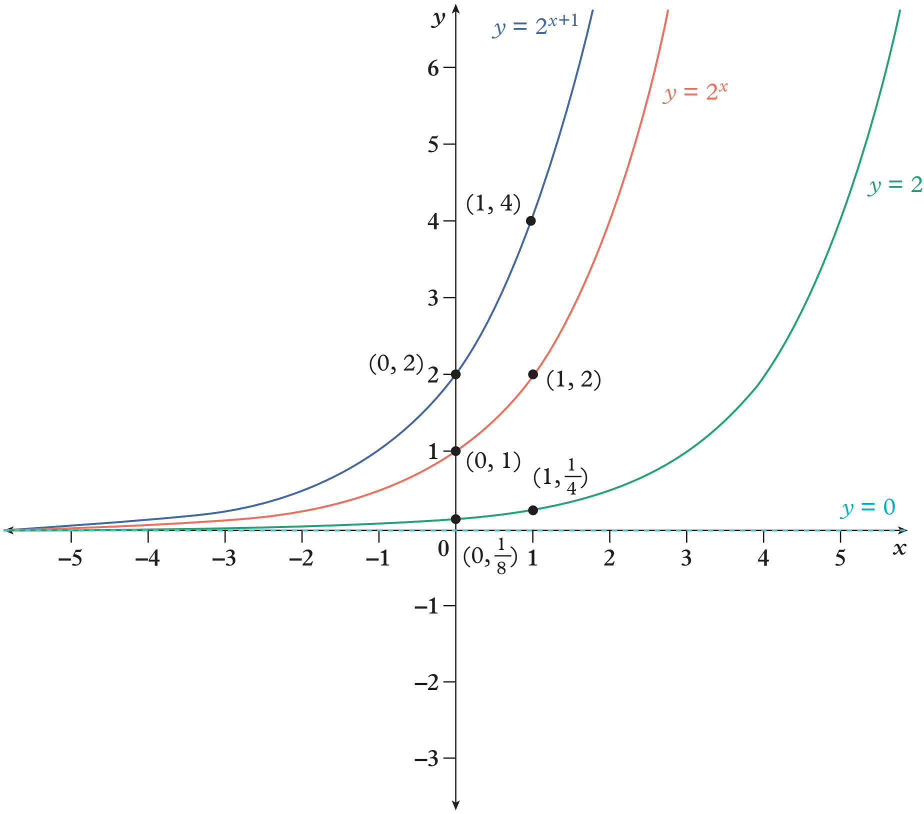
c Horizontal asymptote: $y = 0$

When $x = 0$, $y = 2^{0-3} = 2^{-3} = \frac{1}{8}$.

y-intercept: $\left(0, \frac{1}{8}\right)$

When $x = 1$, $y = 2^{1-3} = 2^{-2} = \frac{1}{4}$.

Point on curve is $\left(1, \frac{1}{4}\right)$.



Example 5F.4 Sketching the graph of a simple exponential relationship using reflections

On the same Cartesian plane, sketch the graphs of these exponential relationships. Label the y-intercept and asymptote for each graph.

a $y = 2^x$ **b** $y = -2^x$ **c** $y = 2^{-x}$

THINK

1 Identify the asymptote. The asymptote for $y = a^x$ is the horizontal line with equation $y = 0$.

2 Find the y-intercept.

3 Label another point on the curve with its coordinates. Find the value of y when $x = 1$.

4 Sketch the curve for an exponential relationship through the y-intercept and the labelled point, using the horizontal asymptote as a boundary. Label the asymptote and the y-intercept for each graph.

WRITE

a Horizontal asymptote: $y = 0$

When $x = 0$, $y = 2^0 = 1$.

y-intercept: (0, 1)

When $x = 1$, $y = 2^1 = 2$.

Point on curve is (1, 2).

b Horizontal asymptote: $y = 0$

When $x = 0$, $y = -2^0 = -1$.

y-intercept: (0, –1)

When $x = 1$, $y = -2^1 = -2$.

Point on curve is (1, –2).

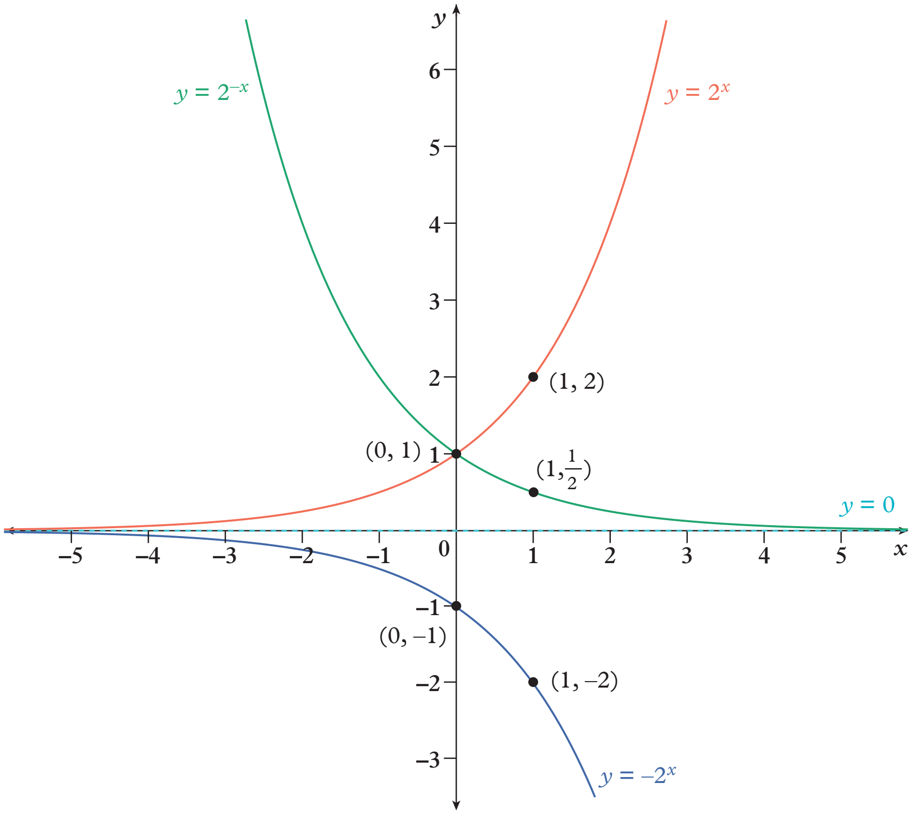
c Horizontal asymptote: $y = 0$

When $x = 0$, $y = 2^{-0} = 1$.

y-intercept: (0, 1)

When $x = 1$, $y = 2^{-1} = \frac{1}{2}$.

Point on curve is $\left(1, \frac{1}{2}\right)$.



Helpful hints

✔ Using your graphing calculator to alter the exponential function will help you to better understand the transformations. You can start by moving a negative sign to different places in the same relationship to increase or decrease values, or you can compare the graphs for the following equations:

- $y = -2^x$
- $y = 2^{-x}$
- $y = -2^{-x}$
- $y = 2^x + 1$
- $y = 2^x - 1$

✔ The graph of an exponential relationship never touches the asymptote line. It can get increasingly closer to the asymptote but can never cross or touch it.

✔ Remember that every exponential graph will have a y-asymptote and you must always include it, even when it is $y = 0$ (which will be on the x-axis).

ANS p709

Exercise 5F Exponential relationships

△ 1–11, 13, 15 □ 2–7, 9–12, 14, 16, 19 ⬠ 1(e, f), 3–7(d–f), 9, 10, 14, 17–19

5F.1 1 Sketch the graph of each of these exponential relationships. Label the y-intercept and asymptote each time.

a $y = 5^x$ **b** $y = 10^x$ **c** $y = 1.5^x$ **d** $y = 3^x$ **e** $y = 4^x$ **f** $y = 6^x$

2 **a** On the same Cartesian plane, draw graphs of $y = 2^x$ and $y = 3^x$ for x values from –3 to 3. You can use a table of values to help you plot the graphs, or use digital technology.

b Compare the two curves. How are they different?

c Which features of the two curves are the same?

d Write the y-intercept for each graph.

e Write the equation for the asymptote of each graph.

f Predict how the graph of $y = 5^x$ would compare to the two curves you have drawn.

3 Sketch the graph of each of these exponential relationships by performing a stretch.

a $y = 2 \times 4^x$ **b** $y = 5 \times 2^x$ **c** $y = 0.5 \times 6^x$

d $y = \frac{1}{4} \times 4^x$ **e** $y = 10 \times 3^x$ **f** $y = 1.5 \times 8^x$

5F.2 4 Sketch the graph of each of these exponential relationships by performing a vertical translation.

a $y = 3^x + 2$ **b** $y = 3^x - 3$ **c** $y = 5^x + 1$

d $y = 2^x - 4$ **e** $y = 6^x + 2$ **f** $y = 10^x - 1$

5F.3 5 Sketch the graph of each of these exponential relationships by performing a horizontal translation.

a $y = 3^{x-2}$ **b** $y = 3^{x+4}$ **c** $y = 5^{x-1}$ **d** $y = 2^{x+2}$ **e** $y = 4^{x-1}$ **f** $y = 7^{x+1}$

5F.4 6 Sketch the graph of each of these exponential relationships by performing a reflection.

a $y = -4^x$ **b** $y = -9^x$ **c** $y = -2.5^x$ **d** $y = -7^x$ **e** $y = -2^x$ **f** $y = -1.2^x$

7 Sketch the graph of each of these exponential relationships, showing all key features including the coordinates of a point on the curve other than the y-intercept.

a $y = 3^{-x}$ **b** $y = -3^x$ **c** $y = -3^{-x}$ **d** $y = 4^{-x}$ **e** $y = -5^x$ **f** $y = -6^{-x}$

8 On the same Cartesian plane, draw the graphs of $y = 2^x$ and $y = 2^{-x}$ for x values from –3 to 3. You can use a table of values to help you plot the graphs, or use digital technology.

a Compare the two curves. How are they different?

b Which features are the same?

c Write the y-intercept for each graph.

d Write the rule for the asymptote of each graph.

e Describe the transformation on the basic graph of $y = 2^x$ that produced the graph of $y = 2^{-x}$.

f What transformations are needed to produce the graph of $y = -2^{-x}$ from the graph of $y = 2^x$?

UNDERSTANDING AND FLUENCY

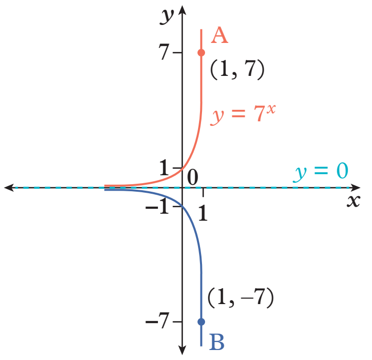
UNDERSTANDING AND FLUENCY

9 For each pair of graphs shown on the Cartesian planes below:

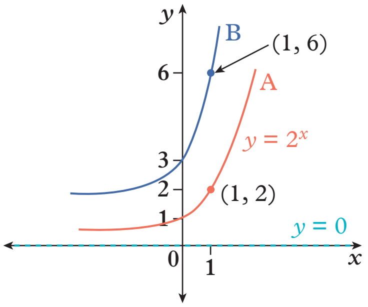
i identify the transformation performed on graph **A** to produce graph **B**

ii write the equation for graph **B**.

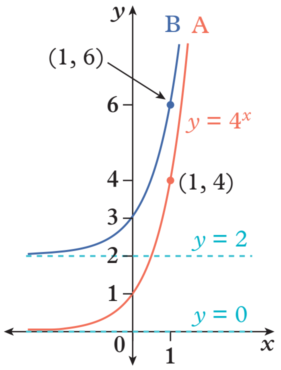
a



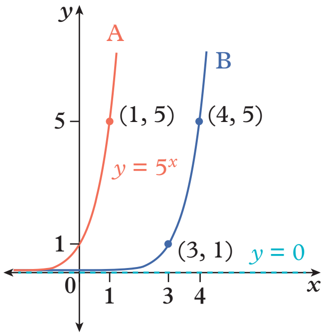
b



c



d



10 Sketch the graph of each of these equations.

a $y = 2^{x-1} + 3$ **b** $y = 3^{x+2} - 5$ **c** $y = 4^{x-3} + 7$

d $y = 3^{x-2} - 4$ **e** $y = 2^{x+4} + 1$ **f** $y = 5^{x+1} + 2$

11 Use digital technology to reproduce the first four graphs from question **10**. Compare those graphs to the ones you sketched.

12 What is the rule obtained when the graph of $y = a^x$ is reflected in:

a the x-axis? **b** the y-axis? **c** the x-axis and the y-axis?

PROBLEM SOLVING AND REASONING

13 Sienna invests $2000 at 10% p.a. interest compounded annually. The amount, A (in dollars), of the investment after n years can be represented by $A = 2000 \times (1.1)^n$.

a What will the graph of this relationship look like? Sketch the graph showing the key features.

b Describe what happens to the value of A as n increases.

c Produce the graph of $A = 2000 \times (1.1)^n$ for $0 \leq n \leq 10$, using a table of values or digital technology to help you. Use the graph to find:

  i the amount of the investment after 4 years

  ii the number of years it takes for the investment amount to be more than $4000.

14 The value of Gabriel's work tools depreciates at the rate of 20% p.a. compounded annually. The asset value, V (in dollars), after n years can be represented by $V = 3500 \times (1.25)^{-n}$.

a What will the graph of this relationship look like? Sketch the graph showing the key features.

b What was the initial value of Gabriel's tools?

c Describe what happens to the value of V as n increases.

d Produce the graph of $V = 3500 \times (1.25)^{-n}$ for $0 \leq n \leq 10$, using a table of values or digital technology to help you. Use the graph to find the number of years it takes for the asset value of the tools to be less than $1000.

15 A common mistake with exponential equations is writing $a \times b^x$ as $(ab)^x$. For example, incorrectly writing 3×2^x as 6^x.

- **a** Show that 3×2^x is not equivalent to 6^x by substituting $x = 2$ into each expression.
- **b** Find, by inspection or by trial and error, the value of x for which 3×2^x is equal to 6^x.
- **c** Considering your answers from parts **a** and **b**, explain the difference between the descriptions ‘equal to’ and ‘equivalent to’ in this context.
- **d** Considering your answers from parts **a** and **b**, explain why checking the value of one variable is not enough to decide whether two expressions are equivalent.

16 The equation $x^2 = 2^x$ has three solutions.

- **a** Complete the following table to find two of the solutions.

x	−2	−1	0	1	2	3	4	5
x^2								
2^x								

- **b** Use the table to explain why the third solution must be between −1 and 0.
- **c** Sketch the graphs of $y = x^2$ and $y = 2^x$ using appropriate technology, and find the value of the third solution to $x^2 = 2^x$, rounded to three decimal places.

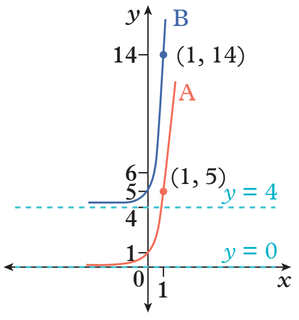
17 For exponential graphs, horizontal translations can be expressed as stretches in the y-direction, and vice versa. For example, using index laws, $y = 2^{x+1}$ can be written as $y = 2^x \times 2^1$, which simplifies to $y = 2 \times 2^x$.

- **a** Sketch $y = 2^{x+1}$ and $y = 2 \times 2^x$ using appropriate technology to confirm that they are the same graph.
- **b** Rewrite each of the following so the horizontal translation is expressed as a stretch in the y-direction.
  - **i** $y = 3^{x+2}$
  - **ii** $y = 2^{x+3}$
  - **iii** $y = 4^{x-1}$
  - **iv** $y = 4^{x+0.5}$
- **c** Rewrite each of the following so the stretch in the y-direction is expressed as a horizontal translation.
  - **i** $y = 3 \times 3^x$
  - **ii** $y = 16 \times 4^x$
  - **iii** $y = \frac{1}{8} \times 2^x$
  - **iv** $y = 8 \times 4^x$

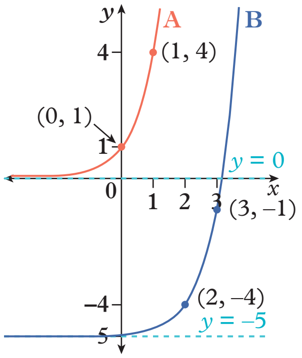
18 For each pair of graphs shown on the Cartesian planes below:

- **i** identify the transformations performed on graph **A** to produce graph **B**
- **ii** write the rules for graphs **A** and **B**.

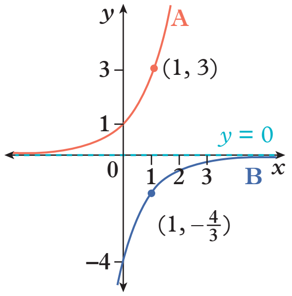
a



b



c



19 Describe the transformations performed on the graph of $y = 2^x$ to produce the graph of $y = -3 \times 2^{x+1} - 2$. Sketch the graph.

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Interactive skillsheet
Exponential relationships

Investigation
The exponential growth of one grain of rice

Topic quiz
5F

5G Hyperbolas

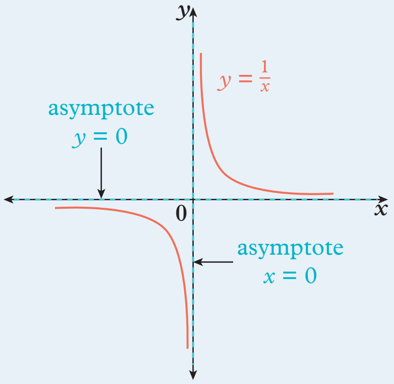
Learning intentions

- ✔ I can sketch simple hyperbolas of the form $y = \frac{a}{x}$.
- ✔ I can sketch hyperbolas using reflections and stretches.
- ✔ I can sketch hyperbolas using vertical translations and horizontal translations.

Inter-year links

Year 7	8D Translations
Year 8	7C Transformations
Year 9	5E Circles and other non-linear relationships

Hyperbolas



- The basic form of a simple **rectangular hyperbola** is the graph of the rule $y = \frac{1}{x}$. The graph has two asymptotes and no x- or y-intercepts.
- The asymptotes for $y = \frac{1}{x}$ lie along the x-axis and along the y-axis; that is, they are the lines whose equations are $y = 0$ and $x = 0$.
- The general equation of a hyperbola is: $y = \frac{a}{x - h} + k$ where the graph has asymptotes at $x = h$ and $y = k$.

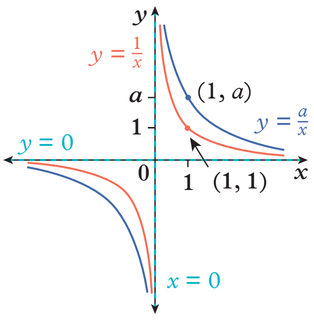
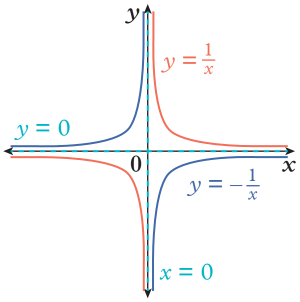
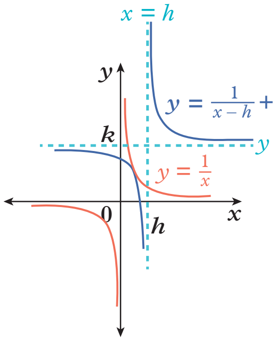
Stretch in the y-direction
If $a > 0$, upright
If $a < 0$, inverted

$$y = \frac{a}{x - h} + k$$

Vertical translation of k units
If $k > 0$, up
If $k < 0$, down

Horizontal translation of h units
If $h > 0$, right
If $h < 0$, left

- The following transformations can be performed on the graph of $y = \frac{1}{x}$ to produce a sketch of $y = \frac{a}{x - h} + k$:
  - → a stretch in the y-direction as a increases
  - → a reflection in the x-axis if $a < 0$
  - → a horizontal translation of h units right or left
  - → a vertical translation of k units up or down.

Stretch	Reflection	Translation
		

- When finding the domain and range of a hyperbolic relationship, remember that the denominator of a fraction cannot equal zero.

Example 5G.1 Sketching the graph of a simple hyperbola using stretching transformations

Perform transformations to sketch the graph of each hyperbola below on the same Cartesian plane. Label the asymptotes.

a $y = \frac{1}{x}$ **b** $y = \frac{3}{x}$ **c** $y = \frac{6}{x}$

THINK

1 Identify the asymptote. The asymptotes for $y = \frac{a}{x}$ are the horizontal line with equation $y = 0$ and the vertical line with equation $x = 0$.

2 Identify any intercepts.

3 To gain an idea of the scale on your sketch graph, label another point on the curve with its coordinates. Find the value of y when $x = 1$.

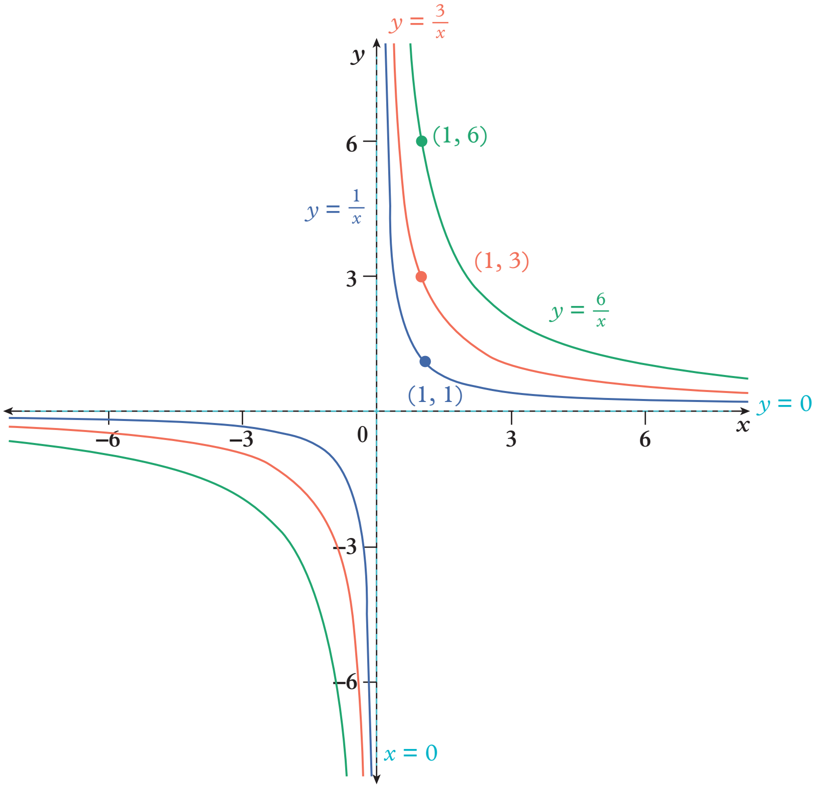
4 Sketch the basic curve for a rectangular hyperbola using the asymptotes as boundaries. Label a known point on the curve to indicate that a stretch has occurred.

WRITE

a Horizontal asymptote: $y = 0$
Vertical asymptote: $x = 0$
No x- or y-intercepts
When $x = 1$, the point on the curve is (1, 1).

b Horizontal asymptote: $y = 0$
Vertical asymptote: $x = 0$
No x- or y-intercepts
When $x = 1$, the point on the curve is (1, 3).

c Horizontal asymptote: $y = 0$
Vertical asymptote: $x = 0$
No x- or y-intercepts
When $x = 1$, the point on the curve is (1, 6).



Example 5G.2 Sketching the graph of a hyperbola using translations

Sketch the graph of $y = \frac{1}{x + 2} - 4$. Clearly show the asymptotes and intercepts.

THINK

1 Identify the translations that need to be performed on $y = \frac{1}{x}$ (translate 2 units left and 4 units down). Compare to $y = \frac{a}{x - h} + k$ to identify the translations: $h = -2$ (move 2 units left) and $k = -4$ (move 4 units down).

WRITE

$$y = \frac{1}{x \boxed{+ 2}} \boxed{- 4}$$

Horizontal translation of two units left

Vertical translation of four units down

2 Identify the rule for each asymptote: $x = h$ and $y = k$.

The vertical asymptote is $x = -2$.

The horizontal asymptote is $y = -4$.

3 Find the x-intercept.

When $y = 0$:

$$\begin{aligned} 0 &= \frac{1}{x + 2} - 4 \\ \frac{1}{x + 2} &= 4 \\ x + 2 &= \frac{1}{4} \\ x &= \frac{1}{4} - 2 \\ &= -1\frac{3}{4} \end{aligned}$$

The x-intercept is $\left(-1\frac{3}{4}, 0\right)$.

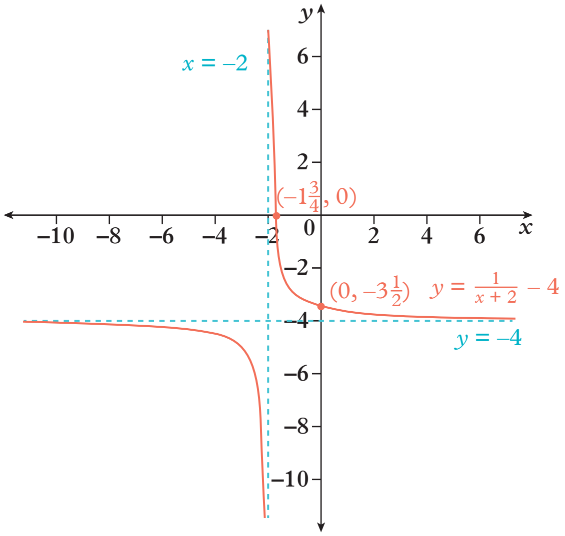
4 Find the y-intercept.

When $x = 0$:

$$\begin{aligned} y &= \frac{1}{0 + 2} - 4 \\ &= \frac{1}{2} - 4 \\ &= -3\frac{1}{2} \end{aligned}$$

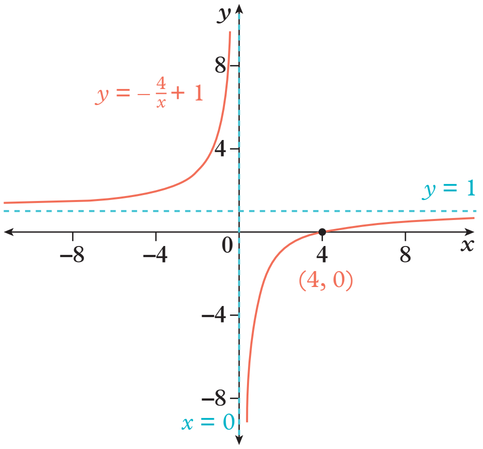
The y-intercept is $\left(0, -3\frac{1}{2}\right)$.

5 Sketch the basic curves for a hyperbola using the asymptotes as boundaries. Label the intercepts and asymptotes.



Example 5G.3 Sketching the graph of a hyperbola using reflections

Sketch the graph of $y = -\frac{4}{x} + 1$. Clearly show the asymptotes and intercepts.

THINK	**WRITE**
1 Identify the three translations that need to be performed on $y = \frac{1}{x}$: reflection in the x-axis; $k = 1$ (move 1 unit up); and a stretch in the y-direction by a factor of 4.	Stretch in the y-direction by a factor of 4 $y = \textcircled{-}\frac{\textcircled{4}}{x} + \textcircled{1}$ Reflection in the x-axis Vertical translation of one unit up
2 Identify the rule for each asymptote: $x = 0$ and $y = k$.	The vertical asymptote is $x = 0$. The horizontal asymptote is $y = 1$.
3 Find the x-intercept.	For the x-intercept, $y = 0$: $y = -\frac{4}{x} + 1$ $0 = -\frac{4}{x} + 1$ $\frac{4}{x} = 1$ $x = 4$ The x-intercept is $(4, 0)$.
4 Find the y-intercept.	There is no y-intercept.
5 Sketch the basic curves for a hyperbola using the asymptotes as boundaries. Label the intercepts and asymptotes.	

Helpful hints

- ✔ Be careful not to interchange h and k. The h values refers to the horizontal translation, and the k value refers to the vertical translation. Think 'h for horizontal' to help you remember.
- ✔ Don't forget to sketch the x- and y-asymptotes, even when they are on an axis.
- ✔ The coordinates (h, k) give the point of intersection for the two asymptotes of the hyperbola.
- ✔ When sketching a hyperbola, start by drawing the asymptotes.

ANS p714

Exercise 5G Hyperbolas

△ 1–9, 11, 12 □ 1–9, 11, 13, 15(a, b) ⬠ 1(d, e), 2, 4–10, 14, 15(c), 16

UNDERSTANDING AND FLUENCY

5G.1 **1** Sketch the graph of each of these hyperbolas. Label the asymptotes.

a $y = \frac{4}{x}$ **b** $y = \frac{7}{x}$ **c** $y = \frac{2}{x}$ **d** $y = \frac{5}{x}$ **e** $y = \frac{10}{x}$

5G.2 **2** Sketch the graph of each of these relationships by performing a vertical translation. Clearly show the asymptotes and intercepts.

a $y = \frac{1}{x} + 4$ **b** $y = \frac{1}{x} - 7$ **c** $y = \frac{1}{x} + 5$ **d** $y = \frac{1}{x} - 1$

3 Sketch the graph of each of these relationships by performing a horizontal translation. Clearly show the asymptotes and intercepts.

a $y = \frac{1}{x-5}$ **b** $y = \frac{1}{x+4}$ **c** $y = \frac{1}{x-7}$ **d** $y = \frac{1}{x+1}$

4 Sketch the graph of each of these equations. Clearly show the asymptotes and intercepts.

a $y = \frac{1}{x-3} - 2$ **b** $y = \frac{1}{x+5} + 3$ **c** $y = \frac{1}{x-2} + 4$ **d** $y = \frac{1}{x+4} - 5$ **e** $y = \frac{1}{x-1} - 8$

5 Use digital technology to produce the first four graphs from question **4**. Compare them to your sketches.

5G.3 **6** Sketch the graph for each equation below, clearly showing the asymptotes and intercepts.

a $y = -\frac{1}{x} + 3$ **b** $y = -\frac{2}{x}$ **c** $y = -\frac{1}{x-2}$ **d** $y = -\frac{5}{x}$ **e** $y = -\frac{1}{x-2} + 3$

7 Use transformations to help you sketch the graph of each of these equations. Clearly show the asymptotes and intercepts. List the domain and range for each graph.

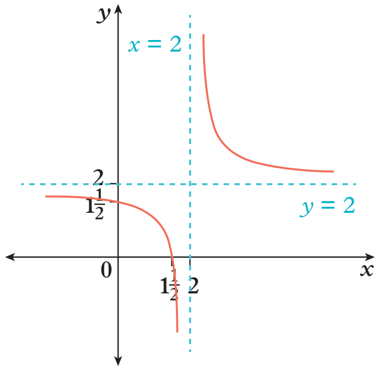
a $y = -\frac{1}{x+4}$ **b** $y = -\frac{1}{x} - 2$ **c** $y = -\frac{1}{x-3} + 5$ **d** $y = -\frac{1}{x+5} - 1$ **e** $y = -\frac{2}{x-1} - 4$

8 Use digital technology to produce the first four graphs from question **7**. Compare them to the ones you sketched.

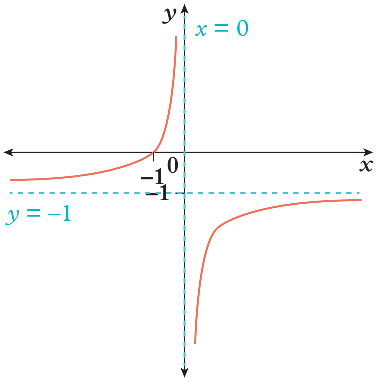
PROBLEM SOLVING AND REASONING

9 Write the equation for each hyperbola shown below.

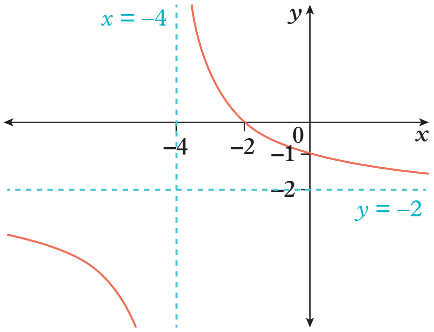
a



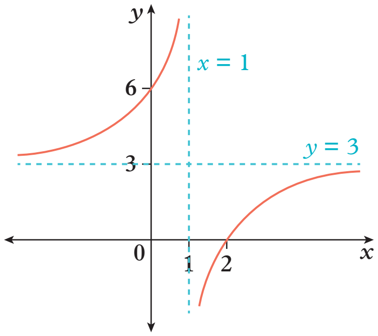
b



c



d



PROBLEM SOLVING AND REASONING

10 Why does reflecting the graph of $y = \frac{1}{x}$ in the x-axis produce the same graph as reflecting $y = \frac{1}{x}$ in the y-axis?

11 One way to describe the graph of $y = \frac{1}{x}$ is to describe the behaviour of the curve as x approaches very large or very small values. Complete each statement below using a term from this list:

- positive infinity
- zero
- negative infinity

a When x approaches infinity, y approaches ____________.

b When x approaches zero from the right, y approaches ____________.

c When x approaches zero from the left, y approaches ____________.

d When x approaches negative infinity, y approaches ____________.

12 At a bank, the average waiting time for customers in a queue is related to the number of bank tellers on duty and can be modelled by the relationship $W_n = \frac{12}{n}$, where W_n is the average waiting time in minutes when there are n bank tellers on duty.

a Sketch a graph of this relationship.

b On average, how long does a customer wait when the number of bank tellers on duty is:

i 2? **ii** 3? **iii** 4? **iv** 6? **v** 12? **vi** 24?

c What is the maximum waiting time for a customer? When does this occur?

d According to this model, will there be occasions when the waiting time is zero? Explain.

13 The graph of a hyperbola will always have an x-asymptote and a y-asymptote.

Consider the hyperbola $y = \frac{1}{x - 2} + 3$, which has asymptotes at $x = 2$ and $y = 3$.

a Substitute $x = 2$ into the equation and explain why this causes the y-value to be undefined.

b Explain why the error from part **a** will not occur for any other value of x that is not 2.

c Substitute $y = 3$ into the equation and attempt to solve it. What goes wrong when you attempt to solve it for x?

d Explain why the error from part **c** will not occur for any other value of y that is not 3.

CHALLENGE

14 For the graphs of each pair of relationships below:

i identify the number of points of intersection by sketching their graphs

ii use an algebraic method to find the points of intersection.

a $y = \frac{1}{x}$ and $y = x$ **b** $y = \frac{1}{x}$ and $y = -x$

c $y = \frac{6}{x + 2}$ and $y = x + 3$ **d** $y = \frac{1}{x - 4}$ and $y = -\frac{1}{x}$

15 a Find the rule for a hyperbola of the form $y = \frac{8}{x - h}$ that passes through (6, 2).

b Find the rule for a hyperbola of the form $y = \frac{3}{x - h} + 2$ that passes through (–2, 1).

c Find the rule for a hyperbola of the form $y = \frac{a}{x - h}$ that passes through (–7, –1) and (1, 1).

16 Sketch the graphs of each of the following. Remember that the x-asymptote occurs for the value of x that would cause the denominator to equal zero.

a $y = \frac{1}{2x - 1}$ **b** $y = \frac{1}{1 - x} + 2$ **c** $y = \frac{-2}{1 - 2x} - 1$

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Interactive skillsheet
Hyperbolas

Investigation
Inverse proportion

Topic quiz
5G

5H Direct and inverse proportion

Learning intentions

- ✔ I can identify when two variables are directly proportional to each other.
- ✔ I can identify when two variables are inversely proportional to each other.
- ✔ I can solve modelling problems involving inverse proportion and solving related equations.

Inter-year links

Year 9 4G Direct proportion

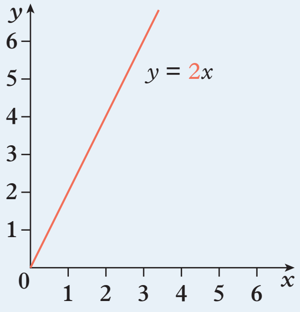
Direct proportion

- y is said to be **directly proportional** to x if:
  - → $y = 0$ when $x = 0$
  - → y increases as x increases
  - → the rate of change of y with respect to x is constant.
- Proportionality is denoted using the symbol '$\propto$'.
- If $y \propto x$:
  - → the equation for the relationship between x and y is $y = kx$ where k, the **constant of proportionality**, is equal to the rate of change of y with respect to x

$$y = kx$$

constant of proportionality
= rate of change
= gradient

  - → the graph of y against x is a straight line with a gradient of k that passes through the origin, (0, 0).
- If $y \propto x$, then $x \propto y$.



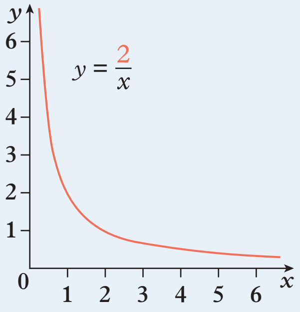
Inverse proportion

- y is said to be **inversely proportional** to x if:
  - → y increases as x decreases at a proportional rate; $y \propto \frac{1}{x}$
  - → the product of x and y is constant.
- If $y \propto \frac{1}{x}$:
  - → the equation for the inverse relationship between x and y is $y = \frac{k}{x}$

$$y = \frac{k}{x}$$

constant of proportionality

  - → the graph of y versus $\frac{1}{x}$ is a hyperbola with asymptotes of $x = 0$ and $y = 0$.
- If $y \propto \frac{1}{x}$, then $x \propto \frac{1}{y}$.
- If y is inversely proportional to x then y is directly proportional to $\frac{1}{x}$.



Example 5H.1 Finding the constant of proportionality for direct proportion

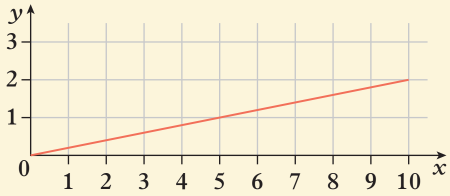
If $y \propto x$, find the constant of proportionality using the information given about each relationship below.

a When $x = 3$, $y = 18$.

b

x	0	2	4	6	8
y	0	24	48	72	96

c



THINK

a If $y \propto x$, then $y = kx$. Substitute the given values into $y = kx$ and solve the equation for k.

b Select any pair of values in the table and substitute the corresponding values of x and y into $y = kx$, then solve the equation for k.

c Because the graph passes through the origin, (0, 0), to find the gradient, k, select any point on the graph, substitute the corresponding values of x and y into $y = kx$, then solve the equation for k.

WRITE

a

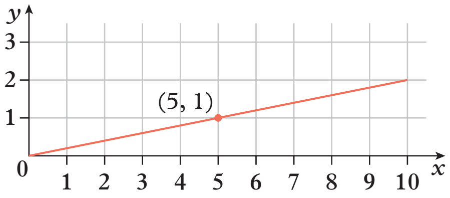
$$y = kx$$
$$18 = k \times 3$$
$$k = 6$$

b

x	0	2	4	6	8
y	0	24	48	72	96

$$y = kx$$
$$24 = k \times 2$$
$$k = 12$$

c



$$y = kx$$
$$1 = k \times 5$$
$$k = \frac{1}{5}$$

Example 5H.2 Finding the constant of proportionality for inverse proportion

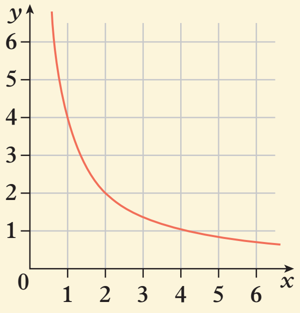
If $y \propto \frac{1}{x}$, use the information given about each relationship below to find the constant of proportionality.

a When $x = 4$, $y = 3$.

b

x	3	4	5	6	7
y	2.3	1.75	1.4	1.17	1

c



THINK

a If $y \propto \frac{1}{x}$, then $y = \frac{k}{x}$. Substitute the given values into $y = \frac{k}{x}$ and solve the equation for k.

b Select any pair of values in the table and substitute the corresponding values of x and y into $y = \frac{k}{x}$, then solve the equation for k.

c To find the constant, k, select any point on the graph, substitute the corresponding values of x and y into $y = \frac{k}{4}$, then solve the equation for k.

WRITE

a

$$y = \frac{k}{x}$$

$$y = \frac{k}{4}$$

$$3 = \frac{k}{4}$$

$$k = 12$$

b

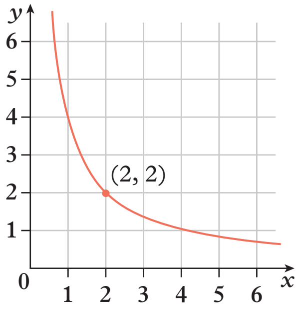
x	3	4	5	6	7
y	2.3	1.75	1.4	1.17	1

$$y = \frac{k}{x}$$

$$1 = \frac{k}{7}$$

$$k = 7$$

c



$$y = \frac{k}{x}$$

$$2 = \frac{k}{2}$$

$$k = 4$$

Helpful hints

- ✔ Direct proportion is an example of linear algebra and linear graphs. Remember that for the relationship $y = kx$:

  k = constant of proportionality = rate of change = gradient
- ✔ For a relationship to be directly or inversely proportional, the constant of proportionality, k, must be a constant, but it can still be fractional or irrational.

ANS p718

Exercise 5H Direct and inverse proportion

△ 1–8, 10 □ 1–7, 9, 11–13 ⬠ 1–7, 11, 13–15

5H.1 1 If $y \propto x$, find the constant of proportionality for each of these relationships, using the information given.

a $y = 12$ when $x = 3$

b

x	1	2	3	4
y	9	18	27	36

c $y = 9$ when $x = 6$

d

x	0.5	1	1.5	2
y	2.5	5	7.5	10

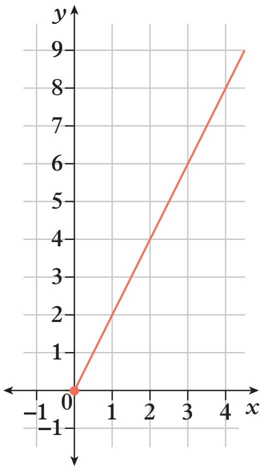
e $y = 10$ when $x = 20$

f

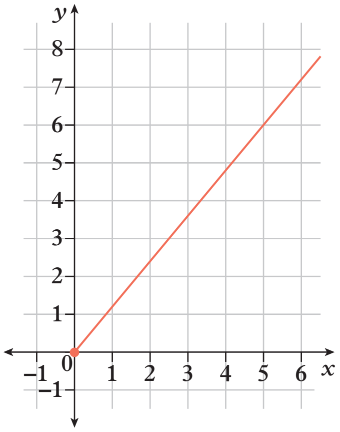
x	5	10	15	20
y	2	4	6	8

2 For each of the following graphs, $y \propto x$. Find the constant of proportionality.

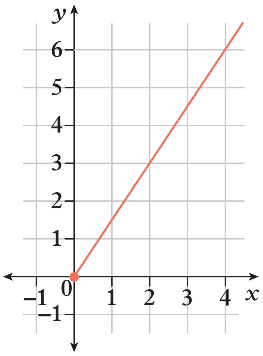
a



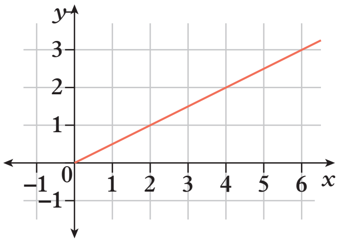
b



c



d



UNDERSTANDING AND FLUENCY

5H.2 **3** If $y \propto \frac{1}{x}$, use the information given to find the constant of proportionality for each of the following relationships.

a $y = 1$ when $x = 6$

b

x	2	4	6	8
y	6	3	2	1.5

c $y = 0.4$ when $x = 10$

d

x	2	3	4	5
y	3	2	1.5	1.2

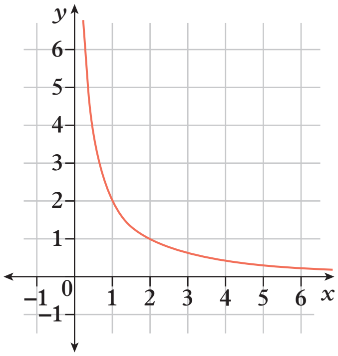
e $y = 3$ when $x = 5$

f

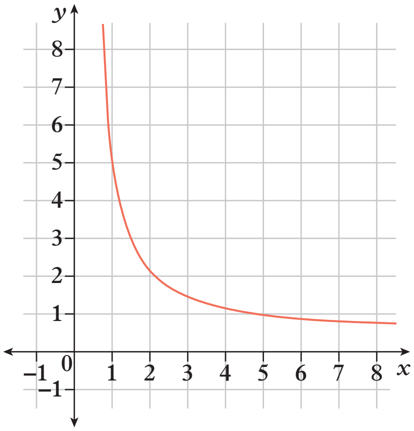
x	2	5	10	20
y	0.5	0.2	0.1	0.05

4 For each of the following graphs, $y \propto \frac{1}{x}$. Find the constant of proportionality.

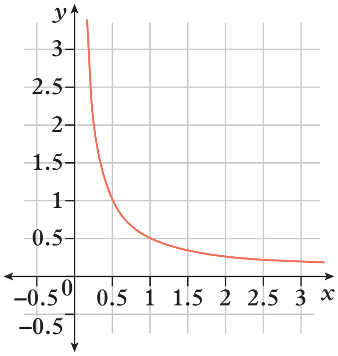
a



b



c



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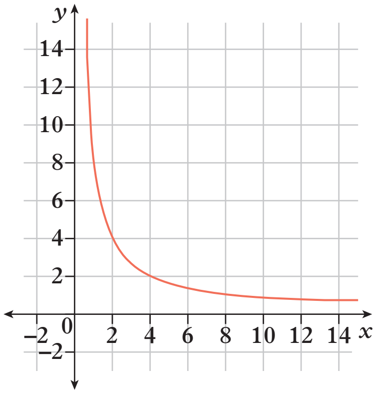


5 Decide whether each of the following relationships are directly proportional, inversely proportional, or neither.

a The height of a tree and the length of its shadow

b The cost of hiring a plumber who charges $70 per hour plus a $90 call out fee

c The time taken to travel a fixed distance and the speed of travel (assuming the speed is constant)

d The area of a square and its side length

e The height of water in a cylindrical glass and the volume of the water

f The number of people paying a bill and the cost per person

6 Consider the incomplete table on the right, which shows a proportional relationship.

x	0.5	1	2	4	8
y	8	4	2		

a Is the relationship directly proportional or inversely proportional?

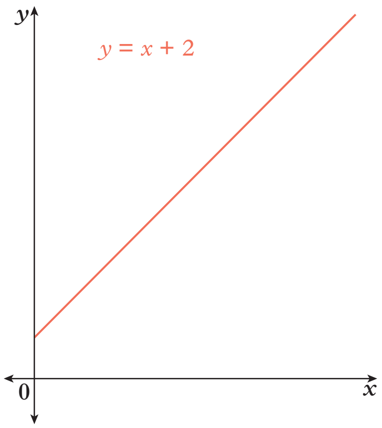
b Find the constant of proportionality.

c Calculate the two y values missing from the table.

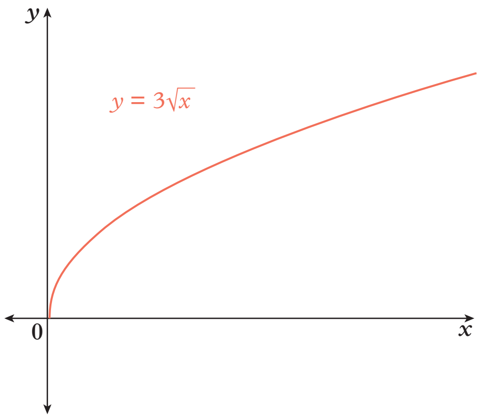
d Plot the points from the table on a graph and draw a smooth curve to join the points.

7 For each of the following graphs, give a reason that the relationship is not directly proportional.

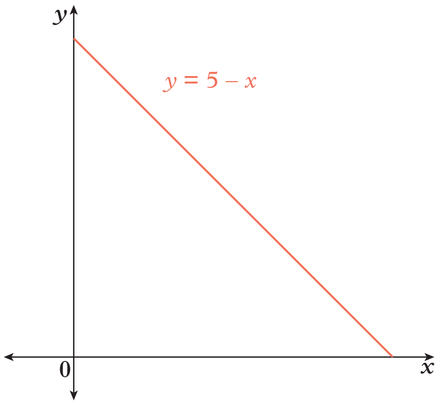
a



b



c



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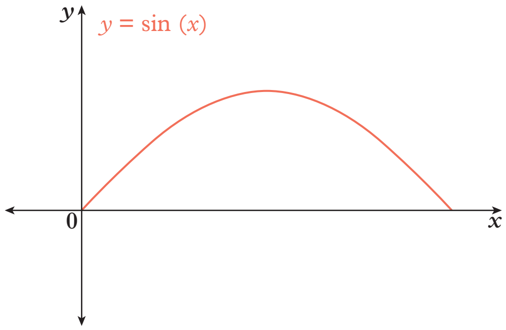


8 An employee at a fast-food restaurant is paid an amount directly proportional to the number of hours she works. This week she worked 16 hours and was paid \$376.

a Find her hourly rate of pay.

b If she works 22 hours next week, how much will she be paid?

9 On Friday, a television was on for 4 hours and used 560 watts of electricity. Let W be the number of watts used and let t be the number of hours the television was on.

a If $W \propto t$, then find the constant of proportionality.

b If the television was on for 6 hours on Saturday, how many watts of electricity did it use?

c If the television used 245 watts on a Sunday, for how many hours was the television on?

10 The number of days it takes to complete a construction job is inversely proportional to the number of people working on the job. If it would take 12 days for 5 workers to complete the job, then how many days would it take for 8 workers to complete the job?

11 Consider the following relationship between x and y.

x	0.5	1	1.5	2
y	12	6	4	3

a If $y \propto \frac{1}{x}$, find the constant of proportionality.

b If $x \propto \frac{1}{y}$, find the constant of proportionality.

c Your answers for parts **a** and **b** should be the same. Will this always be the case for inversely proportional relationships, or were these answers just a coincidence?

12 To attend a formal, a group of Year 10 students are considering hiring a limousine, which costs a fixed amount regardless of how many passengers ride in the limousine. The rental company tells the students that, if 5 of them hire the limousine, it will cost \$34.20 per person.

a What is the total cost of 5 people hiring the limousine for the night?

b What is the constant of proportionality for $C \propto \frac{1}{n}$, where C is the cost per person and n is the number of people?

c Explain why the total cost and the constant of proportionality for $C \propto \frac{1}{n}$ will always be the same for situations like this.

13 The circumference of a circle is directly proportional to the diameter of the circle. Malachi tries to find the constant of proportionality for $C \propto d$ by measuring, as best he can, the circumferences and diameters of several circles. His measurements, in centimetres, are shown in the table below.

d	1	2	3	4	5	6	7
C	3.2	6.2	9.3	13	16.2	18.9	21.7

a For each circle, find the constant of proportionality for $C \propto d$, as a decimal.

b Since there are several different values, Malachi estimates the constant of proportionality by finding the average of the seven ratios. What is his estimation of the constant? Round your answer to two decimal places.

c What famous constant is Malachi trying to approximate? Is he close?

14 The brightness of light decreases inversely proportionally to the square of the distance from the light source. This means that $I \propto \frac{1}{d^2}$ where I is a measure of light intensity (in watts per square metre) and d is the distance (in metres) from the light source at which the measurement is taken.

Consider a lamp with a light intensity of 1 .2 W/m^2 measured at a distance of 2 metres from the lamp.

a Find the constant of proportionality.

b Find:

- **i** the light intensity at a distance of 5 metres from the lamp
- **ii** the distance from the lamp, to the nearest centimetre, if the light intensity is 100 W/m^2.

c By what percentage does the intensity of the light on an object decrease when the object is moved twice as far from the lamp?

15 a Prove that if $x \propto y$ and $y \propto z$, then $x \propto z$.

b Prove that if $a \propto \frac{1}{b}$ and $b \propto \frac{1}{c}$, then $a \propto c$.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Direct proportion

Interactive skillsheet
Inverse proportion

Investigation
Modelling problems involving inverse proportion

Topic quiz
5H

5I Sketching non-linear graphs using transformations

Learning intentions

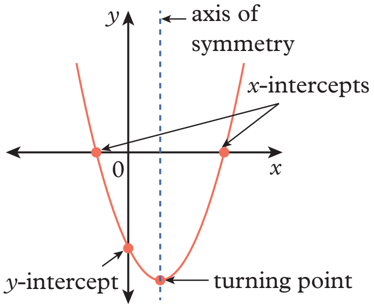
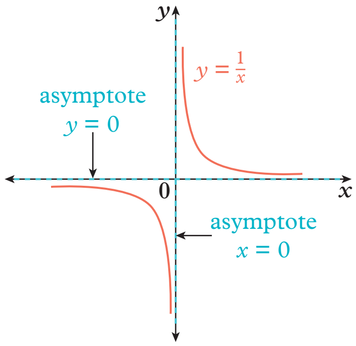
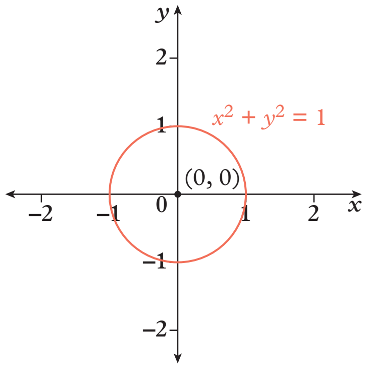
- ✔ I can describe the transformations of non-linear graphs.
- ✔ I can interpret and describe a scenario using a non-linear relationship.

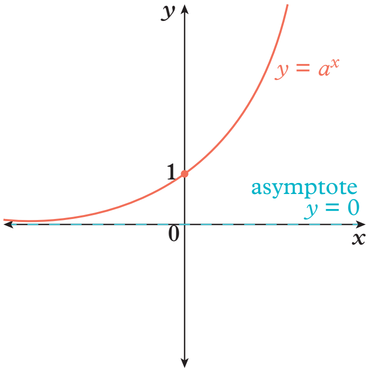
Inter-year links

Years 5/6	Transformations
Year 7	8E Rotations and reflections
Year 8	7C Transformations
Year 9	5D Sketching parabolas using transformations

Sketching graphs using transformations

- Transformations such as stretches, reflections and translations can be performed on the basic graph of many non-linear relationships to produce graphs with the general rules shown in the following table.

Name	Graph	General rule
Parabola		Horizontal translation of h units If $h > 0$, right If $h < 0$, left Vertical translation of k units If $k > 0$, up If $k < 0$, down $y = a(x - h)^2 + k$ Stretch in the y-direction If $a > 0$, vertical stretch up But, if $0 < a < 1$, horizontal stretch outward
Hyperbola		Stretch in the y-direction If $a > 0$, upright If $a < 0$, inverted $y = \frac{a}{x-h} + k$ Vertical translation of k units If $k > 0$, up If $k < 0$, down Horizontal translation of h units If $h > 0$, right If $h < 0$, left
Circle		Vertical translation of k units If $k > 0$, up If $k < 0$, down Radius of r units $(x - h)^2 + (y - k)^2 = r^2$ Horizontal translation of h units If $h > 0$, right If $h < 0$, left

Name	Graph	General rule
Exponential relationship		Horizontal translation of h units If $h > 0$, right If $h < 0$, left If $a > 1$, upright $y = B \times a^{x-h} + k$ Vertical translation of k units If $k > 0$, up If $k < 0$, down If $B > 0$, upright If $B < 0$, inverted

Example 5I.1 Identifying the rule for a non-linear relationship

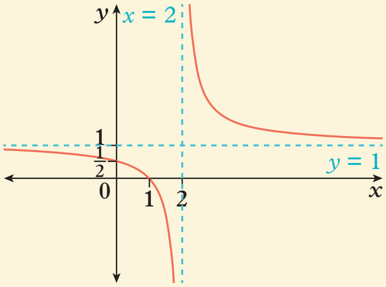
For the graph shown on the right:

a describe the relationship

b identify the basic graph used to produce the graph of this relationship

c identify the transformations that have been performed on the basic graph

d write the equation for the relationship shown.



THINK	WRITE
a Identify the shape. This graph is hyperbolic.	**a** The graph is a hyperbola.
b Use the shape of the graph to identify the equation of the basic type of relationship it represents.	**b** $y = \frac{1}{x}$
c **1** Check if a reflection has been performed.	**c** There has been no reflection, because the curves are in the same quadrants as the basic graph.
2 Identify the translations performed.	There is an asymptote at $y = 1$ so there has been a translation of 1 unit up. There is an asymptote at $x = 2$ so there has been a translation of 2 units right.
3 Check for stretches by substituting known x and y values.	The rule is of the form: $y = \frac{a}{x-2} + 1$ For $x = 1$, $y = 0$: $0 = \frac{a}{1-2} + 1$ $\frac{a}{-1} = -1$ $a = 1$ There has been no stretch.
d Use your answers for parts **b** and **c** to help you write the equation of the graph.	**d** $y = \frac{1}{x-2} + 1$

Helpful hints

- ✔ You may have noticed that the constant k, relating to vertical translations, is **subtracted** in the general rule for a circle, but **added** in the general rule for other graphs. In all cases, a vertical translation of k units is achieved by substituting y for $y - k$. The only difference is how the equations are rearranged.
- ✔ It is a good idea to complete the stretch and reflection transformations before doing any translations, because translating and then stretching or reflecting could affect the final result of the translations.

ANS p718

Exercise 5I Sketching non-linear graphs using transformations

▲ 1–3, 6, 7 ■ 1–4, 7–10, 13 ⬟ 1, 2, 4, 5, 7, 9, 11–13

5I.1 **1** For each of the following graphs:

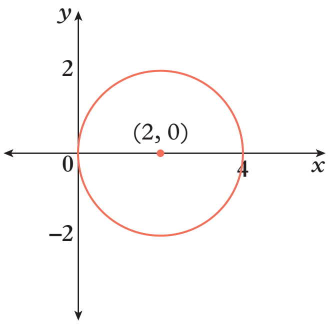
i describe the relationship

ii identify the basic graph used to produce the graph of the relationship

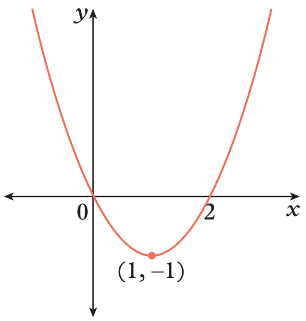
iii identify the transformations that have been performed on the basic graph to produce the graph shown

iv write the rule for the relationship.

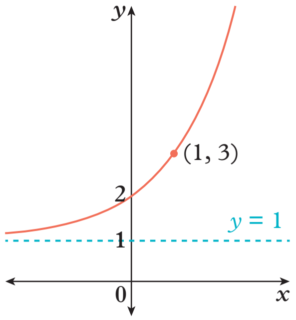
a



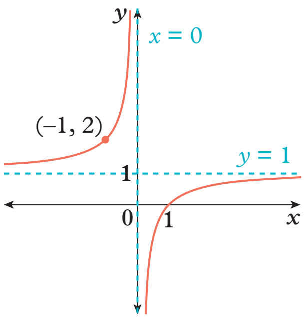
b



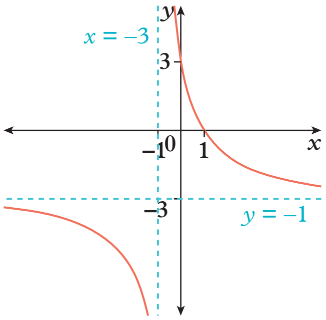
c



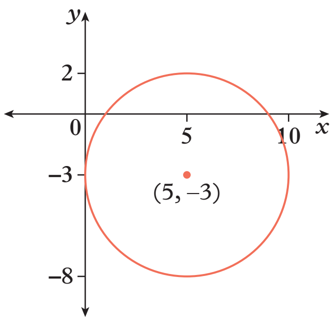
d



e

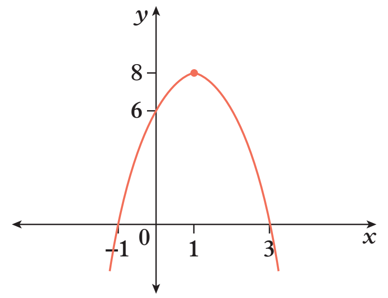


f



UNDERSTANDING AND FLUENCY

g



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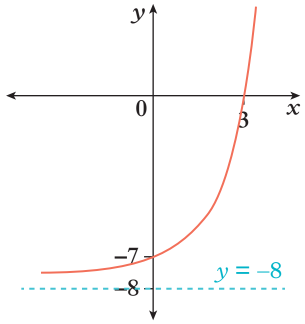


2 For each equation below:

- **i** describe the type of non-linear relationship it represents
- **ii** identify the transformations to be performed on the basic graph to produce the graph of the relationship
- **iii** list key features, including intercepts
- **iv** use transformations to sketch the graph, showing all key features.

a $y = -(x + 3)^2 - 5$

b $y = 3^{x+2} - 1$

c $(x - 4)^2 + (y + 2)^2 = 9$

d $y = \dfrac{1}{x - 6} + 2$

e $y = 3(x - 2)^2 - 12$

f $y = -2^x + 1$

PROBLEM SOLVING AND REASONING

3 The area of grass watered by a sprinkler forms a circle of radius 4 m.

a Write an equation for the relationship that describes this circle if the sprinkler is located at the origin of a Cartesian plane.

b If the sprinkler is moved 5 m north (up) and 7 m west (left) of its original position on the Cartesian plane, write a new equation for the circle that encloses the area being watered.

c Does the new area overlap with the original area? Explain.

4 The amount of time it takes to move a distance of 100 m depends on the average speed of travel.

a Write a rule for the relationship between time t (in seconds, s) and average speed s (in metres per second, m/s) over a distance of 100 m. (Hint: in this situation, the time is dependent on the speed, so t will be the dependent variable on the right-hand side of the equation).

b What type of graph would show this relationship?

c Sketch a graph of this relationship. (Hint: remember to place the dependent variable on the vertical axis.)

d Interpret your graph from part **c** to describe how the amount of time changes when the speed is increased.

5 Explain how the graph of $y = -\frac{1}{3}(x - 4.6)^2 + 7.1$ can relate to the shape of an entrance to a railway tunnel if drawn on the Cartesian plane with x and y as distances in metres. Find the dimensions of the tunnel entrance, correct to one decimal place.

6 Investigate the graphs of $y = 2^{-x}$ and $y = \left(\frac{1}{2}\right)^x$ using technology. What do you notice? Explain.

7 There are many other non-linear relationships. Consider a relationship involving the square root of x.

a Complete the table of values on the right for the relationship, $y = \sqrt{x}$.

x	0	1	4	9	16	25
y	0			3		

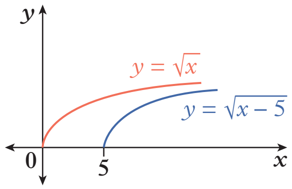
b Plot the points from the table on a Cartesian plane and sketch the graph by drawing a smooth line through each point.

c Describe the shape of the graph you drew for part **b**. Can you plot points for negative values of x? Explain.

8 Graphs of $y = \sqrt{x}$ and $y = \sqrt{x-5}$ are shown on the right.

a On the graph of $y = \sqrt{x-5}$, identify the coordinates of the point where y is a minimum.

b Describe how you could use a translation to produce the graph of $y = \sqrt{x-5}$ from the graph of $y = \sqrt{x}$.



9 For each relationship below:

i use your understanding of translations to describe how its graph can be produced from the graph of $y = \sqrt{x}$

ii identify the end point of the curve

iii sketch the graph

iv use a calculator or other digital technology to produce the same graph.

a $y = \sqrt{x} + 2$ **b** $y = \sqrt{x} - 1$ **c** $y = \sqrt{x-1}$

d $y = \sqrt{x+4}$ **e** $y = \sqrt{x-2} + 3$ **f** $y = -\sqrt{x+3} - 2$

CHALLENGE

10 On the same Cartesian plane, sketch the graphs of $y = -\sqrt{x}$, $y = -\sqrt{x} + 3$ and $y = -\sqrt{x-4}$. Show the shape of each graph and its end point only.

11 Use transformations to sketch the graph of each of these relationships. Show all key features including asymptotes and intercepts (to one decimal place, where necessary).

a $y = -3 \times 2^{x-1} + 7$ **b** $y = -2(x+3)^2 + 5$ **c** $y = \sqrt{x-4} + 3$

12 When multiple transformations are performed on a single graph, the order is very important. For example, stretching the graph of $y = x^2$ by a factor of 2 in the y-direction and then translating it 1 unit up results in the graph of $y = 2x^2 + 1$. However, translating the graph of $y = x^2$ up by 1 unit and then stretching it by a factor of 2 in the y-direction results in the graph of $y = 2x^2 + 2$.

a Find the equation of the graph that results from applying the following transformations to the graph of $y = x^2$, in the given order.

i stretching by a factor of $\frac{1}{2}$ in the y-direction, then translating up by 2 units

ii translating down by 3 units, then stretching by a factor of 2 in the y-direction

b Find the equation of the graph that results from applying the following transformations, in order, to the graph of $y = x^2$.

i reflecting in the x-axis, then translating up by 3 units

ii translating up by 3 units, then reflecting in the x-axis

13 a Track the position of the point $A(1, 2)$ as it is transformed by the following sequence of transformations. What is the final position of the transformed point?

1 reflected in the x-axis
2 translated up by 3 units
3 translated left by 4 units
4 stretched by a factor of 2 in the y-direction
5 translated right by 4 units

b Find the equation of the graph of $y = x^2$ after the following sequence of transformations is applied, in order.

1 reflected in the x-axis
2 translated up by 3 units
3 translated left by 4 units
4 stretched by a factor of 2 in the y-direction
5 translated right by 4 units

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Determining non-linear relationships

Interactive skillsheet
Sketching non-linear relationships

Worksheet
Sketching non-linear relationships

Topic quiz
5I

Chapter summary

Quadratic formula

$ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The discriminant

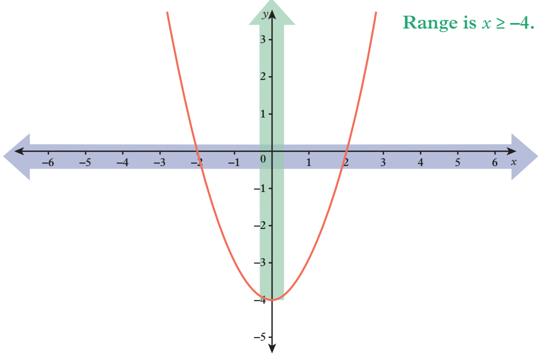
$\Delta = b^2 - 4ac$

- If $\Delta < 0$, there are no real solutions.
- If $\Delta = 0$, there is one real solution.
- If $\Delta > 0$, there are two real solutions.

The Null Factor Law

If $(x - p)(x - q) = 0$, then $x - p = 0$ or $x - q = 0$.

Domain and range



Domain is ℝ.

Range is $x \geq -4$.

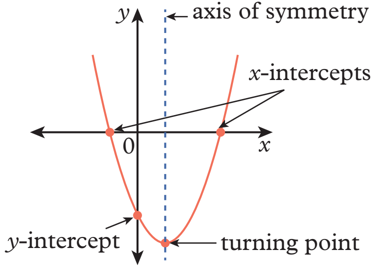
Parabolas

$y = a(x - h)^2 + k$

Vertical translation of k units
If $k > 0$, up
If $k < 0$, down

Horizontal translation of h units
If $h > 0$, right
If $h < 0$, left

Stretch in the y-direction
If $a > 0$, vertical stretch up
If $0 < a < 1$, horizontal stretch outward
But if $a < 0$, invert



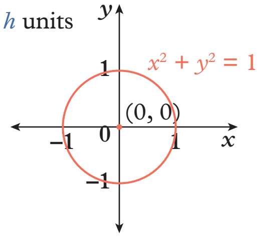
Circles

$(x - h)^2 + (y - k)^2 = r^2$

Radius of r units

Vertical translation of k units
If $k > 0$, up
If $k < 0$, down

Horizontal translation of h units
If $h > 0$, right
If $h < 0$, left



Exponential relationships

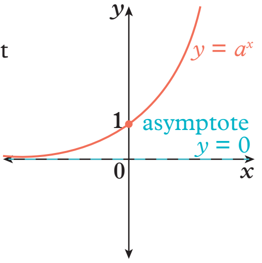
$y = B \times a^{x-h} + k$

Vertical translation of k units
If $k > 0$, up
If $k < 0$, down

Horizontal translation of h units
If $h > 0$, right
If $h < 0$, left

If $B > 0$, upright
If $B < 0$, inverted

If $a > 1$, upright



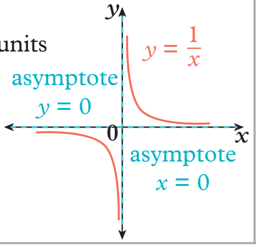
Hyperbolas

Stretch in the y-direction
If $a > 0$, upright
If $a < 0$, inverted

$$y = \frac{a}{x - h} + k$$

Vertical translation of k units
If $k > 0$, up
If $k < 0$, down

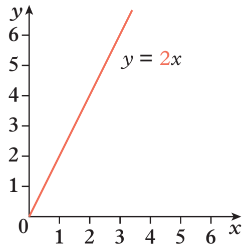
Horizontal translation of h units
If $h > 0$, right
If $h < 0$, left



Direct proportion

$y = kx$

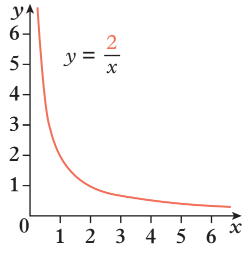
constant of proportionality
= rate of change
= gradient



Inverse proportion

$y = \frac{k}{x}$

constant of proportionality



Chapter review

Chapter review quiz
Take the chapter review quiz to assess your knowledge of this chapter.

Quizlet
Test your knowledge of this topic by working individually or in teams.

Multiple-choice

5A **1** Which of these is a quadratic equation?

A $y - 2 = x^2 + 4$ **B** $x^2 + y^2 = 9$ **C** $y = \frac{5}{x}$ **D** $3^x = 9$ **E** $y = 4x + 1$

5B **2** For the quadratic equation $2x - 4x^2 = 3$, which of these is the value of the discriminant?

A 52 **B** –8 **C** –44 **D** 44 **E** 0

5C **3** The coordinates of the x-intercepts of the graph of $y = x^2 - 6x + 5$ are:

A (1, 0), (5, 0) **B** (0, 1), (0, 5) **C** (–3, 0), (2, 0)
D (–1, 0), (–5, 0) **E** (0, –1), (0, –5)

5C **4** Which of the following statements about the graph of $y = x^2 + 2x - 3$ is *incorrect*?

A It is upright. **B** The graph has two x-intercepts.
C The turning point has coordinates (–4, –1). **D** The graph shows a minimum.
E The y-intercept is (0, –3).

5D **5** Compared with the graph of $y = x^2$, the graph of $y = (x - 2)^2$ is:

A stretched by a factor of 2 in the y-direction. **B** reflected in the x-axis.
C translated 2 units up. **D** translated 2 units right.
E translated 2 units left.

5D **6** The coordinates of the turning point of the graph of $y = (x - 2)^2 - 5$ are:

A (–2, –5) **B** (2, –5) **C** (–2, 5) **D** (2, 5) **E** (0, –1)

5E **7** The coordinates of the centre of the circle $(x - 2)^2 + (y + 3)^2 = 16$ are:

A (2, –3) **B** (–2, 3) **C** (–2, –3) **D** (2, 3) **E** (4,4)

5F **8** Which of these equations represents an exponential relationship?

A $y = 3x^2 + 4$ **B** $x^2 + y^2 = 4$ **C** $y = 3^x + 4$ **D** $y = \frac{1}{x}$ **E** $y = x^3$

5G **9** Which of these equations represents a hyperbola?

A $y = x^2 - 4$ **B** $y = x^4$ **C** $y = 4^x$ **D** $y = \frac{1}{x - 4}$

5H **10** Which of the following represents an example of a directly proportional relationship?

A

x	0	1	2	3
y	2	7	12	17

B

x	0	1	2	3
y	0	1	4	9

C

x	1	2	3	4
y	48	24	16	12

D

x	5	10	15	20
y	12	22	32	42

E

x	0	1	2	3
y	0	6	12	18

10A **5I** **11** The coordinates of the x-intercept of the graph of $y = \frac{1}{x + 2} + 5$ are:

A (0, –2.2) **B** (–2.2, 0) **C** (–0.4, 0) **D** (–7, 0) **E** (5.5, 0)

Short answer

5A **1** Solve each of these equations. Write your solutions as exact values.

a $(x - 2)^2 = 0$
b $(4x - 3)(3x + 4) = 0$
c $(\sqrt{2} - x)(\sqrt{2} + x) = 0$
d $3x^2 - 6x - 24 = 0$
e $5x^2 - 7 = 0$

5A **2** Solve each of these equations by completing the square. Write your solutions as exact values.

a $x^2 + 6x + 4 = 0$ **b** $x^2 - 3x - 3 = 0$

5B **3** Use the discriminant to identify the number of real solutions for each of these quadratic equations.

a $x^2 - 4x + 4 = 0$ **b** $3x^2 = 2x - 4$ **c** $5x - 4x^2 = 2$

d $4 - 3x^2 = 8x$ **e** $1 - x - x^2 = 0$ **f** $6x^2 + 4x = -1$

5B **4** Use the quadratic formula to solve each of these equations. Write your solutions as:

i exact values

ii approximate values, to two decimal places.

a $x^2 + 3x = 10$ **b** $5x^2 = 3x + 1$ **c** $x^2 = 5x - 2$

d $x^2 + 5x + 2 = 0$ **e** $2x^2 = x + 2$ **f** $3x^2 + 7x + 3 = 0$

5C **5** For each of these quadratic relationships, find the coordinates of the:

i x-intercept(s) **ii** y-intercept.

a $y = -x^2 + 25$ **b** $y = x^2 - 3x - 4$ **c** $y = 2x^2 - 8x + 8$ **d** $y = -x^2 - x + 6$

5C **6** Use intercepts to help you sketch the graph of each relationship below. Label the turning point with its coordinates each time.

a $y = x^2 - 4x + 3$ **b** $y = -x^2 - 4x + 5$

5D **7** For each of the quadratic relationships below:

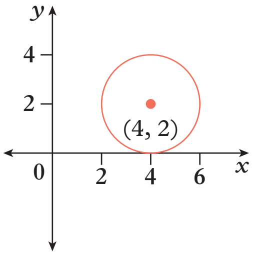
i describe the translations performed on $y = x^2$ **ii** write the coordinates of the turning point.

a $y = (x - 5)^2 + 2$ **b** $y = (x + 6)^2 - 4$

5E **8** Write the equation for the circle on the right.

c $y = (x - 1)^2 - 1$ **d** $y = (x + 3)^2 + 5$

e $y = (x - 4)^2 + 2$ **f** $y = 3(x + 7)^2 - 3$



5F **9** **a** Sketch the graph of $y = 5^x$.

b Describe the transformations performed on the graph of $y = 5^x$ to produce the graph of:

i $y = 5^{x+3}$ **ii** $y = 5^x + 3$

c Sketch all three graphs from parts **a** and **b** on the same Cartesian plane.

5G **10** Describe the transformations performed on the graph of $y = \frac{1}{x}$ to produce the graph of each of these equations:

a $y = \frac{1}{x} - 5$ **b** $y = -\frac{1}{x}$ **c** $y = \frac{4}{x}$ **d** $y = \frac{1}{x + 5} + 6$

5H **11** At a craft store, the cost of fabric is directly proportional to the number of square metres purchased. Mary purchased 7.5 square metres of fabric for $93.75.

a Find the constant of proportionality.

b How much would it cost Mary for 4 square metres of fabric?

c If Mary has $200, how many square metres of fabric can she purchase?

5H **12** Consider the following table, which shows an inversely proportional relationship between F and a.

a Find the constant of proportionality for $F \propto \frac{1}{a}$.

b Find the constant of proportionality for $a \propto \frac{1}{F}$.

c Find the value of F if $a = 2.5$.

a	2	4	5	10
F	10	5	4	2

5I **13** Identify the type of graph each of these equations has.

a $x^2 + y^2 = 2$ **b** $y = \frac{1}{x + 4}$ **c** $y = 5^x$ **d** $y = 3x^2 - 5$

10A 5I **14** The graph of $y = \frac{1}{x}$ is translated 4 units left and 6 units down. What is the equation of the new curve? List the domain and range of the graph.

Analysis

1 The parabolas you have looked at so far are vertical parabolas, because they each have a vertical axis of symmetry. There are also horizontal parabolas, for which the axis of symmetry is horizontal. Similar to the way we have used $y = x^2$ as the basic rule for a vertical parabola, we can use $x = y^2$ as the basic rule for a horizontal parabola.

A horizontal parabola can be represented by the general equation $x - h = a(y - k)^2$, where (h, k) is the turning point of the parabola, and a is a constant.

- **a** Consider the graph of $x + 4 = (y + 1)^2$.
  - **i** Provide values for a, h and k.
  - **ii** What are the coordinates of the turning point?
  - **iii** There is one x-intercept. Calculate its coordinates.
  - **iv** There are two y-intercepts. Calculate the coordinates of both points.
  - **v** Use the coordinates of the turning point and the intercepts to sketch the graph on a Cartesian plane.
  - **vi** Describe the transformations that need to be performed on the graph of $x = y^2$ to produce the same graph you sketched in part **v**.
- **b** Follow the procedure in part **a** to sketch the graph of $x - 4 = -(y + 1)^2$ on the same Cartesian plane as the previous graph.
- **c** Compare the two graphs from parts **a** and **b**. What is the effect of the negative value of a?

2 By now, you are familiar with the quadratic formula expressed in the form $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

An alternative form of the quadratic formula is $x = \dfrac{2c}{-b \pm \sqrt{b^2 - 4ac}}$.

- **a** Show that both quadratic formulas give the same solution for the quadratic equation $x^2 - x - 3 = 0$. Give your solutions correct to one decimal place.
- **10A** **b** Start with the first formula given above. Show how algebraic methods can be used to transform the standard quadratic formula into the second formula. (Hint: Take the positive value of the square root, then rationalise the numerator.)

10A **3** The graph of a hyperbola consists of two branches, which are rotationally symmetrical around a centre point. That is, there is a point in the centre of the hyperbola and when the graph is rotated 180° about this point, the result is the same graph.

The shortest distance between the two branches of a hyperbola can be found by drawing a straight line angled at 45° (that is, with a gradient of 1 or −1) that passes through the centre of the hyperbola.

A hyperbola with equation $y = \dfrac{a}{x - h} + k$ has its centre at (h, k).

Consider the hyperbola described by the equation $y = \dfrac{2}{x - 1} + 2$.

- **a** What is the centre of this hyperbola?
- **b** Sketch the graph of $y = \dfrac{2}{x - 1} + 2$.
- **c** Find, using trial and error, the equation for the circle with the same centre as the hyperbola, that just touches each branch of the hyperbola.
- **d** Use the equation of the circle you found in part **c** to find the shortest distance between the two branches of $y = \dfrac{2}{x - 1} + 2$.
- **e** Consider the hyperbola $y = \dfrac{a}{x}$, where $a > 0$, and the circle $x^2 + y^2 = r^2$.

  Find the relationship between a and r, such that the circle just touches each branch of the hyperbola. It is possible to find this relationship algebraically, but it might be easier to find it with trial and error using technology, by first sketching the circle for specific values of r and then finding corresponding values of a.
- **f** Find the shortest distance between the two branches of the general hyperbola: $y = \dfrac{a}{x - h} + k$, where $a > 0$.

6

Polynomials

Index

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Expanding binomial products
- ✔ Factorising quadratic expressions
- ✔ Solving quadratic equations
- ✔ Sketching parabolas using intercepts
- ✔ Sketching parabolas using transformations

Curriculum links

- Investigate the concept of a polynomial and apply the factor and remainder theorems to solve problems (VCMNA357) [10A]
- Apply understanding of polynomials to sketch a range of curves and describe the features of these curves from their equation (VCMNA361) [10A]
- Use function notation to describe the relationship between dependent and independent variables in modelling contexts (VCMNA363) [10A]

Materials

✔ Calculator

6A Polynomials

Learning intentions

- ✔ I can identify the features of a polynomial.
- ✔ I can evaluate a polynomial using substitution.
- ✔ I can expand and simplify a product of polynomials.

Inter-year links

Year 7	6F Simplifying
Year 8	5F Expanding
Year 9	3B Expanding

The distributive law

- The distributive law states that multiplication can be ‘distributed’ by multiplying one factor across another bracketed factor. This means that large numbers and algebraic expressions can be broken down into the sum of a group of smaller numbers or terms, which can then be multiplied separately.

  For example, the distributive law can be applied to binomials and trinomials.

$$(x^2 + x + 1)(x^2 - 2x - 3) = x^2(x^2 - 2x - 3) + x(x^2 - 2x - 3) + 1(x^2 - 2x - 3)$$

Polynomials

- A **polynomial** is an expression that contains at least one variable, such as x. Each term contains that variable raised to a non-negative integer value $(0, 1, 2, 3, \ldots)$.

  For example, $x^3 + 7x^2 + 3x + 2$ and $3x^2 + 2xy - y^2$ are examples of polynomials, because the index of each variable is a positive integer value.

  $3x^{\frac{1}{3}} - 9x$ and $\frac{2}{x} + x^2$ are not polynomials because the variables are raised to negative indices or indices that are fractions.
- A single variable polynomial function can be written as:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x^1 + a_0x^0$$

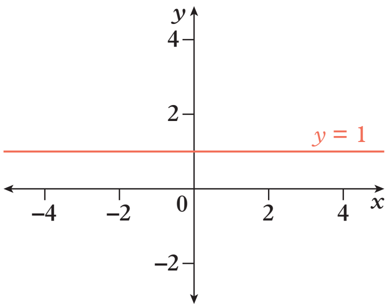
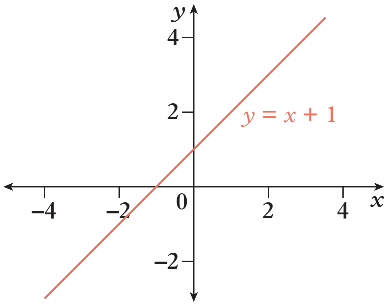
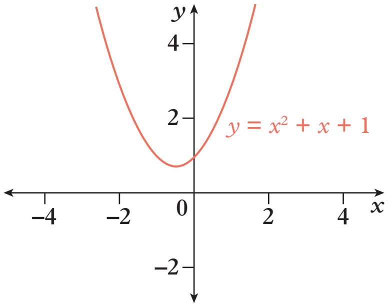
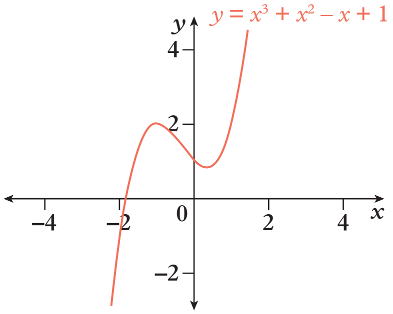
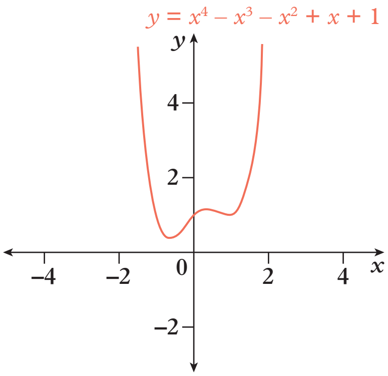
  where n is the highest index, and $a_n, a_{n-1}, a_{n-2}, \ldots a_2, a_1, a_0$ are coefficients.
- $P(x)$ is read as ‘P of x’ meaning the polynomial P in terms of the variable x. This way of writing the equation of a polynomial is also called function notation.
- Function notation is useful for substituting values into an expression.

  For example, if $P(x) = x + 2$, then $P(3) = 3 + 2 = 5$.

$$\mathrlap{\overbrace{}^{\text{function}}}P(x) = \underbrace{\underbrace{x^2 + 2x + 1}_{\text{expression}} = 0}_{\text{equation}}$$

- The **degree of a polynomial (*n*)** is the largest index of the variable. Polynomials are named according to their degree. For example, $P(x) = x^2 - 2x + 1$ is a degree 2 polynomial called a quadratic. Some common polynomials are listed in the table on the next page.
- The **leading term** $(a_n x^n)$ contains the largest index of the variable and is usually written first. The coefficient of the leading term is called the leading coefficient.
- Polynomials are generally written with the variables in descending or ascending order of their indices.

Types of polynomials

Degree	Name	Example	Graph
0	constant	1 (or $1x^0$)	
1	linear	$x + 1$ (or $x^1 + 1$)	
2	quadratic	$x^2 + x +1$	
3	cubic	$x^3 + x^2 - x + 1$	
4	quartic	$x^4 - x^3 - x^2 + x + 1$	

Example 6A.1 Identifying features of a polynomial

For the polynomial $2x^3 - x^2 + 7x + 4$, identify:

a the number of terms

b the degree of the polynomial

c the constant term

d the leading term

e the leading coefficient

f the coefficient of the x^2 term.

THINK	**WRITE**
a Terms are separated by + and – signs.	**a** There are four terms.
b Look for the highest power of x.	**b** The degree is 3.
c Look for a term without a variable (or a term for which the power of the variable is 0).	**c** The constant term is 4.
d Look for the term that has the highest power of x.	**d** The leading term is $2x^3$.
e Write the coefficient of the leading term.	**e** The leading coefficient is 2.
f Write the coefficient of the term containing x^2. The sign shown between it and the term before belongs to the x^2 term. ($-x^2$ means $-1x^2$.)	**f** The coefficient of x^2 is -1.

Example 6A.2 Evaluating a polynomial

If $P(x) = 2x^3 - x^2 + 7x + 4$, evaluate:

a $P(3)$

b $P(0)$

c $P(-1)$

THINK	**WRITE**
a Substitute $x = 3$ into the expression and evaluate.	**a** $P(x) = 2x^3 - x^2 + 7x + 4$ $P(3) = 2(3)^3 - (3)^2 + 7(3) + 4$ $= 54 - 9 + 21 + 4$ $= 70$
b Substitute $x = 0$ into the expression and evaluate.	**b** $P(x) = 2x^3 - x^2 + 7x + 4$ $P(0) = 2(0)^3 - (0)^2 + 7(0) + 4$ $= 0 - 0 + 0 + 4$ $= 4$
c Substitute $x = -1$ into the expression and evaluate.	**c** $P(x) = 2x^3 - x^2 + 7x + 4$ $P(-1) = 2(-1)^3 - (-1)^2 + 7(-1) + 4$ $= -2 - 1 - 7 + 4$ $= -6$

Example 6A.3 Expanding the product of two polynomials

Expand and simplify each of these products.

a $2x^3(x^2 - 3x + 4)$

b $(x^2 - 2)(x^3 - x + 5)$

c $(2x^4 - x^2 + 3)(x^3 + 4x - 1)$

THINK

a Apply the distributive law. Multiply each term inside the pair of brackets by the term outside the brackets.

b Apply the distributive law. Multiply each term in the second pair of brackets by each term in the first pair of brackets. Simplify like terms.

WRITE

a

$$2x^3(x^2 - 3x + 4) = 2x^5 - 6x^4 + 8x^3$$

b

$$\begin{aligned}(x^2 - 2)(x^3 - x + 5) &= x^2(x^3 - x + 5) - 2(x^3 - x + 5)\\ &= x^5 - x^3 + 5x^2 - 2x^3 + 2x - 10\\ &= x^5 - 3x^3 + 5x^2 + 2x - 10\end{aligned}$$

c

$$\begin{aligned}(2x^4 - x^2 + 3)(x^3 + 4x - 1) &= 2x^4(x^3 + 4x - 1) - x^2(x^3 + 4x - 1) + 3(x^3 + 4x - 1)\\ &= 2x^7 + 8x^5 - 2x^4 - x^5 - 4x^3 + x^2 + 3x^3 + 12x - 3\\ &= 2x^7 + 7x^5 - 2x^4 - x^3 + x^2 + 12x - 3\end{aligned}$$

Example 6A.4 Expanding the product of three polynomials

Expand and simplify $(x + 2)(x - 3)(x + 4)$.

THINK	**WRITE**
1 Multiply two of the factors and simplify. It is easier to choose the last two.	$(x + 2)(x - 3)(x + 4) = (x + 2)(x^2 + 4x - 3x - 12)$ $= (x + 2)(x^2 + x - 12)$
2 Multiply the linear factor by the quadratic factor.	$= x(x^2 + x - 12) + 2(x^2 + x - 12)$ $= x^3 + x^2 - 12x + 2x^2 + 2x - 24$
3 Simplify like terms.	$= x^3 + 3x^2 - 10x - 24$

Helpful hints

- ✔ When multiplying through by a negative term, use brackets for clarity. For example,
  $-7p(3q-4) = -7p \times 3q + (-7p \times -4)$
- ✔ Show all your working to avoid making errors in arithmetic and errors related to the signs.
- ✔ After expanding expressions containing brackets, always simplify your results by looking for like terms. Simplify, simplify, simplify!

ANS p722

Exercise 6A Polynomials

△ 1–4, 6–8, 9(a, c, e, g), 10, 11, 12, 14 ☐ 1, 3–8, 9(c, d, f, h), 10, 13, 15, 17(a–d), 19(a, b) ⬠ 1, 5, 6(f, g, j, k), 8, 9(d, f, h), 10, 13, 16–18, 19(d), 20

UNDERSTANDING AND FLUENCY

6A.1 **1** For each polynomial below, identify:

a the number of terms **b** the degree of the polynomial
c the constant term **d** the leading term
e the leading coefficient **f** the coefficient of the x^2 term.

i $2x^3 + 3x^2 + 4x + 5$ **ii** $4x^5 + x^4 + 7x^3 - 2x^2 + 9x - 3$ **iii** $-5x^4 - 2x^3 + 5x^2 + 1$
iv $x^7 + x^6 - x^5 + x^4 + x^3 - x^2 + x$ **v** $9 - 3x - 6x^3 + 2x^2 - 7x^6$ **vi** $3 - 11x^{10} + 5x^8$

2 Simplify each of these polynomials by collecting like terms.

a $4x^3 + 2x^2 - 4x - 1 + x^3 - 5x^2 + 3$ **b** $x^5 - 3x^2 + x^3 - 2x^4 - x^3 + 7x^2$
c $x^4 + 2 + 4x^2 + x^4 - 2x^2 - 5 + 6x^3$ **d** $(2x^2 - 5x + 1) + (3x^2 - 6x + 8)$
e $(5x^3 - 2x + 1) - (2x^3 - 7x + 4)$ **f** $(3x^2 + x + x^3) - (4x^4 - 3x^2 + 5x^3)$

6A.2 **3** If $P(x) = x^3 - 2x^2 + 5x - 8$, evaluate:

a $P(2)$ **b** $P(0)$ **c** $P(3)$ **d** $P(-3)$

4 If $P(x) = 3x^4 - 4x^3 + x^2 - 6x + 1$, evaluate:

a $P(3)$ **b** $P(1)$ **c** $P(-1)$ **d** $P(-2)$

5 If $P(x) = 2x^3 + 3x^2 - 5x + 1$, evaluate:

a $P\left(\frac{1}{2}\right)$ **b** $P\left(-\frac{2}{3}\right)$ **c** $P(\sqrt{2})$ **d** $P\left(-\frac{1}{\sqrt{3}}\right)$

6 If $P(x) = 2x^3 - x^2 + 3x - 6$ and $Q(x) = 3x^2 - 7x + 2$, find:

a $P(x) + Q(x)$ **b** $P(x) - Q(x)$ **c** $Q(x) - P(x)$ **d** $3P(x)$
e $-2Q(x)$ **f** $2P(x) + Q(x)$ **g** $P(x) - 4Q(x)$ **h** $3P(x) - 2Q(x)$
i $2P(-1)$ **j** $-3Q(2)$ **k** $P(3) + Q(3)$ **l** $5P(0) - 4Q(0)$

6A.3 **7** Expand and simplify to find each of these products.

a $x^3(x^2 + 2x + 7)$ **b** $4x^2(x^2 - 5x + 2)$
c $-6x^3(x^4 - 4x^2 + 1)$ **d** $(x + 3)(x^2 - 2x + 4)$
e $(x^2 - 1)(x^2 + 3x - 7)$ **f** $(x^4 - 3x + 2)(x^3 + 4x^2 - 1)$

6A.4 **8** Expand and simplify to find each of these products.

a $3x(x + 2)(x + 3)$ **b** $-7x^2(x + 5)(x - 2)$
c $(x + 3)(x + 1)(x + 6)$ **d** $(x - 3)(x - 4)(x + 1)$
e $(3x - 2)(x - 5)(x - 2)$ **f** $(4x + 1)(2x + 7)(x - 4)$

9 Expand and simplify each of these expressions.

a $(x^3 + 4)^2$ **b** $(x^2 + 3x)^2$ **c** $2x(x^2 - 2)^2$ **d** $(x^3 - 5x^2)^2$
e $(x + 2)^3$ **f** $(2x - 3)^3$ **g** $(x + 3)^4$ **h** $(1 - x)^4$

PROBLEM SOLVING AND REASONING

10 State whether each of the following is a polynomial or is not a polynomial.

a $2x^3 - 3x^2 + 5x - 1$ **b** $2x^{-3} - 3x^{-2} + 5x^{-1} - 1$ **c** $2x^{\frac{3}{7}} - 3x^{\frac{2}{7}} + 5x^{\frac{1}{7}} - 1$

d $\frac{2}{x^3} - \frac{3}{x^2} + \frac{5}{x} - 1$ **e** $\frac{2}{7}x^3 - \frac{3}{7}x^2 + \frac{5}{7}x - \frac{1}{7}$ **f** $2x^3 - 3x^2 + 5x - 1 + \frac{8}{x+1}$

g $\sqrt{2}\,x^3 - \sqrt{3}\,x^2 + \sqrt{5}\,x - 1$ **h** $2\sqrt{x^3} - 3x + 5\sqrt{x} - 1$ **i** $2x^3 - 3x^2 + 5x + 2\sqrt{x} - 1$

11 What is the maximum number of terms in a polynomial of degree 5?

12 What is the minimum number of terms in a polynomial of degree 3?

13 **a** If $P(x)$ is a polynomial of degree n, what is:

i the maximum number of terms in $P(x)$? **ii** the minimum number of terms in $P(x)$?

iii the degree of $2P(x)$? **iv** the degree of $[P(x)]^2$?

b If $Q(x)$ is a polynomial of degree m, where $n > m$, what is:

i the maximum possible degree of $P(x) + Q(x)$? **ii** the maximum possible degree of $P(x) - Q(x)$?

iii the maximum possible degree of $P(x)Q(x)$?

14 If $P(2) = 2$, find the value of k in $P(x) = x^3 - 2x^2 + 3x + k$.

15 If $P(-1) = 7$, find the value of k in $P(x) = 3x^4 + 2x^3 - kx - 5$.

16 If $P(3) = 31$ and $P(-2) = -19$, find the value of a and b in $P(x) = x^3 + ax^2 + bx + 1$.

17 If $P(x) = x^3 - 3x^2 + 2x + 1$, write simplified expressions for:

a $P(a)$ **b** $P(-3a)$ **c** $P(a^2)$ **d** $P(-a^2)$ **e** $P(a + 2)$ **f** $P(2a - 1)$

18 Show that $(2x - 1)^4 = 16x^4 - 32x^3 + 24x^2 - 8x + 1$.

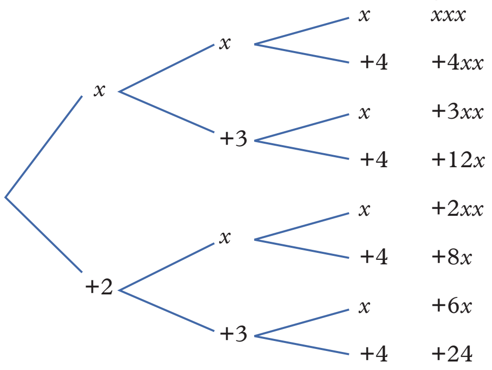
CHALLENGE

19 Expand each of these expressions.

a $(x^2 - 2)^3$ **b** $(x^3 + x - 1)^2$ **c** $(2x^3 + 1)^4$

20 Consider $(x + 2)(x + 3)(x + 4)$. We can expand the expression by adding the products of every combination of one term from each binomial factor.

To ensure all combinations have been multiplied, we can use a tree diagram, as shown below.



The tree diagram shows all the products that need to be added together. The path along each sequence of connected branches on the tree diagram is represented by a set of brackets in this calculation:

$$\begin{aligned}(x + 2)(x + 3)(x + 4) &= (x \times x \times x) + (x \times x \times 4) + (x \times 3 \times x) + (x \times 3 \times 4) + (2 \times x \times x) + (2 \times x \times 4)\\ &\quad + (2 \times 3 \times x) + (2 \times 3 \times 4)\\ &= x^3 + 4x^2 + 3x^2 + 12x + 2x^2 + 8x + 6x + 24\\ &= x^3 + 9x^2 + 26x + 24\end{aligned}$$

a Use a tree diagram to help you expand $(x + 5)(x - 3)(x - 8)$.

b Use a tree diagram to help you expand $(x - 2)(x + 5)(x - 3)(x - 8)$.

Check your Student obook pro for these digital resources and more:

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Polynomials

Interactive skillsheet
Multiplying polynomials

Worksheet
Working with polynomials

CAS instructions
Expanding

Topic quiz
6A

6B Dividing polynomials

Learning intentions

✔ I can divide a polynomial by a linear factor.

✔ I can divide a polynomial that has coefficients of zero by a linear factor.

Inter-year links

Year 7	1F Dividing whole numbers
Year 8	5E Dividing algebraic terms
Year 9	3B Expanding

Division of polynomials

- Long division is used to divide polynomials. We begin by dividing the leading term of the dividend by the leading term of the divisor and then multiplying the resulting term by the whole divisor to find a new expression, which is written below the dividend. We then subtract this new expression from the dividend and write the result below. We continue to do this at each new stage of the long division until we reach a remainder with polynomial degree less than the degree of the divisor.

$$\begin{array}{rrl}
 & x+2 & \leftarrow \text{quotient}\\
\text{divisor} \rightarrow x+1 & \big)\overline{x^2+3x+4} & \leftarrow \text{dividend}\\
 & \underline{x^2+x} & \\
 & 2x+4 & \\
 & \underline{2x+2} & \\
 & 2 & \leftarrow \text{remainder}
\end{array}$$

- The quotient and remainder are important elements to identify. In the example above, the quotient is $x + 2$ and the remainder is 2. We can write this as:

$$\underbrace{(x^2+3x+4)}_{\text{dividend}} \div \underbrace{(x+1)}_{\text{divisor}} = \underbrace{x+2}_{\text{quotient}} \quad \underbrace{\text{remainder } 2}_{\text{remainder}}$$

or as:

$$\frac{\overbrace{x^2+3x+4}^{\text{dividend}}}{\underbrace{x+1}_{\text{divisor}}} = \overbrace{x+2}^{\text{quotient}} + \frac{\overset{\text{remainder}}{2}}{\underbrace{x+1}_{\text{divisor}}}$$

- Alternatively, the polynomials can be written as:

  dividend = divisor × quotient + remainder

  $x^2 + 3x + 4 = (x + 1)(x + 2) + 2$

- In general, if $P(x)$ is divided by $D(x)$ to produce the quotient $Q(x)$ and the remainder $R(x)$, then $P(x) = D(x)Q(x) + R(x)$.
- If the remainder is a negative constant, then it follows that a value must be subtracted from the product of the divisor and the quotient to equal the dividend.

Example 6B.1 Dividing a polynomial by a linear expression

Use long division to find the quotient and remainder for $(x^3 + 2x^2 - 9x - 3) \div (x - 2)$.

THINK

1. Divide the dividend's leading term, x^3, by the divisor's leading term, x, and write the result (x^2) in the quotient line.
2. Expand $x^2(x - 2)$ to give $x^3 - 2x^2$ and write that result aligned under the dividend. Subtract like terms.
3. Divide $4x^2$ by x and write the result ($4x$) in the quotient line.
4. Expand $4x(x - 2)$ to give $4x^2 - 8x$ and write that result aligned under the dividend. Subtract like terms.
5. Divide $-x$ by x and write the result (-1) in the quotient line.
6. Expand $-1(x - 2)$ to give $-x + 2$ and write that result aligned under the dividend. Subtract like terms.
7. Identify the quotient and the remainder.

WRITE

$$\begin{array}{rr}
 & x^2 + 4x - 1 \\
x - 2\,\big) & \overline{x^3 + 2x^2 - 9x - 3} \\
x^2(x-2) \rightarrow & -(x^3 - 2x^2) \\
 & \overline{4x^2 - 9x - 3} \\
4x(x-2) \rightarrow & -(4x^2 - 8x) \\
 & \overline{-x - 3} \\
-1(x-2) \rightarrow & -(-x + 2) \\
 & \overline{-5}
\end{array}$$

The quotient is $x^2 + 4x - 1$.
The remainder is -5.

Example 6B.2 Dividing a polynomial that has coefficients of zero by a linear expression

Use long division to find the quotient and remainder for $(3x^4 + 2x^2 - 5) \div (x + 1)$.

THINK

1. To keep like terms aligned, write the 'missing' terms in the dividend, with zero as the coefficient.
2. Divide $3x^4$ by x and write $3x^3$ in the quotient line.
3. Expand $3x^3(x + 1)$ to be $3x^4 + 3x^3$. Subtract like terms.
4. Divide $-3x^3$ by x and write $-3x^2$ in the quotient line.
5. Expand $-3x^2(x + 1)$ to be $-3x^3 - 3x^2$. Subtract like terms.
6. Divide $5x^2$ by x and write $5x$ in the quotient line.
7. Expand $5x(x + 1)$ to be $5x^2 + 5x$. Subtract like terms.
8. Divide $-5x$ by x and write -5 in the quotient line.
9. Expand $-5(x + 1)$ to be $-5x - 5$. Subtract like terms.
10. Identify the quotient and the remainder.

WRITE

$$\begin{array}{rr}
 & 3x^3 - 3x^2 + 5x - 5 \\
x + 1\,\big) & \overline{3x^4 + 0x^3 + 2x^2 + 0x - 5} \\
3x^3(x+1) \rightarrow & -(3x^4 + 3x^3) \\
 & \overline{-3x^3 + 2x^2 + 0x - 5} \\
-3x^2(x+1) \rightarrow & -(-3x^3 - 3x^2) \\
 & \overline{5x^2 + 0x - 5} \\
5x(x+1) \rightarrow & -(5x^2 + 5x) \\
 & \overline{-5x - 5} \\
-5(x+1) \rightarrow & -(-5x - 5) \\
 & \overline{0}
\end{array}$$

The quotient is $3x^3 - 3x^2 + 5x - 5$.
The remainder is 0.

Helpful hints

- ✔ The leading coefficient is not always the coefficient of the first term of a polynomial. For example, the expression $3 - 3x + x^2$ needs to be rearranged to $x^2 - 3x + 3$ before it can be divided using long division.
- ✔ Before dividing a polynomial, remember to insert the missing terms with zero as the coefficient. It can be easy to make mistakes if you skip this step.
- ✔ The ultimate purpose of dividing polynomials is so that you can fully factorise cubics and quartics. It is best to keep this skill to the front of your mind!

ANS p723

Exercise 6B Dividing polynomials

△ 1–6, 7(a, c), 8(a, c), 12

■ 1(d–f), 2(a, d, e, g, i, j), 3, 6, 7, 8(b, d), 10, 13

⬠ 1(e, f), 2(g–j), 3, 7, 8(b, d), 9, 11, 13, 14

UNDERSTANDING AND FLUENCY

1 Use long division to find the quotient and remainder for each of these division problems.

a $(x^2 + 10x + 26) \div (x + 7)$

b $(x^2 - 12x + 26) \div (x - 4)$

c $(5x^2 - 4x - 2) \div (x - 2)$

d $(9x^2 - 37x + 22) \div (x - 4)$

e $(14x^2 + 53x - 6) \div (2x + 7)$

f $(60x^2 - 52x + 79) \div (10x + 3)$

6B.1 **2** Use long division to find the quotient and remainder for each of these division problems.

a $(x^3 - 2x^2 - 5x + 7) \div (x - 3)$

b $(x^3 + 3x^2 + 7x - 1) \div (x + 1)$

c $(x^3 + 6x^2 - 4x - 15) \div (x - 2)$

d $(x^3 - 3x^2 - 20x + 25) \div (x + 4)$

e $(2x^3 + 3x^2 - 5x - 4) \div (x + 2)$

f $(3x^3 - 2x^2 - 7x + 2) \div (x - 1)$

g $(5x^3 - 12x^2 - 6x - 15) \div (x - 3)$

h $(4x^3 + 9x^2 - 8x - 1) \div (x + 1)$

i $(x^4 + 2x^3 - 4x^2 - 2x + 8) \div (x + 1)$

j $(x^4 - 5x^3 + 2x^2 - x - 3) \div (x - 2)$

6B.2 **3** Use long division to find the quotient and remainder for each of these division problems.

a $(x^3 + x + 21) \div (x + 3)$

b $(2x^3 - 3x^2 - 6) \div (x - 2)$

c $(4x^4 - 3x^2 - 5) \div (x - 1)$

d $(3x^4 - 41) \div (x + 2)$

4 For each of the parts **a**–**f** in question **1**:

i Write the answers for the division problems in the form: quotient + $\frac{\text{remainder}}{\text{divisor}}$.

ii Write the dividends in the form: divisor × quotient + remainder.

5 For each of the parts **a**–**f** in question **2**:

i Write the answers for the division problems in the form: quotient + $\frac{\text{remainder}}{\text{divisor}}$.

ii Write the dividends in the form: divisor × quotient + remainder.

6 Perform the following divisions and write your answers in the form $Q(x) + \frac{R(x)}{D(x)}$, where Q, R, and D are polynomial expressions and the degree of R is less than the degree of D.

a $\frac{3x^3 + 6x^2 + 7x + 20}{x + 2}$

b $\frac{2x^3 - 11x^2 + 15x + 9}{x - 3}$

c $\frac{x^4 + x^3 - 30x^2 - 8x + 2}{x + 6}$

d $\frac{x^4 - 7x^3 + 12x - 34}{x - 7}$

7 Write the following polynomials in the form $(x - 3)Q(x) + R(x)$, where Q and R are polynomial expressions. (Hint: divide each polynomial by $(x - 3)$ to find the quotient and remainder.)

a $x^2 + 2x - 9$

b $x^3 - 7x^2 + 21x - 21$

c $x^3 - 9x^2 + 20x - 17$

d $x^4 + 5x^3 - 15x^2 - 23x - 9$

8 Perform the following divisions and write your answer in the form $Q(x) + \frac{R(x)}{D(x)}$, where Q, R, and D are polynomial expressions and the degree of R is less than the degree of D.

a $(4x + 3) \div (x + 2)$

b $(9x - 4) \div (x - 3)$

c $\frac{x - 8}{x - 2}$

d $\frac{10x + 4}{2x - 5}$

9 **a** Consider the polynomial division $P(x) \div (x - a)$ where the degree of P is n, and n is a positive integer. Determine the degree of the quotient.

b Consider the polynomial division $P(x) \div D(x)$ where the degree of P is n and the degree of D is m, and m and n are positive integers where $n > m$. Determine the maximum degree of the quotient.

10 Consider the division $\frac{x^3 + 9x^2 + 23x + 17}{x + 4}$.

a Rewrite the division with the numerator written in the form $(x + 4)Q(x) + R(x)$, where Q and R are polynomial expressions.

b Write your answer from part **a** in the form $\frac{(x + 4)Q(x)}{x + 4} + \frac{R(x)}{x + 4}$ and explain how $\frac{x^3 + 9x^2 + 23x + 17}{x + 4} = Q(x) + \frac{R(x)}{x + 4}$ for $x \neq -4$.

11 **a** If $P(x) = x^3 - 3x^2 - 10x + k$, for what value of k does $P(x) \div (x - 2)$ give a remainder of zero?

b From the divisor and the quotient, identify the linear factor and the quadratic factor of $P(x)$. Then show that you can write $P(x)$ as the product of three linear factors.

12 **a** Expand and simplify $(x + 2)x^2 + 5(x + 2)x - 36(x + 2)$.

b Factorise $(x + 2)x^2 + 5(x + 2)x - 36(x + 2)$.

13 Use long division to find the quotient and remainder for each of these division problems.

a $(x^3 - 3x^2 + 2x - 4) \div (x^2 + 1)$

b $(x^4 + x^3 - 7x^2 + 3x + 5) \div (x^2 - 3)$

c $(2x^4 + 6x^2 - 1) \div (x^2 + 2)$

14 Suppose $f(x) = \frac{P(x)}{D(x)}$ where P and D are polynomial expressions.

a If $P(x) = (x - a)Q(x)$ and $D(x) = x - a + b$, where $b \neq 0$, write an expression for $f(a)$.

b If $P(x) = (x - a)Q(x) + b$ and $D(x) = x - a$, where $b \neq 0$, explain why $f(a)$ is undefined.

c If $P(x) = (x - a)Q(x)$ and $D(x) = x - a$, write an expression for $f(x)$, where $x \neq a$.

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Topic quiz
6B

6C Remainder and factor theorems

Learning intentions

- ✔ I can use the remainder theorem to find a linear factor of a polynomial.
- ✔ I can use the factor theorem to find a linear factor of a polynomial.
- ✔ I can factorise cubic polynomials.

Inter-year links

Year 7	6E Substitution
Year 8	5B Substitution
Year 9	3E Factorising quadratic expressions

The remainder theorem

- When a polynomial $P(x)$ is divided by $(x - a)$, the remainder is $P(a)$.
  For example, when $P(x)$ is divided by $(x - 2)$, the remainder is $P(2)$ and, when $P(x)$ is divided by $(x + 3)$, the remainder is $P(-3)$.

The factor theorem

- When a polynomial $P(x)$ is divided by $(x - a)$ and the remainder $P(a)$ is zero, then $(x - a)$ is a factor of $P(x)$.
  For example, for $P(x) = x^3 + x^2 - 10x + 8$, since $P(1) = 0$, $(x - 1)$ is a factor of $P(x)$.
- If a linear factor of a cubic polynomial is known, long division can be used to find the quadratic factor.
  For example, the quotient when dividing $P(x) = x^3 + x^2 - 10x + 8$ by $(x - 1)$ is $x^2 + 2x - 8$.
- Factorising the quotient allows us to write the polynomial as a product of its linear factors.
  For example,
  $$\begin{aligned} P(x) &= x^3 + x^2 - 10x + 8 \\ &= (x - 1)(x^2 + 2x - 8) \\ &= (x - 1)(x - 2)(x + 4) \end{aligned}$$
- In general, if $P(x)$ is divided by a factor $(x - a)$ to produce the quotient $Q(x)$, then $P(x) = (x - a)Q(x)$.

Example 6C.1 Using the remainder theorem

Find the remainder when $x^3 + 4x^2 + x - 6$ is divided by:

a $x - 2$ **b** $x + 3$

THINK	WRITE
First, name the polynomial.	$P(x) = x^3 + 4x^2 + x - 6$
a For a divisor of $(x - 2)$, calculate $P(2)$ and write the remainder.	**a** $P(2) = 8 + 16 + 2 - 6$ $= 20$ The remainder is 20.
b For a divisor of $(x + 3)$, calculate $P(-3)$ and write the remainder.	**b** $P(-3) = -27 + 36 - 3 - 6$ $= 0$ The remainder is 0.

Example 6C.2 Deciding whether a divisor is a factor of a polynomial

Decide whether each of these divisors is a factor of $x^3 + 2x^2 - x - 2$.

a $x - 3$ **b** $x + 1$

THINK	WRITE
First, name the polynomial.	$P(x) = x^3 + 2x^2 - x - 2$
a For $(x - 3)$ to be a factor, the remainder should be equal to zero. Check if $P(3) = 0$.	**a** $P(3) = 27 + 18 - 3 - 2$ $= 40$ $(x - 3)$ is not a factor of $P(x)$.
b For $(x + 1)$ to be a factor, the remainder should be equal to zero. Check if $P(-1) = 0$.	**b** $P(-1) = -1 + 2 + 1 - 2$ $= 0$ $(x + 1)$ is a factor of $P(x)$.

Example 6C.3 Using the factor theorem to find a linear factor of a polynomial

Use the factor theorem to find a linear factor for each of the following polynomials.

a $P(x) = x^3 + 5x^2 - 2x - 24$ **b** $P(x) = x^3 - 3x^2 - 9x + 27$

THINK	WRITE
a 1 Write the polynomial.	**a** $P(x) = x^3 + 5x^2 - 2x - 24$
2 Determine a value of x where $P(x) = 0$. Try $P(1)$, $P(-1)$, $P(2)$, etc. That is, substitute x values that are positive or negative factors of the constant term in $P(x)$.	$P(1) = 1 + 5 - 2 - 24$ $= -20$ $\neq 0$ $P(-1) = -1 + 5 + 2 - 24$ $= -18$ $\neq 0$ $P(2) = 8 + 20 - 4 - 24$ $= 0$
3 Use the factor theorem to identify a linear factor of $P(x)$.	So $(x - 2)$ is a factor of $P(x)$.
b 1 Write the polynomial.	**b** $P(x) = x^3 - 3x^2 - 9x + 27$
2 Substitute x values that are positive or negative factors of the constant term in $P(x)$. Try $P(1)$, $P(-1)$, $P(3)$, etc.	$P(1) =1 - 3 - 9 + 27$ $= 16$ $\neq 0$ $P(-1) = -1 - 3 + 9 + 27$ $= 32$ $\neq 0$ $P(3) = 27 -27 - 27 + 27$ $= 0$
3 Use the factor theorem to identify a linear factor of $P(x)$.	So $(x - 3)$ is a factor of $P(x)$.

Example 6C.4 Factorising a cubic polynomial

Factorise $x^3 - 7x^2 + 7x + 15$.

THINK	**WRITE**
1 Name the polynomial.	$P(x) = x^3 - 7x^2 + 7x + 15$
2 Determine a value of x for which $P(x) = 0$. Try $P(1)$ and $P(-1)$.	$P(1) = 1 - 7 + 7 + 15$ $= 16$ $\neq 0$ $P(-1) = -1 - 7 - 7 + 15$ $= 0$
3 Identify a linear factor of $P(x)$.	So $(x + 1)$ is a factor of $P(x)$.
4 Use long division to find the quotient when $P(x)$ is divided by $(x + 1)$. The remainder will be zero if the division has been performed correctly.	$\begin{array}{r} x^2-8x+15 \\ x+1\overline{)x^3-7x^2+7x+15} \\ \underline{x^3+x^2} \\ -8x^2+7x+15 \\ \underline{-8x^2-8x} \\ 15x+15 \\ \underline{15x+15} \\ 0 \end{array}$
5 Write $P(x)$ as the product of the divisor and quotient.	$P(x) = (x + 1)(x^2 - 8x + 15)$
6 Factorise the quadratic factor.	$= (x + 1)(x - 3)(x - 5)$

Helpful hints

✔ Students sometimes prove that a value results in a remainder of 0, but then don't get the same result when they perform long division. Performing the long division can be a good way to check your working because you will have a clear flag that something went wrong!

✔ The leading coefficient of the expanded polynomial expression is not always the first term of the factorised polynomial expression. But, if we write each linear factor in the form $(x - a)$, the leading coefficient will be the constant factor at the front of the factorised expression.

For example, $6x^2 + 7x + 2$ can be factorised to become $(2x + 1)(3x + 2)$, and can be further factorised to become $6\left(x - \frac{1}{2}\right)\left(x + \frac{2}{3}\right)$.

✔ When factorising, try the factors of the constant value of the polynomial. If $P(a)$ is zero, then a is a factor of the constant term in polynomial $P(x)$.

For example, when determining a linear factor of $x^2 + 2x + 6$, you should try substituting the positive and negative factors of the constant term, 6, which are: 1, –1, 2, –2, 3, –3, 6 and –6.

✔ When a linear factor has been established and you are performing a polynomial division, you will know you have performed the division correctly if the remainder is zero.

ANS
p724

Exercise 6C Remainder and factor theorems

△ 1–7, 8(a, c, e), 9, 10, 11, 14, 15

□ 4–6, 7–8(1st column), 10, 12, 13, 16, 18, 19, 21(a)

⬠ 4, 6(e–h), 7–8(1st column), 12, 16–18, 20–22

1 Consider the polynomial $P(x) = x^3 + x^2 - 10x + 8$.

- **a** Use long division to find the remainder when $P(x)$ is divided by $(x - 3)$.
- **b** Evaluate $P(3)$. What do you notice?
- **c** Use long division to find the remainder when $P(x)$ is divided by $(x + 2)$.
- **d** Evaluate $P(-2)$. What do you notice?
- **e** Without using long division, use the pattern you noticed in parts **a–d** to show how to work out the remainder when $P(x)$ is divided by $(x - 4)$.
- **f** Use long division to verify your answer to part **e**.

6C.1 **2** Find the remainder when $x^3 - 5x^2 - 8x + 12$ is divided by:

- **a** $x - 1$
- **b** $x + 1$
- **c** $x - 2$
- **d** $x + 2$

3 Find the remainder when $2x^3 + 3x^2 - 8x + 3$ is divided by:

- **a** $x + 1$
- **b** $x - 2$
- **c** $x + 2$
- **d** $x + 3$

4 Find the remainder when $x^4 - 7x^3 + 5x^2 + 31x - 30$ is divided by:

- **a** $x - 2$
- **b** $x + 1$
- **c** $x - 3$
- **d** $x + 4$

6C.2 **5** Decide whether each of these divisors is a factor of $x^3 + 3x^2 - 6x - 8$.

- **a** $x - 1$
- **b** $x + 1$
- **c** $x - 3$
- **d** $x + 3$
- **e** $x - 4$
- **f** $x + 4$
- **g** $x + 2$
- **h** $x - 2$

6 Decide whether each of these divisors is a factor of $x^3 - 4x^2 - 7x + 10$.

- **a** $x - 2$
- **b** $x - 3$
- **c** $x - 1$
- **d** $x + 3$
- **e** $x + 1$
- **f** $x + 2$
- **g** $x - 5$
- **h** $x - 4$

6C.3 **7** Use the factor theorem to find a linear factor for each of the following polynomials.

- **a** $P(x) = x^3 + 8x^2 + 9x - 18$
- **b** $P(x) = x^3 - x^2 - 14x + 24$
- **c** $P(x) = x^3 - 4x^2 - 9x + 36$
- **d** $P(x) = x^3 - 19x - 30$
- **e** $P(x) = 3x^3 - 10x^2 + x + 6$
- **f** $P(x) = 2x^3 - 5x^2 - 14x + 8$
- **g** $P(x) = 8x^3 - 26x^2 + 17x + 6$
- **h** $P(x) = 4x^3 + 4x^2 - 21x + 9$

6C.4 **8** Factorise each of these polynomials.

- **a** $x^3 + 6x^2 + 3x - 10$
- **b** $x^3 + 3x^2 - 4x - 12$
- **c** $x^3 + 4x^2 - 19x + 14$
- **d** $x^3 + 5x^2 - 4x - 20$
- **e** $x^3 - 5x^2 - 8x + 48$
- **f** $x^3 - 2x^2 - 9x + 18$
- **g** $2x^3 + x^2 - 8x - 4$
- **h** $4x^3 - 9x^2 - 19x + 30$
- **i** $6x^3 - 23x^2 - 6x + 8$
- **j** $12x^3 - 17x^2 + 2x + 3$

9 Factorise each of the polynomials in question **7**.

10 a If $P(x) = x^3 + 9x^2 + 23x + 15$, for what three values of k does $P(x) \div (x + k)$ result in a remainder of zero?

- **b** Write $P(x)$ as the product of three linear factors.

11 What is the maximum number of linear factors $P(x)$ can have if $P(x)$ is:

- **a** a quadratic polynomial?
- **b** a cubic polynomial?
- **c** a quartic polynomial?
- **d** a polynomial of degree 7?
- **e** a polynomial of degree 12?
- **f** a polynomial of degree n?

PROBLEM SOLVING AND REASONING

12 a How many linear factors does the polynomial $P(x)$ have if $P(x) = (x + 2)(x - 3)(x + 4)$?

b Without expanding completely, what is the constant term of $P(x)$? Explain.

c To find linear factors using the factor theorem, we look for values of x for which $P(x) = 0$. What are these x values and how do they relate to the constant term?

13 Consider the polynomial $P(x) = x^3 + 2x^2 - 5x - 6$.

a Use the factor theorem to find a linear factor of $P(x)$.

b Use the factor theorem to find another linear factor of $P(x)$.

c Use the factor theorem to find a third linear factor of $P(x)$.

d Write $P(x)$ as a product of three factors.

e Check your answer to part **d** by expanding and simplifying the product.

14 a Use your answers to question **5** to identify three factors of the polynomial $P(x) = x^3 + 3x^2 - 6x - 8$.

b Write $P(x)$ as a product of three factors.

c Check your answer to part **b** by expanding and simplifying the product.

15 a Use your answers to question **6** to identify three factors of the polynomial $P(x) = x^3 - 4x^2 - 7x + 10$.

b Write $P(x)$ as a product of three factors.

16 Using only the factor theorem, find three factors of $x^3 + 2x^2 - 11x - 12$. Then write the polynomial as a product of three factors.

17 A polynomial, P, can be written in the form $P(x) = (x - a)Q(x) + r$, where Q is a polynomial.

a Solve $x - a = 0$ for x.

b Substitute the solution to part **a** into $P(x)$.

c Explain why your answer from part **b** is equal to the remainder of the polynomial division $P(x) \div (x - a)$.

d Another polynomial, F, can be written in the form $F(x) = (bx - a)G(x) + h$, where G is a polynomial. Solve $bx - a = 0$ for x.

e Substitute your solution for part **d** into $F(x)$.

f Explain why your answer from part **e** is equal to the remainder of the polynomial division $F(x) \div (bx - a)$.

18 a When $x^3 + 2x^2 + ax + 5$ is divided by $x - 3$ the remainder is 5. Find the value of a.

b When $x^3 + bx^2 - 6x - 7$ is divided by $x + 2$ the remainder is 17. Find the value of b.

c If $x + 4$ is a factor of $x^3 + 13x^2 + cx - 20$, find the value of c.

d When $x^3 + mx^2 + nx + 5$ is divided by $x + 1$ the remainder is 13, and when divided by $x - 2$ the remainder is 31. Find the values of m and n.

CHALLENGE

19 Explain why the strategy used in question **16** is not suitable for factorising each of these polynomials.

a $6x^3 + 13x^2 + 4x - 3$ **b** $x^3 + 5x^2 + 10x + 8$

20 Find a polynomial of degree 3 with a leading coefficient of 1 that has a remainder of 5 when divided by $x - 2$.

21 Factorise each of these quartic polynomials.

a $x^4 - x^3 - 7x^2 + x + 6$ **b** $x^4 + 8x^3 + 17x^2 - 2x - 24$ **c** $2x^4 + 13x^3 + 21x^2 + 2x - 8$

22 a Show that $(x^2 + 4)$ is a factor of $x^4 + 13x^2 + 36$.

b Factorise $x^4 + 13x^2 + 36$.

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Remainder and factor theorems

Worksheet
Dividing and factorising polynomials

Investigation
Shortcut method for factorising a polynomial

CAS instructions
Factorising

Topic quiz
6C

Checkpoint

Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this chapter.

10A **6A** **1** State whether or not each of the following is a polynomial. If it is not a polynomial, state the term(s) in the expression that cannot be included in a polynomial. If it is a polynomial, state:

i its degree
ii its leading term
iii its leading coefficient.

a $5x^2 - 3x^5 + 4x^7$
b $9x^{100} - 9x^{-100}$
c $\frac{5y^{81}}{9} - \sqrt{3}y^{42} + \frac{2}{3}y^{27} - 1.2$
d $81y^5 - 42\sqrt{y^3} + 27y^{\frac{2}{3}} - 1.2$

10A **6A** **2** Suppose $P(x) = 5x^3 - 3x^2 + 2x - 1$ and $Q(x) = 2x^2 - 8x + 7$. Write an expression for each of the following.

a $P(x) + Q(x)$ **b** $P(x) - Q(x)$ **c** $6P(x)$ **d** $P(x) \times Q(x)$

10A **6B** **3** Determine the quotient and remainder for each of the following polynomial divisions.

a $(x^2 + 2x - 29) \div (x - 4)$
b $(x^3 - 3x^2 + 2x + 33) \div (x + 2)$
c $\frac{2x^2 - 8}{x + 3}$
d $\frac{6x^3 + 19x^2 + 4x - 8}{3x + 2}$

10A **6B** **4** Write the following results in the form $Q(x) + \frac{R(x)}{D(x)}$, where $D(x)$, $Q(x)$ and $R(x)$ are polynomial expressions and the degree of R is less than the degree of D.

a $(x^2 + 16x - 9) \div (x + 4)$; quotient: $x + 12$, remainder: -57
b $(5x^3 - 9x^2 + 2x + 1) \div (x - 6)$; quotient: $5x^2 + 21x + 128$, remainder: 769
c $\frac{5x^3 - 6x^2 + 7}{x^2 + 4}$; quotient: $5x - 6$, remainder: $-20x + 31$
d $\frac{6x^2 + 8x + 18}{3x^2 - 4x + 1}$; quotient: 2, remainder: $16x + 16$
e $\frac{18x + 7}{6x - 7}$; quotient: 3, remainder: 28

10A **6B** **5** Write each divisor from question **4** in the form $D(x)Q(x) + R(x)$.

10A **6B** **6** Perform the following divisions. Write your answers in the form $Q(x) + \frac{R(x)}{D(x)}$, where $D(x)$, $Q(x)$ and $R(x)$ are polynomial expressions and the degree of R is less than the degree of D.

a $(x^4 + 5x^3 - 6x^2 + 2x + 9) \div (x - 2)$
b $\frac{12x^4 + x^2 + 14}{2x + 1}$

10A **6C** **7** Determine the remainder when $P(x) = 3x^3 + 4x^2 - 13x + 6$ is divided by:

a $x + 1$ **b** $x - 1$ **c** $x - 3$ **d** $3x - 2$

10A **6C** **8** For each of the following, determine whether the linear expression is a factor of the non-linear polynomial.

a $x - 2$ and $x^3 - 5x^2 + 7x + 4$ **b** $x + 1$ and $5x^3 + 13x^2 + 5x - 3$
c $x^3 - 2$ and $x^2 - 11x + 13$ **d** $4x - 3$ and $12x^3 + 11x^2 - 23x + 6$

10A **6C** **9** Factorise each of the following polynomials.

a $x^3 - 4x^2 - 25x + 28$ **b** $x^3 - 7x - 6$

10A **6C** **10** Factorise each of the following polynomials.

a $5x^3 - 15x^2 - 170x + 600$ **b** $x^4 - x^3 - 43x^2 - 23x + 210$

6D Solving polynomial equations

Learning intentions

✔ I can solve polynomial equations using the Null Factor Law.

Inter-year links

Year 7	6C Terms, expressions and equations
Year 8	6A Equations
Year 9	5A Solving quadratic equations

Polynomials and the Null Factor Law

- Null Factor Law: If $a \times b = 0$, then $a = 0$ or $b = 0$, or both a and b are 0.
- The polynomial equation $P(x) = 0$ can be solved by applying the Null Factor Law. To do this, $P(x)$ must be in factor form.
- If $P(x) = (x - a)(x - b)(x - c)$, then the solutions to $P(x) = 0$ are $x = a$, $x = b$ and $x = c$.

$$\underbrace{\overbrace{P(x) = \underbrace{(x-a)(x-b)(x-c)}_{\text{expression}}}^{\text{function}} = 0}_{\text{equation}}$$

$$\begin{aligned} x - a = 0 &\Rightarrow x = a \\ x - b = 0 &\Rightarrow x = b \\ x - c = 0 &\Rightarrow x = c \end{aligned}$$

- A polynomial, $P(x)$, of degree n can have a maximum of n linear factors. Therefore, the number of solutions to the equation $P(x) = 0$ is at most n.
- If a polynomial, $P(x)$, of degree n has repeated factors, then the number of solutions to $P(x) = 0$ will be less than n.

  For example, the factor $(x + 1)$ is repeated twice in $P(x) = x(x + 1)^2$, so even though the degree of the polynomial is 3, the number of solutions will be 2.

Factorising polynomials

- To factorise $P(x)$ so that it is a product of linear factors:
  1. apply the factor theorem to find a linear factor
  2. divide $P(x)$ by the linear factor to obtain the quotient
  3. factorise the quotient.
- If $P(x)$ is a cubic polynomial, the quotient will be a quadratic factor that may be further factorised into two linear factors (making a total of three linear factors).

  For example, $\begin{aligned}[t] P(x) &= (x - 1)(x^2 + 2x - 8) \\ &= (x - 1)(x - 2)(x + 4) \end{aligned}$

Example 6D.1 Solving polynomial equations in factor form

Solve each of these equations.

a $(x-1)(x+3)(x-2)=0$ **b** $(2x+1)(x-5)(x-4)^2=0$

THINK	WRITE
a **1** The left-hand side is in factor form and the right-hand side is equal to 0, so we apply the Null Factor Law.	**a** $(x-1)(x+3)(x-2)=0$
2 Solve each linear equation.	$x-1=0$ or $x+3=0$ or $x-2=0$ $x=1$ or $x=-3$ or $x=2$
3 Write the solutions in ascending order.	$x=-3$, 1 or 2
b **1** Apply the Null Factor Law.	**b** $(2x+1)(x-5)(x-4)^2=0$
2 Solve each linear equation.	$2x+1=0$ or $x-5=0$ or $x-4=0$ $x=-\frac{1}{2}$ or $x=5$ or $x=4$
3 Write the solutions in ascending order.	$x=-\frac{1}{2}$, 4 or 5

Example 6D.2 Solving a cubic polynomial equation

Solve $x^3-7x^2+7x+15=0$.

THINK	WRITE
1 Name the polynomial.	$P(x)=x^3-7x^2+7x+15=0$
2 Use the factor theorem to identify a linear factor of the polynomial.	$P(1)=1-7+7+15$ $=16$ $\neq 0$ $P(-1)=-1-7-7+15$ $=0$ So $(x+1)$ is a factor of $P(x)$.
3 Use long division to find the quotient when $P(x)$ is divided by $(x+1)$.	$\begin{array}{r} x^2-8x+15 \\ x+1\big)\overline{x^3-7x^2+7x+15} \\ \underline{x^3+x^2} \\ -8x^2+7x+15 \\ \underline{-8x^2-8x} \\ 15x+15 \\ \underline{15x+15} \\ 0 \end{array}$
4 Write $P(x)$ as the product of the divisor and quotient.	$P(x)=(x+1)(x^2-8x+15)$
5 Factorise the quadratic factor.	$=(x+1)(x-3)(x-5)$
6 Solve $P(x)=0$ using the Null Factor Law.	For $P(x)=0$: $(x+1)(x-3)(x-5)=0$ $x=-1$, 3 or 5

Helpful hints

✔ Ensure that the equation shows that something is equal to zero before applying the Null Factor Law. Remember that 'null' means zero.

For example, if $(x + 2)(x + 3)(x + 4) = 1$, you do not solve the equations $x + 2 = 1$, $x + 3 = 1$ or $x + 4 = 1$. You need to expand the polynomial, make the equation equal to zero, factorise again, and then apply the Null Factor law.

✔ Remember, sometimes a quadratic equation will have irrational solutions, or no real solutions at all. If you factorise a polynomial, $P(x)$, into the product of linear and quadratic factors, then you might need to use the quadratic formula $\left(x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$ to determine any irrational solutions, if they exist.

For example,
$$P(x) = x^3 - 2x^2 - 6x + 4$$
$$P(x) = (x + 2)(x^2 - 4x + 2)$$
$$P(x) = (x + 2)(x - 2 + \sqrt{2})(x - 2 - \sqrt{2})$$

✔ The discriminant ($\Delta = b^2 - 4ac$) can be used to check whether a quadratic factor can be factorised further. Remember that when $\Delta > 0$ there are two real solutions, when $\Delta = 0$ there is one real solution, and when $\Delta < 0$ there are no real solutions and the expression cannot be further factorised.

ANS
p725

Exercise 6D Solving polynomial equations

△ **1, 2, 3–4(1st, 2nd columns), 5–8, 9(a, b), 10–12, 14**

■ **3–4(2nd, 3rd columns), 5–8, 9(c, d), 12, 13, 15, 17**

⬟ **4(2nd, 3rd columns), 5–7, 9(e, f), 13, 15, 16, 18, 19**

UNDERSTANDING AND FLUENCY

6D.1 **1** Solve each of these equations.

a $(x + 2)(x + 5)(x - 4) = 0$ **b** $(x + 1)(x - 3)(x + 4) = 0$ **c** $(x - 6)(x - 2)(x + 3) = 0$

d $x(x + 2)(x - 9) = 0$ **e** $(2x - 1)(x + 1)(x - 1) = 0$ **f** $(3x - 2)(2x + 7)(x + 5) = 0$

2 Solve each of these equations.

a $(x + 3)(x - 4)(x + 7)(x - 1) = 0$ **b** $(x - 2)(x - 5)(x - 3)(x + 6) = 0$ **c** $(x + 4)(x + 1)(x + 2)^2 = 0$

d $x^2(x + 5)(x - 6) = 0$ **e** $5x(3x + 1)(x + 4)(4 - x) = 0$ **f** $(4x - 3)(2x + 5)(x^2 + 1) = 0$

3 Fully factorise the left-hand side of each of these equations. Then solve each equation.

a $(x + 1)(x^2 + 8x + 12) = 0$ **b** $(x - 3)(x^2 - x - 20) = 0$ **c** $(x - 2)(x^2 - 6x + 8) = 0$

d $(x + 4)(x^2 + 6x - 7) = 0$ **e** $(x + 3)(x^2 - 16) = 0$ **f** $x(x - 1)(x^2 - 1) = 0$

g $(x - 5)(x^2 - 10x + 25) = 0$ **h** $(x + 2)(x^2 + 6x + 9) = 0$ **i** $(x - 4)(x^2 - x - 6) = 0$

j $(x + 1)(x^2 + 2x + 2) = 0$ **k** $3(x + 3)(5x^2 + 30x) = 0$ **l** $(x - 2)(2x^2 + 4x - 6) = 0$

6D.2 **4** Solve each of these equations.

a $x^3 + 6x^2 + 5x - 12 = 0$ **b** $x^3 - 5x^2 - 4x + 20 = 0$ **c** $x^3 + x^2 - 36x - 36 = 0$

d $x^3 + 10x^2 + 21x = 0$ **e** $x^3 - 7x - 6 = 0$ **f** $x^3 + 2x^2 + 5x + 10 = 0$

g $x^3 + 3x^2 - 9x - 27 = 0$ **h** $x^3 + 9x^2 + 24x + 16 = 0$ **i** $x^3 - 3x^2 - 3x - 4 = 0$

5 Solve each of these equations by first taking out a common factor.

a $2x^3 - 2x^2 - 20x - 16 = 0$ **b** $3x^3 - 15x^2 - 3x + 15 = 0$

c $5x^3 + 10x^2 - 20x - 40 = 0$ **d** $4x^3 + 24x^2 + 44x + 24 = 0$

6 Solve each of these equations by first taking out a negative common factor.

a $-x^3 - 2x^2 + 9x + 18 = 0$ **b** $-x^3 + 4x^2 + 17x - 60 = 0$

c $-2x^3 + 8x^2 + 2x - 8 = 0$ **d** $-3x^3 - 3x^2 + 24x + 36 = 0$

UNDERSTANDING AND FLUENCY

7 Solve each of these equations using the quadratic formula after a linear factor has been taken out. Write the solutions as exact values.

a $x^3 + x^2 - 3x + 1 = 0$ **b** $x^3 + 6x^2 + 9x + 2 = 0$

c $x^3 + x^2 - 10x - 12 = 0$ **d** $x^3 + 4x^2 - 27x - 20 = 0$

8 Work through the following steps to solve $P(x) = 0$, where $P(x) = x^4 + x^3 - 7x^2 - x + 6$.

a Identify one of the linear factors of $P(x)$.

b Use long division to find the quotient when $P(x)$ is divided by the linear factor you identified in part **a**. Write $P(x)$ as the product of the divisor and the quotient, $Q(x)$.

c Identify a linear factor of $Q(x)$ and use long division to find its quadratic factor.

d Factorise the quadratic factor you found in part **c**.

e Write $P(x)$ as a product of four linear factors. Then solve $P(x) = 0$ using the Null Factor Law.

9 Solve each of these equations.

a $x^4 - 5x^3 + 5x^2 + 5x - 6 = 0$ **b** $x^4 - 4x^3 - 7x^2 + 22x + 24 = 0$ **c** $x^4 + 7x^3 + 8x^2 - 28x - 48 = 0$

d $x^4 - x^3 - 19x^2 - 11x + 30 = 0$ **e** $x^4 - 5x^3 + 20x - 16 = 0$ **f** $x^4 - 13x^2 + 36 = 0$

PROBLEM SOLVING AND REASONING

10 What is the maximum number of real solutions for $P(x) = 0$ if $P(x)$ is:

a a linear polynomial? **b** a quadratic polynomial? **c** a cubic polynomial?

d a quartic polynomial? **e** a polynomial of degree 6? **f** a polynomial of degree n?

11 a Using only the factor theorem, find three linear factors of $P(x)$ when $P(x) = x^3 - 2x^2 - 5x + 6$.

b Solve $P(x) = 0$.

12 Explain why $x^3 + x^2 - 2x - 8 = 0$ has only one real solution.

13 Explain why $x^3 - 5x^2 + 3x + 9 = 0$ has only two real solutions.

14 The volume of a lamington is 192 cm^3. Its length is twice its height, and its width is 2 cm more than its height.

a Write a polynomial to represent the volume of the slice.

b Solve a polynomial equation to find the dimensions of the slice.

15 The volume of a Toblerone chocolate box is 450 cm^3. The height of the box's triangular face is 1 cm less than the length of the base of that face. The perpendicular length of the box is five times the length of the base of the triangular face. Find the dimensions of the box.

16 A polynomial equation has the solutions $x = 1$, $x = -1$, $x = 2$ and $x = -3$. Write all possible equations, equal to zero and in factorised form, if the polynomial:

a is of degree 4 and its leading coefficient is 1

b is of degree 5 and its leading coefficient is 1

c is of degree 5 and its leading coefficient is 2

d is of degree 6 and its leading coefficient is 5.

e If the polynomial is of degree n and its leading coefficient is a, where n is a non-zero, positive integer and a is a real number, then write a generalised equation for the factorised form.
(Hint: let p, q and r be non-zero positive integers for three of the four possible indices.)

CHALLENGE

17 $P(x) = x^4 - 2x^3 - 13x^2 + 14x + 24$ has the quadratic factor $(x^2 - x - 2)$. Factorise $P(x)$ and solve $P(x) = 0$.

18 Solve each of these equations.

a $x^5 - x^4 - 17x^3 - 19x^2 + 16x + 20 = 0$ **b** $x^5 - 2x^4 - 15x^3 + 20x^2 + 44x - 48 = 0$

19 Solve the equation $(x + 2)(x - 3)(x + 4) = (x + 2)(x - 10)$.

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Solving simultaneous equations numerically using cobweb diagrams

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Topic quiz
6D

6E Sketching graphs of polynomials using intercepts

Learning intentions

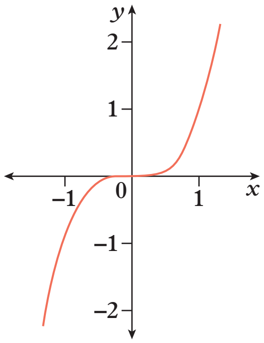
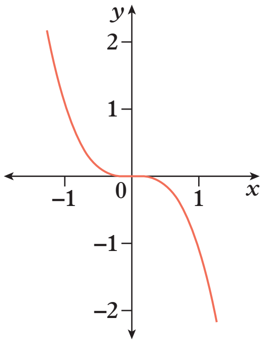
- ✔ I can sketch the graph of a cubic relationship.
- ✔ I can sketch the graph of a cubic relationship with repeated factors.
- ✔ I can sketch the graph of a quartic relationship.
- ✔ I can sketch the graph of a quartic relationship with repeated factors.

Inter-year links

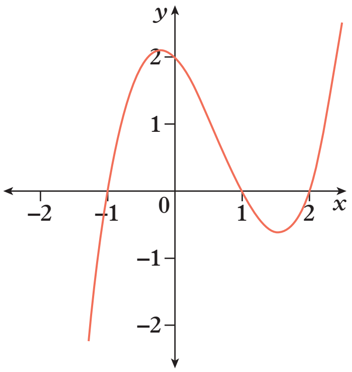
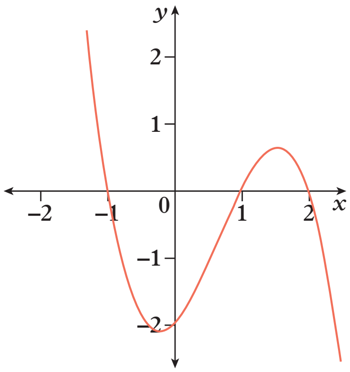
Year 7	6E Substitution
Year 8	5B Substitution
Year 9	5C Sketching parabolas using intercepts

Sketching graphs of cubic relationships

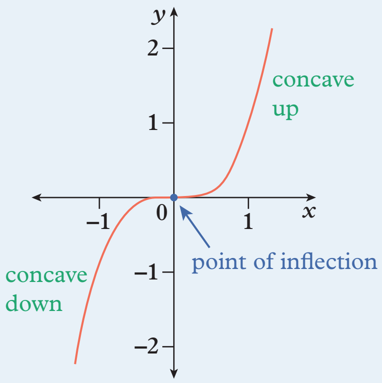
- When the leading coefficient of a cubic function is positive, the curve starts from the bottom left of the Cartesian plane.
- When the leading coefficient of a cubic function is negative, the graph is reflected in the x-axis, and the curve starts from the top left of the Cartesian plane.

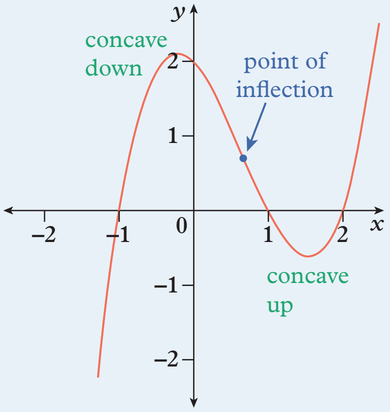
$y = x^3$	$y = -x^3$
	

- The graph of the function $P(x) = (x - a)(x - b)(x - c)$ has x-intercepts at $(a, 0)$, $(b, 0)$ and $(c, 0)$; and has a y-intercept at $(0, abc)$.
- The graph of the function $P(x) = -(x - a)(x - b)(x - c)$ is the reflection in the x-axis of $P(x) = (x - a)(x - b)(x - c)$ and has x-intercepts at $(a, 0)$, $(b, 0)$ and $(c, 0)$; and a y-intercept at $(0, -abc)$.

$y = (x - 2)(x - 1)(x + 1)$	$y = -(x - 2)(x - 1)(x + 1)$
	

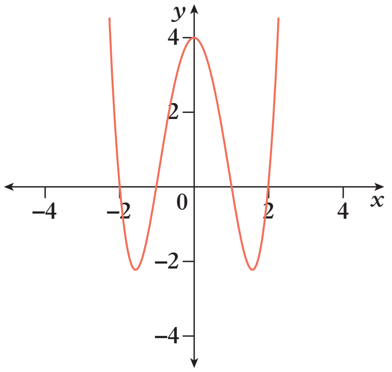
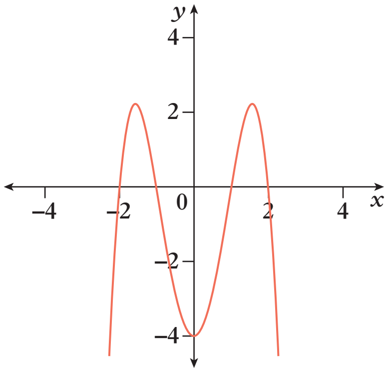
- A **point of inflection** is the point where a curve changes between being ‘concave up’ and ‘concave down’. For example, the point of inflection is shown on each of the curves below.





Sketching graphs of quartic relationships

- The graph of the function $P(x) = (x - a)(x - b)(x - c)(x - d)$ has x-intercepts at $(a, 0)$, $(b, 0)$, $(c, 0)$ and $(d, 0)$; and has a y-intercept at $(0, abcd)$. The curve starts from the top left of the Cartesian plane.
- The graph of the function $P(x) = -(x - a)(x - b)(x - c)(x - d)$ is the reflection in the x-axis of $P(x) = (x - a)(x - b)(x - c)(x - d)$. The curve starts from the bottom left of the Cartesian plane.

$y = (x - 2)(x - 1)(x + 1)(x + 2)$	$y = -(x - 2)(x - 1)(x + 1)(x + 2)$
	

- To sketch the graph of a polynomial relationship:
  1 write the polynomial in factor form
  2 identify the x-intercepts (find x when $y = 0$)
  3 identify the y-intercept (find y when $x = 0$)
  4 draw a smooth curve through the known points
  5 if necessary, find the coordinates of another point to confirm the orientation of the graph.

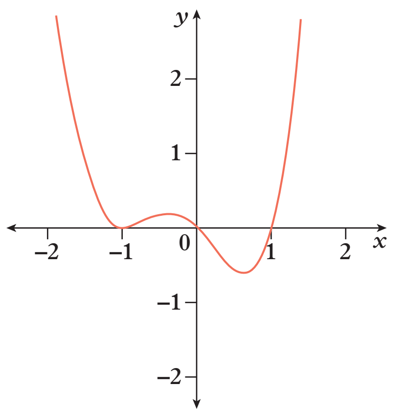
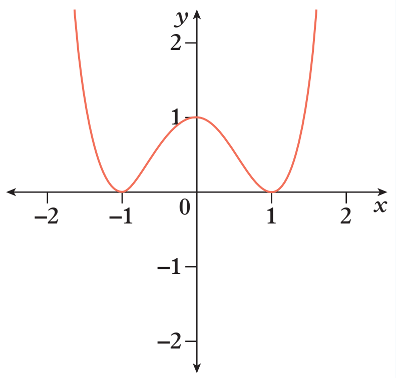
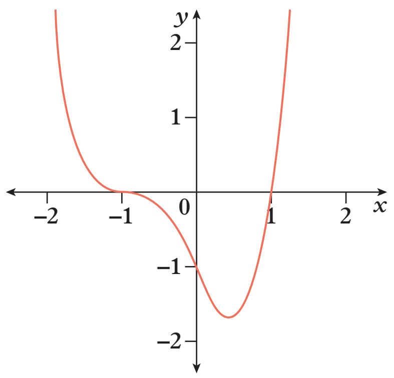
Repeated factors

- When a linear factor, $(x - a)$, is repeated in a polynomial, the factor is raised to the power of an index greater than 1.

  For example, in the cubic function $P(x) = (x + 1)(x - 2)^2$, the linear factor $x - 2$ is repeated 2 times. Or, in the quartic function $P(x) = (x - 3)(x + 5)^3$, the linear factor $x + 5$ is repeated 3 times.
- When a linear factor is repeated in a polynomial, $P(x)$, the number of solutions to the equation $P(x) = 0$ will be less than the degree of the polynomial. This reduces the number of possible x-intercepts in the graph of $y = P(x)$.

- When a linear factor, $(x - a)$, is repeated in a polynomial it means that the x-intercept at a will have a horizontal gradient as it passes through the x-axis. For a non-constant polynomial, points with horizontal gradients will correspond to points of inflection and turning points.

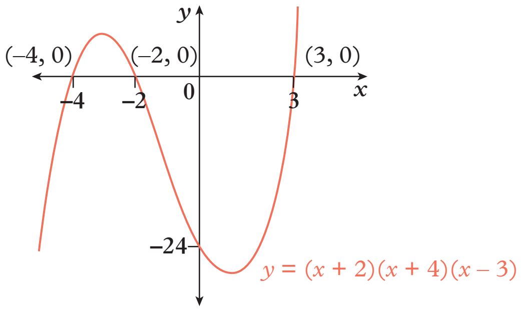
  For example, consider how repeated factors affect the shape of the graphs of the quartic relationships in the table below. Look at the x-intercepts corresponding to each repeated factor.

One factor to the power of 2	Two factors to the power of 2	One factor to the power of 3
$y = x(x - 1)(x + 1)^2$	$y = (x - 1)^2(x + 1)^2$	$y = (x - 1)(x + 1)^3$
		

Example 6E.1 Sketching the graph of a cubic relationship

Sketch the graph of each of these cubic relationships.

a $y = (x + 2)(x + 4)(x - 3)$ **b** $y = -x(x - 1)(x + 2)$

THINK	WRITE
a **1** Substitute $y = 0$ and use the Null Factor Law to find the x-intercepts.	When $y = 0$: $(x + 2)(x + 4)(x - 3) = 0$ $x + 2 = 0$ or $x + 4 = 0$ or $x - 3 = 0$ $x = -2, -4$ or 3 The x-intercepts are at $(-4, 0)$, $(-2, 0)$ and $(3, 0)$.
2 Substitute $x = 0$ to find the y-intercept.	When $x = 0$: $y = (2)(4)(-3)$ $= -24$ The y-intercept is at $(0, -24)$.
3 Mark the four intercepts on a Cartesian plane and draw a smooth curve through them. The leading coefficient is positive, so the curve starts from the bottom left of the Cartesian plane. Label the graph with its equation.	

b **1** Substitute $y = 0$ and use the Null Factor Law to find the x-intercepts.

When $y = 0$:

$-x(x - 1)(x + 2) = 0$

$-x = 0$ or $x - 1 = 0$ or $x + 2 = 0$

$x = -2, 0$ or 1

The x-intercepts are at $(-2, 0)$, $(0, 0)$ and $(1, 0)$.

2 Substitute $x = 0$ to find the y-intercept.

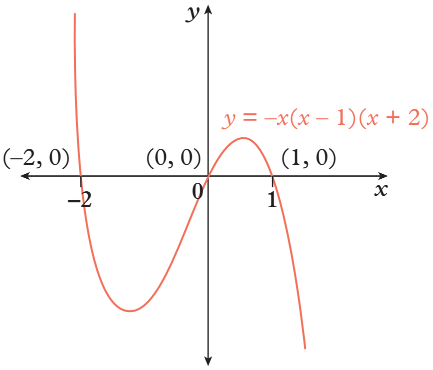
When $x = 0$:

$y = -(0)(-1)(2)$

$= 0$

The y-intercept is at $(0, 0)$.

3 Mark the three intercepts on a Cartesian plane and draw a smooth curve through them. The leading coefficient is negative, so the curve starts from the top left of the Cartesian plane due to the negative sign. Label the graph with its equation.



Example 6E.2 Sketching the graph of a quartic relationship

Sketch the graph of $(x - 2)(x + 3)(x + 1)(x - 4)$.

THINK

WRITE

1 Substitute $y = 0$ and use the Null Factor Law to find the x-intercepts.

When $y = 0$:

$(x - 2)(x + 3)(x + 1)(x - 4) = 0$

$x - 2 = 0$ or $x + 3 = 0$ or $x + 1 = 0$ or $x - 4 = 0$

$x = -3, -1, 2$ or 4

The x-intercepts are at $(-3, 0)$, $(-1, 0)$, $(2, 0)$ and $(4, 0)$.

2 Substitute $x = 0$ to find the y-intercept.

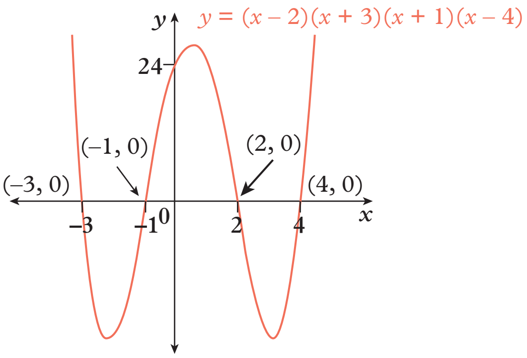
When $x = 0$:

$y = (-2)(3)(1)(-4)$

$= 24$

The y-intercept is at $(0, 24)$.

3 Mark the five intercepts on a Cartesian plane and draw a smooth curve through them. The leading coefficient is positive, so the curve starts from the top left of the Cartesian plane. Label the graph with its equation.



Helpful hints

- ✔ In this section, you don't need to plot the graph, only to sketch it. The only points to plot are the intercepts. To understand the shape of the graph, ask yourself questions like: 'What happens when x is very large or very small?' and 'Which way does the graph need to turn to pass through all of the intercepts without touching an axis in any other places?'
- ✔ When sketching, remember to use an approximate scale to ensure accuracy. The more accurate your scale, the more accurate the shape of your sketch will be.
- ✔ Plot all the intercepts before attempting to sketch the line.
- ✔ Don't forget to check for both x- and y-intercepts! Some students find the x-intercepts and forget to find the y-intercepts.
- ✔ Make sure your lines are smooth rather than dashed lines. And be careful to create a smooth curve rather than a pointy line when approaching turning points.

ANS p726

Exercise 6E Sketching graphs of polynomials using intercepts

△ 1–5, 6(a, b, e, g), 7–9, 10(a, b), 13

□ 3–5, 6(c, d, f, h), 8, 9(b, d), 10(c–f), 12, 14

⬠ 4, 5(d–f), 6(e–h), 9(b, d), 10(b, d, f), 11, 12, 15–17

UNDERSTANDING AND FLUENCY

6E.1 **1** Sketch the graph of each of these cubic relationships.

a $y = (x + 1)(x + 3)(x - 1)$ **b** $y = (x - 2)(x - 5)(x + 1)$ **c** $y = (x - 4)(x + 2)(x + 3)$

d $y = (x - 3)(x - 1)(x - 5)$ **e** $y = x(x + 3)(x - 3)$ **f** $y = 2x(x + 4)(x - 1)$

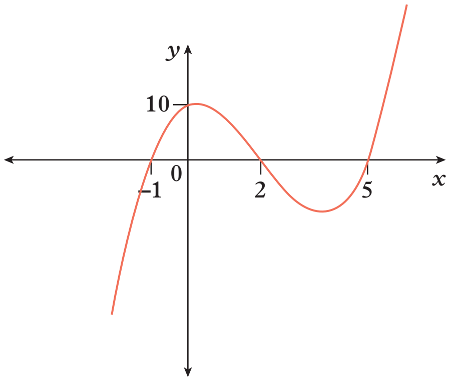
2 For each of the cubic relationships below:

i identify the x-intercepts

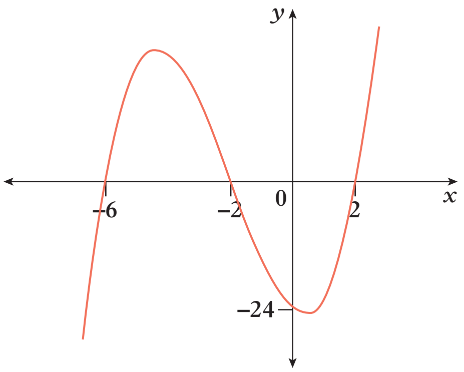
ii identify the y-intercept

iii write the rule in factorised form.

a



b



3 **a** Find the x- and y-intercepts for:

i $y = (x + 7)(x - 2)(x - 3)$

ii $y = -(x + 7)(x - 2)(x - 3)$

b How is the graph of $y = -(x + 7)(x - 2)(x - 3)$ different from the graph of $y = (x + 7)(x - 2)(x - 3)$?

4 Sketch the graph of each of these cubic relationships.

a $y = -(x + 5)(x - 1)(x - 3)$ **b** $y = -(x - 6)(x + 1)(x - 2)$

c $y = -(x + 2)(x + 3)(x - 5)$ **d** $y = -x(x + 7)(x - 4)$

UNDERSTANDING AND FLUENCY

6E.2 **5** Sketch the graph of each of these quartic relationships.

a $y = (x - 1)(x + 2)(x + 3)(x - 3)$

b $y = (x + 4)(x + 1)(x - 1)(x + 3)$

c $y = x(x - 2)(x - 6)(x + 5)$

d $y = -(x - 3)(x + 3)(x + 1)(x - 1)$

e $y = (2x - 3)(x + 2)(x + 1)(x - 4)$

f $y = -(3x - 1)(x + 3)(x + 1)(x + 2)$

6 Sketch the graph of each of these polynomial relationships. (Hint: You will need to first factorise the polynomial.)

a $y = x^3 - 3x^2 - 13x + 15$

b $y = -x^3 - 2x^2 + 16x + 32$

c $y = x^3 - 2x^2 - 3x$

d $y = -2x^3 + 8x^2 - 2x - 12$

e $y = x^4 - 15x^2 - 10x + 24$

f $y = x^4 - 9x^3 + 6x^2 + 56x$

g $y = -x^4 + 10x^2 - 9$

h $y = -2x^4 - 9x^3 + 18x^2 + 71x + 30$

7 Use digital technology to check your graphs from question **6**.

PROBLEM SOLVING AND REASONING

8 Consider the relationship $y = (x + 2)(x - 3)^2$.

a What type of polynomial relationship is this?

b Find the x- and y-intercepts.

c As the leading coefficient of the polynomial is positive, should the graph start from the top left or bottom left of the Cartesian plane?

d Sketch the graph. Use digital technology to verify your answer.

e What effect does the repeated factor of $(x - 3)$ have on the graph?

9 Sketch the graph of each of these cubic relationships.

a $y = (x + 4)(x - 1)^2$

b $y = (x + 2)^2(x - 5)$

c $y = -(x - 2)(x - 7)(x - 2)$

d $y = -(2x + 1)(x - 4)^2$

10 Sketch the graph of each of these quartic relationships.

a $y = (x - 3)(x + 2)(x + 1)^2$

b $y = -(x - 2)^2(x - 4)(x + 2)$

c $y = -x(x - 2)(x + 3)^2$

d $y = x^2(x + 5)(x - 3)$

e $y = (x - 1)^2(x + 2)^2$

f $y = -x^2(x - 4)^2$

11 A squared factor in a polynomial relationship means that an x-intercept is also a turning point. Investigate the effect of having a cubed factor in a quartic relationship. That is, find what happens at $x = a$ for a graph of the form $y = (x - a)^3(x - b)$. Use digital technology to try different values for a and b.

12 The motion of a person on a particular water slide can be approximated using the cubic relationship $h = -\frac{1}{5}(t^3 - 11t^2 + 39t - 45)$, where h is the height above the ground in metres and t is the time in seconds since the start of the slide.

a Sketch a graph of this relationship.

b At what height above the ground does a person start the slide?

c How high is a person after 1 second?

d The ride descends to its lowest point. How long does this take?

e How long does it take for the slide to ascend from its lowest point and then descend again to its lowest point?

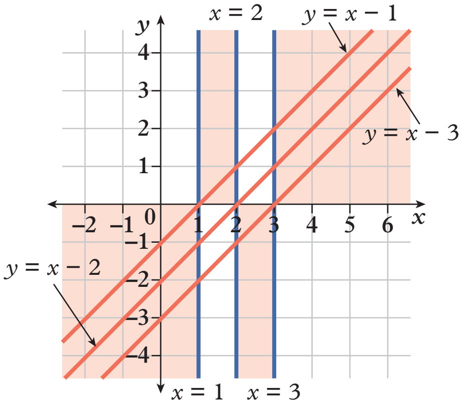
13 Amelia monitors the change in value of shares during the month of June. She finds that the change in value, y (in dollars), after x days can be approximated using a cubic relationship, where y increases from \$0 to \$21 after 5 days and is zero again after 12 days and 20 days.

a Sketch a graph of this cubic relationship.

b Find the equation for this relationship.

c Use this relationship to estimate the change in the value of the shares by the end of June.

14 Shown below are the graphs of the lines $y = x - 1$, $y = x - 2$ and $y = x - 3$ as well as the vertical lines $x = 1$, $x = 2$ and $x = 3$. The shaded regions show where the function of the product of the lines, $y = (x - 1)(x - 2)(x - 3)$, will lie.



a The table below shows if a line will have positive or negative y values for the given x values. Copy the table and complete it by writing 'positive' or 'negative' in each space.

	$x < 1$	$1 < x < 2$	$2 < x < 3$	$x > 3$
$y = x - 1$	negative			
$y = x - 2$				
$y = x - 3$				
$y = (x - 1)(x - 2)(x - 3)$				

b Describe why regions of the graph are shaded.

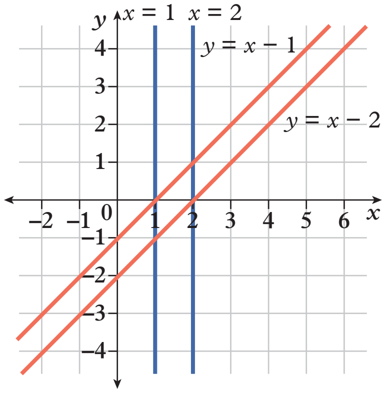
c The table lists inequalities that do not include the values for $x = 1$, $x = 2$, and $x = 3$.

i What is the value of the product $y = (x - 1)(x - 2)(x - 3)$ for these x values?

ii What feature/s of the graph of $y = (x - 1)(x - 2)(x - 3)$ do these x values correspond with?

d Use the information you found in part **c** to help you sketch the graph of $y = (x - 1)(x - 2)(x - 3)$. Label all axis intercepts.

e Shown below are the graphs of the lines $y = x - 1$ and $y = x - 2$ as well as the vertical lines $x = 1$ and $x = 2$.



PROBLEM SOLVING AND REASONING

Copy the table below and complete it by writing ‘positive’ or ‘negative’ in each space.

	$x < 1$	$1 < x < 2$	$x > 2$
$y = x - 1$			
$y = x - 2$			
$y = x - 2$			
$y = (x - 1)(x - 2)^2$			

f Copy the graphs on the previous page in part **e** and shade the regions in which the graph of $y = (x - 1)(x - 2)^2$ will lie.

g Explain why you shaded the regions you did in part **f**.

h Sketch the graph of $y = (x - 1)(x - 2)^2$. Label all axis intercepts.

15 a Sketch $y = x$, $y = x^3$ and $y = x^5$ on the same set of axes.

b Sketch $y = -x$, $y = -x^3$ and $y = -x^5$ on the same set of axes.

c By considering the extreme ends of the graphs from parts **a** and **b**, describe the rough shape of the graphs of polynomials of the form $y = x^n$ and $y = -x^n$ when n is odd.

d Sketch $y = x^2$, $y = x^4$ and $y = x^6$ on the same set of axes.

e Sketch $y = -x^2$, $y = -x^4$ and $y = -x^6$ on the same set of axes.

f By considering the extreme ends of the graphs from parts **d** and **e**, describe the rough shape of the graphs of polynomials of the form $y = x^n$ and $y = -x^n$ when n is even.

g Use digital technology to sketch the following graphs. Do **not** find or plot any intercepts.

i $y = x^2 + x$

ii $y = x^3 + x^2 + x$

iii $y = x^4 + x^3 + x^2 + x$

iv $y = x^5 + x^4 + x^3 + x^2 + x$

v $y = x^6 + x^5 + x^4 + x^3 + x^2 + x$

vi $y = x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x$

h By considering the extreme ends of the graphs from part **g**, describe the rough shape of polynomials with a leading term of x^n and $-x^n$, when n is odd or even.

CHALLENGE

16 a Zara is using the factor theorem to attempt to factorise $P(x) = x^3 - 10x^2 - 11x + 180$ so she can graph $y = P(x)$. So far, she has found the following: $P(-6) = -330$, $P(-2) = 154$, $P(2) = 126$, $P(6) = -30$ and $P(10) = 70$.

i How many x-intercepts does the graph of $y = P(x)$ have?

ii If Zara knows that the x-intercepts have integer x values, then list a possible set of solutions for $P(x) = 0$.

iii Which number(s) can be ruled out from the set? Explain why.

b Sara is using the factor theorem to attempt to factorise $Q(x) = x^4 - 6x^3 - x^2 + 54x - 72$ so she can graph $y = Q(x)$. So far, she has found the following: $Q(1) = -24$, $Q(-1) = -120$, $Q(2) = 0$, $Q(-2) = -120$, $Q(3) = 0$, $Q(-3) = 0$, $Q(4) = 0$ and $Q(-4) = 336$.

i State the coordinates of the x-intercepts of $y = Q(x)$.

ii Explain how Sara could use the remaining substitutions to sketch the graph of $y = Q(x)$ more accurately.

17 a Sketch a rough graph of $y = x(x - 6)(x - 3)(x + 2)(x + 4)(x + 6)$. You do not need to label any intercepts.

b Sketch a rough graph of $y = (x - 8)(x - 5)^3(x + 1)^2(x + 5)^3(x + 10)^2$. You do not need to label any intercepts.

Check your Student obook pro for these digital resources and more:

pro

Interactive skillsheet
Sketching graphs of cubic relationships

Interactive skillsheet
Sketching graphs of quartic relationships

Worksheet
Solving polynomial equations and graphing polynomial relationships

Investigation
Finding an intersection point of two graphs using a convergent set of intervals

CAS instructions
Graphing functions

Topic quiz
6E

6F Sketching graphs of polynomials using transformations

Learning intentions

✔ I can perform transformations on the graph of a polynomial relationship.

Inter-year links

Years 5/6	Transformations
Year 7	8E Rotations and reflections
Year 8	7C Transformations
Year 9	5D Sketching parabolas using transformations

Transformations of a polynomial relationship

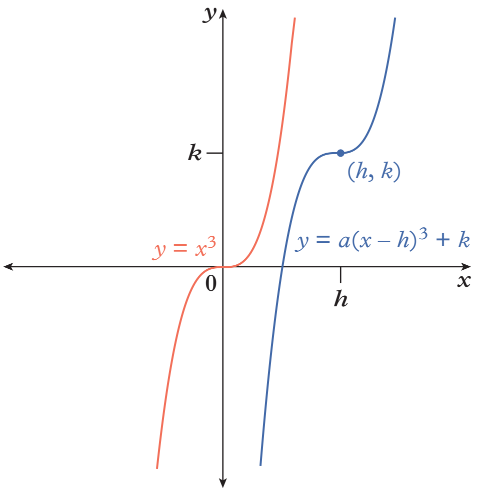
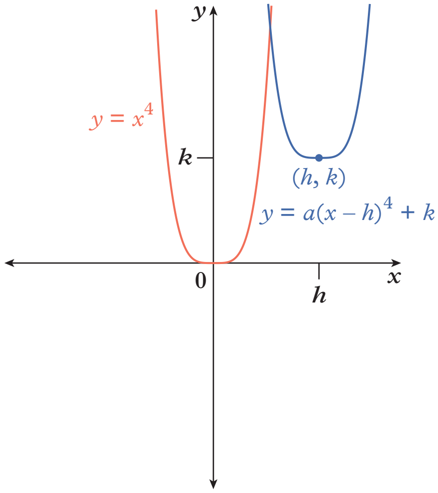
- Transformations such as stretches, reflections and translations can be performed on the graph of $y = P(x)$.
- A stretch by a factor of a in the direction of the y-axis produces $y = aP(x)$.
- Reflection in the x-axis produces $y = -P(x)$.
- A vertical translation of k units produces $y = P(x) + k$.
- A horizontal translation of h units produces $y = P(x - h)$.
- A combination of transformations can be performed on the graph of $y = P(x)$ to produce the graph of $y = aP(x - h) + k$.

$$y = aP(x - h) + k$$

Vertical translation k units
For $k > 0$, move up
For $k < 0$, move down

Stretch by a factor of a in the y-axis
For $a < 0$, reflect in the x-axis

Horizontal translation of h units
For $h > 0$, move right
For $h < 0$, move left

Cubic relationships	Quartic relationships
	

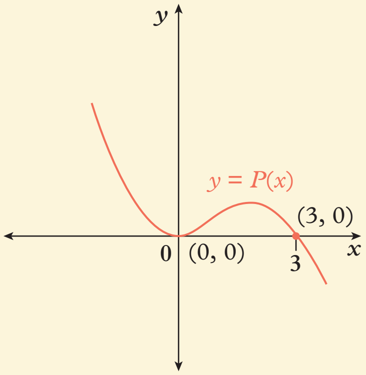
Example 6F.1 Performing transformations on the graph of a polynomial relationship

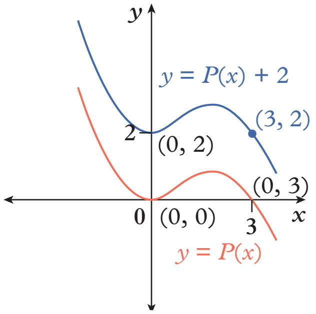
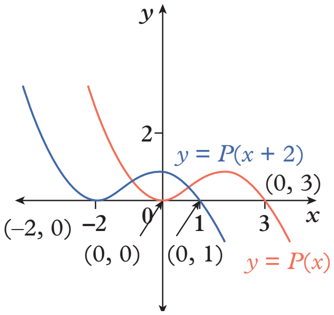
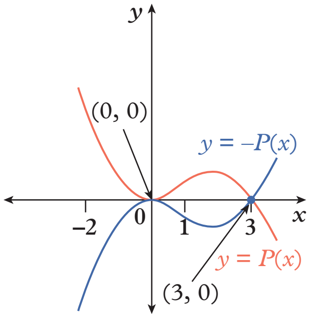
Perform a transformation on the graph of $y = P(x)$ (shown on the right) to produce the graph of:

a $y = P(x) + 2$

b $y = P(x + 2)$

c $y = -P(x)$



THINK	**WRITE**
a **1** Identify the transformation. The original graph needs to be translated 2 units up. **2** Determine the impact of the transformation on the intercepts: (0, 0) moves to (0, 2) and (3, 0) moves to (3, 2). **3** Draw $y = P(x)$ and $y = P(x) + 2$ on the same Cartesian plane.	**a** The graph of $y = P(x)$ will be translated 2 units up. 
b **1** Identify the transformation. The original graph needs to be translated 2 units left. **2** Determine the impact of the transformation on the intercepts. (0, 0) moves to (–2, 0) and (3, 0) moves to (1, 0). **3** Draw $y = P(x)$ and $y = P(x + 2)$ on the same Cartesian plane.	**b** The graph of $y = P(x)$ will be translated 2 units left. 
c **1** Identify the transformation. The original graph needs to be reflected in the x-axis. **2** Determine the impact of the transformation on the intercepts: (0, 0) and (3, 0) remain the same. **3** Draw $y = P(x)$ and $y = -P(x)$ on the same Cartesian plane.	**c** The graph of $y = P(x)$ will be reflected in the x-axis. 

Helpful hints

- ✔ Sketching graphs is an important skill when studying Year 11 and 12 Maths subjects. While you may have access to a graphics calculator to help you, practice makes perfect!
- ✔ You will only become familiar with the shape and turns of a polynomial by sketching them manually. Practice makes permanent!
- ✔ It can be easy to miss the negative sign at the front of a function. Be careful to pay attention, because a negative factor has a large impact on the shape of the graph. It completely flips it!

ANS p732

Exercise 6F Sketching graphs of polynomials using transformations

△ 1–7, 9, 10, 12

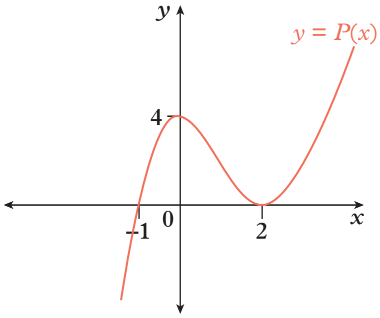
□ 2–8, 11, 12

⬠ 2, 5–8, 11, 13, 14

UNDERSTANDING AND FLUENCY

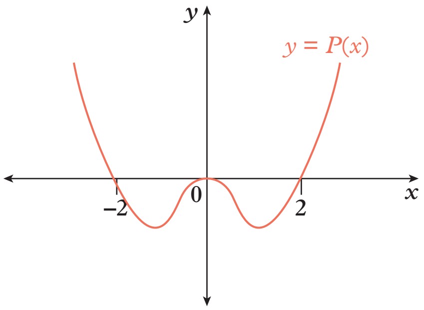
1B.1 **1** Perform a transformation on the graph of $y = P(x)$ (shown below) to produce the graph of:

a $y = P(x) + 1$
b $y = P(x) - 2$
c $y = P(x - 1)$
d $y = P(x + 3)$
e $y = -P(x)$
f $y = 2P(x)$



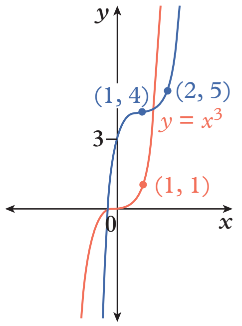
2 Perform a transformation on the graph of $y = P(x)$ (shown below) to produce the graph of:

a $y = P(x) + 3$
b $y = P(x) - 1$
c $y = P(x - 3)$
d $y = P(x + 4)$
e $y = -P(x)$
f $y = 2P(x)$

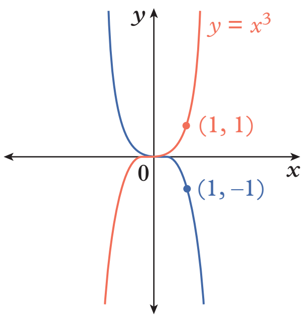


3 Identify the transformations performed on the graph of $y = x^3$ to produce each graph shown in blue below.

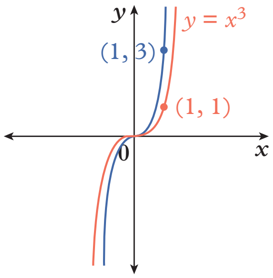
a



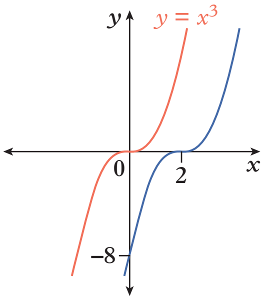
b



c



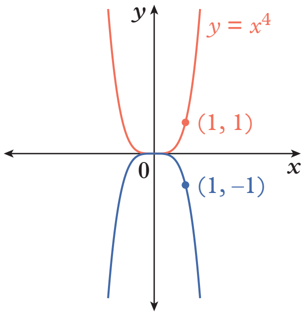
d



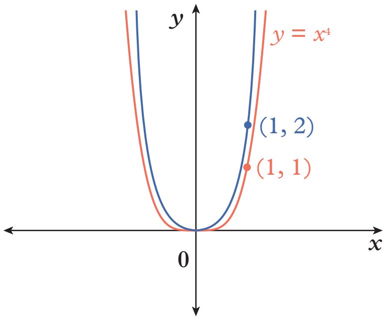
4 Write the equation for each of the graphs in question **3**.

5 Identify the transformations performed on the graph of $y = x^4$ to produce each graph shown in blue below.

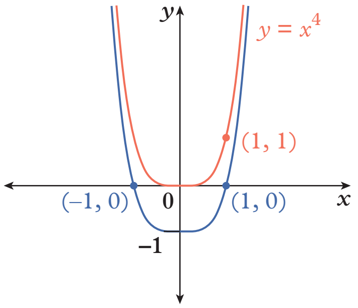
a



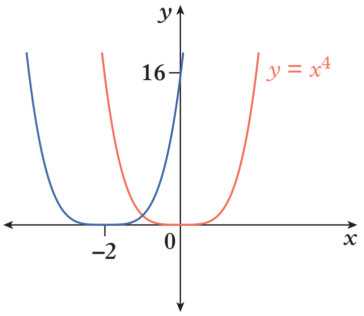
b



c

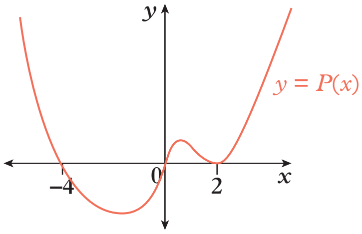


d

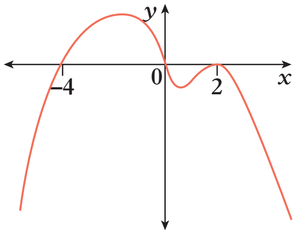


6 Write the equation for each of the transformed graphs in question **5**.

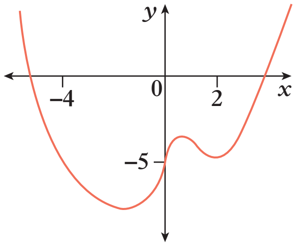
7 Identify the equation for each of these graphs, in terms of $P(x)$, using the graph of $y = P(x)$ shown on the right to help you.



a



b



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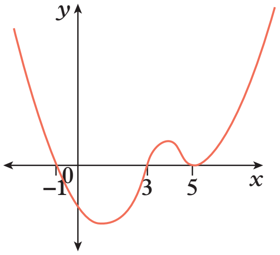


8 Compare the graph produced after reflecting $y = x^3$ in the x-axis with the graph produced after reflecting $y = x^3$ in the y-axis. What is the equation for each of these transformed graphs?

9 For each cubic relationship below:

i describe the transformations that need to be performed on $y = x^3$ to produce the graph of the relationship

ii identify the coordinates of the point of inflection

iii find the x- and y-intercepts

iv sketch the graph.

a $y = \frac{1}{2}(x - 3)^3 + 4$ **b** $y = -2(x + 1)^3 - 2$

10 Use digital technology to help you verify your answers for question **9**.

11 Stretches can also occur in the direction of the x-axis by multiplying the x-coordinates instead of the y-coordinates. Consider the table of values for a polynomial shown on the right.

x	0	1	2	3
$y = P(x)$	5	0	−5	2

When stretching in the x-axis, we replace x with a new expression, then work backwards to find the x values for the transformed graph. This way, the y-coordinates corresponding to the original x values now correspond with the transformed x values.

a Complete each table of values below by performing the appropriate operations to the x-coordinates.

i

$\frac{x}{2}$	0	1	2	3
x				
$y = P\left(\frac{x}{2}\right)$	5	0	−5	2

ii

$2x$	0	1	2	3
x				
$y = P(2x)$	5	0	−5	2

iii

$\frac{x}{5}$	0	1	2	3
x				
$y = P\left(\frac{x}{5}\right)$	5	0	−5	2

iv

$5x$	0	1	2	3
x				
$y = P(5x)$	5	0	−5	2

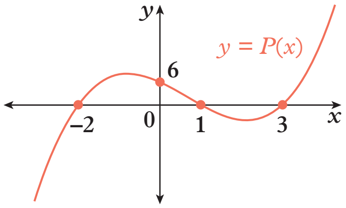
v

$-\frac{x}{2}$	0	1	2	3
x				
$y = P\left(-\frac{x}{2}\right)$	5	0	−5	2

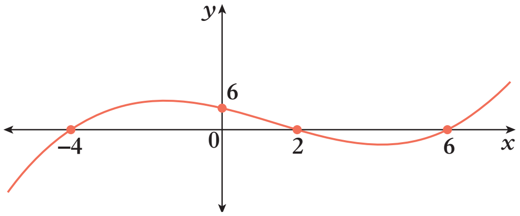
vi

$-2x$	0	1	2	3
x				
$y = P(-2x)$	5	0	−5	2

b Identify the rule for each graph in terms of $P(x)$ using the graph of $y = P(x)$ shown on the right to help you.



i



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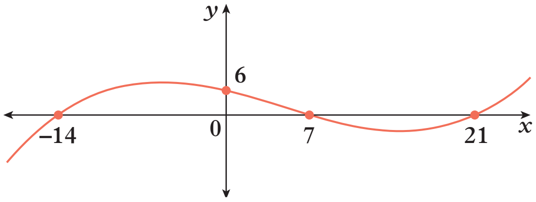

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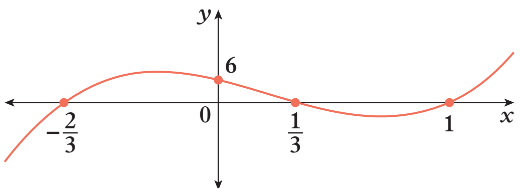

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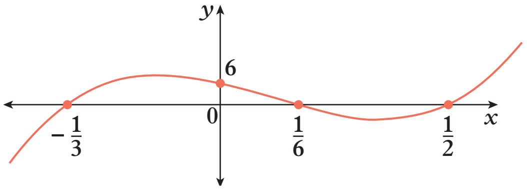
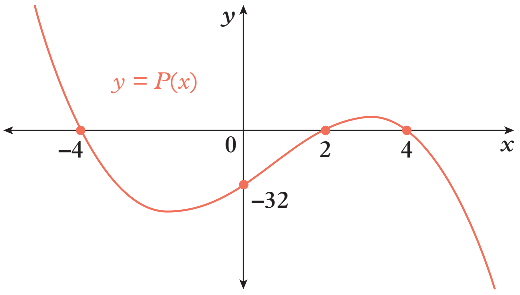
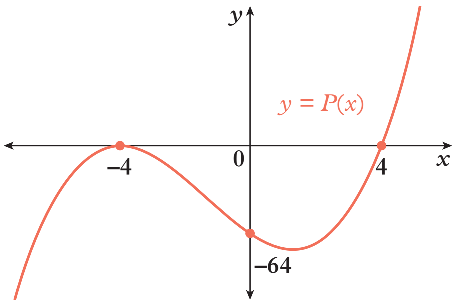


c When multiple transformations occur, we can work backwards to find x and y values. For each of the following polynomials, construct a table of values for the transformed polynomial and sketch the graph.

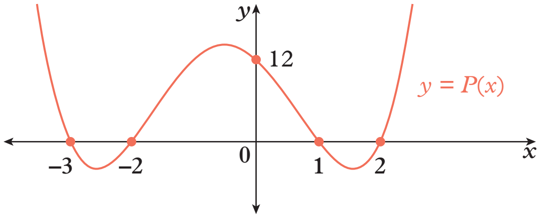
i $y = P(2x - 4)$



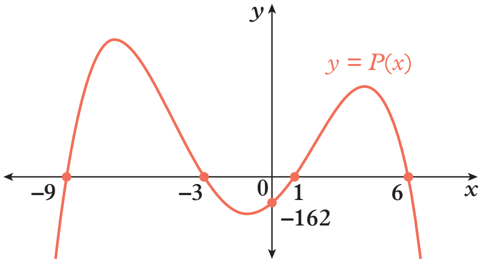
ii $y = P[2(x - 4)]$



iii $y = \frac{1}{3}P\left(\frac{x}{2}\right)$



iv $y = 2P(3x) + 324$



12 Dylan is sketching transformations of the graph of the polynomial $y = P(x)$. He has written the transformation he will apply for each of the following. State whether his descriptions are correct or incorrect. Also state the correct transformation for any incorrect description.

a $y = P(x) + 6$; translate 6 units up

b $y = 4P(x)$; stretch by a factor of 4 in the x-axis

c $y = P(x - 5)$; translate 5 units left

d $y = P(3x)$; stretch by a factor of $\frac{1}{3}$ in the x-axis

13 **a** Expand $(x - 2)^3 + 27$.

b Using what you found in part **a** to help you, sketch the following graphs.

i $y = 4x^3 - 24x^2 + 48x + 76$

ii $y = (x - 2)^3 - 6(x - 2)^2 + 12(x - 2) + 21$

iii $y = -x^3 - 6x^2 - 12x + 19$

14 For polynomials of the form $y = x^n$, a stretch in the x-axis is equivalent to a different stretch in the y-axis. This equivalence can be determined algebraically by expanding or factorising the expression. For example, $y = (2x)^3$ is a stretch of $y = x^3$ by a factor of $\frac{1}{2}$ in the x-axis. When expanded, $y = (2x)^3 = 8x^3$ is a stretch of $y = x^3$ by a factor of 8 in the y-axis.

For each of the following:

a describe the equivalent stretches in both axes of the graph of $y = x^n$ for it to be transformed to the graph of each of the following polynomials

b describe the equivalent stretches in both axes of the graph of each of the polynomials below for it to be transformed to the graph of $y = x^n$.

i $y = (3x)^2$

ii $y = \left(\frac{x}{2}\right)^4$

iii $y = 25x^2$

iv $y = \frac{625}{81}x^4$

Check your Student obook pro for these digital resources and more:

pro

Interactive skillsheet
Sketching polynomials using transformations

Investigation
Maximising the volume of a box

Topic quiz
6F

Chapter summary

Polynomials

$$\overbrace{P(x) = \mathrlap{\underbrace{}_{\text{equation}}}\underbrace{x^2 + 2x + 1}_{\text{expression}}}^{\text{function}} = 0$$

- 1 is a degree 0 polynomial.
- x is a degree 1 polynomial.
- x^2 is a degree 2 polynomial.
- x^3 is a degree 3 polynomial.

The remainder theorem

- When a polynomial $P(x)$ is divided by $(x - a)$, the remainder is $P(a)$.

The distributive law

$$(x^2 + x + 1)(x^2 - 2x - 3) = x^2(x^2 - 2x - 3) + x(x^2 - 2x - 3) + 1(x^2 - 2x - 3)$$

Division of polynomials

$$\begin{array}{rl}
 & x + 2 \leftarrow \text{quotient} \\
\text{divisor} \rightarrow x + 1 \!\!\!\! & \overline{)\, x^2 + 3x + 4} \leftarrow \text{dividend} \\
 & \underline{x^2 + x\phantom{{}+4}} \\
 & \phantom{x^2 + {}} 2x + 4 \\
 & \phantom{x^2 + {}} \underline{2x + 2} \\
 & \phantom{x^2 + 2x + {}} 2 \leftarrow \text{remainder}
\end{array}$$

$$\frac{\overbrace{x^2 + 3x + 4}^{\text{dividend}}}{\underbrace{x + 1}_{\text{divisor}}} = \overbrace{x + 2}^{\text{quotient}} + \frac{\overset{\text{remainder}}{2}}{x + 1}$$

The factor theorem

- When a polynomial $P(x)$ is divided by $(x - a)$ and the remainder $P(a)$ is zero, then $(x - a)$ is a factor of $P(x)$.

The Null Factor Law

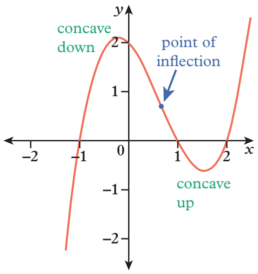
- If $P(x) = (x - a)(x - b)(x - c)$, then $P(x) = 0$ has the solution $x = a$, $x = b$, $x = c$, or a combination of solutions.

$$\overbrace{P(x) = \mathrlap{\underbrace{}_{\text{equation}}}\underbrace{(x - a)(x - b)(x - c)}_{\text{expression}}}^{\text{function}} = 0$$

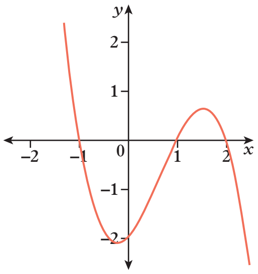
$$\begin{aligned} x - a = 0 &\Rightarrow x = a \\ x - b = 0 &\Rightarrow x = b \\ x - c = 0 &\Rightarrow x = c \end{aligned}$$

Cubics

$P(x) = (x - 2)(x - 1)(x + 1)$

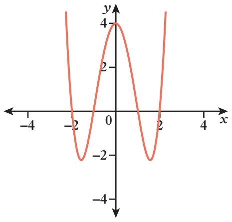


$P(x) = -(x - 2)(x - 1)(x + 1)$

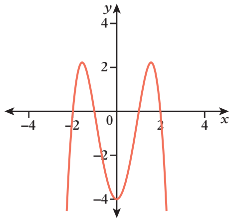


Quartics

$P(x) = (x - 2)(x - 1)(x + 1)(x + 2)$



$P(x) = -(x - 2)(x - 1)(x + 1)(x + 2)$



Sketching polynomials

1. Write the polynomial in factorised form.
2. Identify the x-intercepts (find x when $y = 0$).
3. Identify the y-intercepts (find y when $x = 0$).
4. Draw a smooth curve through the known points.
5. If necessary, find the coordinates of another point to confirm the orientation of the graph.

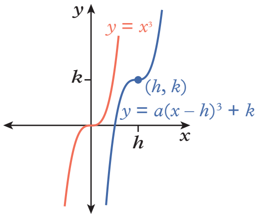
Sketching polynomials using transformations

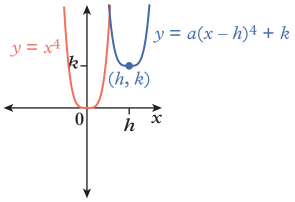
$$y = aP(x - h) + k$$

Vertical translation k units
For $k > 0$, move up
For $k < 0$, move down

Stretch by a factor of a
in the y-axis
For $a < 0$, reflect in the x-axis

Horizontal translation of h units
For $h > 0$, move right
For $h < 0$, move left





Chapter review

Chapter review quiz
Take the chapter review quiz to assess your knowledge of this chapter.

Quizlet
Test your knowledge of this topic by working individually or in teams.

Multiple-choice

10A 6A **1** Which of these expressions is a polynomial?

A $x^2 + \sqrt{x}$ **B** $\frac{3x}{x^2 + 1}$ **C** $x^3 - 2x$ **D** $x^2 + x^{-1}$ **E** $\left(\frac{1}{x} + 2\right)\left(x + \frac{1}{2}\right)$

10A 6A **2** The degree of the polynomial $4x^2 - 3x^4 + x^3 - 2x$ is:

A 0 **B** 1 **C** 2 **D** 3 **E** 4

10A 6A **3** The coefficient of the leading term in the expression $3 - 2x^2 + 5x^3 + 7x$ is:

A 5 **B** −2 **C** 3 **D** 7 **E** 2

10A 6B **4** When $x^2 + 6x - 3$ is divided by $x - 2$, the remainder is:

A 13 **B** −19 **C** 5 **D** −11 **E** $x + 8$

10A 6B **5** Which of these is correct?

A $(5x^2 - x + 2) \div (x - 4) = 5x + 19$, remainder −74
B $(3x^2 - 2x + 1) \div (x - 1) = 3x - 5$, remainder −9
C $(4x^2 - x + 8) \div (x - 2) = 4x - 9$, remainder −10
D $(2x^2 - x + 1) \div (x - 5) = 2x + 9$, remainder 37
E $(6x^2 - 13x + 15) \div (x - 3) = 6x + 5$, remainder 30

10A 6C **6** When $P(x) = x^3 - 2x^2$ is divided by $(x - 1)$, the remainder is:

A $P(0)$ **B** $P(-2)$ **C** $P(2)$ **D** $P(-1)$ **E** $P(1)$

10A 6C **7** Which of these expressions is a factor of $x^3 - 4x^2 + x + 6$?

A $x - 1$ **B** $x - 4$ **C** $x + 3$ **D** $x + 6$ **E** $x - 2$

10A 6D **8** The leading term in a polynomial is raised to the power n. The maximum number of solutions it could have is:

A $n - 1$ **B** $n + 1$ **C** n **D** $\frac{n}{2}$ **E** impossible to tell.

10A 6D **9** How many different solutions does $(x + 1)(x - 1)(x + 2)(x + 1) = 0$ have?

A 0 **B** 1 **C** 2 **D** 3 **E** 4

10A 6E **10** The graph of $y = (x - 3)(x + 2)(x - 5)$ has a y-intercept of:

A 3 **B** −2 **C** 30 **D** −30 **E** 5

10A 6E **11** The graph of $y = (x - 1)^3$ has a point of inflection at:

A $y = -1$ **B** $x = -1$ **C** $x = 0$ **D** $y = 1$ **E** $x = 1$

10A 6F **12** The transformation that needs to be performed on the graph of $y = x^3$ to produce the graph of $y = (x + 5)^3$ is a translation of:

A 5 units right **B** 5 units left **C** 5 units up
D 5 units down **E** 5 units left and 5 units up.

Short answer

10A 6A **1** Decide whether each of these expressions is a polynomial. For those that are polynomials, correctly identify the type of polynomial using its descriptive name: linear, quadratic, cubic or quartic.

a $5 - 3x^2 + 4x - 6x^3$ **b** $1 - 3x$ **c** $\frac{4x}{5}$
d $\frac{6}{7x}$ **e** $6x^2 - 5x + 12$ **f** $\frac{1}{2}x - 3x^4$

10A 6A **2** For the polynomial $3x^4 + 2x^6 - 4x^5 + x - 8x^3 - 7x^2 - 5x^8$, identify:

a the number of terms it has **b** its degree **c** the constant term
d the leading term **e** the leading coefficient **f** the coefficient of the x^2 term.

10A **6B** **3** Use long division to find the quotient and remainder for each of these:

a $(x^2 + 5x - 2) \div (x - 2)$

b $(3x^2 - x + 4) \div (x + 1)$

c $(x^2 - x + 8) \div (x - 4)$

d $(x^3 + 2x^2 - x + 3) \div (x^2 + 1)$

10A **6B** **4** Expand and simplify the right-hand side to verify whether each of these statements is true.

a $2x^2 + 5x - 1 = (2x + 3)(x + 1) - 4$

b $x^2 - 3x + 5 = (x - 1)(x - 2) + 7$

10A **6C** **5** Use the factor theorem to find a linear factor of each of the following polynomials, $P(x)$.

a $P(x) = x^3 - x^2 - 5x - 3$

b $P(x) = x^3 + 4x^2 + 7x + 12$

c $P(x) = x^3 + 3x^2 - 6x + 2$

d $P(x) = x^3 + x^2 + 4$

10A **6C** **6** Fully factorise each of these polynomials.

a $x^3 - 7x^2 + 14x - 8$

b $x^3 - 7x - 6$

10A **6C** **7** Two factors of the polynomial $x^3 - 39x - 70$ are $(x - 7)$ and $(x + 5)$. What is the third factor?

10A **6D** **8** Solve each of these equations.

a $(x - 5)(x + 2)(x - 1)(x + 1) = 0$

b $x(x + 3)(x - 3)(x - 2) = 0$

c $(x + 3)^2(x - 5)^2 = 0$

d $x^2(x + 4)(x - 4) = 0$

9 Fully factorise the left-hand side of each of these equations, then solve.

a $x(x - 2)(x^2 - 4) = 0$

b $(x - 4)(x^2 - 8x + 16) = 0$

10A **6E** **10** Sketch the graph of each of these equations.

a $y = (x - 3)(x + 1)(x - 2)$

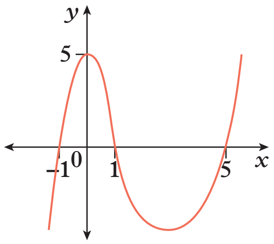
b $y = -(x + 2)^2(x - 4)$

10A **6E** **11** For the cubic relationship shown on the right:

a identify the x-intercepts

b identify the y-intercept

c write the equation as a product of factors.



10A **6E** **12** Consider $y = x^4 - 5x^2 + 4$.

a Write this relationship in terms of its factors.

b Identify the x- and y-intercepts.

c Sketch the graph.

10A **6E** **13** A polynomial relationship is graphed on the right. Write its equation in:

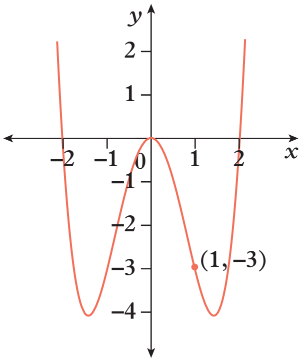
a factor form

b expanded form.

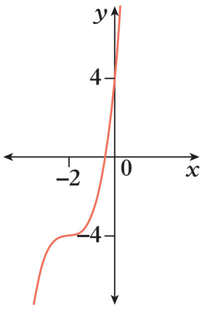
10A **6E** **14** Consider the graph of $y = x(x - 1)^3$.

a Identify the x- and y-intercepts.

b Sketch the graph.



10A **6F** **15** Describe the transformation/s performed on the graph of $y = x^3$ to produce the graph shown on the right.



Analysis

10A You should be familiar with the infinity symbol. It looks like the number 8 on its side. Its shape is similar to the graph of a cubic function with three x-intercepts, combined with its reflection in the x-axis.

a It is possible to model the infinity sign using two cubic relationships graphed for the same set of x values.

i Copy and complete the table below for $y = x^3 - x$.

x	y
–1	
–0.6	
–0.5	
0	
0.5	
0.6	
1	

ii Repeat part **i** for $y = -x^3 + x$.

iii On a Cartesian plane, plot the points from the tables in parts **i** and **ii**. Join the points with a smooth curve.

iv Describe the domain of the two graphs.

v Do the turning points for the two relationships occur at $x = -0.5$ and $x = 0.5$? Explain.

vi Use digital technology to draw the graphs of these two relationships for the set of x values described in part **iv**. Find the coordinates of the turning points.

b It is also possible to model the infinity symbol by tracing the path a point takes as it moves from an angle of 0° with the x-axis to an angle of 360° with the x-axis.

The coordinates of all points (x, y) on the infinity symbol can be represented by the values $\left(\cos[\theta], \frac{\sin[2\theta]}{2}\right)$, where θ is the angle made with the x-axis by the line joining a point with the origin.

i Construct a table like the one below, for x and y using angles of θ increasing by 15°, between 0° and 360°. (This number of points is necessary for an accurate representation of the graph.)

ii Complete the table, giving your answers correct to two decimal places if necessary. You might find some of the trigonometric values to be negative as the angle increases beyond 90°. You will understand the reason for this when you study the trigonometric ratios for angles greater than 90° in Chapter 8.

Angle (θ)	x **($\cos[\theta]$)**	y $\left(\frac{\sin[2\theta]}{2}\right)$
0°		
15°		
30°		
…		
345°		
360°		

iii Plot the points (x, y) on a Cartesian plane. Join them with a smooth curve. (Alternatively, you could use a spreadsheet to plot the graph.)

iv Describe the shape of the graph.

v Describe the domain and range of the graph.

c Compare the two models for infinity symbols from parts **a** and **b**. Do you consider one to be a better model of the infinity symbol than the other? Explain.

Semester 1 review

Short answer

1 Simplify each of the following. Write your answers using positive indices.

a $2x + x^2 + 2 - 5x + 8 - 3x^2 - 5 - 6x$

b $3xy - 4yx^2 + 11xy^2 + 9yx - 5y^2x + 4x^2y + 10xy$

c $\dfrac{x^6y^{-5} \times x^{-4}y^{-11}}{x^{-3}y^3 \times xy^{-2}}$

d $(2x^{-3}y^5)^{-2} \times \left(\dfrac{y^5}{4x^3}\right)^2$

e $\dfrac{12^0x^5y^{-4}}{4^{-2}(xy)^{-4}} \div \dfrac{6^2x^{-5}y}{12^2(x^{-1}y)^3}$

f $\dfrac{3x+2}{5} - \dfrac{8x-4}{3}$

2 Expand and simplify each of the following.

a $3(2x + 5y)$

b $4x^2y(9xy^4 - 8x^3y)$

c $\dfrac{-5x}{4}(3x + 6y - 8)$

d $(2x - 7y)(8a - 7b)$

e $(x - 4)(x + 7) + (x + 4)(x + 3)$

f $(5x + 2)(3x - 4) + 3(2x + 5)$

g $(10x + 3)^2$

h $(x - 5)(3a + 8b + 2c)$

3 Solve each of the following equations.

a $\dfrac{1-2x}{3} = 5$

b $4 + 3(2x - 5) = 11$

c $2(x - 5) - 7(x - 4) = -9$

d $10(3x - 4) = 5(3 - 7x)$

e $\dfrac{4x+5}{5} = \dfrac{6x-5}{3}$

f $\dfrac{x+1}{2} + \dfrac{4-x}{6} = \dfrac{x+4}{4}$

g $\dfrac{9}{3x-1} = -5$

h $\dfrac{12}{x-5} = \dfrac{9}{x+7}$

4 Rearrange each of the following equations to make x the subject.

a $y = mx + c$

b $ax + by = c$

c $rx + t = px - q$

d $A = \frac{1}{2}(x + y)z$

e $A = \pi x^2$

f $V = \frac{4}{3}\pi x^3$

5 **a** Calculate 12% of \$1500.

b Calculate 215% of \$80.

c Write \$30 as a percentage of \$150.

d Write \$400 as a percentage of \$64.

e Calculate the selling price when \$450 is marked up by 22%.

f Calculate the selling price when \$35 is discounted by 75%.

g Write the profit as a percentage of the original cost when the cost price is \$2.50 and the selling price is \$10.

h Write the loss as a percentage of the original cost when the cost price is \$50 and the selling price is \$20.

i Determine the original price if the selling price is \$940 after a mark-up of 25%.

j Determine the original price if the selling price is \$48 after a discount of 36%.

6 Calculate the interest for each of the following, correct to the nearest cent.

a Abdul invested \$3500 at a simple interest rate of 14% per annum for 3 years.

b Nguyen invested \$3200 at an interest rate of 7.2% per annum compounding annually for 3 years.

c Iona borrowed \$5200 for 40 weeks, at a simple interest rate of 24% per annum.

d Ezra borrowed \$8500 at an interest rate of 5.2% per annum compounding monthly for 5 years.

e Louise agrees to pay \$120 per fortnight over 2 years for a \$5000 loan.

7 Calculate the principal for each of the following.

a Huang earns \$500 interest in 5 years at a simple interest rate of 16% per annum.

b Galen borrowed an amount of money for 10 years at an interest rate of 6% per annum compounding annually. At the end of the 10 years, he owed \$20 057.50.

c After 4 years with a simple interest rate of 12% per annum, Zala's investment is worth \$2516.

8 Calculate the annual interest rate for each of the following, correct to one decimal place.

a Oliver borrowed \$850 for 18 months and accrued \$50 of simple interest.

b Dani invested \$1000 for 6 years, with interest compounding annually, and her investment earned a total of \$400 interest.

c Carla invested \$10 000 for 20 years, with interest compounding monthly, and her investment earned a total of \$15 000 interest.

d For a \$900 purchase, Desiree's customers pay instalments of \$40 per month for 30 months.

9 Calculate the term of the loan or investment in each of these cases, correct to the nearest year where necessary.

a Isra invested \$6200 at a simple interest rate of 18% per annum and the investment grew to \$10 664.

b Avoca borrowed \$40 000 at a simple interest rate of 23% per annum and now owes \$18 400 more than the principal amount.

10A **c** Jedda invested \$12 000 at an interest rate of 8.1% per annum compounding annually, and that investment is now worth \$20 000.

10A **d** Christos borrowed \$39 990 at an interest rate of 9.6% per annum compounding monthly. He now owes \$60 000.

10 Factorise each of the following expressions.

a $16xy^2 - 24x^2y^2 + 40xy$

b $20xy - 5x + 16y - 4$

c $25x^2 - 49$

d $(x - 3)^2 - 9$

e $x^2 - 10x + 25$

f $x^2 + 4x + 3$

g $x^2 + 7x - 44$

h $3x^2 - 36x - 135$

11 Simplify each of the following.

a $\dfrac{6x + 3}{2} \times \dfrac{2x}{2x + 1}$

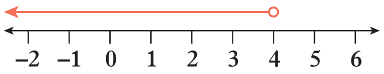
b $\dfrac{(x + 10)(x - 4)}{(x - 4)^2} \times \dfrac{(x + 10)(x - 3)}{(x - 2)(x - 3)}$

c $\dfrac{x^2 + 12x + 35}{x^2 + 7x + 10} \times \dfrac{x^2 - 4}{4x + 28}$

d $\dfrac{x^2 - 5x + 6}{x^2 + 6x - 16} \div \dfrac{x^2 - 9}{x^2 - 6x + 8}$

12 State the inequality represented on each of the following number lines.

a



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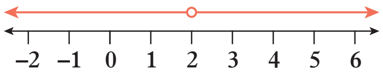

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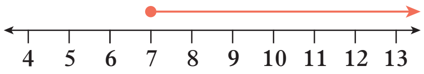

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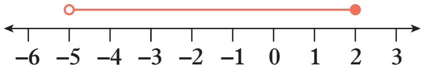


13 Solve each of the following inequalities. Show the solution on a number line.

a $4x + 5 \neq 17$

b $-6x + 4 > 28$

c $8 - 4x \leq 5$

d $5(2x - 3) \leq 4x$

14 Solve each of the following equations.

a $(2x - 1)(3x + 5) = 0$

b $4x^2 = 100$

c $(x - 5)^2 - 81 = 0$

d $x^2 - 10x + 24 = 0$

e $x^2 - 15 = 2x$

f $5x^2 = 10x + 40$

g $x^2 + 5x - 10 = 0$

h $\dfrac{x - 3}{5} = \dfrac{4}{x - 2}$

15 Complete the square to write each of the following expressions in turning point form. Do not fully factorise the expressions.

a $x^2 + 10x$

b $x^2 - 6x + 12$

c $x^2 - 5x + 12$

d $2x^2 - 28x - 18$

16 Sketch the graph of each of the following equations. Write the coordinates of any axis intercepts and turning points and the equations of any asymptotes.

a $y = 2x$

b $y = 2x^2$

c $y = (x + 2)^2$

d $x^2 + y^2 = 2^2$

e $y = 2^x$

f $y = \dfrac{2}{x}$

17 Determine the number of x-intercepts the graphs of each of the following equations will have.

a $y = 2x - 10$
b $3x - 6y = 20$
c $x = 12$
d $y = 3$
e $y = -4(x - 1) + 5$
f $y = 3(x - 4)(x + 6)$
g $y = (x - 4)^2$
h $y = -x^2 - 5x + 6$
i $y = (x - 2)^2 + 1$
j $(x - 5)^2 + (y - 3)^2 = 25$
k $y = 2^x + 1$
l $y = 3 - 3^x$
m $y = \dfrac{1}{x - 3} + 2$
n $y = -\dfrac{3}{x + 2}$

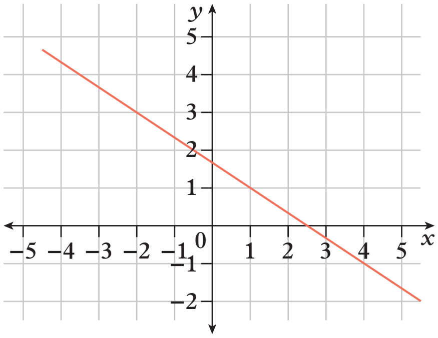
18 Determine the coordinates of the x- and y-intercepts, where they exist, for the graphs of the equations in question **17**.

19 i Sketch the graphs of the equations in question **17**. Write the coordinates of any axis intercepts and turning points and the equations of any asymptotes.

ii State the domain and range for each of the graphs you drew for the equations from question **17**.

20 From the given information, determine the equation of each of the lines **a–f** below. Write each equation in the form $y = mx + c$.

a

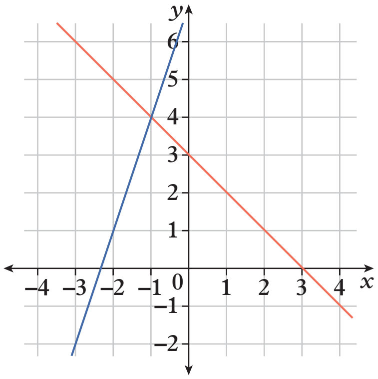


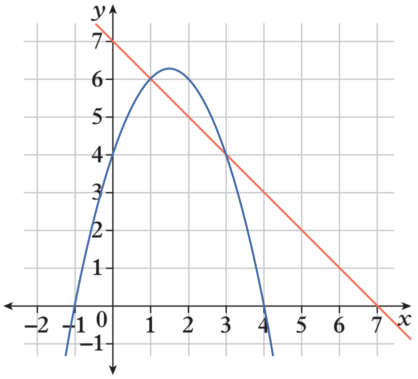
b The gradient of the line is 4 and the line passes through the point (–2, 5).
c The line passes through the points (3, –4) and (8, 12).
d The line is parallel to $3x - 6y = 12$ and passes through the point (4, 3).
e The line is perpendicular to $y = -\frac{4}{3}x + 5$ and passes through the point (–2, 5).
f The line is perpendicular to the line segment between the points (1, 4) and (–2, 5) and passes through the midpoint of the line segment.

21 Use the graphs provided to solve each of the following pairs of simultaneous equations.

a $y = 3x + 7$
$x + y = 3$

b $y = -x^2 + 3x + 4$
$y = -x + 7$





22 Find the intersection point(s) of the graphs of the following pairs of equations.

a $y = 3x - 4$
$y = -7x + 16$

b $y = 2x - 5$
$4x - 5y = 12$

c $2x + 9y = 11$
$2x - 9y = 15$

d $3x - 15y = 10$
$-6x + 5y = 10$

23 Determine the constant of proportionality, k, for each of the following.

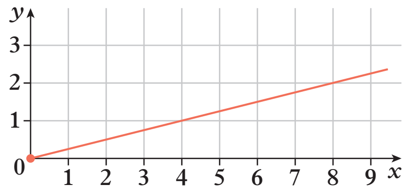
a y is directly proportional to x and, when $x = 4$, $y = 9$.

b y is inversely proportional to x and, when $x = 4$, $y = 9$.

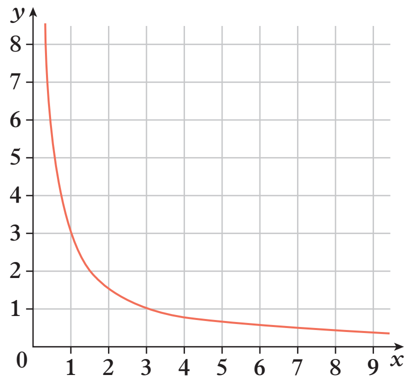
c y is directly proportional to x and, when $x = 10$, $y = -1000$.

d y is inversely proportional to x and, when $x = 5$, $y = 20$.

e



f



24 i For each of the proportional relationships in question **23**, determine the value of y when $x = 2$.

ii For each of the proportional relationships in question **23**, determine the value of x when $y = 2$.

10A **25** Use the substitution method to solve each of these pairs of simultaneous equations.

a $y = x^2 - 4x + 5$
$y = 3x - 1$

b $y = x^2 + 5x - 9$
$y = x^2 - 8x + 43$

c $x^2 + y^2 = 25$
$y = x + 1$

d $y = \frac{6}{x}$
$y = x + 5$

10A **26** For each of the following polynomials, state:

i the leading term **ii** the degree **iii** the leading coefficient.

a $P(x) = -6x^5 + 4x^4 - 10x^3 + 5x^3 + 2x^2 + x - 8$

b $Q(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4$

c $f(x) = \dfrac{3x^2 - 7x^4 - x}{2}$

10A **27** Evaluate each of the following.

a $8^{\frac{2}{3}}$ **b** $81^{\frac{1}{4}}$ **c** $1000^{-\frac{4}{3}}$

d $\log_2 (128)$ **e** $\log_9 (3)$ **f** $\log_{10} \left(\frac{1}{100\,000}\right)$

10A **28** If $P(x) = x^3 - 3x^2 + 5x - 4$, evaluate each of the following.

a $P(-3)$ **b** $P\left(\frac{1}{2}\right)$ **c** $P(\sqrt{2})$ **d** $P(2\sqrt{3})$

10A **29** Evaluate and/or simplify each of the following.

a $\dfrac{a^{\frac{5}{4}} \times a^{\frac{2}{3}}}{a^{\frac{5}{6}}}$

b $\left(b^{\frac{2}{3}} c^{\frac{5}{2}}\right)^6$

c $3 \log_2 (d) + \log_2 (2d) - \log_2 (d^2)$

d $\log_5 \left(\frac{1}{e}\right) + \log_5 (5) + \log_5 (1)$

e $\sqrt{180}$

f $\sqrt{2} + \sqrt{72} + \sqrt{27} + \sqrt{50} + \sqrt{3} + \sqrt{12}$

g $\sqrt{75} \times \sqrt{6}$

h $\dfrac{\sqrt{120}}{\sqrt{5}}$

i $3\sqrt{6}\left(4\sqrt{6} + 9\sqrt{2}\right)$

j $\left(4\sqrt{3} - 2\sqrt{5}\right)\left(2\sqrt{3} + 3\sqrt{5}\right)$

30 Write each of the following with a rational denominator.

a $\dfrac{5}{\sqrt{3}}$ **b** $\dfrac{4 + \sqrt{2}}{\sqrt{6}}$ **c** $\dfrac{3}{5 - 3\sqrt{2}}$ **d** $\dfrac{3 + 4\sqrt{5}}{2\sqrt{5} + 4}$

10A **31** Perform each of the following polynomial operations. Write the answers to parts **e–g** in the form $Q(x) + \dfrac{R(x)}{D(x)}$.

a $(x^3 + 8x^2 + 9x + 5) + (x^3 - 5x^2 + 4x - 2)$

b $(x^3 + 8x^2 + 9x + 5) - (x^3 - 5x^2 + 4x - 2)$

c $(3x^2 + 5x - 7) \times (2x + 3)$

d $(x^2 - 4x + 5) \times (x^2 + 5x - 3)$

e $(3x^2 - 6x + 5) \div (x - 3)$

f $(x^3 + 5x^2 - 7x - 6) \div (x + 2)$

g $(x^3 + 2x - 1) \div (x + 4)$

h $(x^3 + 5x^2 - 7x + 2) \div (x + 5)$

10A **32** **a** Factorise $2x^2 - x - 15$.

b Factorise $6x^2 - 7x - 20$.

c **i** Complete the square for $4x^2 - 3x - 7$.

ii Using your answer from part **i**, factorise $4x^2 - 3x - 7$.

d Given that $x + 3$ is a factor, factorise $x^3 + 9x^2 + 23x + 15$.

e Given that $x - 2$ is a factor, factorise $x^3 - 3x^2 - 28x + 60$.

f Given that $P(5) = 0$, factorise $P(x) = x^3 - 19x - 30$.

10A **33** Solve each of the following equations.

a $2^x = 32$

b $3^x = 30$

c $2 \times 5^{3x+2} - 1 = 0$

d $x^2 - 6x + 4 = 0$

e $4x^2 + 3x - 1 = 0$

f $3x^2 - 4x - 12 = 0$

g $(2x + 5)(3x - 10)(5 - x)(x - 4) = 0$

h $x^3 + 6x^2 + 5x - 12 = 0$

10A **34** Sketch the graph of each of the following equations.

a $y = 6x^2 - 7x - 10$

b $y = (x - 3)(x + 1)(x + 2)$

c $y = (x - 2)^2(x + 2)$

d $y = (x - 4)^2(x + 2)^2$

e $y = (x + 1)^3(x - 1)$

f $y = x^3 + 7x^2 + 7x - 15$

10A **35** Describe a sequence of transformations that could have been applied to make the transformed equation from the original each time.

	Original	**Transformed**
a	$y = x^2$	$y = -2x^2 + 8$
b	$y = x$	$y = \frac{3}{8}(x + 2)$
c	$y = 2^x$	$y = \frac{1}{2} \times 2^x - 4$
d	$y = \frac{1}{x}$	$y = \frac{1}{4(x - 2)} + 1$
e	$y = x^3$	$y = -(x + 1)^3 + 8$
f	$y = (x - 1)(x - 3)(x + 4)$	$y = x(x - 2)(x + 5) + 2$

10A **36** Sketch the graphs of the transformed equations from question **35**.

Analysis

1 Surjan recently received a \$1000 bonus and is considering two investments. They are listed below.

Investment A: A simple interest rate of 7% per annum

Investment B: A compound interest rate of 5% per annum, compounding annually

a If Surjan plans to invest his \$1000 for 10 years, which investment should he choose?

b Write the rule that gives the value, V_A and V_B, of each investment after n years.

c Complete the table of values on the right to show the performance of each investment. Write each value correct to the nearest cent.

n	0	5	10	15	20
V_A					
V_B					

d Use the values from your completed table to help you sketch graphs on the same set of axes showing the value of each investment against the number of years, for $0 \le n \le 20$.

e Use the graphs you drew for part **d** to help you determine after which year Investment B would be worth more than Investment A.

f After how many years will Investment A be worth more than Investment B is at 5 years?

g Surjan is offered an alternative investment, described below.

Investment C: Simple interest rate of 5% per annum, with \$150 added to the account when it is first opened.

i After how many years would Investment A and Investment C be worth the same amount?

ii What would be the value of each investment at that time?

10A **h** After how many years would Investment B be worth the same amount as Investment A was worth after 5 years? Write your answer as:

i an exact value **ii** correct to two decimal places.

2 Air resistance in measured in units of force called Newton (N). The air resistance on an object is directly proportional to the object's velocity when wind is moving smoothly in parallel lines (laminar flow). It is also directly proportional to the square of the object's velocity when wind is moving chaotically (turbulent flow).

a During laminar flow, the air resistance on an object travelling at 10 m/s is 25 N.

i Determine the constant of proportionality and write the equation relating the laminar flow air resistance, R_L, and the velocity, v.

ii Determine the air resistance when the object is travelling at 20 m/s.

b During turbulent flow, the air resistance on the object travelling at 10 m/s is 60 N.

i Determine the constant of proportionality and write the equation relating the turbulent flow air resistance, R_T, and the velocity, v.

ii Determine the air resistance when the object is travelling at 20 m/s.

c Sketch the graphs of the two air resistance equations on the same pair of axes.

d Determine the speeds at which the air resistance during laminar and turbulent flow are the same.

e The time it takes for the object to travel a set distance is inversely proportional to its velocity. If an object moves at 8 m/s and takes 12.5 seconds to travel a set distance, then determine the constant of proportionality and write the equation for the velocity, v, of the object in terms of the time, t, it takes to travel the set distance.

f What is the set distance that is used to determine the relationship between velocity and time in part **e**?

g Sketch the graph of the velocity versus the time it takes to travel the set distance.

h Can the time it takes to travel the set distance ever equal zero? Explain.

3 At a local park, the height of a hill is measured from one side to the other, where height is measured from the park's entrance near the base of the hill. The height is then modelled by an equation which gives the height, h metres, at a horizontal distance, d metres, from the park's entrance. $h = -0.05d^2 + d$, where $h \geq 0$

a Determine the width of the hill.

b Determine the maximum height of the hill and how far, horizontally, it is from the entrance.

c Sketch the graph of the equation for an appropriate domain.

10A **d** Determine the horizontal distance to the points on the hill that are 4 m above the ground level measured at the entrance. Write your answers:

i as exact values **ii** correct to two decimal places.

10A **e** The highest point of the hill does not actually lie half-way between the edges of the base of the hill. To better model the hill, a polynomial model can be used. The polynomial also models the depth of a pond on the other side of the hill, for which the depth is the negative value of the height. The pond is 8 m wide.

Using the width of the hill from part **a**, write the equation of the polynomial in the form $h = \frac{1}{380}d(d-a)(d-b)$, where a and b are integers.

10A **f** Write the polynomial in part **e** in the form $h = \frac{1}{380}(d^3 + pd^2 + qd)$, where p and q are integers.

10A **g** Sketch the graph of the polynomial for an appropriate domain.

10A **h** Using the polynomial model, use technology to determine:

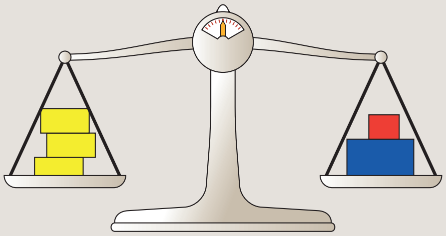
i the maximum height of the hill and the horizontal distance from the entrance to the highest point, correct to two decimal places

ii the maximum depth of the pond and the horizontal distance from the entrance to the deepest point, correct to two decimal places.

EXPLORATIONS 1

1 Balancing bricks

Indy has a collection of coloured plastic bricks and a balance scale. All bricks of the same colour weigh the same, but bricks of different colours weigh different amounts.



a Indy discovers the following about red, blue, yellow and green bricks:

- 1 blue brick weighs the same as 1 yellow and 1 green brick
- 3 yellow bricks weigh the same as 1 blue and 1 red brick
- 1 blue brick weighs more than 1 green and 3 red bricks
- 14 red bricks weigh more than 1 blue brick
- at least one combination of the other bricks weighs the same as 1 yellow brick.

Find all such combinations.

b Indy discovers the following about black, white, orange and purple bricks:

- 1 orange brick weighs the same as 1 white and 1 black brick
- 3 white bricks weigh the same as 1 orange and 1 purple brick
- 2 white bricks weigh less than 5 purple bricks
- **only one** combination of the other bricks weighs the same as 1 black brick. Find that combination.

2 Times tables

In each of the following partially completed multiplication tables, the letters A, B, C, … represent different whole numbers, subject to some extra conditions. Find the value of each letter and complete the table.

a Letters A to J represent the numbers 1 to 10, in some order. The sum of numbers A to E is equal to the sum of numbers F to I.

×	*A*	*B*	*C*	*D*	*E*
F	18				
G				20	
H					
I					24
J			10		

b Letters A to H represent the even numbers from 2 to 16, in some order. The sum of E, G and H is twice the sum of A, B, C and D.

×	*A*	*B*	*C*	*D*
E		40		
F			24	
G				112
H				

c Letters A to I represent the numbers 2 to 10, in some order. The sum of the squares of A, B and C is equal to the sum of the squares of D, E, G, H and I.

×	*A*	*B*	*C*	*D*	*E*
F			30		
G		28			
H				27	
I					

3 How irrational!

a Rationalise the denominator to simplify $\frac{1}{\sqrt{2}+\sqrt{3}}$.

b Let a and b be positive integers and consider the fraction $\frac{1}{\sqrt{a}+\sqrt{b}}$.

i Find, in terms of a and b, an alternative expression for this fraction in which the denominator is an integer.

ii Does your answer always make sense? If not, describe when it fails and find the correct answer in this case.

c Rationalise the denominator twice to simplify $\frac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$.

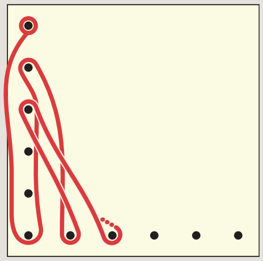
d Let a, b and c be positive integers with $c = a + b$ and consider the fraction $\frac{1}{\sqrt{a}+\sqrt{b}+\sqrt{c}}$.

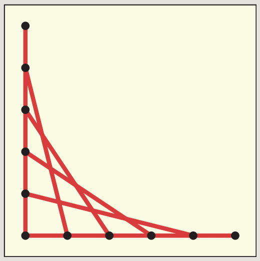
i Find, in terms of a, b and c, an alternative expression for this fraction in which the denominator is an integer.

ii Use your rule to simplify $\frac{1}{2\sqrt{2}+\sqrt{3}+\sqrt{5}}$.

4 String art

If you have access to a pegboard and string, try the following hands-on activity first. Alternatively, you can use free online geometry software, such as GeoGebra *or* Desmos. *For best results, use as many pegs (or virtual pegs) as possible.*





On a pegboard, vertical and horizontal rows of equally spaced pegs are arranged in an L-shape. A piece of string is looped from the first (top) peg of the vertical row to the first (left-most) peg of the horizontal row, then to the second vertical peg, second horizontal peg, third vertical peg and so on – see the first diagram. When the string is pulled tight, it forms the design in the second diagram. The larger the number of pegs used, the smoother the curve formed by the boundary of the intertwined string. What is this curve?

To model this mathematically, it is best to rotate the board 45° anti-clockwise.

a Follow these instructions to make your own 'string art' on a sheet of grid paper:

- ▶ Draw x- and y-axes with the origin in the middle of the page.
- ▶ Plot the lines $y = x - 10$ and $y = -x - 10$, extending them to the edge of the page. (There is nothing special about the number 10. Later we will replace it with an arbitrary number, d.)
- ▶ Pick any point on one of the lines (for example, $(13, 3)$ on $y = x - 10$). Find the point on the other line whose y-coordinate is the negative of your first point's y-coordinate (for example, $(-7, -3)$ on $y = - x - 10$). Draw a line through your pair of points, extending it to the edge of the page.
- ▶ Repeat the previous step several times. As more and more lines are added, a familiar shape should emerge!

Let's prove that the curve produced by the lines in part **a** is the parabola $y = \frac{1}{40}x^2$. To do so, it is enough to show that each such line is a tangent to this parabola. To warm up, we start with a concrete example with two given points.

b **i** Find, in gradient-intercept form, the equation of the line through $(13, 3)$ and $(-7, -3)$.

ii Verify that this line intersects the parabola $y = \frac{1}{40}x^2$ exactly once.

Now try the same strategy in more generality.

c Let $P(k + 10, k)$ be a point on the line $y = x - 10$, where k is a fixed but arbitrary number.

i Find the point Q on the line $y = - x - 10$ whose y-coordinate is $-k$.

ii Find, in gradient-intercept form, the equation of the line through P and Q.

iii Verify that this line intersects the parabola $y = \frac{1}{40}x^2$ exactly once.

Finally, let's see how the parabola changes when the original 'lines of pegs' are changed.

d For some arbitrary fixed number $d \neq 0$, consider the lines $y = x - d$ and $y = -x - d$ with points $P(k + d, k)$ and $Q(k - d, -k)$, respectively. There is some parabola of the form $y = ax^2$ which, for all values of k, intersects the line through P and Q exactly once. Verify this claim by finding an expression for a in terms of d.

5 Fifth degree

a Calculate the first five fifth powers: 1^5, 2^5, 3^5, 4^5 and 5^5. What do you notice about their last digits? Does this always happen?

b Fully factorise the expression $n^5 - n$. Hence show that $n^5 - n$ is divisible by 30 for all integers n. [Hint: if n, $n - 1$ and $n + 1$ are not divisible by 5, then n is of the form $5k \pm 2$ for some integer k.]

c How does the result of part **b** answer the final question of part **a**?

d Show that $n^5 - n$ is divisible by 240 for all odd integers n.

e For what values of n is $n^5 - n$ divisible by 9?

f Deduce from part **b** that every integer has the same final digit as its ninth power. For what other powers is this true?

Explorations inspired by the Australian Maths Trust's competitions and programs: www.amt.edu.au

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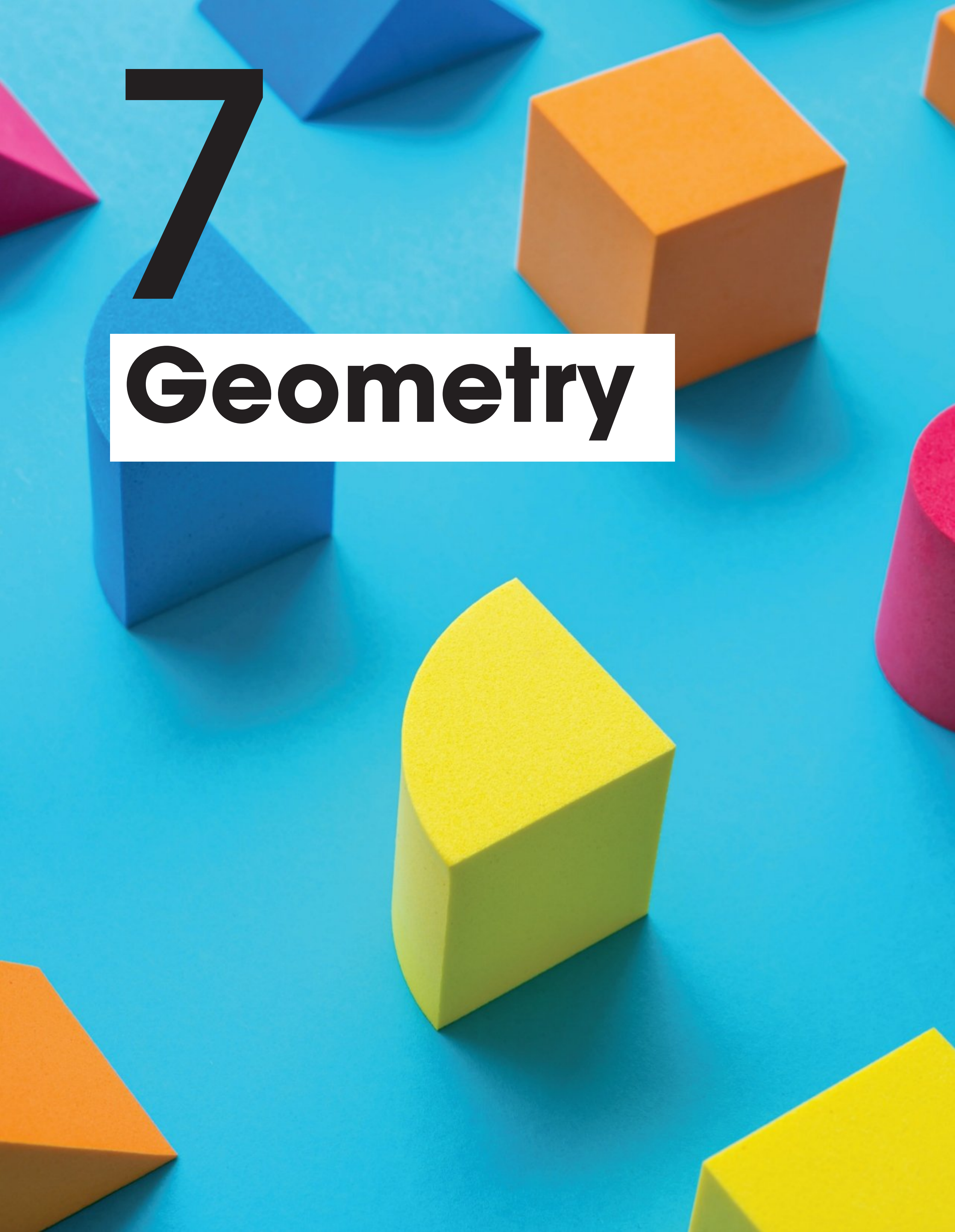
7

Geometry



Index

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

✔ Triangle properties
✔ Quadrilateral properties

Curriculum links

- Formulate proofs involving congruent triangles and angle properties (VCMMG344)
- Apply logical reasoning, including the use of congruence and similarity, to proofs and numerical exercises involving plane shapes (VCMMG345)
- Prove and apply angle and chord properties of circles (VCMMG366) (10A)

Materials

✔ Calculator

7A Geometry review

Learning intentions

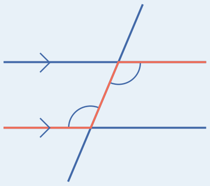
✔ I can determine the size of unknown angles.

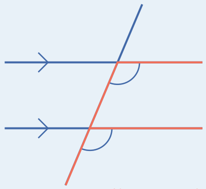
Inter-year links

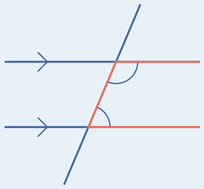
Year 5/6	Understanding angles
Year 7	7B Angles at a point
Year 8	7A Angles
Year 9	7A Angles and lines

Angles

- **Complementary angles** add to 90°.
- **Supplementary angles** add to 180°.
- Angles at a point add to 360°.
- **Vertically opposite angles** are equal.
- When parallel lines are crossed by a **transversal**, a number of equal and supplementary angles are formed.


alternate angles
(equal)


corresponding angles
(equal)

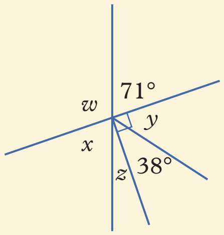

co-interior angles
(supplementary)

Polygons

- A **polygon** is a closed shape with straight sides.
- A triangle is a three-sided polygon with an interior angle sum of 180°.
- A **quadrilateral** is a four-sided polygon with an interior angle sum of 360°.
- The interior angle sum of any polygon is $(n - 2) \times 180°$, where n is the number of sides of the polygon.
- The exterior angle sum of any polygon is 360°.

Example 7A.1 Finding unknown angles using angle properties

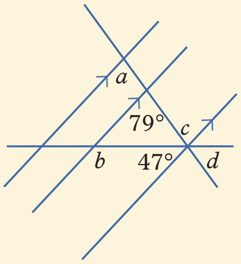
Find the size of the labelled unknown angles in this diagram.



THINK	WRITE
1 Angle w is supplementary to the 71° angle.	$w = 180° - 71°$ $= 109°$
2 Angle x is vertically opposite the 71° angle.	$x = 71°$
3 Angle y is complementary to the 38° angle.	$y = 90° - 38°$ $= 52°$
4 Angle z is supplementary to the angles 71°, y and 38°.	$z = 180° - 71° - 52° - 38°$ $= 19°$

Example 7A.2 Finding sizes of angles using angle relationships with parallel lines

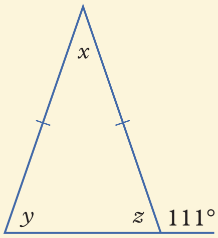
Find the values of the pronumerals in this diagram.



THINK	**WRITE**
1 Angle a is a corresponding angle to the 79° angle. Corresponding angles on parallel lines are equal.	$a = 79^\circ$
2 Angle b is co-interior with the 47° angle. Co-interior angles on parallel lines are supplementary.	$b = 180^\circ - 47^\circ$ $= 133^\circ$
3 Angle c is alternate to the 79° angle. Alternate angles on parallel lines are equal.	$c = 79^\circ$
4 Angle d is supplementary to angle c and the unlabelled angle beside it, which is vertically opposite and, therefore, equal to 47°.	$d = 180^\circ - 79^\circ - 47^\circ$ $= 54^\circ$

Example 7A.3 Using triangle properties to find the sizes of angles

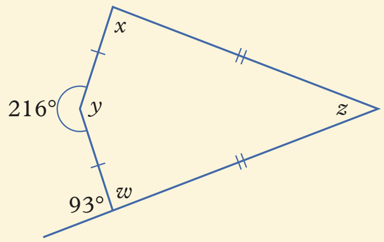
Find the value of each pronumeral in this diagram.



THINK	**WRITE**
1 Angle z is supplementary to the 111° angle.	$z = 180^\circ - 111^\circ$ $= 69^\circ$
2 Angle y is equal to angle z because the triangle is isosceles. Base angles of an isosceles triangle are equal.	$y = z = 69^\circ$
3 Angle x is supplementary to angles y and z because there are 180° in a triangle.	$x = 180^\circ - 69^\circ - 69^\circ$ $= 42^\circ$

Example 7A.4 Using quadrilateral properties to find sizes of angles

Find the value of each pronumeral in this diagram of a kite.



THINK	**WRITE**
1 Angle w is supplementary to 93°.	$w = 180° - 93°$ $= 87°$
2 Angle x is equal to angle w because the opposite angles formed from the unequal sides of a kite are equal.	$x = w = 87°$
3 Angle y and the angle labelled 216° meet at a point, therefore they add to 360°.	$y = 360° - 216°$ $= 144°$
4 Angles w, x, y and z add to 360° because the interior angle sum of a quadrilateral is 360°.	$z = 360° - 144° - 87° - 87°$ $= 42°$

Helpful hints

✔ When calculating unknown angles, you might not be given the required information to determine all the unknown angles (at first). Initially, you may need to use the angle sizes you can calculate to help you find the angle sizes you are unable to determine.

ANS p743

Exercise 7A Geometry review

△ 1–8, 10(a–c), 11

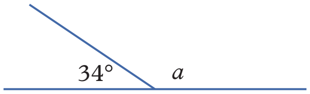
■ 1–7, 9, 10(b, d, e), 11, 12(a, b), 13(b, d), 14(a–e)

⬟ 3, 5, 6, 9, 10(d–f), 11, 12(c), 13, 14

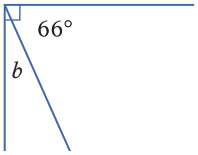
UNDERSTANDING AND FLUENCY

7A.1 **1** Find the sizes of the labelled unknown angles in these diagrams.

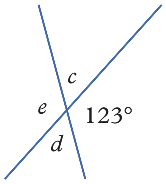
a



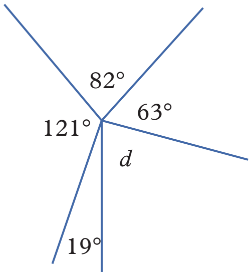
b



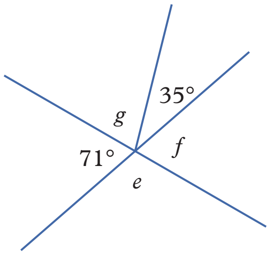
c



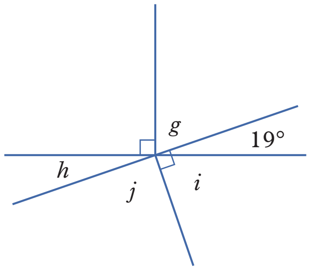
d



e

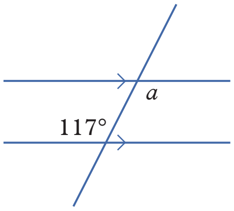


f



7A.2 2 Find the value of each pronumeral in these diagrams.

a



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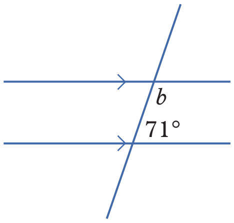

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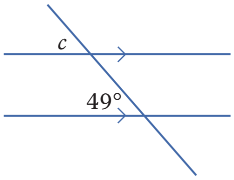
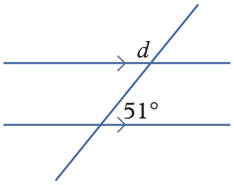


d



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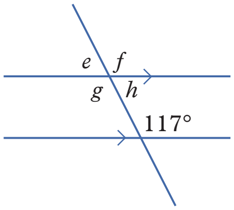

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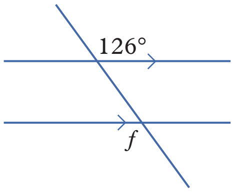
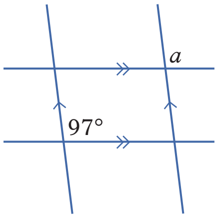
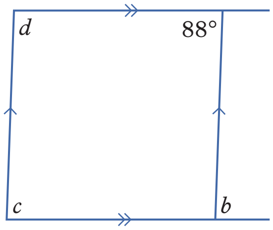


3 Find the value of each pronumeral in these diagrams.

a



b



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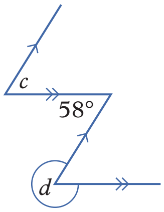

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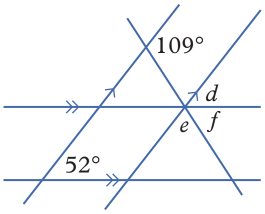

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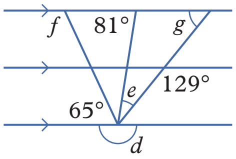

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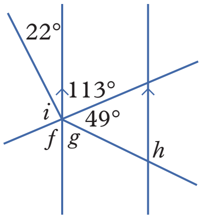
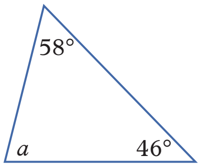


7A.3 4 Find the value of each pronumeral in these diagrams.

a



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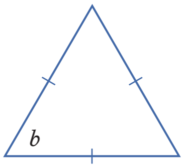

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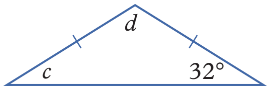

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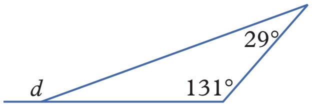

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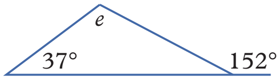

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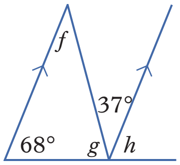
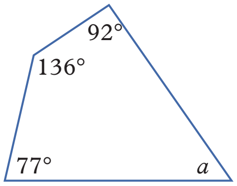


7A.4 5 Find the value of each pronumeral in these diagrams.

a



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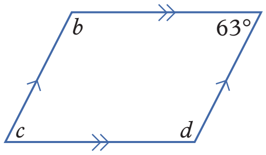

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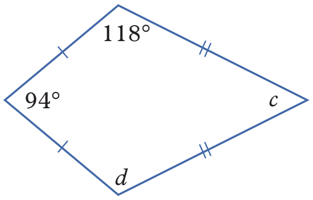

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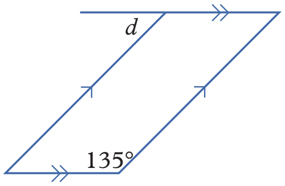

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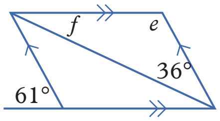

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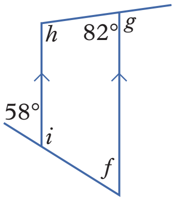


PROBLEM SOLVING AND REASONING

6 Use the expression $(n - 2) \times 180°$ to find the internal angle sum of a polygon with:

a 20 sides **b** 50 sides **c** 100 sides.

7 Explain why you can use the formula $x = \dfrac{(n-2) \times 180°}{n}$ to find the size of an individual internal angle in any regular polygon.

8 Find the size of an individual internal angle for a:

a regular heptagon **b** regular nonagon **c** regular hexagon.

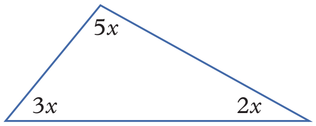
Round your answers to two decimal places.

9 Find the number of sides of a regular polygon that has each of its internal angles equal to:

a 160° **b** 144° **c** 170° **d** 179.64°

10 Find the value of x in each of these diagrams. Round the answer to two decimal places.

a



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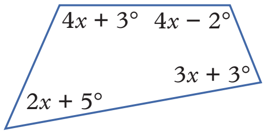

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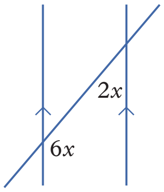

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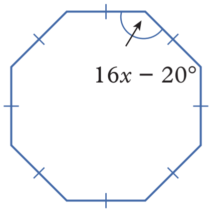

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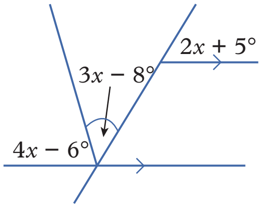

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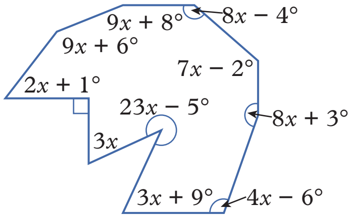


11 Decide whether each of these statements is true or false. If false, write a correct version of the statement.

a A kite has a pair of equal opposite angles.

b Angles at a point add to 180°.

c An isosceles triangle has two pairs of equal angles.

d Corresponding angles on parallel lines are supplementary.

e A square is a rhombus.

f A parallelogram is a rectangle.

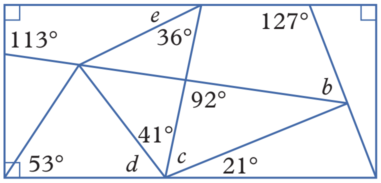
g Complementary angles add to 180°.

h A triangle that has one 60° angle must be equilateral.

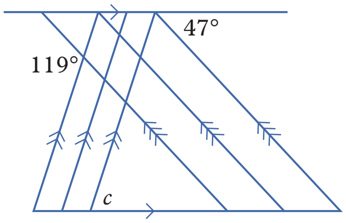
i A quadrilateral can have a maximum of one concave angle.

12 Find the value of each pronumeral in these diagrams.

a



b



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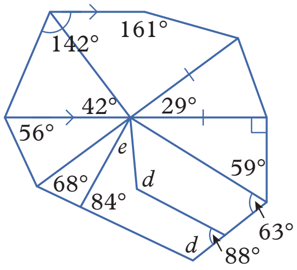
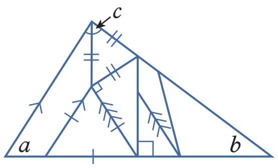


13 Find the value of each pronumeral in these diagrams.

a



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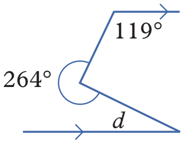

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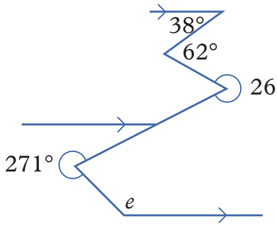

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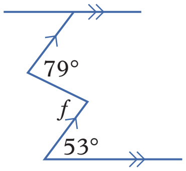

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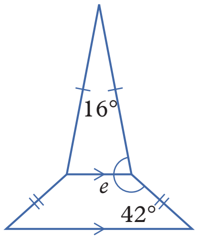

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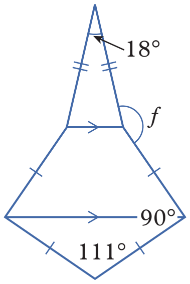


14 Explementary angles are a pair of angles that form a revolution; that is, they add to 360°. Determine the size of each of the following angles:

- **a** the explement of 30°
- **b** the explement of 175°
- **c** the explement of 286°
- **d** the explement of the supplement of 45°
- **e** the explement of the complement of 81°
- **f** the complement of the explement of 324°
- **g** the supplement of the explement of 267°
- **h** the complement of the supplement of the explement of 200°
- **i** the supplement of the complement of the explement of 299°
- **j** the explement of the angle alternate to 64° on parallel lines
- **k** the explement of the angle co-interior to 145° on parallel lines
- **l** the explement of the exterior angle of a regular hexagon.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet Angles at a point

Interactive skillsheet Angles and parallel lines

Topic quiz 7A

7B Geometric proofs

Learning intentions

✔ I can understand what constitutes a mathematical proof.

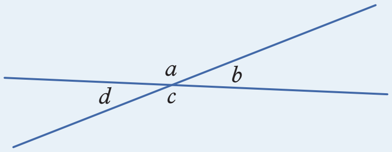
✔ I can create a mathematical proof consisting of a sequence of logically connected steps.

Inter-year links

Year 8	7A Angles
Year 9	7A Angles and lines

Mathematical proofs

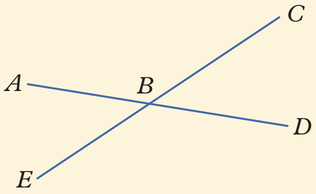
- In mathematics, a practical demonstration can be used to show something is true. For example, we can show that vertically opposite angles are equal by cutting out one of the angles and placing it on top of the other angle.



