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Australian Curriculum

MathsWorld 9

Australian Curriculum edition

Jill Vincent Beth Price Natalie Caruso
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First published 2012 by
MACMILLAN EDUCATION AUSTRALIA PTY LTD
15–19 Claremont Street, South Yarra VIC 3141

Visit our website at www.macmillan.com.au

Associated companies and representatives throughout the world.

National Library of Australia
Cataloguing-in-Publication Data

Title:	Mathsworld 9 Australian curriculum edition / Jill Vincent … [et al.].
Edition:	1st ed.
ISBN:	978 1 4202 2963 9 (pbk.)
Target Audience:	For secondary school age.
Subjects:	Mathematics—Textbooks.
	Mathematics—Problems, exercises, etc
Other Authors/Contributors:	Vincent, Jill.
Dewey Number:	510

Publisher: Colin McNeil
Project editor: Liz Filleul
Editor: Monique Miotto
Illustrators: Sunset Publishing Services, Brisbane, Qld, and Andrew Craig and Nives Porcellato
Cover designer: Dimitrios Frangoulis
Text designer: Dimitrios Frangoulis
Production controller: Loran McDougall
Photo research and permission clearance: Jan Calderwood
Consultant: Tracey MacBeth-Dunn
Typeset in 10pt Times Ten by Sunset Publishing Services, Brisbane, Qld
Cover image: Shutterstock/tonyz20

Printed in Malaysia

Contents

Introduction

The ***MathsWorld Australian Curriculum editions*** have been rigorously developed to cover all the content and requirements of the Australian Curriculum. They are carefully written to provide comprehensive and accessible textbooks which cater for a range of ability levels. The textbooks are accompanied by a complete teacher resource package that provides extra resources, teacher notes and further support.

Access to Macmillan's innovative ebook platform, OneStopDigital, is included with purchase of the student textbook or is available as stand-alone digital access. As part of the teacher resource package teachers have access to the ebook version of the student book.

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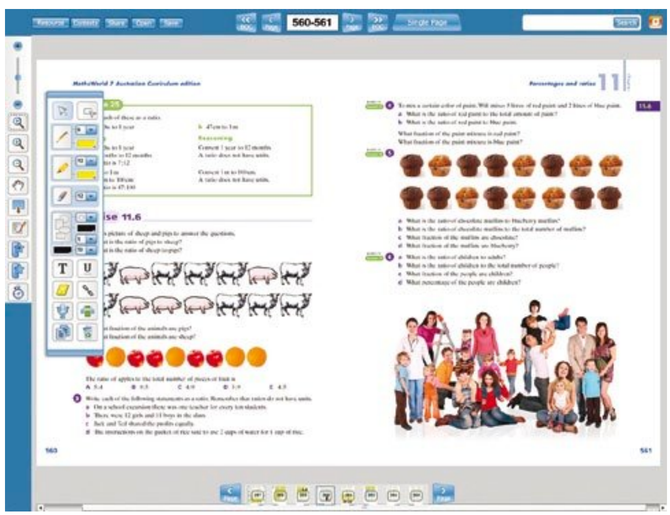
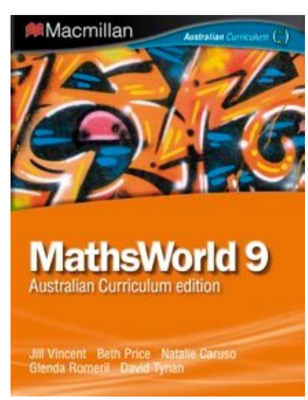


Student book



Theory and worked examples are structured into manageable sections and are designed to illustrate the main concepts and skills for each topic. Solutions to worked examples are structured into two columns, with a working column indicating what students should write and the reasoning column showing the thought process for each major step of working.

Sets of carefully graded sets of exercise questions occur at the end of each section. Exercise questions are cross-referenced to the relevant worked examples to reinforce learning and provide links where help is needed. Each exercise contains one or two clearly marked challenge questions.

At the end of each chapter there is an analysis task which explores and ties together concepts covered in the chapter.

Each chapter ends with a thorough review section, consisting of a concise summary, a visual map task and sets of multiple choice, short answer and extended response questions.

Practice quiz
Chapter 1

A practice quiz is included as a link from the ebook. The quiz can be used for revision at a later stage or could be used for individual student assignments.

Icons throughout the book indicate links to a wide range of worksheets, technology files, tests and quizzes.

Tossing a coin

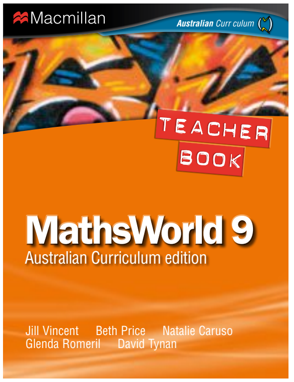
Technology files are indicated by the icon to the left and consist of GeoGebra, HTML, Excel and PowerPoint files. The icon shown here links to an Excel file that simulates the tossing of a coin 10, 100 and 1000 times.

Right-angled triangles

Icons for worksheets, blackline master templates and class activities are shown in the margin to the left and in the ebook these are linked directly to printable files for use as needed.

Teacher book

BLM
Short shoe laces



The teacher book provides a complete package of supplementary material and support with many time-saving and customisable resources available.

Pre-tests and answers in all chapters allow teachers to assess students' prior knowledge before commencing each chapter. These are also provided as PDF files linked from the student ebook.

Chapter warm-ups assist teachers with the introduction of the 'big idea' relevant to each topic. These are also provided as PDF files linked from the student ebook.

Teaching notes are provided for each chapter.

Additional worked examples are provided in the TRB—these can be used as teaching examples during the lesson.

Blackline masters and technology files are provided as links in the ebook where appropriate.

The teacher book also contains answers to all student exercises and all supplementary activities.

The answers to the Analysis tasks are provided in the teacher book. Also provided are additional analysis tasks and answers.

Each chapter includes two tests, in editable word format. Each chapter test includes multiple choice, short answer and extended response questions and is provided with a marking scheme.

Curriculum links and planning documents, including a suggested teaching program, are included in the teacher book.

Acknowledgements

The authors and publishers are grateful to the following for permission to reproduce copyright material:

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OTHER MATERIAL

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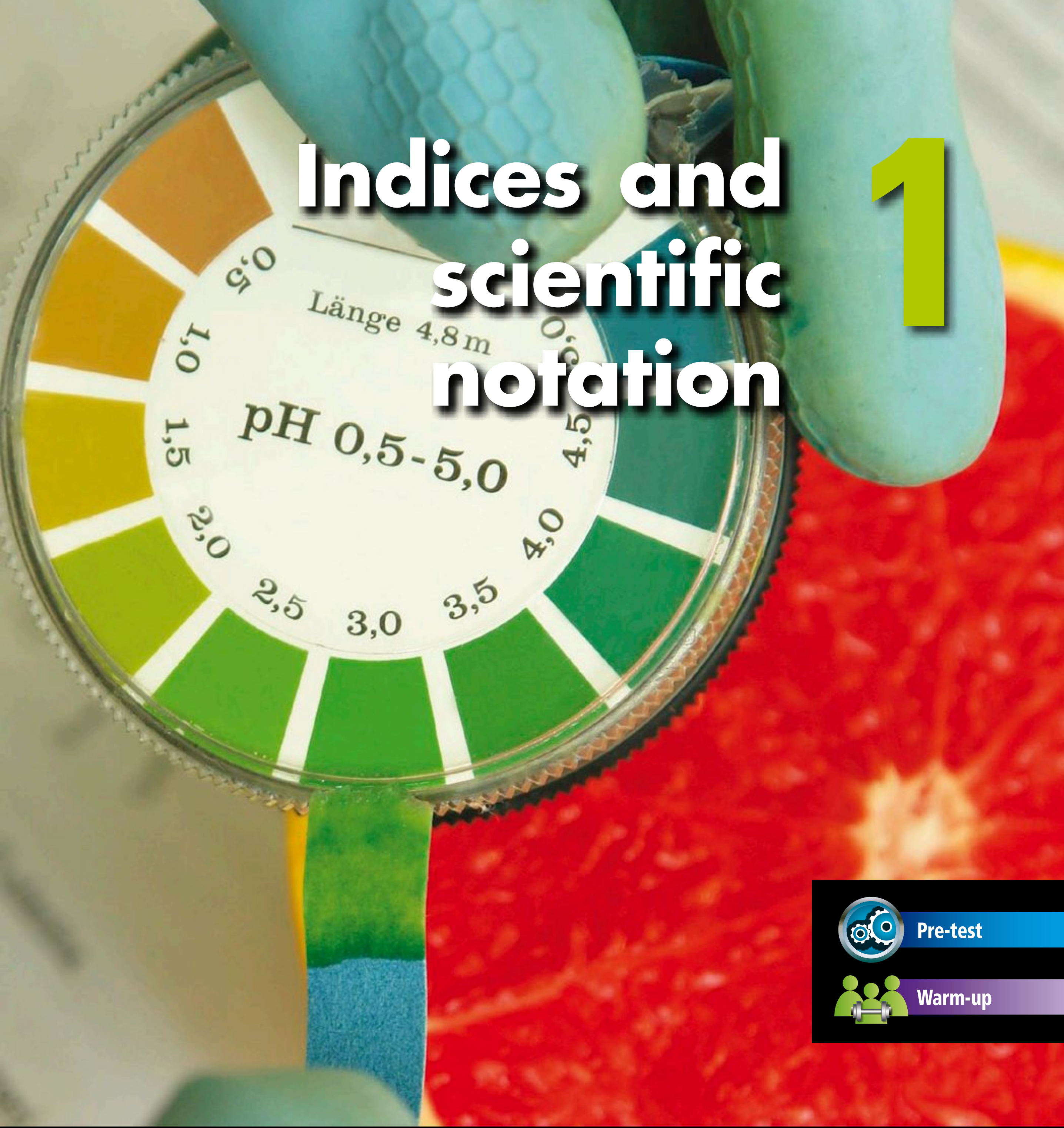
1 Indices and scientific notation



Pre-test

Warm-up

You may have used litmus paper in science to see if substances were acidic or basic. It is often important to know more accurately *how* acidic or basic substances are. For example, the acidity of the water in a fish tank is critical for healthy fish and the acidity of soil affects the types of plants that will grow in the soil. The pH scale is a scale from 0 to 14, where pH 1 is strongly acidic, pH 7 is neutral and pH 14 is strongly basic. Chapter 1 introduces negative indices. In the analysis task at the end of this chapter you will see how the pH scale is based on negative indices.

1.1 Multiplying and dividing expressions in index form

When numbers are written as the product of their prime factors, the same factor may be repeated two or more times. For example, 18 can be written as $2 \times 3 \times 3$ where we see that the factor 3 is repeated. Repeated factors can be written in index form, so we can write 18 as 2×3^2.

BLM
Chapter warm-up

Example 1

Simplify the expression $2 \times 2 \times 2 \times 2 \times 3 \times 3$ by writing it in index form.

Working

$2 \times 2 \times 2 \times 2 \times 3 \times 3$
$= 2^4 \times 3^2$

Reasoning

The repeated factor of 2 occurs four times so we write 2^4. The repeated factor of 3 occurs twice so we write 3^2.

Algebraic expressions with repeated factors can be written in index form in the same way.

Example 2

Simplify $5 \times a \times a \times a \times b \times b \times b \times b$ by writing in index form.

Working

$5 \times a \times a \times a \times b \times b \times b \times b$
$= 5 \times a^3 \times b^4$
$= 5a^3b^4$

Reasoning

$$5 \times \underbrace{a \times a \times a}_{\text{3 factors}} \times \underbrace{b \times b \times b \times b}_{\text{4 factors}}$$

$a \times a \times a = a^3$ and $b \times b \times b \times b = b^4$
$5 \times a^3 \times b^4$ can be written as $5a^3b^4$.

When expressions are written in index form, we can reverse the process and write them as the product of factors (in factor form).

Example 3

Write the following expressions in factor form.

a $3^2 \times 5 \times 7^3$ **b** $2h^2m^4$

continued

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Example 3 continued

Working	Reasoning
a $3^2 \times 5 \times 7^3$ $= 3 \times 3 \times 5 \times 7 \times 7 \times 7$	There are two factors of 3, one factor of 5 and three factors of 7.
b $2h^2m^4$ $= 2 \times h \times h \times m \times m \times m \times m$	There is one factor of 2, and there are two factors of h and four factors of m.

In Year 8 we saw that there are several index laws that make simplification of expressions easier. We can now generalise these laws to include pronumerals as well as numbers.

The first two laws relate to multiplying and dividing numbers or pronumerals that have the same base.

If we expand $2^3 \times 2^5$ by writing each index form in its factor form, we see that we can take a shortcut to the answer by adding the indices 3 and 5 to give 8. Similarly, if we expand $d^2 \times d^3$ we see that we can take a short by adding the indices 2 and 3 to give 5.

$$2^3 \times 2^5 = \underbrace{2 \times 2 \times 2}_{3\ \textit{factors}} \times \underbrace{2 \times 2 \times 2 \times 2 \times 2}_{5\ \textit{factors}}$$

$$= \underbrace{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}_{8\ \textit{factors}}$$

$$= 2^8$$

$$d^2 \times d^3 = \underbrace{d \times d}_{2\ \textit{factors}} \times \underbrace{d \times d \times d}_{3\ \textit{factors}}$$

$$= \underbrace{d \times d \times d \times d \times d}_{5\ \textit{factors}}$$

$$= d^5$$

We can only add indices in this way if the bases are the same.

Multiplication index law

When multiplying numbers or pronumerals that have the same base, add the indices.

$$a^m \times a^n = a^{m+n}$$

Example 4

Use the multiplication index law to write each of the following in simpler index form.

a $10^3 \times 10^4$ **b** $2^4 \times 3^5 \times 2 \times 3^2$

Working	Reasoning
a $10^3 \times 10^4$ $= 10^{3+4}$ $= 10^7$	The bases are the same so the indices can be added.

continued

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Example 4 continued

Working

b $2^4 \times 3^5 \times 2 \times 3^2$

$= 2^4 \times 2^1 \times 3^5 \times 3^2$

$= 2^{4+1} \times 3^{5+2}$

$= 2^5 \times 3^7$

Reasoning

$2 = 2^1$

There are two different bases, 2 and 3.

Add the indices for 2^4 and 2^1.

Add the indices for 3^5 and 3^2.

Example 5

Use the multiplication index law to simplify the following expressions.

a $x^3 \times x^7$ **b** $w^6 \times w^8 \times w$ **c** $3k^4 \times 2k^5$

Working

a $x^3 \times x^7$

$= x^{3+7}$

$= x^{10}$

Reasoning

x^3 and x^7 have the same base, so we can add the indices.

Working

b $w^6 \times w^8 \times w$

$= w^{6+8+1}$

$= w^{15}$

Reasoning

$w = w^1$

The bases are the same, so we can add the indices.

Working

c $3k^4 \times 2k^5$

$= 3 \times 2 \times k^4 \times k^5$

$= 6k^{4+5}$

$= 6k^9$

Reasoning

Multiply the coefficients 3 and 2.

k^4 and k^5 have the same base, so we can add the indices.

If we expand $\frac{3^8}{3^6}$ by writing it in factor form, we see that we can take a shortcut to the answer by subtracting the index 6 from the index 8 to give the index 2. Similarly, if we write $\frac{h^7}{h^3}$ in factor form, we can take a shortcut by subtracting the index 3 from the index 7 to give the index 4.

$$\frac{3^8}{3^6} = \frac{\overbrace{\cancel{3}^1 \times \cancel{3}^1 \times \cancel{3}^1 \times \cancel{3}^1 \times \cancel{3}^1 \times \cancel{3}^1 \times 3 \times 3}^{8\ \textit{factors}}}{\underbrace{\cancel{3}_1 \times \cancel{3}_1 \times \cancel{3}_1 \times \cancel{3}_1 \times \cancel{3}_1 \times \cancel{3}_1}_{6\ \textit{factors}}}$$

$$= 3 \times 3$$

$$= 3^2$$

$$\frac{h^7}{h^3} = \frac{\overbrace{\cancel{h}^1 \times \cancel{h}^1 \times \cancel{h}^1 \times h \times h \times h \times h}^{7\ \textit{factors}}}{\underbrace{\cancel{h}_1 \times \cancel{h}_1 \times \cancel{h}_1}_{3\ \textit{factors}}}$$

$$= h \times h \times h \times h$$

$$= h^4$$

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Division index law

When dividing numbers or pronumerals that have the same base, subtract the indices.

$a^m \div a^n = a^{m-n}$ or $\frac{a^m}{a^n} = a^{m-n}$

Example 6

Use the division index law to write each of the following in simpler index form.

a $\frac{10^9}{10^6}$ **b** $\frac{3^3 \times 5^7}{3^2 \times 5^4}$

Working	Reasoning
a $\frac{10^9}{10^6}$ $= 10^{9-6}$ $= 10^3$	The bases are the same, so the indices can be subtracted.
b $\frac{3^3 \times 5^7}{3^2 \times 5^4}$ $= 3^{3-2} \times 5^{7-4}$ $= 3^1 \times 5^3$ $= 3 \times 5^3$	There are two different bases, 3 and 5. Subtract the indices for 3^3 and 3^2. Subtract the indices for 5^7 and 5^4. 3^1 can be written as 3.

Example 7

Use the division index law to simplify the following expressions.

a $\frac{y^9}{y^7}$ **b** $\frac{14b^{10}}{4b^6}$ **c** $\frac{a^4b^3}{a^2b^2}$

Working	Reasoning
a $\frac{y^9}{y^7}$ $= y^{9-7}$ $= y^2$	y^9 and y^7 have the same base, so we can subtract the indices.

continued

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Example 7 continued

Working	**Reasoning**
b $\dfrac{14b^{10}}{4b^6}$	Factorise the coefficients and cancel the common factors. Then apply the division index law.
$= \dfrac{\cancel{2}^{1} \times 7 \times b^{10}}{\cancel{2}_{1} \times 2 \times b^6}$	$\dfrac{b^{10}}{b^6} = b^{10-6} = b^4$
$= \dfrac{7 \times b^{10-6}}{2}$	
$= \dfrac{7b^4}{2}$	
c $\dfrac{a^4b^3}{a^2b^2}$	There are two different bases.
$= a^{4-2}b^{3-2}$	Subtract the indices for a. Subtract the indices for b.
$= a^2b^1$	b^1 can be written as b.
$= a^2b$	

In the next example, the multiplication and division index laws are both used.

Example 8

Simplify each of the following. Leave the answer to part **a** in index form and evaluate part **b**.

a $\dfrac{2^3 \times 2^5}{2^2}$ **b** $\dfrac{3^2 \times 3^7}{3^4 \times 3^2}$

Working	**Reasoning**
a $\dfrac{2^3 \times 2^5}{2^2} = \dfrac{2^{3+5}}{2^2}$	Use the multiplication index law to simplify the numerator. $2^3 \times 2^5 = 2^{3+5} = 2^8$.
$= \dfrac{2^8}{2^2}$	Use the division index law to simplify $\dfrac{2^8}{2^2}$.
$= 2^{8-2}$	
$= 2^6$	
b $\dfrac{3^2 \times 3^7}{3^4 \times 3^2}$	Use the multiplication index law to simplify the numerator and to simplify the denominator.
$= \dfrac{3^{2+7}}{3^{4+2}}$	
$= \dfrac{3^9}{3^6}$	Use the division index law to simplify $\dfrac{3^9}{3^6}$.
$= 3^{9-6}$	
$= 3^3$	Evaluate 3^3
$= 27$	

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When simplifying expressions that include coefficients in the numerator and denominator, the coefficients should be simplified by dividing by any common factors. If there are products of terms in index form, the numerator and the denominator should each be simplified first before applying the division index.

Example 9

Simplify each of the following, leaving in index form.

a $\dfrac{m^2 \times m^5}{m^3}$ **b** $\dfrac{15x^7y^3}{9x^4y}$ **c** $\dfrac{10a^2b \times 3a^3b^4}{5ab \times 2a^2b^2}$

Working	Reasoning
a $\dfrac{m^2 \times m^5}{m^3} = \dfrac{m^{2+5}}{m^3}$	Use the multiplication index law to simplify the numerator.
$= \dfrac{m^7}{m^3}$	
$= m^{7-3}$	Use the division index law to simplify $\dfrac{m^7}{m^3}$.
$= m^4$	
b $\dfrac{15x^7y^3}{9x^4y} = \dfrac{5x^7y^3}{3x^4y}$	Simplify the coefficients by dividing by the common factor, 3.
$= \dfrac{5x^{7-4}y^{3-1}}{3}$	There are two different bases. Subtract the indices for x. Subtract the indices for y.
$= \dfrac{5x^3y^2}{3}$	
c $\dfrac{10a^2b^3 \times 3a^3b^4}{5ab \times 2a^2b^2}$	
$= \dfrac{10 \times 3 \times a^2 \times a^3 \times b^3 \times b^4}{5 \times 2 \times a^1 \times a^2 \times b^1 \times b^2}$	Group the coefficients. Use the multiplication index law to simplify the numerator. Repeat for the denominator.
$= \dfrac{\overset{3}{\cancel{30}} \times a^{2+3} \times b^{3+4}}{\underset{1}{\cancel{10}} \times a^{1+2} \times b^{1+2}}$	Simplify the coefficients.
$= \dfrac{3 \times a^5 \times b^7}{a^3 \times b^3}$	
$= 3 \times a^{5-3}b^{7-3}$	Use the division index law to simplify the division.
$= 3a^2b^4$	

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Zero index

Any non-zero number divided by itself is one, so it is always true that $\frac{a^m}{a^m} = 1$, if a does not equal zero.

However, applying the division index law shows that $\frac{a^m}{a^m} = a^{m-m} = a^0$.

Since $\frac{a^m}{a^m} = a^0$ and $\frac{a^m}{a^m} = 1$, it follows that $a^0 = 1$ for any non-zero number or pronumeral base a.

For example, we can show that $3^0 = 1$.

$\frac{3^5}{3^5} = 1$ because any non-zero number divided by itself is 1.

But using the division index law,

$\frac{3^5}{3^5} = 3^{5-5} = 3^0$

This means that $\frac{3^5}{3^5} = 1$ and $\frac{3^5}{3^5} = 3^0$.

So $3^0 = 1$.

Zero index

Any non-zero number or pronumeral base, a, raised to the power of 0 is equal to 1.

$a^0 = 1$ where $a \neq 0$

Example 10

Simplify each of the following.

a 7^0

b $a^0 + b^0$ where $a \neq 0$ and $b \neq 0$

c $3x^0$ where $x \neq 0$

d $\frac{a^3 \times a^2}{a^5}$

Working	Reasoning
a $7^0 = 1$	Any non-zero number raised to the power 0 is equal to 1.
b If neither a nor b are zero, $a^0 + b^0 = 1 + 1 = 2$	Any non-zero pronumeral raised to the power 0 is equal to 1.
c If x is not zero, $3x^0 = 3 \times 1 = 3$	x^0 is multiplied by 3. Substitute 1 for x^0.

continued

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Example 10 continued

Working	Reasoning
d $\dfrac{a^3 \times a^2}{a^5}$	Use the multiplication index law to simplify the numerator: $a^3 \times a^2 = a^{3+2} = a^5$.
$= \dfrac{a^{3+2}}{a^5}$	
$= \dfrac{a^5}{a^5}$	Use the division index law: $\dfrac{a^5}{a^5} = a^{5-5} = a^0$.
$= a^{5-5}$	
$= a^0$	Any non-zero number or pronumeral raised to the power 0 is equal to 1.
$= 1$	Note that we could also have written $\dfrac{a^5}{a^5} = 1$ because any non-zero expression divided by itself is equal to 1.

exercise 1.1

LINKS TO Example 1

1 Simplify the following by writing in index form.

- **a** $4 \times 4 \times 4 \times 4 \times 4$
- **b** $10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$
- **c** $3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 7 \times 7 \times 7$
- **d** $2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 3 \times 3 \times 5 \times 5$

LINKS TO Example 2

2 Simplify the following by writing in index form.

- **a** $a \times a \times a \times a$
- **b** $k \times k \times k \times k \times k$
- **c** $x \times x \times x \times y \times y \times y \times y$
- **d** $a \times a \times b \times b \times b \times b \times b$
- **e** $3 \times m \times n \times 3 \times m \times m \times n$
- **f** $2 \times c \times c \times c \times d \times 2 \times c \times d \times d$
- **g** $5 \times a \times a \times b \times 10 \times a \times b \times 4 \times a \times b \times b$
- **h** $6 \times d \times d \times k \times 4 \times d \times k \times k \times 12 \times k \times k$

LINKS TO Example 3

3 Write the following in factor form.

a $2^4 \times 5^3$	**b** $3^5 \times 7^4 \times 11^2$	**c** $3a^3b^2$
d $5m^4n^3$	**e** $2d^5y$	**f** $7a^2m^5$
g $5ab^7$	**h** $7m^3p$	**i** $13x^3y^5$

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LINKS TO Example 4a

Simplify each of the following, expressing your answers in index form.

a $4^3 \times 4^2$ **b** $11^3 \times 11^7$ **c** $2^{13} \times 2^8$ **d** $5^4 \times 5^5$
e $10^{11} \times 10^7$ **f** 5×5^3 **g** $10^2 \times 10^5$ **h** $7^3 \times 7$
i $3^4 \times 3^4$ **j** $10 \times 10^4 \times 10^2$ **k** $6^2 \times 6^2 \times 6^2$ **l** $2 \times 2^3 \times 2^4$

LINKS TO Example 4b

Simplify each of the following, expressing your answers in index form.

a $3^3 \times 2^4 \times 3^2 \times 2^8$ **b** $5^4 \times 2^5 \times 5^2 \times 2$ **c** $2^9 \times 7^3 \times 2^2 \times 7^5$
d $3^4 \times 10^2 \times 3 \times 10^3$ **e** $2^4 \times 5^2 \times 2^3 \times 5$ **f** $2^5 \times 7^2 \times 2^3 \times 7$
g $3^3 \times 5 \times 3^2 \times 5$ **h** $2^4 \times 3^2 \times 5 \times 2 \times 3^3 \times 5^2$ **i** $3 \times 7 \times 7^4 \times 3 \times 7^4$

LINKS TO Example 5

Simplify each of the following.

a $a^5 \times a^2$ **b** $m^4 \times m \times m^2$ **c** $x^2 \times x^3 \times x^2$
d $b \times b^3 \times b$ **e** $r^5s^4 \times r^2s^8$ **f** $v^6w^3 \times v^2w^7$
g $m^3n^4 \times mn$ **h** $x^7y^2 \times x^5y^6$ **i** $g^3h^2 \times g^3h^6$
j $a^4b^4 \times a^7b^2$ **k** $x^5y^3 \times x^2y^5$ **l** $a^2bc^3 \times a^3c^4$

LINKS TO Example 6a

Simplify, leaving in index form.

a $12^9 \div 12^5$ **b** $8^7 \div 8^5$ **c** $10^{19} \div 10^8$
d $3^5 \div 3$ **e** $2^5 \div 2$ **f** $10^{14} \div 10^7$
g $5^6 \div 5^2$ **h** $4^7 \div 4^4 \div 4^2$ **i** $\dfrac{3^7}{3^2}$
j $\dfrac{5^3}{5}$ **k** $\dfrac{2^{12}}{2^4}$ **l** $\dfrac{3^9}{3^3}$

LINKS TO Example 6b

Simplify the following, leaving in index form.

a $\dfrac{2^4 \times 3^8}{2^2 \times 3^5}$ **b** $\dfrac{7^5 \times 5^7}{7 \times 5^5}$ **c** $\dfrac{7^8 \times 2^6}{7^4 \times 2^2}$ **d** $\dfrac{5^9 \times 11^4}{5^3 \times 11}$

LINKS TO Example 7

Simplify the following expressions.

a $\dfrac{a^6}{a^3}$ **b** $\dfrac{k^7}{k^2}$ **c** $\dfrac{m^{12}}{m^4}$ **d** $\dfrac{8y^7}{2y^3}$
e $\dfrac{15x^5}{5x}$ **f** $\dfrac{14b^9}{4b^3}$ **g** $\dfrac{24p^6}{6p^6}$ **h** $\dfrac{36a^8}{24a^2}$
i $\dfrac{m^6n^3}{m^3n^2}$ **j** $\dfrac{a^8b^{12}}{a^5b^7}$ **k** $\dfrac{x^{11}y^4}{x^{10}y^2}$ **l** $\dfrac{h^{11}k^7}{h^7k^3}$

LINKS TO Example 8

Simplify, leaving in index form.

a $\dfrac{3^2 \times 3^5}{3}$ **b** $\dfrac{10^2 \times 10^7}{10^3}$ **c** $\dfrac{4^3 \times 9^6}{4 \times 9^4}$ **d** $\dfrac{5^6 \times 5^3}{5^5 \times 5^2}$
e $\dfrac{8^5}{8^2 \times 8}$ **f** $\dfrac{6^4 \times 6^5}{6^7}$ **g** $\dfrac{7^8 \times 7}{7^5}$ **h** $\dfrac{3^5 \times 3^7}{3^2 \times 3^9}$
i $\dfrac{6^4 \times 6^{12}}{6^6 \times 6^5}$ **j** $\dfrac{2^7 \times 7^6}{2^4 \times 7^3}$ **k** $\dfrac{3^9 \times 11^3}{3^5 \times 11}$ **l** $\dfrac{3^4 \times 3^7}{3^8}$

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Simplify in index form then evaluate.

a $\dfrac{3^4 \times 3^7}{3^8}$ b $\dfrac{5^6 \times 5^3}{5^5 \times 5^2}$ c $\dfrac{8^5}{8^2 \times 8}$ d $\dfrac{6^4 \times 6^5}{6^7}$

LINKS TO Example 9

Use the first and second index laws to simplify each of the following.

a $\dfrac{e^2 \times e^4}{e^3}$ b $\dfrac{t^4 \times t^3}{t^2}$ c $(d^7 \times d^4) \div d^5$ d $(y^5 \times y) \div y^4$

e $\dfrac{u^3 \times u^6}{u^5}$ f $\dfrac{g^9 \times g^3}{g^7}$ g $(e^5 \times e^9) \times e^8$ h $(n^5 \times n^2) \div n^6$

i $\dfrac{a^3 \times a^5}{a^2 \times a^4}$ j $\dfrac{t^2 \times t^6}{t^3 \times t^4}$ k $\dfrac{w^5}{w^2} \times \dfrac{w^7}{w^4}$ l $\dfrac{c^6}{c^4} \times \dfrac{c^5}{c^3}$

Simplify.

a $\dfrac{12a^3b^3}{3a^2b^2}$ b $\dfrac{12ab^3}{4b}$ c $\dfrac{20x^3y^2}{15xy}$ d $\dfrac{a^2b \times 2ab^2}{6ab}$

e $\dfrac{24x^4y}{16xy}$ f $\dfrac{6x^2y}{5} \times \dfrac{10xy^2}{2y^3}$ g $\dfrac{3a^3b \times 3a^2b^2}{6ab}$ h $\dfrac{3x^3y}{10xy^2} \times \dfrac{15x^2y^2}{6x^3}$

LINKS TO Example 10

Evaluate. Simplify first, where necessary.

a 2^0 b $10^0 \times 10^0$ c $10^0 + 10^0$ d $\dfrac{5^4 \times 5^3}{5^5 \times 5^2}$

e $3a^2b \times 3 \div a^2b$ f $\dfrac{2^8 \times 7^5}{2^6 \times 7^2 \times 7^3}$ g $\dfrac{6^4}{6^2 \times 6^2}$ h $5^2 + 6^0$

i $\dfrac{3^5 \times 3}{2^0 \times 3^4}$ j $\dfrac{4^2 \times 4^4 \times 5^0}{4^5 \times 4}$ k ab^0 l $5a^0$

m $3xy^0$ n $a^0 + b^0$ o $7^0 - 4$ p $3^0 - 5^0$

exercise 1.1 challenge question

All the indices in this chapter are integers.

a Use your calculator to find the following.

i $9^{0.5}$ ii $16^{0.5}$

iii $25^{0.5}$ v $36^{\frac{1}{2}}$

b What does an index of 0.5 mean? Test your conjecture with some other numbers raised to the power 0.5, that is, to the power $\dfrac{1}{2}$.

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1.2 Powers with brackets

If we write $(3^2)^4$ in expanded factor form and then write the result in simplest index form, we find that $(3^2)^4$ is equal to 3^8.

$$\begin{aligned}(3^2)^4 &= 3^2 \times 3^2 \times 3^2 \times 3^2\\ &= 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3\\ &= 3^8\end{aligned}$$

Similarly $(a^3)^2$ is equal to a^6.

$$\begin{aligned}(a^3)^2 &= a^3 \times a^3\\ &= a \times a \times a \times a \times a \times a\\ &= a^6\end{aligned}$$

We see that there is a shortcut method of multiplying the two indices.

$$(3^2)^4 = 3^{2\times 4} = 3^8 \text{ and } (a^3)^2 = a^{3\times 2} = a^6$$

Raising a power to a power

When a power is raised to another power, multiply the indices.

$$(a^m)^n = a^{m\times n}$$

Example 11

Express each of the following without brackets.

a $(10^2)^3$

b Express $(x^3)^4$ without brackets.

Working	**Reasoning**
a $(10^2)^3 = 10^{2\times 3}$ $= 10^6$	$(10^2)^3 = 10^2 \times 10^2 \times 10^2$ $= 10 \times 10 \times 10 \times 10 \times 10 \times 10$ $= 10^6$ Multiplying the indices is a shortcut.
b $(x^3)^4 = x^{3\times 4}$ $= x^{12}$	The raising a power to a power law tells us that we can multiply the indices.

When a product in brackets is raised to a power, each number or pronumeral in the brackets is raised to the power.

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$$\begin{aligned}(3 \times 5)^4 &= (3 \times 5) \times (3 \times 5) \times (3 \times 5) \times (3 \times 5) \\ &= 3 \times 3 \times 3 \times 3 \times 5 \times 5 \times 5 \times 5 \\ &= 3^4 \times 5^4\end{aligned}$$

$$\begin{aligned}(ab)^3 &= (ab) \times (ab) \times (ab) \\ &= a \times b \times a \times b \times a \times b \\ &= a \times a \times a \times b \times b \times b \\ &= a^3 \times b^3 \\ &= a^3b^3\end{aligned}$$

Power of a product

If a product of numbers or pronumerals in brackets is raised to a power, each number or pronumeral inside the brackets is raised to the power outside the brackets.

$$(ab)^m = a^m \times b^m$$

We normally evaluate coefficients that are powers, for example, $(2a)^3 = 2^3 \times a^3 = 8a^3$.

Example 12

Express each of the following without brackets.

a $(2a)^3$ **b** $(3xy)^3$ **c** $(2a^2b^3)^4$ **d** $(3x)^0$ where $x \neq 0$

Working	Reasoning
a $(2a)^3 = 2^3a^3$ $= 8a^3$	Using the power of a product index law, raise each term inside the brackets to the power 3. Evaluate: $2^3 = 8$.
b $(3xy)^3$ $= 3^3x^3y^3$ $= 27x^3y^3$	Using the power of a product law, raise each term inside the brackets to the power 3. Evaluate: $3^3 = 27$.
c $(2a^2b^3)^4$ $= 2^4 \times (a^2)^4 \times (b^3)^4$ $= 2^4 \times a^{2\times4} \times b^{3\times4}$ $= 2^4 \times a^8 \times b^{12}$ $= 16a^8b^{12}$	Using the power of a product law, raise each term inside the brackets to the power 4. Using the third index law, simplify $(a^2)^4$ by multiplying the indices. Repeat for $(b^3)^4$. Evaluate: $2^4 = 16$.

continued

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Example 12 continued

Working

d If x is not zero,

$$(3x)^0$$
$$= 3^0 \times x^0$$
$$= 1$$

Reasoning

The brackets indicate that the entire expression $3x$ is raised to the power zero. Compare this with $3x^0$ where it is only x that is raised to the power zero.

$3x^0 = 3 \times x^0 = 3 \times 1 = 3$

Just as for products raised to a power, when a fraction in brackets is raised to a power, each number or pronumeral in the brackets is raised to the power.

$$\left(\frac{2}{3}\right)^3 = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{2 \times 2 \times 2}{3 \times 3 \times 3} = \frac{2^3}{3^3} = \frac{8}{27}$$

$$\left(\frac{x}{2}\right)^4 = \frac{x}{2} \times \frac{x}{2} \times \frac{x}{2} \times \frac{x}{2} = \frac{x \times x \times x \times x}{2 \times 2 \times 2 \times 2} = \frac{x^4}{2^4} = \frac{x^4}{16}$$

Power of a quotient or a fraction

When a fraction is raised to a power, both the denominator and the numerator are raised to the same power.

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

Example 13

Express each of the following without brackets.

a $\left(\frac{2}{5}\right)^3$

b $\left(\frac{2x}{y}\right)^5$

Working

a $\left(\frac{2}{5}\right)^3 = \frac{2^3}{5^3}$

$= \frac{8}{125}$

Reasoning

The numerator and the denominator are each raised to the power 3.

In index form, $\left(\frac{2}{5}\right)^3 = \frac{2^3}{5^3}$

Evaluate 2^3 and $5^3 = \frac{8}{125}$

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Example 13 continued

Working

b $\left(\frac{2x}{y}\right)^5 = \frac{2^5x^5}{y^5}$

$= \frac{32x^5}{y^5}$

Reasoning

The power of a quotient law tells us that we can raise each term inside the brackets to the power 5.
Evaluate $2^5 = 32$

Example 14

Express each of the following without brackets.

a $\left(\frac{3a^2}{b^3}\right)^4$ **b** $\left(\frac{a^5}{2b^3}\right)^3$

Working

a $\left(\frac{3a^2}{b^3}\right)^4 = \frac{3^4(a^2)^4}{(b^3)^4}$

$= \frac{3^4a^{2\times4}}{b^{3\times4}}$

$= \frac{3^4a^8}{b^{12}}$

$= \frac{81a^8}{b^{12}}$

Reasoning

The power of a quotient law tells us that we can raise each term inside the brackets to the power 4.
Using the raising a power to a power law, simplify $(a^2)^4$ by multiplying the indices.
Repeat for $(b^3)^4$.

Evaluate $3^4 = 81$.

b $\left(\frac{a^5}{2b^3}\right)^3 = \frac{(a^5)^3}{2^3(b^3)^3}$

$= \frac{a^{5\times3}}{2^3b^{3\times3}}$

$= \frac{a^{15}}{2^3b^9}$

$= \frac{a^{15}}{8b^9}$

The power of a quotient law tells us that we can raise each term inside the brackets to the power 3.

Using the raising a power to a power law, simplify $(a^5)^3$ by multiplying the indices.

Repeat for $(b^3)^3$.

Evaluate $2^3 = 8$.

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exercise 1.2

LINKS TO Example 11a

1 Express each of the following in simplest index form.

a (3^2) **b** $(3^3)^2$ **c** $(10^5)^2$ **d** $(4^3)^3$
e $(2^4)^5$ **f** $(3^2)^0$ **g** $(3^0)^2$ **h** $(10^4)^3$
i $(5^2)^4$ **j** $(7^4)^0$ **k** $(2^5)^3$ **l** $(6^0)^3$

LINKS TO Example 11b

2 Express each of the following in simplest index form.

a $(a^2)^4$ **b** $(a^4)^2$ **c** $(x^6)^2$ **d** $(x^2)^6$
e $(b^2)^0$ **f** $(b^0)^2$ **g** $(m^3)^5$ **h** $(y^2)^5$
i $(x^5)^3$ **j** $(d^4)^6$ **k** $(k^3)^3$ **l** $(t^8)^2$

3 Which of the following does **not** simplify to x^{12}?

A $x^{10} \times x^2$ **B** $(x^3)^4$ **C** $(x^6)^2$ **D** $x^{12} \times x$ **E** $\left((x^2)^2\right)^3$

LINKS TO Example 12a

4 Express each of the following without brackets, evaluating the coefficients.

a $(2b)^2$ **b** $(3y)^4$ **c** $(2a)^5$ **d** $(5x)^3$
e $(2h)^0$ **f** $(4h)^3$ **g** $(3x)^5$ **h** $(10k)^6$
i $(6a)^3$ **j** $(2x)^8$ **k** $(4n)^4$ **l** $(5n)^0$

LINKS TO Examples 12b–d

5 Express each of the following without brackets.

a $(ab)^3$ **b** $(xy)^4$ **c** $(a^2b)^3$ **d** $(xy^2)^2$
e $(m^3n^2)^4$ **f** $(c^3d^4)^2$ **g** $(2x^2y)^4$ **h** $(3k^4t^3)^3$
i $(3a^2b^3)^4$ **j** $(2mp^3)^5$ **k** $(11x^2y^3)^2$ **l** $(5c^4d)^3$

LINKS TO Example 13a

6 Express each of the following without brackets then evaluate.

a $\left(\frac{1}{2}\right)^3$ **b** $\left(\frac{1}{3}\right)^2$ **c** $\left(\frac{2}{5}\right)^3$ **d** $\left(\frac{3}{5}\right)^4$
e $\left(\frac{3}{4}\right)^3$ **f** $\left(\frac{7}{10}\right)^3$ **g** $\left(\frac{4}{7}\right)^4$ **h** $\left(\frac{4}{5}\right)^3$
i $\left(\frac{1}{4}\right)^3$ **j** $\left(\frac{3}{2}\right)^4$ **k** $\left(\frac{2}{3}\right)^5$ **l** $\left(\frac{5}{8}\right)^2$

LINKS TO Example 13b

7 Express each of the following without brackets, evaluating any coefficients.

a $\left(\frac{a}{b}\right)^3$ **b** $\left(\frac{x}{y}\right)^4$ **c** $\left(\frac{ab}{c}\right)^4$ **d** $\left(\frac{cd}{e^2}\right)^3$
e $\left(\frac{x}{2}\right)^2$ **f** $\left(\frac{a}{3}\right)^3$ **g** $\left(\frac{2y}{5}\right)^2$ **h** $\left(\frac{3a}{4}\right)^2$

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LINKS TO Example 14

Express each of the following without brackets, evaluating any coefficients.

a $\left(\frac{pq^2}{r}\right)^4$ **b** $\left(\frac{2x^2}{y^3}\right)^3$ **c** $\left(\frac{2m}{3n}\right)^2$ **d** $\left(\frac{2a}{3b}\right)^2$

e $\left(\frac{3x^2}{2y}\right)^4$ **f** $\left(\frac{c^2d^3}{2}\right)^3$ **g** $\left(\frac{5e^3h}{2}\right)^4$ **h** $\left(\frac{3xy^2}{4}\right)^3$

Evaluate the following expressions when $a = 3$ and $b = 2$.

a $3a^2b$ **b** $4a^2b^3$ **c** $2ab + b^2$ **d** $(ab)^3$

e $(2ab^2)^2$ **f** $(5a^2b)^4$ **g** $\frac{2a^3b^2}{5ab^3}$ **h** $5ab^0$

exercise 1.2 challenge

In the challenge question in exercise **1.1**, numbers were raised to the power $\frac{1}{2}$.

a Evaluate each of the following:

i $\left(4^{\frac{1}{2}}\right)^2$ **ii** $\left(9^{\frac{1}{2}}\right)^2$ **iii** $\left(16^{\frac{1}{2}}\right)^2$ **iv** $\left(25^{\frac{1}{2}}\right)^2$

b Use the raise a power to a power law to simplify $\left(a^{\frac{1}{2}}\right)^2$.

c What do you think $a^{\frac{1}{3}}$ might mean?

d Test your conjecture by using your calculator to evaluate $8^{\frac{1}{3}}$. Hint: use the 'raised to a power' button on your calculator and put brackets around $\frac{1}{3}$. Now try $27^{\frac{1}{3}}$. Do the results confirm your conjecture in part **c**?

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1.3 Negative indices

What does a negative index mean?

Suppose we are simplifying $\dfrac{a^2}{a^5}$. There are two ways we can do this.

Method 1
Write each term in expanded form. Cancel identical terms in the numerator and denominator.

$$\begin{aligned}\frac{a^2}{a^5} &= \frac{a \times a}{a \times a \times a \times a \times a}\\ &= \frac{1}{a \times a \times a}\\ &= \frac{1}{a^3}\end{aligned}$$

Method 2
The terms have the same base. The second index law tells us to subtract the indices.

$$\begin{aligned}\frac{a^2}{a^5} &= a^{2-5}\\ &= a^{-3}\end{aligned}$$

We can conclude that $\dfrac{1}{a^3}$ and a^{-3} must be equivalent expressions.

In general, we can write

$$\begin{aligned}a^{-n} &= a^{0-n}\\ &= \frac{a^0}{a^n}\\ &= \frac{1}{a^n}\end{aligned}$$

Negative index

A number or pronumeral raised to a negative power is equal to the reciprocal of that number or pronumeral raised to the positive power.

$$a^{-n} = \frac{1}{a^n} \text{ and } \frac{1}{a^{-n}} = a^n$$

$$4^{-3} = \frac{1}{4^3} \text{ and } \frac{1}{3^{-5}} = 3^5$$

The reciprocal of a number is 1 divided by that number. For example the reciprocal of 4 is $\dfrac{1}{4}$.

Example 15

Express the following with a positive index.

a 3^{-7} **b** $\dfrac{1}{4^{-3}}$ **c** a^{-3} **d** $\dfrac{1}{x^{-2}}$

continued

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Example 15 continued

Working	Reasoning
a $3^{-7} = \frac{1}{3^7}$	A number raised to a negative power is equal to the reciprocal of the number raised to the positive power.
b $\frac{1}{4^{-3}} = 4^3$	The reciprocal of a number raised to a negative power is the number raised to a positive power.
c $a^{-3} = \frac{1}{a^3}$	A pronumeral raised to a negative power is the reciprocal of the pronumeral raised to a positive power.
d $\frac{1}{x^{-2}} = x^2$	The reciprocal of a pronumeral raised to a negative power is the pronumeral raised to a positive power.

When evaluating a number raised to a negative power, it is easier to first write it as its reciprocal with a positive index.

Example 16

Evaluate each of the following.

a 3^{-4} **b** $\frac{1}{3^{-2}}$ **c** $\left(\frac{2}{3}\right)^{-2}$

Working	Reasoning
a $3^{-4} = \frac{1}{3^4}$ $= \frac{1}{81}$	3 raised to the power -4 is the reciprocal of 3 raised to the power 4. 3^4 is then evaluated as 81.
b $\frac{1}{3^{-2}} = 3^2$ $= 9$	$1 \div 3^{-2} = 1 \div \frac{1}{3^2} = 1 \times \frac{3^2}{1} = 3^2$
c $\left(\frac{2}{3}\right)^{-2}$ $= 2^{-2} \div 3^{-2}$	The numerator and the denominator of the fraction are both raised to the power -2.
$= \frac{1}{2^2} \div \frac{1}{3^2}$ $= \frac{1}{4} \div \frac{1}{9}$ $= \frac{1}{4} \times \frac{9}{1}$ $= \frac{9}{4}$	2 raised to the power -2 is the reciprocal of 2 raised to the power 2. 3 raised to the power -2 is the reciprocal of 3 raised to the power 2.

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Expressions that include negative indices can be simplified using the same index laws that apply to positive indices.

Example 17

Simplify each of the following, expressing each answer with a positive index.

a $3^4 \times (3^2)^{-3}$ **b** $\dfrac{2^8}{2^{-5}}$ **c** $a^2 \times (a^{-2})^3$ **d** $y^7 \div y^{-3} \div y$

Working	Reasoning
a $3^4 \times (3^2)^{-3}$	Use the power of a power index law.
$= 3^4 \times 3^{-6}$	$(3^2)^{-3} = 3^{2\times(-3)} = 3^{-6}$
$= 3^{-2}$	Use the multiplication index law. $3^4 \times 3^{-6} = a^{4+(-6)} = 3^{4-6} = 3^{-2}$
$= \dfrac{1}{3^2}$	Express with a positive index.
b $\dfrac{2^8}{2^{-5}}$	The bases are the same. When dividing, subtract the indices.
$= 2^{8-(-5)}$	
$= 2^{8+5}$	
$= 2^{13}$	
c $a^2 \times (a^{-2})^{-3}$	Use the power of a power index law.
$= a^2 \times a^{-6}$	$(a^{-2})^3 = a^{-2\times3} = a^{-6}$
$= a^{-4}$	Use one multiplication index law. $a^2 \times a^{-6} = a^{2+(-6)} = a^{-4}$
$= \dfrac{1}{a^4}$	Express with a positive index.
d $y^7 \div y^{-3} \div y = y^{7-(-3)-1}$	When dividing, subtract the indices.
$= y^{7+3-1}$	
$= y^9$	

> Subtracting a negative number is equivalent to adding a positive number.
>
> $8 - (-5) = 8 + 5$

If there are products of terms in index form, the numerator and the denominator should each be simplified first before applying the division index. If there is a power raised to a power, this should be simplified by multiplying the indices.

Example 18

Simplify, expressing answers with positive indices.

a $\dfrac{2^5 \times 2^{-3}}{2^8 \times 2^{-2}}$ **b** $\dfrac{x^4 \times (x^{-2})^3}{x^6 \times x^{-3}}$

continued

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Example 18 continued

Working	Reasoning
a $\dfrac{2^5 \times 2^{-3}}{2^8 \times 2^{-2}} = \dfrac{2^{5+(-3)}}{2^{8+(-2)}}$	Simplify the numerator by adding the indices. $5 + (-3) = 5 - 3 = 2$
$= \dfrac{2^{5-3}}{2^{8-2}}$	Simplify the denominator by adding the indices. $8 + (-2) = 8 - 2 = 6$
$= \dfrac{2^2}{2^6}$	
$= 2^{2-6}$	Simplify the fraction by subtracting the indices.
$= 2^{-4}$	$2 - 6 = -4$
$= \dfrac{1}{2^4}$	Express with a positive index.
b $\dfrac{x^4 \times (x^{-2})^3}{x^6 \times x^{-3}} = \dfrac{x^4 \times x^{-6}}{x^6 \times x^{-3}}$	Simplify the power of a power. $(x^{-2})^3 = x^{-2\times 3} = x^{-6}$
$= \dfrac{x^{4+(-6)}}{x^{6+(-3)}}$	Simplify the numerator by adding the indices. $4 + (-6) = -2$
$= \dfrac{x^{4-6}}{x^{6-3}}$	Simplify the denominator by adding the indices. $6 + (-3) = 3$
$= \dfrac{x^{-2}}{x^3}$	
$= x^{-2-3}$	Simplify the fraction by subtracting the indices.
$= x^{-5}$	$-2 - 3 = -5$
$= \dfrac{1}{x^5}$	Express with a positive index.

When simplifying expressions that include negative indices, there is often more than one way of going about the simplifying.

Example **19** shows two different methods for simplifying $\dfrac{3^5}{3^2} \div \dfrac{3^4}{3^7}$.

Example 19

Simplify $\dfrac{3^5}{3^2} \div \dfrac{3^4}{3^7}$.

continued

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Example 19 continued

Working

Method 1:

$$\frac{3^5}{3^2} \div \frac{3^4}{3^7} = \frac{3^5}{3^2} \times \frac{3^7}{3^4}$$
$$= \frac{3^5 \times 3^7}{3^2 \times 3^4}$$
$$= \frac{3^{5+7}}{3^{2+4}}$$
$$= \frac{3^{12}}{3^6}$$
$$= 3^6$$

Reasoning

The first step in this method is to change the division by $\frac{3^4}{3^7}$ to multiplying by its reciprocal $\frac{3^7}{3^4}$.
The numerator and the denominator are then both simplified before subtracting the indices for the division.

Method 2:

$$\frac{3^5}{3^2} \div \frac{3^4}{3^7} = 3^{5-2} \div 3^{4-7}$$
$$= 3^3 \div 3^{-3}$$
$$= 3^{3-(-3)}$$
$$= 3^{3+3}$$
$$= 3^6$$

Each fraction is simplified first by subtracting indices. The division of 3^3 by 3^{-3} is then done by subtracting the indices.

$3 - (-3) = 3 + 3 = 6$

When simplifying expressions that include coefficients in the numerator and denominator, the coefficients should be simplified by dividing by any common factors.

Example 20

Simplify and express with positive indices.

a $3a^{-2} \times 2a^{-4}$

b $\dfrac{6x^{-3}}{15x^{-5}}$

Working

a $3a^{-2} \times 2a^{-3}$
$$= 3 \times 2 \times a^{-2} \times a^{-3}$$
$$= 6 \times a^{-2+(-3)}$$
$$= 6 \times a^{-2-3}$$
$$= 6 \times a^{-5}$$
$$= 6 \times \frac{1}{a^5}$$
$$= \frac{6}{a^5}$$

Reasoning

When multiplying pronumerals with the same base, add the indices.

$-2 + (-3) = -2 - 3 = -5$

a raised to the power -5 is the reciprocal of a raised to the power 5.

continued

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Example 20 continued

Working

b $\dfrac{6x^{-3}}{15x^{-5}} = \dfrac{2x^{-3}}{5x^{-5}}$

$= \dfrac{2x^{-3-(-5)}}{5}$

$= \dfrac{2x^{-3+5}}{5}$

$= \dfrac{2x^{2}}{5}$

Reasoning

Simplify the division of coefficients.

$\dfrac{6}{15} = \dfrac{2}{5}$

Subtract the indices to simplify $\dfrac{x^{-3}}{x^{-5}}$

$-3 - (-5) = -3 + 5 = 2$

exercise 1.3

LINKS TO Example 15

1 Express each of the following with positive indices.

a 6^{-3} **b** 2^{-8} **c** 4^{-7} **d** 3^{-1}

e $\dfrac{1}{3^{-2}}$ **f** $\dfrac{1}{2^{-5}}$ **g** $\dfrac{1}{5^{-2}}$ **h** $\dfrac{1}{6^{-3}}$

LINKS TO Example 15

2 Write the following with positive indices.

a a^{-2} **b** b^{-6} **c** x^{-3} **d** m^{-5}

e $\dfrac{1}{y^{-4}}$ **f** $\dfrac{1}{n^{-2}}$ **g** $\dfrac{1}{x^{-6}}$ **h** $\dfrac{1}{k^{-3}}$

LINKS TO Example 16

3 Evaluate each of the following.

a 2^{-4} **b** 5^{-2} **c** 10^{-3} **d** 10^{-6}

e $\dfrac{1}{3^{-1}}$ **f** $\dfrac{1}{2^{-3}}$ **g** $\dfrac{1}{3^{-4}}$ **h** $\dfrac{1}{10^{-6}}$

i $\left(\dfrac{1}{2}\right)^{-3}$ **j** $\left(\dfrac{3}{4}\right)^{-2}$ **k** $\left(\dfrac{2}{5}\right)^{-2}$ **l** $\left(\dfrac{1}{2}\right)^{-5}$

LINKS TO Example 17a

4 Simplify each of the following by adding the indices, leaving the answers in index form.

a $2^4 \times 2^{-5}$ **b** $3^2 \times 3^{-7}$ **c** $5^3 \times 5^{-5}$ **d** $10^2 \times 10^{-5}$

e $10^{-6} \times 10^{-3}$ **f** $10^5 \times 10^{-12}$ **g** $7^{-3} \times 7^4 \times 7^{-6}$ **h** $2^8 \times 2^{-5} \times 2^3$

i $2^{-8} \times 2^{-3} \times 2^{11}$ **j** $3 \times 3^4 \times 3^{-2}$ **k** $(2^{-2})^2 \times 2^7$ **l** $5^{-2} \times 3^7 \times 3^{-2} \times 5^{-5}$

LINKS TO Example 17b

5 Simplify each of the following by subtracting the indices.

a $2^7 \div 2^{-5}$ **b** $3^4 \div 3^{-9}$ **c** $\dfrac{5^3}{5^{-5}}$ **d** $\dfrac{10^7}{10^{-5}}$

e $10^{-6} \div 10^3$ **f** $10^7 \div 10^{-12}$ **g** $\dfrac{2^{-7}}{2^4}$ **h** $10^{-15} \div 10^6$

i $3^3 \div 3^7$ **j** $2^{-6} \div 2^{-9}$ **k** $4^{-7} \div 4^{-10}$ **l** $5^4 \div 5^6 \div 5^{-8}$

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LINKS TO Examples 17c, d

Simplify the following. Express answers with positive indices.

a $p^4 \times p^{-3}$ **b** $a^{-4} \times a^{-3} \times a^{-8}$ **c** $m^4 \times m^{-7}$

d $b^{-3} \times b^5$ **e** $a^{-3} \times a^{-3}$ **f** $x^{-2} \times x^{-3} \times x^6$

g $y^{-4} \times y^3 \times y^{-5}$ **h** $(x^{-2})^3$ **i** $(a^{-4})^3 \times a^7$

LINKS TO Examples 17c, d

Simplify the following. Express answers with positive indices.

a $x^{-4} \div x^{-7}$ **b** $y^5 \div y^{-8}$ **c** $\dfrac{m^4}{m^7}$

d $m^5 \div m^7 \div m^{-3}$ **e** $b^4 \div b^{-5} \div b^8$ **f** $m^5 \div m^9$

g $c^2 \div c^{12}$ **h** $p \div p^7$ **i** $a^5 \div (a^4)^2$

LINKS TO Example 18a

Simplify, giving each answer in index form. Hint: where appropriate, simplify the numerator first.

a $\dfrac{2^8 \times 2^{-4}}{2^3}$ **b** $\dfrac{5^{-7} \times 5^3}{5^{-6}}$ **c** $\dfrac{10^{-17} \times 10^9}{10^{-6}}$ **d** $\dfrac{10^{14} \times 10^5}{10^{-11}}$

e $\dfrac{3^{-2} \times 3^8}{3^4}$ **f** $\dfrac{2^{-5} \times 2^{-3}}{2^{-8}}$ **g** $\dfrac{6^4}{6^3 \times 6^{-8}}$ **h** $\dfrac{3^{-2}}{3^8 \times 3^{-4}}$

i $\dfrac{(2^{-2})^4}{2^2 \times (2^3)^{-1}}$ **j** $\dfrac{2^{-3}}{2^6 \times 2^4}$ **k** $\dfrac{3^{-2} \times 3^6}{3^{-8} \times 3^4}$ **l** $\dfrac{5^{-3} \times 5^4}{5^{-2} \times 5^{-6}}$

LINKS TO Example 18b

Simplify each of the following, expressing your answers with positive indices.

a $\dfrac{p^5 \times p^8}{p^{22}}$ **b** $\dfrac{m^{15}}{m^{18} \times m^{-5}}$ **c** $\dfrac{k^2 \times k^{-8}}{k^4}$ **d** $\dfrac{z^{-2} \times z^5}{z^3}$

e $\dfrac{f^{13} \times f^{-8}}{f^6 \times f^{-2}}$ **f** $b^4 \div (b^8 \times b^3)$ **g** $(d^4 \times d^3) \div d^{12}$ **h** $(n^3 \times n^2) \div (n^7 \times n)$

LINKS TO Example 19

Simplify the following, expressing your answer with positive indices, using

i method 1 from worked example **19**

ii method 2 from worked example **19**.

a $\dfrac{p^3}{p^6} \div \dfrac{p^2}{p^8}$ **b** $\dfrac{x}{x^9} \div \dfrac{x^3}{x^{15}}$

LINKS TO Example 20

Simplify, and express with positive indices.

a $4x^{-3} \times 3x^2$ **b** $7m^{-6} \times 2m^3$ **c** $3a^4 \times 5a^{-2} \times 2a^{-6}$ **d** $-3x^4 \times 4x^{-3}$

e $\dfrac{18a^{-2}}{12a^{-4}}$ **f** $\dfrac{24x^7}{20x^{-2}}$ **g** $\dfrac{42b^{-4}}{28b^{-7}}$ **h** $\dfrac{3x^{-2} \times 4x^2}{8x^{-6}}$

exercise 1.3 challenge

Simplify, and express with positive indices.

$$\frac{(-3x^{-2})^3 \times 5(x^{-2})^4}{54x^{-8} \times 15(x^3)^2}$$

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1.4 Scientific notation

Writing very large numbers in scientific notation

Sometimes we have to deal with very large numbers, for example, the distance from Earth to particular stars, the number of cells in the human body, or the number of atoms in a sample of metal. Each of these numbers will include a large number of zeros that are simply placeholders. For example, the speed of light is 300 000 kilometres per second, or $3 \times 100\,000$ km/s. To save writing all the zeros, a shorter form is 3×10^5 km/s. This form of writing numbers is called **scientific notation** or, sometimes, standard form.

Scientific notation

A number in scientific notation is made up of the following two parts multiplied together.

- A number greater than or equal to 1 but less than 10, that is, $1 \leq n < 10$.
- A power of 10.

For example, to write 732 000 in scientific notation, the digits 732 are written as 7.32, which must then be multiplied by 10 five times to get to 732 000.

$$732\,000 = 7.32 \times 10^5$$

7.32 is between 1 and 10 power of 10

Example 21

Write the following numbers in scientific notation.

a 8 070 000 000 000 **b** 114 900 000 000 000 000

c 873.4 **d** 7 million

Working	Reasoning
a 8 070 000 000 000 $= 8.07 \times 10^{12}$	Write the digits 807 as a number between 1 and 10. 8.07 must be multiplied by 10 twelve times to equal to 8 070 000 000 000.
b 114 900 000 000 000 000 $= 1.149 \times 10^{17}$	Write the digits 1149 as a number between 1 and 10. 1.149 must be multiplied by 10 seventeen times to equal 114 900 000 000 000 000.

continued

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Example 21 continued

Working

c $873.4 = 8.734 \times 10^2$

d 7 million $= 7\,000\,000$

$= 7 \times 10^6$

Reasoning

Write the digits 8734 as a decimal number between 1 and 10.
8.734 must be multiplied by 100.

1 million $= 10^6$

Example 22

Write 5.674×10^{13} as an ordinary number.

Working

5.674×10^{13}
$= 56\,740\,000\,000\,000$

Reasoning

5.674 must be multiplied by 10 thirteen times.
Move the decimal point 13 places to the right.

Example 23

Use a calculator to find $14\,500\,000 \times 35\,000$. Give your answer

a in scientific notation
b as an ordinary number.

Working

a $14\,500\,000 \times 35\,000$
$= 5.075 \times 10^{11}$

Reasoning

$14\,500\,000 \times 35\,000$
$= 1.45 \times 10^7 \times 3.5 \times 10^4$
$= 1.45 \times 3.5 \times 10^7 \times 10^4$
$= 5.075 \times 10^{11}$

Your calculator may give the answer as 5.075E11. Another name for an index is an *exponent*. E11 is an abbreviation for *exponent 11*, that is, by *10 raised to the power 11*.

b $5.075 \times 10^{11} = 507\,500\,000\,000$

Move the decimal point 11 places to the right.

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Writing very small numbers in scientific notation

As in the case of large numbers, writing a very small number in scientific notation means expressing the number as the product of

- a number greater than or equal to 1, but less than 10.
- a power of 10.

To write very small numbers in scientific notation we use negative indices.

If we follow through the sequence below we see that $10^{-1} = \dfrac{1}{10^1}$, $10^{-2} = \dfrac{1}{10^2}$ and so on.

$$10\,000 \xrightarrow{\div 10} 1000 \xrightarrow{\div 10} 100 \xrightarrow{\div 10} 10 \xrightarrow{\div 10} 1 \xrightarrow{\div 10} \frac{1}{10} \xrightarrow{\div 10} \frac{1}{100} \xrightarrow{\div 10} \frac{1}{1000}$$

10 000	1000	100	10	1	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$
10^4	10^3	10^2	10^1	10^0	$\frac{1}{10^1}$	$\frac{1}{10^2}$	$\frac{1}{10^3}$
10^4	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}

For example, the very small number 0.000064 can be rewritten as

$$\begin{aligned} 0.000064 &= 6.4 \div 10\,000 \\ &= 6.4 \div 10^5 \\ &= 6.4 \times 10^{-5} \end{aligned}$$

Example 24

Express the following numbers in scientific notation.

a 0.0000001 **b** 0.00000783

Working

a 0.0000001
$= 1.0 \div 10^7$

$= 1 \times 10^{-7}$

Reasoning

In scientific notation, the first part must be a number between 1 and 10.

$0.0000001 = \dfrac{1}{10\,000\,000}$

In scientific notation, the first part must be multiplied by a power of 10. Dividing by 10^7 is the same as multiplying by 10^{-7}.

continued

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Example 24 continued

Working

b 0.00000783
$= 7.83 \div 10^6$

This is equivalent to shifting the decimal point 6 places to the left.

$= 7.83 \times 10^{-6}$

Reasoning

In scientific notation, the first part must be a number between 1 and 10.

$$0.00000783 = \frac{7.83}{1\,000\,000}$$

In scientific notation, the first part must be multiplied by a power of 10. Dividing by 10^6 is the same as multiplying by 10^{-6}.

Example 25

Express as ordinary numbers.

a 2.83×10^{-3} **b** 1.6×10^{-7}

Working

a $2.83 \times 10^{-3} = 0.00283$

b $1.6 \times 10^{-7} = 0.00000016$

Reasoning

$$10^{-3} = \frac{1}{10^3} = \frac{1}{1000}$$

The decimal point in 2.83 must be moved three places to the left.

$$10^{-7} = \frac{1}{10^7} = \frac{1}{10\,000\,000}$$

The decimal point in 1.6 must be moved seven places to the left.

Example 26

Calculate the following. Give your answers

i in scientific notation.

ii as an ordinary number.

a $24.5 \div 800\,000$ **b** 0.006×0.00048

Working

a i $24.5 \div 800\,000$
$= 3.0625 \times 10^{-5}$

ii $3.0625 \times 10^{-5} = 0.000030625$

Reasoning

Written in scientific notation, the answer is 3.0625×10^{-5}

The power of 10 is –5. This indicates that we move the decimal point 5 places to the left.

continued

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1.4

Example 26 continued

Working

b $0.006 \times 0.00048 = 2.88 \times 10^{-6}$
$= 0.00000288$

Reasoning

Your calculator will probably display 2.88E-6. This means 2.88×10^{-6}

Example 27

Calculate each of the following, giving the answers in scientific notation.

a the product of 3.4×10^7 and 7.38×10^5

b $5.32 \times 10^9 \div (8 \times 10^{-4})$

Working

a $3.4 \times 10^7 \times 7.38 \times 10^5$
$= 2.5092 \times 10^{13}$

Reasoning

$3.4 \times 10^7 \times 7.38 \times 10^5$
$= 3.4 \times 7.38 \times 10^7 \times 10^5$
$= 25.092 \times 10^{12}$
$= 2.5092 \times 10^1 \times 10^{12}$
$= 2.5092 \times 10^{13}$

Working

b $5.32 \times 10^9 \div (8 \times 10^{-4})$
$= 6.65 \times 10^{12}$

Reasoning

$\dfrac{5.32 \times 10^9}{(8 \times 10^{-4})}$
$= \dfrac{5.32}{8} \times \dfrac{10^9}{10^{-4}}$
$= 0.665 \times 10^9 \div 10^{-4}$
$= 0.665 \times 10^9 \times 10^4$
$= 0.665 \times 10^{13}$

Written in scientific notation, the answer is 6.65×10^{12}.

exercise 1.4

LINKS TO Examples 21a–d

1 Write each of these numbers in scientific notation.

a 1200
b 425 000
c 36 000 000 000 000
d 1 750 000 000 000 000 000 000
e 86 000 000
f 38 640 000
g 186 478
h 8721.6
i 478.4
j 18.45
k 3926.7
l 26.732
m 468.2
n 3 800 000

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o 17 589.4 **p** 3947.68
q 6 743 000 000 **r** 9 500 000 000 000

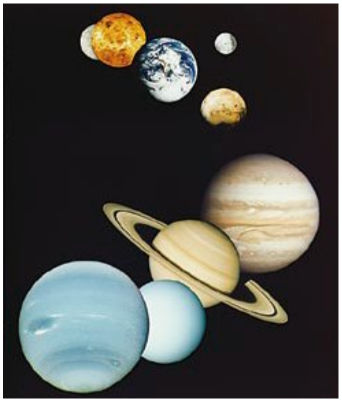
LINKS TO Example 21d

Write the following numbers in scientific notation.

a 5 thousand **b** 6 million **c** 23 million
d 596 thousand **e** 194 thousand **f** 257 million
g 58 billion **h** 17 hundred thousand **i** 18.6 million
j 2.7 million **k** 12.3 billion **l** 5.8 billion

Write each of these distances in scientific notation.

Planet	Average distance from the Sun (km)
Mercury	58 000 000
Venus	108 000 000
Earth	150 000 000
Mars	228 000 000
Jupiter	778 000 000
Saturn	1 429 000 000
Uranus	2 871 000 000
Neptune	4 504 000 000



LINKS TO Example 22

Write each of the following as an ordinary number.

a 6×10^4 **b** 2×10^3 **c** 6×10^5 **d** 9×10^7
e 7.143×10^6 **f** 8.09×10^{10} **g** 4.016×10^7 **h** 1.08×10^4

LINKS TO Example 23

Use a calculator to find each of the following products. Give your answers in scientific notation.

a $178\,000 \times 4\,500\,000$ **b** $3\,600\,000 \times 180\,000$
c $2\,900\,000 \times 84\,500\,000$ **d** $184\,000\,000\,000 \times 17\,000$
e $15\,000 \times 286\,000\,000$ **f** $5860 \times 4\,800\,000\,000$
g 917.5×3000 **h** $13\,700 \times 8000$
i $250\,000\,000 \times 29.48$ **j** $19.8 \times 32\,000$
k $360\,000 \times 2560$ **l** $9750 \times 150\,000$

LINKS TO Example 24

Express these numbers in scientific notation.

a 0.0065 **b** 0.000072 **c** 0.00001
d 0.000000009 **e** 0.00503 **f** 0.00000000073
g 0.00000912 **h** 0.000000000000000834 **i** 0.0000000000073
j 0.0000702 **k** 0.000000084 **l** 0.0000000000955
m 0.0000000102 **n** 0.0000000000000075 **o** 0.0000000032

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LINKS TO Example 25

Write each of the following as an ordinary number.

a 3.76×10^{-4} **b** 4.9×10^{-3} **c** 6.8×10^{-6} **d** 2.185×10^{-4}

e 1.32×10^{-5} **f** 5.1×10^{-8} **g** 4.0×10^{-7} **h** 9.925×10^{-2}

LINKS TO Example 26

Use a calculator to find each of the following. Give your answers in scientific notation.

a $18.6 \div 250\,000\,000$ **b** $427 \div 200\,000\,000$

c 0.0004×0.0000003 **d** 0.0024×0.00056

e $5.4 \div 180\,000\,000$ **f** 0.000346×0.0012

g $9.6 \div 30\,000\,000\,000$ **h** 1.84×0.0000008

i $0.0056 \div 2000$ **j** 0.0074×0.0000005

k $925 \div 250\,000$ **l** 4.62×0.0000000075

LINKS TO Example 27

Calculate each of the following, giving the answers in scientific notation.

a The product of 3.894×10^3 and 2.1×10^8

b The product of 1.95×10^2 and 6.8×10^7

c 8×10^6 divided by 5×10^3

d 4.5×10^7 divided by 2×10^3

The mass of a molecule of hydrogen gas is about 3.34×10^{-24} g. The mass of an oxygen molecule is 16 times the mass of a hydrogen molecule. Give the mass of an oxygen molecule in scientific notation.

There can be as many as 40 million bacteria in a gram of soil. How many could there be in a bucket containing 8 kilograms of soil? Give the number in scientific notation.

A light year is the distance that light travels in a year. The speed of light is 3.0×10^5 km/s. How far (in kilometres) is a light year? Assume there are 365 days in a year and give your answer in scientific notation.

It has been claimed that it would take 1 270 846 648 000 Smarties to fill the Melbourne Cricket Ground. Write this number in scientific notation.

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exercise 1.4 challenge

1. A nanometre is 10^{-9} metres. Viruses range in size from about 20 nanometres to 400 nanometres. Express these sizes in centimetres, using scientific notation.

2. Use scientific notation to express the answer for each of the following.

a There are approximately 5 thousand million red blood corpuscles per millilitre of blood. What is the number of red blood corpuscles per millilitre?

b How many red blood corpuscles are there in one litre of blood?

c The average volume of blood in an adult person is approximately 6 L. Calculate the approximate number of red blood corpuscles in an adult.

d The chemical molecule in our red blood cells that is responsible for carrying oxygen around our bodies is called haemoglobin. Each red blood corpuscle contains about 270 million haemoglobin molecules. Calculate the approximate number of haemoglobin molecules in the blood of an adult.

e The diameter of a red blood corpuscle is approximately 7.9×10^{-3} mm. Write this as an ordinary number.

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1.5 Significant figures

When we perform operations on numbers based on physical quantities such as length or area, it is important to calculate to a sensible number of decimal places. One method of deciding on a sensible number of decimal places is to look at the number of decimal places in the given quantities.

Example 28

The area of a rectangle is 24.6 m^2 and the length is 5.2 m. To calculate the width of the rectangle, we divide the area by the length. Using a calculator to divide 24.6 by 5.2, we obtain 4.730769231.
How many decimal places do you think would be sensible?

Working

One decimal place: 4.7 m

Reasoning

24.6 m^2 and 5.2 m both have one decimal place. So the answer should have no more than one decimal place. The width of the rectangle should therefore be stated as 4.7 m.

Another method is to consider the number of **significant figures** in the given quantities.

Significant figures

Significant figures in a number are:

- all the digits except zeros at the end of a number for numbers greater than 1.
- all the digits except zeros after the decimal point for numbers between 0 and 1.

For example,

- The number 458 000 has three significant figures: 4, 5 and 8. The zeros at the end are simply placeholders.
- The number 0.00002945 has four significant figures: 2, 9, 4 and 5 because the zeros at the beginning are placeholders.
- The number 1.0038 has five significant figures: 1, 0, 0, 3 and 8 because the two zeros are significant. They indicate that the value of the tenths place and hundredths place are each zero rather than some other digit.

Significant figures give us a way of determining the certainty of measurements.

For example, the measurements 16 m and 16.00 m do not mean the same thing.

- The measurement 16 m means that a length was measured to the nearest metre. A more accurate measurement for the actual length might be 15.7 m or 16.45 m.
- A measurement of 16.00 m, however, tells us that the length was measured to the nearest centimetre. This measurement has four significant figures whereas 16 m has only two.

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As a general rule, when using measured quantities in calculations, the number of significant figures in an answer should be no greater than the smallest number of significant figures in any of the quantities used in the calculation.

To determine the number of significant figures in a number, it is easiest to first write the number using scientific notation then count the number of significant digits.

Example 29

Complete the following.

a State the number of significant figures in each of the following numbers.

i 300.69 **ii** 0.000450

b Round the following numbers to three significant figures.

i 80.7083 **ii** 0.08649

Working	Reasoning
a i $300.69 = 3.0069 \times 10^2$ There are five significant figures.	Write the number using scientific notation. Count the number of digits in 3.0069.
ii $0.000450 = 4.50 \times 10^{-4}$ There are three significant figures.	Write the number using scientific notation. Count the number of digits in 4.50.
b i $80.7083 = 8.07083 \times 10^1$ Correct to three significant figures. $80.7083 \approx 8.07 \times 10^1$ or 80.7	Write the number using scientific notation. We require three digits when we round off.
ii $0.08649 = 8.649 \times 10^{-2}$ Now round 8.649 to 8.65. Correct to three significant figures. $0.08649 \approx 8.65 \times 10^{-2}$ or 0.0865	Write the number using scientific notation. We require three digits when we round off.

Example 30

Express these calculations to an appropriate number of significant figures.

a The calculation of circumference of a circle of radius 2.45 m is displayed on a calculator as 15.393804.

b The calculation of the volume of a rectangular prism with length 9.4 cm, width 8.6 cm and height 3.2 cm is displayed on a calculator as 258.688.

Working	Reasoning
a 15.4 m	2.45 has three significant figures.
b 260 cm^3	9.4, 8.6 and 3.2 each have two significant figures.

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exercise 1.5

1. The radius of a circle is measured as 5.6 cm. The area calculation displayed on a calculator is 98.52034562. What would be a sensible number of decimal places to give for the area?

LINKS TO Example 29a

2. How many significant figures are there in each of the following?

a 8.32	**b** 0.00045	**c** 1056.7	**d** 210 000
e 14.08	**f** 0.0601	**g** 7.00	**h** 19
i 1.300	**j** 0.0720	**k** 1.00004	**l** 2.030

LINKS TO Example 29b

3. Express each of the following numbers
   i to two significant figures. **ii** to two decimal places.

a 47.338	**b** 170.875	**c** 5.071	**d** 0.00008759
e 107.649	**f** 3.009	**g** 800.399	**h** 200.188
i 1.00814	**j** 130.418	**k** 5.000	**l** 0.00753

4. Round the following numbers to the number of significant figures shown in brackets.

a 1874 (3)	**b** 2.3792 (2)	**c** 164 (2)	**d** 1 480 000 (1)
e 0.000834 (1)	**f** 20 056 (3)	**g** 4.0083 (4)	**h** 20.08356 (4)
i 11.00 (1)	**j** 6.000 (2)	**k** 0.00059200 (5)	**l** 4.005 (3)

5. Round each of these numbers to three significant figures.

a 0.0006038	**b** 18 005	**c** 195 738	**d** 60.057
e 8.481	**f** 20.1067	**g** 479 800 000	**h** 0.00006072
i 72.084	**j** 10.005	**k** 380 400	**l** 1.000346

6. Use a calculator to evaluate each of the following. Give your answers correct to the number of significant figures shown in brackets.
   a 56.21×34.85 (4) **b** 6.85^2 (3)
   c 0.505^3 (3) **d** $0.265 \div 3.4$ (2)

LINKS TO Example 30

7. Express these calculations to an appropriate number of significant figures.
   a The calculation of the area of a rectangle with length 1.85 cm and width 3.75 cm is displayed as 6.9375
   b The calculation of the area of a circle with radius 2.8 m is displayed as 24.6300864
   c The calculation of the volume of a rectangular prism with length 6.4 cm, width 8.2 cm and height 3.6 cm is displayed as 188.928
   d The calculation of the radius of a circle with circumference 18.6 m is displayed as 2.960281942

exercise 1.5 challenge

1. Evaluate each of the following measurement calculations, giving each answer correct to the same number of significant figures as the measurement containing the least number of significant figures.
   a 6.1408×1.40 **b** $4.3718 \div 1.05$
   c $\dfrac{6.34 \times 0.0056}{2.001}$ **d** $\dfrac{6.314 \times 10^{-3} \times 0.05}{2.5}$

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Analysis task

Acidic or basic?

The pH scale is used to measure how acidic or alkaline a solution is. You may work with this in science this year or in later years. It is based on the index of the hydrogen ion concentration in the solution. If the solution is neutral (neither acidic or alkaline), the concentration of hydrogen ions in the solution is 10^{-7} (measured in a unit called moles/litre). The opposite of -7 is 7 and this is called the pH of the solution. Acidic solutions have pH less than 7, that is they have hydrogen ion concentrations greater than 10^{-7}, for example, 10^{-6} or 10^{-3}. Alkaline solutions have pH greater than 7, up to 14.

a A particular alkaline solution has a hydrogen ion concentration of 10^{-13}. What is its pH?

b A dilute solution of hydrochloric acid had a pH of 4. Express its hydrogen ion concentration in index form.

c Express this as an ordinary number.

d A solution had a pH of 12.

- **i** Was the solution acidic or alkaline?
- **ii** Express its hydrogen ion concentration in index form.
- **iii** Express this as an ordinary number in decimal form.

e The pH indicator paper shown here has been dipped into a solution.

- **i** What was the pH of the solution?
- **ii** Write this in index form.
- **iii** Express this as an ordinary number in decimal form.

f What is the pH of the grapefruit shown at the beginning of this chapter?

g Write this in index form.

h Write this as an ordinary number in decimal form.

Challenge

i What is the pH of this kiwi fruit?

j Write this in index form.

k Use your calculator to express this as an ordinary number in decimal form.

l Which is more acidic: the grapefruit or the kiwi fruit?

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Review Indices and scientific notation

Summary

Index form with pronumerals

$$a^n \leftarrow \text{Index or exponent}$$

$\uparrow$

Base

The index laws

The multiplication index law	$a^m \times a^n = a^{m+n}$
The division index law	$\frac{a^m}{a^n} = a^{m-n}$
The raising a power to a power law	$(a^m)^n = a^{m\times n}$
The power of a product law	$(ab)^m = a^m b^m$
The power of quotient law	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
The negative index law	$a^{-m} = \frac{1}{a^m}, \frac{1}{a^{-m}} = a^m$

The zero index

$a^0 = 1$

Scientific notation

A number in scientific notation has two parts that are multiplied together.

- a number greater than or equal to 1 but less than 10, that is $1 \leq n \leq 10$.
- a power of 10.
- a positive power is a large number.
- a negative power is a small number.

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Significant figures

Significant figures in a number are:

- all the digits except zeros at the end of a number for numbers greater than 1.
- all the digits except zeros after the decimal point for numbers between 0 and 1.

Revision

Multiple-choice questions

$2y^2 \times 3y^5 =$

A $5y^7$ **B** $6y^7$ **C** $5y^{10}$ **D** $6y^{10}$ **E** $10y^7$

$3a^0 + (5a)^0 =$

A 1 **B** 2 **C** 4 **D** 8 **E** 15

A correct simplification of $3x^2 \times 8x^4 \div 6x^{-3}$ is

A $5x^3$ **B** $5x^9$ **C** $4x^3$ **D** $4x^9$ **E** $4x^{-9}$

The value of $\frac{2^{-5}}{2^{-3}}$ is

A -4 **B** $\frac{1}{4}$ **C** $\frac{1}{2}$ **D** -8 **E** -2

Which of the following is an incorrect simplification?

A $m^3 + m^3 = 2m^3$ **B** $3m^2 \times 27m^6 = 3^4m^8$ **C** $36m^{10} \div 18m^5 = 2m^2$
D $(m^3)^2 \times (m^2)^3 = m^{12}$ **E** $(3m)^2 \times 2n^4 = 18m^2n^4$

Short-answer questions

Simplify, leaving in index form.

a $(3^4)^2$ **b** $\frac{3^3}{3^7}$ **c** $\frac{7^{12}}{7^4}$ **d** $\frac{5^{-2} \times 5^5}{5 \times 5^{-3}}$

Evaluate.

a 2^{-1} **b** $(2^2)^3$ **c** 10^{-4} **d** $\frac{1}{5^{-2}}$

Simplify, then evaluate.

a $\frac{2^6}{2^2} + 2^0$ **b** $\frac{2^4 \times 2^3}{2^2 \times 2}$ **c** $10^{-6} \times 10^2$ **d** $\frac{3^3 \times (3^2)^3}{3^5}$

Simplify each of the following.

a $m^2 \times m^3 \div m^4$ **b** $8p^2q^4 \div 4pq$ **c** $\frac{m^4n \times mn^3}{m^2n^2}$

d $\frac{4a^2b \times 3a^5b^7}{6ab^4}$ **e** $\frac{s^4}{s^3} \times \frac{s^7}{s^2}$ **f** $\frac{4a^2b^2 \times 5a^3b^4}{10ab^2 \times 2a^3b}$

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Simplify, leaving your answers in index form.

a $3^7 \times 3^4$ **b** $a^9 \times a^2 \times a$ **c** $\frac{x^9}{x^5}$

d $8^{10} \div 8^5$ **e** $(3^2)^3$ **f** $(m^4)^5$

g $3a^2 \times 5b^5$ **h** $4x^5 \times 3x^2 \times 5y^7$ **i** $\frac{12a^{11}}{3a^9}$

j $24b^5 \div 8b^2$ **k** $(2y)^3$ **l** $3d^4 \times 2e^3 \times d \times 4e^3$

Simplify each of the following.

a a^0 **b** $5b^0$ **c** $2pq^0$

d $(4m^2n)^0$ **e** $\frac{p^6 \times p^4}{p^{10}}$ **f** $x^0 + 3x^0$

Express each of the following in simplest form.

a $(k^5)^4$ **b** $(5xy^4)^2$ **c** $(4x^7y^2)^0$

d $\left(\frac{2a^3}{3b^2}\right)^3$ **e** $\left(\frac{m^3n^4}{10}\right)^2$ **f** $(2a^2b)^3 \times 5(a^4b^2)^2$

Simplify each of the following. Express your answers with positive indices.

a $x^7 \div x^{-6}$ **b** $m^5 \times m^{-8}$ **c** $(a^2b)^{-2}$

d $3a^{-4}b^2 \times 5ab^3$ **e** $\frac{p^7 \times p^4}{p^9 \times p^{-5}}$ **f** $3x^{-5}y^{-3} \times 4x^{-2}y^2$

Simplify each of the following.

a $(x^2y^3)^2 \times xy^2$ **b** $(x^3y)^2 \div xy^2$ **c** $(3a^2)^0 + 3a^0 + a^0$

Express in scientific notation.

a 108.9 **b** 0.00000000076

c 594 000 000 **d** 0.000042

Express as ordinary numbers.

a 1.56×10^7 **b** 4.5×10^{-4}

c 8.67×10^{10} **d** 6.0×10^{-8}

Calculate the following, giving the answers in scientific notation.

a $2.4 \div 3000$ **b** 2560×18500

c 0.000006×17.5 **d** $3.5 \times 10^4 \times 6.4 \times 10^{-7}$

Round to 3 significant figures.

a 28 751 496 **b** 183.85

c 4.0067 **d** 0.00729543

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Extended-response questions

A light year is the distance that light travels in a year. The speed of light is 3.0×10^8 m/s. Neptune moves in an orbit which is an average 4500 million km from the sun. Assuming there are 365 days in a year, answer the following giving your answer in scientific notation.

a How many metres is Neptune from the sun?

b How far is a light year in metres?

c How many light years is Neptune from the sun (to three significant figures)?

d How many seconds would it take for light to travel from the sun to Neptune?

Practice quiz
Chapter 1

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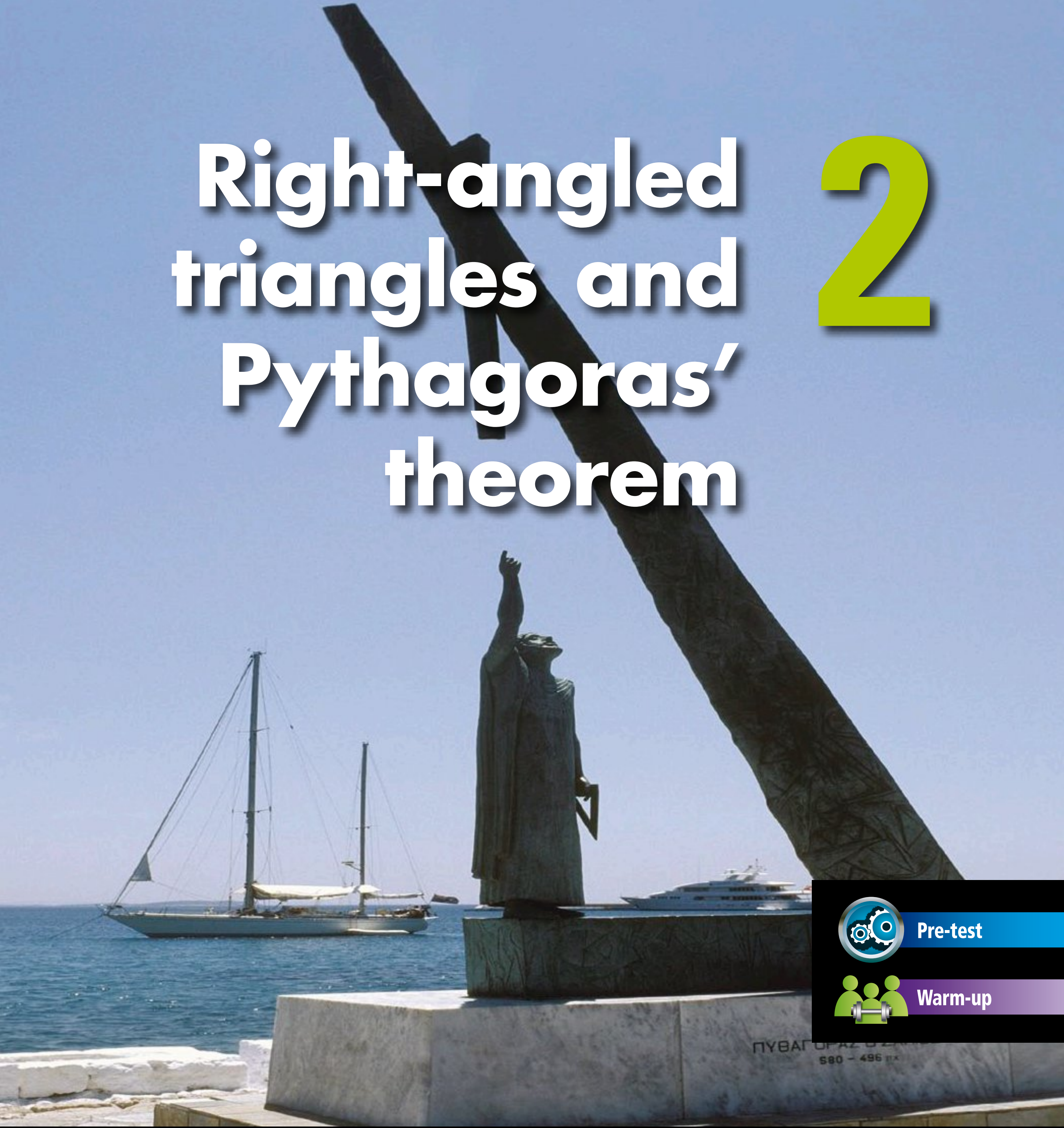
2 Right-angled triangles and Pythagoras' theorem



Pre-test

Warm-up

This statue of the mathematician and philosopher Pythagoras is on the Greek island of Samos, where Pythagoras was born about 2600 years ago. Pythagoras is well known today as the discoverer of the special relationship between the squares of the three sides of any right-angled triangle. The relationship is referred to as Pythagoras' theorem, although it was probably known before his time. Pythagoras may have been the first mathematician to prove the theorem. This chapter introduces Pythagoras' theorem and shows how it is used to calculate unknown side lengths in right-angled triangles.

2.1 Pythagoras' theorem and the hypotenuse

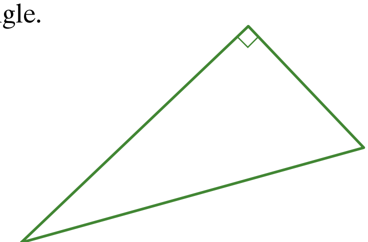
Chapter warm-up

Right-angled triangles

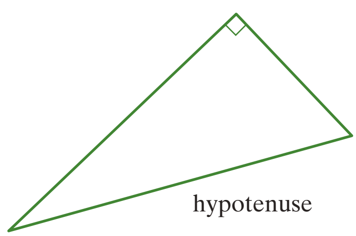
The longest side of any right-angled triangle has been given a special name. This is the side that does not form one of the arms of the right angle. It is called the **hypotenuse**.

Example 1

Label the hypotenuse in this right-angled triangle.



Working



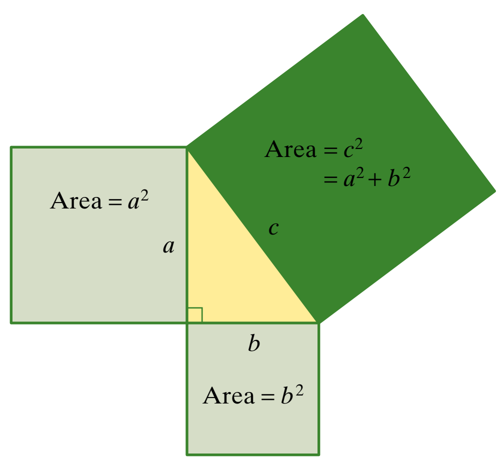
Reasoning

The hypotenuse is the side that does not form one of the arms of the right angle.

Right-angled triangles

Pythagoras' theorem

It has been known for over 2500 years that there is a special relationship between the area of a square drawn on the hypotenuse and the sum of the areas of squares drawn on the other two sides of a right angled triangle. This is known as **Pythagoras' theorem**.



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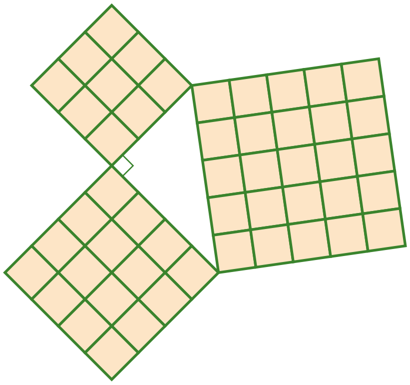
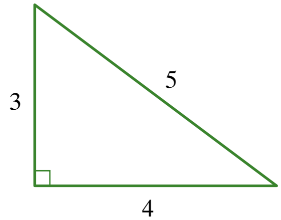
A triangle with sides 3, 4 and 5 units in length, for example, will be a right-angled triangle because

$3^2 + 4^2 = 9 + 16 = 25$ and $5^2 = 25$



The 3, 4, 5 triangle has been used for centuries by builders marking out the position of the foundations and walls for new buildings to make sure that the corners are right angles. It is still used for this purpose today.

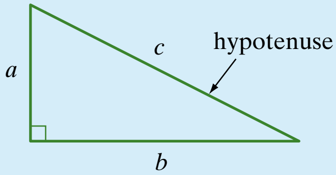


Pythagoras' theorem

For any right-angled triangle

$a^2 + b^2 = c^2$

where a and b are the perpendicular sides of the right-angled triangle and c is the hypotenuse.



The converse of this is also true: if the sum of the squares of the two shorter sides of a triangle is equal to the square of the hypotenuse, then the triangle is a right-angled triangle.

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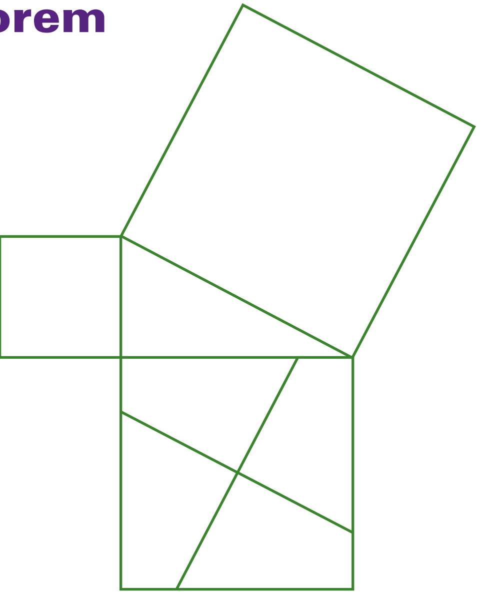
Proving Pythagoras' theorem

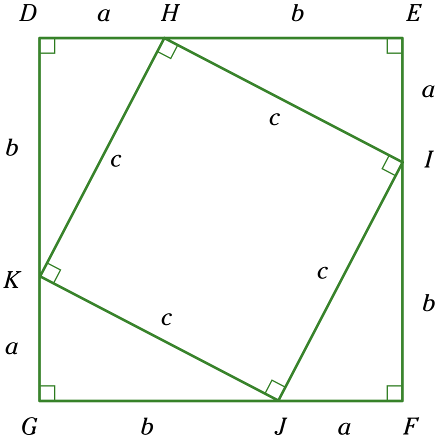
Over the centuries many proofs of Pythagoras' theorem have been developed. One of these proofs, Perigal's proof, is shown here as a diagram.

A square is constructed on each side of the right-angled triangle. Lines parallel to the sides of the large square have been drawn so that they pass through the centre of the middle-sized square. These lines divide the middle-sized square into four congruent pieces. The four pieces, together with the small square can be rearranged to exactly cover the large square. You can try this for yourself using the template or the interactive GeoGebra file, both of which are in the student ebook.



Another Pythagoras proof

DEFG is a square. The points *H*, *I*, *J* and *K* divide each side of the square in the ratio a:b. Four right-angled triangles are formed by joining *HI*, *IJ*, *JK* and *KH*.



Step 1: showing that the quadrilateral *HIJK* is a square.

The straight angle at point *H* is made up of two complementary angles $\angle DHK$ and $\angle EHI$ of the right-angled triangles plus $\angle KHI$ of quadrilateral *HIJK*. This means that $\angle KHI = 90°$. In a similar way we can show that the other angles of quadrilateral *HIJK* are right-angles. We also know that the four sides of quadrilateral *HIJK* are equal because each side is the hypotenuse of one of the congruent right-angled triangles.

Step 2: Expressing the area of *DEFG* in terms of *a* and *b*.

Notice how the distributive law is used to expand $(a + b)(a + b)$.

Length of each side of $DEFG = a + b$.

$$\begin{aligned}\text{Area of } DEFG &= (a + b)^2 \\ &= (a + b)(a + b) \\ &= a(a + b) + b(a + b) \\ &= a^2 + ab + ab + b^2 \\ &= a^2 + 2ab + b^2\end{aligned}$$

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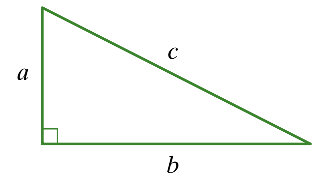
Step 3: Expressing the area of *DEFG* as the sum of the area of the four triangles and *HIJK*

$$\begin{aligned}\text{Area of } DEFG &= \text{Area of the four triangles} + \text{area of } HIJK\\ &= 4 \times \frac{ab}{2} + c^2\\ &= 2ab + c^2\end{aligned}$$

Step 4: Equating the two expressions for the area of *DEFG*

We put the two expressions for the area equal to each other then subtract $2ab$ from both sides.

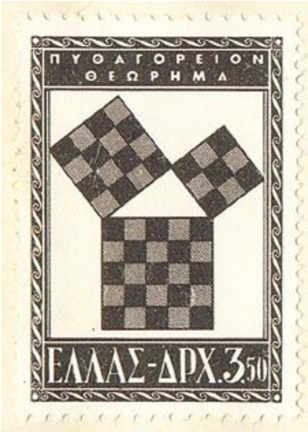
$$\begin{aligned}a^2 + 2ab + b^2 &= 2ab + c^2\\ a^2 + b^2 &= c^2\end{aligned}$$



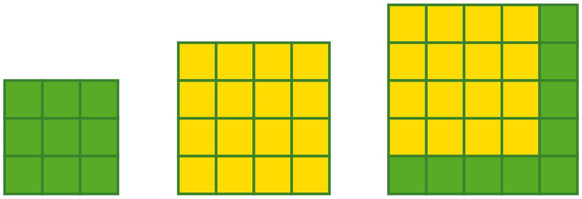
Pythagorean triples

A set of three whole numbers such as 3, 4 and 5 is called a Pythagorean triple because the sum of the squares of the two smaller numbers equals the square of the largest number.

This Greek stamp uses the 3, 4, 5 triple to commemorate Pythagoras. Although the 3, 4, 5 triple is the best known Pythagorean triple, there are many others.



Finding a Pythagorean triple is like finding two squares that can be rearranged to make a third, larger square.



3^2	+	4^2	=	5^2
9	+	16	=	25

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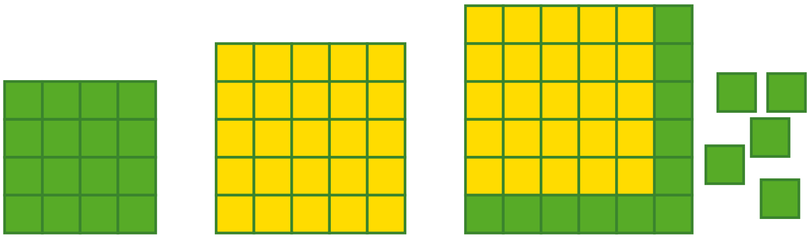
Only certain sets of three whole numbers are of this type. If, for example, we add 4^2 and 5^2, that is 16 and 25, we obtain 41, which is not a square number. As shown below, we cannot arrange the two squares to make a third square without having some leftover pieces.



$4^2 + 5^2$

$16 + 25 = 41$

For example, if we double each of the numbers 3, 4, 5, we obtain the numbers 6, 8, 10. The sum of the squares of the two smaller numbers is equal to the square of the larger number.

$6^2 + 8^2 = 36 + 64 = 100$

$10^2 = 100$

Example 2

Show that

a the set of integers 5, 12, 13 is a Pythagorean triple.

b the set of integers 4, 5, 7 is not a Pythagorean triple.

Working	Reasoning
a $5^2 + 12^2 = 25 + 144$ $= 169$ $13^2 = 169$ so $5^2 + 12^2 = 13^2$ So the set of integers 5, 12, 13 is a Pythagorean triple.	A set of integers is a Pythagorean triple if the sum of the squares of the two smaller numbers is equal to the square of the largest number.
b $4^2 + 5^2 = 16 + 25$ $= 41$ $7^2 = 49$ So the set of integers 4, 5, 7 is not a Pythagorean triple.	The sum of the squares of the two smaller numbers is not equal to the square of the largest number.

Multiples of Pythagorean triples are also Pythagorean triples.

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Example 3

Starting with the Pythagorean triple 3, 4, 5, create another Pythagorean triple by multiplying all the numbers by 3.

Working

$3 \times 3, 4 \times 3, 5 \times 3$
$= 9, 12, 15$

$$\begin{aligned} 9^2 + 12^2 &= 81 + 144 \\ &= 225 \\ 15^2 &= 225 \end{aligned}$$

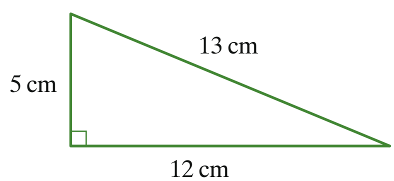
So $9^2 + 12^2 = 15^2$

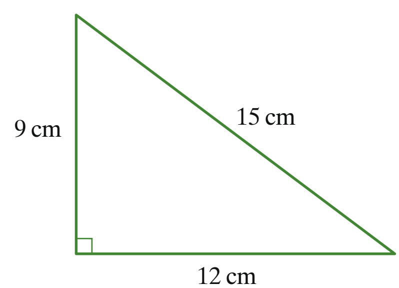
The set of integers 9, 12, 15 is a Pythagorean triple.

Reasoning

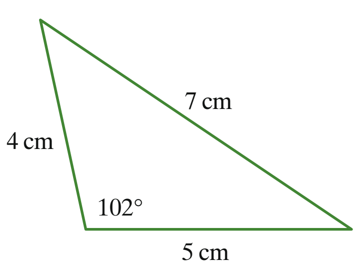
Multiples of Pythagorean triples are also Pythagorean triples.

We have seen in examples **2** and **3** that the integers 5, 12, 13 and 9, 12, 15 are Pythagorean triples. If a set of three integers is a Pythagorean triple, then a triangle with those side-lengths will be a right-angled triangle.





The three integers 4, 5, 7 are not a Pythagorean triple, so a triangle with these side lengths is not a right-angled triangle.



Example 4

If three iron bars of length 2 m, 3 m and 4 m are placed to form a triangle, will the triangle be right-angled?

continued

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Example 4 continued

Working

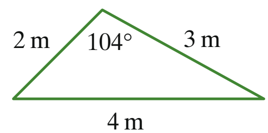
$2^2 + 3^2$
$= 4 + 9$
$= 13$
$\neq 4^2$

So the triangle is not right-angled.

Reasoning

The two short sides are 2 m and 3 m.

Calculate $2^2 + 3^2$ then check to see if it is equal to 4^2.

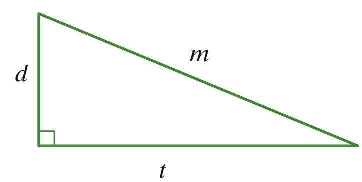


exercise 2.1

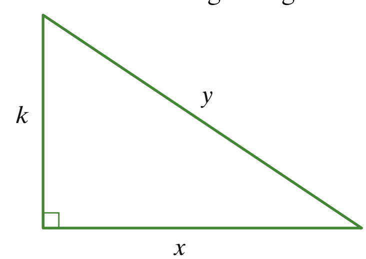
LINKS TO Example 1

State the letter corresponding to the hypotenuse in each of these right-angled triangles.

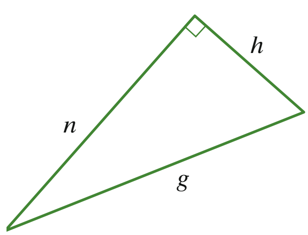
a



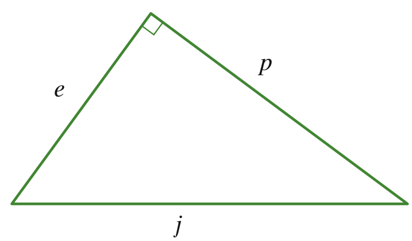
b



c



d



LINKS TO Example 2

Show which of these sets of numbers are Pythagorean triples.

a 8, 15, 17 **b** 3, 5, 8 **c** 7, 24, 25 **d** 7, 9, 12

Which of these sets of integers is a Pythagorean triple?

A 4, 6, 8 **B** 10, 12, 13 **C** 16, 30, 34 **D** 14, 24, 26 **E** 9, 16, 25

LINKS TO Example 3

Multiply each of these Pythagorean triples by the numbers below and show that the multiples of the triples are also Pythagorean triples.

i 2 **ii** 3 **iii** 4 **iv** 5

a 3, 4, 5 **b** 5, 12, 13 **c** 8, 15, 17 **d** 7, 24, 25

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Find the missing integer in each of these Pythagorean triples.

a 24, _____, 51 **b** 27, 36, _____ **c** 14, _____, 50 **d** _____, 72, 75

LINKS TO Example 4

Check each set of side lengths to see if they would form a right-angled triangle.

a 6 cm, 11 cm, 9 cm **b** 7 m, 3 m, 5 m **c** 17 km, 15 km, 8 km

d 10 cm, 24 cm, 26 cm **e** 5 cm, 12 cm, 13 cm **f** 36 cm, 48 cm, 60 cm

exercise 2.1 challenge

This table shows three Pythagorean triples.

a	b	c
3	4	5
5	12	13
7	24	25

a What do you notice about the values of a?

b What do you notice about the values of c compared with the values of b?

c Find $b + c$ for each of the triples. What do you notice? Hint: compare $b + c$ with a.

d Use the patterns to predict the next two triples in this group. Check to see whether those numbers form a Pythagorean triple.

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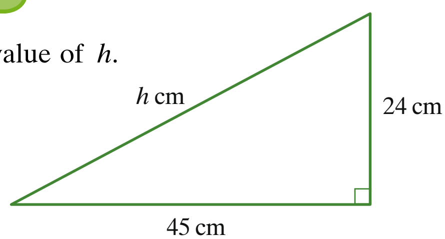
2.2 Calculating the length of the hypotenuse

If we know the lengths of the two perpendicular sides of a right-angled triangle, a and b, we can use $c^2 = a^2 + b^2$ to calculate the length, c, of the hypotenuse.

Example 5

Calculate the value of h.



Working

$$\begin{aligned} c^2 &= a^2 + b^2 \\ h^2 &= 24^2 + 45^2 \\ &= 576 + 2025 \\ &= 2601 \\ h &= \sqrt{2601} \\ h &= 51 \end{aligned}$$

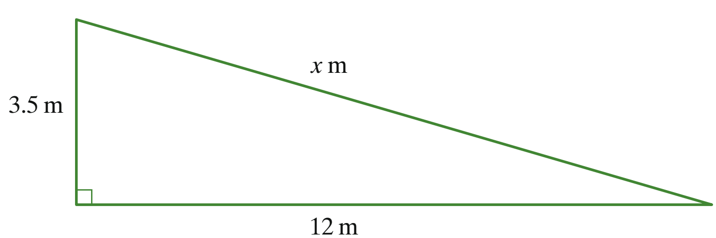
Reasoning

Notice that it does not matter which of the shorter sides we call a and which we call b.
Find the square root of 2601 to find the value of h.

As we have seen, some right-angled triangles have all three sides with integer lengths. In most right-angled triangles, though, this is not the case. In example **6**, the length of the hypotenuse is a rational number, but not an integer.

Example 6

Find the length of the hypotenuse of this triangle, correct to one decimal place.



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Example 6 continued

Working

$$c^2 = a^2 + b^2$$
$$x^2 = 3.5^2 + 12^2$$
$$= 12.25 + 144$$
$$= 156.25$$
$$x = \sqrt{156.25}$$
$$x = 12.5$$

The length of the hypotenuse is 12.5 m.

Reasoning

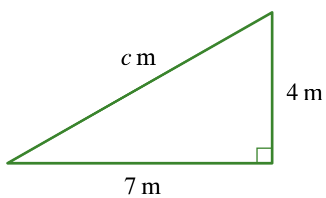
The two shorter sides have lengths 3.5 m and 12 m. The length of the hypotenuse is x m.
Substitute values to find x.

Irrational hypotenuse lengths

Irrational numbers and the Pythagorean spiral

In most right-angled triangles the length of the hypotenuse is an **irrational number**. In the right-angled triangle below, the perpendicular sides a and b of the triangle are 4 m and 7 m. The length of the hypotenuse is $\sqrt{65}$ which is an irrational number. This means that the decimal places continue forever without forming any repeating pattern. When we find that the hypotenuse is irrational, we can either leave it in this exact form, for example, $\sqrt{65}$, or we can find a rational approximation, for example, $\sqrt{65} = 8.1$ correct to one decimal place.

An irrational number is a number that cannot be expressed as a fraction $\frac{a}{b}$ where a and b are integers.



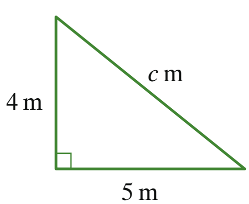
$$c^2 = a^2 + b^2$$
$$= 4^2 + 7^2$$
$$= 16 + 49$$
$$= 65$$
$$c = \sqrt{65}$$
$$c \approx 8.1$$

Example 7

Find the length of the hypotenuse

i in exact form.

ii as an approximate value correct to 1 decimal place.



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Example 7 continued

Working

i
$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= 4^2 + 5^2 \\ &= 16 + 25 \\ c^2 &= 41 \\ c &= \sqrt{41} \end{aligned}$$

ii $c \approx 6.40$

The length of the hypotenuse is $\sqrt{41}$ m. This is 6.4 m correct to one decimal place.

Reasoning

The side labelled c m is the hypotenuse.
Substitute $a = 4$ and $b = 5$.
Note that it would not make any difference if we substituted $a = 5$, $b = 4$.

Take square root to find c.

Use a calculator to find the approximate decimal value of $\sqrt{41}$.
Find c to one more decimal place than is required. Round to one decimal place.

Example 8

In a right-angled triangle the two shorter sides measure 6 cm and 3 cm.

a Draw a labelled diagram.

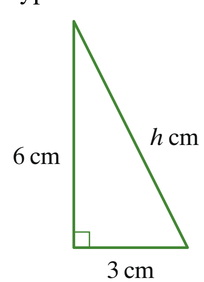
b Calculate the length of the hypotenuse

  i in exact form.

  ii correct to one decimal place.

Working

a Let h cm be the length of the hypotenuse.



b i
$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= 3^2 + 6^2 \\ &= 9 + 36 \\ &= 45 \\ c &= \sqrt{45} \end{aligned}$$

The length of the hypotenuse is $\sqrt{45}$ cm.

Reasoning

Draw a diagram.
Label the lengths of the two shorter sides.
Choose a pronumeral for the length of the hypotenuse.

Write the rule.
Substitute 3 and 6 for a and b (it does not matter whether we substitute $a = 3$, $b = 6$ or $a = 6$, $b = 3$).

continued

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Example 8 continued

Working

ii $h \approx 6.70$
The length of the hypotenuse is 6.7 cm correct to one decimal place.

Reasoning

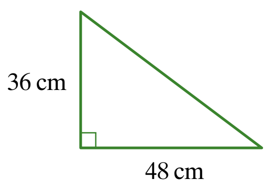
Use a calculator to find the approximate decimal value of $\sqrt{45}$ to one more decimal place than required.
Round to one decimal place.

If two sides of a right-angled triangle are in the same proportion as the matching numbers in a Pythagorean triple, then we can use the proportion to work out the missing side. Sometimes this may be quicker than using Pythagoras' theorem.

Example 9

The two shorter sides of a right-angled triangle have lengths 36 cm and 48 cm. Use a Pythagorean triple to find the length of the hypotenuse.

Working



$36 = 3 \times 12$

$48 = 4 \times 12$

$60 = 5 \times 12$

Using the 3, 4, 5 triple, the length of the hypotenuse is 60 cm.

Reasoning

Multiples of Pythagorean triples are also Pythagorean triples.
36 and 48 are respectively 12 times 3 and 12 times 4.
So the hypotenuse will be 12 times 5.

Example 10

Find the length of timber needed to make a diagonal strut for a gate that is 1.20 m wide and 2.00 m high. Draw a diagram and give your answer correct to two decimal places.

continued

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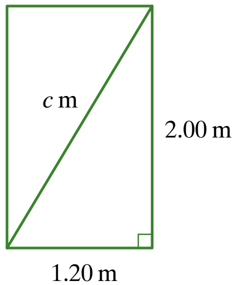
Example 10 continued

Working

Let c m be the length of the strut.



$$c^2 = 1.2^2 + 2^2$$
$$= 1.44 + 4.00$$
$$= 5.44$$
$$c = \sqrt{5.44}$$
$c = 2.33$ correct to two decimal places

The length of timber required is 2.33 m.

Reasoning

The diagonal forms the hypotenuse of a right-angled triangle with sides 1.20 m and 2.00 m.

Pythagoras' theorem is used to calculate the length of the hypotenuse.

$\sqrt{5.44} \approx 2.332$

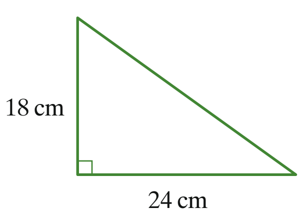
The answer to the question must be given in a sentence.

exercise 2.2

LINKS TO Example 5

1 Calculate the length of the hypotenuse in each of these right-angled triangles.

a



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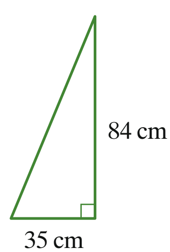

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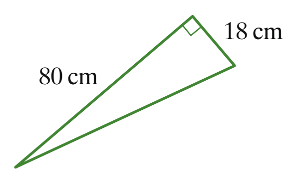

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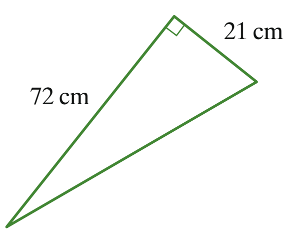

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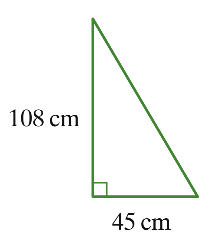

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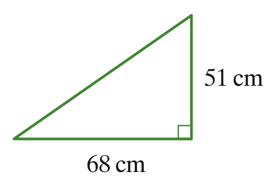

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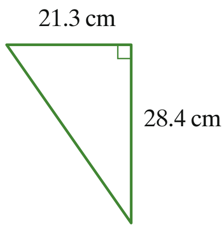

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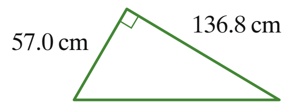

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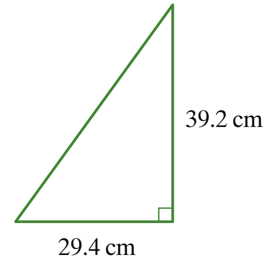

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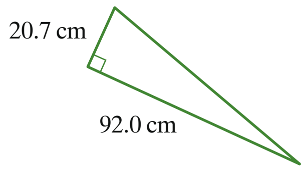

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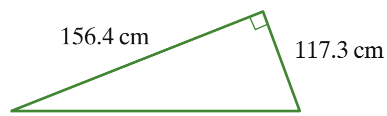

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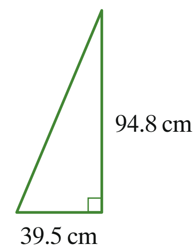


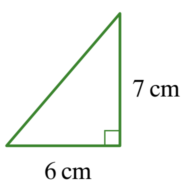
LINKS TO Example 6

Find the length of the hypotenuse

i in exact form.

ii as an approximate value correct to one decimal place.

a



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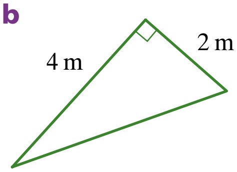

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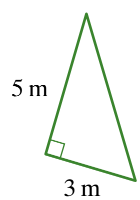

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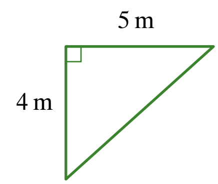

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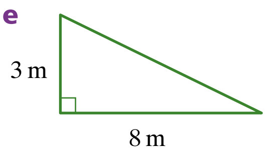

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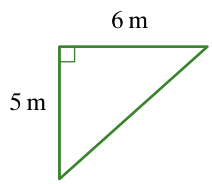

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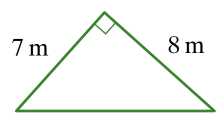

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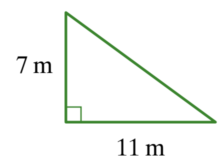

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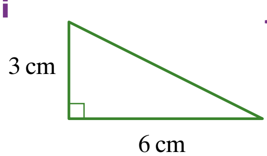

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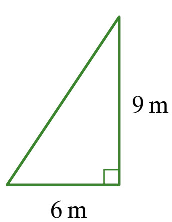

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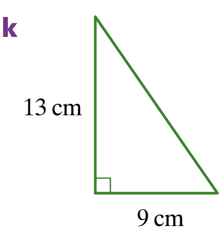

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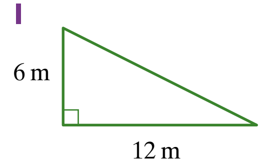

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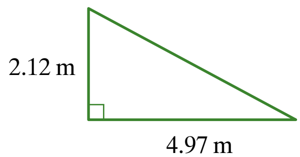
LINKS TO Examples 7, 8

Calculate the length of the hypotenuse for each of the following triangles. Round each length to the same number of decimal places given for the other two sides.

a



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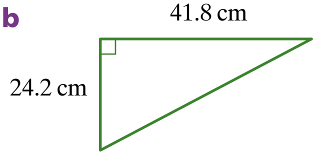

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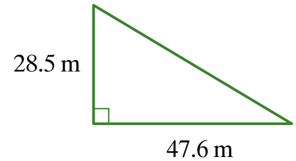

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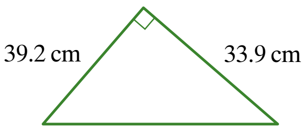

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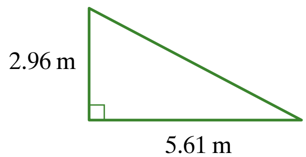

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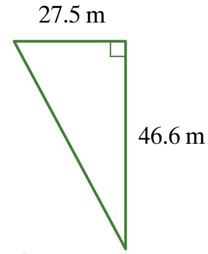

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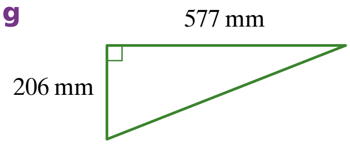

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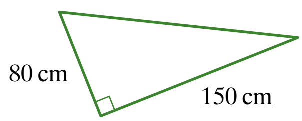

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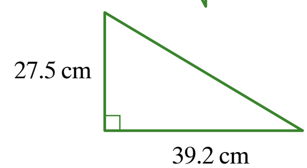

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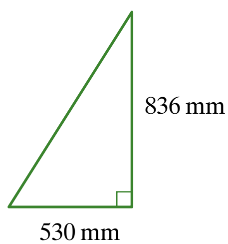

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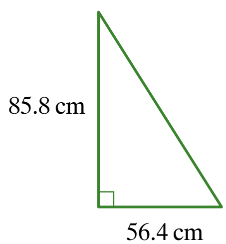

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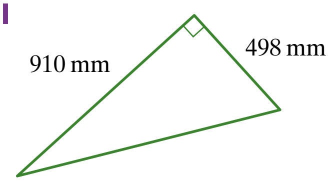
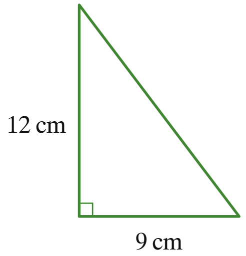


LINKS TO Example 9

Use Pythagorean triples to find the length of the hypotenuse of each of these right-angled triangles.

a



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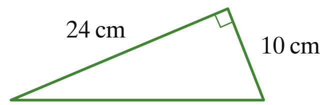

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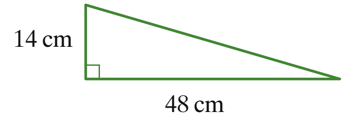

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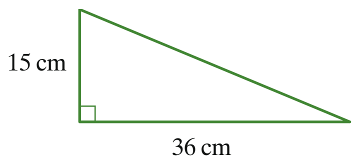

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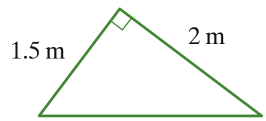

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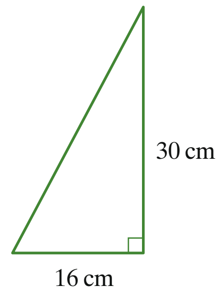
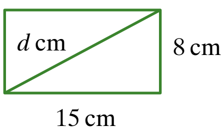
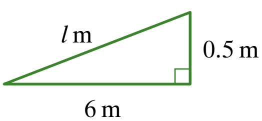


A rectangle is 15 cm long and 8 cm wide. Calculate the length of a diagonal.

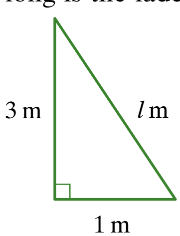


LINKS TO Example 10

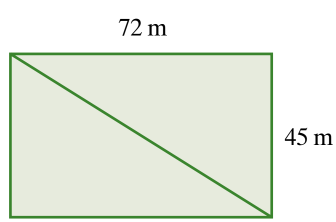
A ramp is to be built to allow a wheelchair to get to the front door of a house. The door is 50 cm above the ground, and the ramp needs to extend 6 m horizontally along the level ground. What will be the length of the ramp? Give your answer correct to the nearest centimetre.



A ladder provides access to a tree house. The door of the tree house is 3 m above the ground, and the base of the ladder extends 1 m horizontally along level ground. How long is the ladder? Give your answer in metres, correct to two decimal places.



Naida noticed that people always take shortcuts by walking diagonally across rectangular parks. She wondered how many metres she saved by cutting across the park near her home instead of walking around the road. Calculate the distance in metres Naida saves by walking across this park. Give your answer correct to one decimal place.



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For each of these shapes

i draw a labelled diagram.

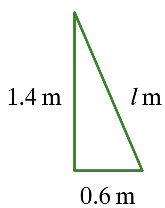
ii calculate the length of a diagonal, correct to one decimal place.

a a rectangle with sides of 15 cm and 36 cm

b a rectangle with length 56 cm and width 18 cm

c a square with sides of 10 m.

A guy rope is to be attached at the top of a vertical tent pole that is 1.4 m long. It is to be pegged 0.6 m from the base of the tent pole along level ground. The rope is 1.5 m long. Is it long enough to be pegged in this position? Explain.



exercise 2.2 challenge

The three lengths 3 m, 4 m and 5 m form a right-angled triangle. Find the side lengths of three other right-angled triangles that have a hypotenuse 5 m long. Give your answers as exact values.

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2.3 Calculating the length of a shorter side

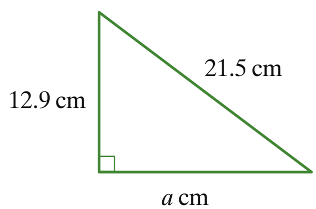
Sometimes we know the length of the hypotenuse and one of the shorter sides. By rearranging the rule for Pythagoras' theorem we can easily calculate the missing side length.

$a^2 + b^2 = c^2$ so $a^2 = c^2 - c^2$

As we saw in the last section, it does not matter which of the shorter sides we call a and which we call b.

Example 11

Find the value of a in this right-angled triangle.



Working

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 &= c^2 - b^2 \\ &= 21.5^2 - 12.9^2 \\ &= 295.84 \\ a &= \sqrt{295.85} \\ a &= 17.2 \end{aligned}$$

Reasoning

Rearrange $a^2 + b^2 = c^2$ then substitute for b and c.
Alternatively, substitute first then rearrange:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + 12.9^2 &= 21.5^2 \\ a^2 &= 21.5^2 - 12.9^2 \end{aligned}$$

Example 12

The hypotenuse of a right-angled triangle measures 9 cm and one of the shorter sides measures 5 cm.

a Draw a labelled diagram.

b Calculate the length of the third side of the triangle

  i in exact form.

  ii as an approximate value correct to one decimal place.

continued

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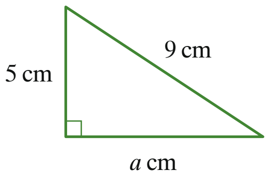
Example 12 continued

Working

a Let a cm be the length of the unknown side.



Reasoning

Choose a pronumeral to represent the length of the shorter side. Draw a diagram. Label the length of one of the shorter sides.

b

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + 5^2 &= 9^2 \\ a^2 &= 9^2 - 5^2 \\ a^2 &= 56 \\ a &= \sqrt{56} \end{aligned}$$

Substitute 5 for b and 9 for c.

Rearrange the equation to make a^2 the subject.

i The length of the third side is $\sqrt{56}$ cm

$a \approx 7.48$

Find the square root of 56 to one more decimal place than required.

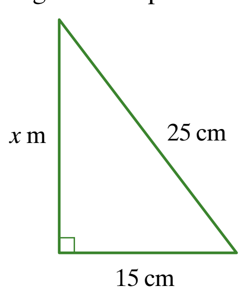
ii The length of the third side is 7.5 cm.

Round to one decimal place.

Example 13

Use Pythagorean triples to find the missing side lengths in these right-angled triangles.

a



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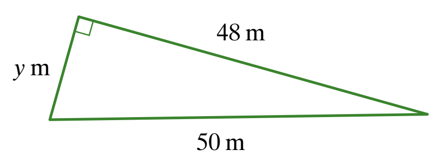

Working

a Using the 3, 4, 5 triple, $x = 20$

Reasoning

$15 = 5 \times 3$
$25 = 5 \times 5$
so $x = 5 \times 4$

b Using the 7, 24, 25 triple, $y = 14$

$48 = 2 \times 24$
$50 = 2 \times 25$
so $y = 2 \times 7$

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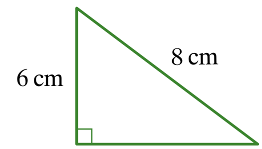
Note that we could not use the Pythagorean triple 6, 8, 10 to find the missing side of this triangle. The 6 cm and 8 cm sides must be the two right-angle sides.

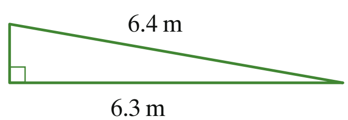


Example 14

The slope of a wheelchair ramp is 6.4 m long. The ramp covers a horizontal distance of 6.3 m.

a Draw a diagram to show the ramp and label the lengths.

b How high above the ground is the top of the ramp? Give the height correct to one decimal place.

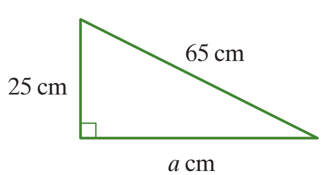
Working	Reasoning
a 	The ramp is the hypotenuse of the right-angled triangle.
b $a^2 + b^2 = c^2$	Substitute 6.4 for c and 6.3 for b.
$a^2 + 6.3^2 = 6.4^2$	
$a^2 = 6.4^2 - 6.3^2$	Rearrange the equation to make a^2 the subject.
$a^2 = 1.27$	
$a = \sqrt{1.27}$	Find the square root of 1.27.
$a \approx 1.12$	
The top of the ramp is 1.1 m above the ground.	Round to one decimal place.

exercise 2.3

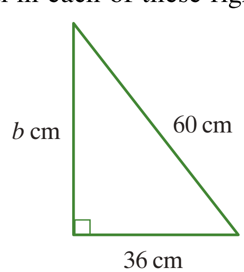
LINKS TO Example 11

1 Find the value of the pronumeral in each of these right-angled triangles.

a



b



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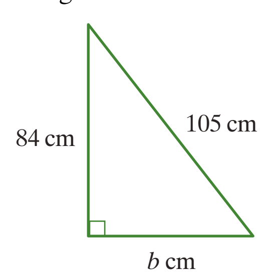

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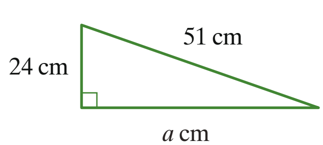
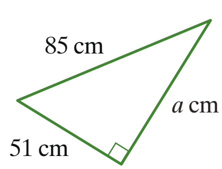


e



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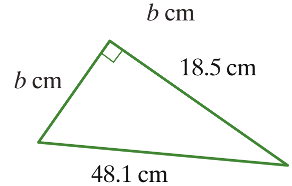

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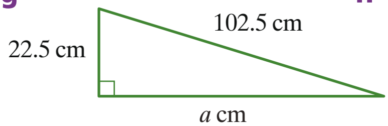
g



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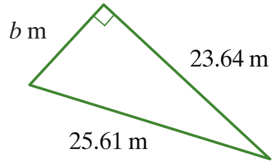
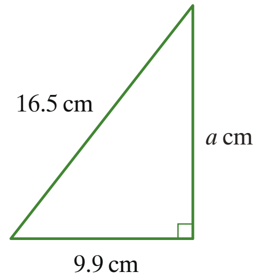


i



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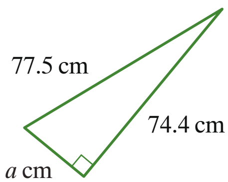

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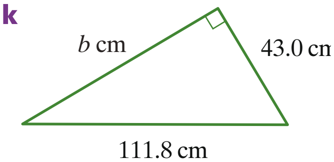

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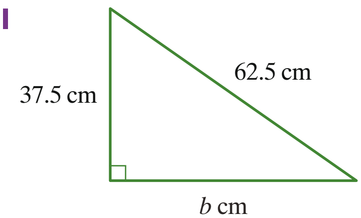


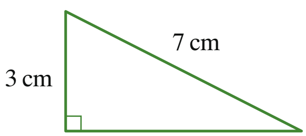
LINKS TO Example 12

Find the missing side length in each of these right-angled triangles

i in exact form.

ii as an approximate value correct to one decimal place.

a



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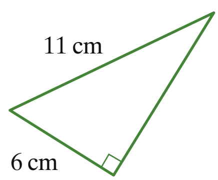

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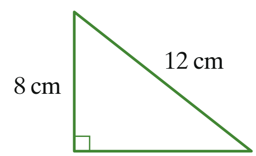

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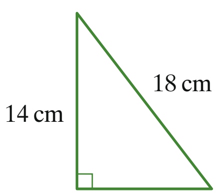

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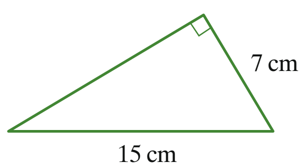

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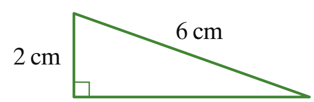

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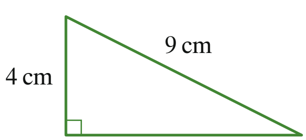

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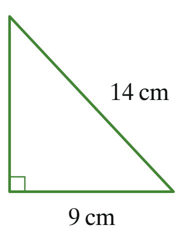
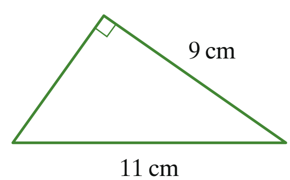


i



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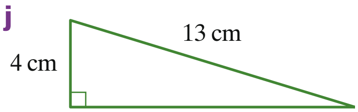

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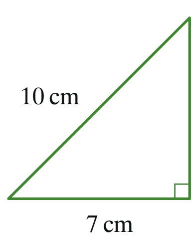

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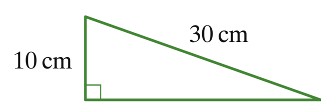

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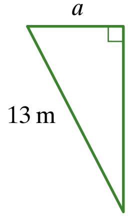
2.3

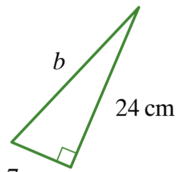
LINKS TO Example 13

Each of the following right-angled triangles has side lengths that are either a Pythagorean triple or a multiple of a triple. Find the missing side lengths.

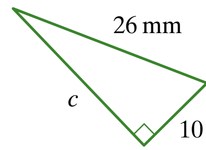
a



b



c



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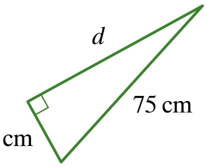
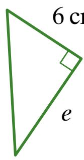


e



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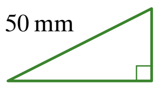
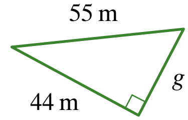


g



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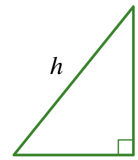


The missing side length in this right-angled triangle is

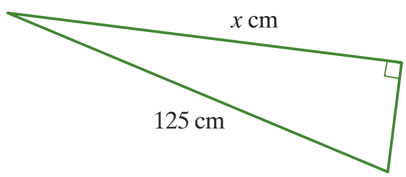
A 90 cm.

B 120 cm.

C 124.9 cm.

D 129.8 cm.

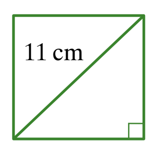
E 160 cm.



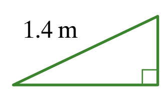
LINKS TO Example 14

A ladder is 2.5 metres long and it is leaning against a vertical wall with its foot 1 metre from the base of the wall on horizontal ground. How far up the wall does the ladder reach? Give the distance in metres, correct to one decimal place.

Find the length of a rectangle that has a width of 8 cm and a 11 cm long diagonal. Give the distance in centimetres, correct to one decimal place.



A piece of steel 1.4 metres long has been used as a ramp to reach a doorway 60 cm above the horizontal ground. How far out from the doorway does this ramp extend? Give the distance correct to the nearest centimetre.



The hypotenuse of a right-angled triangle measures 63 cm and one of the shorter sides measures 35 cm. Find the length of the third side. Give the length correct to one decimal place.

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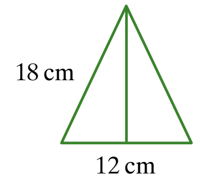
An isosceles triangle has two equal sides of 18 cm. The base of the triangle is 12 cm. Calculate the perpendicular height

a as an exact value.

b correct to one decimal place.



An isosceles triangle has two equal sides of 25 cm. The base of the triangle is 14 cm.

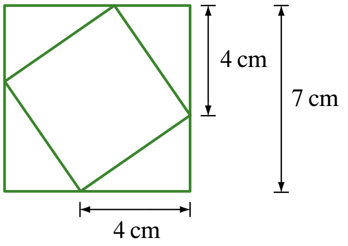
a Draw a labelled diagram.

b Label the perpendicular height of the triangle.

c Calculate the perpendicular height.

exercise 2.3 challenge

This diagram shows a square inside a square. The larger square has a side length of 7 cm. Calculate the side length of the inner square.



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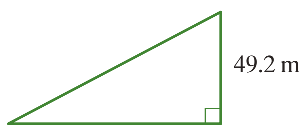
2.4 Mixed practice using Pythagoras' theorem

If any two sides of a right-angled triangle are known, Pythagoras' theorem can be used to find the third side.

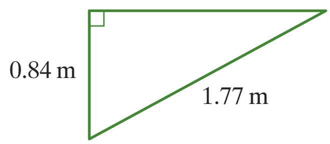
Example 15

Find the unknown side lengths in these right-angled triangles. Give the lengths correct to the same number of decimal places as in the given lengths.

a



b



Working

a $c^2 = 49.2^2 + 93.6^2$

$c^2 = 11\,181.6$

$c = \sqrt{11\,181.6}$

$c \approx 105.74$

The hypotenuse length is 105.7 m.

Reasoning

Let c m be the unknown hypotenuse length. Substitute 49.2 and 93.6 for a and b, respectively.

Evaluate to one more decimal place than required.
Round to one decimal place.

Working

b $a^2 + 0.84^2 = 1.77^2$

$a^2 = 1.77^2 - 0.84^2$

$a^2 = 2.4273$

$a = \sqrt{2.4273}$

$a \approx 1.557$

The unknown side length is 1.56 m.

Reasoning

The unknown side is one of the shorter sides. Let a m be the unknown side length. Substitute 0.84 and 1.77 for b and c, respectively.

Evaluate to one more decimal place than required.
Round to two decimal places.

Example 16

A ladder that is 3.6 m long leans against a wall with the foot of the ladder 80 cm from the wall.

a Draw a labelled diagram to show this information.

b How far up the wall does the ladder reach? Give the height in metres correct to one decimal place.

continued

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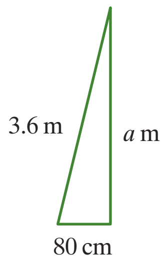
Example 16 continued

Working

Reasoning

a



The ladder represents the hypotenuse of the triangle.
Let a m be the height to which the ladder reaches.

b

$$\begin{aligned} a^2 + 0.8^2 &= 3.6^2 \\ a^2 &= 3.6^2 - 0.8^2 \\ a^2 &= 12.32 \\ a &= \sqrt{12.32} \\ a &\approx 3.50 \end{aligned}$$

The unknown side is one of the shorter sides.

Evaluate to one more decimal place than required.

The ladder reaches 3.5 m up the wall.

Round to one decimal place.

Example 17

A ship sails 56 km due south and then 35 km due west.

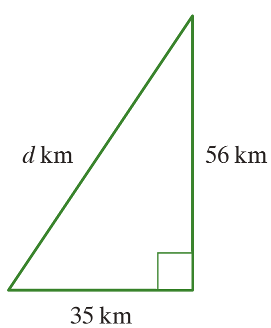
a Draw a diagram to show this information.

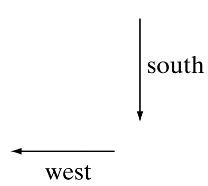
b Calculate the distance of the ship from its starting point. Give the distance to the nearest kilometre.

Working

Reasoning

a





continued

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Example 17 continued

Working

b $d^2 = 56^2 + 35^2$

$= 4361$

$d = \sqrt{4361}$

≈ 66.04

The ship is approximately 66 km from its starting point.

Reasoning

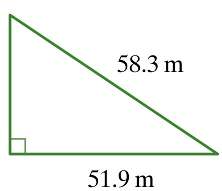
The distance of the ship from its starting point is the hypotenuse of the right-angled triangle.

exercise 2.4

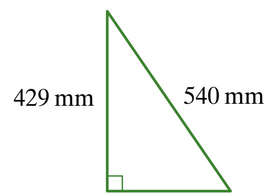
LINKS TO Example 15

1 Find the missing side length in each of these right-angled triangles. Hint: first decide whether it is the length of the hypotenuse or one of the shorter sides you are calculating. Give each length correct to the same number of decimal places as in the given lengths.

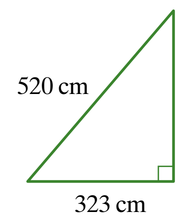
a



b



c



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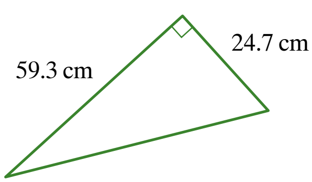

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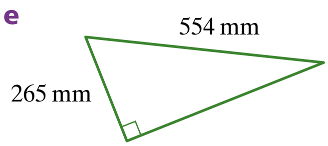

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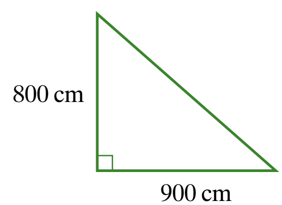

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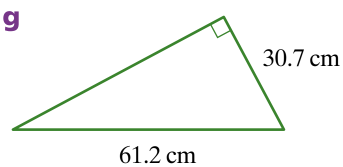

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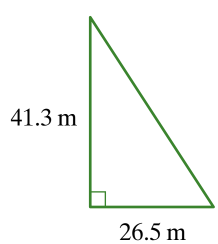

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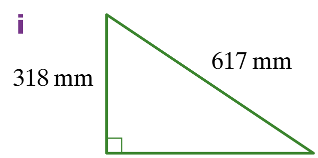

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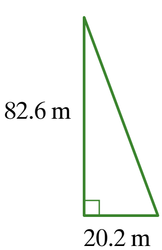

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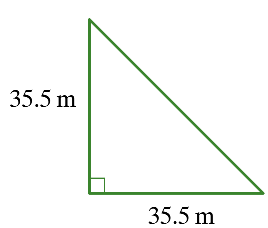

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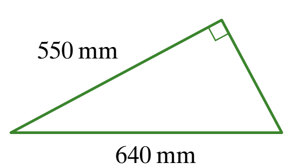

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The hypotenuse of a right-angled triangle is 45 cm long and one of the shorter sides is 24 cm long. The other side is closest to

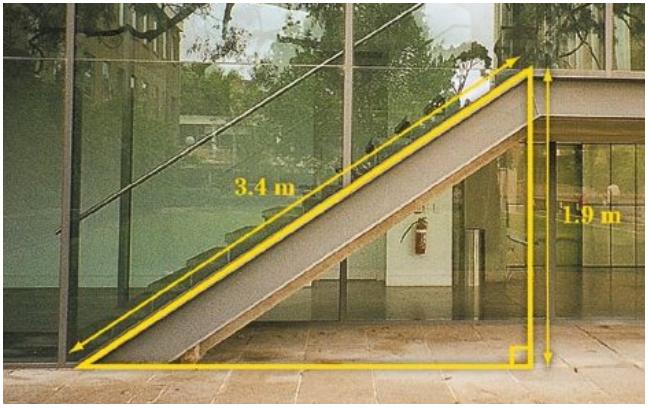
A 21 cm **B** 25 cm **C** 38 cm **D** 51 cm **E** 69 cm

The two shorter sides of a right-angled triangle have lengths 72 cm and 180 cm. The length of the third side of the triangle is closest to

A 73 cm **B** 108 cm **C** 165 cm **D** 194 cm **E** 252 cm

LINKS TO Example 16

The section of stairs shown in the photo rises 1.9 metres, and rests on a beam 3.4 metres long. What is the horizontal distance covered by this section of the stairs? Give your answer to the nearest centimetre.



The sloping ramp of a flat bank skateboard ramp is 200 cm. The top of the ramp is 70 cm above the ground.

a Draw a labelled diagram of the right-angled triangle.

b What horizontal distance does the ramp cover to the nearest centimetre?

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The top of a ramp for loading and unloading a truck is 1.2 m above the ground. The ramp covers a horizontal distance of 3.7 m.

a Draw a labelled diagram of the right-angled triangle formed by the ramp and the ground.

b What is the length of the sloping ramp? Give the length correct to one decimal place.

A ladder 3.2 m long rests against a wall. The foot of the ladder is 1.3 m from the wall.

a Draw a right-angled triangle, labelling the given information on your diagram.

b How far up the wall does the ladder reach? Give the height correct to one decimal place.

A guy rope is attached to a tent 1.35 metres above the ground. The length of the guy rope is 1.55 m

a Draw a right-angled triangle, labelling the given information on your diagram.

b How far from the base of the tent will the guy rope need to be pegged so that it is fully stretched? Give the distance correct to two decimal places.

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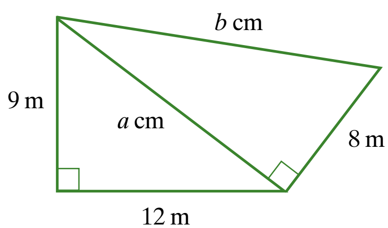
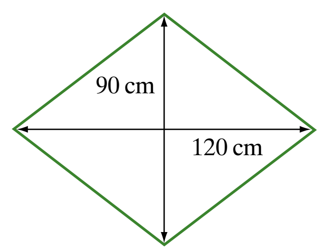
- The Thredbo chair lift travels a sloping distance of 1800 m. The horizontal distance covered is 1710 m.
  - **a** Draw a diagram to show this information.
  - **b** Label the distance to be found.
  - **c** Through what vertical distance does the chair lift travel? Give the height correct to the nearest 10 m.

LINKS TO Example 17

- A ship travels 83 km due south and then 17 km due west.
  - **a** Draw a diagram to show this information.
  - **b** Label the distance to be found.
  - **c** Calculate the distance of the ship from its starting point. Give the distance to the nearest kilometre.
- Some bushwalkers leave their car in a car park, walk 5 km east along a straight path and then 3.6 km in a northerly direction through the bush. How far are the bushwalkers from their car? Give the distance in kilometres correct to one decimal place.
- A ship travels 67 km due north, and then 46 km due west. Calculate its distance from the starting point. Give your answer correct to the nearest kilometre.
- Find the values of a and b, correct to one decimal place.



- Find the lengths of the sides of this rhombus, correct to one decimal place.



exercise 2.4 challenge

- An isosceles right-angled triangle has a hypotenuse of 10 cm. How long are the shorter sides? Give your answer correct to 1 decimal place.
- The diameter of the base of a party hat is 8 cm. The sloping edge of the hat is 17 cm.
  - **a** Draw a labelled right-angled triangle.
  - **b** What is the height of the hat? Give your answer correct to one decimal place.

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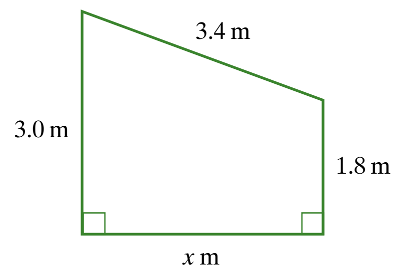
2.5 Using Pythagoras' theorem to find unknown dimensions and areas

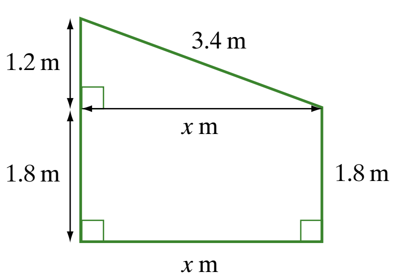
Pythagoras' theorem can be applied in composite shapes to find unknown lengths if the composite shape can be divided to include one or more right-angled triangles. Unknown side lengths in irregular-shaped pieces of land can often be found in this way.

Example 18

Find the width of the shape, x m, correct to two decimal places.



Working



Reasoning

Divide the shape into a right-angled triangle and a rectangle. The height of the right-angled triangle is (3.0 – 1.8) m, that is, 1.2 m.

$$
\begin{aligned}
x^2 + 1.2^2 &= 3.4^2 \\
x^2 &= 3.4^2 - 1.2^2 \\
&= 10.12 \\
x &= \sqrt{10.12} \\
&\approx 3.181
\end{aligned}
$$

Use Pythagoras' theorem to find x.

The width of the shape is 3.18 m.

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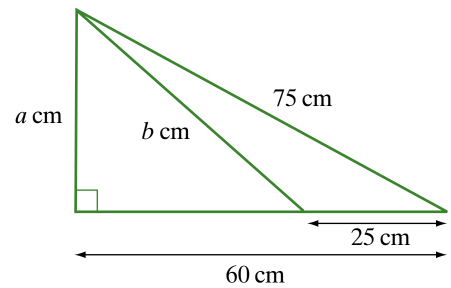
In the following example, there is one right-angled triangle within another right-angled triangle. By working first in one triangle, an unknown side length is found that allows us to work in the second triangle to find another unknown side length.

Example 19

Find, correct to the nearest whole number,

a the length marked a cm.

b the length marked b cm.

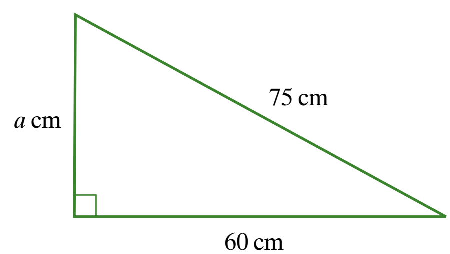


Working

a
$$\begin{aligned} a^2 + 60^2 &= 75^2 \\ a^2 &= 75^2 - 60^2 \\ &= 2025 \\ a &= \sqrt{2025} \\ a &= 45 \end{aligned}$$

The length marked a cm is 45 cm.

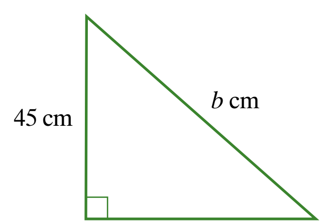
Reasoning



b The base of the smaller triangle
= 60 cm – 25 cm = 35 cm

$$\begin{aligned} b^2 &= 35^2 + 45^2 \\ b^2 &= 3250 \\ b &= \sqrt{3250} \\ b &\approx 57.0 \end{aligned}$$

The length marked b cm is 57 cm.



60 cm – 25 cm = 35 cm

Find the value to one more decimal place than required.

Calculating areas of irregular shapes

The area of irregular shapes, particularly irregular blocks of land, can often be calculated by dividing the shape into rectangles and right-angled triangles. Pythagoras' theorem can be used to calculated required side lengths before calculating the area.

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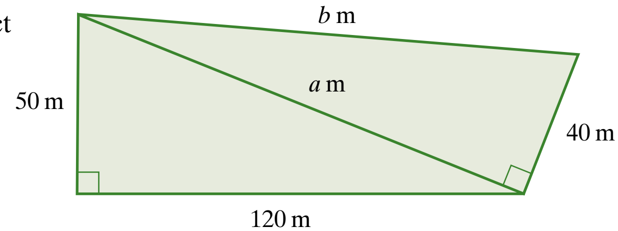
Example 20

The diagram shows an irregular shaped piece of land divided into two right-angled triangles.

a Find the values of a and b, correct to the nearest whole number.

b Calculate the area of the land in square metres.



Working

a

$$a^2 = 50^2 + 120^2$$
$$a^2 = 16\,900$$
$$a = \sqrt{16\,900}$$
$$a = 130$$

$$b^2 = 40^2 + 130^2$$
$$b^2 = 18\,500$$
$$b = \sqrt{18\,500}$$
$$b \approx 136.0$$
$$a = 130, b = 136$$

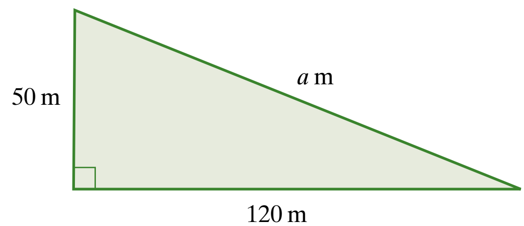
b Area of land

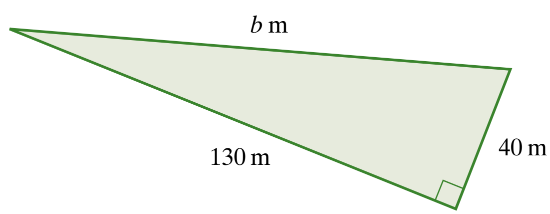
$= $ sum of area of triangles

$$= \frac{120 \times 50}{2} + \frac{130 \times 40}{2}$$
$$= 3000 + 2600$$
$$= 5600$$

The area of the land is $5600\ \text{m}^2$.

Reasoning





Area of triangle $= \dfrac{bh}{2}$

First triangle:
Base = 120 m, perpendicular height = 50 m
Second triangle:
Base = 130 m, perpendicular height = 40 m

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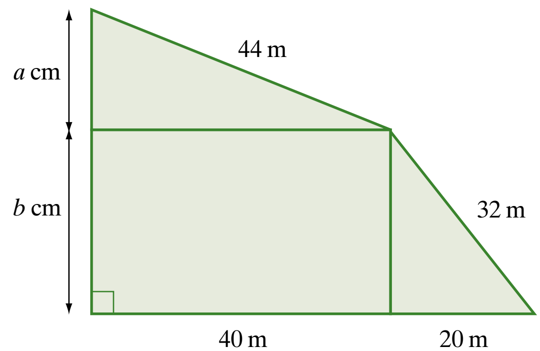
Example 21

The irregular shaped block of land shown here has been divided into a rectangle and two right-angled triangles.

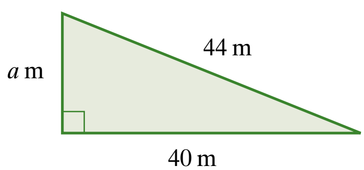


a Find the unknown lengths, a m and b m correct to one decimal place.

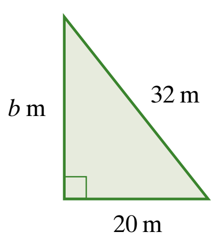
b Calculate the area of the land, correct to the nearest square metre.

Working

a



$$\begin{aligned} a^2 + 40^2 &= 44^2 \\ a^2 &= 44^2 - 40^2 \\ a^2 &= 336 \\ a &= \sqrt{336} \\ a &\approx 18.3 \end{aligned}$$



$$\begin{aligned} b^2 + 20^2 &= 32^2 \\ b^2 &= 32^2 - 20^2 \\ b^2 &= 624 \\ b &= \sqrt{624} \\ b &\approx 25.0 \end{aligned}$$

The length marked a m is 18.3 m and the one marked b m is 25.0 m.