- On the other hand, a mathematical **proof** explains why something is true through a series of logically connected statements. Mathematical proofs are constructed to show that a theory is true in all cases.
- Mathematical proofs begin with self-evident or given statements. For example, you can use the fact that there are 180° on a straight line as a given statement to prove that vertically opposite angles are equal (as shown in Example 7B.1).
- The following list of given statements can be used to create the proofs in this section:
  - → Angles at a right angle add to 90°.
  - → Angles on a straight line add to 180°.
  - → Angles at a point add to 360°.
  - → If point B lies in the interior of $\angle AOC$, then $\angle AOB + \angle BOC = \angle AOC$ (angle addition postulate).
  - → The interior angle sum of a triangle is 180° (as shown in Exercise 7B, question 4).
  - → If $a = b$ and $a = c$, then $b = c$ (transitive law).
  - → The same number can be added or subtracted from both sides of an equation to create an equivalent equation (addition and subtraction properties of equality).
- The symbol $\therefore$ is shorthand for 'therefore' and the symbol $\because$ is shorthand for 'because'.
- The letters 'QED' or the 'tombstone', ■ or □, are often written at the end of a proof. The acronym QED stands for the Latin phrase 'quod erat demonstrandum', which means 'which was to be demonstrated'.
- Mathematical proofs are often laid out in two columns, with mathematical statements given in the left-hand column, and reasons why each statement is true given in the right-hand column.

Example 7B.1 Proving that vertically opposite angles are equal

For the diagram on the right, prove that $\angle ABC = \angle DBE$.



THINK

1 Set up a two-column table with 'Statements' as the heading of the left-hand column and 'Reasons' as the heading of the right-hand column.

2 Use the fact that angles on a straight line add to 180° to help you write two equations, making sure to include a common angle in both equations (for example, $\angle CBD$).

3 The right-hand sides of the equations from step **2** are the same, so the left-hand sides of the equations must also be equal (according to the transitive law). Express this in a new equation.

4 Simplify the new equation by subtracting $\angle CBD$ from both sides of the equation.

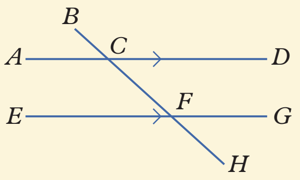
5 Include a symbol (□ or ■) or write 'QED' to show that your proof is complete.

WRITE

Statements	**Reasons**
$\angle ABC + \angle CBD = 180°$	Angles on a straight line add to 180°
$\angle DBE + \angle CBD = 180°$	Angles on a straight line add to 180°
$\angle ABC + \angle CBD = \angle DBE + \angle CBD$	Transitive law
$\angle ABC = \angle DBE$ ■	Subtraction property of equality

Example 7B.2 Proving that alternate angles on parallel lines are equal

For the diagram on the right, prove that $\angle ACF = \angle CFG$, given that $\angle CFG = \angle BCD$.



THINK

1 Set up a two-column table for the proof and write the given information in the first row.

2 Use the fact that angles on a straight line add to 180° to write two equations, making sure to include a common angle in both equations (for example, $\angle DCF$).

3 Equate the left-hand sides of the equations (because the right-hand sides are equal).

4 Simplify the equation by subtracting $\angle DCF$ from both sides of the equation.

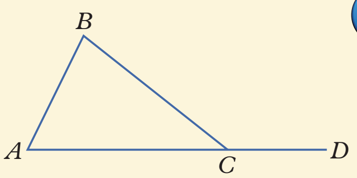
5 Because $\angle BCD = \angle CFG$, substitute $\angle CFG$ for $\angle BCD$ in the equation (using the transitive law).

WRITE

Statements	**Reasons**
$\angle CFG = \angle BCD$	Given
$\angle ACF + \angle DCF = 180°$	Angles on a straight line add to 180°
$\angle BCD + \angle DCF = 180°$	Angles on a straight line add to 180°
$\angle ACF + \angle DCF = \angle BCD + \angle DCF$	Transitive law
$\angle ACF = \angle BCD$ $\angle BCD = \angle CFG$	Subtraction property of equality
$\therefore \angle ACF = \angle CFG$ ■	Transitive law

Example 7B.3 Proving the exterior angle theorem

From the diagram on the right, prove that $\angle BCD = \angle ABC + \angle CAB$.



THINK

1 Set up a two-column table for the proof.

2 Use the fact that angles on a straight line sum to 180° to write an equation linking the exterior angle of the triangle to the adjacent interior angle of the triangle.

3 Write an equation linking the interior angles of the triangle.

4 Equate the left-hand sides of the two equations (because the right-hand sides as equal).

5 Simplify by subtracting $\angle BCA$ from both sides.

WRITE

Statements	Reasons
$\angle BCD + \angle BCA = 180°$	Angles on a straight line add to 180°
$\angle ABC + \angle CAB + \angle BCA = 180°$	Interior angle sum of a triangle is 180°
$\angle BCD + \angle BCA = \angle ABC + \angle CAB + \angle BCA$	Transitive law
$\angle BCD = \angle ABC + \angle CAB$ ■	Subtraction property of equality

Helpful hints

✔ Be patient when constructing mathematical proofs. Start with the information that you are given and build on that information. You may not see the whole proof at first but, by taking it one step at a time, you will work to the required conclusion.

ANS p744

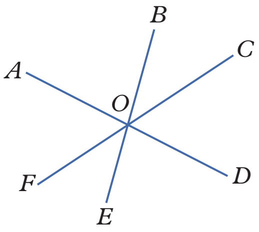
Exercise 7B Geometric proofs

△ 1–6, 10 □ 2, 4–8, 10 ⬠ 4–10

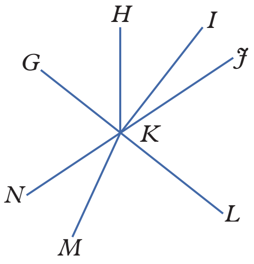
7B.1 1 For the diagram on the right, prove that $\angle EOF = \angle BOC$. Complete the proof twice, starting with each of the following opening statements:

a $\angle EOF + \angle FOB = 180°$

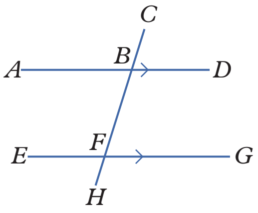
b $\angle EOF + \angle AOF + \angle AOB = 180°$.



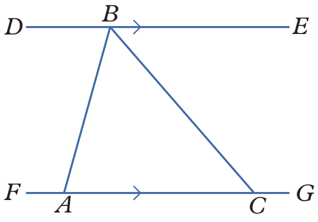
2 For the diagram below, prove that $\angle GKJ = \angle NKL$.



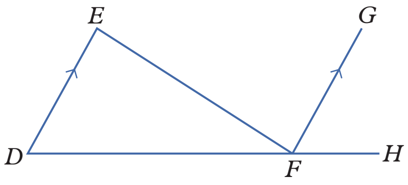
7B.2 **3** For the diagram below, prove that $\angle DBF$ is supplementary to $\angle BFG$, given that $\angle ABC = \angle EFB$.



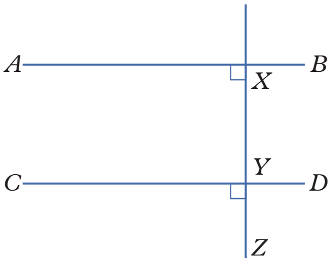
7B.3 **4** Use your knowledge of angles and parallel lines, as well as the diagram below, to prove that the internal angle sum of a triangle is 180°.



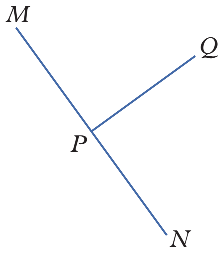
5 For the diagram below, use the fact that DE and FG are parallel to prove that $\angle EFH = \angle DEF + \angle FDE$.



6 Given that X lies on AB and Y lies on CD, prove that AB is parallel to CD.
Note: 'If co-interior angles are supplementary they lie on parallel lines' can be used as a reason.



7 Given that $\angle MPQ$ and $\angle QPN$ are supplementary, prove that MN is a straight line.
Note: 'Supplementary angles sum to 180°' can be used as a reason.



PROBLEM SOLVING AND REASONING

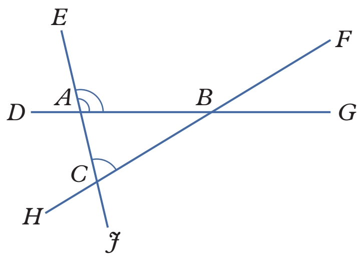
8 There are different forms of mathematical proofs. The proofs you have seen in this section so far are called 'direct proofs', where a statement is established from known facts. Another form of proof is 'proof by contradiction', where we assume that what we want to prove is not true, and then we demonstrate that this assumption is false.

We can use proof by contradiction to prove that, if corresponding angles are equal, the lines being considered are parallel. Start with the statement, 'If two corresponding angles are equal, the lines being considered are not parallel'.

a Why does the lines not being parallel mean that the lines would intersect?

b Consider the diagram on the right, showing intersecting lines *DG* and *HF*, cut by the transversal *EJ*.

Name the two angles in the diagram that you are assuming to be equal.



c The other assumption that you must make is that the length $AC > 0$. Why do you need to make this assumption?

d If $\angle EAB = \angle ACB = x$, show that $\angle BAC = 180° - x$.

e If $\angle ABC$ can be represented by y, write a statement adding together all the angles in ΔABC.

f Show that simplifying this statement leaves you with $y = 0°$.

g How does this imply that *AC* does equal 0?

h Explain how this is 'proof by contradiction' that, if corresponding angles are equal, the lines must be parallel.

i Write the proof in full.

CHALLENGE

9 Another form of proof is 'proof by contrapositive'. The contrapositive of the statement 'if *X*, then *Y*' is 'if not *X*, then not *Y*'. These two statements are logically equivalent. That is, if you are able to prove the contrapositive, the original statement is also proved to be true.

a Write the contrapositive of the statement, 'When two lines are cut by a transversal, if the lines are parallel then the corresponding angles are equal'.

b In the diagram from question **8**, let *EJ* be the transversal. Prove the contrapositive of the statement from part **a** using the diagram and the fact that *AB* is not parallel to *CB*, $AB \nparallel CB$.

Note: The following additional reasons can be used:

- Three points that do not form a line form a triangle.
- The exterior angle of a triangle is equal to the sum of the two non-adjacent interior angles.
- Angles cannot be equal to 0°.

10 It is important not to make errors when completing proofs because, if you are not careful, you may think you have proven something that is actually false. The following example appears to 'prove' that 2 = 1. Find the error.

$a = b$, where a and b are not equal to zero.
So: $a^2 = ab$
and: $a^2 - b^2 = ab - b^2$
Factorising gives: $(a + b)(a - b) = b(a - b)$
Dividing by $(a - b)$ gives: $a + b = b$
Remember that $a = b$.
So: $b + b = b$
$2b = b$
Dividing by b gives: $2 = 1$

Check your Student obook pro for these digital resources and more:

Topic quiz
7B

7C Congruence and similarity

Learning intentions

- ✔ I can find unknown side lengths and angles in congruent and similar triangles.
- ✔ I can identify whether a pair of triangles are congruent or similar.
- ✔ I can prove that a pair of triangles are congruent or similar.

Inter-year links

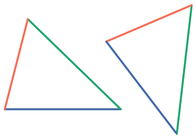
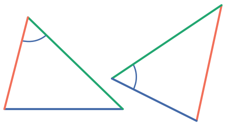
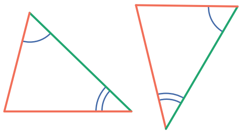
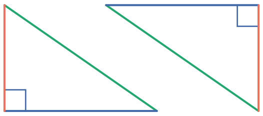
Year 7	7D Classifying triangles
Year 8	7E Congruent triangles
Year 9	6E Similar triangles

Congruent figures

- **Congruent figures** are identical in shape and size but can be in any position or orientation.
- Two figures are congruent if their corresponding sides are the same length and their corresponding angles are the same size.
- The symbol for congruence is ≅ (‘is congruent to’).

Congruent triangles

- If two triangles meet any of the four conditions shown in the table below, they are congruent.

SSS (side–side–side)	**SAS** (side–angle–side)	**AAS** (angle–angle–side)	**RHS** (right angle–hypotenuse–side)
When two triangles have all their corresponding sides the same length (SSS), the corresponding angles will also be the same size. This means that the triangles are congruent. 	Two of the side lengths are equal and the included angle (between those sides) is equal. 	Two of the angles and a corresponding pair of side lengths are equal. 	If the triangles are right-angled, the hypotenuses and another pair of side lengths are equal. 

- If a pair of triangles fails any of the four conditions for congruence shown above, they are not congruent.
- If $\Delta ABC \cong \Delta DEF$, then vertex A will correspond with vertex D, vertex B will correspond with vertex E, and vertex C will correspond with vertex F.

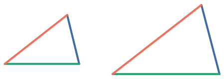
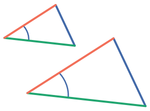
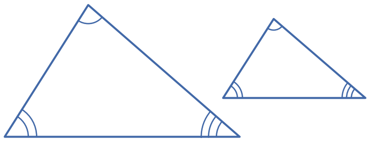
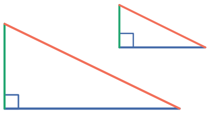
Similar figures

- **Similar figures** are identical in shape but can be different in size.
- For two figures to be similar, all angles must be equal in size and the lengths of all corresponding sides must be in the same ratio.
- If two figures are similar, an unknown side length or angle can be found if the scale factor is known.
- The symbol for similarity is ~ (‘is similar to’).
- The scale factor can be calculated using the formula:

$$\text{scale factor} = \frac{\text{image length}}{\text{original length}}$$

Similar triangles

- For two triangles to be similar, they must meet one of four conditions of similarity.

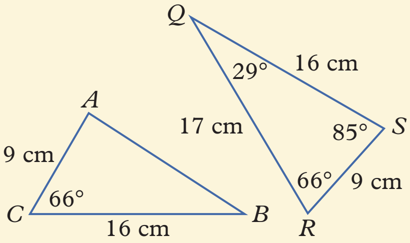
SSS **(side–side–side)**	**SAS** **(side–angle–side)**	**AAA** **(angle–angle–angle)**	**RHS** **(right angle–hypotenuse–side)**
The lengths of all three pairs of corresponding sides are in the same ratio. 	The lengths of two pairs of corresponding sides are in the same ratio and the included angles are equal. 	Each of the three angles in one triangle is equal to an angle in the other triangle. 	In two right-angled triangles, the lengths of their hypotenuses are in the same ratio as the lengths of another pair of corresponding sides. 

Proving triangles are congruent or similar

- In addition to the given statements in 7B, the following statements can be used to create the proofs in this section:
  - → Vertically opposite angles are equal.
  - → Alternate angles on parallel lines are equal.
  - → Corresponding angles on parallel lines are equal.
  - → Co-interior angles on parallel lines are supplementary.
- Properties of different types of triangles can also be used.

Example 7C.1 Identifying whether two triangles are congruent

Decide whether these triangles are congruent, giving a reason for your answer.



THINK	WRITE
1 Look at the given information to determine which congruence condition to use. We are given the sizes of two sides and an included angle for ΔBCA. This fits the SAS condition for congruence.	ΔABC has two sides and an included angle, which fits the SAS condition for congruence.

2 Identify the corresponding sides in the second triangle.

RS corresponds to AC.
BC corresponds to QS.

3 Compare the sizes of the included angles between the matching sides in each triangle.

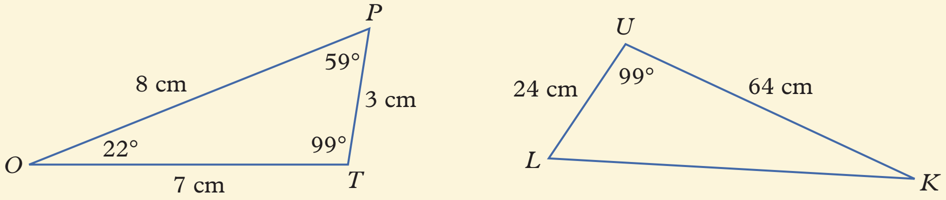
$\angle ACB = 66° \neq \angle RSQ$.

4 Write a conclusion.

The two triangles are not congruent because they fail the SAS condition for congruence.

Example 7C.2 Identifying whether two triangles are similar

Decide whether these triangles are similar, giving a reason for your answer.



THINK

1 Look at the given information to determine which similarity condition to use. ΔLUK has two sides and an included angle, which fits the SAS condition for similarity.

WRITE

ΔLUK has two sides and an included angle, which fits the SAS condition for similarity.

2 Use the given angle in ΔLUK to find the corresponding sides.

$\angle PTO = \angle LUK = 99°$
PT corresponds to LU.
TO corresponds to UK.
OP corresponds to KL.

3 Determine the scale factor used for one of the pairs of matching sides (LU and PT).

$$\begin{aligned}\text{scale factor} &= \frac{\text{image length}}{\text{original length}}\\ &= \frac{LU}{PT}\\ &= \frac{24}{3}\\ &= 8\end{aligned}$$

4 Find the scale factor used for the other pair of matching sides (TO and UK).

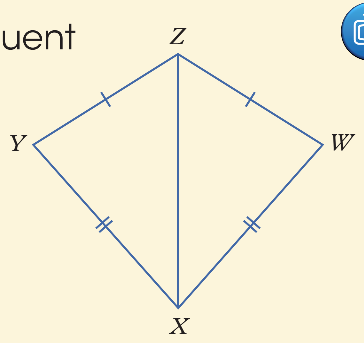
$$\begin{aligned}\text{scale factor} &= \frac{\text{image length}}{\text{original length}}\\ &= \frac{UK}{TO}\\ &= \frac{64}{7}\\ &\approx 9.14\end{aligned}$$

5 Write a conclusion.

The two triangles are not similar because these two scale factors are not the same, meaning their corresponding side lengths are not in the same ratio. They fail the SAS condition for similarity.

Example 7C.3 Proving that two triangles are congruent

For the diagram on the right, prove that $\Delta XYZ \cong \Delta XWZ$ using a congruence condition.



THINK

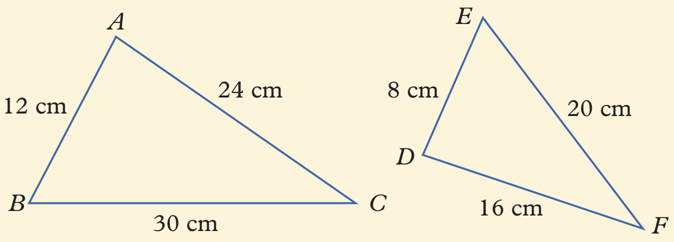
1. Set up a two-column table for your proof.
2. List the given information.
3. Determine whether the given information meets any of the conditions for congruence.
4. Complete the proof.

WRITE

Statements	Reasons
$XY = XW$	Given
$ZY = ZW$	Given
XZ is common to both triangles	Common side
$\therefore \Delta XYZ \cong \Delta XWZ$ ■	SSS (congruent)

Example 7C.4 Proving that two triangles are similar

Prove that ΔABC is similar to ΔDEF using one of the conditions for similarity.



THINK

1. Identify the corresponding side for the two triangles. For example, *AB* is the shortest side in ΔABC and *DE* is the shortest side in ΔDEF.
2. Calculate the scale factor for each pair of corresponding sides.
3. Determine whether the given information meets any of the conditions for similarity.
4. Complete the proof.

WRITE

Statements	Reasons
$\frac{DE}{AB} = \frac{8}{12} = \frac{2}{3}$	Scale factor between corresponding sides
$\frac{EF}{BC} = \frac{20}{30} = \frac{2}{3}$	Scale factor between corresponding sides
$\frac{DF}{AC} = \frac{16}{24} = \frac{2}{3}$	Scale factor between corresponding sides
$\therefore \Delta ABC \sim \Delta DEF$ ■	SSS (similar)

Helpful hints

- ✔ If two triangles meet one of the conditions for congruence, they will meet all the conditions for congruence. Likewise, if two triangles fail one of the conditions for congruence, they will fail all the conditions for congruence.
- ✔ When writing a congruence or similarity statement, the corresponding vertices in each triangle must be written in the same order.

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Exercise 7C Congruence and similarity

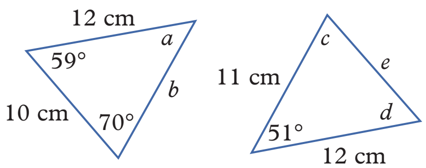
▲ 1–5, 6(a, c, d), 7, 8(a), 9(a), 10(a, b), 12, 15

■ 1(c, d), 2(d–f), 3, 4, 5(d–f), 6, 8, 9(b), 11, 13, 16, 17, 18(a)

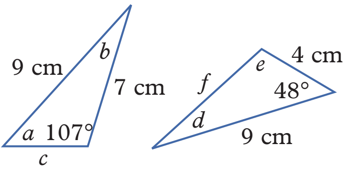
⬟ 3, 4, 6(d–f), 7–9, 11, 14, 16, 18(b), 19, 20

1 Find the unknown side lengths and angles in each of these pairs of congruent triangles.

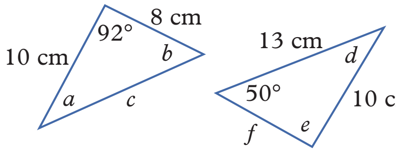
a



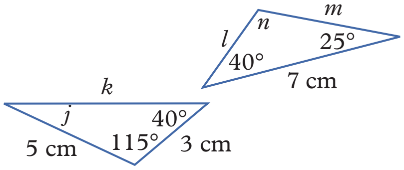
b



c

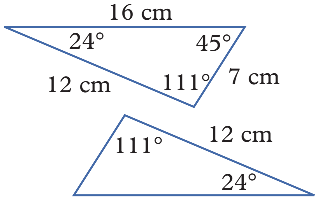


d



7C.1 **2** Decide whether the triangles in each of these pairs are congruent, giving a reason for your answer each time.

a



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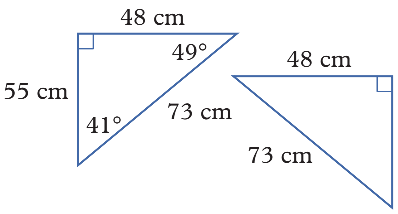

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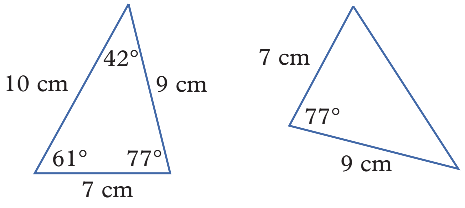

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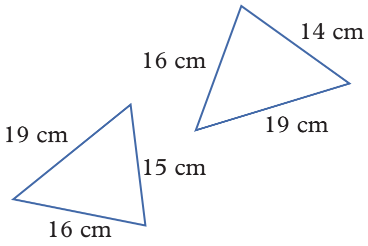
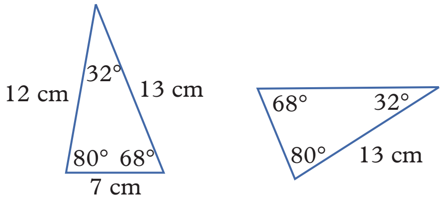
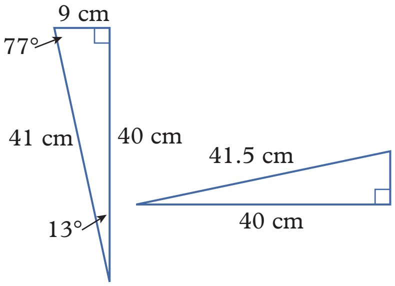


e

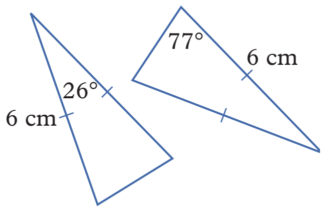


f



3 Use your understanding of triangle properties to decide whether the triangles in each of these pairs are congruent.

a



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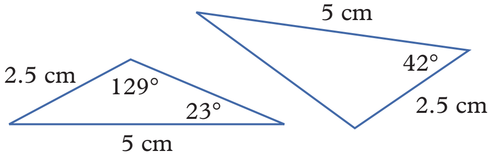

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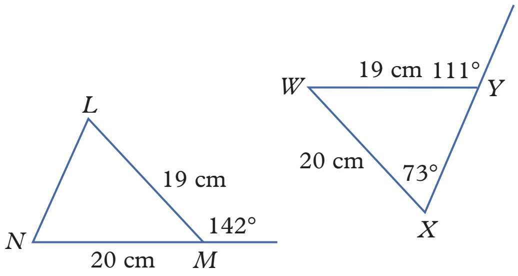

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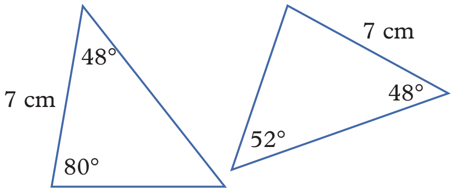
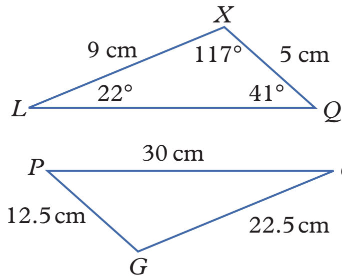


4 Find the scale factor between the triangles in each of these pairs of similar triangles. Assume that the first triangle in each pair is the original.

a



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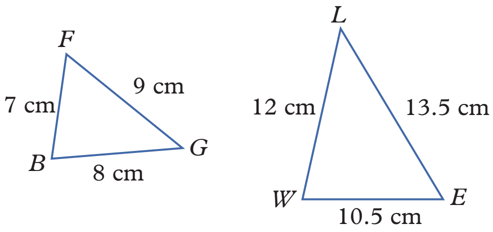

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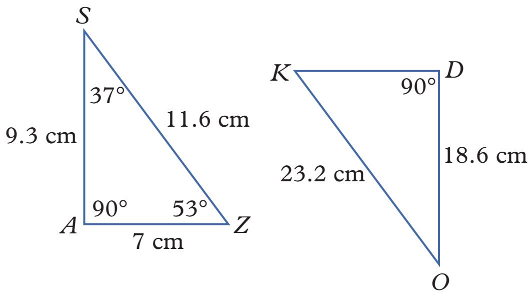

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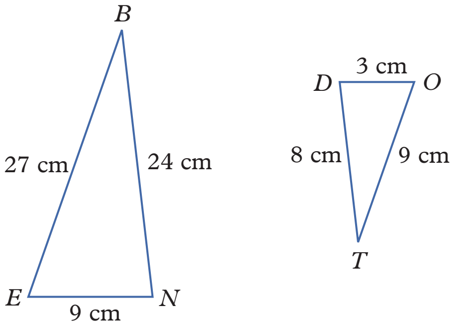
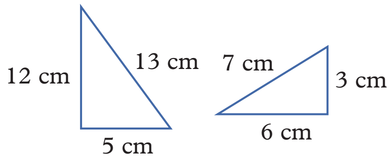


7C.2 **5** Decide whether the triangles in each of these pairs are similar, giving reasons for your answers.

a



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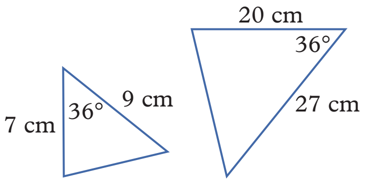

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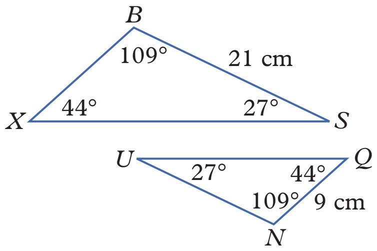

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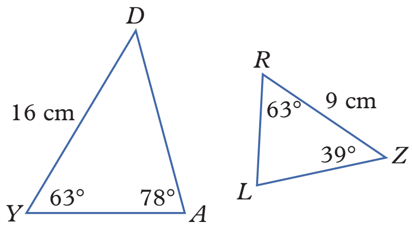

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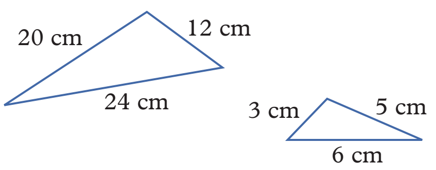

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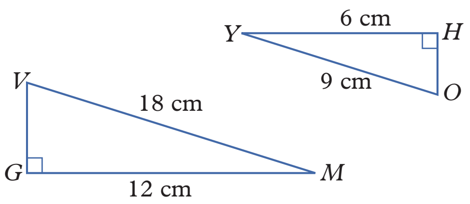
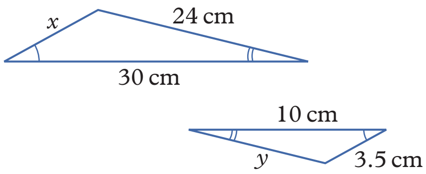


6 Find the unknown side lengths if the triangles in each pair below are similar.

a



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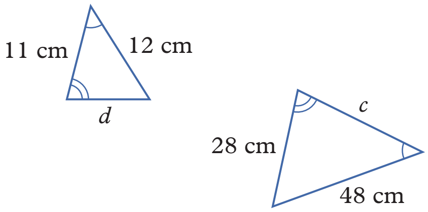

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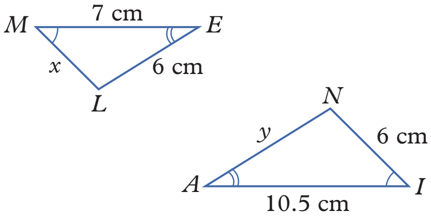

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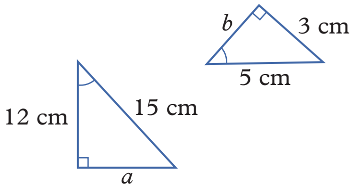

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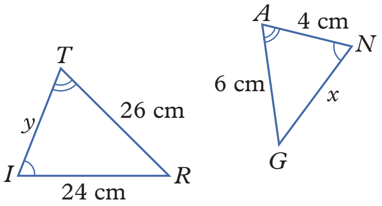

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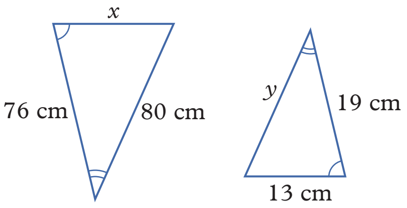
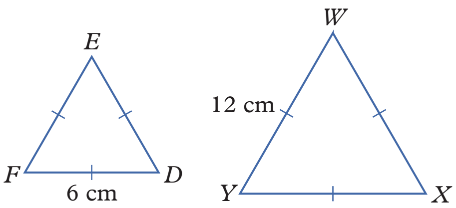


7 Use your understanding of triangle properties to decide whether the triangles in each of these pairs are similar.

a



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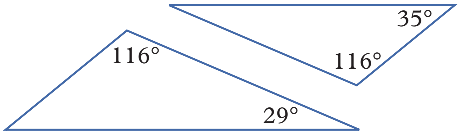

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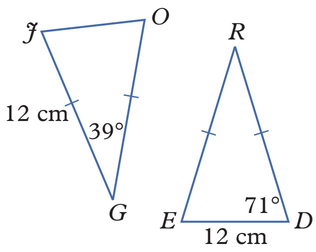

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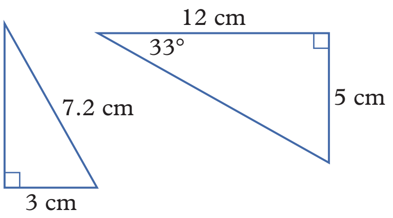
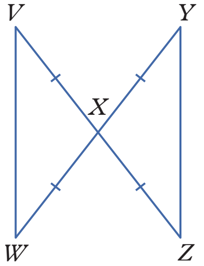
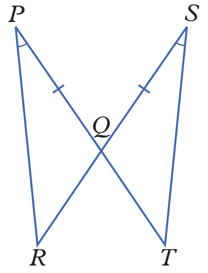


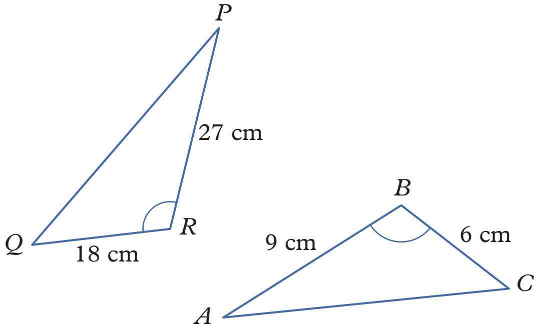
7C.3 **8** **a** For the diagram below, prove that $\Delta VWX \cong \Delta YZX$.



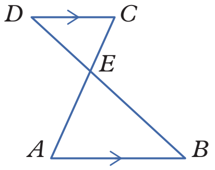
b For the diagram below, given that $\angle RPQ = \angle TSQ$, prove that $\Delta PQR \cong \Delta SQT$.



7C.4 **9** **a** Given that $\angle ABC = \angle PRQ$, prove that $\Delta ABC \sim \Delta PRQ$.



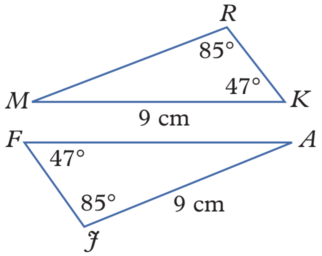
b Given that $AB \parallel CD$, prove that $\Delta ABE \sim \Delta CDE$.



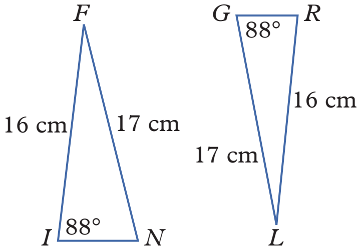
PROBLEM SOLVING AND REASONING

10 Lachlan said that the information given for the pairs of triangles below prove that they are congruent triangles. Explain why he is wrong in each case.

a



b



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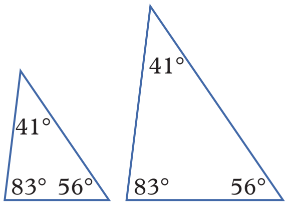

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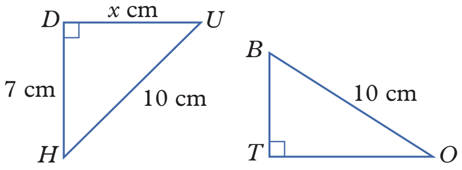


11 Decide whether each of these statements is true or false, giving reasons for your answers.

a If two triangles meet the condition AAA, they are not congruent.

b If two triangles have two corresponding sides the same length and one corresponding angle of the same size, the triangles are congruent.

c A pair of triangles can meet one congruence condition but fail another.

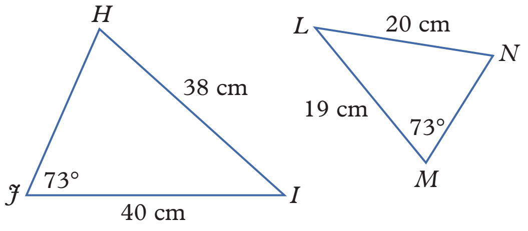
d If ΔABC is congruent to ΔDEF, and ΔDEF is congruent to ΔGHI, then ΔGHI is congruent to ΔABC.

e If two quadrilaterals have all matching sides equal in length, then they are congruent.

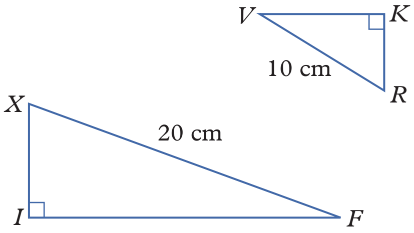
f All equilateral triangles are congruent.

12 Jessica said that the triangles in each pair below could be proven to be similar. Explain why she is wrong in each case.

a

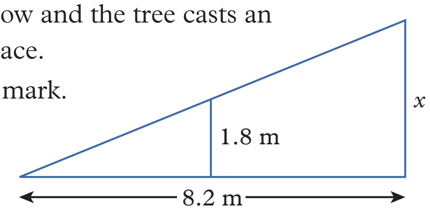


b



13 A 1 m ruler is placed upright next to a tree. If the ruler casts a 2.3 m shadow and the tree casts an 11.2 m shadow, how tall is the tree? Round your answer to one decimal place.

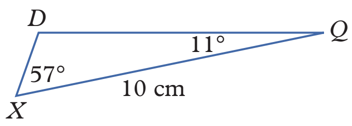
14 A ski jump is 8.2 m long. It has 1.8 m tall vertical supports at the halfway mark. How tall is the ski jump at its highest point?



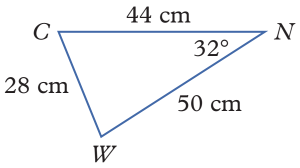
PROBLEM SOLVING AND REASONING

15 Select two pairs of similar triangles from the options below. Provide reasons for your selection.

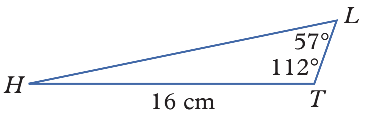
A



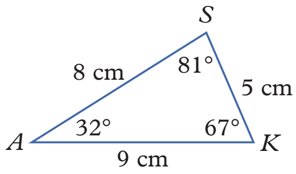
B



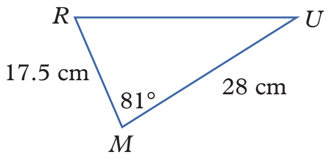
C



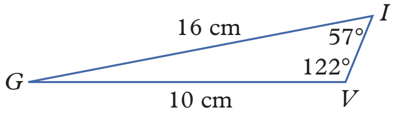
D



E



F

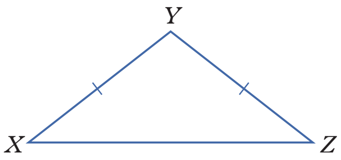


16 Decide whether each of these statements is true or false. Give reasons for your answers.

- **a** If two triangles meet the condition AAA, they are not necessarily similar.
- **b** If ΔABC is similar to ΔDEF, and ΔDEF is similar to ΔGHI, then ΔGHI is similar to ΔABC.
- **c** All isosceles triangles are similar.
- **d** All squares are similar.
- **e** If two triangles fail the condition SAS, they are not necessarily similar.
- **f** If two quadrilaterals have all sides in the same ratio, they are not necessarily similar.

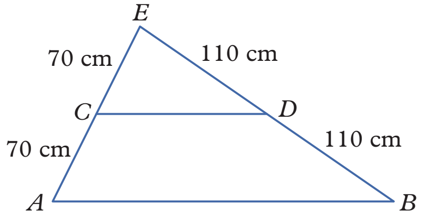
17 If a triangle has two sides that are equal in length (meaning it is isosceles), then the angles opposite those sides are equal.

You can prove this fact using congruent triangles. To help you, consider ΔXYZ on the right.

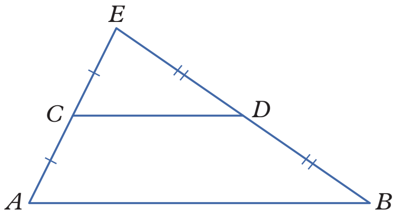


- **a** Draw a line from Y to the midpoint of XZ, and name this point W.
- **b** Name the two triangles you have now formed.
- **c** Which pairs of sides do you know are equal in these two triangles?
- **d** Name the side common to both triangles.
- **e** Use a congruence condition to prove that $\Delta WXY \cong \Delta WZY$.
- **f** Explain why you now know that $\angle WXY = \angle WZY$.

18 **a** Prove that $\Delta ABE \sim \Delta CDE$.



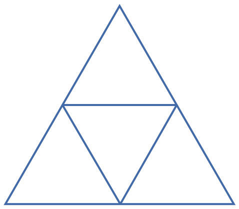
b Prove that $\Delta ABE \sim \Delta CDE$.



CHALLENGE

19 A flagpole has a guide wire attached at its midpoint and tethered to the ground 2.7 m away. A metre ruler, standing vertically, touches the guide wire when placed 75 cm away from where it is tethered to the ground. How tall is the flagpole?

20 Prove that drawing a triangle whose vertices are the midpoints of the sides of an equilateral triangle splits that equilateral triangle into four smaller congruent equilateral triangles. (Hint: You will need to make use of your knowledge of isosceles triangles.)



Check your Student obook pro for these digital resources and more: pro

Interactive skillsheet
Congruent triangles

Interactive skillsheet
Similar triangles

Topic quiz
7C

7D Proofs and quadrilaterals

Learning intentions

✔ I can prove quadrilateral properties.

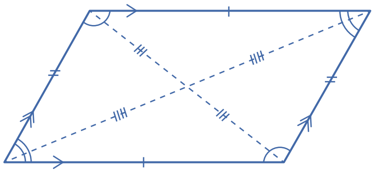
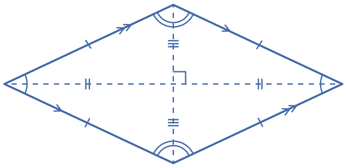
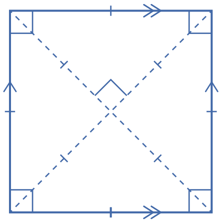
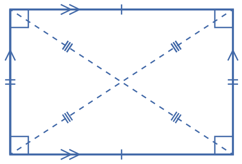
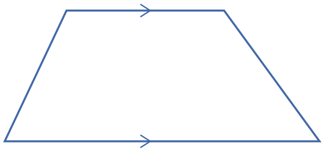
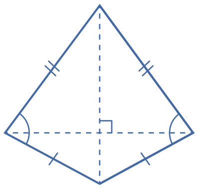
Inter-year links

Year 7	7E Classifying quadrilaterals
Year 8	7G Quadrilaterals
Year 9	7A Angles and lines

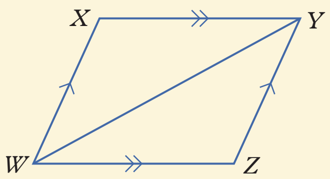
Geometry terminology

- 'Bisect' means to cut in half.
- The symbol for 'is parallel to' is ∥.
- The symbol for 'is perpendicular to' is ⊥.

Quadrilateral properties

Shape	Properties	Example
Parallelogram	Opposite sides are parallel and equal. Opposite angles are equal. Diagonals bisect each other.	
Rhombus	All sides are equal and opposite sides are parallel. Opposite angles are equal. Diagonals bisect each other at right angles. Diagonals bisect the interior angles.	
Square	All sides are equal and opposite sides are parallel. All angles are right angles. Diagonals are equal in length and bisect each other at right angles. Diagonals bisect the interior angles.	
Rectangle	Opposite sides are parallel and equal. All angles are right angles. Diagonals are equal and bisect each other.	
Trapezium	One pair of opposite sides are parallel.	
Kite	Two pairs of sides are equal in length. One pair of opposite angles are equal in size. Diagonals are perpendicular. One diagonal bisects the other diagonal. One diagonal bisects a pair of vertex angles.	

Example 7D.1 Proving opposite sides of a parallelogram are equal



Prove that the opposite sides of a parallelogram are equal in length.

THINK

1 Set up a two-column table for your proof.

2 Use the fact that the diagonal of the parallelogram is common to both ΔWXY and ΔYZW.

3 Use your knowledge of angles and parallel lines to identify equal angles.

4 Use a congruence condition to show that the triangles are congruent.

5 Show that the opposite sides of the parallelogram are equal in length (as corresponding sides of the congruent triangles).

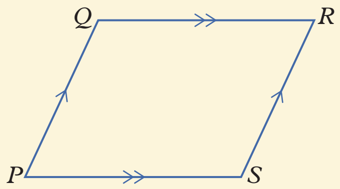
6 Complete the proof with a closing statement.

WRITE

Statements	**Reasons**
YW is common to both ΔWXY and ΔYZW	Common side
$\angle XYW = \angle ZWY$	Alternate angles on parallel lines are equal
$\angle YWX = \angle WYZ$	Alternate angles on parallel lines are equal
$\therefore \Delta WXY \cong \Delta YZW$	Meets the AAS condition of congruence
$WX = YZ$	Corresponding sides of congruent triangles are equal
$WZ = YX$	Corresponding sides of congruent triangles are equal
∴ Opposite sides of a parallelogram are equal in length ■	

Example 7D.2 Proving opposite angles in a parallelogram are equal

Prove that the opposite angles of a parallelogram are equal in size.



THINK

1 Use your understanding of angles and parallel lines to write two equations about co-interior angles, making sure the same angle appears in both equations.

2 Equate the left-hand sides of the equations you wrote (because the right-hand sides are equal).

3 Simplify by subtracting the same angle from both sides.

4 Repeat steps **1–3** for the other pair of co-interior angles.

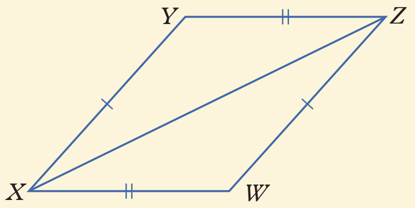
5 Complete the proof with a closing statement.

WRITE

Statements	**Reasons**
$\angle QPS + \angle PSR = 180°$	Co-interior angles on parallel lines are equal
$\angle QPS + \angle PQR = 180°$	Co-interior angles on parallel lines are equal
$\angle QPS + \angle PSR = \angle QPS + \angle PQR$	Transitive law
$\angle PSR = \angle PQR$	Subtraction property of equality
$\angle PSR + \angle QRS = 180°$	Co-interior angles on parallel lines are equal
$\angle PSR + \angle QPS = 180°$	Co-interior angles on parallel lines are equal
$\angle PSR + \angle QRS = \angle PSR + \angle QPS$	Transitive law
$\angle QRS = \angle QPS$	Subtraction property of equality
∴ Both pairs of opposite angles of the parallelogram are equal in size ■	

Example 7D.3 Proving a quadrilateral is a parallelogram

Prove that *WXYZ* shown on the right is a parallelogram.



THINK

1. Show that the two triangles formed by the diagonal are congruent.
2. Identify the corresponding angles in these congruent triangles.
3. If alternate angles are equal, then the lines must be parallel. (Note: || means 'is parallel to'.)
4. Use the definition of a parallelogram to prove that *WXYZ* is a parallelogram.

WRITE

Statements	**Reasons**
$XY = ZW$	Given
$YZ = WX$	Given
XZ is common to both ΔXWZ and ΔZYX	Common side
$\therefore \Delta XWZ \cong \Delta ZYX$	Meets the SSS condition of congruence
$\angle WXZ = \angle YZX$	Corresponding angles of congruent triangles are equal
$\angle XZW = \angle ZXY$	Corresponding angles of congruent triangles are equal
$YZ \parallel XW$	$\angle WXZ$ and $\angle YZX$ are alternate angles. If alternate angles are equal, they lie on parallel lines
$XY \parallel WZ$	$\angle XZW$ and $\angle ZXY$ are alternate angles. If alternate angles are equal, they lie on parallel lines
∴ *WXYZ* is a parallelogram ■	If a quadrilateral has opposite sides that are parallel then it is a parallelogram

Helpful hints

- ✔ If you are unsure where to begin when constructing a mathematical proof, start by writing all the information you are given (using mathematical terminology), as well as what you are trying to prove.
- ✔ Quadrilaterals, and other polygons, are labelled in a cyclic manner around the shape in order. That is, pairs of adjacent letters name the edges of the quadrilateral.

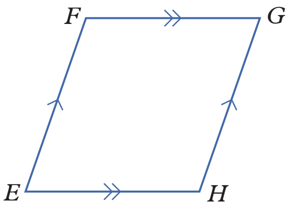
ANS p747

Exercise 7D Proofs and quadrilaterals

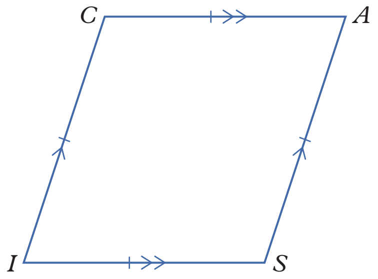
△ 1–4, 6, 8, 9 □ 1–5, 7, 10, 11, 13 ⬠ 3, 4, 7, 9, 11, 12, 14–16

UNDERSTANDING AND FLUENCY

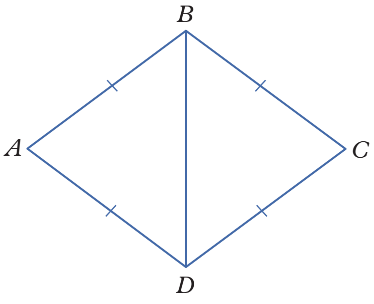
7D.1 **1** If $FG = EF$, prove that the shape on the right is a rhombus by demonstrating that all four side lengths are equal.



7D.2 **2** Prove that the opposite angles of the rhombus *ISAC* on the right are equal.



7D.3 **3** Prove that the opposite sides of the rhombus *ABCD* with four equal sides are parallel.



4 Consider the parallelogram on the right.

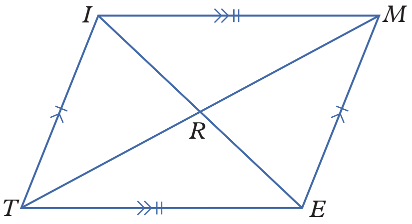
a What angle is equal to $\angle IMR$?

b What angle is equal to $\angle MIR$?

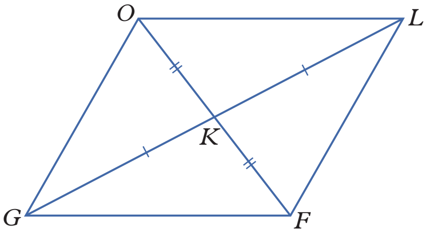
c Prove that $\Delta MIR \cong \Delta TER$.

d Copy the figure and mark the corresponding equal sides of the congruent triangles.

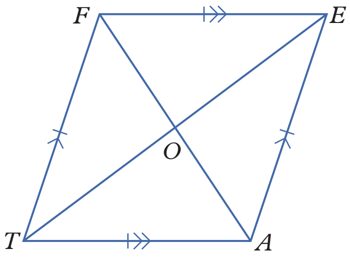
e Explain how this shows that the diagonals of parallelogram *TIME* bisect each other.



5 Prove that the shape below is a parallelogram.



6 Consider the rhombus on the right.



a Follow similar steps to those given in question **4** to prove that the diagonals of the rhombus bisect each other.

b Show that the four triangles formed by the diagonals are congruent to one another.

c Redraw the rhombus with markings showing the corresponding equal sides.

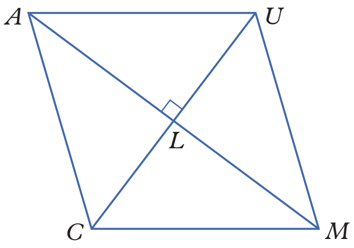
d Explain why $\angle FOE = \angle AOE$.

e Use your answer from part **d** to help you prove that the diagonals of *FEAT* bisect each other at right angles.

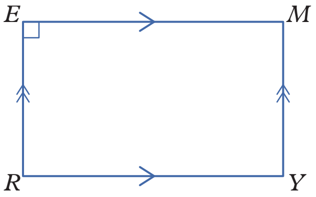
7 Prove that the shape on the right is a rhombus if:

a $AL = ML$ and $CL = UL$

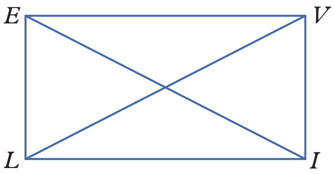
b $AU \parallel CM$ and $AC \parallel UM$.



8 Prove that parallelogram *REMY* is a rectangle if $\angle REM = 90°$.



9 Consider figure *LEVI* below, where $VE = IL$, $VE \parallel IL$ and $\angle EVI = 90°$.



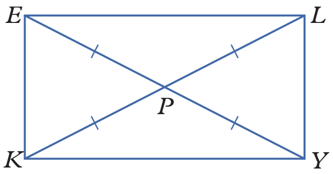
a Prove that $\angle VIL$, $\angle ILE$ and $\angle LEV$ are all 90°.

b Prove that $EL = IV$ and $EL \parallel IV$.

c Use Pythagoras' Theorem to show that $LV = EI$.

d Prove that $\Delta LEV \cong \Delta ELI$.

10 Prove that the shape on the right is a rectangle by following these steps:

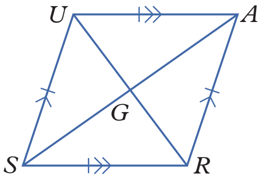


a Use what you know about isosceles triangles and about equal alternate angles to first prove that *ELKY* is a parallelogram.

b Use $\angle LYE + \angle YEL + \angle ELK + \angle KLY$ to prove that $\angle ELY = 90°$.

c Prove that all interior angles are 90° and therefore that *ELKY* is a rectangle.

11 Consider the rhombus below.



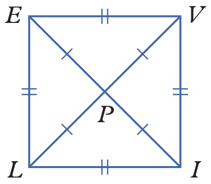
a Prove that $UG = RG$ and that $AG = SG$ (in other words, that the diagonals bisect each other).

b Prove that $\Delta UAG \cong \Delta RAG$.

c Prove that $\angle UAS = \angle RAS$.

d Explain why this method shows that the diagonals of a rhombus bisect its interior angles.

12 For the diagram below, prove that $\angle EVL = 45°$.



13 a Use mathematical reasoning with a square and a rhombus to show that having all corresponding sides the same length is not enough to prove congruence for shapes other than triangles.

b Use mathematical reasoning with a parallelogram and a rhombus to show that meeting the SAS condition of congruence does not prove congruence for shapes other than triangles.

c Use mathematical reasoning with a square and a rectangle to show that meeting the AAS condition of congruence does not prove congruence for shapes other than triangles.

14 a Use mathematical reasoning with a rectangle and a parallelogram to show that having all corresponding sides in the same ratio is not enough to prove similarity for shapes other than triangles.

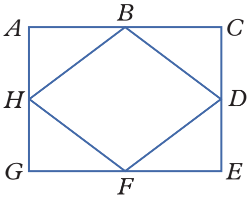
b Use mathematical reasoning with a parallelogram and a rhombus to show that having all corresponding angles the same size is not enough to prove similarity for shapes other than triangles.

15 Use your understanding of the relevant geometric properties to draw each shape described below and show that the triangles within them are congruent.

a A kite is split into two triangles by a line drawn along its line of symmetry.

b A regular hexagon is split into six triangles by drawing three diagonal lines joining opposite vertices.

16 *ACEG* is a rectangle. If the vertices of *HBDF* touch the midpoints of each of the rectangle's sides, prove that *HBDF* is a rhombus and that the opposite angles of *HBDF* are equal.



Check your Student obook pro for these digital resources and more:

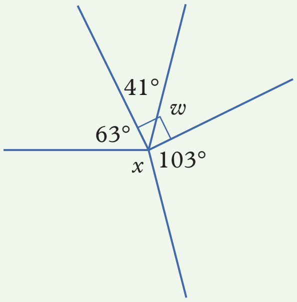
Topic quiz
7D

Checkpoint

Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this chapter.

7A **1** Determine the values of the pronumerals.

a



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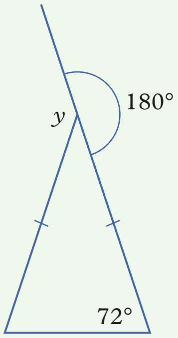
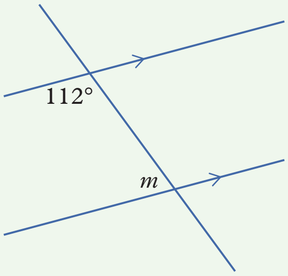


7A **2** Determine the values of the pronumerals.

a



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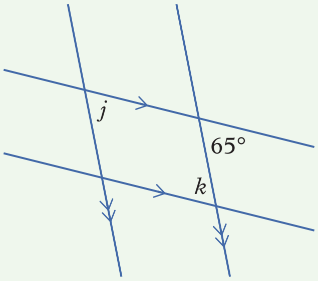
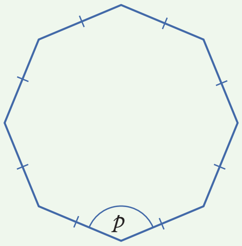
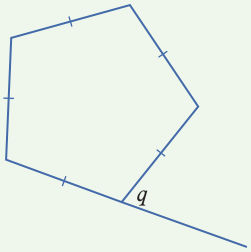


7A **3** Determine the values of the pronumerals.

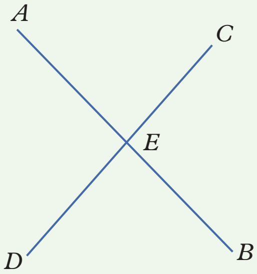
a



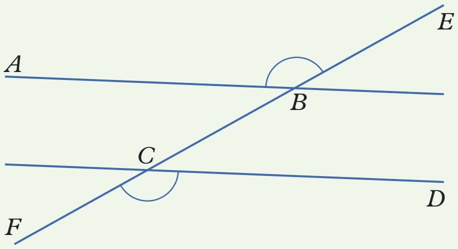
b



7B **4** Given that AB and CD in the diagram on the right are line segments, prove that $\angle AEC = \angle DEB$.

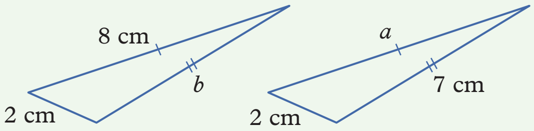


5 Given that $\angle ABE = \angle DCF$, prove that $AB \parallel CD$.

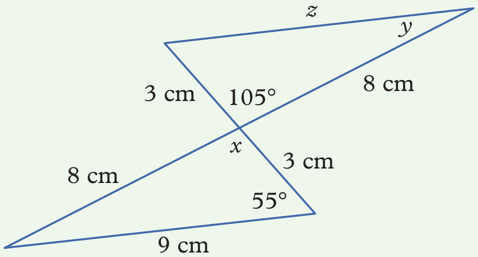


7C **6** Determine the values of the pronumerals.

a

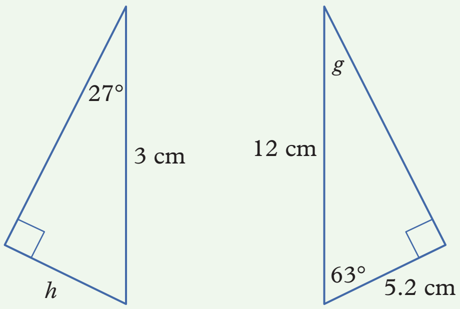


b

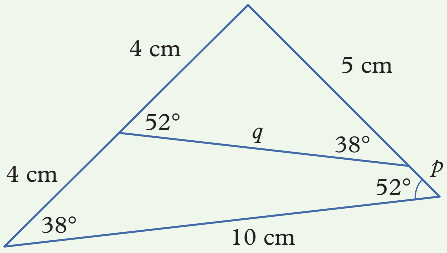


7C **7** Determine the values of the pronumerals.

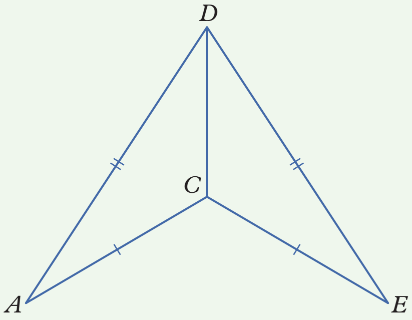
a



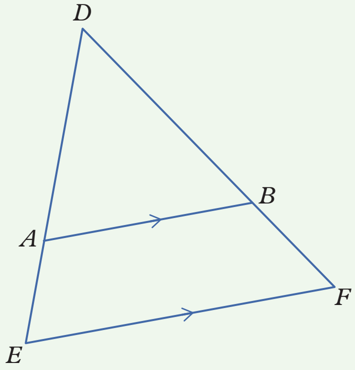
b



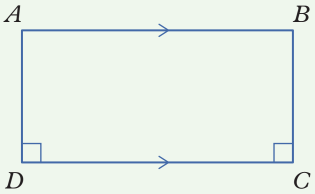
7C **8** **a** Prove that $\Delta ADC \cong \Delta EDC$ and, therefore, that $\angle ACD = \angle ECD$.



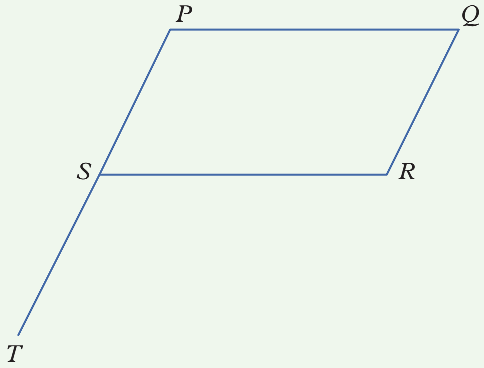
b Prove that $\Delta ADB \sim \Delta EDF$.



7D **9** Given that $AB \parallel DC$, $BC \perp CD$ and $AD \perp CD$, prove that $ABCD$ on the right is a rectangle and, therefore, that $AC = BD$.



7D **10** Given that $\angle SPQ = \angle RST = \angle SRQ$, prove that $PQRS$ is a parallelogram.



7E Circle geometry: circles and angles

Learning intentions

✔ I can understand and apply circle theorems 1–5.

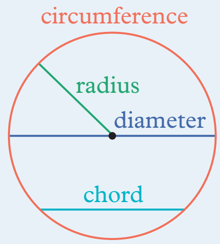
Inter-year links

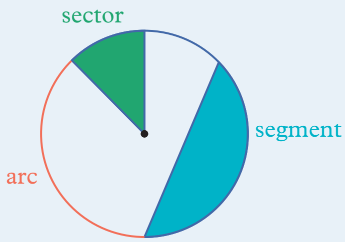
Year 8 7B Triangles

Year 9 7A Angles and lines

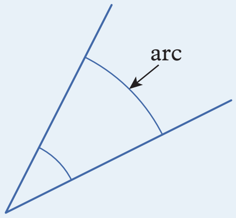
Circle terminology

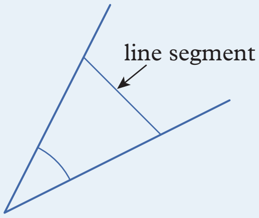
- **Arc**: A part of the **circumference** of a circle.
  A minor arc measures less than a semi-circle (180°). A major arc measures more than a semi-circle.
- **Chord**: A line segment connecting two points on a circle's circumference.
- **Sector**: A portion of a circle enclosed by two radii and an arc on the circumference.
- **Segment**: A region of a circle enclosed by a chord and an arc on the circumference.



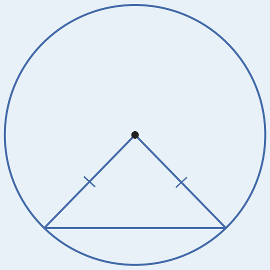


- A subtended angle is an angle whose two rays pass through the endpoints of an arc, line segment or curve. When an arc subtends an angle, the lines drawn from the arc's endpoints form that angle at the point where they meet.



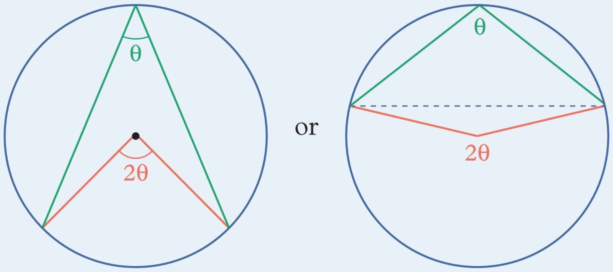


- A cyclic quadrilateral is a quadrilateral whose vertices all lie on the circumference of a single circle.
- An isosceles triangle is formed between two radii of a circle and a chord joining the points where they meet the circumference.



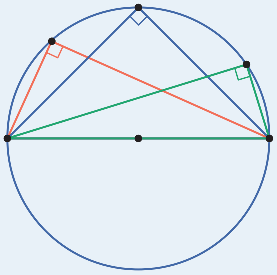
Circle theorems 1–5

- **Circle theorem 1:** The angle subtended by an arc at the centre of a circle is twice the size of the angle subtended by the same arc at the circumference.
  This is also known as the 'angle at the centre' theorem.



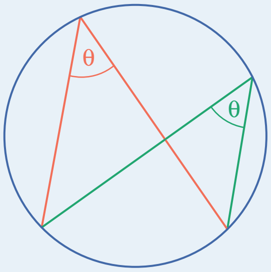
Circle theorem 1

- **Circle theorem 2:** Any angle subtended at the circumference of a circle by the ends of the diameter of the circle is a right angle.
  This is also known as the 'angle in a semi-circle' theorem, or Thales' theorem.



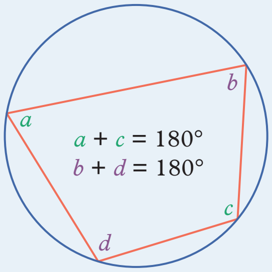
Circle theorem 2

- **Circle theorem 3:** All angles subtended at the circumference by the same arc are equal in size.
  This is also known as the 'angles in the same segment' theorem.



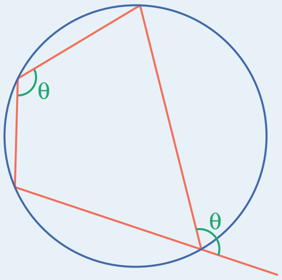
Circle theorem 3

- **Circle theorem 4:** The opposite angles of a cyclic quadrilateral are supplementary.
  This is also known as the 'cyclic quadrilateral' theorem.



Circle theorem 4

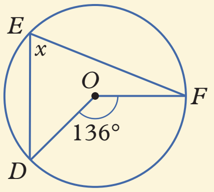
- **Circle theorem 5:** Any exterior angle of a cyclic quadrilateral is equal to the opposite interior angle.
  This is an extension of the 'cyclic quadrilateral' theorem.



Circle theorem 5

Example 7E.1 Using circle theorem 1

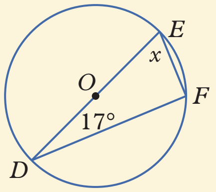
Find the value of x in the circle on the right.



THINK	WRITE
1 The angle subtended by an arc at the centre of a circle is twice the size of the angle subtended at the circumference. So $\angle DOF = 2\angle DEF$.	$136° = 2x$
2 Write your answer.	$x = 68°$

Example 7E.2 Using circle theorem 2

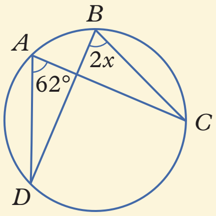
Find the value of x in the circle on the right.



THINK	WRITE
1 Any angle subtended at the circumference by the diameter of a circle is a right angle. So $\angle DFE$ is a right angle.	$\angle DFE = 90°$
2 Use the angle sum of a triangle to help you write an equation to find the value of x.	$x + 17° + 90° = 180°$ $x = 180° - 17° - 90°$
3 Solve the equation and write your answer.	$x = 73°$

Example 7E.3 Using circle theorem 3

Find the value of x in the circle on the right.



THINK	WRITE
1 All angles subtended by the same arc at the circumference are equal in size, therefore $\angle DBC$ and $\angle DAC$ are equal in size.	$\angle DBC = \angle DAC$
2 Substitute values and solve for x.	$2x = 62°$ $x = 31°$

Helpful hints

✔ If you can't remember the circle theorems at first, carefully inspecting the diagrams can be helpful. For example, you might be able to identify by inspection that the angle in a semi-circle is a right angle!

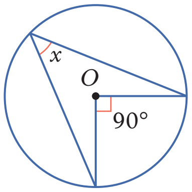
ANS p752

Exercise 7E Circle geometry: circles and angles

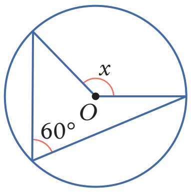
△ 1–4, 5(a–c), 6–8 □ 1–5, 9, 10, 12(a) ⬠ 4, 5, 11, 12(c, d)

7E.1 1 Find the value of x in each of these circles.

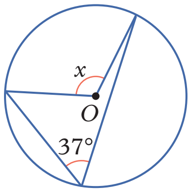
a



b

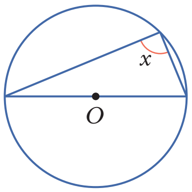


c



7E.2 2 Find the value of x in each of these circles.

a



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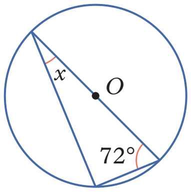

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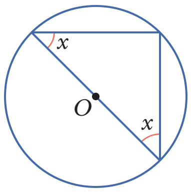
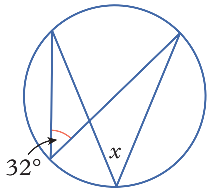


7E.3 3 Find the value of each pronumeral in these circles.

a



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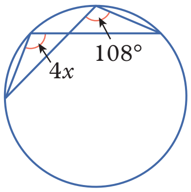

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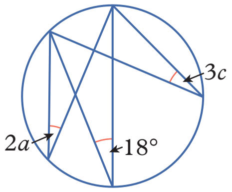
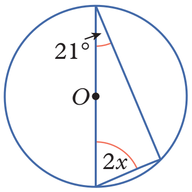


4 Find the value of each pronumeral in these circles (**a–i**), and state which circle theorem is used.

a



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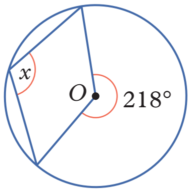
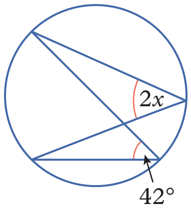


c



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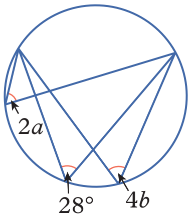

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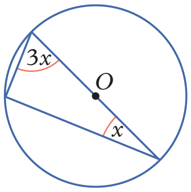

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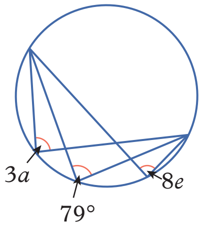
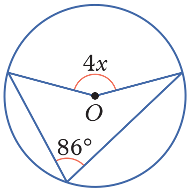


UNDERSTANDING AND FLUENCY

g



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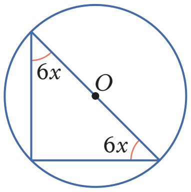

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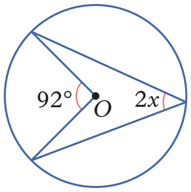
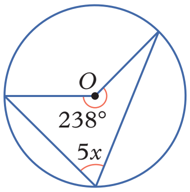


5 Use your understanding of triangle and circle properties and theorems to find the value of x in each of these circles.

a



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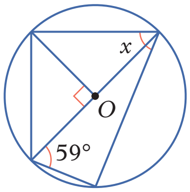

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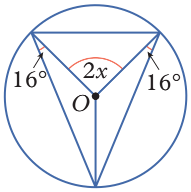

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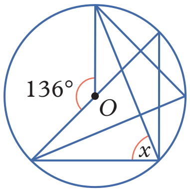

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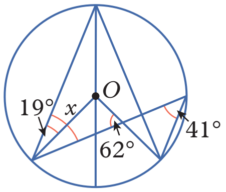

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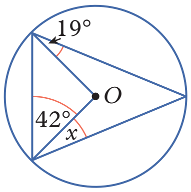


PROBLEM SOLVING AND REASONING

6 This question proves circle theorem 1. Consider the diagram on the right.

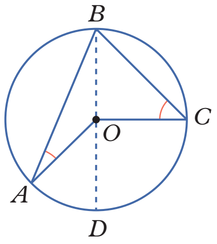
a How do you know that $AO = BO = CO$?

b How do you know that $\angle ABO = \angle OAB$?

c Use the theorem for the exterior angle of a triangle to show that $\angle AOD = 2\angle ABO$.

d Use your answers for parts **b** and **c** to show that $\angle DOC = 2\angle OBC$.

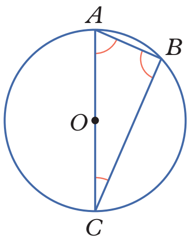
e Explain how this shows that $\angle AOC = 2\angle ABC$.



7 This question proves circle theorem 2. Consider the diagram on the right.

a What is the angle at the centre of the circle?

b Use circle theorem 1 to show that $\angle ABC = 90°$.

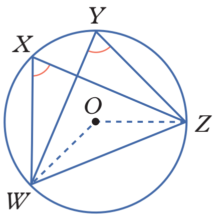


8 This question proves circle theorem 3. Consider the diagram on the right.

a Use circle theorem 1 to explain why $\angle WOZ = 2\angle WXZ$.

b Similarly, show that $\angle WOZ = 2\angle WYZ$.

c What can you say about $\angle WYZ$ and $\angle WXZ$?



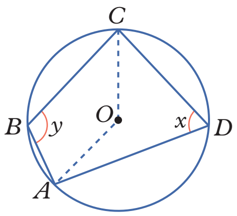
9 This question proves circle theorem 4. Consider the diagram on the right, showing a cyclic quadrilateral.

a Explain why the obtuse angle $\angle AOC = 2x$ and the reflex angle $\angle AOC = 2y$.

b Use what you know about angles at a point to explain why reflex $\angle AOC = 360° - 2x$.

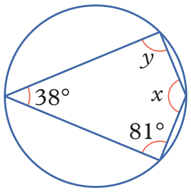
c Use your answers from parts **a** and **b** to show that $y = 180° - x$.

d Copy the diagram, replacing radii OC and OA with OB and OD, then use this to show that $\angle BAD = 180° - \angle BCD$.



e Find the value of each pronumeral in the circles below.

i



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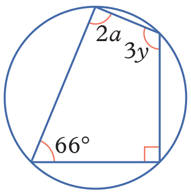
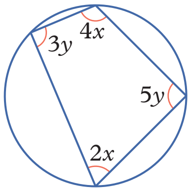
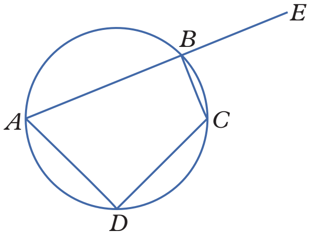


iii



10 This question proves circle theorem 5. Consider the diagram below, showing another cyclic quadrilateral.



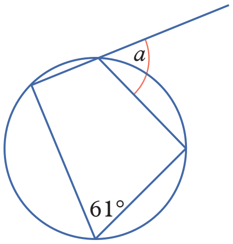
a Use circle theorem 4 to explain why $\angle ADC = 180° - \angle ABC$.

b Why does $\angle EBC = 180° - \angle ABC$?

c Explain why this shows that $\angle ADC = \angle EBC$.

d Find the value of each pronumeral in the diagrams below.

i



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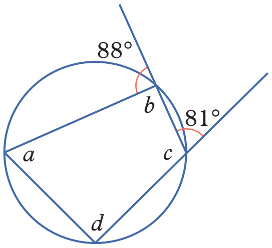
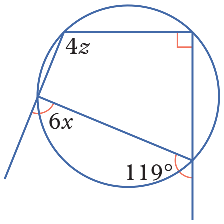


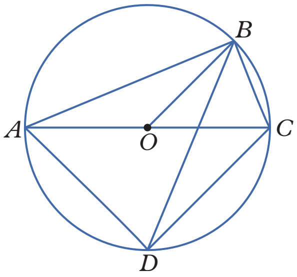
iii



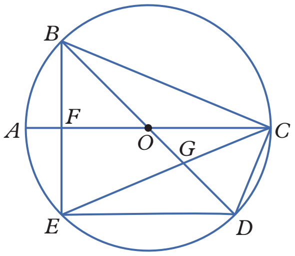
11 Write a complete proof for each of the circle theorems 1–5.

12 Use the given information to answer these questions.

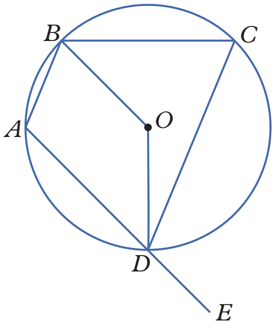
a Find $\angle DBC$, given that $\angle ADB = 68°$, $OA = OB = OC$ and $\angle BCD = 92°$.



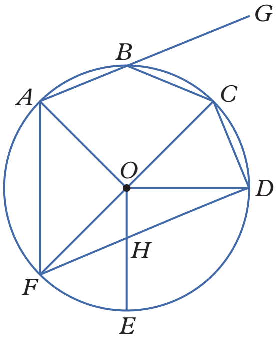
b Find $\angle DEG$, given that $BE \perp AC$, $\angle OCG = 19°$ and $\angle FOG = 142°$.



c Find $\angle ODC$, given that $\angle BAD = 96°$ and $\angle CBO = 46°$.



d Find $\angle OAB$, given that $\angle GBC = 46°$, $\angle BCD = 138°$ and $\angle AFD = 70°$.



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pro

Interactive skillsheet
Circle geometry: chords and angles

Investigation
Clock angles

Topic quiz
7E

7F Circle geometry: chords

Learning intentions

✔ I can understand and apply circle theorems 6–9.

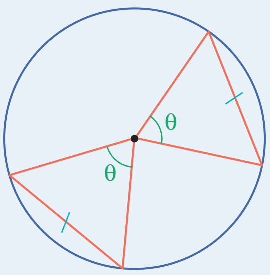
Inter-year links

Year 8	7B Triangles
Year 9	7A Angles and lines

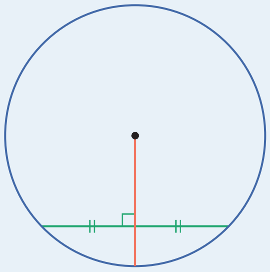
Circle theorems 6–9

- **Circle theorem 6:** Chords of equal length subtend equal angles at the centre of the circle. This is also known as the 'equal chords, equal angles' theorem.
- **Circle theorem 7:** If a radius and a chord intersect perpendicularly, then the radius bisects the chord.
- **Circle theorem 8:** Chords that are equal in length are **equidistant** (at equal distances) from the centre of the circle.
- **Circle theorem 9:** When two chords intersect inside a circle, this divides each chord into two line segments so that the product of the lengths of the line segments for both chords is the same.

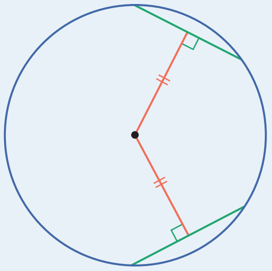
Circle theorem 6



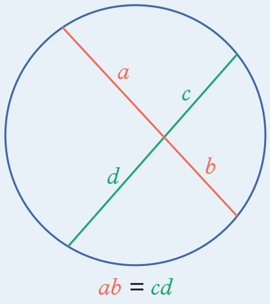
Circle theorem 7



Circle theorem 8

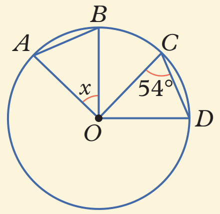


Circle theorem 9



Example 7F.1 Using circle theorem 6

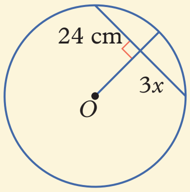
Given $AB = CD$, find the value of x in the circle on the right.



THINK	WRITE
1 Since OC and OD are radii, they are of equal length. This means that ΔCOD is an isosceles triangle.	$OC = OD$
2 Find the two unknown interior angles of ΔCOD.	ΔCOD is isosceles, so $\angle OCD = \angle ODC = 54°$. $\angle COD = 180° - 54° - 54°$ $= 72°$
3 Chords that are equal in length subtend equal angles at the centre of the circle. So $\angle AOB = \angle COD$.	$\angle AOB = \angle COD = 72°$
4 Write your answer.	$x = 72°$

Example 7F.2 Using circle theorem 7

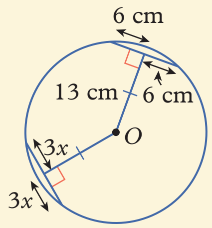
Find the value of x in the circle on the right.



THINK	**WRITE**
1 If a radius and a chord intersect perpendicularly, then the radius bisects the chord. So $3x$ is equal to 24 cm.	$3x = 24$ cm
2 Solve for x.	$x = 8$ cm

Example 7F.3 Using circle theorem 8

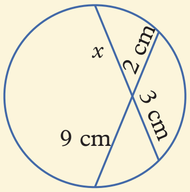
Find the value of x in the circle on the right.



THINK	**WRITE**
1 Chords that are equal in length are equidistant from the centre of the circle. So the two chords must be equal in length.	$3x + 3x = 6 + 6$
2 Solve for x, including units in your answer.	$6x = 12$ $x = 2$ cm

Example 7F.4 Using circle theorem 9

Find the value of x in the circle on the right.



THINK	**WRITE**
1 When two chords intersect, the products of the line segments they are divided into are equal. Write the product of the lengths of the line segments for each chord.	Chord 1: $x \times 3$ cm Chord 2: 2 cm × 9 cm $x \times 3 = 2 \times 9$
2 Simplify the equation.	$3x = 18$
3 Solve for x, including units in your answer.	$x = 6$ cm

Helpful hints

- ✔ All radii of the same circle are equal in size.
- ✔ When chords intersect, each of the equal products result from multiplying the lengths of the two parts of the same line segment.
- ✔ Circle theorems 6 and 8 can be confirmed by rotating the two chords around the centre of the circle until they are aligned.

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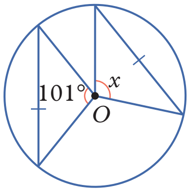
Exercise 7F Circle geometry: chords

△ 1–9, 15 □ 1–4(b, c), 5–7, 10–12, 16, 17 ⬠ 5–7, 13–18

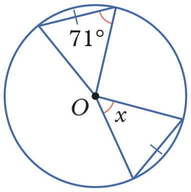
UNDERSTANDING AND FLUENCY

7F.1 **1** Find the value of x in each of these circles.

a



b



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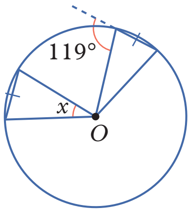
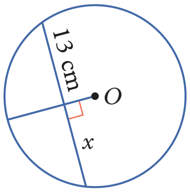


7F.2 **2** Find the value of x in each of these circles.

a



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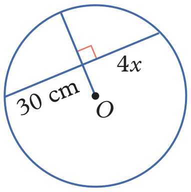

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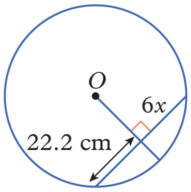
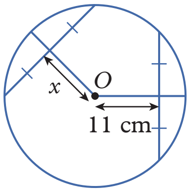


7F.3 **3** Find the value of each pronumeral in these circles.

a



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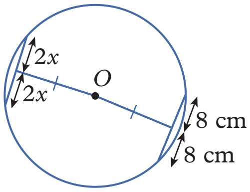

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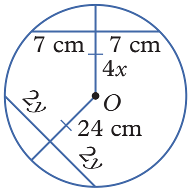
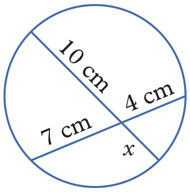


7F.4 **4** Find the value of x in each of these circles, correct to one decimal place.

a



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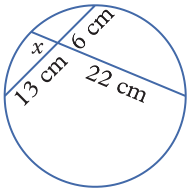

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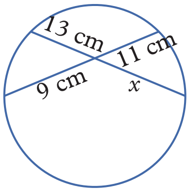
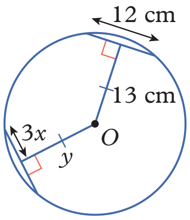


5 Find the value of each pronumeral in these circles. Round your answer to two decimal places.

a



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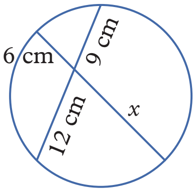

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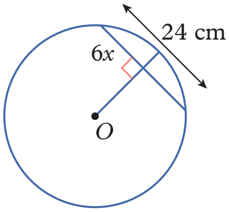

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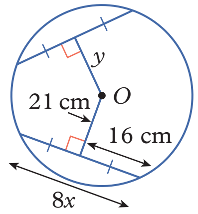

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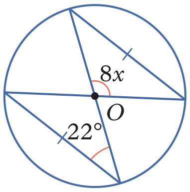

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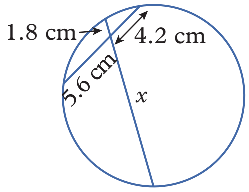

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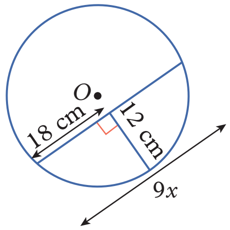

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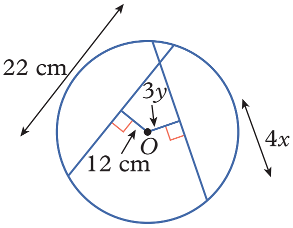

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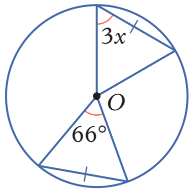
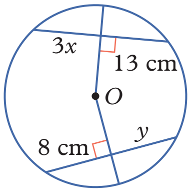
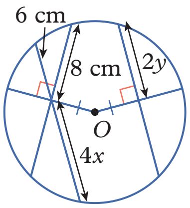


6 Use your understanding of circle properties and theorems, and any other necessary geometric properties, to find the value of each pronumeral in these circles. Round your answer to two decimal places.

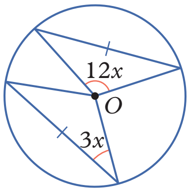
a



b



c



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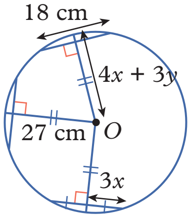

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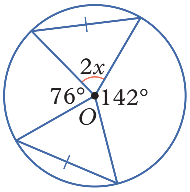

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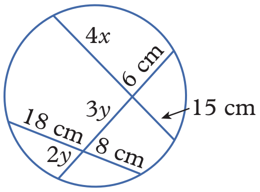
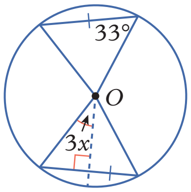


7 Find the value of each pronumeral in these circles. Round your answer to two decimal places.

a



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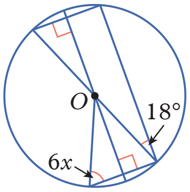

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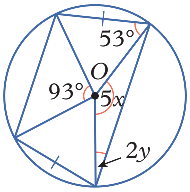

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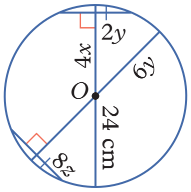

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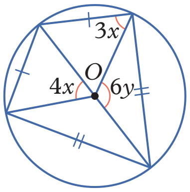

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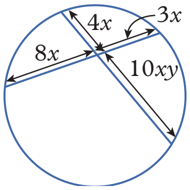


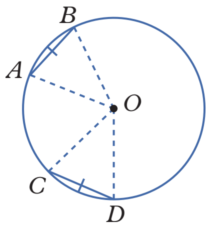
8 This question proves circle theorem 6.
Consider the diagram on the right, in which $AB = CD$.

a Given that OA, OB, OC and OD are radii, what can you say about them?

b What type of triangles does this mean ΔOAB and ΔOCD are?

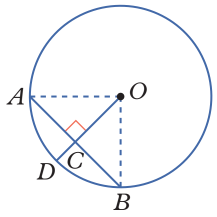
c Explain why $\Delta OAB \cong \Delta OCD$.

d Explain why $\angle COD = \angle AOB$.



PROBLEM SOLVING AND REASONING

9 This question proves circle theorem 7. Consider the diagram on the right.



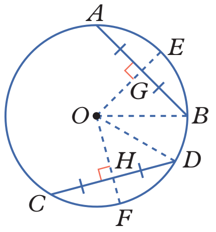
a Explain why $OA = OB$.

b Explain why $\angle OCA = \angle OCB$.

c Use one of the conditions for the congruence of triangles to explain why $\Delta ACO \cong \Delta BCO$.

d Explain why $AC = CB$.

10 This question proves circle theorem 8.
Consider the diagram on the right, in which $AB = CD$.



a Explain why $OE = OB = OD = OF$.

b Use circle theorem 7 to explain why $AG = GB$ and $CH = HD$.

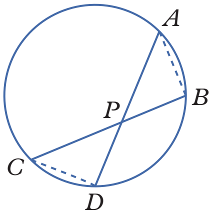
c Likewise, use circle theorem 7 and the fact that $AB = CD$ to explain why $GB = HD$.

d Use one of the conditions for the congruence of triangles to explain why $\Delta OHD \cong \Delta OGB$.

e Explain why $OH = OG$.

f Use your understanding of Pythagoras' Theorem to explain why the shortest distance from the centre of a circle to a chord is perpendicular to that chord.

11 This question proves circle theorem 9.
Consider the diagram on the right.



a Explain why $\angle CPD = \angle APB$.

b Use circle theorem 3 to explain why $\angle DCB = \angle BAD$.

c Explain why $\angle PDC = \angle PBA$.

d Explain how you can tell that ΔCPD is similar to ΔAPB.

e Explain why you can state that $\frac{AP}{BP} = \frac{CP}{DP}$ and, hence, that $AP \times DP = CP \times BP$.

12 Prove that, if a cyclic quadrilateral is a parallelogram, then it must be a rectangle.

13 Draw a circle with two chords. Show that constructing perpendicular bisectors of these two chords locates the centre of the circle.

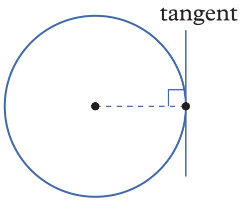
14 Write a complete proof for each of the circle theorems 6–9.

CHALLENGE

15 A tangent is a line that touches the circumference of a circle at one point only.

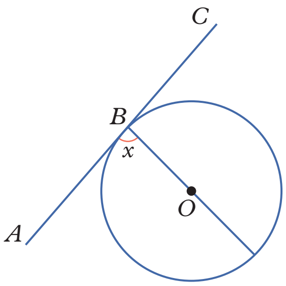
Circle theorem 10 states that, if a tangent to a circle meets a radius of that circle, the tangent and the radius will be perpendicular to each other.

Circle theorem 10

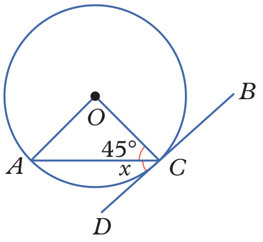


Use circle theorem 10 to find the value of x in each diagram below.

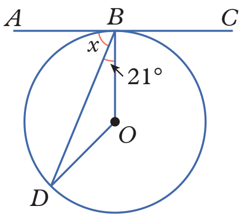
a



b



c

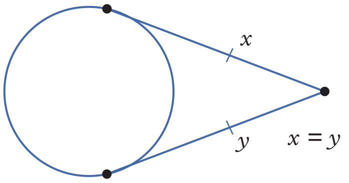


CHALLENGE

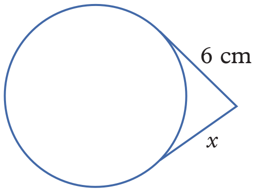
16 Circle theorem 11 states that if two tangents intersect outside a circle, the distances along the tangents from their intersection to the circumference of the circle are equal.

Use circle theorem 11 to find the value of each pronumeral in the diagrams below.

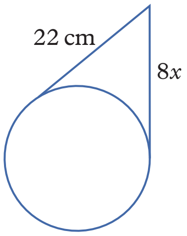
Circle theorem 11



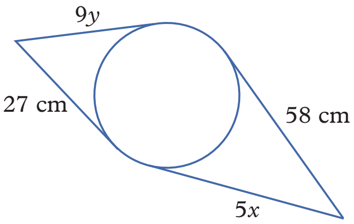
a



b



c

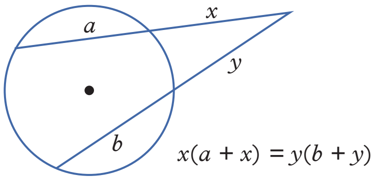


17 A secant is a line that cuts a circle twice.

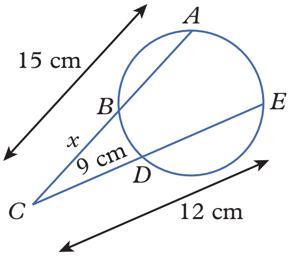
Circle theorem 12 states that if two secants intersect outside a circle, the products of one secant's whole length by its length external to the circle will be equal to the product of the other secant's whole length by its length external to the circle.

Use circle theorem 12 to find the value of x in each diagram below.

Circle theorem 12



a



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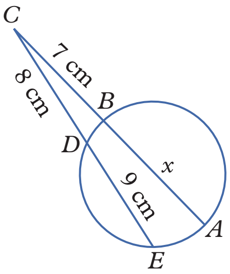
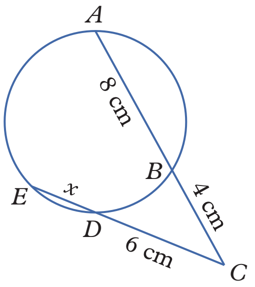


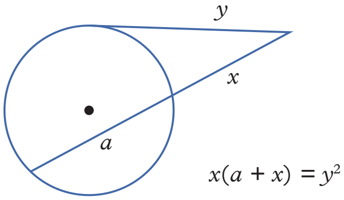
c



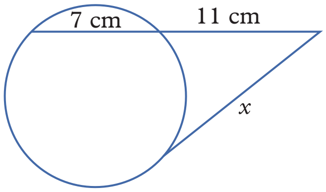
18 Circle theorem 13 states that, if a tangent and a secant intersect outside a circle, the product of the entire secant length by the external secant length will be equal to the square of the tangent length.

Use circle theorem 13 to find the value of x in each diagram below.

Circle theorem 13



a



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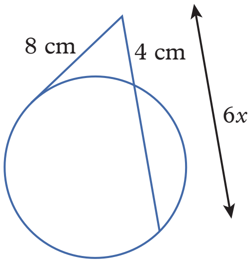

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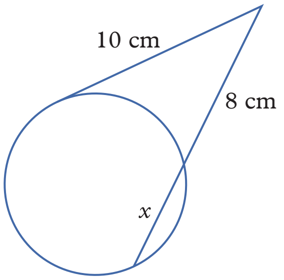


Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Circle geometry: chords

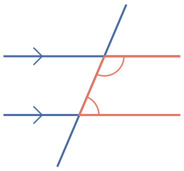
Worksheet
Tangents and secants

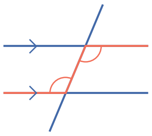
Topic quiz
7F

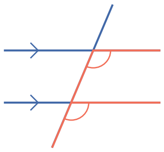
Chapter summary

Angles

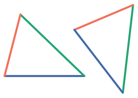
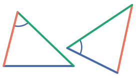
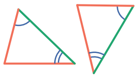
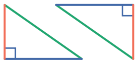
- Complementary angles add to 90°.
- Supplementary angles add to 180°.
- Angles at a point add to 360°.
- Vertically opposite angles are equal.


co-interior angles
(supplementary)

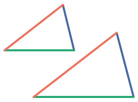
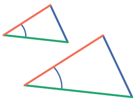
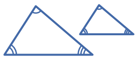
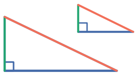

alternate angles
(equal)


corresponding angles
(equal)

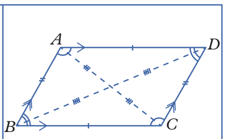
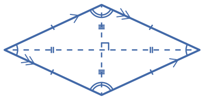
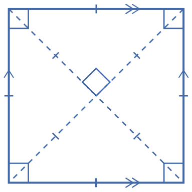
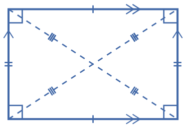
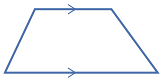
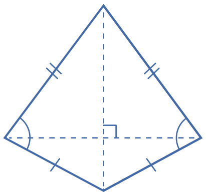
Congruent triangles

SSS (side–side–side)	**SAS** (side–angle–side)	**AAS** (angle–angle–side)	**RHS** (right angle–hypotenuse–side)
			

Similar triangles

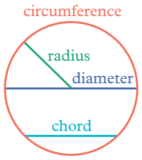
SSS (side–side–side)	**SAS** (side–angle–side)	**AAS** (angle–angle–side)	**RHS** (right angle–hypotenuse–side)
			

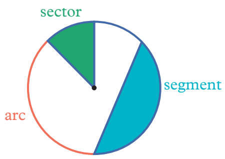
Properties of quadrilaterals

Parallelogram • Opposite sides are parallel and equal. • Opposite angles are equal. • Diagonals bisect each other.	
Rhombus • All sides are equal and opposite sides are parallel. • Opposite angles are equal. • Diagonals bisect each other at right angles. • Diagonals bisect the interior angles.	
Square • All sides are equal and opposite sides are parallel. • All angles are right angles. • Diagonals are equal in length and bisect each other at right angles. • Diagonals bisect the interior angles.	
Rectangle • Opposite sides are parallel and equal. • All angles are right angles. • Diagonals are equal and bisect each other.	
Trapezium • One pair of opposite sides are parallel.	
Kite • Two pairs of sides are equal in length. • One pair of opposite angles are equal in size. • Diagonals are perpendicular. • One diagonal bisects the other diagonal. • One diagonal bisects a pair of vertex angles.	

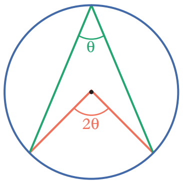
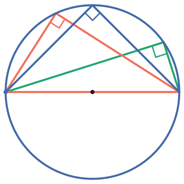
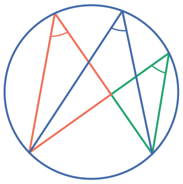
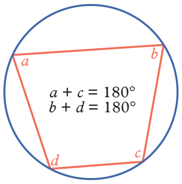
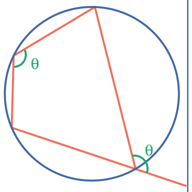
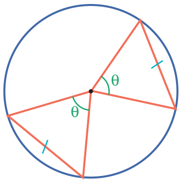
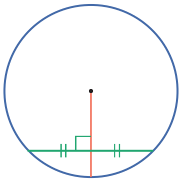
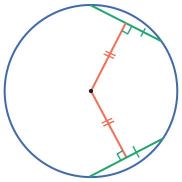
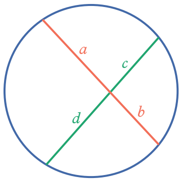
Terminology

- 'Bisect' means to cut in half.
- The symbol for 'is parallel to' is ||.
- The symbol for 'is perpendicular to' is ⊥.





Circle theorems

Circle theorem 1 	**Circle theorem 2** 	**Circle theorem 3** 
Circle theorem 4  $a + c = 180°$ $b + d = 180°$	**Circle theorem 5** 	**Circle theorem 6** 
Circle theorem 7 	**Circle theorem 8** 	**Circle theorem 9**  $ab = cd$

Chapter review

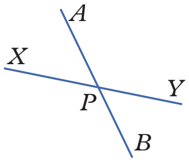
Chapter review quiz
Take the chapter review quiz to assess your knowledge of this chapter.

Multiple-choice

7A **1** Which of these angles is supplementary to 25°?

A 75° **B** 25° **C** 155° **D** 65° **E** 335°

7A **2** Straight lines *AB* and *XY* intersect at *P*. Given that $\angle APY = 2\angle APX$, which of the following statements is true?



A $\angle BPX = 2\angle APY$

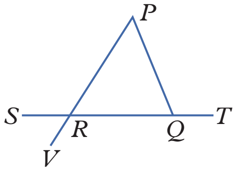
B $\angle XPB = 2\angle APX$

C $\angle YPB = 2\angle APX$

D $\angle APX = 2\angle BPY$

E $\angle APX = 2\angle APY$

7A **3** Which of the following statements is true for the diagram below?



A $\angle PQT = \angle QRV$

B $\angle RPQ = \angle RQP$

C $\angle PRS = \angle PQT$

D $\angle TQP = \angle PRQ + \angle RQP$

E $\angle RPQ + \angle PQR = \angle SRP$

7C **4** Which of these is *not* a test for congruent triangles?

A AAA **B** SSS **C** SAS **D** AAS **E** RHS

7C **5** Which of these is *not* a test for similar triangles?

A AAA **B** SSS **C** SAS **D** AAS **E** RHS

7D **6** Which of the following reasons, by itself, cannot be used to prove that a quadrilateral is a rhombus?

A All sides are equal.

B Opposite angles are equal.

C Diagonals bisect each other at right angles.

D Diagonals bisect the interior angles.

E Parallel opposite sides and adjacent sides are equal.

10A 7E **7** For a particular circle, the angle subtended by an arc at the centre of the circle and the angle subtended by the same arc at the circumference are complementary angles. Which of these values are possible for the two angles?

A 60° and 30° **B** 45° and 45° **C** 90° and 45° **D** 120° and 60° **E** 120° and 240°

10A 7F **8** A circle contains two chords of the same length. Which of these statements is *not* true?

A The arcs on which the two chords stand are the same length.

B The chords are the same perpendicular distance from the centre.

C The chords cannot intersect.

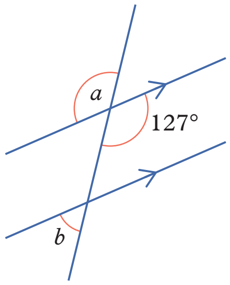
D The chords subtend equal angles at the centre of the circle.

E None of the above.

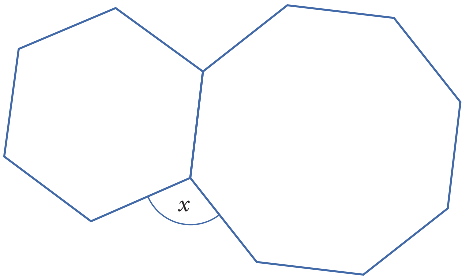
Short answer

7A **1** Determine the values of the pronumerals in these diagrams.

a



b



Note: Both polygons are regular.

7B **2** ΔABC has an internal line drawn from vertex B, meeting the opposite side AC at point D. The following fact is known about this triangle:

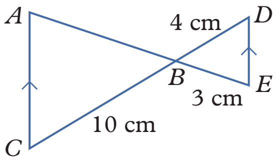
$\angle ABC$ is a right angle.

- **a** Draw a sketch of the triangle.
- **b** Use only the given information to prove that $\angle ADB = \angle DBC + \angle BCD$.

7C **3** Rectangle $ABCD$ has side lengths of 4 cm and 3 cm. Its diagonals intersect at point P.

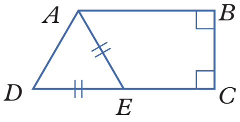
- **a** Draw a diagram to represent this rectangle.
- **b** Explain why $\angle ABP$ is equal to $\angle CDP$, and $\angle PAB$ is equal to $\angle PCD$.
- **c** Explain why $\Delta PAB \cong \Delta PCD$.
- **d** If $\angle BDC = 37°$, find the size of $\angle CBD$.

7C **4**
- **a** For the diagram on the right, provide a reason why ΔABC is similar to ΔEBD.
- **b** If ΔEBD is the reduced image of ΔABC, what is the scale factor?
- **c** What is the length of AB?

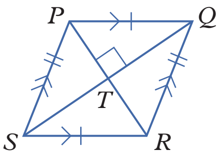


7D **5** Refer to the diagram on the right and write a proof to show that:

- **a** $\angle AEC = 2\angle ADE$
- **b** $2\angle ADE + \angle BAE = 180°$

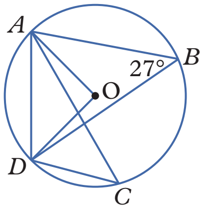


7D **6** Use the information from the diagram below to prove that figure $PQRS$ is a rhombus.

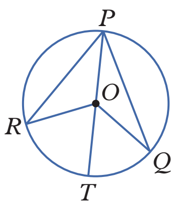


10A 7E **7** Determine the size of these angles in the diagram on the right.

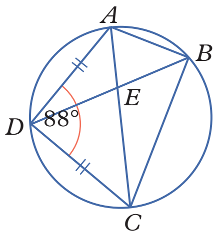
a $\angle AOD$ **b** $\angle ACD$ **c** $\angle OAD$



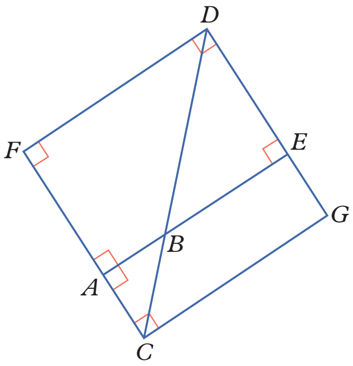
10A 7E **8** For the diagram on the right, prove that
$2\angle RPO + 2\angle QPO + \angle POR + \angle POQ = 360°$.



10A 7F **9**
- **a** Determine the size of $\angle DBC$.
- **b** Prove that $\angle DBC = \angle ABD$.



Analysis



1 Consider the quadrilateral *FDGC*. Any facts proven in parts **a–c** below can be used as given facts in parts **d–f**.

a Prove that $\Delta CFD \cong \Delta CGD$.

b Prove that $\Delta CAB \sim \Delta CFD$.

c Prove that $\Delta CAB \sim \Delta DEB$.

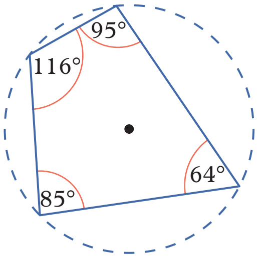
d If $AC = DE$, prove that B is the midpoint of CD.

e If $\dfrac{FC}{AC} = 3$, prove that $\dfrac{DE}{AC} = 2$.

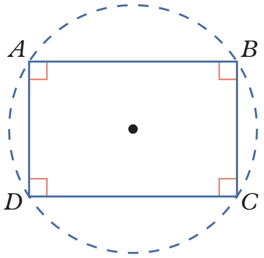
f If $\angle DBE = 45°$, prove that *FDGC* is a square.

10A **2** All of the edges of a cyclic quadrilateral are chords of the same circle. We can consider these chords and the chords that are the diagonals of the cyclic quadrilateral in terms of the circle theorems.

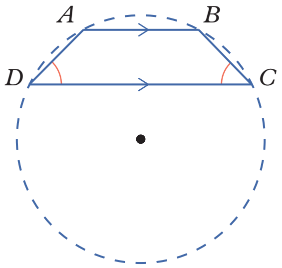
a Explain why this quadrilateral is cyclic.



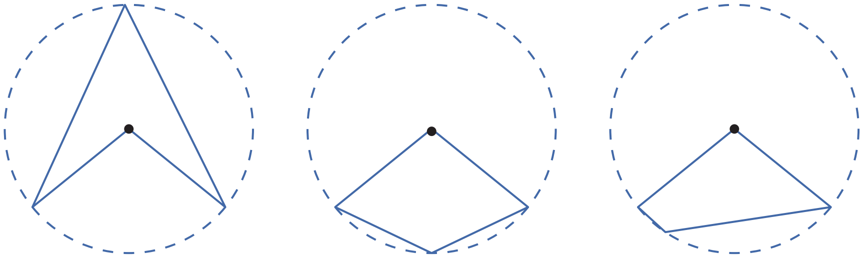
b Prove that all rectangles are cyclic, using *ABCD* shown below.



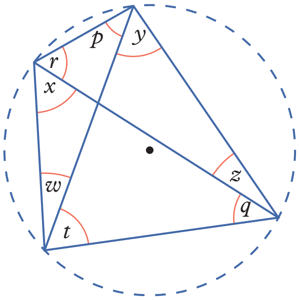
c An isosceles trapezium has equal base angles. Prove that all isosceles trapezia are cyclic, using *ABCD* shown below.



d Explain why quadrilaterals formed using circle theorem 1 (the 'angle at the centre' theorem) are not cyclic.

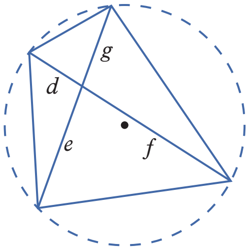


e The diagonals have been added to the cyclic quadrilateral in part **a**. Explain why $p = q$, $r = t$, $x = y$, and $w = z$.

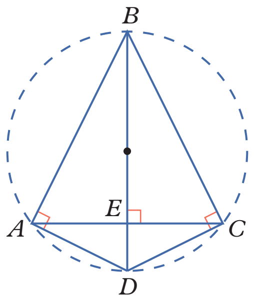


f If $x = 54°$, determine the values of p, q, r, t, w, y and z.

g Considering only the diagonals of the cyclic quadrilateral shown below, what equation can be written in terms of d, e, f and g?

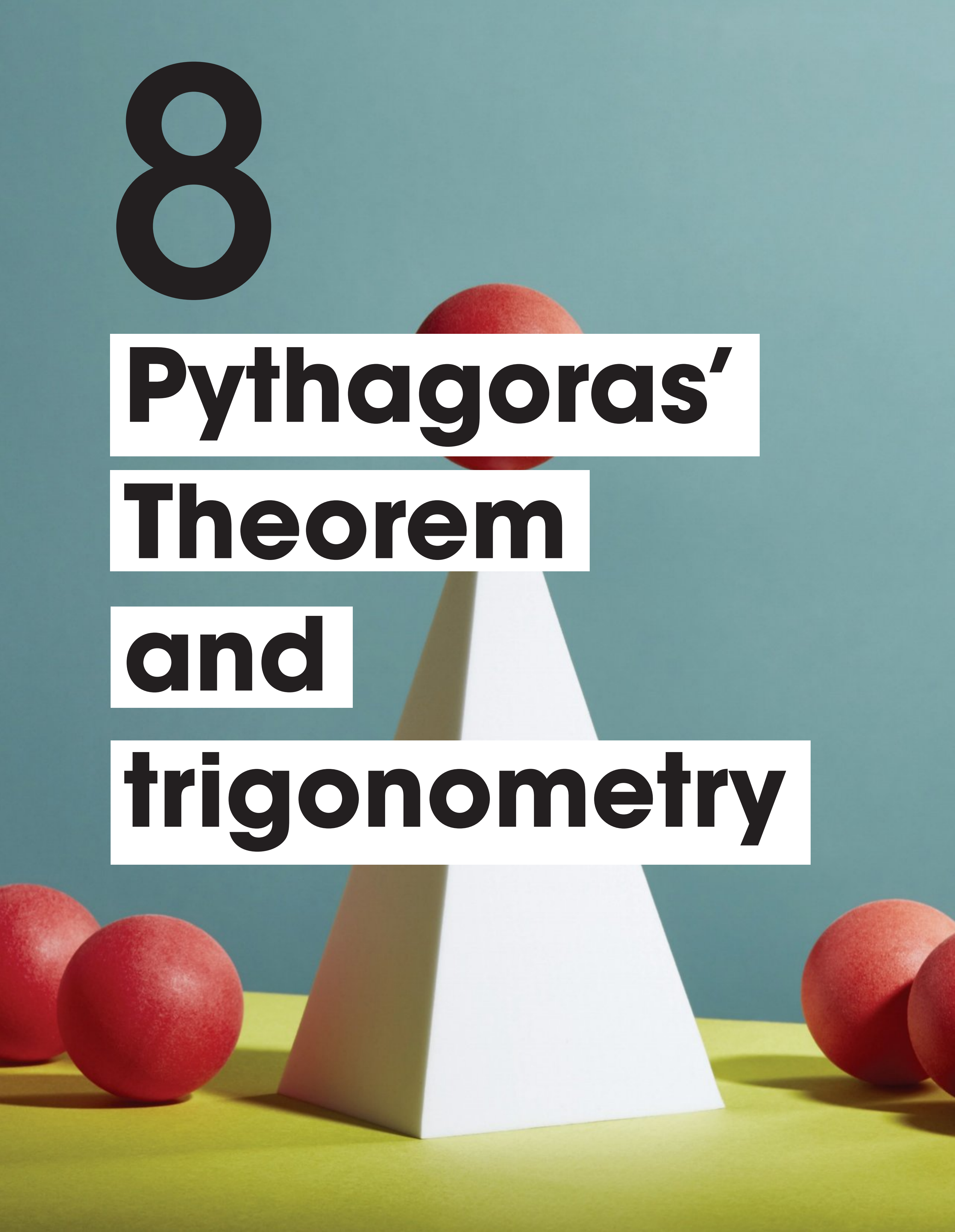


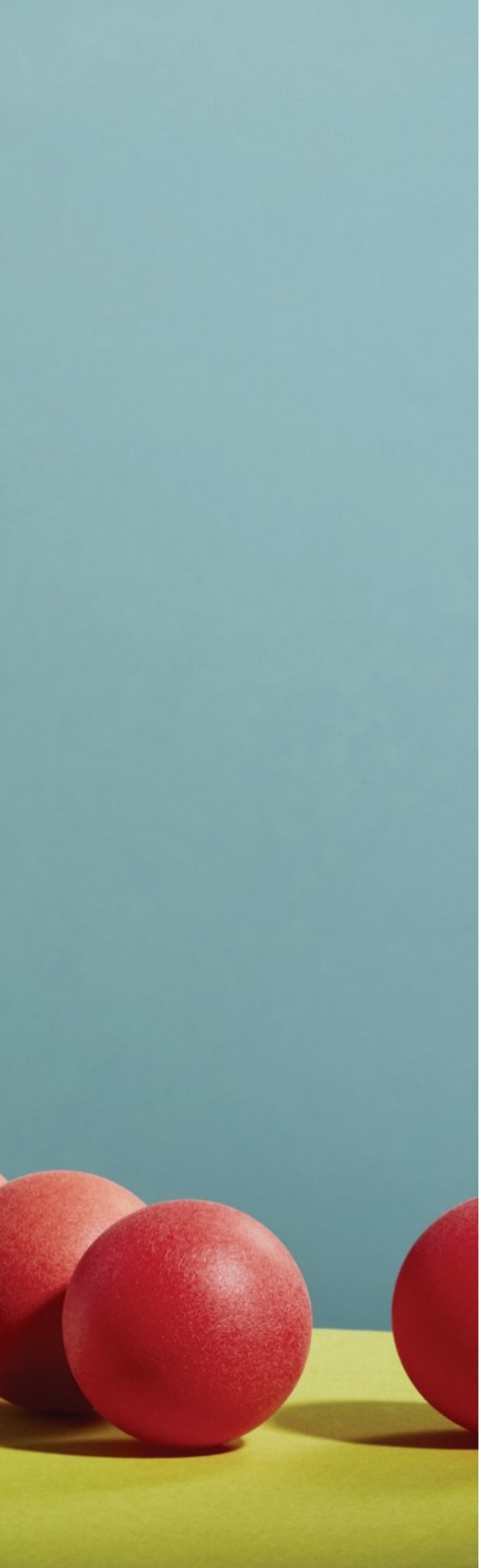
h Prove that all kites with a pair of right angles, as shown in *ABCD* below, are cyclic.



8

Pythagoras’ Theorem and trigonometry





Index

Prerequisite skills

Diagnostic pre-test

Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets

After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Triangle properties
- ✔ Angles and parallel lines
- ✔ Volume of prisms
- ✔ Rounding decimals
- ✔ Simplifying surds

Curriculum links

- Solve right-angled triangle problems including those involving direction and angles of elevation and depression (VCMMG346)
- Establish the sine, cosine and area rules for any triangle and solve related problems (VCMMG367) [10A]
- Use the unit circle to define trigonometric functions as functions of a real variable, and graph them with and without the use of digital technologies (VCMMG368) [10A]
- Solve simple trigonometric equations (VCMMG369) [10A]
- Apply Pythagoras' Theorem and trigonometry to solving three-dimensional problems in right-angled triangles (VCMMG370) [10A]

Materials

- ✔ Protractor
- ✔ Calculator

8A Pythagoras' Theorem

Learning intentions

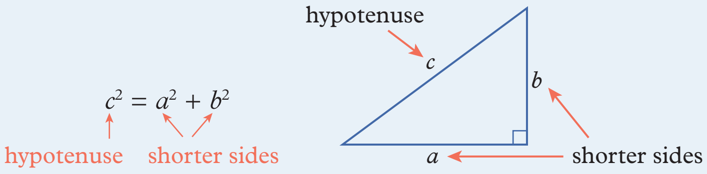
- ✔ I can use Pythagoras' Theorem to determine whether a triangle is right-angled.
- ✔ I can use Pythagoras' Theorem to determine unknown side lengths.

Inter-year links

Year 7	1G Indices and square roots
Year 8	4E Roots
Year 9	7B Pythagoras' Theorem

Pythagoras' Theorem

- The longest side of a right-angled triangle is called the hypotenuse. It is always opposite the right angle.
- **Pythagoras' Theorem** states the relationship between the three side lengths of a right-angled triangle: the sum of the squares of the shorter sides that meet at a right angle is equal to the square of the hypotenuse. The relationship for the triangle shown below is:



- Any set of three whole numbers that satisfies Pythagoras' Theorem is called a **Pythagorean triad** (or **Pythagorean triple**).

  For example, 3, 4, 5 is a Pythagorean triad because $5^2 = 3^2 + 4^2$.
- If the length of the unknown side of a right-angled triangle is found to be an irrational square root, the solution can be written as either an approximate decimal value or an exact value in simplified surd form.

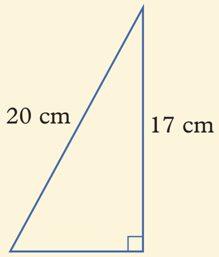
Example 8A.1 Using Pythagoras' Theorem to determine whether a triangle is right-angled

Determine whether a triangle with side lengths of 10 cm, 15 cm, and 12 cm is right-angled.

THINK	WRITE
1 Identify the length of the longest side and call it *c*.	If $c = 15$ cm:
2 Calculate the value of c^2.	$c^2 = (15)^2$ $= 225$
3 Identify the lengths of the two shorter sides and call them *a* and *b*.	If $a = 10$ cm and $b = 12$ cm:
4 Calculate the value of $a^2 + b^2$.	$a^2 + b^2 = (10)^2 + (12)^2$ $= 100 + 144$ $= 244$
5 Compare c^2 to $a^2 + b^2$. If these two values are equal, the triangle is right-angled. If the two values are not equal, the triangle is not a right-angled triangle.	$c^2 \neq a^2 + b^2$ The triangle is not a right-angled triangle.

Example 8A.2 Using Pythagoras' Theorem to calculate an unknown side (decimal value)

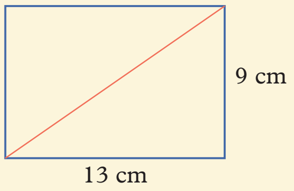
Calculate the length of the unknown side of this triangle, correct to one decimal place.



THINK	**WRITE**
1 Since the triangle is right-angled, state Pythagoras' Theorem and define *a*, *b* and *c*. Remember that *c* is the hypotenuse and use *b* for the unknown side.	If $c^2 = a^2 + b^2$, where $c = 20$ cm and $a = 17$ cm:
2 Substitute the values for *a* and *c* into the equation for Pythagoras' Theorem and simplify the equation.	$(20)^2 = (17)^2 + b^2$ $400 = 289 + b^2$
3 Solve the equation for *b* using inverse operations.	$b^2 = 400 - 289$ $= 111$ $b = \sqrt{111}$
4 Use a calculator to find the approximate length of the unknown side. Round to one decimal place and include units.	$b \approx 10.5$ cm

Example 8A.3 Using Pythagoras' Theorem to calculate an unknown side (exact value)

Calculate the length of the diagonal of this rectangle. Give your answer as an exact value in simplest form.



THINK	**WRITE**
1 Because the shape is a rectangle, the corners are right angles. State Pythagoras' Theorem and define *a*, *b* and *c*. Remember that the diagonal of a rectangle will be the hypotenuse of the two side lengths.	If $c^2 = a^2 + b^2$, where $a = 13$ cm and $b = 9$ cm:
2 Substitute the values for *a* and *b* into the equation for Pythagoras' Theorem. Let *c* be the unknown side and solve the equation for *c*.	$c^2 = (13)^2 + (9)^2$ $= 169 + 81$ $= 250$ $c = \sqrt{250}$
3 Simplify the surd by identifying any factors of 250 that are perfect squares. Include units.	$= \sqrt{25 \times 10}$ $= \sqrt{25} \times \sqrt{10}$ $= 5\sqrt{10}$ cm

Helpful hints

- ✔ Make sure you correctly identify the hypotenuse. It is always the longest side, and is opposite the right angle on the triangle. It may not be obvious, so always look at the numbers carefully.
- ✔ It doesn't matter which of the shorter sides you assign to be a or b, but the hypotenuse must always be c.
- ✔ Pythagoras' Theorem is an excellent formula to memorise. You will use it many times during your VCE Mathematics courses.
- ✔ Remember to always give your answer with correct units.

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Exercise 8A Pythagoras' Theorem

△ **1–6, 8(a, c, f), 9, 10, 12, 13**

■ **1(a, c, e), 2(b, d, f, h), 3(d–f), 4, 7–9, 11, 15, 17**

⬟ **1(a, c, e), 2(e, g, i), 5(b, d), 8(c, d, f), 11, 14, 16, 18, 19**

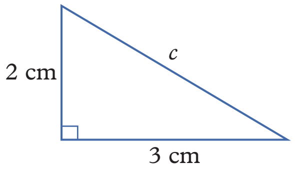
UNDERSTANDING AND FLUENCY

8A.1 **1** Determine whether each set of side lengths form a right-angled triangle.

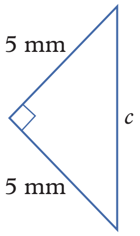
- **a** 26 cm, 24 cm, 10 cm
- **b** 100 cm, 96 cm, 28 cm
- **c** 3.5 cm, 8.5 cm, 9.3 cm
- **d** 1 cm, 2.4 cm, 2.6 cm
- **e** 7.5 cm, 4.2 cm, 8.5 cm
- **f** 5 cm, 7 cm, 6 cm

8A.2 **2** Calculate the length of the unknown side in each of these triangles, correct to one decimal place.

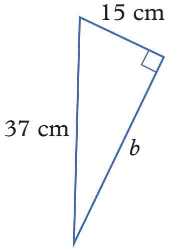
a



b



c



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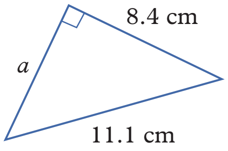

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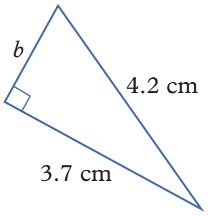

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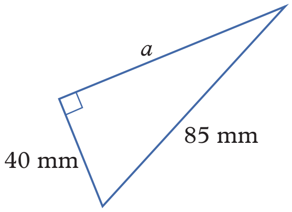

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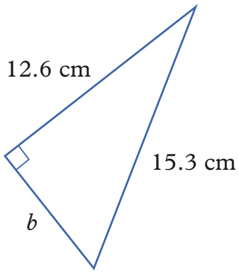

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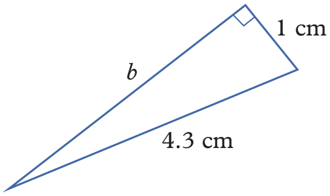

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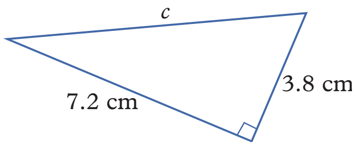
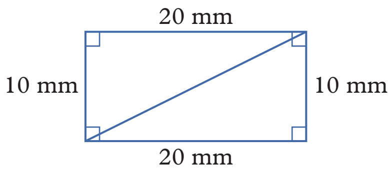
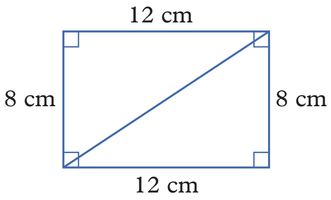


8A.3 **3** Calculate the length of the diagonal in each of these rectangles. Give your answer as an exact value in simplest form.

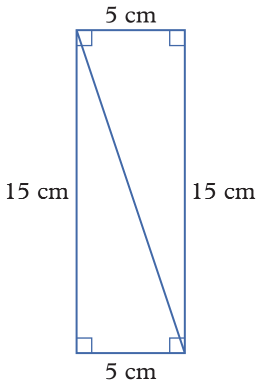
a



b



c



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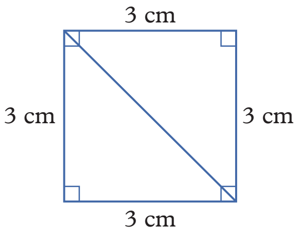

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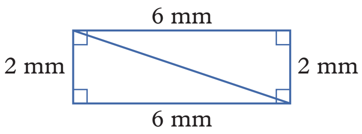
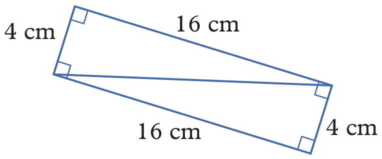


f



4 Rewrite the lengths of the diagonals of the rectangles in question **3**, correct to one decimal place.

5 Calculate the length of the diagonal of a square with each of these side lengths. Give your answer as an exact value in simplest form.

a 5 cm **b** 20 mm **c** 8 cm **d** 25 cm

6 Rewrite the lengths of the diagonals of the squares in question **5**, correct to one decimal place.

7 An equilateral triangle has side lengths of 8 cm.

a Draw a sketch of the triangle. From the top vertex, draw a line that is perpendicular to the base. Label the lengths of all sides of the figure.

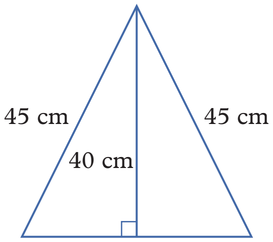
b Use Pythagoras' Theorem to calculate the height of the triangle. Give your answer as:

i an exact value

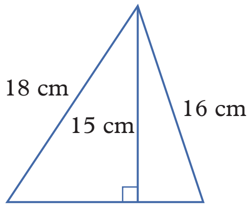
ii an approximate value, correct to one decimal place.

8 Calculate the length of the unknown side for each of these triangles, using the given height of each triangle to help you. Give your answer as an exact value in simplest form.

a



b



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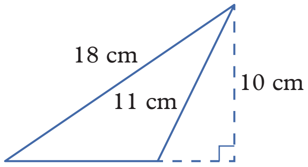

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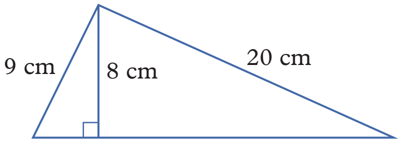

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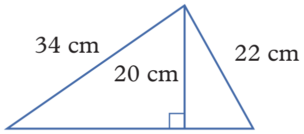
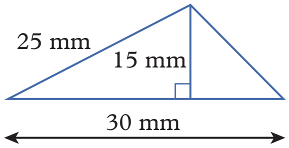


f



9 Rewrite the lengths of the unknown sides for the triangles in question **8**, correct to one decimal place.

10 A squaring tool is used when making picture frames, to ensure that the corners of the frames are true right angles.

Imagine you are constructing a square picture frame with side lengths of 54 cm.

a One of the diagonals of the frame measures 78 cm. Explain how you can be sure that the frame is not truly square.

b The other diagonal measures 76 cm. Draw a labelled sketch of the frame, showing its true shape.

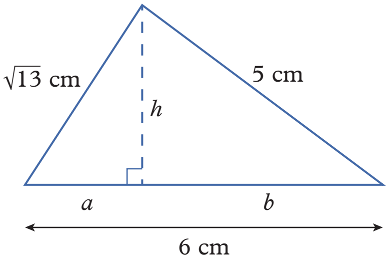
11 A sheet of writing paper is 21 cm wide and 30 cm long. The envelope used for this writing paper has a length of 22 cm and a diagonal of 24.6 cm.

a What is the width of the envelope?

b The writing paper is folded into equal-sized strips along its longer side. What is the minimum number of folds necessary to ensure the paper will fit in the envelope?

c If the paper was folded along its shorter side as well as its longer side, what would be the minimum number of folds needed for it to fit the envelope?

12 Afra wanted to determine the perpendicular height to the 6 cm edge of the triangle on the right but does not know exactly where along the 6 cm edge it intersects. She was able to write the following three equations based on the information in the triangle.



$a^2 + h^2 = 13$ (1)
$b^2 + h^2 = 25$ (2)
$a + b = 6$ (3)

a Explain how Afra was able to write the three equations.

b **i** Rearrange equation (1) to make h^2 the subject.

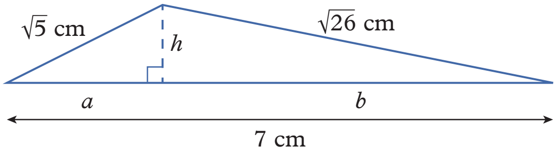
ii Rearrange equation (3) to make b the subject.

c Use substitution of the rearranged equations in part **b** to show that equation (2) can be written as $49 - 12a = 25$.

d Solve the equation in part **c**.

e Hence, determine the values of b and h.

f Repeat this process to find the exact perpendicular height to the 7 cm edge of the triangle shown below.



13 We know that a triangle with side lengths 3 cm, 4 cm and 5 cm is right-angled. The numbers 3, 4, 5 are one example of a Pythagorean triad (or triple). There are many more.

a To investigate other triads, we can start by finding out if multiples of the numbers 3 : 4 : 5 also satisfy Pythagoras' Theorem. Do each of the following sets of numbers satisfy Pythagoras' Theorem?

i 30 : 40 : 50 **ii** 6 : 8 : 10 **iii** 12 : 16 : 20

b We can use the Pythagorean triad 3, 4, 5 to check whether sets of side lengths that are not whole numbers are also Pythagorean triads. Use the 3, 4, 5 triad to show that triangles with the following side lengths are indeed right-angled triangles.

i 1.5 cm, 2 cm, 2.5 cm **ii** 0.6 cm, 0.8 cm, 1 cm **iii** 1.2 cm, 1.6 cm, 2 cm

c What do you conclude from your investigations in parts **a** and **b**?

d All the triangles in parts **a** and **b** are similar triangles. Explain why this is so.

PROBLEM SOLVING AND REASONING

14 There are different formulas that can be used to generate sets of Pythagorean triads. For example, three numbers in a Pythagorean triad can be written as: $x^2 - y^2$, $2xy$ and $x^2 + y^2$, where integers x is greater than integer y.

- **a** Which of the given expressions will represent the hypotenuse? Will this always be the case?
- **b** Choose values for x and y. Note the restriction that has been placed on these values.
- **c** Use the values you chose in part **b** to calculate values for:
  - **i** $x^2 - y^2$
  - **ii** $2xy$
  - **iii** $x^2 + y^2$
- **d** Verify that the set of numbers you generated in part **c** does, in fact, satisfy Pythagoras' Theorem.
- **e** Explain the restriction $x > y$.
- **f** Repeat the procedure from parts **b–d** to generate another Pythagorean triad.
- **g** What values of x and y will generate the triad 3, 4, 5 from the expressions given in part **c**?

15 The two sides of a right-angled triangle, other than the hypotenuse, are such that one of those sides is twice as long as the other. The hypotenuse is 25 cm long. What are the lengths of the two shorter sides? Write your answer

- **a** as a simplified exact value
- **b** correct to one decimal place.

16 Josh has an extension ladder that is 9.8 m long. He needs to clean the windows on the outside of a building. The lower part of the windows are 9.8 m from the base of the building, and the windows are 1.2 m tall. Unfortunately, bushes prevent Josh from placing his ladder any closer to the base of the building than 3 m. Josh is 1.7 m tall.

- **a** Draw a diagram to show the situation.
- **b** Calculate the height the ladder reaches up the building correct to one decimal place.
- **c** Explain whether Josh will be able to clean the windows.

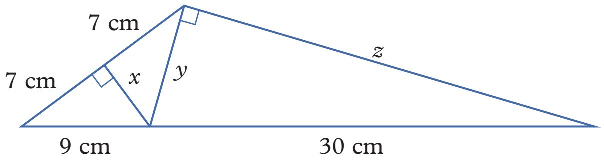
17 A cubby house is built on two levels in a tree. The lower level is 10 m above the ground, and the upper level is a further 5 m higher vertically. The ladder from the ground to the lower level is 12 m long. Sam wants to construct another ladder to go from the ground, 1 m further along the ground from the base of the tree than the first ladder, to the top level. How long should he make this ladder?

CHALLENGE

18 The orthocentre of a triangle is the intersection point of the perpendicular heights of each side of the triangle. Consider ΔABC where $A(-5,0)$, $B(-1,8)$, and $C(2,-1)$.

- **a** Determine the equation of the lines of all three perpendicular heights.
- **b** Find the intersection point of the perpendicular heights.
- **c** Find the exact distance each vertex of ΔABC is from the orthocentre.

19 Determine the exact simplified values of x, y, and z.



Check your Student obook pro for these digital resources and more:

- **Interactive skillsheet** Pythagoras' Theorem.
- **Interactive skillsheet** Finding the length of the hypotenuse
- **Interactive skillsheet** Finding the length of a shorter side
- **Topic quiz** 8A

8B Trigonometry

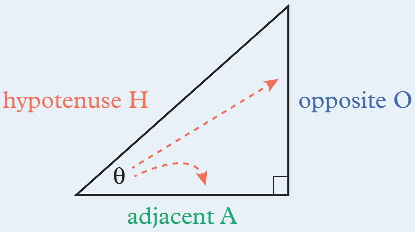
Learning intentions

✔ I can use trigonometry to find unknown side lengths.
✔ I can use trigonometry to find unknown angles.

Inter-year links
Year 9 7D Trigonometry

Finding lengths using trigonometry

- **Trigonometry** describes the relationship between the angles and side lengths of a triangle.
- The Greek letter θ (theta) is often used as a pronumeral for angles.
- The sides of a right-angled triangle can be labelled as:
  - → the hypotenuse: the longest side of the right-angled triangle opposite the right angle
  - → the **opposite side**: the side opposite an interior reference angle
  - → the **adjacent side**: the side adjacent to (or next to) an interior reference angle that is not the hypotenuse.



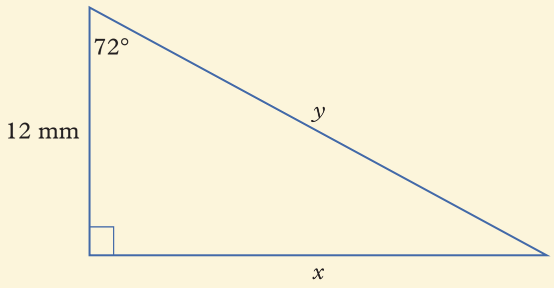
- The trigonometric ratios are known as the **sine** ratio, the **cosine** ratio and the **tangent** ratio. They are the ratios between two side lengths of a right-angled triangle with respect to a given interior reference angle.
- For right-angled triangles with a reference angle of θ:
  - → $\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{O}{H}$
  - → $\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{A}{H}$
  - → $\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{O}{A}$
- A calculator can be used to determine the sine, cosine or tangent values of any angle.
- The equations for the sine, cosine and tangent ratios can be rearranged to determine the side lengths of a right-angled triangle given an angle and one side length.

Using trigonometry to determine unknown angles

- The non-right angles in a right-angled triangle can be determined using the inverse sine inverse sine ($\sin^{-1}$), inverse cosine ($\cos^{-1}$) or inverse tangent ($\tan^{-1}$) of a ratio as they are the opposite operations or inverse functions of the sine, cosine and tangent functions.
  - → $\theta = \sin^{-1}\left(\frac{\text{Opposite}}{\text{Hypotenuse}}\right)$
  - → $\theta = \cos^{-1}\left(\frac{\text{Adjacent}}{\text{Hypotenuse}}\right)$
  - → $\theta = \tan^{-1}\left(\frac{\text{Opposite}}{\text{Adjacent}}\right)$
- A calculator can be used to determine the inverse sine, inverse cosine or inverse tangent values.

Example 8B.1 Using trigonometry to determine unknown side lengths (decimal values)

Find the length of each unknown side in this triangle, correct to one decimal place.



THINK	**WRITE**
1 Identify the hypotenuse (H), opposite side (O) and adjacent side (A) with respect to the reference angle 72°.	$H = y, O = x, A = 12$ mm
2 Decide which trigonometric ratio to use to find the length of y. Since this involves the side adjacent to the reference angle and the hypotenuse (A and H), use cosine.	$\cos(\theta) = \frac{A}{H}$
3 Substitute values for θ, A and H into $\cos(\theta) = \frac{A}{H}$.	$\cos(72°) = \frac{12}{y}$
4 Solve the equation for y.	$y \times \cos(72°) = 12$
5 Use a calculator to divide 12 by cos(72°). Round the value you find for y to one decimal place and include units.	$y = \frac{12}{\cos(72°)}$ $= 38.8328\ldots$ ≈ 38.8 mm
6 Decide which trigonometric ratio to use to find the length of x. Since this involves the sides opposite and adjacent to the reference angle (O and A), use tangent.	$\tan(\theta) = \frac{O}{H}$
7 Substitute values for θ, A and H into $\tan(\theta) = \frac{O}{A}$.	$\tan(72°) = \frac{x}{12}$
8 Solve the equation for x.	$x = 12 \times \tan(72°)$
9 Use a calculator to multiply 12 by tan(72°). Round the value you find for x to one decimal place and include units.	$= 36.9322\ldots$ ≈ 36.9 mm

Example 8B.2 Using the inverse trigonometric functions

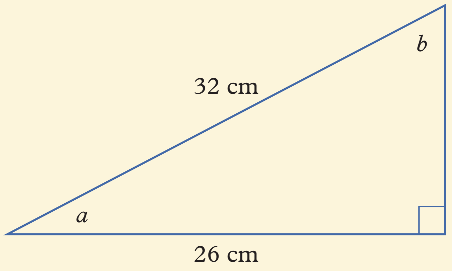
Use a calculator to find the size of θ for these trigonometric ratios to the nearest degree.

a $\sin(\theta) = 0.439$ **b** $\tan(\theta) = \frac{17}{5}$

THINK	**WRITE**
a **1** Rearrange to make θ the subject of the equation using the inverse sine.	**a** $\sin(\theta) = 0.439$ $\theta = \sin^{-1}(0.439)$
2 Use the $\sin^{-1}$ key on your calculator to find the inverse sine of 0.439.	$\theta = 25.8489\ldots$
3 Round the angle to the nearest degree.	$\theta \approx 26°$
b **1** Rearrange to make θ the subject of the equation using the inverse tangent.	**b** $\tan(\theta) = \frac{17}{5}$ $\theta = \tan^{-1}\left(\frac{17}{5}\right)$
2 Use the $\tan^{-1}$ key on your calculator to find the inverse tangent of $\frac{17}{5}$.	$\theta = 73.6104\ldots$
3 Round the angle to the nearest degree.	$\theta \approx 74°$

Example 8B.3 Using trigonometry to determine unknown angles

Find the size of each of the two unknown angles in this triangle. Give your answers correct to two decimal places.



THINK	**WRITE**
1 Decide which angle to calculate first, and identify the known side lengths with respect to that chosen angle.	$\theta = a$, H = 32 cm, A = 26 cm
2 Decide which trigonometric ratio to use. Because the adjacent angle (A) and hypotenuse (H) are involved, use cosine.	$\cos(\theta) = \frac{\text{A}}{\text{H}}$
3 Substitute values for θ, A and H into $\cos(\theta) = \frac{\text{A}}{\text{H}}$.	$\cos(a) = \frac{26}{32}$
4 Rearrange to make a the subject of the equation using the inverse cosine.	$a = \cos^{-1}\left(\frac{26}{32}\right)$
5 Use the $\cos^{-1}$ key on your calculator to find the inverse cosine of $\frac{26}{32}$.	$= 35.659\ldots$
6 Round your answer correct to two decimal places.	$\approx 35.66°$
7 Use the value of a and the right angle to find the value of b. Remember that the sum of the interior angles of a triangle is equal to 180°.	$b = 180° - 90° - a$ $\approx 90° - 35.66°$ $= 54.34°$

Helpful hints

- ✔ When finding an unknown side length, label the triangle to identify which side length you know (O, A or H) and which side length you want to find (O, A or H). Then pick the trigonometric ratio whose formula contains both.
- ✔ When finding an unknown angle, label the triangle to identify which two side lengths you know (O, A or H). Then pick the trigonometric ratio whose formula contains both.
- ✔ Remember that $\sin^{-1}$ does *not* mean sine to the power of -1 (the reciprocal of sine).
- ✔ Remember to write $\theta = \sin^{-1}\left(\frac{4}{5}\right)$, not $\sin^{-1}(\theta) = \frac{4}{5}$.
- ✔ It helps to draw diagrams roughly to scale. Remember, longer sides are opposite larger angles and shorter sides are opposite smaller angles.

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Exercise 8B Trigonometry

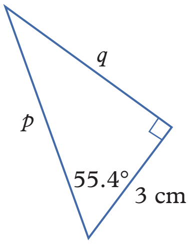
△ 1, 2(1st column), 3–5, 7–9

■ 1, 2(2nd column), 3, 4, 6, 10, 13–15, 17, 19(a, b)

⬠ 1, 3(d–f), 6, 11–16, 18, 19(b, d)

8B.1 **1** Find the lengths of the unknown sides in each of these triangles, correct to one decimal place.

a



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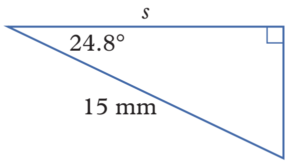
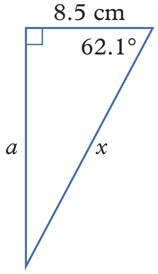
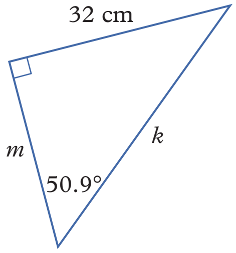


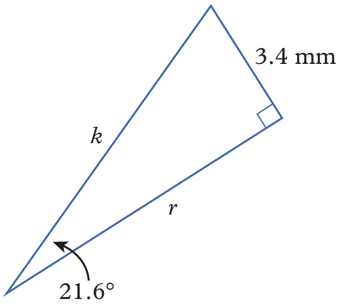
c



d



e



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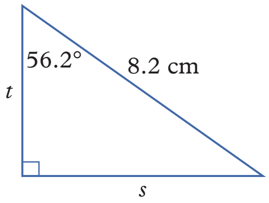


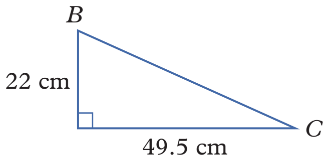
8B.2 **2** Use a calculator to find the size of θ for the trigonometric ratios below. Write your answers to the nearest degree.

a $\sin(\theta) = 0.378$
b $\cos(\theta) = 0.845$
c $\tan(\theta) = 2.376$
d $\tan(\theta) = 0.097$
e $\sin(\theta) = 0.759$
f $\cos(\theta) = 0.238$
g $\tan(\theta) = 45.326$
h $\sin(\theta) = 0.019$
i $\cos(\theta) = 0.777$
j $\cos(\theta) = \frac{4}{7}$
k $\tan(\theta) = \frac{15}{11}$
l $\sin(\theta) = \frac{4}{13}$
m $\tan(\theta) = \frac{57}{72}$
n $\sin(\theta) = \frac{29}{31}$
o $\cos(\theta) = \frac{3}{19}$
p $\sin(\theta) = \frac{10}{13}$
q $\cos(\theta) = \frac{44}{99}$
r $\tan(\theta) = \frac{1}{100}$

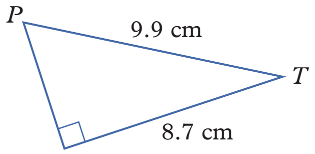
UNDERSTANDING AND FLUENCY

8B.3 **3** Find the size of each of the two unknown angles in each of these triangles. Give your answers to the nearest degree.

a



b



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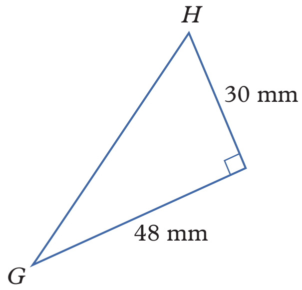

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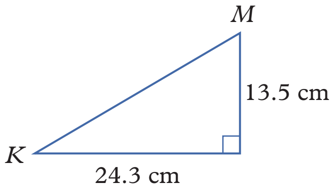

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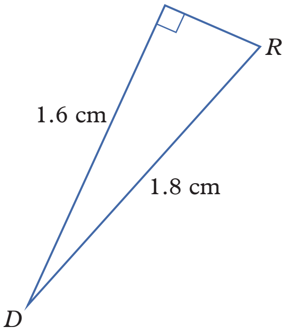

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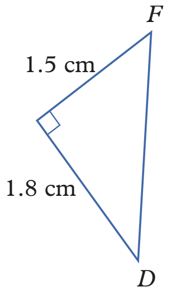
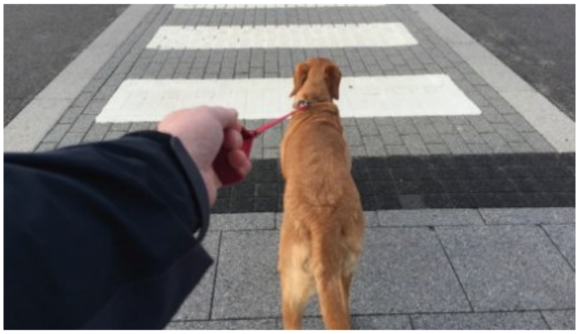


4 Shun is walking his dog on a leash. The leash is 2.5 m long, and his dog is 55 cm tall. Shun's dog is pulling him along so that the taut leash makes an angle of 85° with the vertical.

a Draw a diagram to show this information.

b How far in front of Shun is his dog, correct to the nearest centimetre?

c What is the vertical height of the end of the leash in Shun's hand to the nearest centimetre?



5 The front face of a brick has a diagonal length of 242 mm. This diagonal forms an angle of 18.3° with the longer side of the face. What are the dimensions of the face of the brick, correct to the nearest millimetre?

6 The Leaning Tower of Pisa is a freestanding bell tower, next to the cathedral in the Italian city of Pisa. The tower is 55.86 m tall on its low side and 56.7 m tall on its high side. The top of the tower is displaced 3.9 m horizontally from where it would be if it were standing vertically.

a The 'average' height of the tower above the ground represents the height of the centre of the top of the tower above the ground. Calculate this height correct to two decimal places.

b Draw a labelled sketch of the tower using this average height and its horizontal displacement. Mark the angle of 'lean' of the tower.

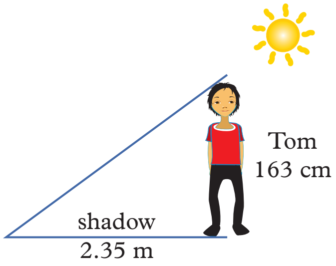
c Describe which trigonometric ratio you would use to calculate the angle of the tower's lean, marked in part **b**.

d Calculate the angle of lean of the tower. Give your answer to the nearest degree.

7 Tom and his brother Sam were standing in the sunshine measuring the length of each other's shadow. They each knew their height, so they collated the measurements (see the table on the right).

	Height	Length of shadow
Tom	163 cm	2.35 m
Sam	141 cm	2.04 m

a Tom drew the diagram on the right to represent his shadow and height. Draw a similar diagram to display Sam's measurements.



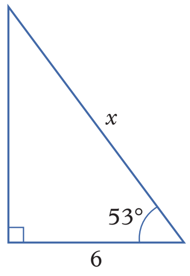
b Calculate the base acute angle for the triangles in each diagram. (Remember to take into account the difference in units.) Give your answers in degrees, correct to one decimal place. What does this base angle represent?

c **i** Explain why the two triangles formed by the boys' heights and their shadows must be similar shapes.

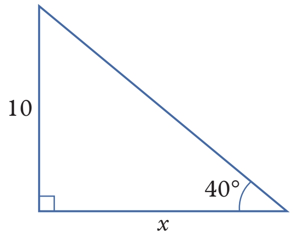
ii What is the linear scale factor from the triangle formed by Sam's shadow to the triangle formed by Tom's shadow?

iii Draw a diagram combining the two triangles from the diagram above and your own diagram from part **a**.

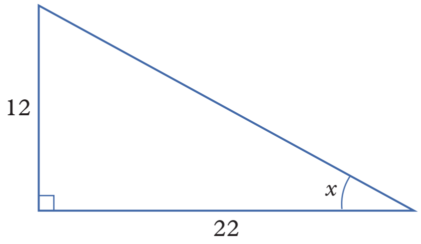
8 **a** Ji-ho has attempted to find the value of x in the following triangles. Explain the mistake Ji-ho made in all three questions.



$$\cos(53°) = \frac{x}{6}$$
$$x = 6\cos(53°)$$
$$x = 3.61089...$$
$$x \approx 3.6$$



$$\tan(40°) = \frac{x}{10}$$
$$x = 10\tan(40°)$$
$$x = 8.390996...$$
$$x \approx 8.4$$



$$\tan(x) = \frac{22}{12}$$
$$x = \tan^{-1}\left(\frac{22}{12}\right)$$
$$x = 61.3895...°$$
$$x \approx 61.4°$$

b Explain how you can ensure you don't make the mistakes in part **a**.

9 Parliament House in Canberra is 300 m long and 300 m wide, being one of the largest buildings in the Southern Hemisphere.

a Draw a labelled diagram to show the shape and size of the building from above.

b Use Pythagoras' Theorem to calculate the diagonal length of the building.

c Use trigonometry to calculate the diagonal length of the building. (Hint: Identify the size of the angle you can use.)

d Comment on your answers for parts **b** and **c**.

10 The Great Hall of Parliament House in Canberra displays a tapestry based on a painting by the Australian artist Arthur Boyd. The tapestry measures 20 m wide and is said to be one of the largest tapestries in the world. The tapestry hangs from the ceiling, and reaches all the way to the floor, with the angle between the diagonal and the floor being 24.2°.

a Draw a diagram to display this information.

b How high is the ceiling in the Great Hall, correct to the nearest metre?

c What is the perimeter of the tapestry, correct to the nearest metre?

11 Sandy is standing 2 m in front of a picture hanging on a wall. The picture is 100 cm wide, and is hung so the bottom of the frame is 1.5 m from the floor. If she lowers her eyes 5.7° from the horizontal, the bottom of the frame of the painting is directly in her line of sight. If she raises her eyes 16.7° from the horizontal, the top of the frame is directly in her line of sight.

a At what height are Sandy's eyes?

b What are the dimensions of the painting?

12 Some road signs display the gradient of a road, particularly when the road is approaching a hill or mountain climb. The gradient can be written in different forms, commonly as a ratio, a percentage or a fraction. For example, 1 in 10, 1:10, 10% and $\frac{1}{10}$ can all be used to represent the same gradient. Pat is a long-distance truck driver delivering goods between various locations. He generally plans his route to avoid steep hill climbs, if possible. His next delivery takes him via a mountainous stretch, where he has a choice of three possible routes, described below.

Route 1: Over a horizontal distance of 12 km, with the rise being 6 km
Route 2: Over a horizontal distance of 18 km, with the rise being 7.5 km
Route 3: Over a horizontal distance of 15 km, with the rise being 6.5 km

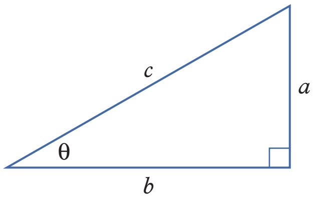
a Write the gradient of each route, correct to one decimal place, as:

i a ratio **ii** a percentage.

b What is the angle of the rise of each road over the distances given, correct to one decimal place?

c Rank the three routes in order of increasing steepness.

13 Investigate what happens to the value of the sine ratio as the angle increases from 0° to 90° in a right-angled triangle. Consider the triangle in the diagram shown on the right.



a The sine of θ is the ratio of the length of the opposite side to the length of the hypotenuse; that is, $a : c$. When θ is 0°:

i What is the length of a?

ii What does this mean for the value of the ratio $a : c$?

b Now consider the situation when θ becomes almost 90°.

i What is the value of a compared with c?

ii What is the value of the ratio $a : c$?

c Describe what happens to the value of the sine ratio as the angle increases from 0° to 90°.

d Repeat parts **a–c** for the cosine ratio. Consider the relevant line segments for that ratio.

e Once more, repeat parts **a–c** for the tangent ratio, considering the relevant line segments for that ratio.

14 From your investigations in question **13**, you can determine the minimum and maximum values for each of the three trigonometric ratios for angles in the range 0° to 90°. Copy and complete the following to summarise your findings.

The sine ratio has a minimum value of _____ and a maximum value of _____.
The cosine ratio has a minimum value of _____ and a maximum value of _____.
The tangent ratio has a minimum value of _____ and a maximum value of _____.

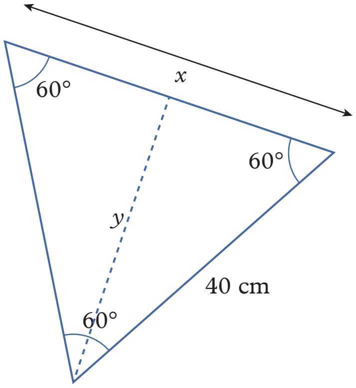
15 Check your answers for question **14** with a calculator. Provide some values to support your conclusions.

16 A hexagon is a six-sided polygon. Consider a regular hexagon with side lengths of 12 cm. Calculate the distance between its parallel sides correct to two decimal places.

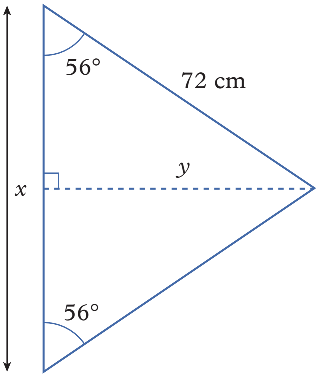
17 a Determine the value of x and y in the following triangles, correct to two decimal places where required.

b Hence, determine the area of each triangle, correct to one decimal place.

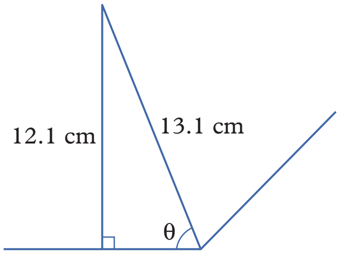
i



ii



18 The vertex of a regular polygon where line segments have been drawn from the polygon's centre to the vertex and the midpoint of the adjacent edge is shown to the right.



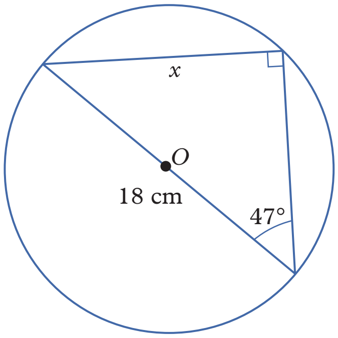
a Determine the value of θ correct to one decimal place.

b Use the angle from part **a** to determine what type of regular polygon the vertex is from.

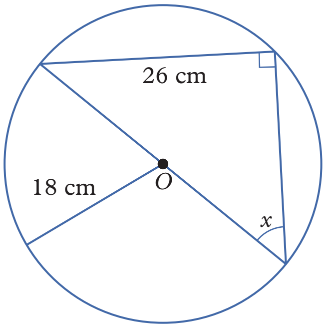
c Determine the perimeter of the regular polygon correct to the nearest centimetre.

19 Determine the value of x in each of the following correct to two decimal places.

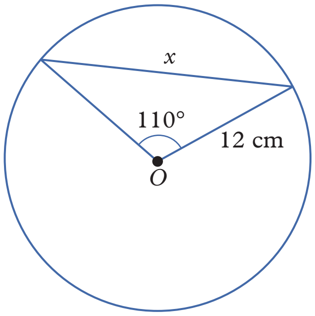
a



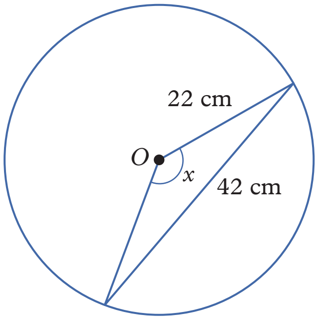
b



c



d



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Trigonometry

Interactive skillsheet
Finding angles by trigonometry

CAS instructions
Trigonometry

Topic quiz
8B

Interactive skillsheet
Finding lengths by trigonometry

8C Applications of trigonometry

Learning intentions

✔ I can use angles of elevation and depression to calculate distances.

✔ I can use compass bearings and true bearings.

Inter-year links

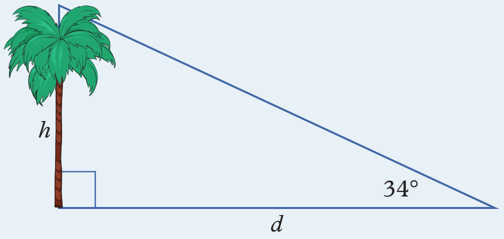
Year 7	6H Solving equations using inverse operations
Year 8	6B Solving equations using inverse operations
Year 9	7E Using trigonometry to find lengths

Angles of elevation and depression

- The **angle of elevation** is the angle measured upward from the horizontal.

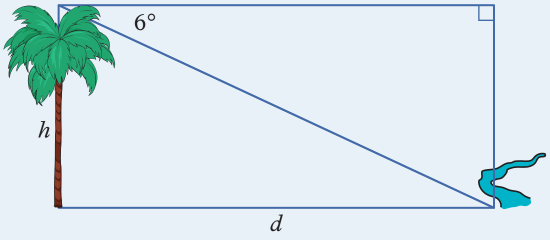
  For example, in the diagram on the right, there is a 34° angle of elevation to the top of a palm tree from a distance d along a flat beach.

  The height of the palm tree is given by $h = d\tan(34°)$.



- The **angle of depression** is the angle measured down from the horizontal.

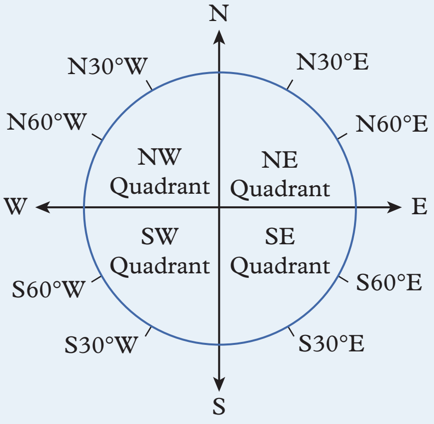
  For example, if someone climbs to the top of the same palm tree and measures a 6° angle of depression from the top of the tree to a nearby ocean inlet, then the distance from the base of the tree to the inlet would be given by $d = \frac{h}{\tan(6°)}$.



Bearings

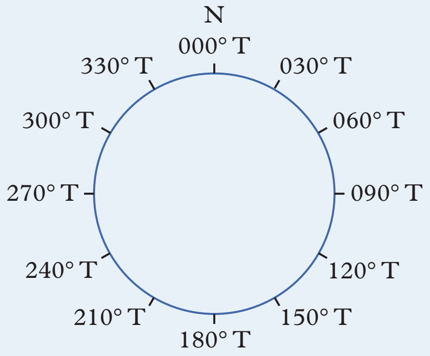
- The four cardinal directions are north, south, east and west.
- **Compass bearings** describe direction using angles between 0° and 90° within one of the four quadrants of a compass (see the diagram on the right). The bearings are written with a capital letter for the north-south direction, followed by the angle, followed by a capital letter for the east-west direction.

  For example, the angle 35° clockwise of north would be written as N35°E, and the opposite direction would be written as S35°W.

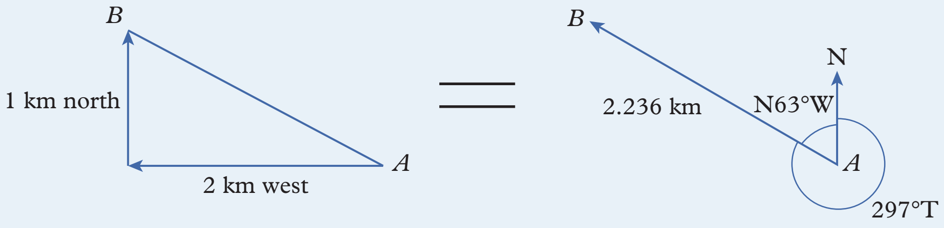


- **True bearings** describe a direction using an angle between 0° and 360° clockwise from north. They are written with three digits for the angle, followed by a capital T.

  For example, the angle 35° clockwise of north would be written as 035°T, and the opposite direction would be written as 215°T.



- It is often useful to know how far a position is in terms of distance along the cardinal directions. For example, if point *B* is 1 km north and 2 km west of point *A*, then Pythagoras' Theorem can be used to find the total distance and bearings from point *A* to point *B* to be approximately 2.236 km N63°W or 297°T.

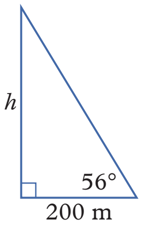
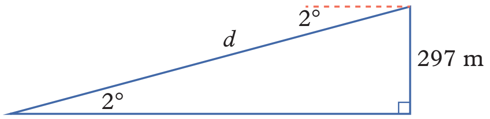


Example 8C.1 Angles of elevation and depression

Cape Pillar in the Tasman National Park is home to the tallest sea cliffs in the Southern Hemisphere.

a If a tourist measures her kayak to be 200 m from the base of the closest cliff face and the angle of elevation to the tip of the cliff is 56°, how tall is the cliff? Round your answer to the nearest metre.

b The next day, the same tourist hikes to the top of the cliff and spots a boat near the horizon at an angle of depression of 2°. How far away is the boat from the tourist? Round your answer to the nearest metre.

THINK	WRITE
a **1** Draw a diagram to represent the situation. It does not need to be to scale.	**a** 
2 Substitute the known values into the equation for the tangent ratio.	$\theta = 56°$, O = h, A = 200 m, $\tan(\theta) = \frac{O}{A}$ $\tan(56°) = \frac{h}{200}$
3 Solve for the height of the cliff and round to the nearest metre.	$h = 200\tan(56°)$ $= 296.5121\ldots$ ≈ 297 m
b **1** Draw a diagram to represent the situation. It does not need to be to scale. Assign the angle of depression to the corresponding angle of elevation, as they are alternate angles.	**b** 
2 Substitute the known values into the equation for the sine ratio.	$\theta = 2°$, A = 297 m, H = d $\sin(\theta) = \frac{O}{H}$ $\sin(2°) = \frac{297}{d}$
3 Solve for the distance from the tourist to the boat and round to the nearest metre.	$d = \frac{297}{\sin(2°)}$ $= 8510.1514\ldots$ ≈ 8510 m

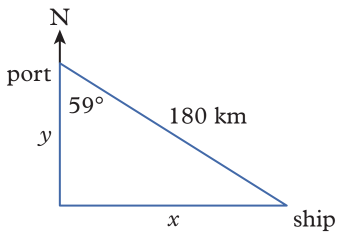
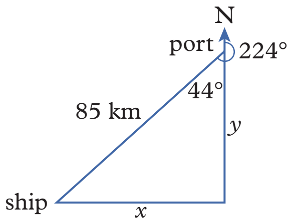
Example 8C.2 Bearings

For ships travelling the distances and directions below:

i Draw a labelled diagram to display the distance travelled along the relevant cardinal directions.

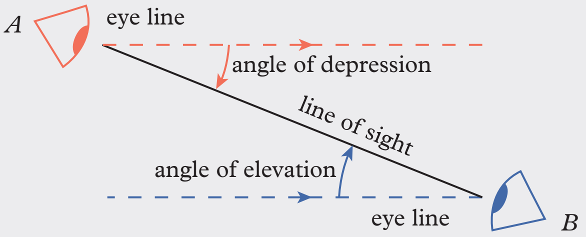
ii Use trigonometry to calculate the distances travelled along the relevant cardinal directions, to the nearest kilometre.

a 180 km on a bearing of S59°E   **b** 85 km on a bearing of 224°T

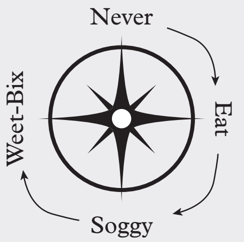
	THINK	WRITE
a i	Draw a diagram to represent the situation. Start with the port and draw an upwards arrow to represent north, then draw a line in the direction towards the ship. Label the distances along the east-west and north-south axes of your diagram.	
ii 1	Use the cosine ratio to calculate the distance south.	$\cos(\theta) = \frac{A}{H}$ $\cos(59°) = \frac{y}{180}$ $y = 180 \times \cos(59°)$ ≈ 93 km The ship is 93 km south of port.
2	Use the sine ratio to calculate the distance east.	$\sin(\theta) = \frac{O}{H}$ $\sin(59°) = \frac{x}{180}$ $x = 180 \times \sin(59°)$ ≈ 154 km The ship is 154 km east of port.
b i 1	A true bearing between 180°T and 270°T will be in the SW quadrant, so calculate the angle measured clockwise from south.	$224° - 180° = 44°$
2	Draw a diagram to represent the situation. Start with the port and draw an upwards arrow to represent north, then a line in the correct direction towards the ship.	
ii 1	Use the sine ratio to calculate the distance west.	$\sin(\theta) = \frac{O}{H}$ $\sin(44°) = \frac{x}{85}$ $x = 85 \times \sin(44°)$ ≈ 59 km The ship is 59 km west of port.
2	Use the cosine ratio to calculate the distance south.	$\cos(\theta) = \frac{A}{H}$ $\cos(44°) = \frac{y}{180}$ $y = 85 \times \cos(44°)$ ≈ 61 km The ship is 61 km south of port.

Helpful hints

- ✔ It always helps to draw a diagram and label the sides and angles with the known and unknown values. It does not need to be exactly to scale.
- ✔ The angle of depression from point *A* to point *B* is equal to the angle of elevation from point *B* to point *A*. This is because they form alternate angles on parallel eye lines.



- ✔ One way of remembering the cardinal directions is the phrase 'Never Eat Soggy Weet-Bix'. The first letter of each word matches the cardinal directions, read clockwise from north.



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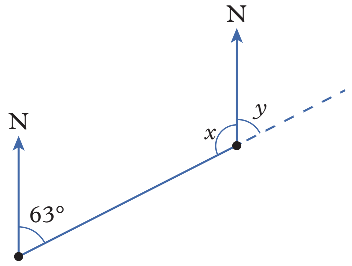
Exercise 8C Applications of trigonometry

△ 1–6, 8, 11, 12, 14, 15 ■ 1, 2, 5–7, 9, 10, 13, 16, 17, 19 ⬟ 1(d–f), 5, 6, 10, 14, 16–18, 20

UNDERSTANDING AND FLUENCY

1 Determine the value of the pronumerals in each of the following.

a



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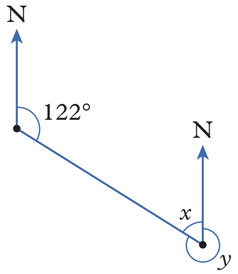
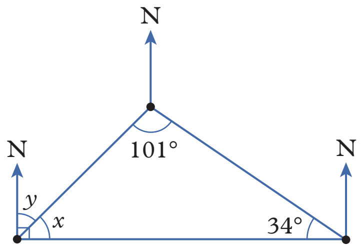
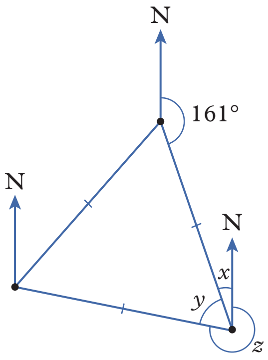


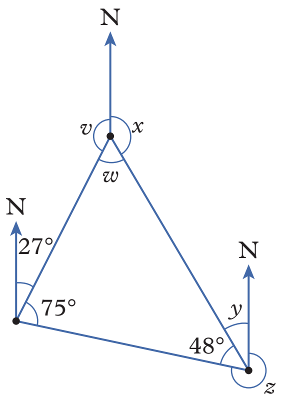
c



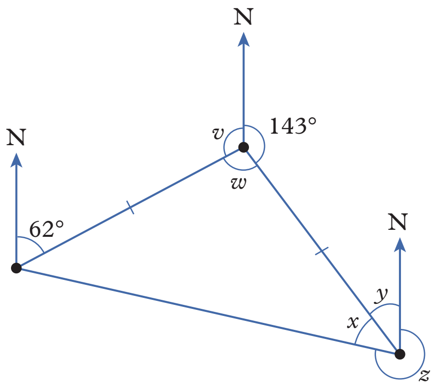
d



e



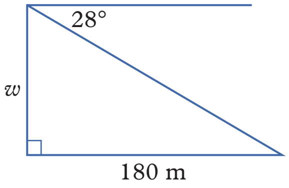
f



UNDERSTANDING AND FLUENCY

2 Use the angle of elevation or depression in each diagram to calculate the side length marked with the pronumeral. Give your answers correct to one decimal place.

a



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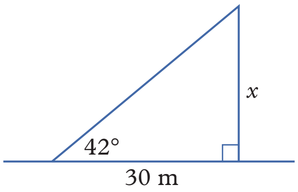

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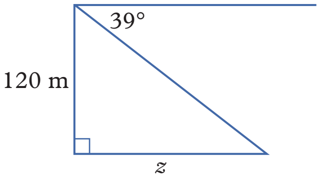

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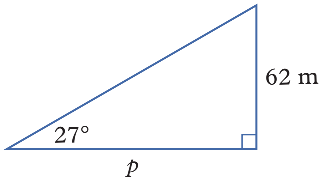

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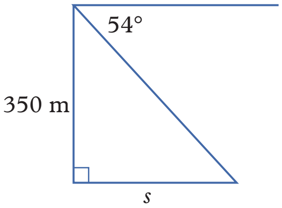

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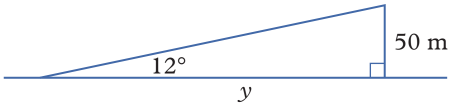


8C.1 **3** A lighthouse stands on a cliff above the sea. The angle of elevation to the top of the lighthouse from a point on level ground 100 m from its base is 14°.

a Find the height of the lighthouse, to the nearest metre.

b The angle of depression from the light at the top of the lighthouse to a ship at sea 2.5 km from the base of the cliff is 3.3°. Find the height of the light above the sea.

4 Convert the following bearings:

a from compass bearings to true bearings

i S30°E **ii** N52°W **iii** N5°E

b from true bearings to compass bearings.

i 200°T **ii** 100°T **iii** 300°T

8C.2 **5** For ships travelling each of the distances and directions below:

i draw a labelled diagram to display the distance travelled along the relevant cardinal directions

ii use trigonometry to calculate these distances travelled along the relevant cardinal directions, to the nearest kilometre.

a 250 km on a bearing of N30°E **b** 100 km on a bearing of 300°T

c 75 km on a bearing of S45°W **d** 210 km on a bearing of 090°T

6 If each ship in question **5** turns around and returns to its original port, write a description of the ship's path with respect to its turn-around position using:

i compass bearings and distance in kilometres

ii true bearings and distance in kilometres.

7 Stavros is descending an escalator at an angle of depression of 30°.

a If the escalator is 6 m high at the top, correct to two decimal places, what horizontal distance does it cover?

b If it takes 30 seconds for Stavros to descend the escalator, correct to one decimal place, at what speed, in m/s, is the escalator moving?

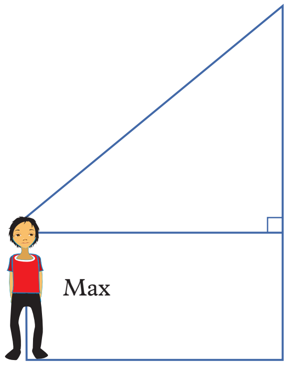
8 Max throws a ball into the air. He views the ball at its highest point to be at an angle of elevation of 40° from him and 3 m horizontally away from him. Max's eye level is 172 cm above ground.

a Copy the diagram on the right and label it with the known values.

b What height does the ball reach above the level of Max's eyes?

c What is the maximum height the ball reaches above the ground?

d Why is it important to consider the height of a person's eyes in a situation like this?



9 Can the angle of elevation or angle of depression be an obtuse angle? Explain.

10 Toshiko is standing at the edge of a 10 m cliff and sees a boat near the base of the cliff. Toshiko measures the angle of depression to the point where the bow of the boat (the front of the boat) touches the water as 24°.

a Correct to two decimal places, how far from the base of the cliff is the boat if Toshiko's eye height is 152 cm?

Toshiko spots a shark directly below the boat and measures the angle of depression to the nose of the shark as 25°.

b If the nose of the shark is directly below where the bow of the boat touches the water, correct to two decimal places, how deep is the shark swimming in the water?

11 Ann is standing 8 m in front of a tree. She observes the angle of elevation to the top of the tree to be 52°. Beth is standing further from the tree than Ann, but in the same line. She observes the angle of elevation to be 40°. (Ignore the heights of the two girls in this problem.)

a Copy the diagram on the right and label it with the known values.

b Find the height of the tree calculated using Ann's observations.

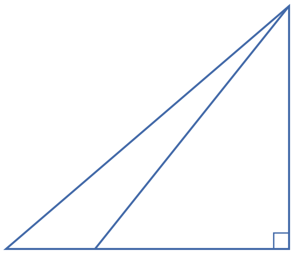
c Use the tree's height, together with Beth's angle of elevation, to calculate Beth's distance from the tree.

d What is the distance between Ann and Beth?

e Suppose Beth had been standing in the same line as Ann, but on the opposite side of the tree.

i Draw a diagram to show this situation.

ii What would have been your answer to part **d** in this case?



12 A lookout tower is built on top of a cliff. At a point 50 m from the base of the cliff, the angle of elevation to the base of the tower is 58°, while the elevation to the top of the tower is 64°.

a Draw a diagram to display this information.

b Find the height of the tower.

13 Two bushwalkers start walking from their campsite. Roger walks due north, while Ben walks N 22.5° E. They both walk 25 km.

a Display the information on a diagram.

b How far north of his starting point is Ben when he completes his walk?

c How far east of his starting point is Ben at this stage?

d If Ben were to walk directly to meet Roger at his finishing point, how far would he have to walk?

14 Jake is riding his bike from home to his friend Luke's house. He rides 1.2 km directly north, then makes a right-angled turn left and continues for 3.2 km west to Luke's house.

a Draw and label a sketch of Jake's route. Mark the north direction at each significant point on his journey.

b Calculate the direct distance between Jake's house and Luke's house.

c Use trigonometry to find the true bearing from Jake's house to Luke's house. (Remember that bearings are measured from the north in a clockwise direction.)

d What is the true bearing from Luke's house to Jake's house?

e The bearing in part **d** is known as the back-bearing of the bearing in part **c** (and vice versa). What do you notice about the relationship between these two bearings?

f Draw a scale drawing of Jake's route. From your drawing:

i find the distance between the two houses

ii use a protractor to measure the bearings in both directions.

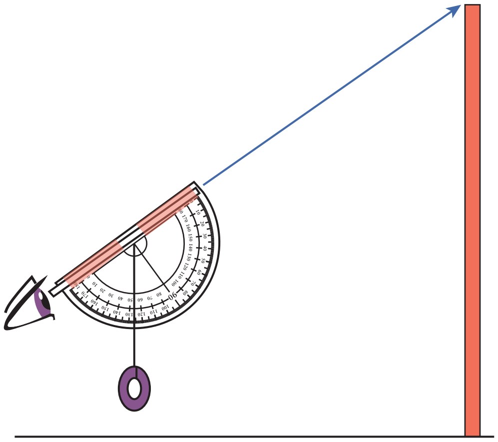
g Comment on any differences between your previous answers and those found using your scale drawing.

PROBLEM SOLVING AND REASONING

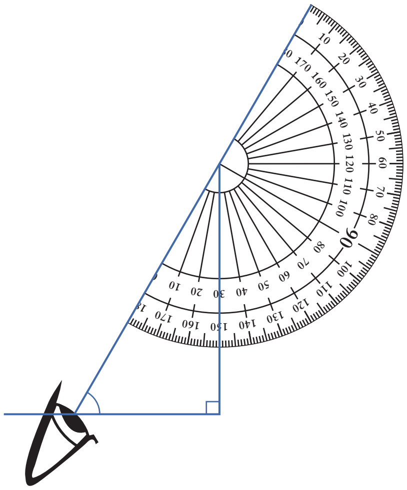
15 A clinometer (or inclinometer) is a tool used to measure angles of inclination. That is, angles of elevation or depression.

A makeshift clinometer can be constructed using a protractor, a straw, some string and a weight. You can then measure angles of elevation and depression by looking at the object through the straw, reading the angle the string makes with the protractor, then using geometry to determine the angle of elevation or depression.

Determine the angle of elevation or depression each of the following clinometers is measuring.



a



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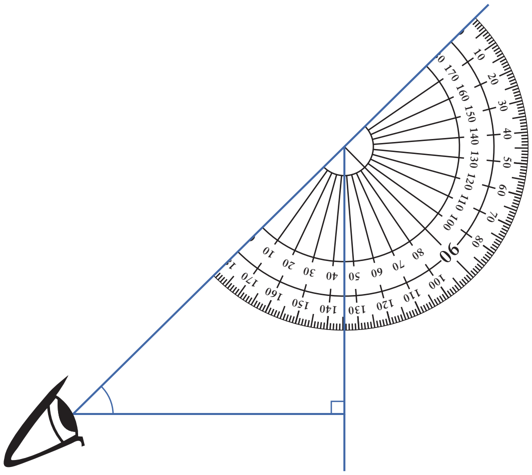

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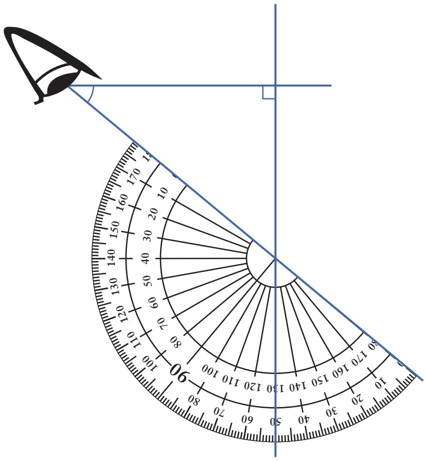

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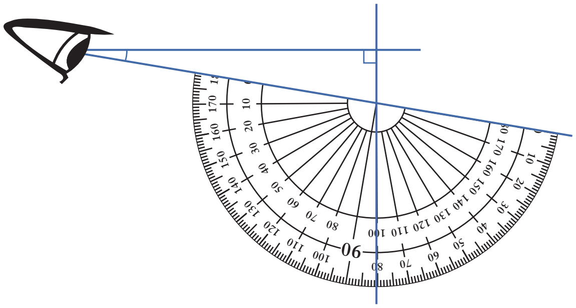


e Describe how you can use the measured angle on the clinometer to determine:

i the angle of elevation

ii the angle of depression.

16 The members of a bushwalking group have been given instructions for their next hike.

- Start from a meeting place, labelled *A*, and walk 35 km at a bearing of 055°T to *B*.
- From B, walk 25 km at a bearing of 015°T to finish the walk at *C*.

a Draw a sketch of the walk. Try to make it roughly to scale because it can lead to incorrect calculations if the relative position of points is misleading.

b Find how far north:

i *B* is from *A*

ii *C* is from *B*

iii *C* is from *A*.

c Find how far east:

i *B* is from *A*

ii *C* is from *B*

iii *C* is from *A*.

d Consider the right-angled triangle in which AC is the hypotenuse. Calculate:

- **i** the distance from A to C
- **ii** the true bearing from A to C
- **iii** the true bearing from C to A.

e Draw a scale drawing of the bushwalkers' route.

- **i** Measure distances and bearings to obtain measured answers for part **d**.
- **ii** Compare your answers with those you already had for part **d**.

17 A new hike for bushwalkers took them along a different route.

- From the starting point, walk 18 km at a bearing of 300°T.
- Change direction and walk 23 km at a bearing of 200°T to reach your endpoint.

a Find the direct distance from the starting point to the endpoint.

b What is the bearing of the starting point from the endpoint?

18 Jan is standing 2 m in front of a painting on a wall. The bottom of the painting's frame is 1.4 m from the floor, and Jan's eyes are 165 cm from the floor. The painting, including the frame, has a height of 70 cm.

a Draw a diagram to show this information.

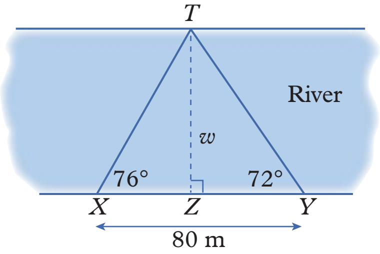
b Find the angle of elevation from Jan's eyes to the top of the painting's frame.

c What is the angle of depression from Jan's eyes to the bottom of the painting's frame?

d If Jan stood 1 m further away from the painting, through what angle would she have to sweep her eyes to look directly at the top of its frame, and then directly at the bottom of its frame?

19 Council surveyors have been set the task of measuring the width of a river along a straight stretch in preparation for building a bridge across it. They have located a tree on the opposite side of the river and have taken angle measurements from two points X and Y, which are 80 m apart, as shown in the diagram on the right.

Calculate the width of the river. (Hint: Let the distance XZ be represented by a pronumeral.)



20 A pilot was flying a routine route from A, directly east to B, a distance of 3500 km. He didn't know his compass was not operating correctly, and he was actually travelling 10° south of his true direction. When he didn't arrive at his destination after travelling 3500 km, the pilot realised he was off course.

a How much further does he need to travel to fly directly to his destination from this point? Give your answer to the nearest kilometre.

b The pilot only has enough fuel for another 60 km, so he decides to land on a nearby runway 29 km away. The pilot measures the angle of depression from the plane to the runway to be 6° at an altitude of 3 km. The runway is a straight line from north to south and so the pilot must approach the runway in that direction. The bearing from the plane to the runway is 115°T. The pilot wants to travel directly east and then make a sharp right-angled turn to land the plane.

- **i** What is the horizonal distance east before the pilot must make the turn? Give your answer to the nearest kilometre.
- **ii** What is the horizontal distance south from the turn to the runway? Give your answer to the nearest kilometre.

c If the pilot had become aware of the error when he was halfway into the flight, and adjusted his direction from that point, would he have made it to his destination?

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How do you get to school?

Topic quiz
8C

8D Three-dimensional problems

Learning intentions

✔ I can find lengths and angles within 3D objects.

Inter-year links

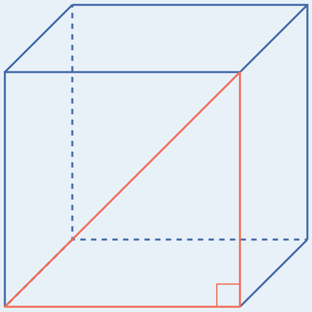
Year 8	6B Solving equations using inverse operations
Year 9	7F Using trigonometry to find angles

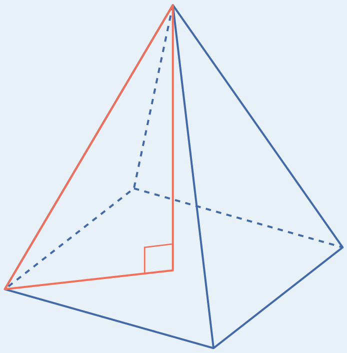
Three-dimensional problems

- Just as 2D shapes can contain many right-angled triangles, there are also many right-angled triangles to be found in 3D objects.

  Some right-angled triangles can be found on the surface of 3D objects, such as the triangle that has as its hypotenuse the diagonal joining vertex to vertex across the face of a cube.

  Other right-angled triangles can be found within a shape, such as the triangle shown in the diagram of the pyramid on the right. That triangle is formed by a slanting edge of the pyramid (the hypotenuse), a perpendicular line from the base of the square pyramid to the apex of the pyramid, and a line on the base joining the perpendicular to the slanting edge.
- Pythagoras' Theorem can be used to find an unknown length, either on the surface of a 3D object or within a 3D object.
- Trigonometry can be used to find an unknown angle or length, either on the surface of a 3D object or within a 3D object.





Example 8D.1 Three-dimensional problems

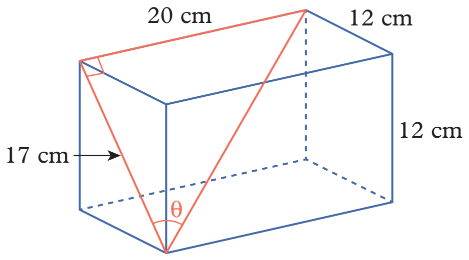
A box 20 cm long is the shape of a square prism, having two square faces with 12 cm sides.

a Calculate the length of the diagonal across each of the square faces of the box, to the nearest centimetre.

b Find the length of the diagonal within the box, to the nearest centimetre.

c Find the angle the internal diagonal from part **b** makes with the diagonal across the square face, to the nearest degree.

THINK	WRITE
a Substitute the side lengths of the square into Pythagoras' Theorem to find the length of the diagonal across the square face.	**a** $c^2 = a^2 + b^2$ $= (12)^2 + (12)^2$ $= 12^2 \times 2$ $c = 12\sqrt{2}$ ≈ 17 cm

b 1 Draw a labelled diagram of the box. Identify the right-angled triangle formed by the two diagonals.	**b** 
2 Substitute the side lengths of the two short sides of the right-angled triangle into Pythagoras' Theorem to find the length of the diagonal within the box (the hypotenuse).	$c^2 = a^2 + b^2$ $= (12\sqrt{2})^2 + (20)^2$ $= 288 + 400$ $= 688$ $c = \sqrt{688}$ ≈ 26 cm
c 1 Choose a trigonometric ratio to use for calculating the angle. Note that all the sides are now known, so any trigonometric ratio can be used.	**c** $\tan(\theta) = \frac{O}{A}$ $\tan(\theta) = \frac{20}{12\sqrt{2}}$
2 Solve for the angle using the inverse trigonometric function.	$\theta = \tan^{-1}\left(\frac{20}{12\sqrt{2}}\right)$ $\approx 50°$

Helpful hints

- ✔ It always helps to draw a diagram and label the sides and angles with the known and unknown values. It does not need to be exactly to scale.
- ✔ Remember, the trick is to find right-angled triangles! This can often be done by:
  - → drawing a line across the diagonal of a rectangle
  - → drawing a line through any triangle from one vertex to the opposite side, to meet that side at a right angle.
- ✔ When being asked to round to a given number of decimal places, the rounding should only take place at the final stage of your work and not in any intermediate steps. For example, rounding the size of angles can lead to considerable differences in the values of trigonometric ratios.

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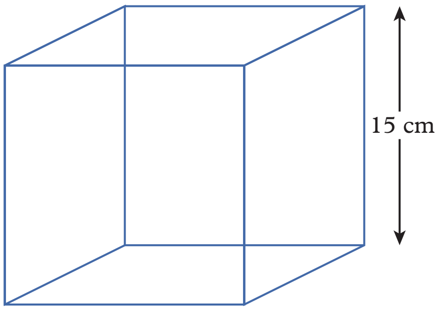
Exercise 8D Three-dimensional problems

△ **1–5, 7, 8, 10, 11**   □ **1–4, 6, 9, 10, 12, 13, 15**   ⬠ **3, 6, 9, 10, 12–14, 16, 17**

2D.1 **1** A box is the shape of a cube with side lengths of 15 cm.



- **a** Calculate the length of the diagonal of the base of the box correct to two decimal places.
- **b** Find the length of the diagonal within the box correct to two decimal places.
- **c** Find the angle the internal diagonal makes with the base diagonal correct to two decimal places.

2 The Pyramid of Giza is the largest stone monument in the world. At one stage, it was also the tallest.

The base of the pyramid is a square with side lengths of 230 m. It stands 140 m tall.

a Draw a labelled sketch of the pyramid.

b Calculate the diagonal length of the base correct to two decimal places.

c The apex of the pyramid lies directly over the midpoint of the base diagonal. Use the vertical right-angled triangle to calculate the angle that the sloping edge makes with the base diagonal, correct to two decimal places.

d A different right-angled triangle can be used to find the angle the sloping sides make with the base by measuring from the midpoint of one of the base sides.

i Draw a labelled diagram to display the dimensions of this triangle.

ii Calculate the angle the sloping side makes with the base, correct to two decimal places.

3 A box has a base 20 cm by 15 cm and is 10 cm tall.

a Provide working to show whether a 30 cm rod will fit in the box.

b If a rod was a snug fit, correct to two decimal places, what angle would it make with the base diagonal of the box?

4 A piece of cheese has the shape of a triangular prism. Its base is an isosceles triangle with side lengths of 10 cm, 10 cm and 4 cm, and the prism's height is 5 cm.

a Draw a labelled diagram of the triangular prism, showing all dimensions.

b Draw a labelled diagram of:

i the front view of the largest side

ii the top view (or base view).

c On your diagram of the front view, draw a diagonal.

i Find the length of the diagonal you drew correct to one decimal place.

ii What angle does the diagonal make with the longer side, correct to two decimal places?

iii Without using trigonometry, explain how you could find the angle the diagonal makes with the shorter side.

d Calculate, correct to one decimal place:

i the perpendicular height of the base triangle **ii** the area of the base.

e The cheese is sliced along the line of the base triangle's perpendicular height.

i Draw the shape of one of the exposed faces. Label its dimensions.

ii Find the area exposed on both sides of the slice.

iii What is the diagonal length of the exposed face, correct to the nearest centimetre?

5 Jack was given a tent for Christmas. He wants to make a cover the same size as the tent to protect it when it rains. The front of the tent forms an equilateral triangle, with a height of 1.5 m.

- **a** Sketch the front of the tent. Label the height 1.5 m, and let the side lengths be x m. Highlight a right-angled triangle, and write the length of its base in terms of x.
- **b** Use Pythagoras' Theorem to find the value of x correct to one decimal place.
- **c** Jack has bought a rectangular tarpaulin sheet with one side long enough to fit the length of the tent from front to back. If the other side length is 3.5 m, explain whether this is sufficient to reach the ground from one side of the tent to the other.

6 The rectangular perimeter of a football field is lined with trees. Ethan is standing 70 m in front of one of the trees. He notices a much taller tree 37° to his left. He measures the angle of elevation to the top of the taller tree as 13°.

- **a** Draw a sketch showing the relative position of the two trees described. Label the known values.
- **b** Find the distance along the ground from Ethan to the base of the taller tree correct to one decimal place.
- **c** Draw a sketch showing the elevation from Ethan to the top of the tall tree. Label the known values.
- **d** Use this diagram to find the height of the tall tree correct to one decimal place.

7 Cooks sometimes find that if they leave their stirring spoon unattended, it falls into the saucepan.

This saucepan is in the shape of a cylinder open at one end, with a circumference of 60 cm and a height of 12 cm.

- **a** Find the diameter of the saucepan correct to one decimal place.
- **b** A stirring spoon is 20 cm in length. Explain whether it could fall below the rim of saucepan.

8 An ice cream cone has a height of 10 cm. The circumference of the circular top of the cone is 15.5 cm.

- **a** Find the radius of the circular top of the cone correct to one decimal place.
- **b** Draw a labelled two-dimensional sketch of the outline of the cone. Define any right angles within the shape.
- **c** Find the slant length of the cone correct to one decimal place.

9 A Christmas tree in the shape of a cone is 3 m tall with a base diameter of 2 m. Sandy wants to put streamers from the top to the base of the tree, along the outer surface of the tree. She has one 30 m long roll of streamer paper that she can cut up to do this. How many streamers the full length of the slant of the tree is she able to attach?

10 A solid cube of side length 5 cm is sliced along its base diagonal.

- **a** What is the shape of the exposed face? Draw a diagram with its dimensions labelled.
- **b** Compare the size of the exposed face with the size of one of the faces of the cube.

11 A regular tetrahedron is a three-dimensional object with four equilateral triangular faces.

a Identify the tetrahedron in the photo below and draw a sketch of it.

b Suppose that the length of each side of the tetrahedron you drew is 10 cm and its height from the base to the upper apex is 8.2 cm. Add these measurements to your sketch from part **a**.

c **i** Draw a diagram of a right-angled triangle with corresponding measurements from the tetrahedron. Label one side of the triangle with the height of the tetrahedron from part **a** and label the hypotenuse with the length of one edge of the tetrahedron.

ii Use your diagram to help you calculate the angle between the edge and the base of the tetrahedron correct to the nearest degree.

d Draw the shape of a face of the tetrahedron. Draw the perpendicular height, and label all known values.

i What is the perpendicular height of the face of the tetrahedron correct to one decimal place?

ii Draw a right-angled triangle showing the perpendicular height of the face and the height of the tetrahedron. Use your diagram to calculate the angle between the base and a face of the tetrahedron correct to the nearest degree.

12 A tower stands at point T on flat ground. From point X on the ground, the angle of elevation to the top of the tower, point A is 54°. At point Y on the ground, in the same straight line as XT, and 25 m beyond X, the angle of elevation to the top of the tower is 28°.

a Draw a labelled diagram to display the given information. Label AT, the height of the tower, as h, and the distance XT along the ground as d.

b Use a trigonometric ratio with ΔAXT to find an equation for h in terms of d.

c Use a trigonometric ratio with ΔAYT to find an equation for h in terms of d. Note that the length of YT is $(25 + d)$ m.

d Equate your two equations from parts **b** and **c**. Solve to find a value for d correct to one decimal place.

e Use the value you found for d in part **d** with ΔAXT to find the height of the tower correct to one decimal place.

13 A rectangular prism has width, w, length, l, and height, h.

a Write expressions for the lengths of the diagonals along each of the faces.

b Write an expression for the length of the diagonal of the rectangular prism from one vertex to the vertex it is furthest away from.

14 A rectangular prism has a base with length, l, twice its width, w. The height of the prism, h, is three times its width.

a Write the length and height of the prism in terms of its width, w.

b Draw a labelled diagram of the prism.

c Find the length of the base diagonal in terms of w.

d Calculate the length of the diagonal within the prism in terms of w.

e Find the angle that the internal diagonal of the prism makes with the diagonal of the base. Give your answer to the nearest degree.

CHALLENGE

15 The length of the diagonal within a cube is 6.2 cm. What is the side length of the cube?

16 Consider a cube of side length x cm.

a Draw a diagram of the cube.

b An ant crawls from the top front left-hand corner to the bottom back right-hand corner along the outer surface of the cube.

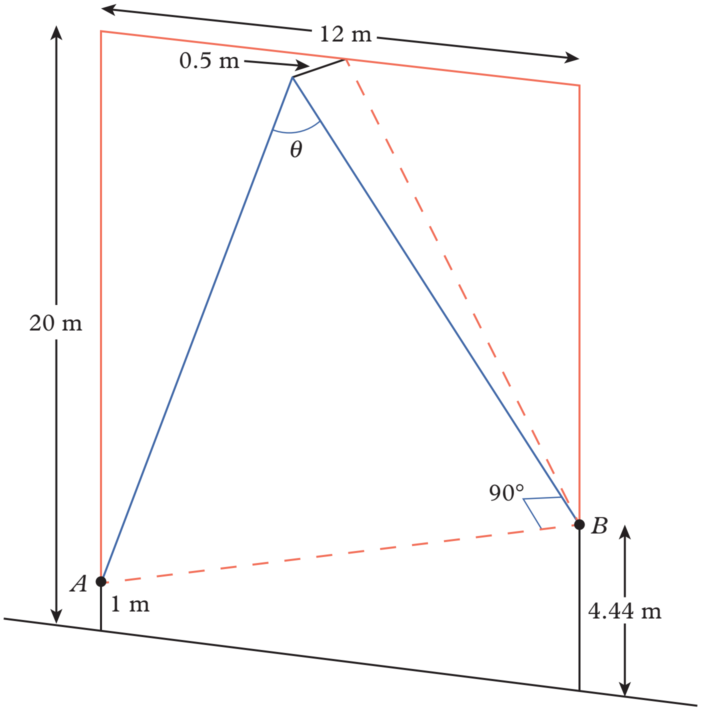
i Draw a labelled net of the cube and label the ant's starting and finishing points. Mark the shortest path the ant could take.

ii Find the length of the shortest path from your answer to part **i**. Give your answer in terms of the side length x.

17 Yolandi goes rock-climbing at an indoor rock-climbing gym. She starts by climbing a rectangular practice wall that is 12 m wide and 20 m high. A belay rope is suspended from the ceiling by a pulley and attached to Yolandi's waist to stop her from falling too far. The pulley is positioned at the centre, 0.5 m away from the wall. Yolandi starts from the left side of the wall with the rope tied to her waist at point A, 1 m above the floor. She then climbs directly to point B, 4.44 m high on the right side of the wall. She then continues directly to the top of the wall at its centre. Given that the angle the rope makes with the line from point A to point B is a right angle, correct to one decimal place, determine:

a the distance Yolandi climbs in total

b the angle, θ, that the rope turns from when Yolandi is on the left side of the wall to when Yolandi is at the right side of the wall.



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- **Interactive skillsheet** Finding angles within 3D objects
- **Topic quiz** 8D

8E The sine and area rules

Learning intentions

✔ I can determine unknown side lengths and angles using the sine rule.

✔ I can determine the two possible solutions for an ambiguous case of the sine rule.

✔ I can determine the area of a triangle using the included angle area formula.

Inter-year links

Year 7	6B Solving equations using inverse operations
Year 8	6B Solving equations using inverse operations
Year 9	4A Solving linear equations

The sine of angles from 0° to 180°

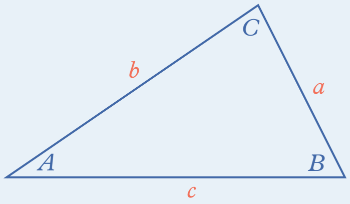
- So far, sine has been defined in terms of the side lengths of a right-angled triangle with reference to an interior angle: $\sin(\theta) = \frac{O}{H}$. However, the interior angles of a right-angled triangle are limited to acute and right angles, $0° \leq \theta \leq 90°$.
- The interior angles of a general triangle can be acute and obtuse, so sine must be defined for $0 \leq \theta \leq 180°$.
- The sine values of supplementary angles are equal, $\sin(180° - \theta) = \sin(\theta)$. For example, $\sin(150°) = \sin(30°)$

The sine rule

- The sides of a non-right-angled triangle are often labelled using lowercase letters a, b and c. The interior angles opposite those side lengths are then labelled using the capital letters: A, B and C.
- The side lengths and interior angles of any triangle, ABC, are related to each other by the **sine rule**:

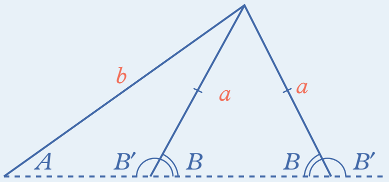
$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

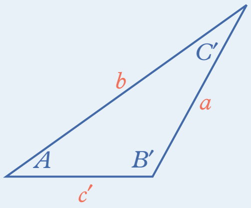
- The sine rule can be used to find missing side lengths or angles in cases where three of the values are already known and two of the known values are an opposite angle-side pair. That is, a pair of $a - A$, $b - B$ or $c - C$.



Ambiguous case of the sine rule

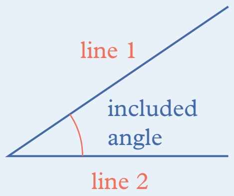
- There are two possible triangles that can be drawn when two side lengths (a and b) of a non-right-angled triangle are known and the angle A opposite the shorter side length is also known.
- The important difference between the two possible triangles is that, for one of them, angle B (opposite side b) is acute while, for the other, angle B is obtuse.
- The acute angle for B can be determined using the sine rule with a calculator.
- The obtuse angle for B (which can be labelled as B') is determined using the fact that the two possible values for B are supplementary: $B + B' = 180°$.
- Once B and B' are both known, they can be used with the sine rule to determine the remaining unknown values for the first triangle (side length c and angle C), as well as for the second triangle (side length c' and angle C').





Included angle area formula

- The included angle is the interior angle made by two lines that meet at a common vertex.
- The area of any triangle can be determined using the included angle of two known side lengths. This relationship is given by the included angle area formula:

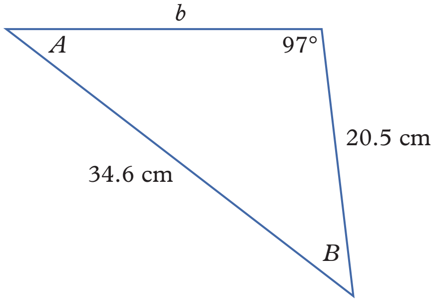


$$\text{area} = \frac{1}{2}\,\underset{\substack{\uparrow\uparrow\\ \text{adjacent}\\ \text{side lengths}}}{bc}\,\sin(\underset{\substack{\uparrow\\ \text{included}\\ \text{angle}}}{A})$$

Example 8E.1 The sine rule

For Δ*ABC*, find the unknown angles and side lengths correct to one decimal place using this information:

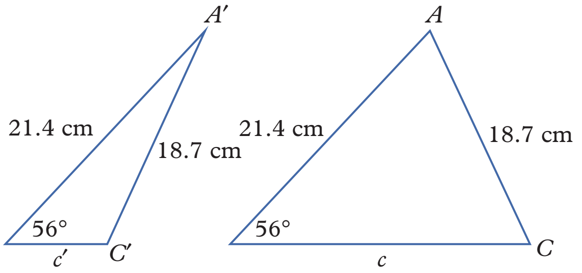
$a = 20.5$ cm, $c = 34.6$ cm, $C = 97^\circ$

THINK	**WRITE**
1 Draw a labelled diagram, roughly to scale.	
2 To solve for *A*, substitute the known values into an equation for the sine rule containing *a* and *c*.	$\frac{a}{\sin(A)}=\frac{c}{\sin(C)}$ $\frac{20.5}{\sin(A)}=\frac{34.6}{\sin(97^\circ)}$ $\frac{\sin(A)}{20.5}=\frac{\sin(97^\circ)}{34.6}$ $\sin(A) = \frac{20.5 \times \sin(97^\circ)}{34.6}$ $= 0.5881\ldots$ $A = \sin^{-1}(0.5881\ldots)$ $\approx 36.0^\circ$
3 The size of angle *B* can now be calculated, because the interior angles of a triangle add to 180°.	$A + B + C = 180^\circ$ $B \approx 180^\circ - 97^\circ - 36.0^\circ$ $\approx 47.0^\circ$
4 To solve for *b* to find the unknown side length, substitute the known values into any equation for the sine rule containing *b*.	$\frac{b}{\sin(B)}=\frac{c}{\sin(C)}$ $\frac{b}{\sin(47^\circ)}=\frac{34.6}{\sin(97^\circ)}$ $b = \frac{34.6 \times \sin(47^\circ)}{\sin(97^\circ)}$ $b \approx 25.5$ cm

Example 8E.2 Ambiguous case of the sine rule

For ΔABC, find the two possible sets of unknown angles and side lengths correct to one decimal place using this information:

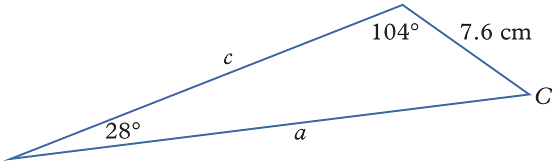
$b = 18.7$ m, $c = 21.4$ m, $B = 56°$

THINK	**WRITE**
1 Draw a labelled diagram roughly to scale. Because this is an ambiguous case of the sine rule, draw both possible triangles and use the convention for differentiating both sets of unknown values from each other (C and C').	
2 To solve for C, substitute the known values into the equation for the sine rule containing b and c.	$\begin{aligned}\frac{c}{\sin(C)} &= \frac{b}{\sin(B)}\\ \frac{21.4}{\sin(C)} &= \frac{18.7}{\sin(56°)}\\ \sin(C) &= \frac{21.4\times\sin(56°)}{18.7}\\ &= 0.9487\ldots\\ C &= \sin^{-1}(0.9487\ldots)\\ &\approx 71.6°\end{aligned}$
3 The size of angle A can now be calculated, because the interior angles of a triangle add to 180°.	$\begin{aligned}A + B + C &= 180°\\ A &\approx 180° - 56° - 71.6°\\ &\approx 52.4°\end{aligned}$
4 To solve for a, substitute the known values into any equation for the sine rule containing a.	$\begin{aligned}\frac{a}{\sin(A)} &= \frac{b}{\sin(B)}\\ \frac{a}{\sin(52.4°)} &= \frac{18.7}{\sin(56°)}\\ a &= \frac{18.7\times\sin(52.4°)}{\sin(56°)}\\ a &\approx 17.9 \text{ m}\end{aligned}$
5 Calculate C' using the fact that C' is supplementary to C.	$\begin{aligned}C + C' &= 180°\\ C' &\approx 180° - 71.6°\\ C' &\approx 108.4°\end{aligned}$
6 A' can now be calculated by subtracting the known angles from 180°.	$\begin{aligned}A' + B' + C' &= 180°\\ A' &\approx 180° - 56° - 108.4°\\ &\approx 15.6°\end{aligned}$
7 To solve for a', substitute the known values into any equation for the sine rule containing a'.	$\begin{aligned}\frac{a'}{\sin(A')} &= \frac{b}{\sin(B)}\\ \frac{a'}{\sin(15.6°)} &= \frac{18.7}{\sin(56°)}\\ a' &= \frac{18.7\times\sin(15.6°)}{\sin(56°)}\\ a' &\approx 6.1 \text{ m}\end{aligned}$

Example 8E.3 The area rule

Find the area of ΔABC correct to one decimal place given the following information:

$b = 7.6$ cmm, $A = 104°$, $B = 28°$

THINK	**WRITE**
1 Draw a labelled diagram. Note that at least two side lengths must be known to calculate the area using the area rule.	
2 To solve for *a*, substitute the known values into an equation for the sine rule containing *a* and *b*.	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$ $\frac{a}{\sin(104^\circ)} = \frac{7.6}{\sin(28^\circ)}$ $a = \frac{7.6 \times \sin(104^\circ)}{\sin(28^\circ)}$ $a \approx 15.7$ cm
3 Calculate the included angle between *a* and *b* using the fact that the interior angles of a triangle add to 180°.	$C + A + B = 180^\circ$ $C = 180^\circ - 104^\circ - 28^\circ$ $= 48^\circ$
4 Substitute the relevant known values into the area rule.	$\text{area} = \frac{1}{2}ab\sin(C)$ $= \frac{1}{2} \times 15.7 \times 7.6 \times \sin(48^\circ)$ $\approx 44.4 \text{ cm}^2$

Helpful hints

✔ When using the sine rule, it helps to label the diagram with side lengths *a*, *b* and *c*, and their opposite angles *A*, *B* and *C*, respectively.

✔ To determine whether the given information for any triangle represents an ambiguous case of the sine rule, run through this short checklist:

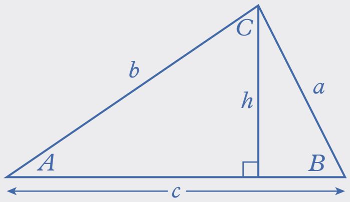
→ Do I only know the values for two side lengths and one angle?

→ Is the known angle opposite the shorter known side?

If the answer is yes to both these questions, it is an ambiguous case.

✔ One way to remember area $= \frac{1}{2}bc\sin(A)$ is using the following diagram, where $\sin(A) = \frac{h}{b}$ multiply both sides by b, we have $h = b \times \sin(A)$

$$\begin{aligned} \text{Area of } \Delta ABC &= \frac{1}{2} \times \text{base} \times \text{height} \\ &= \frac{1}{2} \times c \times h \\ &= \frac{1}{2} \times c \times b \times \sin(A) \\ &= \frac{1}{2}bc\sin(A) \end{aligned}$$



ANS p762

Exercise 8E The sine and area rules

△ 1(a–d), 2–7, 8(a–c), 12, 13, 15, 16 □ 3–7, 8(b, d, f), 9, 11, 12, 14, 17, 19, 21 ⬠ 4, 5, 7, 8(b, f), 10, 14, 16, 17, 18, 20–22

UNDERSTANDING AND FLUENCY

1 Use a calculator to evaluate each of the following correct to four decimal places where necessary.

a $\sin(30°)$ **b** $\sin(150°)$ **c** $\sin(40°)$ **d** $\sin(140°)$

e $\sin(80°)$ **f** $\sin(100°)$ **g** $\sin(22°)$ **h** $\sin(158°)$

2 Use the sine rule to find the unknown value for each of the following triangles, ABC.

a If $A = 45°$, $B = 72°$ and $a = 10$ mm, calculate b correct to one decimal place.

b If $C = 65°$, $a = 12$ cm and $c = 13.4$ cm, calculate A correct to one decimal place.

c If $B = 37°$, $c = 8$ cm and $b = 15$ cm, calculate A correct to one decimal place.

8E.1 **3** For each of these triangles, ABC, use the given information to find the unknown angles and side lengths, correct to one decimal place. Draw a labelled sketch of the triangle.

a $a = 8.5$ cm, $B = 81°$ and $A = 37°$ **b** $c = 37.3$ mm, $b = 33.7$ mm, $C = 85°$

c $a = 22.5$ cm, $c = 28.3$ cm, $A = 49°$ **d** $b = 5.8$ cm, $B = 26°$, $A = 67°$

8E.2 **4** Use the given information to find the two possible sets of unknown angles and side lengths for each of these, correct to one decimal place. Draw a labelled sketch of both possible triangles.

a $c = 33.7$ mm, $b = 37.3$ mm, $C = 40°$ **b** $a = 14.9$ mm, $b = 26$ mm, $A = 32°$

c $b = 12.3$ cm, $c = 32.8$ cm, $B = 16°$ **d** $c = 22.1$ mm, $a = 23.8$ mm, $C = 67°$

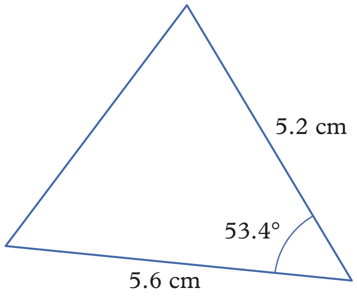
5 Find the area of each of these triangles, ABC, correct to one decimal place.

a $a = 9.5$ cm, $b = 7.6$ cm, $C = 28°$ **b** $b = 2.8$ mm, $c = 8.9$ mm, $A = 87°$

c $a = 4.7$ cm, $c = 9.2$ cm, $B = 57°$ **d** $c = 12.5$ cm, $b = 9.2$ cm, $A = 43°$

8E.3 **6** Find the area of each of these triangles, correct to one decimal place.

a



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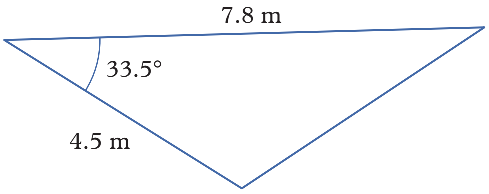

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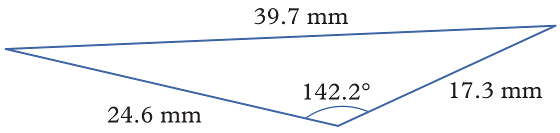

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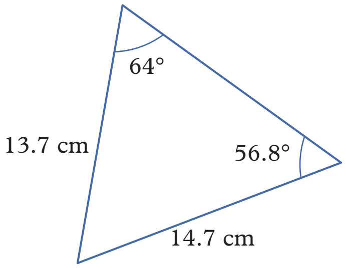


7 For each set of known values for the triangle ABC, explain why or why not:

i the sine rule can be used

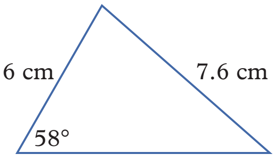
ii the set implies an ambiguous case of the sine rule.

a $c = 65$ cm, $b = 70$ cm, $C = 64°$ **b** $a = 60$ cm, $b = 70$ cm, $C = 64°$

c $b = 60$ cm, $B = 70°$, $A = 64°$ **d** $c = 65$ cm, $b = 70$ cm, $B = 64°$

8 Find the perimeter of each triangle. (Hint: check which triangles result in an ambiguous case of the sine rule and decide which of the two possible triangles match the diagram.)

a



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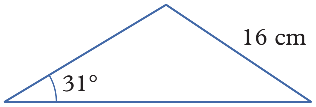

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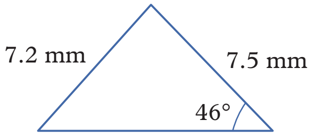

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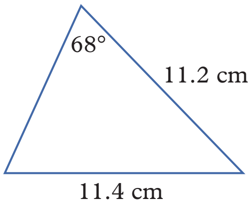

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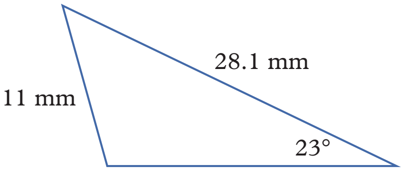

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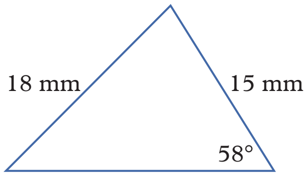


9 **a** Copy the given triangle into your workbook.

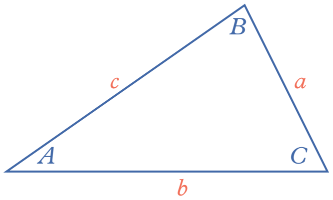
b Draw a perpendicular from vertex B to base b. Label the point where the perpendicular meets the base, D. Label the length of the perpendicular h.

c **i** Use the sine ratio for angle A to write a relationship between h and c. Make h the subject of the formula.

ii Use the sine ratio for angle C to write a relationship between h and a. Make h the subject of the formula.

iii Equate your two answers for parts **i** and **ii** and show how this can be written as $\frac{a}{\sin(A)} = \frac{a}{\sin(C)}$.

d Repeat the same reasoning outlined in parts **a**, **b**, and **c** to find a similar relationship involving angle B. Show that $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$.



10 Consider the right-angled triangle, ΔABC.

The sine rule is quoted as $\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$.

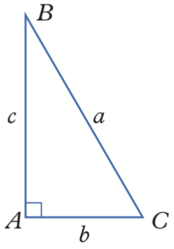
a Evaluate $\sin(A)$.

b Use the triangle to help you write expressions for $\sin(B)$ and $\sin(C)$.

c Substitute your expressions from parts **a** and **b** into the sine rule. Simplify.

d Explain your answer. Does the sine rule hold for right-angled triangles?

e What would have been your answer for part **c** if the triangle had been right-angled at $\angle ABC$ or $\angle BCA$?



11 Refer to the diagram in question **10**. The area of the triangle can be calculated using the formula:
area $= \frac{1}{2}ab\sin(C)$

a Use the diagram to help you obtain an expression for $\sin(C)$.

b Substitute the expression you found in part **a** into the area formula and simplify.

c Explain the resulting formula.

12 Chloe measures the angle of elevation to the top of a building as 27°. When she walks 15 m closer to the building, she measures the angle of elevation to be 47°.

a Find the size of:

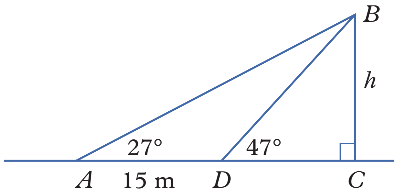
i $\angle ADB$ **ii** $\angle ABD$.

b Use the sine rule in ΔABD to calculate the length of BD, correct to one decimal place.

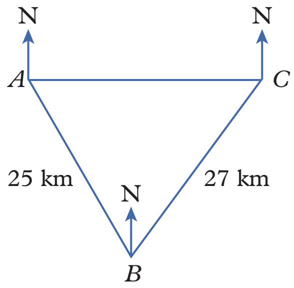
c Use the sine ratio in ΔBCD to find the height of the building.

d Use the cosine ratio in ΔBCD to find how far Chloe was from the building when she took her second reading.

e How far was Chloe from the building initially?



13 A group of bushwalkers started from point A, travelled 25 km to point B at a bearing of S30°E, then continued for 27 km to point C, which lies due east of their starting point.



- **a** What is the size of $\angle BAC$?
- **b** Use the sine rule in ΔABC to find the size of $\angle ACB$ to the nearest degree.
- **c** What is the size of $\angle ABC$?
- **d** Use the sine rule again to find the distance the bushwalkers would need to travel from their finishing point to return directly to their starting point. Give your answer to one decimal place.
- **e** Give the bearing:
  - **i** of C from B
  - **ii** of B from C
  - **iii** the bushwalkers would need to take to return directly to their starting point.
- **f** Calculate the area enclosed by the bushwalkers' hike.

14 The banks of a creek run directly east–west. From where Kim stands on the banks of the creek, the bearing to a tree on the opposite side of the creek is 032°T. If she moves 20 m due east on the bank of the creek, the bearing to the same tree is 310°T.

- **a** Draw a labelled diagram displaying as much information as possible.
- **b** Find the distance (in metres, to one decimal place) of the tree from:
  - **i** Kim's first viewing point
  - **ii** Kim's second viewing point.
- **c** Calculate the width of the creek.

15 A give-way sign is the shape of an equilateral triangle, with side lengths of 40 cm. Calculate the area of the sign, to the nearest square centimetre.

16 A lighthouse stands 50 m tall on top of a cliff. From a ship at sea, the angle of elevation to the base of the lighthouse is 3°, while the angle of elevation to the top of the lighthouse is 4.2°.

- **a** Draw a labelled diagram to display this information.
- **b** Use the sine rule to find the straight-line distance from the ship to:
  - **i** the base of the lighthouse
  - **ii** the top of the lighthouse.
- **c** Calculate the distance in kilometres (to one decimal place) from the ship to the base of the cliff.
- **d** Find the height of the cliff, to the nearest metre.

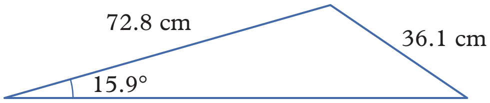
17 A pirate ship departs from Ashy Bay on a bearing of 295°T to Basalt Cove, 23 km away. From Basalt Cove, the pirate ship needs to travel 12.4 km to Coral Quay, but the inexperienced crew have forgotten which bearing they need to travel along. The crew knows that Coral Quay is due west from Ashy Bay but so is Deserted Depths, which is also 12.4 km from Basalt Cove.

- **a** Determine the bearings of Coral Quay and Deserted Depths from Basalt Cove, correct to the nearest degree.
- **b** Determine the distances from Ashy Bay to Coral Quay and Deserted Depths, correct to one decimal place.

One of the crew members incorrectly recalls that Coral Quay is closer to Ashy Bay than Deserted Depths is. The pirate ship travels to the location closer to Ashy Bay but arrives at Deserted Depths instead. The pirate ship now must travel from Deserted Depths to Coral Quay.

- **c** Calculate the minimum total distance the pirate ship will have travelled when it finally reaches Coral Quay, correct to one decimal place.

18 Consider the triangle below. Note that it is not drawn to scale.

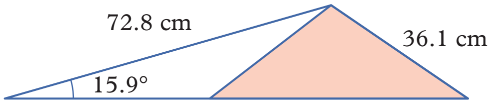


a The triangle is ambiguous. Explain why.

b Correct to one decimal place, determine the two possible lengths of the third side.

c Correct to one decimal place, calculate the area of the two possible triangles.

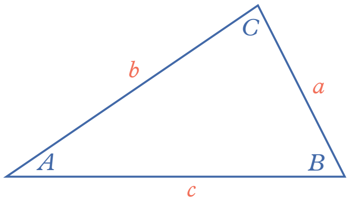
d When the two triangles are overlapped at the 15.9° angle, the excess is an isosceles triangle.



i Determine the area of the isosceles triangle using your answers to part **c** correct to one decimal place.

ii Write an expression for the area of the isosceles triangle using the angle between the equal side lengths.

19 a Consider the triangle *ABC*. Write three different expressions for the area of the triangle using the included angle area formula.



b Equate the expressions from part **a** and show that the resulting equation is equivalent to the sine rule.

All interior angles of an equilateral triangle are 60°.

c Use a calculator to determine the value of $(\sin(60°))^2$ as a fraction.

d Hence, write the exact value of $\sin(60°)$.

e Hence, write a rule for the area of an equilateral triangle with a side length of *l*.

20 Regular *n*-sided polygons can be broken up into *n* congruent, isosceles triangles by joining the centre of the polygon to each vertex. For each of the following regular polygons shown, determine:

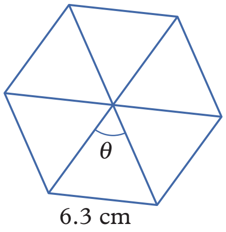
a the size of the angle θ at the centre

b the distance from the centre to each vertex, correct to one decimal place

c the area of each isosceles triangle, correct to one decimal place

d the area of the regular polygon, correct to one decimal place.

i



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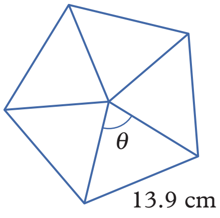

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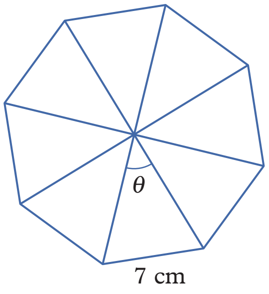

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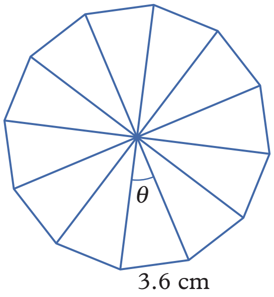
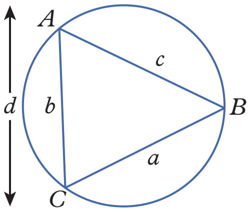


e Write a rule for the area of an *n*-sided regular polygon with side length *l*.

CHALLENGE

21 The circumcircle of a triangle is the circle that passes through the vertices of the triangle. The ratio of a side to the sine of the opposite angle is equal to the diameter of the circumcircle.

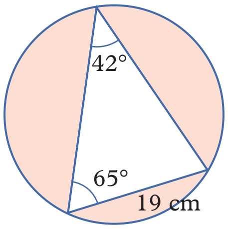
That is, $\dfrac{a}{\sin(A)} = \dfrac{b}{\sin(B)} = \dfrac{c}{\sin(C)} = d$.



a Determine the diameter of the circumcircle of the triangle in the circle below correct to two decimal places.

b Correct to two decimal places, determine the area of:

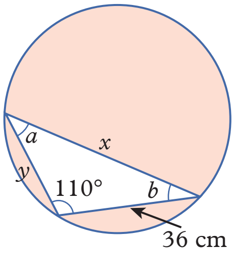
i the triangle **ii** the circumcircle **iii** the shaded area.



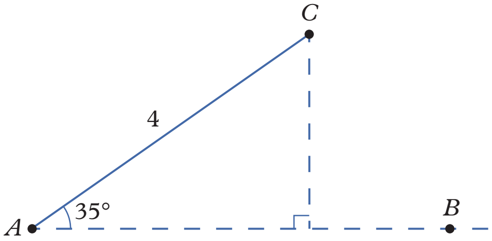
c The diameter of the circle on the right is 56 cm.

i Determine the value of each of the pronumerals correct to two decimal places.

ii Determine the shaded area correct to two decimal places.



22 So far, when dealing with the ambiguous case of the sine rule, we have assumed a triangle (or triangles) can be formed. However, it is possible for the side opposite the known angle to be too short to form a triangle. Consider the diagram below.

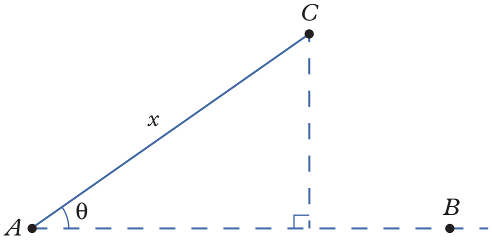


a Determine the perpendicular distance from C to the ray AB, correct to two decimal places.

b What is the minimum length of the line segment BC such that ABC is a triangle? Explain.

c What is the maximum length of the line segment BC such that there are two possible triangles ABC? Explain.

d For the diagram below, write an inequality for the length of line segment BC such that ABC is a triangle and also ambiguous.



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pro

Interactive skillsheet
The sine rule

Interactive skillsheet
The area rule

investigation
Proof of Heron's formula

Topic quiz
8E

Checkpoint

Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this chapter.

8A **1** Determine whether the following lengths can form a right-angled triangle.

a 9 cm, 40 cm, 41 cm

b 39 cm, 89 cm, 80 cm

c 137 cm, 104 cm, 89 cm

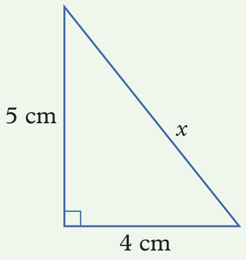
d 21 cm, 221 cm, 220 cm

8A **2** Determine the value of x in each of the following:

i as a simplified exact value

ii correct to two decimal places.

a



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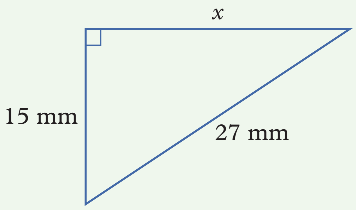

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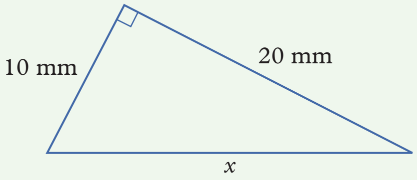

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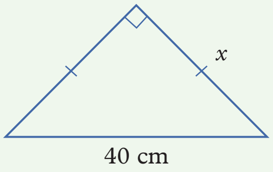
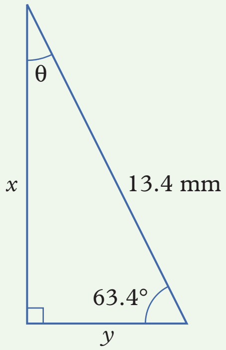


8B **3** Determine the value of each of the pronumerals correct to one decimal place.

a



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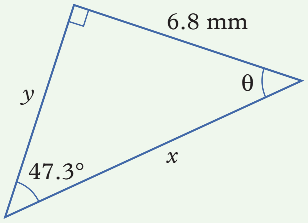

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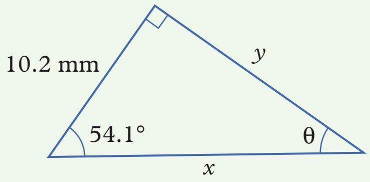
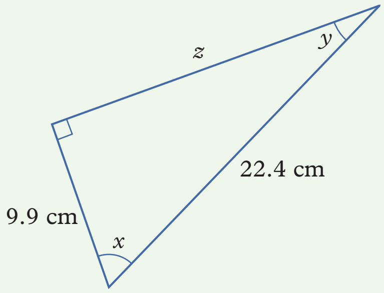


8B **4** Determine the value of each of the pronumerals correct to one decimal place.

a



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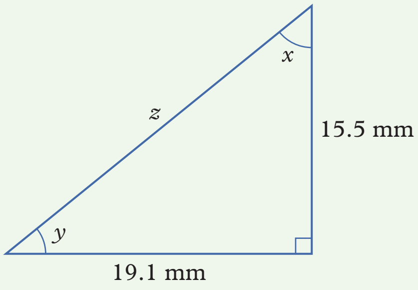
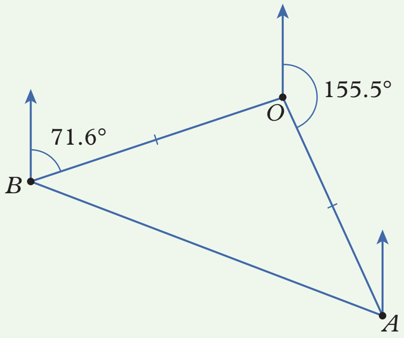


8C **5** Determine the true bearing of B from A in each of the following.

a The bearing of B from A is S85°E.

b The bearing of A from B is N35°W.

c



8C **6** Calculate each of the following correct to one decimal place.

a The angle of elevation of the top of a large tree from a point on the ground 6 m away is 78°. Determine the height of the tree.

b A building is 120 m tall. Another building is 105 m tall. Determine the angle of depression from the top of the taller building to the top of the shorter building if the tops of the buildings are 30 m apart.

c The angle of depression of an overboard passenger from the deck of a ship is 24°. A 12 m rope pulled taut just reaches from the deck of the ship to the overboard passenger in the water. Determine how far, in total, the overboard passenger needs to be pulled towards the boat through the water and then up the side of the ship to get them back onto the deck of the boat.

d A child holds the 8 m string of a kite 130 cm from the ground. If the angle of elevation of the kite is 65° how high above the ground is the kite flying?

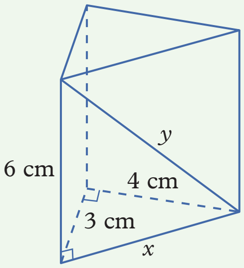
8C **7** Calculate each of the following distances correct to one decimal place and bearings correct to the nearest degree.

a A snorkeller swims to a point 230 m at a bearing of 160° from their boat. Determine how far south and east they are from the boat.

b A farmer walks 1.2 km east and then 800 m north along two sides of a paddock. Determine the true bearing of the location the farmer ended at from their starting point.

10A **8D** **8** Determine the value of each of the pronumerals correct to one decimal place.

a



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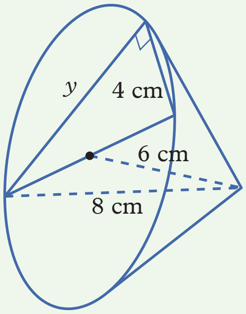
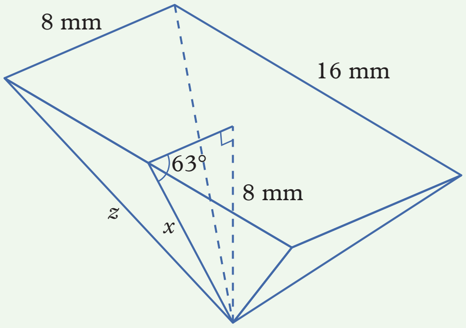


10A **8D** **9** Determine the value of each of the pronumerals correct to one decimal place.

a



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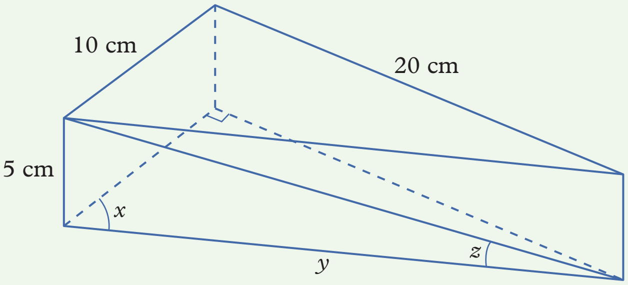
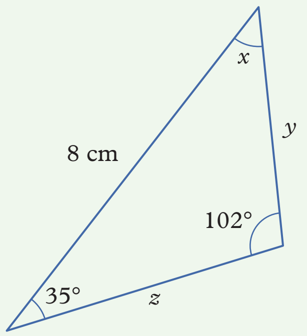


10A **8E** **10** Determine the value of each of the pronumerals correct to one decimal place.

a



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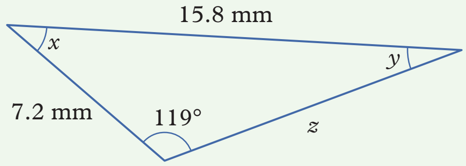

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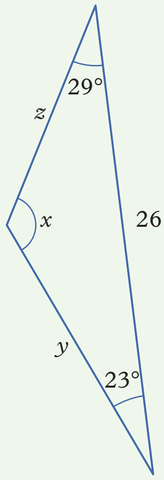

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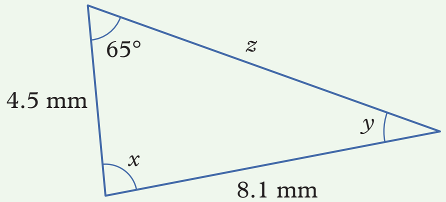
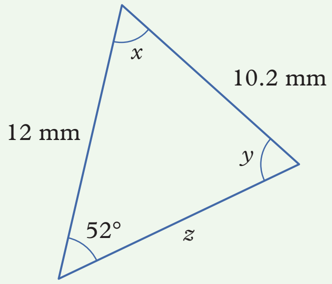


10A **8E** **11** Determine all possible values of each of the pronumerals correct to one decimal place.

a



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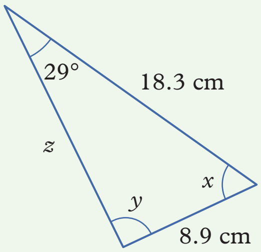
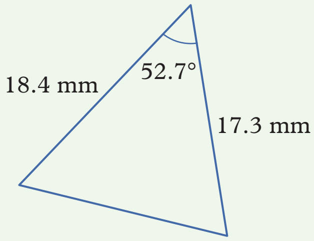


10A **8E** **12** Determine the area of the following triangles correct to two decimal places.

a



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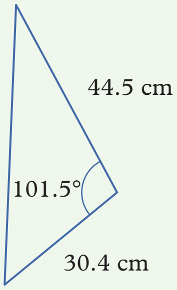
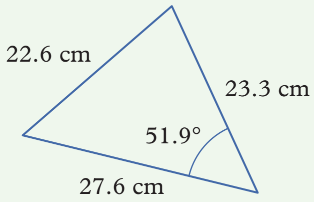


c



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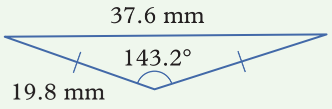


8F The cosine rule

Learning intentions

- ✔ I can determine the unknown side length of a triangle using the cosine rule.
- ✔ I can determine the unknown angles of a triangle using the cosine rule.

Inter-year links

Year 7	6B Solving equations using inverse operations
Year 8	6B Solving equations using inverse operations
Year 9	7F Using trigonometry to find angles

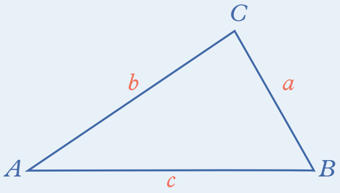
The cosine of angles from 0° to 180°

- So far, cosine has been defined in terms of the side lengths of a right-angled triangle with reference to an interior angle: $\cos(\theta) = \frac{A}{H}$. However, the interior angles of a right-angled triangle are limited to acute and right angles, $0° \leq \theta \leq 90°$.
- The interior angles of a general triangle can be acute and obtuse, so cosine must be defined for $0 \leq \theta \leq 180°$.
- The cosine values of supplementary angles are the negative of each other, $\cos(180° - \theta) = -\cos(\theta)$. For example, $\cos(120°) = -\cos(60°)$.

Using the cosine rule

- The side lengths and interior angles of any triangle, ΔABC, are related to each other by the **cosine rule**:

$$a^2 = b^2 + c^2 - 2bc\cos(A).$$

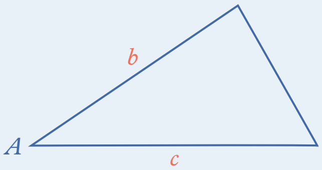


a, b and c are the three side lengths of a triangle.

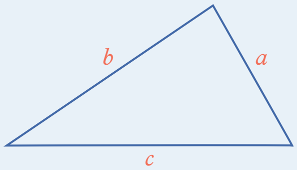
A is the included angle between b and c.

- The cosine rule is particularly useful for determining angles and side lengths in two situations when the sine rule is not suitable for finding the unknowns:

Situation 1: The only known angle is the included angle between the only known side lengths.



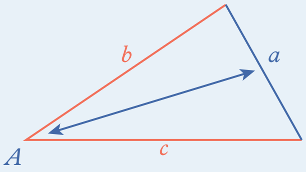
Situation 2: All the side lengths are known, and all the angles are unknown.



- To help remember the cosine rule, remember that the side length and the angle at either end of the equation form an opposite side-angle pair, and the side lengths in the middle are the adjacent side lengths to the included angle.

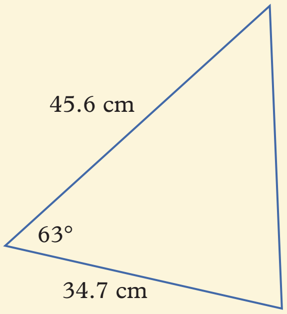
opposite side-angle air

$$a^2 = \underbrace{b^2 + c^2 - 2bc}_{\text{adjacent side lengths where } A \text{ is included}}\cos(A)$$



Example 8F.1 Using the cosine rule to find a side length

Find the unknown side length in this triangle, correct to one decimal place.



THINK	WRITE
1 Identify the known values.	The included angle is 63° between the adjacent side lengths 45.6 cm and 34.7 cm.
2 Substitute the known values into the cosine rule.	$a^2 = b^2 + c^2 - 2bc\cos(A)$ $= (45.6)^2 + (34.7)^2 - 2 \times (45.6) \times (34.7) \times \cos(63°)$ $= 1846.7335\ldots$
3 Find the square root and round the answer to one decimal place. Write the units.	$a = \sqrt{1846.7335\ldots}$ ≈ 43.0 cm

Example 8F.2 Using the cosine rule to find an unknown angle

The following lengths are given for the sides of ΔABC:

$a = 2.8$ m, $b = 5.3$ m, $c = 7.1$ m

Determine the size of the included angle between a and b correct to one decimal place.

THINK	WRITE
1 Substitute the side lengths into the cosine rule.	$c^2 = a^2 + b^2 - 2ab\cos(C)$ $(7.1)^2 = (2.8)^2 + (5.3)^2 - 2 \times (2.8) \times (5.3) \times \cos(C)$
2 Solve for cos(C).	$50.41 = 35.93 - 29.68\cos(C)$ $14.48 = -29.68\cos(C)$ $\cos(C) = -0.4878\ldots$
3 Use the inverse cosine to calculate C.	$C = \cos^{-1}(-0.4879\ldots)$ $\approx 119.2°$

Helpful hints

- ✔ When you are calculating an angle using the cosine rule, always check that the value you have calculated for the cosine value is between −1 and 1.
- ✔ If you are calculating all the interior angles of a triangle using the cosine rule, check your calculations by making sure the three angles add to 180°.

ANS p765

Exercise 8F The cosine rule

△ 1(a–d), 2(a, c, e, g, i), 3, 4, 5(a–c), 6, 9, 10, 13, 15   □ 2(g–i), 3, 5–7, 11–13, 17, 18, 20   ⬠ 3(c, d), 5, 8, 11, 12, 14, 16, 17, 19, 21, 22

UNDERSTANDING AND FLUENCY

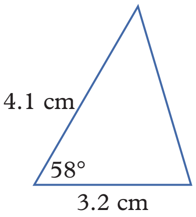
1 Use a calculator to evaluate each of the following correct to four decimal places where necessary.

a $\cos(60^\circ)$ **b** $\cos(120^\circ)$ **c** $\cos(40^\circ)$ **d** $\cos(140^\circ)$

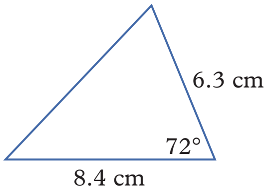
e $\cos(80^\circ)$ **f** $\cos(100^\circ)$ **g** $\cos(22^\circ)$ **h** $\cos(158^\circ)$

8F.1 **2** Find the unknown side length in each of these triangles, correct to one decimal place.

a



b



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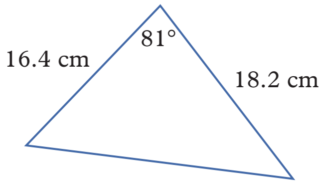

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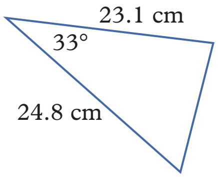

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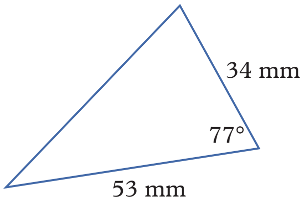

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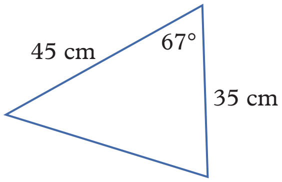

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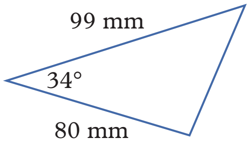

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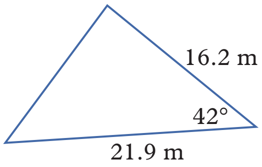
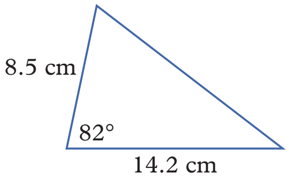


i



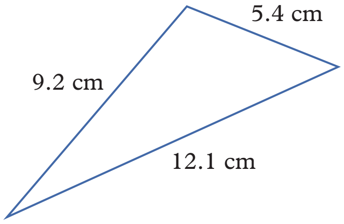
8F.2 **3** Each of the following sets of side lengths are for a different ΔABC. Find the sizes of all the angles of each triangle, correct to one decimal place.

a $a = 4.5$ cm, $b = 6$ cm and $c = 5$ cm **b** $a = 13$ mm, $b = 12$ mm and $c = 7$ mm

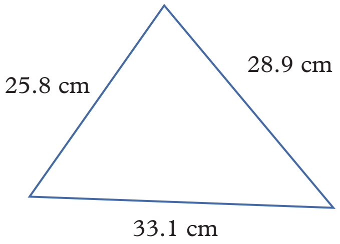
c $a = 1.5$ m, $b = 2.3$ m and $c = 1.8$ m **d** $a = 16$ cm, $b = 19$ cm and $c = 12$ cm

4 Find the sizes of the angles of each of these triangles, correct to one decimal place.

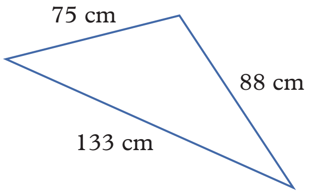
a Find the size of the largest interior angle.



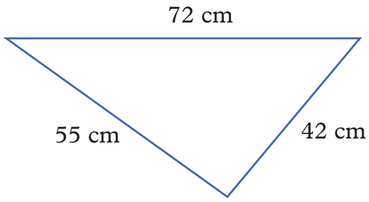
b Find the size of the smallest interior angle.



c Find the size of the median interior angle.

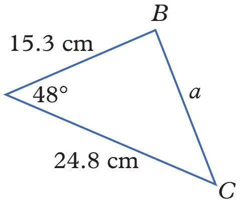


d Find the size of the largest exterior angle.



5 Find the unknown side lengths and angles of each of these triangles.

a



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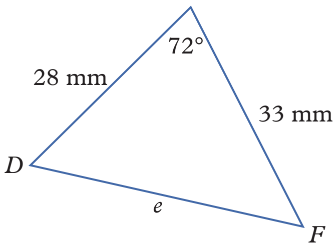
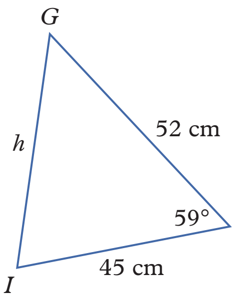


c



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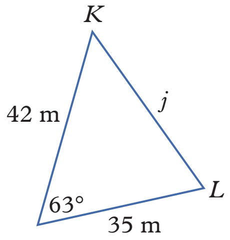

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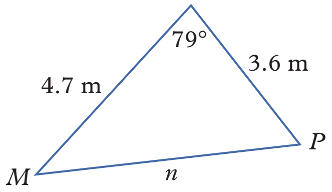

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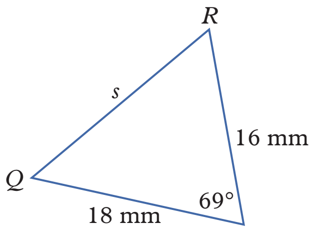
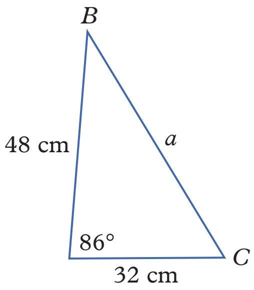
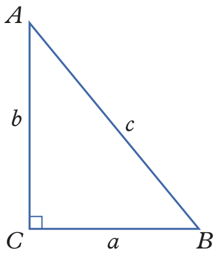


6 Sometimes there is a choice as to whether the cosine rule or the sine rule can be used to solve a problem. Consider the triangle on the right.



- **a** Explain why it is not possible to use the sine rule to calculate the length of a.
- **b** Use the cosine rule to find the length of a.
- **c** **i** Explain how you could now use the sine rule to find the sizes of angles B and C.
  - **ii** Calculate the sizes of those two angles.
- **d** Angles B and C from part **c** could also have been calculated using the cosine rule. Find their sizes using this rule.
- **e** Comment on your answers to parts **c ii** and **d**.

7 Consider the right-angled triangle ABC with the right angle at C.

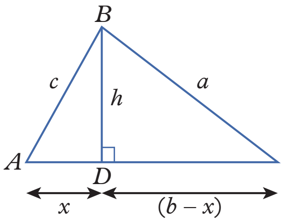


- **a** Copy and complete the cosine rule for triangle ABC: $c^2 =$ __________________.
- **b** Evaluate $\cos(90°)$.
- **c** Substitute the value from part **b** into your equation in part **a** and simplify. Describe the result.

The cosine rule has the extra term $-2ab\cos(C)$ compared to Pythagoras' Theorem. This is the generalised form for using non-right angles.

- **d** If C is an acute angle, explain why the side length opposite to angle C will be shorter than the hypotenuse if C were a right angle, referring to the adjustment term.
- **e** If C is an obtuse angle, explain why the side length opposite to angle C will be longer than the hypotenuse is C were a right angle, referring to the adjustment term.

8 Consider ΔABC, with the perpendicular h from B dividing the base b into two sections of length x and $(b - x)$.



a Use ΔCDB and Pythagoras' Theorem to complete the relationship $a^2 = h^2 +$ __________.

b Use ΔADB and Pythagoras' Theorem to complete the relationship $h^2 =$ ____________.

c Substitute the value of h^2 you wrote for part **b** into your equation in part **a**. Simplify the resulting equation and confirm that it can be written as $a^2 = b^2 + c^2 - 2bx$.

d Use ΔADB and the cosine ratio to write a relationship with x as the subject of the equation.

e Substitute your value for x into the equation you wrote for part **c**. Your resulting equation should be: $a^2 = b^2 + c^2 - 2bc\cos(A)$.

f Use a similar method to that used for writing an equation for a^2 in part **e** to write equations for b^2 and c^2.

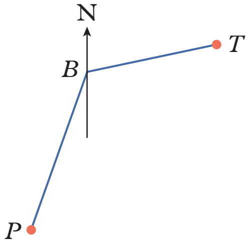
g The cosine rule can be rearranged to find the size of an angle of a triangle. Write three versions of the cosine rule with the following cosine values as the subject.

i $\cos(A)$ **ii** $\cos(B)$ **iii** $\cos(C)$

h Using the cosine rule, what information would you need to calculate:

i a side length? **ii** the size of an angle?

9 From a buoy at sea, Todd rows 2.5 km at a bearing of N80°E, while Pete rows 3.5 km at a bearing of S20°W.



a Copy the diagram and label all known sides and angles.

b What is the size of $\angle TBP$?

c Join TP. This represents the distance between the two rowers. Use the cosine rule with ΔBTP to calculate this distance.

d Use the cosine rule with ΔBTP to calculate the size of $\angle BPT$.

e Use your answer for part **d** to find the bearing of Todd from Pete.

f Calculate the size of $\angle BTP$.

g Use your answer for part **f** to find the bearing of Pete from Todd.

10 A triangular garden bed has side lengths 5.3 m, 4.8 m and 4.5 m.

a Draw a labelled sketch of the garden bed.

b Identify the vertex with the largest angle. (Hint: The largest angle lies opposite the longest side.)

c Use the cosine rule to find the size of the angle you identified in part **b**.

d Explain how, using the angle you identified, you can now calculate the area of the bed. What is its area?

11 A deep-sea diver is attached to two safety ropes from two boats on the surface of the ocean. The boats are 50 m apart. One rope is 110 m long and the other is 100 m long.

a Draw a labelled diagram to describe the situation when the diver is at the maximum depth that is possible while attached to the two ropes.

b Use the cosine rule to find the angles the two ropes make with the surface of the water.

c On your diagram from part **a** construct a right-angled triangle that will enable you to find the maximum depth to which the diver can descend. Explain a technique you could use to find this depth.

d Calculate the maximum depth to which the diver can descend while attached to the two safety ropes.

12 Emily is practising shooting hockey goals. The hockey goal is 3 m wide. She stands 5 m directly in front of one of the posts.

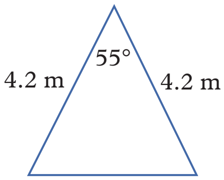
a Draw a diagram to display the situation.

b Use Pythagoras' Theorem to calculate the distance from Emily to the other goalpost.

c Use the cosine rule to find the angle within which Emily must shoot to score a goal.

d Emily moves her position, so she is 5 m from each of the goal posts. Explain whether she has a greater chance of scoring a goal from this position.

13 The frame formed by the end support poles of a swing forms a triangular shape with the ground. The poles are each 4.2 m long with an angle of 55° between them.



a How far apart are the ends of the poles on the ground?

b Find the vertical height of the top of the frame above the ground.

14 The frame for a bridge consists of a series of triangular shapes connected together as shown in the photo on the right.

The sloping steel parts are 3.9 m long and the horizontal steel parts are 2.8 m long.

a Find the angle between any two sloping pieces of steel.

b Draw a sketch of several sections of the bridge, showing the angles and lengths of the connecting pieces.

c Find the height of the frame of the bridge in metres, correct to one decimal point.

15 The diagonal length of an octagon, between vertices that are four edges apart, measures 4 cm.

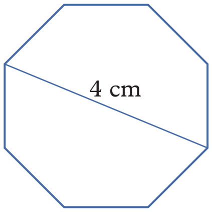
a Draw a sketch of the octagon. Add the three remaining diagonals that join vertices that are four edges apart.

b Consider one of the triangles within your sketch.

i What is the size of the angle at the centre of the octagon?

ii Find the sizes of the other two angles of the triangle.

c Calculate the length of the outer edge (the perimeter) of the whole octagon.



16 A light aircraft is set to fly on a course N78°E for a distance of 350 km. A strong wind moving from east to west blows the plane off course by 6°.

a Draw a labelled diagram to show this situation.

b If the pilot flies the planned distance on this incorrect path, how far off course will he be at the end of his journey?

17 A cube has a side length of 25 cm.

a Draw a labelled sketch of the cube.

b Use the cosine rule to find the length of the diagonal of the base. Leave your answer in a simplified exact form.

c Use the cosine rule again to find the length of the diagonal of the cube. Leave your answer in a simplified exact form.

18 The angles of a triangle are in the ratio 3 : 5 : 7.

a Find the size of all the angles.

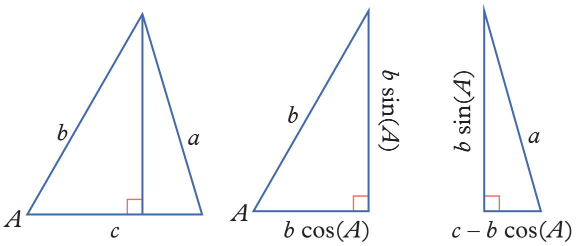
The two sides adjacent to the largest angle measure 2.58 cm and 4.30 cm.

b Correct to two decimal places, calculate the length of the third side.

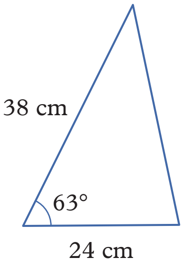
c Draw a labelled diagram to display the measurements of all the angles and sides.

PROBLEM SOLVING AND REASONING

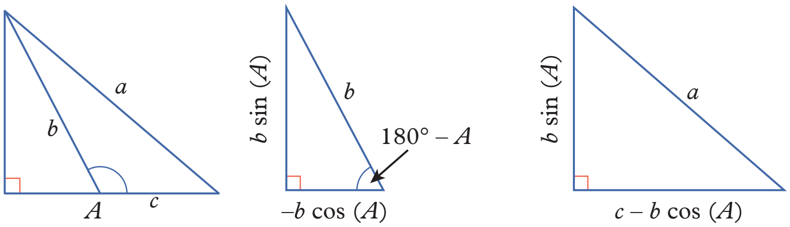
19 Consider the diagrams on the right. Using Pythagoras' Theorem on the rightmost triangle, we can write the following equation: $(b\sin(A))^2 + (c - b\cos(A))^2 = a^2$.



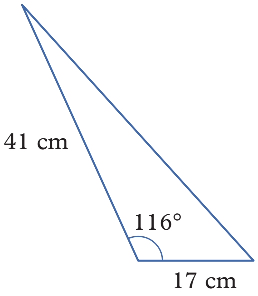
a Substitute the values from the triangle shown below into the equation above and hence determine the unknown side length correct to one decimal place.



The equation also holds true if A is an obtuse angle as $\sin(180°-A) = \sin(A)$, and $\cos(A)$ will be negative and so adds the extra length on.



b Substitute the values from the triangle shown below into the equation above and hence determine the unknown side length correct to one decimal place.



c Use the fact that $(b\sin(A))^2 + (b\cos(A))^2 = b^2$ to show that $(b\sin(A))^2 + (c - b\cos(A))^2 = a^2$ is equivalent to $b^2 + c^2 - 2bc\cos(A) = a^2$.

CHALLENGE

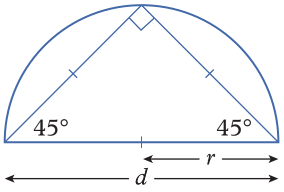
20 A right-angled isosceles triangle is inscribed in a semi-circle of radius r.

a Find the lengths of the triangle's two equal sides, in terms of r, using:

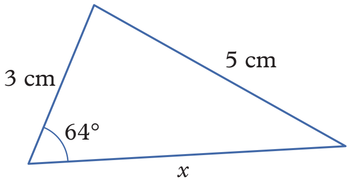
i Pythagoras' Theorem

ii trigonometry.

b Comment on your two answers for part **b**.

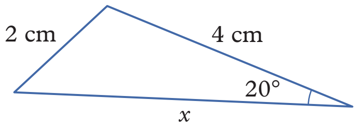


21 Azar has written the equation $5^2 = 3^2 + x^2 - 2(3)x\cos(64°)$ for the triangle on the right.

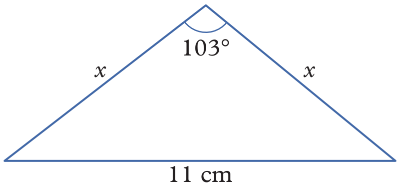


- **a** Write the equation in the form $x^2 + bx + c = 0$, where b and c are real numbers.
- **b** Solve the equation in part **a**. Give your answers correct to one decimal place.
- **c** Are both solutions to the equation in part **b** possible side lengths of the triangle? Explain, stating the length of the unknown side length.

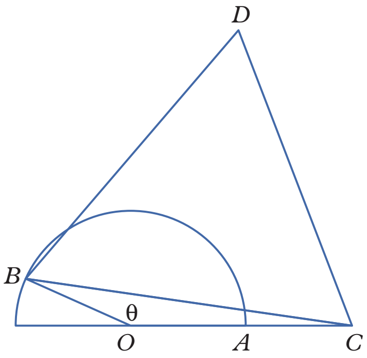
Azar has written the equation $2^2 = 4^2 + x^2 - 2(4)x\cos(20°)$ for the triangle below, which is not drawn to scale.



- **d** Solve this equation. Give your answers correct to one decimal place.
- **e** Are both solutions to the equation in part **d** possible side lengths of the triangle? Explain by referring to ambiguous triangles.
- **f** Compare using the sine and cosine rules to find the third side of a possibly ambiguous triangle.
- **g** Consider the isosceles triangle shown below. Write and solve an equation for the value of the side lengths, x, correct to one decimal place.



22 Point B moves along the arc of semi-circle O. The angle formed between OB and OA is θ. The radius of the semi-circle is 1. C is on the same line as OA and $OA = AC$, connecting BC and creating an equilateral triangle BCD. As B moves along the arc of the semi-circle, the area of ΔBCD also changes. Find the area of ΔBCD in terms of $\cos(\theta)$.



Check your Student obook pro for these digital resources and more:

Interactive skillsheet
The cosine rule

Topic quiz
8F

8G The unit circle

Learning intentions

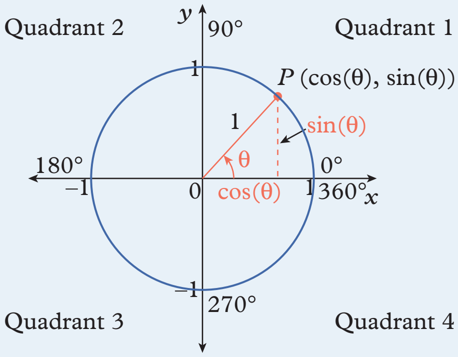
- ✔ I can find supplementary angles using the symmetrical properties of trigonometric functions.
- ✔ I can determine the magnitude of an angle using trigonometric functions.

Inter-year links

Year 9 7D Trigonometric ratios

The unit circle

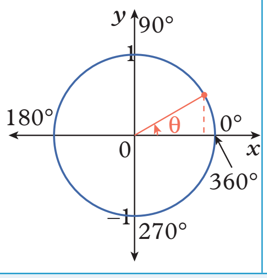
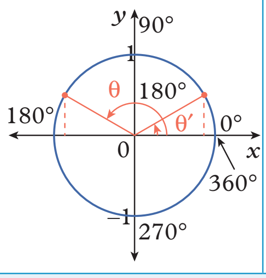
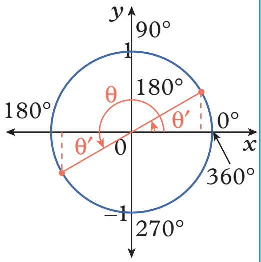
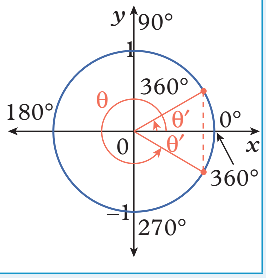
- The **unit circle** is a circle on the Cartesian plane, having its centre at the origin and a radius of 1 unit. The quadrants on the Cartesian plane are labelled in an anticlockwise direction from the positive x-axis.



- If θ is the angle measured anticlockwise from the positive x-axis to the radius that meets the unit circle at point P, then:
  - → The x-coordinate of the point on the unit circle is the cosine of θ.
  - → The y-coordinate of the point on the unit circle is the sine of θ.
  - → The gradient from the origin to the point on the unit circle is the tangent of θ.

The reference angle

- The sine and cosine values for any angle θ can be related by symmetry to an acute reference angle θ′ in quadrant 1.
- The reference angle is the acute angle between the x-axis and the line connecting the origin to the point on the unit circle.

	Quadrant 1	**Quadrant 2**	**Quadrant 3**	**Quadrant 4**
Angle (θ)	$0° < \theta < 90°$	$90° < \theta < 180°$	$180° < \theta < 270°$	$270° < \theta < 360°$
Reference angle (θ′)	$\theta' = \theta$ 	$\theta' = 180° - \theta$ 	$\theta' = \theta - 180°$ 	$\theta' = 360° - \theta$ 

Example 8G.1 Determining the quadrant on an angle

Identify the quadrant in which each of these angles lies.

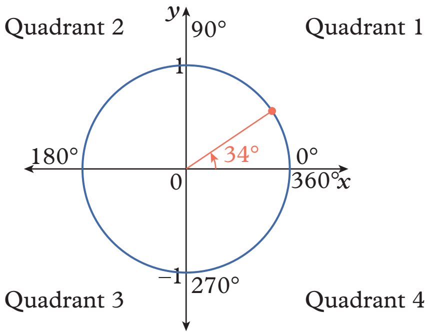
a 34° **b** 122° **c** 230° **d** 325°

THINK

WRITE

a Sketch the unit circle and mark the angle 34°. Because the size of the angle is between 0° and 90°, the angle is in quadrant 1.

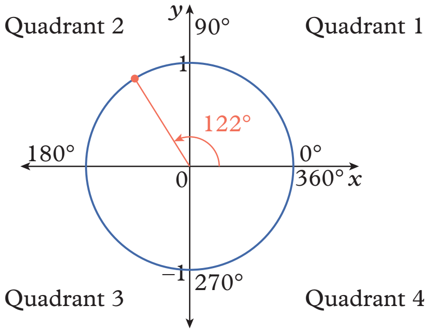
a



The angle is in quadrant 1.

b Sketch the unit circle and mark the angle 122°. Because the size of the angle is between 90° and 180°, the angle is in quadrant 2.

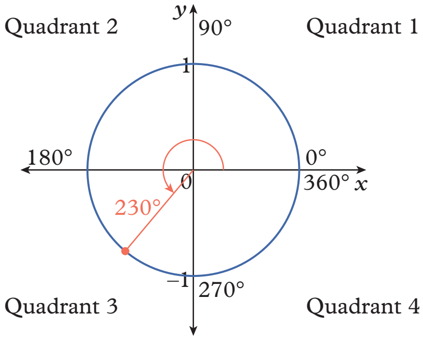
b



The angle is in quadrant 2.

c Sketch the unit circle and mark the angle 230°. Because the size of the angle is between 180° and 270°, the angle is in quadrant 3.

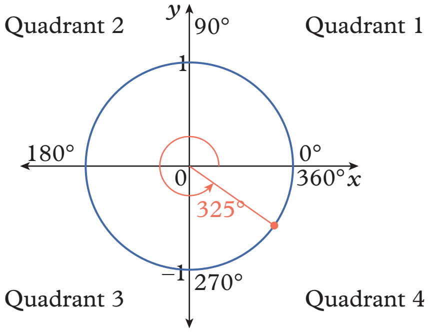
c



The angle is in quadrant 3.

d Sketch the unit circle and mark the angle 325°. Because the size of the angle is between 270° and 360°, the angle is in quadrant 4.

d

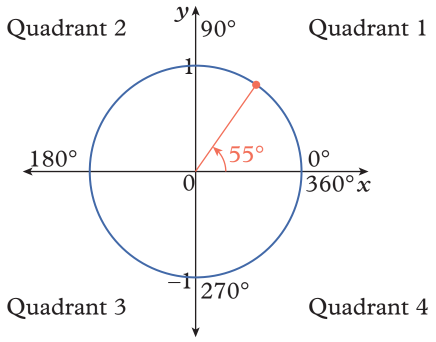
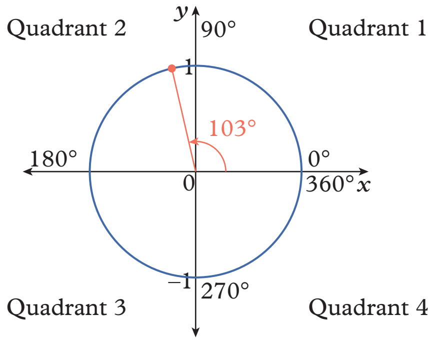
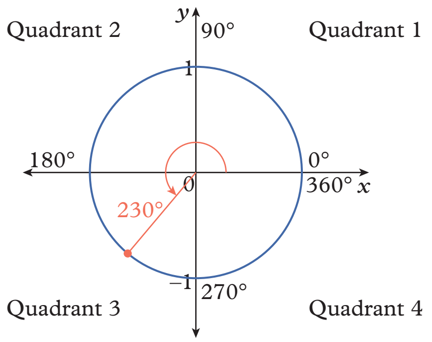
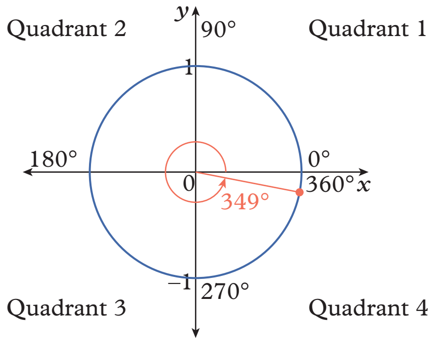


The angle is in quadrant 4.

Example 8G.2 Determining the sign of a trigonometric expression

Identify the sign (positive or negative) of each of these trigonometric expressions.

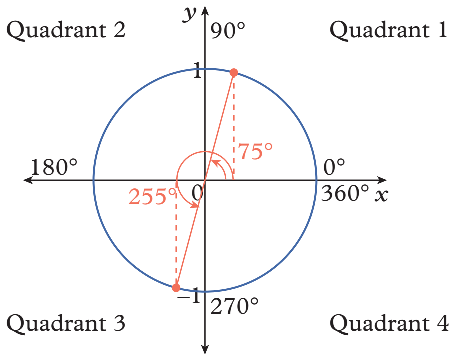
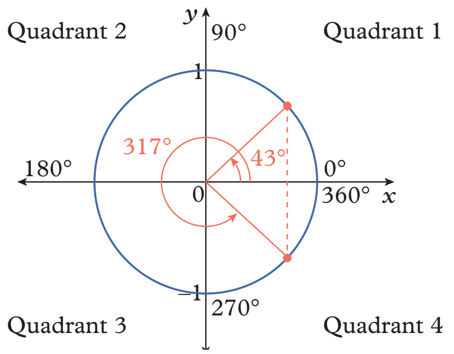
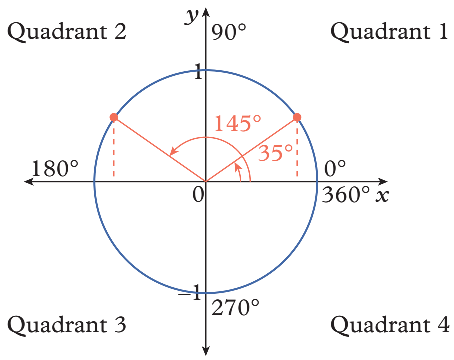
a cos(55°) **b** tan(103°) **c** sin(230°) **d** tan(349°)

THINK	**WRITE**
a Sketch the unit circle and mark the angle 55°. The angle is in quadrant 1, and cos is positive in quadrant 1 because the *x* value is positive.	**a**  cos(55°) is positive.
b Sketch the unit circle and mark the angle 103°. The angle is in quadrant 2, and tan is negative in quadrant 2 because the gradient from the origin is negative.	**b**  tan(103°) is negative.
c Sketch the unit circle and mark the angle 230°. The angle is in quadrant 3, and sin is negative in quadrant 3 because the *y* value is negative.	**c**  sin(230°) is negative.
d Sketch the unit circle and mark the angle 349°. The angle is in quadrant 4, and tan is negative in quadrant 4 because the gradient from the origin is negative.	**d**  tan(349°) is positive.

Example 8G.3 Determining the reference angle

Write each trigonometric expression in terms of the reference angle using the correct sign.

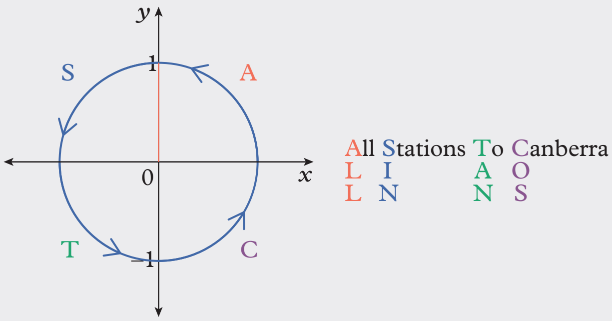
a $\sin(255^\circ)$ **b** $\cos(317^\circ)$ **c** $\tan(145^\circ)$

THINK	**WRITE**
a **1** Sketch the unit circle and mark the angle 255°.	**a** 
2 The angle is in quadrant 3, so the reference angle is $\theta - 180^\circ$.	$255^\circ - 180^\circ = 75^\circ$
3 $\sin(255^\circ)$ is negative because the y values in quadrant 3 are negative.	$\sin(255^\circ) = -\sin(75^\circ)$
b **1** Sketch the unit circle and mark the angle 317°.	**b** 
2 The angle is in quadrant 4, so the reference angle is $360^\circ - \theta$.	$360^\circ - 317^\circ = 43^\circ$
3 $\cos(317^\circ)$ is positive because the x values in quadrant 4 are positive.	$\cos(317^\circ) = \cos(43^\circ)$
c **1** Sketch the unit circle and mark the angle 145°.	**c** 
2 The angle is in quadrant 2, so the reference angle is $180^\circ - \theta$.	$180^\circ - 145^\circ = 35^\circ$
3 $\tan(145^\circ)$ is negative because the gradient from the origin to any point in quadrant 2 is negative.	$\tan(145^\circ) = -\tan(35^\circ)$

Helpful hints

✔ The sine, cosine and tangent values have different signs (positive or negative) depending on the quadrant in which they lie.
  - Sine is positive where the y-coordinate of the point on the unit circle is positive (quadrants 1 and 2), and negative where the y-coordinate is negative (quadrants 3 and 4).
  - Cosine is positive where the x-coordinate of the point on the unit circle is positive (quadrants 1 and 4), and negative where the x-coordinate is negative (quadrants 2 and 3).
  - Tangent is positive where the gradient from the origin to the point on the unit circle is positive (quadrants 1 and 3), and negative where the gradient of the radius is negative (quadrants 2 and 4).

✔ An easy way to remember the sign of the trigonometric functions in the four quadrants is with a mnemonic. Something like **A**ll **S**tations **T**o **C**anberra is good:



ANS p766

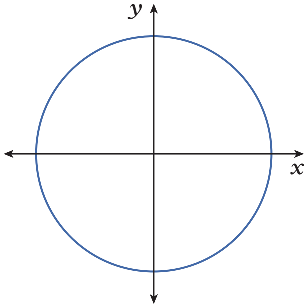
Exercise 8G The unit circle

△ 1–3, 4(a–j), 5, 6, 8, 10 □ 3, 4(f–o), 5, 6, 8, 9, 11, 13 ⬠ 3, 5–9, 12–14

UNDERSTANDING AND FLUENCY

1 For each of the following, shade the appropriate regions of a unit circle diagram.

a Shade the regions where the sine of angles in that region is positive.
b Shade the regions where the cosine of angles in that region is positive.
c Shade the regions where the tangent of angles in that region is positive.
d Shade the regions where the sine of angles in that region is negative.
e Shade the regions where the cosine of angles in that region is negative.
f Shade the regions where the tangent of angles in that region is negative.



8G.1 **2** Identify the quadrant in which each of these angles lies.

a 345° **b** 103° **c** 204°
d 265° **e** 74° **f** 139°

8G.2 **3** Identify the sign (positive or negative) of each of these trigonometric expressions.

a sin(100°) **b** cos(200°) **c** tan(140°) **d** cos(295°) **e** tan(260°)
f sin(340°) **g** cos(94°) **h** sin(130°) **i** cos(190°) **j** tan(350°)

4 Write the reference angle for each of the following.

a 140° **b** 92° **c** 200° **d** 190° **e** 302°
f 315° **g** 260° **h** 100° **i** 280° **j** 150°
k 345° **l** 170° **m** 220° **n** 255° **o** 299°

8G.3 **5** Express each of the following trigonometric expressions in terms of an appropriate reference angle. (Hint: remember to consider whether the sign of each expression will be positive or negative.)

a sin(241°) **b** cos(304°) **c** tan(95°) **d** sin(164°) **e** cos(110°)

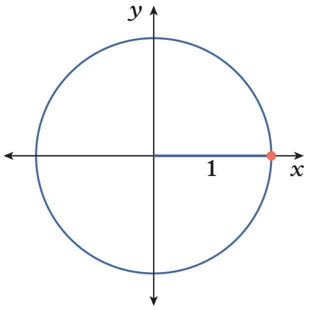
f cos(200°) **g** tan(200°) **h** tan(358°) **i** sin(285°) **j** sin(175°)

6 The unit circle can be used to evaluate the sine and cosine of angles that are multiples of 90° (including 0°) by considering the coordinate on the unit circle. For each of the following identify:

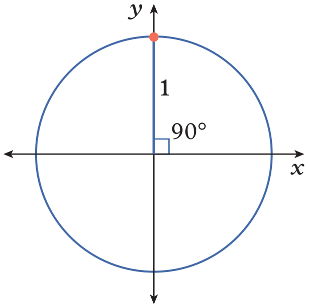
i the y-coordinate of the point shown equal to the sine of the marked angle

ii the x-coordinate of the point shown equal to the cosine of the marked angle.

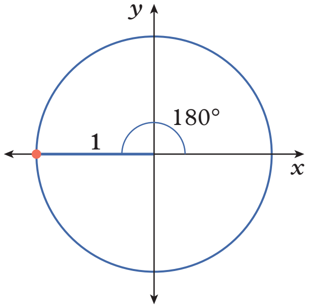
a



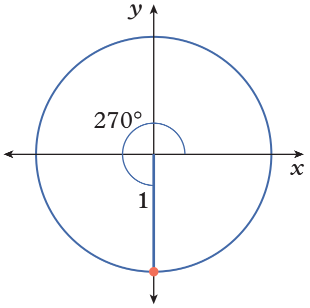
b



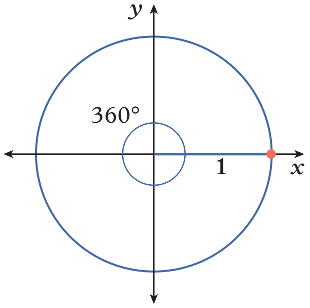
c



d



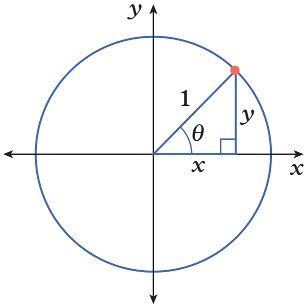
e



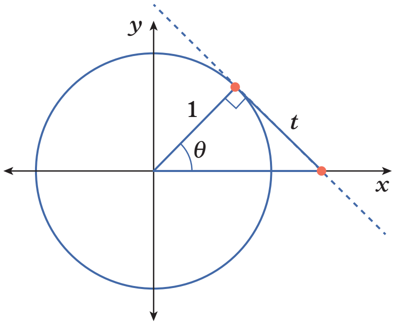
7 The tangent of an angle using the unit circle can be defined in different ways.

a State the tangent of θ in the right-angled triangles shown.

i



ii



b Explain how the definition from part **a i** can be described as ‘the tangent of the angle is equal to the gradient of the line from the origin to the point on the unit circle’.

c The lengths x and y also refer to the coordinate of the point where the radius touches the circle. Write the expression of the tangent of θ from part **a i** in terms of sine and cosine.

d Use your expression from part **c** to write the following in terms of tangent.

i $\frac{\sin(20°)}{\cos(20°)}$ **ii** $\frac{\sin(152°)}{\cos(152°)}$ **iii** $\frac{\cos(218°)}{\sin(218°)}$ **iv** $\frac{\sin(347°)}{\cos(347°)}$

e The unit circle can also be used to evaluate the tangent of angles that are multiples of 90° (including 0°) by considering the gradients from the origin to the point on the unit circle. Since the gradient is either horizontal or vertical at these angles, the tangent value is either zero or undefined, respectively. For each of the following state the gradient of the radius shown equal to the tangent of the marked angle.

i tan(0°) **ii** tan(90°) **iii** tan(180°) **iv** tan(270°) **v** tan(360°)

8 Angles greater than 360° are coterminal with angles between 0° and 360°, as they terminate or end at the same point on the unit circle. That is, every angle greater than 360° corresponds with an angle between 0° and 360° as they point in the same direction. To determine a coterminal angle, add or subtract 360° from the given angle. Write each trigonometric expression in terms of:

i the coterminal angle between 0° and 360°

ii the reference angle using the correct sign, or its value if it is a multiple of 90°.

a $\sin(400^\circ)$ **b** $\cos(530^\circ)$ **c** $\tan(800^\circ)$ **d** $\sin(630^\circ)$

e $\cos(777^\circ)$ **f** $\cos(1024^\circ)$ **g** $\tan(2000^\circ)$ **h** $\cos(900^\circ)$

9 Using the unit circle, we can also consider negative angles as angles measure clockwise from the positive *x*-axis. Negative angles are also coterminal with angles between 0° and 360°. Write each trigonometric expression in terms of:

i the coterminal angle between 0° and 360°

ii the reference angle using the correct sign or its value if it is a multiple of 90°.

a $\sin(-45^\circ)$ **b** $\cos(-225^\circ)$ **c** $\tan(-135^\circ)$ **d** $\sin(-315^\circ)$

e $\cos(-508^\circ)$ **f** $\tan(-436^\circ)$ **g** $\cos(-630^\circ)$ **h** $\sin(-1000^\circ)$

10 Complete the following for angles between 0° and 360°.

a $\sin(158^\circ) = \sin(\underline{\quad}^\circ) = -\sin(\underline{\quad}^\circ) = -\sin(\underline{\quad}^\circ)$

b $\cos(213^\circ) = \cos(\underline{\quad}^\circ) = -\cos(\underline{\quad}^\circ) = -\cos(\underline{\quad}^\circ)$

c $\tan(285^\circ) = \tan(\underline{\quad}^\circ) = -\tan(\underline{\quad}^\circ) = -\tan(\underline{\quad}^\circ)$

11 Consider the following three equalities.

$\sin(234^\circ) = -\sin(126^\circ)$ $\cos(234^\circ) = \cos(126^\circ)$ $\tan(234^\circ) = -\tan(126^\circ)$

a Use a calculator to confirm that the pairs of trigonometric expressions are equal.

b Evaluate $234^\circ + 126^\circ$.

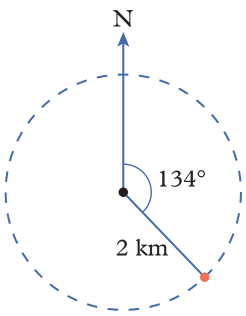
c Explain how these equalities relate to the fourth quadrant.

12 True bearings measure angles from north (the positive *y*-axis) clockwise, rather than from the positive *x*-axis, like angles anticlockwise on the unit circle. The distance travelled along the bearing will scale the coordinates on the circle where that distance is the scale factor. If the centre point of the circle is the origin, state the coordinate of the points shown in terms of

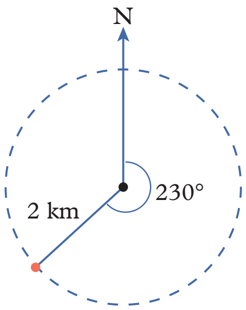
i the true bearings

ii the *x*- and *y*-coordinates in terms of trigonometric expressions containing a reference angle.

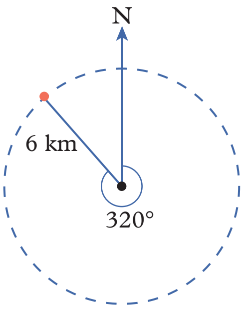
a



b



c



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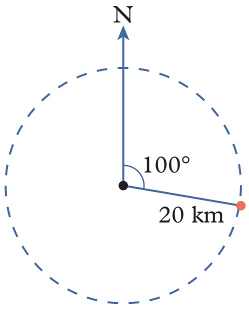

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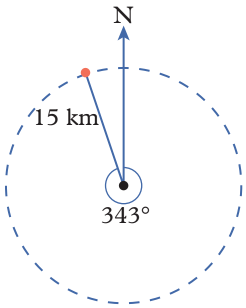

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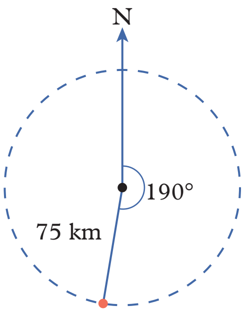
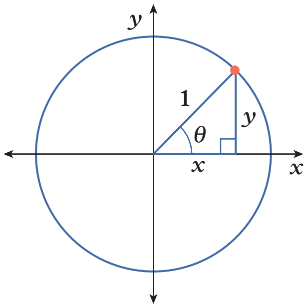


g Compare the coordinates for bearings and the coordinates for the unit circle.

CHALLENGE

13 Using the diagram shown, we can write the equation $x^2 + y^2 = 1$ using Pythagoras' Theorem.



a Write the equation in terms of sine and cosine of θ.

b Rearrange the equation in part **a** to make:

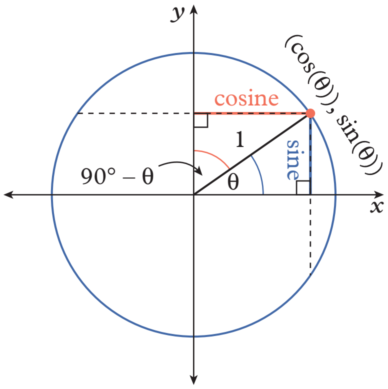
i sin(θ) the subject　　**ii** cos(θ) the subject.

c We can use the equation in part **a** (and its rearrangements in part **b**) to determine the cosine value of an angle if we know the sine of the angle and vice versa. The sign of the solution can be determined by considering which quadrant the angle lies in. Use the sine and cosine values provided to determine the value of the trigonometric expression in brackets correct to four decimal places.

i $\sin(50°) = 0.7660$ $(\cos(50°))$　　**ii** $\cos(116°) = -0.4384$ $(\sin(116°))$

iii $\sin(177°) = 0.0523$ $(\cos(177°))$　　**iv** $\cos(240°) = -0.5$ $(\sin(240°))$

14 The name 'cosine' comes from an abbreviation of the phrase 'the sine of the complementary angle'. That is, $\sin(90° - \theta°) = \cos(\theta°)$. We can use this definition to turn sine expressions into cosine expressions of the complementary angle and vice versa. Write the following sine expressions using cosine, and the following cosine expressions using sine.



a sin(30°)　　**b** sin(52°)　　**c** cos(24°)　　**d** cos(45°)

This is true even when θ° is not an acute angle. Write the following sine expressions using cosine, and cosine expressions using sine, using:

i the angle from the subtraction 90° – θ

ii the coterminal angle between 0° and 360° if applicable

iii the reference angle using the correct sign or its value if it is a multiple of 90° if applicable.

e sin(158°)　　**f** cos(351°)　　**g** sin(211°)

h sin(−15°)　　**i** cos(482°)　　**j** cos(−555°)

Check your Student obook pro for these digital resources and more:

pro

Interactive skillsheet
The sign of trigonometric expressions

Interactive skillsheet
Reference angles

Topic quiz
8G

8H Exact values

Learning intentions

- ✔ I can convert between degrees and radians.
- ✔ I can determine the exact values of the sine, cosine and tangent functions.
- ✔ I can identify points on the unit circle via arc lengths in radians.

Inter-year links

Year 9 7D Trigonometric ratios

Degrees and radians

- A radian is a unit for measuring angles that is related to circles. Radians are the ratio between the arc length and the radius of a circle such that 1 radian is the angle subtended by the arc length that is one radius long.
- There are 2π radians in one revolution of a circle. When written in terms of π, radians describe the fraction of a full turn, 2π.

  For example, $\frac{2\pi}{5}$ is a fifth of a full turn and $\frac{3\pi}{4} = \frac{3(2\pi)}{8}$ is three-eighths of a full turn.
- 180° is equivalent to π radians and 1 radian is exactly $\frac{180°}{\pi}$, or approximately 57.3°. To convert from degrees to radians, we multiply by $\frac{\pi}{180°}$. To convert from radians to degrees, we multiply by $\frac{180°}{\pi}$.

Exact values of trigonometric ratios

- The table below can be used to find exact values of the trigonometric ratios for three possible angles.

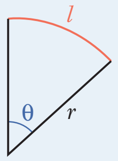
θ	30° or $\frac{\pi}{6}$	45° or $\frac{\pi}{4}$	60° or $\frac{\pi}{3}$
sin(θ)	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cos(θ)	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tan(θ)	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

- There are two special triangles whose side lengths, when trigonometry is applied, provide the exact values of the trigonometric ratios for the three angles specified in the table above. We can use geometry and Pythagoras' Theorem to determine all angles and lengths of the two triangles.

Arc length

- The circumference of the unit circle is 2π.
- The length of the circumference of an arc, l, can be calculated by multiplying the angle θ (in radians) it subtends at the centre of the circle by the circle's radius, r.

$$\underset{\text{arc length}}{l} = \underset{\text{radius}}{r}\underset{\text{angle in radians}}{\theta}$$



- Since the unit circle has a radius of 1, any arc length on the circumference of the unit circle is equal to the radian measure of the angle θ subtending the arc.

Example 8H.1 Converting between radians and degrees

Convert each of the following to the unit indicated in the brackets.

a 110° (convert to radians)

b $\frac{4\pi}{3}$ radians (convert to degrees)

THINK

a To convert from degrees to radians, multiply by $\frac{\pi}{180°}$.

b To convert from radians to degrees, multiply by $\frac{180°}{\pi}$.

WRITE

a $110° \times \frac{\pi}{180°} = \cancel{110°}^{11} \times \frac{\pi}{\cancel{180°}^{18}}$

$= \frac{11\pi}{18}$ radians

b $\frac{4\pi}{3} \times \frac{180°}{\pi} = \frac{\cancel{4\pi}^{4}}{\cancel{3}^{1}} \times \frac{\cancel{180°}^{60°}}{\cancel{\pi}^{1}}$

$= \frac{4 \times 60°}{1}$

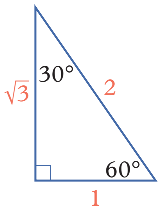
$= 240°$

Example 8H.2 Finding the exact values of a trigonometric ratio

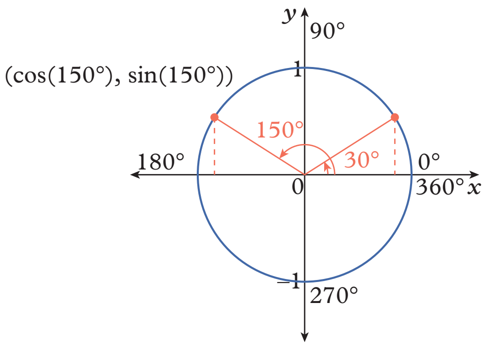
Use the unit circle and exact values for the trigonometric ratios to evaluate:
$\sin(150°)$, $\cos(150°)$, $\tan(150°)$

THINK

1 Sketch the unit circle and mark the angle 150°. It is in quadrant 2, so the reference angle is 180° − 150° = 30°.



WRITE



THINK

2 Calculate $\sin(30°)$ using the side lengths of the special triangle above.

WRITE

$\sin(30°) = \frac{O}{H}$

$= \frac{1}{2}$

THINK

3 Calculate $\cos(30°)$ using the side lengths of the special triangle.

WRITE

$\cos(30°) = \frac{A}{H}$

$= \frac{\sqrt{3}}{2}$

THINK

4 Calculate $\tan(30°)$ using the side lengths of the special triangle. Remember to rationalise the denominator.

WRITE

$\tan(30°) = \frac{O}{A}$

$= \frac{1}{\sqrt{3}}$

$= \frac{\sqrt{3}}{3}$

THINK

5 Apply the correct signs to the trigonometric ratios for quadrant 2.

WRITE

In quadrant 2, cos and tan are both negative, but sin is positive.

$\sin(150°) = \frac{1}{2}$

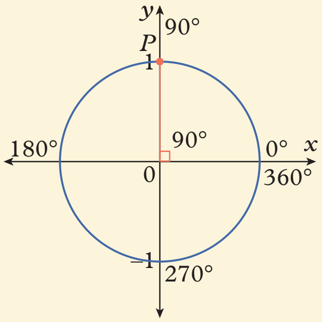
$\cos(150°) = -\frac{\sqrt{3}}{2}$

$\tan(150°) = -\frac{\sqrt{3}}{3}$

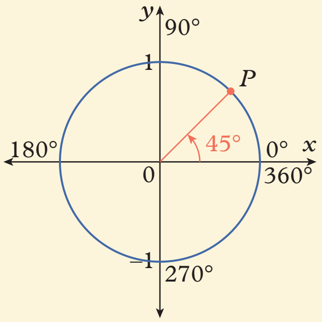
Example 8H.3 Identifying points on the unit circle

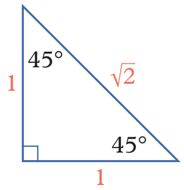
Identify the coordinates of the point *P* in each case below.

a



b

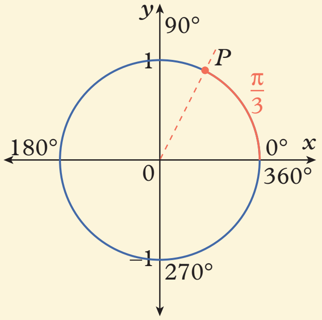


THINK	WRITE
a 1 Any point on the unit circle has an *x*-coordinate of cos(θ) and a *y*-coordinate of sin(θ). Write *P* in coordinate form using θ = 90°.	**a** $P = (\cos(90°), \sin(90°))$
2 Identify the exact values of the trigonometric ratios. Remember that the unit circle has a radius of 1.	$= (0, 1)$
b 1 Write *P* in coordinate form using θ = 45°.	**b** $P = (\cos(45°), \sin(45°))$ 
2 Identify the exact values of the trigonometric ratios. Use the half-square triangle to determine the values.	$\cos(45°) = \frac{\sqrt{2}}{2}$ $\sin(45°) = \frac{\sqrt{2}}{2}$ $P = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

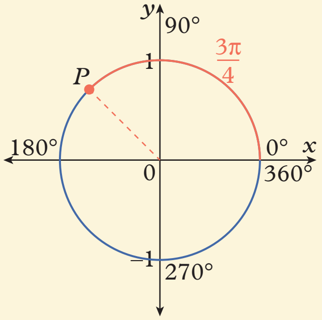
Example 8H.4 Identifying points on the unit circle from an arc length

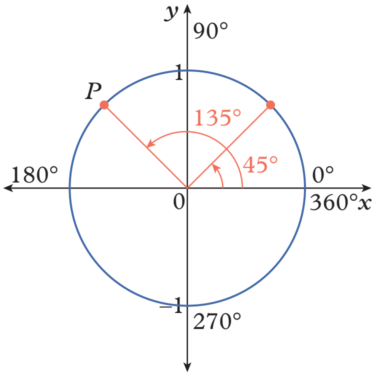
Identify the coordinates of point *P* in each case.

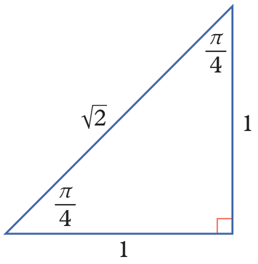
a



b



<table>
<tr><th>THINK</th><th>WRITE</th></tr>
<tr><td>a 1 Given the arc length, determine the angle in radians that identifies the position of point <i>P</i>.</td><td>a $l = r\theta$
$\frac{\pi}{3}$ radians $= \theta$</td></tr>
<tr><td>2 Write <i>P</i> in coordinate form.</td><td>$P = \left(\cos\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{3}\right)\right)$</td></tr>
<tr><td>3 Identify the exact values of the trigonometric ratios. Use the half-equilateral triangle to determine the values.</td><td>$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$
$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$
$P = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$</td></tr>
<tr><td>b 1 Given the arc length, determine the angle in radians that identifies the position of point <i>P</i>.</td><td>b $l = r\theta$
$\frac{3\pi}{4}$ radians $= \theta$</td></tr>
<tr><td>2 Write <i>P</i> in coordinate form.</td><td>$P = \left(\cos\left(\frac{3\pi}{4}\right), \sin\left(\frac{3\pi}{4}\right)\right)$</td></tr>
<tr><td>3 Identify the reference angle and determine the sign in quadrant 2.</td><td>

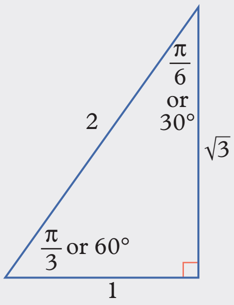
$\cos\left(\frac{3\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right)$
$\sin\left(\frac{3\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right)$</td></tr>
<tr><td>4 Identify the exact values of the trigonometric ratios. Use the half-square triangle to determine the values.</td><td>

$-\cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$
$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$
$P = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$</td></tr>
</table>

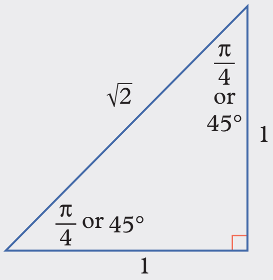
Helpful hints

✔ It is helpful to memorise the special triangles below. You will use them a lot in senior Maths classes.

half-equilateral triangle



half-square triangle



θ	30° or $\frac{\pi}{6}$	45° or $\frac{\pi}{4}$	60° or $\frac{\pi}{3}$
sin(θ)	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cos(θ)	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tan(θ)	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

ANS p768

Exercise 8H Exact values

△ 1–8, 10, 12 ■ 2, 4–7, 9, 10, 11(a–c), 12, 13, 14(a) ⬠ 5, 7, 10–13, 14(c, d), 15

UNDERSTANDING AND FLUENCY

8H.1 **1** Convert each of the following to the unit given in brackets.

a 190° (radians) **b** 156° (radians) **c** $\frac{2\pi}{5}$ (degrees)

d $\frac{17\pi}{12}$ (degrees) **e** 136° (radians) **f** $\frac{143\pi}{90}$ (degrees)

8H.2 **2** Use the unit circle and exact values for the trigonometric ratios to evaluate the sine, cosine and tangent of each of these angles.

a 240° **b** 300° **c** 120°

d 315° **e** 135° **f** 150°

3 For each of the following angles:

i Determine the reference angle.

ii Use a calculator to evaluate the sine, cosine and tangent of the reference angle.

iii Use the unit circle to determine the sine, cosine and tangent of the original angle.

a $\frac{7\pi}{6}$ **b** $\frac{3\pi}{4}$ **c** $\frac{4\pi}{3}$

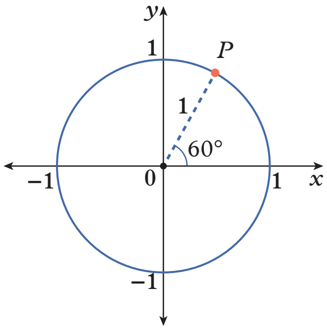
4 Use the unit circle and exact values for the trigonometric ratios to evaluate the sine, cosine and tangent of each angle.

a $\frac{\pi}{6}$ **b** $\frac{5\pi}{6}$ **c** $\frac{5\pi}{3}$

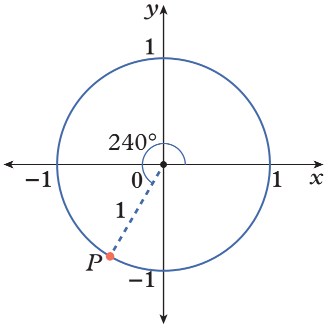
d $\frac{5\pi}{4}$ **e** $\frac{2\pi}{3}$ **f** $\frac{7\pi}{4}$

8H.3 5 Identify the coordinates of the point P in each of these cases.

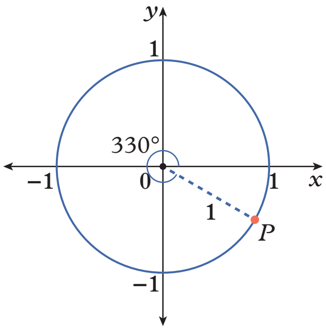
a



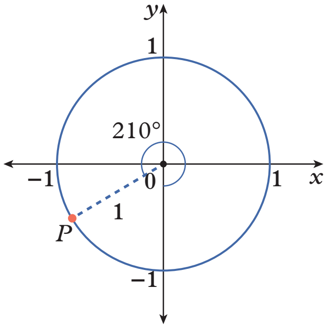
b



c



d



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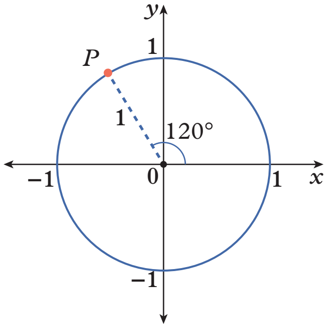
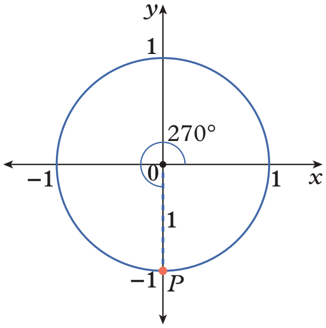


f



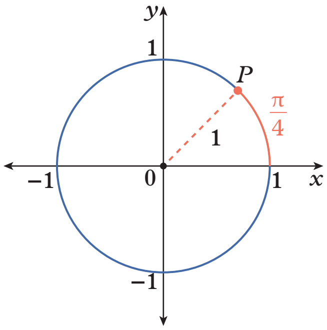
6 Give the exact coordinates of the point on the unit circle when the radial arm is rotated by each of these angles from the positive *x*-axis.

a 225° **b** 300 **c** 150°

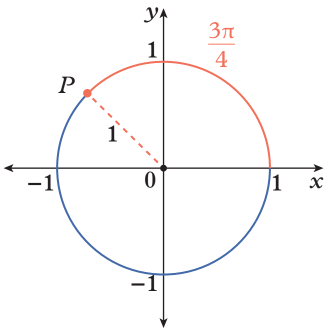
d 135° **e** 315° **f** 180°

8H.4 7 Identify the coordinates of the point *P* in each of these cases.

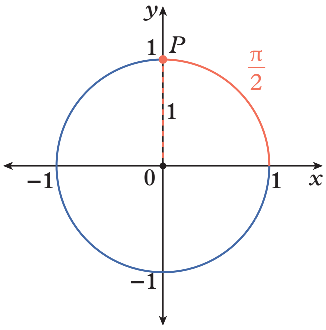
a



b



c



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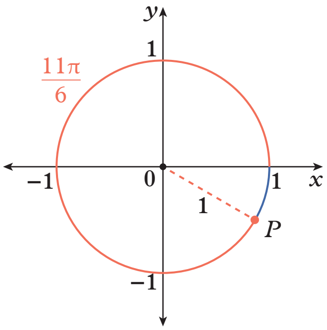

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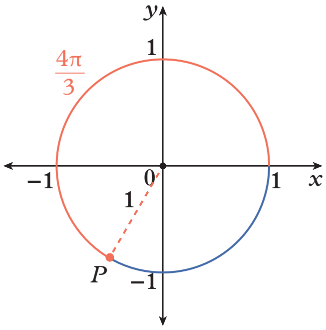

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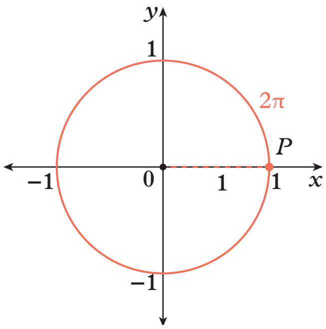


PROBLEM SOLVING AND REASONING

8 Answer each of the following in radians using your knowledge of geometry and without directly converting the angles to or from degrees.

a State the angle of:

i a revolution **ii** a straight angle **iii** a right angle.

b Write an inequality for the angle θ such that it is:

i acute **ii** obtuse **iii** reflex.

c Complete the following sentences.

i Complementary angles (two angles that form a _______________) add to ______ radians.

ii Supplementary angles (two angles that form a _______________) add to ______ radians.

d Determine the complement of the following angles.

i $\frac{\pi}{3}$ **ii** $\frac{3\pi}{8}$ **iii** $\frac{\pi}{4}$ **iv** $\frac{\pi}{10}$

e Determine the supplement of the following angles.

i $\frac{\pi}{3}$ **ii** $\frac{3\pi}{4}$ **iii** $\frac{\pi}{6}$ **iv** $\frac{5\pi}{12}$

f Determine the angle that would form a revolution with each of the following angles:

i $\frac{\pi}{3}$ **ii** $\frac{11\pi}{6}$ **iii** $\frac{5\pi}{4}$ **iv** $\frac{7\pi}{5}$

g State the interior angle sum of:

i a triangle **ii** a quadrilateral **iii** a hexagon **iv** an n-sided polygon.

h State the size of each interior angle in:

i an equilateral triangle **ii** a square

iii a regular hexagon **iv** a regular n-sided polygon.

i Angles are said to be coterminal if the radial arm terminates or ends at the same point on the unit circle. State the coterminal angle one revolution greater than each of the following:

i $\frac{\pi}{3}$ **ii** $\frac{5\pi}{4}$ **iii** $\frac{11\pi}{6}$ **iv** $\frac{\pi}{2}$ **v** π

j State the coterminal angle one revolution less than each of the following:

i $\frac{\pi}{3}$ **ii** $\frac{5\pi}{4}$ **iii** $\frac{11\pi}{6}$ **iv** $\frac{\pi}{2}$ **v** π

9 **a** For each angle, state the fraction of a full turn that it represents. Mark each angle on a unit circle.

i $\frac{2\pi}{3}$ **ii** $\frac{\pi}{4}$ **iii** $\frac{11\pi}{6}$ **iv** $\frac{3\pi}{2}$

v $\frac{8\pi}{7}$ **vi** $\frac{4\pi}{5}$ **vii** π **viii** $\frac{\pi}{180}$

b For each fraction of a full turn, state the angle in radians. Mark each angle on a unit circle.

i $\frac{5}{6}$ **ii** $\frac{3}{5}$ **iii** $\frac{7}{8}$ **iv** $\frac{1}{12}$

v $\frac{6}{11}$ **vi** $\frac{9}{10}$ **vii** $\frac{8}{15}$ **viii** $\frac{5}{16}$

10 The constructions of the half-square and half-equilateral triangles are helpful to learn in case you forget any of the side lengths or angles, as you can then reconstruct the triangle from the name.

a Draw a square, mark the side lengths 1 unit, and add one of the diagonals.

b Use your knowledge of geometry and Pythagoras' Theorem to label all angles in degrees and lengths with their size.

c Repeat parts **a** and **b** using radians.

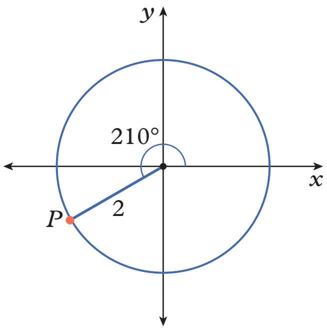
d Draw an equilateral triangle with a horizontal base, mark the side lengths 2 units, and add the perpendicular height to the base such that it bisects the top angle and the base length.

e Use your knowledge of geometry and Pythagoras' Theorem to label all angles in degrees and lengths with their size.

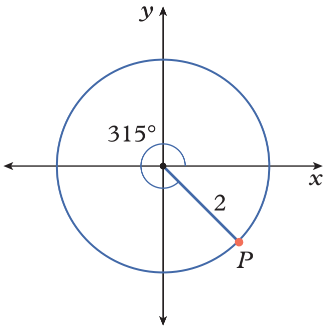
f Repeat parts **d** and **e** using radians.

11 Identify the coordinates of the point P in each case.

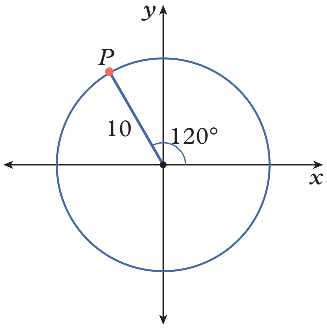
a



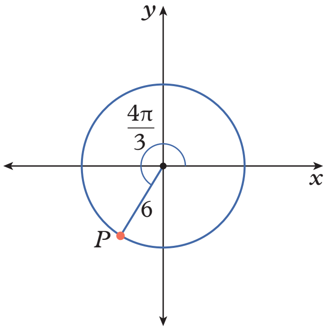
b



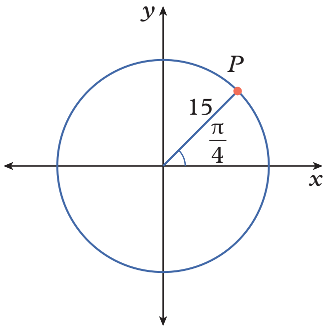
c



d



e



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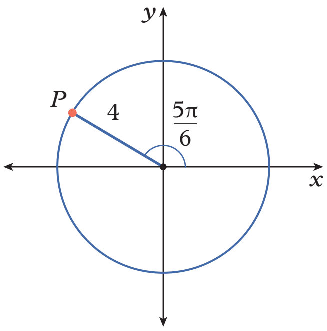


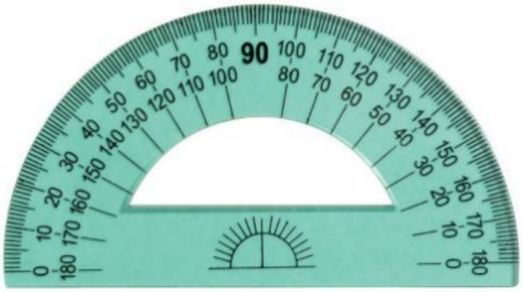
12 Complete the table.

	Radius	Diameter	Angle subtending arc	Arc length
a	3 cm		$\frac{5\pi}{6}$	
b	6 cm		$\frac{2\pi}{3}$	
c		14 cm	$\frac{3\pi}{4}$	
d			$\frac{5\pi}{12}$	$\frac{4\pi}{3}$ cm
e	4 cm			$\frac{9\pi}{2}$ cm
f		5 cm		$\frac{2\pi}{7}$ cm

13 Evaluating trigonometric expressions of angles in radians where the answer is an exact value has a simple method for determining the reference angle and quadrant.

a For each of the following, state the reference angle.

i $\frac{2\pi}{3}$ **ii** $\frac{4\pi}{3}$ **iii** $\frac{5\pi}{3}$
iv $\frac{5\pi}{6}$ **v** $\frac{5\pi}{4}$ **vi** $\frac{7\pi}{4}$
vii $\frac{7\pi}{6}$ **viii** $\frac{11\pi}{6}$ **ix** $\frac{3\pi}{4}$



b Explain how you can determine the reference angle for an angle in radians.

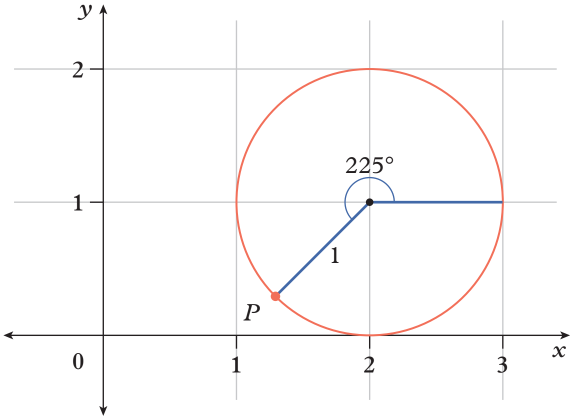
c Determine the quadrant each angle in part **a** is in.

d Explain how you can determine the quadrant for an angle in radians.

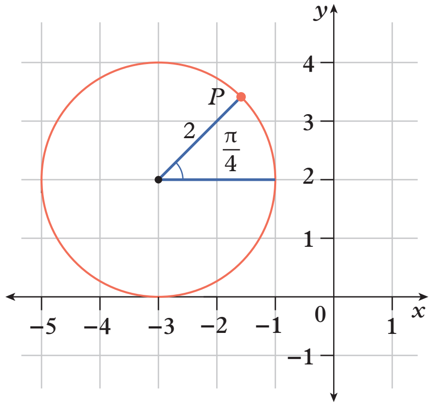
CHALLENGE

14 Identify the coordinates of the point P in each case.

a



b



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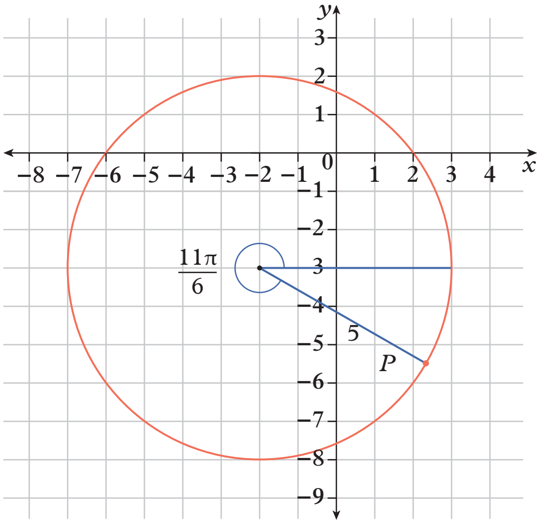
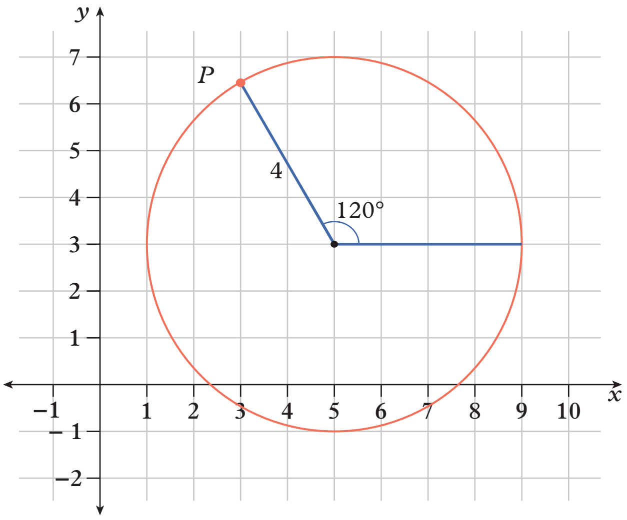
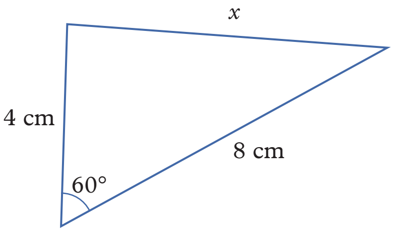


d



15 Determine the exact value of x for both triangles.

a



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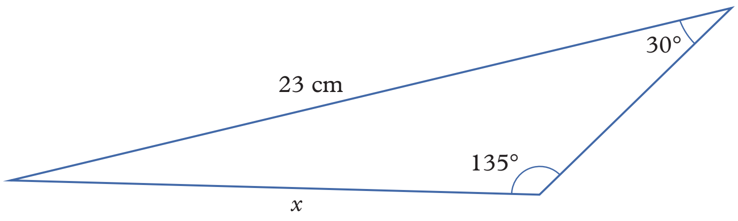


Check your Student obook pro for these digital resources and more:

- **Interactive skillsheet** Converting between degrees and radians
- **Interactive skillsheet** Exact values of trigonometric ratios
- **Interactive skillsheet** Coordinates of points on the unit circle
- **Topic quiz** 8H

8I Solving trigonometric equations

Learning intentions

✔ I can solve simple trigonometric equations.

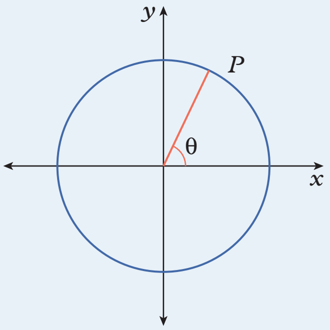
Inter-year links

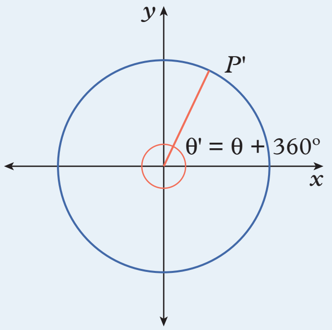
Year 9	7F Using trigonometry to find angles

Trigonometric equations

- A **periodic function** is a function that repeats itself at regular intervals.
- The trigonometric functions sin(θ), cos(θ) and tan(θ) are all periodic, as their values repeat with every revolution of the angle, θ.
- If θ is the angle measured anticlockwise from the positive *x*-axis to the radius that meets the unit circle at point *P*, then:
  - → the *x*-coordinate of the point on the unit circle is cos(θ)
  - → the *y*-coordinate of the point on the unit circle is sin(θ)
  - → the gradient from the origin to the point *P* on the unit circle is tan(θ).
- Adding or subtracting 360° or 2π to θ, the intersection of the new angle θ' meet the unit circle at point *P*' on the unit circle, then:
  - → the *x*-coordinate of *P*' on the unit circle is $\cos(\theta \pm 360°)$ or $\cos(\theta \pm 2\pi)$
  - → the *y*-coordinate of *P*' on the unit circle is $\sin(\theta \pm 360°)$ or $\sin(\theta \pm 2\pi)$
  - → the gradient from the origin to the point *P*' on the unit circle is $\tan(\theta \pm 360°)$ or $\tan(\theta \pm 2\pi)$.
- Because *P* and *P*' would have the same coordinates on the unit circle:

  $\sin(\theta) = \sin(\theta \pm 2\pi)$
  $\cos(\theta) = \cos(\theta \pm 2\pi)$
  $\tan(\theta) = \tan(\theta \pm 2\pi)$



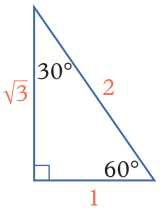


- Equations involving trigonometric functions have infinitely many solutions, as trigonometric functions are periodic.

  For example, $\sin(\theta) = 0$ for $\theta = 0, \pi, 2\pi, 3\pi, 4\pi$ ... or $\theta = 0, 180°, 360°, 540°, 720°$ ...
- The solutions to a trigonometric equation are often restricted to a domain given by an interval, such as $0 \le \theta \le 2\pi$ or $0° \le \theta \le 360°$.
- Most calculators only give one of the solutions for sin(θ), cos(θ) and tan(θ) as the domains are restricted to only half a revolution of the unit circle.
- Once one solution for a trigonometric equation is identified, the rest can be determined using the symmetry of the unit circle within a specified domain.

Example 8I.1 Solving a trigonometric equation within a given domain

Solve the equation $\sin(\theta) = -\frac{1}{2}, 0° \leq \theta \leq 360°$.

THINK	**WRITE**
1 Determine the first solution to the equation. Use the special triangles to determine the angle in the first quadrant that has a sine of $\frac{1}{2}$.	 $\sin(\theta) = \frac{1}{2}$ $\theta = \sin^{-1}\left(\frac{1}{2}\right)$ $= 30°$
2 Identify the quadrants in which sine is negative.	We were given that $\sin(\theta) = -\frac{1}{2}$ and sine is negative in quadrants 3 and 4.
3 Write possible values for θ.	So: $\theta = 180° + 30°$ $= 210°$ or: $\theta = 360° - 30°$ $= 330°$
4 Write the solution to the equation.	There are two solutions: $\theta = 210°$ or $\theta = 330°$

Helpful hints

✔ The trigonometric functions repeat themselves every 360° or 2π radian, because these are always coterminal angles. But don't forget that most values of sin(θ), cos(θ) and tan(θ) will repeat themselves in another quadrant of the unit circle as well.

ANS p771

Exercise 8I Solving trigonometric equations

△ 1–5, 7, 9, 11, 16(a, b), 18(a, e), 19 ■ 1–3, 6, 8, 10(a, e, i), 11, 12, 15, 16–18(a, c, e), 20 ⬟ 1, 2, 6, 10(g–i), 11–14, 17–18(a, c, e), 21, 22

8I.1 **1** Solve each equation for the domain $0° \leq \theta \leq 360°$.

a $\cos(\theta) = \frac{\sqrt{3}}{2}$ **b** $\sin(\theta) = -\frac{1}{\sqrt{2}}$ **c** $\tan(\theta) = -\frac{1}{\sqrt{3}}$

d $\sin(\theta) = -\frac{\sqrt{3}}{2}$ **e** $\cos(\theta) = \frac{1}{\sqrt{2}}$ **f** $\tan(\theta) = 1$

2 Using a calculator to determine the reference angle, solve each equation for the domain $0° \leq \theta \leq 360°$ correct to the nearest degree.

a $\cos(\theta) = -0.4$ **b** $\sin(\theta) = -0.3$ **c** $\tan(\theta) = 0.9$

d $\sin(\theta) = 0.6$ **e** $\cos(\theta) = 0.6$ **f** $\tan(\theta) = -7$

3 Consider the equation $\sin(\theta) = 1, 0° \le \theta \le 360°$.

- **a** What angle has a sine value of 1? (Hint: Refer to the unit circle.)
- **b** Are there any other angles within the domain with a sine value of 1? If so, which one(s)?
- **c** What is the solution to the equation?
- **d** Write a list of all the solutions for the domain $0° \le \theta \le 720°$.

4 Consider the equation $\cos(\theta) = -1$.

- **a** What solution(s) can be found in the domain $0° \le \theta \le 360°$?
- **b** What are the solutions in the domain $0° \le \theta \le 720°$?

5 Consider the equation $\sin(\theta) = 0, 0° \le \theta \le 360°$.

- **a** What angle has a sine value of 0? (Hint: Refer to the unit circle.)
- **b** Are there any other angles within the domain with a sine value of 1? If so, which one(s)?
- **c** What is the solution to the equation?
- **d** Write a list of all the solutions for the domain $0° \le \theta \le 720°$.

6 Solve each equation for the domain $0° \le \theta \le 360°$.

- **a** $\cos(\theta) = 0$
- **b** $\tan(\theta) = 0$
- **c** $\sin(\theta) = -1$
- **d** $\cos(\theta) = 1$

7 Give the smallest domain within $0° \le \theta \le 360°$ for there to be two solutions for each equation.

- **a** $\sin(\theta) = -1$
- **b** $\cos(\theta) = 0$
- **c** $\cos(\theta) = \frac{1}{2}$
- **d** $\sin(\theta) = \frac{\sqrt{3}}{2}$
- **e** $\sin(\theta) = \frac{1}{\sqrt{2}}$
- **f** $\cos(\theta) = -\frac{\sqrt{3}}{2}$

8 Find the smallest domain for each equation to give:

- **i** two solutions within $0° \le \theta \le 360°$
- **ii** four solutions within $0° \le \theta \le 720°$.
  - **a** $\sin(\theta) = -\frac{1}{2}$
  - **b** $\cos(\theta) = -\frac{1}{2}$
  - **c** $\sin(\theta) = -\frac{1}{\sqrt{2}}$
  - **d** $\cos(\theta) = -\frac{1}{\sqrt{2}}$

9 It is possible that a trigonometric equation has no real solution. Explain why each of these equations has no solution within any real domain.

- **a** $3 - \sin(\theta) = 1$
- **b** $2\cos(\theta) = 3$
- **c** $\frac{1}{4}\sin(\theta) = 1$

10 Indicate whether each equation has a real solution within the domain $0° \le \theta \le 360°$. For those without a real solution, explain why this is so. There is no need to solve the equation.

- **a** $\tan(\theta) = 4$
- **b** $1 - \sin(\theta) = 1$
- **c** $2\cos(\theta) = 2$
- **d** $\frac{1}{2}\sin(\theta) = -\frac{1}{2}$
- **e** $1 + \cos(\theta) = -1$
- **f** $4\sin(\theta) = -\frac{1}{4}$
- **g** $3 - \cos(\theta) = \frac{1}{3}$
- **h** $\frac{1}{2}\cos(\theta) = -\frac{1}{4}$
- **i** $\frac{1}{4} + \sin(\theta) = -1$

11 Take care when you consider the domain of a trigonometric equation. Don't always assume that a trigonometric equation has two or more solutions. Consider the following equations with the domains indicated. Write the number of solutions for each. (Don't write the values of the solutions.)

- **a** $\sin(\theta) = \frac{1}{2}, 0° \le \theta \le 90°$
- **b** $\sin(\theta) = -\frac{1}{2}, 0° \le \theta \le 90°$
- **c** $\cos(\theta) = -\frac{1}{2}, 0° \le \theta \le 180°$
- **d** $\tan(\theta) = 0.3, 0° \le \theta \le 180°$
- **e** $\sin(\theta) = -0.4, 0° \le \theta \le 270°$
- **f** $\cos(\theta) = 0.8, 0° \le \theta \le 270°$

12 If the pronumeral inside the brackets of the trigonometric function is multiplied by a coefficient, then start by redefining the specified domain in terms of the expression inside the brackets.

For example, if the equation is $\cos(3\theta) = \frac{1}{2}, 0° \le \theta \le 180°$

Then $\cos(3\theta) = \frac{1}{2}, 0° \le 3\theta \le 540°$

So, solving normally, $3\theta = 60°, 300°, 420°$

Therefore, $\theta = 20°, 100°, 210°$

Solve each equation for the domain $0° \le \theta \le 180°$.

- **a** $\sin(2\theta) = \frac{1}{\sqrt{2}}$
- **b** $\cos(4\theta) = -1$
- **c** $\tan(3\theta) = -1$

PROBLEM SOLVING AND REASONING

13 Solve each equation for the domain indicated.

a $2\sin(\theta) - 3 = \sin(\theta) - 2, 0^\circ \le \theta \le 180^\circ$

b $4\cos(\theta) + 5 = 4 + 5\cos(\theta), 0^\circ \le \theta \le 180^\circ$

c $\sqrt{2}\cos(\theta) + 1 = 0, 0^\circ \le \theta \le 270^\circ$

d $2\sin(\theta) + \sqrt{3} = 0, 0^\circ \le \theta \le 270^\circ$

e $4 - 4\tan(\theta) = 0, 90^\circ \le \theta \le 270^\circ$

f $\sqrt{48}\tan(\theta) + 2 = -3 - \sqrt{3}\tan(\theta), 90^\circ \le \theta \le 270^\circ$

14 Use the method shown in question **11** to solve each equation for the domain $0^\circ \le \theta \le 1080^\circ$.

a $\sin\left(\frac{\theta}{2}\right) = -\frac{1}{2}$

b $\cos\left(\frac{\theta}{3}\right) = \frac{1}{\sqrt{2}}$

c $\tan\left(\frac{2\theta}{5}\right) = -\sqrt{3}$

15 Explain why solving $\tan(\theta) = 1$ is different to solving $\sin(\theta) = 1$ and $\cos(\theta) = 1$.

16 Determine the sum of the solutions of the following equations for the domain $0^\circ \le \theta \le 360^\circ$.

a $\sin(\theta) = \frac{1}{2}$

b $\cos(\theta) = \frac{1}{2}$

c $\tan(\theta) = \frac{1}{\sqrt{3}}$

d $\sin(\theta) = -\frac{1}{2}$

e $\cos(\theta) = -\frac{1}{2}$

f $\tan(\theta) = -\frac{1}{\sqrt{3}}$

17 Solve each equation for the domain $-360^\circ \le \theta \le 0^\circ$.

a $\sin(\theta) = \frac{\sqrt{3}}{2}$

b $\cos(\theta) = -\frac{1}{\sqrt{2}}$

c $\tan(\theta) = \sqrt{3}$

d $\sin(\theta) = -1$

e $\cos(\theta) = 0$

f $\tan(\theta) = 0$

18 Trigonometric equations can be solved in radians as well as degrees. To determine whether you should write your answer in degrees or radians, use the same unit that the domain is written in. Solve the following equations for the domain $0 \le \theta \le 2\pi$.

a $\sin(\theta) = 0$

b $\cos(\theta) = \frac{\sqrt{3}}{2}$

c $\tan(\theta) = -1$

d $\sin(\theta) = \frac{1}{\sqrt{2}}$

e $\cos(\theta) = -1$

f $\tan(\theta) = \frac{1}{\sqrt{3}}$

19 Complete the following sentences:

a The solutions to a sine equation always lie on the same (horizontal, vertical or diagonal) line through the unit circle.

b The solutions to a cosine equation always lie on the same (horizontal, vertical or diagonal) line through the unit circle.

c The solutions to a tangent equation always lie on the same (horizontal, vertical or diagonal) line through the unit circle.

20 The equation $\sin(\theta) = \cos(\theta)$ can be solved by using the identity $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$.

a Show that $\sin(\theta) = \cos(\theta)$ is equivalent to $\tan(\theta) = 1$.

b Solve the following equations for $0^\circ \le \theta \le 360^\circ$.

i $\sin(\theta) = \cos(\theta)$

ii $\sin(\theta) = \sqrt{3}\cos(\theta)$

iii $\sqrt{3}\sin(\theta) + \cos(\theta) = 0$

iv $3\sin(\theta) - \sqrt{3}\cos(\theta) = 0$

CHALLENGE

21 Solve the following equations.

a $10\sin(2\theta - 30^\circ) + 5 = 0, 180^\circ \le \theta \le 540^\circ$

b $12\cos\left(\frac{\theta}{4} + \pi\right) - \sqrt{72} = 0, -5\pi \le \theta \le 12\pi$

22 a Solve the equation $2x^2 - 3x - 2 = 0$.

b Hence, solve the equation $2(\sin(\theta))^2 - 3\sin(\theta) - 2 = 0, 0^\circ \le \theta \le 360^\circ$.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Solving trigonometric equations

CAS instructions
Solving trigonometric equations

Topic quiz
8I

8J Trigonometric graphs

Learning intentions

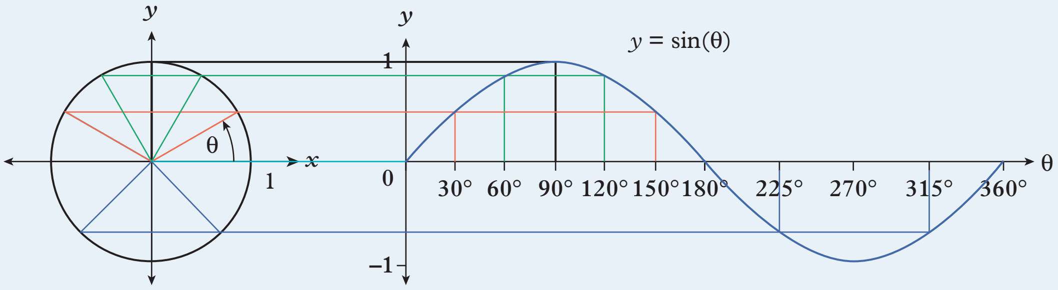
- ✔ I can read a point on a trigonometric graph.
- ✔ I can determine the period and amplitude of a trigonometric function.
- ✔ I can sketch graphs of simple trigonometric functions.

Inter-year links

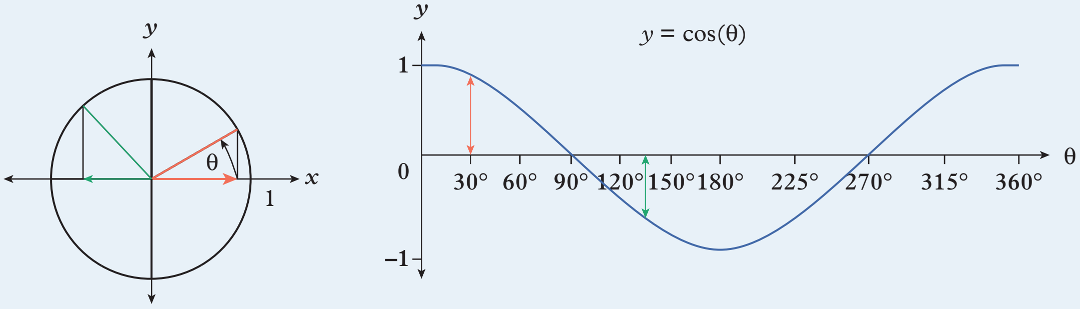
Year 9 5B Plotting quadratic relationships

Trigonometric functions

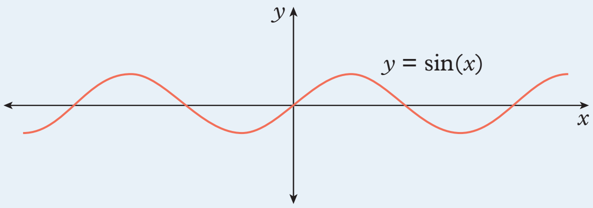
- A trigonometric function can be represented as a graph on the Cartesian plane.
- The x-axis represents the angle and the y-axis represents the value for the trigonometric ratio.
- The graph of $y = \sin(\theta)$ has y values corresponding to the y values of the unit circle for a given angle θ.

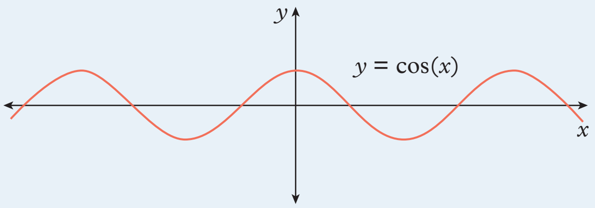


- The graph of $y = \cos(\theta)$ has y values corresponding to the x values of the unit circle for a given angle θ.



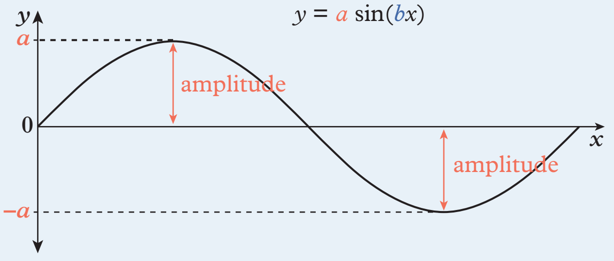
- For sine and cosine functions, the shape of the graph is called a sinusoid. They both look like waves with repeating peaks and troughs going along the horizontal axis.

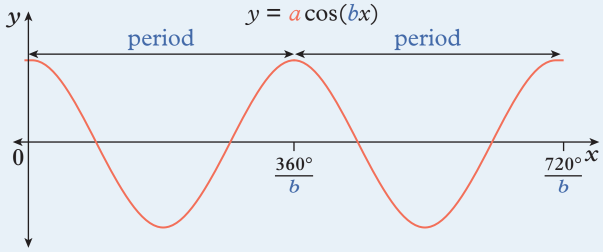




Amplitude and period

- The amplitude of a trigonometric function is the distance from the centre of the curve to the maximum/minimum y-value.
- For trigonometric functions of the form $y = a\sin(bx)$ or $y = a\cos(bx)$, the amplitude is equal to the positive value of a. That means, if $a = 2$ or $a = -2$, then the amplitude is 2.
- The **period** of a periodic function is the length of the interval along the x-axis that is required for the function to repeat itself.
- Trigonometric functions are periodic functions, because their output is repeated as the angle moves around the circle.
- For trigonometric functions of the form $y = a\sin(bx)$ or $y = a\cos(bx)$, the period is given by: period $= \dfrac{360°}{b}$





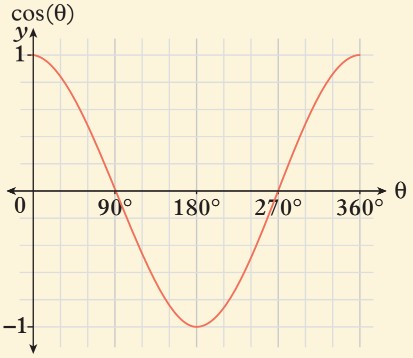
Sketching trigonometric graphs

- For trigonometric functions of the form $y = a\sin(bx)$ or $y = a\cos(bx)$:
  - → as the size of a increases, the amplitude increases, so the sinusoid is stretched on the y-axis.
  - → as b increases, the period decreases, so the sinusoid is compressed on the x-axis.
- For functions of the form $y = a\sin(bx)$, the y-intercept will be at $y = 0$.
- For functions of the form $y = a\cos(bx)$, the y-intercept will be at $y = a$.

Example 8J.1 Reading a trigonometric ratio from a trigonometric graph

Consider the graph of the cosine function on the right. Use the graph to read the cosine value for each of these angles. Give your answers as approximate values, if necessary.

a 360° **b** 80°

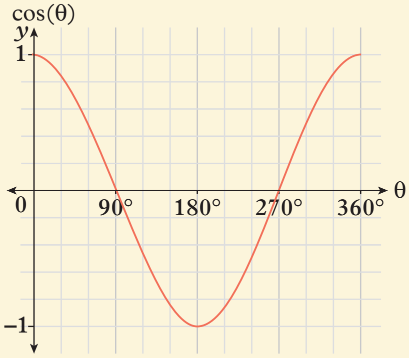


THINK	WRITE
a There is a maximum on the graph at θ = 360°, so the vertical ordinate is exactly 1.	**a** (360°, cos(360°)) = (360°, 1) So: cos(360°) = 1
b Identify an approximate coordinate for the graph just to the left of (90°, 0).	**b** (80°, cos(80°)) ≈ (80°, 0.2) So: cos(80°) ≈ 0.2

Example 8J.2 Reading an angle from a trigonometric graph

Consider this graph of the cosine function on the right. Use the graph to find the angles from 0° to 360° that have the following cosine values. Give your answers to the nearest 10°.

a 0.6 **b** -0.2



THINK	WRITE
a **1** There are two coordinates for the graph at $\cos(\theta) = 0.6$ near 50° and 310°.	**a** $(\theta, 0.6) \approx (50°, 0.6)$ or $(310°, 0.6)$
2 Check that the approximated values for θ have the same cosine value by checking the reference angle and sign are the same.	1st quadrant: $\theta' = \theta$ $= 50°$ 4th quadrant: $\theta' = 360° - \theta$ $= 360° - 310°$ $= 50°$ Cosine is positive in the first and fourth quadrant, so $\cos(50°) = \cos(310°)$. So $\theta \approx 50°$ or $310°$
b **1** There are two coordinates for the graph at $\cos(\theta) = -0.2$ near 100° and 260°.	**b** $(\theta, 0.2) \approx (100°, -0.2)$ or $(260°, -0.2)$
2 Check that the approximated values for θ have the same cosine value by checking the reference angle and sign are the same.	2nd quadrant: $\theta' = 180° - \theta$ $= 180° - 100°$ $= 80°$ 3rd quadrant: $\theta' = \theta - 180°$ $= 260° - 180°$ $= 80°$ Cosine is negative in the second and third quadrant, so $\cos(100°) = \cos(260°)$. So $\theta \approx 100°$ or $260°$

Example 8J.3 Determining the amplitude and period of a trigonometric function

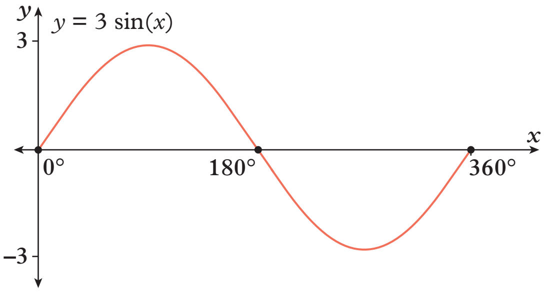
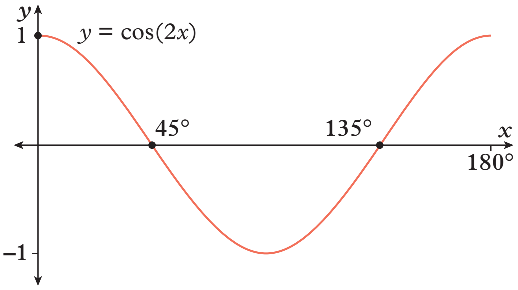
Determine the amplitude and period of $y = -\frac{1}{2}\sin(6x)$.

THINK	WRITE
1 Identify the amplitude as the positive value of the coefficient of the sine function.	amplitude $= \frac{1}{2}$
2 Write the period as 360° divided by the coefficient in front of x and simplify.	period $= \frac{360°}{6}$ $= 60°$

Example 8J.4 Sketching trigonometric graphs

Sketch the graph of each of these functions over a one-period interval.

a $y = 3\sin(x)$ **b** $y = \cos(2x)$

THINK	WRITE
a **1** Identify the amplitude as the coefficient in front of the sine function.	**a** amplitude = 3
2 Identify the period.	period = 360°
3 This is a sine function, so sketch a sinusoid starting at the point (0, 0) and extending for at least one period.	
4 Use the amplitude to label the y-axis with the maximum and minimum values. For a sine function the x-intercepts will be every half period starting from 0.	
b **1** Identify the amplitude as the coefficient in front of the sine function.	**b** amplitude = 1
2 Calculate the period.	period $= \dfrac{360°}{2}$ $= 180°$
3 This is a cosine function, so sketch a sinusoid starting with a maximum at $x = 0$ and extending for at least one period.	
4 Use the amplitude to label the y-axis with the maximum and minimum values. For a cosine function the x-intercepts will be every half period starting from the first quarter of a period.	

Helpful hints

- ✔ Remember to consider all 4 quadrants of the unit circle and convert to the correct reference angle when trying to determine the angles for a given y value on a trigonometric graph. There will usually be 2 angles with the same y value per period.
- ✔ Keep in mind that $\sin(30°) = \frac{1}{2}$ and $\cos(60°) = \frac{1}{2}$. That is, the graphs do not equal half the amplitude at the halfway point between the x-axis and the maximum/minimum.
- ✔ Any sine or cosine will essentially be the same shape when graphed. The only things you need to adjust are the dilations of the amplitude and period.

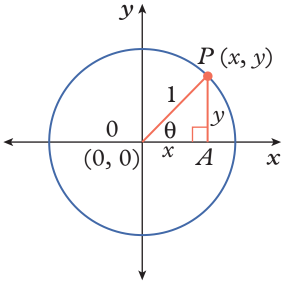
ANS p771

Exercise 8J Trigonometric graphs

△ 1–8, 9(b–e), 10(b–e), 11, 14 □ 3–6, 8, 9–10(c–f), 13, 15, 16 ⬠ 5, 6, 8, 9–10(d–f), 12, 13, 15–17

UNDERSTANDING AND FLUENCY

1 Trace the point P around the unit circle in an anticlockwise direction from its starting point on the positive x-axis back to its starting point. As you trace, consider the size of angle θ as P moves. Also consider what happens to the x-coordinate for $P(x, y)$ as it moves along the x-axis.



a As you traced anticlockwise around the unit circle, you will have noticed that the size of angle θ increased from 0° to 360°, and that the x-coordinate of point P decreased as it moved from 0° to 90°. Describe what happened to the x-coordinate of P when the size of the angle moved between:

i 90° and 180° ii 180° and 270° iii 270° and 360°

b The change in x value from part **a** is the change in the cosine of angle θ as it increases from 0° to 360°. Complete this table with the aid of your calculator to obtain a clearer picture of the values for cosine as the angle increases from 0° to 360°. Write your answers correct to one decimal place.

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
cos(θ)													

c Plot the points from the table on grid or graph paper, with the size of angle θ along the horizontal axis scale and the cosine value, cos(θ), up the vertical axis scale. Join your points with a smooth curve.

d Sketch on graph paper what would happen if you extended the angle values to 720°.

e What is the value of cos(θ) at its first and second maximum. What is the period of the cosine graph?

f Note the difference between the maximums and the minimums of the graph. What is the amplitude of the graph?

g What is the y-intercept of the cosine graph?

2 Consider the change in the y-coordinate as point $P(x, y)$ moves and angle θ increases in size from 0° to 360°. This represents the change in the sine value of the angle.

a Repeat the procedure from question **1b**, constructing a table for the sine values, to obtain a clearer picture of the values for sine as the angle sweeps from 0° to 360°.

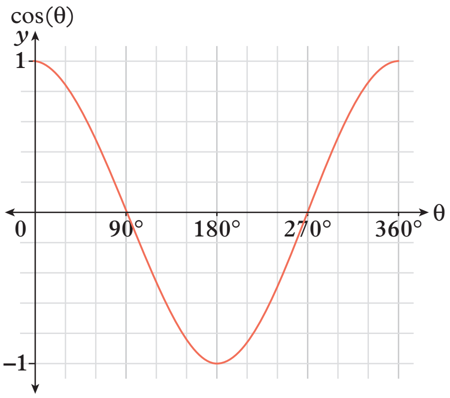
b Sketch the sine graph from 0° to 360°.

c What is the period of the graph?

d What is the amplitude of the graph?

e What is the y-intercept of the sine graph?

UNDERSTANDING AND FLUENCY



8J.1 **3** Consider the graph on the right of the cosine function.

a From the graph, write the cosine value for each of these angles.

i 0° **ii** 90°

iii 270° **iv** 180°

b Use the graph on the right to help you write the approximate cosine value (correct to one decimal place) for each of these angles.

i 40° **ii** 100° **iii** 340°

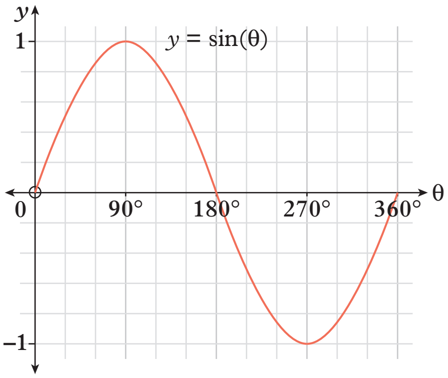
iv 200° **v** 280° **vi** 140°

8J.2 **4** Use the graph from question **3** to help you find the angles from 0° to 360° with the following cosine values. Give your answers as approximate values, if necessary.

a 1 **b** –1 **c** 0

d 0.4 **e** –0.6 **f** 0.8

5 Consider the graph of the sine function on the right.



a From the graph, write the sine value for each of these angles.

i 360° **ii** 90°

iii 270° **iv** 180°

b Use the graph to help you write the approximate sine value (correct to one decimal place) for each of these angles:

i 40° **ii** 100°

iii 350° **iv** 190°

v 280° **vi** 140°

6 Use the graph from question **5** to help you find the angles from 0° to 360° with the following sine values. Give your answers as approximate values, if necessary.

a 1 **b** –1 **c** 0

d 0.4 **e** –0.6 **f** 0.8

7 Consider the function $y = 2\sin(3x)$. Calculate the y value for each of the following x values and plot each pair of coordinates on a Cartesian plane. Draw a line to sketch the graph.

a Complete the table of values below. The first two columns have been completed for you.

b Plot each point (x,y), on a Cartesian plane.

c Sketch the graph of $y = 2\sin(3x)$ for $0° \leq x \leq 240°$.

3*x*	0°	90°	180°	270°	360°	450°	540°	630°	720°
x	0°	30°							
sin(3*x*)	0	1							
***y* = 2 sin(3*x*)**	0	2							

8J.3 **8** Determine the amplitude and period in radians for each of the following.

a $y = 3\sin(2x)$ **b** $y = 5\cos(3x)$ **c** $y = -4\sin\left(\frac{x}{2}\right)$ **d** $y = -6\cos\left(\frac{x}{6}\right)$

e $y = \frac{3}{4}\sin\left(\frac{3x}{4}\right)$ **f** $y = \frac{5}{3}\cos\left(\frac{3x}{5}\right)$ **g** $y = -\frac{7}{8}\sin\left(\frac{4x}{7}\right)$ **h** $y = \frac{5}{12}\cos\left(\frac{24x}{5}\right)$

8J.4 **9** Sketch the graphs of the following equations for 0°≤ x ≤ 360°.

a $y = 4\sin(x)$ **b** $y = \sin(4x)$ **c** $y = \sin\left(\frac{x}{4}\right)$

d $y = -4\sin(x)$ **e** $y = 5\sin(2x)$ **f** $y = -3\sin\left(\frac{x}{2}\right)$

10 Sketch the graphs of the following equations for 0°≤ x ≤ 360°.

a $y = 3\cos(x)$ **b** $y = \cos(3x)$ **c** $y = -3\cos(x)$

d $y = \cos\left(\frac{x}{3}\right)$ **e** $y = -\frac{1}{2}\cos(4x)$ **f** $y = 7\cos\left(\frac{x}{2}\right)$

PROBLEM SOLVING AND REASONING

11 The graphs of the sine function and the cosine function are quite similar. If you were presented with a graph of $y = \sin(x)$ and $y = \cos(x)$, and the functions weren't labelled, how could you tell which graph was which?

12 A grandfather clock has a pendulum that swings back and forth in a regular periodic motion. This motion is so regular that it has been the same method used, for hundreds of years, for keeping time. The weight of the bob on the end of the pendulum does not affect the period, but the period is affected by the length of the pendulum.

a Explain why the motion of a pendulum is periodic.

b The equation used to find the period of the motion of a pendulum is $T = \pi\sqrt{\frac{l}{g}}$, where T is the time (in seconds), l is the length of the pendulum (in metres) and g is the acceleration due to gravity (9.8 m/s^2).

Find the period of a pendulum with each of the following lengths:

i 1 m **ii** 1.5 m **iii** 2 m

c Explain how the length of a pendulum affects the period of its motion.

13 Periodic functions occur in many everyday activities. Consider this table of the sunrise and sunset times recorded for the east coast of Australia on the first day of the month throughout a year. The times are recorded as Eastern Standard Time.

	Jan	**Feb**	**Mar**	**Apr**	**May**	**Jun**	**Jul**	**Aug**	**Sep**	**Oct**	**Nov**	**Dec**
Sunrise (am)	4:55	5:20	5:40	5:57	6:12	6:29	6:38	6:29	6:02	5:28	4:57	4:44
Sunset (pm)	6:46	6:42	6:20	5:46	5:17	5:01	5:04	5:19	5:34	5:47	6:05	6:28

a Plot the sunrise and sunset figures from the table on separate graphs, placing the sunrise graph directly below the sunset graph. Use the horizontal axis scale for the months and the vertical axis scale for the times.

b Describe the shapes of the two curves you graphed.

c Are these two curves similar to those for the sine and cosine functions? Explain any similarities or differences.

d Are the curves periodic? Explain why or why not.

e Because of the way you positioned these two graphs, the trend in daylight hours throughout the year is visually apparent. Explain how this is so.

f If you were not told that these were figures for the east coast of Australia, how could you reason that they must be figures for somewhere in the Southern Hemisphere?

14 State the number of cycles (a repetition of a sinusoid) the graphs of each of the following equations will have in the domain 0°≤ x ≤ 360°.

a $y = \sin(3x)$ **b** $y = \frac{1}{2}\sin(11x)$ **c** $y = 5\sin\left(\frac{x}{2}\right)$ **d** $y = a\sin(bx)$

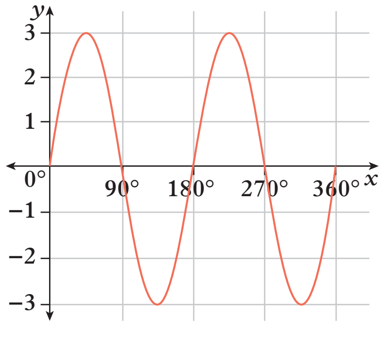
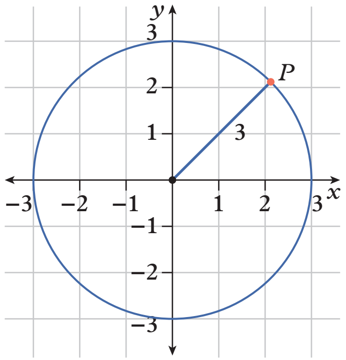
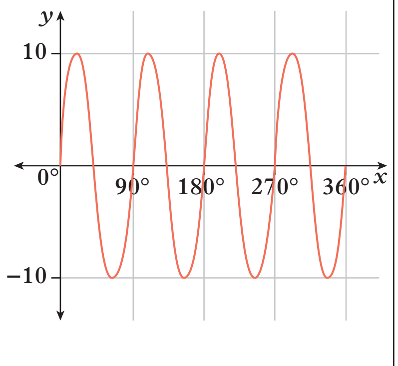
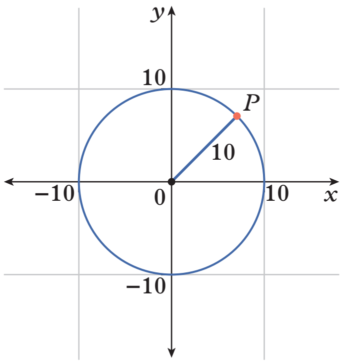
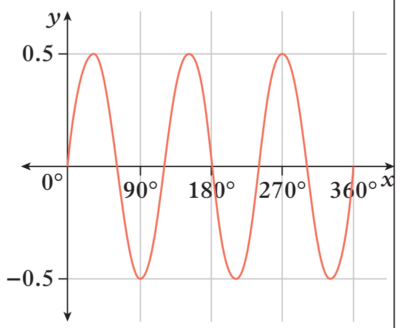
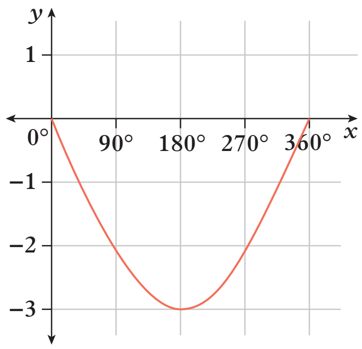
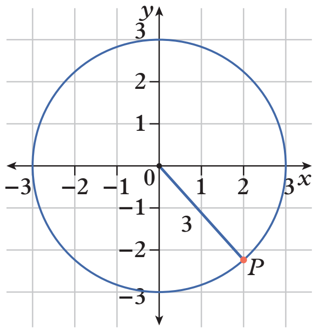
e $y = \cos(4x)$ **f** $y = -4\cos(90x)$ **g** $y = \frac{5}{2}\cos\left(\frac{3x}{2}\right)$ **h** $y = a\cos(bx)$

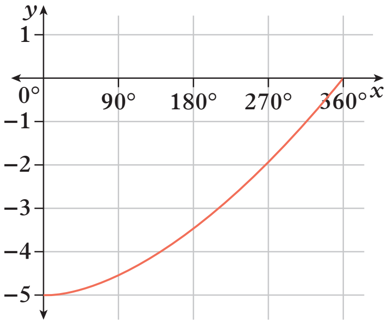
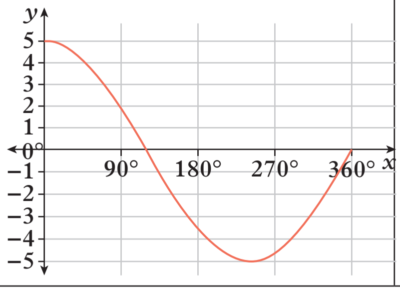
15 The graphs for sine and cosine can be considered a point moving at a constant speed anticlockwise around a circle. The amplitude and period, then, have analogues on the circle that they are related to.

a Complete the table below.

b Determine what feature of the circle the amplitude is related to and how.

c Determine what feature of the circle the period is related to and how.

	Graph	Circle	Radius	Speed of point (revolutions per 360°)
			3	2
i				
ii				
iii				

	Graph	Circle	Radius	Speed of point (revolutions per 360°)
iv			5	
v			3	5
vi				$\frac{3}{4}$

16 The tangent ratio is also a periodic function.

a Complete a table of values, correct to one decimal place, for θ values from 0° to 360° as shown.

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
tan(θ)																	

b Plot these points on a Cartesian plane and join them with a smooth curve.

c **i** Describe the shape of the graph.

ii It is different from the sine curve and the cosine curve. Why is this so?

iii At θ = 90°, a vertical asymptote occurs. Explain what you understand this to mean. Where does the next asymptote occur?

iv Explain why the graph is periodic. What is its period?

v What is the amplitude of the graph?

17 Like other graphs, the graphs of trigonometric functions can be translated vertically and horizontally.

a Complete the table of values below for $y = \sin(x - 30°) + 4$.

x − 30	0°	90°	180°	270°	360°
x	30°	120°			
sin(x − 30°)	0	1			
y = sin(x − 30°) + 4	4	5			

b Sketch the graph of $y = \sin(x - 30°) + 4$ for $30° \le x \le 390°$.

c Sketch the graph of $y = \cos(x - 45°) - 3$ for $45° \le x \le 405°$.

Check your Student obook pro for these digital resources and more:

pro

Interactive skillsheet
Period of a trigonometric function

Interactive skillsheet
Amplitude of a trigonometric function

Interactive skillsheet
Sketching trigonometric graphs

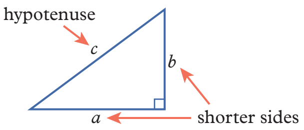
Topic quiz
8J

Chapter summary

Pythagoras' Theorem

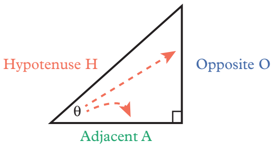
$$c^2 = a^2 + b^2$$

hypotenuse — shorter sides



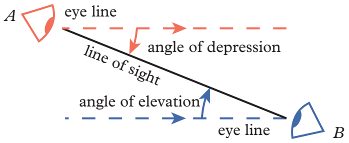
Trigonometric ratios

- $\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{O}{H}$
- $\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{A}{H}$
- $\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{O}{A}$

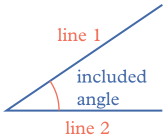


Angles of elevation and depression

- The angle of depression from A to B is equal to the angle of elevation from B to A.



Included angle area formula



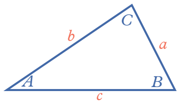
$$\text{area} = \frac{1}{2}bc\sin(A)$$

adjacent side lengths — included angle

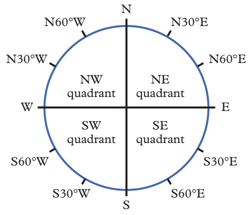
The sine and cosine rules

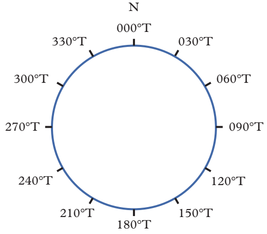
$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$



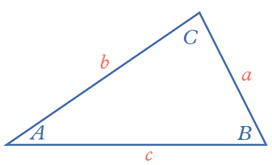
Compass bearings and true bearings

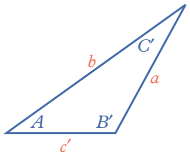




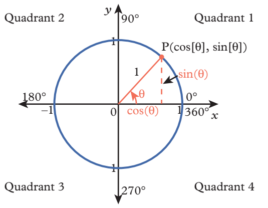
Ambiguous case of the sine rule

- If two side lengths (a and b) are known and the angle (A) opposite the shorter side length is also known, then there are two possible triangles.





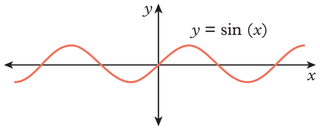
The unit circle

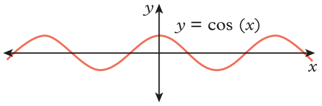


- The x-coordinate of the point on the unit circle is cos (θ).
- The y-coordinate of the point on the unit circle is the sine of θ.
- The gradient of the radial arm is the tangent of θ.
- Every angle has a corresponding reference angle in quadrant 1.

	Quadrant 1	Quadrant 2	Quadrant 3	Quadrant 4
Angle (θ)	$0° < \theta < 90°$	$90° < \theta < 180°$	$180° < \theta < 270°$	$270° < \theta < 360°$
Reference angle (θ′)	$\theta' = \theta$	$\theta' = 180° - \theta$	$\theta' = \theta - 180°$	$\theta' = 360° - \theta$

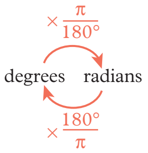
Trigonometric graphs



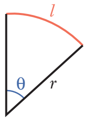


- The values of trigonometric functions can be graphed on a Cartesian plane, where the value on the horizontal axis is the angle.

Radians



- One revolution (360°) is equal to 2π radians.



$$l = r\theta$$

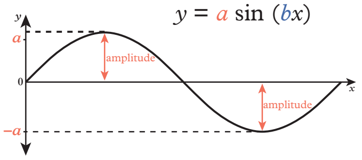
arc length — radius — size of angle in radians

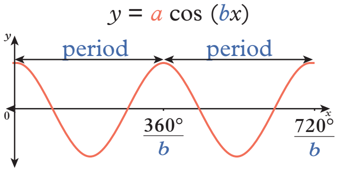
Exact values

θ	30° or $\frac{\pi}{6}$	45° or $\frac{\pi}{4}$	60° or $\frac{\pi}{3}$
sin (θ)	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cos (θ)	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tan (θ)	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Amplitude and period

- For $y = a\sin(bx)$ and $y = a\cos(bx)$, the amplitude is the positive a value and the period is the positive $\frac{360°}{b}$ value.





Chapter review

Chapter review quiz
Take the chapter review quiz to assess your knowledge of this chapter.

Quizlet
Test your knowledge of this topic by working individually or in teams.

Multiple choice

8A **1** Which of these describes a triangle with side lengths measuring 4 cm, 5 cm and 6 cm?

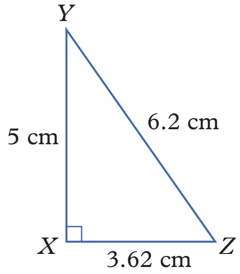
A right-angled **B** obtuse-angled **C** acute-angled **D** isosceles **E** equilateral

8B **2** The ratio $\frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$ of a right-angled triangle is known as which of the following?

A sine **B** cosine **C** tangent **D** π **E** none of these

8B **3** What is the size of $\angle YZX$, to the nearest degree?

A 54° **B** 55° **C** 35° **D** 36° **E** 90°



8B **4** Which of these ratios represents the sine of $\angle ZYX$?

A $\frac{5}{6.2}$ **B** $\frac{3.62}{6.2}$ **C** $\frac{3.62}{5}$ **D** $\frac{5}{3.62}$ **E** $\frac{6.2}{3.62}$

8C **5** What is the equivalent true bearing of N12°W?

A 348°T **B** 012°T **C** 192°T **D** 168°T **E** 078°T

10A 8D **6** Which of these values is closest to the diagonal length within a cube of side length 5 cm?

A 5 cm **B** 8 cm **C** 9 cm **D** 10 cm **E** 7 cm

10A 8E **7** What is the size of angle A in $\frac{10}{\sin(80°)} = \frac{9}{\sin(A)}$ closest to?

A 55° **B** 60° **C** 65° **D** 115° **E** 130°

10A 8F **8** The cosine rule is used to calculate the side length of a non-right-angled triangle ABC. If you know the value of angle B, what do you also need to know the value of?

A a and b **B** a and c **C** c and b **D** angle A **E** angle A and a

10A 8G **9** Which of the following has a different acute reference angle to the others?

A 330° **B** $\frac{5\pi}{6}$ **C** 210° **D** $\frac{13\pi}{6}$ **E** 120°

10A 8G **10** If a point P on the unit circle in quadrant 2 is given in terms of an acute reference angle, θ, then which of these coordinates for P are correct?

A $(-\cos(\theta), \sin(\theta))$ **B** $(\cos(\theta), -\sin(\theta))$ **C** $(-\cos(\theta), -\sin(\theta))$ **D** $(\cos(\theta), \sin(\theta))$ **E** $(-\sin(\theta), \cos(\theta))$

10A 8H **11** Which of these is the exact sine value for 30°?

A $\frac{1}{2}$ **B** $\frac{\sqrt{3}}{2}$ **C** $\frac{1}{\sqrt{3}}$ **D** $\frac{1}{\sqrt{2}}$ **E** $\sqrt{3}$

10A 8I **12** How many solutions does $\sin(\theta) = -\frac{1}{2}$ have in the domain $0° \leq \theta \leq 720°$?

A 0 **B** 1 **C** 2 **D** 3 **E** 4

10A 8J **13** The graph of $y = \sin(\theta)$ has what period?

A 90° **B** 180° **C** 270° **D** 360° **E** 720°

Short answer

For these questions, calculate each angle correct to the nearest degree and each length correct to two decimal places (unless stated otherwise).

8A **1** An equilateral triangle has side lengths of 5 cm. Find the height of the triangle as:

a an exact value **b** an approximate value, correct to one decimal place.

8B **2** A rectangle has a diagonal length of 15.8 cm. The diagonal forms an angle of 31.3° with the longer side. What are the dimensions of the rectangle?

8C **3** A map shows a stretch of road as a horizontal distance of 18.5 km. The road rises 6.2 km from start to finish.

- **a** Find the angle of the rise of the road, to the nearest degree.
- **b** Find the average gradient of the road as a:
  - **i** ratio
  - **ii** percentage.
- **c** What is the actual distance you would travel along this stretch of road?

8C **4** Two buildings stand 20 m apart. One is taller than the other. From the top of the taller building to the base of the shorter building, the angle of depression is 75°. From the top of the shorter building to the top of the taller building, the angle of elevation is 60°. What is the height of the shorter building?

10A 8D **5** Find the length of the longest diagonal within each 3D object.

- **a** a cylinder with base circumference of 20 cm and height of 15 cm
- **b** a box 10 cm long, 8 cm wide and 15 cm tall
- **c** a cube of side length 25 cm.

10A 8D **6** Calculate the angle each diagonal you calculated in question **5** makes with the height of the object.

10A 8E **7** Standing in a park, Anabel takes a bearing of 065°T to a tree. She walks 15 m due east and finds the bearing to the same tree to be 295°T.

- **a** Draw a labelled diagram to represent this situation.
- **b** Find the distance from the tree to Anabel, in both positions.

10A 8F **8** In ΔABC, $a = 5$ cm, $b = 6.1$ cm and $A = 39°$.

- **a** Explain why two distinct triangles can be drawn with these measurements. Draw them.
- **b** Find the dimensions of all the angles and sides of the two triangles you drew in part **a**.

10A 8F **9** A triangular garden bed in a park is 50 m by 40 m by 20 m.

- **a** Draw a sketch of the shape of the garden bed.
- **b** Find the size of the angle at each vertex of the triangular shape.
- **c** Find the area of the garden bed.
- **d** What is the shortest distance from the vertex opposite the longest side to the longest side?

10A 8G **10** **a** Convert the following angles to radians.

- **i** 126°
- **ii** 315°
- **iii** 130°

b Convert the following angles to degrees.

- **i** $\frac{2\pi}{5}$
- **ii** $\frac{8\pi}{3}$
- **iii** $\frac{9\pi}{8}$

10A 8H **11** Show how the unit circle can be used to find each of these values by drawing the corresponding points on a unit circle and labelling their coordinates.

- **a** $\sin(300°)$
- **b** $\cos(225°)$
- **c** $\sin(135°)$
- **d** $\cos\left(\frac{7\pi}{4}\right)$
- **e** $\sin\left(\frac{4\pi}{3}\right)$
- **f** $\cos\left(\frac{11\pi}{6}\right)$

10A 8I **12** Solve each of these equations for the domain $0° \leq \theta \leq 360°$.

- **a** $\cos(\theta) = -1$
- **b** $\sin(\theta) = 0$
- **c** $\cos(\theta) = \frac{\sqrt{3}}{2}$
- **d** $\sin(\theta) = -\frac{1}{2}$
- **e** $\cos(\theta) = -\frac{1}{\sqrt{2}}$
- **f** $\sin(\theta) = -\frac{\sqrt{3}}{2}$

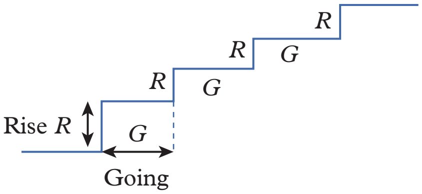
10A 8J **13** Sketch the graph of the following functions for the domain $0° \leq \theta \leq 360°$.

- **a** $y = -3\sin(x)$
- **b** $y = \cos(2x)$
- **c** $y = 4\sin\left(\frac{x}{2}\right)$
- **d** $y = -2\cos(3x)$

Analysis

1 There are building code regulations for the construction of staircases. For safety purposes, the tread cannot be too narrow, nor the staircase too steep. There are particular terms for staircases, as listed below and shown in the diagram on the right.

- The rise (R) must have a minimum height of 115 mm and a maximum height of 190 mm.
- The going (G) must have a length within the range of 250–355 mm.
- The regulation also states that $2R + G$ must lie within the range of 550–700 mm.



a Consider a staircase with the minimum rise measurement of 115 mm and the minimum going measurement of 250 mm.

  i Explain whether this staircase would be within the building regulation guidelines.

  ii What is the slope of the staircase described? Give your answer to the nearest degree.

b Repeat part **a** using the maximum values allowed for the rise and going measurements.

c If the going measurement is the maximum value allowed, of 355 mm, and the value of $2R + G$ falls within the required limits:

  i find a range of acceptable values for the rise

  ii calculate the range of acceptable slope values.

d The number of steps is counted by the number of going from the bottom to the top, starting with a going and ending with a going. Draw a set of five stairs with a going measurement of 300 mm and a rise of 150 mm.

  i What is the total horizontal distance of the five stairs?

  ii Find the total vertical height of the five stairs.

  iii Calculate the direct distance between the base of the first step and the total vertical height calculated in part **ii**.

  iv What is the slope of the staircase?

e What do you consider would be ideal measurements for the going and rise of a comfortable staircase? Draw a sketch of your design, show that it falls within the regulation guidelines, and provide its slope.

f There are different building code regulations for wheelchair ramps. The slope of the ramp is of particular importance, because a wheelchair may tip backwards or run away if the ramp is too steep. The code specifies that an internal ramp should be no steeper than 1 to 12, while an external ramp should have a gentler slope of 1 to 20.

  i Draw diagrams of 20 m long internal and external ramps with these slopes.

  ii For each ramp you drew in part **i**, find the going and rise measurements, rounding to three decimal places. Mark these measurements on your diagrams.

10A **2** Emma rides her bicycle at a constant speed with a tag attached to a spoke at the rim of the wheel.

Tom uses a video to record the height of the tag with respect to its starting position. He also made a record of the results in the table below.

Time (s)	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Height (cm)	20	30	20	0	–20	–30	–20	0	20	30

a Graph the data from the table, showing time on the x-axis.

b Suggest whether the graph models a sine curve or a cosine curve.

c How long does it take for the wheel to rotate once?

d What was the starting position of the tag on the wheel? Explain how you deduced this.

e What is the radius of the wheel?

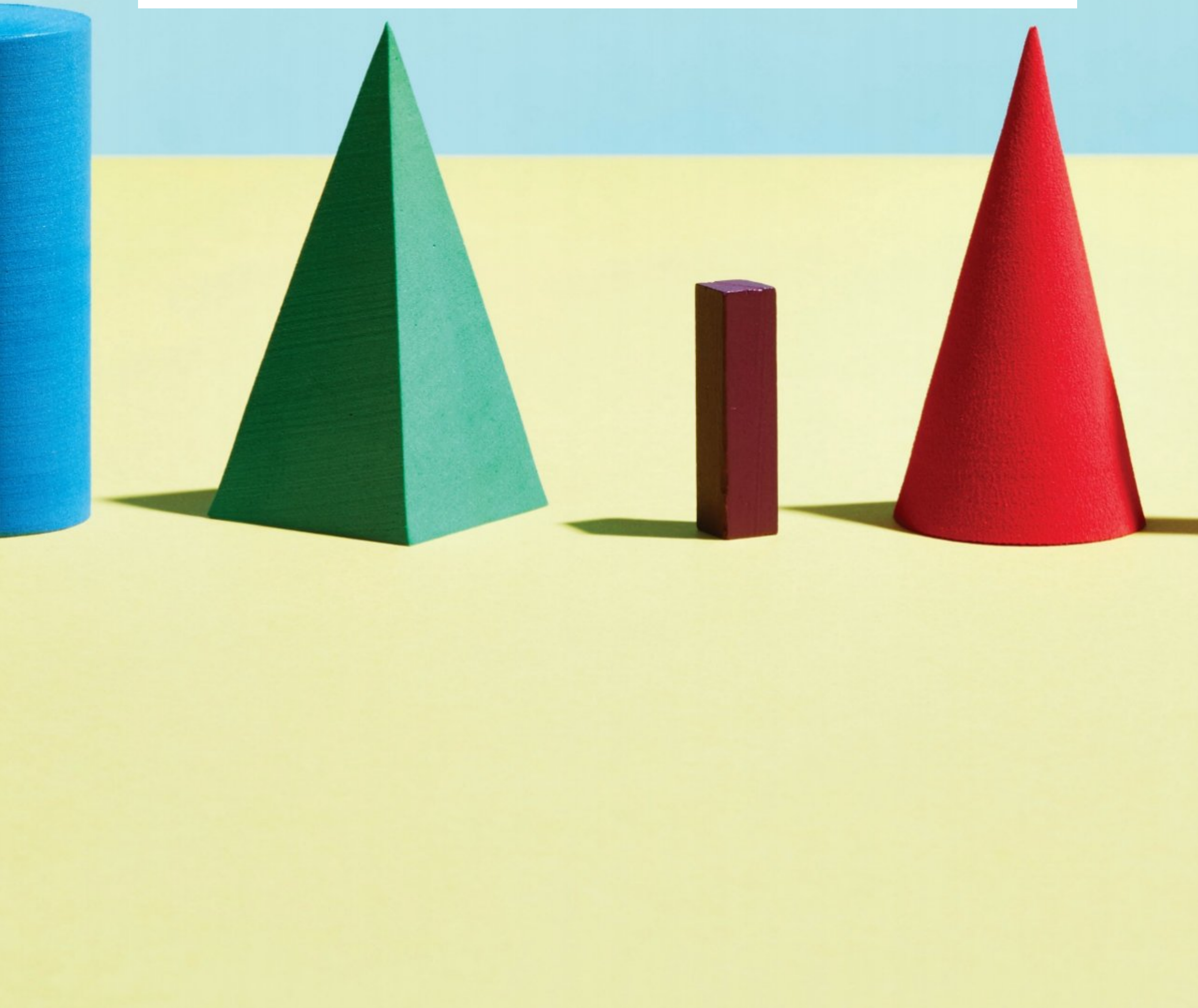
f Calculate the circumference of the wheel.

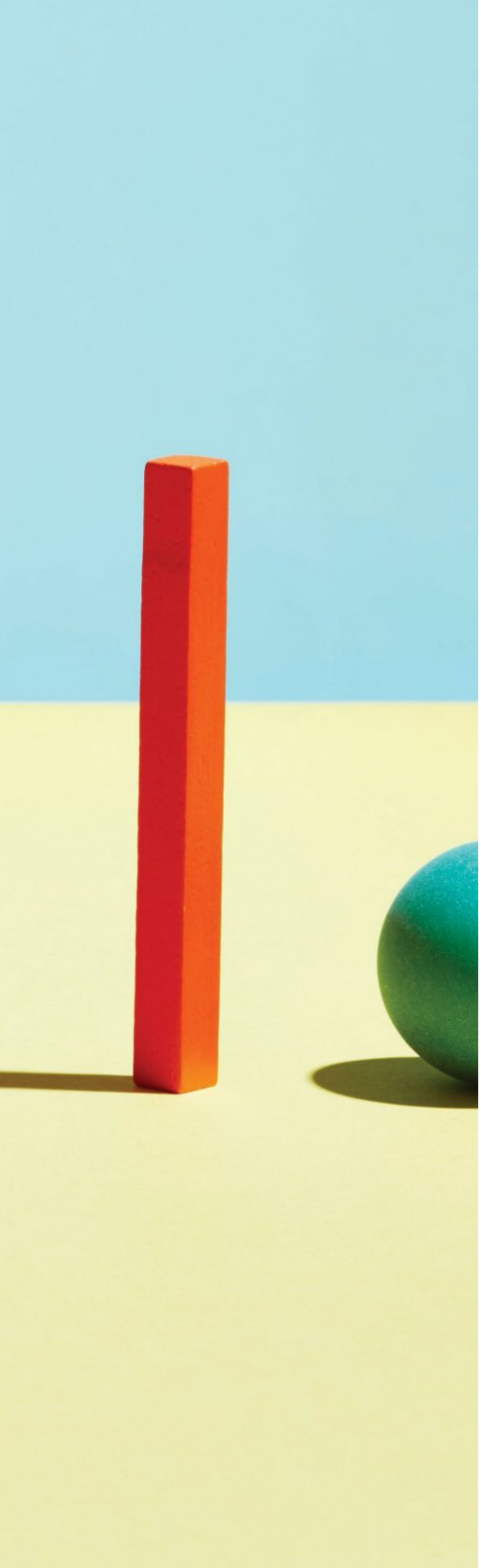
g At what speed does Emma ride? Give your answer in metres per second.

h Draw a diagram to show the position of the tag after 10 seconds of riding at this constant speed. Explain how you arrived at this answer.

9

Measurement





Index

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Converting between units of length
- ✔ Identifying shapes within composite shapes
- ✔ Area of a triangle
- ✔ Area of quadrilaterals
- ✔ Area of a circle

Curriculum links

- Solve problems involving surface area and volume for a range of prisms, cylinders and composite solids (VCMMG343)
- Solve problems involving surface area and volume of right pyramids, right cones, spheres and related composite solids (VCMMG365) [10A]

Materials

✔ Calculator

9A Area review

Leaning intentions

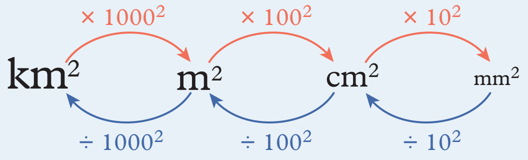
✔ I can calculate the area of composite shapes.
✔ I can calculate the area of sectors.

Inter-year links

Year 5/6	Understanding area
Year 7	9D Area of a triangle
Year 8	8E Area of a circle
Year 9	6A Area of composite shapes

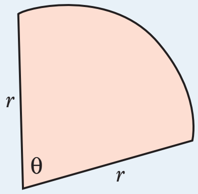
Units of area

- A unit of **area** can be converted to another unit of area by multiplying or dividing by the appropriate conversion factor.
- 1 ha (hectare) = 100 m × 100 m
  $= 10\,000\ \text{m}^2$
  Alternatively, 1 km^2 = 100 ha



Composite shapes

- To find the area of a **composite shape**:
  1 split the shape into individual parts
  2 calculate any missing dimensions
  3 calculate the areas of the individual parts
  4 add or subtract the areas to find the total area.
- The area of the sector of a circle is $\frac{\theta}{360°} \times \pi r^2$, where θ is the angle, in degrees, between the two radii forming the sector.



Example 9A.1 Converting units of length and area

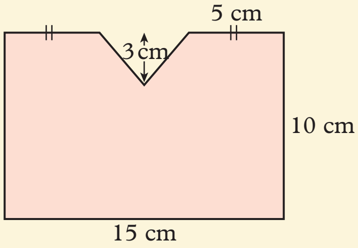
Convert each length or area measurement below to the unit shown in brackets.

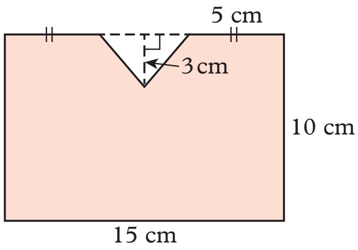
a 5.6 m (cm) **b** 120 mm (m) **c** 1.2 km^2 (m^2) **d** 320 mm^2 (cm^2)

THINK	**WRITE**
a Identify the correct conversion factor. To convert from metres (m) to centimetres (cm), multiply by 100.	**a** 5.6 m = (5.6 × 100) cm = 560 cm
b Identify the correct conversion factor. To convert from millimetres (mm) to metres (m), first divide by 10 (to convert to centimetres), then divide by 100 (to convert to metres).	**b** 120 mm = (120 ÷ 10 ÷ 100) cm = 0.12 m
c Identify the correct conversion factor. To convert from square kilometres (km^2) to square metres (m^2) multiply by 1000^2.	**c** 1.2 km^2 = (1.2×1000^2) m^2 = 1 200 000 m^2
d Identify the correct conversion factor. To convert from square centimetres (cm^2) to square millimetres (mm^2), divide by 10^2.	**d** 320 mm^2 = ($320 \div 10^2$) cm^2 = 3.2 cm^2

Example 9A.2 Calculating the area of a composite shape

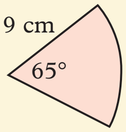
Calculate the area of this composite shape.



THINK	**WRITE**
1 Redraw the shape with dashed lines, splitting it into individual shapes for which you can calculate the area. **2** Label any unlabelled dimensions you know for each individual shape. The total length of the rectangle is 15 cm and the length either side of the triangle is 5 cm. The cut-out triangle must have a base of 5 cm (because 15 – 2 × 5 = 5).	
3 Calculate the area of each individual shape.	rectangle: $A = lw$ $= 10 \times 15$ $= 150\ \text{cm}^2$ triangle: $A = \frac{1}{2}bh$ $= \frac{1}{2} \times 5 \times 3$ $= 7.5\ \text{cm}^2$
4 Calculate the required area by subtracting the area of the triangle from the area of the rectangle.	total area $= 150 - 7.5$ $= 142.5\ \text{cm}^2$

Example 9A.3 Calculating the area of a sector

Calculate the area of this sector correct to one decimal place.



THINK	**WRITE**
1 Identify the values of θ and *r*.	$\theta = 65°$ $r = 9$ cm
2 Substitute the values into the formula for the area of a sector.	$A = \frac{\theta}{360°} \times \pi r^2$ $= \frac{65°}{360°} \times \pi \times 9^2$
3 Calculate the area, correct to one decimal place, including the appropriate unit.	$= 45.945...$ $\approx 45.9\ \text{cm}^2$

Helpful hints

- ✔ There can be more than one way to split a composite shape into individual component shapes. But the way you split up the shape should not change the total area.
- ✔ When being asked to round to a given number of decimal places, the rounding should only take place at the final stage of your work and not in any intermediate steps. For example, if you are calculating the area of a composite shape, use the full numbers on your calculator when finding the areas of the individual shapes and only round after adding or subtracting.

ANS p776

Exercise 9A Area review

△ 1–3, 4(a, c, e), 5, 6(a, c, d), 7–10 ■ 1, 3(c, e, f), 4(b, d, f), 6(b, d, f), 7, 8, 11, 13, 15 ⬟ 1, 3(c, f), 4(d, f), 6(d, f), 8, 10, 12–16

For questions involving π, use the π key on your calculator.

9A.1 **1** Convert each length or area measurement below to the unit shown in brackets.

a 7 m (cm) **b** 4500 m (km) **c** 8.4 km^2 (m^2)

d 29 mm^2 (cm^2) **e** 24 000 m^2 (ha) **f** 0.88 m (mm)

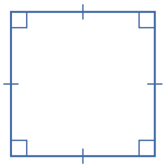
g 34 200 000 cm^2 (km^2) **h** 10.22 km (cm) **i** 3 km^2 (ha)

2 Match each shape below with a formula from this list that could be used for finding its area:

a $A = \pi r^2$ **b** $A = l^2$ **c** $A = \frac{1}{2}(a + b)h$

d $A = bh$ **e** $A = \frac{xy}{2}$

A



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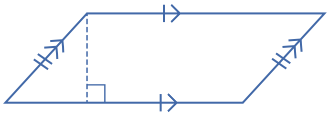

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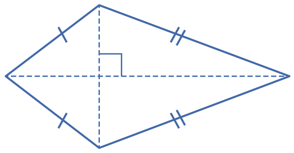
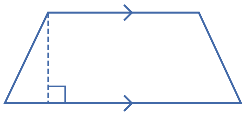
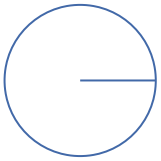


D

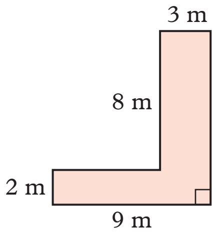


E

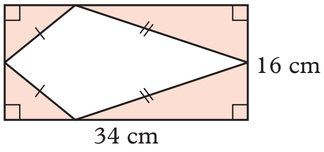


9A.2 **3** Calculate the shaded area of each of these composite shapes. Give your answers to one decimal place where necessary.

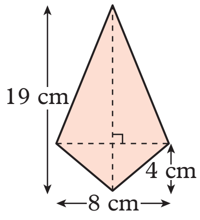
a



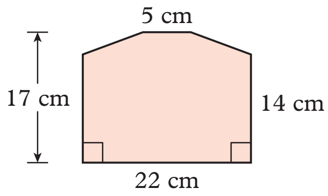
b



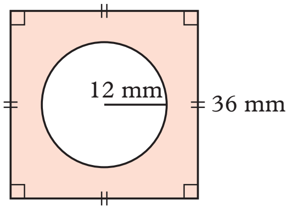
c



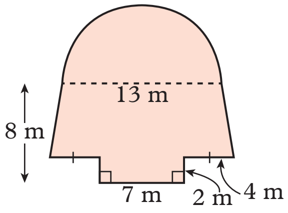
d



e

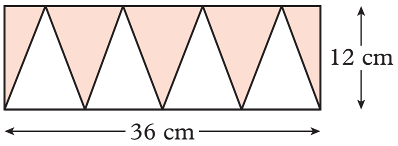


f

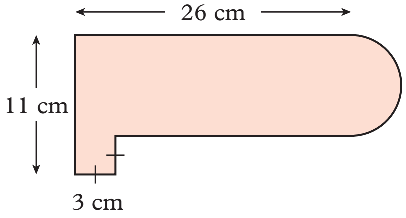


4 Calculate the shaded area of each of these composite shapes. Give your answers in the units given in the brackets, rounded to one decimal place where necessary.

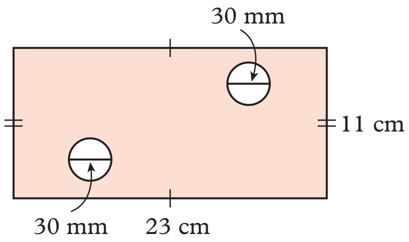
a (cm^2)



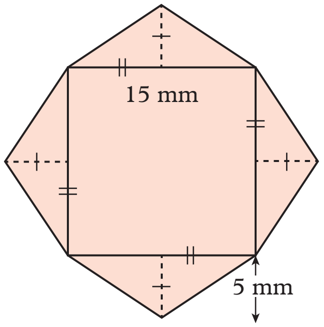
b (cm^2)



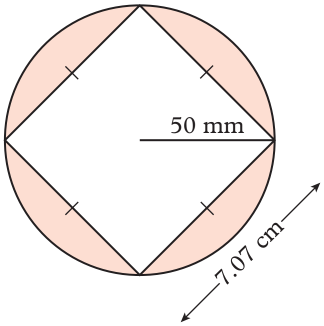
c (cm^2)



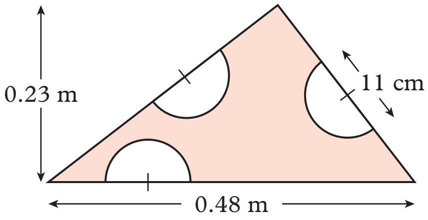
d (mm^2)



e (cm^2)



f (cm^2)



5 The front view of this house is made of a number of basic shapes.

a Identify each shape.

b Calculate the area of each window pane you identified in part **a**.

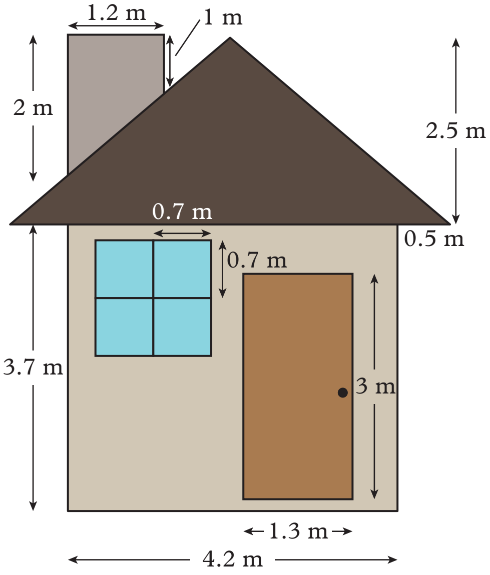
c There are four window panes of equal size. Explain how you can calculate the area of all four windows if you know the area of only one window.

d The owners wish to paint the front of the house (including the door, roof and chimney) and need to find the total area to be painted.

Using your knowledge of how to find the area of a composite shape, explain how to calculate the required area. (Remember that the windows won't need to be painted.)

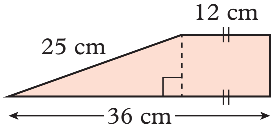
e Use your answers for parts **c** and **d** to help you find the area of the house that will be painted.

f If 1 L of paint covers 15 m^2, how many litres of paint do the owners need to purchase? Give your answer to the nearest half litre.

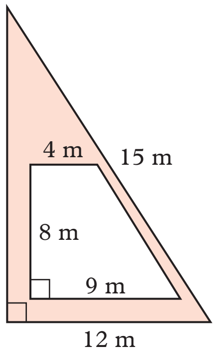


6 Use your knowledge of Pythagoras' Theorem to find any unknown lengths for each of these composite shapes. Then calculate the shaded area of each one. Give your answers to one decimal place where necessary. (Hint: Check that all measurements are written in the same units.)

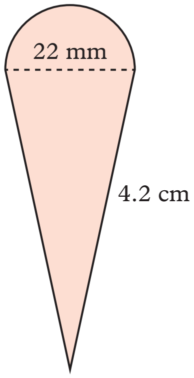
a



b



c



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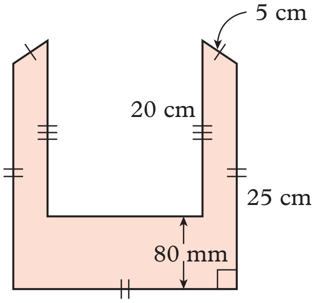

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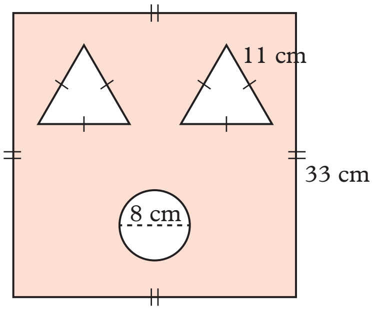
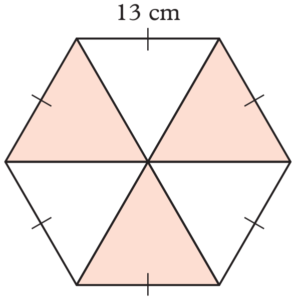


f

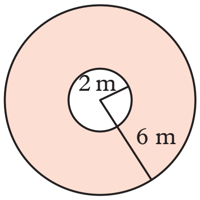


7 The garden bed shown in the image on the right is in the shape of an annulus.

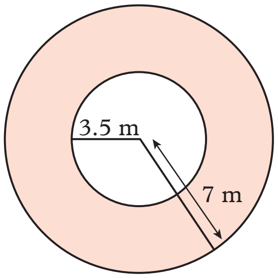
a Explain how you can find the area of this garden bed.

b Find the area of the garden bed correct to one decimal place.

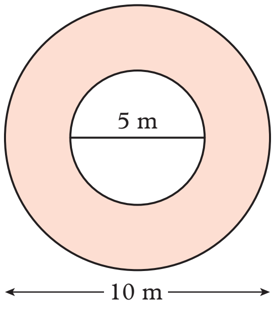
c Three more garden beds, all annuli, are to be added to the same garden. Find the area of each one correct to one decimal place.



i



ii



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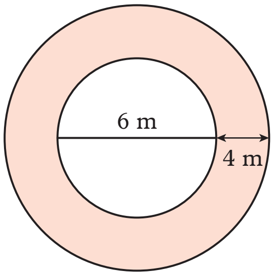
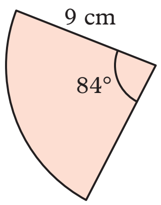


9A.3 **8** Calculate the area of each of these sectors correct to one decimal place.

a



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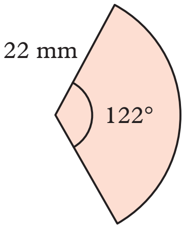
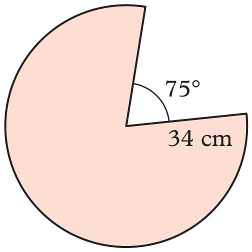


c



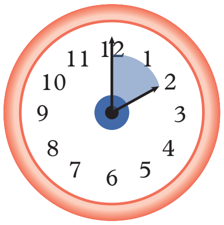
9 A farm is on land in the shape of a trapezium. The length of the northern boundary of the farm is 265 m and the length of the southern boundary is 180 m. The perpendicular distance from the northern boundary to the southern boundary is 90 m.

- **a** Draw a diagram of the farm and find its area.
- **b** Write your answer to part **a** to the nearest hectare.
- **c** The farmer decides to purchase his neighbour's farm, which is in the shape of a rectangle but has the same area as his own farm. What is one set of possible dimensions of his neighbour's farm? Give your answer in metres.

10 A rectangle has an area of 720 cm^2. One of its sides is five times as long as another.

- **a** If x is the length of the smaller side, draw a diagram of the rectangle and label the sides.
- **b** Use this information to find the dimensions of the rectangle.
- **c** If the dimensions are all doubled, what is the area of the new rectangle?
- **d** How does the area of the new rectangle compare to the area of the old rectangle?
- **e** If the dimensions of the original rectangle are all tripled, what is the area of the new rectangle? Compare this area to the original area of 720 cm^2.
- **f** Explain what will happen to the area of the rectangle when the dimensions are all:
  - **i** quadrupled
  - **ii** multiplied by a factor of n.

11 A clock has a minute hand that is 7 cm long and an hour hand that is 5 cm long.

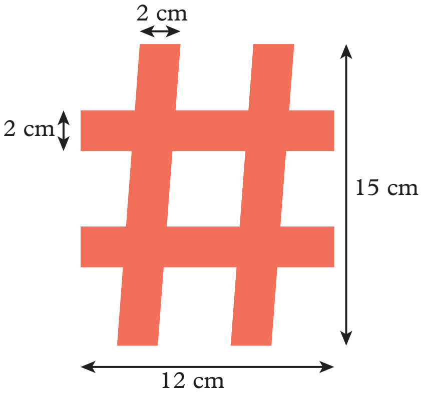


- **a** What is the difference, correct to one decimal place, between the areas covered by the hands when they each complete a full revolution?
- **b** The time shown on the clock face on the right is 2 o'clock. Calculate the angle of the sector between the minute hand and the hour hand.
- **c** Find the difference in the areas that would be covered by the hands if they each moved through the sector from the 12 to the 2 on the clock face. Give your answer correct to one decimal place.
- **d** Find the difference in the area that would be covered by the hands if they each moved through the smaller sector created between them when the time is 7 o'clock. Give your answer correct to one decimal place.

12 A design based on the ancient yin-yang symbol is shown on the right.

- **a** What would be the relationship between the radius of the complete circle and the radius of the semi-circles if they were formed inside it?
- **b** If the radius of the full circle is 24 cm, find the area shaded blue correct to one decimal place (yin).
- **c** Find the area of the circle's white section (yang) without performing any further calculations.

13 The area of a parallelogram is $A = bh$, where b is the base length and h is the perpendicular height. Find the total shaded area of this hash symbol.



14 Consider the target on the right.

The radius of the inner circle is 2 cm and the edge of each ring is a further 2 cm from the centre of the inner circle, as shown.

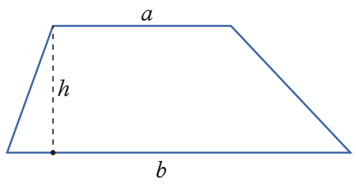
- **a** Find the area, as an exact value, of the shaded region.
- **b** Find the area, as an exact value, of the unshaded region.
- **c** Find the ratio of the shaded area to the unshaded area.

CHALLENGE

15 A regular hexagon can be thought of as the composite of six congruent equilateral triangles.

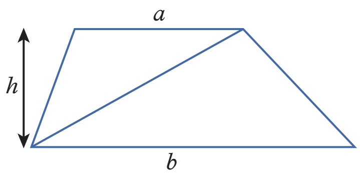
- **a** Use Pythagoras' Theorem to find the height, as an exact value, of an equilateral triangle with side lengths 10 cm.
- **b** Hence, find the area of an equilateral triangle with side lengths 10 cm.
- **c** Hence, find the area of a regular hexagon with side lengths 10 cm.
- **d** Find the area of a regular hexagon with side lengths of a.

16 A trapezium is a quadrilateral with a pair of parallel sides. The area of a trapezium is $A = \frac{h}{2}(a + b)$, as shown below.



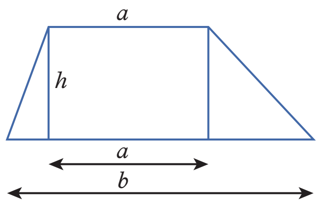
Demonstrate that the trapezium area formula works using the following methods.

- **a** Considering a trapezium as the composite of two triangles, as shown below.

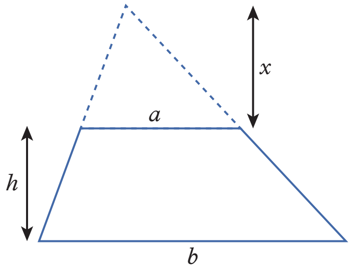


- **b** Considering a trapezium as the composite of one rectangle and two triangles, as shown below.

  Hint: you can combine the two triangles into one triangle, since they have the same height.



- **c** Considering a trapezium as a truncated triangle, as shown below.
  Hint: The two triangles are similar and so you can substitute out x in the area equation by considering ratios.



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Interactive skillsheet
Converting between units of area

Interactive skillsheet
Area of composite shapes

Topic quiz
9A

9B Surface area of prisms and cylinders

Learning intentions

✔ I can calculate the surface area of prisms and cylinders.
✔ I can calculate the surface area of composite solids consisting of prisms and cylinders.

Inter-year links

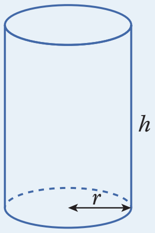
Year 7	9E Surface area
Year 8	8E Area of a circle
Year 9	6B Surface area

Surface area

- The **total surface area** (TSA) of a 3D object is the total area of its outer surface. This is the sum of the areas of the faces (or surfaces) of that object.
- **Prisms** are 3D objects that have a uniform cross-section that is a polygon.
  - → All faces of a prism are flat.
  - → Cubes, rectangular prisms and triangular prisms are all common examples of prisms.
- **Cylinders** are 3D objects with a uniform circular **cross-section** and a curved face. The surface area of the curved face of a cylinder is equal to the circumference ($2\pi r$) of either end multiplied by the height (h):

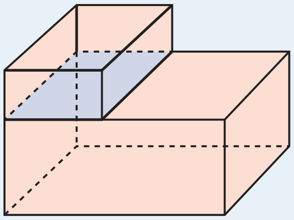
  Area of curved face of a cylinder = $2\pi rh$
- The surface area of a cylinder (the area of the two circular ends and the curved surface) can be found using the formula:

  TSA = $2\pi r^2 + 2\pi rh$



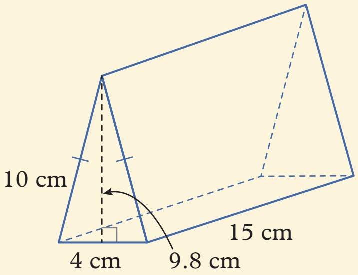
Surface area of composite solids

- A composite solid consists of two or more simple solids.
- To calculate the total surface area of a composite solid, identify the individual component solids and adapt their formulas to find the surface area required. Add or subtract the individual surface areas to find the total surface area of the composite solid.
- The area where two solids join should not be included in surface area calculations. (An example is shaded blue in the diagram below.)



Example 9B.1 Calculating the surface area of a prism

Calculate the surface area of this triangular prism.



THINK

1 Identify the number of faces and the shapes of the faces of the triangular prism.

2 Calculate the area of each face. The triangular faces are isosceles triangles, meaning the two rectangles sharing common sides with them are identical.

3 Add the areas and include the appropriate unit.

WRITE

The prism has five faces: two identical rectangular faces, two identical triangular faces and a rectangular base.

rectangular faces: $A = 10 \times 15$
$= 150\text{ cm}^2$ each

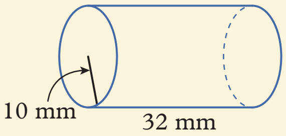
rectangular base: $A = 4 \times 15$
$= 60\text{ cm}^2$

triangular faces: $A = \frac{1}{2} \times 4 \times 9.8$
$= 19.6\text{ cm}^2$ each

$\text{TSA} = 2 \times 150 + 60 + 2 \times 19.6$
$= 399.2\text{ cm}^2$

Example 9B.2 Calculating the surface area of a cylinder

Calculate the surface area of this cylinder, correct to one decimal place.



THINK

1 Identify the values of r and h for this cylinder.

2 Substitute the values you identified into the formula for the surface area of a cylinder and simplify.

3 Calculate the result using the π key on your calculator. Write the answer correct to one decimal place and include the appropriate unit.

WRITE

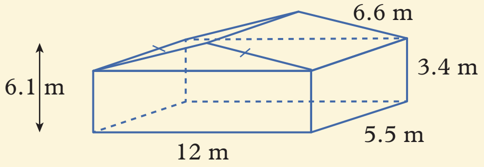
$r = 10\text{ mm} \qquad h = 32\text{ mm}$

$\text{TSA} = 2\pi r^2 + 2\pi rh$
$= 2 \times \pi \times 10 \times 32 + 2 \times \pi \times 10^2$

$= 640\pi + 200\pi$
$= 840\pi$
$= 2638.937\ldots$
$\approx 2638.9\text{ mm}^2$

Example 9B.3 Calculating the surface area of a composite solid

Calculate the surface area of this composite object.

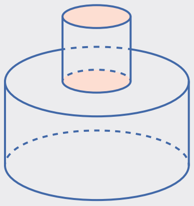


THINK	**WRITE**
1 Identify the number of faces and the shapes of the faces of the solid.	The solid has nine faces: three pairs of identical rectangular faces (including the pair of sloping faces), a pair of identical triangular faces and the rectangular base.
2 Calculate the area of each face.	rectangular faces (front and back): $A = 3.4 \times 12$ $= 40.8\text{ m}^2$ each rectangular faces (sides): $A = 3.4 \times 5.5$ $= 18.7\text{ m}^2$ each rectangular faces (sloping faces): $A = 6.6 \times 5.5$ $= 36.3\text{ m}^2$ each rectangular base: $A = 12 \times 5.5$ $= 66\text{ m}^2$ triangular faces: $A = \frac{1}{2} \times (6.1 - 3.4) \times 12$ $= \frac{1}{2} \times 2.7 \times 12$ $= 16.2\text{ m}^2$ each
3 Add the areas and include the appropriate unit.	$\text{TSA} = 2 \times 40.8 + 2 \times 18.7 + 2 \times 36.3 + 66 + 2 \times 16.2$ $= 81.6 + 37.4 + 72.6 + 66 + 32.4$ $= 290\text{ m}^2$

Helpful hints

- ✔ If you can’t remember the formula for the surface area of a cylinder, remember that a cylinder consists of two circles (the ends) and a rectangle (the curved surface). Calculate the area of each individual face and add the areas together.
- ✔ In some instances, the area where the faces of solids join in a composite solid is the same as the area of the top face of the combined solid. Identifying this can help with your calculations.

  For example, in the diagram on the right the total surface area of the composite solid is the total surface area of the bottom cylinder plus just the curved surface area of the top cylinder because its exposed circular end is the same as the joined surface calculated as part of the surface area of the bottom cylinder.



- ✔ The surface area of a prism can also be found by using the formula:

  TSA = 2 × area of base + perimeter of base × height

ANS
p777

Exercise 9B Surface area of prisms and cylinders

△ 1(a, b, f, g, i), 2–7, 9, 12(a)

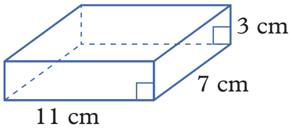
□ 1(d, e, h, i), 2–6, 8, 9, 11, 12(a), 13(a), 14

⬠ 1(f, h, i), 3, 6, 8, 11, 12(b), 13–16

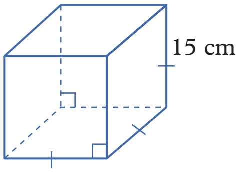
UNDERSTANDING AND FLUENCY

9B.1 **1** Calculate the surface area of each of these prisms.

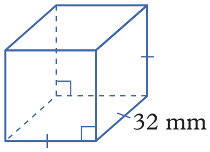
a



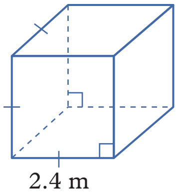
b



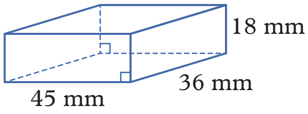
c



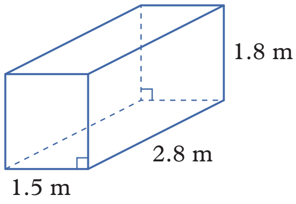
d



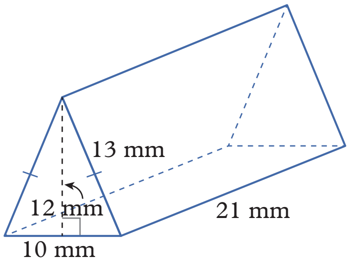
e



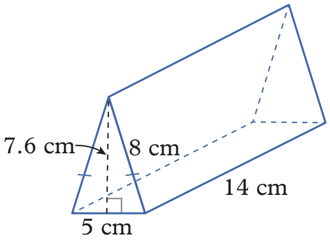
f



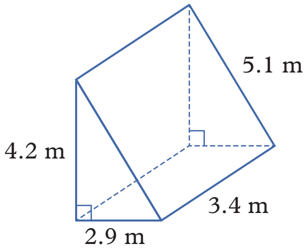
g



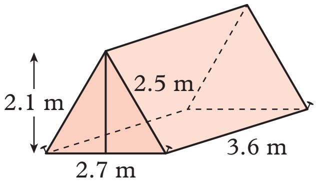
h



i

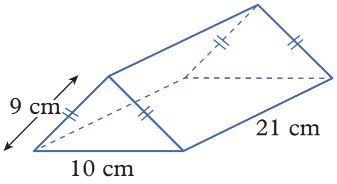


2 The tent shown below has the shape of a triangular prism. Find the amount of canvas required to construct it (assuming it has no floor).

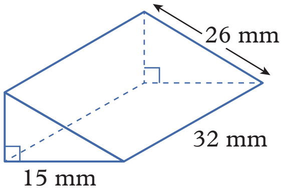


3 Calculate the surface area of each of these triangular prisms correct to one decimal place. (Hint: Use Pythagoras' Theorem.)

a



b



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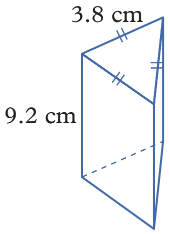
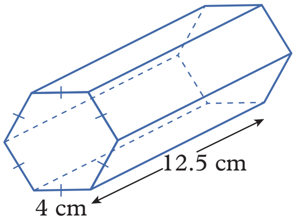
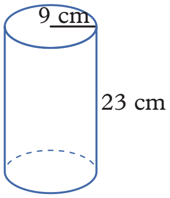


4 If the base of this hexagonal prism has an area of 41.52 cm^2, calculate its total surface area.

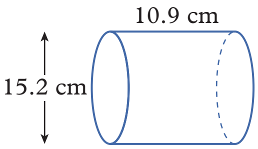


9B.2 **5** Calculate the surface area of each of these cylinders correct to one decimal place.

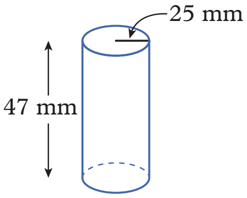
a



b

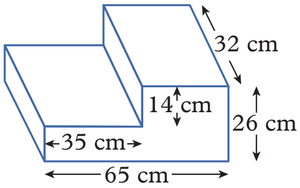


c

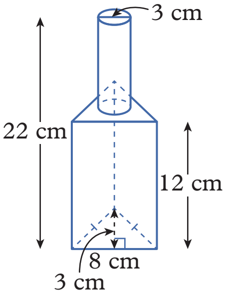


9B.3 **6** Calculate the surface area of each of these composite solids. Give your answers correct to one decimal place where appropriate.

a



b



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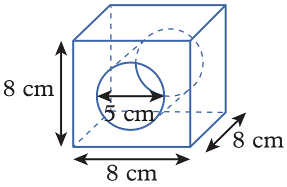


7 Each die in this stack is the shape of a cube with side lengths of 15 mm.

a Calculate the surface area of one die.

b Use your answer for part **a** to calculate the total surface area of nine separate dice.

c When stacked, the nine dice form a rectangular prism. Using the information you have been given about each die, draw the rectangular prism in your books and label its dimensions.

d Calculate the total surface area of the rectangular prism made of nine dice.

e Find the difference between your answers to parts **b** and **d**. Explain why these answers differ. Support your explanation with calculations.

8 A cube has a total surface area of 384 cm^2.

a What is the length of each side of the cube? Explain how you found your answer.

b Use your answer for part **a** to write a formula for finding the side length of a cube when the total surface area is known.

9 An open pipe has a length of 33 cm and a diameter of 55 mm.

a Calculate the surface area of the outer surface of the pipe correct to the nearest cm^2.

b To find your answer for part **a**, did you need to use the entire formula for the surface area of a cylinder: TSA = $2\pi rh + 2\pi r^2$? Explain.

10 Find the surface area of the outside of an open cylinder that has a radius of 4.5 cm and a height of 9.9 cm. Give your answer correct to one decimal place.

11 The teachers at a kindergarten want to add two cylindrical poles to the playground. One pole is to be painted yellow and the other red. The vertical poles have the same diameter of 125 mm, but they vary in height.

a The yellow pole is 1.8 m tall. Calculate the area needing to be painted yellow correct to one decimal place. (Assume that the top end and the side of the pole needs painting but the end in the ground does not.)

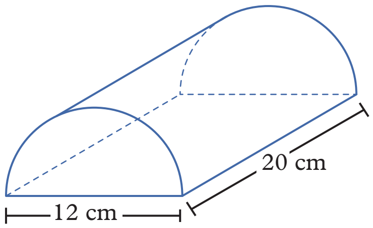
b The red pole is double the height of the yellow pole. Compared to the amount of yellow paint needed, will the teachers need to purchase double the amount of red paint in order to paint this second pole? Why or why not?

c Calculate the area of the pole needing to be painted red correct to one decimal place. Does your result support your answer to part **b**?

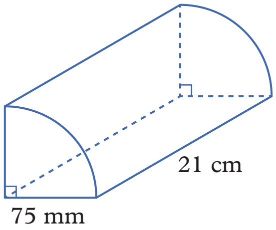
PROBLEM SOLVING AND REASONING

12 Find the total surface area of the following cut cylinders. Give your answers in cm^2 correct to one decimal place.

a

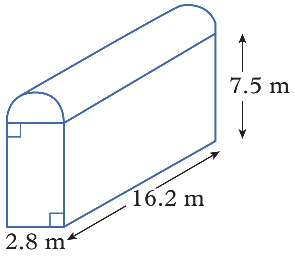


b

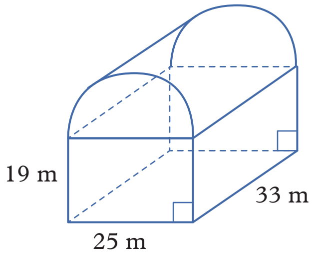


13 Calculate the surface area of each of these composite solids correct to one decimal place.

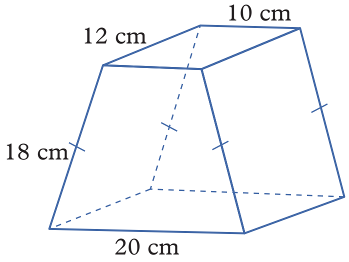
a



b

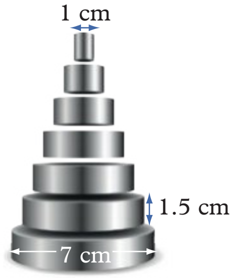


14 Calculate the surface area of this 3D solid.



CHALLENGE

15 Seven cylinders with the same height but different diameters are stacked together as shown on the right. The top cylinder has a diameter of 1 cm and the diameter of each cylinder beneath it increases by 1 cm each time. The bottom cylinder has a diameter of 7 cm. If each cylinder is the same height, 1.5 cm, calculate the total surface area when the cylinders are sitting one on top of the other, as shown. Give your answer correct to one decimal place.

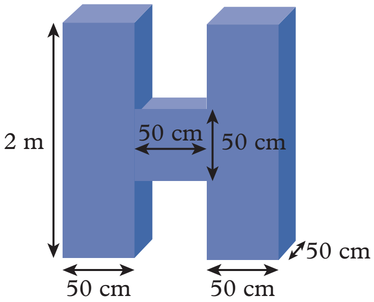


16 A company commissions an artist to make a giant version of their logo, an H, for the foyer of their office building. The specifications are in the diagram on the right.

The H will be made of wood and then every face of the H will be painted, except the two rectangles that will be on the ground.

a Calculate the surface area of the H, not including the two rectangles that will be touching the ground. Answer in square metres.

b The H will be painted blue, as that is the company's colour. The company will use a premium blue paint which is sold in 2 L cans. Each litre of this paint is enough to paint one coat on 4 square metres. How many cans of paint must they buy if they will paint the H with a triple coat?



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Surface area of prisms

Interactive skillsheet
Surface area of cylinders

Topic quiz
9B

9C Volume of prisms and cylinders

Learning intentions

✔ I can calculate the volume of prisms and cylinders.

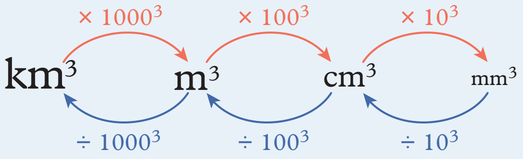
✔ I can calculate the volume of composite solids consisting of prisms and cylinders.

Inter-year links

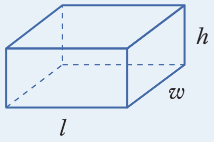
Year 5/6	Volume and capacity
Year 7	9F Volume and capacity
Year 8	8F Volume of prisms
Year 9	6C Volume and capacity

Volume

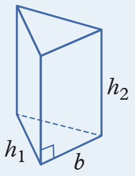
- **Volume** is measured in cubic units, such as mm^3, cm^3 and m^3.
- A unit of volume can be converted to another unit of volume by multiplying or dividing by the appropriate conversion factor.



- The volume of any prism can be found using the formula $V = Ah$, where A is the area of the base and h is the height of the prism.
  - → The height is perpendicular to the base.
  - → When dealing with a triangular prism, subscripts can be used to represent the two heights, for example h_1 and h_2, as shown in the diagram of a triangular prism on the right, where h_1 is the height of the triangular base.



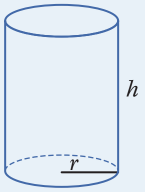
$V = lwh$



$V = \frac{1}{2}bh_1h_2$

- The volume of a cylinder can be found by multiplying the area of the base (πr^2) by the height (h).

  $V = \pi r^2 h$



Volume of composite solids

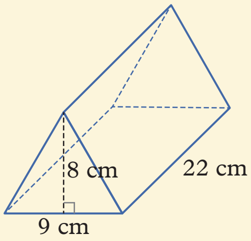
- The volume of a composite solid can be found by calculating the volumes of the individual component solids and adding them together.

Capacity

- The **capacity** of a 3D object is a measure of how much the object can hold. A container with an inside volume of:
  - → 1 cm^3 holds 1 mL of liquid
  - → 1000 cm^3 holds 1 L of liquid
  - → 1 m^3 holds 1 kL of liquid.

Example 9C.1 Calculating the volume of a prism

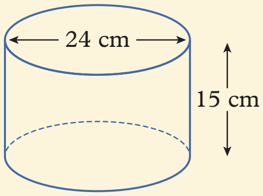
Calculate the volume of this prism.



THINK	WRITE
a Identify the base of the triangular prism and the perpendicular height.	The base is a triangle with base length 9 cm and height 8 cm. The height (h) of the prism is 22 cm.
b Calculate the area of the base.	$A = \frac{1}{2}bh$ $= \frac{1}{2} \times 9 \times 8$ $= 36\text{ cm}^2$
c Substitute the value of the area of the base and the perpendicular height into the formula for the volume of a prism.	$h = 22\text{ cm}$ $V = bh$ $= 36 \times 22$
d Include the appropriate unit in your answer.	$= 792\text{ cm}^3$

Example 9C.2 Calculating the volume of a cylinder

Calculate the volume of this cylinder. Give your answer to one decimal place.



THINK	WRITE
a Calculate the radius measurement of the circular base.	$r = D \div 2$ $= 24 \div 2$ $= 12\text{ cm}$
b Substitute the values for r and h into the formula for the volume of a cylinder.	$h = 15\text{ cm}$ $V = \pi r^2 h$ $= \pi \times 12^2 \times 15$
c Calculate using the π key on your calculator, round to one decimal place, and include the appropriate unit in your answer.	$= 2160\pi$ $= 6785.840...$ $\approx 6785.8\text{ cm}^3$

Helpful hints

✔ If the dimensions of a 3D object are given using different units, convert all lengths to the required unit specified in the question before you do anything else.

ANS p778

Exercise 9C Volume of prisms and cylinders

△ 1, 2(c–i), 3, 4, 5(a, c), 6(a, c, d), 7, 8, 9(b), 12, 15

□ 1(b, d, f, h, j), 2(d, e, i, j, k, l), 3(c–e), 5(c), 6(a, d, e), 9(a, b), 10, 11, 17

⬠ 1(g–j), 2(i–l), 6(d, f), 9(b, c), 13, 14, 16, 18, 19

UNDERSTANDING AND FLUENCY

1 Convert each of these volume measurements to the unit shown in brackets.

a 12.5 m^3 (cm^3)

b 240 000 mm^3 (cm^3)

c 34 200 000 cm^3 (m^3)

d 0.55 m^3 (mm^3)

e 67 200 mm^3 (m^3)

f 0.000 009 m^3 (mm^3)

g 7520 mm^3 (cm^3)

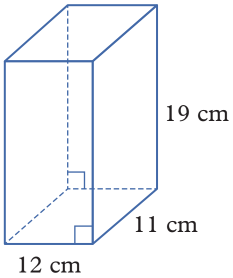
h 8.74 m^3 (cm^3)

i 142 900 cm^3 (m^3)

j 73 000 000 mm^3 (m^3)

9C.1 2 Calculate the volume of each of these prisms.

a



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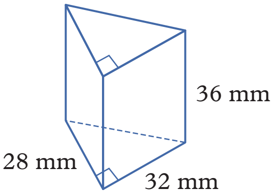

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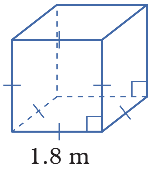

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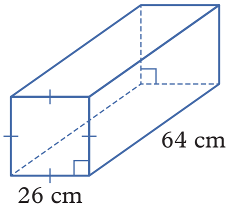

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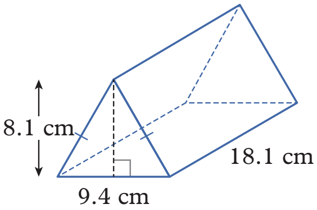

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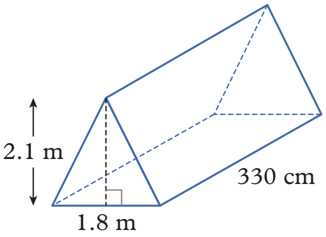

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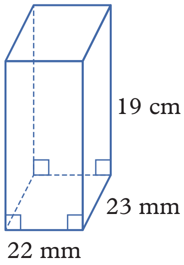

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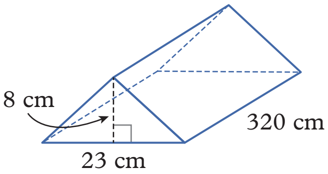
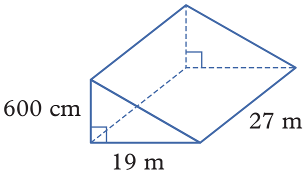


i



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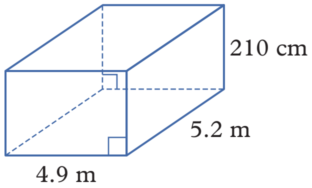
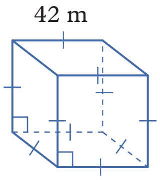
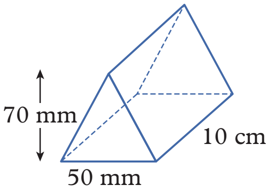


k

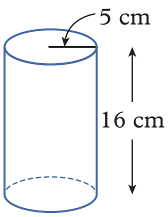


l



9C.2 **3** Calculate the volume of each of these cylinders. Give your answers correct to one decimal place.

a



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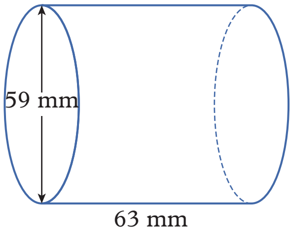

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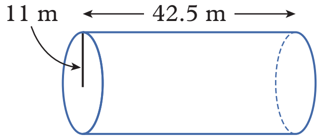

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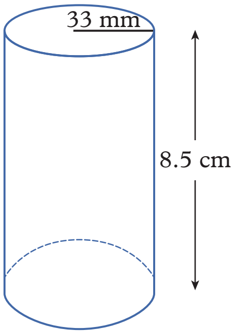

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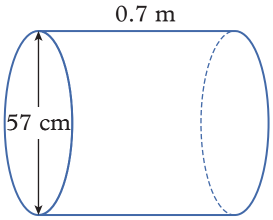

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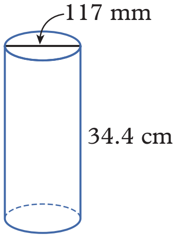


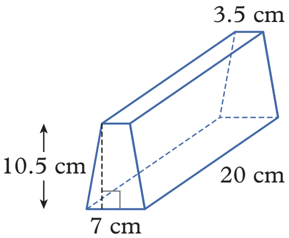
4 Paula is using odd pieces of colourful fabric to construct a child's cushion. The cushion is in the shape of a cube with side lengths of 0.4 m.

a How much polyester fibre does Paula need to fill the cushion?

b Convert your answer to cubic centimetres.

5 Calculate the volume of each of these prisms.

a



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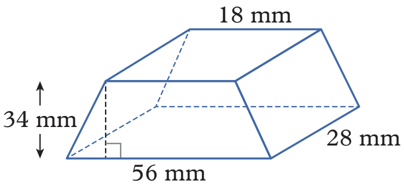

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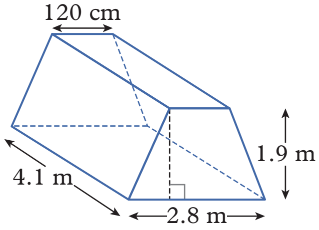
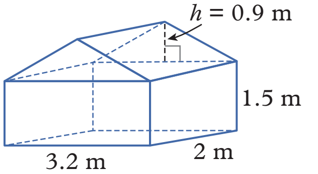


6 Calculate the volume of each of these composite solids. Where appropriate, give your answers correct to one decimal place. (Hint: Identify the individual solids first.)

a



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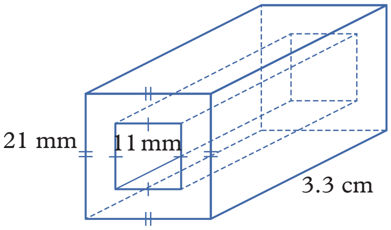

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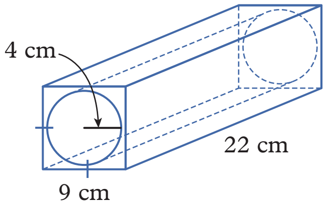

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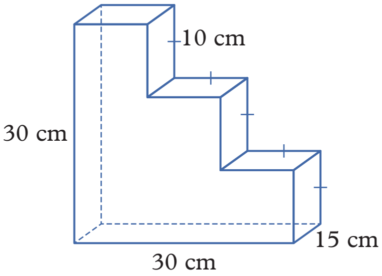

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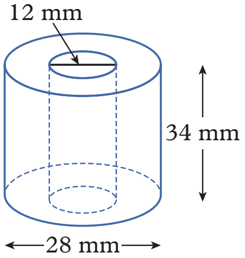

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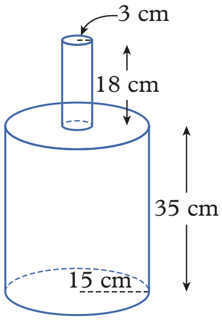


7 A round hatbox has a diameter of 23 cm and a height of 12.5 cm. Calculate its volume correct to one decimal place.

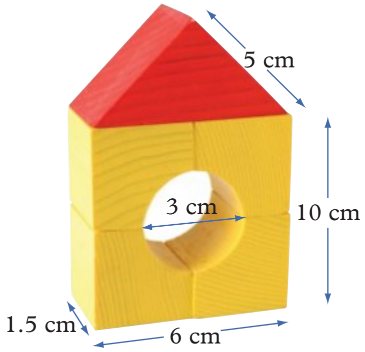
8 In this toy house, constructed from different shaped blocks, there are three common solids that can be identified.

a Identify the three solids.

b Explain how you can calculate the total surface area of the toy house.

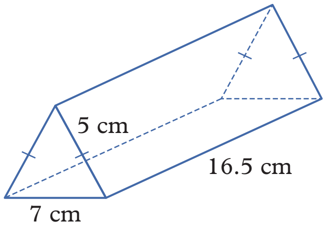
c Find the total surface area of the toy house correct to the nearest cm^2.

d Find the volume of wood used to make this toy house, correct to the nearest cm^3.



9 Use Pythagoras' Theorem to help you calculate the volume of each of these prisms. Give your answers correct to one decimal place where appropriate.

a



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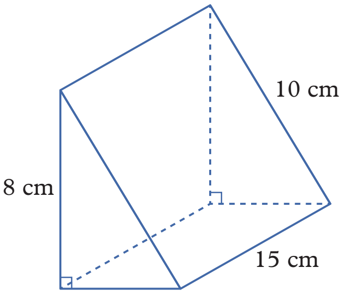

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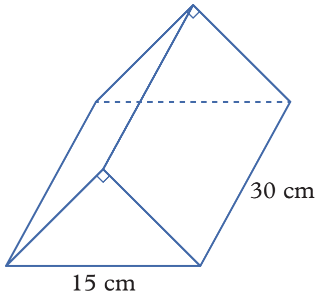
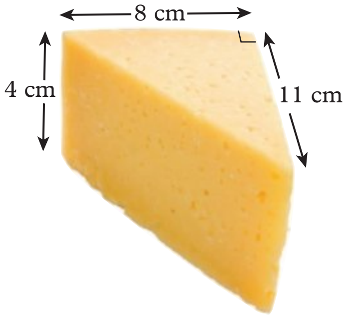


10 This piece of cheese is in the shape of a right-angled triangular prism.

a Calculate its volume correct to one decimal place.

b If 1 cm^3 of cheese has a mass of 0.001 kg, find the mass of the cheese, in kilograms, correct to four decimal places.

c This particular cheese costs $21 per kilogram. What is the cost of the piece of cheese shown to the nearest cent?



11 A rectangular fish tank is 45 cm long, 20 cm wide and 50 cm tall.

a Calculate the volume of the fish tank.

b What is the capacity of the fish tank if it is filled to the brim? Give your answer in millilitres and litres. (Remember: 1 cm^3 = 1 mL)

c The fish tank should only be filled to a level 6 cm from the top. How many litres of water will be required to fill the tank to that height?

12 The volume of a particular prism is 2744 cm^3. If the prism is a cube, what is its side length?

13 a If a cylinder has a volume of 1649.34 cm^3 and a radius of 5 cm, what is the height of the cylinder, to the nearest centimetre?

b If a cylinder has a volume of 207 cm^3 and a height of 5.7 cm, find the radius of its circular base, to the nearest millimetre.

14 Jack installs two cylindrical water tanks. The first tank is 2 m tall and the diameter of its base is 1.5 m. The second tank is 1.5 m tall and the diameter of its base is 2 m.

a Which tank has the larger capacity?

b What is the difference in the capacity of the two tanks? Give your answer to the nearest litre.

PROBLEM SOLVING AND REASONING

15 **a** A particular cylinder has a radius of 8 cm and a height of 15 cm. Calculate its volume.

b If the radius of the cylinder from part **a** is halved but the height remains the same:

i calculate the volume

ii describe how this volume compares to the volume of the original cylinder.

c If the radius of the cylinder from part **a** is doubled but the height remains the same:

i calculate the volume

ii describe how this volume compares to the volume of the original cylinder.

d If the radius of the cylinder from part **a** remains the same but the height is halved:

i calculate the volume

ii describe how this volume compares to the volume of the original cylinder.

e If the radius of the cylinder from part **a** remains the same but the height is doubled:

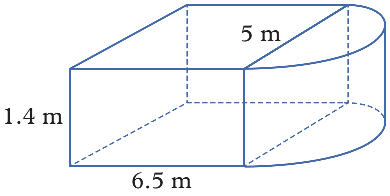
i calculate the volume

ii describe how this volume compares to the volume of the original cylinder.

16 A swimming pool is the shape of a rectangular prism. A semi-circular spa is built onto one end of the pool, as shown in the diagram on the right.

a What amount of water is required if the pool and spa are filled to capacity? Give your answer to the nearest litre.

b To make the pool and spa functional, the water levels must be below the top edge. If the pool is filled to 70% of its depth and the spa is filled to 60% of its depth, what amount of water is required, to the nearest litre?

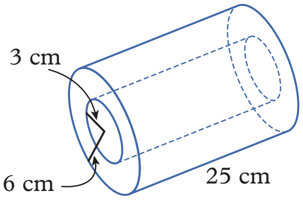


17 The plastic pipe on the right is hollow.

a What shape is the base of the pipe? Calculate its area correct to one decimal place.

b Use the area of the base to calculate the volume of plastic used for this pipe.

c Can you think of a different method for finding the volume? Explain your answer.



CHALLENGE

18 Nicko wishes to make an open box from a 30 cm by 30 cm piece of red cardboard. He decides to cut out a square of length 2 cm from each corner and fold up the sides to form the box. The edges of the box will be joined by sticky tape, so no tabs are needed to create the box.

a Draw a diagram showing the piece of cardboard with the corners cut out. Label the dimensions.

b Determine the length, width and height of the box once Nicko has folded up the sides.

c Calculate the volume of the box.

d Would the volume of the box increase or decrease if he cut out 3 cm square corners from the piece of cardboard before forming the rectangular prism? Explain your answer.

19 These nuts have the same dimensions but different shapes. One is circular and the other is in the shape of a regular hexagon. They both have a depth of 6.2 mm.

a What amount of steel is required to manufacture each nut? Give your answers to two decimal places.

b Which nut requires the greater amount of steel?

Check your Student obook pro for these digital resources and more:

pro

Interactive skillsheet
Volume of prisms

Interactive skillsheet
Volume of cylinders

Topic quiz
9C

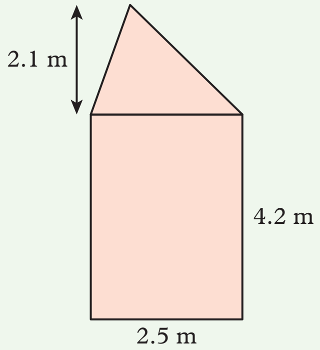
Checkpoint

Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this chapter.

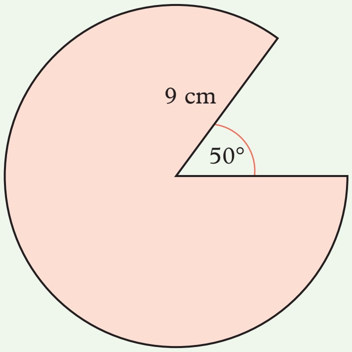
9A **1** Convert each length and area measurement to the unit shown in brackets.

a 45 mm (cm) **b** 400 cm (m)
c 4.53 m (mm) **d** 4000 mm^2 (cm^2)
e 5.2 km^2 (m^2) **f** 68 000 cm^2 (m^2)

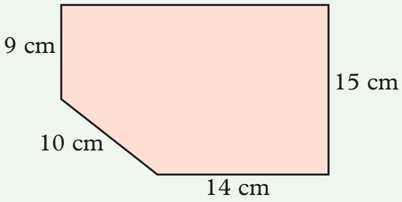
9A **2** Find the area of the following composite shape.



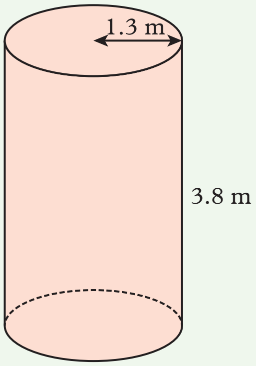
9A **3** Find the area of the following circle sector. Give your answer correct to the nearest whole number.



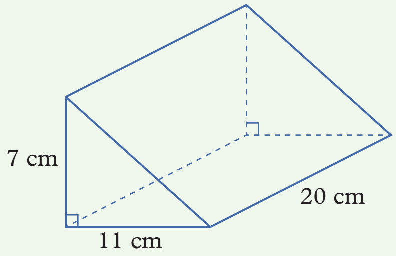
9A **4** Use Pythagoras' Theorem to help determine the area of the following shape.



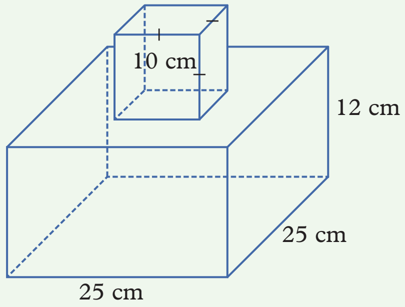
9B **5** Find the surface area of the following cylinder correct to one decimal place.



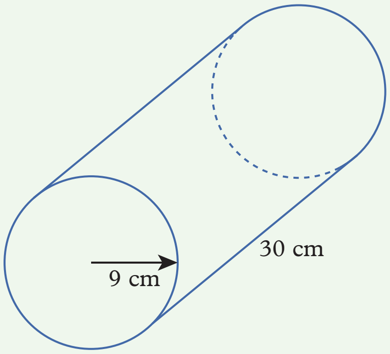
9B **6** Use Pythagoras' Theorem to help find the total surface area of the following prism correct to one decimal place.



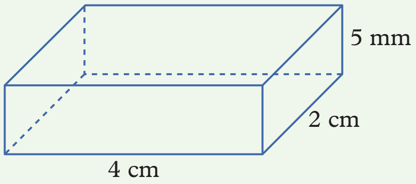
9B **7** A cube with side lengths of 10 cm sits on top of a square-based rectangular prism, as shown below. Find the total surface area of the composite shape.



9C **8** Calculate the volume of the following cylinder correct to one decimal place.



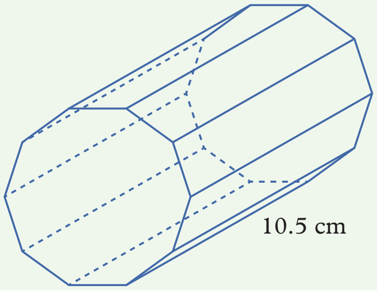
9C **9** Consider the following shape.



Find the volume of this shape in:

a mm^3 **b** cm^3

9C **10** The following prism has a cross-sectional area of 123.11 cm^2. Find the volume of the prism.



9D Surface area of pyramids and cones

Learning intentions

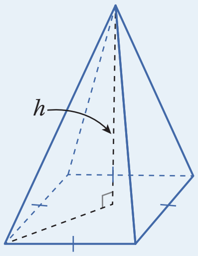
- ✔ I can calculate the surface area of right pyramids and right cones.

Inter-year links

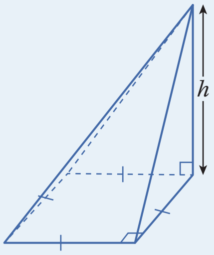
Year 8	8C Area of triangles and rectangles
Year 9	6B Surface area

Pyramids

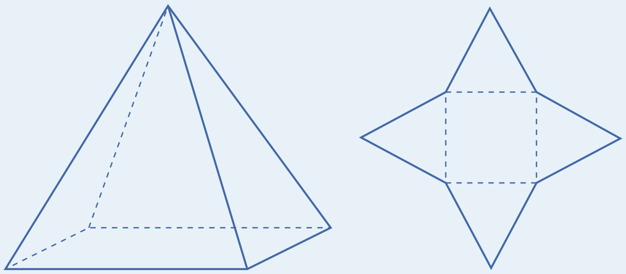
- A **pyramid** is a 3D object with a polygon as its base. All other faces of a pyramid are triangles that meet at the apex (highest point) of the pyramid.
  - → Pyramids are named according to the shape of their base.
  - → A **right pyramid** has an apex directly above the centre of its base.
  - → An **oblique pyramid** does not have its apex over the centre of its base.
- The total surface area of a pyramid is the sum of the areas of all of its faces.



right square pyramid



oblique square pyramid



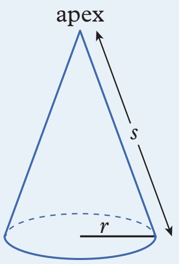
- A **tetrahedron** is a pyramid with a triangular base.

  A **regular tetrahedron** has four triangular faces that are all congruent equilateral triangles.
- If the length of the base and the slant edge of a pyramid are known, Pythagoras' Theorem can be used to find the height of a triangular face.

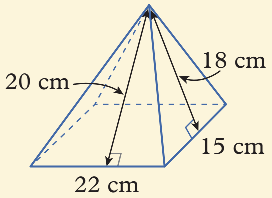
Cones

- A **cone** is a 3D object with a circular base and a curved surface that tapers from the circular base to the apex.
- A **right cone** has its apex directly above the centre of its base.
- The total surface area of a right cone with radius r and slant height s is given by the formula: TSA = $\pi r^2 + \pi rs$.
- The net of a cone consists of a circle and a sector of a circle.

Note: In this topic, only right pyramids and right cones will be considered.



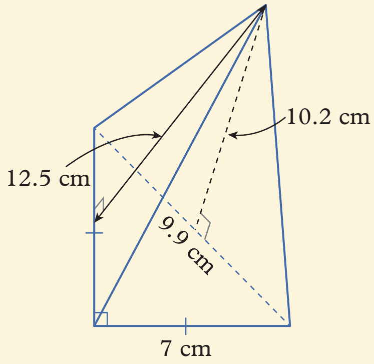
Example 9D.1 Calculating the surface area of a pyramid



Calculate the surface area of the pyramid on the right.

THINK	**WRITE**
1 Identify the number and shape of the faces of the pyramid.	The pyramid has five faces: one rectangular face and two pairs of identical triangular faces.
2 Calculate the area of each face.	rectangular face: $A = 22 \times 15$ $= 330\text{ cm}^2$ front triangular face: $A = \frac{1}{2} \times 22 \times 20$ $= 220\text{ cm}^2$ side triangular face: $A = \frac{1}{2} \times 15 \times 18$ $= 135\text{ cm}^2$
3 Add the areas of the faces together and include the appropriate unit.	$\text{TSA} = 330 + 2 \times 220 + 2 \times 135$ $= 1040\text{ cm}^2$

Example 9D.2 Calculating the surface area of a tetrahedron

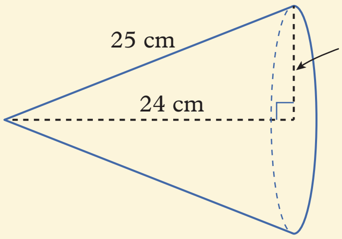


Calculate the surface area of the tetrahedron on the right.

THINK	**WRITE**
1 Identify the number and shapes of the faces of the tetrahedron.	The base is an isosceles right-angled triangle: $b = 7$ cm, $h = 7$ cm There are two identical triangular faces: $b = 7$ cm, $h = 12.5$ cm There is one other triangular face: $b = 9.9$ cm, $h = 10.2$ cm
2 Calculate the area of each face.	base: $A = \frac{1}{2} \times 7 \times 7$ $= 24.5\text{ cm}^2$ triangular face 1: $A = \frac{1}{2} \times 7 \times 12.5$ $= 43.75\text{ cm}^2$ each triangular face 2: $A = \frac{1}{2} \times 9.9 \times 10.2$ $= 50.49\text{ cm}^2$
3 Add the areas of the faces together and include the appropriate unit.	$\text{TSA} = 24.5 + 2 \times 43.75 + 50.49$ $= 162.49\text{ cm}^2$

Example 9D.3 Calculating the surface area of a cone

Calculate the surface area of this cone correct to one decimal place.



THINK	WRITE
1 Determine the radius and the slant height of the cone.	$r = 7$ cm, $s = 25$ cm
2 Write the formula for the surface area of a cone. Substitute values for r and s and simplify.	$\text{TSA} = \pi r^2 + \pi rs$ $= \pi \times 7^2 + \pi \times 7 \times 25$ $= 49\pi + 175\pi$ $= 224\pi$
3 Calculate the result using the π key on your calculator. Write the answer correct to one decimal place and include the appropriate unit.	$= 703.716...$ $\approx 703.7 \text{ cm}^2$

Helpful hints

- ✔ If a tetrahedron is not regular, you'll need to work out the area of more than one triangular face to calculate the total surface area.
- ✔ If the base of a right pyramid is a regular polygon, then the total surface area can also be found using the formula:

$$\text{TSA} = \text{base area} + \frac{1}{2} \times \text{perimeter of base} \times \text{slant height.}$$

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Exercise 9D Surface area of pyramids and cones

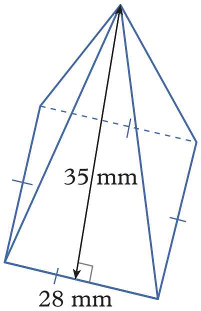
△ 1–4, 5(a, d, e), 6, 7, 10

■ 1(c, e, f), 2–4, 5(d–f), 6–8, 11, 12(b), 13

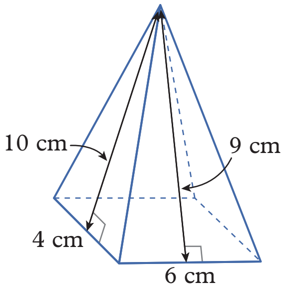
⬟ 1(e, f), 2, 5(d–f), 6, 9, 11–15

9D.1 **1** Calculate the surface area of each of these pyramids.

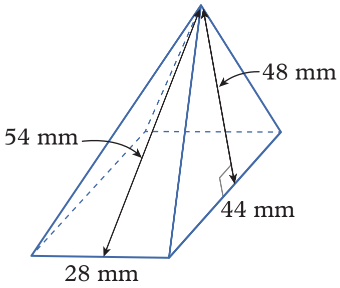
a



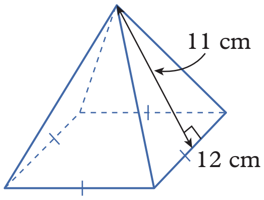
b



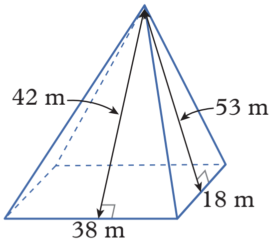
c



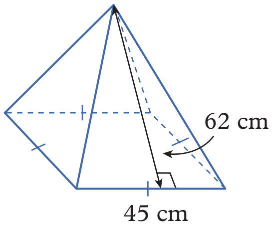
d



e

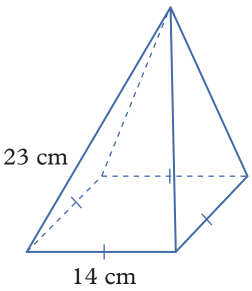


f



2 To calculate the surface area of each of these right pyramids, first find the height of a triangular face. Give your answers to one decimal place where necessary.

a



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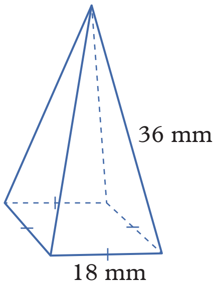

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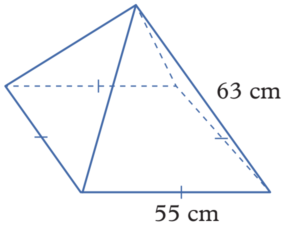


3 At the main entrance to the Louvre Museum in Paris, France, a pyramid has been constructed with a square base of length 35 m. Each identical sloping triangular face has a height of 27 m and is made of glass segments. Find the total surface area of the glass walls of the pyramid.

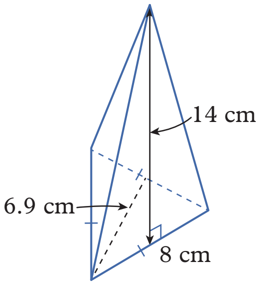
4 The Red Pyramid in Dahshur, Egypt, was originally covered in white limestone. It has a slant edge of length 187 m and a square base of side length 220 m.

a Draw a diagram of this pyramid and label its dimensions.

b Find the surface area of the pyramid that was covered with limestone. (Hint: Do not include the base in your calculations.)

9D.2 5 Calculate the surface area of each of these right tetrahedrons correct to one decimal place. (Hint: Use Pythagoras' Theorem to find any missing heights.)

a



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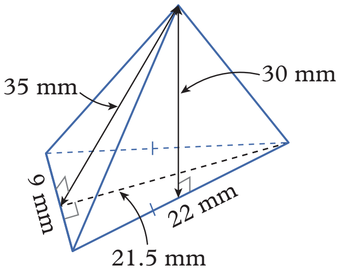

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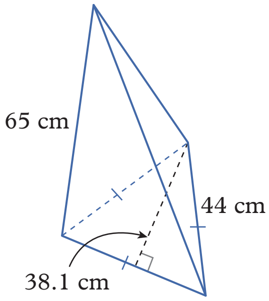

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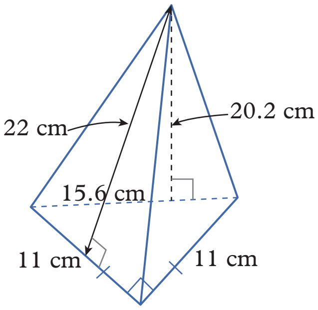

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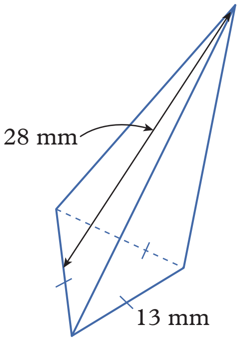

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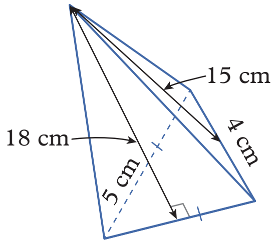
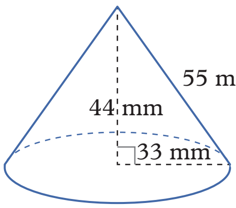


9D.3 **6** Calculate the surface area of each of these cones correct to one decimal place.

a



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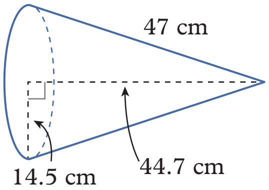

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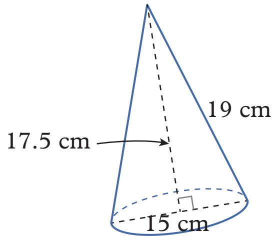

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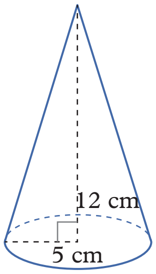

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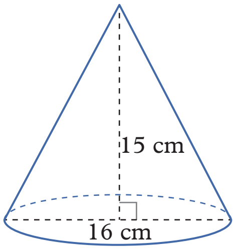

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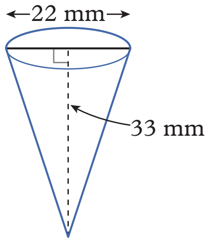


7 A square-based right pyramid has a surface area of 561 cm^2. If the length of the base is 11 cm, find the height of all the triangular faces.

8 A cone has a radius of 15.5 cm and a height of 9.5 cm. Use Pythagoras' Theorem to find the slant height of the cone correct to one decimal place and its total surface area.

9 **a** What is the surface area of a regular tetrahedron with a side length of 4 cm?

b Develop a formula to determine the surface area of a regular tetrahedron with a side length of x cm.

10 A party hat is in the shape of a cone. The diameter of the base is 12 cm and the slant height is 18 cm. Find the amount of cardboard needed to make this party hat assuming there is no overlap. Give your answer correct to one decimal place. (Hint: Do you need to use all parts of the formula for the total surface area of a cone?)

11 An eight-sided die is in the shape of an octahedron.

a How many equilateral triangles form the octahedron?

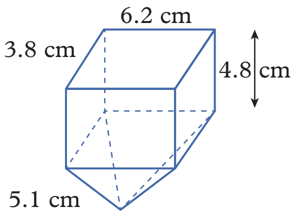
b If the sides of the triangular faces measure 14 mm, find the total surface area of the octahedron correct to one decimal place.

c The octahedron resembles two identical square pyramids sitting base to base. Calculate the surface area of one pyramid and double your answer.

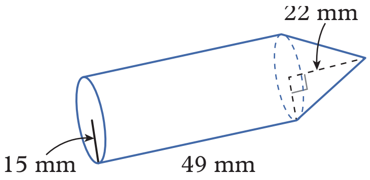
d Are your answers for parts **b** and **c** equal? Why or why not?

12 Calculate the surface area of each of these composite solids. Give your answers correct to one decimal place.

a



b



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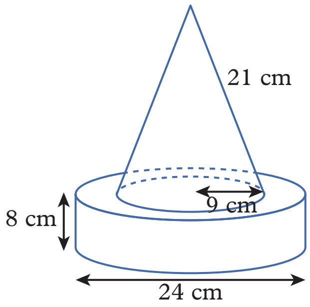
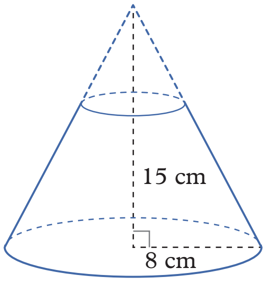


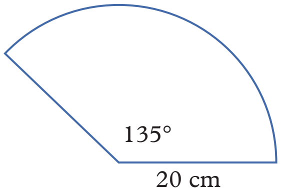
CHALLENGE

13 The tip of a hollow cone is sliced through and removed. The remaining object is called a truncated cone. If the removed tip was one-third of the height of the whole cone, find the outer surface area of the truncated cone correct to one decimal place. (Assume the cone includes a base.)



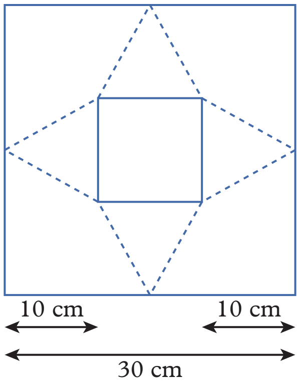
14 An open cone can be made from a circle sector by curving the shape to join the two straight edges. Then the centre of the circle becomes the apex of the cone and the radius of the circle sector becomes the slant length of the open cone.

- **a** Consider the following sector:
  - **i** Find the area of this circle sector (and therefore the surface area of the open cone made from this sector). Give your answer to one decimal place.
  - **ii** Hence, find the radius of the open cone made from this sector correct to one decimal place.

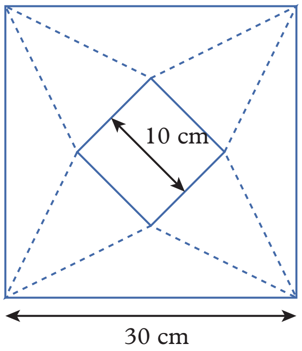


- **b** A circle sector with radius r and interior angle θ is made into an open cone.
  - **i** Find the value of θ if a radius of the cone is half the radius of the circle segment.
  - **ii** Find the value of θ if a radius of the cone is one third the radius of the circle segment.

15 Rhonda wants to create a square-based right pyramid with a 30 cm by 30 cm square sheet of paper. She draws the following dotted lines on the paper to cut out.



- **a** Find the surface area of the pyramid that Rhonda will create by cutting the dotted lines and folding the triangles up.
- **b** Find the fraction of the square paper that will go to waste.
- **c** Before cutting, Rhonda's friend Daniele suggests a different way to cut the paper in order to create a taller right pyramid with the same square base. She draws her design on the back of the paper, as shown below.
  - **i** Use Pythagoras' Theorem to find the height of the triangles that will become the slanted faces of the pyramid. Give your answer correct to one decimal place.
  - **ii** Hence, find the total surface area of the pyramid that is created by cutting the dotted lines this way. Give your answer correct to one decimal place.
  - **iii** Find the percentage of the square paper that will go to waste with this design. Give your answer correct to one decimal place.



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Interactive skillsheet
Surface area of pyramids and cones

Topic quiz
9D

9E Volume of pyramids and cones

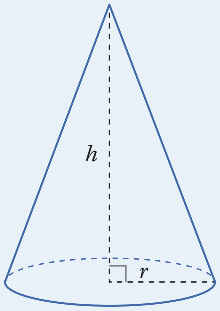
Learning intentions

✔ I can calculate the volume of pyramids and cones.

Inter-year links
Year 9 6C Volume and capacity

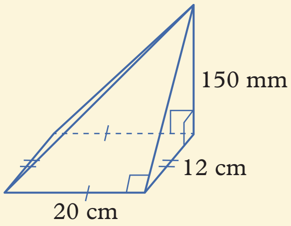
Volume of pyramids and cones

- The volume of any pyramid (oblique or right) can be found using the formula $V = \frac{1}{3}Ah$, where A is the area of the base and h is the perpendicular height of the pyramid.
- The volume of a cone (oblique or right) can be found using $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the perpendicular height of the cone.



Example 9E.1 Calculating the volume of a pyramid

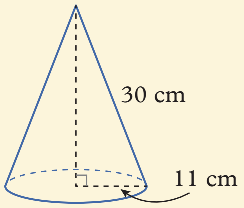
Find the volume of this pyramid.



THINK	WRITE
1 Identify the shape of the base of the pyramid and calculate the area of the base.	The pyramid has a rectangular base. $l = 20$ cm $\quad w = 12$ cm $A = lw$ $= 20 \times 12$ $= 240\text{ cm}^2$
2 Convert the measurement given for the perpendicular height to the same units as the sides of the base.	$h = 150$ mm $= 15$ cm
3 Substitute the values of the perpendicular height and the area of the base into the formula for the volume of a pyramid.	$V = \frac{1}{3}Ah$ $= \frac{1}{3} \times 240 \times 15$
4 Calculate the result and include the appropriate unit.	$= 1200\text{ cm}^3$

Example 9E.2 Calculating the volume of a cone

Calculate the volume of this cone, correct to one decimal place.



THINK	**WRITE**
1 Calculate the value of the cone's perpendicular height by using Pythagoras' Theorem.	$h^2 + 11^2 = 30^2$ $h^2 = 900 - 121$ $h^2 = 779$ $h = \sqrt{779}$
2 Substitute the values of the radius of the base and the perpendicular height into the formula for the volume of a cone.	$r = 11$ cm $V = \frac{1}{3}\pi r^2 h$ $= \frac{1}{3} \times \pi \times 11^2 \times \sqrt{779}$ $= 3536.573...$
3 Write the answer correct to one decimal place and include the appropriate unit.	≈ 3536.6 cm^3

Helpful hints

- ✔ If the perpendicular height of a pyramid or cone is not given, you may need to calculate it using Pythagoras' Theorem.
- ✔ The formulas for the volume of a pyramid or a cone are essentially the same, with πr^2 representing the area of the base of a cone.

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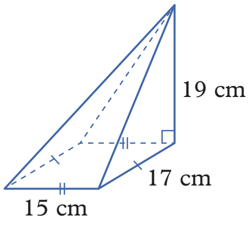
Exercise 9E Volume of pyramids and cones

△ 1–5, 8, 10, 11 ■ 1(d–i), 2, 4, 6, 9, 12, 15, 16 ⬠ 1(e, h, i), 2, 4(e, f), 6, 7, 9, 13–18

UNDERSTANDING AND FLUENCY

9E.1 **1** Calculate the volume of each of these pyramids. Give your answers correct to one decimal place where appropriate.

a



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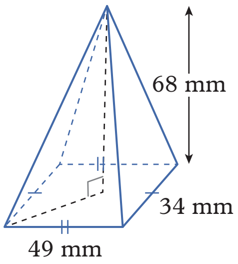

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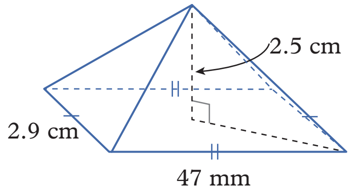

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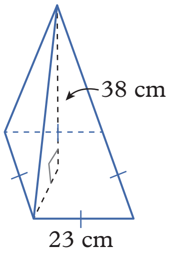

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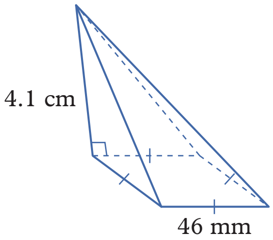

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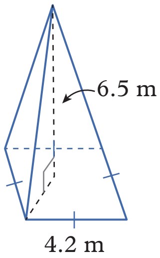

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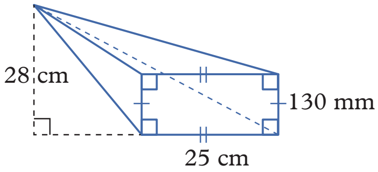

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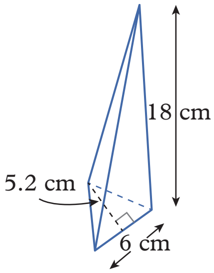
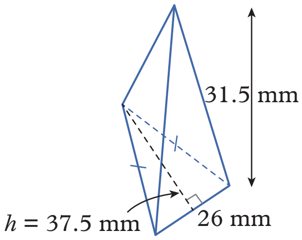
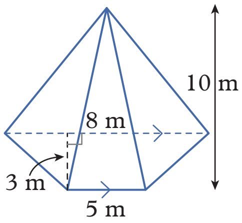


i



2 Find the volume of each of these pyramids by first calculating the area of the base. Give your answers correct to one decimal place where appropriate.

a



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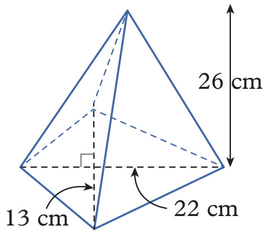

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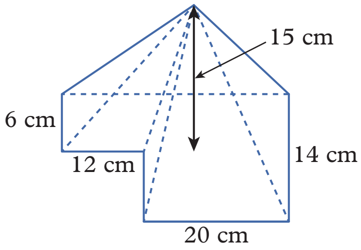
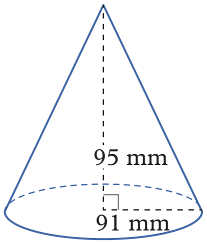


3 The Great Pyramid of Cheops in Giza, Egypt, is a square-based pyramid. Its perpendicular height is approximately 147 m and the length of its base is approximately 230 m. Calculate its volume.

9E.2 **4** Find the volume of each of these cones. Give your answers correct to one decimal place.

a



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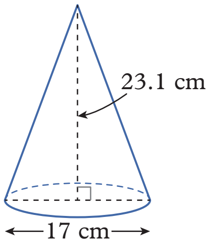

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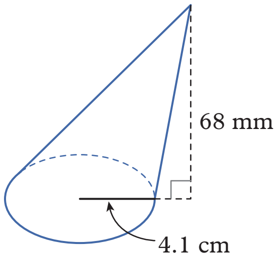

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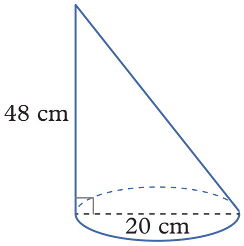

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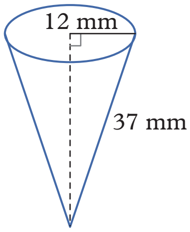

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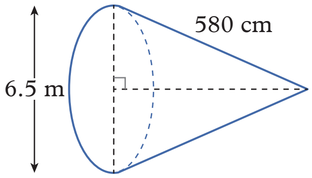
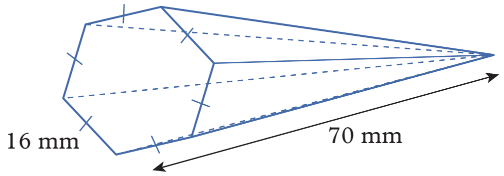


5 A traffic cone has a base with a diameter of 270 mm and it is 850 mm tall. Find the volume of the cone, giving your answer in cubic centimetres correct to one decimal place.

PROBLEM SOLVING AND REASONING

6 Calculate the volume of this hexagonal pyramid correct to one decimal place.



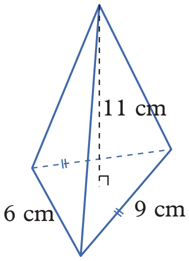
7 **a** Use Pythagoras' Theorem to find the height of the triangular base of this tetrahedron. Then find the area of the base. Give your answers correct to one decimal place.

b Use your answer for part **a** to help you find the volume of the tetrahedron. Give your answer in:

i cubic centimetres

ii cubic millimetres.

c Write a formula for finding the volume of a tetrahedron.



8 **a** If a cone has a volume of 89.8 cm^3 and the radius of its base is 3.5 cm, find the height of the cone correct to one decimal place.

b If a second cone has the same volume as the cone from part **a**, but has a height of 10 cm, find the radius of this cone correct to one decimal place.

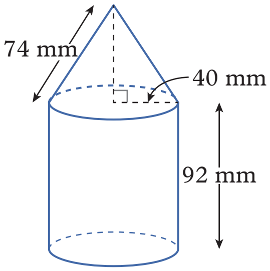
9 A vase has the shape of an open inverted cone sitting in a frame. The circular top of the vase has a radius of 12 cm and the height of the vase is 46 cm.

If the vase is filled to three-quarters of its capacity with water, find the amount of water, in litres, that the vase contains. Give your answer correct to one decimal place.

10 **a** Calculate the total surface area of this solid, correct to one decimal place.

b Which unknown dimension is also needed to calculate the volume of this solid?

c Use Pythagoras' Theorem to find the unknown length. Calculate the volume of the solid. Give your answers correct to one decimal place.



11 A cone has a base with a radius of 3 cm. The cone is 5 cm tall.

a Calculate the volume of the cone correct to one decimal place.

b Double the radius of the cone.

i What is the new radius?

ii If the height does not change, calculate the volume of the new cone correct to one decimal place.

c Double the radius once again. Using the same height, find the volume of the new cone correct to one decimal place.

d Compare your answers for parts **a**, **b** and **c**. Describe what happens to the volume of a cone if the radius is doubled and the height remains the same.

e This time, double the height of the original cone and leave the radius unchanged. Calculate the volume of the cone if the radius of the base is still 3 cm but the cone is now 10 cm tall. Give your answer correct to one decimal place.

f Double the height of the cone again, but do not change the radius. Calculate the volume of this new cone correct to one decimal place.

g Can you predict what the volume of the cone will be if the height is doubled once again but the radius of the base remains unchanged? Calculate the volume to see if your prediction is correct.

h Write a sentence describing the effect on the volume of a cone when the height is doubled but the radius remains the same.

i Why is the volume increased by a different factor when the radius is doubled compared to when the height is doubled?

12 This pyramid framework is 8 cm high, has a base length of 9 cm and a width of 5 cm. A coin of diameter 3 cm and thickness 0.2 cm sits in the centre of the pyramid. What percentage of the space within the pyramid does the coin occupy? Give your answer correct to one decimal place.

13 A cone of height 40 cm and radius 12 cm is placed inside a square-based pyramid. The pyramid has a height of 40 cm and the edges of its square are 24 cm. This means that the pyramid is just big enough to contain the cone.

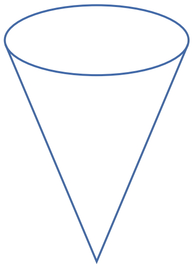
a Find the volume of the cone correct to one decimal place.

b Find the volume of the pyramid correct to one decimal place.

c Hence, find the volume of empty space between the pyramid and the cone.

d Find the exact ratio of the space in the pyramid taken up by the cone to the space in the pyramid that is empty.

14 A paper cup is in the shape of an open cone. The radius of the cone is 5 cm and the height of the cone is 10 cm.

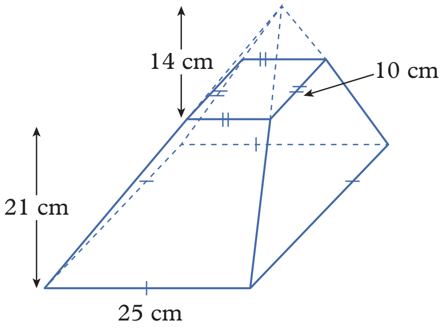


a Find the volume of the cone correct to one decimal place.

b Find the height of the water in the cone when the cup is half full. Give your answer correct to one decimal place.

Hint: The shape of the water will be a similar cone to the paper cup, meaning it will have the same ratio between the radius and the height.

15 The tip or apex of a square pyramid is cut off parallel to the base and is removed. The object remaining is called a truncated pyramid.



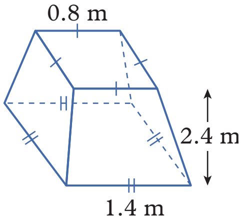
a What shapes are the faces of the truncated pyramid?

b Calculate the volume of the square pyramid before the tip has been removed.

c Calculate the volume of the smaller pyramid that has been cut off correct to one decimal place.

d Use your answers to parts **b** and **c** to calculate the volume of the truncated pyramid on the right.

16 The base of a statue was cut from a concrete block formed as a square-based pyramid. If the height of the original block was 3.1 m before it was truncated, what total volume of concrete was needed to construct this base? Give your answer in cubic metres correct to one decimal place.



17 A drinking glass is 120 mm tall and is in the shape of a truncated cone. The base of the glass has a diameter of 55 mm and the top of the glass has a diameter of 75 mm.

a If the height of the conical piece of glass used to construct the drinking glass was 192 mm, calculate the volume of the drinking glass correct to one decimal place.

b How many millilitres of iced tea can this glass hold when it is filled to its brim? Give your answer correct to the nearest whole number.

18 It is possible to write a formula for finding the volume of a truncated square-based pyramid and a truncated cone. Use your answers to questions **15–17** to assist you to find these formulas.

Check your Student obook pro for these digital resources and more:

pro

Interactive skillsheet
Volume of pyramids and cones

Investigation
Ice cream cone designs

Topic quiz
9E

9F Surface area and volume of spheres

Learning intentions

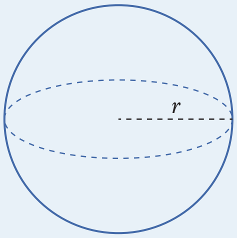
✔ I can calculate the surface area and volume of spheres.

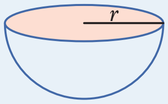
Inter-year links

Year 8	8E Area of a circle
Year 9	6B Surface area

Spheres

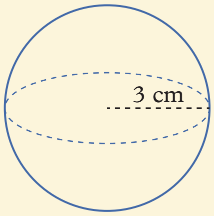
- A **sphere** is a round 3D object, with every point on its surface equidistant from its centre. The size of a sphere is determined by its radius (or diameter).
- The formula for the total surface area of a sphere is:
  TSA = $4\pi r^2$
- The formula for the volume of a sphere is:
  $V = \frac{4}{3}\pi r^3$
- A hemisphere is half of a sphere.
  In this book, closed hemispheres are drawn with a shaded circular face.





Example 9F.1 Calculating the surface area of a sphere

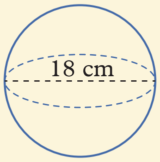
Calculate the total surface area of a sphere with a radius of 3 cm, correct to one decimal place.



THINK	WRITE
1 Substitute the length of the radius into the formula for the surface area of a sphere.	$r = 3$ cm TSA = $4\pi r^2$ $= 4 \times \pi \times 3^2$
2 Calculate the result, using the π key on your calculator.	$= 4 \times \pi \times 9$ $= 36\pi$ $= 113.097...$
3 Write your answer correct to one decimal place and include the appropriate unit.	≈ 113.1 cm^2

Example 9F.2 Calculating the volume of a sphere

Calculate the volume of this sphere correct to one decimal place.



THINK	WRITE
1 Calculate the length of the radius of the sphere.	$r = 18 \div 2$ $= 9 \text{ cm}$
2 Substitute the length of the radius into the formula for the volume of a sphere.	$V = \frac{4}{3}\pi r^3$ $= \frac{4}{3} \times \pi \times 9^3$
3 Calculate the result using the π key on your calculator.	$= 972\pi$ $= 3053.628...$
4 Write the answer correct to one decimal place and include the appropriate unit.	$\approx 3053.6 \text{ cm}^3$

Example 9F.3 Calculating the radius of a sphere, given the volume

Calculate the radius of a sphere that has a volume of 500 cm^3. Give your answer correct to one decimal place.

THINK	WRITE
1 Substitute the given volume into the formula for the volume of a sphere.	$V = \frac{4}{3}\pi r^3$ $500 = \frac{4}{3}\pi \times r^3$
2 Manipulate the equation so r^3 is alone on one side of the equation.	$500 \div \frac{4}{3}\pi = r^3$ $119.366... = r^3$
3 Use a calculator to find the cube root of the other side of the equation.	$r = \sqrt[3]{119.366...}$ $r = 4.923...$
4 Write the answer correct to one decimal place and include the appropriate unit.	$r \approx 4.9 \text{ cm}$

Helpful hint

✔ When working with surface areas and volumes, always remember to include units with your final answers.

ANS
p779

Exercise 9F Surface area and volume of spheres

▲ **1–9, 11, 15, 16(a, b)**

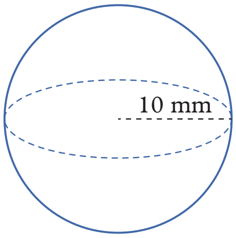
■ **1–4, 6, 8, 12, 14, 16(b, c), 18**

⬟ **1(c, d), 2(c, d), 3, 4, 8(b, e, f), 10, 13, 14, 17, 19, 20**

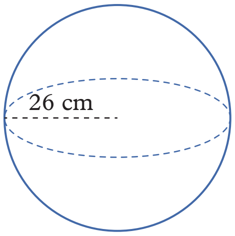
UNDERSTANDING AND FLUENCY

9F.1 **1** Calculate the total surface area of each of these spheres correct to one decimal place.

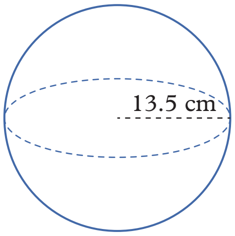
a



b



c



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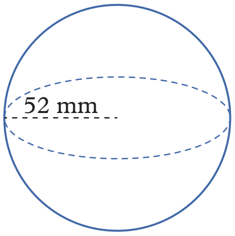
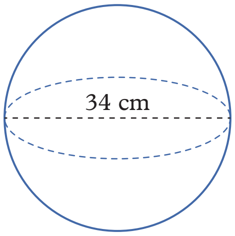


2 Calculate the total surface area of each of these spheres by first calculating the radius. Give your answers correct to one decimal place.

a



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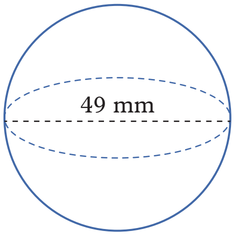

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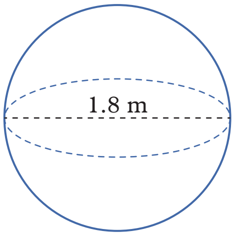

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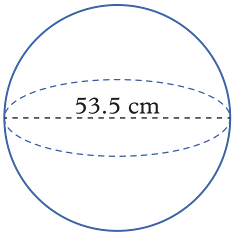


9F.2 **3** Calculate the volume of each sphere in question **1** correct to one decimal place.

4 Calculate the volume of each sphere in question **2** correct to one decimal place.

9F.3 **5** Calculate the radius of a sphere that has a volume of 5575 cm^3. Give your answer correct to the nearest cm.

6 **a** Find the radius of a sphere that has a surface area of 144π cm^2.

b Find the radius of a sphere that has a volume of 972π cm^3.

c Find the radius of a sphere that has a surface area of 2124 cm^2.

d Find the diameter of a hemisphere that has a volume of 1072 cm^3.

7 Consider the following solid sphere on the right.

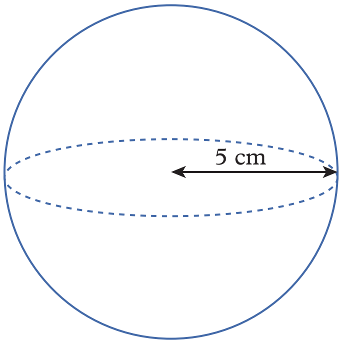
a Find the surface area of the sphere, as an exact value.

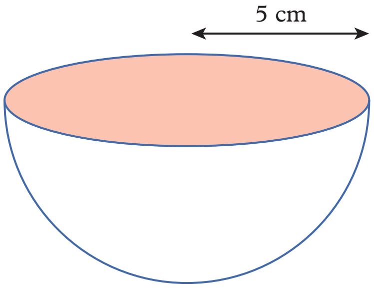
b The sphere is cut exactly in half, as shown below.

i Find the area of the circular face, as an exact value.

ii Find the area of the curved surface, as an exact value.

iii Hence, find the total surface area of the solid hemisphere, as an exact value.





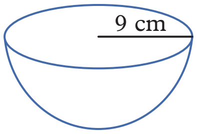
8 For each hemisphere below, calculate:

i its total outer surface area correct to one decimal place

ii its volume correct to one decimal place.

(Hint: First determine whether each hemisphere is open or closed. In this book closed hemispheres are drawn with a shaded circular face.)

a



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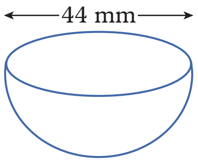

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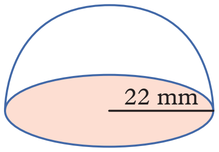

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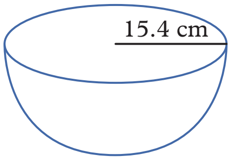

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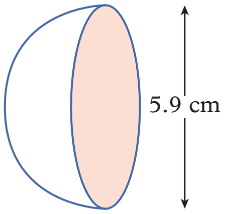

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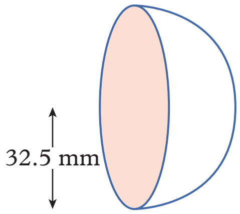


9 When inflated, the beach ball on the right has a diameter of 76 cm.

a Calculate its total surface area correct to one decimal place.

b Find the volume of air, correct to one decimal place, that it can hold.

10 a The volume of a fully inflated volleyball is 0.15 m^3. Find the radius of the volleyball, to the nearest centimetre.

b The total surface area of a tennis ball is approximately 564 cm^2. Find the radius of the tennis ball correct to one decimal place.

11 The approximate diameters of the planets in our solar system are listed in the table below.

Planet	Diameter (km)	Total surface area	Volume
Mercury	4878		
Venus	12 104		
Earth	12 756		
Mars	6787		
Jupiter	142 800		
Saturn	120 000		
Uranus	51 118		
Neptune	49 528		

a Copy and complete the table. Write the values in scientific notation correct to two significant figures ($a \times 10^m$ where $1 \le a < 10$ and m is an integer).

b Approximately how many times greater is Earth's volume compared to the volume of Mercury?

c Approximately how many times greater is Jupiter's surface area compared to Earth's surface area?

d Approximately 71% of Earth's surface is covered by water. To the nearest square kilometre, how much of Earth's surface area is covered by water?

12 The total surface area of the rind of this blood orange is 116.9 cm^2.

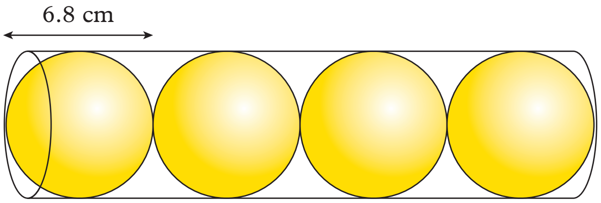
- **a** What is the diameter of the orange correct to one decimal place?
- **b** The thickness of the rind is approximately 1 cm. What volume does the edible part of the orange occupy to the nearest cubic centimetre.

13 A balloon is inflated to become a sphere with a radius of 9.5 cm.

- **a** What is the volume of the balloon correct to one decimal place?
- **b** Two more balloons are inflated to exactly the same size as the first balloon. What is the total volume of the three balloons correct to one decimal place?
- **c** When objects are filled with a liquid or gas, you can refer to their capacity rather than volume.

  What is the capacity of the three balloons, in litres? Give your answer correct to one decimal place.
- **d** If the rate that the balloons can be inflated is 2.5 L/s, how long will it take to inflate all three balloons? Give your answer correct to one decimal place.
- **e** What assumptions do you need to make when answering part **d**?

14 Tennis balls are sold in canisters. Each canister holds four tennis balls.

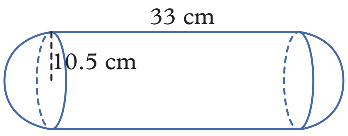


- **a** If each tennis ball has a diameter of 6.8 cm, calculate the volume of one tennis ball correct to one decimal place.
- **b** What is the combined volume of all four tennis balls correct to one decimal place?
- **c** Determine the dimensions of the canister and find its volume correct to one decimal place. (Assume that the tennis balls fit perfectly into the canister.)
- **d** What amount of the space in the canister is not taken up by tennis balls? Give your answer correct to one decimal place.
- **e** If the tennis balls were to be sold in a container with the same dimensions but in the shape of a rectangular prism instead of a cylinder, without any further calculations, would there be more or less unused space?

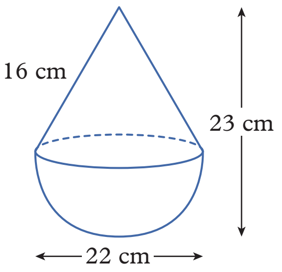
15 Rebecca buys an ice-cream cone. As well as the ice cream on top of the cone, the cone is completely filled with ice cream. Calculate the total amount of ice cream Rebecca has if the top forms a perfect hemisphere. Give your answer to the nearest millilitre.

16 Calculate the volume of each of these composite solids correct to one decimal place.

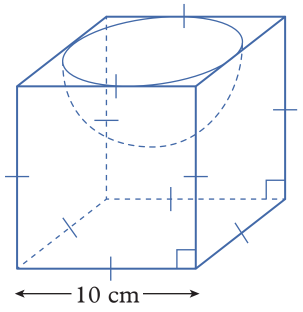
a



b



c



17 The bricks shown are all equal in size and measure 23 cm by 11.5 cm by 7.5 cm. The holes on each brick are hemispheres and the diameter of each hole is 2.5 cm. Calculate the volume of clay required to make the five bricks. Give your answer to the nearest cm^3.

18 a A ping pong ball has a diameter of 40 mm. Find its volume correct to one decimal place.

b It is also possible to find the density of a solid. Density is defined as the mass per unit volume and can be found using the formula:

$$\text{density} = \frac{\text{mass}}{\text{volume}}$$

If the mass of the ping pong ball is 2.7 g, calculate its density in g/cm^3. Give your answer correct to two decimal places.

19 Investigate the effect that each of the following situations has on:

i the total surface area of a sphere

ii the volume of a sphere.

- **a** doubling the radius
- **b** tripling the radius
- **c** halving the radius
- **d** dividing the radius by 3.

20 a A solid sphere of radius 10 cm is placed inside the smallest cube that can contain it.

- **i** Find the side length of the cube.
- **ii** Find the volume of empty space in the cube correct to one decimal place.

b A solid cube of side length 10 cm is enclosed in the smallest sphere that can contain it.

- **i** Find the side length of the sphere correct to one decimal place.
- **ii** Find the volume of empty space in the sphere correct to one decimal place.

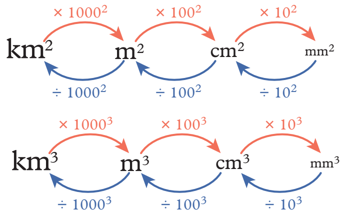
Check your Student obook pro for these digital resources and more:

pro

Interactive skillsheet Surface area of spheres	Interactive skillsheet Volume of spheres	Topic quiz 9F

Chapter summary

Units

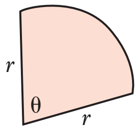


Area of a composite shape

1. Split the shape into individual parts.
2. Calculate any missing dimensions.
3. Calculate the areas of the individual parts.
4. Add or subtract the areas to find the total area required.

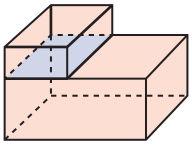
Area of a sector

- Area = $\frac{\theta}{360^\circ} \times \pi r^2$



Total surface area

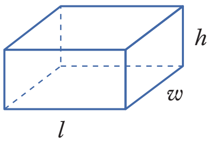
- The total surface area (TSA) of a 3D object is the total area of its outer surface. This is the sum of the areas of the faces (or surfaces) of that object.



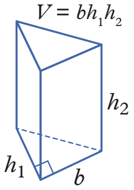
Volume of a prism

- For any prism, $V = Ah$, where A is the area of the base and h is the height of the prism.

$V = lwh$



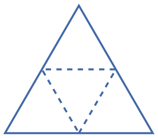
$V = bh_1h_2$



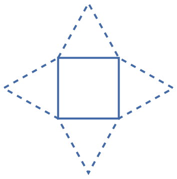
Pyramids

Surface area

The TSA is the sum of the areas of all the faces.



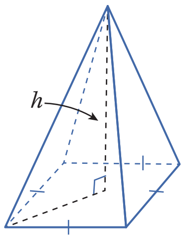
net of a regular tetrahedron



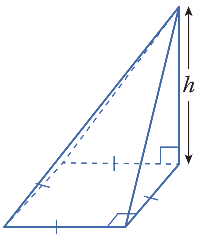
net of a square pyramid

Volume

$V = \frac{1}{3}Ah$, where A is the area of the base and h is the perpendicular height of the pyramid.



right square pyramid

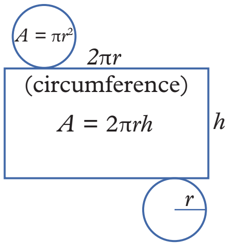


oblique square pyramid

Cylinders

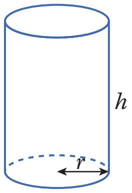
Surface area

TSA = $2\pi r^2 + 2\pi rh$



Volume

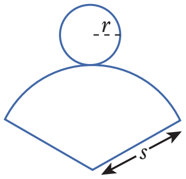
$V = \pi r^2 h$



Cones

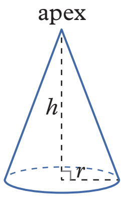
Surface area

TSA = $\pi r^2 + \pi rs$



Volume

$V = \frac{1}{3}\pi r^2 h$



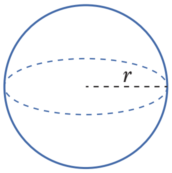
Spheres

Surface area

TSA = $4\pi r^2$

Volume

$V = \frac{4}{3}\pi r^3$



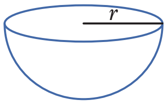
Hemispheres

Surface area

TSA = $3\pi r^2$

Volume

$V = \frac{2}{3}\pi r^3$



Chapter review

Chapter review quiz
Take the chapter review quiz to assess your knowledge of this chapter.

Quizlet
Test your knowledge of this topic by working individually or in teams.

Multiple-choice

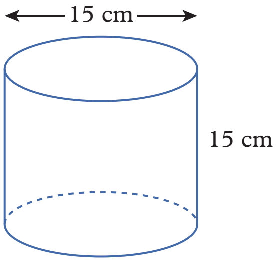
9A **1** Which of the following is equivalent to 300 cm^2?

A 3 m^2 **B** 0.3 m^2 **C** 3000 mm^2 **D** 0.03 m^2 **E** 30 mm^2

9B **2** What is the TSA of a cube with side lengths of 8 cm?

A 384 cm^2 **B** 48 cm^2 **C** 64 cm^2 **D** 512 cm^2 **E** 288 cm^2

9B **3** A closed cylinder has a base diameter of 15 cm and a height of 15 cm. Which of the following is closest to the total surface area of the cylinder?



A 225 cm^2 **B** 700 cm^2 **C** 750 cm^2 **D** 1060 cm^2 **E** 2800 cm^2

9C **4** A cylinder has a base diameter of 6 cm and a height of 20 cm. Which of the following is closest to the volume of the cylinder?

A 120 cm^3 **B** 188 cm^3 **C** 377 cm^3 **D** 565 cm^3 **E** 2262 cm^3

9C **5** A solid wooden cylinder has a base with radius 10 cm and is 10 cm tall. It has a cylinder of radius 2 cm drilled through its centre, from top to base. What is the volume of timber remaining (in cubic centimetres)?

A 640π cm^3 **B** 1000π cm^3 **C** 960π cm^3 **D** 80π cm^3 **E** 40π cm^3

10A 9D **6** A square-based pyramid has a total surface area of 224 cm^2. If the length of the base is 8 cm, what is the height of each triangular face?

A 56 cm **B** 14 cm **C** 10.5 cm **D** 10 cm **E** 3.5 cm

10A 9D **7** How many triangular faces does a tetrahedron have?

A 1 **B** 3 **C** 4 **D** 5 **E** 6

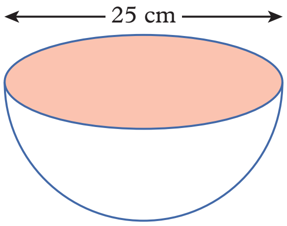
10A 9E **8** A square-based pyramid has the same base length and height as a cube with side lengths of 6 cm. What is the volume of the pyramid?

A 216 cm^3 **B** 72 cm^3 **C** 12 cm^3 **D** 108 cm^3 **E** 144 cm^3

10A 9E **9** Which of the following is closest to the volume of a cone with a diameter of 2.5 cm and a height of 10 cm?

A 15 cm^3 **B** 25 cm^3 **C** 50 cm^3 **D** 65 cm^3 **E** 200 cm^3

10A 9F **10** Which of the following is closest to the total exposed surface area if a solid sphere with a diameter of 25 cm is sliced in half?



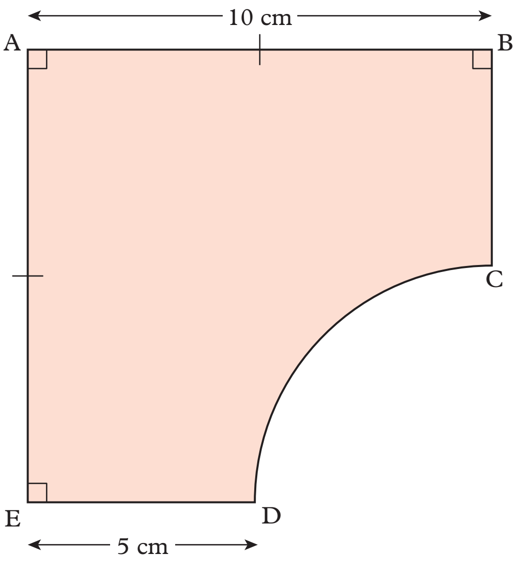
A 500 cm^2 **B** 1500 cm^2 **C** 3000 cm^2 **D** 4000 cm^2 **E** 6000 cm^2

Short answer

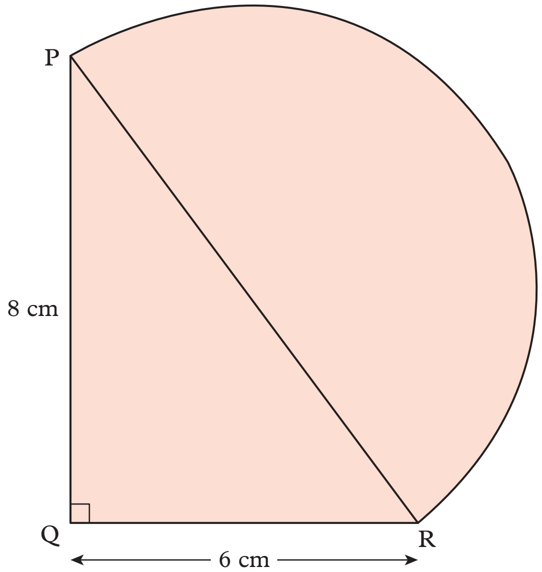
9A **1** A flat metal washer is in the form of an annulus. The radius of the inner ring of the washer is 1 cm, while the outer ring has a radius of 1.5 cm. What is the area of the annulus correct to one decimal place?

9A **2** Determine the area of each of these composite shapes. Give your answers correct to one decimal place.

a



b



9B **3** The base of a triangular prism is a right-angled triangle with a base length of 25 mm and a height of 18 mm. The height of the prism is 30 mm. What is the TSA of the prism?

9B **4** A set of three wooden cubes was made for a playground. The smallest cube had a side length of 40 cm. The next sized cube had side lengths twice as long as the smallest cube, while the side lengths of the largest cube were twice as long as those of the middle cube.

a Calculate the total surface area of each cube.

b If 1 L of paint covers an area of 15 m^2, and the paint is only available in 5 L cans, how many cans of paint would be required to paint all three cubes?

9C **5** A can of baked beans has a cylindrical shape with a base diameter of 7 cm and a height of 10 cm. What is the capacity of the can correct to the nearest millilitre?

9C **6** Calculate the volume of each cube in question **4**.

10A 9D **7** A square-based pyramid has a base with sides 5 cm in length. The height of each triangular face is 20 cm. Calculate the total surface area of the pyramid.

10A 9D **8** The diameter of a closed cone is 10 cm and the cone's height is 16 cm. Calculate the total surface area of the cone correct to one decimal place.

10A 9E **9** A square-based pyramid has a base with sides 30 cm in length. The triangular faces are equilateral. Find the volume of the pyramid.

10A 9F **10** A sphere fits neatly inside a cylinder with a base radius of 8 cm. Find the volume of the cylinder and the sphere in terms of π.

10A 9F **11** **a** Calculate the total surface area, correct to one decimal place where appropriate, of:

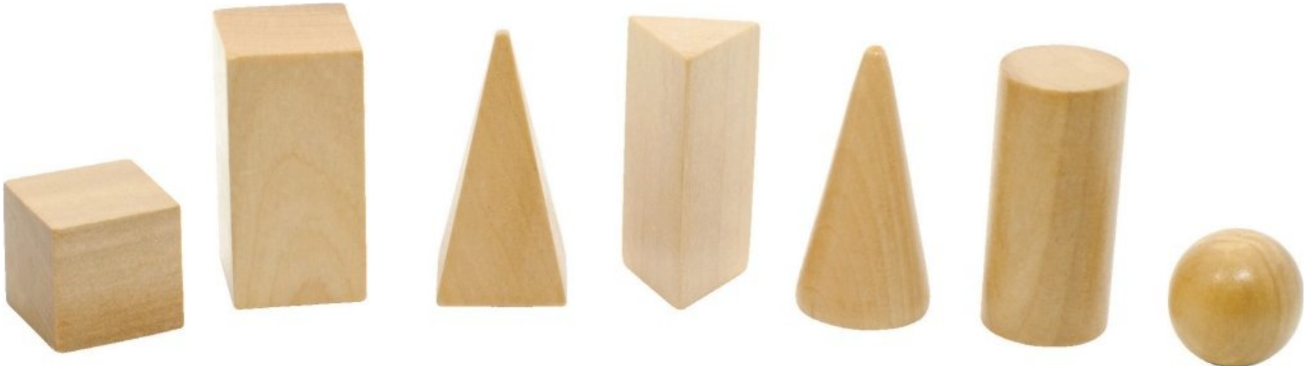
i a solid hemisphere with a diameter of 40 cm

ii a cube with sides 40 cm in length.

b The hemisphere is placed on top of the cube. What is the total surface area of this object?

Analysis

Ethan bought this set of wooden blocks for his young cousins.



a Name the solids in order, from left to right.

b The cube has side lengths of 3 cm. Calculate its:
  - **i** total surface area
  - **ii** volume.

c The rectangular prism has a base the same size as the cube, and is twice the height of the cube.
  - **i** Compare the surface area of the rectangular prism with that of the cube.
  - **ii** Compare the volume of the rectangular prism with that of the cube.

d The triangular prism is the same shape as the rectangular prism cut vertically along the diagonal of its base.
  - **i** Draw a labelled diagram showing the shape of the triangular prism's base.
  - **ii** Calculate the total surface area of the triangular prism correct to one decimal place.
  - **iii** Is the TSA of the triangular prism half that of the rectangular prism that you calculated in part **c**? Explain.
  - **iv** Compare the volumes of the triangular prism and the rectangular prism.

e The base of the cylinder has a diameter the same length as the side of the cube, and the cylinder is the same height as the rectangular prism. Calculate, correct to one decimal place, the cylinder's:
  - **i** total surface area
  - **ii** volume.

f Create two tables listing the solids in parts **b–e** in increasing order of:
  - **i** surface area
  - **ii** volume.

10A **g** The square base of the pyramid is the same shape as that of the base of the cube. Its height is the same as that of the rectangular prism. Calculate, correct to one decimal place, the pyramid's:
  - **i** total surface area
  - **ii** volume.

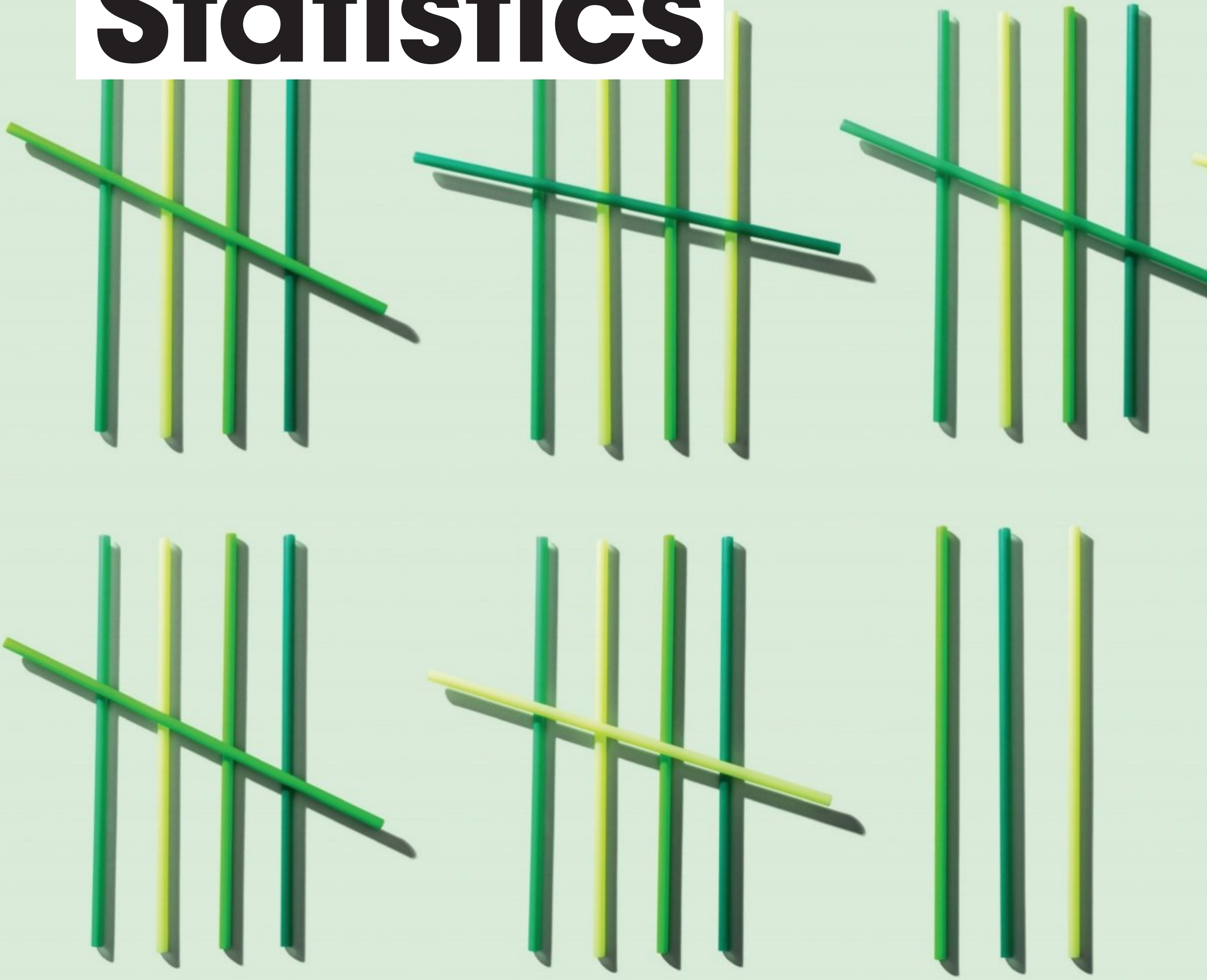
10A **h** The cone has the same base diameter and height as the cylinder. Calculate, correct to one decimal place, the cone's:
  - **i** total surface area
  - **ii** volume.

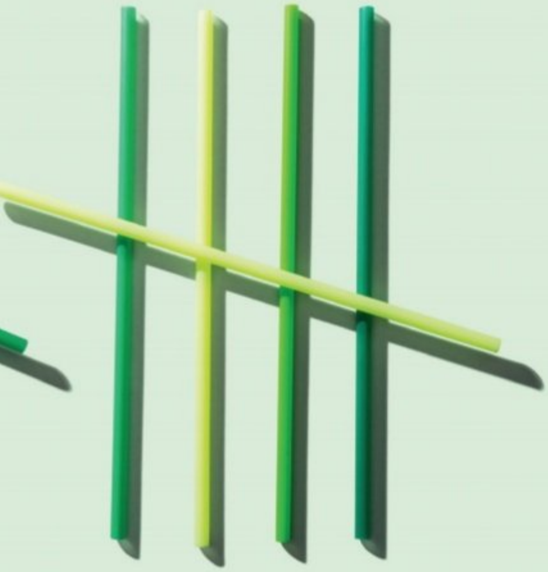
10A **i** The sphere has the same diameter as the cone. Calculate, correct to one decimal place, the sphere's:
  - **i** total surface area
  - **ii** volume.

10A **j** Using your answers for parts **g**, **h** and **i**, update your tables from part **f**.

10

Statistics





Index

Prerequisite skills

Diagnostic pre-test

Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets

After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✓ The mean
- ✓ The median
- ✓ Interpreting graphs
- ✓ Histograms

Curriculum links

- Determine quartiles and interquartile range and investigate the effect ... (VCMSP349)
- Construct and interpret box plots and use them to compare data sets (VCMSP350)
- Compare shapes of box plots to corresponding histograms and dot plots ... (VCMSP351)
- Use scatter plots to investigate and comment on relationships ... (VCMSP352)
- Investigate and describe bivariate numerical data ... (VCMSP353)
- Evaluate statistical reports in the media and other places ... (VCMSP354)
- Investigate reports of studies in digital media and elsewhere ... (VCMSP371) [10A]
- Calculate and interpret the mean and standard deviation of data ... (VCMSP372) [10A]
- Use digital technology to investigate bivariate numerical data sets ... (VCMSP373) [10A]

10A Five-number summary and interquartile range

Learning intentions

- ✔ I can find the five-number summary for a numerical data set.
- ✔ I can calculate the interquartile range of a numerical data set.