Reasoning

Use Pythagoras' theorem to calculate the unknown side lengths in the right-angled triangles.

continued

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Example 21 continued

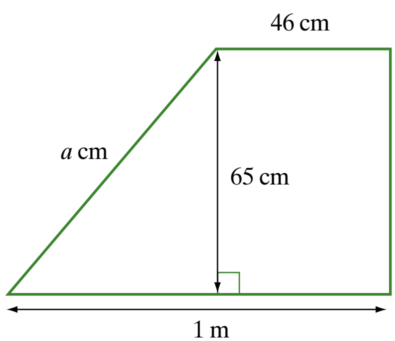
Working	Reasoning
b Area of land = area of triangles + area of rectangle	Area of triangle = $\dfrac{bh}{2}$
$= \dfrac{40 \times 18.3}{2} + \dfrac{20 \times 25}{2} + 40 \times 25$	First triangle: Base = 40 m, perpendicular height = 18.3 m
$= 366 + 250 + 1000$	Second triangle: Base = 20 m, perpendicular height = 25.0 m
$= 1616$	
The area of the land is $1616\ \text{m}^2$.	

exercise 2.5

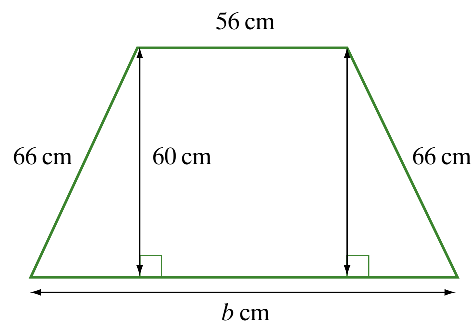
LINKS TO Example 18

1. Find the unknown lengths in each of the following, giving your answers correct to one decimal place.

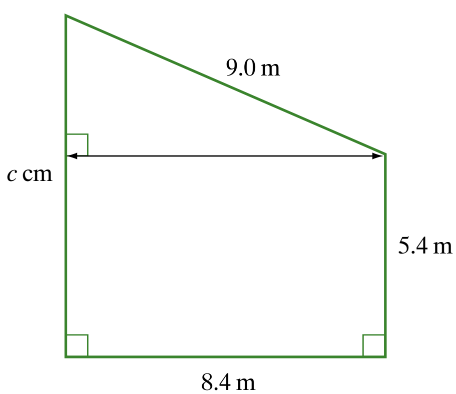
a



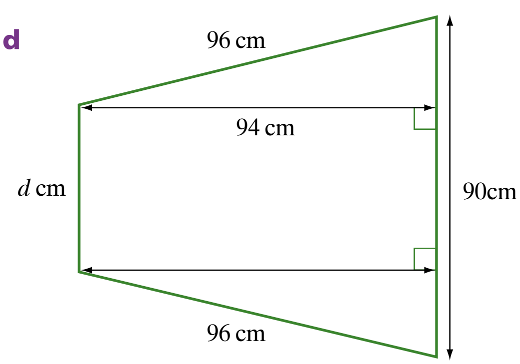
b



c



d



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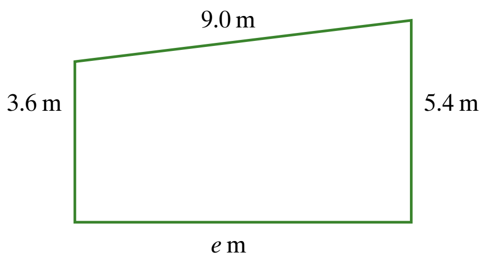
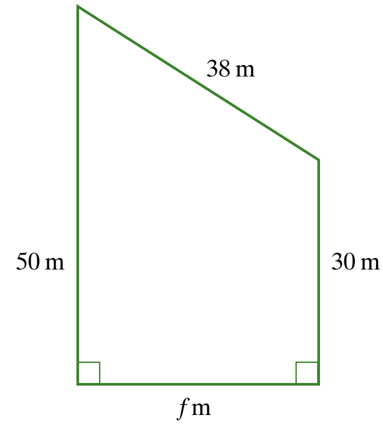
e



f



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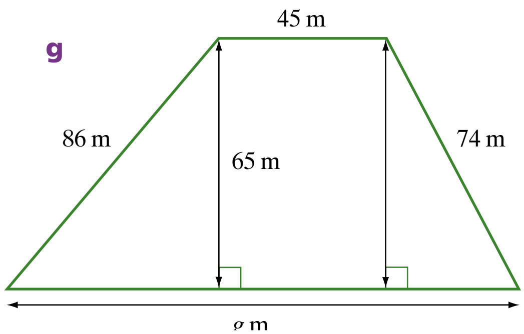

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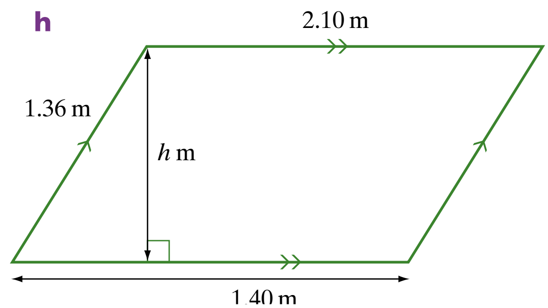
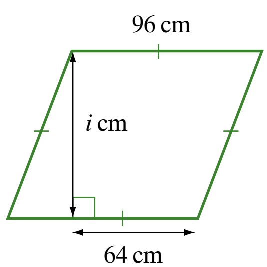


i



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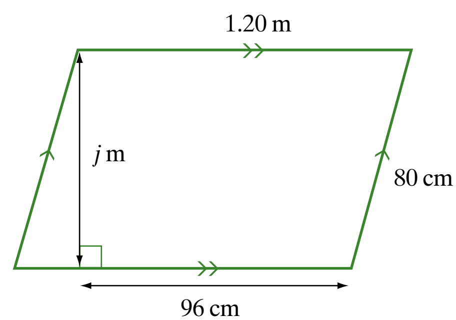
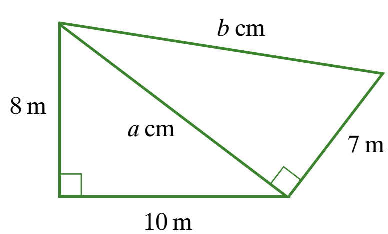


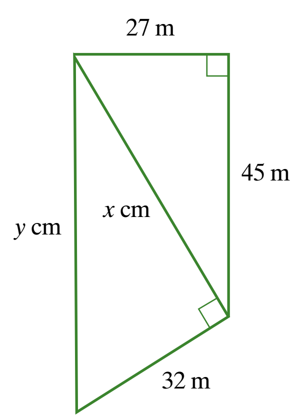
LINKS TO
Examples 19, 20a

Find the unknown lengths in each of the following, giving your answers correct to one decimal place.

a



b



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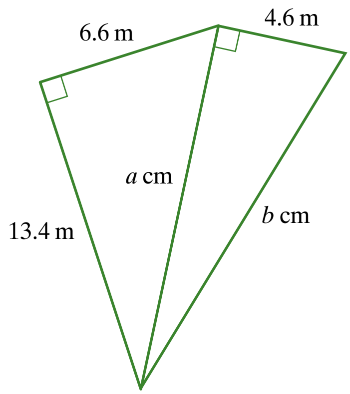
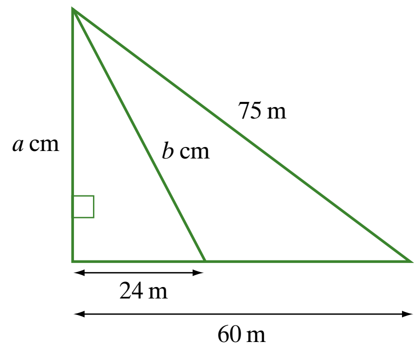
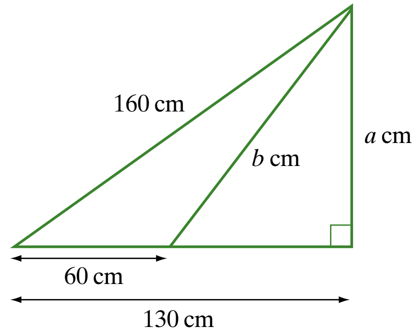
c



d



e



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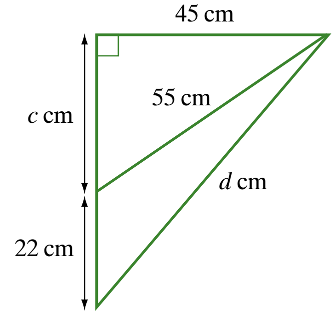

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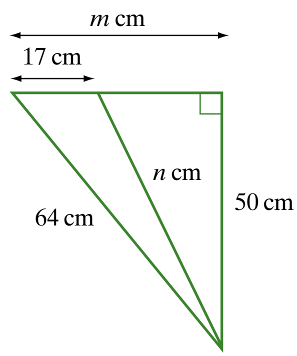

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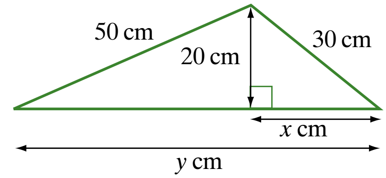
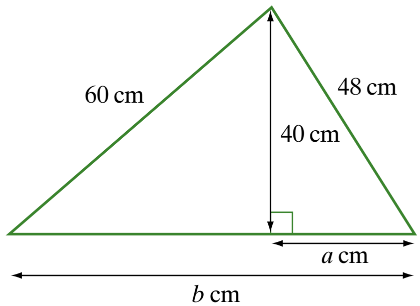


i



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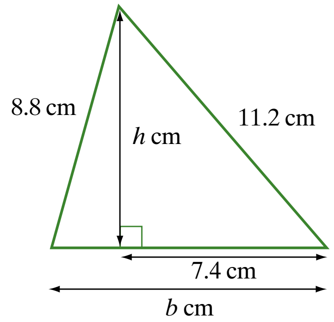

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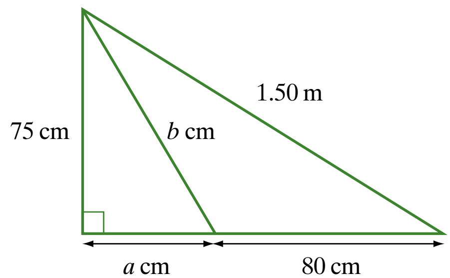
The values of a and b to the nearest whole numbers are

A $a = 102$, $b = 127$
B $a = 88$, $b = 150$
C $a = 88$, $b = 116$
D $a = 50$, $b = 90$
E $a = 50$, $b = 110$



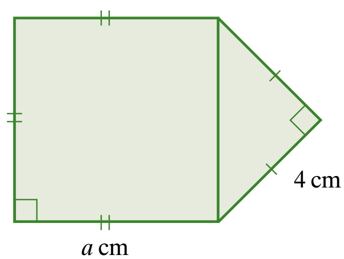
LINKS TO Examples 20b. 21

For each of the following shapes

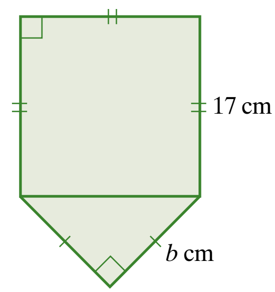
i use Pythagoras' theorem to find the unknown lengths, correct to two decimal places, where necessary.

ii calculate the area to the nearest whole number.

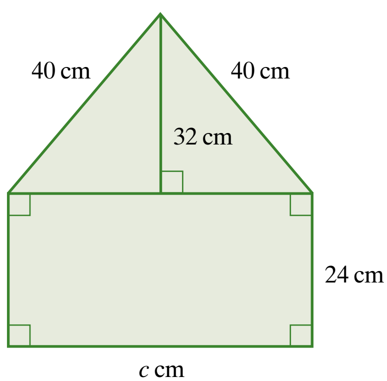
a



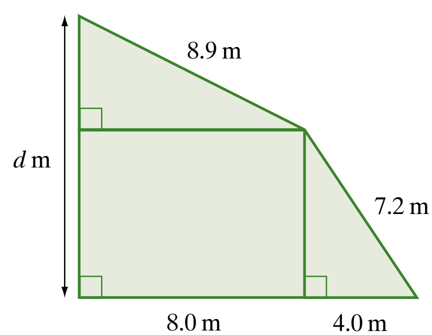
b



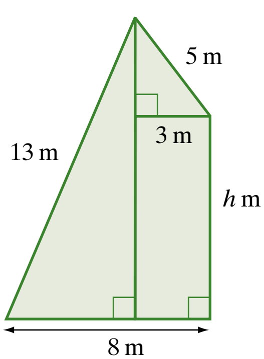
c



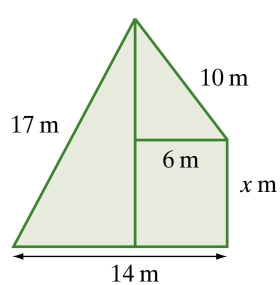
d



e



f



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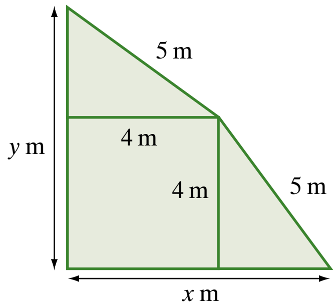
g



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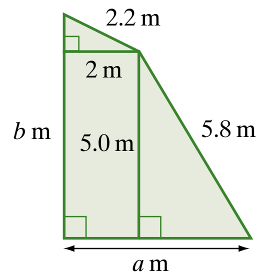
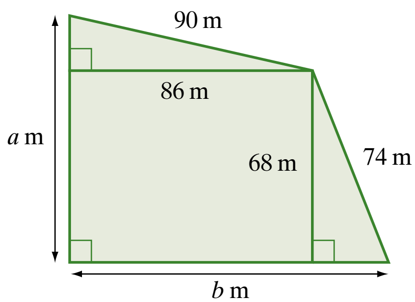
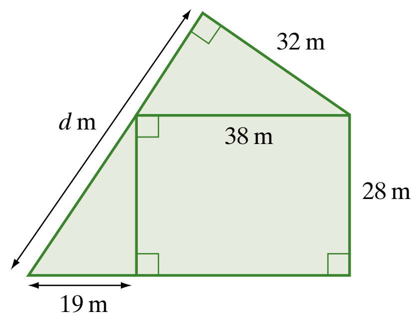


i

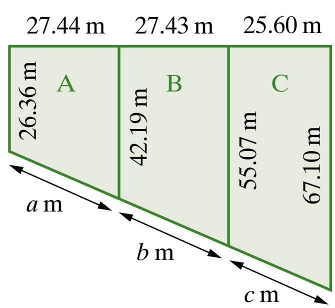


j



exercise 2.5 challenge

● A block of land was divided into three smaller building blocks as shown.



a Find the values of a, b and c, correct to two decimal places.

b Calculate the area of each of the blocks A, B and C, correct to the nearest square metre.

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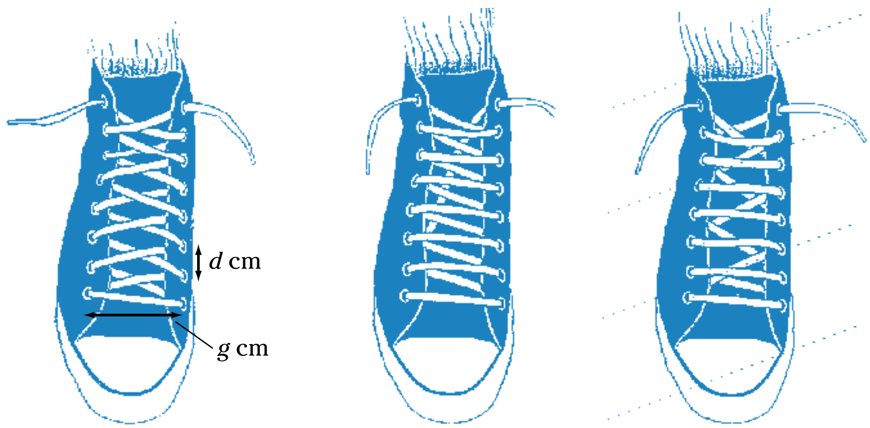
Analysis task

Short shoelaces!

Have you ever tried to tie your shoelaces and found they were too short? Is there a way of re-threading your laces to save some length for tying? Here are three different methods for threading laces. There are three variables involved for a particular shoe. They are

- the number of pairs of holes (n)
- the distance between holes (d cm)
- the gap between the two rows of holes (g cm).

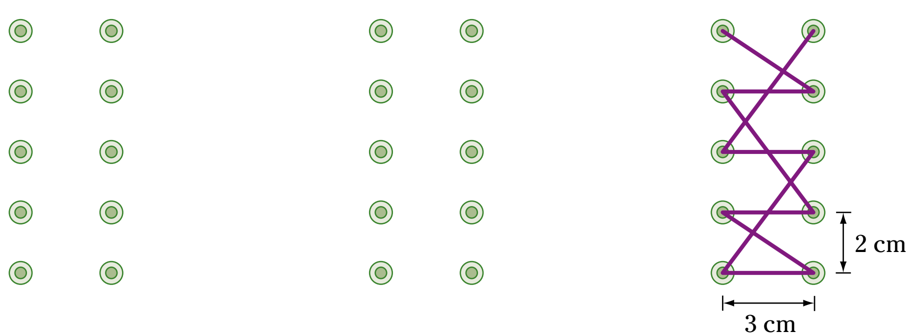


American method   Shoestore method   European method

To simplify the problem we will start by making some assumptions.

- There are 5 pairs of holes, that is, $n = 5$.
- The distance d cm between holes is 2 cm.
- The gap g cm between rows of holes is 3 cm.

a Copy the pairs of holes and the pattern of lacing for the European method shown below. Then draw the pattern of lacing for the American and shoestore methods.



American method   Shoestore method   European method

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b **i** Show that the length of lacing in the European method with 5 pairs of holes is $4 \times 3 + 2\sqrt{2^2 + 3^2} + 3\sqrt{3^2 + 4^2}$ cm. Hint: use Pythagoras' theorem to find the lengths of diagonal sections.

ii Simplify this expression, leaving it in surd form.

iii Evaluate the expression correct to one decimal place.

c Using Pythagoras' theorem, find expressions for the length of lacing in the American and shoestore methods with 5 pairs of holes. Evaluate each of your expressions correct to one decimal place.

d Which method uses

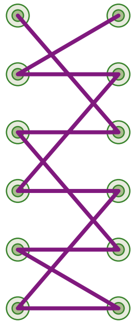
i the shortest length of shoelace?

ii the longest length of shoelace?

Now see if the lengths are in the same order for 6 pairs of holes.

e Draw the pattern of lacing for the American and shoestore methods for 6 pairs of holes.

f Using the diagram below, show that the length of lacing in the European method with 6 pairs of holes (with gap 3 cm and distance between holes 2 cm as before) is $5 \times 3 + 2\sqrt{2^2 + 3^2} + 4\sqrt{3^2 + 4^2}$



g Simplify and evaluate this expression correct to one decimal place.

h Repeat parts **c** and **d** for 6 pairs of holes.

i Is the order for the three methods the same for 6 pairs of holes as for 5 pairs of holes?

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Review Right-angled triangles and Pythagoras' theorem

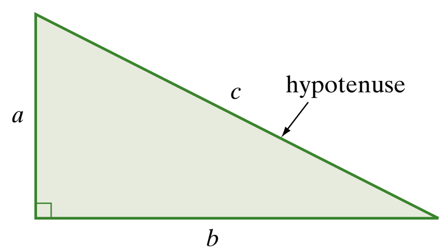
Summary

Pythagoras' theorem

- For any right-angled triangle

  $a^2 + b^2 = c^2$

  where a and b are the perpendicular sides of the right-angled triangle and c is the hypotenuse.



- Finding the hypotenuse: $c^2 = a^2 + b^2$
- Finding one of the shorter sides: $a^2 = c^2 - b^2$

Pythagorean triples

- A set of three whole numbers such as 3, 4 and 5 is called a Pythagorean triple because the sum of squares of the two smaller numbers equals the square of the largest number.

Visual map

diagonal
exact value
hypotenuse
irrational number
Pythagoras' theorem
Pythagorean triple
right-angled triangle
square root
surd

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Revision

Multiple-choice questions

- Which of these sets of integers is a Pythagorean triple?

  A 4, 5, 8 **B** 10, 24, 25 **C** 8, 16, 17 **D** 21, 28, 35 **E** 9, 21, 33

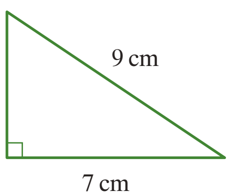
- The two shorter sides of a right-angled triangle are 9 cm and 40 cm. The length of the hypotenuse in centimetres is

  A 11 **B** 41 **C** 49 **D** 121 **E** 1681

- Which of the following is a Pythagorean triple?

  A 8, 12, 15 **B** 7, 20, 25 **C** 8, 15, 17 **D** 7, 12, 13 **E** 6, 9, 12

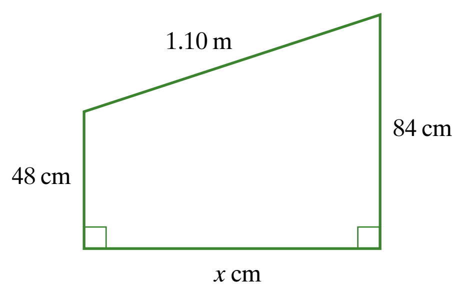
- The value of x in this right-angled triangle is



  A 2 cm **B** 4 cm **C** 16 cm **D** $\sqrt{32}$ cm **E** $\sqrt{130}$ cm

- The value of x is closest to

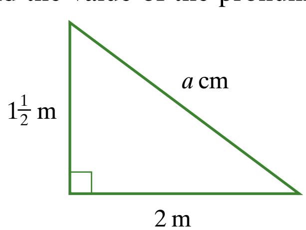
  A 36 **B** 42
  C 66 **D** 88
  E 104



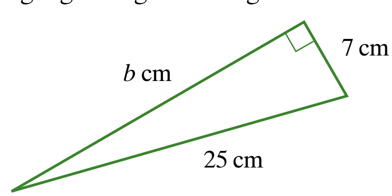
Short-answer questions

- Find the value of the pronumerals in the following right-angled triangles.

  a



  b



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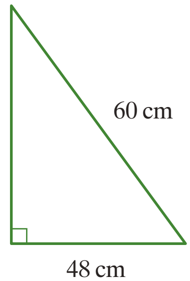
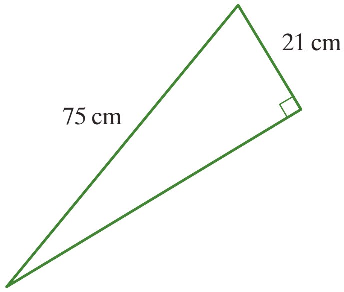
Find the unknown side lengths in these right-angled triangles.

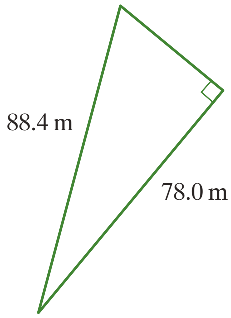
a



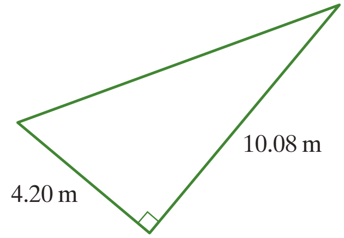
b



c

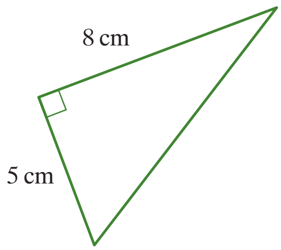


d

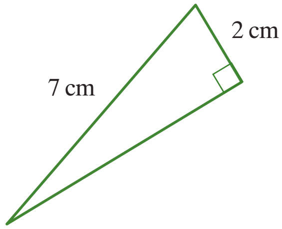


Find the unknown side lengths in these right-angled triangles. Give your answer correct to one decimal place.

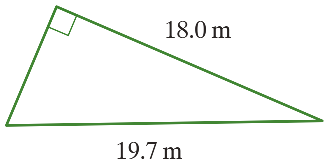
a



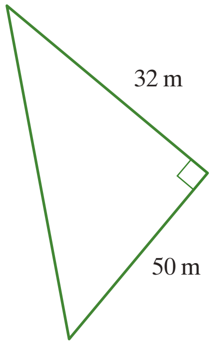
b



c



d



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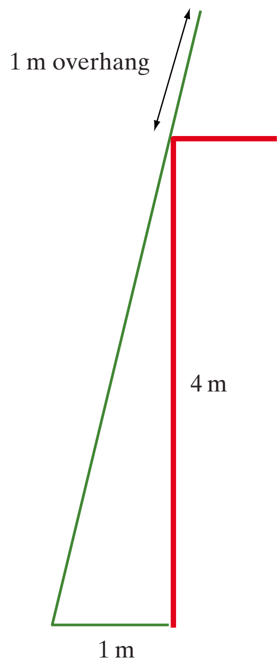
Find the length of the diagonal of these rectangles correct to one decimal place.

A ladder leans against a wall as shown in the diagram below. If the ladder is required to reach 1 metre above the roof for safety as shown, what total length does the ladder need to be? Give the length in metres correct to one decimal place.



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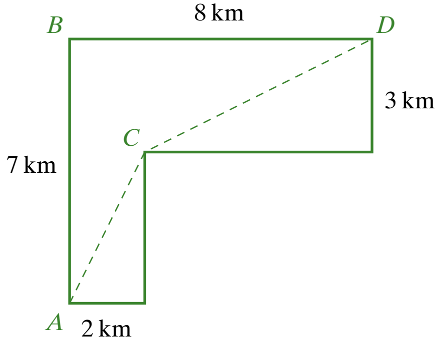
Extended-response questions

The diagram below shows an L-shaped forest, with measurements as marked. There is a track around the edge from A to D via B, but there is also a track that goes directly from A to D via C.



a What is the direct distance from A to C? Give your answer in exact form.

b What is the direct distance from C to D? Give your answer in exact form.

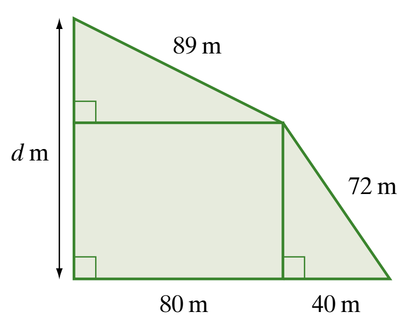
c What is the difference between the distance from A to D via C, and the distance from
A to D via B? Give your answer

i in exact form. **ii** correct to one decimal place.

The diagram shows a block of land.

a Find the missing dimension, d m correct to the nearest metre.

b Calculate the area of the land to the nearest square metre.



Practice quiz
Chapter 2

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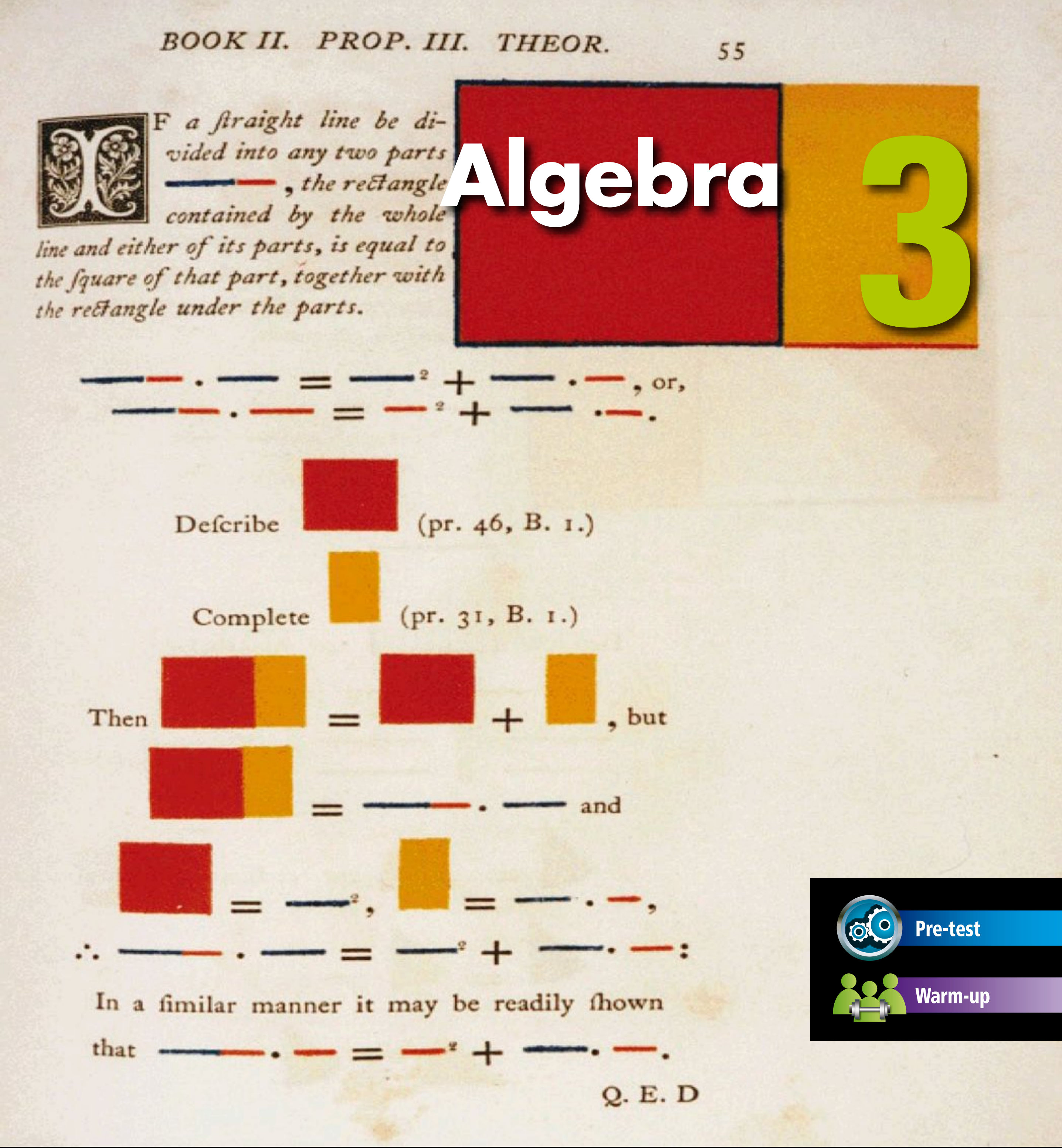
Algebra 3



- Pre-test
- Warm-up

The page shown here is from a 19th century illustrated version of Euclid's famous set of 13 books titled The Elements. Euclid was an ancient mathematician who lived about 2500 years ago in the city of Alexandria, which was then part of the Greek empire. The pages show a proof of the Distributive law using areas of rectangles. The whole area is made up of the red square plus the yellow rectangle. In the diagram a multiplication sign is represented by a dot. In algebra we could write $(a + b) = a^2 + ab$.

3.1 Working with pronumerals

BLM
Chapter warm-up

An algebraic expression contains pronumerals. A pronumeral is a letter used in place of a number. For example, a might be used to represent an **unknown number** and 5 less than a would be represented by the **expression** $a - 5$.

If we let $a - 5 = 20$ we now have an **equation** that can be solved. The presence of the equals sign makes this an equation rather than an expression. When we solve this equation we find the value of a that makes the equation true. In this case, $a = 25$. If we evaluate $a - 5$ when $a = 25$, we get 20.

Pronumerals represent numbers so they can be added, subtracted, multiplied and divided in the same way as numbers. When multiplying pronumerals we leave out the multiplication signs so $6 \times x \times y$ is written as $6xy$.

Each part of an algebraic expression is referred to as a term. In the expression $6xy + 5y - 2$ there are three terms; $6xy$, $5y$ and -2. The number in front of a term is referred to as the coefficient of that term. So, in the term $6xy$, the number 6 is the coefficient and in the term $5y$, the number 5 is the coefficient. A constant term does not contain any pronumerals so -2 is the constant term in this expression.

If a pronumeral can take on different values, it is called a **variable**. In the equation $a = 2b + 5$, both a and b are variables.

Example 1

a Simplify the following expression $2 \times p + 3 \times q$.
b How many terms are there in the expression $5p + 2ab - 7q - 6$?
c What is the constant term in the expression in part **b**?
d What is the coefficient of p in the expression in part **b**?
e What is the coefficient of q in the expression in part **b**?

Working	Reasoning
a $2 \times p + 3 \times q = 2p + 3q$	Remove multiplication signs.
b There are four terms in the expression $5p + 2ab - 7q - 6$.	Count terms. The terms are separated by + or −.
c In $5p + 2ab - 7q - 6$ the constant term is -6	The constant term is the term that does not contain a pronumeral.
d The coefficient of p in $5p + 2ab - 7q - 6$ is 5	This is the number multiplying p.
e The coefficient of q in $5p + 2ab - 7q - 6$ is -7	The coefficient of q is the number multiplying q. This includes the sign.

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Simplifying expressions

Like terms

Like terms are terms which contain the same pronumeral part, for example,

- $5x$ and $6x$ are like terms but $5x$ and $6y$ are unlike terms.
- $2ab$ and $-3ab$ are like terms but $2ab$ and $-3a$ are unlike terms.
- $6x^2$ and $8x^2$ are like terms but $6x^2$ and $8x$ are unlike terms.
- $ab = ba$

Example 2

Identify the like terms in the following: $3x$, $-5x^2$, $2xy$, $-5y$, 10, $-8yx$, $6x^2$, 2

Working	Reasoning
$-5x^2$ and $6x^2$ are like terms.	These two terms are like terms because they both contain x^2.
$2xy$ and $-8yx$ are like terms.	These two terms are like terms because they both contain xy. Note that $xy = yx$.
10 and 2 are like terms.	These two terms are like terms because they are both constants.

Collecting like terms

Simplifying an expression by addition and subtraction of algebraic terms is possible only if there are like terms.

Example 3

Simplify the following expressions.

a $2x^2 + 5xy - 3x^2 + 10xy$

b $4a^2b - 2a + 3ab^2 - 5a$

Working	Reasoning
a $2x^2 + 5xy - 3x^2 + 10xy$ $= 5xy + 10xy + 2x^2 - 3x^2$ $= 15xy - x^2$	Rearrange the expression to group the like terms. Simplify the expression, $5xy + 10xy = 15xy$ $2x^2 - 3x^2 = -x^2$
b $4a^2b - 2a + 3ab^2 - 5a$ $= 4a^2b - 2a - 5a + 3ab^2$ $= 4a^2b - 7a + 3ab^2$	Rearrange the expression to group the like terms. Simplify the expression. $-2a - 5a = -7a$ $4a^2b$ and $3ab^2$ are not like terms and thus cannot be added.

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Multiplying and dividing

When dealing with positive and negative pronumerals the same rules apply as for numbers; for example,

$-2 \times 3 = -6$ and $-a \times b = -ab$.

Similarly, $-2 \times (-3) = 6$ and $-a \times (-b) = ab$.

When dividing with pronumerals, it is easier to write the expression as a fraction; for example,

$$\begin{aligned} 12x \div 10 &= \frac{12x}{10} \\ &= \frac{6x}{5} \end{aligned}$$

Once again, the same rules apply for division with pronumerals as for division with numbers; for example, $\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$.

Example 4

Simplify.

a $2x \times 3y \times 2x$

b $5m \times (-3m)$

Working	Reasoning
a $2x \times 3y \times 2x$ $= 2 \times 3 \times 2 \times x \times x \times y$ $= 12x^2y$	Multiply the coefficients and multiply the pronumerals.
b $5m \times (-3m) = -15m^2$	Multiply the coefficients and multiply the pronumerals.

Example 5

Simplify the following expressions:

a $-32a \div 8$

b $\frac{35a}{5ab}$

Working	Reasoning
a $-32a \div 8 = \frac{-32a}{4}$ $= -4a$	Write as a fraction. Cancel the common factor 4.
b $\frac{35a}{5ab} = \frac{7}{b}$	Cancel common factors in the numerator and denominator.

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exercise 3.1

LINKS TO Example 1

1 For each of the following expressions, state

i the number of terms.
ii the coefficient of b.
iii the coefficient of c.
iv the constant term.

a $3a + 7b - 2c + 8$ b $3c^2 + b + 5c - 8$ c $3b^2 - b + 10$ d $a - b - 2c + 18$

2 The coefficient of x in the expression $3x^2 - x + 4xy$ is

A -1 B 1 C 2 D 3 E 4

3 Which one of the following statements about the expression $y - 3xy + 4x - 10$ is true?

A It contains three terms.
B It has only one variable.
C The coefficient of x is -3.
D The constant term is -10.
E Zero is the coefficient of y.

LINKS TO Example 2

4 Identify the like terms in each of the following sets.

a $3x, 6x, 7y, 8x$
b $y^3, 3y, -4y^2, -7y$
c $5d^2, 2d, -3d, d^3$
d $4gh, 5g, 7gh, -2h, -3gh$
e $a^2, 3a, -5a^2, 2a^3, -3$
f $4cd, 3d, 5dc, -4c, -8cd$
g $5, 5a, 5ab, 10ba, -15b$
h $xy, 2x, 4xy, -5yx, 3y$
i $23wx, 45zy, 14z, 18w, -3x, 2yz$
j $3rst, -4st, 5str, 17sr, -3srt, 2tr$
k $7, -4ab^2, 4a, 2a^2b, 9b$
l $h^2m, hm, 3mh^2, -2m^2h$

5 Which one of the following is *not* a pair of like terms?

A $-2x$ and x
B $-u^2v$ and $5vu^2$
C $3e^2$ and $-5e^2$
D $3a^2b$ and $4ab^2$
E $-7m^2np^2$ and $14m^2p^2n$

LINKS TO Example 3

6 Simplify each of the following.

a $5a + 3b + 2a + b$
b $4x + 10x + 12y + 2x$
c $9 - 9b + 17b - 6$
d $xy + 5x - 2xy - 8x$
e $5m^2 - 8n + 4m^2 - 6n$
f $-2k - 2km + 4k - 8km$
g $7 + 5k - 4k^3 - 3k + 8$
h $x^2 - 5x - 3x + 2$
i $8a^2 - 9ab + 4ab - 6b^2$
j $5x^4 - 3x^2 + 2 - 8x^2$
k $10ab - 4ba + 9ab$
l $4abc - 5acb + 2abc - 8bac$
m $2x^2y - 6xy^2 + 3x^2y + 9xy^2$
n $6kl + 2k + 5k - 8lk$
o $8cd + 2d - 5dc - 4d$
p $x^3 - 5x^2 + 3x - 7 + x^2 - 8x + 4$

LINKS TO Example 4

7 Simplify the following expressions.

a $3a \times 4ba \times 6ab$
b $10x \times 3y \times 2$
c $2a \times 2b \times 3c$
d $-2a \times (-3ab) \times 2b$
e $7ab \times (-5a)$
f $-5mn \times 3m^2$
g $2a \times 3b \times 5ab$
h $3x \times 2y$
i $10p \times 5q$
j $4 \times (-5a) \times (-2b) \times a$

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k $6ab \times 2bc \times 2 \times a$

l $5m \times 2mn \times 3mn$

m $2ab \times (-3bc) \times 2cd \times (-ad) \times (-bd)$

n $4x \times 5xy \times 2xz$

o $2xy \times (-4y) \times 3x^2$

p $3a \times 3a \times 2b \times 3a$

LINKS TO Example 5

Simplify each of the following.

a $56x \div 8$

b $3x^2 \div 3$

c $3x^2 \div x$

d $28x^2 \div 4x$

e $\dfrac{12a}{4}$

f $\dfrac{27a}{9ab}$

g $\dfrac{15xy}{-5y}$

h $\dfrac{12x}{2x}$

i $\dfrac{45a^2}{-9a}$

j $\dfrac{-72m}{12}$

k $\dfrac{-54x^2y}{-9y}$

l $\dfrac{36a^2b^2}{4ab}$

exercise 3.1 challenge

Simplify each of the following.

a $\dfrac{15ab^2}{3a} \times \dfrac{24a^2b^2}{8b}$

b $\dfrac{45xy}{4x^2} \times \dfrac{24x^2y}{10y^2}$

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3.2 Formulas and substitution

Algebraic expressions and rules involve numbers, **pronumerals** and operations. A formula is a rule for calculating a particular quantity. For example, $C = \pi d$ is the formula for calculating the circumference, C, of a circle if we know the diameter, d.

When working with expressions and rules, we often need to **evaluate** an expression for particular values of the pronumerals in the expression. Replacing the pronumerals with particular numbers is called **substitution**.

Example 6

Substitute $a = 2$, $b = 5$ and $c = -3$ to find the value of the following expressions.

a $a + b + c$ **b** $3a + b$ **c** ac
d $a^2 - 2c$ **e** $3a^2b^2c$

Working	Reasoning
a $a + b + c$ $= 2 + 5 + (-3)$ $= 4$	Replace each pronumeral with its numerical value.
b $3a + b$ $= 3 \times 2 + 5$ $= 11$	Remember that $3a$ means $3 \times a$. Be careful with the order of operations.
c ac $= 2 \times (-3)$ $= -6$	The product of a positive number and a negative number is negative.
d $a^2 - 2c$ $= 2^2 - 2 \times (-3)$ $= 4 + 6$ $= 10$	The product of two negative numbers is positive: $-2 \times (-3) = +6$.
e $3a^2b^2c$ $= 3 \times 2^2 \times 5^2 \times (-3)$ $= 3 \times 4 \times 25 \times (-3)$ $= -900$	Be careful with the negative signs and the order of operations.

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Example 7

Complete each table of values.

a $y = 3x - 7$

x	-2	0	1	3
y				

b $a = 10 - 2b$

b	-5	-1	0	3
a				

c $s = 20t - 5t^2$

t	0	1	2	4
s				

Working

a $y = 3x - 7$

x	-2	0	1	3
y	-13	-7	-4	2

Reasoning

Substitute each value in turn for x.
When $x = -2$, $y = 3 \times (-2) - 7 = -13$,
when $x = 0$, $y = 3 \times 0 - 7 = -7$,
and so on.

b $a = 10 - 2b$

b	-5	-1	0	3
a	20	12	10	4

Substitute each value in turn for b.
When $b = -5$, $a = 10 - 2 \times (-5) = 20$,
and so on.

c $s = 20t - 5t^2$

t	0	1	2	4
s	0	15	20	0

Substitute each value in turn for t.
When $t = 0$, $s = 20 \times 0 - 5 \times 0^2 = 0$,
when $t = 1$, $s = 20 \times 1 - 5 \times 1^2 = 15$,
and so on.

Formulas

You should already be familiar with some formulas. A **formula** is a rule that is used for calculating a particular quantity. For example, the area of a circle with radius r is given by the formula $A = \pi r^2$, and the circumference is given by $C = 2\pi r$.

We can use these formulas to find the area or circumference of a circle, given its radius.

Example 8

Find the area of a circle with radius

a 3 m

b 12 cm

Give your answers as exact values (in terms of π), correct to one decimal place.

continued

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Example 8 continued

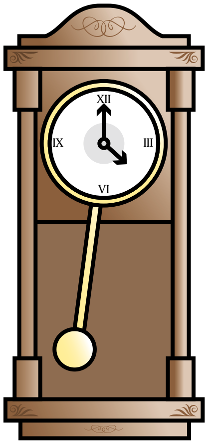
Working	Reasoning
a $A = \pi r^2$ $A = \pi \times 3^2$ $= \pi \times 9$ $= 9\pi$	Identify the formula you need to use. Substitute $r = 3$.
The exact area of the circle is 9π m^2.	The mathematical convention is to write the coefficient (in this case, 9) before the pronumeral or symbol.
$A = 9\pi$ ≈ 28.27 m^2 The area is 28.3 m^2 correct to one decimal place.	Use the π key on your calculator. Write to one more decimal place than you require. Then round.
b $A = \pi r^2$ $A = \pi \times 12^2$ $= \pi \times 144$ $= 144\pi$	Identify the formula you need to use. Substitute $r = 12$.
The exact area of the circle is 144π cm^2. $A = 144\pi$ ≈ 452.38 cm^2 The area is 452.4 cm^2 correct to one decimal place.	Use the π key on your calculator. Write to one more decimal place than you require. Then round.

Example 9

The time, T seconds, that the bob of a pendulum takes to swing and return to its starting point is related to the length, l metres, of the pendulum by the formula

$$T = 2\pi\sqrt{\frac{l}{9.8}}$$

What is the difference in the time taken for one swing of a pendulum of length 1 m and a pendulum of length 40 cm? Give your answer to the nearest tenth of a second.



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Example 9 continued

Working	Reasoning
For the pendulum of length 1 m	Identify the formula you need to use.
$T = 2\pi\sqrt{\dfrac{l}{9.8}}$	
Substitute $l = 1$,	Substitute the value of the variable.
$T = 2\pi\sqrt{\dfrac{l}{9.8}}$	
$T = 2.007$	Use a calculator to evaluate. At this stage, work to several more decimal places than you need in the final answer.
For the pendulum of length 40 cm,	
$T = 2\pi\sqrt{\dfrac{l}{9.8}}$	Begin with the formula again.
Substitute $l = 0.4$,	Convert 40 cm to metres.
$T = 2\pi\sqrt{\dfrac{0.4}{9.8}}$ $T = 1.269$	Substitute the value of the variable. Evaluate to several decimal places.
Time difference $= 2.007 - 1.269$ $= 0.738$ The difference in time correct to one tenth of a second is 0.7 seconds.	Subtract to find the difference between the times. Round your answer.

exercise 3.2

LINKS TO Example 6

1. Evaluate each of the following if $a = -5$ and $b = 3$.

a $3a - 4b$ **b** $-4a + 2b$ **c** $5a - 7b$ **d** $5ab - b$

e $\dfrac{a+b}{2}$ **f** $2a(a - b)$ **g** $-a(b - a)$ **h** $(a + b)(a - b)$

i $a^2 - b^2$ **j** $2a^2 - 3b^2$ **k** $a^2 + 2ab + b^2$ **l** $3a^2 - ab - 2b^2$

2. Find the value of the unknown in each of the following.

a $y = 3x + 5$ where $x = 10$ **b** $v = 15 - 10t$ where $t = 0.3$

c $s = 12t + 5t^2$ where $t = 3$ **d** $f = \dfrac{9c}{5} + 32$ where $c = 35$

3. If $y = 2x(x - z)$, the value of y when $x = 5$ and $z = 2$ is

A 13 **B** 21 **C** 30 **D** 48 **E** 70

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LINKS TO Example 7

Copy and complete each table of values.

a $y = 2x + 3$

x	–4	–3	–1	0	2
y					

b $y = 5 - 4x$

x	–3	–1	0	2	5
y					

c $y = -3x + 5$

x	–8	–5	–1	0	4
y					

d $y = x^2 + 4$

x	–3	–1	0	1	3
y					

e $y = x^2 - 3$

x	–5	–2	0	1	2
y					

f $y = x^2 + x + 4$

x	–6	–4	–2	0	2
y					

g $y = 2x^2 - 3x + 5$

x	–2	–1	0	1	2
y					

h $y = \dfrac{1}{x}$

x	0.1	0.2	0.5	2	10
y					

i $y = \dfrac{3}{x+1}$

x	–2	0	2	5	9
y					

j $y = \dfrac{6}{2-x}$

x	–10	–1	0	1	3
y					

LINKS TO Example 8

The exact area of a circle with diameter 8 cm is

A 8π cm^2 **B** 16π cm^2 **C** $8\pi^2$ cm^2 **D** $16\pi^2$ cm^2 **E** 64π cm^2

LINKS TO Example 9

The formula $T = 2\pi\sqrt{\dfrac{l}{9.8}}$, represents the time taken for a pendulum of length l m to complete one swing. The difference in time taken for a pendulum of length 1 m and a pendulum of length 60 cm to complete one swing is closest to

A 4.52 seconds. **B** 1.27 seconds. **C** 0.46 seconds. **D** 0.45 seconds. **E** 0.14 seconds.

If $I = PRT$, find I when

a $P = 1000$, $R = 0.12$ and $T = 5$ **b** $P = 550$, $R = 0.08$ and $T = 20$

Find the value of the unknown in each of the following.

a $P = r(\pi+2)$ where $r = 4.5$ **b** $V = \dfrac{4\pi r^3}{3}$ where $r = 5.6$

c $S = 4\pi r^2$ where $r = 8.4$ **d** $r = \sqrt{\dfrac{A}{\pi}}$ where $A = 60$

If $A = \dfrac{(a+b)h}{2}$, find A when

a $a = 18.5$, $b = 10.5$, $h = 11$ **b** $a = 7.2$, $b = 16.4$, $h = 8.3$

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- If $s = ut + \frac{gt^2}{2}$, find s when
  - **a** $u = 0, t = 5, g = 9.8$
  - **b** $u = 10, t = 4, g = 9.8$
- If $\Delta = b^2 - 4ac$, find Δ when
  - **a** $a = 2, b = -5, c = 3$
  - **b** $a = 3, b = 7, c = -5$
  - **c** $a = -3, b = 4, c = 2$
  - **d** $a = -4, b = -1, c = 0$
  - **e** $a = -1, b = -2, c = -1$
  - **f** $a = 1, b = -5, c = 3$
  - **g** $a = -1, b = 4, c = 4$
  - **h** $a = 3, b = -2, c = 5$

exercise 3.2 challenge

- A sequence of numbers that changes in equal steps (for example, 1, 4, 7, 10, 13, …) is called an **arithmetic sequence**. Each number in the sequence is called a **term**. The formula for the nth term of an arithmetic sequence is $t = a + (n - 1)d$, where a is the first term and d is the difference between consecutive terms.
  - **a** Find the 20th term of the arithmetic sequence 1, 4, 7, 10, 13, …
  - **b** Find the 32nd term of the arithmetic sequence 5, 11, 17, 23, …
  - **c** Find the 15th term of the sequence 3, 14, 25, 36, 47, …
  - **d** Find the 50th term of the sequence 1, 2, 3, 4, 5, …
- The formula for the sum S of the first n terms of an arithmetic sequence is $S = \frac{n}{2}(2a + (n - 1)d)$, where a is the first term and d is the difference between consecutive terms.
  - **a** Find the sum of the numbers 5, 13, 21, 29, 37, 45, 53, 61.
  - **b** Find the sum of 25 terms of the sequence 2, 5, 8, 11, … .
  - **c** Find the sum of the first 100 positive integers.
  - **d** Find the sum of the first 200 even integers.

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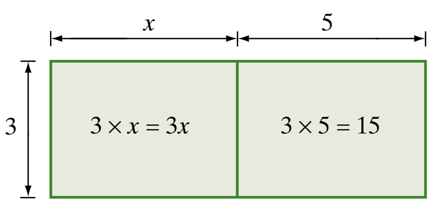
3.3 Expansion and the distributive law

Equivalent expressions give the same result for every value of the pronumerals. For example, $3(x + 5)$ is equivalent to $3x + 15$ for any value of x. We refer to $3(x + 5)$ as the **factorised form** of the expression and $3x + 15$ as the **expanded form**.

The process of going from the factorised form to the expanded form of an expression is called **expansion**.

We can use a rectangle area diagram to show visually that $3(x + 5)$ and $3x + 15$ are equivalent expressions. The width of the rectangle below is $x + 5$ and its height is 3, so the area is $3(x + 5)$. By dividing the rectangle into two regions as shown, we see that the area can also be written as $3x + 15$.



Distributive law

This is an example of the **distributive law** for multiplication, where 3 is distributed over x and 5.

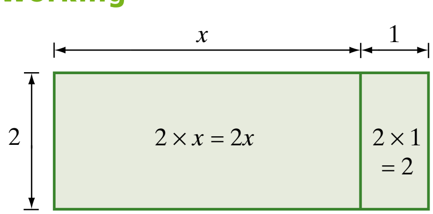
$$\begin{aligned} 3(x + 5) &= 3 \times x + 3 \times 5 \\ &= 3x + 15 \end{aligned}$$

Factorised form Expanded form

Example 10

Use a rectangle area diagram to show that $2(x + 1)$ and $2x + 2$ are equivalent expressions.

Working



$2(x + 1) = 2x + 2$

Reasoning

The width of this rectangle is $x + 1$ and its height is 2. The area of the rectangle is $2(x + 1)$. By dividing the rectangle into two regions as shown, we see that the area can also be written as $2x + 2$. So $2x + 2$ is an equivalent expression to $2(x + 1)$.

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Example 11

Expand each of the following.

a $3(2x + 1)$ **b** $5(3x - 2)$ **c** $a(6 - 5a)$

d $-2(b + 7)$ **e** $-5(2b - 3)$

Working	Reasoning
a $3 \times (2x + 1)$ $= 3 \times 2x + 3 \times 1$ $= 6x + 3$	$3(2x + 1) = 3 \times (2x + 1)$ Use the distributive law. Multiply each term in the brackets by the number before the bracket. Simplify the terms.
b $5(3x - 2)$ $= 5 \times 3x - 5 \times 2$ $= 15x - 10$	Multiply each term in the brackets by 5. Simplify the terms.
c $a(6 - 5a)$ $= a \times 6 - a \times 5a$ $= 6a - 5a^2$	Multiply each term in the brackets by a.
d $-2(b + 7)$ $= -2b - 14$	Multiply each term in the brackets by -2. When you can do this mentally, you can leave out the step $-2 \times b - 2 \times 7$.
e $-5(2b - 3)$ $= -10b + 15$ $-5 \times (-3) = 15$	Multiply each term in the brackets by -5.

After expanding, any like terms should be collected.

Example 12

Expand and simplify each of the following.

a $5(2x - 3) + 7x - 2$ **b** $10(y + 2) - 4(2y + 1)$ **c** $3(2w - 5) - 2(w - 1)$

Working	Reasoning
a $5(2x - 3) + 7x - 2$ $= 10x - 15 + 7x - 2$ $= 17x - 17$	Multiply each term in the brackets by 5. Simplify the expression by grouping like terms.

continued

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Example 12 continued

Working

b $10(y + 2) - 4(2y + 1)$
$= 10y + 20 - 8y - 4$
$= 2y + 16$

$-4 \times 1 = -4$

Reasoning

Multiply each term in the first set of brackets by 10 and each term in the second set of brackets by –4.

Simplify the expression by grouping like terms.

c $3(2w - 5) - 2(w - 1)$
$= 6w - 15 - 2w + 2$
$= 4w - 13$

$-2 \times (-1) = +2$

Multiply each term in the first set of brackets by 3 and each term in the second set of brackets by –2.

Simplify the expression by grouping like terms.

Example 13

Expand and simplify each of the following expressions.

a $6x(2y - 1) - (4xy - 3x)$

b $10a(5a + 3) - 6a + 3(4 - 3a) - 17$

Working

a $6x(2y - 1) - (4xy - 3x)$
$= 12xy - 6x - 4xy + 3x$

Reasoning

Multiply each term in the first set of brackets by $6x$. Although there is no number before the second bracket, the subtraction sign indicates that each term must be multiplied by –1.

$$-(4xy - 3x) = -4xy - (-3x)$$
$$= -4xy + 3x$$

Alternatively you could write $-(4xy - 3x) = -1(4xy - 3x)$ and expand by multiplying each term in the brackets by –1.

$= 8xy - 3x$

Simplify the expression by grouping like terms.

b $10a(5a + 3) - 6a + 3(4 - 3a) - 17$
$= 50a^2 + 30a - 6a + 12 - 9a - 17$

Multiply each term in the first set of brackets by $10a$ and each term in the second set of brackets by 3.

$= 50a^2 + 15a - 5$

Simplify the expression by grouping like terms.

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exercise 3.3

LINKS TO Example 10

1. Use rectangle diagrams to show equivalent expressions for each of the following.
   a $4(x + 3)$ **b** $2(x + 2)$ **c** $3(2x + 1)$

LINKS TO Example 11

2. Expand each of the following.
   a $2(x + 4)$ **b** $3(a - 5)$ **c** $4(2b - 1)$
   d $6(3m + 2)$ **e** $-3(x + 2)$ **f** $-a(a + 3)$
   g $3(x - 2)$ **h** $5(2a + 1)$ **i** $p(p + 3)$
   j $2x(3 - 4x)$ **k** $-2(a + 6)$ **l** $-m(2m + 7)$
   m $-h(h - 7)$ **n** $-3(5 - 3x)$ **o** $xy(x + 2y)$

LINKS TO Example 12

3. Expand and simplify each of the following.
   a $3(a + 6) + 2a - 5$ **b** $2(b + 5) + 3(b - 3)$
   c $3(a - 6) + 4(a - 8)$ **d** $5(c - 2) + 2(c + 5)$
   e $5(x + 1) - 2(x + 2)$ **f** $6(2y - 3) - 2(3y + 1)$
   g $4(2d + 3) - 3(d - 5)$ **h** $7(x - 5) + 2(x + 3)$
   i $3(x + 9) - 5(2 + x)$ **j** $5(3 - 2a) - 4(6 - 3a)$
   k $-3(2k + 7) - 5(k - 8)$ **l** $-4(n - 9) - (n - 15)$

LINKS TO Example 13

4. Expand and simplify each of the following.
   a $2p(p - 7) + 3p^2 - 4$ **b** $k(k + 7) + 2(k + 2)$
   c $a(3 - 4a) + a(2a - 5)$ **d** $3p(2p - 7) + 3(9 - 4p)$
   e $d(d - 2) - 3(2d + 5)$ **f** $p(5 + 2p) - 3(4 - p)$
   g $x(2x + 3) - 5(x - 4)$ **h** $2x(x - 7) - 3(5x - 2)$
   i $2p(3q - p) - q(4p - q)$ **j** $2a(4a - 3) - a(7 - a)$
   k $-2n(n - 9) - 5(3n + 4)$ **l** $4x(2y - 3x) - y(x - 4y)$

5. A correct simplification of $3(x - 5) - 2(x - 3)$ is
   A $x - 9$ **B** $x - 21$ **C** $x + 9$ **D** $5x - 9$ **E** $5x - 21$.

6. $2x(3x - 2) - (x + 5)$ simplifies to
   A $6x^2 - 5x - 5$ **B** $6x^2 - 5x + 5$ **C** $6x^2 - 4x - 5$
   D $6x^2 - 4x + 5$ **E** $6x^2 - 3x + 5$.

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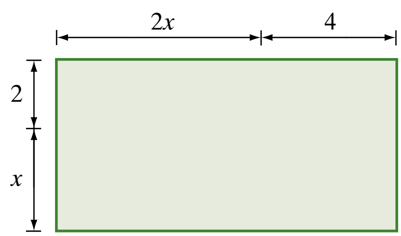
For each rectangle,

i write an expression for the perimeter.

ii simplify by grouping like terms.

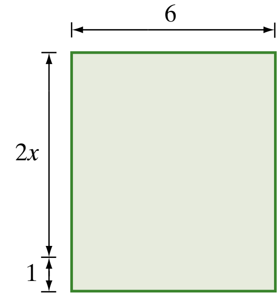
iii state whether the rectangle has the same perimeter for all values of x.

a



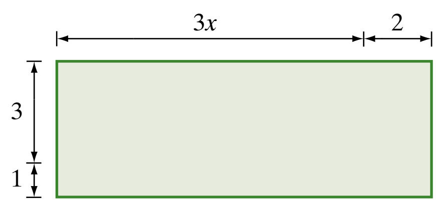
Rectangle A

b



Rectangle B

c



Rectangle C

exercise 3.3 challenge

For each of the following, find the values of a and b that makes the equation true.

a $3x(x + 2) = ax^2 + bx$

b $x(x - 5) + 2(x + 1) = x^2 + ax + b$

c $a - bx = -2(x + 7)$

d $3x(x - 4) - x(4x + 1) = ax^2 + bx$

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3.4 Binomial expansion

Binomial expansions: a matching activity

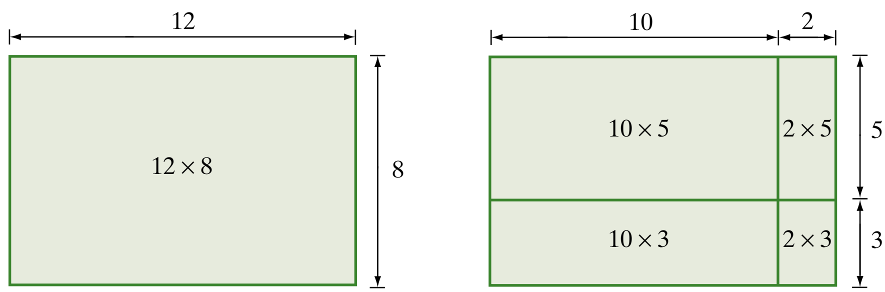
A **binomial expression** is an expression with two terms, for example, $x + 2$ and $2a - 5$ are both binomial expressions.

'bi' means two and 'nomial' refers to the names or terms.

Binomial expansion

In the previous section a binomial expression was multiplied by a single term, for example, $-3(x + 2)$ or $a(2a - 5)$. In this section we look at the multiplication of two binomial expressions. Using an area diagram with numbers shows a method that we can use with algebraic expressions.

On the left is a rectangle with area 12×8 square units. On the right, the width, 12, is represented as $10 + 2$, and the height, 8, is represented as $5 + 3$. There are now four regions within the 12×8 rectangle.



From this we can see that

$$\begin{aligned} 12 \times 8 &= (10 + 2)(5 + 3) \\ &= 10 \times 5 + 10 \times 3 + 2 \times 5 + 2 \times 3 \end{aligned}$$

Using the distributive law, we can see that this can also be written as

$$\begin{aligned} 12 \times 8 &= (10 + 2)(5 + 3) \\ &= 10(5 + 3) + 2(5 + 3) \end{aligned}$$

The same method is now used with the binomial expression $(x + 5)(x + 3)$. The area of the rectangle on the next page is $(x + 5)(x + 3)$.

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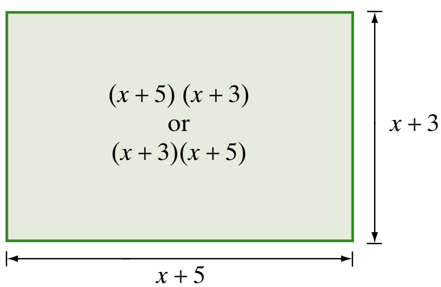
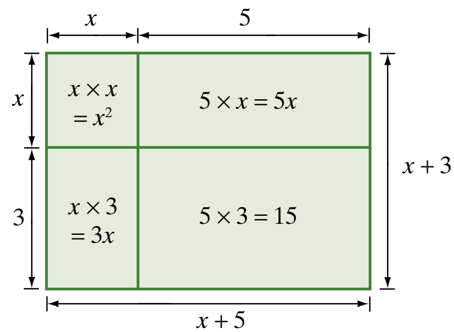
We can also calculate the area of the rectangle by dividing it into four smaller rectangles as shown. The area is equal to the sum of the four separate areas.





$$(x+5)(x+3) = x^2 + 3x + 5x + 15$$
$$= x^2 + 8x + 15$$

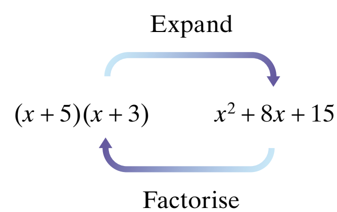
If we use the distributive law, we can write

$$(x+5)(x+3) = x(x+3) + 5(x+3)$$
$$= x^2 + 3x + 5x + 15$$
$$= x^2 + 8x + 15$$

The expansion of $(x + 5)(x + 3)$ also shows that the factors of $x^2 + 8x + 15$ are $(x + 5)$ and $(x + 3)$.

Expanding an expression is the reverse process of writing the expression as a product of factors.

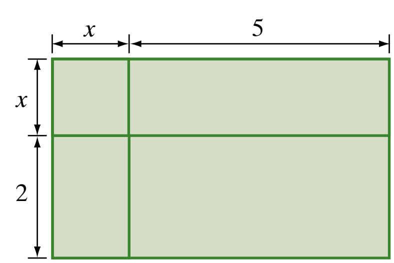
Expressions of the type $x^2 + 8x + 15$ are called **quadratic trinomial** expressions.



'Quadratic' comes from a Latin word meaning square or four and refers to the squared term, x^2. 'Trinomial' means that there are three terms.

Example 14

Use the diagram to expand $(x + 5)(x + 2)$.



continued

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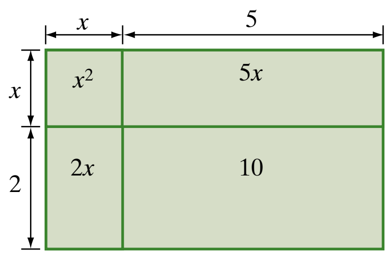
Example 14 continued

Working



$$(x + 5)(x + 2) = x^2 + 2x + 5x + 10$$
$$= x^2 + 7x + 10$$

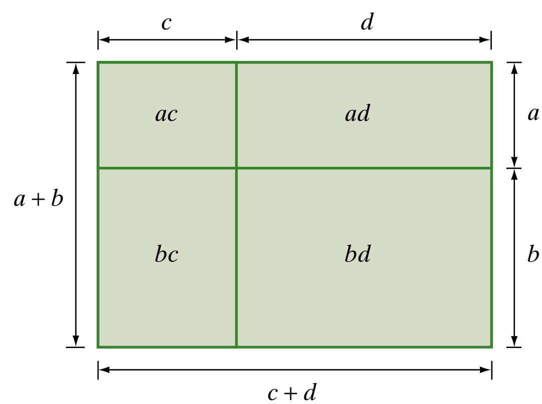
Reasoning

Calculate the areas of the four rectangular regions by multiplying the length by the width in each case.

$x \times x = x^2$
$x \times 5 = 5x$
$2 \times x = 2x$
$2 \times 5 = 10$

The expansion of $(x + 5)(x + 2)$ is equal to the sum of the four smaller areas.
Simplify by adding like terms.

We can use a diagram to find a general expression for the expansion of any binomial $(a + b)(c + d)$.



$$(a + b)(c + d) = ac + ad + bc + bd$$

We obtain the same result by applying the distributive law and multiplying each term in the second bracket by each term in the first bracket.

$$(a + b)(c + d) = a(c + d) + b(c + d)$$
$$= ac + ad + bc + bd$$

This results in four terms. Sometimes, there are like terms which can be collected.

General binomial expansion

$$(a + b)(c + d) = a(c + d) + b(c + d)$$
$$= ac + ad + bc + bd$$

Example 15

Expand and simplify each of the following.

a $(x + 5)(x + 2)$ **b** $(a + 2)(a - 4)$ **c** $(2y - 7)(y - 4)$

continued

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Example 15 continued

Working

a $(x+5)(x+2) = x(x+2) + 5(x+2)$
$= x^2 + 2x + 5x + 10$
$= x^2 + 7x + 10$

b $(a+2)(a-4) = a(a-4) + 2(a-4)$
$= a^2 - 4a + 2a - 8$
$= a^2 - 2a - 8$

c $(2y-7)(y-4) = 2y(y-4) - 7(y-4)$
$= 2y^2 - 8y - 7y + 28$
$= 2y^2 - 15y + 28$

Reasoning

a Multiply each term in the first bracket by each term in the second bracket.
Expand $x(x+2)$ and $5(x+2)$.
Simplify by collecting like terms:
$2x + 5x$.

b Multiply each term in the second bracket by each term in the first bracket.
Simplify by collecting like terms:
$-4a + 2a$.

c Multiply each term in the second bracket by each term in the first bracket.
Simplify by collecting like terms:
$-8y - 7y$.

FOIL

The word FOIL helps you remember how to expand a binomial expression.

Firsts: multiply the first term in each bracket

Outers: multiply the outer terms (the first term in the first bracket and the last term in the last bracket)

Inners: multiply the inner terms (the last term in the first bracket and the first term in the last bracket)

Lasts: multiply the last term in each bracket

$(a+b)(c+d)$

Example 16

Use FOIL to help expand each of the following. Simplify where possible.

a $(x+2)(y+3)$ b $(x+4)(2x+1)$
c $(a-3)(a+7)$ d $(2z-3)(z-4)$

Working

a $(x+2)(y+3) = xy + 3x + 2y + 6$

Reasoning

$(x+2)(y+3)$

continued

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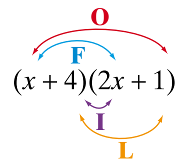
Example 16 continued

Working

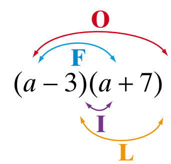
b $(x + 4)(2x + 1) = 2x^2 + x + 8x + 4$
$= 2x^2 + 9x + 4$

Reasoning



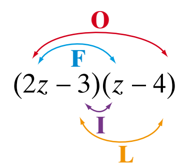
Collect like terms: $x + 8x$.

c $(a - 3)(a + 7) = a^2 + 7a - 3a - 21$
$= a^2 + 4a - 21$



Collect like terms: $7a - 3a$.

d $(2z - 3)(z - 4) = 2z^2 - 8z - 3z + 12$
$= 2z^2 - 11z + 12$



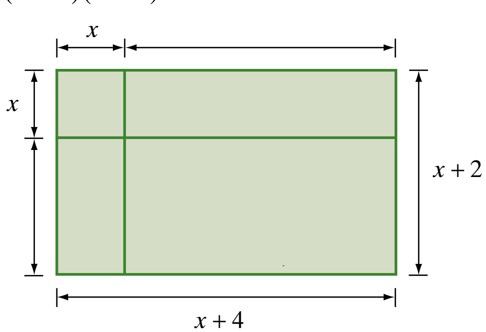
Collect like terms: $-8z - 3z$.

exercise 3.4

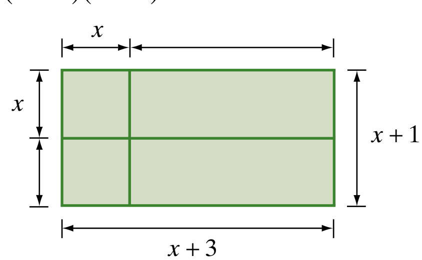
LINKS TO Example 14

1 For each of the following, copy and complete the diagram and use it to find the expansion.

a $(x + 2)(x + 4)$



b $(x + 1)(x + 3)$



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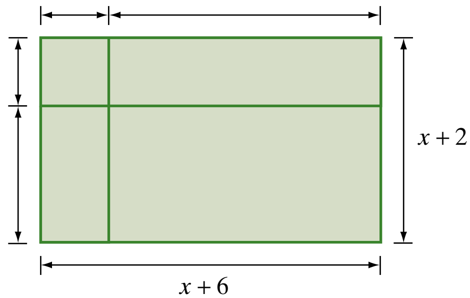
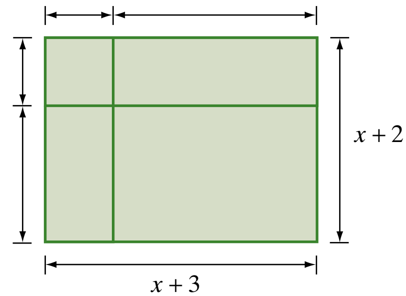
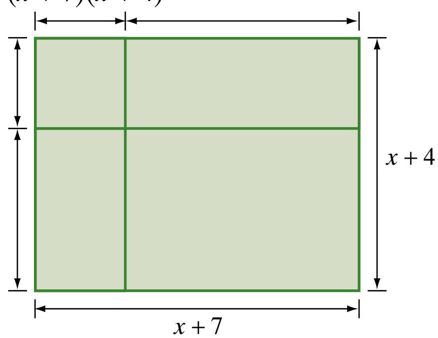
c $(x + 6)(x + 2)$



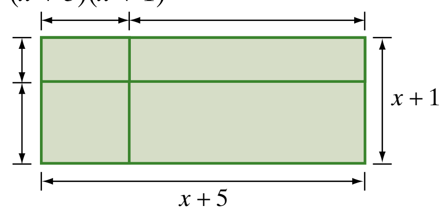
d $(x + 3)(x + 2)$



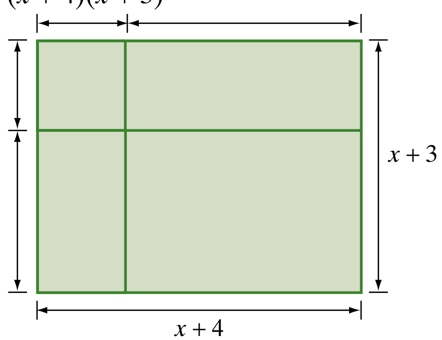
e $(x + 7)(x + 4)$



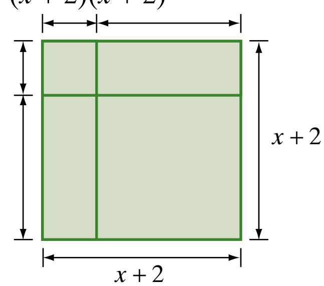
f $(x + 5)(x + 1)$



g $(x + 4)(x + 3)$



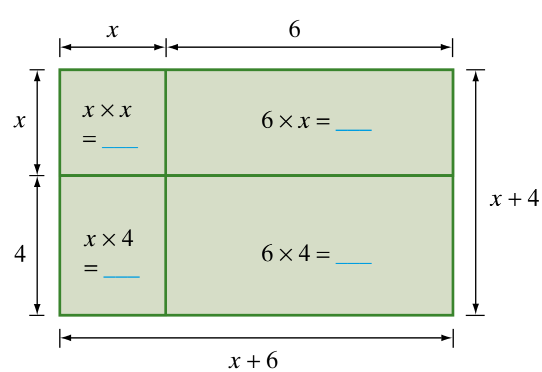
h $(x + 2)(x + 2)$



Consider the diagram at right.

a Copy and complete the diagram. Use it to find the expansion of $(x + 4)(x + 6)$.

b Can you expand $(d + 3)(d + 4)$ without using a diagram?



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LINKS TO Example 15

Expand and simplify each of the following expressions.

a $(x + 1)(x + 2)$ **b** $(y + 3)(y + 6)$ **c** $(z + 4)(z + 8)$ **d** $(a + 5)(a + 6)$
e $(b - 5)(b + 2)$ **f** $(c - 7)(c + 1)$ **g** $(d - 3)(d + 5)$ **h** $(e - 5)(e + 3)$
i $(f + 4)(f - 1)$ **j** $(g + 7)(g - 9)$ **k** $(h - 2)(h - 5)$ **l** $(k - 7)(k - 1)$
m $(l - 4)(l - 7)$ **n** $(m - 3)(m - 6)$ **o** $(x - 4)(x + 5)$ **p** $(x - 7)(x - 2)$

LINKS TO Example 16

Use **FOIL** to expand each of the following.

a $(a + 2)(b + 1)$ **b** $(s + 2)(t + 4)$ **c** $(m - 3)(n + 2)$ **d** $(x + 3)(y - 2)$
e $(a + b)(x + y)$ **f** $(m - n)(p + 2q)$ **g** $(c + d)(g - h)$ **h** $(2m + n)(a - b)$
i $(3k - 1)(2l - 1)$ **j** $(2p - 3)(4 - q)$ **k** $(ab - 3)(b + 2c)$ **l** $(a + 2b)(3c - 2d)$

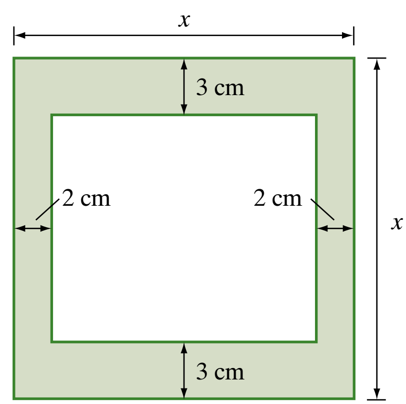
Expand and simplify each of the following expressions.

a $(2a + 1)(a + 3)$ **b** $(3b + 2)(2b + 5)$ **c** $(3c - 4)(c + 2)$ **d** $(d + 5)(4d - 1)$
e $(3e - 7)(2e - 3)$ **f** $(4f - 3)(f - 3)$ **g** $(7 - 4g)(g + 2)$ **h** $(3 - 5g)(1 - g)$
i $(2x + 3)(2x - 1)$ **j** $(3a - 4)(a + 5)$ **k** $(b - 4)(2b + 7)$ **l** $(5x - 7)(2x + 3)$

In the expansion of $(5 + 3a)(2a + 3)$, the coefficient of a^2 is

A 15 **B** 9 **C** 7 **D** 3 **E** 6.

A square panel is made up of a rectangular pane of glass surrounded by a wooden frame as shown in the diagram.



a Write expressions for the length and width of the glass pane, in terms of x.
b Write an expression for the area of the glass pane, using expanded form.
c Find the area of the glass when $x = 30$ cm.

A square of side length x m has its length increased by 2 metres and its width increased by 5 metres.

a Draw a diagram of the new shape, labelling the length and width.
b What is the area of the new shape?
c What is the change in the area?

Expand and simplify each of the following expressions.

a $(x - 2)(x - 3)$ **b** $(x - 2)(x + 3)$
c $(x + 2)(x - 3)$ **d** $(x + 2)(x + 3)$

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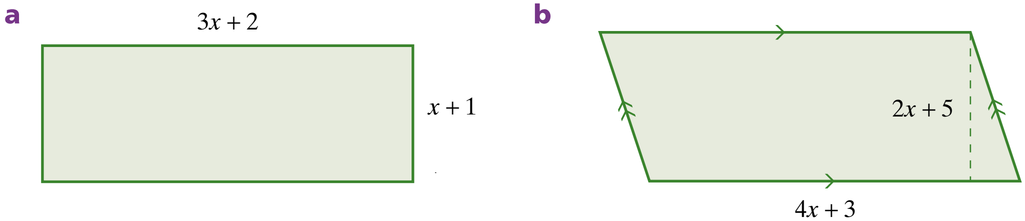
Find the areas of each of the following shapes giving your answers in expanded form.



A rectangular painting 10 cm long and 15 cm wide is surrounded by a frame of width x cm. Write an expression for

a the length of the picture, including the frame

b the width of the picture, including the frame

c the area of the picture, including the frame, using expanded form.

Jane expands $(x - 3)(x - 2)$ and obtains the result $x^2 - x + 6$. She checks her result by substituting $x = 0$ into each of the expressions and decides that she is correct.

a If Jane had chosen a different value for x, such as $x = 4$ would she still have thought she was correct? Explain.

b What is the correct expansion of $(x - 3)(x - 2)$?

exercise 3.4 challenge

Expand and simplify each of the following.

a $(x + 3)(x^2 + 2x + 1)$

b $(a - b)(a^2 + ab + b^2)$

c $(2p + 1)(p^2 + p - 1)$

d $(a^2 + 2a + 1)(a^2 - 2a + 1)$

e $(a^2 - b^2)(a^2 - 2ab + b^2)$

f $(y^2 + y + 2)(y^2 - 3y + 1)$

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3.5 Factorising expressions

In section **3.3** we expanded expressions that were written as the product of factors. In this section we are looking at the reverse process where we write expressions as the product of factors.

Prime numbers such as 7 can be written as the product of themselves and 1. All composite numbers have at least three factors. For example, the factors of 18 are 1, 2, 3, 6, 9, and 18. We can write 18 as 1×18, 2×9, 3×6 or $2 \times 3 \times 3$.

Algebraic expressions and terms also have factors and can be written as the product of their factors. For example, the factors of $3xy$ are 1, 3, x, y, $3x$, $3y$, xy and $3xy$. We can write $3xy$ as $1 \times 3xy$, $3 \times xy$, $3x \times y$ or $x \times 3y$.

Example 17

List all the factors of $2ab^2$.

Working	Reasoning
$2ab^2$ can be written as $1 \times 2ab^2$ $2 \times ab^2$ $a \times 2b^2$ $2a \times b^2$ $2ab \times b$ $2b \times ab$	Systematically write $2ab^2$ as a product of two factors until all combinations are exhausted.
The factors of $2ab^2$ are 1, 2, a, b, $2a$, $2b$, ab, $2ab$, b^2, $2b^2$, $2ab^2$	List the factors of $2ab^2$. There is no correct order.

Highest common factors

Supposing we want to know the highest common factor of 12 and 32. We can list all the factors of each number then see which is the largest number in both lists.

Factors of 12: 1, 2, 3, 4, 6, 12

Factors of 32: 1, 2, 4, 8, 16, 32

The highest common factor is 4.

We can use the same method to find the highest common factor of two algebraic terms, as shown here for $3xy$ and $6x^2y$.

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Factors of $3xy$: 1, 3, x, y, $3x$, $3y$, xy and $3xy$

Factors of $6x^2y$:

1, 2, 3, 6, x, $2x$, $3x$, $6x$, x^2, $2x^2$, $3x^2$, $6x^2$, y, $2y$, $3y$, $6y$,
xy, $2xy$, $3xy$, $6xy$, x^2y, $2x^2y$, $3x^2y$ and $6x^2y$

We see that the highest common factor (HCF) of $3xy$ and $6x^2y$ is $3xy$.

Example 18

Find the highest common factor of the following pairs of terms.

a $3b$ and 12
b $4x$ and $6x^2$
c $15x^2y$ and $20xy^2$

Working	**Reasoning**
a The HCF of $3b$ and 12 is 3.	Factors of $3b$: 1, 3, b, $3b$ Factors of 12: 1, 2, 3, 4, 6, 12
b The HCF of $4x$ and $6x^2$ is $2x$.	Factors of $4x$: 1, 2, 4, x, $2x$, $4x$ Factors of $6x^2$: 1, 2, 3, 6, x, $2x$, $3x$, $6x$, x^2, $2x^2$, $3x^2$, $6x^2$
c The HCF of $15x^2y$ and $20xy^2$ is $5xy$.	Factors of $15x^2y$ 1, 3, 5, 15, x, $3x$, $5x$, $15x$, x^2, $3x^2$, $5x^2$, $15x^2$, y, $3y$, $5y$, $15y$, xy, $3xy$, $5xy$, $15xy$, x^2y, $3x^2y$, $5x^2y$ and $15x^2y$ Factors of $20xy^2$ 1, 2, 4, 5, 10, 20, x, $2x$, $4x$, $5x$, $10x$, $20x$, y, $2y$, $4y$, $5y$, $10y$, $20y$, y^2, $2y^2$, $4y^2$, $5y^2$, $10y^2$, $20y^2$, xy, $2xy$, $4xy$, $5xy$, $10xy$, $20xy$, xy^2, $2xy^2$, $4xy^2$, $5xy^2$, $10xy^2$ and $20xy^2$

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Factorising by taking out a common factor

We have shown that the expressions $2x + 6$ and $2(x + 3)$ are equivalent. We refer to $2x + 6$ as the **expanded form**, while $2(x + 3)$ is the **factorised form** because it is written as the product of the factors 2 and $x + 3$. The process of going from expanded form to factorised form is called **factorisation**.

To factorise $2x + 6$, we identify the HCF (2) and write

$$2x + 6 = 2 \times x + 2 \times 3$$

Then the HCF is written before the brackets to give $2(x + 3)$. We say that $2(x + 3)$ is the factorised form of $2x + 6$.

Example 19

Factorise each of the following.

a $6a - 2m$ **b** $3w^2 + 5w$ **c** $4ab + 6bc$ **d** $-2a^2 - 4ab$

Working	Reasoning
a $6a - 2m$ $= 2(3a - m)$	The HCF of $6a$ and $2m$ is 2. $6a - 2m = 2 \times 3a - 2 \times m$ So 2 goes before the brackets and $3a - m$ goes inside the brackets.
b $3w^2 + 5w$ $= w(3w + 5)$	The HCF of $3w^2$ and $5w$ is w. $3w^2 + 5w = w \times 3w + w \times 5$ So w goes before the brackets and $3w + 5$ goes inside the brackets.
c $4ab + 6bc$ $= 2b(2a + 3c)$	The HCF of $4ab$ and $6bc$ is $2b$. $4ab + 6bc = 2b \times 2a + 2b \times 3c$ So $2b$ goes before the brackets and $2a + 3c$ goes inside the brackets.
d $-2a^2 - 4ab$ $= -2a(a + 2b)$	The HCF of $-2a^2$ and $-4ab$ is $-2a$. $-2a^2 - 4ab = -2a \times a + (-2a) \times 2b$ So $-2a$ goes before the brackets and $a + 2b$ goes inside the brackets.

To check that you have factorised correctly, expand your answer and see if you get the original expression.

In part **d** of the previous example,

$$\begin{aligned} -2a(a + 2b) &= -2a \times a + (-2a) \times 2b \\ &= -2a^2 - 4ab \end{aligned}$$

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exercise 3.5

LINKS TO Example 17

Copy and complete the following table.

Term	Factors
9	
12	
x	
$7b$	
	$1, 2, 4, 8, p, 2p, 4p, 8p$
$4b$	

Term	Factors
$6x$	
xy	
$7gh$	
x^2	
	$1, 3, a, 3a, a^2, 3a^2$
$2x^2y$	

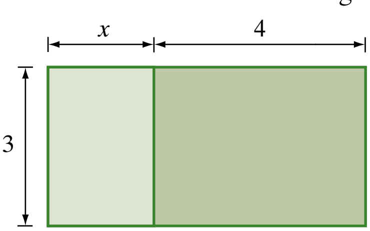
LINKS TO Example 18

Write the highest common factor of each of the following.

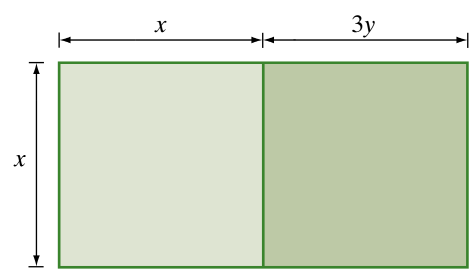
a $2a$ and $6a$
b ab and ac
c $10p$ and $25q$
d $15d^2$ and $5de$
e $-14x$ and $-12y$
f $-6x^2y$ and $-9xy^2$
g $5abc$ and $15ab$
h $4x^2$ and $6x$
i $4a^2$ and $8ab$ and $14ab^2$
j $-45a^2b$ and $-20ab^2$
k $18x^2$ and $12x$
l $6x^2$ and $18x$ and 27

Write an equivalent expression of the type 'factorised form = expanded form' to show the area for each of the following.

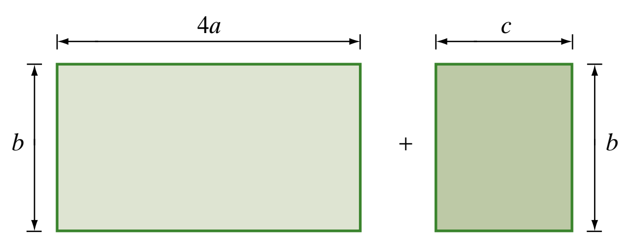
a



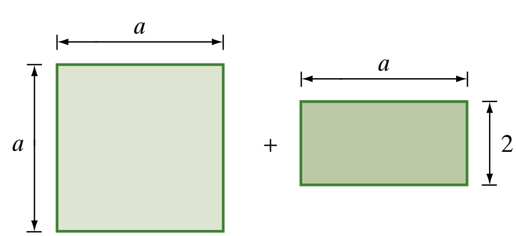
b



c



d



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LINKS TO Example 19

Copy and complete each of these equivalent expressions.

a $(3x + 6) = 3(\_\_ + \_\_)$

b $5a^2 - 8ab = a(\_\_ - \_\_)$

c $6a + 21b = \_\_(2a + \_\_)$

d $-15p^2q - 18pq^2 = -3pq(\_\_\_\_\_\_)$

e $-7x - 14 = -\_\_(\_\_ + \_\_)$

f $-3x - 12 = -\_\_(\_\_ + \_\_)$

g $-5x - 20 = -5(\_\_\_\_\_\_)$

h $-6x^2 - 10x = -\_\_(\_\_\_\_\_\_)$

i $-12xy + 20x = -\_\_(\_\_\_\_\_\_)$

j $-28 - 21x = -\_\_(\_\_\_\_\_\_)$

k $-15x + 12x^2 = -\_\_(\_\_\_\_\_\_)$

l $-24a^2 + 16a = -\_\_(\_\_\_\_\_\_)$

Factorise each of the following expressions.

a $3a + 9$ **b** $2y - 4$ **c** $x^2 + xy$

d $b^2 + 3b$ **e** $5 - 10x$ **f** $4x - 14$

g $24a - 20b$ **h** $6m - 27$ **i** $3x + x^2$

j $3x + 3x^2$ **k** $5x - 15x^2$ **l** $3x - 8x^2$

m $4x^2y - 6xy^2$ **n** $a^2b - ab$ **o** $-2x - 4$

p $-6p - 18p^2$ **q** $-3 - 6x$ **r** $-12xy - 8x^2$

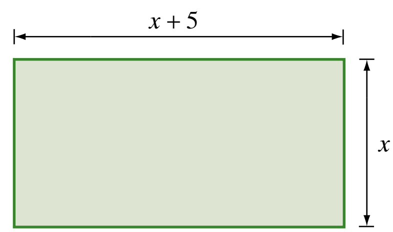
s $-20a^2b - 16ab^2$ **t** $-36xy - 42x^2$ **u** $18mn^2 + 12m^2n$

v $15pq - 10p^2q + 5pq^2$ **w** $-6x^2 - 4x + 10$ **x** $-12a^2 - 20ab - 15a$

This diagram shows the dimensions of a backyard lawn.

a Write two equivalent expressions for the perimeter of the lawn.

b Write two equivalent expressions for the area of the lawn.



a Complete the following $6 - 3x = 3\ (\_\_\_\_\_\_)$.

b Glenda used her calculator to factorise $6 - 3x$ and got the result $-3(x - 2)$. Is this expression equivalent to your answer to part **a**?

exercise 3.5 challenge

Factorise each of the following expressions.

a $2\pi r^2 + 2\pi rh$ **b** $\sqrt{2}p - \sqrt{2}q$ **c** $\dfrac{a^2b}{7} - \dfrac{3ab}{14}$ **d** $\dfrac{5}{x} - \dfrac{5}{y}$

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3.6 Binomial factors

In the expression $5(x + 3) + x(x + 3)$, the common factor is $x + 3$. This binomial factor can be taken out and placed before the bracket like any other common factor.

$$\begin{aligned}&5(x + 3) + x(x + 3)\\&= (x + 3) \times 5 + (x + 3) \times x\\&= (x + 3) \times (5 + x)\\&= (x + 3)(5 + x)\end{aligned}$$

Note that $(5 + x)$ can be written as $(x + 5)$ but $(5 - x)$ cannot be written as $(x - 5)$.

Example 20

Factorise each of the following.

a $x(x + 3) + 4(x + 3)$

b $a(2a - 1) - 2(2a - 1)$

c $x(x - 2) + x - 2$

d $x(x - 3) + 3 - x$

Working	Reasoning
a $x(x + 3) + 4(x + 3)$ $= (x + 3)(x + 4)$	The common factor is $x + 3$. So $(x + 3)$ goes before a bracket containing the other factor $(x + 4)$.
b $a(2a - 1) - 2(2a - 1)$ $= (2a - 1)(a - 2)$	The common factor is $2a - 1$. So $(2a - 1)$ goes before a bracket containing the other factor $(a - 2)$.
c $x(x - 2) + x - 2$ $= x(x - 2) + 1(x - 2)$ $= (x - 2)(x + 1)$	$+x - 2$ can be written as $+1(x - 2)$. The common factor is $(x - 2)$. So $(x - 2)$ goes before a bracket containing the other factor $(x + 1)$.
d $x(x - 3) + 3 - x$ $= x(x - 3) - x + 3$ $= x(x - 3) - 1(x - 3)$ $= (x - 3)(x - 1)$	Write 3 and $-x$ in the same order as in $(x - 3)$. The signs in 3 and $-x$ are the opposites of the signs in $(x - 3)$ so a negative sign must be put before the bracket: $-x + 3 = -1(x - 3)$. Take out the common factor of $(x - 3)$. The other factor is $(x - 1)$.

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Factorisation by grouping 'two and two'

Algebra review 1

Consider the expression $mx + mn + 5x + 5n$. The terms can be grouped in pairs so that each pair has a common factor.

$mx + mn + 5x + 5n = m(x + n) + 5(x + n)$.

Algebra review 2

We can then take out the binomial factor $x + n$, so that

$$\begin{aligned} mx + mn + 5x + 5n &= m(x + n) + 5(x + n) \\ &= (x + n)(m + 5) \end{aligned}$$

Example 21

Factorise each of the following by grouping 'two and two'.

a $a^2 + ab + 2a + 2b$ **b** $ab + 7b - 2a - 14$ **c** $x^2 - 3x - 7x + 21$

Working	Reasoning
a $a^2 + ab + 2a + 2b$ $= a(a + b) + 2(a + b)$ $= (a + b)(a + 2)$	The first pair has a common factor of a and the second pair has common factor of 2. The binomial factor of $(a + b)$ is then taken out as a common factor. The other factor is $a + 2$.
b $ab + 7b - 2a - 14$ $= b(a + 7) - 2(a + 7)$ $= (a + 7)(b - 2)$	The first pair has a common factor of b and the second pair has common factor of -2. The binomial factor of $(a + 7)$ is then taken out as a common factor. The other factor is $b - 2$.
c $x^2 - 3x - 7x + 21$ $= x(x - 3) - 7(x - 3)$ $= (x - 3)(x - 7)$	The first pair has a common factor of x and the second pair has common factor of -7. The binomial factor of $(x - 3)$ is then taken out as a common factor. The other factor is $x - 7$.

exercise 3.6

LINKS TO Example 20

1 Factorise each of the following by taking out a binomial factor.

a $a(a + 2) + 6(a + 2)$ **b** $3(x - 5) + x(x - 5)$
c $y(2y + 1) - 2(2y + 1)$ **d** $x(x - 3) - (x - 3)$
e $a(x + 1) + 3(x + 1)$ **f** $4(x - 7) + x(x - 7)$
g $p(2p + 3) - 2(2p + 3)$ **h** $a(a + 3) + a + 3$
i $3w(2w - 5) + 4(2w - 5)$ **j** $x(x + 3) + 2(x + 3)$
k $k(k - 7) - 2(k - 7)$ **l** $m(m - 5) + 3(m - 5)$

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m $7(4 - p) - p(4 - p)$ **n** $3a(a - 5) - 4(a - 5)$
o $4n(n^2 + 1) + 3(n^2 + 1)$ **p** $2m(2m + 3) + 3(2m + 3)$
q $p(p - 1) - p + 1$ **r** $x(x + 5) + 5(x + 5)$

LINKS TO Example 21

Copy and complete each of the following.

a $bx + by + 7x + 7y$
$= \_\_(x + y) + \_\_(x + y)$
$= (x + y)(\_\_\_\_\_\_)$

b $k^2 - k + pk - p$
$= \_\_(k - 1) + \_\_(k - 1)$
$= (k - 1)(\_\_\_\_\_\_)$

c $6z + 36 + az + 6a$
$= 6(\_\_\_\_\_\_) + a(\_\_\_\_\_\_)$
$= (\_\_\_\_\_\_)(\_\_\_\_\_\_)$

d $ab + 7b - 2a - 14$
$= b(\_\_\_\_\_\_) - 2(\_\_\_\_\_\_)$
$= (\_\_\_\_\_\_)(\_\_\_\_\_\_)$

e $2ab + 4b + 3a + 6$
$= \_\_(\_\_\_\_\_\_) + \_\_(\_\_\_\_\_\_)$
$= (\_\_\_\_\_\_)(\_\_\_\_\_\_)$

f $4c - 12 + bc - 3b$
$= \_\_(\_\_\_\_\_\_) + \_\_(\_\_\_\_\_\_)$
$= (\_\_\_\_\_\_)(\_\_\_\_\_\_)$

g $3b^2 - 6b + 4b - 8$
$= \_\_(\_\_\_\_\_\_) + \_\_(\_\_\_\_\_\_)$
$= (\_\_\_\_\_\_)(\_\_\_\_\_\_)$

h $2x^2 - 10x - 3x + 15$
$= \_\_(\_\_\_\_\_\_) - \_\_(\_\_\_\_\_\_)$
$= (\_\_\_\_\_\_)(\_\_\_\_\_\_)$

i $x^2y + 6x + xy^2 + 6y$
$= \_\_(\_\_\_\_\_\_) + \_\_(\_\_\_\_\_\_)$
$= (\_\_\_\_\_\_)(\_\_\_\_\_\_)$

j $3m^2 - m^2n + 3k - kn$
$= \_\_(\_\_\_\_\_\_) + \_\_(\_\_\_\_\_\_)$
$= (\_\_\_\_\_\_)(\_\_\_\_\_\_)$

Factorise each of the following by grouping ‘two and two’.

a $x^2 + 5x + 3x + 15$ **b** $mn + 2n + 9m + 18$ **c** $4a + 24 + ba + 6b$
d $ab + 2a + cb + 2c$ **e** $xy - 5y + 2x - 10$ **f** $ax + 4x + 6a + 24$
g $x^2 + 2x - 8x - 16$ **h** $k^2 + 4k - 6k - 24$ **i** $x^2 - 8x - 3x + 24$
j $14 - 7m + 2n - mn$ **k** $a^2 - 6a - 5a + 30$ **l** $cd + bd - c - b$
m $ab + a - b - 1$ **n** $b^2 + 2b - ab - 2a$ **o** $x^2 + 11x - 3x - 33$

For each expression,
i write an algebraic expression.
ii factorise.
a x times $(x + y)$ plus 3 times $(x + y)$
b $(3x - 4)$ subtracted from 3 times $(3x - 4)$

exercise 3.6 challenge

Factorise each of the following.
a $x^2 - 4y + xy - 4x$
b $ab - 12 - 3b + 4a$

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3.7 Factorising quadratic trinomials

As we saw on page 105 expressions of the type $x^2 + 5x + 6$ and $x^2 - 7x + 10$ are called quadratic trinomials. Note that each quadratic trinomial has an x^2 term, an x term, and a constant term.

In this section we see how the process of expanding two binomial factors can be reversed to find the factors of a quadratic trinomial.

For example, $(x + 3)(x + 2) = x^2 + 5x + 6$. In this section we see how this process can be reversed to factorise a quadratic trinomial into two binomial factors.

$x^2 + 5x + 6$	$=$	$(x + 3)(x + 2)$
expanded form (quadratic trinomial)		**factorised form (two binomial factors)**

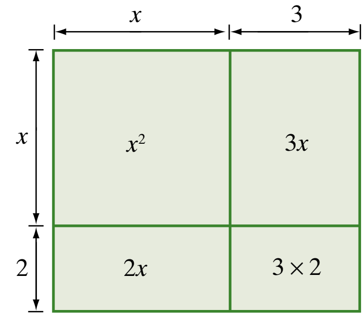
Example 22

Which of the following expressions are quadratic trinomials?

a $3a^2 + 2a$ **b** $2b^2 - 5b + 2$ **c** $c^2 + c^3 + 5$ **d** $x^2 - 10x + 7$

Working	Reasoning
a $3a^2 + 2a$ This is not a quadratic trinomial.	There are only two terms, so the expression is not a trinomial, although it is a quadratic expression.
b $2b^2 - 5b + 2$ This is a quadratic trinomial.	This expression has three terms and fits the definition.
c $c^2 + c^3 + 5$ This is not a quadratic trinomial.	This has three terms but the term c^3 means that it is not a quadratic expression.
d $x^2 - 10x + 7$ This is a quadratic trinomial.	This expression has three terms and fits the definition.

To work out how we can reverse the process of expanding two binomial factors, consider the area diagram for the expansion $(x + 2)(x + 3) = x^2 + 5x + 6$.



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The middle term, $5x$, comes from adding $2x$ and $3x$ so we can write $(x+2)(x+3) = x^2 + 2x + 3x + 6$.

Notice that $x^2 + 2x + 3x + 6$ resembles the expressions that you factorised in the previous section.

$$\begin{aligned} x^2 + 2x + 3x + 6 &= x(x+2) + 3(x+2) \\ &= (x+2)(x+3) \end{aligned}$$

So we can now write $x^2 + 5x + 6 = (x+2)(x+3)$. This means that we have reversed the process of expanding to find the factors of $x^2 + 5x + 6$.

We can see that $2 + 3 = 5$ and $2 \times 3 = 6$.

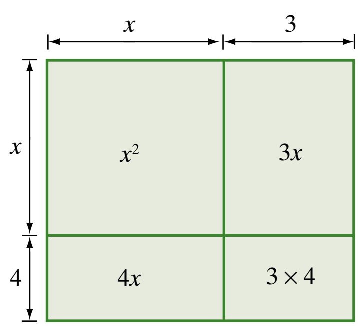
Applying the same method to factorising $x^2 + 7x + 12$, we want two numbers that add to 7 and multiply to 12. We choose 3 and 4 because $3 + 4 = 7$ and $3 \times 4 = 12$.

Notice that $2 \times 6 = 12$ and $1 \times 12 = 12$ but $2 + 6 \neq 7$ and $1 + 12 \neq 7$.
So 3 and 4 are the correct factors of 12 to choose.

So,

$$\begin{aligned} x^2 + 7x + 12 &= x^2 + 3x + 4x + 12 \\ &= x(x+3) + 4(x+3) \\ &= (x+3)(x+4) \end{aligned}$$

The rectangle area diagram confirms that $(x+3)(x+4)$ and $x^2 + 7x + 12$ are equivalent expressions.



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Example 23

Factorise each of the following.

a $x^2 + 5x + 4$ **b** $x^2 + 9x + 20$ **c** $x^2 + 9x + 8$

Working	Reasoning
a $x^2 + 5x + 4$ $= (x + 4)(x + 1)$	Factors of 4 are 4×1 and 2×2. We need factors that have a sum of 5. $4 + 1 = 5$ So 4 and 1 are the correct factors of 4 to choose.
b $x^2 + 9x + 20$ $= (x + 5)(x + 4)$	Factors of 20 are 20×1, 10×2 and 5×4. We need factors with a sum of 9. $5 + 4 = 9$ So 5 and 4 are the correct factors of 20 to choose.
c $x^2 + 9x + 8$ $= (x + 1)(x + 8)$	Factors of 8 are 1×8 and 2×4. We need factors with a sum of 9. $1 + 8 = 9$ So 1 and 8 are the correct factors of 8 to choose.

The numbers in the brackets may be positive or negative, so look for clues in the numbers in the expanded form.

As we have seen, the constant number in a quadratic trinomial is the product of the numbers in the brackets. In the examples we have looked at so far, all the signs have been positive. There are three possibilities for the signs in the brackets. These signs determine the sign of the constant term in the quadratic trinomial.

both positive	$(x + a)(x + b)$	$(+a) \times (+b) = +ab$
both negative	$(x - a)(x - b)$	$(-a) \times (-b) = +ab$
one positive and one negative	$(x + a)(x - b)$	$(+a) \times (-b) = -ab$

So, if the *constant term* in a quadratic trinomial is *positive*, the signs in the brackets could be *both positive* or *both negative*. We then need to look at the sign of the middle term (that is, the x term). If this is negative, then both brackets must contain a negative sign. For example, compare the factors of $x^2 + 9x + 14$ and $x^2 - 9x + 14$.

$x^2 + 9x + 14$	$x^2 - 9x + 14$
$= x^2 + 7x + 2x + 14$	$= x^2 - 7x - 2x + 14$
$= x(x + 7) + 2(x + 7)$	$= x(x - 7) - 2(x - 7)$
$= (x + 7)(x + 2)$	$= (x - 7)(x - 2)$

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If the *constant term* in a quadratic trinomial is *negative*, the signs in the brackets must be *different*. Looking at the sign of the middle term helps us work out which bracket contains the negative sign and which contains the positive sign.

Compare the factors of $x^2 + 5x - 14$ and $x^2 - 5x - 14$. Notice that if the middle term is positive, it is the bracket with the larger number that is positive. If the middle term is negative, it is the bracket with the larger number that is negative.

$$\begin{aligned} & x^2 + 5x - 14 \\ &= x^2 + 7x - 2x - 14 \\ &= x(x + 7) - 2(x + 7) \\ &= (x + 7)(x - 2) \end{aligned} \qquad \begin{aligned} & x^2 - 5x - 14 \\ &= x^2 - 7x + 2x - 14 \\ &= x(x - 7) + 2(x - 7) \\ &= (x - 7)(x + 2) \end{aligned}$$

With plenty of practice you will become skilled at recognising the required factors and signs.

Example 24

Factorise each of the following.

a $x^2 - 3x + 2$ **b** $x^2 + 3x - 4$ **c** $x^2 - 3x - 4$

Working	**Reasoning**
a $x^2 - 3x + 2$ $= (x - 2)(x - 1)$	The constant (or last) term (+2) is positive, so the numbers will have the *same* sign. The middle term ($-3x$) is negative, so both numbers must be negative. $-2 \times (-1) = 2$ $-2 + (-1) = -3$ so the required numbers are -2 and -1.
b $x^2 + 3x - 4$ $= (x + 4)(x - 1)$	The last term (-4) is negative, so the numbers will have *opposite* signs. The middle term ($+3x$) shows that the sum is positive, so the greater number must be positive. $4 \times (-1) = -4$ $4 + (-1) = 3$ so the required numbers are $+4$ and -1.

The two numbers in the brackets must have a product of +2 and a sum of –3.

continued

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Example 24 continued

Working

c $x^2 - 3x - 4$
$= (x - 4)(x + 1)$

Reasoning

The last term (–4) is negative, so the numbers will have *opposite* signs.
The middle term ($-3x$) shows that the sum is negative, so the greater number must be negative.
$-4 \times 1 = -4$
$-4 + 1 = -3$ so the required numbers are –4 and +1.

Example 25

Factorise the following quadratic trinomials.

a $x^2 - 10x - 24$ **b** $x^2 - 10x + 24$

Working

a $x^2 - 10x - 24 = (x - 12)(x + 2)$

Reasoning

The two numbers have a product of –24 and a sum of –10.
The constant term is negative so the numbers have opposite signs.
The numbers are –12 and +2.

Working

b $x^2 - 10x + 24 = (x - 6)(x - 4)$

Reasoning

The two numbers have a product of +24 and a sum of –10 .
The constant term is positive so the numbers are both negative.
The numbers are –6 and –4.

exercise 3.7

LINKS TO Example 22

1. Which of the following are quadratic trinomials?

   a $x^2 + 2$ **b** $2x^3 - 4x + 7$ **c** $\frac{1}{2}x^2 - 3x + 2$

   d $2 - x^2 - 3x$ **e** $5x^2$ **f** $3x^2 - 2x + \frac{1}{x}$

   g $10x^2 + 12 + 6x$ **h** $3x^2 + 2x$ **i** $-5x^2 + 3x^{-7}$

2. Which one of the following is a quadratic trinomial?

   A $x^2 + 5x$ **B** $x^2 - 4$ **C** $xy + 2y + 3x$

   D $x^2 - 4x + 5$ **E** $x^2 + 5x - 3x$

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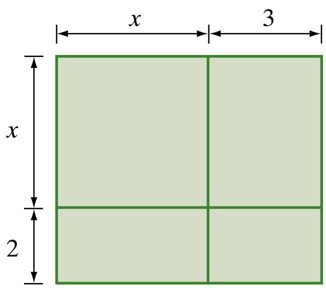
Write an expression for the area of the rectangle

a in expanded form

b in factorised form.



LINKS TO Example 23

Factorise each of the following.

a $x^2 + 3x + 2$ **b** $x^2 + 6x + 8$ **c** $x^2 + 11x + 10$

d $x^2 + 9x + 18$ **e** $x^2 + 10x + 16$ **f** $x^2 + 11x + 24$

g $x^2 + 17x + 30$ **h** $x^2 + 6x + 5$ **i** $x^2 + 15x + 56$

j $x^2 + 16x + 28$ **k** $x^2 + 10x + 24$ **l** $x^2 + 10x + 21$

LINKS TO Example 24

Factorise each of the following.

a $x^2 - 12x - 13$ **b** $x^2 - 6x + 8$ **c** $x^2 - 7x + 10$

d $x^2 - 9x - 22$ **e** $x^2 - 6x - 7$ **f** $x^2 - 11x + 28$

g $x^2 + 13x + 22$ **h** $x^2 + 7x + 6$ **i** $x^2 + 11x + 18$

j $x^2 + 12x + 35$ **k** $x^2 - 8x + 7$ **l** $x^2 - 9x + 14$

LINKS TO Example 25

Fill in the missing numbers.

a $x^2 + \_\_x + 16 = (x + 8)(x + 2)$ **b** $x^2 + 10x + \_\_ = (x + 6)(x + 4)$

c $x^2 + 7x + 12 = (x + 4)(x + \_\_)$ **d** $x^2 - 12x - 13 = (x - 13)(x + \_\_)$

e $x^2 - 11x + 24 = (x - \_\_)(x - \_\_)$ **f** $x^2 - 6x + 8 = (x - \_\_)(x - \_\_)$

Factorise each of the following.

a $x^2 + 14x - 32$ **b** $x^2 + x - 20$ **c** $x^2 - 9x - 70$ **d** $x^2 + 8x - 48$

e $x^2 - 15x + 26$ **f** $x^2 - 10x + 25$ **g** $x^2 + 16x + 63$ **h** $x^2 + 13x + 36$

i $x^2 - 8x + 15$ **j** $x^2 - 14x + 33$ **k** $x^2 - 7x + 12$ **l** $x^2 - 8x + 12$

m $x^2 - 13x + 12$ **n** $x^2 - 14x + 24$ **o** $x^2 + 10x - 11$ **p** $x^2 - 10x - 11$

q $x^2 + 2x - 24$ **r** $x^2 - x - 30$ **s** $x^2 + 2x - 48$ **t** $x^2 + 10x - 56$

u $x^2 - x - 42$ **v** $x^2 + 13x - 48$ **w** $x^2 - 13x + 40$ **x** $x^2 - 13x - 30$

Factorise each of the following.

a $x^2 + 5x + 6$ **b** $x^2 - 5x + 6$ **c** $x^2 - 7x + 6$ **d** $x^2 + 7x + 6$

e $x^2 - 5x - 6$ **f** $x^2 + 5x - 6$ **g** $x^2 - x - 6$ **h** $x^2 + x - 6$

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exercise 3.7 challenge

1. Jack has an older sister and a younger sister. If Jack's age is x years, the product of his sisters' ages is given by the expression $x^2 - 2x - 24$.
   a Factorise $x^2 - 2x - 24$ to find out how many years older and younger than Jack his sisters are.
   b What is the product of the sisters' ages when Jack is 14?

2. The area of a rectangle is given by $A = x^2 + 6x - 16$.
   a Find expressions for the length and width of the rectangle, in terms of x.
   b Find the length and width of the rectangle when $x = 4$.
   c Do all values of x make sense in this situation? Hint: think about what happens if $x = 1$.

LINKS TO Example 20

3. Factorise each of the following. Hint: first take out the common factor.
   a $2x^2 + 8x + 6$ **b** $3x^2 + 21x + 30$ **c** $2x^2 + 16x + 14$
   d $5x^2 - 25x - 30$ **e** $4x^2 + 4x - 48$ **f** $2x^2 - 16x - 40$
   g $2x^2 + 8x - 10$ **h** $-x^2 - 7x - 12$ **i** $2x^2 - 30x + 52$

4. Find the values of a, b and c if $3x^2 + 27x + 24 = a(x + b)(x + c)$.

5. Factorise each of the following.
   a $x^2 + 3ax + 2a^2$ **b** $x^2 + (a - b)x - ab$
   c $x^2 - (p + q)x + pq$ **d** $(a + 2)^2 + 5(a + 2) + 6$

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3.8 Algebraic fractions

Algebraic fractions can be added and subtracted in the same way as we add and subtract number fractions.

If the denominators are the same, add the numerators and use the same denominator.

For example,

$$\frac{3}{5} + \frac{1}{5} = \frac{4}{5} \qquad \frac{3x}{5} + \frac{x}{5} = \frac{4x}{5}$$

If the denominators are different, find equivalent fractions with the same denominator then add the numerators and use the common denominator.

For example,

$$\begin{aligned}\frac{3}{4} + \frac{2}{3} &= \frac{9}{12} + \frac{8}{12} \\ &= \frac{17}{12}\end{aligned} \qquad \begin{aligned}\frac{3a}{4} + \frac{2a}{3} &= \frac{9a}{12} + \frac{8a}{12} \\ &= \frac{17a}{12}\end{aligned}$$

Example 26

Express as single fractions.

a $\frac{4x}{7} - \frac{x}{7}$ **b** $\frac{4a}{9} + \frac{2a}{9}$ **c** $\frac{5x}{7} + \frac{x}{3}$

Working	Reasoning
a $\frac{4x}{7} - \frac{x}{7}$ $= \frac{3x}{7}$	The denominators are the same. Subtract x from $4x$ to give the numerator.
b $\frac{4a}{9} + \frac{2a}{9}$ $= \frac{\cancel{6}^{2}a}{\cancel{9}_{3}}$ $= \frac{2a}{3}$	The denominators are the same. Add $4a$ and $2a$ to give the numerator. Divide the numerator and denominator by the common factor 3.

continued

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Example 26 continued

Working	Reasoning
c $\frac{5x}{7}+\frac{x}{3}=\frac{15x}{21}+\frac{7x}{21}$ $=\frac{22x}{21}$	The lowest common denominator is 21. Express the fractions as equivalent fractions with the common denominator 21. Add $15x$ and $7x$ to give the numerator.

exercise 3.8

LINKS TO Examples 26a, b

1. Express as single fractions.

a $\frac{x}{5}+\frac{x}{5}$ **b** $\frac{3x}{8}+\frac{2x}{8}$ **c** $\frac{4x}{7}-\frac{3x}{7}$

d $\frac{6x}{11}-\frac{5x}{11}$ **e** $\frac{7b}{9}-\frac{5b}{9}$ **f** $\frac{a}{5}+\frac{2a}{5}$

g $\frac{3x}{7}+\frac{3x}{7}$ **h** $\frac{3b}{8}+\frac{5b}{8}-\frac{b}{8}$ **i** $\frac{2a}{7}+\frac{5a}{7}-\frac{3a}{7}$

j $\frac{4x}{5}-\frac{x}{5}+\frac{3x}{5}$ **k** $-\frac{a}{3}+\frac{5a}{3}-\frac{2a}{3}$ **l** $\frac{7x}{11}-\frac{4x}{11}+\frac{2x}{11}$

LINKS TO Example 26c

2. Express as single fractions.

a $\frac{x}{3}+\frac{x}{4}$ **b** $\frac{a}{2}+\frac{2a}{3}$ **c** $\frac{2a}{5}+\frac{2a}{3}$

d $\frac{4b}{3}-\frac{b}{8}$ **e** $\frac{3x}{8}+\frac{5x}{16}$ **f** $\frac{4b}{5}-\frac{b}{3}$

g $\frac{7x}{8}-\frac{3x}{4}$ **h** $\frac{2x}{3}-\frac{5x}{6}$ **i** $\frac{3x}{4}+\frac{x}{2}-\frac{2x}{3}$

j $\frac{5a}{3}+\frac{2a}{5}-\frac{6a}{5}$ **k** $\frac{3x}{8}+\frac{2x}{3}+\frac{x}{6}$ **l** $\frac{3x}{4}+\frac{2x}{3}-\frac{5x}{6}$

3. Simplify.

a $\frac{x}{6}+\frac{x}{6}$ **b** $\frac{7x}{9}-\frac{x}{9}$ **c** $\frac{3a}{10}+\frac{a}{5}$

d $\frac{4a}{7}+\frac{3a}{7}$ **e** $\frac{5x}{9}-\frac{2x}{9}$ **f** $\frac{b}{6}+\frac{b}{3}$

g $\frac{2x}{3}+\frac{x}{5}-\frac{3x}{15}$ **h** $\frac{5a}{12}+\frac{2a}{3}+\frac{a}{6}$ **i** $\frac{7x}{3}+\frac{x}{2}-\frac{5x}{6}$

j $\frac{8x}{5}-\frac{x}{4}+\frac{3x}{10}$ **k** $\frac{2x}{3}+\frac{3x}{2}-\frac{5x}{6}$ **l** $\frac{7x}{8}-\frac{2x}{3}+\frac{x}{6}$

exercise 3.8 challenge

1. Simplify.

$$-\frac{5x}{8}+\frac{3x}{4}-\frac{x}{6}+\frac{4x}{3}-\frac{7}{24}$$

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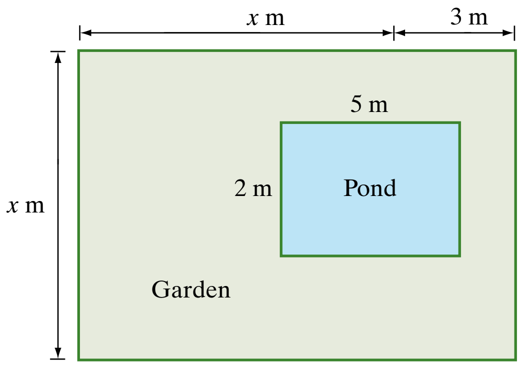
Analysis task

Garden factors

A landscape gardener is designing a rectangular area with a garden and a rectangular pond. Both the pond and the garden are to have whole number side-lengths measured in metres.



a Write an expression for the total area of the garden and pond

  i in factorised form.

  ii in expanded form.

b Write an expression for the area of garden surrounding the pond.

c Factorise the expression for the area of the garden.

d Calculate the area of the garden for the following values of x.

  i 5

  ii 7

e Calculate the area of the garden if $x = 2$. Explain what this means in terms of the garden and the pond.

f The landscape gardener experiments with other whole number dimensions for the pond. Write an expression for the area of garden surrounding the pond if the pond measures 1 m by 4 m. Can you factorise your expression?

g If the pond measures 3 m by 5 m will you be able to factorise the expression for the garden area?

h Find three more sets of whole number dimensions for the pond that will give a garden area that you can factorise.

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Review Algebra

Summary

Algebraic expressions and the distributive law

- Equivalent expressions are true for all values of the pronumerals involved.
- According to the distributive law: $a(b+c) = ab + ac$.
- The factorised form of an expression is written as the product of factors. For example, $a(b+c)$ is the factorised form of $ab + ac$.
- The process of going from expanded form to factorised form is called factorisation.
- Expressions of the type $x^2 + 8x + 15$ are called quadratic trinomial expressions.
- Expressions such as $(x + 3)$ are called binomial expressions because they have two terms.
- The diagram shows the expanded form and factorised form of an algebraic expression.

  FOIL reminds us to multiply

  First terms
  Outer terms
  Inner terms
  Last terms

  $$(a+b)(c+d)$$

  $$\begin{tikzcd}(x+5)(x+3) \arrow[r, bend left, "\text{Expand}"] & x^2+8x+15 \arrow[l, bend left, "\text{Factorise}"]\end{tikzcd}$$

- Algebraic fractions can be added and subtracted by converting to equivalent fractions with the same denominator.

Visual map

Using the following terms (and others if you wish), construct a visual map that illustrates your understanding of the key ideas covered in this chapter.

algebraic fraction	equivalent expressions	factor	quadratic trinomial
binomial expansion	expanded form	factorisation	substitution
coefficient	expansion	factorised form	
common factor	expression	grouping 'two and two'	

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Revision

Multiple-choice questions

When expanded, $-2x(5 - x)$ is equal to

A $-10x - x^2$ **B** $-10x - 2x^2$ **C** $2x^2 - 10x$ **D** $-10x - 4x^2$ **E** $2x^2 + 3x$

A correct simplification of the expression $4x - 2(x - 5) + 3(4x + 5)$ is

A $14x + 5$ **B** $-10x - 25$ **C** $-10x - 5$ **D** $14x + 25$ **E** $4x^2 - 34x - 5$

The highest common factor of $9a^2b^3$ and $15a^4b$ is

A ab **B** $3ab$ **C** a^2b **D** $3a^2b$ **E** $3a^4b^3$

Which one of the following is a quadratic trinomial?

A $3a + 2b - c$ **B** $a^2 - 2ab + b$ **C** $y^2 + 15 - 4y$ **D** $x^2 - 9$ **E** $x^2 + 3x$

The factorised form of $x^2 - 10x - 24$ is

A $(x - 2)(x - 12)$ **B** $(x + 2)(x - 12)$

C $(x - 4)(x - 6)$ **D** $(x + 4)(x - 6)$

E $(x - 4)(x + 6)$

Short-answer questions

Simplify the following expressions.

a $7a + 2b - 3ab - 5b - 6ba$ **b** $2x^2y - 4xy + 3xy^2 - 3x^2y + 5xy$

c $5x \times (-4y) \times 3$ **d** $\dfrac{36mn}{24n}$

If $y = ax + b$, find

a y when $a = 3, x = 2, b = 5$ **b** y when $a = 1\frac{1}{2}, x = -7$ and $b = 3$

v when $u = 2$, $a = -3$ and $s = -2$.

Expand the following expressions and simplify if possible.

a $3(2x - 5)$ **b** $-2(5 - y)$ **c** $2(x + 11) - 3(2x + 5)$

Factorise each of the following.

a $x^2 + 5x$ **b** $2a^2b - 4ab^2$ **c** $3x^2y - 9xy^2 + 15xy$

Factorise each of the following.

a $x(x + 3) - 2(x + 3)$ **b** $a^2 + ab + 2a + 2b$

c $m^2 + 5m - mn - 5n$ **d** $xy - 9x + 2y - 18$

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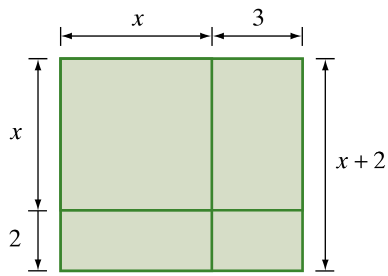


a Write expressions for the area of each of the four regions of the rectangle.

b Write an expression for the total area

i in factorised form.

ii in expanded form.

Expand and simplify.

a $(x+2)(x+5)$

b $(a-3)(a+2)$

c $(p-5)(p-8)$

d $(2y+1)(y-3)$

e $(4e-3)(2e+3)$

f $(x-5)(2x+3)$

Factorise each of the following quadratic trinomials.

a $x^2+7x+10$

b $y^2-8y+12$

c m^2-m-12

d $k^2+2k-15$

Simplify.

a $\dfrac{7a}{11}-\dfrac{4a}{11}$

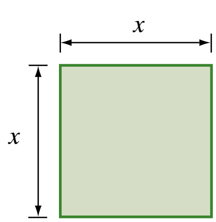
b $\dfrac{2x}{3}+\dfrac{x}{6}$

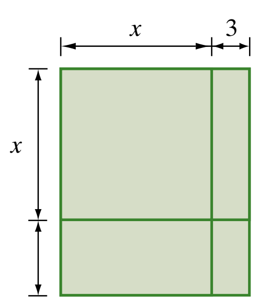
c $\dfrac{5x}{6}+\dfrac{3x}{4}$

d $\dfrac{3x}{5}+\dfrac{4x}{3}+\dfrac{x}{15}$

Extended-response questions

Freya grows vegetables in two garden beds. One bed is square, with side length x metres. The other is rectangular, with width $x-2$ metres and length $x+4$ metres.





a Find the area of the rectangular bed

i in factorised form.

ii in expanded form.

Practice quiz
Chapter 3

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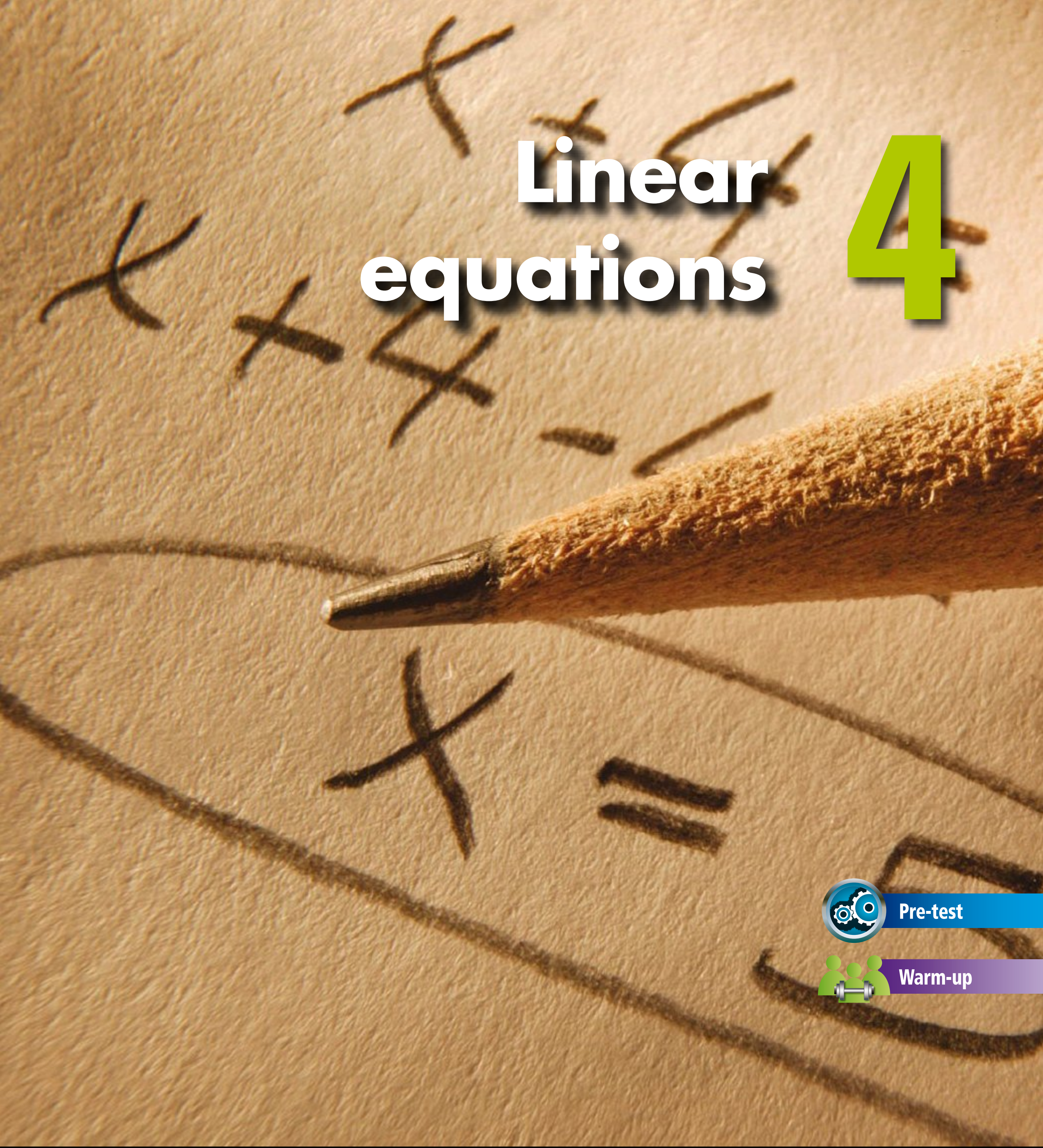
4 Linear equations



Pre-test

Warm-up

We solve linear equations by doing the same to both sides. In the solution shown here, we can see that 4 has been subtracted from both sides of the equation $x + 4 = 9$ to give $x = 5$. In the step before this, 54 on the right side has been divided by 6 to give 9. What must the left side of the equation have been? In this chapter skills are developed for solving linear equations. These skills can then be applied in problem solving.

4.1 Solving linear equations

BLM Chapter warm-up

Simple equations can be solved mentally by finding the value for the pronumeral that makes the equation a true statement.

Example 1

Solve the following equations mentally.

a $4x = 48$ **b** $\frac{p}{7} = 8$

Working	Reasoning
a $4x = 48$	Write down the equation.
$4 \times 12 = 48$	Think of a number that can be multiplied by 4 to give 48. Try 12.
$x = 12$	Write down the solution to the equation.
b $\frac{p}{7} = 8$	Write down the equation.
$\frac{56}{7} = 8$	Think of a number that can be divided by 7 to give 8. Try 56.
$p = 56$	Write down the solution to the equation.

Algebraic methods apply a logical sequence of operations to both sides of the equation. The idea is to simplify the equation, step by step, until it the pronumeral is on its own on one side of the equation. The equation is then solved.

There may be more than one sequence of operations that will successfully solve an equation. However, some sequences will be more efficient than others. As the following examples highlight, the clear recording of the sequence of operations is important.

Example 2

Solve each of the following equations for x. Check your answers using substitution.

a $5x - 8 = 14$ **b** $7 - 2.4x = 13$

Working	Reasoning
a $5x - 8 = 14$	
$5x - 8 + 8 = 14 + 8$	Add 8 to both sides to eliminate −8 on the left side and leave the x term on its own.
$5x = 22$	
$\frac{5x}{5} = \frac{22}{5}$	Divide both sides by 5 to obtain x on its own.
$x = \frac{22}{5}$	
$x = 4\frac{2}{5}$	

continued

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Example 2 continued

Working

Check:

$$\begin{aligned} LS &= 5x - 8 \\ &= 5 \times \frac{22}{5} - 8 \\ &= 22 - 8 \\ &= 14 \\ LS &= RS \end{aligned}$$

Reasoning

Substituting $x = \frac{22}{5}$ into the LS of the equation gives the same value as the RS.

b

$$7 - 2.4x = 13$$

$$7 - 7 - 2.4x = 13 - 7$$

Subtract 7 from both sides.

$$-2.4x = 6$$

$$\frac{-2.4x}{-2.4} = \frac{6}{-2.4}$$

Divide both sides by -2.4.

$$x = -2.5$$

Check:

$$\begin{aligned} LS &= 7 - 2.4x \\ &= 7 - 2.4 \times (-2.5) \\ &= 7 + 6 \\ &= 13 \\ LS &= RS \end{aligned}$$

Substituting -2.5 for x in the LS of the equation gives the same value as the RS.

Example 3

Solve each of the following equations for x. Check your solutions using substitution.

a $-\frac{2x}{3} = 14$ **b** $\frac{x}{4} + 7 = -3$

Working

a $-\frac{2x}{3} = 14$

Reasoning

The equation could also be solved in a single step by multiplying both sides by $-\frac{3}{2}$.

$$-\frac{2x}{3} = 14$$

$$-\frac{2x}{3} \times \left(-\frac{3}{2}\right) = \left(14 \times -\frac{3}{2}\right)$$

$$x = -21$$

continued

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Example 3 continued

Working	Reasoning
$-\frac{2x}{3} \times 3 = 14 \times 3$	Multiply both sides by 3.
$-2x = 42$	
$\frac{-2x}{-2} = \frac{42}{-2}$	Divide both sides by –2.
$x = -21$	
Check:	Substituting –21 for x in the LS of the equation gives the same value as the RS.
$\text{LS} = -\frac{2x}{3}$	
$= -\frac{2 \times (-21)}{3}$	
$= 14$	
$\text{LS} = \text{RS}$	
b $\frac{x}{4} + 7 = -3$	Subtract 7 from both sides to eliminate +7 on the left side and leave the x term on its own.
$\frac{x}{4} + 7 - 7 = -3 - 7$	
$\frac{x}{4} = -10$	
$\frac{x}{4} \times 4 = -10 \times 4$	Multiply both sides by 4 to remove the fraction and obtain x on its own.
$x = -40$	
Check:	Substituting $x = -40$ into the LS of the equation gives the same value as the RS.
$\text{LS} = \frac{x}{4} + 7$	
$= \frac{-40}{4} + 7$	
$= -10 + 7$	
$= -3$	
$\text{LS} = \text{RS}$	

Example 4

Solve the equation $\frac{3x + 3}{2} = 18$ algebraically.

continued

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Example 4 continued

Working

$$\frac{3x+3}{2} = 18$$

$$\frac{3x+3}{2} \times 2 = 18 \times 2$$

$$3x + 3 = 36$$

$$3x + 3 - 3 = 36 - 3$$

$$3x = 33$$

$$\frac{3x}{3} = \frac{33}{3}$$

$$x = 11$$

Check:

$$\begin{aligned} LS &= \frac{3x+3}{2} \\ &= \frac{3 \times 11 + 3}{2} \\ &= \frac{36}{2} \\ &= 18 \\ LS &= RS \end{aligned}$$

Reasoning

Doing the same thing to both sides ensures that each of the equations is equivalent to the original equation. First eliminate the denominator, 2, by multiplying both sides by 2.

Eliminate the 3 by subtracting 3 from both sides.

Divide by 3 to obtain x on its own.

Substituting $x = 11$ into the LS of the equation gives the same value as the RS.

Equations with brackets

Brackets in an equation should be expanded and like terms collected before proceeding with solving the equation.

Example 5

Solve the equation $5(x + 2) - 2(x - 3) = 49$.

Working

$$5(x + 2) - 2(x - 3) = 49$$

$$5x + 10 - 2x + 6 = 49$$

$$3x + 16 = 49$$

$$3x + 16 - 16 = 49 - 16$$

$$3x = 33$$

$$\frac{3x}{3} = \frac{33}{3}$$

$$x = 11$$

Reasoning

Expand the brackets.
Collect like terms.

Subtract 16 from both sides.

Divide both sides by 3.

continued

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Example 5 continued

Working	Reasoning
Check:	
$\text{LS} = 5(x + 2) - 2(x - 3)$	
$= 5(11 + 2) - 2(11 - 3)$	
$= 5 \times 13 - 2 \times 8$	
$= 65 - 16$	
$= 49$	
$\text{RS} = 49$	
$\text{LS} = \text{RS}$	

Equations with fractions

When an equation includes the addition or subtraction of fractions, the fractions are eliminated by multiplying both sides by the lowest common multiple of the two denominators.

Example 6

Solve the equation $\frac{x}{4} + \frac{2x}{3} = 33$ for x.

Working	Reasoning
$\frac{x}{4} + \frac{2x}{3} = 33$	
$\frac{x}{4} \times 12 + \frac{2x}{3} \times 12 = 33 \times 12$	Eliminate the fractions by multiplying both sides by the LCM of 4 and 3, which is 12.
$3x + 8x = 396$	Simplify by collecting like terms.
$11x = 396$	
$\frac{11x}{11} = \frac{396}{11}$	Divide both sides by 11.
$x = 36$	
Check:	Substituting $x = 36$ into the LS of the equation gives the same value as the RS.
$\text{LS} = \frac{x}{4} + \frac{2x}{3}$	
$= \frac{36}{4} + \frac{2 \times 36}{3}$	
$= 9 + 24$	
$= 33$	
$\text{LS} = \text{RS}$	

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Equations may include expressions rather than single terms in the numerator of fractions. Special care needs to be taken with signs, particularly if there is a negative sign before the fraction.

Solving skills 1

Example 7

Solve the following equation for the unknown variable x.

$$\frac{x+2}{3} - \frac{x-1}{5} = 1$$

Working

$$\frac{x+2}{3} - \frac{x-1}{5} = 1$$

$$\frac{x+2}{3} \times 15 - \frac{x-1}{5} \times 15 = 1 \times 15$$

$$5(x+2) - 3(x-1) = 15$$

$$5x + 10 - 3x + 3 = 15$$

$$2x + 13 = 15$$

$$2x + 13 - 13 = 15 - 13$$

$$2x = 2$$

$$x = 1$$

Check:

$$\text{LS} = \frac{x+2}{3} - \frac{x-1}{5}$$

$$= \frac{1+2}{3} - \frac{1-1}{5}$$

$$= 1 - 0$$

$$= 1$$

$$\text{LS} = \text{RS}$$

Reasoning

Eliminate the fractions by multiplying both sides by the LCM of 3 and 5, which is 15.

Expand the brackets.

Simplify.

Subtract 13 from both sides.

Divide both sides by 2.

Substituting $x = 1$ into the LS of the equation gives the same value as the RS.

Solving skills 2

exercise 4.1

LINKS TO Example 1

1. Solve these equations mentally, and then check each answer by substitution.

a $x - 5 = 9$ **b** $\frac{x}{4} = 8$ **c** $11x = 77$

d $x + 11 = 9$ **e** $-3x = 27$ **f** $-\frac{x}{6} = 7$

g $8x = 56$ **h** $x + 15 = 7$ **i** $2x - 3 = 9$

j $4(x + 1) = 28$ **k** $-\frac{x}{5} = -60$ **l** $\frac{8}{x} = -2$

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For which of the following equations is $x = -2$ a solution? (Try to answer this question mentally.)

a $-x - 2 = 2$ b $-2 = -x - 4$ c $3x + 2 = 4$

d $2(2 + x) = 0$ e $-\frac{x}{2} = 1$ f $\frac{2 - x}{4} = 1$

LINKS TO Example 2

Solve each of the following equations by doing the same thing to both sides.

a $2x + 3 = 17$ b $5x - 4 = 11$ c $7x + 2 = 37$

d $4x + 7 = 43$ e $-4x - 5 = 13$ f $-5x - 13 = 8$

g $-2x + 4 = 10$ h $-3x - 5 = 22$ i $7 - 3x = 31$

j $8 - x = 17$ k $11 - 9x = 43$ l $3.8 - x = 5.2$

m $7x - 15 = 20$ n $6 - 3x = 12$ o $20 - 11x = 42$

Thuynh solves the equation $4x + 8 = 11$ as shown at the right.

a Is the solution correct?

b Suggest an alternative way of solving the equation.

$$4x + 8 = 11$$
$$\frac{4x + 8}{4} = \frac{11}{4}$$
$$x + 2 = 2\frac{3}{4}$$
$$x = \frac{3}{4}$$

The solution to the equation $\frac{y}{2} + 8 = 24$ is

A $y = 12$ B $y = 16$ C $y = 20$ D $y = 32$ E $y = 40$

LINKS TO Example 3

Solve each of the following equations by doing the same thing to both sides. Check your answers with a calculator.

a $\frac{4a}{7} = 32$ b $\frac{7x}{5} = 21$ c $\frac{-3x}{4} = 15$

d $\frac{-5x}{3} = 4$ e $\frac{9a}{11} = 18$ f $\frac{m}{5} - 1 = 3$

g $\frac{x}{3} - 5 = 2$ h $\frac{5a}{7} + 3 = 10$ i $\frac{4a}{3} + 11 = 5$

j $5 + \frac{3x}{4} = -7$ k $17 - \frac{6x}{7} = 14$ l $4 - \frac{5x}{2} = 11$

m $\frac{x}{6} + 6 = 17$ n $\frac{x}{4} - 11 = 5$ o $9 - \frac{x}{8} = 4$

p $7 - \frac{x}{3} = 9$ q $13 - \frac{x}{3} = 8$ r $23 - \frac{4x}{5} = 17$

LINKS TO Example 4

Solve each of the following linear equations.

a $\frac{a - 4}{2} = -5$ b $\frac{4x + 3}{3} = 2$ c $\frac{2c + 3}{5} = 3$

d $\frac{5 - 2x}{7} = 12$ e $\frac{2y - 3}{4} = 1$ f $\frac{3 + 5x}{7} = 9$

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g $\frac{4-6n}{5} = -7$ **h** $\frac{3x-5}{7} = 4$ **i** $\frac{7-2b}{3} = -13$

j $\frac{4x+5}{3} = -1$ **k** $\frac{34-3t}{7} = 4$ **l** $\frac{68-7x}{11} = 3$

LINKS TO Example 5

Solve each of the following equations.

a $6(x-3) = 12$ **b** $7(a+4) = 42$ **c** $2(x+3) = 7$

d $5(m-5) + 6 = 11$ **e** $3(2-3x) = -4$ **f** $5(b+3) - 9b = 11$

g $2(6x-3) = 15$ **h** $6(a-3) = 7$ **i** $5(4-d) = 8$

j $9(x-3) - 4x = 2$ **k** $4(2k-3) - 9k = 14$ **l** $12x - 7(x-8) = -35$

LINKS TO Example 6

Solve each of the following linear equations.

a $\frac{x}{5} + \frac{x}{2} = 21$ **b** $\frac{x}{2} + \frac{x}{3} = 1$ **c** $\frac{3x}{4} + \frac{4x}{5} = 62$

d $\frac{4x}{3} - \frac{3x}{5} = 33$ **e** $\frac{2x}{3} - \frac{x}{6} = \frac{1}{4}$ **f** $\frac{x}{7} + \frac{2x}{5} = \frac{57}{2}$

g $\frac{7x}{12} - \frac{x}{6} = \frac{1}{12}$ **h** $\frac{2x}{3} + \frac{5x}{4} = 46$ **i** $\frac{3x}{7} + \frac{2x}{5} = 87$

j $\frac{7x}{8} - \frac{3x}{5} = 22$ **k** $\frac{8x}{3} - \frac{2x}{5} = 17$ **l** $\frac{3x}{8} - \frac{2x}{3} = -35$

LINKS TO Example 7

Solve the following linear equations.

a $\frac{a+2}{5} + \frac{a-1}{2} = 2$ **b** $\frac{x+3}{3} + \frac{x-1}{4} = 6$ **c** $\frac{x-4}{3} + \frac{3x+1}{2} = 12$

d $\frac{k-3}{7} + \frac{k+5}{3} = 6$ **e** $\frac{x-7}{3} + \frac{x+2}{5} = 5$ **f** $\frac{2x+1}{7} - \frac{3x+1}{5} = -1$

Solve the following equation.

$$\frac{2(2x-1)}{7} - \frac{4(4x \times 3)}{3} = \frac{4(5-6x)}{5}$$

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4.2 Solving equations with pronumerals on both sides

To solve equations with pronumerals on both sides of the equals sign, it is first necessary to get all of the pronumerals onto one side of the equation.

Example 8

Solve the equation $2x + 3 = 2 + x$ algebraically. Then check that the solution is correct by substituting this value back into the original equation.

Working

$$
\begin{aligned}
2x + 3 &= 2 + x \\
2x + 3 - 3 &= 2 + x - 3 \\
2x &= x - 1 \\
2x - x &= x - 1 - x \\
x &= -1
\end{aligned}
$$

Check:

$$
\begin{aligned}
\text{LS} &= 2x + 3 \\
&= 2 \times (-1) + 3 \\
&= -2 + 3 \\
&= 1 \\
\text{RS} &= 2 + x \\
&= 2 + (-1) \\
&= 1 \\
\text{LS} &= \text{RS}
\end{aligned}
$$

Reasoning

Doing the same thing to both sides ensures that each of the equations is equivalent to the original equation.
First the 3 is eliminated by subtracting 3 from both sides.
Subtracting x from each side presents the solution.

Example 9

Solve each of the following equations for x.

a $3(x - 5) = -2x - 3$

b $5(2x - 1) = 4(x + 10)$

continued

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Example 9 continued

Working	Reasoning
a $3(x-5) = -2x - 3$	Expand the brackets.
$3x - 15 = -2x - 3$	
$3x + 2x - 15 = -2x - 3 + 2x$	Eliminate $-2x$ on the right side by adding $2x$ to both sides.
$5x - 15 = -3$	
$5x - 15 + 15 = 15 = -3 + 15$	Eliminate -15 on the left side by adding 15 to both sides.
$5x = 12$	
$\frac{5x}{5} = \frac{12}{5}$	Divide both sides by 5 to obtain x on its own.
$x = 2\frac{2}{5}$	

Check:

$$\begin{aligned} \text{LS} &= 3(x-5) \\ &= 3\left(2\frac{2}{5} - 5\right) \\ &= 3 \times \left(-2\frac{3}{5}\right) \\ &= -7\frac{4}{5} \end{aligned}$$

$$\begin{aligned} \text{RS} &= -2x - 3 \\ &= -2\left(2\frac{2}{5}\right) - 3 \\ &= -4\frac{4}{5} - 3 \\ &= -7\frac{4}{5} \end{aligned}$$

RS = LS

Working	Reasoning
b $5(2x-1) = 4(x+10)$	Expand the brackets.
$10x - 5 = 4x + 40$	Subtract $4x$ from both sides.
$10x - 5 - 4x = 4x + 40 - 4x$	
$6x - 5 = 40$	Simplify.
$6x - 5 + 5 = 40 + 5$	Add 5 to both sides.
$6x = 45$	Simplify.
$\frac{6x}{6} = \frac{45}{6}$	Divide both sides by 6.
$x = 7\frac{1}{2}$	Simplify.

continued

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Example 9 continued

Working

Reasoning

Check:

$$\begin{aligned}\text{LS} &= 5(2x-1)\\ &= 5\left(2\times 7\frac{1}{2}-1\right)\\ &= 5\times(15-1)\\ &= 70\\ \text{RS} &= 4(x+10)\\ &= 4\left(7\frac{1}{2}+10\right)\\ &= 4\times 17\frac{1}{2}\\ &= 70\\ \text{RS} &= \text{LS}\end{aligned}$$

Example 10

Solve the following equation for the unknown valuable x.

$$\frac{x-3}{2}=\frac{x+6}{3}$$

Working

Reasoning

$$\frac{x-3}{2}=\frac{x+6}{3}$$

The LCM of 2 and 3 is 6 so multiply both sides by 6 to eliminate the fractions.

$$\frac{(x-3)}{2}\times 6=\frac{(x+6)}{6}\times 6$$

$$3(x-3)=2(x+6)$$

Expand brackets.

$$3x-9=2x+12$$

Numerators must be equal.

$$3x-9-2x=2x+12-2x$$

Subtract $2x$ from both sides.

$$x-9=12$$

Simplify.

$$x-9+9=12+9$$

Add 9 to both sides.

$$x=21$$

continued

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Example 10 continued

Working	Reasoning
Check:	
$\text{LS} = \dfrac{(x-3)}{2}$	
$= \dfrac{21-3}{2}$	
$= 9$	
$\text{RS} = \dfrac{x+6}{3}$	
$= \dfrac{(21+6)}{3}$	
$= 9$	
$\text{RS} = \text{LS}$	

exercise 4.2

LINKS TO Example 8

1. Solve these linear equations.

a $3a - 2 = a + 6$ **b** $5 - 8e = 2 - 3e$ **c** $2d + 5 = 7d$

d $5x - 2 = 3x - 3$ **e** $2m + 5 = 7m + 6$ **f** $16k + 8 = 8k - 12$

g $12 - 9x = 8 - 12x$ **h** $4 - 12x = 6 - 7x$ **i** $7 - 8a = 4 - 3a$

j $3y - 2 = y$ **k** $4 - 8j = j$ **l** $7 - 3e = 2e + 3$

m $7a - 21 = 75 - 5a$ **n** $-12 - 2b = 4b + 2$ **o** $a - 7 = -5a - 2$

2. If $2 - m = 3m + 8$, then $m =$

A 3 **B** $\dfrac{5}{2}$ **C** $\dfrac{2}{3}$ **D** $-\dfrac{3}{2}$ **E** -3

LINKS TO Example 9

3. Solve for x in each of the following.

a $4(x - 3) = -2(3x + 7)$ **b** $-2(2 - a) = -3(a + 4)$

c $8\left(x + 2\right) = 2\left(2x - 5\right)$ **d** $4(2x - 3) = 9x + 14$

e $6(3 - 2x) = 5(4 - 3x)$ **f** $7(m + 2) = 5(4 + m)$

g $2(x - 10) = 4x$ **h** $21 + 3x = 3(3x - 47)$

i $2(x + 1.3) = x - 3.2$ **j** $5(3x - 4) = 4(2x - 3)$

k $-7(x - 3) = 5(x - 15)$ **l** $14(x + 2) = 4(7 - 2x)$

m $4.3x + 7.1 = 0.3(8x + 30)$ **n** $5(2x - 1.3) = 3(3x - 4.2)$

LINKS TO Example 10

4. Solve the following linear equations.

a $\dfrac{7x-1}{5} = \dfrac{x+5}{3}$ **b** $\dfrac{x+2}{3} = \dfrac{x+4}{4}$

c $\dfrac{a+5}{3} = \dfrac{a-7}{2}$ **d** $\dfrac{3-x}{2} = \dfrac{5+2x}{4}$

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e $\dfrac{5-4m}{6} = \dfrac{3m-2}{2}$

f $\dfrac{2-x}{5} = \dfrac{3+x}{2}$

g $\dfrac{p-3}{5} = \dfrac{p+7}{3}$

h $\dfrac{x+2}{5} = \dfrac{x-4}{2}$

i $\dfrac{2y+7}{2} = 3y-5$

j $\dfrac{x-2}{3} = \dfrac{x+2}{5}$

k $\dfrac{3m+1}{5} = \dfrac{2m+3}{4}$

l $\dfrac{8-p}{2} = p+1$

m $\dfrac{2b-7}{5} = \dfrac{b-3}{4}$

n $\dfrac{2-k}{3} = \dfrac{3k-14}{5}$

- Shana is asked to solve the linear equation $3(x-3) = 4x$, and her attempt is shown at the right.

  a Is Shana's solution correct?

  b Try to solve the problem in fewer steps.

  c How would you advise Shana to solve another equation presented this way?

$$
\begin{aligned}
3(x-3) &= 4x \\
x-3 &= \frac{4x}{3} \\
x &= \frac{4x}{3}+3 \\
x &= \frac{4x}{3}+\frac{9}{3} \\
x &= \frac{4x+9}{3} \\
3x &= 4x+9 \\
x &= -9
\end{aligned}
$$

- Solve the following linear equations.

  a $\dfrac{x+2}{3} + \dfrac{x-2}{4} = x+2$

  b $\dfrac{2x-1}{2} - \dfrac{3-x}{3} = \dfrac{4x}{5}$

exercise 4.2 challenge

- Explain why the equation $x - 3 = 1 + x$ has no solutions.
- Try to solve the equation $\dfrac{6x+2}{2} = 3(x-1)$. What happens and what does this tell you about the equation?

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4.3 Problem solving with linear equations

Many problems can be solved by using algebraic techniques. The process of translating the words of a problem into algebra is called **formulation**.

As a starting point, it is important to read the question carefully and identify what you are being asked to find. Choose a pronumeral to represent the unknown, then construct an equation that can be solved.

The four-step approach to solving algebraic word problems

- **Step 1: Translate the words into algebra.** Decide on the unknown variable and give it a letter, then formulate an equation that involves this variable.
- **Step 2: Solve the equation.** Solve for the variable by 'doing the same to both sides'.
- **Step 3: Check the solution.** Substitute your solution back into the original equation to check that the LS = RS.
- **Step 4: Translate the algebra back to words.** Express your solution in terms of the original problem wording checking that you have answered the question asked and that your solution fits the given information.

Example 11

There are 24 company cars in a parking lot. The cars are either black or white. There are 4 more black cars than white cars. How many white cars are in the parking lot?

Working	Reasoning
Let w be the number of white cars in the parking lot.	**Step 1: Words into Algebra** Select a pronumeral to represent the unknown.
There must be $w + 4$ black cars.	There are 4 more black cars than white cars.
$w + w + 4 = 24$ $2w + 4 = 24$	The total number of cars in the parking lot is 24.

continued

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Example 11 continued

Working

$$2w + 4 = 24$$
$$2w + 4 - 4 = 24 - 4$$
$$2w = 20$$
$$\frac{2w}{2} = \frac{20}{2}$$
$$w = 10$$

Check:

$$\begin{aligned} LS &= 2w + 4 \\ &= 2 \times 10 + 4 \\ &= 24 \\ RS &= 24 \\ RS &= LS \end{aligned}$$

There are 10 white cars in the parking lot.

Reasoning

Step 2: Solve the equation
Solve for w.
Subtract 4 from both sides.
Simplify.

Divide both sides by 2.

Simplify.

Step 3: Check the solution

Step 4: Algebra into words
Express the answer in terms of the original problem.

Example 12

One third of a class studies Japanese and one fifth of the same class studies Indonesian. This gives a total of 16 students. If no student in the class studies both languages, how many students are in the class?

Working

Let x be the number of students in the class.

$$\frac{x}{3} + \frac{x}{5} = 16$$

Reasoning

Step 1: Words into Algebra
Select a pronumeral to represent the unknown.

One third of the class is $\frac{x}{3}$ students.

One fifth of the class is $\frac{x}{5}$ students.

The total of these fractions is 16.

continued

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Example 12 continued

Working	Reasoning
$\frac{x}{3} \times 15 + \frac{x}{5} \times 15 = 16 \times 15$	**Step 2: Solve the equation** Solve for x. The LCM of 3 and 5 is 15 so multiply every term by 15 to eliminate the fractions.
$5x + 3x = 240$	
$8x = 240$	Simplify.
$\frac{8x}{8} = \frac{240}{8}$	Divide both sides by 8.
$x = 30$	
Check: $\text{LS} = \frac{x}{3} + \frac{x}{5} = \frac{30}{3} + \frac{30}{5} = 10 + 6 = 16$; LS = RS	**Step 3: Check the solution**
There are 30 students in the class.	**Step 4: Algebra into words** Express the answer in terms of the original problem.

Problems involving consecutive integers

Consecutive integers are integers that come one after the other, for example, 3 and 4 are consecutive integers. 7, 8 and 9 are consecutive integers. Notice that each consecutive integer is one more than the integer before it. If we let n stand for an integer, then the next integer will be $n + 1$, the next integer will be $n + 2$, and so on.

Consecutive even integers or consecutive odd integers are two apart, for example, 6, 8, 10, or 11, 13, 15. If we let n stand for an even integer, then the next even integer will be $n + 2$, the next even integer will be $n + 4$, and so on, and similarly for odd integers.

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Example 13

The sum of three consecutive even integers is 216. Find the three integers.

Working

Let n be the smallest of the even integers.
The other two even integers are $n + 2$ and $n + 4$.

$$\begin{aligned} n + (n + 2) + (n + 4) &= 216 \\ 3n + 6 &= 216 \\ 3n &= 210 \\ n &= 70 \end{aligned}$$

Check:

$$\begin{aligned} \text{LS} &= n + (n + 2) + (n + 4) \\ &= 70 + 72 + 74 \\ &= 216 \\ \text{LS} &= \text{RS} \end{aligned}$$

The three even integers are 70, 72 and 74.

Reasoning

Step 1: Words into Algebra
If n is the smallest of the three even integers, the next even integer will be 2 more than n and the next will be 4 more than n.
Write an equation to show the sum of the three even integers.

Step 2: Solve the equation
Solve for n.

Step 3: Check the solution
Check $70 + 72 + 74 = 216$.

Step 4: Algebra into words
Write a sentence to answer the question.

Finding unknown dimensions

Perimeters of rectangles and other shapes can be used to formulate equations and solve problems involving unknown dimensions.

Example 14

The length of a rectangle is 2.8 m longer than the width. The perimeter is 14.8 m. Find the width and length of the rectangle.

Working

Let x m be the width of the rectangle.
Length = $(x + 2.8)$ m

$$\begin{aligned} 2(x + x + 2.8) &= 14.8 \\ 2(2x + 2.8) &= 14.8 \\ 2x + 2.8 &= 7.4 \\ 2x &= 4.6 \\ x &= 2.3 \end{aligned}$$

Reasoning

Step 1: Words into Algebra
Define a variable for the width.
Express the length in terms of the width.
Write an equation using the perimeter.

Step 2: Solve the equation
Solve the equation for x.

continued

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Example 14 continued

Working	Reasoning
Check:	**Step 3: Check the solution**
$LS = 2(2x + 2.8)$ $= 2(2 \times 2.3 + 2.8)$ $= 2(4.6 + 2.8)$ $= 2 \times 7.4$ $= 14.8$ $LS = RS$	
The width of the rectangle is 2.3 m and the length is 5.1 m.	**Step 4: Algebra into words** If width is 2.3 m, length is $2.3 + 2.8 = 5.1$ m. Write a sentence answering the question.

Calculating ages

In problems involving people's ages, the problem may refer, for example, to the ages of two people in 4 years time. It is important to remember that *both* people will be 4 years older in 4 years time.

Example 15

Lachlan is 5 years older than his brother Alexander and Robbie is 4 years younger than Alexander. In three years' time Lachlan's age will be equal to the sum of the ages of Alexander and Robbie in three years' time. Find the ages now of the three brothers.

Working	Reasoning
Let x years be the age of Alexander now. Lachlan is now $x + 5$ and Robbie is now $x - 4$. In three years' time the ages will be: Alexander: $x + 3$ Lachlan: $(x + 5) + 3 = x + 8$ Robbie: $(x - 4) + 3 = x - 1$	**Step 1: Words into Algebra** The ages of Lachlan and Robbie are given in terms of the age of Alexander, so define a variable for Alexander's age. Express Lachlan's and Robbie's ages in terms of x. Write an equation showing the relationship between the ages in 3 years' time.
i.e. $x + 8 = (x + 3) + (x - 1)$ $x + 8 = 2x + 2$ $x = 6$	**Step 2: Solve the equation**

continued

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Example 15 continued

Working	Reasoning
Check: LS $= x + 8$ $= 6 + 8$ $= 14$ RS $= 2x + 2$ $= 2 \times 6 + 2$ $= 14$ LS = RS	**Step 3: Check the solution** In 3 years' time Angus will be 9, Lachlan will be 14 and Robbie will be 5. $5 + 9 = 14$
Alexander is 6, Lachlan is 11 and Robbie is 2.	**Step 4: Algebra into words** Write a sentence showing the three ages.

exercise 4.3

LINKS TO Example 11

1. For each of the following, formulate an equation and then solve for the unknown. Check your solution.
   - **a** 9 is subtracted from a to give a result of 12.
   - **b** 5 is added to twice b to give a result of 19.
   - **c** The difference between three times c and 7 is 20.
   - **d** 6 more than the quotient of d and 2 is 10.
   - **e** The sum of 4 times e and 8 is 40.
   - **f** The product of 5 and three more than f is 25.
   - **g** The product of 2 and the sum of g and 8 is –6.
   - **h** The product of 3 and the difference between h and 4 is 12.

For questions **2** to **16**, formulate an equation, solve the equation, check your solution and write a sentence to answer the question.

2. There are 51 lollies in a lolly jar. The lollies are either red or yellow. There are 5 more red lollies than yellow lollies. How many yellow lollies are in the jar?

3. A class sat a test on equations. The highest grade was 40 marks higher that the lowest grade. The sum of these two grades was 136. What was the lowest grade?

4. Marisa has a piece of string that is 40 centimetres long. She cuts it into two pieces so that one piece is 6 centimetres longer than the other. How long is each piece of string?

5. Natasha had n pot plants on the window sill. She added 3 new ones, but then half of her plants died, so she added 2 more. She now has 7 plants. How many pot plants did Natasha have to start with?

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LINKS TO Example 12

- One fifth of the students in a class caught the bus to school and two fifths of the students in the same class caught the train to school. A total of 18 students caught either the bus or the train. If no student in the class used both methods of transport, how many students are in the class?
- One quarter of the cars in a car park are silver and one fifth are white. If there are 45 silver and white cars in the car park, how many cars are in the car park altogether?
- Carla received her pay for the week and spent $\frac{1}{3}$ on rent and $\frac{1}{4}$ paying off bills. She had \$280 left. How much was she paid?

LINKS TO Example 13

- The sum of two consecutive whole numbers is 43. Find the numbers.
- The sum of two consecutive even numbers is 70. Find the numbers.
- The sum of three consecutive odd numbers is 255. Find the numbers.

LINKS TO Example 14

- The length of a rectangular field is 15 metres more than its width. If the perimeter of the field is 70 metres, find its width.
- The length of a rectangular room is 12 metres more than twice its width. If the perimeter of the room is 144 metres, find the length and width of the room.

LINKS TO Example 15

- Let j be Justine's age now.
  - **a** Write an expression for her age in 12 years' time.
  - **b** In 12 years, Justine will be 3 times as old as she was 28 years ago. Write an equation that represents this information.
  - **c** Solve this equation to find Justine's age now.
  - **d** In how many years will Justine turn 70?
- The sum of the ages of Anna and Bella is 32. In two years Bella will be three times as old as Anna. How old are they now?

exercise 4.3 challenge

- The Cheap Cars company charges \$50 per day and \$2.40 per kilometre to hire a car. The Best Hire company charges \$40 per day plus \$2.70 per kilometre. Will wishes to rent a car for 3 days. How far can he travel so that the cost would be the same from either company?

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4.4 Transposing formulas

Transposing formulas

We call this process **transposing** the formula to make r the **subject**.

The steps taken to transpose a formula are the same as those used to solve an equation.

We perform the same operations on both sides of the formula.

Example 16

Transpose each of the following to make the pronumeral in brackets the subject.

a $P = 2(l + w) \quad (l)$ **b** $F = \frac{GmM}{r^2} \quad (r)$ **c** $S = 2\pi r(r + h) \quad (h)$

Working	Reasoning
a $P = 2(l + w)$	Making l the subject is like solving the equation to find l.
$\frac{P}{2} = l + w$	Divide both sides by 2.
$\frac{P}{2} - w = l$	Subtract w from both sides.
$l = \frac{P}{2} - w$	Make l the subject.
b $F = \frac{GmM}{r^2}$	This is Newton's famous formula for calculating the gravitational force F between two masses m and M, where r is the distance between their centres and G is a constant. Making r the subject is like solving the equation to find r.
$Fr^2 = GmM$	Multiply both sides by r^2, so that r is no longer in the denominator.
$r^2 = \frac{GmM}{F}$	Divide both sides by F.
$r = \sqrt{\frac{GmM}{F}}$	Take the square root of both sides. In this case we take only the positive square root because r represents a distance and cannot be negative.

continued

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Example 16 continued

Working	Reasoning
c $S = 2\pi r(r + h)$	This is the formula for the total surface area S of a cylinder with radius r and height h. Making h the subject is like solving the equation to find h.
$\frac{S}{2\pi r} = r + h$	Divide both sides by $2\pi r$.
$\frac{S}{2\pi r} - r = h$	Subtract r from both sides.
$h = \frac{S}{2\pi r} - r$	Make h the subject.

exercise 4.4

LINKS TO Example 16

1. Transpose each of the following formulas, making the pronumeral in brackets the subject.

a $a = x + b \quad (b)$ **b** $F = ma \quad (a)$ **c** $D = ST \quad (T)$

d $I = PRT \quad (T)$ **e** $C = 2\pi r \quad (r)$ **f** $E = mc^2 \quad (m)$

g $V = \frac{lwh}{3} \quad (w)$ **h** $M = \frac{a + b}{2} \quad (b)$ **i** $P = 2(l + h) \quad (h)$

j $A = \pi r^2 \quad (r)$ **k** $v = u + at \quad (t)$ **l** $P = \frac{k}{V} \quad (V)$

m $A = \frac{(a + b)h}{2} \quad (b)$ **n** $S = b(2b + h) \quad (h)$ **o** $F = \frac{9C}{5} + 32 \quad (C)$

p $s = ut + \frac{1}{2}at^2 \quad (a)$ **q** $V = \frac{\pi r^2 h}{3} \quad (h)$ **r** $V = \frac{\pi r^2 h}{3} \quad (r)$

exercise 4.4 challenge

1. Solve the following equations for x in terms of the given pronumerals.

a $a + b + x = c$ **b** $\frac{ax}{b} = c$ **c** $\frac{ax + b}{c} = d$

d $\frac{a(bx + c)}{d} = e$ **e** $a(x + c) = bx$ **f** $\frac{ax}{b} + \frac{cx}{d} = e$

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Analysis task

Diophantus and the donkey and the mule

For thousands of years, people have enjoyed creating and solving number puzzles. Diophantus was a Greek mathematician who was born about 250 CE (just over 1750 years ago). A story similar to the following one was written on his tombstone:

Diophantus spent $\frac{1}{6}$ of his life as a child, $\frac{1}{12}$ as a young man, then another $\frac{1}{7}$ of his life as a single man before he married. Five years after he married, he left his hometown and spent $\frac{1}{2}$ his life in another town. He returned to his hometown 4 years before he died. How old was Diophantus when he died?

a Using x to represent Diophantus' age in years when he died, write an expression in terms of x for the sum of the stages of Diophantus' life.

b Put your expression equal to x (why?) and solve the equation to find x.

c How old was Diophantus when he died?