Inter-year links

Year 5/6	Understanding data
Year 7	10B Summary statistics
Year 8	9B Summary statistics
Year 9	8C Summary statistics from tables and displays

Measures of data distribution

- **Quartiles** divide a numerical data set into four quarters.
- The **median** is the middle value of an ordered numerical data set. The median divides a data set into two halves.
  - → The median is also known as the second quartile, Q_2.
- The **lower quartile** (Q_1) is the median of the lower half of an ordered numerical data set.
- The **upper quartile** (Q_3) is the median of the upper half of an ordered numerical data set.
- When calculating the quartiles for a data set with an odd number of data points, the median is not included in either half of the data set.

$$\underbrace{1\ \ 2\ \ \overset{Q_1}{\boxed{3}}\ \ 4\ \ 5}_{\text{lower half}}\ \ \overset{Q_2}{\boxed{6}}\ \ \underbrace{7\ \ 8\ \ \overset{Q_3}{\boxed{9}}\ \ 10\ \ 11}_{\text{upper half}}$$

- If there is an even number of data points, both middle values are included in the quartile calculations.

$$\underbrace{1\ \ 2\ \ 3\overset{Q_1}{\downarrow}4\ \ 5\ \ 6}_{\text{lower half}}\overset{Q_2}{\downarrow}\underbrace{7\ \ 8\ \ 9\overset{Q_3}{\downarrow}10\ \ 11\ \ 12}_{\text{upper half}}$$

$$\begin{aligned} Q_1 &= \frac{(3+4)}{2} \\ &= 3.5 \\ Q_2 &= \frac{(6+7)}{2} \\ &= 6.5 \\ Q_3 &= \frac{(9+10)}{2} \\ &= 9.5 \end{aligned}$$

- The **five-number summary** for a numerical data set consists of the minimum and maximum values, the lower and upper quartiles and the median.
  - → The five values that make up the five-number summary are displayed in ascending order: minimum, Q_1, Q_2 (median), Q_3, maximum
- The **range** of a data set can be found by subtracting the smallest value (minimum) from the largest value (maximum).
- The **interquartile range** (IQR) represents the range of the middle 50% of a data set. It can be found by subtracting the lower quartile from the upper quartile.

$$\text{IQR} = Q_3 - Q_1$$

- An **outlier** is an extreme value that is much higher or lower than the other values in a data set.
  - → Outliers do not have an impact on the median or interquartile range.

Example 10A.1 Finding the five-number summary

Find the five-number summary for this ordered data set: 2, 2, 3, 4, 5, 6, 6, 6, 8.

THINK	WRITE
1 The data set is already ordered, so find the median by either circling the middle number or finding the average of the two middle numbers.	$\underbrace{2\ \ 2\ \ 3\ \ 4}\ \textcircled{5}\ \underbrace{6\ \ 6\ \ 6\ \ 8}$ $Q_2 = 5$
2 Divide the data set into two equal halves, remembering not to include the median in either half.	
3 Find the upper quartile (Q_1) and the lower quartile (Q_3) by finding the median of each half of the set.	$\underbrace{2\ \ 2\ \ 3\ \ 4}_{Q_1 = 2.5}\ \textcircled{5}\ \underbrace{6\ \ 6\ \ 6\ \ 8}_{Q_3 = 6}$
4 Identify the minimum and maximum scores from the data set.	minimum = 2, maximum = 8
5 Write your answer.	2, 2.5, 5, 6, 8

Example 10A.2 Calculating the interquartile range

Find the interquartile range for this data set:
6, 32, 7, 2, 8, 9, 9, 4, 6, 6, 21, 19, 13, 5

THINK	WRITE
1 Arrange the scores in order.	$\underbrace{2\ \ 4\ \ 5\ \ \textcircled{6}\ \ 6\ \ 6\ \ 7}_{Q_1}\ \underbrace{8\ \ 9\ \ 9\ \ \textcircled{13}\ \ 19\ \ 21\ \ 32}_{Q_3}$
2 Divide the data set into two halves. Because there are 14 data values, each half will contain 7 values.	$Q_1 = 6, Q_3 = 13$
3 Find the value of the lower and upper quartiles by finding the median of each half of the data set.	
4 Subtract Q_1 from Q_3 to obtain the IQR.	$\text{IQR} = Q_3 - Q_1$ $= 13 - 6$ $= 7$

Helpful hints

✔ When placing a set of data in order, count to make sure the ordered set has the same amount of data values as the unordered set.

✔ If there are n values in an ordered data set, the median will be in the $\frac{n+1}{2}$th position. For example, if there are 11 values the median will be in the $\frac{11+1}{2} = 6$th position.

ANS p781

Exercise 10A Five-number summary and interquartile range

△ 1–8, 12, 13 ■ 2, 3, 5, 8–10, 12, 14, 16(a, b) ⬠ 3(d–f), 5(d–f), 8, 10–12, 14–16

UNDERSTANDING AND FLUENCY

1 Find the range for each of these data sets.

a 5, 9, 6, 12, 16, 3, 8, 19, 11, 8, 12 **b** 48, 3, 7, 3, 8, 4, 12, 19, 6, 6, 8, 19, 13
c 110, 167, 189, 102, 144, 117, 166 **d** 54, 46, 78, 11, 19, 22, 26, 57, 67

10A.1 **2** Find the five-number summary for each of these ordered data sets.

a 3, 3, 3, 3, 4, 5, 6, 6, 7, 8, 9
b 7, 9, 11, 12, 13, 13, 18, 19, 35, 42, 45, 46, 61, 62, 78
c 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 4, 5, 6, 6, 6, 6, 7, 7, 7, 8, 8, 9, 9
d 2, 4, 4, 6, 7, 7, 8, 8, 9, 11, 16, 16, 18, 19, 19, 19, 21, 27, 32

3 Find the five-number summary for each of these data sets.

a 4, 8, 9, 12, 7, 6, 6, 6, 7, 4, 3, 2, 8, 9, 11
b 46, 28, 16, 23, 8, 4, 46, 49, 55, 13
c 72, 81, 16, 22, 42, 55, 67, 73, 19, 24, 16, 77, 37
d 724, 169, 843, 120, 564, 786, 341, 186
e 42, 18, 32, 18, 64, 39, 42, 18, 64, 21, 27, 34
f 8, 19, 24, 26, 17, 13, 8, 4, 13, 9, 14, 21, 17, 32, 8, 4, 3, 19

4 Given the following five-number summaries, find the interquartile range each time.

a 2, 6, 10, 16, 19 **b** 17, 22, 25, 31, 33 **c** 7, 9, 13, 19, 24 **d** 34, 40, 52, 59, 66

10A.2 **5** Find the interquartile range for each of these data sets.

a 46, 48, 32, 12, 62, 27, 19, 34, 86, 16, 16
b 8, 1, 6, 4, 23, 5, 7, 16, 3, 3, 14, 13, 7, 5
c 123, 189, 146, 179, 98, 165, 108, 142, 152
d 19, 62, 53, 14, 8, 43, 62, 9, 46, 13, 19, 20
e 5, 9, 11, 16, 13, 7, 13, 18, 21, 34, 12, 8, 7
f 125, 465, 132, 99, 164, 222, 378

PROBLEM SOLVING AND REASONING

6 The stem-and-leaf plot below shows the number of cupcakes a local bakery sold per day over the course of a month. Find the range and the interquartile range and describe the spread of the set.

Stem	Leaf
4	1 2 2 6 7 8
5	1 6 6 7 9
6	1 2 3 3 6 6 8 9
7	2 4 8 9 9
8	0 1 5 5 7
9	1 2

Key: 4 | 2 = 42

7 Consider these two data sets.

Set A: 9, 31, 46, 11, 39, 24, 3, 43, 25, 10, 19, 4, 36, 14, 16

Set B: 16, 26, 29, 18, 14, 23, 14, 57, 26, 21, 16, 18, 14, 19, 19

a Find the range for each data set. What do you notice?

b Find the interquartile range for each data set.

c Explain why the interquartile range should be used to describe the difference between the spread of these two sets.

8 Consider this data about the number of times a year a sample of Year 10 students have dyed their hair.

Number of times hair has been dyed	0	1	2	3	4	5	6
Frequency	10	2	3	7	11	9	10

a Find the range of the data.

b Find the five-number summary.

c Find the interquartile range.

9 Explain what it means if a data set has:

a a small interquartile range

b a small interquartile range and a large range

c a large interquartile range

d an interquartile range that is very similar to its range.

10 Consider this data set, which is the income (in dollars) per week of some Year 10 students.

10	120	50	60	45	85	250	80	100	65	75	20
35	45	50	80	25	10	20	50	100	45	80	

a Calculate the range and the interquartile range of this data.

b Why is the range so much larger than the interquartile range?

c If the data in a set is spread evenly, the range will be roughly twice the size of the interquartile range. Explain why.

d The presence of an outlier can affect the range. Explain why for data sets with more than 5 values an outlier would not affect the interquartile range.

11 For each data set below:

i calculate the range and interquartile range

ii describe the spread of the data.

a 11, 8, 19, 9, 11, 15, 7, 2, 6, 17, 13, 13, 12, 16, 8, 7, 16, 5, 4, 19

b 32, 68, 43, 31, 77, 44, 67, 42, 37, 55, 59, 79, 81, 83, 11, 56, 57, 61, 82, 63

c 123, 118, 131, 141, 142, 136, 128, 127, 119, 129, 138, 127, 191, 132, 134

12 Consider this back-to-back stem-and-leaf plot showing the heights of students in two classes.

a For each class, find:

i the median

ii the range

iii the interquartile range.

b Compare the heights of the students in the two classes. Which class would you say is taller? Which class would you say is more diverse?

Leaf Class A	**Stem**	**Leaf Class B**
	14	9
9	15	6 8
9 8 7 7 7 6 5 4 3 3 2 1 1	16	0 1 3 4 5 5 7 9
9 9 8 7 6 5 5 4 3 2 2 1	17	2 3 4 4 5 5 5 7 9
6 5 4 1	18	1 4 8
	19	7

Key: 1 | 6 = 16

PROBLEM SOLVING AND REASONING

13 Consider the set of data: 80, 50, 60, 66, 74, 71, 83, 55, 70, 79.

a Determine the five-number summary for the data.

b Determine the interquartile range and range for the data.

It turns out the final value, 79, was instead meant to be 9.

c Determine the correct five-number summary for the data.

d Determine the correct interquartile range and range for the data.

e Describe the effect changing 79 to 9 had on the:

i median **ii** range **iii** interquartile range.

f Explain why outliers and other extreme values have little to no effect on the interquartile range.

14 Consider this five-number summary created from a list of 10 numbers.

10, 10, 12, 14, 14

a Determine the number of possible original data sets that would have this five-number summary.

b If the maximum was increased to 15, determine the number of possible original data sets that would have this five-number summary.

CHALLENGE

15 a Determine the value of the upper and lower quartile for each of the following:

i 1, 2, 3, 4, 5, 6, 7, 8

ii 1, 2, 3, 4, 5, 6, 7, 8, 9

iii 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

iv 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11

Wally believes that he has another method to determine the location of the upper and lower quartiles:

1. Divide the number of data values, n, by 2.
2. Divide the ordered data list into groups of the quotient from step **1**, starting from each end.
   If there was no remainder, place a vertical line between the two groups.
   If there was a remainder, put a box around the data value that lies between the two groups.
3. Divide the integer quotient part (excluding the remainder/decimal part) from step **1** by 2.
4. Divide each group from step **2** into groups of the quotient from step **3**, starting from each end of each group.
   If there was no remainder, place a vertical line between each new pair of groups.
   If there was a remainder, put a box around the data value that lies between each new pair of groups.
5. The lower quartile is either the boxed number or the average of the two numbers either side of the vertical line between group 1 and 2.
6. The upper quartile is either the boxed number or the average of the two numbers either side of the vertical line between group 3 and 4.

b Show that all four sets of data from part **a** work with Wally's method.

c Explain why Wally's method works.

d Wally also believes that the remainder, when dividing the number of data values, n, by 4, tells you the number of the median and quartiles that will be data values in the data set while the quotient tells you the number of data values in each group. Explain why this is true.

16 Determine the interquartile range for each of the following.

a 21, 4, −11, −2, −24, −18, −3, −1

b $\frac{13}{6}, \frac{17}{3}, \frac{10}{3}, \frac{7}{3}, \frac{1}{4}, 5, \frac{17}{3}, \frac{2}{3}$

c $\frac{18}{3}, \frac{8}{19}, \frac{8}{3}, \frac{13}{10}, \frac{12}{13}, \frac{14}{13}, \frac{9}{11}, \frac{6}{7}$

d $-\frac{17}{18}, -\frac{1}{2}, -\frac{5}{13}, -\frac{11}{8}, -\frac{20}{7}, \frac{9}{10}, \frac{19}{14}, -\frac{13}{17}$

Check your Student obook pro for these digital resources and more: pro

Interactive skillsheet
Five-number summary and interquartile range

Topic quiz
10A

10B Box plots

Learning intentions

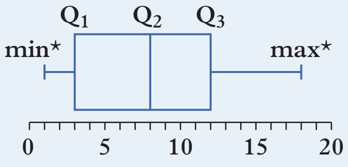
- ✔ I can create and interpret box plots.
- ✔ I can use parallel box plots to compare data sets.

Inter-year links

Year 5/6	Understanding data
Year 7	10B Summary statistics
Year 8	9B Summary statistics
Year 9	9E Comparing data

Box plots

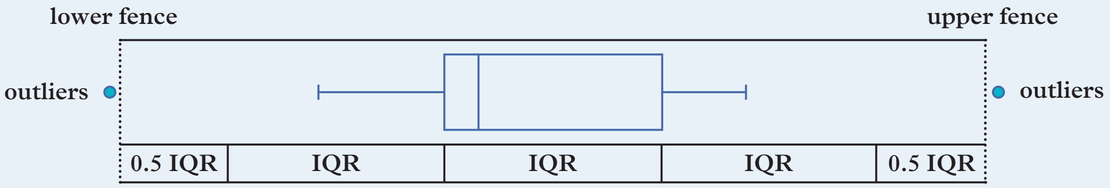
- A **box-and-whisker plot** or **box plot** can be used to represent the five-number summary visually and to display the centre (median) and spread (range and interquartile range) of a numerical data set.
- A box plot consists of a central box with horizontal lines called 'whiskers' on either side of the box.
  - → Box plots should always be accompanied by a clear and even scale.
  - → If the minimum or maximum value is equal to the value of the lower or upper quartile, there will be no whisker on that side of the box plot.
  - → If the median is equal to either quartile, the median line will not be visible.



*excluding outliers

Identifying outliers

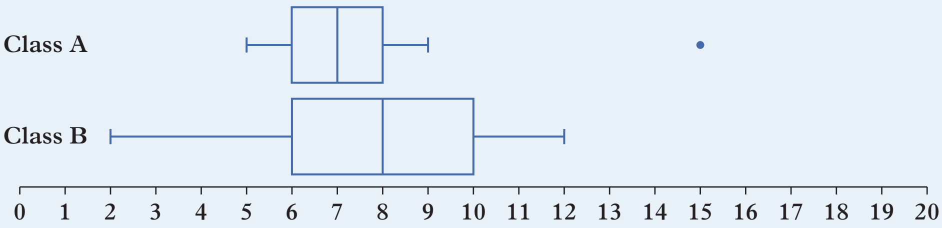
- The quartiles and the interquartile range can be used to identify outliers. The **lower fence** lies 1.5 × IQR below the lower quartile, and the **upper fence** lies 1.5 × IQR above the upper quartile. Values beyond these fences are classified as outliers.
- Outliers should be identified before box plots are drawn. The values shown by the ends of the whiskers are the minimum and maximum values excluding any outliers. Outliers should be represented on box plots as single points using a cross lying beyond the whiskers (sometimes a dot is used).



Using box plots to compare data sets

- **Parallel box plots** can be used to compare data sets on the same scale. Place one box plot directly above the other box plot on the same scale to directly compare key points from each set of data.

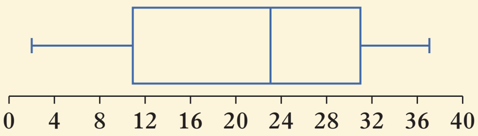
  For example, in the parallel box plots below, the scores of Class A are more consistent than the scores of Class B, except for the outlier in Class A.



Example 10B.1 Understanding box plots

Use this box plot to state the:

a five-number summary **b** interquartile range.

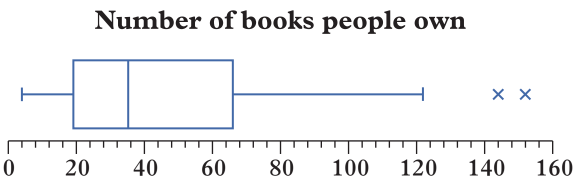


THINK	WRITE
a The end of the left-hand whisker represents the minimum value, the left-hand side of the box represents the lower quartile, the line inside the box represents the median, the right-hand side of the box represents the upper quartile and the end of the right-hand whisker represents the maximum value.	**a** min = 2, $Q_1 = 11$, $Q_2 = 23$, $Q_3 = 31$, max = 37
b Subtract Q_1 from Q_3 to obtain the IQR.	**b** $\text{IQR} = Q_3 - Q_1$ $= 31 - 11$ $= 20$

Example 10B.2 Creating a box plot

Draw a box plot to represent this data about the number of books that people own.

45	22	19	4	14	39	152	108	39	19	66	81
122	42	38	22	18	35	144	33	27	13	31	

THINK	WRITE
1 Arrange the scores in order and find the value of the median, the lower quartile and the upper quartile.	4, 13, 14, 18, 19, 19, 22, 22, 27, 31, 33, 35, 38, 39, 39, 42, 45, 66, 81, 108, 122, 144, 152 $Q_1 = 19$, $Q_2 = 35$, $Q_3 = 66$
2 Calculate the values of the lower fence (LF) and upper fence (UF).	$\text{LF} = 19 - 47 \times 1.5$ $= 19 - 70.5$ $= -51.5$ $\text{UF} = 66 + 47 \times 1.5$ $= 66 + 70.5$ $= 136.5$
3 Check for outliers by determining whether any values lie below the lower fence or above the upper fence.	144 and 152 are outliers.
4 Identify the minimum and maximum values, excluding outliers.	min = 4, max = 122
5 Draw an even scale that covers all values, including outliers.	
6 Mark in the position of the minimum and maximum values, the quartiles, the median and any outliers. Complete your box plot using a ruler, providing a title if appropriate.	

Helpful hints

✔ The ends of the whiskers do not represent the minimum and maximum values in a data set if there are outliers. If there are outliers, the left whisker will start at the value directly after the closest outlier to the lower fence, and the right whisker will finish at the value directly before the closest outlier to the upper fence.

ANS p782

Exercise 10B Box plots

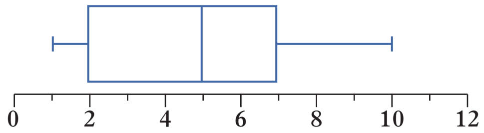
▲ 1, 2, 3(a–d), 4–6, 8　■ 1, 3(e–h), 4, 5, 7, 8, 11, 12　⬟ 3(e–h), 4, 5, 9, 10, 12–14

UNDERSTANDING AND FLUENCY

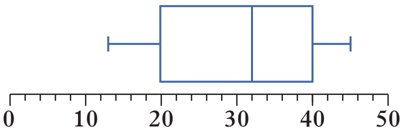
10B.1 **1** For each box plot below, state the:

i five-number summary　**ii** interquartile range.

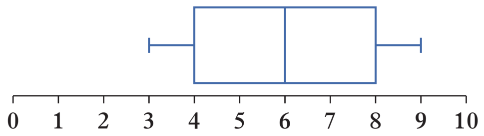
a



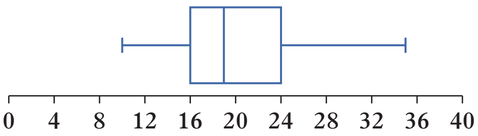
b



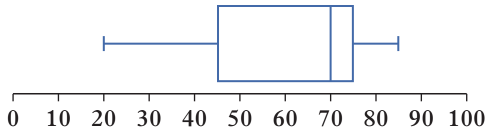
c



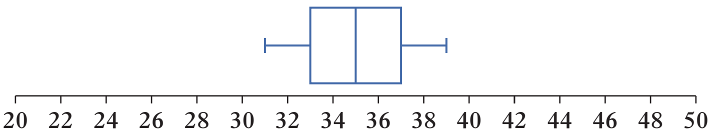
d



e



f



2 Consider this data set: 2, 6, 3, 8, 9, 10, 14, 19, 4, 8, 22, 11, 13, 17, 4, 19, 7, 7, 10.

a Arrange the data in order.

b Find the lower quartile, median and upper quartile.

c Calculate the values of the lower and upper fences.

d Determine whether there are any outliers in this data set.

e Create a box plot for this data.

10B.2 **3** Draw a box plot to represent each of these data sets.

a 3, 6, 2, 8, 4, 8, 10, 15, 6, 8, 11

b 11, 15, 14, 19, 20, 26, 17, 13, 6, 22

c 4, 7, 1, 8, 3, 7, 6, 6, 9, 3, 5, 1, 6, 9, 7, 5, 3

d 12, 36, 45, 92, 42, 85, 49, 66, 51, 19, 27, 78

e 16, 28, 9, 16, 13, 27, 18, 22, 33, 24, 9, 18, 8

f 5, 7, 9, 19, 6, 9, 8, 5, 5, 4, 3, 9, 9, 1, 6, 5, 4

g 11, 18, 49, 23, 24, 26, 37, 31

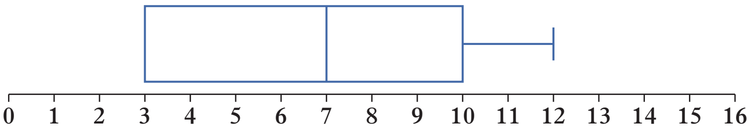
h 8, 11, 19, 21, 6, 7, 12, 18, 19, 31, 4, 6, 24, 7, 9

4 For each box plot, state the:

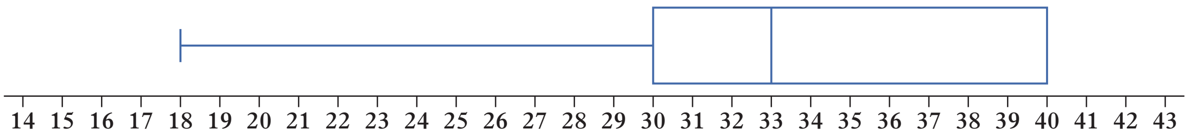
i five-number summary

ii interquartile range.

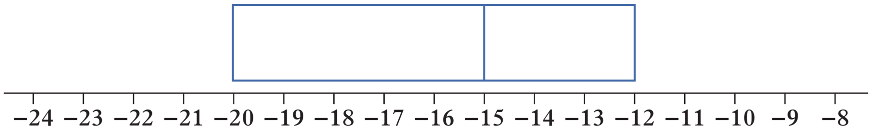
a



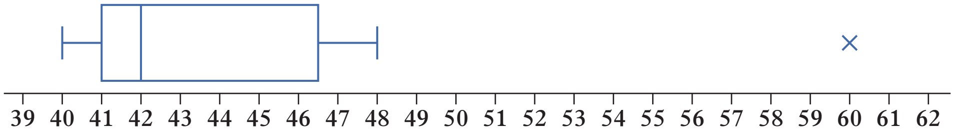
b



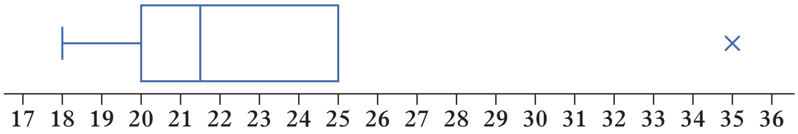
c



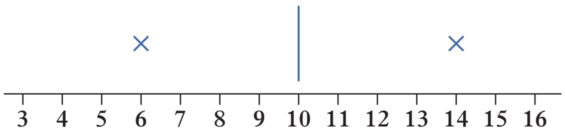
d



e

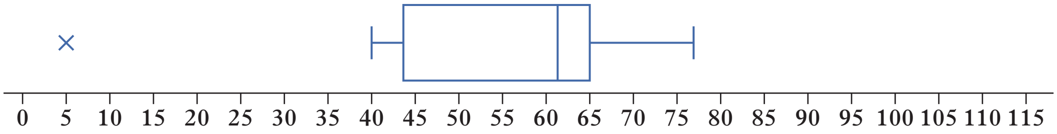


f

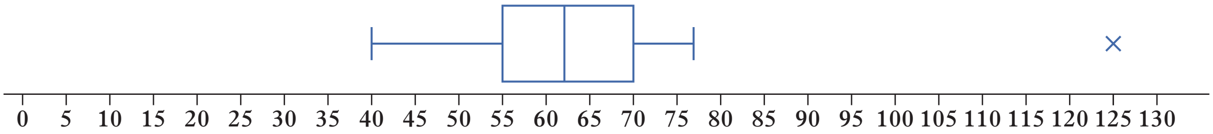


5 For each of the following box plots, add a vertical dashed line where the upper and lower fences lie.

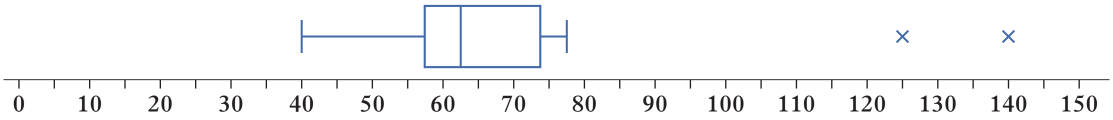
a



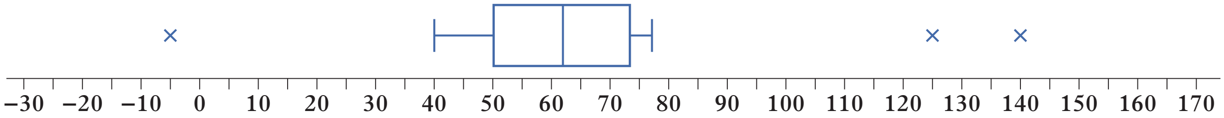
b



c

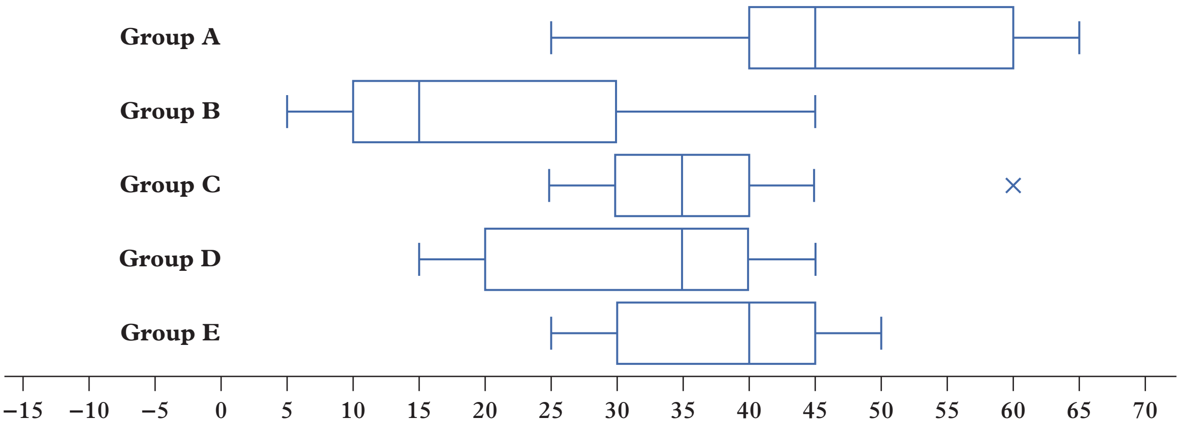


d



UNDERSTANDING AND FLUENCY

6 Consider the parallel box plots below.



a State the group(s) with the lowest:

i minimum

ii maximum.

b At 45, state the statistic (if any) that group:

i A has

ii B has

iii C has

iv D has

v E has.

c Which group has a range equal to the interquartile range of another group?

PROBLEM SOLVING AND REASONING

7 This data was collected about the number of puppies in a litter:

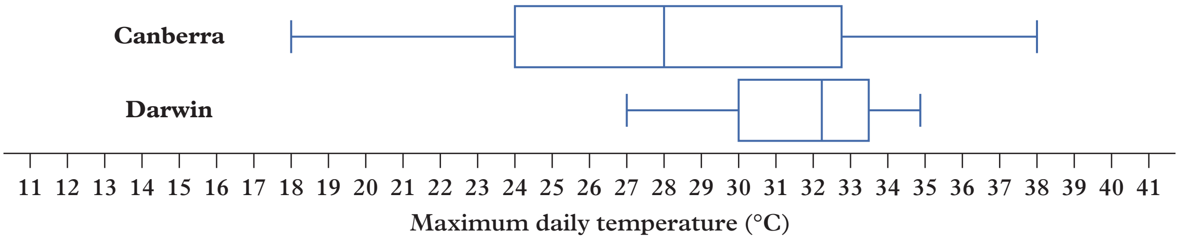
1	6	4	2	5	4	8	6	4	5	8
3	2	4	6	5	8	7	5	6	5	6
12	4	6	5	7	6	5	6	5		

a Find the lower quartile, median and upper quartile of the data set.

b Determine if there are any outliers in this data set.

c Draw a box plot to represent the data.

8 Consider these parallel box plots, showing the distribution of the maximum daily temperatures for a particular month in Darwin and Canberra.



a What is the centre (median) and spread (IQR) of the temperature in Darwin?

b What is the centre (median) and spread (IQR) of the temperature in Canberra?

c Write a paragraph comparing the distributions of the two sets of temperature data. Which would you say is the hotter city?

PROBLEM SOLVING AND REASONING

9 Data was collected on the ages of customers at a laser tag facility on weekdays (set A) and at weekends (set B).

Set A: 19, 18, 21, 22, 23, 18, 24, 25, 22, 21, 20, 19, 18, 18, 19, 20, 19, 20, 24, 22, 21, 23, 27, 19, 21

Set B: 21, 26, 23, 19, 18, 17, 15, 6, 10, 43, 36, 26, 23, 49, 7, 11, 14, 13, 51, 38, 32, 27, 18, 12, 19, 33

a Draw a set of parallel box plots to represent the data.

b Describe each data set in terms of its centre (median) and spread (range).

c Compare the two sets. What differences do you notice?

10 The five-number summary for a set of data that shows the number of customers in a shoe shop at 1 pm each day for a month is 4, 6, 7, 8, 10.

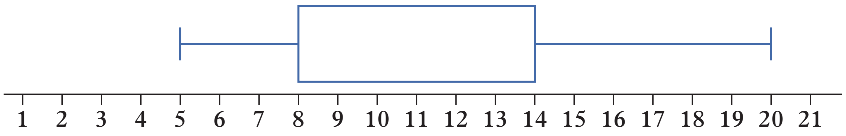
a Draw a possible box plot for this five-number summary.

b Are other box plots possible? Explain.

The five-number summary for another set of data is 2, 6, 7, 8, 12.

c Explain why you cannot draw the box plot for this five-number summary with only the information provided.

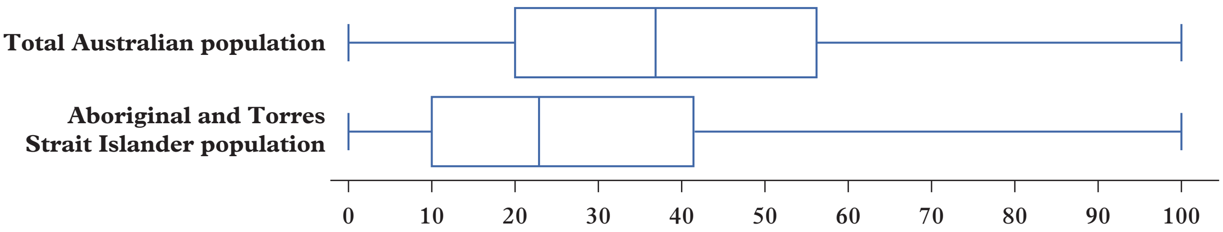
11 Consider this box plot.



a Explain why there are two possible five-number summaries when only the box plot (and not the original data) is provided.

b Provide the two possible five-number summaries for the data the box plot represents.

12 The parallel box plots below show the distribution of the ages of the total Australian population and the distribution of ages of Aboriginal and Torres Strait Islanders in 2016.



Compare the distributions of the ages of all Australians and the ages of Aboriginal and Torres Strait Islanders.

CHALLENGE

13 Draw a set of 3 parallel box plots, A, B and C, such that:

- the maximum of A and B are equal
- 75% of the data in B lies within the same values that 25% of the data in A lies within
- 100% of the data in C, excluding an outlier, lies within the same values that the middle 50% of the data of A lies within
- the median of B is equal to the value of the outlier of C
- the medians of A and C are equal to the minimum value of B
- A is perfectly symmetric and each quartile is equally spaced, B is negatively skewed, and C is symmetric.

14 For a particular set of data, the lower fence is 15 and the upper fence is 51. Determine the:

a interquartile range

b the upper and lower quartiles.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet Box plots	**Investigation** How varying data values affect the interquartile range	**CAS instructions** Box plots	**Topic quiz** 10B

10C Distributions of data

Learning intentions

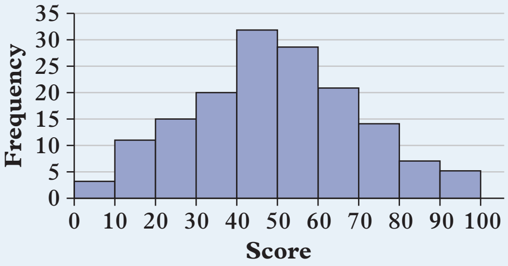
- ✔ I can accurately describe distributions of data.
- ✔ I can create and interpret a cumulative frequency distribution.

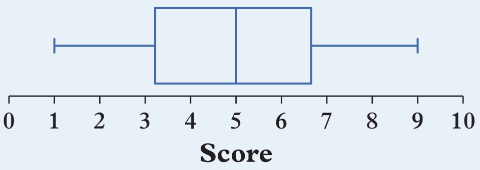
Inter-year links

Year 9 8D Describing data

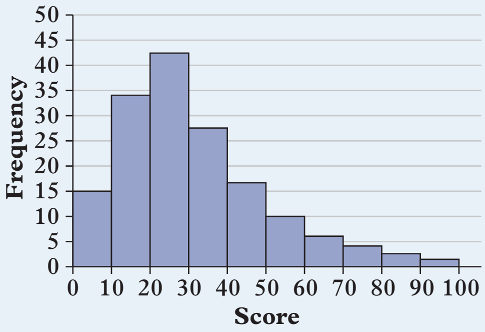
Describing data distribution

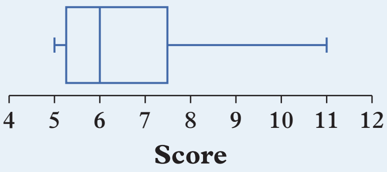
- **Symmetrical** or symmetric distributions are evenly spread on either side of a central peak. The **histogram** and box plot below are examples of symmetric distributions.



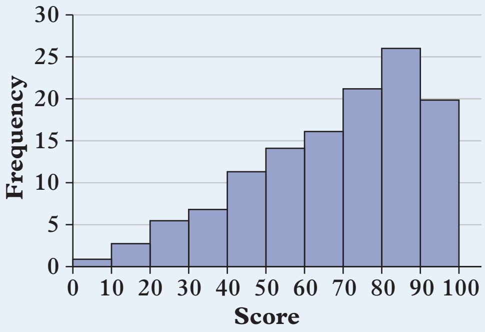


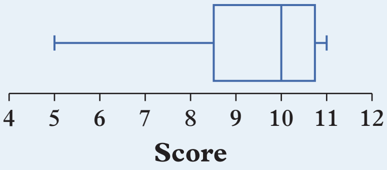
- A **skewed** distribution has a bigger cluster of data points on one side of the distribution. The other side of the distribution has a tail.
  - → A **positively skewed** distribution has a cluster of values around the left-hand side of the distribution and a tail tapering to the right. The histogram and box plot below are examples of positively skewed distributions.





  - → A **negatively skewed** distribution has a cluster of values around the right-hand side of the distribution and a tail tapering to the left. The histogram and box plot below are examples of negatively skewed distributions.

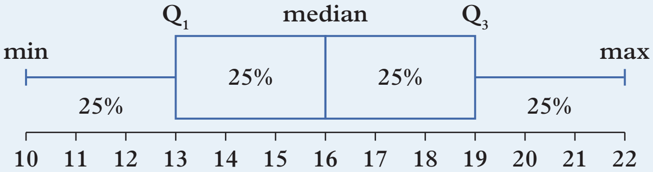




- To describe the **spread** of a data set, consider where the scores in the data set lie across the distribution. Remember that 25% of a data set lies between each pair of consecutive terms in a five-number summary, so if any two values in a five-number summary are close together, 25% of scores will be covered in a small range of data (i.e. packed closely together).
- To describe the centre of a data set, consider where the median score in the data set lies in the distribution.

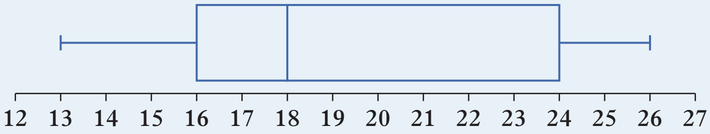
Cumulative frequency distributions

- Each section of a box plot represents approximately 25% of a data set.

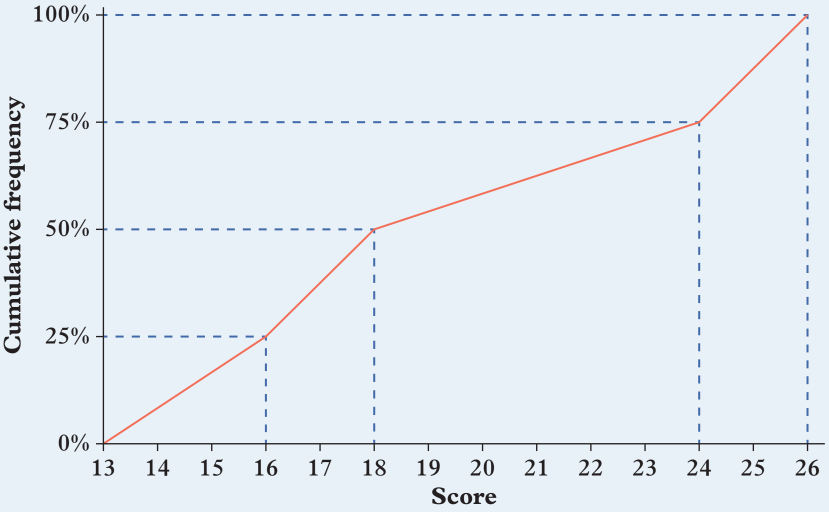


- The **cumulative frequency** of a numerical data set can be calculated by adding the frequency of a particular score to the sum of the frequencies of all its preceding scores.
  → Cumulative frequencies can be expressed as raw numbers or as a percentage of the data set.
- A **cumulative frequency polygon**, also called an **ogive** (pronounced: 'oh-jive'), is a graph plotting the cumulative frequency of the scores in a data set (on the vertical axis) against those scores (on the horizontal axis).

  The data displayed in the following box plot and cumulative frequency table are plotted on the ogive below.



Score	Cumulative frequency
13	0%
16	25%
18	50%
24	75%
26	100%

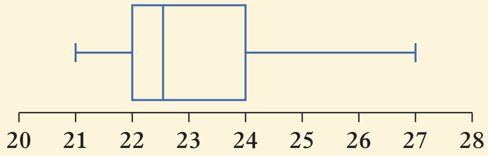


- Cumulative frequency polygons can be used to plot grouped data. Each section of the cumulative frequency polygon should start at the lowest value of a class interval and end at the highest value of the same class interval (as shown in Example 10C.3).

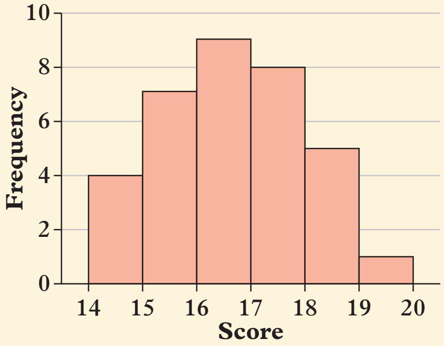
Example 10C.1 Describing a data distribution

Describe the type of distribution of data shown on each of the following.

a



b



THINK

a Each section of a box plot represents approximately 25% of the data set. Identify whether there is a cluster of points on one side of the box plot, or whether the distribution is symmetric.

b Look at the shape of the histogram. Is there a skew towards one side of the distribution, or is the data approximately symmetrical?

WRITE

a Positively skewed

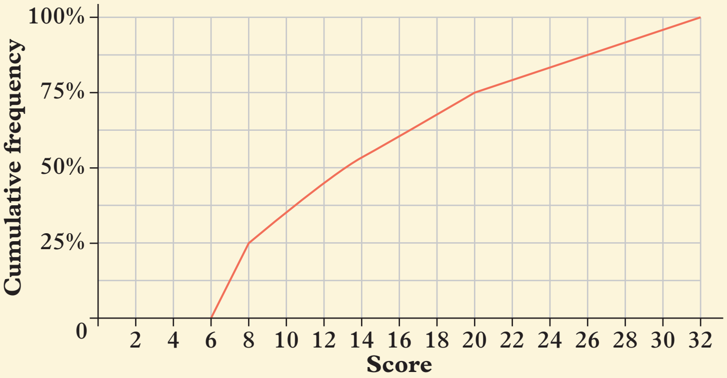
b Symmetrical

Example 10C.2 Interpreting a cumulative frequency polygon

Use this cumulative frequency distribution to write the:

a five-number summary for the data displayed

b interquartile range for the data.



THINK

a The value at 0% represents the minimum value, the value at 25% represents the lower quartile, the value at 50% represents the median, the value at 75% represents the upper quartile and the value at 100% represents the maximum value.

b Subtract Q_1 from Q_3 to obtain the interquartile range.

WRITE

a min = 6, Q_1 = 8, Q_2 = 13, Q_3 = 20, max = 32

b $\text{IQR} = Q_3 - Q_1$
$= 20 - 8$
$= 12$

Example 10C.3 Creating a cumulative frequency polygon

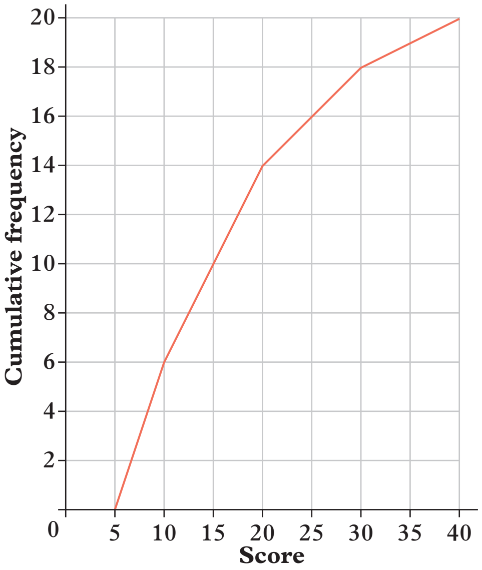
Using the data from this table, create a cumulative frequency polygon.

Score	Frequency	Cumulative frequency
5–<10	6	6
10–<15	4	10
15–<20	4	14
20–<25	2	16
25–<30	2	18
30–<35	1	19
35–<40	1	20

THINK

1. Draw and label axes to cover the full range of scores on the horizontal axis and the cumulative frequency on the vertical axis.
2. Mark the first point with a frequency of 0 at the start of the first class interval. The next point to mark will be the point at the end of the first class interval, with a cumulative frequency equal to that of the first class interval.
3. Mark each subsequent point from the table on your graph.
4. Join the points together with straight lines.

WRITE



Helpful hints

- ✔ Symmetric data does not need to be perfectly symmetric; it only needs to be approximately symmetric.
- ✔ The cumulative frequency of the final score will equal the number of pieces of data, or 100% of the data set.

ANS p783

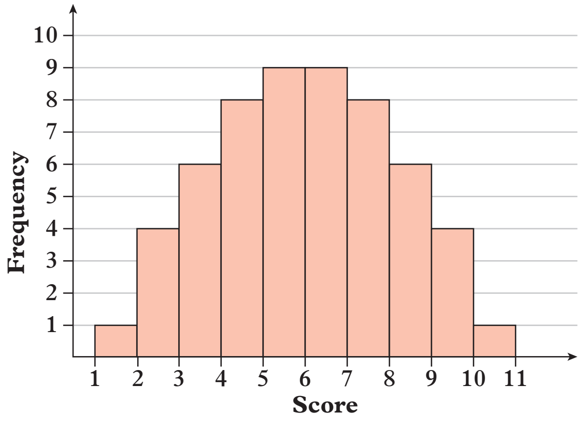
Exercise 10C Distributions of data

▲ 1–3, 4(a, b, d), 5–9, 11(a, c) ■ 1–3, 4(c–e), 5, 7–9, 11(b, d), 13, 14 ⬟ 1, 2, 4(c–e), 7, 10, 11(b, d), 12, 13, 15

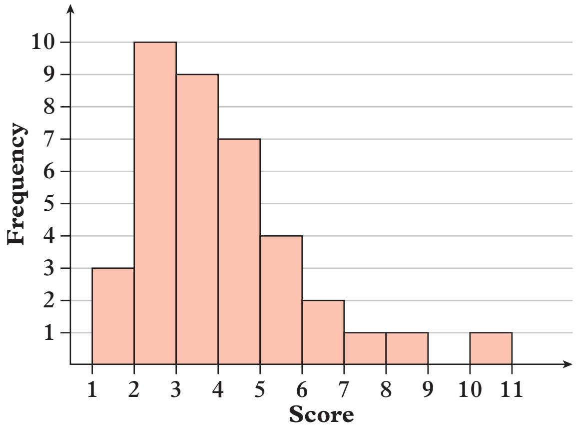
UNDERSTANDING AND FLUENCY

10C.1 **1** Describe each of the following distributions of data.

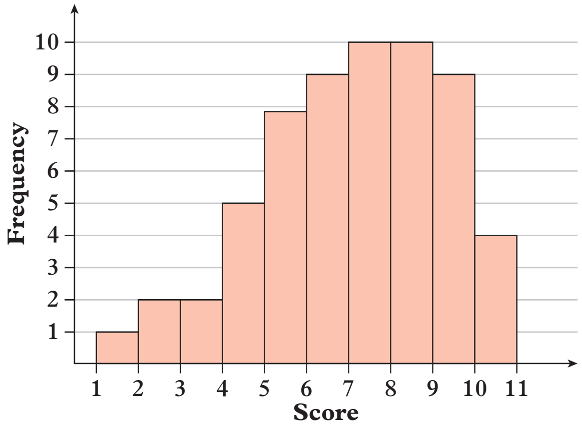
a



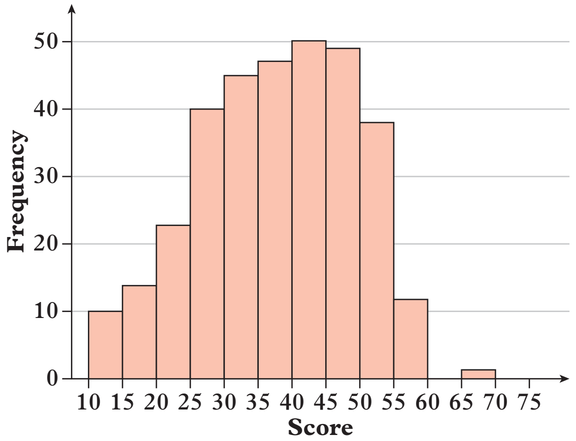
b



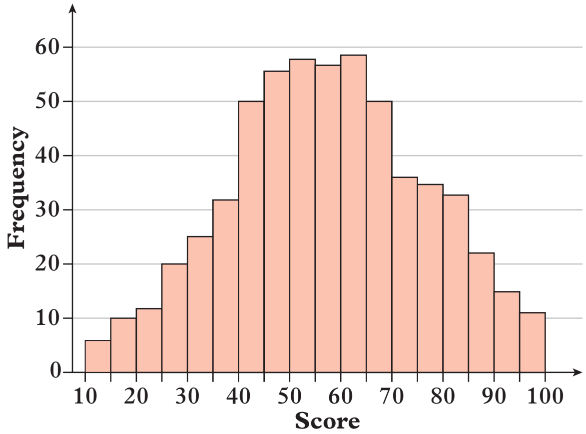
c



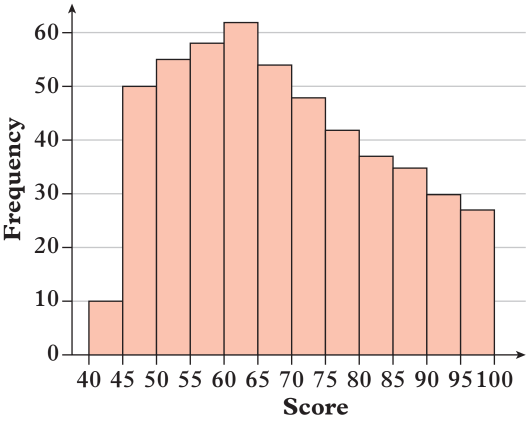
d



e

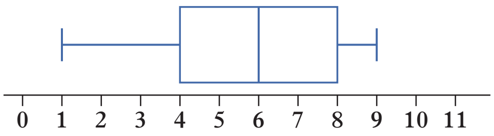


f

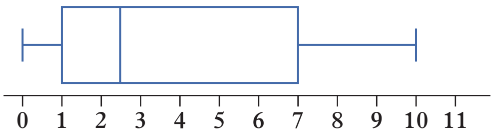


2 Describe each of the following distributions of data.

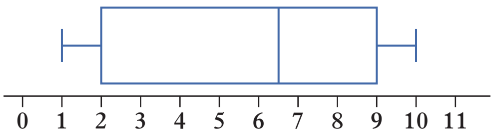
a



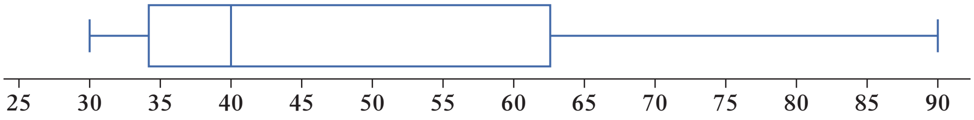
b



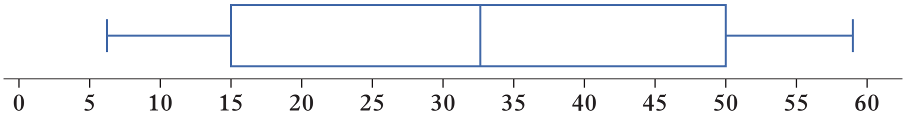
c



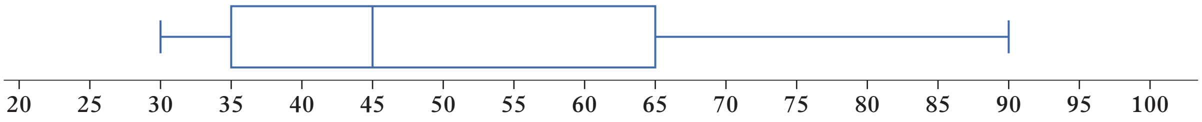
d



e



3 State the percentage of the data that would occur in the following ranges of values for this box plot.

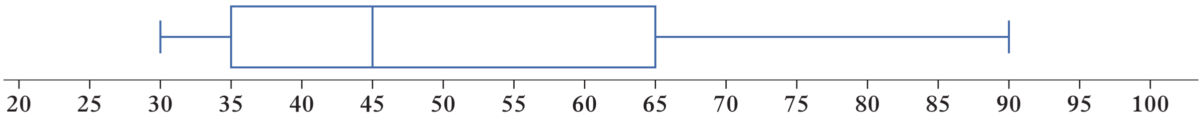


a between 35 and 45 **b** between 35 and 65 **c** between 65 and 90

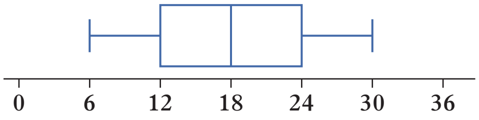
d less than 65 **e** greater than 35 **f** less than 35 or greater than 65

4 Construct a cumulative frequency polygon for the following box plots.

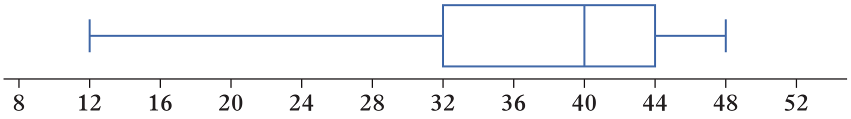
a



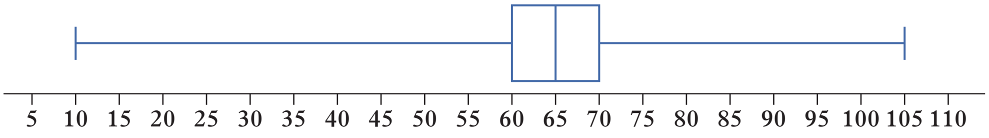
b



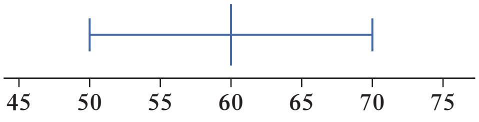
c



d



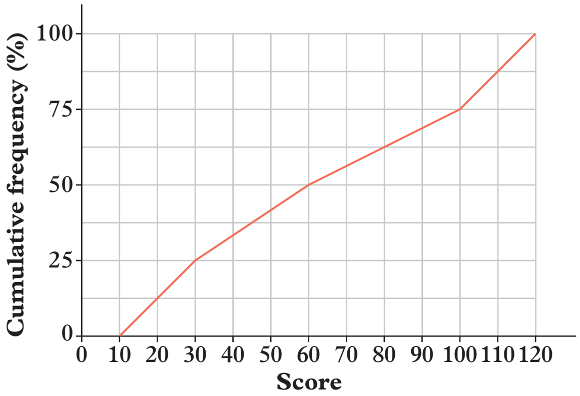
e



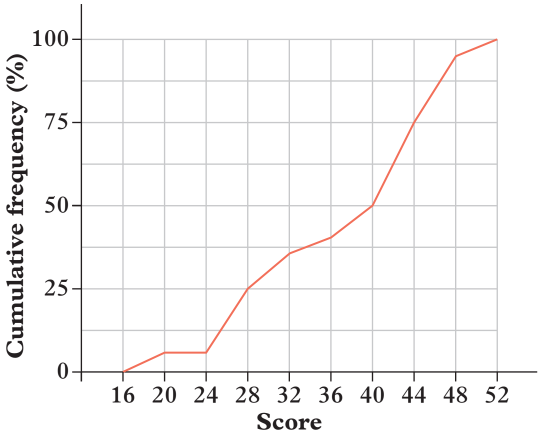
10C.2 **5** For each cumulative frequency polygon, state the:

i five-number summary **ii** interquartile range.

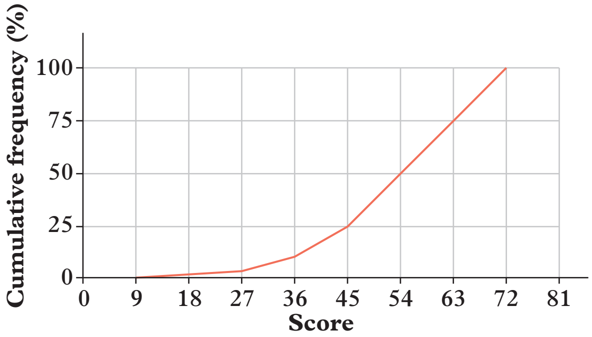
a



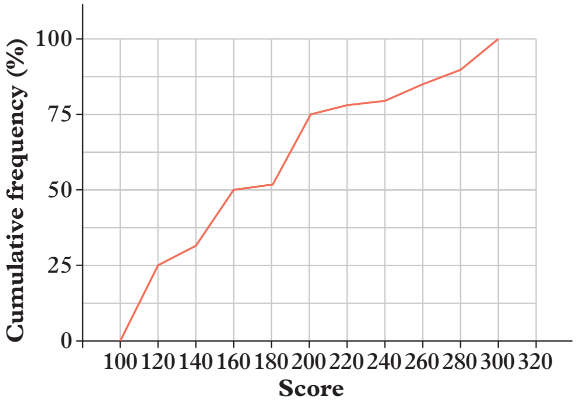
b



c



d



10C.3 **6** Create a cumulative frequency polygon using the data in the table below.

Score	Frequency	Cumulative frequency	Cumulative percentage
7	5	5	25
8	3	8	40
9	2	10	50
10	5	15	75
11	3	18	90
12	2	20	100

7 For this table:

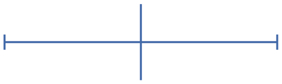
a complete the cumulative frequency and cumulative percentage columns

b construct a cumulative frequency polygon.

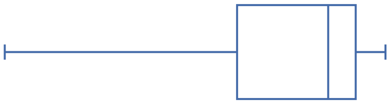
Class interval	Frequency	Cumulative frequency	Cumulative percentage
20–<30	10		
30–<40	40		
40–<50	50		
50–<60	20		
60–<70	10		
70–<80	20		
80–<90	10		
90–<100	40		

8 Match each of these box plots (**i–iv**) with the correct graph (**A–D**).

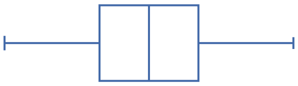
i



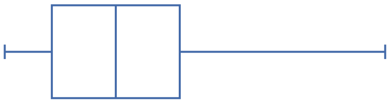
ii



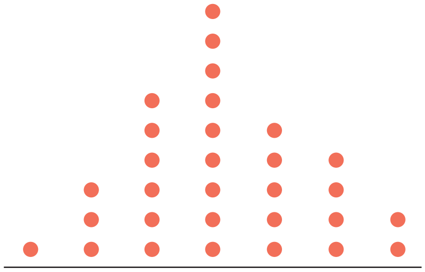
iii



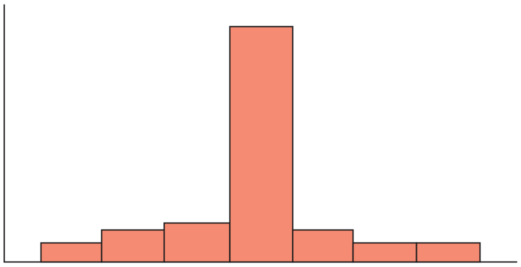
iv



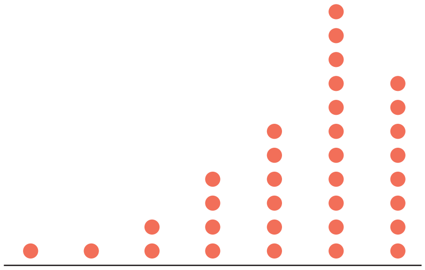
A



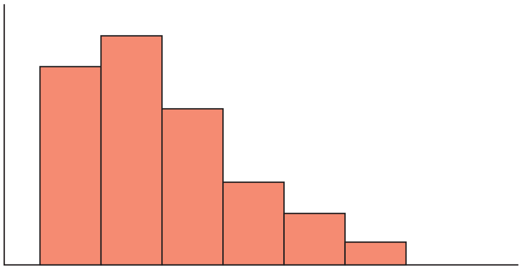
B



C



D



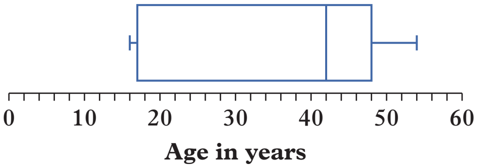
9 **a** Describe what the box plot below tells you about the ages of people attending the Year 10 orientation night, by describing its:

i shape

ii spread

iii centre.

Ages of people at a Year 10 orientation night



b Can you suggest a reason for the unusual shape of the distribution?

10 A vet collected the following data about the weights (in kilograms) of dogs.

5.1	11.2	35.5	17.9	21.1	84.4	42.2	51.0	9.2	8.6	7.5	18.5	19.2	21.8	22.6
24.8	13.7	13.1	16.9	29.8	58.3	42.7	2.1	3.8	9.6	10.8	46.0	1.9	35.0	39.4

a Draw a box plot to represent the data.

b Draw a histogram with intervals of width 10 to represent the data.

c Describe the distribution of the data.

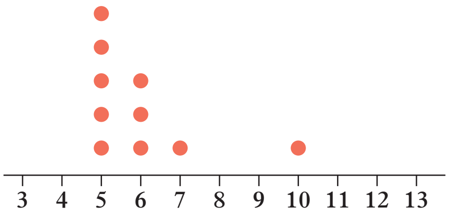
d Which of the displays you drew in parts **a** and **b** do you think represents the data best? Explain.

e Why would a dot plot not be as useful for looking at the distribution of the weights of these dogs?

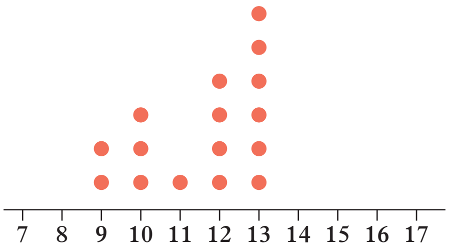
PROBLEM SOLVING AND REASONING

11 Construct a cumulative percentage frequency polygon for the each of the following displays.

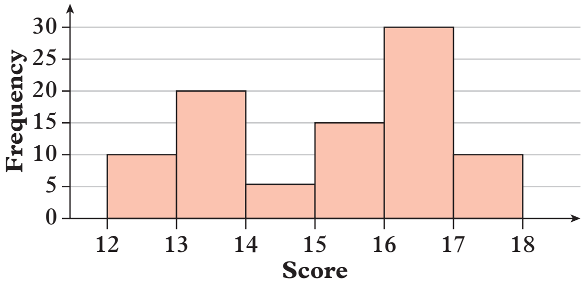
a



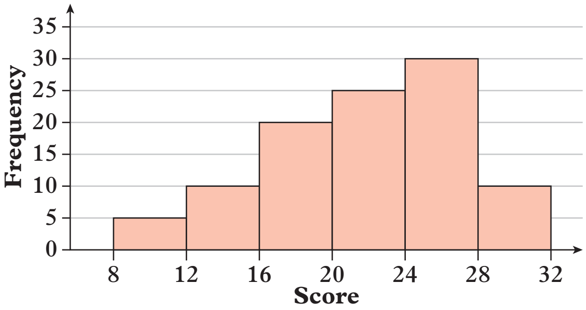
b



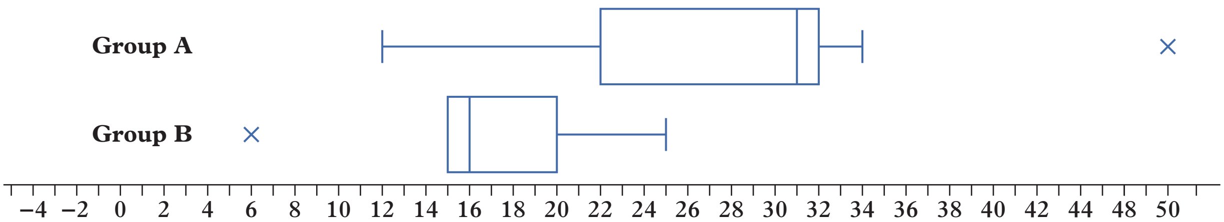
c



d



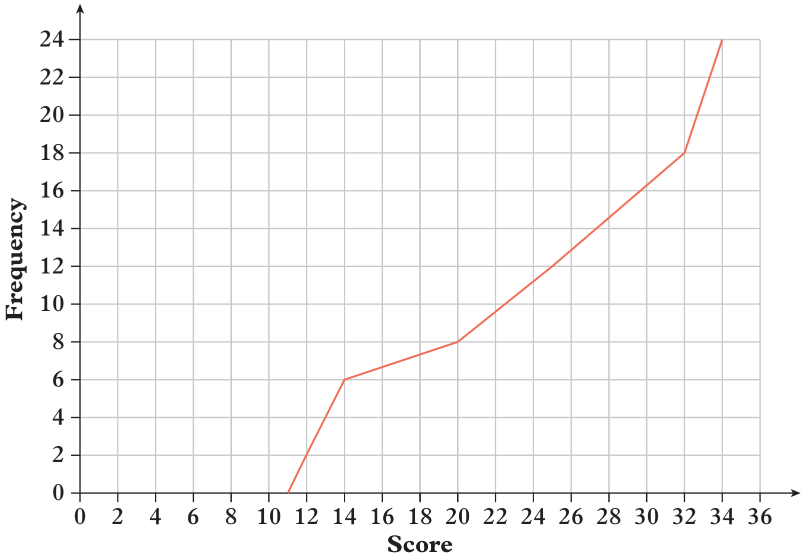
12 Consider the parallel box plots below.



a Describe the shape of the box plots.

b Give three reasons why the median would be a better measure of centre to compare the box plots with than the mean.

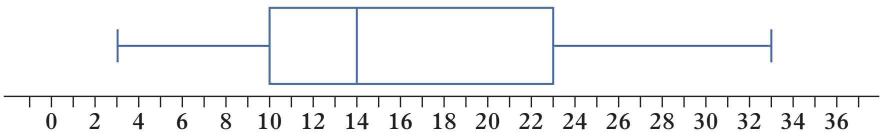
13 Consider the cumulative frequency plot shown with a frequency scale. Determine the five-number summary for the data.

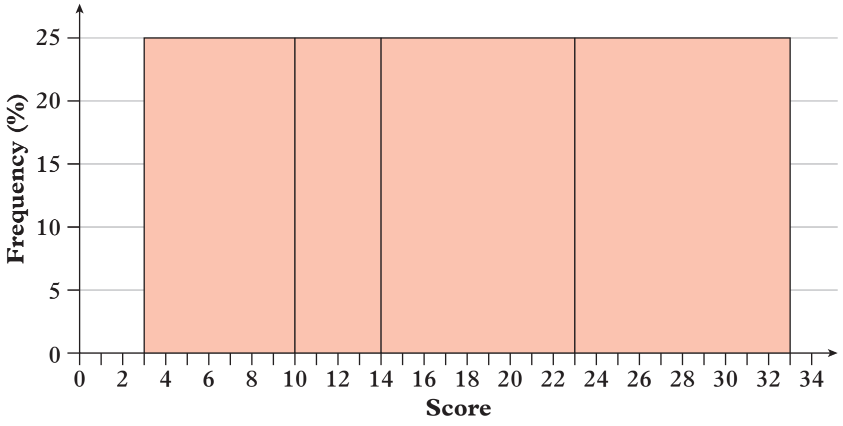


CHALLENGE

14 Explain why no section of a cumulative frequency polygon can have a negative gradient.

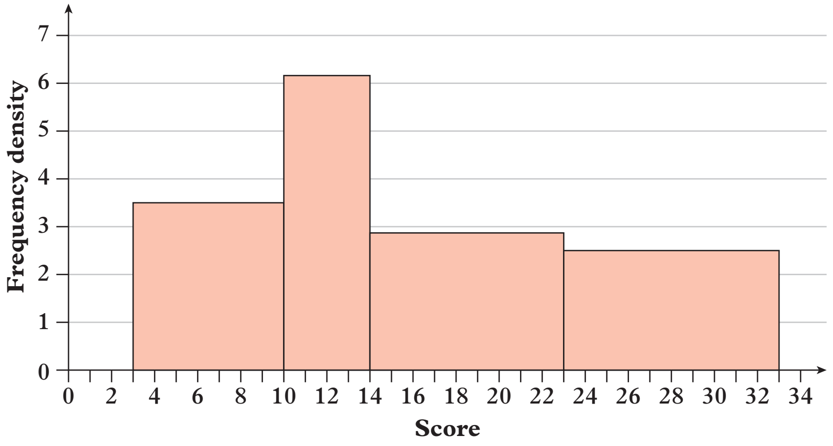
15 Sylvia created the following histogram using the provided box plot.



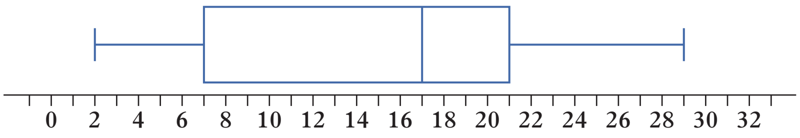


a Describe why Sylvia's histogram is not well drawn.

When histograms are drawn with the same width intervals (called bin widths), the height is proportional to the frequency. If we need to use different sized bin widths, then the area of each bar is proportional to the frequency. For example, Sylvia's histogram with the area of each bar being proportional to the percentage frequency of that bar would look like this.



b Construct a variable bin width histogram for the following box plot.



Check your Student obook pro for these digital resources and more:

Interactive skillsheet Describing data

Interactive skillsheet Cumulative frequency polygons

CAS instructions Histograms

Topic quiz 10C

10D The mean and standard deviation

Learning intentions

✔ I can calculate the standard deviation of a population and sample.

✔ I can use the mean and standard deviation to compare data sets.

Inter-year links

Year 8	9B Summary statistics
Year 9	8C Summary statistics from tables and displays

Populations and samples

- When working with data, the **population** refers to all potential pieces of data under consideration.
- A **sample** is a selection of data that is part of the population.
  - → Random samples from a population will vary with differing characteristics.

Mean and standard deviation

- The **mean** is a numerical average of all values in a data set.
  - → The Greek letter μ (mu) is used to represent the population mean.
  - → $\bar{x}$ (x-bar) is used to represent a sample mean.
  - → The population mean and the sample mean can be calculated in the same way.
- The **standard deviation** is a measure of how much the scores in a set differ (or deviate) from the mean.
  - → A small standard deviation means that the scores in a data set are close to the mean.
  - → A large standard deviation means that the scores in a data set are spread out from the mean.
  - → The Greek letter σ (sigma) is used to represent the population standard deviation.
- s is used to represent a sample standard deviation.
- A population's standard deviation (σ) can be calculated using the formula:

$$\sigma = \sqrt{\frac{\left(x_1 - \mu\right)^2 + \left(x_2 - \mu\right)^2 + \ldots + \left(x_n - \mu\right)^2}{n}}$$

- A sample's standard deviation (s) can be calculated using the formula:

$$s = \sqrt{\frac{\left(x_1 - \bar{x}\right)^2 + \left(x_2 - \bar{x}\right)^2 + \ldots + \left(x_n - \bar{x}\right)^2}{n - 1}}$$

  - → When calculating the standard deviation from a sample mean, the sum is divided by $n - 1$ rather than n because this provides a better estimate of the standard deviation for the population as the squared deviations from the sample mean underestimate the squared deviations from the population mean.

Using the mean and standard deviation to describe and compare data sets

- The mean describes a central point in a data set. If a data set is positively skewed the mean will be greater than the median, and if the data is negatively skewed the mean will be less than the median.
- When the data is skewed, the median represents the centre of the data better than the mean.
- The standard deviation describes the spread of a data set. When comparing data sets, a larger standard deviation will have scores that are more spread out than a smaller standard deviation.

Example 10D.1 Finding the standard deviation for a list of data

Find the standard deviation for this sample of data, correct to one decimal place:

4, 8, 12, 19, 5

THINK

1 Calculate the mean of the sample.

2 Substitute the given values into the formula for the standard deviation of a sample and round your answer to one decimal place.

WRITE

$$\bar{x} = \frac{4 + 8 + 12 + 19 + 5}{5}$$
$$= 9.6$$

$$s = \sqrt{\frac{(4 - 9.6)^2 + (8 - 9.6)^2 + (12 - 9.6)^2 (19 - 9.6)^2 + (5 - 9.6)^2}{5 - 1}}$$
$$= \sqrt{\frac{(-5.6)^2 + (-1.6)^2 + (2.4)^2 + (9.4)^2 + (-4.6)^2}{4}}$$
$$\approx 6.1$$

Example 10D.2 Using a spreadsheet to find the standard deviation

Use Excel (or alternative spreadsheet software) to help you find the standard deviation for the data sample below. Give your answer correct to one decimal place.

10, 15, 11, 23, 18, 9, 22, 17, 14, 10

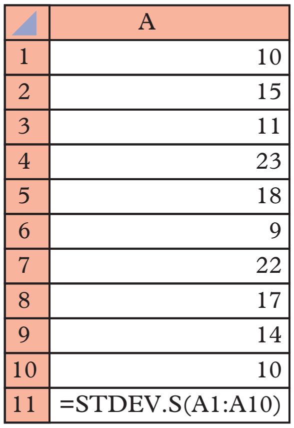
THINK

1 Enter each value from the data sample into a separate cell in column A, in a vertical list.

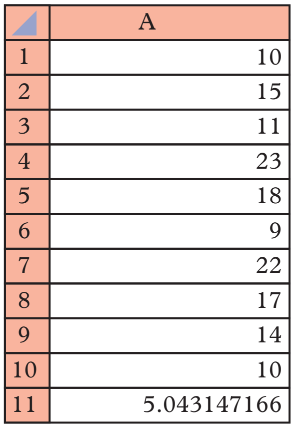
2 Use the formula =STDEV.S(A1:A10) to calculate the standard deviation of cells A1 to A10.

3 Round the standard deviation to one decimal place.

WRITE



	A
1	10
2	15
3	11
4	23
5	18
6	9
7	22
8	17
9	14
10	10
11	=STDEV.S(A1:A10)



	A
1	10
2	15
3	11
4	23
5	18
6	9
7	22
8	17
9	14
10	10
11	5.043147166

$s \approx 5.0$

Note: The formula for finding the standard deviation might vary if you are using a different spreadsheet program.

Helpful hints

- ✔ The formula for the population standard deviation using Excel is =STDEV.P(:), where the range of cells containing the data is placed within the brackets. The cell names of the first and last cells from the listed data are separated by the colon.
- ✔ Most calculators have a function for calculating the standard deviation of a list of data.

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Exercise 10D The mean and standard deviation

▲ 1, 2, 3(a, d–f), 4(a, b, d, f), 5, 7, 8, 10, 14, 15

■ 1, 2, 3(b, e, f), 4(b, e, f), 5–7, 9, 11, 13, 15, 16

⬟ 1, 2, 5, 9, 11, 12, 15–18

10D.1 1 Using a calculator, find the standard deviation for each of these sample data sets. Give your answers to one decimal place.

- **a** 6, 3, 8, 65, 4
- **b** 8, 11, 15, 3, 9, 5
- **c** 7, 7, 10, 12, 17, 15
- **d** 13, 29, 21, 22, 22, 11, 19
- **e** 140, 156, 120, 99, 187, 147
- **f** 159, 166, 122, 171, 136, 129

2 Using a calculator, find the standard deviation for each of these population data sets. Give your answers to one decimal place.

- **a** 2, 2, 2, 2, 2
- **b** 2, 2, 2, 8, 8, 8
- **c** 20, 60, 70, 80, 100
- **d** −11, −9, −17, −23, −15
- **e** 2.2, 5.0, −0.2, −0.8, 5.0, 3.2
- **f** 1, 2, 4, 16, 32, 62

10D.2 3 Calculate the standard deviation for these samples of data using technology. Give your answers to one decimal place.

- **a** 1, 2, 3, 4, 5, 6, 7, 8, 9, 10
- **b** 39, 24, 34, 34 30, 32, 38, 36, 36, 32
- **c** 10, 100, 1000, 10 000, 100 000, 1 000 000
- **d** 20, 200, 2000, 20 000, 200 000, 2 000 000
- **e** −1 000 000, −100 000, −10 000, −1000, −100, −10
- **f** $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{9}{10}, \frac{10}{11}$

4 Calculate the standard deviation for these populations of data using technology. Give your answers to one decimal place.

- **a** 89, 104, 97, 114, 97, 90, 107, 82, 109, 103
- **b** 1776.6, 2521.4, 2110.1, 606.5, 926.6, 2395.4, 1572.5, 12.03, 2496.3, 123.7
- **c** 1, 2, 3, 4, 5, 6, 7, 8, 9, 10
- **d** 81, 82, 83, 84, 85, 86, 87, 88, 89, 90
- **e** 5, 6, 7, 8, 9, 10, 11, 12, 13, 14
- **f** 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256

5 For each data set below:

- **i** decide whether the data set is from a sample or a population.
- **ii** calculate the appropriate standard deviation.

a This data was collected about the ages of students attending a school formal by surveying everybody who attended.

16	17	18	16	17	16	15	14	16	17	18	16	17	17
16	17	16	17	15	16	17	16	17	16	16	16	15	17
18	18	17	16	17	16	17	16	17	16	16	16	17	17

b This data was collected about the number of hours 20 students in a Year 10 class (of 30 students) spent per week using social networks.

5	10	7	1	19	17	16
7	10	11	8	9	15	13
14	6	21	18	14	13	

c This data was collected about the weights of newborn babies in South Australia:

3.2	4.1	2.9	3.4	3.5	3.4	3.1
3.8	4.0	3.3	3.6	2.9	3.1	3.4
3.5	4.2	2.6	3.0	3.6	3.4	4.8
2.1	3.5	3.3	3.6			

6 Consider this data set, which shows the ages of every player in two cricket teams.

16	16	15	17	15	16	14	15	16
16	17	16	15	17	16	15	16	16
17	16	17	15	16	16	17	16	

a Find the standard deviation for the data set. Give your answers to two decimal places.

b Write a sentence about the spread of ages of players in the cricket teams.

7 Consider this data, about the number of text messages sent by a sample of Year 10 students per day.

1	5	11	17	21	9	8	25	26
19	5	2	8	9	13	34	16	
29	31	26	19	18	15	20	10	

a Find the mean and standard deviation for the data. Give your answers to two decimal places.

b Write a sentence describing the centre and spread of the data set.

8 The following is a list of the number of people at a shoe sale each hour over a 24-hour period:

46	26	35	49	58	13
27	48	46	39	42	48
46	51	35	29	47	41
42	42	49	45	46	51

a Find the mean for this data. Give your answers to one decimal place.

b Find the standard deviation. Give your answers to two decimal places.

c Comment on the centre and spread of the data set.

9 Data was collected about the number of minutes a sample of passengers were kept waiting for their trains on two different train lines over the course of a day.

Train line A: 4, 2, 3, 5, 6, 7, 4, 4, 6, 5, 6, 7, 9, 4, 5, 6, 6, 7, 6, 7, 9, 4, 6, 6, 1

Train line B: 2, 9, 15, 3, 4, 3, 11, 1, 1, 6, 2, 3, 4, 2, 3, 19, 2, 3, 2, 2, 2, 3, 1, 4

a For each train line, find the mean waiting time. Give your answers to one decimal place.

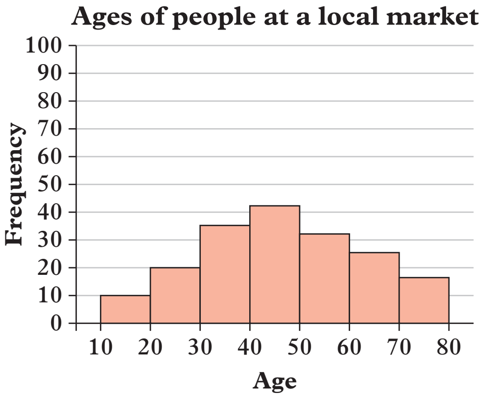
b Find the standard deviation for the data from each train line. Give your answers to two decimal places.

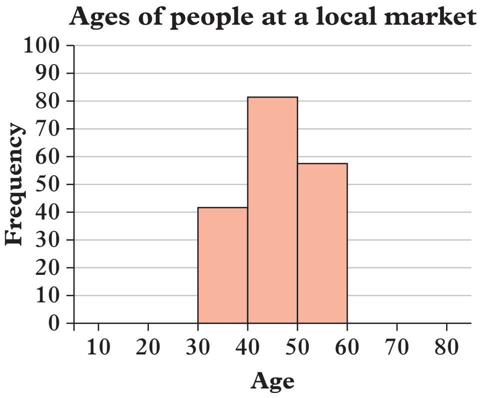
c Write a sentence comparing the centre and spread of the two data sets.

d On which train line would you say that the trains run late more significantly?

10 The two histograms below show data collected about the ages of people attending a local market on two different days. The data sets have the same mean and the same number of scores, and have been shown on axes with the same scale for ease of comparison.

Which data set has a small standard deviation and which has a large standard deviation? Explain how you know.





11 Consider this data, collected about the price (in dollars) of a particular brand of electronic tablet bought from all stores where it is available in a city.

499 519 499 475 469 479 509 549 529 525
475 499 499 199 489 479 499 475 515 499

a Find the standard deviation for this data set.

b You may notice that one of the prices is distinctly different from the rest (an outlier). Provide a possible reason for this.

c Recalculate the standard deviation for the data set excluding the outlier.

d What do you notice about your answers for parts **a** and **c**?

e What measure of spread might you use instead of standard deviation when there is an outlier in the data set?

12 This back-to-back stem-and-leaf plot shows the ages of all people auditioning for a talent show in two different cities. Find the mean (answer to one decimal place) and standard deviation (answer to two decimal places) for both data sets and write a sentence comparing the centre and spread of the ages of people auditioning in the two cities.

Leaf City A	**Stem**	**Leaf City B**
9 8 7 7 7 6	1	5 6 7 7 8 8 8 9
9 8 8 7 5 4 2 1	2	0 0 1 1 2 2 2 3 4 5 6 6 7 9
9 8 7 6 4 3 2 2 2 0	3	1 5 6 7 9
6 5 4 2 1 1	4	2 6
3 2 1	5	1
5	6	
1	7	

Key: 1 | 6 = 16

13 For a school camp, a class of 25 students will hike and camp at various locations in the bush. The mean weight of the class is 55 kg with a standard deviation of 5 kg. Each student must carry a bag that contains 8 kg of camping supplies. Determine the mean weight and standard deviation of the weights of this class while the students are wearing their bags.

14 a When is the median a better measure of the centre of a distribution of data than the mean? Explain.

b When is the interquartile range a better measure of the spread of a distribution of data than the standard deviation? Explain.

15 Compare the following pairs of symmetric sample data in terms of their mean and standard deviation. Use Excel (or alternative spreadsheet software) to help you calculate the statistics.

a A: 20, 24, 28, 28, 30, 30, 31, 32, 35, 40
B: 24, 25, 25, 25, 26, 27, 29, 31, 34, 36, 36, 36, 37

b A: 75, 81, 83, 84, 88, 92, 95, 101, 105, 109, 113
B: 64, 70, 71, 71, 72, 74, 80, 81, 83, 84, 86, 90

16 We can compare individual data values by determining the number of standard deviations, s_x, the value, x, is above or below the mean, $\bar{x}$. That is $z = \frac{x - \bar{x}}{s_x}$. These values are called standard scores (or z-scores).

For example, if Julia scored 15 on a test where the mean score was 20 and the standard deviation was 2, then Julia's standard score is $z = \frac{15 - 20}{2} = -2.5$ meaning Julia's score is 2.5 standard deviations below the mean score.

i Calculate the standard score for each pair of students' results, correct to two significant figures.

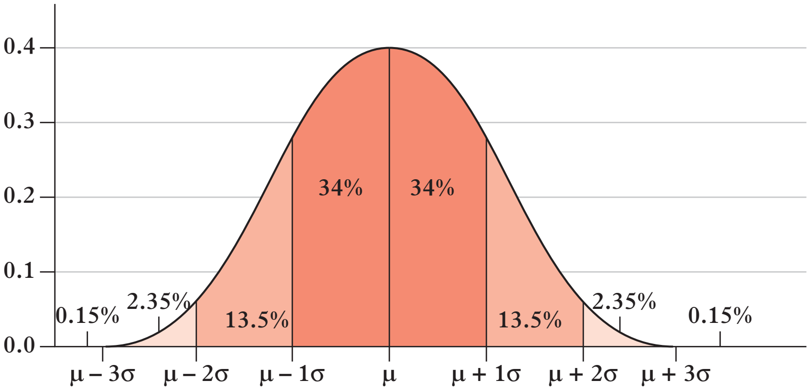
ii Compare the scores of each pair of students from different classes using their standard scores.

	Class A			**Class B**		
	Mean	**SD**	**Student A**	**Mean**	**SD**	**Student B**
a	70	10	85	55	16	85
b	70	20	85	70	15	85
c	50	25	30	75	5	50
d	50	8	26	50	16	2

CHALLENGE

17 A population that has a symmetric distribution where the majority of the data is centred around the mean in a rough bell shape is said to be normally distributed (also known as forming a bell curve). Standard deviation plays a useful role in making predictions about populations that are normally distributed. A normal distribution will have nearly all values (99.7%) within three standard deviations (also known as three sigma) of the mean.

Consider the graph below with approximate percentages of the amount of data between different standard deviations using the 68–95–99.7% rule.



a Show that 99.7% of values lie within three sigma (three standard deviations) of the mean (μ).

b What percentage of values lie within:

  i one sigma of the mean

  ii two sigma of the mean?

c A typical example of this kind of data distribution can be seen in the data for IQ (intelligence quotient). The average (μ) IQ score is defined as 100, and the data about IQ scores is thought to have a standard deviation of 15.

Redraw the graph above so that the scale on the *x*-axis uses these values (that is, μ = 100, μ + 1σ = 115, etc.).

d What is the IQ range of 99.7% of the population?

e What IQ would you need to have in order to be in the top 0.15% of the population?

f What IQ would you need to have in order to be in the bottom 2.5% of the population?

g In what upper percentage of the population is an IQ of 135?

h In what lower percentage of the population is an IQ of 80?

i A group of highly intelligent people had their IQs tested and the distribution was found to be normal, with a mean in the middle of these scores. If 95% of this group had an IQ between 120 and 152, find the mean and standard deviation of the group.

18 If we were to take different samples from a population, the sample mean in each sample would vary in value but would approximate the mean of the population. Regardless of the shape of the distribution of the original data, the distribution of the sample means would be approximately normally distributed with a mean of $\mu_{\bar{x}}$ and a standard deviation of $s_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$ where n is the size of each sample. A particular population has a mean of 40 and a standard deviation of 9.

a Find the mean and standard deviation of the distribution of sample means when taking samples of 25.

Use the diagram in question **17** to answer the following.

b Determine the probability, as a percentage, that a randomly selected sample will be:

  i between 38.2 and 41.8

  ii greater than 38.2

  iii less than 34.6

  iv greater than 43.6 or less than 36.4.

c Determine the value(s) of the sample mean such that the probability of selecting a sample mean:

  i less than this sample mean is 84%

  ii greater than this sample mean is 50%

  iii between these two sample means is 95%

  iv between these two sample means is 15.85%.

Check your Student obook pro for these digital resources and more:

pro

Interactive skillsheet
Standard deviation

CAS instructions
Standard deviation

Topic quiz
10D

Checkpoint

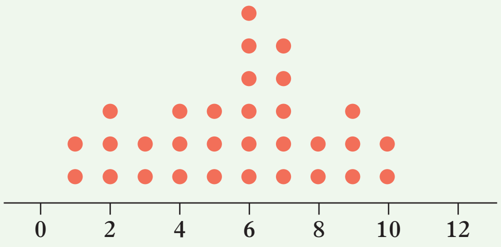
Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this chapter.

10A **1** Determine the five-number summary for each of the following.

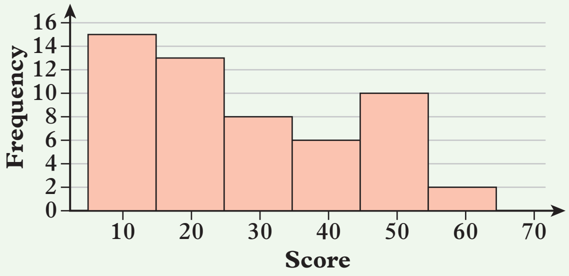
a 14, 16, 50, 23, 17, 24, 32, 19, 22, 32

b 40, 35, 38, 54, 51, 37, 48, 54, 55, 39, 42, 60

c



d



10A **2** Determine the interquartile range and range for the following five-number summaries:

a 14, 18, 23, 29, 32

b 58, 64, 85, 91, 93

10B **3** Construct a box plot for each of the following five-number summaries. These box plots will have no outliers.

a 120, 130, 145, 150, 175

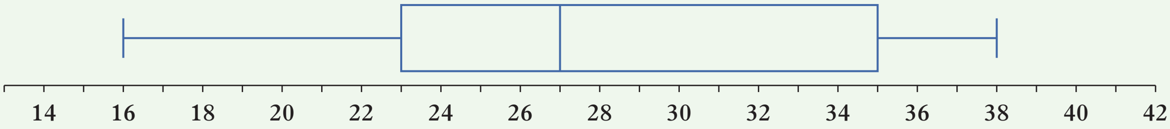
b 7, 12, 16, 22, 30

10B **4** For each of the following, state the:

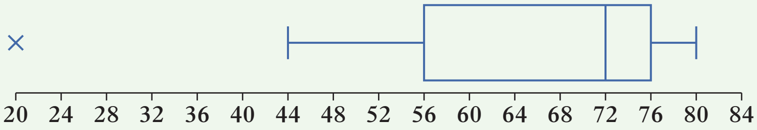
i five-number summary

ii interquartile range.

a



b

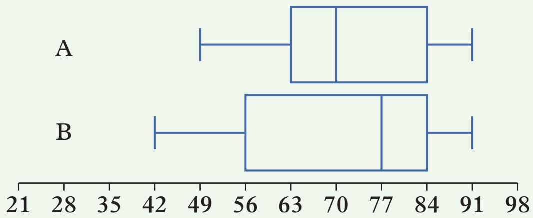


10B **5** Construct a box plot for each set of data. Check each set for potential outliers.

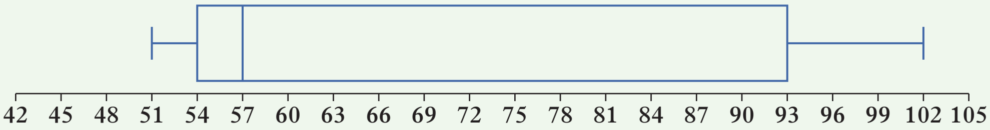
a 41, 85, 74, 67, 79, 75, 22, 84, 80, 66, 41, 79, 45, 54, 47, 84

b 56, 68, 68, 71, 56, 60, 79, 68, 76, 74, 64, 75, 62, 77, 80, 72, 76, 55, 62, 71, 72

10B **6** Consider the parallel box plots shown below. Compare the distributions of the data in terms of the centre and spread.



10C **7** State the percentage of the data that lies in the following ranges for the box plot shown below.



a less than 54

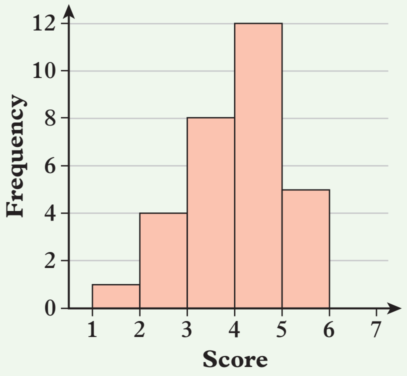
b between 54 and 93

c less than 93

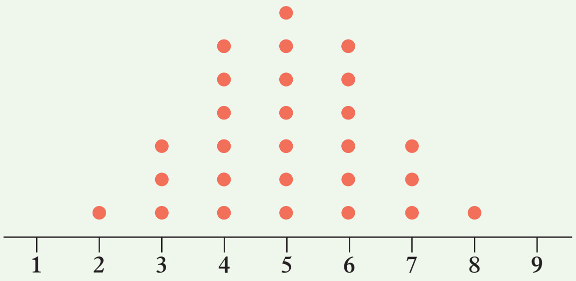
d greater than 57

10C **8** State the shape of the following distributions.

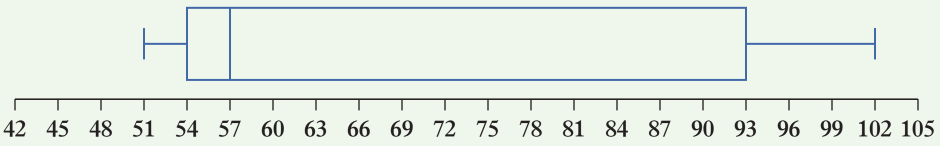
a



b



c



d

Stem	**Leaf**
1	2 5 8
2	0 1 4 4 5 6 9
3	1 2 3 5 7
4	2 2 9

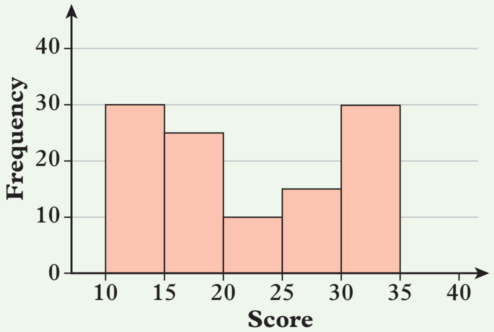
Key 1 | 2 = 12

10C **9** Construct a cumulative frequency polygon for each of the following.

a

Score	**Frequency**
8	5
9	10
10	13
11	0
12	4
Total	**32**

b



10C **10** Determine the five-number summary for the cumulative frequency plot shown.

10D **11** Using a calculator, find the mean and standard deviation of the following populations. Write your answers as exact values.

a 7, 5, 1, 5, 5, 7

b 17, 11, 8, 15, 9, 18

10D **12** Calculate the mean and standard deviation of the following samples using technology. Write your answers correct to two decimal places.

a 42, 61, 38, 55, 19, 46, 15, 75, 98, 36

b 103.5, 115.2, 98.6, 108.3, 120.1, 95.9, 109.8

10E Scatterplots and bivariate data

Learning intentions

✔ I can interpret and draw scatterplots.

✔ I can determine the strength and direction of a relationship from a scatterplot.

Inter-year links

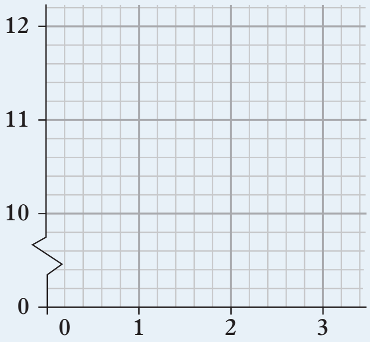
Year 8	9C Presenting data
Year 9	8A Classifying and displaying data

Bivariate data

- **Bivariate data** is data that shows the relationship between two variables.
- When investigating numerical bivariate data, we usually look at how one variable (the **independent** or **explanatory variable**) affects another variable (the **dependent** or **response variable**).

Scatterplots

- A **scatterplot** can be used to display numerical bivariate data.
  - → When plotting numerical bivariate data on a scatterplot, the independent variable is plotted on the x-axis and the dependent variable is plotted on the y-axis.
- Scatterplots are usually drawn focusing on the area of the axes between which the data points lie. A break in the scale on the axes, indicated by a zigzag, can be used to indicate that the focus is on the axes a distance away from the origin.

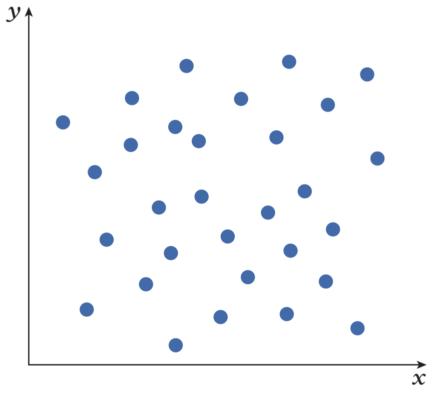
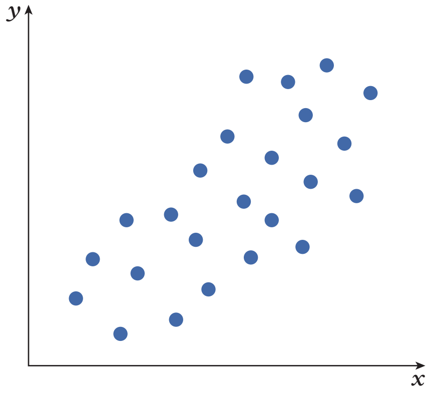
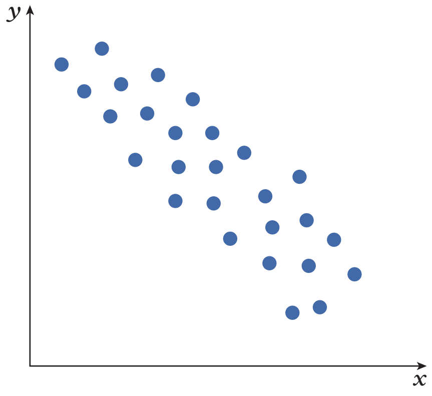
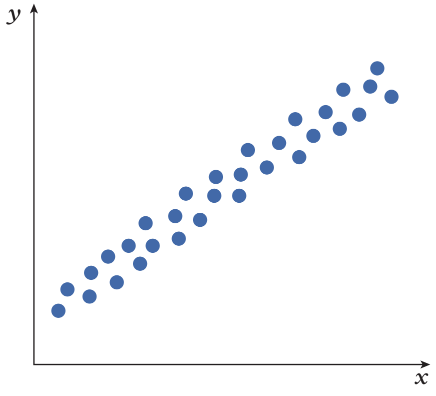


The direction of the relationship between bivariate data

- If there is a **positive relationship** between two numerical variables, as one variable increases in value the other variable also increases in value.
- If there is a **negative relationship** between two numerical variables, as one variable increases in value the other variable decreases in value.

The strength of the relationship between bivariate data

- The **correlation** between two numerical variables is a measure of the strength of the relationship between those variables. If the relationship is strong, the points will lie close to a straight line.

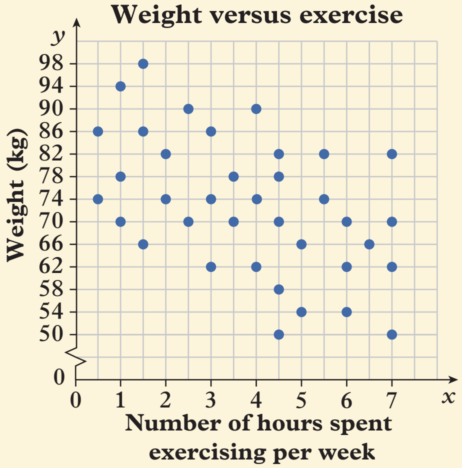
Sample scatterplot	Description	Strength of relationship	Direction of sample
	no obvious pattern in the data	no correlation	not applicable
	a weak pattern in the data	weak	positive
	a clear pattern in the data	moderate	negative
	a strong pattern in the data	strong	positive

Example 10E.1 Describing the relationship in a scatterplot

Describe the relationship shown in this scatterplot:

a in terms of its strength and direction

b in the context of the data it represents.



THINK

a There is only a weak correlation pattern in the data but, as the independent variable increases, the dependent variable does decrease.

b Describe the relationship shown in the context of what it tells you about the data.

WRITE

a The scatterplot shows a weak, negative relationship between the two variables.

b As the number of hours spent exercising increases, weight generally decreases.

Example 10E.2 Drawing a scatterplot

Draw a scatterplot for this data set.

Hours of sleep per night	8	6	7	9	10	11	8	7	6	5	4	8	7	6	10
Exam marks	72	61	82	92	68	72	49	68	61	56	32	62	41	45	55

THINK

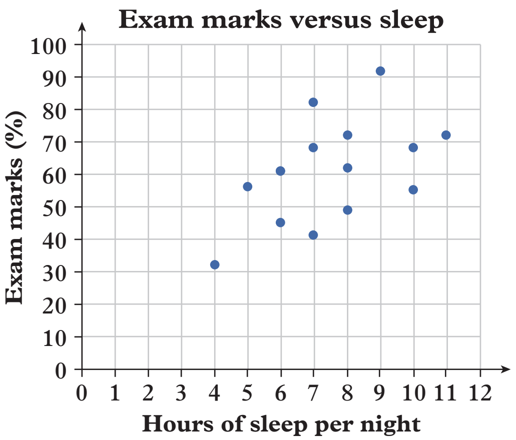
1 Determine which of the two variables is the independent variable and which variable is the dependent variable. A change in the number of hours of sleep per night could affect the exam marks, so the number of hours of sleep per night is the independent variable.

2 Use the horizontal axis for the independent variable (hours of sleep per night) and the vertical axis for the dependent variable (exam mark).

3 Identify the smallest and largest value for each variable and mark scales on the axes to cover these values. (Show breaks in the axes if the values on either axis do not begin at zero.)

4 Create a scatterplot by plotting each pair of variables from the table on the axes.

WRITE



Helpful hints

✔ When creating a scatterplot by hand, use graph paper to ensure the scatterplots you create are appropriately scaled and the points are accurately plotted.

ANS p788

Exercise 10E Scatterplots and bivariate data

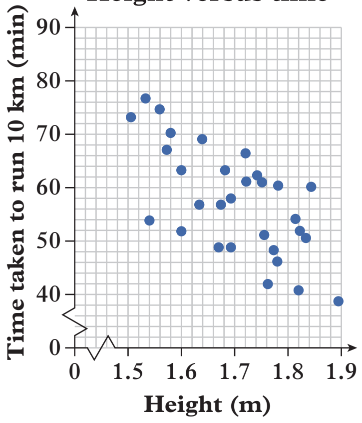
△ 1–3, 5–8 ▢ 1–5, 7, 9, 11, 12 ⬠ 1–3, 7–12

UNDERSTANDING AND FLUENCY

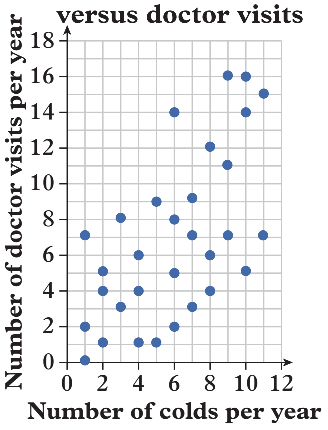
10E.1 **1** Describe the relationship shown in each of the scatterplots below:

i in terms of strength and direction **ii** in the context of what it shows about the data it represents.

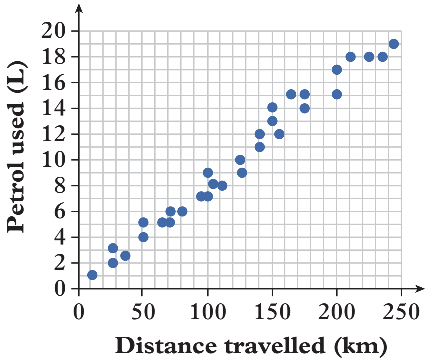
a **Height versus time**



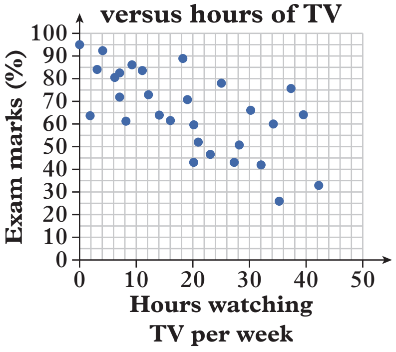
b **Number of colds versus doctor visits**



c **Distance versus petrol used**



d **Exam marks versus hours of TV**



10E.2 **2** Draw a scatterplot for each of these data sets.

a

Height (cm)	150	181	166	168	175	178	172	155	161	168	179	184	188	159	167
Shoe size	4	9	9	7	9	8	7	4	5	7	7	10	12	7	6

b

Temperature (°C)	33	15	16	25	27	36	41	11	8	32	20	25	31	18	28
Number of pies sold	5	25	33	16	19	2	3	42	35	12	21	11	10	51	15

c

Number of customers	45	56	16	32	78	98	26	66	45	76	86	42	59	31	67
Daily profit ($100s)	12	16	8	19	21	48	7	32	22	29	36	13	18	16	12

3 Describe the relationship shown in each scatterplot you drew for question **2** in the context of what it tells you about the bivariate data it represents.

4 Kieran and Marie want to investigate the relationship between the number of different types of animals in zoos and the number of visitors that the zoos receive per day.

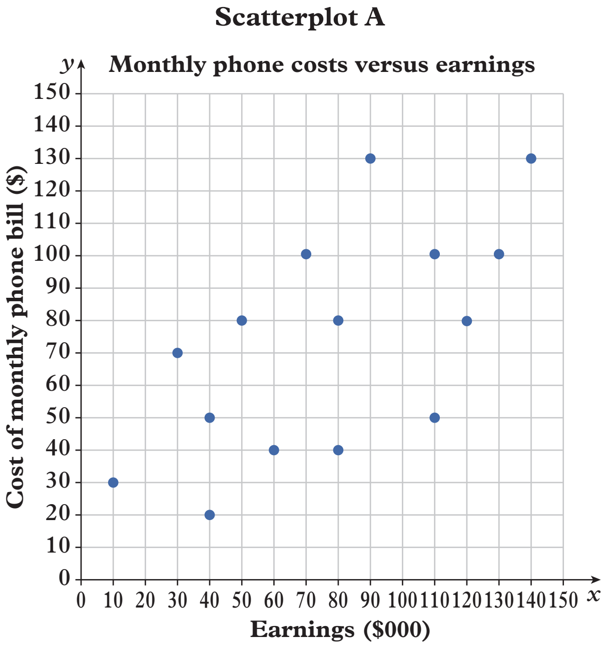
a For this investigation, predict the correlation you expect to see between the two variables in terms of strength and direction.

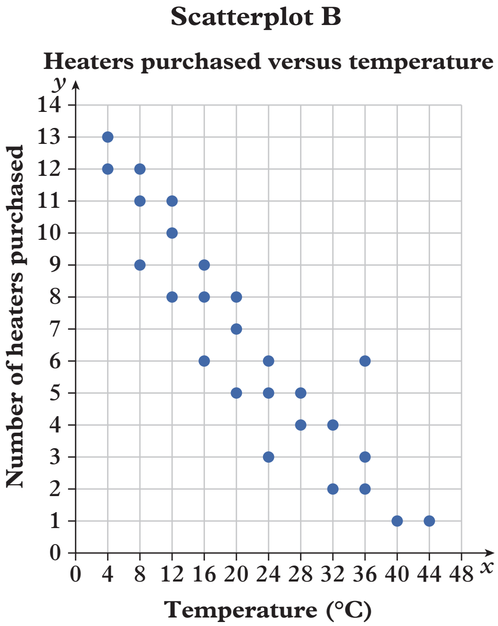
b Kieran and Marie collected the data in the table below. Draw a scatterplot to represent this data.

Number of different animal types	9	26	11	37	19	42	15	14	25	32	47	34	31	37	22	18
Number of visitors per day	42	261	150	541	247	387	226	291	317	401	761	396	309	311	177	151

c Describe the trend shown by the scatterplot you drew for part **b**. Compare the trend to the prediction you made in your answer to part **a**.

5 Consider the two scatterplots below.





a For each of the scatterplots, describe the strength and direction of the relationship shown in the context of the data it shows.

b If the temperature on a day was 26°C, how many heaters do you predict would be bought?

c If somebody had earnings of $100 000, what would you predict their monthly phone bill to be?

d How confident are you of your predictions in parts **b** and **c**? Explain why you would not be as confident predicting a value from scatterplot A as you would be predicting a value from scatterplot B.

6 When examining the relationship between two variables in real life, we think about how one variable changes depending on how the other changes. Height and weight are a good example of this.

a Would you say that the height of a person depends on how much he or she weighs? Or does the weight of a person depend on how tall the person is?

b Another way to look at this would be to think about what effect changes in one variable would have.

Would you say that, as a person gets taller, this generally makes them heavier? Or, as a person gets heavier, this generally makes them taller?

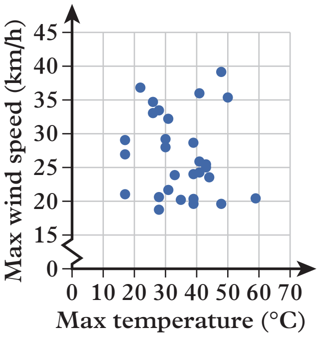
c If we consider height and weight, which is:

i the dependent variable **ii** the independent variable?

7 For each of the pairs of variables **a–d** below, determine:
  i if there might be a relationship between them
  ii which is the independent variable and which is the dependent variable, if a correlation does exist between them
  iii the strength and direction of any correlation that exists.
    a time (years) and height of a tree (metres)
    b value of a car (dollars) and age (years)
    c weight of a person (kilograms) and number of languages he or she speaks
    d shoe size and height of a person (metres)

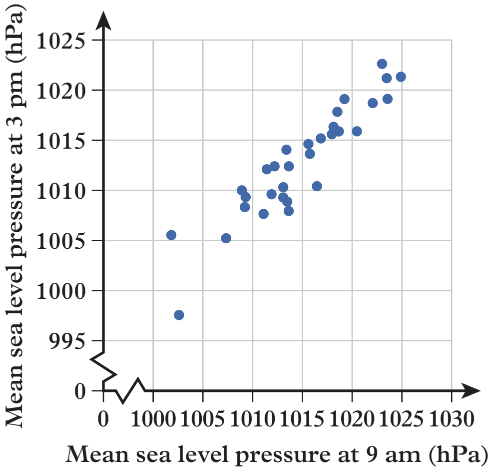
8 A strong positive correlation was found between the number of hours studied and the result achieved on a test.
  a Does this mean that if you study for many hours you will get high marks? Explain.
  b To help prevent assumptions like the one made in part **a**, there is a saying: ‘correlation does not mean causation’. What does this mean?
  c Explain why the saying quoted in part **b** is useful to remember, particularly when dealing with strong correlations.

9 For each of the following, state whether there is a correlation between the two variables and, if there is, describe it in terms of strength and direction.
  a The scatterplot below plots the maximum wind speed, in kilometres per hour, against the maximum temperature, in °C, for each day in January 2021 in Coldstream, Victoria.



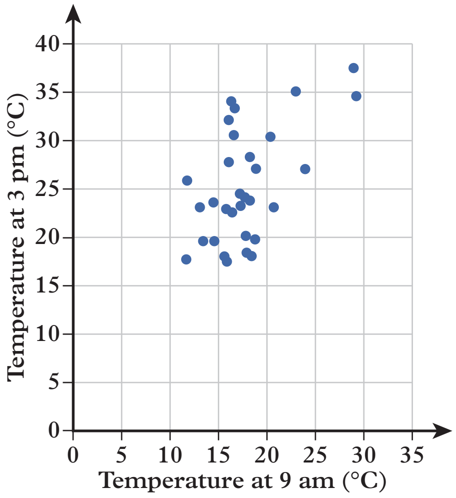
(Source: BOM)

  b The scatterplot below plots the mean sea level pressure at 3 pm, in hectopascals, against the mean sea level pressure at 9 am, in hectopascals, for each day in January 2021 in Coldstream, Victoria.



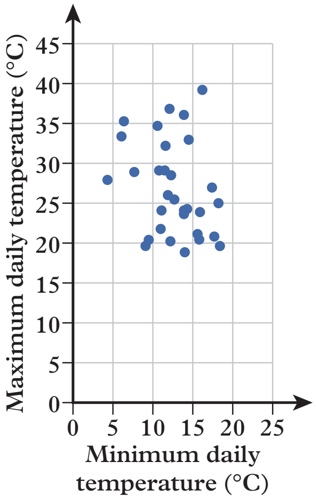
(Source: BOM)

c The scatterplot below plots the temperature at 3 pm, in °C, against the temperature at 9 am, in °C, for each day in January 2021 in Coldstream, Victoria.



(Source: BOM)

d The scatterplot below plots the maximum daily temperature, in °C, against the minimum daily temperature, in °C, for each day in January 2021 in Coldstream, Victoria.



(Source: BOM)

10 Look up the weather data for the previous month in your local area. Determine any correlations between the numerical variables given by constructing a variety of scatterplots with the data.

11 Can a scatterplot be drawn with a negative section of the *x*- and/or *y*-axis? Explain.

12 Explain why it is important to plot a scatterplot with the original data rather than with grouped data.

PROBLEM SOLVING AND REASONING

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Scatterplots

Topic quiz
10E

10F Time series

Learning intentions

✔ I can plot time series data.
✔ I can identify the trends shown in time series data.
✔ I can describe time series data.

Inter-year links

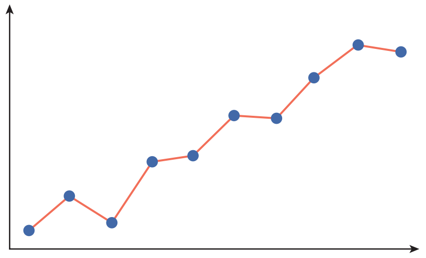
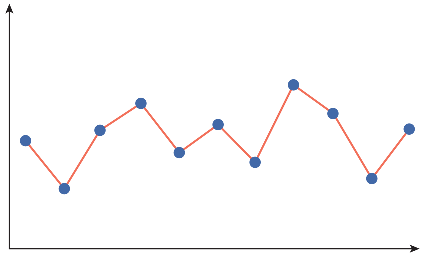
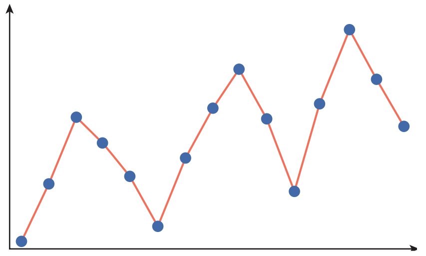
Year 8	9C Presenting data
Year 9	8A Classifying and displaying data

Time series

- A **time series** is a sequence of numerical data taken at regular intervals.
- Time series data is bivariate data for which the independent variable is time.
- Time series data is usually shown as a **line graph**.

Identifying features in time series

- Time series plots can be described in terms of their trend (long term direction: upward, downward or stationary) as well as whether they have seasonality (repeated patterns).
  - → When describing the trend of a time series, you are looking for a general direction, taking into account any seasonality.
  - → Seasonality can be identified by spotting a regular repeating pattern in the points for example every 4th point being higher than average.
- Time series plots will almost always have some natural variation (bouncing around) and may also have irregular fluctuations.
- Some time series follow a non-linear trend, for example there may be a steady increase followed by a steady decrease.

Sample time series graph	Trend	Seasonality
	upwards	no
	stationary	no
	upwards	yes

Example 10F.1 Plotting time series data

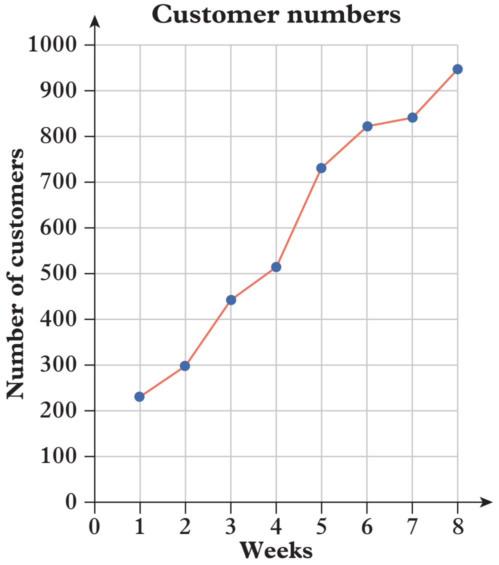
Plot the time series data from this table, showing the number of customers in a store over an 8-week period.

Week	1	2	3	4	5	6	7	8
Number of customers	229	298	442	512	731	822	841	945

THINK

Plot the data on an appropriate set of axes, joining successive points with a straight line. Ensure your graph has a title and an even scale on the axes.

WRITE

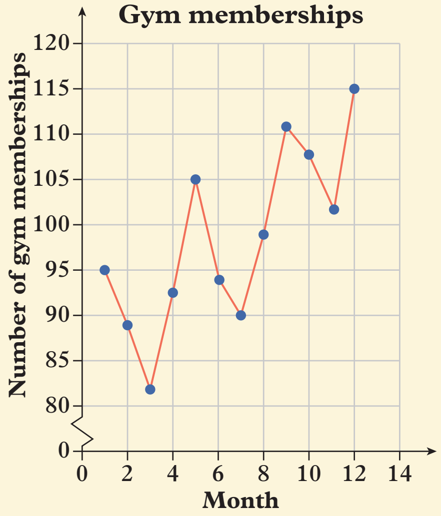


Example 10F.2 Identifying the features of a time series

For the time series graph on the right:

a state whether the time series has seasonality

b describe the trend.



THINK	WRITE
a Look at the pattern of the data. Is there seasonality (showing regular peaks and troughs)?	**a** The time series has seasonality.
b Look at the graph. Is the overall trend upward, downward, or staying roughly the same?	**b** There is an upward trend.

Example 10F.3 Describing time series data

Describe the time series data from Example 10F.1 in the context of what it shows about customer numbers during 8 weeks.

THINK	WRITE
1 Look at the graph. Determine the trend shown in the data and whether there is seasonality. 2 Interpret the trend the graph shows in the context of the data it represents.	As time increases, the number of customers is steadily increasing with no seasonality.

Helpful hints

✔ The time intervals for a time series can be any units, for example hours, days, weeks or years, but the dependent variable must be measured after the same amount of time has passed for every interval.

✔ The natural variation in time series data should be considered when identifying features of a time series plot.

ANS p790

Exercise 10F Time series

△ 1–7, 9 ■ 1(d–f), 2, 4, 5, 8–10, 12 ⬠ 1(d–f), 4, 8–13

UNDERSTANDING AND FLUENCY

10F.1 **1** Plot each of these time series data sets.

a

Hours after midday	1	2	3	4	5	6	7	8	9	10
Temperature	34	36	37	35	34	31	27	22	19	17

b

Years of experience	1	2	3	4	5	6	7	8	9	10	11	12
Salary ($1000s)	42	44	46	49	51	54	56	59	62	65	68	72

c

Days after rain	0	5	10	15	20	25	30	35
Water storage level (L)	456	421	362	321	298	254	221	182

d

Weeks of experience	0	1	2	3	4	5	6	7
Time to complete task (min)	98	71	62	55	50	46	43	42

e

Month	1	2	3	4	5	6	7	8	9	10	11	12
Sales ($100s)	26	32	29	31	35	32	31	34	37	39	36	40

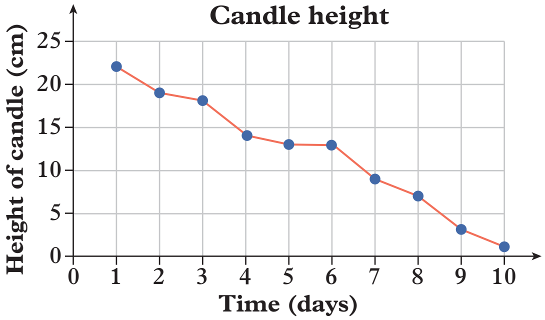
f

Week	1	2	3	4	5	6	7	8	9
Hours worked	36	42	38	35	38	41	48	31	38

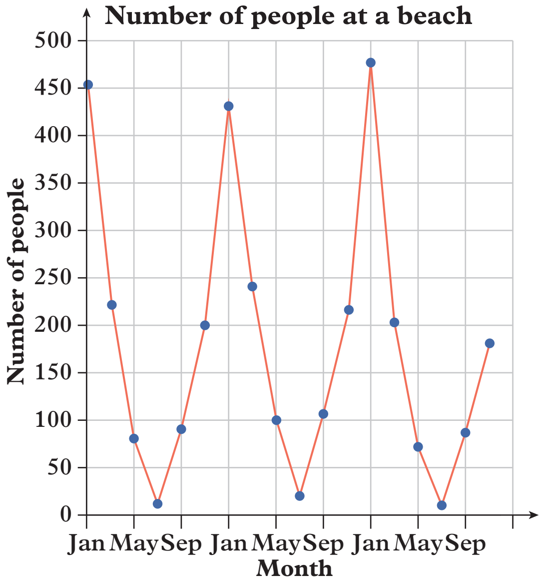
10F.2 **2** For each of the following time series graphs:

i state whether the time series has seasonality **ii** describe the trend.

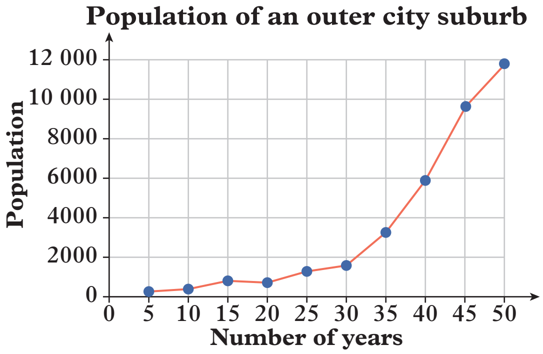
a



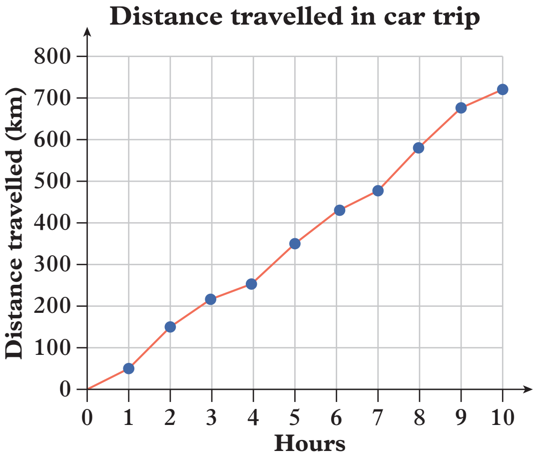
b



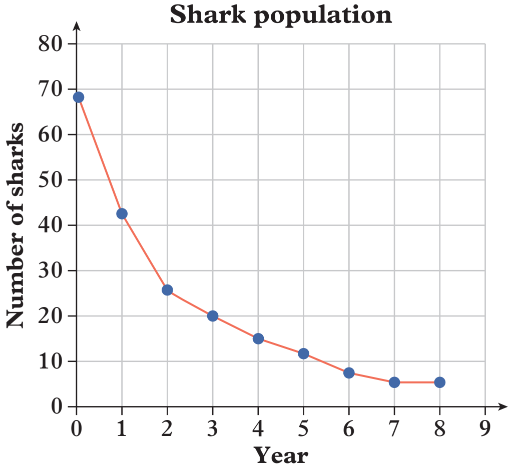
c



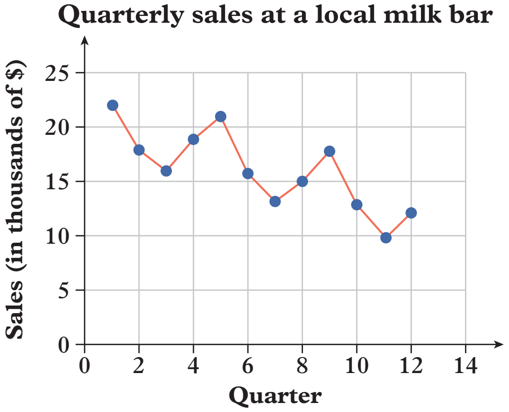
d



e



f

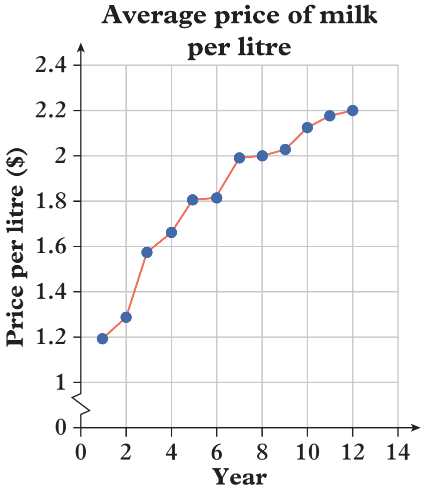


3 Describe the trends shown in the time series plots from question **1**.

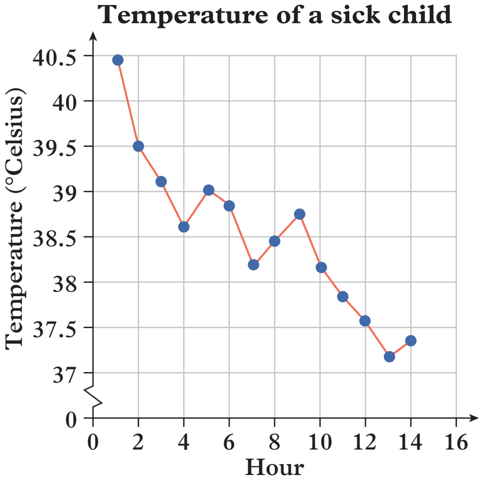
UNDERSTANDING AND FLUENCY

10F.3 **4** Describe the time series data from each of these graphs in context.

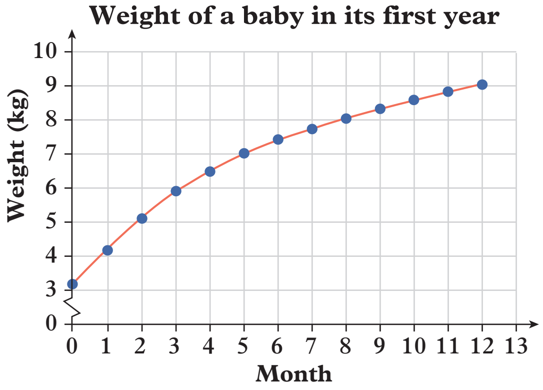
a



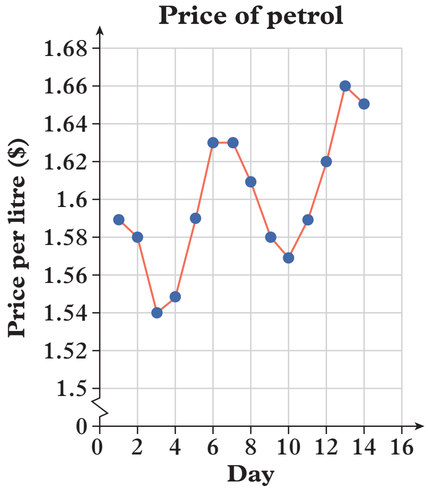
b



c



d



5 Describe each of the time series plots from question **2** in context.

6 Describe each of the time series plots from question **1** in context.

PROBLEM SOLVING AND REASONING

7 Consider this data set, showing the balance (to the nearest dollar) of a bank account over a 10-week period.

Week	1	2	3	4	5	6	7	8	9	10
Balance ($)	787	542	312	116	816	678	457	298	998	749

a Plot the time series data and describe the features shown.

b Over the 10-week time period, what was:

i the minimum balance

ii the maximum balance?

c Suggest a reason for the seasonality shown in your graph.

8 Consider the data set shown in this table. It records the growth of a tree over a number of years.

Time (years)	1	2	3	4	5	6	7	8
Tree height (m)	0.4	1.2	2.3	3.5	4.5	5.6	6.8	7.9

a Plot the time series data from the table.

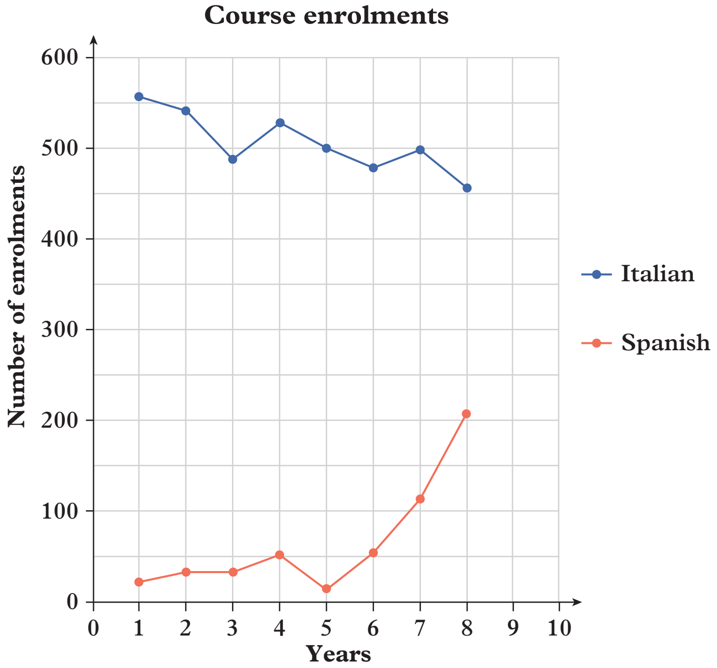
b Describe the features shown in your graph.

c How tall would you predict the tree to be when it is:

i 10 years old **ii** 20 years old?

d Explain why you would be more confident about your first prediction than your second prediction.

9 Consider the graph below.



a What is the graph showing?

b Describe the trend shown by each of the graph lines.

c At what point would you predict that the Spanish course would have more enrolments than the Italian course?

10 Consider the data set in this table, showing the number of hours students spent studying for a Maths exam and the marks they obtained for the exam.

Time spent studying (hours)	0	1	2	3	4	5	6	7	8	2	6
Maths exam mark (%)	35	42	56	83	71	76	88	92	70	67	72

a Explain why a scatterplot must be used to plot this data.

b Explain why this is not a time series plot, even though time is the independent variable.

c Plot the data and describe the trend shown.

d Explain why, even if the last two data points were not included in the set, a scatterplot must still be used to plot this data.

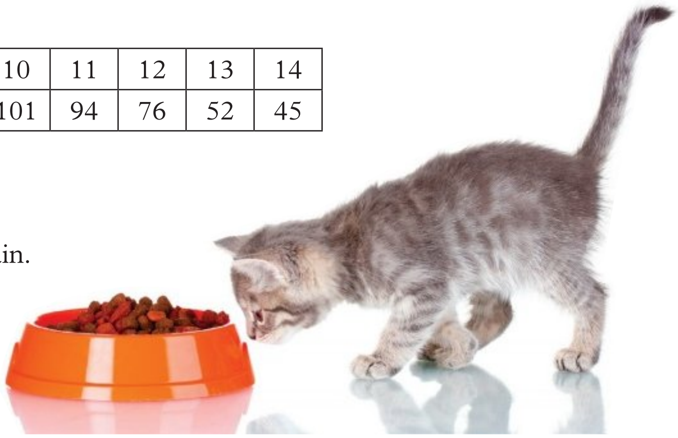
11 Consider the data set in this table, showing the number of pieces of dry cat food in a bowl after a number of hours.

Time (hours)	5	6	7	8	9	10	11	12	13	14
Cat food pieces	189	167	167	132	128	101	94	76	52	45

a Plot the time series data from the table.

b Describe the trend shown in your graph.

c Would you expect this trend to continue? Explain.

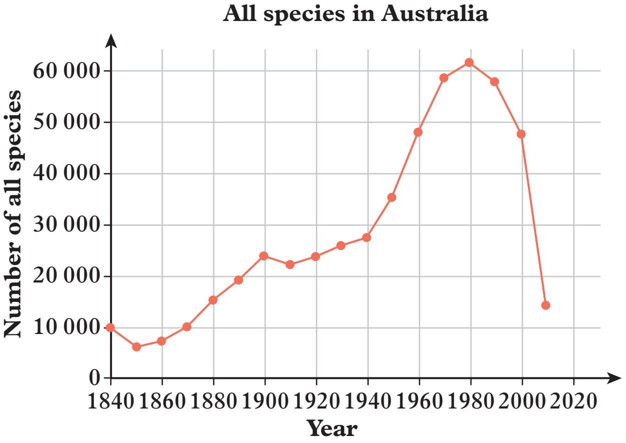


CHALLENGE

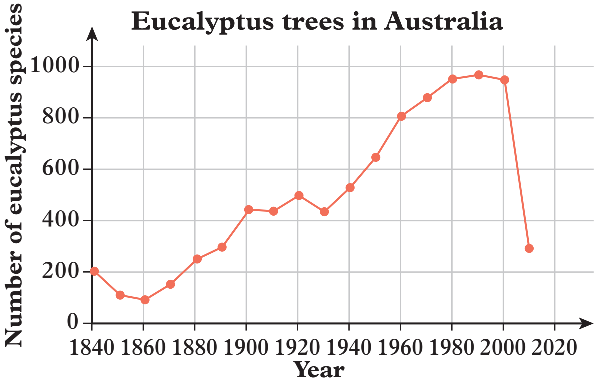
12 Below are two time series. The first shows the change in the number of all species in Australia and the second shows the change in the number of eucalyptus species in Australia.

a Describe the changing trends of the number of all species over time.

b Is the number of species of eucalyptus associated with the number of all species? Explain.

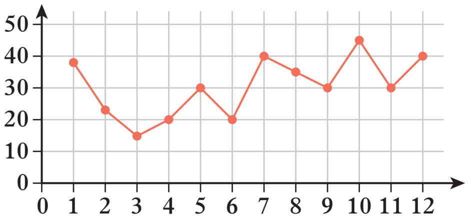


(Source: BOM)



(Source: BOM)

13 Consider the time series with seasonality shown below.



To better understand the data, we can compare each seasonal value to the mean value of the seasons by finding the percentage each value is above or below the mean of all the values. For example, if the seasonal value is 250 and the mean value is 200, then the season is $\frac{250}{200} = 1.25$ times greater than the mean and so is 25% greater than the mean. This multiplier, 1.25, is called the seasonal index.

a Determine the mean value of the seasons.

b Hence, correct to two decimal places, determine the seasonal index for each season.

Season	1	2	3	4	5	6	7	8	9	10	11	12
Value	38	23	15	20	30	20	40	35	30	45	30	40
Seasonal index												

c State the percentage and direction, above or below the mean, of the following seasons compared to the mean.

i 2 **ii** 7 **iii** 10 **iv** 11

To correct for the variation caused by the seasonality, we can increase or decrease the seasonal value by a per cent to make the value average. For example, for a value that is 25% above the mean, with a seasonal index of 1.25, multiplying by the reciprocal $\frac{1}{1.25} = 0.8$ makes the seasonal value average. Hence, decreasing by 20% will correct for the seasonality.

d Determine, correct to two decimal places, the percentage and the direction (increase or decrease) the following seasons must be increased or decreased by to correct for seasonality.

i 2 **ii** 7 **iii** 10 **iv** 11

Check your Student obook pro for these digital resources and more: pro

Interactive skillsheet
Time series

CAS instructions
Line graphs

Topic quiz
10F

10G Lines of best fit

Learning intentions

✔ I can draw a line of best fit by eye.
✔ I can calculate the least-squares regression line using technology.
✔ I can use a line of best fit to make predictions.

Inter-year links

Year 8	9C Presenting data
Year 9	8A Classifying and displaying data

Using a line of best fit to make predictions

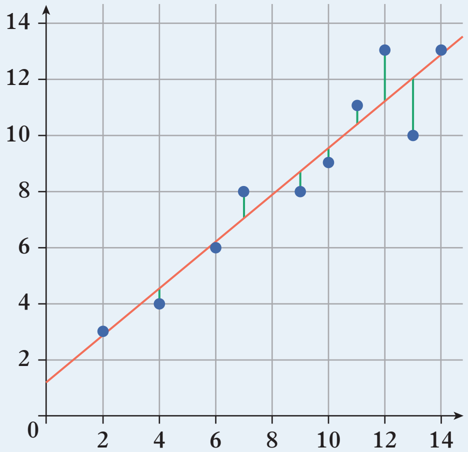
- A **line of best fit** is a line that best represents the relationship between the two variables on a scatterplot. Lines of best fit are usually drawn as straight lines.
- A line of best fit can help predict the value of one variable when the other variable is known.
  - → If the strength of the correlation between two plotted variables increases, the predictions made using a line of best fit become more reliable.
  - → Predictions made within the original range of the data plotted are more reliable than predictions made outside the original range of data.
- A known variable can be substituted into the equation of the line of best fit to predict the value of the related unknown variable.

Lines of best fit by eye

- To draw a line of best fit by eye, draw a straight line on your scatterplot so that it lies as close to as many points as possible.
  - → There should be approximately the same number of points on each side of the line of best fit.
- When a line of best fit has been drawn by eye, the equation of the line can be calculated by identifying two points on the line and using the formula $y = m(x - x_1) + y_1$.

The least-squares regression line

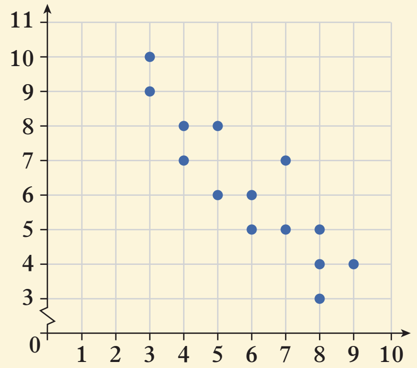
- A **least-squares regression line** is a mathematically calculated line of best fit for numerical bivariate data.
- Least-squares regression lines minimise the sum of the square of the errors between a line of best fit and the points on a scatterplot.



- The equation for a least-squares regression line can be found easily using spreadsheet software (like Excel), a calculator or other technology.

Example 10G.1 Drawing a line of best fit by eye

Draw a line of best fit by eye for the data shown on this scatterplot.

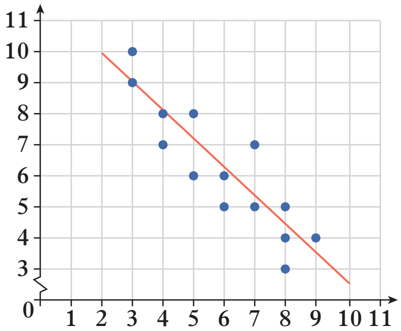


THINK

1 Look at the direction of the plotting of the points. There is a clear downward trend, so the line of best fit will need to be drawn in this direction.

2 Draw a straight line on the scatterplot in the direction of the data, with an approximately equal number of points below and above the line.

WRITE



Example 10G.2 Creating a scatterplot with a line of best fit

Use Excel (or alternative spreadsheet software) to graph the data from this table and find the least-squares regression line, correct to one decimal place.

x	1	5	8	3	10	2	7	4	1	9	10	3	5	2	1	8	4
y	10	4	3	6	1	8	5	7	8	3	2	7	6	9	6	4	5

THINK

1 Enter the data into an Excel spreadsheet. The values can be put in rows or columns, with each cell containing a single value.

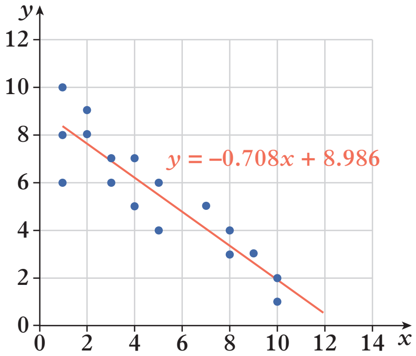
2 To create a scatterplot, highlight the cells containing the data and select Insert → Scatter.

3 Select the graph, then use the toolbar to select Chart Design → Add Chart Element → Trendline → Linear.

4 Click on the trendline and select Trendline Options → Display Equation on chart.

5 Round the values in the equation to one decimal place.

WRITE



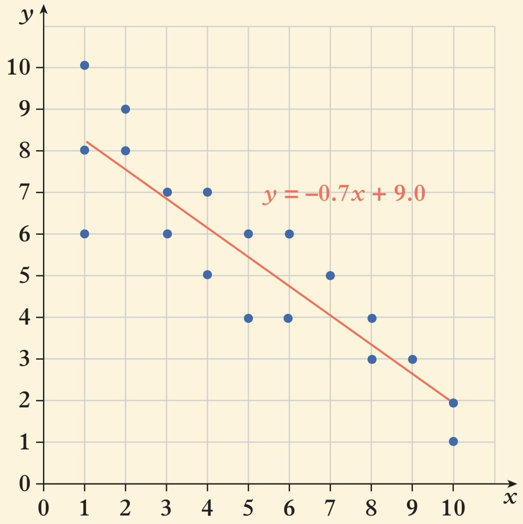
$y = -0.7x + 9.0$ (correct to one decimal place)

Example 10G.3 Using a line of best fit to make predictions

Use the equation for the line of best fit given in Example 10G.2 (reproduced here) to:

a predict the value of y when $x = 3$

b predict the value of x, to one decimal place, when $y = 5$.



THINK	WRITE
a **1** Write the equation for the line of best fit.	**a** $y = -0.7x + 9.0$
2 Substitute the value $x = 3$ into the equation manually, or use your calculator, to find the predicted value of y.	$= -0.7 \times 3 + 9.0$ $= 6.9$ The predicted value of y is 6.9.
b **1** Write the equation for the line of best fit.	**b** $y = -0.7x + 9.0$
2 Substitute the value $y = 5$ into the equation manually, or use your calculator, to find the predicted value of x.	$5 = -0.7x + 9.0$ $0.7x = 9.0 - 5$ $0.7x = 4$ $x = \frac{4}{0.7}$
3 Round the value of x to one decimal place.	$x \approx 5.7$ The predicted value of x is 5.7.

Helpful hints

✔ A line of best fit may, but does not need to, pass through any of the points on the scatterplot.

ANS p792

Exercise 10G Lines of best fit

△ 1(a–c), 2(a–c), 3–6, 8, 10 □ 1(c–e), 2(b–d), 3, 4, 7, 8, 10, 11 ⬠ 1(d, e), 2(c, d), 7, 9–13

UNDERSTANDING AND FLUENCY

10G.1 **1** For each of these data sets, construct a scatterplot and draw a line of best fit by eye.

a

x	3	2	7	5	6	3	1	4	10	9	7	4	8	10	1	5	7	3	8	6
y	5	4	7	6	8	2	2	3	10	9	9	8	6	8	5	4	4	6	10	5

b

x	4	7	1	2	10	5	8	6	2	9	4	3	10	7	1	9	8	5
y	5	4	9	9	2	6	4	5	7	4	8	8	1	6	8	1	5	7

c

x	6	8	3	9	11	14	5	1	15	13	8	5	11	12	7	4
y	5	10	5	6	11	9	3	2	11	13	8	7	8	10	3	1

d

x	2	8	4	6	7	7	5	1	4	5	3	9	4	8	6	2
y	4	8	9	5	4	6	7	2	8	9	4	10	5	6	9	4

e

x	4	9	13	15	7	6	18	11	13	7	9	10	18	12	16	17
y	42	29	15	6	38	36	1	13	18	40	20	12	4	16	9	7

10G.2 **2** For each of these data sets, use technology to graph the data and find the equation of the line of best fit (least-squares regression line). Round the values in the equation to one decimal place.

a

x	11	6	14	8	13	7	16	3	19	8	18	3	6	4	7	10	12	1
y	32	16	49	27	40	29	58	24	62	36	64	12	23	17	24	37	42	6

b

x	3	6	4	1	2	8	9	7	4	8	7	19	9	4	3	6	2	1
y	5	11	7	2	4	10	12	12	10	15	10	16	15	5	9	9	6	5

c

x	2	9	7	8	6	4	4	6	7	3	8	2	1	6	4	9	8	3
y	19	2	5	7	9	11	12	22	2	15	4	17	23	7	15	1	3	18

d

x	2	3	4	8	6	4	7	9	2	9	6	5	3	7	8	5	1	2
y	10	13	16	22	19	14	25	33	13	26	8	20	15	20	26	22	9	8

10G.3 **3** Use the line of best fit shown on this scatterplot to predict the value of the following (correct to one decimal place where necessary):

a x when $y = 36$

b x when $y = 73$

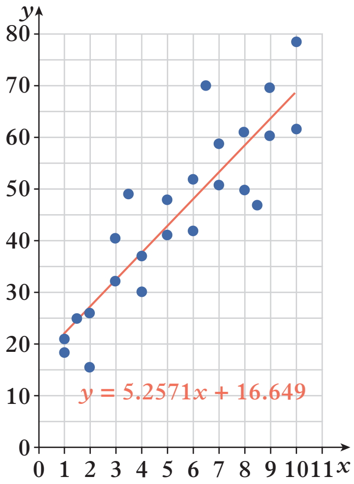
c y when $x = 4$

d y when $x = 9$

e y when $x = 0$

f x when $y = 30$

g y when $x = 10$

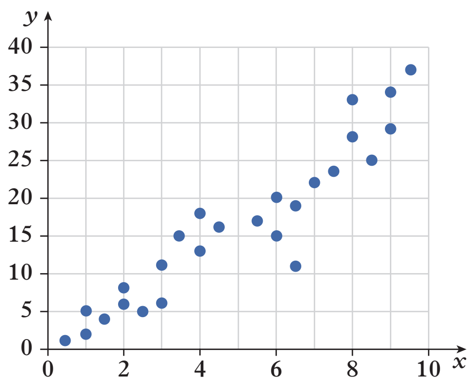


4 Use each line of best fit you found in question **2** to predict the value of y when $x = 10$.

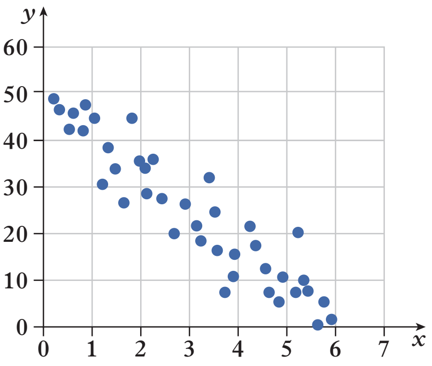
5 For each scatterplot below:

i draw a line of best fit by eye **ii** use the line of best fit you drew to predict the value of y when $x = 5$.

a



b



Note: Copies of these scatterplots can be found on your obook pro.

6 Andrew wanted to investigate the relationship between the number of books students read in a year and their scores in an English exam.

a What relationship do you predict will exist between the number of books read and the English mark obtained?

b State which variable (number of books read, or English mark obtained) is the independent variable and which is the dependent variable.

c Andrew collected the data in the table below. Create a scatterplot to represent the data.

Number of books read	11	6	21	8	2	7	9	15	16	1	0	4	1	0	12	29
English exam mark (%)	70	60	89	70	47	67	65	86	75	45	55	52	62	30	77	95

d Use technology to find a line of best fit (least-squares regression line) for this data.

e Andrew wants to know how many books he should read in a year if he is aiming for a mark of 70% in the English exam. What would you tell him?

7 Consider the scatterplot below right, showing the relationship between the heights and weights of the students in a Year 10 class.

a Describe the trend shown.

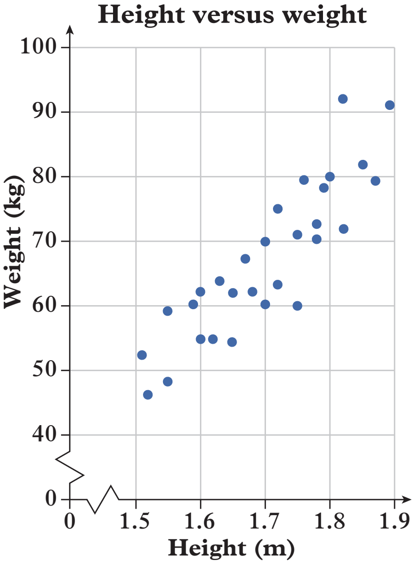
b Draw a line of best fit by eye for this scatterplot.

c Use your line of best fit to predict the weight of a person who is 185 cm tall.

d It is usually best to find the equation of the straight line and substitute values into that to make predictions about values from the graph. When drawing the line of best fit by eye, why is it usually more accurate to choose two points in a straight line that are as far apart as possible and join them with the line:

e The line of best fit for this scatterplot passes through the points (1.65, 62) and (1.85, 82). Use these two given points and the formula $y - y_1 = m(x - x_1)$ to find the equation of the line of best fit.

f Use the equation you found in part **e** to predict the weight of a person who is 185 cm tall. How does this compare to your answer for part **c**?



8 Consider the data in this table.

Number of videos downloaded	21	6	17	21	8	49	2	31	22	13	4	6	42	36	25	9	30	17	21	14
Space available on device (GB)	13	10	8	9	18	2	14	1	5	21	22	18	9	2	10	11	13	15	3	10

a Draw a scatterplot to represent the data.

b From your scatterplot, describe the relationship between the two variables.

c On your scatterplot, draw a line of best fit by eye and find its equation.

d If somebody had downloaded 30 videos, how much space would you predict there would still be available on the device?

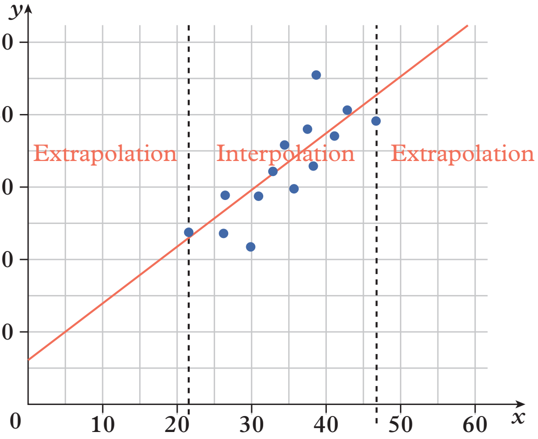
e Explain why you can't have much confidence in your prediction from part **d**.

f Can you provide a reason why the relationship between the variables in this data set is moderate?

g Predicting a value within a data set, as you did in part **d**, is called interpolation. Sometimes you may want to predict a value outside the data set. This is called extrapolation.

Identify whether each of these predictions uses interpolation or extrapolation.

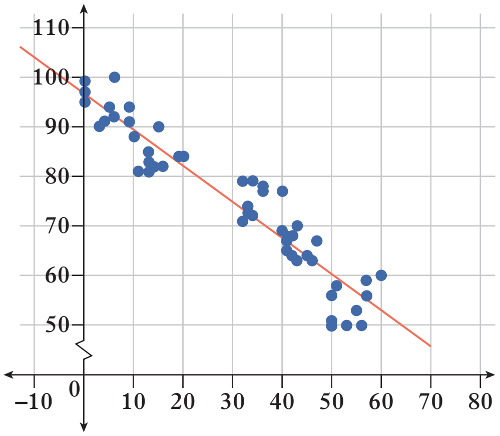
- **i** There will be 5 GB available on a device when 40 videos have been downloaded.
- **ii** There will be 2 GB available on a device when 60 videos have been downloaded.
- **iii** There will be 15 GB available on a device when 20 videos have been downloaded.
- **iv** There will be 1 GB available on a device when 50 videos have been downloaded.



h Imagine that somebody had downloaded 200 videos. If the graph continues its current downward trend, what prediction would you make about available space on the device?

i Comment on your answer to part **h**. How confident can you be about making a prediction outside the given data set?

9 Consider the scatterplot and line of best fit shown.



a Predict the dependent value for each of the following correct to the nearest 5. State whether the prediction is interpolation or extrapolation.

- **i** $x = 10$ **ii** $x = -10$
- **iii** $x = 25$ **iv** $x = 65$

b Predict the independent value for each of the following. State whether the prediction is interpolation or extrapolation.

- **i** $y = 75$ **ii** $y = 55$
- **iii** $y = 40$ **iv** $y = 100$

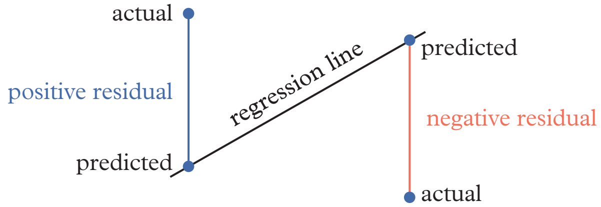
c For which values of x will predictions of the dependent variable be an:

- **i** interpolation? **ii** extrapolation?

d For which values of y will predictions of the independent variable be:

- **i** interpolation? **ii** extrapolation?

10 A residual is the signed difference between the dependent value from the original data and the predicted value from the regression line of the same independent value. That is, residual value = actual value – predicted value.



a Determine the least-squares regression line for the data below using technology. Write the coefficients correct to three significant figures.

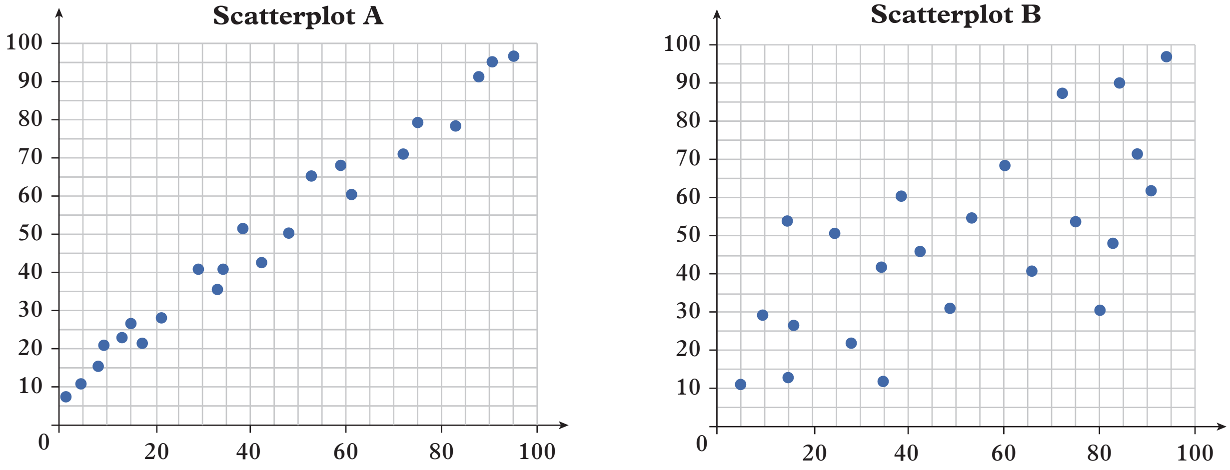
x	1	1.2	2	1.4	2.4	3	1.6	3	2.3
y	10	16.9	20	24.2	15.6	20	14.5	30	25.8

b Determine the residual values of the following data values correct to two significant figures.

i $x = 1.2$ **ii** $x = 1.6$ **iii** $x = 2$ **iv** $x = 2.3$

c Write each residual as a percentage of its respective actual data value correct to one decimal place.

11 Consider these two scatterplots.

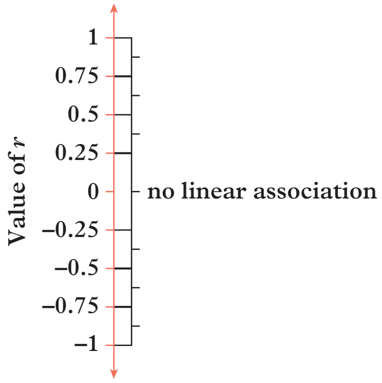


a Describe the difference between the two scatterplots.

b Draw a line of best fit by eye for each scatterplot and use it to predict the value of y when $x = 8$.

c In which prediction from part **b** do you have more confidence? Explain.

d The strength of a bivariate relationship (its correlation) can be described using the Pearson's product-moment correlation coefficient (represented by the pronumeral r). For scatterplot A, $r = 0.9$ and, for scatterplot B, $r = 0.6$.



Copy and complete this figure using the words strong, moderate or weak and either 'positive linear association' or 'negative linear association'.

PROBLEM SOLVING AND REASONING

12 The equation of the least-squares regression line for the following set of data is $y = \frac{27}{11} + \frac{7}{11}x$.

x	1	2	3	2	5	3	6	6	4	8
y	2	6	3	4	4	5	5	6	6	9

a Determine the mean of the x-coordinates, $\overline{x}$, and the mean of the y-coordinates, $\overline{y}$.

b Hence, show that the point $(\overline{x},\overline{y})$ lies on the least-squares regression line.

c For the equation of the least-squares regression line, $y = a + bx$, show that the y-intercept, a, can be given by $a = \overline{y} - b\overline{x}$.

d Determine the exact value of the standard deviation of the x-coordinates, s_x, and the mean of the y-coordinates, s_y. Assume the data is a sample.

The slope of the least-squares regression line, b, is given by $b = r\frac{s_y}{s_x}$, where r is the correlation coefficient.

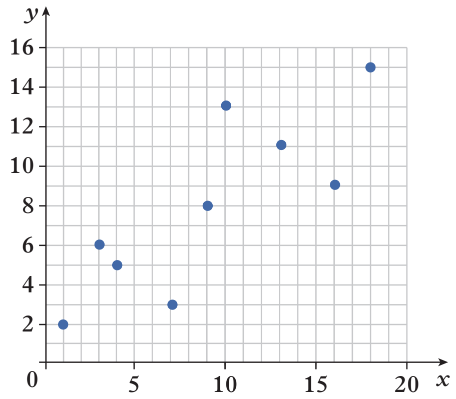
e Determine the value of the correlation coefficient correct to four decimal places.

f Hence, state the strength of the relationship between the two variables.

CHALLENGE

13 Another common method for finding a line of best fit is called median–median regression or three-median regression.

a Copy this scatterplot and divide it vertically into three sections so that each section has the same number of points.



b For the median–median regression method, you need to find the median of each section. This is obtained by finding the median x-coordinate and the median y-coordinate and marking those points on the graph.

Show that the points (3, 4), (9, 8) and (16, 11) are the medians of each section.

c Draw in the line of best fit by placing your ruler on the lower median and the upper median, then moving it $\frac{1}{3}$ of the way towards the middle median before you draw the line.

d Find the equation of the line of best fit you drew in part **c**.

e Use your calculator to find the equation of the line of best fit using linear regression. How does this equation compare to what you calculated in part **d**?

f Explain why the three-median regression line is best used when a scatterplot has one or more outliers.

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Interactive skillsheet
Lines of best fit

BLM
Exercise 10G

Investigation
Our expanding universe

CAS instructions
Scatterplots and lines of best fit

Topic quiz
10G

10H Evaluating statistical reports

Learning intentions

- ✔ I can evaluate statistical reports.

Inter-year links

Year 8	9A Collecting data and sampling methods

Evaluating statistical reports

- When evaluating a report or claim, you need to consider how the information was collected and how the information has been interpreted.
- A **biased** sample is a sample that is not reflective of the population. Samples should be selected randomly and be large enough to represent the whole population.
  - → The sample size should be at least equal to the square root of the population. For example, if the population is 10 000, the sample size should be at least $\sqrt{10\,000} = 100$.
- Data should be fairly represented, so as not to skew the outcome. Presenting only some of the information, or selecting inaccurate statistics to describe the data, is misleading.

Example 10H.1 Evaluating surveys

Decide whether each of these surveys will provide fair or biased results, giving a reason for your answer.

a Asking every 20th person in the local phone book if they enjoy running, to determine how many people in the local community exercise.

b Surveying 1000 random households in Sydney in order to find the average number of people living in residential homes in NSW.

THINK	WRITE
a **1** Consider the surveying technique. Is the sample large enough and gathered in an unbiased way so that it theoretically represents the population (the local community)? **2** Consider the question asked. Does it answer what the questioner is trying to discover (how many people exercise)?	**a** The sample is randomly selected from the community and is of sufficient size. However, this survey is likely to provide biased results, because the question does not ask if people exercise, just if they enjoy running.
b Consider the surveying technique. Is the sample large enough and gathered in an unbiased way so that it theoretically represents the population (people in NSW)?	**b** This survey is likely to provide biased results because, although the sample is of a reasonable size, it only samples from one city in NSW and is therefore unlikely to be representative of the whole population.

Example 10H.2 Identifying how a statement might misrepresent data

Explain how the statement 'Cricket is deemed the world's most popular sport' might be a misrepresentation of the data that 67% of people surveyed at cricket matches in 15 countries answered that cricket was their favourite sport.

THINK	**WRITE**
1 Look at the sample chosen. Does it reflect the population (the world)? **2** Look at the question or result given. Is the conclusion fair?	This statement misrepresents the data by using a biased sample (people at cricket matches), who are much more likely to answer that cricket is their favourite sport than people not attending a cricket match. The question was also only asked in 15 countries, not every country in the world.

Example 10H.3 Analysing surveys and results

Caleb wanted to know the average age of people at a skate park. He surveyed every 10th person he saw there over one week, surveying a total of 20 people. Summary statistics gave a mode of 17, a median of 18, a mean of 22 and a range of 80. Caleb concluded that the average age of people at the skate park was 22.

a Decide whether Caleb's data collection method is fair.

b Decide whether the interpretation of the data is fair.

c If appropriate, provide a suggestion to improve the survey.

THINK	**WRITE**
a Is the sampling method random, with an equal chance of selecting every person in the population?	**a** The method appears fair, although it could be biased depending on the times that Caleb visited the skate park. The sample could be larger.
b Does the conclusion fairly interpret the statistics?	**b** The interpretation could be improved. The sample size is small and the mean is distinctly larger than the median, implying either the presence of an outlier or positively skewed data that skews the mean. The median would be a better statistic to use.
c Are there any ways to improve this survey?	**c** Caleb could take a larger sample and ensure that he visits the skate park at different times of the day. His interpretation could also be improved by using the median as the measure of centre.

Helpful hints

✔ Make sure samples are taken from the whole population, not from a subset of the population. A sample will only be random if it is randomly selected from the whole population.

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Exercise 10H Evaluating statistical reports

△ 1–4, 6, 7 ▢ 2–6, 8 ⬠ 3, 6–10

10H.1 1 Decide whether each survey below will provide fair or biased results, giving reasons for your answers.

- **a** Asking 500 random people at an AFL match their favourite sport, in order to determine the country's favourite sport
- **b** Asking everybody at your school their opinion on a school issue
- **c** Asking every 10th person on the electoral roll if they like dogs, to determine the most popular pet
- **d** Asking a random sample of 100 people (stratified by age) from around the country who they think should be prime minister at the next election
- **e** Asking 10 000 people randomly selected around the country their opinion on an issue in your community
- **f** Asking everybody you see at a local shopping mall their opinion on a global issue

10H.2 2 Explain how each of the following headlines might be misrepresenting the data described in brackets.

- **a** 'Soft rock is the most popular music genre.' (70% of people surveyed said they enjoy listening to soft rock.)
- **b** 'New Apple tablet tops popularity stakes.' (48 people out of 50 surveyed in an Apple store said the Apple tablet was their favourite.)
- **c** 'Women now earn more than men.' (10 people were randomly surveyed on a street.)
- **d** 'New anti-ageing cream is unbeatable.' (100% of the anti-ageing cream company's shareholders were surveyed and agreed.)

10H.3 3 For each scenario below:

- **i** decide whether the data collection method is fair
- **ii** decide whether the interpretation of the data collected is fair
- **iii** if appropriate, provide a suggestion to improve the survey.
  - **a** Tess wanted to know which is the most popular beach in NSW. She asked every 10th person on the electoral roll in her town, 'What is your favourite beach in NSW?' Seventy-five per cent of 200 people said that Bondi was their favourite beach, so Tess concluded that Bondi was the most popular beach in NSW.
  - **b** Stuart wanted to know the average height of basketball players in his town. He surveyed 10 players at random. Summary statistics gave a mode of 1.82 m, a median of 1.80 m, a mean of 1.79 m and a range of 15 cm. Stuart concluded the average height of basketball players in his town was 1.80 m.
  - **c** Isaac wanted to know the most popular computer brands. He asked a random sample of 1000 people, 'Do you like Macs?' He found that 60% of people said yes. Isaac concluded that Mac computers were the most popular brand of computer.
  - **d** Chelsea wanted to know the most popular brand of soft drink in town. She asked all the local supermarkets which brand they sold the most of, and found that Coke made up 37% of the sales, Pepsi 29%, Schweppes 17% and all other brands made up the remainder. She concluded that no single brand was the most popular.
  - **e** Sienna wanted to know the average number of hours people spent sitting down every day. She asked over 100 people, chosen at random from people she saw walking through a mall, how many hours they sat down per day, and came up with an average of 8.5 hours.
  - **f** Kane wanted to know the most popular party foods to serve at an upcoming school dance. He surveyed every second person in his neighbourhood about their favourite party foods, and decided to serve party pies, chocolates and sandwiches.

4 The marketing manager of a drug company claims that the company's new pain relief drug blocks pain for up to 18 hours. This is based on a sample of 15 people recording the number of hours they were pain-free. Here is that data:

2	18	3	2	2
1	2.5	2	1.5	2.5
2	1.5	2	1.5	1.5

a Why is the company's marketing manager making this claim? Explain why it is misleading.

b Calculate the three measures of centre (mean, median and mode) for the given data.

c The marketing manager changes tactics and decides to claim that the pain relief drug blocks pain for an average of 3 hours. Comment on this claim. Is the claim now fair? Explain.

5 What questions should you ask yourself when you are presented with a claim that uses statistics?

6 Each of these media statements is followed by a 'fine print' statement in brackets qualifying the original claim. Explain why the claims are not so 'amazing' when you read the fine print.

a An advertisement for a phone's battery life claims it 'lasts up to 7 days' (based on standby power consumption).

b An advertisement for toilet paper claims it is 'Voted Australia's favourite!' (a total of 42 people were surveyed).

c An advertisement for a moisturising lotion states '67% of women notice a difference after just one week!' (compared to the use of regular soap).

7 Use the statistics shown in this table to comment on the number of refugees granted asylum by Australia to 2018, in relation to the population as a whole compared with other countries. (Hint: Find the number of refugees per million of the country's population.)

Country	Refugee population*	Estimated total population in 2018
Australia	56 934	24 992 370
Germany	1 063 835	82 927 920
Jordan	715 298	9 956 011
Pakistan	1 438 955	212 215 000
Turkey	3 652 362	82 319 730

*Refers to the number of people residing in the country of asylum. For most industrialised countries, this refers to the number of people recognised as refugees after arriving as asylum seekers over the past 10 years. Excludes resettled refugees.

(Source: UNHCR)

8 Consider the table and text below, which were published online.

a From the table, where is Australia ranked in terms of their response to handling the coronavirus pandemic?

b Why would it be misleading to say that Australia ranked 8th out of 10 countries?

c Why is it not useful to know that Australia ranked 8th out of 10 countries?

d How does the data missing from the table make it difficult to determine where Australia sits in the world rankings?

e The article does not directly provide the data to support its claims that 'Australia has been ranked among the top 10 countries'. What could you do to research this?

> 'Australia has been ranked among the top 10 countries for its handling of the coronavirus pandemic, with New Zealand taking out the top spot in a poll conducted by a prominent Australian think tank.
>
> The Lowy Institute assessed the response of 98 countries in how they managed the pandemic in the 36 weeks following their hundredth confirmed case of the virus.
>
> It judged countries that had fewer reported cases and deaths (both in aggregate and per capita basis), as well as nations where testing rates were high.
>
> China was excluded from the ranking because of a lack of publicly available data on testing.
>
> The United States, Brazil, Iran and Mexico were ranked the worst handlers of the pandemic.
>
> Countries that proved more successful in containing the virus were largely in the Asia-Pacific region.'

Country	Refugee population	Estimated total population in 2018
Australia	56 934	24 898 000
Canada	114 101	37 075 000
Germany	1 063 835	83 124 000
Jordan	715 294	9 965 000
New Zealand	1 545	4 743 000
Pakistan	1 404 008	212 228 000
Turkey	3 681 688	83 340 000
United Kingdom	126 708	67 142 000
United States of America	313 242	327 069 000

(Source: SBS news)

9 Find a media article that uses statistics to support its claims (including at least one visual display). Write a paragraph analysing these claims and commenting on how trustworthy you think the article is.

10 Write an informative guide to teach somebody how and what to look for when reading statistics and the claims that they are supposedly supporting.

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Worksheet
Analysing statistics

Investigation
Conducting your own statistical report

Topic quiz
10H

10I Sampling and reporting

Learning intentions

✔ I can evaluate the appropriateness of sampling methods.
✔ I can evaluate where graphs or claims may be misleading.

Inter-year links
Year 8 9A Collecting data and sampling methods

Sampling methods

- For a sample to be representative of a population, it must be collected using a random sampling method. Otherwise, bias will occur when the data is collected.
- **Stratified sampling** divides the population into small groups known as strata (for example, based on age) and takes a random sample from each stratum. The size of each sample is proportional to how large the category is.
- **Systematic sampling** selects data at fixed intervals (for example, every fifth person). The starting point should be random.
- Be aware that bias can easily occur when collecting data by sampling.
- When collecting a random sample from a population, natural variation means the sample statistics are unlikely to exactly match the population parameters.
  - → The degree to which the sample statistics and population parameters differ should be considered when looking for bias.
  - → Large samples are more accurate and should better reflect the population parameters than small samples (assuming they are random).

Misleading graphs and claims

- Reports may contain graphs or data displays that are misleading. Look at the scales used on the axes of graphs to be certain they are evenly spaced with consistent numbering.
  - → Look at the value the vertical axis starts at – if it does not start at 0, vertical differences will be enlarged and the differences in heights will be exaggerated.
  - → Where only 'averages' are provided, the distribution of the data is not known, so the average may not be very informative. It is also important to know if the stated average is the mean or the median.
- Graphs and axes on graphs should be properly labelled. By not labelling the graph correctly, incorrect assumptions can be made about the data being presented.
- Look at the range of data being presented and consider whether any data has been purposefully omitted. For example, if you are looking at temperature data over a year, all months should be considered, as temperature varies due to the seasons in a year.
- Consider why the data has been presented as it has, and what may have been omitted. For example, if only summary statistics are provided, this may not provide adequate information about the distribution of the data.
- Be prepared to question the claims made in reports. Some reports are written motivated by an agenda that can result in an unwarranted claim being pushed to further that agenda.

Example 10I.1 Reviewing a published report

A new toll tunnel has just been opened to allow cars to travel from one side of the river to the other. The alternative route is via a no-toll bridge. Shortly after the tunnel opened, a newspaper reported:

'The tunnel is now so popular that the probability of a car using the tunnel rather than the bridge is 85%'. Public perception was that this is not the case. Some residents decided to do a car count over a long time and found 967 cars from a total of 2472 went over the bridge. Comment on the newspaper report.

THINK	WRITE
1 From the collected data, calculate the probability of a car using the tunnel.	Using the collected data: $\Pr(\text{car uses bridge}) = \frac{967}{2472} \times 100\%$ $\approx 39\%$ So: $\Pr(\text{car uses tunnel}) \approx 61\%$
2 Compare the collected data with the reported data, and comment on the comparison.	The collected data shows that 61% of cars use the tunnel, while the newspaper reports this figure to be 85%. It seems likely that the newspaper report is exaggerated as this gap is unlikely to occur from natural variation.

Example 10I.2 Identifying misleading features in graphs

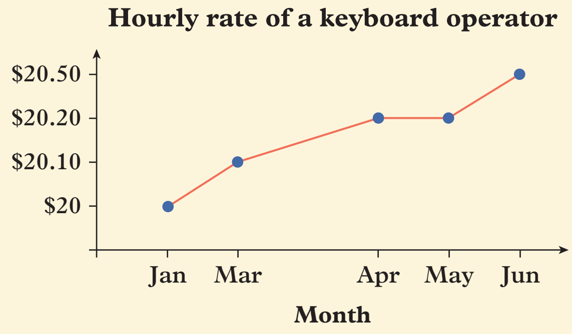
This graph displays the hourly pay rate for a keyboard operator employed by a large company. The time period covered is the first 6 months of the year.

a This graph is misleading. Give reasons why.

b The company manager claims that the hourly rate of the keyboard operator has increased dramatically over the last 6 months. The graph appears to indicate this.

i Is this true?

ii How does the graph seem to support the manager's interpretation of the data?



THINK	WRITE
a Look for anything that might make the graph misleading.	**a** This graph is misleading for the following reasons: • The scale on the x-axis is not uniform. • The month of February is missing. • The scale on the y-axis does not start at zero, and no break is indicated. • The scale on the y-axis is not uniform.
b i Look at the values on the y-axis.	**b i** The keyboard operator's hourly pay rate has only increased from \$20 to \$20.50 over the 6-month period, an increase of 50c per hour. This is not a 'dramatic' increase.
ii Consider the scale on the y-axis.	**ii** Because the scale on the y-axis does not start at zero, the range of \$20 to \$20.50 has been stretched out to appear larger than it would on a complete scale. This gives the impression that the increase is greater than it really is.

Helpful hints

✔ When reviewing graphs, look at where the numbering on the axes starts. Heights of columns in a column graph may not be proportional to their frequency unless the scale is accurate.

ANS p796

Exercise 10I Sampling and reporting

△ 1–5, 7, 8, 10, 11, 15 ■ 1–3, 5, 6, 8, 11–15 ⬠ 1, 2, 5, 6, 8, 12–17

UNDERSTANDING AND FLUENCY

10I.1 **1** Critically analyse each of these reports.

a A survey conducted at a gym revealed that 90% of the gym's clients felt they were getting enough exercise each week. It was later quoted that: 'Australians are getting fitter, as 90% of them say they exercise regularly each week.'

b Four teenage boys were still left on a bus when it reached the end of the line. A reporter asked them if they had part-time jobs. Two of them said they did. It was later reported that, '50% of teenagers today have part-time jobs.'

c A survey showed that four out of five children believe they do their fair share of chores at home. A report later appeared in a newspaper with the heading, 'Families lucky: 80% of children do fair share of chores.'

d The children in the stands watching a school swimming carnival, held indoors, were asked whether they were wearing sunscreen to prevent them getting sunburnt. They all said that they had not worn sunscreen. It was later reported that 'School children ignore warnings about wearing sunscreen to prevent skin cancer.'

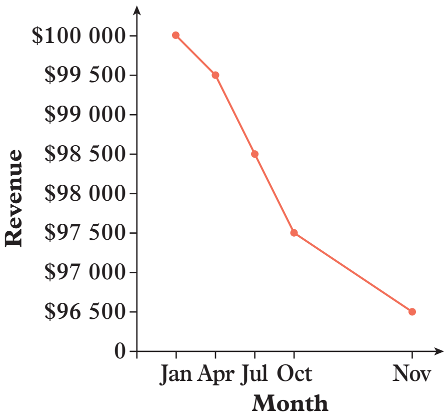
2 Critically analyse each of the media reports below. Suggest what research would need to have been done to make each claim valid.

a In a study of women, having coffee with lunch seemed to reduce the risk of diabetes more than having coffee at other times of day.

b Studies suggest that physically active people have a reduced risk of Alzheimer's disease in later life.

c Studies suggest that happiness wards off heart disease.

10I.2 **3** The graph on the right displays the revenue a company has made during a year.



a This graph is misleading. Give reasons why.

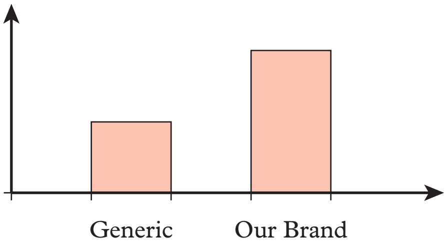
b The chief finance officer claims that the company's revenue has declined significantly this year, but declined at a slower rate at the end of the year. The graph appears to indicate this.

i Is this true?

ii How was this effect achieved?

iii By what percentage has the revenue declined from January to November?

4 The graph on the right displays the average time a brand of paracetamol takes before relief compared to a generic competitor.



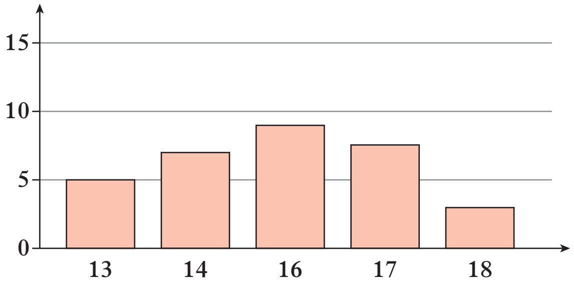
a This graph is misleading. Give reasons why.

b The company claims their brand of paracetamol is twice as effective as their generic competitor's. The graph appears to indicate this.

i Is this true?

ii How was this effect achieved?

5 This graph displays the number of secondary students of different ages that joined the school band.



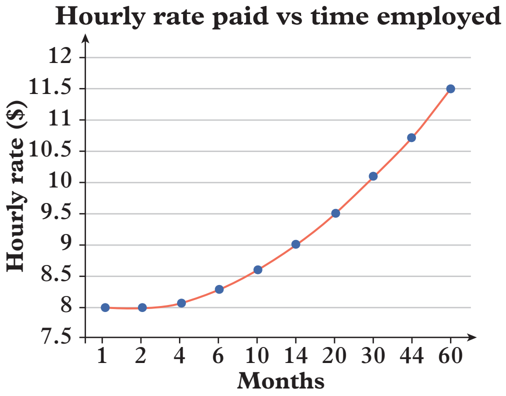
a This graph is misleading. Give reasons why.

b The band conductor claims that the number of students that joined the band is approximately symmetric and students of all ages joined. The graph appears to indicate this.

i Is this true?

ii How was this effect achieved?

6 Fatima's employer constructed the graph on the right.



a What is the graph supposed to represent?

b Explain how the graph misrepresents the data.

c Redraw the graph as best you can so that it is fair.

7 Consider the proportions of the Australian population in 2019 shown in the tables below.

Age group (years)	0–4	5–9	10–14	15–19	20–24	25–29	30–34	35–39	40–44	45–49	50–54
Percentage	6.179	6.382	6.133	5.923	6.935	7.525	7.458	7.021	6.291	6.616	6.051
Number to survey											

Age group (years)	55–59	60–64	65–69	70–74	75–79	80–84	85–89	90–94	95–99	100 and over
Percentage	6.091	5.473	4.829	4.171	2.895	1.993	1.234	0.605	0.174	0.02
Number to survey										

(Source: ABS, Australian Demographic Statistics Tables)

Determine the number of people in each stratum that should be surveyed for a stratified sample of 10 000 people.

UNDERSTANDING AND FLUENCY

8 Consider the number of grey kangaroos in Victoria in 2020.

Zone	Frequency	Number to survey
Mallee	37 300	
Upper Wimmera	138 100	
Lower Wimmera	427 700	
Central	658 950	
Otway	236 950	
North east	239 850	
Gippsland	172 700	
State-wide total	1 911 550	

(Source: Arthur Rylah Institute for Environmental Research)

Determine the number of grey kangaroos that would need to be surveyed for a stratified sample of 1400 grey kangaroos.

PROBLEM SOLVING AND REASONING

9 Some advertisements claim that eating a good breakfast leads to success at school. Consider an alternative argument:

'Students who don't eat breakfast are more likely to be absent from school or experience lethargy.'

Does this statement mean the same thing as eating a good breakfast leading to success at school? Why do the advertisements use one statement rather than the other?

10 We are often unaware of companies sponsoring studies.

Consider this headline: 'A hotdog a day increases risk of cancer by 21%.'

Whether this statement is true or not is open to debate. However, if the study was later revealed to have been conducted by an animal-rights group whose purpose is to convince people to be vegan, would this affect the credibility of the study? Suggest how this type of reporting of information from a biased source could be monitored.

11 Celebrities often promote products on social media. Do these celebrities really believe in the products, or are they simply paid to advertise them? Are they given these products free for their own use?

What are your thoughts when you see advertisements such as these? Do the advertisements prompt you to purchase a product?

12 Online surveys are very common. However, for an online survey to have any credibility, many issues need to be considered. For example:

- How can we ensure that a random sample has been surveyed that is representative of the target audience?
- How can we ensure that people completing the survey are who they claim to be?
- How do we deal with the non-response rate?

Write some guidelines you feel would be important to follow when choosing a sample for an online survey.

13 After answering an online survey, you frequently receive unwanted advertising emails from companies unknown to you, 'Great holiday deals on Hayman Island', 'Take advantage of this special deal', etc. Some groups make money by selling email addresses to interested companies. Is this simply part of the digital age, or can you somehow monitor this behaviour? What are your thoughts?

14 A television channel invites its viewers to vote on a topic that has just been discussed on air. Later in the program, the results of the votes are revealed, such as, '98% of viewers agree that …' Discuss whether the reported figures in this type of survey are valid.

15 The CEO of a company has constructed this table showing the composition of her company.

	Total employees		**Full-time**		**Part-time**	
	25 and over	**Under 25**	**25 and over**	**Under 25**	**25 and over**	**Under 25**
Managers	18	4	14	4	4	0
Non-managers	42	61	40	31	2	30

She made these claims about her employees.

a 82% of 25-and-over employees are managers.

b 6% of under-25s in the company are managers.

c 29% of employees work part-time.

d 49% of under-25 non-managers work full-time.

Discuss whether each statement is correct. Provide mathematical evidence to support your answers. If a particular claim is incorrect, supply the correct figure.

16 The top 10 Facebook users in 2020 came from the countries listed in the table below. The table shows the number of users (in millions) and the population (in millions) of these countries.

Number	**Country**	**Users (millions)**	**Population (millions)**	**Per cent users**
1	India	320	1380	
2	USA	190	331.0	
3	Indonesia	140	273.5	
4	Brazil	130	212.6	
5	Mexico	93	128.9	
6	Philippines	83	109.6	
7	Vietnam	68	97.34	
8	Thailand	51	69.80	
9	Egypt	45	102.3	
10	Bangladesh	41	164.7	

a Complete the last column of the table (correct to one decimal place).

b Which of these countries has the:

- **i** highest percentage of users
- **ii** lowest percentage of users?

c China has the largest population of any country (1.4 billion), yet it does not make the top 10 list of Facebook users. Suggest some reasons for this.

d Using the data from the table, draw different graphs to support these two statements:

- **i** Far more people in the USA and India use Facebook than in other countries.
- **ii** People in Indonesia don't use Facebook as much as those in other countries.

e Explain why you chose to draw your graphs the way you did in part **d**.

17 Choose a topic and conduct a survey among the members of your class. Report on the results of your survey, justifying any claims with mathematical evidence.

Check your Student obook pro for these digital resources and more:

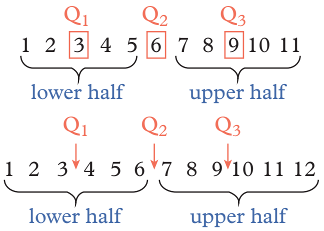
Interactive skillsheet
Sampling methods

Topic quiz
10I

Chapter summary

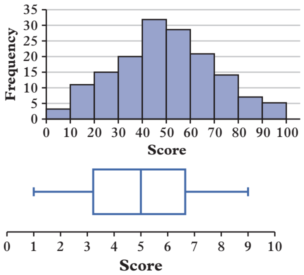
Five number summary

minimum, Q_1, Q_2 (median), Q_3, maximum

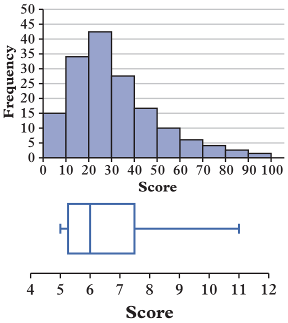


Distributions

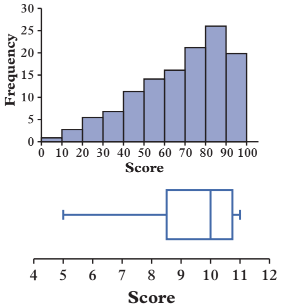
- Symmetric



- Positively skewed



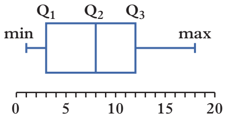
- Negatively skewed



Line of best fit

- Draw a straight line on your scatter plot so that it lies as close to as many points as possible.
- There should be approximately the same number of points on each side of a line of best fit.
- The equation of the line can be calculated by using the formula $y = m(x - x_1) + y_1$.

Box plots



- The lower fence lies 1.5 × IQR below the lower quartile.
- The upper fence lies 1.5 × IQR above the upper quartile.
- Outliers are values that lie beyond these fences.

Standard deviation

- Population's standard deviation:

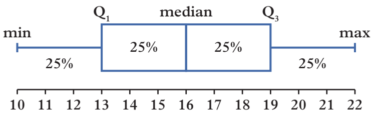
$$\sigma = \sqrt{\frac{(x_1 - \mu)^2 + (x_2 - \mu)^2 + \ldots + (x_n - \mu)^2}{n}}$$

- Sample's standard deviation:

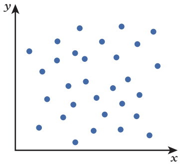
$$s = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2}{n - 1}}$$

Cumulative frequency

- The cumulative frequency of a numerical data set can be calculated by adding the frequency of a particular score to the sum of the frequencies of all its preceding scores.

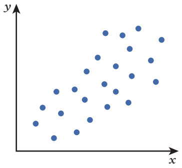


Scatterplots



No obvious pattern in the data

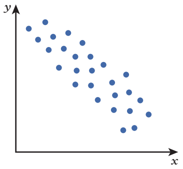
No correlation



A weak pattern in the data

Weak correlation

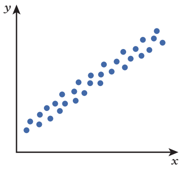
Positive direction



A clear pattern in the data

Moderate correlation

Negative direction

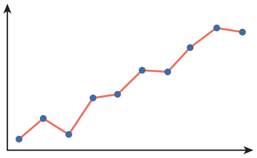


A strong pattern in the data

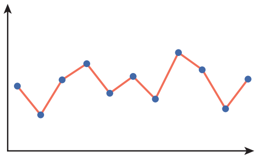
Strong correlation

Positive direction

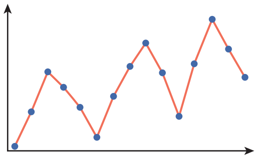
Time series graphs



Upward trend
No seasonality



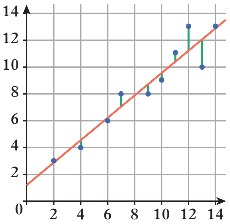
Stationary trend



Upwards trend

Seasonality

The least-squares regression line



- Least-squares regression lines minimise the sum of the square of the errors between a line of best fit and the points on a scatterplot.

Sampling

- Stratified sampling divides the population into small groups known as strata (for example, based on age) and takes a random sample from each strata.
- Systematic sampling selects data at fixed intervals (for example, every fifth person).

Chapter review

Chapter review quiz
Take the chapter review quiz to assess your knowledge of this chapter.

Quizlet
Test your knowledge of this topic by working individually or in teams.

Multiple-choice

10A **1** A data set has 14 values. Which of the following is correct?

A The median and both upper and lower quartiles are values in the set.
B The median is a value in the set, but the upper and lower quartiles are not.
C The upper and lower quartiles are values in the set, but the median is not.
D None of the median, upper or lower quartiles are values in the set.
E The median and the upper quartile are values in the set, but the lower quartile is not.

10B **2** In a box plot, what does the 'box' part represent?

A mean **B** interquartile range **C** range **D** median **E** standard deviation

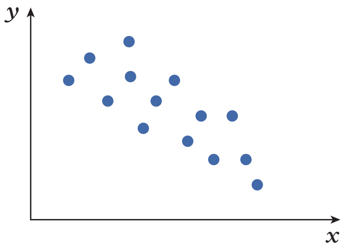
10C **3** The length from the lower quartile to the upper quartile in a box plot represents what percentage of the scores of the data set?

A 25% **B** 50% **C** 75% **D** 90% **E** 100%

10A **10D** **4** What does the standard deviation in a set of data indicate?

A how large the mean is
B how the scores are spread out from the mean
C how many scores there are in the data set
D where the median lies in the set
E where the mean lies in the set

Questions **5–6** refer to the scatterplot on the right. It shows the relationship between two variables, x and y.



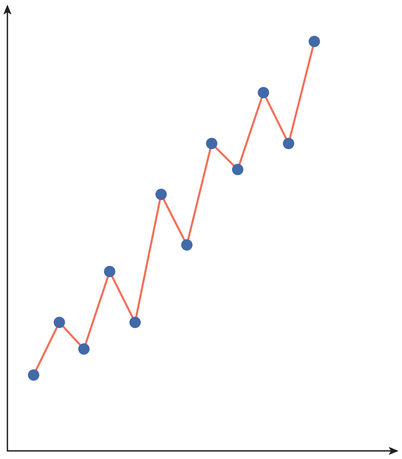
10E **5** How could you describe the relationship between the two variables?

A strong **B** weak **C** moderate
D no correlation **E** impossible to determine

10E **6** Which of these terms also describes the relationship between the two variables displayed on the graph?

A positive **B** negative **C** horizontal **D** vertical **E** random

10F **7** The best description of this time series is:



A increasing **B** decreasing **C** seasonal
D increasing and seasonal **E** irregular fluctuations.

10A 10G **8** The line of best fit for a scatterplot has the equation $y = 1.7x + 1.3$.
When x has a value of 1.5, what is the predicted value for y?

A 1.25 **B** 1.41 **C** 2.55 **D** 3.85 **E** 4.5

10H **9** You are taking a sample from a population of approximately 4000. Which of the following is the smallest size of the sample that should be taken to give reliable results?

A 10 **B** 60 **C** 63 **D** 64 **E** 2000

10A 10I **10** If a representative sample of 20 people is to be taken from a group of 280 children and 120 adults, how many adults should be chosen?

A 6 **B** 10 **C** 14 **D** 20 **E** 60

Short answer

10A **1** For the data in the stem-and-leaf plot on the right showing the number of ice creams sold per day during the month of January, find:

a the range **b** the median

c the five-number summary **d** the interquartile range.

Stem	**Leaf**
4	0 1 3 5 7 9
5	2 5 5 7 8
6	2 3 3 5 7 7 9 9
7	3 5 6 7 8
8	2 4 6 7 8
9	3 6

Key: 4 | 0 = 40

10B **2** Draw a box plot for the data from question **1** and comment on its shape.

10C **3** **a** Create a grouped frequency table using class intervals of 10 with cumulative frequency and cumulative percentage columns for the data in question **1**. Round percentages correct to the nearest integer.

b Use the table in part **a** to construct a cumulative percentage plot.

10A 10D **4** Below are the results Lily and Cassy obtained for five tests.

Lily: 72, 77, 67, 69, 75

Cassy: 47, 92, 87, 77, 52

a For each student, calculate:

i the mean result

ii the standard deviation (to two decimal places).

b Which of these two students performed more consistently?

10E **5** This table shows the number of children enrolled at a day-care centre over the 12 years since its opening.

Years since opening	1	2	3	4	5	6	7	8	9	10	11	12
Number of children	25	27	26	33	32	35	37	39	40	39	40	45

a Create a scatterplot for the data.

b Describe the relationship between the two variables.

10F **6** This table shows a town's mean monthly temperature (°C) in the years 2000 and 2015.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
2000	28.8	29.0	27.4	26.9	23.1	21.5	19.8	20.5	24.5	24.5	27.3	27.6
2015	28.3	29.2	27.8	25.3	22.4	20.3	20.8	21.7	23.6	24.6	27.5	28.4

a Plot the two sets of data as time series on the same graph.

b Describe the trend of each time series.

c Compare the two graphs to describe the differences between the data.

10A 10G **7** **a** Draw a line of best fit by eye on the scatterplot you drew for question **5**.

b Use the line of best fit to predict the number of children the day-care centre could expect to have 20 years after opening.

c Explain the problems in making predictions of this kind.

10H **8** On their packaging, two brands of yoghurt display nutrition facts per 100 g serving.

a Compare the information supplied.

b What would you consider the term 'light' to mean? Would you consider Brand X lives up to this classification?

c The manufacturers of Brand X defend the claim that their product is 'light' by producing a bar graph comparing the carbohydrate content of their product and their rival's product. Draw a graph that they could use to support the claim. Explain.

	Brand X 'light'	**Brand Y '98% fat free'**
Energy	425 kJ	397 kJ
Protein	5.8 g	5.0 g
Total fat	4.8 g	1.9 g
Carbohydrate	9.0 g	14.2 g
Sodium	80 mg	63 mg
Calcium	206 mg	177 mg

10A 10I **9** **a** A survey was conducted by selecting the first person in the top left-hand column of every page of the local telephone book. What type of sampling would this be?

b A survey was conducted by randomly selecting a proportionate amount of students in each level. What type of sampling would this be?

Analysis

Usain Bolt of Jamaica became a triple Olympic champion in the 100 m sprint at the Rio 2016 Olympic Games. A record of all the winning times for this race has been kept since the early 1900s. The Olympic Games usually take place every four years but were not held during the years of World War I or II, or in 2020 due to COVID-19, so there are some gaps in the record.

Consider this table showing winning times for the 100 m sprint since 1952.

Year	1952	1956	1960	1964	1968	1972	1976	1980	1984
Time (s)	10.79	10.62	10.32	10.06	9.95	10.14	10.06	10.25	9.99

Year	1988	1992	1996	2000	2004	2008	2012	2016
Time (s)	9.92	9.96	9.84	9.87	9.85	9.69	9.63	9.81

a From the raw data, describe the winning times over the period shown.

b From the data in the table, calculate (to two decimal places) the:

- **i** mean, median and mode
- **ii** range and interquartile range
- **iii** five-number summary.

c **i** Find the standard deviation of the data set.

- **ii** Comment on the value you found for the standard deviation.

d **i** Draw a box plot to represent the data from the table.

- **ii** Comment on the shape of the box plot.
- **iii** Explain how you think the shape of the box plot will change as future winning times are added to the data.

e Draw a scatterplot of the data.

10A **f** Draw the line of best fit by eye and choose two points to calculate the equation of that line.

g Use technology to calculate the least squares regression line of best fit and write its equation.

10A **h** Compare the equation you wrote for part **g** with the one you wrote for part **f**.

i In the 1960s, speculation centred on the question, 'When will the 10 second barrier be broken for the 100 m sprint?' Today, the talk centres around the question, 'When will the 9.5 second barrier be broken?' Use an equation from part **g** to make a prediction for when this might occur. (Remember that the Olympic Games are only held every four years.)

11

Probability

Index

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Calculating theoretical probability
- ✔ Fractions, decimals and percentages
- ✔ Adding and subtracting fractions
- ✔ Multiplying and dividing fractions

Curriculum links

- Describe the results of two- and three-step chance experiments, both with and without replacements, assign probabilities to outcomes and determine probabilities of events. Investigate the concept of independence (VCMSP347)
- Use the language of 'if . . . then', 'given', 'of', 'knowing that' to investigate conditional statements and identify common mistakes in interpreting such language (VCMSP348)

Materials

- ✔ Calculator

11A Theoretical probability

Learning intentions

- ✔ I can calculate the probability of experiments with and without equally likely outcomes.
- ✔ I can calculate theoretical probabilities using tree diagrams and arrays.

Inter-year links

Year 5/6	Understanding probability
Year 7	10F Theoretical probability
Year 8	9E Theoretical probability
Year 9	9A Two-step chance experiments

Theoretical probability

- The **sample space** of an **experiment** is a list of all the different possible **outcomes** of that experiment.
- The different outcomes of an experiment may not be equally likely to occur.
- An **event** is a set of one or more outcomes from the sample space of an experiment.
- The probability of all events within the sample space will add to 1.
- If all the outcomes of an experiment are equally likely to occur, the probability of an event occurring can be calculated using the formula:

$$\Pr(\text{event}) = \frac{\text{number of outcomes in event}}{\text{number of outcomes in the sample space}}$$

- The **complement** of an event A is the event that A does not occur.
- If an event is identified as A, its complementary event is identified as A′. A′ is sometimes referred to as ‘not A’.
- The sum of the values of the probabilities of complementary events is 1.

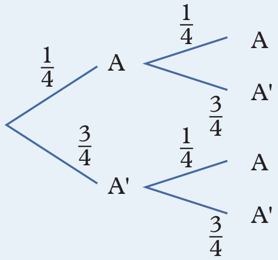
$$\Pr(A) + \Pr(A') = 1$$

Mutually exclusive events

- If two events are mutually exclusive then they cannot both occur at the same time. For example, when rolling a standard six-sided die, the events ‘rolling an odd number’ and ‘rolling an even number’ are mutually exclusive.

Tree diagrams

- **Tree diagrams** can be used to display the outcomes of multi-step experiments.
  - → The possible outcomes for each step of the experiment are represented by the branches.
  - → The branches might not all have the same probability.
  - → The final outcomes are listed at the ends of the branches.
  - → The probability of the final outcomes can be calculated by multiplying together the individual probabilities along the branches.



Outcomes	Probability
AA	$\frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$
AA'	$\frac{1}{4} \times \frac{3}{4} = \frac{3}{16}$
A'A	$\frac{3}{4} \times \frac{1}{4} = \frac{3}{16}$
A'A'	$\frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$

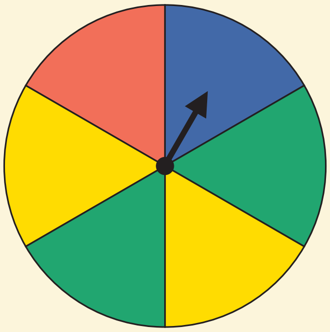
Arrays

- The sample space of a two-step experiment can be displayed in an **array** or table.
- Arrays can be used to calculate the probability of an experiment. To do this, all the cells in the array must be equally likely to occur. Divide the number of cells that satisfy the given event by the total number of cells in the array.
- The table below shows the outcomes for the two-step experiment of rolling a die and then flipping a coin.
  - → The outcomes of the first step are the headings of the columns and the outcomes of the second step are the headings of the rows. Reading across the columns and then down the rows to each cell will help you list all the possible combinations of outcomes.

		Rolling a standard six-sided die					
		1	**2**	**3**	**4**	**5**	**6**
Flipping a coin	**H**	(1, H)	(2, H)	(3, H)	(4, H)	(5, H)	(6, H)
	T	(1, T)	(2, T)	(3, T)	(4, T)	(5, T)	(6, T)

Example 11A.1 Identifying the sample space and calculating theoretical probability

This spinner divided into equally sized sectors (two yellow, two green, one red, one blue) is spun and the colour that the pointer lands on is recorded.



a List the sample space for this experiment.

b Calculate the probability of the spinner landing on a green sector.

THINK	**WRITE**
a List all of the possible outcomes for this experiment inside curly brackets (also called braces).	**a** {red, blue, green, yellow}
b **1** The spinner has six equally sized sections, each of which has an equally likely chance of the pointer landing on it.	**b** $\Pr(\text{green}) = \frac{2}{6}$
2 Divide the number of equally sized sections that are green by the total number of equally sized sections and simplify.	$= \frac{1}{3}$

Example 11A.2 Calculating the probability of multi-step experiments with equally likely outcomes

A coin is flipped twice. Calculate the probability that the outcome will be:

a exactly one head **b** at least one head **c** no heads.

THINK

1 Draw a tree diagram to represent all the different outcomes of this experiment.

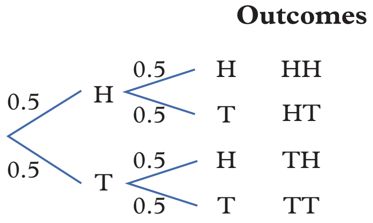
2 Write the individual probabilities on the branches of the tree diagram. For each flip of the coin there are two different possible outcomes which are equally likely to occur, therefore they each have a $\frac{1}{2}$ or 0.5 chance of occurring.

3 The probabilities of the final outcomes can be determined by multiplying the probabilities along the related branches together.

4 Identify the required final outcomes.

5 All final outcomes have the same probability of occurring, so the required probability can be determined by multiplying the number of final outcomes by their probability.

WRITE



All final outcomes have a probability of:

$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

a Favourable outcomes are HT and TH.

$$\text{Pr(exactly one head)} = 2 \times \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

b The favourable outcome is TT.

$$\text{Pr(no heads)} = \frac{1}{4}$$

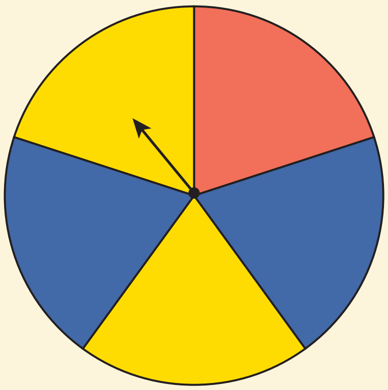
c Favourable outcomes are HH, HT and TH.

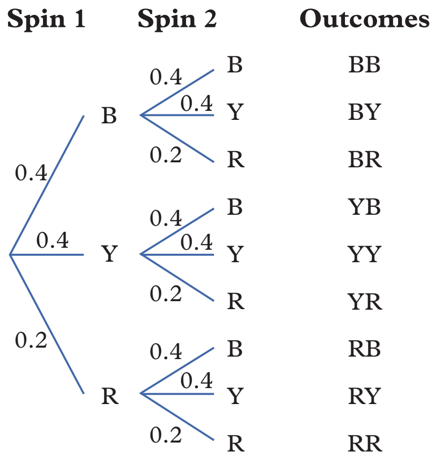
$$\text{Pr(at least one head)} = 3 \times \frac{1}{4} = \frac{3}{4}$$

Note: The probability of this event could also be found by subtracting the probability of the complementary event ('no heads') from 1.

Example 11A.3 Calculating the probability of multi-step experiments without equally likely outcomes

This spinner divided into equally sized sectors (two blue, two yellow, one red) is spun twice, with the colour the pointer lands on being recorded both times. Calculate the probability of landing on a blue sector both times.



THINK	WRITE
1 Calculate the probabilities of landing on blue. There are 5 equally sized sectors, so the probability of landing on a yellow sector is $\frac{2}{5}$, and so on.	$\Pr(\text{blue}) = \Pr(\text{B})$ $= \frac{2}{5}$ $= 0.4$ $\Pr(\text{yellow}) = \Pr(\text{Y})$ $= \frac{2}{5}$ $= 0.4$ $\Pr(\text{red}) = \Pr(\text{R})$ $= \frac{1}{5}$ $= 0.2$
2 Draw a tree diagram. Mark the individual probabilities on the branches and show all outcomes.	
3 Identify the outcome with two blue sectors.	$\Pr(\text{two blues}) = 0.4 \times 0.4$
4 Calculate the probability of this outcome by multiplying the probabilities on each branch together.	$= 0.16$

Helpful hints

- ✔ The word ‘event’ has a specific meaning in probability that is separate to its meaning in everyday English. Read the definition carefully and make sure you do not get confused.
- ✔ Remember that you can only use the theoretical probability formula, $\Pr(\text{event}) = \frac{\text{number of outcomes in event}}{\text{number of outcomes in the sample space}}$, if the outcomes you are dealing with are equally likely to occur.
- ✔ Addition of decimals is often easier than addition of fractions, and multiplication of fractions is often easier than addition of multiplication of decimals. Therefore, depending on the context, it may be useful to write your probabilities as decimals or fractions.
  For example, evaluating $0.25 \times 0.7 \times 0.8$ is more difficult than $\frac{1}{4} \times \frac{7}{10} \times \frac{4}{5}$ whereas $0.15 + 0.125$ is easier to evaluate than $\frac{3}{20} + \frac{1}{8}$.
- ✔ Be careful not to represent decimals incorrectly. For example, $\frac{1}{3}$ is not equal to 0.33.

ANS p799

Exercise 11A Theoretical probability

△ 1–4, 6–8, 10, 12, 13, 17 ■ 1, 2, 6, 7, 9, 10, 14, 17, 18 ⬟ 1, 2, 8, 10, 11, 14, 15–17, 19, 20

UNDERSTANDING AND FLUENCY

11A.1 **1** List the sample space for each of these experiments.

- **a** rolling an eight-sided die (with sides numbered 1–8) and recording the uppermost number
- **b** randomly selecting a letter from the word AUSTRALIA
- **c** spinning a spinner with two red, three white and four blue sections (all sections being equal in size) and recording the colour

2 State whether the outcomes in the sample spaces for the experiments in question **1** are equally likely to occur or not.

3 A letter is selected at random from the word MATHEMATICS.

- **a** List the sample space for this experiment.
- **b** Calculate the probability of selecting an M.

11A.2 **4** A multiple-choice question consists of two parts. Each part has an equally likely chance of being true or false. What is the probability that:

- **a** the answers are both true?
- **b** the answers are both false?
- **c** one answer is true, and one answer is false?

5 In a room there are two lights, each with its own on/off switch.

- **a** Draw a tree diagram to show all the possible combinations of on or off the two lights could be.
- **b** If it is equally likely that each light is on or off, what is the probability that the lights will:
  - **i** both be on?
  - **ii** both be off?
  - **iii** not be in the same mode?

11A.3 **6** Two fair six-sided dice are rolled. Each time a 6 is rolled on either of the dice, it is recorded.

- **a** If A represents the event ‘rolling a 6’, what does A′ represent?
- **b** Draw a tree diagram showing the probabilities of the different outcomes of the experiment. Write the probabilities for each step on the branches.
- **c** Use your tree diagram to find the probability of rolling:
  - **i** a double 6
  - **ii** at least one 6.

7 The experiment in question **6** is repeated. This time, all the numbers rolled on both dice are recorded.

- **a** Explain how this experiment differs from the experiment in question **6**.
- **b** How many outcomes are possible in this case?
- **c** Draw an array to represent the experiment, and use it to determine the probability that the two numbers rolled are:
  - **i** the same
  - **ii** both prime numbers.

8 Three eight-sided dice (numbered 1–8) are thrown and the number rolled with each die is noted. It is recorded whether the number is even or odd.

Use a tree diagram to find the probability of rolling:

a no odd numbers

b one or two odd numbers

c more odd numbers than even numbers.

9 'Unders and overs' is a game that is played by rolling two six-sided dice. The numbers rolled are added together to give a total. Players predict whether the total will be under 7, equal to 7 or over 7.

a Draw an array to show all the possible outcomes when rolling two dice.

b Colour your array from part **a** to show which outcomes are:

i under 7 **ii** equal to 7 **iii** greater than 7.

c Calculate the probability of the prediction being correct if you predicted:

i under 7 **ii** equal to 7 **iii** greater than 7.

d Describe what your strategy would be if you played this game.

10 A ten-sided die (numbered 1–10) is rolled and the number facing up is recorded. Let A represent the event 'rolling an even number', B represent the event 'rolling a multiple of 4', and C represent the event 'rolling a prime number'.

a List the outcomes in each event: A, B, and C.

b Which two events are mutually exclusive?

c Find each of the following:

i Pr(A) **ii** Pr(B) **iii** Pr(C)

11 When rolling two unbiased dice of the same kind, the probability of rolling a double depends on the number of sides on each die.

a Consider rolling a pair of standard six-sided dice.

i How many outcomes are possible when the two dice are rolled?

ii How many of these outcomes are doubles? List them.

iii What is the probability of rolling a double?

b Consider rolling a pair of eight-sided dice. Repeat the questions from part **a**.

c Write a general statement for the probability of rolling a double using a pair of dice with *n* sides.

12 In a group of students there are 15 boys. The probability of choosing a girl from the group is $\frac{2}{5}$. How many girls are in the group?

13 Two dice are thrown and the product of the two numbers rolled is recorded.

a What is the probability that this number is:

i odd? **ii** prime? **iii** a square number?

b For the product of the two numbers rolled, what is:

i the most likely outcome? **ii** the least likely outcome?

14 A game involves a regular six-sided die and a fair coin. The die is rolled, and the number recorded. The coin is then flipped. If the coin lands on heads, then the die score is doubled, whereas if the coin lands on tails, then the die score remains the same.

a List the sample space of possible scores for this game.

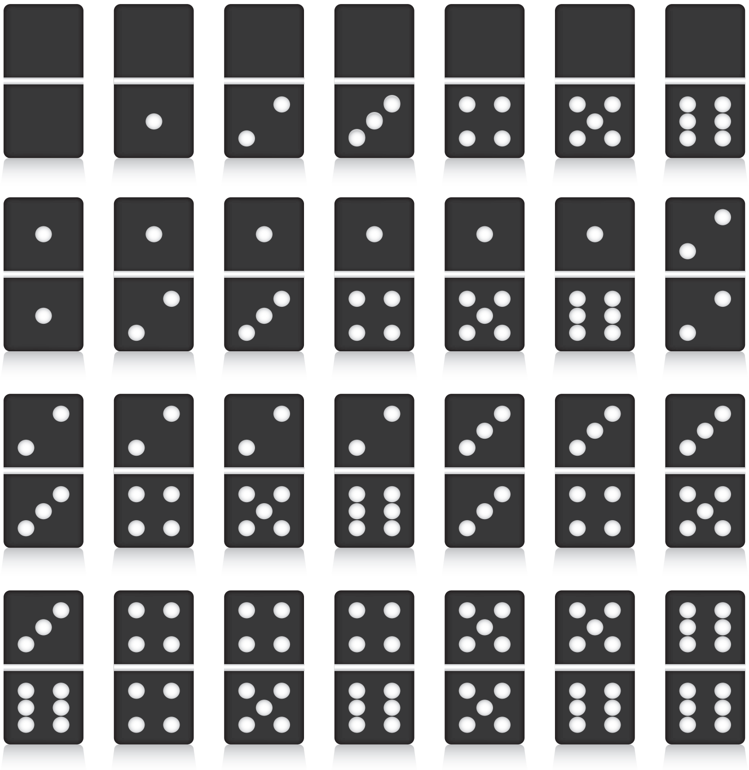
b Explain why each outcome in the sample space is not equally likely to occur.

c Find the probability that a player achieves a score of:

i 10 **ii** 4 **iii** at least 8.

15 Two standard six-sided dice are rolled and the two numbers facing up are recorded. What is the probability that the sum of the numbers is greater than the product of the numbers?

16 A set of dominoes contains tiles, each tile being divided in two and showing dots (similar to dice) to represent numbers. A blank space on a tile represents 0.



- **a** How many tiles are in the complete set of dominoes?
- **b** Explain the order of the tiles in the photo.
- **c** For each of the values (0, 1, 2, 3, 4, 5 and 6), count the number of tiles on which the value appears. If the value appears on both halves of a tile, count the tile once. You should get the same number of tiles for each of the values from 0 to 6. What is this number?
- **d** One tile is chosen at random. What is the probability that it shows:
  - **i** double 6?
  - **ii** the same number twice?
  - **iii** two different numbers?
  - **iv** one even number and one odd number?
  - **v** at least one prime number?
  - **vi** two numbers, one of which is a multiple of the other (not including zero as a multiple)?

17 A coin is biased so that the chance of it showing a tail when flipped is 0.7.

- **a** Draw a tree diagram to show the outcomes and their probabilities if the coin is flipped three times.
- **b** From your tree diagram, what is:
  - **i** the most likely outcome?
  - **ii** the least likely outcome?
- **c** What is the probability of flipping:
  - **i** only one head?
  - **ii** only one tail?
  - **iii** more heads than tails?

PROBLEM SOLVING AND REASONING

18 Snakes and ladders is played on a board with numbered grid squares. Ladders connect certain pairs of squares together, and snakes connect other pairs of squares together. The aim of the game is to progress from square 1 to square 100 by rolling a die and moving the number of squares rolled. You must move up a ladder or down a snake if you land on the squares that show the base of a ladder or the head of a snake.

Imagine a grid on which the first ladder base is on square 7.

a What is the minimum number of die rolls which would take you to square 7?

b What is the maximum number of die rolls which could take you to square 7?

c Consider the minimum number of rolls required.

- **i** List the die rolls which would take you to square 7.
- **ii** How many different options are there?

CHALLENGE

19 A bag contains Christmas decorations of four different colours: red, green, gold and silver. There are four silver decorations and eight gold decorations. One decoration is drawn at random from the bag. The probability it is silver or gold is 0.6. If there are three green decorations, how many red decorations are in the bag?

20 Consider the target on the right.

a Discuss whether the chance of landing on a red section is the same as that of landing on a white section.

b The diameter of the innermost red circle (the bullseye) is 2 cm, and the widths of each of the subsequent rings is 2 cm. What is the radius of each of the rings?

c Calculate the area of each of the rings in terms of pi.

d Assuming the arrow is equally likely to hit any point within the target, calculate the probability of the arrow landing:

- **i** in the bullseye
- **ii** in the outermost red ring.

Round your answers to three decimal points.

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Interactive skillsheet Complementary events

Interactive skillsheet Arrays

Investigation Sicherman dice

Topic quiz 11A

Interactive skillsheet Tree diagrams

11B Experiments with and without replacement

Learning intentions

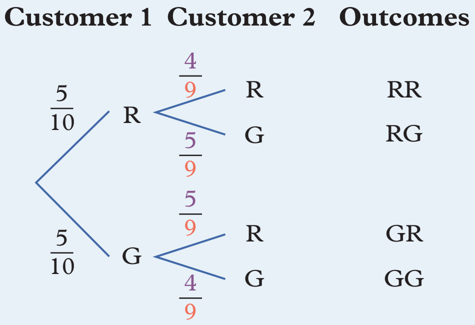
✔ I can calculate probabilities of multi-step experiments with and without replacement.

Inter-year links

Year 7	10F Theoretical probability
Year 8	9E Theoretical probability
Year 9	9B Experiments with replacement

Experiments with and without replacement

- When multi-step experiments involve items being selected from a group, the item can either be replaced or not replaced after each selection.
- If the items in a multi-step experiment involving selection are replaced, the probabilities for each step of the experiment will remain the same.
- If the items in a multi-step experiment involving selection are not replaced, the probabilities for each step of the experiment will change.
- The probability of the second step will depend on the item selected at the first step.
- The number of items to select from will decrease by 1 after each selection.



The number of outcomes in the sample space is reduced for the second selection.
Depending on the first selection, the number of successful outcomes may also be reduced.

- Arrays can be used to show the reduced sample space for experiments without replacement. For example, if two letters are picked at random from the word 'SUN' without replacement, the sample space would be:

		Second letter		
		S	**U**	**N**
First letter	**S**	–	(S, U)	(S, N)
	U	(U, S)	–	(U, N)
	N	(N, S)	(N, U)	–

Example 11B.1 Calculating probability for experiments with replacement

A barrel contains coloured balls: two red, one blue and three green. One ball is drawn from the barrel, its colour is noted, then it is replaced. A second ball is then chosen and its colour is also noted.

a Draw a tree diagram to represent the situation.

b Determine the probability that the two balls selected are the same colour.

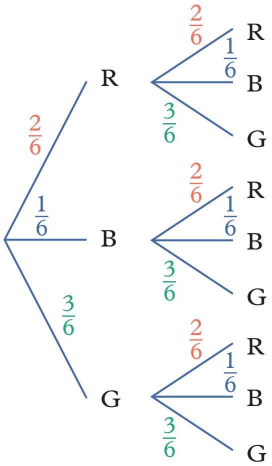
c Determine the probability that a red and green ball are selected in any order.

THINK

a Draw a tree diagram, writing all possible outcomes along with their probabilities. Because the selected ball is replaced after the first selection, the probabilities for the second selection will remain the same.

WRITE

a



Outcomes	**Probability**
RR	$\frac{2}{6}\times\frac{2}{6}=\frac{4}{36}$
RB	$\frac{2}{6}\times\frac{1}{6}=\frac{2}{36}$
RG	$\frac{2}{6}\times\frac{3}{6}=\frac{6}{36}$
BR	$\frac{1}{6}\times\frac{2}{6}=\frac{2}{36}$
BB	$\frac{1}{6}\times\frac{1}{6}=\frac{1}{36}$
BG	$\frac{1}{6}\times\frac{3}{6}=\frac{3}{36}$
GR	$\frac{3}{6}\times\frac{2}{6}=\frac{6}{36}$
GB	$\frac{3}{6}\times\frac{1}{6}=\frac{3}{36}$
GG	$\frac{3}{6}\times\frac{3}{6}=\frac{9}{36}$

b On your tree diagram, identify the final outcomes in which the two balls are the same colour. Add the probabilities for those outcomes to find the required probability, and then simplify.

b Favourable outcomes are RR, BB and GG.

$$\Pr(RR) = \frac{4}{36}$$

$$\Pr(BB) = \frac{1}{36}$$

$$\Pr(GG) = \frac{9}{36}$$

$$\begin{aligned}\Pr(\text{same colour}) &= \frac{4}{36}+\frac{1}{36}+\frac{9}{36}\\ &= \frac{14}{36}\\ &= \frac{7}{18}\end{aligned}$$

c Identify the final outcomes in which a red and a green ball are selected. Add the probabilities for these outcomes to find the required probability, and then simplify.

c Favourable outcomes are RG and GR.

$$\Pr(RG) = \frac{6}{36}$$

$$\Pr(GR) = \frac{6}{36}$$

$$\begin{aligned}\Pr(\text{red and green}) &= \frac{6}{36}+\frac{6}{36}\\ &= \frac{12}{36}\\ &= \frac{1}{3}\end{aligned}$$

Example 11B.2 Calculating probability for experiments without replacement

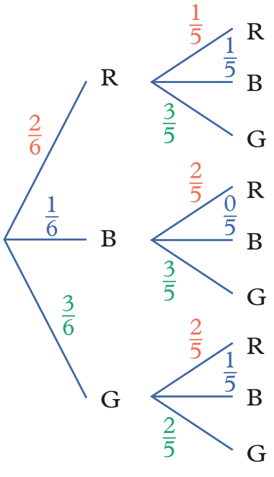
Repeat the experiment from Example 11B.1, but this time assume the first ball is not replaced before the second ball is chosen.

THINK

a Draw a tree diagram. The number of balls from which the second ball is chosen decreases by 1, so the possible outcomes at the second step are different from the possible outcomes for the first step.

WRITE

a



Outcomes	Probability
RR	$\frac{2}{6} \times \frac{1}{5} = \frac{2}{30}$
RB	$\frac{2}{6} \times \frac{1}{5} = \frac{2}{30}$
RG	$\frac{2}{6} \times \frac{3}{5} = \frac{6}{30}$
BR	$\frac{1}{6} \times \frac{2}{5} = \frac{2}{30}$
BB	$\frac{1}{6} \times \frac{0}{5} = 0$
BG	$\frac{1}{6} \times \frac{3}{5} = \frac{3}{30}$
GR	$\frac{3}{6} \times \frac{2}{5} = \frac{6}{30}$
GB	$\frac{3}{6} \times \frac{1}{5} = \frac{3}{30}$
GG	$\frac{3}{6} \times \frac{2}{5} = \frac{6}{30}$

b On your tree diagram, identify the final outcomes in which the two colours are the same. Add the probabilities for those outcomes to find the required probability and simplify. (Note the required the outcome BB is not possible, because there is only one blue ball.)

b Favourable outcomes are RR and GG.

$$\Pr(\text{RR}) = \frac{2}{30}$$

$$\Pr(\text{GG}) = \frac{6}{30}$$

$$\begin{aligned}\Pr(\text{same colour}) &= \frac{2}{30} + \frac{6}{30} \\ &= \frac{8}{30} \\ &= \frac{4}{15}\end{aligned}$$

c Identify the final outcomes in which a red and a green ball are selected. Add the probabilities for these outcomes to find the required probability, and then simplify.

c Favourable outcomes are RG and GR.

$$\Pr(\text{RG}) = \frac{6}{30}$$

$$\Pr(\text{GR}) = \frac{6}{30}$$

$$\begin{aligned}\Pr(\text{red and green}) &= \frac{6}{30} + \frac{6}{30} \\ &= \frac{12}{30} \\ &= \frac{2}{5}\end{aligned}$$

Helpful hints

- ✔ The sum of the probabilities at each step of a multi-step experiment must always be equal to 1.
- ✔ For multi-step experiments, it is often helpful to leave fractions unsimplified until the final answer. In Example 11B.1, the probability that the balls are the same colour was calculated as $\frac{4}{36} + \frac{1}{36} + \frac{9}{36}$, which is easy to add. However, if each fraction were simplified then the denominators would all be different, which would make addition more difficult.

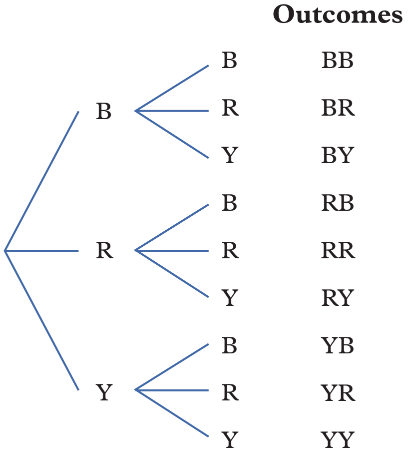
ANS p800

Exercise 11B Experiments with and without replacement

△ 1–3, 5–7, 10, 12 ◻ 3, 4, 8, 9, 11, 13, 14, 16, 17 ⬠ 4, 7, 9, 11, 13–18

1 A bag contains 22 coloured marbles: 6 blue, 8 red and 8 yellow. One marble is drawn from the bag, its colour is noted, then it is replaced. A second marble is then chosen and its colour is also noted.

a Complete the tree diagram representing this situation by adding probabilities to the branches.



b Determine the probability that the two chosen marbles are the same colour.

11B.1 2 A standard deck of playing cards consists of four suits (hearts, diamonds, clubs and spades) with 13 cards in each suits (52 cards in total). One card is chosen from a deck and its suit is noted. It is replaced, then a second card is chosen and the suit noted.

a Draw a tree diagram to represent this experiment.

b Determine the probability that two hearts are selected.

c Determine the probability that a club and a diamond are selected in any order.

3 John removes the loose coins from his pockets each night and throws them into a tin. There are 75 silver coins and 25 gold coins in his tin. He picks out a coin from the tin and notes its colour. After replacing it in the tin, he repeats this process another two times.

a Determine the probability of John drawing out:

i a silver coin

ii a gold coin.

b Draw a tree diagram to display this experiment. Write the individual probabilities on the branches of the tree diagram.

c Use your tree diagram to calculate the probability of John drawing out:

i all silver coins

ii no silver coins

iii at least one coin of each colour.

4 Consider the digits 1, 2, 3, 4 and 5. You are asked to write a two-digit number using these digits. The digits may be used more than once.

a Draw a tree diagram to represent this situation and use it to help you write all possible outcomes.

b How many outcomes are there?

c How many of these outcomes have two digits which are the same?

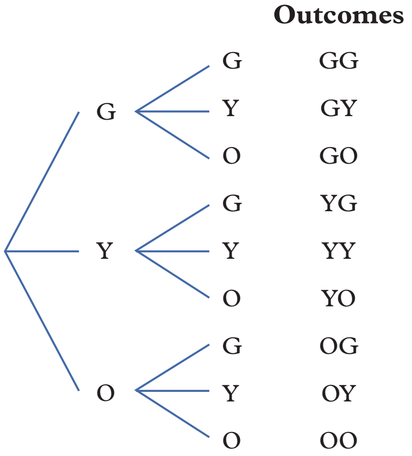
d Calculate the probability that the number has:

i two different digits **ii** two digits the same

iii the first digit smaller than the second.

5 A bag contains 16 coloured marbles: 6 green, 5 yellow and 5 orange. One marble is drawn from the bag, its colour is noted and it is not replaced. A second marble is then chosen and its colour is also noted.

a Complete the tree diagram representing this situation by adding probabilities to the branches.



b Determine the probability that the two chosen marbles are the same colour.

11A.2 **6** A bag contains 8 cubes: two black, three white, and three grey. One cube is drawn and its colour is noted. A second cube is drawn without the first cube being replaced and its colour is noted.

a Draw a tree diagram to represent the situation.

b Determine the probability that the two cubes selected are the same colour.

c Determine the probability that a black and a white cube are selected in any order.

7 The letters of the word PROBABILITY are written on separate pieces of paper and placed in a box. One letter is drawn from the box and it is recorded whether the letter is a consonant or a vowel. A second letter is drawn without replacing the first letter. Determine the probability that:

a the letters are both vowels

b one letter is a vowel and the other is a consonant.

8 From a set of all 26 letters in the alphabet, two letters are drawn:

a with replacement **b** without replacement.

For each of these situations, what is the probability that the letters are both vowels or both consonants?

9 Consider again the situation described in question **4** but, this time, you are only able to form a two-digit number without repeated digits.

a Draw a tree diagram to represent this situation and use it to help you write all possible outcomes.

b How many outcomes are there?

c Calculate the probability that the number has:

i two digits the same

ii two different digits

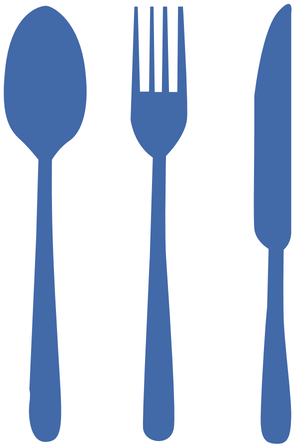
iii the first digit smaller than the second.

10 Sam has a small bag of lollies. The bag contains 10 red, 8 blue, 6 orange, 4 yellow and 12 green lollies.

a Sam prefers the red lollies. What is the probability that, if he chooses one lolly at random, it will be red?

b Sam eats all the red lollies. He then chooses another lolly at random, eats it, then chooses another one and eats it. What is the probability that the two lollies Sam ate were the same colour?

11 You are setting the table for you and your friend to have dinner. You each need a knife, fork and spoon. The cutlery drawer in the kitchen contains 15 knives, 10 forks and 15 spoons.



Without looking in the drawer, you choose three pieces of cutlery.

a What is the chance that your first choice was a knife?

b If your first choice was a knife, what is the probability that your next choice will be a fork, and the next choice a spoon?

c What is the probability of choosing a knife, a fork and a spoon in any order?

d Assume that you have already chosen a knife, fork and spoon for yourself. What is the probability that you will now choose these three items for your friend on your next selection?

12 Consider a pencil case containing 30 pencils: 10 black, 2 red, 4 blue, 3 green and 11 yellow.

Two pencils are taken out one after the other. The chance that the first pencil chosen is black (and is not replaced), then the next one is red is calculated as:

$$\begin{aligned}\text{Pr(black then red)} &= \text{Pr(black)} \times \text{Pr(red)}\\ &= \frac{10}{30} \times \frac{2}{29}\\ &= \frac{2}{87}\end{aligned}$$

a Use this information to help you determine the probability that:

- **i** the first pencil chosen is black, then the next one is not red
- **ii** the first pencil chosen is black, then the next one is red or blue
- **iii** the first pencil chosen is black, then the next one is red, blue or green
- **iv** both pencils are black
- **v** both pencils are red
- **vi** the first pencil chosen is red, then the next one is blue.

b Investigate to see whether undertaking the two events in the reverse order affects the probability by determining the probability that:

- **i** the first pencil chosen is red, then the next one is black
- **ii** the first pencil chosen is blue, then the next one is red.

c Comment on your answers to part **b**.

13 A bag contains 5 balls: three white and two black. Three balls are drawn, and each ball is placed back in the bag immediately after it has been drawn.

a Find the probability that all three balls drawn are black.

b Find the probability that all three balls drawn are white.

PROBLEM SOLVING AND REASONING

14 As with question **13**, a bag contains 5 balls: three white and two black. This time, three balls are drawn without replacement.

a Explain why at least one white ball will be drawn.

b Find the probability that all three balls drawn are white.

15 Five balls labelled with numbers 1–5 are placed in a bag. Two balls are selected, at random, one at a time.

a Write the sample space for the sum of the two numbers if:

i the first ball is replaced after being selected

ii the first ball is not replaced after being selected.

b Find the probability that the sum of the two balls chosen is 4 if:

i the first ball is replaced after being selected

ii the first ball is not replaced after being selected.

c Find the probability that the sum of the two balls chosen is 6 if:

i the first ball is replaced after being selected

ii the first ball is not replaced after being selected.

16 It's not always necessary to draw tree diagrams to determine the number of outcomes of an experiment.

First, consider forming a two-digit number from the digits 1 to 5 with replacement (as was described in question **4**). There are five digits to choose from when writing the first digit. You can use all the digits again for the second digit, so there are five from which to choose again. This means that there should be 5×5 possible outcomes.

Next, consider forming a two-digit number from the digits 1 to 5 *without* replacement. There are five digits to choose from when writing the first digit. But, since you are not able to repeat any of the digits, there are only four digits from which to choose the second digit. This means that there should be 5×4 possible outcomes.

a Use this method to determine the number of possible outcomes when forming a three-digit number from the digits 1 to 5:

i with replacement **ii** without replacement.

b Repeat part **a** using the digits 1 to 9 to form the three-digit number.

CHALLENGE

17 A restaurant offers a menu of six entrees, six main meals and six desserts.

a How many different combinations of courses does the restaurant offer?

b During the evening, the kitchen runs out of one of the entrees and two of the desserts. What is the decrease in the number of options available?

18 a Suppose that the number plates for cars contain six digits and digits can be repeated. Remember that a number plate can start with the digit 0.

i How many different digits can be used in each place of the six-digit number?

ii How many different number plates could be produced using this system?

b A new system was introduced whereby each number plate contained three digits followed by three letters.

i How many number plates would be possible using this system?

ii Explain why the system with a mixture of digits and letters for number plates is better than the system with six digits.

Check your student obook pro for these digital resources and more: pro

Interactive skillsheet
Experiments with replacement

Interactive skillsheet
Experiments without replacement

Investigation
A card trick

Topic quiz
11B

11C Two-way tables and Venn diagrams

Learning intentions

- ✔ I can understand and use set notation.
- ✔ I can convert two-way tables to Venn diagrams and vice versa.
- ✔ I can use the addition rule for probability.

Inter-year links

Year 5/6	Understanding probability
Year 7	10F Theoretical probability
Year 8	9G Venn diagrams
Year 9	9F Venn diagrams

Set notation

- A **set** is a collection of distinct objects.
- An **element** is a member of a set. The number of elements in set A is written as $n(\text{A})$.
- The **universal set** contains all the elements under consideration and is represented by ξ (the Greek letter xi, pronounced 'zai').
- The **intersection** of two sets, A and B, is the set of overlapping elements of A and B and is denoted by $\text{A} \cap \text{B}$.
- The **union** of two sets, A and B, includes all elements from both sets and is denoted by $\text{A} \cup \text{B}$.
- The complement of set A is written as A′. This includes everything from the universal set that is not in set A (not A).
- A **subset** is a smaller set within a set.
  - → If set A is a subset of set B, this can be written as $\text{A} \subset \text{B}$.
  - → If set A is not a subset of set B, this can be written as $\text{A} \not\subset \text{B}$.
- A null set, or the empty set, is a set with no elements; that is, $\{\}$. Null sets can be symbolised by Ø.
- If two sets, A and B, are mutually exclusive, their intersection has no elements: $\text{A} \cap \text{B} = \varnothing$.

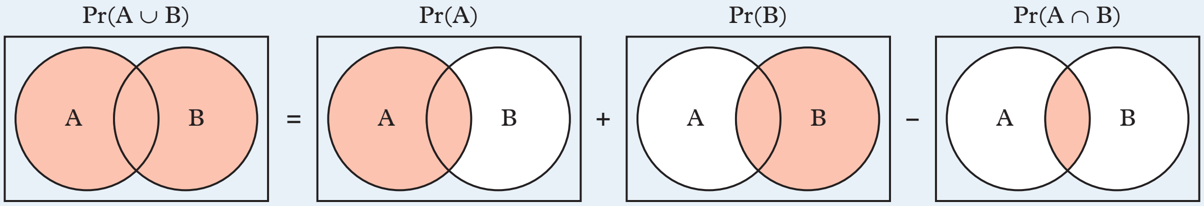
Venn diagrams

- Each circle in a **Venn diagram** represents a different set of data, and the rectangle represents the entire data set (the universal set, ξ).
- Sometimes Venn diagrams may be drawn with the circles being completely separated, which should be interpreted as meaning that the events are mutually exclusive.

The addition rule for probability

- When two sets (A and B) overlap, the following formula can be used:

$$\Pr(\text{A} \cup \text{B}) = \Pr(\text{A}) + \Pr(\text{B}) - \Pr(\text{A} \cap \text{B})$$



Two-way tables

- A **two-way table** is a means of displaying the relationship between two sets of data. The sets are represented by the rows or columns of the two-way table.

	School A	**School B**	**Total**
Musical instrument	10	4	14
No musical instrument	3	13	16
Total	13	17	30

Example 11C.1 Converting a two-way table to a Venn diagram

This two-way table details information about the cards found in a standard deck of playing cards. Convert the two-way table into a Venn diagram.

	Ace (A)	**Not ace (A′)**	**Total**
Heart (H)	1	12	13
Not heart (H′)	3	36	39
Total	4	48	52

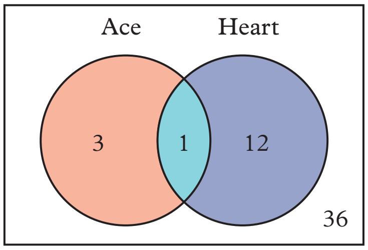
THINK

1 Draw two circles on your Venn diagram to represent the two sets. Label each circle with one of the possible options from each set.

2 Fill in the sections of the Venn diagram with the information from the two-way table.

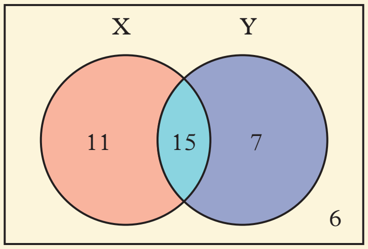
3 Check that the total of the numbers in the Venn diagram equals the total in the bottom right-hand corner of the two-way table.

WRITE



Example 11C.2 Converting a Venn diagram to a two-way table

Convert this Venn diagram into a two-way table.



THINK

1 Create a table with columns and rows for the sets in the Venn diagram and their complements.

2 Add a row and a column for the totals.

3 Fill in the inner cells of the two-way table with the information from the Venn diagram.

4 Calculate the values in the total row and column, and check the bottom right-hand corner equals the total of the values in the Venn diagram.

WRITE

	X	**X′**	**Total**
Y	15	7	22
Y′	11	6	17
Total	26	13	39

Example 11C.3 Using the addition rule for probability

Given that Pr(A) = 0.35, Pr(B) = 0.75 and $\Pr(A \cap B) = 0.2$, find $\Pr(A \cup B)$.

THINK

1 Substitute the given values into the addition rule for probability.

2 Simplify and write your answer.

WRITE

$$\begin{aligned}\Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B)\\ &= 0.35 + 0.75 - 0.2\\ &= 0.9\end{aligned}$$

Helpful hints

✔ Converting between two-way tables and Venn diagrams can help you to visualise problems in a different way.

✔ The addition rule for probability can be rearranged to find the values of Pr(A), Pr(B) or $\Pr(A \cap B)$ as necessary.

✔ Two-way tables can provide information that can't be displayed in a Venn diagram. For example, Venn diagrams do not display the total values in a two-way table.

ANS p802

Exercise 11C Two-way tables and Venn diagrams

▲ 1–11, 14 ■ 2, 4, 5, 7, 9–12, 16, 18 ⬟ 2, 7, 9–13, 15, 17, 19, 20

UNDERSTANDING AND FLUENCY

1 Consider the whole numbers from 1 to 12, inclusive. Then the universal set, ξ, is {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}. Let A be the set of odd numbers and B be the set of prime numbers.

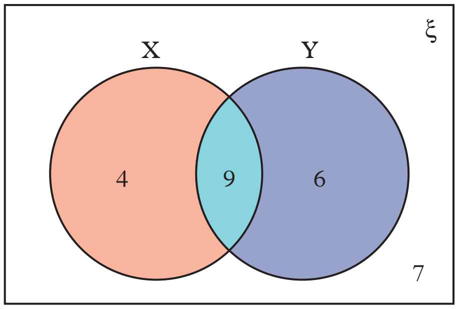
a Write the list of elements in the set:

i A **ii** B **iii** $A \cap B$ **iv** $A \cup B$

b One number from the universal set is selected at random. Find:

i Pr(A) **ii** Pr(B′) **iii** $\Pr(A \cap B)$

2 Consider this Venn diagram.



a Find:

i n(X) **ii** $n(X \cup Y)$

iii n(X′) **iv** $n(X \cap Y)$

v n(ξ) **vi** $n(X' \cap Y)$

b Calculate each of these probabilities.

i Pr(X) **ii** Pr(X′)

iii $\Pr(X \cap Y)$ **iv** $\Pr(X \cup Y)'$

v $\Pr(X' \cap Y')$ **vi** $\Pr(X \cap Y')$

vii $\Pr(X \cup Y')$ **viii** $\Pr(Y \cup Y')$

3 Explain how you could complete all the entries in this two-way table correctly. Then complete the table.

	Primary school	**High school**	**Total**
Catch public transport	22		40
Driven	37	11	
Walk/ride			
TOTAL		38	102

4 A survey of 50 people was conducted to collect data relating to the types of movies those people like. The movie categories were comedy, drama and animation. The results were as follows.

- 26 like comedy
- 24 like drama
- 33 like animation
- 13 like comedy and drama
- 17 like drama and animation
- 14 like comedy and animation
- 8 like all three types of movies.

Draw a Venn diagram to display this data.

5 Use your Venn diagram from question **4** to help you find the probability that a person chosen:

a likes comedy
b likes comedy, but doesn't like drama
c likes drama, but doesn't like comedy
d likes comedy or drama, but doesn't like animation
e likes any of these movie forms.

11C.1 **6** The following two-way table details information about the sports that 140 students play. Convert the two-way table into a Venn diagram.

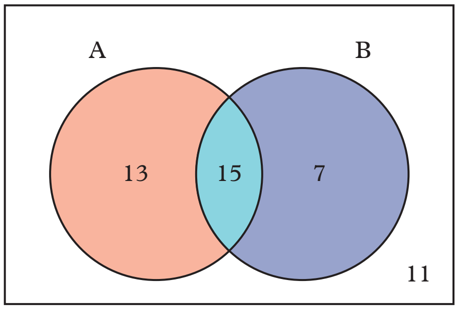
	Netball (N)	**Not netball (N′)**	**Total**
Field hockey (F)	22	10	32
Not field hockey (F′)	45	63	108
Total	67	73	140

7 200 Year 10 students were asked if they have subscriptions to Netflix and Spotify. The results are partially displayed in the two-way table below.

	Netflix (N)	**Not Netflix (N′)**	**Total**
Spotify (S)	85	62	147
Not Spotify (S′)			53
Total	126		200

a Fill in the missing numbers in the two-way table.
b Convert the completed two-way table into a Venn diagram.

11C.2 **8** Convert the following Venn diagram into a two-way table.

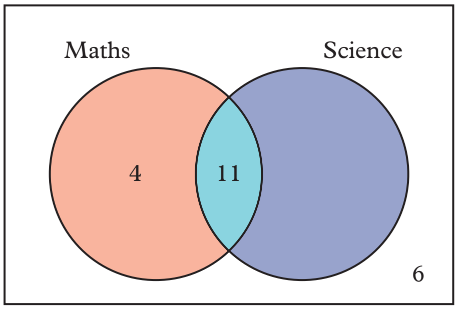


UNDERSTANDING AND FLUENCY

9 A class of 28 students were each asked if they enjoyed studying mathematics and science. The results are partially displayed in the Venn diagram on the right.

  a The number of students who enjoy studying science, but not maths, is missing. What is this number?

  b Convert the completed Venn diagram into a two-way table.



11C.3 10 Given that $\Pr(A) = 0.6$, $\Pr(B) = 0.3$ and $\Pr(A \cap B) = 0.2$, find $\Pr(A \cup B)$.

11 Find the probability of A, given that $\Pr(B) = \frac{1}{3}$, $\Pr(A \cup B) = \frac{4}{9}$, and $\Pr(A \cap B) = \frac{1}{6}$.

PROBLEM SOLVING AND REASONING

12 Consider the Venn diagram on the right.

  a Decide whether each of the following statements is true for the Venn diagram and explain your reasoning.

    i $R \subset P$ **ii** $R \subset Q$

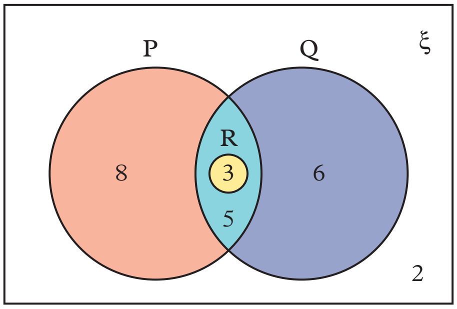
    iii $R \not\subset (P \cap Q)$ **iv** $R \subset (P \cup Q)$

    v $R \not\subset \xi$ **vi** $R \subset Q \subset \xi$

  b Use the Venn diagram to help you calculate each of these probabilities.

    i $\Pr(P \cap R)$ **ii** $\Pr(P \cap Q')$

    iii $\Pr(P' \cap Q')$ **iv** $\Pr(P \cap Q \cap R)$



13 One hundred people took part in a survey to determine the readership of three monthly magazines (A, B and C). The results were as follows:

- 59 read A
- 68 read B
- 59 read C
- 42 read A and B
- 45 read B and C
- 39 read A and C
- 27 read all three magazines

  a Draw a Venn diagram to show the results of the survey.

  b Find the number of people who:

    i read magazines A or B

    ii read magazine C but not magazine B

    iii read exactly two of the magazines

    iv read two magazines at most

    v do not read magazine A

    vi read only one magazine

    vii do not read B or C

    viii do not read three magazines.

14 As with question **1**, let the universal set, ξ, be $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, let A be the set of odd numbers, and let B be the set of prime numbers.

  a Write the list of elements in the set:

    i A'

    ii $A' \cap B$

    iii $(A \cap B)'$, the complement of the set $A \cap B$

  b One number from the universal set is selected at random. Find:

    i $\Pr(A' \cap B)$ **ii** $\Pr(A' \cap B')$ **iii** $\Pr(A' \cup B')$

PROBLEM SOLVING AND REASONING

15 In the theory for this section, Venn diagrams were used in the proof of the addition rule. Use Venn diagrams to help you prove that $\Pr((A \cap B)') = \Pr(A' \cup B')$.

16 For two events, A and B, it is known that $\Pr(A) = 0.75$, $\Pr(B) = 0.3$, and $\Pr(A' \cap B) = 0.1$.

a Find $\Pr(A')$.

b Use the addition rule to find $\Pr(A' \cup B)$.

17 For two events, A and B, it is known that $\Pr(A) = 0.3$, $\Pr(B) = 0.6$, and $\Pr(A \cap B) = 0.2$. These values have been entered into the following two-way probability table.

	A	**A′**	**Total**
B			0.6
B′	0.2		
Total	0.3		1

a Complete the missing values in the two-way probability table.

b Use the table from part **a** to help you find:

i $\Pr(A' \cap B')$

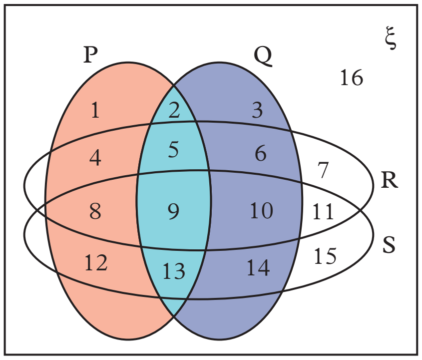
ii $\Pr(A' \cap B)$

iii $\Pr(A' \cup B')$

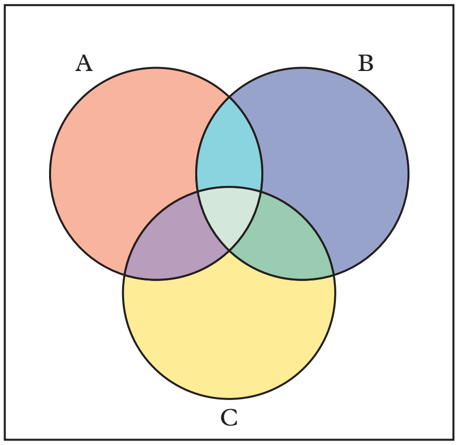
CHALLENGE

18 In a room there are 60 children, each of whom is either left-handed or right-handed. There are 30 right-handed boys, and 11 left-handed children. There are 25 girls among the 60 children. Complete a two-way table to find how many left-handed girls are in the room.

19 This Venn diagram shows the relationship between four sets P, Q, R and S. Determine the sum of the elements within the region $(P \cap Q) \cup (R \cap S')$.



20 Use the following Venn diagram to find an additional rule formula for $\Pr(A \cup B \cup C)$. That is, express $\Pr(A \cup B \cup C)$ in terms of the probabilities of A, B, C and their intersections.



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Interactive skillsheet Two-way tables

Interactive skillsheet Set notation

Interactive skillsheet The addition rule for probability

Topic quiz 11C

Interactive skillsheet Venn diagrams

Checkpoint

Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this chapter.

11A **1** List the sample space for each of the following experiments.

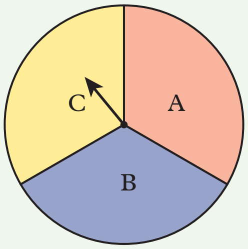
- **a** A fair coin is tossed.
- **b** A standard six-sided die is rolled.
- **c** A card is picked from a standard 52-card deck and its suit noted.
- **d** A letter is selected at random from the word LOLLIPOP.

11A **2** A letter from the word CHANCE is selected at random. Find the probability that the selected letter is

a ‘H’ **b** ‘C’ **c** a vowel **d** not a vowel.

11A **3** The spinner on the right is spun twice.

- **a** Draw a tree diagram to represent the possible outcomes of the two spins.
- **b** Find the probability the spinner lands on ‘A’ both times.

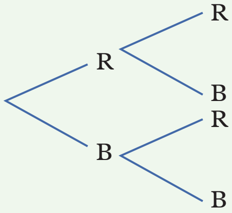


11A **4** A biased coin, which has a 60% chance of landing on heads, is tossed twice.

- **a** Draw a tree diagram to represent this situation.
- **b** Find the probability the coin lands on:
  - **i** heads both times **ii** heads once and tails once.

11B **5** A bag contains 3 red balls and 4 black balls. A ball is randomly selected from this bag and its colour is noted. The ball is then replaced before another ball is randomly selected from the bag. The tree diagram on the right shows the possible outcomes of the experiment.

- **a** Complete the tree diagram by writing the probabilities on the branches.
- **b** Find the probability that both balls selected are red.



11B **6** Two balls are selected at random from a bag containing 2 black balls and 3 white balls. Find the probability that both balls selected are black if:

- **a** the first ball is replaced before the second ball is selected
- **b** the first ball is not replaced before the second ball is selected.

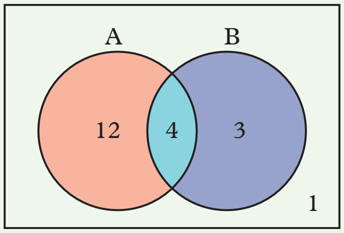
11B **7** A card is randomly selected from a standard 52-card deck, and its suit is noted. The card is then replaced, and another card is drawn at random. Find the probability that the two cards are:

a both hearts **b** the same suit **c** different suits.

11C **8** Consider the Venn diagram on the right.

Find:

- **a** $\Pr(A)$
- **b** $\Pr(B)$
- **c** $\Pr(A \cap B)$
- **d** $\Pr(A' \cap B)$



11C **9** Consider the incomplete two-way table on the right.

- **a** Fill in the missing numbers in the two-way table.
- **b** Convert the two-way table to a Venn diagram.