The Donkey and Mule Problem

This old problem is attributed to the ancient Greek mathematician Euclid:

The mule says to the donkey, 'If you give me one of your sacks, I would have as many as you.'

The donkey says to the mule, 'If you give me one of your sacks, I would have twice as many as you.'

d Let x be the number of sacks that the donkey has and let y be the number of sacks that the mule has.

In parts **e** and **f**, each of the equations will include both x and y.

e Write an equation to represent what the mule says to the donkey.

f Write an equation to represent what the donkey says to the mule.

g The two equations you have written must be true at the same time. Work out a way of solving the two equations to find how many sacks the donkey and the mule each have.

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Review *Linear equations*

Summary

Substitution and formulas

- To evaluate an expression or rule for particular values of the pronumerals, replace the pronumerals with those numbers. This is called substitution.
- A pronumeral that may take many values is called a variable.
- A formula is a rule that is used for calculating a particular quantity.
- Rearranging a formula to make a different pronumeral the subject is called transposing the formula. Transposing a formula involves the same steps as solving an equation.

Solving equations and inequalities

- Solving an equation means finding the value of the pronumeral that makes the equation true.
- Numeric solving strategies include mental strategies, guess, check and improve and using a table of values.
- Equations can be solved algebraically by doing the same thing to both sides.
- Constructing an algebraic expression or equation from a worded problem is called formulation.

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Visual map

Using the following terms (and others if you wish), construct a visual map that illustrates your understanding of the key ideas covered in this chapter.

equation	formulation	subject
evaluate	pronumeral	substitution
expression	solving	transposing
formula	solution	variable

Revision

Multiple-choice questions

1. If 7 is subtracted from the quotient of x and 2 the result is 5. The equation to solve for x is

A $\frac{x}{2} - 7 = 5$
B $7 - \frac{x}{2} = 5$
C $\frac{x-7}{2} = 5$
D $\frac{7-x}{2} = 5$
E $\frac{2}{x} - 7 = 5$

2. If $\frac{a+1}{2} - \frac{a-2}{3} = 1$ then $a =$

A 13
B 7
C 1
D $\frac{7}{5}$
E −1

3. The solution to the equation $\frac{2x}{3} - 10 = 3(x-1)$ is

A $x = 3$
B $x = 2\frac{1}{3}$
C $x = -\frac{3}{7}$
D $x = -3$
E $x = -2\frac{6}{7}$

4. If $A = 5B - C$ then

A $B = \frac{A+C}{5}$
B $B = \frac{A-C}{5}$
C $B = \frac{5+C}{A}$
D $B = \frac{C-A}{5}$
E $B = \frac{A}{5} + C$

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Short-answer questions

Solve the following equations for x.

a $3x + 7 = 1$

b $\frac{x}{3} + 4 = 6$

c $3(x - 7) = 39$

d $3x - 20 = x + 4$

e $4x - 6 = 15 - 3x$

f $-12 = 2 - 7x$

g $3x + 17 = 2$

h $15 + 4x = 10 - 6x$

i $\frac{2(x - 1)}{5} - 5 = 7$

j $3\left(\frac{2x}{5} - 1\right) = 1\frac{4}{5}$

k $\frac{x}{4} + \frac{x}{3} = 21$

l $\frac{2x}{5} + \frac{3x}{4} = 46$

m $\frac{2x + 7}{3} = \frac{3}{4}$

n $\frac{x + 6}{4} = \frac{x + 7}{3}$

o $\frac{3x + 4}{5} = \frac{5 + x}{3}$

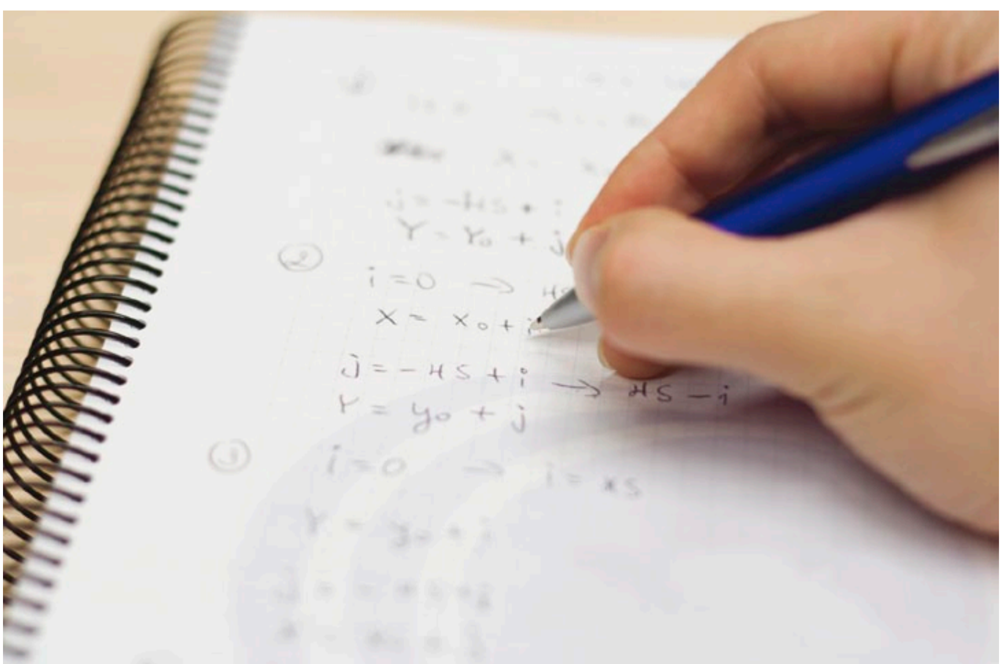
p $\frac{x + 1}{2} + \frac{x + 1}{2} = 10$

Transpose each formula to make the pronumeral in brackets the subject.

a $v = u + at \quad (a)$

b $S = 4\pi r^2 \quad (r)$

c $T = \sqrt{\frac{3W}{5P}} \quad (P)$



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Extended-response questions

The temperature $c°$ Celsius can be converted to $f°$ Fahrenheit using the formula $f = 32 + \frac{9c}{5}$.

a Find the temperature f °F for a day with a temperature of 25 °C, that is, $c = 25$.

b What is the temperature c °C when the temperature is 100 °F?

c For what temperature is the number of degrees the same for both Celsius and Fahrenheit scales?

To book tickets over the internet there is a \$7 booking fee. The cost of a ticket is \$48. Find purchases x tickets over the internet.

a Write an expression for the total cost of the tickets.

b The total cost of Fin's purchase is \$535. Write an equation to show this.

c Solve the equation to find the number of tickets that Fin purchased.

Practice quiz
Chapter 4

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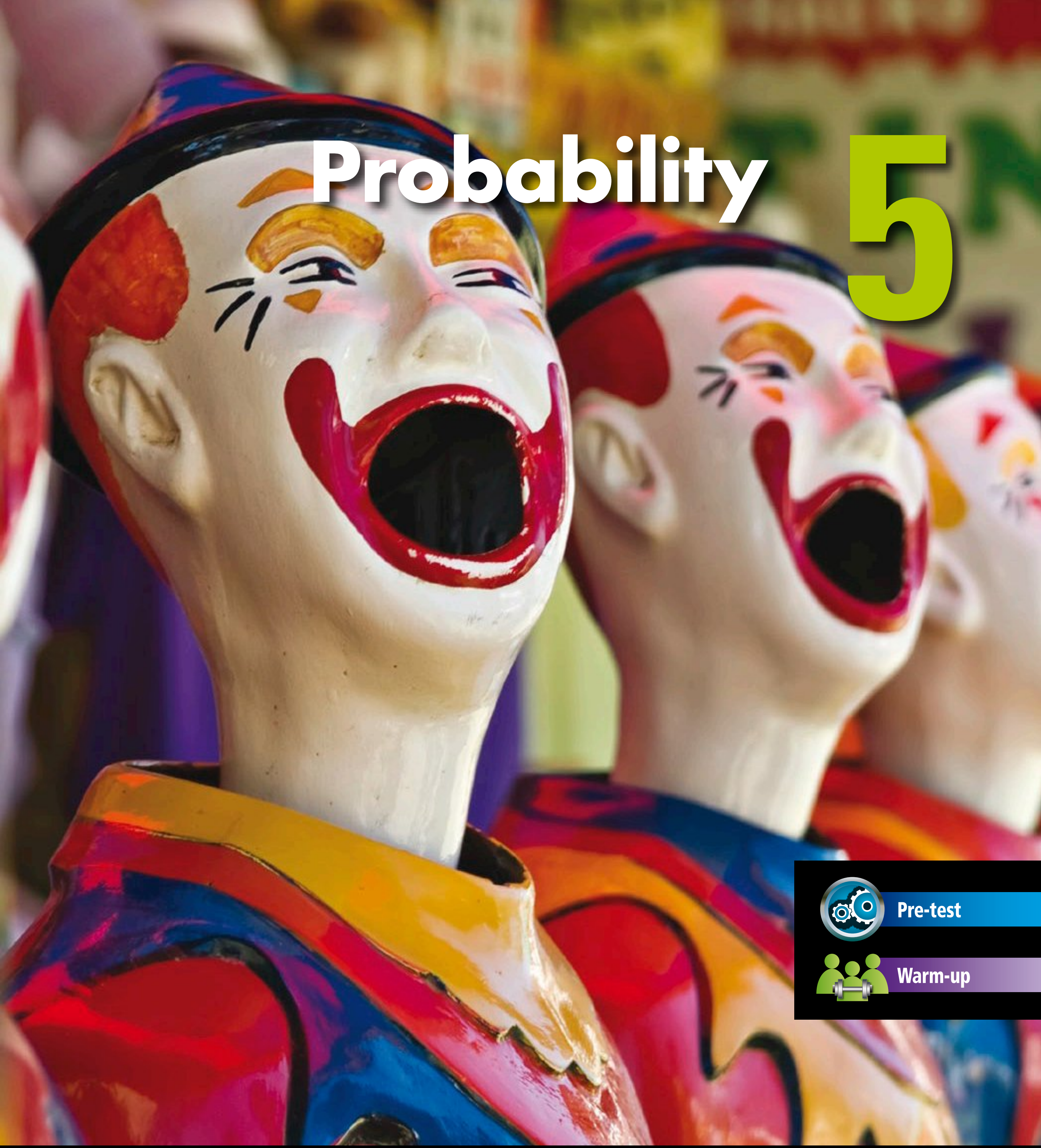
Probability 5



Pre-test

Warm-up

Many games depend on chance events, such as the rolling of a particular number on a die or drawing a particular card from a pack. Inside these laughing clowns there are several alternate pathways along which a ball can roll. What is the chance of a ball rolling along a winning pathway? Is every pathway equally likely? What is your chance of winning compared with your chance of not winning? Are games of chance fair?

5.1 Reviewing theoretical probability

BLM Chapter warm-up

In mathematics, we use the word **random** to refer to situations where the **outcomes**, or things that can happen, are **uncertain**. The **probability** of a particular outcome is a measure of how likely that outcome is to occur.

Probabilities are measured on a scale from 0 to 1, with 0 representing an **impossible** outcome and 1 representing a **certain** outcome.

A particular set of outcomes that we are interested in is referred to as an **event**. For example, when rolling a die, we might be interested in the event of rolling a 6, or the event of rolling an even number, or the event of rolling a number less than 3.

Sometimes, especially when all possible outcomes are **equally likely**, it is possible to calculate the theoretical probability of an event.

Theoretical probability of an event

$$\Pr(\text{event}) = \frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$$

The set of all possible outcomes, S is called the **sample space** or the **event space**.

The **complement** of an event is the set of outcomes that are *not* in that event. For example, if the event is 'rolling a 6' on a die, then the complement of the event is 'not rolling a 6', that is, rolling 1, 2, 3, 4 or 5.

The probabilities of complementary events always add up to 1, because it is certain that one event or the other must occur. In probability notation, A' means the complement of A.

Complementary events

$\Pr(A') = 1 - \Pr(A)$
where A' is the complement of event A.

Example 1

A spinner has eight equal sections numbered 1 to 8. Carla spins the spinner once.

- **a** List the sample space.
- **b** If X is the event 'spinning a number less than 4', list the set of outcomes in this event.
- **c** What is the event X'?
- **d** List the set of outcomes in the event X'.
- **e** Find $\Pr(X)$ and $\Pr(X')$.
- **f** Which is the more likely event, X or X'?

continued

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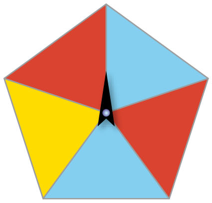
Example 1 continued

Working	Reasoning
a $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$	The sample space is the set of 8 possible outcomes from spinning the spinner.
b $X = \{1, 2, 3\}$	The possible outcomes are 1, 2 or 3.
c The event X' is 'spinning a number greater than or equal to 4'.	X' is the complement of X.
d $X' = \{4, 5, 6, 7, 8\}$	The outcomes in X' are the possible outcomes that are not 1, 2 or 3.
e $\Pr(X) = \frac{3}{8}$ $\Pr(X') = \frac{5}{8}$	For the event X there are three favourable outcomes and a total of eight possible outcomes. For the event X' there are five favourable outcomes and a total of eight possible outcomes.
f X' is the more likely event.	$\frac{5}{8} > \frac{3}{8}$ so the probability of X' is greater than the probability of X.

Theoretical probabilities can be used to determine how many times a particular event is likely to occur when conducting an experiment. This is sometimes called the **expected value**.

For the spinner shown, the probability of spinning blue is $\frac{2}{5}$. This means that with every five spins of the spinner you would expect to spin blue twice. In general, you would expect $\frac{2}{5}$ of all spins to be blue. If you spin the spinner 100 times you would expect $\frac{2}{5} \times 100 = 40$ blues.



So, in an experiment, the expected number of a particular outcome can be determined by multiplying the probability of that outcome by the number of trials.

Expected frequency of an event

Expected frequency of an event = Pr(Event) × number of trials.

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Example 2

A fair die is rolled 240 times. How many 6s could you expect to roll?

Working

$\Pr(6) = \frac{1}{6}$

Reasoning

Determine the probability of rolling a 6 with a fair die first.

A 'fair die' is a die that is not weighted towards any of the six numbers.

Working

Expected number of 6's $= \frac{1}{6} \times 240$

$= 40$

Reasoning

Use the equation:
expected number
= Pr (Event) × number of trials
to determine the expected number of 6's.

exercise 5.1

LINKS TO Example 1

A letter is to be selected from the set {A, B, C, D, E, F, G}. Find the probability that it is a vowel.

LINKS TO Example 2

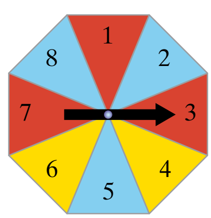
One card is selected at random from a standard pack of 52 playing cards. Find each of the following probabilities.

a Pr(heart)
b Pr(red or black)
c Pr(not a spade)
d Pr(picture card)

The picture cards are the jack, queen and king in each suit.

For this spinner, find the following probabilities.

a Pr(blue)
b Pr(odd number)
c Pr(blue and odd)
d Pr(blue or odd)



The letters of the word PROBABILITY are written on slips of paper and put into a box. One slip is chosen at random. Find the following probabilities.

a Pr(The letter chosen is B)
b Pr(The letter chosen is P)
c Pr(The letter chosen is a consonant)
d Pr(The letter chosen is a vowel or Y)

A group of lockers numbered 1 to 28 is reserved for your class. If you are allocated one of those lockers at random, calculate the probability that your locker number will be

a less than 5
b even

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c a multiple of 7

d a factor of 28

e a multiple of 7 and a factor of 28

f a multiple of 7 or a factor of 28.

g the same as the date (day) of your birthday

Hint: first list all the possible locker numbers for each of the above.

● The heights of students in a class are shown in the table at the right.
Find the probability that a student chosen at random from the class has a height that is

a less than 160 cm.

b between 155 cm and 165 cm.

Height (cm)	Number of students
145–<150	1
150–<155	2
155–<160	9
160–<165	5
165–<170	2
170–<175	1

● The word 'MATHEMATICS' is printed and then cut up into individual letters. These are placed in a bowl and a letter is selected at random. Find the probability that the letter will be

a an A.

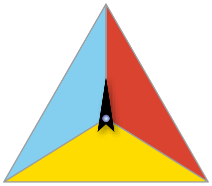
b a vowel.

c one of the first 5 letters in the alphabet.

d a letter from the word ALGEBRA.

LINKS TO Example 2

● Hien tosses a coin 200 times; how many heads could she expect to toss?
Jake spins this spinner 600 times. How many times would he expect to spin yellow?



exercise 5.1 challenge

● In a Scrabble game the bag contains each letter of the alphabet and 2 blank tiles (these can represent any letter). Julio has R, A, V, A, G, E and can draw one letter. What is the probability that he can make either RAVAGED or RAVAGES?

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5.2 Reviewing frequency and experimental probability

In the previous section we considered theoretical probabilities. In many situations we cannot assume that different outcomes are equally likely so we may need to estimate the probability of a particular outcome. Sometimes we may estimate a probability by carrying out an experiment with a number of trials. We can then calculate the **relative frequency** of a particular event occurring.

Tossing a coin

Relative frequency

$$\text{Relative frequency of event } A = \frac{\text{number of times } A \text{ occurs}}{\text{total number of trials}}$$

We can then use this relative frequency as an **estimated probability** or **experimental probability**.

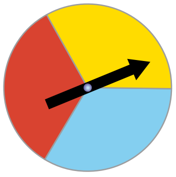
Rolling a die

Estimated or experimental probability

$$\Pr(A) = \frac{\text{number of times } A \text{ occurs}}{\text{total number of trials}}$$

Example 3

Cait spins the spinner shown 20 times. The spinner lands on red 7 times.



a Calculate the relative frequency of the spinner landing on red.

b Estimate the probability of the spinner landing on red.

Working

a

$$\text{Relative frequency of event } A = \frac{\text{number of times } A \text{ occurs}}{\text{total number of trials}}$$

$$\text{Relative frequency of red} = \frac{\text{number of times spinner lands on red}}{\text{total number of spins}}$$

$$= \frac{7}{20}$$

Reasoning

The spinner landed on red 7 times. The number of trials is 20.

continued

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Example 3 continued

Working

b $\Pr(A) = \dfrac{\text{number of times } A \text{ occurs}}{\text{total number of trials}}$

$\Pr(R) = \dfrac{\text{number of times spinner lands on red}}{\text{total number of spins}}$

$= \dfrac{7}{20}$

Reasoning

Let R represent the event 'landing on red'.

Example 4

Horatio has a six-sided die. He conducts an experiment to determine if the die is fair; that is, if each side has the same chance of being rolled. He rolls the die 180 times and records the numbers rolled. Horatio's results are shown below.

Number	Tally	Frequency
1	卌 卌 卌 卌 卌 ||	27
2	卌 卌 卌 卌 卌 卌 ||||	34
3	卌 卌 卌 卌 卌 |||	28
4	卌 卌 卌 卌 卌 卌 ||	32
5	卌 卌 卌 卌 卌 卌 |||	33
6	卌 卌 卌 卌 卌 |	26

Each tally mark represents the result from a trial (a roll).

a What is the theoretical probability of rolling each number?

b Add a relative frequency column to the table and calculate the relative frequency of each number as a decimal to three decimal places.

c Is this likely to be a fair die?

Working

a $\Pr(A) = \dfrac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$

Theoretical probability is $\dfrac{1}{6}$, or approximately 0.167.

Reasoning

There are 6 equally likely outcomes on a fair die.

continued

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Example 4 continued

10 trial probability simulation

Working

b

Number	Tally	Frequency	Relative Frequency
1	卌 卌 卌 卌 卌 II	27	$\frac{27}{180} = \frac{3}{20} = 0.150$
2	卌 卌 卌 卌 卌 卌 IIII	34	$\frac{34}{180} = \frac{17}{90} \approx 0.189$
3	卌 卌 卌 卌 卌 III	28	$\frac{28}{180} = \frac{7}{45} \approx 0.156$
4	卌 卌 卌 卌 卌 卌 II	32	$\frac{32}{180} = \frac{8}{45} \approx 0.178$
5	卌 卌 卌 卌 卌 卌 III	33	$\frac{33}{180} = \frac{11}{60} \approx 0.183$
6	卌 卌 卌 卌 卌 I	26	$\frac{26}{180} = \frac{13}{90} \approx 0.144$

Reasoning

The total for each number is divided by the total of 180.

c The die is probably fair.

The numbers are all close to 0.167.

50 trial probability simulation

The estimated or experimental probability of an event can be improved by extending an experiment to include more trials. The relative frequency tends to move closer to the expected long-run probability. This can be shown on a graph of the relative frequency against the number of trials.

Example 5

Amanda tosses a coin 20 times and records her results in a table.

Outcome	Frequency	Relative frequency
Head	8	0.4
Tail	12	0.6

She then tosses the coin another 20 times and adds these to those previously obtained.

Outcome	Frequency	Relative frequency
Head	22	0.55
Tail	18	0.45

continued

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Example 5 continued

She tosses the coin another 20 times and adds these results to her table.

Outcome	Frequency	Relative frequency
Head	28	0.47
Tail	32	0.53

She repeats the experiment again, adding the results to her table.

Outcome	Frequency	Relative frequency
Head	38	0.475
Tail	42	0.525

a Use Amanda's results to complete the table below.

Number of throws	Relative frequency of number of tails
20	
40	
60	
80	

b Plot a graph with number of throws on the horizontal axis and relative frequency of number of tails on the vertical axis. Join your points using straight lines.

c Does the coin appear to be fair?

For a fair coin, the probability of tossing a head and the probability of tossing a tail is 0.5.

Working

a

Number of throws	Relative frequency of number of tails
20	0.6
40	0.45
60	0.53
80	0.525

Reasoning

This data is read directly from the relative frequency column and the tail row of each of Amanda's tables of results.

continued

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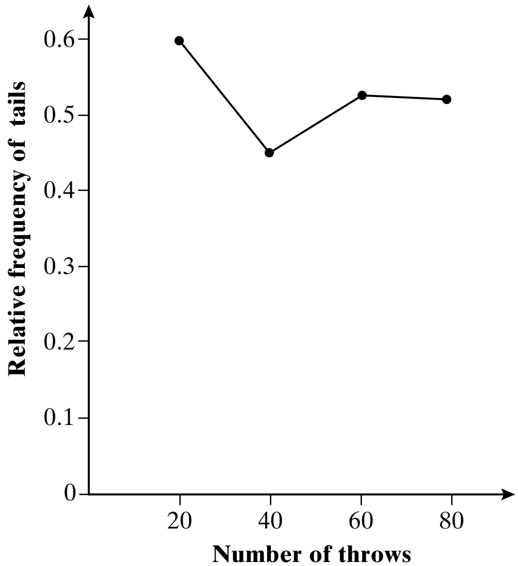
Example 5 continued

Working

b



Reasoning

Plot the ordered pairs (20, 0.6), (40, 0.45), (60, 0.53) and (80, 0.525). Connect the points using straight lines.

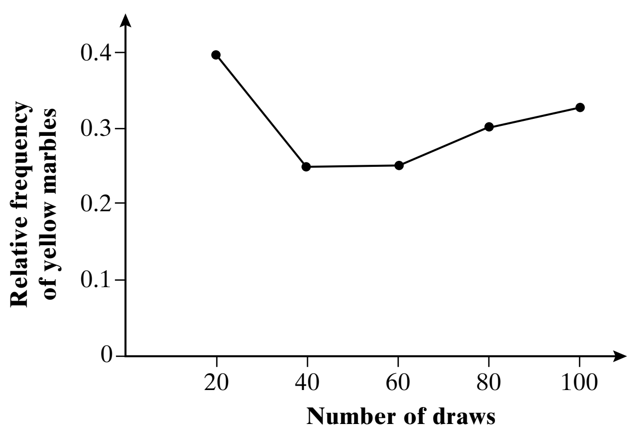
Note: this idea can be simulated with the **Probability simulator** spreadsheet in the student ebook.

c The coin appears to be fair because with more trials the relative frequency of number of tails is approaching 0.5.

Relative frequency gives an estimate of the theoretical probability. For a fair coin the theoretical probability of tossing a tail is 0.5.

Example 6

Jie has a bag containing a red marble, a white marble and a yellow marble. She draws a marble from the bag, notes its colour then replaces it. She does this 100 times. Every 20 times she determines the relative frequency of drawing a yellow marble. Her results are graphed on the right.



Use the graph to determine the number of yellow marbles obtained in the

a first 20 draws **b** first 40 draws **c** second 20 draws.

continued

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Example 6 continued

Working

a Relative frequency = $\dfrac{\text{frequency of favourable outcomes}}{\text{total frequency}}$

When the number of draws is 20, the relative frequency is 0.4.

$$\frac{\text{Number of yellow marbles drawn}}{20} = 0.4$$

Number of yellow marbles drawn $= 20 \times 0.4$
$= 8$

b When the number of draws is 40, the relative frequency is 0.25.

$$\frac{\text{Number of yellow marbles drawn}}{40} = 0.25$$

Number of yellow marbles drawn $= 40 \times 0.25$
$= 10$

c We know from part **b** that 10 yellow marbles are drawn in total in the first 40 draws and from that, eight yellow marbles are drawn in the first 20 draws.
The number of yellow marbles in the second 20 draws is $10 - 8 = 2$.

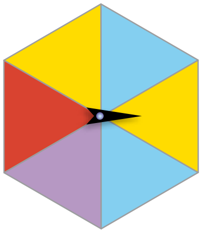
Reasoning

In this case the total frequency is the number of draws. Points on the graph give the number of draws and the corresponding relative frequency.

Subtracting the number of yellow marbles drawn in the first 20 draws from the number in the first 40 draws gives the number in the second 20 draws.

Example 7

Galina spins the spinner shown 60 times, recording the number of blues after each ten spins. Her results are shown in the table below.



continued

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Example 7 continued

Number of spins	Number of Blues	Relative frequency
10	5	
20	9	
30	13	
40	17	
50	20	
60	22	

a Find the theoretical probability of each colour to three decimal places.

b Complete the relative frequency column. Give your answers correct to three decimal places.

c Plot the relative frequency of blue against number of rolls; joining points with straight lines.

d Is the relative frequency approaching the theoretical probability of spinning blue?

Working

a $\Pr(\text{Red}) = \frac{1}{6} \approx 0.167$

$\Pr(\text{Purple}) = \frac{1}{6} \approx 0.167$

$\Pr(\text{Blue}) = \frac{2}{6} \approx 0.333$

$\Pr(\text{Yellow}) = \frac{2}{6} \approx 0.333$

b

Number of spins	Number of blues	Relative frequency
10	5	$\frac{5}{10} = 0.500$
20	9	$\frac{9}{20} = 0.450$
30	13	$\frac{13}{30} = 0.433$
40	17	$\frac{17}{40} = 0.425$
50	20	$\frac{20}{50} = 0.400$
60	22	$\frac{22}{60} = 0.367$

Reasoning

One of the six sectors is red and one is purple.

Two of the six are blue and two are yellow.

Divide each frequency by the number of spins.

continued

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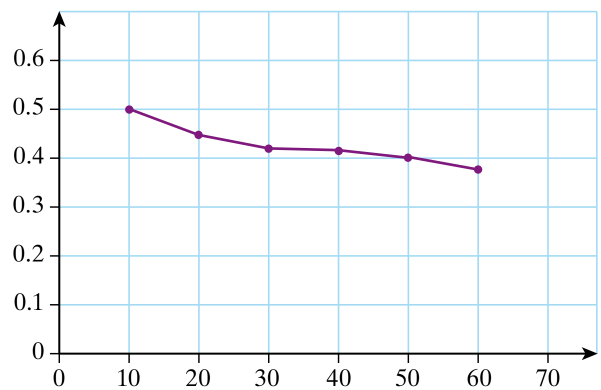
Example 7 continued

Working

Reasoning

c



d Yes, the relative frequency is approaching the theoretical probability at 0.333.

The graph appears to be approaching 0.333.

We now look at how data from surveys and opinion polls can be used to estimate probabilities and predict outcomes.

Example 8

The probability of rain in Darwin in January is approximately 0.68. In a particular week in Darwin in January, on how many days would you expect rain?

Working

$\Pr(R) = 0.68$

Number of wet days expected in a week

$= 0.68 \times 7$

$= 4.76$

You would expect that it would rain on 5 days.

Reasoning

Use R to represent the event of rain on any day in January in Darwin.

Multiply the probability of rain on any day by the number of days.

Example 9

At Sunlands Secondary College, the student representative council asked 600 students whether they supported the introduction of a compulsory sun hat. Of 600 students, 350 voted 'Yes', 150 voted 'No' and 100 were undecided.

a Find the probability that a student chosen at random from this group

i voted 'Yes' **ii** voted 'No' **iii** was undecided.

b The college has 1500 students. How many students would you expect to vote yes?

continued

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Example 9 continued

Working	Reasoning
a i $\Pr(\text{Yes}) = \dfrac{\text{number who voted yes}}{\text{total number}} = \dfrac{350}{600} = \dfrac{7}{12}$	The proportion of the group who voted 'yes' gives the probability.
ii $\Pr(\text{No}) = \dfrac{\text{number who voted no}}{\text{total number}} = \dfrac{150}{600} = \dfrac{1}{4}$	This is the proportion of the group who voted 'no'. The probability is the same as the proportion.
iii $\Pr(\text{No}) = \dfrac{\text{number who were undecided}}{\text{total number}} = \dfrac{100}{600} = \dfrac{1}{6}$	The probability is the proportion who were undecided.
b $\dfrac{7}{12}$ of $1500 = \dfrac{7}{12} \times 1500 = 875$	The expected number in the whole population is in the same proportion as in the sample who were surveyed.

Quality control in production often use probability experiments to estimate the proportion of products that are faulty or to estimate variation.

Example 10

A machine fills bottles with mineral water. Each bottle should contain 1000 mL. Nick randomly chooses 100 bottles of mineral water and tests the machine by measuring the volume of mineral water in each bottle. The results of his tests are shown in the table below.

Volume of water	Less than 1000 mL	Exactly 1000 mL	More than 1000 mL
Frequency	10	42	48

Use the data from Nick's tests to estimate the probability that the next bottle he selects will

a contain exactly 1000 mL.

b not contain exactly 1000 mL.

c be underfilled.

d be overfilled.

continued

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Example 10 continued

Working	Reasoning
a $\Pr(A) = \dfrac{\text{number of times } A \text{ occurs}}{\text{total number of trials}}$ $= \dfrac{\text{number of times 1000 mL}}{\text{number of bottles tested}}$ $= \dfrac{42}{100}$ $= 0.42$ Probability that the next bottle contains 1000 mL is 0.42.	We can use sample data to estimate probabilites. The relative frequency of an event *A* can be used to estimate the probability of *A* occurring.
b $\Pr(\text{not } 1000 \text{ mL}) = 1 - 0.42$ $= 0.58$	The events containing 1000 mL and not containing 1000 mL are complementary. Their probabilities add to 1.
c $\Pr(A) = \dfrac{\text{number of times } A \text{ occurs}}{\text{total number of trials}}$ $= \dfrac{\text{number of times less than 1000 mL}}{\text{number of bottles tested}}$ $= \dfrac{10}{100}$ $= 0.1$ Probability that the next bottle contains less than 1000 mL is 0.1.	If a bottle is underfilled it contains less than 1000 mL.
d $\Pr(A) = \dfrac{\text{number of times } A \text{ occurs}}{\text{total number of trials}}$ $= \dfrac{\text{number of times more than 1000 mL}}{\text{number of bottles tested}}$ $= \dfrac{48}{100}$ $= 0.48$ Probability that the next bottle contains more than 1000 mL is 0.48.	If a bottle is overfilled it contains more than 1000 mL.

exercise 5.2

LINKS TO Example 1

1 In a survey of 200 people, 125 said that they approved of a new bridge. Calculate, in simplest form, the relative frequency of a person approving of the bridge.

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Marielle tosses a coin 1000 times. The results of her experiment are shown in the table at right.

a Calculate the experimental probability of tossing a

i head. **ii** tail.

b Do you think this is a fair coin? Give a reason for your answer.

Outcome	Frequency
Head	330
Tail	670

LINKS TO Example 4

Ari records the colours of cars driving in one lane on a freeway. He tallies his results in a table.

a Add a frequency column and a relative frequency column to the table. Fill in these columns giving relative frequencies as decimals.

b How many cars did Ari count in his survey?

c What is the most likely colour of the next car to drive along this lane of the freeway?

d Give an estimate of the probability that the next car in the lane will be black.

Car colour	Tally
White	卌 卌 卌 卌 卌 IIII
Silver	卌 卌 卌 III
Red	卌 卌 卌 I
Black	卌 卌 卌 卌 I
Blue	卌 I
Green	III
Other	卌 II

e Give an estimate of the probability that the next car in the lane will not be red.

f How can the estimate of the probability that the next car will be silver be made more accurate?

LINKS TO Example 5

Niles rolls a die 60 times and records the number of times each number occurs.

Number on die	Tally	Frequency
1	卌 III	
2	卌 IIII	
3	卌 II	
4	卌 卌 II	
5	卌 卌 III	
6	卌 卌 I	

a Complete the frequency column for the table.

b Determine the relative frequency of each outcome, expressing your answer as a simplified fraction.

c Based on the results of Niles' experiment, how many 4s would you expect to roll in

i 120 rolls of this die **ii** 600 rolls of this die?

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d Based on the theoretical probability of obtaining a 4 when you roll a die, how many 4s would you expect to roll in

i 120 rolls of the die **ii** 600 rolls of the die?

LINKS TO Example 6

Grainne rolls a die 100 times and works out the relative frequency of rolling a 2 after every 20 rolls. Her results are recorded in the table below.

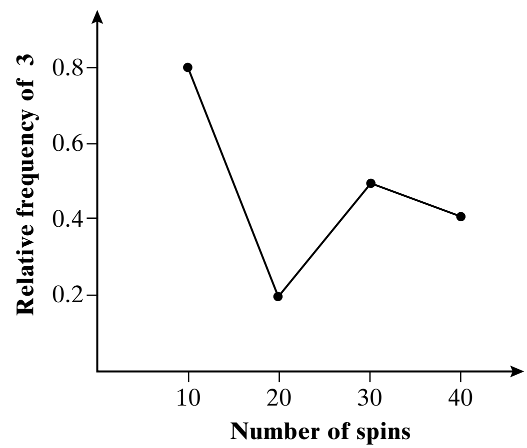
Number of rolls	20	40	60	80	100
Relative frequency of a 2	0.34	0.30	0.23	0.20	0.24

a Plot a graph of relative frequency of a 2 against number of rolls joining the points using straight lines.

b Do you think that this die is fair? Give a reason for your answer.

LINKS TO Example 6

Hugo takes a spinner from a board game and tests it to see if it is fair. The graph shows the relative frequency of getting a 3.



a Write down the number of 3s obtained in the first

i 10 spins **ii** 20 spins

iii 30 spins **iv** 40 spins.

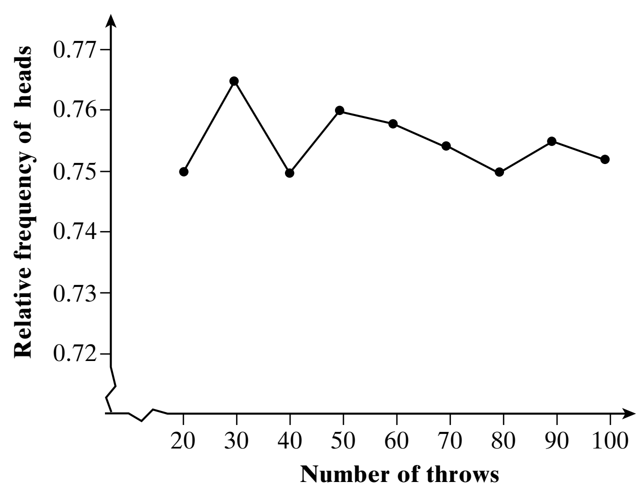
b Use the answers from part **a** to explain why one of the points on the graph must be incorrect.

Ben threw a coin 20 times and recorded the number of heads and tails obtained in a table.

a Draw Ben's table and complete the frequency column in Ben's table.

He threw the coin another 80 times and recorded the total number of heads and tails at the end of every 10 throws.

The graph shows the relative frequency of number of heads against number of throws.



b How many heads were obtained in the first 50 throws?

c How many tails were obtained in the first 50 throws?

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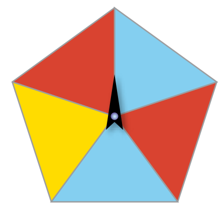
d Use the graph to estimate the probability of obtaining a head on one throw. Give the probability correct to two decimal places.

e Use your result from part **d** to estimate the probability of obtaining a tail with one throw.

f Do you think that the coin Ben used in his experiment was a fair coin? (Give a reason for your answer.)

LINKS TO Example 7

Tom spins the spinner shown 100 times.



He records the number of reds for every 20 spins in a table as shown.

Number of spins	Number of reds	Relative frequency
20	12	
40	26	
60	33	
80	40	
100	45	

a Complete the relative frequency column of this table.

b Calculate the theoretical probability of rolling each colour.

c Plot the relative frequency of red, against number of rolls; joining each of your points with straight lines.

d Is the relative frequency approaching the theoretical probability of spinning red?

LINKS TO Example 8

The probability that a letter posted before 6 pm on a weekday will be delivered to its destination the following day is 0.85. Jenny posts 120 birthday invitations. How many would she expect to be delivered the following day?

An insurance company expects 25% of its motor vehicle policy holders to make a claim in the next 12 months. If the company has 24 360 motor vehicle policy holders, how many of these people could they expect to make a claim in the next 12 months?

A machine produces plastic cups. Philip takes a sample of 50 plastic cups and checks them for faults. He finds 4 cups that have a fault.

a The machine produces 12 000 cups each day. Based on Philip's sample, how many would you expect the machine to produce that are faulty?

b The next day, a team of quality controllers check all 12 000 cups produced by the machine and find 480 cups with faults. Is this a better or worse result than expected?

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A class experimented with dropping a buttered slice of bread and noting whether it landed buttered side down or buttered side up. They found it landed buttered side down 72 times and buttered side up 18 times.

a Find Pr(buttered side down).

b If 500 slices were dropped, how many would you expect to land buttered side down?

Ethan rolls a die that is biased in such a way that in 100 rolls he expects to roll 20 ones.

a What is the probability of rolling 1 with this die?

b What is the probability of not rolling 1 with this die?

c Ethan rolls the die 400 times. How many ones would he expect to roll?

Sandhiya notes that she has walked to school 8 times in the last 48 school days. What is the probability that she will not walk to school tomorrow?

LINKS TO Example 9

Aaron has won 10 of the last 15 games of Scrabble that he has played against Jay.

a What is the probability that he will win the next game?

b How many games would he expect to win in 60 games?

LINKS TO Example 9

Fifty Year 9 students were surveyed about the introduction of a new elective subject. 28 were in favour, 16 were against and 6 had no opinion.

a Find the probability that a student is

i in favour. **ii** against. **iii** has no opinion.

b There are 225 year 9 students at the school. How many would you expect to be in favour of the new elective?

Priyanka surveyed 12 of her friends and found that 6 loved the new Pink album, 2 disliked it and 4 had never heard it.

a What is the probability that a friend loved the album?

b In her class of 27 students, how many would you expect to have not heard the album?

A poll taken of 100 people at a supermarket found that 68 thought that the new, smaller trolleys were an improvement over the old trolleys, 24 thought they were worse and the remainder never used a trolley.

a What is the probability that a customer never uses a trolley?

b On an average day 800 customers use the store. How many of these would not be expected to use a trolley?

LINKS TO Example 10

A tissue box packing machine fills boxes with tissues. Each box is expected to contain 200 tissues. James randomly selects 100 boxes of tissues and counts the number of tissues in each box. His results are shown in the table below.

Number of tissues	Less than 200	Exactly 200	More than 200
Frequency	18	44	38

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Use the data from James' tests to estimate the probability that the next box he selects will

a contain exactly 200 tissues.

b not contain exactly 200 tissues.

c contain less than 200 tissues.

d contain 200 tissues or more.

exercise 5.2 challenge

● As a fund-raising activity, a group buys bulk chocolates to sell in boxes of 12. Since the chocolates all look the same, they decide to taste test a sample of 48 and record the centres that they find. The results are shown in the table.

Centre	Number
Chewy	11
Nut	9
Fruit	23
Mint	5

Use the table to estimate the probable number of each type that would be found in each of the boxes sold.

● The last digit in Katie's phone number is 1. She wants to estimate the probability that a randomly selected number from the phone book will end in the same number. She opens the phone book to a random page and reads off the last digit of each phone number in batches of 50 numbers for 300 phone numbers. Her results are shown in the table below.

Number of phone numbers	50	100	150	200	250	300
Number of 1s	8	13	18	25	28	33

a Determine the relative frequency of a 1 for each number of phone numbers.

b Plot the relative frequency of a 1 against number of phone numbers, joining each of your points with straight lines.

c Use your graph to estimate the probability of the last digit being 1.

d What is the theoretical probability of the last digit in a phone number being a 1?

e Compare your answers to parts **c** and **d**.

  i Does the data from Katie's experiment give a good estimate of the theoretical probability?

  ii How could you improve the accuracy of this estimate?

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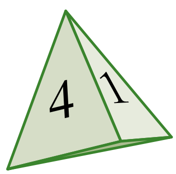
5.3 Two-step events: tree diagrams and lattice diagrams

When we are concerned with the outcomes of tossing a coin, we are considering a **simple event**. Similarly, rolling a die is a simple event. However, is we toss a coin and roll a die, the sample space consists of outcomes involving both events. We call this a **compound event**.

$$S = \{(H, H), (H, T), (T, H), (T, T)\}$$

Example 11

A coin is tossed and this tetrahedral die numbered 1 to 4 is rolled.



List the sample space.

Working

$S = \{(H,1),(H,2),(H,3),(H,4),(T,1),(T,2),(T,3),(T,4)\}$

Reasoning

If the coin lands heads, the tetrahedron may land on either 1, 2, 3 or 4. Similarly if the coin lands tails.

For two-step events, a tree diagram or table helps to show the outcomes and calculate probabilities.

Tree diagrams

A tree diagram is one way of showing the possible outcomes at each stage of an experiment. Each branch of the tree diagram represents an outcome.

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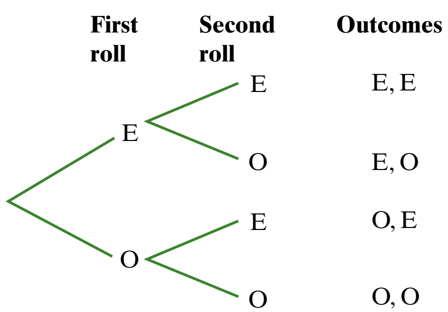
Example 12

Amy rolls a die twice and notes whether the result is odd (O) or even (E).

a Use a tree diagram to list all possible outcomes.
b What is the probability that both rolls are even?
c What is the probability that exactly one roll is odd?
d What is the probability that at least one roll is even?
e What is the probability of getting two even numbers or two odd numbers.

Working



Reasoning

The first two branches represent the possible outcomes of the first roll, even or odd.
Amy can get even or odd on the second roll, so draw two branches after each of the outcomes from the first roll.

List all the possible outcomes by following each pathway through the tree.

b $\text{Pr(both even)} = \dfrac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$

$= \dfrac{1}{4}$

One of the four possible outcomes has two even numbers.
E, E
E, O
O, E
O, O

c $\text{Pr(exactly one odd)} = \dfrac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$

$= \dfrac{2}{4}$

$= \dfrac{1}{2}$

Two of the four possible outcomes have exactly one odd number.
E, E
E, O
O, E
O, O

continued

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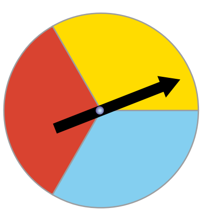
Example 12 continued

Working	Reasoning
d $\text{Pr(at least one even)} = \dfrac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}} = \dfrac{3}{4}$	Three of the four possible outcomes have at least one even number. **E, E** **E, O** **O, E** O, O
e $\text{Pr(both even or both odd)} = \dfrac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}} = \dfrac{2}{4} = \dfrac{1}{2}$	Of the four possible outcomes, one has both numbers even and one has both numbers odd. **E, E** E, O O, E **O, O**

The different stages of the experiment can also be different actions. In the following example, the first stage of the experiment is tossing a coin and the second stage is spinning a spinner.

Example 13

Steven tosses a coin and spins the spinner.



a Draw a tree diagram to list the possible outcomes.

b Find Pr(head and blue).

c Find Pr(tail or red).

continued

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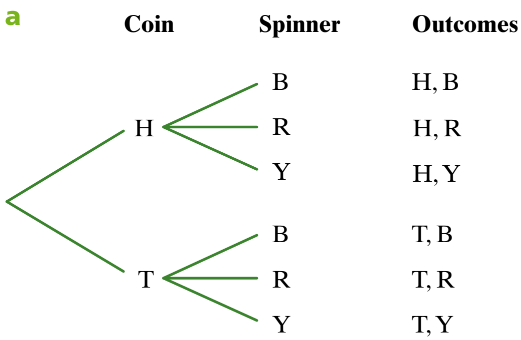
Example 13 continued

Working

Reasoning

a



There are two possible outcomes from tossing the coin: heads (H) or tails (T).

For each of these outcomes, there are 3 possible outcomes for the spinner: blue (B), red (R) or yellow (Y).

We show these three possible outcomes by drawing three branches from each of the H and T branches.
List all the possible outcomes by following each pathway through the tree.
This gives a total of 6 possible outcomes.

b Pr(head and blue)

$= \dfrac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$

$= \dfrac{1}{6}$

There is 1 outcome with a head for the coin and blue for the spinner.
H, B
H, R
H, Y
T, B
T, R
T, Y

c Pr(tail or red)

$= \dfrac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$

$= \dfrac{4}{6}$

$= \dfrac{2}{3}$

There are 4 outcomes with a tail for the coin or green for the spinner (or both).
H, B
H, **R**
H, Y
T, B
T, **R**
T, Y

Example 14

In a game of chance, a die is rolled and a coin is tossed.

a List all possible outcomes for this game using a tree diagram.

b What is the total number of possible outcomes?

c What is the probability of obtaining a head and a 3?

d What is the probability of obtaining a head and an even number?

continued

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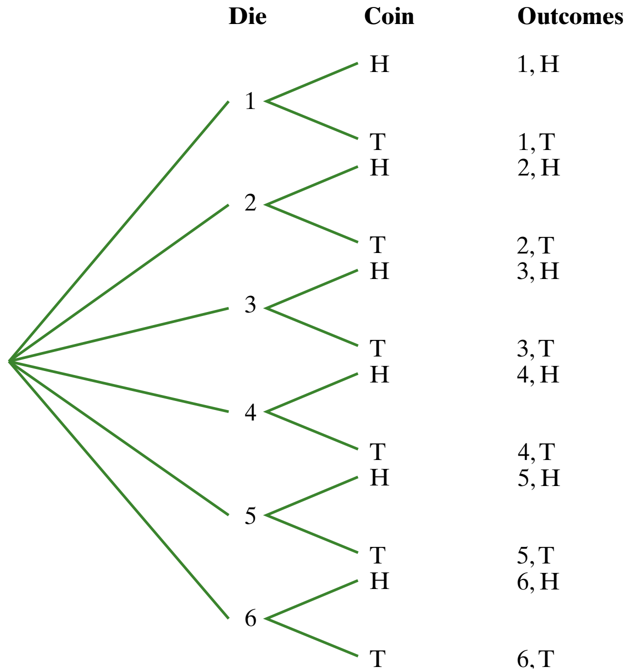
Example 14 continued

Working

a



Reasoning

The first six branches represent the possible outcomes from rolling the die.
We show the two possible outcomes from tossing the coin (H) or (T) by drawing two branches from each of the six branches for the die.
List all the possible outcomes by following each pathway through the tree.

b There are 12 possible outcomes.

Count the number of outcomes

c Pr(head and 3)

$$= \frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$$

$$= \frac{1}{12}$$

One of the 12 possible outcomes is a head and a 3.

1, H	4, H
1, T	4, T
2, H	5, H
2, T	5, T
3, H	6, H
3, T	6, T

d Pr(head and even)

$$= \frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$$

$$= \frac{3}{12}$$

$$= \frac{1}{4}$$

Three of the 12 possible outcomes are a head and an even number.

1, H	**4, H**
1, T	4, T
2, H	5, H
2, T	5, T
3, H	**6, H**
3, T	6, T

The fraction can be simplified.

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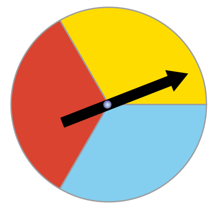
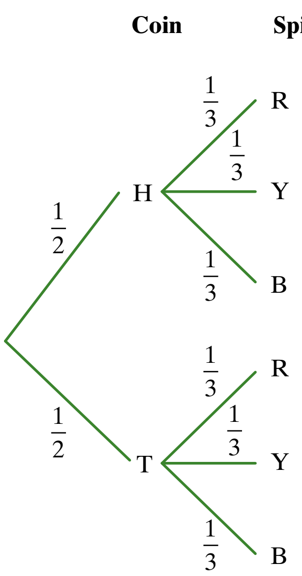
Calculating probabilities from a tree diagram

We look again at the coin and spinner in example **13**. We can write the probability for each event on the branches. Each of the six outcomes is equally likely so the probability of each outcome is $\frac{1}{6}$. Note that this probability can be found by multiplying the two probabilities on each branch of the tree: $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$





Outcomes	Probabilities
H, R	$Pr(H, R) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$
H, Y	$Pr(H, Y) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$
H, B	$Pr(H, B) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$
T, R	$PR(T, R) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$
T, Y	$Pr(T, Y) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$
T, B	$Pr(T, B) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

Multiplication of probabilities

Probabilities of a combination of outcomes along the branches of a tree diagram can be **multiplied** to find the probability of one outcome **and** the other outcome occurring together.

We associate multiplication of probabilities with the word *and*.

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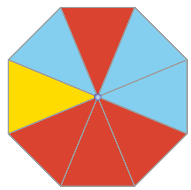
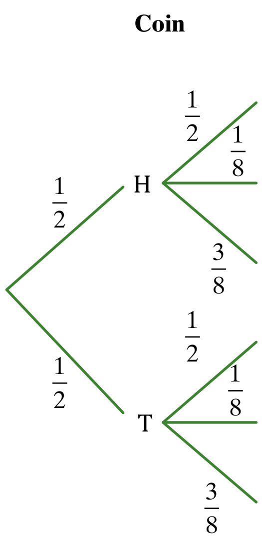
Example 15

Eleni tosses a coin and spins this spinner.



a Draw a tree diagram showing all the possible outcomes.
b Write the probabilities along the branches.
c Calculate the probability of each outcome and write beside the outcomes.

Working



Outcomes	Probabilities
H, R	$\Pr(H, R) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
H, Y	$\Pr(H, Y) = \frac{1}{2} \times \frac{1}{8} = \frac{1}{16}$
H, B	$\Pr(H, B) = \frac{1}{2} \times \frac{3}{8} = \frac{3}{16}$
T, R	$\Pr(T, R) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
T, Y	$\Pr(T, Y) = \frac{1}{2} \times \frac{1}{8} = \frac{1}{16}$
T, B	$\Pr(T, B) = \frac{1}{2} \times \frac{3}{8} = \frac{3}{16}$

The probabilities of the 8 outcomes must add to 1.
Check:

$$\frac{1}{4} + \frac{1}{16} + \frac{3}{16} + \frac{1}{4} + \frac{1}{16} + \frac{3}{16}$$
$$= \frac{4}{16} + \frac{1}{16} + \frac{3}{16} + \frac{4}{16} + \frac{1}{16} + \frac{3}{16}$$
$$= \frac{16}{16}$$
$$= 1$$

Reasoning

Four of the 8 sections of the spinner are red (R) so $\Pr(R) = \frac{4}{8} = \frac{1}{2}$.

One of the 8 sections of the spinner is yellow (Y) so $\Pr(Y) = \frac{1}{8}$.

Three of the 8 sections of the spinner are blue (B) so $\Pr(B) = \frac{3}{8}$.

Multiply the probabilities along each branch of the tree diagram.

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For the tree diagram in example **15**, suppose we wish to know the probability of getting either ‘tails and red’ or ‘heads and blue’. These are separate branches of the tree diagram so we can add the two probabilities.

$$
\begin{aligned}
\Pr(\text{T, R or H, B}) &= \frac{1}{4} + \frac{3}{16} \\
&= \frac{4}{16} + \frac{3}{16} \\
&= \frac{7}{16}
\end{aligned}
$$

Example 16

For the tree diagram in example **15**, calculate the following probabilities.

a ‘tails and yellow’ or ‘heads and blue’

b ‘heads and yellow’ or ‘tails and red’

Working

a
$$
\begin{aligned}
\Pr(\text{T, Y or H, B}) &= \frac{1}{16} + \frac{3}{16} \\
&= \frac{4}{16} \\
&= \frac{1}{4}
\end{aligned}
$$

b
$$
\begin{aligned}
\Pr(\text{H, Y or T, R}) &= \frac{1}{16} + \frac{1}{4} \\
&= \frac{1}{16} + \frac{4}{16} \\
&= \frac{5}{16}
\end{aligned}
$$

Reasoning

The outcomes ‘tails and yellow’ and ‘heads and blue’ are on different branches of the tree diagram so we can *add* the probabilities to find the probability of ‘tails and yellow’ *or* ‘heads and blue’.

The outcomes ‘heads and yellow’ and ‘tails and red’ are on different branches of the tree diagram so we can *add* the probabilities to find the probability of ‘heads and yellow’ *or* ‘tails and red’.

Addition of probabilities

Probabilities of two outcomes that are on different branches of a tree diagram can be *added* to find the probability of one outcome *or* the other.

We associate addition of probablities with the word *or*.

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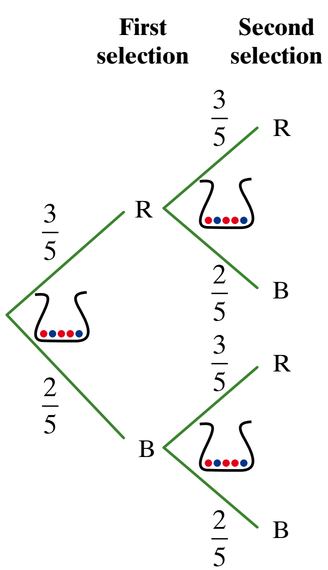
Independent events

So far we have considered compound events where the outcome of the second event is not influenced by the outcome of the first event. We say that the events are **independent**. For example, when a die is rolled and a coin is tossed, the number obtained on the die does not affect whether the outcome of tossing the coin is a head or a tail.

With replacement

Independent events occur when repeated selections are made from a set of objects and the selected object is replaced before making a second selection. Suppose a bag contains 3 red balls and 2 blue balls. Jim randomly selects one ball from the bag. The probability of selecting a red ball is $\frac{3}{5}$ and the probability of selecting a blue ball is $\frac{2}{5}$. If Jim returns the ball he selects to the bag and then randomly selects a ball from the bag again, the probabilities of selecting a red ball or a blue ball are the same as for the first selection. That is, the two events are independent. The tree diagram for these two selections is shown below.



Outcomes	Probabilities
R, R	$\Pr(R,R) = \frac{3}{5} \times \frac{3}{5} = \frac{9}{25}$
R, B	$\Pr(R,B) = \frac{3}{5} \times \frac{2}{5} = \frac{6}{25}$
B, R	$\Pr(B,R) = \frac{2}{5} \times \frac{3}{5} = \frac{6}{25}$
B, B	$\Pr(B,B) = \frac{2}{5} \times \frac{2}{5} = \frac{4}{25}$

Independent events

When repeated selections are made with replacement, the outcome of the first selection does not influence the outcome of the next selection so the events are independent.

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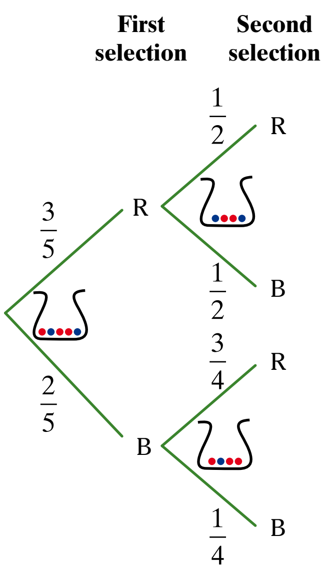
Dependent events

In some compound events the outcome of the second event is influenced by the outcome of the first so the two events are not indpendent. In some compound events the outcome of the second event is influenced by the outcome of the first so the two events are not independent, that is they are **dependent**.

Without replacement

Dependent events occur when repeated selections are made from a set of objects and the selected object is *not* replaced before making a second selection.

Suppose Jim selects a red ball and does not replace it before he randomly selects a second ball. There are now only 4 balls, 2 red and 2 blue. The probability of selecting a red ball is now $\frac{2}{4}$, that is, $\frac{1}{2}$, and the probability of selecting a blue ball is also $\frac{2}{4}$ or $\frac{1}{2}$. On the other hand, if Jim selects a blue ball the first time, 3 of the 4 remaining balls are red and only one is blue. The probability of selecting a red ball is now $\frac{3}{4}$ and the probability of selecting a blue ball is $\frac{1}{4}$. So the outcome of the second selection depends on what colour Jim selected the first time. This means the outcome of the second selection is not independent of the first selection.



Outcomes	Probabilities
R, R	$\Pr(R,R) = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10}$
R, B	$\Pr(R,B) = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10}$
B, R	$\Pr(B,R) = \frac{2}{5} \times \frac{3}{4} = \frac{3}{10}$
B, B	$\Pr(B,B) = \frac{2}{5} \times \frac{1}{4} = \frac{1}{10}$

Without replacement: dependent events

When repeated selections are made without replacement, the outcome of the first selection *influences* the outcome of the next selection and the events are *dependent*.

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Example 17

Anna selects a card at random from a standard pack of 52 playing cards. The card she selects is the queen of spades. Anna does not replace the card. She randomly selects a second card from the pack.
Find the probability that the second card is

a a queen **b** a spade **c** the queen of spades.

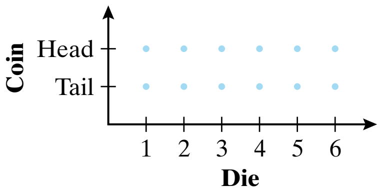
Working	Reasoning
a $\Pr(\text{queen}) = \dfrac{3}{51}$	One card has been selected without replacement so there are only 51 cards left. One queen has been selected so there are only 3 queens left.
b $\Pr(\text{spade}) = \dfrac{12}{51}$	One card has been selected without replacement so there are only 51 cards left. One spade has been selected so there are only 12 spades left.
c $\Pr(\text{queen of spades}) = 0$	The queen of spades has been selected and not replaced. So there is no chance of drawing the queen of spades in the second selection.

Lattice diagrams

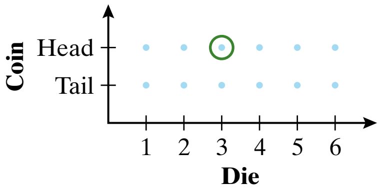
Lattice diagrams, sometimes called **dot diagrams**, are particularly useful when a compound event involving two activities has a larger number of outcomes.

Each activity is represented by an axis, and each outcome is represented by a point.

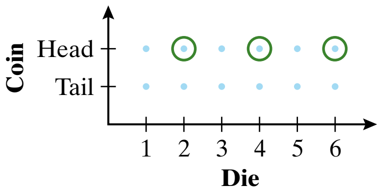
The information in example **15** is shown here as a lattice diagram. The 12 possible outcomes are represented as dots.



One of the 12 outcomes is 'head and 3'. This is shown by the circle around the dot corresponding to 'head' and 'three'.



Three of the 12 outcomes are 'head and even'.



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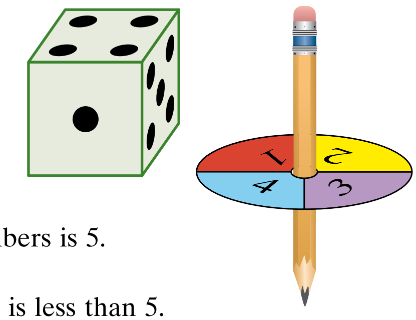
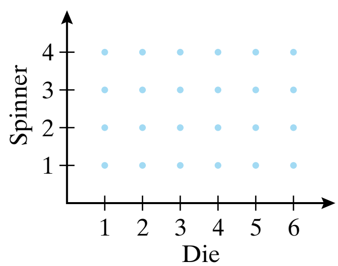
Example 18

Matthew rolls a regular six-sided die and spins the spinner shown.



a Represent the outcomes on a lattice diagram.
b How many outcomes are there?
c Find the probability that the number on the die and the spinner are the same.
d Find the probability that the sum of the two numbers is 5.
e Find the probability that both numbers are even.
f Find the probability that the sum of the numbers is less than 5.

Working	Reasoning
a 	Show the outcomes for the die along one axis and the outcomes for the spinner along the other. Draw a dot to correspond with each outcome from rolling the die and spinning the spinner.

b There are $6 \times 4 = 24$ outcomes.

There are 6 outcomes for the die and 4 possible for the spinner.

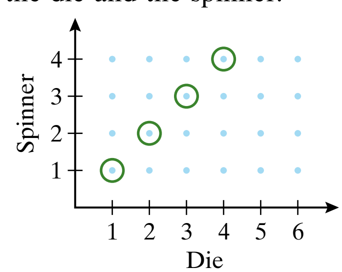
c Pr(same number)

$$= \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$$

$$= \frac{4}{24}$$

$$= \frac{1}{6}$$

There are 4 points with the same number for the die and the spinner.



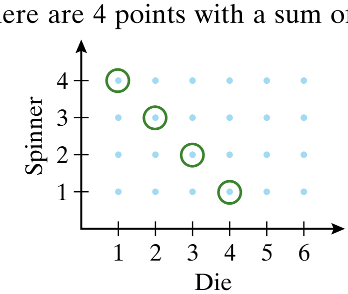
d Pr(sum of 5)

$$= \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$$

$$= \frac{4}{24}$$

$$= \frac{1}{6}$$

There are 4 points with a sum of 5.



continued

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Example 18 continued

Working

e Pr(both even)

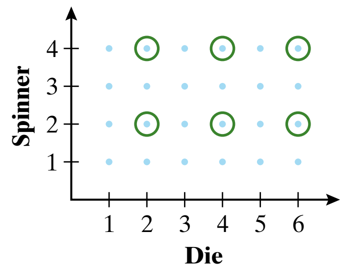
$$= \frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$$

$$= \frac{6}{24}$$

$$= \frac{1}{4}$$

Reasoning

There are 6 points with both numbers even.



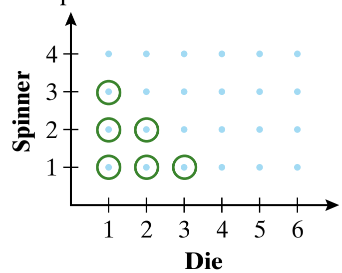
f Pr(sum < 5)

$$= \frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$$

$$= \frac{6}{24}$$

$$= \frac{1}{4}$$

There are 6 points with a sum less than 5.



Example 19

A green die and a blue die are tossed together.

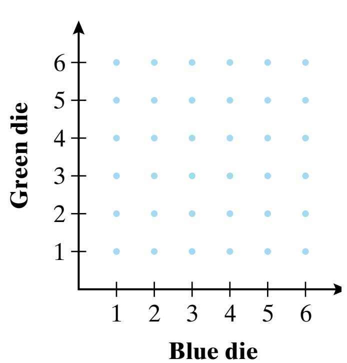
a Use a lattice diagram to show all the possible outcomes.

b Use the lattice diagram to find

- **i** the total number of outcomes.
- **ii** the number of outcomes where the numbers rolled are the same.
- **iii** the probability of rolling the same number on each die.
- **iv** the probability of rolling a total less than 7.

Working

a



Reasoning

There are 6 possible outcomes for the green die and 6 possible outcomes for the blue die. Each point in the lattice represents an outcome.

continued

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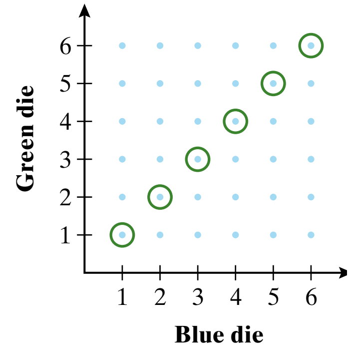
Example 19 continued

Working | **Reasoning**

b i There are 36 outcomes.

There are $6 \times 6 = 36$ points on the lattice.

ii There are 6 outcomes where the numbers rolled are the same.



iii Pr(same number on each die)

$$= \frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$$

$$= \frac{6}{36}$$

$$= \frac{1}{6}$$

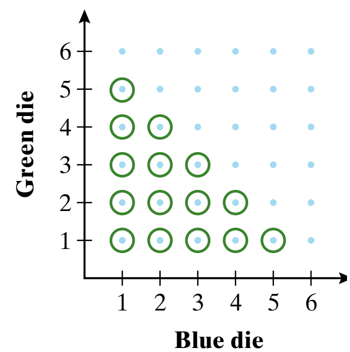
iv Pr(total less than 7)

$$= \frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$$

$$= \frac{15}{36}$$

$$= \frac{5}{12}$$

There are 15 favourable outcomes.



exercise 5.3

LINKS TO Example 11

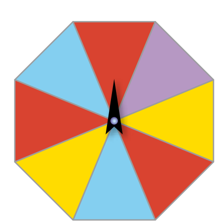
1 List the sample spaces for each of the following compound events.

a tossing two coins

b rolling a die and tossing a coin

c spinning this spinner twice

d drawing two playing cards from a pack of 52 cards and noting the suit of each card.



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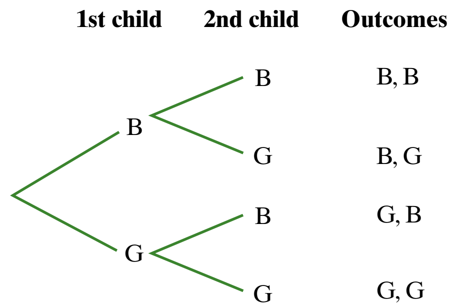
LINKS TO Example 12

This tree diagram shows the possible outcomes for a family of two children. Use it to answer the following questions.



a What is the probability that the older child is a boy?

b What is the probability that both children are the same sex?

c What is the probability that there is at least one girl?

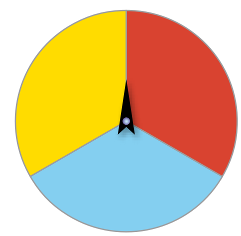
d Draw a similar diagram for three children.

e What is the probability that the children are all of the same sex?

f What is the probability that there are at least two girls?

LINKS TO Examples 13, 14

Oscar tosses the coin then spins the spinner.



a Draw a tree diagram showing all the possible outcomes.

b Write the probabilities along the branches.

c What is the probability of tails and red?

d What is the probability of tails and red or heads and blue?

There are three true or false questions on Liam's science test that he just doesn't know the answer to, so he decides to guess the answer. For each question he will either be correct or incorrect.

a Use a tree diagram to list the possible outcomes for Liam's guesses.

b What is the probability that he guesses all three answers correctly?

c What is the probability that he has no more than one incorrect guess?

LINKS TO Example 15

Goran has two pairs of pants, one black and one blue in his wardrobe. He also has three shirts: one white, one blue and one green.

a Draw a tree diagram to show all possible combinations of Goran's pants with shirts, writing the probabilities along the branches.

b If he randomly selects one pair of pants and one shirt in the dark one morning, what is the probability that the pants will be black and the shirt will not be white?

LINKS TO Example 16

Angelo rolls a 4-sided die then tosses a coin.

a Draw a tree diagram to show all possible combinations, writing the probabilities along the branches.

b Find the probability of getting

i a 4 and a head. **ii** an even number and a tail.

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LINKS TO Example 17

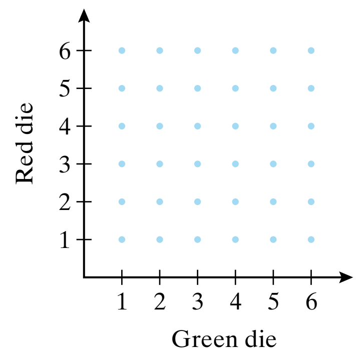
A drawer contains four different-coloured pairs of socks: black, brown, white and blue. (Each pair is folded together.) Laurinda reaches into the drawer and takes out one pair of socks, without looking. She wants to wear the black socks, but the pair she selects is the wrong colour. So she puts the socks back into the drawer and selects another pair, again, without looking.

a What is the probability that Laurinda selects the white pair both times?

b If Laurinda did not return the white socks to the drawer after her first selection, what is the probability that she would select the black pair for her second selection?

LINKS TO Examples 18, 19

A red die and a green die are rolled and the outcomes are represented by dots in this lattice diagram.



a Use the diagram to find each of the following.

i Pr(sum of 3) **ii** Pr(sum of 5)

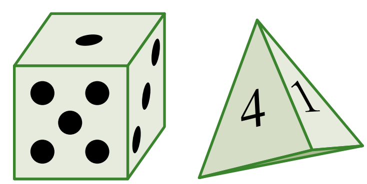
iii Pr(sum of 12) **iv** Pr(at least one 3)

v Pr(same number on each die)

vi Pr(a greater number on the green die than red die).

b What is the most likely sum?

A tetrahedron labelled 1 to 4 and a standard six-sided die are rolled.



a Draw a lattice diagram to show all the ordered pairs of outcomes in the event space.

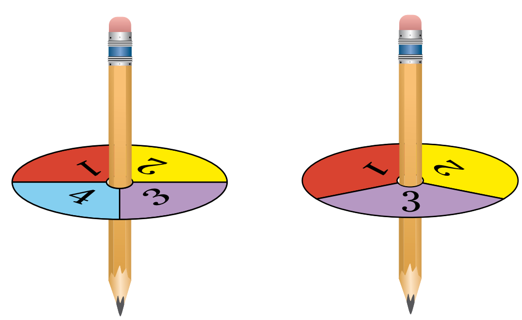
b Use your diagram to find the probability of each of the following outcomes.

i A double six **ii** At least one 4

iii A sum of 6 **iv** Both odd numbers

v At least one odd number **vi** no odd numbers.

Peter spins each of these spinners once.



a Use a lattice diagram to show all the possible outcomes.

b Find the probability of

i spinning an even number on both spinners.

ii spinning the same number on each spinner.

iii spinning different numbers on the spinners.

Elsa is sitting a test where she is required to answer true or false to each question. She doesn't know the answer to three of the questions and decides to guess each of those answers. She is equally likely to pick the correct answer or the incorrect answer.

a Use a tree diagram to list the sample space.

b What is the probability that Elsa guesses only one correct answer?

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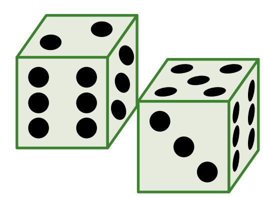
A die is rolled twice.

- **a** Use a lattice diagram to show all the possible outcomes.
- **b** Find the probability that
  - **i** the first roll is an odd number.
  - **ii** the second roll is a 1 or a 2.
  - **iii** the sum of the two rolls is 9.
  - **iv** the number on the first roll is greater than the number on the second roll.
  - **v** the number on the first roll is greater than 2 and the number on the second roll is less than 4.
  - **vi** the sum of the two rolls is even.
- **c** For two rolls of a die, which is more likely: a sum of 7 or a sum greater than 10?

A six-sided die is tossed twice.

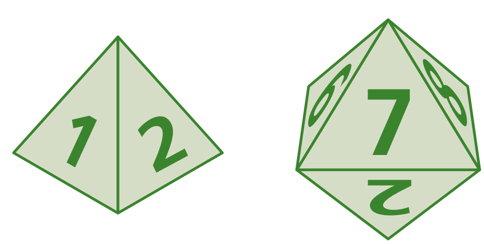
- **a** Complete a lattice diagram.
- **b** What is the total number of outcomes?
- **c** Find the probability that the outcome is
  - **i** (1, 1)
  - **ii** a double
  - **iii** (3, 1), (2, 2) or (1, 3)
  - **iv** any outcome containing a 1 or a 6.

The lattice diagram in question **11** can also be used to analyse the outcomes of rolling two dice at the same time.



- **a** Use the diagram to help find the probability that the sum of the two dice is:
  - **i** 12
  - **ii** 2 or 3
  - **iii** 11 or 12
  - **iv** less than or equal to 7
  - **v** less than 7
  - **vi** at least 10
  - **vii** at most 4.
- **b** Which total sum has the highest probability and what is the probability of tossing that sum?

Charlotte rolls a tetrahedral die (with faces numbered 1 to 4) and an octahedral die (with faces numbered 1 to 8).
What is the probability that Charlotte rolls



- **a** a 1 on both dice?
- **b** an even number on the tetrahedral die and an odd number on the octahedral die?
- **c** a 2 on the tetrahedral die and a number greater than 6 on the octahedral die?

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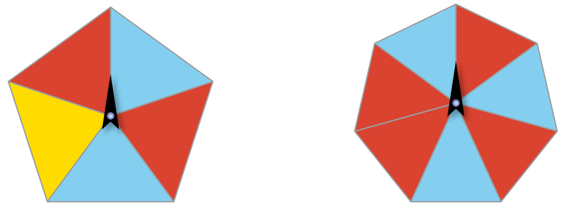
Each of these spinners is spun once. What is the probability of spinning red on the first spinner and blue on the second?



A fair die is rolled.
What is the probability of rolling

- **a** an even number?
- **b** an odd number or a factor of 4?
- **c** a prime number or a 6?
- **d** a factor of 9 or an even number?

exercise 5.3 challenge

Jason tosses a coin to decide whether to go swimming (a head) or stay home (a tail). He decides to choose the options for which he obtains two favourable outcomes out of three.

- **a** Use a tree diagram to find the probability that
  - **i** he goes swimming.
  - **ii** he stays home.
- **b** What is the probability that he only had to toss the coin twice?

A random number generator is set up to generate the digits 1, 2, 3, 4, 5. Jane uses this random number generator to generate two numbers. What is the probability that

- **a** the first number is 1 and the second number is 2?
- **b** the first number is even and the second number is odd?
- **c** the first number is greater than 2 and the second number is even?

Huang takes two calculators into a Maths exam. The probability that the calculator batteries will go flat during the exam is 0.15 for one calculator and 0.03 for the other. What is the probability that

- **a** the batteries in both calculators will go flat during the exam?
- **b** the batteries in both calculators will last the length of the exam?

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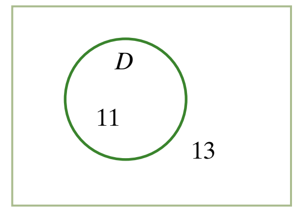
5.4 Venn diagrams and two-way tables

Venn diagrams

Venn diagram

In Year 8, we saw how Venn diagrams are useful for organising data. In a Venn diagram, a rectangle represents the total number of data values. Inside the rectangle, data for an event is shown in a circle. In this Venn diagram, the rectangle represents all the students in a particular class of 24 students. The circle represents 'students who have a dog'. The complement, 'students who do not have a dog', is shown in the rectangle outside the circle.

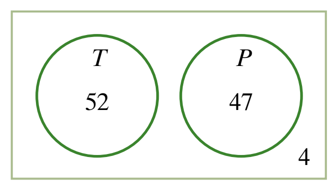


$n = 24$

In a Venn diagram, a rectangle represents the total number of data values. Inside the rectangle, data for an event is shown in a circle.

Mutually exclusive and non-mutually exclusive events

Mutually exclusive events are events that cannot occur at the same time. One event automatically excludes the other. Suppose a school offers students a choice of Textiles or Painting but not both. Because students cannot study both, being in the Textiles class and being in the Painting class are mutually exclusive. In the Venn diagram here we show this as two separate circles, where T represents the 52 students who choose Textiles and P represents the 47 students who choose Painting. There are 4 students who study neither.



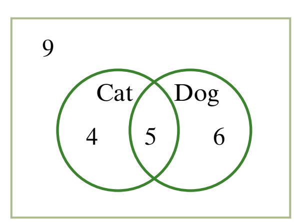
$n = 103$

In many situations, events are **not mutually exclusive**. In these cases, the circles in the Venn diagram overlap.

The Venn diagram below represents a class of 24 students. The students who have a dog as a pet and the students who have a cat as a pet are shown as overlapping circles. This overlapping region represents students who have a cat *and* a dog. The rest of the rectangle outside the circles represents the 9 students who have neither a dog nor a cat.

We can see from the Venn diagram

- 4 students have a cat only
- 6 have a dog only
- 5 students have both a cat and a dog
- 9 students have neither a cat nor a dog
- a total of $4 + 5 = 9$ students have a cat
- a total of $5 + 6 = 11$ students have a dog.



$n = 24$

The four numbers 9, 4, 5, and 6 must add to the total number of students in the class: $9 + 4 + 5 + 6 = 24$.

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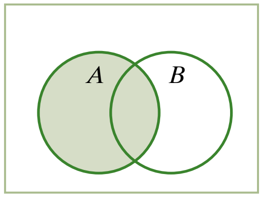
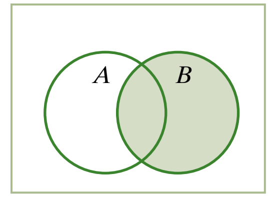
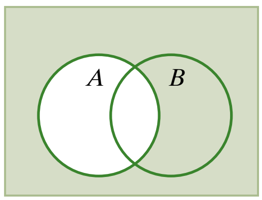
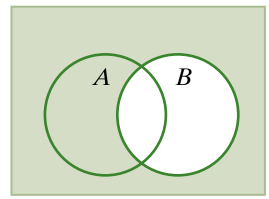
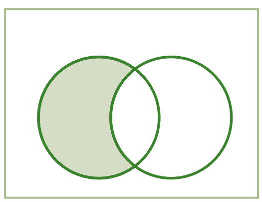
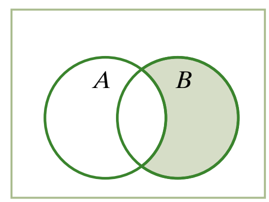
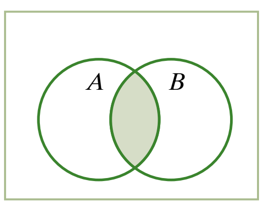
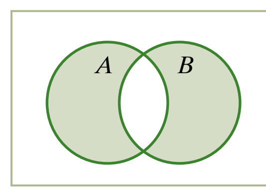
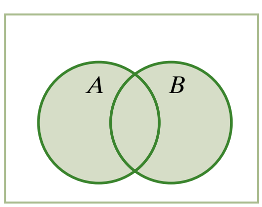
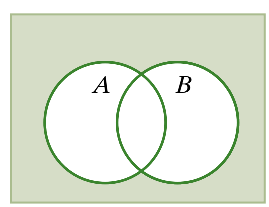
The diagrams below show how the different parts of the Venn diagram relate to the words 'and', 'or' and 'not'.

A		B	
Not A (that is, the complement of A)		Not B (that is, the complement of B)	
A but not B		B but not A	
A and B		A or B but not both	
A or B or both		Neither A nor B nor both	

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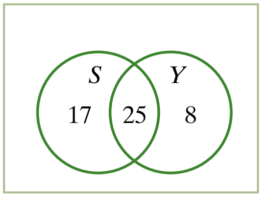
Example 20

At a fitness club the 50 members choose either swimming (S) or yoga (Y) or both or neither. This Venn diagram shows the members' choices. Calculate the probability that a club member participates in

a both swimming and yoga
b swimming but not yoga
c yoga but not swimming
d neither swimming nor yoga.

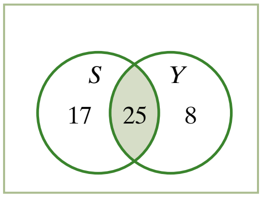


$n = 50$

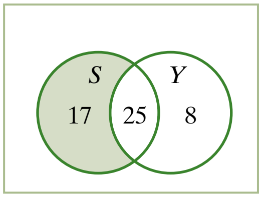
Working

a $\Pr(\text{swimming and yoga}) = \dfrac{25}{50} = 0.5$

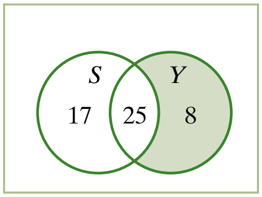
Reasoning



b $\Pr(\text{swimming but not yoga}) = \dfrac{17}{50} = 0.34$



c $\Pr(\text{yoga but not swimming}) = \dfrac{8}{50} = 0.16$



d $\Pr(\text{neither swimming nor yoga}) = \dfrac{0}{50} = 0$

17 + 25 + 8 = 50 so all club members participate in either swimming or yoga or both.

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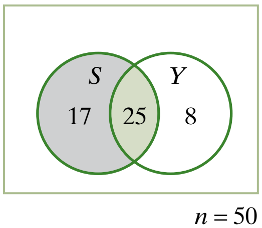
Venn diagrams are also useful for situations involving the word 'given'. For example, we could use the Venn diagram in example **20**, to answer the following question:

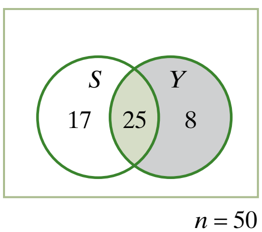
Given that a member has chosen swimming, what is the probability that they also do yoga?

The Venn diagram shows that a total of 42 members do swimming. Of these, 25 do yoga as well.



$$\text{Pr(yoga given they do swimming)} = \frac{25}{42}$$

Similarly, the Venn diagram shows that a total of 33 members do yoga. Of these, 25 do swimming as well.



$$\text{Pr(swimming given they do yoga)} = \frac{25}{33}$$

Example 21

Of a group of 30 students, 15 play tennis, 20 play cricket and 5 play neither sport.

a Show the information on a Venn diagram.

b What is the probability of a student from the group playing tennis given that they also play cricket?

c What is the probability of a student from the group playing cricket given that they also play tennis?

continued

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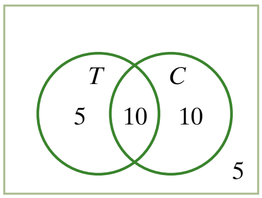
Example 21 continued

Working

a



$n = 30$

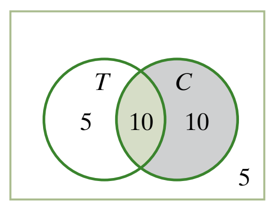
Reasoning

15 play tennis and 20 play cricket, and 5 play neither, making a total of 40 students.
As there are only 30 students in the group, 10 students must play both sports. Write this in the intersection.
If 15 students play tennis but 10 of them play both sports, then 5 students must play tennis only.
If 20 students play cricket but 10 of them play both sports, then 10 students must play cricket only.

b Pr(tennis given they play cricket) $= \dfrac{10}{20}$

$= \dfrac{1}{2}$

20 students play cricket. Of these, 10 also play tennis.

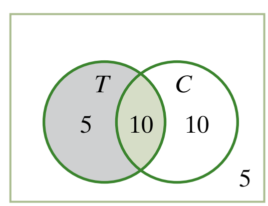


$n = 30$

c Pr(cricket given they play tennis) $= \dfrac{10}{15}$

$= \dfrac{2}{3}$

15 students play tennis. Of these, 10 also play cricket.



$n = 30$

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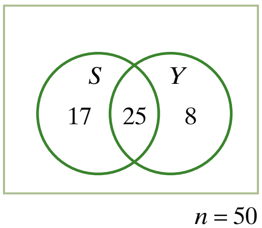
Two-way tables

Information shown in a Venn diagram can also be shown in a two-way table. The Venn diagram from example **20** is presented here in a two-way table.



	Y	*Y′*	Total
S	25	17	42
S′	8	0	8
Total	33	17	50

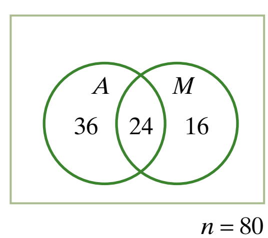
where S = members who take swimming
S' = members who do not take swimming
Y = members who take yoga
Y' = members who do not take yoga

The probabilities calculated in example **20** can also be represented in a two-way table. Notice that the probabilities in each row add to the total probability for that row. Similarly, the probabilities in each column add to the total probability for that column. The four probabilities must add to 1 because it is certain that one of the four combinations will occur.

	Y	*Y′*	Total
S	0.5	0.34	0.84
S′	0.16	0	0.16
Total	0.66	0.34	1

Example 22

Students in Year 9 at Cambrook High School can take art (A), music (M), both or neither. Their choices are shown in this Venn diagram.



a Show the information in this Venn diagram as a two-way table.

b A student is chosen at random from the Year 9 students at this school. Calculate the probability that the student does

i music but not art
ii art but not music
iii neither art nor music
iv music and art
v art or music
vi music given that they do art.

c Show the information in the Venn diagram as a two-way probability table.

continued

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Example 22 continued

Working

a

	M	M'	Total
A	24	36	60
A'	16	4	20
Total	40	40	80

b **i** $\Pr(\text{music but not art}) = \dfrac{16}{80} = 0.2$

ii $\Pr(\text{art but not music}) = \dfrac{36}{80} = 0.45$

iii $\Pr(\text{neither music nor art}) = \dfrac{4}{80} = 0.05$

iv $\Pr(\text{music and art}) = \dfrac{24}{80} = 0.3$

v $\Pr(\text{music or art}) = \dfrac{76}{80} = 0.95$

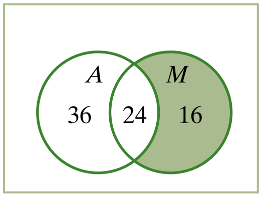
vi $\Pr(\text{music given that they do art}) = \dfrac{24}{60} = 0.4$

c

	M	M'	Total
A	0.3	0.45	0.75
A'	0.2	0.05	0.25
Total	0.5	0.5	1

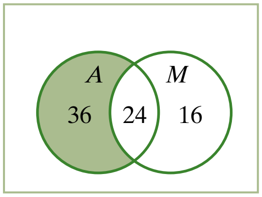
Reasoning

i



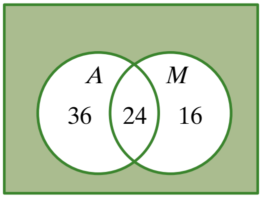
$n = 80$

ii



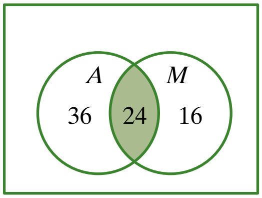
$n = 80$

iii



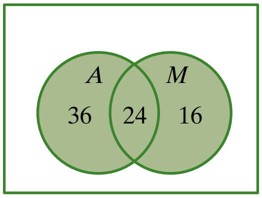
$n = 80$

iv



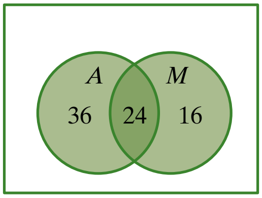
$n = 80$

v



$n = 80$

vi



$n = 80$

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Example 23

In a survey, 100 people were asked whether smoking in public places should be banned. The results are shown in this table.

	Male	Female	Total
Yes	24	22	
No	21	33	
Total			100

a How many surveyed were male?

b How many people voted Yes?

c One of the people surveyed is chosen at random. What is the probability that this person

i voted No? **ii** was female? **iii** was male and voted No?

d One of the males surveyed is chosen at random. What is the probability that he voted No?

e One of the females surveyed is chosen at random. What is the probability that she voted No?

Working

a The total number of males surveyed is 45.

Reasoning

	Male	Female	Total
Yes	24	22	
No	21	33	
Total	**45**		100

The blue column shows the data for males.

Working

b The total number of people who voted Yes is 46.

Reasoning

	Male	Female	Total
Yes	24	22	**46**
No	21	33	
Total	45		100

The green row shows the data for people who voted Yes.

continued

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Example 23 continued

Working

c i Pr(person voted No)

$= \frac{\text{number who voted No}}{\text{total number of people}}$

$= \frac{54}{100}$

$= \frac{27}{50}$

Reasoning

	Male	Female	Total
Yes	24	22	46
No	21	33	**54**
Total	45		100

Add the numbers in the red row to find how many people voted No.

ii Pr(person was female)

$= \frac{\text{number of females}}{\text{total number of people}}$

$= \frac{55}{100}$

$= \frac{11}{20}$

	Male	Female	Total
Yes	24	22	46
No	21	33	54
Total	45	**55**	**100**

Add the numbers in the pink column to find the number of females.

iii Pr(person was male and voted No)

$= \frac{\text{number of males who voted No}}{\text{total number of people}}$

$= \frac{21}{100}$

	Male	Female	Total
Yes	24	22	46
No	**21**	33	54
Total	45	55	**100**

The number in the Male column and the No row shows how many males voted No.

d Pr(a male who voted No)

$= \frac{\text{number of males who voted No}}{\text{total number of males}}$

$= \frac{21}{45}$

$= \frac{7}{15}$

	Male	Female	Total
Yes	24	22	46
No	**21**	33	54
Total	**45**	55	100

Considering only males, the total is 45. Of those, 21 voted No.

continued

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Example 23 continued

Working

e Pr(female and voted No)

$= \dfrac{\text{number of females who voted No}}{\text{total number of females}}$

$= \dfrac{33}{55}$

$= \dfrac{3}{5}$

Reasoning

	Male	Female	Total
Yes	24	22	46
No	21	**33**	54
Total	45	**55**	100

Considering only females, the total is 55. Of those, 33 voted No.

Example 24

Pr(A) = 0.35, Pr(B) = 0.52 and Pr(A and B) = 0.21.

a Copy and complete this two-way probability table.

	A	A'	Total
B			
B'			
Total			

b What is the probability of neither A nor B occurring?

c What is the probability of A but not B occurring?

Working

a

	A	A'	Total
B	0.21		0.52
B'			
Total	0.35		1

	A	A'	Total
B	0.21	0.31	0.52
B'	0.14		
Total	0.35		1

Reasoning

Enter the given probabilities.
Enter 1 in the bottom right cell.

0.52 – 0.21 = 0.31
0.35 – 0.21 = 0.14

continued

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Example 24 continued

Working

	A	A'	Total
B	0.21	0.31	0.52
B'	0.14		0.48
Total	0.35	0.65	1

Reasoning

$1 - 0.52 = 0.48$
$1 - 0.35 = 0.65$

	A	A'	Total
B	0.21	0.31	0.52
B'	0.14	0.34	0.48
Total	0.35	0.65	1

$0.48 - 0.14 = 0.34$
Check:
$0.65 - 0.31 = 0.34$

b Pr(neither A nor B) = 0.34

	A	A'	Total
B	0.21	0.31	0.52
B'	0.14	0.34	0.48
Total	0.35	0.65	1

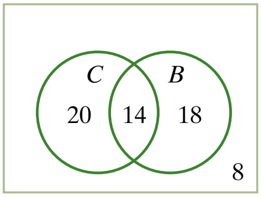
c Pr(A but not B) = 0.14

	A	A'	Total
B	0.21	0.31	0.52
B'	0.14	0.34	0.48
Total	0.35	0.65	1

exercise 5.4

LINKS TO Example 20

1 In a survey people were asked whether they owned a car (C), a bike (B) or neither. The results are shown in the Venn diagram.



Calculate the probability that a person chosen at random from the group

a does not own a bike or a car.
b owns a car.
c owns a car but not a bike.
d owns a bike but not a car.
e owns a car or a bike.
f owns a car and a bike.

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LINKS TO Example 21

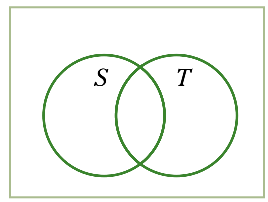
Use the information shown in the Venn diagram in question **1** to calculate

a the probability that a person chosen at random from the group owns a bike given that they also own a car.

b the probability that a person chosen at random from the group owns a car given that they also own a bike.

The letters A to Z are written on separate pieces of paper and placed in a box. Let S be the set of letters in the word 'FEBRUARY' and T be the set of letters in the word 'BIRTHDAY'.

a Copy and complete this Venn diagram, showing the number of letters in each section.



b One letter is selected at random from the box. Calculate the probability that the letter is in the set

i T. **ii** S.

iii S and T. **iv** S or T.

v neither S nor T.

c Given that a randomly selected letter is in the set S, what is the probability that it is not in the set T?

LINKS TO Example 22

A video library surveyed 200 people regarding their movie preferences. The survey showed that 140 people liked adventure movies, 185 people liked comedies, and 8 people liked neither adventure nor comedy movies.

a Represent the data in a Venn diagram.

b Represent the data in a two-way table where

A = people who like adventure movies

A' = people who do not like adventure movies

C = people who like comedy movies

C' = people who do not like comedy movies.

	C	C'	Total
A			
A'			
Total			

c Construct a two-way probability table to show the probabilities for a randomly selected user of the video library.

d If one of the people surveyed is selected at random, what is the probability that this person liked

i both comedy and adventure movies?

ii adventure movies but not comedy movies?

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LINKS TO Example 23

In a survey, 200 people were asked if they approved of longer school lessons. The results are shown in the table.

	Student	Teacher	Total
Approved	44	45	
Disapproved	86	25	
Total			200

a How many teachers were surveyed?

b How many people approved?

c One of the people surveyed was chosen at random. What is the probability that this person

i approved? **ii** was a student?

iii was a student who disapproved?

d One of the students was chosen at random. What is the probability that they approved?

e One of the teachers was chosen at random. What is the probability that they approved?

LINKS TO Example 24

In a study of people using a local library it was found that 70% of the library users were students (S), 60% were female (F) and 42% were female students.

	S	S'	Total
F			
F'			
Total			1

a Use the information to fill in this probability table.

b What is the probability that a randomly selected library user is neither female nor a student?

c Given that a randomly selected library user is a student, what is the probability that they are female?

$\Pr(A) = 0.45$, $\Pr(B) = 0.36$ and $\Pr(A \text{ and } B) = 0.28$.

a Copy and complete this two-way probability table.

	A	A'	Total
B			
B'			
Total			

b What is the probability of neither A nor B occurring?

c What is the probability of B but not A occurring?

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At a particular dog show, the dogs are classified according to whether or not they are a large breed (L) and whether or not their coat is short (S). This two-way table summarises the data for all dogs at the show.

One dog at the show is selected at random. What is the probability that this dog

a has a short coat?

b is a large breed with a short coat?

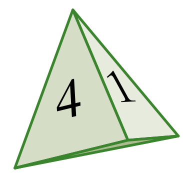
c is a large breed without a short coat?

d is a small breed with a short coat?

	L	L'	Total
S	50	120	170
S'	70	60	130
Total	120	180	300

The faces of a tetrahedron are labelled with the digits 1 to 4. The total sum is recorded from tossing the tetrahedron twice.

		Toss 1			
		1	2	3	4
Toss 2	1	2	3		
	2				
	3				
	4				



a Copy and complete this table, showing all possible totals that can be obtained.

b Find the probability that the total sum is

i 3 | **ii** 3 or 4

iii less than or equal to 4 | **iv** more than 6

v at most 6 | **vi** at least 6

Olympia rolls a tetrahedral (four-sided) die and an octahedral (eight-sided) die.

a Use a table to list all the possible outcomes.

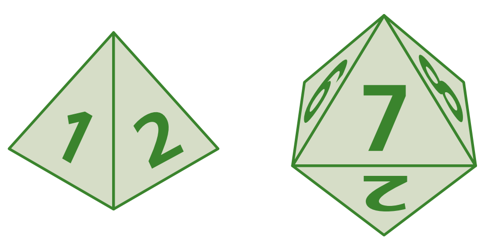
b **i** What is the total number of possible outcomes?

ii How does this total relate to the number of sides on each die?

c What is the probability that the number rolled on each die is even?

d What is the probability of rolling a 4 on the tetrahedral die and an odd number on the octahedral die?

e What is the probability of rolling the same number on each die?



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LINKS TO Example 12

● The following table shows all possible outcomes when two dice are rolled.

a What is the probability of rolling a double 1?

b What is the probability of rolling the same number on each die?

1st die (columns), 2nd die (rows)

	1	2	3	4	5	6
1	1,1	2,1	3,1	4,1	5,1	6,1
2	1,2	2,2	3,2	4,2	5,2	6,2
3	1,3	2,3	3,3	4,3	5,3	6,3
4	1,4	2,4	3,4	4,4	5,4	6,4
5	1,5	2,5	3,5	4,5	5,5	6,5
6	1,6	2,6	3,6	4,6	5,6	6,6

● One bag contains two white beads, one red bead and one yellow bead. Another bag contains two red beads, one white bead and one yellow bead. One bead is selected from each bag.

a Fill in this table to list all the possible outcomes.

b What is the probability of drawing two beads of the same colour?

c What is the probability of drawing exactly one red bead?

d What is the probability of drawing at least one yellow bead?

Bag 1 (columns), Bag 2 (rows)

	W	W	R	Y
R				
R				
W				
Y				

exercise 5.4 challenge

● Brett surveyed a class of 28 students and found that 12 were taking a drama elective and 16 were taking a language elective. He called a meeting of all these students and found that there were 20 in the group.

a What is the probability that a student chosen at random from this group takes both electives?

b Brett then called a meeting of the drama students. What is the probability that a student chosen at random from this group takes both electives?

● At a school fete, children can pay $1 and toss a coin four times. If they toss four heads or four tails they win a prize.

a Use a tree diagram to list all possible outcomes when a coin is tossed four times.

b What is the probability of winning a prize at the fete?

c Julia gets heads with her first two tosses of the coin.

  i Draw a new tree diagram to show the possible outcomes for her second two tosses.

  ii What is the probability that she wins a prize?

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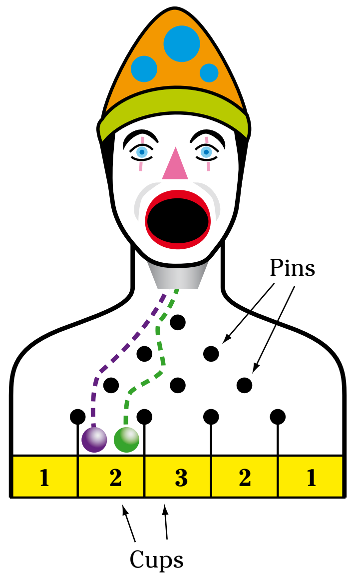
Analysis task

BLM At the fair

At the fair

In a fairground game, a ball is dropped into the mouth of a clown. Inside the clown, the ball rolls down between several rows of pins and drops into a cup labelled 1, 2 or 3. For example, here are the paths of two balls that each scored 2.



Assume that each ball has an equal chance of rolling to the left or the right of each pin.

a How many different paths are possible for a ball that lands in the middle cup and scores 3?

b How many paths result in a score of 2?

c How many paths result in a score of 1?

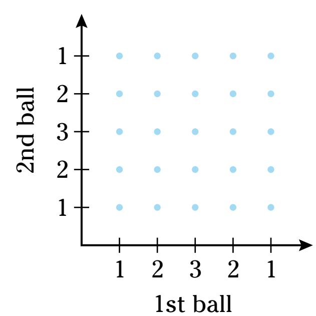
d Find the probability of each score.

Supposing at a fair, customers pay \$1 to play one ball. They win \$3 if they score 1 and get their money back if they score 3.

e If there were 800 customers in a day, how much would the stallholder expect to win or lose on average?

f Determine the payout for a score of 1 that would result in no overall profit or loss.

Suppose that the pins are adjusted so that each cup is equally likely to receive the ball, and each player drops two balls into the clown's mouth. We can show all the equally-likely outcomes on this lattice diagram.



g Copy the diagram and write the total of the two scores at each point.

h List the possible sums and calculate the probability of each.

Customers pay \$2 to play two balls. They win \$10 if they score a total of 6 and get their money back if they score a total of 2.

i If there were 500 customers, how much profit or loss would the stallholder expect to make?

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Review Probability

Summary

- The sample space (S) is the set of all possible outcomes for an event.
- If all outcomes are known and equally likely, the theoretical probability of an event can be calculated as

$$\text{Pr(event)} = \frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$$

- Probabilities are numbers from 0 to 1.
- If an event is impossible, then Pr(event) = 0.
- If an event is certain, then Pr(event) = 1.
- The more likely an event is to occur, the closer its probability is to 1.
- Tree diagrams, two-way tables, Venn diagrams and lattice or dot diagrams are all useful in organising information relating to probability experiments.
- Relative frequency is a measure of experimental probability.
  - The greater the number of trials in an experiment, the closer the relative frequency should be to the theoretical probability.
- Two-step events involve more than one event.
  - Tree diagrams are used to show the outcomes at each stage of an experiment involving two or more events.
  - Tables can be used to show the outcomes of an experiment involving two events only.

Visual map

Using the following terms (and others if you wish), construct a visual map that illustrates your understanding of the key ideas covered in this chapter.

complement	long-run probability	theoretical probability
dependent events	mutually exclusive	tree diagram
equally likely	outcome	two-step event
experimental probability	probability table	two-way table
event	random	Venn diagram
independent events	relative frequency	with replacement
lattice diagram	sample space	without replacement

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Revision

Multiple-choice questions

A card is drawn at random from a pack of 52 playing cards. The probability that it is an ace or a club is

A $\frac{1}{4}$ **B** $\frac{1}{13}$ **C** $\frac{1}{52}$ **D** $\frac{4}{13}$ **E** $\frac{17}{52}$

Two fair dice are rolled. The probability that the sum of the numbers rolled is more than 8 is

A $\frac{1}{6}$ **B** $\frac{1}{3}$ **C** $\frac{7}{9}$ **D** $\frac{11}{18}$ **E** $\frac{5}{18}$

Amanda tosses a coin and then rolls a die. The chance of getting an odd number together with a tail is

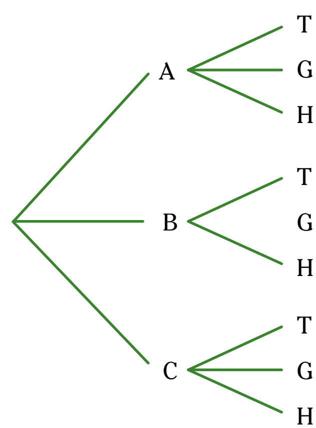
A $\frac{1}{2}$ **B** $\frac{1}{4}$ **C** $\frac{1}{6}$ **D** $\frac{1}{8}$ **E** $\frac{3}{8}$

Two four-sided dice (numbered 1, 2, 3 and 4) are tossed and the total sum is recorded. The probability that the sum is 8 is

A $\frac{1}{16}$ **B** $\frac{1}{8}$ **C** $\frac{1}{4}$ **D** $\frac{1}{2}$ **E** 0

Dario plans to visit a friend to play sport. He has three friends, Alex, Beata and Chris, and they may play tennis, golf or hoops. The tree diagram shows the different combinations.

The probability that he ends up playing either tennis or golf, but not with Beata is



A $\frac{4}{9}$ **B** $\frac{5}{9}$

C $\frac{6}{9}$ **D** $\frac{3}{9}$

E $\frac{2}{9}$

Short-answer questions

In a class of 32 students, 12 study Mathematics and History, 8 study History only and 3 study neither History nor Mathematics. Draw a Venn diagram to help you answer the following questions.

a How many students study Mathematics but not History?

b How many students study Mathematics?

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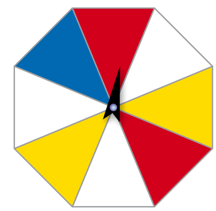
A card is drawn at random from a pack of 52 playing cards. Determine the probability that it is

a black. **b** not a club. **c** a picture card.
d a queen or a king. **e** a picture card and a heart.

a List the possible outcomes for this spinner.
b Are all the outcomes equally likely?
c Find the probability of spinning
  i yellow. **ii** red.
  iii white. **iv** not blue.
d In 300 spins, how many times would you expect to spin yellow?



In a group of 30 adults and 20 children, 15 of the adults and 12 of the children have brown hair.

a Construct a two-way table to show this information.
b If one member of the group is selected at random, what is the probability that this person
  i is an adult? **ii** has brown hair? **iii** is an adult with brown hair?

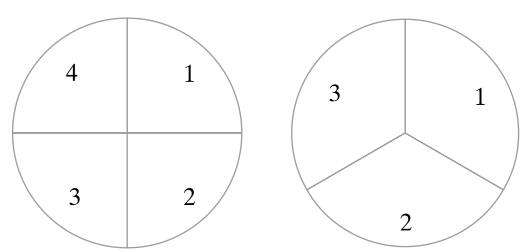
A game is played using a pack containing 40 cards. There are 10 blue, 10 yellow, 10 white and 10 black cards in the pack.

a If one card is drawn at random from the pack, what is the probability that it is blue?
b Jacinta removes two white cards from the pack and sets them aside. If one card is now drawn at random from the remaining cards in the pack, what is the probability that it is
  i yellow? **ii** white? **iii** not black?

Extended-response questions

Danny conducts an experiment where he spins two spinners. One has the numbers 1 to 4 and the other has the numbers 1 to 3 as shown.

a Use a lattice diagram to list the event space (i.e. all possible outcomes) of Danny's experiment.
b Use a tree diagram to show all the possible outcomes.
c What is the probability of spinning an even number with the first spinner and an odd number with the second spinner?



Spinner 1 Spinner 2

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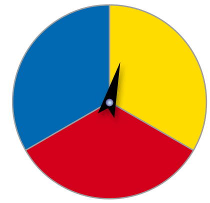
Marika spins the spinner shown, then tosses a coin.



a Use a tree diagram to list all possible outcomes of her experiment.

b Find the probability of getting

i a head and red. ii a tail and yellow.

iii a head or blue. iv a tail

A bowl contains an apple, a banana and a pear. In the door of the refrigerator there are four juice boxes: pineapple, orange and mango, apricot nectar and tropical. Rushing off to school, Melinda grabs a piece of fruit and a juice box.

a Draw a tree diagram to find all the possible combinations she could have.

b Use it to find the probability that Melinda grabbed

i a banana and pineapple juice. ii an apple and any juice except tropical.

iii either an apple or a pear, and either orange and mango juice or apricot nectar.

Practice quiz
Chapter 5

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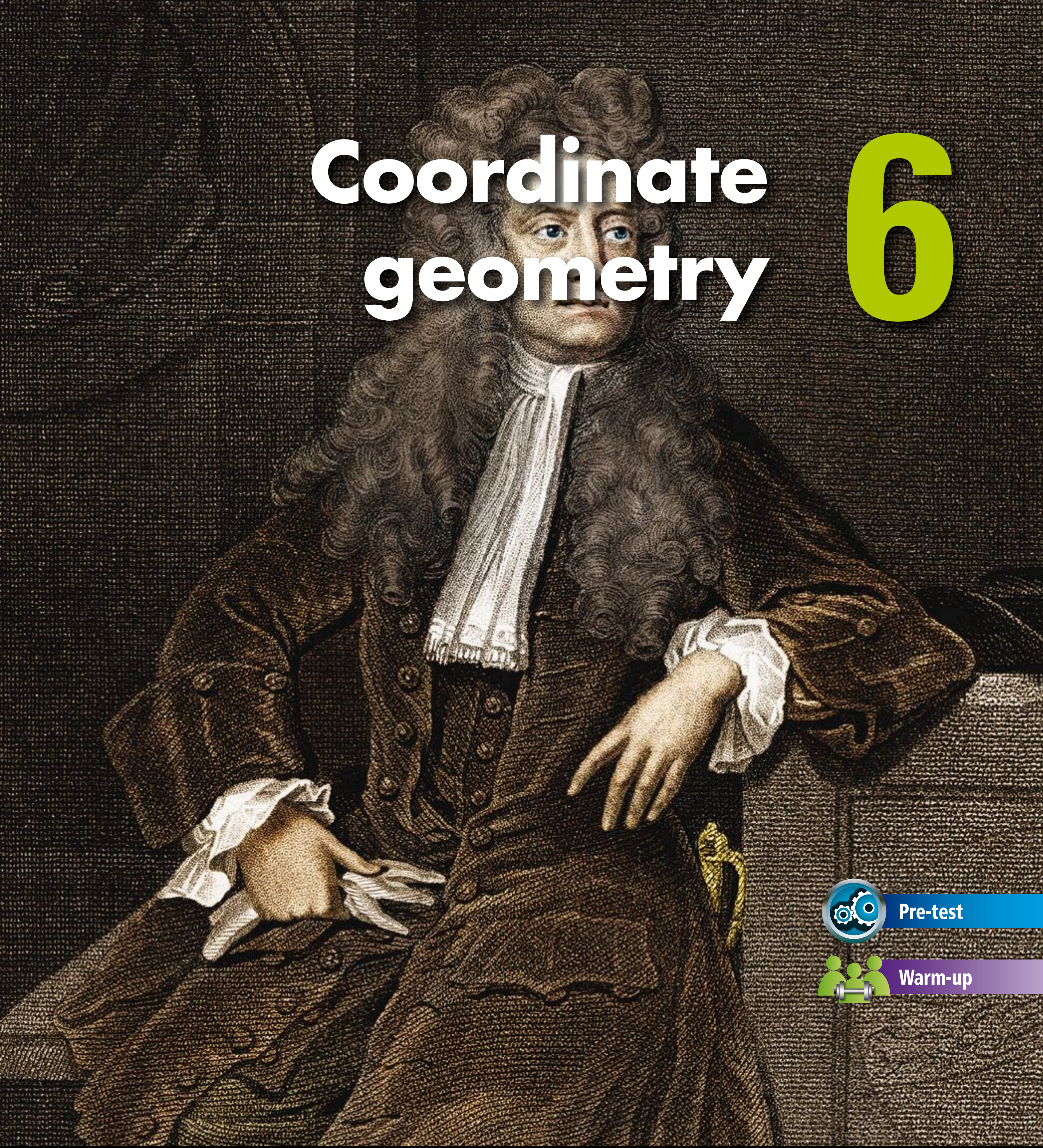
6 Coordinate geometry



Pre-test

Warm-up

The French mathematician and philosopher René Descartes, who lived in the first half of the 17th century developed the idea of pairs of coordinates to describe the position of a point on a plane with respect to two perpendicular axes. His work led to our system of Cartesian coordinates. However, it was the English mathematician and scientist Sir Isaac Newton (1642–1727), shown above, who was the first to use negative coordinates.

6.1 Cartesian coordinates and midpoints

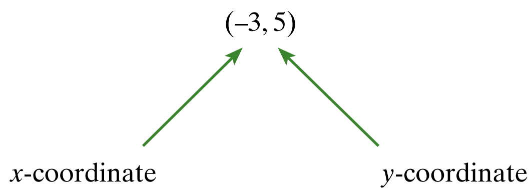
The Cartesian plane

BLM
Chapter warm-up

In Years 7 and 8 you have seen that points on the **Cartesian plane** can be represented by pairs of numbers in brackets called **Cartesian coordinates**, for example, (–3, 5).

The horizontal position of the point on the Cartesian plane is referred to as the ***x*-coordinate** of the point. The vertical position of the point is called the ***y*-coordinate**.

The first number in the brackets is always the x-coordinate and the second number is the y-coordinate.

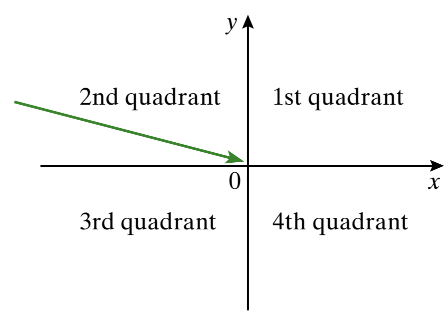


The order of the coordinates in the brackets is easy to remember because the coordinates are in alphabetical order; that is, x, then y.

The x-axis and the y-axis include negative numbers as well as positive numbers. The x-axis and the y-axis cross at the point (0, 0). This point is called the **origin**.

Anywhere along the x-axis, $y = 0$. Anywhere along the y-axis, $x = 0$.

The two axes divide the Cartesian plane into four sections called **quadrants**.

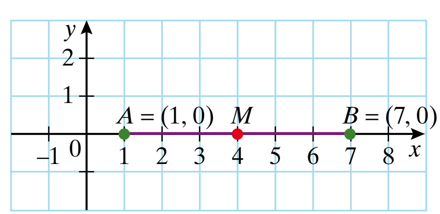


Finding the coordinates of the midpoint of two points

On the Cartesian plane below M is the midpoint of the line segment (also called a line interval) AB. We can see that M is the point (4, 0). The x-coordinate of M is the mean of the x-coordinates of A and B.

$$\frac{1+7}{2} = 4$$

A and B both have the same y-coordinate so M also has y-coordinate 0.



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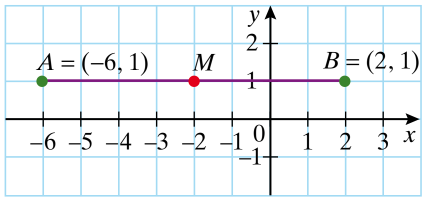
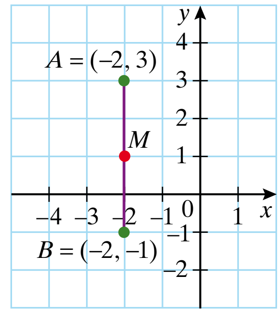
Example 1

Find the coordinates of M, the midpoint of AB.

a



b



c $A\ (4, 6)$ and $B\ (8, 6)$

d $A\ (-3, -5)$ and $B\ (-3, 7)$

Working

a x-coordinate of $M = \dfrac{-6 + 2}{2}$

$= \dfrac{-4}{2}$

$= -2$

M is the point $(-2, 1)$

b y-coordinate of $M = \dfrac{3 + (-1)}{2}$

$= \dfrac{2}{2}$

$= 1$

M is the point $(-2, 1)$

c $A\ (4, 6)$ and $B\ (8, 6)$

x-coordinate of $M = \dfrac{4 + 8}{2}$

$= \dfrac{12}{2}$

$= 6$

M is the point $(6, 6)$

Reasoning

a The x-coordinate of M is the mean of the x-coordinates of A and B.

A and B both have the same y-coordinate, 1.

b A and B both have the same x-coordinate, –2.
The y-coordinate of M is the mean of the y-coordinates of A and B.

c A and B have the same y-coordinate.
The x-coordinate of M is the mean of the x-coordinates of A and B.

continued

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Example 1 continued

Working

d $A\ (-3, -5)$ and $B\ (-3, 7)$

$$y\text{-coordinate of } M = \frac{-5+7}{2} = \frac{2}{2} = 1$$

M is the point $(-3, 1)$

Reasoning

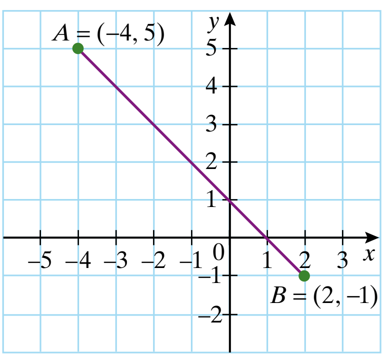
A and B have the same x-coordinate.
The y-coordinate of M is the mean of the y-coordinates of A and B.

When both the x-coordinates and the y-coordinates of the two points are different

- the x-coordinate of the midpoint is the mean of the two x-coordinates and
- the y-coordinate of the midpoint is the mean of the two y-coordinates.

Example 2

Find the coordinates of the midpoint of AB?



continued

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Example 2 continued

Working	Reasoning
x-coordinate of M: $\frac{-4+2}{2} = \frac{-2}{2} = -1$	Find the mean of the two x-coordinates.
y-coordinate of M: $\frac{5+(-1)}{2} = \frac{4}{2} = 2$	Find the mean of the two y-coordinates.
M is the point $(-1, 2)$	

Midpoint formula

For two points (x_1, y_1) and (x_2, y_2), the midpoint is the point $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Don't confuse the numbers with indices, for example, x_2 does not mean x^2. The **2** is simply a label to say x_2 is the x-coordinate of the **second** point.

Example 3

Find the midpoint of the line segment joining each of these pairs of points

a $(1, 1)$ and $(6, 5)$ **b** $(-3, 8)$ and $(5, -5)$

continued

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Example 3 continued

Working

a $M = \left(\dfrac{1+6}{2}, \dfrac{1+5}{2}\right)$

$= \left(\dfrac{7}{2}, \dfrac{6}{2}\right)$

$= \left(3\dfrac{1}{2}, 3\right)$

M is the point $\left(3\dfrac{1}{2}, 3\right)$.

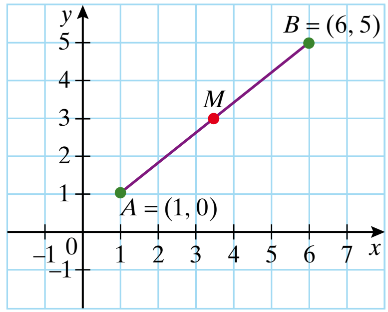
b $M = \left(\dfrac{-3+5}{2}, \dfrac{8+(-5)}{2}\right)$

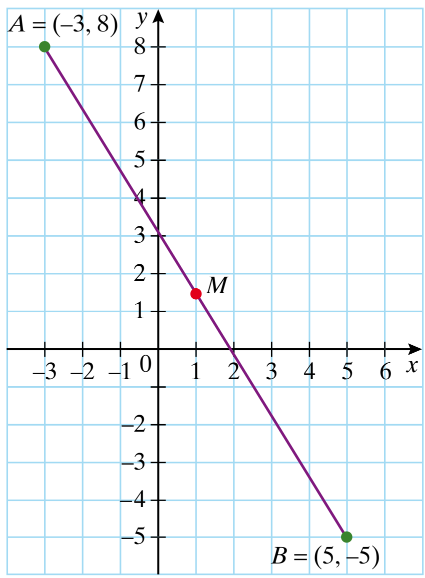
$= \left(\dfrac{2}{2}, \dfrac{3}{2}\right)$

$= \left(1,\ 1\dfrac{1}{2}\right)$

M is the point $\left(1,\ 1\dfrac{1}{2}\right)$.

Reasoning





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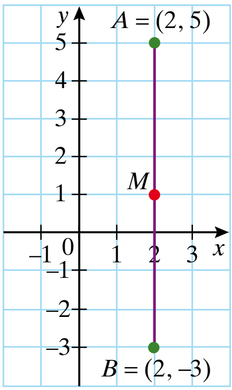
exercise 6.1

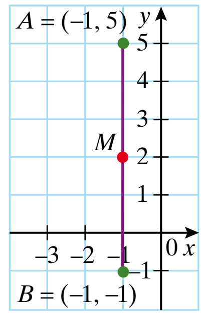
LINKS TO Example 1

1 Write the coordinates of M, the midpoint of AB.

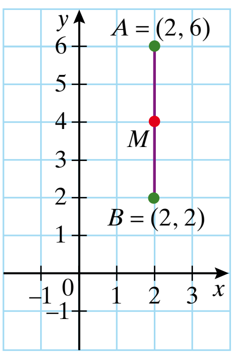
a



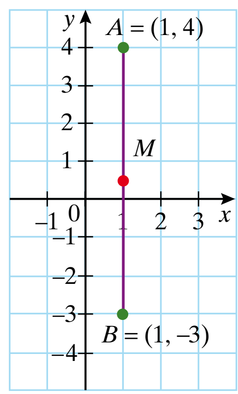
b



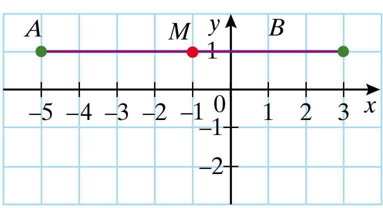
c



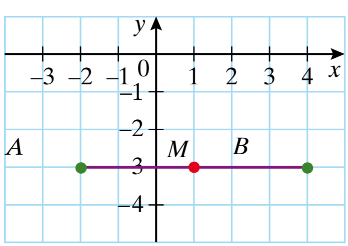
d



e



f



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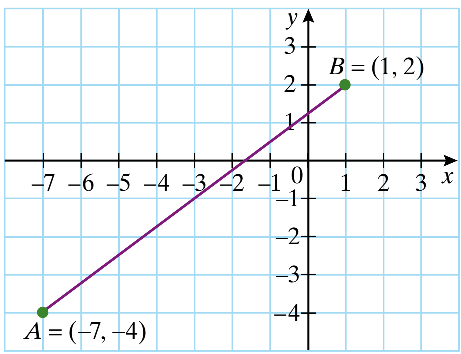
Find the midpoint of the line segment joining each of the following pairs of points.

a $(3,6)$ and $(9,6)$
b $(4,3)$ and $(12,3)$
c $(5,8)$ and $(5,2)$
d $(-4,1)$ and $(-8,1)$
e $(-6,0)$ and $(4,0)$
f $(-5,2)$ and $(-5,-4)$
g $(0,-5)$ and $(0,7)$
h $(3,2)$ and $(-7,2)$
i $(3,4)$ and $(3,9)$
j $(-4,-3)$ and $(5,-3)$
k $(4,3)$ and $(11,3)$
l $(-3,-6)$ and $(-3,-10)$

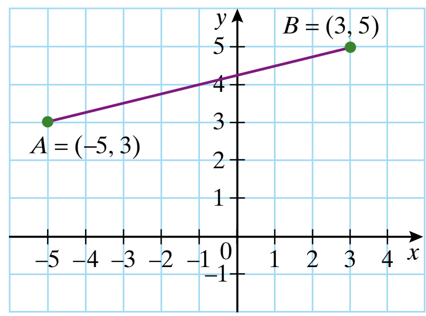
LINKS TO Example 2

Find the coordinates of the midpoint of AB.

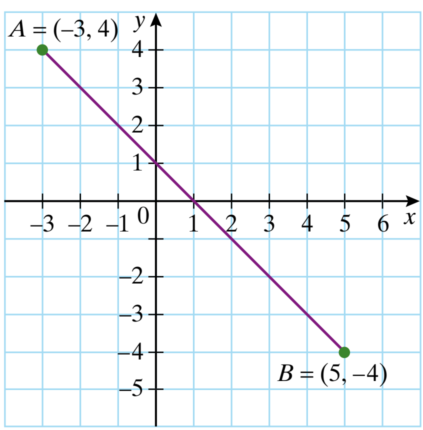
a



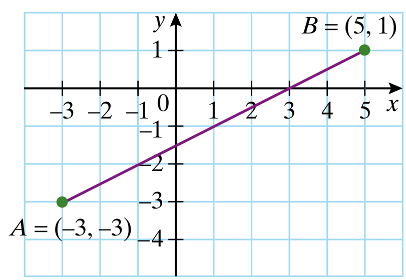
b



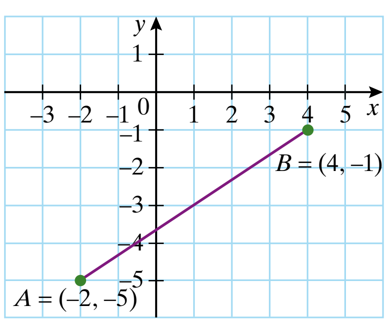
c



d



e



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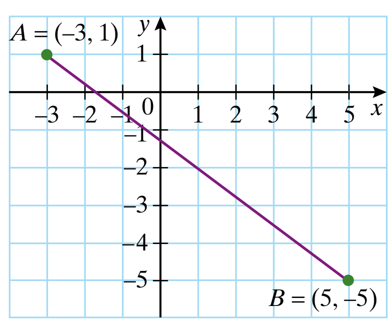

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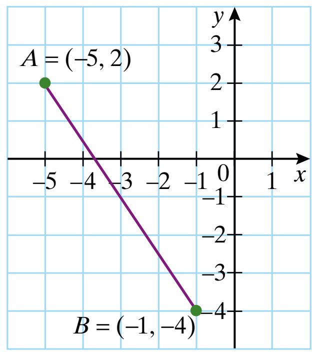
g



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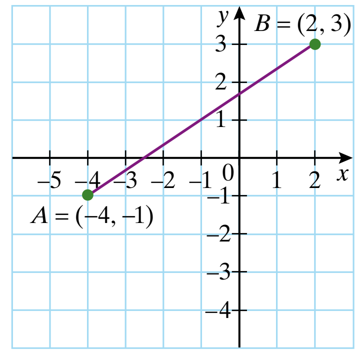
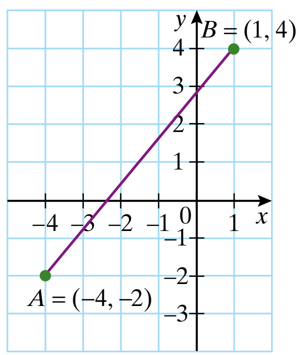


i



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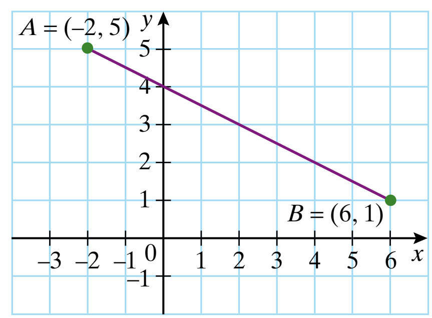

LINKS TO Example 3

Find the midpoint of the line segment joining each of these pairs of points.

a (4, 6) and (10, 8)
b (3, 5) and (7, 13)
c (–3, 8) and (7, –2)
d (–4, 3) and (–8, 9)
e (–1, 0) and (5, 6)
f (–5, 2) and (5, –4)
g (–3, –7) and (–5, 7)
h (3, 4) and (8, 10)
i (3, –7) and (–1, 9)
j (–4, 5) and (5, 8)
k (0, 4) and (–6, 9)
l (–10, –6) and (4, –1)

exercise 6.1 challenge

The midpoint of two points, A and B, is the point (–4, 6). If A is the point (3, –11), find the coordinates of B.

The midpoint of the point (5, –7) and the point (a, b) is the point (–9, –5). Find the values of a and b.

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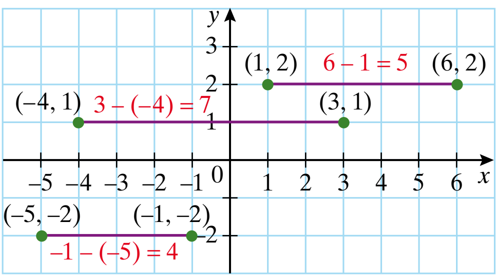
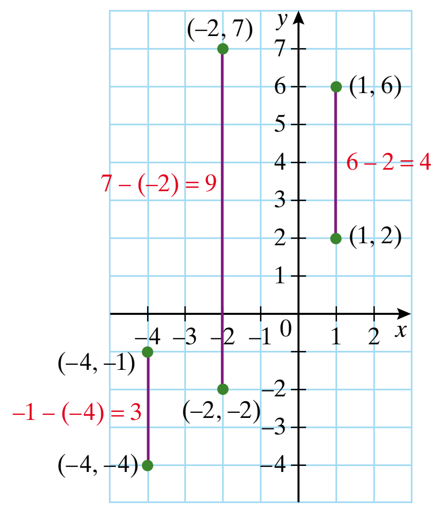
6.2 Distance between two points

The coordinates of two points can be used to calculate the distance between the two points.

> If two points have the same *y*-coordinate, the distance between them is simply the difference between their *x*-coordinates.

> Similarly if two points have the same *x*-coordinate, the distance between them is the difference between their *y*-coordinates.





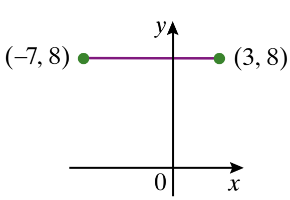
Example 4

Find the distance, d, between the following pairs of points.

a (3, 8) and (–7, 8) **b** (–3, 5) and (–11, 5) **c** (2, –6) and (2, 1)

Working

a



(3, 8) and (–7, 8)

$d = 3 - (-7)$

$d = 10$

Reasoning

The *y*-coordinates are the same.
Subtract the smaller *x*-coordinate from the larger *x*-coordinate.

continued

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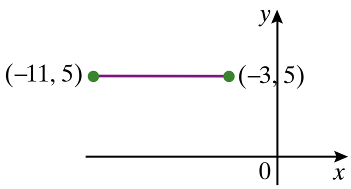
Example 4 continued

Working

b



(–3, 5) and (–11, 5)

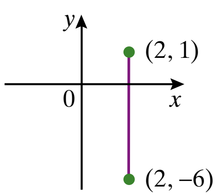
$d = -3 - (-11)$

$d = 8$

Reasoning

The y-coordinates are the same.
Subtract the smaller x-coordinate from the larger x-coordinate.

b



(2, –6) and (2, 1)

$d = 1 - (-6)$

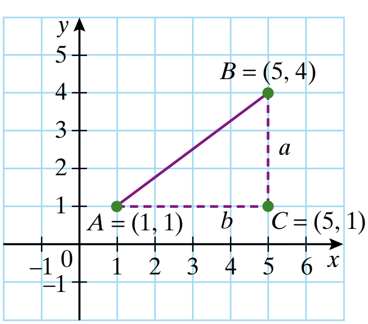
$d = 7$

The x-coordinates are the same.
Subtract the smaller y-coordinate from the larger y-coordinate.

We now consider cases where neither the x-coordinates nor the y-coordinates are the same.

In the diagram below we see that a right-angled triangle has been drawn so that the line segment AB forms the hypotenuse of the triangle. We can then use Pythagoras' theorem to find the length of AB.

The sides labelled a and b have lengths 3 units and 4 units respectively, so using the Pythagorean triple 3, 4, 5, we know that the length of AB is 5 units.



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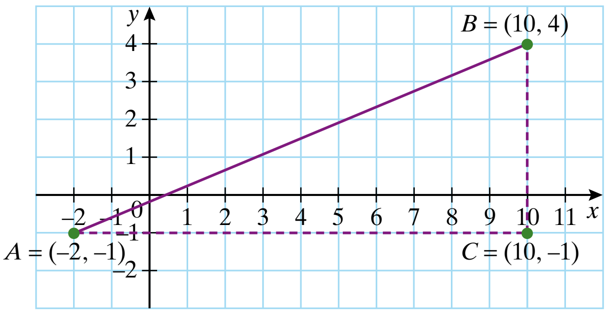
Example 5

Consider $\triangle ABC$ on the Cartesian plane below.

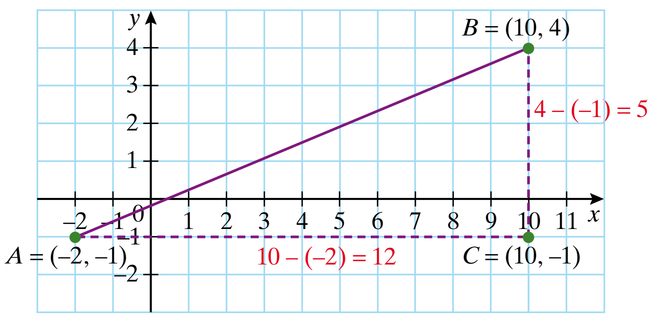
a Find the lengths of AC and BC and label them on the diagram.

b Find the length of AB.



Working

a



$$AC = 10 - (-2) = 12$$

$$BC = 4 - (-1) = 5$$

b

$$AB^2 = 12^2 + 5^2 = 144 + 25 = 169$$

$$AB = \sqrt{169} = 13$$

Reasoning

The distance between A and C is the difference between the x-coordinates. The distance between B and C is the difference between the y-coordinates.

AB is the hypotenuse of the right-angled triangle. 5, 12, 13 is a Pythagorean triple.

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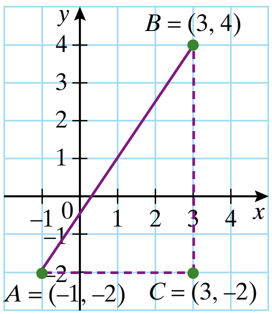
Example 6

Consider $\triangle ABC$ on the Cartesian plane below.

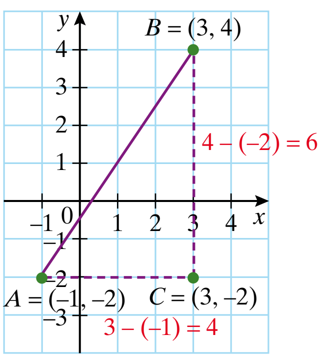
a Find the lengths of AC and BC.

b Calculate the distance AB correct to one decimal place.



Working

a



$$\begin{aligned} AC &= 3 - (-1) \\ &= 4 \\ BC &= 4 - (-2) \\ &= 6 \end{aligned}$$

b

$$\begin{aligned} AB^2 &= 4^2 + 6^2 \\ &= 16 + 36 \\ &= 52 \\ AB &= \sqrt{6^2 + 4^2} \\ &= \sqrt{52} \\ &\approx 7.21\ldots \end{aligned}$$

The distance AB is 7.2 correct to one decimal place.

Reasoning

a The distance between A and C is the difference between their x-coordinates.
The distance between B and C is the difference between their y-coordinates.

b AB is the hypotenuse of the right-angled triangle.
Use Pythagoras' theorem to calculate the distance AB.

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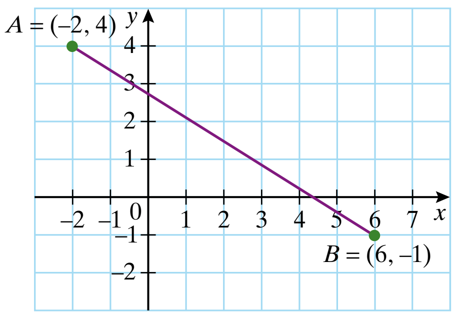
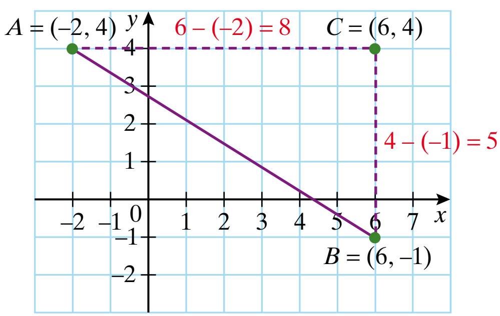
Example 7

Calculate the distance between A and B, correct to one decimal place.



Working



$$AC = 6 - (-2)$$
$$= 8$$
$$BC = 4 - (-1)$$
$$= 5$$

Reasoning

Draw a right-angled triangle, $\triangle ABC$, with AB as the hypotenuse.
Label the lengths of AC and BC.
The distance between A and C is the difference between their x-coordinates.
The distance between B and C is the difference between their y-coordinates.

$$AB^2 = 8^2 + 5^2$$
$$AB = \sqrt{8^2 + 5^2}$$
$$= \sqrt{89}$$
$$\approx 9.43...$$

Use Pythagoras' theorem to calculate the length of AB.

The distance AB is 9.4 correct to one decimal place.

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Distance between two points

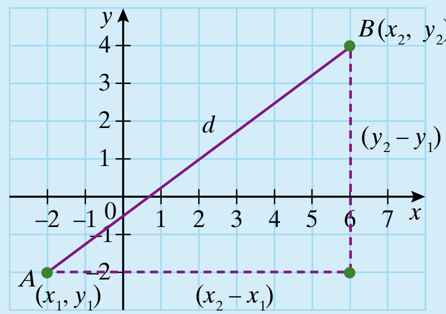
In general, for two points $A(x_1, y_1)$ and $B(x_2, y_2)$, the sides of the right-angled triangle are $(x_2 - x_1)$ and $(y_2 - y_1)$, where $x_2 - x_1$ is the difference in x-coordinates and $y_2 - y_1$ is the difference in y-coordinates.

If d is the distance between A and B, then by Pythagoras' theorem,

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

and

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Example 8

Find the distance between these pairs of points correct to one decimal place.

a (2, 6) and (7, 13) **b** (–4, 6) and (3, –2)

Working	Reasoning	
a $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$	(2, 6)	(7, 13)
$d^2 = (7 - 2)^2 + (13 - 6)^2$	(x_1, y_1)	(x_2, y_2)
$= 5^2 + 7^2$		
$= 74$		
$d = \sqrt{74}$		
$d \approx 8.60...$		
The distance between the points is 8.6 correct to one decimal place.		
b $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$	(–4, 6)	(3, –2)
$d^2 = (3 - (-4))^2 + (-2 - 6)^2$	(x_1, y_1)	(x_2, y_2)
$= 7^2 + (-8)^2$		
$= 113$		
$d = \sqrt{113}$		
$d \approx 10.63...$		
The distance between the points is 10.6 correct to one decimal place.		

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exercise 6.2

LINKS TO Example 4

Find the distance between A and B for each of the line intervals AB in question **1** of Exercise **6.1**

LINKS TO Example 4

For each of these pairs of points

i draw the points on the Cartesian plane.

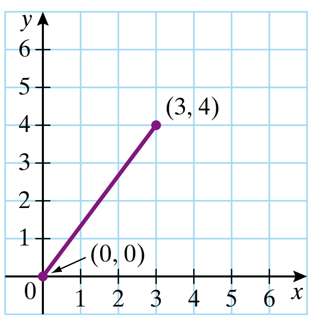
ii find the distance between the points.

a $(3, 6)$ and $(9, 6)$	**b** $(4, 3)$ and $(12, 3)$	**c** $(5, 8)$ and $(5, 2)$
d $(-4, 1)$ and $(-8, 1)$	**e** $(-6, 0)$ and $(4, 0)$	**f** $(-5, 2)$ and $(-5, -4)$
g $(0, -5)$ and $(0, 7)$	**h** $(3, 2)$ and $(-7, 2)$	**i** $(3, 4)$ and $(3, 9)$
j $(-4, -3)$ and $(5, -3)$	**k** $(4, 3)$ and $(11, 3)$	**l** $(-3, -6)$ and $(-3, -10)$

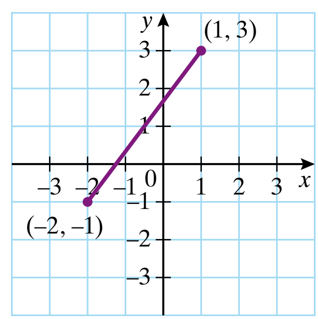
LINKS TO Example 5

Find the distance between each pair of points by first drawing a right-angled triangle and finding the length of the two shorter sides. Then use Pythagoras' theorem to calculate the hypotenuse.

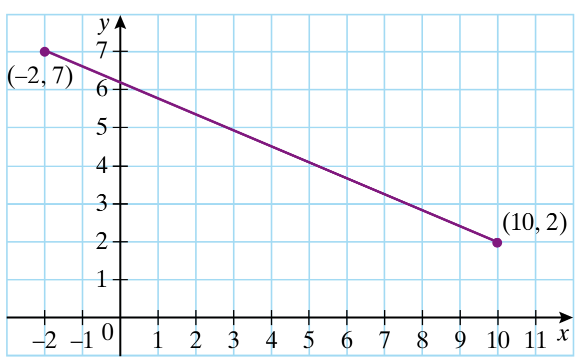
a



b



c



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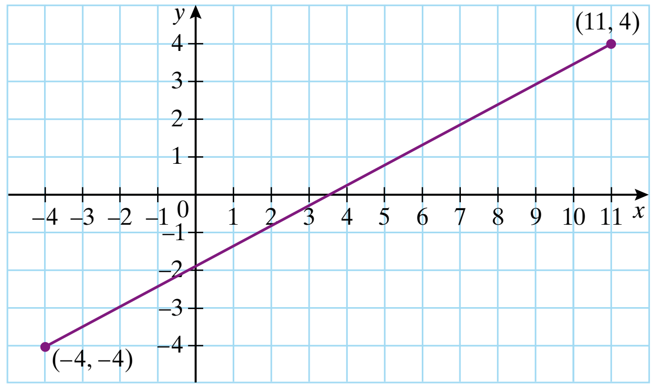
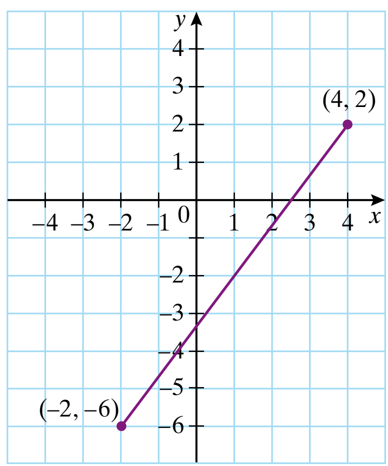
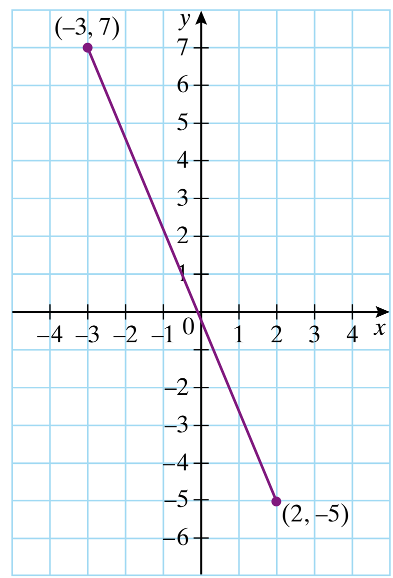
d



e



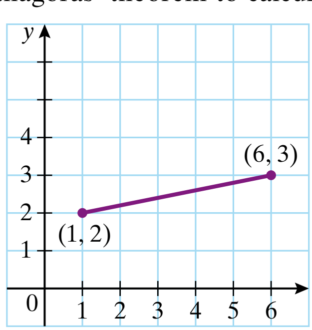
f



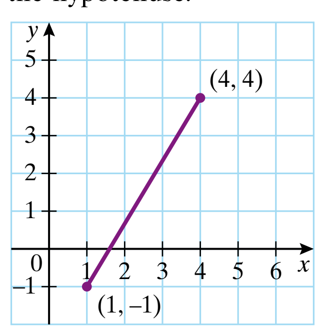
LINKS TO Examples 6, 7

Find the distance between these pairs of points, correct to one decimal place by first drawing a right-angled triangle and finding the length of the two shorter sides. Then use Pythagoras' theorem to calculate the hypotenuse.

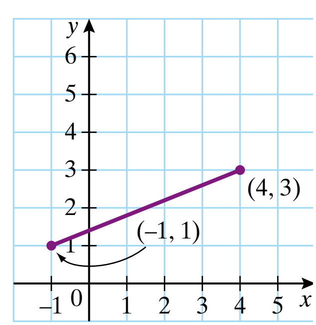
a



b



c



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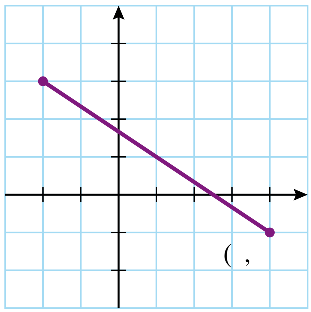
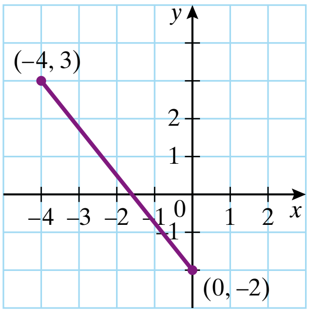
d



e



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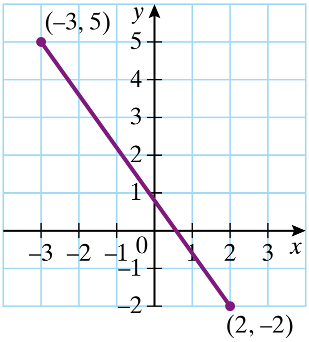
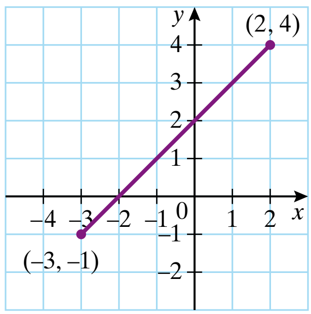
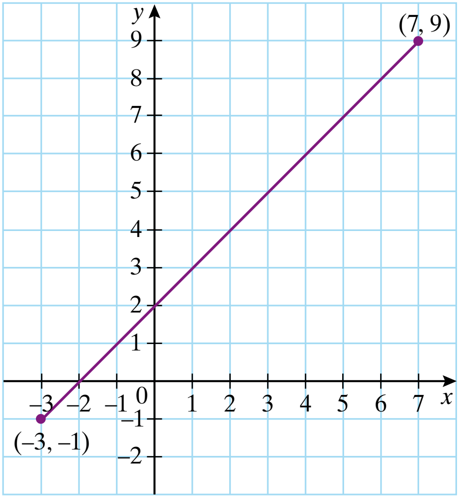


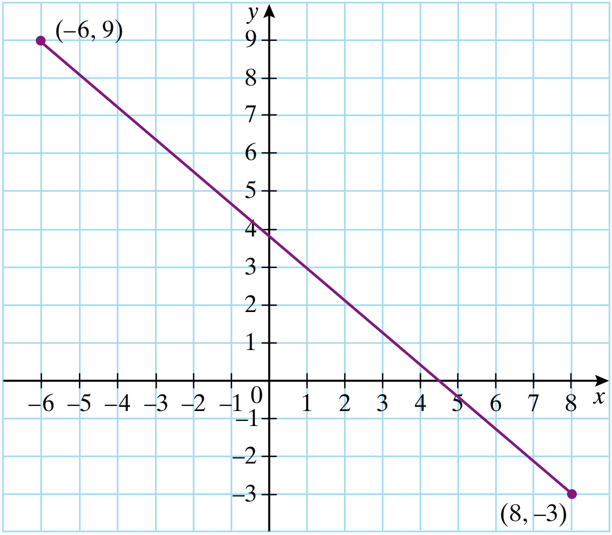
g



h



i



LINKS TO Example 8

For each of these pairs of points,

i draw a diagram to show the points on the Cartesian plane.

ii calculate the distance between the points. Give each distance correct to one decimal place.

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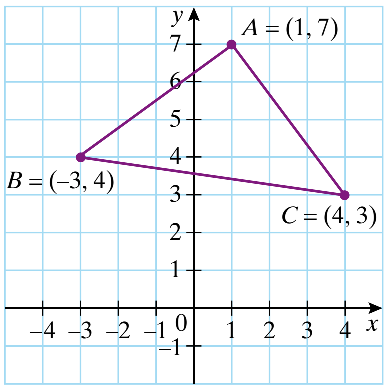
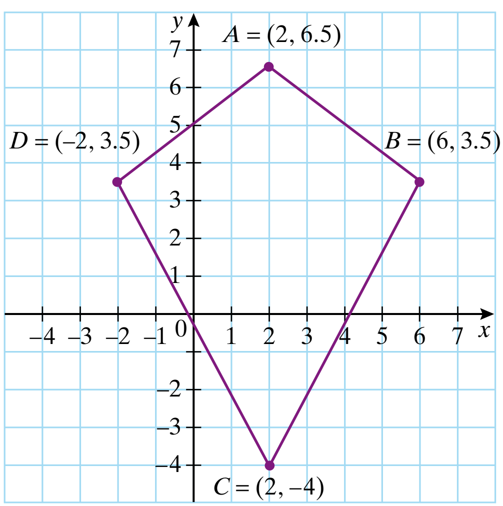
a (3, 2) and (8, 10) b (2, 5) and (7, 8) c (–1, –3) and (4, 7)
d (–6, 8) and (5, –4) e (–3, 4) and (–5, 0) f (2, –5) and (–6, 3)
g (–3, –7) and (–1, 5) h (–7, 3) and (2, 5) i (0, –7) and (8, 3)

Find the distances between *A* and *B* and between *A* and *C* and, hence, show that *ABC* is an isosceles triangle.



By calculating the distances between points *A* and *D*, *A* and *B*, *D* and *C* and *B* and *C*, show that *ABCD* is a kite.



Plot the points *A* (0, 5), *B* (4, 7), *C* (2, 3) and *D* (–2, 1). Show that the points are the vertices of a rhombus.

Plot the points *A* (1, 0), *B* (4, 6), *C* (5, 3) and *D* (2, –3). Show that the points are the vertices of a parallelogram.

exercise 6.2 challenge

Plot the points *A* (2, 4), *B* (11, 16) and *C* (18, –8). Show that the points are the vertices of a right-angled triangle.

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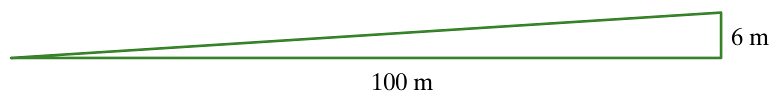
6.3 Gradient of a line

Gradient as a measure of slope

Whenever we walk or cycle uphill, we are aware that some hills are steeper than others.

This road sign tells us the steepness of the road. The road rises 6 m for every 100 m that it goes across horizontally. This is the same as saying that the road rises 0.06 m for every 1 m it goes across horizontally.



Example 9

Consider the road sign at right.

a Explain what this road signs tell us about the steepness of the road.

b How many metres would the road rise in a horizontal distance of 100 m?

c Write this as a percentage.

d How much does the road rise for every 1 m it goes across?

Working	Reasoning
a The road rises 1 metre for every 20 metres it goes across horizontally	
b 1 m to 20 m 5 m to 100 m The road would rise 5 m in a horizontal distance of 100 m.	Write an equivalent ratio to 1 : 20.
c This would be a slope of 5%.	$\frac{5}{100} \times \frac{100}{1}\% = 5\%$
d The road rises 0.05 m for every 1 m it goes across horizontally.	5 m to 100 m Divide both sides by 100 0.05 m to 1 m

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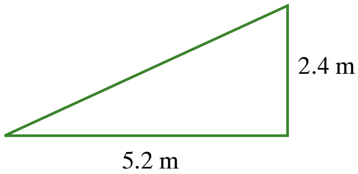
In mathematics we calculate steepness by dividing the vertical rise by the horizontal distance. This tells us the vertical rise for every 1 unit of horizontal distance. Steepness is called the **gradient**.

$$\text{gradient} = \frac{\text{rise}}{\text{run}}$$

Example 10

Calculate the gradient of this slope.



Working

$$\begin{aligned}\text{Gradient} &= \frac{\text{Rise}}{\text{Run}}\\ &= \frac{2.4\,\text{m}}{5.2\,\text{m}}\\ &\approx 0.46\end{aligned}$$

Reasoning

The rise is 2.4 m
The run is 5.2 m

Gradient of a linear graph

The gradient of a road changes along its length as it goes up hill, levels out, and so on. A linear graph has the same gradient all along its length.

If we take any two points on the linear graph, the vertical distance between them on the graph is referred to as the rise and the horizontal distance is called the run. The gradient of the linear graph is calculated by dividing the vertical rise by the horizontal run to the right. Traditionally the symbol m is used for the gradient of a linear graph. For every 1 unit to the right, there is a rise of m units.

$$m = \frac{\text{rise}}{\text{run}}$$

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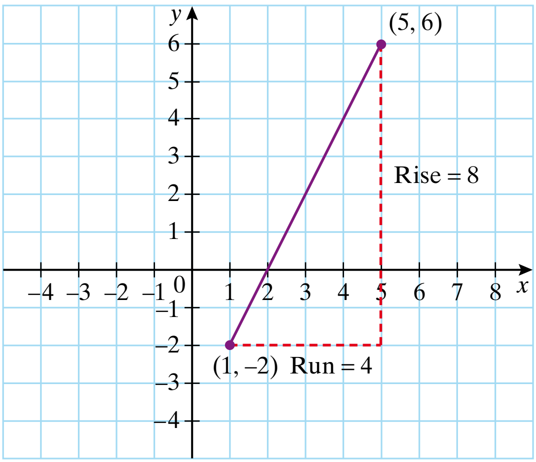
Example 11

Calculate the gradient of this line segment.



Working

$$m = \frac{\text{rise}}{\text{run}}$$
$$= \frac{8}{4}$$
$$m = 2$$

Reasoning

The right-angled triangle shows that the rise is 8 units when the run is 4 units.
The pronumeral m is used for gradient.

Positive and negative gradients

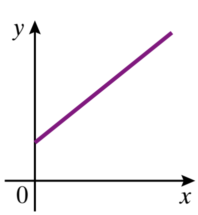
If a line graph slopes upwards to the right, the gradient is positive.

If a line graph slopes downwards to the right, the gradient is negative. We interpret the fall as a negative rise.

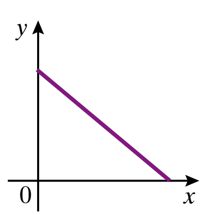
Example 12

State whether the gradient of each of these line segments is positive or negative. Explain.

a



b



continued

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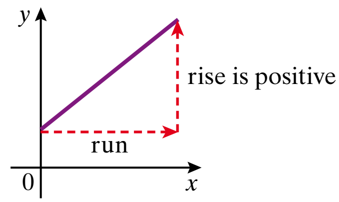
Example 12 continued

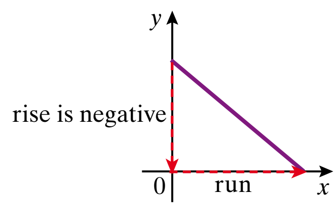
Working

a Positive. The graph slopes upwards to the right.

Reasoning

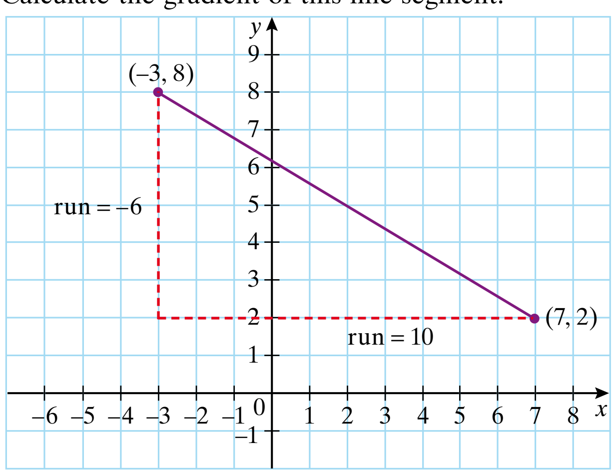


b Negative. The graph slopes downwards to the right.



Example 13

Calculate the gradient of this line segment.



Working

$$m = \frac{\text{rise}}{\text{run}}$$

$$= \frac{-6}{10}$$

$$m = -\frac{3}{5}$$

Reasoning

As the line segment 'runs' to the right, it falls.
The rise is negative.

The gradient is negative.

We can use the coordinates of two points on a line or line segment to calculate the rise, run and gradient.

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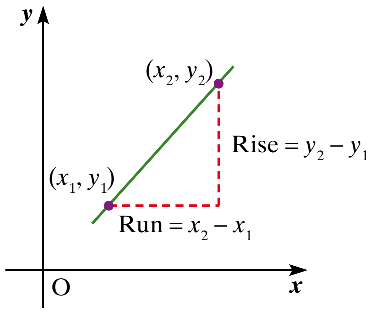

It doesn't matter which point we call the first point and which we call the second point, as shown below for the two points (1, –2) and (5, 6).

(1, –2)	(5, 6)
(x_1, y_1)	(x_2, y_2)

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{6 - (-2)}{5 - 1} \\ &= \frac{8}{4} \\ m &= 2 \end{aligned}$$

(5, 6)	(1, –2)
(x_1, y_1)	(x_2, y_2)

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-2 - 6}{1 - 5} \\ &= \frac{-8}{-4} \\ m &= 2 \end{aligned}$$

Calculating gradient

In general, for two points (x_1, y_1) and (x_2, y_2) on a straight line,

- rise is the difference between the y-coordinates, that is, $y_2 - y_1$
- run is the difference between the x-coordinates, that is, $x_2 - x_1$
- $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

Don't confuse the numbers with indices, for example, x_2 does not mean x^2. The **2** is simply a label to say x_2 is the x-coordinate of the **second** point.

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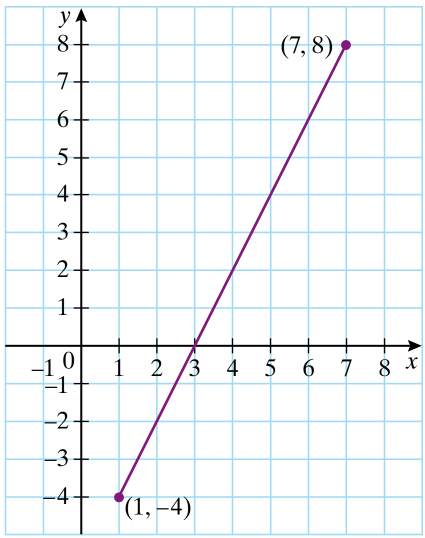
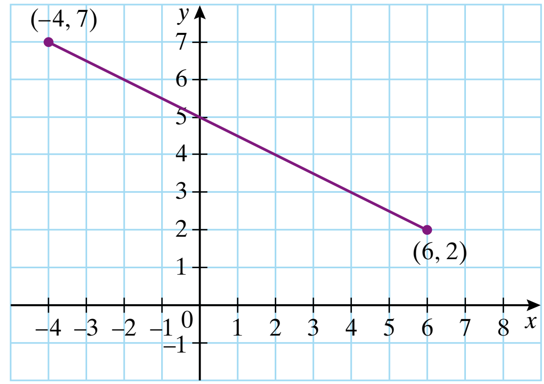
Example 14

Calculate the gradient of each of these line segments.

a



b



Working

a (1, –4)   (7, 8)

(x_1, y_1) (x_2, y_2)

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{8 - (-4)}{7 - 1}$$

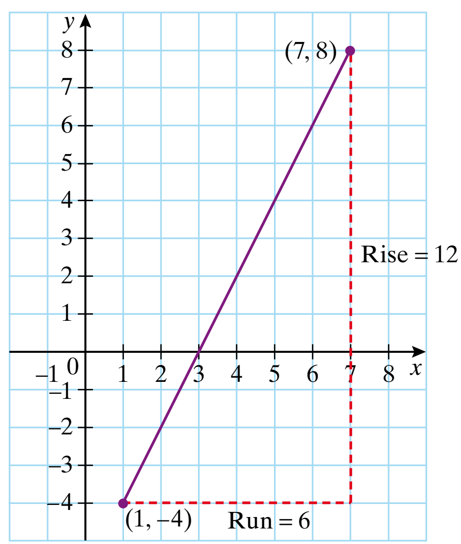
$$= \frac{12}{6}$$

$$= 2$$

The gradient is 2.

Reasoning

Label the points (x_1, y_1) and (x_2, y_2).



The graph slopes upwards to the right. The gradient is positive.

continued

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Example 14 continued

Working

b $(-4, 7)$ $(6, 2)$

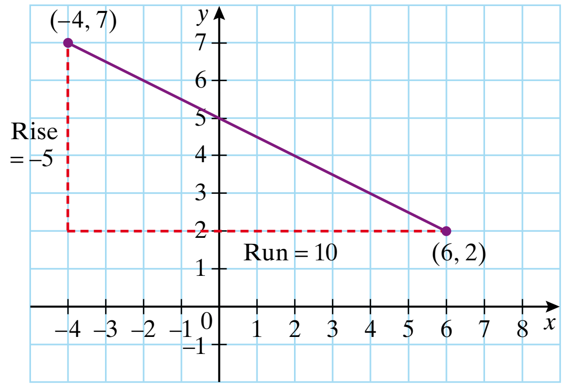
(x_1, y_1) (x_2, y_2)

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{2 - 7}{6 - (-4)}$$
$$= \frac{-5}{10}$$
$$= -\frac{1}{2}$$

The gradient is $-\frac{1}{2}$.

Reasoning

Label the points (x_1, y_1) and (x_2, y_2).

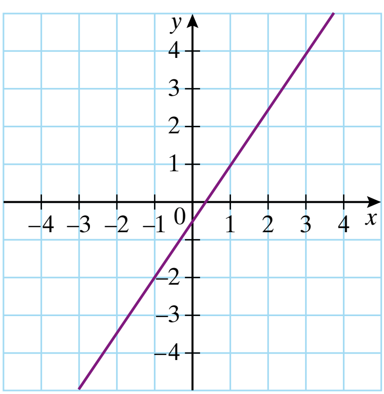


The graph slopes downwards to the right. The gradient is negative.

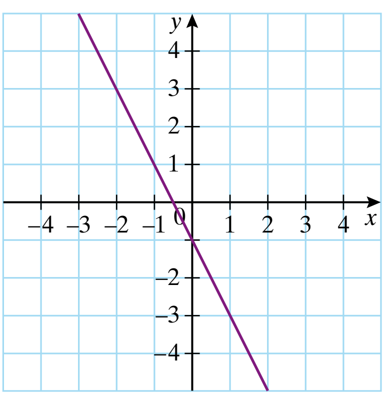
Example 15

Calculate the gradient of each of these lines.

a



b



continued

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Example 15 continued

Working

a (1, 1) and (3, 4)
(x_1, y_1) (x_2, y_2)

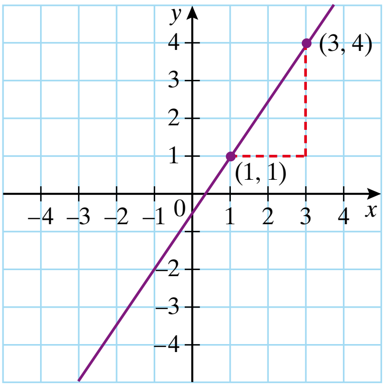
To find gradient use

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-1}{3-1} = \frac{3}{2} = 1\frac{1}{2}$$

The gradient is $1\frac{1}{2}$.

Reasoning

Choose any two points on the line that are at an intersection of gridlines. This means the coordinates will be whole numbers. Label the points (x_1, y_1) and (x_2, y_2).



Working

b (–3, 5) and (2, –5)
(x_1, y_1) (x_2, y_2)

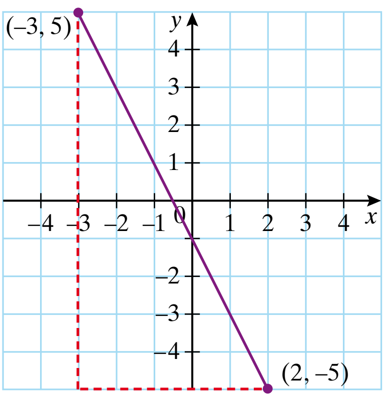
To find gradient use

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-5-5}{2--3} = \frac{-10}{5} = -2$$

The gradient is –2.

Reasoning

Choose any two points on the line that are at an intersection of gridlines. Label the points (x_1, y_1) and (x_2, y_2).



Example 16

Calculate the gradient of lines passing through the following points.

a (–3, –4) and (2, 6) **b** (–4, 5) and (2, 2) **c** (–2, 2) and (3, 8)

continued

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Example 16 continued

Working | **Reasoning**

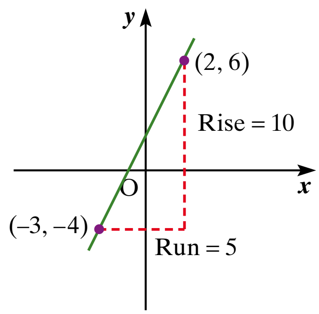
a $(-3, -4) \quad (2, 6)$

$(x_1, y_1) \quad (x_2, y_2)$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{6 - (-4)}{2 - (-3)} \\ &= \frac{10}{5} \\ &= 2 \end{aligned}$$

The gradient is 2.

Label the points (x_1, y_1) and (x_2, y_2).



The gradient is positive.

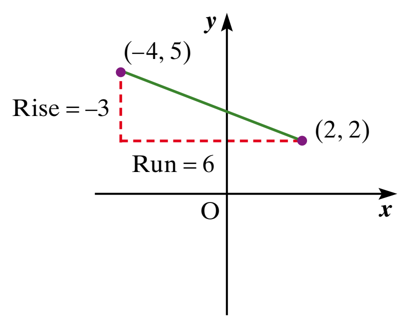
b $(-4, 5) \quad (2, 2)$

$(x_1, y_1) \quad (x_2, y_2)$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2 - 5}{2 - (-4)} \\ &= \frac{-3}{6} \\ &= -\frac{1}{2} \end{aligned}$$

The gradient is $-\frac{1}{2}$.

Label the points (x_1, y_1) and (x_2, y_2).



The gradient is negative.

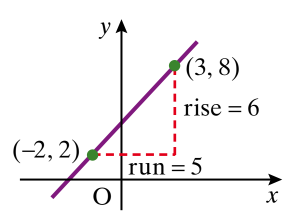
c $(-2, 2) \quad (3, 8)$

$(x_1, y_1) \quad (x_2, y_2)$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{8 - 2}{3 - (-2)} \\ &= \frac{6}{5} \\ &= 1\frac{1}{5} \\ &= 1.2 \end{aligned}$$

The gradient is 1.2.

Label the points (x_1, y_1) and (x_2, y_2).



The gradient is positive.

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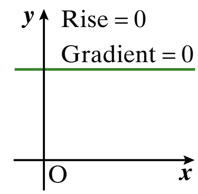
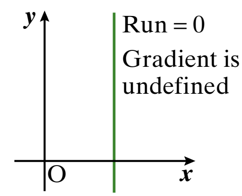
Gradients of horizontal and vertical lines



For a horizontal line, the gradient is zero because the rise is zero, regardless of the run.

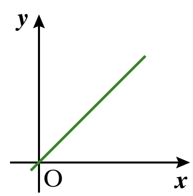
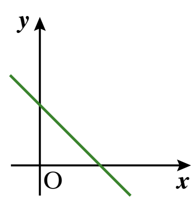
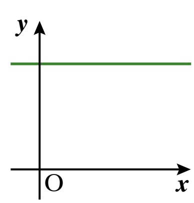
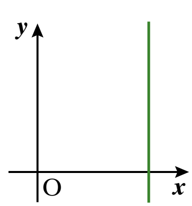
$$\frac{\text{rise}}{\text{run}} = \frac{0}{\text{run}} = 0$$



For a vertical line the run is zero, so the gradient is undefined as any number divided by zero is undefined.

$$\frac{\text{rise}}{\text{run}} = \frac{\text{rise}}{0} = \text{undefined}$$

Summary of gradients

Gradient	**Graph**	$\frac{\textbf{rise}}{\textbf{run}}$
Positive		$\frac{\text{rise}}{\text{run}} = \frac{+}{+} = +$
Negative		$\frac{\text{rise}}{\text{run}} = \frac{-}{+} = -$
Zero		$\frac{\text{rise}}{\text{run}} = \frac{0}{+} = 0$
Undefined		$\frac{\text{rise}}{\text{run}} = \frac{+}{0} = \text{undefined}$

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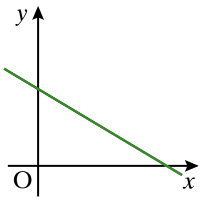
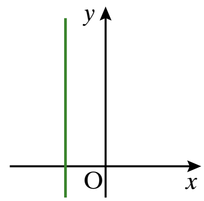
Example 17

For each of the following graphs state whether the gradient is positive, negative, zero or undefined. Explain.

a



b



Working	Reasoning
a The gradient is negative because the graph slopes downwards to the right so the rise is negative.	$\dfrac{\text{rise}}{\text{run}} = \dfrac{-}{+} = -$
b The gradient is undefined because the run is zero.	$\dfrac{\text{rise}}{\text{run}} = \dfrac{+}{0} = \text{undefined}$

exercise 6.3

LINKS TO Example 9

For each of these road signs

i How many metres would the road rise in a horizontal distance of 100 m?

ii How much does the road rise for every 1 m it goes across?

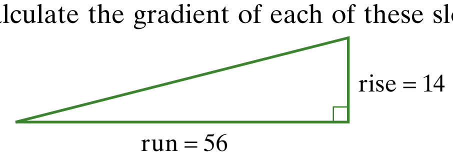
a

b

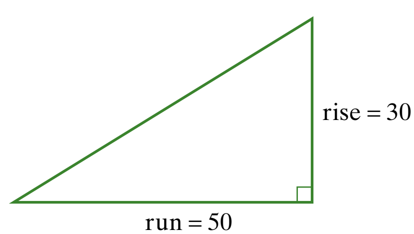
LINKS TO Example 10

Calculate the gradient of each of these slopes.

a



b



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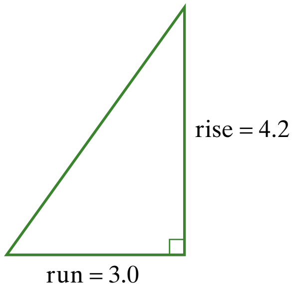
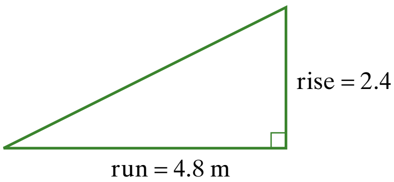
c



d



LINKS TO Example 11

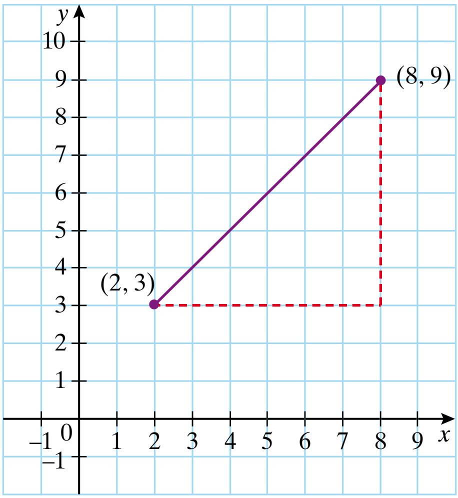
For each of these line segments, calculate

i the rise.

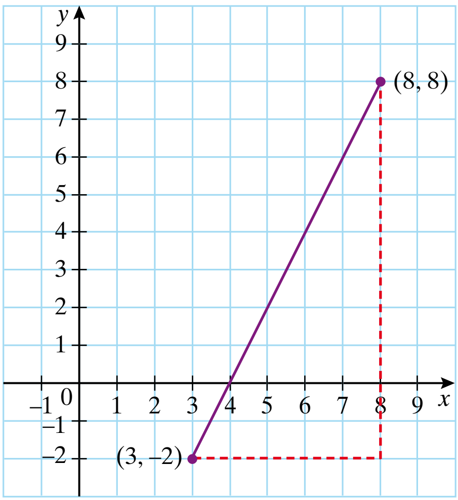
ii the run.

iii the gradient (expressing it as a fraction, where appropriate).

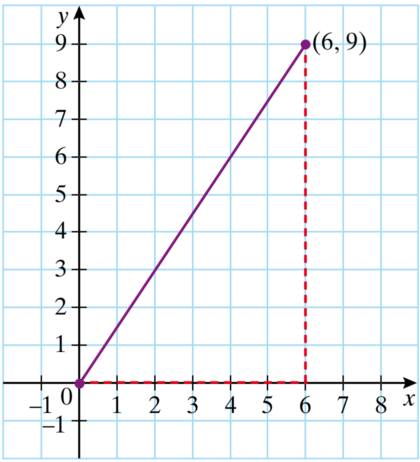
a



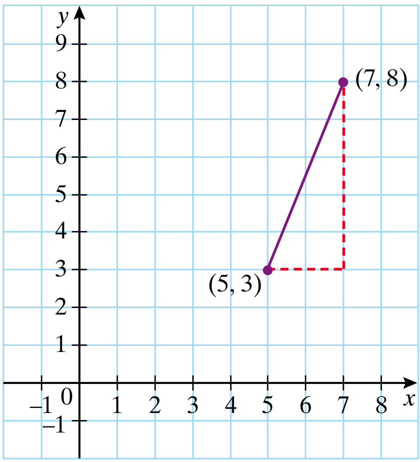
b



c



d



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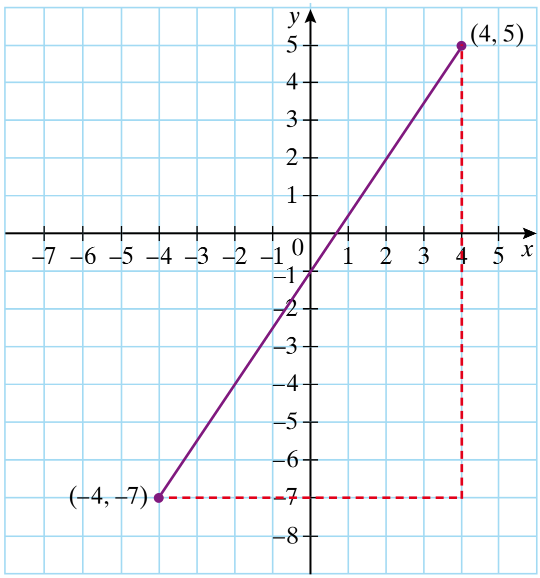
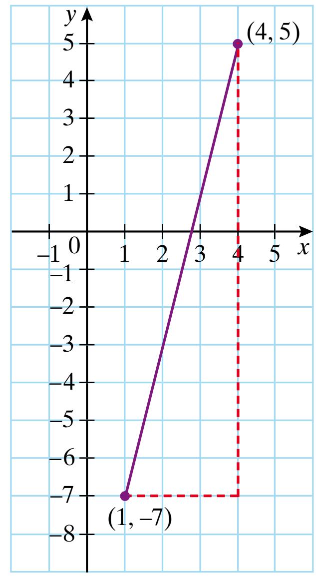
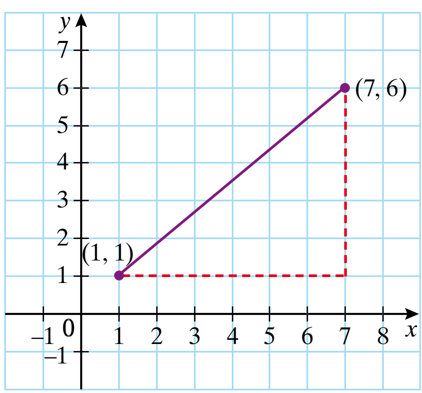
e



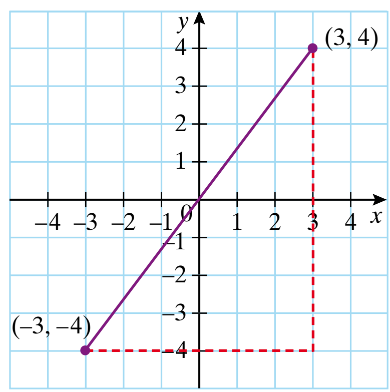
f



g



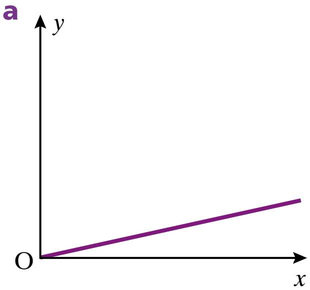
h



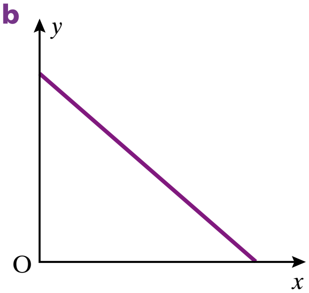
LINKS TO Example 12

For each of the following graphs, state whether the gradient is positive or negative. Explain.

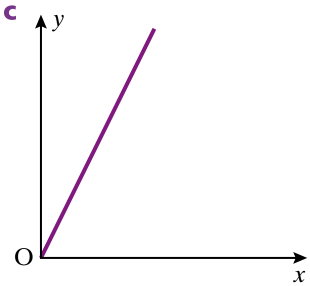
a



b



c



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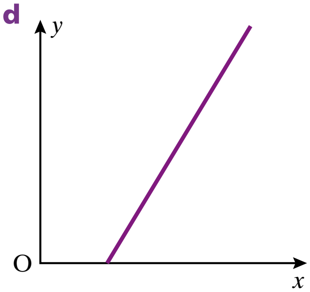
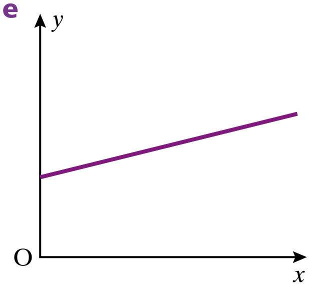
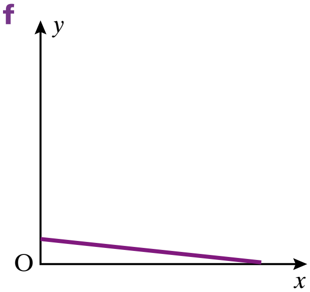
d



e



f



LINKS TO Example 13

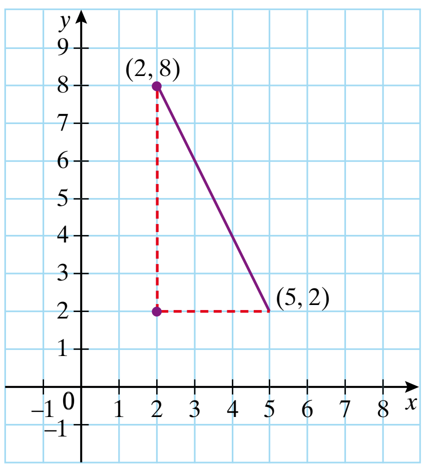
For each of these line segments, calculate

i the rise.

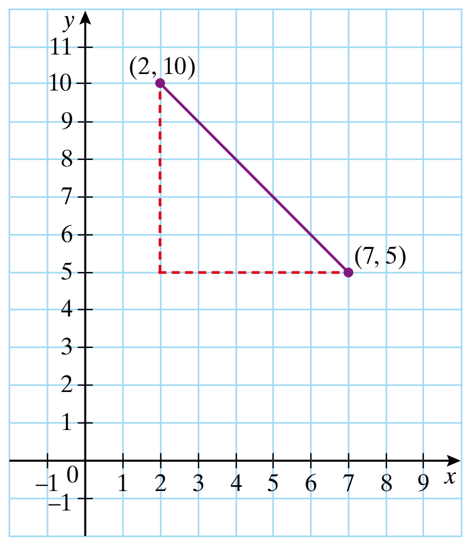
ii the run.

iii the gradient (expressing it as a fraction, where appropriate).

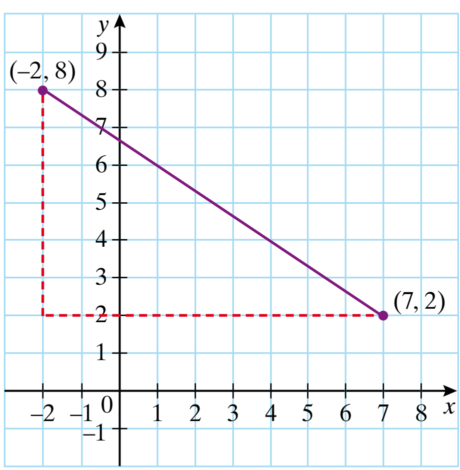
a



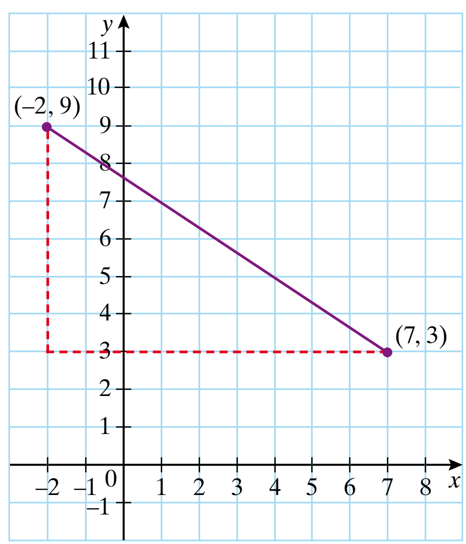
b



c



d



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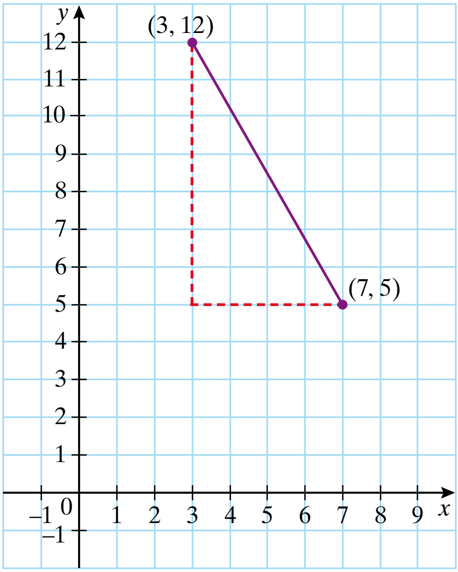
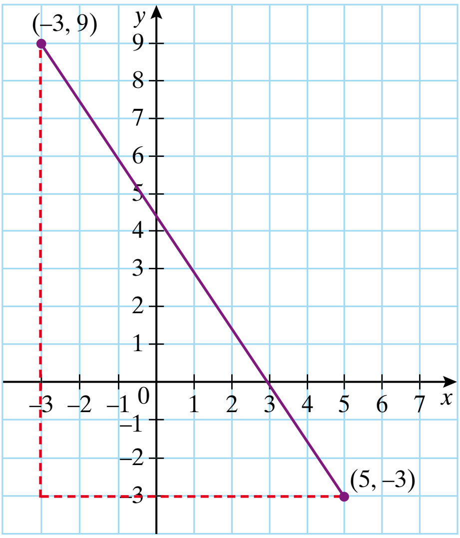
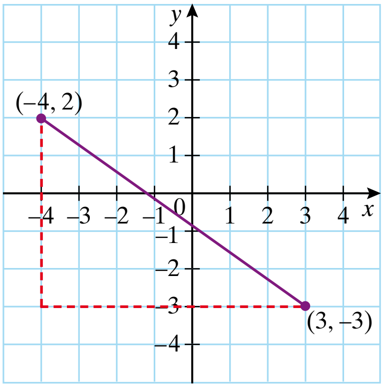
e



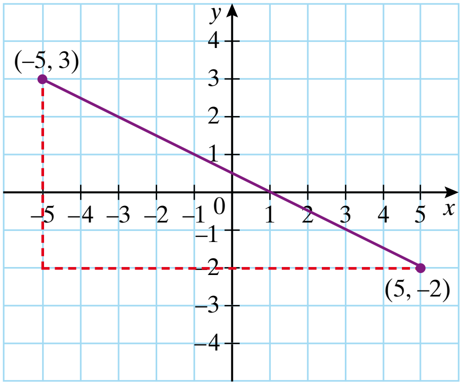
f



g



h



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LINKS TO Example 14

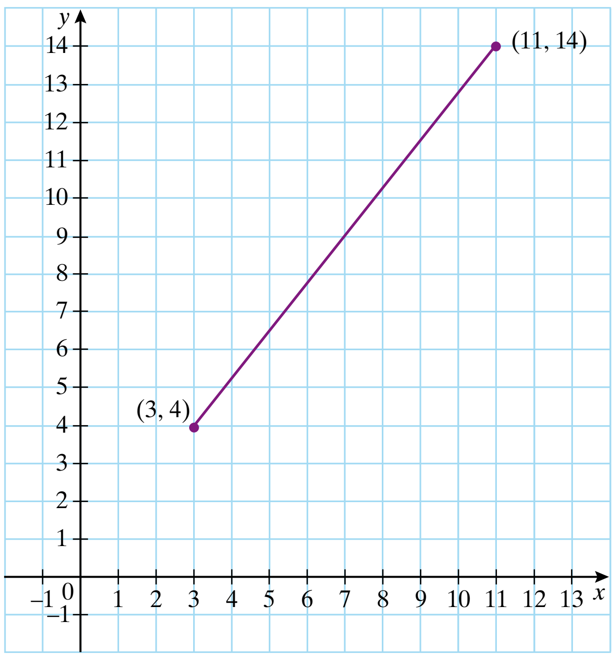
For each of these line segments, calculate

i the rise.

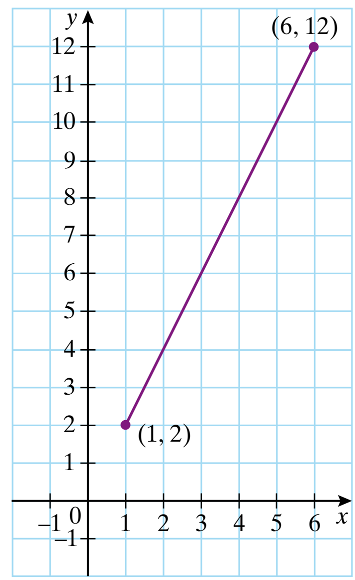
ii the run.

iii the gradient.

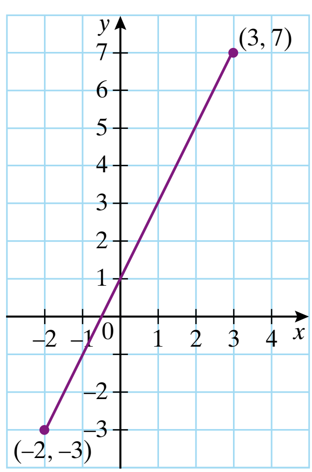
a



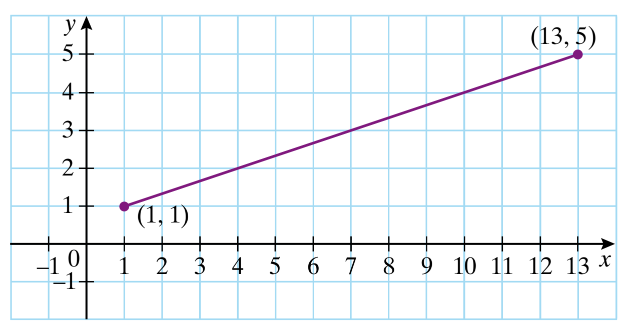
b



c



d



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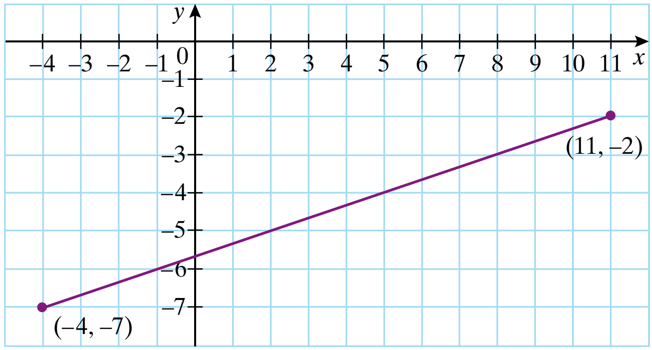
e



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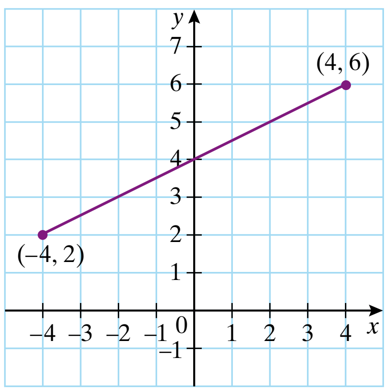
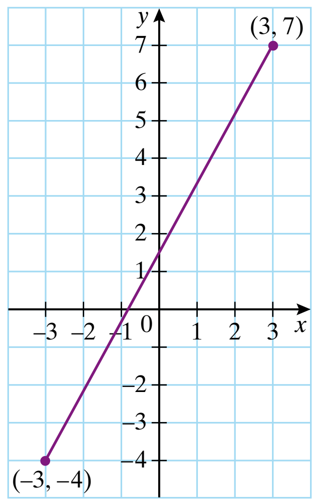
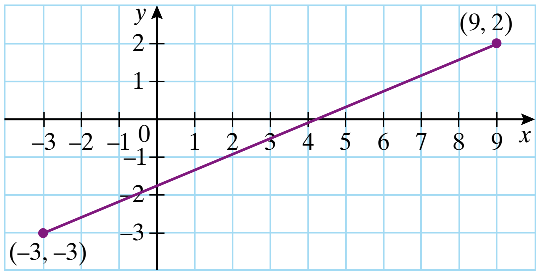


g



h



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LINKS TO Example 14b

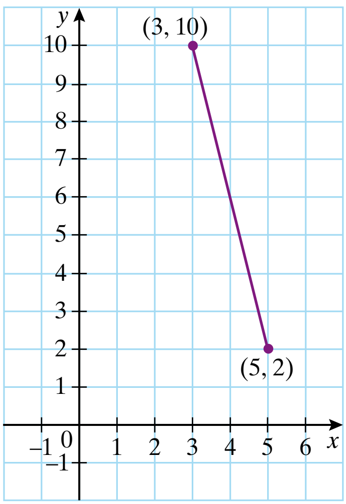
For each of these line segments, calculate

i the rise.

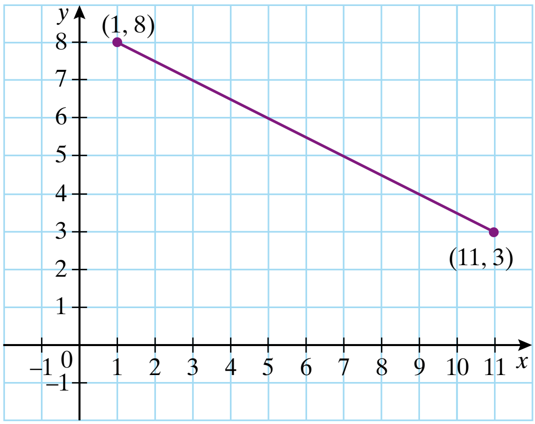
ii the run.

iii the gradient.

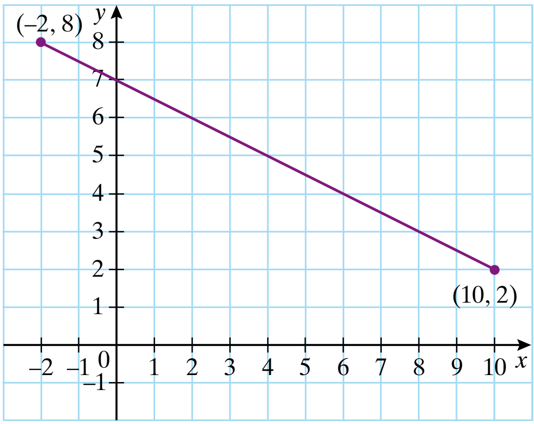
a



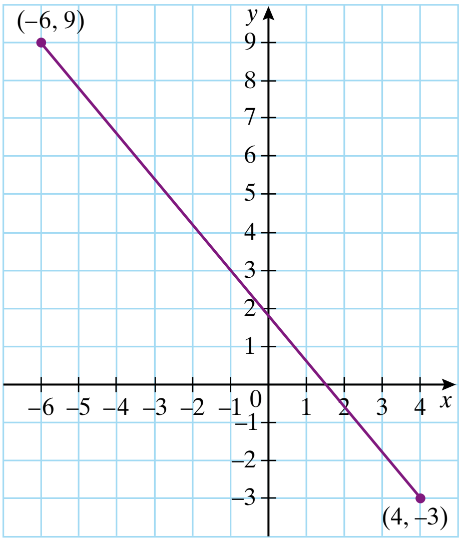
b



c



d



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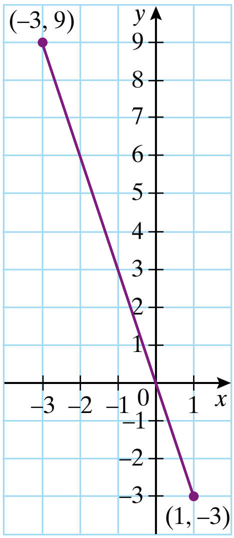
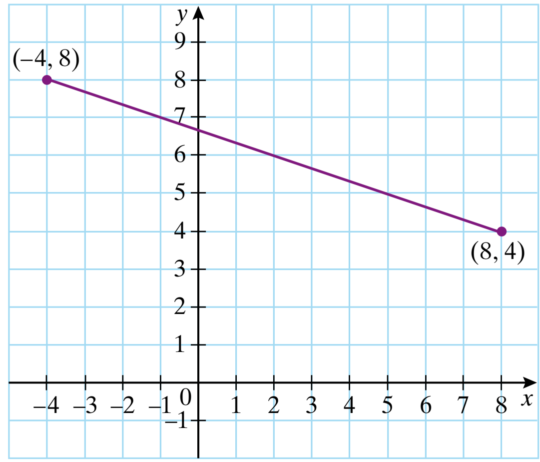
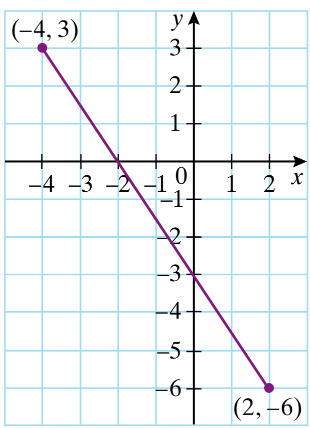
e



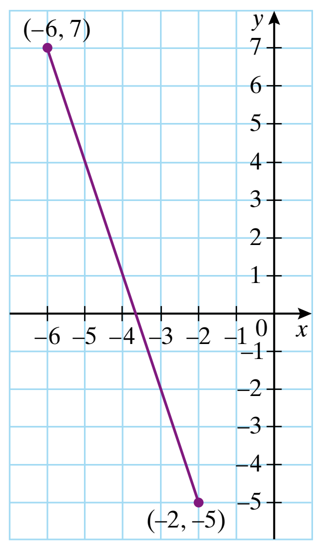
f



g



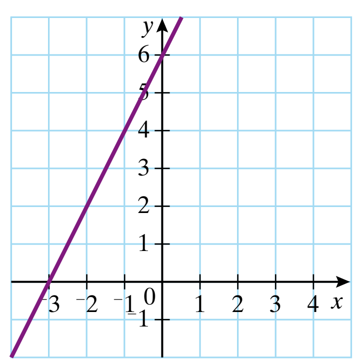
h



The gradient of this graph is

A 6 **B** –6 **C** 2

D 3 **E** –3



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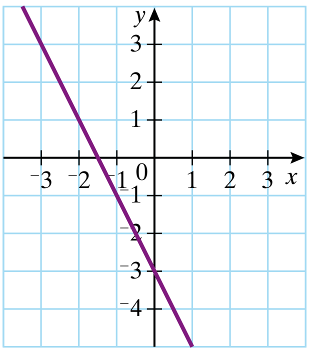
The gradient of this graph is

A 2 **B** –1.5 **C** –3

D 3 **E** –2



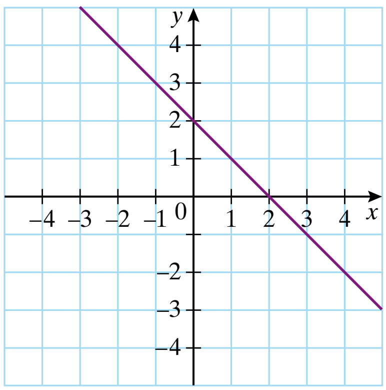
LINKS TO Example 15

For each of these linear graphs

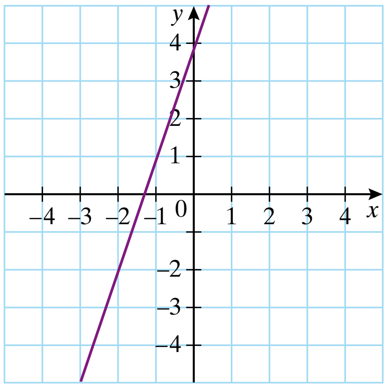
i state whether the gradient is positive or negative.

ii choose two appropriate points on the line and use the gradient formula to calculate the gradient.

a



b



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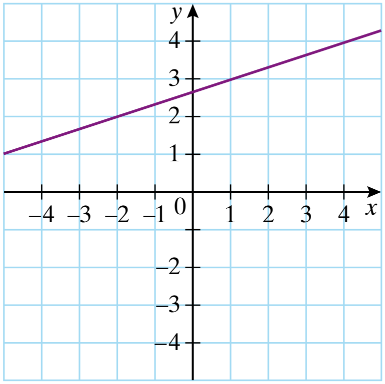
c



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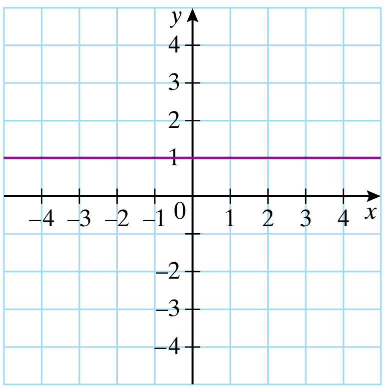

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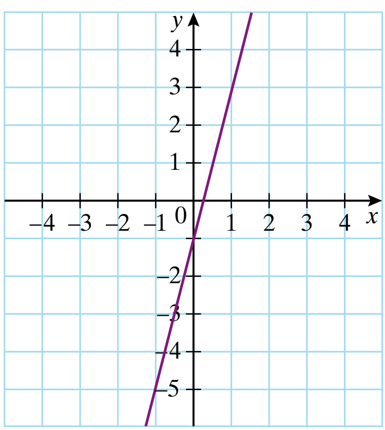

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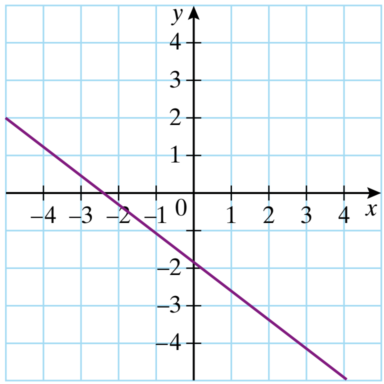
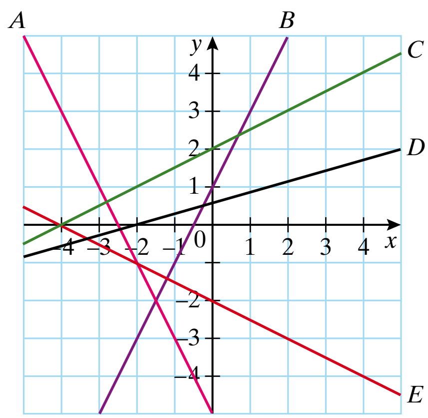


Which of the straight line graphs shown on the grid has a gradient of –2?



LINKS TO Example 16

Using the formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, calculate the gradient of the line passing through each pair of given points.

a (2, 3) and (6, 11)
b (1, 5) and (4, 8)
c (2, 1) and (0, 9)
d (3, 7) and (6, 16)
e (3, 4) and (1, 10)
f (1, 6) and (7, 3)
g (–1, 4) and (4, 9)
h (–3, 0) and (0, –9)
i (–2, –3) and (4, 7)
j (–3, 8) and (2, –1)
k (–3, 7) and (5, 17)
l (–1, 4) and (5, –5)
m (8, 3) and (3, –1)
n (–5, 3) and (7, –3)
o (–6, –5) and (6, 2)
p (–7, 4) and (8, –2)
q (9, –4) and (–7, 12)
r (–2, –7) and (4, 8)

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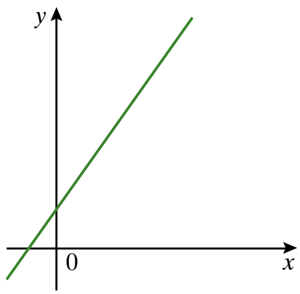
LINKS TO Example 17

For each of the following graphs state whether the gradient is positive, negative, zero or undefined.

a



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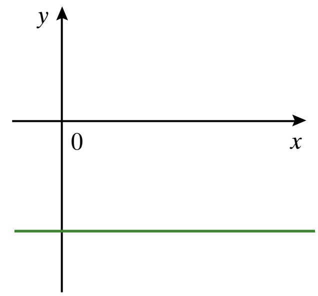

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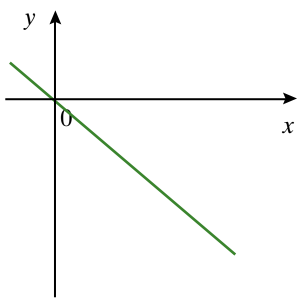

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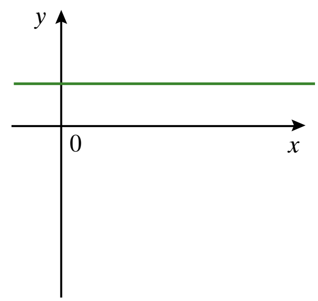

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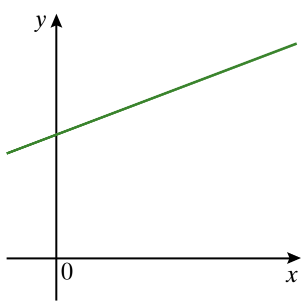

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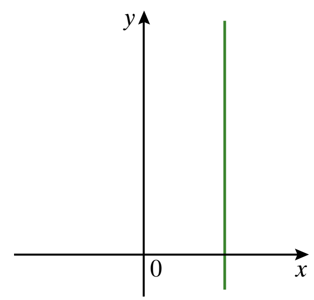

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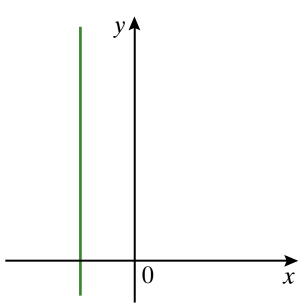

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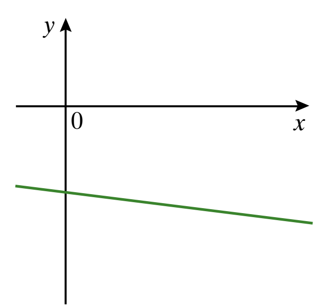

exercise 6.3 challenge

The line segment joining the points (–3, –5) and (2, a) has a gradient of 3. What is the value of a?

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Analysis task

Quadrilateral midpoints

When the midpoints of the four sides of a quadrilateral are joined in order, another quadrilateral is formed. Coordinate geometry can be used to investigate the properties of this quadrilateral.

a Using graph paper or GeoGebra, plot and label the points A (–6, 7), B (4, 5), C (6, –1) and D (–8, –5). Join the points in order to form the quadrilateral $ABCD$.

b Find the coordinates of the midpoint of each side of the quadrilateral. Label the midpoints E, F, G, H, with E as the midpoint of AB, F as the midpoint of BC, and so on.

c Make a conjecture about quadrilateral $EFGH$.

d Calculate the gradient of

- **i** EF
- **ii** FG
- **iii** GH
- **iv** HE.

e What do you notice about the gradients you have calculated in part **d**? Explain.

f State what this tells you about

- **i** EF and GH
- **ii** FG and HE.

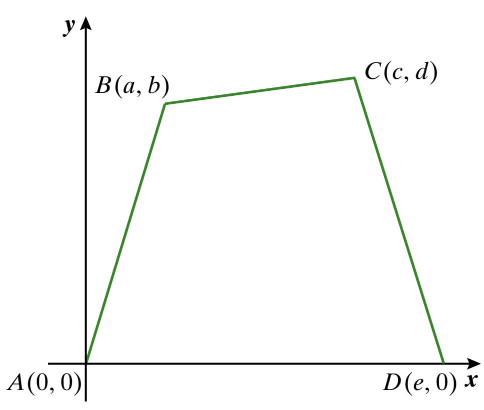
g What can you conclude about quadrilateral $EFGH$?

Challenge

Coordinate geometry can be used to prove the general case for any quadrilateral $ABCD$.

h Using the following diagram

- **i** repeat part **b**.
- **ii** repeat part **d**.



i Write a statement stating what you have proved.

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Review Coordinate geometry

Summary

Midpoint

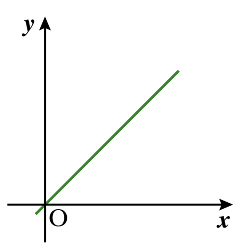
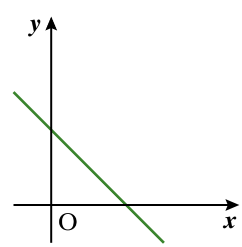
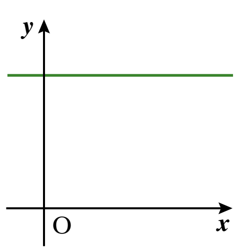
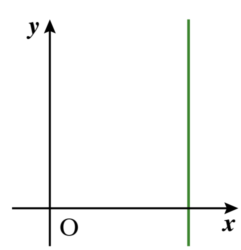
- The coordinates of the midpoint of two points (x_1, y_1) and (x_2, y_2) are
  $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

Distance between points

- The distance, d, between two points (x_1, y_1) and (x_2, y_2) is given by
  $$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$
  $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Gradient

- Gradient $= \dfrac{\text{rise}}{\text{run}}$
- The gradient, m, of a line passing through the points (x_1, y_1) and (x_2, y_2) is given by
  $$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Gradient	Graph	Gradient	Graph
Positive		Negative	
Zero		Undefined	

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Visual map

coordinate geometry	midpoint	rise
gradient	ordered pairs	run
line segment	Pythagoras' theorem	undefined

Revision

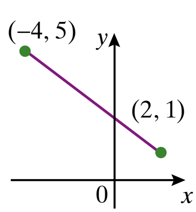
Multiple-choice questions

1. The midpoint of the points (–4, 8) and (2, 8) is
A (4, 10) **B** (–1, 6) **C** (–2, 8) **D** (–1, 8) **E** (–3, 8)

2. The midpoint of the points (–5, –7) and (3, –1) is
A (–4, –4) **B** (4, –4) **C** (–1, –4) **D** (–2, –6) **E** (–6, 1)

3. The distance between the points (–3, 5) and (2, –7) is
A 2 **B** 5 **C** 8 **D** 12 **E** 13

4. What is the gradient of the linear graph shown here?
A $-\frac{3}{2}$
B –1
C $-\frac{2}{3}$
D $\frac{2}{3}$
E $\frac{3}{2}$



5. The gradient of the line joining (2, –5) and (–4, 7) is
A –2 **B** –1 **C** $-\frac{1}{2}$ **D** 1 **E** 2

Short-answer questions

1. Find the midpoint of the line segment joining each of these pairs of points.
a (4, 7) and (8, 13)
b (–5, 7) and (3, –9)

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For each of these pairs of points

- **i** draw a diagram to show the points on the Cartesian plane.
- **ii** calculate the distance between the points.

a $(-2, -5)$ and $(7, 7)$

b $(-4, 6)$ and $(4, -9)$

Calculate the distance between the points $(-4, 8)$ and $(5, -9)$, correct to one decimal place.

For each of these pairs of points, calculate the gradient of the straight line that passes through the two points.

a $(-2, -5)$ and $(4, 13)$

b $(-3, 7)$ and $(5, 3)$

The line segment joining the points $(-1, 4)$ and $(5, a)$ has gradient -2. What is the value of a?

Extended-response questions

Show that the points $A(-3, -2)$, $B(1, 5)$ and $C(4, 2)$, are the vertices of an isosceles triangle.

A quadrilateral has the vertices (in clockwise order) $A(-3, 2)$, $B(3, 4)$, $C(6, -3)$ and $D(0, -5)$.

a Plot the points.

b Prove that $ABCD$ is a parallelogram.

Practice quiz
Chapter 6

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7 Linear and non-linear relationships

Pre-test

Warm-up

In everyday life there are many relationships between variables. If each taxi is allocated a certain length for parking on a taxi rank, the number of taxis that can fit depends on the length of parking strip available. The distance that a taxi travels depends on the speed and on the time it has been travelling. At constant speed, the relationship between time and distance is linear. If we were to plot a graph of distance against time for the constant speed, the graph would be a straight line. The taxi fare depends on the distance travelled and on a fixed charge called a flagfall. In this chapter we look at the characteristics of linear relationships and also at some non-linear relationships.

7.1 Plotting linear graphs

BLM Chapter warm-up

A relationship between variables that gives a straight line graph is called a **linear relationship**. As we saw in Year 8 a relationship can be represented in several ways:

- Story or description in words of some situation
- Table of values
- Rule
- Ordered pairs
- Graph

A story can be written as a rule. The rule for a linear relationship can be used to construct a table of values. The ordered pairs can then be plotted to construct the graph.

Example 1

Carry out the following steps for the relationship with the rule $y = -2x + 3$.

a Construct a table of values for the rule, using integer values of x between -2 and 2.

b Use this table of values to plot the graph. Join the points.

c Use the graph to find

  i the value of y if $x = 1.5$ **ii** the value of x if $y = 2$.

d Use algebra to check your answers for part **d**.

Working

a

x	-2	-1	0	1	2
y	7	5	3	1	-1

Reasoning

To find each y-value, substitute the corresponding x-value into the formula $y = -2x + 3$. For example, when $x = -2$,

$$y = 2 \times (-2) + 3$$
$$= 4 + 3$$
$$= 7.$$

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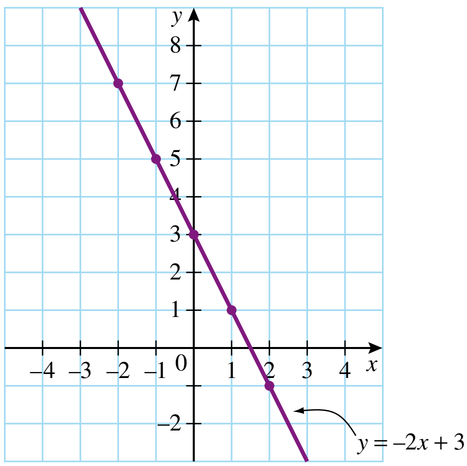
Example 1 continued

Working

b



Reasoning

Use the values in the table to choose appropriate scales for the axes.

- Since the x-values range from –2 to 2, the horizontal scale should range at least from –2 to 2.
- Since the y-values range from –1 to 7, the vertical scale should range at least from –1 to 7.

c **i** From the graph, when $x = 1.5$, $y = 0$

Locate the point on the graph where the x-coordinate is 1.5, and read the corresponding y-coordinate.

ii From the graph, when $y = 2$, $x = 0.5$

Locate the point on the graph where the y-coordinate is 2, and read the corresponding x-coordinate.

d **i** If $y = -2x + 3$ and $x = 1.5$ then

$$\begin{aligned} y &= -2 \times (1.5) + 3 \\ &= -3 + 3 \\ &= 0 \end{aligned}$$

Substituting the given value for x into the equation allows us to solve for the corresponding y-value. This confirms the solution found graphically.

ii If $y = -2x + 3$ and $y = 2$ then

$$\begin{aligned} 2 &= -2x + 3 \\ 2 - 3 &= -2x + 3 - 3 \\ -1 &= -2x \\ \frac{-1}{-2} &= \frac{-2x}{-2} \\ \frac{1}{2} &= x, \text{ that is, } x = \frac{1}{2} \end{aligned}$$

Substituting the given value for y into the equation allows us to solve for the corresponding x-value. This confirms the solution found graphically.

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exercise 7.1

LINKS TO Example 1

1. For the relationship with rule $y = 2x + 4$,
   - **a** construct a table of values for the rule, using integer values of x between –2 and 2
   - **b** use this table of values to plot the graph. Join the points.
   - **c** use the graph to find
     - **i** the value of y if $x = 1.5$
     - **ii** the value of x if $y = 2$
   - **d** use algebra to check your answers for part **c**.

2. For the relationship with rule $y = -x + 1$,
   - **a** construct a table of values for the rule, using integer values of x between –2 and 2
   - **b** use this table of values to plot the graph. Join the points.
   - **c** use the graph to find
     - **i** the value of y if $x = 0.5$
     - **ii** the value of x if $y = 3$
   - **d** use algebra to check your answers for part **c**.

3. For the relationship with rule $y = -2x + 5$,
   - **a** construct a table of values for the rule, using integer values of x between –3 and 1
   - **b** use this table of values to plot the graph. Join the points.
   - **c** use algebra to find
     - **i** the value of y if $x = -1.5$
     - **ii** the value of x if $y = 6$
   - **d** use the graph to check your answers for part **c**.

4. For the relationship with rule $y = 3x + 2$,
   - **a** construct a table of values for the rule, using integer values of x between –2 and 2
   - **b** use this table of values to plot the graph. Join the points.
   - **c** use the graph to find
     - **i** the value of y if $x = 0$
     - **ii** the value of x if $y = 0$
   - **d** use algebra to check your answers for part **c**.

5. I think of an integer, x, then I double it and subtract 4. The answer is y.
   - **a** Write a rule to represent the story.
   - **b** Using integer values for x from –3 to 5, construct a table of values.
   - **c** Plot a graph of the points.
   - **d** If we draw a straight line through the points, what is its gradient?
   - **e** What is the significance of the gradient in terms of the story?
   - **f** What is the value of y when $x = 0$?
   - **g** What is the significance of this value of y in terms of the story?

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7.2 Finding linear rules from tables of values

It is possible to tell, without actually plotting the points, whether a table of values will give a straight line when graphed.

When the x-values increase in steps of 1, the y-values of a linear relationship will change in steps of a constant size. The y-values leading to other sorts of graphs will not change in this way.

For example, the table opposite will lead to a linear graph.

The x values increase by 1.

At the same time the y-values increase in constant steps of 2.

+1 +1 +1 +1

x	–1	0	1	2	3
y	–5	–3	–1	1	3

+2 +2 +2 +2

Example 2

Decide whether each table represents a linear relationship.

a

x	0	1	2	3	4
y	13	9	5	1	–3

b

x	0	1	2	3	4
y	4	12	16	18	19

Working

+1 +1 +1 +1

a

x	0	1	2	3	4
y	13	9	5	1	–3

– 4 – 4 – 4 – 4

The relationship is linear because the y-values are changing in equal steps as the x-values increase in steps of 1.

Reasoning

A relationship is linear if the y-values increase or decrease in equal steps as the x-values increase in steps of 1.

continued

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Example 2 continued

Working

b

+1 +1 +1 +1

x	0	1	2	3	4
y	4	12	16	18	19

+8 +4 +2 +1

The relationship is not linear because the y-values are not changing in equal steps as the x-values increase in steps of 1.

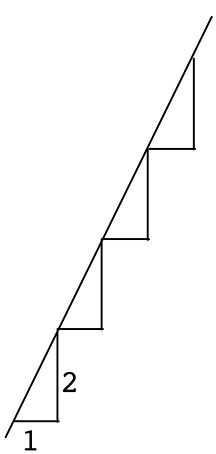
Reasoning

A relationship is linear if the y-values increase or decrease in equal steps as the x-values increase in steps of 1.

As we saw in section **6.3**, the steepness of a linear graph is called the gradient and the gradient is calculated by dividing the rise by the run. Consider again the table at the top of page 269. For each horizontal run of 1 step in the x-values, there is a rise of 2 in the y-values.

$$m = \frac{\text{rise}}{\text{run}}$$
$$= \frac{2}{1}$$
$$m = 2$$

So the gradient of the graph of this relationship is 2.



The value of y when $x = 0$ is also useful in finding a linear rule from a table of values. We refer to this value as b.

The general rule for a linear relationship is $y = mx + b$.

In the table of values above, $m = 2$ and $b = -3$ (because $y = -3$ when $x = 0$). The rule for the linear relationship is $y = 2x - 3$.

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Example 3

Find the rule for the relationship represented by each of these tables of values and state whether the graph will slope upwards or downwards to the right.

a

x	0	1	2	3	4
y	3	7	11	15	19

b

x	0	1	2	3	4
y	2.0	–0.5	–3.0	–5.5	–8.0

Working

a The gradient is 4.

Reasoning

First check that the relationship is linear.
When x goes up in steps of 1, the y-values go up in constant steps of 4, so this is a linear relationship.
As y-values are going up by 4, the gradient is 4.
The gradient is 4, that is, $m = 4$.

+1 +1 +1 +1

x	0	1	2	3	4
y	3	7	11	15	19

+4 +4 +4 +4

$b = 3$
The rule is $y = 4x + 3$.

The ordered pair (0, 3) tells us that $b = 3$.
The general rule for a linear relationship is $y = mx + b$, where m is the gradient and b is the value of y when $x = 0$.

The graph will slope upwards to the right.

The gradient is positive.

continued

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Example 3 continued

Working

b The gradient is –2.5.

Reasoning

First check that the relationship is linear. When x goes up in steps of 1, the y-values go down in constant steps of 2.5, so this is a linear relationship.
The gradient is –2.5, that is, $m = -2.5$.

$+1$ $+1$ $+1$ $+1$

x	0	1	2	3	4
y	2.0	–0.5	–3.0	–5.5	–8.0

-2.5 -2.5 -2.5 -2.5

$b = 2$
The rule is $y = -2.5x + 2$

The ordered pair (0, 2) tells us that $b = 2$.
$y = mx + b$, where m is the gradient and b is the value of y when $x = 0$.

The graph will slope downwards to the right.

The gradient is negative.

Example 4

Find the rule for the relationship represented by each of these tables of values and state whether the graph will slope upwards or downwards to the right.

a

x	–3	–2	–1	0
y	–3	–1	1	3

b

x	–3	–2	–1	0
y	$6\frac{2}{3}$	6	$5\frac{1}{3}$	$4\frac{2}{3}$

Working

a The gradient is 2.

Reasoning

First check that the relationship is linear. When x goes up in steps of 1, the y-values go up in constant steps of 2, so this is a linear relationship.
The gradient is 2, that is, $m = 2$.

$b = 3$
The rule is $y = 2x + 3$.

The ordered pair (0, 3) tells us that $b = 3$.
The general rule for a linear relationship is $y = mx + b$, where m is the gradient and b is the value of y when $x = 0$.

The graph will slope upwards to the right.

The gradient is positive.

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Example 4 continued

Working	Reasoning
b The gradient is $-\frac{2}{3}$.	First check that the relationship is linear. When x goes up in steps of 1, the y-values go down in constant steps of $\frac{2}{3}$, so this is a linear relationship. The gradient is $-\frac{2}{3}$, that is, $m = -\frac{2}{3}$.
$b = 4\frac{2}{3}$	The ordered pair $(0, 4\frac{2}{3})$ tells us that $b = 4\frac{2}{3}$.
The rule is $y = -\frac{2}{3}x + 4\frac{2}{3}$	$y = mx + b$, where m is the gradient and b is the value of y when $x = 0$.
The graph will slope downwards to the right.	The gradient is negative.

exercise 7.2

LINKS TO Example 2

1. Decide whether or not each table of values will give a linear graph.

a

x	–2	–1	0	1	2
y	–5	–3	–1	1	3

b

x	–2	–1	0	1	2
y	4	1	0	1	4

c

x	–3	–2	–1	0	1
y	15	12	9	6	3

d

x	–1	0	1	2	3
y	–10	–4	2	8	14

e

x	–2	–1	0	1	2
y	1	2	4	7	11

f

x	–4	–3	–2	–1	0
y	22	17	12	7	2

g

x	0	1	2	3	4
y	3	6	9	12	15

h

x	0	1	2	3	4
y	2	4	8	8	10

i

x	0	1	2	3	4
y	–4	–1	2	5	8

j

x	0	1	2	3	4
y	–5	–3	–1	1	3

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LINKS TO Examples 3, 4

For each of the linear relationships represented in these tables of values, find

i the gradient.

ii the y-intercept.

iii the linear rule.

a

x	–3	–2	–1	0	1	2	3
y	–4	–3	–2	–1	0	1	2

b

x	–3	–2	–1	0	1	2	3
y	–6	–4	–2	0	2	4	6

c

x	–3	–2	–1	0	1	2	3
y	2	3	4	5	6	7	8

d

x	–3	–2	–1	0	1	2	3
y	–12	–8	–4	0	4	8	12

e

x	–3	–2	–1	0	1	2	3
y	–1	1	3	5	7	9	11

f

x	–3	–2	–1	0	1	2	3
y	3	2	1	0	–1	–2	–3

g

x	–3	–2	–1	0	1	2	3
y	5	4	3	2	1	0	–1

h

x	–3	–2	–1	0	1	2	3
y	–17	–11	–5	1	7	13	19

i

x	–3	–2	–1	0	1	2	3
y	15	10	5	0	–5	–10	–15

j

x	–3	–2	–1	0	1	2	3
y	–1.5	–1	–0.5	0	0.5	1	1.5

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● The rule for the table shown here is:

x	−4	−3	−2	−1	0
y	−0.8	−0.5	−0.2	0.1	0.4

A $y = -0.8$ **B** $y = -0.8x + 0.4$ **C** $y = 0.3x + 0.4$
D $y = 0.3x - 0.8$ **E** $y = -0.4x - 0.8$.

exercise 7.2 challenge

● For each of the relationships described below (a to c), find

i the rule expressed in the form $y = mx + b$

ii the gradient of the straight line graph.

iii the y-intercept.

a There is a set of points where the y-coordinate is always three more than twice the x-coordinate.

b There is a set of points where the y-coordinate is always one less than half the x-coordinate.

c There is a set of points where the sum of the x- and y-coordinates is always three.

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7.3 Gradient and y-intercept

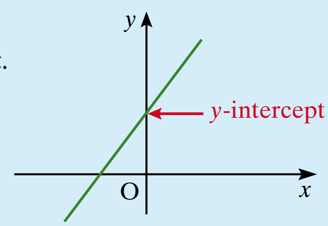
Linear graph

We have already seen that a linear relationship can be represented by the rule $y = mx + b$, where m is the gradient and b is the value of y when $x = 0$. We will now consider the rule $y = mx + b$ in terms of the graphs of linear relationships. The point where a graph crosses the y-axis, that is, the value of y when $x = 0$, is called the y-intercept. This means that we can refer to b in the equation $y = mx + b$ as the y-intercept.

Intercepts

A point where a graph crosses an axis is called an **intercept**.

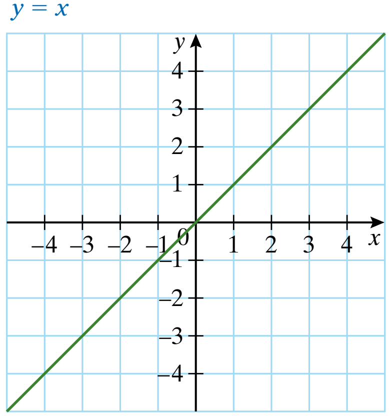
- The y-intercept is the point where the graph crosses the y-axis.



For each of the six linear graphs below, we can calculate the gradient, m, and we can find b by locating the y-intercept.

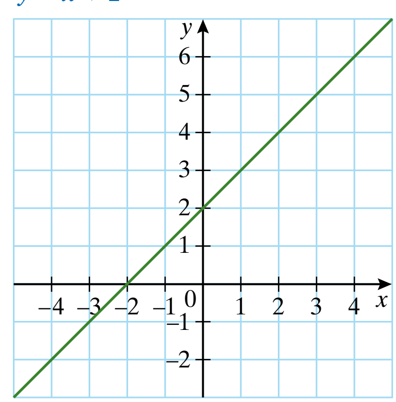
The table on the next page shows the linear rule, gradient and y-intercept for each of the six graphs.

$y = x$



Graph 1

$y = x + 2$



Graph 2

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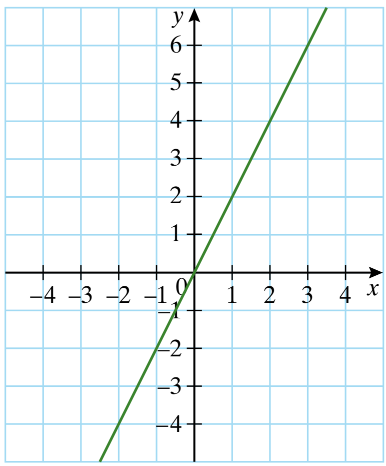
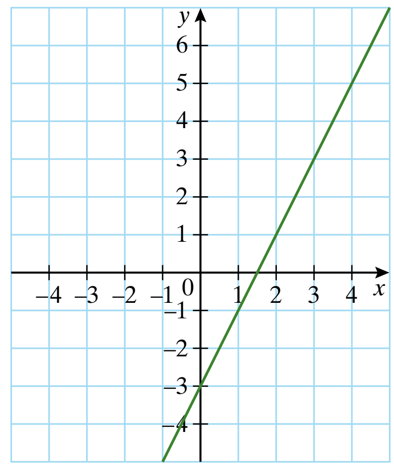
$y = 2x$



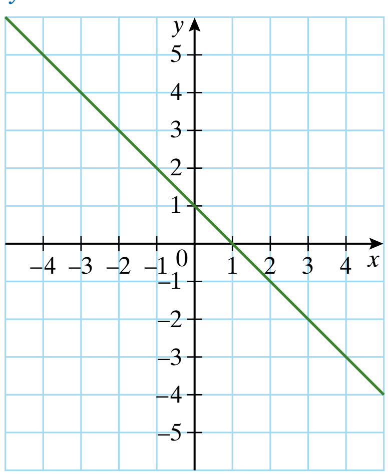
Graph 3

$y = 2x - 3$



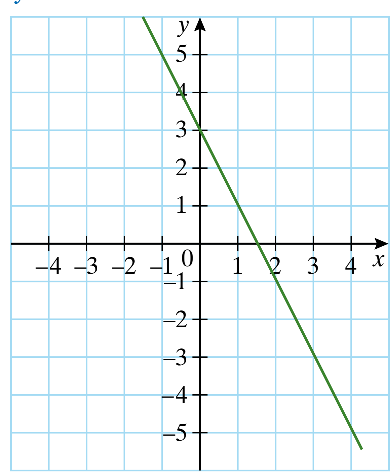
Graph 4

$y = -x + 1$



Graph 5

$y = -2x + 3$



Graph 6

We can see a pattern, where the gradient is the coefficient of x in the rule and the y-intercept is the constant number in the rule.

Linear rule	Gradient	y-intercept
$y = x$	1	0
$y = x + 2$	1	2
$y = 2x$	2	0
$y = 2x - 3$	2	–3
$y = -x + 1$	–1	1
$y = -2x + 3$	–2	3

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Gradient-intercept form of a linear relationship

We refer to $y = mx + b$ as the **gradient-intercept form** of a linear rule.

The equation of a straight line can be written as $y = mx + b$, where

- m is the gradient.
- b is the y-intercept. We can also say that the y-intercept is the point $(0, b)$.

Traditionally, the letter m has been used for gradient and c for the y-intercept so that in older textbooks the general rule for a linear relationship is $y = mx + c$. Calculators and computer algebra systems generally use $y = ax + b$.

Example 5

For each of these linear rules (in gradient-intercept form) state

i the gradient.

ii the coordinates of the y-intercept.

a $y = 3x + 5$ **b** $y = 2x - 4$ **c** $y = \frac{2x}{5} - 1$

d $y = -5x$ **e** $y = x + 7$

Working	Reasoning
a i For $y = 3x + 5$ the gradient is 3. **ii** The y-intercept is $(0, 5)$.	The gradient is the coefficient of x. The constant at the end is the y-intercept. Its x-coordinate is always 0.
b i For $y = 2x - 4$ the gradient is 2. **ii** The y-intercept is $(0, -4)$.	The gradient is the coefficient of x. The constant at the end is the y-intercept. In this case it is negative.
c i For $y = \frac{2x}{5} - 1$ the gradient is $\frac{2}{5}$. **ii** The y-intercept is $(0, -1)$.	The gradient is the coefficient of x. In this case it is a fraction. It could also have been written as $y = \frac{2}{5}x - 1$. The constant at the end is the y-intercept.
d i For $y = -5x$ the gradient is -5. **ii** The y-intercept is $(0, 0)$.	The gradient is the coefficient of x. In this case it is negative. The constant at the end is the y-intercept. If there is no constant the y-intercept is $(0, 0)$.
e i For $y = x + 7$ the gradient is 1. **ii** The y-intercept is $(0, 7)$.	The gradient is the coefficient of x. In this case the coefficient 1 is understood but not written. The constant at the end is the y-intercept.

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To write the equation $y = mx + b$ we need the values of m and b. The method we use for finding m and b depends on the information we are given.

Finding the equation given the gradient and the y-intercept

If we know the values of m and b we substitute them directly to the rule $y = mx + b$.

Example 6

Write equations for the following lines.

a Gradient 3 and y-intercept 5

b Gradient –4 and y-intercept is the point (0, 6)

c Gradient $\frac{2}{3}$ and y-intercept –3

d Gradient $-\frac{7}{2}$ and y-intercept is the point (0, 0)

Working	Reasoning
a $y = mx + b$ $y = 3x + 5$	$m = 3, b = 5$
b $y = mx + b$ $y = -4x + 6$	$m = -4, b = 6$
c $y = mx + b$ $y = \frac{2}{3}x - 3$	$m = \frac{2}{3}, b = -3$ We could also write $y = \frac{2x}{3} - 3$
d $y = mx + b$ $y = -\frac{7}{2}x + 0$ $y = -\frac{7}{2}x$	$m = -\frac{7}{2}, b = 0$ We could also write $y = -\frac{7x}{2}$

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Finding the equation from the graph

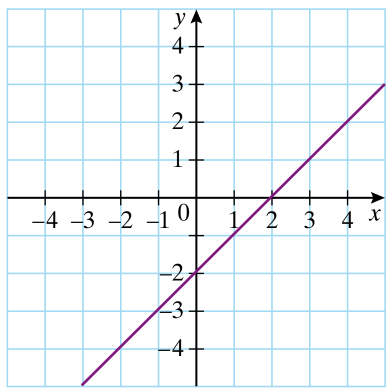
From a linear graph we can find

- the gradient, m.
- the y-intercept, b.

We can then write the equation for the line.

Example 7

Find the equation of this linear graph.



Working

Finding the gradient
(0, –2) and (4, 2)
(x_1, y_1) (x_2, y_2)

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$m = \dfrac{2 - (-2)}{4 - 0}$

$m = \dfrac{4}{4}$

$m = 1$

$b = -2$

The equation is $y = x - 2$

Reasoning

We need to find the gradient, m, and the y-intercept, b.
Choose any two points on the line that are at an intersection of gridlines.
Here, the points $(0,-2)$ and $(4,2)$ have been used.

The y-intercept is –2.

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Finding the equation given the gradient and one point on the line

The gradient formula applies to any two points on a line. If we know one point, (x_1, y_1), we can use the (x, y) to represent any other point on the line. If we also know the gradient, we can find the equation for the line by substituting for m and (x_1, y_1) in

$$m = \frac{y - y_1}{x - x_1}$$

This can be rearranged to give: $y - y_1 = m(x - x_1)$

Equation of a line given the gradient and one point

$y - y_1 = m(x - x_1)$

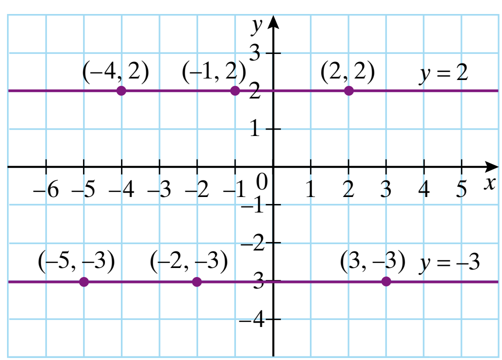
Example 8

Find the equation of the line with gradient -2 and passing through the point $(3, -5)$.

Working	Reasoning
$y = y_1 = m(x - x_1)$	Substitute $m = -2$, $x_1 = 3$ and $y_1 = -5$ in $y - y_1 = m(x - x_1)$
$y - (-5) = -2(x - 3)$	
$y + 5 = -2x + 6$	
$y = 2x + 1$	

Equations of horizontal and vertical lines

For a horizontal line, every point on the line has the same y-coordinate, regardless of its x-coordinate. The gradient of a horizontal line is zero. Using the general equation for a straight line $y = mx + b$, we see that when $m = 0$, the equation is simply $y = b$, where every point on the line has y-coordinate b.



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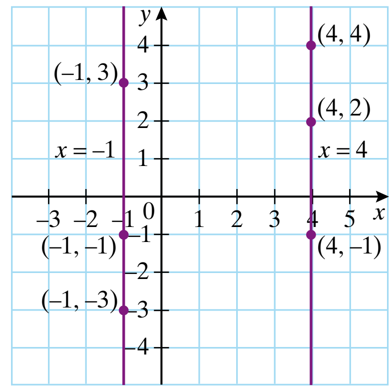
The gradient of a vertical line is undefined so we cannot use $y = mx + b$. Every point on the line has the same x-coordinate, regardless of its y-coordinate, so the equation is of the form $x = a$ where every point on the line has x-coordinate a.



The x-*axis* is the line $y = 0$

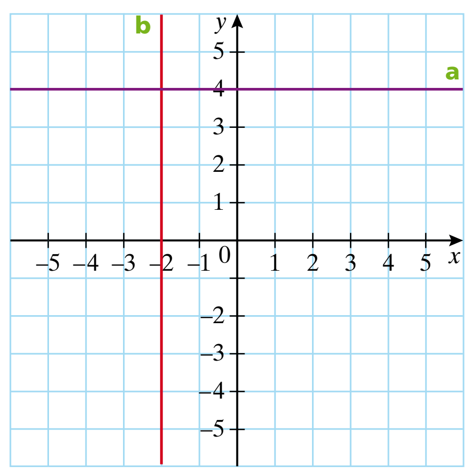
The y-*axis* is the line $x = 0$

Example 9

Write the rule for

a the horizontal linear graph.

b the vertical linear graph.



Working

a This graph has a y-intercept at (0, 4). Its rule is $y = 4$.

b This graph has an x-intercept at (–2, 0). Its rule is $x = -2$.

Reasoning

Every point on this line is 4 units up from the x-axis. Thus each point has $y = 4$.

Every point on this line is 2 units to the left of the y-axis. Thus each point has $x = -2$.

Example 10

Sketch each of the following graphs.

a $y = 5$ **b** $y = -3$ **c** $x = 2$ **d** $x = -4$

continued

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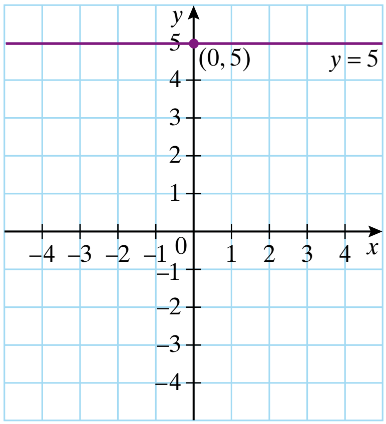
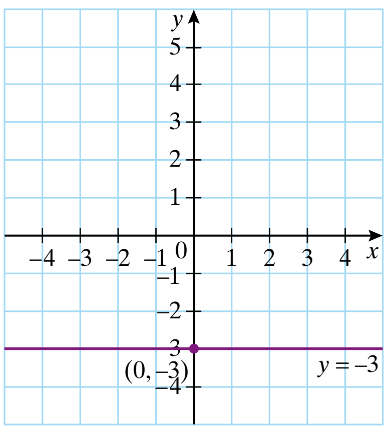
Example 10 continued

Working

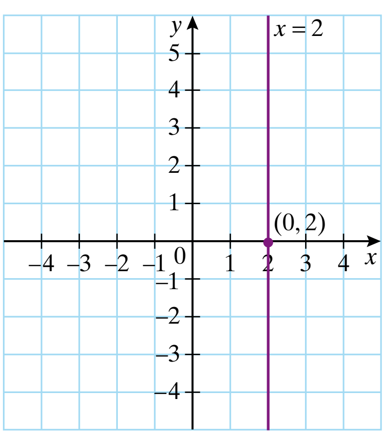
a



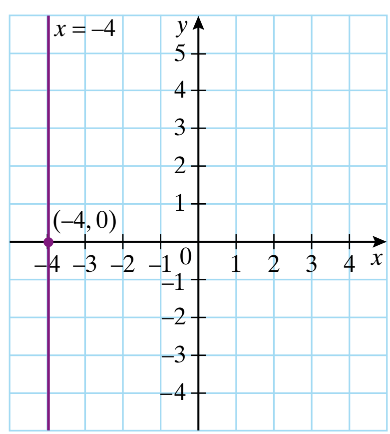
b



c



d



Reasoning

The line $y = 5$ connects all points that have a y-coordinate of 5, regardless of their x-coordinate.

The line $y = -3$ connects all points that have a y-coordinate of -3.

The line $x = 2$ connects all points that have an x-coordinate of 2, regardless of their y-coordinate.

The line $x = -4$ connects all points that have an x-coordinate of -4.

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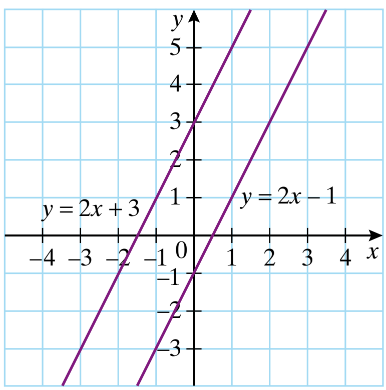
Parallel lines

Families of linear graphs

Linear graphs that have the same gradient are parallel lines.

Each of these linear graphs has gradient 2. Conversely, linear graphs that are parallel have the same gradient.



Example 11

For which of these linear rules will the graphs be parallel?
$y = 2x + 4$, $y = -3x + 4$, $y = 2$, $y = 2x$, $y = -2x + 4$, $y = 2x - 5$

Working

The graphs of $y = 2x + 4$, $y = 2x$, $y = 2x - 5$ will be parallel lines as they all have gradient 2.

Reasoning

Parallel lines have the same gradient.

General equation for a straight line

The equation $ax + by + c = 0$ is often used as a general equation for a straight line so that all possible cases can be covered by a single general equation. In this general form, a, b and c are constants. Note that b in this general form is not the same b as in $y = mx + b$. Either a or b can be 0, but obviously not in the same equation as there would then be no equation!

If $a = 0$, the equation reduces to the equation for a horizontal line. If $b = 0$, the equation reduces to the equation for a vertical line.

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Example 12

a Convert the equation $y = -\frac{3x}{4} + 3$ into the general form for a straight line.

b Convert the equation $5x - 4y + 8 = 0$ into the gradient-intercept form.

Working	Reasoning
a $y = -\frac{3x}{4} + 3$	
$4y = -3x + 12$	Multiply both sides by 4 to remove the fractions.
$3x + 4y - 12 = 0$	Take all terms to the left side, as the x term needs to be positive.
b $5x - 4y + 8 = 0$	
$4y = 5x + 8$	Isolate the y term.
$y = \frac{5x}{4} + \frac{8}{4}$	Divide both sides by 4.
$y = \frac{5x}{4} + 2$	Simplify.

Example 13

For each of these linear rules

i rearrange into gradient-intercept form, $y = mx + b$

ii state the gradient

iii state the coordinates of the y-intercept.

a $y = 4 - 6x$ **b** $2x + y = 3$

c $4x - y = 5$ **d** $3x + 2y = 7$

Working	Reasoning
a i $y = 4 - 6x$ $y = -6x + 4$ **ii** The gradient is –6. **iii** The y-intercept is (0, 4).	The constant and the x term can be swapped. The – sign before the $6x$ stays before it. The 4 with no sign is actually positive so becomes + 4.
b i $2x + y = 3$ $y = -2x + 3$	Subtract $2x$ from both sides.
ii The gradient is –2. **iii** The y-intercept is (0, 3).	

continued

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Example 13 continued

Working	Reasoning
c i $4x - y = 5$	
$-y = -4x + 5$	Subtract $4x$ from both sides.
$y = 4x - 5$	Then multiply both sides by -1.
ii The gradient is 4.	
iii The y-intercept is $(0, -5)$.	
d i $3x + 2y = 7$	Subtract $3x$ from both sides.
$2y = -3x + 7$	
$\frac{2y}{2} = \frac{-3x}{2} + \frac{7}{2}$	Then divide both sides by 2.
$y = -\frac{3}{2}x + \frac{7}{2}$	Improper fractions can be left, or changed to mixed numbers, or changed to decimals if they do not recur.
ii The gradient is $-\frac{3}{2}$ or -1.5.	
iii The y-intercept is $(0, \frac{7}{2})$ or $(0, 3.5)$.	

exercise 7.3

LINKS TO Example 5

1. Each of the following equations represents a linear relationship. Copy and complete the table to show the gradient and the y-intercept of each relationship.

Equation	Gradient	y-intercept
$y = 3x - 5$		
$y = -5x + 4$		
$y = x + 3$		
$y = -2x + 1$		
$y = x$		
$y = 3x + 2$		
$y = -4x - 1$		
$y = -x - 5$		
$y = 2x + 7$		
$y = 0.5x + 3.1$		

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LINKS TO Example 6

Write equations for the following lines.

a gradient 2 and y-intercept –6

b gradient –3 and y-intercept 4

c gradient $\frac{2}{3}$ and y-intercept –1

d gradient $-\frac{4}{5}$ and y-intercept 9

e gradient $\frac{5}{7}$ and y-intercept –3

f gradient –3 and y-intercept $\frac{1}{2}$

LINKS TO Example 7

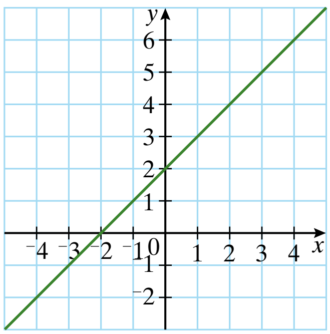
For each of these graphs

i find the gradient and hence write the value of m

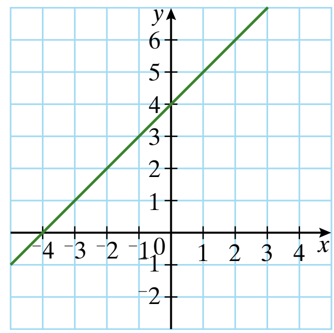
ii state the y-intercept, that is, the value of b

iii write the rule for the relationship in the form $y = mx + b$

a



b



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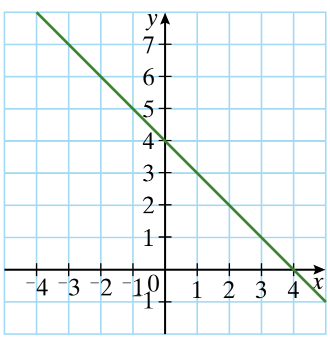
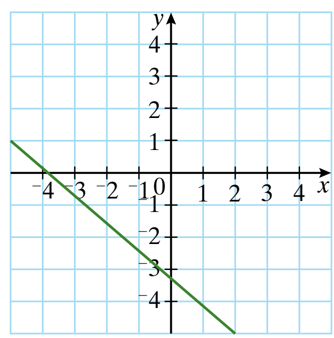
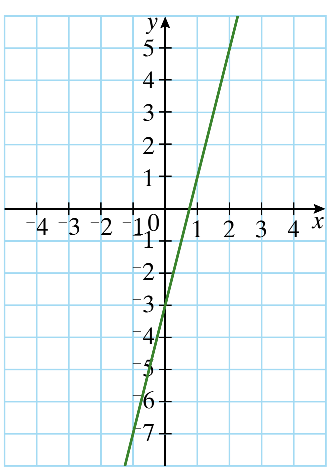


d



e



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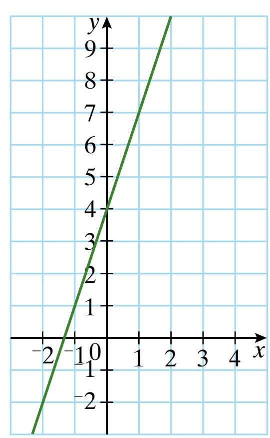

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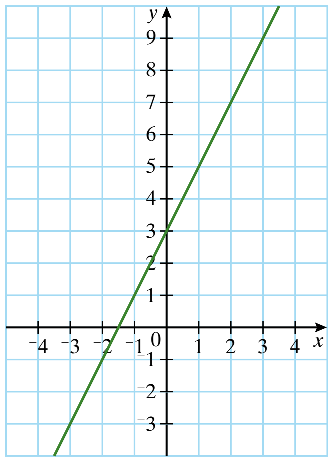
g



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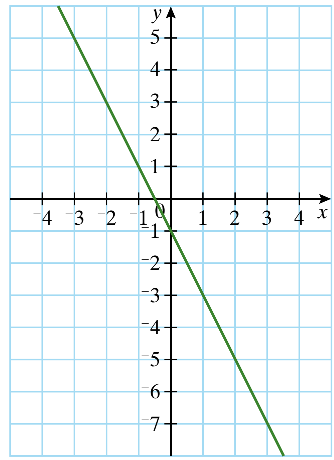
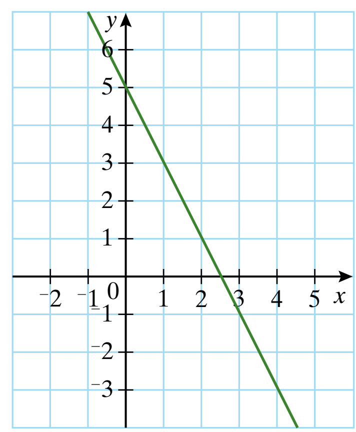


The graph of a linear relationship has gradient 4 and y-intercept –0.5. The rule for the relationship is

A $y = 0.5x + 4$ **B** $y = -0.5x + 4$
C $y = -4x - 0.5$ **D** $y = 4x - 0.5$
E $y = -4x + 0.5$

The rule for the relationship represented by this graph is



A $y = 2x + 5$
B $y = 5x + 2.5$
C $y = -2x + 5$
D $y = 5x - 2$
E $y = -0.5x + 5$

LINKS TO Example 8

Find the equation for each of these lines.

a gradient 2 and passing through the point $(-1, 5)$
b gradient -5 and passing through the point $(6, 8)$
c gradient -3 and passing through the point $(-2, 4)$
d gradient 4 and passing through the point $(2, -5)$
e gradient $-\frac{5}{4}$ and passing through the point $(3, 8)$
f gradient $\frac{2}{3}$ and passing through the point $(-4, -5)$
g gradient $\frac{3}{4}$ and passing through the point $(-3, -5)$
h gradient $-\frac{1}{2}$ and passing through the point $(5, 0)$

Which of these equations will have the steepest graph?

A $y = 2x$ **B** $y = 2x + 7$ **C** $y = 3x + 2$
D $y = 4x + 1$ **E** $y = x + 11$

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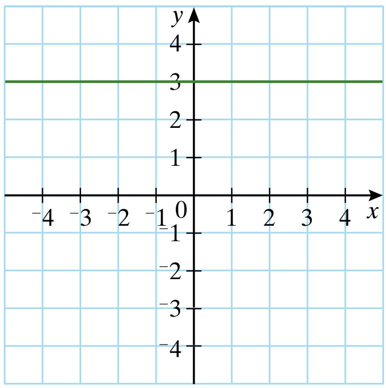
- **a** Explain how we can tell from the rule of a linear relationship whether it passes through the origin.
  b Explain how we can tell from the rule of a linear relationship whether it will slope upwards to the right or downwards to the right.

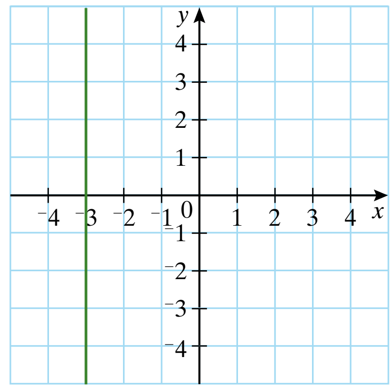
LINKS TO Example 9

- Write the rule for each of the graphs shown here.

a



b



LINKS TO Example 10

- Draw each of the following graphs on the same set of axes.
  a $y = -5$
  b $x = 7$
  c $y = 6$
  d $x = -2$

LINKS TO Example 11

- For which of these linear rules will the graphs be parallel?
  $y = -3x + 2,\ y = 3,\ y = -3x,\ y = -2x + 2,\ y = -3x + 7,\ y = -3x + 2,\ y = 3x$

LINKS TO Example 12

- Write each of these equations in the general form for a straight line.
  a $y = \frac{7}{3}x - 8$ **b** $y = \frac{2}{5}x - 7$ **c** $y = -\frac{2}{11}x - 1$
  d $y = \frac{3}{2}x + \frac{5}{2}$ **e** $y = -\frac{4}{5}x - \frac{9}{5}$ **f** $y = -\frac{3}{4}x - \frac{5}{2}$

LINKS TO Example 13

- Write each of these equations in gradient-intercept form and hence state the gradient and y-intercept.
  a $2x + 4y - 5 = 0$ **b** $5x - 2y - 10 = 0$ **c** $3y - 4x - 15 = 0$
  d $5x - 8y - 20 = 0$ **e** $6x - 4y - 25 = 0$ **f** $7x + 4y - 10 = 0$

exercise 7.3 challenge

- Find the equation of a line that is parallel to the line $6x - 5y = 3$ and passes through the point (5, 4).

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7.4 Sketching linear graphs

Sketching linear graphs given the intercepts

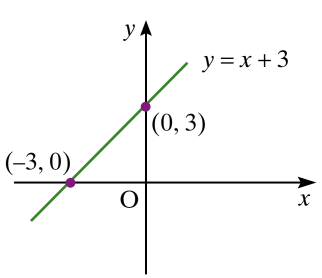
Only two points are needed to specify the position of a straight line. We can sketch the graph of any linear equation if we find two points on the line. The easiest two points to find are the x-intercept and the y-intercept.

- The x-intercept is the point where the graph crosses the x-axis. At this point, $y = 0$, so substitute $y = 0$ in the equation for the line.
- The y-intercept is the point where the graph crosses the y-axis. At this point, $x = 0$, so substitute $x = 0$ in the equation for the line. If the equation is in the form $y = mx + b$ we know the y-intercept.

Example 14

Sketch these linear graphs by first finding the intercepts.

a $y = x + 3$ **b** $y = -2x + 5$

Working	Reasoning
a x-intercept: When $y = 0$, $0 = x + 3$ $x = -3$ So the x-intercept is the point $(-3, 0)$.	Find the x-intercept by substituting $y = 0$. Solve the equation for x.
y-intercept: When $x = 0$, $y = 3$ So the y-intercept is the point $(0, 3)$.	The equation is in gradient-intercept form, so y-intercept is 3.
	Draw and label the axes. Mark the two intercepts and draw a line through them. Label the intercepts.

continued

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Example 14 continued

Working

b x-intercept:

When $y = 0$,

$0 = -2x + 5$

$2x = 5$

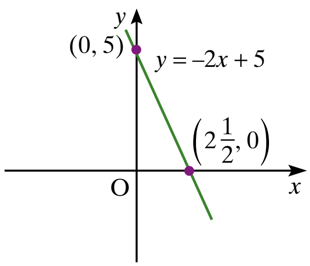
$x = 2\frac{1}{2}$

So the x-intercept is the point $\left(2\frac{1}{2}, 0\right)$,

y-intercept:

When $x = 0$, $y = 5$

So the y-intercept is the point $(0, 5)$,



Reasoning

Find the x-intercept by substituting $y = 0$.

Solve the equation for x.

The equation is in gradient-intercept form, so y-intercept is 5.

Draw and label the axes. Mark the two intercepts and draw a line through them. Label the intercepts.

Example 15

Sketch these linear graphs by first finding the intercepts.

a $x + 3y = 4$

b $3x + 4y = 7$

Working

a x-intercept:

When $y = 0$,

$x + 3 \times 0 = 4$

$x + 0 = 4$

$x = 4$

So the x-intercept is the point $(4, 0)$.

y-intercept:

When $x = 0$,

$0 + 3y = 4$

$y = \frac{4}{3}$

$y = 1\frac{1}{3}$

Reasoning

Find the x-intercept by substituting $y = 0$.

Solve the equation for x.

Find the y-intercept by substituting $x = 0$.

Solve the equation for y.

continued

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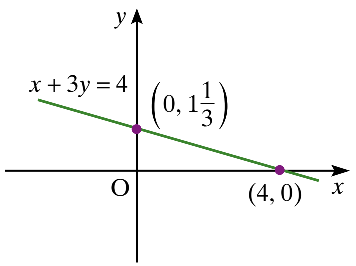
Example 15 continued

Working	**Reasoning**
So the y-intercept is the point $\left(0, 1\frac{1}{3}\right)$.	Rule and label the axes. Mark the two intercepts and rule a line through them. Label the intercepts.



b x-intercept: — Find the x-intercept by substituting $y = 0$.

When $y = 0$,

$3x + 0 = 7$ — Solve the equation for x.

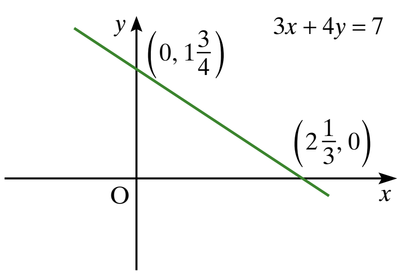
$$x = \frac{7}{3}$$
$$= 2\frac{1}{3}$$

y-intercept: — Find the y-intercept by substituting $x = 0$.

When $x = 0$,

$0 + 4y = 7$ — Solve the equation for y.

$$y = \frac{7}{4}$$
$$= 1\frac{3}{4}$$



Draw and label the axes. Mark the two intercepts and draw a line through them. Label the intercepts.

For equations of the form $y = mx$, the graph passes through the origin. Using the intercepts gives only one point, so we need to choose a second point when sketching the graph. We can choose any value of x to substitute, such as $x = 1$ or $x = 2$.

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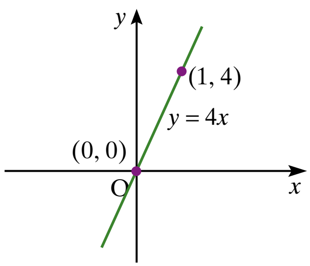
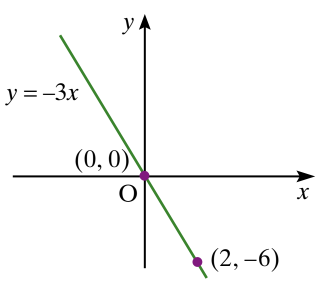
Sketching linear graphs passing through the origin

Example 16

Sketch the graphs of

a $y = 4x$ **b** $y = -3x$

Working	Reasoning
a The graph passes through (0, 0). When $x = 1$, $y = 4 \times 1 = 4$. The graph passes through (1, 4).	Any value may be chosen for x to find a second point. In this example $x = 1$ was chosen.
	Rule and label the axes. Rule a line passing through the origin and through the point (1, 4). Label the points (0, 0) and (1, 4).
b The graph passes through (0, 0). When $x = 2$, $y = -3 \times 2 = -6$. The graph passes through (2, –6).	Any value may be chosen for x to find a second point. In this example $x = 2$ was chosen.
	Rule and label the axes. Rule a line passing through the origin and through the point (2, –6). Label the points (0, 0) and (2, –6).

exercise 7.4

LINKS TO Example 15

1 For each of the following straight lines

i determine the x- and y-intercepts. **ii** sketch the graph.

a $y = x + 6$ **b** $y = -x + 3$ **c** $y = x + 5$

d $y = 2x + 8$ **e** $y = 2x - 6$ **f** $y = -2x + 4$

g $y = -2x - 5$ **h** $y = -4x - 4$ **i** $y = 3x + 12$

j $y = 3x - 6$ **k** $y = -3x + 6$ **l** $y = -5x + 3$

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For each of the following straight lines

i determine the x- and y-intercepts **ii** sketch the graph.

a $2x + y = 6$ **b** $2x - y = 6$ **c** $x + 2y = 6$

d $x - 2y = 6$ **e** $3x + y = 9$ **f** $x - 3y = 12$

g $4x + y = 12$ **h** $x - 4y = 6$ **i** $3x - 4y = 6$

j $4x + 5y = 10$ **k** $5x + 6y = 30$ **l** $2x + 5y = 8$

LINKS TO Example 16

Sketch the graphs of the following equations, labelling the coordinates of two points.

a $y = 3x$ **b** $y = -x$ **c** $y = 2x$

d $y = -2x$ **e** $y = 4x$ **f** $y = -4x$

g $y = 0.5x$ **h** $y = -0.5x$ **i** $y = \frac{x}{3}$

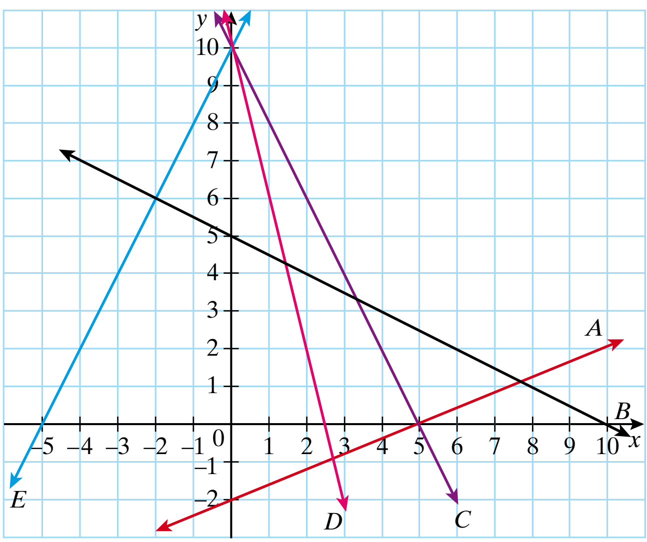
j $y = -\frac{x}{3}$ **k** $y = \frac{x}{4}$ **l** $y = -3.5x$

The linear rule that gives a graph with intercepts of (3, 0) and (0, –4) is

A $3x - 4y = 0$ **B** $3x - 4y = 12$ **C** $-4x + 3y = 0$

D $4x + 3y = 12$ **E** $4x - 3y = 12$

Which of the graphs shown here is a sketch of $y = -2x + 10$?



Sketch the graph of each of the following rules, labelling the coordinates of two points.

a $y = x$ **b** $y = 3x$ **c** $y = -4x$

d $y = \frac{2x}{3}$ **e** $2x + y = 0$ **f** $3x + 5y = 0$

The linear rule that gives a graph passing through (0, 0) and (1, –6) is

A $y = x - 6$ **B** $y = -6x + 1$ **C** $y = -6x$

D $y = -\frac{1}{6}x$ **E** $y = -x + 6.$

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The graph that passes through the points (0, –7) and (1, –7) has the rule

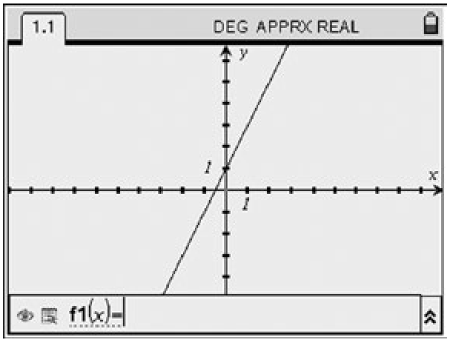
A $y = x - 7$ **B** $y = -7x$ **C** $y = -7x + 1$ **D** $x = -7$ **E** $y = -7$

exercise 7.4 challenge

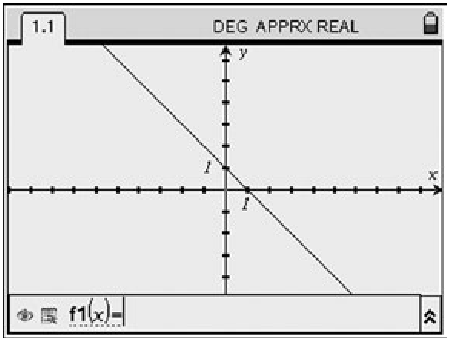
The window for each of the CAS calculators below has the same scale. Match each of the graphs (a to d) with its rule (A to D).

A $y = x + 1$ **B** $y = 2x + 1$ **C** $y = -x + 1$ **D** $y = -2x + 1$

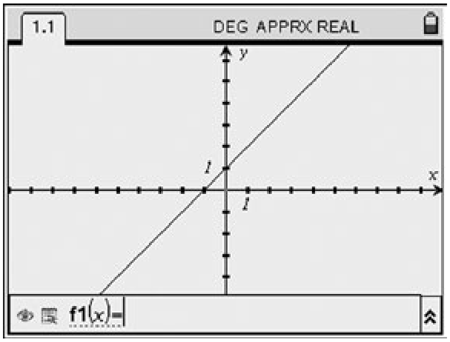
a



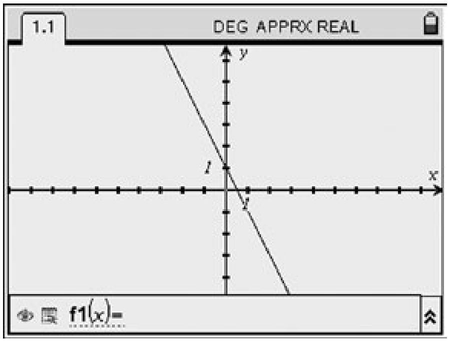
b



c



d



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7.5 Applications of linear relationships

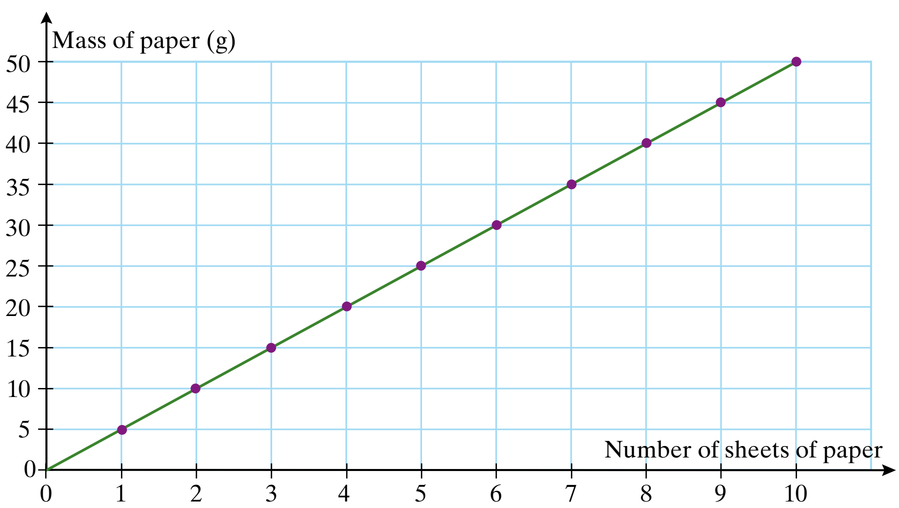
Gradient as a rate

Although we often think of gradient in terms of the steepness of a slope, gradient is simply a **rate**. It tells how one quantity is changing compared with another quantity.

A standard sheet of A4 paper has a mass of 5 g, so two sheets will have a mass of 10 g, three sheets will have a mass of 15 g and so on. We can express this as a rate of 5 g/sheet. The table and the graph below show how the mass changes as the number of sheets of paper increases. The number of sheets is shown on the horizontal axis of the graph and the mass is shown on the vertical axis. The points on the graph all lie along a straight line and we can calculate the gradient of this line by dividing the rise by the run.

$$\begin{aligned}\text{gradient} &= \frac{\text{rise}}{\text{run}}\\ &= \frac{50\text{ g}}{10\text{ sheets}}\\ &= 5\text{ g/sheet}\end{aligned}$$

It can be seen that the gradient represents the rate of 5 g/sheet. Each sheet of paper has a mass of 5 g, so each time we add another sheet of paper, the mass increases by 5 g. The rate is constant, that is the gradient is constant, so the graph is a straight line.



Gradient and rate

The gradient of a linear graph represents a rate.

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Independent and dependent variables

7.5

In the example of the sheets of paper, the mass of the paper depends on how many sheets of paper we have. We say that the number of sheets of paper is the **independent variable** and the mass is the **dependent variable**. It is the custom when drawing graphs to put the independent variable on the horizontal axis and the dependent variable on the vertical axis.

Example 17

Sam had \$50. When he got an after school job he saved \$30 each week. Let n be the number of weeks and A be the number of dollars that Sam has after n weeks.

a State the independent variable.
b State the dependent variable.
c How much did Sam have after 2 weeks?
d Complete the table of values.

n	0	1	2	3	4	5
A	50					

e Plot the graph, choosing a suitable scale for each axis.
f How much did Sam have after n weeks?
g Write the rule for A in terms of n in the form $y = mx + b$
h State the gradient. What does the gradient represent in terms of the story?
i State the A-intercept. What does the A-intercept represent in terms of the story?

Working | **Reasoning**

a The number of weeks, n

The number of dollars that Sam has depends on the number of weeks he has been saving.

b The number of dollars, A

The number of dollars that Sam has depends on the number of weeks he has been saving.

c $\$50 + \$30 \times 2 = \$110$

After 2 weeks, Sam has two lots of \$30 plus \$50 he started with.

d

n	0	1	2	3	4	5
A	50	80	110	140	170	200

continued

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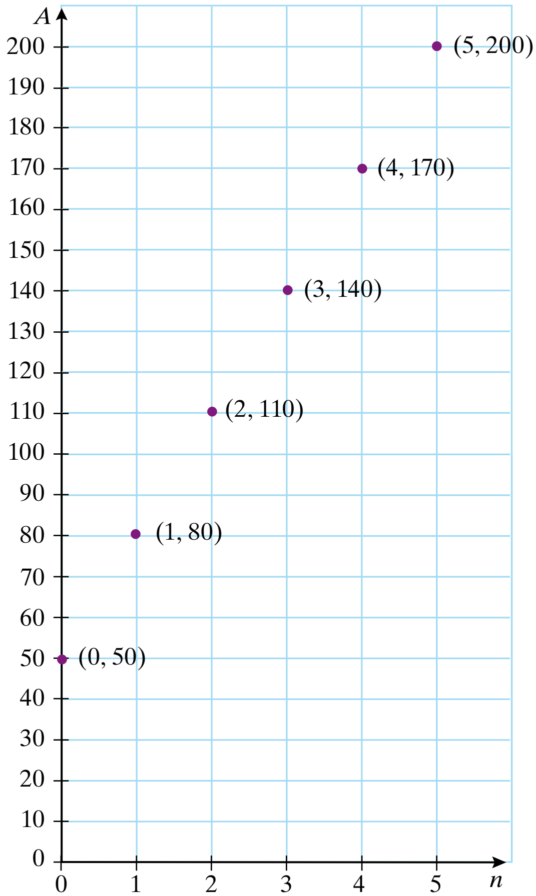
Example 17 continued

Working | **Reasoning**

e



f $30n + 50$

After n weeks, Sam has n lots of \$30 plus \$50 he started with.

g $A = 30n + 50$

Sam has \$$A$ after n weeks.
Note that we do not write dollar signs in the rule.

h The gradient is 30. It represents the number of dollars that Sam saves per week.

Comparing $A = 30n + 50$ with the rule for a linear relationship, $y = mx + b$, we see that the gradient is 30. Gradient is a rate.

i The A-intercept is 50. This represents the number of dollars that Sam started with.

Comparing $A = 30n + 50$ with the rule for a linear relationship, $y = mx + b$, the A-intercept is 50.

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Example 18

Mandy is 450 km from home. She is driving home at an average speed of 90 km/h. After t hours, Mandy is d km from home.

a What is the independent variable?
b What is the dependent variable?
c Make a table of values showing the distance d km from home after time t hours. Use values of t from $t = 0$ to $t = 5$.
d Explain how the table shows that this is a linear relationship.
e Plot the graph of distance versus time.
f Calculate the gradient. How does this relate to the journey?
g Find the d-intercept. How does this relate to the journey?
h State the rule connecting d and t.

Working

a The number of hours, t, that Mandy has been travelling.

Reasoning: Mandy's distance from home depends on how long she has been travelling.

b Mandy's distance from home, d km.

Reasoning: Mandy's distance from home depends on how long she has been travelling.

c

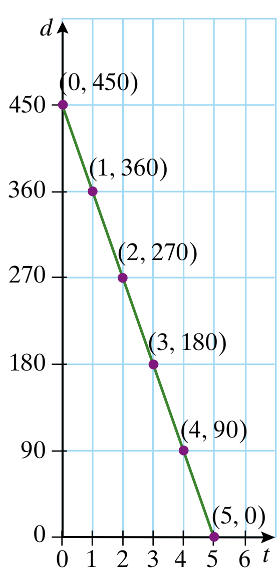
t	0	1	2	3	4	5
d	450	360	270	180	90	0

Reasoning: Mandy is 450 km from home at the start. This distance decreases by 90 km every hour.

d As the values of t increase in equal steps of 1, the values of d decrease in equal steps of 90.

Reasoning: A relationship is linear if the values of the dependent variable increase or decrease in equal steps when the values of the independent variable increase in equal steps.

e



Reasoning: The horizontal axis needs to go to $t = 6$ and the vertical axis needs to go to $d = 450$. The distance decreases by 90 km each hour so it is convenient to label the vertical axis in steps of 90.

continued

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Example 18 continued

Working	Reasoning
f Gradient $= \dfrac{y_2 - y_1}{x_2 - x_1}$ $= \dfrac{d_2 - d_1}{t_2 - t_1}$ $= \dfrac{360 - 450}{1 - 0}$ $= -90$	The graph slopes downwards to the right so the gradient is negative.
Mandy is travelling at 90 km/h so her distance from home is decreasing by 90 km every hour.	We can also say that Mandy's distance from home is increasing at the rate of −90 km/h since a fall of 90 km each hour is a rise of −90 km each hour.
g The d-intercept is 450. This represents Mandy's distance from home in kilometres at the start of her journey home.	The point (0, 450) is on the graph.
h The rule is $d = -90t + 450$	Mandy is 450 km from home at the start. This distance decreases by 90 km every hour. We could also write the rule as $d = 450 - 90t$.

Example 19

When a technician from Rob's Repairs makes a house call, he charges a service fee of $60, and then $40 per hour.

a Write the rule for the cost $$C$ in terms of the number of hours, t.

b State the value of C when

  i $t = 0$

  ii $t = 1$

c Using graph paper mark the scale on the t-axis from 0 to 6, and the scale on the C-axis from 0 to 300. Use the values from part **b** to draw a graph of cost versus time. Extend your graph to $t = 6$.

d Use your graph to find the cost when $t = 3$.

e Check your answer to part **d** by substituting $t = 3$ in the rule.

f Use your graph to find the number of hours worked if the total cost is $260.

g Check your answer to part **d** by substituting in the rule.

h What is the gradient of the graph?

i What does the gradient represent?

continued

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Example 19 continued

Working

a $C = 40t + 60$

Reasoning

The total cost is made up of a charge of \$40 for each hour plus a fixed charge of \$60.

b i When $t = 0$,

$C = 40 \times 0 + 60$

$= 60$

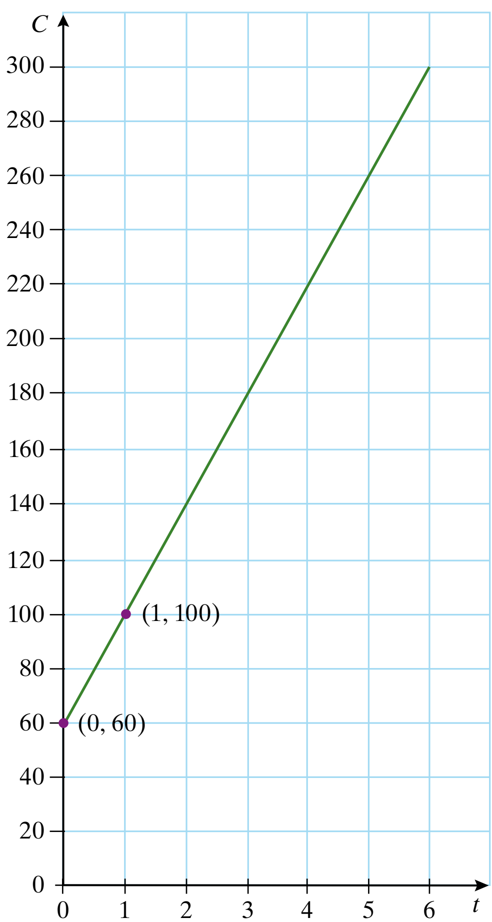
ii When $t = 1$,

$C = 40 \times 1 + 60$

$= 100$

Substitute the values for t into the rule $C = 40t + 60$

c



The rule shows that the relationship is linear. Only two points are needed to draw a straight line.

continued

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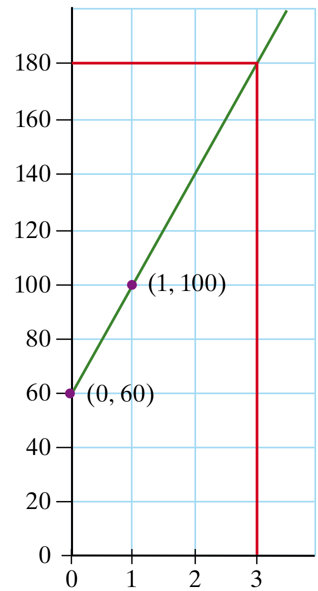
Example 19 continued

Working	Reasoning
d When $t = 3$, $C = 180$	Find the point on the graph where $t = 3$ then move across to the C-axis to find the corresponding value of C. 
e $C = 40t + 60$ When $t = 3$, $C = 40 \times 3 + 60$ $= 180$	Substitute $t = 3$ into the rule $C = 40t + 60$
f When $C = 260$, $t = 5$	Find the point on the graph where $C = 260$ then move down to the t-axis to find the corresponding value of t.
g $C = 40t + 60$ When $C = 260$, $260 = 40t + 60$ $200 = 40t$ $t = 5$	Substitute $C = 260$ into the rule $C = 40t + 60$ and solve the equation for t.

continued

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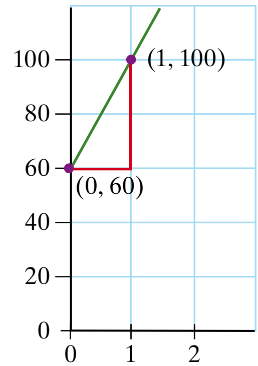
Example 19 continued

Working	Reasoning
h Gradient $= \dfrac{y_2 - y_1}{x_2 - x_1}$ $= \dfrac{100 - 60}{1 - 0}$ $= 40$	
i The gradient is the rate that is charged per hour.	Rob's Repairs charges $40 per hour.

exercise 7.5

LINKS TO Example 17

1 Write the rule for each of the following situations.

a The total cost charged by a plumber consists of a call-out fee of $50 and an hourly rate of $40 per hour. Write the rule for the total cost, T, after h hours.

b The total cost of hiring a limousine consists of a flag fall of $10 and a rate of $5 per kilometre. Write the rule for the total cost, C, where k kilometres are travelled.

c The amount raised by a student in a walk for charity consisted of donations of $40 and sponsorship at the rate of $6 per kilometre. Write the rule for the amount, A, raised after d kilometres have been walked.

d A cyclist begins 30 km from Camperdown and travels towards the town at a steady speed of 5 km/h for t hours. Write the rule for the distance, d, from Camperdown at any time t hours.

e The club fees for Saturday tennis are $55 per season plus $5 per match played. Write the rule for the total fee, F, for a season in which n matches are played.

LINKS TO Example 18

2 Harry is 700 km from Darwin. He is driving towards Darwin at an average speed of 100 km/h.

a Make a table of values showing the distance d km from home after time t hours. Use values of t from $t = 0$ to $t = 7$.

b Explain how the table shows that this is a linear relationship.

c Calculate the gradient. How does this relate to the journey?

d Find the y-intercept. How does this relate to the journey?

e What is the rule?

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LINKS TO Example 19

An electrician charges a fee of \$60, then \$55 per hour after this.

- **a** Make a table of values showing the total cost, d dollars, after time t hours. Use values of t from $t = 0$ to $t = 4$.
- **b** Explain how the table shows that this is a linear relationship.
- **c** Calculate the gradient. How does this relate to the way the electrician charges?
- **d** Find the y-intercept. How does this relate to the way the electrician charges?
- **e** What is the rule?

When a computer technician from Compu Calls repairs a home computer at the home, he charges a service fee of \$80, and then \$60 per hour. Sketch a graph to show the total charge, \$$C$, for calls of time t hours.

The Dodgy car company hires cars for a fee of \$200 plus \$50 per day. Sketch a graph to show the total charge \$$C$ for n days.

In a certain city, taxi charges between 6 am and 6 pm are a Flag fall (or fixed fee) of \$3.60 and a fee of \$1.50 per kilometre. Sketch a graph to show the total charge, \$$C$, for trips of distance d kilometres.

The rule $V = 8000 - 150t$ describes the volume of water, V litres, in a tank t hours after a pump started pumping water from the tank.

- **a** How much water did the tank hold at the start?
- **b** What is the steady rate at which the water is being pumped out?

The rule $S = 60 + 20w$ describes the amount, \$$S$, of Julia's savings w weeks after the start of the year.

- **a** How much money did Julia have at the start of the year?
- **b** What was the steady rate at which Julia saved money after that?

The rule $L = 5.5n + 1.5$ is used to calculate the length of roadway needed, in metres, for a taxi rank that will hold n taxis.

- **a** What length of roadway is needed per taxi?
- **b** How much extra roadway is needed?

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- At the Raging Rides fun park, entry is \$12 per head and each ride costs \$3.
  a What is the rule that gives the total cost (entry and rides), T, for a person who has n rides?
  b Sketch a graph to show this relationship.
  c Use your graph to find the total cost if a person has
    i 6 rides.
    ii 9 rides.
  d Check your answers to part **c** by substitution in the rule.
  e Use your graph to find the number of rides the person has had if the total cost is \$45.
  f Check your answer to part **e** by substitution in the rule.

- Jo is 80 km from Perth. He is driving away from Perth along a straight road at an average speed of 90 km/h.
  a Make a table of values showing the distance d km from Perth after time t hours. Use values of t from $t = 0$ to $t = 5$.
  b Explain how the table shows that this is a linear relationship.
  c Calculate the gradient and the y-intercept.
  d What is the rule?
  e When he has been driving for $3\frac{1}{2}$ hours, how far from Perth is he?
  f How long will it take him to get to a town 800 km from Perth along the same road (not counting rest breaks)?

exercise 7.5 challenge

- Janet is a travelling salesperson who uses her own vehicle. She is paid travel expenses at the rate of \$155 per week plus 21 cents per kilometre that she travels for work.
  a What is the independent variable?
  b What is the dependent variable?
  c If \$$T$ is the amount paid in a week for travel expenses and k kilometres is the distance travelled for work, write a rule for T in terms of k.
  d How much would Janet receive for travel expenses in a week where she travelled 580 km for work?

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7.6 Non-linear graphs

The graphs of many relationships are not straight lines. In this section we look at two different types of non-linear graphs.

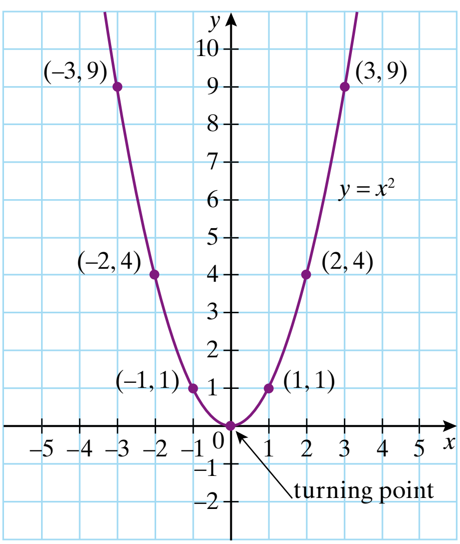
Parabola

Parabola

In chapter 3, the term 'quadratic' was introduced to describe expressions in which the highest power of x was x^2. Consider the following table of values for the rule $y = x^2$.

x	−3	−2	−1	0	1	2	3
y	9	4	1	0	1	4	9

As the x values increase from −3 to 0, the y-values decrease to a minimum value (0) and then increase. The curved graph is called a **parabola**. The **turning point** or **vertex** of the graph is at the point (0, 0). Notice that the parabola is **symmetrical**. The y-axis is the **axis of symmetry**.



The parabola

The graph of $y = x^2$ is called a parabola.

The graph of $y = x^2$ is symmetrical about the y-axis.

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Example 20

For each of these quadratic rules,

i complete a table of values with x values from -3 to 3.
ii plot the points and join with a smooth curve.
iii compare the parabola with the parabola $y = x^2$.

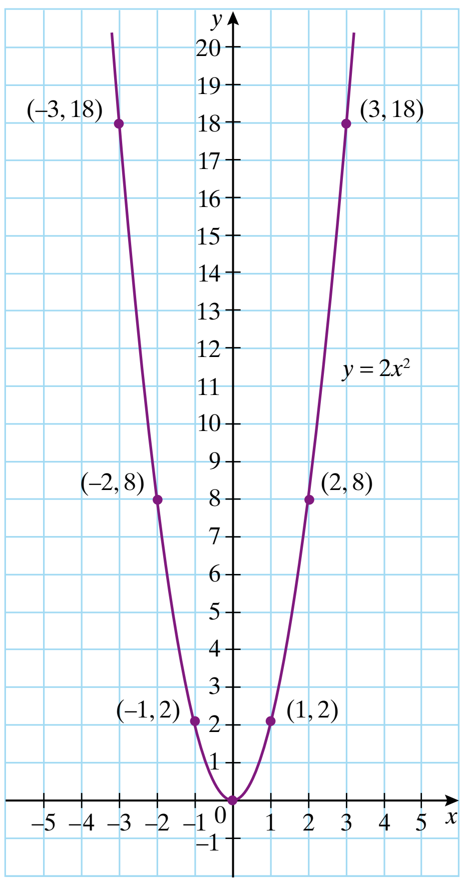
a $y = 2x^2$ **b** $y = -x^2$

Working

a i $y = 2x^2$

x	–3	–2	–1	0	1	2	3
y	18	8	2	0	2	8	18

ii



Reasoning

$2 \times (-3)^2 = 18$
$2 \times (-2)^2 = 8$
$2 \times (-1)^2 = 2$
$2 \times 0^2 = 0$
$2 \times 1^2 = 2$
$2 \times 2^2 = 8$
$2 \times 3^2 = 18$

continued

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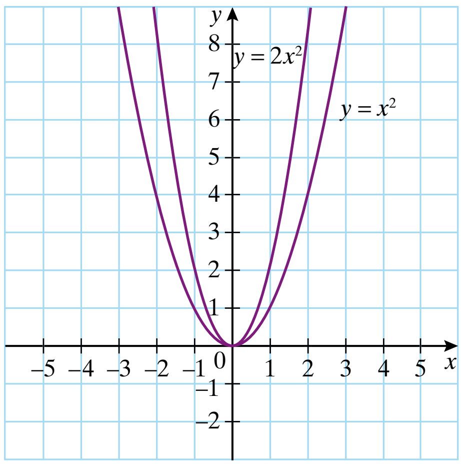
Example 20 continued

Working

Reasoning

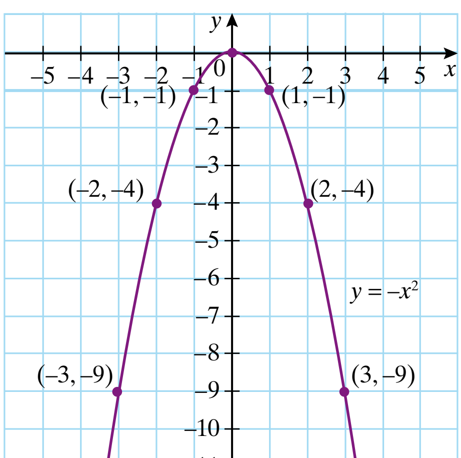
iii The graph of $y = 2x^2$ has the same basic shape as the graph of $y = x^2$ but it is narrower, that is, it rises more steeply.



b i $y = -x^2$

x	–3	–2	–1	0	1	2	3
y	–9	–4	–1	0	–1	–4	–9

$-(-3)^2 = -9$
$-(-2)^2 = -4$
$-(-1)^2 = -1$
$-0^2 = 0$
$-1^2 = -1$
$-2^2 = -4$
$-3^2 = -9$



ii The graph of $y = -x^2$ is the same shape as the graph of $y = x^2$ but it is inverted. It is a reflection of the graph of $y = x^2$ in the x-axis.

The path of a soccer ball after it is kicked and a jet of water both have parabolic shapes.

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Parabola vertical translation

Translating the parabola vertically

Adding a constant term to x^2 results in a vertical translation of the parabola. Consider the graph of the quadratic relationship $y = x^2 + 2$.

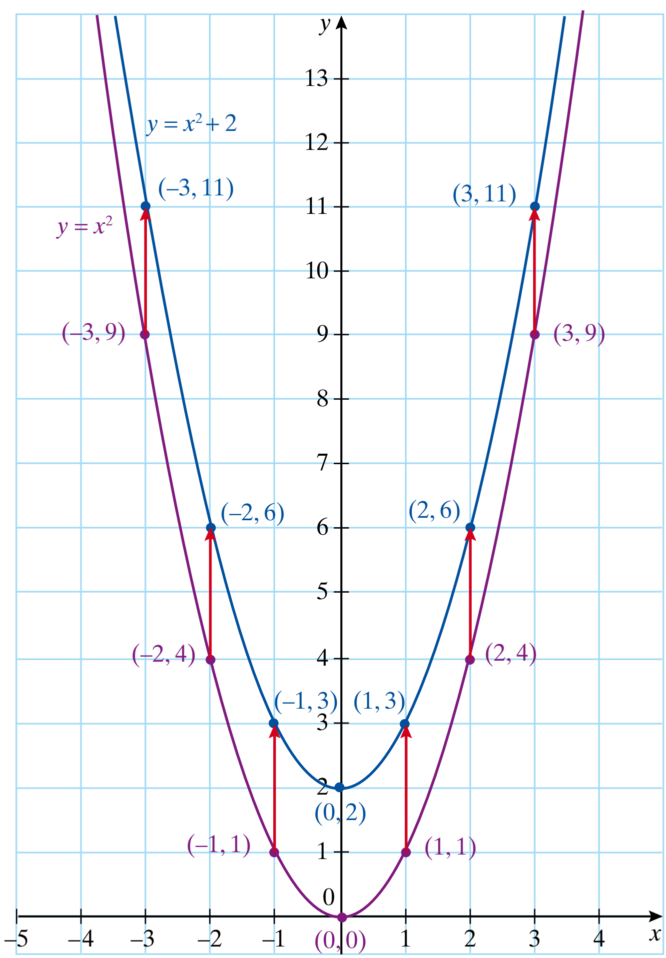
$y = x^2$

x	3	2	1	0	1	2	3
y	9	4	1	0	1	4	9

$y = x^2 + 2$

x	3	2	1	0	1	2	3
y	11	6	3	2	3	6	11

Comparing the graph of $y = x^2 + 2$ with the graph of $y = x^2$, we see that each point on the parabola $y = x^2$ is translated vertically upwards by 2 units. The graphs are the same shape but the turning point of $y = x^2 + 2$ is at (0, 2). The line $x = 0$ is still the axis of symmetry.



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When we sketch the graphs of the parabolas in example **2**, we show the position of the turning point and the points where $x = -1$ and $x = 1$. A sketch graph shows the labelled axes but does not need to be exactly to scale.

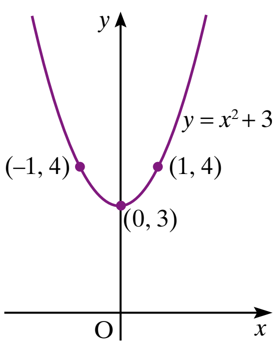
Example 21

Sketch the graphs of the following quadratic relationships, showing the position of the turning point and the value of y when $x = -1$ and $x = 1$.

a $y = x^2 + 3$ **b** $y = x^2 - 2$ **c** $y = -x^2 + 1$

Working

a

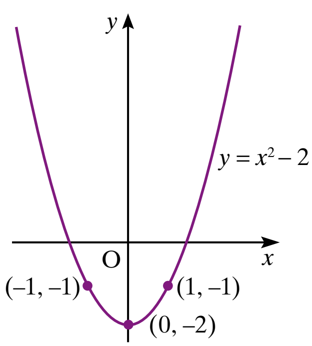


Reasoning

The graph of $y = x^2 + 3$ is the graph of $y = x^2$ translated up 3 units.
The turning point is at (0, 3).

x	–3	–2	–1	0	1	2	3
y	12	7	4	3	4	7	12

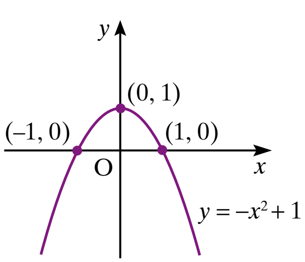
b



The graph of $y = x^2 - 2$ is the graph of $y = x^2$ translated down 2 units.
The turning point is at (0, –2).

x	–3	–2	–1	0	1	2	3
y	7	2	–1	–2	–1	2	7

c



The graph of $y = -x^2 + 1$ is the graph of $y = -x^2$ translated up 1 unit.
The turning point is at (0, 1).

x	–3	–2	–1	0	1	2	3
y	–8	–3	0	1	0	–3	–8

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Vertical translation

The graph of $y = x^2 + k$ has its turning point at $(0, k)$. If $k > 0$, the parabola $y = x^2$ is shifted up. If $k < 0$, the parabola $y = x^2$ is shifted down.

The graph of $y = -x^2 + k$ is 'upside down', with its turning point at $(0, k)$.

Translating the parabola horizontally

We now consider the equation $y = (x - 3)^2$ and compare the graph with the graph of $y = x^2$.

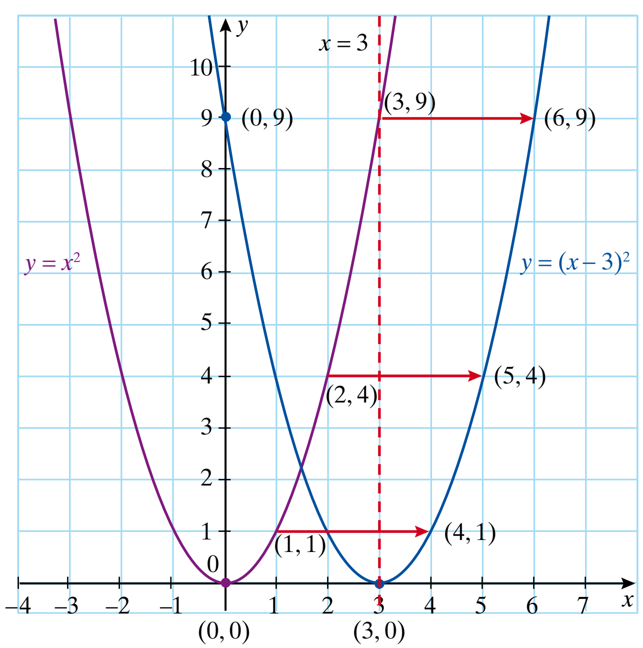
$y = x^2$

x	–3	–2	–1	0	1	2	3
y	9	4	1	0	1	4	9

$y = (x - 3)^2$

x	0	1	2	3	4	5	6
y	9	4	1	0	1	4	9

Notice how each point on the parabola $y = x^2$ has been translated 3 units to the right. The turning point is now at (3, 0) and the axis of symmetry is the line $x = 3$. The parabola crosses the y-axis at $y = 9$, that is, the y-intercept is the point (0, 9).



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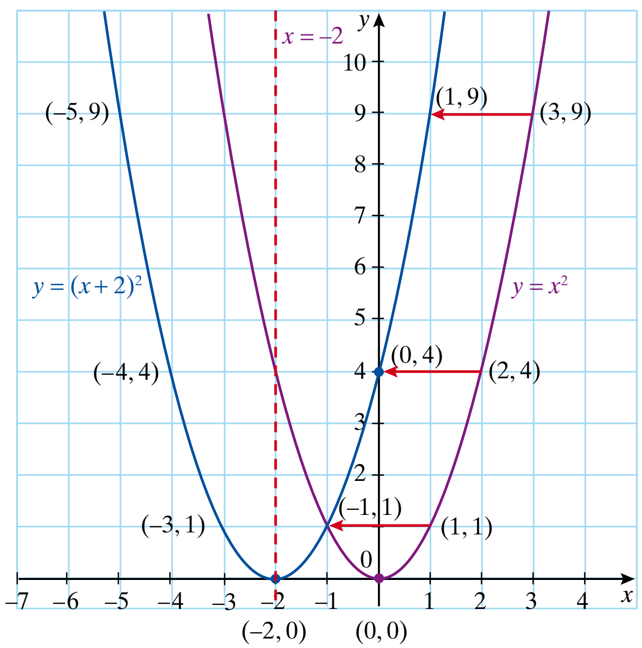
We now look at the graph of $y = (x + 2)^2$. Each point on the parabola $y = x^2$ has been translated 2 units to the left. The turning point is now at (–2, 0) and the axis of symmetry is the line $x = -2$. The y-intercept is the point (0, 4).

$y = x^2$

x	–3	–2	–1	0	1	2	3
y	9	4	1	0	1	4	9

$y = (x + 2)^2$

x	–5	–4	–3	–2	–1	0	1
y	9	4	1	0	1	4	9



Parabola horizontal translation

Horizontal translation

In general, the graph of $y = (x - h)^2$ is the same shape as the graph of $y = x^2$ but the parabola is translated h units to the right. If h is negative, the parabola is translated to the left. The turning point is at $(h, 0)$ and the axis of symmetry is the line $x = h$.

We can find the y-intercept by substituting $x = 0$ in the equation.

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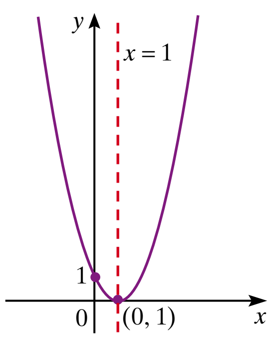
Example 22

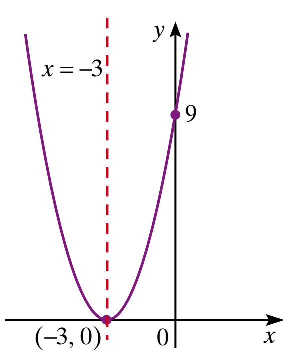
Sketch the graphs of the following quadratic relationships, showing the coordinates of the turning point, the y-intercept and the equation of the axis of symmetry.

a $y = (x - 1)^2$ **b** $y = (x + 3)^2$ **c** $y = -(x + 3)^2$

Working	**Reasoning**
a $y = (x - 1)^2$ y-intercept: When $x = 0$, $y = (0 - 1)^2$ $y = 1$	To find the y-intercept, put $x = 0$ in the equation.
The y-intercept is the point (0, 1). Turning point: (1, 0) Axis of symmetry: $x = 1$	The parabola $y = x^2$ is translated 1 unit to the right.



Working	**Reasoning**
b $y = (x + 3)^2$ y-intercept: When $x = 0$, $y = (0 + 3)^2$ $= 9$	To find the y-intercept, put $x = 0$ in the equation.
The y-intercept is the point (0, 9). Turning point: (–3, 0) Axis of symmetry: $x = -3$	The parabola $y = x^2$ is translated 3 units to the left. $y = (x - h)^2$ is the equation of a parabola that has been translated h units to the right. If the parabola is translated to the left, then $h = -3$ so $x - h$ becomes $x + 3$.



continued

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Example 22 continued

Working

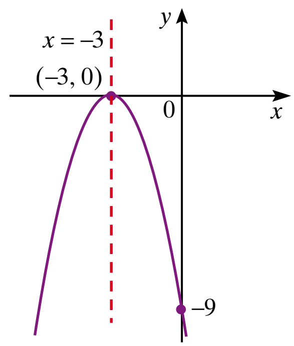
c $y = -(x + 3)^2$

y-intercept:
When $x = 0$,

$y = -(0 + 3)^2$
$= -9$

The y-intercept is the point $(0, -9)$.

Turning point:
$(-3, 0)$
Axis of symmetry:
$x = -3$



Reasoning

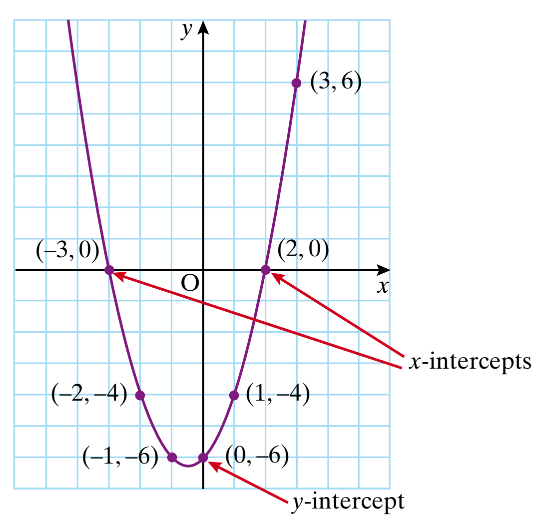
To find the y-intercept, put $x = 0$ in the equation.

The parabola $y = -x^2$ is translated 3 units to the left.

Graphs of quadratic equations in factorised form

In chapter 3 we saw how binomial products such as $(x + 3)(x - 2)$ could be expanded to give a quadratic trinomial, for example, $(x + 3)(x - 2) = x^2 + x - 6$. We now look at the graphs of quadratic relationships that are in simple factorised form. We start with the graph of $y = (x + 3)(x - 2)$. A table of values for $-3 \le x \le 3$ is shown below.

x	–3	–2	–1	0	1	2	3
y	0	–4	–6	–6	–4	0	6



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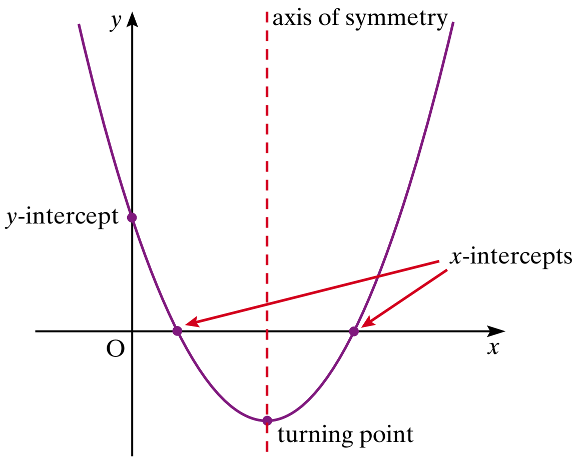
We can see that the parabola crosses the x-axis twice, at $x = -3$ and at $x = 2$. So the graph of $y = (x + 3)(x - 2)$ has x-intercepts at $(-3, 0)$ and at $(2, 0)$. The y-intercept is at $(0, -6)$.

What the factors tell us

The graph of $y = (x + a)(x + b)$ has x-intercepts at $(-a, 0)$ and at $(-b, 0)$.

The y-intercept is found by substituting $x = 0$ in the rule.

The axis of symmetry is a line parallel to the y-axis halfway between the two x-intercepts.



Example 23

For each of the following quadratic rules

i state the x-intercepts.
ii substitute $x = 0$ to find the y-intercept.
iii sketch the graph, labelling the intercepts and the axis of symmetry.

a $y = (x + 2)(x + 8)$
b $y = (x - 5)(x - 1)$
c $y = x(x - 8)$

continued

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Example 23

Working

Reasoning

a $y = (x + 2)(x + 8)$

i x-intercepts are at $(-2, 0)$ and at $(-8, 0)$

For the general rule $y = (x + a)(x + b)$, the x-intercepts at $(-a, 0)$ and at $(-b, 0)$.

ii When $x = 0$,

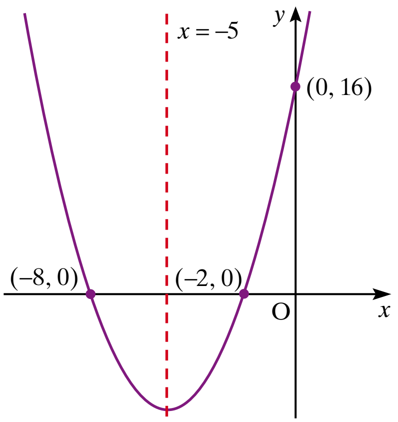
$y = (0 + 2)(0 + 8)$

$y = 16$

The y-intercept is at $(0, 16)$.

Substitute $x = 0$ in the rule to find the y-intercept.

iii



Mark and label the intercepts. Draw a smooth parabola through the intercepts. The axis of symmetry passes halfway between the two x-intercepts. Halfway between $x = -8$ and $x = -2$ is $x = -5$.

b $y = (x - 5)(x - 1)$

i x-intercepts are at $(5, 0)$ and at $(1, 0)$

For the general rule $y = (x + a)(x + b)$, the x-intercepts at $(-a, 0)$ and at $(-b, 0)$.

Note that if a is negative, then $-a$ is positive. Similarly, if b is negative, $-b$ is positive.

ii When $x = 0$,

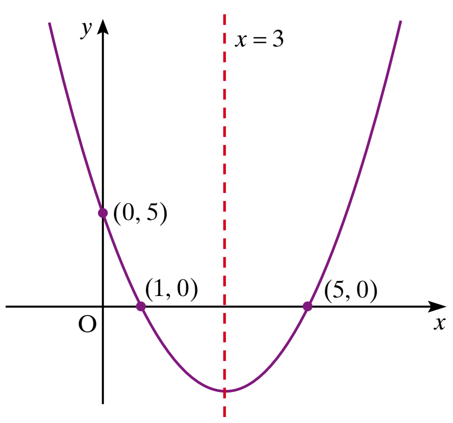
$y = (0 - 5)(0 - 1)$

$y = 5$

The y-intercept is at $(0, 5)$.

Substitute $x = 0$ in the rule to find the y-intercept.

iii



Mark and label the intercepts. Draw a smooth parabola through the intercepts. The axis of symmetry passes halfway between the two x-intercepts. Halfway between $x = 1$ and $x = 5$ is $x = 3$.

continued

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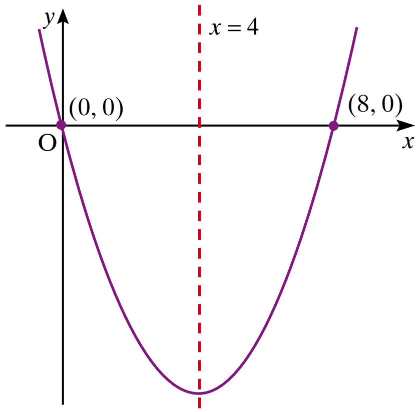
Example 23 continued

Working	Reasoning
c $y = x(x - 8)$	The factor x can be thought of as $x - 0$ so the equation is $y = (x - 0)(x - 8)$
i x-intercepts are at (0, 0) and at (8, 0)	
ii When $x = 0$, $y = 0(0 - 8)$, $y = 0$. The y-intercept is at (0, 0).	Substitute $x = 0$ in the rule to find the y-intercept.
iii 	Mark and label the intercepts. Draw a smooth parabola through the intercepts. The axis of symmetry passes halfway between the two x-intercepts. Halfway between $x = 0$ and $x = 8$ is $x = 4$.

Circle

Circle

We define a circle as the set of all points in a plane that are the same distance from a fixed point. This fixed point is the centre of the circle, and the distance of the points from the centre is the radius of the circle.

The circle on the following page has centre at (0, 0) and radius 3 units. P is any point on the circle, with coordinates (x, y). The radius OP is the hypotenuse of the right-angled triangle. Using Pythagoras' theorem we can see that $x^2 + y^2 = 3^2$. We can write this as $x^2 + y^2 = 9$. This is the equation for the circle.

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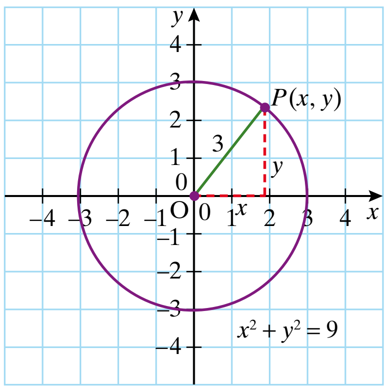
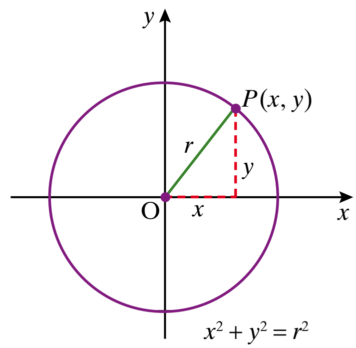
General equation for a circle

The general equation for a circle with centre (0,0) and radius r is $x^2 + y^2 = r^2$.





Example 24

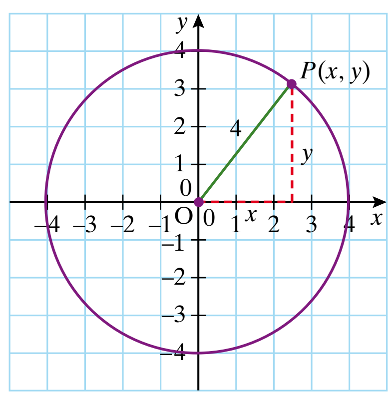
What is the equation of a circle with centre (0, 0) and radius 4 units?

Working

$x^2 + y^2 = 4^2$

$x^2 + y^2 = 16$

Reasoning



The radius of the circle is the hypotenuse of the right-angled triangle.

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Example 25

For each of these circles
- **i** find the radius.
- **ii** sketch the graph, labelling the coordinates of the four intercepts.

a $x^2 + y^2 = 144$ **b** $x^2 + y^2 = 6$

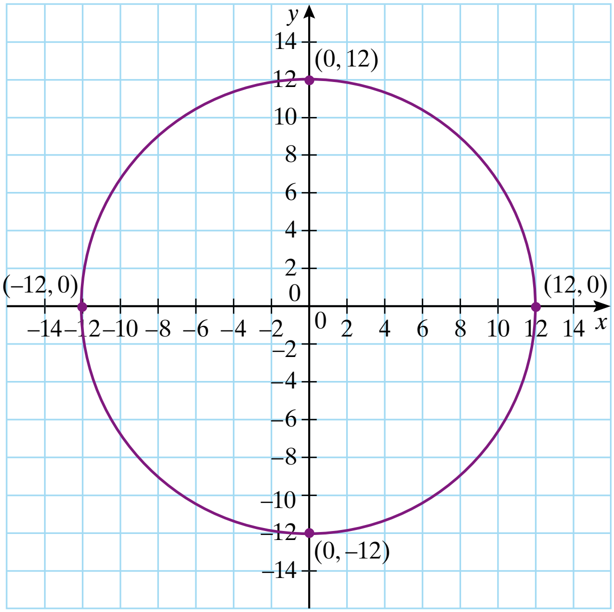
Working

a i $x^2 + y^2 = 144$

$x^2 + y^2 = 12^2$

radius = 12

ii



Reasoning

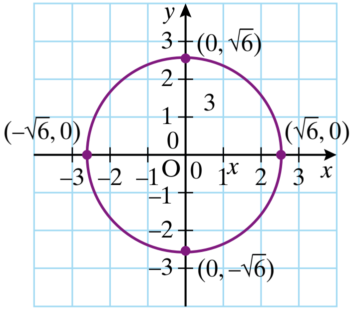
Put the equation in the form $x^2 + y^2 = r^2$ where r is the radius.
The square root of 144 is 12. Mark the positions of the intercepts and draw a circle with centre (0, 0).

b i $x^2 + y^2 = 6$

$x^2 + y^2 = \left(\sqrt{6}\right)^2$

radius = $\sqrt{6}$

ii



Put the equation in the form $x^2 + y^2 = r^2$ where r is the radius.

Give the coordinates of the intercepts as exact values.
The square root of 6 is between 2 and 3.
$\sqrt{6} \approx 2.4$
Mark approximate position of intercepts and draw a circle with centre (0, 0).

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exercise 7.6

1. For each of these quadratic rules
   i complete a table of values with x values from –3 to 3.
   ii plot the points and join with a smooth curve.

   a $y = 2x^2$ **b** $y = 0.5x^2$ **c** $y = -0.5x^2$ **d** $y = 4x^2$

LINKS TO Example 21

2. Sketch graphs of the following parabolas, labelling
   i the turning point.
   ii the axis of symmetry.
   iii the points where $x = 1$ and $x = -1$

   a $y = x^2 + 1$ **b** $y = x^2 - 3$ **c** $y = x^2 + 6$ **d** $y = x^2 - 4$
   e $y = -x^2 + 1$ **f** $y = -x^2 - 3$ **g** $y = -x^2 + 6$ **h** $y = -x^2 - 4$

LINKS TO Example 22

3. Sketch graphs of the following parabolas, labelling
   i the y-intercept.
   ii the turning point.
   iii the axis of symmetry.

   a $y = (x + 1)^2$ **b** $y = (x + 4)^2$ **c** $y = (x - 2)^2$ **d** $y = (x - 4)^2$
   e $y = (x + 3)^2$ **f** $y = (x - 5)^2$ **g** $y = -(x + 1)^2$ **h** $y = -(x - 2)^2$

LINKS TO Example 23

4. Sketch graphs of the following parabolas, labelling
   i the x-intercepts.
   ii the y-intercept.
   iii the axis of symmetry.

   a $y = (x + 6)(x - 2)$ **b** $y = (x + 6)(x + 2)$ **c** $y = (x - 6)(x - 2)$
   d $y = (x - 6)(x + 2)$ **e** $y = (x - 3)(x + 3)$ **f** $y = (x - 1)(x - 7)$
   g $y = (x + 3)(x - 7)$ **h** $y = (x + 2)(x - 6)$ **i** $y = x(x + 5)$
   j $y = x(x - 6)$ **k** $y = x(x + 4)$ **l** $y = x(x - 3)$

LINKS TO Example 24

5. Write the equation for each of the following circles.

   a centre (0, 0), radius 3 **b** centre (0, 0), radius 7
   c centre (0, 0), radius 6 **d** centre (0, 0), radius 11
   e centre (0, 0), radius 15 **f** centre (0, 0), radius $\sqrt{12}$
   g centre (0, 0), radius $\sqrt{50}$ **h** centre (0, 0), radius $\sqrt{18}$

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LINKS TO Example 25

For each of these circles,

i find the radius.

ii sketch the graph, labelling the four intercepts.

a $x^2 + y^2 = 64$ **b** $x^2 + y^2 = 81$ **c** $x^2 + y^2 = 1$

d $x^2 + y^2 = 25$ **e** $x^2 + y^2 = 10$ **f** $x^2 + y^2 = 5$

g $x^2 + y^2 = 8$ **h** $x^2 + y^2 = 20$ **i** $x^2 + y^2 = 400$

exercise 7.6 challenge

Sketch the graphs of the following quadratic rules.

a $y = -(x + 3)^2$

b $y = (x - 4)^2 + 3$

c $y = (x + 2)^2 - 4$

d $y = -(x - 5)^2 + 3$

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Analysis task

Phone cards

Phone cards

Families who regularly make telephone calls to other countries can save on call charges by using pre-paid phone cards. Some phone cards have a fixed charge plus a certain amount per minute, while others just have an amount per minute without a fixed charge. Different phone cards may vary in their rates per minute for different countries. Use websites to find the rates for different phone cards or use the data given here (the names of the cards here are not actual card names).

	Lucky talk	Golden minutes	Sunshine
Fixed charge	**18 cents for each call**	–	–
Country	**Cost per minute (cents)**		
USA	1.7	2.2	3.0
Vietnam	4.0	8.0	8.0
UK	1.6	2.1	3.0
China	1.7	2.6	4.0
India	2.6	5.0	5.0
Greece	1.9	2.6	4.0
Indonesia	5.0	7.0	11.0
Japan	2.4	3.2	4.0
New Zealand	2.2	2.9	4.0
Singapore	1.8	2.2	3.3

a Select a country and set up a spreadsheet to determine the cost of various lengths of calls (say, from 1 minute up to 60 minutes) for each of the three different cards.

b Construct an XY scatter graph for each of the three cards on the same set of axes.

c Do the graphs for the three cards have the same gradient? Which graph has the greatest gradient?

d What does the gradient of each graph represent?

e If you needed to make a very brief call, say 1 minute, which would be the best card to use?

f If your call was 5 minutes, which would be the best card?

g If you regularly made calls of about 60 minutes to relatives in your selected country, which would be the best card?

h Do any of your graphs intersect? What is the significance of points where two graphs intersect?

i Repeat for a second country. Are your findings the same?

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Review Linear and non-linear relationships

Summary

Gradient intercept form of straight-line equation

- The gradient-intercept form of a straight line is $y = mx + b$, where m is the gradient and b is the intercept.

Equation of a line through a point

- $y - y_1 = m(x - x_1)$

General form of equation for a straight line

- $ax + by + c = 0$

Parallel lines

- Lines parallel to the axes
  - A line parallel to the x-axis has the equation $y =$ a constant.
  - The x-axis is the line $y = 0$.
  - A line parallel to the y-axis has the equation $x =$ a constant.
  - The y-axis is the line $x = 0$.
- Parallel lines have the same gradient.

Parabola

- The graph of $y = x^2$ is called a parabola.
- The graph of $y = x^2$ is symmetrical about the y-axis.
- Vertical translation
  - The graph of $y = x^2 + k$ and $y = -x^2 + k$ has its turning point at $(0, k)$. If $k > 0$, the parabola $y = x^2$ is shifted up. If $k < 0$, the parabola $y = x^2$ is shifted down.
  - The graph of $y = -x^2 + k$ is 'upside down', with its turning point at $(0, k)$.
- Horizontal translation
  - In general, the graph of $y = (x - h)^2$ is the same shape as the graph of $y = x^2$ but the parabola is translated h units to the right. If h is negative, the parabola is translated to the left. The turning point is at $(h, 0)$ and the axis of symmetry is the line $x = h$.
  - We can find the y-intercept by substituting $x = 0$ in the equation.

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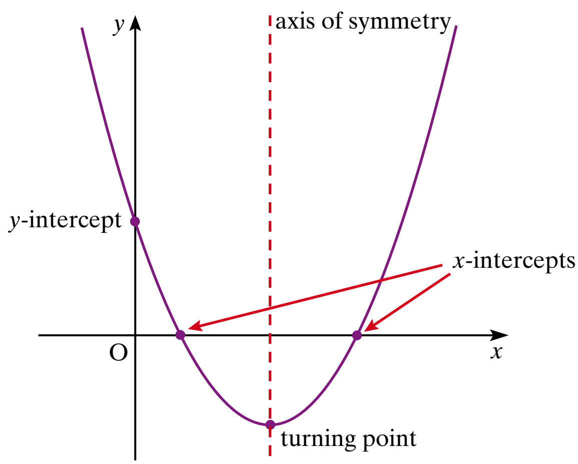
- What the factors tell us
  - The graph of $y = (x + a)(x + b)$ has x-intercepts at $(-a, 0)$ and at $(-b, 0)$.
  - The y-intercept is found by substituting $x = 0$ in the rule.
  - The axis of symmetry is a line parallel to the y-axis halfway between the two x-intercepts.

Circle

- The equation of a circle with centre (0, 0) and radius r is $x^2 + y^2 = r^2$.

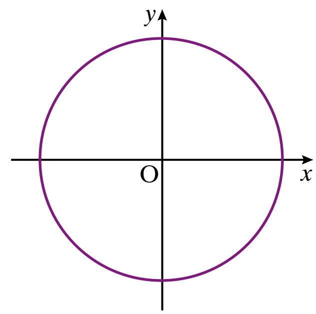
Parabola



Circle

$x^2 + y^2 = r^2$

The radius of the circle is r



Visual map

axis of symmetry	ordered pairs	quadrant	table of values
gradient	parabola	rectangular hyperbola	turning point
intercept	parallel lines	sketch graph	x-intercept
linear graph	plot		y-intercept

Revision

Multiple-choice questions

1. The linear graph that crosses the axes at (–2, 0) and (0, 10) has the equation:

A $y = -5x + 10$ **B** $y = -2x + 10$ **C** $y = 5x + 10$
D $y = 10x - 2$ **E** $y = 10x + 5$.