	M	**M'**	**Total**
N	6	11	
N'		15	
Total			40

11C **10** The following two-way table shows data gathered from a class of 24 students about which brand of phone they have and which operating system they use on their home computer.

One of the 24 students is selected at random. Find the probability that this selected student:

- **a** has an Apple phone
- **b** has Windows on their home computer
- **c** has an Apple phone and Macintosh on their home computer
- **d** has Windows on their home computer but not a Samsung phone.

	Apple	**Samsung**	**Other**	**Total**
Macintosh	8	1	0	9
Windows	6	4	5	15
Total	14	5	5	24

11D Conditional probability

Learning intentions

- ✔ I can understand conditional statements.
- ✔ I can calculate conditional probability using tree diagrams.
- ✔ I can calculate conditional probability using two-way tables and Venn diagrams.

Inter-year links

Year 8	9E Theoretical probability
Year 9	9C Experiments without replacement

Conditional probability

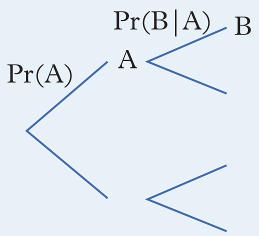
- **Conditional probability** is the probability of an event occurring, given the knowledge that another event has occurred.
  - → Conditional probability is often about a single trial of an experiment. For example, if a standard six-sided die is rolled, what is the probability the outcome is odd, given that the outcome is prime?
  - → Conditional probability restricts the sample space to a subset of the universal set. In the above example, the information 'given that the outcome is prime' restricts the sample space to $\{2, 3, 5\}$ from $\{1, 2, 3, 4, 5, 6\}$. So the Pr(odd given prime) $= \frac{2}{3}$, since two of the three primes are odd.
- $\Pr(A|B)$ is the probability that event A occurs given that event B has occurred. This can be read as 'the probability of A given B'.
- $\Pr(B|A)$, which represents the probability that event B occurs given that event A has occurred, is different from $\Pr(A|B)$.
- $\Pr(A \cap B)$ is the probability that events A and B both occur.
  - → It can be read as 'the probability of A and B occurring'.
- Conditional probability can be calculated by using the following formulas:

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} \qquad \text{or} \qquad \Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

- The above formulas can be rearranged to give:

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B|A) \qquad \text{or} \qquad \Pr(A \cap B) = \Pr(B) \times \Pr(A|B)$$

  - → This explains why we multiply the probabilities on branches in tree diagrams:



$\Pr(A \cap B) = \Pr(A) \times \Pr(B|A)$

- Conditional probability can be identified when any of the following terms are used in a probability question: 'if ... then ...', 'given ...' or 'knowing that ...'
- When calculating conditional probability using two-way tables or Venn diagrams, the words 'of' and 'from' indicate the set from which the choice is being made. For example, 'from the group who …'

Example 11D.1 Calculating conditional probability using a tree diagram

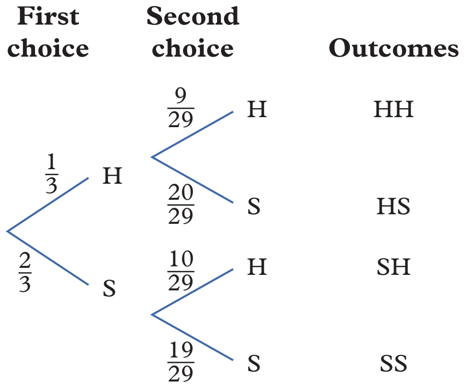
A sandwich tray contains 30 sandwiches, 10 of which are ham (H) and 20 of which are salad (S). Jack takes two sandwiches at random, one after the other.

a Draw a tree diagram to display the possible outcomes of Jack's two choices.

b If Jack's first sandwich is ham, what is the probability that his second sandwich will also be ham?

c Knowing that Jack's first choice is a salad sandwich, what is the probability that his second one will be ham?

d What is the probability of Jack's second sandwich being salad, given that his first sandwich is ham?

THINK	**WRITE**
a Draw a tree diagram to display all the outcomes. The probabilities for the second choice will depend on the outcome of the first choice. Write the probabilities on the branches of your tree diagram.	**a** H represents ham and S represents salad. 
b In this case, you are looking for the probability that the second choice is H, given that the first choice is H. Follow the branches H then H on the tree diagram. The probability on the second branch is the probability of the first and second sandwiches both being ham.	**b** $\Pr(H\|H) = \frac{9}{29}$
c In this case, you are looking for the probability that the second choice is H, given that the first choice is S. Follow the branches S then H on the tree diagram. The probability on the second branch is the probability of the second sandwich being ham if the first sandwich was salad.	**c** $\Pr(S\|H) = \frac{10}{29}$
d In this case, you are looking for the probability that the second choice is S, given that the first choice is H. Follow the branches H then S on the tree diagram. The probability on the second branch is the probability of the second sandwich being salad if the first sandwich was ham.	**d** $\Pr(S\|H) = \frac{20}{29}$

Example 11D.2 Calculating conditional probability using a two-way table

The table below shows the results of a survey taken at a library to find whether people under 18 and people 18 and over prefer using Facebook or Twitter (or neither).

	Under 18	**18 and over**	**Total**
Facebook	20	30	50
Twitter	5	10	15
Neither	10	5	15
Total	35	45	80

Use the two-way table to find the relative frequency that a person:

a prefers Facebook, given they are 18 and over

b is 18 and over, given they prefer Facebook.

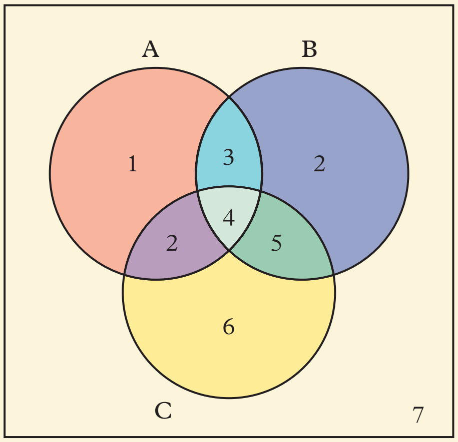
THINK	**WRITE**
a **1** Identify the total number of people surveyed who are 18 and over by looking at the total number in the '18 and over' column.	**a** total number of people 18 and over surveyed = 45
2 Identify the number of people who prefer Facebook in the '18 and over' column.	number of people 18 and over who prefer Facebook = 30
3 Divide the number of people who are 18 and over who prefer Facebook by the total number of people who are 18 and over and give the answer as a fraction, simplifying if possible.	$\Pr(\text{prefer Facebook} \mid \text{18 and over}) = \frac{30}{45}$ $= \frac{2}{3}$
b **1** Identify the total number of people who prefer Facebook by looking at the Facebook row.	**b** total number of people who prefer Facebook = 50
2 Identify the number of people who are 18 and over in the 'Facebook' row.	number of the people who prefer Facebook who are over 18 = 30
3 Divide the number of people who prefer Facebook who are 18 and over by the total number of people who prefer Facebook and give the answer as a fraction, simplifying if possible.	$\Pr(\text{18 and over} \mid \text{prefer Facebook}) = \frac{30}{50}$ $= \frac{3}{5}$

Example 11D.3 Calculating conditional probability using a Venn diagram

This Venn diagram shows the distribution of a group of students who study Art (A), Business Studies (B) and Computing (C). Use the diagram to determine the relative frequency that a student studies:

a all three of these subjects, given that they study two of the subjects

b Art and Computing, given that they do not study Business Studies.



THINK	**WRITE**
a **1** Identify the number of students in the 'given' set. This is the total in all the overlapping sections of the diagram.	**a** number of students studying two or more subjects $= 3 + 4 + 5 + 2$ $= 14$
2 Identify the number of students within this group who study all three subjects.	number of students studying all three subjects $= 4$
3 Express the two numbers you identified in parts **1** and **2** as a fraction, simplifying if possible.	Pr(all three subjects | two subjects) $= \frac{4}{14}$ $= \frac{2}{7}$
b **1** Identify the number who do not study Business Studies. This is all those outside the circle B.	**b** number of students not studying Business Studies $= 1 + 2 + 6 + 7$ $= 16$
2 Identify the number of students within this group who study Art and Computing.	number of students in this group studying Art and Computing $= 2$
3 Express the two numbers you identified in parts **1** and **2** as a fraction, simplifying if possible.	Pr(Art and Computing | not Business Studies) $= \frac{2}{16}$ $= \frac{1}{8}$

Helpful hints

- ✔ When working with conditional probability, remember to determine the number of elements in the conditional (or given) set first.
- ✔ When calculating conditional probability, the given probability always goes on the denominator.
- ✔ 'Given A, the probability of B …' is Pr(B | A), because A is given.

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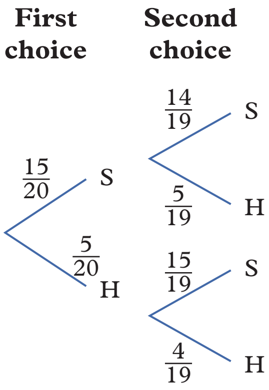
Exercise 11D Conditional probability

▲ 1, 3–8, 10, 12, 14, 15 ■ 2, 3, 5–7, 9–11, 13, 15, 17 ⬟ 3, 5, 6, 9, 11, 15, 16, 18, 19

UNDERSTANDING AND FLUENCY

1 A box of 20 chocolates contains 15 chocolates with soft centres and 5 chocolates with hard centres. Two chocolates are taken at random, one after the other. The tree diagram on the right represents the outcomes of the two choices.



- **a** Explain the probabilities of $\frac{15}{20}$ and $\frac{5}{20}$ on the first two branches of the tree diagram.
- **b** Explain the probabilities on the four branches for the four possible second choices.
- **c** If the first chocolate is hard, what is the probability that the second chocolate will also be hard?
- **d** Knowing that the first chocolate is a soft one, what is the probability the second chocolate will be hard?
- **e** What is the probability of the second chocolate being soft, given the first chocolate is hard?

11D.1 **2** A bag contains 6 balls: two black and four white. A ball is selected at random, its colour noted, and then another ball is selected without the first ball being replaced.

- **a** Draw a tree diagram to display this situation.
- **b** If the first ball is white, what is the probability the second ball will also be white?
- **c** Knowing that the first ball is black, what is the probability the second ball is white?
- **d** What is the probability that the second ball is black, given that the first ball is white?

3 A box contains 7 balls: four green and three red. Three balls are selected without replacement.

- **a** Draw a tree diagram to display this situation.
- **b** If the first ball is green, what is the probability the second ball will also be green?
- **c** Knowing that the first two balls are both green, what is the probability that the third ball will also be green?
- **d** What is the probability that the third ball is green, given that one of each colour is drawn in the first two selections?

11D.2 **4** Consider the two-way table from Example 11C.1 again. Use this two-way table to determine the relative frequency of a card being:

- **a** not an Ace, given it is a heart
- **b** not an Ace, given it is not a heart.

	Ace (A)	**Not ace (A′)**	**Total**
Heart (H)	1	12	13
Not heart (H′)	3	36	39
Total	4	48	52

5 The two-way table on the right shows the modes of transport a group of students use to get to school. State the number of:

- **a** primary school students who catch public transport to school
- **b** high school students who are driven to school
- **c** students who walk or ride to school
- **d** students who are in high school.

	Primary school	**High school**	**Total**
Catch public transport	15	27	42
Driven	22	18	40
Walk/ride	4	14	18
Total	41	59	100

6 Using the table from question **5**, calculate the relative frequency of a student:

a catching public transport to school, given they are at primary school

b being at high school, given they are driven to school

c walking or riding to school

d being at high school

e being driven to school, given they are at high school.

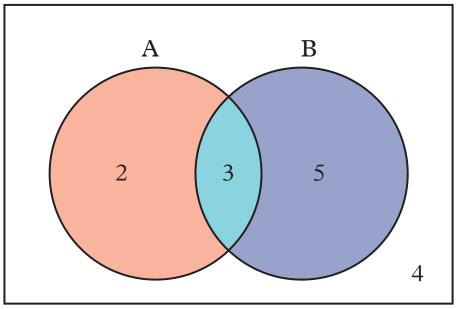
7 Consider the Venn diagram on the right.

a How many elements are in set B?

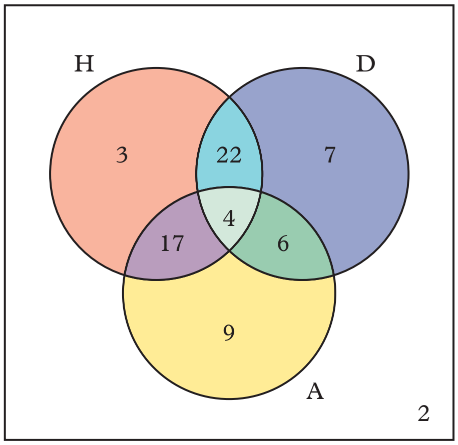
b How many of the elements in set B are also in set A?

c Determine $\Pr(A|B)$.

d By a similar reasoning, determine $\Pr(B|A)$.



11D.3 8 The following Venn diagram shows the distribution of a group of 70 students who were asked if they enjoy watching horror films (H), drama films (D), and action films (A).

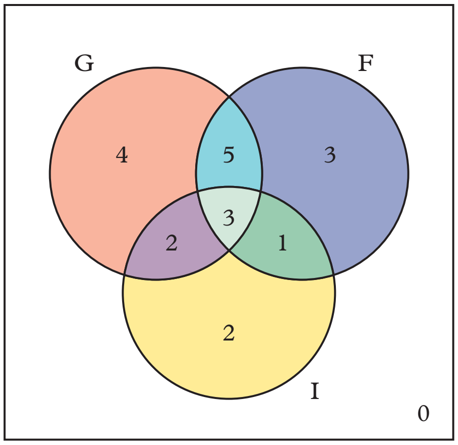


Use the Venn diagram to determine the relative frequency that a student enjoys:

a all three genres of film, given that they enjoy two of the film genres

b horror films, given that they enjoy action films.

9 Twenty members of the Swiss national soccer team were asked which languages they speak fluently, out of German (G), French (F), and Italian (I). Their responses are shown in the Venn diagram on the right.



a One of the twenty players is selected at random. Find the probability that they speak:

- **i** German
- **ii** German, but not French
- **iii** German, given that they speak French
- **iv** French, given that they speak German
- **v** All three languages, given that they speak at least two of the languages.

b An Italian-speaking player is selected at random. Find the probability that this player can also speak French.

PROBLEM SOLVING AND REASONING

10 For two events, A and B, it is known that $\Pr(A|B) = 0.4$, $\Pr(A \cap B) = 0.1$, and $\Pr(A) = 0.3$.

a Find $\Pr(B|A)$. Give your answer as a fraction.

b Find $\Pr(B)$. Give your answer as a decimal.

11 A computer generates random questions for students to use for practice. From collected data, the chance that a student gets the first question correct is 0.8. If the student gets this question correct, the computer generates a harder question, with the probability of the student getting this one correct being reduced by 0.1. If the student gets the first question wrong, the computer gives the student a question of the same level of difficulty as the first question.

a Draw a tree diagram to display the possible outcomes of the two practice questions.

b Find the probability, expressed as a decimal, that the student gets the second question:

i correct, given they got the first question correct

ii correct, given they got the first question wrong

iii wrong, given they got the first question correct.

12 At a small firm with 50 employees, some work full-time and others work part-time. There are 15 female employees in total and 30 of the 34 full-time employees are men.

a Draw a two-way table to record this data. Complete all the entries in the table.

b How many employees are:

i full-time employees who are females

ii part-time?

iii male?

c Find the relative frequency of an employee:

i working part-time, given they are male

ii being male, given they work part-time

iii being female, given they work full-time

iv working full-time, given they are female.

d Explain the differences in your answers for the following parts of part **c**:

i parts **i** and **ii**

ii parts **iii** and **iv**.

13 The two-way table below shows the number and types of paper clips in a pile on a desk.

	Small	**Large**	**Total**
Silver	27	35	
Coloured	18	10	
Total			

a Complete the totals for the rows and columns.

b Determine whether each of these statements is true, showing the calculations you used. (Percentages are rounded to the nearest whole number.) For those statements which are not true, show working to provide a correction.

i There is an equal number of small and large paper clips.

ii There are fewer coloured paper clips than non-coloured ones.

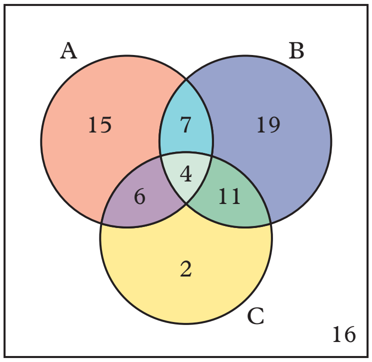
iii The percentage of small paper clips which are silver is 44%.

iv The percentage of coloured paper clips which are large is 22%.

v The whole pile contains 69% silver paper clips.

14 A jar contains black and white balls. Two balls are drawn without replacement. The probability of selecting a black ball on the first draw is 0.375, and the probability of selecting a second black ball, given that the first ball is black, is $\frac{1}{3}$. What is the probability of selecting a white ball on the second draw, given that the first ball is a black one? Give your answer as a decimal.

15 Consider the following Venn diagram, which shows survey data gathered from 80 people.



a Find each of the following.

i $n(\mathrm{A})$ **ii** $n(\mathrm{B})$ **iii** $n(\mathrm{C}')$

b One person from the 80 is selected at random. Find:

i $\Pr(\mathrm{B}|\mathrm{A})$ **ii** $\Pr(\mathrm{A}|\mathrm{B})$ **iii** $\Pr(\mathrm{A}'|\mathrm{C}')$

c Explain how the answers to part **a** are helpful in answering part **b**.

16 For two events, X and Y, it is known that $\Pr(\mathrm{X}|\mathrm{Y}) = 0.2$ and $\Pr(\mathrm{Y}|\mathrm{X}) = 0.5$.

a Find:

i $\Pr(\mathrm{X}'|\mathrm{Y})$ **ii** $\Pr(\mathrm{Y}'|\mathrm{X})$.

b If $\Pr(\mathrm{X} \cap \mathrm{Y}) = 0.05$, then find Pr(X) and Pr(Y).

17 There are three cards in a box: one is green on both sides, one is red on both sides, and one has a red side and a green side. A card is randomly selected from the box and placed on the table. The visible side of the selected card is green. What is the probability that the other side is also green?

18 If an event, A, relates to the second step of a multi-step experiment, then it may be necessary to split A into cases in order to determine its probability. That is, $\Pr(\mathrm{A}) = \Pr(\mathrm{A} \cap \mathrm{B}) + \Pr(\mathrm{A} \cap \mathrm{B}')$, meaning that A can either occur with B or without B, where B relates to the first step of the experiment. Then, using the conditional probability formula, $\Pr(\mathrm{A}) = \Pr(\mathrm{B}) \times \Pr(\mathrm{A}|\mathrm{B}) + \Pr(\mathrm{B}') \times \Pr(\mathrm{A}|\mathrm{B}')$.

Imagine that a soccer team has a 40% chance of winning their upcoming game if it rains and a 70% chance of winning their upcoming game if it does not rain.

a Find the probability that the team wins their upcoming game if there is a 20% chance of rain. Give your answer as a decimal.

b Find the probability that it rains on the day of their upcoming game if their chance of winning the game is 0.5. Give your answer as a fraction.

19 Baye's rule is a useful conditional probability formula which states that $\Pr(\mathrm{A}|\mathrm{B}) = \dfrac{\Pr(\mathrm{B}|\mathrm{A}) \times \Pr(\mathrm{A})}{\Pr(\mathrm{B})}$. Baye's rule can be proved using the conditional probability formulas for $\Pr(\mathrm{A}|\mathrm{B})$ and $\Pr(\mathrm{B}|\mathrm{A})$.

a Prove Baye's rule using $\Pr(\mathrm{A}|\mathrm{B}) = \dfrac{\Pr(\mathrm{A} \cap \mathrm{B})}{\Pr(\mathrm{B})}$ and $\Pr(\mathrm{B}|\mathrm{A}) = \dfrac{\Pr(\mathrm{A} \cap \mathrm{B})}{\Pr(\mathrm{A})}$.

b Two boxes each contain white balls and black balls. Box X contains 7 white and 3 black, whereas Box Y contains 2 white and 3 black. One box is chosen at random and then a ball is selected from it at random. Let X be the event 'selecting box X' and W be the event 'selecting a white ball.'

i Find Pr(W). Give your answer as a decimal.

ii Use Baye's rule to find $\Pr(\mathrm{X}|\mathrm{W})$, which is the probability that Box X was chosen, given that the selected ball is white. Give your answer as a fraction.

PROBLEM SOLVING AND REASONING

CHALLENGE

Check your Student obook pro for these digital resources and more:

Interactive skillsheet Conditional probability

Investigation Bertrand's box

Topic quiz 11D

11E Independence

Learning intentions

✔ I can understand the difference between independent and dependent events.

✔ I can calculate the probability of independent and dependent events.

Inter-year links

Year 9 9E Two-way tables

Independent events

- Two events are **independent** if one event occurring has no effect on the probability of the other event occurring.
- The concept of independence can be used to determine if the different steps of a multi-step chance experiment are independent of each other; for example, if the first step has an impact or not on the outcome of the second step.
  - → The outcomes for each step of a chance experiment with replacement are independent, as the probabilities for the second step are not dependent on the outcome of the first step.
  - → The outcomes for each step of a chance experiment without replacement are not independent, as the probabilities for the second step depend on the outcome of the first step.
- The concept of independence can also be used to determine if real-life events are independent or not; for example, if the weather outside has an effect on whether people take public transport on their commute to work.
- If A and B are independent events, then $\Pr(A|B) = \Pr(A)$ and $\Pr(B|A) = \Pr(B)$, since the fact that A or B occurred does not change the probability of the other event happening.
  - → Natural variation in real-life events means it can be difficult to prove independence, but you can state whether real-life events appear to be independent or not.
- If $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$, then A and B are independent events.

Example 11E.1 Testing for independence

For two events, A and B, it is known that $\Pr(A) = 0.6$, $\Pr(B) = 0.75$ and $\Pr(A \cap B) = 0.5$.

a Evaluate:

i $\Pr(A|B)$

ii $\Pr(B|A)$

b Are A and B independent or dependent events?

THINK	WRITE
a Use the conditional probability formula to evaluate each probability.	**a i** $\Pr(A\|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)} = \dfrac{0.5}{0.75} = \dfrac{2}{3}$

ii $\Pr(B|A) = \dfrac{\Pr(A \cap B)}{\Pr(A)}$

$$= \frac{0.5}{0.6}$$

$$= \frac{5}{6}$$

b Use your answers to part **a** to check if Pr(A|B) is equal to Pr(A), or if Pr(B|A) is equal to Pr(B). If yes, then the events are independent. If not, the events are dependent.

b A and B are dependent events, since Pr(A|B) is not equal to Pr(A) and Pr(B|A) is not equal to Pr(B).

Example 11E.2 Using a two-way table to determine whether two events are independent

The two-way table below shows whether people who prefer cats or dogs have a preference for reading or exercise.

	Cats	Dogs	Total
Reading	31	18	49
Exercise	16	35	51
Total	47	53	100

Are the events 'prefers cats' and 'prefers exercise' independent?

THINK

1 Calculate the probability of a person chosen at random preferring exercise.

2 Calculate the probability of a person chosen at random preferring exercise, given that they prefer cats.

3 Compare the two probabilities. If they are similar, the two events are independent. If there is a significant difference, the two events are dependent.

WRITE

$$\Pr(\text{exercise}) = \frac{51}{100} = 0.51$$

$$\Pr(\text{exercise}|\text{cats}) = \frac{16}{47} \approx 0.34$$

The events 'prefers cats' and 'prefers exercise' are dependent.

Helpful hints

✔ The equation $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$ is not a formula, but rather a test. This means that you can use the equation to test if A and B are independent. If you are told that A and B are independent, then you can use the equation as a formula. However, if you are not told that A and B are independent, then you can find $\Pr(A \cap B)$ using the addition rule formula or the conditional probability formulas, which are always true. These are:

- → $\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B)$
- → $\Pr(A \cap B) = \Pr(B) \times \Pr(A|B)$
- → $\Pr(A \cap B) = \Pr(A) \times \Pr(B|A)$

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Exercise 11E Independence

△ 1–7, 10, 12 □ 1, 2, 4, 7–13 ⬠ 2, 4, 9–15

UNDERSTANDING AND FLUENCY

11E.1 **1** For two events, A and B, it is known that $\Pr(A) = 0.4$, $\Pr(B) = 0.5$, and $\Pr(A \cap B) = 0.2$.

a Evaluate:

i $\Pr(A|B)$ **ii** $\Pr(B|A)$

b Are A and B independent or dependent events?

2 For two independent events, M and N, it is known that $\Pr(M) = \frac{2}{3}$ and $\Pr(N) = \frac{1}{4}$. Find each of the following.

a $\Pr(M|N)$ **b** $\Pr(N|M)$ **c** $\Pr(M \cap N)$

3 The two-way table on the right shows data gathered from 21 students about their preferred sport (out of cricket and football) and their preferred subject at school (out of Maths and English). Are the events 'prefers cricket' and 'prefers Maths' independent?

	Cricket	**Football**	**Total**
Maths	6	8	14
English	3	4	7
Total	9	12	21

11E.2 **4** The two-way table below shows data gathered from 40 people about their preferred ice cream flavour (out of chocolate, strawberry, vanilla) and their preferred chocolate variety (out of milk, dark, white).

	Chocolate	**Strawberry**	**Vanilla**	**Total**
Milk	8	7	10	25
Dark	4	2	1	7
White	3	0	5	8
Total	15	9	16	40

Decide whether the following pairs of events are independent or dependent.

a 'Prefers vanilla' and 'prefers milk' **b** 'Prefers chocolate' and 'prefers milk'

c 'Prefers strawberry' and 'prefers dark' **d** 'Prefers white' and 'prefers chocolate'

5 Consider the two-way table on the right.

a Evaluate:

i $\Pr(A)$ **ii** $\Pr(A|B)$

b Based on the results from part **a**, are A and B independent or dependent events?

	A	**A′**	**Total**
B	9	3	12
B′	6	2	8
Total	15	5	20

6 Consider the two-way table on the right.

a Evaluate:

i $\Pr(Y)$ **ii** $\Pr(Y|X)$

b Based on the results from part **a**, are X and Y independent or dependent events?

	X	**X′**	**Total**
Y	14	6	20
Y′	4	6	10
Total	18	12	30

7 For two events, A and B, it is known that $\Pr(A) = 0.6$ and $\Pr(B) = 0.8$. If A and B are independent events, then find:

a $\Pr(A \cap B)$ **b** $\Pr(A \cup B)$

8 Alexei is told that $\Pr(A) = 0.6$, $\Pr(B) = 0.7$, and $\Pr(A \cup B) = 0.8$. He is asked to find $\Pr(A \cap B)$ and calculates the answer to be 0.42.

a What was Alexei's mistake?

b What is the actual value of $\Pr(A \cap B)$, and how must it be calculated?

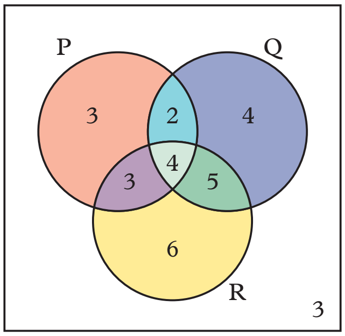
9 If A and B are independent events, then are A and B′ also independent events? Explain your answer.

10 In each case, decide if A and B are independent events or dependent events.

a $\Pr(A) = 0.4$, $\Pr(B) = 0.3$, $\Pr(A \cap B) = 0.2$ **b** $\Pr(A) = \frac{3}{5}$, $\Pr(B) = \frac{2}{3}$, $\Pr(A \cap B) = \frac{2}{5}$

c $\Pr(A) = 0.5$, $\Pr(B) = 0.8$, $\Pr(A \cap B') = 0.1$ **d** $\Pr(A) = 0.6$, $\Pr(B) = 0.5$, $\Pr(A|B) = 0.3$

PROBLEM SOLVING AND REASONING



11 Consider the Venn diagram on the right.

Which pair of events is independent? Justify your answer with appropriate calculations.

12 Independent events in statistics are rare, due to random variations.

For example, outside of the Mathematics classroom we would say that someone's mode of transport to school and their favourite school subject are 'independent,' since it does not follow that knowing that a student takes a train to school would tell us anything about their favourite subject. However, it does not follow that the events 'takes train' and 'prefers Maths' would be independent events in a survey of Year 10 students.

Consider the following data which shows the results of a survey from 30 students who were asked their preferred movie genre (out of action, comedy, drama) and whether their birthday occurs on an even calendar date or an odd calendar date.

	Action	**Comedy**	**Drama**	**Total**
Odd	7	5	2	14
Even	5	3	2	10
Total	12	8	4	24

a One student is selected at random. Find:

i Pr(Action)

ii Pr(Action | Odd)

iii Pr(Comedy)

iv Pr(Comedy | Even)

b Explain why 'prefers action movies' and 'born on an odd date' are independent events in this survey.

c Explain why 'born on an even date' and 'prefers comedy movies' are dependent events in this survey.

d Explain why it does not make sense to claim that being born on an even date makes someone more or less likely to prefer comedy movies.

13 'Independent events' and 'independent trials' are often confused in probability. Independent trials are separate steps of a multi-step experiment where each step does not affect the probabilities at any other step. If the trials in a multi-step experiment are independent, then this will inevitably lead to an event relating to one trial being independent of an event relating to another trial.

Consider a bag containing six balls: 4 white and 2 black. Two balls are randomly selected from the bag. Let W1 be the event 'the first ball is white' and W2 be the event 'the second ball is white.'

a Explain why W1 and W2 are independent events if the first ball is replaced before choosing the second ball.

b Find Pr(W2 | W1) if:

i the first ball is replaced after being selected

ii the first ball is not replaced after being selected.

c Find the probability that the two balls selected are white if:

i the first ball is replaced after being selected

ii the first ball is not replaced after being selected.

14 A and B are independent events with Pr(A) = Pr(B). Find Pr(A) if $\Pr(A \cup B) = 0.84$.

15 If A, B, and C are all independent events, then does it follow that $\Pr(A \cap B \cap C) = \Pr(A) \times \Pr(B) \times \Pr(C)$? Justify your answer.

Check your Student obook pro for these digital resources and more:

Interactive skillsheet
Independence

Topic quiz
11E

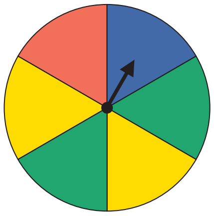
Chapter summary

Theoretical probability

$$\Pr(\text{event}) = \frac{\text{number of outcomes in event}}{\text{number of outcomes in the sample space}}$$

$$\Pr(\text{green}) = \frac{2}{6}$$

$$= \frac{1}{3}$$



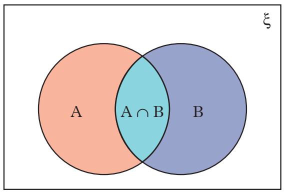
- The sum of probabilities of complementary events is 1: $\Pr(A) + \Pr(A') = 1$
- If two events are mutually exclusive then they cannot both occur at the same time.

Conditional probability

- $\Pr(A|B)$ = the probability that event A occurs given that event B has occured.

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} \qquad \Pr(A|B) \neq \Pr(B|A)$$

Venn diagrams



Independent events

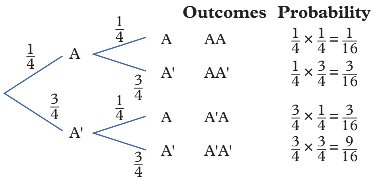
- Two events are independent if the outcome of one event has no effect on the outcome of the other event.
- If A and B are independent events, then $\Pr(A|B) = \Pr(A)$.
- If $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$, then A and B are independent events.
- The outcomes for each step of a chance experiment with replacement are independent.
- The outcomes for each step of a chance experiment without replacement are not independent.

Set notation

- A set is a collection of distinct objects.
- ξ = the universal set containing all the elements under consideration
- $A \cap B$ = the intersection of two sets, A and B
- $A \cup B$ = the union of two sets, A and B
- A' = the complement of set A, everything from ξ that is not in A
- $n(A)$ = the numbers of elements in set A
- $A \subset B$ = 'set A is a subset of set B'

Tree diagrams

- Each branch represents the possible outcomes at each step.
- The final outcomes are listed at the end.



Arrays

- The sample space of a two-step experiment can be displayed in an array or table.

		Rolling a standard six-sided die					
		1	**2**	**3**	**4**	**5**	**6**
Flipping a coin	**H**	(1, H)	(2, H)	(3, H)	(4, H)	(5, H)	(6, H)
	T	(1, T)	(2, T)	(3, T)	(4, T)	(5, T)	(6, T)

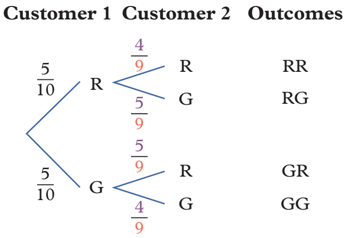
Two way tables

- A two-way table is a means of displaying the relationship between two sets of data.

	School A	**School B**	**Total**
Musical instrument	10	4	14
No musical instrument	3	13	16
Total	13	17	30

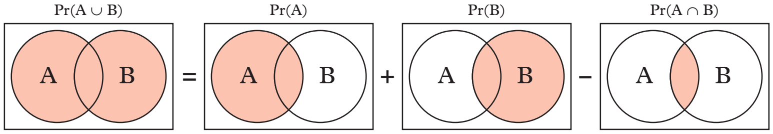
Experiments without replacement

- When multi-step experiments involve items being selected from a group, if the items are not replaced after each selection, then the probabilities will change.



The number of outcomes in the sample space is reduced for the second selection.
Depending on the first selection, the number of successful outcomes may also be reduced.

Additional rule of probability



Chapter review

Chapter review quiz
Take the chapter review quiz to assess your knowledge of this chapter.

Quizlet
Test your knowledge of this topic by working individually or in teams.

Multiple-choice

11A **1** A coin is flipped three times. How many outcomes would appear on a tree diagram representing this experiment?

A 2 **B** 4 **C** 6 **D** 8 **E** 10

11A **2** A letter is randomly selected from a word. The sample space for the experiment is {A, E, L, P}. Which of the following could be the word?

A HELP **B** PLEASE **C** PLEAD **D** APPLE **E** ELLA

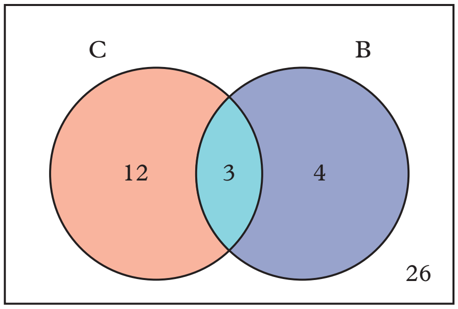
11B **3** A standard deck of cards contains 52 cards, with 13 cards of each of four suits (clubs, diamonds, hearts and spades). One card is drawn from a deck of playing cards and its suit is noted. It is replaced, then a second card is drawn. What is the probability two diamonds are drawn?

A $\frac{1}{4}$ **B** $\frac{1}{13}$ **C** $\frac{1}{16}$ **D** $\frac{1}{17}$ **E** $\frac{3}{52}$

11B **4** Repeat the experiment from question **3** but, this time, assume the first card is not replaced before the second card is drawn. What is the probability that two diamonds are drawn?

A $\frac{1}{4}$ **B** $\frac{1}{13}$ **C** $\frac{1}{16}$ **D** $\frac{1}{17}$ **E** $\frac{3}{52}$

11C **5** The following Venn diagram shows the number of students who play chess and badminton.



One of these students is selected at random. What is the probability that they play chess?

A $\frac{1}{15}$ **B** $\frac{12}{19}$ **C** $\frac{1}{3}$ **D** $\frac{4}{15}$ **E** $\frac{7}{45}$

11C **6** Based on the following incomplete two-way table, what is Pr(A)?

	A	**A′**	**Total**
B	12		24
B′		16	
Total			50

A 0.44 **B** 0.5 **C** $\frac{6}{11}$ **D** $\frac{14}{25}$ **E** 0.48

11D **7** Based on the Venn diagram in question **5**, what is the probability that a student plays badminton given that they also play chess?

A $\frac{1}{3}$ **B** $\frac{7}{45}$ **C** $\frac{1}{15}$ **D** $\frac{3}{7}$ **E** $\frac{1}{5}$

11D **8** For two events, A and B, it is known that $\Pr(A) = 0.45$, $\Pr(B) = 0.6$, and $\Pr(A \cup B) = 0.85$. What is $\Pr(A|B)$?

A $\frac{19}{45}$ **B** 0.27 **C** $\frac{12}{17}$ **D** $\frac{4}{9}$ **E** $\frac{3}{4}$

11E **9** If A and B are independent events with $\Pr(A) = 0.7$ and $\Pr(B) = 0.4$, then what is $\Pr(A \cup B)$?

A 0.28 **B** 1.1 **C** 0.55 **D** 0.82 **E** 0.3

11E **10** The two-way table on the right shows the data gathered from 50 repetitions of a scientific experiment. Which pair of events is independent?

A 'Cold' and 'Success'
B 'Warm' and 'Inconclusive'
C 'Hot' and 'Failure'
D 'Hot' and 'Success'
E 'Warm' and 'Failure'

	Cold	**Warm**	**Hot**	**Total**
Success	12	8	0	20
Inconclusive	4	5	1	10
Failure	3	12	5	20
Total	19	25	6	50

Short answer

11A **1** A ten-sided die (with sides numbered 1–10) is rolled and the number facing up is recorded.

a List the sample space.

b If A is the event that the number facing up is odd and B is the event that the number facing up is a multiple of 3, then list the elements in the set:

i A **ii** B **iii** $A \cap B$ **iv** $A \cup B$ **v** $A \cap B'$

11A **2** A bag contains the letters of the alphabet. One letter is randomly chosen, it's recorded whether it's a vowel or consonant, then replaced. This is repeated another two times. Draw a tree diagram to find the probability of choosing:

a no vowels
b at least one vowel
c three consonants
d no more than two consonants.

11B **3** A three-digit number is made using the digits 0 to 9. The digits can be repeated.

a How many numbers are possible?
b Describe any restrictions on the formation of the number.
c What is the probability the number will be 111?
d What is the probability that all the digits in the number will be the same?

11B **4** The experiment in question **3** is repeated, this time not allowing any repetition of the digits.

a How many numbers are possible in this case?
b Describe any restrictions on the formation of the number.
c What is the probability the number will be 107?
d How many numbers with a value greater than 500 are possible?

11C **5** Sets A, B and C overlap, and can be described as follows:

- $n(A) = 14$
- $n(B) = 13$
- $n(C) = 15$
- $n(A \cap B) = 6$
- $n(B \cap C) = 5$
- $n(A \cap C) = 7$
- $n(A \cap B \cap C) = 4$

Draw a Venn diagram to display this information.

11C **6** This table shows the number of iPhone users in a group of students.

State the number of:

a Year 8 students who do not use an iPhone
b students who do not use an iPhone
c Year 10 students
d students who use an iPhone.

	Year 8	**Year 10**	**Total**
iPhone	20		
Other type of phone		12	
Total	35		82

11D **7** Two pens are taken from a box containing four blue pens, five black pens and two red pens. Calculate the probability that:

a the second pen is red, given the first pen is blue
b the second pen is red, given the first pen is black
c the two pens are the same colour
d the two pens are different colours.

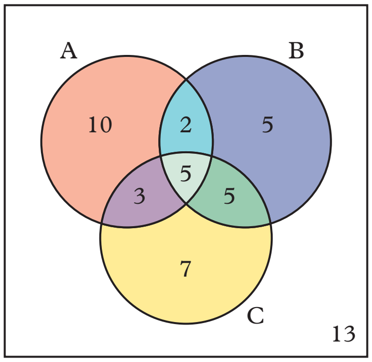
11D **8** A letter is chosen at random from the word CONDITIONAL. Find the probability that the chosen letter is:

a C
b N
c O, given that it is a vowel
d A vowel, given that it is O
e A vowel, given that it is adjacent to an N.

11E **9** For two events, A and B, it is known that $\Pr(A) = \frac{1}{6}$ and $\Pr(A \cap B) = \frac{1}{12}$. If A and B are independent events, then find:

a $\Pr(A | B)$ **b** $\Pr(B)$ **c** $\Pr(A \cup B)$

11E **10** Consider the Venn diagram on the right.



a Calculate the probability of:

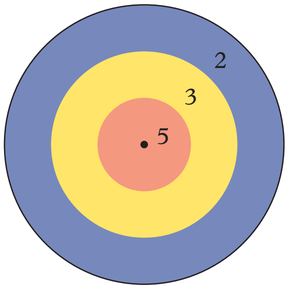
i A **ii** $A \cap B$

iii $A \cup B$ **iv** $B' \cap C$

b Which pair of events is independent? Justify your answer.

Analysis

1 An archery target consists of an inner circle of radius 10 cm, encircled by two rings, each with width 10 cm. An arrow which lands in the outermost ring is worth 2 points, an arrow which lands in the middle ring is worth 3 points, and an arrow which lands in the inner circle is worth 5 points, as shown in the diagram.



An archer shoots two arrows, both of which hit this target.

a Find the fraction of the target area that is taken up by the region worth:

i 5 points **ii** 3 points **iii** 2 points.

b Write the sample space for the sum of the two scores from the archer's two shots.

Assume that each arrow has an equally likely chance of hitting anywhere on the target.

c Find the probability that, from their two shots, the archer scores:

i 10 points **ii** exactly 5 points **iii** at least 8 points.

It is not realistic to assume that the archer has an equally likely chance to hit any point on the target. Instead, assume that, for each shot, the archer has a 20% chance of hitting the outermost ring, a 50% chance of hitting the middle ring, and a 30% chance of hitting the inner circle.

d Find the probability that, from their two shots, the archer scores:

i 10 points **ii** exactly 6 points **iii** no more than 5 points.

It is reasonable to think that the number of points the archer scores on the second shot is dependent on their score from the first shot. Assume that the archer learns their first shot and will therefore score at least as many points on the second shot as they did on their first shot.

If the archer scores 2 points on their first shot, then the probabilities for their second shot remain 20%, 50%, and 30% for scoring 2, 3, or 5 points, respectively. However, if the archer scores 3 points on their first shot, then they have an even chance of scoring 3 or 5 points on their next shot. If the archer scores 5 points on their first shot, then they will score 5 points again on their second shot.

e Under these conditions, find the probability that, from their two shots, the archer scores:

i 10 points **ii** exactly 8 points **iii** at least 6 points.

2 Two boxes each contain balls. Box A contains 5 green, 2 red and 3 yellow. Box B contains 1 green, 1 red, and 2 yellow.

a If one box is chosen at random and then one ball is chosen at random from it, then find the probability that:

i box B was chosen **ii** the ball is green

iii the ball is green and was chosen from box B **iv** the ball is green or box B was chosen.

b If two balls are randomly selected, one from each box, then find the probability that:

i both balls are red **ii** at least one red ball is chosen

iii two red balls are chosen, given that at least one red ball was chosen.

c If one box is chosen at random and then two balls are chosen from that box, without replacement, find the probability that:

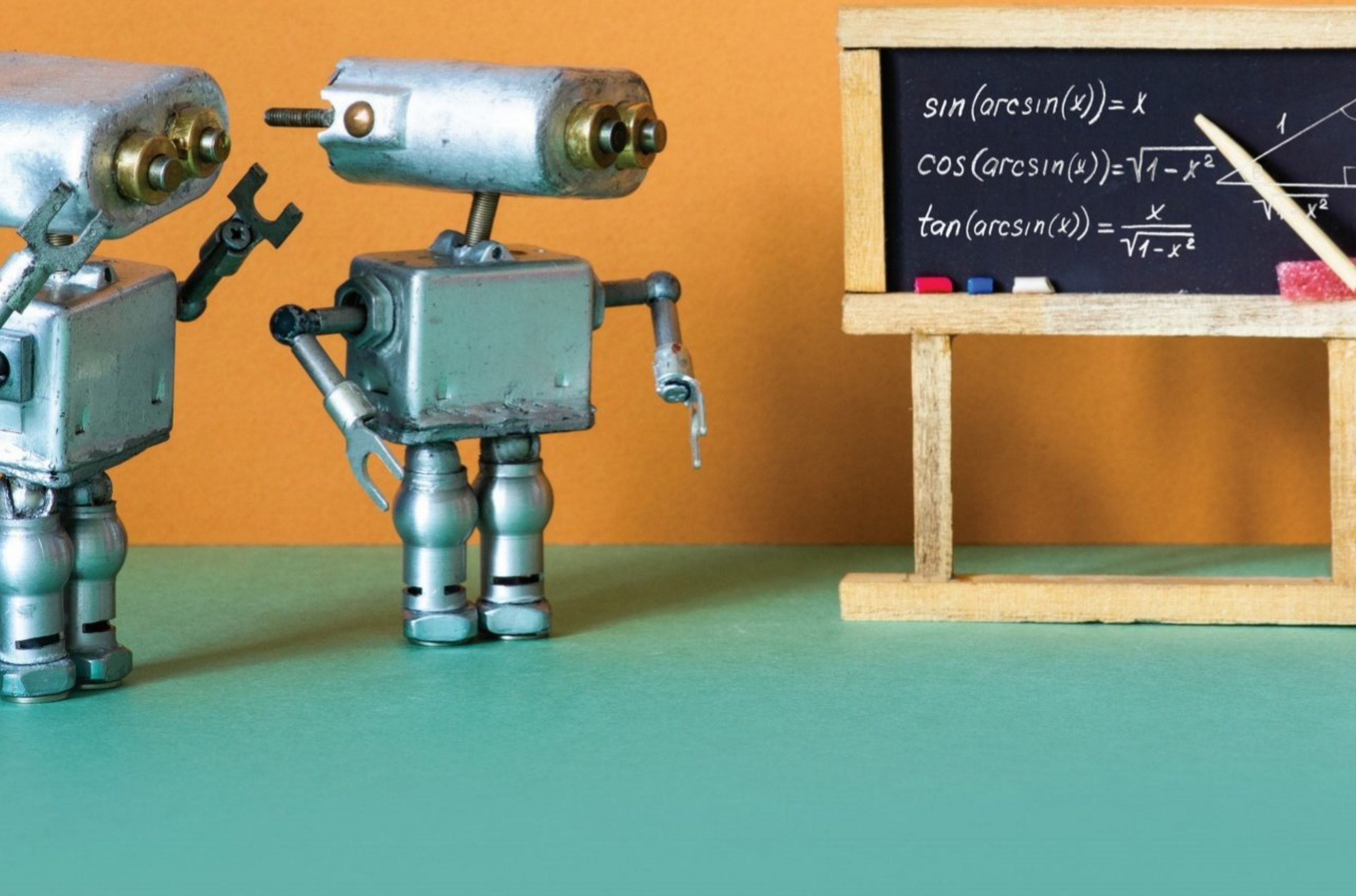
i both balls are yellow

ii both balls are yellow, given that box B was chosen

iii box B was chosen, given that both balls are yellow.

12

Computational thinking



Index

Prerequisite skills

Before starting this chapter, please ensure you have completed the Computational thinking chapters from Oxford Maths 7, Oxford Maths 8 and Oxford Maths 9.

Inter-year links

Year 7	11 Computational thinking
Year 8	10 Computational thinking
Year 9	10 Computational thinking

Curriculum links

- Implement algorithms using data structures in a general-purpose programming language (VCMNA334)

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Materials

✔ Computer
✔ Python (online or download)

Note: Python is used as the coding language in this chapter. Instructions will vary for other coding languages.

12A Two-dimensional lists

Learning intentions

✔ I can define a two-dimensional list and look up specific items.
✔ I can change elements in a two-dimensional list.

Two-dimensional lists

- A **two-dimensional list** is a data structure that allows for the representation of data that might normally be found in a table.

  Consider the example of the two-way table below showing the results of a round-robin tennis tournament where, in each row, a 1 represents a win for the player listed, a 0 represents a draw and a –1 represents a loss.

	Player 1	**Player 2**	**Player 3**	**Player 4**
Player 1	0	1	–1	–1
Player 2	–1	0	1	–1
Player 3	1	–1	0	–1
Player 4	1	1	1	0

  Here is the same data represented using a two-dimensional list:

```
Table = [[ 0,  1, -1, -1],
         [-1,  0,  1, -1],
         [ 1, -1,  0, -1],
         [ 1,  1,  1,  0]]
```

- A two-dimensional list is also called a 'list of lists', which provides us with a good way to understand its structure. Consider the following code that first defines each player's results separately, and then combines them into the same two-dimensional list shown above:

```
Player1 = [ 0,  1, -1, -1]
Player2 = [-1,  0,  1, -1]
Player3 = [ 1, -1,  0, -1]
Player4 = [ 1,  1,  1,  0]
Table = [Player1, Player2, Player3, Player4]
```

  The table variable here looks very much like a standard list, the difference being that, in this case, the items in the list are not numbers or words, they are themselves lists; meaning this is a 'list of lists'.

- Any table can be represented by a two-dimensional list as follows:

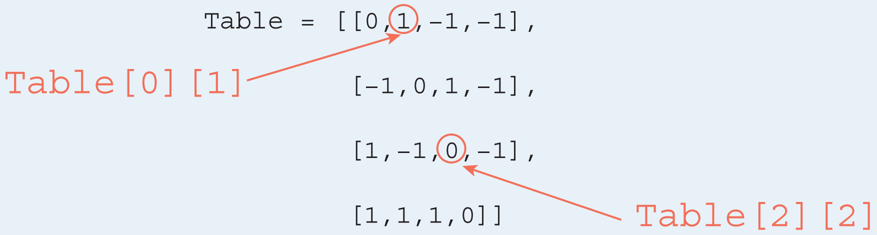
  `Table = [row1, row2, row3, ...]` where each of the rows is itself represented as a list.

Elements in a two-dimensional list

- To access a specific element in a two-dimensional list, two indices are required:
  - → a row index; and
  - → a column index, in that order (where indices are always integers).

- Recall that, in the Python programming language, indices always start at 0 to access the first row or the first column.

  For example, `Table[0][1]` accesses row 0 (the first row) and column 1 (the second column), giving the value 1. `Table[2][2]` accesses row 2 (the third row) and column 2 (the third column), giving the value 0.



```
Table = [[0,1,-1,-1],
         [-1,0,1,-1],
         [1,-1,0,-1],
         [1,1,1,0]]
```

- More generally, `Table[m][n]` will return the value stored in row m, which is the $(m + 1)$th row, and column (n), which is the $(n + 1)$th column.
- It is also possible to specify the value of a whole row by using `Table[m]`, without a second index. In the example above, `Table[2]` identifies the list `[1, -1, 0, -1]`.
- The two-dimensional list above can be written more concisely as:

  `Table = [ [0, 1, -1, -1], [-1, 0, 1, -1], [1, -1, 0, -1], [1, 1, 1, 0] ]`

Changing elements in two-dimensional lists

- The same method used to access elements in a two-dimensional list may also be used to change the value of an element.

  For example, `Table[0][1] = -1` would overwrite the value of 1 found in that position and would assign it the value of −1 instead.

  `Table[2] = [1, 0, -1, 1]` would overwrite the whole of row 2 (the third row), changing it from `[1, -1, 0, -1]` to `[1, 0, -1, 1]`.

Example 12A.1 Defining a two-dimensional list

Consider this table, showing how four students rated three films (from 0 to 5). Construct a two-dimensional list to represent the numerical information from the table.

	Film rating		
	Iron Man	***Black Panther***	***The Avengers***
Margot	3	4	4
Eren	3	5	3
Ahmad	5	3	3
Tabitha	2	4	5

THINK

There are four rows in the table, each with three entries. The row and column names won't be included in the two-dimensional list.

WRITE

```
Films = [ [3, 4, 4],
          [3, 5, 3],
          [5, 3, 3],
          [2, 4, 5] ]
```

Example 12A.2 Accessing elements in a two-dimensional list

The table and the two-dimensional list used to represent the data from Example 12A.1 are both given below.

	Film rating		
	Iron Man	***Black Panther***	***The Avengers***
Margot	3	4	4
Eren	3	5	3
Ahmad	5	3	3
Tabitha	2	4	5

```
Films = [[3,4,4],
         [3,5,3],
         [5,3,3],
         [2,4,5]]
```

Using the appropriate row and column index for the two-dimensional list, identify the location of the elements corresponding to:

a the score that Margot gave *Black Panther*

b the score that Tabitha gave *The Avengers*

c all of Eren's scores.

THINK	**WRITE**
a Margot's ratings are in the first row, which is index 0. The ratings for *Black Panther* are in the second column, which is identified by column index 1.	**a** `Films[0][1]`
b Tabitha's ratings are in the fourth row, which is identified by row index 3. The ratings for *The Avengers* are in the third column, which is identified by column index 2.	**b** `Films[3][2]`
c To identify an entire row, use the appropriate row index without the column index. Eren's ratings are in the second row, which is identified by row index 1.	**c** `Films[1]`

Example 12A.3 Changing elements in a two-dimensional list

The four people from examples 12A.1 and 12A.2 re-watched each film and submitted updated scores. The scores after the second viewing are given below, with the ratings that have been updated highlighted in red.

	Film rating		
	Iron Man	***Black Panther***	***The Avengers***
Margot	3	4	3
Eren	4	5	3
Ahmad	5	3	3
Tabitha	3	5	4

Assuming that the previous two-dimensional list, ‘Films’, is still already defined, write the lines of code necessary to change ‘Films’ to include the updated scores.

THINK

For each of Margot and Eren’s changes, a single score (element) needs to change. But Tabitha's changes can be made more efficiently by changing the whole row at once.

- Margot’s changed score is in the position identified by row index 0, column index 2.
- Eren’s changed score is in the position identified by row index 1, column index 0.
- Tabitha’s changed scores are in the row identified by row index 3.

WRITE

```
Films[0][2] = 3

Films[1][0] = 4

Films[3] = [3, 5, 4]
```

Helpful hints

✔ Two-dimensional lists always need an open square bracket at the very beginning and a closed square bracket at the very end.

✔ Each row in a two-dimensional list should be separated by a comma:

```
Table = [row1, row2, row3, row4]
```

and

```
Table = [ [ 0,  1, -1, -1],
          [-1,  0,  1, -1],
          [ 1, -1,  0, -1],
          [ 1,  1,  1,  0] ]
```

✔ Each row of a two-dimensional list is itself a list, so will require an open square bracket at the beginning and a closed square bracket at the end, with commas separating each element.

ANS p807

Exercise 12A Two-dimensional lists

1 Consider this table showing three students' test scores for three subjects.

	Test scores		
Student	**Maths**	**English**	**Science**
Thelma	82	75	90
Zhi	74	86	95
Andrea	93	95	88

12A.1 **a** Construct a two-dimensional list to represent the table.

12A.2 **b** Use the appropriate row and column index to locate the following data:

- **i** Thelma's English score
- **ii** Andrea's Maths score
- **iii** All of Zhi's scores.

12A.3 **c** The students' Maths teacher realised that she hadn't marked the last question on their tests. After she re-marked the tests, the students' Maths scores changed. Write the lines of code necessary to make the changes described below to the two-dimensional list you constructed in part **a**.

- **i** Thelma's score increased by 3 points.
- **ii** Zhi's score increased by 5 points.
- **iii** Andrea's score decreased by 1 point as the teacher realised she has marked the last question incorrectly.

2 Consider this multiplication table containing the products of all pairs of integers between 0 and 5.

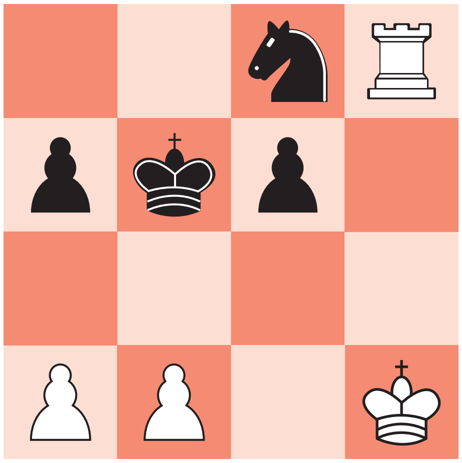
Multiplication						
	0	**1**	**2**	**3**	**4**	**5**
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	6	8	10
3	0	3	6	9	12	15
4	0	4	8	12	16	20
5	0	5	10	15	20	25

a Construct a two-dimensional list to represent the table.

b Identify the code, containing the appropriate row and column index, to locate each of the following pieces of data:

- **i** the product of 2 and 3
- **ii** the product of 1 and 5
- **iii** all products for the number 3.

3 Consider the following illustration of a section of a chessboard.



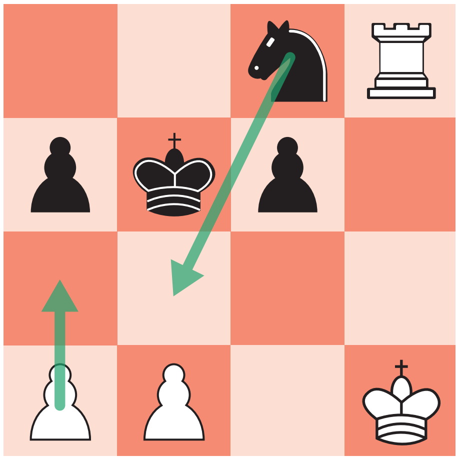
♖ is a white Rook (WR)

♞ is a black Knight (BN)

♚ is a black King (BK)

♟ is a black Pawn (BP)

and so on …



a Use the given abbreviations, and ES for empty space, to construct a two-dimensional list representing this part of the chessboard.

b Two moves are made in the game, shown by the arrows on the board on the right above.

Write the code that would change the two-dimensional list you constructed in part **a** to represent the board resulting after those two moves.

12B Transformations of points on a plane

Learning intentions

✔ I can write a transformation on a plane as a matrix.
✔ I can use a transformation matrix to transform a set of points on a plane.
✔ I can describe transformations on a plane.

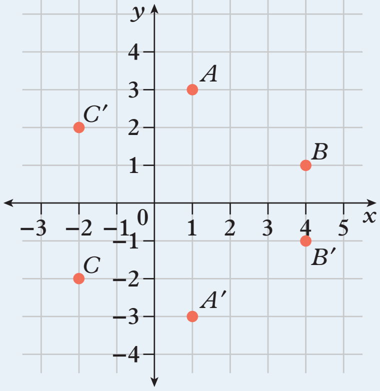
Matrices

- A **matrix** (plural matrices) is a rectangular mathematical object with a number of rows and columns.
- Matrices are enclosed in large square brackets, as shown on the right. This matrix contains the same data as the two-dimensional list called 'Table' in section 12A.

$$\begin{bmatrix} 0 & 1 & -1 & -1 \\ -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

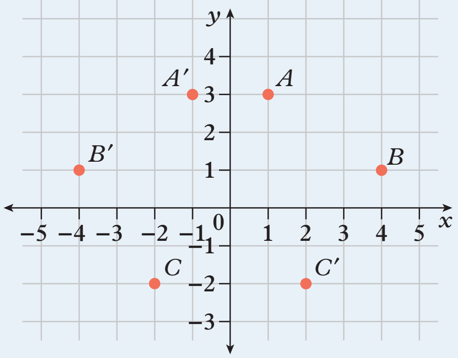
Reflection in the *x*-axis

- When a point is reflected across the x-axis, its x value remains unchanged and the sign of the y value changes: $(x, y) \rightarrow (x, -y)$. For example, consider the points $A(1, 3)$, $B(4, 1)$ and $C(-2, -2)$, shown on the right, being reflected in the x-axis:
  $A(1, 3) \rightarrow A'(1, -3)$
  $B(4, 1) \rightarrow B'(4, -1)$
  $C(-2, -2) \rightarrow C'(-2, 2)$



Reflection in the *y*-axis

- When a point is reflected in the y-axis, the sign of its x value changes and its y value remains unchanged: $(x, y) \rightarrow (-x, y)$. For example, consider the same points $A(1, 3)$, $B(4, 1)$ and $C(-2, -2)$ being reflected in the y-axis, as shown on the right.
  $A(1, 3) \rightarrow A'(-1, 3)$
  $B(4, 1) \rightarrow B'(-4, 1)$
  $C(-2, -2) \rightarrow C'(2, -2)$



Rotation around the origin

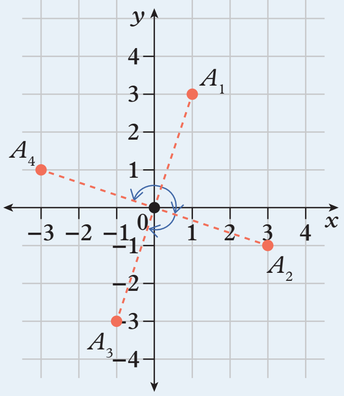
- A rotation of 180° around the origin changes the sign of both the x and y values: $(x, y) \rightarrow (-x, -y)$.
- A clockwise rotation of 90° around the origin changes the x and y values in this way: $(x, y) \rightarrow (y, -x)$.
- An anticlockwise rotation of 90° around the origin changes the x and y values in this way: $(x, y) \rightarrow (-y, x)$.

  For example, consider rotating the point $A(1, 3)$ around the origin, as shown on the right.

  Rotate clockwise 90°: $A_1(1, 3) \rightarrow A_2(3, -1)$

  Rotate 180°: $A_1(1, 3) \rightarrow A_3(-1, -3)$

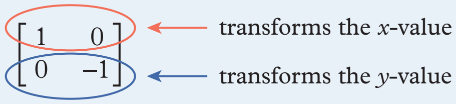
  Rotate anti-clockwise 90°: $A_1(1, 3) \rightarrow A_4(-3, 1)$



Transformation matrices

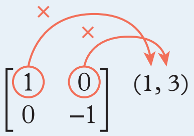
- A **transformation matrix**, T, is a matrix that describes the transformation of a point. To describe transformations on the Cartesian plane, a matrix with 2 rows and 2 columns is needed.

  For example, $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
- The first row of the transformation matrix is used to determine the x value of the transformed point, and the second row of the transformation matrix is used to determine the y value of the transformed point.

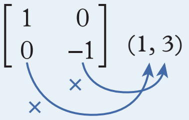


- To apply a transformation matrix to a point, multiply the first value in each row by the x value and the second value in each row by the y value. Then, for each row, add the x and y values together.

  For example, applying the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ to the point $(1, 3)$ generates the equations:



$$\begin{aligned} x' &= (1 \times 1) + (0 \times 3) \\ &= 1 \end{aligned}$$



$$\begin{aligned} y' &= (0 \times 1) + (-1 \times 3) \\ &= 0 \times 1 - 1 \times 3 \\ &= -3 \end{aligned}$$

So, the transformed point would be $(1, -3)$.

Example 12B.1 Describing transformations

For each of the transformations given below, describe the single transformation (reflection across an axis or rotation around the origin) that has occurred.

a $A(-5, 3) \rightarrow A'(5, 3)$ **b** $B(1, 2) \rightarrow B'(-1, -2)$ **c** $C(-4, 3) \rightarrow C'(3, 4)$

d $D(2, 6) \rightarrow D'(2, -6)$ **e** $E(1, -4) \rightarrow E'(-4, -1)$

THINK	WRITE
Identify how the values have changed in the given transformations. The first value in an ordered pair is the x value and the second value is the y value. If necessary, draw the points on a Cartesian plane to help you identify each transformation.	**a** reflection in the y-axis **b** rotation of 180° around the origin **c** rotation of 90° clockwise around the origin **d** reflection in the x-axis **e** rotation of 90° anticlockwise around the origin

Example 12B.2 Applying transformation matrices

a Apply the transformation matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ to each of the following points.

i (–2, –4) **ii** (6, 1) **iii** (2, 5)

b Apply the transformation matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ to each of the following points.

i (3, 2) **ii** (–5, 4) **iii** (0, 7)

c What single transformations are parts **a** and **b** describing?

THINK	**WRITE**
a Rewriting the transformation matrix in the form of equations gives us: $x' = -1 \times x + 0 \times y$ $y' = 0 \times x - 1 \times y$ Substitute the *x*- and *y*-coordinates of each of the given points (**i–iii**) into both equations.	**a** **i** $x' = -1 \times -2 + 0 \times -4$ $= 2$ $y' = 0 \times -2 - 1 \times -4$ $= 4$ $(x', y') = (2, 4)$ **ii** $x' = -1 \times 6 + 0 \times 1$ $= -6$ $y' = 0 \times 6 - 1 \times 1$ $= -1$ $(x', y') = (-6, -1)$ **iii** $x' = -1 \times 2 + 0 \times 5$ $= -2$ $y' = 0 \times 2 - 1 \times 5$ $= -5$ $(x', y') = (-2, -5)$
b Rewriting the transformation matrix in the form of equations gives us: $x' = 0 \times x + 1 \times y$ $y' = -1 \times x + 0 \times y$ Substitute the *x*- and *y*-coordinates of each of the given points (**i–iii**) into both equations.	**b** **i** $x' = 0 \times 3 + 1 \times 2$ $= 2$ $y' = -1 \times 3 + 0 \times 2$ $= -3$ $(x', y') = (2, -3)$ **ii** $x' = 0 \times -5 + 1 \times 4$ $= 4$ $y' = -1 \times -5 + 0 \times 4$ $= 5$ $(x', y') = (4, 5)$ **iii** $x' = 0 \times 0 + 1 \times 7$ $= 7$ $y' = -1 \times 0 + 0 \times 7$ $= 0$ $(x', y') = (7, 0)$
c Identify how the *x* and *y* values for each point have changed. If necessary, draw the points on a Cartesian plane to help you identify each transformation.	**c** Part **a** is describing a rotation of 180° around the origin and part **b** is describing a rotation of 90° clockwise around the origin.

Helpful hints

✔ Remember that a coefficient in an equation includes the sign in front of it. For example, in the equation $y' = 0 \times x - 1 \times y$ it is crucial that you do not forget to include the negative sign in front of the 1.

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Exercise 12B Transformations of points on a plane

12B.1 **1** For each of the transformations given below describe the single transformation that has occurred (reflection in an axis or rotation around the origin).

a $(-4, 3) \rightarrow (4, -3)$
b $(7, 2) \rightarrow (-7, 2)$
c $(1, 6) \rightarrow (6, -1)$
d $(-3, -2) \rightarrow (-3, 2)$
e $(-3, 3) \rightarrow (-3, -3)$
f $(0, 3) \rightarrow (-3, 0)$

2 Write the points after each transformation below is carried out on the point $(-2, 3)$.

a reflection in the x-axis
b reflection in the y-axis
c rotation by 90° in the anticlockwise direction around the origin
d reflection in the x-axis followed by a rotation of 90° around the origin in the clockwise direction
e rotation by 180° around the origin followed by a reflection in the y-axis

12B.2 **3** Apply the transformation matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ to each of the following points.

a $(-4, 1)$
b $(5, -2)$
c $(1, 1)$
d $(0, 0)$

4 Apply the transformation matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ to each of the following points.

a $(2, 6)$
b $(-1, 4)$
c $(11, -5)$
d $(-2, 0)$

5 What transformations do the matrices in questions **3** and **4** describe? Plot both the original and transformed points on a Cartesian plane to see the transformation each time.

6 The transformation matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the **identity matrix**. Try applying this matrix to a set of coordinates of your choosing. Why do you think it is called the identity matrix?

7 Try applying the transformation matrix $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ to a set of coordinates of your choosing. Describe the transformation that occurs when $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ is applied to a point.

8 If the general transformation matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is applied to the point (x, y) to produce the new point (x', y'), write the coordinates for (x', y') in terms of a, b, c, d, x, and y.

12C Using two-dimensional lists to implement transformation matrices

Learning intentions

✔ I can use two-dimensional lists to model transformation matrices.

✔ I can implement transformations of sets of points on a plane.

Transformation matrices as two-dimensional lists

- Two-dimensional lists can be used to represent transformation matrices.

  For example,

transformation matrix → **two-dimensional list**

$$T = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \longrightarrow$$

```
T = [[-1,0],
      [0,-1]]
```

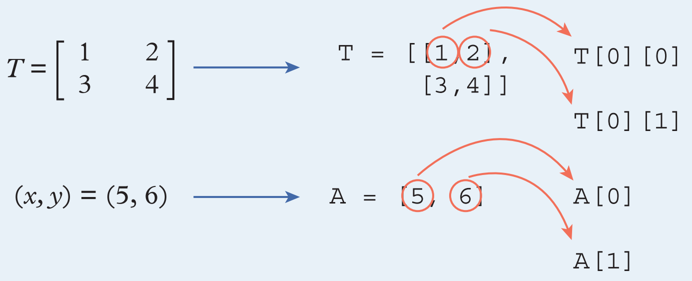
Using a two-dimensional list to apply transformations to a point on a plane

- Any point can be represented in the Python programming language using a simple list with two elements.

  For example, the point $A(2, 5)$ can be represented as the list: `A = [2, 5]`

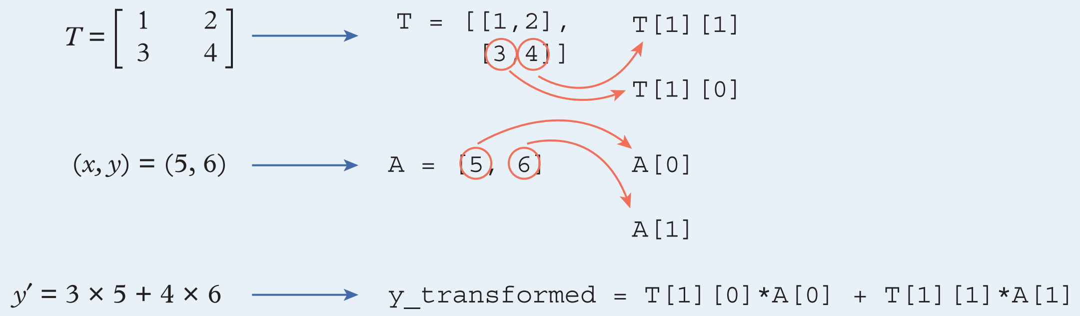
  `A[0]` can then be used to locate the x value, and `A[1]` can be used to locate the y value.
- To apply a transformation in Python programming, consider how the algebraic description of the transformation is constructed.

Transforming the *x* values



$x' = 1 \times 5 + 2 \times 6 \longrightarrow$ `x_transformed = T[0][0]*A[0] + T[0][1]*A[1]`

Transforming the *y* values



Replacing the coefficients and the x, y values in the equation with the appropriate indices from each list, we get the general Python programming equations:

```
x_transformed = T[0][0]*A[0] + T[0][1]*A[1]
y_transformed = T[1][0]*A[0] + T[1][1]*A[1]
```

Applying transformations to a set of points in the plane using a two-dimensional list

- To represent a set of points in the Cartesian plane, we can also use a two-dimensional list.

set of points

$S = \{(1, 3), (4, 1), (-2, 2)\}$

table

1	3
4	1
–2	2

two-dimensional list

```
S = [[1,3],
     [4,1],
     [-2,2]]
```

- In this representation, each row in the two-dimensional list represents a single point. Therefore, a simple 'for loop' can be used to access each point, one by one.
- The general structure for transforming a list of points, *S*, using a transformation matrix, *T*, is:

```
for point in S:
    'apply T to the current point and store the transformed point'
```

- In order to store transformed points, two more Python commands will be needed.
- To define an empty list, use `S_new = []`.
- To add a point to the list, use `S_new.append(point)`.

For example, the following sample code defines an empty list and then adds the points (1, 3) and (5, –2) to it.

```
S_new = []
S_new.append([1,3])
S_new.append([5,-2])
```

If `S_new` were now printed, we would see:

```
[[1,3], [5,-2]]
```

Example 12C.1 Transformation matrices as two-dimensional lists

For the transformation matrix $T = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$, write a code that defines the two-dimensional matrix and then apply the transformation matrix to the point (2, –5). Print the result.

THINK

Define the transformation matrix as a two-dimensional list, `T`, and the point as a list, `A`. Consider the two general formulas for the coordinates of a transformed point:

```
x_transformed = T[0][0]*A[0]
+ T[0][1]*A[1]
y_transformed = T[1][0]*A[0]
+ T[1][1]*A[1]
```

Use these formulas to define the coordinates of the transformed point, then print the coordinates as a list defined as `new_point`.

WRITE

```
T = [[1,0], [1,-1]]
A = [2,-5]
x_transformed = T[0][0]*A[0] + T[0][1]*A[1]
y_transformed = T[1][0]*A[0] + T[1][1]*A[1]
new_point = [x_transformed, y_transformed]

print(new_point)
```

(Note: The output for this code should give the transformed coordinate point, (1, 5).)

Example 12C.2 Applying a transformation to a set of points

Consider the following set of points: $S = \{(2, 4), (4, 2), (6, 2), (8, 4), (8, 6), (6, 8), (4, 8), (2, 6)\}$

a Write a transformation matrix that will reflect this set of points across the x-axis.

b Use the code for a two-dimensional list to apply the transformation matrix to the set of points. Store the transformed points in a new two-dimensional list called `S_new`.

THINK

a Recall that reflecting in the x-axis involves leaving the x value unchanged and changing the sign of the y value.

b Before points can be added to a new list, the new list must be defined as an empty list: `S_new = []`
Recall that to add elements to a list we use `S_new.append(element)`.

WRITE

a $T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

b

```
T = [[1,0], [0,-1]]

S = [[2,4], [4,2], [6,2], [8,4], [8,6], [6,8], [4,8], [2,6]]
S_new = []

for point in S:
    x_transformed = T[0][0]*point[0] + T[0][1]*point[1]
    y_transformed = T[1][0]*point[0] + T[1][1]*point[1]
    new_point = [x_transformed, y_transformed]
    S_new.append(new_point)

print(S_new)
```

Example 12C.3 Applying a sequence of transformations to a set of points

Consider the following sequence of transformations, defined individually and stored in a list named ‘Transformations’.

```
T1 = [[-1,0],             T2 = [[ 0,1],             T3 = [[-1, 0],
      [ 0,1]]                   [-1,0]]                   [ 0,-1]]
Transformations = [T1, T2, T3]
```

Apply all three transformations to the set of points $S = \{(2, 4), (4, 2), (6, 2), (8, 4), (8, 6), (6, 8), (4, 8)\}$.

THINK

Define the list of transformations named `Transformations` and the list of points named `S`. An empty list `S_new` must be defined to store the transformed points.

Create a for loop to transform each point in the list, `S`. Then, within that loop, create a nested loop to cycle through each of the three transformation matrices in `Transformations`. The code will be saying: ‘For every point in S and for every T in Transformations; apply T to each point’.

It will also be important to consider how we store interim values of points. For example, consider the point (2, 4) undergoing all three transformations in order: $(2, 4) \rightarrow (-2, 4) \rightarrow (4, 2) \rightarrow (-4, -2)$.

A variable is needed to store the interim points from $(2, 4)$ to $(-4, -2)$. Use `x_trans` and `y_trans`.

After each of the transformations have been applied to a point, it must be added to `S_new` using `S_new.append(point)`. Once all points in `S` have been transformed, print `S_new`.

WRITE

```
T1 = [[-1,0],[0,1]]
T2 = [[0,1],[-1,0]]
T3 = [[-1,0],[0,-1]]
Transformations = [T1,T2,T3]

S = [[2,4],[4,2],[6,2],[8,4],[8,6],[6,8],[4,8]]
S_new = []

for point in S:
    for T in Transformations:
        x_trans=T[0][0]*point[0] + T[0][1]*point[1]
        y_trans=T[1][0]*point[0] + T[1][1]*point[1]
        point=[x_trans,y_trans]
    S_new.append(point)

print(S_new)
```

Helpful hints

- ✔ A ‘set’ in Python is a data structure very similar to the mathematical idea of a set you have dealt with before. However, Python sets are not used to store points because Python sets are unordered data structures, so there is no way to retain the order of the points in a set. Using lists instead of sets enables us to keep the points in order.
- ✔ If you use round brackets instead of square brackets to write points in Python you may be surprised to see that this doesn’t break the code. This is because, in Python, round brackets define a data structure called tuples and these are commonly used to denote points in Python. Instead, in this chapter, we used a list of two items to reduce the number of different data structures we are using.

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Exercise 12C Using two-dimensional lists to implement transformation matrices

12C.1 **1** For each of the transformation matrices below, write a code that defines the two-dimensional matrix and apply that to the point (–4, 1). Print the result.

a $T = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ **b** $T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **c** $T = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$

12C.2 **2** Consider this set of points: $S = \{(-1, -3), (-2, -5), (-4, -8), (-6, 1), (-3, 5)\}$.

a Write a transformation matrix that will reflect this set of points in the y-axis.

b Write code to apply the transformation and store the transformed points in a new two-dimensional list called `S_new`.

12C.3 **3** Consider this set of points: $S = \{(0, 0), (3, 0), (3, 2)\}$.

a Write a transformation matrix that will rotate the set of points 90° anticlockwise around the origin.

b Write code to apply the transformation and store the transformed points in a new two-dimensional list called `S_new`.

4 Consider the following sequence of transformations, defined individually and stored in a list named ‘Transformations’.

```
T1 = [[1, 0],
      [0,-1]]
T2 = [[-1,0],
      [-1,0]]
Transformations = [T1, T2]
```

a Apply the two transformations, in order, to the point (–3, –2).

b Apply the two transformations to the set of points $S = \{(-3, -2), (2, 7), (5, -3), (-4, 1), (1, 5)\}$.

5 Consider the following sequence of transformations, defined individually and stored in a list named ‘Transformations’.

```
T1 = [[0, 0],                T2 = [[ 0,1],                T3 = [[1,1],
      [1,-1]]                      [-1,0]]                      [1,1]]
Transformations = [T1, T2, T3]
```

a Apply these three transformations, in order, to the point (6, –6).

b Apply these three transformations to the set of points $P = \{(6, -6), (6, 6), (-6, 6), (-6, -6)\}$.

6 Write a sequence of transformation matrices that will perform a rotation of 90° clockwise followed by a reflection in the x-axis. Test your matrices by writing code and applying the transformations to a series of points, then plotting the original against the transformed points.

7 Write a sequence of transformation matrices that will perform a rotation by 180° followed by a reflection in the y-axis, followed by a rotation of 90° anticlockwise. Test your matrices by writing code and applying the transformations to a series of points, then plotting the original points against the transformed points.

8 A student applies the following five transformations to a set of points. Unfortunately, after applying the transformations, the new set of points looks to be identical to the original set of points. Has the student made an error? Explain your answer.

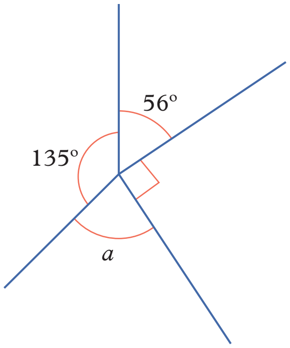
```
T3 = [[0,-1],                T1 = [[-1,0],                T2 = [[ 0,1],
      [1, 0]]                      [ 0,1]]                      [-1,0]]

T5 = [[1, 0],                T4 = [[-1, 0],
      [0,-1]]                      [ 0,-1]]
Transformations = [T1, T2, T3, T4, T5]
```

Semester 2 review

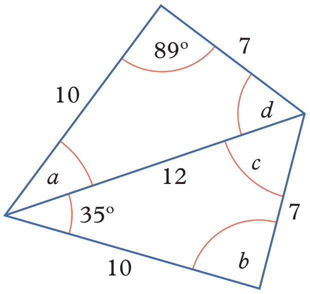
Short answer

1 Determine the value of each of the pronumerals in these diagrams. Write your answers correct to one decimal place where necessary.

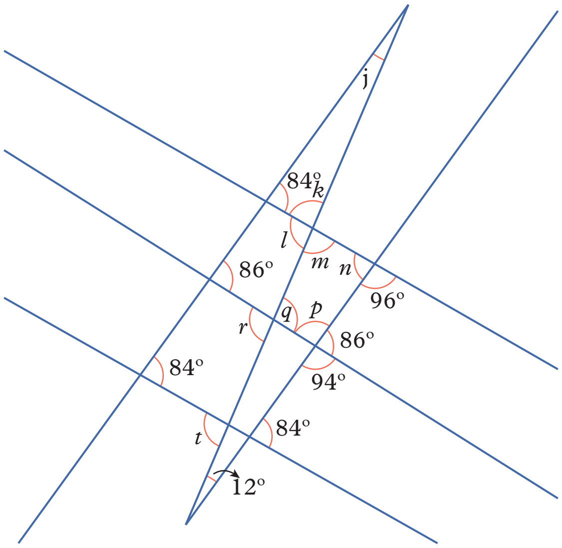
a



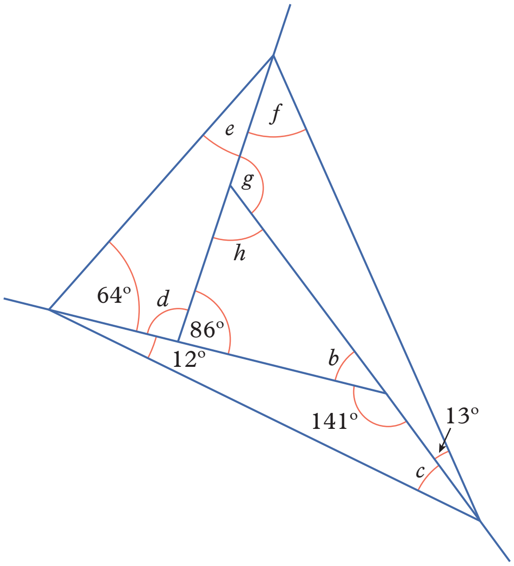
b



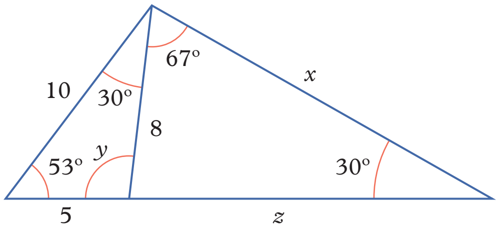
c



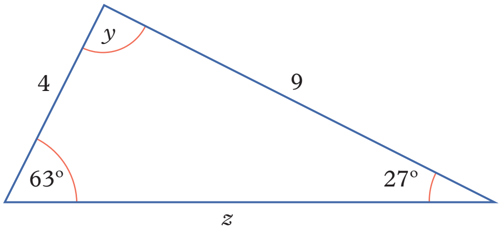
d



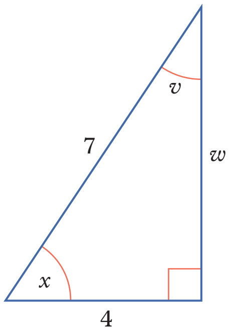
e



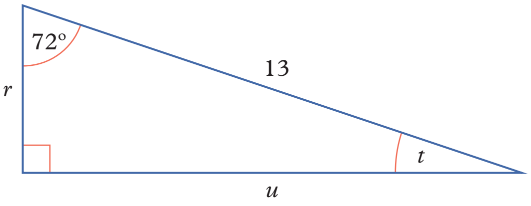
f



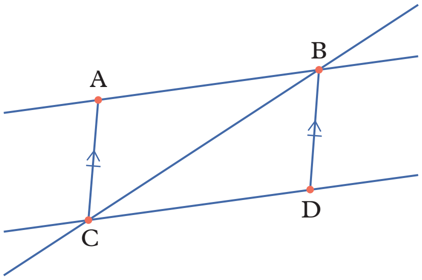
g



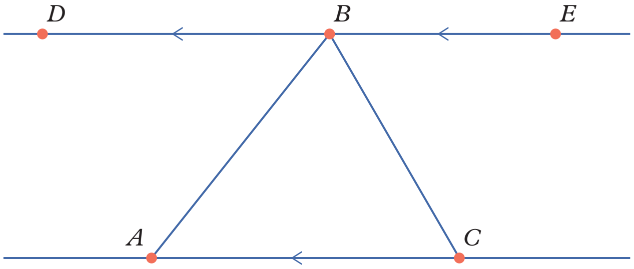
h



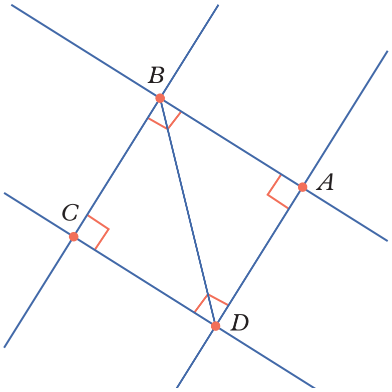
2 **a** Prove that $AB \parallel CD$ in this diagram.



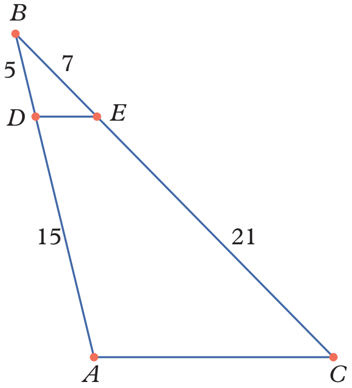
b Prove that $\angle BAC + \angle ABC + \angle BCA = 180°$:



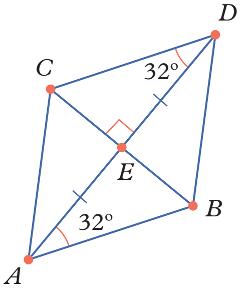
3 **a** Prove that $\Delta ABD \cong \Delta CDB$ in this diagram.



b Prove that ΔABC is similar to ΔDBE in this diagram.



c Given that E is the intersection point of CB and AD, prove that $ABDC$ is a rhombus.



4 Determine if triangles with the following side lengths are right-angled.

a 6 cm, 8 cm, 10 cm **b** 12 cm, 22 cm, 25 cm **c** 15 cm, 36 cm, 39 cm

5 **a** The observation deck of the Eureka Tower (the Eureka Skydeck) is 285 m above sea level, and the Aurora Melbourne Central skyscraper is 287 m above sea level at its highest point. If the two skyscrapers are 1.33 km apart, determine the angle of elevation from the Eureka Skydeck to the top of the Aurora skyscraper, correct to four decimal places.

b Melbourne's Parliament Station's famous 30 m long escalators were the longest single-span uninterrupted escalators in the Southern Hemisphere from January 1983 until the Forrestfield-Airport Link escalator was installed in Perth, Western Australia. The Forrestfield-Airport Link escalator is 15 m high and covers a horizontal distance of 35 m.

i Determine the angle of elevation of the Forrestfield-Airport Link escalators correct to one decimal place.

ii Determine how much longer the Forrestfield-Airport Link escalator is than the Parliament Station escalator, correct to one decimal place.

c Victoria's Twelve Apostles are giant, natural limestone structures that are approximately 45 m tall, standing in the water off the Great Ocean Road in Victoria. If you were standing at the lookout on the adjacent 70 m high cliff, how far from the nearest of the Twelve Apostles would you be (measuring horizontally) if the angle of depression from you to the top of the nearest of the Twelve Apostles was 18°? Give your answer correct to the nearest metre.

6 **a** A hiker walks 2.5 km due north and then walks 1.2 km due west of that point.

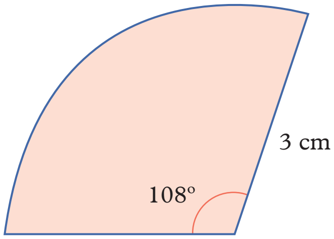
- **i** Determine the true bearing of the hiker's final position from the starting point, correct to the nearest degree.
- **ii** Determine the true bearing of the starting point from the hiker's final position, correct to the nearest degree.
- **iii** Determine the straight-line distance the hiker is from the starting point correct to one decimal place.

b A snorkeller swims, from a boat at point B, 35 m on a bearing of 120° T to point C, then continues for 20 m on a bearing of 200°T to point D.

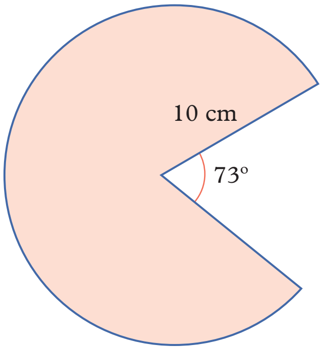
- **i** Draw a diagram showing the snorkeller's swimming path. Include all the bearings and distances provided.
- **ii** How far south of the boat is the snorkeller at point C, correct to one decimal place?
- **iii** How far east of the boat is the snorkeller at point C, correct to one decimal place?
- **iv** How far south of point C is the snorkeller at point D, correct to one decimal place?
- **v** How far west of point C is the snorkeller at point D, correct to one decimal place?
- **vi** What is the true bearing from point D on which the snorkeller must swim in a straight line to return to the boat, correct to the nearest degree?
- **vii** To return to the boat from point D, how far does the snorkeller need to swim along the bearing you wrote for part **vi**, correct to one decimal place?

7 Determine the area of each of the following shapes, correct to two decimal places.

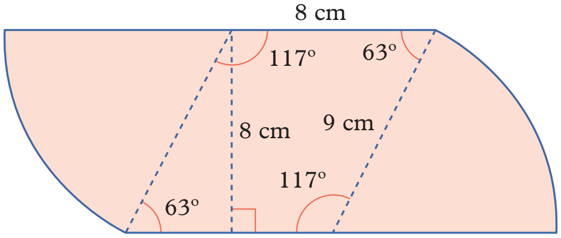
a



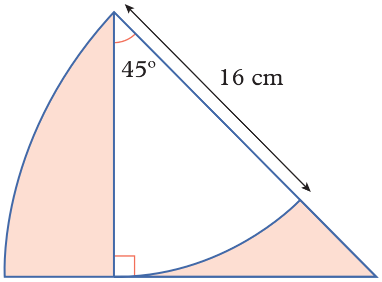
b



c



d

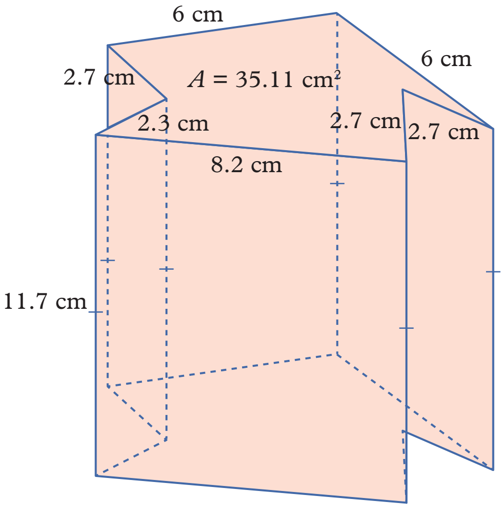


8 For each of the following objects, determine:

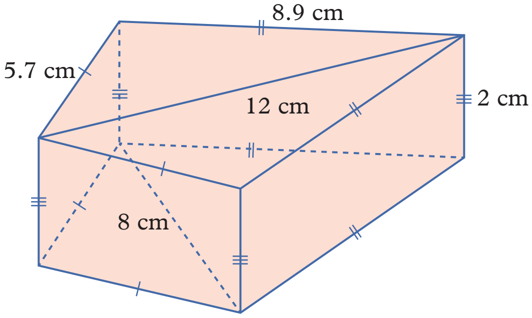
i the surface area **ii** the volume.

Write your answers correct to two decimal places where required.

a



b



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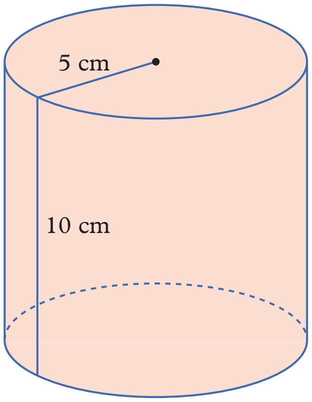
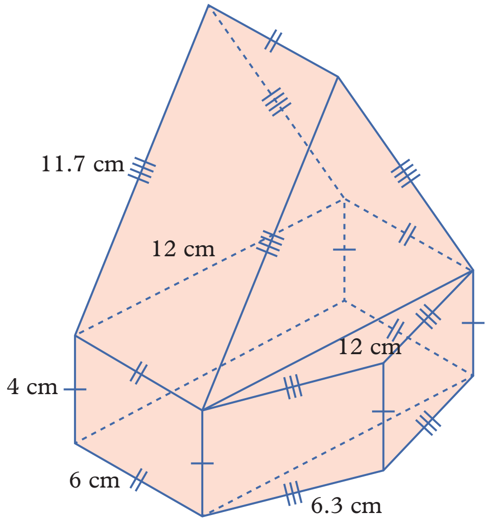


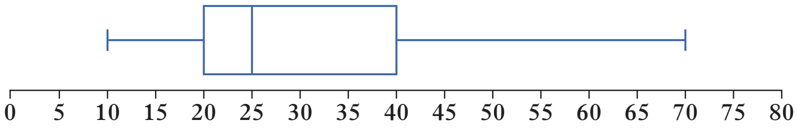
d



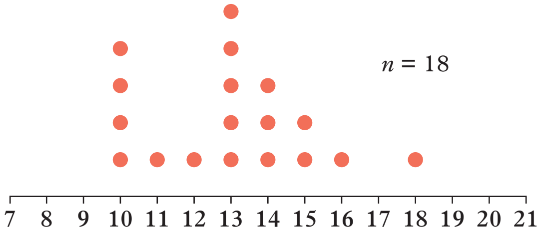
9 Determine the five-number summary and interquartile range for each of the following data sets.

a 68, 32, 62, 56, 59, 75, 38, 76, 50, 64, 48, 48, 55, 41

b



c

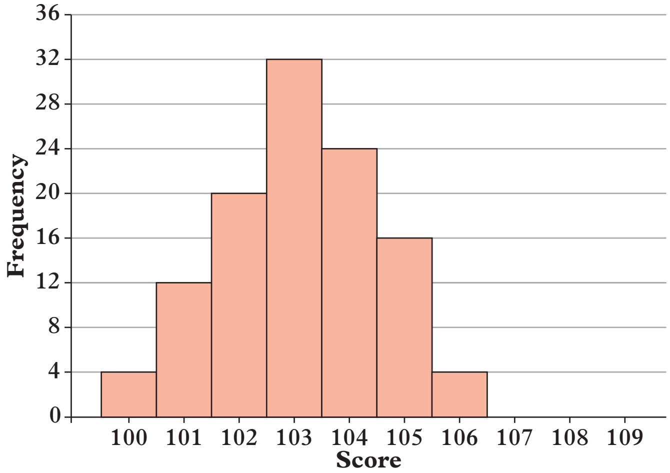


d

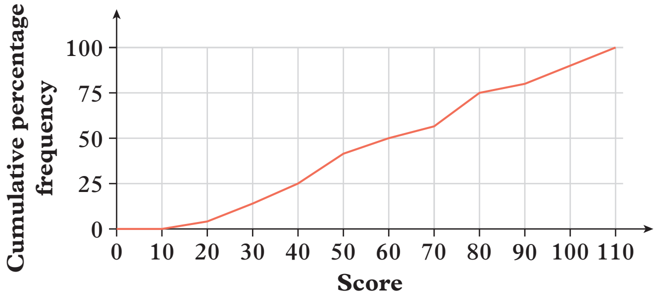
Stem	**Leaf**
1	2 8 9
2	0 1 3 4 7
3	1 2 2 2 4 5 6 7 8 9
4	0 0 0 1 2 3 4

Key: 1 | 2 = 1.2
n = 25

e



f



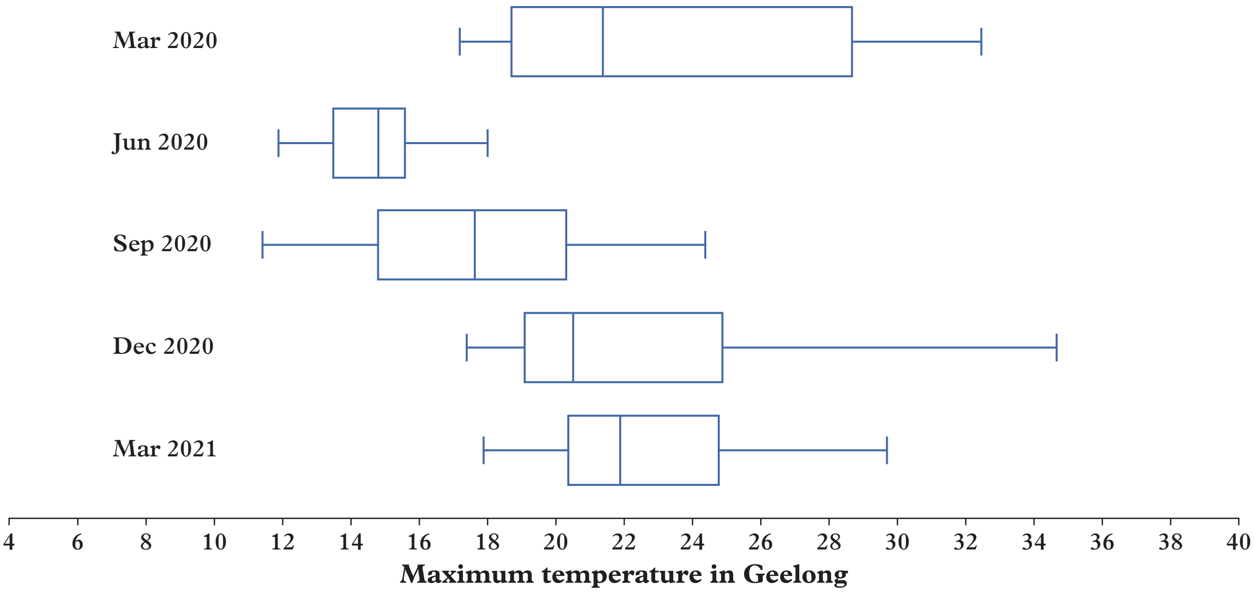
10 Construct a box plot for each of the following sets of data. Remember to check for outliers.

a 105, 110, 123, 117, 121, 120, 125, 117, 122, 113, 118, 116, 120, 125, 122, 100, 110, 112, 115, 103

b 4, 1, 2, 7, 2, 4, 6, 21, 9, 4, 1, 1, 3, 6, 3, 2, 1, 5, 8, 4, 2, 4, 2, 5, 2

c 49, 42, 44, 45, 41, 43, 44, 47, 42, 43, 46, 58, 42, 49, 20, 45, 41, 44

11 Consider the parallel box plots shown below and state whether each of the following statements is true or false.



a The maximum daily temperatures in June 2020 were all less than the lowest of the maximum daily temperatures in March 2021.

b The median maximum daily temperature in March 2020 and March 2021 are both approximately 22°C.

c The maximum daily temperatures in December 2020 were positively skewed.

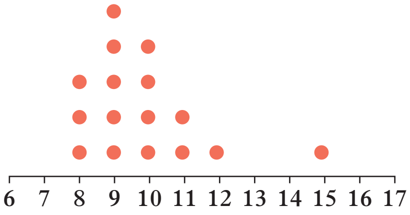
d In September 2020, approximately 50% of the days had maximum daily temperatures between 14°C and 20°C.

e In March 2020, approximately 25% of the days had maximum daily temperatures between 25°C and 30°C.

f Approximately 50% of the days in December 2020 had maximum daily temperatures greater than the median maximum daily temperature in September 2020.

12 Describe the shape of each of the following data displays.

a

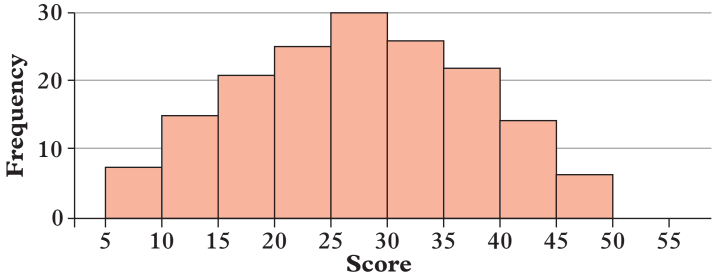


b

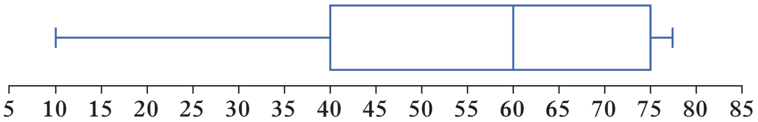
Stem	Leaf
1	9 0
2	1 2 4 6 8
3	0 1 2 3 4 5 5 6 7 8 8
4	0 0 1 1 4 7 9
5	1 2 3

Key: 1 | 2 = 1200

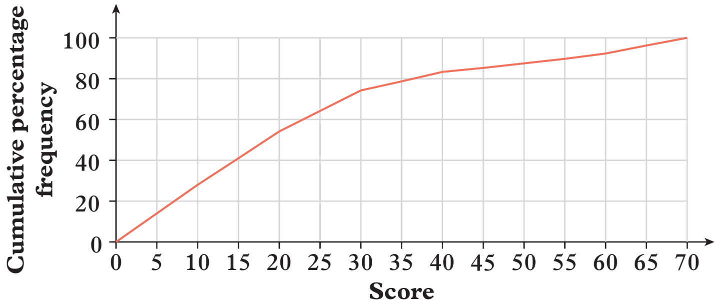
c



d

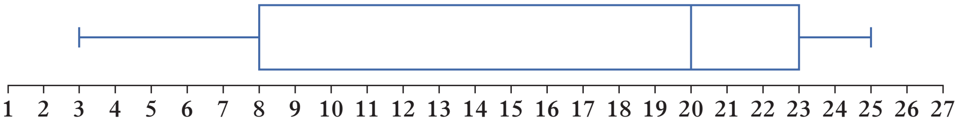


e

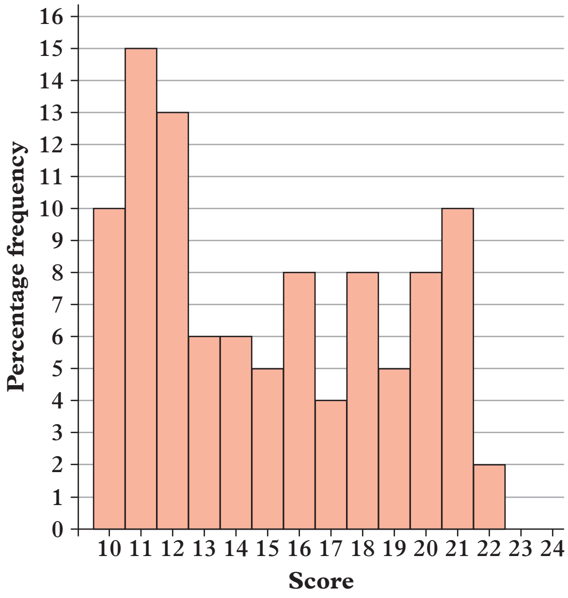


13 Construct a cumulative frequency plot (ogive) for each of the following.

a



b

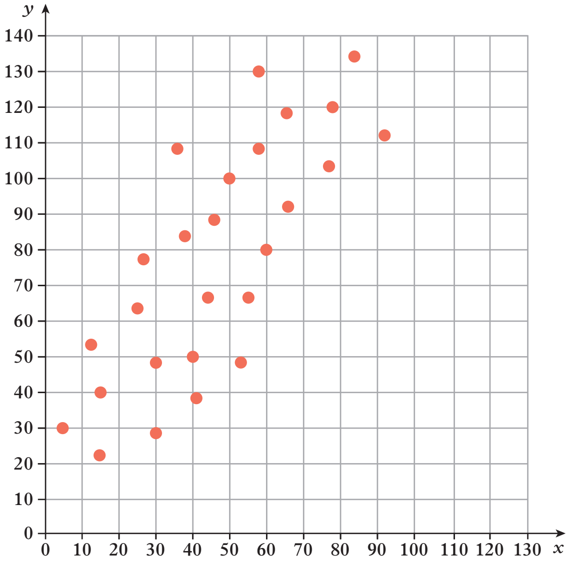


14 Construct a scatterplot for the following data set.

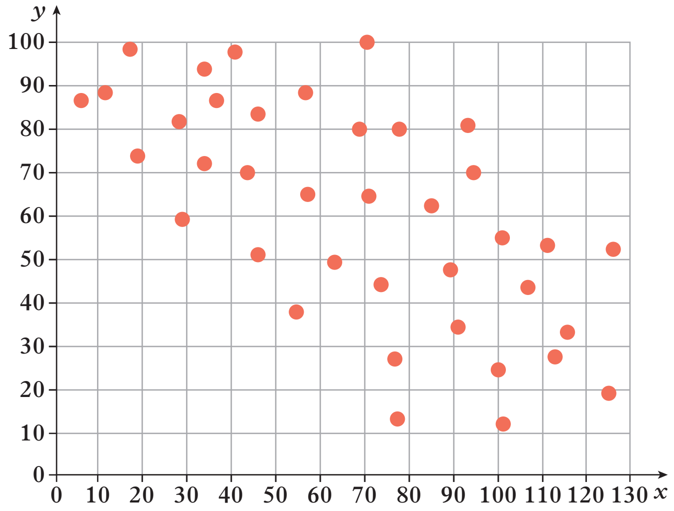
x	2	5	4	7	3	8	0	6	6	4
y	7	9	6	10	8	7	6	7	9	5

15 For each of the following scatterplots describe the correlation between the variables in terms of strength and direction.

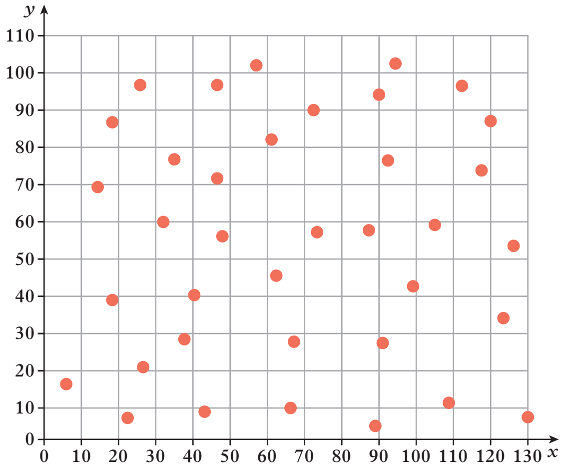
a



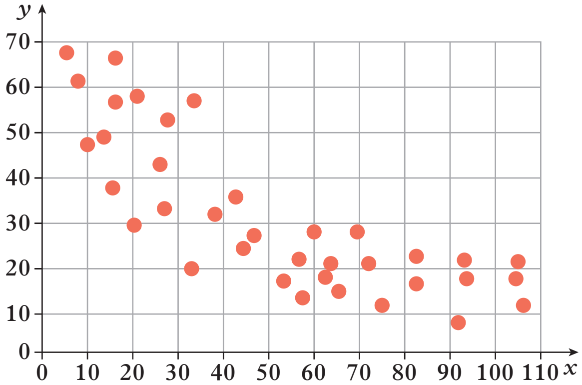
b



c



d

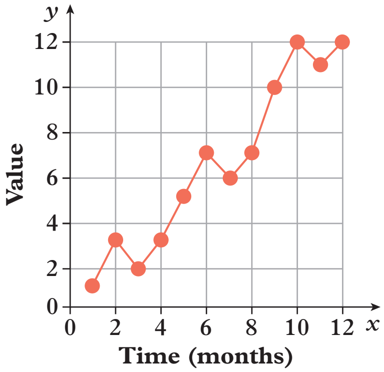


16 Construct a time series graph for the following data set.

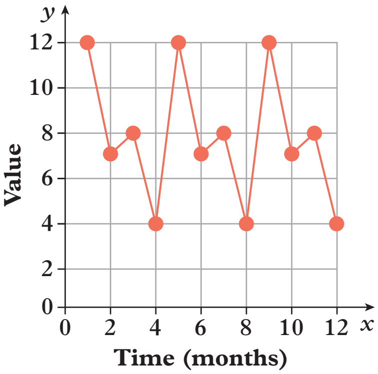
Time (months)	1	2	3	4	5	6	7	8	9	10	11	12
Value	7	4	3	8	5	4	10	7	6	12	9	8

17 Describe the trend shown in each of the following time series graphs. State if there is seasonality or irregular fluctuations.

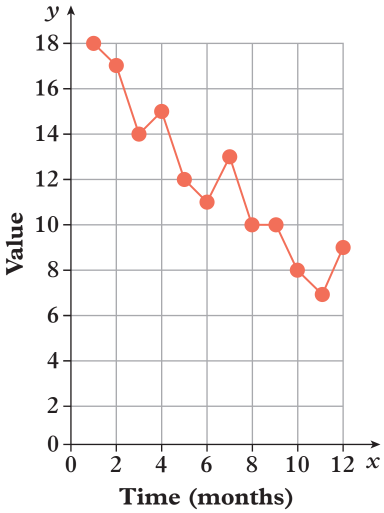
a



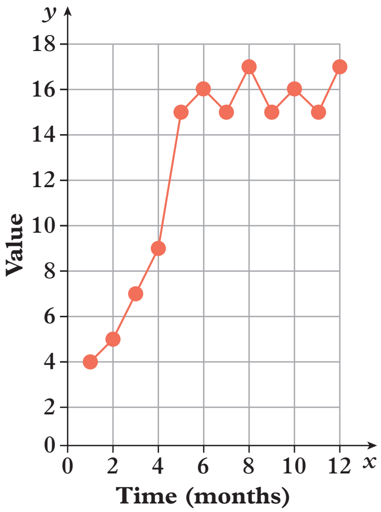
b



c



d



18 Decide whether each of these surveys will provide fair or biased results, giving reasons for your answers:

a determining the most popular video game by asking every fifth competitor of the 128 competitors in a video game tournament (in which they are all playing the same game) what their favourite game is

b asking a random sample of 50 students from a school with 2000 students whether a tie should be required to be worn with their school uniform, in order to determine student opinion about whether ties should be included in the school uniform policy or not

c asking a stratified sample (stratified by age) of 1000 people in Victoria (population 7 million) whether they think Victoria should keep Daylight Saving Time

d asking everyone in Australia what languages they speak in order to determine the most spoken language in the Southern Hemisphere.

19 Explain how each of the following headlines might be misrepresenting the given statistic.

a 'The majority of people prefer name brands to generic brands.' (51% of people surveyed said they preferred name brands to generic brands.)

b '9 out of 10 dentists recommend our toothpaste.' (10 dentists were surveyed, nine of whom are sponsored by the toothpaste company.)

20 For each of the following:

i decide whether the data collection method is fair

ii decide whether the interpretation of the data is fair

iii if appropriate, provide a suggestion to improve the survey.

a Peta wanted to know what would make more people travel on public transport. She asked every fifth person entering her local shopping centre in a five-hour period on a busy Saturday, 'What would make you more likely to take public transport?' Of the people surveyed, 60% responded that they would take public transport if it was more reliable and ran more frequently. Peta concluded that more people would take public transport if it was more reliable and ran more frequently.

b Rob wanted to know the average height of professional basketball players. He researched the heights of the players of all 30 teams in the NBA and calculated an average height of 2.01 m. Rob concluded that the average height of a basketball player is 2.01 m.

21 Simone was organising her book collection, which contained 65 books. She found there were 40 fiction books, and 24 books that were part of a series of which 14 were fiction.

a Construct a two-way table to represent this information.

b Convert the two-way table into a Venn diagram.

c Determine the probability of each of the following:

i Simone randomly selects a fiction book that is not part of a series.

ii Simone randomly selects a non-fiction book that is part of a series.

iii Given that Simone has selected a fiction book, it is part of a series.

iv Given that Simone has selected a book that is not part of a series, it is non-fiction.

22 A printing company is hired by two other companies to print trading cards. Company A orders twice as many cards to be printed as Company B. Due to differences in the trading card designs, there is an 8% chance that any one card for company A is misprinted compared to the 4% chance of a misprinted card for company B. A printed trading card is selected at random from a recent print run.

a Construct a tree diagram to represent the situation.

b Determine the probability of each of the following:

i The card is for company A and is misprinted.

ii The card is for company B and is not misprinted.

iii Given that the card is for company A, it is misprinted.

iv Given that the card is misprinted, it is for company A.

23 While driving, Fergus is listening to his playlist. It contains 52 tracks and Fergus knows that three songs will play at random without repeats before he reaches his destination. The playlist contains songs from four albums:

- Album A has 13 tracks.
- Album B has 11 tracks.
- Album C has 12 tracks.
- Album D has 16 tracks.

Fergus is thinking about the probability of tracks from album A playing or not playing.

a Construct a tree diagram to represent the situation.

b Determine the probability of each of the following:

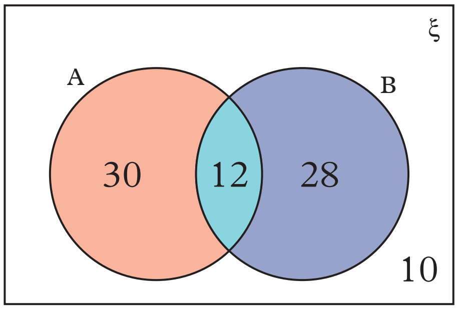
i All three songs that play are from album A.

ii Two songs that play are from album A.

iii Given that the first song that plays is from album A, the next two songs are not from album A.

iv Given that the first two songs that play are not from album A, the third song is from album A.

24 Consider the Venn diagram shown on the right. Determine the probability of:



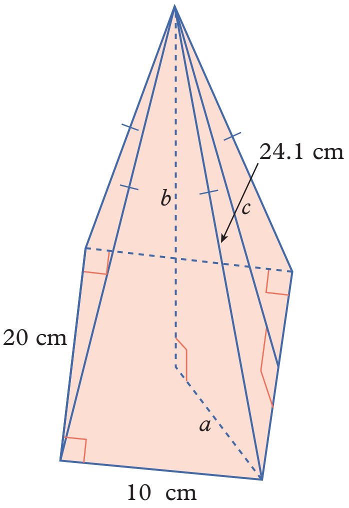
- **a** Pr(A)
- **b** Pr(B′)
- **c** Pr(A ∩ B)
- **d** Pr(A ∪ B′)
- **e** Pr(A | B)
- **f** Pr(B | A′)

25 Determine if events A and B in each of the following situations are independent or dependent.

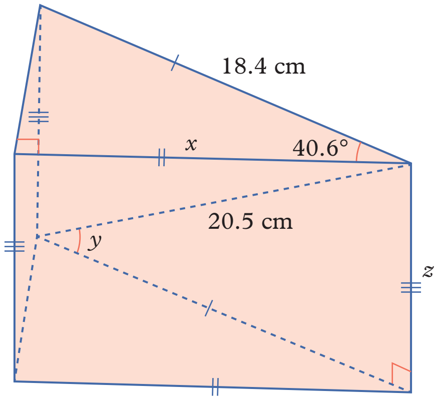
- **a** $\Pr(A) = 0.16$ $\Pr(B) = 0.25$ $\Pr(A \cap B) = 0.4$
- **b** $\Pr(A) = \frac{2}{3}$ $\Pr(B) = \frac{5}{6}$ $\Pr(A \cap B) = \frac{5}{9}$
- **c** $\Pr(A) = \frac{7}{15}$ $\Pr(A|B) = \frac{7}{15}$ $\Pr(A \cap B) = \frac{1}{3}$
- **d** $\Pr(B) = 0.625$ $\Pr(B|A) = 0.35$ $\Pr(A \cap B) = 0.35$

10A **26** Determine the value of the each of the pronumerals in these diagrams. Write your answers correct to two decimal places where required.

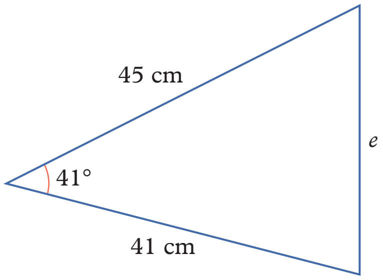
a



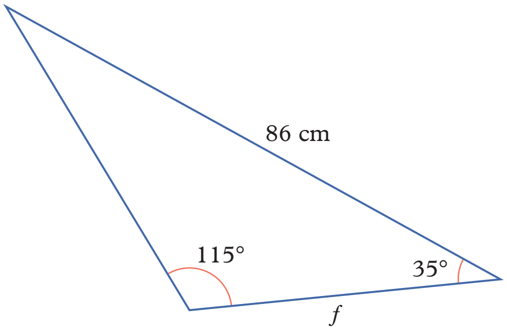
b



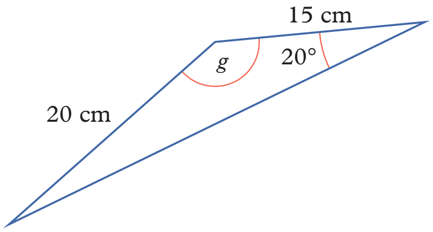
c



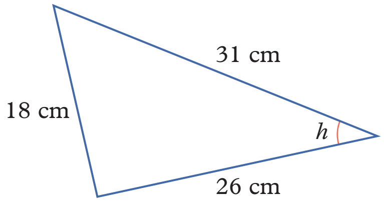
d



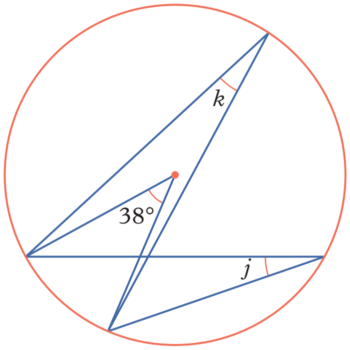
e



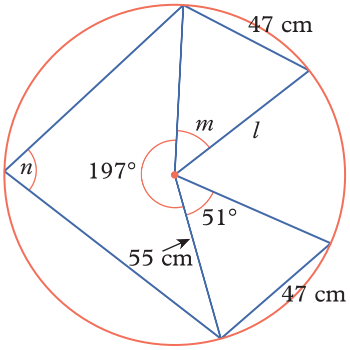
f



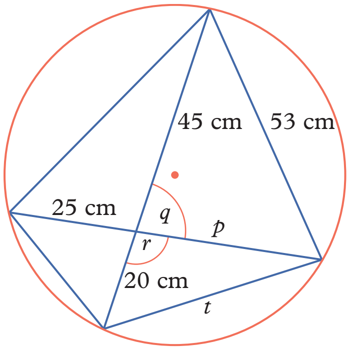
g



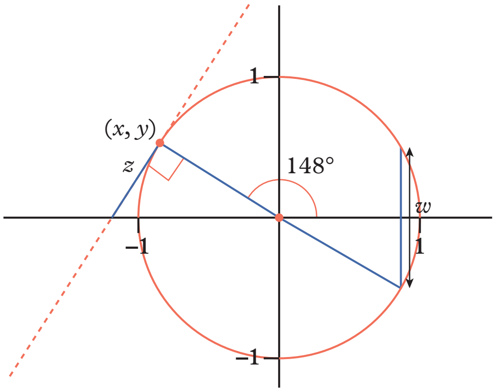
h



i

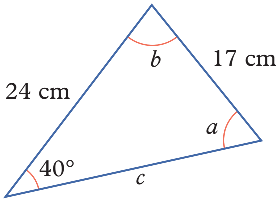


j



10A **27** Determine all possible values for the pronumerals in each of the following diagrams.

a



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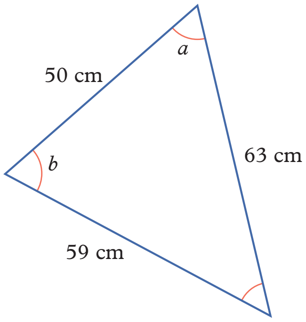
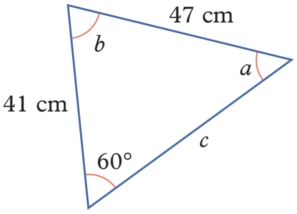
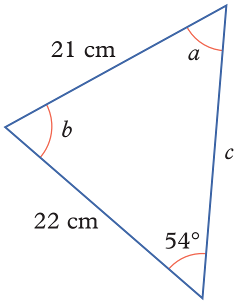


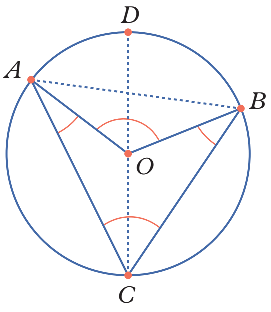
c



d

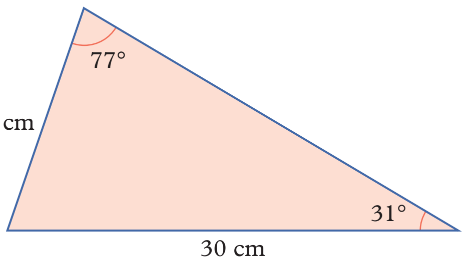


10A **28** Given that CD is a diameter of the circle on the right and O is the centre point, prove that $\angle AOB = 2\angle ACB$ (without using any circle theorems).

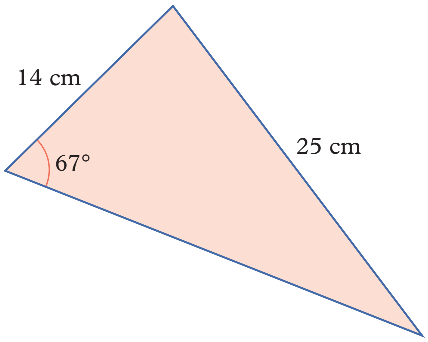


10A **29** Determine the area of each of the following. Write your answers correct to two decimal places.

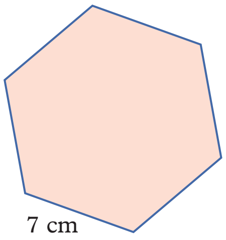
a



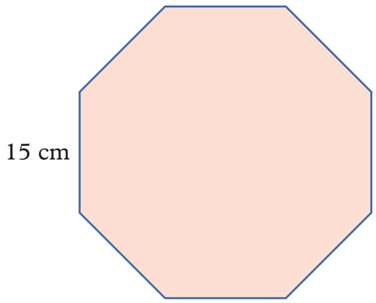
b



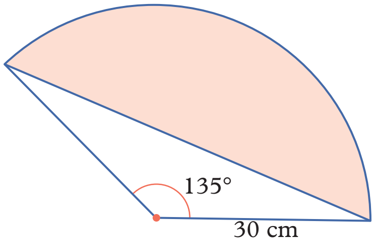
c



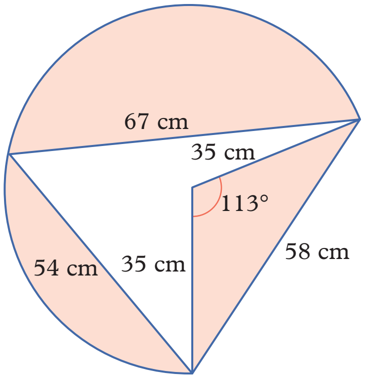
d



e



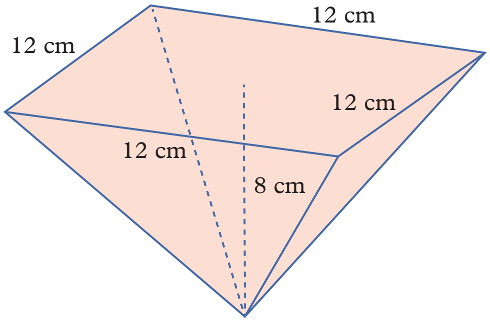
f



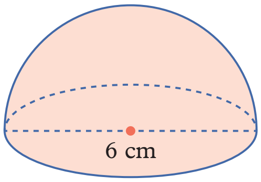
10A **30** For each of the following, determine:

i the surface area **ii** the volume.

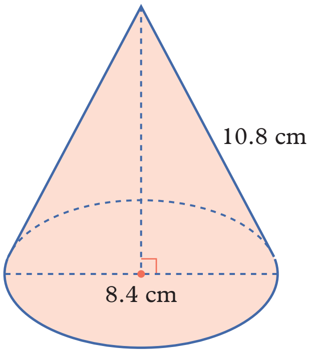
a



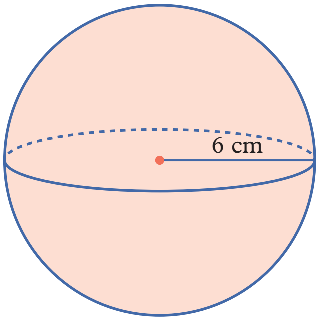
b



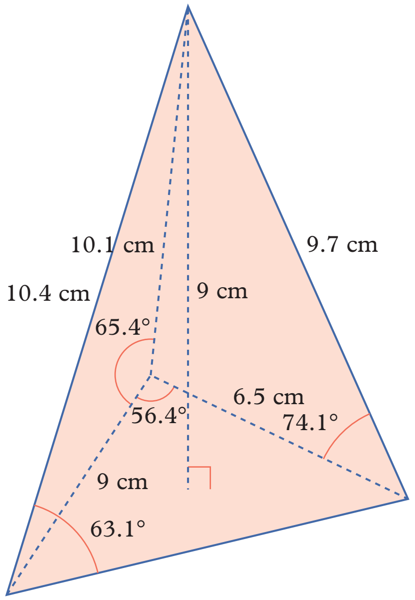
c



d



e



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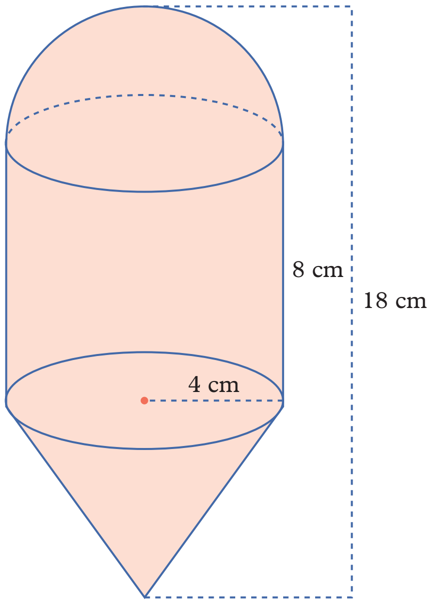


10A **31** Consider the angle measurements below. Convert those expressed in degrees to radians and those expressed in radians to degrees.

a 72° **b** $\frac{3\pi}{4}$ **c** 225° **d** $\frac{7\pi}{18}$ **e** 330° **f** $\frac{13\pi}{12}$

10A **32** Evaluate each of the following by first writing it in terms of a positive acute angle (the reference angle). Write your answer as an exact value.

a $\sin(120°)$ **b** $\cos\left(\frac{7\pi}{6}\right)$ **c** $\tan(315°)$ **d** $\sin\left(\frac{3\pi}{4}\right)$ **e** $\cos(300°)$ **f** $\tan\left(\frac{11\pi}{6}\right)$

10A **33** Solve each of the following equations.

a $\sin(x) = -\frac{1}{2}, 0° \le x \le 360°$

b $\cos(x) = \frac{\sqrt{2}}{2}, 0 \le x \le 2\pi$

c $\tan(x) = \sqrt{3}, 0° \le x \le 360°$

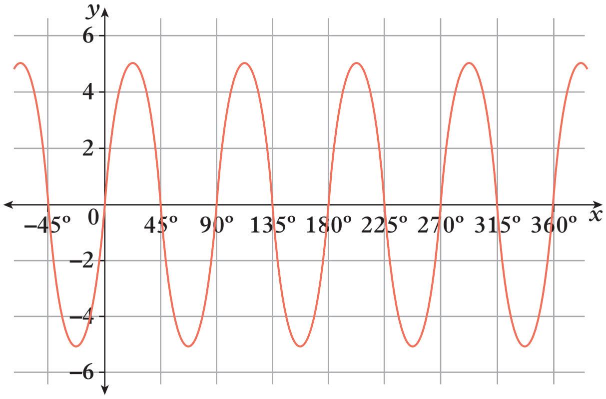
d $\sin(x) = -1, 0 \le x \le 2\pi$

e $\cos(x) = 1, 0° \le x \le 360°$

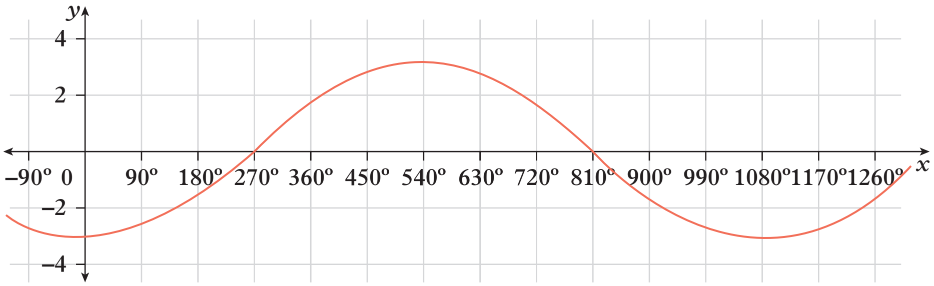
f $\tan(x) = -1, 0 \le x \le 2\pi$

10A **34** State the amplitude and period of each of the following graphs.

a



b



10A **35** Calculate the mean ($\bar{x}$), sample standard deviation (s), and population standard deviation (σ) for each of the following data sets by hand. Then use a calculator to write your answers correct to two decimal places.

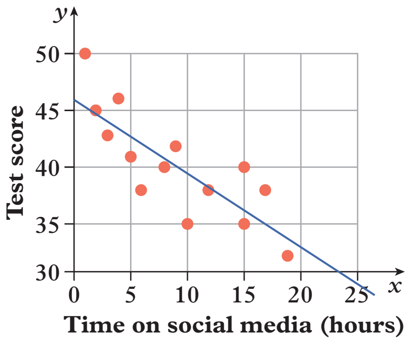
a 32, 36, 40, 43

b 20, 18, 28, 21, 24

c 4, 4, 4, 4, 4, 4, 4, 4, 4, 4

10A **36** The test scores out of 50 of a group of students have been plotted against the time, in hours, those students spent on social media during the two days prior to the test.

a Find the equation of the line of best fit shown below.



b Use your equation from part **a** to predict, correct to one decimal place:

- **i** a student's test score if the student spent 20 hours on social media in the two days before the test
- **ii** a student's test score if the time the student spent on social media is 11 hours
- **iii** the time spent on social media if the student's test score is 44
- **iv** the time spent on social media if the student's test score is 30.

c State the type of prediction (extrapolation or interpolation) used for each answer in part **b** and whether that type of prediction is reliable or not.

Analysis

1 Erica is planning a hike through a local national park. She plans to:

- drive to the carpark, at point P
- then walk on a bearing of 036.9°T for 10 km towards the lower bridge, at point L
- then walk on a bearing of 108.4°T for 6.3 km towards the upper bridge, at point U
- then walk on a bearing of 233.1°T for 10 km towards the new bridge, at point N.

From the new bridge there is then a 4 km track due west back to the carpark.

a Draw a diagram to show Erica's hike plan.

b Determine the length of the planned hike.

c What is the average minimum speed Erica must walk to make it back to her car in 6 hours?

d Erica reaches the upper bridge and sees a warning sign that says it is a 50 m drop to the ground below. From the upper bridge, Erica can see a kangaroo on the ground at an angle of depression of 12°. If Erica's eye height is 172 cm, determine the horizontal distance to the kangaroo, correct to the nearest metre.

e For emergency purposes, the park has a straight-line emergency route to point U from the carpark. Determine the straight-line distance from point P to point U, correct to two decimal places.

f Determine the true bearing of point U from point P, correct to the nearest degree.

g The park has multiple tracks that can be taken from the carpark. The tracks vary in difficulty. Hikers often mark which way they went on a signpost and this has allowed the park rangers to determine the following:

- From point P, 20% of hikers walk to point R and 80% walk to point L.
- From point L, 5% of hikers walk to point U and 95% walk to point A.
- From point R, 10% of hikers walk to point U and 90% walk to point A.
- They all then follow tracks that lead to point N.

Determine the probability that:

i a randomly selected hiker took the path through points P, R, U and N
ii a randomly selected hiker took the path through points P, L, A and N
iii a randomly selected hiker took the path through points P, R, A and N
iv given that they walked to point U, a randomly selected hiker took the path through points P, L, U and N.

10A **h** Determine the area of the region Erica hiked around, correct to two decimal places.

2 A teacher decided to retest the class two weeks after a recent test to see how much the students were able to remember. The scores the 24 students scored on the two tests are given in the table on the right.

The teacher hoped to be able to predict the retest scores from the test scores.

Test score (%)	95	90	90	85	85	80	80	80	75	75	75	70	70	70	65	65	65	60	60	60	55	50	40
Retest score (%)	80	85	70	50	65	65	50	90	65	60	40	80	60	75	60	65	35	35	70	50	50	30	30

a State the independent variable.

b Construct a scatterplot of the scores from the retest (vertical axis) against the scores from the original test (horizontal axis).

c Describe the correlation between the scores for the retest and the scores for the original test.

10A **d** Add a line of best fit to your scatterplot from part **b**.

e For your scatterplot, determine the equation of the line of best fit. Write the coefficient correct to two decimal places.

10A **f** **i** If a student scored 70% for the original test, use the equation from your answer to part **e** to predict the student's score for the retest correct to 2 decimal places.

ii State whether your prediction was made using interpolation or extrapolation and whether or not it is a reliable prediction.

g The teacher claims that students who scored 75% or more originally did better on the retest than students who scored less than 75%. Evaluate the teacher's claim.

3 A special Ferris wheel with enclosed, air-tight carriages has been designed so that half of the wheel is submerged underwater. Passengers board the carriage at sea level, the Ferris wheel rotates upwards to its highest point, then back to sea level, then underwater to the deepest point, then back to sea level. A full rotation of the Ferris wheel takes one hour to complete. The deepest point the Ferris wheel can reach is 20 m.

10A **a** State the amplitude and period of the Ferris wheel's height in relation to time.

10A **b** Write the equation of the height, h m, of the carriage t minutes after boarding using an appropriate function.

10A **c** Sketch the graph of the height of the carriage against the time it takes the Ferris wheel to complete one full rotation.

10A **d** Determine the height the carriage is above sea level after 20 minutes:

i as an exact value **ii** correct to two decimal places.

10A **e** Determine, as an exact value, the depth the carriage is below sea level after 35 minutes.

10A **f** Determine the time it takes after boarding for the carriage to be $10\sqrt{2}$ m below sea level.

10A **g** People on a nearby boat watch the Ferris wheel go around. Describe the vertical and horizontal location of a carriage relative to the centre of the Ferris wheel 22.5 minutes after it was boarded:

i using exact values **ii** correct to two decimal places.

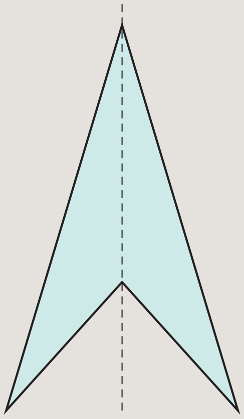
10A **h** Determine the distance the carriage has travelled during the time described in part **g**:

i as an exact value **ii** correct to two decimal places.

EXPLORATIONS 2

1 Thorn polygons

A *thorn polygon* is a shape that has at least one axis of symmetry and the maximum possible number of interior reflex angles. The symmetrical dart shown is an example of a thorn quadrilateral.



a How many reflex angles can a triangle have? What is another name for a thorn triangle?

b There are two different types of thorn pentagon: one with the reflex angles adjacent and one with them separated. Draw an example of each type.

c Explain why every thorn heptagon has at least one pair of adjacent reflex angles.

d Explain why no thorn octagon can have two obtuse angles.

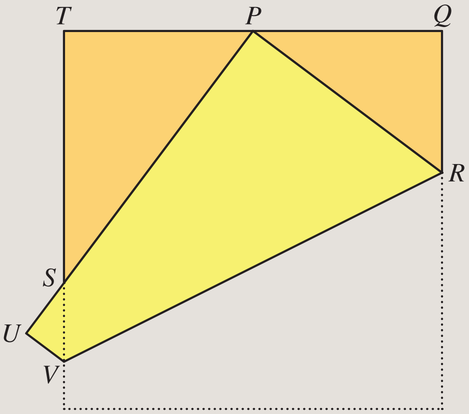
e Show that an n-sided thorn polygon has exactly $n - 3$ reflex angles. [Hint: This requires two steps: 1) show that a symmetrical n-sided polygon with $n - 3$ reflex angles always exists, and 2) show that more than $n - 3$ reflex angles is impossible.]

f Find the maximum number of sides in a thorn polygon in which no three consecutive angles are all reflex angles.

g Show that, for every $n \geq 3$, there exists an n-sided thorn polygon that can be bisected along its line of symmetry to form two congruent thorn polygons.

2 Know when to fold 'em

A square piece of origami paper has side length 1 unit. The bottom-right corner is folded to a point P on the top edge, as shown, creating triangles PQR, PTS and SUV.



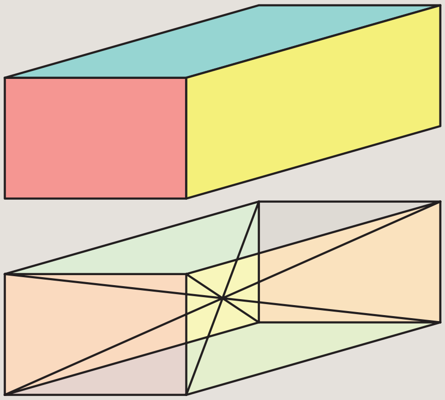
a Suppose P is at the midpoint of the top edge of the square.

- **i** Set up an equation to find the length of QR.
- **ii** Use similarity to find the side lengths of triangle SUV.
- **iii** Calculate the area of the folded yellow region of the paper.

b Find all possibilities for the length of PQ given that triangle SUV has side lengths in the ratio 7 : 24 : 25.

c Show that, for any Pythagorean triple (a, b, c) with $a^2 + b^2 = c^2$, there are two different positions for P such that triangle SUV has sides in the ratio $a : b : c$. What is the second position for the ratio in part **a**?

3 Six pyramids



a A 4 cm × 6 cm × 12 cm rectangular prism is divided into six right rectangular-based pyramids, as shown. Each face of the prism is the base of one pyramid and the centre of the prism is the apex of each pyramid.

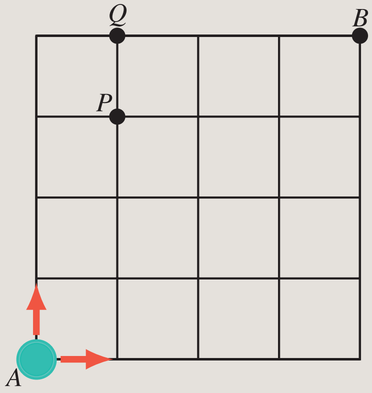
- **i** Find the volume of each pyramid.
- **ii** For each pyramid, find the sum of the lengths of its eight edges.
- **iii** Find the surface area of each pyramid, to 2 decimal places.

b Given an 85 cm × 132 cm × 720 cm rectangular prism, show that its six pyramids each have integer surface area, in square centimetres.

c Let (a, b, c) be any Pythagorean triple satisfying $a^2 + b^2 = c^2$ and define $u = 2a(4b^2 - c^2)$, $v = 2b(4a^2 - c^2)$ and $w = 8abc$.

Show that the six pyramids of a u cm × v cm × w cm rectangular prism each have integer surface area, in square centimetres.

4 Chance encounters



In the 4 × 4 grid shown, a counter is placed at *A*. Each second, the counter is moved along a grid line to an adjacent grid point, always increasing its distance from *A*. Where there are two possible moves, they are equally likely. After 8 seconds, the counter reaches point *B*.

a Explain why there is a one-in-four chance that the counter starting at *A* will be at *P* after 4 seconds.

b What is the probability that the counter starting at *A* will be at *Q* after 5 seconds?

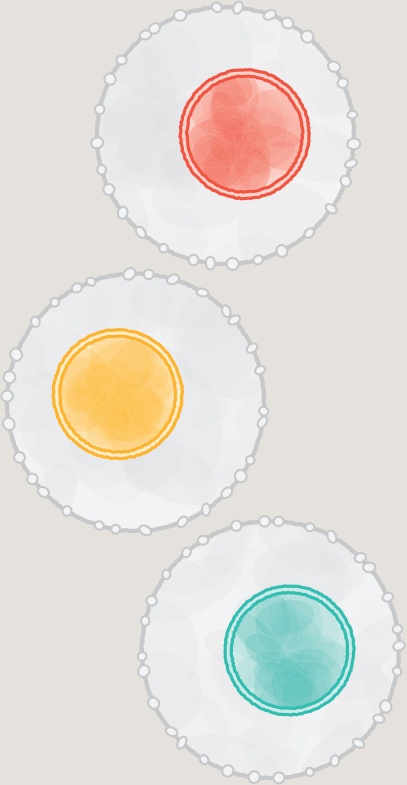
A second counter is placed at *B*. Each second, it is moved along a grid line, always increasing its distance from *B* and with each move equally likely. The two counters start moving from *A* and *B* at the same time.

c Show on a diagram all the grid points at which the counters can meet. Explain why there are no others.

d What is the probability that the counters meet?

e The game is changed so that each counter has probability $\frac{1}{3}$ of moving vertically and probability $\frac{2}{3}$ of moving horizontally. Is it more or less likely that the counters will meet?

5 Algal bloom



Dr Al Goldblum discovered that a certain species of poisonous singled-celled algae has three different forms, called kirons, lirons and mirons. A kiron lives for one week and then changes into a liron. A liron also lives for one week and then changes into a miron. One week later, a miron divides into one kiron and one liron.

A spreadsheet is a useful way to track the numbers of kirons, lirons, mirons and total algae over time. Dr Al always takes measurements at the same time of the week: 12 pm on Wednesday.

a In one experiment, Dr Al started in week 1 with a single kiron in a flask of water.

  i What will be the total number of algae in the flask in week 10?

  ii In which week will the total number of algae first exceed 100?

b Show that, from week 6 onwards, the number of kirons is equal to the sum of the numbers of kirons one week earlier and five weeks earlier. Is this also true for lirons and mirons?

c In another experiment, Dr Al started in week 1 with an equal number of kirons and mirons, but no lirons. Some weeks later, there were exactly 100 algae in total. What is the smallest possible number of algae at the beginning of the experiment?

d Dr Al's goal is to develop a herbicide to kill the poisonous algae. Batch X31 destroys all lirons within one day, but it does not affect the other two forms. It remains active for several weeks.

  i On Wednesday in week 11, the X31 herbicide is added to the flask that started with a single kiron in week 1. During which week will all forms of algae be completely eradicated?

  ii If X31 is added to a lake infested with all three forms, what is the maximum number of days it will take to eradicate the algae?

Explorations inspired by the Australian Maths Trust's competitions and programs: www.amt.edu.au

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[STEAM project 1]

How can we use technology so that we improve the lives of people in the world's poorest nations?

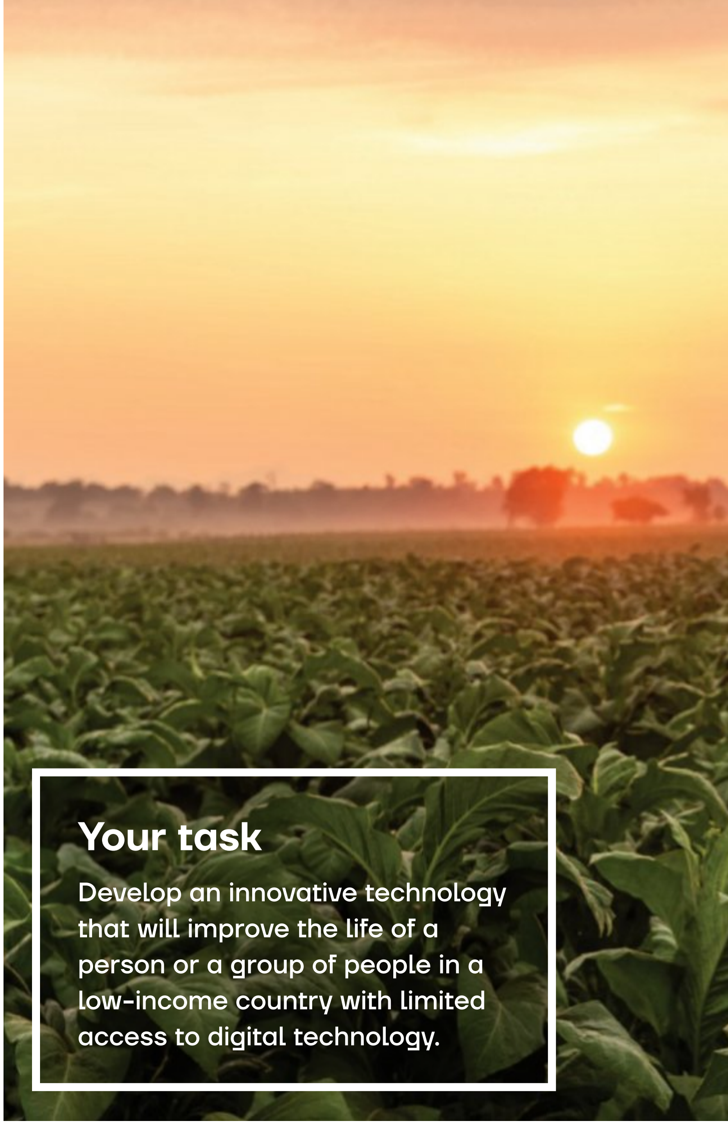
In Australia we are surrounded by technology every day. It is in the phones we use, the televisions we watch and the cars we drive. The term 'technology' is used for any machinery or equipment that applies the scientific knowledge we have discovered. Wheels and computers are both examples of technology.

In high-income countries, emergency response teams often rely on technological data supplied by electronic sensors to respond to natural disasters such as storms, fires and plagues. Drones might be used to conduct search and rescue operations. Doctors can use technology to remotely diagnose people who are sick and to perform operations that save lives.

At the end of 2017, the number of high-speed mobile subscriptions in member countries of the Organisation for Economic Co-operation and Development (OECD) reached a milestone: more subscriptions than the number of people. These mobile phones have been used to alert people to natural disasters, or to call for help in the event of floods and fires.

Technology is not just used for communication during natural disasters. It is also used to create medicine, improve farming practices and for education.

However, not everyone has access to technology.



Your task

Develop an innovative technology that will improve the life of a person or a group of people in a low-income country with limited access to digital technology.

Figure 1 Technology can be used to enhance and improve agricultural practices.

The digital divide

The term 'digital divide' is used to describe the gap between those who have access to digital technology – such as mobiles, computers and the internet – and those who do not. The Australian Bureau of Statistics has identified that almost 2.6 million Australians do not use the internet and cannot access technology in an emergency. Access is even lower in lower-income countries throughout Africa and Asia.

The OECD has identified that targeted innovation that uses technology can boost productivity, increase economic growth and help solve problems in society.

Figure 2 Technology has made attending a doctor's appointment easier and more accessible for people who may have difficulty attending in person.

HUMANITIES

In Geography this year, you will be learning about the spatial variations in human wellbeing globally. You will need to explore a range of factors that lead to inequalities, such as social, political, economic and technological differences. In Economics and Business, you will develop skills in business reasoning, and examine the trade-offs required when innovating a product or service and deciding on a course of action.

To complete this task successfully, you will need to research the initiatives of international governments and non-government organisations (NGOs) aimed at improving human wellbeing, particularly regarding health. You should consider how technology could be effectively accessed, resourced and used by a group of people to address their health concerns.

You will find more information on this in Chapter 5 'Inequalities in wellbeing' and Chapter 20 'The business environment' of *Oxford Humanities 10 Victorian Curriculum*.

MATHS

In Maths this year, you will extend your skills in representing, comparing and interpreting data. You will use digital technology to work with data, but also perform calculations by hand.

To complete this task successfully, you will need to find data to quantify the problem, to cost your interventions and to calculate a quantitative, evidence-based estimate of the likely benefits of your interventions. You will need to use skills in performing proportionality and other calculations with very large numbers, using scientific notation. You may also require your knowledge of financial mathematics.

You will find relevant mathematical and statistical concepts in Chapter 1 'Financial mathematics' and Chapter 10 'Statistics' of *Oxford Maths 10/10A Victorian Curriculum*.

SCIENCE

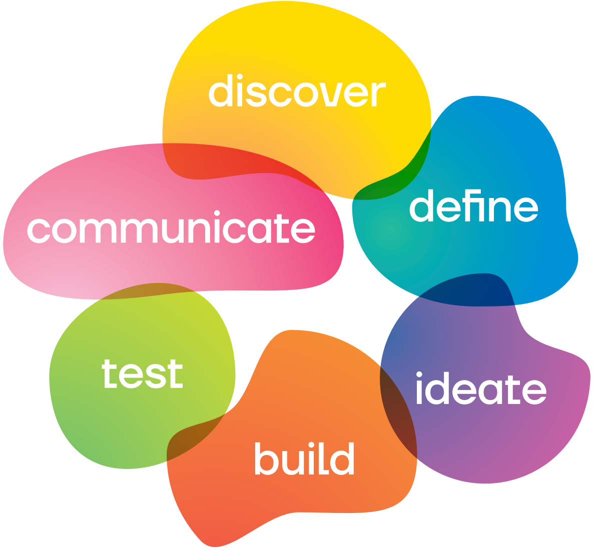
In Science this year, you will learn how an understanding of evolution can contribute to the selection of desired traits (such as drought resistance) in plants and animals. You will also learn how genetic engineering can be used to develop medicines that will cure cancers and prevent disease.

To complete this task successfully, you will need to consider how the values and needs of different societies can influence the focus of scientific research. You will also need to consider the ethics of the technology that you will be offering to your selected individual or group of people.

You will find more information on this in Chapter 2 'Genetics' and Chapter 3 'Evolution' of *Oxford Science 10 Victorian Curriculum*.

The design cycle

To successfully complete this task, you will need to complete each of the phases of the design cycle.



Discover

When designing solutions to a problem, you need to know who you are helping and what they need. The people you are helping, who will use your design, are called your end-users.

Consider the following questions to help you empathise with your end-users:

- Who am I designing for?
- What problems are they facing? Why are they facing them?
- What do they need? What do they not need?
- What does it feel like to face these problems? What words would you use to describe these feelings?

To answer these questions, you may need to investigate using different resources, or even conduct interviews or surveys.

Define

Before you start to design your innovative technology, you need to define the parameters you are working towards.

Define your version of the problem

Rewrite the problem so that you describe the group you are helping, the problem they are experiencing and the reason it is important to solve it. Use the following phrase as a guide.

'How can we help (the group) to solve (the problem) so that (the reason)?'

Determine the criteria

1 Describe the limitations in energy, communications, transport and support personnel that will need to be considered as part of the solution.
2 Describe how many copies of the solution prototype will need to be made to make a difference in the country you have chosen.
3 Identify who could pay for the construction of the solution prototypes.
4 Describe the social culture that is experienced by the individuals and groups who are affected by the problem. Why might some technologies be viewed as unwanted or even dangerous?

Ideate

Once you know who you're designing for, and you know what the criteria are, it's time to get creative!

Outline the criteria or requirements your technological design must fulfil (i.e. cost, size and weight for transportation, and cultural appropriateness).

Brainstorm at least one idea per person that fulfils the criteria.

Remember that there are no bad ideas at this stage. One silly thought could lead to a genius innovation!

Build

Each group member should select one design to draw. Label each part of the design. Include the material that will be used for its construction.

Include in the individual designs:

- a a detailed diagram of the design
- b a description of how it will change the life of your selected individual or group
- c an outline of any similar designs that are already available to buy
- d an outline of why your idea or design is better than others that are already available.

Present your design to your group.

Build the prototype

Choose one solution and build two or three prototypes. The prototype may be full size, or it may be a scale model (10 cm = 1 m).

Use the following questions as a guideline for your prototype solution.

- What materials or technology will you need to build or represent your prototype solution?
- What skills will you need to construct your prototype design? Does your group have these skills, or who can teach you those skills?
- How will you make sure your prototype design is able to be used by your selected individual or group? Will they need training?
- How will you display or describe the way the prototype design will work?

Test

Prototype 1

Use the scientific method to design an experiment that will test the effectiveness and strength of your first prototype solution. You will test the prototype more than once to compare results, so you will need to control your variables between tests.

What criteria will you use to determine the success of your prototype? Conduct your tests and record your results in an appropriate table.

Prototype 2

If your prototype will be used to help an individual, then you will need to generate a survey to test whether the prototype is appropriate for their use. (How would they use it? Would it make their work easier or harder? Would they consider buying it?)

Prototype 3

Use the information you have obtained from testing the first two versions to adapt your last prototype to be more effective and usable for the group you are helping. You may want to use the first two prototypes to demonstrate how the design has been improved over time.

Communicate

Present your design to the class as though you are trying to get your peers to invest in your designed solution.

In your presentation, you will need to:

- outline the situation of the country in which your selected individual or group lives
- outline the challenges faced by your selected individual or group
- include a working model or a detailed series of diagrams with a description of how the prototype of the solution will be used
- include a description of how you changed your design prototype as a result of testing or feedback
- include a description of how the design prototype will improve the lives of your selected individual or group.

Check your Student obook pro for the following digital resources to help you with this STEAM project: pro

Student guidebook
This helpful booklet will guide you step-by-step through the project.

What is the design cycle?
This video will help you to better understand each phase in the design cycle.

How to manage a project
This 'how-to' video will help you to manage your time throughout the design cycle.

How to pitch your idea
This 'how-to' video will help you with the 'Communicate' phase of your project.

Check your Teacher obook pro for these digital resources and more: pro

Implementation advice
Find curriculum links and advice for this project.

Assessment resources
Find information about assessment for this project.

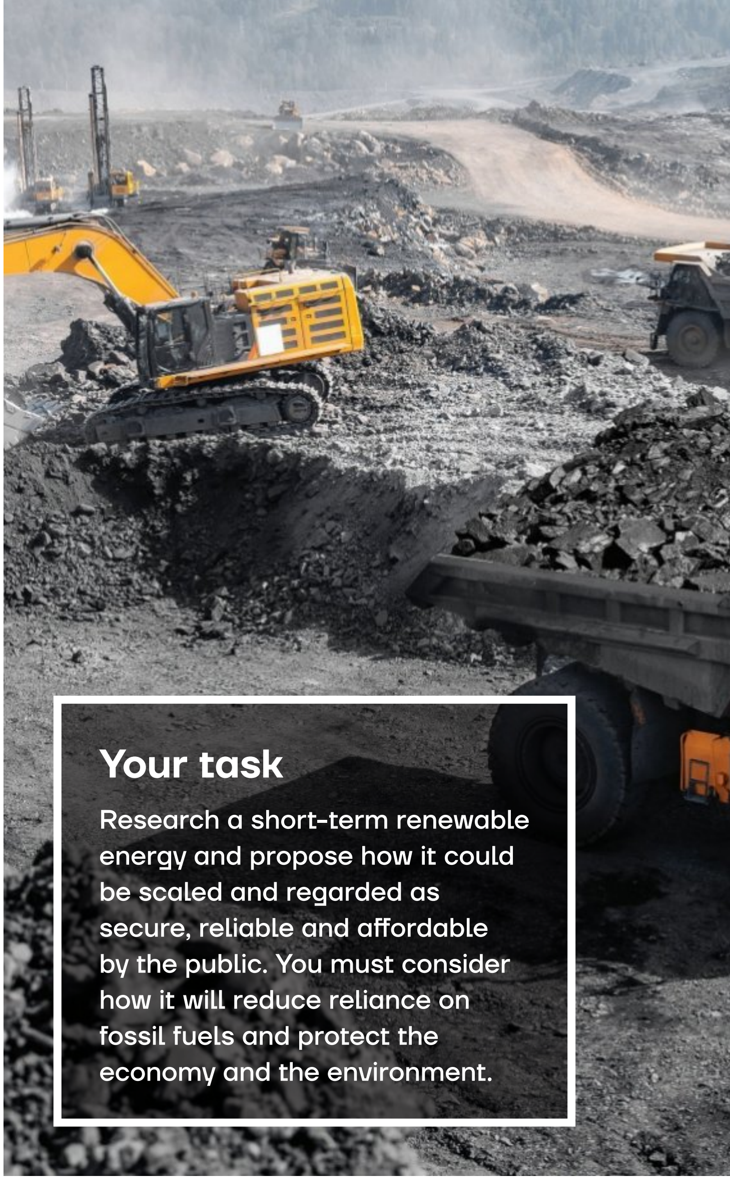
[STEAM project 2]

How can Australia reduce its reliance on fossil fuels so that we protect the environment and the economy?

Fossil fuels such as coal, oil and gas are made from fossilised, decomposed organisms that aged over millions of years in the Earth's crust. They contain carbon, which can be burned for energy. Due to the length of time it takes for these fuels to form as part of the carbon cycle, they are classified as long-term renewable sources of energy, sometimes called non-renewable.

Australia is a major user, producer and exporter of fossil fuels. Nearly 80 per cent of Australia's electricity is generated from coal and gas. Seventy-five per cent of coal mined in Australia is exported, making Australia the largest net exporter of this fuel in the world. In 2019, it was reported that Australia was the world's third-largest exporter of fossil fuels. Economically, the production and export of fossil fuels contributes hugely to Australia's GDP, and the mining industry is an important employer.

Coal emits higher amounts of CO_2 than oil or gas when used to produce energy. Measuring fossil fuel exports according to their potential to emit CO_2 makes Australia's carbon footprint per capita one of the largest in the world. This is contentious globally, particularly for nations most affected by a changing climate.



Your task

Research a short-term renewable energy and propose how it could be scaled and regarded as secure, reliable and affordable by the public. You must consider how it will reduce reliance on fossil fuels and protect the economy and the environment.

Figure 1 Most of Australia's energy is generated from coal. Almost 80 per cent of the coal produced in Australia is from open-cut mines.

Renewable alternatives

To protect both the environment and the economy, Australia needs to focus more on short-term renewable energy. When deciding on energy alternatives, it is important that energy supply be secure, reliable and affordable.

Short-term renewable energy sources include hydropower, solar power, wind power, bioenergy and ocean energy. Australia's landscape is suitable for many of these alternatives, but large investments in technology are required. As we invest more in the technology that makes short-term renewable energy possible, the better we get at making it, the more affordable it becomes and the more demand for renewable energy grows (a cyclical process).

Figure 2 A solar farm in Canberra. Solar energy is a source of renewable energy.

HUMANITIES

In Geography this year, you will learn about environmental change and management. You will explore the environmental, technological and economic factors that have influenced the change and the consequences of human actions on the sustainability of the environment.

In Economics and Business, you will investigate how the performance of Australia's economy is measured and how Australia's economic growth has depended on natural resources. You will explore the impact that environmental policies can have on Australia's economy and living standards.

To complete this task successfully, you will need to understand how stakeholders such as governments, communities and businesses can work together to initiate environmental change and management plans that protect both the environment and the economy.

You will find more information on this in Chapter 2 'Changing and managing the environment', Chapter 18 'Measuring the performance of Australia's economy' and Chapter 19 'Living standards' of *Oxford Humanities 10 Victorian Curriculum*.

MATHS

In Maths this year, you will extend your skills in representing and interpreting data, including multivariate data sets and time-series data. This will include critical consideration of media reports that use statistics and present graphs. You will use digital technology to work with data but also perform calculations by hand.

To complete this task successfully, you will need to find data to quantify the problem, to cost your interventions and to calculate a quantitative, evidence-based estimate of the likely benefits of your interventions. You will need to have skills in performing proportionality and other calculations with very large numbers, using scientific notation.

You will find relevant mathematical and statistical concepts in Chapter 1 'Financial mathematics' and Chapter 10 'Statistics' of *Oxford Maths 10/10A Victorian Curriculum*.

SCIENCE

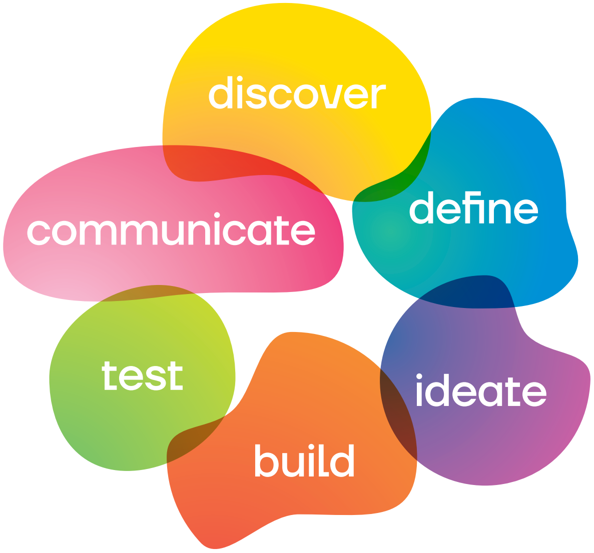
In Science this year, you will learn about the impacts of fossil fuel combustion reactions in the production of carbon dioxide and carbon monoxide. You will also examine how increased reliance on this form of energy has affected the way carbon cycles through Earth's spheres, and how the resulting increase in greenhouse gases (including carbon dioxide) has led to enhanced global warming, which is contributing to melting sea ice and permafrost, rising sea levels and an increased number of extreme weather events.

To complete this task successfully, you will need to consider how energy that is generated can be used efficiently.

You will find more information on this in Chapter 6 'Global systems' of *Oxford Science 10 Victorian Curriculum*.

The design cycle

To successfully complete this task, you will need to complete each of the phases of the design cycle.



Discover

When designing solutions to a problem, you need to know who you are helping (your end-users) and what they need.

Consider the following questions to help you empathise with your end-users:

- Who am I designing for? Will I be helping the government or members of the public?
- What problems are they facing? Why are they facing them?
- What do they need? What do they not need?

To answer these questions, you may need to investigate using different resources, or even conduct interviews or surveys.

Define

Before you start to design your solution for the potential replacement of fossil fuels, you need to define the parameters you are working towards.

Define your version of the problem

Rewrite the problem so that you describe the group you are helping, the problem they are experiencing and the reason it is important to solve it. Use the following phrase as a guide.

‘How can we help (the group) to solve (the problem) so that (the reason)?’

Determine the criteria

1 What type of energy source are you trying to replace? How much of it is currently used and how is it used?

2 How will the renewable energy be used? Will it be easy for the user to access?

3 Will the renewable energy require many changes in the vehicles or equipment being used? Who will pay for this change in infrastructure? How much will it cost?

4 How long will it take to generate the resources needed to make this renewable energy resource accessible and affordable for most people?

Ideate

Once you know who you’re designing for, and you know what the criteria are, it’s time to get creative!

Outline the criteria or requirements your design must fulfil (i.e. type of equipment, number and amount of materials, area that needs to be covered).

Brainstorm at least one idea per person that fulfils the criteria.

Remember that there are no bad ideas at this stage. One silly thought could lead to a genius innovation!

Build

Each team member should draw one individual design. Label each part of the design. Include the material that will be used to construct a model of the design.

Include in the individual designs:

a a description of what you see as the biggest problem with the energy source that you are replacing

b a description of the renewable energy that you propose could be used instead

c a description of how this renewable energy could be scaled up so that it can be used more effectively.

Present your design to your group.

Build the prototype

As a group, choose one design and plan how to model or build it. You may need to produce two or three to-scale prototypes for your group's design. Keep each iteration so that you can show the progress of your ideas.

Use the following questions as a guideline for your prototype.

- How will you replicate or model the renewable energy source?
- How will you model how the renewable energy will be used?
- What are the limitations of the renewable energy source? Will it produce enough energy for the equipment that currently uses fossil fuels?
- Calculate the number, density or requirements of energy sources in your area. How will your model provide for these demands?

Test

Use the scientific method to design an experiment that will test the limitations of your renewable energy prototype idea. You will need to model and test more than one prototype to compare results, so you will need to consider all variables between tests.

What criteria will you use to determine the success of your renewable energy prototype?

If your prototype will be used by a particular group of individuals, then you will need to generate a survey to test whether the prototype is appropriate for their use. (How would they use the alternative energy source? Would it make their life easier or harder? Would they consider buying it? How much would they be prepared to pay to access this form of energy?)

Conduct your tests and record your results in an appropriate table.

Communicate

Present your design to the class as though you are trying to get your peers to invest in your alternative energy design.

In your presentation, you will need to:

- outline the energy needs of the selected individual or group you are supporting
- outline the energy challenges faced by your selected individual or group
- create a working model or a detailed series of diagrams, with a description of how the design prototype will be used to replace the current energy demands
- describe how you changed your design prototype as a result of testing or feedback
- describe how the renewable energy prototype will improve the life of your selected individual or group
- estimate the cost of production for each element of your energy design
- estimate the number of each element of your energy design required in your local government area
- estimate the total implementation cost to individuals, or to local, state or national government bodies
- compare how this energy system could be implemented in developed and developing countries.

Check your Student obook pro for the following digital resources to help you with this STEAM project: pro

Student guidebook
This helpful booklet will guide you step-by-step through the project.

What is the design cycle?
This video will help you to better understand each phase in the design cycle.

How to manage your project
This 'how-to' video will help you to manage your time throughout the design cycle.

How to define a problem
This 'how-to' video will help you to narrow your ideas down and define a specific problem.

Check your Teacher obook pro for these digital resources and more: pro

Implementation advice
Find curriculum links and advice for this project.

Assessment resources
Find information about assessment for this project.

Answers

CHAPTER 1 Financial mathematics

EX p6

1A Calculating percentages

1 **a** 0.15 **b** 0.9 **c** 1.12
d 0.755 **e** 0.188 **f** 0.0025
g 0.384 **h** 2.03

2 **a** $\frac{21}{100}$ **b** $\frac{17}{20}$ **c** $\frac{9}{4} = 2\frac{1}{4}$
d $\frac{16}{25}$ **e** $\frac{19}{50}$ **f** $\frac{401}{500}$
g $\frac{11}{10} = 1\frac{1}{10}$ **h** $\frac{21}{400}$ **i** $\frac{63}{400}$
j $\frac{11}{2000}$ **k** $\frac{21}{250}$ **l** $\frac{1}{25\,000}$

3 **a** 60% **b** 37.5% **c** 70%
d 65% **e** 68.75% **f** 62.5%
g 115% **h** 250%

4 **a** $16\frac{2}{3}\%$ **b** $66\frac{2}{3}\%$ **c** $116\frac{2}{3}\%, 1.1\dot{6}$
d $\frac{511}{600}, 0.851\dot{6}$ **e** $\frac{7}{15}, 0.4\dot{6}$ **f** $\frac{37}{300}, 0.12\dot{3}$
g $91\frac{2}{3}\%, 0.91\dot{6}$ **h** $388\frac{8}{9}\%, 3.\dot{8}$

5 **a** \$57.60 **b** \$600 **c** \$198.45
d \$1332 **e** \$1248 **f** \$700
g \$4200 **h** \$10 980

6 **a** \$16.05 **b** \$76.92 **c** \$11.94
d \$387.66 **e** \$271.29 **f** \$2.90
g \$0.81 **h** \$46.16

7 **a** 20% **b** 61.54% **c** 19.33%
d 25.17% **e** 51.64% **f** 234.62%
g 13.60% **h** 54.78% **i** 191.20%

8 **a** $\frac{131}{399}$ **b** 32.83%, 0.33

9 **a** $\frac{13}{20}$ **b** $\frac{3}{40}$ **c** 7.5%
d $\frac{3}{10} \times 65\% = 19.5\%$
e credit card: 40.625%; debit card 4.875%

10 **a** \$2.50
b $\frac{28}{5}$
c 560%, Bruno's business will make a large profit if he can sell all his roasted beans.

11 **a** \$109.99 **b** \$145.74 **c** $\frac{1}{5}, \frac{53}{200}$

12 **a** \$133.33 **b** \$133.33
c They are the same because $33\frac{1}{3}\% = \frac{1}{3}$.

13 **a** **i** \$35.60 **ii** \$17.80 **iii** \$3.56
b Move the decimal point one place left.
c Find 10% and halve it.
d The decimal point is one place further left. When calculating 1% of any amount, move the decimal point two places left.

14 **a** \$43.85 **b** \$21.93 **c** \$4.39
d \$2.30 **e** \$72.95 **f** \$14.84
g \$2.95 **h** \$9.57 **i** \$122.92

15 **a** **i** \$53.40 **ii** \$71.20 **iii** \$1.78
b **i** Find 10%, then halve it and add that on.
ii Find 10% and double it.
iii Find 1% and halve it.

16 **a** \$126.87 **b** \$169.16 **c** \$4.23
d \$50.54 **e** \$6.27 **f** \$53
g \$956.19 **h** \$0.23 **i** \$57.66

17 **a** \$4.50 **b** \$450 **c** \$450

18 **a** $12.5\% = 0.125, 0.125\,x = \$56.25$ **b** $\$56.25 \div 0.125$
c \$450 **d** the same
e Finding 12.5% is the same as multiplying by 0.125. Dividing by 0.125 is the same as dividing by 12.5% in order to find 1% in the unitary method.

19 **a** \$64 **b** \$86.40
c Normal price is \$65. He paid \$52.

20 **a** \$48 **b** \$58.50
c $S = P(100\% - D\%)$
d $P = \frac{S}{100\% - D\%}$ **e** \$70
f \$180

21 **a** \$39.10 **b** \$27.50
c \$16.80 **d** \$27.72

EX p13

1B Financial calculations

1 **a** \$22.50 **b** \$1120 **c** \$81.40
d \$490 **e** \$45 **f** \$172.50

2 **a** \$459 **b** \$21 483 **c** \$1186.10
d \$788.63 **e** \$5186.29 **f** \$1807.92

3 **a** \$138.75 **b** \$7590 **c** \$445.90
d \$144.81 **e** \$233.08 **f** \$272.20

4 **a** **i** \$229.25 **ii** \$139.35
b **i** \$1308.83 **ii** \$186.98
c **i** \$32.44 **ii** \$17.47
d **i** \$16.53 **ii** \$13.68

5 **a** **i** loss, \$31.35 **ii** 36.90%
b **i** profit, \$4.25/kg **ii** 250%
c **i** loss, \$20 545 **ii** 54.16%
d **i** profit, \$104 **ii** 18.98%
e **i** profit, \$14.85 **ii** 45.07%
f **i** loss, \$26.95 **ii** 62.75%

6 **a** **i** \$1028 **ii** \$257
b **i** \$1521 **ii** \$279
c **i** \$1539 **ii** \$999
d **i** \$1014.75 **ii** \$564.75
e **i** \$286.23 **ii** \$60.72
f **i** \$318.90 **ii** \$531.50

g **i** $6485.61 **ii** $3026.62
h **i** $22 153.95 **ii** $15 913.40

7 **a** $330.85 **b** 12.80%

8 **a** $42.45 **b** $\frac{283}{943}$ **c** 30.01%

9 **a** **i** $24 **ii** $\frac{6}{17}$ **iii** 35%
b **i** $17 150 **ii** $\frac{343}{600}$ **iii** 57%
c **i** $46.50 **ii** $\frac{31}{97}$ **iii** 32%
d **i** $25.45 **ii** $\frac{509}{759}$ **iii** 67%
e **i** $526.05 **ii** $\frac{3507}{4663}$ **iii** 75%
f **i** $52.41 **ii** $\frac{5241}{29458}$ **iii** 18%

10 **a** 82% **b** $69.66 **c** $15.29
d $1624.67 **e** $64.99

11 **a** 112.5% **b** $145.63 **c** $10.63

12 **a** $1023.36 **b** $977.31
c No, the discounted price would have been $967.20.
d $270.69
e 21.69%
f It is less than the total 22.5% because the extra 4.5% is applied to a smaller (already discounted) amount.

13 **a** $350 **b** $45.45
c $66.65 **d** $8.90

14 **a** $71.30 **b** $87.60
c 81.39% of $87.60

15 $792.43

16 **a** $925.50 **b** $1544.25 **c** $932.72
d $1815.96 **e** $993.52 **f** $1188.73
g $945.67 **h** $1274.81

17 **a** Option 1: $6000
Option 2: $6100
Option 3: $6040
He should choose option 2.
b Option 1: $72 000
Option 2: $72 500
Option 3: $75 600
He should choose option 3.
c Yes. It is stable and reliable.

18 **a** $720 and $720 **b** $640 and $648
c 5 weeks. After 4 weeks: $524.88.
After 5 weeks: $472.39

19 **a** $44.80 **b** $16.80
c 44% **d** $115
e The discount would be the same 0.8 × 0.7 = 0.56 and 0.7 × 0.8 = 0.56.

20 **a** $990
b Because the 10% is taken off the original amount, whereas the increase of 10% is on the new, lower amount.

21 **a** The first increase of 50% is on the $3000 but the second increase of 50% will be on the new amount, which will be more than $3000.
b $6750
c 125%

EX p19

1C Simple interest

1 **a** 2 **b** 4 **c** 2.25
d 3.75 **e** $\frac{7}{12}$ **f** 1
g $1\frac{1}{4}$ **h** $\frac{1}{13}$ **i** 6
j 1.5 **k** 2 **l** $\frac{1}{5}$

2 **a** $1200 **b** $204 **c** $31 500
d $11 520 **e** $2387.50 **f** $8250

3 **a** **i** $640 **ii** $8640 **iii** $360
b **i** $6000 **ii** $21 000 **iii** $437.50
c **i** $202.50 **ii** $1452.50 **iii** $40.35
d **i** $2870 **ii** $16 870 **iii** $562.33
e **i** $2550 **ii** $27 550 **iii** $1530.56
f **i** $589.50 **ii** $7139.50 **iii** $297.48

4 **a** **i** $1080 **ii** $13 080
b **i** $6600 **ii** $72 600
c **i** $18 750 **ii** $143 750
d **i** $231 **ii** $5731
e **i** $876.56 **ii** $9126.56
f **i** $494.64 **ii** $5074.64

5 **a** 4 years **b** 2.5 years **c** 3 years

6 **a** $15 000 **b** $6500 **c** $20 000

7 **a** 4.4% **b** 10.5% **c** 4.5%

8 **a** $2160 **b** 4 years **c** $18 000
d 8.5% p.a. **e** $2592 **f** 5 years
g $20 000 **h** 2.4% p.a.

9 **a** $10 160 **b** $6000 **c** $22 680
d $72 160 **e** $14 592 **f** $13 064
g $24 320 **h** $23 175

10 **a** $624.75 **b** $3564.75 **c** $178.24
d $2866.50 **e** $698.25

11 **a** $1615.50, $1546, $2421.55 **b** 8, 2, 9, 2, 3, 7
c 0.30, 0.07, 0.62, 0.12, 0.18, 0.65
d $1.94
e $2423.49

12 **a** **i** $5000 **ii** no
iii between 2 months and less than 6 months
b 5.4%
c $2430
d $2812.50; $382.50 more
e It increases by $19.53.
f three 4-month investments with interest added to the investment at the end of each term

13 a Yes. there were no withdrawals and over $250 has been deposited this month.
b $42.02
c $9994.52
d No bonus interest would have been calculated; final balance is $9861.27.

14 a The interest rate won't change over that time.
b $204 **c** $29 411.76 **d** $12 107.32

15 a 2.6 years **b** 12.5% **c** 28%
d She will have trouble finding a financial institution offering such high interest rates.

16 a $120.00 for 4 days
$476.00 for 3 days
$515.90 for 7 days
$700.90 for 6 days
$850.90 for 5 days
$916.75 for 1 day
b $925.69
c 21.9% p.a.

17 a $45, $43.33
b $135, $130
c $10 043.33
d $43.52, $10 086.85
e $43.71, $10 130.56
f $130.56
g different, because the interest earns interest

18 a $10 135.61 **b** 61 cents more
c $10 045.19 **d** 5.4244% p.a.

19 a i $204 **ii** $168 **iii** $96
b Because *D* = $252, meaning its value after 7 years would be −$12. An item cannot be worth a negative amount.
c Sample answer: After 7 years the value is $0 and then remains $0 forever. Alternatively, the printer could decrease towards a value of $0 at a slower rate as the years go on.

20 a i $60 **ii** $60.09
b i $315 **ii** $317.22
c i $1487.50 **ii** $1511.39
d i $451.50 **ii** $459.63

21 a 3.61% **b** 8.46%
c $12.95% **d** $21.89

1 Checkpoint

1 a $90 **b** $12
c $150 **d** $195

2 a 30% **b** 40%
c 27.5% **d** 175%

3 a $\frac{3}{10}$ **b** 70%

4 a $180 **b** $136
c $28.50 **d** $90

5 a $45 **b** $1400
c $9.35 **d** $58.24

6 a $0.75 **b** $42 **c** 233.3%

7 a $45 **b** $\frac{3}{8}$ **c** 37.5%

8 a $600 **b** $2100 **c** $1615

9 a i $600 **ii** $5600
b i $840 **ii** $12 840
c i $29 216 **ii** $162 016

10 a $900 **b** $5400 **c** $112.50

11 a $320 **b** $9600 **c** 5%

EX p30

1D Compound interest

1

	Years	Simple interest	Value of principal + simple interest	Compound interest	Value of principal + compound interest
a	3	$10	$130	*I* = $121 × 10% = $12.10	*A* = $121 + $12.10 = $133.10
b	4	$10	$140	$13.31	$146.41
c	5	$10	$150	$14.64	$161.05
d	6	$10	$160	$16.11	$177.16

2 a i $42 000 **ii** $44 100 **iii** $46 305
b i $6120 **ii** $6242.40 **iii** $6367.25
c i $1040 **ii** $1081.60 **iii** $1124.86
d i $20 700 **ii** $21 424.50 **iii** $22 174.36
e i $6955 **ii** $7441.85 **iii** $7962.78
f i $16 179.20 **ii** $16 567.50 **iii** $16 965.12

3 a i 2 **ii** 1 year
b i 3 **ii** 1 year
c i 8 **ii** 6 months
d i 24 **ii** 1 month

4 a i 3 **ii** 1 year
b $442 **c** $8942 **d** $464.98
e $9406.98 **f** $489.16 **g** $9896.14
h Final value with simple interest is $9826, which is $70.14 less.

5 a $3401.40 **b** $2809.61 **c** $2919.29
d $2008.85 **e** $19 335.94 **f** $38 421.57

6 a $12 155.06 **b** $5955.08 **c** $816.29
d $53 019.97 **e** $112 649.26 **f** $19 993.02

7 a i $8682.19 **ii** $1182.19
b i $10 270.50 **ii** $1270.50
c i $31 263.59 **ii** $8763.59
d i $53 022.34 **ii** $13 022.34

8 a $4101.25 **b** $7325.28 **c** $16 925
d $34 945.12 **e** $127 567.39 **f** $507 382.93

9 **a** $901.25 **b** $2735.28 **c** $925
d $9045.12 **e** $27 567.39 **f** $252 382.93

10 **a** **i** 1.2% **ii** $R = 0.012$, $n = 12$
b **i** 2.95% **ii** $R = 0.0295$, $n = 8$
c **i** 0.7% **ii** $R = 0.007$, $n = 24$
d **i** 0.8% **ii** $R = 0.008$, $n = 60$

11 **a** $6071.85 **b** $13 406.88 **c** $22 895.02
d $68 216.57 **e** $61 373.74 **f** $162 619.66
g $551 913.02 **h** $1 017 670.16

12 **a** Multiplying by 1 keeps the amount and multiplying by 0.05 evaluates 5% of the amount. Therefore, multiplying by 1.05 increases the amount by 5%.
b $10\,000 \times (1.05)^n$
c $10\,000 \times (1 + R)^n$
d $P(1 + R)^n$

13 **a** 3 **b** 1 year
c $P = \$16\,000$, $R = 0.0575$, $n = 3$
d $18 921.74 **e** $2921.74 **f** Yes

14 **a** $63 404.70 **b** $18 404.70

15 **a** $10 444.50, $10 403.75
b Option 1, since it yields greater interest
c $11407.50, $11459.21. Option 2 is better since it yields greater interest.

16 **a** 2.15% **b** $R = 0.0215$ **c** 20
d $12 854.25 **e** $4454.25

17 **a** $13 360
b **i** $13 897.66 **ii** $13 944.79
iii $13 960.82 **iv** $13 977.02
c It increases. Yes.

18 fewer

19 **a** **i** 8 **ii** 0.017
b A
c $6500 = P(1 + 0.017)^8$; $P = \$5679.97$
d $5680

20 **a** $6000 **b** 8%
c (3, 7558.27) and (4, 8162.93) **d** $11 105.58

21 **a** The reducing balance depreciation formula subtracts the rate per compounding period, R, whereas the compound interest formula adds R. This is so that the value will decrease over time, rather than increase, as the $(1 - R)$ factor is less than 0.

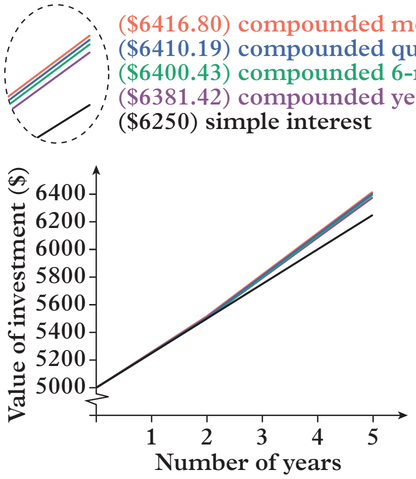
b

Years since purchase	Value
0	$2000
1	$1600
2	$1280
3	$1024
4	$819.20

c 14 years
d 80% is percentage of its current value that it retains each year.
e Decreasing a positive amount by a percentage results in another positive amount. Unless the value decreases by 100%, it can never reach $0.
f For most situations, yes. An item will usually drop a lot of value in the first few years and have its value drop towards $0 slowly. In this example, the value difference between a 1-year-old computer and a 2-year-old computer is large, but the value difference between a 10-year-old computer and 11-year-old computer is very small, which reflects what you would expect in real life.

22 **a** $5250, $5512.50, $5788.13, $6077.54, $6381.42
b $250, $262.50, $275.63, $289.41, $303.88
c See graph below
d **i** **a** $5254.73, $5522.43, $5803.77, $6099.45, $6410.19
b $254.73, $267.70, $281.34, $295.68, $310.74
c See graph below
ii **a** $5253.13, $5519.07, $5798.47, $6092.02, $6400.43
b $253.13, $265.94, $279.40, $293.55, $308.41
c See graph below
iii **a** $5255.81, $5524.71, $5807.36, $6104.48, $6416.80
b $255.81, $268.90, $282.65, $297.12, $312.32
c See graph below
e It would be a straight line with a positive gradient, while all of the compound interest graphs would be increasing curves.
f See graph below

300% enlargement of ends of graph lines



EX
p38

1E Compound interest calculations

1 **a** $20 122.05 **b** $8522.70
c $4003.95 **d** $6930.08

2 **a** $4274.64 **b** $7775.87
c $13 545.38 **d** $10 550.71
e $145 348.17

3 **a** \$925.36 **b** \$2224.13
c \$8454.62 **d** \$6099.29
e \$79 651.83

4 **a** **i** 0.0817 **ii** 16.3%
b **i** 0.0583 **ii** 5.8%
c **i** 0.0392 **ii** 15.7%

5 **a** R
b $15\,000 = 12\,000(1 + R)^3$

$$\frac{15\ 000}{12\ 000} = (1 + R)^3$$

$$\frac{15}{12} = (1 + R)^3$$

$$\frac{5}{4} = (1 + R)^3$$

c $\sqrt[3]{1.25}$, $\sqrt[3]{1.25}$, 1.077 217...
d 7.7% p.a.

6 **a** **i** 0.1086 **ii** 10.9%
b **i** 0.0226 **ii** 4.5%
c **i** 0.0323 **ii** 6.5%

7 **a** $A = \$8000, P = \$5500, n = 4$
b 0.098 **c** 9.8%

8 **a** **i** 0.145 **ii** 14.5%
b **i** 0.107 **ii** 10.7%
c **i** 0.044 **ii** 8.8%
d **i** 0.034 **ii** 13.6%
e **i** 0.055 **ii** 10.9%
f **i** 0.021 **ii** 8.4%

9 **a** 36 **b** 0.006
c P **d** \$12 093.83

10 **a** n
b $30\,000 = 25\,000(1 + 0.05)^n$;
$30\,000 \div 25\,000 = (1.05)^n$
$1.2 = (1.05)^n$
c 4
d \$25 000 must be invested for 4 years at 5.0% compounded annually to increase to \$30 000.
$1.2 = (1.05)^n$

11 **a** $A = \$1100, P = \$850, R = 0.042$
b $\frac{22}{17} = (1.042)^n$ **c** 7
d 7

12 **a** 6 **b** 5 **c** 9
d 13 **e** 6 **f** 6

13 **a** 6 **b** 5 **c** 4.5
d 3.25 **e** 3 **f** 1.5

14 **a** \$8548.04 **b** \$8523.71
c **i** 5.74% **ii** 5.62%
iii 5.59% **iv** 5.58%

15 **a** 3.6% **b** 10.8% **c** 46.8%
d Quarters. 3.6% is a reasonable interest rate for a savings account.
e Months. 3.6% is too low because credit card interest rates are usually quite high. However, they are not as high as 46.8%.

16 **a** $A = P(1 + Cr)^{\frac{T}{C}}$
b As C decreases, R decreases, because $R = rC$.
c As C decreases, n increases, because $n = \frac{T}{C}$.
d As C decreases, A increases. While $R = rC$ decreases, $n = \frac{T}{C}$ increases, and because $\frac{T}{C}$ is in the exponent, the exponential increase more than makes up for the decrease in R so that the net effect is that A increases.
e When $C = T$, $A = P(1 + Tr)^{\frac{T}{T}} = P(1 + Tr)^1 = P + PrT$, which is the formula for the value of a simple interest investment. Therefore, if the compounding period equals the term of investment, then it is just simple interest.

17 **a** **i** 6 **ii** 25
b The n value must be rounded up to the next interest payment for the account balance to be above the required minimum amount.
c **i** 13 **ii** 34 **iii** 43

CHAPTER 1 review

Multiple-choice

1 C **2** E **3** B **4** E
5 C **6** C **7** A **8** D
9 C **10** B

Short answer

1 **a** 0.24 **b** 0.358 **c** 0.1625 **d** 0.002 72
2 **a** $\frac{43}{50}$ **b** $\frac{47}{20}$ **c** $\frac{19}{125}$ **d** $\frac{173}{2000}$
e $\frac{41}{300}$ **f** $\frac{8}{15\,625}$
3 **a** \$102.75 **b** \$16150 **c** \$17.24
4 **a** 12.5% **b** 55.10%
5 $51\frac{471}{479}\%$
6 **a** **i** \$2154.75 **ii** \$395.25
b **i** \$2023.88 **ii** \$1124.38
c **i** \$24 974.28 **ii** \$6929.28
d **i** \$1216.19 **ii** \$729.71
7 **a** **i** profit, \$31.45 **ii** 22%
b **i** loss, \$17 676.05 **ii** 64%
8 **a** \$450 **b** \$5950
9 **a** **i** \$25 200 **ii** \$175 200
b **i** \$116.16 **ii** \$9566.16
10 **a** 1.25 **b** \$18 000 **c** 11.25%

11 a Interest: \$675.00, \$705.38, \$737.12
Investment: \$15 675, \$16 380.38, \$17 117.50

b Interest: \$438.40, \$466.46, \$496.31, \$528.07
Investment: \$7288.40, \$7754.86, \$8251.17, \$8779.24

12 a \$2600.16 **b** \$26 978.10 **c** \$191 793.77

13 a \$6691.22 **b** \$14 700.60 **c** 0.04

14 a 0.003

b i 0.3% **ii** 1.2%
iii 3.6% **iv** 15.6%

c 1.2% or 3.6%

Analysis

1 a 16

b i 0.013 75 **ii** 0.014 25 **iii** 0.0155

c \$12 673.74

d $P = \$12\,673.74$, $R = 0.014\,25$, $n = 8$

e \$14 192.70 **f** \$15 093.32 **g** \$3093.32

h 25.78% **i** 6.45% **j** 6.42%

2 a i \$4160 **ii** \$4140

b i \$5600 **ii** \$5642.40

c 9 years **d** \$4138.14

e No. The effective interest rate will be lower. This can be seen by observing that the gap between the balance for the monthly compounded rate and the annually compounded rate is increasing each year. So, assuming this trend continues, the annually compounded rate will always give the higher balance.

Years since investment	Balance for 3.5% interest compounded annually (\$)	Balance for 3.4% interest compounded monthly (\$)	Difference between balances (\$)
0	4000	4000	0
1	4140	4138.14	1.86
2	4284.90	4281.05	3.85
3	4434.87	4428.90	5.97

CHAPTER 2 Algebra

2A Indices

EX p50

1 a a^4 **b** b^{20} **c** c^{15} **d** d
e y^{10} **f** m^{14} **g** n^{14} **h** h^7
i x^6 **j** p^{18} **k** w^3 **l** a^{10}

2 a 1

b i 1 **ii** 1 **iii** 1 **iv** 1

3 a 3 **b** 1 **c** 8 **d** 1
e 1 **f** 2 **g** 0 **h** 2
i 6 **j** 1 **k** 1 **l** 1

4 a a^5b^5 **b** k^9p^9 **c** $64c^3$ **d** $7^{13}y^{13}$
e x^4y^4 **f** c^7d^7 **g** $9w^2$ **h** 6^6g^6
i 2^8p^8

5 a $\frac{a^4}{b^4}$ **b** $\frac{f^7}{g^7}$ **c** $\frac{k^6}{64}$
d $\frac{5^6}{x^6}$ **e** $\frac{d^{-2}}{c^{-2}}$ **f** $\frac{m^{-5}}{3^{-5}}$

6 a $5x^5$ **b** $6a^7$ **c** $9x^2$ **d** $8x^{12}$
e $4a^3$ **f** $2x^5$ **g** 2 **h** $30x^{12}$
i $3a^3$

7 a $8y^4$ **b** $24x^9$ **c** $72k^5$ **d** $45b^{12}$
e $60a^{20}$ **f** $10c^5$ **g** $2n^2$ **h** v^6
i $25m^{13}$

8 a p^9 **b** $\frac{2t^{20}}{6561}$ **c** $25m^{39}n^{48}$ **d** $\frac{7}{j^6}$

9 a false; $x^7 \times x \times x^7 = x^{7+1+7} = x^{15}$

b false; $(5a)^3 = 5^3 \times a^3 = 125 \times a^3$

c true; $-k^0 = -(1) = -1$

d false; $a^2b^3 \times a^3b^4 = a^{2+3}b^{3+4} = a^5b^7$

e true; $x^7 \times y^4 \div y^6 = \frac{x^7y^4}{y^6} = \frac{x^7\overset{1}{\cancel{y^4}}}{\underset{y^2}{\cancel{y^6}}} = \frac{x^7}{y^2}$

f false; $\left(\frac{m}{n}\right)^5 = \frac{m^5}{n^5}$

g false; $\frac{w^6 \times w^6}{w^{12}} = \frac{w^{6+6}}{w^{12}} = \frac{w^{12}}{w^{12}} = 1$

h true; $\frac{(b^5)^4 \times b^2}{(b^3)^7} = \frac{b^{20} \times b^2}{b^{21}} = \frac{b^{22}}{b^{21}} = b$

i false; $\frac{a^8b^5}{a^4b^3} \times \frac{a^2b^2}{a^9b^4} = \frac{a^{8+2}b^{5+2}}{a^{4+9}b^{3+4}} = \frac{a^{10}b^7}{a^{13}b^7} = \frac{1}{a^3}$

10 a 9 **b** 0 **c** 10 **d** 4
e 7 **f** 9 **g** 6 **h** –5
i 8

11 2^5

12 3^{2x}

13 $\frac{5^{4x}}{3^{3x}}$

14 a $x = 4$ and $y = 8$

b $x = -2$, $y = 12$

15 a

$$\begin{aligned}\frac{2^{13} \times 3^{11} \times 5^9}{30^7} &= \frac{2^{13} \times 3^{11} \times 5^9}{(2 \times 3 \times 5)^7}\\ &= \frac{2^{13} \times 3^{11} \times 5^9}{2^7 \times 3^7 \times 5^7}\\ &= 2^6 \times 3^4 \times 5^2\\ &= (2^3 \times 3^2 \times 5)^2\end{aligned}$$

b $a = 5 \times 7$ or $a = 5 \times 7^3$ or $a = 5 \times 7^5$ or $a = 5 \times 7^7$ or $a = 5^3 \times 7$ or $a = 5^3 \times 7^3$ or $a = 5^3 \times 7^5$ or $a = 5^3 \times 7^7$ or $a = 5^5 \times 7$ or $a = 5^5 \times 7^3$ or $a = 5^5 \times 7^5$ or $a = 5^5 \times 7^7$

16 50

17 a i 4 **ii** 4 **iii** 8
iv –8 **v** 100 000 **vi** –100 000

b i –32 **ii** 81 **iii** –125

c i positive, since the power is even
ii negative, since the power is odd
iii positive, since the power is even

EX

2B Negative indices

p54

1 **a** $\frac{1}{x^5}$ **b** $\frac{d^4}{c^9}$ **c** $\frac{j^5}{k^3}$ **d** $\frac{y^8}{x}$

e $\frac{2a^3}{b^6}$ **f** $\frac{3}{p^4q^5}$ **g** $\frac{w^4y^7}{x^5}$ **h** $\frac{m^5}{k^3n^8}$

i $\frac{6d^2f}{e^7}$ **j** $\frac{11}{a^3b^5c^9}$ **k** x^3 **l** b^4c^6

m $4m^3p^7$ **n** $\frac{b^2}{a^4}$ **o** $\frac{4y^5}{x^2}$ **p** $\frac{d^8e^4}{c}$

2 **a** x^3 **b** m^7 **c** k^{12} **d** y^4

e a^5 **f** d^{-15} **g** y **h** n^2

3 **a** $\frac{1}{x^4y^4}$ **b** $\frac{1}{4^6a^6}$ **c** $\frac{1}{9n}$

d $\frac{1}{5^3p^3} = \frac{1}{125p^3}$ **e** $\frac{1}{9p^2}$ **f** $\frac{1}{k^5m^5}$

g $\frac{64}{x^3}$ **h** $\frac{d}{7}$

4 **a** $14x^9$ **b** $3x^2$ **c** x^{15} **d** $20x^8$

e $4x^3$ **f** $6x^6$ **g** $\frac{3x}{7}$ **h** $\frac{3x^5}{2}$

i x^{17} **j** x^{34} **k** $2x^{23}$ **l** $5x^{35}$

5 **a** $\frac{1}{x^2}$ **b** $\frac{2}{x^5}$ **c** $42x$ **d** $\frac{20}{x^4}$

e $\frac{1}{x^2}$ **f** $2x^9$ **g** $\frac{1}{4x^9}$ **h** $\frac{3}{5x^4}$

i $\frac{1}{x^{11}}$ **j** $\frac{10}{x^6}$ **k** $\frac{3}{x^7}$ **l** $\frac{1}{x^{21}}$

6 **a** $\frac{1}{x^6}$ **b** $\frac{4}{y^6}$ **c** w^{12} **d** $4x^{14}$

e $\frac{9}{a^8}$ **f** $2m$ **g** $8h^{12}$ **h** $\frac{30}{c^9}$

i $2d^3$ **j** $\frac{2}{t^{16}}$ **k** x **l** $\frac{12}{a^8}$

7 **a** $a^{11}b^7$ **b** $2304p^5$ **c** $32y^{15}$ **d** $3m^{12}n^6$

e $\frac{d^{12}}{h^{11}}$ **f** $\frac{125w^3}{y^3}$ **g** $a^{30}b^{12}$ **h** $\frac{b^8}{a^{16}c^{28}}$

i u^3wx^{17} **j** $125c^4n$ **k** $m^{34}n$ **l** $\frac{1}{y^7}$

8 **a** false, $\frac{a^5}{a^9} = \frac{1}{a^4}$ **b** false, $\frac{x^3}{x^6} = \frac{1}{x^3}$

c true **d** false, $\left(\frac{3a^2b^{-1}}{c^3}\right)^2 = \frac{9a^4}{b^2c^6}$

e true **f** true

g false, $\frac{3(x^{-2}y^1)^3}{x^3y^{-2}z} = \frac{3y^5}{x^9z}$ **h** true

9 **a** 3 **b** −4 **c** −2

d 3 **e** 1 **f** 5

10 **a** $x^{2a+5}y^{4b+1}$ **b** $m^{2x+4}n^3$

11 **a** $\frac{1}{25}$ **b** $\frac{1}{27}$ **c** 40

d 144 **e** $\frac{1}{8}$ **f** $\frac{13}{9}$

12 **a** x^{3a-4} **b** y^{3m+5}

c a^{x+3} **d** b^{x+4}

13 **a** $\frac{3a^2b^2}{2c}$ **b** $\frac{3}{2b^8c^2}$ **c** $\frac{1}{y}$

14 **a** $\frac{8c^{12}}{27a^6b^9}$

15 $2c$

16 **a**

$$\begin{aligned} a^m &= 1 \times a^m \\ \frac{a^m}{a^m} &= 1 \\ a^{m-m} &= 1 \\ a^0 &= 1 \end{aligned}$$

b

$$\begin{aligned} a^0 &= 1 \\ a^{1+(-1)} &= 1 \\ a^1a^{-1} &= 1 \\ a \times a^{-1} &= 1 \\ a^{-1} &= \frac{1}{a} \end{aligned}$$

c $a^{-1} = \frac{1}{a}$ only works for non-zero values of a because you cannot define a number divided by 0, so $\frac{1}{0}$ is undefined.

d

$$\begin{aligned} a^{-1} &= \frac{1}{a} \\ (a^{-1})^m &= \left(\frac{1}{a}\right)^m \\ a^{-1\times m} &= \frac{1^m}{a^m} \\ a^{-m} &= \frac{1}{a^m} \end{aligned}$$

e

$$\begin{aligned} a^{-m} &= \frac{1}{a^m} \\ a^{-m}a^m &= 1 \\ a^m &= \frac{1}{a^{-m}} \end{aligned}$$

f

$$\begin{aligned} a^{-m} &= \frac{1}{a^m} \\ a^{-m} \times b^m &= \frac{1}{a^m} \times b^m \\ a^{-m} \times \frac{1}{b^{-m}} &= \frac{1}{a^m} \times b^m \\ \frac{a^{-m}}{b^{-m}} &= \frac{b^m}{a^m} \\ \left(\frac{a}{b}\right)^{-m} &= \left(\frac{b}{a}\right)^m \end{aligned}$$

EX

2C Simplifying

p59

1 **a** **i** 3 **ii** Sample answer: $-4abcd$

b **i** −4 **ii** Sample answer: $2mn$

c **i** 1 **ii** Sample answer: $5xy^2$

d **i** 9 **ii** Sample answer: $-3k^2m^4p$

2 **a** −6 **b** −3 **c** 10 **d** 9

e −52 **f** −24 **g** −15 **h** −9

3 **a** $9x$ **b** $-10ab$ **c** $10y^2$

d $8m^2n$ **e** $12a + 9d$ **f** $8k + 2m$

g $2 - 3y$ **h** $7a - 4a^2$ **i** $3x^2 + 2xy$

4 **a** $11a$ **b** $-4n + 2n^2 + m - mn$

c $3x^2 - 2x + 1$ **d** $-3x^2y + 3xy - 5$

e $2p^3 + 15p^2 - 9$ **f** $-a^2 - b^2 + 3ab^2 + 9a^2b$

5 **a** $10abcd$ **b** $-21mnxy$ **c** $4k^2p$

6 **a** ac **b** $6n$ **c** $\frac{2x}{y}$

7 **a** $4x^5y^3$ **b** $6a^6b^3$ **c** $\frac{24m^7}{n^2}$

d $\frac{10x^7y^2}{z^3}$ **e** $\frac{2y^2}{x^2}$ **f** $4x$

g $\frac{36x^5z^7}{y^4}$ **h** $\frac{18y^{11}}{x}$ **i** $\frac{4a^4}{b^4}$

8 **a** 2 **b** −69 **c** 246

d −20 736 **e** −18 **f** 4

9 **a** 1550 cm^2 **b** 0.62 m^2

10 25π cm^3

11 **a** $7x + 6y$ **b** $6x + 4y - 6$ **c** $20y - 12$

12 **a** 61 cm **b** 44 cm **c** 28 cm

13 The answers are square units:

a $42ab$ **b** $4a^2$ **c** $10ab$

d $9\pi a^2$ **e** $24ab + \frac{9}{8}\pi b^2$ **f** $37ab$

14 **a** 504 m^2 **b** 64 m^2 **c** 120 m^2

d $144\pi\text{ m}^2$

e $288 + \frac{81\pi}{8}$

f 444 m^2

15 **a** $3\pi x^2$

b Any value larger than about 6.5147 mm and smaller than 7.2836 mm. Some possible values are: 6.7 mm, 7 mm and 7.1 mm.

c $6\pi x$

d 126.3 mm, 131.9 mm, 133.8 mm

16 **a** $12a$ **b** $32ab$

c $-3a + 15ab$ **d** $24xy^2 - x$

e $-4a^2 + 25a^2b^6$ **f** $16x^2y^2 + 13y^2$

17 **a** $7ab$ **b** $7abc$

c $5x^2y + 3xy$ **d** 0

e $23a^2b^2$ **f** $11y^2z^2 + 3x^2z$

18 **a** $abc - 6c$ **b** $2ab + 2ac + 2bc - 12$

EX p63

2D Expanding

1 **a** $3a + 21$ **b** $5b - 20$ **c** $12 - 2c$

d $4 + 4d$ **e** $-6e - 18$ **f** $-7f + 14$

g $-3g + 3$ **h** $-45 + 9h$ **i** $x^2 + 2x$

j $2y^2 - 8y$ **k** $-k^2 - 5k$ **l** $-3p^2 + 3p$

m $3mn + km$ **n** $12ab + 24ac$ **o** $-w^2 + 8wx$

p $-4p - 10p^2$

2 **a** $3a - 12$ **b** $-5b + 15$

c $8x + 12$ **d** $-14y - 28$

e $10a - 5a^2$ **f** $2m^3 + 6m$

g $24a^2b + 8ab$ **h** $8wx - 4w^2$

i $15k^2 - 10k^4$ **j** $12x^3y - 6x^2y^4$

k $15c^2d - 10cd^2$ **l** $-8a^3c + 6a^2bc$

3 **a** $9x + 10$ **b** $4y - 13$ **c** $mn + 3m$

d $8x^2 - 5x$ **e** $2p^3 - 3p$ **f** $x^7 - 3x^5$

4 **a** $6x + 2$ **b** $m^2 + 14m - 18$

c $6a^2 + 14a - 6$ **d** $k^5 + 7k^2 + 5$

e $3p^6 - 4p^4 + 12p$ **f** $10y^8 + y^3 + 8y$

5 **a** $ab + 2a + 4b + 8$ **b** $cd + 6c + 3d + 18$

c $mn + 5m + 7n + 35$ **d** $xy + 8x + 6y + 48$

e $kp - k + 5p - 5$ **f** $fg + 3f - 2g - 6$

g $xy - 4x - 9y + 36$ **h** $2mn + 2m + 3n + 3$

i $12xy - 28x + 15y - 35$

j $15ac - 12ad - 10bc + 8bd$

k $x^2 + 7x + 10$ **l** $y^2 + 8y - 9$

m $m^2 - 14m + 33$ **n** $p^2 - 6p + 8$

o $30 - 11k + k^2$ **p** $4x^2 + 19x + 21$

q $3a^2 - 13a - 10$ **r** $8w^2 - 23w + 14$

s $12p^2 + 52p + 35$ **t** $8x^2 + 10x - 3$

u $15y^2 - 26y + 8$

6 **a** $2a^2 + 6a - ab - 3b$ **b** $4t^2 - 3tu - u^2$

c $2x^2 + 11xy + 12y^2$ **d** $-x^2 + x + 20$

e $6x^3 + 18x^2y + xy + 3y^2$

f $ab^4 - 2a^2b^2 + b^2 - 2a$

g $ab + ac + b^2 + bc$

h $b^2 + bd + 3bc + 3cd$

i $3a - ac + 6b - 2bc$

j $3a^4b + 2a^3 + 3ab^3 + 2b^2$

k $5x^5 - 2x^4y - 15xz + 6yz$

l $c^4 - 16a^6b^2$

7 **a** **i** $x^2 - 4$ **ii** $k^2 - 49$ **iii** $m^2 - 64$

iv $w^2 - 36$ **v** $y^2 - 1$ **vi** $a^2 - b^2$

b The product of the two binomial factors produces an expression that is the difference of two squared numbers or pronumerals. The first term in each binomial factor is the same, and the second term in one factor is the negative of the second term in the other factor.

c In the expression $a^2 - b^2$, the terms a^2 and b^2 are both squares, and there is a minus sign between them indicating 'find the difference'.

d No; multiplication is commutative.

e **i** $a^2 - 9$ **ii** $x^2 - 100$ **iii** $m^2 - n^2$

iv $9 - x^2$ **v** $1 - d^2$ **vi** $4x^2 - 25$

vii $4 - 9k^2$ **viii** $16g^2 - h^2$ **ix** $25y^2 - 4w^2$

8 **a** **i** $x^2 + 6x + 9$ **ii** $y^2 + 16y + 64$

iii $a^2 + 2ab + b^2$ **iv** $k^2 - 10k + 25$

v $p^2 - 12p + 36$ **vi** $x^2 - 2xy + y^2$

b The two binomial factors are the same. The product of factors like these produces an expression made up of the squares of the two terms in the factor, and twice the product of the two terms in the factor.

c The expression $(a + b)^2$ is the square of $(a + b)$.

d **i** $a^2 + 4a + 4$ **ii** $x^2 + 10x + 25$

iii $p^2 + 8p + 16$ **iv** $y^2 + 20y + 100$

v $k^2 + 18k + 81$ **vi** $m^2 + 2mn + n^2$

vii $25 + 10x + x^2$ **viii** $1 + 4d + 4d^2$

ix $9w^2 + 42w + 49$

e The expression $(a - b)^2$ is the square of $(a - b)$.

f **i** $a^2 - 6a + 9$ **ii** $b^2 - 8b + 16$

iii $x^2 - 18x + 81$ **iv** $n^2 - 22n + 121$

v $4 - 4x + x^2$ **vi** $g^2 - 2gh + h^2$

vii $16w^2 - 8w + 1$ **viii** $25 - 20y + 4y^2$

ix $9m^2 - 12mp + 4p^2$

9 **a** $x^4 - 4$ **b** $x^4 - 25$ **c** $x^4 - y^4$

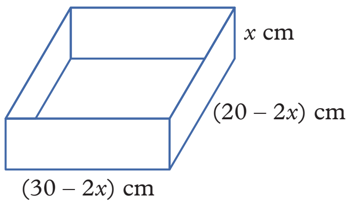
d $x^4 + 8x^2 + 16$ **e** $x^4 - 6x^2 + 9$ **f** $x^6 - 2x^3 + 1$

10 **a** **i** $(30 - 2x)$ cm

ii $(20 - 2x)$ cm

iii x cm

b



c **i** $x(30 - 2x)(20 - 2x) = 4x^3 - 100x^2 + 600x$

ii $30 \times 20 - 4x^2 = 600 - 4x^2$ (using the net)

d **i** 1008 cm^3 **ii** 564 cm^2

11 **a** $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$

b $x^2 + 4xy + 6x + 4y^2 + 12y + 9$

c $4a^2 - 9b^2 - c^2 + 6bc$

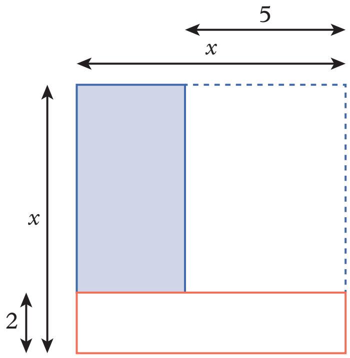
12 **a** $x^3 + 3x^2 + 3x + 1$

b $x^3 - 9x^2 + 27x - 27$

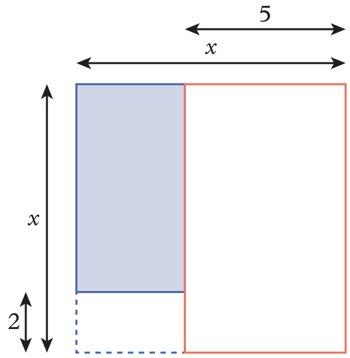
c $x^4 + 8x^3 + 24x^2 + 32x + 16$

d $x^4 - 2x^2y^2 + y^4$

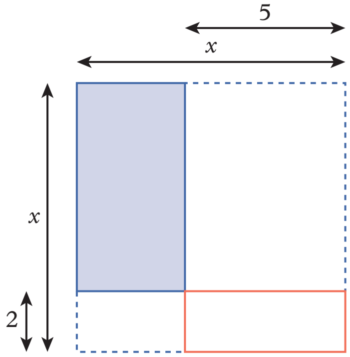
13 **a** **i**



ii



iii



b The rectangle with area 10 is subtracted as part of the 2x rectangle and as part of the 5x rectangle. Since the rectangle with area 10 is subtracted twice when it should only be subtracted once, it must be added back once.

EX p70

2E Algebraic fractions

1 **a** $\frac{x}{2}$ **b** $\frac{1}{y}$ **c** 4 **d** $\frac{2x}{3}$

e $\frac{x+3}{5}$ **f** $2(x - 5)$ **g** $x - 4$ **h** $x + 1$

2 **a** $\frac{x+5}{x+1}$ **b** $\frac{x-6}{x-2}$ **c** $\frac{x-1}{x+1}$ **d** $\frac{x+2}{x-8}$

e 1 **f** 1

3 **a** $\frac{2}{y}$ **b** $2b$ **c** $\frac{5}{a}$ **d** $\frac{x}{2}$

e $\frac{1}{3m}$ **f** $\frac{n+7}{2}$ **g** $3y$ **h** $\frac{x}{x+1}$

i $\frac{5(y-3)}{y-4}$ **j** $\frac{t+2}{r+2}$ **k** $\frac{x-1}{x-4}$ **l** $\frac{p-2}{3(p+2)}$

4 **a** $\frac{1}{2}$ **b** 5 **c** 1 **d** 3

e 5 **f** x

5 **a** $\frac{2a+1}{a-4}$ **b** $\frac{x+1}{x-3}$

c $\frac{2}{t}$ **d** $\frac{m+2}{(m-3)(m+7)}$

e $\frac{(x+3)(x+5)}{(x-3)}$ **f** $\frac{1}{x+5}$

g $\frac{3(2y+3)}{y+3}$ **h** $\frac{m(m+1)}{(m-3)(m-1)}$

i $\frac{c^2+6}{c+2}$ **j** $\frac{n-p}{n}$

k $\frac{y^2}{x^2(x+1)}$ **l** $\frac{10(a^2+b^2)}{a-b}$

6 **a** $\frac{2(x-5)}{x-2}$ **b** $\frac{(x+8)(x-6)}{(x-3)(x+6)}$

c $\frac{(3x+2)(5x-1)}{(x+1)(x+2)}$ **d** $\frac{x}{2(x+8)}$

7 **a** $\frac{9x}{20}$ **b** $\frac{5x}{14}$ **c** $\frac{x}{2}$ **d** $\frac{x}{40}$

e $\frac{7x}{6}$ **f** $\frac{5x}{8}$ **g** $\frac{43x}{35}$ **h** $\frac{7x^2}{18}$

i $\frac{34a}{15}$ **j** $\frac{2b^2-b}{4}$ **k** $\frac{5c+3}{12}$ **l** $\frac{7m-5}{12}$

m $\frac{3x-10}{10}$ **n** $\frac{12k+5}{20}$ **o** $\frac{29x-7}{40}$ **p** $\frac{7y}{8}$

q $\frac{-x+2}{6}$ **r** $\frac{b+10}{24}$ **s** $\frac{-5}{12}$ **t** $\frac{23x+18}{20}$

8 **a** $\frac{5x+13}{6}$ **b** $\frac{11x-8}{30}$ **c** $\frac{3x-7}{8}$ **d** $\frac{x+10}{12}$

e $\frac{x-12}{24}$ **f** $\frac{4x+1}{10}$ **g** $\frac{13x-6}{15}$ **h** $\frac{7x+17}{18}$

i $\frac{23x+5}{6}$ **j** $\frac{7x+15}{20}$ **k** $\frac{40-10x}{21}$ **l** $\frac{3x^2+1}{40}$

9 **a** $\frac{2}{x}$ **b** $\frac{4}{3a}$

c $\frac{b-5}{b^2}$ **d** $\frac{3-x^2}{3x}$

e $\frac{5m+3}{m(m+1)}$ **f** $\frac{4k^2+3}{k(k^2+1)}$

g $\frac{4y-7}{(y+2)(y-3)}$ **h** $\frac{x^2+2x+6}{x(x+3)}$

i $\frac{t^2+t+10}{(t+5)(t-1)}$ **j** $\frac{8x+5}{x(2x+1)}$

k $\frac{2x-1}{(x+2)(x+1)}$ **l** $\frac{-2h+11}{(2h-1)(h+2)}$

m $\frac{2m^2+3}{(m-3)(2m+1)}$ **n** $\frac{a^3+12a+4}{a(3a+1)}$

o $\frac{x^2-3x-6}{2(x+1)}$ **p** $\frac{2x+1}{x(x+1)}$

10 $a = x - 3$

11 $b = 3x + 1$

12 a $\frac{37x}{30}$ **b** $\frac{7x}{8}$ **c** $\frac{x}{8}$

d $\frac{3}{7}$ **e** $-\frac{8x}{21}$ **f** $\frac{7x-13}{12}$

13 a $\frac{(1-x^4)}{x^2}$ **b** $\frac{(x^4+2x^2+1)}{x^2}$

c $\frac{1}{x-1}$ **d** $\frac{1}{2x^2}$

14 a $\frac{1}{x(x+1)}$

b i $\frac{1}{42}$ **ii** $\frac{1}{90}$

15 a $\frac{x}{y(y+1)}$

b i $\frac{5}{42}$ **ii** $\frac{7}{110}$

16 a -1 **b** -4 **c** $\frac{x-3}{x+4}$

17 a $\frac{yz+xz+xy}{xyz}$ **b** $\frac{a^2c+ab^2+bc^2}{abc}$

18 a $\frac{(x+4)(5x-2)}{3x}$ m^2 **b** $\frac{2x^2+38x-12}{3x}$ m

2 Checkpoint

1 a x^9 **b** y^4 **c** $2m$

d a^{10} **e** $9x^2y^4$ **f** $\frac{8a^3}{b^6}$

2 a 1 **b** 2 **c** 1 **d** 3

3 a $\frac{1}{y^3}$ **b** $\frac{4}{x^2}$ **c** $\frac{1}{6a^7}$

d $2b^3$ **e** $\frac{1}{2a^2b^5}$ **f** $\frac{5d^4}{c^3}$

4 $\frac{12}{a^7b}$

5 a $6x$ **b** $8m^2$

c $7a+5b$ **d** $-3c+5d$

e $5a^2-2a$ **f** $-2x^2y+3xy+xy^2$

6 a $3ab$ **b** $8xy$

c $3ab^2$ **d** $-3x+12$

7 a $3a+21$ **b** $6y+7$

c $3x-8$ **d** $-7x+10$

8 a a^2+3a+2 **b** $x^2+2x-24$

c k^2-25 **d** $2x^2+x-6$

9 $x^3+2x^2y-xy-2y^2$

10 a $\frac{5x}{6}$ **b** $\frac{7y+2}{6}$

c $\frac{5m^2-4m+5}{20}$ **d** $\frac{11x+1}{21}$

11 a $\frac{a}{3}$ **b** $\frac{x+4}{5}$

c $\frac{1}{x+7}$ **d** $\frac{m-4}{m}$

12 $\frac{x^2-2x+6}{(x-2)(x+1)}$

EX p78

2F Factorising

1 a $5(a+7)$ **b** $6(5b+1)$ **c** $3(8+11c)$

d $4(3d+2)$ **e** $2(4e-9)$ **f** $7(5f-3)$

g $8(3+2g)$ **h** $10(1-4h)$ **i** $9(j+k)$

j $4(4w-7x)$ **k** $m(m+2)$ **l** $n(n-6)$

m $a(5+a)$ **n** $p(9-p)$ **o** $3q(q+1)$

p $2r(r-4)$

2 a $3(x-4)$ **b** $3(2y-5)$

c $8x(x-2)$ **d** $2k(2k+3)$

e $c(5+11c)$ **f** $4u(2-u)$

g $3m(2m-1)$ **h** $y^2(1+5y)$

i $2t(7-4t)$ **j** $3p^2(3p+1)$

k $a(4b+3)$ **l** $tu(5t-3)$

m $2mn(2m+n)$ **n** $3xy(2x+1)$

o $3c(5c+7d)$ **p** $6a^2bc(3b^2-2a^2c)$

3 a b, b

b $-5x, -5x, -5x$

4 a $-6(mn+3)$ **b** $-3b(a+c)$

c $-y(y+10)$ **d** $-9x(2x+1)$

e $-4k(1-2k)$ **f** $-10a(2a^2+3)$

g $-2xy(6x-7)$ **h** $-w^5(1-w^2)$

i $-2b^2(2b-1+3b^2)$

5 a $y(x-5)+2(x-5) = (x-5)\times y+(x-5)\times 2$
$= (x-5)(y+2)$

b $4a(3+k^2)-9(3+k^2) = (3+k^2)\times 4a+(3+k^2)\times(-9)$
$= (3+k^2)(4a-9)$

6 a $(x+3)(a+5)$ **b** $(k-2)(m+4)$

c $(y+5)(y+2)$ **d** $(x-1)(x+9)$

e $(k+6)(k-3)$ **f** $(p-9)(p-6)$

g $(4-a)(7+a)$ **h** $(2n-5)(3n+4)$

i $(y^2+2)(x+8)$ **j** $(a-4)(a^2+3)$

k $(y^2+1)(2x-5)$ **l** $(7-x^3)(5x+2)$

7 a $(a+4)(b+3)$ **b** $(x-7)(x+2y)$

c $(n+5)(m-6)$ **d** $(a+2)(a+5)$

e $(x-1)(x+4)$ **f** $(c+8)(c-3)$

g $(2-y)(3y+4)$ **h** $(p+2)(p^2+5)$

i $(x^2+1)(x^2+3)$ **j** $(m^2+2)(2m-3)$

k $(1+k^2)(4+k^3)$ **l** $(y-3)(x+5)$

8 a x^2-25

b Write each term as a square; that is, $x^2-25 = x^2-5^2$. Write the product of the sum and difference of the two values that are squared, x and 5. The factorised form is $(x+5)(x-5)$.

9 a $(x+2)(x-2)$ **b** $(a+6)(a-6)$

c $(10+y)(10-y)$ **d** $(8b+3)(8b-3)$

e $(5+7p)(5-7p)$ **f** $(2a+9b)(2a-9b)$

g $3(m+1)(m-1)$ **h** $2(2k+3)(2k-3)$

i $(xy+4w)(xy-4w)$ **j** $(m^3+n)(m^3-n)$

k $4(a^5+3b^3)(a^5-3b^3)$ **l** $(x^2y+w^2)(x^2y-w^2)$

m $(h+8)(h-2)$ **n** $(c-1)(c-7)$

o $4(1-x)$ **p** $(1+a+b)(1-a-b)$

q $(y-3)(7-y)$ **r** $9(2x-3)$

10 a $(x+\sqrt{7})(x-\sqrt{7})$ **b** $(a+\sqrt{13})(a-\sqrt{13})$

c $(\sqrt{19}+y)(\sqrt{19}-y)$ **d** $(4k+\sqrt{5})(4k-\sqrt{5})$

e $(p+8+\sqrt{2})(p+8-\sqrt{2})$

f $(m-4+\sqrt{6})(m-4-\sqrt{6})$

g $(x+7+\sqrt{3})(x+7-\sqrt{3})$

h $(y-1+\sqrt{19})(y-1-\sqrt{19})$

i $(d-17+\sqrt{17})(d-17-\sqrt{17})$

11 a $x^2+10x+25$

b Find the square root of the first and last terms in the expansion ($\sqrt{x^2}=x$ and $\sqrt{25}=5$). Note that the middle term is twice the product of these two values, and is positive because it is preceded by a '+' sign ($+10x$). Write the factorised form as the square of the sum of x and 5; that is, $(x+5)^2$.

12 a $y^2-16y+64$

b Find the square root of the first and last terms in the expansion ($\sqrt{y^2}=y$ and $\sqrt{64}=8$). Note that the middle term is twice the product of these two values, and is negative because it is preceded by a '–' sign ($-16y$). Write the factorised form as the square of the difference between y and 8; that is, $(y-8)^2$.

13 a $(x+6)^2$ **b** $(m+4)^2$ **c** $(p+10)^2$

d $(y-3)^2$ **e** $(x-7)^2$ **f** $(g-2)^2$

g $(1+a)^2$ **h** $(5-x)^2$ **i** $(9-b)^2$

j $(w+x)^2$ **k** $(k-m)^2$ **l** $(3x+2)^2$

14 a i $(x+10)$ cm **ii** $(2x+3)$ cm

b i $4x$ cm **ii** $6(x+1)$ cm

15 a $\frac{a}{b}$ **b** $\frac{x}{3y}$ **c** c^2-4b

d mn **e** $\frac{x-y}{2}$ **f** $\frac{3}{x}$

g $\frac{3x}{4}$ **h** $\frac{m+7}{5m}$ **i** $\frac{a+b}{a-b}$

j $\frac{x+3}{x-3}$ **k** $\frac{4a+5}{2a}$ **l** $\frac{2x+7}{7-2x}$

16 a $\frac{6}{x-7}$ **b** 1

17 a $(n+2)$ and $(n+4)$ **b** $3n+6$

c $3(n+2)$; this expression is 3 times the middle odd number.

d $425 \times 3 = 1275$

e $n+(n+2)+(n+4)+(n+6)+(n+8)=5n+20$
$=5(n+4)$

The expression describes a value that is 5 times the middle number.

f $2675 \times 5 = 13\,375$

18 a The sum of k odd numbers (where k is an even number)

$= k \times$ (the average of the two middle odd numbers)

or $k \times (n+k-1)$

or $k \times$ (the average of the first and last of the odd numbers)

b 10 (the average of 81 and 83) $= 10 \times 82 = 820$

or $10(73+10-1)=820$ or $10([73+91] \div 2)=820$

19 15

20 a 399 **b** 896 **c** 6391 **d** 2484

21 a 101 **b** 50

c $50 \times 101 = 5050$ **d** $\frac{n}{2}(n+1)$

EX p83

2G Factorising quadratic expressions

1 a 7 and 1 **b** 8 and 2 **c** 9 and 5

d 10 and 6 **e** 9 and 4 **f** 8 and 3

2 a $(x+7)(x+1)$ **b** $(x+8)(x+2)$

c $(x+9)(x+5)$ **d** $(x+10)(x+6)$

e $(x+9)(x+4)$ **f** $(x+8)(x+3)$

3 a $(x+2)(x+1)$ **b** $(x+3)(x+2)$

c $(x+10)(x+1)$ **d** $(x+9)(x+3)$

e $(x+5)(x+2)$ **f** $(x+7)(x+2)$

g $(x+3)(x+3)=(x+3)^2$ **h** $(x+11)(x+1)$

i $(x+9)(x+2)$ **j** $(x+8)(x+4)$

k $(x+9)(x+7)$ **l** $(x+10)(x+4)$

m $(x+6)(x+5)$ **n** $(x+11)(x+2)$

o $(x+12)(x+3)$

4 a 8 and –1 **b** –6 and –1 **c** –7 and 5

d 5 and –3 **e** –4 and 2 **f** –6 and –5

5 a $(x+8)(x-1)$ **b** $(x-6)(x-1)$

c $(x-7)(x+5)$ **d** $(x+5)(x-3)$

e $(x-4)(x+2)$ **f** $(x-6)(x-5)$

6 a $(x+6)(x-2)$ **b** $(x-10)(x+2)$

c $(x-3)(x-2)$ **d** $(x+9)(x-3)$

e $(x-8)(x-5)$ **f** $(x-5)(x+4)$

g $(x+7)(x-3)$ **h** $(x-11)(x-3)$

i $(x-7)(x+4)$ **j** $(x+4)(x-3)$

k $(x-9)(x+2)$ **l** $(x-8)(x-7)$

7 a $2(x+8)(x+1)$ **b** $3(x+3)(x+2)$

c $7(x+4)(x-2)$ **d** $-5(x+3)(x+4)$

e $-4(x-5)(x-2)$ **f** $-1(x+6)(x-1)$

g $2(x-2)(x+9)$ **h** $19(x-3)(x+1)$

i $-3(x-13)(x+2)$ **j** $-2(x-1)(x-2)$

k $\frac{1}{2}(x-1)(x+4)$ **l** $\frac{1}{3}(x+6)(x-3)$

8 a $x+4$ **b** $\frac{1}{x-2}$ **c** $\frac{x}{x-2}$

d $\frac{x-5}{x+3}$ **e** $\frac{2(x-5)}{x+2}$ **f** $\frac{x}{x+2}$

g $\frac{x+1}{1-x}$ **h** $\frac{2(x-3)}{x+4}$

9 a 2 **b** $\frac{x-6}{3}$ **c** $\frac{5x-2}{6}$ **d** $\frac{29x-19}{24}$

10 a $(x+11)(x+4)$ **b** $(x-5)(x+3)$

c $(x+2)^2$ **d** $3(x-4)(x-2)$

e $5(x+3)(x-3)$ **f** $2x(x-4)$

11 a x^2+3^2 is the sum of two squares.

b $(x-3)^2+4^2$ is the sum of two squares.

c $(x+1)^2+2^2$ is the sum of two squares.

d There are no numbers that add to –4 and multiply to 5.

e The expression simplifies to $x^2 + 2^2$, which is the sum of two squares.

f There are no numbers that add to 12 and multiply to 40.

12 a $2x^2$ square metres and $(12x + 18)$ square metres

b $(2x^2 + 12x + 18)$ square metres

c $(x + 3)$ m

13 $(x + 6)$ m

14 a i $x^2 + 2xy + y^2$ **ii** $x^2 + 5xy + 6y^2$

b i $(x + 2)(x + 8)$ **ii** $(x + 2y)(x + 8y)$

c i $(x + 5)(x + 2)$ **ii** $(x + 5y)(x + 2y)$

d i $(x - 4y)(x - 2y)$

ii $(y - 7x)(y - 2x)$ or $(7x - y)(2x - y)$

iii $(4x + y)(x + y)$

15 a $x(x + 6)$ **b** not possible

c not possible **d** not possible

e not possible **f** $(x + 1)(x + 5)$

g not possible **h** not possible

i $(x + 2)(x + 4)$ **j** $(x + 3)^2$

16 a $k = 5, 8, 9$ (positive) or $k = 0$ or $k = -7, -16, -27, -40\ldots$ (negative)

The negative case can be expressed as any number of the form $n(-n - 6) = -n(n + 6)$ where n is any positive integer.

b $k = 19, 36, 51, 64, 75, 84, 91, 96, 99, 100$ (positive) or $k = 0$ or $k = -21, -44, -69, -96\ldots$ (negative)

The negative case can be expressed as any number of the form $-n(n + 20)$ where n is any positive integer.

The positive case can be expressed as any number of the form $n(20 - n)$ where n is positive and less than 20.

EX p89

2H Completing the square

1 a i $x^2 + 6x + 9$ **ii** $(x + 3)(x + 3) = (x + 3)^2$

b i $x^2 - 18x + 81$ **ii** $(x - 9)(x - 9) = (x - 9)^2$

c i $(x + 4)^2$ **ii** $(x + 2)^2$

iii $(x - 6)^2$ **iv** $(x + 8)^2$

v $(x - 5)^2$ **vi** $(x - 10)^2$

2 a 9 **b** 25 **c** 16

d 4 **e** 1 **f** 64

3 a $x^2 + 14x + 49$ **b** $x^2 - 6x + 9$

c $x^2 + 18x + 81$ **d** $x^2 + 20x + 100$

e $x^2 - 12x + 36$ **f** $x^2 + 22x + 121$

4 a $(x + 7)^2$ **b** $(x - 3)^2$ **c** $(x + 9)^2$

d $(x + 10)^2$ **e** $(x - 6)^2$ **f** $(x + 11)^2$

5 $(x + 3)^2 - (\sqrt{2})^2 = (x + 3 + \sqrt{2})(x + 3 - \sqrt{2})$

6 a $(x + 4 + \sqrt{7})(x + 4 - \sqrt{7})$

b $(x - 5 + \sqrt{11})(x - 5 - \sqrt{11})$

c $(x + 2 + \sqrt{10})(x + 2 - \sqrt{10})$

d $(x + 1 + \sqrt{6})(x + 1 - \sqrt{6})$

e $(x - 3 + \sqrt{13})(x - 3 - \sqrt{13})$

f $(x - 6 + \sqrt{23})(x - 6 - \sqrt{23})$

g $(x + 5 + \sqrt{29})(x + 5 - \sqrt{29})$

h $(x - 7 + \sqrt{3})(x - 7 - \sqrt{3})$

i $(x + 7)(x + 1)$

j $\left(x + \frac{3}{2} - \frac{\sqrt{5}}{2}\right)\left(x + \frac{3}{2} + \frac{\sqrt{5}}{2}\right)$

k $\left(x - \frac{5}{2} - \frac{\sqrt{13}}{2}\right)\left(x - \frac{5}{2} + \frac{\sqrt{13}}{2}\right)$

l $\left(x + \frac{11}{2} - \frac{\sqrt{85}}{2}\right)\left(x + \frac{11}{2} + \frac{\sqrt{85}}{2}\right)$

7 a $2(x + 2 - \sqrt{3})(x + 2 + \sqrt{3})$

b $-(x + 3 - \sqrt{11})(x + 3 + \sqrt{11})$

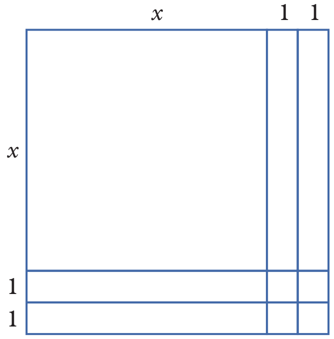
c $-5(x - 1 - \sqrt{2})(x - 1 + \sqrt{2})$

d $3\left(x - \frac{5}{2} - \frac{\sqrt{17}}{2}\right)\left(x - \frac{5}{2} + \frac{\sqrt{17}}{2}\right)$

8 a x by $y - x$ **b** x^2

c y by $y - x$ **d** y^2

9 a



b $x^2 + 4x + 4$

c $(x + 2)^2$

10 a $x(x + 6)$

b $(x + 3 + \sqrt{8})(x + 3 - \sqrt{8})$

c $(x + 3 + \sqrt{7})(x + 3 - \sqrt{7})$

d $(x + 3 + \sqrt{6})(x + 3 - \sqrt{6})$

e $(x + 3 + \sqrt{5})(x + 3 - \sqrt{5})$

f $(x + 1)(x + 5)$

g $(x + 3 + \sqrt{3})(x + 3 - \sqrt{3})$

h $(x + 3 + \sqrt{2})(x + 3 - \sqrt{2})$

i $(x + 2)(x + 4)$

j $(x + 3)^2$

11 a
$$\begin{aligned} x^2 + 7x + 6 &= \left(x + \frac{7}{2}\right)^2 - \left(\frac{5}{2}\right)^2 \\ &= \left(x + \frac{7}{2} + \frac{5}{2}\right)\left(x + \frac{7}{2} - \frac{5}{2}\right) \\ &= (x + 6)(x + 1) \end{aligned}$$

b $6 \times 1 = 6$ and $6 + 1 = 7$, so the factors are $(x + 6)$ and $(x + 1)$.

12 a $c > 16$ **b** $c > 25$ **c** $-2 < b < 2$

13 a Can be written as $(x + 3)^2 + 1$, which is the sum of two squares.

b Can be written as $(x - 4)^2 + 5$, which is the sum of two squares.

c The leading coefficient is not a highest common factor.

d The leading coefficient is not a highest common factor.

e Can be written as $2(x+1)^2+2$, is the sum of two squares.

14 a $2(x+7)(x-4)$

b Not possible as it is the sum of two squares: $\left(x-\frac{3}{2}\right)^2+\left(\frac{\sqrt{11}}{2}\right)^2$

c $x(x+6)(x-3)$

d $-(x-4)(x+3)$

e $-2(x+7)(x-7)$

f Not possible as it is the sum of two squares: $(2x)^2+5^2$

15 a $2\left(x+3-\sqrt{\frac{13}{2}}\right)\left(x+3+\sqrt{\frac{13}{2}}\right)$

b $2\left(x+\frac{7}{4}-\frac{\sqrt{41}}{4}\right)\left(x+\frac{7}{4}+\frac{\sqrt{41}}{4}\right)$

c $3\left(x+\frac{11}{6}-\frac{\sqrt{133}}{6}\right)\left(x+\frac{11}{6}+\frac{\sqrt{133}}{6}\right)$

16 a $\left(x+\frac{b}{2}+\sqrt{\frac{b^2}{4}-c}\right)\left(x+\frac{b}{2}-\sqrt{\frac{b^2}{4}-c}\right)$

b $\frac{b^2-4c}{4}$

EX p94

2I Factorising non-monic quadratic expressions

1 a i $ac=4, b=5$ **ii** 1, 4 **iii** $x+4x$

b i $ac=18, b=11$ **ii** 2, 9 **iii** $2x+9x$

c i $ac=16, b=17$ **ii** 1, 16 **iii** $x+16x$

d i $ac=-24, b=10$ **ii** $-2, 12$ **iii** $-2x+12x$

e i $ac=-18, b=-7$ **ii** $-9, 2$ **iii** $-9x+2x$

f i $ac=30, b=-13$ **ii** $-3, -10$ **iii** $-3x-10x$

2 a $(3x+2)(x+3)$ **b** $(4x+1)(x+4)$

c $(2x+1)(x+2)$ **d** $(2x-3)(3x+1)$

e $(3x-2)(x+4)$ **f** $(5x-3)(x-2)$

3 a $(2x+3)(x+2)$ **b** $(3x+2)(x+1)$

c $(2x+1)(2x+5)$ **d** $(7x+2)(x+1)$

e $(5x+4)(x+2)$ **f** $(2x+3)(x+3)$

g $(3x+2)(2x+1)$ **h** $(5x+1)(2x+3)$

i $(4x+5)(2x+1)$

4 a $(3x+1)(x-3)$ **b** $(4x-3)(x+2)$

c $(2x-5)(x-2)$ **d** $(5x+4)(x-1)$

e $(4x-1)(2x+3)$ **f** $(2x-3)(2x+5)$

g $(7x-4)(x-2)$ **h** $(4x-1)(3x+1)$

i $(3x+4)(x-5)$

5 a $2(2x+3)(x+1)$ **b** $3(4x+1)(x+2)$

c $2(5x-1)(x+4)$ **d** $(4x+5)(2x+3)$

e $3(2x+5)(x-2)$ **f** $4(3x+1)(x-1)$

g $-2(2x+1)(x+1)$ **h** $-3(4x-1)(x+2)$

i $-(3x+4)(2x+3)$

6 a $2x+3$ **b** $\frac{1}{x+3}$ **c** $2x+7$

d $\frac{2x-1}{x+5}$ **e** $\frac{5x+2}{2x-3}$ **f** -4

7 a $\frac{5x-3}{x-3}$ **b** $\frac{2x+7}{2x+1}$

c 1 **d** $\frac{(2x+7)(x+4)}{(2x-7)(x-4)}$

e $\frac{5x+2}{2}$ **f** $\frac{x+1}{x-3}$

8 a $3(x+3+\sqrt{2})(x+3-\sqrt{2})$

b $2(x-5+\sqrt{5})(x-5-\sqrt{5})$

c $5(x+1+\sqrt{7})(x+1-\sqrt{7})$

d $3(x+2+\sqrt{3})(x+2-\sqrt{3})$

e $2\left(x-3+\sqrt{\frac{21}{2}}\right)\left(x-3-\sqrt{\frac{21}{2}}\right)$

f $4\left(x+1+\frac{\sqrt{5}}{2}\right)\left(x+1-\frac{\sqrt{5}}{2}\right)$

g $6\left(x-2+\sqrt{\frac{29}{6}}\right)\left(x-2-\sqrt{\frac{29}{6}}\right)$

h $3\left(x+4+\sqrt{\frac{47}{3}}\right)\left(x+4-\sqrt{\frac{47}{3}}\right)$

i $9\left(x-1+\frac{\sqrt{7}}{3}\right)\left(x-1-\frac{\sqrt{7}}{3}\right)$

9 $\frac{-14}{3x+4}$

10 a $(4x-1)(x+2)$

b
$$\begin{aligned}-4x^2-7x+2=-(4x^2+7x-2)&=-(4x-1)(x+2)\\&=(-4x+1)(x+2)\\&=(1-4x)(x+2)\end{aligned}$$

c i $(3-2x)(x+1)$

ii $(1-3x)(x-5)=(3x-1)(5-x)$

iii $(7-5x)(x+2)$ **iv** $(2-x)(4x+3)$

v $(1-2x)(3x+2)$ **vi** $(1-5x)(5x-1)$

11 a $\sqrt{9x^2-4x-5}$ metres **b** $(9x+5)(x-1)$

12 a $(x+4-\sqrt{11})(x+4+\sqrt{11})$

b $2\left(x+2-\sqrt{\frac{3}{2}}\right)\left(x+2+\sqrt{\frac{3}{2}}\right)$

c $3(x+1)\left(x+\frac{5}{3}\right)$ or $(x+1)(3x+5)$

13 Parts **a**, **d** and **e** cannot be factorised.

b $(5x-4)(3x+2)$

c $3(x-2+\sqrt{3})(x-2-\sqrt{3})$

f $-3(2x-9)(x-7)$

14 a Room A: $2x+3$ by $4x-5$ metres

Room B: $4x-5$ by $4x-5$ metres

b $(24x^2-38x+10)$ m^2

c $6x-2$ by $6x-2$ metres

d $(6x-2)^2-(4x-5)^2$ m^2

e $(10x-7)(2x+3)$ m^2

f $36-\frac{24}{x}+\frac{4}{x^2}$

g 28

15 a i $(x^2+5)(x^2+2)$

ii $(x^2+12)(x^2+2)$

iii $(x^2+3)(x^2+1)$

b i $(x^2+1)(x-1)(x+1)$

ii $(x^2+4)(x-2)(x+2)$

iii $(x^2+9)(x-3)(x+3)$

c **i** $(x-2)(x+2)(x-3)(x+3)$
ii $(x-1)(x+1)(x-5)(x+5)$
iii $(x-1)(x+1)\left(x-\sqrt{10}\right)\left(x+\sqrt{10}\right)$

16 **a** $a\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a}+c$

b $a\left(x+\frac{b+\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b-\sqrt{b^2-4ac}}{2a}\right)$

c b^2-4ac

CHAPTER 2 review

Multiple-choice

1 D **2** B **3** E **4** C
5 D **6** B **7** B **8** A
9 E **10** D **11** E **12** C

Short answer

1 **a** a^7 **b** $24x^7y^5$ **c** 3 **d** $3ac^3$
2 **a** $\frac{x^3}{y^4}$ **b** $\frac{a^4}{3b^5}$ **c** $\frac{6p^4r^7}{5q^5}$
3 **a** x^8 **b** $\frac{1}{pq^2}$ **c** $\frac{1}{864a^8b^3}$
4 **a** $-6xy-3x^2+2x^2y-5xy^2$
b $-3ab^2-2a^2b^2+2a^2b$
c $-4pq-8p^2+3q^2$
5 **a** $a^2b^2c^2$ and $(2abc)^2$
b $5xyz^2, -2z^2xy$ and $3yz^2x$
c $2p^2q^2, -(pq)^2$ and $p^2 \times q \times -q$
6 **a** 14 **b** 39 **c** –16
7 **a** $x^2+4x-21$ **b** x^2+4x+4
c $x^2-8x+16$ **d** $2x^2-7x-4$
e a^2-b^2 **f** $a^3+a^2b^2+ab+b^3$
8 **a** $6x^4-2x^2-20$ **b** $4x^2-9y^2$
c $9x^4-24x^2+16$
9 **a** $2x^2-5x-6$ **b** $4x-13x^2$
c x^2-x
10 **a** $\frac{25x}{12}$ **b** $\frac{9x-22}{20}$ **c** $\frac{14x-11}{24}$
d $\frac{31}{12x}$
11 **a** $(x-3)(x-4)$ **b** $(x+1)^2(x-1)$
c $2(8+3x)(x-2)$
12 **a** $2(3x+2y)(3x-2y)$ **b** $-3(2x-3)$
c $8x$
13 **a** $(xy+\sqrt{13}\,)(xy-\sqrt{13})$
b $(x+1+\sqrt{3})(x+1-\sqrt{3})$
c $(\sqrt{7}+a-1)(\sqrt{7}-a+1)$
14 **a** $(x+2)(x+6)$ **b** $(x-9)(x+7)$
c $-(x-4)(x-2)$ **d** $-6(x+3)(x-2)$
15 **a** $\frac{x+4}{x+3}$ **b** $\frac{(x-3)(x+1)}{(x+4)(x+3)}$
16 **a** $(x+5+\sqrt{5})(x+5-\sqrt{5})$
b $(x-4+\sqrt{17})(x-4-\sqrt{17})$
17 **a** $(3x+2)(x+9)$ **b** $(4x-1)(2x+3)$
c $(2x-11)(x+2)$ **d** $(4x-3)(3x-2)$
18 **a** $2(x+3+\sqrt{11})(x+3-\sqrt{11})$
b cannot be factorised
c $3(2x-5)(x+6)$
d $-4\left(x-\frac{3}{2}+\frac{\sqrt{7}}{2}\right)\left(x-\frac{3}{2}-\frac{\sqrt{7}}{2}\right)$

Analysis

a **i** $4x^2+12x+9$ **ii** $x^2+8x+16$
b $3x^2+4x-7$
c $(3x+7)(x-1)$
d $(2x+3+(x+4))(2x+3-(x+4))$
$=(3x+7)(x-1)$
e **i** $(4x+9)(2x-1)$ **ii** $(3x-2)(x+4)$
iii $4(x+3)$ **iv** $3(x-1)(x-3)$

CHAPTER 3 Real numbers

EX p104

3A Rational and irrational numbers

1 **a** D **b** F **c** B **d** E **e** A **f** C
2 **a** rational **b** rational **c** irrational
d rational **e** irrational **f** rational
g rational **h** irrational **i** rational
j rational **k** rational **l** irrational
m rational **n** irrational **o** irrational
3 **a**, **c**, **d**, **f**, **g**, **i**, **j**, **k**, **l**, **n**, **o**, **p**
4 **a** 4.5826 **c** 2.8284 **d** –2.4495
f 11.1803 **g** –15.5885 **i** 1.4422
j 5.1962 **k** 4.7321 **l** 1.2361
n 0.4899 **o** 0.4082 **p** –0.7071
5 $\sqrt{3}, -3\sqrt{5}, \sqrt{8}, \sqrt[4]{32}$
6 **a** 11 **b** 30 **c** –32 **d** –1
e 24 **f** 14 **g** 10 **h** 5
i 13 **j** –6 **k** 11 **l** 9
7 rational numbers: **a**, **b**, **c**, **f**, **g**, **k**, **l**
irrational numbers: **d**, **e**, **h**, **i**, **j**
8 **d**, **h**, **i**
9 **a** rational **b** irrational **c** irrational **d** rational
e irrational **f** irrational **g** irrational **h** irrational
10 Irrational, since it cannot be written exactly as a terminating or recurring decimal.
11 **a** 4.6856 **b** 6.7807 **c** 1.6042
d 4.9892 **e** 5.9161 **f** 54.9909
g 3.4641 **h** 2.6788 **i** 4.6291
j 0.3769 **k** 1.5157 **l** 0.1135
12 **a** $\sqrt{\frac{5}{8}}\approx 0.790\,569, \frac{\sqrt{5}}{\sqrt{8}}\approx 0.790\,569$
b yes
c $\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}$
d Irrational, because it cannot be written exactly as a terminating or recurring decimal.

e **i** $\sqrt{\frac{32}{2}} = \sqrt{16} = 4$, rational

ii $\sqrt{\frac{15}{3}} = \sqrt{5} = 2.236\ \ldots$, irrational

iii $\sqrt{\frac{7}{28}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$, rational

13 **a** **i** 2 **ii** 3

b 2, 3

c By choosing between which two perfect squares the number lies.

d **i** 3, 4 **ii** $-4, -5$ **iii** 11, 12

iv $-6, -7$ **v** 13, 14 **vi** 17, 18

14 **a** **i** 8, 9 **ii** $-5, -4$ **iii** 4, 5

iv 15, 18 **v** 5, 8 **vi** $-15, -12$

b **i** $\frac{3}{5}, \frac{4}{5}$ **ii** $\frac{5}{4}, \frac{5}{3}$ **iii** $\frac{13}{2}, 7$

iv $-1, -\frac{1}{2}$ **v** $-\frac{7}{2}, -\frac{10}{3}$ **vi** $-\frac{13}{3}, -\frac{22}{5}$

15 **a** 2, 3 **b** 1, 2 **c** 3, 4 **d** 13, 14

e 14, 21 **f** 2, 3 **g** $-27, -22$ **h** $6, \frac{20}{3}$

16 **a** $\sqrt{2^2 + 1^2} = \sqrt{5}$ **b** 2.24

17 **a** 8 cm

b **i** $\sqrt{128}$ cm **ii** 11.31 cm

18 **a** side = $\sqrt{35}$ cm, 5.92 cm; diagonal = $\sqrt{70}$ cm, 8.37 cm

b side = $\sqrt{51}$ mm, 7.14 mm; diagonal = $\sqrt{102}$ mm, 10.10 mm

c side = $\sqrt{95}$ cm, 9.75 cm; diagonal = $\sqrt{190}$ cm, 13.78 cm

19 **a** **i** $8\sqrt{2}$ cm **ii** $\sqrt{70}$ cm

iii $\sqrt{102}$ cm **iv** $\sqrt{190}$ cm

The diagonal length is equal to the side length multiplied by $\sqrt{2}$.

b The diagonal of a square with a side length of *a* is:

If *a* and *c* are a side and the diagonal respectively, then

$$c^2 = a^2 + a^2 = 2a^2$$
$$\left(\sqrt{2a}\right)^2 = \sqrt{2a} \times \sqrt{2a} = 2a^2$$
$$c^2 = \left(\sqrt{2a}\right)^2$$
$$c = \sqrt{2\text{a}}$$
$$= \sqrt{2} \times a$$

So, the diagonal of any square is equal to $\sqrt{2}$ multiplied by the side length.