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- The equation of the graph shown here is

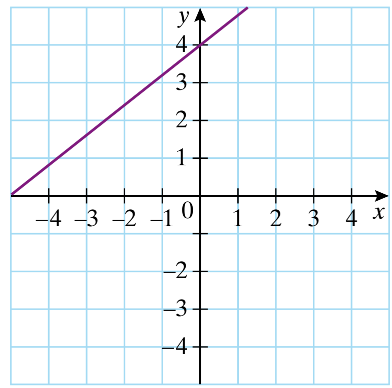
  A $y = 4$

  B $y = -5x + 4$

  C $y = -\frac{4}{5}x + 4$

  D $y = \frac{4}{5}x + 4$

  E $y = 4x - 5$



- When rearranged into gradient form, the equation $3x + 2y = 5$ becomes:

  A $y = -\frac{3}{2}x + \frac{5}{2}$ **B** $y = -\frac{3}{2}x + 5$ **C** $y = -3x + \frac{5}{2}$

  D $y = -3x + 5$ **E** $y = 3x + 5$.

- The equation of the linear graph with a y-intercept of (0, 1) and passing through (–3, 7) is:

  A $y = -2x + 1$ **B** $y = -3x + 7$ **C** $y = -\frac{8x}{3} + 1$ **D** $y = x - 3$ **E** $y = 2x + 1$.

- The rule for the table shown here is:

x	0	1	2	3	4
y	7	2	–3	–8	–13

  A $y = 7$ **B** $y = 7x$ **C** $y = 7x - 5$ **D** $y = -5x$ **E** $y = -5x + 7$.

Short-answer questions

- Write equations for the following lines.

  a gradient –3 and y-intercept 5

  b gradient $\frac{3}{4}$ and y-intercept –7

- For each of the following linear relationship rules, state the gradient and the y-intercept.

  a $y = 2x - 7$

  b $y = -1.5x - 3.5$

- Find the equation for each of these lines.

  a gradient 4 and passing through the point (–3, –6)

  b gradient $-\frac{1}{2}$ and passing through the point (–3, 8)

  c passing through the points (–7, –2) and (5, 4)

  d passing through the points (–4, 9) and (12, –3)

- Write the equation for each of these lines.

  a parallel to the x-axis and passing through the point (–2, –5)

  b parallel to the y-axis and passing through the point (–1, 7)

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- Write each of these equations in gradient-intercept form and hence state the gradient and y-intercept.

  a $3x - 4y - 18 = 0$ **b** $5x - 6y - 24 = 0$

- Write each of these equations in the general form for a linear equation.

  a $y = -\frac{2x}{5} + 6$ **b** $y = \frac{3x}{8} - 5$

- **a** Construct a table of values with x values from –2 to 2 for the rule $y = 4x - 2$.

  b Plot the graph on graph paper, using a suitable scale.

  c From the graph find

  i the y-value when $x = -1.5$ **ii** the x-value when $y = 4$

- Sketch graphs of the following equations.

  a $y = x + 5$ **b** $y = -2x + 4$ **c** $3x + y = 6$

  d $y - x = 0$ **e** $y = 3x$ **f** $y = -4x$

  g $x = -3$ **h** $y = 6$ **i** $x = 0$

- Sketch graphs of the following quadratic rules, labelling the position of the turning point.

  a $y = x^2 + 4$ **b** $y = -x^2 + 3$

  c $y = (x - 3)^2$ **d** $y = (x + 2)^2$

- Sketch graphs of the following quadratic rules, labelling

  i the x-intercepts.

  ii the y-intercept.

  iii the axis of symmetry.

  a $y = (x - 4)(x + 4)$

  b $y = (x - 3)(x + 7)$

- Sketch a graph of the circle $x^2 + y^2 = 16$, labelling the coordinates of all the intercepts.

- Give the equation of a circle with centre (0, 0) and radius 5.

Extended-response questions

- Guy has \$100 in the bank at the moment. He has just started a part-time job, earning \$40 per week. Guy is planning to save all of the money he earns so that he can buy a MP3 player.

  a How much money will Guy have in the bank after his first week at work?

  b How much will he have saved after two weeks?

  c Which is the independent variable: the amount of money saved or the number of weeks?

  d Construct a table of values to show Guy's savings after each of the first four weeks.

  e Find the rule for this relationship.

  f Use the rule to find how many weeks it will take Guy to save enough money to buy a \$300 MP3 player.

Practice quiz
Chapter 7

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Proportion

8

Pre-test

Warm-up

Hour glasses were probably invented in the 14th century to measure time. They depend on the fact that the amount of sand that runs through is proportional to the time. If a certain amount of sand runs through from the top to the bottom in one minute, then twice as much sand will run through in two minutes. Egg timers are still used for this purpose to time the cooking of an egg. There are many other instances where one variable is directly proportional to another. For example, the force of gravity acting on an object is proportional to its mass and the pressure exerted by water is proportional to the depth below the surface. There are other cases where the proportion is not direct. In inverse proportion, the value of one variable is halved as the other doubles.

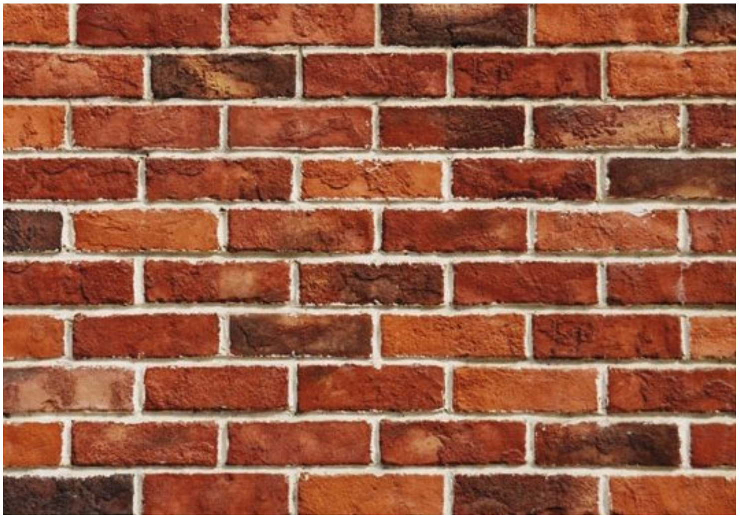
8.1 Direct proportion

BLM
Chapter warm-up

There are many relationships between quantities where one quantity depends directly on the other. For example, the circumference of a circle depends on the diameter, the weight of a load of bricks depends on the number of bricks, or the cost of muffins depends on how many muffins are bought. In each of these examples, there is a linear relationship between the two variables – if the value of one variable increases, the value of the other variable increases in proportion. We say that one variable is **directly proportional** to the other.

Many relationships between physical quantities are examples of direct variation. For example, the pressure in a liquid increases in proportion to the depth below the surface, the electric current flowing through a circuit is proportional to the voltage.

We will start by looking at the simple example of the number of bricks required to build walls of different areas. The photograph shows a section of a brick wall with an area of one square metre. There are 45 whole bricks and 10 half bricks, making up the equivalent of 50 bricks to one square metre.



If we double the area of the wall, we double the number of bricks required. If we multiply the area of the wall by 4, four times as many bricks are required.

$\times 2$ (from 1 to 2) $\times 4$ (from 5 to 20)

Area of wall (A m^2)	0	1	2	3	4	5	20
Number of bricks (n)	0	50	100	150	200	250	1000

$\times 2$ (from 50 to 100) $\times 4$ (from 250 to 1000)

We say that the number of bricks is directly proportional to the area of the wall. If A m^2 is the area of the wall and n is the number of bricks, we can write

$n = kA$ where k is the **constant of proportionality**.

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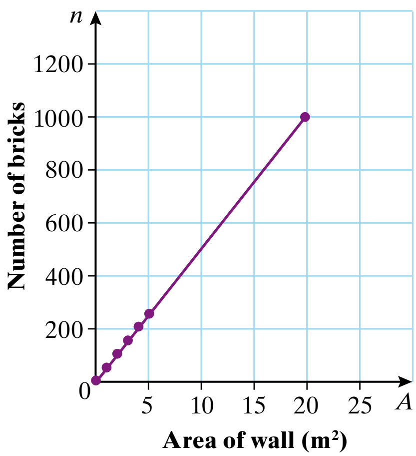
This means that $k = \frac{n}{A}$ so in this example, $k = \frac{50}{1} = \frac{100}{2} = \frac{1000}{20} = 50$. We can see that k is the number of bricks per square metre.

We can also write $n \propto A$ where the mathematical symbol $\propto$ means 'is proportional to'.

The graph of n against A is a linear graph passing through the origin, with gradient 50.



Direct proportion

In general, direct proportion can be represented as $y \propto x$ or $y = kx$ where k is the constant of proportionality. Direct proportionality can stated in several ways.

- y is directly proportional to x.
- y is proportional to x.
- y varies directly as x.

The graph of $y = kx$ is a straight line passing through the origin, with gradient k. The converse also applies. If the graph of two variables is linear and passes through the origin, we know that one variable is directly proportional to the other.

Example 1

The number of paving stones required is proportional to the area to be paved. If n is the number of paving stones and A m^2 is the area, write the relationship in the following forms

i $y \propto x$

and

ii $y = kx$ where k is the constant of proportionality.

continued

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Example 1 continued

Working

i $n \propto A$

ii $n = kA$

Reasoning

The symbol $\propto$ means 'is directly proportional to'.

Dependent and independent variables

When one variable is directly proportional to another, we usually identify one of the variables as being independent and the other variable as dependent. In the rule $y = kx$, we take x to be the independent variable and y as the dependent variable, that is y depends on x. In the following worked example, Dino's earnings depend on how many matches he referees so n is the independent variable and E is the dependent variable.

- In a table of values the independent variable usually goes in the top row (or in the left hand column if the table is vertical).
- The independent variable goes on the horizontal axis of the graph and the dependent variable on the vertical axis.

Example 2

Dino referees basketball matches and he is paid $15 per match.

a Using n for the number of matches and $\$E$ dollars for Dino's earnings in a season, write this as a rule.

b Complete the table of values to show Dino's seasonal earnings for different numbers of matches refereed.

n	0	1	2	5	10
E					

c Plot a graph to show Dino's seasonal earnings.

d Explain why this is an example of direct variation.

e How many matches would Dino have to referee to earn $135?

Working

a $E = 15n$

b

n	0	1	2	5	10
E	0	15	30	75	150

Reasoning

Dino earns $15 for every match he referees.

Substitute $n = 0$, $n = 1$, and so on, into $E = 15n$.

continued

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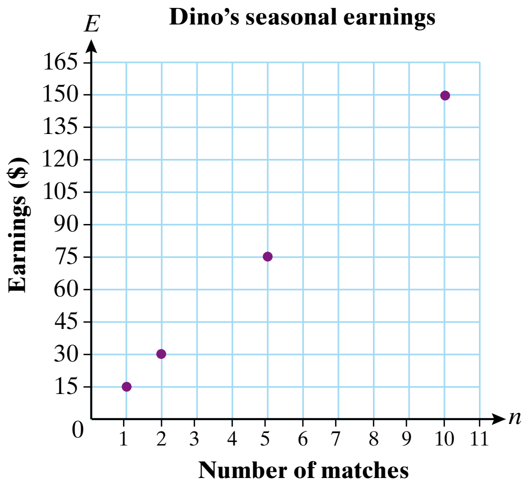
Example 2 continued

Working

c



Reasoning

Dino referees whole matches so it does not make sense to join the points.

d The graph is a straight line passing through zero.

If the number of matches doubles, Dino's earnings double.

e If $E = 135$,

$$135 = 15n$$

$$n = \frac{135}{15}$$

$$n = 9$$

Dino would need to referee 9 matches to earn $135.

Substitute 135 for E to find n.

If we know that one variable is directly proportional to another, we can calculate the constant of proportionality from one pair of values. We can then write the rule linking the two variables and use the rule to calculate other values.

Example 3

T is directly proportional to L.

a Using T and L, write the rule in the forms $y \propto x$ and $y = kx$.

b Find the value of k

c Rewrite the rule with the calculated value for k.

d Complete the table.

L	0	1	2	3	4
T	0			18.75	

Working

a $T \propto L$

$T = kL$

Reasoning

T is directly proportional to L.

continued

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Example 3 continued

Working / **Reasoning**

b When $L = 3$, $T = 18.75$
$18.75 = 3k$
$$k = \frac{18.75}{3}$$
$k = 6.25$

Substitute the given values of T and L to find k.

c $T = 6.25L$

Substitute 6.25 for k

d

L	0	1	2	3	4
T	0	6.25	12.50	18.75	25.00

Multiply each L value by 6.25.

Example 4

The amount of petrol used is directly proportional to the distance travelled. A car uses 27 L of petrol in traveling 360 km.

a Using p for the litres of petrol and d km for the distance travelled, write the rule in the forms $y \propto x$ and $y = kx$.
b Find the value of k.
c Rewrite the rule with the calculated value for k.
d How much petrol (to the nearest litre) would be used for 560 km?
e How far could the car travel with 34 L of petrol?

Working / **Reasoning**

a $p \propto d$
$p = kd$

The amount of petrol used depends on the distance travelled so d is the independent variable and p is the dependent variable.

b When $d = 360$, $p = 27$.
$27 = 360k$
$$k = \frac{27}{360}$$
$k = 0.075$

Substitute the given values for p and d to find k. (k has been calculated by dividing the amount of petrol in litres by the distance in kilometres, so k represents the litres of petrol per kilometre. Because it is a small number, the petrol consumption of cars is usually stated in litres per 100 km. In this case the petrol consumption would be 7.5 L/100 km.)

c $p = 0.075d$

Substitute the calculated value of k in the rule.

d When $d = 560$,
$p = 0.075 \times 560$
$p = 42$
42 L would be used.

Substitute 560 for d to find p.

continued

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Example 4 continued

Working

e When $p = 34$,

$$34 = 0.075d$$

$$d = \frac{34}{0.075}$$

$$d = 453.\dot{3}$$

The car could travel approximately $453\frac{1}{3}$ km.

Reasoning

Substitute 34 for p to find d.

Example 5

For a given speed, the distance travelled is proportional to the length of time travelling.

a Using d km for the distance and t h for the time, express this in the forms $y \propto x$ and $y = kx$.

b A car travels 102 km in 1 hr 12 minutes. Calculate the value of k.

c What does k represent in this situation?

d Rewrite the rule using the calculated value of k.

e How far would the car travel in 2 hours at the same average speed?

f How long would it take for the car to travel 212 km at the same average speed?

Working	**Reasoning**
a $d \propto t$ $d = kt$	The distance travelled depends on the travel time. t is the independent variable and d is the dependent variable.
b $102 = 1.2k$ $k = \frac{102}{1.2}$ $k = 85$	$1 \text{ h } 12 \text{ min} = 1\frac{12}{60} \text{ h} = 1.2 \text{ h}$
c k represents the speed of the car in km/h.	k is calculated by dividing the distance in kilometres by the time in hours, so k represents the number of kilometres per hour, that is, the speed.
d $d = 85t$	Substitute 85 for k in the rule.
e When $t = 2$, $d = 85 \times 2$ $d = 190$ The car would travel 190 km.	Substitute 2 for t in the rule.

continued

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Example 5 continued

Working	Reasoning
f When $d = 212$, $212 = 85 \times t$ $t = \frac{212}{85}$ $t = 2.49...$ It would take approximately $2\frac{1}{2}$ hours to travel 212 km.	Substitute 212 for d in the rule.

Many everyday problems involve direct proportion. Problems of this type can be solved using the following steps.

- Express the relationship in the form $y = kx$.
- Solve for k.
- Write the rule using the value of k.
- Substitute to find the required quantity.

Example 6

A 2.5 L tin of paint covered an area of 35 m^2 of wall. How much paint would be required to paint 64 m^2? Give your answer to one decimal place.

Working

Let x m^2 be the area of wall and y L be the amount of paint.

When $x = 35$, $y = 2.5$

$$y = kx$$
$$2.5 = 35k$$
$$k = \frac{2.5}{35}$$
$$k = \frac{1}{4}$$
$$y = 0.07142x$$

When $x = 64$,

$$y = \frac{1}{4} \times 64$$
$$y = 4.57...$$

Approximately 4.6 L of paint is required.

Reasoning

This problem could be solved in other ways:

2.5 L	35 m^2
p L	64 m^2

Writing a proportion equation, we obtain

$$\frac{p}{2.5} = \frac{64}{35}$$
$$p = \frac{64 \times 2.5}{35}$$
$$p \approx 4.6$$

Approximately 4.6 L of paint is required.

The unitary method could also be used:

35 m^2	2.5 L
1 m^2	$\frac{2.5}{35}$ L
64 m^2	$\frac{2.5 \times 64}{35} \approx 4.6$ L

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exercise 8.1

LINKS TO Example 1

1. Rewrite each of the following in the form
   i $y \propto x$.
   ii $y = kx$.
   a W is directly proportional to m.
   b C is directly proportional to n.
   c p varies directly as t.
   d C is proportional to r.
   e V varies directly as h.
   f I is proportional to t.

LINKS TO Example 2

2. Each of the following tables of values represents the numbers of tiles, n, of a particular size, needed to pave an area, A m^2. Complete each table.

a

A	0	1	2	3	4	5
n			76			

b

A	0	1	2	3	4	5
n				57		

c

A	0	1	2	3	4	5
n					20	

d

A	0	1	2	3	4	5
n					25	

3. A concreting contractor charges \$55 per square metre to concrete a driveway.
   a Using A m^2 for the area of the driveway and \$$C$ for the cost, write this as a rule.
   b Complete the table of values to show the cost for different areas.

A	0	10	20	30	40	50	60
C	0						

   c Use a spreadsheet to graph the cost against area.
   d Explain why this is an example of direct variation.
   e How many squares metres could be concreted for \$935?
   f What would it cost to have 26 m^2 concreted?

LINKS TO Example 4

4. A fencing contractor charges \$4480 for 56 m of paling fence.
   a Using L m for the length of the fence and \$$C$ for the cost, write this as a rule in the form $y \propto x$ and $y = kx$.
   b Calculate the value of k.
   c Rewrite the rule with the calculated value of k.

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d Copy and complete the table to show the cost for fences of different lengths.

Length of fence (L m)	0	10	20	30	40	50	56
Cost (\$$C$)	0						4480

e Calculate the cost of 68 m of fencing.

To build a wall from these blocks, 27 blocks per square metre are required.

a Copy and complete the table to show the number of blocks required for walls with different areas.

Area of wall, A (m^2)	0	1	2	3	4	5	10
Number of blocks, n	0						

b Using A m^2 for the area of the wall and n for the number of blocks, write the rule in the form $y \propto x$ and $y = kx$.

c Write the rule with the calculated value of k.

d What area of wall could be built using 500 blocks?

LINKS TO Example 5

A car travelling at constant speed travels 144 km in 1 hour 30 minutes.

a Using d km for the distance travelled and t hr for the time, write the rule in the forms $y \propto x$ and $y = kx$.

b Find the value of k.

c What is the speed of the car?

d Write the rule with the calculated value of k.

e How far would the car travel in 2 hours 30 minutes at the same speed?

f How long would it take for the car to travel 312 km at the same speed?

A car travelling at constant speed travels 133 km in 1 hour 24 minutes.

a Using d km for the distance travelled and t hr for the time, write the rule in the forms $y \propto x$ and $y = kx$.

b Find the value of k.

c What is the speed of the car?

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d Write the rule with the calculated value of k.

e How far would the car travel in 2 hours 15 min at the same speed?

f How long would it take for the car to travel 304 km at the same speed?

LINKS TO Example 6

Jeremy and Claire are making mini pizzas for a party. The topping for 32 pizzas requires 120 mL tomato paste. How much tomato paste will they require for 200 mini pizzas?

For a given voltage, the electrical power W watt is proportional to the current I amp that flows through it.

a Write this as a rule in the form and $y = kx$.

b When $W = 1200$, $I = 5$. Find the value of k.

c Rewrite the rule with the calculated value of k.

d Find I when $W = 60$

The circumference, C, of a circle is proportional to the diameter, d.

a Write this as a rule in the form $y \propto x$ and $y = kx$.

b What is the constant of proportionality?

A cake recipe includes 90 g of butter and 150 g flour. How much flour would be required if a larger cake is to be made with 120 g butter?

A car uses 9.5 L of petrol for 100 km. How much petrol would be required for 230 km in similar driving conditions?

The amount of a particular medicinal drug given to children in a hospital is directly proportional to the child's weight. A child weighing 34 kg is given 51 mg of the medicine.

a Using m kg for the child's mass and d mg for the amount of drug, write the rule in the forms $y \propto x$ and $y = kx$.

b Find the value of k.

c Rewrite the rule using the calculated value of k.

d How much of the drug should a child who weighs 42 kg be given?

In everyday language we often use the word *weight* when we mean *mass*. The weight of an object is the force of gravity acting on it and is measured in a unit called the Newton (named after Isaac Newton). The weight of an object, W Newtons, is proportional to its mass, m kg.

a Express this in the forms $y \propto x$ and $y = kx$.

b A brick has mass 2.7 kg and on Earth its weight is 26.46 Newton. Find the value of k.

c Rewrite the rule with the calculated value for k.

d What is the weight in Newtons of a person who has a mass of 68 kg?

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exercise 8.1 challenge

● The gas ethane burns in oxygen to produce carbon dioxide and water. If 60 g of ethane and 224 g of oxygen theoretically produce 176 g of carbon dioxide and 108 g of water. If the amount of ethane is changed, the other substances change in the same proportion.

Copy and complete this table.

ethane	60	120	200	500
oxygen	224			
carbon dioxide	176			
water	108			

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8.2 Inverse proportion

We now look at a different type of proportionality. Suppose, for example, a 48 cm length of liquorice is shared between several people. The *greater* the number of people, the *shorter* the length each will receive. We say that the length of liquorice received is **inversely proportional** to the number of people.

In the following table, x is the number of people and y is the length of liquorice in centimetres that each person receives.

x	1	2	3	4	6	8	12	16	24	48
y	48	24	16	12	8	6	4	3	2	1

We can write $y \propto \frac{1}{x}$ or $y = \frac{k}{x}$ where k is the constant of proportionality.

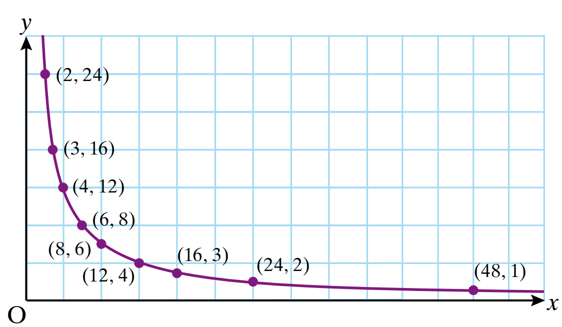
Note that this can also be written in the form $xy = k$, that is, the product of each x value and y value is constant.

In the case of the liquorice, when $x = 3$, for example, $y = 16$. So $16 = \frac{k}{3}$, that is, $k = 48$.

We can then write $y = \frac{48}{x}$.

Multiplying both sides of this rule by x gives $xy = 48$.

The graph of this relationship is called a hyperbola.



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Inverse variation

If two quantities, x and y, vary inversely, then they can be represented by $y \propto \frac{1}{x}$ or $y = k \times \frac{1}{x}$ or $y = \frac{k}{x}$ or $xy = k$. These relationships can also be called **reciprocal relationships** because y is proportional to the reciprocal of x.

The product xy is constant.

We can express inverse proportion as

- y is inversely proportional to x.
- y is proportional to the reciprocal of x.
- y varies inversely as x.

Example 7

The table of values represents inverse proportion. Show the doubling and halving of values.

x	1	2	3	4	5	6
y	30	15	10	7.5	6	5

Working

$\times 2$ $\times 2$

x	1	2	3	4	5	6
y	30	15	10	7.5	6	5

$\div 2$ $\div 2$

Reasoning

Whatever factor the x-value is multiplied by, the y-value is divided by that factor.

$\times 3$

x	1	2	3	4	5	6
y	30	15	10	7.5	6	5

$\div 3$

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Example 8

v is inversely proportional to m. Write this in the forms $v \propto \frac{1}{x}$ and $y = \frac{k}{x}$.

Working

$$v \propto \frac{1}{m}$$

$$v = \frac{k}{m}$$

Reasoning

Use k as the constant.

Example 9

x and y are inversely proportional. When $x = 3$, $y = 16$.

a Express the relationship using the constant k.
b Find the value of k.
c Write the rule with the calculated value for k.
d Complete the table.

x	1	2	3	4	5	6
y			16			

Working

a $y = \frac{k}{x}$

Reasoning

y is inversely proportional to x means that y is proportional to the reciprocal of x.

Working

b When $x = 3$, $y = 16$

$$16 = \frac{k}{3}$$

$$k = 48$$

Reasoning

Substitute the given values for x and y into $y = \frac{k}{x}$ and solve for k.

Working

c $y = \frac{48}{x}$

Reasoning

Substitute 48 for k in $y = \frac{k}{x}$.

Working

d

x	1	2	3	4	5	6
y	48	24	16	12	9.6	8

Reasoning

Substitute each of the values of x in $y = \frac{48}{x}$ to find the y values.
For example, when $x = 2$.

$$y = \frac{48}{2} = 24$$

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Example 10

One bottle of orange juice is shared equally between drinking glasses. The number of glasses of juice that can be poured from the bottle is inversely proportional to the volume of juice in each glass.

a Using n for the number of glasses and V mL for the volume of juice in each glass, write the relationship in the forms $y \propto \frac{1}{x}$ and $y = \frac{k}{x}$ where k is the constant of proportionality.

b If each glass has 150 mL of juice, 8 glasses can be poured. Find the value of k.

c Rewrite the rule using the value of k.

d What does k represent in this situation?

e How much would each glass contain if the juice was distributed between 6 glasses?

Working	Reasoning
a $n \propto \frac{1}{V}$ $n = \frac{k}{V}$	The symbol $\propto$ means 'is directly proportional to'. n is inversely proportional to V.
b $8 = \frac{k}{150}$ $k = 8 \times 150$ $k = 1200$	Substitute 8 for n and 150 for V to find k.
c $n = \frac{1200}{V}$	The rule could be written as $nV = 1200$. For inverse variation there is a constant product of the two variables.
d k is the volume of juice in the bottle before any is poured into the glasses. There must be 1200 mL of juice in the bottle.	8 glasses each with 150 mL means that there was 1200 mL in the bottle.
e $n = \frac{1200}{V}$ When $n = 6$,	Substitute $n = 6$ in the rule and solve for V.
$6 = \frac{1200}{V}$	Multiply both sides by V.
$6V = 1200$	Divide both sides by 6.
$V = 200$ There would be 200 mL in each glass.	

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Example 11

If 16 people work on a particular task it can be finished in 2 days. The task will take twice as long if the number of people is halved.

a Using p for the number of people working and d for the number of days taken, express this relationship in the forms $y \propto \frac{1}{x}$ and $y = \frac{k}{x}$.

b What is the value of k?

c Complete the table of values.

p	1	2	4	8	16	32
d		16				

d Graph the values.

Working

a $d \propto \frac{1}{p}$

$d = \frac{k}{p}$

b $d = \frac{k}{p}$

$2 = \frac{k}{16}$

$k = 32$

c

p	1	2	4	8	16	32
d	32	16	8	4	2	1

Reasoning

d is inversely proportional to p.

k is the constant of proportionality.

Substitute the given values for p and d into $d = \frac{k}{p}$ and solve for k.

Substitute each of the values of p in $d = \frac{48}{p}$ to find the d values.

For example, when $p = 4$,

$d = \frac{32}{4}$

$= 8$

continued

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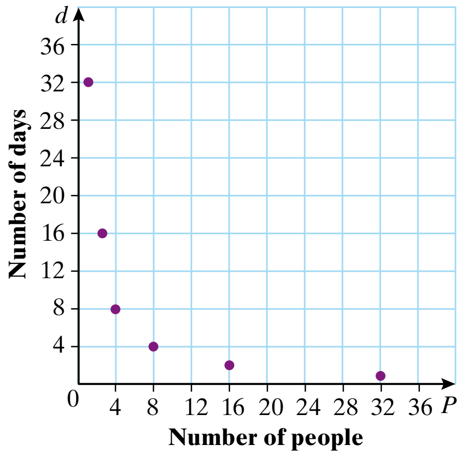
Example 11 continued

d



In this example, it is not appropriate to join the points as the number of people must be a whole number.
The graph approaches each axis but does not touch the axes. If there were zero people it is meaningless to talk about the number of days to complete the task. Similarly, the task could not be completed in zero days. With a large number of people, the task may be completed in a fraction of a day, but never in zero time. So the graph approaches both axes without actually touching them.

exercise 8.2

LINKS TO Example 7

1 For each of these tables of values,

i copy and use arrows to show the doubling and halving.
ii find the value of k.
iii write the rule using the value of k.

a

x	1	2	3	4	5
y	30	15	10	7.5	6

b

x	1	2	4	5	10
y	40	20	10	8	4

c

x	0.5	1	2	4	8
y	32	16	8	4	2

d

x	2	3	4	8	15
y	60	40	30	15	8

LINKS TO Example 8

2 Write each of these in the forms $y \propto \frac{1}{x}$ and $y = \frac{k}{x}$.

a W is inversely proportional to A.
b T is inversely proportional to S.
c I varies inversely as R.
d n is inversely proportional to C.
e n varies inversely as A.
f V is inversely proportional to P.

LINKS TO Example 9

3 These tables of values represent inverse proportion.

i Copy and complete each table.
ii Find the value of k.
iii Write the rule using the value of k.

a

x	1	2	5	8	10	20
y				15		

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b

x	1	2	3	4	5	6
y			12			

c

x	1	2	3	5	6	10
y				30		

d

A	1	2	3	4	5	6
n				36		

Tyson had a box of 24 chocolates. If he ate all the chocolates himself, he would get 24 chocolates. If he shared them equally with his sister, he would get 12 chocolates.

a Construct a table of values to show the number, c, of chocolates that each person would receive if the chocolates were divided between n people, where $n = 1, 2, 3, 4, 5$ and 6.

b Explain why this is an example of inverse proportion.

c Write the rule in the forms $y \propto \frac{1}{x}$ and $y = \frac{k}{x}$.

d What is the value of k?

e Rewrite the rule using the value of k.

LINKS TO Example 10

Maddy has made 720 mL of muffin mixture. She has tins for six different sizes of muffin. The number of muffins Maddy can make from the 720 mL of mixture is inversely proportional to the size of each muffin.

a Complete the table to show the number of muffins that can be made for each different size.

Size (S mL)	30	60	90	120	144	180
Number (n)						

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b Write the rule in the forms $y \propto \frac{1}{x}$ and $y = \frac{k}{x}$.

c What is the value of k?

d Rewrite the rule using the value of k.

LINKS TO Example 11

Ahmed has \$60 and plans to buy several CDs. He decides to check out the prices in five different shops before he spends any money.

a At the first shop, CDs are \$15 each. How many could Ahmed buy?

b At the second shop, CDs are \$12 each. How many could Ahmed buy at this shop?

c At the other three shops, Ahmed finds that the price of a CD is \$20, \$10 and \$6 respectively. Make a table of values to show the number of CDs (n) that Ahmed could buy if one CD costs p dollars.

d Draw a graph of the number, n, of CDs that Ahmed could buy against the cost, \$$p$, per CD. Is it appropriate to join the points?

Sketch the graph for each of the following rules. Make sure that the axes are labelled correctly and mark the coordinates of two points on each graph.

a $y = \frac{24}{x}$ **b** $t = \frac{20}{s}$ **c** $P = \frac{16}{V}$ **d** $c = \frac{36}{p}$

For a given area of 36 cm^2, there are many different rectangles that could be drawn.

a Make a table of ten possible values for the length, L cm, with the corresponding values for the width, W cm.

b Express the proportionality in the forms $y \propto \frac{1}{x}$ and $y = \frac{k}{x}$.

c What is the constant of proportionality?

d Rewrite the rule using the value of k.

e Graph W against L.

f Is it appropriate to join the points? Explain.

The exterior angle size of regular polygons is inversely proportional to the number of sides of the regular polygon. The table shows the size of each exterior angle, $e°$, for regular polygons with n sides.

n	3	4	5	6	8	10
e	120	90	72			

a Express the proportionality in the forms $y \propto \frac{1}{x}$ and $y = \frac{k}{x}$.

b What is the constant of proportionality?

c Rewrite the rule using the value of k.

d Copy and complete the table.

e What is the size of each exterior angle of a 12-sided regular polygon?

f The size of each exterior angle of a regular polygon is 15°. How many sides does the polygon have?

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exercise 8.2 challenge

1. The electrical current, I amp, flowing through an appliance is inversely proportional to its resistance, R ohm. $I = \frac{V}{R}$ where the constant of proportionality is the voltage.

The voltage of electricity supplied to homes in Australia is 240 volts. The table shows the resistance of some domestic appliances.

Appliance	**Resistance** **R ohm**	**Current** **I amp**
Fan heater	25	
Toaster	50	
Refrigerator	60	
Heated towel rail	120	
TV	300	
Desktop computer	400	
Incandescent light globe	600	
Compact fluorescent light	5000	

a Calculate the current flowing in each of the appliances when operating.

b Use a spreadsheet or graphing calculator to graph current against resistance. Join the points to make a smooth curve.

c Find the resistance if the current is 0.2 amp.

2. In the word game Scrabble™, letters of the alphabet are printed on tiles that are used to build words. The number of tiles for each letter is related to the frequency of that letter in the English language. For example, E is the most commonly occurring letter, so there are more E's than any other letter in a set of Scrabble tiles.

On the other hand, a higher score is obtained for using a less common letter, such as Q. In general, the more commonly used the letter, the lower the score for that letter. Amanda is designing her own version of Scrabble. For each letter of the alphabet, the product of the number of tiles, n, and the score for that letter, S, will be 42. (Different letters may have the same number of tiles and the same score.) Both n and S are whole numbers.

a Copy and complete the table show all the possible values for n and S.

n	1							
S	42							

b Write a rule for the relationship between n and S in the form $y = \frac{k}{x}$.

c Draw a graph of S against n. Is it appropriate to join the points?

d Would it be possible for any letter to have a zero score? Explain.

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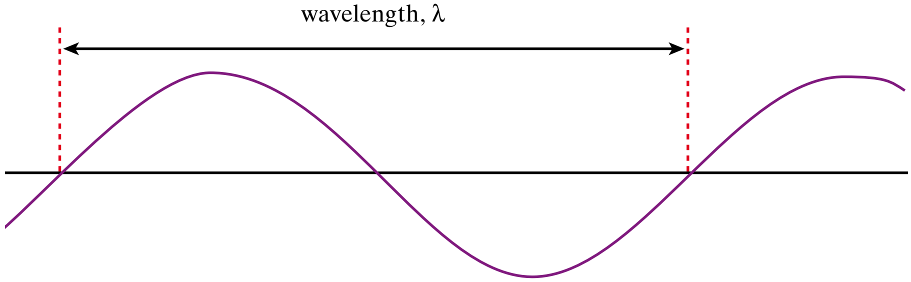
Analysis task

Radio waves

Radio waves represent just one section of the whole electromagnetic spectrum. Electromagnetic radiation also includes light, ultra-violet radiation and X-rays. All these forms of radiation can be represented as wave motion. The wavelength is the length of one complete wave and is represented by the Greek letter, λ, pronounced 'lambda'.



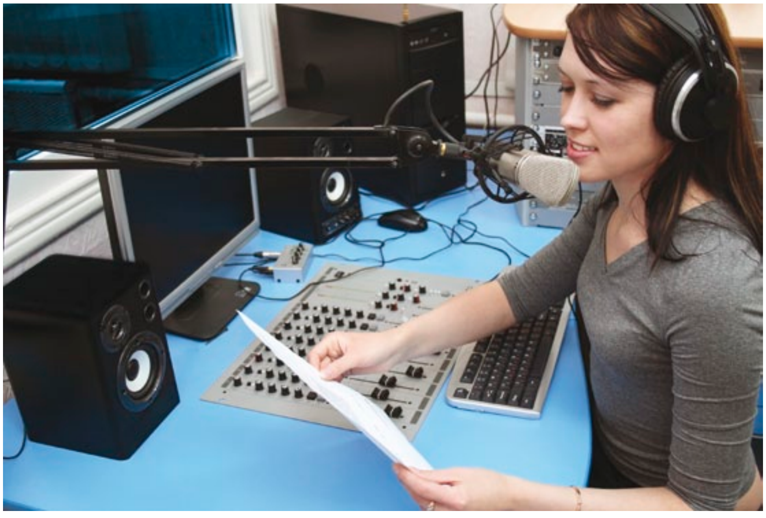
The frequency, f, of a wave is how many waves are passing a point in a second. We can see that if the wavelength is doubled, only half as many waves will pass the point in a second. So wavelength is inversely proportional to frequency.

Frequency is measured in a unit called the hertz (Hz) and wavelength is measured in metres.

a Express the inverse relationship between frequency and wavelength in the forms $y \propto \frac{1}{x}$ and $y = \frac{k}{x}$ using the symbols f and λ.

b k represents the speed of electromagnetic radiation, which is approximately 3×10^8 m/sec. Rewrite the rule using the value of k.

Radio stations transmit waves in the radio region of the spectrum and each radio station is assigned a particular frequency.



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The table below shows the frequencies in megahertz of several Melbourne radio stations. (1 MHz = 10^6 Hz)

c Convert each frequency into hertz.

d Calculate the wavelength in metres correct to three decimal places.

Radio station	Frequency (MHz)	Frequency (Hz)	Wavelength (m)
MW 9 3ppS Radio	93.1		
ABC Classic FM	105.9		
Triple J	107.5		
Classic Rock 91.5	91.5		
Nova 100	100.3		
Mix 101.1	101.1		
Fox	101.9		
Triple M	105.1		

e Using the website www.ausradiostations.com, find the frequencies of radio stations in your capital city or country region.

f Calculate the wavelength for each as in parts **c** and **d**.

g Ham radio operators often broadcast on the 6.0 metre band. What is the frequency in MHz of this wavelength?

h In Australia beginning amateur radio operators (with a Foundation Licence) are permitted to use the following wavelengths 40 m, 15 m, 10 m, 2 m and 70 cm. What frequencies correspond to these wavelengths? Give your answers in MHz to one decimal place, where required.

Wavelength	Frequency (MHz)
40 m	
15 m	
10 m	
2 m	
70 cm	

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i Amateur radio operators with an Advanced Licence can use many other wavelengths including the following. Calculate the frequency for each in MHz, current to one decimal place where required.

Wavelength	Frequency (MHz)
6 cm	
3 cm	
1.25 cm	
7.5 mm	
3.7 mm	
2.5 mm	
2 mm	
1.25 mm	

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Review Proportion

Summary

Direct proportion

- $y \propto x$, $y = kx$
- The graph of $y = kx$ is a straight line passing through the origin, with gradient k.

Inverse proportion

- $y \propto \frac{1}{x}$, $y = k \times \frac{1}{x} = \frac{k}{x}$
- The product xy is constant and equal to k.
- The graph of $y = \frac{k}{x}$ is a hyperbola.

Visual map

constant of proportionality	inversely proportional
directly proportional	proportion
hyperbola	variation

Revision

Multiple-choice questions

1. If T is directly proportional to R and $R = 11.6$ when $T = 174$, the relationship between x and y is given by
   A $R = 15T$ **B** $T = 15R$ **C** $T = \frac{15}{R}$ **D** $R = \frac{15}{T}$ **E** $y = -\frac{0.4}{x}$
2. If x and y are directly proportional and $x = 15$ when $y = 2$, the value of x when $y = 24$ is
   A 1.2 **B** 1.25 **C** 3.2 **D** 112.5 **E** 180
3. If x and y vary inversely and $x = 1.2$ when $y = 3$, the relationship between x and y is given by
   A $y = 3.6x$ **B** $y = 0.4x$ **C** $y = \frac{3.6}{x}$ **D** $y = \frac{0.4}{x}$ **E** $y = -\frac{0.4}{x}$
4. If a and b are inversely proportional and $b = 63$ when $a = 2.4$, the value of b when $a = 5.6$ is
   A 26.25 **B** 27 **C** 147 **D** 25.875 **E** 151.2

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The frequency, f vibrations per second, of a musical note varies inversely as its wavelength, w m. One particular musical note has a frequency of 321 vibrations per second and a wavelength of 3.2 m. The wavelength of another musical note that has a frequency of 374 vibrations per second is closest to

A 2.0 m. **B** 2.3 m. **C** 2.7 m. **D** 3.7 m. **E** 4.0 m

Short-answer questions

Rewrite each of these in the form

- **i** $y \propto x$
- **ii** $y = kx$.

- **a** C is directly proportional to n.
- **b** F is directly proportional to m.
- **c** A varies directly as w.
- **d** P is proportional to T.

A is directly proportional to n.

n	0	1	2	3	4	5
A			54			

- **a** Complete the table.
- **b** Calculate the value of k.
- **c** Rewrite the rule with the calculated value of k.

A painter charges \$14 per square metre to paint the walls and ceiling inside a house.

- **a** Using A m^2 for the area of to be painted and \$$C$ for the cost, write this as a rule.
- **b** Complete the table of values to show the cost for different areas.

A	0	100	200	250	400	500	600
C	0						

- **c** Use a spreadsheet to graph the cost against area.
- **d** Explain why this is an example of direct variation.
- **e** How many square metres (to the nearest square metre) could be painted for \$5000?
- **f** What would it cost to have 640 m^2 painted?

The cost of putting guttering around a house is proportional to the length of guttering.

- **a** Using L m for the length of the guttering and \$$C$ for the cost, write this as a rule in the form $y \propto x$ and $y = kx$.
- **b** A roofing contractor charges \$4104 for 108 m of guttering around a roof. Calculate the value of k.
- **c** Rewrite the rule with the calculated value of k.

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d Copy and complete the table to show the cost of different lengths of guttering.

Length of guttering, L (m)	0	50	108	120	150	200
Cost, \$ (C)	0					

The number, n, of Frequent Flyer points earned is directly proportional to the distance travelled, d km.

***d* (km)**	1000	2000	3600	8000	12 000
n				12 872	

a Calculate the value of k.
b Complete the table.
c Write the rule with the calculated value of k.
d How many points would be earned for 15 400 km of travel?
e How far would you need to fly to earn 30 000 points?

A bank offers rewards for paying by credit card. The points earned can be redeemed for cash or for travel or shopping vouchers. 18 334 points can be redeemed for a \$50 voucher.

a How many points would be needed for
  i a \$100 voucher?
  ii a \$20 voucher?
b 8367 points can be redeemed for two child cinema passes. How much are these two cinema passes worth?

Write each of these in the forms $y \propto \frac{1}{x}$ and $y = \frac{k}{x}$.

a L is inversely proportional to W.
b R is inversely proportional to I.
c T varies inversely as n.
d n is inversely proportional to R

Extended-response questions

In this table, y is inversely proportional to x.

x	1	2	4	5	8	10
y		2.5		1		

a Copy and complete the table, using arrows to show the doubling and halving.
b Find the value of k.
c Write the rule using the value of k.
d Sketch the graph of y against x, labeling two points on the graph.
e What is the value of y when $x = 25$

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An events organizer has \$1200 to spend on food for a conference dinner. She needs to choose between several different menus that cost \$24, \$25, \$30, \$40 and \$48 per head. The number, n, of people who can be invited is inversely proportional to the cost per head, \$$C$.

a Copy and complete the table.

\$*C*	24	25	30	40	48
n					

b Write the rule in the forms $y \propto \frac{1}{x}$ and $y = \frac{k}{x}$.

c What is the value of k?

d Rewrite the rule using the value of k.

e Sketch a graph of n against C, labelling two points on the graph.

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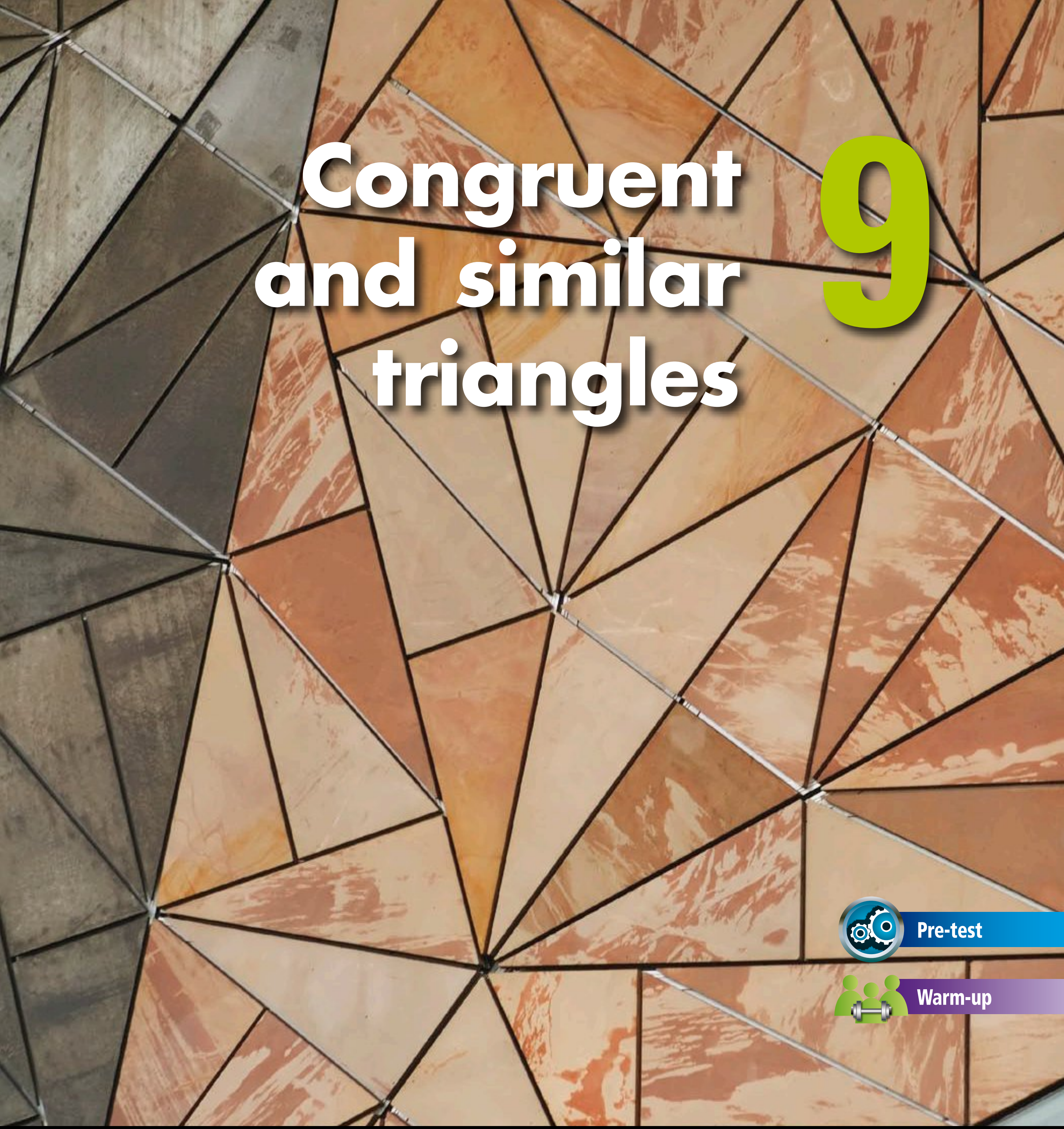
9 Congruent and similar triangles



Pre-test

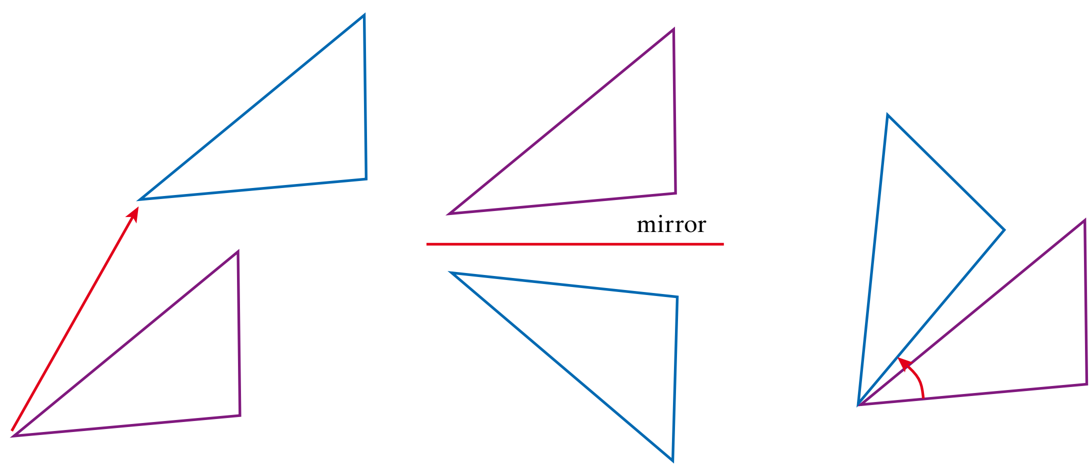
Warm-up

The photograph shows part of the outside of the buildings at Federation Square in Melbourne. The design is made up of congruent triangles. Five triangles fit together to make up a larger triangle that has exactly the same shape. Notice also how two triangles join to form a kite. A larger kite can also be seen that has exactly the same shape as the small kite. How many small triangles make up this kite? How many times longer are the sides of the large kite compared with the small kite? How does the area of the large kite compare with the area of the small kite?

9.1 Review of congruent triangles

BLM Chapter warm-up

Congruent figures have exactly the same size and shape. In Year 8 we saw that congruent figures are produced by the isometric transformations of translation, reflection and rotation. If two figures are congruent we can use one or more of the isometric transformations to move one figure exactly on top of the other.

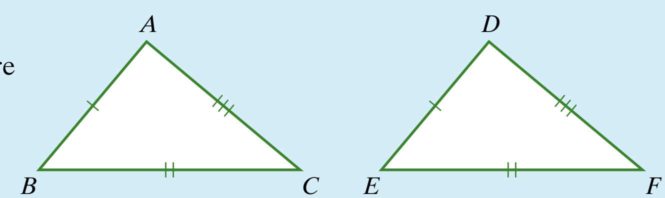


Translation Reflection Rotation

There are certain sets of information that enable us to decide if two triangles are congruent.

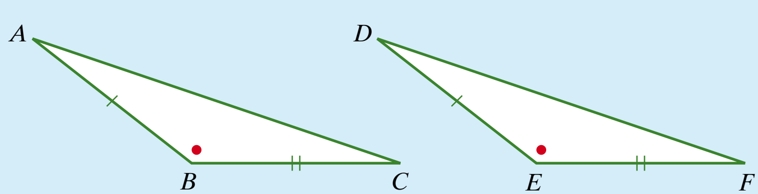
Side–Side–Side (SSS)

The three sides of one triangle are equal in length to the three sides of the other triangle.



Side–Angle–Side (SAS)

Two sides of one triangle are equal to the corresponding two sides of the other triangle, and the angles in between these two sides are equal.



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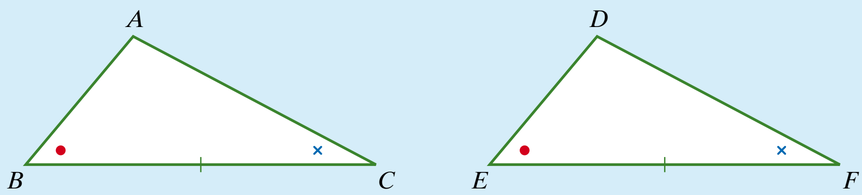
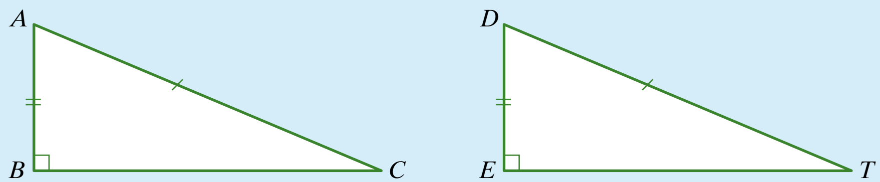
Angle–Side–Angle (ASA)

Two angles and a side of one triangle are equal to two angles and the matching side in the other triangle.



Right angle–Hypotenuse–Side (RHS)

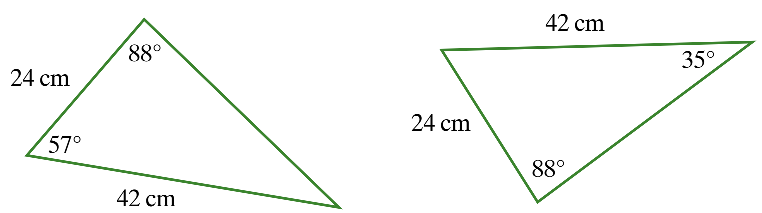
Both triangles are right-angled and the hypotenuse and another side of one triangle are equal to the hypotenuse and the matching side of the other triangle.



Example 1

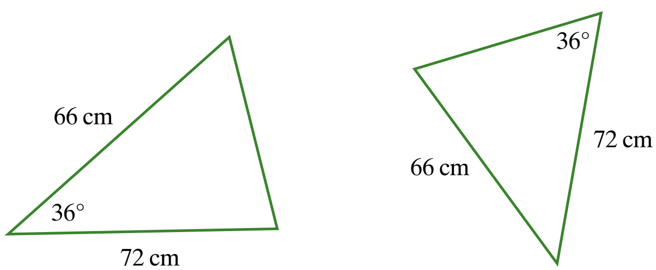
Are the following pairs of triangles congruent?

a



Triangle 1 Triangle 2

b



Triangle 1 Triangle 2

continued

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Example 1 continued

Working

a The third angle of triangle 2 is 57° as angles in a triangle add to 180°.
The triangles are congruent because two sides and the included angle of triangle 1 are equal to two sides and the included angle of triangle 2. (SAS)

Reasoning

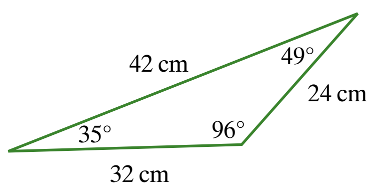
Two triangles are congruent if two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle.

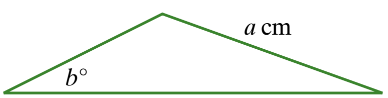
b Triangle 1 and triangle 2 are not necessarily congruent. Although they have two sides and an angle equal, the 36° angle in triangle 2 is not between the 66 cm and 72 cm sides.

Matching sides and angles must satisfy the conditions for triangle congruency.

Example 2

Find the values of the pronumerals in this pair of congruent triangles.





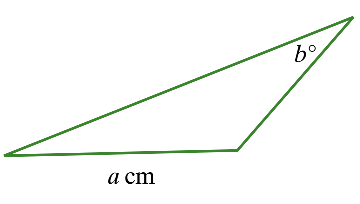
Working

$a = 32$

$b = 49$

Reasoning

Rotate the right-hand triangle so it has the same orientation as the left-hand triangle.



As we saw in Year 8, many proofs in geometry are based on showing that two triangles are congruent. We review two of these proofs here. Starting with the definition of an isosceles triangle as a triangle with two equal sides, we can use congruent triangles to prove other properties of isosceles triangles. In example **3**, a line segment AM is drawn, where M is the midpoint of the base, BC. The example demonstrates how we can then prove that AM is perpendicular to BC.

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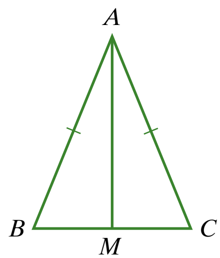
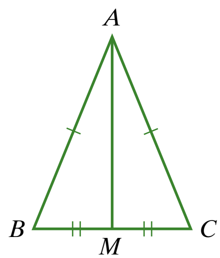
Example 3

ΔABC is an isosceles triangle. M is the midpoint of BC. Prove that $AM \perp BC$.



Working



In $\triangle AMB$ and $\triangle AMC$

1. $AB = AC$ (definition of isosceles triangle)
2. $BM = CM$ (M is midpoint of BC)
3. AM is common to both triangles.

Therefore, $\triangle AMB \equiv \triangle AMC$ (SSS)

So $\angle AMB = \angle AMC$

But $\angle AMB + \angle AMC = 180°$ (straight angle)

So $\angle AMB = \angle AMC = 90°$

So $AM \perp BC$.

Reasoning

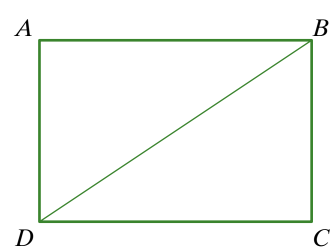
Two triangles are congruent if three sides of one triangle are equal to three sides of the other triangle.

$\equiv$ is the symbol for 'is congruent to'

If two triangles are congruent, the angles of one triangle are equal to the corresponding (matching) angles of the other triangle.

Example 4

Show that the diagonal BD of rectangle $ABCD$ divides the rectangle into two congruent triangles.



continued

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Example 4 continued

Working

In $\triangle ABD$ and $\triangle CDB$,

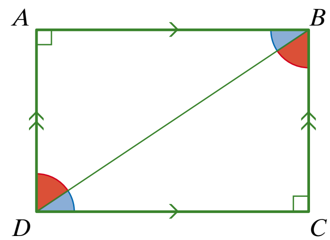
1. $\angle ABD = \angle CDB$ (alternate angles, $AB \parallel DC$)
2. $\angle ADB = \angle CBD$ (alternate angles, $AB \parallel DC$)
3. BD is shared by both triangles.

So $\triangle ABD = \triangle CDB$ (ASA)

So, the diagonal BD of rectangle $ABCD$ divides the rectangle into two congruent triangles.

Recall from that $\parallel$ is the symbol for 'is parallel to'.

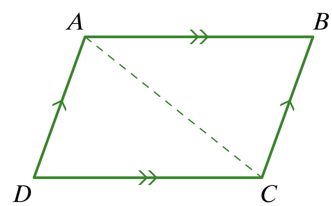
Reasoning



Example **5** shows a similar proof that the diagonals of any parallelogram divide the parallelogram into two congruent triangles.

Example 5

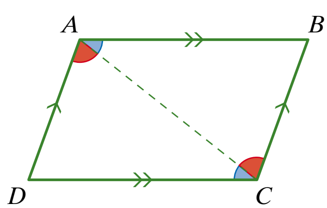
In this figure, $ABCD$ is a parallelogram and AC is a diagonal. From the definition of a parallelogram, we know that $AB \parallel DC$ and $AD \parallel BC$. Using this information and your knowledge of angles associated with parallel lines and transversals, show that $\triangle ABC$ is congruent to $\triangle CDA$. Justify each statement that you make.



Working

In $\triangle ABC$ and $\triangle CDA$,

1. AC is common to both triangles.
2. $\angle DAC = \angle BCA$ (alternate angles, $AD \parallel BC$)
3. $\angle ACD = \angle CAB$ (alternate angles, $AB \parallel DC$)



So, $\triangle ABC \equiv \triangle CDA$ (ASA)
Hence, we have shown that $\triangle ABC$ is congruent to $\triangle CDA$.

Reasoning

Look for clues: $\triangle ABC$ and $\triangle CDA$ share the side AC.

Diagonal AC is a transversal cutting across the parallel sides. This suggests that we could find equal angles in $\triangle ABC$ and $\triangle CDA$.

Two triangles are congruent if two angles of one triangle are equal to two angles of the other triangles and the sides joining the two equal angles are equal.

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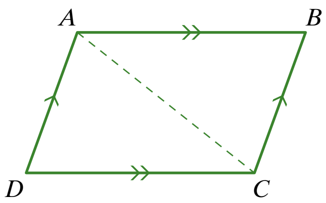
We define a parallelogram as 'a quadrilateral with opposite sides parallel'.

We can use this definition to show other properties of a parallelogram.

Example 6

Using what was shown in example 5, show that both pairs of opposite sides of a parallelogram are equal.

Working



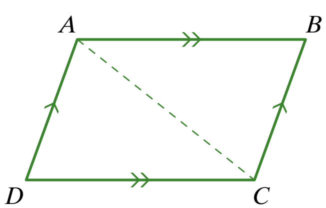
$\triangle ABC \equiv \triangle CDA$ (shown in example 5)
So,
$AB = DC$
$AD = BC$
So both pairs of opposite sides of a parallelogram are equal.

Reasoning

Matching sides of congruent triangles are equal.

Example 7

AC is a diagonal of the parallelogram $ABCD$. Prove that the opposite angles, $\angle ABC$ and $\angle CDA$ are equal.



Working

1. $\triangle ABC = \triangle CDA$
2. $\angle ABC = \angle CDA$

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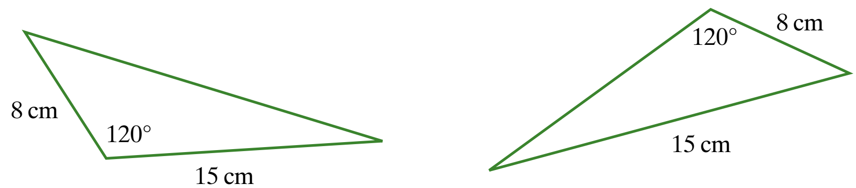
exercise 9.1

LINKS TO Example 1

1. State whether the following pairs of triangles are congruent, giving your reasons.

a



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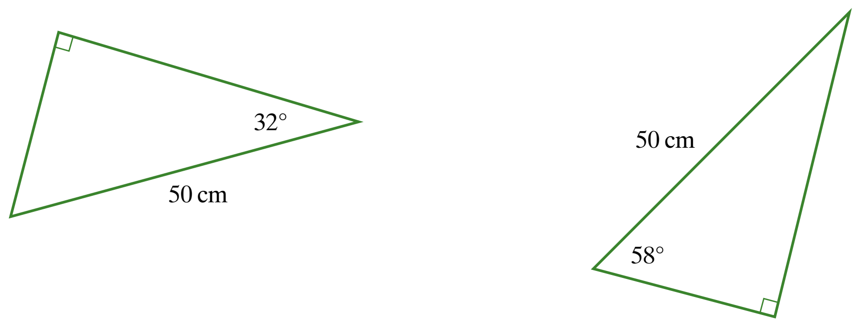
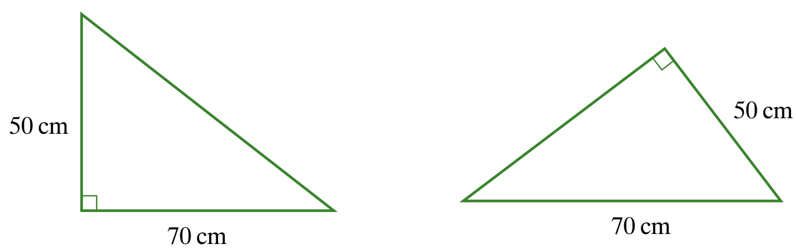


c



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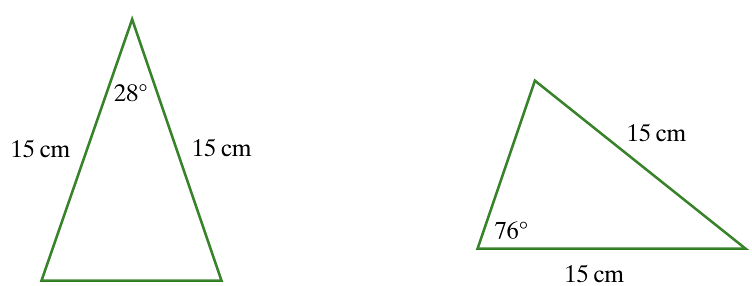
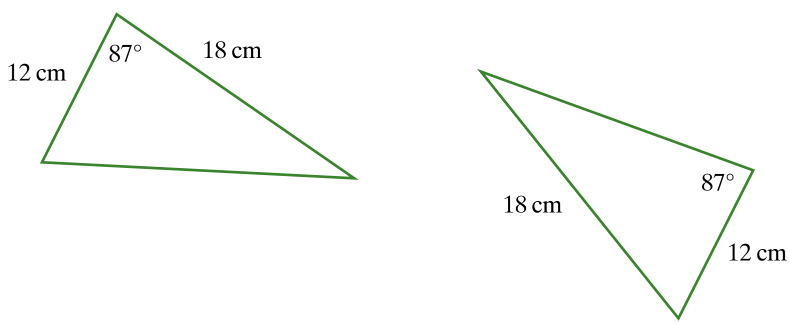


e



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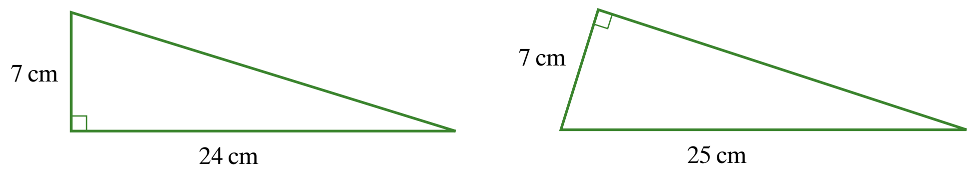

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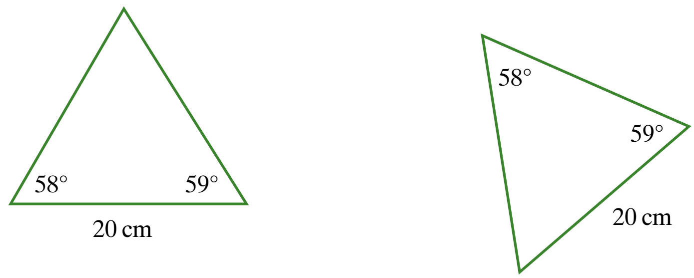

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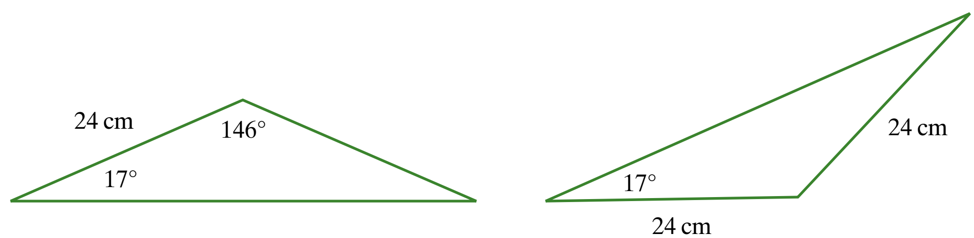
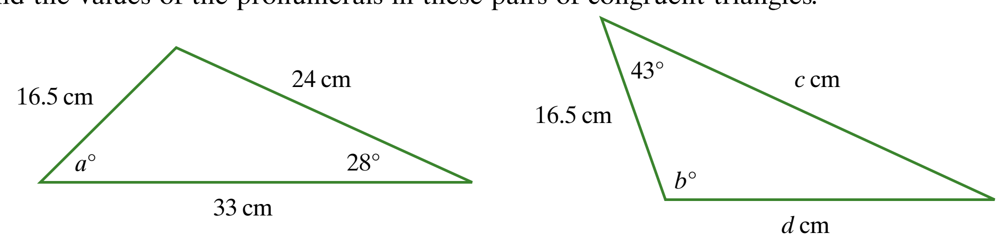


LINKS TO Example 2

Find the values of the pronumerals in these pairs of congruent triangles.

a



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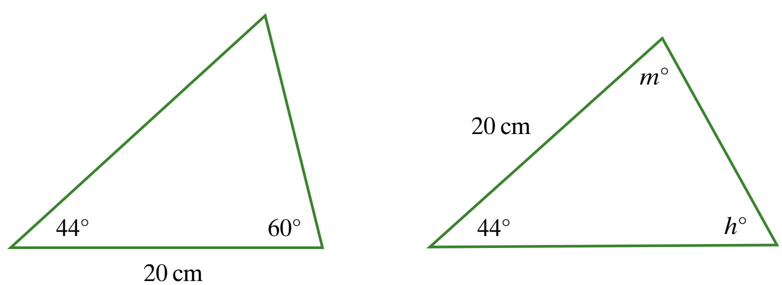

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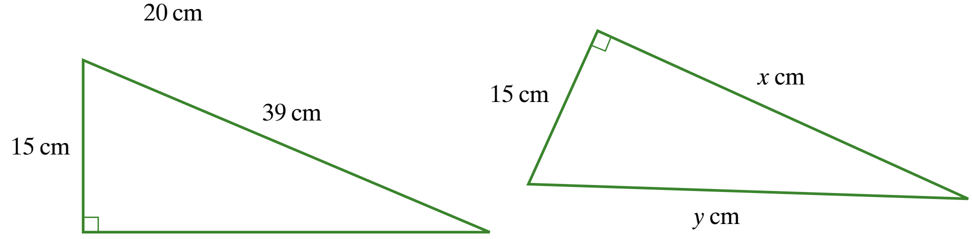

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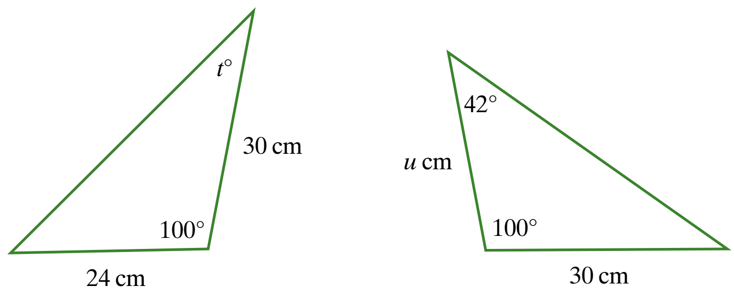
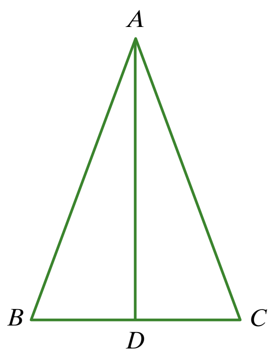


LINKS TO Example 3

$\triangle ABC$ is an isosceles triangle with $AB = AC$. AD is perpendicular to BC. Prove that AD bisects BC.

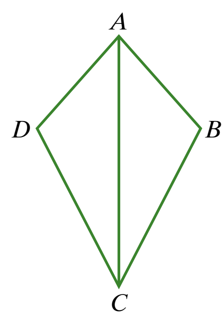


LINKS TO Example 3

$ABCD$ is a kite.

a Use your knowledge of the properties of the sides of a kite to prove that $\triangle ADC \equiv \triangle ABC$.

b Hence prove that the kite has one pair of opposite angles equal.

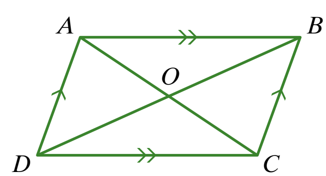


LINKS TO Example 4

In example **6** we proved that the opposite sides of a parallelogram are equal. Using this fact and your knowledge of angles associated with parallel lines,

a prove that $\triangle AOB \equiv \triangle COD$

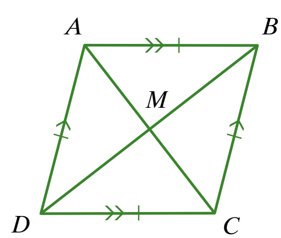
b hence prove that the diagonals AC and BD bisect each other.



Using the property you proved in question **5**,

a prove that the four triangles $\triangle AMD$, $\triangle AMB$, $\triangle CMD$ and $\triangle CMB$ are congruent.

b hence prove that the diagonals of a rhombus intersect at right angles.



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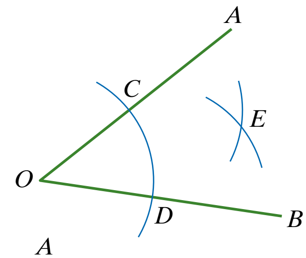
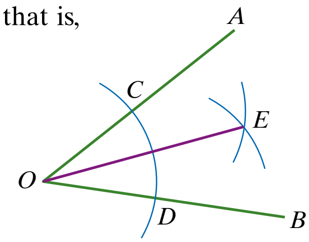
- The following steps and diagrams show one method for bisecting an angle.
  - The point of a compass is placed at O. An arc (part of a circle) is drawn so that it cuts the arms of $\angle AOB$ at C and D.
  - The compass point is then placed at C and an arc is drawn in the space between OA and OB.
  - Without changing the opening of the compass, the point is placed at D and another arc is drawn to intersect the previous arc (point E).



  - The segment OE bisects $\angle AOB$, that is, $\angle AOE = \angle EOB$.

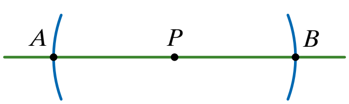


a Explain why this method works, that is, why OE bisects $\angle AOB$.

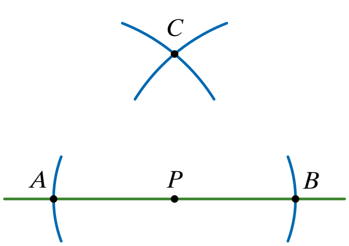
b Why is it important to use the same compass opening when drawing the arcs from C and D? That is, why must the arcs have the same radius?

exercise 9.1 challenge

- These three diagrams show a method for constructing a perpendicular line passing through a point P on a line. The compass point is placed at P and arcs are drawn to cut the line at A and B.



The compass point is then placed at A and an arc with radius greater than AP is drawn above the line. Keeping the compass at the same opening, another arc is drawn from B to intersect the previous arc (point C).



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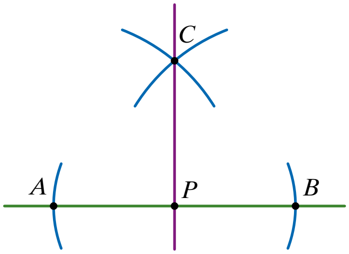
The line CP is perpendicular to AB.



a Why does the radius of the two arcs that intersect at C need to be greater than AP?

b Explain why this method works, that is, why is CP perpendicular to AB? Set out a logical explanation, justifying each statement you make.

c Why is it important to keep the compass open the same amount (that is, draw arcs with the same radius) when drawing the arcs from A and B?

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9.2 Similarity

Enlarging and reducing

In section **9.1** we noted that reflection, rotation and translation are called isometric transformations because they produce congruent shapes. Enlarging or reducing a shape is not an isometric transformation. Although the image formed has the same proportions as the original shape, it is a different size.

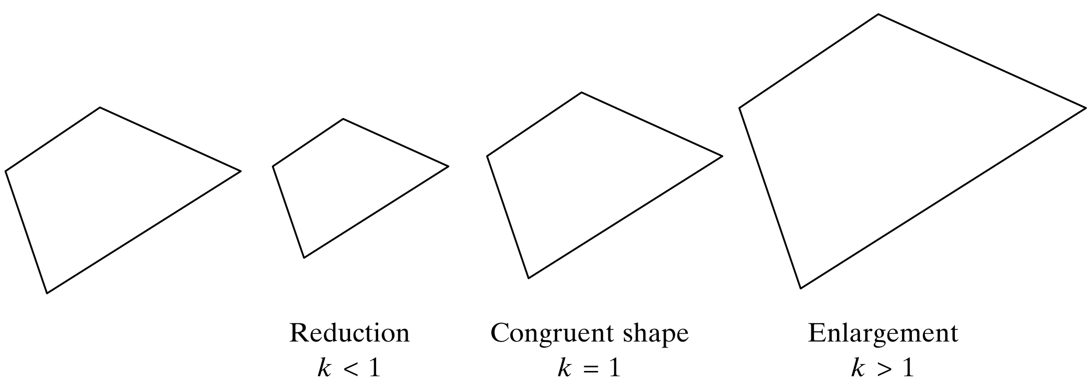
If we use a photocopier to enlarge or reduce a shape, the image produced is in proportion to the original shape. The image is like a scale drawing of the original shape. The scale factor indicates the factor by which each dimension of the shape has been enlarged or reduced.

For a scale factor k,

if $k > 1$, the figure is enlarged

if $k = 1$, a congruent figure is produced

if $k < 1$, the figure is reduced.



Shapes that are in proportion are said to be **similar**. Notice that the word *similar* has a special meaning in mathematics. In everyday language we say that things are similar if they are alike in some way. In the mathematical use of the word, there is a special way in which two geometric figures are alike.

Similar shapes

Shapes are similar only if *both* of the following conditions are satisfied:

- matching sides are in the same ratio
- matching angles are equal.

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Calculating the scale factor

To calculate the scale factor for an enlargement or reduction, we compare the height or width of the image with the same dimension for the original figure.

If two figures are similar then the scale factor for each dimension of the figures must be the same.

Scale factor

We use the pronumeral k for the scale factor.

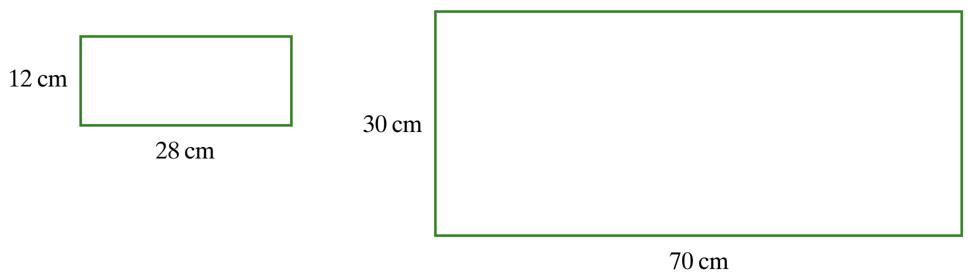
$$k = \frac{\text{length of side of large shape}}{\text{length of matching side of small shape}}$$

The value of k is greater than 1 because it shows how many times larger the larger shape is than the smaller shape.

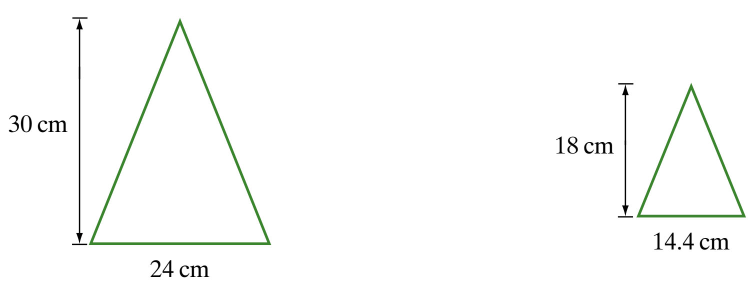
Example 8

Calculate the scale factor for these pairs of similar shapes.

a



b



continued

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Example 8 continued

Working

a Scale factor $= \dfrac{70}{28}$

$= 2.5$

Reasoning

Scale factor for enlargement

$= \dfrac{\text{width of enlarged rectangle}}{\text{width of original rectangle}}$

We could also compare the heights:

Scale factor $= \dfrac{30}{12}$

$= 2.5$

Working

b Scale factor $= \dfrac{30}{18}$

$= 1\dfrac{2}{3}$

Reasoning

Scale factor for enlargement

$= \dfrac{\text{height of enlarged triangle}}{\text{height of original triangle}}$

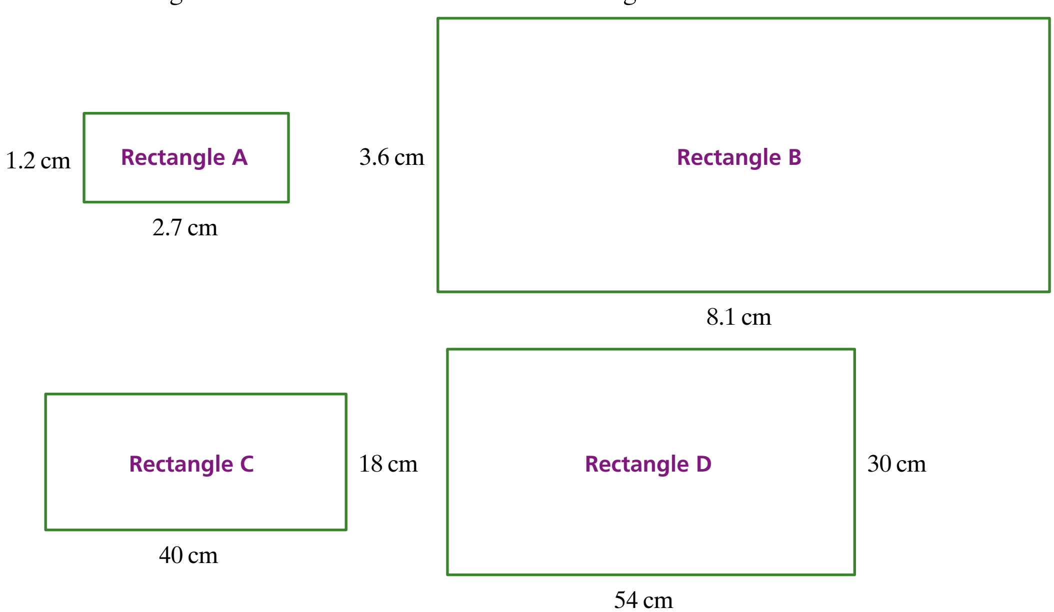
We could also compare the bases:

$\dfrac{24}{14.4} = 1.6 = 1\dfrac{2}{3}$

We have seen that similar shapes have their matching sides in the same ratio and their matching angles equal. The converse is also true. If two or more shapes have their matching sides in the same ratio and their matching angles equal then the shapes are similar.

Example 9

a Are rectangles A and B similar? **b** Are rectangles C and D similar?



continued

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Example 9 continued

Working

a Scale factor for width of rectangles A and B

$$k = \frac{\text{length of side of large shape}}{\text{length of matching side of small shape}}$$
$$= \frac{\text{width of rectangle B}}{\text{width of rectangle A}}$$
$$= \frac{8.1}{2.7}$$
$$= 3$$

Scale factor for height of rectangles A and B

$$k = \frac{\text{length of side of large shape}}{\text{length of matching side of small shape}}$$
$$= \frac{\text{height of rectangle B}}{\text{height of rectangle A}}$$
$$= \frac{3.6}{1.2}$$
$$= 3$$

The matching sides are in the same ratio.
The matching angles are equal because they are right angles.
Rectangles A and B are similar.

b $\frac{30}{18} = 1.\dot{6}$

$\frac{54}{40} = 1.35$

Rectangles C and D are not similar because their sides are not in the same ratio.

Reasoning

Similar shapes must have sides in proportion and the matching angles must be equal.

All angles are right angles.

Matching angles are equal because they are right angles, but similar shapes must also have sides in proportion.

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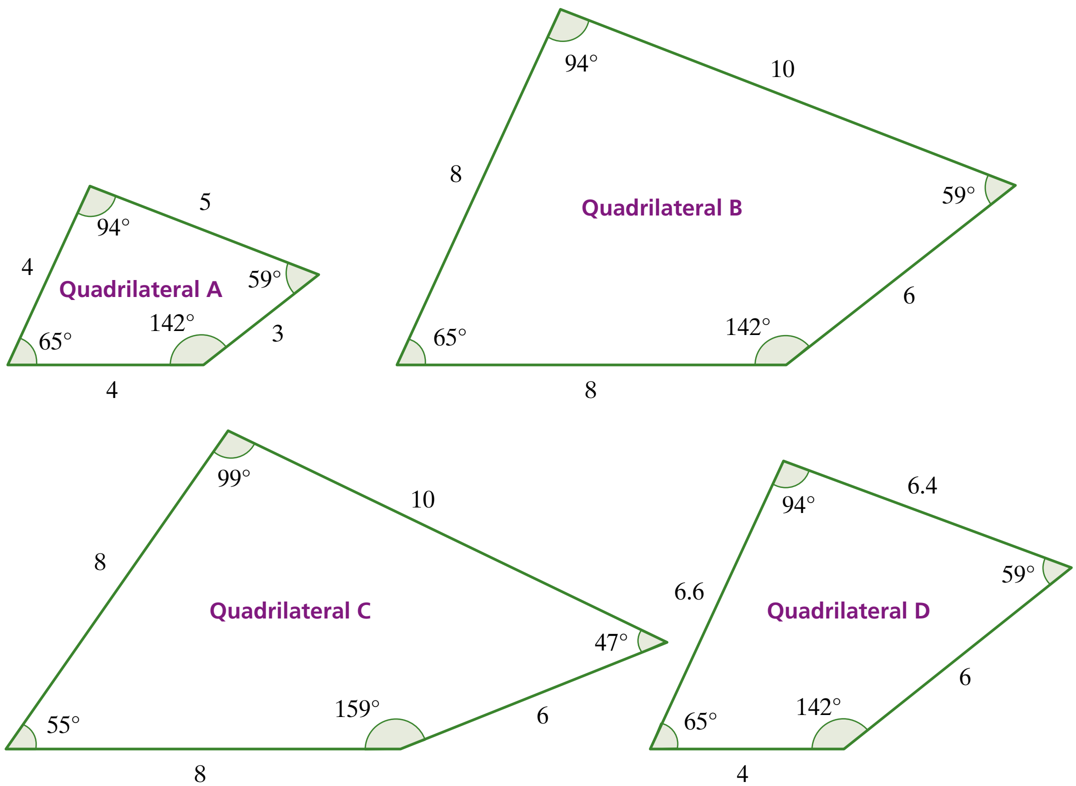
Example 10

Are quadrilaterals B, C and D similar to quadrilateral A?



Working

Quadrilaterals A and B:
The side lengths of quadrilateral B are each twice the length of the matching sides of quadrilateral A ($k = 2$).
The corresponding angles are equal.
So quadrilaterals A and B are similar.

Quadrilaterals A and C:
The side lengths of quadrilateral C are each twice the length of the matching sides of quadrilateral A ($k = 2$).
The corresponding angles are not equal.
So quadrilaterals A and C are not similar.

Quadrilaterals A and D:
The scale factor for the sides quadrilateral D and quadrilateral A is not the same for each pair of sides.
So quadrilaterals A and D are not similar.
So only quadrilateral B is similar to quadrilateral A.

Reasoning

Similar shapes must have their sides in the same proportion *and* their corresponding angles equal.

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Using the scale factor to find unknown sides

If we know that two figures are similar, we can calculate the scale factor from known side lengths then use the scale factor to find missing side lengths.

Scale factor

$$k = \frac{\text{length of side of large shape}}{\text{length of matching side of small shape}}$$

$$\text{length of side of large shape} = \text{length of matching side of small shape} \times k$$

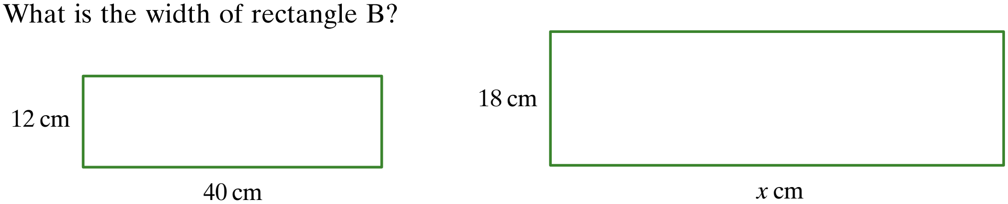
$$\text{length of matching side of small shape} = \frac{\text{length of side of large shape}}{k}$$

Example 11

Rectangles A and B are similar.

a Calculate the scale factor for the enlargement.

b What is the width of rectangle B?



Rectangle A

Rectangle B

Working

a

$$k = \frac{\text{height of rectangle B}}{\text{height of rectangle A}}$$
$$= \frac{18}{12}$$
$$= 1.5$$

Reasoning

The heights are given for both rectangles. Divide the height of rectangle by the height of rectangle A.

b Width of rectangle B:

The width of rectangle B is 60 cm.

$$\begin{aligned}\text{length of side of large shape} &= \text{length of matching} \\ &\quad \text{side of small shape} \times k \\ \text{width of rectangle B} &= \text{width of rectangle A} \times k \\ x &= 40 \times 1.5 \\ x &= 60\end{aligned}$$

Multiply the width of rectangle A by the scale factor.

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Using proportion statements to find unknown sides

A **proportion statement** is an equation showing that two ratios are equal. If we have two similar rectangles, A and B, we know that their sides are in proportion so we can write

$$\frac{\text{height of rectangle B}}{\text{height of rectangle A}} = \frac{\text{width of rectangle B}}{\text{width of rectangle A}}$$

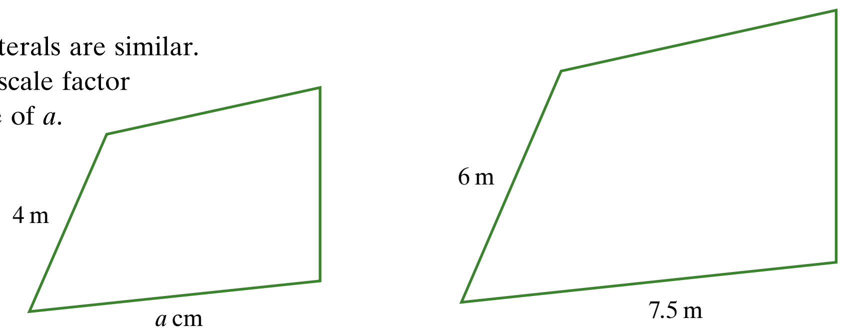
This gives us an alternative method for finding unknown side lengths in similar figures.

Example 12

The two quadrilaterals are similar.

a Calculate the scale factor

b Find the value of *a*.



Working

a $k = \frac{\text{length of side of large shape}}{\text{length of matching side of small shape}}$

$= \frac{6}{4}$

$k = 1.5$

b length of side of small shape

$= \frac{\text{length of matching side of large shape}}{k}$

$a = \frac{7.5}{1.5}$

$a = 5$

Reasoning

Use the given pair of matching sides to calculate the enlargement scale factor.

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Example 13

For rectangles A and B from example **11**,

i write a proportion statement

ii solve to find x.

Working

$$\frac{x}{40} = \frac{18}{12}$$
$$x = \frac{18}{12} \times 40$$
$$x = 60$$

Reasoning

Write the proportion statement so that the unknown is in the numerator of the fraction. This makes it easier to solve the equation.

Working

Alternatively:

$$\frac{\text{height of rectangle B}}{\text{width of rectangle B}} = \frac{\text{height of rectangle A}}{\text{width of rectangle A}}$$
$$\frac{x}{18} = \frac{40}{12}$$
$$x = \frac{40}{12} \times 18$$
$$x = 60$$

Reasoning

Alternatively, because the sides of the similar figures are in proportion, we could write the proportion statement. Again we write the proportion statement so that the unknown is in the numerator of the fraction.

exercise 9.2

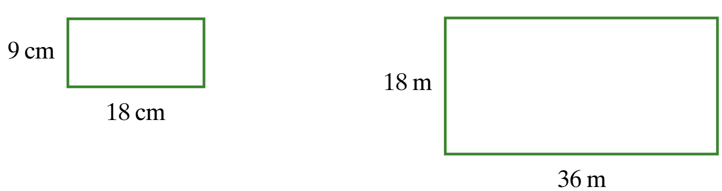
LINKS TO Example 8

For each of these pairs of similar figures,

i state whether the right hand figure is an enlargement or reduction.

ii calculate the scale factor.

a



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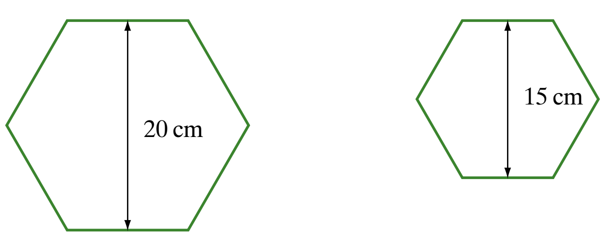

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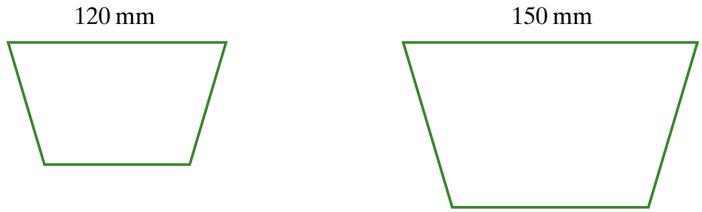
c



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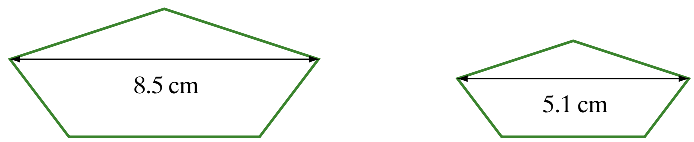
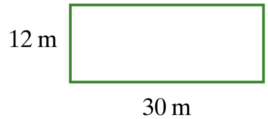


LINKS TO Example 9

Which of these rectangles are similar?

a



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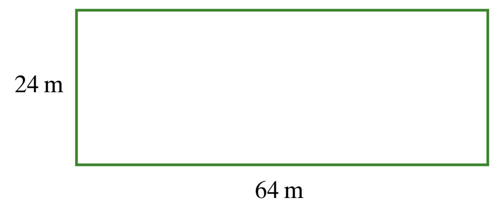

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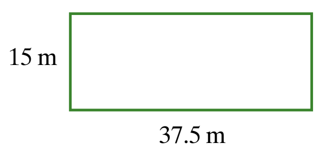
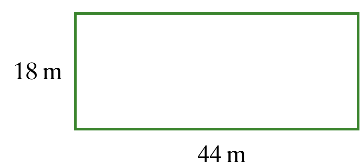
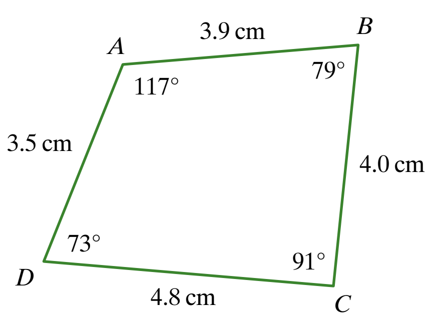


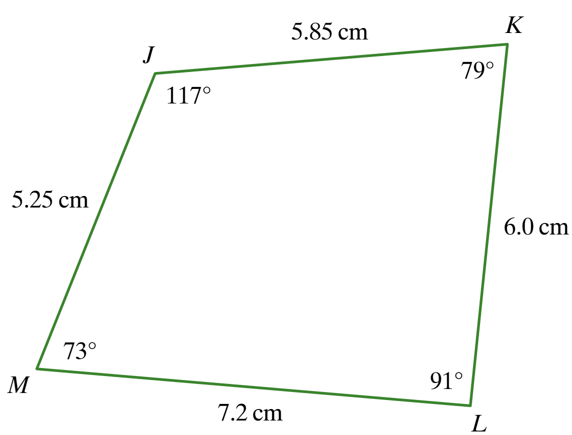
d



LINKS TO Example 10

Show that quadrilateral *ABCD* is similar to quadrilateral *JKLM*.





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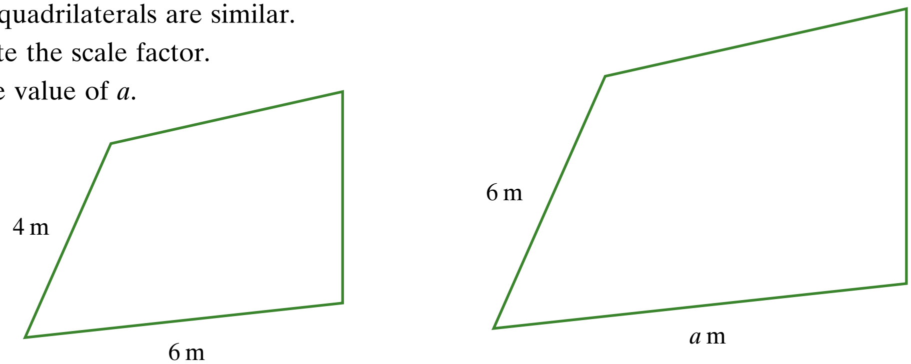
LINKS TO Examples 11, 12, 13

These two quadrilaterals are similar.

a Calculate the scale factor.

b Find the value of a.

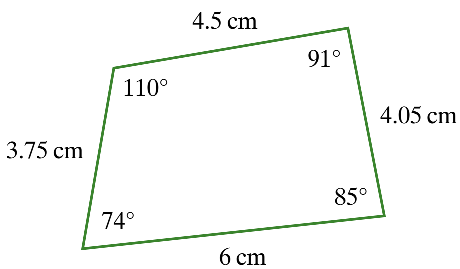


A picture measures 36 cm by 20 cm. It is copied on a photocopier so that its length and width are each 125% of the original size.

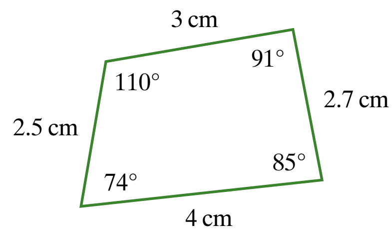
a Explain why the copy is similar to the original picture.

b What is the size of the copy?

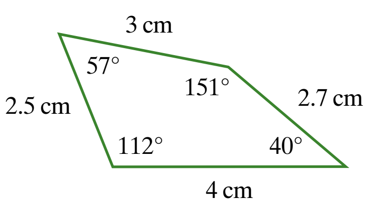
Are any of these quadrilaterals similar?



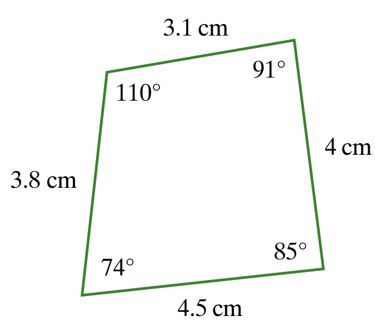
Quadrilateral A



Quadrilateral B



Quadrilateral C



Quadrilateral D

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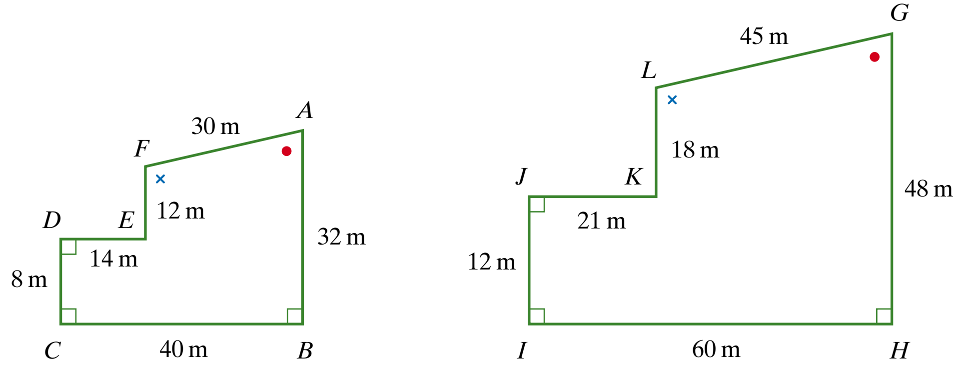
Show that these two polygons are similar.



exercise 9.2 challenge

Explain, with the aid of diagrams, whether you think each of the following statements is true or false.

a All squares are similar.

b All rectangles are similar.

c All isosceles triangles are similar.

d All equilateral triangles are similar.

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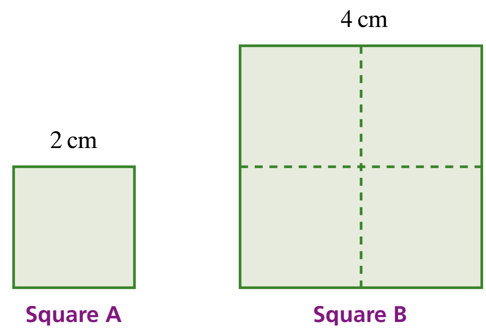
9.3 Similar figures and area

Square B is an enlargement of square A. The scale factor is 2 because the sides of square B are twice the length of the sides of square A.

Perimeter of square A = 4×2 cm = 8 cm

Perimeter of square B = 4×4 cm = 16 cm

So the perimeter is increased in the same proportion as the sides.



However, we can see that the area of square B is four times the area of square A.
Area of square A = 2^2 cm^2 = 4 cm^2
Area of square A = 4^2 cm^2 = 16 cm^2

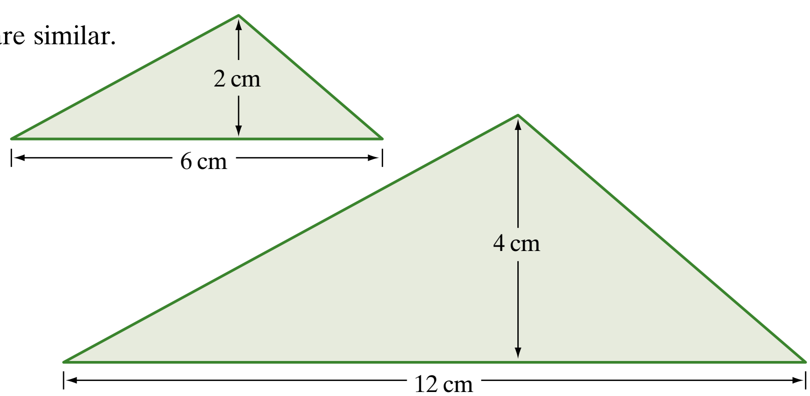
Perimeter and area of similar figures

In general, when the side lengths of a two-dimensional shape are increased by a scale factor k,

- the perimeter increases by a factor k
- the area increases by a factor k^2.

Example 14

These two triangles are similar.



a What is the scale factor?
b Calculate the area of the small triangle.
c Calculate the area of the enlarged triangle
 i using the area formula for a triangle.
 ii using the area scale factor, k^2.

continued

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Example 14 continued

Working

a $k = \dfrac{\text{length of side of large shape}}{\text{length of matching side of small shape}}$

$= \dfrac{\text{base of large triangle}}{\text{base of small triangle}}$

$= \dfrac{12}{6}$

$= 2$

Reasoning

Divide the base of the larger triangle by the base of the smaller triangle.

b Area of small triangle $= \dfrac{1}{2}bh$

$= \dfrac{1}{2} \times 6 \times 2$

$= 6 \text{ cm}^2$

Area of a triangle equals half the product of the base and the perpendicular height.

c i Area of enlarged triangle $= \dfrac{1}{2}bh$

$= \dfrac{1}{2} \times 12 \times 4$

$= 24 \text{ cm}^2$

ii $\dfrac{\text{area of large triangle}}{\text{area of small triangle}} = k^2$

$\dfrac{\text{area of large triangle}}{6} = 2^2$

area of large triangle $= 4 \times 6$

$= 24 \text{ cm}^2$

The scale factor for the sides is 2 but the area has increased by 4 times.

If we know the scale factor, k, for the side lengths of two similar shapes and the area of one of the shapes, we can use the area scale factor k^2, to find the area of the other shape.

Example 15

A polygon is enlarged by a factor of 1.5 to produce a similar polygon. How many times larger is the area?

Working

Area enlargement

$= k^2$

$= 1.5^2$

$= 2.25$

The area is 2.25 times larger.

Reasoning

When the side lengths of a figure are enlarged by a factor k the area increases by a factor k^2.

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Example 16

The sides of two similar figures are in the ratio 2 : 5.

a Find the scale factor, k.

b What is the area scale factor, k^2?

c If the area of the smaller figure is 24 cm^2, what is the area of the larger figure?

Working	Reasoning
a $k = \dfrac{\text{length of side of large shape}}{\text{length of matching side of small shape}} = \dfrac{5}{2} = 2.5$	The ratio of side lengths can be used to calculate k.
b $k^2 = \left(\dfrac{5}{2}\right)^2 = \dfrac{25}{4} = 6.25$	The ratio of areas is 4 : 25.
c Area larger figure $= k^2 \times$ area of smaller figure $= \dfrac{25}{4} \times 24 = 150\text{ cm}^2$	Multiply by the area scale factor.

exercise 9.3

LINKS TO Example 14

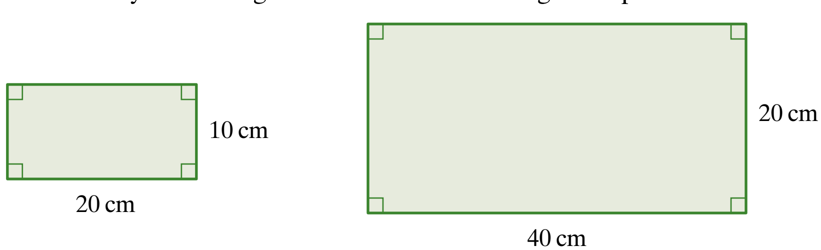
1 For each of these pairs of similar shapes answer the following questions.

i What is the scale factor?

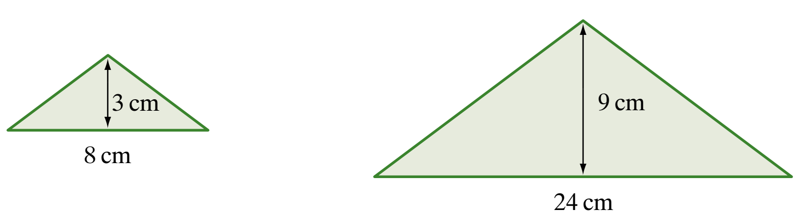
ii What is the area of each shape?

iii How many times larger is the area of the larger shape?

a



b



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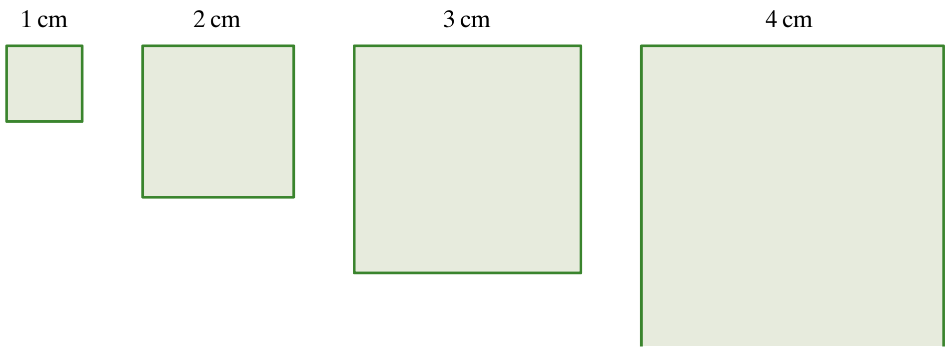
LINKS TO Example 15

Consider the squares below.

a Knowing that the area of the small square is 1 cm^2, use the area scale factor k^2 to calculate the areas of the other squares.



b Complete the following sentences:

When the length of the side of a square is doubled, the area is multiplied by ______.

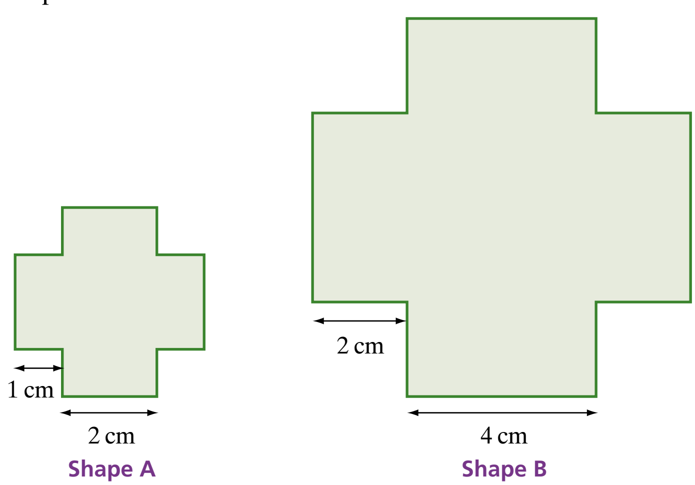
When the length of the side of a square is multiplied by 3, the area is multiplied by ______.

The area of a square with sides of length 5.6 cm would be ______ times the area of a square with sides of length 2.8 cm.

The area of a square with sides of length 7.2 cm would be ______ times the area of a square with sides of length 2.4 cm.

LINKS TO Example 16

Shapes A and B are similar.



a State the scale factor, k.

b Calculate the perimeter of shape A.

c Calculate the perimeter of shape B

i by adding the side lengths.

ii by using the length scale factor, k.

d Calculate the area of shape A.

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c Calculate the area of shape B

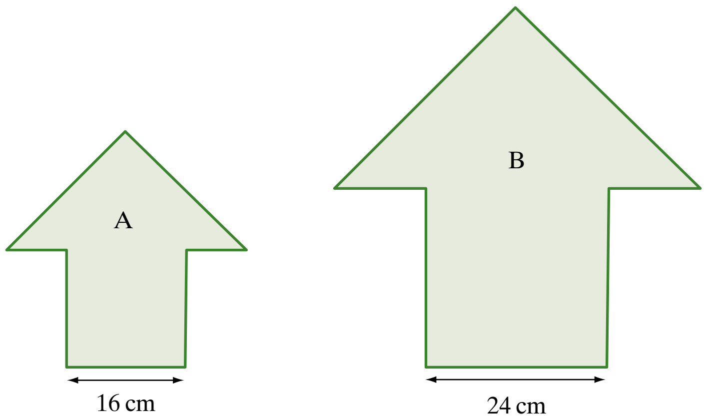
i using the same method as in part **d**.

ii by using the area scale factor, k^2.

The two arrows, A and B, are similar.

a What is the scale factor, k?

b How many times larger is the area of arrow B than the area of arrow A?



LINKS TO Example 16

The sides of two similar triangles are in the ratio 5 : 6.

a What is the scale factor, k?

b What is the ratio of their areas?

c If the area of the smaller triangle is 50 cm^2, what is the area of the larger triangle?

exercise 9.3 challenge

Two similar polygons have areas of 64 cm^2 and 169 cm^2.

a What is the ratio of their side lengths?

b If a side of the smaller polygon is 20 cm, what is the length of the corresponding side of the larger polygon?

Two similar polygons have their areas in the ratio 5 : 8. If a side length in the smaller polygon is 10 cm what is the length of the corresponding side in the larger polygon? Give the length

a as an exact value.

b correct to one decimal place.

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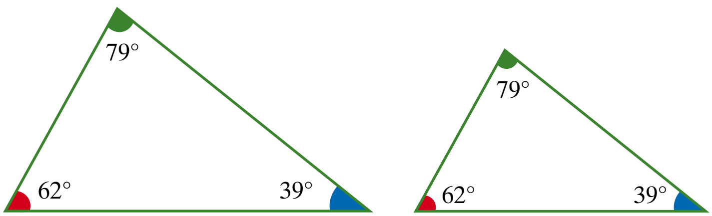
9.4 Similar triangles

Section **9.1** reviewed the sets of conditions for triangles to be congruent. We will now look at the conditions for triangles to be similar.

Three angles

Given the three angles of a triangle, the shape of the triangle is determined, but not its size. This means that two triangles that have the same set of angle sizes will always be similar triangles. In fact, we need only know that two angles of one triangle are equal to two angles of the other triangle to be able to say that the triangles are similar because we know that the three angles must add to 180°.



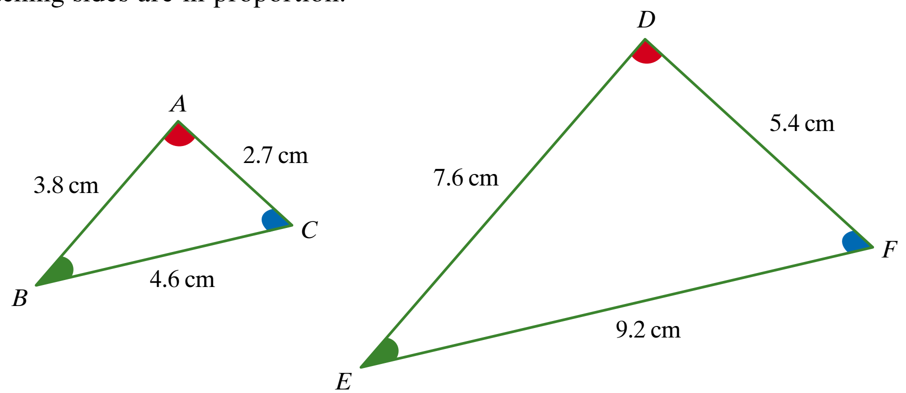
If the three angles of one triangle are equal to the three angles of another triangle, the triangles are similar.

Three sides

We can see that triangles ABC and DEF have the same shape, although their sizes are different.

$\frac{DE}{AB} = \frac{7.6}{3.8} = 2$ $\qquad \frac{EF}{BC} = \frac{9.2}{4.6} = 2$ $\qquad \frac{DF}{AC} = \frac{5.4}{2.7} = 2$

The matching sides are in proportion.



If the three sides of one triangle are proportional to the three sides of another triangle, the triangles are similar.

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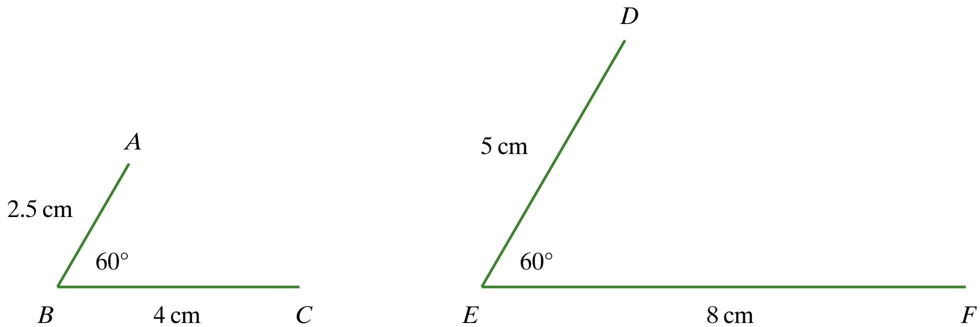
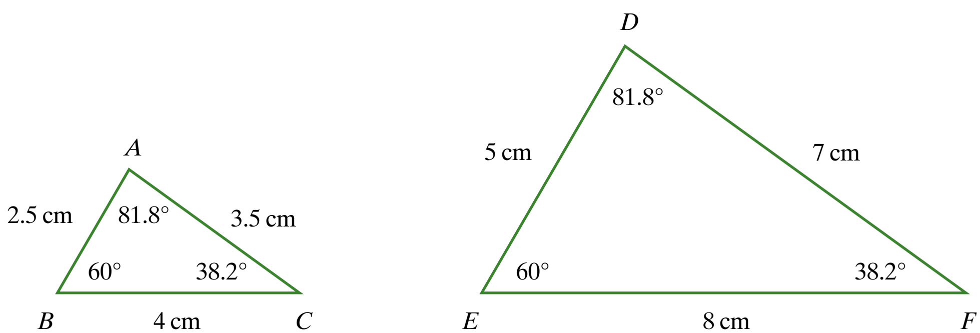
Two sides and the included angle

In the figures below, $\dfrac{DE}{AB} = \dfrac{5}{2.5} = 2$ and $\dfrac{EF}{BC} = \dfrac{8}{4} = 2$. Therefore, the two sets of line segments are **in proportion**. In each case, the angle between the two segments (the **included angle**) is equal to 60°.



If we draw the segments AC and DF, we find that they are also in proportion:
$\dfrac{DF}{AC} = \dfrac{7}{3.5} = 2$. The three angles of $\triangle ABC$ are equal to the three angles of $\triangle DEF$.



So $\triangle ABC$ is similar to $\triangle DEF$.

If two sides of one triangle are proportional to two sides of another triangle and the angles between those sides are equal, the triangles are similar.

Right-angled triangles: hypotenuse and another side

The side lengths of right-angled triangles have a special relationship with each other, as we know from Pythagoras' theorem. For right-angled triangles, we need know only that two matching sides are in proportion to be able to say that the triangles are similar.

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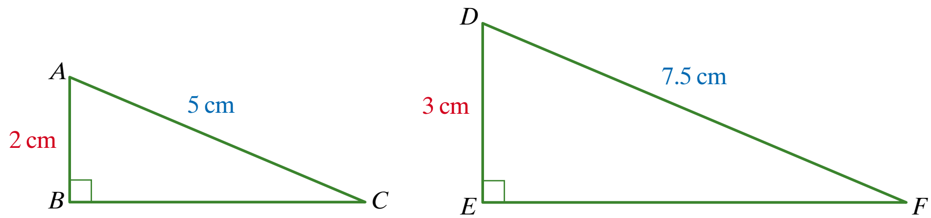

If two right-angled triangles have the hypotenuse and one other side of each in proportion then they will be similar.

Comparing the conditions for congruent and similar triangles

Congruent triangles	Similar triangles
Congruent triangles must satisfy both of these conditions. ■ Matching angles must be equal. ■ Matching sides must be *equal in length*.	Similar triangles must satisfy both of these conditions. ■ Matching angles must be equal. ■ Matching sides must be *in the same proportion*.
Conditions for congruent triangles	**Conditions for similar triangles**
■ SSS (three sides of one triangle equal to three sides of the other)	■ SSS (three sides of one triangle in proportion to three sides of the other)
■ AAS (two angles of one triangle equal to the two angles of the other triangle and a matching side equal) Note: two angles equal means that three angles are equal.	■ AAA (three angles of one triangle equal to three angles of the other)
■ SAS (two sides of one triangle equal to two sides of the other triangle and the angle between them the same)	■ SAS (two sides of one triangle in proportion to two sides of the other triangle and the angle between them the same)
■ RHS (both triangles right-angled, hypotenuse and another side of one triangle equal to the hypotenuse and another side of the other triangle)	■ RHS (both triangles right-angled, hypotenuse and another side of one triangle in proportion to the hypotenuse and another side of the other triangle)

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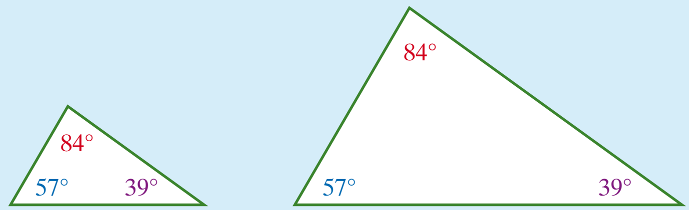
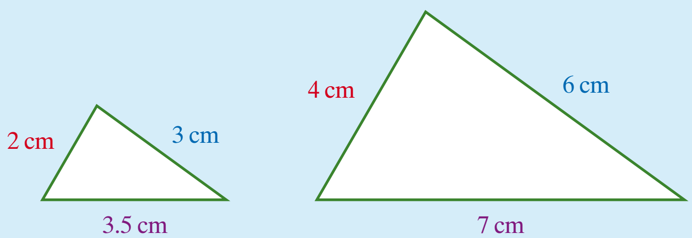
Conditions for triangles to be similar

AAA The three angles of one triangle are equal to the three angles of the other triangle. For example

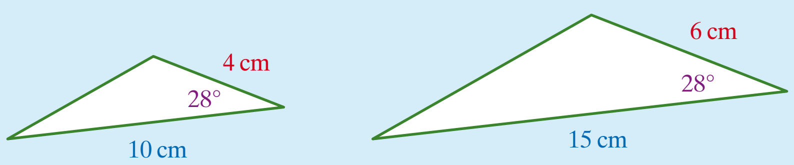


(In fact we only need to know that two angles of one triangle are equal to two angles of the other triangle.)

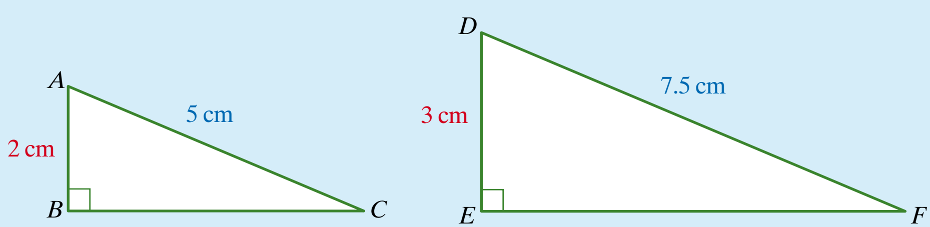
SSS Three sides of one triangle are in the same ratio as (that is, are proportional to) the three sides of the other triangle. For example



SAS Two sides of one triangle are in the same ratio as (that is, are proportional to) two sides of another triangle, and the included angles are equal. For example



RHS Both triangles are right-angled and the hypotenuse and another side of one triangle are in proportion to the hypotenuse and another side of the other triangle.



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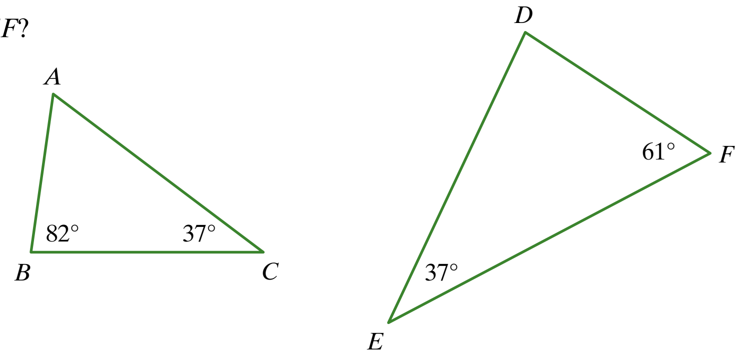
Example 17

Is $\triangle ABC$ similar to $\triangle DEF$?



Working

$82° + 37° = 119°$
$\angle BAC = 180° - 119°$
$= 61°$

$61° + 37° = 98°$
$\angle EDF = 180° - 98°$
$= 82°$

The angle sizes of the two triangles are the same. Each triangle has angles of 82°, 61° and 37°.

So $\triangle ABC$ is similar to $\triangle DEF$ (AAA).

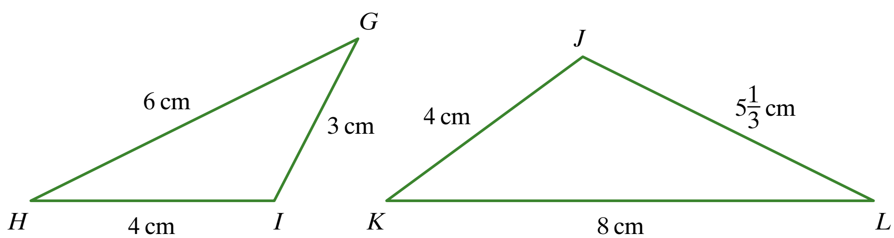
Reasoning

The sum of the angles of a triangle is 180°.

Two triangles are similar if the sizes of the three angles of one triangle are the same as the sizes of the three angles of the other triangle.

Example 18

Are these two triangles similar?



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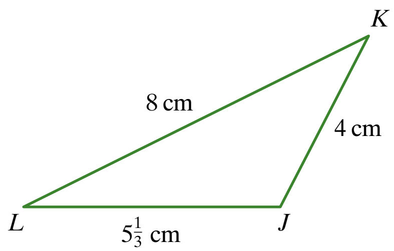
Example 18 continued

Working



Scale factor

$$= \frac{\text{length of side of second triangle}}{\text{length of corresponding side of first triangle}}$$

$$\begin{aligned}\frac{KL}{GH} &= \frac{8}{6}\\ &= \frac{4}{3}\\ &= 1\frac{1}{3}\end{aligned}$$

$$\begin{aligned}\frac{LJ}{HI} &= \frac{5\frac{1}{3}}{4}\\ &= \frac{16}{3} \div \frac{4}{1}\\ &= \frac{4}{3}\\ &= 1\frac{1}{3}\end{aligned}$$

$$\begin{aligned}\frac{KJ}{GI} &= \frac{4}{3}\\ &= 1\frac{1}{3}\end{aligned}$$

The scale factors for each of the pairs of sides of the triangles are the same so the sides of the triangles are in proportion. So $\triangle GHI$ and $\triangle KLJ$ are similar (SSS).

Reasoning

Draw $\triangle KLJ$ in the same position as $\triangle GHI$, that is, with the obtuse angles at I and J in the same positions.
Side GH corresponds with side KL (longest sides).
Side HI corresponds with side LJ.
Side GI corresponds with side KJ (shortest sides).

If the scale factors are the same, the matching sides are all in proportion.

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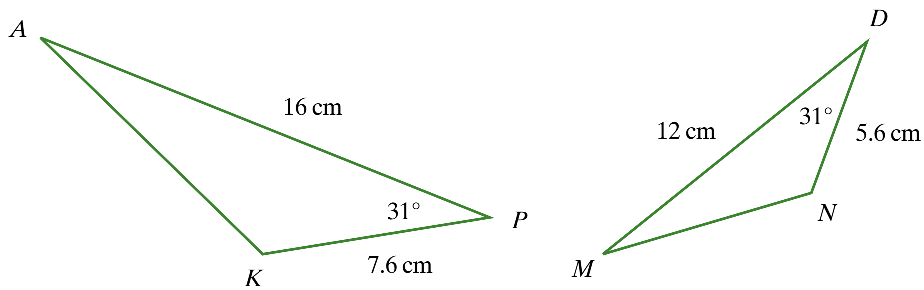
Example 19

Are the following triangles similar?



Working	Reasoning
$\frac{MD}{AP} = \frac{12}{16}$ $= 0.75$	Side MD of $\triangle MDN$ corresponds to side AP of $\triangle APK$.
$\frac{DN}{PK} = \frac{5.7}{7.6}$ $= 0.75$	Side DN of $\triangle MDN$ corresponds to side PK of $\triangle APK$.
$\angle APK = \angle MDN = 31°$	Angles between these sides are equal.
$\triangle APK$ is similar to $\triangle MDN$ (SAS).	One of the conditions for similarity of triangles is that two pairs of sides are in the same ratio and the included angles are equal.

In solving similar triangles problems we can use either of two methods:

1 Scale factor method
- Use a pair of matching sides to calculate the scale factor.
- Use the scale factor to calculate an unknown side length.

2 Proportion method
- Use two pairs of matching sides to write a proportion equation involving the unknown side length.
- Solve to find the unknown length.

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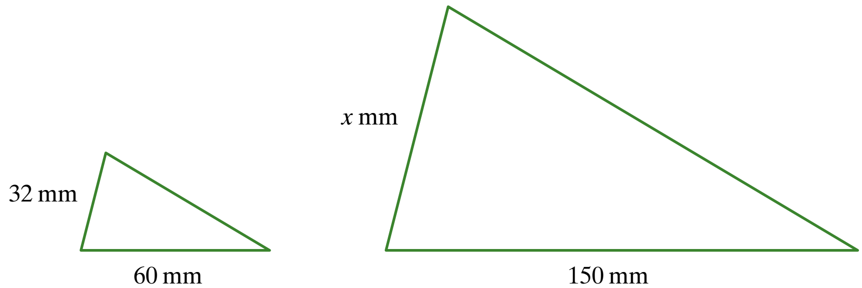
Example 20

The two triangles below are similar. Find the value of x.



Working

Scale factor method:

$$\text{Scale factor} = \frac{\text{Length of side of second triangle}}{\text{Length of corresponding side of first triangle}}$$

$$\begin{aligned}\text{Scale factor} &= \frac{150}{60}\\ &= 2.5\end{aligned}$$

$$\begin{aligned}x &= 2.5 \times 32\\ &= 80\end{aligned}$$

Proportion method:

$$\frac{x}{32} = \frac{150}{60}$$

$$x = \frac{\cancel{150}^{5}}{\cancel{60}_{2}} \times 32$$

$$\begin{aligned}x &= \frac{5 \times 32}{2}\\ &= 80\end{aligned}$$

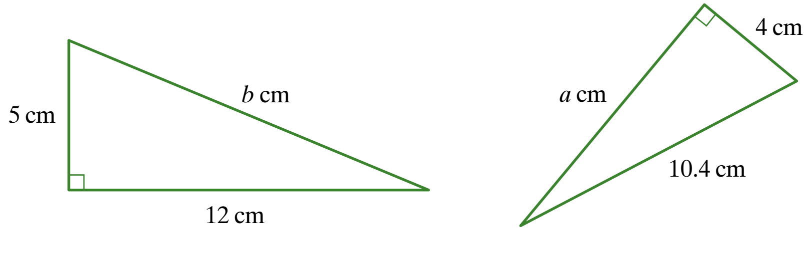
Reasoning

Similar triangles have their sides in the same ratio. The scale factor is 2.5, that is, the lengths of the sides of the second triangle are 2.5 times the lengths of the sides of the first triangle.

Matching sides of similar figures are in the same ratio.

Example 21

The two triangles below are similar. Find the values of a and b.



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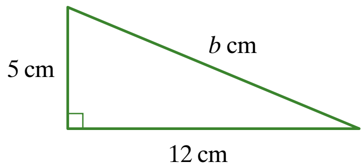
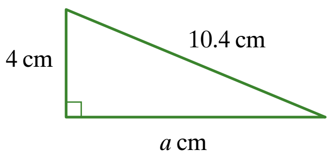
Example 21 continued

Working	Reasoning
	Draw triangles in the same orientation. When triangles are in the same orientation it is easier to match the sides.
	
Scale factor method:	
Scale factor = $\dfrac{\text{Length of side of second triangle}}{\text{Length of corresponding side of first triangle}}$	
Scale factor $= \dfrac{4}{5}$ $= 0.8$	Similar triangles have their sides in the same ratio. The scale factor is 0.8, that is, the lengths of the sides of the second triangle are 0.8 times the lengths of the sides of the first triangle.
$a = 12 \times 0.8$ $a = 9.6$	
$b = 10.4 \div 0.8$ $= 13$	To find the length of a side in the first triangle, we must divide by the scale factor. Check: b is in the larger triangle, so b must be greater than 10.4.
Proportion method:	
$\dfrac{a}{12} = \dfrac{4}{5}$ $a = \dfrac{4}{5} \times 12$ $= \dfrac{48}{5}$ $= 9.6$	Matching sides of similar figures are in the same proportion.
$\dfrac{b}{10.4} = \dfrac{5}{4}$ $b = \dfrac{5}{4} \times 10.4$ $= \dfrac{52}{4}$ $= 13$	Matching sides of similar figures are in proportion.

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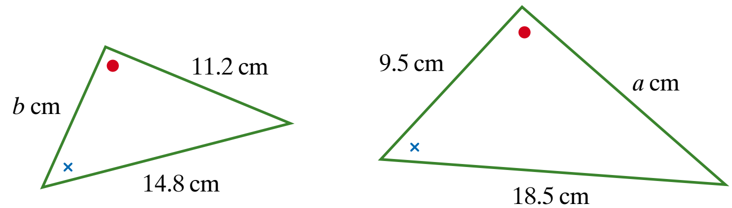
Example 22

The following triangles have two pairs of equal angles as shown. Find the values of the pronumerals a and b.



Working

Scale factor method:

$$\text{Scale factor} = \frac{\text{Length of side of second triangle}}{\text{Length of corresponding side of first triangle}}$$

$$\text{Scale factor} = \frac{18.5}{14.8}$$
$$= 1.25$$

$$a = 11.2 \times 1.25$$
$$= 14.0$$

$$b = \frac{9.5}{1.25}$$
$$= 7.6$$

Reasoning

Match the sides:

$a \leftrightarrow 11.2$

$9.5 \leftrightarrow b$

$18.4 \leftrightarrow 14.8$

Calculate the scale factor.
Use the scale factor to find the unknown side lengths.

Proportion method:

$$\frac{a}{11.2} = \frac{18.5}{14.8}$$

$$a = \frac{18.5}{14.8} \times 11.2$$
$$= 14.0$$

$$\frac{b}{9.5} = \frac{14.8}{18.5}$$

$$b = \frac{14.8}{18.5} \times 9.5$$
$$= 7.6$$

Match the sides:

$a \leftrightarrow 11.2$

$9.5 \leftrightarrow b$

$18.4 \leftrightarrow 14.8$

Write a proportion equation with the pronumeral in the numerator.
Solve the equation.

Shadows can be used to find unknown heights of objects such as trees and poles. At a given time the sun's rays striking the top of nearby objects make the same angle with the ground. If we have two nearby vertical objects we can identify similar right-angled triangles. In each triangle the hypotenuse represents the sun's rays. The vertical side of the triangle represents the object and the horizontal side of the triangle represents the shadow of the object on the ground.

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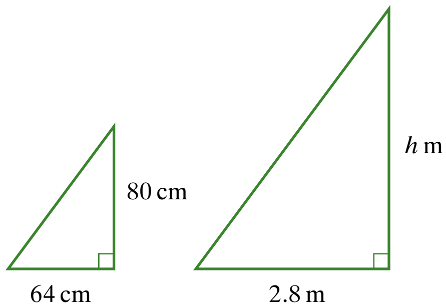
Example 23

A stick which is 80 cm high has a shadow 64 cm long. At the same time, the shadow from a pole is 2.8 m long.

a Draw two triangles to show the information.

b Find the height of the pole.

Working	Reasoning
a	



Working	Reasoning
b Scale factor method:	Lengths must be in the same unit. The unknown height is in metres. Convert 80 cm to 0.8 m. Convert 64 cm to 0.64 m.
$\text{Scale factor} = \dfrac{\text{length of large triangle}}{\text{length of matching side of small triangle}}$	
$k = \dfrac{2.8}{0.64}$ $= 4.375$	
$h = 0.8 \times 4.375$ $= 3.5$	Calculate the scale factor, k. Use the scale factor to find the unknown side length.
The height of the pole is 3.5 m.	
c Proportion method:	Lengths must be in the same unit. The unknown height is in metres. Convert 80 cm to 0.8 m. Convert 64 cm to 0.64 m.
$\dfrac{h}{0.8} = \dfrac{2.8}{0.64}$	
$h = \dfrac{2.8}{0.64} \times 80$	
$h = 3.5$	Write a proportion equation with the pronumeral in the numerator. Solve the equation.
The height of the pole is 3.5 m	

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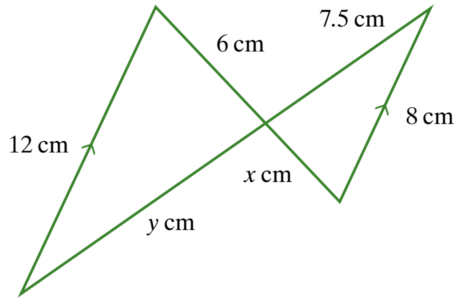
Example 24

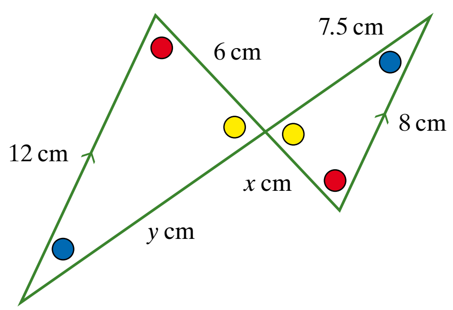
Use the information in the diagram to answer the questions.



a Explain why the two triangles are similar.
b Find the values of x and y.

Working

a



The two red angles are equal (alternate angles between parallel lines).
The two blue angles are equal (alternate angles between parallel lines).
The two yellow angles are equal (vertically opposite).
The two triangles are similar (AAA).

b **Scale factor method:**

$$\text{Scale factor} = \frac{\text{length of side of large triangle}}{\text{length of matching side of small triangle}}$$
$$= \frac{12}{8}$$
$$= 1.5$$

length of side of small triangle
= length of matching side of large triangle ÷ k

$x = 6 \div 1.5$

$x = 4$

Reasoning

Use the clue that there are two parallel line segments to look for angles that are equal.

We actually need only to show that there are two pairs of angles equal. We automatically know then that the third pair will be equal.

Use the given information for two matching sides to calculate the scale factor.

The unknown length, x cm, is in the small triangle.

continued

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Example 24 continued

Working

length of side of large triangle

= length of matching side of small triangle × k

$$x = 7.5 \times 1.5$$

$$x = 11.25$$

Reasoning

The unknown length, y cm, is in the large triangle.

c **Proportion method:**

$$\frac{x}{6} = \frac{8}{12}$$

$$x = \frac{8}{12} \times 6$$

$$x = 4$$

Write a proportion equation using the matching known sides, the unknown side length, x cm, and the length of the matching side in the large triangle.

Solve for x.

$$\frac{y}{7.5} = \frac{12}{8}$$

$$7 = \frac{12}{8} \times 7.5$$

$$y = 11.25$$

Write a proportion equation using the matching known sides, the unknown side length, y cm, and the length of the matching side in the small triangle.

Solve for y.

exercise 9.4

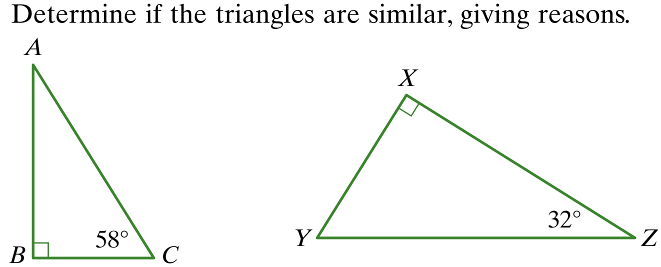
Consider each pair of triangles.

i Redraw one triangle so it is in the same orientation as the other.

ii Determine if the triangles are similar, giving reasons.

LINKS TO Example 17

a



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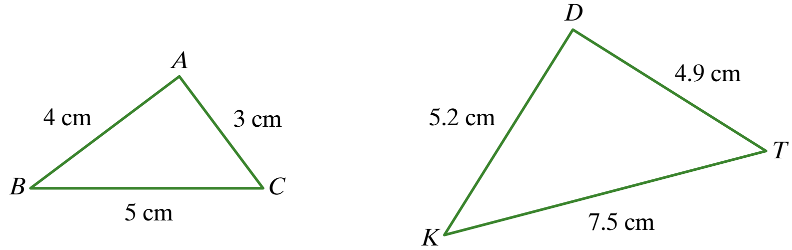
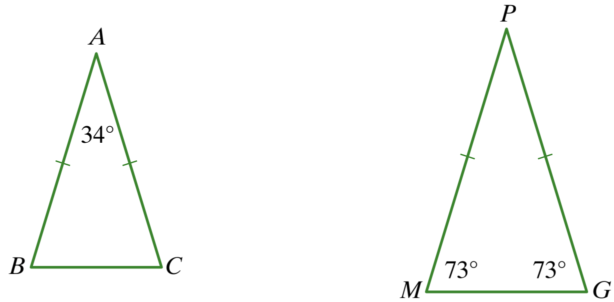
LINKS TO Example 18

b

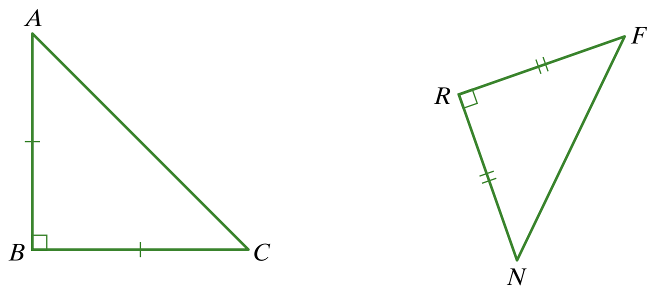


LINKS TO Example 19

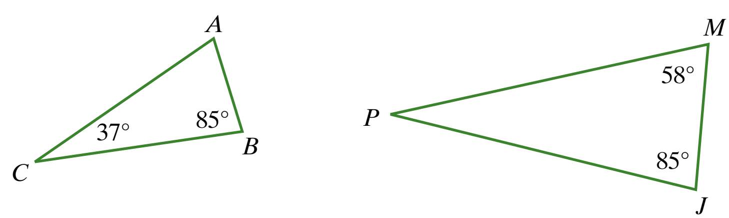
c



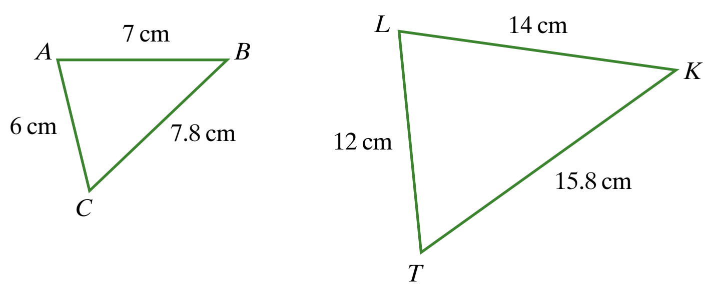
d



e



f



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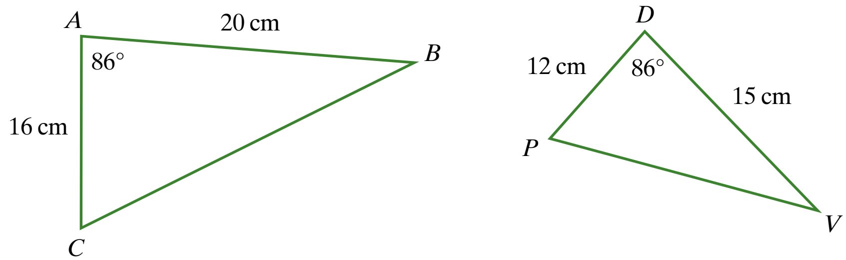
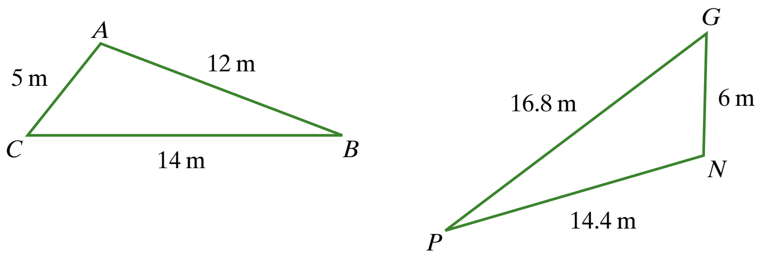
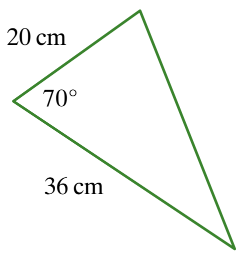
g



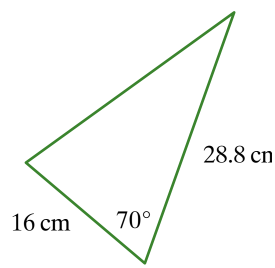
h



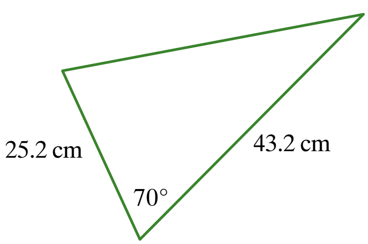
Which of these triangles are similar?



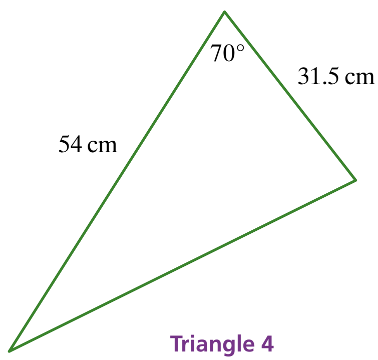
Triangle 1



Triangle 2



Triangle 3



Triangle 4

A 1 and 2 **B** 1 and 3 **C** 1 and 4 **D** 1, 2 and 3 **E** 1, 3 and 4

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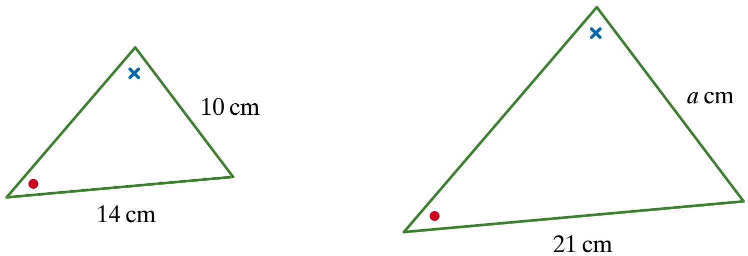
LINKS TO Example 20

The triangles in each pair are similar. The matching angles that are equal are marked. For each pair of triangles,

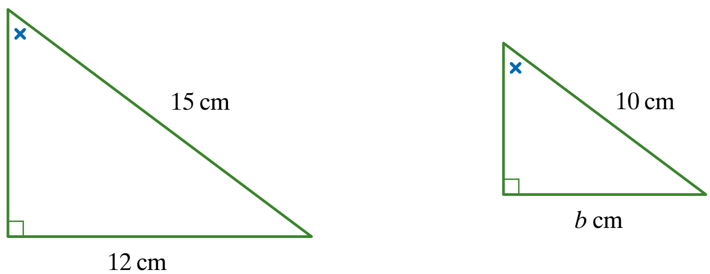
i calculate the scale factor.

ii find the value of the pronumeral.

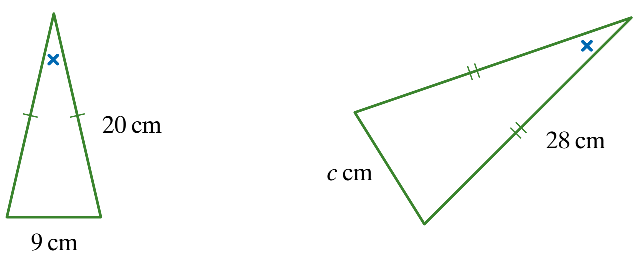
a



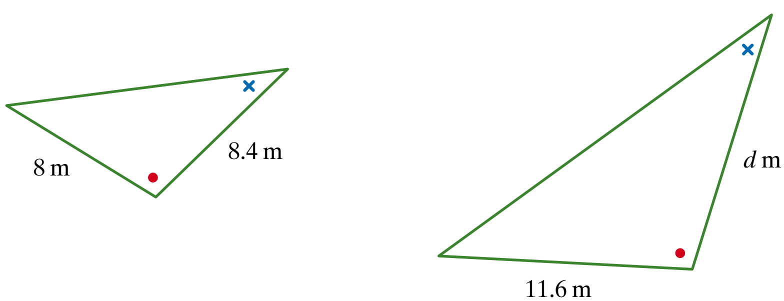
b



c



d



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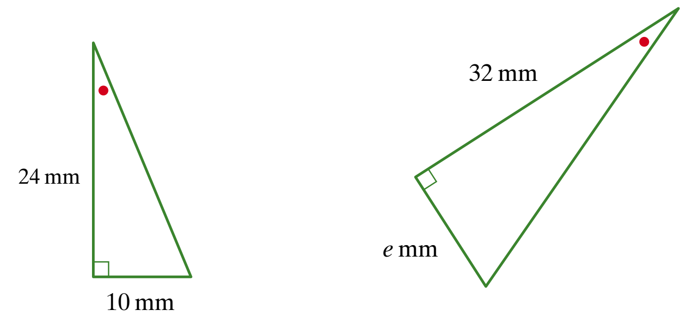
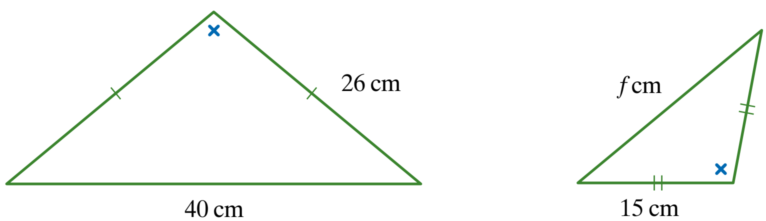
e

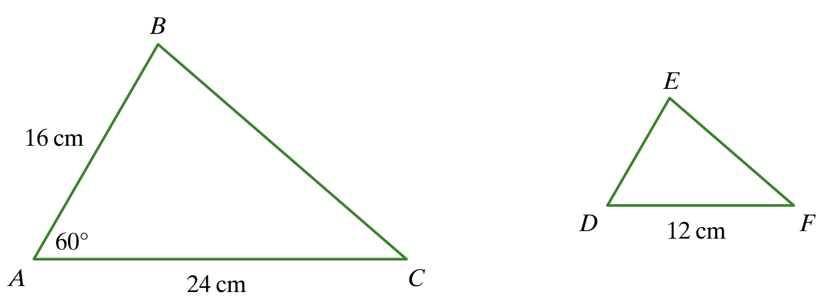


f



LINKS TO Example 21

Triangle ABC is similar to triangle DEF and the triangles are drawn so that matching sides are in matching positions.



a How long is ED? Explain your answer.

b What is the size of $\angle EDF$? Explain your answer.

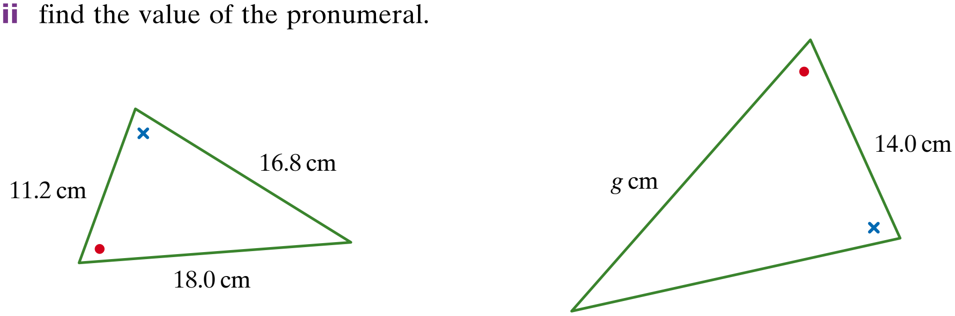
LINKS TO Example 22

The triangles in each pair are similar. The matching angles that are equal are marked. For each pair of triangles,

i calculate the scale factor.

ii find the value of the pronumeral.

a



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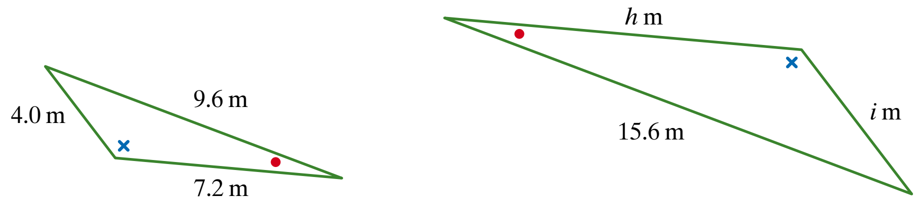
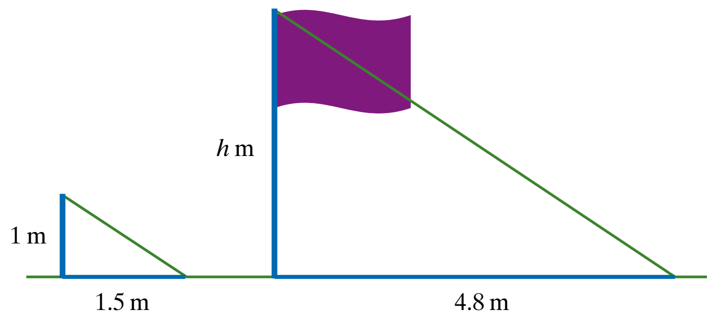
b



LINKS TO
Example 23

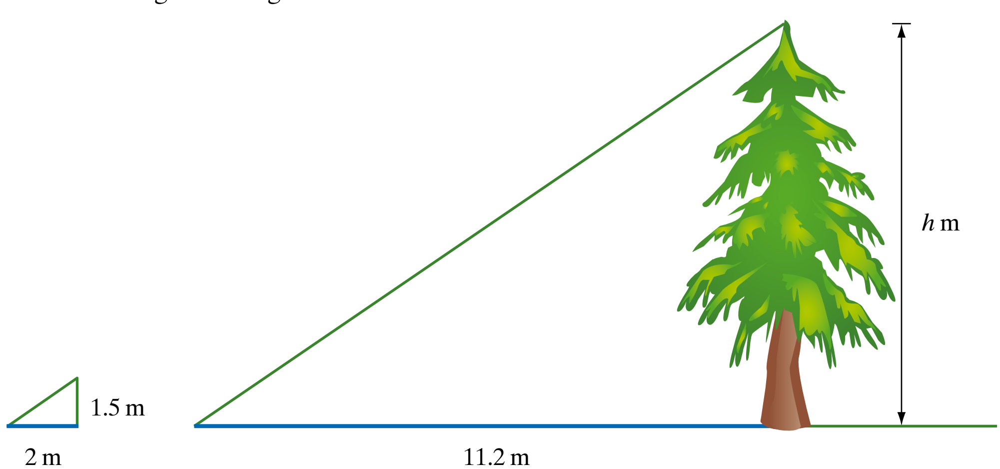
The shadow of a flagpole is 4.8 m long. A nearby post 1 m high has a shadow 1.5 m long. We can assume that the sun's rays are parallel and therefore that they make the same angle with the ground for the flagpole and for the post. Assuming also that both the flagpole and the post are perpendicular to the ground, we can then say that the two triangles are similar because their angle sizes are the same.



a What is the scale factor?

b How high is the flagpole?

A post 1.5 m high casts a shadow 2 m long. At the same time the shadow of a tree is 11.2 m long. How high is the tree?



A tower casts a shadow 16 m long at the same time as the shadow of a nearby post is 0.8 m. The post is 140 cm high.

a Draw a labelled diagram to show the information.

b How high is the tower?

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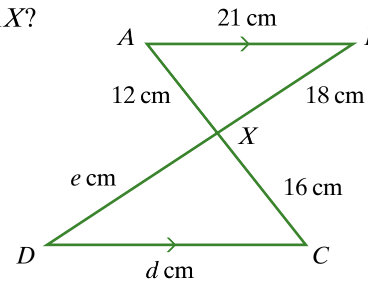
LINKS TO Example 24

Consider the figure below.

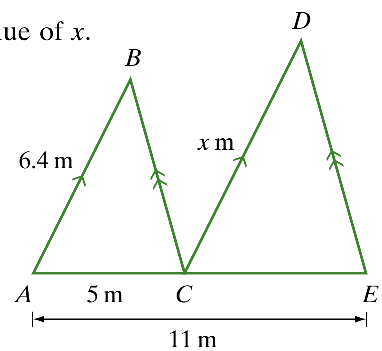
a Explain, giving reasons, why $\triangle ABX$ and $\triangle CDX$ are similar.

b Which side of $\triangle CDX$ corresponds to AX?

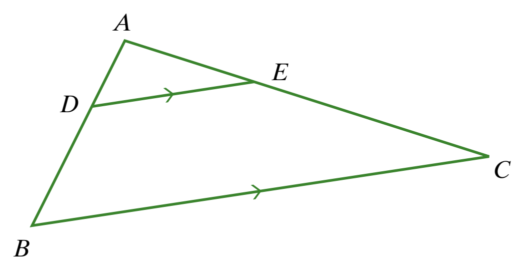
c Find the value of d and e.



Find the value of x.



In $\triangle ABC$, segment DE is parallel to BC. Show that $\triangle ADE$ is similar to $\triangle ABC$.



Using diagrams to help, explain whether each of these statements is true or false.

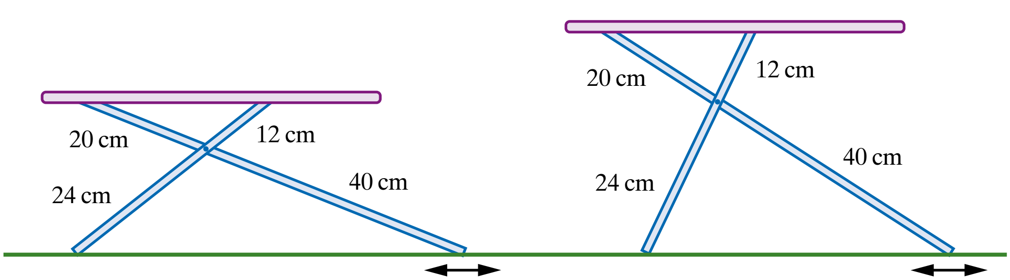
a All equilateral triangles are similar.

b All right-angled triangles are similar.

c All isosceles triangles are similar.

d All right-angled isosceles triangles are similar.

The ironing table shown below has legs that pivot. As the ironing table is raised or lowered, the top always stays parallel to the floor.



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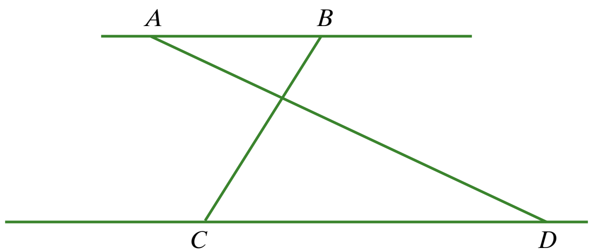
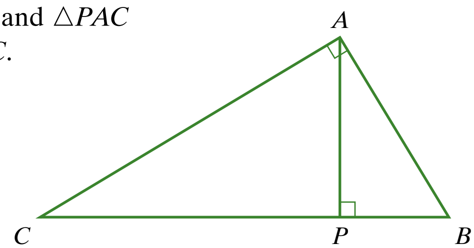
a Explain why the two triangles formed by the legs are similar.

b Explain why the top always stays parallel to the floor. Use this diagram in your explanation.

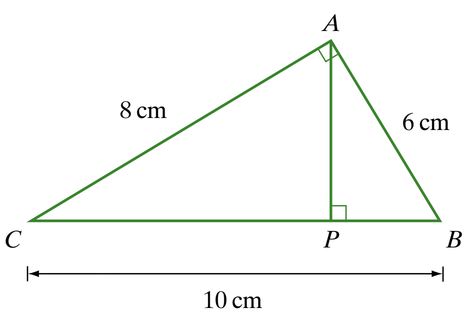


Consider the shape below.

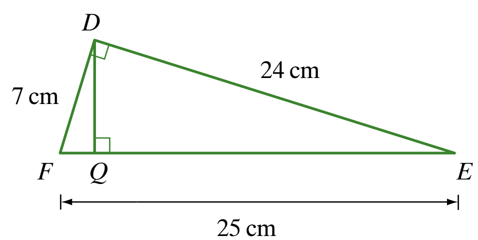
a Explain why $\triangle PBA$ and $\triangle PAC$ are similar to $\triangle ABC$.



b Find the lengths of AP and PB.

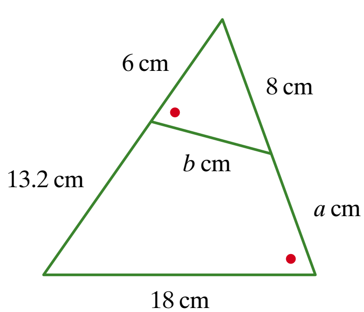


c Find the lengths of DQ and EQ.



exercise 9.4 challenge

Find the values of a and b.