20 **a** $\frac{2}{5}$ **b** $\frac{13}{20}$ **c** $\frac{49}{100}$

d $\frac{8}{5}$ **e** $\frac{257}{100}$ **f** $\frac{851}{1000}$

21 **a** 1D, 2C, 3E, 4F, 5B, 6A

b To find another number with the same recurring decimal part, so that we could subtract to eliminate the decimal part.

22 **a** $\frac{2}{9}$ **b** $\frac{7}{9}$ **c** $\frac{7}{3}$ **d** $\frac{17}{3}$

23 **a** Let $x = 0.363\,636\ldots$

$100x = 36.363\,636\ldots$

$100x - x = 36.363\,636\ldots - 0.363\,636\ldots$

$99x = 36$

$x = \frac{36}{99}$

$= \frac{4}{11}$

So $0.\overline{36} = \frac{4}{11}$

b Because the recurring part of the decimal contained two digits it was necessary to shift the decimal point two places to obtain the same recurring decimal again.

24 **a** $\frac{14}{99}$ **b** $\frac{7}{33}$ **c** $\frac{16}{33}$ **d** $\frac{172}{99}$

25 **a** $\frac{1}{3}$ **b** $\frac{8}{33}$ **c** $\frac{785}{999}$ **d** $\frac{41}{99}$

e $\frac{17}{30}$ **f** $\frac{35}{11}$ **g** $\frac{383}{90}$ **h** $\frac{3899}{1110}$

EX p109

3B Simplifying surds

1 **a** 4 **b** 9 **c** 7 **d** 2 **e** 10 **f** 5

g 18 **h** 44 **i** 1 **j** 2

2 **a** $9\sqrt{7}$ **b** $4\sqrt{3}$ **c** $2\sqrt{6}$ **d** $5\sqrt{10}$

e $3\sqrt{2}$ **f** $6\sqrt{5}$ **g** $8\sqrt{11}$ **h** $30\sqrt{5}$

i $6\sqrt{10}$

3 **a** $2\sqrt{7}$ **b** $5\sqrt{2}$ **c** $2\sqrt{6}$ **d** $3\sqrt{3}$

e $4\sqrt{3}$ **f** $5\sqrt{3}$ **g** $3\sqrt{7}$ **h** $2\sqrt{15}$

i $2\sqrt{14}$ **j** $3\sqrt{5}$ **k** $8\sqrt{2}$ **l** $6\sqrt{3}$

m $6\sqrt{5}$ **n** $7\sqrt{3}$ **o** $9\sqrt{2}$ **p** $5\sqrt{5}$

q $15\sqrt{2}$ **r** $12\sqrt{2}$ **s** $3\sqrt{30}$ **t** $9\sqrt{7}$

4 **a** $10\sqrt{5}$ **b** $6\sqrt{2}$ **c** $12\sqrt{3}$ **d** $27\sqrt{2}$

e $16\sqrt{5}$ **f** $9\sqrt{10}$ **g** $60\sqrt{6}$ **h** $-12\sqrt{3}$

i $28\sqrt{6}$ **j** $-14\sqrt{5}$ **k** $64\sqrt{3}$ **l** $60\sqrt{7}$

m $-20\sqrt{7}$ **n** $21\sqrt{11}$ **o** $-24\sqrt{17}$ **p** $1000\sqrt{3}$

q $4\sqrt{5}$ **r** $3\sqrt{6}$ **s** $16\sqrt{62}$ **t** $3\sqrt{11}$

5
$$5\sqrt{27} = 5 \times \sqrt{27}$$
$$= 5 \times \sqrt{9 \times 3}$$
$$= 5 \times \sqrt{9} \times \sqrt{3}$$
$$= 5 \times 3 \times \sqrt{3}$$
$$= 15\sqrt{3}$$

6 **a** $2\sqrt{11}$ **b** $5\sqrt{5}$ **c** $2\sqrt{30}$ **d** $6\sqrt{2}$

e $5\sqrt{6}$ **f** $-6\sqrt{6}$ **g** $3\sqrt{14}$ **h** $4\sqrt{15}$

i $-9\sqrt{2}$ **j** $7\sqrt{5}$ **k** $4\sqrt{3}$ **l** $2\sqrt{3}$

m $-8\sqrt{3}$ **n** $60\sqrt{5}$ **o** $6\sqrt{21}$ **p** $9\sqrt{2}$

q $-160\sqrt{2}$ **r** -20 **s** $-10\sqrt{10}$ **t** $3\sqrt{2}$

u $5\sqrt{7}$ **v** $-4\sqrt{6}$ **w** $\frac{7\sqrt{5}}{2}$ **x** $\frac{27\sqrt{2}}{2}$

7 **a** $4\sqrt{5}$ cm **b** $6\sqrt{5}$ cm **c** $2\sqrt{13}$ cm **d** $12\sqrt{11}$ m

8 **a**
$$3\sqrt{6} = \sqrt{3^2} \times \sqrt{6}$$
$$= \sqrt{9} \times \sqrt{6}$$
$$= \sqrt{9 \times 6}$$
$$= \sqrt{54}$$

b To be simplified to a whole number coefficient it must have first been expressed as the square root of a perfect square.

c They are equal.

d Put the entire surd and the original surd into your calculator and press equal. You should get the same answer.

e $\sqrt{48}$

9 **a** $\sqrt{45}$ **b** $\sqrt{12}$ **c** $\sqrt{32}$ **d** $\sqrt{252}$
e $\sqrt{160}$ **f** $\sqrt{275}$ **g** $\sqrt{384}$ **h** $\sqrt{125}$
i $\sqrt{240}$ **j** $-\sqrt{128}$ **k** $\sqrt{1008}$ **l** $-\sqrt{75}$

10 **a** $\sqrt{a^2 b}$

c Square the whole part then multiply it by the surd part to find the number that will be under the root sign in the entire surd.

11 **a** $\sqrt{m^2 n}$ **b** $\sqrt{49a}$ **c** $\sqrt{5x^2}$ **d** $\sqrt{100xy}$
e $\sqrt{2cd^2}$ **f** $\sqrt{36abc}$ **g** $\sqrt{a^3b^2}$ **h** $\sqrt{x^4 y^3 z^2}$

12 **a** $x\sqrt{y}$ **b** xy **c** $b\sqrt{a}$ **d** $xz\sqrt{y}$
e x^2 **f** $x^2\sqrt{y}$ **g** x^2y **h** $a^2b\sqrt{c}$

13 **a** $2^2 \times 3$
b $\sqrt{2^2 \times 3} = \sqrt{2^2} \times \sqrt{3} = 2 \times \sqrt{3} = 2\sqrt{3}$

14 **a** 2×3^3 **c** $3\sqrt{6}$

15 **a** $3\sqrt{2}$ **b** $2\sqrt{7}$ **c** $4\sqrt{2}$ **d** $3\sqrt{3}$
e $2\sqrt{6}$ **f** $5\sqrt{2}$ **g** $6\sqrt{5}$ **h** $16\sqrt{2}$
i $15\sqrt{7}$ **j** $16\sqrt{10}$ **k** $28\sqrt{3}$ **l** $18\sqrt{5}$
m $6\sqrt{5}$ **n** $35\sqrt{3}$ **o** $54\sqrt{2}$ **p** $5\sqrt{5}$
q $30\sqrt{2}$ **r** $12\sqrt{2}$

16 **a** $\sqrt{18}\,\text{m} = 3\sqrt{2}\,\text{m}$
b $\sqrt{2}\,\text{m}$
c The side length of the 18 m^2 square is three times the size of the 2 m^2 square. The side length of a square divided into three smaller squares has a total area of $3 \times 3 = 9$ smaller squares which is the specified amount.
d **i** $\sqrt{50}\,\text{m} = 5\sqrt{2}\,\text{m}$
ii $\sqrt{75}\,\text{m} = 5\sqrt{3}\,\text{m}$
iii $\sqrt{27}\,\text{m} = 3\sqrt{3}\,\text{m}$
iv $\sqrt{48}\,\text{m} = 4\sqrt{2}\,\text{m}$

17 **a** **i** $2\sqrt{2}$ **ii** $3\sqrt{3}$ **iii** $5\sqrt{5}$ **iv** $a\sqrt{a}$
b **i** $4\sqrt{2}$ **ii** $9\sqrt{3}$ **iii** $25\sqrt{5}$ **iv** $a^2\sqrt{a}$
c **i** $2\sqrt[3]{2}$ **ii** $3\sqrt[3]{3}$ **iii** $5\sqrt[3]{5}$ **iv** $a\sqrt[3]{a}$
d **i** $2\sqrt[4]{2}$ **ii** $3\sqrt[4]{3}$ **iii** $5\sqrt[4]{5}$ **iv** $a\sqrt[4]{a}$

18 **a** $\sqrt{5}, \sqrt{10}, \sqrt{21}, \sqrt{22}, \sqrt{27}, \sqrt{30}, \sqrt{42}, \sqrt{46}$
b $\sqrt{6}, 3, \sqrt{11}, 2\sqrt{3}, 2\sqrt{6}, 6, \sqrt{39}, 4\sqrt{3}$
c $4, 2\sqrt{5}, \sqrt{22}, 3\sqrt{3}, \sqrt{43}, 2\sqrt{11}, 3\sqrt{6}, \sqrt{83}$
d $2\sqrt{6}, 4\sqrt{5}, 3\sqrt{14}, 2\sqrt{57}, 12\sqrt{2}, 3\sqrt{43}, 20, 7\sqrt{10}$

EX p114

3C Multiplying and dividing surds

1 **a** $\sqrt{35}$ **b** $\sqrt{33}$ **c** $-\sqrt{14}$ **d** $-\sqrt{105}$
e $\sqrt{65}$ **f** $-\sqrt{22}$ **g** $\sqrt{21}$ **h** $\sqrt{95}$
i $3\sqrt{77}$ **j** $20\sqrt{6}$ **k** $-28\sqrt{22}$ **l** $-36\sqrt{30}$
m $66\sqrt{34}$ **n** $64\sqrt{42}$ **o** $-24\sqrt{33}$ **p** $132\sqrt{15}$

2 **a** 7 **b** 3 **c** 2 **d** 6
e 5 **f** 11 **g** 10 **h** 9

3 **a** The answer is rational.
b a

4 **a** 9 **b** -77 **c** 40 **d** 60
e $3\sqrt{10}$ **f** $30\sqrt{30}$ **g** $-32\sqrt{26}$ **h** $3\sqrt{33}$
i 84 **j** 15 **k** $-6\sqrt{26}$ **l** $\sqrt{105}$
m $72\sqrt{42}$ **n** $\frac{3\sqrt{195}}{8}$ **o** $-5\sqrt{70}$ **p** $-11\sqrt{85}$

5 **a** $\sqrt{17}$ **b** $\sqrt{3}$ **c** 1 **d** $-\sqrt{5}$
e $\sqrt{6}$ **f** $\sqrt{7}$ **g** $4\sqrt{6}$ **h** $2\sqrt{5}$
i $-3\sqrt{5}$ **j** $2\sqrt{5}$ **k** $-4\sqrt{10}$ **l** $\frac{\sqrt{5}}{2}$

6 **a** 4 **b** 2 **c** 3 **d** 2

7 **a** $-12\sqrt{5}$ **b** $5\sqrt{3}$ **c** $-3\sqrt{2}$ **d** $2\sqrt{5}$
e $4\sqrt{7}$ **f** $-\frac{7\sqrt{5}}{3}$ **g** $\frac{3}{\sqrt{3}}$ **h** $\frac{\sqrt{5}}{5}$
i 3

8
$$\begin{aligned}6\sqrt{3} \times 3\sqrt{2} &= 6 \times \sqrt{3} \times 3 \times \sqrt{2}\\ &= 6 \times 3 \times \sqrt{3} \times \sqrt{2}\\ &= 18 \times \sqrt{3 \times 2}\\ &= 18\sqrt{6}\end{aligned}$$

9
$$\begin{aligned}\frac{24\sqrt{40}}{8\sqrt{20}} &= \frac{24}{8} \times \frac{\sqrt{40}}{\sqrt{20}}\\ &= 3 \times \sqrt{\frac{40}{20}}\\ &= 3\sqrt{2}\end{aligned}$$

10 **a** 6 **b** $\frac{3\sqrt{5}}{4}$ **c** $\frac{11\sqrt{17}}{7\sqrt{21}}$ **d** $-\frac{4\sqrt{15}}{15}$
e $\frac{9\sqrt{5}}{4}$ **f** $-\frac{2}{3}$ **g** -48 **h** $\frac{\sqrt{15}}{16\sqrt{2}}$
i 1 **j** $\frac{8}{\sqrt{2}}$ **k** 15 **l** $\frac{2\sqrt{3}}{3\sqrt{2}}$

11 **a** 3 **b** $\frac{9}{10}$ **c** 24 **d** $\frac{1}{3}$
e -60 **f** 2 **g** 84 **h** 576
i $\frac{24}{5}$ **j** $\frac{1}{8}$ **k** $-\frac{4}{3}$ **l** $-\frac{7}{3}$

12 **a** $2\sqrt{10}$ **b** $\frac{\sqrt{3}}{5}$ **c** $\frac{8}{5}$
d $-\frac{11\sqrt{2}}{8}$ **e** $\frac{8\sqrt{2}}{3}$ **f** $-\frac{2\sqrt{6}}{5}$
g $\frac{\sqrt{5}}{9}$ **h** $-\frac{9\sqrt{7}}{5}$ **i** $-\frac{10\sqrt{11}}{9}$

13 **a** 5 **b** 8 **c** 2 **d** 7
e 9 **f** 80 **g** 54 **h** 192
i 891

14 **a** rational
b $\sqrt{a} \times \sqrt{a} = (\sqrt{a})^2 = \sqrt{a^2} = a$
c $\sqrt{80}, \sqrt{54}, \sqrt{192}, \sqrt{891}$
d The number beneath the square root of an entire surd is equal to the square of the surd.

15 **a** x **b** ab **c** $6y$ **d** $x + y$

16 **a** $\frac{25}{7}$ **b** $\frac{1}{5}$ **c** $\frac{3}{2}$ **d** $\frac{10}{13}$

17 **a** Method 1: $1680\sqrt{6}$ Method 2: $42\sqrt{2} \times 40\sqrt{3}$
b yes
c Use Method 2 only when the surd part can be simplified. Method 2 has the added advantage that the numbers are smaller.

18 **a** $6\sqrt{2}$ **b** $\sqrt{2}$ **c** $\frac{3\sqrt{13}}{5}$ **d** $\frac{6\sqrt{5}}{7}$
e $-7\sqrt{3}$ **f** $\frac{8\sqrt{3}}{3}$ **g** $-16\sqrt{5}$ **h** $\frac{\sqrt{6}}{5}$
i $\frac{-2\sqrt{21}}{3}$ or $\frac{-2\sqrt{7}}{\sqrt{3}}$

19 **a** $\frac{11\sqrt{2}}{8}$ **b** $8\sqrt{6}$ **c** $\frac{-7\sqrt{2}}{16}$ **d** $8\sqrt{3}$

e $-\frac{15\sqrt{3}}{2}$ **f** $\frac{6\sqrt{2}}{5}$ **g** -8 **h** $\frac{32\sqrt{6}}{11}$

20 $ab\sqrt{bc}$

21 **a** **i** 50.00 cm^2 **ii** 50 cm^2

b **i** 137.48 cm^2 **ii** $30\sqrt{21}\text{ cm}^2$

c **i** 19.44 cm^2 **ii** $3\sqrt{42}\text{ cm}^2$

d **i** 169.71 mm^2 **ii** $120\sqrt{2}\text{ mm}^2$

e **i** 23.24 m^2 **ii** $6\sqrt{15}\text{ m}^2$

f **i** 6.00 cm^2 **ii** 6 cm^2

22 **a** **i** 2.51 cm^2 **ii** $\frac{4\pi}{5}\text{cm}^2$

b **i** 37.70 cm^2 **ii** $12\pi\text{ cm}^2$

c **i** 0.39 m^2 **ii** $\frac{\pi}{8}\text{m}^2$

23 **a** $240\sqrt{6}\text{ cm}^2$ **b** $96\sqrt{10}\text{ cm}^2$

c $\frac{24\sqrt{11}}{5}\text{mm}^2$ **d** $33\sqrt{30}\text{ m}^2$

24 The perimeter of the square is $4\sqrt{a}$ and the area is a. Equating the perimeter and area to form the equation $4\sqrt{a} = a$, which has the solutions $a = 0$, $a = 16$. Since the square roots of 0 and 16 are not surds, there is no value of a such that the perimeter and area are equal if the side length is a surd.

25 **a** $4\sqrt{5}\text{ m}$ **b** $4\sqrt{17}\text{ m}$ **c** $12\sqrt{2}\text{ m}$

d $8\sqrt{5}\text{ m}$ **e** $2\sqrt{3}\text{ m}$ **f** $\frac{10\sqrt{7}}{3}\text{m}$

26 **a** $\sqrt{5}$ **b** $\sqrt{6}$ **c** $\sqrt{35}$

d $\sqrt{14}$ **e** $\sqrt{39}$ **f** $\sqrt{66}$

EX
p120

3D Adding and subtracting surds

1 **a** yes **b** no **c** no **d** yes

e no **f** no **g** yes **h** yes

2 **a** $7\sqrt{3} + \sqrt{2}$ **b** $4\sqrt{3} - \sqrt{2}$

c $-3\sqrt{10} - 4\sqrt{7} - \sqrt{6} + 2\sqrt{3}$ **d** $8\sqrt{13} + 4\sqrt{39} - \sqrt{3}$

e $13\sqrt{2} - 6\sqrt{5}$ **f** $4\sqrt{6} + 4\sqrt{5} + 1$

g $3\sqrt{11} + 4\sqrt{55} - \sqrt{5}$ **h** $\sqrt{21} - \sqrt{3}$

i $7\sqrt{6} + 3\sqrt{5} + 5$ **j** $-7\sqrt{15} - 3\sqrt{7}$

3 **a** $\sqrt{2}, 4\sqrt{2}, 3\sqrt{2}$ **b** none

c $3\sqrt{12}, \sqrt{3}, 2\sqrt{27}$ **d** $9\sqrt{32}, 4\sqrt{48}, \sqrt{128}, 5\sqrt{200}$

4 **a** $13\sqrt{2}$ **b** $24 - 4\sqrt{3}$ **c** $6\sqrt{6} + 3\sqrt{5}$

d $28\sqrt{7}$ **e** $59\sqrt{2}$ **f** $28 - 4\sqrt{2}$

5 **a** $-15\sqrt{3}$ **b** $-6\sqrt{5}$ **c** $-4\sqrt{7}$

d $5\sqrt{3}$ **e** $23\sqrt{2}$ **f** 0

6 **a** $4\sqrt{5}$ **b** 0

c $4\sqrt{6}$ **d** $\sqrt{2}$

e $-19\sqrt{6} + 8\sqrt{5}$ **f** $-25\sqrt{3} + 2\sqrt{2}$

g $16\sqrt{2} - 30\sqrt{3}$ **h** 0

i $6\sqrt{3} - 2\sqrt{7} + 10\sqrt{2} - 4\sqrt{5}$ **j** $3\sqrt{5} - 2\sqrt{6} - 6\sqrt{3}$

7 **a** $5\sqrt{5}$

b $42\sqrt{3}$

c $29\sqrt{2} - 80\sqrt{3}$

d $18\sqrt{6} - 5\sqrt{5} - 14\sqrt{3}$

e $66\sqrt{7} - 8\sqrt{6} + 112\sqrt{3}$

f $10\sqrt{2} - 24\sqrt{5} + 60\sqrt{6} - 12\sqrt{10}$

g $3\sqrt{3} - 25\sqrt{5} - 61$

h $88\sqrt{2} - 113\sqrt{3}$

8 **a** 0 **b** $-22\sqrt{x}$ **c** $7\sqrt{x} - 6\sqrt{3x}$

d $-5\sqrt{2xy}$ **e** $20\sqrt{6xy}$ **f** $-x\sqrt{6y}$

g $12y\sqrt{3x}$ **h** $34x\sqrt{y} - 5x\sqrt{3y}$

i $8x\sqrt{5xy} + 10\sqrt{3xy} - 5x\sqrt{5y} - 24x\sqrt{3y}$

9 **a** $12 + 6\sqrt{3}$ **b** $8\sqrt{5} - 32$

c $20 - 30\sqrt{2}$ **d** $12\sqrt{2} + 8\sqrt{3}$

e $10\sqrt{6} - 6\sqrt{5}$ **f** $-10\sqrt{42} + 15\sqrt{70} - 10\sqrt{7}$

g $45\sqrt{2} + 12\sqrt{15}$ **h** $70 - 60\sqrt{7}$

i $12\sqrt{33} - 30\sqrt{77} + 66$ **j** $3\sqrt{5} - 10 + 25\sqrt{3}$

k $-36\sqrt{2} + 6\sqrt{30} - 81\sqrt{5}$

10 **a** $10 + 20\sqrt{6} + 6\sqrt{3} + 36\sqrt{2}$

b $15\sqrt{10} - 24\sqrt{2} + 30\sqrt{5} - 48$

c $28\sqrt{10} + 56\sqrt{30} - 24 - 48\sqrt{3}$

d $12\sqrt{3} + 15\sqrt{2} + 12\sqrt{6} + 30$

e $24\sqrt{6} - 288 - 15\sqrt{15} + 90\sqrt{10}$

f $56\sqrt{5} - 48\sqrt{30} + 21\sqrt{10} - 36\sqrt{15}$

11 **a** $31 + 10\sqrt{6}$ **b** $16 - 6\sqrt{7}$

c $57 + 12\sqrt{15}$ **d** $159 - 24\sqrt{42}$

12 **a**
$$\begin{aligned}(a + b)^2 &= (a + b)(a + b)\\ &= a^2 + ab + ba + b^2\\ &= a^2 + 2ab + b^2\end{aligned}$$

b
$$\begin{aligned}(a - b)^2 &= (a - b)(a - b)\\ &= a^2 - ab - ba + b^2\\ &= a^2 - 2ab + b^2\end{aligned}$$

13 **a** 10 **b** 116 **c** 10 **d** -67

14 **a** The terms in the second bracket are exactly the same as the terms in the first bracket but the operation sign between the terms in the second bracket is the inverse of the sign between the terms in the first bracket.

b Rational, because the surd terms add to zero each time.

c
$$\begin{aligned}(a + b)(a - b) &= a^2 - ab + ba - b^2\\ &= a^2 - b^2\\ (a - b)(a + b) &= a^2 + ab - ba - b^2\\ &= a^2 - b^2\end{aligned}$$

15 **a** $\left(32\sqrt{10} + 15\sqrt{30} - 12\sqrt{6}\right)\text{ m}^2$

b $\left(72\sqrt{5} - 40 + 10\pi\right)\text{ cm}^2$

c $\left(15 + 3\sqrt{3} + 2.5\sqrt{2} - 0.5\sqrt{6}\right)\text{ m}^2$

16 **a** $180\sqrt{2} - 18\sqrt{5}\text{ cm}^2$ **b** $409 + 80\sqrt{6}\text{ m}^2$

c $(151 - 24\sqrt{7})\pi\text{ cm}^2$ **d** $(7 + 4\sqrt{2})\pi\text{ cm}^2$

17 **a** $\left(10\sqrt{10} + 16\sqrt{5}\right)\text{ m}$ **b** $\left(36 + 2\pi\sqrt{5}\right)\text{ cm}$

18 **a** $8\sqrt{6}$ **b** $3\sqrt{15}$ **c** $\frac{73\sqrt{3}}{18}$

d $2\sqrt{10} + 6\sqrt{5}$ **e** $\sqrt{11} - 3\sqrt{7}$ **f** $6\sqrt{6} - 6$

19 $25 + 3\sqrt{2} + 3\sqrt{3} + \sqrt{5} + \sqrt{6} + \sqrt{7} + \sqrt{10} + \sqrt{11} + \sqrt{13} + \sqrt{14} + \sqrt{15}$

20 **a** **i** add $5\sqrt{5}$ **ii** add $\sqrt{3} + 7$

iii multiply $\sqrt{2}$ **iv** multiply $2\sqrt{5}$

b **i** $23\sqrt{5}, 28\sqrt{5}, 33\sqrt{5}, 38\sqrt{5}, 43\sqrt{5}, 48\sqrt{5}$

ii $6\sqrt{3} + 28, 7\sqrt{3} + 35, 8\sqrt{3} + 42, 9\sqrt{3} + 49,$
$10\sqrt{3} + 56, 11\sqrt{3} + 63$

iii $12\sqrt{2}, 24, 24\sqrt{2}, 48, 48\sqrt{2}, 96$

iv $400\sqrt{5}, 4000, 8000\sqrt{5}, 80\,000, 160\,000\sqrt{5}, 1\,600\,000$

c **i** $255\sqrt{5}$

ii $65\sqrt{3} + 315$

iii $93\sqrt{2} + 186$

iv $168\,421\sqrt{5} + 1684\,210$

21 **a** **i** $\frac{15\sqrt{2} + 30}{8} = \frac{15\sqrt{2}}{8} + \frac{15}{4}$

ii $2 + 2\sqrt{2}$

b **i** $\frac{3\sqrt{5} + 4\sqrt{3} + 6\sqrt{2}}{6} = \frac{\sqrt{5}}{2} + \frac{2\sqrt{3}}{3} + \sqrt{2}$

ii $\frac{3\sqrt{5}}{2}$

EX 3E Rationalising the denominator

p126

1 **a** **i** $\frac{1}{2}$ **ii** $\frac{3}{4}$ **iii** $\frac{3}{4}$ **iv** $\frac{2}{5}$

b The fraction appearing first in the multiplication remained unchanged.

c It is the same as 1, so each fraction has been multiplied by 1.

2 **a** **i** $\frac{2}{\sqrt{3}}$ **ii** $\frac{6}{3\sqrt{3}}$ **iii** $\frac{2\sqrt{3}}{3}$ **iv** $\frac{4}{2\sqrt{2}}$

b part **iii**

c Multiplication of both the numerator and the denominator by the surd in the denominator.

3 **a** $\frac{\sqrt{6}}{2}$ **b** $\frac{2\sqrt{7}}{7}$ **c** $\sqrt{3}$ **d** $\frac{4\sqrt{35}}{7}$

e $\frac{5\sqrt{2}}{6}$ **f** $\frac{7\sqrt{5}}{20}$ **g** $\frac{2\sqrt{15}}{5}$ **h** $-\frac{\sqrt{5}}{2}$

i $\frac{\sqrt{7}}{49}$ **j** $\frac{2\sqrt{110}}{5}$ **k** $\frac{4\sqrt{10}}{3}$ **l** $\frac{9\sqrt{30}}{20}$

4 **a** $\sqrt{2}$ **b** $\sqrt{2}$ **c** $\frac{2\sqrt{15}}{3}$ **d** $2\sqrt{6}$

e $\frac{\sqrt{3}}{3}$ **f** $\frac{5\sqrt{5}}{7}$ **g** $\frac{4\sqrt{15}}{5}$ **h** $\frac{\sqrt{2}}{4}$

i $\frac{\sqrt{10}}{2}$ **j** $-\frac{7\sqrt{5}}{6}$ **k** $\frac{1}{3}$ **l** $\frac{3\sqrt{6}}{5}$

5 **a** $-\sqrt{3}$ **b** $-\frac{\sqrt{10}}{2}$ **c** $\frac{4\sqrt{14}}{21}$ **d** $\sqrt{3}$

e $\frac{6\sqrt{5}}{7}$ **f** -6 **g** $\frac{5\sqrt{x}}{x}$ **h** $\frac{\sqrt{y}}{y}$

i $\frac{2x\sqrt{2y}}{y}$ **j** $\frac{\sqrt{xy}}{xy}$ **k** $\frac{\sqrt{2xy}}{2xy}$ **l** $-\frac{\sqrt{xyz}}{2xyz}$

6 **a** $3 - \sqrt{2}$ **b** $2 + \sqrt{3}$ **c** $\sqrt{6} + 5$

d $-4 - 3\sqrt{5}$ **e** $-3\sqrt{5} - 2$ **f** $4\sqrt{7} - 1$

g $1 - 2\sqrt{3}$ **h** $\sqrt{2} - \sqrt{3}$ **i** $\sqrt{5} + \sqrt{3}$

j $5\sqrt{6} + 2$ **k** $-3\sqrt{7} - 2$ **l** $-5 + 2\sqrt{5}$

7 **a** $2 - \sqrt{3}$ **b** $\frac{9 + 3\sqrt{2}}{7}$ **c** $\frac{30 - 5\sqrt{5}}{31}$

d $\frac{-9 - 12\sqrt{5}}{71}$ **e** $\frac{5\sqrt{2} - 6}{7}$ **f** $\frac{12 - 2\sqrt{3}}{11}$

g $\frac{3\sqrt{5} + 3}{4}$ **h** $-6\sqrt{42} + 15\sqrt{7}$ **i** $4\sqrt{6} + 8\sqrt{2}$

j $\sqrt{15} - 2\sqrt{3}$ **k** $\frac{10\sqrt{5} - 3\sqrt{30}}{23}$ **l** $\frac{6\sqrt{10} + 12\sqrt{5}}{5}$

8 **a** $\frac{2\sqrt{6} - 3\sqrt{2} + 2\sqrt{3} - 3}{3}$

b $\frac{20 - 5\sqrt{2} + 8\sqrt{3} - 2\sqrt{6}}{14}$

c $\frac{3 + \sqrt{5} + 3\sqrt{3} + \sqrt{15}}{4}$

d $\frac{3\sqrt{6} - 3 + \sqrt{30} - \sqrt{5}}{5}$

e $\frac{6\sqrt{15} - 4\sqrt{3} - 6\sqrt{5} + 4}{41}$

f $\frac{10\sqrt{10} - 15\sqrt{2} - 2\sqrt{5} + 3}{11}$

9 **a** $17 - 12\sqrt{2}$ **b** -1

c $\frac{x^2 + x\sqrt{y}}{x^2 - y}$ **d** $\frac{x + y\sqrt{x}}{x - y^2}$

e $\frac{x^2\sqrt{x} + xy\sqrt{x} - xy - y^2}{x^3 - y^2}$ **f** $\frac{x^2 - 2x\sqrt{y} + y}{x^2 - y}$

10 **a** $\frac{3\sqrt{2} + 1}{2}$ **b** $\frac{3\sqrt{2} + 1}{2}$

11 **a** $\frac{3\sqrt{5} + 1}{5}$ **b** $\frac{4\sqrt{3} - 1}{3}$ **c** $\frac{9\sqrt{2} + 4\sqrt{3}}{6}$

d $\frac{\sqrt{3} + 9\sqrt{2}}{12}$ **e** $\frac{9\sqrt{5} - \sqrt{6}}{9}$ **f** $\frac{12\sqrt{6} + 5\sqrt{15}}{60}$

12 **a** $\frac{\sqrt{5} + 1}{4} + \frac{3\sqrt{5} - 3}{4}$ **b** $\frac{2\sqrt{5} - 1}{2}$

13 **a** $2\sqrt{6}$ **b** -8

c $\frac{2\sqrt{6} + 5\sqrt{3} - 3}{5}$ **d** $5\sqrt{2} - 4\sqrt{5} - 13$

e $\frac{106\sqrt{3} - 5\sqrt{6}}{238}$ **f** $\frac{16\sqrt{3} - 25}{13}$

14 **a** $\frac{\sqrt{2}}{2}$ **b** $\frac{\sqrt{3}}{15}$ **c** $\frac{\sqrt{7}}{2}$

d $\frac{4\sqrt{3}}{9}$ **e** $\frac{\sqrt{6} - 2}{2}$ **f** $\frac{7 - 3\sqrt{5}}{2}$

15 **a** 2 cm^2 **b** 1 cm^2 **c** $\frac{1}{\sqrt{2}}$ cm **d** $\frac{1}{2}$

e $\frac{1}{2}\sqrt{2}$ cm

f The area of the shaded rectangle is given by both $\frac{1}{\sqrt{2}}$ and $\frac{1}{2}\sqrt{2}$. Therefore, they are equal because rationalising $\frac{1}{\sqrt{2}}$ gives $\frac{1}{2}\sqrt{2}$.

g $\frac{7}{\sqrt{10}} = \frac{7}{10}\sqrt{10}$

16 **a** **i** $\sqrt{2} - \sqrt{3}, -\sqrt{2} + \sqrt{3}$

ii $\sqrt{5} + \sqrt{3}, -\sqrt{5} - \sqrt{3}$

iii $2\sqrt{5} - 5\sqrt{6}, -2\sqrt{5} + 5\sqrt{6}$

iv $2\sqrt{5} + 3\sqrt{7}, -2\sqrt{5} - 3\sqrt{7}$

b **i** -1 **ii** 1 **iii** $-2\sqrt{6} - 5$

iv $2\sqrt{6} - 5$ **v** 1

c **i** $\sqrt{5} + \sqrt{3}$ or $-\sqrt{5} - \sqrt{3}$

ii $2\sqrt{2} - 5\sqrt{5}$ or $-2\sqrt{2} + 5\sqrt{5}$

iii $\sqrt{11} + 3\sqrt{2}$ or $-\sqrt{11} - 3\sqrt{2}$

iv $5\sqrt{13} - 4\sqrt{3}$ or $-5\sqrt{13} + 4\sqrt{3}$

d **i** $\dfrac{\sqrt{5}+\sqrt{3}}{2}$ **ii** $\dfrac{5\sqrt{5}-2\sqrt{2}}{117}$

iii $\dfrac{\sqrt{11}+3\sqrt{2}}{7}$ **iv** $\dfrac{4\sqrt{3}-5\sqrt{13}}{277}$

e **i** Lucy multiplied by the negative of the denominator, so it did not rationalise. The product is not the difference of two squares, it is the negative of the square of the denominator.

ii Lucy could use the conjugate of her new denominator $-2\sqrt{6}-5$ and multiply her final line by $\dfrac{-2\sqrt{6}-5}{-2\sqrt{6}-5}$.

f **i** $\sqrt{5}-\sqrt{3}$

ii $-4\sqrt{6}-8$

iii $\dfrac{\sqrt{15}-\sqrt{6}}{3}$

iv $\dfrac{3\sqrt{30}-10\sqrt{2}}{7}$

v $2\sqrt{5}+2\sqrt{2}$

vi $\dfrac{15\sqrt{6}-10\sqrt{10}-3\sqrt{3}+2\sqrt{5}}{7}$

vii -1

viii $\dfrac{x^2\sqrt{x}-x\sqrt{y}+xy\sqrt{x}-y\sqrt{y}}{x^3-y}$

g $2\sqrt{6}$

h $\dfrac{\sqrt{6}(\sqrt{2}+\sqrt{3}-\sqrt{5})}{3}=\dfrac{2\sqrt{3}+3\sqrt{2}-\sqrt{30}}{3}$

3 Checkpoint

1 **a** rational **b** rational **c** irrational

d rational **e** irrational **f** irrational

2 **a** surd **b** not surd **c** not surd

d not surd **e** surd **f** not surd

3 **a** $2\sqrt{2}$ **b** $3\sqrt{5}$ **c** $5\sqrt{7}$ **d** $10\sqrt{6}$

4 **a** $10\sqrt{3}$ **b** $21\sqrt{6}$ **c** $150\sqrt{2}$ **d** $80\sqrt{5}$

5 **a** $\sqrt{35}$ **b** $-2\sqrt{77}$ **c** $-90\sqrt{2}$ **d** $12\sqrt{14}$

6 **a** $\sqrt{6}$ **b** $-\dfrac{2\sqrt{17}}{3}$ **c** $-\dfrac{3}{2}$ **d** $\dfrac{5\sqrt{3}}{2}$

e $-\dfrac{\sqrt{30}}{3}$ **f** $\dfrac{6\sqrt{74}}{37}$

7 **a** 6 **b** $\dfrac{2\sqrt{6}}{9}$

8 **a** $12\sqrt{2}$ **b** $-6\sqrt{2}$

c $12\sqrt{3}+18\sqrt{5}$ **d** $17\sqrt{11}-2\sqrt{7}$

9 **a** $13\sqrt{2}$ **b** $2\sqrt{3}$

c $8\sqrt{6}+14\sqrt{3}$ **d** $17\sqrt{10}-17\sqrt{2}$

10 **a** $6\sqrt{3}$

b $18\sqrt{6}-18\sqrt{2}$

c $2\sqrt{7}+3\sqrt{5}-\sqrt{35}-6$

d $15\sqrt{15}+10\sqrt{10}+8\sqrt{6}+36$

e $30\sqrt{2}-54\sqrt{30}$

f $10\sqrt{154}-12\sqrt{14}+160\sqrt{11}-192$

11 **a** $\dfrac{5\sqrt{2}}{2}$ **b** $4\sqrt{3}$ **c** $\dfrac{2\sqrt{35}}{7}$ **d** $\dfrac{\sqrt{66}}{6}$

12 **a** $5\sqrt{2}-5$ **b** $\dfrac{\sqrt{3}+1}{2}$

c $\dfrac{4\sqrt{5}+30}{41}$ **d** $\dfrac{-13\sqrt{7}-35}{14}$

e $\dfrac{9\sqrt{5}-47}{164}$ **f** $\dfrac{-\sqrt{10}-8\sqrt{5}-2\sqrt{3}-8\sqrt{6}}{62}$

EX
p132

3F Fractional indices

1 **a** $x^{\frac{3}{5}}$ **b** $x^{\frac{1}{2}}$ **c** $x^{\frac{7}{8}}$ **d** $x^{\frac{17}{30}}$

e x **f** $x^{\frac{11}{12}}$ **g** $x^{\frac{1}{2}}$ **h** $x^{\frac{1}{2}}$

2 **a** $x^{\frac{4}{11}}$ **b** $x^{\frac{1}{4}}$ **c** $x^{\frac{1}{14}}$ **d** $x^{\frac{1}{20}}$

e $x^{\frac{1}{6}}$ **f** $x^{\frac{1}{10}}$ **g** $x^{\frac{7}{4}}$ **h** 1

3 **a** $x^{\frac{15}{28}}$ **b** $x^{\frac{1}{5}}$ **c** 1 **d** $x^{\frac{3}{10}}$

e $x^{\frac{1}{18}}$ **f** $x^{\frac{1}{2}}$ **g** x^{12} **h** $x^{\frac{1}{35}}$

4 **a** $10x^{\frac{7}{12}}$ **b** $30x^{\frac{31}{40}}$ **c** $-36x^{\frac{41}{63}}$ **d** $4x^{\frac{1}{12}}$

e $2x^{\frac{9}{40}}$ **f** $4x^{\frac{1}{30}}$ **g** $6x^{\frac{5}{6}}$ **h** $\frac{1}{4}x^{\frac{1}{10}}$

5 **a** $5^{\frac{3}{5}}x^{\frac{1}{5}}$ **b** $16^{\frac{1}{4}}x^{\frac{1}{10}}$ **c** $27^{\frac{2}{3}}x^{\frac{1}{4}}$ **d** $2x^{\frac{1}{5}}y^{\frac{3}{8}}$

e $4x^{\frac{2}{15}}y^{\frac{2}{7}}$ **f** $5x^{\frac{1}{4}}y^{\frac{3}{10}}$ **g** $5xy^{\frac{3}{10}}$ **h** $7x^{\frac{1}{3}}y^{\frac{20}{33}}$

6 **a** $28x^{\frac{7}{12}}y^{\frac{19}{20}}$ **b** $-72x^{\frac{3}{2}}y$ **c** $45x^{\frac{19}{24}}y^{\frac{7}{5}}$

d $5x^{\frac{2}{5}}y^{\frac{1}{6}}$ **e** $2x^{\frac{7}{40}}y^{\frac{4}{33}}$ **f** $x^{\frac{1}{14}}y^{\frac{1}{35}}$

7 **a** $a^m \times a^n = a^{m+n}$ **b** 3

c 3 **d** The answers are the same.

e **i** 2 **ii** 5 **iii** 10 **iv** x

f The fractional power of $\frac{1}{2}$ performs the same operation as finding the square root ($\sqrt{\ }$).

8 **a** 3

b 3

c The answers are the same.

d **i** 2 **ii** 5 **iii** 10 **iv** x

e $\sqrt[3]{\ }$ is a cube root and $\sqrt{\ }$ is a square root.

f The fractional power of $\frac{1}{3}$ performs the same operation as finding the cube root ($\sqrt[3]{\ }$).

9 **a** $(x^2)^{\frac{1}{3}} = x^{2\times\frac{1}{3}} = x^{\frac{2}{3}}$ and $\left(x^{\frac{1}{3}}\right)^2 = x^{\frac{1}{3}\times 2} = x^{\frac{2}{3}}$

b Since $(x^2)^{\frac{1}{3}}$ and $\left(x^{\frac{1}{3}}\right)^2$ are both equal to $x^{\frac{2}{3}}$, we only need to show that $x^{\frac{2}{3}}$ is equal to $\sqrt[3]{x^2}$ and $\left(\sqrt[3]{x}\right)^2$. So $x^{\frac{2}{3}} = x^{\frac{1}{3}\times 2} = \left(x^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{x}\right)^2$ and $x^{\frac{2}{3}} = x^{2\times\frac{1}{3}} = (x^2)^{\frac{1}{3}} = \sqrt[3]{x^2}$.

10 $x^{\frac{m}{n}} = x^{m\times\frac{1}{n}} = (x^m)^{\frac{1}{n}} = \sqrt[n]{x^m}$ and $x^{\frac{m}{n}} = x^{\frac{1}{n}\times m} = \left(x^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{x}\right)^m$

11 **a** 5 **b** 6 **c** 1 **d** 12

e 10 **f** 14 **g** 2 **h** 3

i 5 **j** 1 **k** 1 **l** 2

12 **i** The answers would be the reciprocal of the original answers.

ii **a** $\frac{1}{5}$ **b** $\frac{1}{6}$ **c** 1 **d** $\frac{1}{12}$

e $\frac{1}{10}$ **f** $\frac{1}{14}$ **g** $\frac{1}{2}$ **h** $\frac{1}{3}$

i $\frac{1}{5}$ **j** 1 **k** 1 **l** $\frac{1}{2}$

iii Parts **g**, **h**, **i**, **j**, **k**, and **l** can have a negative base as the root is odd (3 or 5). The cube root and 5th root of a negative number are real numbers as $(-a)^3 = -a$ so $\sqrt[3]{-a} = -a$, but the square root of a negative number is not a real number.

iv **g** $-\frac{1}{2}$ **h** $-\frac{1}{3}$ **i** $-\frac{1}{5}$

j -1 **k** -1 **l** $-\frac{1}{2}$

13 a $\sqrt{5}$ **b** $2\sqrt{2}$ **c** $2\sqrt{6}$ **d** $6\sqrt{3}$
e $2\sqrt{38}$ **f** $3\sqrt[3]{2}$ **g** $4\sqrt[3]{2}$ **h** $8\sqrt{5}$
i $9\sqrt{3}$ **j** $12\sqrt{3}$ **k** $5\sqrt{5}$ **l** $6\sqrt{6}$

14 a $x^{\frac{3}{2}}$ **b** $6^{\frac{1}{2}}$ **c** $x^{\frac{3}{2}}y^{\frac{5}{2}}$
d $3^{\frac{1}{2}}x^{\frac{1}{2}}y^{\frac{3}{2}}$ **e** $4x^{-\frac{1}{2}}y^{\frac{1}{2}}$ **f** $x^{\frac{4}{3}}y^{\frac{2}{3}}$

15 a $108x^{\frac{9}{20}}y^{\frac{13}{15}}$ **b** $\frac{1}{5}x^{\frac{2}{9}}y^{\frac{7}{20}}$ **c** $2x^{\frac{1}{5}}y^{\frac{1}{4}}$
d $6x^4y^2$ **e** $x^{\frac{3}{2}}y^{\frac{1}{2}}$ **f** $x^{\frac{1}{2}}y^{\frac{5}{6}}$
g $96^{\frac{1}{2}}x^2y^{\frac{1}{2}}z^4$ **h** $x^4y^{\frac{2}{9}}z^{\frac{2}{3}}$

16 a $4x\sqrt{y}$ **b** $\frac{2y^3\sqrt{3y}}{x\sqrt{x}}$ **c** $xy\sqrt{3x}$
d $3\sqrt[3]{xy^2}$ **e** $2\sqrt[4]{x^3y^2z}$ **f** $\frac{7y^2\sqrt{7y}}{\sqrt{x}}$

17 a $18x^{\frac{5}{12}}y^{\frac{7}{15}}$ **b** $\frac{72y^{\frac{1}{12}}}{x^{\frac{5}{12}}}$ **c** $\frac{x^{\frac{2}{3}}}{y^{\frac{1}{2}}}$
d $\frac{1}{4x^{\frac{1}{15}}y^{\frac{7}{20}}}$ **e** $\frac{1}{x^{\frac{1}{12}}y^{\frac{3}{4}}}$ **f** $\frac{16}{x^{\frac{8}{5}}y^5}$
g $10x^{\frac{7}{30}}$ **h** $\frac{y^{\frac{13}{30}}}{3^{\frac{1}{2}}x^{\frac{1}{24}}}$

18 a $\frac{3^{\frac{2}{3}}}{9}$ or $3^{-\frac{4}{3}}$ **b** $x = -\frac{2}{3}$

19 a $x = -\frac{5}{2}$ **b** $x = 11$ **c** $x = -\frac{1}{2}$

20 a Both denominators specify how many repeated parts the base is broken into. When multiplying the parts, are added; when raising to the power, the parts are multiplied. Both numerators describe the number of those repeated parts to operate together.

b i 3 **ii** 27 **iii** 243
iv 823 543 **v** 59 049 **vi** 8

c i $81^{\frac{1}{4}} = \sqrt[4]{81}$ **ii** $81^{\frac{3}{4}} = \left(\sqrt[4]{81}\right)^3$
iii $81^{\frac{5}{4}} = \left(\sqrt[4]{81}\right)^5$ **iv** $343^{\frac{7}{3}} = \left(\sqrt[3]{343}\right)^7$
v $81^{\frac{5}{2}} = \left(\sqrt[2]{81}\right)^5 = \left(\sqrt{81}\right)^5$ **vi** $256^{\frac{3}{8}} = \left(\sqrt[8]{256}\right)^3$

d The denominator of the index and the root of a surd both specify how many repeated parts the base is broken into. The numerator of the index and the index of a surd describe the number of those repeated factors of the base to multiply together.

21 a
$$\begin{aligned}\sqrt{ab} &= (ab)^{\frac{1}{2}}\\ &= a^{\frac{1}{2}} \times b^{\frac{1}{2}}\\ &= \sqrt{a} \times \sqrt{b}\end{aligned}$$

b
$$\begin{aligned}\sqrt{\frac{a}{b}} &= \left(\frac{a}{b}\right)^{\frac{1}{2}}\\ &= \frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}\\ &= \frac{\sqrt{a}}{\sqrt{b}}\end{aligned}$$

22 a $\sqrt{10^5}$ **b** $\sqrt[4]{2^{17}}$ **c** $\sqrt[25]{3^4}$ **d** $\sqrt[200]{5^{1111}}$

EX
p137

3G Logarithms

1 a $\log_3(27) = 3$ **b** $\log_4(4) = 1$ **c** $\log_5(625) = 4$
d $\log_{10}(1000) = 3$ **e** $\log_6(36) = 2$ **f** $\log_9(729) = 3$
g $\log_3(81) = 4$ **h** $\log_2(128) = 7$

2 a $2^5 = 32$ **b** $3^2 = 9$ **c** $5^2 = 25$
d $7^2 = 49$ **e** $4^2 = 16$ **f** $2^{10} = 1024$
g $6^3 = 216$ **h** $10^2 = 100$

3 a 4 **b** 4 **c** 3 **d** 3 **e** 2 **f** 3
g 2 **h** 5 **i** 4 **j** 3 **k** 2 **l** 4
m 3 **n** 3 **o** 2 **p** 5

4 a i $\frac{1}{6}$ **ii** $\log_6\left(\frac{1}{6}\right) = -1$
b i $\frac{1}{4}$ **ii** $\log_4\left(\frac{1}{4}\right) = -1$
c i $\frac{1}{4}$ **ii** $\log_2\left(\frac{1}{4}\right) = -2$
d i $\frac{1}{32}$ **ii** $\log_2\left(\frac{1}{32}\right) = -5$
e i $\frac{1}{25}$ **ii** $\log_5\left(\frac{1}{25}\right) = -2$
f i $\frac{1}{10}$ **ii** $\log_{10}\left(\frac{1}{10}\right) = -1$
g i $\frac{1}{1000}$ **ii** $\log_{10}\left(\frac{1}{1000}\right) = -3$
h i $\frac{1}{27}$ **ii** $\log_3\left(\frac{1}{27}\right) = -3$
i i 2 **ii** $\log_{\frac{1}{2}}(2) = -1$
j i 9 **ii** $\log_{\frac{1}{3}}(9) = -2$
k i 125 **ii** $\log_{\frac{1}{5}}(125) = -3$
l i 100 **ii** $\log_{0.01}(100) = -1$

5 a -3 **b** -3 **c** -2 **d** -1
e -4 **f** -6 **g** -3 **h** -2

6 a i 16 **ii** 1.1892
iii $\frac{1}{16} = 0.0625$ **iv** 0.8409
b no real number
c i no real number **ii** no real number
iii no real number **iv** no real number
v no real number **vi** no real number
d There is no real logarithm of a negative number to any positive integer base.

7 a no
b i no **ii** no **iii** no
iv no **v** no
c There is no power of base 1 that gives an answer other than 1, so there is no logarithm of any other number to base 1.

8 a Raise 2 to the power of 7. Then, take the logarithm with base 2 of 2^7. The result is 7.
b i 7 **ii** 5 **iii** 9 **iv** 140
v -6 **vi** 123 **vii** $\frac{5}{7}$ **viii** $-\frac{37}{19}$
c The logarithm with base a is the index to raise a to get a^x. x is the index to raise a to get a^x. Therefore, $\log_a(a^x) = x$.

9 a Take the logarithm to base 2 of 7. Then, raise 2 to the power of that. The result is 7.
b i 7 **ii** 5 **iii** 11 **iv** 531
v 8 **vi** 147 **vii** $\frac{6}{17}$ **viii** $\frac{514}{21}$
c The logarithm to base a is the index, y, to raise a to get x. We raise a to that index, y. Therefore, $a^{\log_a(x)} = x$.

10 a $\log_x(64) = 2$

$$x^{\log_x(64)} = x^2$$
$$64 = x^2$$
$$x = 8$$

b 8, as $x > 0$ so $x = -8$, while a solution to $x^2 = 64$, is not a solution for $\log_x(64) = 2$.

c $x^2 = 8^2$

d It is the same.

11 a 2 **b** 5 **c** 4 **d** 2

e 2 **f** 3 **g** 4 **h** 2

12 a $\log_4(x) = 3$

$$4^{\log_4(x)} = 4^3$$
$$x = 4^3$$

b 64

c It can be easily solved by multiplying the base by itself the number of times indicated by the index term.

13 a 27 **b** 25 **c** 6 **d** 10 000

e 32 **f** 64 **g** $\frac{1}{27}$ **h** $\frac{1}{4}$

14 a **i** 5.5 **ii** 3.1 **iii** 4.9

iv 2.8 **v** 7.3 **vi** 8.2

b **i** 100 000 μm **ii** 1 000 000 μm

iii 3 162 000 μm **iv** 31.62 μm

v 63 100 000 μm **vi** 19 950 μm

c **i** 10 **ii** 3.162 **iii** 31.62

iv 63.10 **v** 1 259 000 **vi** 1.259

15 a The number of cases becomes 10 times greater every 100 days.

b **i** 50 days **ii** 200 days **iii** 500 days

iv 31 623 cases **v** 1000 cases

c The growth in cases has slowed down below the point where the cumulative number of cases in country B is the same as if the number of cases had been increasing at a rate of 10 times the amount of cases every 100 days.

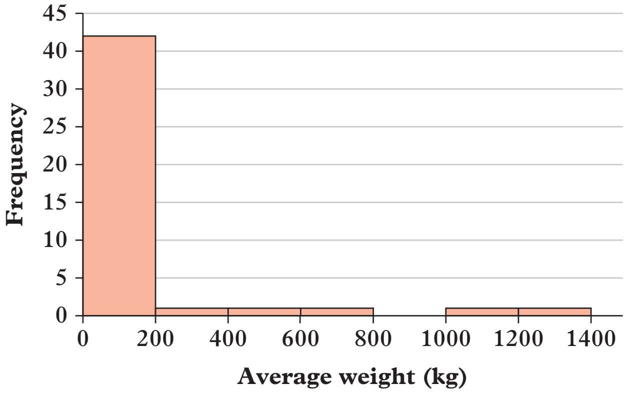
16 a **i**

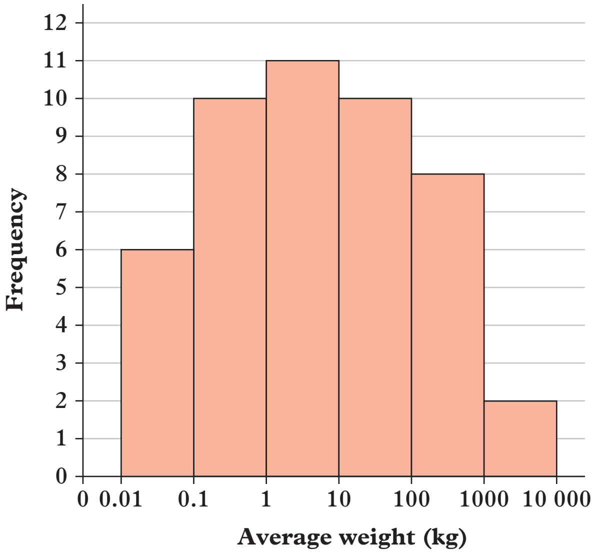
Class interval	Frequency
0–<200	42
200–<400	1
400–<600	1
600–<800	1
800–<1000	0
1000–<1200	1
1200–<1400	1

ii

Class interval	Frequency
0.01–<0.1	6
0.1–<1	10
1–<10	11
10–<100	10
100–<1000	8
1000–<10 000	2

b





c Linear scale: positively skewed
Logarithmic scale: approximately symmetric

d The logarithmic scale spreads out the average weights in the 0–<200 column (42 out of 47 weights) into five different columns and groups the data in the remaining columns (with only 5 weights) into two columns.

17 a $\frac{1}{2}$ **b** 2 **c** $-\frac{3}{2}$ **d** $\frac{7}{3}$

e $\frac{3}{2}$ **f** $\frac{3}{8}$ **g** 2 **h** $-\frac{1}{3}$

18 a **i** 10 **ii** 10 **iii** 10 000

iv 100 000 **v** 1 000 000 000 **vi** 1.259

b **i** −10 dB **ii** 10 dB **iii** −20 dB

iv −6 dB **v** 0 dB **vi** −20 dB

c **i** 5000 W **ii** 10.0 W

iii 99.8 W **iv** 500 000 W

v 240 000 000 W

vi 24 000 W

EX
p144

3H Logarithm laws

1 a 1 **b** 1 **c** 1 **d** 4 **e** 3 **f** 0

g 0 **h** 0

2 a $\log_{10}(3) + \log_{10}(5)$ **b** $\log_2(6) + \log_2(3)$

c $\log_5(2.5) + \log_5(10)$ **d** $\log_2(x) + \log_2(y)$

3 **a** $\log_3(12)$ **b** $\log_{10}(15)$ **c** $\log_2(50)$
d $\log_4(72)$ **e** $\log_a(15)$ **f** $\log_2(xy)$

4 **a** $\log_{10}(5) - \log_{10}(7)$ **b** $\log_2(5) - \log_2(3)$
c $\log_5(11) - \log_5(20)$ **d** $\log_2(a) - \log_2(b)$

5 **a** $\log_3\left(\frac{12}{5}\right)$ **b** $\log_{10}\left(\frac{50}{15}\right)$ **c** $\log_2\left(\frac{25}{10}\right)$
d $\log_4\left(\frac{36}{9}\right)$ **e** $\log_a\left(\frac{3}{15}\right)$ **f** $\log_2\left(\frac{x}{y}\right)$

6 **a** 4 **b** 2 **c** 5 **d** 4 **e** 2 **f** 3
g 2 **h** 2 **i** 9 **j** 2 **k** 2 **l** 4

7 **a** 2 **b** 2 **c** 3

8 **a** $2 \times \log_3(7)$ **b** $9 \times \log_5(2)$
c $3 \times \log_4(8)$ **d** $4 \times \log_9(11)$

9 **a** 5 **b** 2 **c** 5 **d** 8 **e** 9 **f** 12
g 10 **h** 2

10 **a** −2 **b** −1 **c** −2
d −14 **e** −30 **f** −21
g $-\frac{1}{3}$ **h** −5

11 **a** 1 **b** 4 **c** 4
d $\frac{1}{2}$ **e** 1 **f** 1

12 **a** 3 **b** 2 **c** 2
d −1 **e** −2 **f** 3

13 **a** 3 **b** 2 **c** 6 **d** 4
e 1 **f** 7 **g** 5 **h** 3
i 3 **j** 3

14 **a** 3 **b** 3 **c** 9 **d** 4 **e** 216 **f** 64
g −3 **h** 36 **i** 1 **j** $\frac{1}{2}$ **k** $\frac{1}{2}$ **l** 4

15 **a** 0.5
b $\log_{10}(9) = \log_{10}(3^2) = 2\log_{10}(3)$
c $\frac{\log_{10}(3)}{\log_{10}(9)} = \frac{\log_{10}(3)}{2\log_{10}(3)} = \frac{1}{2} = 0.5$

16 Hailey, because $\frac{\log_3(8)}{\log_3(2)} = \frac{3\log_3(2)}{\log_3(2)} = 3$

17 **a** 4 **b** $\frac{3}{2}$ **c** 3
d $\frac{3}{4}$ **e** $\frac{2}{3}$ **f** 3

18 **a** **i**

$$\begin{aligned} \log_5(2x) + \log_5(3) - \log_5(12) &= 0 \\ \log_5(6x) - \log_5(12) &= 0 \\ \log_5\left(\frac{6x}{12}\right) &= 0 \\ \log_5\left(\frac{x}{2}\right) &= 0 \end{aligned}$$

ii $5^0 = \frac{x}{2}, \frac{x}{2} = 1, x = 2$

b **i** $\log_5(2x) + \log_5(3) = \log_5(6x) = \log_5(12)$
ii $6x = 12, x = 2$

19 **a** 3 **b** 4
c 20 **d** $\frac{2}{3}$
e $\frac{1}{14}$ **f** $\frac{25\,000}{3}$, or $8333\frac{1}{3}$

20 $x = 4$ or $x = 5$

21

$$\begin{aligned} \log_{10}(2^{2x+1}) &= \log_{10}(4) \\ (2x+1)\log_{10}(2) &= \log_{10}(2^2) \\ (2x+1)\log_{10}(2) &= 2\log_{10}(2) \\ 2x+1 &= 2 \\ 2x &= 1 \\ x &= \frac{1}{2} \end{aligned}$$

22 $y = 77$

EX
p148

3I Exponential equations

1 **a** $x = 9$ **b** $x = 6$ **c** $x = -3$ **d** $x = 1$
e $x = 8$ **f** $x = 4$ **g** $x = 7$ **h** $x = -10$
i $x = 3$ **j** $x = 5$ **k** $x = 6$ **l** $x = 0$

2 **a** $x = 3$ **b** $x = 2$ **c** $x = 5$ **d** $x = 4$
e $x = 7$ **f** $x = 3$ **g** $x = 2$ **h** $x = -1$
i $x = \frac{7}{3}$ **j** $x = \frac{3}{2}$ **k** $x = \frac{3}{4}$ **l** $x = \frac{5}{4}$

3 **a** $x = \log_2(11)$ **b** $x = \log_5(92)$
c $x = \log_{92}(5)$ **d** $x = -\log_{14}(4)$
e $x = -\log_3(2)$ **f** $x = \log_7\left(\frac{53}{2}\right)$
g $x = \frac{1}{2}\log_5(17)$ **h** $x = \frac{1}{3}\log_{10}(75)$
i $x = \log_8(33)$ **j** $x = \frac{1}{6}\log_{11}(55)$
k $x = \log_{17}(12)$ **l** $x = -\frac{1}{4}\log_{12}\left(\frac{46}{9}\right)$

4 **a** $x = \log_8(10) + 1$ **b** $x = \log_3(12) - 1$
c $x = \log_5(2) - 3$ **d** $x = \log_2(6) + 5$
e $x = -\log_2(48) + 3$ **f** $x = -\log_9(20) + 2$
g $x = -\log_6(4) + 3$ **h** $x = -\log_2(30) + 6$
i $x = \frac{1}{2}\log_4(24) - \frac{3}{2}$ **j** $x = \frac{1}{4}\log_3(87) - \frac{1}{4}$
k $x = \frac{1}{5}\log_7(3) - \frac{2}{5}$ **l** $x = \frac{1}{4}\log_{10}\left(\frac{3}{2}\right) - \frac{3}{4}$

5 **a** $x = \log_8(80)$ **b** $x = \log_3(4)$
c $x = \log_5\left(\frac{2}{125}\right)$ **d** $x = \log_2(192)$
e $x = -\log_2(6)$ **f** $x = \log_9\left(\frac{20}{27}\right)$
g $x = \log_6(54)$ **h** $x = \log_2\left(\frac{32}{15}\right)$
i $x = \frac{1}{2}\log_4\left(\frac{3}{8}\right)$ **j** $x = \frac{1}{4}\log_3(29)$
k $x = \frac{1}{5}\log_7\left(\frac{3}{49}\right)$ **l** $x = \frac{1}{4}\log_{10}\left(\frac{3}{2000}\right)$

6 **a** 1.05 **b** 2.11 **c** 13.0 **d** 0.726
e −3.18 **f** 1.03 **g** 0.377 **h** 4.87

7 **a** $x = 2$ **b** $x = 3, x = -3$ **c** $x = 2, x = -2$

8 **a** $x = 2, x = 3$ **b** $x = 3, x = 4$
c $x = 6, x = 4$ **d** $x = 0, x = 1, x = 2$
e $x = -5, x = -3$ **f** $x = 1, x = \frac{5}{3}$

9 **a** 2 and 3 **b** 5 and 6 **c** 1 and 2
d 3 and 4 **e** 1 and 2 **f** 2 and 3
g 1 and 2 **h** 4 and 5

10 **a** **i** 8.5 billion **ii** 11.9 billion
b 2053. It will reach 15 billion after 103.96 years, or very close to the end of 2053.

c The model predicted 7.1 billion, so the 2010 figure was slightly less than the prediction; there have been changes to birth and death rates so, despite the population continuing to grow, the overall rate of population growth has decreased slightly since 1950.

11 a 100°C **b** 61°C **c** 4.4 **d** 20°C

e decay; temperature decreases as the amount of time increases

12 a 2 **b** 82 **c** 3.14

d population growth; population increases as the amount of time increases

13 a $\frac{9}{32} \times 64^x = 27$ **b** $x = \log_{64}(96)$

14 a $k = 0.13$ **b** 11 minutes

c 1.5 min or 1 min 30 sec or 90 sec

d $m = 0.28$

15 a The solution to the equation would be $x = \log_3(-1)$. However, the logarithm of a negative number is not a real number. Therefore, the equation has no real solution.

b The solution to the equation would be $x = \log_3(0)$. However, the logarithm of 0 is not a defined value. Therefore, the equation has no real solution.

16 a $x = 3$ **b** $x = 1$

c no real solutions **d** $x = \log_{10}(15)$

e $x = 1, x = 2$ **f** $x = \log_3(7)$

17 a 45 bacteria

b i $3\log_2(10)$ hours **ii** 10 hours

c 750 000 000 bacteria

d 5025 bacteria

e i $\log_{0.99}\left(\frac{1}{2}\right) - 1 = \log_{0.99}\left(\frac{50}{99}\right)$ hours

ii 68 hours

18 $x = -4, x = -2, x = 3, x = 7$

19 a i $\frac{\log_{10}(7)}{\log_{10}(3)}$ **ii** $\frac{\log_{10}(3)}{\log_{10}(7)}$ **iii** $\frac{\log_{10}(45)}{\log_{10}(11)}$

iv $\frac{\log_{10}(12)}{\log_{10}(101)}$ **v** $\frac{\log_{10}(24)}{\log_{10}(19)}$ **vi** $\frac{\log_{10}(b)}{\log_{10}(a)}$

b i $\log_9(8)$ **ii** $\log_8(9)$ **iii** $\log_5(18)$

iv $\log_{15}(6)$ **v** $\log_2(75)$ **vi** $\log_t(v)$

c $2^x = 54$

d

$$\begin{aligned} x &= \log_2(54) \\ 2^x &= 54 \\ \log_{10}(2^x) &= \log_{10}(54) \\ x\log_{10}(2) &= \log_{10}(54) \\ x &= \frac{\log_{10}(54)}{\log_{10}(2)} \end{aligned}$$

e

$$\begin{aligned} x &= \log_b(a) \\ b^x &= b^{\log_b(a)} \\ b^x &= a \\ \log_{10}(b^x) &= \log_{10}(a) \\ x\log_{10}(b) &= \log_{10}(a) \\ x &= \frac{\log_{10}(a)}{\log_{10}(b)} \end{aligned}$$

CHAPTER 3 review

Multiple-choice

1 A **2** D **3** C **4** D **5** B

6 B **7** E **8** A **9** B **10** E

Short answer

1 a rational **b** irrational **c** rational

d irrational **e** irrational **f** rational

2 a, b, d, f

3 a 4.899 **b** 0.059

4 a $0.65 = \frac{65}{100}$ which is a fraction and therefore rational.

b $0.2175 = \frac{2175}{10\,000}$ which is a fraction and therefore rational.

c $0.5\dot{2} = \frac{47}{90}$ which is a fraction and therefore rational.

d $1.\overline{23} = 1\frac{23}{99}$ which is a fraction and therefore rational.

5 a $4\sqrt{6}$ **b** $15\sqrt{5}$ **c** $3\sqrt{3}$

6 a $72\sqrt{2}$ **b** -150

7 a $24\sqrt{3}$ m² **b** $90\sqrt{2}$ cm²

8 a $\sqrt{77}$ **b** $27\sqrt{30}$

9 a $\sqrt{3}$ **b** $4\sqrt{3}$

10 a $-2\sqrt{2}$ **b** $2\sqrt{7} + 5\sqrt{5}$

c $2\sqrt{10} + 4\sqrt{5}$ **d** $3\sqrt{6} + 2 + 3\sqrt{5}$

11 a $2\sqrt{5} - 22\sqrt{6}$ **b** $4\sqrt{3} + 6\sqrt{2}$

c $15\sqrt{5}$ **d** $6\sqrt{2} + 24\sqrt{5} + 4\sqrt{10}$

12 a $6 - 8\sqrt{2} + 12\sqrt{3} - 16\sqrt{6}$ **b** $92 + 16\sqrt{15}$

13 a $\frac{5\sqrt{6}}{12}$ **b** $\frac{3\sqrt{3}}{2}$

c $\frac{8\sqrt{30} - 4\sqrt{15}}{21}$

d $\frac{-6\sqrt{7} - 10\sqrt{3} - 3\sqrt{42} - 15\sqrt{2}}{12}$

14 a i $x^{\frac{3}{2}}$ **ii** $\sqrt{x^3}$

b i $x^{\frac{5}{6}}$ **ii** $\sqrt[6]{x^5}$

c i $\frac{1}{9}x^{\frac{5}{12}}$ **ii** $\frac{1}{9}\sqrt[12]{x^5}$

d i $24^{\frac{1}{2}}x^2y^{\frac{3}{2}}$ **ii** $2x^2y\sqrt{6y}$

e i $3x^{\frac{1}{8}}y^{\frac{1}{5}}$ **ii** $3\sqrt[8]{x}\sqrt[5]{y}$

f i $\left(\frac{1}{5}\right)\left(\frac{3}{8}\right)^{\frac{1}{2}}x^{\frac{3}{2}}y^{\frac{1}{2}}z^{-\frac{1}{2}}$

ii $\frac{x}{10}\sqrt{\frac{3xy}{2z}}$ or $\frac{x\sqrt{6xyz}}{20z}$ (rationalising the denominator)

15 a $\log_3 81 = 4$ **b** $\log_6 216 = 3$

c $\log_2 32 = 5$ **d** $\log^x w = y$

16 a $2^3 = 8$ **b** $4^3 = 64$

c $10^{-4} = \frac{1}{10\,000}$ **d** $x^b = z$

17 a 3 **b** 2 **c** 3 **d** 3

18 a 2 **b** $\frac{1}{125}$ **c** 4

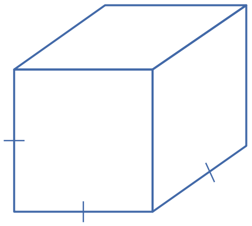
d $\frac{1}{2}$ **e** $\frac{12}{5}$ **f** $\frac{625}{4}$

19 a $x = 0$ **b** $x = -\frac{1}{4}$

c $x = \frac{1}{3}\log_5 15$ **d** $x = \frac{1}{2}\log_6 7$

Analysis

1 **a**



$7\sqrt{10} + 11\sqrt{7}$ cm

b $(1337 + 154\sqrt{70})$ cm^2

c irrational

d $(8022 + 924\sqrt{70})$ cm^2

e \$20.16, correct to the nearest cent

f You need to show that a 1 L can of the protective paint contains enough paint to cover $(8022 + 924\sqrt{70})$ cm^2 three times, an area of $1.575\ldots \times 3 \approx 4.725\ldots < 5$; it does.

g 94.5% correct to one decimal place

h The diagonal of the base is

$$\begin{aligned}\sqrt{\left(7\sqrt{10} + 11\sqrt{7}\right)^2 + \left(7\sqrt{10} + 11\sqrt{7}\right)^2} &= \sqrt{2} \times \left(7\sqrt{10} + 11\sqrt{7}\right)\\ &= 7\sqrt{20} + 11\sqrt{14}\\ &= 14\sqrt{5} + 11\sqrt{14}\\ &= 72.463\ldots\end{aligned}$$

72.463 cm < 80 cm, so it won't fit.

i Yes, diagonally from bottom corner to opposite top corner.

j You need to assume that the interior measurements are the same as exterior measurements.

2 **a** 10 000

b **i** $\log_{10}(2)$ **ii** 0.30

c $B = 10\,000 \times 2^t$

d 64 000 bees

e **i** $\log_2(100)$ **ii** 7

f $B = 10\,000\,000 \times 0.91^n$

g 74 years

CHAPTER 4 Linear relationships

4A Solving linear equations

EX
p161

1 **a** $x = \frac{19}{3}$ **b** $x = -40$ **c** $x = \frac{2}{7}$

d $x = \frac{24}{5}$ **e** $x = -53$ **f** $x = -3$

g $x = -\frac{5}{3}$ **h** $x = -8$ **i** $x = \frac{29}{12}$

j $x = -\frac{19}{2}$ **k** $x = -\frac{43}{6}$ **l** $x = \frac{62}{5}$

m $x = -\frac{32}{9}$ **n** $x = \frac{15}{2}$ **o** $x = -\frac{7}{3}$

2 **a** $x = -8\frac{1}{2}$ **b** $x = 1.7$ **c** $x = 4$

d $x = -2.464$ **e** $x = 2.4$ **f** $x = \frac{15}{77}$

g $x = \frac{1}{2}$ **h** $x = -\frac{1}{3}$ **i** $x = -\frac{20}{21}$

3 **a** $x = -5$ **b** $x = 3$ **c** $x = -2$

d $x = 6$ **e** $x = -\frac{1}{3}$ **f** $x = 7$

g $x = 3$ **h** $x = \frac{1}{9}$ **i** $x = 6$

4 **a** $x = 2$ **b** $x = -5$ **c** $x = -6$

d $x = 4$ **e** $x = 4$ **f** $x = \frac{15}{2}$

g $x = \frac{10}{3}$ **h** $x = \frac{2}{35}$ **i** $x = 2$

j $x = -1$ **k** $x = \frac{1}{4}$ **l** $x = -\frac{12}{5}$

m $x = -\frac{11}{8}$ **n** $x = \frac{35}{2}$ **o** $x = \frac{5}{6}$

5 **a** $x = 2$ **b** $x = 5$ **c** $x = -3$

d $x = 1$ **e** $x = -7$ **f** $x = -2$

g $x = 3$ **h** $x = 4$ **i** $x = \frac{3}{7}$

j $x = -4$ **k** $x = 2$ **l** $x = -5$

6 **a** $x = 5$ **b** $x = -2$ **c** $x = 3$

d $x = -1$ **e** $x = 6$ **f** $x = 2$

g $x = -4$ **h** $x = 8$ **i** $x = 4$

j $x = -1$ **k** $x = 5$ **l** $x = 9$

7 **a** $x = -8$ **b** $x = 4$ **c** $x = -2$

d $x = 2$ **e** $x = -1$ **f** $x = 4$

8 **a** $x = -8$ **b** $x = 31$ **c** $x = 8$

d $x = 1$ **e** $x = 17$ **f** $x = 13$

g $x = -4\frac{2}{3}$ **h** $x = 6\frac{1}{7}$ **i** $x = -2\frac{1}{8}$

9 **a** $x = 3$ **b** $x = -3$ **c** $x = 12$

d $x = -\frac{3}{7}$ **e** $x = \frac{14}{15}$ **f** $x = \frac{1}{11}$

10 **a** $x = b - a$ **b** $x = k + p$ **c** $x = \frac{d}{c}$

d $x = \frac{h - g}{3}$ **e** $x = \frac{y - 5}{4}$ **f** $x = mn - a$

g $x = \frac{5 + 2w}{7}$ **h** $x = a(b - c)$ **i** $x = \frac{h}{k}$

j $x = \frac{e + f}{2}$ **k** $x = \frac{v - 2y}{2}$

l $x = p(w - q)$ **m** $x = \frac{a}{b + c}$

n $x = \frac{2ny + m}{k}$ **o** $x = \frac{5c - 3d}{8}$

11 **a** Decide that n = the number of weeks of saving. Any letter will do.

b $24n + 105 = 219$

c $n = 4.75$

d It will take Liam 5 weeks to save for his new tennis racquet.

12 **a** Let m be the cost of a makeup kit.

$4m + 29 = 387$

$m = 89.5$

Cost: \$89.50

b Let the length be l m and the width w m.

$l = w + 12$

$2l + 2w = 88$

$2(w + 12) + 2w = 88$

$4w + 24 = 88$

$w = 16, l = 28$

length: 28 m, width: 16 m

c Let their scores be l, s and b respectively.

$l = s + 5, b = s - 3$

$l + s + b = (s + 5) + s + (s - 3)$

$3s + 2 = 29$

$s = 9, l = 14, b = 6$

Lisa: 14 goals, Sara: 9 goals, Bindi: 6 goals

d Let n = the number of people

$27n + 400 = 1800; n = 51.851\ldots$

51 people

13 a $6d + 3.8 = 4d + 9.2$, where d is the cost of a dumpling

b $2.70

14 a $A = 50\text{ m}^2$ **b** $h = 7$ mm

c $a = 7$ m **d** $a = \frac{2A}{h} - b$

e i $a = 52$ cm **ii** $a = 7.6$ m

15 a $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$

$$\frac{1}{c} = \frac{1}{a} + \frac{1}{b}$$
$$\frac{1}{c} = \frac{b}{ab} + \frac{a}{ab}$$
$$\frac{1}{c} = \frac{a+b}{ab}$$
$$c = \frac{ab}{a+b}$$

b i 3 cm **ii** 400 ohms

c $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$

$$\frac{1}{a} = \frac{1}{c} - \frac{1}{b}$$
$$\frac{1}{a} = \frac{b}{bc} - \frac{c}{bc}$$
$$\frac{1}{a} = \frac{b-c}{bc}$$
$$a = \frac{bc}{b-c}$$

d i 40 cm **ii** 16 ohms

16 $5x + 20 = 3x + 50; x = 15$; 15, 17, 19, 21, 23

17 a $x = \frac{33}{5}$ **b** $x = -\frac{41}{34}$ **c** $x = \frac{33}{52}$

18 a $x = -6$ **b** $x = 8$ **c** $x = -1$

d $x = 13$ **e** $x = \frac{1}{8}$ **f** $x = -2\frac{7}{10}$

19 a $x = \frac{p - qr}{nr - m} = \frac{qr - p}{m - nr}$

b $x = \frac{b^2 - 2aby - a^2}{a^2y - 2ab - b^2y} = \frac{a^2 + 2aby - b^2}{b^2y + 2ab - a^2y}$

c $x = -\frac{1}{y}$

EX
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4B Solving linear inequalities

1 a $2 < 3$ **b** $3 > 2$

c $3 \geq 2$ **d** $-3 \leq -2$

e $3 \neq -3$ **f** $-3 \neq 3$

g $4 < 5 < 6$ **h** $1 < 5 \leq 10$

i $-10 \leq -5 < -1$

2 a $x \leq 5$ **b** $x > 1$

c $x \geq -4$ **d** $x < -2$

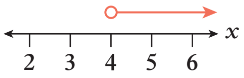
e $3 \leq x \leq 7$ **f** $-5 < x \leq 0$

g $-1 < x < 5$ **h** $x < 0$

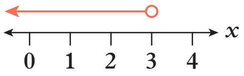
i $x \neq 3$ **k** $x \geq 0$

j $-10 \leq x < -9$ **l** $x \neq -1$

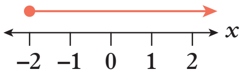
3 a $x > 4$



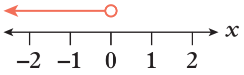
b $x \leq 3$



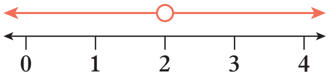
c $x \geq -2$



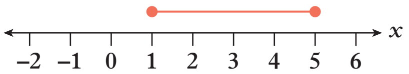
d $x < 0$



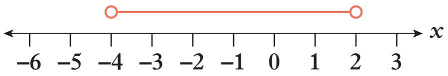
e $x \neq 2$



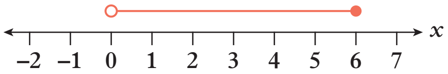
f $1 \leq x \leq 5$



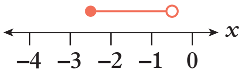
g $-4 < x < 2$



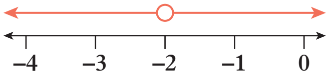
h $0 < x \leq 6$



i $-2.5 \leq x < -0.5$



j $x \neq -2$



4 a i $1 < 14$ **ii** $-13 < 0$

iii $-12 < 27$ **iv** $-2 < \frac{9}{2}$

v $12 > -27$ **vi** $2 > -\frac{9}{2}$

vii $-\frac{4}{3} < 3$ **viii** $\frac{4}{3} > -3$

b i $x + 5 \geq 11$ **ii** $x - 9 \geq -3$

iii $3x \geq 18$ **iv** $\frac{x}{2} \geq 3$

v $-3x \leq -18$ **vi** $-\frac{x}{2} \leq -3$

vii $\frac{x}{3} \geq 2$ **viii** $-\frac{x}{3} \leq -2$

c i $4 < x + 5 \leq 9$ **ii** $-10 < x - 9 \leq -5$

iii $-3 < 3x \leq 12$ **iv** $-\frac{1}{2} < \frac{1}{2}x \leq 2$

v $-12 \leq -3x < 3$ **vi** $-2 \leq -\frac{1}{2}x < -\frac{1}{2}$

vii $-\frac{1}{3} < \frac{x}{3} \leq \frac{4}{3}$ **viii** $-\frac{4}{3} \leq -\frac{x}{3} < \frac{1}{3}$

5 **a** $x > 4$ **b** $x \leq 30$ **c** $x < -5$

d $x \leq 5$ **e** $x < -70$ **f** $x \leq 15$

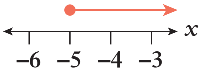
g $x \geq 2$ **h** $x > -3$ **i** $x \leq -9$

j $x \leq 2$ **k** $x > -28$ **l** $x > -\frac{9}{10}$

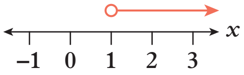
m $x < -\frac{1}{7}$ **n** $x \leq \frac{29}{6}$ **o** $x > \frac{11}{6}$

p $x \leq -\frac{95}{33}$

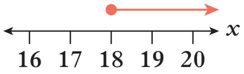
6 **a** $x \geq -5$



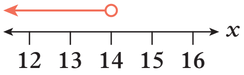
b $x > 1$



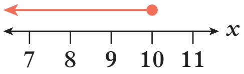
c $x \geq 18$



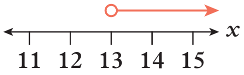
d $x < 14$



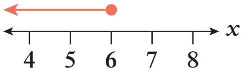
e $x \leq 10$



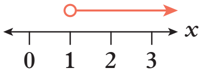
f $x > 13$



g $x \leq 6$



h $x > 1$



7 **a** $x > 4$ **b** $x \leq -2$

c $x \geq 3$ **d** $x > 4$

e $x < 1$ **f** $x \geq -1$

g $x \leq 5$ **h** $x < 2$

8 **a** $x > 11$ **b** $x \leq 13$ **c** $x \geq -2$

d $x > -19$ **e** $x > -15$ **f** $x \leq 1$

g $x \leq 3$ **h** $x > -\frac{1}{4}$ **i** $x \geq -\frac{13}{2}$

9 **a** $x \leq -7$ **b** $x > 1$ **c** $x \leq -3$

d $x < 1$ **e** $x \geq 4$ **f** $x > -5$

10 **a** one

b Linear inequalities have an infinite number of solutions within a given range.

11 **a** $p \geq 650\,000$, where p is the selling price in dollars

b $h > 97$, where h is the height of a person in centimetres

c $s \leq 60$, where s is the speed in km/h

d $125 \leq h \leq 196$, where h is the height of a person in centimetres

12 **a** $3p \leq 20$, where p is the cost of a pack of gum in dollars; $p \leq 6\frac{2}{3}$

b Todd could buy 1, 2, 3, 4, 5 or 6 packs of gum.

13 **a** $m = \frac{1}{2}(20 - x)$, where m is the number of watermelons they will each take home.

b $\frac{1}{2}(20 - x) \leq 3$; $x \geq 14$

c 14 or more watermelons

14 **a** 25 min

b 16 min

c 19 min to 29 min

15 **a** **i** $4 < 9$ **ii** $4 < 9$

iii $4 < 9$ **iv** $4 < 9$

v $\frac{1}{4} > \frac{1}{9}$ **vi** $\frac{1}{4} > \frac{1}{9}$

vii $0 < 4 < 9$ **viii** $0 < 4 < 9$

ix $4 = 4$

b The square of a negative and a positive number are both positive, so when two values in an inequality have different signs, then the order of the inequality may change.

c **i** $\frac{1}{3} < 2$ **ii** $-\frac{1}{3} < 2$

iii $-\frac{1}{3} < -2$ **iv** $\frac{1}{3} > -2$

v $3 > 2$ **vi** $-3 < 2$

vii $-2 < \frac{1}{3} < 1$ **viii** $-2 < -1 < -\frac{1}{3}$

d The greater the positive number, the smaller its reciprocal will be. The smaller the negative number, the greater its reciprocal will be. Therefore, when the signs of the values are the same, the order will reverse. However, when the signs of the values are different, the order does not change as the negative value will still be smaller than the positive value.

16 **a** $4x + 5y \leq 100$

b $x \geq 0$ and $y \geq 0$

c $x = 25$

d $y = 20$

e $x = 15$

17 **a** $x \geq -11$ **b** $x < 4$ **c** $x \geq 2$

d $x < -8$ **e** $x \leq 2$ **f** $x \leq -3$

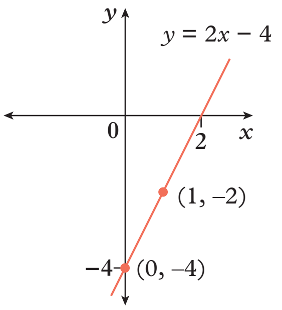
18 **a** $1 \leq x \leq 6$

b $-4 > x > -7$ or $-7 < x < -4$

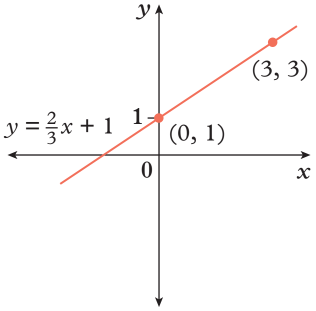
EX p175

4C Sketching linear graphs

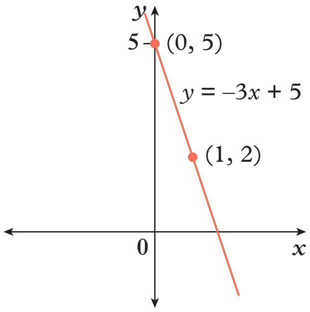
1 a



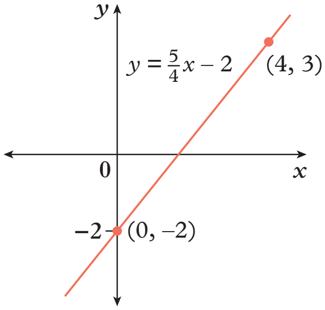
b



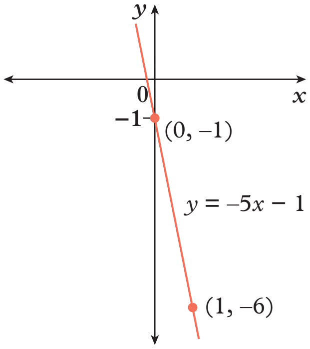
c



d



e



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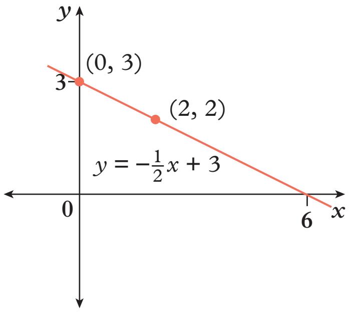
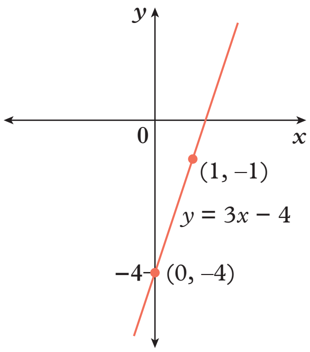
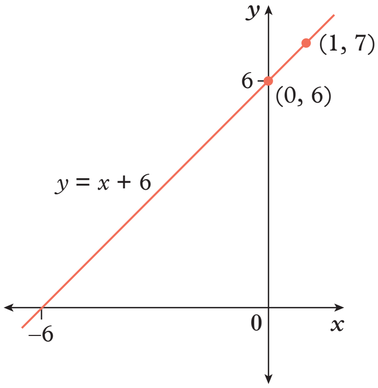


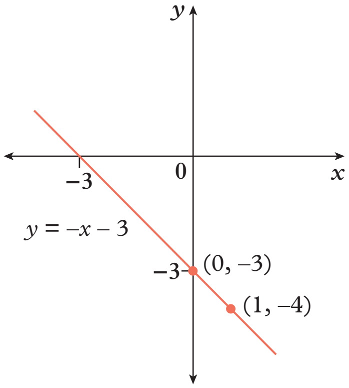
g



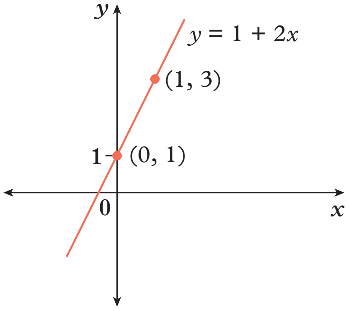
h



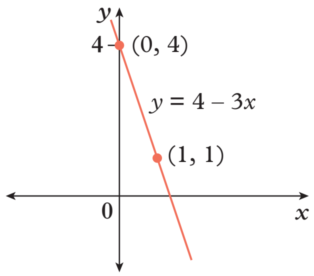
i



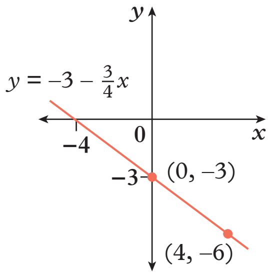
2 a $y = 2x + 1$, $m = 2$, $c = 1$



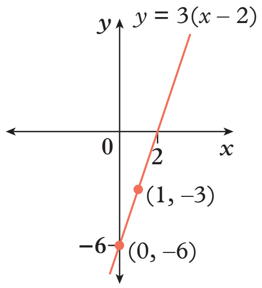
b **i, ii, iii** $y = -3x + 4$, $m = -3$, $c = 4$



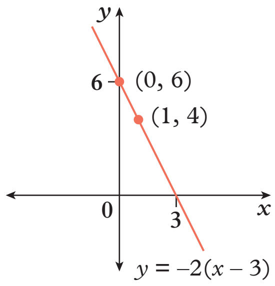
c **i, ii, iii** $y = -\frac{3}{4}x - 3$, $m = -\frac{3}{4}$, $c = -3$



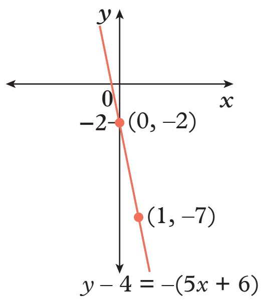
d **i, ii, iii** $y = 3x - 6$, $m = 3$, $c = -6$



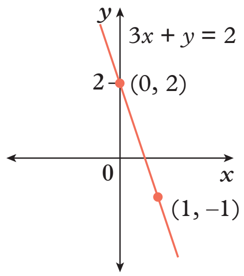
e **i, ii, iii** $y = -2x + 6$, $m = -2$, $c = 6$



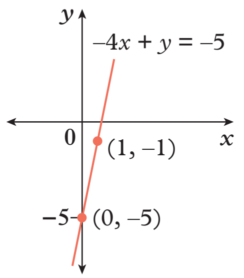
f **i, ii, iii** $y = -5x - 2$, $m = -5$, $c = -2$



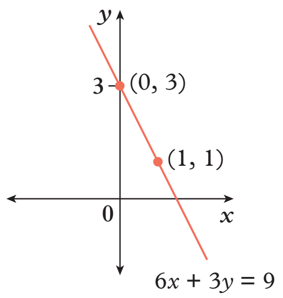
g **i, ii, iii** $y = -3x + 2$, $m = -3$, $c = 2$



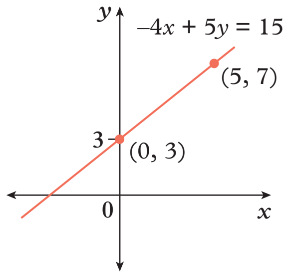
h **i, ii, iii** $y = 4x - 5$, $m = 4$, $c = -5$



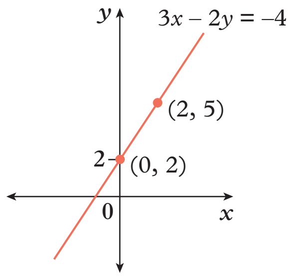
i **i, ii, iii** $y = -2x + 3$, $m = -2$, $c = 3$



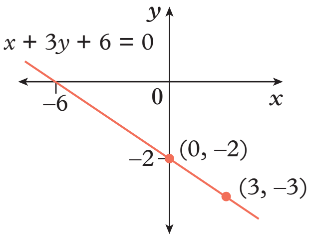
j **i, ii, iii** $y = \frac{4}{5}x + 3$, $m = \frac{4}{5}$, $c = 3$



k **i, ii, iii** $y = \frac{3}{2}x + 2$, $m = \frac{3}{2}$, $c = 2$

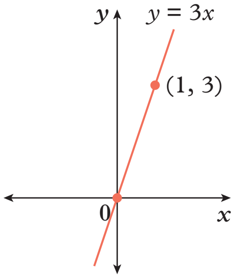


1 **i, ii, iii** $y = -\frac{1}{3}x - 2, m = -\frac{1}{3}, c = -2$

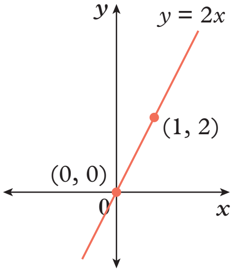


3 **a** **i** $m = 3$ **ii** $c = 0$

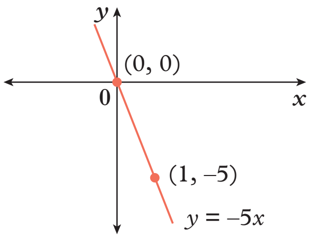
b



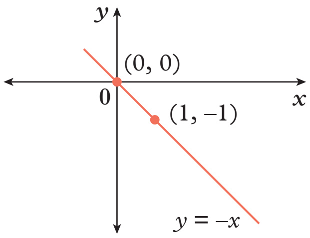
4 **a**



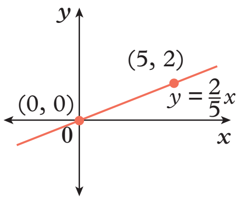
b



c

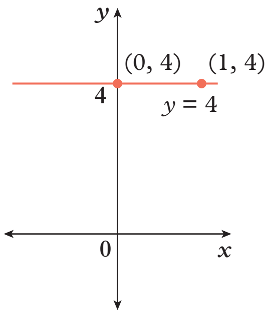


d

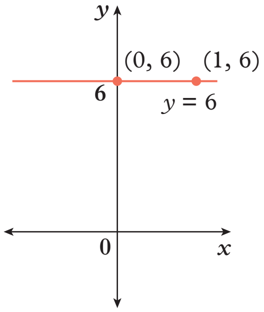


5 **a** **i** $m = 0$ **ii** $c = 4$

b



6 **a**



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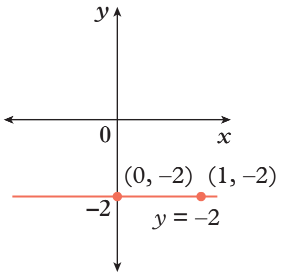

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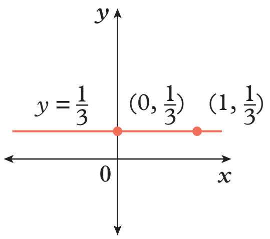

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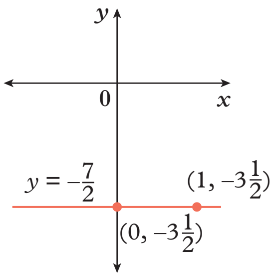
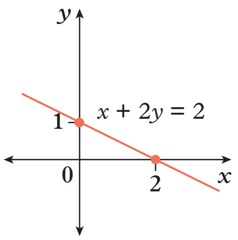


7 **a**



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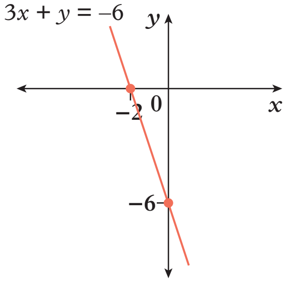

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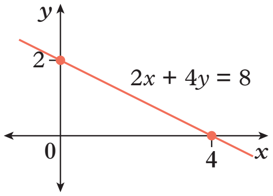

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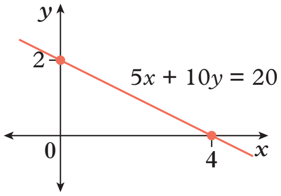
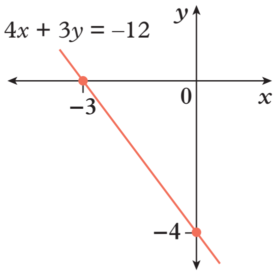


e



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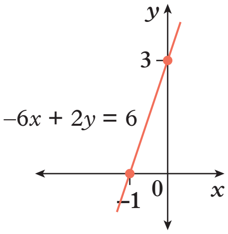

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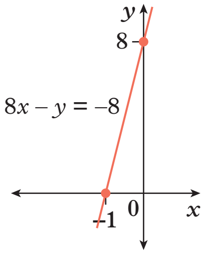

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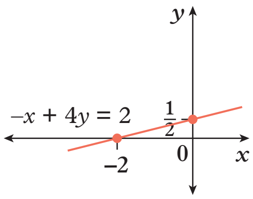

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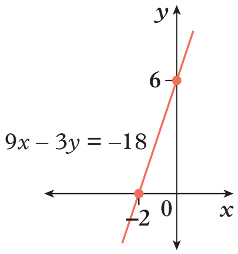

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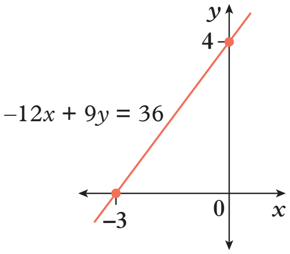

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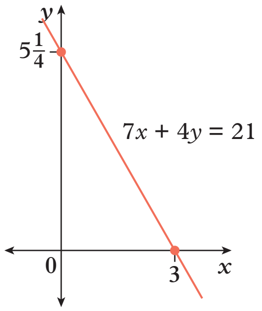

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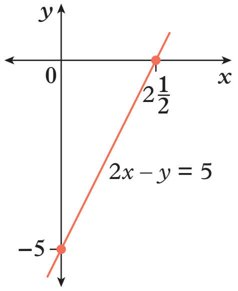
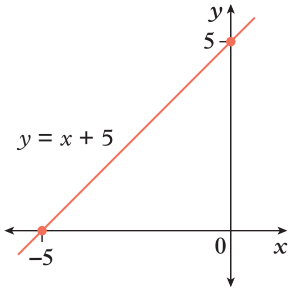
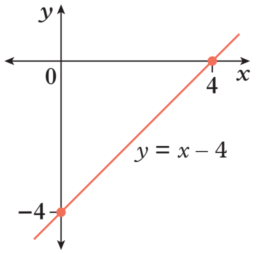


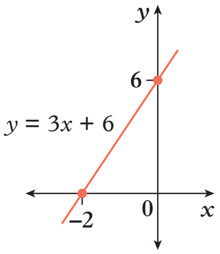
8 **a** **i, ii, iii** x-intercept: $(-5, 0)$, y-intercept: $(0, 5)$



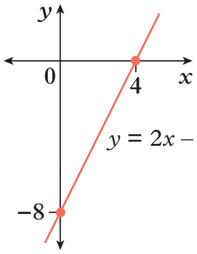
b **i, ii, iii** x-intercept: $(4, 0)$, y-intercept: $(0, -4)$



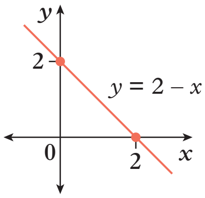
c **i, ii, iii** x-intercept: $(-2, 0)$, y-intercept: $(0, 6)$



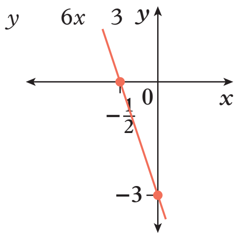
d **i, ii, iii** x-intercept: $(4, 0)$, y-intercept: $(0, -8)$



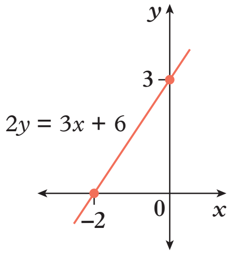
e **i, ii, iii** x-intercept: $(2, 0)$, y-intercept: $(0, 2)$



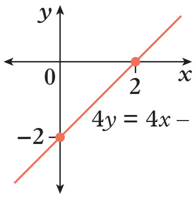
f **i, ii, iii** x-intercept: $\left(-\frac{1}{2}, 0\right)$, y-intercept: $(0, -3)$



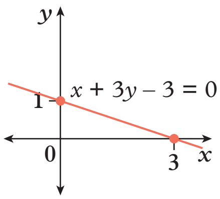
9 **a**



b



c



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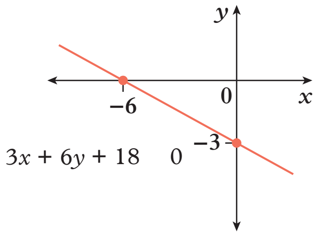

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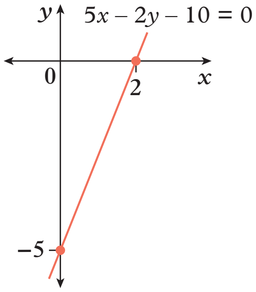

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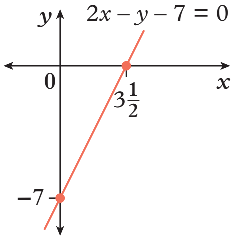


10 **a** **i** x-intercept: $(0, 0)$

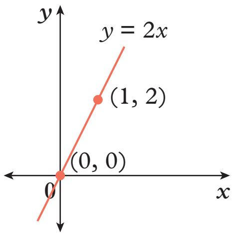
ii y-intercept: $(0, 0)$

b No; you only have one point.

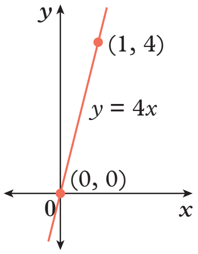
c You need the coordinates of one other point on the line.

d $y = 2$ **e** $(0, 0)$ and $(1, 2)$

f



11 **a**



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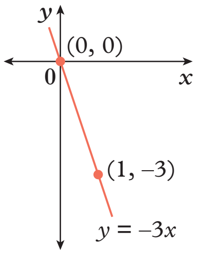
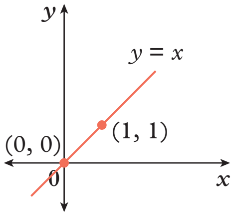


c



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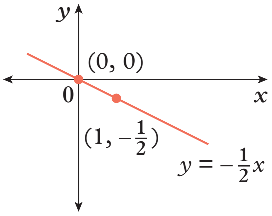
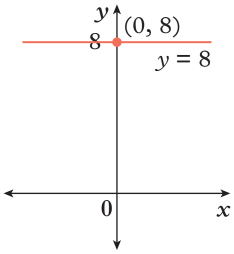


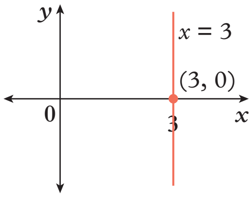
12 a i y-intercept: $(0, 8)$, no x-intercept

ii



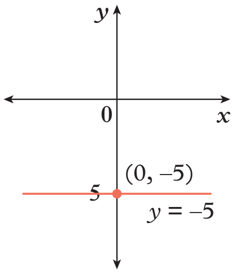
b i x-intercept: $(3, 0)$, no y-intercept

ii



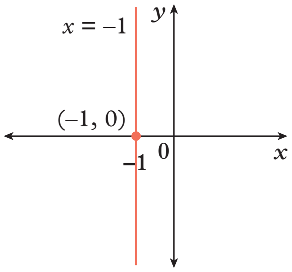
c i y-intercept: $(0, -5)$, no x-intercept

ii



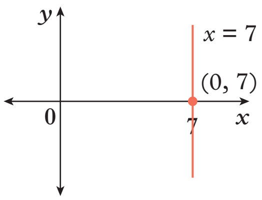
d i x-intercept: $(-1, 0)$, no y-intercept

ii



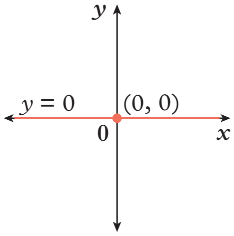
e i x-intercept: $(7, 0)$, no y-intercept

ii



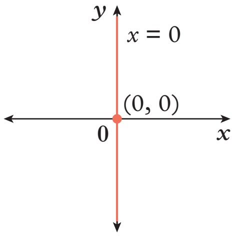
f i y-intercept: $(0, 0)$, no x-intercept

ii



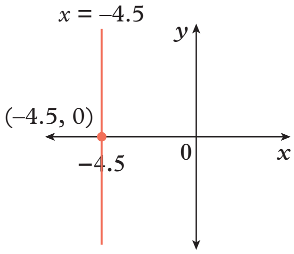
g i x-intercept: $(0, 0)$, no y-intercept

ii



h i x-intercept: $(-4.5, 0)$, no y-intercept

ii



13 A linear graph can have one or two axis intercepts (or can perhaps lie along the x-axis or y-axis).

a x-intercept: $(2, 0)$, y-intercept: $(0, -4)$, two intercepts

b no x-intercept, y-intercept: $(0, -7)$, one intercept

c x-intercept: $(3, 0)$, y-intercept: $(0, 2)$, two intercepts

d x-intercept: $(0, 0)$, y-intercept: $(0, 0)$, one intercept

e x-intercept: $(10, 0)$, y-intercept: $(0, -50)$, two intercepts

f x-intercept: $(-5, 0)$, y-intercept: $(0, 3)$, two intercepts

g x-intercept: $(28, 0)$, no y-intercept, one intercept

h x-intercept: $(6, 0)$, y-intercept: $(0, -2)$, two intercepts

i no x-intercept, y-intercept: $(0, 5.6)$, one intercept

14 a i $y = \frac{1}{p}(x - q)$

ii $y = \frac{1}{p}x - \frac{q}{p}$

b p is the reciprocal of the gradient $\left(\frac{\text{run}}{\text{rise}}\right)$ and q is the x-intercept.

c $x = 0y + 3$

d i $p = 0$

ii $q = 3$

e **i** $y = \frac{1}{0}(x - 3)$ and $y = \frac{1}{0}x - \frac{3}{0}$

ii The gradient of $x = 3$ is $\frac{1}{0}$ which is undefined. In general, the gradient of $x = py + q$ is $\frac{1}{p}$ which will be undefined if $p = 0$ which occurs when there is no y term.

15 **a** gradient–intercept method

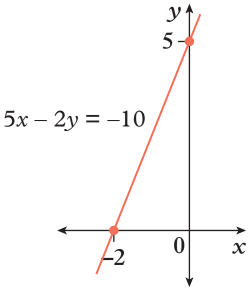
b x-and-y-intercept method

c gradient–intercept method

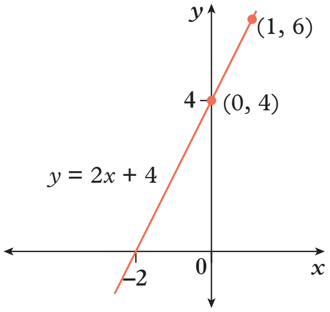
d Locate the y-intercept, and draw a line through that point parallel to the x-axis.

e Locate the x-intercept, and draw a line through that point parallel to the y-axis.

16 **a**



b



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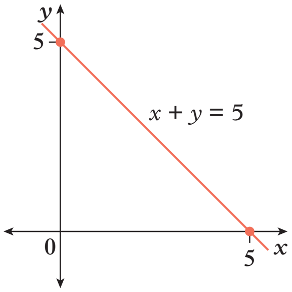

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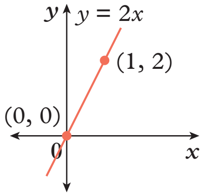

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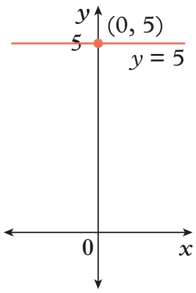

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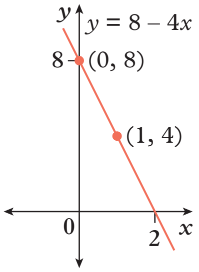

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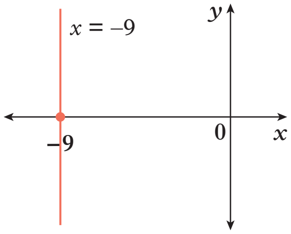

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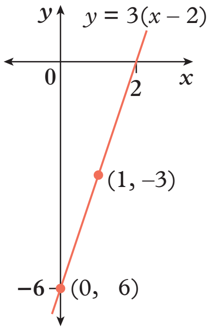

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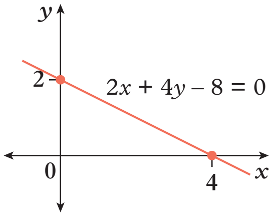

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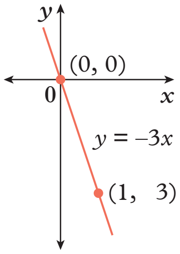

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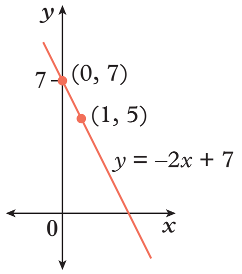

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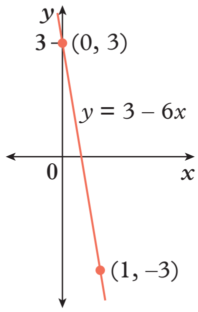
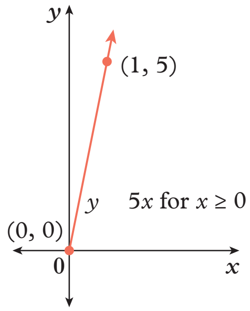


17 a



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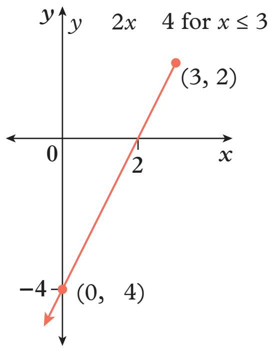

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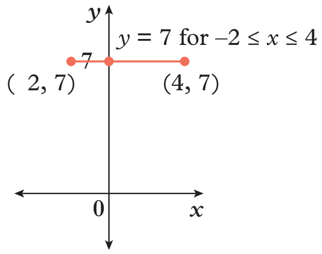

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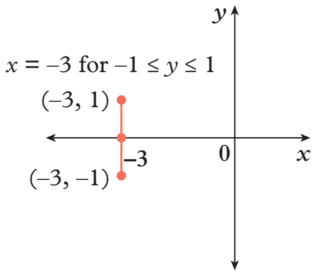

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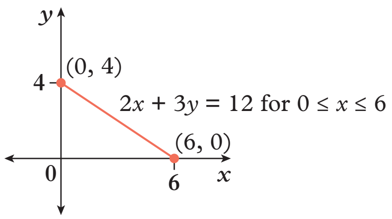

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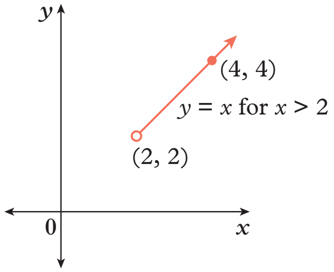
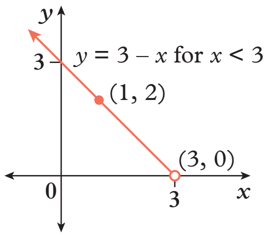


g



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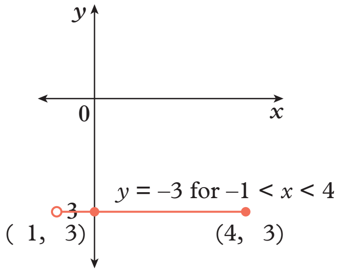
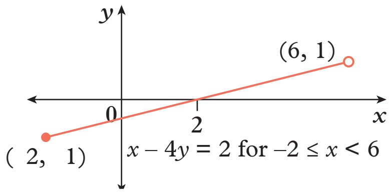


i



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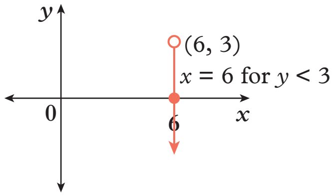
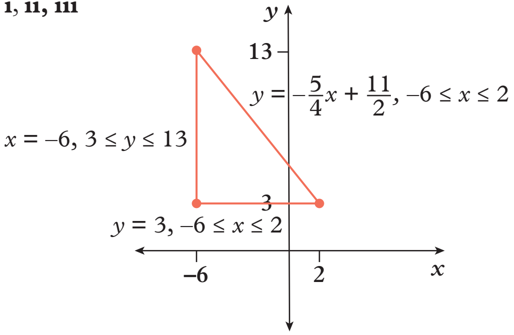


18 a **i, ii, iii**

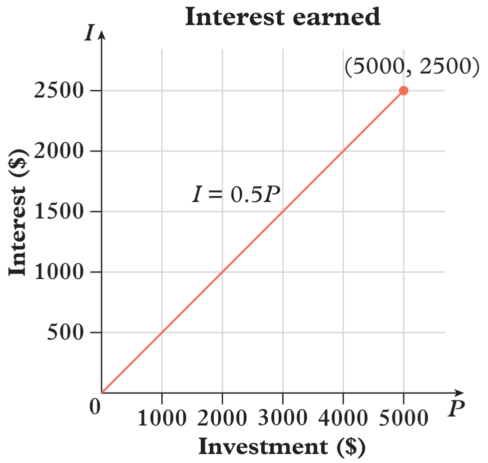


b Perimeter = 30.81 units

c Area = 40 units^2

19 a $I = 0.5P$

b

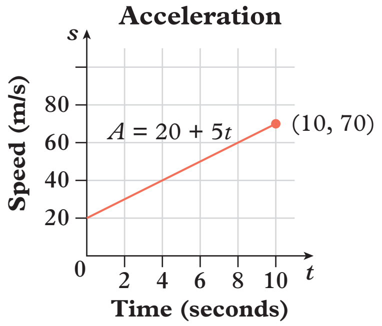


c **i** \$500 **ii** \$1250 **iii** \$2125

d **i** \$1500 **ii** \$2000 **iii** \$4500

e The gradient represents the value of rT (the rate at which the investment grows over a 10-year period).
$rT = 0.05 \times 10 = 0.5$

20 a



b The initial constant speed

c The rate of increase in speed, which is the acceleration

d **i** 35 m/s **ii** 45 m/s
iii 57.5 m/s

e **i** 126 km/h **ii** 162 km/h
iii 207 km/h

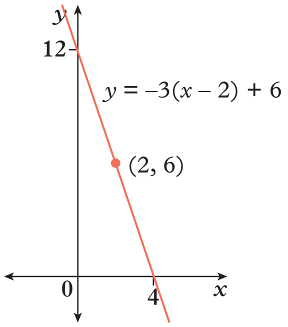
f **i** 2 s **ii** 7 s **iii** 9 s

g 8 s

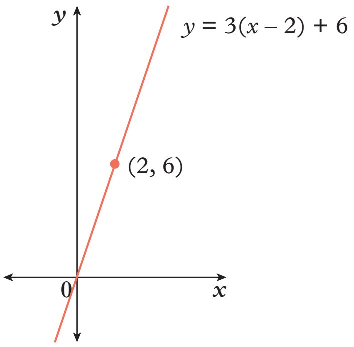
h 252 km/h reached after 10 s

21 a $y = m(x - h) + k$
$y = m(x - 0) + c$
$y = mx + c$

b **i**



ii



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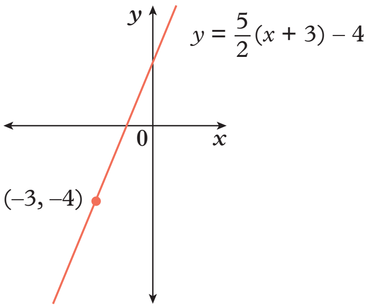

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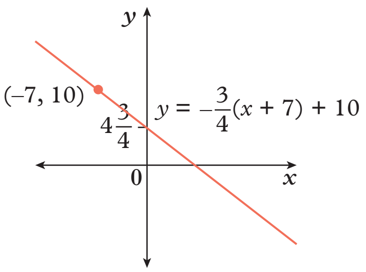


c $(0, k - mh)$ and $\left(h - \frac{k}{m}, 0\right)$

d **i** $y = m(x - b)$; $x = \frac{y}{m} + b$
This is the x-intercept version of $y = mx + c$.

ii Take the gradient, m, out as if it was a factor of the expression $mx + c$ by dividing each term by m to write the equation as $y = m\left(x + \frac{c}{m}\right)$.

iii The x-intercept is the negative of the value added to x, $\frac{c}{m}$.

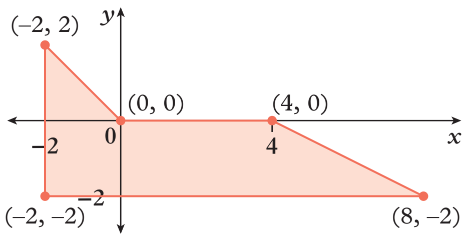
e **i** $y = 3(x - 4)$, $(4, 0)$

ii $y = 2\left(x - \frac{5}{2}\right)$, $\left(\frac{5}{2}, 0\right)$

iii $y = -3(x + 4)$, $(-4, 0)$

iv $y = 2\left(x + \frac{5}{2}\right)$, $\left(-\frac{5}{2}, 0\right)$

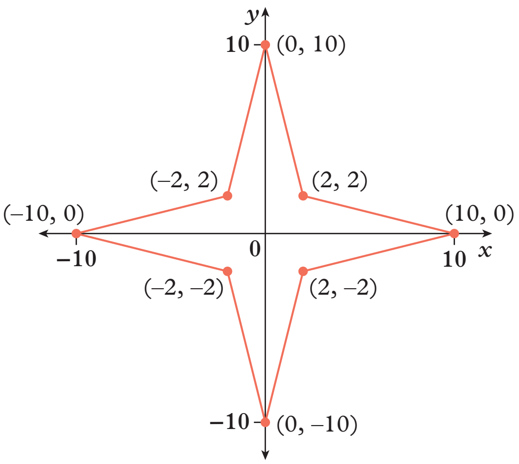
22 a



b 25.3 units

c 18 square units

23 a



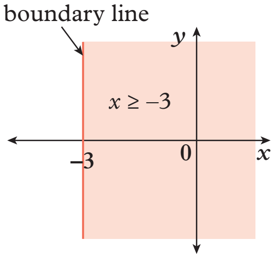
b 66 units

c 80 square units

d $y = x$

e $y = -x$

24 a i



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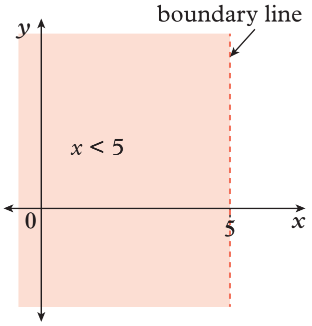

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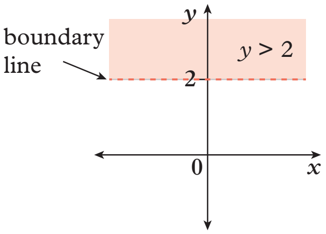

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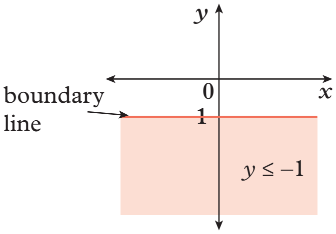
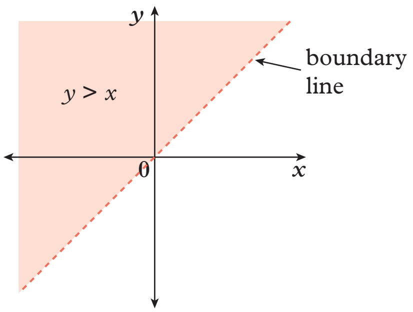


b i



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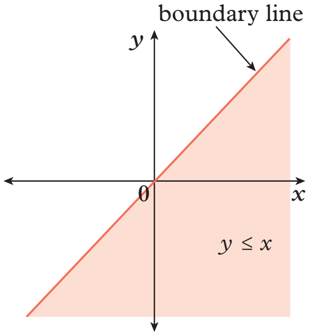

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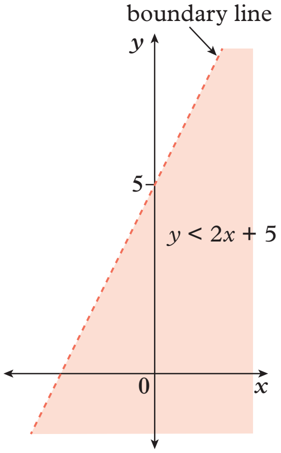

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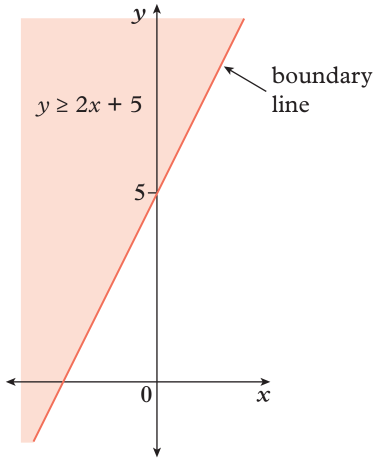

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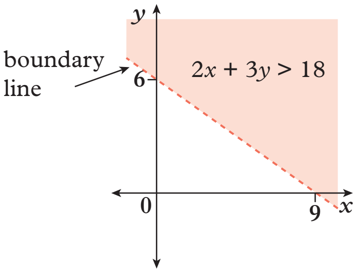

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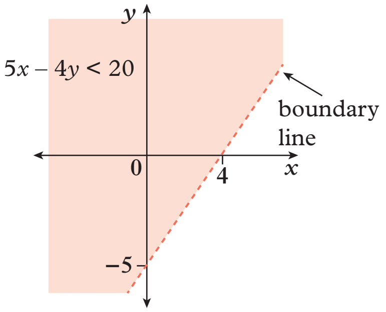

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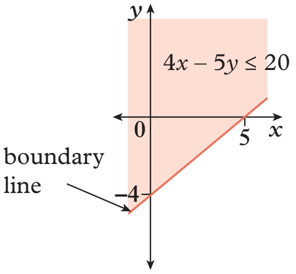

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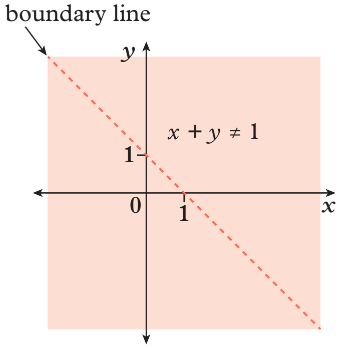


25 **a** $\left(\frac{c}{a}, 0\right)$ and $\left(0, \frac{c}{b}\right)$

b $\frac{a}{c}x + \frac{b}{c}y = 1$

c The reciprocals of the coefficients of x and y $\left(\frac{a}{c} \text{ and } \frac{b}{c}\right)$ are the corresponding x and y values for the x- and y-intercept of the line. This is because when the equation is equal to 1, we multiply both sides of the equation by the reciprocal of the coefficient of either term when solving for the intercept.

d **i** $\frac{2}{5}x + \frac{3}{5}y = 1$, $\left(\frac{5}{2}, 0\right)$, $\left(0, \frac{5}{3}\right)$

ii $\frac{4}{11}x - \frac{7}{11}y = 1$, $\left(\frac{11}{4}, 0\right)$, $\left(0, -\frac{7}{11}\right)$

iii $\frac{1}{6}x - \frac{1}{4}y = 1$, $(6, 0)$, $(0, 4)$

iv $\frac{1}{4}y - \frac{1}{18}x = 1$, $(-18, 0)$, $(0, 4)$

v $-\frac{1}{4}x + \frac{1}{8}y = 1$, $(-4, 0)$, $(0, 8)$

vi $2x + 5y = 1$, $\left(\frac{1}{2}, 0\right)$, $\left(0, \frac{1}{5}\right)$

4 Checkpoint

1 **a** $x = 21$ **b** $x = -9$

c $x = -\frac{31}{2}$ **d** $x = -\frac{3}{7}$

2 **a** $x = \frac{5}{2}$ **b** $x = \frac{1}{3}$

c $x = -7$ **d** $x = \frac{41}{21}$

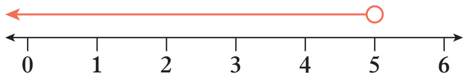
3 **a** $x = 19$ **b** $x = \frac{5}{7}$

c $x = \frac{1}{30}$ **d** $x = \frac{57}{10}$

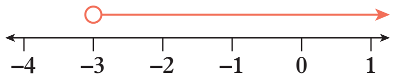
4 **a** $x = \frac{c - by}{a}$ **b** $x = \frac{f}{dg} + \frac{e}{d} = \frac{f + eg}{dg}$

c $x = \frac{jk + mn}{m - j}$ **d** $x = \frac{rs + q}{rt + p}$

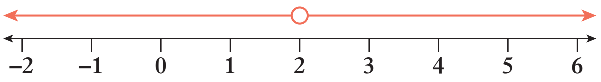
5 **a**



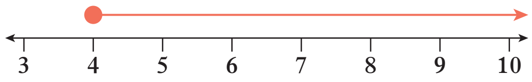
b



c



d



6 **a** $x < 5$ **b** $x \neq 6$

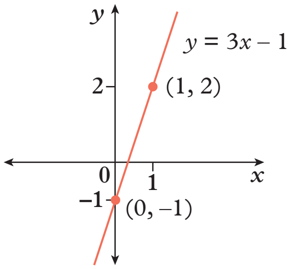
c $x \geq -5$ **d** $x \leq -2$

7 **a** $x > 3$ **b** $x \leq -5$

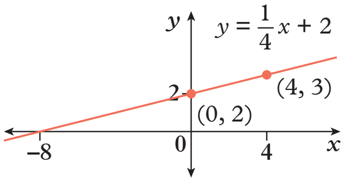
c $x > -25$ **d** $x > 1$

e $x \neq -3$ **f** $x \leq \frac{22}{3}$

8 **a**



b



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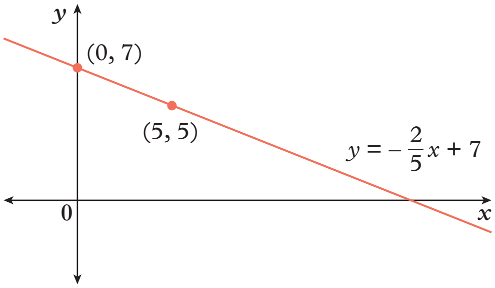

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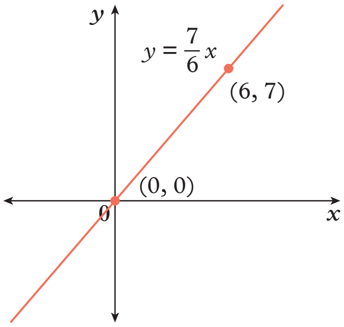
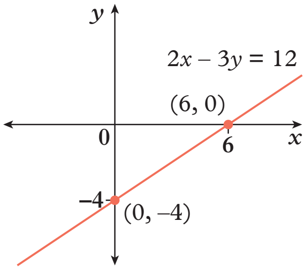
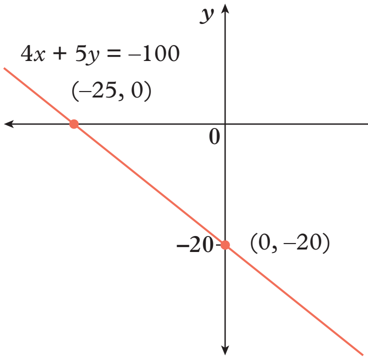


9 a



b



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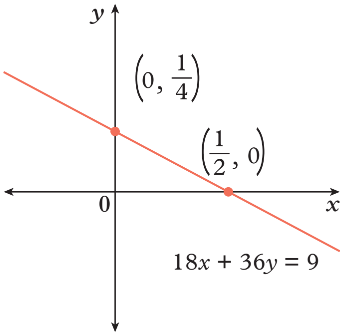

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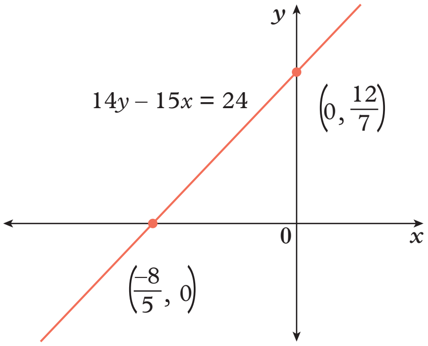
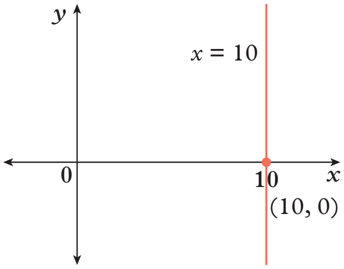
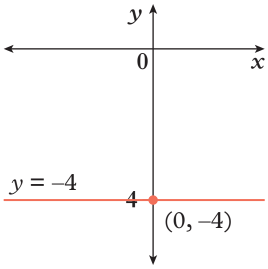


10 a



b



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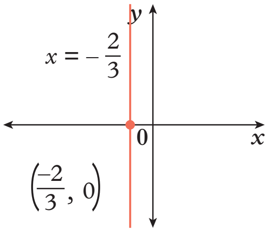
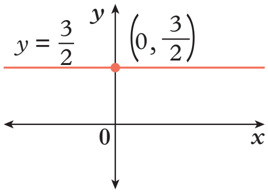


d



EX p184

4D Determining linear equations

1 **a** 4 **b** −2 **c** $\frac{1}{2}$

d −1 **e** 5 **f** −3

2 **a** **i** 4 **ii** $y = 4x + 2$

b **i** 1 **ii** $y = x - 7$

c **i** $\frac{1}{2}$ **ii** $y = \frac{1}{2}x + 5$

d **i** 10 **ii** $y = 10x + 6$

e **i** −3 **ii** $y = -3x - 5$

f **i** $-\frac{1}{4}$ **ii** $y = -\frac{1}{4}x + 4$

3 **a** $y = -2x + 4$ **b** $y = x + 3$

c $y = 2x - 6$ **d** $y = -5x + 5$

e $x = 5$ **f** $y = -2$

4 a $y = 3x + 1$ **b** $y = 2x + 3$

c $y = -4x - 2$ **d** $y = -x + 9$

e $y = 5x + 9$ **f** $y = -3x - 1$

g $y = \frac{6}{5}x - 14$ **h** $y = -\frac{3}{4}x + 1$

i $y = \frac{5}{2}x + \frac{3}{2}$ **j** $y = -\frac{7}{9}x - \frac{2}{3}$

5 a $y = 2x + 1$ **f** $y = -x - 6$

b $y = 6x - 28$ **g** $y = \frac{7}{3}x - \frac{16}{3}$

c $y = -3x + 10$

d $y = -2x + 4$ **h** $y = -\frac{7}{5}x - \frac{24}{5}$

e $y = x - 1$ **i** $y = \frac{1}{6}x + \frac{23}{6}$

6 a $y = 5x - 3$

b $y = -2x + 4$

c $y = \frac{4}{5}x + 2$

7 a $y = 3x - 11$

b $y = -2x + 6$

c $y = x + 2$

8 a $y = 2x - 5$ **b** $y = \frac{1}{2}x - 5$

c $y = -5x + 3$ **d** $y = -\frac{1}{5}x + 3$

e $y = 4x - 2$ **f** $y = \frac{3}{2}x + \frac{1}{2}$

g $y = 4$ **h** $y = 3x$

i $x = -3$ **j** $y = 6x - 12$

k $y = \frac{5}{12}x + 10$ **l** $y = -\frac{4}{7}x + \frac{31}{7}$

9 a $m = \frac{5-2}{4-(-2)}$ and $m = \frac{2-5}{-2-4}$

$m = \frac{3}{6}$ $m = \frac{-3}{-6}$

$m = \frac{1}{2}$ $m = \frac{1}{2}$

b Juan has not vertically aligned the corresponding x- and y-coordinates of the same point, so he will end up with the negative of the actual gradient, $-m = -\frac{1}{2}$.

c
$$\begin{aligned} m &= \frac{y_1 - y_2}{x_1 - x_2} \\ &= \frac{y_1 - y_2}{x_1 - x_2} \times \frac{-1}{-1} \\ &= \frac{-(y_1 - y_2)}{-(x_1 - x_2)} \\ &= \frac{-y_1 + y_2}{-x_1 + x_2} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \end{aligned}$$

10 a $2 = \frac{1}{2}(-2) + c$ and $5 = \frac{1}{2}(4) + c$

$2 = -1 + c$ $5 = 2 + c$

$c = 3$ $c = 3$

b Tate has substituted the x-coordinate of one of the points for the gradient and the corresponding y-coordinate for the y-intercept.

11 a $c = k - mh$

b $y = mx + k - mh$

c $y = mx + k - mh$

$y = mx - mh + k$

$y = m(x - h) + k$

12 a i $y = \frac{3}{4}(x - 8) + 2$

ii $y = \frac{3}{4}x - 4$

b i $y = -\frac{7}{3}(x - 15) - 6$

ii $y = -\frac{7}{3}x + 29$

c i $y = \frac{1}{4}(x + 3) + 5$ or $y = \frac{1}{4}(x - 1) + 6$

ii $y = \frac{1}{4}x + \frac{23}{4}$

d i $y = -\frac{8}{5}(x + 6) + 5$ or $y = -\frac{8}{5}(x + 1) - 3$

ii $y = -\frac{8}{5}x - \frac{23}{5}$

e i $y = -\frac{1}{2}(x + 4) + 2$ or $y = -\frac{1}{2}(x - 2) - 1$

ii $y = -\frac{1}{2}x$

f i $y = 0(x - 4) + 3$ or $y = 0(x + 7) + 3$

ii $y = 3$

g i $y = -\frac{3}{4}(x - 4) + 0$ or $y = -\frac{3}{4}(x - 0) + 3$

ii $y = -\frac{3}{4}x + 3$

h i $y = \frac{2}{5}(x + 5) + 0$ or $y = \frac{2}{5}(x - 0) + 2$

ii $y = \frac{2}{5}x + 2$

13 a i translation of 6 units up

ii translation of 3 units left

iii translation of 2 units left and 2 units up

b i $y = 2x + 6$

ii $y = 2(x + 3)$

iii $y = 2(x + 2) + 2$

c i translation of 1 unit left and 2 units down

ii translation of 1 unit right and 2 units up

iii translation of 1 unit right and 2 units up

iv translation of 4 units left and 6 units up

v translation of $\frac{3}{2}$ units right and $\frac{4}{3}$ units up

vi translation of 4 units right and 2 units up

14 a $y = \frac{1}{200}x + 50$ **b** $y = -\frac{1}{15}x + 6$

c $y = \frac{1}{5}x$

15 a $a = 20n + 80$ **b** 21 weeks

16 a $w = -195n + 1560$

b $975

c 8 months

17 **a** **i** $y = 2x + 5$

ii $-4 \le x \le -2$

iii $-3 \le y \le 1$

b **i** $y = \frac{1}{4}x + \frac{3}{2}$

ii $-2 \le x \le 2$

iii $1 \le y \le 2$

$y = -\frac{7x}{3} - 6$

c **i** $y = 3x - 4$

ii $2 \le x \le 3$

iii $2 \le y \le 5$

d **i** $y = -\frac{5}{2}x + \frac{25}{2}$

ii $3 \le x \le 7$

iii $-5 \le y \le 5$

e **i** $y = -3x + 11$

ii $x \le 4$

iii $y \ge -1$

f **i** $y = -\frac{1}{4}x + \frac{11}{4}$

ii $x \le 3$

iii $y \ge 2$

g **i** $y = \frac{5}{2}x + \frac{11}{2}$

ii $x \le -1$

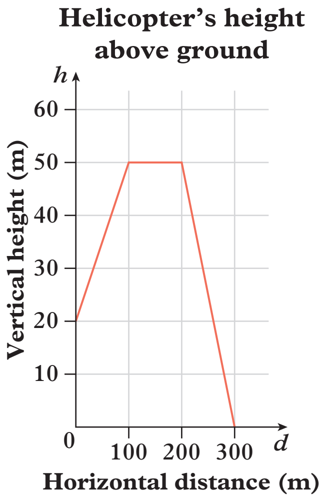
iii $y \le 3$

h **i** $y = \frac{1}{7}x - \frac{11}{7}$

ii $x \ge -3$

iii $y \ge -2$

18 **a**



b In the order of the flight: $h = 0.3d + 20$, $h = 50$, $h = -0.5d + 150$

c **i** 29 m **ii** 44 m

iii 50 m **iv** 30 m

d $66\frac{2}{3}$ m and 220 m

19

$y = \frac{y_2 - y_1}{x_2 - x_1}x + c$

$c = y - \frac{y_2 - y_1}{x_2 - x_1}x$

Substituting (x_1, y_1) and (x_2, y_2) for (x, y) gives:

$c = y_1 - \frac{y_2 - y_1}{x_2 - x_1}x_1$ and $c = y_2 - \frac{y_2 - y_1}{x_2 - x_1}x_2$

$c = y_1 - \frac{x_1y_2 - x_1y_1}{x_2 - x_1}$ $c = y_2 - \frac{x_2y_2 - x_2y_1}{x_2 - x_1}$

$c = \frac{y_1(x_2 - x_1)}{x_2 - x_1} - \frac{x_1y_2 - x_1y_1}{x_2 - x_1}$ $c = \frac{y_2(x_2 - x_1)}{x_2 - x_1} - \frac{x_2y_2 - x_2y_1}{x_2 - x_1}$

$c = \frac{x_2y_1 - x_1y_1}{x_2 - x_1} - \frac{x_1y_2 - x_1y_1}{x_2 - x_1}$ $c = \frac{x_2y_2 - x_1y_2}{x_2 - x_1} - \frac{x_2y_2 - x_2y_1}{x_2 - x_1}$

$c = \frac{x_2y_1 - x_1y_2}{x_2 - x_1}$ $c = \frac{x_2y_1 - x_1y_2}{x_2 - x_1}$

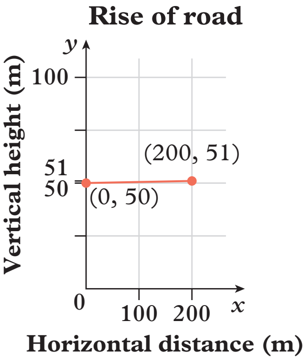
20 **a** $x - 2y = 4$

b $y = -23$

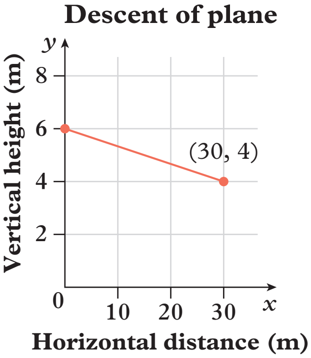
c $2x + 3y + 20 = 0$

d $12x - 17y = 0$ or $17y = 12x$

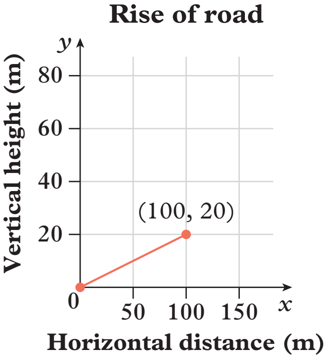
21 **a** $y = \frac{1}{200}x + 50$, for $0 \le x \le 200$



b $y = -\frac{1}{15}x + 6$, for $0 \le x \le 30$



c $y = \frac{1}{5}x$, for $0 \le x \le 100$



22 **a** **i** $\frac{3}{2}x + \frac{7}{2}y = 1$ **ii** $3x + 7y = 2$

b **i** $-\frac{5}{3}x + \frac{2}{3}y = 1$ **ii** $-5x + 2y = 3$

c **i** $\frac{5}{3}x - \frac{5}{2}y = 1$ **ii** $10x - 15y = 6$

d **i** $-2x - 3y = 1$ **ii** $2x + 3y = -1$

e **i** $\frac{1}{7}x + \frac{1}{11}y = 1$ **ii** $11x + 7y = 77$

f **i** $x - y = 1$ **ii** $x - y = 1$

EX
p193

4E Parallel and perpendicular lines

1 **a** **i** gradients: 2 and 2

ii gradients: $-\frac{1}{3}$ and $-\frac{1}{3}$

b Both pairs of lines are parallel.

c The gradients of parallel lines are always equal.

d **i** parallel

ii parallel

iii not parallel

2 **a** **i** gradients: 3 and $-\frac{1}{3}$

ii gradients: $\frac{3}{4}$ and $-\frac{4}{3}$

b Both products are –1.

c Both pairs of lines are perpendicular.

d The product of the gradients of two perpendicular lines is always –1.

e $m_2 = -\frac{1}{m_1}$

f **i** perpendicular

ii not perpendicular

iii perpendicular

3 **a** $-\frac{1}{5}$ **b** $\frac{1}{8}$ **c** 1

d –6 **e** $-\frac{5}{4}$ **f** $\frac{2}{3}$

4 **a** **i** $y = -3x + 11$ **ii** -3 **iii** $\frac{1}{3}$

b **i** $y = 4x + 7$ **ii** 4 **iii** $-\frac{1}{4}$

c **i** $y = \frac{5}{3}x + \frac{2}{3}$ **ii** $\frac{5}{3}$ **iii** $-\frac{3}{5}$

d **i** $y = -\frac{3}{2}x + 4$ **ii** $-\frac{3}{2}$ **iii** $\frac{2}{3}$

e **i** $y = \frac{5}{2}x - 2$ **ii** $\frac{5}{2}$ **iii** $-\frac{2}{5}$

f **i** $y = -\frac{8}{5}x - \frac{11}{5}$ **ii** $-\frac{8}{5}$ **iii** $\frac{5}{8}$

g **i** $y = \frac{1}{3}x + \frac{5}{3}$ **ii** $\frac{1}{3}$ **iii** -3

h **i** $y = -\frac{1}{5}x + \frac{3}{5}$ **ii** $-\frac{1}{5}$ **iii** 5

i **i** $y = x + 7$ **ii** 1 **iii** -1

5 **a** parallel; same gradient of 4

b perpendicular; negative reciprocal gradients of $\frac{2}{3}$ and $-\frac{3}{2}$

c perpendicular; negative reciprocal gradients of $\frac{3}{8}$ and $-\frac{8}{3}$

d neither; gradients of –2 and 2

e neither; gradients of –5 and $-\frac{1}{5}$

f parallel; same gradient of $\frac{2}{7}$

g neither; gradients of $\frac{2}{3}$ and $-\frac{2}{3}$

h perpendicular; negative reciprocal gradients of –2 and $-\frac{1}{2}$

6 **a** $y = 4x - 13$ **b** $y = -3x - 6$

c $y = -x - 2$ **d** $y = \frac{1}{2}x - 3$

e $y = -\frac{1}{3}x + 5$ **f** $y = \frac{2}{7}x + 11$

g $y = -\frac{5}{4}x - 5$ **h** $y = \frac{5}{6}x + \frac{39}{2}$

7 **a** $y = -4x + 5; y = -4x + 18$

b $y = -\frac{1}{2}x + 3; y = -\frac{1}{2}x - 3$

c $y = \frac{7}{3}x - \frac{5}{3}; y = \frac{7}{3}x + 6$

d $y = \frac{3}{5}x + \frac{3}{20}; y = \frac{3}{5}x - 3$

8 **a** $y = -\frac{1}{5}x - 7$ **b** $y = \frac{1}{7}x + 3$

c $y = -x - 3$ **d** $y = \frac{3}{2}x + 10$

e $y = -\frac{9}{4}x + 8$ **f** $y = 3x + 30$

g $y = \frac{6}{5}x - \frac{54}{5}$ **h** $y = -\frac{7}{4}x - 46$

9 **a** $y = -5x + 4; y = \frac{1}{5}x + 9$

b $y = \frac{1}{3}x - 2; y = -3x - 33$

c $y = -\frac{9}{7}x + \frac{13}{7}; y = \frac{7}{9}x + 47$

d $y = \frac{2}{5}x + \frac{7}{5}; y = -\frac{5}{2}x - 100$

10 **a** $y = -\frac{1}{2}x + 3$

b $y = x + 3$

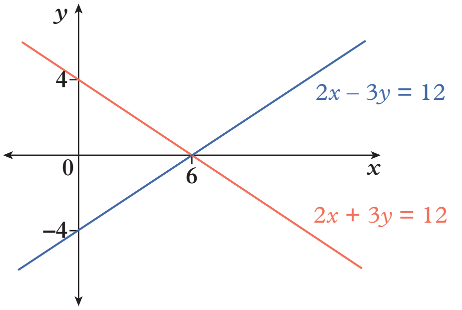
c $y = -\frac{2}{3}x + 3$

11 **a** **i** $y = 2x - 4$ **ii** $y = -\frac{1}{2}x - 4$

b **i** $y = -x - 4$ **ii** $y = x - 4$

c **i** $y = \frac{3}{2}x - 4$ **ii** $y = -\frac{2}{3}x - 4$

12 **a** **i**



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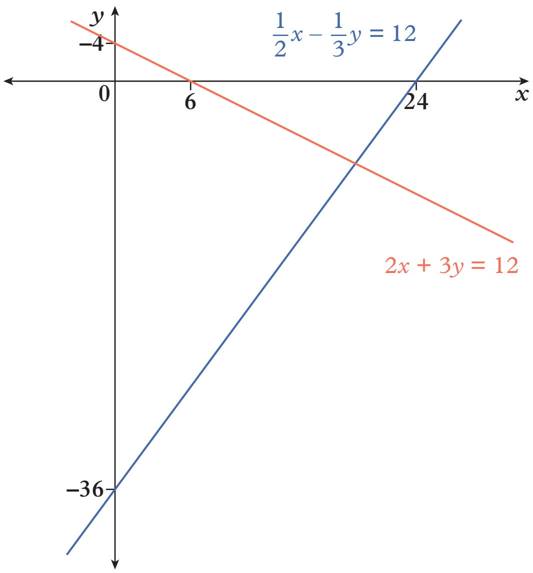

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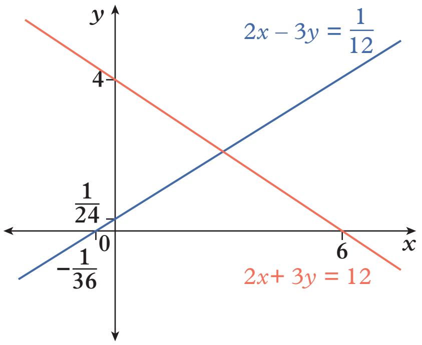

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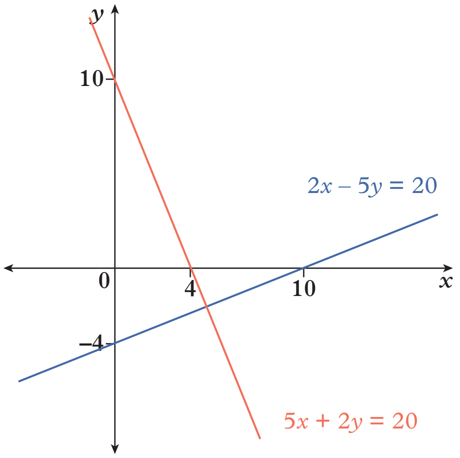
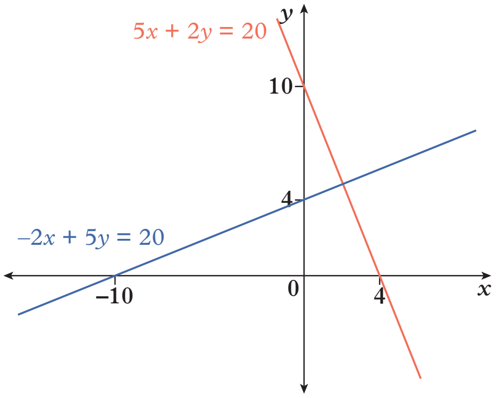
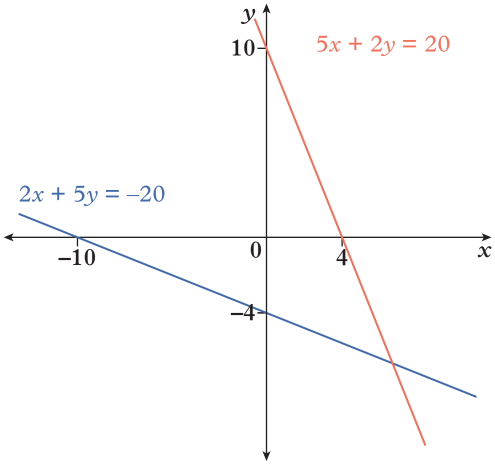


v



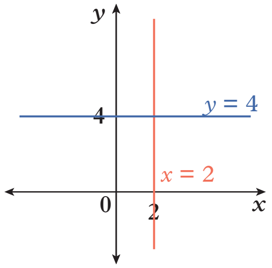
vi



b ii, iv, v

c $\frac{1}{a}x - \frac{1}{b}y = 1$ or $-\frac{1}{a}x + \frac{1}{b}y = 1$ or $bx - ay = 1$ or $-bx + ay = 1$

13 a



b i Aamira is correct because lines that meet at right angles are, by definition, perpendicular.

ii Jack is incorrect as you cannot divide by zero, even if you divide out that factor.

c i Vertical lines are perpendicular to horizontal lines because they meet at a 90° angle.

ii Vertical lines are perpendicular to horizontal lines but we cannot use the formula $m_1 \times m_2 = -1$ because the gradient of a vertical line is undefined.

iii Vertical lines are perpendicular to horizontal lines, so the equations $x = a$ and $y = c$ are perpendicular.

14 The gradient of the line joining (2, –3) and (4, 5) is $\frac{5-(-3)}{4-2} = \frac{8}{2} = 4$.

The gradient of $y = 4x - 7$ is 4.

So the lines are parallel.

15 The gradient of the line joining (–11, –7) and (–1, –2) is $\frac{-2-(-7)}{-1-(-11)} = \frac{5}{10} = \frac{1}{2}$.

The gradient of $y = -2x + 5$ is –2.

The product of the gradients is –1, so the lines are perpendicular.

16 a $\frac{2}{3}$

b i $y = \frac{2}{3}x - 5$ **ii** $y = -\frac{3}{2}x + 2$

17 a $y = -3x$

b $y = \frac{1}{3}x + 9$

c $y = \frac{1}{3}x - 1$

18 $y = -x + 1, y = -x + 2, y = x + 1, y = x + 2$

19 a Yes, both have a gradient of $\frac{3}{7}$.

b No; AC has a gradient of $\frac{5}{4}$, while BD has a gradient of $-\frac{2}{3}$.

20 a The gradient of AB is 3; the gradient of BC is $-\frac{1}{3}$, so line segments AB and BC are perpendicular. This makes triangle ABC right-angled.

b The length of AB is $\sqrt{90}$ units; the length of BC is $\sqrt{40}$ units. So the triangle is not isosceles right-angled.

c 27.2 units

d 30 square units

21 a $AD \parallel BC$, each line segment has a gradient of $\frac{1}{2}$.

$AB \parallel DC$, each line segment has a gradient of 2.

length of AD = length of BC = length of AB = length of $DC = \sqrt{5}$ units

Note: It is only necessary to prove all sides equal in length, but it can also be shown by proving both pairs of opposite sides parallel and two adjacent sides equal in length.

b $EF \parallel HG$; both line segments have a gradient of $\frac{3}{7}$.

$HE \parallel GF$; both line segments have a gradient of 2.

22 gradient of KN = gradient of $LM = -\frac{1}{2}$

gradient of MN = gradient of $LK = 2$

So opposite sides are parallel, and adjacent sides are at right angles.

$KL = LM = MN = NK = 2\sqrt{5} \approx 4.5$ units,

perimeter $= 8\sqrt{5} \approx 17.9$ units,

area $= \left(2\sqrt{5}\right)^2 = 20$ square units

23 gradient of PS = gradient of $QR = 1$

gradient of PQ = gradient of $SR = -1$

So opposite sides are parallel, and adjacent sides are at right angles.

$PS = QR = 3\sqrt{2}$ units, $PQ = SR = 4\sqrt{2}$ units,

perimeter $= 14\sqrt{2} \approx 19.8$ units

area $= 3\sqrt{2} \times 4\sqrt{2} = 24$ square units

24 $y = x - 5$

25 length of YX = length of $YZ = 7.1$ units

$WX = WZ = \sqrt{425} = 5\sqrt{17} \approx 20.6$ units

This is all that is required. Extra information is as follows: The gradient of $XZ = -\frac{3}{4}$ and the gradient of $WY = \frac{4}{3}$, so the diagonals are at right angles. The midpoint of XZ is (3, 5). This point also lies on the diagonal WY, which has the equation $y = \frac{4}{3}x + 1$.

26 Gradients of the line segments are: $AB = \frac{1}{2}$, $BC = -3$, $CD = \frac{1}{2}$, $DA = \frac{6}{5}$, $AC = 0$, $BD = \frac{9}{11}$

This indicates that the quadrilateral is a trapezium (one pair of parallel sides).

EX
p202

4F Simultaneous linear equations

1 a is a solution **b** is a solution
c is not a solution **d** is a solution
e is not a solution **f** is not a solution

2 a i (4, 3) **ii** $x = 4, y = 3$
b i (2, –4) **ii** $x = 2, y = -4$
c i (–5, –2) **ii** $x = -5, y = -2$
d i (–2, 3) **ii** $x = -2, y = 3$

3 a $x = 5, y = 4$ **b** $x = 2, y = 6$
c $x = 6, y = 4$ **d** $x = -2, y = 5$
e $x = -1, y = 3$ **f** $x = -3, y = 1$
g $x = 3, y = 2$ **h** $x = 4, y = -2$
i $x = -2, y = -3$ **j** $x = -5, y = 6$
k $x = 2, y = -2$ **l** $x = 3, y = 5$
m $x = -4, y = -1$ **n** $x = 1, y = -4$
o $x = -6, y = -5$ **p** $x = -1, y = 2$

4 a $x = 2, y = 7$ **b** $x = 4, y = 1$
c $x = -1, y = 3$ **d** $x = 6, y = 5$
e $x = -4, y = -2$ **f** $x = 2, y = 12$
g $x = 1, y = 3$ **h** $x = -3, y = 11$
i $x = 5, y = -5$ **j** $x = 5, y = 3$
k $x = -4, y = -1$ **l** $x = 5, y = 6$

5 a $x = 1, y = 8$ **b** $x = 6, y = 4$
c $x = -3, y = -5$ **d** $x = 5, y = -1$
e $x = -1, y = 4$ **f** $x = 4, y = -7$
g $x = 2, y = 1$ **h** $x = -2, y = 6$

6 a $x = 2, y = 5$
b $x = -3, y = 1$
c $x = 1, y = -5$

7 a $x = 5, y = 3$
b $x = -2, y = 9$
c $x = 3, y = -7$

8 a $x = -4, y = -2$ **b** $x = 6, y = 2$
c $x = -1, y = 2$ **e** $x = -1, y = 5$
d $x = 5, y = 3$ **f** $x = -3, y = -4$

9 a $x = 3, y = 2$ **b** $x = 2, y = -1$
c $x = -4, y = 7$ **d** $x = 5, y = 3$
e $x = -3, y = -4$ **f** $x = -2, y = 6$
g $x = 1, y = 4$ **h** $x = -3, y = 1$
i $x = 2, y = -1$ **j** $x = -4, y = 2$
k $x = 5, y = 5$ **l** $x = -3, y = -2$

10 a i The graphical method is most appropriate when the equations of the lines are easy to sketch accurately on the same set of axes and you have a grid on which to read the intersection point.

ii The substitution method is most appropriate when one or both of the equations have a variable as the subject of the equation.

iii The elimination method is most appropriate when the terms with the same variable in each equation have the same coefficient or the negative value of the other coefficient.

b i The graphical method is least appropriate when the equations of the lines are not easy to sketch accurately on the same set of axes or the intersection point cannot be read off accurately.

ii The substitution method is least appropriate when neither equation has a variable the subject of the equation.

iii The elimination method is least appropriate when the coefficients of the same variable cannot be simply multiplied by a number to make them the same or the negative value of each other.

c i The graphical method can be made more appropriate by using digital technology to sketch and locate the intersection point.

ii The substitution method can be made more appropriate by writing one or both equations with a variable as the subject.

iii The elimination method can be made more appropriate by multiplying the equations by a number to make the coefficients of the same variable the same or the negative value of each other.

11 a $x = 2, y = 0$

b $x = \frac{1}{3}, y = -3$

c $x = \frac{3}{2}, y = \frac{2}{5}$ or $x = 1.5, y = 0.4$

d $x = -9, y = -7$

e $x = 4, y = 2$

f $x = -2, y = -3$

g $x = 3, y = 10$

h $x = 5, y = -2$

i $x = 2, y = -1$

j $x = -2, y = 2$

k $x = 7, y = 4$

l $x = -4, y = 1$

m $x = -0.8, y = 2.3$

n $x = -\frac{1}{4}, y = 5$ or $x = -0.25, y = 5$

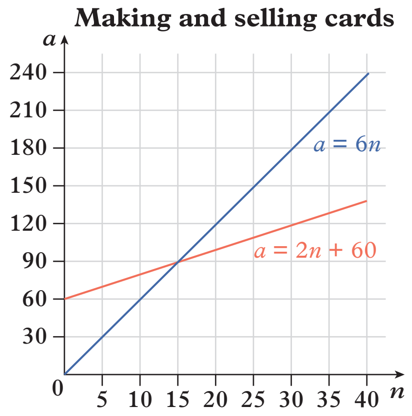
o $x = 28, y = 41$

12 Both graphs have a gradient of 2. This means the lines are parallel, so there is no point of intersection.

13 a To make n cards, the cost of plain white cards is $2 \times n$ or $2n$. So, the amount of money spent (in \$) is the cost of the plain white cards plus the start-up cost of \$60.

b Each card sells for \$6, so for selling n cards, the amount of money received (in \$) is $6 \times n$ or $6n$.

c



d (15, 90); this shows the number of cards that need to be sold (15) so that the amount of money received is the same as the amount of money spent (\$90).

e 15 cards

f 16 to 40 cards

14 a $2l + 2w = 810$ or $2(l + w) = 810$ or $l + w = 405$

b $l = 2w$

c 270 cm long and 135 cm wide

15 a $15 + (-5) = 10$ and $15 + 5 = 20$

b $22 + (-12) = 10$ and $22 + (-2) = 20$

c Since there are three variables but only two equations, if we add, subtract or substitute, the resulting equation will still have two variables in it when we need it to have only one.

16 a

$$\begin{aligned}
ax + by &= c \\
ax + by - (ax + qy) &= c - (r) \\
by - qy &= c - r \\
(b - q)y &= c - r \\
y &= \frac{c - r}{b - q} \\
ax + b\left(\frac{c - r}{b - q}\right) &= c \\
ax + \frac{bc - br}{b - q} &= \frac{c(b - q)}{b - q} \\
ax &= \frac{cb - cq}{b - q} - \frac{bc + br}{b - q} \\
ax &= \frac{cb - cq - bc - br}{b - q} \\
ax &= \frac{br - cq}{b - q} \\
x &= \frac{br - cq}{ab - aq}
\end{aligned}$$

b $x = \frac{4 \times 7 - 9 \times 6}{3 \times 4 - 3 \times 6} = \frac{13}{3}$ and $y = \frac{9 - 7}{4 - 6} = -1$

c

$$\begin{aligned}
ax + by &= c \quad (\times p)\\
apx + bpy &= cp\\
px + qy &= r \quad (\times a)\\
apx + aqy &= ar\\
apx + bpy - (apx + aqy) &= cp - (ar)\\
bpy - aqy &= cp - ar\\
(bp - aq)y &= cp - ar\\
y &= \frac{cp - ar}{bp - aq}\\
ax + b\left(\frac{cp - ar}{bp - aq}\right) &= c\\
ax + \frac{bcp - abr}{bp - aq} &= \frac{c(bp - aq)}{bp - aq}\\
ax &= \frac{bcp - acq}{bp - aq} - \frac{bcp - abr}{bp - aq}\\
ax &= \frac{bcp - acq - bcp + abr}{bp - aq}\\
ax &= \frac{abr - acq}{bp - aq}\\
x &= \frac{br - cq}{bp - aq}
\end{aligned}$$

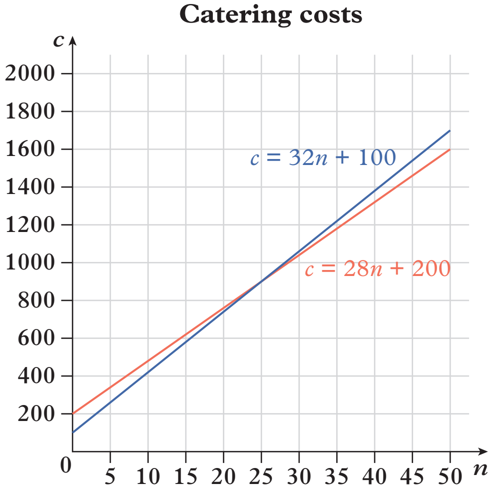
d $x = \frac{4 \times 7 - 9 \times 6}{4 \times 5 - 3 \times 6} = -13$ and $y = \frac{9 \times 5 - 3 \times 7}{4 \times 5 - 3 \times 6} = 12$

17 n = number of people and c = catering cost

a $c = 28n + 200$

b $c = 32n + 100$

c



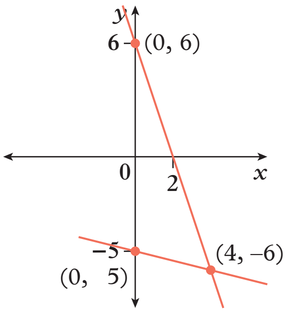
d Cool Food Club is cheaper for 18 people; at $n = 18$, as the line for Cool Food Club is lower than the line for Angie's Catering. This represents a lower cost for 18 people.

e Angie's Catering

f 25 people at a cost of $900

g When catering for fewer than 25 people, Cool Food Club is cheaper. When catering for more than 25 people, Angie's Catering is cheaper. The cost is the same when catering for 25 people.

18 a



b $y = -\frac{1}{4}x - 5, y = -3x + 6$

c $x = 4, y = -6$

d If $x = 4$ and $y = -6$:

For $y = -\frac{1}{4}x - 5$, RHS $= -\frac{1}{4} \times 4 - 5 = -6 =$ LHS.

For $y = -3x + 6$, RHS $= -3 \times 4 + 6 = -6 =$ LHS.

19 a $y = x - 7$ and $y = -x + 1$

b $x = 4$ and $y = -3$

c $y = \frac{a}{b}(x - 4) - 3$ and $y = -\frac{b}{a}(x - 4) - 3$

20 They are equivalent equations. When graphed, the position of the lines is identical; that is, one line lies on top of the other. There are an infinite number of points where the lines intersect so there are infinite solutions.

21 a Strategy A: Equation ③ is $10x - 6y = -2$.

① – ③: $-3x = -12, x = 4$

Substituting $x = 4$ into ②: $20 - 3y = -1, y = 7$

The solution is $x = 4, y = 7$.

Strategy B: Equation ④ is $-10x + 6y = 2$.

① + ④: $-3x = -12$, so $x = 4$

Substituting $x = 4$ into ②: $20 - 3y = -1, y = 7$

The solution is $x = 4, y = 7$.

Strategy C: Equation ⑤ is $35x - 30y = -70$ and equation ⑥ is $-35x - 21y = -7$.

⑤ – ⑥: $-9y = -63$, so $y = 7$

Substituting $y = 7$ into ②: $5x - 21 = -1$, so $x = 4$

The solution is $x = 4, y = 7$.

The strategies all provide the same solution of $x = 4$ and $y = 7$.

b Some possible answers are:

Multiply equation ① by 5 to form equation ⑦ and multiply equation ② by –7 to form equation ⑧. Add equations ⑦ and ⑧.

Graph equations ① and ② and identify the coordinates of the point of intersection.

Rearrange equation ② to obtain $3y = 5x - 1$ and substitute that into equation ①. In other words, substitute $5x - 1$ for $3y$ in $7x - 2(3y) = -14$.

CHAPTER 4 review

Multiple-choice

1 B	**2** A	**3** C	**4** D	**5** E
6 B	**7** C	**8** D	**9** E	**10** C
11 E	**12** D	**13** E	**14** B	**15** A

Short answer

1 **a** $x = -1\frac{1}{2}$ **b** $x = 9\frac{2}{3}$

2 **a** $x = 9$ **b** $x = \frac{5}{13}$

3 **a** $b = \frac{2A}{h} - a$ **b** $h = \frac{A}{2(l + w)}$

4 **a** $1\frac{5}{8}, 2.6, 8.5, \frac{3}{4}$

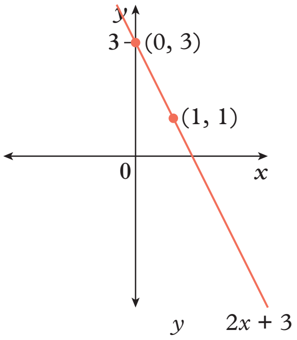
b $-3.9, -8.5, \frac{3}{4}$

c $2.6, 8.5$

5 **a** $x \geq -3$ **b** $x < -10$ **c** $x \leq 2.5$

6 **a** $x > -1$ **b** $x < -2\frac{2}{3}$

7 **a**



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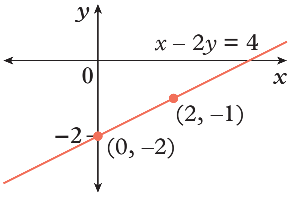

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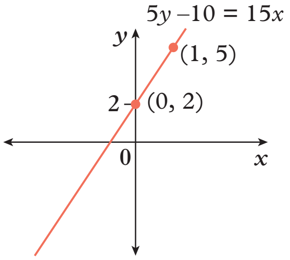
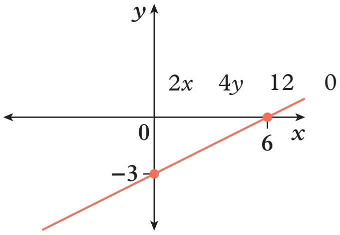


8 **a**



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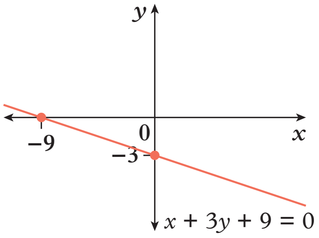

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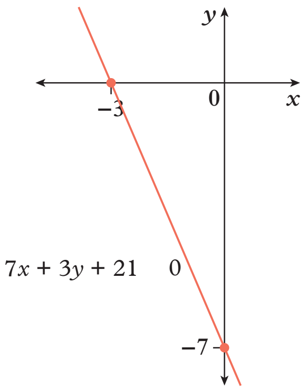


9 **a** $y = 2x - 4$

b $y = 6$

c $x = -3$

10 **a** $y = 2x + 3$

b $y = -2x + 7$

11 **a** $y = x + 2$

b $y = -3x + 2$

c $y = x$

12 **a** $y = -4x - 3$

b $y = -\frac{1}{2}x - 4$

13 **a** $y = -\frac{1}{3}x + 3$

b $y = 3x + 7$

14 **a** $x = 1, y = 2$

b $x = 6, y = 17$

15 **a** $x = 1, y = -1$

b $x = -2, y = -5$

c $x = 2, y = 10$

16 **a** $x = 2, y = 4$

b $x = -2, y = 3$

c $x = 3, y = -1$

Analysis

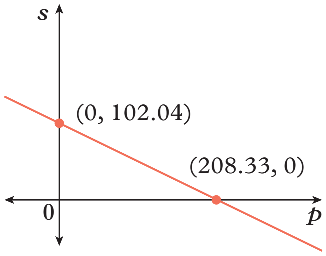
1 **a** $0.71s + 0.26p \leq 60$

b $p \leq 94$ or $p \leq 94.23$ (2 decimal places)

c $s \leq 29$ or $s \leq 29.58$ (2 decimal places)

d $P = 0.49s + 0.24p$

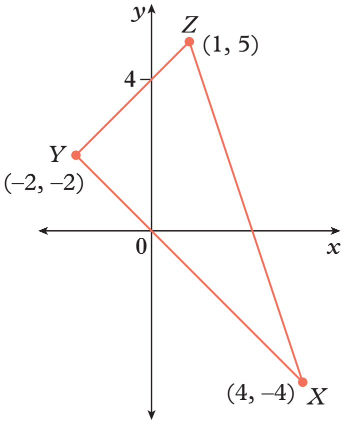
e



f 208 party pies

g 32 sausage rolls and 143 party pies

2 **a**



b For XY: $y = -x, -2 \le x \le 4$

For YZ: $y = x + 4, -2 \le x \le 1$

For ZX: $y = -3x + 8, 1 \le x \le 4$

c The gradient of XY is –1, the gradient of YZ is 1. The product of the two gradients is –1, so XY is perpendicular to YZ. This means triangle XYZ is right-angled, with the right angle at vertex Y. The hypotenuse is ZX.

d $x = \sqrt{18} = 3\sqrt{2}(\approx 4.24)$ units

$y = \sqrt{90} = 3\sqrt{10}\ (\approx 9.49)$ units

$z = \sqrt{72} = 6\sqrt{2}(\approx 8.49)$ units

e $x^2 + z^2 = \left(\sqrt{18}\right)^2 + \left(\sqrt{72}\right)^2 = 18 + 72 = 90$

$= \left(\sqrt{90}\right)^2 = y^2$

Using approximations gives:

$4.24^2 + 8.49^2 = 90.06$

$9.49^2 = 90.06.$

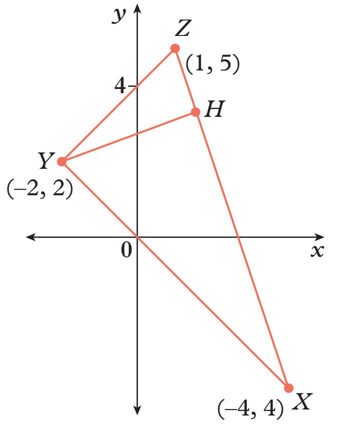
This shows that the square of the hypotenuse = the sum of the squares of the other two sides.

f M is the point (2.5, 0.5).

g $\frac{3\sqrt{10}}{2} \approx 4.74$ units

h Since $MX = MZ = \frac{\sqrt{90}}{2} = MY$, these are both isosceles triangles.

i $y = \frac{1}{3}x + 2\frac{2}{3}$



j $\left(-\frac{12}{11}, \frac{7}{11}\right)$

k $1.2\sqrt{10}$ or approximately 3.79 units

l Using YX as the base and YZ as the perpendicular height:

Area $= \frac{1}{2} \times 6\sqrt{2} \times 3\sqrt{2} \approx 18$ square units

Using XZ as the base and YH as the perpendicular height:

Area $= \frac{1}{2} \times 9.49 \times 3.79 = 18.0$ square units or

Area $= \frac{1}{2} \times 3\sqrt{10} \times 1.2\sqrt{10} = 18$ square units

m Using the two methods in **l**,

$2 \times$ Area of $XYZ = y \times YH = xz$,

so $YH = \frac{xz}{y}$

$$\frac{xz}{y} = \frac{3\sqrt{2} \times 6\sqrt{2}}{3\sqrt{10}} = \frac{12}{\sqrt{10}} = \frac{12}{\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}} = \frac{6\sqrt{10}}{5}$$

$= 1.2\sqrt{10} = YH$ or

$$\frac{xz}{y} \approx \frac{4.24 \times 8.49}{9.49} \approx 3.79 \text{ units} = YH$$

n With centre M, and radius $MX = MY = MZ$, a circle can be drawn to touch the vertices of the right-angled triangle. This is one of the geometric facts which states that the angle subtended by the diameter of a semi-circle is a right angle.

CHAPTER 5 Non-linear relationships

EX p214

5A Solving quadratic equations

1 **a** $x = -2$ or $x = 4$ **b** $x = -7$ or $x = -1$

c $x = 3$ or $x = 6$ **d** $x = \pm 9$

e $x = 0$ or $x = 5$ **f** $x = 0$ or $x = \frac{1}{2}$

g $x = 0$ or $x = -1$ **h** $x = 0$ or $x = -2$

i $x = 0$ or $x = \frac{3}{4}$ **j** $x = \pm\sqrt{2}$

k $x = \pm\sqrt{5}$ **l** $x = \pm 4\sqrt{3}$

m $x = -1\frac{1}{2}$ or $x = 4$ **n** $x = \pm\frac{1}{3}$

o $x = -\frac{2}{7}$ or $x = 1\frac{1}{4}$

2 **a** $x = -3$ or $x = -4$ **b** $x = -5$ or $x = 2$

c $x = 1$ or $x = 8$ **d** $x = \pm 5$

e $x = \pm 9$ **f** $x = \pm 2$

g $x = 0$ or $x = 3$ **h** $x = 0$ or $x = -1$

i $x = 0$ or $x = -4$ **j** $x = 0$ or $x = \frac{2}{5}$

k $x = 0$ or $x = 6$ **l** $x = 0$ or $x = -10$

4 **a** Both sides of the original equation have been divided by 3, so the equation remained in balance and the values of the variables are unchanged.

b $x = -6$ or $x = 8$

5 **a** $x = -5$ or $x = 6$ **b** $x = 2$ or $x = 4$

c $x = -8$ or $x = -1$ **d** $x = 3$ or $x = -9$

e $x = \pm 4$ **f** $x = 0$ or $x = -7$

g $x = 0$ or $x = 2$ **h** $x = 0$ or $x = -1\frac{2}{3}$

i $x = 1\frac{1}{2}$ or $x = -10$

6 **a** $x = -2$ or $x = -1$ **b** $x = -5$ or $x = 3$

c $x = 1$ or $x = 4$ **d** $x = -3$ or $x = 2$

e $x = -2$ or $x = 6$ **f** $x = -8$ or $x = 7$

g $x = \pm 2$ **h** $x = \pm 3$

i $x = 0$ or $x = \frac{1}{2}$

7 **a** $x = \pm\sqrt{3}$ **b** $x = \pm\sqrt{11}$ **c** $x = \pm\sqrt{7}$

d $x = \pm 2\sqrt{2}$ **e** $x = \pm 3\sqrt{3}$ **f** $x = \pm\sqrt{2}$

g $x = \pm\sqrt{5}$ **h** $x = \pm\sqrt{6}$ **i** $x = \pm 4\sqrt{2}$

j $x = \pm\sqrt{19}$ **k** $x = \pm 2\sqrt{5}$ **l** $x = \pm 2\sqrt{13}$

8 **a** $x = -1 \pm \sqrt{7}$ **b** $x = 2 \pm \sqrt{2}$

c $x = -4 \pm \sqrt{5}$ **d** $x = 6 \pm 2\sqrt{2}$

e $x = -10 \pm 5\sqrt{3}$ **f** $x = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$

9 **a** $x \approx -3.65$ or $x \approx 1.65$ **b** $x \approx 0.59$ or $x \approx 3.41$

c $x \approx -6.24$ or $x \approx -1.76$ **d** $x \approx 3.17$ or $x \approx 8.83$

e $x \approx -18.66$ or $x \approx -1.34$ **f** $x \approx -0.62$ or $x \approx 1.62$

10 Dividing both sides of the first equation by 3 gives the equivalent equation $x - 2 = 0$ and this does not affect the solution (with $x = 2$ being a solution for both equations). The solution for the second equation is $x = 0$ or $x = 2$. But dividing both sides of the second equation by x does not produce an equivalent equation because, when $x = 0$ is a solution, we cannot divide by 0.

11 **a** two **b** one **c** two

d one **e** two **f** none

12 **a** $x = \pm 3$ **b** $x = -11$ or $x = 7$

c $x = 4$ **d** no solutions

e $x = 0$ or $x = -4$ **f** $x = 3 \pm \sqrt{10}$

13 **a** $x = -6$ or $x = 2$ **b** $x = 1$ or $x = 4$

c $x = \pm 3$ **d** $x = -5$ or $x = 8$

e $x = 0$ or $x = 9$ **f** $x = 2 \pm \sqrt{3}$

g $x = 0$ or $x = 4$ **h** $x = 2$ or $x = 9$

i $x = -3$ or $x = 6$ **j** $x = -7$ or $x = 3$

k $x = 4$ or $x = 6$ **l** $x = -3$

14 **a** **i** 8 m **ii** 6 m

b 6 m

c 3 s

d It is not possible to have negative time values.

15 **a** 25 kg **b** 9 kg **c** 5 weeks

d No. If the amounts were the same each week, the relationship would be linear.

16 **a** $x(x + 3) = 70$ or $x^2 + 3x - 70 = 0$

b $x = -10$ or $x = 7$. The feasible solution is $x = 7$, while $x = -10$ is not feasible because it is not possible to have a negative length.

c Width 7 m, length 10 m

17 $x(x - 5) = 66$, where x is the length in centimetres. The dimensions are length 11 cm, width 6 cm.

18 **a** $x = -3 \pm \sqrt{2}$

b $x = -1 \pm \sqrt{5}$

c $x = 2 \pm \sqrt{3}$

d $x = -4 \pm \sqrt{10}$

e $x = 3 \pm \sqrt{\frac{21}{2}}$ or $x = 3 \pm \frac{\sqrt{42}}{2}$

f $x = -\frac{1}{2} \pm \sqrt{\frac{13}{20}}$ or $x = -\frac{1}{2} \pm \frac{\sqrt{65}}{10}$

19 **a** $x = -5$ or $x = -\frac{3}{2}$ **b** $x = \frac{2}{5}$ or $x = 4$

c $x = -\frac{2}{3}$ or $x = \frac{1}{2}$ **d** $x = -7$ or $x = \frac{3}{4}$

e $x = -1$ or $x = \frac{1}{3}$ **f** $x = -\frac{3}{7}$ or $x = 2$

EX
p220

5B The quadratic formula

1 **a** none **b** two **c** two

d none **e** one **f** two

2 **a** **i** 64 **ii** two **iii** $x = -3$ or $x = 5$

b **i** 0 **ii** one **iii** $x = -2$

c **i** 196 **ii** two **iii** $x = -7$ or $x = 7$

d **i** 324 **ii** two **iii** $x = 0$ or $x = 6$

e **i** 8 **ii** two **iii** $x = 3 - \sqrt{2}$ or $x = 3 + \sqrt{2}$

f **i** –8 **ii** none **iii** no solutions

3 **a** **i** $x = \frac{7 \pm \sqrt{29}}{2}$ **ii** $x \approx 0.81$ or $x \approx 6.19$

b **i** $x = \frac{1 \pm \sqrt{13}}{6}$ **ii** $x \approx -0.43$ or $x \approx 0.77$

c **i** $x = \frac{-1 \pm \sqrt{33}}{4}$ **ii** $x \approx -1.69$ or $x \approx 1.19$

d **i** $x = \frac{-1 \pm \sqrt{109}}{6}$ **ii** $x \approx -1.91$ or $x \approx 1.57$

e **i** $x = \frac{-5 \pm \sqrt{97}}{12}$ **ii** $x \approx -1.24$ or $x \approx 0.40$

f **i** $x = 1 \pm \sqrt{2}$ **ii** $x \approx -0.41$ or $x \approx 2.41$

g **i** $x = -4 \pm 2\sqrt{3}$ **ii** $x \approx -7.46$ or $x \approx -0.54$

h **i** $x = \frac{3 \pm 2\sqrt{2}}{2}$ **ii** $x \approx 0.09$ or $x \approx 2.91$

i **i** $x = 2 \pm \sqrt{6}$ **ii** $x \approx -0.45$ or $x \approx 4.45$

j **i** $x = \frac{-5 \pm 2\sqrt{10}}{5}$ **ii** $x \approx -2.26$ or $x \approx 0.26$

k **i** $x = \frac{-3 \pm \sqrt{97}}{4}$ **ii** $x \approx -3.21$ or $x \approx 1.71$

l **i** $x = -1 \pm 2\sqrt{2}$ **ii** $x \approx -3.83$ or $x \approx 1.83$

4 **a** $x = \frac{-3 \pm \sqrt{13}}{2}$ **b** no solutions **c** $x = -3 \pm \sqrt{5}$

d no solutions **e** $x = \frac{-5 \pm \sqrt{17}}{4}$ **f** $x = 4$

g $x = -1 \pm \sqrt{5}$ **h** $x = \frac{7 \pm \sqrt{61}}{2}$ **i** $x = -4 \pm \sqrt{26}$

j $x = \frac{1 \pm \sqrt{21}}{2}$ **k** no solutions **l** $x = \frac{-\sqrt{5} \pm 3}{2}$

5 **a** $m = \frac{-3 \pm \sqrt{37}}{2}$ **b** $u = -2 \pm \sqrt{5}$

c $r = \frac{-9 \pm \sqrt{77}}{2}$ **d** $y = -1$ or $y = 3$

e $h = \frac{-7 \pm \sqrt{61}}{6}$ **f** $w = 3 \pm \sqrt{14}$

6 $x = \frac{3 \pm \sqrt{13}}{2}$

7 Step 3: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

Step 4: $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$

Step 5: $\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$

Step 7: $\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right)^2 = 0$

Step 10: $\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right) = 0$

Step 11: $x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} = 0$ or $x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} = 0$

Step 12: $x = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$ or $x = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$

8 **a, b** $x = 2$ or $x = 3$

c In cases such as this, the method described in part **a** is often found to be quicker and easier.

d If integer solutions exist, the method described in part **a** is usually quicker.

9 **a** \$620 **b** 59

10 length 81.4 m, width 61.4 m

11 length 369 mm, width 131 mm

12 **a** There is no real number that is the square root of a negative number, so we cannot obtain a result using the quadratic formula. There are no solutions.

b The square root of zero is zero, so there will be only one distinct value obtained using the quadratic formula: $x = \frac{-b \pm 0}{2a} = -\frac{b}{2a}$

c The square root of a positive number gives a positive result so there will be two distinct solutions:

$x = \frac{-b - \text{positive}}{2a}$ or $x = \frac{-b + \text{positive}}{2a}$.

13 **a** **i** $\Delta = 33$. Solutions will be irrational.

ii $\Delta = 25$. Solutions will be rational.

iii $\Delta = 9$. Solutions will be rational.

iv $\Delta = 41$. Solutions will be irrational.

v $\Delta = 49$. Solutions will be rational.

b The quadratic formula is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. If $\sqrt{b^2 - 4ac}$ is an integer and a, b and c are integers then the solutions are rational. $\sqrt{b^2 - 4ac}$ is an integer if $b^2 - 4ac$, the discriminant, is a square number, since the square root of a square number is an integer.

14 **a** **i** $(-2,0)$ and $(-5,0)$ **ii** $(2,0)$ and $(4,0)$

iii no x-intercepts **iv** $(4,0)$

v no x-intercepts

b The discriminant relates to the number of solutions to a quadratic when it equals zero, and therefore the number of x-intercepts of its graph.

15 **a** $k = 3$

b **i** $k < 3$ **ii** $k > 3$

16 **a** $k = 1$ **b** $k < 2$ **c** $k < -9$ **d** $k = \frac{9}{4}$

e $k < -6$ and $k > 6$ **f** $-10 < k < 10$

EX p228

5C Sketching parabolas using intercepts

1 **a** **i** $(-6, 0)$ **ii** $(0, 0)$

b **i** $(0, 10)$ **ii** $(0, 0)$

c **i** $(1, 5)$ **ii** 5

d **i** $(-1, 1)$ **ii** -1

e **i** $(-6, 2)$ **ii** 12

f **i** $(-3, 3)$ **ii** 9

2 **a** upright **b** inverted **c** upright

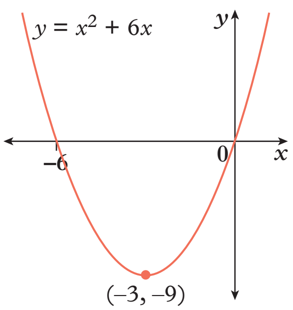
d upright **e** inverted **f** inverted

3 **a** $(-3, -9)$ **b** $(5, 25)$ **c** $(3, -4)$

d $(0, -1)$ **e** $(-2, 16)$ **f** $(0, 9)$

4 **a** $(-1, -16)$ **b** $\left(\frac{1}{2}, 12\frac{1}{4}\right)$

5 **a**



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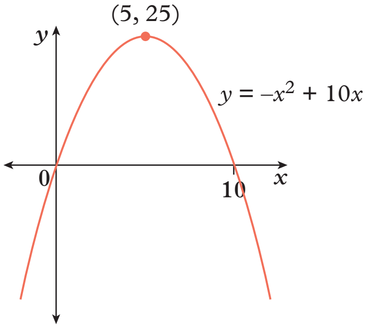

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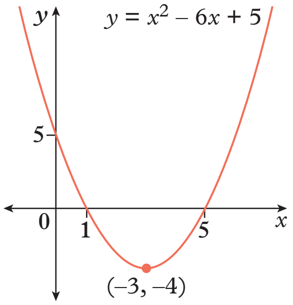

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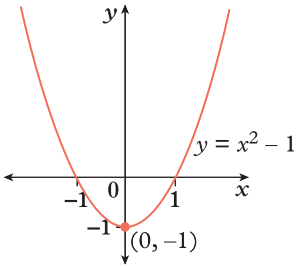

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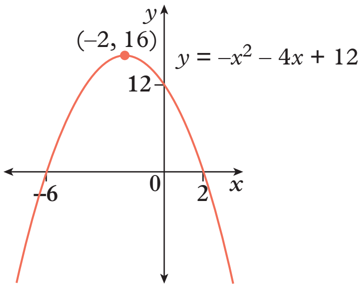

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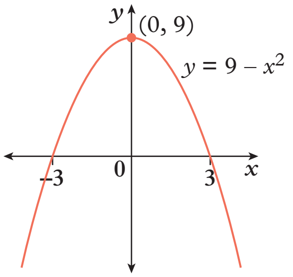
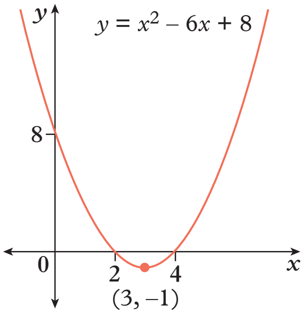


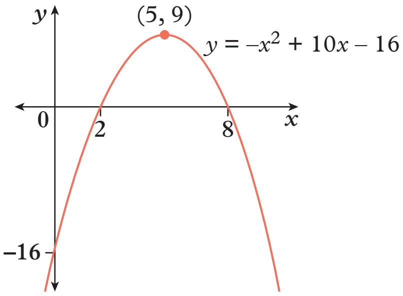
6 **a** Domain is ℝ.

Range is $y \geq -1$.



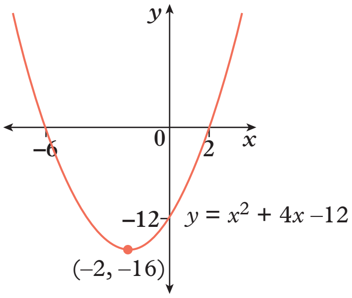
b Domain is ℝ.

Range is $y \leq 9$.



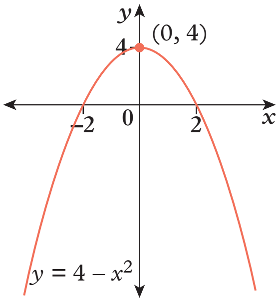
c Domain is ℝ.

Range is $y \geq -16$.



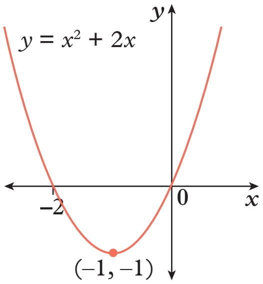
d Domain is ℝ.

Range is $y \leq 4$.



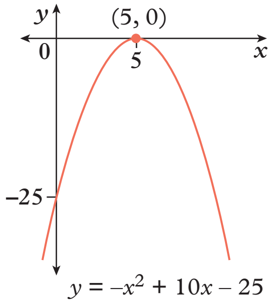
e Domain is ℝ.

Range is $y \geq -1$.



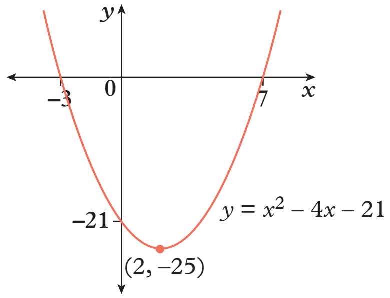
f Domain is ℝ.

Range is $y \leq 0$.



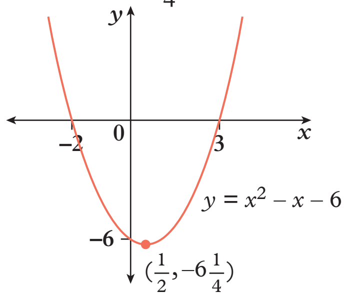
g Domain is ℝ.

Range is $y \geq -25$.



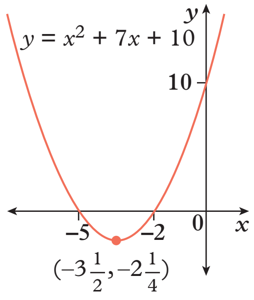
h Domain is ℝ.

Range is $y \geq -6\frac{1}{4}$.



i Domain is ℝ.

Range is $y \geq -2\frac{1}{4}$.



7 **a** For all these graphs, the x-intercepts are at $x = -5$ and $x = 1$.

b The x-intercepts are the points when $y = 0$. For these equations, when $y = 0$ the equations all have the same solution for x. But, for other y values, the x values differ.

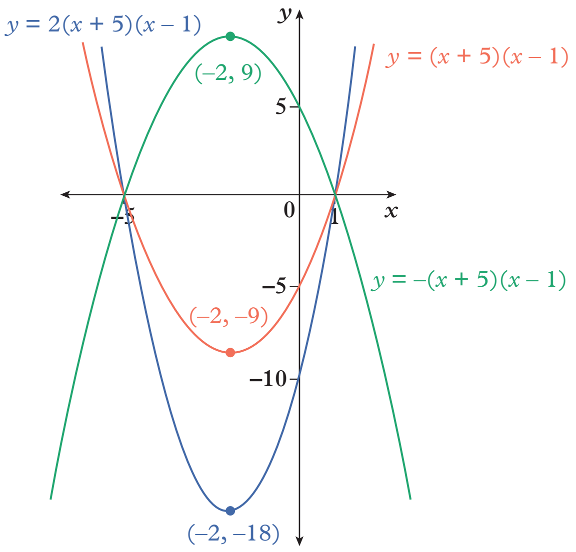
c For $y = (x + 5)(x - 1)$, the y-intercept is at $(0, -5)$.

For $y = 2(x + 5)(x - 1)$, the y-intercept is at $(0, -10)$.

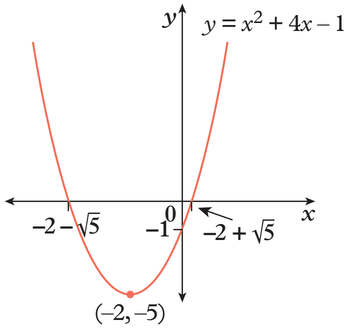
For $y = -(x + 5)(x - 1)$, the y-intercept is at $(0, 5)$.

d For $y = (x + 5)(x - 1)$, the turning point is $(-2, -9)$.
For $y = 2(x + 5)(x - 1)$, the turning point is $(-2, -18)$.
For $y = -(x + 5)(x - 1)$, the turning point is $(-2, 9)$.

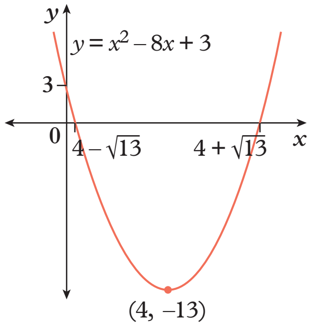
e



8 a



b



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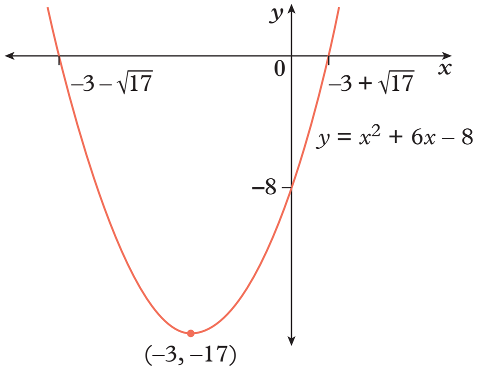

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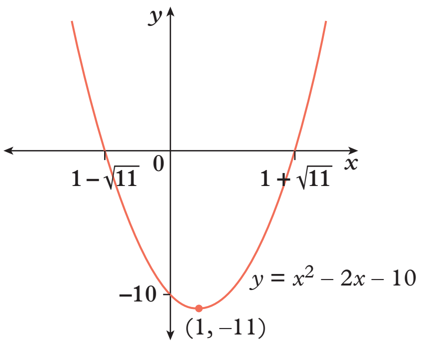

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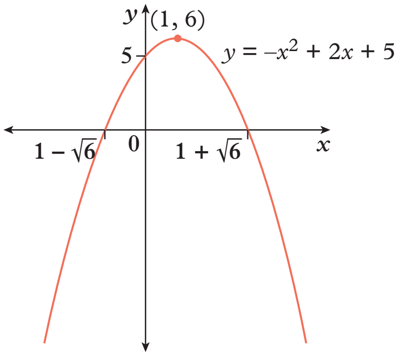
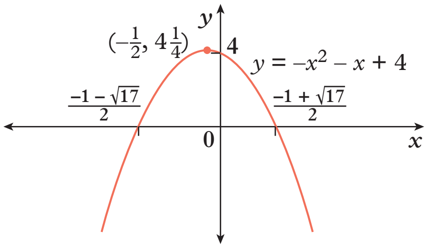


f



9 a i two **ii** one **iii** none

b i $x = -1$ or 2 **ii** $x = 2$ **iii** no solution

c i $\Delta = (-1)^2 - 4(1)(-2) = 9$. Because $\Delta > 0$, there are two solutions to $x^2 - x - 2 = 0$ and, therefore, two x-intercepts.

ii $\Delta = 4^2 - 4(-1)(-4) = 0$. Because $\Delta = 0$, there is one solution to $-x^2 + 4x - 4 = 0$ and, therefore, one x-intercept.

iii $\Delta = 0^2 - 4(1)(\ 2) = -8$. Because $\Delta < 0$, there are no solutions to $x^2 - x - 2 = 0$ and, therefore, no x-intercepts.

d i $x = \dfrac{-(-1) \pm \sqrt{1 - 4(1)(-2)}}{2} = \dfrac{1 \pm \sqrt{9}}{2} = \dfrac{1 \pm 3}{2}$
$= -1$ or 2

ii $x = \dfrac{-4 \pm \sqrt{16 - 4(-1)(-4)}}{-2} = \dfrac{-4 \pm \sqrt{0}}{-2}$
$= \dfrac{-4 \pm 0}{-2} = 2$

iii $x = \dfrac{0 \pm \sqrt{0 - 4(1)(2)}}{2} = \dfrac{\pm\sqrt{-8}}{2} =$ undefined

10 a i $\Delta = 0$, one x-intercept

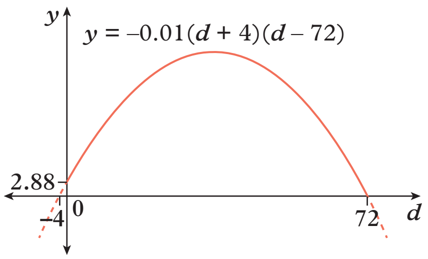
ii $(2, 0)$

b i $\Delta = -8$, no x-intercepts

c i $\Delta = 8$, two x-intercepts

ii $(3 - \sqrt{2}, 0), (3 + \sqrt{2}, 0)$

11 a

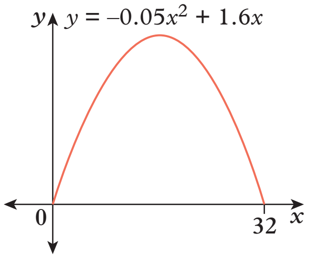


b 2.88 m

c 72 m

d 14.44 m

12 a



b height of 12.8 m at horizontal distance of 16 m

c 32 m

d 30.8 m

13 a $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$ or $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$

b Average of x-intercepts

$= \frac{1}{2}\left(\frac{-b-\sqrt{b^2-4ac}}{2a} + \frac{-b+\sqrt{b^2-4ac}}{2a}\right)$

$= \frac{1}{2}\left(\frac{-2b}{2a}\right)$

$= \frac{-b}{2a}$ which is the x-coordinate of the turning point

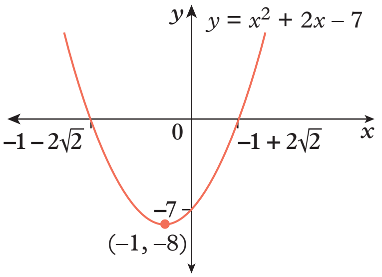
c When $x = \frac{-b}{2a}$, $y = a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c$

$= \frac{b^2}{4a} - \frac{b}{2a}^{2} + c$

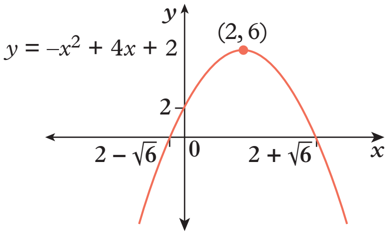
$= -\frac{b^2}{4a} + c$ or $c - \frac{b^2}{4a}$ which is the y-coordinate of the turning point

The turning point is $\left(\frac{-b}{2a}, -\frac{b^2}{4a} + c\right)$ or $\left(\frac{-b}{2a}, c - \frac{b^2}{4a}\right)$.

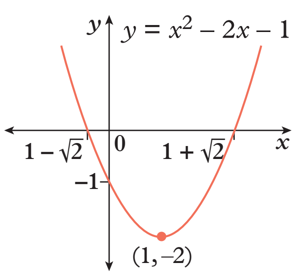
14 a



b



c



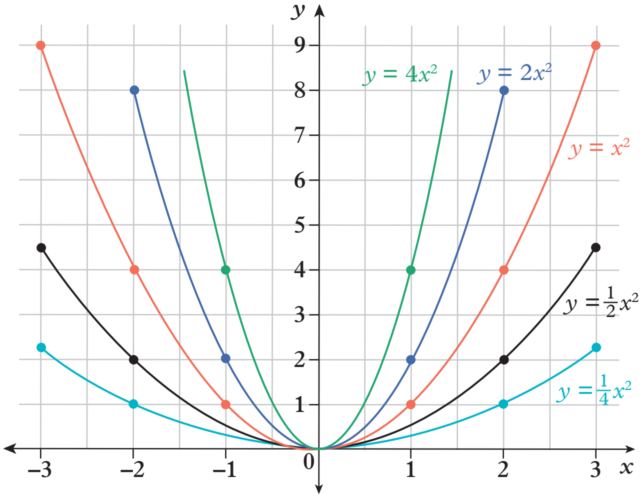
15 a $y = (x-1)(x-3)$ **b** $y = 2(x+1)(x-2)$

c $y = -2(x-2)(x+3)$ **d** $y = \frac{1}{2}(x-2)(x+6)$

EX
p235

5D Sketching parabolas using transformations

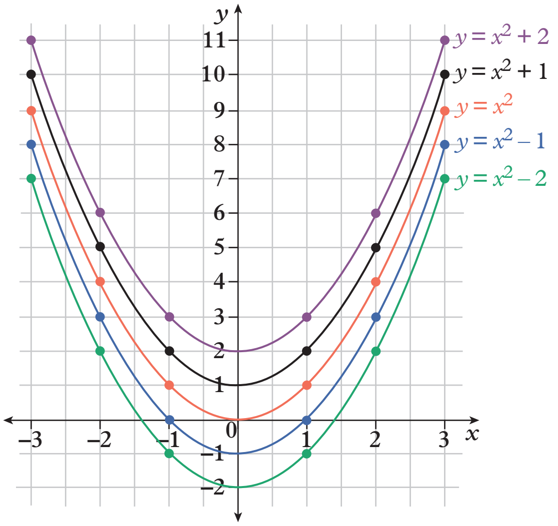
1 a i



ii All the graphs have the same minimum turning point (0, 0), the same axis of symmetry and the same x- and y-intercepts [also (0, 0)]. The graphs of $y = 2x^2$ and $y = 4x^2$ are narrower than the graph of $y = x^2$; and the graphs of $y = \frac{1}{2}x^2$ and $y = \frac{1}{4}x^2$ are wider than the graph of $y = x^2$.

iii stretching; a graph is narrower when the coefficient of x^2 is greater than 1; a graph is wider when the coefficient of x^2 is between 0 and 1.

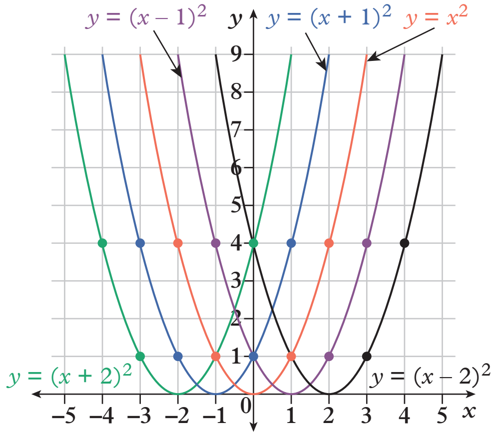
b i



ii All graphs have the same axis of symmetry and the same shape but they have different turning points.

iii vertical translation; the graph of $y = x^2$ moves up when a positive constant is added; the graph of $y = x^2$ moves down when a negative constant is added (or a positive constant is subtracted)

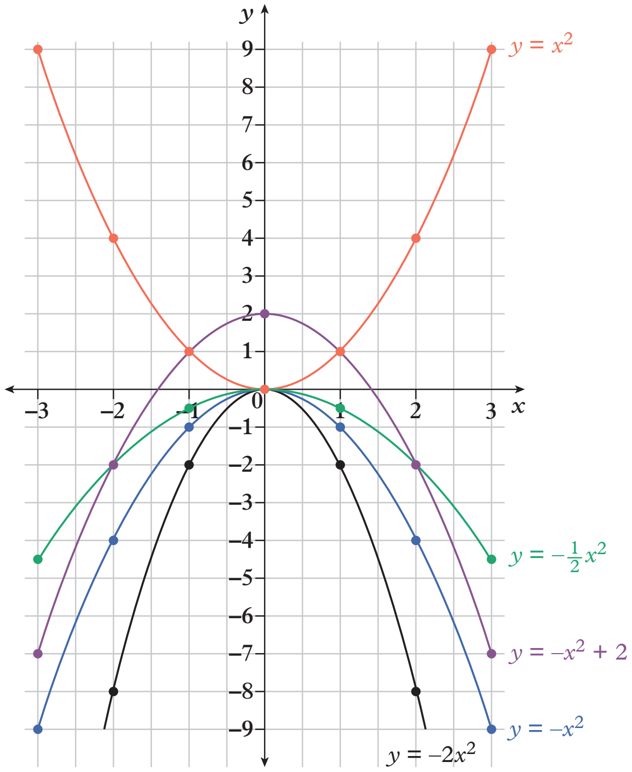
c **i**



ii All the graphs have the same shape but different turning points.

iii horizontal translation; the graph of $y = x^2$ moves to the right when a positive constant is subtracted from x before squaring; the graph of $y = x^2$ moves to the left when a negative constant is subtracted from x (or a positive constant is added to x) before squaring

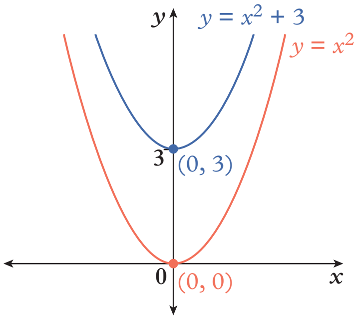
d **i**



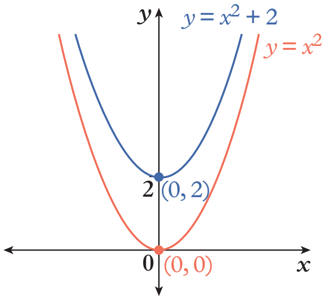
ii The graph of $y = -x^2$ is a reflection in the x-axis of the graph of $y = x^2$, but it has the same shape, the same axis of symmetry and the coordinates of its turning point (0, 0) are the same as for the graph of $y = x^2$. The graph of $y = -x^2 + 2$ results from the graph of $y = x^2$ being reflected in the x-axis and moved 2 units up. The graph of $y = -2x^2$, compared to the graph of $y = x^2$, is reflected in the x-axis and is narrower. The graph of $y = -\frac{1}{2}x^2$, compared to the graph of $y = x^2$, is reflected in the x-axis and is wider.

iii All the graphs have been reflected in the x-axis when compared to the graph of $y = x^2$ because each has a negative coefficient of x^2. There is also a vertical translation for $y = -x^2 + 2$ because a constant is added to x^2; there is stretching for $y = -2x^2$ (it is narrower than $y = x^2$ because its coefficient of x^2 is greater than 1); and there is stretching for $y = -\frac{1}{2}x^2$ (it is wider than $y = x^2$ because its coefficient of x^2 is between 0 and 1).

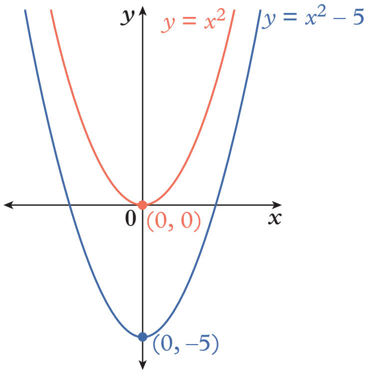
2 **a**



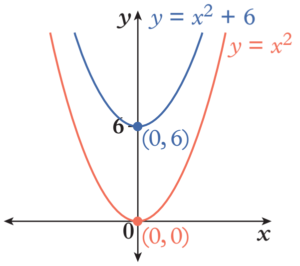
b



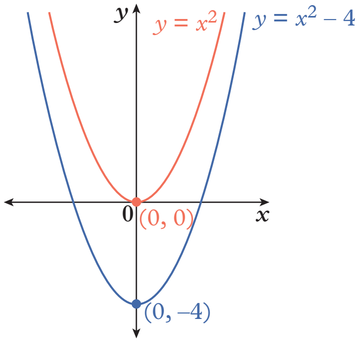
c



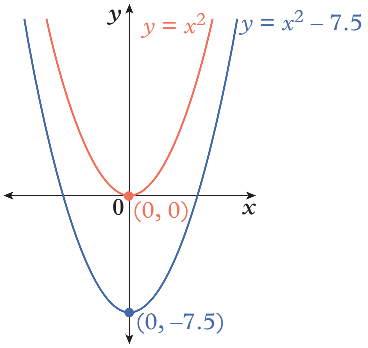
d



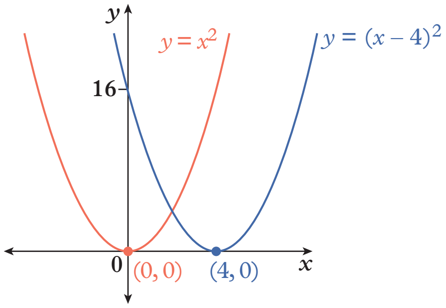
e



f



3 **a**



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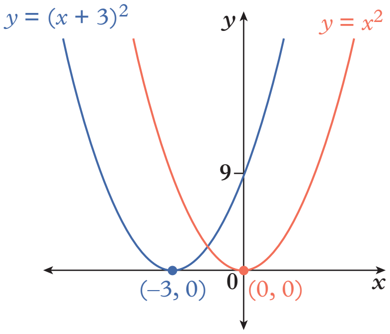
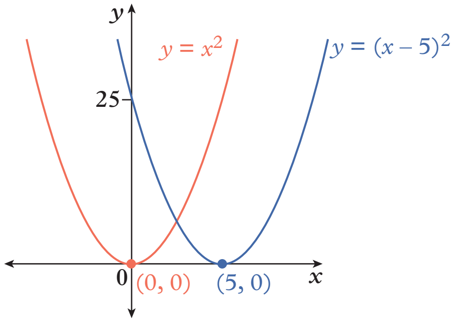
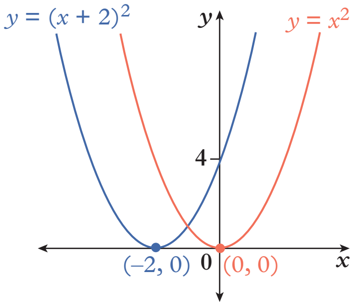


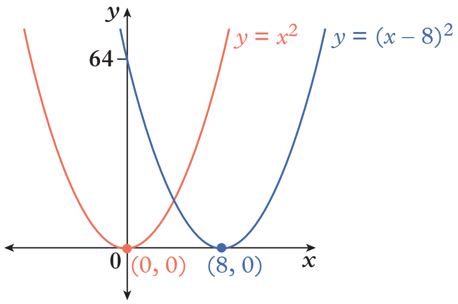
c



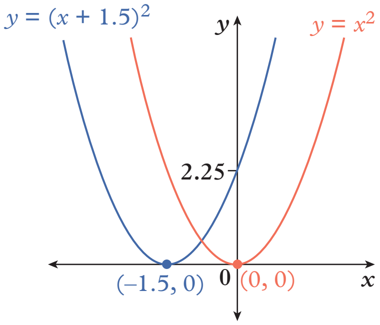
d



e



f



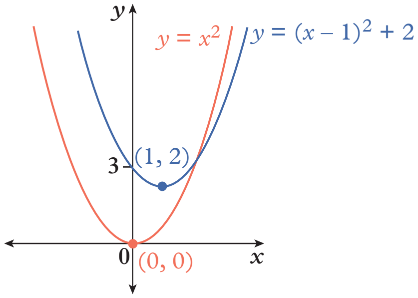
4 **a** B **b** D **c** A

d F **e** E **f** C

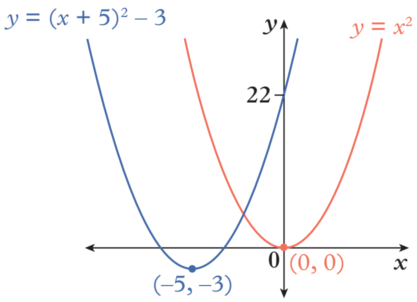
5 **a** $y = 6x^2$ **b** $y = -x^2$ **c** $y = (x - 5)^2$

d $y = x^2 - 2$ **e** $y = -\frac{1}{4}x^2$ **f** $y = -x^2 + 3$

6 **a**



b



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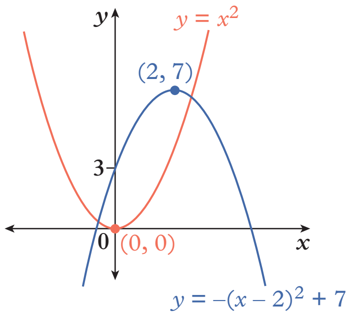
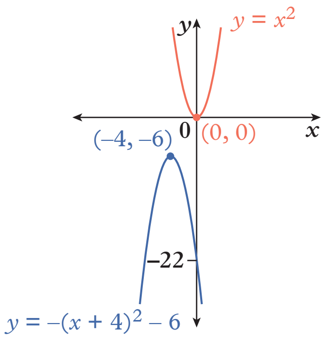


d



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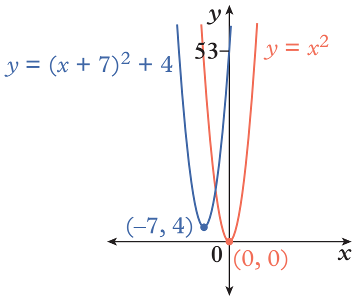
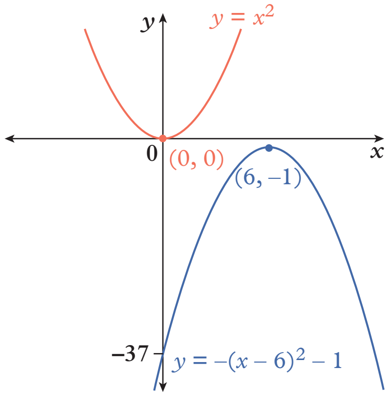
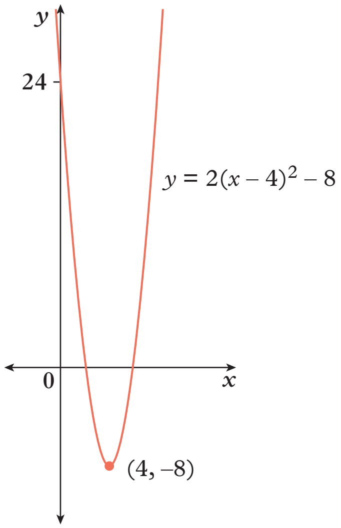


f



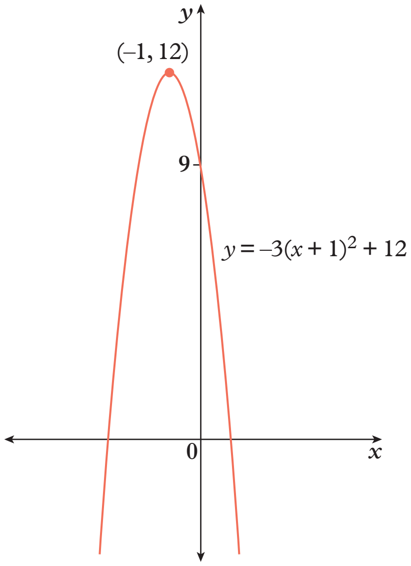
8 **a** **i** narrower **ii** upright **iii** (4, −8)

iv



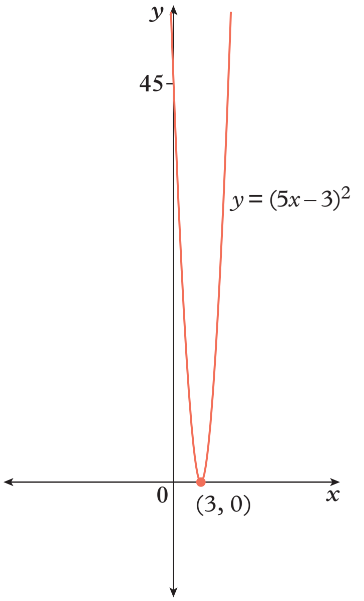
b **i** narrower **ii** inverted **iii** (−1, 12)

iv



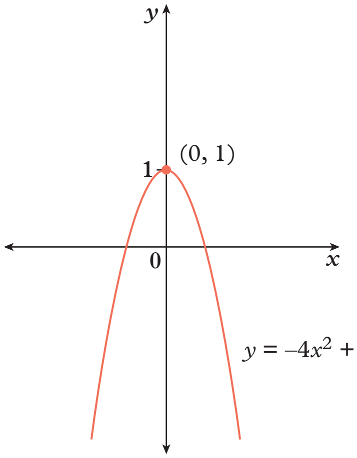
c **i** narrower **ii** upright **iii** (3, 0)

iv



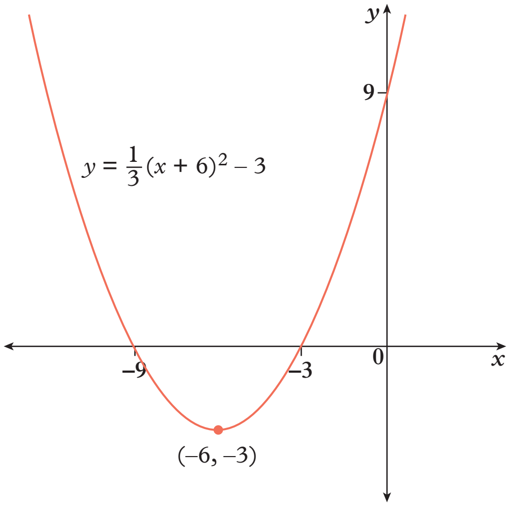
d **i** narrower **ii** inverted **iii** (0, 1)

iv



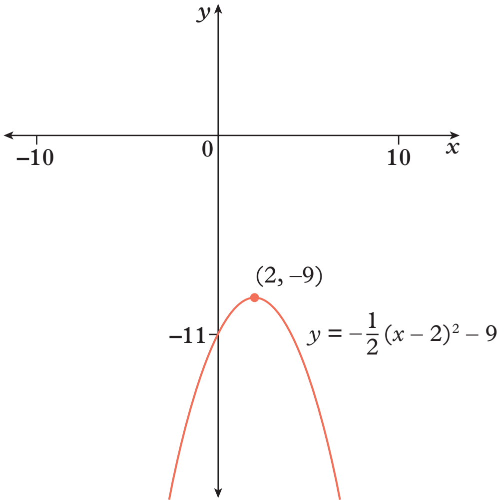
e **i** wider **ii** upright **iii** (–6, –3)

iv



f **i** wider **ii** inverted **iii** (2, –9)

iv



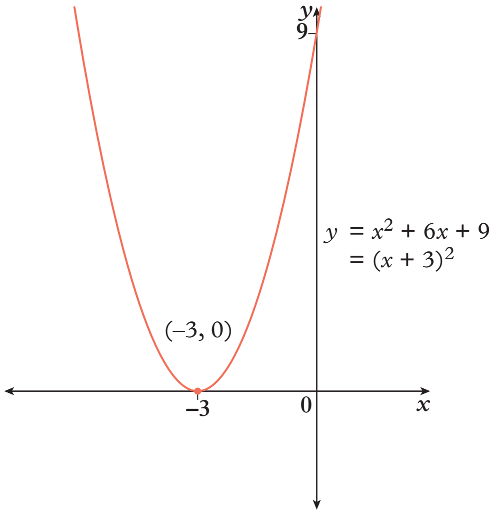
9 **a** $y = 2(x - 5)^2$ **b** $y = -(x + 4)^2 - 3$

c $y = -\frac{1}{3}x^2$ **d** $y = -5x^2 + 7$

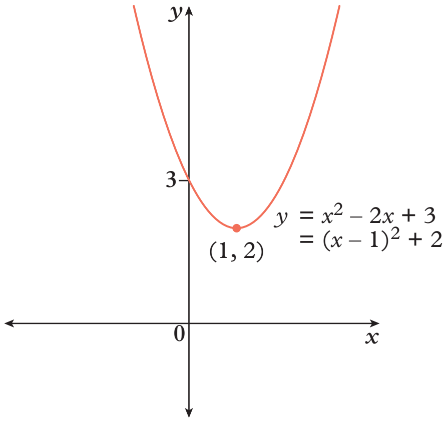
10 4 is the smallest, because the graph is upright and has a minimum turning point at (–2, 4). There is no largest value because the y values increase as the x values move away from –2.

11 –1 is the largest, because the graph is inverted and has a maximum turning point at (6, –1). There is no smallest value because the y values decrease as the x values move away from 6.

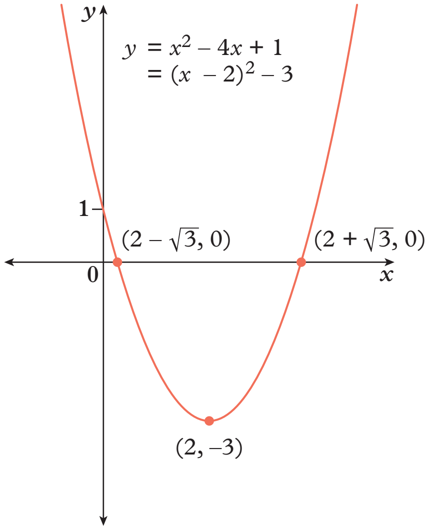
12 **a**



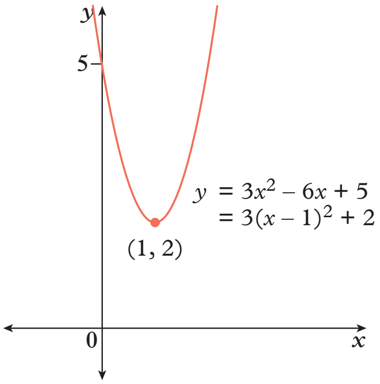
b



c

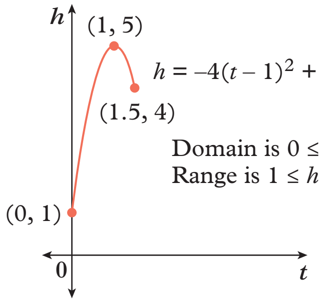


d



13 a (1, 5)

b



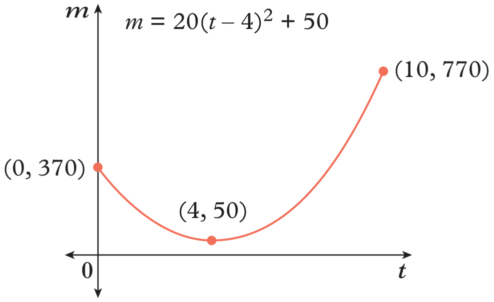
Domain is $0 \leq t \leq 1.5$.
Range is $1 \leq h \leq 5$.

c **i** 1 m **ii** 5 m **iii** 4 m

d 2.1 seconds

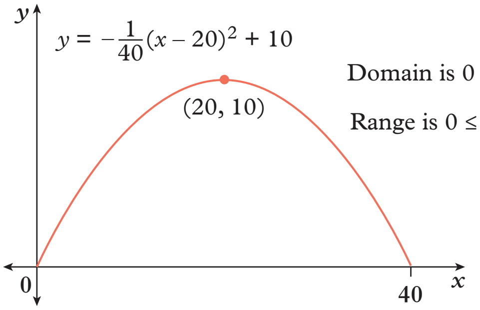
14 a The domain is $0 \leq t \leq 10$.

The range is $50 \leq m \leq 770$.



b \$370 **c** \$50 **d** 8 weeks **e** \$770

15 a



Domain is $0 \leq x \leq 40$.

Range is $0 \leq y \leq 10$.

b 10 m **c** 40 m

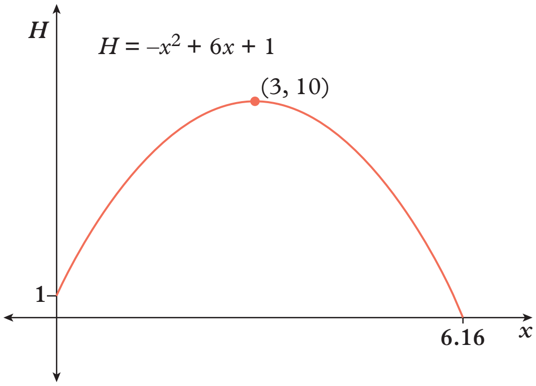
16 a 1 metre

b $-(x - 3)^2 + 10$

c 10 metres

d 6.16 metres

e



17 a $y = (x - 2)^2 + 6$ **b** $y = 2(x + 4)^2 - 12$

c $y = -3(x + 1)^2 + 7$ **d** $y = -\frac{1}{3}(x - 3)^2 - 5$

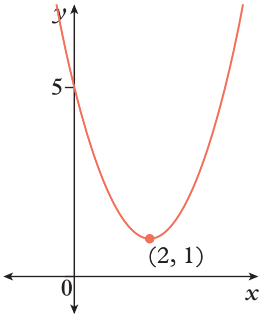
18 a $y = (x - 1)^2 + 2$ **b** $y = \frac{1}{2}(x + 4)^2 + 1$

c $y = -2(x - 2)^2 + 3$ **d** $y = (x + 1)^2 - 4$

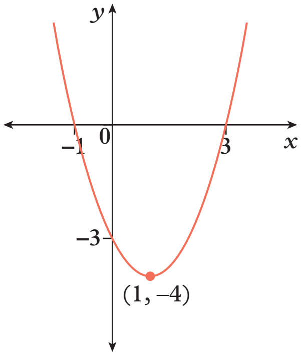
19 a Sample answer:

$$\begin{aligned}(2 - x)^2 &= (2 - x)(2 - x)\\ &= 4 - 2x - 2x + x^2\\ &= x^2 - 4x + 4\\ &= (x - 2)(x - 2)\\ &= (x - 2)^2\end{aligned}$$

b **i**



ii



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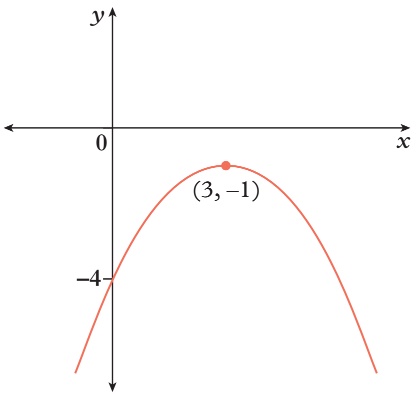

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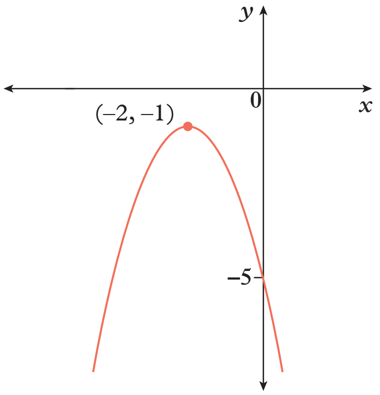


5 Checkpoint

1 **a** $x = 0$ or $x = 2$ **b** $x = -3$ or $x = 4$

c $x = -1$ or $x = 1$ **d** $x = -\frac{3}{2}$ or $x = 5$

2 **a** $x = 0$ or $x = -5$ **b** $x = 0$ or $x = 6$

c $x = -2$ or $x = -4$ **d** $x = 2$ or $x = 20$

e $x = -17$ or $x = 1$ **f** $x = -3$ or $x = 2$

3 **a** $x = 2$ or $x = -2$ **b** $x = \sqrt{7}$ or $x = -\sqrt{7}$

c $x = \sqrt{22}$ or $x = -\sqrt{22}$ **d** $x = 3$ or $x = -3$

e $x = \sqrt{5}$ or $x = -\sqrt{5}$ **f** $x = \sqrt{6}$ or $x = -\sqrt{6}$

4 **a** $\Delta = -3$ **b** $\Delta = 24$

c $\Delta = -4$ **d** $\Delta = 169$

5 **a** $\Delta = 13$. Therefore, there are two solutions.

b $\Delta = 0$. Therefore, there is one solution.

c $\Delta = 129$. Therefore, there are two solutions.

d $\Delta = -7$. Therefore, there are no solutions.

6 **a** $x = \frac{-3 \pm \sqrt{5}}{2}$ **b** $x = 1$ **c** $x = \frac{3 \pm \sqrt{13}}{2}$

d $x = \frac{-1 \pm \sqrt{41}}{10}$ **e** $x = -\frac{3}{2}$ **f** $x = -5 \pm \sqrt{31}$

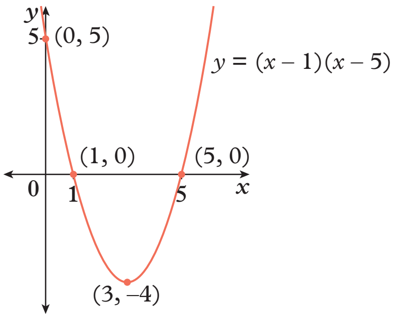
7 **a** x-intercepts: $(0, 0)$ and $(4, 0)$, y-intercept: $(0, 0)$

b x-intercepts: $(-3, 0)$ and $(1, 0)$, y-intercept: $(0, -3)$

c x-intercepts: $(-1, 0)$ and $(2, 0)$, y-intercept: $(0, -2)$

d x-intercepts: $(-8, 0)$ and $(2, 0)$, y-intercept: $(0, -16)$

8



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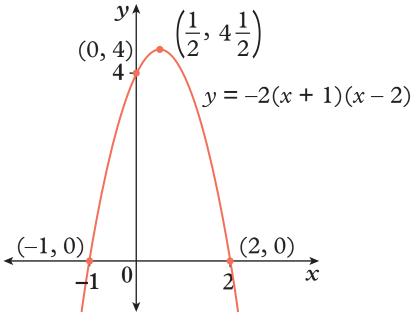
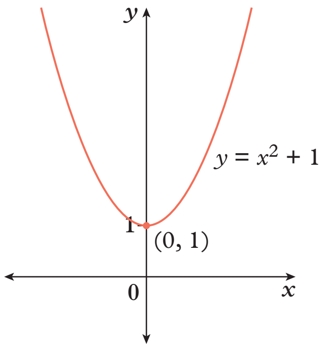
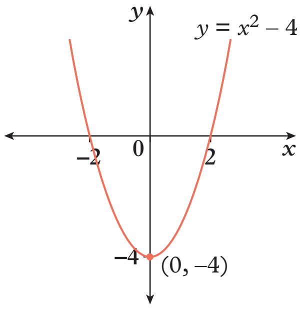


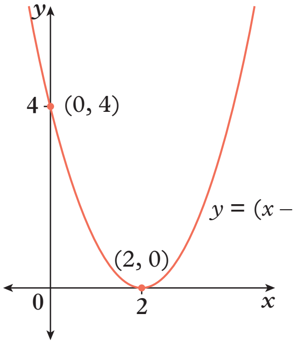
10 a



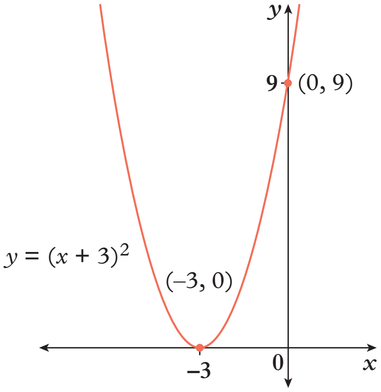
b



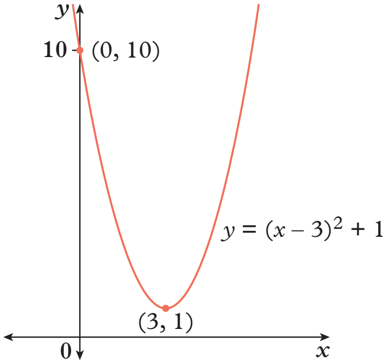
c



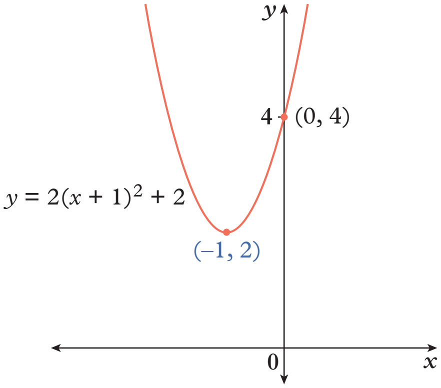
d



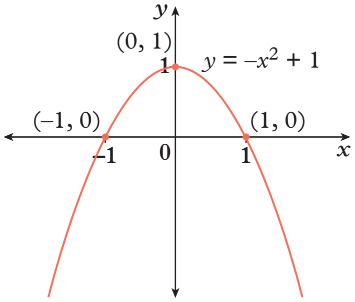
11 a



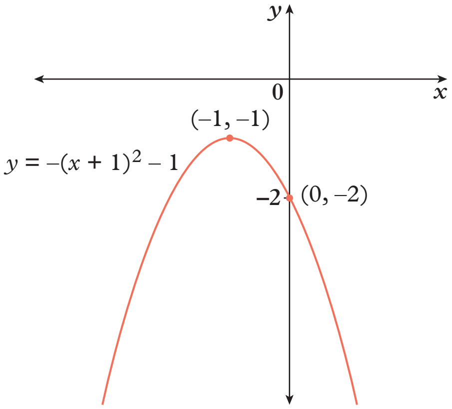
b



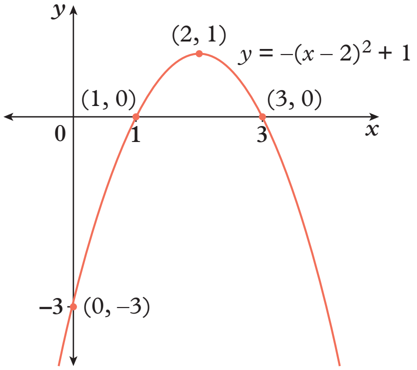
c



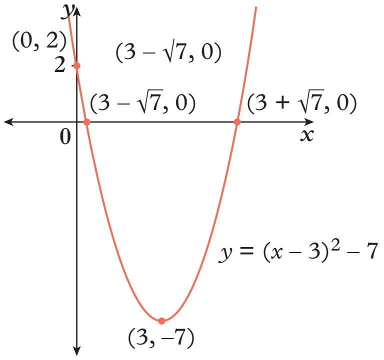
d



e



f



12 a $y = (x - 2)^2 + 1$ **b** $y = -(x + 3)^2 + 1$

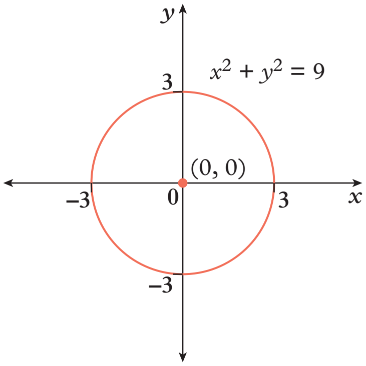
c $y = 2x^2 - 3$ **d** $y = -3(x - 4)^2 - 2$

EX p244

5E Circles

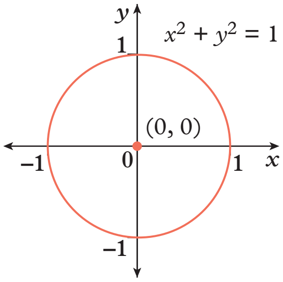
1 **a** Domain is $-3 \le x \le 3$.

Range is $-3 \le y \le 3$.



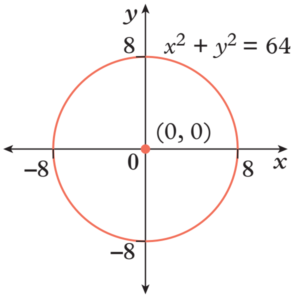
b Domain is $-1 \le x \le 1$.

Range is $-1 \le y \le 1$.



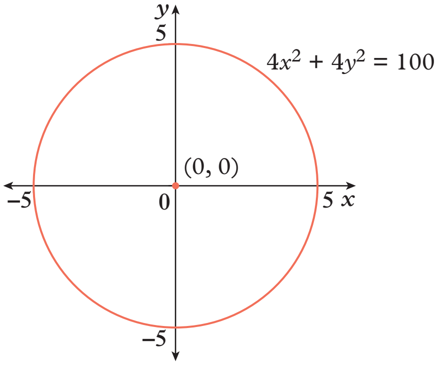
c Domain is $-8 \le x \le 8$.

Range is $-8 \le y \le 8$.



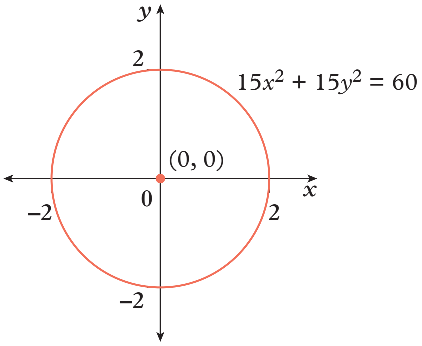
d Domain is $-5 \le x \le 5$.

Range is $-5 \le y \le 5$.



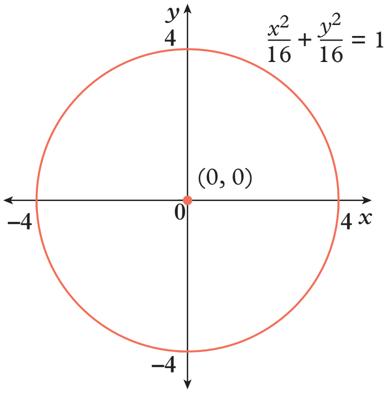
e Domain is $-2 \le x \le 2$.

Range is $-2 \le y \le 2$.



f Domain is $-4 \le x \le 4$.

Range is $-4 \le y \le 4$.



2 **a** $x^2 + y^2 = 9^2$ or $x^2 + y^2 = 81$

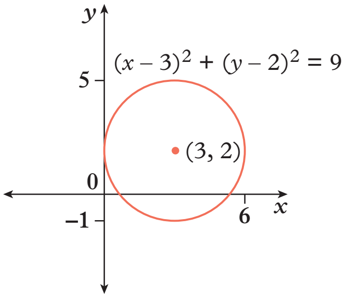
b $x^2 + y^2 = 36$

c $x^2 + y^2 = 400$

3 **a** centre (3, 2), radius 3 units

Domain is $0 \le x \le 6$.

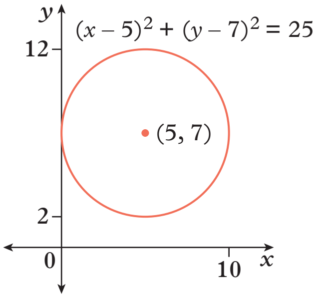
Range is $-1 \le y \le 5$.



b centre (5, 7), radius 5 units

Domain is $0 \le x \le 10$.

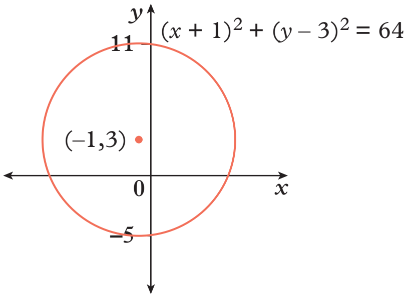
Range is $2 \le y \le 12$.



c centre (–1, 3), radius 8 units

Domain is $-9 \le x \le 7$.

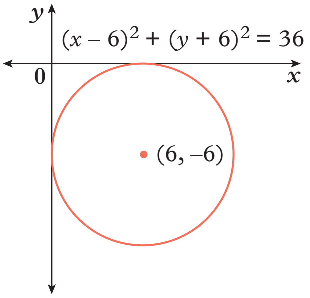
Range is $-5 \le y \le 11$.



d centre (6, –6), radius 6 units

Domain is $0 \le x \le 12$.

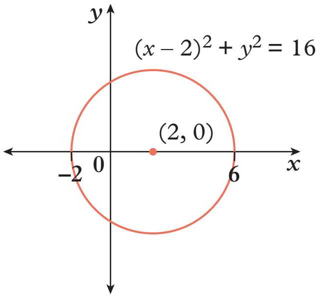
Range is $-12 \le y \le 0$.



e centre (2, 0), radius 4 units

Domain is $-2 \le x \le 6$.

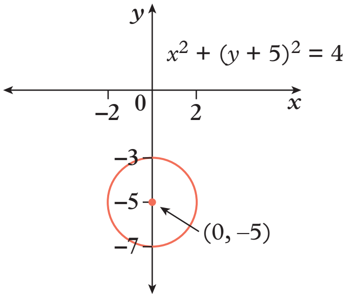
Range is $-4 \le y \le 4$.



f centre (0, –5), radius 2 units

Domain is $-2 \le x \le 2$.

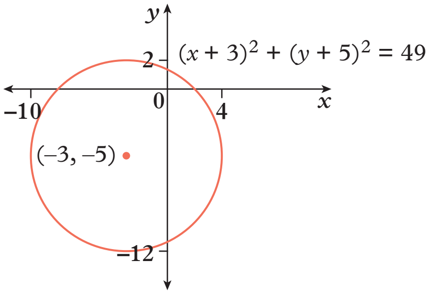
Range is $-7 \le y \le -3$.



g centre (–3, –5), radius 7 units

Domain is $-10 \le x \le 4$.

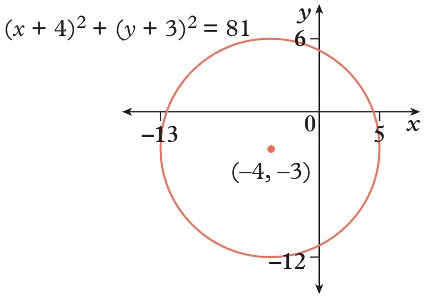
Range is $-12 \le y \le 2$.



h centre (–4, –3), radius 9 units

Domain is $-13 \le x \le 5$.

Range is $-12 \le y \le 6$.



5 **a** $x^2 + y^2 = 144$

b $x^2 + y^2 = 225$

c $(x - 4)^2 + (y + 2)^2 = 1$

d $(x + 5)^2 + (y + 6)^2 = 49$

e $(x - 4)^2 + (y - 4)^2 = 9$

f $x^2 + (y + 5)^2 = 36$

g $(x + 1)^2 + (y - 2.5)^2 = 2.25$

h $(x - 2)^2 + (y + 3)^2 = 5$

6 **a** $(x - 7)^2 + (y - 3)^2 = 9$ **b** $(x + 1)^2 + (y + 4)^2 = 25$

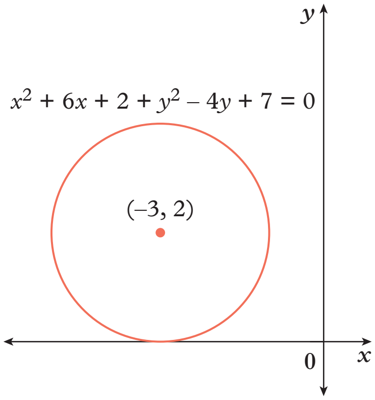
c $(x + 6)^2 + (y - 5)^2 = 4$ **d** $(x - 2)^2 + (y + 8)^2 = 16$

7 **a** $(x - 2)^2 + (y + 1)^2 = 9$

b A stretch in all directions by a factor of 5, then a translation of 7 units left and 4 units up

8 **a** **i** $(x + 3)^2 - 7$ **ii** $(y - 2)^2 + 3$

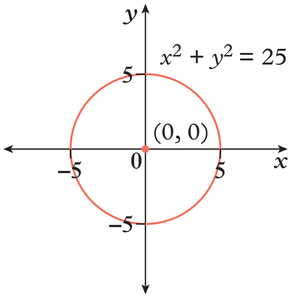
b



9 **a** a is the horizontal distance between the coordinates $(3,4)$ and (x,y). This distance is measured as $x - 3$, hence $a = x - 3$. b is the vertical distance between the coordinates $(3,4)$ and (x,y). This distance is measured as $y - 4$, hence $b = y - 4$.

b Pythagoras' theorem, $a^2 + b^2 = c^2$, is useful for determining the side lengths of a right-angled triangle. The right-angled triangle in the diagram can be explained by $a^2 + b^2 = 2^2$. Substituting a and b into the equation gives $(x - 3)^2 + (y - 4)^2 = 2^2$. Since every point (x,y) on the circle can be found by this process, this is the equation of the circle.

10 **a**



b $y = \pm\sqrt{25 - x^2}$

c **i** B; because $y \leq 0$ for $-5 \leq x \leq 5$

ii A; because $y \geq 0$ for $-5 \leq x \leq 5$

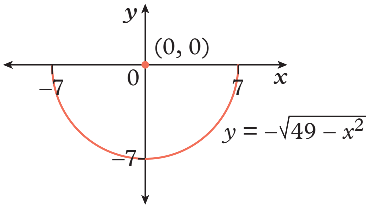
d $y = \sqrt{25 - x^2}$ gives the upper half of the circle and $y = -\sqrt{25 - x^2}$ gives the lower half of circle. Together, $y = \pm\sqrt{25 - x^2}$, or $x^2 + y^2 = 25$, gives the whole circle.

11 **a** $y = \sqrt{9 - x^2}$ **b** $y = -\sqrt{64 - x^2}$

c $y = \sqrt{1 - x^2}$

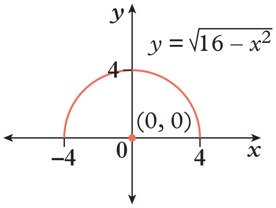
12 **a** Domain is $-7 \leq x \leq 7$.

Range is $-7 \leq y \leq 0$.



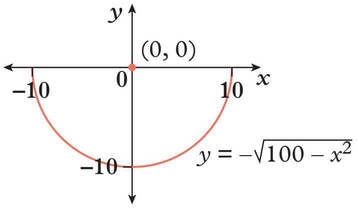
b Domain is $-4 \leq x \leq 4$.

Range is $0 \leq y \leq 4$.



c Domain is $-10 \leq x \leq 10$.

Range is $-10 \leq y \leq 0$.

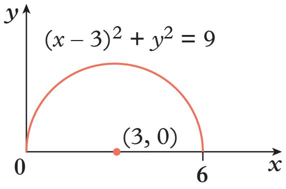


13 **a** $x = \pm\sqrt{25 - y^2}$

b For $x = \sqrt{25 - y^2}$, the graph is the right-hand half of the circle with equation $x^2 + y^2 = 25$.

For $x = -\sqrt{25 - y^2}$, the graph is the left-hand half of the circle with equation $x^2 + y^2 = 25$.

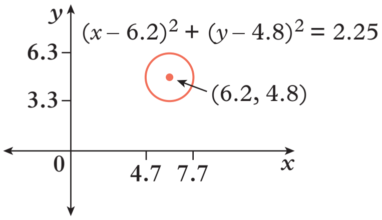
14 **a**



b 6 m **c** 3 m

d When $y = 1.5$, $x \approx 0.4$ and 5.6, so the arch is at half its maximum height at a distance of 0.4 m from either end.

15 **a**



b 3.3 m **c** 7 m² **d** 4.7%

16 $(-4, 3)$, $(-3, 4)$

17 **a** $(0, -3)$, $(3, 0)$ **b** $(-3, 6)$, $(-1, 8)$

18 $(x + 4)^2 + (y - 5)^2 = 36$

19 **a** $(1,2)$

b Substituting in $(3,3)$ gives $(3 - 1)^2 + (3 - 2)^2 = 5$, which is true. Therefore, since $(3, 3)$ satisfies the equation, it is a point on the circle.

c $\frac{1}{2}$

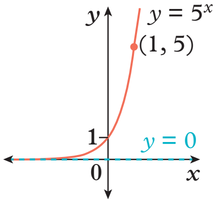
d -2

e $y = -2x + 9$

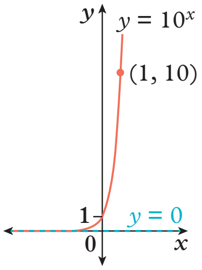
EX p253

5F Exponential relationships

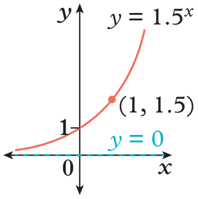
1 **a**



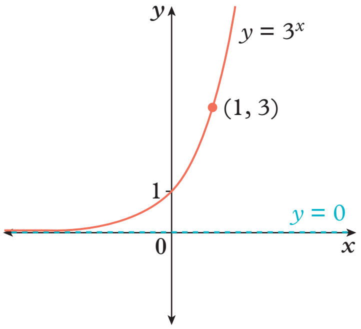
b



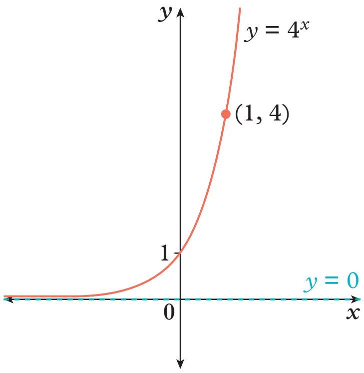
c



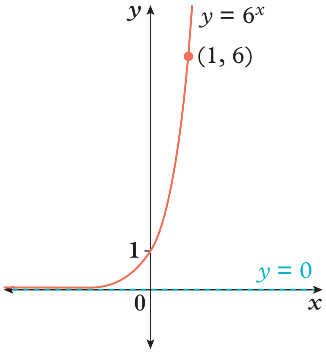
d



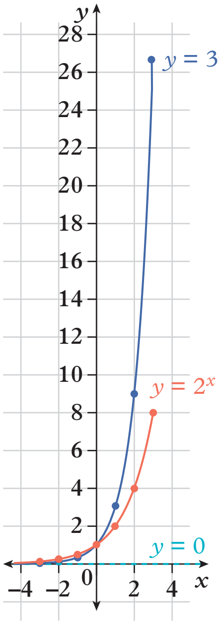
e



f



2 a



b The curve for $y = 3^x$ becomes steeper as the positive x values increase. That is, the y values for $y = 3^x$ are bigger than those for $y = 2^x$ when $x > 0$. When $x < 0$, the y values for $y = 3^x$ are smaller than those for $y = 2^x$.