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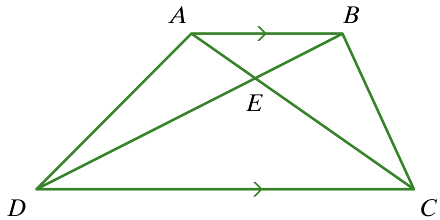
$ABCD$ is a trapezium. AC and BD are the diagonals. $AB = 12$ cm, $DC = 30$ cm, $AE = 5$ cm and $DE = 20$ cm. Find the lengths of BE and CE.



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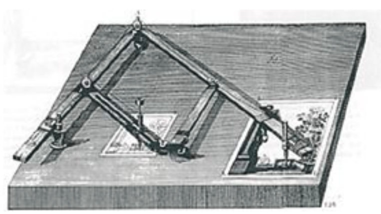
Analysis task

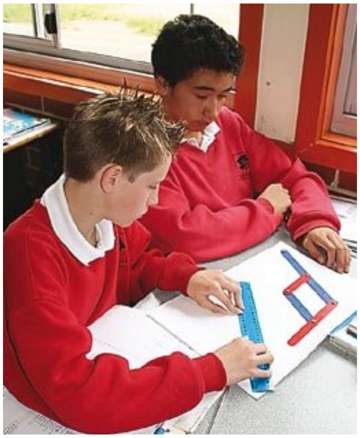
Enlarging pantograph

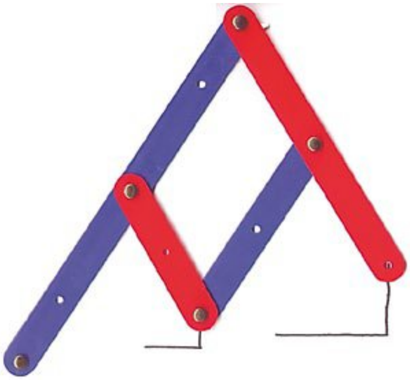
Enlarging pantographs

Before photocopiers were invented, pantographs were used to make exact copies of a drawing, or to enlarge or reduce a drawing. Pantographs have hinged bars with a pointer to trace around the drawing to be copied, and a pencil that traces out the copy. Although pantographs are no longer used as drawing tools, computer-controlled versions are now used in cutting tools, for example, for cutting metal shapes for car bodies.

The illustration below shows a 17th century pantograph designed for enlarging. This analysis task investigates why the pantograph produces enlarged copies of a drawing. The two students are using a model of this pantograph made from plastic geostrips and paper fasteners.

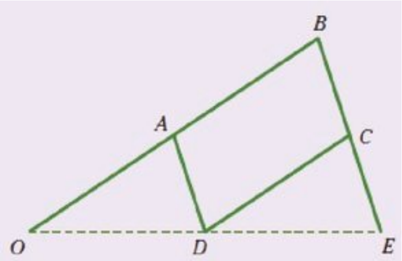






a Measure and compare the sizes of the two L shapes. Remember to measure from the hole.

In the diagram of the pantograph, $OA = AB = CD$ and $AC = BD = DE$.



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b What special quadrilateral is $ABCD$? Explain.

c So as well as knowing that $AB = DC$ and $AD = BC$, what else do you know about AB and DC, and AD and BC?

d What do you now know about $\angle OAD$ and $\angle OBE$? Why?

e What do you now know about $\angle ADO$ and $\angle BEO$? Why?

f $\angle BOE = \angle AOD$. Why?

g What do you now know about $\triangle OAD$ and $\triangle OBE$?

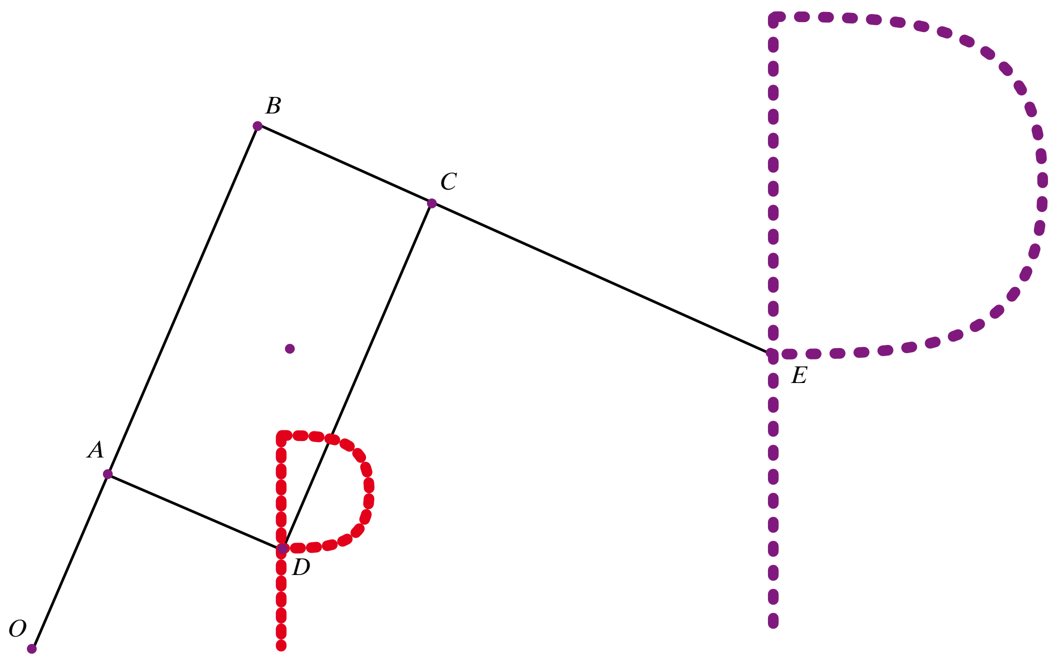
h $OB = \ldots\ldots \times OA$ and $BE = \ldots\ldots \times AD$

i What can you now say about the lengths of OD and OE?

j Because the pantograph pivots around O, it is the distance from O to D compared with the distance from O to E which determines the size of the image. Can you now explain why the image at E is always twice the size of the drawing at D?

k What do you think might happen if the original drawing was at E instead of D and the pencil was placed at E to trace the copy?

l The pantograph below has a different scale factor from the pantograph on the previous page. Compare the height of the letter **P** drawn at D with its image at E. How many times larger is the image? Round to one decimal place. Explain your observation in part **l** with what you showed in part **i**.



m Compare the sides of $\angle OAD$ and $\angle OBE$. What is the relationship between OD and OE?

n Explain your observation in part **l** using what you showed in part **m**.

o Make careful drawings of pantographs that would give the following scale factors.

i 4

ii $1\frac{1}{2}$

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Review Congruent and similar triangles

Summary

Congruent triangles

Congruent triangles must satisfy *both* of these conditions:

- matching angles must be equal
- matching sides must be equal in length.

Conditions for congruent triangles

- SSS (three sides of one triangle **equal** to three sides of the other)
- AAS (two angles of one triangle equal to the two angles of the other triangle and a matching side equal)

Note: two angles equal means that three angles are equal.

- SAS (two sides of one triangle **equal** to two sides of the other triangle and the angle between them the same)
- RHS (both triangles right-angled, hypotenuse and another side of one triangle **equal** to the hypotenuse and another side of the other triangle)

Using congruent triangles in proofs

Look for clues to show that triangles are congruent.

- Already proved properties can be used.
- Parallel lines cut by a transversal can be used to show that angles are equal.
- Shared sides.

Scale factor

- $k = \dfrac{\text{length of side of enlarged shape}}{\text{length of matching side of original shape}}$

  length of side of enlarged shape $= k \times$ length of matching side of original shape

  length of matching side of original shape $= \dfrac{\text{length of side of enlarged shape}}{k}$

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Similar triangles

Similar triangles must satisfy *both* of these conditions:

- matching angles must be equal
- matching sides must be in proportion, that is, in the same ratio.

Conditions for similar triangles

- SSS (three sides of one triangle **in proportion** to three sides of the other)
- AAA (three angles of one triangle equal to three angles of the other)
- SAS (two sides of one triangle **in proportion** to two sides of the other triangle and the angle between them the same)
- RHS (both triangles right-angled, hypotenuse and another side of one triangle **in proportion** to the hypotenuse and another side of the other triangle)

Similar figures and area

When the side lengths of a two-dimensional shape are enlarged by an enlargement factor k

- the perimeter is enlarged by a factor of k
- the area is increased by a factor of k^2.

Visual map

Using the following terms (and others if you wish), construct a visual map that illustrates your understanding of the key ideas covered in this chapter.

alternate angles	proportion
congruent	ratio
corresponding angles	scale factor
hypotenuse	similar
parallel	transversal

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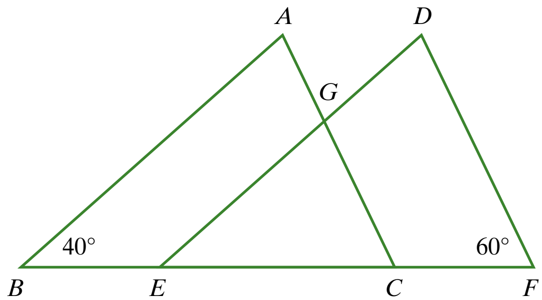
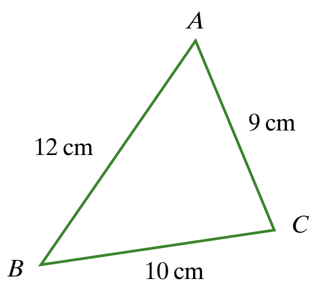
Revision

Multiple-choice questions



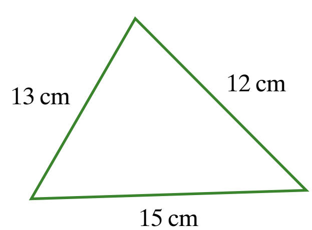
$\triangle ABC \equiv \triangle DEF$. The size of $\angle EGC$ is

A 20° **B** 40° **C** 60° **D** 80° **E** 100°

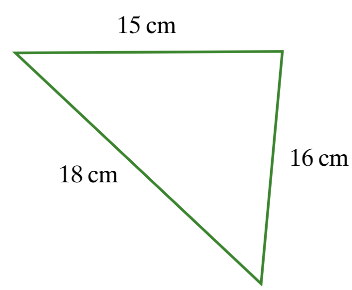


Which of the triangles is similar to $\triangle ABC$?

A



B



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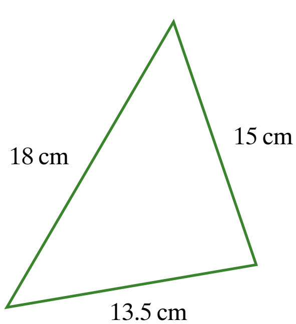

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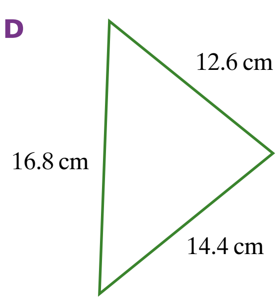

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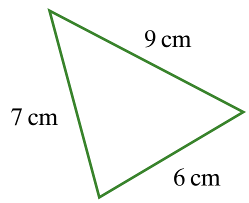
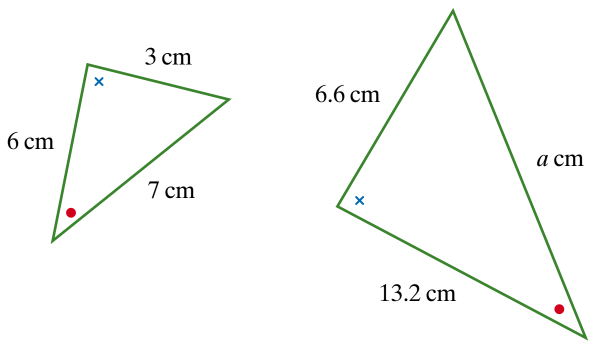
E



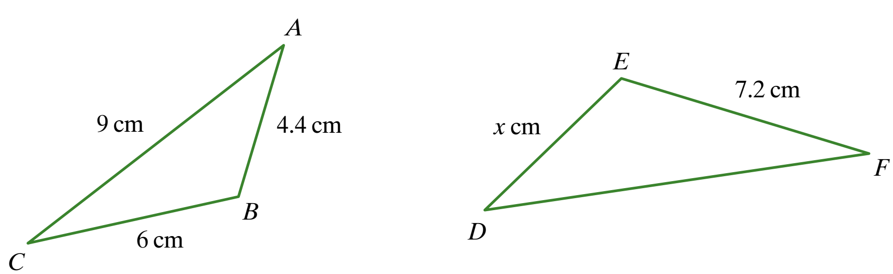


Which equation would you use to find the value of a?

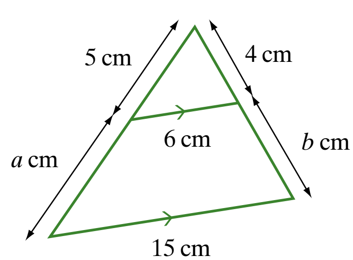
A $\frac{a}{6} = \frac{7}{13.2}$ **B** $\frac{a}{7} = \frac{13.2}{3}$ **D** $\frac{a}{7} = \frac{13.2}{6}$

D $\frac{a}{6} = \frac{13.2}{7}$ **E** $\frac{a}{13.2} = \frac{6}{7}$

Triangles ABC and DEF are similar. What is the value of the pronumeral x?



A 1.2 **B** 1.25 **C** 5.28 **D** 5.5 **E** 10.8



In the diagram above

A $a = 14$ and $b = 13$ **B** $a = 12.5$ and $b = 10$ **C** $a = 10$ and $b = 12.5$

D $a = 6$ and $b = 7.5$ **E** $a = 7.5$ and $b = 6$.

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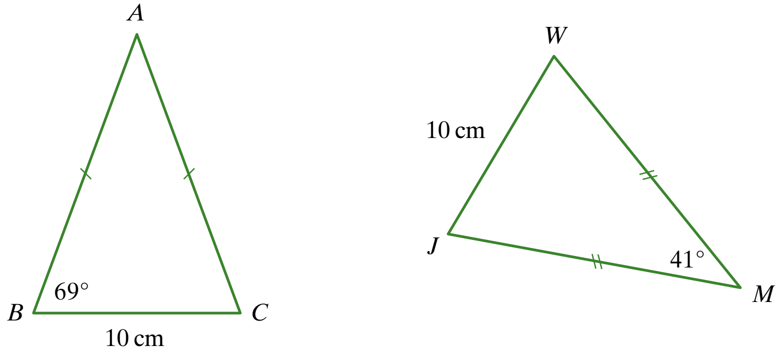
Short-answer questions

● For each of these pairs of triangles, decide if the two triangles are congruent, justifying your answer in each case. For those pairs that you decide are congruent, name the triangles according to their matching vertices.

a



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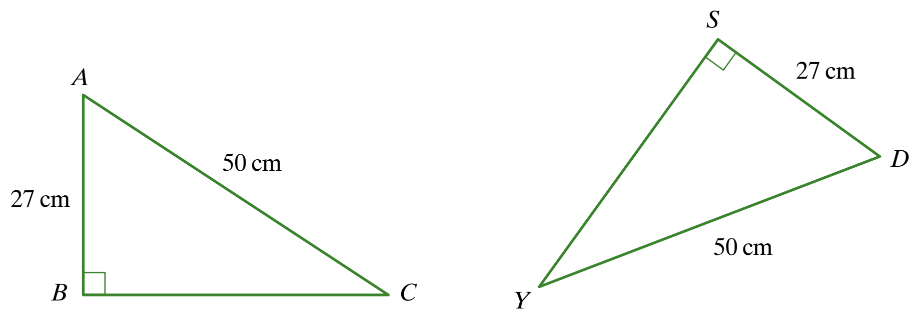
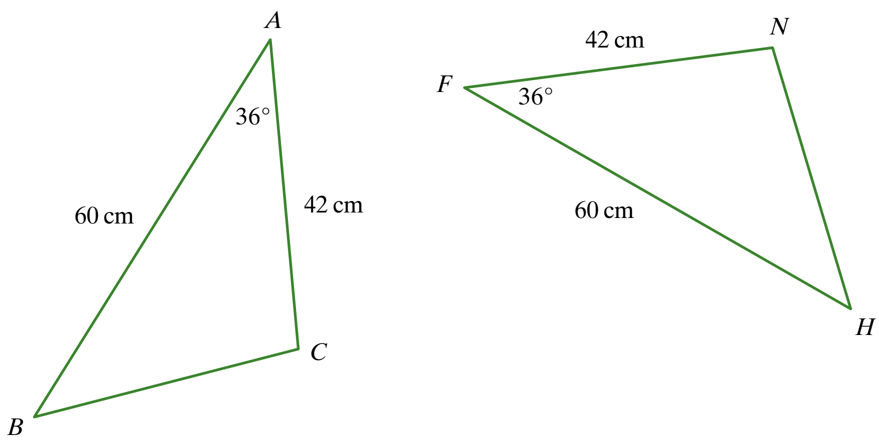


c



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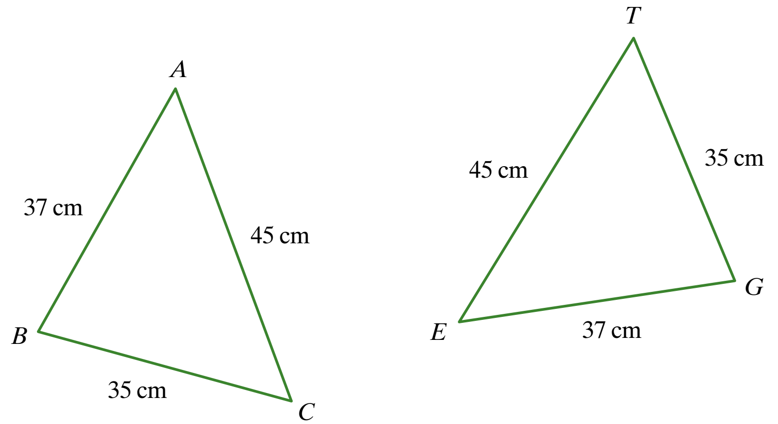
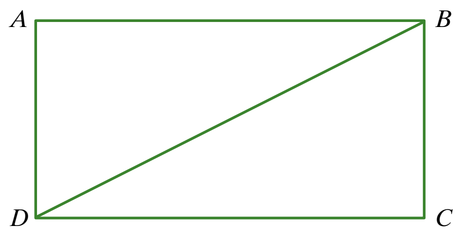
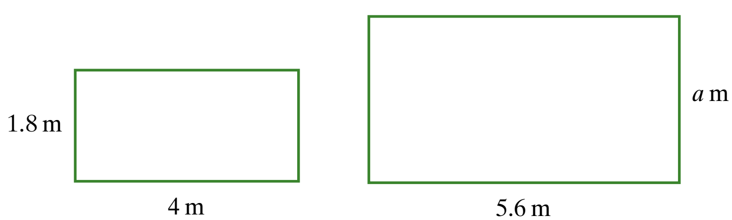
d



Prove that the diagonal BD of the rectangle $ABCD$ divides it into two congruent triangles.



Rectangles A and B are similar.



Rectangle A **Rectangle B**

a Calculate the scale factor, k.
b Find the value of a.
c Calculate the perimeter of rectangle A.
d Calculate the perimeter of rectangle B
 i using the side lengths of rectangle B.
 ii using the perimeter of rectangle A and the scale factor, k.
e Calculate the area of rectangle A.
f Calculate the area of rectangle B
 i using the side lengths of rectangle B.
 ii using the area of rectangle A and the area scale factor, k^2.

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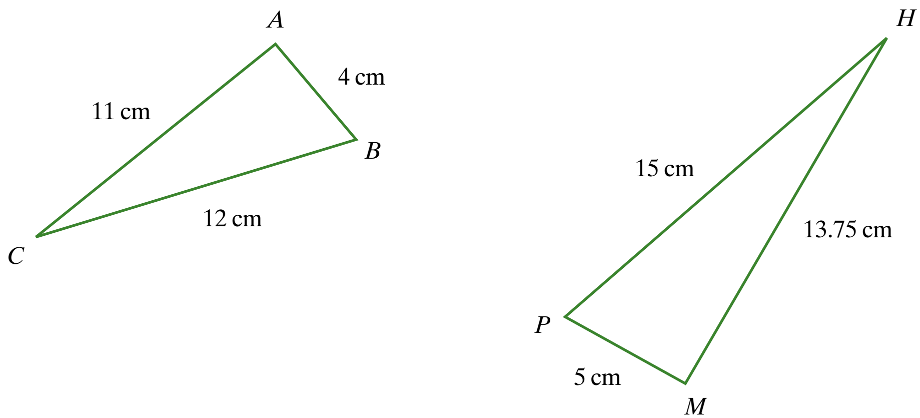
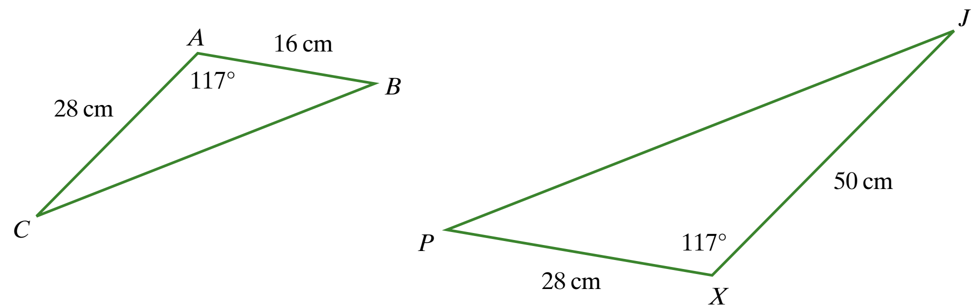
For each of these pairs of triangles, decide if the two triangles are similar, justifying your answer in each case. For those pairs that you decide are similar, name the triangles according to their matching vertices.

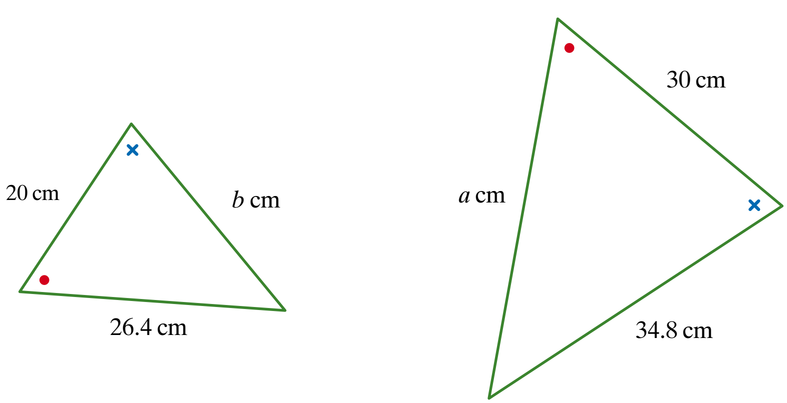
a



b



The following two triangles are similar.



a What is the scale factor?

b Find the values of a and b.

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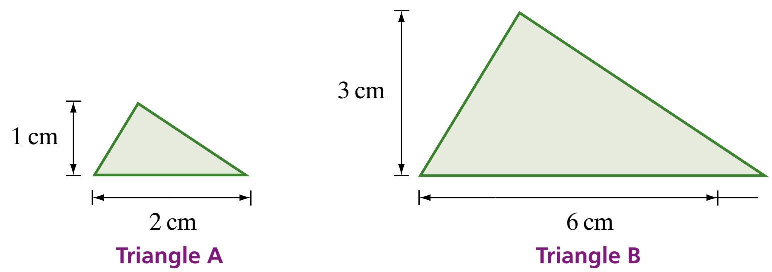
Triangles A and B are similar.

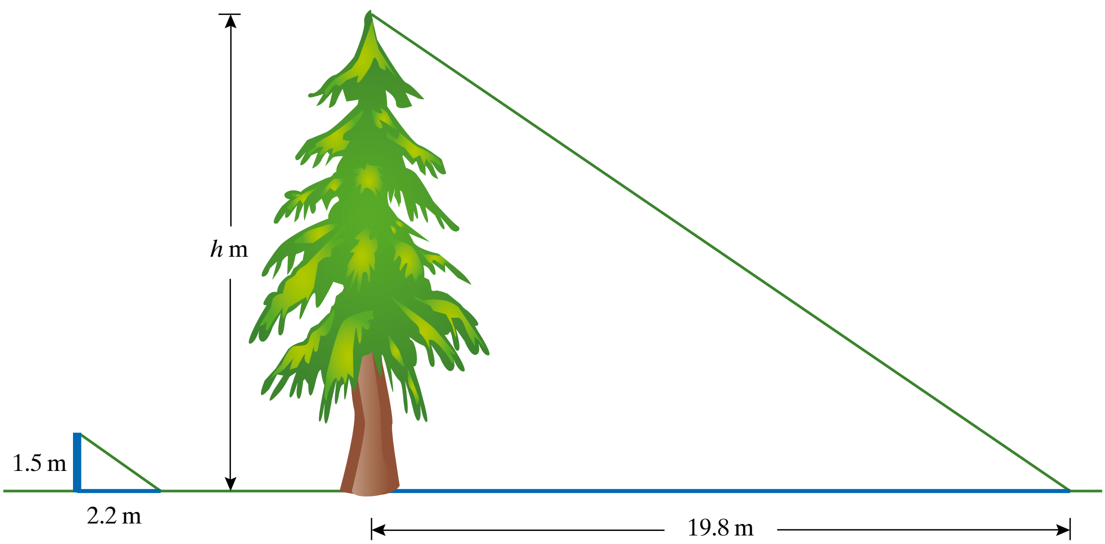
a What is the scale factor?

b Calculate the area of each triangle.

c How many times larger is the area of the larger triangle?



The shadow of a tree is 19.8 m long. A nearby post 1.5 m high has a shadow 2.2 m long. Find the height of the tree, *h* m.



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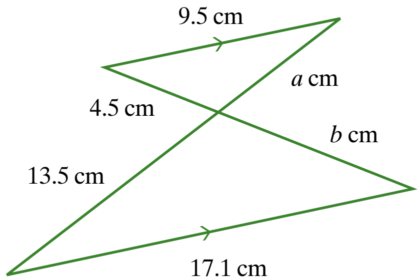
Extended-response questions

Find the values of a and b.



A stick which is 1 m long has a shadow 60 cm long. At the same time, the shadow from a flagpole is 2.1 m long.

a Draw two triangles to show the information.

b Find the height of the flagpole.

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Trigonometry 10

Pre-test

Warm-up

These surveyors are using a theodolite during construction of tunnels for a new freeway. Theodolites are used to measure angles. Calculations based on accurate measurements of angles and distances are important in constructing roads and buildings. How can angle measurements be used in calculating distances?

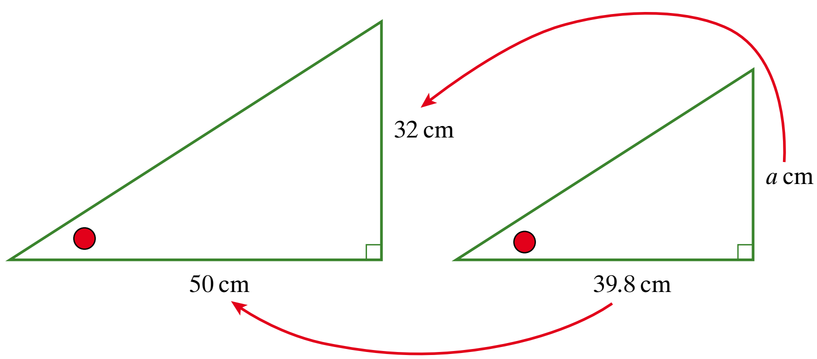
10.1 Trigonometric ratios

BLM
Chapter warm-up

The use of ratios of sides in right-angled triangles to find unknown sides and angles in similar triangles is the basis of a branch of mathematics called **trigonometry**. The word trigonometry comes from the Greek words *trigonon* (triangle) and *metron* (measure).

Consider the two triangles below. Each triangle has a right angle and the angles marked with the red dot are equal. We automatically know that the third angles of the triangles are also equal. So the two triangles are similar (AAA).

In chapter **9** we used the proportion method to find unknown side lengths in similar triangles.



We can write

$$\frac{a}{32} = \frac{39.8}{50}$$

Solving this equation for a, we obtain

$$a = \frac{39.8 \times 32}{50}$$

Now let us look at the triangles again to see if we could write a different proportion equation. Because the triangles are similar, the ratio of two sides of one triangle is equal to the ratio of the two matching sides of the other triangle, that is, we can write

$$\frac{a}{39.8} = \frac{32}{50}$$

Solving this equation for a, we obtain

$$a = \frac{32 \times 39.8}{50}$$

Notice that this is exactly the same expression for a as we obtained before.

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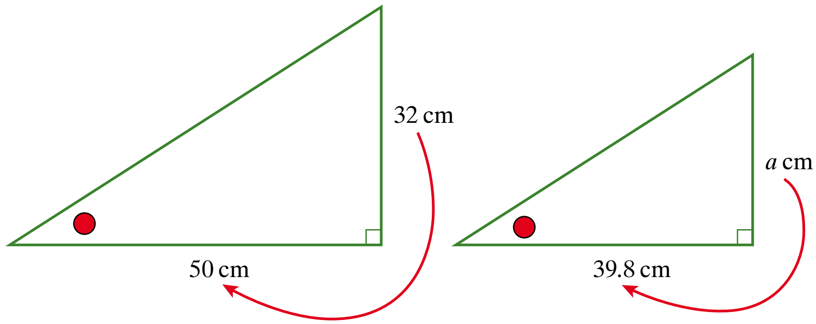
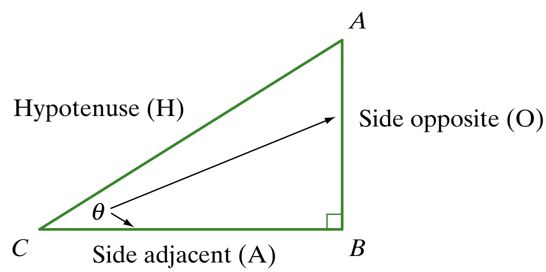


It is this comparison of the ratio of sides of similar right-angled triangles that forms the basis of trigonometry.

Naming sides

When we are working with right-angled triangles, we need some way of naming the sides we are referring to. The convention is to look at the angle we are considering and use the terms **opposite side**, **adjacent side** and **hypotenuse** to refer to the three sides of the triangle. Just as we commonly use the pronumeral x to represent a number in algebra, we often use the Greek letter θ (*theta*) to represent an angle. In the triangle below, suppose we are considering the angle θ.

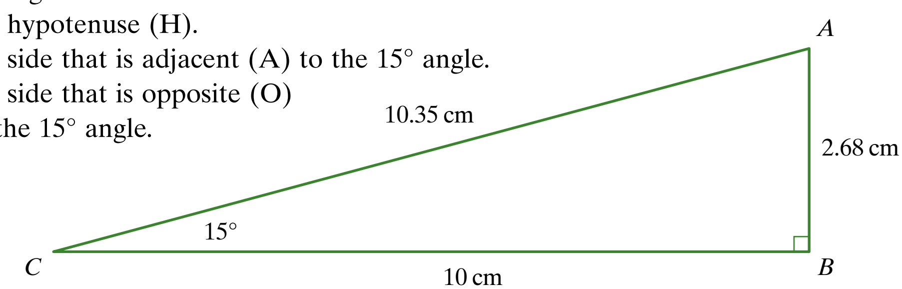


- Side AB is the **opposite** side of the triangle from angle θ.
- Side BC is the side **adjacent** or next to angle θ.
- Side AC is opposite the right angle and is the longest side of the triangle. It is the **hypotenuse**.

Example 1

a Copy triangle ABC and label

- **i** the hypotenuse (H).
- **ii** the side that is adjacent (A) to the 15° angle.
- **iii** the side that is opposite (O) to the 15° angle.



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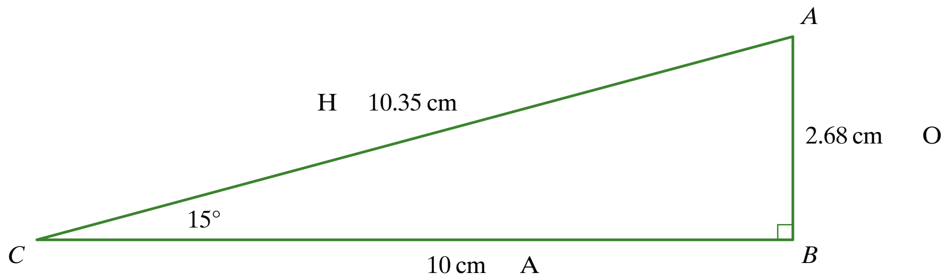
Example 1 continued

b Calculate the following ratios correct to three decimal places.

i $\dfrac{\text{opposite}}{\text{adjacent}}$ **ii** $\dfrac{\text{opposite}}{\text{hypotenuse}}$ **iii** $\dfrac{\text{adjacent}}{\text{hypotenuse}}$

Working and reasoning

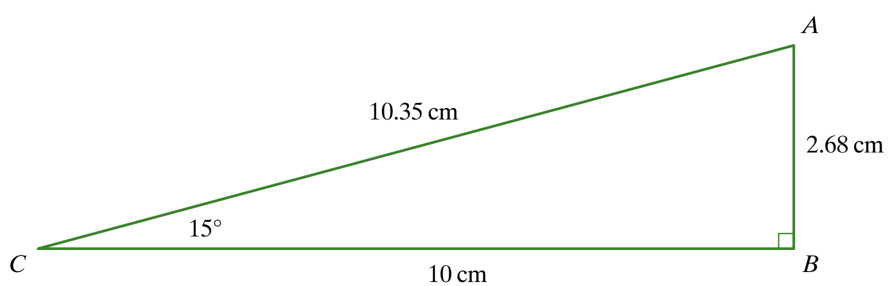
a

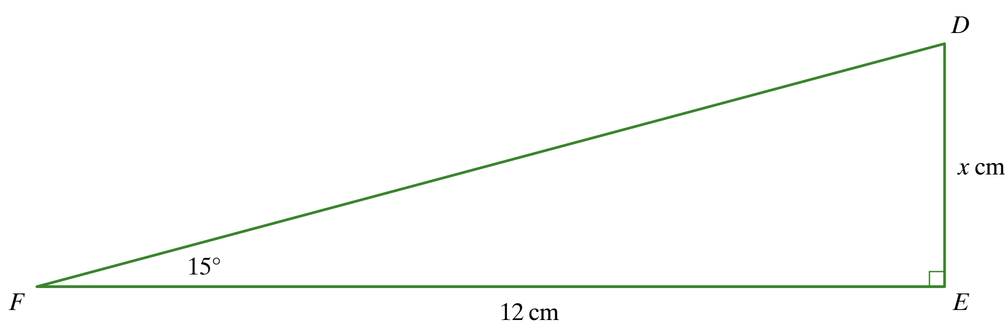


b **i** $\dfrac{\text{Opp}}{\text{Adj}} = \dfrac{2.68}{10} = 0.268$ **ii** $\dfrac{\text{Opp}}{\text{Hyp}} = \dfrac{2.68}{10.35} = 0.259$ **iii** $\dfrac{\text{Adj}}{\text{Hyp}} = \dfrac{10}{10.35} = 0.966$

Now that we have calculated the three ratios in example **1**, we can use these ratios to calculate unknown side lengths in other triangles that are similar to $\triangle ABC$.

Example 2





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Example 2 continued

a Explain why $\triangle DEF$ is similar to $\triangle ABC$.

b Which of the ratios in example **1** is equal to $\frac{x}{12}$?

c Using the appropriate ratio from example **1**, write an equation in the form $\frac{x}{12} = \ldots$ and solve it to find x.

d Measure the length of DE and check with the calculated value of x.

Working and reasoning

a The three angles of each triangle are 15°, 75° and 90° so the triangles are similar (AAA).

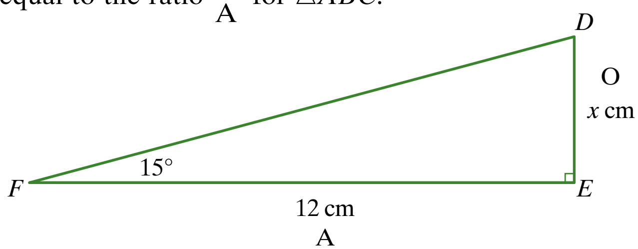
b $\frac{\text{O}}{\text{A}} = \frac{2.68}{10} = 0.268$

Because the triangles are similar, their sides are in the same proportion. The ratio $\frac{\text{O}}{\text{A}}$ for $\triangle DEF$ is equal to the ratio $\frac{\text{O}}{\text{A}}$ for $\triangle ABC$.

c
$$\begin{aligned}\frac{x}{12} &= \frac{2.68}{10}\\ &= 0.268\\ x &= 0.268 \times 12\\ x &\approx 3.2\end{aligned}$$

d $DE = 3.2$ cm

The measured value is the same as the calculated value.



Every right-angled triangle with an angle of 15° will be similar to every other right-angled triangle with an angle of 15°, so the ratios of sides will always be the same.

In the same way, we could find the ratios for right-angled triangles with other angle measures and use these ratios to calculate unknown sides in similar triangles.

Special names have been given to the ratios $\frac{\text{opposite}}{\text{hypotenuse}}$, $\frac{\text{adjacent}}{\text{hypotenuse}}$ and $\frac{\text{opposite}}{\text{adjacent}}$ as shown in the following table.

Ratio	Name of ratio	Abbreviation	How we say it
$\frac{\text{Opposite side}}{\text{Hypotenuse}}$	**sine**	$\sin\,\theta$	'sine theta'
$\frac{\text{Adjacent side}}{\text{Hypotenuse}}$	**cosine**	$\cos\,\theta$	'coz theta'
$\frac{\text{Opposite side}}{\text{Adjacent side}}$	**tangent**	$\tan\,\theta$	'tan theta'

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The sine, cosine and tangent of an angle are simply numbers, each representing the ratio of one side to another side in a right-angled triangle. The mnemonic **SOH-CAH-TOA** is used as a reminder of the ratios.

SOH-CAH-TOA

$$\sin\theta = \frac{\text{Opposite side}}{\text{Hypotenuse}}$$

$$\cos\theta = \frac{\text{Adjacent side}}{\text{Hypotenuse}}$$

$$\tan\theta = \frac{\text{Opposite side}}{\text{Adjacent side}}$$

Trigonometric ratios

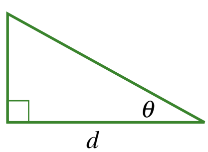
The method used in example **2** is not very accurate as it depends on how accurately we can measure the sides of the triangles. However, mathematicians have calculated accurate values for the trigonometric ratios for all angles and these values are built in to your calculator. In section **10.2** you will use your calculator to find values of the trigonometric ratios for angles of different sizes.

exercise 10.1

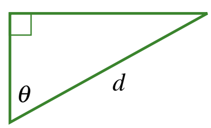
LINKS TO Example 1

In each of the following, look at the angle marked θ. State if the side labelled d represents the hypotenuse (H) or the opposite (O) or the adjacent (A) side with respect to angle θ.

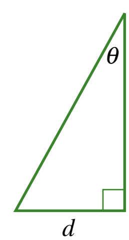
a



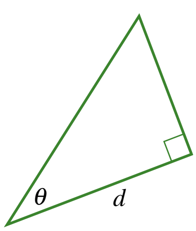
b



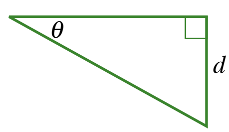
c



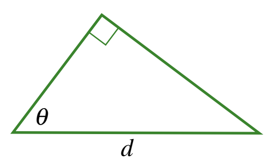
d



e

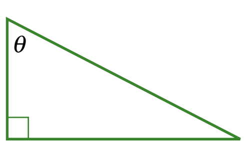


f

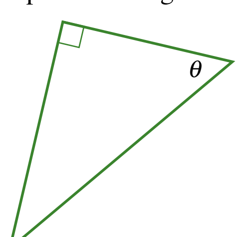


Copy each of the following triangles and label the hypotenuse (H), the opposite side (O) and the adjacent side (A) with respect to the given angle.

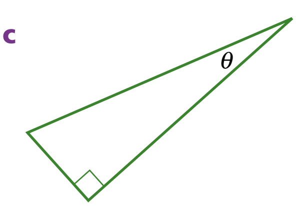
a



b



c



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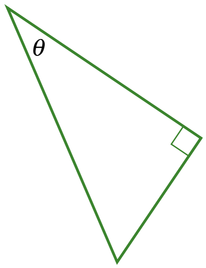
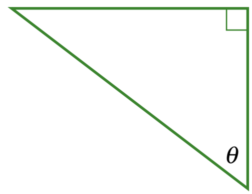
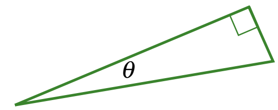
d



e

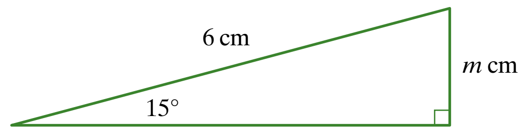


f



LINKS TO Example 2

The following triangle is similar to $\triangle ABC$ in example **1**.



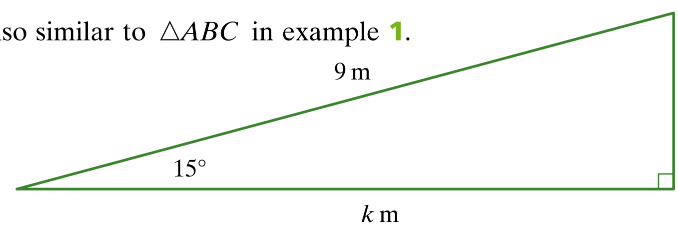
a Copy the triangle and label the hypotenuse (H), the opposite side (O) and the adjacent side (A) for the given angle.

b Which of the ratios in example **1** is equal to $\frac{m}{6}$?

c Using the appropriate ratio from example **1**, write an equation in the form $\frac{m}{6} = \ldots$ and solve it to find m correct to one decimal place.

d Measure the length of the side labeled m cm and see if your answer to part **c** is correct.

The triangle below is also similar to $\triangle ABC$ in example **1**.

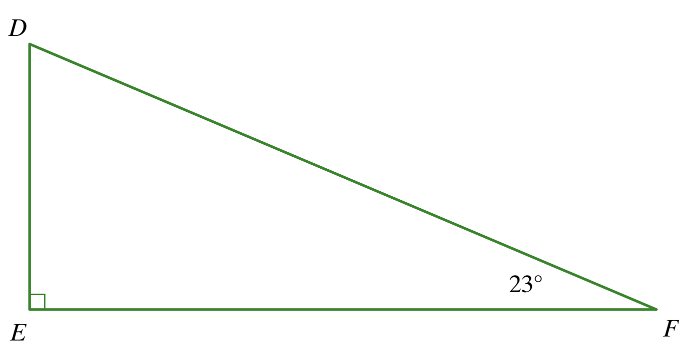


a Copy the triangle and label the hypotenuse (H), the opposite side (O) and the adjacent side (A) for the given angle.

b Which of the ratios in example **1** is equal to $\frac{k}{9}$?

c Using the appropriate ratio from example **1**, write an equation in the form $\frac{k}{9} = \ldots$ and solve it to find k correct to one decimal place.

d Measure the length of the side labeled k m and see if your answer to part **c** is correct.



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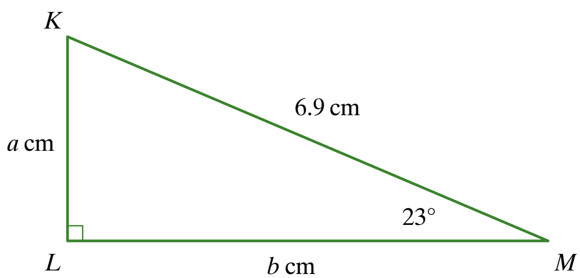
a Carefully measure the three sides of the triangle to the nearest tenth of a centimetre. Copy and complete the table, calculating the ratios correct to two decimal places.

$\frac{\text{Opp}}{\text{Adj}} = \frac{\_\_\_}{\_\_\_}$ $= \_\_\_\_\_$	$\frac{\text{Opp}}{\text{Hyp}} = \frac{\_\_\_}{\_\_\_}$ $= \_\_\_\_\_$	$\frac{\text{Adj}}{\text{Hyp}} = \frac{\_\_\_}{\_\_\_}$ $= \_\_\_\_\_$

b Explain why ΔKLM is similar to ΔDEF.



c Which of the ratios in part **a** is equal to $\frac{a}{6.9}$?

d Use this ratio from part **a** to find the value of a correct to one decimal place.

e Which of the ratios in part **a** is equal to $\frac{b}{6.9}$?

f Use this ratio from part **a** to find the value of b correct to one decimal place.

g Measure KL and LM and see if your answers to parts **d** and **f** are correct.

Copy and complete the following table to show the calculated values for the ratios for 15° in example **1** and for 23° in question **5**. Then use your calculator to find the values of sin 15°, cos 15° and tan 15°, and sin 23°, cos 23° and tan 23°, rounding the values to 3 decimal places. Are your calculated values for the ratios close to the values given by your calculator?

Calculated ratio from measurements for 15° Example **1**	$\frac{\text{O}}{\text{A}}$ $= 0.268$	$\frac{\text{O}}{\text{H}}$ $= 0.259$	$\frac{\text{A}}{\text{H}}$ $= 0.966$
Calculator value	tan 15° =	sin 15° =	cos 15° =
Calculated ratio from measurements for 23° question **5a**	$\frac{\text{O}}{\text{A}}$ $= \_\_\_\_\_$	$\frac{\text{O}}{\text{H}}$ $= \_\_\_\_\_$	$\frac{\text{O}}{\text{A}}$ $= \_\_\_\_\_$
Calculator value	tan 23° =	sin 23° =	cos 23° =

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10.2 Calculator skills with the trigonometric ratios

This section demonstrates the calculator skills and solving skills needed for working with the trigonometric ratios. It is important to be competent with these skills before starting section **10.3**.

The sine, cosine and tangent of most angles are irrational numbers, that is, they cannot be represented as exact decimal values. The calculator values given by your calculator are decimal approximations. We always have to round the results of calculations in trigonometry to a sensible number of decimal places.

Find the keys for the tangent, sine, and cosine of angles on your calculator. It is important to make sure your calculator is in ***degree*** mode as there is an alternative unit for measuring angles called the radian. If your calculator is set in radian mode, you will obtain incorrect values for the sine, cosine and tangent of angles as we will be working only in degrees.

Just as we would not use a square root sign in a calculation without putting a number after it, The sine, cosine and tangent ratios are meaningless in a calculation unless they are followed by an angle measure.

For example,

$$\begin{aligned}\sin 15^\circ &\approx 0.259\\ \cos 15^\circ &\approx 0.966\\ \tan 15^\circ &\approx 0.268\end{aligned}$$

Example 3

Evaluate the following trigonometric ratios correct to four decimal places.

a $\sin 48^\circ$
b $\tan 75^\circ$
c $\cos 23^\circ$

Working	Reasoning
a $\sin 48^\circ = 0.7431$	Press the following calculator keys: $\boxed{\sin}\boxed{4}\boxed{8}$
b $\tan 75^\circ = 3.7321$	Press $\boxed{\tan}\boxed{7}\boxed{5}$
c $\cos 23^\circ = 0.9205$	Press $\boxed{\cos}\boxed{2}\boxed{3}$

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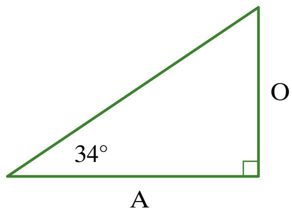
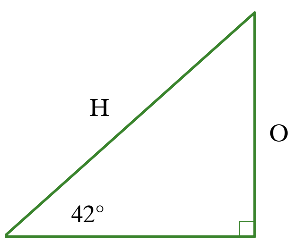
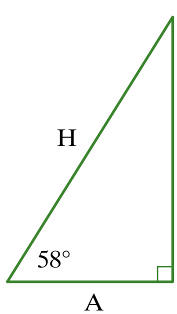
Example 4

For each of the following

i use a protractor to draw a right-angled triangle with the given angle.
ii label the sides.
iii write the given trigonometric ratio as the ratio of sides.
iv use your calculator to find the value, correct to three decimal places.

a tan 34° **b** sin 42° **c** cos 58°

Working	**Reasoning**
a i  $\tan 34° = \dfrac{O}{A}$	SOH-CAH-**TOA** reminds us that in a right-angled triangle, the **tan**gent of an angle is the ratio of the side **opp**osite the angle to the side **adj**acent to the angle.
ii tan 34° = 0.675 (correct to three decimal places)	
b i  $\sin 42° = \dfrac{O}{H}$	**SOH**-CAH-TOA reminds us that in a right-angled triangle, the **sin**e of an angle is the ratio of the **opp**osite side to the **hyp**otenuse.
ii sin 42° = 0.669 (correct to three decimal places)	
c i  $\cos 58° = \dfrac{A}{H}$	SOH-**CAH**-TOA reminds us that in a right-angled triangle, the **cos**ine of an angle is the ratio of the **adj**acent side to the **hyp**otenuse.
ii cos 58° = 0.530 (correct to three decimal places)	

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If we are using the trigonometric ratios in calculations, we often need to multiply a number by a trigonometric ratio, for example, $16 \times \sin 20°$. Just as with pronumerals, we leave out the multiplication sign and write $16 \sin 20°$.

Example 5

Evaluate the following, correct to one decimal place.

a $16 \sin 20°$ **b** $42 \tan 67°$ **c** $53 \cos 19°$

Working

a $16 \sin 20° = 5.5$

b $42 \tan 67° = 98.9$

c $53 \cos 19° = 50.1$

Reasoning

Calculator keys:

a $\boxed{1}\boxed{6}\boxed{\times}\boxed{\sin}\boxed{2}\boxed{0}$

b $\boxed{4}\boxed{2}\boxed{\times}\boxed{\tan}\boxed{6}\boxed{7}$

c $\boxed{5}\boxed{3}\boxed{\times}\boxed{\cos}\boxed{1}\boxed{9}$

When we are finding unknown side lengths, we obtain equations that need to be solved to evaluate the unknown. We do this in the same way as solving algebraic equations, by doing the same to both sides.

Example 6

Find the value of x (correct to one decimal place) in each of the following.

a $27 = x \sin 67°$ **b** $\cos 42° = \dfrac{8.5}{x}$ **c** $\tan 72° = \dfrac{124}{x}$

Working / **Reasoning**

a
$$27 = x \sin 67°$$
Divide both sides by sin 67°.
$$x = 27 \div \sin 67°$$
$$x \approx 29.33$$
Round to one decimal place.
$$x = 29.3 \text{ to one decimal place}$$

b
$$\cos 42° = \frac{8.5}{x}$$
Multiply both sides by x.
Divide both sides by cos 42°.
$$x \cos 42° = 8.5$$
$$x = 8.5 \div \cos 42°$$
Round to one decimal place.
$$x \approx 11.4379$$
$$x = 11.4 \text{ correct to one decimal place}$$

c
$$\tan 72° = \frac{124}{x}$$
Multiply both sides by x.
Divide both sides by tan 72°.
$$x \tan 72° = 124$$
$$x = 124 \div \tan 72°$$
Round to one decimal place.
$$x \approx 40.29$$
$$x = 40.3 \text{ correct to one decimal place}$$

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exercise 10.2

LINKS TO Example 3

1. Use your calculator to find the values of the following correct to four decimal places.

a sin 37° **b** cos 28° **c** tan 82° **d** cos 30°
e sin 57° **f** tan 43° **g** sin 5° **h** cos 5°
i sin 88° **j** tan 12° **k** cos 63° **l** tan 80°
m sin 30° **n** tan 42° **o** sin 54.6° **p** cos 15.3°
q tan 23.7° **r** cos 33.4° **s** sin 65.1° **t** tan 75.9°

LINKS TO Example 4

2. For each of the following
   i use a protractor to draw a right-angled triangle with the given angle. For the given trigonometric ratio (sin, cos, or tan) label the two appropriate sides on the triangle
   ii use your calculator to find the value of the trigonometric ratio correct to three decimal places.

a tan 70° **b** sin 30° **c** cos 45° **d** sin 27° **e** cos 36°
f tan 15° **g** tan 64° **h** cos 85° **i** sin 75° **j** cos 15°

LINKS TO Example 5

3. Use your calculator to find the following correct to one decimal place.

a 4 tan 15° **b** 12 sin 72° **c** 25 cos 24° **d** 9 sin 37°
e 30 tan 55° **f** 28 cos 18° **g** 36 tan 40° **h** 75 sin 8°
i 4.5 sin 18° **j** 3.7 cos 24° **k** 7.6 tan 27.5° **l** 16.4 sin 56.4°

4. Use your calculator to evaluate each of the following correct to one decimal place.

a $24 \div \tan 50°$ **b** $115 \div \cos 24°$ **c** $\dfrac{48}{\tan 8°}$ **d** $\dfrac{250}{\sin 64°}$

e $35 \div \cos 23.5°$ **f** $18.6 \div \tan 23°$ **g** $\dfrac{20}{\cos 18°}$ **h** $\dfrac{56}{\sin 27°}$

i $8.9 \div \cos 75°$ **j** $14.5 \div \sin 18°$ **k** $\dfrac{10}{\sin 30°}$ **l** $\dfrac{60}{\cos 50°}$

LINKS TO Example 6

5. Solve each of the following for x. Give each value correct to one decimal place.

a $\dfrac{x}{12} = \sin 40°$ **b** $\cos 55° = \dfrac{x}{18}$ **c** $\tan 18° = \dfrac{x}{15.6}$ **d** $\dfrac{x}{4.8} = \sin 72°$

e $x \cos 27° = 18$ **f** $125 = x \tan 48°$ **g** $x \sin 25° = 95$ **h** $x \cos 60° = 3.2$

i $\sin 10.5° = \dfrac{125}{x}$ **j** $\cos 4.6° = \dfrac{140}{x}$ **k** $\tan 9° = \dfrac{54}{x}$ **l** $\sin 12° = \dfrac{2.8}{x}$

exercise 10.2 challenge

1. Complete the following.
   a Use your calculator to find the value of tan 45°. What do you notice?
   b Use a diagram to explain your observation from part **a**.

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10.3 Using trigonometric ratios to calculate lengths of sides

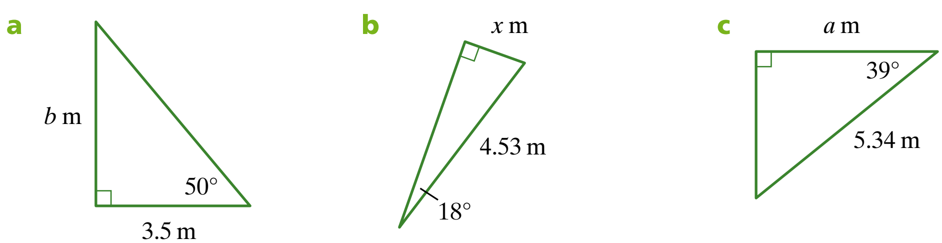
Example 7

For each of the following right-angled triangles,

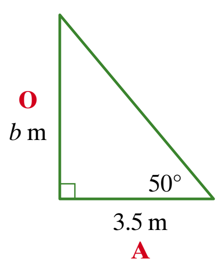
i copy the diagram and mark the given and required sides as either *hypotenuse* or *opposite* or *adjacent* to the given angle.

ii state the appropriate trigonometric ratio, that is, sine, cosine or tangent.

iii use the ratio to write an equation, then solve the equation to find the value of the pronumeral correct to two decimal places.



Working	Reasoning
a i	The required side is **opp**osite the given angle. The given side is the side **adj**acent to the given angle. SOH-CAH-**TOA** reminds us to use **tan**gent.



ii tangent

Working | **Reasoning**

a iii

$$\tan\theta = \frac{O}{A}$$

$$\tan 50^\circ = \frac{b}{3.5}$$

$$b = 3.5 \times \tan 50^\circ$$

$$= 4.171...$$

$$b = 4.17 \text{ to two decimal places}$$

continued

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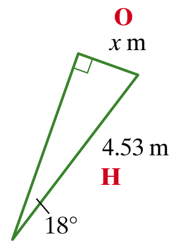
Example 7 continued

Working

b i



ii sine

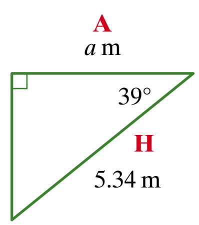
iii $\sin\theta = \frac{O}{H}$

$$\sin 18° = \frac{x}{4.53}$$
$$x = 4.53 \times \sin 18°$$
$$= 1.399...$$
$$x = 1.40 \text{ to two decimal places}$$

Reasoning

The required side is **opp**osite the given angle. The given side is the **hyp**otenuse. **SOH**-CAH-TOA reminds us to use **sin**e.

c i



ii cosine

c iii $\cos\theta = \frac{A}{H}$

$$\cos 39° = \frac{a}{5.34}$$
$$a = 5.34 \times \cos 39°$$
$$= 4.149...$$
$$a = 4.15 \text{ to two decimal places}$$

The required side is the **adj**acent side to the given angle. The given side is the **hyp**otenuse.

SOH-**CAH**-TOA reminds us to use **cos**ine.

In each of the triangles in example **7**, the unknown side was in the numerator. The next example shows the method of working if the unknown side is in the denominator.

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Example 8

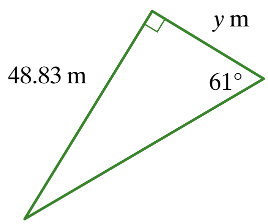
For each of the following right-angled triangles,

i copy the diagram and mark the given and required sides as either *hypotenuse* or *opposite* or *adjacent* to the given angle.

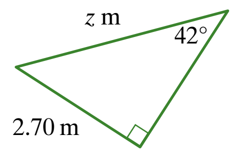
ii state the appropriate trigonometric ratio, that is, sine, cosine or tangent.

iii use the ratio to write an equation, then solve the equation to find the value of the pronumeral correct to two decimal places.

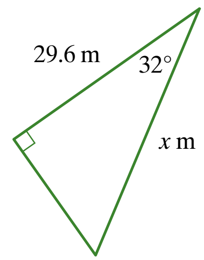
a



b

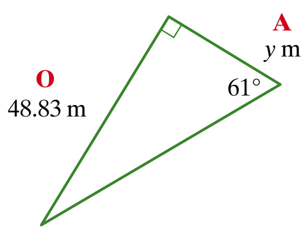


c



Working

a i



Reasoning

The given side is **opp**osite the given angle. The required side is the side **adj**acent to the given angle.

a ii tangent

SOH-CAH-**TOA** reminds us to use **tan**gent.

iii

$$\tan\theta = \frac{O}{A}$$

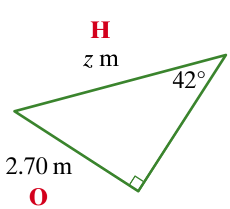
$$\tan 61^\circ = \frac{48.83}{y}$$

$$y \times \tan 61^\circ = 48.83$$

$$y = \frac{48.83}{\tan 61^\circ}$$

$$y = 27.07 \text{ to two decimal places}$$

b i



The given side is **opp**osite the given angle. The required side is the **hyp**otenuse.

SOH-CAH-TOA reminds us to use sine.

ii sine

continued

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Example 8 continued

Working

iii

$$\sin\theta = \frac{O}{H}$$

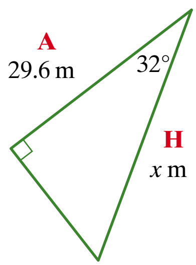
$$\sin 42° = \frac{2.7}{z}$$

$$z \times \sin 42° = 2.7$$

$$z = \frac{2.7}{\sin 42°}$$

$z = 4.04$ correct to two decimal places

c i



Reasoning

The required side is the **adj**acent side to the given angle. The given side is the **hyp**otenuse.

SOH-**CAH**-TOA reminds us to use **cos**ine.

ii cosine

c iii

$$\cos\theta = \frac{A}{H}$$

$$\cos 32° = \frac{29.6}{x}$$

$$x \times \cos 32° = 29.6$$

$$x = \frac{29.6}{\cos 32°}$$

$$= 34.903$$

$x = 34.90$ to two decimal places

Once we know two sides of a right-angled triangle we can use Pythagoras' theorem to calculate the third side. If we know an angle in the triangle as well, we have a choice of using Pythagoras' theorem or trigonometry to find the third side.

Example 9

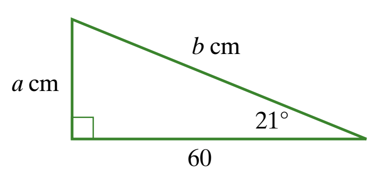
Consider the triangle at right.

a Use trigonometry to find the value of *a* correct to one decimal place.

b Find the value of *b* correct to one decimal place

i using Pythagoras' theorem.

ii using trigonometry.



continued

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Example 9 continued

Working

a $\tan\theta = \dfrac{O}{A}$

$\tan 21° = \dfrac{a}{60}$

$a = 60\tan 21°$

$a = 23.03...$

$a = 23.0$ correct to one decimal place.

$b^2 = 23^2 + 60^2$

$b^2 = \sqrt{23^2 + 60^2}$

$b = 64.3$

$\cos 21° = \dfrac{60}{b}$

$b\cos 21° = 60$

$b = \dfrac{60}{\cos 21°}$

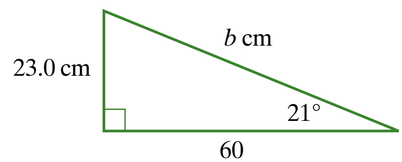
$b = 64.3$

Reasoning

The given side is **adj**acent to the given angle. The required side is the side **opp**osite the given angle.

SOH-CAH-**TOA** reminds us to use **tan**gent.

Round to one decimal place.



The required side is the hypotenuse.

The given side is **adj**acent to the given angle. The required side is the **hyp**otenuse.

SOH-**CAH**-TOA reminds us to use **cos**ine.

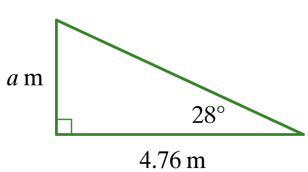
Round to one decimal place.

exercise 10.3

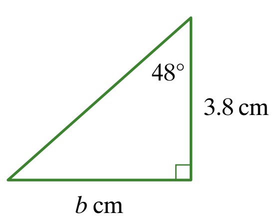
LINKS TO Example 7a

● Each of the following questions requires the use of the *tangent* of the given angle. In each triangle the unknown side is the side opposite the given angle. Calculate the value of the pronumeral to one decimal place.

a



b



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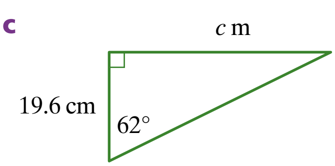
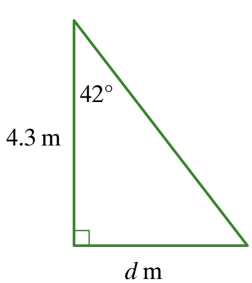
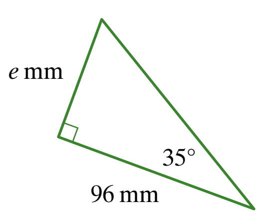


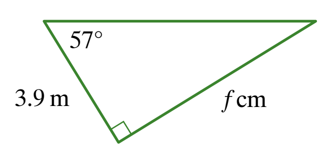
d



e



f



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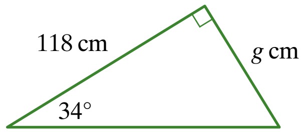
g



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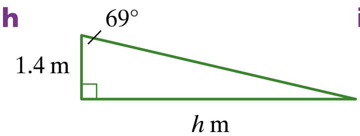
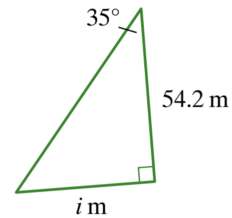


i



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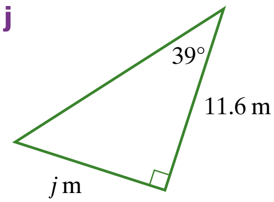

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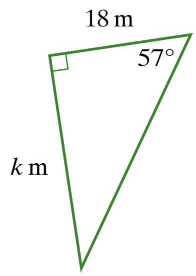

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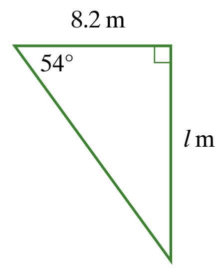
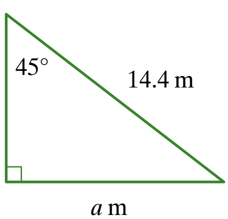


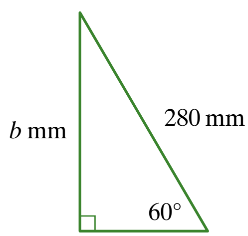
LINKS TO Example 7b

Each of the following questions requires the use of the *sine* of the given angle. In each triangle the unknown side is the side opposite the given angle. Calculate the value of the pronumeral to one decimal place.

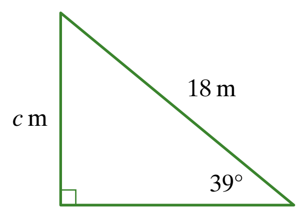
a



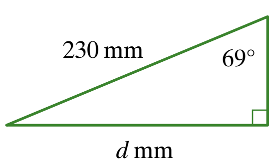
b



c



d



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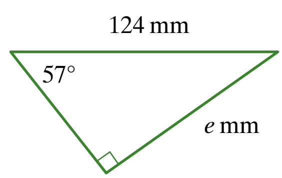

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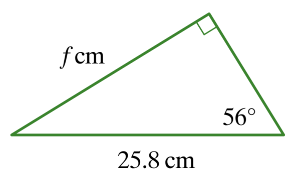

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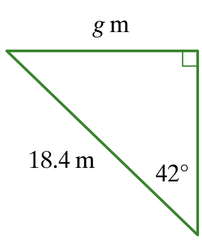

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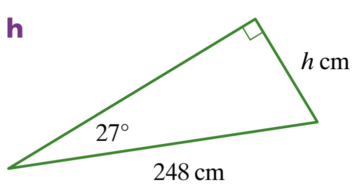
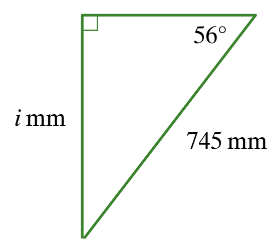
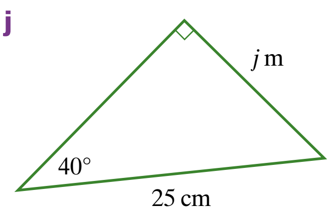


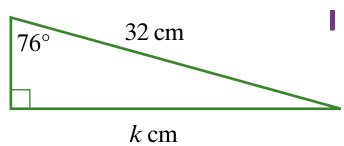
i



j



k



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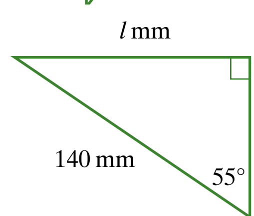

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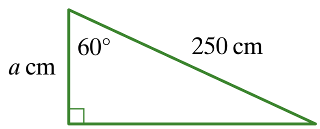
LINKS TO Example 7c

Each of the following questions requires the use of the *cosine* of the given angle. In each triangle the unknown side is the side adjacent to the given angle. Calculate the value of the pronumeral to one decimal place.

a



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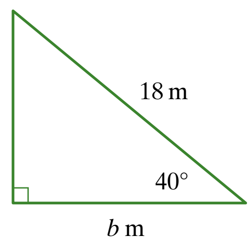

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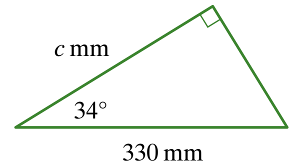

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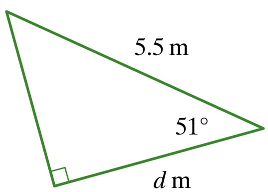

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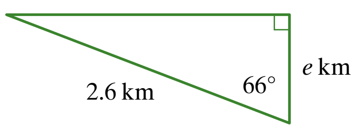

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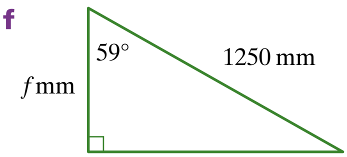

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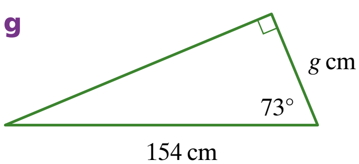

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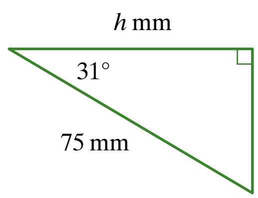

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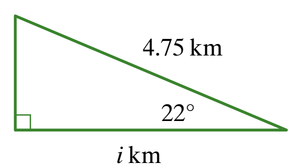

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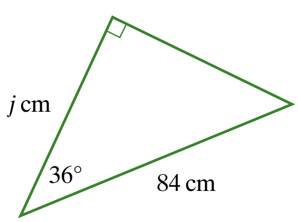

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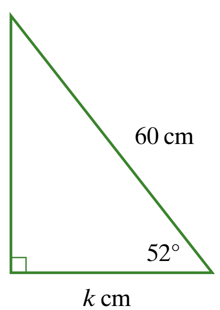

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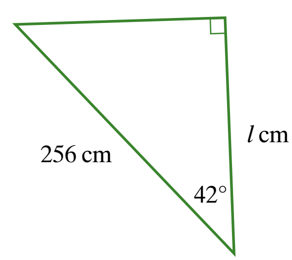

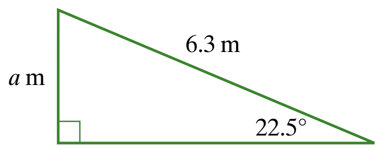

The value of a is given by

A $6.3\cos 22.5°$ **B** $\dfrac{6.3}{\cos 22.5°}$ **C** $6.3\tan 22.5°$ **D** $6.3\sin 22.5°$ **E** $\dfrac{6.3}{\sin 22.5°}$

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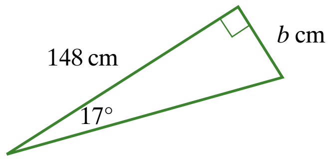


The value of b is given by

A $148\cos 17°$ **B** $\dfrac{148}{\tan 17°}$ **C** $148\tan 17°$ **D** $148\sin 17°$ **E** $\dfrac{148}{\sin 17°}$

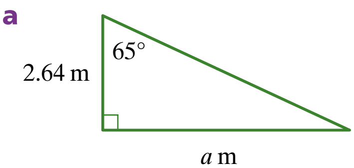
LINKS TO Example 8

For each of the following triangles,

i copy the diagram and mark the given and required sides as either *hypotenuse* or *opposite* or *adjacent* to the given angle.

ii state the appropriate trigonometric ratio, that is, sine, cosine or tangent.

iii use the ratio to write an equation, then solve the equation to find the value of the pronumeral correct to one decimal place.

a


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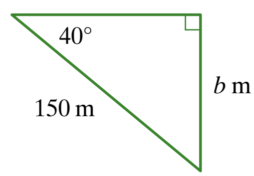

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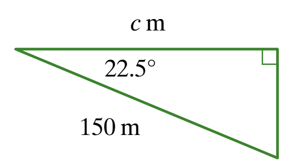

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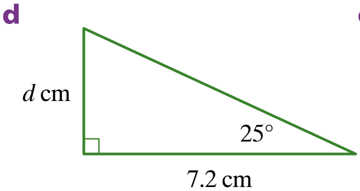

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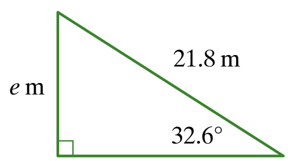

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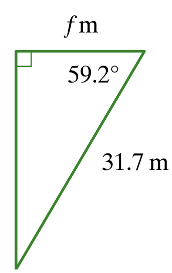

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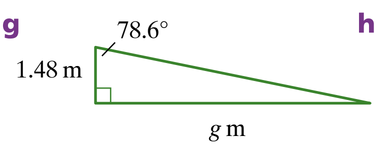

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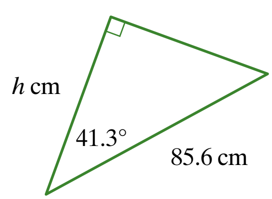

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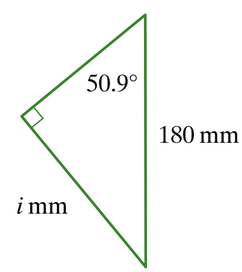

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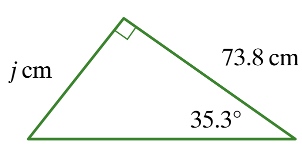

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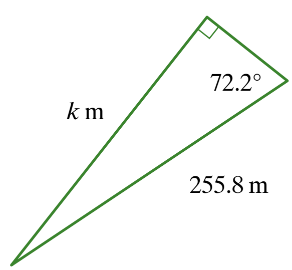

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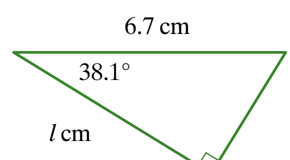

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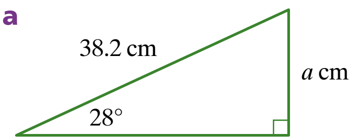
For each of the following triangles,

i identify the trigonometric ratio to use.

ii write an equation.

iii solve the equation to find the value of the pronumeral, giving your answer to one decimal place.

a



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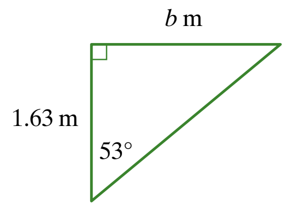

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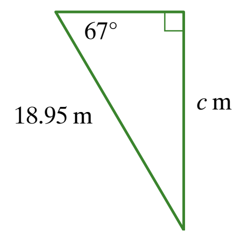

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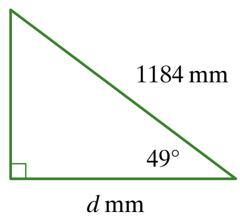

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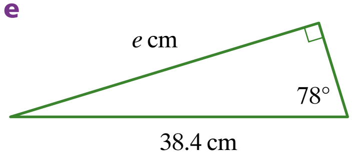

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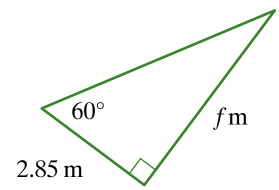

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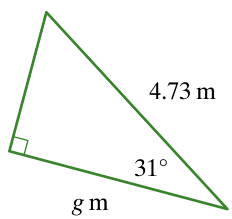

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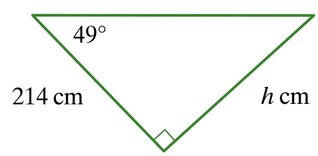

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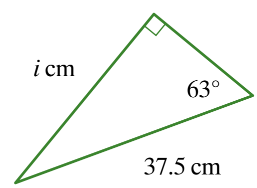

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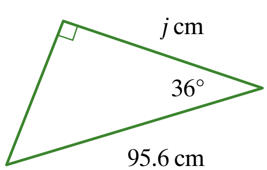

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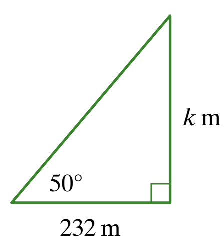

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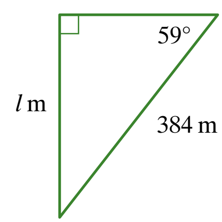

For each of the following triangles,

i copy the diagram and mark the given and required sides as either *hypotenuse* or *opposite* or *adjacent* to the given angle.

ii state the appropriate trigonometric ratio, that is, sine, cosine or tangent.

iii use the ratio to write an equation, then solve the equation to find the value of the pronumeral correct to one decimal place.

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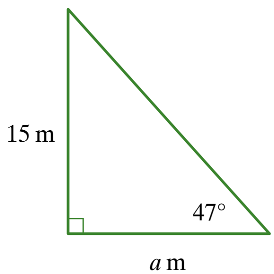
a



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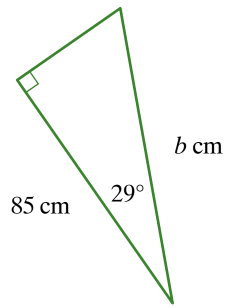

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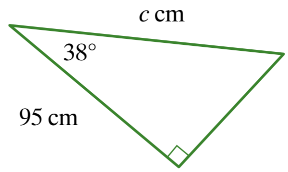

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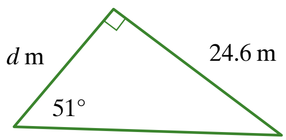

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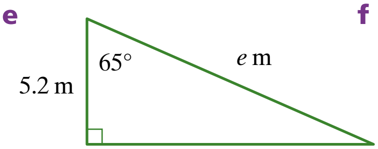

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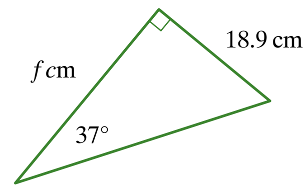

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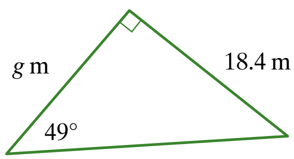

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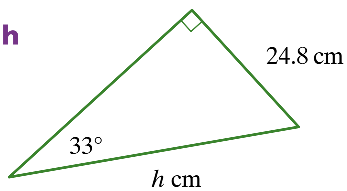

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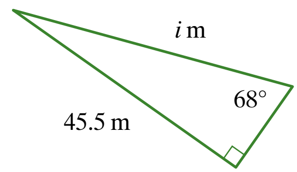

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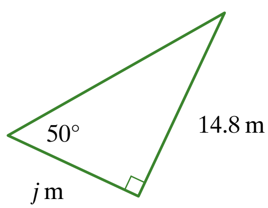

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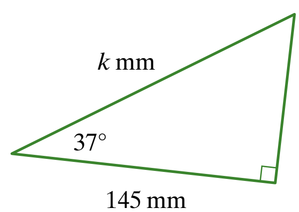

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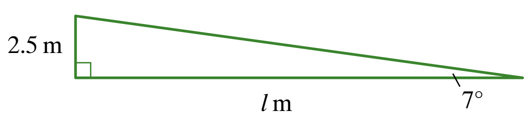

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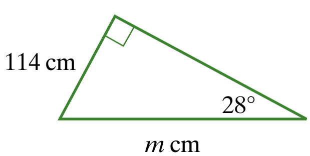

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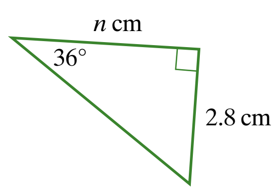

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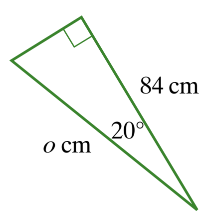

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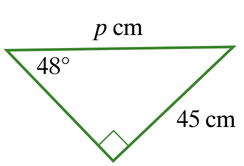

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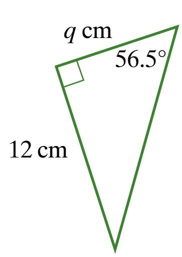

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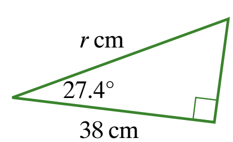

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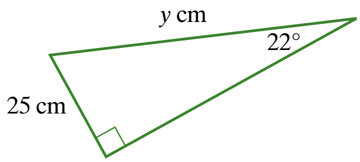
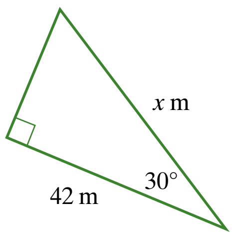


The value of y is given by

A $25 \sin 22°$ **B** $\dfrac{25}{\sin 22°}$ **C** $25 \cos 22°$ **D** $\dfrac{25}{\cos 22°}$ **E** $\dfrac{25}{\tan 22°}$



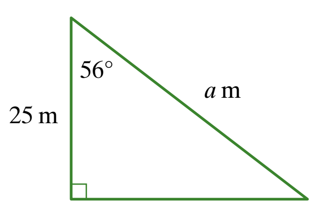
The value of x is given by

A $42 \sin 30°$ **B** $\dfrac{42}{\sin 30°}$ **C** $42 \tan 30°$ **D** $42 \cos 30°$ **E** $\dfrac{42}{\cos 30°}$

For each of the following triangles,

- **i** write an equation.
- **ii** solve the equation to find the value of the pronumeral, giving your answer to one decimal place.

a



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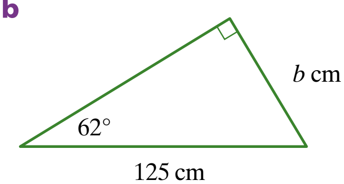

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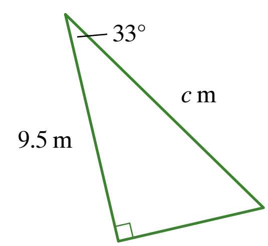

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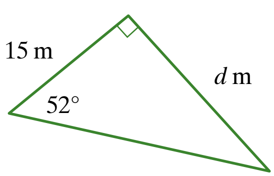

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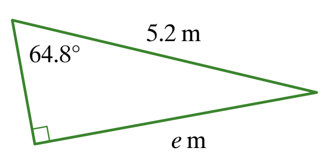

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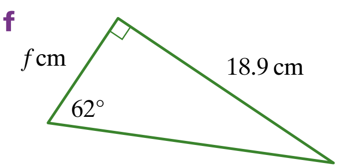

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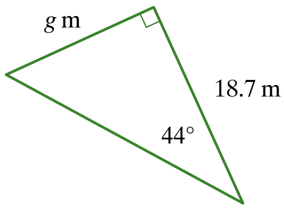

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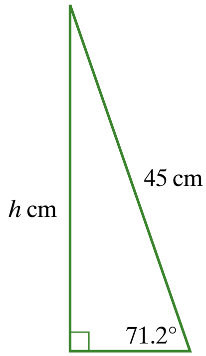

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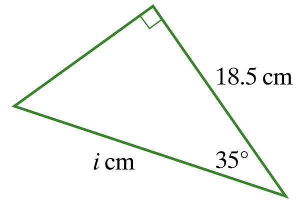

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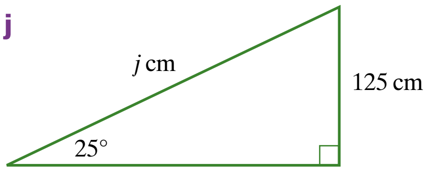

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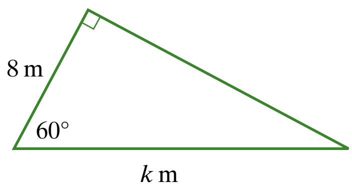

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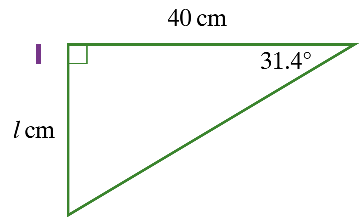


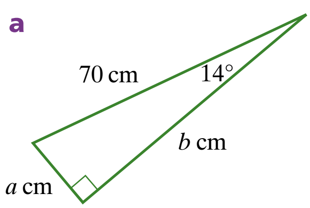
LINKS TO Example 9

For each of these triangles,

a calculate the value of a correct to one decimal place,
  i using Pythagoras' theorem.
  ii using trigonometry.

b calculate the value of b correct to one decimal place,
  i using Pythagoras' theorem.
  ii using trigonometry.

a



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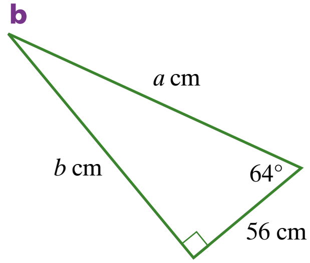

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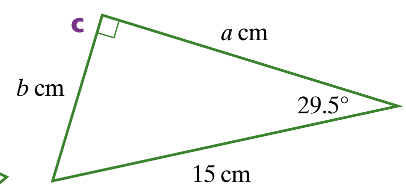

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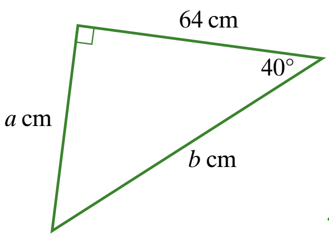

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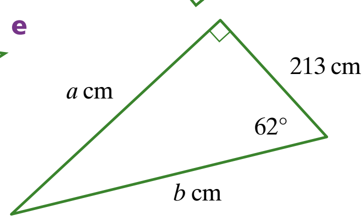

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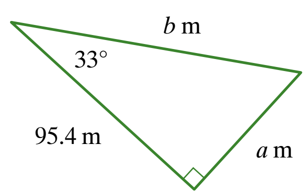

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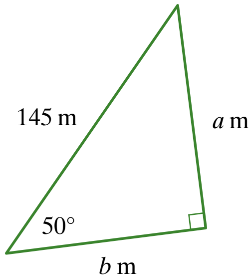
g



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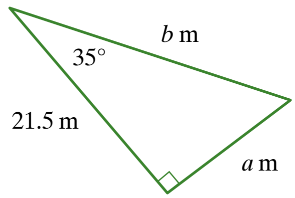

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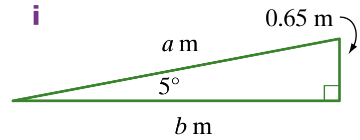
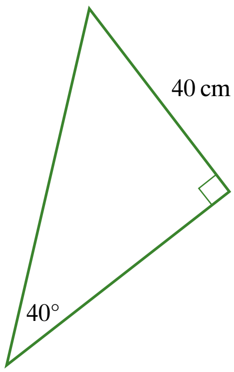


● Find out as much as you can about the sizes of the angles and sides of these triangles.

a



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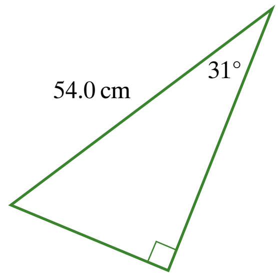
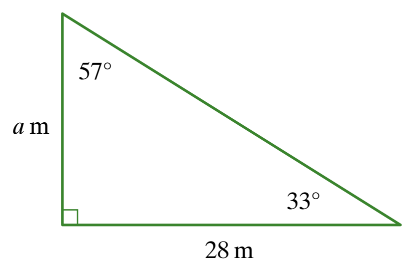


● **a** Consider the triangle at right. Find the value of a correct to one decimal place

i by solving the equation $\tan 33° = \dfrac{a}{28}$

ii by solving the equation $\tan 57° = \dfrac{28}{a}$

b Comment on your answers.



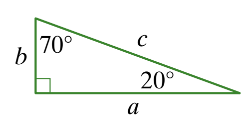
● Angles with a sum of 90° are complementary.

a Use your calculator to find correct to four decimal places.

i cos 20° **ii** sin 70°

b Use this diagram to explain what you observed in part **a**.

c Using the word ‘complementary’, write a sentence stating your observation.



d Copy and complete the following equivalences, giving the ratios correct to three decimal places. The first one is done for you.

i sin 63° = cos 27° = 0.891

ii cos 85° = sin __ = _____

iii sin 0° = cos __ = _____

iv cos 45° = ______ = _____

v sin 58° = ______ = _____

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The length of the hypotenuse of a right-angled triangle is 20 cm. The hypotenuse makes angles of 55° and 35° respectively with the other two sides.

a Draw a labelled diagram of the triangle.

b Find the lengths of the other sides, correct to one decimal place.

exercise 10.3 challenge

A diagonal of a rectangle is 30 cm long and makes an angle of 38° with one of the sides of the rectangle.

a Draw a labelled diagram to show the given information.

b Find the length and the width of the rectangle, correct to one decimal place.

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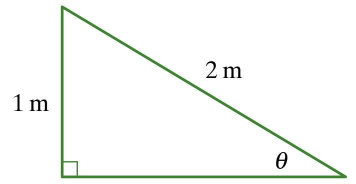
10.4 Calculating angles in right-angled triangles

If we know two of the sides of a right-angled triangle, we can use the trigonometric ratios to find unknown angles. For example, in the following triangle



$$\sin\theta = \frac{1}{2}$$
$$= 0.5$$

> -1 in $\sin^{-1}\theta$ is not an index. It is just a short way of writing 'the angle whose sine is'.

This means that θ is the angle whose sine is 0.5.

The shorthand way of writing this is $\sin^{-1}\theta = 0.5$.

If we know (or can calculate) the value of the sine, cosine or tangent of an angle, we can use a calculator to find the size of the angle.

To find the value of θ we use **inverse sine** on the calculator.

Example 10

For each of the following, θ is an angle between 0° and 90°. Find the value of θ in degrees correct to one decimal place if

a $\theta = \sin^{-1} 0.7145$

b $\tan^{-1}\left(\frac{64}{15}\right)$

c $\cos^{-1}\left(\frac{8.4}{14.6}\right)$

Working	Reasoning
a $\theta = \sin^{-1} 0.7145$ $\theta = 45.6°$ correct to one decimal place	$\theta = \sin^{-1} 0.7145$ means θ is the angle whose sine is 0.7145. $\theta = 45.60\ldots$
b $\theta = \tan^{-1}\left(\frac{64}{15}\right)$ $\theta = 76.8°$ correct to one decimal place	$\theta = \tan^{-1}\left(\frac{64}{15}\right)$ means θ is the angle whose tangent is $\left(\frac{64}{15}\right)$. $\theta = 76.80\ldots$

continued

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Example 10 continued

Working

c $\theta = \cos^{-1}\left(\frac{8.4}{14.6}\right)$

$\theta = 55°$ correct to the nearest degree

Reasoning

$\theta = \cos^{-1}\left(\frac{8.4}{14.6}\right)$ means θ is the angle whose cosine is $\left(\frac{8.4}{14.6}\right)$.

$\theta = 54.87\ldots$

Example 11

For each of the following, θ is an angle between 0° and 90°. Find the value of θ correct to the nearest degree.

a $\sin\theta = 0.4798$ **b** $\cos\theta = \frac{7}{10}$ **c** $\tan\theta = \frac{7.24}{11.5}$

Working

a $\sin\theta = 0.4798$

$\theta = \sin^{-1} 0.4798$

$\theta = 29°$ correct to the nearest degree

Reasoning

$\sin^{-1} 0.4798$ means θ is the angle whose sine is 0.4798.

$\theta = 28.67\ldots$

Working

b $\cos\theta = \frac{7}{10}$

$\theta = \cos^{-1}\left(\frac{7}{10}\right)$

$\theta = 46°$ correct to the nearest degree

Reasoning

$\theta = \cos^{-1}\left(\frac{7}{10}\right)$ means θ is the angle whose cosine is $\frac{7}{10}$.

$\theta = 45.57\ldots$

Working

c $\tan\theta = \frac{72.4}{11.5}$

$\theta = \tan^{-1}\left(\frac{72.4}{11.5}\right)$

$\theta = 81°$ correct to the nearest degree

Reasoning

$\tan^{-1}\left(\frac{72.4}{11.5}\right)$ means θ is the angle whose tangent is $\frac{72.4}{11.5}$.

$\theta = 80.97\ldots$

If we know two side lengths for a right-angled triangle we can calculate the angles.

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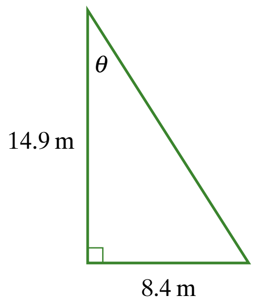
Example 12

For each of the triangles below,

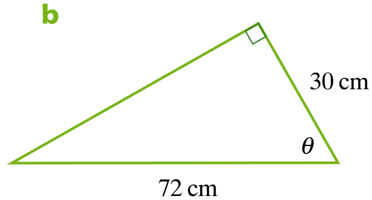
i write an equation.

ii solve the equation to find θ, giving the angle correct to the nearest degree.

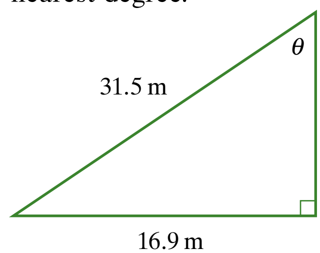
a



b



c



Working	Reasoning
a $\tan\theta = \frac{O}{A}$	The given sides are **o**pposite and **a**djacent to the required angle. SOH-CAH-**TOA** reminds us to use **t**angent.
$\tan\theta = \frac{8.4}{14.9}$	θ is the angle whose tangent is $\frac{8.4}{14.9}$.
$\theta = \tan^{-1}\left(\frac{8.4}{14.9}\right)$	
$\theta \approx 29°$	$\theta = 29.4\ldots$
b $\cos\theta = \frac{A}{H}$	The given sides are the **h**ypotenuse and the side **a**djacent to the required angle. SOH-**CAH**-TOA reminds us to use **c**osine.
$\cos\theta = \frac{30}{72}$	θ is the angle whose cosine is $\frac{30.2}{37.5}$.
$\theta = \cos^{-1}\left(\frac{30}{72}\right)$	
$\theta \approx 65°$	$\theta = 65.3\ldots$
c $\sin\theta = \frac{O}{H}$	The given sides are **o**pposite and **h**ypotenuse. **SOH**-CAH-**TOA** reminds us to use **s**ine.
$\sin\theta = \frac{16.9}{31.5}$	θ is the angle whose sine is $\frac{16.9}{31.5}$.
$\theta = \sin^{-1}\left(\frac{16.9}{31.5}\right)$	
$\theta \approx 32°$	$\theta = 32.44\ldots$

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exercise 10.4

LINKS TO Example 10

1. For each of the following, θ is an angle between 0° and 90°. Find the value of θ in degrees correct to one decimal place if

a $\theta = \sin^{-1} 0.36$	**b** $\theta = \cos^{-1} 0.45$	**c** $\theta = \sin^{-1} 0.84$	**d** $\theta = \tan^{-1} 0.6583$
e $\theta = \tan^{-1} 18.4783$	**f** $\theta = \cos^{-1} 0.06$	**g** $\theta = \sin^{-1} 0.9950$	**h** $\theta = \tan^{-1} 56.3417$
i $\theta = \cos^{-1} 0.5$	**j** $\theta = \tan^{-1} 1$	**k** $\theta = \sin^{-1} 0.7183$	**l** $\theta = \cos^{-1} 0.8582$
m $\theta = \cos^{-1} 0.6$	**n** $\theta = \sin^{-1} 0.75$	**o** $\theta = \sin^{-1} 0.9961$	**p** $\theta = \tan^{-1} 4.1734$
q $\theta = \tan^{-1} 38.1568$	**r** $\theta = \cos^{-1} 0.8$	**s** $\theta = \sin^{-1} 0.9950$	**t** $\theta = \tan^{-1} 56.5$
u $\theta = \cos^{-1}\left(\frac{17}{20}\right)$	**v** $\theta = \tan^{-1}\left(\frac{17}{20}\right)$	**w** $\theta = \sin^{-1}\left(\frac{16}{25}\right)$	**x** $\theta = \cos^{-1}\left(\frac{18.6}{23.4}\right)$

LINKS TO Example 11

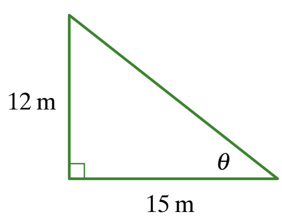
2. For each of the following, θ is an angle between 0° and 90°. Find the value of θ in degrees correct to the nearest degree.

a $\tan\theta = 1.1918$	**b** $\cos\theta = 0.8480$	**c** $\sin\theta = 0.2928$	**d** $\cos\theta = 0.8829$
e $\tan\theta = 1.9626$	**f** $\sin\theta = 0.9945$	**g** $\tan\theta = 3.7321$	**h** $\cos\theta = 0.5592$
i $\sin\theta = 0.4909$	**j** $\tan\theta = 16.3499$	**k** $\cos\theta = 0.9003$	**l** $\tan\theta = 0.1495$
m $\sin\theta = \frac{5}{12}$	**n** $\cos\theta = \frac{16}{25}$	**o** $\sin\theta = \frac{8.6}{14.2}$	**p** $\tan\theta = \frac{45.8}{27.5}$

LINKS TO Example 12

3. For each of the following right-angled triangles, the sides *opposite* and *adjacent* to the angle marked θ have been given.
   - **i** Write an equation for $\tan\theta$.
   - **ii** Solve the equation to find θ, giving the angle correct to the nearest degree.

a



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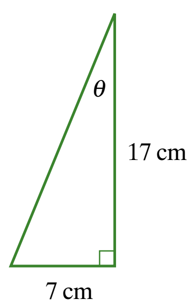

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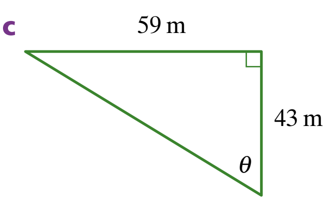

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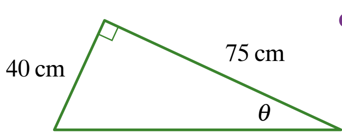

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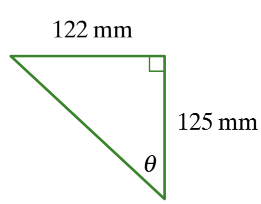

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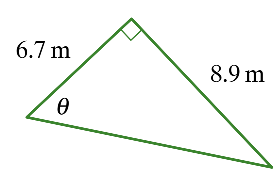

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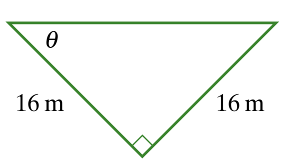

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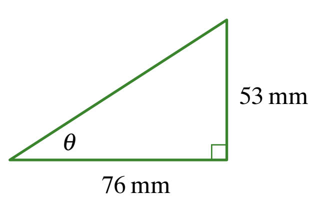
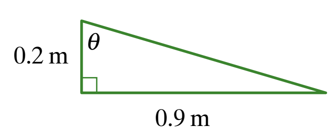


i



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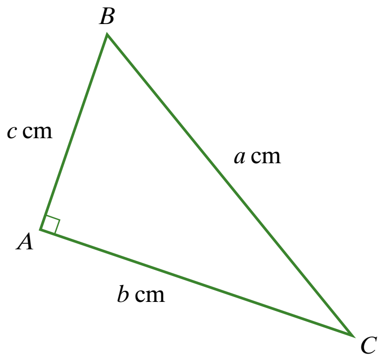


In $\triangle ABC$

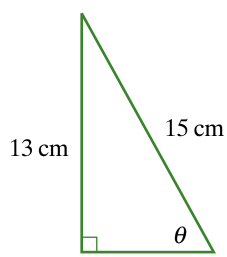
A $\sin B = \dfrac{2}{c}$ **B** $\cos B = \dfrac{b}{c}$ **C** $\tan C = \dfrac{b}{c}$ **D** $\cos C = \dfrac{b}{a}$ **E** $\tan B = \dfrac{c}{a}$

For each of the following right-angled triangles, the *hypotenuse* and the side *opposite* to the angle marked θ have been given.

i Write an equation for $\sin\theta$.

ii Solve the equation to find θ, giving the angle correct to the nearest degree.

a



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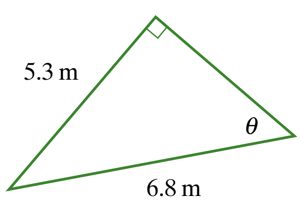

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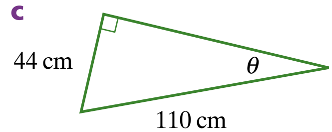

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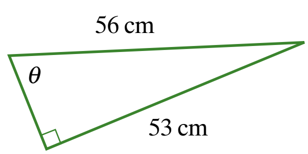

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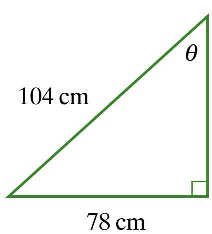

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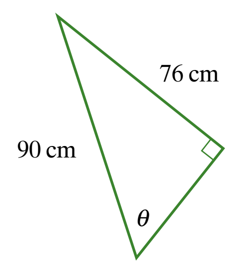

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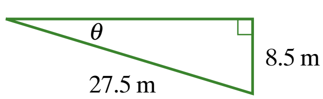

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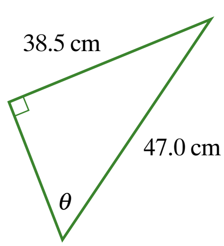

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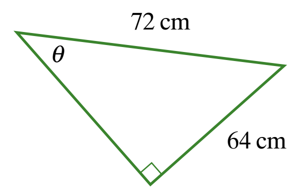

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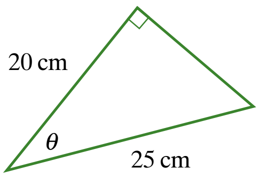
For each of the following right-angled triangles, the *hypotenuse* and the side *adjacent* to the angle marked θ have been given.

i Write an equation for $\cos\theta$.

ii Solve the equation to find θ, giving the angle correct to the nearest degree.

a



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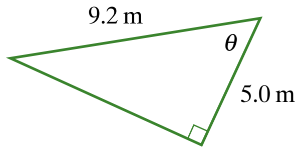

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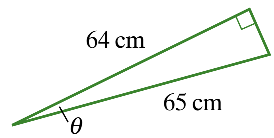

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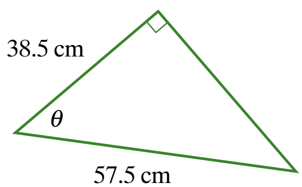

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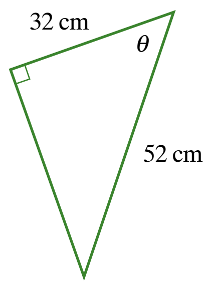

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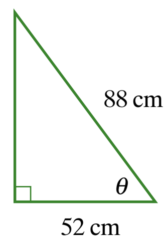

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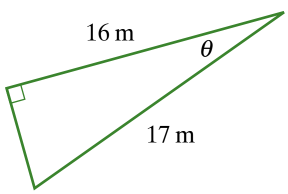

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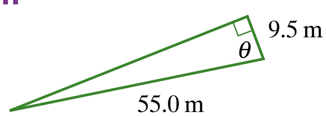
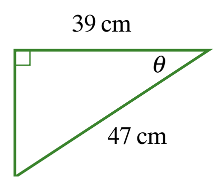
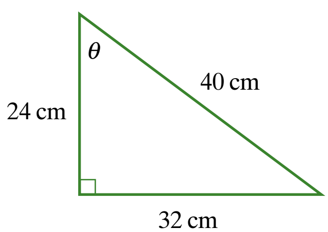


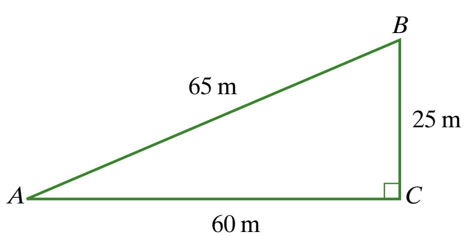
i





Correct to two decimal places, $\tan\theta$ is equal to

A 0.60 **B** 0.75 **C** 0.80 **D** 1.25 **E** 1.33



The size of $\angle ABC$ to the nearest degree is

A 67° **B** 65° **C** 47° **D** 25° **E** 23°

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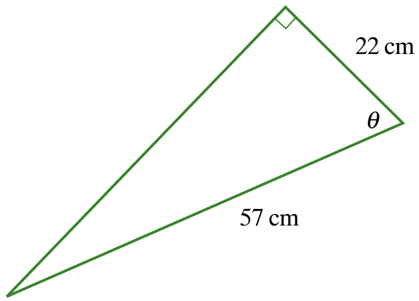


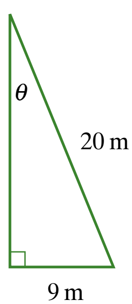
The angle labelled θ is equal to

A $\sin^{-1}\left(\frac{22}{57}\right)$ **B** $\cos^{-1}\left(\frac{57}{22}\right)$ **C** $\cos^{-1}\left(\frac{22}{57}\right)$ **D** $\sin^{-1}\left(\frac{57}{22}\right)$ **E** $\tan^{-1}\left(\frac{22}{57}\right)$

For each of the following right-angled triangles,

i write an equation.
ii solve the equation to find θ, giving the angle correct to the nearest degree.
iii solve the equation to find θ, giving the angle correct to one decimal place.

a



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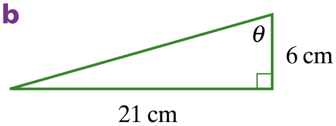

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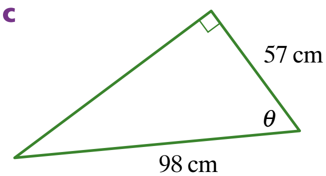

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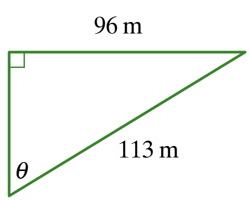

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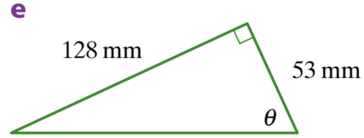

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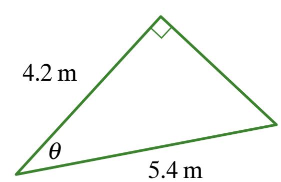

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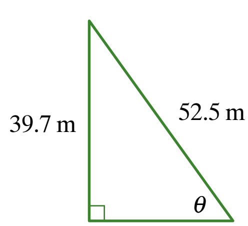

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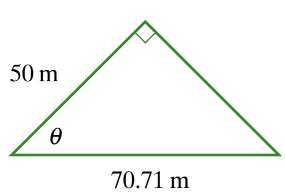
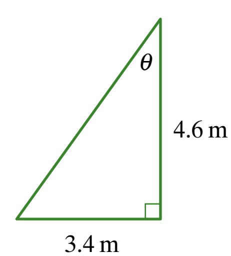


i



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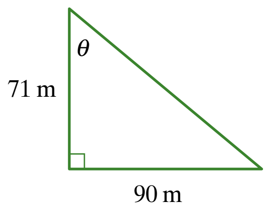
For each of the following right-angled triangles,

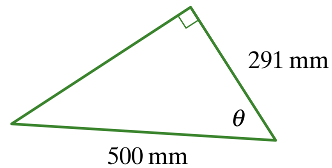
i calculate the size of angle θ correct to one decimal place.

ii use Pythagoras' theorem to calculate the missing side length correct to three significant figures.

a



b



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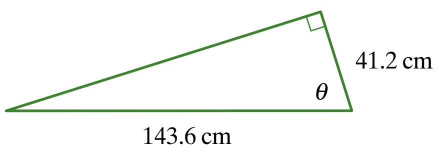

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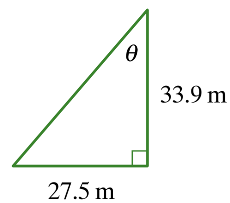

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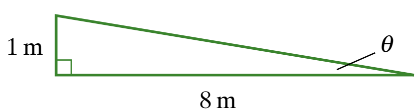

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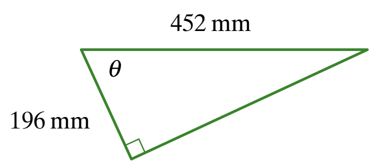

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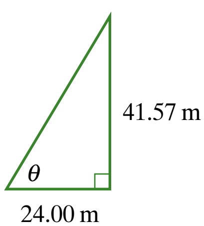

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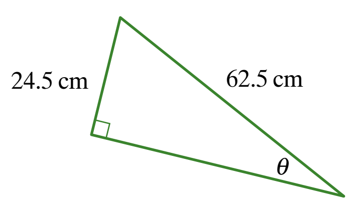
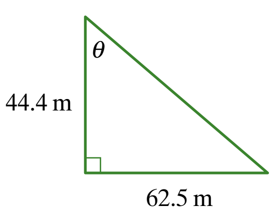


i



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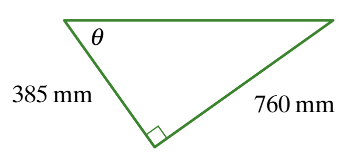

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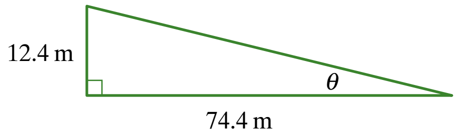
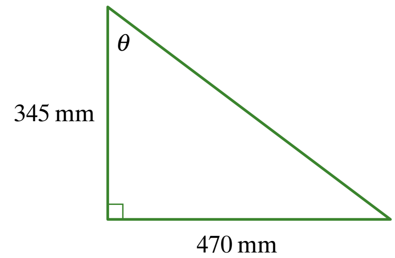
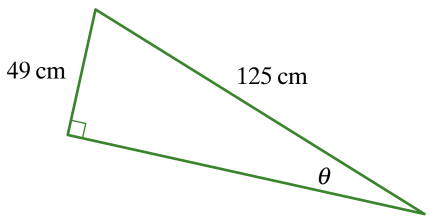
k



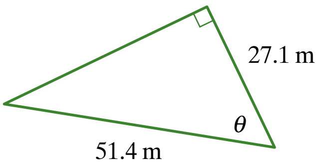
l



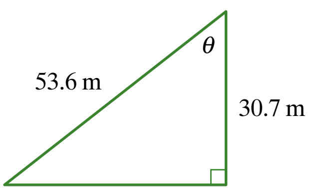
m



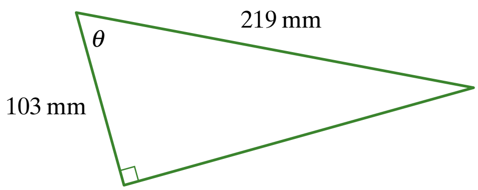
n



o



p



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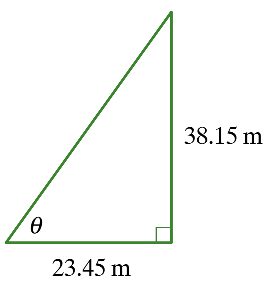
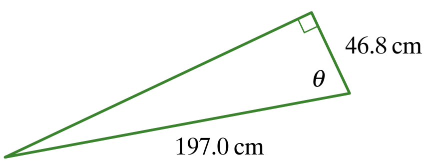
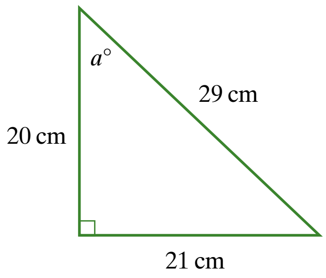


r



Find the value of a correct to the nearest whole number of degrees in three different ways.



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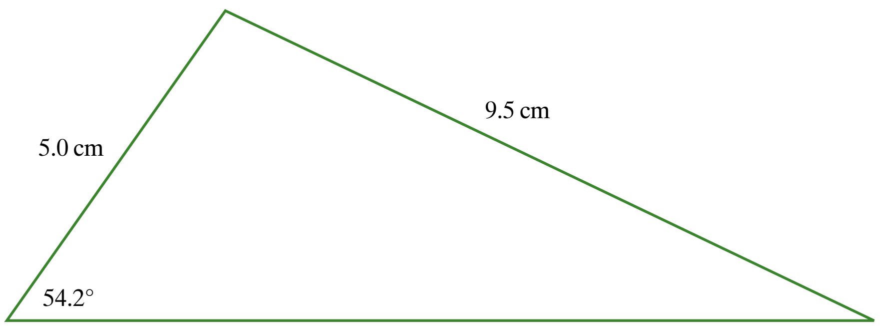
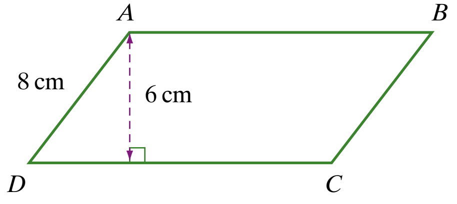
exercise 10.4 challenge

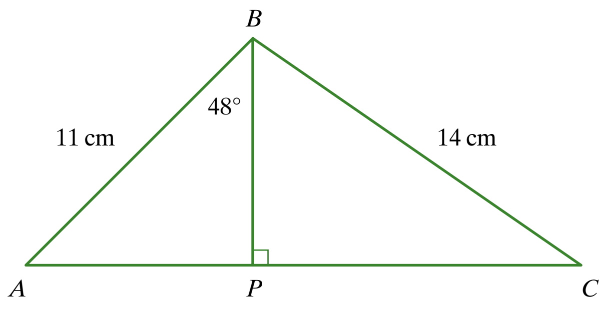
- Find the size of the other angles and the lengths of the other side of this triangle (not to scale). Give the angles and the side length correct to one decimal place. Hint: It is not a right-angled triangle.



- Find the sizes of the angles of this parallelogram, correct to the nearest degree.



- Calculate
  - **a** the length of BP correct to one decimal place.
  - **b** the size of $\angle BCA$ correct to the nearest degree.
  - **c** the length of AC correct to one decimal place.



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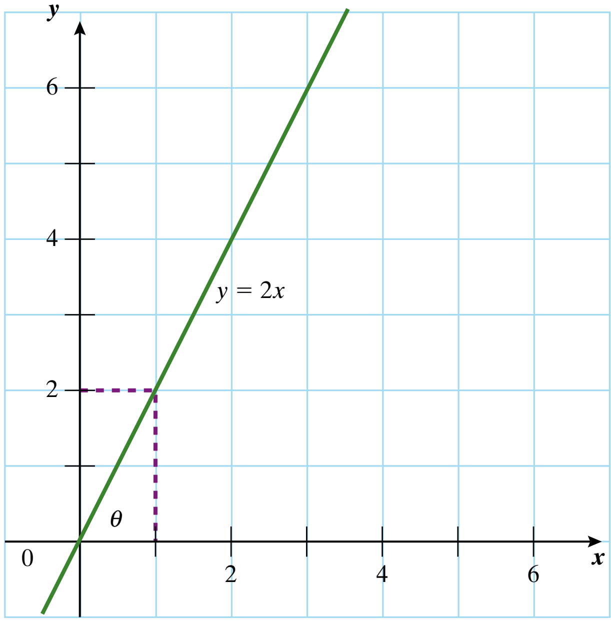
In chapter **6**, we saw that gradient $= \dfrac{\text{rise}}{\text{run}}$.

a If we consider the acute angle θ that the graph of $y = 2x$ makes with the x-axis, which of the three trigonometric ratios is equivalent to the gradient?

b Find acute angle θ that the line $y = 2x$ makes with the x-axis. Answer correct to the nearest degree.

c Find the acute angle correct to the nearest degree that each of these linear graphs makes with the x-axis.



i $y = x$

ii $y = x + 1$

iii $y = 3x$

iv $y = 2x - 3$

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10.5 Applications of trigonometry

Steps for trigonometry problems

Step 1 Draw a diagram and label the given and required sides as A, O or H.

Step 2 Decide which trigonometric ratio to use.

Step 3 Write an equation including the unknown side and solve the equation to find the unknown.

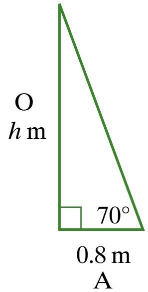
Step 4 Write a sentence to answer the question, rounding your answer to the required number of decimal places.

Example 13

A ladder leaning against a wall makes an angle of 70° with the ground. The foot of the ladder is 0.8 m from the wall. How far up the wall does the ladder reach? Give the height correct to one decimal place.
Use the four steps:

Step 1 Draw a diagram and label the given and required sides as A, O or H.
Step 2 Decide which trigonometric ratio to use.
Step 3 Write an equation including the unknown side and solve the equation to find the unknown.
Step 4 Write a sentence to answer the question, rounding your answer to the required number of decimal places.

Working



Reasoning

Step 1

continued

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Example 13 continued

Working	Reasoning
$\tan\theta = \dfrac{O}{A}$	**Step 2**
$\tan 70° = \dfrac{h}{0.8}$	**Step 3** SOH-CAH-**TOA** reminds us to use **tan**gent.
$h = 0.8 \times \tan 70°$	
$h \approx 2.19$	**Step 4**
The ladder reaches 2.2 m up the wall.	

In some problems, the required length is in the denominator when we write the trigonometric ratio.

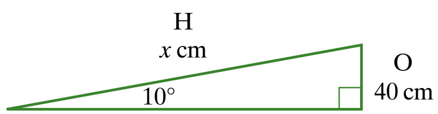
Example 14

A ramp makes an angle of 10° with the horizontal. The top of the ramp is 40 cm above the ground. How long is the sloping ramp? Give the length to the nearest centimetre.
Use the four steps:

Step 1 Copy the diagram and label the given and required sides as A, O or H.
Step 2 Decide which trigonometric ratio to use.
Step 3 Write an equation including the unknown side and solve the equation to find x.
Step 4 Write a sentence, stating the length of the ramp, rounding your answer to the nearest centimetre.

Working — **Reasoning**

Step 1



Step 2

$$\sin\theta = \frac{O}{H}$$

$$\sin 10° = \frac{40}{x}$$

Step 3

$$x \times \sin 10° = 40$$

$$x = \frac{40}{\sin 10°}$$

$$x \approx 230.3$$

Step 4

The length of the ramp is 230 cm.

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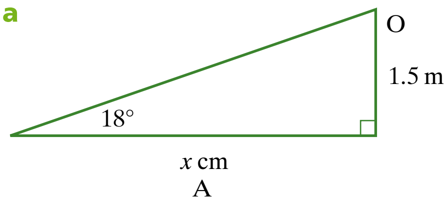
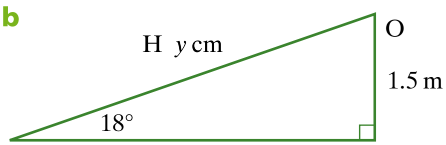
Example 15

A sloping shed roof makes an angle of 18° with the horizontal as shown in the diagram. Use the four steps at the beginning of this section to calculate (correct to two decimal places)

a the width of the shed (labelled x cm).

b the distance up the slope of the roof (labelled y cm).

Working	**Reasoning**
a 	**Step 1**
$\tan\theta = \frac{O}{A}$	**Step 2**
$\tan 18° = \frac{1.5}{x}$	**Step 3**
$x \tan 18° = 1.5$	
$x = \frac{1.5}{\tan 18°}$	
$x \approx 4.616$	
The width of the shed is 4.62 m.	**Step 4**
b 	**Step 1**
$\sin\theta = \frac{O}{H}$	**Step 2**
$\sin 18° = \frac{1.5}{y}$	**Step 3**
$y \sin 18° = 1.5$	
$y = \frac{1.5}{\sin 18°}$	
$= 4.854...$	
The distance up the slope of the roof is 4.85 m.	**Step 4**

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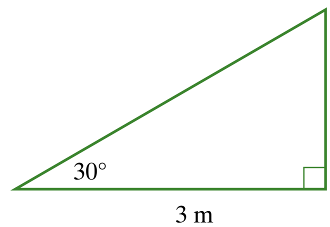
exercise 10.5

For each of the questions in this exercise, use the four steps on page 452 to answer the question.

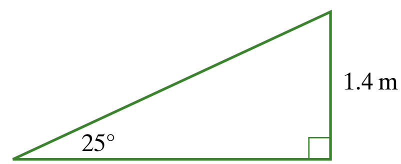
LINKS TO Example 13

A skateboard ramp makes an angle of 30° with the horizontal. The ramp covers a horizontal distance of 3 m. Find the height of the top of the ramp above the ground, correct to one decimal place.



LINKS TO Example 14

A children's slide makes an angle of 25° with the horizontal ground. The top of the slide is 1.4 m above the ground. Find the distance up the slope of the slide, correct to one decimal place.



A ladder leans against a wall, with the foot of the ladder 1.0 m from the ground. The ladder makes an angle of 70° with the ground. How high up the wall does the ladder reach? Give the height correct to one decimal place.

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LINKS TO
Example 15

The bicycle ramp in the photograph makes an angle of 30° with the horizontal and the top is 90 cm above the ground. How long is the ramp? What horizontal distance does it cover, correct to the nearest centimetre?

The top of a ramp for loading and unloading a truck is 1.2 m above the ground. The ramp covers a horizontal distance of 3.7 m. What angle (to the nearest degree) does the ramp make with the horizontal?

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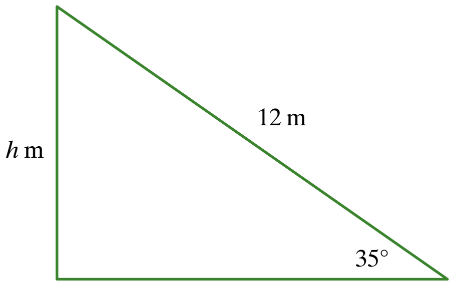
LINKS TO Example 16

The sloping ramp of a flat bank skateboard ramp is 200 cm. The top of the ramp is 70 cm above the ground. What angle (correct to the nearest degree) does the ramp make with the horizontal?

The photograph shows a giant sundial at Singleton in New South Wales. The sloping part of the sundial is called a gnomon (pronounced 'no-mon'). The gnomon is 12 metres long and makes an angle of 35° with the ground. How far above the ground is the upper end of the gnomon? Give your answer correct to one decimal place.



LINKS TO Example 15

An escalator makes an angle of 30° with the horizontal. The escalator covers a horizontal distance of 10 m.

a Find the height of the top of the escalator, correct to two decimal places.

b Find the distance up the slope of the escalator, correct to two decimal places,

  i using Pythagoras' theorem.

  ii using trigonometry.

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The yellow beam of the 'Gateway to Melbourne' on the Tullamarine Freeway is 70 m long, and it makes an angle of 30° with the horizontal. How high above the road is the upper end of the beam?

It is recommended that wheelchair ramps should make an angle of approximately 5° with the horizontal. The wheelchair ramp for a certain building needs to rise 50 cm.

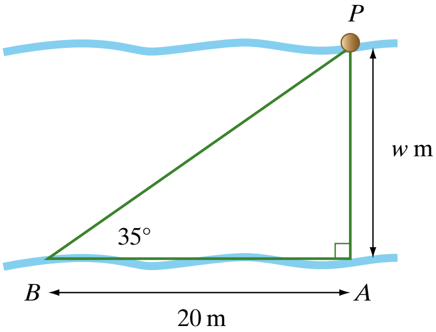
a Draw and label a right-angled triangle to represent the ramp.

b What should the horizontal length of the ramp be?

c What is the length of the slope of the ramp?

Give your answers in metres, correct to two decimal places.

Some scouts are trying to estimate the width of a river. They are at point A directly opposite a post P on the other side of the river. They measure 20 m along the river bank from A to a point B. From here, they measure the angle PBA as 35°. How wide is the river? Give your answer correct to one decimal place.



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exercise 10.5 challenge

1. Alicia is flying a kite. The string of the kite is 27 metres long and makes an angle of 42° with the horizontal when it is stretched out in the wind. Alicia's hand holding the kite string is 1.2 m above the ground.
   - **a** Draw a labeled diagram to show the right-angled triangle formed by the kite string (when stretched out) and the horizontal.
   - **b** What is the vertical height of the kite above the ground? Give the height correct to one decimal place.

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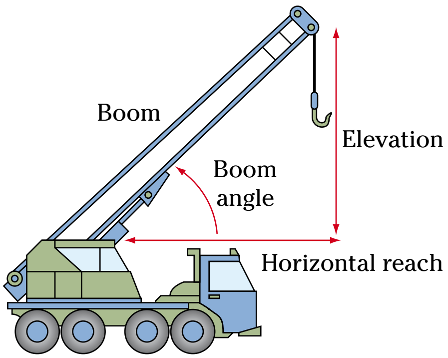
Analysis task

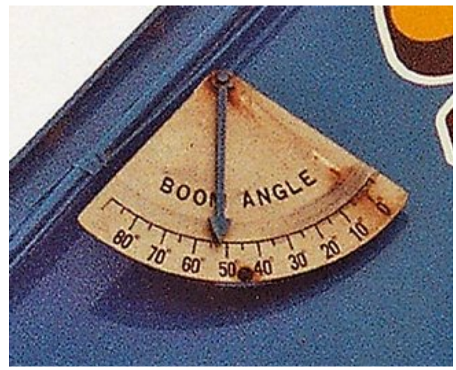
BLM Boom angles

Boom angles

Boom angles

The part of a crane that can be tilted at different angles is called the boom. An angle measurer on the boom indicates the boom angle, that is, the angle the boom makes with the horizontal.



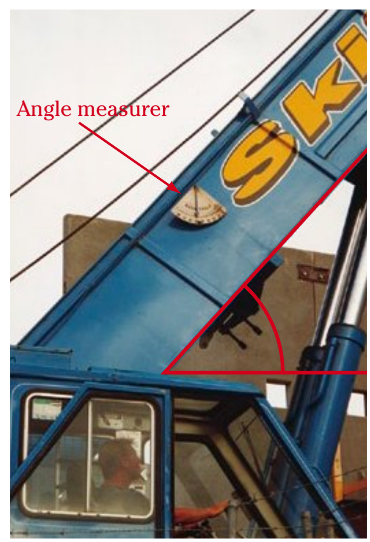


BLM Clinometer

The boom is telescopic and its length can be changed, depending on how far it must reach to set down its load. To move a heavy load, the crane is set at a certain angle, and the load is raised or lowered on a cable at the end of the boom.

If the angle is too small for a certain load, the crane can tip over, especially when the boom is fully extended. The crane operator refers to a **load chart** to find the boom angle that is safe for a particular load, and how far the crane can reach horizontally for that angle and that boom length.

The crane shown in this photograph is lifting concrete walls into position on a building site. The telescopic boom is extended to 31.6 m.



a The following table shows the loads which can be safely lifted for different boom angles when the boom is fully extended to 31.6 m. Use appropriate trigonometric ratios to calculate the horizontal reach and the elevation of the top of the boom for each of the given angles. A table is provided in the student ebook.

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Boom length 31.6 m			
Boom angle (°)	**Load (kg)**	**Horizontal reach (m)**	**Elevation (m)**
79.0	10 000		
77.0	10 000		
75.0	10 000		
73.5	9 000		
71.5	8 100		
67.5	6 550		
63.5	5 000		
59.0	4 000		
55.0	3 200		
50.0	2 500		
45.5	1 800		
40.0	1 300		
33.5	900		
25.5	650		

b After you calculated the horizontal reach, what other method could you have used to calculate the elevation without using trigonometry? Use this method for the first three angles and confirm that you obtain the same value for the elevation.

c Use a printed copy of the graph grid provided in the student ebook to plot a graph of the elevation against horizontal reach. The graph represents the path of the end of the boom as the boom angle changes.

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The following table shows the loads which can be safely lifted at different boom angles for other lengths of the telescopic boom.

Boom length 10.3 m		Boom length 17.4 m		Boom length 24.5 m	
Boom angle (°)	Load (kg)	Boom angle (°)	Load (kg)	Boom angle (°)	Load (kg)
67.5	41 000	77.5	23 500	79.0	15 000
64.5	35 000	75.5	23 500	78.0	15 000
61.5	31 200	74.0	23 500	77.0	15 000
58.0	27 800	72.0	23 000	74.5	15 000
55.0	24 600	70.5	21 500	72.0	13 700
47.5	20 800	67.0	18 000	69.5	12 000
39.5	16 200	63.0	15 000	67.0	10 000
30.0	12 800	59.0	12 200	64.5	8 450
15.0	10 100	55.5	10 100	59.0	6 000
		51.5	8 100	53.5	4 600
		42.5	5 650	47.0	3 400
		31.5	3 950	40.5	2 400
		14.0	2 550	33.0	1 700
				22.0	1 100

d For each boom length, calculate the horizontal reach and the elevation of the top of the boom for each of the given angles. A table is provided in the student e-book.

e Using the same graph grid as for part **c**, plot the paired values for horizontal range and elevation for each of the three boom lengths. You will then have a range diagram similar to the one used by the crane operator. Label each of the four sets of points with the correct boom length.

f Suppose a crane needs to be able to reach 8 m horizontally. Compare the approximate loads that the crane would be able to safely lift for each of the four boom lengths.

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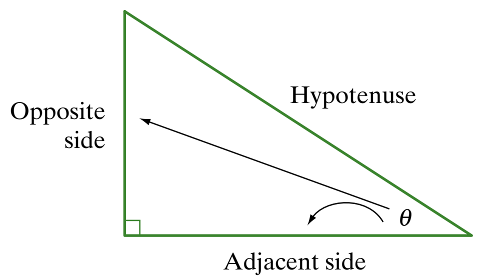
Review Trigonometry

Summary

Trigonometric ratios

- $\sin\theta = \dfrac{\text{Opposite side}}{\text{Hypotenuse}}$
- $\cos\theta = \dfrac{\text{Adjacent side}}{\text{Hypotenuse}}$
- $\tan\theta = \dfrac{\text{Opposite side}}{\text{Adjacent side}}$



Finding sides

- If one side and an angle of a right-angled triangle are known, trigonometry can be used to find another side.
- If two sides and an angle of a right-angled triangle are known, the third side can be found either by trigonometry or by Pythagoras' theorem.

Finding angles

- $\theta = \sin^{-1} 0.4$ means 'the angle whose sine is 0.4'. It works the same way for cosine and tangent ratios.

Steps for using trigonometry to solve word problems

Step 1 Draw a diagram and label the given and required sides as A, O or H.

Step 2 Decide which trigonometric ratio to use.

Step 3 Write an equation including the unknown side and solve the equation to find the unknown.

Step 4 Write a sentence to answer the question, rounding your answer to the required number of decimal places.

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Visual map

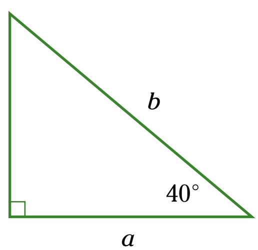
Using the following terms (and others if you wish), construct a visual map that illustrates your understanding of the key ideas covered in this chapter.

adjacent side	Pythagoras' theorem
cosine	sine
hypotenuse	tangent
opposite side	trigonometric ratio

Revision

Multiple-choice questions

1 The value of a in the diagram below is given by

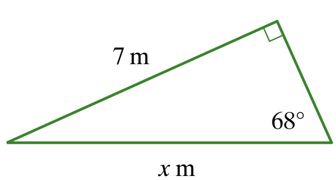


A $b\cos 40°$ **B** $\dfrac{b}{\cos 40°}$

C $b\sin 40°$ **D** $\dfrac{b}{\sin 40°}$

E $\dfrac{\sin 40°}{b}$

2



The pronumeral x is equal to

A $7\tan 68°$ **B** $7\sin 68°$ **C** $\dfrac{7}{\tan 68°}$

D $\dfrac{7}{\cos 68°}$ **E** $\dfrac{7}{\sin 68°}$

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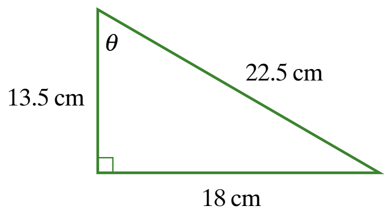

In this triangle, sin θ is equal to

A $\dfrac{13.5}{18}$ **B** $\dfrac{18}{22.5}$ **C** $\dfrac{13.5}{22.5}$

D $\dfrac{18}{13.5}$ **E** $\dfrac{22.5}{18}$

The length of the shadow of a post is 5 m when the angle of elevation of the sun is 37°. The height of the post in metres is equal to

A $5 \times \tan 37°$ **B** $5 \times \sin 37°$ **C** $\dfrac{5}{\tan 37°}$

D $\dfrac{\tan 37°}{5}$ E $\dfrac{5}{\sin 37°}$

A ladder that is 2.6 m long has its foot resting on horizontal ground, with the other end leaning against a wall. The top of the ladder is 1.9 m from the ground. The angle that the ladder makes with the vertical wall is closest to

A 36° **B** 43° **C** 47°

D 54° **E** 57°

Short-answer questions

Use your calculator to find the value of each of the following correct to four decimal places.

a tan 48° **b** sin 17° **c** cos 38.5° **d** sin 60°

Evaluate each of the following correct to one decimal place.

a 12 sin 65° **b** 33 cos 57°

c $\dfrac{18.5}{\tan 35°}$ **d** $\dfrac{72.5}{\cos 15°}$

Find the value of x correct to one decimal place for each of the following.

a $\tan 36° = \dfrac{x}{15}$ **b** $\cos 28° = \dfrac{x}{120}$

c $\sin 48° = \dfrac{12}{x}$ **d** $\tan 5° = \dfrac{2.6}{x}$

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For each of the following, θ is an angle between 0° and 90°. Find the value of θ in degrees correct to the nearest degree.

a $\theta = \cos^{-1} 0.09$

b $\theta = \sin^{-1} 0.65$

c $\theta = \tan^{-1} 6.32$

d $\theta = \sin^{-1} 0.12$

For each of the following, θ is an angle between 0° and 90°. Find the value of θ in degrees correct to the nearest degree.

a $\tan\theta = 2.1315$

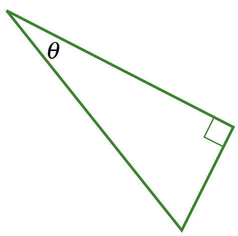
b $\cos\theta = 0.7640$

c $\sin\theta = 0.3148$

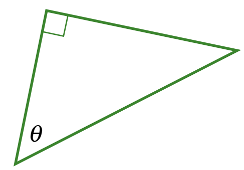
d $\cos\theta = 0.8264$

Label the sides hypotenuse (H),opposite (O) and adjacent (A) with respect to the labelled angle.

a

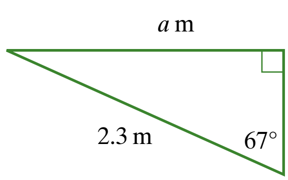


b

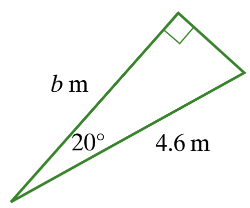


Find the value of the pronumerals in each of the following, correct to one decimal place.

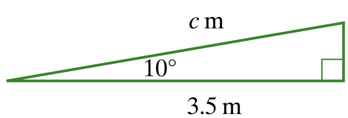
a



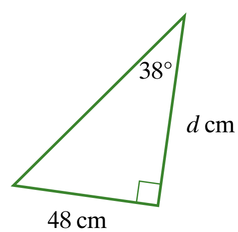
b



c

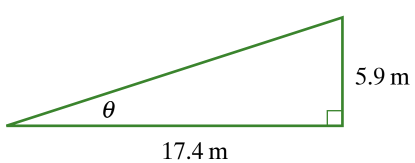


d

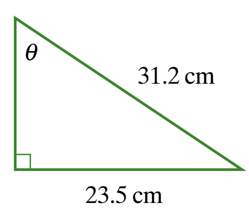


Find the size of the angle θ in each of the following triangles. In each case, give your answer correct to the nearest degree.

a



b



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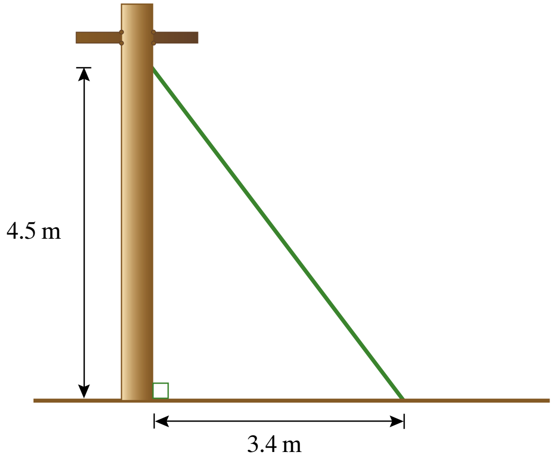
The stairs in the photograph make an angle of 35° with the horizontal. Find the height, h m, correct to one decimal place.

A cable is attached to a pole 4.5 m above the ground as shown. The cable is attached to a peg in the ground 3.4 m from the base of the pole.



a What angle does the cable make with the ground? Answer correct to the nearest degree.

b Calculate the length of the cable, correct to one decimal place,

 i using Pythagoras' theorem.

 ii using trigonometry and the angle you calculated in part **a**.

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Extended-response questions

A kite is flying at the end of a 40 m string. The string makes an angle of 36° with the ground.

a Draw a labelled diagram.

b How high, to the nearest metre, is the kite flying?

A 2.5 m ladder rests against a wall so that it makes an angle of 65° with the ground.

a Draw a labeled diagram.

b How far is the foot of the ladder from the wall? Give the distance correct to one decimal place.

Practice quiz
Chapter 10

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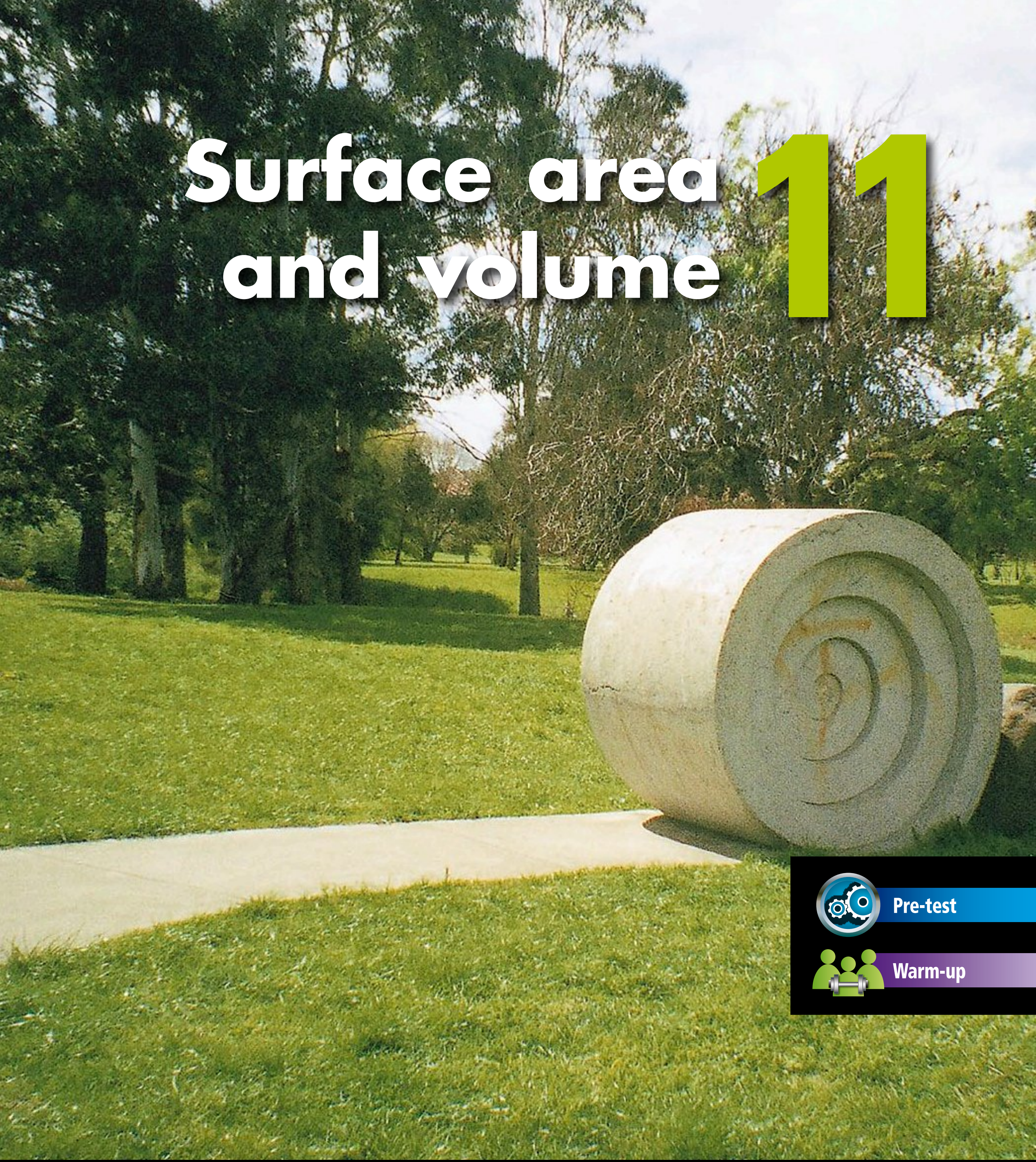
Surface area and volume 11



Pre-test

Warm-up

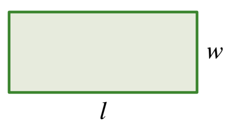
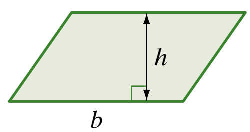
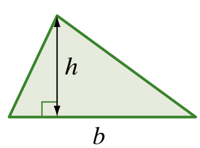
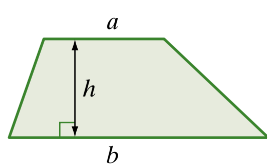
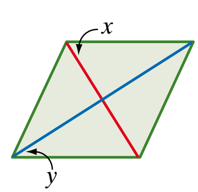
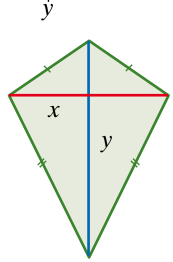
This sculpture by Simon Perry is called Rolling Path. What length of footpath might be rolled up? How could you work out the relationship between the length of the rolled-up path and the diameter of the roll? How would the sculptor calculate the volume of concrete required to make the sculpture?

11.1 Reviewing area calculation

BLM
Chapter warm-up

In Year 8, the areas of triangles, special quadrilaterals (rectangle, parallelogram, trapezium, rhombus and kite) and circles were calculated. The formulas for these areas are summarised below.

Exercise **11.1** provides some revision of working with these formulas for simple and compound shapes. The formulas will be used again in this chapter in calculating surface area and volume of three-dimensional shapes.

	Shape	Dimensions	Area formula
Rectangle		l = length w = width	$A = lw$
Parallelogram		b = base h = perpendicular height	$A = bh$
Triangle		b = base h = perpendicular height	$A = \frac{bh}{2}$
Trapezium		a and b = lengths of parallel sides h = perpendicular height	$A = \frac{(a + b)h}{2}$
Rhombus and kite		x and y = lengths of the diagonals.	$A = \frac{xy}{2}$

continued

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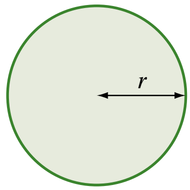
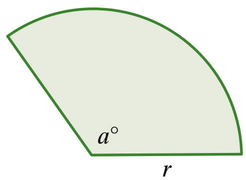
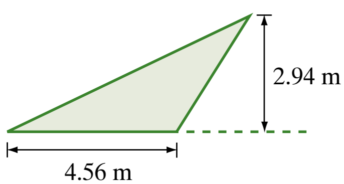
	Shape	Dimensions	Area formula
Circle		r = radius	$A = \pi r^2$
Sector			$A = \pi r^2 \times \dfrac{\alpha}{360}$

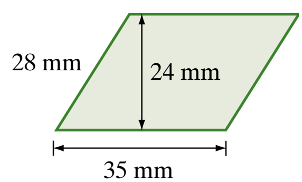
Example 1

Calculate the area of each of the following, giving each area correct to the same number of decimal places as in the given dimensions.

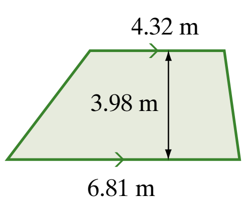
a



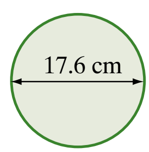
b



c



d



Working

a $A = \dfrac{bh}{2}$

$= \dfrac{4.56 \times 2.94}{2}$

$= 6.7032$

The area is 6.70 m^2.

b $A = bh$

$= 35 = 24$

$= 840$

The area is 840 mm^2.

Reasoning

The base of the triangle is 4.56 m.
The height of the triangle is 2.94 m.

The base of the parallelogram is 35 mm.
The height of the parallelogram is 24 mm.
We do not need to use the side length 28 mm.

continued

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Example 1 continued

Working

c $A = \dfrac{(a+b)h}{2}$

$= \dfrac{(4.32 + 6.81) \times 3.98}{2}$

$= 22.1487$

The area is 22.15 m^2.

d Radius $= 17.6 \div 2$

$= 8.8$ cm

$A = \pi r^2$

$= \pi \times 8.8^2$

≈ 243.285

The area is 243.3 cm^2.

Reasoning

Find the average of the two parallel sides, 4.32 m and 6.81 m.
Multiply by the height, 3.98 m.
Round to two decimal places because the given dimensions have two decimal places.

The radius is half the diameter,
Round to one decimal place because the diameter is given to one decimal place.

The area of a kite or rhombus can be calculated if we know the lengths of the two diagonals. (Note that a square is also a rhombus so the same method can be used to calculate the area of a square. In the case of the square the two diagonals have the same length.)

Example 2

Calculate the area of a rhombus if the lengths of the diagonals are 46 cm and 28 cm.

Working

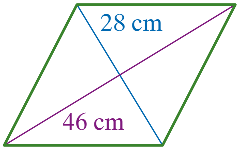
$A = \dfrac{xy}{2}$

$= \dfrac{28 \times 46}{2}$

$= 644$

The area is 644 cm^2.

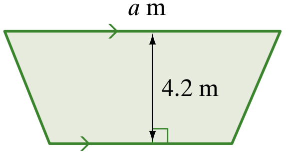
Reasoning



Example 3

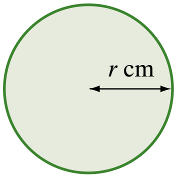
Calculate the unknown dimension for each of these shapes.

a



Area = 28.35 m^2

b



Area = 240 cm^2

continued

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Example 3 continued

Working

a $A = \dfrac{(a+b)h}{2}$

$\dfrac{(a+4.8)\times 4.2}{2}$

$\dfrac{28.35\times 2}{4.2} = a + 4.8$

$13.5 = a + 4.8$

$13.5 - 4.8 = a$

$8.7 = a$

The unknown side is 8.7 m.

Reasoning

The area of a trapezium is given by the formula $A = \dfrac{(a+b)h}{2}$, where a and b are the lengths of the two parallel sides and h is the perpendicular height. Substitute the values for A, b and h.

Solve the equation to find a.

Working

b $A = \pi r^2$

$240 = \pi r^2$

$r^2 = \dfrac{240}{\pi}$

≈ 76.39

$r \approx \sqrt{76.39}$

$= 8.7$

The radius is approximately 8.7 cm.

Reasoning

Substitute 240 for A in the formula for the area of a circle.

Divide both sides by π to find r^2.

To find r, take the square root of both sides.

To calculate the area of a sector of a circle, we use the sector angle to calculate the area as a fraction of the area of a whole circle with the same radius.

Example 4

Calculate the area of a sector correct to the nearest square centimetre.

a radius 15 cm and angle 48° **b** radius 2.6 m and angle 1.8c.

Working

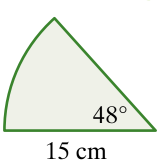
$A = \pi r^2 \times \dfrac{\alpha}{360}$

$= \pi \times 15^2 \times \dfrac{48}{360}$

≈ 94.2

The area is approximately 94 cm^2.

Reasoning



The area is the fraction $\dfrac{48}{360}$ of the area of a circle with radius 15.

To calculate the area that is enclosed between two shapes, that is, one or more regions appear as 'holes' within an outer region, we subtract the 'holes' from the outer region.

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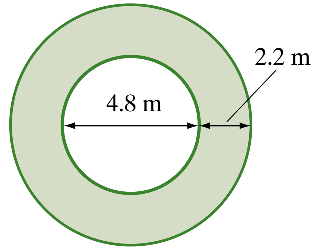
Example 5

Find the area of this annulus correct to the nearest square metre (an annalus is a ring).



Working

Diameter of inner circle = 4.8 m
Radius of inner circle (r) = 2.4 m
Radius of outer circle (R) = 2.4 + 2.2 m = 4.6 m

$A = \pi R^2 - \pi r^2$

$A = \pi \times 4.6^2 - \pi \times 2.4^2$

≈ 48.38

The area is approximately 48 m^2.

Reasoning

To calculate the area of a circle, we need to know the radius.

The shaded area is calculated by subtracting the area of the inner circle from the area of the outer circle.

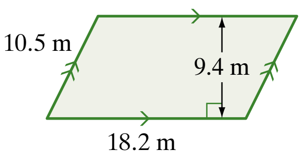
Round to the nearest square metre.

exercise 11.1

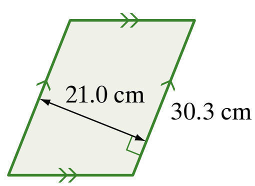
LINKS TO Examples 1, 2

1. Calculate the area of each of these quadrilaterals, giving each area correct to the same number of decimal places as in the given dimensions.

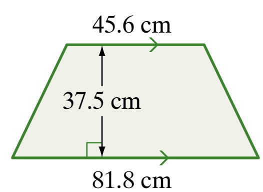
a



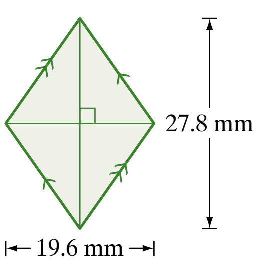
b



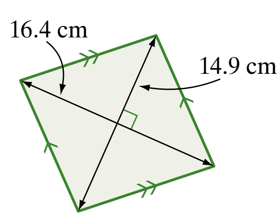
c



d



e



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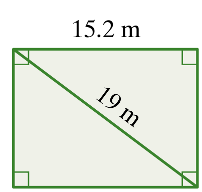
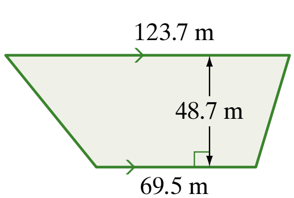
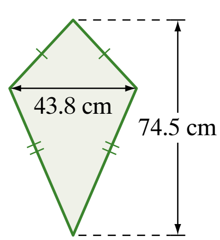


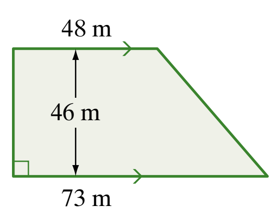
g



h



i



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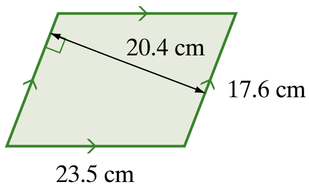
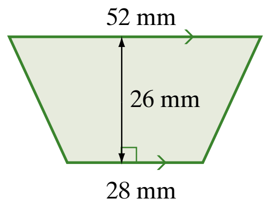
LINKS TO Examples 1, 2

Calculate the area of each of the following shapes. Give your answers correct to the same number of decimal places as for the given dimensions.

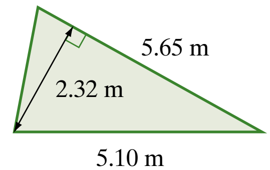
a



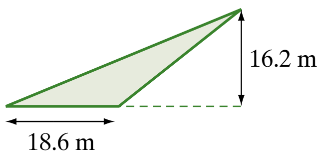
b



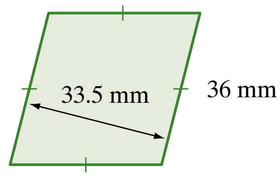
c



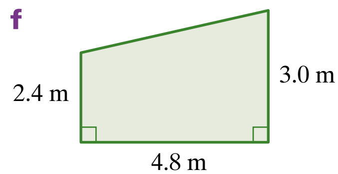
d



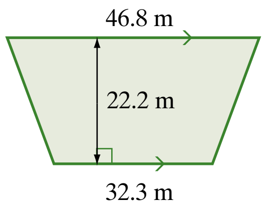
e



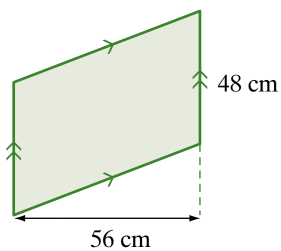
f



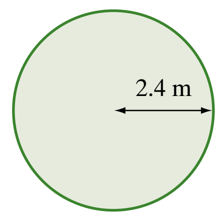
g



h



i



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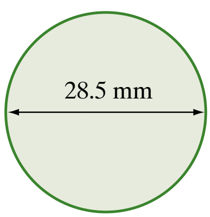
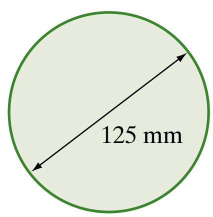
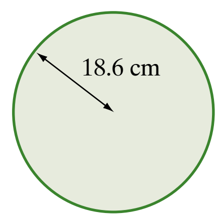


k



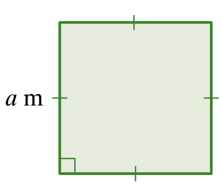
l



LINKS TO Example 3

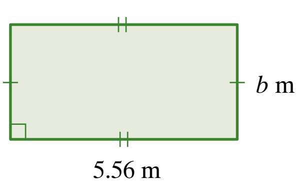
Calculate the labelled dimension for each of the following shapes, giving your answers to the same number of decimal places as for the given area.

a



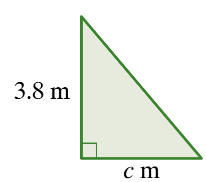
Area = 24.01 m^2

b



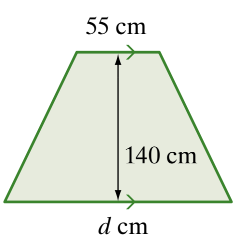
Area = 10.90 m^2

c



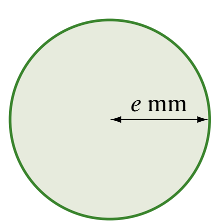
Area = 5.51 m^2

d



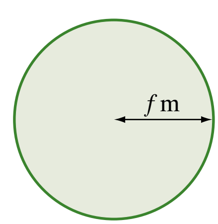
Area = 16 100 cm^2

e



Area = 707 mm^2

f



Area = 55.4 m^2

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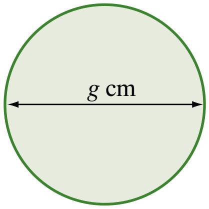
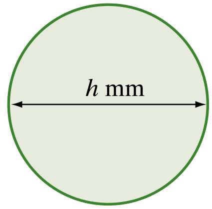
g



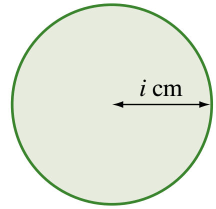
Area = 1809 cm²

h



Area = 4540 mm²

i

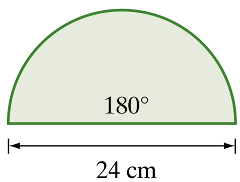


Area = 250 cm²

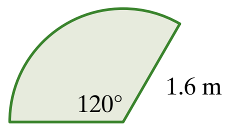
LINKS TO Example 4

Find the area of each of these sectors. Give your answers correct to the same number of decimal places as for the given dimensions.

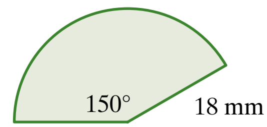
a



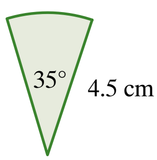
b



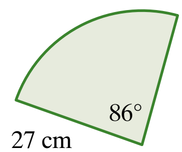
c



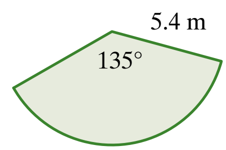
d



e



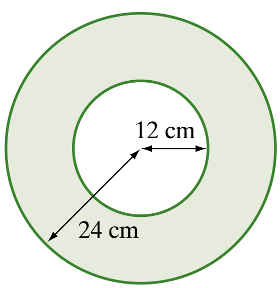
f



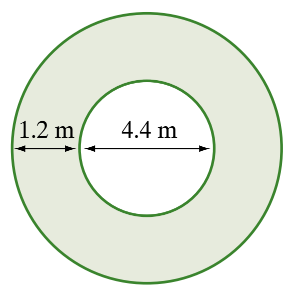
LINKS TO Example 5

Calculate the area of each annulus correct to one decimal place.

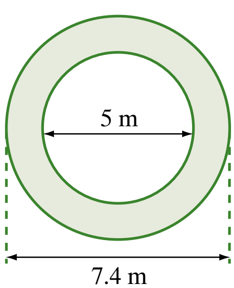
a



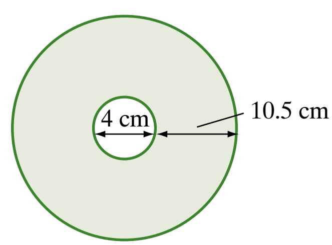
b



c



d



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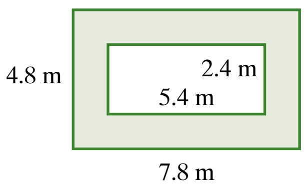
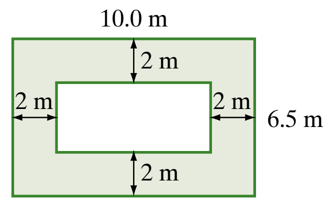
11.1

Calculate the following shaded areas, giving each answer correct to one decimal place.