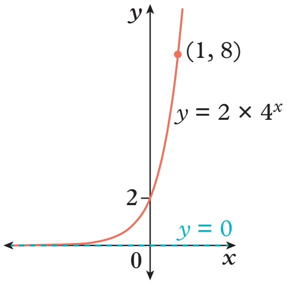
c They have the same asymptote and y-intercept. They are the same general shape.

d (0, 1) for both graphs

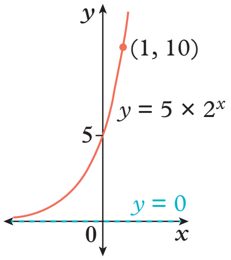
e $y = 0$ for both graphs

f The curve for $y = 5^x$ becomes steeper than both as the positive x values increase. That is, the y values for $y = 5^x$ are bigger than those for both $y = 2^x$ and $y = 3^x$ when $x > 0$. For $x < 0$, the y values for $y = 5^x$ are smaller than those for $y = 2^x$ and $y = 3^x$. However, the asymptote and y-intercept would be the same, and the general shape would be the same.

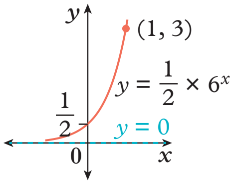
3 a



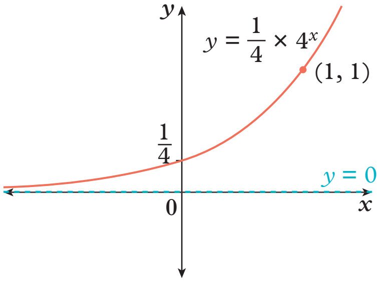
b



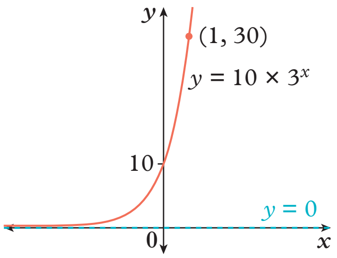
c



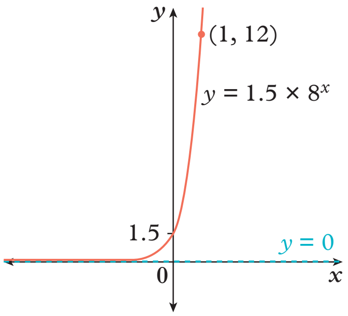
d



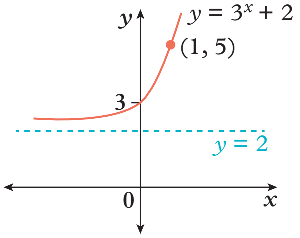
e



f



4 a



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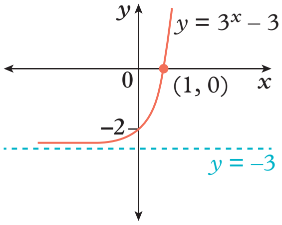

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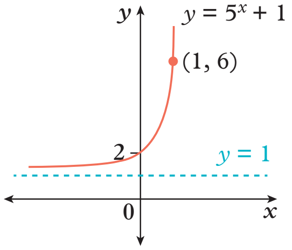

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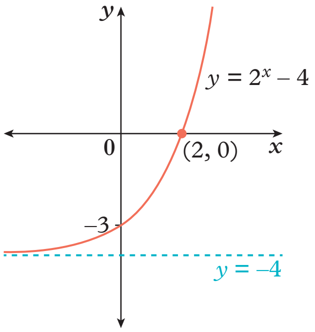

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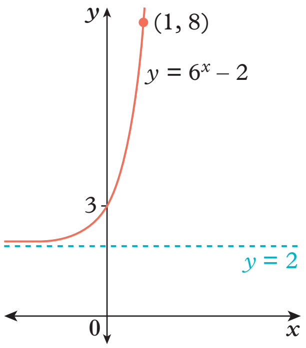

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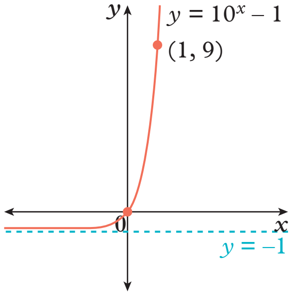

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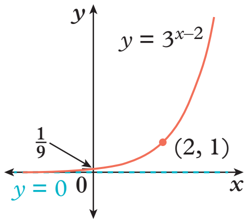

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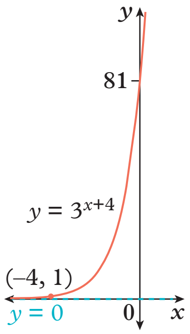

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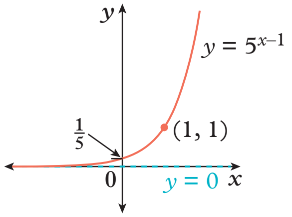

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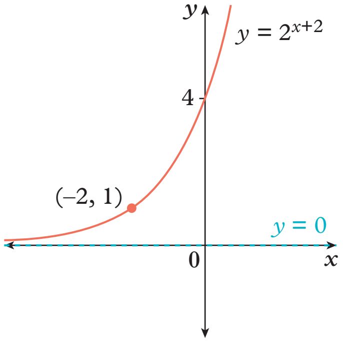

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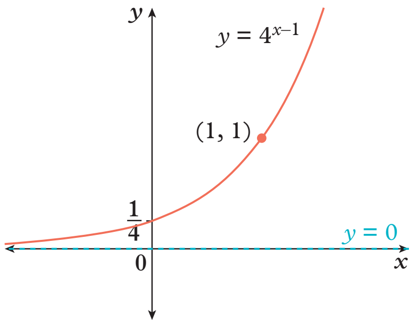

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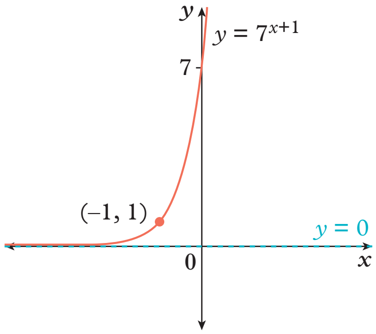
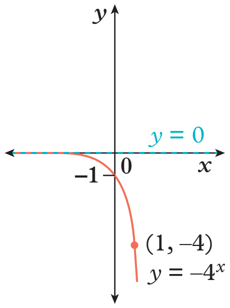


6 **a**



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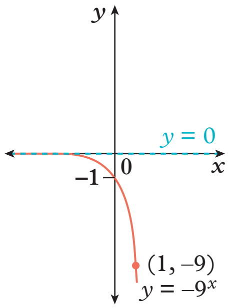

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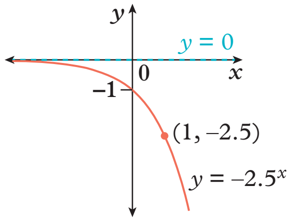

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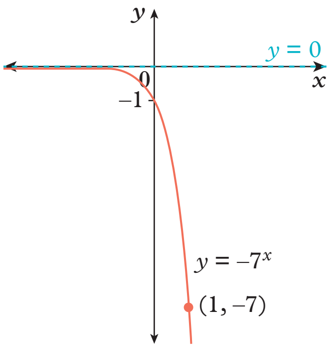

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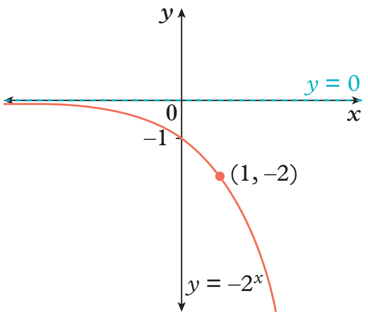

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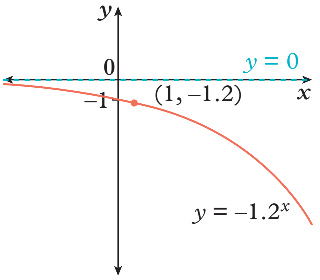
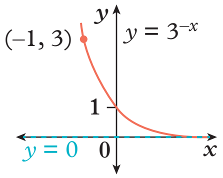


7 **a**



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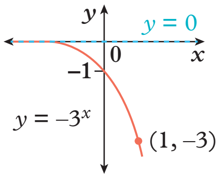

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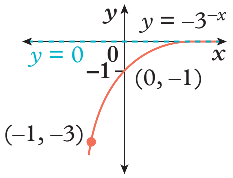

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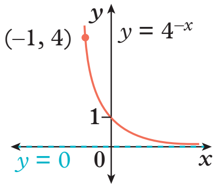

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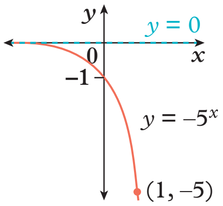

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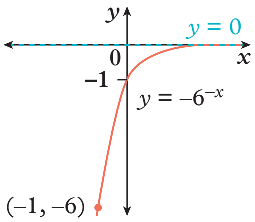

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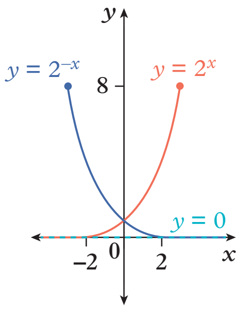


a $y = 2^x$ has positive y values that increase as x increases.
$y = 2^{-x}$ has positive y values that decrease as x increases.

b They have the same asymptote and y-intercept, and the same general shape (reflected in the y-axis).

c 1 for both

d $y = 0$ for both

e reflection in y-axis

f a reflection in the x-axis and then a reflection in the y-axis

9 **a** **i** reflection in the x-axis **ii** $y = -7^x$

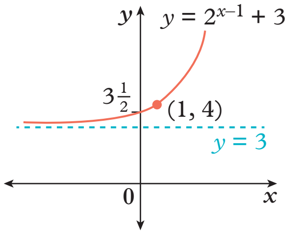
b **i** stretch in the y-direction by a factor of 3

ii $y = 3 \times 2^x$

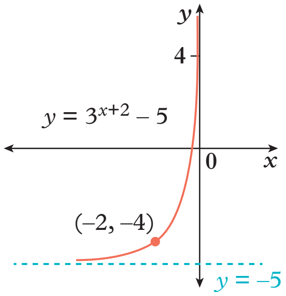
c **i** translation 2 units up **ii** $y = 4^x + 2$

d **i** translation 3 units right **ii** $y = 5^{x-3}$

10 **a**



b



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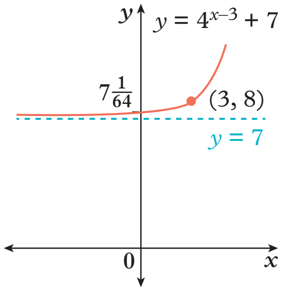

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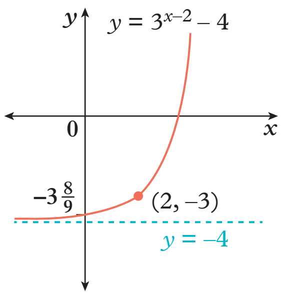

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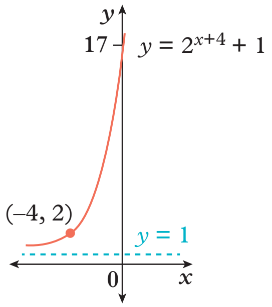

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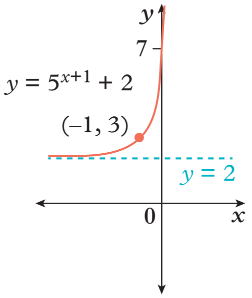
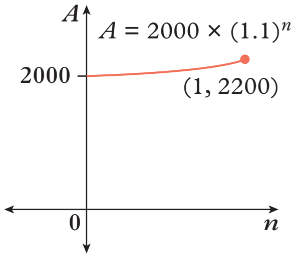


12 **a** $y = -a^x$ **b** $y = a^{-x}$ **c** $y = -a^{-x}$

13 **a**

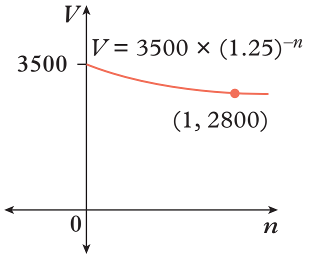


b A increases

c **i** approximately \$2900 read from the graph (the exact value is \$2928.20)

ii 8 years

14 **a** The graph will decrease as time increases.



b \$3500 **c** V decreases **d** 6 years

15 a $3 \times 2^2 = 12$ whereas $6^2 = 36$

b $x = 1$

c 'Equivalent to' means the expressions are exactly the same for all values of x, whereas 'equal to' means solving for a particular instance when the expressions evaluate to the same number.

d Two expressions may be equal for some values, but that does not mean they are necessarily equal for all values.

16 a

x	−2	−1	0	1	2	3	4	5
x^2	4	1	0	1	4	9	16	25
2^x	0.25	0.5	1	2	4	8	16	32

Therefore, $x = 2$ and $x = 4$.

b As at $x = -1$, $x^2 > 2^x$ but at $x = 0$, $x^2 < 2^x$. This means the expressions must be equal at some value in between −1 and 0.

c $x = -0.767$

17 b i $y = 9 \times 3^x$ **ii** $y = 8 \times 2^x$

iii $y = \frac{1}{4} \times 4^x$ **iv** $y = 2 \times 4^x$

c i $y = 3^{x+1}$ **ii** $y = 4^{x+2}$

iii $y = 2^{x-3}$ **iv** $y = 4^{x+1.5}$

18 a i stretch in the y-direction by factor of 2, translation 4 units up

ii The equation for A is $y = 5^x$. The equation for B is $y = 2 \times 5^x + 4$.

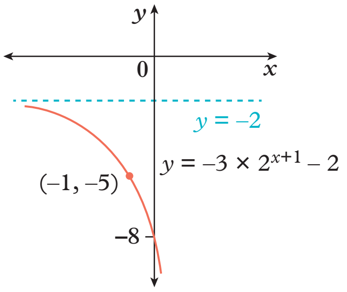
b i translation of 2 units right and 5 units down

ii The equation for A is $y = 4^x$. The equation for B is $y = 4^{x-2} - 5$.

c i stretch in the y-direction by factor of 4, reflection in x-axis and y-axis

ii The equation for A is $y = 3^x$. The equation for B is $y = -4 \times 3^{-x}$.

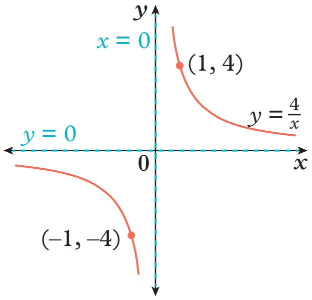
19 stretch in the y-direction by factor of 3, reflection in the x-axis, translation of 1 unit left and 2 units down



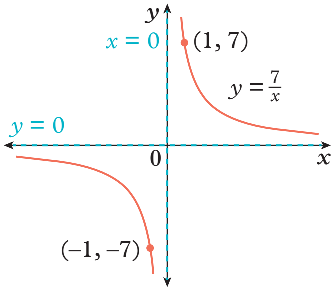
EX
p260

5G Hyperbolas

1 a



b



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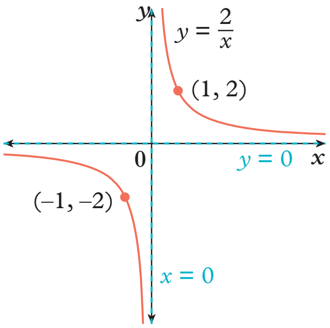

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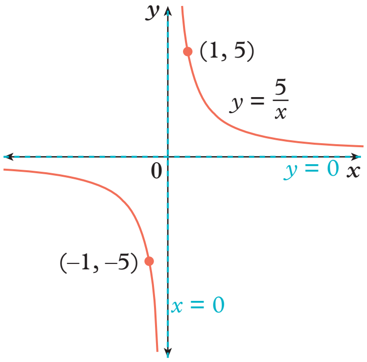

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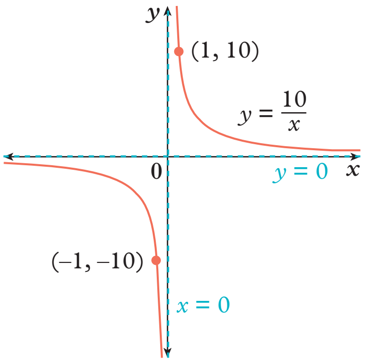
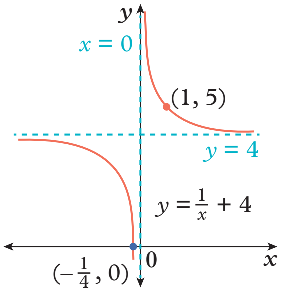


2 a



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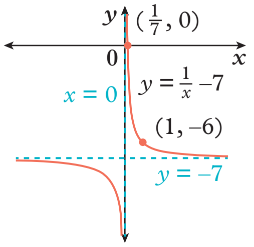

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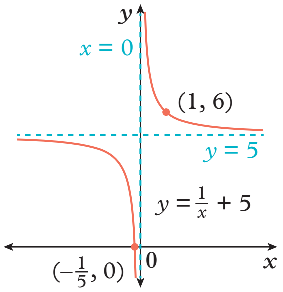

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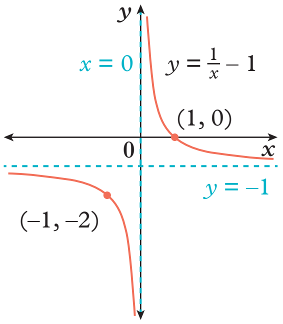
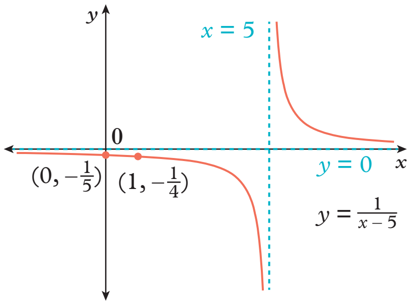


3 a



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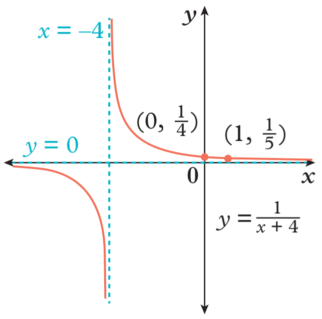

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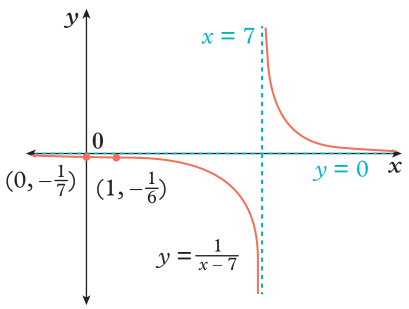

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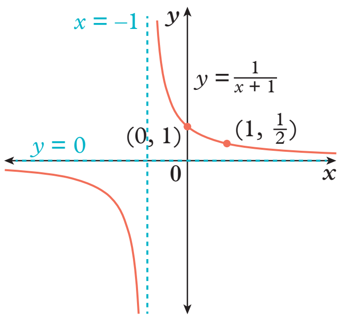
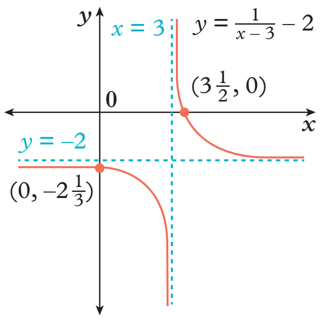


4 a



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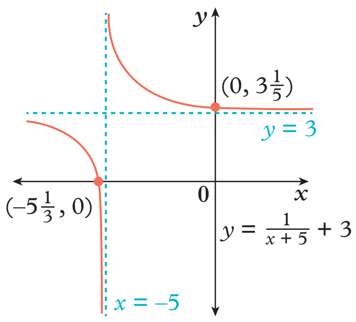

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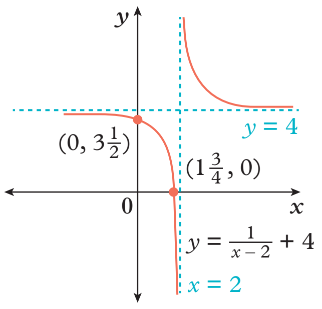

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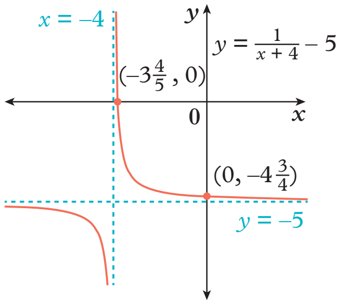

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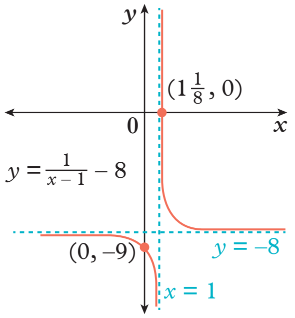
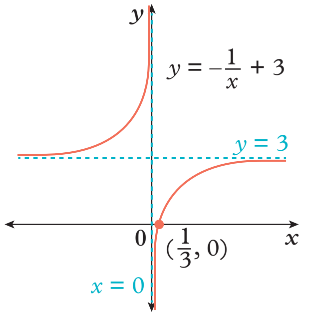


6 **a**



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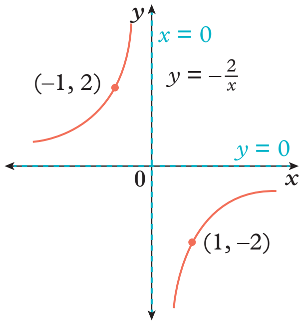

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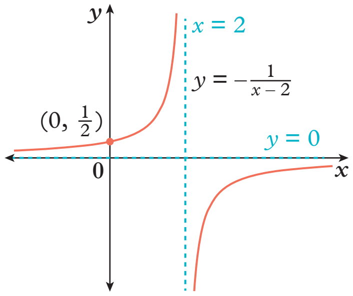

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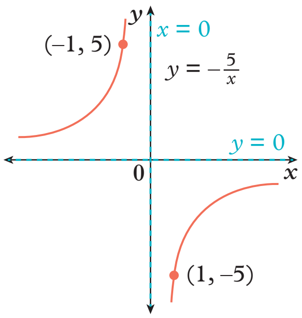

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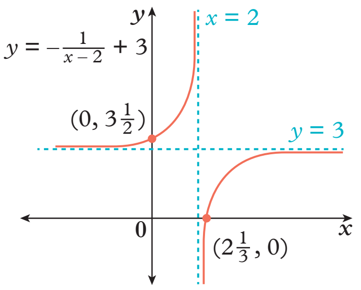
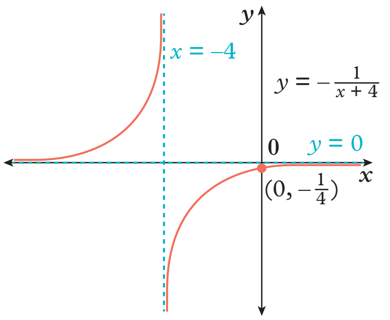


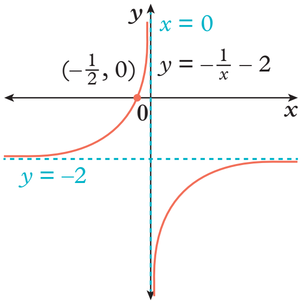
7 **a**



The domain is $x < -4$ and $x > -4$.

The range is $y < 0$ and $y > 0$.

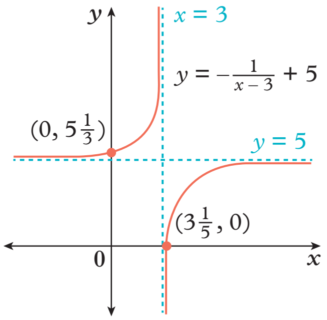
b



The domain is $x < 0$ and $x > 0$.

The range is $y < -2$ and $y > -2$.

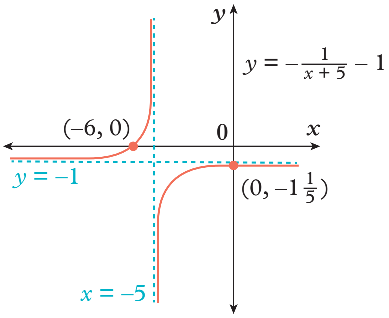
c



The domain is $x < 3$ and $x > 3$.

The range is $y < 5$ and $y > 5$.

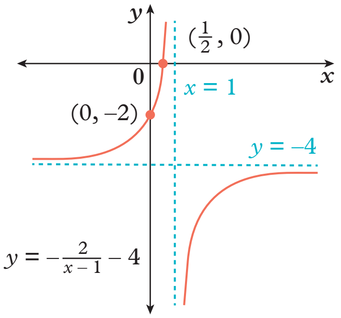
d



The domain is $x < -5$ and $x > -5$.

The range is $y < -1$ and $y > -1$.

e



The domain is $x < 1$ and $x > 1$.

The range is $y < -4$ and $y > -4$.

9 **a** $y = \frac{1}{x-2} + 2$ **b** $y = -\frac{1}{x} - 1$

c $y = \frac{4}{x+4} - 2$ **d** $y = -\frac{3}{x-1} + 3$

10 Reflecting in the x-axis: the curve in the first quadrant is reflected to the fourth quadrant; the curve in the third quadrant is reflected to the second quadrant.

The equation becomes $y = -\left(\frac{1}{x}\right)$ or $y = \frac{1}{x}$.

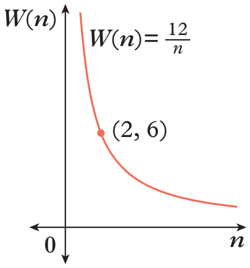
Reflecting in the y-axis: the curve in the first quadrant is reflected to the second quadrant; the curve in the third quadrant is reflected to the fourth quadrant.

The equation becomes $y = \frac{1}{(-x)}$ or $y = -\frac{1}{x}$.

11 **a** zero **b** positive infinity

c negative infinity **d** zero

12 **a**



b **i** 6 min **ii** 4 min **iii** 3 min

iv 2 min **v** 1 min **vi** 30 s

c 12 minutes when there is one bank teller on duty

d No, because $\frac{12}{n} \neq 0$; however, W approaches 0 as n increases.

13 **a** $y = \frac{1}{0} + 3$, the y value is undefined since it is not possible to divide by 0.

b If x equals any other value than 2 then the denominator will not be 0.

c Substituting $y = 3$ into the equation gives $0 = \frac{1}{x-2}$, which is impossible since a fraction cannot equal 0 unless its numerator is 0 (and its denominator is not 0).

d If y equals any other value than 3 then the fraction $\frac{1}{x-2}$ will equal a non-zero number, which can be solved for x.

14 **a** **i** two **ii** (−1, −1) and (1, 1)

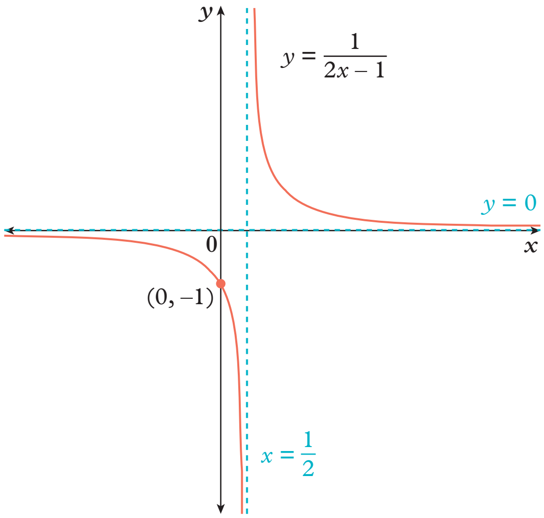
b **i** none

c **i** two **ii** (0, 3) and (−5, −2)

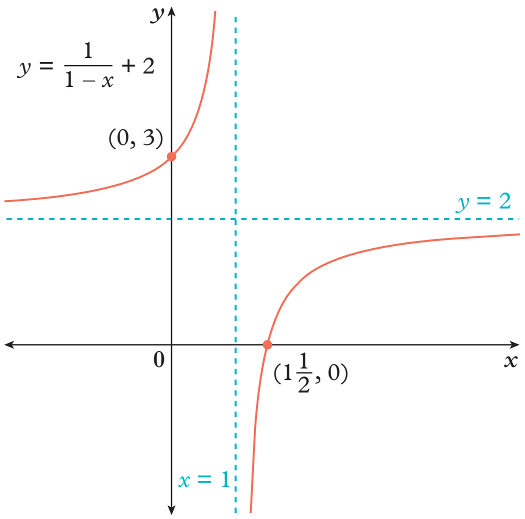
d **i** one **ii** $(2, -\frac{1}{2})$

15 **a** $y = \frac{8}{x-2}$ **b** $y = \frac{3}{x-1} + 2$ **c** $y = \frac{4}{x+3}$

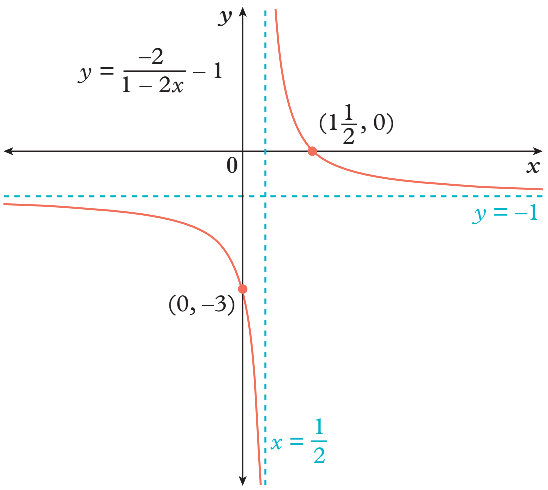
16 **a**



b



c



EX p265

5H Direct and inverse proportion

1 **a** 4 **b** 9 **c** 1.5
d 5 **e** 0.5 **f** 0.4

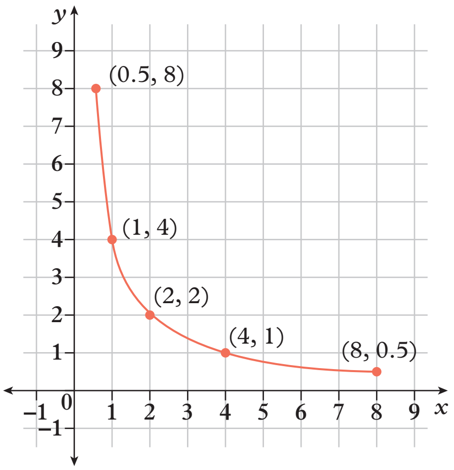
2 **a** 2 **b** 1.2 **c** 0.5 **d** 0.5

3 **a** 6 **b** 12 **c** 4
d 6 **e** 15 **f** 1

4 **a** 2 **b** 5 **c** 0.5 **d** 8

5 **a** directly proportional
b neither
c inversely proportional
d neither
e directly proportional
f inversely proportional

6 **a** inversely proportional
b 4
c 1, 0.5
d



7 **a** The graph does not pass through (0,0).
b The graph is not linear.
c The graph does not pass through (0,0) or y decreases as x increases.
d The graph is not linear.

8 **a** \$23.50 **b** \$517

9 **a** 140 **b** 840 watts **c** 1.75 hours

10 7.5 days

11 **a** 6 **b** 6
c It will always be the same. If y is directly proportional to x then $y = \frac{k}{x}$ for some real number, k. Rearranging to make x the subject gives $x = \frac{k}{y}$, which is the same constant of proportionality.

12 **a** \$171 **b** 171
c The total cost, T, is fixed. Therefore, the cost per person, C, will be $\frac{T}{n}$ where n is the number of people. This gives the equation $C = \frac{T}{n}$, which shows that C is inversely proportional to n and that T is the constant of proportionality.

13 **a** 3.2, 3.1, 3.1, 3.25, 3.24, 3.15, 3.1
b 3.16
c He is trying to approximate π, which is roughly 3.14. Therefore, he is relatively close.

14 **a** 4.8
b **i** 0.192 W/m² **ii** 22 cm
c 75% (i.e. the light intensity is one quarter of its previous value)

15 **a** If $x \propto y$ and $y \propto z$, then $x = k_1 y$ and $y = k_2 z$. Therefore, $x = k_1 k_2 z$, which shows that $x \propto z$.
b If $a \propto \frac{1}{b}$ and $b \propto \frac{1}{c}$, then $a = \frac{k_1}{b}$ and $b = \frac{k_2}{c}$. Therefore, $a = \frac{k_1}{k_2/c} = \frac{k_1 c}{k_2} = \frac{k_1}{k_2}c$, which shows that $a \propto c$.

EX p271

5I Sketching non-linear graphs using transformations

1 **a** **i** circle with centre at (2, 0) and radius of 2 units
ii $x^2 + y^2 = 1$
iii stretch in all directions by factor of 2, then a translation of 2 units left
iv $(x - 2)^2 + y^2 = 4$
b **i** parabola with a minimum turning point at (1, −1)
ii $y = x^2$
iii translation of 1 unit right and 1 unit down
iv $y = (x - 1)^2 - 1$
c **i** exponential relationship with an asymptote at $y = 1$
ii $y = 2^x$
iii translation of 1 unit up
iv $y = 2^x + 1$
d **i** exponential with asymptotes at $x = 0$ and $y = 1$
ii $y = \frac{1}{x}$
iii reflection in the x-axis (or y-axis), then a translation of 1 unit up
iv $y = -\frac{1}{x} + 1$
e **i** hyperbola with asymptotes at $x = -1$ and $y = -3$
ii $y = \frac{1}{x}$
iii stretch in the y-direction by a factor of 6, then a translation of 1 unit left and 3 units down
iv $y = \frac{6}{x + 1} - 3$

f **i** circle with centre at (5, −3) and radius of 5 units

ii $x^2 + y^2 = 1$

iii stretch in all directions by a factor of 5, then a translation of 5 units right and 3 units down

iv $(x - 5)^2 + (y + 3)^2 = 25$

g **i** parabola with a maximum turning point at (1, 8)

ii $y = x^2$

iii reflection in the x-axis and a stretch in the y-direction by a factor of 2, then a translation 1 unit right and 8 units up

iv $y = -2(x - 1)^2 + 8$

h **i** exponential relationship with an asymptote at $y = -8$

ii $y = 2^x$

iii translation of 8 units down

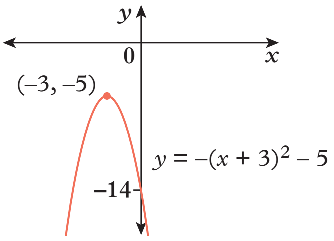
iv $y = 2^x - 8$

2 **a** **i** parabola with a maximum turning point

ii reflection in the x-axis and translation of 3 units left and 5 units down to be performed on the basic graph of $y = x^2$

iii maximum turning point at (–3, –5), no x-intercept, y-intercept of (0, –14)

iv

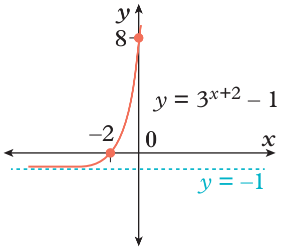


b **i** exponential relationship

ii translation of 2 units left and 1 unit down to be performed on the basic graph of $y = 3^x$

iii asymptote at $y = -1$, x-intercept of (–2, 0), y-intercept of (0, 8)

iv

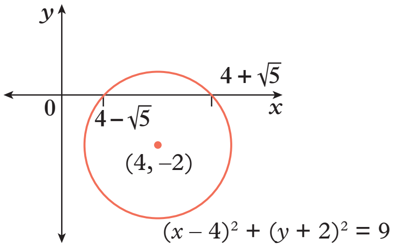


c **i** circle with radius 3 units

ii Translations of 4 units right and 2 units down to be performed on the basic graph of $x^2 + y^2 = 9$

iii centre at (4, –2), radius of 3 units, x-intercepts of $(4 - \sqrt{5}, 0)$ and $(4 + \sqrt{5}, 0)$, no y-intercepts

iv

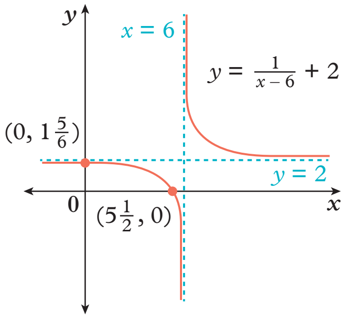


d **i** hyperbola with two asymptotes

ii Translation of 6 units right and 2 units up to be performed on the basic graph of $y = \frac{1}{x}$

iii asymptotes at $y = 2$ and $x = 6$, x-intercept of $\left(5\frac{1}{2}, 0\right)$, y-intercept of $\left(0, 1\frac{5}{6}\right)$

iv

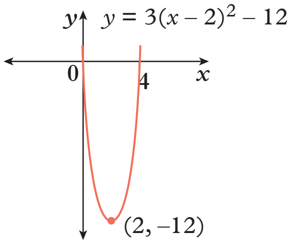


e **i** parabola with a minimum turning point

ii A stretch in the y-direction by a factor of 3 and translation of 2 units right and 12 units down to be performed on the basic graph of $y = x^2$

iii minimum turning point at (2, –12), x-intercepts of (0, 0) and (4, 0), a y-intercept of (0, 0)

iv

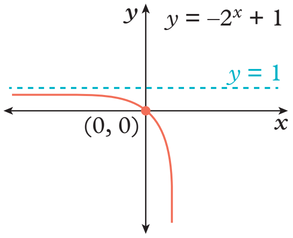


f **i** exponential relationship

ii Reflection in the x-axis and translation of 1 unit up to be performed on the basic graph of $y = 2^x$.

iii asymptote at $y = 1$, an x-intercept of (0, 0), a y-intercept of (0, 0)

iv



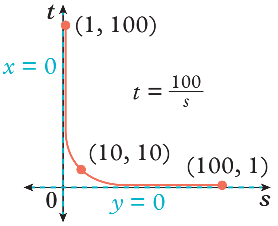
3 **a** $x^2 + y^2 = 16$ **b** $(x + 7)^2 + (y - 5)^2 = 16$

c The radius of both circles is 4 m, so the sum of their radii is 8 m. Using Pythagoras' Theorem, the distance between the centres of both circles is $\sqrt{7^2 + 5^2} \approx 8.6$ m. The distance between the centre of both circles is greater than sum of their radii, so they do not overlap.

4 **a** $t = \frac{100}{s}$

b rectangular hyperbola

c



d as s increases, t decreases

5 The arch is an inverted parabola with its origin located close to where the left-hand edge of the arch meets the ground. The turning point (4.6, 7.1) is at the highest point of the arch. The arch is 7.1 m high and 9.2 m wide at the base.

6 Both graphs are the same, a reflection in the y-axis of the basic graph $y = 2x$.

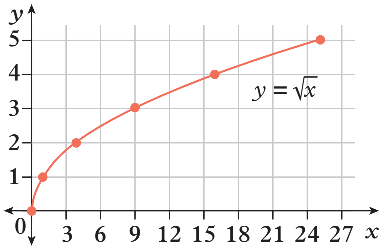
$y = \left(\frac{1}{2}\right)^x = \frac{1^x}{2^x} = \frac{1}{2^x} = 2^{-x}$

The graphs are the same because their equations are equivalent.

7 **a**

x	0	1	4	9	16	25
y	0	1	2	3	4	5

b

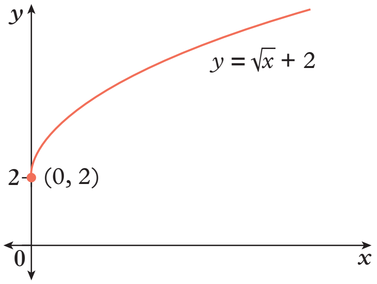


c It is a smooth curve with a minimum value at (0, 0); the curve is half a parabola on its side. No points can be graphed for negative values of x because it is not possible to find the square root of a negative number.

8 **a** (5, 0) **b** translate the graph 5 units right

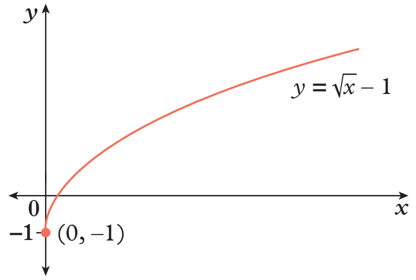
9 **a** **i** translate 2 units up **ii** (0, 2)

iii



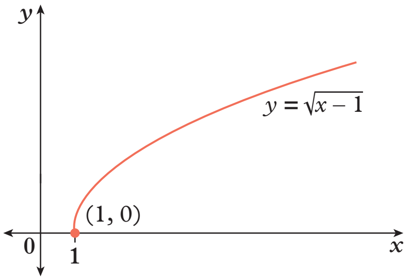
b **i** translate 1 unit down **ii** (0, –1)

iii



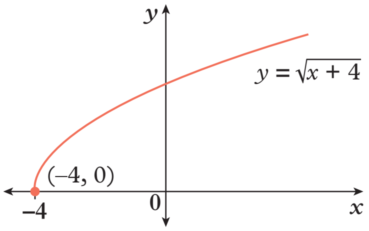
c **i** translate 1 unit right **ii** (1, 0)

iii



d **i** translate 4 units left **ii** (–4, 0)

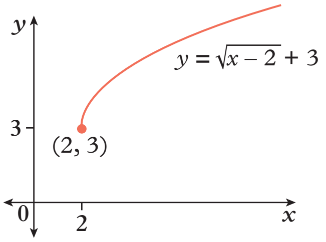
iii



e **i** translate 2 units right and 3 units up

ii (2, 3)

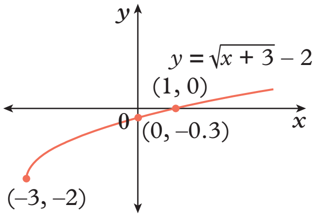
iii



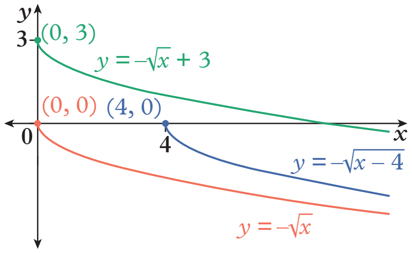
f **i** translate 3 units left and 2 units down

ii (–3, –2)

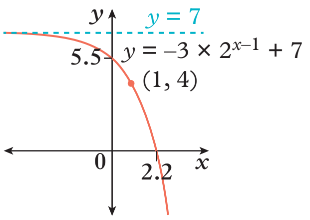
iii



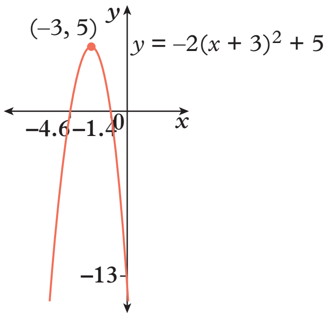
10



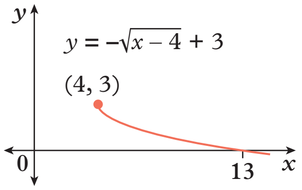
11 **a**



b



c



12 **a** **i** $y = \frac{1}{2}x^2 + 2$ **ii** $y = 2x^2 - 6$

b **i** $y = -x^2 + 3$ **ii** $y = -x^2 - 3$

13 **a**

Transformation	Result
Start	(1,2)
1. Reflected in the x-axis	(1,−2)
2. Translated up by 3 units	(1,1)
3. Translated left by 4 units	(−3,1)
4. Stretched by a factor of 2 in the y-direction	(−3,2)
5. Translated right by 4 units	(1,2)

Final position: (1,2)

b $y = -2x^2 + 6$

CHAPTER 5 review

Multiple-choice

1 A **2** C **3** A **4** C **5** D

6 B **7** A **8** C **9** D **10** E

11 B

Short answer

1 **a** $x = 2$ **b** $x = \frac{3}{4}$ or $x = -\frac{4}{3}$ **c** $x = \pm\sqrt{2}$

d $x = -2$ or $x = 4$ **e** $x = \pm\sqrt{\frac{7}{5}}$

2 **a** $x = -3 \pm \sqrt{5}$ **b** $x = \frac{3 \pm \sqrt{21}}{2}$

3 **a** 1 **b** 0 **c** 0 **d** 2 **e** 2 **f** 0

4 **a** **i** $x = -5$ or $x = 2$ **ii** $x \approx -5.00$ or $x \approx 2.00$

b **i** $x = \frac{3 \pm \sqrt{29}}{10}$ **ii** $x \approx -0.24$ or $x \approx 0.84$

c **i** $x = \frac{5 \pm \sqrt{17}}{2}$ **ii** $x \approx 0.44$ or $x \approx 4.56$

d **i** $x = \frac{-5 \pm \sqrt{17}}{2}$ **ii** $x \approx -4.56$ or $x \approx -0.44$

e **i** $x = \frac{1 \pm \sqrt{17}}{4}$ **ii** $x \approx -0.78$ or $x \approx 1.28$

f **i** $x = \frac{-7 \pm \sqrt{13}}{6}$ **ii** $x \approx -1.77$ or $x \approx -0.57$

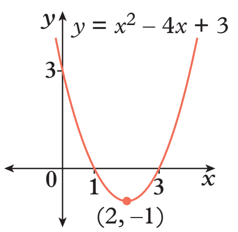
5 **a** **i** (–5, 0) and (5, 0) **ii** (0, 25)

b **i** (–1, 0) and (4, 0) **ii** (0, –4)

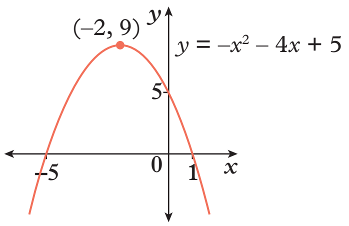
c **i** (2, 0) **ii** (0, 8)

d **i** (–3, 0) and (2, 0) **ii** (0, 6)

6 **a**



b

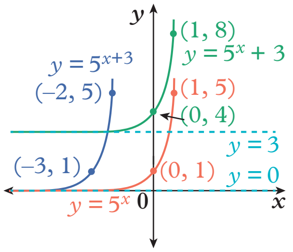


7 **a** **i** 5 units right, 2 units up **ii** (5, 2)

b **i** 6 units left, 4 units down **ii** (–6, –4)

8 $(x - 4)^2 + (y - 2)^2 = 4$

9 **a, c**



b **i** translate 3 units left **ii** translate 3 units up

10 **a** translate 5 units down **b** reflect in the x-axis

c stretch in the y-direction by a factor of 4

d translate 5 units left and 6 units up

11 **a** 12.5 **b** \$50 **c** 16 square metres

12 **a** 20 **b** 20 **c** 8

13 **a** circle **b** hyperbola

c exponential relationship **d** parabola

14 $y = \frac{1}{x+4} - 6$; the domain is $x < -4$ and $x > 4$; the range is $y < -6$ and $y > -6$.

Analysis

1 **a** **i** $a = 1, h = -4, k = -1$ **ii** (–4, –1)

iii (–3, 0) **iv** (0, –3) and (0, 1)

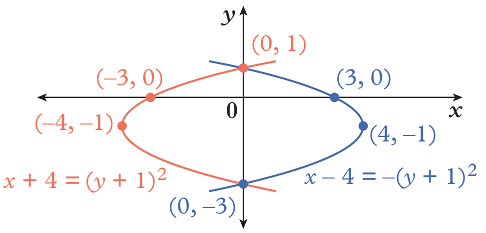
v See graph below.

vi translation of 4 units left and 1 unit down

b **i** $a = -1, h = 4, k = -1$

ii (4, –1) **iii** (3, 0) **iv** (0, –3) and (0, 1)

v



vi reflection in the y-axis and translation of 4 units right and 1 unit down

c A positive value for a results in a parabola with its open end towards the right, while a negative value for a results in the open end being towards the left. The graphs from parts **a** and **b** are a reflection of each other in the y-axis.

2 a Using the first formula:

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times -3}}{2 \times 1}$$
$$= \frac{1 \pm \sqrt{13}}{2}$$
$$\approx -1.3 \text{ or } 2.3$$

Using the second formula:

$$x = \frac{2 \times -3}{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times -3}}$$
$$= \frac{-6}{1 \pm \sqrt{13}}$$
$$\approx -1.3 \text{ or } 2.3$$

The two formulas give the same answer.

b Start with the positive value of the square root:

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

Rationalise the numerator:

$$= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \times \frac{-b - \sqrt{b^2 - 4ac}}{-b - \sqrt{b^2 - 4ac}}$$

Multiply the terms in the numerator, leaving the denominator in factorised form.

$$= \frac{b^2 - (b^2 - 4ac)}{2a(-b - \sqrt{b^2 - 4ac})}$$

Simplify the numerator.

$$= \frac{4ac}{2a(-b - \sqrt{b^2 - 4ac})}$$

Cancel the common factor of $2a$ in the numerator and denominator.

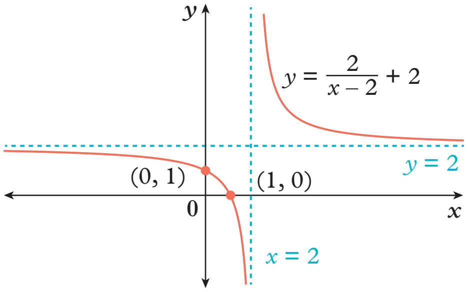
$$= \frac{2c}{-b - \sqrt{b^2 - 4ac}}$$

A similar formula would result by finding the negative value of the square root in the original formula.

So, the two formulas are equivalent.

3 a (1,2)

b



c $(x - 1)^2 + (y - 2)^2 = 2^2$

d 4 **e** $a = \frac{r^2}{2}$ **f** $2\sqrt{2a}$

CHAPTER 6 Polynomials

EX p284

6A Polynomials

1 a **i** 4 **ii** 3 **iii** 5
iv $2x^3$ **v** 2 **vi** 3

b **i** 6 **ii** 5 **iii** –3
iv $4x^5$ **v** 4 **vi** –2

c **i** 4 **ii** 4 **iii** 1
iv $-5x^4$ **v** –5 **vi** 5

d **i** 7 **ii** 7 **iii** 0
iv x^7 **v** 1 **vi** –1

e **i** 5 **ii** 6 **iii** 9
iv $-7x^6$ **v** –7 **vi** 2

f **i** 3 **ii** 10 **iii** 3
iv $-11x^{10}$ **v** –11 **vi** 0

2 a $5x^3 - 3x^2 - 4x + 2$ **b** $x^5 - 2x^4 + 4x^2$
c $2x^4 + 6x^3 + 2x^2 - 3$ **d** $5x^2 - 11x + 9$
e $3x^3 + 5x - 3$ **f** $-4x^4 - 4x^3 + 6x^2 + x$

3 a 2 **b** –8 **c** 16 **d** –68

4 a 127 **b** –5 **c** 15 **d** 97

5 a $-\frac{1}{2}$

b $\frac{137}{27} = 5\frac{2}{27}$

c $7 - \sqrt{2}$

d $\frac{13\sqrt{3} + 18}{9} = \frac{13\sqrt{3}}{9} + 2$

6 a $2x^3 + 2x^2 - 4x - 4$ **b** $2x^3 - 4x^2 + 10x - 8$
c $-2x^3 + 4x^2 - 10x + 8$ **e** $-6x^2 + 14x - 4$
d $6x^3 - 3x^2 + 9x - 18$ **f** $4x^3 + x^2 - x - 10$
g $2x^3 - 13x^2 + 31x - 14$ **h** $6x^3 - 9x^2 + 23x - 22$
i –24 **j** 0
k 56 **l** –38

7 a $x^5 + 2x^4 + 7x^3$
b $4x^4 - 20x^3 + 8x^2$
c $-6x^7 + 24x^5 - 6x^3$
d $x^3 + x^2 - 2x + 12$
e $x^4 + 3x^3 - 8x^2 - 3x + 7$
f $x^7 + 4x^6 - 4x^4 - 10x^3 + 8x^2 + 3x - 2$

8 a $3x^3 + 15x^2 + 18x$ **b** $-7x^4 - 21x^3 + 70x^2$
c $x^3 + 10x^2 + 27x + 18$ **d** $x^3 - 6x^2 + 5x + 12$
e $3x^3 - 23x^2 + 44x - 20$ **f** $8x^3 - 2x^2 - 113x - 28$

9 a $x^6 + 8x^3 + 16$
b $x^4 + 6x^3 + 9x^2$
c $2x^5 - 8x^3 + 8x$
d $x^6 - 10x^5 + 25x^4$
e $x^3 + 6x^2 + 12x + 8$
f $8x^3 - 36x^2 + 54x - 27$
g $x^4 + 12x^3 + 54x^2 + 108x + 81$
h $x^4 - 4x^3 + 6x^2 - 4x + 1$

10 a polynomial **b** not a polynomial
c not a polynomial **d** not a polynomial
e polynomial **f** not a polynomial
g polynomial **h** not a polynomial
i not a polynomial

11 6

12 1

13 a **i** $n + 1$ **ii** 1 **iii** n **iv** $2n$
b **i** n **ii** n **iii** $n + m$

14 $k = -4$

15 $k = 11$

16 $a = -1, b = 4$

17 **a** $a^3 - 3a^2 + 2a + 1$ **b** $-27a^3 - 27a^2 - 6a + 1$

c $a^6 - 3a^4 + 2a^2 + 1$ **d** $-a^6 - 3a^4 - 2a^2 + 1$

e $a^3 + 3a^2 + 2a + 1$ **f** $8a^3 - 24a^2 + 22a - 5$

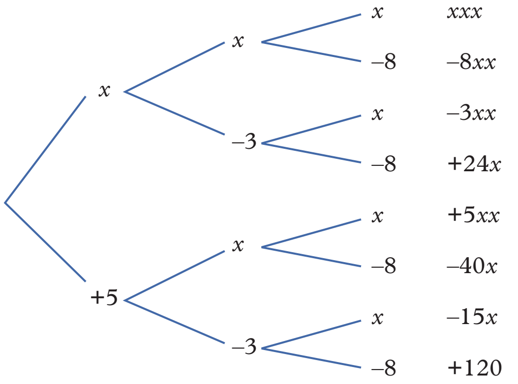
18
$$\begin{aligned}(2x-1)^4 &= (2x-1)(2x-1)(2x-1)(2x-1)\\ &= (2x-1)(2x-1)(4x^2-4x+1)\\ &= (2x-1)(8x^3-8x^2+2x-4x^2+4x-1)\\ &= (2x-1)(8x^3-12x^2+6x-1)\\ &= 16x^4-24x^3+12x^2-2x-8x^3+12x^2-6x+1\\ &= 16x^4-32x^3+24x^2-8x+1\end{aligned}$$

19 **a** $x^6 - 6x^4 + 12x^2 - 8$

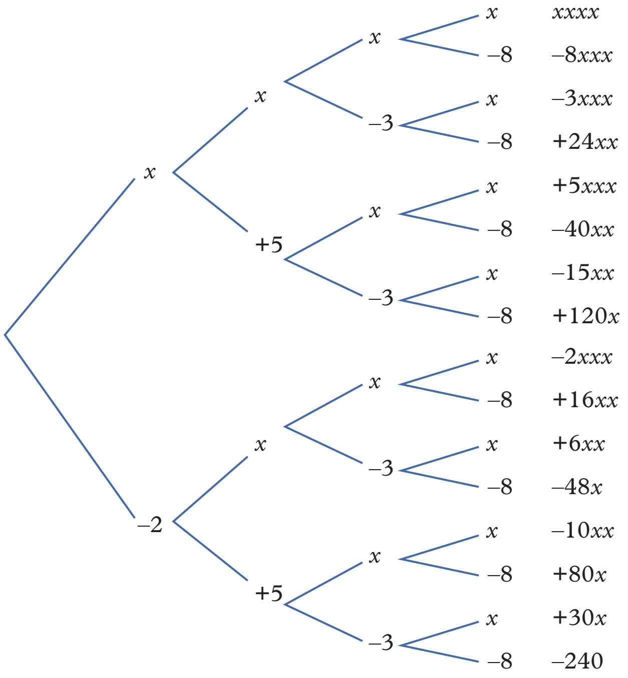
b $x^6 + 2x^4 - 2x^3 + x^2 - 2x + 1$

c $16x^{12} + 32x^9 + 24x^6 + 8x^3 + 1$

20 **a** $x^3 - 6x^2 - 31x + 120$



b $x^4 - 8x^3 - 19x^2 + 182x - 240$



EX p288

6B Dividing polynomials

1 **a** quotient $x + 3$, remainder 5

b quotient $x - 8$, remainder -6

c quotient $5x + 6$, remainder 10

d quotient $9x - 1$, remainder 18

e quotient $7x + 2$, remainder -20

f quotient $6x - 7$, remainder 100

2 **a** quotient $x^2 + x - 2$, remainder 1

b quotient $x^2 + 2x + 5$, remainder -6

c quotient $x^2 + 8x + 12$, remainder 9

d quotient $x^2 - 7x + 8$, remainder -7

e quotient $2x^2 - x - 3$, remainder 2

f quotient $3x^2 + x - 6$, remainder -4

g quotient $5x^2 + 3x + 3$, remainder -6

h quotient $4x^2 + 5x - 13$, remainder 12

i quotient $x^3 + x^2 - 5x + 3$, remainder 5

j quotient $x^3 - 3x^2 - 4x - 9$, remainder -21

3 **a** quotient $x^2 - 3x + 10$, remainder -9

b quotient $2x^2 + x + 2$, remainder -2

c quotient $4x^3 + 4x^2 + x + 1$, remainder -4

d quotient $3x^3 - 6x^2 + 12x - 24$, remainder 7

4 **a** **i** $x + 3 + \frac{5}{x+7}$

ii $(x+7)(x+3) + 5$

b **i** $x - 8 - \frac{6}{x-4}$

ii $(x-4)(x-8) - 6$

c **i** $5x + 6 + \frac{10}{x-2}$

ii $(x-2)(5x+6) + 10$

d **i** $9x - 1 + \frac{18}{x-4}$

ii $(x-4)(9x-1) + 18$

e **i** $7x + 2 - \frac{20}{2x+7}$

ii $(2x+7)(7x+2) - 20$

f **i** $6x - 7 + \frac{100}{10x+3}$

ii $(10x+3)(6x-7) + 100$

5 **a** **i** $x^2 + x - 2 + \frac{1}{x-3}$

ii $(x-3)(x^2+x-2) + 1$

b **i** $x^2 + 2x + 5 - \frac{6}{x+1}$

ii $(x+1)(x^2+2x+5) - 6$

c **i** $x^2 + 8x + 12 + \frac{9}{x-2}$

ii $(x-2)(x^2+8x+12) + 9$

d **i** $x^2 - 7x + 8 - \frac{7}{x+4}$

ii $(x+4)(x^2-7x+8) - 7$

e **i** $2x^2 - x - 3 + \frac{2}{x+2}$

ii $(x+2)(2x^2-x-3) + 2$

f **i** $3x^2 + x - 6 - \frac{4}{x-1}$

ii $(x-1)(3x^2+x-6) - 4$

6 **a** $3x^2 + 7 + \frac{6}{x+2}$

b $2x^2 - 5x + \frac{9}{x-3}$

c $x^3 - 5x^2 - 8 + \frac{50}{x+6}$

d $x^3 + 12 + \frac{50}{x-7}$

7 **a** $(x-3)(x+5)+6$

b $(x-3)(x^2-4x+9)+6$

c $(x-3)(x^2-6x+2)-11$

d $(x-3)(x^3+8x^2+9x+4)+3$

8 **a** $4 - \frac{5}{x+2}$ **b** $9 + \frac{23}{x-3}$

c $1 - \frac{6}{x-2}$ **d** $5 + \frac{29}{2x-5}$

9 **a** $n-1$

b $n-m$

10 **a** $\frac{(x+4)(x^2+5x+3)+5}{x+4}$

b $\frac{(x+4)(x^2+5x+3)}{x+4} + \frac{5}{x+4} = x^2+5x+3+\frac{5}{x+4}$

The fraction $\frac{(x+4)(x^2+5x+3)}{x+4}$ can be simplified by dividing the numerator and denominator by their common factor $(x+4)$.

11 **a** $k = 24$

b linear factor $(x-2)$; quadratic factor (x^2-x-12)

$P(x) = (x-2)(x^2-x-12)$

$= (x-2)(x-4)(x+3)$

12 **a** $x^3+7x^2-26x-72$

b $(x+2)(x^2+5x-36) = (x+2)(x-4)(x+9)$

13 **a** quotient $x-3$, remainder $x-1$

b quotient x^2+x-4, remainder $6x-7$

c quotient $2x^2+2$, remainder -5

14 **a** $f(a) = 0$

b $f(a) = \frac{P(a)}{D(a)} = \frac{b}{0}$ which is undefined as the denominator is zero.

c $f(x) = Q(x)$

EX p293

6C Remainder and factor theorems

1 **a** 14

b 14; same answer

c 24

d 24; same answer

e $P(4) = 64 + 16 - 40 + 8 = 48$; the remainder is 48.

f

$$\begin{array}{r} x^2+5x+10 \\ x-4\overline{)x^3+x^2-10x+8} \\ \underline{x^3-4x^2} \\ 5x^2-10x+8 \\ \underline{5x^2-20x} \\ 10x+8 \\ \underline{10x-40} \\ 48 \end{array}$$

Long division produces a quotient of $x^2+5x+10$ and remainder of 48.

2 **a** 0 **b** 14 **c** –16 **d** 0

3 **a** 12 **b** 15 **c** 15 **d** 0

4 **a** 12 **b** –48 **c** 0 **d** 630

5 **a** no **b** yes **c** no **d** no

e no **f** yes **g** no **h** yes

6 **a** no **b** no **c** yes **d** no

e no **f** yes **g** yes **h** no

7 **a** $(x-1)$ or $(x+3)$ or $(x+6)$

b $(x-2)$ or $(x-3)$ or $(x+4)$

c $(x-3)$ or $(x+3)$ or $(x-4)$

d $(x+2)$ or $(x+3)$ or $(x-5)$

e $(x-1)$ or $(x-3)$

f $(x+2)$ or $(x-4)$

g $(x-2)$

h $(x+3)$

8 **a** $(x-1)(x+2)(x+5)$

b $(x-2)(x+2)(x+3)$

c $(x-1)(x-2)(x+7)$

d $(x-2)(x+2)(x+5)$

e $(x+3)(x-4)(x-4)$ or $(x+3)(x-4)^2$

f $(x-2)(x-3)(x+3)$

g $(x-2)(2x+1)(x+2)$

h $(x+2)(4x-5)(x-3)$

i $(x-4)(3x+2)(2x-1)$

j $(x-1)(4x-3)(3x+1)$

9 **a** $(x-1)(x+3)(x+6)$

b $(x-2)(x-3)(x+4)$

c $(x-3)(x+3)(x-4)$

d $(x+2)(x+3)(x-5)$

e $(x-1)(3x+2)(x-3)$

f $(x+2)(2x-1)(x-4)$

g $(x-2)(4x+1)(2x-3)$

h $(x+3)(2x-1)(2x-3)$

10 **a** $k = 1, 3$ or 5

b $P(x) = (x+1)(x+3)(x+5)$

11 **a** 2 **b** 3 **c** 4

d 7 **e** 12 **f** n

12 **a** 3

b –24; the constant term of $P(x)$ will be the product of the constant terms of the three linear factors, i.e. $2 \times -3 \times 4 = -24$

c –2, 3, –4; they are factors of –24

13 **a** $P(-1) = 0$, so $(x+1)$ is a factor

b $P(2) = 0$ so $(x-2)$ is a factor

c $P(-3) = 0$ so $(x+3)$ is a factor

d $P(x) = (x+1)(x-2)(x+3)$

e $(x+1)(x-2)(x+3)$

$= (x+1)(x^2+x-6)$

$= x^3+x^2-6x+x^2+x-6$

$= x^3+2x^2-5x-6$

14 **a** $(x + 1)$, $(x + 4)$ and $(x - 2)$

b $P(x) = (x + 1)(x + 4)(x - 2)$

c $(x + 1)(x + 4)(x - 2)$
$= (x + 1)(x^2 + 2x - 8)$
$= x^3 + 2x^2 - 8x + x^2 + 2x - 8$
$= x^3 + 3x^2 - 6x - 8$

15 **a** $(x - 1)$, $(x + 2)$ and $(x - 5)$

b $P(x) = (x - 1)(x + 2)(x - 5)$

16 $(x + 1)(x - 3)(x + 4)$

17 **a** $x = a$

b $P(a) = r$

c The product $(a - a)Q(a)$ in the expression for $P(a)$ is equal to zero, so the value of $P(a)$ is equal to the remaining term, r, which is the remainder of the division $P(x) \div (x - a) = \dfrac{(x - a)Q(x) + r}{x - a}$
$= Q(x) + \dfrac{r}{x - a}$

d $x = \dfrac{a}{b}$

e $F\left(\dfrac{a}{b}\right) = h$

f The product $\left(b \times \dfrac{a}{b} - a\right)G\left(\dfrac{a}{b}\right)$ in the expression for $F\left(\dfrac{a}{b}\right)$ is equal to zero, so the value of $F\left(\dfrac{a}{b}\right)$ is equal to the remaining term, h, which is the remainder of the division $F(x) \div (bx - a) = \dfrac{(bx - a)G(x) + h}{bx - a}$
$= G(x) + \dfrac{h}{bx - a}$

18 **a** $a = -15$

b $b = 5$

c $c = 31$

d $m = 6$ and $n = -3$

19 **a** You need to substitute x values that are fractions $\left(\dfrac{1}{3}\text{ and } -\dfrac{3}{2}\right)$; the polynomial factorises to $(3x - 1)(2x + 3)(x + 1)$.

b There is only one linear factor; the polynomial factorises to $(x + 2)(x^2 + 3x + 4)$.

20 Sample answer: $x^3 + x^2 + 2x - 11$

21 **a** $(x - 1)(x + 1)(x + 2)(x - 3)$

b $(x - 1)(x + 2)(x + 3)(x + 4)$

c $(x + 1)(x + 2)(2x - 1)(x + 4)$

22 **a**

$$
\begin{array}{lrl}
 & & x^2+9 \\
 & x^2+4\,\big) & x^4+13x^2+36 \\
x^2(x^2+4) & & -(x^4+4x^2) \\
 & & +9x^2+36 \\
9(x^2+4) & & -(9x^2+36) \\
 & & 0
\end{array}
$$

b $(x^2 + 4)(x^2 + 9)$

6 Checkpoint

1 **a** polynomial
i 7 **ii** $4x^7$ **iii** 4

b not a polynomial; $-9x^{-100}$

c polynomial
i 81 **ii** $\dfrac{5y^{81}}{9}$ **iii** $\dfrac{5}{9}$

d not a polynomial; $\sqrt{y^3}$, $y^{\frac{2}{3}}$

2 **a** $5x^3 - x^2 - 6x + 6$

b $5x^3 - 5x^2 + 10x - 8$

c $30x^3 - 18x^2 + 12x - 6$

d $10x^5 - 46x^4 + 63x^3 - 39x^2 + 22x - 7$

3 **a** quotient: $x + 6$, remainder: -5

b quotient: $x^2 - 5x + 12$, remainder: 9

c quotient: $2x - 6$, remainder: 10

d quotient: $2x^2 + 5x - 2$, remainder: -4

4 **a** $x + 12 - \dfrac{57}{x + 4}$

b $5x^2 + 21x + 128 + \dfrac{769}{x - 6}$

c $5x - 6 + \dfrac{-20x + 31}{x^2 + 4}$

d $2 + \dfrac{16x + 16}{3x^2 - 4x + 1}$

e $3 + \dfrac{28}{6x - 7}$

5 **a** $(x + 4)(x + 12) - 57$

b $(x - 6)(5x^2 + 21x + 128) + 769$

c $(x^2 + 4)(5x - 6) - 20x + 31$

d $2(3x^2 - 4x + 1) + 16x + 16$

e $3(6x - 7) + 28$

6 **a** $x^3 + 7x^2 + 8x + 18 + \dfrac{45}{x - 2}$

b $6x^3 - 3x^2 + 2x - 1 + \dfrac{15}{2x + 1}$

7 **a** 20 **b** 0 **c** 84 **d** 0

8 **a** is not a factor **b** is a factor
c is not a factor **d** is a factor

9 **a** $(x - 1)(x + 4)(x - 7)$

b $(x + 1)(x + 2)(x - 3)$

10 **a** $5(x - 4)(x - 5)(x + 6)$

b $(x - 2)(x + 3)(x + 5)(x - 7)$

EX p298

6D Solving polynomial equations

1 **a** $x = -5, -2$ or 4 **b** $x = -4, -1$ or 3
c $x = -3, 2$ or 6 **d** $x = -2, 0$ or 9
e $x = -1, \dfrac{1}{2}$ or 1 **f** $x = -5, -3\dfrac{1}{2}$ or $\dfrac{2}{3}$

2 **a** $x = -7, -3, 1$ or 4 **b** $x = -6, 2, 3$ or 5
c $x = -4, -2$ or -1 **d** $x = -5, 0$ or 6
e $x = -4, -\dfrac{1}{3}, 0$ or 4 **f** $x = -2\dfrac{1}{2}$ or $\dfrac{3}{4}$

3 **a** $(x + 1)(x + 2)(x + 6) = 0$; $x = -6, -2$ or -1

b $(x - 3)(x - 5)(x + 4) = 0$; $x = -4, 3$ or 5

c $(x - 2)(x - 4)(x - 2) = 0$ or $(x - 2)^2(x - 4) = 0$; $x = 2$ or 4

d $(x + 4)(x + 7)(x - 1) = 0; x = -7, -4$ or 1

e $(x + 3)(x + 4)(x - 4) = 0; x = -4, -3$ or 4

f $x(x - 1)(x + 1)(x - 1) = 0$ or $x(x - 1)^2(x + 1) = 0;$
$x = -1$, 0 or 1

g $(x - 5)(x - 5)(x - 5) = 0$ or $(x - 5)^3 = 0; x = 5$

h $(x + 2)(x + 3)(x + 3) = 0$ or $(x + 2)(x + 3)^2 = 0;$
$x = -3$ or -2

i $(x - 4)(x + 2)(x - 3) = 0; x = -2$, 3 or 4

j $(x+1)(x+1)(x+1) = 0$ or $(x+1)^3$

k $15x(x + 3)(x + 6) = 0; x = -6, -3$ or 0

l $2(x - 2)(x - 1)(x + 3) = 0; x = -3$, 1 or 2

4 **a** $x = -4, -3$ or 1 **b** $x = -2$, 2 or 5

c $x = -6, -1$ or 6 **d** $x = -7, -3$ or 0

e $x = -2, -1$ or 3 **f** $x = -2$

g $x = -3$ or 3 **h** $x = -4$ or -1

i $x = 4$

5 **a** $x = -2, -1$ or 4 **b** $x = -1$, 1 or 5

c $x = -2$ or 2 **d** $x = -3, -2$ or -1

6 **a** $x = -3, -2$ or 3 **b** $x = -4$, 3 or 5

c $x = -1$, 1 or 4 **d** $x = -2$ or 3

7 **a** $x = 1, -1 - \sqrt{2}$ or $-1 + \sqrt{2}$

b $x = -2, -2 - \sqrt{3}$ or $-2 + \sqrt{3}$

c $x = -3, 1 - \sqrt{5}$ or $1 + \sqrt{5}$

d $x = 4, -4 - \sqrt{11}$ or $-4 + \sqrt{11}$

8 **a** $(x - 1)$ or $(x + 1)$ or $(x + 3)$ or $(x - 2)$.

b Sample answer: $P(x) = (x - 1)(x^3 + 2x^2 - 5x - 6)$

c Sample answer: The linear factor is $(x + 1)$, and the quadratic factor is $(x^2 + x - 6)$.

d Sample answer: $(x + 3)(x - 2)$

e $P(x) = (x - 1)(x + 1)(x + 3)(x - 2); x = -3, -1$, 1 or 2

9 **a** $x = -1$, 1, 2 or 3 **b** $x = -2, -1$, 3 or 4

c $x = -4, -3, -2$ or 2 **d** $x = -3, -2$, 1 or 5

e $x = -2$, 1, 2 or 4 **f** $x = -3, -2$, 2 or 3

10 **a** 1 **b** 2 **c** 3

d 4 **e** 6 **f** n

11 **a** $P(x) = (x - 1)(x + 2)(x - 3)$

b $x = -2$, 1 or 3

12 $x^3 + x^2 - 2x - 8 = 0$ has factorised form $(x - 2)(x^2 + 3x + 4) = 0$, because the quadratic factor $x^2 + 3x + 4$ cannot be factorised.
So, $(x - 2)(x^2 + 3x + 4) = 0$ has one solution $(x = 2)$ because there is only one linear factor.

13 $x^3 - 5x^2 + 3x + 9 = 0$ has factorised form $(x + 1)(x - 3)(x - 3) = 0$, which has two solutions $(x = -1, 3)$ because two of the linear factors are the same.

14 **a** $x(2x)(x + 2) = 2x^3 + 4x^2$

b $2x^3 + 4x^2 = 192; x^3 + 2x^2 - 96 = 0;$
$(x - 4)(x^2 + 6x + 24) = 0; x = 4;$
height 4 cm, length 8 cm, width 6 cm

15 $\frac{1}{2}x(5x)(x - 1) = 450; x = 6;$
base 6 cm, height 5 cm, length 30 cm

16 **a** $(x - 1)(x + 1)(x - 2)(x + 3) = 0$

b $(x - 1)^2(x + 1)(x - 2)(x + 3) = 0$ or
$(x - 1)(x + 1)^2(x - 2)(x + 3) = 0$ or
$(x - 1)(x + 1)(x - 2)^2(x + 3) = 0$ or
$(x - 1)(x + 1)(x - 2)(x + 3)^2 = 0$

c $2(x - 1)^2(x + 1)(x - 2)(x + 3) = 0$ or
$2(x - 1)(x + 1)^2(x - 2)(x + 3) = 0$ or
$2(x - 1)(x + 1)(x - 2)^2(x + 3) = 0$ or
$2(x - 1)(x + 1)(x - 2)(x + 3)^2 = 0$

d $5(x - 1)^3(x + 1)(x - 2)(x + 3) = 0$ or
$5(x - 1)(x + 1)^3(x - 2)(x + 3) = 0$ or
$5(x - 1)(x + 1)(x - 2)^3(x + 3) = 0$ or
$5(x - 1)(x + 1)(x - 2)(x + 3)^3 = 0$ or
$5(x - 1)^2(x + 1)^2(x - 2)(x + 3) = 0$ or
$5(x - 1)^2(x + 1)(x - 2)^2(x + 3) = 0$ or
$5(x - 1)^2(x + 1)(x - 2)(x + 3)^2 = 0$ or
$5(x - 1)(x + 1)^2(x - 2)^2(x + 3) = 0$ or
$5(x - 1)(x + 1)^2(x - 2)(x + 3)^2 = 0$ or
$5(x - 1)(x + 1)(x - 2)^2(x + 3)^2 = 0$

e $a(x - 1)^p(x + 1)^q(x - 2)^r(x + 3)^{n-p-q-r} = 0$, where p, q, r are non-zero, positive integers such that $p + q + r \leq n$.

17 $P(x) = (x + 1)(x - 2)(x - 4)(x + 3); x = -3, -1$, 2 or 4

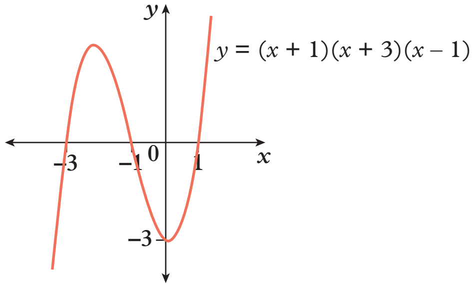
18 **a** $x = -2, -1$, 1 or 5 **b** $x = -3, -2$, 1, 2 or 4

19 $x = \sqrt{2}, -\sqrt{2}$, or -2

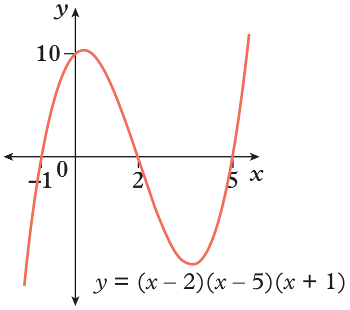
EX
p304

6E Sketching graphs of polynomials using intercepts

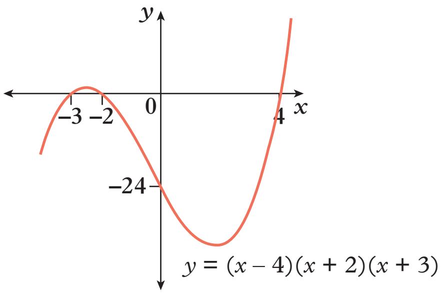
1 **a**



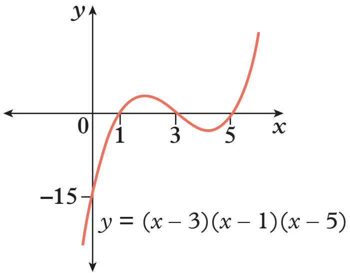
b



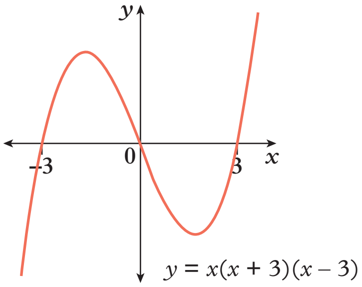
c



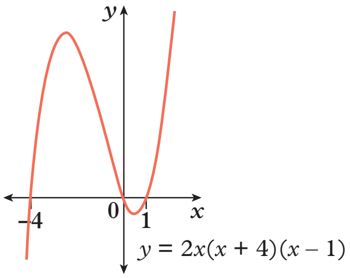
d



e



f



2 **i** **a** (–1, 0), (2, 0) and (5, 0) **b** (0, 10)

c $y = (x + 1)(x - 2)(x - 5)$

ii **a** (–6, 0) (–2, 0) and (2, 0) **b** –24

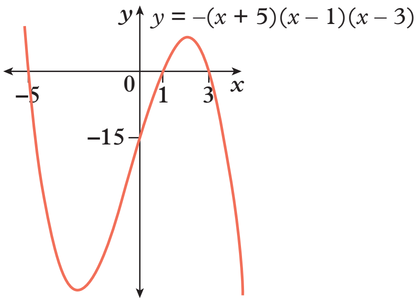
c $y = (x + 6)(x + 2)(x - 2)$

3 **a** **i** (–7, 0), (2, 0) and (3, 0)

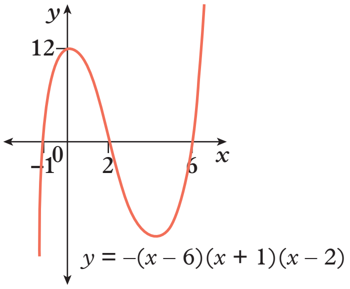
ii (–7, 0), (2, 0) and (3, 0)

b One graph is the reflection of the other graph in the x-axis.

4 **a**



b



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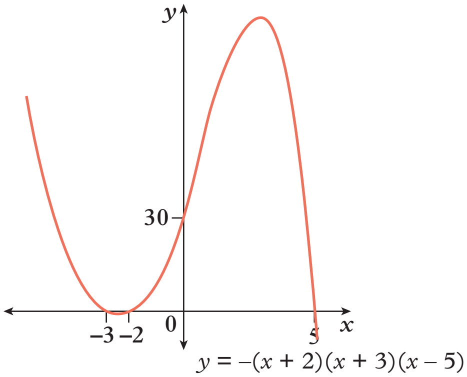

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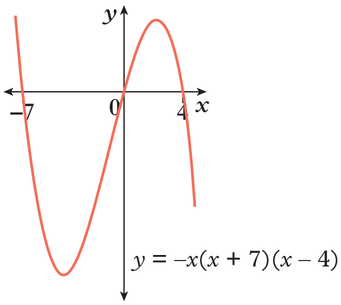
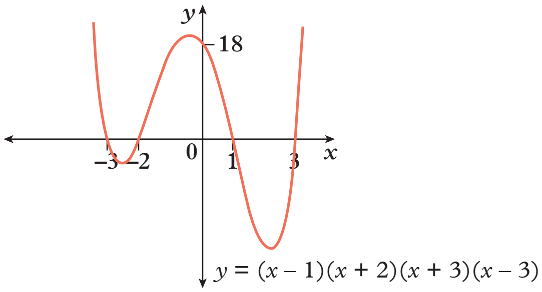


5 **a**



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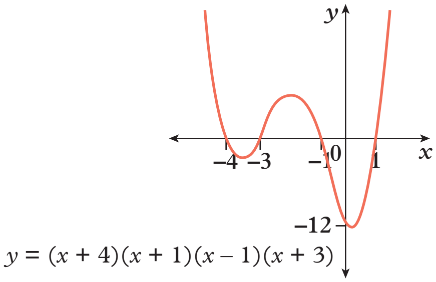

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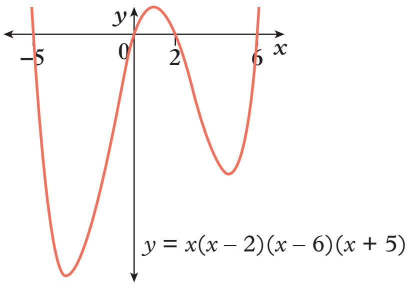

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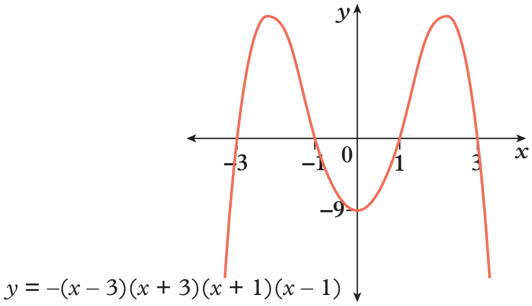

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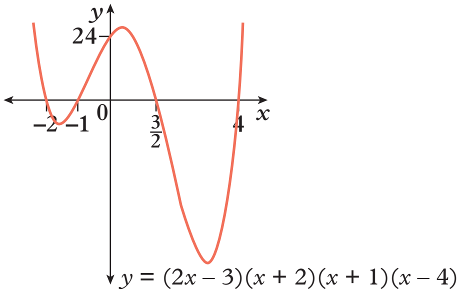

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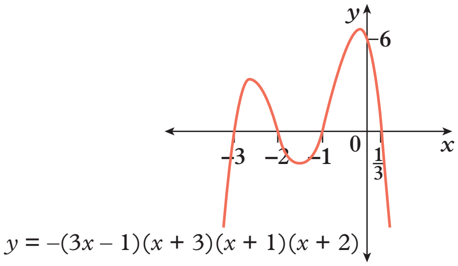
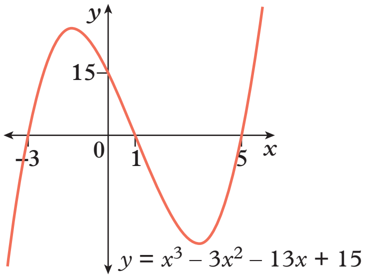


6 **a**



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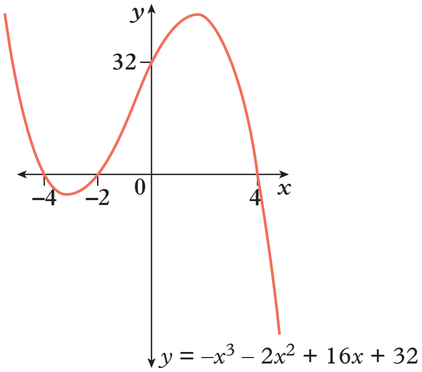

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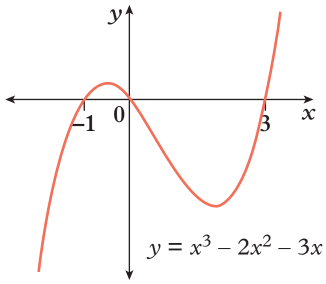

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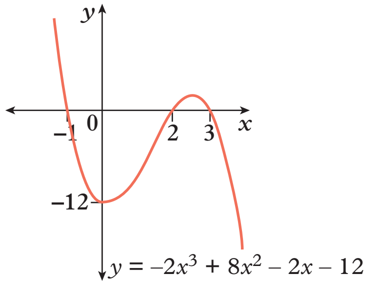

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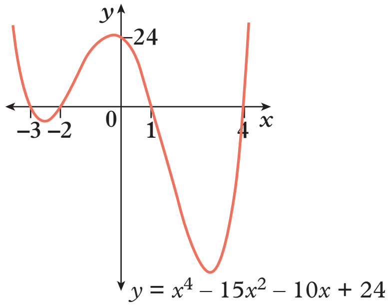

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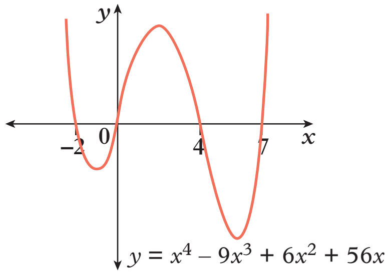

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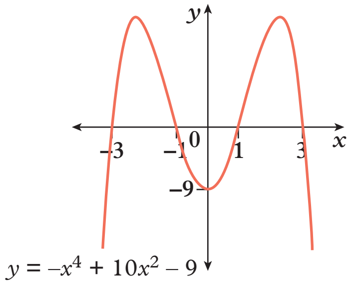

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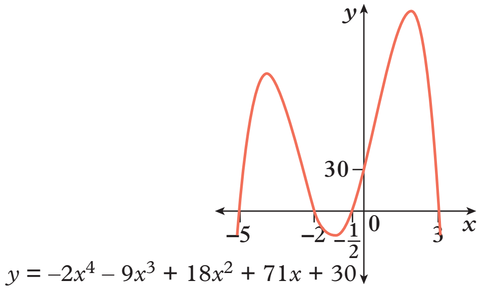
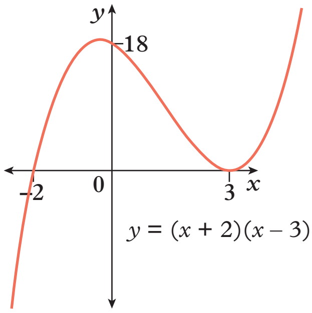


8 **a** cubic

b The x-intercepts are (–2, 0) and (3, 0); the y-intercept is (0, 18).

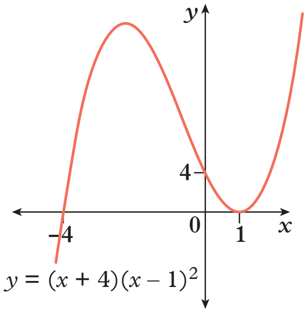
c bottom left

d



e The x-intercept at (3, 0) is also a turning point.

9 **a**



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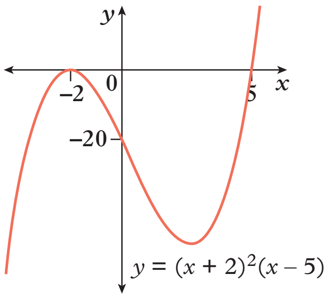

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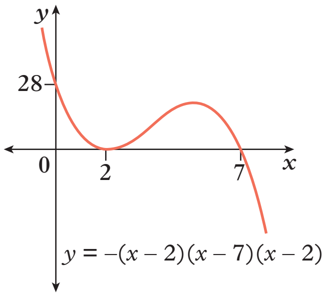

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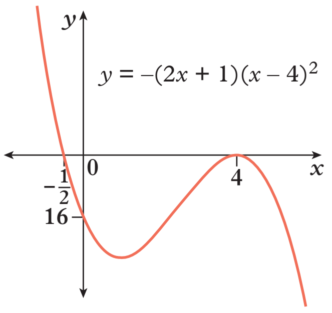
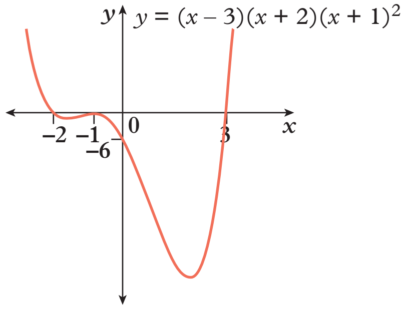
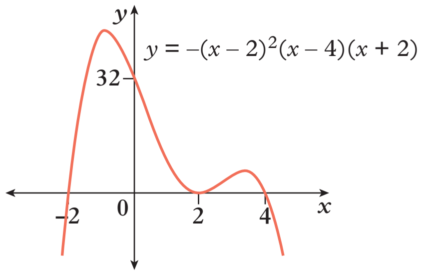


10 **a**



b



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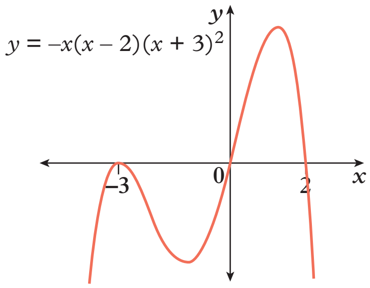

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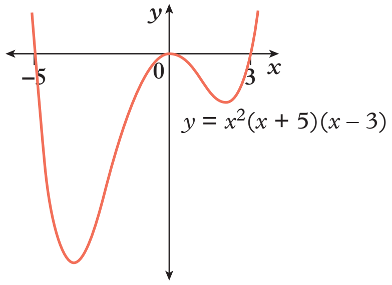

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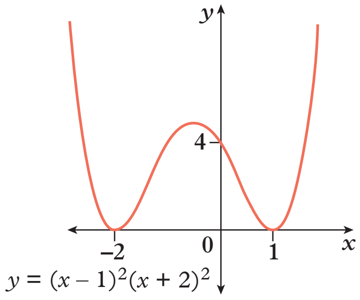

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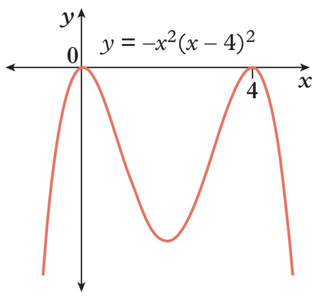
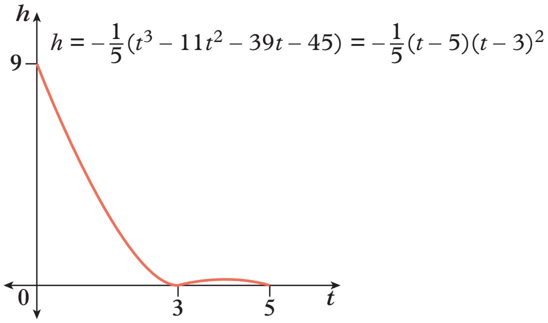


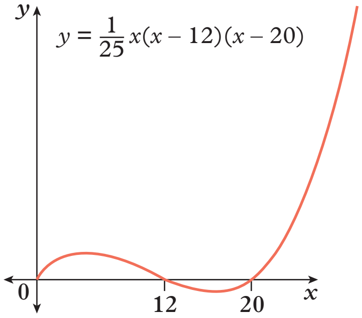
11 The x-intercept at $(a, 0)$ is also a point of inflection.

12 a



b 9 m **c** 3.2 m **d** 3 s **e** 2 s

13 a



b $y = \frac{1}{25}x(x - 12)(x - 20)$ or

$y = 0.04x^3 - 1.28x^2 + 9.6x$

c \$216

14 a

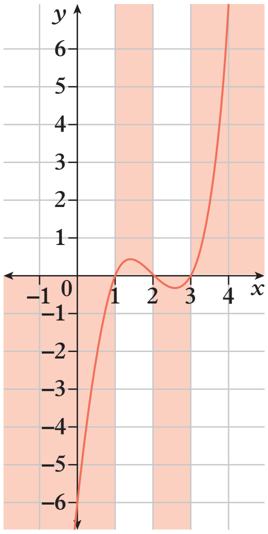
	$x < 1$	$1 < x < 2$	$2 < x < 3$	$x > 3$
$y = x - 1$	negative	positive	positive	positive
$y = x - 2$	negative	negative	positive	positive
$y = x - 3$	negative	negative	negative	positive
$y = (x - 1)(x - 2)(x - 3)$	negative	positive	negative	positive

b The shaded regions are where the product $y = (x - 1)(x - 2)(x - 3)$ is positive (above the y-axis) and negative (below the y-axis).

c i 0

ii x-intercepts

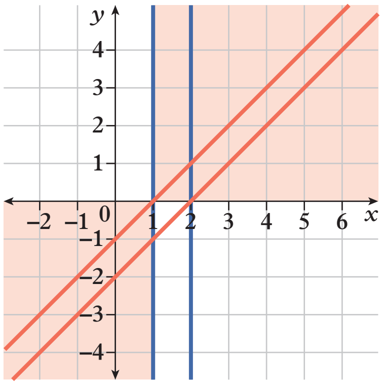
d



e

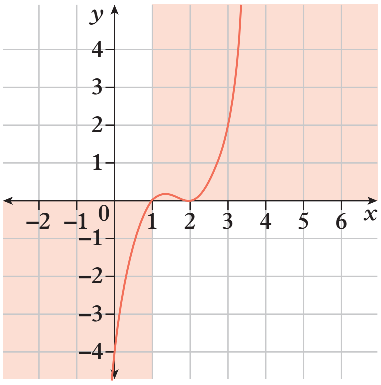
	$x < 1$	$1 < x < 2$	$x > 2$
$y = x - 1$	negative	positive	positive
$y = x - 2$	negative	negative	positive
$y = x - 2$	negative	negative	positive
$y = (x - 1)(x - 2)^2$	negative	positive	positive

f

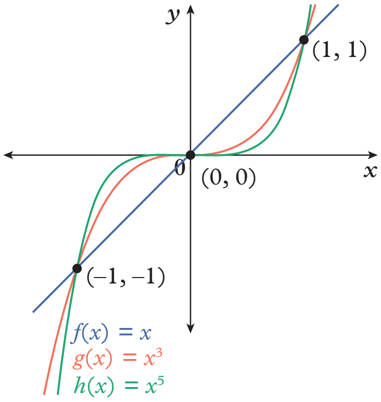


g The product, $y = (x - 1)(x - 2)^2$, is negative for $x < 1$ so below the y-axis was shaded. The product, $y = (x - 1)(x - 2)^2$, is positive for $1 < x < 2$ and $x > 2$ so above the y-axis was shaded.

h



15 a



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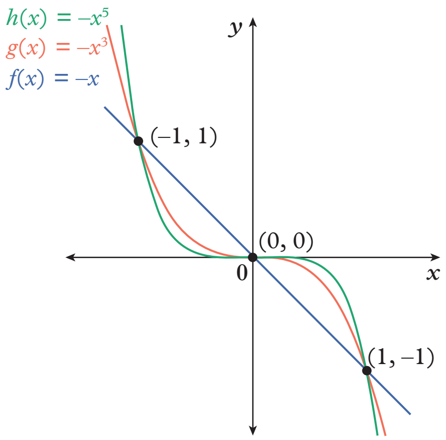
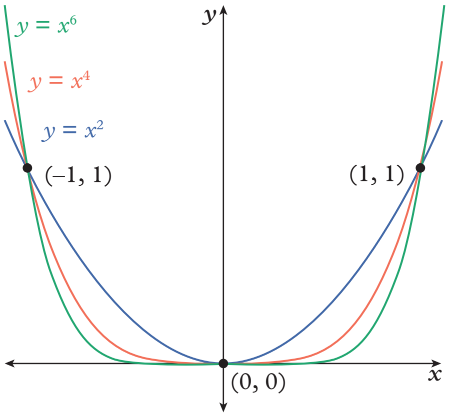


c For $y = x^n$ where n is odd, the graphs have roughly the same shape as $y = x$. For $y = -x^n$ where n is odd, the graphs have roughly the same shape as $y = -x$.

d



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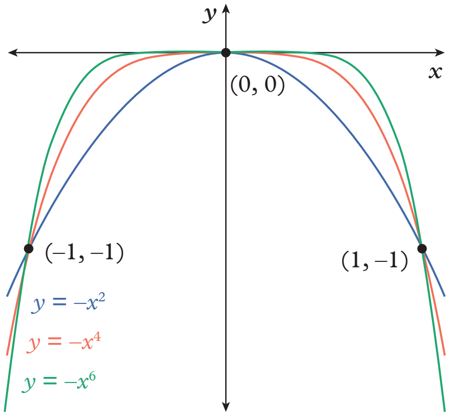
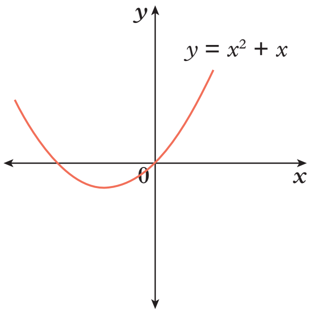


f For $y = x^n$ where n is even, the graphs have roughly the same shape as $y = x^2$. For $y = -x^n$ where n is even, the graphs have roughly the same shape as $y = -x^2$.

g **i**



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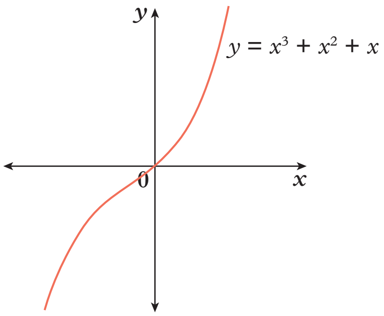

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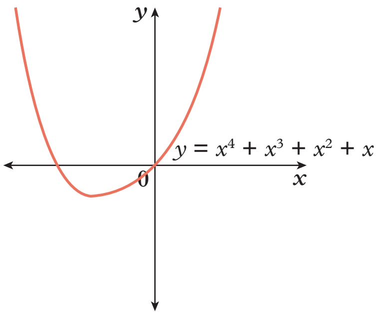

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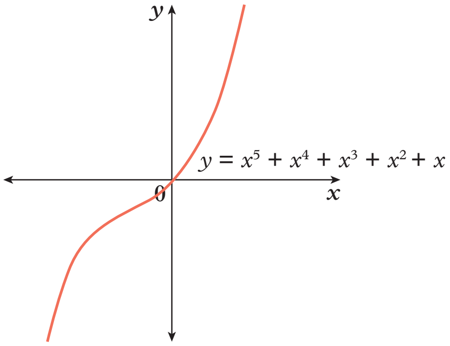

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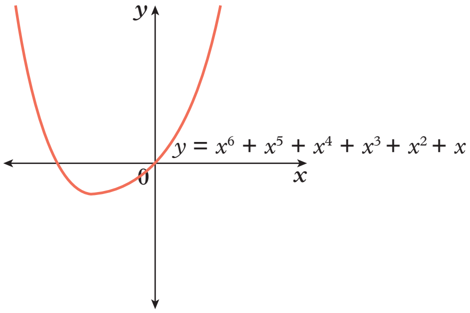

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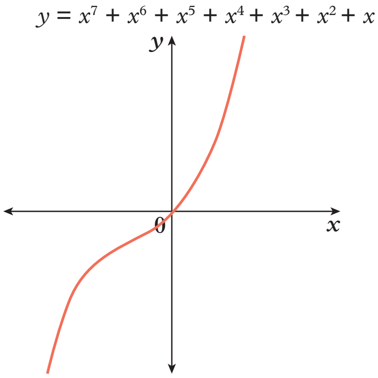


h For a leading term of x^n where n is odd, the graphs have roughly the same shape as $y = x$. For $y = -x^n$ where n is odd, the graphs have roughly the same shape as $y = -x$. For x^n where n is even, the graphs have roughly the same shape as $y = x^2$. For $y = -x^n$ where n is even, the graphs have roughly the same shape as $y = -x^2$.

16 a i 3

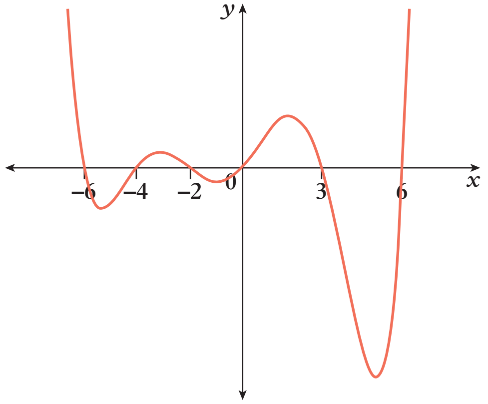
ii $\{-5, -4, -3, 3, 4, 5, 7, 8, 9\}$

iii 7, as it is not a factor of 180

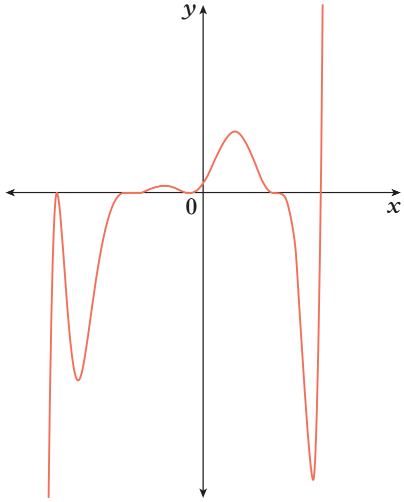
b i $(-3,0), (2,0), (3,0), (4,0)$

ii The remaining substitutions correspond with the coordinates $(1,-24), (-1,-120), (-2,-120), (-4,336)$. Sara can plot these to sketch the graph of $y = Q(x)$ more accurately.

17 a $y = x(x - 6)(x - 3)(x + 2)(x + 4)(x + 6)$



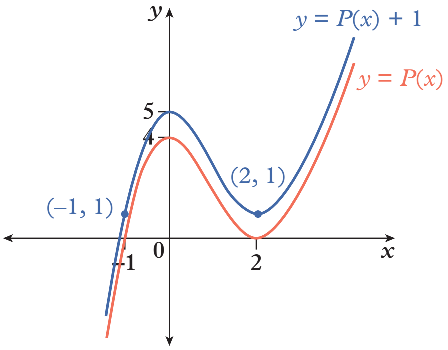
b $y = (x - 8)\ (x - 5)^3\ (x - 1)^2\ (x + 5)^3\ (x + 10)^2$



EX p310

6F Sketching graphs of polynomials using transformations

1 a



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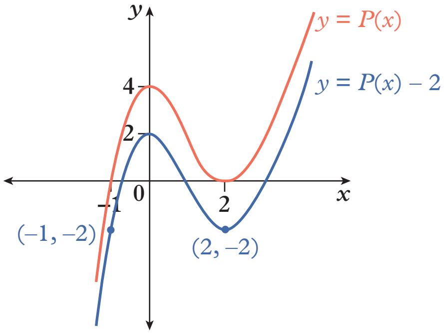

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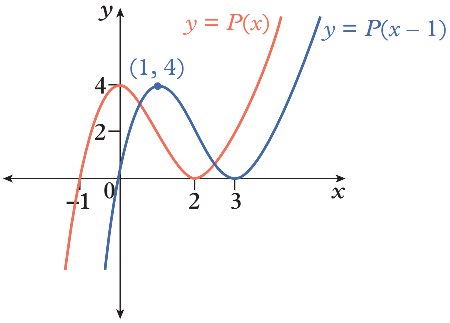

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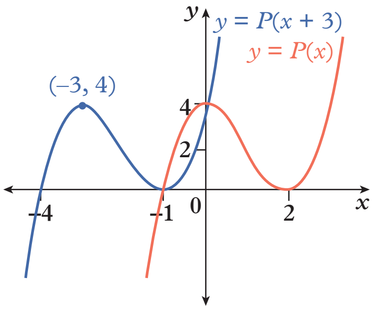

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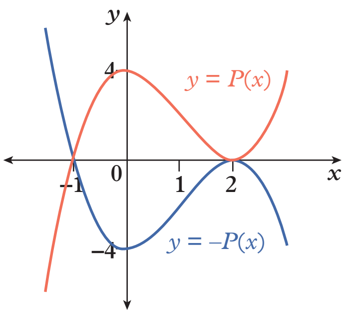

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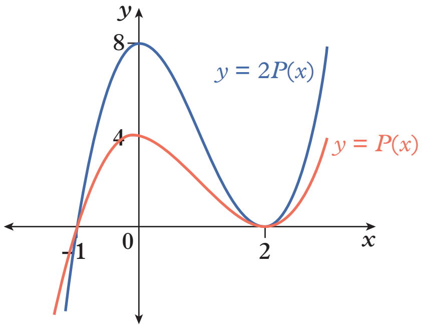
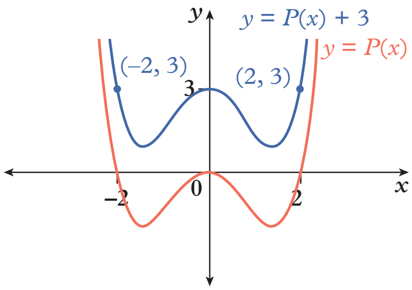


2 **a**



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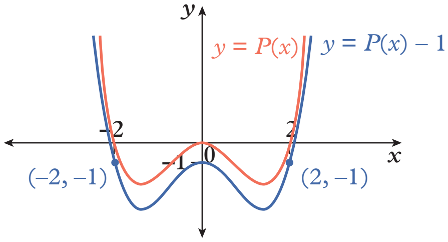

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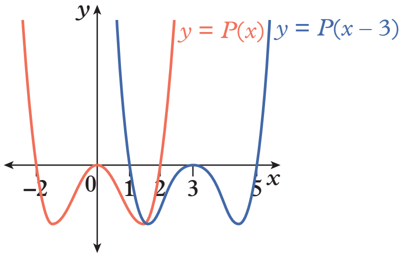

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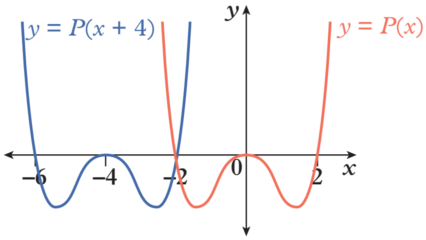

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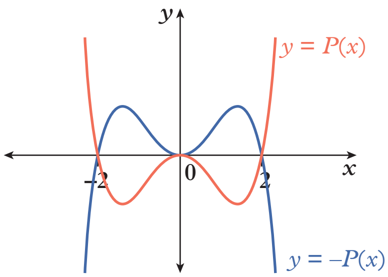

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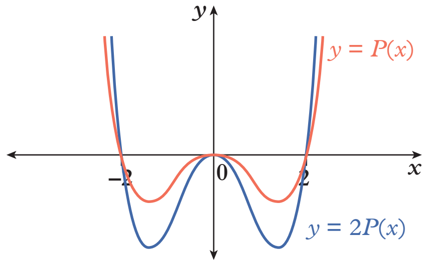


3 **a** translation of 4 units up and 1 unit right

b reflection in the x-axis

c vertical stretch by a factor of 3

d translation of 2 units right

4 **a** $y = (x - 1)^3 + 4$

b $y = -x^3$

c $y = 3x^3$

d $y = (x - 2)^3$

5 **a** reflection in the x-axis

b vertical stretch by a factor of 2

c translation of 1 unit down

d translation of 2 units left

6 **a** $y = -x^4$

b $y = 2x^4$

c $y = x^4 - 1$

d $y = (x + 2)^4$

7 **a** $y = -P(x)$

b $y = P(x) - 5$

c $y = P(x - 3)$

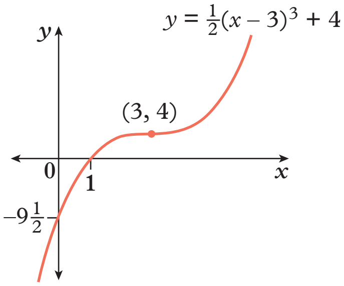
8 Both are the same graph, with the same equation, $y = -x^3$.

9 **a** **i** a stretch in the y-axis by a factor of $\frac{1}{2}$ and a translation of 3 units right and 4 units up

ii (3, 4)

iii The x-intercept is (1, 0); the y-intercept is $\left(0, -9\frac{1}{2}\right)$.

iv

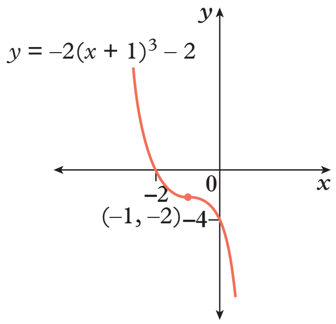


b **i** a stretch in the y-axis by a factor of 2, a reflection in the x-axis and a translation of 1 unit left and 2 units down

ii (−1, −2)

iii The x-intercept is (−2, 0); the y-intercept is (0, −4).

iv



11 a Complete the tables of values by performing the appropriate operations to the x-coordinates.

i

$\frac{x}{2}$	0	1	2	3
x	0	2	4	6
$y = P\left(\frac{x}{2}\right)$	5	0	−5	2

ii

$2x$	0	1	2	3
x	0	$\frac{1}{2}$	1	$\frac{3}{2}$
$y = P(2x)$	5	0	−5	2

iii

$\frac{x}{5}$	0	1	2	3
x	0	5	10	15
$y = P\left(\frac{x}{5}\right)$	5	0	−5	2

iv

$5x$	0	1	2	3
x	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$
$y = P(5x)$	5	0	−5	2

v

$-\frac{x}{2}$	0	1	2	3
x	0	−2	−4	−6
$y = P\left(-\frac{x}{2}\right)$	5	0	−5	2

vi

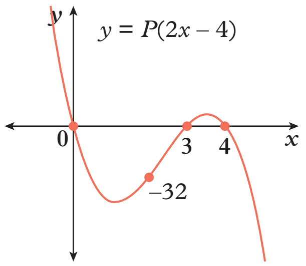
$-2x$	0	1	2	3
x	0	$-\frac{1}{2}$	−1	$-\frac{3}{2}$
$y = P(-2x)$	5	0	−5	2

b **i** $y = P\left(\frac{x}{2}\right)$ **ii** $y = P\left(\frac{x}{7}\right)$

iii $y = P(3x)$ **iv** $y = P(6x)$

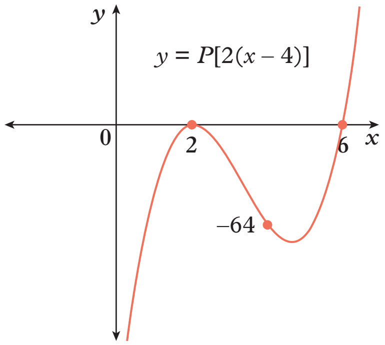
c **i**

$2x - 4$	−4	0	2	4
$2x$	0	4	6	8
x	0	2	3	4
$y = P(2x - 4)$	0	−32	0	0



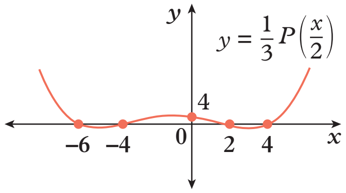
ii

$2(x - 4)$	−4	0	4
$x - 4$	−2	0	2
x	2	4	6
$y = P[2(x - 4)]$	0	−64	0



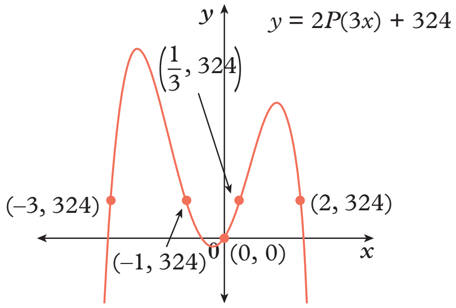
iii

$\frac{x}{2}$	−3	−2	0	1	2
x	−6	−4	0	2	4
$P\left(\frac{x}{2}\right)$	0	0	12	0	0
$y = \frac{1}{3}P\left(\frac{x}{2}\right)$	0	0	4	0	0



iv

$3x$	−9	−3	0	1	6
x	−3	−1	0	$\frac{1}{3}$	2
$P(3x)$	0	0	−162	0	0
$2P(3x)$	0	0	−324	0	0
$y = 2P(3x) + 324$	324	324	0	324	324



12 **a** correct

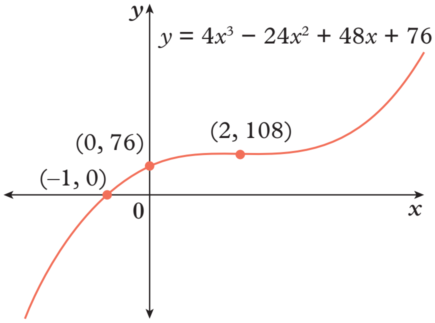
b incorrect, stretch by a factor of 4 in the y-axis

c incorrect, translate right 5 units

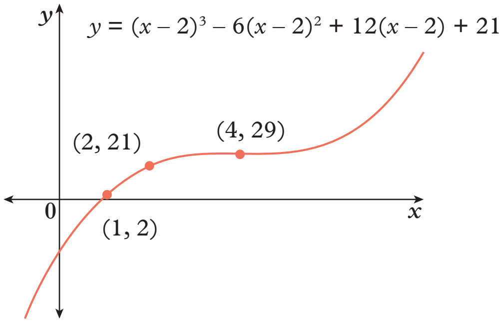
d correct

13 **a** $y = x^3 - 6x^2 + 12x + 19$

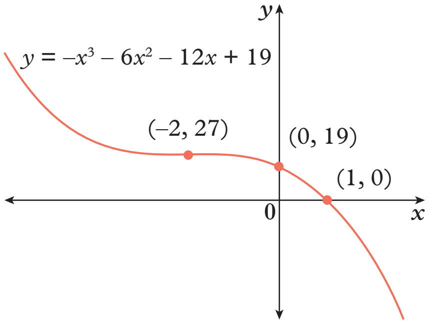
b **i**



ii



iii



14 **a** **i** a stretch by a factor of $\frac{1}{3}$ in the x-axis

a stretch by a factor of 9 in the y-axis

ii a stretch by a factor of 2 in the x-axis

a stretch by a factor of $\frac{1}{16}$ in the y-axis

iii a stretch by a factor of $\frac{1}{5}$ in the x-axis

a stretch by a factor of 25 in the y-axis

iv a stretch by a factor of $\frac{3}{5}$ in the x-axis

a stretch by a factor of $\frac{625}{81}$ in the y-axis

b **i** a stretch by a factor of 3 in the x-axis

a stretch by a factor of $\frac{1}{9}$ in the y-axis

ii a stretch by a factor of $\frac{1}{2}$ in the x-axis

a stretch by a factor of 16 in the y-axis

iii a stretch by a factor of 5 in the x-axis

a stretch by a factor of $\frac{1}{25}$ in the y-axis

iv a stretch by a factor of $\frac{5}{3}$ in the x-axis

a stretch by a factor of $\frac{81}{625}$ in the y-axis

CHAPTER 6 review

Multiple-choice

1	C	**2**	E	**3**	A	**4**	A	**5**	D	**6**	E
7	E	**8**	C	**9**	D	**10**	C	**11**	E	**12**	B

Short answer

1 **a** yes, cubic **b** yes, linear

c yes, linear **d** not a polynomial

e yes, quadratic **f** yes, quartic

2 **a** 7 **b** 8 **c** 0

d $-5x^8$ **e** –5 **f** –7

3 **a** $x + 7$, remainder 12

b $3x - 4$, remainder 8

c $x + 3$, remainder 20

d $x + 2$, remainder $1 - 2x$

4 **a** true **b** false

5 **a** $x + 1$ or $x - 3$ **b** $x + 3$

c $x - 1$ **d** $x + 2$

6 **a** $(x - 1)(x - 2)(x - 4)$

b $(x - 3)(x + 1)(x + 2)$

7 $x + 2$

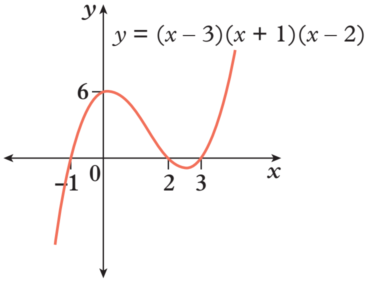
8 **a** $x = -2, -1, 1$ or 5 **b** $x = -3, 0, 2$ or 3

c $x = -3$ or 5 **d** $x = -4, 0$ or 4

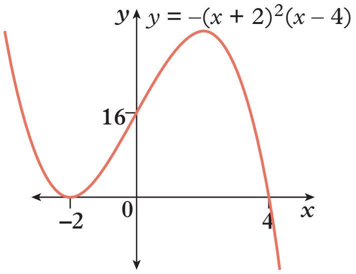
9 **a** $x(x - 2)(x + 2)(x - 2)$ or $x(x - 2)^2(x + 2)$; $x = -2, 0$ or 2

b $(x - 4)(x - 4)(x - 4)$ or $(x - 4)^3$; $x = 4$

10 **a**



b



11 **a** (–1, 0), (1, 0) and (5, 0)

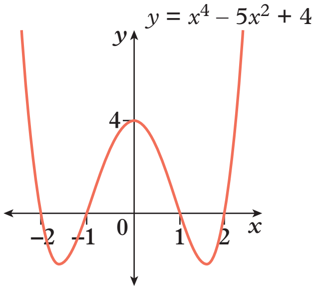
b (0, 5)

c $y = (x - 1)(x + 1)(x - 5)$

12 a $y = (x - 1)(x + 2)(x + 1)(x - 2)$

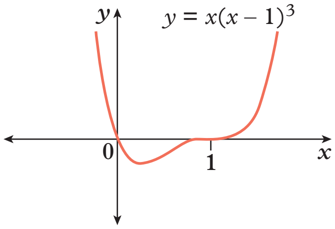
b The x-intercepts are $(-2, 0)$, $(-1, 0)$, $(1, 0)$ and $(2, 0)$; the y-intercept is $(0, 4)$.

c



13 a The x-intercepts are $(0, 0)$ and $(1, 0)$; the y-intercept is $(0, 0)$.

b



14 a $y = x^2(x - 2)(x + 2)$

b $y = x^4 - 4x^2$

15 The graph of $y = x^3$ is translated 2 units left and 4 units down.

Analysis

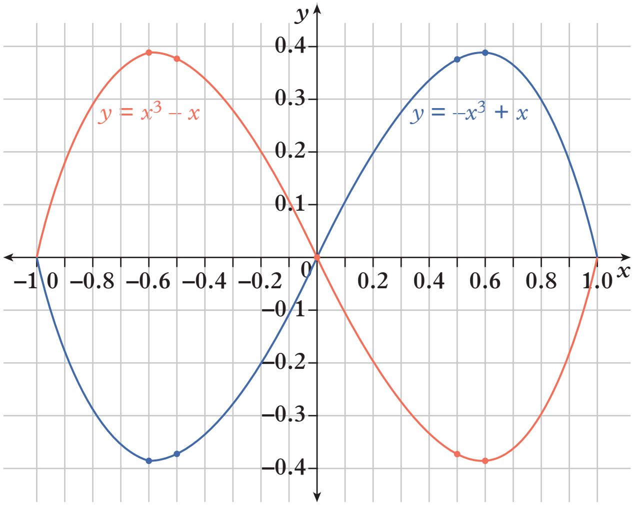
a i $y = x^3 - x$

x	–1	–0.6	–0.5	0	0.5	0.6	1
y	0	0.384	0.375	0	–0.375	–0.384	0

ii $y = -x^3 + x$

x	–1	–0.6	–0.5	0	0.5	0.6	1
y	0	–0.384	–0.375	0	0.375	0.384	0

iii



iv $-1 \le x \le 1$

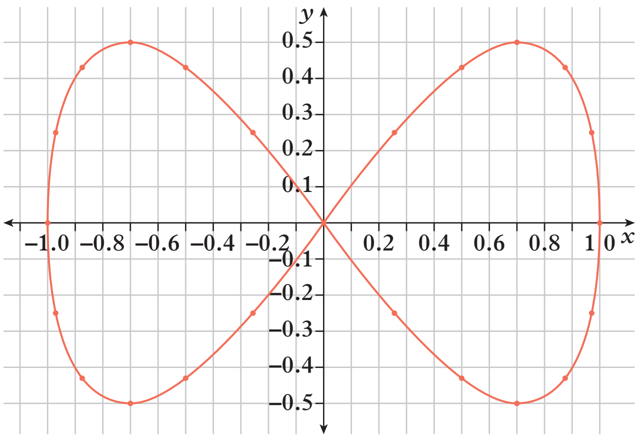
v No, the turning points do not occur midway between the x-intercepts. They occur at an x value between –0.6 and –0.5 (close to –0.6), and between 0.5 and 0.6 (close to 0.6).

vi The turning points are at $\left(-\frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{9}\right)$, $\left(-\frac{\sqrt{3}}{3}, -\frac{2\sqrt{3}}{9}\right)$, $\left(\frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{9}\right)$ and $\left(\frac{\sqrt{3}}{3}, -\frac{2\sqrt{3}}{9}\right)$ or, to three decimal places, $(-0.577, 0.385)$, $(-0.577, -0.385)$, $(0.577, 0.385)$ and $(0.577, -0.385)$.

b i, ii

Angle (θ)	**x cos [θ]**	**y** $\left(\frac{\sin[2\theta]}{2}\right)$
0°	1	0
15°	0.97	0.25
30°	0.87	0.43
45°	0.71	0.5
60°	0.5	0.43
75°	0.26	0.25
90°	0	0
105°	–0.26	–0.25
120°	–0.5	–0.43
135°	–0.71	–0.5
150°	–0.87	–0.43
165°	–0.97	–0.25
180°	–1	0
195°	–0.97	0.25
210°	–0.87	0.43
225°	–0.71	0.5
240°	–0.5	0.43
255°	–0.26	0.25
270°	0	0
285°	0.26	–0.25
300°	0.5	–0.43
315°	0.71	–0.5
330°	0.87	–0.43
345°	0.97	–0.25
360°	1	0

iii



iv It looks like the infinity symbol.

v $-1 \le x \le 1$, $-0.5 \le y \le 0.5$

c The model in part **b** is a better model for the infinity symbol, because the curve at $x = -1$ and $x = 1$ is less pointy.

EX
p318

SEMESTER 1 review

Short answer

1 **a** $-2x^2-9x+5$ **b** $6xy^2+22xy$
c $\frac{x^4}{y^{17}}$ **d** $\frac{1}{64}$
e $64x^{11}y^2$ **f** $\frac{26-31x}{15}$

2 **a** $6x+15y$
b $36x^3y^5-32x^5y^2$
c $-\frac{15x^2}{4}-\frac{15xy}{2}+10x$
d $16ax-14bx-56ay+49by$
e $2x^2+10x-16$
f $15x^2-8x+7$
g $100x^2+60x+9$
h $3ax+8bx+2cx-15a-40b-10c$

3 **a** $x=-7$ **b** $x=\frac{11}{3}$ **c** $x=\frac{27}{5}$
d $x=\frac{11}{13}$ **e** $x=\frac{20}{9}$ **f** $x=-2$
g $x=-\frac{4}{15}$ **h** $x=-43$

4 **a** $x=\frac{y-c}{m}$ **b** $x=\frac{c-by}{a}$ **c** $x=\frac{t+q}{p-r}$
d $x=\frac{2A}{z}-y$ **e** $x=\pm\sqrt{\frac{A}{\pi}}$ **f** $x=\sqrt[3]{\frac{3V}{4\pi}}$

5 **a** \$180 **b** \$172 **c** 20%
d 625% **e** \$549 **f** \$8.75
g 300% **h** 60% **i** \$752
j \$75

6 **a** \$1470 **b** \$742.16 **c** \$960
d \$2517.72 **e** \$1240

7 **a** \$625 **b** \$11 200 **c** \$1700

8 **a** 3.9% **b** 5.8%
c 4.6% **d** 13.3%

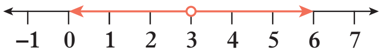
9 **a** 4 years **b** 2 years
c 7 years **d** 4 years

10 **a** $-8xy(3xy-2y-5)=8xy(2y-3xy+5)$
b $(5x+4)(4y-1)$
c $(5x-7)(5x+7)$
d $x(x-6)$
e $(x-5)^2$
f $(x+1)(x+3)$
g $(x+11)(x-4)$
h $3(x-15)(x+3)$

11 **a** $3x$ **b** $\frac{(x+10)^2}{(x-4)(x-2)}$
c $\frac{x-2}{4}$ **d** $\frac{(x-4)(x-2)}{(x+3)(x+8)}$

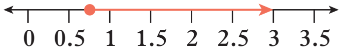
12 **a** $x<4$ **b** $x\neq 2$
c $x\geq 7$ **d** $-5\leq x<2$

13 **a** $x\neq 3$

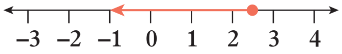


b $x<-4$

c $x\geq\frac{3}{4}$



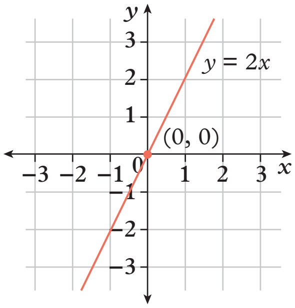
d $x\leq\frac{5}{2}$



14 **a** $x=\frac{1}{2}, x=-\frac{5}{3}$
b $x=5, x=-5$
c $x=-4, x=14$
d $x=4, x=6$
e $x=-3, x=5$
f $x=-2, x=4$
g $x=\frac{-5\pm\sqrt{65}}{2}$
h $x=-2, x=7$

15 **a** $(x+5)^2-25$
b $(x-3)^2+3$
c $\left(x-\frac{5}{2}\right)^2+\frac{23}{4}$
d $2(x-7)^2-116$

16 **a**



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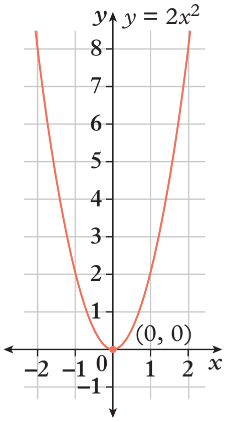

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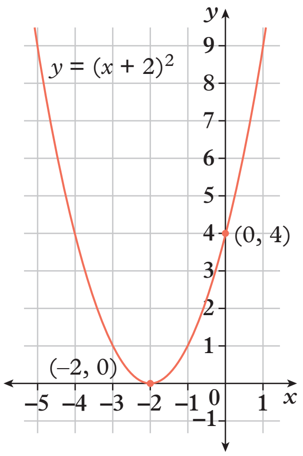

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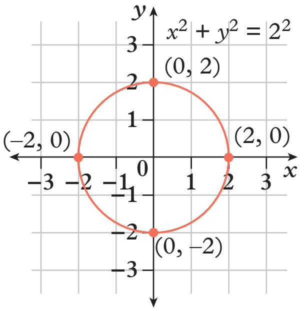

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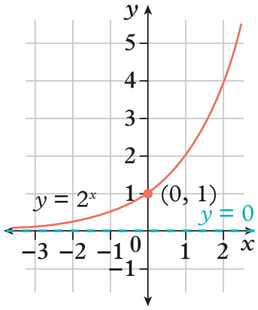

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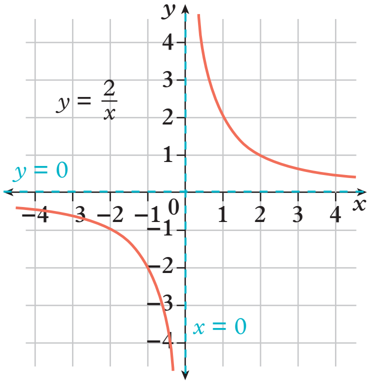


17 **a** 1 **b** 1 **c** 1 **d** 0 **e** 1

f 2 **g** 1 **h** 2 **i** 0 **j** 2

k 0 **l** 1 **m** 1 **n** 0

18 **a** x-intercept (5, 0); y-intercept (0, –10)

b x-intercept $\left(\frac{20}{3}, 0\right)$; y-intercept $\left(0, -\frac{10}{3}\right)$

c x-intercept (12, 0)

d y-intercept (0, 3)

e x-intercept $\left(\frac{9}{4}, 0\right)$; y-intercept (0, 9)

f x-intercepts (4, 0), (–6, 0); y-intercept (0, –72)

g x-intercept (4, 0); y-intercept (0, 16)

h x-intercepts (–6, 0), (1, 0); y-intercept (0, 6)

i y-intercept (0, 5)

j x-intercepts (–9, 0), (1, 0); y-intercept (0, 3)

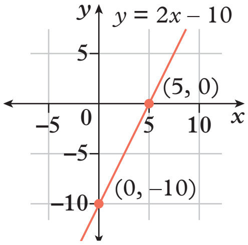
k y-intercept (0, 2)

l x-intercept (1, 0); y-intercept (0, 2)

m x-intercept $\left(\frac{5}{2}, 0\right)$; y-intercept $\left(0, \frac{5}{3}\right)$

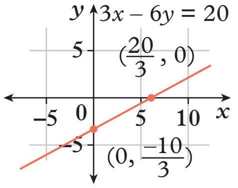
n y-intercept $\left(0, -\frac{3}{2}\right)$

19 **a** **i**



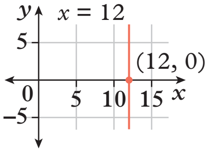
ii Domain is ℝ; range is ℝ.

b **i**



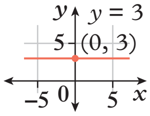
ii Domain is ℝ; range is ℝ.

c **i**



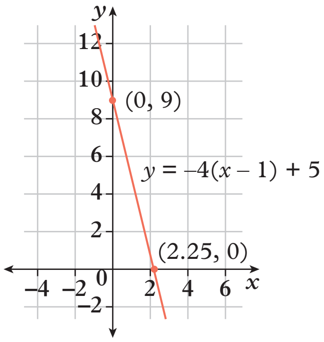
ii Domain is {12}; range is ℝ.

d **i**



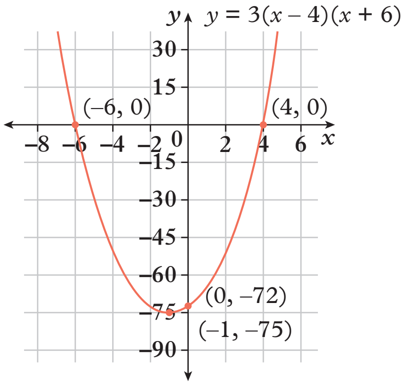
ii Domain is ℝ; range is {3}.

e **i**



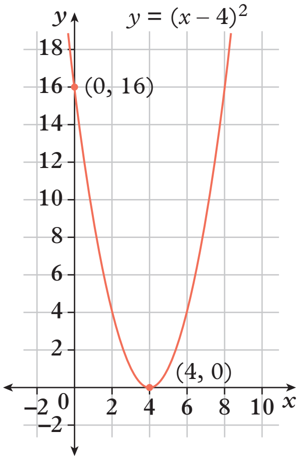
ii Domain is ℝ; range is ℝ.

f **i**



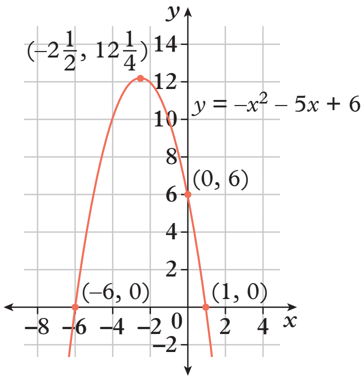
ii Domain is ℝ; range is $y \ge -75$.

g **i**



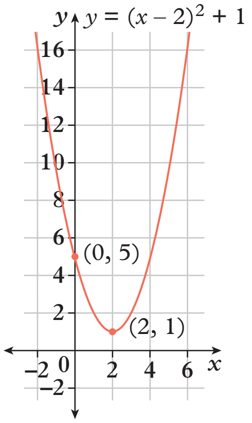
ii Domain is ℝ; range is $y \ge 0$.

h **i**



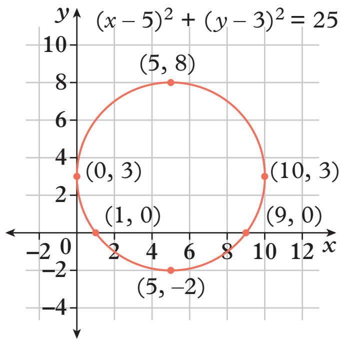
ii Domain is ℝ; range is $y \le \frac{49}{4}$.

i **i**



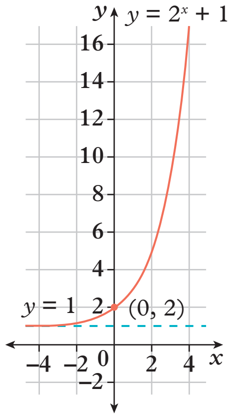
ii Domain is ℝ; range is $y \ge 1$.

j **i**



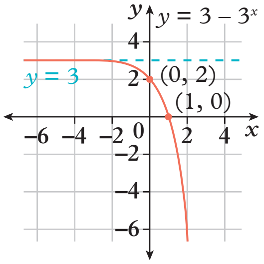
ii Domain is $0 \le x \le 10$; range is $-2 \le y \le 8$.

k **i**



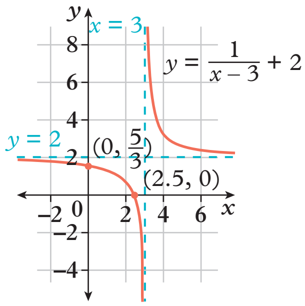
ii Domain is ℝ; range is $y > 1$.

l **i**



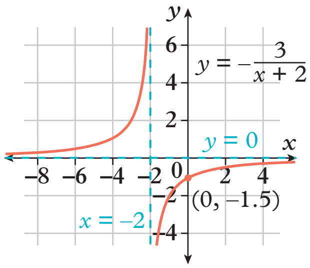
ii Domain is ℝ; range is $y < 3$.

m **i**



ii Domain is $x < 3$ and $x > 3$; range is $y < 2$ and $y > 2$.

n i



ii Domain is $x < -2$ and $x > -2$; range is $y < 0$ and $y > 0$.

20 a $y = -\frac{2}{3}x + \frac{5}{3}$ **b** $y = 4x + 13$

c $y = \frac{16}{5}x - \frac{68}{5}$ **d** $y = \frac{1}{2}x + 1$

e $y = \frac{3}{4}x + \frac{13}{2}$ **f** $y = 3x + 6$

21 a $x = -1, y = 4$

b $x = 1, y = 6$ or $x = 3, y = 4$

22 a $x = 2, y = 2$ **b** $x = \frac{13}{6}, y = -\frac{2}{3}$

c $x = \frac{13}{2}, y = -\frac{2}{9}$ **d** $x = -\frac{8}{3}, y = -\frac{6}{5}$

23 a $k = \frac{9}{4}$ **b** $k = 36$

c $k = -100$ **d** $k = 100$

e $k = \frac{1}{4}$ **f** $k = 3$

24 a i $y = \frac{9}{2}$ **ii** $x = \frac{8}{9}$

b i $y = 18$ **ii** $x = 18$

c i $y = -200$ **ii** $x = -\frac{1}{50}$

d i $y = 50$ **ii** $x = 50$

e i $y = \frac{1}{2}$ **ii** $x = 8$

f i $y = \frac{3}{2}$ **ii** $x = \frac{3}{2}$

25 a $x = 1, y = 2$ or $x = 6, y = 17$

b $x = 4, y = 27$

c $x = -4, y = -3$ or $x = 3, y = 4$

d $x = 1, y = 6$ or $x = -6, y = -1$

26 a i $-6x^5$ **ii** 5 **iii** −6

b i $5x^4$ **ii** 4 **iii** 5

c i $-\frac{7}{2}x^4$ **ii** 4 **iii** $-\frac{7}{2}$

27 a 4 **b** 3 **c** $\frac{1}{10\,000}$

d 7 **e** $\frac{1}{2}$ **f** −5

28 a −73 **b** $-\frac{17}{8}$

c $7\sqrt{2} - 10$ **d** $34\sqrt{3} - 40$

29 a $a^{\frac{13}{12}}$

b b^4c^{15}

c $\log_2(2d^2) = 2\log_2(d) + 1$

d $1 - \log_5(e)$

e $6\sqrt{5}$

f $6\sqrt{3} + 12\sqrt{2}$

g $15\sqrt{2}$

h $2\sqrt{6}$

i $54\sqrt{3} + 72$

j $8\sqrt{15} - 6$

30 a $\frac{5\sqrt{3}}{3}$ **b** $\frac{2\sqrt{6} + \sqrt{3}}{3}$

c $\frac{9\sqrt{2} + 15}{7}$ **d** $\frac{14 - 5\sqrt{5}}{2}$

31 a $2x^3 + 3x^2 + 13x + 3$

b $13x^2 + 5x + 7$

c $6x^3 + 19x^2 + x - 21$

d $x^4 + x^3 - 18x^2 + 37x - 15$

e $3x + 3 + \frac{14}{x - 3}$

f $x^2 + 3x - 13 + \frac{20}{x + 2}$

g $x^2 - 4x + 18 - \frac{73}{x + 4}$

h $x^2 - 7 + \frac{37}{x + 5}$

32 a $(2x + 5)(x - 3)$

b $(3x + 4)(2x - 5)$

c i $4\left(x - \frac{3}{8}\right)^2 - \frac{121}{16}$

ii $4(x + 1)\left(x - \frac{7}{4}\right) = (x + 1)(4x - 7)$

d $(x + 3)(x + 1)(x + 5)$

e $(x - 2)(x + 5)(x - 6)$

f $(x - 5)(x + 2)(x + 3)$

33 a $x = 5$

b $x = \log_3(30)$ or $\log_3(10) + 1$

c $x = \frac{1}{3}\log_5\left(\frac{1}{50}\right)$ or $-\frac{1}{3}(\log_5(2) + 2)$

d $x = 3 \pm \sqrt{5}$

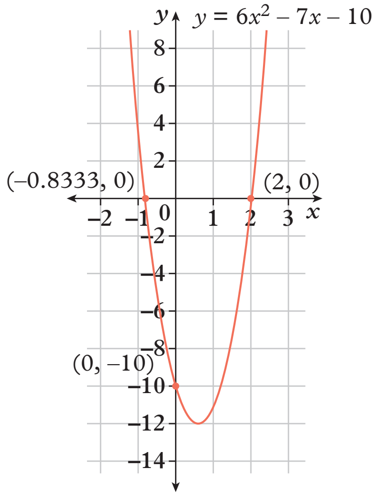
e $x = -1, x = \frac{1}{4}$

f $x = \frac{2 \pm 2\sqrt{10}}{3}$

g $x = -\frac{5}{2}, x = \frac{10}{3}, x = 5, x = 4$

h $x = 1, x = -3, x = -4$

34 a



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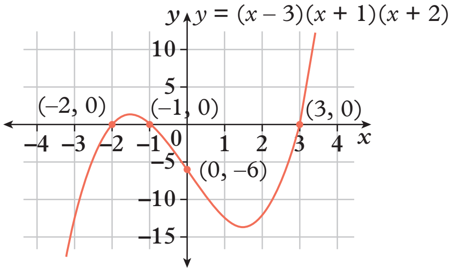

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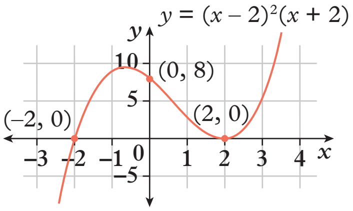

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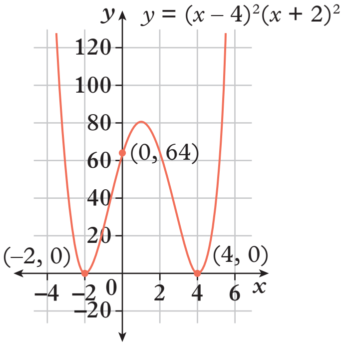

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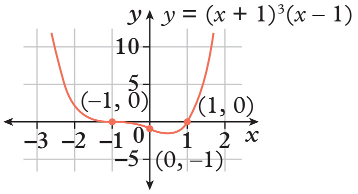

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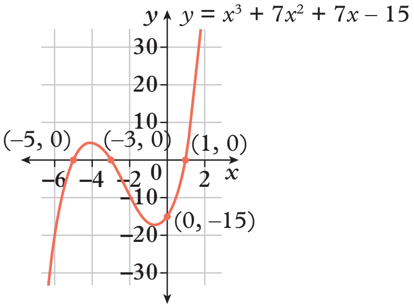


35 **a** stretched by a factor of 2 from the x-axis

reflected in the x-axis

translated 8 units up

b stretched by a factor of $\frac{3}{8}$ from the x-axis

translated 2 units left

c stretched by a factor of $\frac{1}{2}$ from the x-axis, or translated 1 unit right

translated 4 units down

d stretched by a factor of $\frac{1}{4}$ from the x-axis

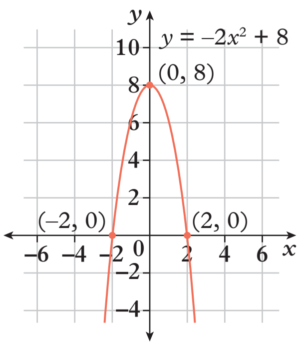
translated 2 units right and 1 unit up

e reflected in the x-axis

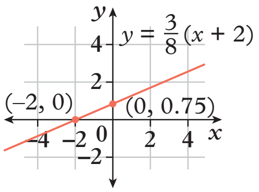
translated 1 unit left and 8 units up

f translated 1 unit left and 2 units up

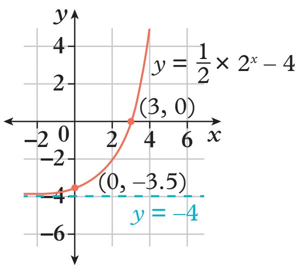
36 **a**



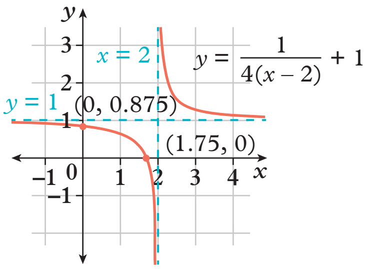
b



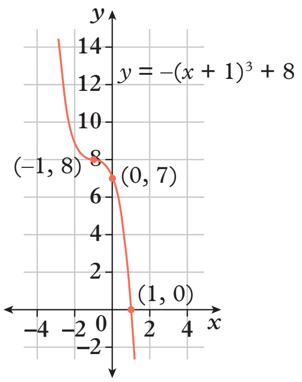
c



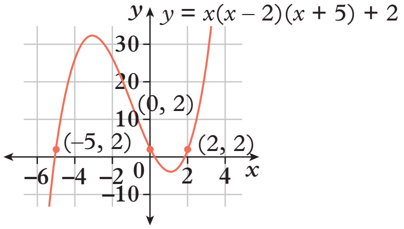
d



e



f



Analysis

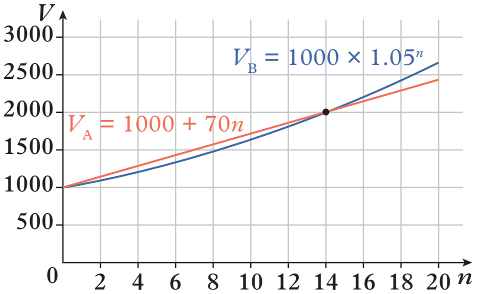
1 **a** Investment A

b $V_A = 1000 + 70n$ and $V_B = 1000 \times 1.05^n$

c

n	0	5	10	15	20
V_A	1000	1350	1700	2050	2400
V_B	1000	1276.28	1628.89	2078.93	2653.30

d



e After the 14th year

f After 4 years

g **i** After 12 years

ii $1840

h **i** $\log_{1.05}\left(\frac{27}{20}\right)$ years or $\frac{\log(1.05)}{\log 1.35}$

ii 6.15 years

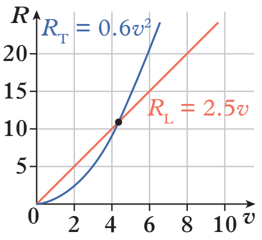
2 **a** **i** $k = 2.5$; $R_L = 2.5v$

ii $R_L = 50$ N

b **i** $k = 0.6$; $R_T = 0.6v^2$

ii $R_T = 240$ N

c

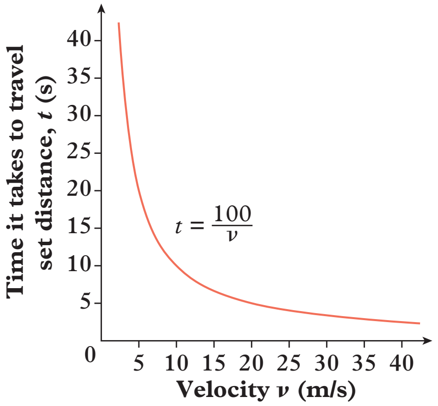


d $v = 0$ m/s; $v = \frac{25}{6} = 4\frac{1}{6}$ m/s

e $k = 100$; $v = \frac{100}{t}$

f 100 m

g

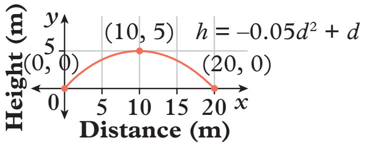


h As the velocity increases the time it takes to travel 100 m approaches zero, but it never reaches zero as $v = \frac{100}{t}$ and $\frac{100}{0}$ is undefined.

3 **a** 20 m wide

b maximum height = 5 m, 10 m away from the entrance

c



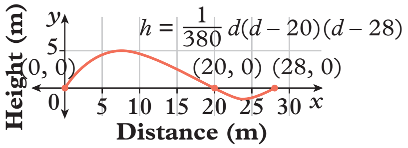
d **i** $10 - 2\sqrt{5}$ m or $10 + 2\sqrt{5}$ m

ii 5.53 m or 14.47 m

e $h = \frac{1}{380}d(d-20)(d-28)$

f $h = \frac{1}{380}(d^3 - 48d^2 + 560d)$

g



h **i** maximum height = 5.06 m, 7.67 m away from the entrance

ii maximum depth = 1.02 m, 24.33 m away from the entrance

EX
p324

EXPLORATIONS 1

1 **a** 4 red bricks

b 2 purple bricks

2 **a**

×	2	7	10	5	3
9	18	63	90	45	27
4	8	28	40	20	12
6	12	42	60	30	18
8	16	56	80	40	24
1	2	7	10	5	3

b

×	6	4	2	8
10	60	40	20	80
12	72	48	24	96
14	84	56	28	112
16	96	64	32	128

c

×	10	7	5	3	2
6	60	42	30	18	12
4	40	28	20	12	8
9	90	63	45	27	18
8	80	56	40	24	16

3 **a** $\sqrt{3} - \sqrt{2}$

b **i** $\dfrac{\sqrt{a} - \sqrt{b}}{a - b}$

ii It fails when $a = b$. Then $\dfrac{\sqrt{a}}{2a}$ is the correct simplified fraction.

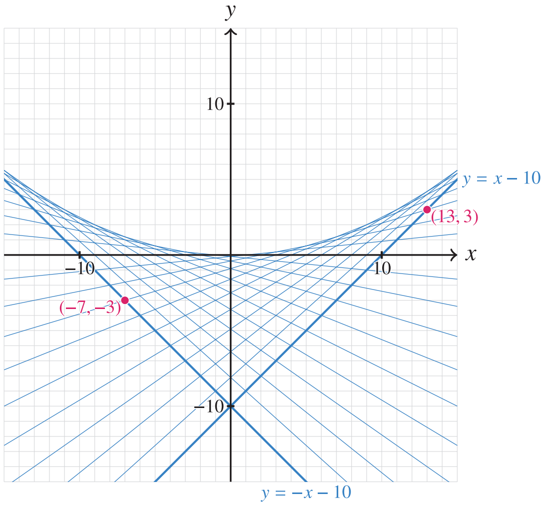
c $\dfrac{2\sqrt{3} + 3\sqrt{2} - \sqrt{30}}{12}$

d **i** $\dfrac{a\sqrt{b} + b\sqrt{a} - \sqrt{abc}}{2ab}$

ii It works for all a and b.

iii $\dfrac{1}{\sqrt{3} + \sqrt{5} + \sqrt{8}} = 3\dfrac{\sqrt{5} + 5\sqrt{3} - 2\sqrt{30}}{30}$

4 **a**



b **i** $y = \dfrac{3}{10}x - \dfrac{9}{10}$

ii Show that $\dfrac{1}{40}x^2 = \dfrac{3}{10}x - \dfrac{9}{10}$ simplifies to $(x - 6)^2 = 0$, which has exactly one solution.

c **i** $Q(k - 10, -k)$

ii $y = \dfrac{k}{10}x - \dfrac{k^2}{10}$

iii Simplify to $(x - 2k)^2 = 0$ or show that the discriminant is zero.

d Show that $ax^2 = \dfrac{k}{d}x - \dfrac{k^2}{d}$ has discriminant equal to zero when $a = \dfrac{1}{4d}$.

5 **a** 1, 32, 243, 1024, 3125. The last digit of the power is equal to the base. Yes, this also works for other fifth powers.

b $n(n - 1)(n + 1)(n^2 + 1)$. Show that 2, 3 and 5 each divide at least one of the four factors. If n is of the form $5k \pm 2$, show that 5 divides $n^2 + 1$.

c Since $n^5 - n$ ends in 0, n^5 and n must have the same last digit.

d Explain why the second and third factors are divisible by 2 and 4, or vice versa.

e Any n that is equal to, one more than, or one less than a multiple of 9.

f $n^9 - n = (n^5 - n)(n^4 + 1)$ is a multiple of 30. It is true for all 1st, 9th, 13th, …, $(4m + 1)$th, … powers.

CHAPTER 7 Geometry

EX
p330

7A Geometry review

1 **a** $a = 146°$

b $b = 24°$

c $c = 57°, d = 57°, e = 123°$

d $d = 75°$

e $e = 109°, f = 71°, g = 74°$

f $g = 71°, h = 19°, i = 71°, j = 90°$

2 **a** $a = 117°$

b $b = 109°$

c $c = 49°$

d $d = 129°$

e $e = 63°, f = 117°, g = 117°, h = 63°$

f $f = 126°$

3 **a** $a = 97°$

b $b = 88°, c = 88°, d = 92°$

c $c = 58°, d = 302°$

d $d = 52°, e = 71°, f = 57°$

e $d = 180°, e = 30°, f = 115°, g = 51°$

f $f = 67°, g = 64°, h = 116°, i = 91°$

4 **a** $a = 76°$

b $b = 60°$

c $c = 32°, d = 116°$

d $d = 160°$

e $e = 115°$

f $f = 37°, g = 75°, h = 68°$

5 **a** $a = 55°$

b $b = 117°, c = 63°, d = 117°$

c $c = 30°, d = 118°$

d $d = 45°$

e $e = 119°, f = 25°$

f $f = 58°, g = 98°, h = 98°, i = 122°$

6 **a** 3240° **b** 8640° **c** 17 640°

7 The internal angle sum of a polygon = $(n - 2) \times 180°$

A regular polygon with n equal sides has n equal angles.

So the size of each angle = $\dfrac{(n - 2) \times 180°}{n}$

8 **a** $128\frac{4}{7}°$ **b** 140° **c** 120°

9 **a** 18 **b** 10 **c** 36 **d** 1000

10 **a** 18° **b** 27° **c** 22.5°

d 9.6875° **e** 21° **f** 17.63°

11 **a** true

b false; angles at a point add to 360°.

c false; an isosceles triangle has one pair of equal angles.

d false; corresponding angles on parallel lines are equal.

e true

f false; a rectangle is a parallelogram, but not all parallelograms are rectangles.

g false; complementary angles add to 90°.

h false; a triangle that has two 60° angles must be equilateral

i true

12 a $b = 30°, c = 44°, d = 74°, e = 29°$

b $c = 72°$

c $d = 109°, e = 8°$

13 a $a = 60°, b = 30°, c = 90°$

b $d = 35°$

c $e = 163°$

d $f = 79°$

e $e = 220°$

f $f = 154.5°$

14 a 330° **b** 185° **c** 74° **d** 225°

e 351° **f** 54° **g** 87° **h** 70°

i 151° **j** 296° **k** 325° **l** 300°

EX
p336

7B Geometric proofs

1 a

Statements	Reasons
$\angle EOF + \angle FOB = 180^\circ$	Angles on a straight line sum to 180°
$\angle BOC + \angle FOB = 180^\circ$	Angles on a straight line sum to 180°
$\angle EOF + \angle FOB = \angle BOC + \angle FOB$	Transitive law
$\angle EOF = \angle BOC$ □	Subtraction property of equality

b

Statements	Reasons
$\angle EOF + \angle AOF + \angle AOB = 180^\circ$	Angles on a straight line sum to 180°
$\angle BOC + \angle AOB + \angle AOF = 180^\circ$	Angles on a straight line sum to 180°
$\angle EOF + \angle AOF + \angle AOB = \angle BOC + \angle AOF + \angle AOB$	Transitive law
$\angle EOF = \angle BOC$ □	Subtraction property of equality

2

Statements	Reasons
$\angle GKJ + \angle JKL = 180^\circ$	Angles on a straight line sum to 180°
$\angle NKL + \angle JKL = 180^\circ$	Angles on a straight line sum to 180°
$\angle GKJ + \angle JKL = \angle NKL + \angle JKL$	Transitive law
$\angle GKJ = \angle NKL$ □	Subtraction property of equality

3

Statements	Reasons
$\angle ABC = \angle EFB$	Corresponding angles on parallel lines are equal
$\angle ABC + \angle CBD = 180^\circ$	Angles on a straight line sum to 180°
$\angle DBF + \angle CBD = 180^\circ$	Angles on a straight line sum to 180°
$\angle ABC + \angle CBD = \angle DBF + \angle CBD$	Transitive law
$\angle ABC = \angle DBF$	Subtraction property of equality
$\angle EFB + \angle BFG = 180^\circ$ $\angle EFB = \angle ABC = \angle DBF$	Angles on a straight line sum to 180°
$\therefore \angle DBF + \angle BFG = 180^\circ$ □	Transitive law

4

Statements	Reasons
$\angle DBA + \angle ABC + \angle CBE = 180^\circ$	Angles on a straight line sum to 180°
$\angle DBA = \angle BAC$	Alternate angles on parallel lines are equal
$\angle CBE = \angle BCA$	Alternate angles on parallel lines are equal
$\therefore \angle BAC + \angle ABC + \angle BCA = 180^\circ$ □	Transitive law

5

Statements	Reasons
$\angle EFH + \angle DFE = 180^\circ$	Angles on a straight line sum to 180°
$\angle DFE + \angle DEF + \angle FDE = 180^\circ$	Interior angle sum of a triangle is 180°
$\angle EFH + \angle DFE = \angle DFE + \angle DEF + \angle FDE$	Transitive law
$\angle EFH = \angle DEF + \angle FDE$ □	Subtraction property of equality

6

Statements	Reasons
X lies on $\overleftrightarrow{AB}$	Given
Y lies on $\overleftrightarrow{CD}$	Given
$\angle AXY$ is a right angle	Given
$\angle CYX$ is a right angle	Given
$\angle AXY = 90^\circ$	All right angles are equal to 90°
$\angle CYX = 90^\circ$	All right angles are equal to 90°
$\angle AXY + \angle CYX = 180^\circ$	Angle addition postulate
$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ □	If co-interior angles are supplementary, they lie on parallel lines

7

Statements	Reasons
$\angle MPQ + \angle QPN = 180^\circ$	Supplementary angles sum to 180°
$\angle MPQ + \angle QPN = \angle MPN$	Angle addition postulate
$\angle MPN = 180^\circ$	Transitive law
MN is a straight line □	Angles on a straight line sum to 180°

8 **a** Parallel lines will never meet, no matter how far they extend. Lines which are not parallel will meet at some point.

b $\angle EAB$ and $\angle ACB$

c If AC is equal to 0, the points A and C would be the same point. Any value less than zero is meaningless.

d $\angle EAB + \angle BAC = 180°$

But: $\angle EAB = x$

So: $x + \angle BAC = 180°$

That is: $\angle BAC = 180° - x$

e $(180° - x) + x + y = 180°$

f $180° + y = 180°$ simplifies to $y = 0°$

g If $\angle ABC$ is 0°, line BC would lie on top of line BA, so points A and C would lie on top of each other.

h This contradicts the assumption that $AC > 0$. In other words, the lines must be parallel.

i Assume that $\angle EAB$ and $\angle ACB$ are equal corresponding angles, and that DG is not parallel to HF.

Also assume that the length of AC is greater than zero.

In ΔABC: $\angle ABC + \angle ACB + \angle CAB = 180°$

But: $\angle CAB = 180° - \angle EAB$

So: $\angle ABC + \angle ACB + (180° - \angle EAB) = 180°$

It was assumed that $\angle EAB = \angle ACB$.

This relationship then reduces to $\angle ABC = 0°$.

This implies that $AC = 0$.This contradicts the statement that $AC > 0$. So the lines DG and HF must be parallel.

9 **a** If two lines cut by a transversal are not parallel, then the corresponding angles are not equal.

b

Statements	**Reasons**
$AB \nparallel CB$	Given
$\angle EAB$ and $\angle ACB$ are corresponding angles	Given
ABC is a triangle	Three points that do not form a line form a triangle
$\angle EAB = \angle ABC + \angle ACB$	The exterior angle of a triangle is equal to the sum of the two non-adjacent interior angles
$\angle ABC \neq 0°$	Angles cannot be equal to 0°
$\angle ABC + \angle ACB \neq 0° + \angle ACB$	Angle addition postulate
$\angle EAB \neq 0° + \angle ACB$	Transitive law
$\angle EAB \neq \angle ACB$ □	Subtraction property of equality

10 If $a = b$, then $(a - b) = 0$. Dividing by $(a - b)$ is actually dividing by 0, which gives an answer which is undefined.

EX
p343

7C Congruence and similarity

1 **a** $a = 51°$, $b = 11$ cm, $c = 70°$, $d = 59°$, $e = 10$ cm

b $a = 48°$, $b = 25°$, $c = 4$ cm, $d = 25°$, $e = 107°$, $f = 7$ cm

c $a = 38°$, $b = 50°$, $c = 13$ cm, $d = 38°$, $e = 92°$, $f = 8$ cm, $g = 10$ cm

d $j = 25°$, $k = 7$ cm, $l = 3$ cm, $m = 5$ cm, $n = 115°$

2 **a** congruent; AAS

b congruent; RHS

c congruent; SAS

d not congruent; SSS does not apply

e not congruent; AAS does not apply

f not congruent; RHS does not apply

3 **a** SAS or AAS **b** SAS does not apply

c SAS **d** AAS

4 **a** $\frac{5}{2}$ **b** $\frac{3}{2}$ **c** 2 **d** $\frac{1}{3}$

5 **a** not similar; does not meet the requirements of the SSS condition for similarity

b not similar; does not meet the requirements of the SAS condition for similarity

c similar; meets the requirements of the AAA condition for similarity

d similar; meets the requirements of the AAA condition for similarity

e similar; meets the requirements of the SSS condition for similarity

f similar; meets the requirements of the RHS condition for similarity

6 **a** $x = 10.5$ cm, $y = 8$ cm

b $c = 44$ cm, $d = 7$ cm

c $x = 4$ cm, $y = 9$ cm

d $a = 9$ cm, $b = 4$ cm

e $x = 5\frac{7}{13}$ cm, $y = 17\frac{1}{3}$ cm

f $x = 52$ cm, $y = 20$ cm

7 **a** similar; passes SSS condition

b similar; meets the requirements of the AAA condition for similarity

c not similar; does not meet the requirements of the AAA condition for similarity

d not similar; does not meet the requirements of the RHS condition for similarity

8 **a**

Statements	**Reasons**
$VX = YX$	Given
$WX = ZX$	Given
$\angle VXW = \angle YXZ$	Vertically opposite angles are equal
$\therefore \Delta VWX \sim \Delta YZX$ □	SAS (congruent)

b

Statements	Reasons
$QR = QT$	Given
$\angle PQR = \angle SQT$	Vertically opposite angles are equal
$\angle RPQ = \angle TSQ$	Given
$\therefore \Delta PQR \cong \Delta SQT$ □	AAS (congruent)

9 a

Statements	Reasons
$\angle ABC = \angle PRQ$	Given
$\frac{PR}{AB} = \frac{27}{9} = 3$	Scale factor between corresponding sides
$\frac{RQ}{BC} = \frac{18}{6} = 3$	Scale factor between corresponding sides
$\therefore \Delta ABC \sim \Delta PRQ$ □	SAS (similar)

b

Statements	Reasons
$\angle AEB = \angle CED$	Vertically opposite angles are equal
$\angle EAB = \angle ECD$	Alternate angles on parallel lines are equal
$\angle EBA = \angle EDC$	Alternate angles on parallel lines are equal
$\therefore \Delta ABE \sim \Delta CDE$ □	AAA (similar)

10 a The 9 cm side length is opposite the 85° angle in one figure, and adjacent to the 85° angle in the other figure. None of the angles are equal, so the case of an isosceles triangle with two equal sides is ruled out.

b The 88° angle is opposite the 17 cm side in one figure and opposite the 16 cm side in the other figure.

c Three pairs of equal angles is not a condition for congruence. One triangle could be a dilation of the other.

d The congruence condition RHS does not apply because the only side measurement given is the hypotenuse.

11 a False; the triangles may or may not be congruent, depending on their side lengths.

b False; the corresponding angle must be between the two corresponding sides for these triangles to be congruent.

c False; if a pair of triangles meets one condition of congruence, they will meet all conditions of congruence (but, perhaps, further calculations are necessary to show the other conditions of congruence are met).

d True; figures which are congruent to each other all have the same shape and size.

e False; a quadrilateral with all sides of equal length could be a square or a rhombus. A square and a rhombus are only ever congruent if all the interior angles are right angles.

f False; all equilateral triangles are similar, but they are only congruent if their side lengths are the same.

12 a It appears at first glance that the scale factor is $\frac{1}{2}$. However, the sides compared are not corresponding sides.

b The RHS test requires a right angle the hypotenuse and one of the shorter sides (which is not provided). This means that the pair of triangles fails the RHS test.

13 4.9 m

14 3.6 m

15 Pairs of similar triangles: **A** and **C** (AAA), **D** and **E** (SAS)

16 a false; they must be similar if their corresponding angles are equal

b true; the corresponding angles in these three triangles must be equal

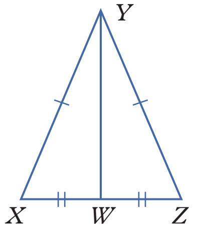
c false; the angles in isosceles triangles are not fixed

d true; all squares have all angles equal to 90° and all four lengths equal to each other, which makes them all the same shape

e false; if they fail SAS they cannot be similar

f true; consider a rhombus and a square with the same side lengths.

17 a



b ΔYWX and ΔYWZ

c $YX = YZ$ and $WX = WZ$

d YW is common to both triangles

e Because the angles at the base of an isosceles triangle are equal.

f

Statements	Reasons
$YX = YZ$	Equal sides in an isosceles triangle
$WX = WZ$	By construction
YW is common to both triangles	Common side
$\therefore \Delta WXY \cong \Delta WZY$	SSS (congruent)
$\therefore \angle WXY = \angle WZY$ □	Corresponding angles in congruent triangles

g $\angle WXY = \angle WZY$ as they are corresponding sides in congruent triangles.

18 a

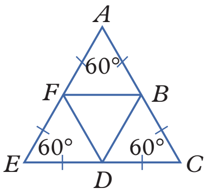
Statements	Reasons
$\angle AEB = \angle CED$	Common angle to both triangles
$AE = 70 + 70 = 140$	Substitution
$BE = 110 + 110 = 220$	Substitution
$\frac{DE}{BE} = \frac{110}{220} = \frac{1}{2}$	Scale factor between corresponding sides
$\frac{CE}{AE} = \frac{70}{140} = \frac{1}{2}$	Scale factor between corresponding sides
$\therefore \Delta ABE \sim \Delta CDE$ □	SAS (similar)

b

Statements	Reasons
$\angle AEB = \angle CED$	Common angle
$AC = CE$	Given
$AE = AC + CE$	Given
$AE = CE + CE = 2AE$	Substitution
$BD = DE$	Given
$BE = BD + DE$	Given
$BE = DE + DE = 2DE$	Substitution
$\frac{DE}{BE} = \frac{DE}{2DE} = \frac{1}{2}$	Scale factor between corresponding sides
$\frac{CE}{AE} = \frac{CE}{2CE} = \frac{1}{2}$	Scale factor between corresponding sides
$\therefore \Delta ABE \sim \Delta CDE$ □	SAS (similar)

19 7.2 m

20



Statements	Reasons
$AC = AE = CE$	Side lengths of an equilateral triangle are equal
$AB = BC = CD = DE = EF = FA$	Points B, D and F lie at the midpoints of equal side lengths
$\angle ACE = \angle CEA = EAC = 60°$	Interior angles of an equilateral triangle are 60°
$\Delta ABF \cong \Delta CDB \cong \Delta EDF$	SAS (congruent)
$FB = BD = DF$	Corresponding sides on congruent triangles
$\therefore \Delta FBD$ is equilateral	A triangle with equal side lengths is equilateral
$\angle FBD = \angle BDF = \angle DFB = 60°$	Interior angles of an equilateral triangle are 60°
ΔABF is isosceles	A triangle with two equal side lengths is isosceles
$\angle ABF = \angle AFB$	Angles opposite equal sides of an isosceles triangle are equal
$\angle ABF = \angle AFB = 60°$	Interior angle sum of a triangle is 180°
$\therefore \Delta ABF$ is equilateral	A triangle with equal angles is equilateral
$AB = FA = FB$	Side lengths on equilateral triangles are equal
$AB = BC = CD = DE = EF = FA = FB = BD = DF$	Transitive law
$\therefore \Delta CDB$ is equilateral	A triangle with equal side lengths is equilateral
$\therefore \Delta EDF$ is equilateral	A triangle with equal side lengths is equilateral
$\therefore \Delta ABF \cong \Delta CDB \cong \Delta EDF \cong FBD$ □	SSS (congruent)

EX p351

7D Proofs and quadrilaterals

1 Join EG.

Statements	Reasons
$\angle FGE = \angle HEG$	Alternate angles on parallel lines are equal
$\angle FEG = \angle HGE$	Alternate angles on parallel lines are equal
EG is common to both ΔFGE and ΔHEG	Common side
$\therefore \Delta FGE \cong \Delta HEG$	AAS (congruence)
$FG = HE$	Corresponding sides in congruent triangles are equal
$EF = GH$	Corresponding sides in congruent triangles are equal
$FG = EF$	Given
$\therefore FG = EF = HE = GH$	Transitive law
$\therefore FEGH$ is a rhombus □	A quadrilateral with four equal side lengths is a rhombus

2

Statements	Reasons
$\angle ACI + \angle CIS = 180°$	Co-interior angles on parallel lines are supplementary
$\angle ASI + \angle CIS = 180°$	Co-interior angles on parallel lines are supplementary
$\angle ACI + \angle CIS = \angle ASI + \angle CIS$	Transitive law
$\angle ACI = \angle ASI$	Subtraction property of equality
$\angle ICA + \angle CAS = 180°$	Co-interior angles on parallel lines are supplementary
$\angle ICA + \angle CIS = 180°$	Co-interior angles on parallel lines are supplementary
$\angle ICA + \angle CAS = \angle ICA + \angle CIS$	Transitive law
$\angle CAS = \angle CIS$	Subtraction property of equality
$\therefore$ the opposite angles of a rhombus are equal □	

3

Statements	Reasons
$AB = CD$	Given
$AD = CB$	Given
BD is common to both ΔABD and ΔCDB	Common side
$\Delta ABD \cong \Delta CDB$	SSS (congruence)
$\angle ABD = \angle CDB$	Corresponding angles in congruent triangles are equal
$\angle ADB = \angle CBD$	Corresponding angles in congruent triangles are equal
$AB \parallel CD$	If alternate angles are equal, they lie on parallel lines
$AD \parallel CB$	If alternate angles are equal, they lie on parallel lines
$\therefore ABCD$ is a rhombus □	A quadrilateral with all sides equal in length and opposite sides being parallel is a rhombus

4 **a** $\angle ETR$ **b** $\angle TER$

c

Statements	**Reasons**
$\angle MIR = \angle TER$	Alternate angles on parallel lines are equal
$\angle IMR = \angle ETR$	Alternate angles on parallel lines are equal
$IM = ET$	Given
$\therefore \Delta MIR \cong \Delta TER$ □	AAS (congruent)

d It follows that: $IR = ER$ and $MR = TR$.

e IR and ER together form the parallelogram's diagonal IE, while MR and TR form the diagonal TM.
The diagonals bisect each other because we know that $IR = ER$ and $MR = TR$.

5

Statements	**Reasons**
$OK = FK$	Given
$LK = GK$	Given
$\angle OKL = \angle FKG$	Vertically opposite angles are equal.
$\therefore \Delta OKL \cong \Delta FKG$	SAS (congruence)
$\angle LOK = \angle GFK$	Corresponding angles in congruent triangles are equal
$\angle OLK = \angle FGK$	Corresponding angles in congruent triangles are equal
$OL \parallel FG$	If alternate angles are equal, they lie on parallel lines
$\angle OKG = \angle FKL$	Vertically opposite angles are equal
$\therefore \Delta OKG \cong \Delta FKL$	SAS (congruence)
$\angle GOK = \angle LFK$	Corresponding angles in congruent triangles are equal
$\angle FLK = \angle OGK$	Corresponding angles in congruent triangles are equal
$OG \parallel FL$	If alternate angles are equal, they lie on parallel lines
$\therefore$ *GOLF* is a parallelogram □	A quadrilateral with parallel opposite sides is a parallelogram

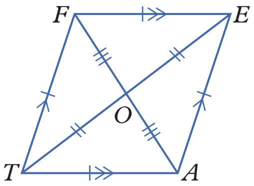
6 **a**

Statements	**Reasons**
$\angle EFO = \angle TAO$	Alternate angles on parallel lines are equal
$\angle FEO = \angle ATO$	Alternate angles on parallel lines are equal
$FE = AT$	Given
$\therefore \Delta FEO \cong \Delta ATO$	AAS (congruence)
$FO = AO$	Corresponding sides in congruent triangles are equal
$EO = TO$	Corresponding sides in congruent triangles are equal
$\therefore$ the diagonals bisect each other □	

b

Statements	**Reasons**
$\Delta FEO \cong \Delta ATO$	Proven in part **a**
$\angle AEO = \angle FTO$	Alternate angles on parallel lines are equal
$\angle EAO = \angle TFO$	Alternate angles on parallel lines are equal
$AE = FT$	Given
$\therefore \Delta AEO \cong \Delta FTO$	AAS (congruence)
$FE = AE$	Given
$FO = AO$	Proven in part **a**
EO is common to both ΔHEO and ΔAEO	Common
$\Delta FEO \cong \Delta AEO$	SSS (congruence)
$\therefore \Delta HEO \cong \Delta AEO \cong \Delta HTO \cong \Delta ATO$ □	Transitive law

c



d $\angle FOE$ and $\angle AOE$ are corresponding angles in congruent triangles, therefore they are equal.

e

Statements	**Reasons**
$\angle FOE + \angle AOE = 180°$	Angles on a straight line sum to 180°
$\angle FOE = \angle AOE$	As shown in part **d**
$\angle FOE + \angle FOE = 180°$	Substitution
$\angle FOE = 90°$	Solving equation
$\angle AOE = 90°$	Transitive law
$\therefore$ the diagonals of a rhombus bisect each other at right angles □	

7 **a**

Statements	**Reasons**
$AL = ML$	Given
$\angle ALU = 90°$	Given
$\angle ALU + \angle MLU = 180°$	Angles on a straight line sum to 180°
$\angle MLU = 90°$	Substitution and solving equation
UL is common to both ΔAUL and ΔMUL	Common
$\therefore \Delta AUL \cong \Delta MUL$	SAS (congruence)
$CL = UL$	Given
$\angle ALU = \angle MLC$	Vertically opposite angles are equal
$\therefore \Delta MUL \cong \Delta MCL$	SAS (congruence)
$\angle MLU = \angle ALC$	Vertically opposite angles are equal
$\therefore \Delta MUL \cong \Delta ACL$	SAS (congruence)
$\therefore \Delta\, AUL \cong \Delta MUL \cong MCL \cong \Delta ACL$	Transitive law

Statements	Reasons
$AU = MU = MC = AC$	Corresponding sides in congruent triangles are equal
$\therefore AUMC$ is a rhombus □	If a quadrilateral has all sides equal in length it is a rhombus

b

Statements	Reasons
$AUMC$ is a parallelogram	If a quadrilateral has opposite sides that are parallel (given), it is a parallelogram
$AU = MC$	Opposite sides of a parallelogram are equal in length
$\angle UAL = \angle CML$	Alternate angles on parallel lines are equal
$\angle AUL = \angle MCL$	Alternate angles on parallel lines are equal
$\therefore \Delta ALU \cong \Delta MLC$	AAS (congruence)
$AL = ML$	Corresponding sides in congruent triangles are equal
UL is common to both ΔALU and ΔMLU.	Common side
$\angle ULA = \angle ULM$	Angles on a straight line sum to 180°
$\therefore \Delta ALU \cong \Delta MLU$	SAS (congruence)
$AU = MU$	Corresponding sides in congruent triangles are equal
$AC = MU$	Opposite sides of a parallelogram are equal in length
$AU = MU = MC = AC$	Transitive law
$\therefore AUMC$ is a rhombus □	If a quadrilateral has all sides equal in length it is a rhombus

8

Statements	Reasons
$RE \parallel YM$	Opposite sides of a parallelogram are parallel
$\angle REM + \angle EMY = 180°$	Co-interior angles on parallel lines are supplementary
$\angle REM = 90°$	Given
$\angle EMY = 90°$	Substitution and solving equation
$RY \parallel EM$	Opposite sides of a parallelogram are parallel
$\angle MYR + \angle EMY = 180°$	Co-interior angles on parallel lines are supplementary
$\angle MYR = 90°$	Substitution and solving equation
$\angle MYR + \angle YRE = 180°$	Co-interior angles on parallel lines are supplementary
$\angle YRE = 90°$	Substitution and solving equation
$\therefore REMY$ is a rectangle □	A quadrilateral with four right angles is a rectangle

9 a

Statements	Reasons
$EV = IL$	Given
EI is common to both ΔEVI and ΔILE	Common side
$\angle VEI = \angle LIE$	Alternate angles on parallel lines are equal
$\therefore \Delta EVI \cong \Delta ILE$	SAS (congruence)
$\angle EVI = 90°$	Given
$\therefore \angle ILE = 90°$	Corresponding angles in congruent triangle are equal
$VE \parallel IL$	Given
$\angle EVI + \angle VIL = 180°$	Co-interior angles on parallel lines are supplementary
$\angle VIL = 90°$	Substitution and solving equation
$\angle ILE + \angle LEV = 180°$	Co-interior angles on parallel lines are supplementary
$\angle LEV = 90°$ □	Substitution and solving equation

b

Statements	Reasons
$EL = IV$	Corresponding sides in congruent triangles are equal
$\angle LEV = \angle EVI = 90°$	Proved in part **a**
$\angle LEV + \angle EVI = 180°$	Substitution
$\therefore EL \parallel IV$ □	If co-interior angles are supplementary, they are on parallel lines

c

Statements	Reasons
$EI^2 = EL^2 + IL^2$	Pythagoras' Theorem
$LV^2 = EL^2 + EV^2$	Pythagoras' Theorem
$EV = IL$	Given
$LV^2 = EL^2 + IL^2$	Substitution
$LV^2 = EI^2$	Transitive law
$\therefore LV = EI$ □	Transitive law

d

Statements	Reasons
$EV = IL$	Given
$LV = EI$	Proved in part **c**
EL is common to both ΔLEV and ΔELI	Common side
$\therefore \Delta LEV \cong \Delta ELI$ □	SSS (congruence)

10 a

Statements	**Reasons**
$PE = PY$	Given
$PL = PK$	Given
$\angle EPL = \angle YPK$	Vertically opposite angles are equal
$\therefore \Delta ELP \cong \Delta YKP$	SAS (congruence)
$\angle PEL = \angle PYK$	Corresponding angles in congruent triangles are equal
$EL \parallel YK$	If alternate angles are equal, they lie on parallel lines
$\angle EPK = \angle YPL$	Vertically opposite angles are equal
$\therefore \Delta EPK \cong \Delta YPL$	SAS (congruence)
$\angle PEK = \angle PYL$	Corresponding angles in congruent triangles are equal
$EK \parallel YL$	If alternate angles are equal, they lie on parallel lines
$\therefore$ *ELYK* is a parallelogram □	If a quadrilateral has two pairs of parallel sides, it is a parallelogram

b

Statements	**Reasons**
$\angle LYE + \angle YEL + \angle ELK + \angle KLY = 180°$	Interior angle sum of a triangle is 180°
$\angle YEL = \angle ELK$	Angles opposite equal sides in an isosceles triangle are equal
$\angle LYE = \angle KLY$	Angles opposite equal sides in an isosceles triangle are equal
$\therefore \angle KLY + \angle ELK + \angle ELK + \angle KLY = 180°$	Substitution
$2\angle ELK + 2\angle KLY = 180°$	Solving equation
$\angle ELK + \angle KLY = 90°$	Solving equation
$\angle ELY = 90°$ □	Transitive law

c Since *ELYK* is a parallelogram,

$\angle ELY + \angle LYK = 180°$ (co-interior angles)

Since $\angle ELY = 90°$, $\angle LYK = 90°$.

So $\angle YKE = 90°$ and $\angle KEL = 90°$, since opposite angles in a parallelogram are equal. Since *ELYK* is a parallelogram, and all its interior angles are 90°, *ELYK* is a rectangle.

11 a

Statements	**Reasons**
$UA = RS$	Sides of a rhombus are equal
$\angle AUG = \angle SRG$	Alternate angles on parallel lines are equal
$\angle UAG = \angle RSG$	Alternate angles on parallel lines are equal
$\therefore \Delta GUA \cong \Delta GRS$	AAS (congruence)
$UG = RG$	Corresponding sides in congruent triangles are equal
$AG = SG$ □	Corresponding sides in congruent triangles are equal

b

Statements	**Reasons**
$UA = RA$	Given
$UG = RG$	Proved in part **a**
GA is common to both ΔUAG and ΔRAG	Common side
$\therefore \Delta UAG \cong \Delta RAG$ □	SSS (congruence)

c

Statements	**Reasons**
$\angle UAG = \angle RAG$	Corresponding angles in congruent triangles are equal
$\angle UAS = \angle RAS$ □	Transitive law

d The two equal angles are the angles at vertex *A*. The angle at *A* has been bisected by the diagonal *SA* to produce them. The same argument from part **b** can be applied to the other interior triangles to show, as was shown in question **8b**, that all four interior triangles of a rhombus are congruent to each other. The same argument from part **c** can then be applied at each vertex of the rhombus to show that the diagonals of a rhombus bisect the interior angles of a rhombus.

12

Statements	**Reasons**
$EP = IP$	Given
$EV = IV$	Given
PV is common to both ΔEVP and ΔIVP	Common side
$\therefore \Delta EVP \cong \Delta IVP$	SSS (congruence)
$\angle VPE = \angle VPI$	Corresponding angles in congruent triangles are equal
$\angle VPE + \angle VPI = 180°$	Angles on a straight line sum to 180°
$\therefore \angle VPE = \angle VPI = 90°$	Solving equation
$\angle EVP = \angle VEP$	Angles opposite equal sides in an isosceles triangle are equal
$\therefore \angle EVP = \angle VEP = 45°$	Angles in a triangle sum to 180° and solving equation
$\therefore \angle EVL = 45°$ □	Transitive law

13 a Consider a square and a rhombus which have equal side lengths. The interior angles of the square are all 90°, but those in the rhombus are not. This means the two figures do not have the same shape, so they are not congruent.

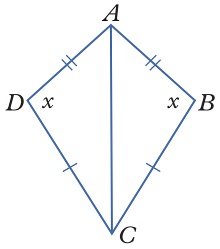
b A rhombus has four sides of equal length. If the SAS condition for congruence held for these shapes, the adjacent sides of the parallelogram would be equal in length. This would make it a rhombus.

c The AAS condition for congruence considers two angles (in this case, right angles) and one side. This could hold true for the square and rectangle for two angles and the included side, but the other sides do not need to be equal lengths unless the rectangle is also square.

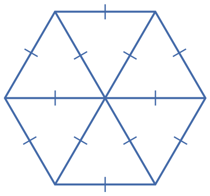
14 **a** A rectangle and a parallelogram can have all corresponding sides in the same ratio without being similar. They will not be the same shape unless all the interior angles of the parallelogram are right angles, which would make the shape a rectangle.

b A parallelogram and a rhombus might have their corresponding angles all the same size, but they only have the same shape if the sides of the parallelogram are all equal. This would make the parallelogram a rhombus.

15 **a** A kite has two pairs of adjacent sides equal, and one side in common to both triangles. The SSS condition for congruence applies.



b The six equilateral triangles formed all have the same side length. The SSS condition for congruence applies.



16

Statements	Reasons
$\angle HAB = \angle DCB = \angle DEF = \angle HGF = 90^\circ$	Interior angles of a rectangle are 90°
$AB = CB = EF = GF$	Opposite sides of a rectangle are equal and B and F lie at the midpoints of these sides
$AH = CD = ED = GH$	Opposite sides of a rectangle are equal and H and D lie at the midpoints of these sides
$\therefore \Delta ABH \cong \Delta CBD \cong \Delta EFD \cong \Delta FGH$	SAS (congruence)
$BH = BD = FD = FH$	Corresponding sides in congruent triangles are equal
$\therefore$ *HBDF* is a rhombus	A quadrilateral with four equal sides is a rhombus
$\angle ABH = \angle CBD = \angle EFD = \angle GFH$	Corresponding angles in congruent triangles are equal
$\angle ABH + \angle HBD + \angle CBD = 180^\circ$	Angles on a straight line sum to 180°
$\angle GFH + \angle HFD + \angle DFE = 180^\circ$	Angles on a straight line sum to 180°
$\therefore \angle HBD = \angle HFD$	Substitution and transitive law
$\angle BHA = \angle BDC = \angle FDE = \angle FHG$	Corresponding angles in congruent triangles are equal
$\angle AHB + \angle BHF + \angle GHF = 180^\circ$	Angles on a straight line sum to 180°
$\angle CDB + \angle BDF + \angle EDF = 180^\circ$	Angles on a straight line sum to 180°
$\therefore \angle BHF = \angle BDF$ □	Substitution and transitive law

7 Checkpoint

1 **a** $w = 49^\circ$, $x = 104^\circ$ **b** $y = 144^\circ$

2 **a** $m = 68^\circ$ **b** $j = 65^\circ$, $k = 65^\circ$

3 **a** $p = 135^\circ$ **b** $q = 72^\circ$

4

Statements	Reasons
AB is a line segment	Given
CD is a line segment	Given
$\angle AEC + \angle CEB = 180^\circ$	Angles on a straight line sum to 180°
$\angle CEB + \angle DEB = 180^\circ$	Angles on a straight line sum to 180°
$\angle AEC + \angle CEB = \angle CEB + \angle DEB$	
$\angle AEC = \angle DEB$ □	Transitive law

5

Statements	Reasons
$\angle ABE = \angle DCF$	Given
$\angle ABE = \angle CBE$	Vertically opposite angles are equal
$AB \parallel CD$	If two lines are cut by a transversal such that corresponding angles are equal, then the lines are parallel
$\angle DCF = \angle CBE$	Transitive law

6 **a** $a = 8$, $b = 7$ cm

b $x = 105^\circ$, $y = 20^\circ$, $z = 9$ cm

7 **a** $g = 27^\circ$, $h = 1.3$ cm

b $p = 1.4$ cm, $q = 6.25$ cm

8 **a**

Statements	Reasons
$AD = DE$	Given
$AC = CD$	Given
CD is common to both ΔADC and ΔEDC	Common side
$\Delta ADC \cong \Delta EDC$	SSS (congruence)
$\angle ACD = \angle ECD$	Corresponding angles in congruent triangles are equal

b

Statements	**Reasons**
$AB \parallel EF$	Given
$\angle DAB = \angle DEF$	Corresponding angles on parallel lines are equal
$\angle DBA = \angle DFE$	Corresponding angles on parallel lines are equal
$\angle ADB = \angle EDF$	Common angle
$\therefore \Delta ADB \sim \Delta EDF$ □	AAA (similarity)

9

Statements	**Reasons**
$\angle ADC = 90°$	Given $AD \perp CD$
$\angle BCD = 90°$	Given $BC \perp CD$
$\angle DAB + \angle ADC = 180°$	Co-interior angles on parallel lines are supplementary
$\angle DAB = 90°$	Substituting and solving equation
$\angle ABC + \angle BCD = 180°$	Co-interior angles on parallel lines are supplementary
$\angle ABC = 90°$	Substituting and solving equation
$\therefore$ $ABCD$ is a rectangle	A quadrilateral with all interior angles of 90° is a rectangle
$AC = BD$ □	Diagonals of a rectangle are equal

10

Statements	**Reasons**
$\angle SPQ = \angle RST$	Given
$PQ \parallel SR$	If two lines are cut by a transversal such that corresponding angles are equal, then the lines are parallel
$PS \parallel QR$, $\angle RST = \angle SRQ$	If alternate angles are congruent, they lie on parallel lines
$PQRS$ is a parallelogram □	If a quadrilateral has two pairs of opposite sides that are parallel, it is a quadrilateral

EX p359

7E Circle geometry: circles and angles

1 **a** 45° **b** 120° **c** 74°

2 **a** 90° **b** 18° **c** 45°

3 **a** $x = 32°$ **b** $x = 27°$

c $a = 9°$, $c = 6°$

4 In the following answers, CT stands for circle theorem.

a $x = 34.5°$, CT2 and angle sum of a triangle

b $x = 109°$, CT1

c $x = 21°$, CT3

d $a = 14°$, $b = 7°$, CT3

e $x = 22.5°$, CT2 and triangle angle sum

f $a = 26\frac{1}{3}°$, $e = 9\frac{7}{8}°$, CT3

g $x = 43°$, CT1

h $x = 7.5°$, CT2 and triangle angle sum

i $x = 23°$, CT1

5 **a** 12.2° **b** 76° **c** 32°

d 68° **e** 55° **f** 29°

6 **a** Each one is a radius of the circle.

b ΔAOB is isosceles.

c $\angle AOD = \angle ABO + \angle OAB = 2\angle ABO$

d ΔAOB is isosceles, so $\angle OBC = \angle OCB$.

$\angle DOC = \angle OBC + \angle OCB = 2\angle OBC$

e $\angle AOC = \angle AOD + \angle DOC = 2\angle ABO + 2\angle OBC = 2(\angle ABO + \angle OBC) = 2\angle ABC$

7 **a** 180°

b Since the angle at the centre is twice the angle at the circumference subtended by the same arc, the angle at the circumference is equal to $\frac{1}{2}$ of 180°, or 90°.

8 **a** $\angle WOZ$ is an angle at the centre of the circle, while $\angle WXZ$ is an angle at the circumference. Both are subtended by the same arc, and so circle theorem 1 applies so that $\angle WOZ = 2\angle WXZ$.

b Similarly, $\angle WOZ$ and $\angle WYZ$ are both subtended by the same arc, so circle theorem 1 applies and $\angle WOZ = 2\angle WYZ$.

c It follows that $\angle WYZ = \angle WXZ$.

9 **a** Obtuse $\angle AOC$ is an angle at the centre of the circle, subtended by the same arc as angle x at the circumference.

Similarly, the reflex angle $\angle AOC$ is an angle at the centre of the circle, subtended by the same arc as angle y at the circumference.

In both cases, circle theorem 1 applies, so that $\angle AOC = 2x$ and $\angle AOC = 2y$.

b obtuse $\angle AOC$ + reflex $\angle AOC = 360°$

That is: $2x$ + reflex $\angle AOC = 360°$

or: reflex $\angle AOC = 360° - 2x$

c Since: reflex $\angle AOC = 2y$

$2y = 360° - 2x$

Simplifying: $y = 180° - x$

d obtuse $\angle BOD = 2\angle BCD$

reflex $\angle BOD = 2\angle BAD$

obtuse $\angle BOD$ + reflex $\angle BOD = 360°$ (angles at a point)

So: $2\angle BAD + 2\angle BCD = 360°$

Rearranging and simplifying: $\angle BAD = 180° - \angle BCD$

e **i** $x = 142°$, $y = 99°$

ii $a = 45°$, $y = 38°$

iii $x = 30°$, $y = 22.5°$

10 **a** $\angle ADC + \angle ABC = 180^\circ$ (opposite angles of a cyclic quadrilateral)

So: $\angle ADC = 180^\circ - \angle ABC$

b supplementary angles on a straight line

c both are equal to the same expression ($180^\circ - \angle ABC$)

d **i** $a = 61^\circ$

ii $a = 81^\circ$, $b = 92^\circ$, $c = 99^\circ$, $d = 88^\circ$

iii $x = 15^\circ$, $z = 29.75^\circ$

11 Teacher to check.

12 **a** $\angle DBC = 66^\circ$

b $\angle DEG = 19^\circ$

c $\angle ODC = 38^\circ$

d $\angle OAB = 62^\circ$

EX
p364

7F Circle geometry: chords

1 **a** 101° **b** 38° **c** 58°

2 **a** 13 cm **b** 7.5 cm **c** 3.7 cm

3 **a** $x = 11$ cm

b $x = 4$ cm

c $x = 6$ cm, $y = 3.5$ cm

4 **a** 2.8 cm **b** 3.5 cm **c** 7.6 cm

5 **a** $x = 2$ cm, $y = 13$ cm

b $x = 18$ cm

c $x = 2$ cm

d $x = 4$ cm, $y = 21$ cm

e $x = 17^\circ$

f $x = 13.07$ cm

g $x = 4$ cm

h $x = 2.75$ cm, $y = 4$ cm

i $x = 19^\circ$

6 **a** $x = 4.33$ cm, $y = 8$ cm

b $x = 2.67$ cm, $y = 4$ cm

c $x = 10^\circ$

d $x = 3$ cm, $y = 5$ cm

e $x = 35.5^\circ$

f $x = 2$ cm, $y = 4$ cm

7 **a** $x = 19^\circ$

b $x = 12^\circ$

c $x = 23.8^\circ$, $y = 15.25^\circ$

d $x = 5.66$ cm, $y = 4$ cm, $z = 1$ cm

e $x = 18^\circ$, $y = 18^\circ$

f x can take on any positive value, $y = 0.6$

8 **a** They are all equal in length.

b isosceles triangles

c $OA = OC$ (radii)

$OB = OD$ (radii)

$AB = CD$ (given)

So: $\Delta OAB \cong \Delta OCD$ (using SSS)

d It follows that $\angle COD = \angle AOB$, because they are corresponding angles in congruent triangles.

9 **a** They are radii of the circle.

b Since $\angle OCA$ is given as 90°, $\angle OCB$ is also equal to 90°, because they are supplementary angles on the chord AB.

c $OA = OB$ (radii)

$\angle OCA = \angle OCB = 90^\circ$ (supplementary angles)

OC is a side common to both triangles.

So: $\Delta ACO \cong \Delta BCO$ (using RHS)

d AC and BC are corresponding sides in congruent triangles.

10 **a** They are all radii of the circle.

b Since OG is perpendicular to AB, it bisects the chord.

So: $AG =$ GB.

Similarly, since OH is perpendicular to chord CD:

$CH = HD$

c If chords AB and CD are same length, and are both bisected by radii that are perpendicular to them, it follows that $AG = GB = CH = HD$.

d $OD = OB$ (radii)

$HD = GB$ (from **c**)

$\angle OHD = \angle OGB = 90^\circ$

So: $\Delta OHD \cong \Delta OGB$ (using RHS)

e $OH = OG$ because they are corresponding sides of congruent triangles.

f In a right-angled triangle, the longest side is the hypotenuse (in this case, the radius). The other two sides are shorter in length than that. Comparing any other line segment from the centre to the chord to the line segment which is perpendicular to the chord will always result in a right-angled triangle with the perpendicular line shorter than the other line.

11 **a** They are vertically opposite angles

b They are angles at the circumference subtended by the same arc, DB.

c They are angles at the circumference subtended by the same arc, CA.

d The two triangles have three equal corresponding angles, AAA.

e In similar triangles, the ratio of the lengths of corresponding sides is constant:

$$\frac{AP}{CP} = \frac{BP}{DP}$$

Rearranging these ratios: $\frac{AP}{BP} = \frac{CP}{DP}$

or: $AP \times DP = CP \times BP$

12 Since the quadrilateral is cyclic, opposite angles are supplementary.

Since the quadrilateral is a parallelogram, opposite angles are equal.

If two angles are equal and supplementary, they must each be 90°.

With all angles equal to 90°, the quadrilateral must be a rectangle.

13 Teacher to check

14 Teacher to check

15 **a** 90° **b** 45° **c** 69°

16 **a** $x = 6$ cm

b $x = 2.75$ cm

c $x = 11.6$ cm, $y = 3$ cm

17 **a** 7.2 cm **b** 19.4 cm **c** 2 cm

18 **a** 14.1 cm **b** 2.7 cm **c** 4.5 cm

CHAPTER 7 review

Multiple-choice

1 C **2** B **3** E **4** A

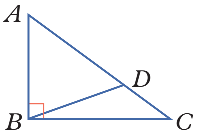
5 D **6** B **7** A **8** C

Short answer

1 **a** $a = 127°, b = 53°$

b $x = 105°$

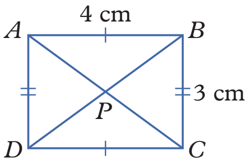
2 **a**



b

Statements	Reasons
$\angle ADB + \angle BDC = 180°$	Angles on a straight line sum to 180°
$\angle DBC + \angle BCD + \angle BDC = 180°$	Interior angle sum of a triangle is 180°
$\angle ADB + \angle BDC = \angle DBC + \angle BCD + \angle BDC$	Transitive law
$\angle ADB = \angle DBC + \angle BCD$ □	Subtraction property of equality

3 **a**



b They are alternate angles between parallel lines AB and CD.

c AAS **d** 53°

4 **a** AAA (alternate and vertically opposite angles)

b $\frac{2}{5}$

c 7.5 cm

5 **a**

Statements	Reasons
$\angle AEC = \angle ADE + \angle DAE$	Exterior angle of a triangle is equal to the sum of the non-adjacent interior angles
$\angle ADE = \angle DAE$	Angles opposite equal sides in an isosceles triangle are equal
$\therefore \angle AEC = \angle ADE + \angle ADE$ $= 2\angle ADE$ □	Substitution and rearrangement

b

Statements	Reasons
$\angle ABC = \angle BCE = 90°$	Given
$\angle ABC + \angle BCE = 180°$	Substitution
$\therefore \angle ADE + \angle BAD = 180°$	Interior angles of a quadrilateral sum to 360°
$\angle AEC = 2\angle ADE$	Given
$2\angle ADE + \angle BAE = 180°$ □	Substitution

6

Statements	Reasons
$PQ = RS$	Given
$\angle PQT = \angle RST$	Alternate angles on parallel lines are equal
$\angle PTQ = \angle RTS$	Vertically opposite angles are equal
$\therefore \Delta PQT \cong \Delta RST$	AAS (congruence)
$TQ = TS$	Corresponding sides in congruent triangles are equal
$TQ^2 + PT^2 = TS^2 + PT^2$	Transitive law
$PQ = PS$	Pythagoras' Theorem
$QR = PS$ and $SR = PQ$	Given
$QR = SR = SP = QP$	Transitive law
$\therefore PQRS$ is a rhombus □	A quadrilateral with four equal sides is a rhombus

7 **a** 54° **b** 27° **c** 63°

8

Statements	Reasons
$\angle RPQ = \angle RPO + \angle QPO$	Angle addition postulate
$\angle ROQ = 2\angle RPQ$	Circle theorem 1
$\therefore \angle ROQ = 2\angle RPO + 2\angle QPO$	Substitution
$\angle ROQ = 360° - \angle POR - \angle POQ$	Angles at a point sum to 360°
$\therefore 2\angle RPO + 2\angle QPO = 360° - \angle POR - \angle POQ$	Transitive law
$2\angle RPO + 2\angle QPO + \angle POR + \angle POQ = 360°$ □	Rearranging

9 **a** 46°

b

Statements	Reasons
$\angle DAC = \angle DBC$	Circle theorem 6
$\angle ACD = \angle ABD$	Circle theorem 6
$\angle DAC = \angle ACD = \angle ABD$	Angles opposite equal sides in an isosceles triangle are equal
$\therefore \angle DBC = \angle ABD$ □	Transitive law

Analysis

1 a

Statements	**Reasons**
$\angle DGC = 90^\circ$	Interior angles of quadrilateral *FDGC* add to 360°
$\angle DFC = \angle DGC = 90^\circ$	Transitive law
$FC \parallel DG$	If co-interior angles are supplementary, the lines must be parallel
$\angle FCD = \angle CDG$	Alternate angles on parallel lines are equal
CD is common to both ΔCFD and ΔCGD	Common side
$\therefore \Delta CFD \cong \Delta CGD$ □	AAS (congruence)

b

Statements	**Reasons**
$\angle CAB = \angle CFD = 90^\circ$	Given
$\angle CBA = \angle CDF$	The angle sum of triangles is 180°
$\angle ACB = \angle FCD$	Common to both triangles
$\therefore \Delta CAB \sim \Delta CFD$ □	AAA (similar)

c

Statements	**Reasons**
$\angle CAB = \angle DEB = 90^\circ$	Given
$\angle CBA = \angle DBE$	Vertically opposite angles are equal
$\angle ACB = \angle BDE$	The angle sum of triangles is 180°
$\therefore \Delta CAB \sim \Delta DEB$ □	AAA (similar)

d

Statements	**Reasons**
$AC = DE$	Given
$\angle ACB = \angle BDE$	Given
$\angle CBA = \angle DBE$	Given
$\therefore \Delta CAB \cong \Delta DEB$ □	AAS (congruence)
$CB = BD$	Corresponding sides in congruent triangles are equal
$\therefore$ *B* is the midpoint of *CD* □	Definition of midpoint (divides a line segment into two congruent pieces)

e

Statements	**Reasons**
$\frac{FC}{AC} = 3$	Given
$FC = FA + AC$	They make up the same line
$FC = 3AC$	Rearranging equation
$FA + AC = 3AC$	Transitive law
$FA = 2AC$	Rearranging the equation
$\angle AFD = \angle FDE = \angle DEA = \angle EAF = 90^\circ$	Given
$\therefore$ *AFDE* is a rectangle	A quadrilateral with four right interior angles is a rectangle
$FA = DE$	The opposite sides of a rectangle are equal
$DE = 2AC$	Substitution
$\frac{DE}{AC} = 2$ □	Rearrangement

f

Statements	**Reasons**
$\angle DBE = 45^\circ$	Given
$\angle DBE + \angle BDE + \angle DEB = 180^\circ$	The interior angle sum of a triangle is 180°
$\angle BDE = 45^\circ$	Substitution and solving equation
$\angle CDG = \angle BDE$	Given
$\angle FCD = \angle CDG$	Alternate angles on parallel lines are equal
$\angle DCG = 45^\circ$	The interior angle sum of ΔCGD is 180°
ΔCGD is isosceles with $CG = DG$.	If two angles in a triangle are equal, then the sides opposite the equal angles will be equal
$DG = CG$	Equal sides of an isosceles triangle
$CG = FD$	Opposite sides of rectangle *FDGC*
$CG = FD$	Opposite sides of rectangle *FDGC*
$FC = FD = DG = GC$	Sides of a triangle opposite equal angles are equal
FDGC is a rectangle □	A rectangle with equal sides must be a square
$\angle CFD = \angle FDG = \angle CGD = \angle GCF = 90^\circ$	Given
$\therefore$ *FDGC* is a square □	A quadrilateral with four equal side lengths and four right angles is a square

2 a The opposite angles are supplementary:
85° + 95° = 180° and 116° + 64° = 180°.

b

Statements	**Reasons**
ABCD is a rectangle.	Given
$\angle ABC = \angle BCD = \angle CDA = \angle DAB = 90^\circ$	Interior angles of a rectangle are 90°
$\angle ABC$ and $\angle CDA$ are supplementary.	Supplementary angles sum to 180°
$\angle BCD$ and $\angle DAB$ are supplementary.	Supplementary angles sum to 180°
$\therefore$ *ABCD* is a cyclic quadrilateral. □	If the opposite angles in a quadrilateral are supplementary, the quadrilateral is cyclic

c

Statements	**Reasons**
$AB \parallel DC$	Given
$\angle CDA = \angle BCD$	Given
$\angle ABC + \angle BCD = 180°$	Co-interior angles on parallel lines are supplementary
$\angle ABC + \angle CDA = 180°$	Substitution
$\angle DAB + \angle CDA = 180°$	Co-interior angles on parallel lines are supplementary
$\angle DAB + \angle BCD = 180°$	Substitution
$\therefore ABCD$ is a cyclic quadrilateral. □	If the opposite angles in a quadrilateral are supplementary, the quadrilateral is cyclic

d Two of the edges of these quadrilaterals are radii of the circle, not chords.

e Each pair of equal pronumerals are angles in the same segment subtended by the same chord.

f $p = 41°, q = 41, r = 62°, t = 62°, w = 23°, y = 54°, z = 23°$

g $df = eg$

h

Statements	**Reasons**
$\angle BAD = \angle BCD = 180°$	Given
$\angle ABC + \angle ADC + \angle BAD + \angle BCD = 360°$	Angle sum of a quadrilateral = 360°
$\angle ABC + \angle ADC + 180° = 360°$	Substitution
$\angle ABC + \angle ADC = +180°$	Simplifying
$ABCD$ is a cyclic quadrilateral.	If the opposite angles in a quadrilateral are supplementary, the quadrilateral is cyclic

CHAPTER 8 Pythagoras' Theorem and trigonometry

EX p376

8A Pythagoras' Theorem

1 **a** right-angled **b** right-angled **c** not right-angled
d right-angled **e** not right-angled
f not right-angled

2 **a** 3.6 cm **b** 7.1 mm **c** 33.8 cm
d 7.3 cm **e** 2.0 cm **f** 75.0 mm
g 8.7 cm **h** 4.2 cm **i** 8.1 cm

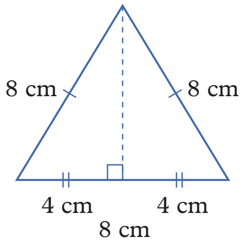
3 **a** $10\sqrt{5}$ cm **b** $4\sqrt{13}$ mm **c** $5\sqrt{10}$ cm
d $3\sqrt{2}$ mm **e** $2\sqrt{10}$ cm **f** $4\sqrt{17}$ mm

4 **a** 22.4 cm **b** 14.4 cm **c** 15.8 cm
d 4.2 mm **e** 6.3 cm **f** 16.5 mm

5 **a** $5\sqrt{2}$ cm **b** $20\sqrt{2}$ mm **c** $8\sqrt{2}$ cm
d $25\sqrt{2}$ cm

6 **a** 7.1 cm **b** 28.3 mm **c** 11.3 cm
d 35.4 cm

7 **a**



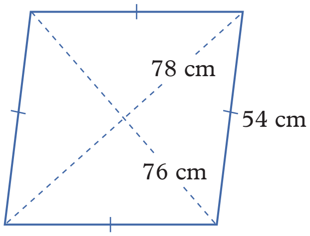
b **i** $4\sqrt{3}$ cm **ii** 6.9 cm

8 **a** $10\sqrt{17}$ cm **b** $(3\sqrt{11} + \sqrt{31})$ cm
c $(4\sqrt{14} - \sqrt{21})$ cm **d** $(4\sqrt{21} + \sqrt{17})$ cm
e $8\sqrt{21}$ cm **f** 13

9 **a** 41.2 cm **b** 15.5 cm **c** 10.4 cm
d 22.5 cm **e** 36.7 cm **f** 18.0 cm

10 **a** $78^2 > 54^2 + 54^2$, so the frame is not truly square. The two sides meet at an obtuse angle.

b



11 **a** 11 cm **b** 2 folds **c** 2 folds

12 **a** (1) Afra used Pythagoras' Theorem on the left right-angled triangle. (2) Afra used Pythagoras' Theorem on the right right-angled triangle. Afra showed the 6 cm edge is the sum of the lengths of a and b.

b **i** $h^2 = 13 - a^2$ **ii** $b = 6 - a$

c

$$\begin{aligned} b^2 + h^2 &= 25 \\ (6 - a)^2 + (13 - a^2) &= 25 \\ 36 - 12a + a^2 + 13 - a^2 &= 25 \\ 49 - 12a &= 25 \end{aligned}$$

d $a = 2$ cm

e $b = 4$ cm, $h = 3$ cm

f $h = 1$ cm ($a = 2, b = 5$)

13 **a** All three sets of numbers (**i–iii**) satisfy Pythagoras' Theorem.

b All three sets of numbers (**i–iii**) satisfy Pythagoras' Theorem so all three triangles described are right-angled triangles.

c If the three numbers of a Pythagorean triad are multiplied or divided by the same number, the resulting numbers are also a Pythagorean triad.

d Each side is multiplied by the same scale factor, so the corresponding sides are in the same ratio.

14 **a** $x^2 + y^2$ will be the hypotenuse; yes, that is always the case because $x^2 + y^2$ will always be greater than $x^2 - y^2$ and $2xy$ (because $x^2 - 2xy + y^2 > 0$ in this case; $x^2 - 2xy + y^2 = 0$ implies $x = y$ and a side of length zero)

b Answers will vary. Any values can be chosen as long as $x > y$, for example $x = 3, y = 2$.

c Answers will vary; one possible answer is, if $x = 3$ and $y = 2$:

i 5 **ii** 12 **iii** 13

d Using Pythagoras' Theorem to verify the numbers from the answer for part **c**, as an example: $13^2 = 5^2 + 12^2 = 169$.

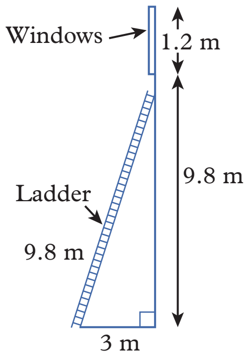
e The restriction ensures that the number that results from $x^2 - y^2$ is positive.

g $x = 2, y = 1$

15 a $5\sqrt{5}$ cm, $10\sqrt{5}$ cm

b 11.2 cm, 22.4 cm

16 a



b 9.3 m

c The top of the window is 9.8 + 1.2 = 11 m above the ground. If Josh stands on the very top of the ladder he is the same height above the ground as the top of the window (9.3 m + 1.7 m = 11 m). He will be able to clean the windows.

17 16.8 m

18 a $y = -\frac{1}{2}x, y = \frac{1}{3}x + \frac{5}{3}, y = 7x + 15$

b $(-2, 1)$

c $2\sqrt{5}$ units, $\sqrt{10}$, units, $5\sqrt{2}$ units

19 $x = 4\sqrt{2}$ cm, $y = 9$ cm, $z = 3\sqrt{91}$ cm

8B Trigonometry

EX p383

1 a $p = 5.3$ cm, $q = 4.3$ cm

b $t = 6.3$ mm, $s = 13.6$ mm

c $x = 18.2$ cm, $a = 16.1$ cm

d $k = 41.2$ mm, $m = 26.0$ mm

e $r = 8.6$ mm, $k = 9.2$ mm

f $t = 6.8$ cm, $s = 4.6$ cm

2 a 22° **b** 32°

c 67° **d** 6°

e 49° **f** 76°

g 89° **h** 1°

i 39° **j** 55°

k 54° **l** 18°

m 38° **n** 69°

o 81° **p** 50°

q 64° **r** 1°

3 a $B = 66°, C = 24°$

b $T = 29°, P = 61°$

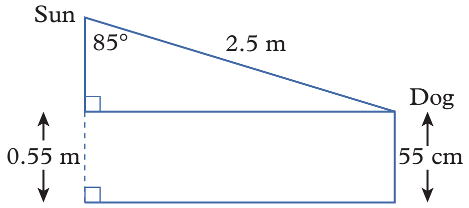
c $G = 32°, H = 59°$

d $K = 29°, M = 61°$

e $D = 27°, R = 63°$

f $F = 40°, D = 50°$

4 a

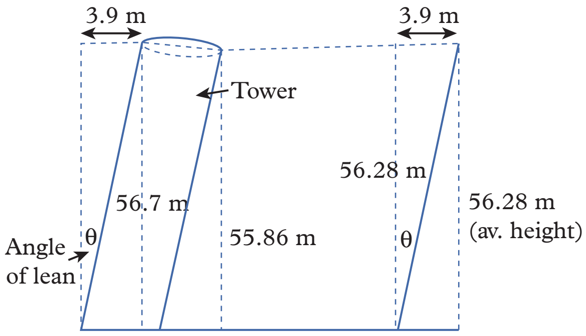


b 249 cm **c** 77 cm

5 $l = 230$ mm, $w = 76$ mm

6 a 56.28 m

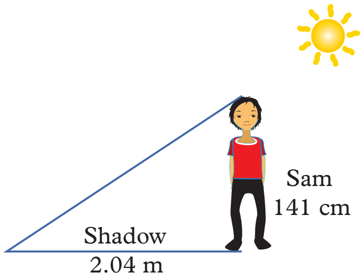
b



c $\tan(\theta) = \dfrac{\text{opposite}}{\text{adjacent}}$

d 4°

7 a

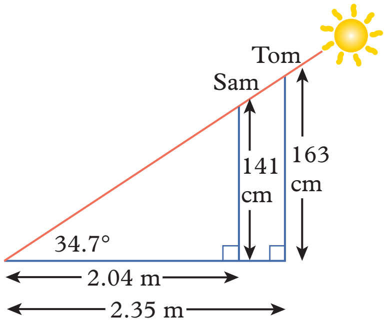


b Tom 34.7°, Sam 34.7°. The base angle represents the angle of elevation of the sun from the ground at the given time.

c i If the two triangles have the same base angle and a right angle then all three corresponding angles are equal, meaning the triangles are similar.

ii 1.16 ≈ 1.2

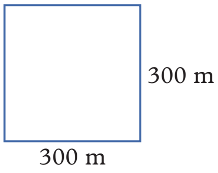
iii



8 **a** Ji-ho has written the reciprocals of the ratios (hypotenuse on the numerator, opposite on the denominator), such that, for their answers, the hypotenuse is not the longest side in the first two and the longer of the perpendicular sides is opposite the smaller angle in the third.

b Ensure that the hypotenuse is the longest side of the triangle. If the hypotenuse is not given, that the longer of the perpendicular sides is opposite the larger angle.

9 **a**

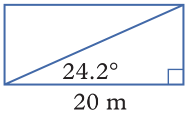


b 424.3 m

c 424.3 m

d The answers are identical.

10 **a**



b 8.99 m ≈ 9 m **c** 58 m

11 **a** 1.7 m **b** $l = 100$ cm, $w = 80$ cm

12 **a** For Route 1 **i** 1:2 **ii** 50%

For Route 2 **i** 5:12 **ii** 41.7%

For Route 3 **i** 13:30 **ii** 43.3%

b Route 1, 26.6°; Route 2, 22.6°; Route 3, 23.4°

c Route 2, Route 3, Route 1

13 **a** **i** 0 **ii** It is 0 : 1.

b **i** a is equal in length to c. **ii** 1

c The sine increases from 0 : 1 at 0° to 1 : 1 at 90°.

d The cosine ratio has a value of 1 : 1 at 0°, becoming 0 : 1 at 90°.

e The tangent is 0 : 1 at 0°, becoming infinitely large as the angle approaches 90°.

14 The sine ratio has a minimum value of 0 and a maximum value of 1.

The cosine ratio has a minimum value of 0 and a maximum value of 1.

The tangent ratio has a minimum value of 0 and a maximum value of ∞.

16 20.78 cm

17 **a** **i** $x = 40$ cm, $y = 34.64$ cm

ii $x = 80.52$ cm, $y = 59.69$ cm

b **i** 692.8 cm^2

ii 2403.3 cm^2

18 **a** 67.5° **b** octagon **c** 80 cm

19 **a** $x = 13.16$ cm

b $x = 46.24°$

c $x = 19.66$ cm

d $x = 145.32°$

EX
p391

8C Applications of trigonometry

1 **a** $x = 117°$, $y = 63°$

b $x = 58°$, $y = 302°$

c $x = 45°$, $y = 45°$

d $x = 19°$, $y = 60°$, $z = 281°$

e $v = 153°$, $w = 57°$, $x = 150°$, $y = 30°$, $z = 282°$

f $v = 118°$, $w = 99°$, $x = 40.5°$, $y = 37°$, $z = 282.5°$

2 **a** $w = 95.7$ m **b** $x = 27.0$ m **c** $z = 148.2$ m

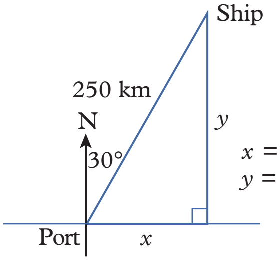
d $p = 254.3$ m **e** $s = 121.7$ m **f** $y = 235.2$ m

3 **a** 25 m **b** 144 m

4 **a** **i** 150°T **ii** 308°T **iii** 005°T

b **i** S20°W **ii** S80°E **iii** N60°W

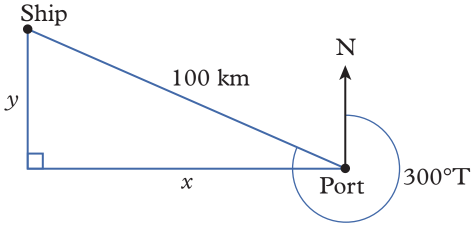
5 **a** **i**



x = distance travelled east
y = distance travelled north

ii 125 km north, 217 km east

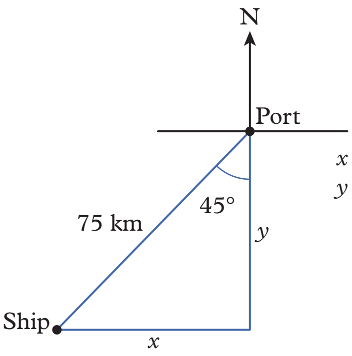
b **i**



x = distance travelled west
y = distance travelled north

ii 87 km west, 50 km north

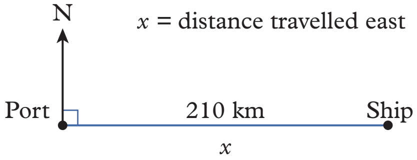
c **i**



x = distance travelled west
y = distance travelled south

ii 53 km west, 53 km south

d **i**



x = distance travelled east

ii 210 km east

6 **a** **i** 250 km on a bearing of S30°W

ii 250 km on a bearing of 210°T

b **i** 100 km on a bearing of S60°E

ii 100 km on a bearing of 120°T

c **i** 75 km on a bearing of N45°E

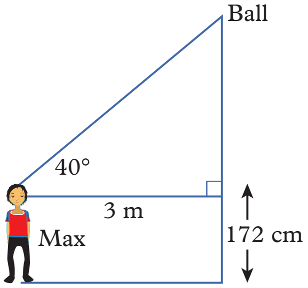
ii 75 km on a bearing of 045°T

d **i** 90 km on a bearing west

ii 90 km on a bearing of 270°T

7 **a** 10.39 m **b** 0.4 m/s

8 **a**



b 2.52 m

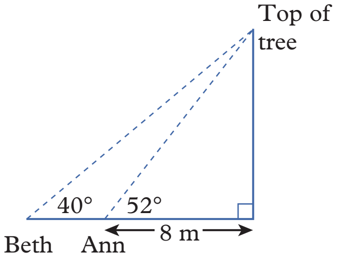
c 4.24 m

d The ball will always be above the person's horizontal line of sight and this affects the length of the sides of the triangles in the calculation.

9 The angle of elevation and the angle of depression cannot be obtuse, because they are always measured from the line of sight pointing towards the other point. An obtuse angle would point backwards and therefore be measured facing the opposite direction.

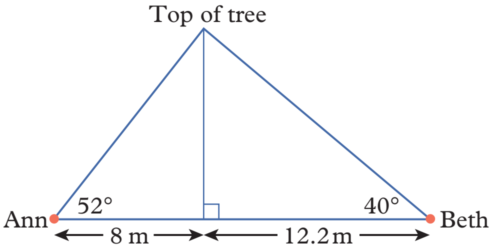
10 **a** 25.87 m **b** 1.17 m

11 **a**



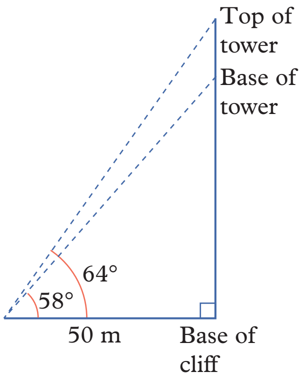
b 10.2 m **c** 12.2 m **d** 4.2 m

e **i**



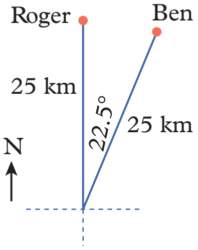
ii 20.2 m

12 **a**



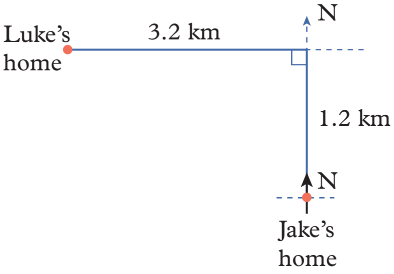
b 22.5 m

13 **a**



b 23.1 km **c** 9.6 km **d** 9.8 km

14 **a**



b 3.4 km **c** 291°T **d** 111°T

e The bearing and its back-bearing differ by 180°.

f **i** 3.4 km

ii The bearing from Jake's house to Luke's house is 291°T.

The bearing from Luke's house to Jake's house is 111°T.

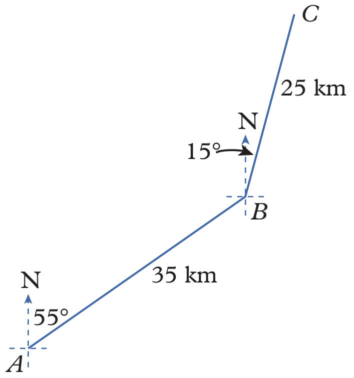
g The answers are the same.

15 **a** 60° **b** 45° **c** 40° **d** 10°

e **i** The angles in a triangle add to 180°; subtracting the acute measured angle and the right angle from 180° gives the angle of elevation. Alternatively, the acute measured angle and the angle of elevation are complementary as they are the non-right angles in a right-angled triangle.

ii The acute measured angle is vertically opposite the other non-right angle in the right-angled triangle. Alternatively, the obtuse measured angle forms a straight angle with the other non-right angle, so they are supplementary and must add to 180°. Then find the angle of depression by subtracting the acute measured angle and the right angle from 180° (or by finding the complement of the acute measured angle).

16 a



b **i** 20.1 km **ii** 24.1 km **iii** 44.2 km

c **i** 28.7 km **ii** 6.5 km **iii** 35.2 km

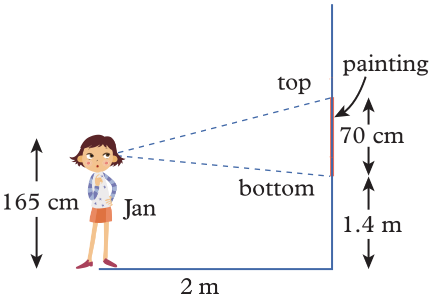
d **i** 56.5 km **ii** 039°T **iii** 219°T

e **i** 56 km, bearing of *C* from *A* is 039°T, bearing of *A* from *C* is 219°T

ii The answers are almost the same.

17 a 26.6 km **b** 062°T

18 a



b 12.7° **c** 7.1° **d** 19.8°

19 139 m

20 a 610 km

b **i** 26 km **ii** 12 km

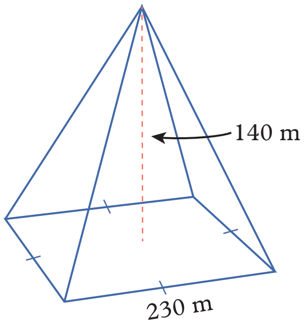
c Yes, total distance would be 3552 km, total fuel was enough to fly 3560 km.

EX p397

8D Three-dimensional problems

1 a 21.21 cm **b** 25.98 cm **c** 35.26°

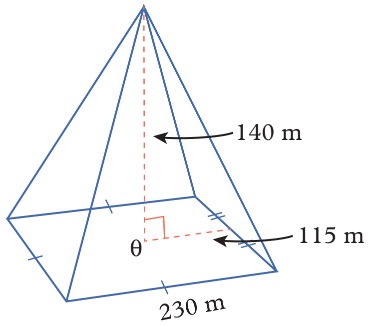
2 a



b 325.27 m

c 40°43′

d **i**



ii 50.60°

3 a diagonal of base (*d*):

$d = \sqrt{20^2 + 15^2}$

$d = \sqrt{625}$

$d = 25$ cm

diagonal of the box (*e*):

$e = \sqrt{10^2 + 25^2}$

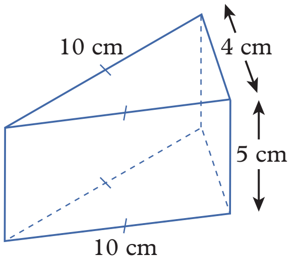
$e = \sqrt{725}$

$e = 26.93$ cm

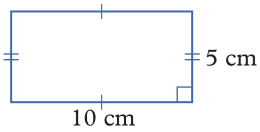
The length of the rod is greater than the length of the diagonal of the box, so a 30 cm rod will not fit into the box.

b 21.80°

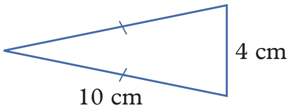
4 a



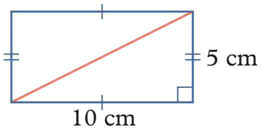
b **i**



ii



c



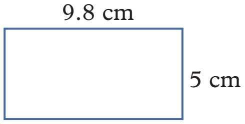
i 11.2 cm

ii 26.57°

iii The angles are complementary.
90° – 26.57° = 63.43°

d **i** 9.8 cm **ii** 19.6 cm^2

e **i**

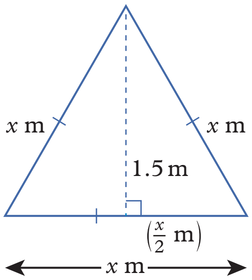


ii Cut side: $9.8 \times 5 = 49$ cm^2

Other side: $10 \times 5 = 50$ cm^2

iii 11 cm

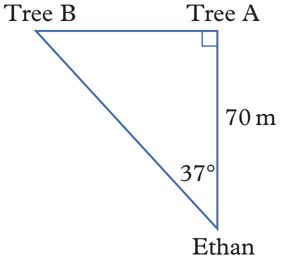
5 **a**



b 1.73 m

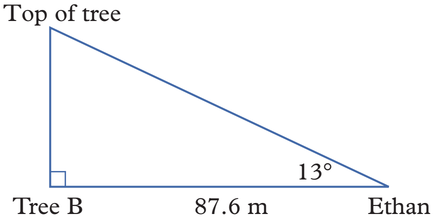
c There is sufficient material. Jack requires $2 \times 1.73 = 3.46$ m of material to cover the tent.

6 **a**



b 87.6 m

c



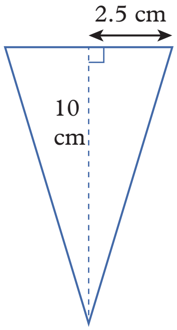
d 20.2 m

7 **a** 19.1 cm

b Internal diagonal = 22.6 cm. The spoon could fall in, because the internal diagonal of the saucepan is longer than the spoon.

8 **a** 2.5 cm

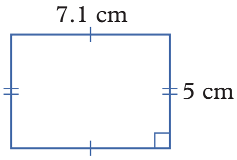
b



c 10.3 cm

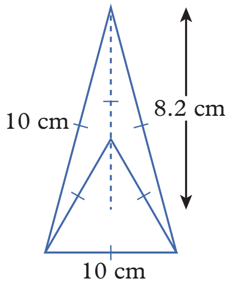
9 9 streamers equalling the full slant length of the tree

10 **a** rectangle:

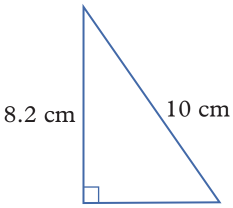


b Area of exposed face = 35.5 cm^2; area of cube face = 25 cm^2. So the area of the exposed face is 1.4 times greater than the area of one of the cube's faces.

11 **a, b**

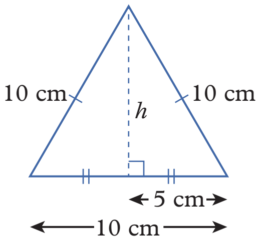


c **i**



ii 55°

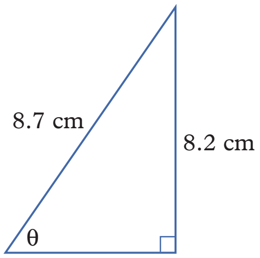
d



h = height of face

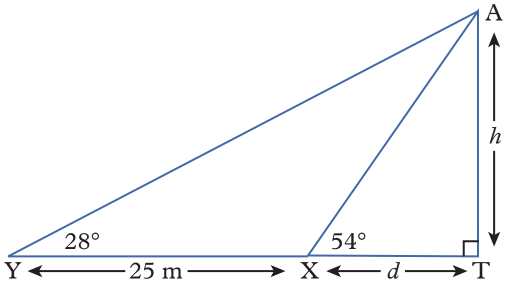
i 8.7 cm

ii



The angle (θ) between the base and a face of the tetrahedron is 70°.

12 **a**



b $h = d \times \tan(54°)$

c $h = (25 + d) \times \tan(28°)$

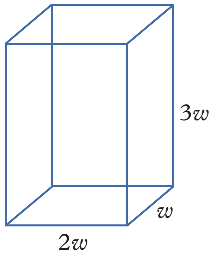
d 15.7 m **e** 21.7 m

13 **a** $\sqrt{w^2 + l^2}, \sqrt{l^2 + h^2}, \sqrt{w^2 + h^2}$

b $\sqrt{w^2 + l^2 + h^2}$

14 **a** $l = 2w$; $h = 3w$

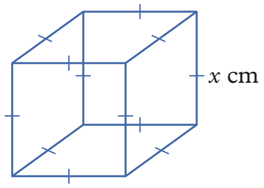
b



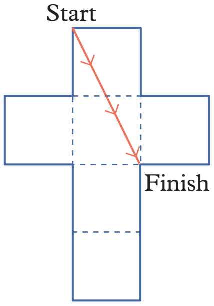
c $\sqrt{5}w$ **d** $\sqrt{14}w$ **e** 53°

15 3.6 cm

16 **a**



b **i**



ii $\sqrt{5}x$ cm

17 **a** 29.47 m **b** 39.69°

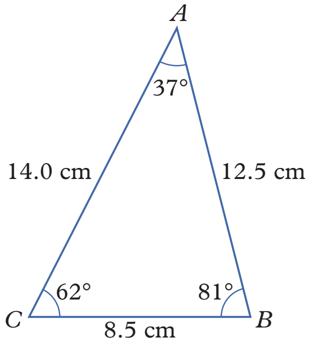
EX
p406

8E The sine and area rules

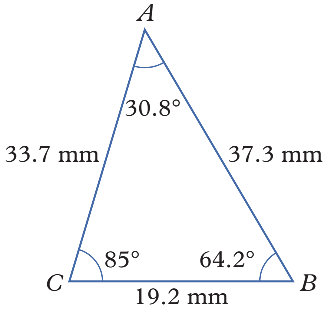
1 **a** 0.5 **b** 0.5 **c** 0.6428
d 0.6428 **e** 0.9848 **f** 0.9848
g 0.3746 **h** 0.3746

2 **a** 13.4 mm **b** 54.3° **c** 124.3°

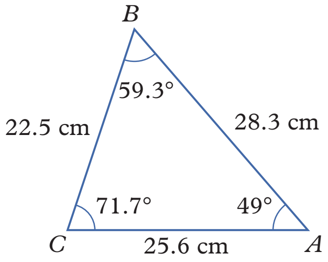
3 **a** $b = 14.0$ cm, $C = 62°$, $c = 12.5$ cm



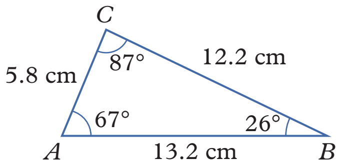
b $B = 64.2°$, $A = 30.8°$, $a = 19.2$ mm



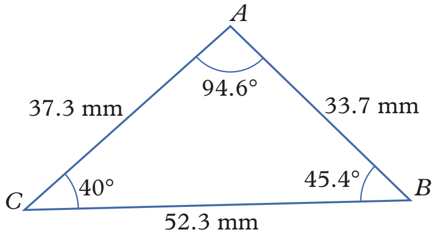
c $C = 71.7°$, $B = 59.3°$, $b = 25.6$ cm

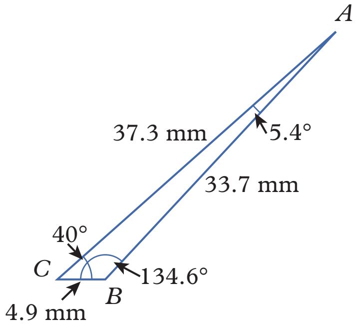


d $C = 87.0°$, $a = 12.2$ cm, $c = 13.2$ cm

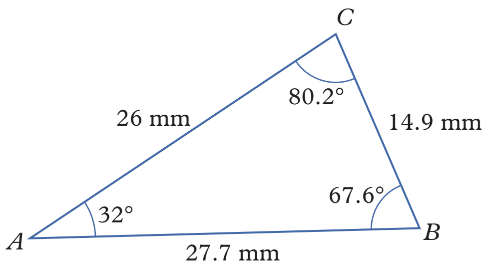


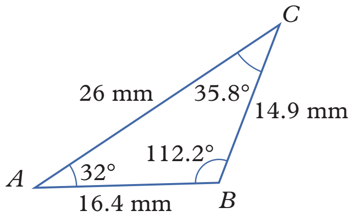
4 **a** $B = 45.4°$, $A = 94.6°$, $a = 52.3$ mm or $B = 134.6°$, $A = 5.4°$, $a = 4.9$ mm



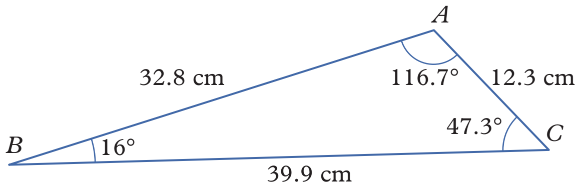


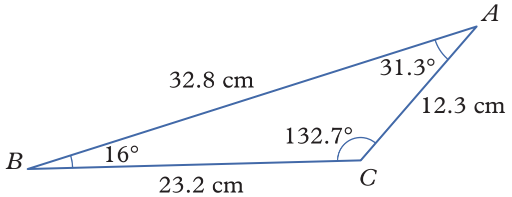
b $B = 67.6°, C = 80.4°, c = 27.7$ mm or $B = 112.4°$, $C = 35.6°, c = 16.4$ mm



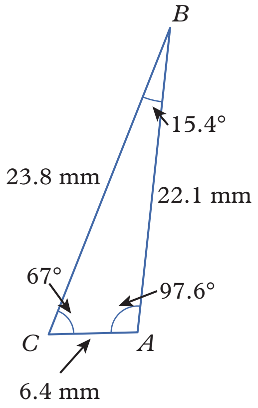


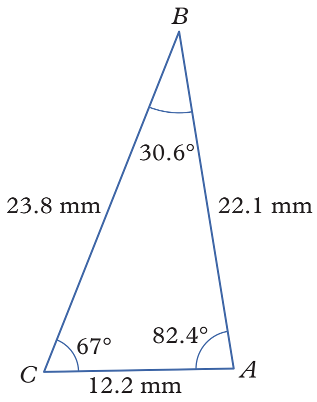
c $C = 47.3°, A = 116.7°, a = 39.9$ cm or $C = 132.7°$, $A = 31.3°, a = 23.2$ cm





d $A = 97.6°, B = 15.4°, b = 6.4$ mm or $A = 82.4°$, $B = 30.6°, b = 12.2$ mm





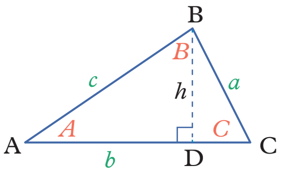
5 **a** 16.9 cm^2 **b** 12.4 mm^2 **c** 18.1 cm^2
d 39.2 cm^2

6 **a** 11.7 cm^2 **b** 9.7 cm^2 **c** 130.4 cm^2
d 86.5 cm^2

7 **a** **i** Yes, an angle-side pair is known (c-C).
ii Yes, as the side opposite the angle, c, is shorter than the other length, b.
b **i** No, an angle-side pair is not known.
ii N/A
c **i** Yes, an angle-side pair is known (b-B).
ii No, two angles are given so it is not ambiguous.
d **i** Yes, an angle-side pair is known (b-B).
ii No, the side opposite the angle, b, is not shorter than the other length, c.

8 **a** 22.4 cm **b** 61.1 cm **c** 24.7 mm
d 31.5 cm **e** 64.3 mm **f** 53.7 mm

9 **a, b**



c **i** $h = c\sin(A)$
ii $h = a\sin(C)$
iii $a\sin(C) = c\sin(A)$

$$a = \frac{c\sin(A)}{\sin(C)}$$

$$\frac{a}{\sin(A)} = \frac{c}{\sin(C)}$$

d $a\sin(B) = b\sin(A)$

$$a = \frac{b\sin(A)}{\sin(B)}$$

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$$

So, with what was found in part **c**, we know that:

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

10 **a** 1
b $\sin(B) = \frac{b}{a}, \sin(C) = \frac{c}{a}$
c $\frac{1}{a} = \frac{1}{a}$
d In the right-angled triangle, the sine rule simplified to $\frac{1}{a}$, where a is the length of the hypotenuse. Yes, the sine rule holds for right-angled triangles.

e If the right angle was at ∠*ABC*, the answer would be $\frac{1}{b}$.

If the right angle was at ∠*BCA*, the answer would be $\frac{1}{c}$.

11 **a** $\sin(C) = \frac{c}{a}$ **b** $A = \frac{1}{2}bc$

c *A* represents the area, *b* represents the base of the triangle, *c* represents the height of the triangle. The formula from part **b** is written in the same form as $A = \frac{1}{2}bh$, the formula for the area of any right-angled triangle.

12 **a** **i** 133° **ii** 20°

b 19.9 m **c** 14.6 m **d** 13.6 m

e 28.6 m

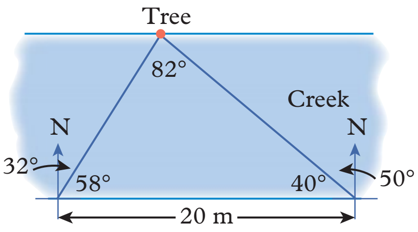
13 **a** 60° **b** 53° **c** 67°

d 28.6 km

e **i** N37°E **ii** S37°W **iii** west

f 310.0 km²

14 **a**

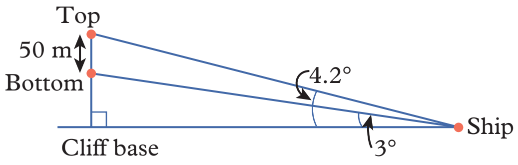


b **i** 13.0 m **ii** 17.1 m

c 11 m

15 693 cm²

16 **a**



b **i** 2381.1 m **ii** 2384.2 m

c 2.4 km **d** 125 m

17 **a** 142° T and 218° T

b 13.1 km and 28.5 km

c 50.8 km

18 **a** The shorter known side is opposite the known angle.

b 100.1 cm, 39.9 cm

c 998.3 cm², 398.1 cm²

d **i** 600.1 cm²

ii $\frac{1}{2} \times 36.1 \times 36.1 \times \sin(112.9°)$

$= \frac{1}{2} \times 36.1^2 \times \sin(112.9°)$

19 **a** $\text{area} = \frac{1}{2}bc\sin(A) = \frac{1}{2}ac\sin(B) = \frac{1}{2}ab\sin(C)$

b $\frac{1}{2}bc\sin(A) = \frac{1}{2}ac\sin(B) = \frac{1}{2}ab\sin(C)$

$$\frac{bc\sin(A)}{abc} = \frac{ac\sin(B)}{abc} = \frac{ab\sin(C)}{abc}$$

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

c $(\sin(60°))^2 = \frac{3}{4}$

d $\sin(60°) = \frac{\sqrt{3}}{2}$

e $\text{area} = \frac{\sqrt{3}}{4}l^2$

20 **a** **i** 60° **ii** 6.3 cm **iii** 17.2 cm² **iv** 103.1 cm²

b **i** 72° **ii** 11.8 cm **iii** 66.5 cm² **iv** 332.4 cm²

c **i** 45° **ii** 9.1 cm **iii** 29.6 cm² **iv** 236.6 cm²

d **i** 30° **ii** 7.0 cm **iii** 12.1 cm² **iv** 145.1 cm²

e $$A = \frac{l^2\left(\sin\left(\frac{90(n-2)}{n}\right)\right)^2}{2\sin\left(\frac{360}{n}\right)}$$

21 **a** 28.40 cm

b **i** 232.47 cm² **ii** 633.25 cm² **iii** 400.78 cm²

c **i** $x = 52.62$ cm, $A = 40.01°$, $B = 29.99°$, $y = 28.00$ cm

ii 1989.48 cm²

22 **a** 2.29 cm

b 2.29 cm where *B* is at the right angle as the perpendicular distance from the ray to the point is the shortest distance to the ray *AB*.

c 4 cm, where one of the two positions for *B* is no longer creating a triangle.

d $x\sin(\theta) < BC < x$

8 Checkpoint

1 **a** yes **b** yes **c** no **d** yes

2 **a** **i** $\sqrt{41}$ cm **ii** 6.40 cm

b **i** $6\sqrt{14}$ cm **ii** 22.45 cm

c **i** $10\sqrt{5}$ cm **ii** 22.36 cm

d **i** $20\sqrt{2}$ cm **ii** 28.28 cm

3 **a** $x = 12.0$ cm, $y = 6.0$ cm, $\theta = 26.6°$

b $x = 9.3$ cm, $y = 6.3$ cm, $\theta = 42.7°$

c $x = 17.4$ cm, $y = 14.1$ cm, $\theta = 35.9°$

4 **a** $x = 63.8°$, $y = 26.2°$, $z = 20.1$ cm

b $x = 50.9°$, $y = 39.1°$, $z = 24.6$ cm

5 **a** 095° T **b** 145° T **c** 293.55° T

6 **a** 28.2 m **b** 26.6° **c** 15.8 m

d 8.6 m

7 **a** 78.7 m east and 216.1 m south **b** 034°

8 **a** $x = 5$ cm, $y = 7.8$ cm

b $r = 5.3$ cm, $y = 9.8$ cm

9 **a** $x = 9.0$ cm, $z = 12.0$ cm

b $x = 63.4°$, $y = 22.4$ cm, $z = 12.6°$

10 **a** $x = 43°$, $y = 4.7$ cm, $z = 5.6$ cm

b $y = 23.5°$, $x = 37.5°$, $z = 11.0$ cm

c $x = 128°$, $y = 16.4$ cm, $z = 13.2$ cm

d $y = 30.2°$, $x = 84.8°$, $z = 8.9$ cm

11 **a** $y = 68.0°$, $x = 60.0°$, $z = 11.2$ cm or $y = 112.0°$, $x = 16.0°$, $z = 3.6$ cm

b $y = 85.5°$, $x = 65.5°$, $z = 16.7$ cm or $y = 94.5°$, $x = 56.5°$, $z = 15.3$ cm

12 **a** 126.61 cm^2 **b** 662.82 cm^2 **c** 253.03 cm^2
d 117.42 cm^2

EX
p416

8F The cosine rule

1 **a** 0.5 **b** −0.5 **c** 0.7660
d −0.7660 **e** 0.1736 **f** −0.1736
g 0.9272 **h** −0.9272

2 **a** 3.6 cm **b** 8.8 cm **c** 22.5 cm
d 13.7 cm **e** 56.2 mm **f** 44.9 cm
g 55.4 mm **h** 14.7 m **i** 15.5 cm

3 **a** $A = 47.2°, B = 78.1°, C = 54.6°$
b $A = 81.8°, B = 66.0°, C = 32.2°$
c $A = 40.7°, B = 87.9°, C = 51.5°$
d $A = 56.9°, B = 84.2°, C = 38.9°$

4 **a** 109.2° **b** 48.6°
c 38.7° **d** 94.9°

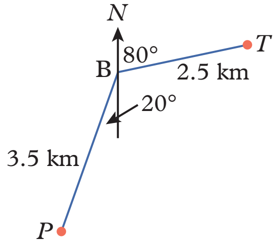
5 **a** $a = 18.5$ cm, $B = 94°$, $C = 38°$
b $e = 36.1$ mm, $D = 60°$, $F = 48°$
c $h = 48.2$ cm, $G = 53°$, $I = 68°$
d $j = 40.7$ m, $K = 50°$, $L = 67°$
e $n = 5.3$ m, $M = 41°$, $P = 60°$
f $s = 19.3$ mm, $Q = 51°$, $R = 60°$

6 **a** We would need to know the magnitude of a second angle, either B or C.
b 55.8 cm
c **i** Sample answer: use $\frac{\sin A}{a} = \frac{\sin B}{b}$ to find B. Subtract the sum of A and B from 180° to find C.
ii $B = 35°, C = 59°$
d $B = 35°, C = 59°$
e The answers are the same.

7 **a** $c^2 = a^2 + b^2 - 2ab\cos(C)$
b 0
c $c^2 = a^2 + b^2$; this is Pythagoras' Theorem.
d If C is an acute angle, then $0 < \cos(C) < 1$, so the adjustment term, $-2ab\cos(C)$, will be negative. Therefore, the adjustment term will decrease the sum of the squares of a and b, so $a^2 + b^2 - 2ab\cos(C) < a^2 + b^2$.
e If C is an obtuse angle, then $-1 < \cos(C) < 0$, so the adjustment term, $-2ab\cos(C)$, will be positive as the product of two negative numbers is positive. Therefore, the adjustment term will increase the sum of the squares of a and b, so: $a^2 + b^2 - 2ab\cos(C) > a^2 + b^2$.

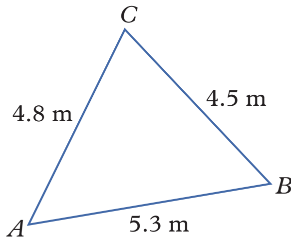
8 **a** $a^2 = h^2 + (b - x^2)$ **b** $h^2 = c^2 - x^2$
c $a^2 = c^2 - x^2 + (b - x)^2$
$a^2 = c^2 - x^2 + b^2 - 2bx + x^2$
$a^2 = b^2 + c^2 - 2bx$
d $x = c\cos(A)$
f $b^2 = a^2 + c^2 - 2ac\cos(B)$
$c^2 = a^2 + b^2 - 2ab\cos(C)$
g **i** $\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$
ii $\cos(B) = \frac{a^2 + c^2 - b^2}{2ac}$
iii $\cos(C) = \frac{a^2 + b^2 - c^2}{2ab}$
h **i** two side lengths and the angle between them
ii three side lengths

9 **a**



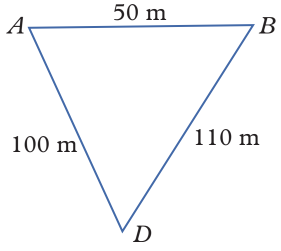
b 120° **c** 5.2 km **d** 25°
e N45°E **f** 35° **g** S45°W

10 **a**

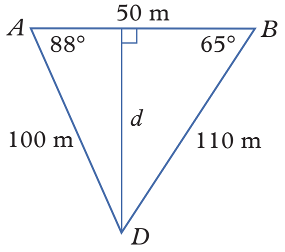


b C
c 69°
d The following formula can be used to calculate the area:
area $= \frac{1}{2}ab\sin(C)$.
$A = 10.1$ m^2

11 **a**

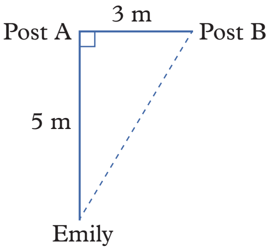


b 65° and 88°
c Use the sine ratio with either of the two right-angled triangles created.



d 99.9 m

12 a



b 5.8 m

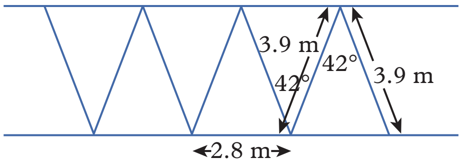
c 31°

d Emily has a greater chance of scoring a goal. She now has a larger angle of 35° within which she can score a goal.