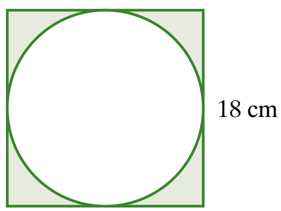
a



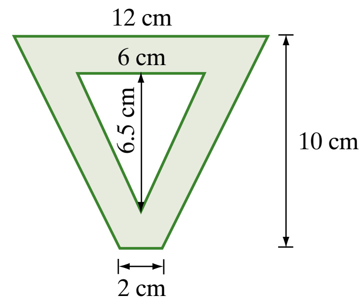
b



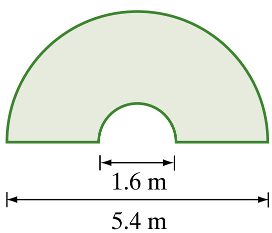
c



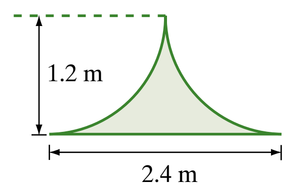
d



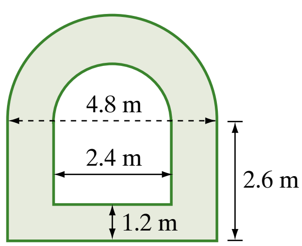
e



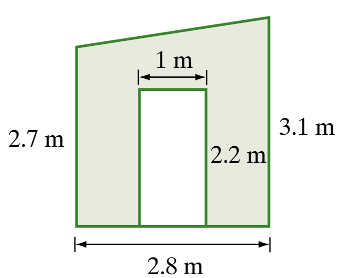
f



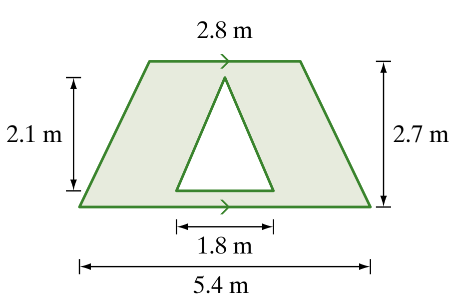
g



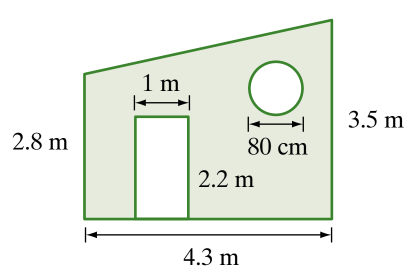
h



i



j



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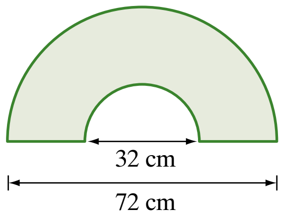
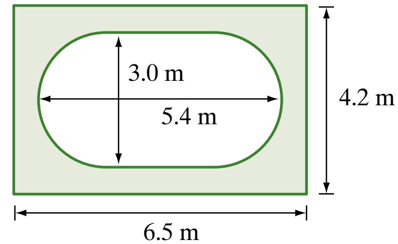
k



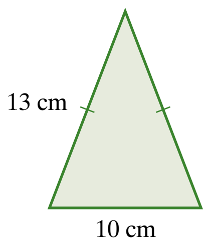
l



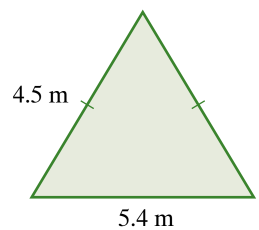
exercise 11.1 challenge

Calculate the area of each of these isosceles triangles. Hint: you will need to use Pythagoras' theorem before you can calculate the area.

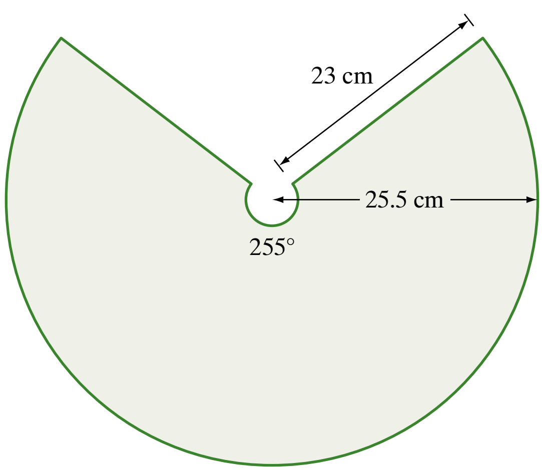
a

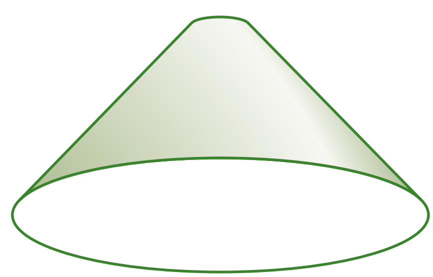


b



A factory makes lightshades from sheet metal by stamping out sectors as shown in the diagram, with a smaller sector cut from the centre to allow for the electrical fittings. Each metal piece is welded into a cone shape and sprayed with coloured enamel paint on the outside and white inside.





a Calculate the area of the outside of the shade, that is sprayed with coloured paint.

b What is the diameter of the 'base' of the finished lightshade?

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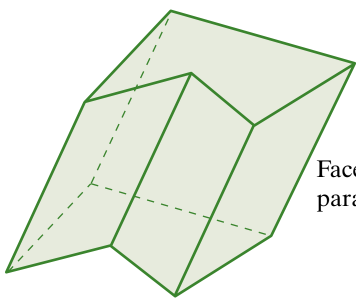
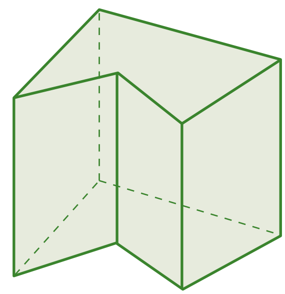
11.2 Prisms: cross-sections and nets

A prism is a 3-dimensional shape with two congruent ends that are polygons. The two polygon ends are connected by faces that are parallelograms. If the two ends are directly one above the other, the prism is called a **right prism** and the faces connecting them are rectangles. In this chapter we are dealing only with right prisms.



Faces are parallelograms

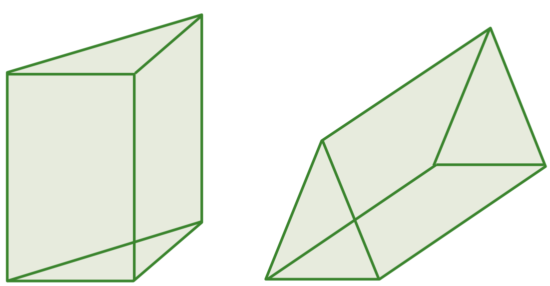
A prism that is not a right prism



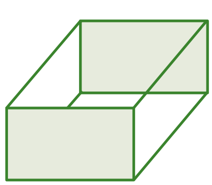
Faces are rectangles

A right prism

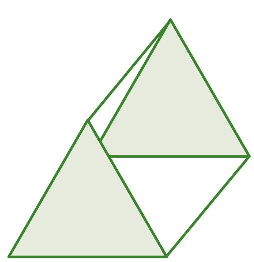
A prism may stand on one of its congruent polygon ends or it may stand on one of the rectangle faces. These two different views of a triangular prism are shown below.



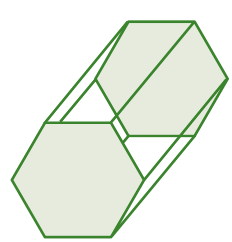
Prisms are named according to the polygon shape of the ends.



Rectangular prism



Triangular prism



Hexagonal prism

A rectangular prism is also called a **cuboid**, and a square prism is called a **cube**.

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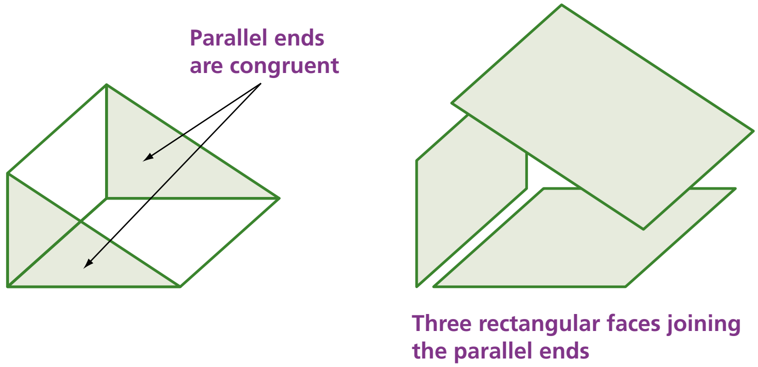
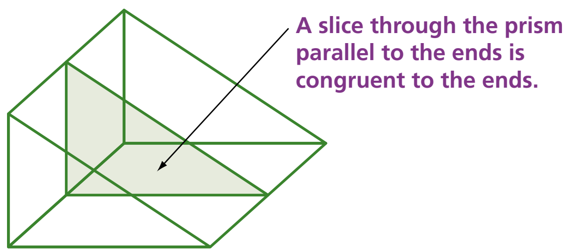
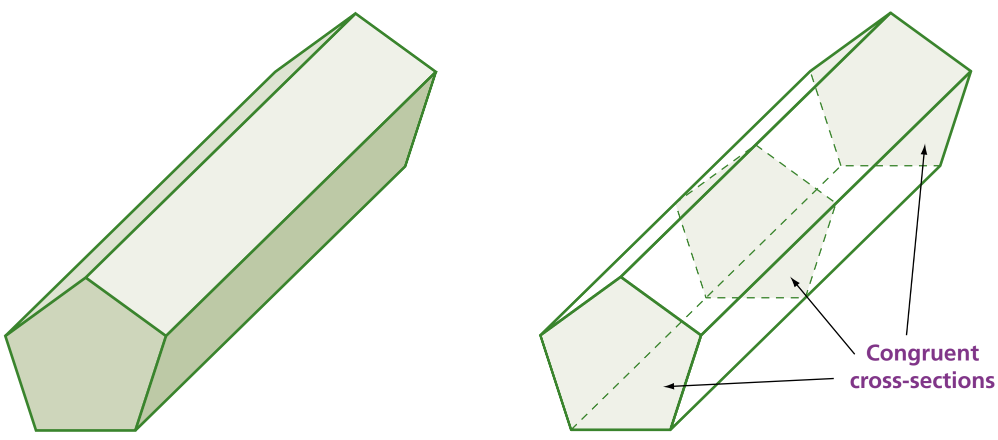
The prism shown below has congruent right-angled triangles at the ends, and three rectangular faces joining the triangles.



If we slice through a prism parallel to the ends, the shape of the cross-section will be congruent to the shape of the ends.



Any cross-section of this pentagonal prism parallel to the ends is a regular pentagon congruent to the regular pentagons at each end.



A pentagonal prism

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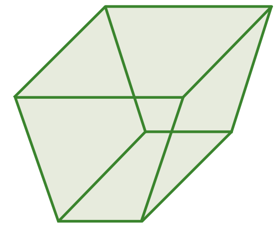
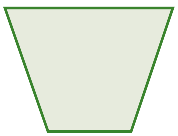
Example 6

Draw a cross-section of this prism parallel to the ends.



Working

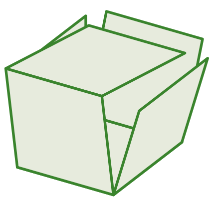


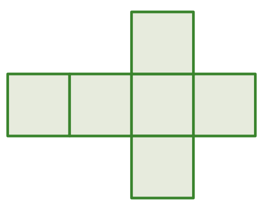
Reasoning

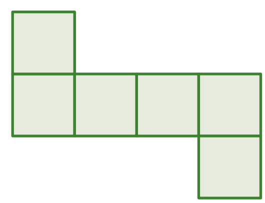
Any cross-section through the prism parallel to the ends will be the same shape as the congruent ends.

Nets of prisms

The **net** of a 3-dimensional shape is a 2-dimensional figure that can be folded up to form the 3-dimensional shape. Two of the eleven possible nets for a cube are shown here.

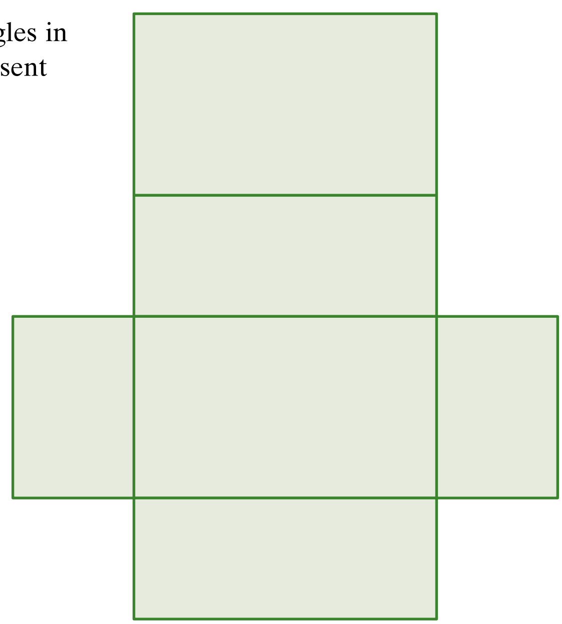


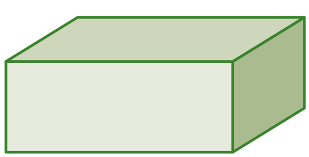


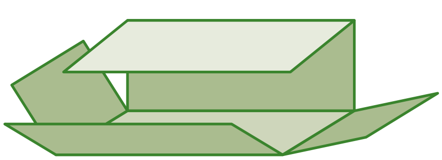


The net for a rectangular prism has six rectangles in three pairs. The three pairs of rectangles represent

- top and bottom
- front and back
- the two ends.







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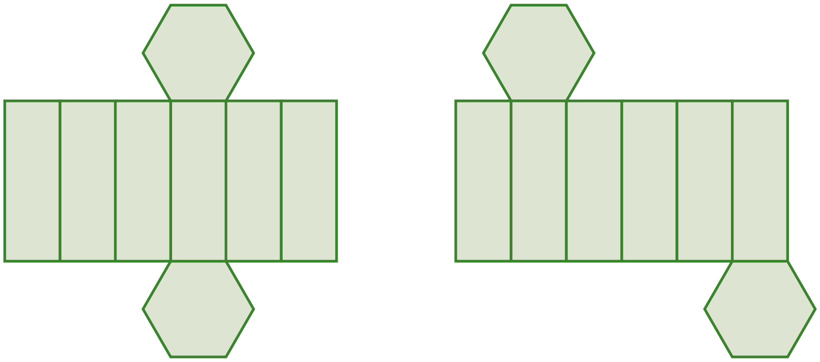
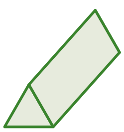
In general, the net of a prism must have two identical polygon faces for the ends, and three or more rectangular faces, depending on the number of sides of the polygonal ends. Two possible nets for a regular hexagonal prism are shown here. The hexagons could be joined to any of the rectangles, and do not both need to be joined to the same rectangle.



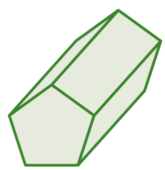
Example 7

Draw nets for these prisms.

a Equilateral triangle prism

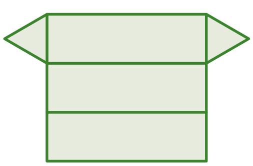


b Regular pentagon prism



Working

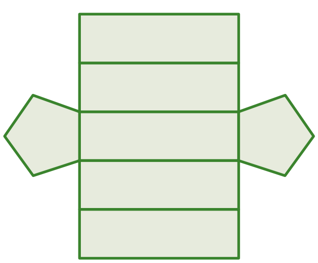
a



Reasoning

The ends are equilateral triangles.
There are three congruent rectangle faces. The length of the rectangles is equal to the length of the prism.

b



The ends are regular pentagons.
There are five congruent rectangle faces. The length of the rectangles is equal to the length of the prism.

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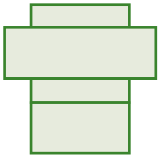
We can visualise how a net can be folded to form a 3-dimensional shape.

Example 8

Draw the 3-dimensional shape represented by each of these nets.

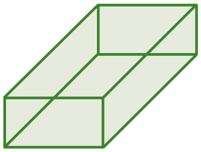
a



b

Working

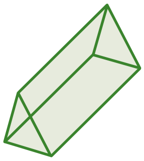
a



Reasoning

The shape is a rectangular prism.

b

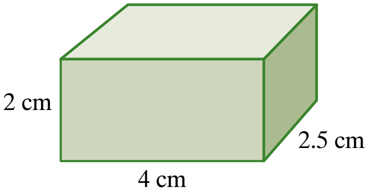


The shape is a triangular prism. It is also a right prism as the faces joining the right-angled triangles are rectangles.

exercise 11.2

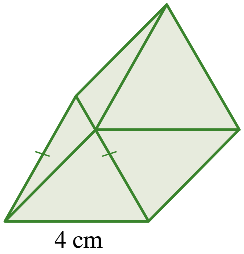
LINKS TO Example 6

1. Draw an accurate labelled diagram of a cross-section through this rectangular prism perpendicular to the base and parallel to the front face.



LINKS TO Example 7

2. Draw a cross-section of this triangular prism parallel to the ends.



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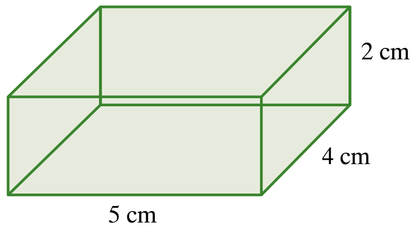
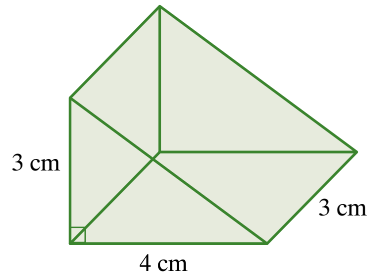
LINKS TO Example 8

Draw accurate labelled nets for these prisms.

a



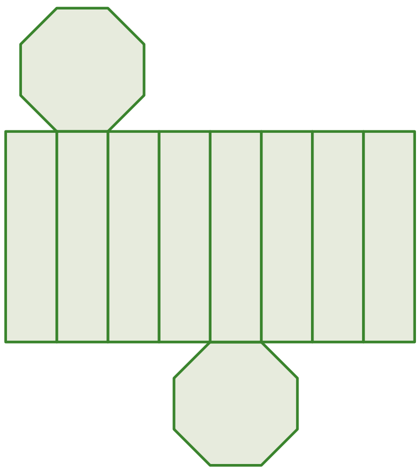
b



The diagram is the net of a prism.

a Name the prism.

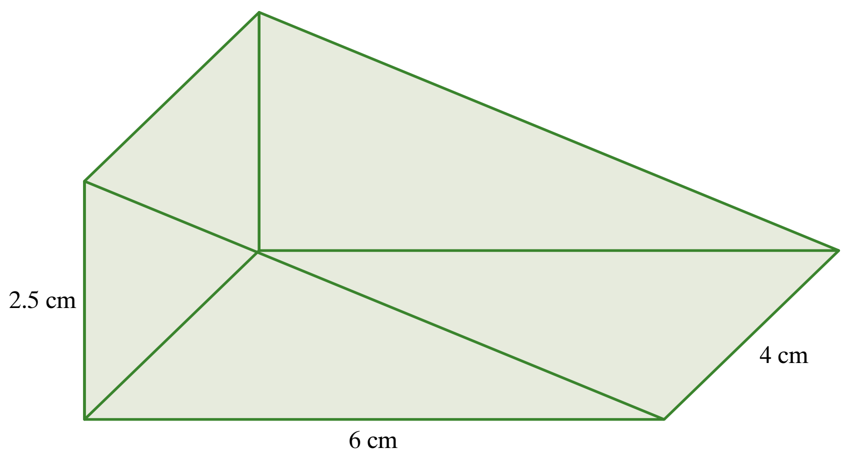
b Make a 3-dimensional drawing of the prism, either by hand or using computer drawing tools.



exercise 11.2 challenge

The drawing shows a prism with right-angled triangle ends.

a Use computer drawing tools (for example, Microsoft Word or PowerPoint or GeoGebra) to construct a copy of the drawing.



b Calculate the missing side length.

c Use computer drawing tools to construct an accurate net for the prism.

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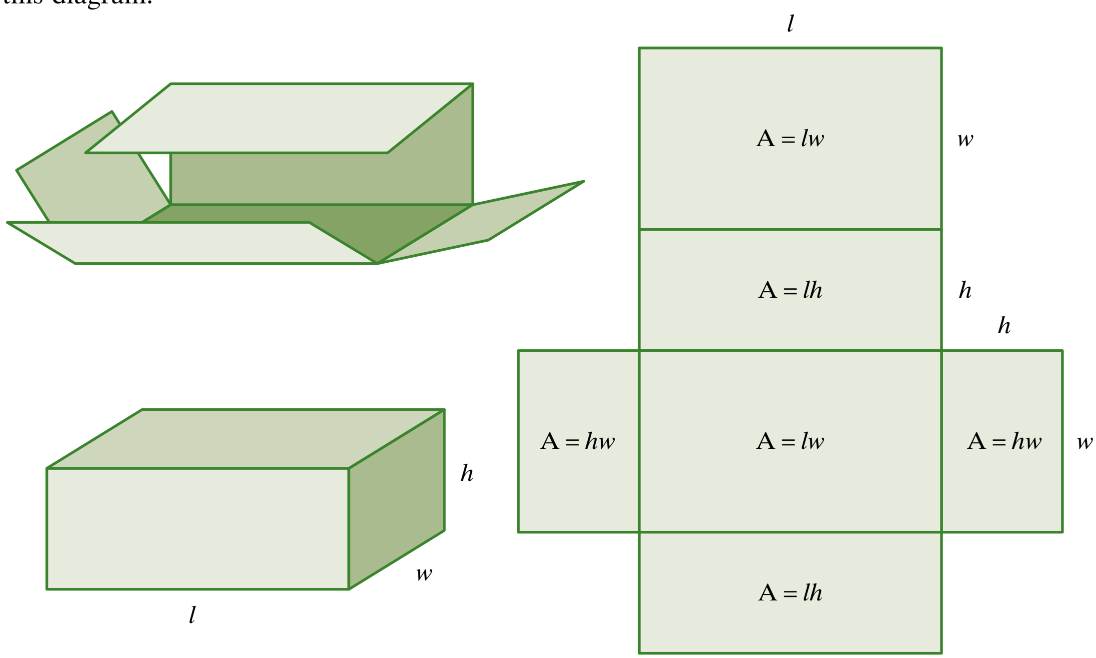
11.3 Calculating surface area: prisms

To find the surface area of a 3-dimensional shape, we need to consider the **net** of the shape. We use the term **total surface area (TSA)** to refer to the entire surface. Sometimes a 3-dimensional shape is open at one or both ends, so parts of the total surface area are not included. Drawing a net of the surface to be included is a useful way of identifying the required areas.

Rectangular prisms

For a rectangular prism, the total surface area includes three pairs of rectangles, as shown in this diagram.



We don't need to remember a formula for the surface area. It is easier just to add the areas of the faces as shown in the diagram.

Surface area: rectangular prism

To find the surface area of a rectangular prism, add the areas of the rectangular faces.

Instead of drawing the entire net of the prism, it is useful to draw each of the different shaped faces and indicate how many of each face is included in the required surface area.

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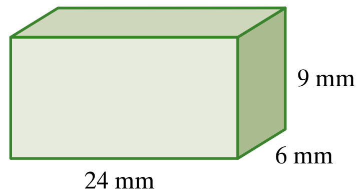
Example 9

Calculate the total surface area of this rectangular prism.

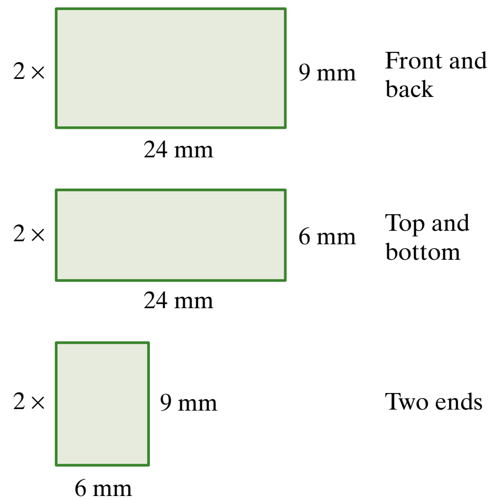


Working

Surface area
$= 2 \times 24 \times 9 + 2 \times 24 \times 6 + 2 \times 6 \times 9$
$= 432 + 288 + 108$
$= 828$
The total surface area is 828 mm^2.

Reasoning

The total surface area is made up of two lots of three different rectangles.



In the special case of a cube, the length, width and height are all the same. The area of each of the six faces is l^2, where l is the length of each edge of the cube.

Example 10

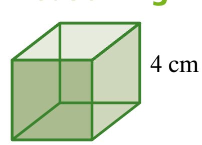
Calculate the surface area of a cube with edges of length 4 cm.

Working

Surface area $= 6 \times 4^2$
$= 96$

The surface area of the cube is 96 cm^2.

Reasoning



There are six faces and each face has an area of 4×4 cm^2.

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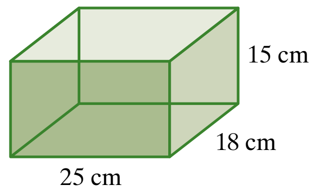
For an open rectangular container, we include the base but not the top.

Example 11

Find the surface area of the outside of this box that is open at the top.



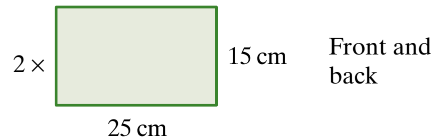
Working

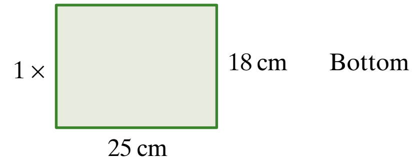
Surface area
$= 25 \times 18 + 2 \times 18 \times 15 + 2 \times 25 \times 15$
$= 450 + 540 + 750$
$= 1740 \text{ cm}^2$

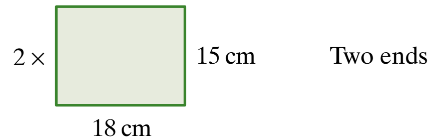
The surface area of the outside of the box is 1740 cm^2.

Reasoning

The box is a rectangular prism, but with only five surfaces: the bottom, the front and back, and two ends.







Other prisms

Just as for rectangular prisms, we find the surface area of any prism by adding the areas of all the faces.

Surface area: any prism

To find the surface area of any prism, add the areas of all the faces.

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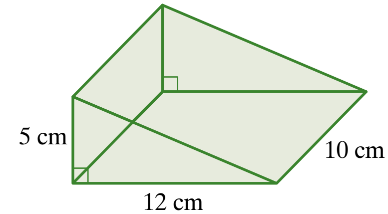
Example 12

Consider the triangular prism at right.

a Draw a labelled diagram of the net of this triangular prism.

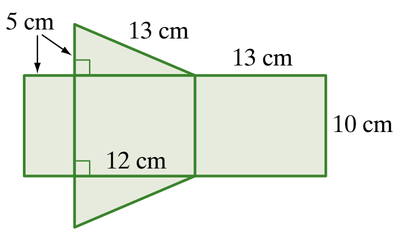
b Calculate the total surface area.



The height of the face of the prism is 36 cm.

Working

a



Reasoning

Using Pythagoras' theorem, the missing side length of the right-angled triangle is 13 cm.

b Total surface area

$= 10 \times 5 + 10 \times 12 + 10 \times 13 + 2 \times \dfrac{5 \times 12}{2}$

$= 50 + 120 + 130 + 60$

$= 360$

$= 3400$

The total surface area is 360 cm^2.

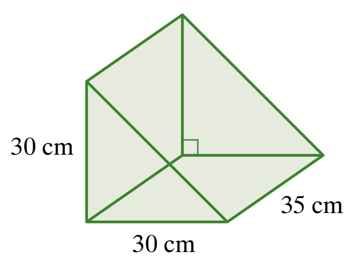
The total surface area equals the areas of three rectangles plus two congruent triangles.

Example 13

The ends of this prism are right-angled triangles.

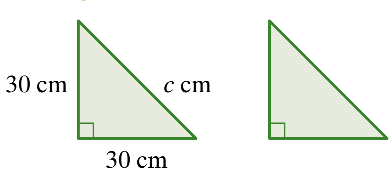
a Draw a diagram showing the dimensions of each of the faces of the prism.

b Find the total surface area of the prism.



Working

a



Reasoning

The prism has five faces.
The 'ends' are two identical triangles.
These triangles are isosceles and right-angled.

continued

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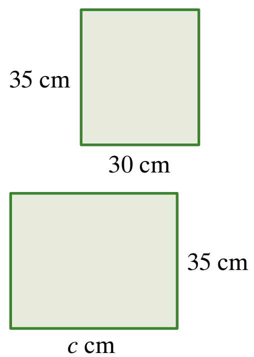
Example 13 continued

Working



b $c^2 = 30^2 + 30^2$

$= 1800$

$c = \sqrt{1800}$

Total surface area

= area of triangles + area of rectangles

$= 2 \times \frac{30 \times 30}{2} + 2 \times 30 \times 35 + \sqrt{1800} \times 35$

$= 900 + 2100 + 1485$

$= 4485$

Total surface area is 4485 cm^2.

Reasoning

There are three rectangular faces joining the triangles. Two of these rectangles are identical because the triangles have two equal sides of length 30 cm.

The width of the third rectangular face is equal to the length of the hypotenuse of the right-angled triangle.
The length of each of the rectangles is equal to the length of the prism, that is, 35 cm.

Use Pythagoras' theorem to find the hypotenuse of the right-angled triangle ends.
Use the exact value for c.

The base and the height of the triangles are each 30 cm. Two of the rectangles have length 35 cm and width 30 cm. The third rectangle has length 35 cm and width $\sqrt{1800}$.

Add the area of the two triangles and three rectangles.

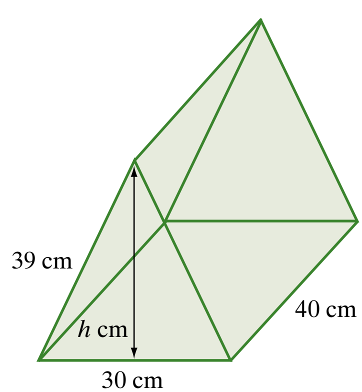
Example 14

The ends of this prism are isosceles triangles.

a Use Pythagoras' theorem to find the height, h cm, of the face of the prism.

b Draw a net of the prism.

c Calculate the total surface area of the prism.



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Example 14 continued

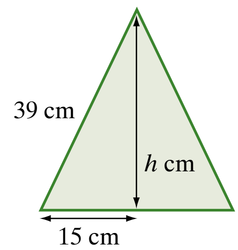
Working | **Reasoning**

a $h^2 = 39^2 - 15^2$

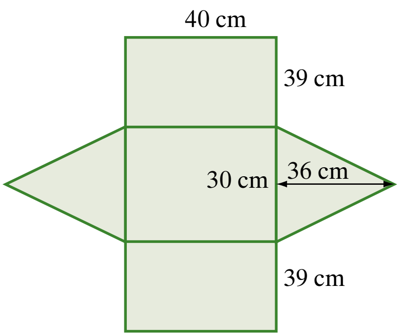
$= 1296$

$h = \sqrt{1296}$

$h = 36$



b



There are two congruent isosceles triangles and three rectangles, two of which are congruent.

c Total surface area

= area of triangles + area of rectangles

$= 2 \times \dfrac{30 \times 36}{2} + 2 \times 40 \times 39 + 40 \times 30$

$= 1080 + 3120 + 1200$

The total surface area is $5400\,\text{cm}^2$.

Add the areas of the two triangular ends and the three rectangular faces.

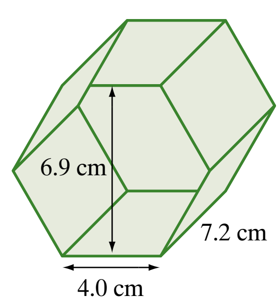
To calculate the area of the polygon end of a prism we may have to divide the polygon into simpler shapes such as triangles and rectangles. In the following example, the regular hexagon end can be divided into six isosceles triangles.

Example 15

A hexagonal prism is shown at right.

a Draw a net of this regular hexagonal prism.

b Calculate the total surface area of the prism correct to the nearest square centimetre.



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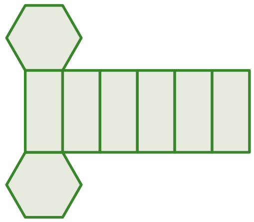
Example 15 continued

Working

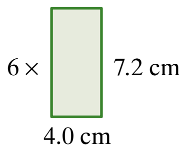
Reasoning

a



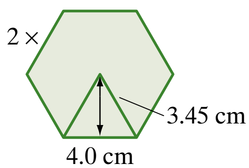
The prism has two regular hexagonal faces and six identical rectangular faces.

b



Area of six rectangular faces
$= 6 \times 4.0 \times 7.2$
$= 172.8\ \text{cm}^2$

The width of each rectangular face is equal to the side length of the regular hexagons that form the ends. The length of each rectangular face is equal to the length of the prism.



Each hexagonal face can be divided into six triangles. The base of each triangle is 4 cm and the height is half of 6.90 cm, that is, 3.45 cm.

Divide the hexagon into 6 triangles.
Area of each triangle
$= \dfrac{4 \times 3.45}{2}$
$= 6.9\ \text{cm}^2$

Area of each hexagon
$= 6 \times 6.9$
$= 41.4\ \text{cm}^2$

Area of two hexagons $= 82.8\ \text{cm}^2$

Total surface area $= 172.8 + 82.8$
$= 255.6\ \text{cm}^2$

Add the area of the two hexagons and the six rectangles.

The total surface area is $256\ \text{cm}^2$.

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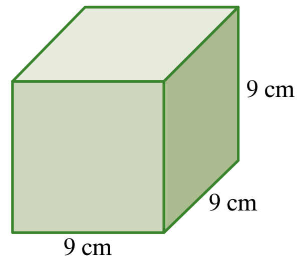
exercise 11.3

LINKS TO Examples 9, 10

1 Calculate the total surface area of each of these prisms.

a



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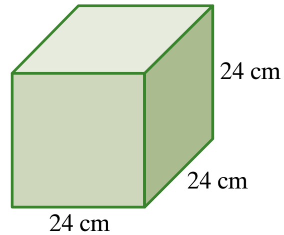
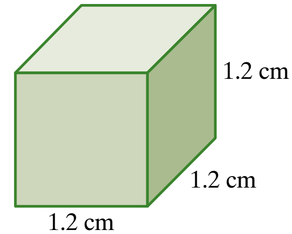


c



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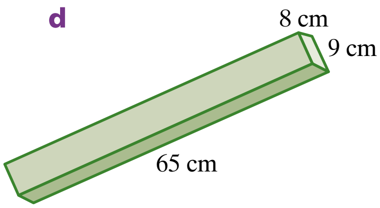

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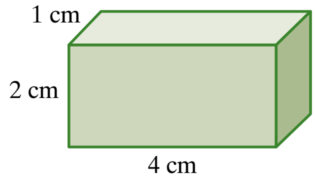

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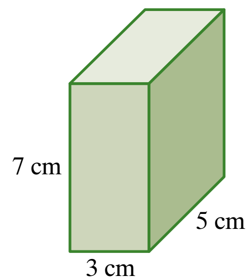

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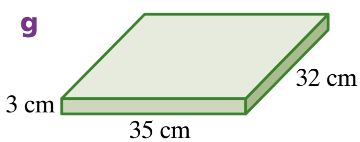

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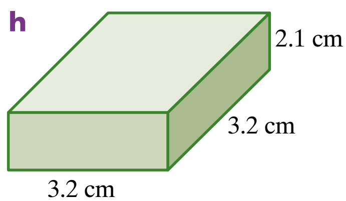

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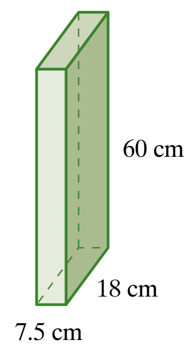

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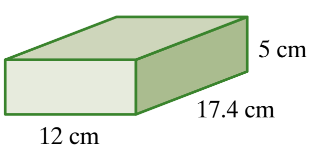

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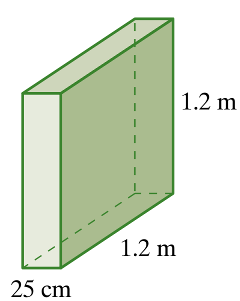

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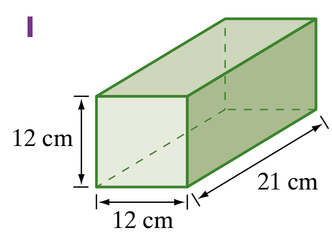

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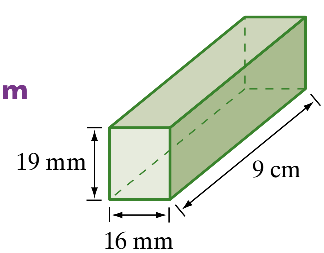

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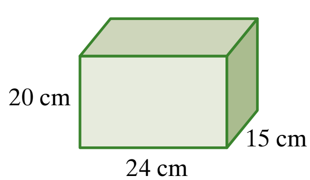

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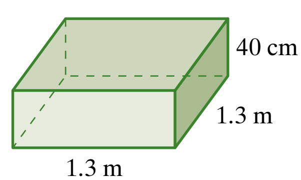

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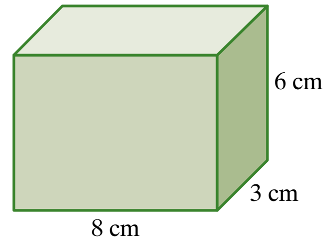
The total surface area of this rectangular prism is

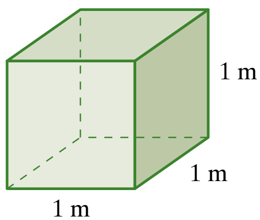
A 90 cm^2. **B** 108 cm^2. **C** 144 cm^2.
D 180 cm^2. **E** 288 cm^2.



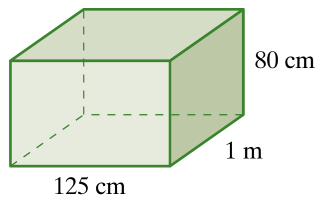
LINKS TO Example 11

Storage bins are being constructed from sheet metal by welding rectangular sheets together at the edges. The bins are to have a volume of 1 m^3 and are open at the top. Two different shapes have been suggested.

Shape 1



Shape 2



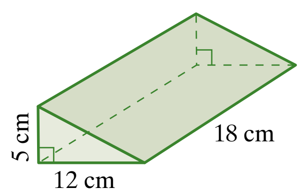
a Calculate the amount of sheet metal used in each bin.
b Which bin uses the least metal?

LINKS TO Examples 12, 13

For each of these triangular prisms,

i use Pythagoras' theorem to find the missing side length of the right-angled triangle ends.
ii draw a net of the prism and label the dimensions.
iii calculate the total surface area.

a



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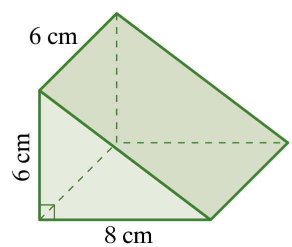

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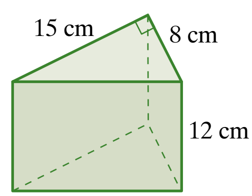

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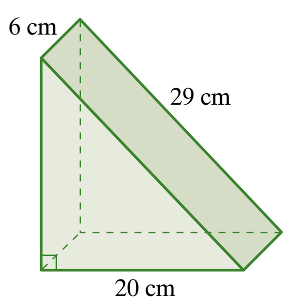

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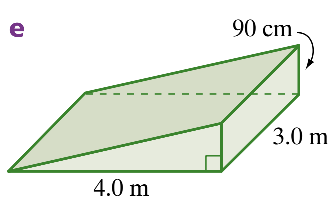

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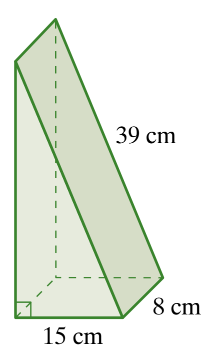

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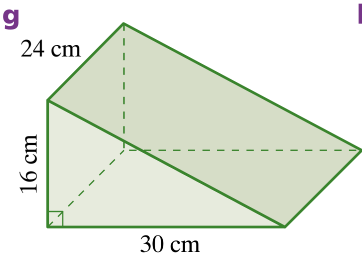

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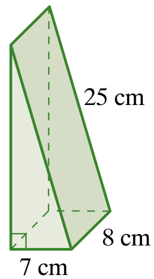

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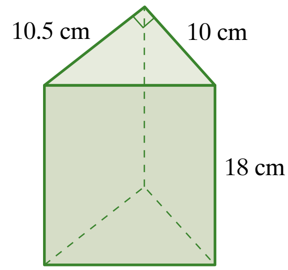


LINKS TO Example 14

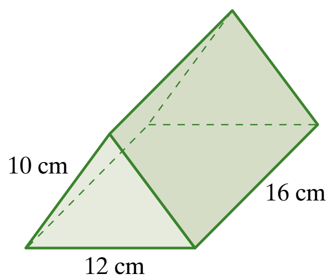
For each of these isosceles triangle prisms,

i use Pythagoras' theorem to find the perpendicular height.

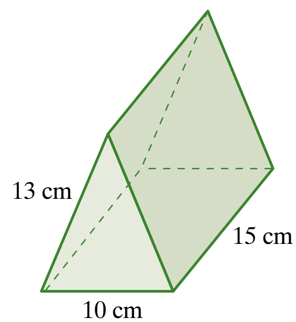
ii draw the net of the prism and label the dimensions.

iii calculate the total surface area.

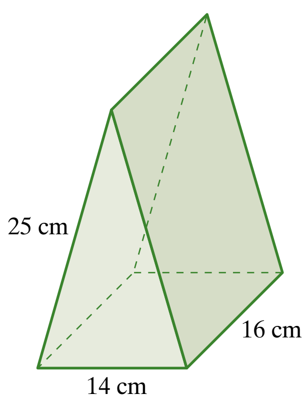
a



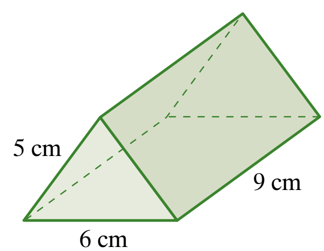
b



c



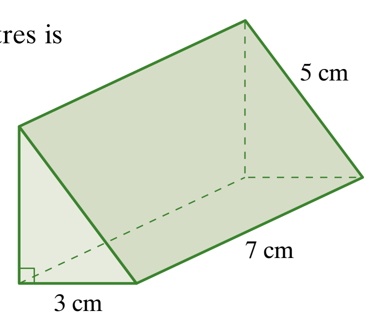
d



The total surface area of this prism in square centimetres is

A 80 **B** 96 **C** 105

D 108 **E** 192



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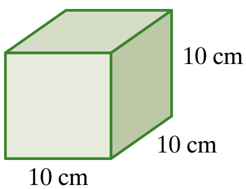
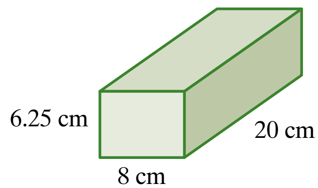
Mint chocolates are to be packed in cardboard packets with a volume of 1000 cm^3. Three possible shapes for the packet are shown below.

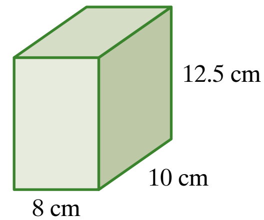
Shape 1



Shape 2



Shape 3



a Calculate the total surface area for each packet.

b Which shape has the least surface area?

LINKS TO Example 15

In each of these regular polygon prisms, the polygon end has been divided into congruent isosceles triangles.

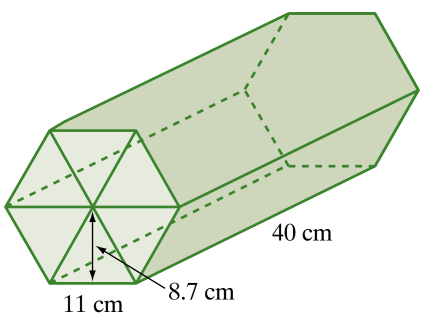
For each prism, find

i the area of one of the isosceles triangles

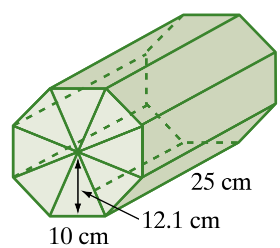
ii the area of the polygon

iii the total surface area.

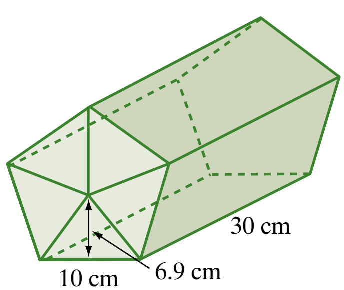
a



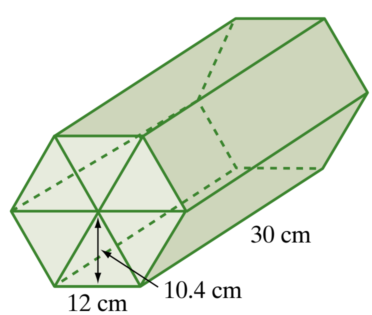
b



c



d



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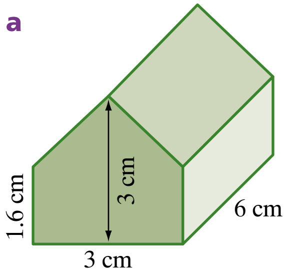
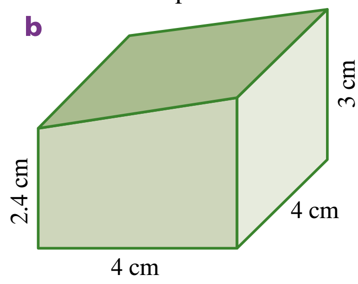
exercise 11.3 challenge

Calculate the total surface area of these shapes. Do not include the floor of the shapes.

a



b



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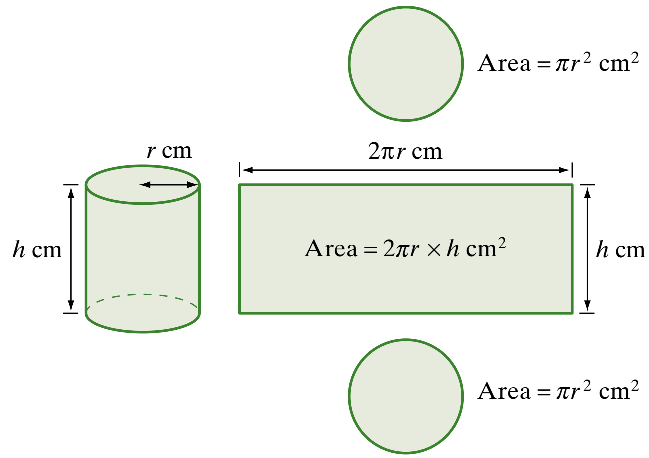
11.4 Calculating surface area: cylinders

The surface area of a cylinder is made up of three parts: the flat top and bottom, and the curved part. The top and bottom are circles. If the curved part is 'unwrapped', it forms a rectangle. The long sides of this rectangle are the same length as the circumference of the top and bottom circles.



Surface area cylinder

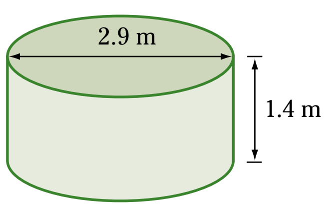
To find the total surface area of a cylinder add the area of the circular end(s) and the area of the curved surface.

Example 16

A cylindrical water tank has the dimensions shown.

a Draw a labelled diagram of the curved part of the tank 'opened out' to form a rectangle.

b Calculate the area of the curved surface of the tank correct to one decimal place.



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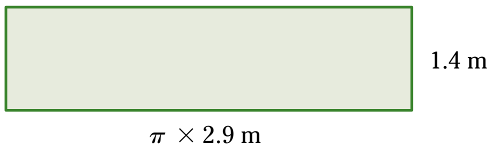
Example 16 continued

Working

a



b Curved surface area

$= \pi \times 2.9 \times 1.4$

≈ 12.75 m^2

The area of the curved surface of the tank is 12.8 m^2 correct to one decimal place.

Reasoning

The width of the rectangle is equal to the circumference of the tank.
The height of the rectangle is the height of the tank.

The area of the rectangle is the product of its width and height.
Find the area to one more decimal place than is needed than round to the required number of decimal places.

Example 17

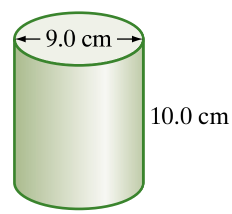
A cylinder is shown at right.

a Draw labelled diagrams of the top and base of the cylinder and of the rectangle representing the curved surface

b Calculate to the nearest square centimetre

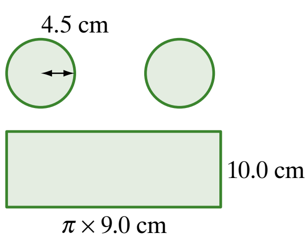
i the surface area of the curved surface of this cylinder

ii the total surface area of the cylinder.



Working

a



b **i** Area of curved surface

$= \pi \times 9.0 \times 10.0$

≈ 282.7

The area of the curved surface is 283 cm^2.

Reasoning

Radius $= 9.0 \div 2 = 4.5$ cm

The curved surface 'unwraps' to a rectangle with side lengths equal to the height and circumference of the cylinder.

The length of the rectangle is the circumference of the circle and the width is equal to the height of the cylinder.

continued

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Example 17 continued

Working

b ii Total surface area

$= 2\pi r^2 + 2\pi rh$

$= 2 \times \pi \times 4.5^2 + 282.7$

$\approx 127.2 + 282.7$

$= 409.9 \text{ cm}^2$

The total surface area of the can is approximately 410 cm^2.

Reasoning

The total surface area is made up of the area of the curved surface (part **b**) and the top and bottom circles.

If the cylindrical container is open at the top, then only one of the circular ends is included in the surface area calculation.

Example 18

An open cylindrical storage bin has diameter 80 cm and height 50 cm.

a Make a labelled diagram of the bin.

b Draw labelled diagrams of the base and of the rectangle representing the curved surface.

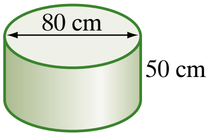
c Calculate the surface area of the bin

  i in square centimetres to the nearest one hundred square centimetres.

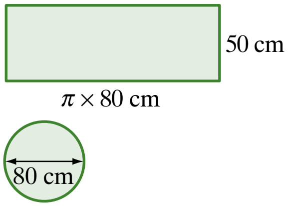
  ii in square metres.

Working

a



b



c Surface area $= \pi \times 40^2 + \pi \times 80 \times 50$

$\approx 5026.5 + 12\,566.4$

$= 17\,592.9$

  i The surface area of the bin is 17 600 cm^2.

  ii The surface area is 1.76 m^2.

Reasoning

b The width of the rectangle is equal to the circumference of the cylinder.

c The radius of the cylinder is 40 cm.

ii $10\,000 \text{ cm}^2 = 1 \text{ m}^2$

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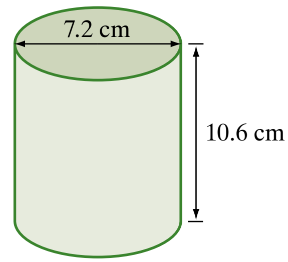
exercise 11.4

LINKS TO Example 16

The label on this can of apricots fits around the can with an extra 1 cm for overlapping.



a Draw a diagram to show the shape of the label before it was glued around the can.

b Find the dimensions of the label correct to one decimal place.

c Find the area of the label. Give your answer to the nearest square centimetre.

A can with label is shown at right.

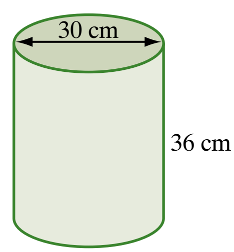
a Draw a diagram of the paper label opened out flat. Label the dimensions on your diagram.

b Calculate the area of the paper label around this can to the nearest square centimetre.

c Calculate the total surface area of the cylindrical can to the nearest square centimetre.

LINKS TO Example 17

The total surface area of this cylinder is



A $2 \times \pi \times 30^2 + 2 \times \pi \times 30 \times 36$

B $\pi \times 15^2 + \pi \times 15 \times 36$

C $2 \times \pi \times 15^2 + 2 \times \pi \times 36$

D $2 \times \pi \times 15^2 + 2 \times \pi \times 15 \times 36$

E $2 \times \pi \times 15^2 + \pi \times 15 \times 36$.

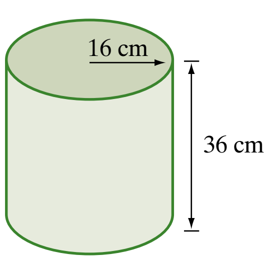
LINKS TO Example 17

For each of these cylinders, calculate, correct to the nearest square centimetre or square metre depending on the given units,

i the area of the curved surface.

ii the total surface area.

a



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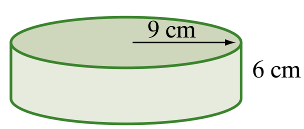

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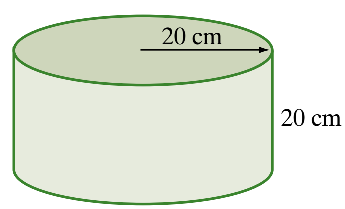

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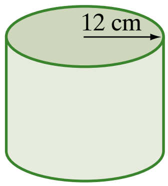
d



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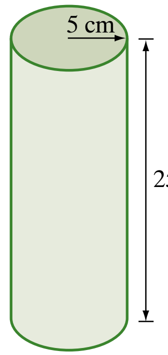

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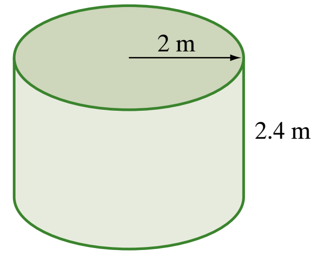

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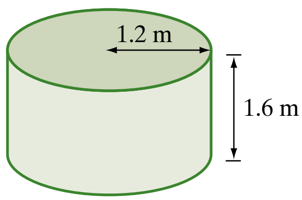

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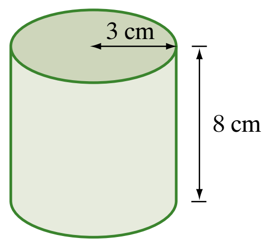
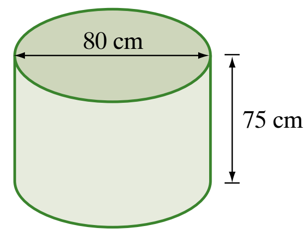


i



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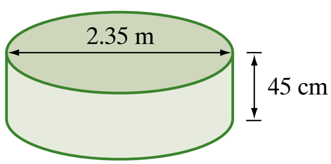

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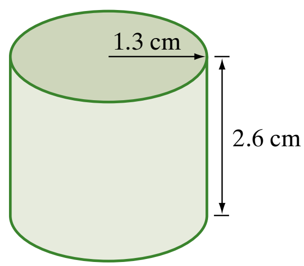

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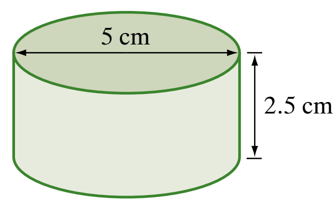


A cylinder has height 8 cm and diameter 10 cm. The total surface area of the cylinder in square centimetres is

A 80π **B** 90π **C** 104π **D** 130π **E** 140π

A cylindrical can has diameter 24 cm and height 18 cm. The area of the paper label around the curved part of the can in square centimetres is

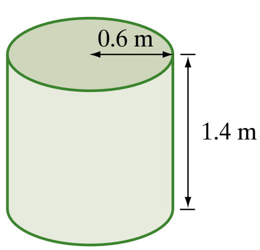
A 24×18 **B** $\pi \times 24 \times 18$ **C** $\pi \times 12 \times 18$

D $\pi \times 12^2 \times 18$ **E** $\pi \times 24^2 \times 18$

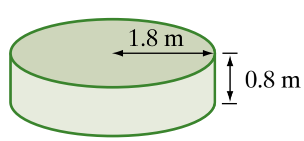
LINKS TO Example 18

Each of these cylindrical containers is open at the top. Calculate the surface area, including the bottom. Answer correct to one decimal place.

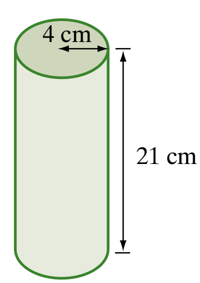
a



b



c



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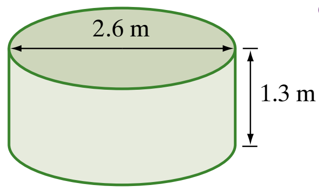
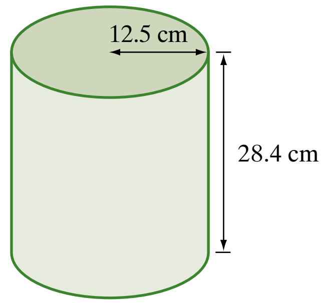
d



e



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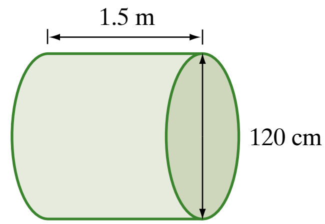
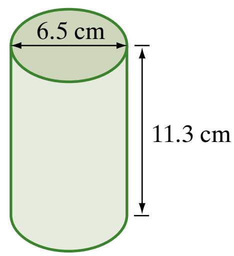


Calculate the total area correct to the nearest square centimetre of aluminium sheet used in this drink can.



Gary is making a cake in a round tin with a diameter of 21 cm. The recipe says to line the tin with paper to stop the outside of the cake burning. Gary has cut a circle of paper to cover the bottom of the tin, but he now needs a strip to wrap around inside the curved part of the tin. How long should Gary cut the strip of paper if he allows an extra centimetre to overlap the ends? Give the length correct to the nearest centimetre.

Marcel is icing a round birthday cake. The icing will cover the top and curved part of the cake, but not the base. If the diameter of the cake is 30 cm and the height is 12 cm, calculate the area to be iced correct to the nearest square centimetre.

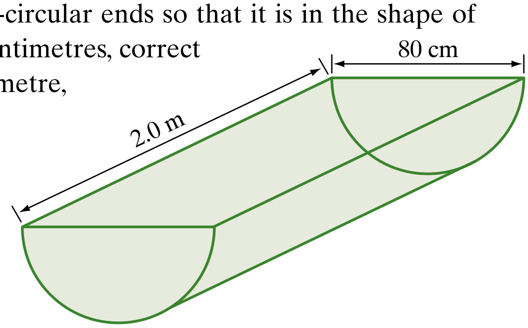
exercise 11.4 challenge

A metal drinking trough for cattle has semi-circular ends so that it is in the shape of a half cylinder. Give each area, in square centimetres, correct to the nearest square centimetre or square metre, depending on the given units.

a Calculate the total area of the two ends.

b Calculate the area of the curved surface.

c What is the total area of metal in the trough?



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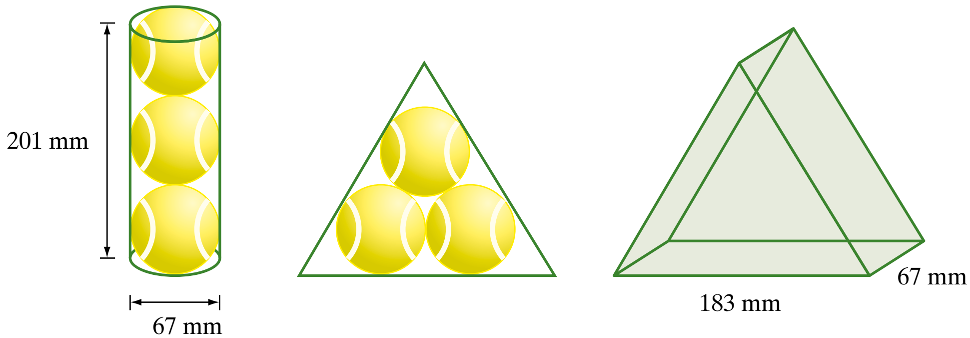
Tennis balls are often sold in sets of three in a cylindrical container. Another possible shape for packing three tennis balls is an equilateral triangle prism. The cost of packaging is related to the amount of material required to make the packaging.

a Calculate the total surface area of each container. Give each area correct to the nearest square millimetre.

b Which container is the most economical to make?



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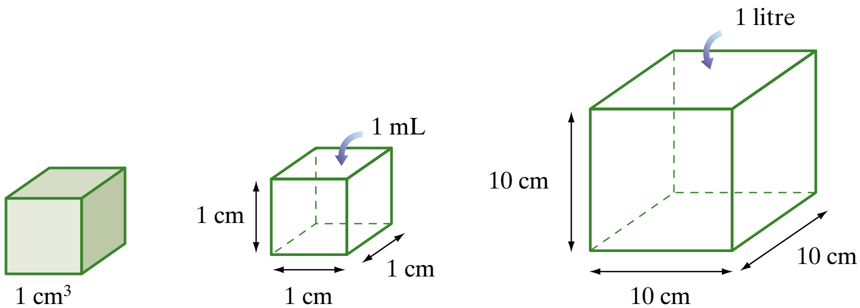
11.5 Calculating volume: prisms

The **volume** of a 3-dimensional object is the amount of space it occupies. The **capacity** of a container is the amount that it will hold.

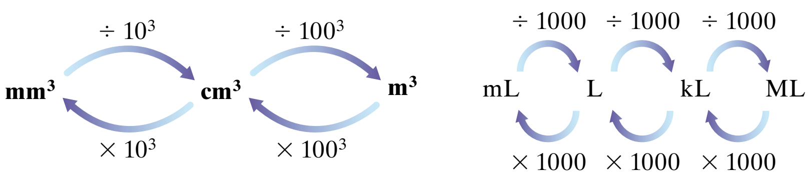
Volume is measured in cubic units or in units based on the litre.

Units of volume and capacity

The volumes of liquids and gases are often measured in litres (or millilitres, kilolitres, megalitres or gigalitres). These units are usually used when we are talking about the **capacity** of a container, that is, how much liquid or gas the container will hold. Garden supplies such as bags of mulch are also often measured in litres.



1 cm^3	1 millilitre (1 mL)
1000 cm^3	1 litre (1 L)
1 m^3	1 kilolitre (1 kL)



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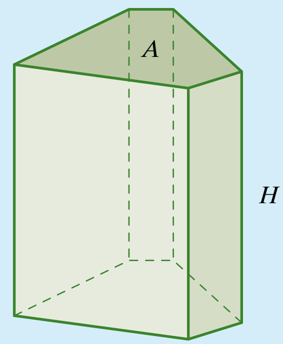
Volume of prisms

Volume of a prism

For any right prism, the volume V is given by

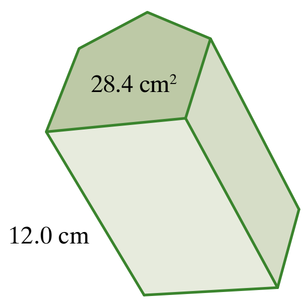
$$V = AH$$

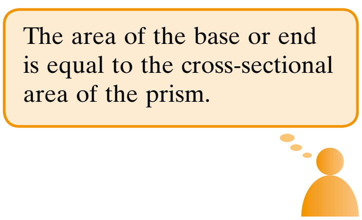
where A is the area of the base (or end) and H is the height (or length).



Example 19

Find the volume of each of this prisms.





Working

$V = AH$
$= 28.4 \times 12.0$
$= 340.8$

The volume is 340.8 cm^3.

Reasoning

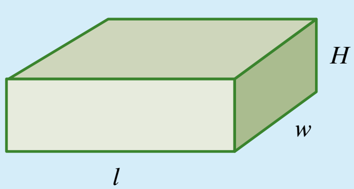
The volume of a prism is calculated by multiplying the area of the end (the cross-sectional area) by the height.

Volume of a rectangular prism

The volume V of a rectangular prism is given by

$$V = AH = lwH$$

where A is the area of the rectangular base of length l and width w, and H is the height.



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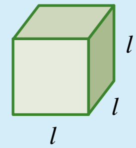
Volume of a cube

Each of the edges of a cube is of length l.

$$\begin{aligned} V &= AH = l^2 \times l \\ &= l^3 \end{aligned}$$



Example 20

Find the volume of

a a rectangular box 10 mm by 23.5 mm by 12 mm.

b a cube with edges of length 12.4 cm correct to the nearest cubic centimetre.

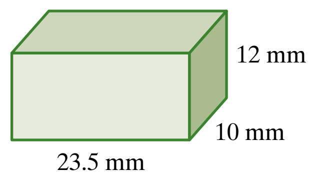
Working

a $V = AH$

$$\begin{aligned} V &= lwH \\ &= 23.5 \times 10 \times 12 \\ &= 2820 \end{aligned}$$

The volume is 2820 mm^3.

Reasoning

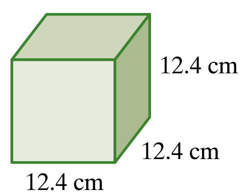


To find the volume of a rectangular prism, multiply the length by the width by the height.

b $V = l^3$

$$\begin{aligned} &= 12.4^3 \\ &= 1906.624 \end{aligned}$$

The volume is 1907 cm^3.



The volume of a cube is found by cubing the length of an edge.

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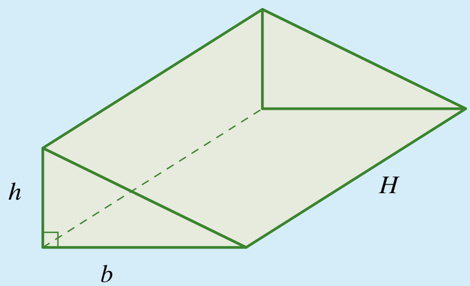
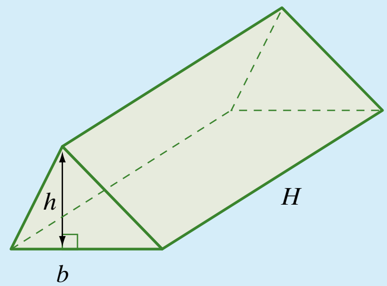
Volume of a triangular prism





The volume V of a triangular prism is given by

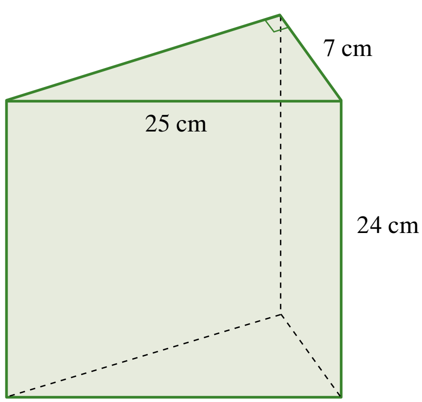
$$V = AH = \frac{b \times h}{2} \times H$$

where A is the area of the triangular base of the base b and height h, and l is the height of the prism.

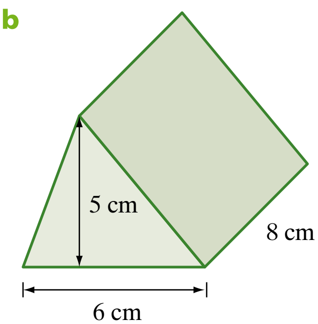
Example 21

Find the volume of each of the following prisms.

a



b



Working

a Let the missing side length of the right-angled triangle end be a cm.

$$\begin{aligned} a^2 + 7^2 &= 25^2 \\ a^2 &= 625 - 49 \\ a^2 &= 576 \\ a &= 24 \end{aligned}$$

Reasoning

The length of the other side of the right-angled triangle end is required before the area can be calculated. Use Pythagoras' theorem.

Note that 7, 24, 25 is a Pythagorean triple.

continued

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Example 21 continued

Working

Area of triangle end:

$$A = \frac{1}{2}bh$$
$$= \frac{1}{2} \times 7 \times 24$$
$$= 84$$

Volume of prism:

$$V = AH$$
$$= 84 \times 24$$
$$= 2016$$

The volume of the prism is 2016 cm^2.

b $V = AH$

$$= \frac{6 \times 5}{2} \times 8$$
$$= 120$$

The volume is 120 cm^3.

Reasoning

Calculate the area of the right-angled triangle.

Calculate the volume of the prism with area 84 cm^2 and height 24 cm.

Each end of the prism is a triangle with base 6 cm and perpendicular height 5 cm.

Area of triangle $= \frac{\text{base} \times \text{height}}{2}$

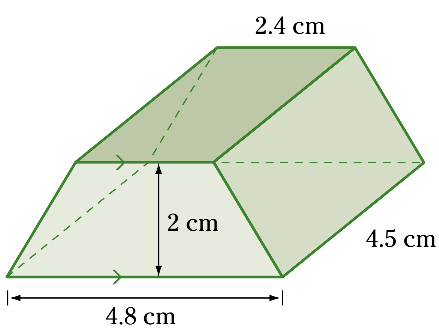
The following example shows a trapezium-shaped prism. The trapezium end can be divided into two triangles, or alternatively, the formula for the area of a trapezium can be used.

Example 22

The ends of this prism are trapeziums.

a Draw diagrams to show the dimensions of the end of the prism.

b Find the volume of the prism.



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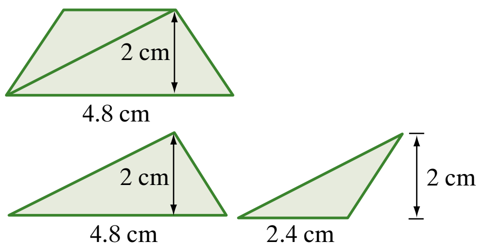
Example 22 continued

Working



Area of trapezium end:

$$\begin{aligned} A &= \frac{(a+b)h}{2} \\ &= \frac{1}{2}bh + \frac{1}{2}bh \\ &= \frac{1}{2} \times 4.8 \times 2 + \frac{1}{2} \times 2.4 \times 2 \\ &= 7.2\ \text{cm}^2 \end{aligned}$$

Volume of prism

$$\begin{aligned} V &= AH \\ &= 7.2 \times 4.5\ \text{cm}^3 \\ &= 32.4\ \text{cm}^3 \end{aligned}$$

Volume of the prism is 32.4 cm^3.

Reasoning

Divide the trapezium into two triangles.
Both triangles have height of 2 cm.
Area of trapezium = sum of the areas of the two triangles

Alternatively, using the formula for the area of a trapezium:

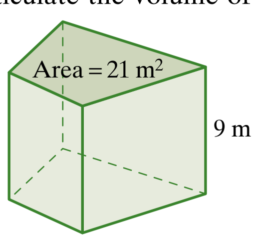
$$\begin{aligned} A &= \frac{(a+b)h}{2} \\ &= \frac{(2.4+4.8) \times 2}{2} \\ A &= 7.2 \end{aligned}$$

exercise 11.5

LINKS TO Example 19

1. Calculate the volume of each of these prisms.

a



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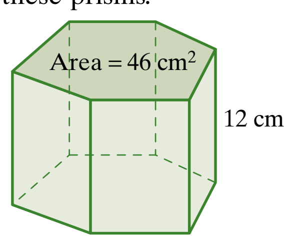

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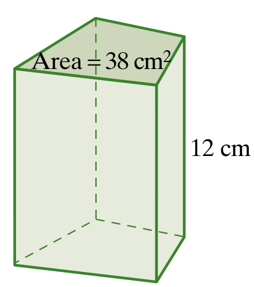

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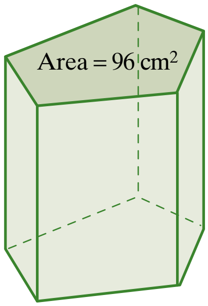

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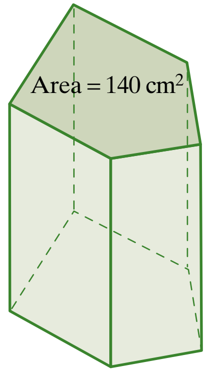

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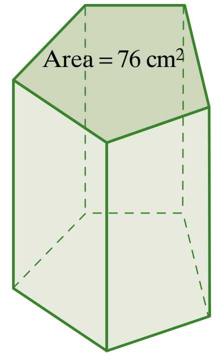
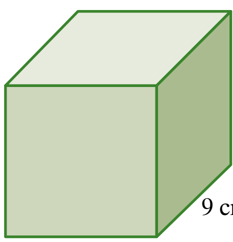


LINKS TO Example 20

Calculate the volume of each of these rectangular prisms.

a



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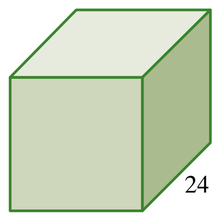

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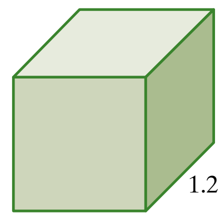

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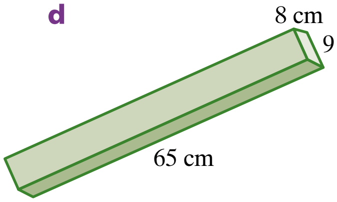

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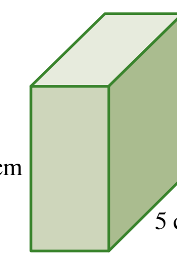

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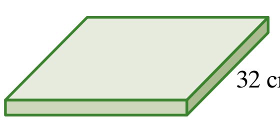

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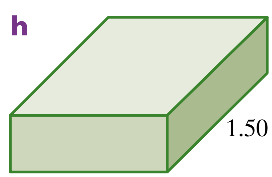

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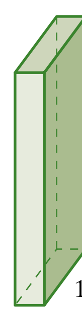

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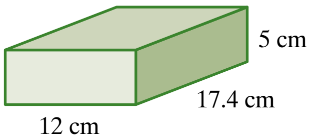

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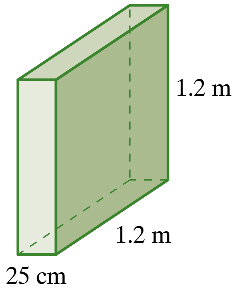

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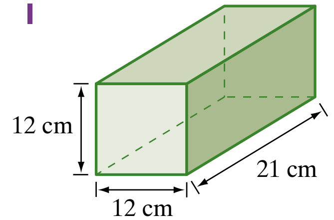

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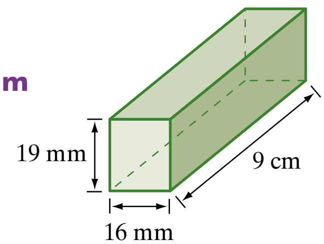

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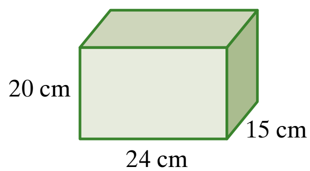

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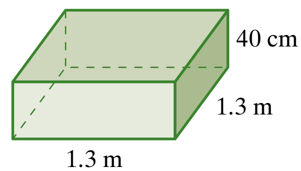
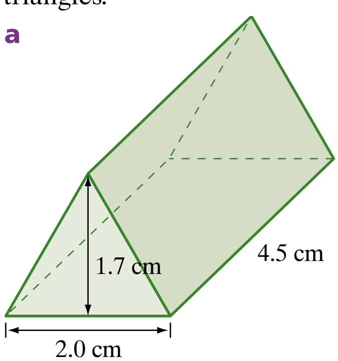


LINKS TO Example 21

Calculate the volume of each of these right-angled triangle prisms. You will need to use Pythagoras' theorem to find the perpendicular height of some of the right-angled triangles.

a



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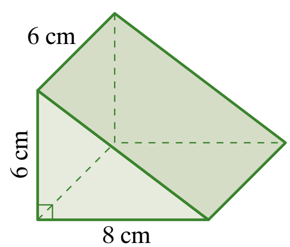

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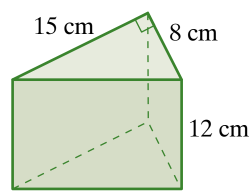

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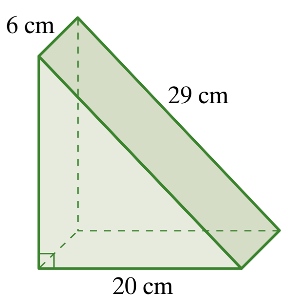

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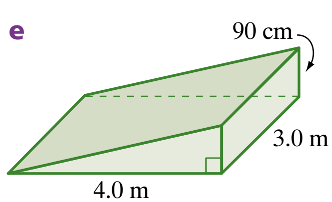

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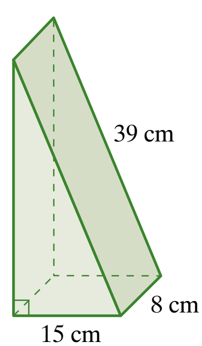

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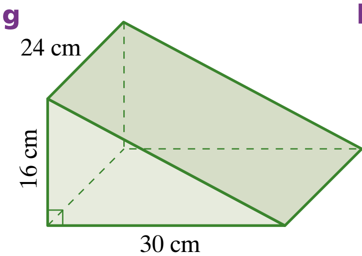

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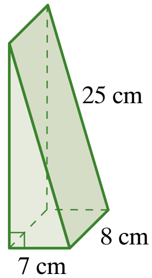

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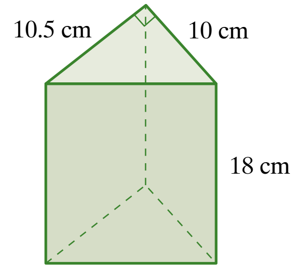
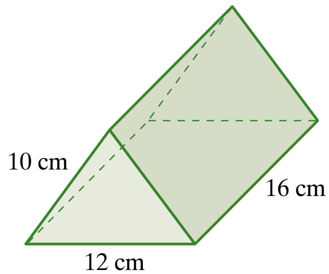


Calculate the volume of each of these isosceles triangle prisms. You will need to use Pythagoras' theorem to find the perpendicular height of the isosceles triangles.

a



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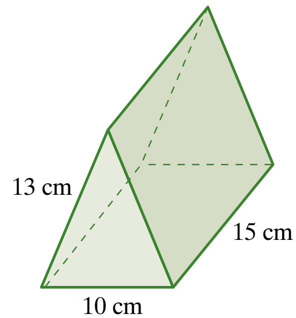

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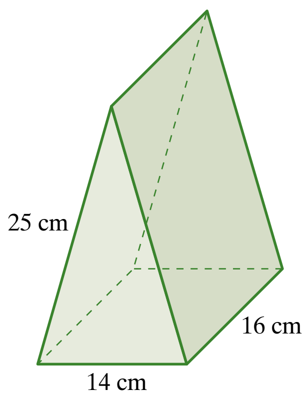

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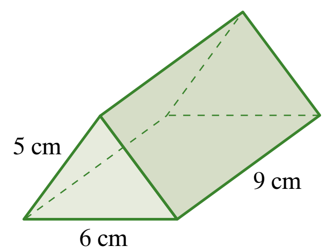
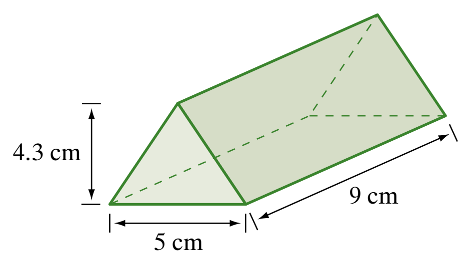


Find the volume of chocolate in the triangular prism-shaped chocolate bar on the right.



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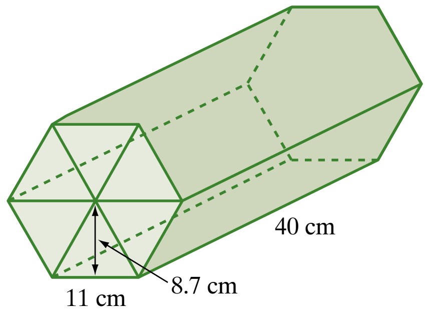
Calculate the volume of each of the prisms in question **9** of exercise **11.3**. Remember that you have already calculated the area of the polygon end for each prism.

a



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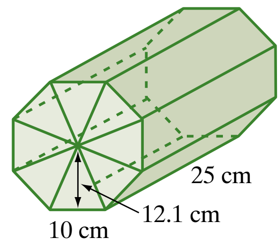

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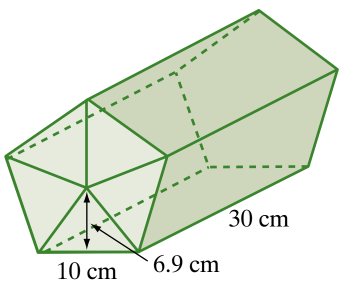

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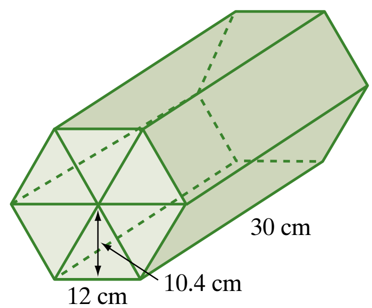
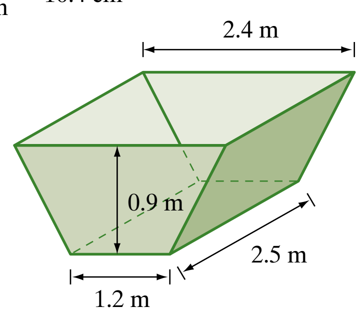


LINKS TO Example 22

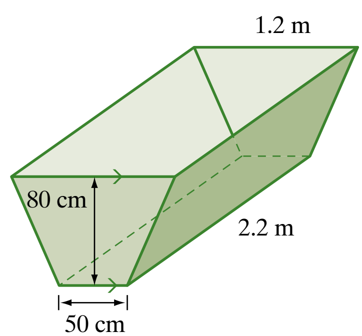
A waste disposal bin is in the shape of a prism with trapezium shaped ends, as shown in the diagram. Calculate the volume of the bin.



A drinking trough for cattle is in the shape of a prism, as shown in the diagram.

a Calculate the volume.

b How much water would the trough hold if filled to the top? Give your answer in litres.



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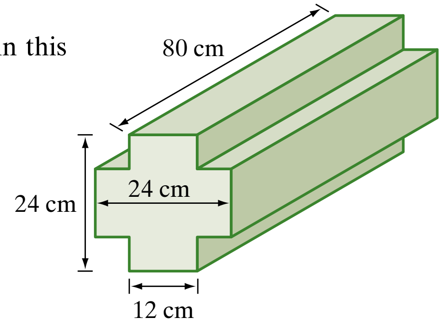
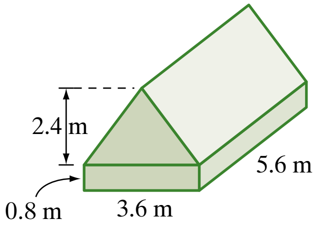
Find the volume of steel in this metal beam.



Calculate the volume of this prism.



exercise 11.5 challenge

Chris has had 1 cubic metre of soil delivered. He is going to spread it over part of his garden. If he spreads the soil to a thickness of 8 cm, what area of garden will the soil cover? Hint: think of the soil as a rectangular prism when it is spread on the garden.

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11.6 Calculating volume: cylinders

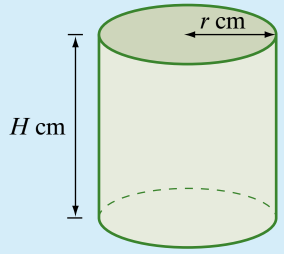
Just as for prisms, the volume of a cylinder is found by multiplying the area of the base by the height. In this case, the base is a circle with area πr^2.

Volume of cylinder

The volume V of a cylinder is given by

$$V = AH$$
$$V = \pi r^2 H$$

where r is the radius and H is the height.



Example 23

Find the volume of a solid cylinder with radius 18 mm and height 70 mm

a in cubic millimetres, correct to the nearest thousand cubic millimetres.

b in cubic centimetres, correct to the nearest cubic centimetre.

Working

a $V = AH$

$V = \pi r^2 H$

$= \pi \times 18^2 \times 70$

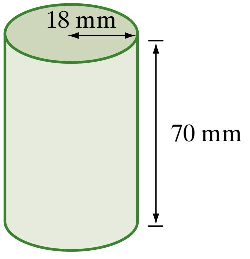
$= 22\,680\pi$

$\approx 71\,251$

The volume is approximately 71 000 mm³.

b The volume is approximately 71 cm³.

Reasoning



Each dimension has two significant figures, so the volume should not be given to more than two significant figures.

If we know the volume of a cylinder in cubic units, we can calculate its capacity.

Example 24

A cylindrical can has a diameter of 8 cm and a height of 8.5 cm.

a Find the volume of the can. Give your answer to the nearest cubic centimetre.

b How many millilitres of juice will the can hold?

continued

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Example 24 continued

Working

a $V = AH$

$V = \pi r^2 \times H$

Volume $= \pi \times 4^2 \times 8.5\ \text{cm}^3$

$\approx 427.25\ \text{cm}^3$

$= 427\ \text{cm}^3$ to the nearest cm^3

b The can will hold 427 mL of juice.

Reasoning

Radius is 4 cm.

$1\ \text{cm}^3 = 1\ \text{mL}$

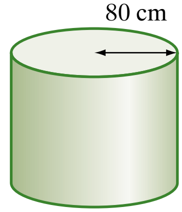
A hollow cylinder such as a concrete pipe has an annulus as its cross-section. To find the volume of material in the pipe we find the area of the annulus then multiply by the length of the pipe.

Example 25

Find the volume of this cylindrical tank

a in cubic metres.

b in kilolitres.



Working

a $V = \pi r^2 H$

$= \pi \times 0.80^2 \times 1.25$

≈ 2.513

The volume of the tank is approximately $2.51\ \text{m}^3$.

b The capacity of the tank is 2.51 kL.

Reasoning

The dimensions must be converted to the same unit, so the radius is 0.80 m.

1 cubic metre is equivalent to 1 kilolitre.

When shapes are made up of more than one simple shape, the volumes are added to find the total volume.

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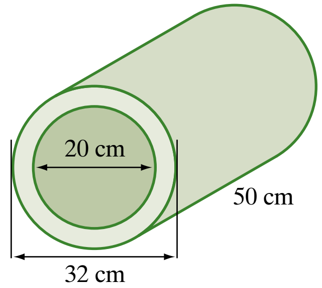
11.6

Example 26

Find the volume of concrete in this pipe correct to the nearest 100 cubic centimetres.



Working

$V = AH$

$= (\pi \times 16^2 - \pi \times 10^2) \times 50$

$V = 24\,504$

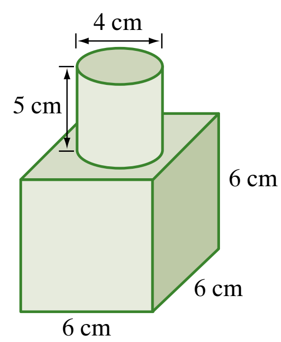
The volume of concrete in the pipe is 24 500 cm^3.

Reasoning

The radius of the outer circle is 16 cm.
The radius of the inner circle is 10 cm.
Area of the annulus
= area of outer circle − area of inner circle.

Example 27

Find the volume of this solid correct to the nearest cubic centimetre.



Working

Volume of cube:

$V = AH$

$V = l^3$

$= 6^3$

$V = 216$

Reasoning

The three dimensions of the shape are each 6 cm.

Working

Volume of cylinder:

$V = AH$

$V = \pi r^2 H$

$= \pi \times 2^2 \times 5$

$= 20\pi$

$V \approx 62.8$

Reasoning

Diameter = 4 cm
$r = 2, H = 5$

continued

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Example 27 continued

Working

Total volume = $216 + 62.8 \text{ cm}^3$
$= 278.8 \text{ cm}^3$

The volume of the shape is 279 cm^3.

Reasoning

Add the two volumes to find the total volume.

Round to the nearest cubic centimetre.

If we know the required volume and radius or diameter of a cylinder we can calculate the required height.

Example 28

If a new drink can is designed with a diameter of 6 cm, what height would be required so that the volume of the can is 375 mL? Give the height correct to one decimal place.

Working

$$V = \pi r^2 H$$
$$375 = \pi \times 3^2 \times H$$
$$H = \frac{375}{\pi \times 3^2}$$
$$H = 13.26...$$

The height of the can needs to be 13.3 cm

Reasoning

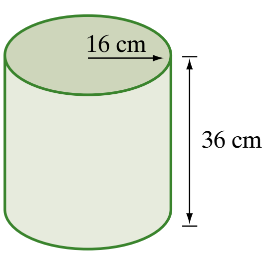
375 mL = 375 cm^2.
Diameter = 6 cm so radius = 3 cm
Substitute $V = 375$, $r = 3$
Solve equation for H.
Round to one decimal place.

exercise 11.6

LINKS TO Example 23

1. Calculate the volume of each of these cylinders. In each case give the volume correct to the nearest cubic centimetre or cubic metre, where appropriate.

a



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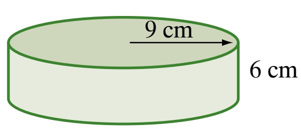

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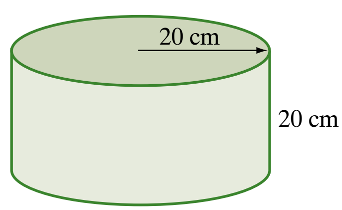

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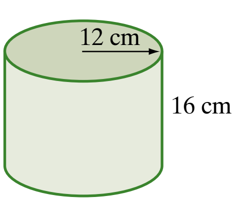

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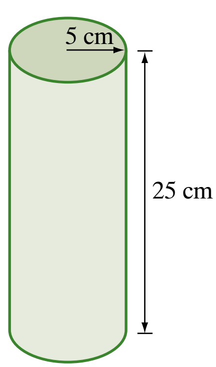

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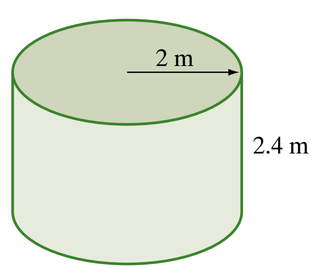

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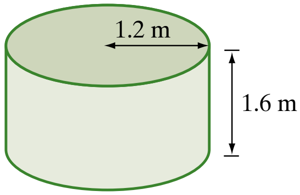
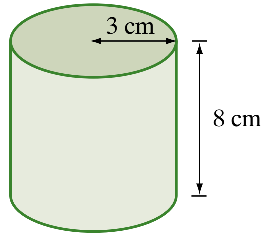
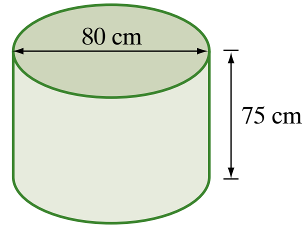
g



h



i



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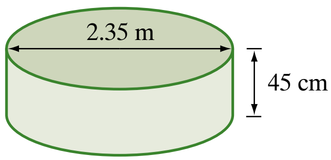

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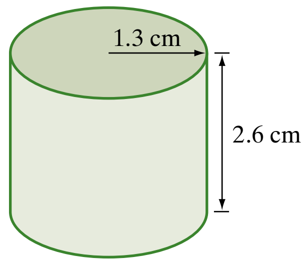
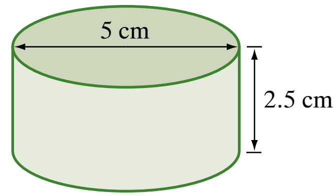


l



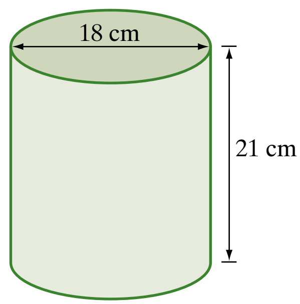
A cylinder has height 8 cm and radius 5 cm. The volume is

A 40π **B** 80π **C** 160π **D** 200π **E** 320π

The volume of this cylinder is given by

A $\pi \times 18 \times 21$ **B** $\pi \times 9 \times 21$ **C** $\pi \times 9^2 \times 21$

D $\pi \times 18^2 \times 21$ **E** $\pi \times (9 \times 21)^2$



LINKS TO Example 24

A 375 mL can of lemonade has a diameter of 6.5 cm and a height of 11.3 cm.

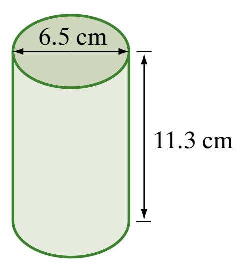
a Calculate the volume

i in cubic centimetres and

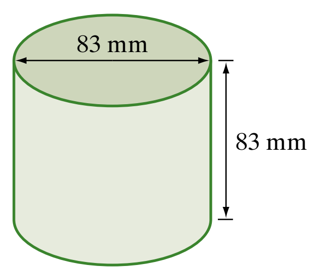
ii in millilitres.

Give your answers correct to the nearest whole number.

b Compare your calculated volume with the labelled volume of 375 mL.



A 450 g can of sliced pineapple has a diameter of 83 mm and a height of 83 mm. Find the volume of the can in cubic centimetres. Give the volume correct to the nearest cubic centimetre.



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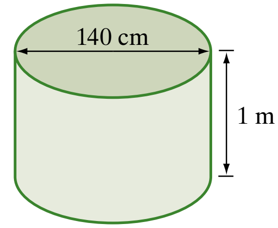
LINKS TO Example 25

A rainwater tank is 1 metre high and has a diameter of 140 cm.

a Find the volume of the tank in cubic metres correct to one decimal place.

b What is the volume of the tank in kilolitres?



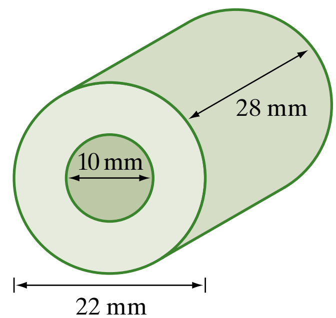
LINKS TO Example 26

A hollow cylinder with inside diameter 10 mm, outside diameter 22 mm and length 28 mm.

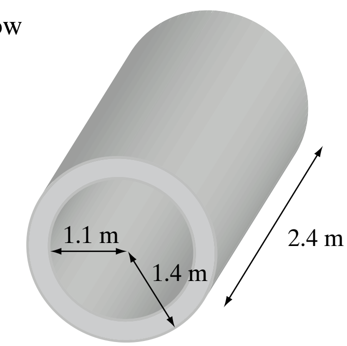
a Give the volume of metal in the cylinder in cubic millimetres.

b Convert your answer to part **a** into cubic centimetres.

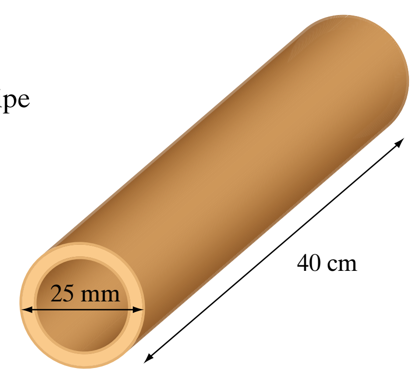
Give your answers correct to the nearest whole number.



Calculate the volume of concrete in this hollow pipe in cubic centimetres correct to one decimal place.



A copper pipe has an outside diameter of 25 mm, and the copper is 1 mm thick. If the pipe is 40 cm long, what is the volume of copper in the pipe in cubic centimetres correct to two decimal places?



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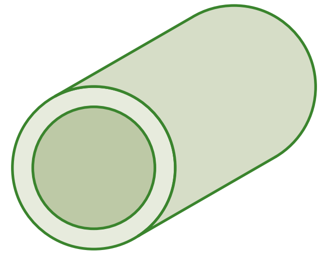
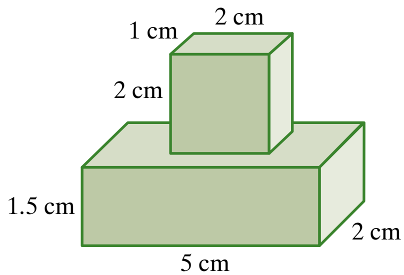
A concrete pipe is 1 m long. It has an inside diameter of 40 cm and an outside diameter of 50 cm. Find the volume of concrete in the pipe in cubic centimetres correct to two decimal places.



LINKS TO Example 27

Find the volume of each of these shapes correct to the nearest cubic centimetre.

a



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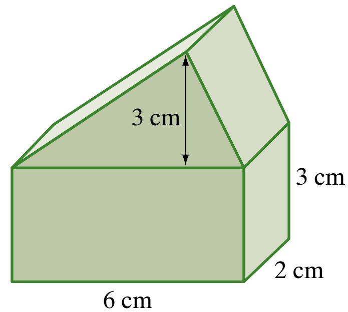

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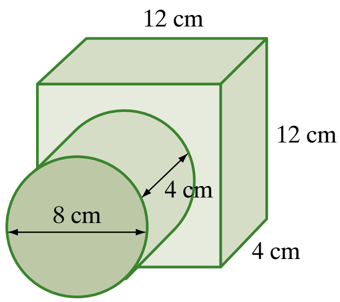

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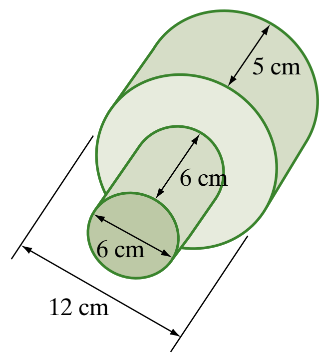

LINKS TO Example 28

Golden Juice packages pineapple juice in 875 mL cylindrical cans. The cans have a diameter of 80 mm. How high are they? Give your answer to one decimal place.

A chemical storage tank is a cylinder with height 8.6 m and radius 4.3 m.

a Calculate the volume of the tank to the nearest cubic metre.

b What is the capacity of the tank in kilolitres?

Rainwater from a flat rectangular roof 44 m by 12 m flows into a cylindrical tank with diameter 4 m. On a particular day 25 mm of rain falls.

a Calculate the volume of water that runs off the roof into the tank.

b By how much will the water level in the tank increase to the nearest centimetre?

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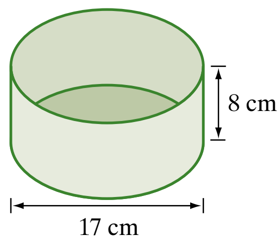
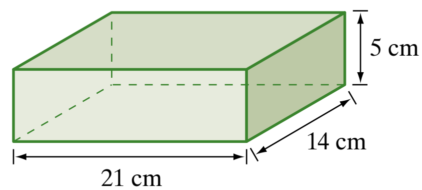
exercise 11.6 challenge

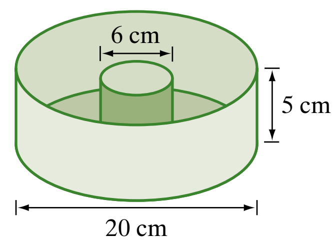
- Lachlan has two recipes for a chocolate cake. One recipe says to use a round tin 17 cm in diameter and 8 cm deep. The other recipe says to use a rectangular tin 21 cm by 14 cm by 5 cm.





a Calculate the volumes of the round tin and the rectangular tin to the nearest cubic centimetre.

Lachlan wants to make his cake in the ring tin shown here. The cake will then be in the shape of a ring, with a hole in the centre. Lachlan needs to know whether the round tin or the rectangular tin has a similar volume to his ring tin.



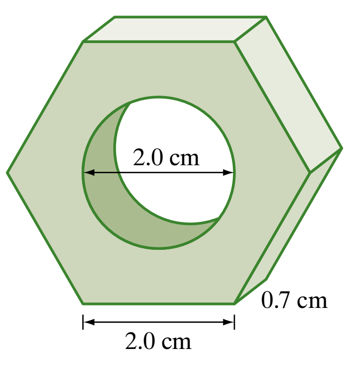
b Calculate the volume of the ring tin to the nearest cubic centimetre.

c Which recipe should Lachlan use—the round tin recipe or the rectangular tin recipe?

- Steel hexagonal nuts have the dimensions shown.

a Calculate the volume of metal in each nut, correct to two decimal places.

b If the density of steel is 7.8 g/cm^3, calculate the mass of each nut.



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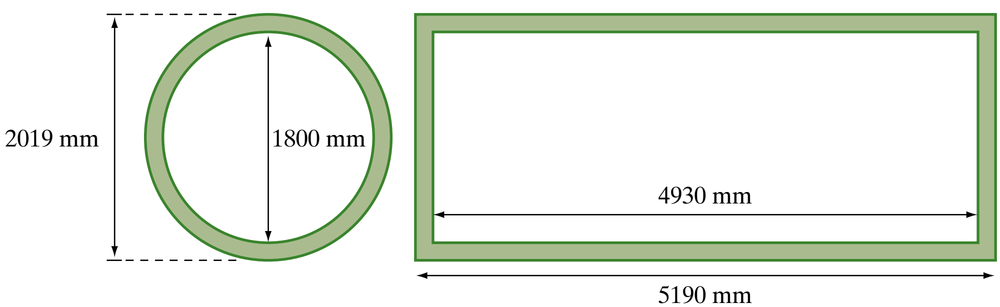
The photograph shows an underground concrete rainwater water storage tank being delivered to a newly constructed house. The diagram on the next page shows the external and internal dimensions of the tank.



End view **Side view**

a Calculate the overall volume of the tank, using the external measurements. Give the volume in cubic metres correct to one decimal place.

b What is the thickness of the concrete in the circular wall of the tank? Give the thickness to the nearest centimetre.

c Calculate the internal volume of the tank. Give the volume in cubic metres correct to one decimal place.

d What is the capacity of the tank in kilolitres? Answer to one decimal place.

e What is the volume of concrete in the tank? Give the volume in cubic metres correct to one decimal place.

f If the density of the concrete is approximately 2500 kg/m^3, calculate the approximate mass in tonnes of the empty tank. Answer to one decimal place.

g What is the approximate mass of the tank if it contains 10 000 litres of water?

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Analysis task

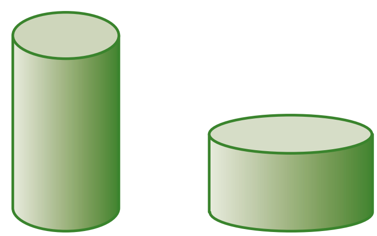
Chemical storage tanks

Chemical storage tanks

Many industrial chemicals are fire hazards, and large-scale explosions could result if a fire spread through a storage area. Liquid chemicals are stored in large tanks. Government safety regulations control the size and shape of tanks and the spacing between the tanks.

a Find the volume of a cylindrical tank with a radius of 10 m and height 2 m, correct to one decimal place.

A liquid chemical is to be stored in a tank with a volume of 500 m^3. Tank manufacturers make different shaped tanks, and a tall narrow tank could have the same volume as a low wide tank.



b If the height of a tank is 2 m, what radius would give the tank a volume of 500 m^3? Give your answer to two decimal places.

c Set up a spreadsheet to show different combinations of height and radius for a tank with volume 500 m^3. Show integer heights from 2 m to 20 m in column A. In column B, calculate each radius (correct to two decimal places), using an appropriate formula. For π, use **pi( )**.

	A	B
1	Height (m)	Radius (m)
2	2	
3	3	
4	4	
5	5	
6	6	
7	7	
8	8	
9	9	
10	10	
11	11	

The design engineer must take several factors into account in choosing a suitable shape for the tank.

- Tanks with smaller diameters are cheaper to construct as they can be manufactured in a factory and transported to the site. Larger tanks have to be built on the site and therefore cost more. However, tall tanks can be unstable in strong winds.

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- The pressure in a liquid increases with depth. The taller the tank, the greater the pressure at the bottom, so the tank has to be constructed to withstand the greater pressure.
- The cost of the tank depends partly on its surface area.

d Why would the cost of a tank depend on the surface area?

e Calculate the surface area for a tank with height 5 m. (Use the value for the radius from your spreadsheet.)

f In column C of your spreadsheet, calculate the surface area for each height and radius, using an appropriate formula.

g Which height and radius in your spreadsheet gives the smallest surface area?

h Can you find a smaller surface area by using intervals of 0.1 between the heights of 8 m and 10 m?

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Review Surface area and volume

Summary

Cross sections of prisms and cylinders

- A cross-section is found by cutting through a solid shape in a given plane. A prism has a constant cross-section congruent to the ends of the prism when it is cut parallel to the ends.
- A cylinder has a constant circular cross-section with the same diameter as the ends of the cylinder when it is cut parallel to the ends.

Nets of prisms and cylinders

- A net of a three-dimensional object is a flat shape that can be folded to make the three-dimensional object.
- The net of a prism consists of two congruent polygons for the end faces and three or more rectangles for the faces joining the ends.
- The net of a cylinder consists of two circles and a rectangle. The width is equal to the circumference of the cylinder and the height of the rectangle is equal to the height of the cylinder.

Surface area

- Prism: add the areas of all the faces.
- Rectangular prism: add the areas of the rectangular surfaces.
- Triangular prism: add the areas of the triangular ends and the rectangular surfaces.
- Cylinder: add the area of the circular ends and the area of the curved surface.
  Area of curved surface = $2\pi rh$

Volume

- Prism: $V = AH$ where A is the cross-sectional area and H is the height
- Rectangular prism: $V = l \times w \times H$ where l is the length, w is the width and H is the height
- Cylinder: $V = \pi r^2 H$ where r is the radius and H is the height
- Composite solids: add the volumes of the component parts of the solid.

Conversion of volume and capacity units

$$\begin{tikzcd}
\mathbf{mm^3} \arrow[r, bend left, "\div 10^3"] & \mathbf{cm^3} \arrow[l, bend left, "\times 10^3"] \arrow[r, bend left, "\div 100^3"] & \mathbf{m^3} \arrow[l, bend left, "\times 100^3"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{mL} \arrow[r, bend left, "\div 1000"] & \mathrm{L} \arrow[l, bend left, "\times 1000"] \arrow[r, bend left, "\div 1000"] & \mathrm{kL} \arrow[l, bend left, "\times 1000"] \arrow[r, bend left, "\div 1000"] & \mathrm{ML} \arrow[l, bend left, "\times 1000"]
\end{tikzcd}$$

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1 cm^3	1 millilitre (1 mL)
1000 cm^3	1 litre (1 L)
1 m^3	1 kilolitre (1 kL)

Visual map

Using the following terms (and others if you wish), construct a visual map that illustrates your understanding of the key ideas covered in this chapter.

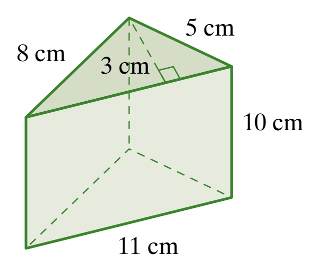
cross-section	cylinder	prism	right prism
cube	diameter	radius	surface area
cuboid	net	rectangular prism	volume

Revision

Multiple-choice questions

The total surface area of the prism is

A 1200 cm^2. **B** 330 cm^2.
C 240 cm^2. **D** 273 cm^2.
E 13 200 cm^2.



The volume of the prism in question **1** is:

A 330 cm^3. **B** 13 200 cm^3.
C 4400 cm^3. **D** 270 cm^3.
E 165 cm^3.

The volume of this prism in cubic centimetres is

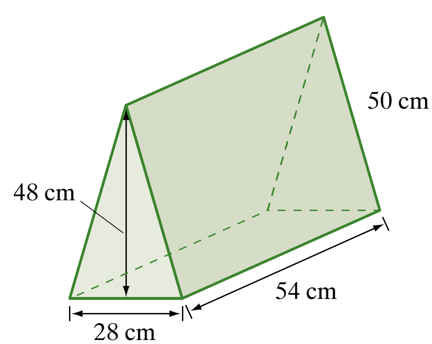
A $\frac{1}{2} \times 28 \times 48 \times 50 \times 54$

B $\frac{1}{2} \times 28 \times 48 \times 50$

C $\frac{1}{2} \times 28 \times 48 \times 54$

D $28 \times 54 \times 50$

E $28 \times 48 \times 50 \times 54$



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The total surface area of this cylinder is given by

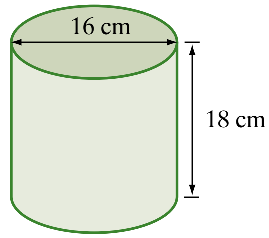
A $2\pi \times 16 \times 18 + \pi \times 16^2$ cm^2.

B $\pi \times 16 \times 18 + \pi \times 16^2$ cm^2.

C $2\pi \times 16 \times 18 + 2\pi \times 16^2$ cm^2.

D $2\pi \times 8 \times 18 + \pi \times 8^2$ cm^2.

E $\pi \times 16 \times 18 + 2\pi \times 8^2$ cm^2.

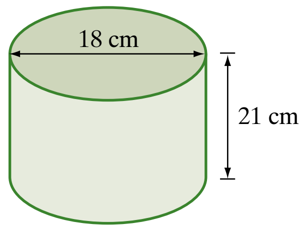


The volume of this cylinder is given by

A $\pi \times 18 \times 21$ **B** $\pi \times 9 \times 21$

C $\pi \times 9^2 \times 21$ **D** $\pi \times 18^2 \times 21$

E $\pi \times (9 \times 21)^2$



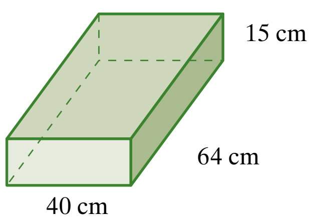
Short-answer questions

For each rectangular prism,

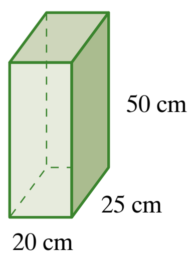
i draw a net

ii calculate the total surface area.

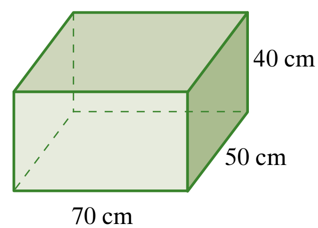
a



b



c



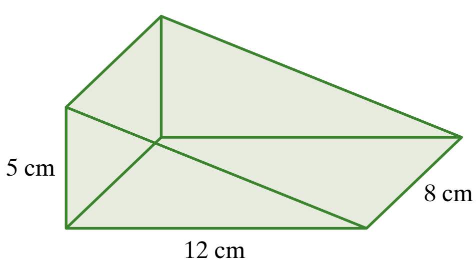
Calculate the volume of each rectangular prism in question **6**.

For this triangular prism,

a draw the net.

b calculate the total surface area.

c calculate the volume.



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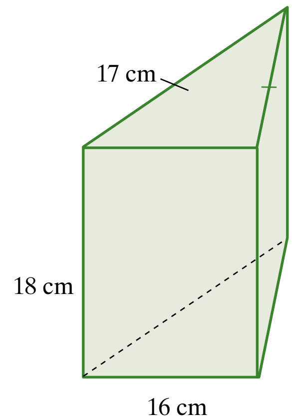
Correct to one decimal place, calculate

a the surface area.

b the volume.

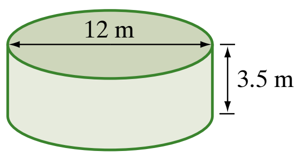


For each cylinder calculate to the nearest square metre (part **a**) or the nearest square centimetre (parts **b** and **c**),

i the area of the curved surface.

ii the total surface area.

a



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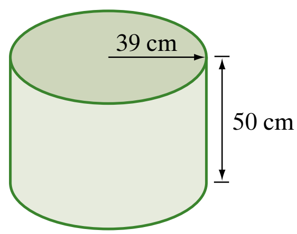
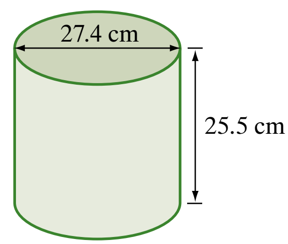
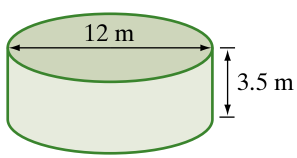


c

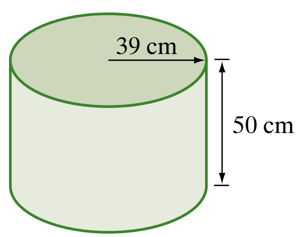


Calculate the volume of each cylinder to the nearest cubic metre (part **a**) or the nearest cubic centimetre (parts **b** and **c**).

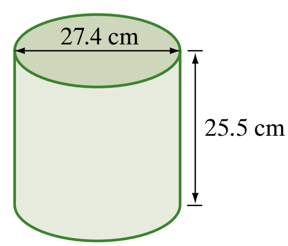
a



b



c



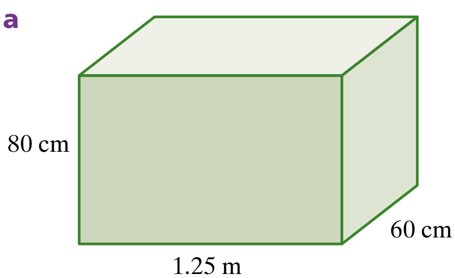
For each of these water tanks,

i draw the net.

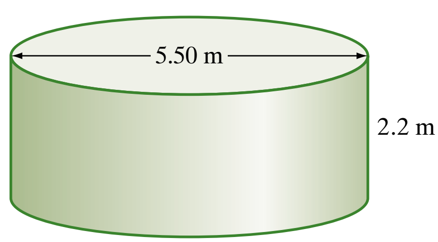
ii calculate the volume in cubic metres.

iii calculate the capacity in kilolitres.

a



b



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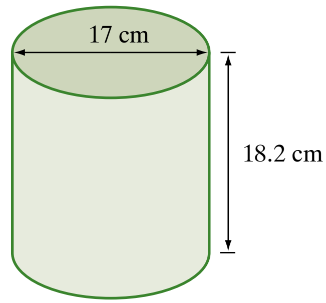
- **a** The inside of a fish tank is 36 cm long, 12 cm wide and 20 cm high. What is the volume in cubic centimetres?
  b How many litres of water would the tank hold if filled to the top?

- **a** Calculate the volume of the can of paint shown here, giving your answer correct to the nearest cubic centimetre.
  b What is the volume of the can in litres? Give your answer correct to the nearest litre.
  c Calculate the area of the curved surface of the can, giving your answer correct to the nearest square centimetre.
  d Estimate the total area of the sheet metal in the can, including the lid. Answer correct to the nearest square centimetre.



Extended-response questions

- The tank in the photograph below collects the rainwater that runs off the roof of the shed. The water collected in the tank supplies three holiday cabins.

  a If the area of the shed roof is approximately 90 m^2 how much water would be collected in the tank on a day when there was 12 mm of rain? Give your answer in kilolitres.
  b The tank has a diameter of 4 m and a height of 2.4 m. What is its capacity to the nearest kilolitre?
  c Assuming that all the water falling on the roof runs into the tank, what is the total amount of water that could be collected from the roof per year if the annual rainfall is 110 cm? Give your answer in kilolitres.
  d If the three holiday units were each occupied on average for 200 days per year, estimate how many litres of water would be available for each holiday unit per day.

Practice quiz
Chapter 11

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12 Percentages and simple interest

Pre-test

Warm-up

People borrowing money from banks or other financial institutions are charged interest. Savers who deposit money into a bank account are paid interest. The amount deposited is called the principal and the amount of interest is a percentage of the principal. Simple interest is calculated each year or month on the principal and earns a constant amount over that period. This means that the total amount of money grows steadily. In some types of investment you receive interest on your previous interest as well as on your principal. This is called compound interest. This chapter focuses on simple interest calculations and also reviews percentage calculations from Year 8.

12.1 Reviewing percentage calculations

BLM
Chapter warm-up

Percentages express a quantity as the number of parts out of 100 parts. Percentages can be converted into fractions or decimals.

Example 1

The percent by weight of sulphur in sodium sulphate is approximately 32%. Write this

a as a fraction **b** as a decimal.

Working	Reasoning
a $32\% = \frac{32}{100} = \frac{8}{25}$	32% is 32 parts out of every 100 parts.
b $32\% = 0.32$	$32 \div 100 = 0.32$

Five important types of calculations with percentages are reviewed in this section:

- Finding a percentage of a quantity
- Expressing one quantity as a percentage of another
- Finding the whole.
- Calculating percentage increase and decrease
- Increasing or decreasing an amount by a given percentage

Finding a percentage of a quantity

Data is often given as percentages, for example, 41.7% of a sample of cars tested for standard safety items (tyres, brakes, steering, lights and seat belts) did not meet safety standards. Unless we know the size of the sample, we have no way of knowing the actual number of cars that did not meet the safety standards.

Example 2

In a sample of 1177 cars, 41.7% did not comply with the safety standards. What number of cars in the sample did not comply?

Working	Reasoning
41.7% of 1177 $= 0.417 \times 1177$ $= 490.8...$ 491 cars did not comply with the safety standards.	$41.7\% = \frac{41.7}{100} = 0.417$

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Expressing one quantity as a percentage of another

By expressing quantities as percentages, it is much easier to make comparisons between groups of different sizes.

Example 3

In a sample of 6180 secondary school students, 5560 students stated that they participated in sport. What percentage of the students is this?

Working	Reasoning
a $\frac{5569}{6180}$	Write 5569 out of 6180 as a fraction.
$= \frac{5569}{6180} \times 100\%$	Multiply by 100%.
$= 90.11...\%$	
90% of students participated in sport.	

Finding the whole

Sometimes we know the value of a certain percentage of a quantity but we may want to know the value of the whole amount. One way to do this is to find the value of 1% of the quantity then multiply by 100 to find 100%, that is, the whole quantity. However, it is simpler to turn the percentage to a decimal and divide the known amount by this decimal number.

Example 4

19% of Australia's total land area is forest (either native or plantation). If 147 million hectares is forest, what is the total land area of Australia?

Working

Let A million hectares be the total land area of Australia.

19% of A = 147

$$0.19A = 147$$

$$A = \frac{147}{0.19}$$

$$= 773.68...$$

The total land area of Australia is approximately 774 million ha.

Reasoning

Alternative method:

19% of A = 147

$$1\% \text{ of } A = \frac{147}{19}$$

$$100\% \text{ of } A = \frac{147}{19} \times \frac{100}{1}$$

$$= 773.68...$$

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Calculating percentage increase and decrease

When quantities such as populations, house prices or occurrence of diseases change, the changes are usually expressed as percentage increases or decreases.

Percentage increase or decrease is expressed as a percentage of the original value.

$$\text{Percentage increase or decrease} = \frac{\text{Change in value}}{\text{Original value}} \times \frac{100}{1}\%$$

Example 5

a On a particular day there was an increase from \$22.12 to \$22.59 in the value of Tasman Bank shares. What was the percentage increase?

b An airline reported that the number of passengers in September 2011 was 1 272 000 compared with 1 210 000 in September 2010. Calculate the percentage decrease correct to one decimal place.

Working

a

$$\begin{aligned}\text{Percentage increase} &= \frac{\text{increase}}{\text{original value}} \times \frac{100}{1}\% \\ &= \frac{22.59 - 22.12}{22.12} \times \frac{100}{1}\% \\ &= \frac{0.47}{22.12} \times \frac{100}{1}\% \\ & 2.12...\%\end{aligned}$$

There was been a 2.1% increase in the value of the shares.

b

$$\begin{aligned}\text{Percentage decrease} &= \frac{\text{decrease}}{\text{original value}} \times 100\% \\ &= \frac{1\,272\,000 - 1\,210\,000}{1\,272\,000} \times 100\% \\ &= \frac{62\,000}{1\,272\,000} \times 100\% \\ &\approx 4.87\%\end{aligned}$$

The percentage decrease in passengers was approximately 4.9%

Reasoning

a The value has increased. Subtract the old value from the new value to find the change. Express the change as a percentage of the old value.

b Percentage decrease and increase are normally calculated as a percentage of the original value.

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Increasing or decreasing an amount by a given percentage

To increase a quantity by a percentage, we add the percentage to 100% and convert to a decimal then multiply the original quantity by this decimal number.

To decrease a quantity by a percentage, subtract the percentage from 100% and convert to a decimal. Multiply the original quantity by this decimal number.

Example 6

a Increase \$240 by 8%.
b Decrease 1300 by 20%.

Working	Reasoning
a $\$240 \times 1.08 = \$259.20$	100% of \$240 + 8% of \$240 = 108% of \$240 108% = 1.08
b $1300 \times 0.80 = 1040$	100% of 1300 – 20% of 1300 = 80% of 1300 80% = 0.80

Example 7

In 2011, the number of students enrolled at a school was 765. This number was up by 4.5% on the 2010 number. How many students were at the school in 2010?

Working

Let n be the number of students in 2010.

$$n \times 1.045 = 765$$
$$n = 765 \div 1.045$$
$$n = 732.057...$$

There were 732 students at the school in 2010.

Reasoning

An increase of 4.5% means that the 2011 enrolment was 104.5% of the 2010 enrolment.
104.5% = 1.045

exercise 12.1

LINKS TO Example 1

1. Minna pays 28% of her income on rent. Write this as
   a a fraction in its simplest form. **b** a decimal.

LINKS TO Example 2

2. **a** A water reservoir can hold 138 gigalitres of water. If it is 33.1% full, how many gigalitres of water does it contain? Give the amount in gigalitres correct to one decimal place.
   b Another reservoir currently holds 25.6 gigalitres. If it is 28.4% full, what is the total amount of water it can hold when full? Give the capacity to the nearest gigalitre.

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Information about home energy use and conservation was collected by the Australian Bureau of Statistics from 12 841 Australian households during March 2011. The results of the survey showed that 61.8% of the sample households had a laptop or notebook computer. How many households had a laptop or notebook computer?

LINKS TO Example 3

In a sample of 1400 secondary school students, 1180 stated that they had a mobile phone. What percentage of the students is this? Give your answer to the nearest whole number percentage.

LINKS TO Example 4

In a particular school 486 students lived within 2 km of the school. If this represented 72% of students, how many students attended the school?

LINKS TO Example 5

The population of a town increased from 23 487 to 24 219. What was the percentage increase?

LINKS TO Example 6a

Increase each of these amounts by the percentage shown.

a 1800 by 35% **b** 480 by 16% **c** 9500 by 25% **d** 285 by 40%

LINKS TO Example 6b

Decrease each of these amounts by the percentage shown.

a 1500 by 65% **b** 120 by 30% **c** 7600 by 18% **d** 590 by 2%

Shares in Western Mineral Exploration rose by 1.8%. If the shares were worth $5.43 before the rise, what was their new price?

The population of a town of 14 750 increased by 0.4%. What was the new population?

Georgie has had a salary increase of 3%. If her current salary is $62 500 per year, what is her new salary?

Rick paid $27 450 for his new car one year ago. After one year of use, its value had dropped by 15%. What is the value of the car now? Give the value to the nearest one hundred dollars.

A company's profit rose by 58% since the previous year. If the profits in the previous year were $1 470 000, what was the new profit?

LINKS TO Example 7

A cricket club has 52 members. This is 18% up on last year's membership. How many members did the club have last year? (Round your answer to the nearest whole number.)

The US Energy Information Administration reported that US carbon dioxide emissions in 2010 were 5638 million tonnes. This was up by 3.9% on 2009 emissions. Calculate the 2009 emissions.

exercise 12.1 challenge

The population of a town increased by 4% from 2009 to 2010 and then decreased by 2% in 2011. The population is now 6370 people. What was the population at the beginning of 2009?

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12.2 GST, mark-ups and discounts

Goods and Services Tax (GST)

In Australia, a Goods and Services Tax (GST) of 10% is added to most goods and services. Certain items are exempt from GST, including unprocessed food such as fresh fruit and vegetables. The displayed price of goods must include GST.

A retailer must set the price of goods then add a further 10% GST. This means that the final price is 110% of the retailer's price.

Example 8

The pre-GST price of a concert ticket is \$12.00. What is the price including GST?

Working

$$\begin{aligned}\text{Price including GST} &= 110\% \text{ of pre-GST price}\\ &= 1.1 \times \$12.00\\ &= \$13.20\end{aligned}$$

The price including GST is \$13.20.

Reasoning

10% GST is added to the pre-GST price.

Example 9

An electrician's charge of \$495 for some work includes GST.

a What is the electrician's charge before GST is added?

b How much of the charge is GST?

Working

a

$$\begin{aligned}\text{Price including GST} &= 110\% \text{ of pre-GST price}\\ \$495 &= 1.1 \times \text{pre-GST price}\\ \text{Pre-GSTprice} &= \frac{\$495}{1.1}\\ &= \$450\end{aligned}$$

The electrician's charge before GST is \$450.

Reasoning

10% GST is added to the pre-GST price.
Price including GST = 110% of price before GST is added.

continued

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Example 9 continued

Working

b GST = \$495 – \$450
= \$45
\$45 of the charge is GST.

Reasoning

Alternatively:

$$\text{GST} = \frac{1}{11} \text{ of price including GST}$$
$$= \frac{1}{11} \times \$495$$
$$= \$45$$

Mark-ups and discounts

Mark-ups and discounts are special cases of increasing or decreasing a quantity by a given percentage. The amount that the retailer adds to the **wholesale price** to set the **retail price** is called the **mark-up**.

The **percentage mark-up** is normally calculated as a percentage of the wholesale price.

Mark-up = percentage mark-up + wholesale price

Retail (or selling) price = wholesale price + mark-up

Example 10

A shop buys jeans from a wholesaler for \$45 and marks them up to \$76.50.

a What is the mark-up?

b What is the percentage mark-up?

Working

a Mark-up = selling price – wholesale price
= \$76.50 – \$45
= \$31.50
The mark-up is \$31.50.

b $$\% \text{ mark-up} = \frac{31.50}{45} \times 100\%$$
$$= 70\%$$
The percentage mark-up is 70%.

Reasoning

The mark-up is the amount that the shop adds to the wholesale price.

Write the mark-up as a fraction of the wholesale price.
Multiply by 100%.

Example 11

A shop puts a 60% mark-up on shirts bought from a wholesaler for \$17.50.

a What is the retailer's price before GST is added?

b What is the final price after GST is added?

continued

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Example 11 continued

Working

a $\text{Mark-up} = 60\% \text{ of } \$17.50$

$\text{Retail price} = 160\% \text{ of } \$17.50$
$= 1.6 \times \$17.50$
$= \$28$

Reasoning

Mark-up is 60% of the wholesale price
Add 60% to 100% to find the retail price.

b $\text{Price including GST} = \$28 \times 1.1$
$= \$30.80$

The final price including GST is $30.80.

Price including GST = 110% of pre-GST price.

Example 12

A shop puts a 75% mark-up on goods it buys from a wholesaler. If a digital camera is marked up to $154 (before GST) what was the wholesale price?

Working

$175\% \text{ of wholesale price} = \$154$

$1.75 \times \text{wholesale price} = \$154$

$\text{wholesale price} = \dfrac{\$154}{1.75}$
$= \$88$

Reasoning

The selling price is 175% of the wholesale price.
Divide the selling price by 1.75 to find the wholesale price.

Percentage discount is calculated as a percentage of the retail (selling) price.

Discount = percentage discount × normal retail price

Sale price = normal retail price − discount

Example 13

An on-line computer store offers 8% discount on laptop computers. What is the discounted price of a laptop that normally sells at $860?

Working

$\text{Discounted price} = (100\% - 8\%) \text{ of } \$860$
$= 0.92 \times \$860$
$= \$791.20$

The discounted price is $791.20.

Reasoning

The normal price represents 100%.
Subtract 8% from 100%.

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When an item is discounted we calculate the percentage discount by expressing the discount as a percentage of the normal price.

Finding the normal price of discounted goods

Finding the normal price of discounted goods is an example of finding the whole when a percentage is known.

Example 14

Find the normal price of a television if it costs \$1020 after a 15% discount.

Working

$$\begin{aligned}\text{Discount price} &= 85\% \text{ of normal price}\\ \text{Normal price} &= \frac{\text{discount price}}{0.85}\\ &= \frac{\$1020}{0.85}\\ &= \$1200\end{aligned}$$

Reasoning

Normal price represents 100%.
Subtract 15% discount from 100% to find the discount price as a percentage of the normal price.

exercise 12.2

LINKS TO Example 8
1. The pre-GST price of a laptop computer is \$580. What is the price including GST?

LINKS TO Example 9
2. What is the amount of a gas bill before GST is added if the amount including GST is \$118.16?

LINKS TO Example 9
3. A widescreen LCD TV is advertised at \$2739. How much of this cost represents GST?

LINKS TO Example 10
4. A shop buys wholesale bananas for \$78 per 13 kg box and sells the bananas for \$7.89 per kilogram. What is the percentage mark-up?

LINKS TO Example 11
5. A retailer buys DVDs for \$9.60 and puts a 75% mark-up on them.
   - **a** What is the marked-up price before GST is added?
   - **b** What is the price including GST if the retailer rounds down the price to the nearest 5 cents?

LINKS TO Example 11
6. A computer shop buys a particular model of computer for \$450 from a wholesaler and marks up the price by 70%.
   - **a** What is the retail price of the computers before GST is added?
   - **b** What is the retail price after GST is added?

LINKS TO Example 12
7. A shop puts an 80% mark-up on clothing it buys from a wholesaler. If shirts are marked up to \$36.90 (before GST) what was the wholesale price?

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LINKS TO Example 13

- A department store offers a 15% discount on all goods on a particular day. What is the saving on purchases totalling $314?

LINKS TO Example 14

- An airline advertises a 20% discount on airfares.
  a What was the original price of a fare that is reduced to $720?
  b By how much has the fare been reduced?

exercise 12.2 challenge

- A clothing shop advertises a 20% discount on all goods.
  a Calculate the sale price of a business shirt that was previously $48.
  b The shirt is actually on sale for $40. What percentage of the original price is this?
  c What discount does this represent?
  d Why is the shop claiming that they are giving a 20% discount?

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12.3 Simple interest

Interest rates

When you deposit money into a savings account with a bank, the bank pays you **interest** in return for 'lending' them the money. The bank is able to do this because it lends some of its deposits at a higher rate of interest.

Bank interest rates are a percentage of the **principal**, that is, the amount of money deposited or borrowed. The interest rate is usually advertised as a certain percentage **per annum** (p.a.).

> *Annum* is the Latin word for year, so per annum means per year.

If, for example, a bank is offering interest of 5.50% p.a., then at the end of each year you would be paid interest of 5.50% of the amount you invested. At the end of the year you would still have your principal, but you would also have an extra 5.5% of your principal. For each year of your investment, you would receive the same amount of interest.

This type of interest is called **simple interest**.

Simple interest

$I = PRT$ where

$\$I$ is the interest,

$\$P$ is the principal,

R is the annual interest rate and

T is the number of years.

Note that R is a percentage, for example, if the interest rate is 6%, then

$R = 6\% = \frac{6}{100} = 0.06.$

Provided we know three of the four variables, I, P, R or T, then we can calculate the fourth variable.

Calculating the amount of interest earned

The values of P, R and T are substituted to calculate I.

Example 15

Hannah invests \$5600 for 5 years at 6.25% p.a. interest.
How much interest will Hannah earn?

continued

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Example 15 continued

Working

$P = 5600$, $R = 0.0625$, $T = 5$

$$\begin{aligned} I &= PRT \\ &= 5600 \times 6.25\% \times 5 \\ &= 5600 \times 0.0625 \times 5 \\ &= 1750 \end{aligned}$$

Hannah earns \$1750 interest on the investment.

Reasoning

The principal is \$5600, the interest rate is 6.25% and the time is 5 years.

Substitute for P, R and T to find I.

Example 16

Alfred invests \$4200 for 3 years at 5.25% p.a. interest.

a How much interest will Alfred earn?

b How much will Alfred have altogether after 3 years?

Working

a $P = 4200$, $R = 5.25\%$, $T = 3$

$$\begin{aligned} I &= PRT \\ &= 4200 \times 5.25\% \times 3 \\ &= 4200 \times 0.0525 \times 3 \\ &= 661.50 \end{aligned}$$

Alfred earns \$661.50 interest on the investment.

Reasoning

The principal is \$4200, the interest rate is 5.25% and the time is 3 years.

Substitute for P, R and T to find I.

Working

b \$4200 + \$661.50 = \$4861.50

Alfred will have \$4861.50 after 3 years.

Reasoning

Add the interest to the principal.

If the time of an investment is given in months but the interest rate is per annum, then the number of months is converted to years by dividing by 12. Similarly, if the length of an investment is given in days, for example, 90 days, the number of days is converted to years by dividing by 365.

Example 17

Calculate the interest earned on a term deposit of \$6000 invested at 5.8% for 9 months.

continued

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Example 17 continued

Working

$P = 6000,\ R = 5.8\%,\ T = \frac{9}{12}$

$$\begin{aligned} I &= PRT \\ &= 6000 \times 5.8\% \times \frac{9}{12} \\ &= 6000 \times 0.058 \times \frac{9}{12} \\ I &= 261 \end{aligned}$$

The interest earned is \$261.

Reasoning

9 months is $\frac{9}{12}$ of year.

Substitute for P, R and T to find I.

Calculating the time taken to earn a given amount of interest

The simple interest formula can be used to calculate how long it would take to earn a certain amount of interest. The values of P, R and I are substituted to calculate T.

Example 18

Harry invests \$4800 at 5.0% per annum interest. How many years will it be before Harry's investment has grown to \$6000?

Working

$$\begin{aligned} \text{Interest} &= \$6000 - \$4800 \\ &= \$1200 \end{aligned}$$

$$\begin{aligned} I &= P \times R \times T \\ I &= 1200,\ P = 4800,\ R = 5.0 \\ 1200 &= 4800 \times 5\% \times T \\ 1200 &= 4800 \times 0.05 \times T \\ T &= \frac{1200}{4800 \times 0.05} \\ T &= 5 \end{aligned}$$

It would take 5 years for the investment to grow to \$6000.

Reasoning

Find how much interest would be needed for the investment to grow from \$4800 to \$6000.

Example 19

How many years would it take to earn \$1000 interest if \$3000 is invested at 4.5% p.a. interest?

continued

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Example 19 continued

Working	Reasoning
$I = 1000$, $P = 3000$, $R = 4.5\%$ $I = PRT$	
$1000 = 3000 \times 4.5\% \times T$ $1000 = 3000 \times 0.045 \times T$ $T = \dfrac{1000}{3000 \times 0.045}$	Substitute for I, P and R to find T.
$T \approx 7.4$ It would take 8 years to earn \$1000 interest.	The interest is added each year, so 7 years would not be long enough to earn \$1000. 7.4 years must be rounded up to 8 years.

Example 20

How long would it take for an investment of \$800 to increase to \$1000 if the simple interest rate is 5%?

Working	Reasoning
Interest earned = \$1000 – \$800 = \$200	Calculate the amount of interest earned by subtracting the original investment from the total amount.
$P = 800$, $R = 5\%$, $I = 200$	The principal is \$800, the interest rate is 5% and the interest is \$200.
$I = PRT$ $200 = 800 \times 0.05 \times T$ $T = \dfrac{200}{800 \times 0.05}$ $= 5$	Substitute for I, P, and R to find T.
It would take 5 years for \$800 to grow to \$1000 at 5% interest.	

Calculating the required principal to earn a given amount of interest

The simple interest formula can be used to find the required principal for a given time, interest rate and amount of interest earned. The values of R, T and I are substituted to calculate P.

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Example 21

What principal would earn \$600 interest at 5% interest for 4 years.

Working

$I = 600, R = 5\%, T = 4$

$$I = PRT$$

$$600 = P \times 5\% \times 4$$

$$600 = P \times 0.05 \times 4$$

$$P = \frac{600}{0.05 \times 4}$$

$$P = 3000$$

The required principal is \$3000.

Reasoning

Substitute for I, R, and T to find P.

Calculating the required interest rate to earn a given amount of interest

For a given principal and time we can calculate the interest rate that will result a given amount of interest. The values of P , T and I are substituted to calculate R.

Example 22

What interest rate per annum would give \$450 interest if \$2500 is invested for 3 years.

Working

$I = 450, P = 2500, T = 3$

$$I = PRT$$

$$450 = 2500 \times R \times 3$$

$$R = \frac{450}{2500 \times 3}$$

$$R = 0.06$$

The required interest rate is 6% p.a.

Reasoning

Substitute for I, P and T to find K.

exercise 12.3

LINKS TO Example 15

1. Calculate the simple interest earned on these investments.
   - **a** \$580 at 4.25% p.a. interest for 4 years
   - **b** \$300 at 5.8% p.a. interest for 3 years
   - **c** \$5000 at 6.4% p.a. interest for 5 years
   - **d** \$12 000 at 6.25% p.a. interest for 8 years

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LINKS TO Example 16

Complete the following table.

	Principal	Interest rate	Time (years)	Interest	Principal + interest
a	$2000	5%	2		
b	$600	4.5%	5		
c	$750	1.4%	4		
d	$15 000	4.8%	10		
e	$8500	3.75%	3		
f	$24 000	6.25%	5		

Calculate the simple interest earned if $2000 is invested at 1.45% simple interest for

a 1 year **b** 3 years.

Calculate the simple interest earned on $285 in a savings account at 1.5% p.a. interest for 3 years.

What is the simple interest earned on $8000 invested for 2 years at 5.4% p.a. interest?

Michael puts $45 000 in a term deposit for 3 years at 5.65% p.a. simple interest. How much money will Michael have at the end of the 3 years?

Maddy invests $3600 for 4 years at 5.75% p.a. simple interest. How much money will Maddy have at the end of four years?

LINKS TO Example 17

Copy and complete the following table.

	Principal	Interest rate (p.a.)	Time	Interest
a	$800	0.75%	3 months	
b	$5000	0.5%	18 months	
c	$18 000	2.3%	90 days	
d	$500 000	2.7%	90 days	
e	$7500	2.3%	9 months	
f	$6000	5.5%	6 months	

Calculate the interest earned for each of the following investments.

a $5000 is invested at 6.4% p.a. interest for 90 days.

b $2400 is invested at 4.8% p.a. interest for 30 days.

Jack puts $4000 into a term deposit at 2.3% p.a. interest for 9 months.

a How much interest will he earn?

b What is the total value of his investment at the end of the 9 months?

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Annika puts $2500 into a term deposit at 1.25% p.a. interest for 15 months.

a How much interest will she earn?

b What is the total value of his investment at the end of the 15 months?

Archie puts $30 000 into a term deposit at 2.4% p.a. interest for 90 days.

a How much interest will she earn?

b What is the total value of his investment at the end of the 90 days?

LINKS TO Example 18

Copy and complete the following table to show the number of years that would be required to give the amounts of interest shown for the given principals and interest rates.

	Principal	Interest rate	Interest	Time (years)
a	$4000	5%	$800	
b	$1600	4%	$320	
c	$2000	4.5%	$720	
d	$2700	5.2%	$702	

LINKS TO Examples 19, 20

Carly invests $4000 at 5.25% simple interest per annum. How long will it take for Carly's money to grow to $6000?

Alex deposits $1500 into a fixed-interest savings account, which receives 4.75% simple interest per annum. The interest is paid at the end of each year.

a How much interest will Alex earn after one year?

b How many years will it take for Alex to earn over $300 interest?

Oliver invested $6000 at 5% simple interest per annum. How long will it be before Oliver's money has grown to $7500?

How many years would it take for $4000 invested at 5% p.a. simple interest to grow to $8000?

How many years would it take for $250 invested at 4% p.a. simple interest to grow to $500?

Will invests $2000 at 6.4% interest p.a. How many years will it before Will has earned more than $1000 interest?

A 4 years **B** 5 years **C** 7 years **D** 8 years **E** 12 years

How long would it take for an investment of $1400 to increase to over $1800, if the simple interest rate is 5.5% p.a.?

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LINKS TO Example 21

Copy and complete the following table to show the principals that would be required to give the amounts of interest shown for the given time and interest rates.

	Interest	Time (years)	Interest rate	Principal
a	\$54	2	4%	
b	\$120	4	6%	
c	\$2000	8	5%	
d	\$2700	5	5.4%	

What principal would be required to give the following amounts of interest over a period of 4 years if the interest rate is 5%?

a \$80 **b** \$200 **c** \$1500 **d** \$2400

Jason wants to earn \$300 interest over 2 years. The best interest rate he can find is 6%. What principal will Jason need?

LINKS TO Example 22

Copy and complete the following table to show the interest rates that would be required to give the amounts of interest shown for the given principal and time.

	Principal	Time (years)	Interest	Interest rate
a	\$8000	5	\$1200	
b	\$450	3	\$54	
c	\$1500	6	\$450	
d	\$24 000	8	\$6000	

Celia has \$4800 to invest. What interest rate would she need in order to earn \$2000 interest over 6 years? Give the rate as a percentage correct to one decimal place.

exercise 12.3 challenge

Lara invests \$5000 in an account which earns 5% p.a. interest. At the end of the first year the interest Lara has earned during the year is added to the principal so the next year she receives interest on the total amount.

a How much will Lara have at the end of the second year?

b How much will Lara have at the end of the third year of this process is repeated?

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12.4 Repaying interest on loans

When you put money into a bank account you are paid interest, but when you borrow money you have to pay interest.

Example 23

Joan borrows \$1500 at a flat rate of interest of 9% per annum for 2 years. Calculate the interest Joan pays on the loan.

Working	Reasoning
$P = 1500$, $R = 0.09$, $T = 2$	Substitute for P, R and T.
$I = PRT$	
$= 1500 \times 0.09 \times 2$	
$I = 270$	
Joan pays \$270 interest.	

Loans are usually paid off by making equal weekly or monthly instalments. To calculate the amount of the instalments, we calculate the total amount, including interest, to be paid and divide it by the number of instalment periods.

Example 24

\$6000 is borrowed at 9.6% p.a. simple interest over 3 years. Calculate

a the interest paid.
b the monthly repayments.

Working	Reasoning
a $P = 6000$, $R = 9.6\%$, $T = 3$	Substitute for P, R and T.
$I = PRT$	
$= 6000 \times 9.6\% \times 3$	
$= 6000 \times 0.096 \times 3$	
$I = 192$	
b Total repayment $= \$6000 + \$192$	Add the interest to the amount borrowed.
$= \$6192$	
Monthly repayment $= \$6192 \div 36$	3 years = 36 months.
$= \$172$	Divide the total amount to be paid by the number of months.

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Example 25

A widescreen LCD TV is advertised at \$3600. Anna buys the TV under a scheme where she pays a deposit of 20% and 24 equal monthly instalments of \$144.

a What deposit does Anna pay?

b What is the total amount Anna pays?

c How much more does Anna end up paying than if she had paid the full amount of \$3600 when she bought the TV?

d The amount in part **c** represents the interest on the \$3600 paid by Anna over the two years. What annual rate of simple interest on \$3600 does this represent?

Working	Reasoning
a 20% of \$3600 = \$720 Anna paid a deposit of \$720.	$3600 \times 0.20 = 720$
b \$720 + \$144 × 24 = \$4176 Anna pays a total of \$4176.	The total amount is made up of the deposit of \$720 plus the 24 instalments of \$144.
c \$4176 – \$3600 = \$576 Anna pays an extra \$576 for the TV.	Subtract the price of the TV from the total amount Anna paid.
d $I = 576,\ P = 3600,\ T = 2$ $I = PRT$ $576 = 3600 \times R \times 2$ $R = \dfrac{576}{3600 \times 2}$ $R = 0.08$ This represents an annual interest rate of 8%.	The extra \$576 that Anna paid represents interest on \$3600 over 2 years. Substitute for I, P, and T. Solve equation to find R.

exercise 12.4

LINKS TO Example 23

1. Lynn borrows \$6000 at a rate of 9% p.a. simple interest for 2 years.
   a Calculate the amount of simple interest that Lynn pays on the loan.
   b How much altogether does Lynn have to pay back?

2. Jason borrowed \$6000 to pay for a car. If he had to pay 8.8% per annum interest on the loan and Jason paid off the loan in one year, what is the total amount (loan plus interest) that Jason had to pay?

3. Cait buys a computer by obtaining a simple interest loan. The computer costs \$850 and the interest rate on the loan is 9.90% p.a. Cait pays the loan back in weekly instalments over two years. Calculate
   a the amount of interest paid over the 2 years.

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b the total amount to be paid back.

c the weekly amount that Cait pays.

James borrowed \$8000 at 12% p.a. simple interest calculated yearly to buy a car. He makes monthly payments for three years. Calculate

a the amount of interest to be paid.

b the total amount that James must pay back.

c the monthly payment that James pays.

LINKS TO Example 24

Calculate the monthly repayments on a loan of \$32 000 at 9.6% p.a. over 5 years.

Calculate the monthly repayments on a loan of \$7500 at 11.2% p.a. simple interest over 5 years.

Jasmine borrows \$3000 from her parents and repays \$75 per month for 4 years. How much interest does she pay her parents?

LINKS TO Example 25

To purchase a car that costs \$33 960, Sam pays 20% deposit and then makes 24 monthly payments of \$1273.50.

a What deposit does Sam pay?

b What is the total amount Sam pays for the car?

c How much more does Sam end up paying for the car than if he had paid the full amount of \$33 960 when he bought the car?

d The extra amount worked out in part **c** represents the interest Sam was charged on the \$33 960 over the two years. What simple interest rate per year does this represent?

exercise 12.4 challenge

Lin and Mai went shopping for a new refrigerator and washing machine. They decided that an interest-free deal would be good because they didn't have enough cash to pay for them. Lin also wanted to get a new plasma TV, because the salesperson told them they could have the interest free deal on all three items. The total cost was \$2352 to be paid in weekly instalments over 24 weeks.

a How much did Lin and Mai have to pay each week to ensure that they paid off the total amount within the interest-free period?

b After 17 weeks Lin was made redundant and Lin and Mai could no longer afford the weekly payments. How much did Lin and Mai still owe?

c When Lin and Mai asked what would happen if they weren't able to pay off the whole amount within the interest-free period, they were shocked to hear that they would have to pay 28% p.a. interest, calculated daily, on what they owed at the end of the 24 weeks interest-free period. After the end of the 24 weeks interest-free period, it was another 60 days before Lin and Mai were able to pay off the amount they owed. How much interest did they have to pay?

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12.5 Credit card interest

Interest is sometimes calculated on a monthly or daily basis. Credit card interest is calculated daily.

Example 26

Express the interest rate of 7.2% p.a. as

a a monthly rate.

b a daily rate.

Working

a $7.2\% \text{ p.a.} = \frac{7.2}{12}\% \text{ per month}$
$= 0.6\% \text{ per month}$

b $7.2\% \text{ p.a.} = \frac{7.2}{365}\% \text{ per day}$
$\approx 0.019726\% \text{ per day}$

Reasoning

Monthly interest rates are calculated on the basis of one-twelfth of the annual rate even though the months vary in length.

If the daily rate is quoted, then it is likely to be rounded to 6 decimal places. Sometimes the exact rate of $\frac{7.2}{365}\%$ will be used.

Credit card payment

Credit card and store card statements are usually sent out each month. The amount of debt shown on this statement is called the **opening balance**. There is normally a **minimum required payment** which must be paid. This minimum required payment is made up of two parts,

- a payment off the opening balance
- and interest on the amount owing.

The minimum required payment may be, for example, 2.5% of the opening balance or $10 whichever is the greater. It is important to understand though, that if only the minimum required amount is paid off, it may take many months or years to pay off a large debt. During this time the interest paid each month will accumulate to a large amount.

Example 27

A credit card has a minimum repayment of 2% of the opening balance or $25, whichever is the greater. For each of the these credit card opening balances

i calculate the minimum payment required on these opening balances.

ii calculate the amount still owing after this minimum payment is made.

a $87

b $2400

continued

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Example 27 continued

Working	Reasoning
a i 2% of \$87	Calculate 2% of the opening balance.
= 0.02×87	Decide which is the greater of \$1.74 or \$25.
= \$1.74	
\$1.74 is less than \$25 so minimum payment required is \$25.	
ii Amount still owing	
= \$87 − \$25	Subtract the amount paid from the opening balance.
= \$62	
b i 2% of \$2400	Calculate 2% of the opening balance.
= 0.02×2400	Decide which is the greater of \$48 or \$25.
= \$48	
Minimum payment required is \$48.	
ii Amount still owing	
= \$2400 − \$48	Subtract the amount paid from the opening balance.
= \$2352	

Credit card interest on overdue payments

When you make purchases using a credit card, the following conditions usually apply.

- Purchases appear on a statement at the end of the month. No interest is charged if you pay the amount due by a certain date.
- If you pay the minimum required payment by the due date, interest is charged at a daily rate for the rest of the month on the balance.
- If you do not pay the minimum required payment by the due date, interest is charged on the debt at a daily rate from the time of your first purchase that contributed to the debt.

This means that, even if your payment is only one day late, you may be charged interest for more than 30 days. The interest rate for credit cards is much higher than the rate for a normal bank loan.

Suppose that an annual rate of 16.650% interest will be charged if the amount due is not paid on time. Dividing the annual rate by 365 gives the daily rate that is charged on overdue amounts.

Annual rate (%)	Daily rate (%)
16.650	0.045616

$16.650 \div 365 = 0.045616$

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Example 28

Carlo used his credit card to buy a computer, and the purchase price of \$2874 was the only item on his next statement. Carlo did not pay this account by the due date. If interest is charged at the daily rate of 0.045616%, how much interest would Carlo owe 36 days after he bought the computer?

Working	Reasoning
$P = 2874,\ R = 0.00045616,\ T = 36$	Substitute for P, R and T.
$I = PRT$	
$= 2874 \times 0.00045616 \times 36$	$0.045616\% = 0.00045616$
$I = 47.196...$	
The interest charged would be \$47.20.	Round to two decimal places for money.

Interest on credit cards is charged on a daily basis, so the annual rate of interest is divided by 365.

Example 29

A credit card company has an annual interest rate of 17.74%. Calculate the interest owing on a cash advance of \$800 if the credit card account is not paid for 23 days after the cash advance was obtained.

Working	Reasoning
$I = PRT$	$17.74\% = 0.1774$
$= 800 \times \dfrac{0.1774}{365} \times 23$	$I = 800 \times \dfrac{17.74\%}{365} \times 23$
$I = 8.9429...$	
\$8.94 interest is owed.	Daily interest rate $= \dfrac{0.1774}{365}$

exercise 12.5

LINKS TO Example 26

1 Calculate each of these annual interest rates
  - **i** as a monthly rate.
  - **ii** as a daily rate correct to six decimal places.

a 17.5% **b** 15% **c** 18.5%
d 21.45% **e** 22.8% **f** 19.990%

LINKS TO Example 27

2 A store card has a minimum payment requirement of 2% of the debt or \$25, whichever is greater. For each of these store card debts,
  - **i** calculate the minimum repayment required.
  - **ii** how much is still owed after the first minimum payment is made.

a \$715 **b** \$2300 **c** \$215 **d** \$1850

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LINKS TO Example 28

- Nita's credit card account of $954 was overdue by 31 days. How much interest would Nita be charged if the daily rate is 0.045616%?
- Matt was 26 days late paying his department store card account of $1365. Interest was charged at the rate of 23.9% p.a., calculated daily. How much interest did Matt have to pay?
- A bank charges interest at the rate of 0.04835% per day for credit card accounts that are not paid by the due date. Jenny had a credit card bill for $4291 which she forgot to pay until 8 days after the due date. How much interest did she have to pay?
- Glen had a department store account of $2487 which he could not afford to pay off until the following month. Calculate the interest Glen had to pay for the 30 days if the interest rate was 0.06548% per day.

LINKS TO Example 29

- A bank has an annual credit card interest rate of 19.99%. Calculate the interest owing on a cash advance of $1400 if the credit card account is not paid for 12 days after the cash advance was obtained.
- A bank has an annual credit card interest rate of 18.85%. Calculate the interest owing on a cash advance of $3700 if the credit card account is not paid for 28 days after the cash advance was obtained.

exercise 12.5 challenge

- Barry has a credit account with a department store. The store charges interest for overdue payments, as shown on this account. The opening balance is the unpaid amount from the previous account, which Barry forgot to pay. Barry has been charged interest on this opening amount right back to the day he made the purchase (36 days before this account). He has also been charged interest on a new purchase made since the previous account (28 days before this account).

The interest is composed of two parts:

- interest on the opening balance of $139 for a total of 36 days
- interest on a new purchase for the 28 days since that purchase was made.

Your annual percentage rate: 23.9%	
ACCOUNT SUMMARY	
Opening balance	**$139.00**
Payments and credits	$0.00
Purchases and other charges	$90.35
Interest	$4.93
Closing balance	**$234.28**

a Calculate the daily interest rate for this annual rate of interest correct to six decimal places.

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Rather than rounding off a daily rate of interest, this retailer uses the daily rate of interest $\frac{23.5}{365}\%$.

b Using this rate, calculate the interest on the opening balance of \$139 for 36 days.

c Calculate the interest on Barry's new purchase of \$90.35 for 28 days.

d Does the total amount of interest you calculated match the amount of interest shown on the account? Explain.

e If Barry failed to pay the account for another 30 days, how much interest would be included in the next account?

Kath has a \$3000 store card debt. She can afford to pay off only the minimum requirement of 2.5% of the debt or \$10, whichever is greater. The interest charged on the outstanding amount each month is 16%.

a How much interest does Kath owe at the end of the first month?

b What is the minimum amount she must pay?

c How much of this minimum amount comes off the amount Kath owes?

d What does Kath owe at the end of the next month (assuming she has made no more purchases)?

e Set up a spreadsheet to calculate how long it will take Kath to pay off the credit card if she continues to pay off only the minimum requirement of 2.5% of the debt or \$10, whichever is greater. Give the time to the nearest year.

f How much interest (to the nearest dollar) does Kath end up paying?

g Now suppose Kath pays off \$200 every month instead of the minimum requirement. Use the spreadsheet to find how long it will take Kath to pay off her debt. Give the time to the nearest month.

h How much interest (to the nearest dollar) does Kath pay?

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Analysis task

Compound interest

With simple interest, the same amount of interest is earned each year and the investment grows at a constant rate.

Suppose that \$100 is invested at 6% p.a. simple interest. After one year, the interest earned is \$6. There is now a total of \$106, but simple interest is calculated only on the original \$100 invested. So after the second year, there is a total of \$112, and so on.

a Copy and complete the following table, extending it to 10 years.

Year	Interest \$	Investment at end of year \$
1	6	106
2	6	112

With **compound interest**, the interest is added to the principal at the end of each interval. In the next interval, interest is calculated on the new principal, that is, on the previous principal plus the interest.

In other words, the interest as well as the principal earns interest, so the amount of interest keeps increasing each year.

Suppose that \$100 is invested at 6% p.a. compound interest.

b How much interest will be earned during the first year?

c If the interest is added to the principal, what is the value of the principal at the start of the second year?

d Calculate the interest earned in the second year.

e What is the value of the principal at the start of the third year?

f Copy and complete the following table and extend it to 10 years.

Year	Investment at start of year \$	Interest \$	Investment at end of year \$
1	100	6	106
2	106		
3			

g Compare the total amount of interest earned for compound interest and for simple interest.

h On the same axes, graph the investment at the start of each year against the year for simple interest and then again for compound interest. Compare the rates of change for simple interest and for compound interest.

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Review Percentages and simple interest

Summary

Finding a percentage of a quantity

- Multiply the quantity by the percentage.

Expressing one quantity as a percentage of another

- Write the quantities as a fraction then convert to a percentage by multiplying by 100.

Finding the whole

- Divide the given amount by the given percentage to find 1% then multiply by 100 to find 100%.

Percentage increase and decrease

- Increased amount = (100% + percentage increase) × original amount
- Decreased amount = (100% + percentage decrease) × original amount

Mark-up

The amount by which a retailer increases the price of goods bought at wholesale price.

- Mark-up = wholesale price × percentage mark-up
- Marked-up price = wholesale price + mark-up

Discount

The amount by which a retailer reduces the cost of goods

- Discount = normal selling price × percentage discount
- Discounted price = normal selling price − discount

GST (Goods and services tax of 10%)

- Price including GST = Pre-GST price + 10% of pre-GST price = 110% of pre-GST price
- GST = $\frac{1}{11}$ of selling price

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Simple interest

- $I = PRT$
  where I is the interest, P is the principal, R is the annual interest rate and T the number of years.
- If we know three of the variables, I, P, R or T, then we can calculate the fourth variable.

Daily and monthly interest rates

- Monthly interest rate is found by dividing the annual interest rate by 12.
- Daily interest rate is found by dividing the annual interest rate by 365.

Credit card interest

- Interest is calculated daily.
- Minimum required payment is made up of two parts: a payment off the opening balance and interest on the amount owing.
- The minimum required payment may be, for example, 2% of the opening balance or $25, whichever is greater.

Simple interest loans

- Payment is usually by equal monthly or weekly instalments
- To find the size of each instalment, calculate the interest to be paid on the loan over the period of the loan then divide by the number of instalments

Visual map

Using the following terms (and others if you wish), construct a visual map that illustrates your understanding of the key ideas covered in this chapter.

discount	interest	loan	principal
GST	interest rate	mark-up	simple interest

Revision

Multiple-choice questions

1. The simple interest on $720 invested at 6.5% per annum for 4 years is
   A $46.80 **B** $1170 **C** $443 **D** $187.20 **E** $907.20
2. Bella invested $3000 at 5.0% p.a. interest. How long will it be before Bella's money has grown to $3600?
   A 3 years **B** 4 years **C** 5 years **D** 6 years **E** 10 years

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- What principal would be needed to give simple interest of \$96 after 3 years at 5% interest?

  A \$640 **B** \$5760 **C** \$1440 **D** \$16 000 **E** \$1562.50

- The simple interest rate that would give \$1080 interest when \$6000 is invested for 4 years is

  A 4% **B** 4.2% **C** 4.5% **D** 4.8% **E** 5%

- \$40 000 is invested at 4.5% p.a. interest for 30 days. The interest earned is

  A \$54 000 **B** \$14 000 **C** \$147.95 **D** \$40 147.95 **E** \$14 795

Short-answer questions

- Calculate the following.
  - **a** What percentage of 56 is 21?
  - **b** What is 6.25% of \$2400?
  - **c** Increase 600 by 18%.
  - **d** 24% of a number is 648. What is the number?
  - **e** The population of a town increases from 4925 to 5065. What is the percentage increase? Answer correct to one decimal place.
- During a sale, a shop advertises 15% discount on all goods.
  - **a** If the normal price of a camera is \$285, what is the discount price?
  - **b** If the discount price of a backpack is \$27.20, what is the normal price?
- Surfboards are bought from a wholesaler for \$420 and marked up 45%. GST is then added. What is the final selling price?
- Calculate the simple interest earned on each of these investments.
  - **a** \$5000 is invested at 5.2% p.a. for 4 years.
  - **b** \$12 000 is invested at 5.6% p.a. for 6 years
- \$20 000 is invested for 60 days at 5.25% p.a. interest. How much interest is earned in the 60 days?
- How many years would it take for an investment of \$2000 to reach \$3500 at a simple interest rate of 6.25% per annum?
- Calculate the interest earned on each of these investments.
  - **a** \$8000 is invested at 5.4% p.a. interest for 3 months.
  - **b** \$25 000 is invested at 6.1% p.a. interest for 90 days.
  - **c** \$20 000 is invested for 30 days at 5.55% p.a.
- Alice invests \$1400 at 6.4% simple interest. How much will Alice have at the end of 5 years?
- Shawn borrows \$600 to be paid back in one year with 8% simple interest.
  - **a** How much interest will Shawn pay?