13 a 3.9 m **b** 3.7 m

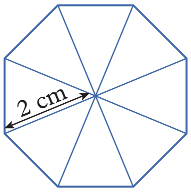
14 a 42°

b



c 3.6 m

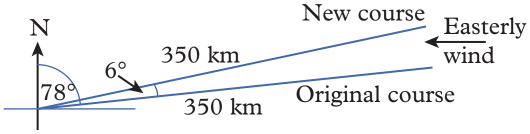
15 a



b i 45° **ii** 67.5°

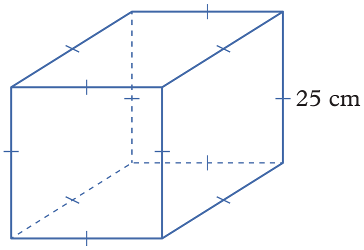
c 12.2 cm

16 a



b 36.6 km

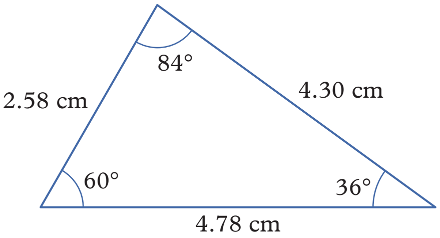
17 a



b $25\sqrt{2}$ cm **c** $25\sqrt{3}$ cm

18 a 36°, 60°, 84° **b** 4.78 cm

c



19 a $(38\sin(63°))^2 + (24 - 38\cos(63°)) = x^2$

$x = 34.5$ cm

b $(41\sin(116°))^2 + (17 - 41\cos(116°))^2 = x^2$

$x = 50.8$ cm

c $(a\sin(C))^2 + (b - a\cos(C))^2 = c^2$

$(a\sin(C))^2 + b^2 - 2b(a\cos(C)) + (a\cos(C))^2 = c^2$

$(a\sin(C))^2 + (a\cos(C))^2 + b^2 - 2ab\cos(C) = c^2$

$a^2 + b^2 - 2ab\cos(C)^2 = c^2$,

since $(a\sin(C))^2 + (a\cos(C))^2 = a^2$

20 a i, ii $\sqrt{2}r$ units

b Both methods give the same answer.

21 a $x^2 - 6\cos(64°)x - 16 = 0$

b $x = -2.9$, $x = 5.5$

c No, as one of the solutions, $x = -2.9$, is negative. Therefore, the unknown side length is 5.5 cm.

d $x = 2.3$, $x = 5.2$

e Yes, as both are positive values. This triangle has two possible lengths for the third side, 2.3 cm and 5.2 cm, as it is an ambiguous triangle as the shorter known side length is opposite the known angle.

f The sine rule would first need to determine the possible angles opposite the other known side length, then determine the corresponding third angle(s) in the triangle using the interior angle sum of a triangle, and then determine the possible lengths of the unknown side length separately. The cosine rule can directly find both possible lengths of the unknown side by solving a trinomial quadratic equation which is a harder process than using the sine rule multiple times. The other angles in the triangle can then be determined if needed.

g $11^2 = x^2 + x^2 - 2xx\cos(103°)$

$11^2 = 2x^2 - 2x^2\cos(103°)$

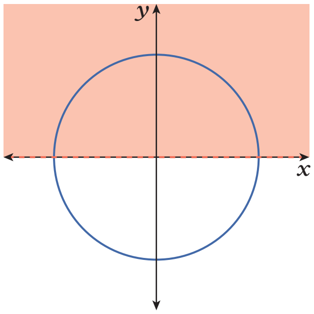
$x = 7.0$ cm since x must be positive

22 Area $= 5\frac{\sqrt{3}}{4} - \sqrt{3}\cos(\theta)$

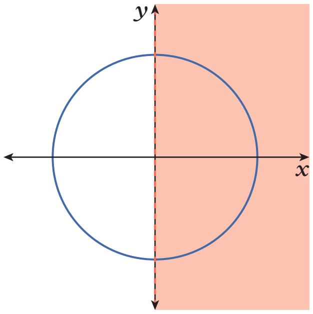
EX p426

8G The unit circle

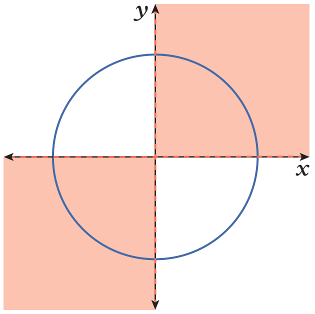
1 a



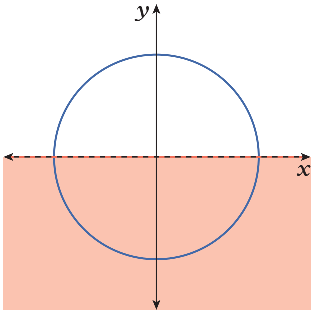
b



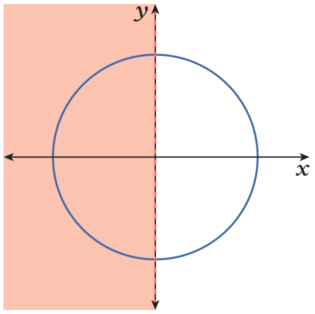
c



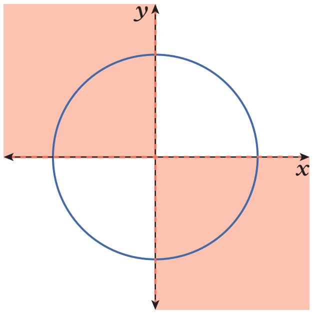
d



e



f



2 **a** quadrant 4 **b** quadrant 2 **c** quadrant 3
d quadrant 3 **e** quadrant 1 **f** quadrant 2

3 **a** positive **b** negative **c** negative
d positive **e** positive **f** negative
g negative **h** positive **i** negative
j negative

4 **a** 40° **b** 88° **c** 20°
d 10° **e** 58° **f** 45°
g 80° **h** 80° **i** 80°
j 30° **k** 15° **l** 10°
m 40° **n** 75° **o** 61°

5 **a** −sin(61°) **b** cos(56°) **c** −tan(85°)
d sin(16°) **e** −cos(70°) **f** −cos(20°)
g tan(20°) **h** −tan(2°) **i** −sin(75°)
j sin(5°)

6 **a** **i** 0 = sin(0°) **ii** 1 = cos(0°)
b **i** 1 = sin(90°) **ii** 0 = cos(90°)
c **i** 0 = sin(180°) **ii** −1 = cos(180°)
d **i** −1 = sin(270°) **ii** 0 = cos(270°)
e **i** 0 = sin(360°) **ii** 1 = cos(360°)

7 **a** **i** $\tan(\theta) = \frac{y}{x}$ **ii** $\tan(\theta) = t$

b The gradient of the radius is its rise over its run, $\frac{y}{x}$. The tangent of θ is also equal to this ratio; therefore, the tangent of the angle is equal to the gradient of the radius.

c $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$

d **i** tan(20°) **ii** tan(152°) **iii** $\dfrac{1}{\tan(218°)}$
iv tan(347°)

e **i** 0 **ii** undefined **iii** 0
iv undefined **v** 0

8 **a** **i** sin(40°) **ii** sin(40^o)
b **i** cos(170°) **ii** −cos(10°)
c **i** tan(80°) **ii** tan(80^o)
d **i** sin(270°) **ii** −1
e **i** cos(57°) **ii** cos(57^o)
f **i** cos(304°) **ii** cos(56°)
g **i** tan(200°) **ii** tan(20°)
h **i** cos(180°) **ii** −1

9 **a** **i** sin(315°) **ii** −sin(45°)
b **i** cos(135°) **ii** −cos(45°)
c **i** tan(225°) **ii** tan(45°)
d **i** sin(45°) **ii** sin(45^o)
e **i** cos(212°) **ii** −cos(32°)
f **i** tan(284°) **ii** −tan(76°)
g **i** cos(90°) **ii** 0
h **i** sin(80°) **ii** sin(80°)

10 **a** sin(158°) = sin(22°) = −sin(202°) = −sin(338°)
b cos(213°) = cos(147°) = −cos(33°) = −cos(327°)
c tan(285°) = tan(105°) = −tan(75°) = −tan(255°)

11 **a** −0.8090... = −0.8090..., −0.587 785... = −0.587 785..., 1.376 38... = 1.376 38...

b 360°

c The reference angle, when the angle is in the fourth quadrant, is calculated using 360°−θ°. The same has been applied to 234° as 360°−234°= 126°. The RHS of each equation uses the sign of the trigonometric function in the fourth quadrant to ensure the expressions are equal.

12 **a** **i** (2 sin(134°),2 cos sin(134°))
ii (2 sin(46°), −2 cos(46°))
b **i** (2 sin(230°),2 cos(230°))
ii (−2 sin(50°),−2 cos(50°))

c **i** $(6\sin(320^\circ), 6\cos(320^\circ))$

ii $(-6\sin(40^\circ), 6\cos(40^\circ))$

d **i** $(20\sin(100^\circ), 20\cos(100^\circ))$

ii $(20\sin(80^\circ), -20\cos(80^\circ))$

e **i** $(15\sin(343^\circ), 15\cos(343^\circ))$

ii $(-15\sin(17^\circ), 15\cos(17^\circ))$

f **i** $(75\sin(190^\circ), 75\cos(190^\circ))$

ii $(-75\sin(10^\circ), -75\cos(10^\circ))$

g The coordinates for bearings are $(r\sin(\theta^\circ), r\cos(\theta^\circ))$ whereas the coordinates for the unit circle are $(\cos(\theta^\circ), \sin(\theta^\circ))$. Both systems have positive values for sine and cosine in quadrant 1 and negative values in quadrant 3 but have the sign of sine and cosine swapped in quadrants 2 and 4 relative to the unit circle. For quadrants 2 and 4 of the bearing diagram the signs are unchanged. Quadrants 1 and 3 are the same for either method of measuring angles (but the angle sizes are different).

13 **a** $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$

b **i** $\sin(\theta) = \pm\sqrt{1 - (\cos(\theta))^2}$

ii $\cos(\theta) = \pm\sqrt{1 - (\sin(\theta))^2}$

c **i** $\cos(50^\circ) = 0.6428$

ii $\sin(116^\circ) = 0.8988$

iii $\cos(177^\circ) = -0.9986$

iv $\sin(240^\circ) = -0.8660$

14 **a** $\cos(60^\circ)$ **b** $\cos(38^\circ)$

c $\sin(66^\circ)$ **d** $\sin(45^\circ)$

e **i** $\cos(-68^\circ)$ **ii** $\cos(292^\circ)$ **iii** $\cos(68^\circ)$

f **i** $\sin(-261^\circ)$ **ii** $\sin(99^\circ)$ **iii** $\sin(81^\circ)$

g **i** $\cos(-121^\circ)$ **ii** $\cos(239^\circ)$ **iii** $-\cos(59^\circ)$

h **i** $\cos(105^\circ)$ **ii** n/a **iii** $-\cos(75^\circ)$

i **i** $\sin(-392^\circ)$ **ii** $\sin(328^\circ)$ **iii** $-\sin(32^\circ)$

j **i** $\sin(645^\circ)$ **ii** $\sin(285^\circ)$ **iii** $-\sin(75^\circ)$

EX p434

8H Exact values

1 **a** $\frac{19\pi}{18}$ **b** $\frac{13\pi}{15}$ **c** 72°

d 255° **e** $\frac{34\pi}{45}$ **f** 286°

2

	θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
a	240°	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$
b	300°	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
c	120°	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
d	315°	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
e	135°	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
f	150°	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$

3

	θ	**Reference angle** θ'	**sin** (θ')	**cos** (θ')	**tan** (θ')	**sin** (θ)	**cos** (θ)	**tan** (θ)
a i **ii** **iii**	$\frac{7\pi}{6}$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
b i **ii** **iii**	$\frac{3\pi}{4}$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
c i **ii** **iii**	$\frac{4\pi}{3}$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$

4

	θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
a	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
b	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
c	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
d	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1
e	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
f	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1

5 **a** $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ **b** $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ **c** $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

d $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ **e** $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ **f** $(0, -1)$

6 **a** $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ **b** $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ **c** $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

d $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ **e** $\left(\frac{\sqrt{2}}{2}, -\frac{1}{2}\right)$ **f** $(-1, 0)$

7 **a** $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ **b** $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ **c** $(0, 1)$

d $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ **e** $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ **f** $(1, 0)$

8 **a** **i** 2π **ii** π **iii** $\frac{\pi}{2}$

b **i** $0 < \theta < \frac{\pi}{2}$ **ii** $\frac{\pi}{2} < \theta < \pi$ **iii** $\pi < \theta < 2\pi$

c **i** Complementary angles (two angles that form a right angle) add to $\frac{\pi}{2}$ radians

ii Supplementary angles (two angles that form a straight angle) add to π radians

d **i** $\frac{\pi}{6}$ **ii** $\frac{\pi}{8}$ **iii** $\frac{\pi}{4}$ **iv** $\frac{2\pi}{5}$

e **i** $\frac{2\pi}{3}$ **ii** $\frac{\pi}{4}$ **iii** $\frac{5\pi}{6}$ **iv** $\frac{7\pi}{12}$

f **i** $\frac{5\pi}{3}$ **ii** $\frac{\pi}{6}$ **iii** $\frac{3\pi}{4}$ **iv** $\frac{3\pi}{5}$

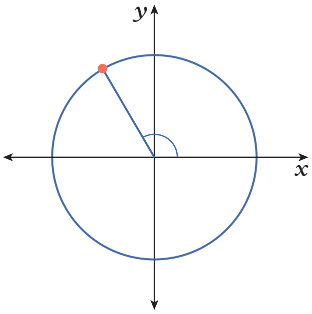
g **i** π **ii** 2π **iii** 4π **iv** $(n-2)\pi$

h **i** $\frac{\pi}{3}$ **ii** $\frac{\pi}{2}$ **iii** $\frac{2\pi}{3}$ **iv** $\frac{(n-2)\pi}{n}$

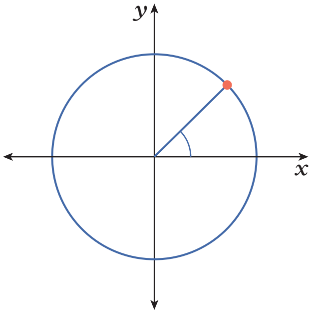
i **i** $\frac{7\pi}{3}$ **ii** $\frac{13\pi}{4}$ **iii** $\frac{23\pi}{6}$ **iv** $\frac{5\pi}{2}$ **v** 3π

j **i** $-\frac{5\pi}{3}$ **ii** $-\frac{3\pi}{4}$ **iii** $-\frac{\pi}{6}$ **iv** $-\frac{3\pi}{2}$ **v** $-\pi$

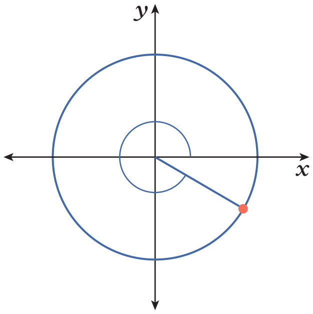
9 **a** **i** $\frac{1}{3}$



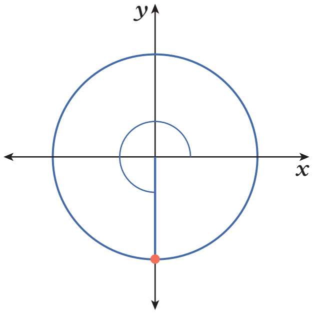
ii $\frac{1}{8}$



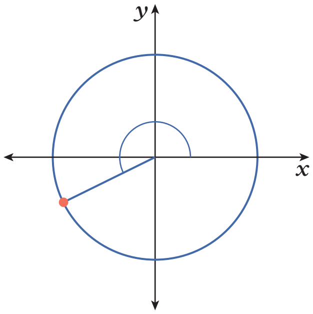
iii $\frac{11}{12}$



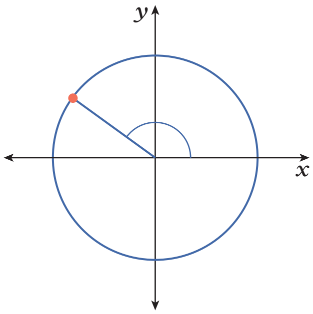
iv $\frac{3}{4}$



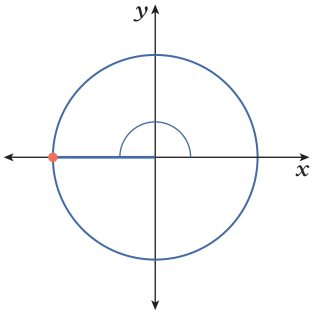
v $\frac{4}{7}$



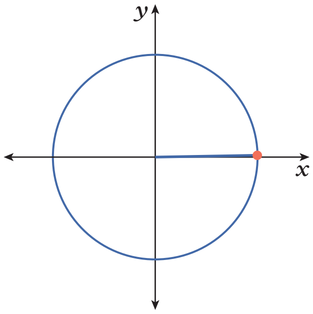
vi $\frac{2}{5}$



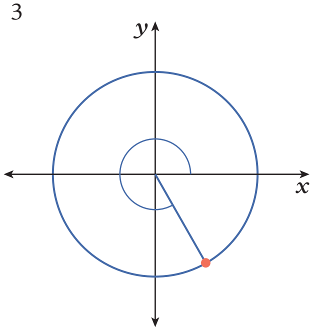
vii $\frac{1}{2}$



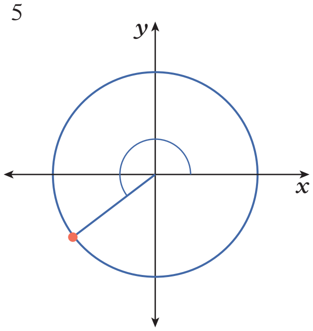
viii $\frac{1}{360}$



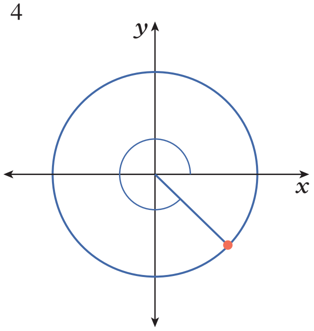
b **i** $\frac{5\pi}{3}$



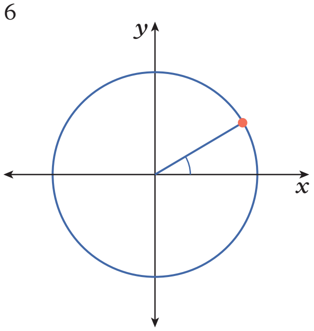
ii $\frac{6\pi}{5}$



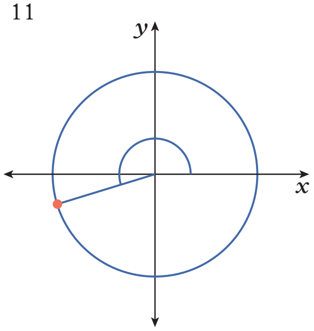
iii $\frac{7\pi}{4}$



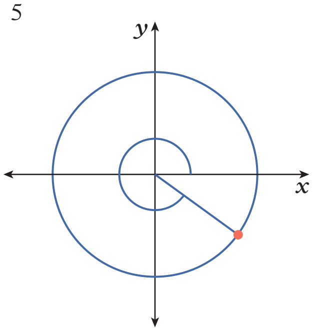
iv $\frac{\pi}{6}$



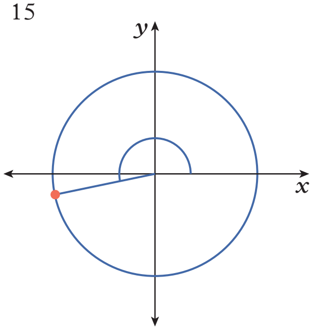
v $\frac{12\pi}{11}$



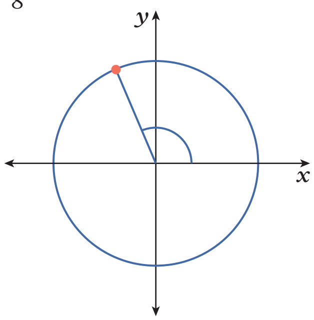
vi $\frac{9\pi}{5}$



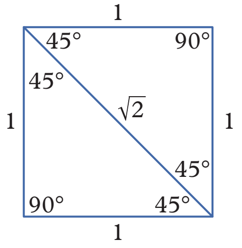
vii $\frac{16\pi}{15}$



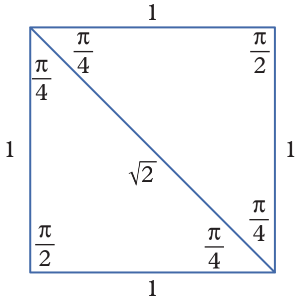
viii $\frac{5\pi}{8}$



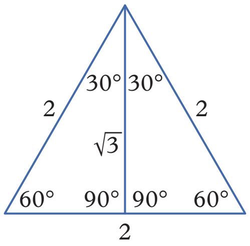
10 a, **b**



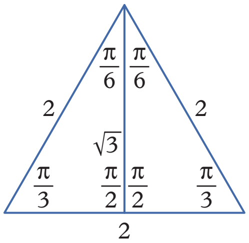
c



d, **e**



f



11 a $\left(-\sqrt{3}, -1\right)$ **b** $\left(\sqrt{2}, -\sqrt{2}\right)$ **c** $\left(-5, 5\sqrt{3}\right)$

d $\left(-3, -3\sqrt{3}\right)$ **e** $\left(\frac{15}{\sqrt{2}}, \frac{15}{\sqrt{2}}\right)$ **f** $\left(-2\sqrt{3}, 2\right)$

12

	Radius	**Diameter**	**Angle subtending arc**	**Arc length**
a	3 cm	6 cm	$\frac{5\pi}{6}$	$\frac{5\pi}{2}$ cm
b	6 cm	12 cm	$\frac{2\pi}{3}$	4π cm
c	7 cm	14 cm	$\frac{3\pi}{4}$	$\frac{21\pi}{4}$ cm
d	3.2 cm	6.4 cm	$\frac{5\pi}{12}$	$\frac{4\pi}{3}$ cm
e	4 cm	8 cm	$\frac{9\pi}{8}$	$\frac{9\pi}{2}$ cm
f	2.5 cm	5 cm	$\frac{4\pi}{35}$	$\frac{2\pi}{7}$ cm

13 a **i** $\frac{\pi}{3}$ **ii** $\frac{\pi}{3}$ **iii** $\frac{\pi}{3}$ **iv** $\frac{\pi}{6}$ **v** $\frac{\pi}{4}$
vi $\frac{\pi}{4}$ **vii** $\frac{\pi}{6}$ **viii** $\frac{\pi}{6}$ **ix** $\frac{\pi}{4}$

b The denominator of the angle and the reference angle are the same.

c **i** Q2 **ii** Q3 **iii** Q4 **iv** Q2 **v** Q3
vi Q4 **vii** Q3 **viii** Q4 **ix** Q2

d If the coefficient of π is less than one-half, the angle is in quadrant 1. If it is between one-half and one, the angle is in quadrant 2. If it is between one and three-halves, the angle is in quadrant 3. If it is between three-halves and two, the angle is in quadrant 4. If it is not between zero and two, then add or subtract two from the coefficient to determine a coterminal angle where the coefficient is between zero and two.

14 a $\left(2-\frac{\sqrt{2}}{2}, 1-\frac{\sqrt{2}}{2}\right)$ **b** $(\sqrt{2}-3, \sqrt{2}+2)$
c $\left(\frac{5\sqrt{3}}{2}-2, -\frac{11}{2}\right)$ **d** $(3, 2\sqrt{3}+3)$

15 a $x = 4\sqrt{3}$ cm **b** $x = \frac{23\sqrt{2}}{2}$cm

EX
p440

8I Solving trigonometric equations

1 a $\theta = 30°, 330°$ **b** $\theta = 225°, 315°$
c $\theta = 150°, 330°$ **d** $\theta = 240°, 300°$
e $\theta = 45°, 315°$ **f** $\theta = 45°, 225°$

2 a $\theta = 114°, 246°$ **b** $\theta = 197°, 343°$
c $\theta = 42°, 222°$ **d** $\theta = 37°, 143°$
e $\theta = 53°, 307°$ **f** $\theta = 98°, 278°$

3 a 90° **b** no
c $\theta = 90°$ **d** $\theta = 90°, 450°$

4 a $\theta = 180°$ **b** $\theta = 180°, 540°$

5 a 0° **b** 180°, 360°
c $\theta = 0°, 180°, 360°$ **d** $\theta = 0°, 180°, 360°, 540°, 720°$

6 a $\theta = 90°, 270°$ **b** $\theta = 0°, 180°, 360°$
c $\theta = 270°$ **d** $\theta = 0°, 360°$

7 a no domain within the stated range
b $90° \le \theta \le 270°$ **c** $60° \le \theta \le 300°$
d $60° \le \theta \le 120°$ **e** $45° \le \theta \le 135°$
f $150° \le \theta \le 210°$

8 a **i** $210° \le \theta \le 330°$ **ii** $210° \le \theta \le 690°$
b **i** $120° \le \theta \le 240°$ **ii** $120° \le \theta \le 600°$
c **i** $225° \le \theta \le 315°$ **ii** $225° \le \theta \le 675°$
d **i** $135° \le \theta \le 225°$ **ii** $135° \le \theta \le 585°$

9 a–c When each of these equations is simplified the sine or cosine value is greater than 1. Sine and cosine can only take values in the range −1 to 1.

10 a yes **b** yes **c** yes **d** yes
e No, when the equation is simplified the cosine value is less than −1.
f yes
g No, when the equation is simplified the cosine value is greater than 1.
h yes
i No, when the expression is simplified the sine value is less than −1.

11 a 1 **b** 0 **c** 1
d 1 **e** 1 **f** 1

12 a $\theta = 22.5°, 67.5°$ **b** $\theta = 45°, 135°$
c $\theta = 45°, 105°, 165°$

13 a $\theta = 90°$ **b** $\theta = 0°$ **c** $\theta = 135°, 225°$
d $\theta = 240°$ **e** $\theta = 225°$ **f** $\theta = 150°$

14 a $\theta = 420°, 660°$ **b** $\theta = 135°, 945°$ **c** $\theta = 300°, 750°$

15 Solving $\sin(\theta) = 1$ and $\cos(\theta) = 1$ involves solutions that are multiples of 90° whereas solving $\tan(\theta) = 1$ involves 45°. That is, the sine and cosine equations will only have one solution for $0° \le \theta \le 180°$. While the tangent equation will have two.

16 a 180° **b** 360° **c** 240°
d 540° **e** 360° **f** 480°

17 a $\theta = -300°, -240°$ **b** $\theta = -225°, -135°$
c $\theta = -300°, -120°$ **d** $\theta = -90°$
e $\theta = -180°$ **f** $\theta = 0°, -180°, -360°$

18 a $x = 0, \pi, 2\pi$ **b** $x = \frac{\pi}{6}, \frac{11\pi}{6}$ **c** $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$
d $x = \frac{\pi}{4}, \frac{3\pi}{4}$ **e** $x = \pi$ **f** $x = \frac{\pi}{6}, \frac{7\pi}{6}$

19 a The solutions to a sine equation always lie on the same horizontal line through the unit circle.
b The solutions to a cosine equation always lie on the same vertical line through the unit circle.
c The solutions to a tangent equation always lie on the same diagonal line through the unit circle.

20 a
$$\sin(\theta) = \cos(\theta)$$
$$\frac{\sin(\theta)}{\cos(\theta)} = \frac{\cos(\theta)}{\cos(\theta)}$$
$$\tan(\theta) = 1$$

b **i** $\theta = 45°, 225°$ **ii** $\theta = 60°, 240°$
iii $\theta = 150°, 330°$ **iv** $\theta = 30°, 210°$

21 a $\theta = 180°, 300°, 360°, 480°, 540°$
b $\theta = -5\pi, -3\pi, 3\pi, 5\pi, 11\pi$

22 a $x = -\frac{1}{2}, 2$ **b** $\theta = 210°, 330°$

EX
p447

8J Trigonometric graphs

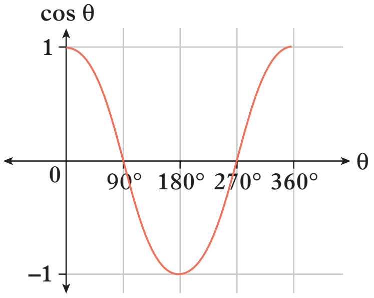
1 a **i** decreased and approached −1
ii decreased and approached 0
iii decreased and approached 1

b

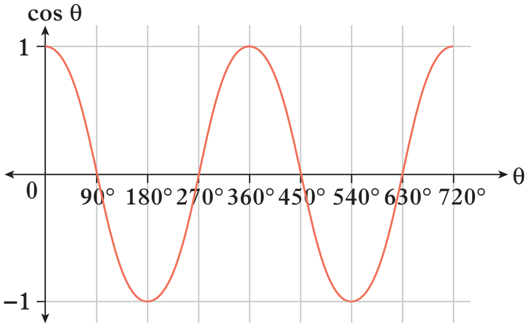
θ	0°	30°	60°	90°	120°	150°	180°
cos(θ)	1	0.9	0.5	0	–0.5	–0.9	–1

θ	210°	240°	270°	300°	330°	360°
cos(θ)	–0.9	–0.5	0	0.5	0.9	1

c



d



e 0°, 360°; the period of the cosine graph is 360°

f amplitude = 1

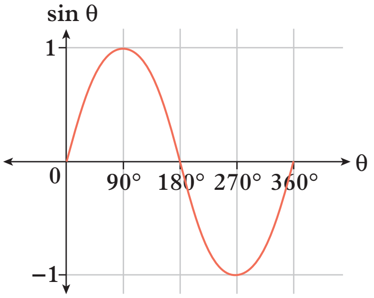
g (0, 1).

2 a

θ	0°	30°	60°	90°	120°	150°
sin θ	0	0.5	0.9	1	0.9	0.5

θ	180°	210°	240°	270°	300°	330°	360°
sin θ	0	–0.5	–0.9	–1	–0.9	–0.5	0

b



c 360° **d** 1 **e** (0,0)

3 a i 1 **ii** 0 **iii** 0
iv –1

b i 0.8 **ii** –0.2 **iii** 0.9
iv –0.9 **v** 0.2 **vi** –0.8

4 a 0°, 360° **b** 180° **c** 90°, 270°
d 70°, 290° **e** 130°, 230° **f** 40°, 320°

5 a i 0 **ii** 1 **iii** –1
iv 0

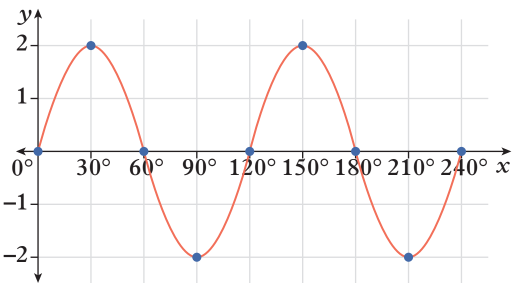
b i 0.6 **ii** 1.0 **iii** –0.2
iv –0.2 **v** –1.0 **vi** 0.6

6 a 90° **b** 270° **c** 0°, 180°, 360°
d 20°, 160° **e** 220°, 320° **f** 50°, 130°

7 a

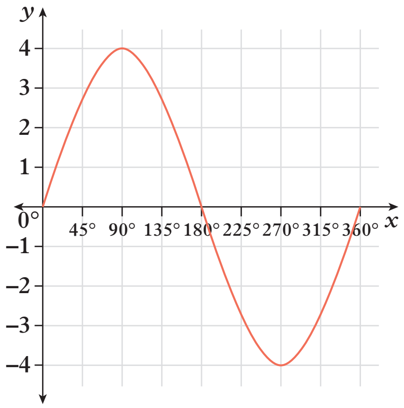
3x	0°	90°	180°	270°	360°	450°	540°	630°	720°
x	0°	30°	60°	90°	120°	150°	180°	210°	240°
sin(3x)	0	1	0	–1	0	1	0	–1	0
y = 2 sin(3x)	0	2	0	–2	0	2	0	–2	0

b, c

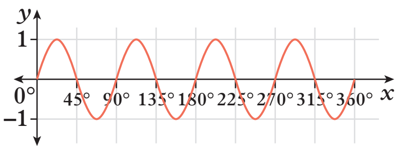


8 a amplitude = 3, period = π
b amplitude = 5, period = $\frac{2\pi}{3}$
c amplitude = 4, period = 4π
d amplitude = 6, period = 12π
e amplitude = $\frac{3}{4}$, period = $\frac{8\pi}{3}$
f amplitude = $\frac{5}{3}$, period = $\frac{10\pi}{3}$
g amplitude = $\frac{7}{8}$, period = $\frac{7\pi}{2}$
h amplitude = $\frac{5}{12}$, period = $\frac{5\pi}{12}$

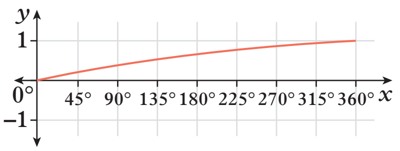
9 a



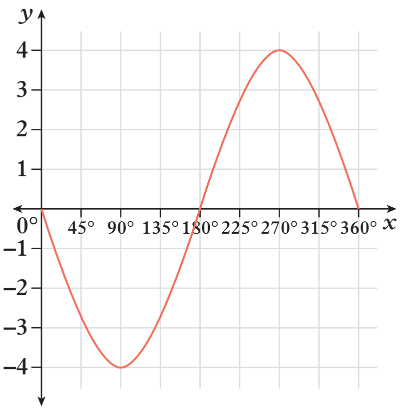
b



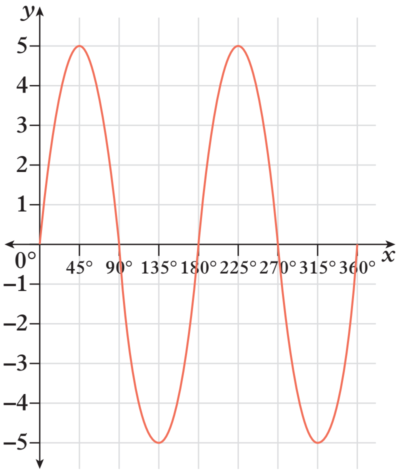
c



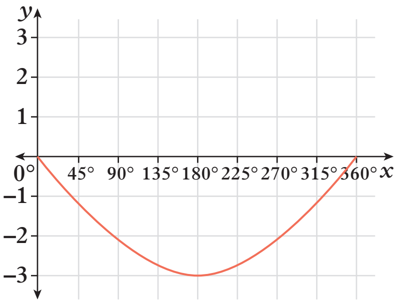
d



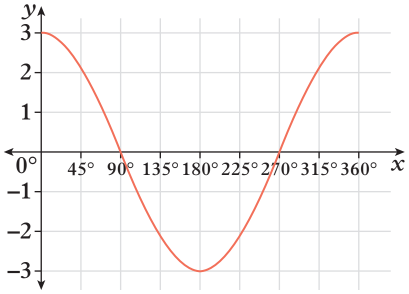
e



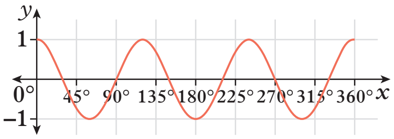
f



10 a



b



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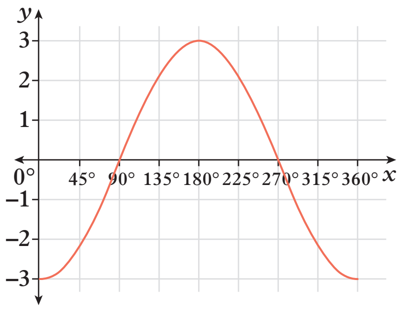

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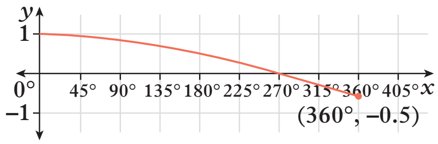
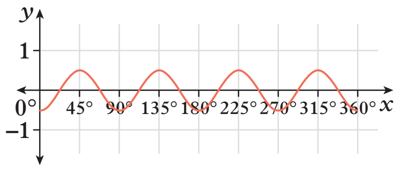
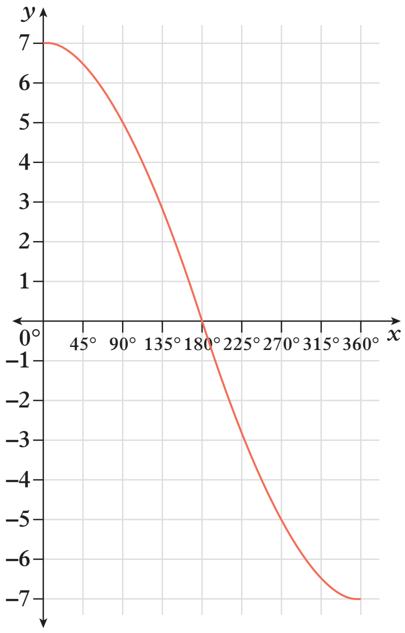


e



f



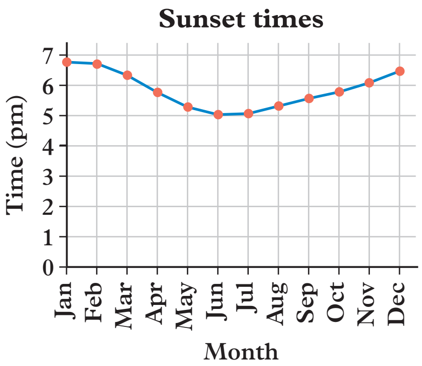
11 The sine graph starts at (0, 0) and the cosine graph starts at (0, 1).

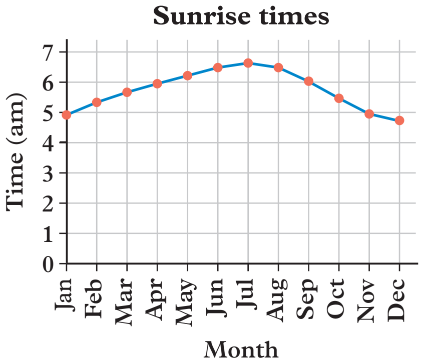
12 a A pendulum swings back and forth the same distance. Its motion is repetitive and regular, which means it is periodic.

b **i** 1.0 s **ii** 1.2 s **iii** 1.4 s

c The longer the pendulum, the longer the period of its motion.

13 a





b Answers may vary. Comments could include: The graphs are cyclic or periodic. The graphs have peaks and troughs. The graphs are both U-shaped (with one being an inverted U shape).

c Similarities to sine and cosine graphs: The graphs have a repeating pattern. The graphs are cyclic or periodic. The graphs have peaks and troughs.

Differences from sine and cosine graphs: The graphs do not start at 1 or 0. The maximum point is not 1 and the minimum point is not –1. The graphs are completely above the *x*-axis. The shape of the graphs resembles a cosine graph more than a sine graph. The graphs are not perfect curves.

d Yes, the values on the graphs would repeat at regular intervals. In this case the interval is 1 year. But the repeated values would be approximately the same, not exactly the same.

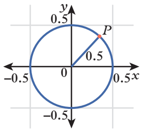
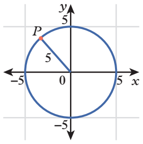
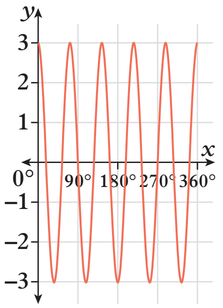
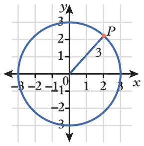
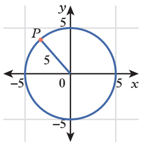
e Daylight hours are the hours which lie between the two graphs. These hours increase in summer and decrease in winter.

f Winter has the shortest days. These graphs indicate the least amount of daylight during the middle of the year. This is when winter occurs in the southern hemisphere.

14 a 3 **b** 11 **c** $\frac{1}{2}$ **d** b

e 4 **f** 90 **g** $\frac{3}{2}$ **h** b

15 a

	Graph	**Circle**	**Radius**	**Speed of point (revolutions per 360°)**
i			10	4
ii			0.5	3
iii			3	$\frac{1}{2}$
iv			5	$\frac{1}{4}$
v			3	5
vi			5	$\frac{3}{4}$

b The amplitude is related to the length of the radius. The amplitude is equal to the length of the radius.

c The period is related to the speed the point travels around the circle. The period is the time it takes for the point to complete a revolution of the circle travelling at its constant speed. That is,

$$\text{period(time)} = \frac{\text{distance(360°)}}{\text{speed}}.$$

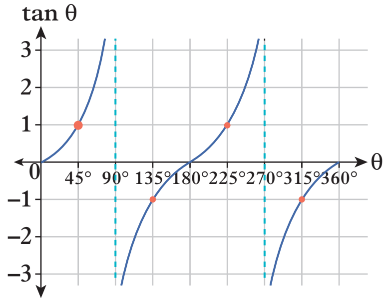
16 a

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°
tan(θ)	0	0.6	1	1.7	⋆	–1.7	–1	–0.6	0

θ	210°	225°	240°	270°	300°	315°	330°	360°
tan(θ)	0.6	1	1.7	⋆	–1.7	–1	–0.6	0

⋆undefined

b



c **i** The graph is a curving line which extends from –∞ to +∞ with a point of inflection on the x-axis.

ii The graph line is not continuous; its cycle repeats after 180° (not 360°); it has values for which tan θ is undefined; the values approach ∞ and –∞.

iii At θ = 90° the value of tan θ is undefined so a vertical asymptote is shown at this point. An asymptote is a line which the graph approaches but never reaches. The next asymptote is at θ = 270°.

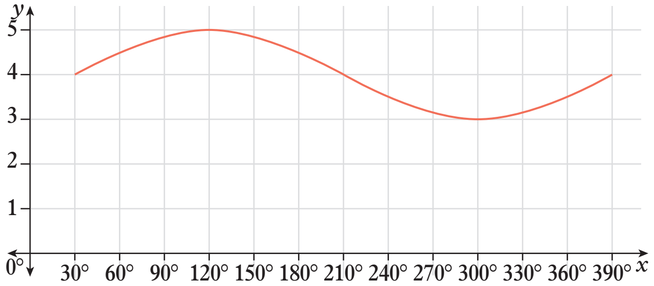
iv The graph repeats after a certain period. Period = 180°.

v The graph has no amplitude. The tan θ values approach ∞ and –∞.

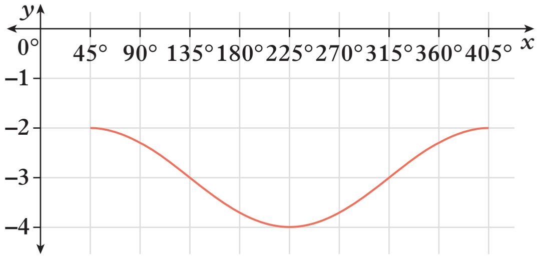
17 **a**

$x - 30$	0°	90°	180°	270°	360°
x	30°	120°	210°	300°	390°
$\sin(x - 30°)$	0	1	0	–1	0
$y = \sin(x - 30°) + 4$	4	5	4	3	4

b



c



CHAPTER 8 review

Multiple-choice

1 C 2 B 3 A 4 B 5 A 6 C
7 B 8 B 9 E 10 A 11 A 12 E
13 D

Short answer

1 **a** $\frac{5\sqrt{3}}{2}$ cm **b** 4.3 cm

2 length = 13.50 cm, width = 8.21 cm

3 **a** 19°

b **i** $\frac{6.2}{18.5} = \frac{62}{185}$ or 62 : 185 **ii** 33.51%

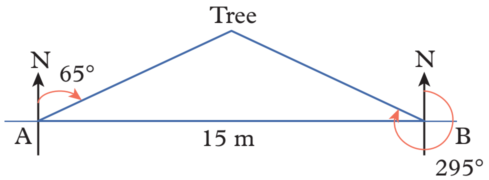
c 19.51 km

4 40 m

5 **a** 16.30 cm **b** 19.72 cm **c** 43.30 cm

6 **a** 23° **b** 40° **c** 55°

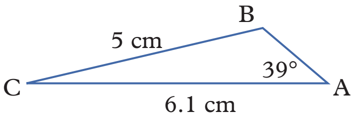
7 **a**



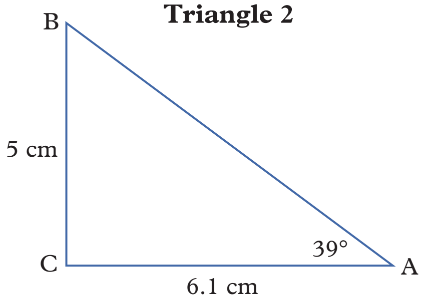
b 8.28 m in both positions

8 **a**

Triangle 1



Triangle 2

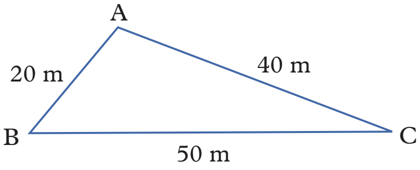


Because there are two sides and an angle given and the shorter side is opposite the angle it is possible to draw two distinct diagrams.

b Triangle 1: $B = 130°$, $C = 11°$, $c = 1.5$ cm

Triangle 2: $B = 50°$, $C = 91°$, $c = 7.9$ cm

9 **a**



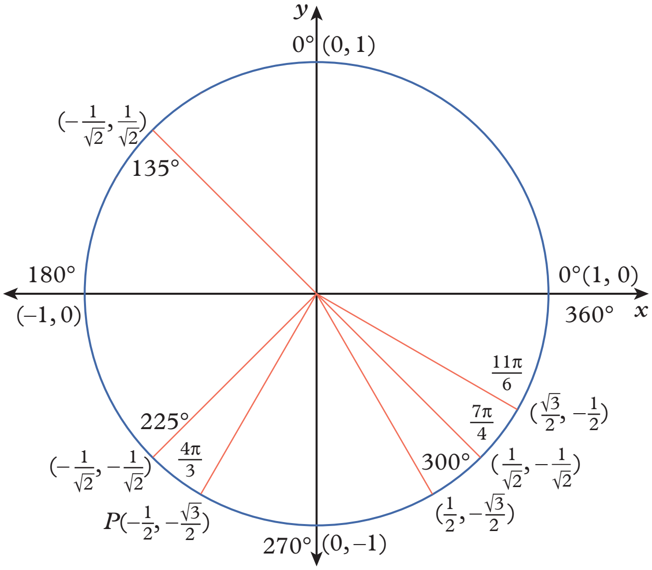
b 380 m^2 **c** $A = 108°$, $B = 49°$, $C = 22°$

d 15.20 m

10 **a** **i** $\frac{7\pi}{10}$ **ii** $\frac{7\pi}{4}$ **iii** $\frac{13\pi}{18}$

b **i** 72° **ii** 480° **iii** 202.5°

11



a $-\frac{\sqrt{3}}{2}$ **b** $-\frac{1}{\sqrt{2}}$ **c** $-\frac{1}{\sqrt{2}}$

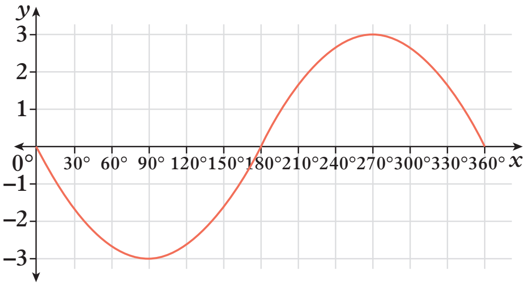
d $\frac{1}{\sqrt{2}}$ **e** $-\frac{\sqrt{3}}{2}$ **f** $\frac{\sqrt{3}}{2}$

12 a $\theta = 180°$ **b** $\theta = 0°, 180°, 360°$

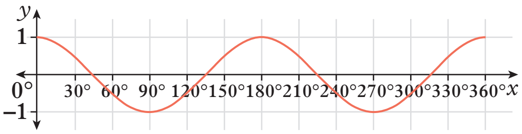
c $\theta = 30°, 330°$ **d** $\theta = 210°, 330°$

e $\theta = 135°, 225°$ **f** $\theta = 240°, 300°$

13 a

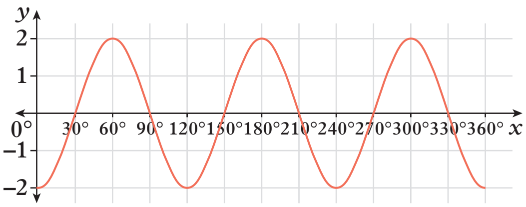


b



c

d



Analysis

1 a i $2R + G = 480$ mm; this is below the required range, so the staircase does not fall within building regulation guidelines.

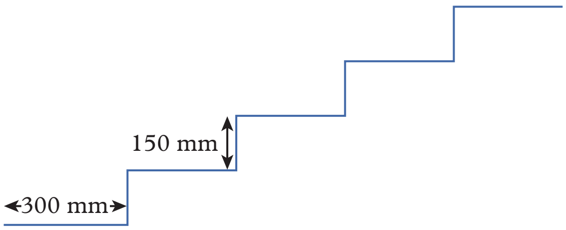
ii 25°

b i $2R + G = 735$ mm; this is above the required range, so the staircase does not fall within building regulation guidelines.

ii 28°

c i 115 mm to 172.5 mm **ii** 18° to 26°

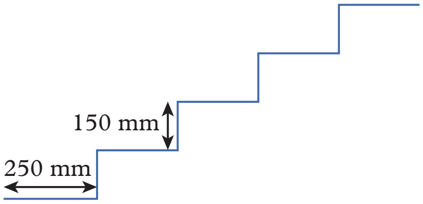
d



i 1500 mm **ii** 600 mm

iii 1615.5 mm **iv** 27°

e One possible solution is given.



115 mm ≤ 150 mm ≤ 190 mm

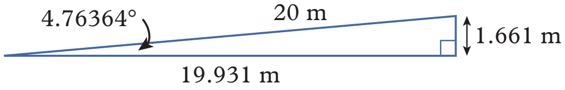
250 mm ≤ 250 mm ≤ 355 mm

$2R + G = 2 \times 150 + 250 = 300 + 250 = 550$ mm

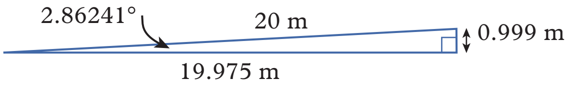
550 mm ≤ 550 mm ≤ 700 mm

f i

Internal ramp 1:12

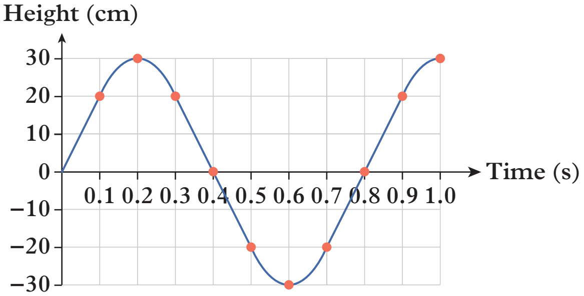


External ramp 1:20



ii Internal: rise = 1.661 m, going = 19.931 m
External: rise = 0.999 m, going = 19.975 m

2 a



b sine curve

c 0.8 seconds

d Starting position was 0 cm. It takes 0.1 seconds for the tag to go from 20 cm to 0 cm in height.

e 30 cm **f** 188.5 cm **g** 2.4 m/s

h After 10 seconds, the wheel will complete 12.5 rotations. If the tag's starting position was at 0 cm, then the tag will be at the same height of 0 cm but on the opposite side of the wheel.

CHAPTER 9 Measurement

EX
p460

9A Area review

1 a 700 cm **b** 4.5 km **c** 8 400 000 m^2

d 0.29 cm^2 **e** 2.4 ha **f** 880 000 mm

g 0.003 42 km^2 **h** 1 022 000 cm **i** 300 ha

2 a E **b** A **c** D

d B **e** C

3 a 42 m^2 **b** 272 cm^2 **c** 76 cm^2

d 348.5 cm^2 **e** 843.6 mm^2 **f** 164.4 cm^2

4 **a** 216 cm^2 **b** 242.1 cm^2 **c** 238.9 cm^2
d 375 mm^2 **e** 28.5 cm^2 **f** 409.5 cm^2

5 **a, b** chimney: trapezium, 1.8 m^2
roof: triangle, 6.5 m^2
front wall: rectangle, 15.54 m^2
front door: rectangle, 3.9 m^2
windows: square, 0.49 m^2 each window

c multiply by 4

d area to be painted = area of chimney + area of roof
+ area of front wall
– area of windows

e 21.88 m^2

f 1.5 L

6 **a** 168 cm^2 **b** 2 m^2 **c** 6.4 cm^2
d 348 cm^2 **e** 933.9 cm^2 **f** 219.5 cm^2

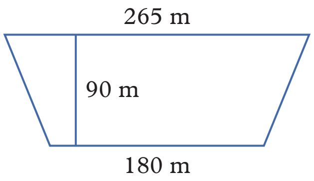
7 **a** Subtract the area of the inner circle from the area of the outer circle.

b 100.5 m^2

c **i** 115.5 m^2 **ii** 58.9 m^2 **iii** 125.7 m^2

8 **a** 59.4 cm^2 **b** 515.3 mm^2 **c** 2875.1 cm^2

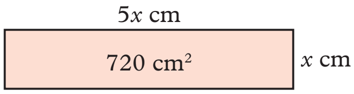
9 **a** 20 025 m^2



b 2 ha

c One possible set of dimensions is 225 m by 89 m.

10 **a**



b The rectangle is 60 cm by 12 cm.

c 2880 cm^2

d 4 times the area

e 6480 cm^2, 9 times the area

f **i** When the dimensions are quadrupled, the area is 16 times as large.

ii When the dimensions are multiplied by a factor of n, the area becomes n^2 times as large.

11 **a** 75.4 cm^2 **b** 60°
c 12.6 cm^2 **d** 31.4 cm^2

12 **a** The radius of the full circle is twice the radius of the semi-circles.

b 904.8 cm^2

c The area of the white section is the same as that of the blue section: 904.8 cm^2.

13 92 cm^2

14 **a** 60π cm^2 **b** 40π cm^2 **c** 3 : 2

15 **a** $5\sqrt{3}$ cm **b** $25\sqrt{3}$ cm^2

c $150\sqrt{3}$ cm^2 **d** $\frac{3\sqrt{3}}{2}a^2$

16 **a**
$$\begin{aligned} A &= \frac{1}{2}(a)(h) + \frac{1}{2}(b)(h) \\ &= \frac{h}{2}(a+b) \end{aligned}$$

b
$$\begin{aligned} A &= ah + \frac{1}{2}(b-a)(h) \\ &= \frac{ah}{2} + \frac{bh}{2} \\ &= \frac{h}{2}(a+b) \end{aligned}$$

c The two triangles are similar, and so $\frac{x}{x+h} = \frac{a}{b}$.
Solving for x gives:
$$\begin{aligned} xb &= a(x+h) \\ xb &= ax + ah \\ xb - ax &= ah \\ x(b-a) &= ah \\ x &= \frac{ah}{b-a} \end{aligned}$$

Therefore:
$$\begin{aligned} A &= \frac{1}{2}(b)(h+x) - \frac{1}{2}(a)(x) \\ &= \frac{bh}{2} + \frac{bx}{2} - \frac{ax}{2} \\ &= \frac{bh}{2} + \frac{x}{2}(b-a) \\ &= \frac{bh}{2} + \frac{ah}{2(b-a)}(b-a) \\ &= \frac{bh}{2} + \frac{ah}{2} \\ &= \frac{h}{2}(a+b) \end{aligned}$$

EX
p468

9B Surface area of prisms and cylinders

1 **a** 262 cm^2 **b** 1350 cm^2 **c** 6144 mm^2
d 34.56 m^2 **e** 6156 mm^2 **f** 23.88 m^2
g 876 mm^2 **h** 332 cm^2 **i** 53.66 m^2

2 23.67 m^2

3 **a** 662.8 cm^2 **b** 2310.1 mm^2 **c** 117.4 cm^2

4 383.04 cm^2

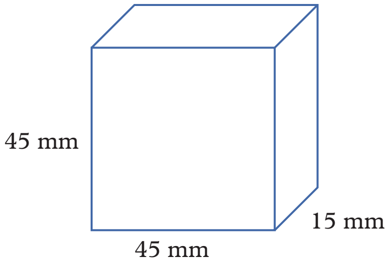
5 **a** 1809.6 cm^2 **b** 883.4 cm^2 **c** 11 309.7 mm^2

6 **a** 8224 cm^2 **b** 334.2 cm^2 **c** 470.4 cm^2

7 **a** 1350 mm^2

b 12 150 mm^2

c



d 6750 mm^2

e 5400 mm^2; 24 surfaces of the dice are hidden within the structure ($24 \times 15^2 = 5400$).

8 **a** 8 cm; divide total surface area by 6 to find the area of one face, then find the square root of the area answer to find the length of one side.

b $l = \sqrt{\frac{\text{TSA}}{6}}$

9 **a** 570.2 cm^2

b No. The pipe is open, so there is no top or bottom. This means $2\pi r^2$ is removed from formula.

10 279.9 cm^2

11 a 7191.3 cm^2

b No. The curved surface area will be double, but the one circular end which needs painting has the same area for both poles.

c 14 259.9 cm^2

2 × 7191.3 = 14 382.6 cm^2

This is 122.7 cm^2 larger than 14 259.9 cm^2.

This difference is the area of one end which is not doubled.

This supports the answer to part **b**.

12 a 730.1 cm^2 **b** 650.8 cm^2

13 a 407.8 m^2 **b** 4815.8 m^2

14 1311 cm^2

15 208.9 cm^2

16 a 9 m^2 **b** 4 cans

EX p473

9C Volume of prisms and cylinders

1 a 12 500 000 cm^3 **b** 240 cm^3

c 34.2 m^3 **d** 550 000 000 mm^3

e 0.000 067 2 m^3 **f** 9000 mm^3

g 7.52 cm^3 **h** 8 740 000 cm^3

i 0.1429 m^3 **j** 0.073 m^3

2 a 2508 cm^3 **b** 16 128 mm^3

c 5.8 m^3 **d** 43 264 cm^3

e 689.1 cm^3 **f** 6.2 m^3

g 96 140 mm^3 or 96.1 cm^3 **h** 29 440 cm^3

i 1539 m^3 **j** 53.5 cm^3

k 74 088 m^3 **l** 175 cm^3

3 a 1256.6 cm^3 **b** 172 240.2 mm^3

c 16 155.6 m^3 **d** 290.8 cm^3

e 178 623.1 cm^3 or 0.2 m^3 **f** 3698.5 cm^3

4 a 0.064 m^3 **b** 64 000 cm^3

5 a 1102.5 cm^3 **b** 35 224 mm^3 **c** 15.58 m^3

6 a 12.48 m^3 **b** 10 560 mm^3 **c** 676.2 cm^3

d 9000 cm^3 **e** 17 090.3 mm^3 **f** 25 249.0 cm^3

7 5193.4 cm^3

8 a rectangular prism, triangular prism, cylinder

b Find the total area of all surfaces showing

c 198 cm^2

d 97 cm^3

9 a 206.2 cm^3 **b** 360 cm^3 **c** 1687.5 cm^3

10 a 176 cm^3 **b** 0.176 kg **c** \$3.70

11 a 45 000 cm^3 **b** 45 000 mL, 45 L **c** 39.6 L

12 14 cm

13 a 21 cm **b** 34 mm

14 a second tank **b** 1178 L

15 a 3015.9 cm^3

b i 754 cm^3 **ii** $\frac{1}{4}$ the volume

c i 12 063.7 cm^3 **ii** 4 times the volume

d i 1508 cm^3 **ii** $\frac{1}{2}$ the volume

e i 6031.9 cm^3 **ii** 2 times the volume

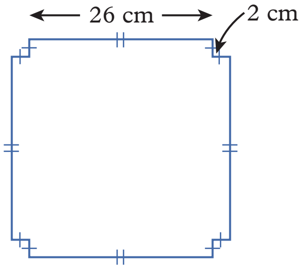
16 a 59 244 litres **b** 40 097 litres

17 a annulus, area = 84.8 cm^2

b 2121 cm^3

c Subtract the volume of the inner cylinder from the volume of the outer cylinder.

18 a



b length = 26 cm, width = 26 cm, height = 2 cm

c 1352 cm^3

d The box would be 24 cm × 24 cm × 3 cm, giving a volume of 1728 cm^3. The volume of the box would increase.

19 a circular nut: 107.71 mm^3

hexagonal nut: 144.68 mm^3

b hexagonal nut

9 Checkpoint

1 a 4.5 cm **b** 4 m **c** 4530 mm

d 40 cm^2 **e** 5 200 000 m^2 **f** 6.8 m^2

2 13.125 m^2

3 219.1 cm^2

4 306 cm^2

5 41.7 m^2

6 697.8 cm^2

7 2850 cm^2

8 7634.1 cm^3

9 a 4000 mm^3 **b** 4 cm^3

10 1292.655 cm^3

EX p481

9D Surface area of pyramids and cones

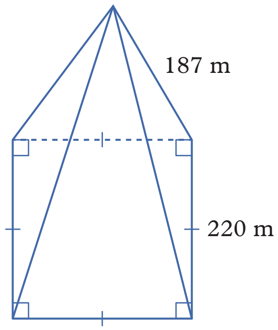
1 a 2744 mm^2 **b** 118 cm^2 **c** 4856 mm^2

d 408 cm^2 **e** 3234 m^2 **f** 7605 cm^2

2 a 809.4 cm^2 **b** 1578.8 mm^2 **c** 9259.9 cm^2

3 1890 m^2

4 a



b 66 539.0 m^2

5 a 195.6 cm^2 **b** 914.3 mm^2 **c** 4875.0 cm^2

d 460.1 cm^2 **e** 619.2 mm^2 **f** 129.2 cm^2

6 **a** 9123.2 mm^2 **b** 2801.5 cm^2 **c** 624.4 cm^2
d 282.7 cm^2 **e** 628.3 cm^2 **f** 1582.2 mm^2

7 20 cm

8 slant height = 18.2 cm, TSA = 1640.0 cm^2

9 **a** 27.7 cm^2 **b** surface area = $x^2\sqrt{3}$

10 339.3 cm^2

11 **a** 8
b 679.0 mm^2
c Each pyramid, including the base, has TSA of 535.5 mm^2. So, TSA of two pyramids is 1071.0 mm^2.
d The answers to parts **b** and **c** are not equal in this form. If you subtract the areas of the bases of the two pyramids (which are covered) the answers are the same.
1071.0 mm^2 – 2 × 196 mm^2 = 679 mm^2

12 **a** 162.7 cm^2 **b** 6579.8 mm^2
c 1847.3 cm^2

13 756.9 cm^2

14 **a** **i** 471.2 cm^2 **ii** 7.5 cm
b **i** $\theta = 180°$ **ii** $\theta = 120°$

15 **a** 300 cm^2 **b** $\frac{2}{3}$
c **i** 16.2 cm **ii** 424.3 cm^2 **iii** 52.9%

EX p486

9E Volume of pyramids and cones

1 **a** 1615 cm^3
b 37 762.7 mm^3
c 11.4 cm^3
d 6700.7 cm^3
e 28.9 cm^3 or 28 918.7 mm^3
f 38.2 m^3
g 3033.3 cm^3
h 93.6 cm^3
i 5118.8 mm^3

2 **a** 65 m^3 **b** 1239.3 cm^3 **c** 1760 cm^3

3 2 592 100 m^3

4 **a** 823 825.1 mm^3
b 1747.7 cm^3
c 119.7 cm^3 or 119 703.1 mm^3
d 5026.5 cm^3
e 5277.9 mm^3
f 53.1 m^3 or 53 136 105.4 cm^3

5 16 222.4 cm^3

6 15 519.2 mm^3

7 **a** 8.5 cm; 20.1 cm^2
b **i** 73.8 cm^3
ii 62.3 mm, 73 700 mm^3
c $V = \frac{1}{3} \times \frac{1}{2}bh_1h_2$, where h_1 and h_2 represent the height

8 **a** 7.0 cm **b** 2.9 cm

9 5.2 L

10 **a** 37 447.8 mm^2 **b** height of cone
c 566 755.9 mm^3

11 **a** 47.1 cm^3
b **i** 6 cm **ii** 188.5 cm^3
c 754.0 cm^3
d When the radius is doubled and the height remains the same, the volume increases by a factor of 4.
e 94.2 cm^3
f 188.5 cm^3
g 377 cm^3
h When the height is doubled and the radius remains the same, the volume is doubled.
i The formula for the volume of a cone involves r^2 so, when r is doubled, r^2 and the volume increase by a factor of 2^2 (or 4). The formula also involves h so, when h is doubled, this increases the volume by a factor of 2.

12 1.2%

13 **a** 6031.9 cm^3 **b** 7680 cm^3
c 1648.1 cm^3 **d** $4 : 4 - \pi$

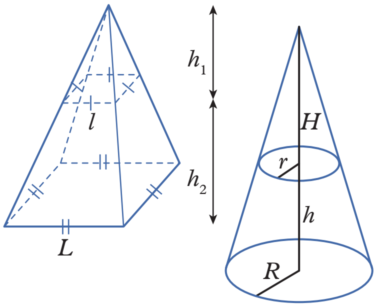
14 **a** 261.8 cm^3 **b** 7.9 cm

15 **a** Four faces are trapeziums; two faces are squares.
b 7291.7 cm^3 **c** 466.7 cm^3 **d** 6825 cm^3

16 3.0 m^3

17 **a** 401 338.5 mm^3 **b** 401 mL

18 One possible answer is shown here.
For a truncated square-based pyramid:
$$V = \frac{1}{3}L^2(h_1 + h_2) - \frac{1}{3}l^2h_1$$
For a truncated cone:
$$V = \frac{1}{3}\pi R^2(h_1 + h_2) - \frac{1}{3}\pi r^2 h_1$$
or: $$V = \frac{\pi}{3}([h_1 + h_2]R^2 - h_1r^2)$$



EX p492

9F Surface area and volume of spheres

1 **a** 1256.6 mm^2 **b** 8494.9 cm^2
c 2290.2 cm^2 **d** 33 979.5 mm^2

2 **a** 3631.7 cm^2 **b** 7543.0 mm^2
c 10.2 m^2 **d** 8992.0 cm^2

3 **a** 4188.8 mm^3 **b** 73 622.2 cm^3
c 10 306.0 cm^3 **d** 588 977.4 mm^2

4 **a** 20 579.5 cm^3 **b** 61 600.9 mm^3
c 3.1 m^3 **d** 80 178.9 cm^3

5 11 cm

6 **a** 6 cm **b** 9 cm
c 13.0 cm **d** 16.0 cm

7 **a** 100π cm^2

b **i** 25π cm^2 **ii** 50π cm^2 **iii** 75π cm^2

8 **a** **i** 508.9 cm^2 **ii** 1526.8 cm^3

b **i** 3041.1 mm^2 **ii** 22 301.1 mm^3

c **i** 4561.6 mm^2 **ii** 22 301.1 mm^3

d **i** 1490.1 cm^2 **ii** 7649.3 cm^3

e **i** 82.0 cm^2 **ii** 53.8 cm^3

f **i** 9954.9 mm^2 **ii** 71 896.7 mm^3

9 **a** 18 145.8 cm^2 **b** 229 847.3 cm^3

10 **a** 33 cm **b** 6.7 cm

11 **a**

Planet	Diameter (km)	Total surface area (km^2)	Volume (km^3)
Mercury	4878	7.5×10^7	6.1×10^{10}
Venus	12 104	4.6×10^8	9.3×10^{11}
Earth	12 756	5.1×10^8	1.1×10^{12}
Mars	6787	1.4×10^8	1.6×10^{11}
Jupiter	142 800	6.4×10^{10}	1.5×10^{15}
Saturn	120 000	4.5×10^{10}	9.0×10^{14}
Uranus	51 118	8.2×10^9	7.0×10^{13}
Neptune	49 528	7.7×10^9	6.4×10^{13}

b The volume of Earth is about 18 times the volume of Mercury.

c The surface area of Jupiter is about 125 times the surface area of Earth.

d 362 942 012 km^2

12 **a** 8.6 cm **b** 75 cm^3

13 **a** 3591.4 cm^3 **b** 10 744.2 cm^3 **c** 10.7 L

d approximately 4.3 s

e It needs to be assumed that the rate of inflation is a constant rate; and that there is no time lost between filling one balloon and the next.

14 **a** 164.6 cm^3 **b** 658.5 cm^3

c dimensions of canister: diameter = 6.8 cm, height = 27.2 cm, volume = 987.8 cm^3

d 329.3 cm^3

e There would be more unused space because the cylinder would fit inside the rectangular prism.

15 132 mL

16 **a** 16 278.9 cm^3 **b** 4308.2 cm^3

c 738.2 cm^3

17 9714 cm^3

18 **a** 33 510.3 mm^3 or 33.5 cm^3 **b** 0.08 g/cm^3

19 **a** **i** Doubling the radius makes the surface area 4 times as large.

ii Doubling the radius makes the volume 8 times as large.

b **i** Tripling the radius makes the surface area 9 times as large.

ii Tripling the radius makes the volume 27 times as large.

c **i** Halving the radius makes the surface area $\frac{1}{4}$ the size.

ii Halving the radius makes the volume $\frac{1}{8}$ the size.

d **i** Dividing the radius by 3 makes the surface area $\frac{1}{9}$ the size.

ii Dividing the radius by 3 makes the volume $\frac{1}{27}$ the size.

20 **a** **i** 20 cm **ii** 3811.2 cm^3

b **i** 8.7 cm **ii** 1720.7 cm^3

CHAPTER 9 review

Multiple-choice

1	D	**2**	A	**3**	D	**4**	D	**5**	C
6	D	**7**	C	**8**	B	**9**	A	**10**	B

Short answer

1 3.9 cm

2 **a** 80.4 cm^2 **b** 63.3 cm^2

3 2664.2 mm^2

4 **a** 9600 cm^2, 38 400 cm^2, 153 600 cm^2

b 1 can

5 385 mL

6 64 000 cm^3, 512 000 cm^3, 4 096 000 cm^3

7 225 cm^2

8 341.9 cm^2

9 2548.8 cm^3

10 cylinder: 1024π cm^3

sphere: $\frac{2048\pi}{3}$ cm^3

11 **a** **i** 3769.9 cm^2 **ii** 9600 cm^2

b 10 856.6 cm^2

Analysis

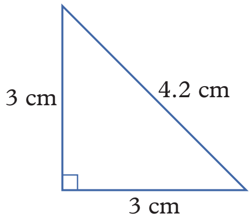
a cube, rectangular prism, square pyramid, triangular prism, cone, cylinder, sphere

b **i** 54 cm^2 **ii** 27 cm^3

c **i** The rectangular prism has a surface area of 90 cm^2, but the surface area of the cube is 54 cm^2. The rectangular prism is $1\frac{2}{3}$ times as large.

ii The volume of the rectangular prism is 54 cm^3. Its volume is twice the volume of the cube.

d **i**



ii 70.5 cm^2

iii No; the total surface area of the triangular prism is more than half the TSA of the rectangular prism because two new faces have been created by cutting the prism in half.

iv The volume of the triangular prism is half that of the rectangular prism, because it is the same size as the rectangular prism cut in half.

e **i** 70.7 cm^2 **ii** 42.4 cm^3

f **i**

Solid	Surface area
Cube	54 cm^2
Triangular prism	70.5 cm^2
Cylinder	70.7 cm^2
Rectangular prism	90 cm^2

ii

Solid	Volume
Cube and triangular prism	27 cm^3
Cylinder	42.4 cm^3
Rectangular prism	54 cm^3

g **i** 46.1 cm^2 **ii** 18 cm^3

h **i** 36.2 cm^2 **ii** 14.1 cm^3

i **i** 28.3 cm^2 **ii** 14.1 cm^3

j **i**

Solid	Surface area
Sphere	28.3 cm^2
Cone	36.2 cm^2
Square pyramid	46.1 cm^2
Cube	54 cm^2
Triangular prism	70.5 cm^2
Cylinder	70.7 cm^2
Rectangular prism	90 cm^2

ii

Solid	Volume
Sphere and cone	14.1 cm^3
Square pyramid	18 cm^3
Cube and triangular prism	27 cm^3
Cylinder	42.4 cm^3
Rectangular prism	54 cm^3

CHAPTER 10 Statistics

EX
p504

10A Five-number summary and interquartile range

1 **a** 16 **b** 45 **c** 87 **d** 67

2 **a** 3, 3, 5, 7, 9
b 7, 12, 19, 46, 78
c 1, 3, 5, 7, 9
d 2, 7, 11, 19, 32

3 **a** 2, 4, 7, 9, 12
b 4, 13, 25.5, 46, 55
c 16, 20.5, 42, 72.5, 81
d 120, 177.5, 452.5, 755, 843
e 18, 19.5, 33, 42, 64
f 3, 8, 13.5, 19, 32

4 **a** 10 **b** 9 **c** 10 **d** 19

5 **a** 32 **b** 9 **c** 56.5 **d** 36
e 9.5 **f** 253

6 Range = 51; IQR = 23. The difference between the least number of cupcakes sold in the month and the most number of cupcakes sold in the month is 51. On half the days, the number of cupcakes sold falls within a range of 23 (from 56 to 79 cupcakes sold).

7 **a** Both sets have a range of 43.
b IQR of set A = 26, IQR of set B = 10
c Because they have the same data range

8 **a** 6
b minimum = 0, Q_1 = 2, Q_2 (median) = 4, Q_3 = 5, maximum = 6
c 3

9 **a** If the IQR is small, 50% of scores are grouped closely around the median.
b If the IQR is small and the range is large, 50% of the scores are grouped closely around the median, while the 25% of scores lying below the lower quartile and the 25% of scores above the upper quartile are spread out.
c If the IQR is large, 50% of the scores are spread widely around the median.
d If the IQR is similar to the range, the range of the 25% of scores lying below the lower quartile plus the range of the 25% of scores above the upper quartile is the same as the range of the middle 50% of scores.

10 **a** range = \$240, IQR = \$45
b because there is an outlier
c The IQR represents the middle 50% of scores; those scores between Q_1 and Q_3. If the spread is even, the remaining 50% of scores will have a spread of the same size as the IQR.
d The IQR indicates a region within which the middle 50% of scores lie and the outlier will fall outside this range.

11 **a** **i** range = 17, IQR = 8.5
ii The data are spread fairly uniformly throughout the range.
b **i** range = 72, IQR = 30
ii The scores are packed more closely in the IQR. There appears to be an outlier of 11 in the lower quarter of the scores.
c **i** range = 73, IQR = 11
ii Scores are packed more closely in the IQR. There appears to be an outlier of 191 in the upper quarter of the scores.

12 **a** class A **i** 171.5 **ii** 27 **iii** 12
class B **i** 172.5 **ii** 48 **iii** 12.5
b The median and IQR of heights in the two classes is quite similar, with class B having a slightly larger median; however, the range of heights in class B is greater. Class B has the tallest and shortest members of the group, indicating it is a more diverse group.

13 a 50, 60, 70.5, 79, 83

b IQR = 19, range = 33

c 9, 55, 68, 74, 83

d IQR = 19, range = 74

e i The median decreased slightly from 70.5 to 68.

ii The range increased significantly from 33 to 74.

iii The interquartile range remained at 19.

f Since outliers and extreme values can only be at the ends of a list of data, they are not close enough to the position of the quartiles to impact the interquartile range.

14 a 13 **b** 26

15 a i $Q_1 = 2.5, Q_3 = 6.5$ **ii** $Q_1 = 2.5, Q_3 = 7.5$

iii $Q_1 = 3, Q_3 = 8$ **iv** $Q_1 = 3, Q_3 = 9$

b i $\frac{8}{2} = 4r0, \frac{4}{2} = 2r0$: 1 2 | 3 4 | 5 6 | 7 8,
$Q_1 = 2.5, Q_3 = 6.5$

ii $\frac{9}{2} = 4r1, \frac{4}{2} = 2r0$: 1 2 | 3 4 $\boxed{5}$ 6 7 | 8 9,
$Q_1 = 2.5, Q_1 = 7.5$

iii $\frac{10}{2} = 5r0, \frac{5}{2} = 2r1$: 1 2 $\boxed{3}$ 4 5 | 6 7 $\boxed{8}$ 9 10,
$Q_1 = 3, Q_3 = 8$

iv $\frac{11}{2} = 5r1, \frac{5}{2} = 2r1$: 1 2 $\boxed{3}$ 4 5 $\boxed{6}$ 7 8 $\boxed{9}$ 10 11,
$Q_1 = 3, Q_3 = 9$

c Steps 1 and 2 locate the median by breaking the data set into two equal sized groups and removing the median if it is one of the data values for the next steps. Steps 3 and 4 locate the quartiles in the same manner as the median by breaking the data set into a total of four groups.

d There can be 0, 1, 2, 3 of the median and the lower and upper quartile that are values in the data set, the same values that the remainder can be when dividing by 4. Removing these data values leaves a number of data values that is divisible by 4 where each group will be the size of the quotient.

16 a 16 **b** $\frac{47}{12}$ **c** $\frac{337}{352}$ **d** $\frac{13267}{9360}$

EX p509

10B Box plots

1 a i 1, 2, 5, 7, 10 **ii** IQR = 5

b i 13, 20, 32, 40, 45 **ii** IQR = 20

c i 3, 4, 6, 8, 9 **ii** IQR = 4

d i 10, 16, 19, 24, 35 **ii** IQR = 8

e i 20, 45, 70, 75, 85 **ii** IQR = 30

f i 31, 33, 35, 37, 39 **ii** IQR = 4

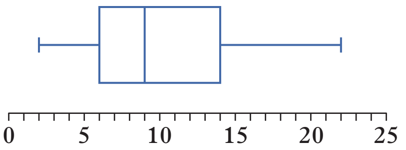
2 a 2, 3, 4, 4, 6, 7, 7, 8, 8, 9, 10, 10, 11, 13, 14, 17, 19, 19, 22

b $Q_1 = 6$, median = 9, $Q_3 = 14$

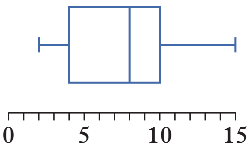
c lower fence = –6, upper fence = 26

d no outliers

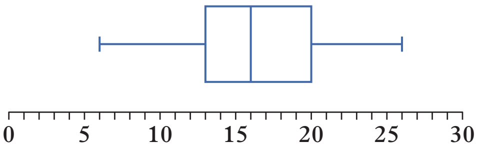
e



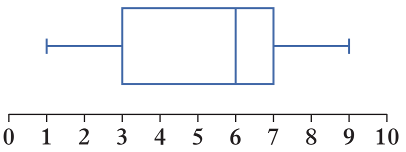
3 a



b



c



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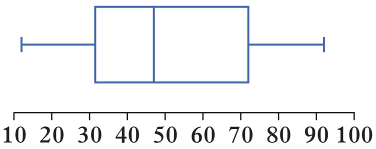

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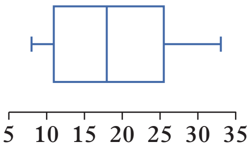

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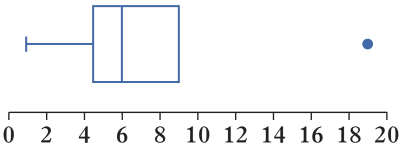

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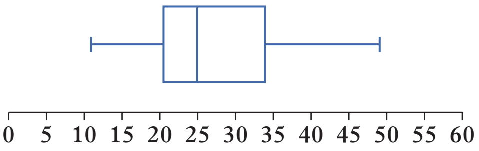
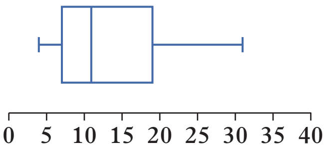


h



4 a i 3, 3, 7, 10, 12 **ii** 7

b i 18, 30, 33, 40, 40 **ii** 10

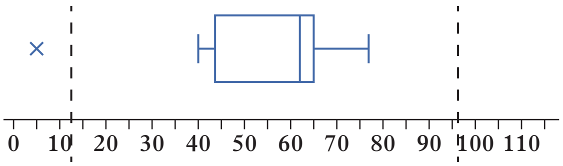
c i −20, −20, −15, −12, −12 **ii** 8

d i 40, 41, 42, 46, 60 **ii** 4

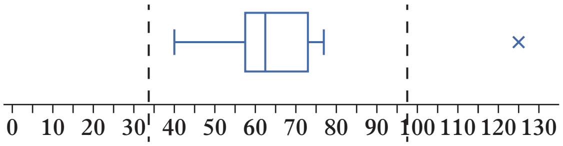
e i 18, 20, 21.5, 25, 35 **ii** 5

f i 6, 10, 10, 10, 14 **ii** 0

5 a



b



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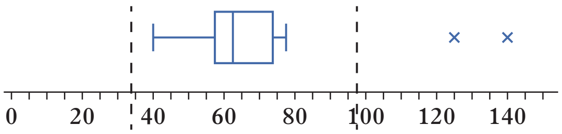

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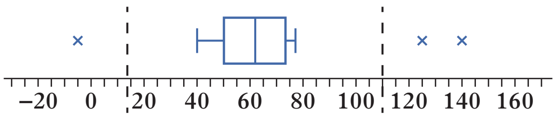


6 **a** **i** B **ii** B, D

b **i** median **ii** maximum **iii** none

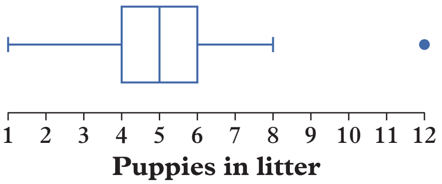
iv maximum **v** upper quartile

c Group C

7 **a** $Q_1 = 4$, median = 5, $Q_3 = 6$

b 12 is an outlier.

c

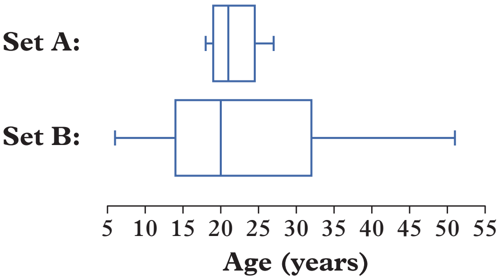


8 **a** median ≈ 32.5 °C, IQR ≈ 3.5 °C

b median ≈ 28 °C, IQR ≈ 8.5 °C

c Darwin's median maximum daily temperature during this month was approximately 32.5 °C which was hotter than Canberra's median maximum daily temperature, 28 °C. Darwin's maximum daily temperature was less variable with an interquartile range of approximately 3.5 °C compared to Canberra's larger interquartile range of approximately 8.5 °C. While Canberra had the largest maximum daily temperature for the month, Darwin is, on average, hotter than Canberra.

9 **a**

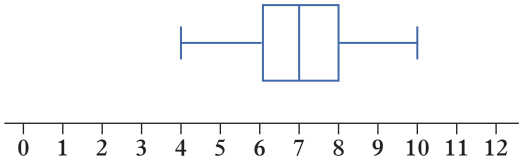


b **Set A:** The centre (median) = 21, the spread is 18–27 years, positively skewed. The range of ages is small, indicating that, during the week, the centre is used mainly by young people.

Set B: The centre (median) = 20, the spread is 6–51 years old, positively skewed. The range of ages is much greater during the weekend, indicating that families most probably visit then.

c The range of ages is much greater in set B than in set A, although the median ages are quite similar. More people in a younger and older age range visit during the weekend than attend on weekdays.

10 **a**



b No, as the minimum value (4) is greater than the lower fence (3) and the maximum value (10) is less than the upper fence (11), so the data contains no outliers.

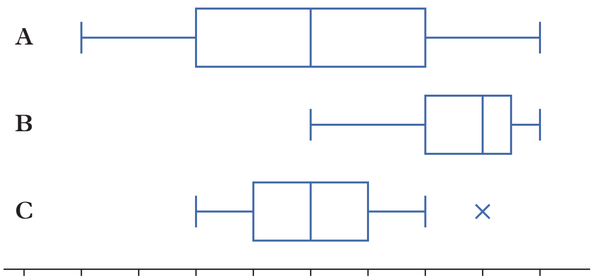
c The minimum value (2) is less than the lower fence (3) and the maximum value (12) is greater than the upper fence (11) so they are both outliers. Therefore, we would need to know at least the next value above the minimum and less than the maximum value (provided they are not also outliers) to know the location of the ends of the whiskers of the box plot.

11 **a** Since the median line is not visible, it is equal to either the upper quartile or lower quartile so that it overlaps the line for either quartile.

b 5, 8, 8, 14, 20 or 5, 8, 14, 14, 20

12 The median age of all Australians is approximately 37 years which is greater than the median age of Aboriginal and Torres Strait Islanders (approximately 23 years). The interquartile range of the ages of all Australians is approximately 37 years which is greater than the interquartile range of the ages of Aboriginal and Torres Strait Islander peoples (approximately 31 years).

13



14 **a** IQR = 9 **b** $Q_1 = 28.5$, $Q_3 = 37.5$

EX
p517

10C Distributions of data

1 **a** symmetric

b positively skewed

c negatively skewed

d negatively skewed

e symmetric

f positively skewed

2 **a** symmetric

b positively skewed

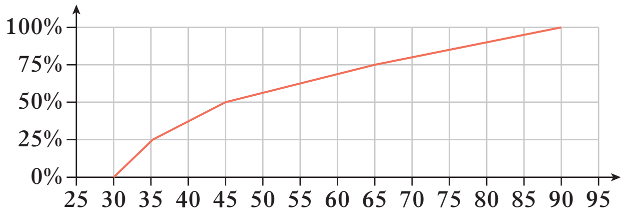
c negatively skewed

d positively skewed

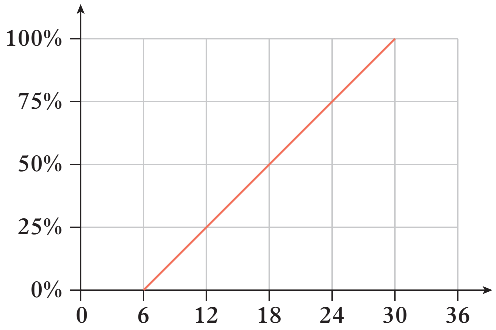
e symmetric

3	**a** 25%		**b** 50%		**c** 25%	
	d 75%		**e** 75%		**f** 50%	

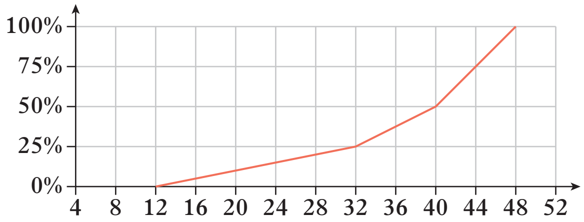
4 **a**



b



c



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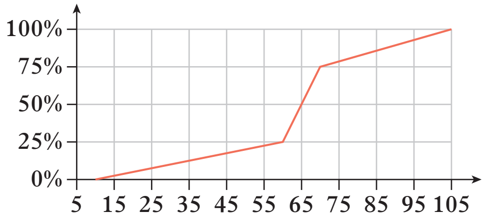

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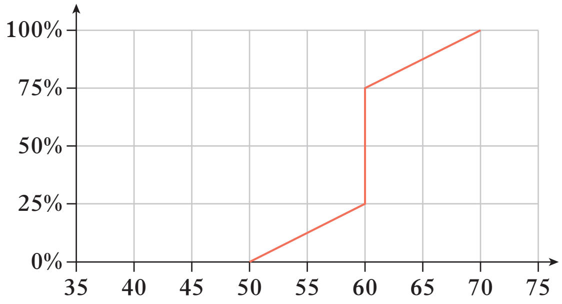
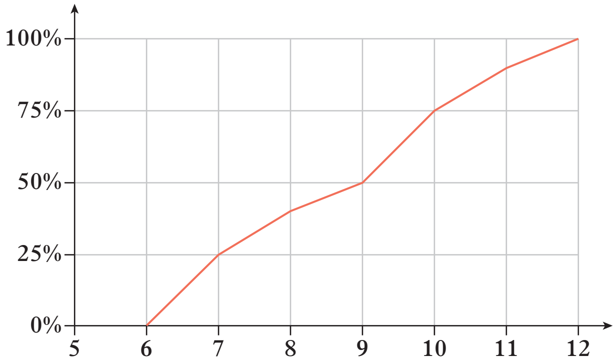


5	**a**	**i**	10, 30, 60, 100, 120	**ii**	70
	b	**i**	16, 28, 40, 44, 52	**ii**	16
	c	**i**	9, 45, 54, 63, 72	**ii**	18
	d	**i**	100, 120, 160, 200, 300	**ii**	80

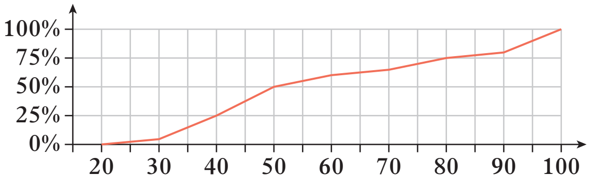
6



7 **a**

Class interval	Frequency	Cumulative frequency	Cumulative percentage
20–<30	10	10	5
30–<40	40	50	25
40–<50	50	100	50
50–<60	20	120	60
60–<70	10	130	65
70–<80	20	150	75
80–<90	10	160	80
90–<100	40	200	100

b



8 **A** and **iii**; **B** and **i**; **C** and **ii**; **D** and **iv**

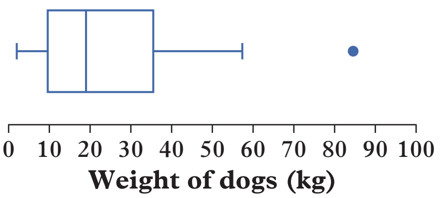
9 **a** **i** Box plot appears negatively skewed.

ii Scores in the first quartile are very closely packed, making the lower whisker very short. In the second quartile, scores are spread far apart, making the lower half of the box long. Scores in the third and fourth quartiles appear spread out to about the same degree: less closely packed than those in the bottom whisker, but more densely packed than those in the lower half of the box.

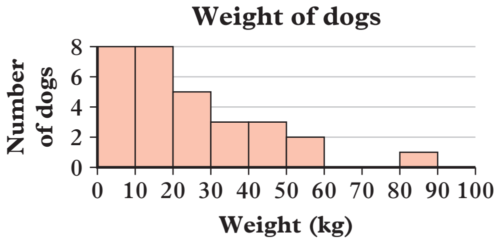
iii The median score is close to the values of data at the top end of the distribution.

b There may be multiple scores of the same value in the lower whisker.

10 a



b

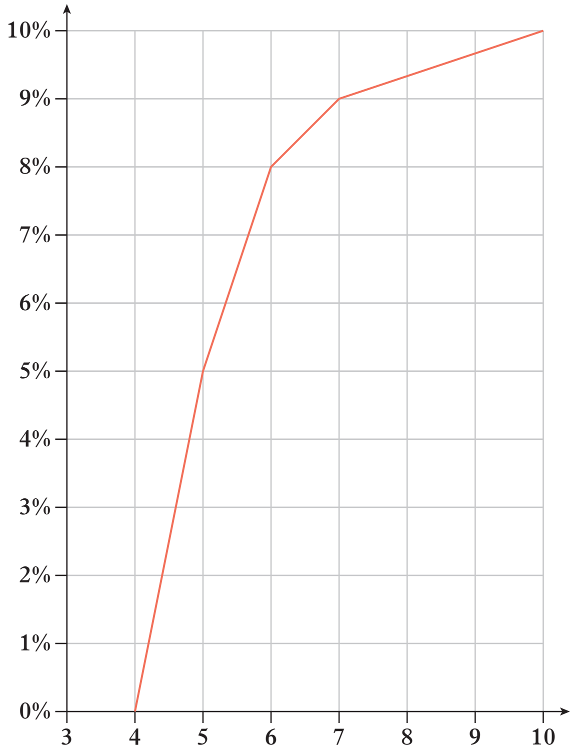


c positively skewed with an outlier

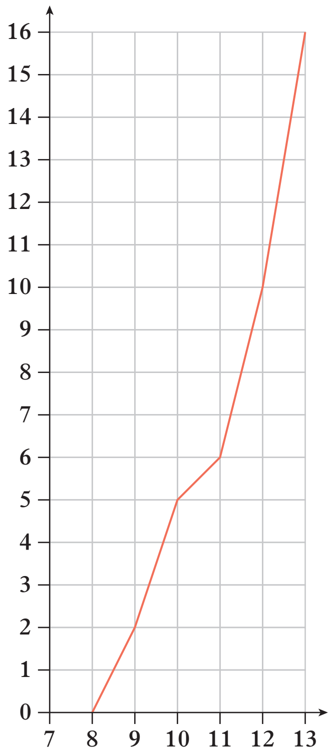
d The histogram shows more detail than the box plot. The box plot shows the positive skew and the median, but not the fine detail.

e To one decimal place, the weights are not repeated and so the dot plot would be a long row of scattered dots which would be difficult to interpret.

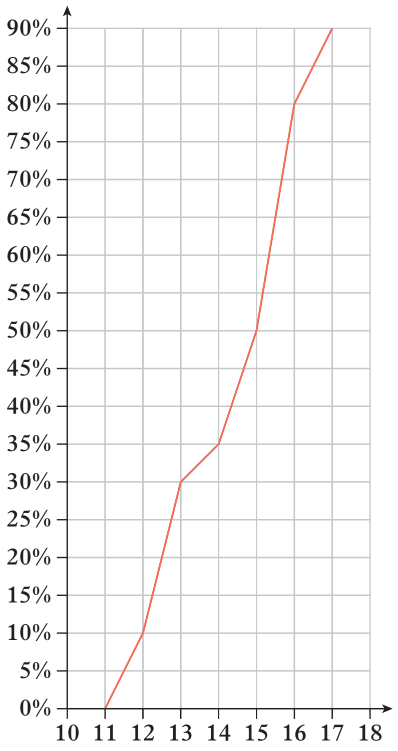
11 a



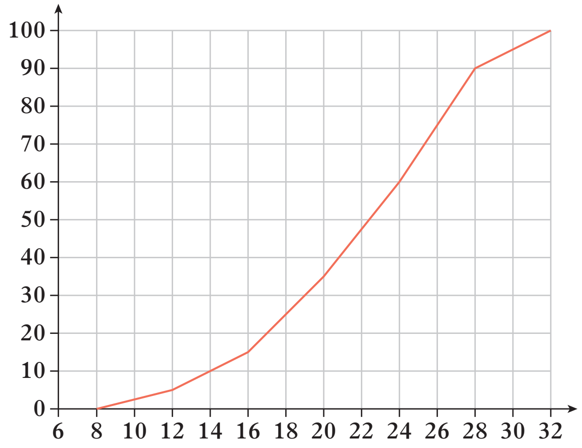
b



c



d



12 a Group A: negatively skewed; Group B: positively skewed

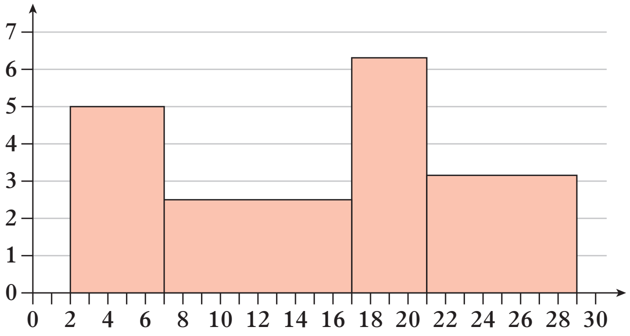
b The mean is affected by both the skew and the outliers the data contains whereas the median is not affected by these. The mean is also not able to be calculated directly from a box plot, whereas the median can be read off easily.

13 6, 14, 24, 32, 34

14 The cumulative frequency always adds to the previous amount therefore the cumulative frequency cannot decrease. This means the gradient cannot be negative.

15 a All of the bars in Sylvia's histogram have the same height as they each represent 25% of the data. This makes it difficult to see the distribution of the data.

b



EX
p525

10D The mean and standard deviation

1 a 26.8 **b** 4.3 **c** 4.1
d 6.1 **e** 30.3 **f** 20.7

2 a 0.0 **b** 3.0 **c** 26.5
d 4.9 **e** 2.3 **f** 21.8

3 a 3.0 **b** 4.4 **c** 401 083.6
d 802 167.1 **e** 401 083.6 **f** 0.1

4 a 9.5 **b** 905.8 **c** 2.9
d 2.9 **e** 2.9 **f** 55.1

5 a i sample **ii** 0.85
b i population **ii** 5.20
c i sample **ii** 0.54

6 a 0.78

b The spread of ages of players in the cricket teams are tightly packed around the mean value, with all bar one player being between 15 and 17 years of age.

7 a mean = 15.88, standard deviation = 8.99

b The data are grouped pretty evenly with a large standard deviation and the mean being very close to halfway between the minimum and maximum.

8 a 41.7 **b** 9.18

c The spread of data is 13 to 58, with the mean at about 42. The mean is towards the higher end of the data, indicating that the data has a negative skew. The standard deviation of scores from the mean is about 10.

9 a the mean for train line A = 5.4
the mean for train line B = 4.5

b the standard deviation for train line A = 1.89
the standard deviation for train line B = 4.57

c The mean for train line A is higher than that for train line B, indicating trains are generally later on line A. Standard deviation for train line A is lower than that for train line B, indicating scores are bunched closely around the mean, so that line A is consistently late. Line B has four outliers of 9, 11, 15 and 19 minutes which have increased the mean and the standard deviation. Without these outliers, the standard deviation decreases to 1.23 and mean for train line B decreases to 2.65.

d The outliers for train line B might have been caused by unusual circumstances. If these outliers are disregarded, it could be said that the train on line A runs late more often. If the outliers are not disregarded, the train on Line B, on average, runs earlier but is very unpredictable.

10 The first histogram has a larger standard deviation than the second, because the scores are more spread out.

11 a 68.51

b The shop may be a discount shop or might be having a sale; the tablet might be a discontinued old model.

c 20.98

d Standard deviation is much lower with the outlier excluded.

e interquartile range

12 City A: mean = 34.3 years, standard deviation = 13.33
City B: mean = 26.1 years, standard deviation = 9.30
People auditioning in City B are generally younger than those auditioning in City A. The standard deviation in City B is also lower than for City A, indicating the spread of ages in City B is lower than for City A. Generally, a group of young people auditioned in City B, and people with a wide range of ages auditioned in City A.

13 mean: 63 kg, standard deviation: 5 kg

14 a The median is a better measure of the centre than the mean when the data has outliers or is skewed as both of these affect the value of the mean.

b The interquartile range is a better measure of the spread than the standard deviation when the data has outliers or is skewed as both of these affect the value of the mean and hence the standard deviation.

15 a Group A has a mean of 29.8 which is less than group B's mean of 30.1. Group A has a standard deviation of 5.5 which is greater than group B's standard deviation of 5.1. Therefore, on average, group A is less than and more variable than group B.

b Group A has a mean of 93.3 which is greater than group B's mean of 77.2. Group A has a standard deviation of 12.4 which is greater than group B's standard deviation of 7.9. Therefore, on average, group A is greater than and more variable than group B.

16 a i $z_A = 1.5, z_B = 1.875$

ii Student B scored 1.875 standard deviations above their class mean, which is greater than the 1.5 standard deviations Student A scored above their class mean.

b i $z_A = 0.75, z_B = 1$

ii Student B scored 1 standard deviation above their class mean, which is greater than the 0.75 standard deviations Student A scored above their class mean.

c i $z_A = -0.8, z_B = -5$

ii Student B scored 5 standard deviations below their class mean, which is less than the 0.8 standard deviations Student A scored below their class mean.

d i $z_A = -3, z_B = -3$

ii Student B scored 3 standard deviations below their class mean, which is the same as the 3 standard deviations Student A scored below their class mean.

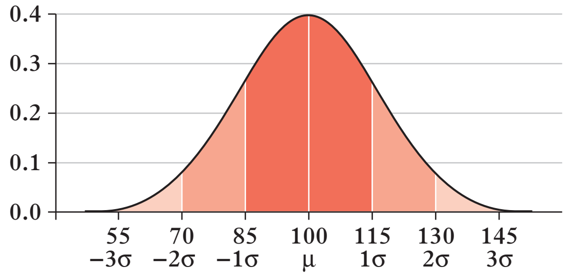
17 a Adding the percentages shown in the given graph:

2.12% + 13.6% + 34.13% + 34.13% + 13.6% + 2.12% = 99.7%

This covers the region of three sigma either side of the mean.

b i 68% **ii** 95%

c



d 55 to 145 **e** higher than 145 **f** less than 70

g 2.5% **h** 16%

i mean = 136, standard deviation = 8

18 a $\mu_{\bar{X}} = 40, s_{\bar{X}} = \frac{9}{5} = 1.8$

b i 68% **ii** 84% **iii** 0.15%

iv 5%

c i 41.8 **ii** 40 **iii** 36.4 and 43.6

iv 36.4 and 38.2 or 41.8 and 43.6

10 Checkpoint

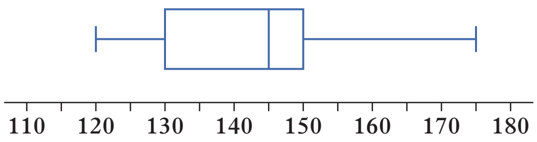
1 a 14, 17, 22.5, 32, 50 **b** 35, 38.5, 45, 54, 60

c 2, 4, 6, 8, 10 **d** 10, 10, 20, 40, 60

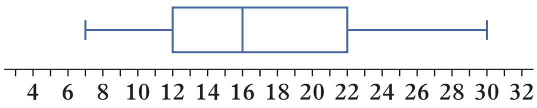
2 a IQR = 11, range = 28

b IQR = 27, range = 35

3 a



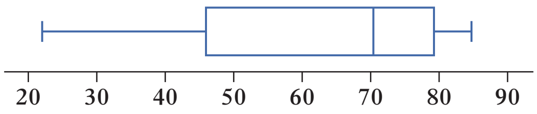
b



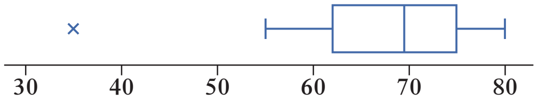
4 a i 16, 23, 27, 35, 38 **ii** 12

b i 20, 56, 72, 76, 80 **ii** 20

5 a



b



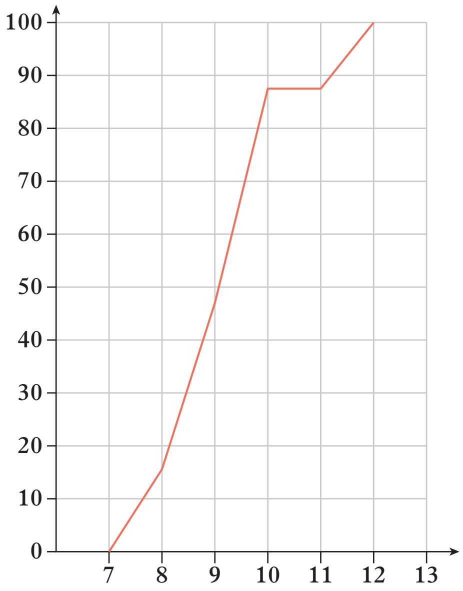
6 The median of group A is 70, which is less than the median of group B, 77. The interquartile range of group of A is 21, which is less than the interquartile range of group B, 28. Therefore, group A is, on average, less than group B and less variable.

7 a negatively skewed **b** symmetric

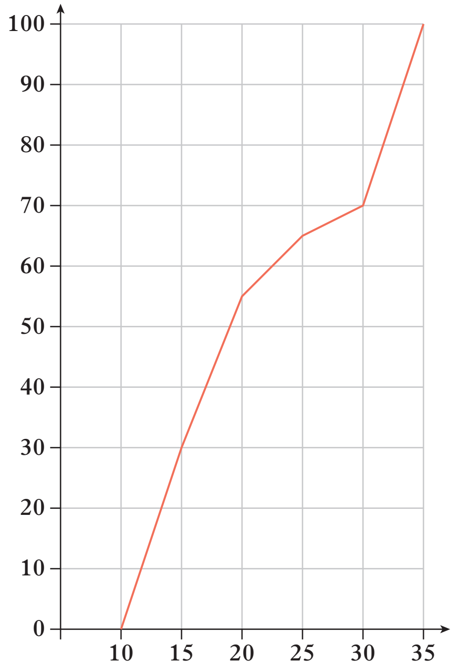
c positively skewed **d** positively skewed

8 a 25% **b** 50% **c** 75% **d** 50%

9 a



b



10 1, 3, 6, 9, 11

11 a mean = 5, sd = 2 **b** mean = 13, sd = $\sqrt{15}$

12 a mean = 48.5, sd = 25.07 **b** mean 107.34, sd = 8.69

EX p534

10E Scatterplots and bivariate data

1 a i a moderate to weak negative trend

ii With increasing height, the time taken to run 10 km decreases moderately.

b i weak positive trend

ii The number of visits to the doctor per year increases slightly as the number of colds per year increases.

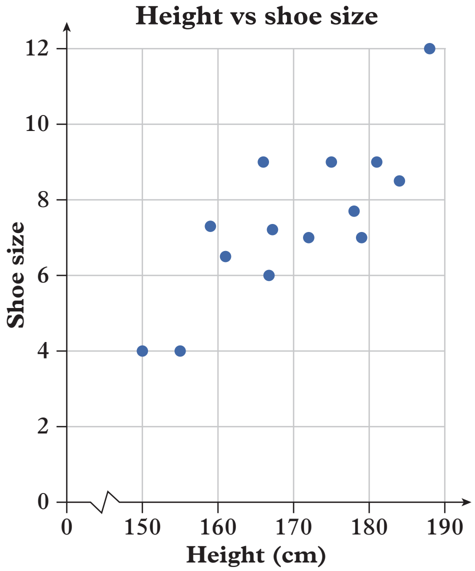
c i strong positive trend

ii The amount of petrol used increases greatly with an increase in the distance travelled.

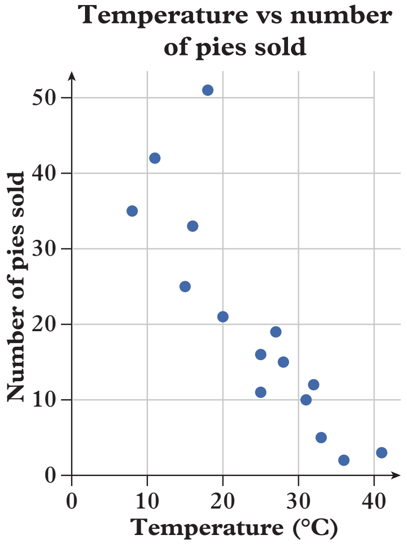
d i moderate negative trend

ii Exam marks decrease moderately with an increase in the hours of television watched per week.

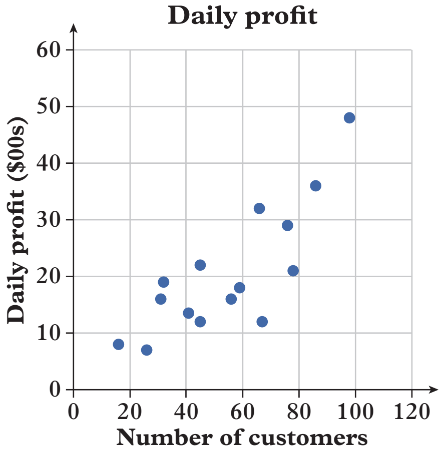
2 a



b



c



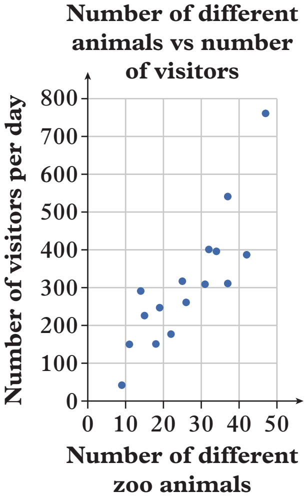
3 **a** There is a moderate increase in shoe size with an increase in people's height.

b There is a strong negative correlation, so the number of pies sold decreases greatly as temperature increases.

c There is a moderate increase in daily profit as the number of customers increases.

4 **a** moderate positive correlation

b



c There is a moderately strong increase in the number of visitors per day as the number of different zoo animals increases.

5 **a** As temperature increases, the number of heaters purchased decreases strongly. There is a strong negative correlation between the temperature and the purchase of heaters. But there is a weak positive correlation between someone's earnings and their monthly phone bill. As the earnings increase, the phone bill does also tend to increase.

b about 4

c about $90

d The prediction from scatterplot A is not very reliable because there is not a strong relationship between the variables. For scatterplot B, the relationship is much stronger, so a prediction would be more reliable.

6 **a** The weight depends on height.

b As a person gets taller, he/she generally becomes heavier.

c **i** weight **ii** height

7 **a** **i** Up to a certain time (the age of the tree) there is a relationship.

ii The independent variable is time, the dependent variable is height.

iii The correlation is strong and positive in the early years, but not so as the tree ages.

b **i** Up to a certain age of car there is a relationship between the value of a car and its age. If the car becomes an antique, or a collectable item, this relationship changes.

ii The independent variable is the age of the car; the dependent variable is the value of the car.

iii Correlation is strong and negative, car losing value as it ages.

c There is no relationship between the weight of a person and the number of languages spoken.

d **i** There is a relationship between a person's shoe size and height.

ii The independent variable is height; the dependent variable is shoe size.

iii The correlation is strong and positive. Generally, as a person grows and gets taller, his/her shoe size increases.

8 **a** The strong positive correlation between the hours studied and the marks achieved supports the view that more study time contributes to achieving a higher percentage result on a test. However, extra study time does not always guarantee a higher test score; the correlation is for a sample and for an individual the result can vary. Also, there may have been another underlying reason that explains the high marks and the extra time studying may have just been a coincidence.

b A high correlation does not imply that one variable 'caused' another to happen. There could be an underlying reason that is hidden and not known, and the correlation seen may actually just be a coincidence or due to choosing a particular sample.

c It is commonly thought if the relationship between two variables is strong, one must have a strong influence on the other. Sometimes this is true, but the particular circumstances need to be examined to determine this.

9 **a** no correlation

b strong, positive

c moderate, positive

d no correlation

10 Answers will vary.

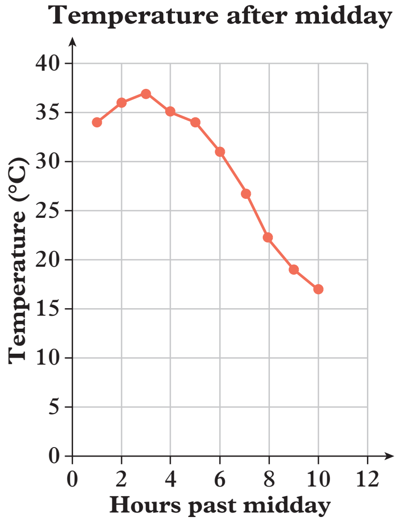
11 While the previous examples have used only positive values for the dependent and independent variables, it is possible to plot data that uses negative values, such as temperature, and end up with a negative section on either axis.

12 To plot data values in a scatterplot you need to know the exact values of each variable. When dealing with grouped data, the exact values are no longer known, so you would not be able to accurately plot each point.

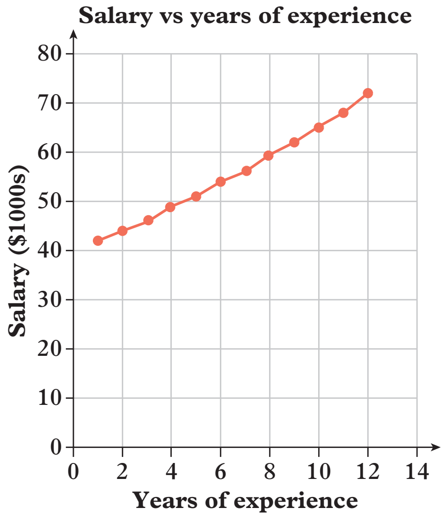
EX p540

10F Time series

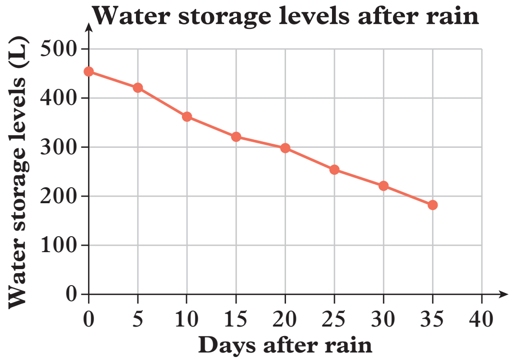
1 a


Temperature after midday

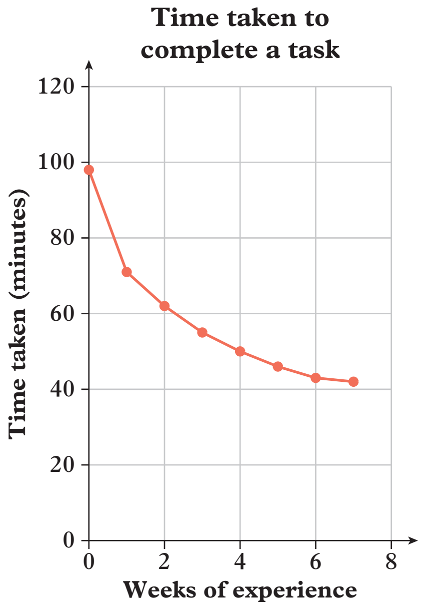
b


Salary vs years of experience

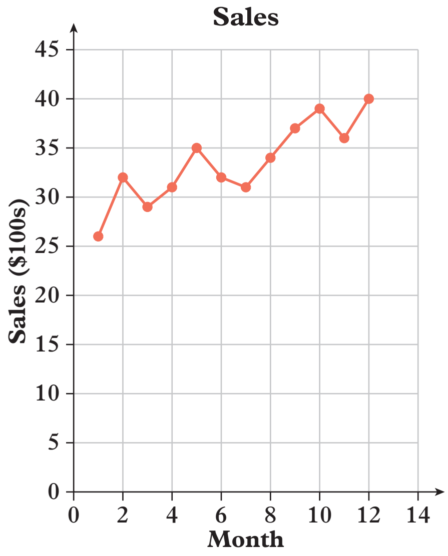
c


Water storage levels after rain

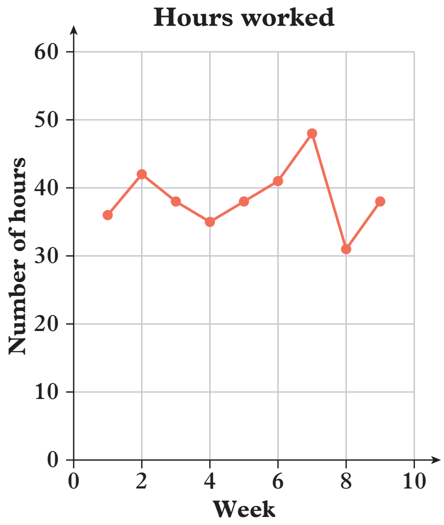
d


Time taken to complete a task

e


Sales

f


Hours worked

2 **a** **i** no seasonality **ii** downward

b **i** seasonality **ii** stationary

c **i** no seasonality **ii** upward

d **i** no seasonality **ii** upward

e **i** no seasonality **ii** downward

f **i** seasonality **ii** downward

3 **a** downward

b upward

c downward

d downward

e upward

f stationary

4 **a** The price of milk per litre increases as time increases with no seasonality.

b The temperature of the child decreases as time increases, with some random fluctuations.

c The weight of the baby increases as time increases.

d The price of petrol shows a seasonal trend which appears to be increasing as time increases.

5 **a** As time increased, the height of the candle steadily decreased.

b Throughout the year, the number of people at the beach decreased with the approach of winter, then increased as summer approached. It reached its minimum in the middle of winter and its maximum in the middle of summer.

c Over the 50 years represented by the graph, the population of the outer city suburb increased, with population growth being greatest in later years.

d Over the 10 hours of the car trip, the distance travelled increased steadily as time increased.

e Over the 8 years the shark population was monitored, the population decreased with a non-linear trend, decreasing more sharply initially.

f Over the 3 years that sales were recorded, they appear to have decreased with a seasonal trend.

6 **a** Temperature increased slightly until it reached a peak at 3 pm, then it decreased steadily until 10 pm.

b With increasing years of experience, salary increased steadily.

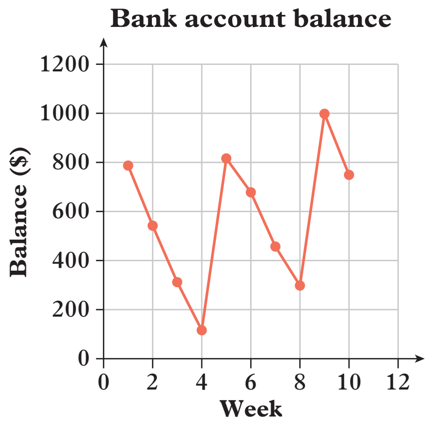
c As the number of days after rain increased, the water storage level decreased steadily.

d The time taken to complete a task decreased as the weeks of experience increased, decreasing more sharply initially.

e Sales increased as time increased with some random fluctuations.

f There does not appear to be a trend between the week and the hours worked, varying around a stationary value.

7 **a**

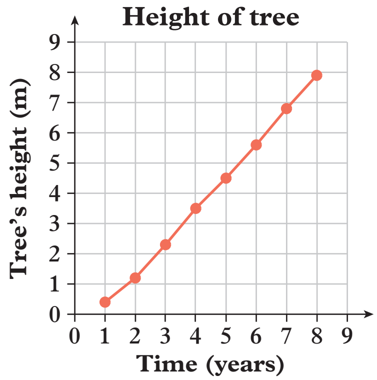


The time series is seasonal with an upward trend.

b **i** $116 **ii** $998

c Increases in bank balance could be caused by the periodic crediting of a work payment into the account. The decreases in the balance indicated by the downward movement of the graph are probably caused by living expenses being withdrawn from the account.

8 **a**



b The trend is steadily increasing; as the tree grew older (as the time passed increased), the tree grew taller.

c **i** about 10 m

ii If the tree continues to grow at the same rate, it should be about 21 m tall.

d Both of the predictions made in part **c** are extrapolated values. In the first case, the prediction is for a time (10 years) just beyond the recorded data (which makes the prediction reasonably likely to be accurate). In the second case, the prediction is for a time (20 years) well after any of the recorded values (making it very unreliable). The tree could, in fact, die or reach its maximum height before it reached 20 years of age!

9 **a** The graph shows the number of yearly enrolments in two courses, Italian and Spanish, over a period of 8 years.

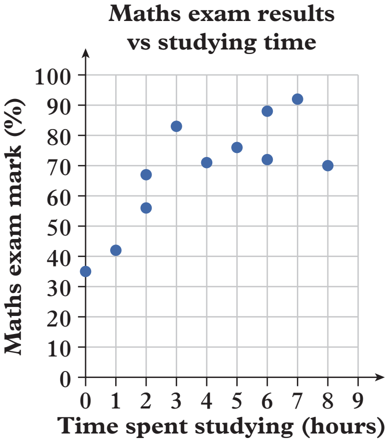
b The trend for course enrolments in Italian shows fluctuation, increasing and decreasing over years, but the general trend indicates a decrease. Enrolments in Spanish remained fairly stationary for the first 5 years, then started to increase quite rapidly.

c If the trends for both graphs continue, Spanish could have more enrolments than Italian in 2 years from the data displayed (at the 10-year mark).

10 a The data represents the results of a number of different individuals rather than changes in the performance of a single individual over time.

b This is not time series data, because the time referred to is not time passing, measured in equal intervals, but rather the time each student studied.

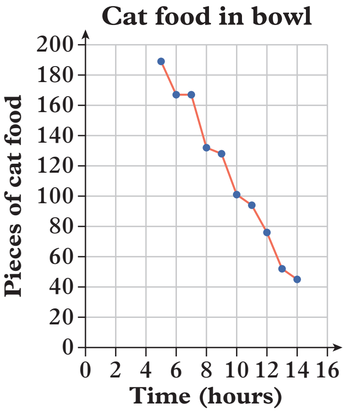
c



The trend is upward. The maths marks generally increased for students who spent more time studying.

d A time series follows data of a particular category over time. These data are not continuous observations over a period of time.

11 a



b The trend is downward. As time increases, the number of pieces of cat food in the bowl decreases.

c No, at some time in the future, either it will reach and stay at zero until it's refilled, or else the cat's bowl will be refilled with food earlier before it reaches zero.

12 a After decreasing from 1840 to 1850, the number of species generally increases from 1850 to 1980, increasing faster between 1940 and 1980 than 1840 to 1940. After 1980 to 2010, the number of species decreases rapidly.

b Yes, because the trends of both all species and eucalyptus over time are similar in direction and in proportional change in the same time periods.

13 a 30.5

b

t	1	2	3	4	5	6
y	38	23	15	20	30	20
Seasonal index	1.25	0.75	0.49	0.66	0.98	0.66

t	7	8	9	10	11	12
y	40	35	30	45	30	40
Seasonal index	1.31	1.15	0.98	1.48	0.98	1.31

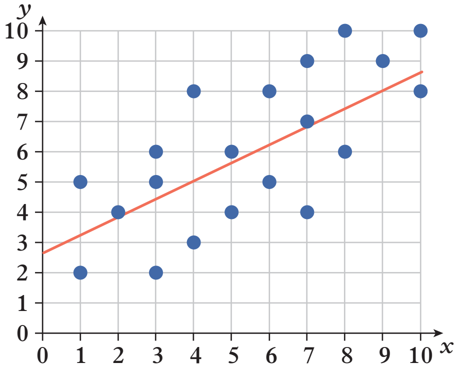
c **i** 25% below **ii** 31% above
iii 48% above **iv** 2% below

d **i** increase by 33.33% **ii** decrease by 23.66%
iii decrease by 32.43% **iv** increase by 2.04%

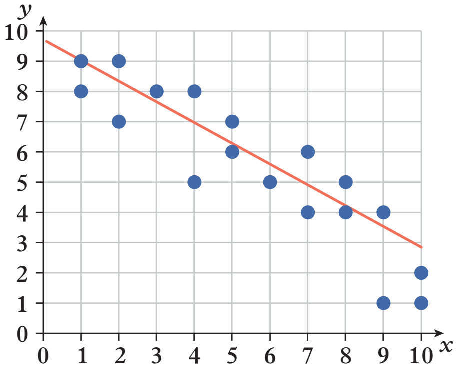
EX p548

10G Lines of best fit

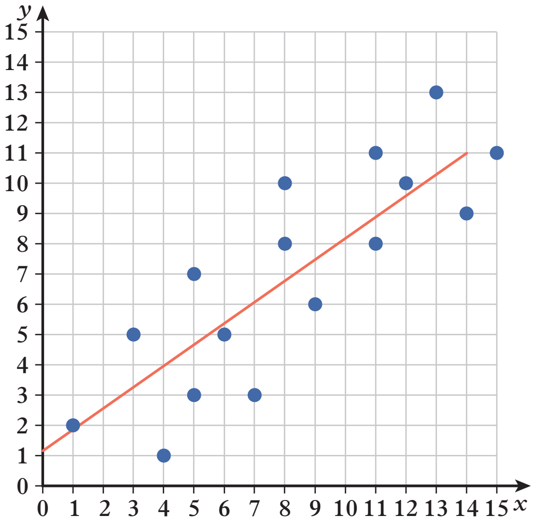
1 a



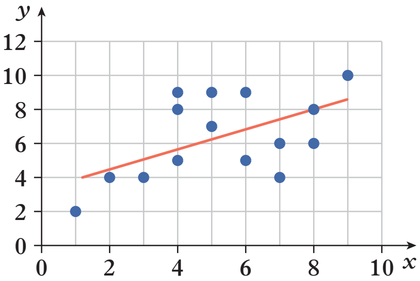
b



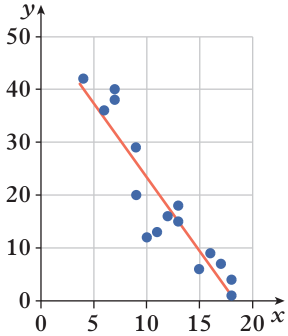
c



d



e



2 **a** $y = 3.1x + 4.6$

b $y = 0.8x + 4.5$

c $y = –2.4x + 23.8$

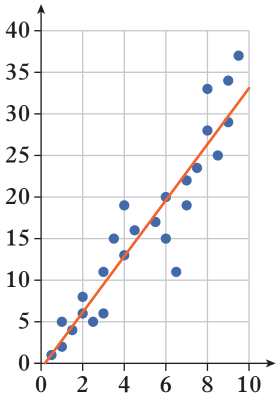
d $y = 2.3x + 5.9$

3 **a** 3.7 **b** 10.7 **c** 37.7 **d** 63.9 **e** 5.4

f 2.5 **g** 69.2

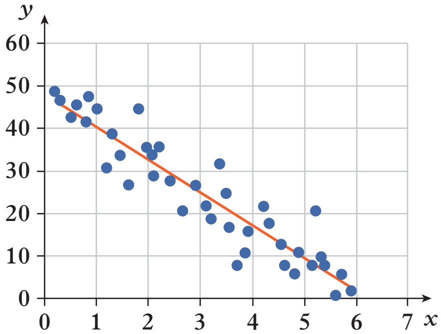
4 **a** 35.6 **b** 12.5 **c** –0.2 **d** 28.9

5 **a** **i**



ii 16 (answers may vary)

b **i**



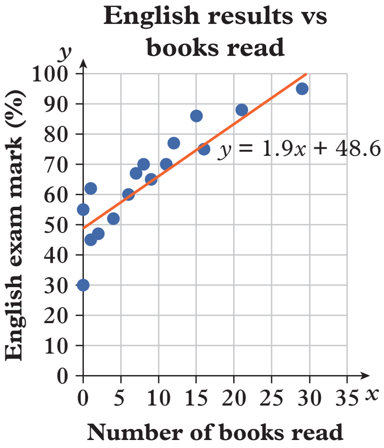
ii 10 (answers may vary)

6 **a** a strong, positive relationship

b independent variable = number of books read

dependent variable = English exam mark

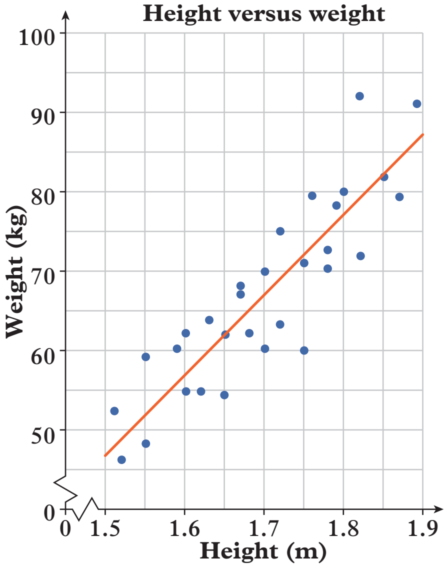
c, d



e The prediction based on the data would be that he should read about 11 books, but this does not guarantee an English exam mark of 70%.

7 **a** The trend is moderately strong and positive. As height increases, weight also increases.

b



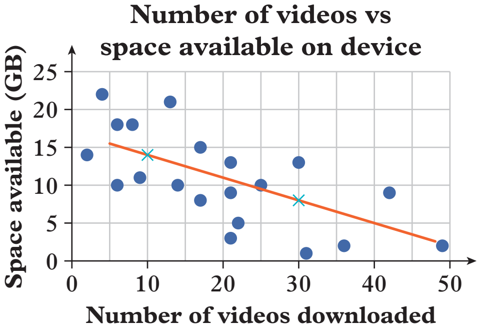
c 82 kg (answers may vary)

d If the chosen points are spread far apart, this minimises error in guesswork and creates a more accurate equation.

e $y = 100x – 103$

f 82 kg. It gives the same result (answers may vary).

8 **a, c**



b The is a weak negative relationship. Generally, as the number of videos increases, the space available on the device decreases.

c $y = -0.3x + 17$ (answers may vary)

d 8 GB

e The relationship is not very strong, so predictions are not reliable.

f Videos do not all require the same space on a device, so the relationship is not linear.

g **i** interpolation **ii** extrapolation

iii interpolation **iv** extrapolation

h If the trend continues, downloading 200 videos would mean there was negative space (–43 GB) available. However, there would be no space available before this point could be reached.

i Making predictions outside the given data set can give unrealistic results; for example, in part **h**, you cannot have negative space available. This means we cannot be confident about making a prediction outside this data set.

9 a **i** $y = 90$, interpolation

ii $y = 105$, extrapolation

iii $y = 80$, interpolation

iv $y = 90$, extrapolation

b **i** $x = 30$, interpolation

ii $x = 55$, interpolation

iii $x = 80$, extrapolation

iv $x = -5$, extrapolation

c **i** $0 \leq x \leq 60$

ii $x < 0$ or $x > 60$

d **i** $50 \leq y \leq 100$

ii $y < 50$ or $y > 100$

10 a $y = 9.69 + 5.01x$

b **i** 1.2 **ii** –3.2 **iii** 0.29 **iv** 4.6

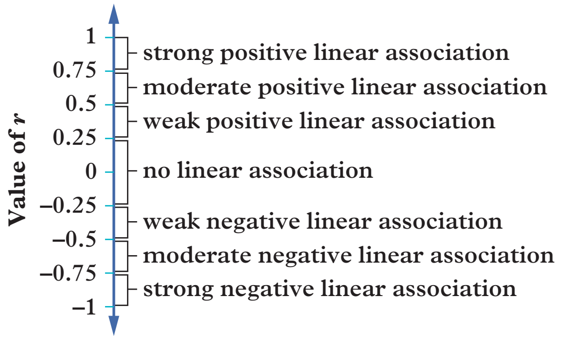
c **i** 7.1% **ii** –22.1% **iii** 1.45% **iv** 17.8%

11 a Relationship between the two variables is strong and positive in scatterplot A, but weak and positive in scatterplot B.

b scatterplot A = 18, scatterplot B = 15 (answers may vary)

c You should have more confidence in your prediction from scatterplot A because it's a stronger correlation and $x = 8$ is close to several other points that lie close to the line of best fit.

d



12 a $\bar{x} = 4, \bar{y} = 5$

b
$$\begin{aligned} y &= \frac{27}{11} + \frac{7}{11}x \\ &= \frac{27}{11} + \frac{7}{11} \times 4 \\ &= \frac{27}{11} + \frac{28}{11} \\ &= \frac{55}{11} \\ &= 5 \end{aligned}$$

c $y = a + bx$

substitute $x = \bar{x}$ and $y = \bar{y}$

$\bar{y} = a + b\bar{x}$

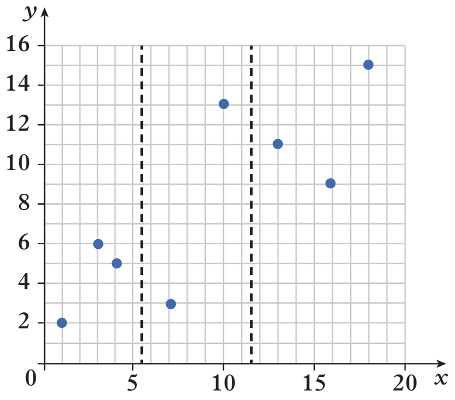
$a = \bar{y} - b\bar{x}$

d $s_x = \frac{\sqrt{44}}{3} = \frac{4\sqrt{11}}{3}, s_y = \frac{\sqrt{34}}{3}$

e $r = 0.7239$

f moderate

13 a

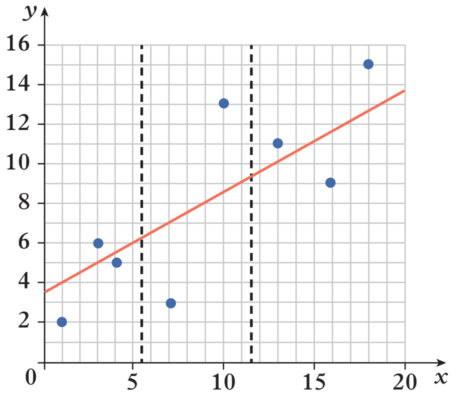


b For first section: median x-coordinate = 3, median y-coordinate = 5

For second section: median x-coordinate = 9, median y-coordinate = 8

For third section: median x-coordinate = 16, median y-coordinate = 11

c



d $y = 0.5x + 3.5$

e $y = 0.5x + 3.7$; this equation is very close to the equation you found in part **d**.

f The effect of extreme values (outliers) is minimised by finding and using the median value of that particular third.

EX 10H Evaluating statistical reports

p555

1 **a** biased; people at an AFL football match are not a representative sample of the whole country.

b fair; this should provide an answer representative of the views of all the people in school.

c biased; the question is not fair and leads to only responses regarding people's attitude to dogs.

d biased; a sample of 100 is too small to represent the population.

e biased; people outside your community would probably not have any appreciation of particular community issues, so their views could skew the results.

f biased; a small sample of one section of the population is not likely to be representative of the entire population.

2 **a** There is no indication of how people were chosen for survey or exactly what they were asked.

b People shopping in an Apple store are probably Apple customers/supporters, so this is a biased sample.

c The sample is too small, and there is no indication of the gender or employment status of people making up the sample or whether it was their opinions or facts about their personal earnings that they were questioned about.

d Shareholders in a company have a vested interest in seeing that company's product succeed.

3 **a** **i** The data collection method is not fair. Tess only surveyed people from her town, when the population needing to be considered is the entire poulation of NSW.

ii The interpretation is biased because the sample is biased.

iii Tess should have surveyed people across all of NSW rather than just in her town.

b **i** The sample is too small for any results to be reliable.

ii Because the sample is so small, the median would be the most reliable result to take.

iii Survey a larger sample, and watch out for outliers.

c **i** The question only asked if the people surveyed like Mac computers, but not if they prefer them to other types of computers. This is not a fair data collection method.

ii Because the question is biased, no assumption can be made.

iii The question could have been, 'Which brand of computer do you prefer?'

d **i** The data collection method is fair.

ii The interpretation of the data is poor. The figure does not need to be above 50% for the soft drink to be the most popular. It just needs to be a greater percentage than any other soft drink.

iii She should conclude Coke is the most popular.

e **i** The sample only consisted of people walking through the mall, so this could lead to biased results (especially when the survey is about time spent sitting down). It's also not clear who her population is.

ii A poor method of collection means the interpretation could be biased.

iii Sienna should have specified her population and have sampled people in various locations for her survey to be representative of the population. It might also be best to use the median as the measure of centre for this data.

f **i** Kane's survey method is biased because his neighbourhood contains only a small section of the target population and also includes people who are not part of the target population.

ii The interpretation is not fair because the sample is biased.

iii Kane should have surveyed a sample of people who will be attending the school dance.

4 **a** Only one person claimed 18 hours pain free. Most of the claims are far less than 18 hours but, because one person said 18 hours, the claim of 'up to 18 hours' has been made.

b mean = 3.1, median = 2, mode = 2

c This claim is not too far from the truth. However, the median or the mode would be a better representative value.

5 How was the data collected? How big was the sample? Does the data represent a sample of the population? How was the questionnaire designed? Is the interpretation fair?

6 **a** If a phone is in stand-by mode, its battery could last up to 7 days. But, in normal use, a phone would not be in stand-by mode for 7 days!

b The 42 people surveyed could have been chosen for the survey because they were purchasing that brand of toilet paper at a supermarket. It is also a very small sample size that is not representative of all of Australia.

c This moisturising lotion was not compared with another moisturising lotion, but with soap.

7 The refugee population in Australia as a percentage of the total Australian population in 2018 is approximately 0.23%. This is more than the refugee population as a percentage of the respective populations in New Zealand (0.03%), the United States (0.10%) and the United Kingdom (0.19%), but less than the refugee populations as a percentage of the respective total population in Canada (0.31%), Germany (1.28%), Turkey (4.44%), Pakistan (6.78%) and Jordan (7.18%).

8 **a** 8th

b Because 98 countries were assessed

c Because many countries were not assessed, and China was specifically excluded

d Other countries that were not included may have handled the coronavirus pandemic better than Australia did.

e Look up the Lowy Institute and find the data they used to rank the countries.

9 Teacher to check.

10 Teacher to check.

EX p560

10I Sampling and reporting

1 **a** This survey has used results from one gym to represent the entire population of Australia, meaning it is extremely biased.

b Two boys out of four does not represent a sufficient sample on which to base the claim. Information such as socio-economic backgrounds, ages, suburbs are not considered.

c If only five children were surveyed, the sample is not large enough to represent the larger population claimed. If the reference is to four-fifths of a larger population, more details are needed about what is considered a 'fair share' of chores and of whom that population was made up. Also, the children stated they 'believe they do' their fair share, but the report simply says they 'do' their fair share.

d Children in the stands at a swimming carnival do not represent an entire population. It is not necessary for people to wear sunscreen indoors so this would be a biased result.

2 **a** They have provided no evidence to support this claim and no sample size is given. They should be able to answer the following questions: Did any of the women surveyed have diabetes, and if so, what type and to what degree? Were women who had coffee at other times of day surveyed too? Is there a biological link between coffee and diabetes?

b No evidence has been provided to support this claim and no sample size is given. They should be able to answer the following questions: How was the survey conducted? If the information was collected from a sample containing some people with Alzheimer's disease, is the information reliable? What timeline was followed? Was it conducted over many years and were the results monitored? Is there a biological link between being physically active and Alzheimer's disease?

c In order to determine the accuracy of data, information is required about the source of the survey results, who conducted the survey, how it was conducted, the sample size surveyed and the population from which the sample was chosen.

3 **a** The scale on the *x*-axis is not uniform, and the scale on the *y*-axis does not start at zero, with no break indicated.

b **i** The revenue declined from \$100 000 to \$96 500, a loss of \$3500 which is not a significant decline. The decrease from October to November is a decrease, of \$1000 over the course of a month. This is about 30% of the total loss over eleven months, which is not a slower rate.

ii Since the scale on the *y*-axis does not start at zero, the range of \$20 to \$20.50 can be stretched out. This gives the impression that the increase is greater than it really is.

iii 3.5%

4 **a** This graph is misleading as the scale on the *y*-axis is not labelled so we do not know if it starts at zero or not or if the scale on the *y*-axis is uniform or not.

b **i** The graph is said to display the time taken before relief, therefore the company's brand takes longer before relief and is therefore less effective.

ii By not labelling the *y*-axis, the company can make the graph look like a bigger bar represents a more effective product as most people assume taller or bigger is better. The bar appears twice as large but with the lack of scaling we cannot verify if the value it actually represents is twice as large.

5 **a** This graph is misleading as the ages of 12 and 15 (at least) are missing.

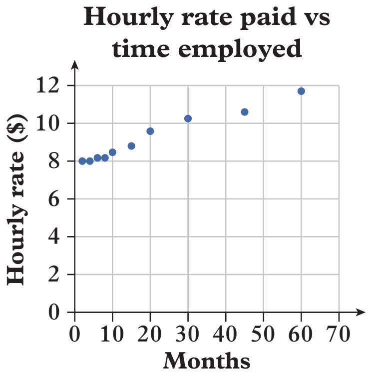
b **i** Since we do not know how many (if any) 12- and 15-year-old students joined the school band, we cannot accurately determine the shape of the distribution or know if there are students in all year levels.

ii By omitting the columns for 12- and 15-year-old students, the band conductor can make the data appear more symmetric with an average age of 16.

6 **a** Fatima's hourly rate of pay over 60 months of employment

b Scale on the *x*-axis is not uniform and the scale on the *y*-axis doesn't start at the origin.

c



7

Age group (years)	**Percentage**	**Number to survey**
0–4	6.179	618
5–9	6.382	638
10–14	6.133	613
15–19	5.923	592
20–24	6.935	694
25–29	7.525	753
30–34	7.458	746
35–39	7.021	702
40–44	6.291	629
45–49	6.616	662
50–54	6.051	605
55–59	6.091	609
60–64	5.473	547
65–69	4.829	483
70–74	4.171	417
75–79	2.895	290
80–84	1.993	199
85–89	1.234	123
90–94	0.605	61
95–99	0.174	17
100 and over	0.02	2

8

Zone	**Frequency**	**Number to survey**
Mallee	37 300	27
Upper Wimmera	138 100	101
Lower Wimmera	427 700	313
Central	658 950	483
Otway	236 950	174
North east	239 850	176
Gippsland	172 700	126
State-wide total	**1 911 550**	**1400**

9 No. Success in school depends on more variables than just whether or not a student eats breakfast. Those who use the first statement may be trying to sell breakfast foods.

10 The source of information, the sample used to obtain the results and the source of funding for such surveys should be stated with any reported results.

11 Teacher to check.

12 Teacher to check.

13 Teacher to check.

14 Not necessarily. The sample only includes people who watch the show and from those people it only includes those viewers who decide to call.

15 **a** incorrect, $\frac{18}{60} \times 100\% = 30\%$

b correct, $\frac{4}{65} \times 100\% = 6\%$

c correct, $\frac{36}{125} \times 100\% = 29\%$

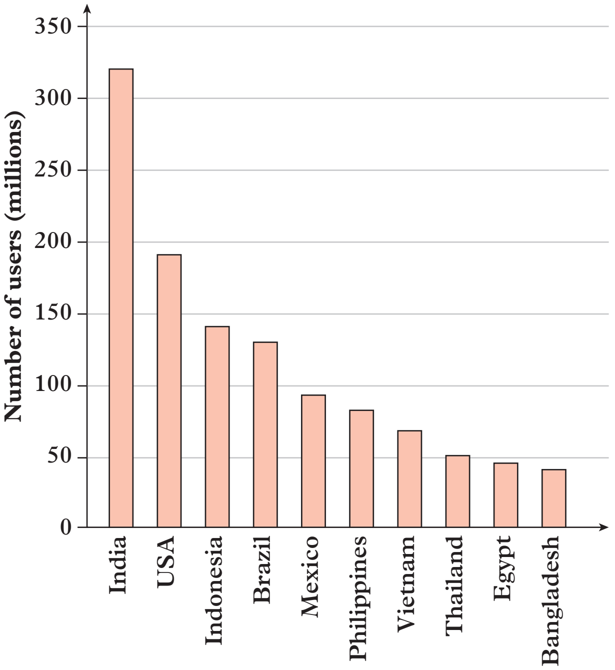
d incorrect (but close to correct), $\frac{31}{61} \times 100\% = 51\%$

16 **a** India: 23.2%, USA: 57.4%, Indonesia: 41.2%, Brazil: 61.1%, Mexico: 72.1%, Philippines: 75.7%, Vietnam: 69.9%, Thailand: 73.1%, Egypt: 44.0%, Bangladesh: 24.9%

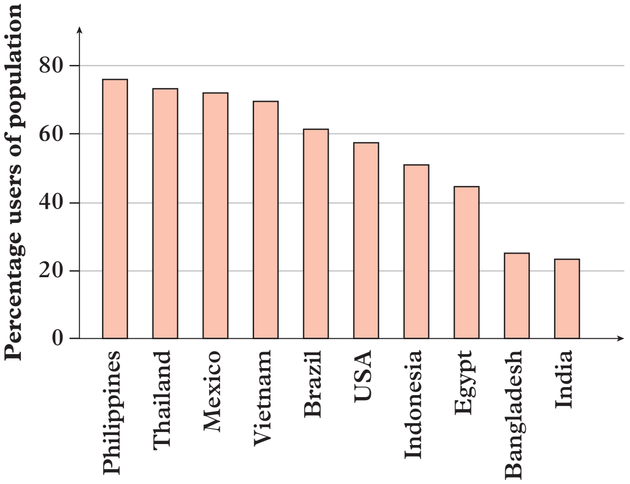
b **i** Philippines **ii** India

c Facebook is banned in China, and China has its own social networks (e.g. Weibo).

d **i**



ii



e **i** By ordering by number of users, it shows India and USA have the most number of users compared to the other countries.

ii By ordering by the percentage of the population that are users of Facebook, India is at the bottom of this group of countries.

17 Teacher to check.

CHAPTER 10 review

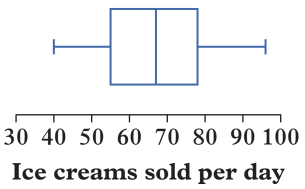
Multiple-choice

1 C **2** B **3** B **4** B **5** C
6 B **7** D **8** D **9** C **10** A

Short answer

1 **a** 56 **b** 67 **c** 23
d 40, 55, 67, 78, 96

2

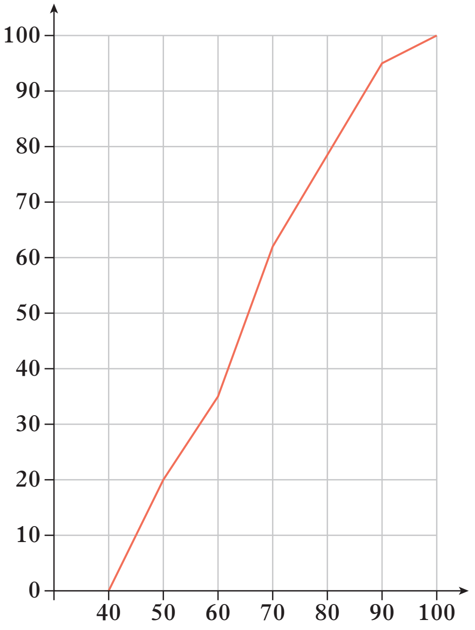


The box plot shows that 50% of the time between 55 and 78 ice creams are sold each day and 67 ice creams are sold half of the time. The data are pretty evenly spread.

3 **a**

Class interval	Frequency	Cumulative frequency	Cumulative percentage
40–<50	6	6	19
50–<60	5	11	35
60–<70	8	19	61
70–<80	5	24	77
80–<90	5	29	94
90–<100	2	31	100
Total	**31**	**31**	**100**

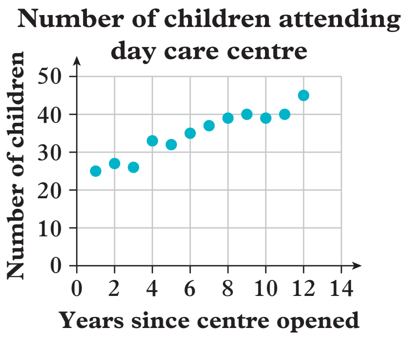
b



4 **a** Lily: **i** 72 **ii** 3.69
Cassy: **i** 71 **ii** 18.28

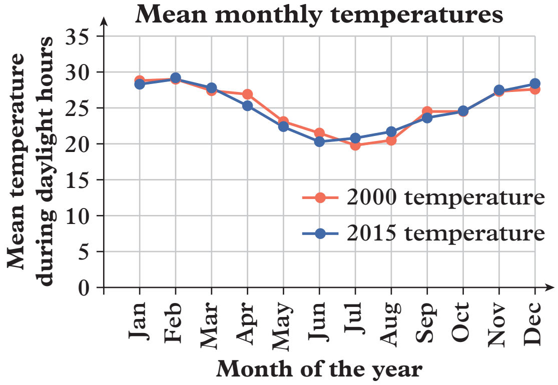
b The lower the standard deviation, the closer the scores lie to the mean. A low standard deviation indicates consistency, so Lily has performed more consistently than Cassy. This is also obvious from the raw scores, where it can be seen that Lily's scores are grouped closer together than Cassy's.

5 **a**



b The number of children in the day care centre increased the more years the centre was open.

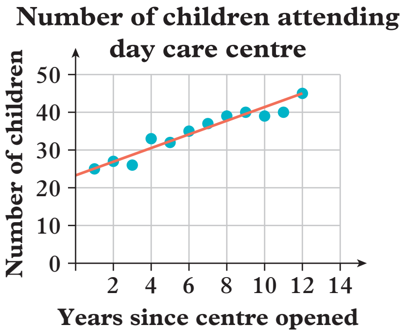
6 **a**



b The trend of both graphs is cyclical.

c Some of the temperatures are a little warmer in 2015 than they were in the corresponding months of 2000, and some are a little cooler. Overall, there appears to have not been too much change over the years.

7 **a, b**



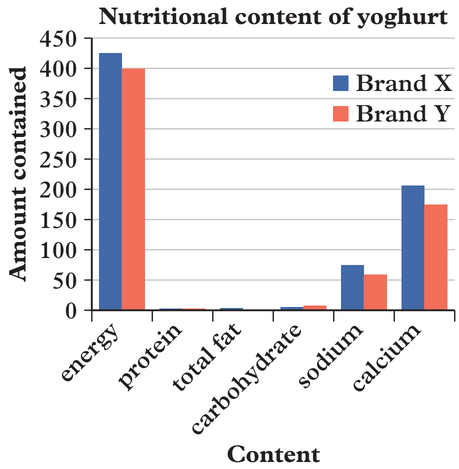
b 58 (answers may vary)

c Predictions made using the line of best fit assume that the trend will continue in the same way indefinitely. This is not always the case. For example, the day care centre could close within that time.

8 **a** Brand X has more energy, protein, total fat, sodium and calcium than Brand Y, but it has less carbohydrate.

b Carbohydrate is the only category in which Brand X is lower and it is significantly lower in carbohydrate than Brand Y. It would appear that we should assume 'light' refers to the amount of carbohydrate contained in Brand X, but 'light' is more commonly used to mean 'low fat'.

c



The graph is misleading. All the figures (regardless of their units) are graphed on the same axes, so total fat content appears to be almost zero.

9 **a** systematic sampling

b stratified sampling

Analysis

a Over the years, the time for winning the event has generally decreased, although there has not always been a decrease from one Olympic year to the next.

b **i** mean = 10.04, median = 9.96, mode = 10.06

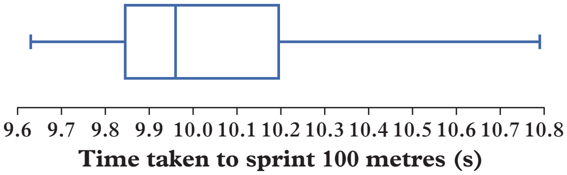
ii range = 1.16, IQR = 0.35

iii 9.63, 9.845, 9.96, 10.195, 10.79

c **i** standard deviation = 0.298

ii The standard deviation value is quite low, indicating all times are close to the mean value.

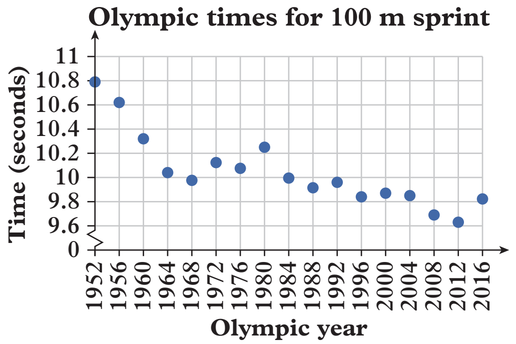
d **i**



ii Times are positively skewed, being packed more tightly below the median than above the median. This indicates it is becoming increasingly difficult to lower the time for the race.

iii The lower whisker will move to the left, and the lower half of the box plot will move left and shrink in size as the times become more densely packed in that region of the plot.

e



f Teacher to check.

g $y = -0.013x + 36.0$

h Teacher to check.

i 2040

CHAPTER 11 Probability

EX
p574

11A Theoretical probability

1 **a** {1, 2, 3, 4, 5, 6, 7, 8}

b {A, U, S, T, R, L, I}

c {red, white, blue}

2 **a** equally likely

b not equally likely

c not equally likely

3 **a** {M, A, T, H, E, I, C, S}

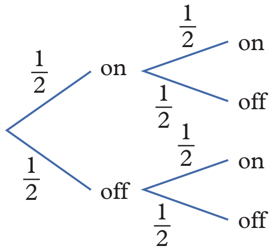
b $\frac{2}{11}$

4 **a** 0.25

b 0.25

c 0.5

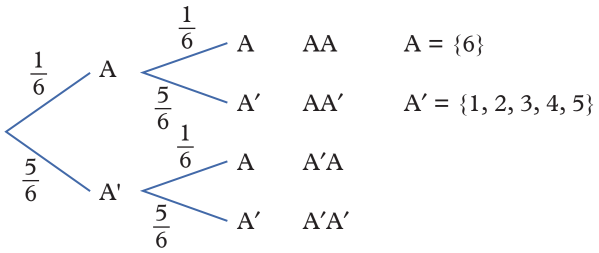
5 **a**



b **i** 0.25 **ii** 0.25 **iii** 0.5

6 **a** (not a 6)

b



c **i** $\frac{1}{36}$ **ii** $\frac{11}{36}$

7 **a** Each outcome now has an equal chance of occurring.

b 36

c **i** $\frac{1}{6}$ **ii** $\frac{1}{4}$

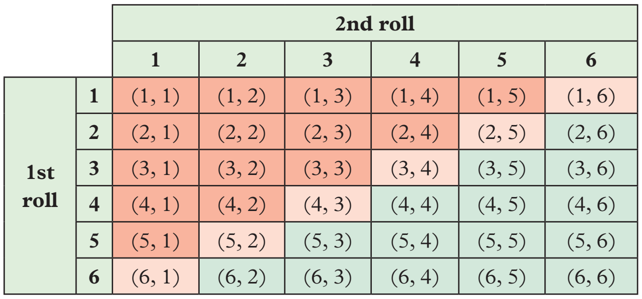
		2nd roll					
		1	**2**	**3**	**4**	**5**	**6**
1st roll	**1**	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

8 **a** 0.125 **b** 0.75 **c** 0.5

9 **a, b** **i** Cells coloured red

ii Cells coloured orange

iii Cells coloured green



		2nd roll					
		1	**2**	**3**	**4**	**5**	**6**
1st roll	**1**	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

c **i** There are 15 favourable outcomes out of 36 possible outcomes; $\frac{15}{36} = \frac{5}{12}$

ii There are 6 favourable outcomes out of 36 possible outcomes; $\frac{6}{36} = \frac{1}{6}$

iii There are 15 favourable outcomes out of 36 possible outcomes; $\frac{15}{36} = \frac{5}{12}$

d The most successful strategy would be to predict a number greater than (or less than) 7.

10 **a** A = {2, 4, 6, 8, 10}, B = {4, 8}, C = {2, 3, 5, 7}

b B and C

c **i** $\frac{1}{2}$ **ii** $\frac{1}{5}$ **iii** $\frac{2}{5}$

11 **a** **i** 36

ii 6 outcomes are doubles; (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)

iii $\frac{1}{6}$

b **i** 64

ii 8 outcomes are doubles, (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (8, 8)

iii $\frac{1}{8}$

c $\Pr(n) = \frac{1}{n}$

12 10 girls

13 **a** **i** $\frac{1}{4}$ **ii** $\frac{1}{6}$ **iii** $\frac{7}{36}$

b **i** For the outcomes from part **a**, an odd number is most likely.

ii For the outcomes from part **a**, a prime number is least likely.

14 **a** {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

b Some scores can only be achieved one way, whereas others can be achieved two ways.

c **i** $\frac{1}{12}$ **ii** $\frac{1}{6}$ **iii** $\frac{1}{4}$

15 $\frac{11}{36}$

16 **a** 28

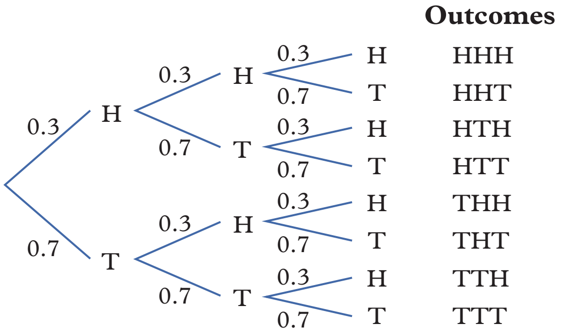
b The rows of tiles are grouped, in order from zero to six, according to the value shown on the top half of the tiles. Within each of these ordered groups, the values of the bottom half of the tiles have also been placed in order.

c 7

d **i** $\frac{1}{28}$ **ii** $\frac{1}{4}$ **iii** $\frac{3}{4}$

iv $\frac{3}{7}$ **v** $\frac{9}{14}$ **vi** $\frac{13}{28}$

17 **a**



b **i** TTT **ii** HHH

c **i** 0.441 **ii** 0.189 **iii** 0.216

18 **a** 2

b 7

c **i** (6, 1), (5, 2), (4, 3), (3, 4), (2, 5), (1, 6)

ii 3

19 5 red decorations

20 **a** The chance of landing on a red section is not the same as the chance of landing on a white section. There are more red bands than white and the combined area of the red bands appears larger than the combined area of the white bands.

b radii of circles from inner circle out: 1 cm, 3 cm, 5 cm, 7 cm, 9 cm

c bullseye = 3.14 cm^2, inner white band = 25.13 cm^2, middle red band = 50.27 cm^2, outer white band = 75.40 cm^2, outermost red band = 100.53 cm^2

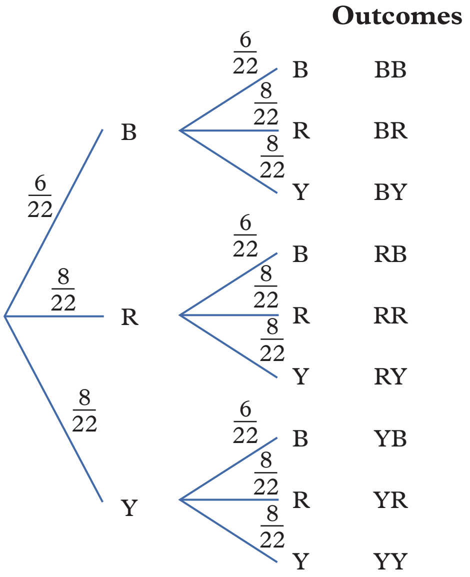
d **i** 0.012

ii 0.395

EX p581

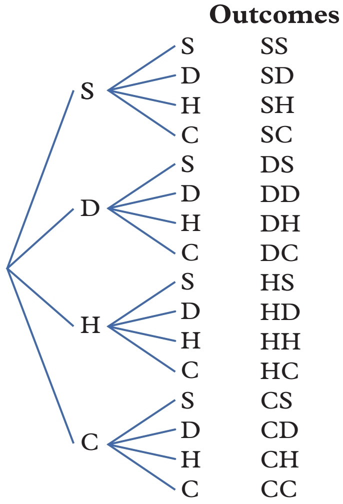
11B Experiments with and without replacement

1 **a**



b $\frac{41}{121}$

2 a



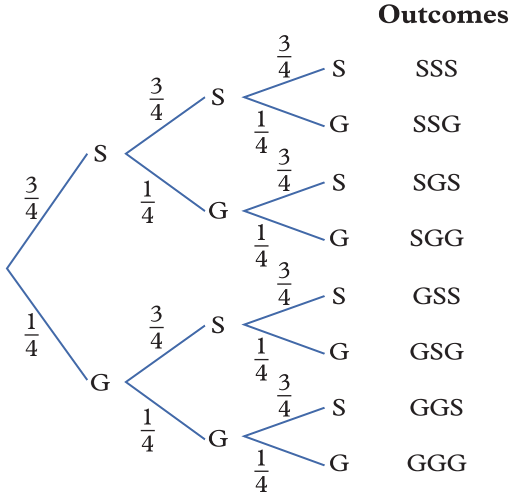
b $\frac{1}{16}$

c $\frac{1}{8}$

3 a i $\frac{3}{4}$

ii $\frac{1}{4}$

b

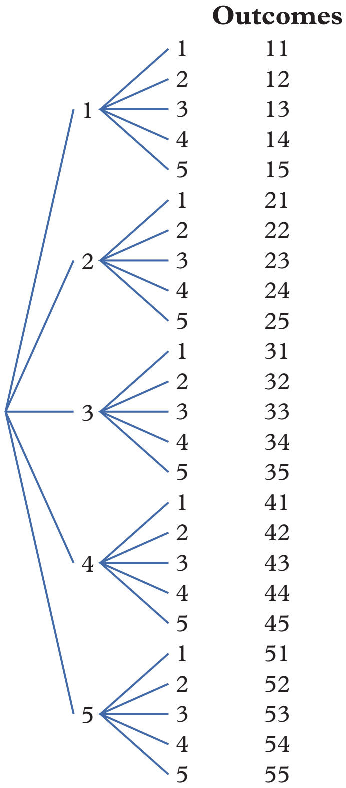


c i $\frac{27}{64}$

ii $\frac{1}{64}$

iii $\frac{9}{16}$

4 a



b 25

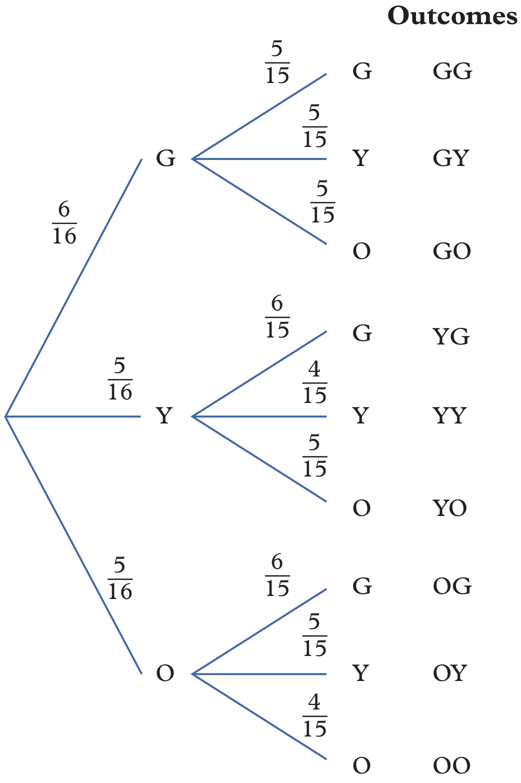
c 5

d i $\frac{1}{5}$

ii $\frac{4}{5}$

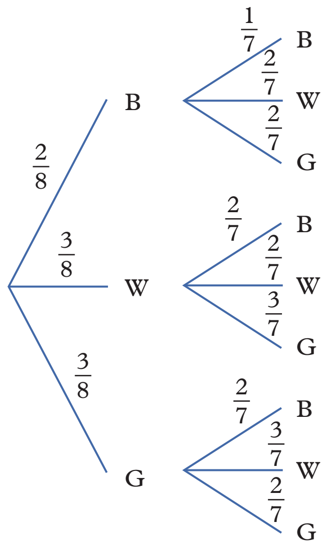
ii $\frac{2}{5}$

5 a



b $\frac{7}{24}$

6 **a**

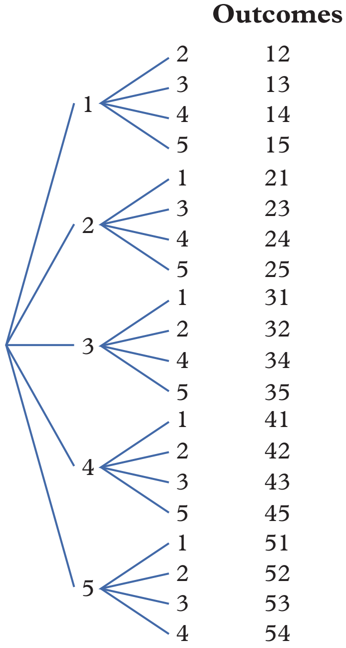


b $\frac{13}{56}$ **c** $\frac{3}{14}$

7 **a** 0.109 **b** 0.509

8 **a** $\frac{466}{676}$ **b** $\frac{44}{65}$

9 **a**



b 20

c **i** 0 **ii** 1 **iii** $\frac{1}{2}$

10 **a** $\frac{1}{4}$ **b** $\frac{23}{87}$

11 **a** $\frac{3}{8}$ **b** $\frac{25}{247}$ **c** $\frac{225}{988}$ **d** $\frac{42}{185}$

12 **a** **i** $\frac{9}{29}$ **ii** $\frac{2}{29}$ **iii** $\frac{3}{29}$

iv $\frac{3}{29}$ **vi** $\frac{4}{435}$

v $\frac{1}{435}$

b **i** $\frac{2}{87}$ **ii** $\frac{4}{435}$

c Undertaking the events in reverse order does not affect the overall probability of the events occurring.

13 **a** $\frac{8}{125}$ **b** $\frac{27}{125}$

14 **a** There are only two non-white balls in total.

b $\frac{1}{10}$

15 **a** **i** {2, 3, 4, 5, 6, 7, 8, 9, 10}

ii {3, 4, 5, 6, 7, 8, 9}

b **i** $\frac{3}{25}$ **ii** $\frac{1}{10}$

c **i** $\frac{1}{5}$ **ii** $\frac{1}{5}$

16 **a** **i** 125 **ii** 60

b **i** 729 **ii** 504

17 **a** 216 **b** 96

18 **a** **i** 10

ii 1 000 000

b **i** 17 576 000

ii You can form more possibilities with this system, because the alphabet has 26 letters, enabling more combinations than the 10 digits.

EX
p587

11C Two-way tables and Venn diagrams

1 **a** **i** {1, 3, 5, 7, 9, 11}

ii {2, 3, 5, 7, 11}

iii {3, 5, 7, 11}

iv {1, 2, 3, 5, 7, 9, 11}

b **i** $\frac{1}{2}$ **ii** $\frac{7}{12}$ **iii** $\frac{1}{3}$

2 **a** **i** 13 **ii** 19 **iii** 13

iv 9 **v** 26 **vi** 6

b **i** $\frac{1}{2}$ **ii** $\frac{1}{2}$ **iii** $\frac{9}{26}$

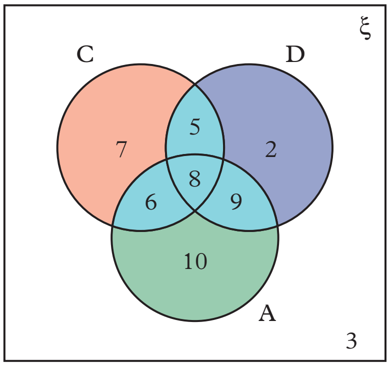
iv $\frac{7}{26}$ **v** $\frac{7}{26}$ **vi** $\frac{2}{13}$

vii $\frac{20}{26}$ **viii** 1

3 Calculate the missing entries in the rows and columns by adding or subtracting existing entries, as required.

	Primary school	**High school**	**Total**
Catch public transport	22	18	40
Driven	37	11	48
Walk/ride	5	9	14
Total	64	38	102

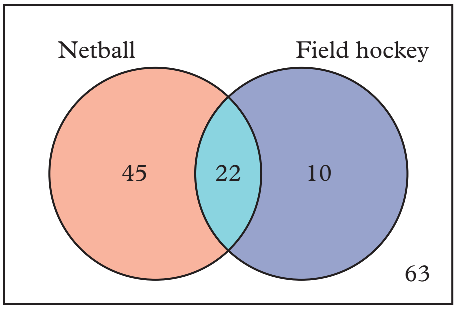
4



5 **a** $\frac{13}{25}$ **b** $\frac{13}{50}$ **c** $\frac{11}{50}$

d $\frac{7}{25}$ **e** $\frac{47}{50}$

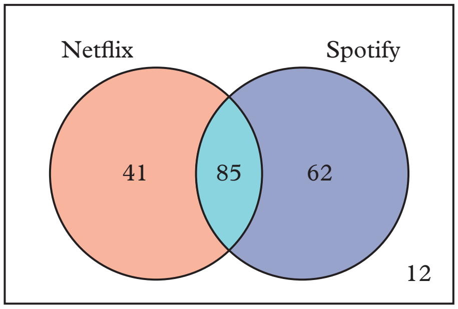
6



7 **a**

	Netflix (N)	Not Netflix (N′)	Total
Spotify (S)	85	62	147
Not Spotify (S′)	41	12	53
Total	126	74	200

b



8

	A	A′	Total
B	15	7	22
B′	13	11	24
Total	28	18	46

9 **a** 7

b

	Maths (M)	Not maths (M′)	Total
Science (S)	11	7	18
Not science (S′)	4	6	10
Total	15	13	28

10 0.7

11 $\frac{5}{18}$

12 **a** **i** true; set R is completely within set P so it is a subset of set Q.

ii true; set R is completely within set Q so it is a subset of set Q.

iii false; set R is completely within the intersection of sets P and Q so it is a subset of the intersection of sets P and Q.

iv false; set R is completely within the union of sets P and Q so it is a subset of the union of sets P and Q.

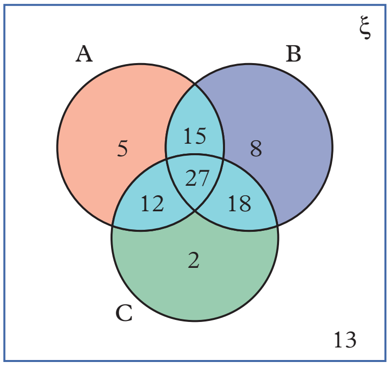
v false; set R is within the universal set, ξ, so it is a subset of the universal set.

vi true; set R is within set Q, which is within the universal set so it is a subset of both set Q and the universal set, ξ.

b **i** $\frac{1}{8}$ **ii** $\frac{1}{3}$ **iii** $\frac{1}{12}$

iv $\frac{1}{8}$

13 **a**



b **i** 85 **ii** 14 **iii** 72

iv 45 **v** 41 **vi** 15

vii 18 **viii** 73

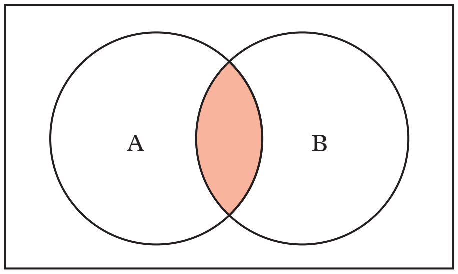
14 **a** **i** {2, 4, 6, 8, 10, 12}

ii {2}

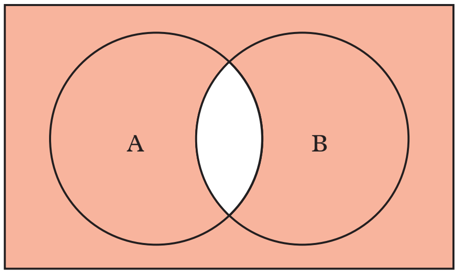
iii {1, 2, 4, 6, 8, 9, 10, 12}

b **i** $\frac{1}{12}$ **ii** $\frac{5}{12}$ **iii** $\frac{2}{3}$

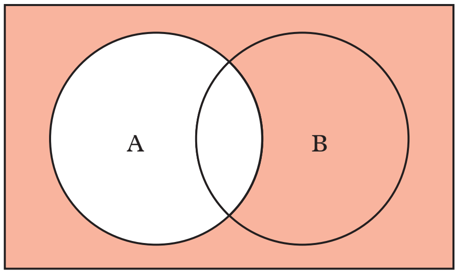
15 $\Pr(A \cap B)$ is:

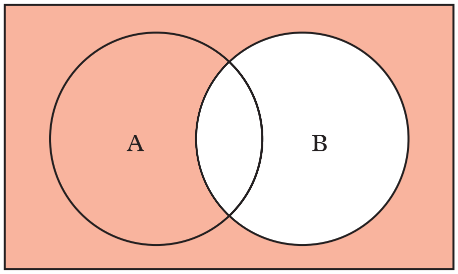


So $\Pr((A \cap B)')$ is:

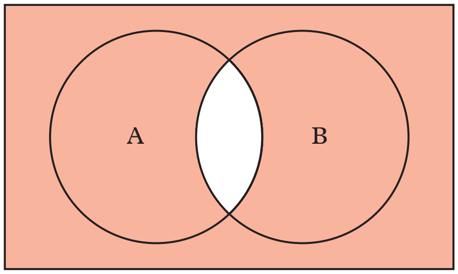


$\Pr(A')$ and $\Pr(B')$ are:





Hence, $\Pr(A' \cup B')$ is:



16 **a** 0.25 **b** 0.95

17 **a**

	A	**A′**	**Total**
B	0.1	0.5	0.6
B′	0.2	0.2	0.4
Total	0.3	0.7	1

b **i** 0.2 **ii** 0.5 **iii** 0.9

18

	Boys	**Girls**	**Total**
Left-handed	5	6	11
Right-handed	30	19	49
Total	35	25	60

There are 6 left-handed girls.

19 $P \cap Q = \{2, 5, 9, 13\}$

$R \cap S' = \{4, 5, 6, 7\}$

$\{2, 5, 9, 13\} \cup \{4, 5, 6, 7\} = \{2, 4, 5, 6, 7, 9, 13\}$

$2 + 4 + 5 + 6 + 7 + 9 + 13 = 46$

20 $\Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(A \cap B)$
$- \Pr(A \cap C) - \Pr(B \cap C) + \Pr(A \cap B \cap C)$

11 Checkpoint

1 **a** {H, T}

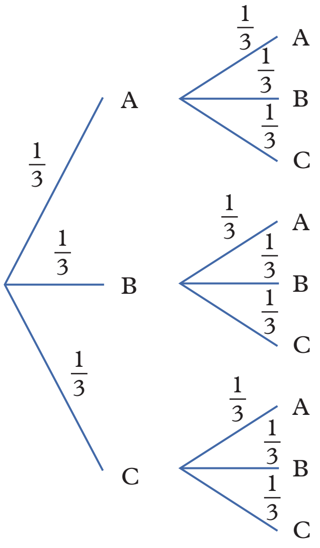
b {1, 2, 3, 4, 5, 6}

c {club, diamond, heart, spade}

d {L, O, I, P}

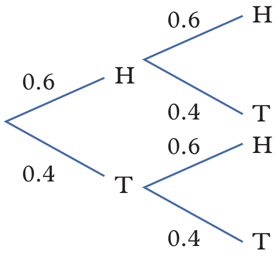
2 **a** $\frac{1}{6}$ **b** $\frac{1}{3}$ **c** $\frac{1}{3}$ **d** $\frac{2}{3}$

3 **a**



b $\frac{1}{9}$

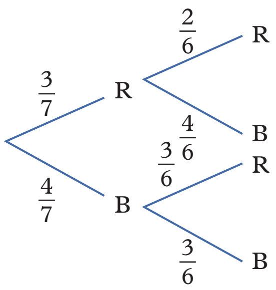
4 **a**



b **i** 0.36

ii 0.48

5 **a**



b $\frac{1}{7}$

6 **a** $\frac{4}{25}$ **b** $\frac{1}{10}$

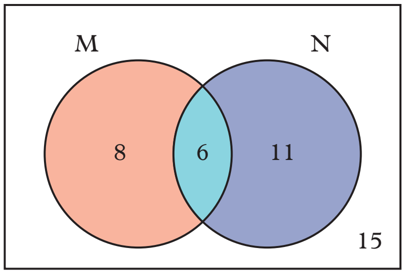
7 **a** $\frac{1}{16}$ **b** $\frac{1}{4}$ **c** $\frac{3}{4}$

8 **a** $\frac{4}{5}$ **b** $\frac{7}{20}$ **c** $\frac{1}{5}$ **d** $\frac{3}{20}$

9 **a**

	M	**M′**	**Total**
N	6	11	**17**
N′	**8**	15	**23**
Total	**14**	**26**	40

b



10 **a** $\frac{7}{12}$ **b** $\frac{5}{8}$ **c** $\frac{1}{3}$ **d** $\frac{11}{24}$

EX
p596

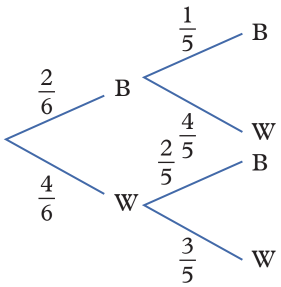
11D Conditional probability

1 **a** The probability $\frac{15}{20}$ means there are 15 soft centres out of a total of 20 chocolates. The probability $\frac{5}{20}$ means there are 5 hard centres out of the 20 chocolates.

b The branches on the tree indicate that if 1 soft centre is eaten, there are 14 soft centres and 5 hard centres left out of the remaining 19 chocolates. They also show that, if it had been 1 hard centre that had been eaten, there would have been 15 soft centres and 4 hard centres left out of the remaining 19 chocolates.

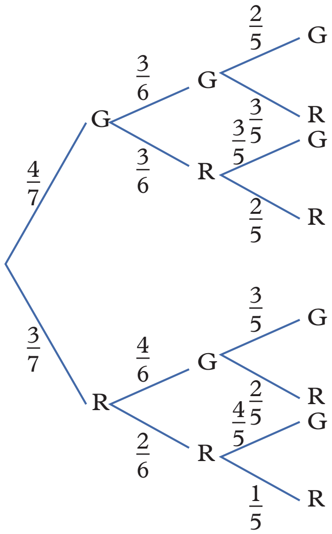
c $\frac{4}{19}$ **d** $\frac{5}{19}$ **e** $\frac{15}{19}$

2 **a**



b $\frac{3}{5}$ **c** $\frac{4}{5}$ **d** $\frac{2}{5}$

3 **a**



b $\frac{1}{2}$ **c** $\frac{2}{5}$ **d** $\frac{3}{5}$

4 **a** $\frac{12}{13}$ **b** $\frac{12}{13}$

5 **a** 15 **b** 18 **c** 18

d 59

6 **a** $\frac{15}{41}$ **b** $\frac{18}{40}$ **c** $\frac{9}{50}$

d $\frac{59}{100}$ **e** $\frac{18}{59}$

7 **a** 8

b 3 **c** $\frac{3}{8}$ **d** $\frac{3}{5}$

8 **a** $\frac{4}{49}$ **b** $\frac{7}{12}$

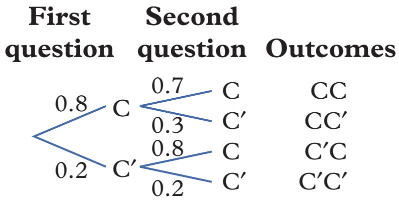
9 **a** **i** $\frac{7}{10}$ **ii** $\frac{3}{10}$ **iii** $\frac{2}{3}$

iv $\frac{4}{7}$ **v** $\frac{3}{11}$

b $\frac{1}{2}$

10 **a** $\frac{1}{3}$ **b** 0.25

11 **a**



b **i** 0.7 **ii** 0.8 **iii** 0.3

12 **a**

	Male	**Female**	**Total**
Full-time employees	30	4	34
Part-time employees	5	11	16
Total	35	15	50

b **i** 4 **ii** 16 **iii** 35

c **i** $\frac{5}{35}$ **ii** $\frac{5}{16}$

iii $\frac{4}{34}$ **iv** $\frac{4}{15}$

d **i** In part **i**, the selection is made from males (total of 35), whereas in part **ii**, the selection is made from part-time employees (total of 16).

ii In part **iii**, the selection is made from full time employees (total of 34), whereas in part **iv**, the selection is made from female employees (total of 15).

13 **a**

	Small	**Large**	**Total**
Silver	27	35	62
Coloured	18	10	28
Total	45	45	90

b **i** True, there are 45 of each.

ii True, there are 28 coloured paper clips but there are 62 silver paper clips.

iii False, there are 27 small silver paper clips out of a total of 45 small paper clips. $\frac{27}{45} \approx 60\%$

iv False, there are 10 large coloured paper clips out of a total of 28 coloured paper clips. $\frac{10}{28} \approx 36\%$

v True, there are 62 silver paper clips out of a total of 90 paper clips. $\frac{62}{90} \approx 69\%$

14 0.78

15 **a** **i** 32 **ii** 41 **iii** 57

b **i** $\frac{11}{32}$ **ii** $\frac{11}{41}$ **iii** $\frac{35}{57}$

c They help to restrict the sample space, giving the appropriate denominator for the conditional probability.

16 **a** **i** 0.2 **ii** 0.5

b $\Pr(X) = 0.1$ and $\Pr(Y) = 0.25$

17 $\frac{2}{3}$

18 **a** 0.64 **b** $\frac{2}{3}$

19 **a** Start with $\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)}$

Multiplying through by Pr(A) gives $\Pr(A \cap B) = \Pr(B|A) \times \Pr(A)$.

Substituting this result into

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$ gives

$\Pr(A|B) = \frac{\Pr(B|A) \times \Pr(A)}{\Pr(B)}$, which is Baye's rule.

b **i** 0.55 **ii** $\frac{7}{11}$

EX p602

11E Independence

1 **a** **i** 0.4 **ii** 0.5

b Independent

2 **a** $\frac{2}{3}$ **b** $\frac{1}{4}$ **c** $\frac{1}{6}$

3 Yes. $\Pr(\text{cricket}) = \frac{9}{21} = \frac{3}{7}$ and $\Pr(\text{cricket}|\text{Maths}) = \frac{6}{14} = \frac{3}{7}$

4 **a** Independent **b** Dependent

c Dependent **d** Independent

5 **a** **i** $\frac{3}{4}$ **ii** $\frac{3}{4}$

b Independent

6 **a** **i** $\frac{2}{3}$ **ii** $\frac{7}{9}$

b Dependent

7 **a** 0.48 **b** 0.92

8 **a** He assumed A and B were independent events and evaluated $\Pr(A) \times \Pr(B)$.

b 0.38. It must be calculated using the addition rule: $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

9 Yes. If A and B are independent events, then B occurring has no effect on the probability of A occurring. Therefore, B not occurring will also have no effect on the probability of A occurring.

10 **a** Dependent **b** Independent

c Independent **d** Dependent

11 Q and R

$\Pr(Q) = \frac{15}{30} = \frac{1}{2}$ and $\Pr(Q|R) = \frac{9}{18} = \frac{1}{2}$

Alternatively, $\Pr(R) = \frac{18}{30} = \frac{3}{5}$ and

$\Pr(R|Q) = \frac{9}{15} = \frac{3}{5}$

12 **a** **i** $\frac{1}{2}$ **ii** $\frac{1}{2}$

iii $\frac{1}{3}$ **iv** $\frac{3}{10}$

b because $\Pr(\text{Action}|\text{Odd}) = \Pr(\text{Action})$

c because $\Pr(\text{Comedy}|\text{even}) \neq \Pr(\text{Comedy})$

d There is no logical reason that these two things should be connected. They appear dependent due to random variation.

13 **a** The colour of the first ball selected will not affect the probability that the second ball is white.

b **i** $\frac{2}{3}$ **ii** $\frac{3}{5}$

c **i** $\frac{4}{9}$ **ii** $\frac{2}{5}$

14 $\Pr(A) = 0.6$

15 Yes. Since C occurring is independent of whether A occurs or B occurs, it follows that C is independent of both A and B occurring, meaning that $A \cap B$ and C are also independent events.

Therefore, $\Pr(A \cap B \cap C) = \Pr(A \cap B) \times \Pr(C) = \Pr(A) \times \Pr(B) \times \Pr(C)$.

CHAPTER 11 review

Multiple-choice

1 D **2** D **3** C **4** D **5** C

6 A **7** E **8** A **9** D **10** B

Short answer

1 **a** {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

b **i** {1, 3, 5, 7, 9}

ii {3, 6, 9}

iii {3, 9}

iv {1, 3, 5, 6, 7, 9}

v {1, 5, 7}

2 **a** $\frac{1000}{2197}$ **b** $\frac{1197}{2197}$

c $\frac{1000}{2197}$ **d** $\frac{1197}{2197}$

3 **a** 900

b You can’t use zero as the first digit (the hundreds digit), because this will produce a two-digit number, not a three-digit number.

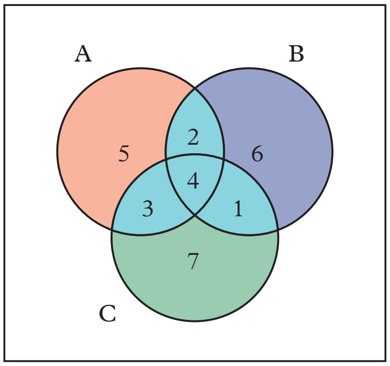
c $\frac{1}{900}$ **d** $\frac{1}{100}$

4 **a** 648

b You can’t use zero as the first digit (the hundreds digit), because this will produce a two-digit number, not a three-digit number. There also can’t be any repetitions of digits.

c $\frac{1}{648}$ **d** 359

5



6 **a** 15 **b** 27 **c** 47 **d** 55

7 **a** 43% **b** 36% **c** 44% **d** 74%

8 **a** $\frac{1}{11}$ **b** $\frac{2}{11}$ **c** $\frac{2}{5}$

d 1 **e** $\frac{3}{4}$

9 **a** $\frac{1}{6}$ **b** $\frac{1}{2}$ **c** $\frac{7}{12}$

10 **a** **i** $\frac{2}{5}$ **ii** $\frac{7}{50}$

iii $\frac{16}{25}$ **iv** $\frac{1}{5}$

b A and C. $\Pr(A) = \frac{20}{50} = \frac{2}{5}$ and $\Pr(A|C) = \frac{8}{20} = \frac{2}{5}$

Analysis

1 **a** **i** $\frac{1}{9}$ **ii** $\frac{1}{3}$ **iii** $\frac{5}{9}$

b {4, 5, 6, 7, 8, 10}

c **i** $\frac{1}{81}$ **ii** $\frac{10}{27}$ **iii** $\frac{7}{81}$

d **i** 9% **ii** 25% **iii** 24%

e **i** 30% **ii** 25% **iii** 86%

2 **a** **i** $\frac{1}{2}$ **ii** $\frac{3}{8}$ **iii** $\frac{1}{8}$ **iv** 0

b **i** $\frac{1}{20}$ **ii** $\frac{2}{5}$ **iii** $\frac{1}{8}$

c **i** $\frac{7}{60}$ **ii** $\frac{1}{6}$ **iii** $\frac{1}{5}$

CHAPTER 12 Computational thinking

EX p614

12A Two-dimensional lists

1 **a**

```
Scores = [ [82, 75, 90],
           [74, 86, 95],
           [93, 95, 88] ]
```

b **i** `Scores[0][1]`

ii `Scores[2][0]`

iii `Scores[1]`

c **i** `Scores[0][0] = 85`

ii `Scores[1][0] = 79`

iii `Scores[2][0] = 92`

2 **a**

```
Multiplication = [ [0,0,0,0,0,0],
                   [0,1,2,3,4,5],
                   [0,2,4,6,8,10],
                   [0,3,6,9,12,15],
                   [0,4,8,12,16,20],
                   [0,5,10,15,20,25] ]
```

b **i** `Multiplication[2][3]` or `Multiplication[3][2]`

ii `Multiplication[1][5]` or `Multiplication[5][1]`

iii `Multiplication[3]`

3 **a**

```
Board = [ [‘ES’, ‘ES’, ‘BN’, ‘WR’],
          [‘BP’, ‘BK’, ‘BP’, ‘ES’],
          [‘ES’, ‘ES’, ‘ES’, ‘ES’],
          [‘WP’, ‘WP’, ‘ES’, ‘WK’] ]
```

b

```
Board[0][2] = ‘ES’
Board[2][1] = ‘BN’
Board[3][0] = ‘ES’
Board[2][0] = ‘WP’
```

EX p618

12B Transformations of points on a plane

1 **a** rotation of 180° around the origin

b reflection across the *y*-axis

c rotation of 90° clockwise around the origin

d reflection across the *x*-axis

e reflection across the *x*-axis

f rotation of 90° anticlockwise around the origin

2 **a** (–2, –3) **b** (2, 3)

c (–3, –2) **d** (–3, 2)

e (–2, –3)

3 **a** (–1, –4) **b** (2, 5)

c (–1, 1) **d** (0, 0)

4 **a** (2, –6) **b** (–1, –4)

c (11, 5) **d** (–2, 0)

5 The transformation matrix in question **3** describes a 90° rotation around the origin in an anticlockwise direction.

The transformation matrix in question **4** describes a reflection across the *x*-axis.

6 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the identity matrix because, when it is applied to a point, it does not change the coordinates.

7 The *x* value is multiplied (or stretched) by a factor of 2 and the *y* value is multiplied (or stretched) by a factor of 3.

8 $(x', y') = (ax + by, cx + dy)$

EX
p623

12C Using two-dimensional lists to implement transformation matrices

1 **a**

```
T = [ [-1, 1], [1, -1] ]
A = [-4, 1]
x_transformed = T[0][0]*A[0]
+ T[0][1]*A[1]
y_transformed = T[1][0]*A[0]
+ T[1][1]*A[1]
new_point = [x_transformed,
y_transformed]
print(new_point)
```

b

```
T = [ [1, 0], [0, 1] ]
A = [-4, 1]
x_transformed = T[0][0]*A[0]
+ T[0][1]*A[1]
y_transformed = T[1][0]*A[0]
+ T[1][1]*A[1]
new_point = [x_transformed,
y_transformed]
print(new_point)
```

c

```
T = [ [0, 0], [-1, 1] ]
A = [-4, 1]
x_transformed = T[0][0]*A[0]
+ T[0][1]*A[1]
y_transformed = T[1][0]*A[0]
+ T[1][1]*A[1]
new_point = [x_transformed,
y_transformed]
print(new_point)
```

2 **a** $T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

b

```
T = [ [-1, 0], [0, 1] ]
S = [ [-1, -3], [-2, -5], [-4, -8],
    [-6, 1], [-3, 5] ]
S_new = []
for point in S:
   x_transformed = T[0][0]*point[0] +
   T[0][1]*point[1]
   y_transformed = T[1][0]*point[0] +
   T[1][1]*point[1]
   new_point = [x_transformed,
   y_transformed]
   S_new.append(new_point)
print(S_new)
```

The transformed set of points is {(1,–3), (2, –5), (4, –8), (6, 1), (3, 5)}.

3 **a** $T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

b

```
T = [ [0, -1], [1, 0] ]
S = [ [0, 0], [3, 0], [3, 2] ]
S_new = []
for point in S:
   xtransformed = T[0][0]*point[0] +
   T[0][1]*point[1]
   ytransformed = T[1][0]*point[0] +
   T[1][1]*point[1]
   newpoint = [xtransformed,
   ytransformed]
   S_new.append(newpoint)
print(S_new)
```

The transformed set of points is {(0, 0), (0, 3), (–2, 3)}.

4 **a**

```
T1 = [ [1, 0], [0, -1] ]
T2 = [ [-1, 0],  [-1, 0] ]
Transformations = [T1, T2]
point = [-3, -2]
for T in Transformations:
   xtransformed = T[0][0]*point[0] +
   T[0][1]*point[1]
   ytransformed = T[1][0]*point[0] +
   T[1][1]*point[1]point =
   [xtransformed, ytransformed]
   point = [x_transformed,
   y_transformed]
print(point)
```

The transformed point is (3, 3).

b

```
T1 = [ [1, 0], [0, -1] ]
T2 = [ [-1, 0], [-1, 0] ]
Transformations = [T1, T2]
S = [ [-3, -2], [2, 7], [5, -3],
[-4, 1], [1, 5] ]
S_new = []
for point in S:
   for T in Transformations:
      xtransformed = T[0][0]*point[0] +
      T[0][1]*point[1]
      ytransformed = T[1][0]*point[0] +
      T[1][1]*point[1]
      point = [xtransformed,
      ytransformed]
   S_new.append(point)
print(S_new)
```

The transformed set of points is {(3, 3), (–2, –2), (–5, –5), (4, 4), (–1, –1)}.

5 a

```
T1 = [ [0, 0], [1, -1] ]
T2 = [ [0, 1],  [-1, 0] ]
T3 = [ [1, 1],  [1, 1] ]
Transformations = [T1, T2, T3]
point = [6, -6]
for T in Transformations:
   xtransformed = T[0][0]*point[0] +
   T[0][1]*point[1]
   ytransformed = T[1][0]*point[0] +
   T[1][1]*point[1]
   point = [xtransformed,
   ytransformed]
print(point)
```

The transformed point is (12, 12).

b

```
T1 = [ [0, 0], [1, -1] ]
T2 = [ [0, 1], [-1, 0] ]
T3 = [ [1, 1], [1, 1] ]
Transformations = [T1, T2, T3]
P = [ [6, -6], [6, 6], [-6, 6],
[-6, -6] ]
P_new = []
for point in P:
      for T in Transformations:
         xtransformed = T[0]
         [0]*point[0] + T[0]
         [1]*point[1]
         ytransformed = T[1]
         [0]*point[0] + T[1]
         [1]*point[1]
         point = [xtransformed,
         ytransformed]
      P_new.append(point)
print(P_new)
```

The transformed set of points is {(12, 12), (0, 0), (–12, –12), (0, 0)}.

6

```
T1 = [ [0, 1], [-1, 0] ]
T2 = [ [1, 0], [0, -1] ]
```

7

```
T1 = [ [-1, 0], [0, -1] ]
T2 = [ [-1, 0], [0, 1] ]
T3 = [ [0, -1], [1, 0] ]
```

8 The new set of points is correct. Working through the five transformations returns the point to its original starting point. The transformations, in order, are:

- reflection across the *y*-axis
- rotation of 90° in a clockwise direction around the origin
- rotation of 90° in an anticlockwise direction around the origin
- rotation of 180° around the origin
- reflection across the *x*-axis.

EX p624

SEMESTER 2 review

Short answer

1 a $a = 79°$

b $a = 35°, b = 89°, c = 56°, f = 56°$

c $j = 12°, k = 84°, l = 96°, m = 84°, n = 84°, p = 94°, q = 98°, r = 98°, t = 96°$

d $b = 39°, c = 27°, d = 94°, e = 22°, f = 42°, g = 125°, h = 55°$

e $x = 15.9, y = 97°, z = 14.7$

f $y = 90°, z = 9.8$

g $v = 55.2°, w = 5.7, x = 34.8°$

h $r = 4.0, t = 18°, u = 12.4°$

2 a

Statements	Reasons
AC ∥ BD	Given
AC = BD	Given
∠ACB ≅ ∠DBC	Alternate angles on parallel lines
BC = BC	Side common to ρACB and ρDBC
ΔACB ≅ ΔDBC	The SAS condition for congruence
∠ABC ≅ ∠DCB	Corresponding angles of congruent triangles, alternate angles
∴ AB ∥ CD □	Alternate angles formed by a transversal cutting AB and CD are equal.

b

Statements	Reasons
AC ∥ DE	Given
∠DBA + ∠ABC + ∠EBC = 180°	Angles on a straight line
∠DBA = ∠BAC	Alternate angles on parallel lines
∠EBC = ∠BCA	Alternate angles on parallel lines
∴∠BAC + ∠ABC + ∠BCA = 180° □	By substituting the equal angles

3 a

Statements	Reasons
∠ABC = ∠BCD = ∠CDA = ∠DAB = 90°	Given
BC ∥ AD	Co-interior angles are supplementary.
∠CBD = ∠ADB	Alternate angles on parallel lines
BD = BD	Side common to both triangles
∠BCD = ∠BAD	Given; both 90°
∴ΔABD ≅ ΔCDB □	Meets the AAS condition for congruence

b

Statements	Reasons
AD = 15	Given
DB = 5	Given
BE = 7	Given
CE = 21	Given
∠ABC = ∠DBE	Different names for the same angle
AD + DB = AB	Addition of line segments on the same line
AB = 15 + 5 = 20	Substitution of given measurements
CE + EB = CB	Addition of line segments on the same line
CB = 21 + 7 = 28	Substitution of given measurements
$\frac{AB}{DB} = \frac{20}{5} = 4$	Scale factor of corresponding sides
$\frac{CB}{EB} = \frac{28}{7} = 4$	Scale factor of corresponding sides
∴ ΔABC ~ΔDBE □	Meets the SAS condition for similarity

c

Statements	Reasons
∠CDA = ∠DAB = 32°	Given
AE = ED	Given
∠CED = 90°	Given
∠AEB = 90°	Vertically opposite angles are congruent.
ΔAEB = ΔDEC	Meets the ASA condition for congruence
CE = BE	Corresponding sides of congruent triangles
∠BED = 90°	Supplementary with ∠CED; angles on a straight line
∠CEA = 90°	Vertically opposite ∠BED
ΔBED ≅ ΔCEA	Meets the SAS condition for congruence
ΔAEB ≅ ΔCEA	Meets the SAS condition for congruence
ΔAEB ≅ ΔDEB ≅ ΔDEC ≅ ΔAEC	All have been proven to be congruent
So: CD =DB = AB = AC	Corresponding sides of congruent triangles are equal.
∴ ABDC is a rhombus □	Diagonals are perpendicular to each other, and all four sides are equal. A kite with four equal sides is a rhombus.

4 **a** yes

b no

c yes

5 **a** 0.0862°

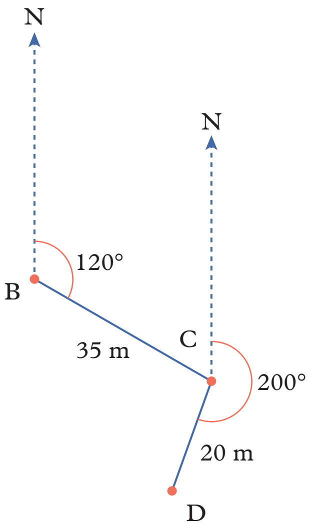
b **i** 23.2° **ii** 8.1 m

c 77 m

6 **a** **i** 334°T **ii** 154°T

iii 2.8 km

b **i**



ii 17.5 m

iii 30.3 m

iv 18.8 m

v 6.8 m

vi 327°T

vii 43.2 m

7 **a** 8.48 cm^2

b 250.45 cm^2

c 153.06 cm^2

d 100.53 cm^2

8 **a** **i** 428.24 cm^2

ii 410.79 cm^3

b **i** 212.80 cm^2

ii 192.00 cm^3

c **i** 471.24 cm^2

ii 785.40 cm^3

d **i** 502.38 cm^2

ii 695.70 cm^3

9 **a** minimum = 32, Q_1 = 48, mean = 55.5, Q_2 = 64, maximum = 76; IQR = 16

b minimum = 10, Q_1 = 20, mean = 25, Q_2 = 40, maximum = 70; IQR = 20

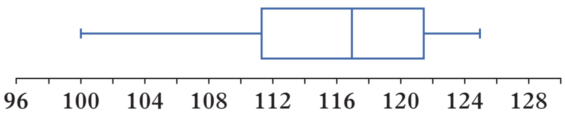
c minimum = 10, Q_1 = 10.5, mean = 13, Q_2 = 14, maximum = 18; IQR = 3.5

d minimum = 1.2, Q_1 = 2.35, mean = 3.4, Q_2 = 4, maximum = 4.4; IQR = 1.65

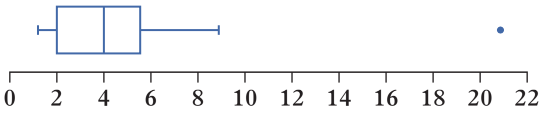
e minimum = 100, Q_1 = 102, mean = 103, Q_2 = 104, maximum = 106; IQR = 2

f minimum = 10, Q_1 = 40, mean = 60, Q_2 = 80, maximum = 110; IQR = 40

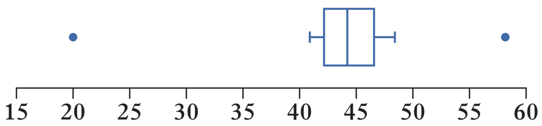
10 **a**



b



c



11 **a** false

b true

c true

d true

e false

f false

12 **a** positively skewed

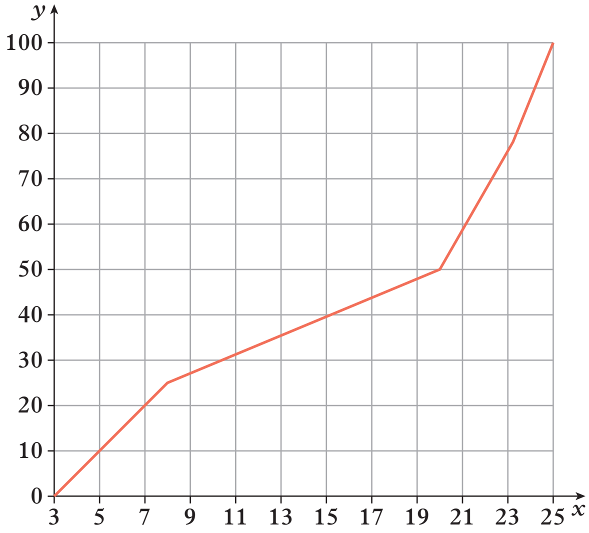
b negatively skewed

c symmetric

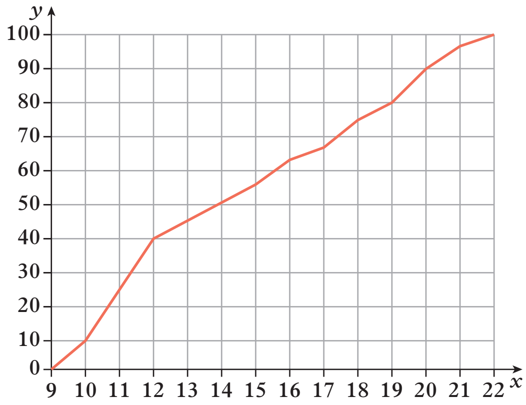
d negatively skewed

e positively skewed

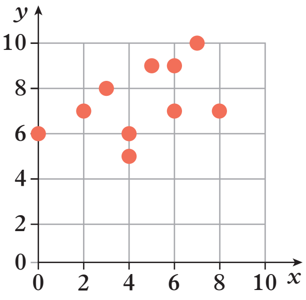
13 **a**



b



14



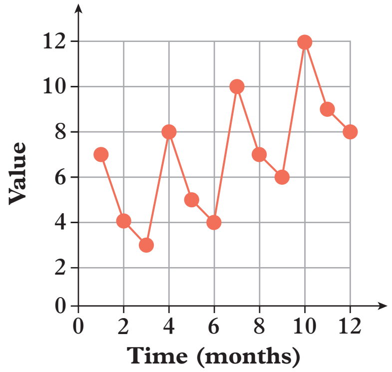
15 **a** moderate, positive, linear

b weak, negative linear

c no correlation

d moderate, negative, non-linear

16



17 **a** increasing with seasonality

b constant with seasonality

c decreasing with random fluctuations

d increasing for the first 5 units, then constant with random fluctuations

18 **a** Biased: the competitors at the tournament are all likely to favour the game they are playing in the tournament over other fighting games, and not enough people are being sampled.

b Fair: the sample size is large enough for the population; the students are questioned about what is being investigated.

c Biased: the sample size should be at least equal to the square root of the population so this sample size is too small (less than $\sqrt{7\,000\,000} \approx 2646$).

d Biased: the sample of all Australians does not represent all the people in the Southern Hemisphere.

19 **a** While 51% is the majority, it is only just over half of the responses to a two-option question and there's no indication of the reliability of the result.

b The fraction '9 out of 10' sounds like it is a simplified fraction from a large sample. However, it is only 9 from a sample of 10, and those 9 were biased because they wanted their answer to positively reflect on their sponsor.

20 **a** **i** unfair

ii unfair

iii Peta should either sample people from a larger population than her local area or should have made a conclusion only about people in her local area who go to the local shopping centre in a five-hour period on Saturday.

b **i** fair

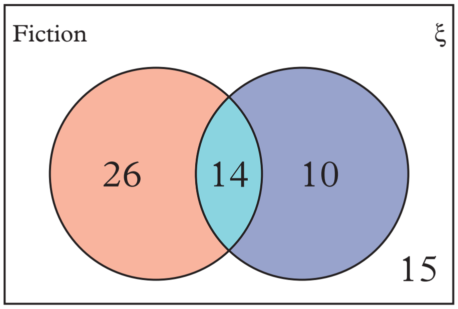
ii unfair

iii Rob should have made his conclusions about basketball players in the NBA, not about all basketball players.

21 a

	Fiction	Non-Fiction	Total
In a series	14	10	24
Not in a series	26	15	41
Total	40	25	65

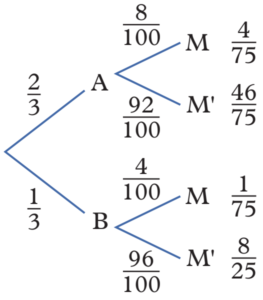
b



c **i** $\frac{2}{5}$ **ii** $\frac{2}{13}$

iii $\frac{7}{20}$ **iv** $\frac{15}{41}$

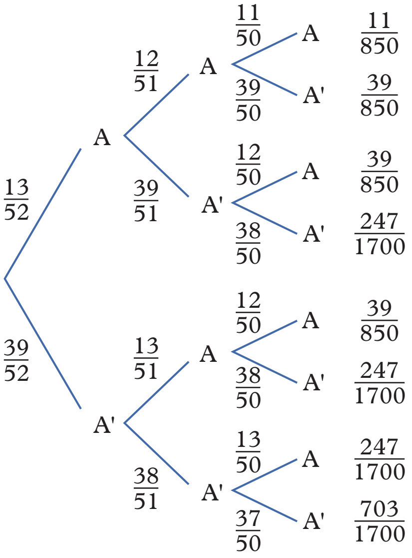
22 a



b **i** $\frac{4}{75}$ **ii** $\frac{8}{25}$

iii $\frac{2}{25}$ **iv** $\frac{4}{5}$

23 a



b **i** $\frac{11}{850}$ **ii** $\frac{117}{850}$

iii $\frac{247}{425}$ **iv** $\frac{13}{50}$

24 a $\frac{21}{40}$ **b** $\frac{1}{2}$ **c** $\frac{3}{20}$

d $\frac{13}{20}$ **f** $\frac{14}{19}$

e $\frac{3}{10}$

25 a dependent

b independent

c independent

d dependent

26 a $a = 11.18$ cm, $b = 21.35$ cm, $c = 21.93$ cm

b $x = 13.97$ cm, $y = 26.16°$, $z = 9.04$ cm

c $e = 30.35$ cm

d $f = 47.45$ cm

e $g = 145.14°$

f $h = 35.46°$

g $j = 19°$, $k = 19°$

h $m = 51°$, $n = 81.5°$, $l = 55$ cm

i $p = 36$ cm, $q = 80.91°$, $r = 99.09°$, $t = 43.86$ cm

j $w = 1.06$, $x = -0.85$, $y = 0.53$, $z = 0.62$

27 a $a = 65.16°$, $b = 74.84°$, $c = 25.53$ cm

or: $a = 114.84°$, $b = 25.16°$, $c = 11.24$ cm

b $a = 61.69°$, $b = 70.31°$

c $a = 49.07°$, $b = 70.93°$, $c = 51.29$ cm

d $a = 57.95°$, $b = 68.05°$, $c = 24.08$ cm

or: $a = 122.05°$, $b = 3.95°$, $c = 1.79$ cm

28 Proof that ∠AOB = 2∠ACB:

Statements	Reasons
CD is a diameter.	Given
O is the centre of the circle.	Given
OA = OB = OC	All radii of the circle
∠OAC = ∠OCA	Base angles of an isosceles triangle
∠AOD = 180° − ∠AOC	Angles on a straight line sum to 180°.
∠AOC = 180° − ∠OAC − ∠OCA	Interior angles of a triangle sum to 180°.
∠AOD = 180° − (180° − ∠OAC − ∠OCA)_	Substitution of equal angles
∠AOD = ∠OAC + ∠OCA	Simplifying
∠AOD = ∠OCA + ∠OCA	Substitution of the equal angle
∠AOD = 2∠OCA	Grouping like angles
∠OBC = ∠OCB	Base angles of an isosceles triangle
∠BOD = ∠OBC + ∠OCB	The exterior angle of a triangle is equal to the sum of the two opposite interior angles.
∠BOD = ∠OCB + ∠OCB	Substitution of the equal angle
∠BOD = 2∠OCB	Grouping like angles
∠AOB = ∠AOD + ∠BOD	
∠AOB = 2∠OCA + 2∠OCB	Substitution of equal angle measures
∠AOB = 2(∠OCA + ∠OCB)	Factorising the RHS
but ∠OCA + ∠OCB = ∠ACB	Sum of adjacent angles
∴ ∠AOB = 2∠ACB □	Substitution of the equal angle measure

29 a 228.25 cm^2

b 173.28 cm^2

c 127.31 cm^2

d 1086.40 cm^2

e 742.09 cm^2

f 1695.77 cm^2

30 **a** **i** 384 cm^2

ii 384 cm^3

b **i** 84.82 cm^2

ii 56.55 cm^3

c **i** 197.92 cm^2

ii 183.80 cm^3

d **i** 452.39 cm^2

ii 904.78 cm^3

e **i** 131.48 cm^2

ii 73.09 cm^3

f **i** 392.21 cm^2

ii 636.70 cm^3

31 **a** $\frac{2\pi}{5}$ **b** 135° **c** $\frac{5\pi}{4}$

d 70° **e** $\frac{11\pi}{6}$ **f** 195°

32 **a** $\sin(60°) = \frac{\sqrt{3}}{2}$

b $-\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

c $-\tan(45°) = -1$

d $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

e $\cos(60°) = \frac{1}{2}$

f $-\tan\left(\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$

33 **a** $x = 210°, 330°$ **b** $x = \frac{\pi}{4}, \frac{7\pi}{4}$

c $x = 60°, 240°$ **d** $x = \frac{3\pi}{2}$

e $x = 0°, 360°$ **f** $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

34 **a** amplitude = 5, period = 90

b amplitude = 3, period = 1080

35 **a** $\bar{x} = 37.75, s = 4.79, \sigma = 4.15$

b $\bar{x} = 22.2, s = 3.90, \sigma = 3.49$

c $\bar{x} = 4, s = 0, \sigma = 0$

36 **a** test score = 45 – 0.6 × (hours on social media)

b **i** test score = 33.0

ii test score = 38.4

iii hours on social media = 1.7

iv hours on social media = 25.0

c **i** extrapolation, unreliable

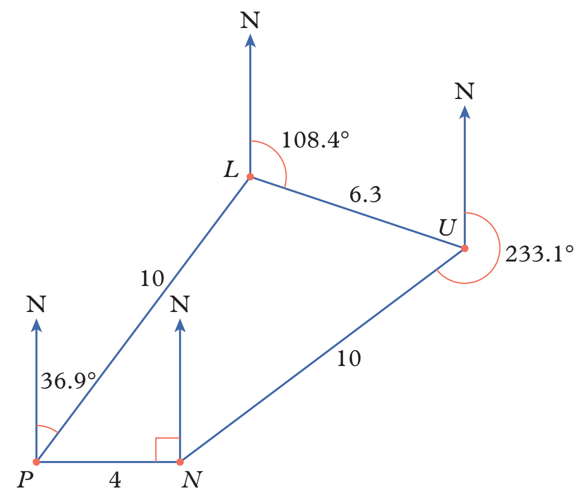
ii interpolation, reliable

iii interpolation, reliable

iv extrapolation, unreliable

Analysis

1 **a**



b 60.6 cm

c 5.05 km/h

d 243 m

e 13.40 km

f 063°T

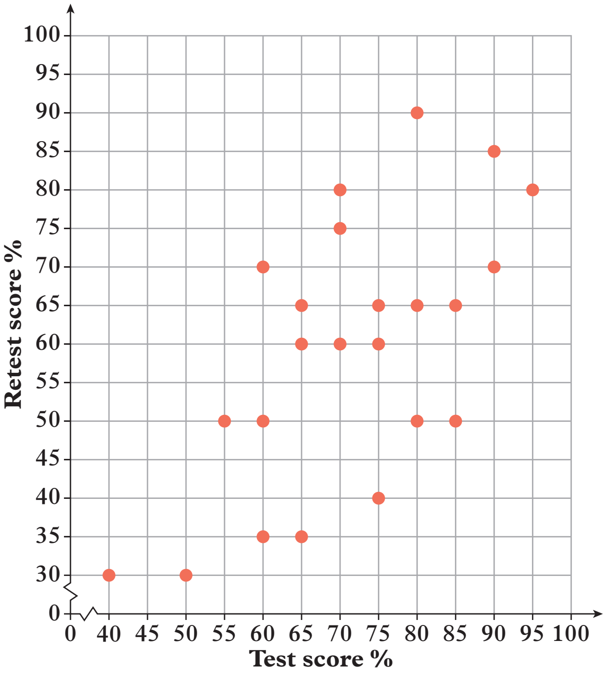
g **i** 2% **ii** 76%

iii 18% **iv** $\frac{2}{3}$

h 41.88 km^2

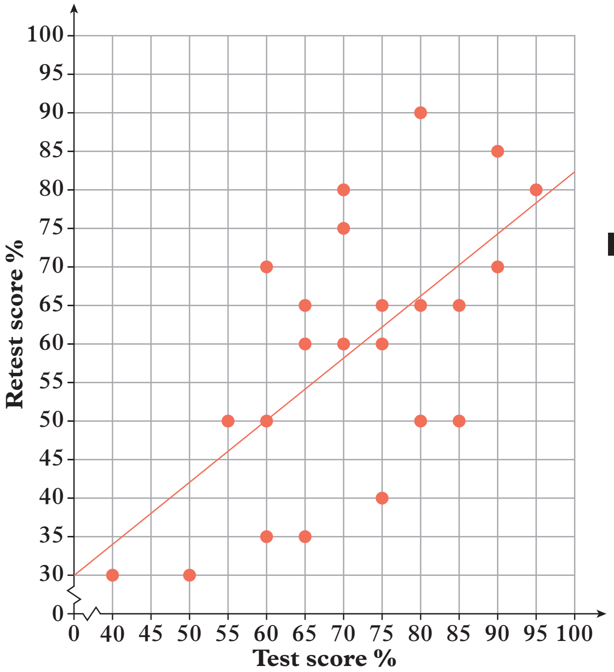
2 **a** test scores

b



c moderate, positive, linear

d



e retest score = 1.44 + 0.81 × (test score)

f **i** 58.14%

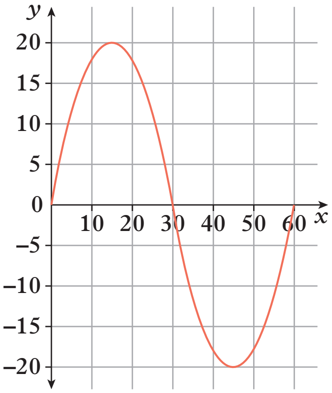
ii interpolation, reliable

g There are 11 students who scored at or above 75% on the original test. Of these, 8 scored better on the retest than the median of those who scored less than 75% on the original test. The remaining 3 students of the 11 who scored at or above 75% on the original test received scores on the retest that were worse than a majority of those who scored less than 75% on the original test. Therefore, while the teacher's claim is somewhat accurate, the moderate correlation means that there are students whose results do not fit the teacher's claim.

3 **a** amplitude = 20 m, period = 1 hour (60 minutes)

b $h = 20\sin\left(\frac{t\pi}{30}\right)$

c



d **i** $10\sqrt{3}$ m

ii 17.32 m

e 10 m

f 37.5 minutes and 52.5 minutes after boarding

g **i** $10\sqrt{2}$ m left and $10\sqrt{2}$ m above

ii 14.14 m left and 14.14 m above

h **i** 15π m

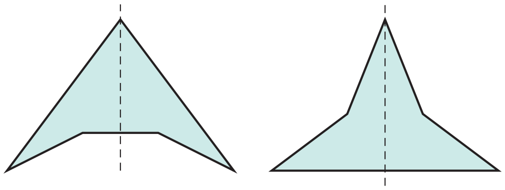
ii 47.12 m

EX EXPLORATIONS 2

p638

1 **a** None, so a thorn triangle is just an isosceles triangle.

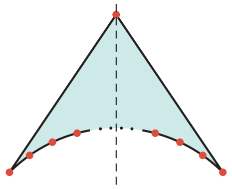
b



c Thorn heptagons have 4 reflex angles. Trying to keep them all separated would require at least 4 non-reflex angles.

d Thorn octagons have 5 reflex angles, leaving less than 180° to share among the remaining 3 angles.

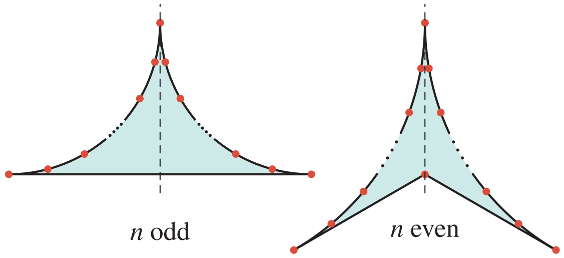
e 1) Place $n - 1$ equally spaced points along a circular arc, as shown. There are other strategies.



2) The angle sum of an n-gon is $180(n - 2)$, which is strictly less than the sum of $n - 2$ reflex angles.

f 9

g Consider equally spaced points along two identical circular arcs. Treat the odd and even cases separately.



2 **a** **i** $\frac{3}{8}$

ii $SU = \frac{1}{6}, SV = \frac{5}{24}, UV = \frac{1}{8}$

iii $\frac{3}{8}$

b $PQ = \frac{1}{7}$ or $\frac{3}{4}$

c $PQ = \frac{a}{b + c}$, so for 4 : 3 : 5, $PQ = \frac{1}{3}$

3 **a** **i** 48 cm^3

ii 48 cm, 60 cm, 64 cm

iii 88.78 cm^2, 118.10 cm^2, 153.21 cm^2

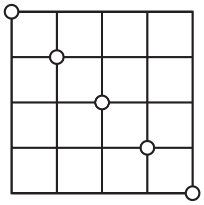
b It is enough to show the triangular faces have areas 14 130 cm^2, 7777.5 cm^2 and 11 962.5 cm^2.

c Show that the perpendicular heights of the triangular faces are the integers c^3, $a(4b^2 + c^2)$ and $b(4a^2 + c^2)$.

4 **a** There are $2^4 = 16$ combinations of horizontal and vertical moves after 4 seconds. Four of these finish at P.

b $\frac{3}{16}$

c The counters can only meet when they are both halfway; that is, after 4 moves:



d $\frac{35}{128} \approx 0.273$

e Less likely, with probability $\frac{1120}{6561} \approx 0.171$

5 **a** **i** 9

ii Week 19

b If there were k kirons, ℓ lirons and m mirons 5 weeks ago, then 1 week ago there were $\ell + m$ kirons and this week there are $k + \ell + m$ kirons. Yes, it is also true for lirons and mirons.

c 8 algae (4 kirons and 4 mirons)

d **i** Week 13 **ii** 15 days

Glossary

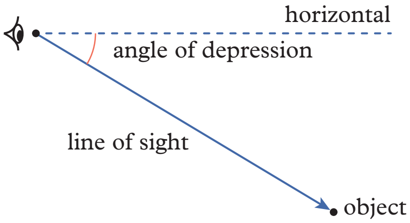
AAA
the abbreviation for angle–angle–angle. Triangles are similar if all three angles are equal. The condition AAA does not necessarily mean the triangles are congruent.

AAS
the abbreviation for angle–angle–side. Triangles are congruent if there are two pairs of corresponding equal angles and two corresponding sides are the same length.

adjacent side
the side next to the angle being considered. In trigonometry, when dealing with a right-angled triangle, the adjacent side is the side that is next to the specified angle but is not the hypotenuse.

algebraic fractions
fractions that contain at least one pronumeral

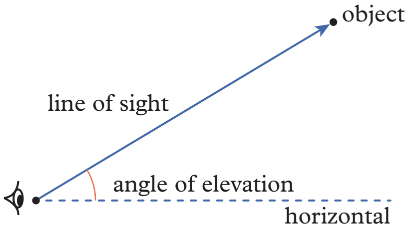
angle of depression
the angle between a horizontal line and an observer's line of sight to an object that is below the horizontal



angle-angle-angle
see AAA

angle-angle-side
see AAS

angle of elevation
the angle between a horizontal line and an observer's line of sight to an object that is above the horizontal



arc
the curved edge of a sector; part of a circle's circumference; the shape usually used to mark angles on diagrams

area
the amount of space enclosed by a two-dimensional (2D) shape

array
a collection of objects or numbers arranged in ordered rows and columns

asymptote
an imaginary line that a graph line approaches but never reaches

axis of symmetry
an imaginary line that divides a symmetrical shape or graph so that one side is a reflection of the other

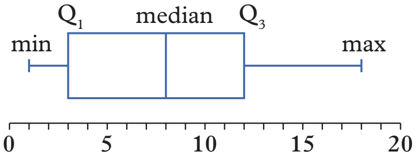
base (index form)
the number that is repeatedly multiplied by itself;
for example, in 24 the base is 2.

biased
a sample is considered biased if the method of collecting data produces a sample that is not an accurate representation of the population.

binomial
an algebraic expression for the sum or the difference of two terms

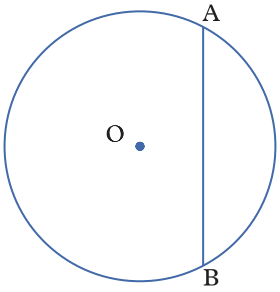
bivariate data
data showing the relationship between two variables

box plot (also box-and-whisker plot)
a visual representation representing the five-number summary of a data set along a number line. The 'box' represents the middle 50% of ordered data, with vertical lines indicating the first quartile, median and third quartile. The 'whiskers' extend from the left of the box to the minimum value and from the right of the box to the maximum value. *see also* five-number summary



capacity
the maximum amount (usually of fluid or gas) that a three-dimensional object (container) can hold; measured in litres (L) or millilitres (mL)

chord
a line segment joining two points on the circumference of a circle. In the diagram below, AB is a chord.



circumference
the perimeter of a circle

coefficient
the number by which a pronumeral is multiplied (usually placed before the pronumeral in an algebraic expression);
for example, in $3x$, the coefficient is 3

compass bearings
directions describing one position from another by using the number of degrees from north or south towards east or west that the new position is from the first;
for example, S43°E represents a direction that is 43° east of due south

complement
the complement of an event A is the event that A does not occur. The sum of the probabilities of an event and its complement is 1.

complementary angles
angles that add to make a right angle (90°)

completing the square
a method for factorising quadratic trinomial expressions by changing the form of the equation so that the left-hand side is a perfect square trinomial or is the difference of two squares

composite shapes

shapes made up of more than one regular shape

compound interest

interest that is added to the principal amount at regular intervals during the term of an investment or loan and is included when calculating the interest for the following period of the loan

compounded value

the final amount of an investment or loan after a given number of compounding periods have elapsed (that is, the total of the principal plus the interest accrued for the term of the investment or loan): $A = P + I$

compounding period

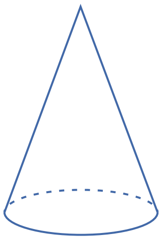
the regular interval at which compound interest is calculated during the term of an investment or loan

conditional probability

the likelihood of an event or outcome occurring based on the occurrence of a previous event or outcome

cone

a three-dimensional (3D) object with a curved surface tapering from its circular base to its apex



congruent figures

figures that are identical in shape and size, but not necessarily in position or orientation

conjugate

the conjugate of a binomial surd expression is formed by changing the operation sign between the two terms to the opposite operation sign (+ to – and – to +);
for example, the conjugate of $(3 + \sqrt{2})$ is $(3 - \sqrt{2})$. When a binomial surd expression is multiplied by its conjugate, the product is a rational number.

constant of proportionality

if the relationship between two quantities x and y is $y = kx$, k is the constant of proportionality. The value k is equal to the rate of change of y with respect to x and is the gradient of the graph of y versus x.

constant

a term without any pronumeral component that has a fixed value in a given context

correlation

the description of the strength (weak, moderate, strong) and direction (positive, negative), of the relationship between bivariate data displayed as a scatterplot or line graph

cosine (cos)

in trigonometry, the cosine of an angle in a right-angled triangle equals the ratio of the length of the adjacent side to the length of the hypotenuse.

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

cosine rule

a rule linking three side lengths (for example, a, b and c) of a triangle (ΔABC) and one of the triangle's angles (A): $a^2 = b^2 + c^2 - 2bc\cos(A)$.

Variations of this rule apply to other angles of the triangle:
$b^2 = a^2 + c^2 - 2ac\cos(B)$; and
$c^2 = a^2 + b^2 - 2ab\cos(C)$.

cross-section

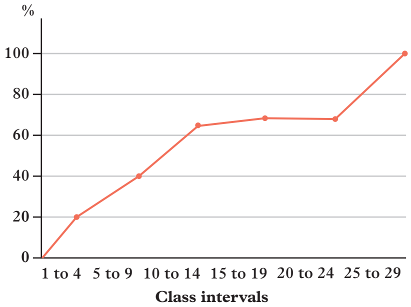
the shape of the surface revealed when we cut straight through an object. When the 'slice' is cut parallel to the base of a prism or cylinder, the cross-section has the same shape and size as that base.

cumulative frequency

for any frequency distribution, this is the sum of all the frequencies up to the current point. Cumulative frequency can be displayed in a table or on a graph (*see* cumulative frequency polygon).

cumulative frequency polygon (ogive)

a graph of a cumulative distribution, displaying data values or class intervals on the horizontal axis and the cumulative frequencies on the vertical axis



cylinder

a three-dimensional (3D) object with a uniform circular cross-section parallel to its circular base

degree of a polynomial (*n*)

the highest power of any variable in a polynomial;
for example, the polynomial $4x^3 - 2x^2 + 5x + 6$ has degree 3

denominator

the number below the horizontal line of a fraction, indicating the number of equal parts into which the whole has been divided. In the fraction $\frac{a}{b}$, b is the denominator.

dependent (response) variable

a variable whose value depends on the value of another variable in a relationship between two variables. The dependent variable is listed second in a table of values and in pairs of coordinates, and is shown on the vertical axis of the Cartesian plane. It is commonly referred to as the y value. (*see also* independent variable)

difference of two squares

when an expression is seen to be the difference of two perfect squares ($a^2 - b^2$), according to this rule it can be factorised as $(a + b)(a - b)$: $a^2 - b^2 = (a + b)(a - b)$

directly proportional

two variables are said to be directly proportional if $y = kx$, where k is the constant of proportionality. An increase in one quantity causes a corresponding increase in the other quantity, or a decrease in one quantity results in a corresponding decrease in the other quantity. Proportionality is denoted using the symbol $\propto$

discount

an amount subtracted from the original price to determine a new selling price

distributive law

the rule indicating that the result of first adding several numbers and then multiplying the sum by another number is the same as first multiplying each of the several numbers separately by that number and then adding the products: $a(b + c) = ab + ac$

domain

the complete set of possible values for the independent variable, shown on the horizontal axis of the Cartesian plane (the x or θ values), to produce a graph or define a rule

element

a member of a set;
for example, 5 is an element of $\{1, 3, 5, 7, 9\}$

elimination method

the algebraic method used to solve simultaneous equations containing more than one variable by forming a new equation with all but one of the variables removed

equidistant

at equal distances from a point

event

a subset of the sample space for a random experiment. It is one of the possible outcomes or sets of outcomes for the experiment;
for example, for the experiment of rolling a die, a single-outcome event might be ‘rolling a 2’; and an event that includes a set of outcomes might be ‘rolling a 2 or a 4’.

expansion of a perfect square

a pattern observable when expanding a factorised expression in which a factor is multiplied by itself, represented as:

$$(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2$$

or:

$$\begin{aligned}(a - b)^2 &= (a - b)(a - b)\\ &= a^2 - 2ab + b^2\end{aligned}$$

experiment

a repeatable procedure with a clearly defined set of possible results

exponent

see index

exponential relationship

a relationship between two variables, one of which is an exponent (power or index);
for example, $y = 2^x$

five-number summary

five pieces of information related to a set of data: minimum value, first quartile, median, third quartile and maximum value

gradient (*m*)

the measure of the steepness of a straight line; or the numerical measure of the slope of a graph

$$\text{gradient} = \frac{\text{rise}}{\text{run}} \qquad m = \frac{y_2 - y_1}{x_2 - x_1}$$

gradient–intercept form

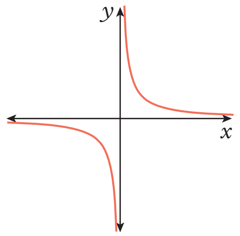
the general form of the equation of a line; $y = mx + c$, where m is the gradient and c is the y value of the y-intercept of the corresponding linear graph

histogram

a graphical display of data using bars or columns of different heights, similar to a bar chart, but showing the frequency of continuous data grouped into ranges or class intervals, with the horizontal scale boundaries located at the edges of the columns, rather than at their centres

hyperbola

an open curved graph line consisting of two branches whose ends continue to move apart from each other and begin to approximate, but never reach, two lines known as asymptotes;
for example, the graph of $y = \frac{1}{x}$ is a hyperbola (see below)



independent events

events in a probability experiment are considered independent when the outcome of one event does not affect the outcome of another event

independent (explanatory) variable

the quantity in a relationship between two variables which does not rely on any other variable for its value and is not affected by changes in another variable. The independent variable is the value listed first in a table of values and in pairs of coordinates, and the value shown along the horizontal axis on a Cartesian plane.

index (exponent)

(*plural*: indices) for a value expressed in index form (index notation), the index indicates the number of times the base is multiplied by itself as a repeated multiplication;
for example, in 24 the index is 4 and 24 = 2 × 2 × 2 × 2

index form

see index notation

index laws

the set of laws that can be used to simplify calculations involving numbers in index form

index notation (index form)

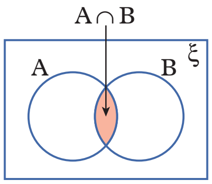
a shorter form of writing a repeated multiplication, in which a number is written with a base and an index;
for example, 24 is written in index form and means 2 × 2 × 2 × 2

interest
a fee paid for an amount borrowed or invested, usually calculated as a percentage of the amount that has been borrowed or invested

interest rate
the rate, usually expressed as a percentage, at which interest is charged for a loan or paid for an investment

interquartile range (IQR)
a measure of spread for a set of data, calculated by subtracting the lower quartile (Q_1) from the upper quartile (Q_3); that is, IQR = $Q_3 - Q_1$. The interquartile range indicates the spread or range of the middle 50% of the data.

intersection
the intersection of two sets ($A \cap B$) is the set of elements that appear in both set A and set B.



inversely proportional
if two variables are inversely proportional, then as one variable increases the other variable decreases at a proportional rate. The product of the two variables will be constant.

investment
money deposited into a bank, financial institution or other business for which interest is paid to the person depositing it (the investor)

irrational numbers
numbers that cannot be written in fraction form; they are non-terminating decimals that are not recurring

leading coefficient
the coefficient of the leading term in a polynomial;
for example, the leading coefficient in the polynomial $4x^3 - 2x^2 + 5x + 6$ is 4
see also coefficient
see also leading term

leading term
the term with the highest power in a polynomial;
for example, the leading term in the polynomial $4x^3 - 2x^2 + 5x + 6$ is $4x^3$

least-squares regression line
a straight line that fits better than any other the numerical bivariate data graphed as a scatterplot. Formulas are used to calculate the line's slope and y-intercept mathematically from the recorded data.

like surds
surd terms that have the same number or pronumeral/s under the root sign;
for example, $2\sqrt{5}$, $-\sqrt{5}$ and $7\sqrt{5}$ are like surds

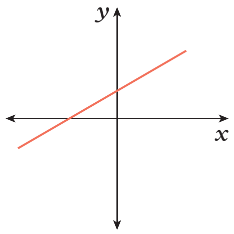
like terms
terms that contain the same pronumerals

line graph
a graph that uses plotted points joined by a line to display the relationship between two variables. The variable displayed on the horizontal axis must be for continuous data (for example, time).

line of best fit
a line that best represents the relationship between the two variables on a scatterplot. Lines of best fit are usually drawn as straight lines.

linear equation
an equation that contains only a pronumeral term (or terms) for which the highest power is one;
for example, $2x + 3 = 6$ is a linear equation

linear graph
the graph of a linear relationship. The graph is a straight line.



linear relationship
a relationship between two variables that produces a linear graph

linear term
a term where the power of the pronumeral is 1

loan
an amount of money lent by a bank or other institution to a person for a limited time (the term of the loan), for which the recipient (the borrower) must pay interest to the lender

logarithm of a power
the logarithm of a term that is expressed as a power. The related logarithm law is: $\log_a x^n = n \times \log_a x$.

logarithm of a product
the logarithm of the product of two terms. The related logarithm law is: $\log_a (xy) = \log_a x + \log_a y$.

logarithm of a quotient
the logarithm of the quotient of two terms. The related logarithm law is: $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$.

logarithm (log)
the power (index, or exponent) to which a given number (the base) must be raised in order to produce the required number;
for example, in $2^3 = 8$, the logarithm is 3 and can be written as $\log_2 8 = 3$

loss
if the selling price is less than the original cost price, this is the difference between the cost price and the selling price

lower fence
the lower end point in a set of data beyond which any lower data points are considered outliers. The lower fence lies 1.5 × IQR below the lower quartile.
also interquartile range (IQR).

lower quartile
the median of the first half (or lower half) of ordered data, denoted by Q_1

mark-up
an amount added to the original price to determine a new selling price

matrix
a rectangular array of values expressed in rows and columns

maximum turning point
a point at which a parabola changes direction and at which it has its maximum y value

mean
a measure of the centre of a data set, calculated by adding all of the data values together and dividing by the number of values

median
a measure of the centre of a data set (the 'middle value'), determined by placing the data in ascending order (smallest to largest), and finding the middle number of the ordered set if there is an odd number of values, and the average of the two middle numbers if there is an even number of values

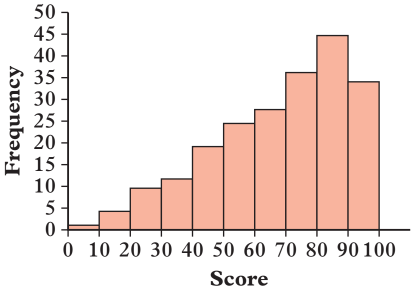
minimum turning point
a point at which a parabola changes direction and at which it has its minimum y value

monic quadratic
a quadratic in which the leading coefficient is equal to one
see also leading coefficient

negative index
an index with a negative value, indicating repeated division

negative relationship
(between bivariate data) a relationship between two sets of numerical variables such that as one variable increases in value the other variable decreases in value

negatively skewed
the description of a distribution that is skewed away from what would be the symmetrical normal distribution, having a tail to the lower end of the distribution (which is the left of the distribution when graphed). The mean of this type of distribution will be less than the median.



non-monic quadratic
a quadratic in which the leading coefficient is not equal to one
see also leading coefficient

Null Factor Law
the law stating that, if the product of two factors is 0, one or both of the factors must be equal to 0

numerator
the number above the horizontal line in a fraction, representing the number of the fractional parts that are being considered. In the fraction $\frac{a}{b}$, a is the numerator.

oblique pyramid
a pyramid with its apex (highest point) not directly above the centre of its base.
see also pyramid

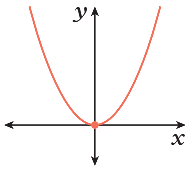
ogive
see cumulative frequency polygon

opposite side
the side that is opposite the angle being considered. In trigonometry, when dealing with a right-angled triangle, the side opposite the right angle is the hypotenuse.

outcome
a possible result of a probability experiment

outlier
extreme piece of data that is much higher or lower than the rest of the data in the data set

parabola
the approximately U-shaped graph of a quadratic relationship. A parabola can open up, down, left, right, or in any other direction, but will always be symmetrical.



parallel
the description of lines (or rays, or segments) that are always the same distance apart so that they never meet. Parallel lines are labelled with matching arrowheads.

parallel box plot
two or more box plots placed directly above each other on the same number line for comparison

percentage
a portion of a whole that has been divided into one hundred parts, shown using the % symbol

percentage decrease
the percentage by which an original amount has been decreased. Found by calculating:

$$\frac{\text{original amount} - \text{new amount}}{\text{original amount}} \times 100\%$$

percentage increase
the percentage by which an original quantity has been increased. Found by calculating:

$$\frac{\text{new amount} - \text{original amount}}{\text{original amount}} \times 100\%$$

percentage loss
the amount by which the selling price is less than the cost price, expressed as a percentage of the cost price

percentage of a quantity
a part of a whole expressed as a percentage

percentage profit
the amount by which the selling price is more than the cost price, expressed as a percentage of the cost price

period
horizontal distance covered by one full cycle of a trigonometric graph

periodic function
function or relationship that repeats itself continuously in cycles;
for example, the sine, cosine and tangent functions are periodic functions

perpendicular
a description of two lines (or rays, or segments) that meet to form a right angle (90°). Perpendicular lines are labelled using a small box at the right angle.

point of inflection
the point on a graph where the curve changes between being 'concave up' and 'concave down' (the point where the gradient will be at a local maximum or a local minimum)

point of intersection
the point on a Cartesian plane where two graphs meet

polygon
the general name for a two-dimensional (2D) shape with straight sides

polynomial

an expression with terms containing one variable only. Each term in a polynomial contains the variable raised to a power that is a positive integer or zero; for example, $4x^3 - 2x^2 + 5x + 6$ is a polynomial $(4x^3 - 2x^2 + 5x^1 + 6^0)$

population

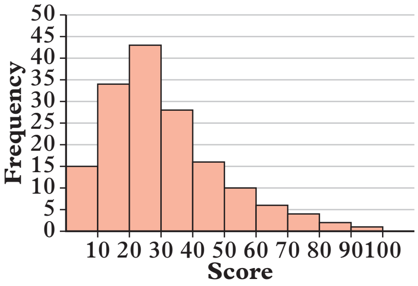
the complete set of individuals or things that statistical information is being sought about

positive relationship

(between bivariate data) a relationship between two sets of numerical variables such that as one variable increases in value the other variable also increases in value

positively skewed

the description of a distribution that is skewed away from what would be the symmetrical normal distribution, having a tail to the upper end of the distribution (which is the right of the distribution when graphed). The mean of this type of distribution will be more than the median.



principal

a sum of money invested, or an amount of money borrowed as a loan

prism

object with two ends that are identical polygons and joined by straight edges
see also polygon

profit

the gain made when the selling price is greater than the original price

pronumeral

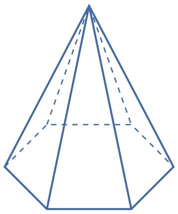
a letter or symbol that takes the place of a number

proof

series of statements showing that a theory is true in all cases

pyramid

a three-dimensional object whose base is a polygon that is joined to triangular faces which all meet at a single point (the apex). A hexagonal pyramid is shown below.



Pythagoras' Theorem

the theorem stating that the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the two other sides: $c^2 = a^2 + b^2$, where c is the length of the hypotenuse and a and b are lengths of the other two sides

Pythagorean triad (Pythagorean triple)

any set of three whole numbers that satisfy Pythagoras' Theorem

quadratic formula

the formula used to solve equations of the form $ax^2 + bx + c = 0$, where a, b and c are constants and $a \neq 0$. Solutions are found using $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$.

quadratic trinomial

any expression of the form $ax^2 + bx + c$, where a, b and c are constants

quadrilateral

a two-dimensional shape with four straight sides

quartiles

the values that divide a number set into four quarters

quotient

the result of a division calculation; for example, in $28 \div 4 = 7$, the quotient is 7

range

1 the complete set of possible values for the dependent variable, shown on the vertical axis of the Cartesian plane (the y values)

2 a measure of spread determined by finding the difference between the highest value and lowest value in a data set

rational number

a number that can be written as a fraction with both the numerator and denominator being integers (whole numbers). This includes whole numbers, fractions, terminating decimals and recurring decimals.

rectangular hyperbola

a hyperbola with perpendicular asymptotes; for example, the graph of $y = \frac{1}{x}$ is a rectangular hyperbola.
see also hyperbola
see also asymptote

reflection

a transformation in which a shape or object is reflected in a mirror line to produce its exact mirror image (sometimes referred to as a 'flip')

regular tetrahedron

a tetrahedron with four faces that are all congruent equilateral triangles
see also tetrahedron

RHS

1 abbreviation for right angle–hypotenuse–side. Right-angled triangles are congruent if their hypotenuses are the same length and another pair of side lengths are equal.

2 abbreviation for right angle–hypotenuse–side. Right-angled triangles are similar if their hypotenuses are in the same ratio as another pair of sides.

right-angle-hypotenuse-side

see RHS

right cone

a cone with its apex (highest point) directly above the centre of the base.
see cone

right pyramid

a pyramid with its apex (highest point) directly above the centre of the base
see pyramid

rise

the vertical distance between two points on a line

run

the horizontal distance between two points on a line

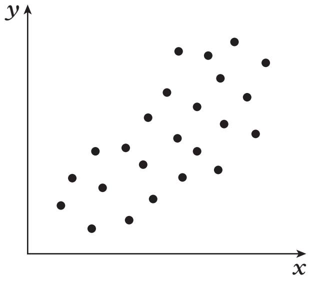
sample
a subset of a population. In statistical investigations data, this type of small selection from a population is sometimes used to represent the whole population.

sample space
the list of all the different outcomes possible for a probability experiment; for example, the sample space when rolling a die is $\{1, 2, 3, 4, 5, 6\}$

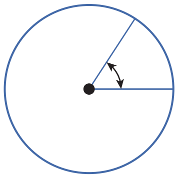
SAS
1 abbreviation for side–angle–side. Triangles are congruent if two corresponding side lengths are equal and the angle between them is equal.

2 abbreviation for side–angle–side. Triangles are similar if two corresponding side lengths are in the same ratio and angle between them is equal.

scatterplot
a graph that displays a relationship between two variables in which the points are plotted but not joined



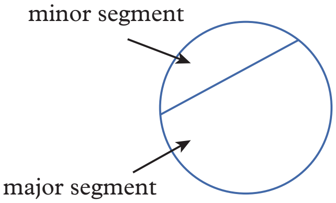
sector
a portion of a circle formed by two radii and part of the circumference



segment
the region enclosed in a circle by a chord and an arc

see also chord

see also arc



set
a collection of distinct objects or elements *see also* element

side-angle-side
see SAS

side-side-side
see SSS

similar figures
figures having the same shape but not necessarily the same size. An image produced after dilation is similar to the original figure before it was dilated.

simple interest
a quantity paid on a loan or investment. The formula for calculating simple interest is $I = PrT$, where I is the amount of interest, P is the principal, r is the interest rate and T is time.

simultaneous linear equations
equations involving two or more unknowns that have the same values in those equations

sine (sin)
in trigonometry, the sine of an angle in a right-angled triangle equals the ratio of the length of the opposite side to the length of the hypotenuse

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

sine rule
a rule linking the side lengths (for example, a, b and c) and angles (A, B and C) of a triangle (ΔABC):

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

Usually only two of these equivalent ratios need to be used in the calculation of an unknown side or angle.

skewed
a description of a data set in which data points are clustered to the left or right of the distribution, unlike a symmetrical normal distribution

sphere
a three-dimensional (3D) object shaped like a ball

spread
a statistic describing how widely the data values in a set are distributed. Measures of spread include the range and interquartile range.

SSS
1 abbreviation for side–side–side. Triangles are congruent if all three side lengths are equal.

2 abbreviation for side–side–side. Triangles are similar if all three side lengths are in the same ratio.

standard deviation
a measure of spread representing the difference between individual data values and the mean of the data set

stratified sampling
dividing the population into separate categories (such as gender or age) of sizes proportional to the size of each category in the population, so that a representative sample of the population is surveyed

subject of an equation
the value that appears by itself on one side of an equation

subset
a set contained within another set

substitution
replacement of a pronumeral with a number

substitution method
the algebraic method used to solve simultaneous equations containing more than one variable, by rearranging to make one of the variables the subject of an equation equivalent to the first equation and substituting the value of that variable from that equation into the second equation, resulting in an equation that has only one variable. This equation can then be solved and its solution substituted back into the first equation so it too can be solved.

supplementary angles
angles that add to make a straight angle (180°)

surd
an irrational root of an integer that cannot be simplified to remove the root; for example, $\sqrt{2}$

symmetrical distributions
having a similar distribution of frequencies either side of a central peak

systematic sampling

selecting a sample from a population at fixed intervals (such as every third person). The starting point should be random.

tangent (tan)

the tangent (tan) of an angle in a right-angled triangle equals the ratio of the length of the opposite side to the length of the adjacent side

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

terms

single pronumerals, single numbers, or the product of a number and one or more pronumerals; for example, the expression $3b - 4c + 10$ has three terms

tetrahedron

a pyramid with a triangular base

see pyramid

time

(in interest calculations) the period for which a loan amount is borrowed or an investment amount is invested

time series

bivariate data for which the independent variable is time. The data is recorded at regular time intervals.

see bivariate data

total surface area (TSA)

area of the total surface of an object. The total surface area of a prism is the sum of the areas of all the faces of that prism, commonly measured in square millimetres (mm^2), square centimetres (cm^2), or square metres (m^2).

transformation matrix

a matrix that describes the transformation of a point, with the first row determining the x value of the transformed point and the second row determining the y value of the transformed point

transformations

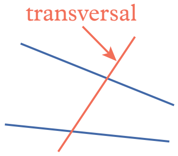
the general name for translations, reflections, rotations and dilations

translation

a transformation in which a shape or object is moved in a straight line without turning or changing size. The movement is often described as a number of units up, down, left or right.

transversal

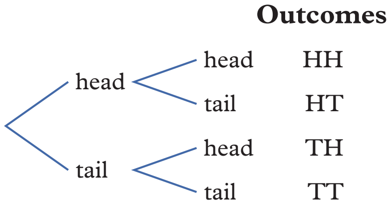
a line that crosses (intersects) two or more lines



tree diagram

a display of outcomes for a multi-step experiment;

for example, a tree diagram can be used to display the possible outcomes of flipping two coins (as shown below)



trigonometry

the study of the relationships between the angles and side lengths of a triangle

trinomial

an expression containing three terms

true bearing

the direction from one position to another, described by a number of degrees from north in a clockwise direction. The angle of a true bearing is written as three digits followed by T (or true);

for example, 090°T is due east, 180°T is due south, and 270°T is due west.

turning point

the point on a graph where a parabola changes direction

turning point form

the general form of a quadratic equation, $y = a(x - h)^2 + k$, where a is the dilation factor, and the coordinates of the turning point are (h, k)

two-dimensional list

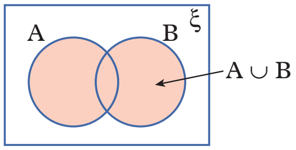
a data structure that can represent the data that might normally be found in a table

two-way table

a display of the outcomes for a two-step probability experiment in the form of a table

union

the union of two sets ($A \cup B$) is the set of elements that appear in A or B or both



unit circle

a circle drawn on the Cartesian plane with its centre at the origin and a radius of 1 unit. This circle is used to define trigonometric functions.

universal set

the set containing all elements. Everything inside the rectangle of a Venn diagram belongs to the universal set. The symbol for the universal set is ξ

upper fence

the upper end point in a set of data beyond which any greater data points are considered outliers. The upper fence lies 1.5 × IQR above the upper quartile.

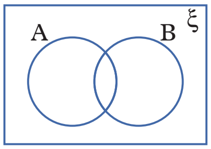
see also interquartile range (IQR)

upper quartile

the median of the upper half of an ordered set of data, denoted by Q_3

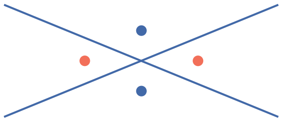
Venn diagram

a display of the relationship between different sets of data. A rectangle represents the universal set, ξ, and each set of data within the universal set is represented by a circle.



vertically opposite angles

a pair of equal angles that are not adjacent to (next to) each other, formed where two lines intersect. They are sometimes called 'X angles'. There are two pairs of vertically opposite angles in the diagram below.



volume

the quantity of space that a three-dimensional (3D) object occupies

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'If' statements and loops are important when building computer games. Most games work within one main loop, which is full of many smaller loops! You can build complex games with only a few simple statements.

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