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b If Shawn pays back the money in 12 equal monthly instalments, how much will he pay each month?

Harvey invests \$6000 at 6% per annum interest. How many years will it be before Harvey's investment has reached \$8000?

What principal would earn \$450 interest if invested at 6.25% interest for 3 years.

What interest rate per annum would give \$300 interest if \$800 is invested for 5 years.

Alf's credit card account is overdue so that he must pay interest for 42 days on amount of \$16 850 at the rate of 16.650% p.a. Interest is calculated daily. How much interest will Alf have to pay?

Extended-response question

Nancy bought a new car that cost \$36 000. She paid a deposit of 15% and monthly instalments of \$817.50 for 4 years.

a What deposit did Nancy pay on the car?

b What is the total amount Nancy paid for the car?

c How much more than \$36 000 did Nancy pay for the car than if she had paid the full amount of \$36 000 at the time she bought the car?

d The amount in part **c** represents the simple interest on the \$36 000 that Nancy paid over the four years. What annual rate of simple interest does this represent?

Practice quiz
Chapter 12

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13 Statistics

Pre-test

Warm-up

The Australian population is made up of people from many parts of the world, who contribute to our cultural diversity. How many different countries are represented in the countries of birth of your class? The 2011 Australian Government Census collected information about many aspects of Australia's population, such as country of birth, employment, education and housing. This data can be compared with data from previous censuses to investigate any changing patterns. Where are the main growth areas in Australia? Is there a change in the proportion of people with tertiary education? How far do people travel to work?

13.1 Classifying and displaying statistical data

BLM
Chapter warm-up

The study of **statistics** uses mathematics to make sense of data. **Data** is anything that can be collected—when something is observed through the human senses—and counted, measured or ranked.

Data can take many forms—numbers or words, pictures or objects, sounds or smells—anything that can be sensed by human beings! Data can be represented in many ways including tables, graphs and diagrams. By categorising, analysing and displaying data so that it becomes more useful and understandable, statistics help us to transform data into **information**.

'Data' is a plural word, so we should really say 'Data are …' not 'Data is …'. However, in everyday speech, most people use 'data' as a singular word.

Any set of data contains information about some group of **individuals**. The characteristic being measured, counted or observed is a **statistical variable**, since it is a characteristic that can vary, or take different values.

In this chapter we will often use the term

- 'variable' to refer to the statistical characteristic being analysed
- 'data' to describe the sets of values collected for these variables.

Types of statistical variables

You will recall from Year 8 that statistical variables may be categorical or numerical.

Categorical and numerical variables

A **numerical variable** takes numerical values for which mathematical operations such as adding and averaging make sense.

A **categorical variable** takes values that place an individual into one of several groups or categories.

Example 1

Consider the following survey questions from the Australian Bureau of Statistics CensusAtSchool. State whether the data obtained from each question would be categorical or numerical.

a In which state/territory or country were you born?

continued

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Example 1 continued

b What is your height?
c In what year were you born?
d How long does it take you to travel to school?

Working	**Reasoning**
a Categorical	The categories are simply names for groups, for example, Australia, Iran, Tonga.
b Numerical	Heights are measured and can be added and averaged.
c Categorical	Birth years are categories, for example,1995, 1996, 1997. It would not make sense to add the years or average them.
d Numerical	Travel times are measured and can be added and averaged.

Nominal and ordinal categorical data

Consider again the categorical data in example **1**.

The data for year of birth has an implied order, and can be ranked, whereas the data for place of birth does not. This provides two ways of classifying categorical data.

- If categories have an obvious order or ranking (for example, year levels such as Year 8, Year 9, Year 10), the data is described as **ordinal**.
- If categories are simply names for subgroups of the data (for example, New South Wales, Northern Territory, Taiwan), the data is described as **nominal**.

The following examples show how such data might be displayed.

Nominal categorical data	Ordinal categorical data
The bar chart below shows sample responses to the survey question: *In which state/territory or country were you born?* The order of the columns is not relevant.	The bar chart below shows sample responses to the survey question: *In what year were you born?* The order of the columns is relevant.

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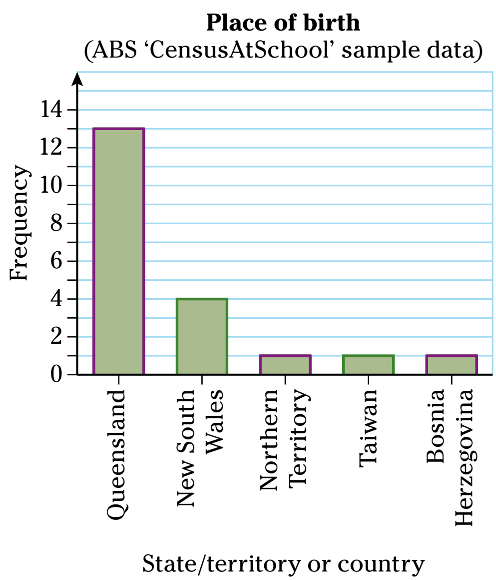
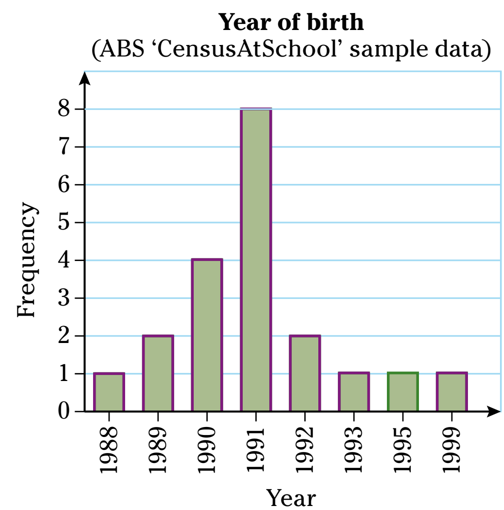
Nominal categorical data



Ordinal categorical data



Categorical data can include numbers, such as year of birth, postcode or room number. Therefore we should not assume that any data made up of numbers is numerical data. To decide whether a statistical variable is numerical, it is helpful to consider whether it would be meaningful to perform mathematical operations on the data values.

For example, if you asked a class of Year 9 students how many phones the members of their family own, it would be meaningful to add or average the resulting data to provide information such as the following.

- Altogether, our families have 71 phones.
- On average, there are 3.7 phones per family.

It would not be meaningful to perform similar calculations with numbers that are years, postcodes or room numbers.

Example 2

For each of the following survey questions, decide whether the responses are likely to be values of an ordinal categorical variable or a nominal categorical variable.

a Which political party do you intend to vote for at the next federal election?

b How would you rate the service at this shop? (Circle your response.)

Excellent *Good* *Fair* *Poor*

c Which state in Australia would you most like to visit this year?

continued

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Example 2 continued

d Studying Mathematics in Year 12 is a good decision for me. (Circle your response.)
Strongly agree *Agree* *Not sure* *Disagree* *Strongly disagree*

e On which floor of the hotel is your room?

Working	Reasoning
a Nominal	The categories would be the *names* of the various political parties.
b Ordinal	The four responses have a clear *order* or rank, from highest to lowest perceived level of service.
c Nominal	The categories would be the *names* of the various states.
d Ordinal	The responses have a clear *order* or rank, in terms of agreement with the statement.
e Ordinal	The floors of a multistorey building have a clear *order* or rank.

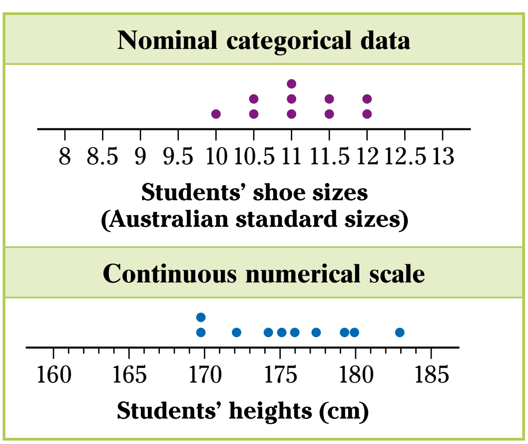
Continuous and discrete numerical data

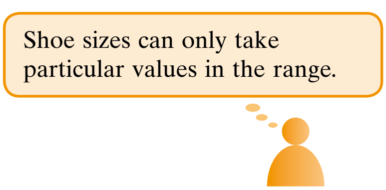
Like categorical data, numerical data can also be classified into two types.

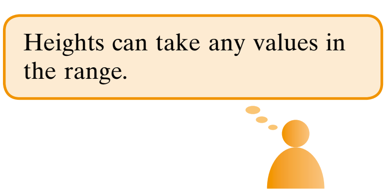
- If a numerical statistical variable can only take particular values within a range (for example, shoe sizes), then the values of the variable are **discrete** numerical data.
- If the variable can take any values within a range (for example, height) then the values of the variable are **continuous** numerical data.

The axis labels for a continuous variable mark *intervals* along the axis, rather than specifying which values the variable can take. Data values can occur at an infinite number of places between such markings.

The dot plots below illustrate the main difference between data obtained from discrete and continuous numerical variables.







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Example 3

State whether each of the following are discrete or continuous data.

a Daily temperatures recorded by the Bureau of Meteorology.
b The number of passengers in each car on a freeway.
c The number of goals scored in a hockey match.
d The weight of pumpkins on a market stall.

Working	Reasoning
a Continuous	Daily temperatures vary continuously between the daily maximum and minimum values for the day.
b Discrete	The number of passengers can only be certain whole number values.
c Discrete	The number of goals must be a whole number.
d Continuous	The weight of the pumpkins varies continuously between the weight of the smallest and largest pumpkins.

Summary of data types

- **Categorical data**
  Data that can be grouped into categories.
  - **Nominal data**
    Data is in named groups.
  - **Ordinal data**
    Categories have a definite order.
- **Numerical data**
  Data that is obtained by measuring or counting. It can be added and averaged.
  - **Discrete data**
    Data can have only particular values.
  - **Continuous data**
    Data can have any values within a given range.

Primary and secondary sources of data

Primary data is original data that is in the form it was originally collected. For example the census forms that all householders complete in the Australian census are a primary data source. Once the data is processed and reported it has become **secondary data**. If your class conducted a survey within your school, the data you collected would be primary data. If you then prepared a report based on the results of your survey, your report would be a secondary data source. Most of the statistics we see reported in the media are secondary data.

In question **6** of exercise **13.1** you will have an opportunity to analyse a primary data source: a record from the 1842 census of Tasmania, which was called Van Diemen's Land at that time. In the analysis task at the end of the chapter you will be accessing primary data on the Australian CensusAtSchool website. Other data in this chapter has been obtained from secondary data sources.

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Displaying statistical variables

13.1

Data is collected by various methods, such as conducting a survey, making observations or using some kind of equipment that records or logs data. Raw data is often recorded in an unstructured form. As you have seen in previous years, the data can then be organised and displayed in **frequency tables** and **visual displays** which make it easier to identify and describe the characteristics of the statistical variables involved.

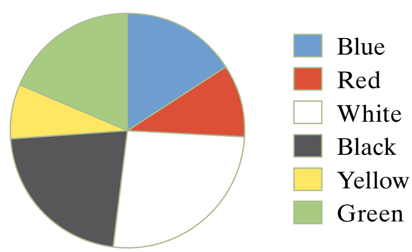
Looking at a visual display, such as a column graph, a dot plot or a stem plot, is an important first step in **analysing data**. In other words, we look at a visual display of the data and describe what it shows us about the **distribution** of the variable.

Visual displays allow us to see

- how widely the data is spread (or distributed)
- where the 'centre' of the data is
- whether there are any **clusters** or **outliers** in the data values.

As you have found in previous years, the type of graph that is used to represent a certain set of data depends on whether the data is **categorical** or **numerical**.

Below is a summary of visual plots.

Pie chart	Example
■ Pie charts are suitable for visually comparing categorical data. Discrete numerical data can also be displayed in a pie chart. ■ They are used to display each value or category of the data as *part of a whole*. This makes it easy for comparisons to be made. ■ The sectors of the pie chart may be labelled with actual numbers or with percentages. ■ If the data takes too many values then the pie chart becomes too complex to interpret easily. ■ If the data include some very small values, for example, 0.5%, it will be difficult to show these.	**Cars sold by colour in March** 

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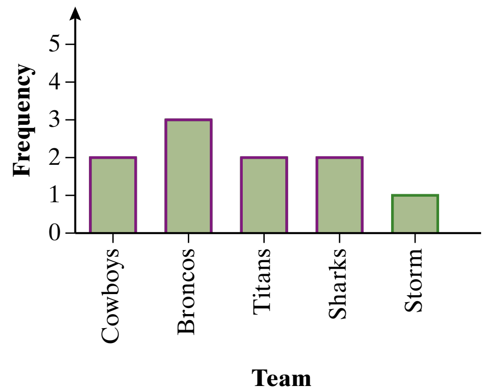
<table>
<tr><th>Bar chart or bar graph</th><th>Example</th></tr>
<tr><td>

- Often the best choice for *categorical* variables. Also appropriate for *discrete numerical* variables.
- Each category has its own labelled 'bar' or column.
- It is usual to show the *category* along the horizontal axis and the *frequency* along the vertical axis.

</td><td>

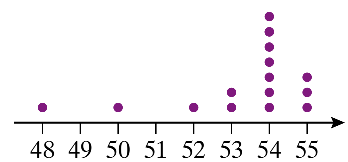
Favourite NRL team



</td></tr>
<tr><th>Dot plot</th><th>Example</th></tr>
<tr><td>

- Best for a small set of *discrete numerical* data where the values are reasonably close together in size.
- Provides a quick view of the 'shape' of a distribution.
- Makes it easy to identify any clusters or outliers.

</td><td>



Number of matches in a match box

</td></tr>
<tr><th>Stem-and-leaf plot</th><th>Example</th></tr>
<tr><td>

- Appropriate for discrete numerical data, where the values are reasonably close together in size.
- The data is 'grouped' (into rows) but each of the actual values can still be read from the plot.
- Provides a quick view of the 'shape' of a distribution; looks a little like a histogram on its side.
- Use split stems to provide more detail.
- To show large or small data values in a stem plot, the values may first need to be rounded.

</td><td>

Number of words per sentence

<table>
<tr><th>Stem</th><th>Leaf</th></tr>
<tr><td>1</td><td>0 1 3 4 4</td></tr>
<tr><td>1</td><td>5 6 6 8 9</td></tr>
<tr><td>2</td><td>2 3 3 4 4</td></tr>
<tr><td>2</td><td>5 7 7</td></tr>
<tr><td>3</td><td>2 2 3</td></tr>
<tr><td>3</td><td>7</td></tr>
<tr><td>4</td><td>0</td></tr>
</table>

(Key: 1|5 means 15)

</td></tr>
</table>

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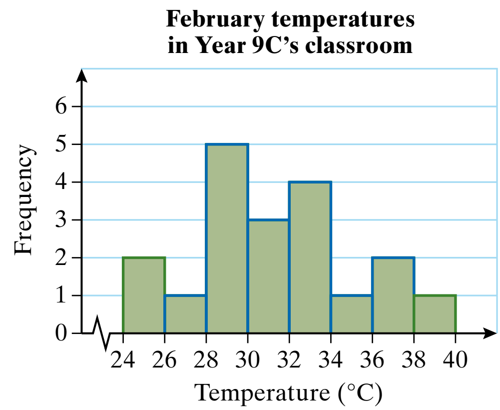
Histogram	Example
■ Appropriate for *continuous* or *discrete numerical* data. ■ The horizontal axis has a continuous numerical scale. The vertical axis shows the *frequency*. ■ The data is grouped into *intervals* (for example 24–<26, 26–<28, 28–<30, and so on). ■ The numbers along the horizontal axis label the marks *between* intervals, not the columns themselves. ■ To display more detail, decrease the size of the class interval.	

Example 4

Harriet's parents run the local news agency and Harriet helps them with the paper deliveries before school. She decides to keep a record of how long it takes to complete the paper round. The times (in minutes) for the month of October are shown below.
28, 40, 31, 22, 38, 26, 24, 37, 42, 23, 35, 32, 27, 43, 33, 28, 25, 41, 33, 38, 45, 32
Construct an ordered stem plot for this data using

a one stem for each multiple of 10 **b** two stems for each multiple of 10.

Working	Reasoning
a Arrange the data in order: 22, 23, 24, 25, 26, 27, 28, 28, 31, 32, 32, 33, 33, 35, 37, 38, 38, 40, 41, 42, 43, 45	For an ordered stem plot, the data first needs to be arranged in order.
Times taken to complete paper round (minutes)	Next, split each piece of data into a stem and a leaf, starting with: 22 has stem 2 and leaf 2.
(stem plot below) (Key: 2|2 means 22)	A stem plot usually has a title and a key.

Stem	Leaf
2	2 3 4 5 6 7 8 8
3	1 2 2 3 3 5 7 8 8
4	0 1 2 3 5

(Key: 2|2 means 22)

continued

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Example 4 continued

Working

b **Times taken to complete paper round (minutes)**

Stem	Leaf
2	2 3 4
2	5 6 7 8 8
3	1 2 2 3 3
3	5 7 8 8
4	0 1 2 3
4	5

(Key: 2|2 means 22)

Reasoning

Where two stems are used for each multiple of 10, the digits 0, 1, 2, 3 and 4 are placed with the first stem and the digits 5, 6, 7, 8 and 9 are placed with the second stem.

This stem plot uses 'split' stems.

exercise 13.1

LINKS TO Example 1

- Classify the following data as either categorical (C) or numerical (N).
  - **a** Age of a child
  - **b** Number of goals scored by a soccer team
  - **c** Styles of music
  - **d** Movie genres
  - **e** School subjects
  - **f** Volumes of various beakers

- Reword the following survey questions so that responses would contain numerical data rather than categorical data.
  - **a** How often do you send text messages? (never/rarely/sometimes/often)
  - **b** How fit are you?
  - **c** Is your height greater than your arm span?

- Reword the following survey questions so that responses would contain categorical data rather than numerical data.
  - **a** How many glasses of water do you drink each day, on average?
  - **b** What is the total number of minutes that you spend each week in internet chat rooms?
  - **c** What is the length of your foot?

LINKS TO Examples 2, 3

- For each of the following survey items, decide whether the data obtained would be nominal categorical (NC), ordinal categorical (OC), discrete numerical (DN) or continuous numerical (CN). In each case, also give a sample response.
  - **a** How many females are there in your family?
  - **b** What is your favourite colour?
  - **c** What is the area of the room in which you sleep?
  - **d** List your three favourite colours.
  - **e** How often do you go to the cinema? (never/rarely/sometimes/often)
  - **f** How many colours are there on your favourite T-shirt?
  - **g** In which year was your mother born?

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h How many times have you been to the snow?

i How far did you travel to come to school today?

j How many text messages have you sent this week?

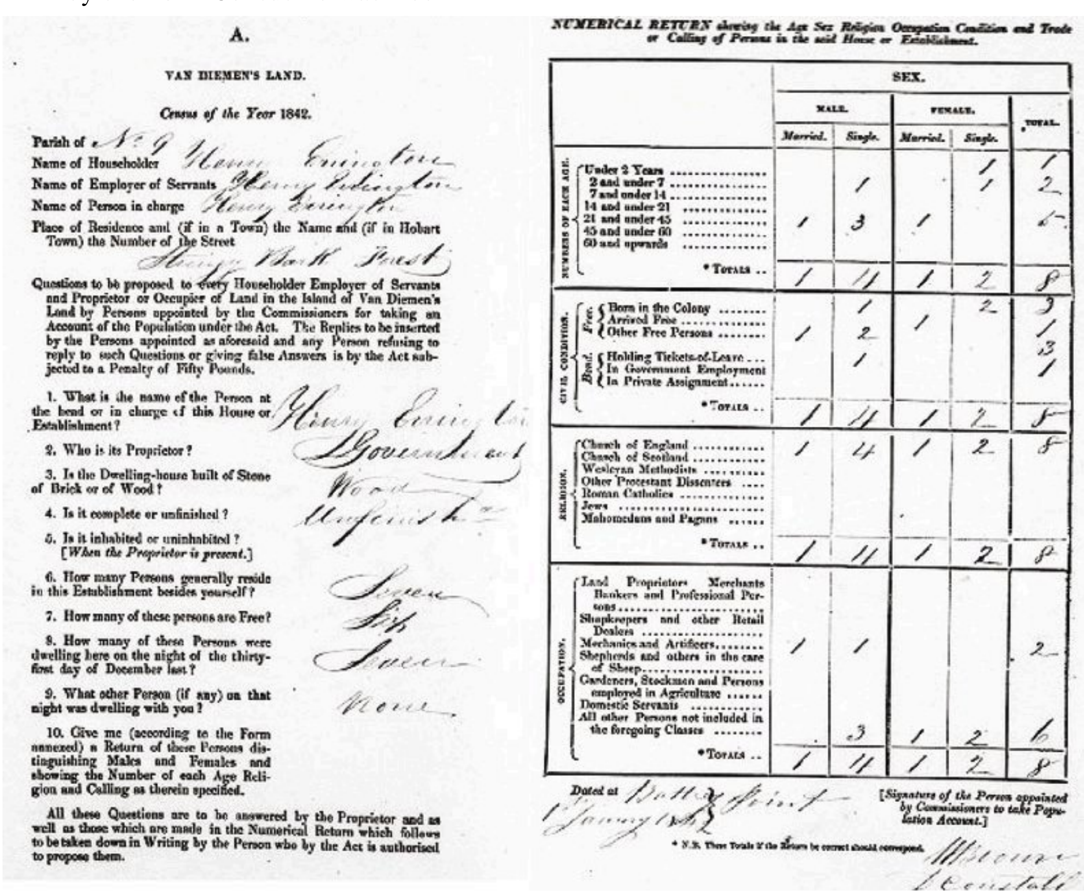
With a partner, design a four-question survey. It must include a question that will collect data of the following type:

- nominal categorical
- ordinal categorical
- discrete numerical
- continuous numerical.

Conduct the survey with a sample of 10 students (randomly chosen) and comment on the quality of your questions.

exercise 13.1 challenge

The following record is from the 1842 Census of Van Diemen's Land (Tasmania). It provides information about the dwelling place and occupants of the house of Henry Errington of Stringy Bark Forest near Launceston. Henry was transported to Van Diemen's land from England as a convict in 1820 for being caught in London with forged bank notes in his possession. He had served his 14 years as a convict in 1834 so by the 1842 Census he was free.



A.

VAN DIEMEN'S LAND.

Census of the Year 1842.

Parish of No 9

Name of Householder Henry Errington

Name of Employer of Servants

Name of Person in charge

Place of Residence and (if in a Town) the Name and (if in Hobart Town) the Number of the Street

Stringy Bark Forest

Questions to be proposed to every Householder Employer of Servants and Proprietor or Occupier of Land in the Island of Van Diemen's Land by Persons appointed by the Commissioners for taking an Account of the Population under the Act. The Replies to be inserted by the Persons appointed as aforesaid and any Person refusing to reply to such Questions or giving false Answers is by the Act subjected to a Penalty of Fifty Pounds.

1. What is the name of the Person at the head or in charge of this House or Establishment? Henry Errington
2. Who is its Proprietor? Government
3. Is the Dwelling-house built of Stone of Brick or of Wood? Wood
4. Is it complete or unfinished? Unfinished
5. Is it inhabited or uninhabited? [*When the Proprietor is present.*]
6. How many Persons generally reside in this Establishment besides yourself? Seven
7. How many of these persons are Free? Six
8. How many of these Persons were dwelling here on the night of the thirty-first day of December last? Seven
9. What other Person (if any) on that night was dwelling with you? None
10. Give me (according to the Form annexed) a Return of these Persons distinguishing Males and Females and showing the Number of each Age Religion and Calling as therein specified.

All these Questions are to be answered by the Proprietor and as well as those which are made in the Numerical Return which follows to be taken down in Writing by the Person who by the Act is authorised to propose them.

NUMERICAL RETURN showing the Age Sex Religion Occupation Condition and Trade or Calling of Persons in the said House or Establishment.

		SEX.				
		MALE.		FEMALE.		TOTAL.
		Married.	*Single.*	*Married.*	*Single.*	
NUMBERS OF EACH AGE.	Under 2 Years				1	1
	2 and under 7		1		1	2
	7 and under 14					
	14 and under 21					
	21 and under 45	1	3	1		6
	45 and under 60					
	60 and upwards					
	*TOTALS	1	4	1	2	8
CIVIL CONDITION. *Free.*	Born in the Colony		1		2	3
	Arrived Free			1		1
	Other Free Persons	1	2			3
Bond.	Holding Tickets-of-Leave		1			1
	In Government Employment					
	In Private Assignment					
	*TOTALS	1	4	1	2	8
RELIGION.	Church of England	1	4	1	2	8
	Church of Scotland					
	Wesleyan Methodists					
	Other Protestant Dissenters					
	Roman Catholics					
	Jews					
	Mahomedans and Pagans					
	*TOTALS	1	4	1	2	8
OCCUPATION.	Land Proprietors Merchants Bankers and Professional Persons					
	Shopkeepers and other Retail Dealers					
	Mechanics and Artificers	1	1			2
	Shepherds and others in the care of Sheep					
	Gardeners, Stockmen and Persons employed in Agriculture					
	Domestic Servants					
	All other Persons not included in the foregoing Classes		3	1	2	6
	*TOTALS	1	4	1	2	8

Dated at

[*Signature of the Person appointed by Commissioners to take Population Account.*]

* N.B. These Totals if the Return be correct should correspond.

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a List all the categorical variables recorded in the census information.
b How many people were living in the house?
c How many were adults and how many were children?
d How many of the occupants were married?
e Who are the most likely occupants to have been born in 'the colony', that is, in Van Diemen's Land?
f Which person arrived free?
g Write a paragraph describing what you know of the occupants from the census record.

LINKS TO Example 4

Twenty people applied for a position that required good word processing skills. Each applicant was required to type a long document. The numbers of errors made by the applicants are as follows.

25, 16, 28, 35, 26, 38, 9, 32, 35, 62, 24, 57, 48, 7, 28, 42, 38, 37, 27, 22

a Display this data using an ordered stem plot with one stem for each multiple of 10.
b What percentage of applicants made 30 or more errors?

In the 'CensusAtSchool' project, students measured their reaction times (in seconds). The data values for a sample of 30 students using their right hands were as follows.

0.38, 0.37, 0.55, 0.37, 0.65, 0.39, 0.3, 0.37, 0.26, 0.66,

0.39, 0.34, 0.37, 0.35, 2.96, 0.40, 0.30, 0.42, 0.25, 0.81,

0.28, 0.29, 0.33, 0.34, 0.33, 0.32, 0.33, 0.36, 0.39, 0.32

a Create a stem plot with an appropriate stem size and comment on its shape.
b One of the points is an outlier. How does the stem plot handle this value, and how might it have occurred?
c Many statistical software packages automatically eliminate outliers from plots. What do you think of this idea?

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13.2 Reviewing centre and spread

There are a number of ways of summarising the 'shape' or distribution of a set of numerical data. In this section, we look at statistics that measure the **centre** and **spread** of a set of data. These statistics provide answers to the following questions.

- Where is the middle (or centre) of the data?
- How varied (or spread out) is the data?

Measures of centre

There are three ways of measuring the centre or middle of a set of data. These are the **mean**, the **median** and the **mode**.

Mean

If x represents the data values, the mean is represented by the symbol $\bar{x}$.

The mean $\bar{x}$ of a set of data is calculated by

$$\bar{x} = \frac{\sum x}{n}$$

where Σ means 'the sum of', x represents the data values and n is the number of data values. We read the formula as

$$\text{Mean} = \frac{\text{Sum of data values}}{\text{Number of data values}}$$

Median

The median of a set of data is the middle value of the data when it is ordered. The data can either be arranged in ascending order (smallest to largest) or descending order (largest to smallest).

The median is the middle $\left(\frac{n+1}{2}\right)^{th}$ value when the data is ordered.

Note that

- if n is odd, the median is the middle value of the data set.
- if n is even, the median is half-way between the two middle values of the data set.

Mode

Finding the mode

The mode is the most frequently observed or most common value of the data.

Sometimes a set of data will have more than one mode. A set of data with two modes is said to be **bimodal**.

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Comparing the measures of centre

Measure of centre	Advantages	Disadvantages
Mean	Every piece of data is used in calculating the mean. If one data value changes, the mean will also change. In this way, the mean represents all the data.	The mean can be affected by extreme values in the data. For example, adding one extremely high house price to a set of data can make it appear that the mean cost of housing has risen significantly.
Median	When there are a few extreme values in a set of data, the median can be a fairer measure of centre than the mean. For example, is the mean average salary a good measure of centre if the managing director is the only person in a company who earns more than that amount?	Although all of the data values are put in order to find the median, changing one or more of the values might not affect the median. Therefore the median does not 'include' each individual value in the way that the mean does.
Mode	The mode is useful for analysing data that cannot be 'averaged', such as postcodes, eye colour or yes–no responses to a question.	When a set of data has more than one mode, this might not provide a useful measure of centre. The mode does not 'include' each individual value in the way that the mean does.

Example 5

The following data shows, in ascending order, the ages of all the members of Amy's family who attended a birthday party.
1, 5, 7, 15, 22, 28, 29, 32, 35, 35, 38, 40, 41, 42, 42, 43, 55, 65, 73, 88

a Calculate the mean age.
b Find the median age.
c What is the modal age?

Working

a

$$\bar{x} = \frac{\sum x}{2}$$
$$= \frac{1 + 5 + 7 + 15 + \ldots + 73 + 88}{20}$$
$$= \frac{736}{20}$$
$$= 36.8$$

The mean age is 36.8.

Reasoning

Add all the data values and divide by 20 because there are 20 data values.

continued

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Example 5 continued

Working	Reasoning
b Use current working.	
c The ages 35 and 42 each occur twice so there are two modes.	This data has two modes, each of which occurs only twice. This is a disadvantage of the mode as a measure of centre.

Example 6

The numbers of children in the families of a class of Year 9 students are as follows.
1, 3, 4, 2, 2, 5, 1, 3, 2, 2, 2, 3, 1, 4, 3, 3, 2, 3, 2, 1, 4, 2, 3, 1, 2

a Calculate the mean, median and modal number of children per family.

b Kayla was absent when this data was collected. There are 12 children in her family.

  i If Kayla's data is included, calculate the new mean, median and mode.

  ii How did this value affect the measures of centre?

Working

a **Mean**

$$\begin{aligned}\bar{x} &= \frac{\sum x}{n}\\ &= \frac{61}{25}\\ &\approx 2.4\end{aligned}$$

The mean number of children in a family is approximately 2.4.

Median

1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, $\boxed{2}$, 2, 3, 3, 3, 3, 3, 3, 3, 4, 4, 4, 5

The median is 2 children.

Mode

The mode is 2 children.

b **i** **Mean**

$$\begin{aligned}\bar{x} &= \frac{\sum x}{n}\\ &= \frac{73}{26}\\ &\approx 2.8\end{aligned}$$

The mean number of children in a family is approximately 2.8.

Reasoning

The mean is equal to the sum of all the data values divided by the number of data values.

The sum of all the data values is 61 and there are 25 data values.

Arrange the data in order.
There are 25 values so the median is the $\frac{25+1}{2}$ = 13th value.

The value that occurs most frequently is 2.

To calculate the new mean, simply add 12 to the previous sum of the values (in the numerator) and 1 to the number of data values (in the denominator).

There are now 26 data values so the median is the $\frac{26+1}{2}$ = 13.5th value.

continued

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Example 6 continued

Working

Median

1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, | 2, 3, 3, 3, 3, 3, 3, 3, 4, 4, 4, 5, 12

The median is 2 children.

Reasoning

This is the average of the 13th and 14th data values. As these values are both 2, the average is 2.

Mode

The mode is 2 children.

The value 2 still occurs more frequently than any other data value.

ii The median and the mode are unaffected by the addition of this one piece of data. However, it has had a significant effect on the mean, which has increased by 0.4.

An increase of 0.4 in the mean can be considered significant since the data values are small, with all but one data value lying in the range 1 to 5.

Finding measures of centre from a frequency table

The measures of centre, mode, median and mean, can be found from a frequency table. Example **7** demonstrates how the three measures of centre are determined.

Example 7

The data for the numbers of children in the families from example **6** (including Kayla's data) are shown in this frequency table. Complete the table and calculate the mean, median and modal number of children in the families.

Number of children in family (x)	Frequency (f)	xf
1	5	
2	9	
3	7	
4	3	
5	1	
12	1	
Totals	$\sum f =$	$\sum xf =$

continued

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Example 7 continued

Working

Number of children in family (x)	Frequency (f)	xf
1	5	5
2	9	18
3	7	21
4	3	12
5	1	5
12	1	12
Totals	$\sum f = 26$	$\sum xf = 73$

Reasoning

5 families each have 1 child (5 children), 9 families have 2 children (18 children), and so on. In the xf column, each number of children (x) is multiplied by its frequency (f). $\sum xf$ means the sum of all the xf values.

To find the mean number of children per family, we divide the total number of children $\sum xf = 73$ by the total number of families, $\sum f = 26$.

$$\bar{x} = \frac{\sum f}{\sum f}$$
$$= \frac{73}{26}$$
$$\approx 2.8$$

The mean number of children per family is 2.8.

Median:
The median number of children per family is 2.

There are 26 families so to find the median we want the number of children for the family halfway between the 13th and 14th families. Numbering from the top of the table, the family halfway between the 13th and 14th families will have 2 children.

Mode:
The modal number of children per family is 2.

The number of children per family corresponding to the highest frequency is 2.

Measuring spread: range

The **range** of a set of data is the difference between the highest and lowest values of the data.

Range

Range = Highest value – Lowest value

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Example 8

Find the range for the data in Example 7.

Working

Range = 12 − 1
= 11

The range for the number of children in the families is 11.

Reasoning

range = highest value − lowest value

exercise 13.2

For this exercise, first attempt the questions without using the statistical features of your calculator. Then use those features to check your answers, where relevant.

LINKS TO Example 5

1. Find the mean, median and mode for each of the following sets of data.
   - **a** 6, 8, 10, 10, 12, 14, 18, 20, 20, 20, 24, 30, 34, 40
   - **b** 14, 10, 35, 22, 15, 18, 42, 60, 15, 34, 53, 42, 61, 21, 14
   - **c** 32.6, 45.4, 23.7, 34.2, 46.8, 33.1, 27.5, 41.7, 38.4, 27.5, 25.2
   - **d** 5, 26, 24, 3, 1, 5, 28, 23, 1, 11, 28, 9, 23, 4, 5, 1, 22, 6, 2, 0, 5

LINKS TO Example 6

2. Sarah tossed 10 coins and recorded the number of heads. She repeated this until she had made 20 tosses of the 10 coins. The number of heads were
   5, 3, 6, 4, 5, 7, 6, 5, 5, 5, 4, 5, 2, 4, 4, 4, 4, 2, 3, 4
   - **a** Find the median.
   - **b** Find the mean number of heads.

3.
   - **a** The mean of four numbers is 10. What might the numbers be?
   - **b** A set of 8 different numbers has a median of 5. What could the numbers be?
   - **c** A set of 6 numbers has a mean of 0 and a range of 10. What might the numbers be?

4. The marks of 7 students in a test are 88, 75, 93, 98, 27, 80, and 36. If 8 students sat the test and the average mark was 65, what score did the eighth student achieve on the test?

5. The most commonly-used letter in the English language is E. The following data shows the number of Es in each of 25 lines of text from a book.
   7, 10, 4, 10, 6, 4, 9, 8, 10, 8, 8, 5, 5, 5, 10, 9, 7, 7, 4, 9, 8, 4, 9, 5, 6
   - **a** Find the mean, median, mode(s) and range for the data.
   - **b** What do these statistics tell you?

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13.2

The scores obtained by a class of 30 students on a test are shown in the frequency table. Calculate the mean score.

Mark out of 10	Frequency
4	1
5	2
6	3
7	4
8	5
9	6
10	7

LINKS TO Example 7

The number of library books borrowed by 25 students during a particular hour in a library is shown in the frequency table. Calculate the mean number of books per student.

Number of books	Frequency
1	7
2	9
3	5
4	2
5	2

LINKS TO Example 8

Calculate the range for each of the sets of data in question **1**.

In the 12 months to August 2007, the mean price of a house in a certain suburb was \$425 000. The median price was \$342 000.

a Calculate the difference between the mean house price and the median house price in this suburb.

b Can you suggest a possible reason for this difference?

c Why do you think the median is used when comparing housing prices in different suburbs, rather than the mean?

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This stem-and-leaf plot shows class results for a maths test.

a What is the mode of the data?

b What is the mean test score?

c What is the median test score?

d What is the range?

Test scores out of 60

Stem	Leaf
2	8
3	2 5 8 8 9
4	0 0 2 3 3 4 5 5 5 6 7 8
5	0 2 4 5 5 6 6 7 7 9 9
6	0 0

(Key: 2|8 means 28)

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13.3 Histograms and the shape of distributions

Grouped frequency tables

In situations where there is a large amount of numerical data or where the data is widely spread, it is often convenient to group data in **class intervals**. For example, if you are recording the lengths of jumps in a long jump competition, you might use class intervals of 5 cm, that is

- 170–<175 cm for jumps from 170 cm up to but not including 175 cm
- 175–<180 cm for jumps from 175 cm up to but not including 180 cm

and so on.

Frequency tables with class intervals such as these are called **grouped frequency tables**. They are particularly useful when the numerical data is **continuous**, that is, when the data can take any values within a particular range. Distance, weight and time are examples of **continuous** data.

Example 9

Twenty students entered a long jump competition. Their best jumps (in centimetres) were as follows.
171, 186, 191, 173, 182, 176, 178, 182, 190, 183, 179, 174, 178, 175, 187, 176, 179, 175, 184, 181
Copy and complete the frequency table.

Class interval (cm)	Frequency
170–<175	3
175–<180	
180–<185	
185–<190	
190–<195	
Total	

Working

Class interval (cm)	Frequency
170–<175	3
175–<180	8
180–<185	5
185–<190	2
190–<195	2
Total	20

Reasoning

Class interval (cm)	Data values (cm)
170–<175	171, 173, 174
175–<180	176, 178, 179, 178, 175, 176, 179, 175
180–<185	182, 182, 183, 184, 181
185–<190	186, 187
190–<195	190, 191

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Notice that the total of the frequencies equals the number of data values. This can be used as a check that no data values have been left out.

Histograms

A **histogram** is used to display numerical data that has been divided into class intervals. Histograms are usually used for continuous data, but can also be used for discrete data that has been grouped in class intervals.

The shape obtained by turning the stem and leaf plot on its side resembles a histogram. In appearance, a histogram is similar to a bar chart or column graph but there are no gaps between the columns. This is because the class intervals are designed with no gaps between them; for example, 170–<175, 175–<180, 180–<185 and so on. The horizontal axis is a continuous numerical scale. The height of each column shows the frequency of the class interval.

The numbers on the horizontal axis label the *divisions between intervals*. This is different from a column graph, where the *columns* are labelled rather than the divisions between the columns.

Example 10

Use the frequency table in example 9 to construct a histogram.

Class interval (cm)	Frequency
170 – <175	3
175 – <180	8
180 – <185	5
185 – <190	2
190 – <195	2
Total	20

continued

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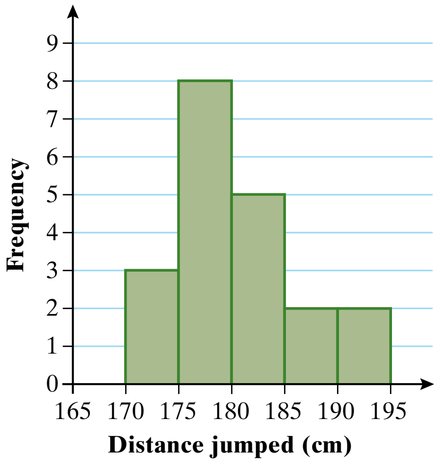
Example 10 continued

Working

Long jump distances for 20 students



Reasoning

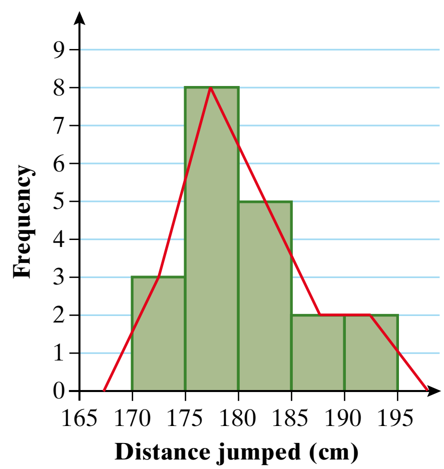
Each class interval is represented by a column. For example, the column between 170 and 175 is 3 units high to indicate that three students had distances in the interval 170 cm up to but not including 175 cm.

The sum of the heights of all the columns equals the total number of data values.

Frequency polygons and the shape of a distribution

A **frequency polygon** is constructed by joining the midpoints of all the columns of a histogram. The graph is extended to finish on the horizontal axis, half a column width to the left of the leftmost column and half a column width to the right of the rightmost column. A frequency polygon has been added to the histogram shown on the right.

Long jump distances for 20 students



The overall shape of a histogram and its frequency polygon help us to analyse the shape or distribution of the data.

When the data has a **symmetrical distribution**, the histogram has a vertical axis of symmetry. A distribution that is **asymmetrical**—that is, not symmetrical—is described as having

- **positive skew** if the data is spread out in the positive direction.
- **negative skew** if the data is spread out in the negative direction

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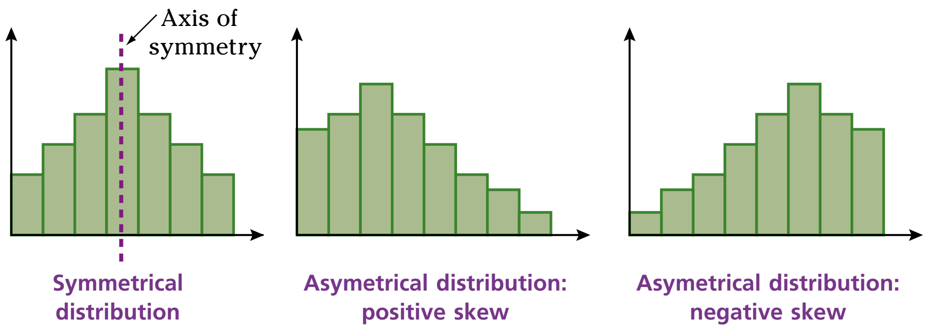
The shape of a distribution



The histogram and frequency polygon shown on the previous page for the long jump data indicate that the data is positively skewed, that is, the distances are spread out in the positive direction.

Example 11

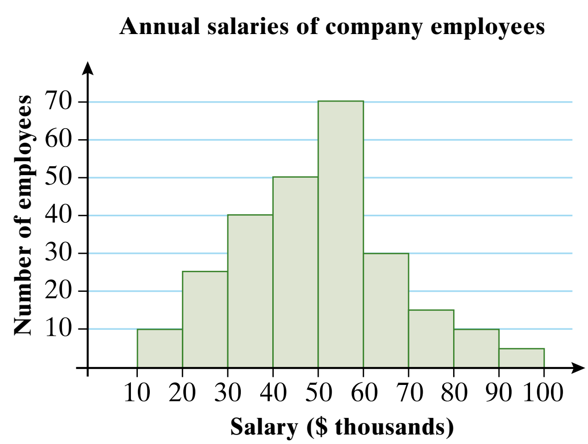
The histogram below shows the salaries of the employees of a large company.

a How many employees earned between \$20 000 and \$30 000?

b What class interval contained the greatest number of employees?

c How many employees earned \$50 000 or more?

d Add a frequency polygon to the histogram.



Working	Reasoning
a 25 employees earned between \$20 000 and \$30 000.	Read the height of the 20–30 column from the frequency axis.
b The class interval with the greatest number of employees is \$50 000 to \$60 000.	This is the interval with the highest column. Salaries are in thousands of dollars so 50–60 corresponds to \$50 000 to \$60 000.
c 70 + 30 +15 + 10 + 5 =130 employees earned \$50 000 or more.	Determine the frequencies of each of the columns to the right of 50 on the horizontal axis. Add these frequencies to find the number of employees who earned more than \$50 000.

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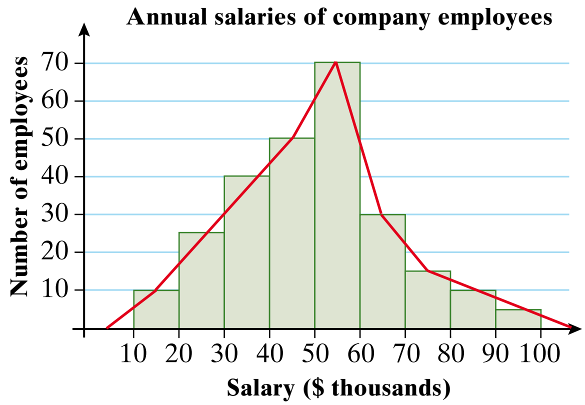
Example 11 continued

Working

d



Reasoning

Join the midpoints of all the columns of the histogram. Extend the graph to finish on the horizontal axis, half a column width to the left of the leftmost column and half a column width to the right of the rightmost column.

The following example compares the same data displayed as a stem-and-leaf plot and as a histogram. Both displays show the shape of the data distribution. The stem-and-leaf plot has the advantage that each individual data value is displayed.

Example 12

Students participating in the 'CensusAtSchool' project were asked to complete a concentration task (unassisted, under supervision) and to record the time they took. The times (in seconds) for a sample of 30 students were:
40, 42, 129, 98, 68, 40, 43, 66, 57, 35, 51, 46, 76, 68, 37, 56, 46, 51, 23, 44, 34, 64, 52, 33, 50, 33, 42, 50, 42, 33

a What type of statistical variable do the values represent?
b Create a histogram with an appropriate class interval to show the data.
c Comment on the distribution of this sample data.
d Represent the data as a stem-and-leaf plot
e What advantages does the stem-and-leaf plot have compared with the histogram?

Working

a The time taken to complete the task could take any value within appropriate limits. Therefore, this is a continuous numerical variable.

Reasoning

Although the sample values have been rounded to the nearest second, it is possible for the task to take 'in between times' such as 65.5 seconds. So the data is not restricted to particular values.

continued

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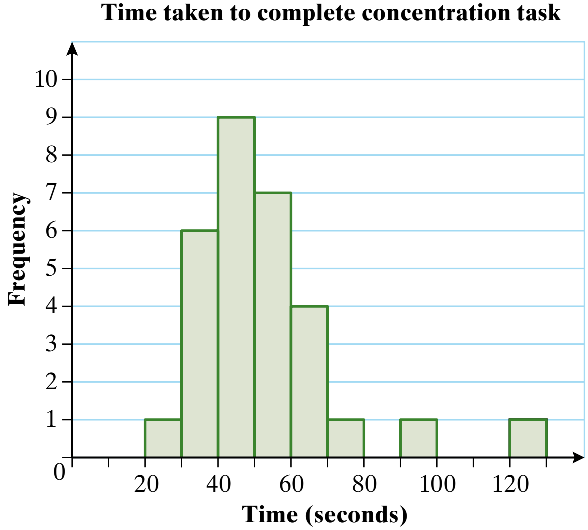
Example 12 continued

Working

b An interval of 10 or 20 seconds seems appropriate. The frequency table at the right and the histogram below show intervals of 10 seconds.



c The majority of the data appears to be centred fairly symmetrically around times of 50 seconds, although there is an outlier in the 120−<130 seconds interval.

Reasoning

The time taken to complete the task can be tabulated in a frequency table.

Time taken (seconds)	Frequency
0−<10	0
10−<20	0
20−<30	1
30−<40	6
40−<50	9
50−<60	7
60−<70	4
70−<80	1
80−<90	0
90−<100	1
100−<110	0
110−<120	0
120−<130	1

To describe the distribution, identify the centre and spread of the data, and any unusual features. For example, outliers, if there are two modes, etc.

continued

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Example 12 continued

Working

d

Stem	Leaf
2	3
3	3 3 3 4 5 7
4	0 0 2 2 2 3 4 6 6
5	0 0 1 1 2 6 7
6	4 6 8 8
7	6
8	
9	8
10	
11	
12	9

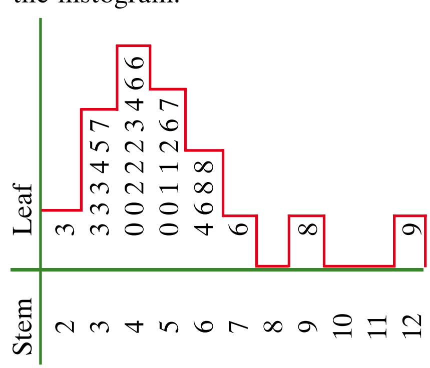
(Key: 2|3 means 23)

Reasoning

Using one stem for each interval of 10 seconds clearly shows the distribution.

e The advantage of the stem and leaf plot is that it displays every data value.

The histogram shows only the number of data values in each class interval. The individual data values are 'hidden' in a histogram.
Tipped sideways, the stem and leaf plot resembles the histogram.



When grouping data, small class intervals give a more detailed picture of the data, whereas larger class intervals consolidate and simplify the data. When deciding upon class intervals, locate the maximum and minimum values of the data and then choose an interval that will result in no more than 10 classes to account for all the data.

Example 13

a Use the data in example **12** to construct histograms with a class interval of

i 20 seconds

ii 40 seconds.

b Compare the information provided by each histogram.

continued

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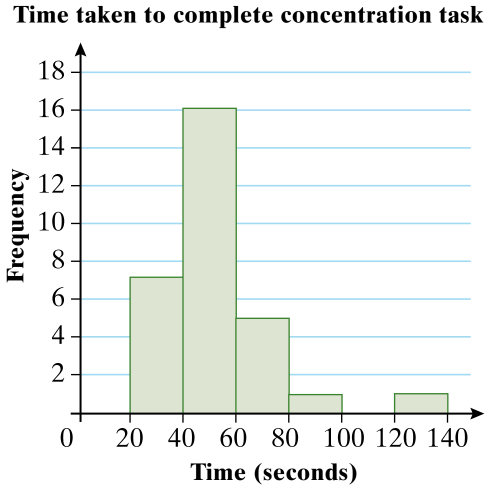
Example 13 continued

Working

a i Class interval of 20:

Class interval (cm)	Frequency
0−<20	0
20−<40	7
40−<60	16
60−<80	5
80−<100	1
100−<120	0
120−<140	1
Total	30



Reasoning

The intervals are 0–<20, 20–<40, and so on. There are seven intervals.

ii Class interval of 40:

Class interval (cm)	Frequency
0−<40	7
40−<80	21
80−<120	1
120−<160	1
Total	30

The intervals are 0-<40, 40-<80, and so on. There are only four intervals.

continued

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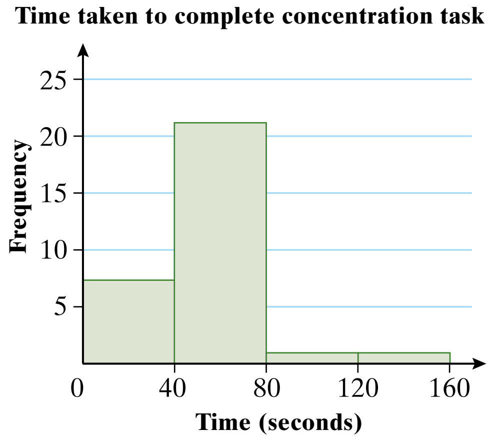
Example 13 continued

Working

Reasoning

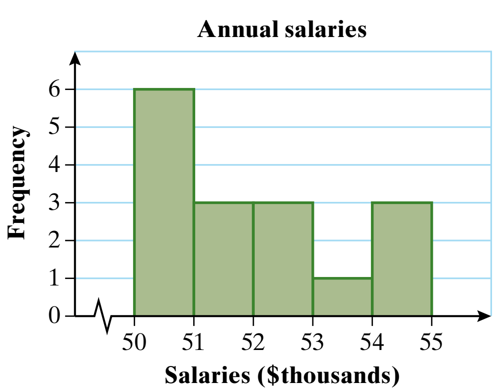


b Using intervals of 20 seconds, the histogram still shows the distribution clearly. For intervals of 40 seconds, the class interval is too large, and the histogram does not show the distribution as clearly and the outlier is not obvious.

Example 14

The following set of data shows the annual salaries (in dollars) of 16 workers.
51 585, 50 226, 52 740, 50 869, 50 299, 54 981, 51 069, 52 950,
50 142, 52 427, 50 477, 51 688, 50 919, 54 359, 54 656, 53 117
Display the data as a histogram, choosing an appropriate class interval.

Working



Reasoning

Intervals of $1000 would be suitable.

Annual salaries ($)	Frequency
50 000–<51 000	6
51 000–<52 000	3
52 000–<53 000	3
53 000–<54 000	1
54 000–<55 000	3

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Describing the shape of a histogram

The overall shape of a histogram helps us analyse the 'shape' or distribution of the data.

When the data has a symmetrical distribution, the histogram has a vertical axis of symmetry. A distribution that is **asymmetrical**—that is, *not* symmetrical—is described as having either **positive skew** if it shows a 'lean' to the left or **negative skew** if it shows a lean to the right.

Estimating the mean from a frequency table of grouped data

If data is organised into class intervals, it is not possible to calculate the *exact* mean but it is possible to obtain an *estimate* of the mean by using the midpoint of each class interval.

Example 15

The ages of 40 members of a junior swimming club are summarised in the table on the right. Estimate the mean age of a member of this club?

Age (years)	Frequency
0−<5	3
5−<10	14
10−<15	19
15−<20	4
Total	40

Working

Age	Frequency f	Midpoint x	xf
0−<5	3	2.5	7.5
5−<10	14	7.5	105.0
10−<15	19	12.5	237.5
15−<20	4	17.5	70.0
Total	40		420.0

$$\bar{x} = \frac{\sum xf}{\sum f} = \frac{420}{40} = 10.5$$

The mean age of members is approximately 10.5 years.

Reasoning

To find the midpoints of each class interval, add the endpoints and divide by 2. For example,

$\frac{0+5}{2} = 2.5, \frac{5+10}{2} = 7.5$

and so on.

Using the midpoints as the values of x, calculate the mean as shown previously.

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Cumulative frequency

Cumulative means 'increasing or growing by successive additions'. For example, in games like Scrabble®, each player's score is a cumulative total—after each turn, a player's score is added to their previous total.

Cumulative frequency is a progressive total of the frequencies. It is used to find how many of the data values lie below a particular value.

The word cumulative comes from the Latin *cumulare*, which means 'to heap up'.

Cumulative frequency is the sum of all the frequencies up to a particular data value.

Cumulative frequency tables are useful for answering questions such as 'How many people achieved a score less than 60?'

Example 16

This table shows students' test results (out of 100) organised into class intervals of 10 marks.

a Add another column to the table and calculate the cumulative frequencies.

b Use the table to find the number of students who scored less than 80 out of 100.

Mark	Frequency
50–<60	2
60–<70	13
70–<80	11
80–<90	3
90–<100	1

Working

Mark	Frequency	Cumulative frequency
50–<60	2	2
60–<70	13	15
70–<80	11	26
80–<90	3	29
90–<100	1	30

b 26 students scored less than 80 out of 100.

Reasoning

For the first class interval, the cumulative frequency is the same as the frequency. For subsequent class intervals, add the frequency for that interval to the previous total.

$2 + 13 = 15$

$15 + 11 = 26$

$26 + 3 = 29$

$29 + 1 = 30$

The cumulative frequency 26 in the table represents the total number of students in the class intervals 50–<60, 60–<70 and 70–<80 .

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Cumulative frequencies are sometimes expressed as percentages—by dividing by the total frequency and multiplying by 100. The cumulative percentage frequency tells us what percentage of the data lies below a particular value.

Example 17

A researcher interviewed people in the city, asking how often they caught a train in the last week. This table shows the results of the survey.

Number of days	Number of people (frequency)
0	8
1	16
2	22
3	14
4	20
5	32
6	8
7	4

a Add a cumulative frequency column to the table.
b How many people caught a train on four days or less?
c Add a cumulative percentage frequency column to the table.
d What percentage of people caught a train on five days or less?

Working

a

Number of days	Number of people (frequency)	Cumulative frequency
0	8	8
1	16	24
2	22	46
3	14	60
4	20	80
5	32	112
6	8	120
7	4	124

Reasoning

For the first data value, the cumulative frequency is the same as the frequency.

For subsequent values, add the frequency for that value to the previous total.
8
$8 + 16 = 24$
$24 + 22 = 46$
$46 + 14 = 60$
$60 + 20 = 80$
$80 + 32 = 112$
$112 + 8 = 120$
$120 + 4 = 124$

continued

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Example 17 continued

Working

b The relevant row from the table is

4	20	60 + 20 = 80

80 people caught the train on four days or less.

Reasoning

Look up the answer in the cumulative frequency column.

c

Number of days	Number of people (frequency)	Cumulative frequency	Cumulative frequency (%)
0	8	8	6.5
1	16	24	19.4
2	22	46	37.1
3	14	60	48.4
4	20	80	64.5
5	32	112	90.3
6	8	120	96.8
7	4	124	100.0

The total of the frequencies is 124. This is the last entry in the cumulative frequency column.

To calculate the cumulative percentage frequencies, divide by 124 and multiply by 100. For example:

$$\frac{8}{124} \times 100 = 6.451$$

$$\frac{24}{124} \times 100 = 19.354$$

d The relevant row from the table is

5	32	112	90.3

90.3% of people caught the train on five days or less.

Look up the answer in the cumulative percentage frequency column.

Cumulative frequency polygons

A **cumulative frequency polygon** shows how the cumulative frequency changes as the values of the data increase.

To construct a cumulative frequency polygon

- label the horizontal axis to represent the data values.
- label the vertical axis to show the cumulative frequency.
- plot points and join them with straight line segments.

Example 18

Using the data from example **17** to

a construct a cumulative frequency polygon.

b estimate the median score.

continued

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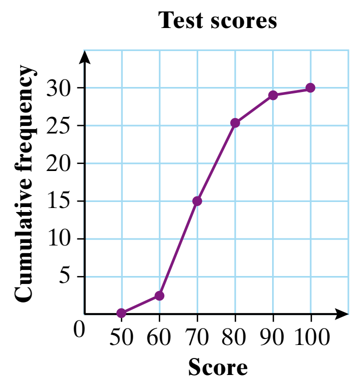
Example 18 continued

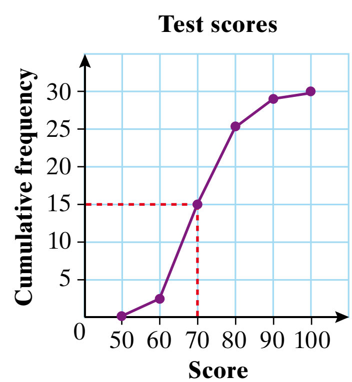
Working

Reasoning

a



b



There are 30 students. Locate 15 on the cumulative frequency axis and draw a horizontal line to meet the cumulative frequency polygon. Draw a vertical line to meet the score axis.

The median score is 70.

exercise 13.3

LINKS TO Examples 9, 10

1. As part of a statistics project, 50 Year 9 students with mobile phones kept a record of the number of SMS messages they sent in one month. Their data is as follows.

108, 124, 97, 108, 95, 112, 80, 114, 123, 71, 85, 74, 117, 103, 116, 114, 82, 119, 137, 95, 83, 107, 113, 128, 139, 105, 97, 83, 116, 83, 97, 92, 85, 114, 108, 117, 132, 74, 105, 107, 121, 128, 113, 102, 100, 93, 117, 81, 76, 108

a Organise this data in a grouped frequency table, using class intervals of 10.

b Display the data using a histogram.

c From the shape of the histogram, estimate the mean number of SMS messages.

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In certain parts of Australia, giant worms can be found. Kattina is a scientist who studies such worms. She breeds such worms in captivity so that she can learn more about them. In one of her studies she measures the lengths of 45 worms and summarises her data in a frequency table. Construct a histogram representing this data.

Length interval (centimetres)	Frequency
20–<40	6
40–<60	16
60–<80	13
80–<100	10
Total	45

The heights of swimmers, correct to the nearest centimetre, at a swimming competition are shown in the frequency table below.

a Construct a histogram that represents this data.
b What is the modal height interval?
c How many swimmers had heights less than 160 cm?

Height (cm)	Frequency
135–<140	1
140–<145	2
145–<150	4
150–<155	5
155–<160	7
160–<165	9
165–<170	10
170–<175	9
175–<180	11
180–<185	2
Total	60

A speed camera recorded the speeds of cars passing a school one morning. The speeds are shown in this table.

a Construct a histogram to display the data.
b What is the modal speed interval?
c How many cars were travelling faster than this?

Speed (km/h)	25–<30	30–<35	35–<40	40–<45	45–<50
Number of cars	8	22	32	7	5

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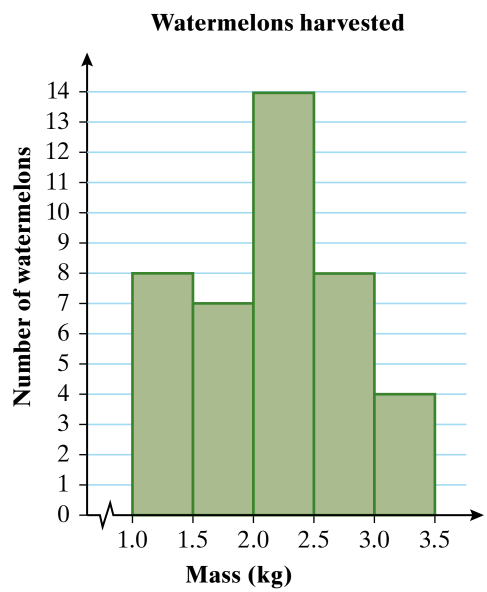
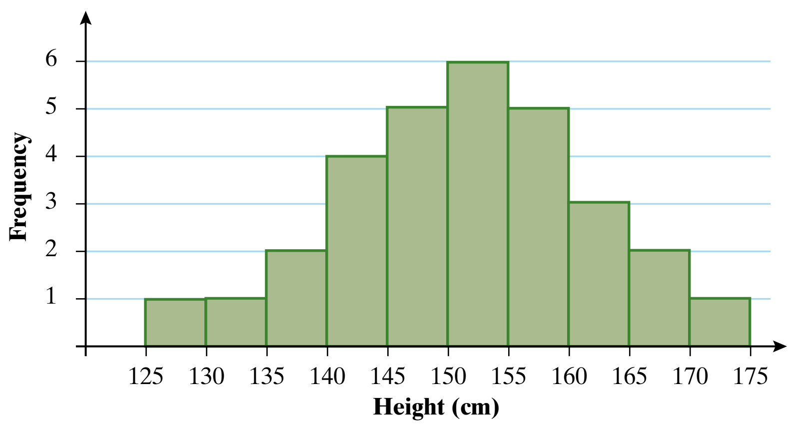
This histogram shows the masses of watermelons harvested from a market garden.

Construct a frequency table from this histogram.



The histogram below shows the heights of a group of students.



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a Complete the frequency table.

b How many students were measured?

Height	Frequency
125–<130	
130–<135	
135–<140	
140–<145	
145–<150	
150–<155	
155–<160	
160–<165	
165–<170	
170–<175	
Total	

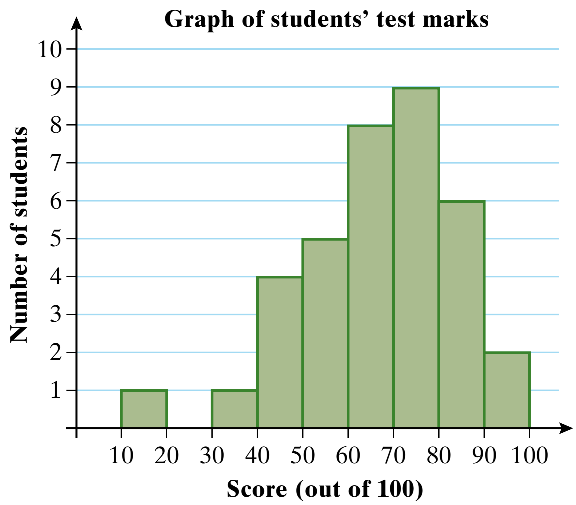
LINKS TO Example 11

The graph on the right shows the results of some Year 9 students on a mathematics test.

a Describe the shape of the histogram.

b Add a frequency polygon to the histogram.

c Describe the shape of the distribution.



LINKS TO Example 12

The data below shows the masses of 40 people, measured correct to the nearest kilogram, who visited a doctor's surgery on a particular day.

52, 68, 91, 23, 64, 59, 43, 37, 44, 35, 106, 88, 34, 8, 65, 48, 82, 63, 65, 59, 81, 63, 101, 29, 62, 56, 48, 60, 71, 78, 82, 64, 32, 57, 63, 24, 81, 73, 56, 97

a Organise this data using a grouped frequency table with class intervals of 10 kg.

b Construct a histogram to display this data.

c Organise the data into a stem-and-leaf plot with stems of 10.

d Comment on the two different visual displays.

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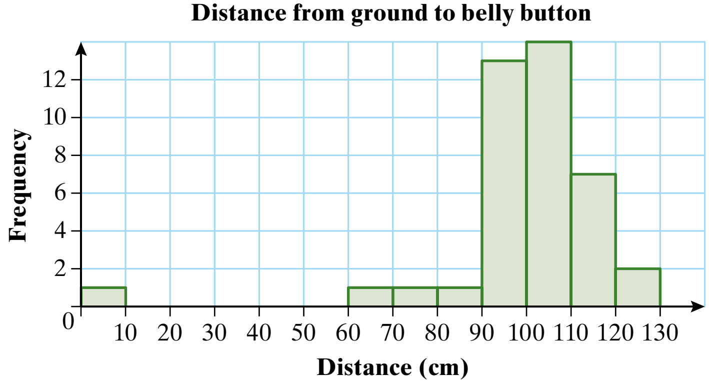
LINKS TO Example 13

In the 'CensusAtSchool' project, one question asked students to measure the distance (to the nearest centimetre) from the ground to their belly button. The data from a sample of 40 students is displayed in this histogram.



a Explain why this is an example of data from a continuous numerical variable.

b What proportion of students had a measurement greater than 1 m?

c How do you think the measurement of 0–10 cm might be explained?

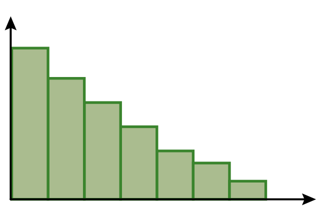
d Redraw the histogram using

  i intervals of 20 cm   **ii** intervals of 40 cm.

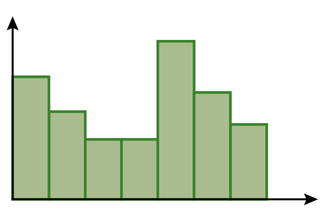
e Explain what happens to the display of the distribution as the size of the interval is increased.

For each of the following histograms, state whether the distribution is symmetrical, positively skewed, negatively skewed or none of these.

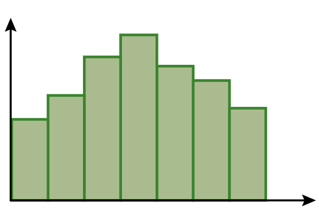
a



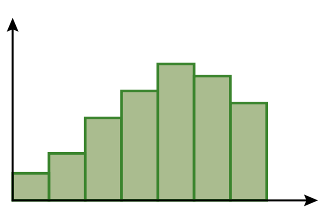
b



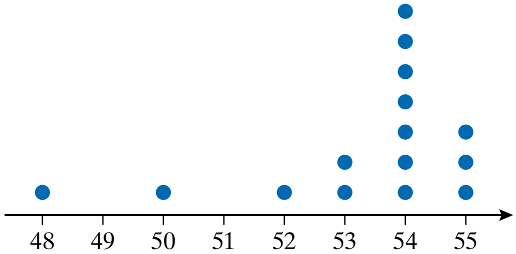
c



d



Fifteen boxes of Smarties are opened and the number of Smarties in each is displayed in the following dot plot. Describe the shape of the distribution.



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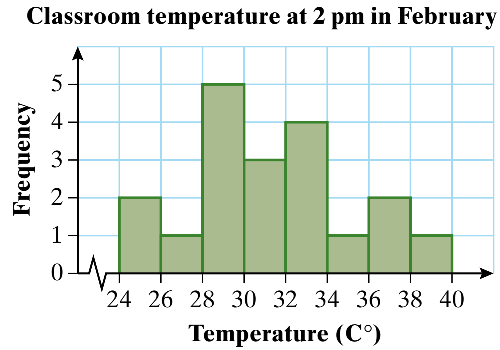
Students at a certain school are trying to convince the principal that one of the classrooms needs an air conditioner. To support their case, they monitor the classroom temperature at 2 pm on each school day during February. This histogram shows the recorded temperatures (to the nearest degree Celsius).

a What class interval is used here?

b On how many days were data recorded?

c On what proportion of the days was the temperature measured at 32°C or above?

d Redraw the histogram using a different class interval, so that the days in part **c** are shown in a single column.



LINKS TO Example 14

The graph below shows the recorded speeds of cars travelling on a freeway.

a Describe the shape of the histogram.

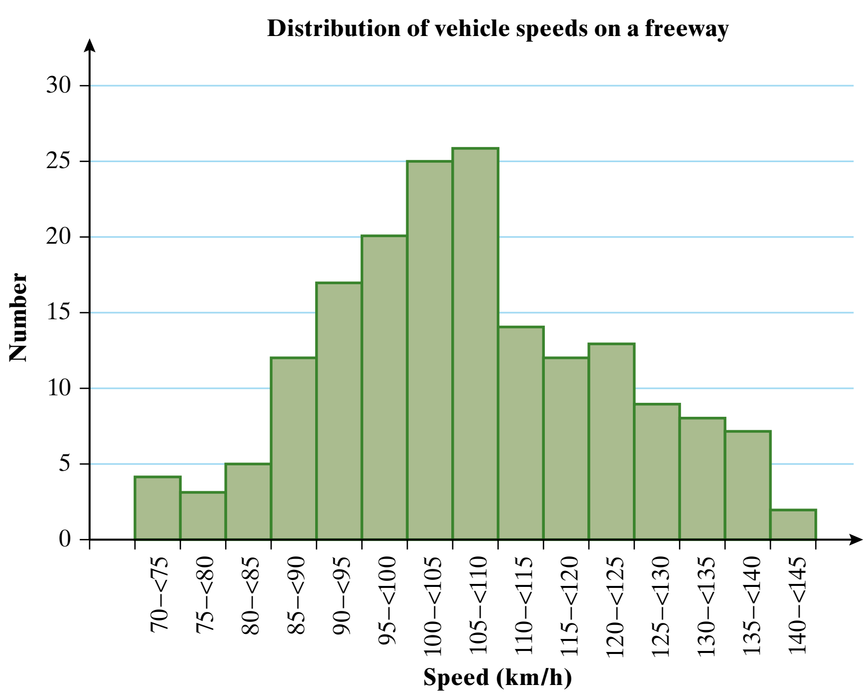
b Determine the number of vehicles travelling

i between 70 and 74 km/h. **ii** at less than 100 km/h.

c What class interval contains the greatest number of vehicles?

d A vehicle travelling at 115 km/h or more is likely to receive a speeding ticket. How many vehicles are in danger of receiving a ticket for speeding?

e Is speed a continuous or discrete variable?



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The data below shows the masses of 40 people, measured correct to the nearest kilogram, who visited a doctor's surgery on a particular day.

52, 68, 91, 23, 64, 59, 43, 37, 44, 35, 106, 88, 34, 8, 65, 48, 82, 63, 65, 59, 81, 63, 101, 29, 62, 56, 48, 60, 71, 78, 82, 64, 32, 57, 63, 24, 81, 73, 56, 97

a Construct a new grouped frequency table with class intervals of 20 kg.

b Construct a histogram from your grouped frequency table in part **a**.

c Compare this histogram with the histogram in part **b** of question **13**. Which of your two histograms do you think shows the distribution of the data more effectively?

Anthony is the junior captain of a tennis club. He records the ages of the junior members in the following table. Construct a histogram showing the ages of the junior members of the tennis club.

Age (years)	8 − <10	10 − <12	12 − <14	14 − <16
Number of members	24	20	14	16

LINKS TO Example 15

The frequency table on the right summarises the data obtained for one fisherman in a fishing competition. The lengths of 30 fish caught by the fisherman were measured correct to the nearest millimetre.

Length interval (cm)	**Frequency**
0–<10	9
10–<20	13
20–<30	4
30–<40	3
40–<50	1
Total	30

a Write down a set of 30 lengths of fish that could be represented by this frequency table.

b Represent this data using a histogram.

c Estimate the mean length of the fish by using the midpoint of each class interval.

LINKS TO Example 16

Use the frequency table from question **16** for this question.

a Copy the table and add a cumulative frequency column.

b How many fish were shorter than 30 cm?

LINKS TO Example 17

Use your grouped frequency table from question **6** for this question.

a Add a cumulative frequency column.

b Add a cumulative percentage frequency column.

c What percentage of students had heights less than 150 cm?

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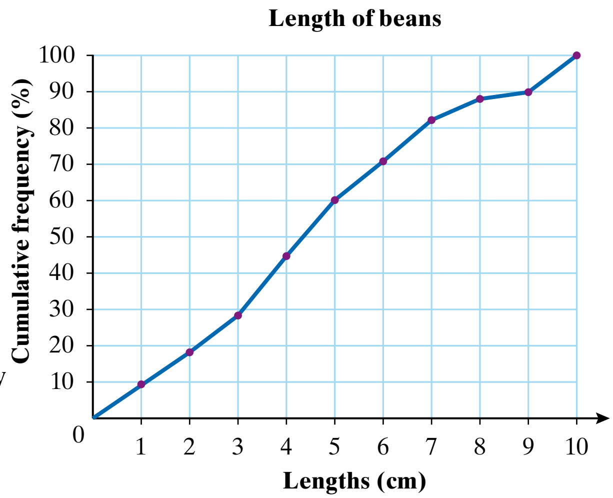
LINKS TO Example 18

The agricultural researcher measures the lengths of beans growing on several plants in another trial plot. She presents the results in this cumulative frequency graph.

a What percentage of beans are less than 3 cm in length?

b If beans are harvested when they are 5 cm or longer, what percentage of the beans are ready for harvesting?

c Estimate the median length of the beans.



exercise 13.3 challenge

The times (in seconds) taken by 50 children to complete a maze are as follows.

31	46	22	37	92	76	53	28	34	85
73	51	64	32	28	95	46	103	128	30
56	93	38	49	72	85	81	29	98	101
29	59	175	41	146	34	24	98	91	75
93	74	42	36	94	71	30	64	67	51

a What was the mean time for completing the maze?

b Organise the data into a frequency table with class intervals of 5 seconds. From the table, calculate the mean time again.

c Organise the data into class intervals of 10 minutes and calculate the mean time again.

d Organise the data into class intervals of 20 minutes and calculate the mean time again.

e **i** Compare your answers to **a**, **b**, **c** and **d**. Are they the same?

ii How does grouping the data affect the accuracy of the estimate for the mean?

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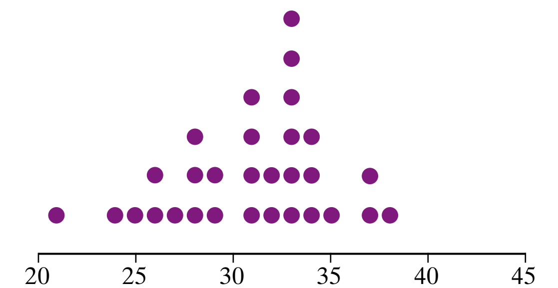
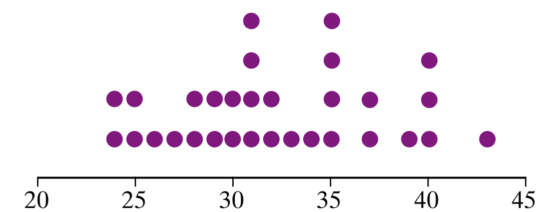
13.4 Relationships between categorical and numerical variables

Sometimes we wish to compare the distribution of a numerical variable such as height against a categorical variable such as gender or year level. The two dot plots below compare the daily maximum temperature for the 31 days of December for 1910 and 2010. The dot plots show clearly the different shapes of the distributions. In this comparison, temperature is the numerical variable and year is the categorical variable.



Daily maximum temperature (°C) December 1910



Daily maximum temperature (°C) December 2010

Back-to-back stem-and-leaf plots

The relationship between a categorical and a numerical variable can displayed with a **back-to-back stem-and-leaf plot**. In a back-to-back stem-and-leaf plot, the leaves of one set of data are placed to the right of the stem while those of the other set of data are placed to the left.

A back-to-back stem-and-leaf plot is restricted to data where the categorical variable has only two possible values. For example, the following stem plot displays the number of sit-ups per minute for two classes of students.

Note that the values on the left side of a back-to-back stem-and-leaf plot are read in reverse order. In the example above, the minimum sit-up rate for Class 1 is 32, not 23. It is helpful to check the key to be sure that you interpret the data correctly. In this case, the number of sit-ups in two minutes is the numerical variable and class is the categorical variable.

Comparison of sit-ups in 2 minutes

Class 1		**Class 2**
Leaf	Stem	Leaf
3 2	3	4
5 5	3	5 6 8
3 2 0	4	0 1 1 2 2 3 3
8 8 7 6 6 5	4	5 5 6 6 7 7 8 9
4 3 3 2 0 0	5	0 0 1
9 7 6 5 5	5	6 7 8
3 1 0	6	4

(Key: 2|3| means 32 and 3|4| means 34)

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Example 19

Josh reads that mothers who smoke give birth to babies with lower birth weights, and he decides to carry out a survey to check this claim. He starts by asking the mothers he knows whether or not they smoke, and recording the birth weights of their babies.
Josh soon finds that most of the women he asks don't smoke. Therefore, after he feels he has enough data for non-smokers, he only keeps recording data for smokers—until he has an equal number of values in each group.
The birth weights (in kg) for babies of smokers and non-smokers in Josh's sample are as follows.

Non-smokers	3.65	3.9	3.45	3.8	4.33	3.36	3.32	4.07	3.66	3.69	3.66	3.69	3.62	3.89
Smokers	3.1	2.87	3.41	2.73	2.27	2.67	3.21	3.55	2.9	2.76	2.87	3.15	2.73	2.93

a What are the two variables, and what types of variable are they?
b Comment on Josh's selection of the sample. Is the sample likely to be representative of the population in his town, state and country?
c Construct a back-to-back stem-and-leaf plot for the data, rounding values to the nearest 0.1 kg.

Working

a The smoking status of the mother is a categorical variable, whereas the birth weight is a continuous numerical variable.

Reasoning

Smoking status can take two categorical values: smoker or non-smoker.
Birth weight can take any numerical value within reasonable bounds.

Working

b The sample is not random, as Josh has simply chosen people he knows. Also, the sample size is quite small. Therefore, it may not be representative of the wider population.

Reasoning

It is important that the sampling method contains no obvious bias, otherwise no clear conclusions can be made. More data would be helpful here.

Working

c

Birth weight

Non-smokers Leaf	Stem	**Smokers** Leaf
	2	3
	2	7 7 7 8 9 9 9 9
4 3	3	1 2 2 4
9 9 8 7 7 7 7 7 6 5	3	6
3 1	4	
	4	

(Key: 4|3 means 3.4 kg and 2|3 means 2.3 kg)

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Side-by-side column graphs

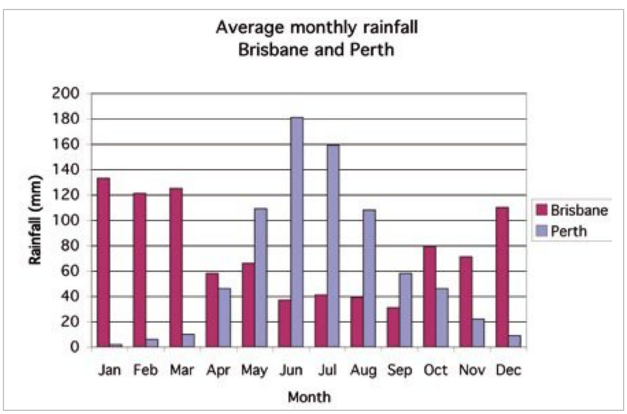
A numerical variable and a categorical variable can be compared using a side-by-side column graph. In the following example, the monthly rainfall in millimetres is the numerical variable and the location is the categorical variable. A spreadsheet has been used to graph the data.

Example 20

The following spreadsheet shows the average monthly rainfall in millimetres for the cities shown. Compare the rainfall patterns for Brisbane and Perth.

	A	B	C	D	E	F	G	H	I	J	K	L	M
1		Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
2	Perth	2	6	10	46	109	181	159	108	58	46	22	9
3	Darwin	389	328	286	79	4	0	0	0	7	54	136	202
4	Alice Springs	15	14	12	4	8	4	3	2	1	19	19	22
5	Adelaide	11	8	21	32	56	41	63	47	38	39	20	20
6	Brisbane	133	121	125	58	66	37	41	39	31	79	71	110
7	Sydney	85	77	102	82	84	101	49	56	47	49	68	59
8	Canberra	37	50	48	48	53	38	38	46	64	49	54	35
9	Melbourne	37	32	40	52	56	43	47	50	53	68	52	50
10	Hobart	39	33	38	48	39	45	51	43	42	57	49	47
11													

Working



Brisbane is wetter than Perth from October to April. Perth is wetter than Brisbane in winter and Brisbane is wetter in summer.

Reasoning

Either summary statistics or graphs can be used to compare the rainfall patterns for Brisbane and Perth.

A column graph can be used to compare the rainfalls for Brisbane and Perth.

Draw graphs for Brisbane and Perth on the same set of axes. This can be done by hand or using technology as shown. Then describe the similarities and differences.

continued

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Example 20 continued

Working

Totals

Brisbane: 911 mm

Perth: 756 mm

Annually, Brisbane receives more rain than Perth.

Reasoning

For each city, add the rainfall figures for all 12 months.

Compare the totals.

On a spreadsheet, you can use the **SUM** function to find the total of a set of data.

Means

Brisbane: $\frac{911}{12} \approx 75.9$ mm

Perth: $\frac{756}{12} \approx 630$ mm

Perth has a higher average monthly rainfall than Brisbane.

To find the mean, divide the total rainfall by 12.

On a spreadsheet, you can use the **AVERAGE** function for each set of data.

Ranges

Brisbane: $133 - 31 = 102$ mm

Perth: $181 - 2 = 179$ mm

The spread of Perth's rainfall figures is greater than that of Brisbane's.

To find the range, subtract the lowest rainfall from the highest rainfall.

On a spreadsheet, you can use the **MAX** function and the **MIN** function to find the range of a set of data.

exercise 13.4

LINKS TO Example 19

1. This back-to-back stem plot shows the waist measurements (in cm) of 20 men aged in their thirties and 20 men in their fifties.

   a For each age group, find the percentage of men with a waist measurement less than 90 cm.

   b Compare the distribution of waist measurements for the two age groups.

Waist measurements (cm)

Fifties Leaf	Stem	Thirties Leaf
7	6	8
9 7 3 2	7	0 1 1 2 3 4 4 7 8 9
9 4 4 2 1	8	0 0 1 4 4 7 8
9 8 7 5 3 0	9	5 7
7 6 2	10	
3		

(Key: 7|6| means 67 and |6|8 means 68)

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The following back-to-back stem and leaf plot shows the times in seconds of 45 Year 9 girls and 45 Year 9 boys in a CensusAtSchool test of reaction time.

Girls		Boys
Leaf	Stem	Leaf
	2	0
9 9 8 6 6 5	2	5 6 6 8 8 8 8 9 9 9
4 4 4 4 3 2 2 2 1 1 1 1	3	0 1 1 1 1 2 2 2 2 4
9 8 8 7 7 7 77 7 7 6 6 6 6 5 5 5 5	3	5 5 6 6 7 7 7 7 7 8 9 9
3 3 3 2 1 1 1 0 0	4	0 0 0 0 2 3
	4	6 8
	5	4 4
	5	
	6	4
	6	9

(Key: 1|3| means 13 and |3|0 means 30)

a Compare the median times for girls and for boys.

b Compare the range for girls and for boys.

c Describe the shape of the distributions.

Vowels occur in almost all words. The data in set A below shows the number of vowels in 20 sets of 10 English words. Set B shows the number of vowels in 20 sets of 10 Italian words.

Set A: 15, 15, 11, 13, 15, 24, 18, 27, 26, 22, 21, 27, 22, 20, 20, 19, 11, 14, 19, 22

Set B: 30, 30, 20, 28, 35, 15, 21, 21, 7, 9, 35, 14, 16, 17, 23, 23, 37, 24, 30, 23

a Find the median and range for each group.

b Construct a back-to-back stem and leaf plot for the data.

c Compare the shapes of each distribution.

d What do the results tell you about the use of vowels in these languages?

The numbers in sets A and B are the heights (in cm) of 14-year-olds. One set shows boys' heights and the other girls' heights.

Set A: 171, 158, 167, 173, 149, 163, 169, 176, 176, 176, 153

Set B: 153, 165, 160, 156, 157, 154, 157, 164, 152, 155

a Find the median and range for each group.

b Use the results to suggest which set is the girls' data. Explain how the statistics helped you to decide.

The results for two classes on their statistics tests are shown below. The results are rounded off to the nearest 10%.

Class 1: 20, 30, 40, 50, 50, 50, 60, 60, 60, 60, 70, 70, 70, 70, 70, 70, 80, 80, 80, 80, 80, 80, 90, 90, 90, 90, 100

Class 2: 10, 30, 30, 40, 50, 50, 50, 50, 60, 60, 60, 60, 60, 60, 70, 70, 70, 70, 70, 80, 80, 80, 90, 90, 90, 100, 100

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a Use a back-to-back stem and leaf plot to display these two sets of data.
b Which class had the greatest range of results?
c Which class had more results above 80%?
d Which class performed better on the test? Why?

LINKS TO Example 20

The results of several rounds of soccer matches are shown below. The number of goals scored by the team playing at home is shown first.

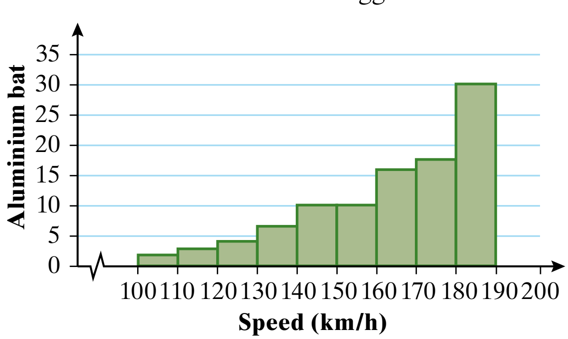
4−1	2−0	1−1	2−1	0−1	1−1	0−3	5−2	2−1	1−3
0−5	0−5	4−0	4−0	0−0	1−2	1−2	1−0	1−3	0−4
0−4	1−1	1−3	5−2	2−1	4−0	2−5	1−1	1−3	1−3

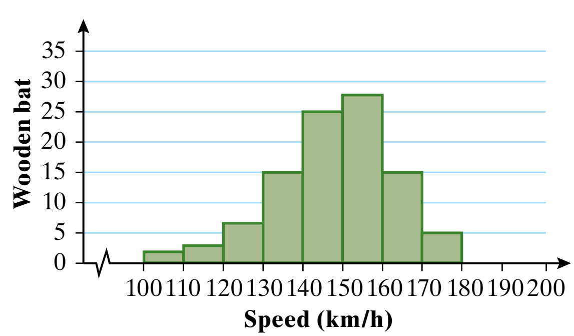
a Copy and complete the following frequency table. For example, for the score 4–1, make a tally mark to show 4 home goals and another tally mark to show 1 away goal.

Number of goals	Home goals tally	Home goals frequency	Away goals tally	Away goals frequency
0				
1				
2				
3				
4				
5				

b Construct a side-by-side column graph to compare the numbers of home goals and away goals.

The two histograms show the ball speeds for a wooden bat and an aluminium bat for the same group of 20 baseball players. The players ranged from high school students to professional players. Electronic equipment was used to record the ball speed immediately after collision with the bat.
a Describe the shape of each distribution.
b What does the data suggest about aluminium baseball bats?





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13.5 Sampling a population

When data about a particular variable is gathered, it is often impractical to survey the entire **population**. In such cases, data is usually obtained from a **sample** of the known population. For example, if you wanted to find out the average daily number of text messages sent by students at your school (the population), you would probably survey a small sample of students rather than every student.

Example 21

For each of the following questions, decide whether it would be more appropriate to use a sample or a census to collect the relevant data.

a How pure is Perth's water supply?

b What proportion of Australian families have private health insurance?

Working	Reasoning
a Sample	Data to answer this question would be more appropriately obtained from testing a sample of water.
b Census	Data to answer this question would be obtainable from census data or taxation returns.

It is often assumed that a sample displays the same characteristics as the entire population. In other words, it is often assumed that the population has the same centre and spread as the sample. Whether this is in fact likely to be the case depends on

- the way the sample was chosen
- the size of the sample.

Consider the following common sampling scenarios, and think about the way the samples are chosen. Are they likely to be representative of the larger population?

Example 22

Explain why the following methods of sampling may not produce samples that reflect the larger population.

a A web poll asks users whether they think that downloading music illegally is unfair to musicians.

b Sara eats two chocolates from a box of 50 assorted chocolates and decides, 'They're all horrible!'

c A radio station conducts an SMS poll, asking listeners whether they approve of the current state premier.

d A student surveys class members about their favourite mobile phone brand.

continued

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Example 22 continued

Working	Reasoning
a The respondents to this poll would not be representative of the general population.	People answering a web poll are internet users, so they are more likely than non-internet users to have downloaded music themselves.
b Sara's conclusion is based on too small a sample.	Sara may have picked the only two chocolates in the box that she doesn't like!
c The respondents to this poll would not be representative of the general population.	Older people would probably be under-represented in an SMS poll. Particular radio stations or shows may attract people with certain political views. People with strong positive or negative opinions about the premier are more likely to respond than people with more moderate views.
d The opinions of students in one class might not represent the student population or larger population.	If someone in the class had recently bought a 'cool' new phone, that might influence other students' responses. The views of one popular (or unpopular) student could influence others to vote a certain way.

Predictions and claims about the whole population are often made on the basis of sampling methods like those in the above example. We need to be wary of such claims, especially when no details are provided about the sampling methods used.

Simple random samples

If a sample does not represent the population fairly, we say that it is **biased**. The sampling methods discussed in example **22** are likely to result in biased samples.

An easy way to minimise bias is to take a **random sample**, where every member of the population has an equal **chance** of being selected. That way, no particular response or group is favoured or represented unfairly in the sample.

The simplest way to select a random sample is to place names in a hat (the population) and draw out a handful (the sample). We use such a method when we draw winners for a raffle. This method produces a **simple random sample**.

When the population is large, it is impractical to draw names out of a hat. Instead, a **random number generator** is used.

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Random number generators

When a standard die is rolled, it randomly **generates** whole numbers between 1 and 6. In that sense, a die is a simple random number generator. However, we need some way of generating a wider range of numbers.

Until relatively recently, tables of numbers were used to **simulate** the generation of random numbers. For example, if you wanted to choose six random numbers between 0 and 99, you could use the following extract from a random number table to 'generate' the numbers: 61, 42, 4, 19, 86, 54.

61424	20419	86546	00517
90222	27993	04952	66762
50349	71146	97668	86523
85676	10005	08261	25906
02429	19761	15370	43882
90519	61988	40164	15815
20631	88967	19660	89624
89990	78733	16447	27932

Note that the repeat of 42 was ignored.

Today, random numbers can be generated by spreadsheet software packages and by most calculators. The advantage of modern number generators is that they can generate a huge quantity of random numbers in a very short time.

Example 23

Use an Excel spreadsheet to generate

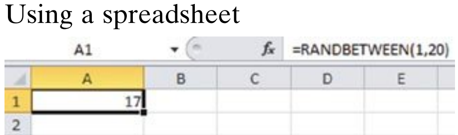
a a random integer between 1 and 20

b 5 random integers between 1 and 20

c 50 random integers between 1 and 500.

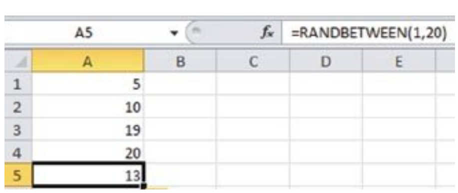
Working

a Using a spreadsheet



The formula =RANDBETWEEN(1,20) in cell A1 generates a random integer between 1 and 20.

b



Dragging the formula down to cell A5 gives five random integers between 1 and 20.

Reasoning

continued

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Example 23 continued

Working

c

A1 f_x =RANDBETWEEN(1,500)

	A	B	C	D	E
1	468				
2	273				
3	232				
4	298				
5	425				
6	405				
7	91				

Reasoning

The formula =RANDBETWEEN(1,500) in cell A1 generates a random integer between 1 and 500.

Dragging the formula down to cell A50 gives fifty random integers between 1 and 500.

Example 24

Igor wants to find out the views of the 239 students in his year level on a range of issues. He decides to survey a sample of 40 students. How could Igor use the RANDBETWEEN function in a spreadsheet to help him choose a simple random sample, so that the results he obtains are likely to represent the views of the entire year level?

Working

Igor should obtain a list of all 239 students and assign each student a number between 1 and 239.

To select a random sample of the 239 students Igor should generate a list of 40 random numbers between 1 and 239 inclusive. Sorting the numbers will show if any of the 40 integers has been repeated. For each repeated integer, Igor should generate another integer between 1 and 239. When Igor has a list of 40 different integers, he should interview the students with those numbers.

A1 f_x =RANDBETWEEN(1,239)

	A	B	C	D	E
1	124				
2	25				
3	212				
4	129				
5	21				
6	119				
7	204				

Reasoning

The formula =RANDBETWEEN(1,239) in cell A1 will generate a random integer between 1 and 239. Dragging the formula down to cell A40 will give 40 random integers.

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Systematic sampling

For a **systematic sample**, the starting point is chosen at random. Then the other members of the sample are chosen at regular intervals from the population.

Example 25

A street has 160 houses and a sample of 10 is required. Use a systematic sample to choose 10 houses from the street.

Working	Reasoning
$\frac{160}{10} = 16$ One house in 16 must be selected. The starting point must be between 1 and 16 (inclusive).	To determine the size of the interval, divide the population (160 houses) by the sample size (10 houses). The starting point needs to be in the first interval.
The random number 4 was generated. Start at the 4th house.	Generate a random number between 1 and 16 to determine the starting point.
Selecting every 16th house, the sample consists of the houses numbered: 4, 20, 36, 52, 68, 84, 100, 116, 132, 148.	Use the interval size to determine the houses in the sample.

Convenience sampling

As the name suggests, **convenience sampling** is based on convenience. Rather than selecting a random sample from the entire population, the researcher uses whatever individuals are conveniently available.

For example, a company wants to determine how to package their cheese so that it appeals to customers. A researcher stands inside a supermarket during the day and shows customers samples of five different types of packages. She asks 150 customers for their preference.

In this type of sampling, some members of the population do not have a chance of being sampled. Therefore, it is not possible to determine how accurately the sample obtained represents the actual population.

Stratified sampling

Often, a population is made up of separate groups or strata. A **stratified sample** is selected in such a way that each group is represented fairly. This involves making a random selection from each group in the population. The number selected from each group in the sample is proportional to the number in each group in the population.

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Example 26

Students in Years 9, 10 and 11 at a particular school are surveyed regarding the number of hours they watch television each week. There are 164 students in Year 9, 148 students in Year 10 and 112 students in Year 11. A sample of 60 students is to be taken. How many students from each year level should be included in this stratified sample?

Working

$164 + 148 + 112 = 424$

Year 9:

$$\frac{164}{424} \times 60 = 23$$

(to the nearest whole number)

Year 10:

$$\frac{148}{424} \times 60 = 21$$

(to the nearest whole number)

Year 11:

$$\frac{112}{424} \times 60 = 16$$

(to the nearest whole number)

The stratified sample should consist of 23 Year 9 students, 21 Year 10 students and 16 Year 11 students.

Reasoning

Find the total number of students in the population.

To find the number of students from each level, multiply the fraction of students in that year level by the size of the sample. Answers need to be rounded to the nearest whole number.

Check that the total number of students in the sample is 60.

exercise 13.5

LINKS TO Example 21

For each of the following questions, decide whether it would be more appropriate to use a sample or a census to collect the relevant data.

a What proportion of households in Tasmania were tuned in to a particular television program?

b What proportion of tennis players in Australia are left-handed?

c How many students are enrolled at each year level of non-government schools in South Australia?

d What proportion of elastic bands from a given machine in a certain factory are faulty?

To determine the average length of life of AAA batteries, why is it necessary to use sampling?

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Student opinion is being sought about proposed changes to the school's discipline policy. Why might it be preferable to obtain a sample of 100 students rather than surveying the entire school population of 1300 students?

LINKS TO Example 22

In the following scenarios, the samples are biased and do not represent the population fairly. In each case, describe what is wrong with the way the sample has been selected.

a A PE teacher wants to determine how many hours students spend exercising each week. She surveys the students who attend hockey training. From this information she calculates the average number of hours per week that students spend exercising.

b A Mathematics teacher wants to find out how much money high school students spend on mobile phone calls each month. He asks 12 students in Year 9 to answer the question: 'How much do you spend on mobile phone calls each month?'

c A local council has received complaints from residents about the number of vehicles using a side street. An engineer has been employed to determine whether the road is busy enough to warrant the installation of speed humps. The engineer counts the number of cars using the street between 9 am and 10 am for two weeks. He uses this information to determine whether or not it is advisable to install speed humps.

d A writer for a magazine is conducting research for an article about pet ownership. To determine the average number of pets per household in a typical suburb in Brisbane, the writer asks members of the local dog obedience club how many pets they own.

LINKS TO Example 23

Use the random integer function of your calculator to generate 10 random numbers between 0 and 99.

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LINKS TO Example 24

The following table shows the ages and heights of 40 members of a scout group.

Scout	Age	Height (m)	Scout	Age	Height (m)
1	5	0.95	21	6	1.02
2	17	1.56	22	16	1.47
3	13	1.27	23	14	1.58
4	11	1.31	24	13	1.65
5	7	1.12	25	11	1.67
6	8	1.24	26	18	1.59
7	13	1.47	27	5	1.03
8	12	1.68	28	14	1.62
9	15	1.75	29	20	1.71
10	12	1.54	30	17	1.58
11	7	1.06	31	15	1.54
12	9	1.28	32	14	1.48
13	10	1.42	33	12	1.39
14	21	1.82	34	9	1.36
15	18	1.78	35	7	1.25
16	16	1.65	36	12	1.34
17	14	1.57	37	12	1.28
18	12	1.51	38	15	1.64
19	7	1.16	39	18	1.62
20	5	1.08	40	10	1.42

a Using a random number generator, choose a sample of 8 scouts from this population of 40. Record the age and height of each scout in the sample.

b Select two more random samples of 8 scouts and write down the heights and ages of the scouts in each sample.

c Were any of the scouts selected in all three samples?

d Were any of the scouts selected in two of the three samples?

LINKS TO Example 25

Use a systematic sampling method to select a sample of 8 scouts from the list in question **6**. Record the age and height of your sample.

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LINKS TO Example 26

● Hoa is collecting data for a project. She is interested in the number of hours students in Year 9 spend on the telephone each week. There are 142 students in Year 9: 80 girls and 62 boys. Hoa wants to take a stratified sample of 40 students.

a How many boys should be in the sample?

b How many girls should be in the sample?

● For each of the following scenarios, decide whether the sampling used is random, systematic, convenience or stratified.

a Amanda is researching the level of customer satisfaction with banks. She surveys 100 people who come into a bank.

b Antonio uses random numbers to choose a sample of 20 students studying science at university.

c Genevieve works for a large catering company. She needs to estimate how many plates need replacing due to cracks. She starts with the first plate and then checks every tenth plate for cracks.

d Kevin is researching the most popular type of car on the market. He stands in a shopping centre and asks 200 passers-by to complete a questionnaire.

e All classes at a particular school have 20 students. A sample of students in a school is obtained by randomly selecting two students from each class.

exercise 13.5 challenge

● A tennis club is planning to upgrade its clubhouse facilities. The table below gives the number of club members divided into age groups.

Age (years)	0–19	20–39	40–59	60–79	80–
Number of members	140	184	126	78	16

The committee decides to select a stratified sample of 50 people to survey about the facilities. How many people should the committee select at random from each age group?

● A company has three offices in Brisbane. The Milton office employs 50 men and 50 women. The Newmarket office employs 30 men and 20 women and the Albion office employs 20 men and 80 women.

The owner of the company wants to survey 60 workers regarding their working conditions.

a What is one advantage of using a sample rather than a census in this situation?

b Explain why random sampling might not be the most suitable survey method in this situation.

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The owner uses stratified sampling to select the sample.

c How many workers should be selected from each of the offices?

d How many women should be selected from the Albion office so that the women in this office are fairly represented in the sample?

e After the sample size for each group has been determined, how would the actual workers to be surveyed be selected?

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13.6 Interpreting statistics in secondary data

Some articles in the media refer to the results of surveys and include many statistics based on these. When reading such articles it is worth considering the following questions.

- Is the article about a sample of the population or the population itself?
- What difference is there between the conclusions that can be drawn from samples and populations?
- Is the sample representative or non-representative of the associated population?
- Is the sample voluntary in nature? Why would this be important?
- Does the article explain well the nature of the sample?
- For surveys, who paid for them? Could this be significant for the results reported?

Newspapers and other media frequently use statistics in the form of tables, graphs or isolated numbers. Statistics are often used in advertisements or political articles that are trying to persuade us to buy something or believe something. Sometimes statistics may be used to confuse or mislead us, but they can also play an important role in informing us. We, therefore, need to be able to interpret data and make reasoned judgements about what statistics are telling us. Does the data match the headline? If numbers are given as percentages, are we told what they are percentages of? Do table row and column labels make sense?

exercise 13.6

1. The following article appeared in the New Zealand Herald on May 26, 2011.

If you're checking your Facebook page today, have a look at how many friends you have—fewer than 124 and the average Facebook user in New Zealand is more popular than you.

A survey by UMR research has found the average Facebook user in New Zealand has 124 friends—but that term should be applied loosely. Seventy per cent of respondents were friends with people they had not seen since school, and 35 per cent were friends with people they had never met in person. And chances are the 124 figure will seem either large or small, depending on your age. Respondents aged 18 to 29 had an average of 213 friends, while those aged more than 60 only had an average of 44.

UMR research director Gavin White said because Facebook now cuts across such a wide section of society, there would be inevitably be divisions in activity between age and other groups. The survey was based on a sample group of 1000 New Zealanders aged 18 and over, and questioned the 687 respondents who used the social networking site.

a Describe the sample on which the research was based.
b What percentage of the sample group responded?
c Suggest how the results might have been different if the sample group were aged 15 and over rather than 18 and over.
d What does the article not tell us about the sample or about those who responded?
e Would similar results be obtained with a sample of 1000 Australians?

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High school students graduating to be tomorrow's cancer and cardiovascular patients

9 February 2011

Results of a national diet and physical activity survey of high school students, released today (9 February 2011), will ring alarm bells among educators, health professionals and parents.

The research, by Cancer Council and the Heart Foundation, reveals excessive prevalence of overweight and obesity among students (highest in low SES areas), inadequate rates of physical activity, insufficient fruit and vegetable intake and a high proportion of students making food choices based on advertising.

The participation of 12,000 students in years eight to 11 across 237 schools provides the first truly national sample for a physical activity survey of young Australians since 1985.

Key findings:

- One in four students are overweight or obese, with a significantly higher rate in low SES areas.
- Eighty-five per cent of students don't engage in sufficient activity to provide a health benefit.
- Low fruit and vegetable intake, with 76% not meeting the daily recommended intake of four vegetable servings daily and 59% not meeting the daily recommended intake of three servings of fruit daily.
- One third drink four or more cups of soft drink, cordial or sports drink a week
- More than half (51%) tried a new food or drink product in the past month they had seen advertised.

http://www.cancer.org.au/Newsmedia/mediareleases/mediareleases2011/9February2011.htm

a Describe the sample used in the study.

b What other information would you like to know about how the sample of students was determined?

c Would you expect the results of the study to represent the students in your school?

The following report was published by PerthNow on 24 January 2011.

PERTH residents have ranked the capital city as an average place to live, with taxes and housing affordability weighing on the city's standing, a liveability survey shows.

According to a national survey of more than 4000 residents by the Property Council of Australia (PCA), Perth was overall ranked fourth on the liveability survey with a score of 60.6. Adelaide was the number one place to live with 63.4 points, followed by Canberra and Melbourne. Hobart came in fifth, then Brisbane and Darwin while Sydney was ranked as the least liveable city in Australia at 55.1 points.

Source: PerthNow

a What do we know about the sample of people in the national survey?

b List some questions that you would like to ask the Property Council of Australia about their sampling method.

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The Sydney Morning Herald conducts online opinion polls. Their website states the following: 'These polls are not scientific and reflect the opinion only of visitors who have chosen to participate.'

a Who are the people most likely to respond to these opinion polls?

b What does the Sydney Morning Herald mean by stating that the polls are not scientific?

exercise 13.6 challenge

The following article was published in 2007 before the devastating floods in Brisbane and the Lockyer Valley.

City supports water subsidies for farmers: survey

EIGHTY per cent of Brisbane residents say they would eat food grown with recycled water while 90 per cent favour subsidisation of water for farmers to ensure locally grown produce, according to a survey conducted by a UQ research centre.

The survey, by University of Queensland (UQ) PhD student, Claudia Baldwin, compared attitudes to water of Lockyer Valley region residents with their counterparts in the City of Brisbane.

Approximately 800 people were surveyed for a study funded by the Queensland Government Growing the Smart State PhD Funding Program for evidence-based policy research through the Cooperative Research Centre for Irrigation Futures at UQ. City and country respondents agreed on most topics, particularly that water usage continued to be restricted and farmers should be subsidised for efficient water use.

Ninety per cent of respondents said they would support subsidisation of water to local farmers for irrigation, in order to have locally grown fruit and vegetables; to maintain a viable local agriculture industry; and, if irrigation water was used efficiently.

With southeast Queensland currently in the grip of a serious water shortage, more than 80 per cent of Brisbane City residents said they would eat food grown with recycled water while 70 per cent said they would drink recycled water if health standards were met. Figures were slightly lower for country residents.

Other survey results showed that contrary to expectations about the impact of water restrictions, concern for the current and future availability of water and for the environment were the main reasons residents had reduced their water use, rather than restrictions or peer pressure. According to Ms Baldwin [the survey], 35 per cent indicated that they would not return to previous levels of water use once restrictions were lifted.

Source: *UQ News Online*
14 February 2007

a If you wanted to find out whether or not the sample was representative of the Queensland population, what questions would you ask the surveyor?

b What kind of information about the survey would you need to know if you were going to make recommendations to the Queensland government?

c Using at least one statistic, write a paragraph to encourage Queenslanders that it is safe to drink recycled water.

d Data is often relevant at the time it is collected but attitudes may change over time. Do you think any of the statistics reported in this article could have changed since 2007?

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Analysis task
sample data

Analysis task

Travelling to school

This analysis task involves comparing the times taken to travel to school by a sample of girls and a sample of boys in Year 9. Random samples of data are obtained from the CensusAtSchool website www.cas.abs.gov.au/cgi-local/cassampler.pl.

a Selecting the data for a sample of 30 girls

After accepting the conditions on the Student page on the website, you are able to choose the type of sample you want. Choose the following to obtain your data for a sample of 30 girls:

Reference year	2010
Questions	All
Sample size	30
Post code range	0000 – 9999
Year level	7
Sex	Female

Now click on **Select sample data**. Click on the Excel spreadsheet icon beside **Download data xls sample file**. Open the spreadsheet and save it as 'Data girls' as you will need the spreadsheet again later in this investigation. Now scroll across to **column AK**.

Copy and paste the column of data into a new spreadsheet and label the column Female.

b Selecting the data for a sample of 30 boys

Repeat all the above steps, choosing Male so that you have data for 30 boys to paste into a column labelled Male.

c Unordered stem-and-leaf plot

Make sure that you have the same numbers of females and males, as some students may not have answered the question. Try another random sample if you have missing data. Using a table as shown, enter all your data into the tables so that you have an unsorted back-to-back stem-and-leaf plot.

Unordered travel time to school

Males		**Females**
Leaves	Stem	Leaves

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d Ordered stem-and-leaf plot

Using your unordered data, sort each leaf into ascending order, with the smallest values closest to the stem on each side. Give your stem-and-leaf plot a title and a key.

Ordered travel time to school

Males		**Females**
Leaves	Stem	Leaves

e Comparing male and female students

Use the shape, centre and spread of the stem and leaf plot to compare the distribution of times for males and females.

- How are the distributions similar and how are they different?
- Do you think different results would be obtained with a different year level?

f Now combine all your data for boys and girls for travel time to school so that you have 60 data values. Group the data into suitable intervals and construct a frequency table for your grouped data.

g Construct a histogram of the data.

h What is the median time taken to travel to school?

i Collect data for the travel times to school for students in your class and construct a histogram. Compare your class histogram with the histogram of the CensusAtSchool sample data.

j–l Go back to your two downloaded spreadsheets and repeat parts **c**–**e** for question **19** of the CensusAtSchool: the usual number of hours of sleep before a school day.

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Review Statistics

Summary

Types of data

- **Categorical data** Data that can be grouped into categories
  - **Nominal data** Data is in named groups
  - **Ordinal data** Categories have a definite order
- **Numerical data** Data that is obtained by measuring or counting. It can be added and averaged.
  - **Discrete data** Data can have only particular values
  - **Continuous data** Data can have any values within a given range

Displaying data

Categorical data

For categorical data, **bar charts** and **pie charts** highlight

- which categories have the highest and lowest frequencies
- how the frequencies vary among the categories.

Numerical data

For numerical data, **dot plots**, **stem plots** and **histograms** give a useful overview of the distribution, by providing a quick picture of

- which values occur more frequently than others
- how much variation there is among the data values
- the range of the values (that is, the difference between the smallest and largest values).

Summarising data: measures of centre and spread

- The mean $\bar{x}$ is given by $x = \dfrac{\sum x}{n}$, where $\sum x$ is the sum of all the data values and n is the number of data values.
- For data displayed in a frequency table, $\bar{x} = \dfrac{\sum xf}{\sum f}$, where f is the frequency of each data value.

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- For data grouped into class intervals, an estimate of the mean is calculated using the midpoint of each interval.
  - Median: the middle value when the data set is arranged in order
  - Mode: the most frequent data value
- The range is a measure of spread.
  - Range: the difference between the smallest and largest data values

Sampling

- In statistics, the population is the total set of people, things or events under investigation.
- Taking a census involves collecting information about the whole population.
- Taking a sample involves collecting information about part of the population.
- Sampling techniques include
  - random sampling
  - systematic sampling
  - convenience sampling
  - stratified sampling.

Visual map

Using the following terms (and others if you wish), construct a visual map that illustrates your understanding of the key ideas covered in this chapter.

back-to-back stem-and-leaf plot
census
centre
convenience sampling
cumulative frequency
cumulative percentage frequency
dot plot
grouped data
grouped frequency table
frequency
frequency polygon
histogram
interval
mean
median
midpoint
mode
population
questionnaire
random sampling
range
sample
spread
stem plot
stratified sampling
survey
systematic sampling

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Revision

Multiple-choice questions

When every member of a population is surveyed, this is referred to as a

A sample. **B** statistic. **C** survey.
D census. **E** questionnaire.

The masses of 100 students are shown in this table.

Mass (kg)	40−<44	45−<49	50−<54	55−<59	60−<64
Frequency	12	24	36	20	8

The mean mass is

A 51.1 kg. **B** 51.4 kg. **C** 51.7 kg. **D** 52.0 kg. **E** 52.3 kg.

Use the following back-to-back stem-and-leaf plot to answer questions **3** and **4**. It shows the amounts of cash carried by a group of teenagers at a cinema.

Girls		**Boys**
Leaf	Stem	Leaf
5 2	0	8
8 6 5 3 0	1	1 5
9 8 8 6 5 3 1 0	2	2 3 6
7 5 4 2 1	3	0 5 5 8 9 9
8 5 3	4	0 3 5 5
5 0	5	2 7 9
	6	5 2
	7	3
	8	5
	9	2
	10	0

(Key: 5|9| means 95 and |9|7 means 97)

The number of teenagers who were surveyed is

A 10 **B** 11 **C** 17 **D** 25 **E** 50.

Which one of the following statements is correct?

A The boys generally carried less cash than the girls.
B There was more variation in the amounts of cash carried by the girls than the boys.
C The distribution of the amounts carried by the girls is roughly symmetrical.
D The distribution of the amounts carried by the boys is negatively skewed.
E The range of the girls' amounts is $48.

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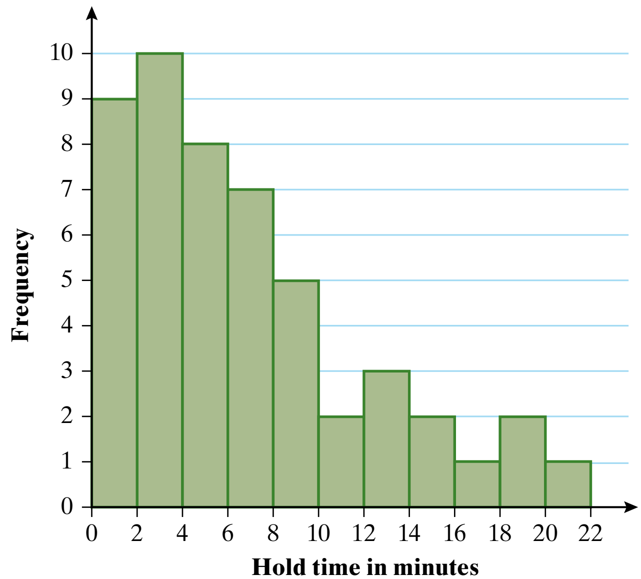
The time, in minutes, spent on hold by 50 callers to an insurance company was recorded and summarised in the form of a histogram as shown.

Time kept on hold for 50 callers



The median time spent on hold was somewhere between

A 2 and 4 minutes. **B** 4 and 6 minutes. **C** 6 and 8 minutes.
D 8 and 10 minutes. **E** 10 and 12 minutes.

Short-answer questions

The data showing the number of minutes that 30 members of a gym spent on a treadmill are listed below.

24, 49, 54, 21, 24, 57, 28, 33, 27, 34, 22, 32, 38, 43, 28, 26, 23, 46, 31, 60, 51, 53, 39, 52, 36, 38, 40, 50, 37, 46

a Find the mean and median times.

b What is the range?

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c Use the data to complete the grouped frequency table.

Time (minutes)	Tally	Frequency
20–<30		
30–<40		
40–<50		
50–<60		
60–<70		

d Construct a histogram representing this data.

e Describe the shape of the distribution.

State the sampling method used in each of the following.

a Samara asks 100 people entering a shopping centre whether they watch a particular show on television.

b A small school has 60 students in Year 8 and 40 students in Year 9. Katia selects a random sample of 12 students from Year 8 and 8 students from Year 9 to survey.

c Joshua has an alphabetical list of all students in Year 12 and he assigns each one a number from 1 to 120. He interviews student number 5 and every 10th student on the list after that.

d Jane uses a random number generator to choose the numbers of five houses in a street, and interviews the residents of those houses.

Extended-response questions

Twelve students in a Year 9 class study both Mathematics and French. Here are their scores (out of 100) for a test in each subject.

Mathematics: 88, 77, 86, 75, 100, 87, 78, 97, 76, 82, 92, 78

French: 69, 76, 89, 55, 87, 76, 82, 86, 62, 74, 81, 75

a Use an ordered back-to-back stem-and-leaf plot to display both sets of data.

b Which subject had the greater range of marks?

c Determine the median score for each subject.

d Overall, did students perform better in Mathematics or French?

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The table below shows the heights of students in Year 9 at a large regional high school.

Height (cm)	Frequency
130–<140	5
140–<150	27
150–<160	75
160–<170	98
170–<180	38
180–<190	12
190–<200	2

a How many students are in Year 9 at the school?
b Estimate the mean height of the Year 9 students.
c Construct a histogram for the data.
d Use the shape of the histogram to describe the distribution of the data.

The following data from the Australian Bureau of Statistics CensusAtSchool website (2006) shows the reaction times in seconds for 50 Year 9 students who completed an online exercise. The students were required to click on a Start button with their right hand and then click again when they saw a square on the screen fill with colour.

0.50	0.35	0.29	0.34	0.26
0.28	0.29	0.39	0.25	0.29
0.31	0.98	0.42	0.29	0.32
0.35	0.40	0.36	0.36	0.32
0.39	0.30	0.39	0.34	0.32
0.31	0.31	0.36	0.32	0.54
0.41	0.37	0.32	0.32	0.31
0.97	0.39	0.39	0.29	0.41
0.32	0.26	0.31	0.35	0.42
0.29	0.25	0.23	0.33	0.51

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a Make an ordered stem plot of the data.

Stem	Leaf
1	
2	
3	
⋮	
9	

(Key: 2|3 means 0.23)

b What is the median time?

c What is the mode time?

d Calculate the mean time.

e Copy and complete the following frequency table.

Reaction time (x seconds)	**Tally**	**Frequency (f)**	xf
0.00–<0.10			
0.10–<0.20			
0.20–<0.30			
0.30–<0.40			
0.40–<0.50			
0.50–<0.60			
0.60–<0.70			
0.70–<0.80			
0.80–<0.90			
0.90–<1.00			
Totals		$\sum f =$	$\sum xf =$

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f Use the frequency table to estimate the mean reaction time of the students who completed the exercise. Compare it with your answer to part **d**.

g Construct a frequency histogram of the data.

h Are there any outliers?

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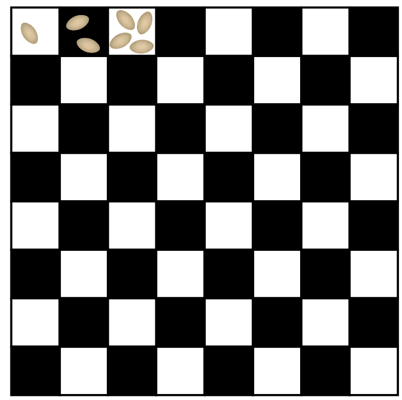
14 Extending and investigating

Pre-test

Warm-up

In this chapter you will have the opportunity to use your mathematics to make connections between mathematics and real world contexts. The first investigation relates to the image on this page—an old story in which a king is supposed to have offered a reward to the inventor of chess. The inventor could choose gold or instead could choose one grain of rice for the first chess board square, two for the second square, four for the third square and so on. Other investigations include the Richter scale of earthquake magnitudes, the braking distance required for a vehicle to stop, standard drinks, and the relationship between rational and irrational numbers and music.

14.1 How many grains of rice?



According to a story, a king was so pleased with the man who invented the game of chess that he offered to reward him in gold. However, the inventor said that he would rather have rice than gold, provided it was given to him in the following way: one grain of rice for the first square of the chess board, two grains for the second square, four for the third square and so on, each time doubling the number of grains for the next square. The king accepted this alternative reward, mistakenly believing this would be less than his offer of gold. How many total grains of rice would the king have to pay the inventor? The answer is 18 446 744 073 709 551 615, a really huge number!

a Copy and complete the table below for the first eight squares.

Square number (n)	**Number of grains on square n**		**Calculating the total number of grains on n squares**			**A pattern for the total number of grains**
1	1	2^0	1	2^0	1	$1 = 2^1 - 1$
2	2	2^1	1+2	2^0+2^1	3	$3 = 2^2 - 1$
3	4	2^2	1+2+4	$2^0+2^1+2^2$	7	$7 = 2^3 - 1$
4	8	2^3	1+2+4+8	$2^0+2^1+2^2+2^3$		
5						
6						
7						
8						

b Use the pattern to calculate the number of grains on square 32.

c Use the pattern to calculate the total number of grains up to and including square 32.

d Write an expression using a power of 2 for the total number of grains on the chessboard.

e Write a general rule for the total number of grains up to and including square n.

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14.2 The golden ratio

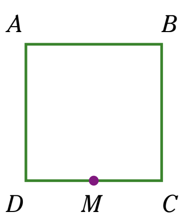
The ancient Greeks placed great importance on creating buildings with pleasing proportions. The particular proportion that they believed to be most pleasing to the eye became known as the golden ratio. A rectangle with its length and width in this ratio is known as a **golden rectangle**.

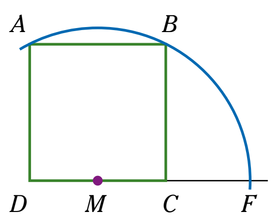
The following diagrams show how the ancient Greeks may have constructed a golden rectangle.

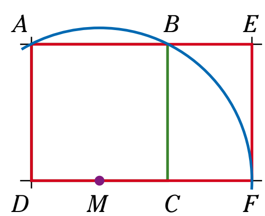
Step 1: Draw a square ($ABCD$) and mark in the midpoint (M) of one side.

Step 2: Place a compass point at M, open the compass and draw a circle through point B. Extend the segment DC to meet the circle at F.

Step 3: Draw a line through F perpendicular to DF and extend the segment AB to meet this line at E. $AEFD$ is a golden rectangle.







a Use interactive geometry software to construct a golden rectangle, following the steps above.

b Taking the side length of the square $ABCD$ to be 2 units, use Pythagoras' theorem to calculate the radius MB. Hint: use the triangle MBC. Give the radius in exact form.

c Since MF is the same length as radius MB, write an expression for DF, the length of the rectangle $AEFD$.

d Write the ratio $\dfrac{\text{length of rectangle } AEFD}{\text{width of rectangle } AEFD}$ in exact surd form. This ratio is called the golden ratio. It is an irrational number and is represented by the Greek letter ϕ (*phi*).

e Give the value of ϕ as a decimal correct to five decimal places.

f The golden ratio has been used by artists, architects and musicians. Use a library or the internet to find examples of where the golden ratio or golden rectangles have been used.

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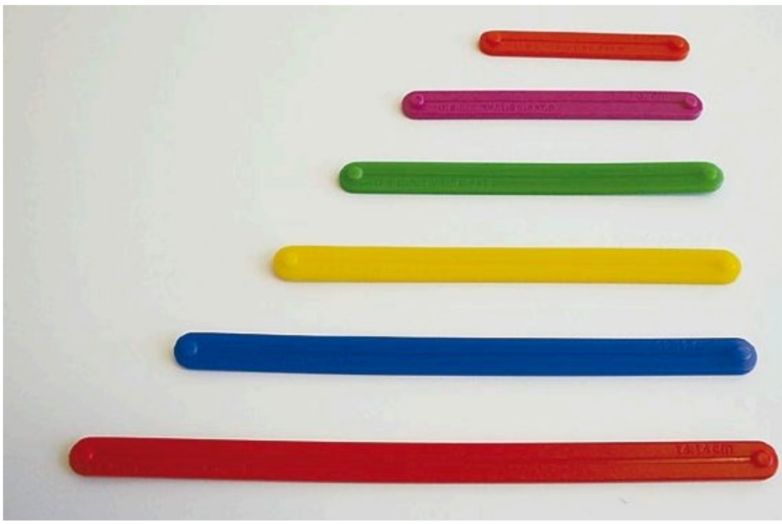
14.3 Geometry strips

A set of clip-together plastic strips has been designed to help young students explore geometric shapes. The lengths of the strips have been chosen so that right-angled triangles and the diagonals of squares can be constructed. Below, each strip has been labelled with its length. Notice that some of the strips have integer lengths. The lengths of the other strips are rational approximations for irrational lengths.



5 cm
7.07 cm
8.66 cm
10 cm
12.25 cm
14.14 cm

a The length of a magenta strip is shown as 7.07 cm. The number 7.07 is a rational approximation for the square root of which integer?

b Suppose that a square is made by clipping together four orange (5 cm) strips. Explain why a magenta (7.07 cm) strip could be used for the diagonal.

c Express each of the other rational approximations for irrational lengths (8.66, 12.24, 14.14) as the square root of an integer. Use a calculator to help.

d Which strip would you use for the diagonal of a square made from four yellow (10 cm) strips?

e Is it possible to make the hypotenuse of a right-angled triangle that uses an orange (5 cm) strip and yellow (10 cm) strips as its two shorter sides. Explain.

f How many different right-angled triangles could be constructed from the strips? (Each strip may be used more than once but only three strips can be used for each triangle.)

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14.4 Binary numbers

Our decimal number system uses 10 as the base, but there are also number systems with other bases. Binary numbers, for example, have 2 as the base.

In base 10 we have ten digits: 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9.

In the binary (base 2) system there are only two digits: 0 and 1.

In base 10, the place-values are 10^0 (units), 10^1 (tens), 10^2 (hundreds), and so on, but in the binary system, the place-values are 2^0, 2^1, 2^2, and so on.

Just as 1101 in the base 10 system means $1 \times 10^3 + 1 \times 10^2 + 0 \times 10^1 + 1 \times 10^0$,
in the binary system, 1101 means $1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$.

Light-emitting diodes (LEDs) can be either off or on and can be used to represent the binary digits 0 (off) or 1 (on). This is the basis of many electronic devices.

In computer technology, a binary digit is referred to as a **bit**. A set of 8 binary digits is called a **byte**. For example, 00111011 is a byte. Each character on the computer keyboard (letters, numbers, spaces, punctuation) can be represented as a byte and stored in the computer's memory.

Converting binary numbers into decimal numbers

To turn a binary (base 2) number into a decimal (base 10) number, we evaluate the powers of 2. For example, the binary number 1101 is equivalent to the decimal number 13.

1101 (base 2)
$= 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$
$= 1 \times 8 + 1 \times 4 + 0 \times 2 + 1 \times 1$ (base 10)
$= 8 + 4 + 0 + 1$ (base 10)
$= 13$ (base 10)

The binary number 110111 is equivalent to the decimal number 23.

10111 (base 2)
$= 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$
$= 1 \times 16 + 0 \times 8 + 1 \times 4 + 1 \times 2 + 1 \times 1$ (base 10)
$= 16 + 4 + 2 + 1$ (base 10)
$= 23$ (base 10)

a Convert each of these binary numbers into decimal numbers by first writing in expanded index notation, then evaluating the powers of 2.

i 101 **ii** 10011

iii 101010 **iv** 11111

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Converting decimal numbers into binary numbers

Multiples of 2	Index form
1	2^0
2	2^1
4	2^2
8	2^3
16	2^4
32	2^5
64	2^6

To convert decimal (base 10) numbers into binary (base 2) numbers we start by writing the number as the sum of multiples of 2 and then convert these into index form.

It is helpful to have a list of the multiples of 2 and their index forms.

57 (base 10)
$= 32 + 16 + 8 + 1$
$= 2^5 + 2^4 + 2^3 + 2^0$
$= 1 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^0$
$= 1 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$
= 111001 (base 2)

81 (base 10)
$= 64 + 16 + 1$
$= 2^6 + 2^4 + 2^0$
$= 1 \times 2^6 + 1 \times 2^4 + 1 \times 2^0$
$= 1 \times 2^6 + 0 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$
= 1010001 (base 2)

b Convert each of these decimal numbers into binary numbers by first writing the number as the sum of powers of 2, for example, $7 = 4 + 2 + 1$.

i 7 **ii** 15
iii 4 **iv** 20
v 72 **vi** 53

There is actually an easier way of turning decimal numbers into binary numbers. If we keep dividing by 2, then the list of remainders gives us the binary number.

81 (base 10)
$\underline{\div 2}$
$40 \rightarrow$ remainder 1
$\underline{\div 2}$
$20 \rightarrow$ remainder 0
$\underline{\div 2}$
$10 \rightarrow$ remainder 0
$\underline{\div 2}$
$5 \rightarrow$ remainder 0
$\underline{\div 2}$
$2 \rightarrow$ remainder 1
$\underline{\div 2}$
$1 \rightarrow$ remainder 0
$\underline{\div 2}$
$0 \rightarrow$ remainder 1
81 (base 10) = 1010001 (base 2)

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c Convert each of the decimal numbers in part **b** into binary numbers by the method of repeated division by two and recording the remainders. Compare your answers with the answers you obtained in part **b**.

d How old are you, if you express your age in years as a binary number?

e Expressed as a binary number, how old will you be in 3 years time?

Adding binary numbers

When we add two binary digits, there are only four possibilities:

$$\begin{array}{r} 0 \\ +\ 0 \\ \hline 0 \end{array} \qquad \begin{array}{r} 0 \\ +\ 1 \\ \hline 1 \end{array} \qquad \begin{array}{r} 1 \\ +\ 0 \\ \hline 1 \end{array} \qquad \begin{array}{r} 1 \\ +\ 1 \\ \hline 10 \end{array}$$

Notice that when we add the binary digits 1 and 1 we get 2. This means that we must carry 1 into the next place value. In any place value column we can have only a 0 or a 1.

This is just like the addition of base 10 numbers—we cannot have numbers greater than 9 in any place-value columns. If we add 6 and 4, for example, we must carry 1 into the next place value.

$$\begin{array}{r} {}^{1}1\,{}^{1}0\,{}^{1}1\,{}^{1}0\ 1 \\ +\quad 1\ 1\ 1\ 1 \\ \hline 1\ 0\ 0\ 1\ 0\ 0 \end{array}$$

Check in base 10:

10101 (base 2) = 21 (base 10)

1111 (base 2) = 15 (base 10)

21 + 15 = 36

100100 (base 2) = 36 (base 10)

f Add the following pairs of binary numbers. Check your answers by converting each of the numbers and your answer into decimal numbers.

i $\begin{array}{r} 100 \\ +\ \ 11 \\ \hline \end{array}$ **ii** $\begin{array}{r} 110 \\ +\ \ 11 \\ \hline \end{array}$ **iii** $\begin{array}{r} 110 \\ +\ 111 \\ \hline \end{array}$ **iv** $\begin{array}{r} 1010 \\ +\ 111 \\ \hline \end{array}$

v $\begin{array}{r} 1100 \\ +\ 101 \\ \hline \end{array}$ **vi** $\begin{array}{r} 11110 \\ +\ 101 \\ \hline \end{array}$ **vii** $\begin{array}{r} 101100 \\ +\ 101011 \\ \hline \end{array}$ **viii** $\begin{array}{r} 111111 \\ +\ \ 11111 \\ \hline \end{array}$

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14.5 Mersenne primes

Marin Mersenne was a French monk and mathematician who lived from 1588 to 1648. His room in a monastery in Paris became a meeting place for several philosophers and mathematicians, including the famous mathematicians Fermat and Pascal. Mersenne played an important part in spreading mathematical knowledge throughout Europe.

Mersenne investigated prime numbers and tried to find a formula that would represent all primes.

He did not succeed, but he found that if p is a prime number, $2^p - 1$ is often a prime number.

Mersenne's method was:

Raise 2 to any prime number power, then subtract 1.

For example, $2^2 - 1 = 3$, $\quad 2^5 - 1 = 31$,
$2^7 - 1 = 127$, etc.

However, Mersenne found that $2^p - 1$ does not always give a prime number.

Marin Mersenne

Mersenne tested all prime values of p up to 257. Checking to see whether large numbers have factors is not easy, particularly in Mersenne's time when there were no calculators or computers. Mersenne thought, for example, that $2^{67} - 1$ and $2^{257} - 1$ were prime, but these numbers have since been shown to be composite.

a Use your calculator to find the value of $2^p - 1$ for each of the values of p shown in the first column of the table on the next page. For the larger primes from 37 onwards you will need to use a calculator such as the Microsoft computer calculator that can display 32 digits. (Remember that in Mersenne's method, p had to be a prime number.)

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b Use the prime number calculator on the website http://www.math.princeton.edu/math_alive/Crypto/Lab2/Factorization.html to see if the Mersenne numbers in column 2 are prime. The calculator displays how long it takes to find if the number is prime and to display the factors if it is not prime. Don't be surprised if the calculator takes a few minutes to test the very large numbers.

Prime power, p	Mersenne number $2^p - 1$	Is the Mersenne number a prime number?
2		
3		
5		
7		
11		
13		
17		
19		
23		
29		
31		
37		
41		
43		
47		
53		
59		
61		
67		
71		
73		
79		
83		
89		
97		

c For which values of p in the table is Mersenne's conjecture not true?

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The Great Internet Mersenne Prime Search (GIMPS)

GIMPS, the Great Internet Mersenne Prime Search, was formed in January 1996 to use computers to discover new world-recordsize Mersenne primes.

You can find out more about the GIMPS on the website www.mersenne.org/prime.htm.

- **d** Find out the largest Mersenne prime that has been discovered. Write it in the form $2^p - 1$.
- **e** How many digits does it have?
- **f** When was this Mersenne prime found?

Why are prime numbers important?

When information is sent over the internet it can be sent in a coded 'scrambled' form to make it secure. This is called encryption. However, the person receiving the information must be able to decode the information. To encrypt information, the coded information could be 'locked' with a number key that only the sender and receiver know.

One way that this is done is to use two very large prime numbers as the *private keys* to the encryption. This is called RSA encryption after the three people who invented the method. The two prime numbers are multiplied together to give an even larger composite number (called the *public key*). If the prime numbers are large enough, the chance of someone working out the two factors (the private keys) is very small.

Using two small prime numbers as an example, multiplying 137 and 271 gives 37 127. The number 37 127 could then be used as the public key to 'scramble' or encrypt information. To unscramble the information you would need to know the private keys 137 and 271.

- **g** My public key is 209. Find my two 2-digit private keys.
- **h** My public key is 1517. Find my two 2-digit private keys.
- **i** Choose two 2-digit prime numbers and multiply them together using your calculator. This is your public key. See if your partner can find the two private keys without being told. If your partner cannot work out the private keys, tell them one of the two numbers and see if they can find the other one.
- **j** Now repeat with two 3-digit numbers.

Mersenne primes and perfect numbers

A perfect number is defined as a number whose factors (apart from itself) add up to itself. For example, 6 is a perfect number because $1 + 2 + 3 = 6$.

It has been found that for the Mersenne primes (prime numbers found by using the rule $2^p - 1$ where p is a prime number),

$(2^p - 1) \times (2^{p-1})$ is always a perfect number.

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For example, when $p = 2$,

$$2^p - 1 = 2^2 - 1$$
$$= 3 \text{ (the first Mersenne prime)}$$

and $(2^p - 1) \times (2^{p-1}) = (2^2 - 1) \times (2^{2-1})$

$$= 3 \times 2$$
$$= 6 \text{ (a perfect number)}$$

k The following table shows the Mersenne primes 3, 7, 31, 127 and 8191. Use a factor ladder to find the factors of the product $(2^p - 1) \times (2^{p-1})$ for each of the values of p. Add the factors to show that the product is a perfect number. Remember not to include the number itself. For example, for the factors of 6, add only 1, 2 and 3, even though 6 is a factor of 6.

p	Mersenne prime	$(2^p - 1) \times (2^{p-1})$	Factors of product	Sum of factors	Is $(2^p - 1) \times (2^{p-1})$ a perfect number?
2	3	$3 \times 2 = 6$	1, 2, 3	6	Yes
3	7	$7 \times 4 = 28$			
5	31				
7	127				
13	8191				

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14.6 Music fractions and decimals

The just music scale

The ancient Greek mathematicians had stringed musical instruments which were played by plucking the strings as on a guitar. They noticed that if a particular length of string gave a certain note, then if the string was shortened, the frequency of the note increased, that is, the note sounded higher in pitch. They also noticed that certain fractions of the string length seemed to give pleasing sounds when they were played together with the original note. Other fractions of the length gave unpleasant sounds when played with the original note.

The frequency of a note is related to the *reciprocal* of the length of the vibrating string.

- When the string was $\frac{1}{2}$ of the original length, the frequency was $\frac{2}{1}$ times the frequency of the original note.
- When the string was $\frac{2}{3}$ of the original length, the frequency was $\frac{3}{2}$ times the frequency of the original note.

  a If a string is $\frac{3}{4}$ of the length of the original string, what is the frequency of the note played by the shorter string compared with the original string?

If we start with the musical note C, then the frequency of the note F, for example, is $\frac{4}{3}$ times the frequency of C and the frequency of G is $\frac{3}{2}$ times the frequency of C. The difference between the sounds of the notes F and G is called a tone. The diagram below shows that there are also tones between C and D, D and E, G and A, and A and B. The note C' is one octave higher in pitch than C. The intervals between E and F and between B and C' are semitones.

C	D	E	F	G	A	B	C'
1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{2}{1}$

Tone Tone Semitone Tone Tone Tone Semitone

We can see that the frequency ratio for the tone C to D is $\frac{9}{8}$. We can calculate the frequency ratio for the other tones, as shown here for F to G.

$$\begin{aligned}\frac{\text{frequency of G}}{\text{frequency of F}} &= \frac{3}{2} \div \frac{4}{3} \\ &= \frac{3}{2} \times \frac{3}{4} \\ &= \frac{9}{8}\end{aligned}$$

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b Using the same method as above, calculate the frequency ratios for the following tones.

i D to E

ii G to A

iii A to B

c Is the frequency ratio the same for every tone interval?

d Using the frequency ratios in the above diagram, calculate the frequency ratio for the semitones

i E to F

ii B to C'

In between the notes that are separated by a tone, there are semitones. The diagram below shows all the semitones and their frequency ratios. The symbol # is read as 'sharp' so C# is read as 'C sharp'. This musical scale based on fractions is called the **just scale**.

C	C#	D	D#	E	F	F#	G	G#	A	A#	B	C'
1	$\frac{16}{15}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{9}{5}$	$\frac{15}{8}$	$\frac{2}{1}$

Semitone Semitone Semitone Semitone Semitone Semitone Semitone Semitone Semitone Semitone Semitone Semitone

e Calculate the frequency ratios for each of the following semitones.

i C to C# **ii** C# to D **iii** D to D#

iv D# to E **v** F to F# **vi** F# to G

vii G to G# **viii** G# to A **ix** A to A#

x A# to B

f Is the frequency ratio the same for every semitone interval?

The equal-tempered scale

A problem with the just scale is that there are different fraction relationships between equivalent intervals of notes. In the 17th century, in the time of the composer Bach, musicians became concerned that if they tuned their keyboards for one key, then it was not in tune for another key. It was decided that the intervals between all the semitones should be the same. This was achieved by making each semitone interval equal to the 12th root of 2. We write this as $\sqrt[12]{2}$. Like many square roots, cube roots, and so on, the 12th root of 2 is an irrational number.

The 12th root of 2 = 1.05946…

Scales based on the 12th root of 2 were called **equal-tempered scales**. This method of tuning was a compromise which allowed keyboards to be tuned for all keys without too much distortion to the pleasing sound of chords. Bach wrote *The well-tempered clavier* to prove that the well-tempered tuning worked.

g The y^x key on your calculator is used to raise numbers to powers. Check the value of $\sqrt[12]{2}$ by using the sequence of keys 2 y^x (1/12)

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For the *just scale* (based on fractions) the frequency ratio for the interval C to C# is $\frac{16}{15} \approx 1.067$. For the *equal-tempered scale*, the frequency ratio for C to C# is $\sqrt[12]{2} \approx 1.059$. This means that C# on the equal-tempered scale would not be quite as high in pitch as on the just scale.

h Copy and complete the following table to compare the frequencies of the notes for the just scale and the equal-tempered scale.

Note	Frequency ratio for notes compared with C correct to three decimal places	
	Just scale	Equal-tempered scale
C	1	1
C#	$\frac{16}{15} = 1.059$	$\sqrt[12]{2} = 1.067$
D	$\frac{9}{8} = 1.125$	$\left(\sqrt[12]{2}\right)^2 = 1.122$
D#	$\frac{6}{5} =$	$\left(\sqrt[12]{2}\right)^3 =$
E	$\frac{5}{4} =$	$\left(\sqrt[12]{2}\right)^4 =$
F	$\frac{4}{3} =$	$\left(\sqrt[12]{2}\right)^5 =$
F#	$\frac{45}{32} =$	$\left(\sqrt[12]{2}\right)^6 =$
G	$\frac{3}{2} =$	$\left(\sqrt[12]{2}\right)^7 =$
G#	$\frac{8}{5} =$	$\left(\sqrt[12]{2}\right)^8 =$
A	$\frac{5}{3} =$	$\left(\sqrt[12]{2}\right)^9 =$
A#	$\frac{9}{5} =$	$\left(\sqrt[12]{2}\right)^{10} =$
B	$\frac{15}{8} =$	$\left(\sqrt[12]{2}\right)^{11} =$
C'	$\frac{2}{1} =$	$\left(\sqrt[12]{2}\right)^{12} =$

i List the notes for which the equal-tempered frequency is higher, that is the note is sharper, than for the just scale.

j List the notes for which the equal-tempered frequency is lower, that is the note is flatter, than for the just scale.

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14.7 Water tank costs

Water tank costs

The table below shows the sizes and prices of a number of cylinderical polyethylene water tanks. Your task is to find out how the price of a water tank is determined. Does the price depend on how much water the tank can hold or does it depend on how much polyethylene is needed to make the tank?

A spreadsheet *Water tanks* in the student and teacher ebooks includes the data in the table in columns A, B and C.

- **a** For the first tank in the table, work out the volume in cubic millimetres. Hence, find the capacity in litres to the nearest litre.
- **b** For the first tank in the table, work out the surface area in square metres, correct to two decimal places.
- **c** Enter the heading *Capacity (L)* in column D. In the cell below the heading, enter a formula for finding the capacity of a tank, then fill down. Hint: think about the steps you used to find the capacity in part **a**.
- **d** Enter the heading *Surface area (sq m)* in column E. In the cell below the heading, enter a formula for finding the surface area of a tank, then fill down. Hint: think about the steps you used to find the surface area in part **b**.
- **e** Graph the cost in dollars against the capacities in litres. Use a scatter plot where the points are not connected.

Cost of water tanks

Diameter (mm)	Height (mm)	Cost ($)
500	670	148.50
600	1150	255.00
800	1220	268.40
800	1550	368.50
1130	1300	434.50
1200	1760	572.00
1300	1800	619.30
1400	1800	616.00
1800	1750	869.00
2110	1655	944.90
2110	2260	1212.75
2900	2700	1969.00
3500	2400	2227.50
3500	3000	2640.00

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f Is the graph of cost against capacity linear?

> Capacity is the independent variable, so it is shown on the horizontal axis.

g Graph the cost in dollars against the surface area in square metres.

> Here, the surface area is the independent variable.

h Is the graph of cost against surface area linear?

i Find the rule for this graph. This rule tells how the cost of a tank can be calculated.

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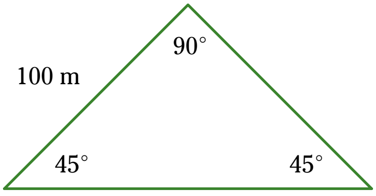
14.8 Fencing the quarry

A land developer has been asked by the local council to fence off a disused stone quarry to prevent people falling into it. The fence has to be at least 30 m from the edge of the quarry all the way round. The land developer wants a fence that is exactly 30 m from the edge so that he has the most land left unfenced.

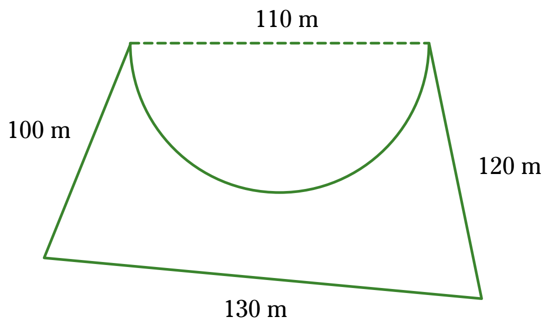


a Copy the triangle and draw the fence around it.

b Find the total length of the fence.

c If the edge of the quarry were a quadrilateral with all its interior angles less than 180° and with sides 100 m, 110 m, 120 m and 130 m, what would be the length of the fence?

d If the edge of the quarry were this shape, where the curved edge is a semicircle, what would be the length of the fence?



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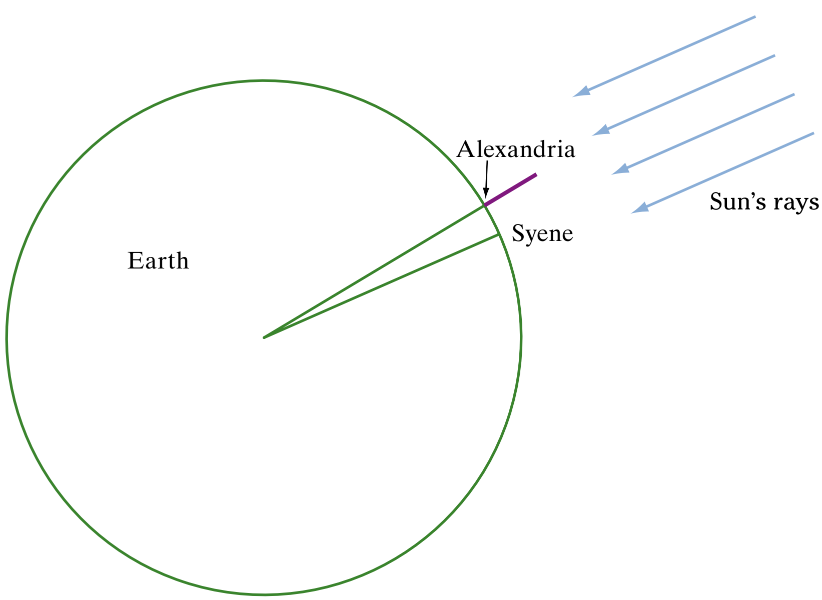
14.9 Eratosthenes and the radius of the Earth

Eratosthenes was an astronomer and mathematician who lived from about 270 BCE to 194 BCE (about 2200 years ago) in Alexandria in northern Egypt. At that time Alexandria was part of the Greek empire. You have probably come across Eratosthenes' sieve for finding prime numbers.

Eratosthenes is also known for his calculation of the circumference and radius of the Earth. When visiting the town of Syene (now called Aswan) he noticed that the sun was directly overhead on a certain day of the year. On this day, when Eratosthenes looked down a well in Syene at noon, the Sun's rays reflected straight back into his eyes. He made the assumption that, because the Sun was so far away, he could regard the Sun's rays as parallel.

With these two pieces of information, Eratosthenes devised a method for calculating the radius of the Earth. He placed a stick vertically in the ground at Alexandria on the day when he knew the Sun was directly overhead in Syene. At noon, he measured the angle subtended by the shadow of the stick and found that it was approximately one-fiftieth of a circle, that is, 7.2°.



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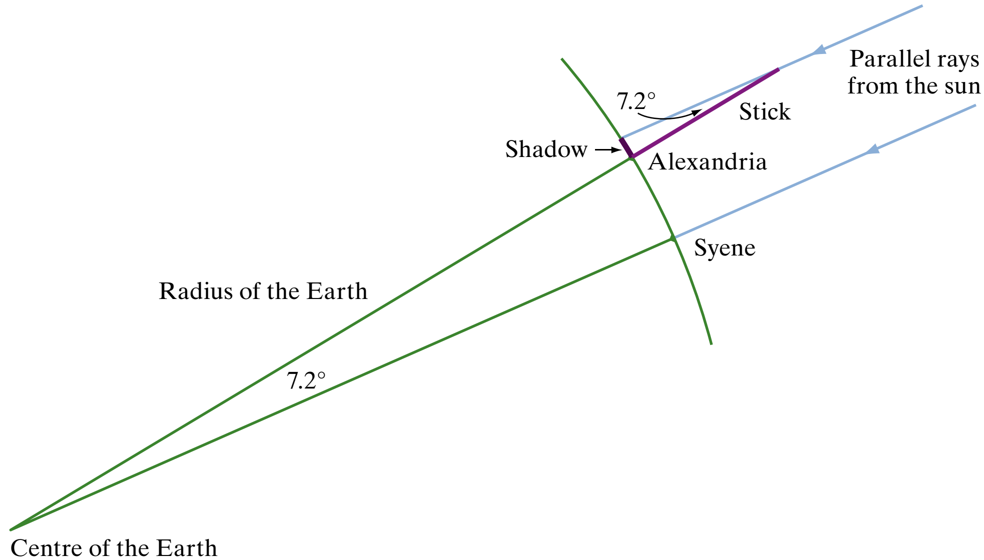

Eratosthenes was then able to say that the angle at the centre of the Earth subtended by the arc from Alexandria to Syene was also one-fiftieth of a circle (that is, 7.2°). As Eratosthenes knew the distance from Alexandria, he was then able to calculate the circumference and radius of the Earth.

a Having measured the angle subtended by the shadow as one-fiftieth of a circle, why was Eratosthenes also able to say that the angle subtended at the centre of the Earth was one-fiftieth of a circle?

b At the time of Eratosthenes, distances were measured in units called stades. One stade is equivalent to a distance of approximately 5000 stades. What is the distance from Alexandria to Syene in kilometres?

c Using the angle measured by Eratosthenes and the distance in kilometres between Syene and Alexandria, what value would Eratosthenes have obtained for the circumference of the Earth?

d Based on this circumference, what value would Eratosthenes have obtained for the radius of the Earth?

e If the radius of the Earth is now known to be approximately 6370 km, what was the percentage error in Eratosthenes' estimate?

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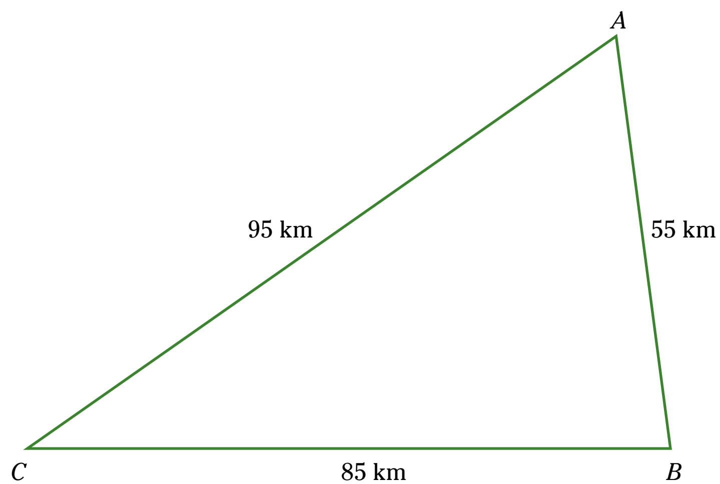
14.10 Locating a power station

BLM
Locating the power station

There are many situations where it is important to find the shortest paths linking a number of places. The lengths of electric transmission lines and oil or gas pipe lines, for example, need to be kept as short as possible to minimise costs. In underground mines the tunnels linking the shafts are expensive to construct so it is important to make them as short as possible. Sometimes, the shortest paths can be calculated mathematically, but in some situations it is very difficult to work out shortest paths.

Supposing a new power station is to be constructed to provide electricity for towns A, B and C. The length of the power lines connecting each of the towns to the power station needs to be minimised to reduce the cost. A scale map of the three towns is shown below.



Scale 1 mm:1 km

a Copy the map carefully into the centre of a piece of A3 paper and decide where you think the power station ought to be. When you have chosen a position for the power station, draw line segments from your power station to each of the three towns and measure their total length.

b Use the scale to find the actual length of power lines. Compare your lengths with those of others in your group or class.

One way of investigating shortest paths makes use of the fact that soap films forming between a set of vertices always represent the shortest path between the points. The following method was used to obtain the photographs on the next page. You may like to try this yourself, either in your mathematics lesson or in the kitchen at home.

- Make three small holes in identical positions in each of two clear plastic lids such as those used on cream pots.

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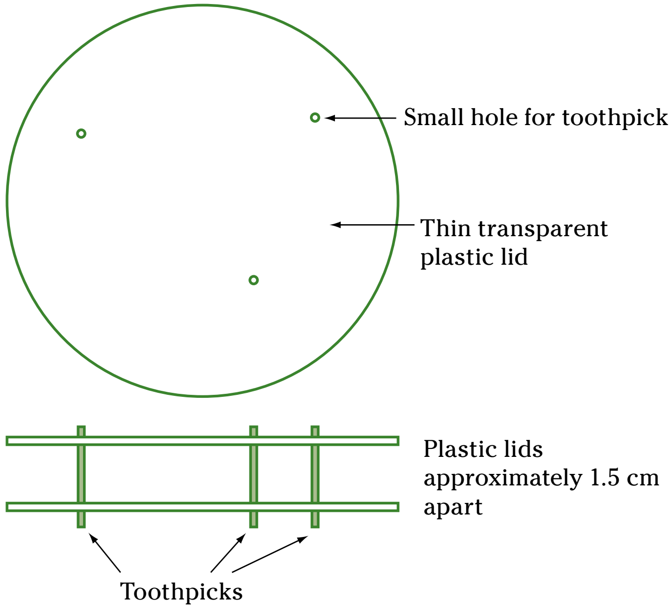
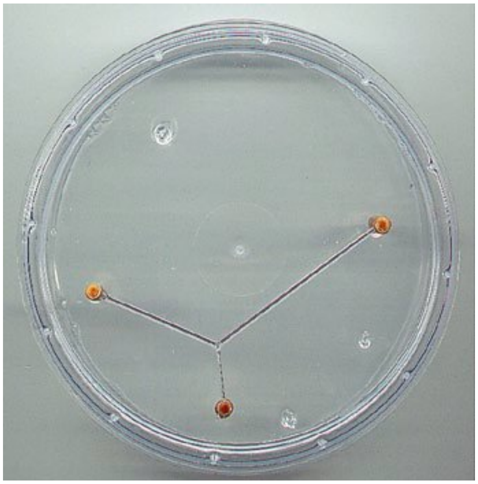
- Push pointed wooden toothpicks through the holes so that the two lids are about 1.5 cm apart as shown in the diagram below. (It's a good idea to snip the sharp ends from the toothpicks first so that you don't injure your hands.)



- Fill a dish with water and add a few drops of dishwashing liquid. Gently mix the detergent and water, being careful not to make the detergent foam.
- Gently immerse the pair of lids into the water so that they are completely covered with water, then lift out carefully. There should be a soap film connecting the three toothpicks. You may need to try several times as sometimes the film joins only two of the vertices instead of all three.

The photographs below show the soap films formed for two different sets of triangle vertices.



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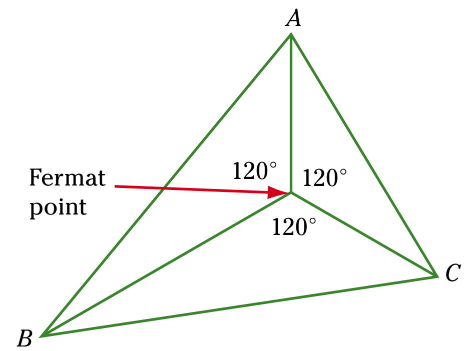
c Notice how the soap films connected to each vertex of the triangle meet at a point (a 'node') inside the triangle. Use a protractor to measure the three angles at the point where the soap films meet. This is the point in the triangle where the sum of the lengths of the three straight sections of the soap film is shortest.

The point in a triangle where the sum of the distances to each of the three vertices is the shortest is called the Fermat point of the triangle. It is the point where the line segments from the three vertices meet at angles of 120°.

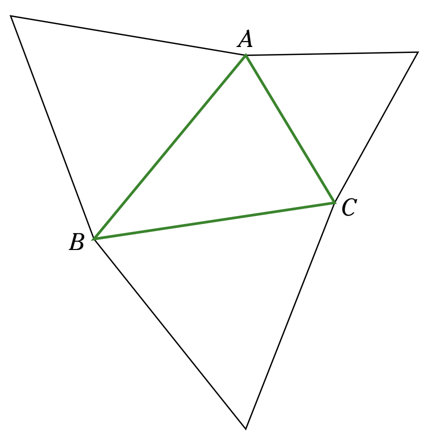


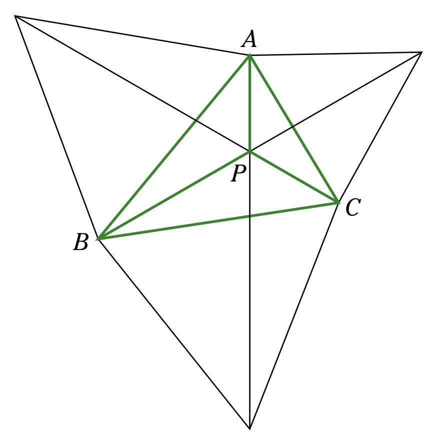
Finding the Fermat point of a triangle

BLM
Proof of Fermat's theorem

To find the Fermat point of a triangle, we construct an equilateral triangle on each of the three sides of the triangle, then join the outer vertex of each equilateral triangle to the opposite vertex of $\triangle ABC$ as shown. The point, P, where these three line segments intersect is the Fermat point of the triangle.

$AP\ BP\ CP$ represents the shortest total distance to each of the three vertices.





Locating the power station

d Go back to the map of the three towns. Use the method described above to locate the Fermat point for the triangle and then measure the total distance of power lines on your map.

e Use the scale to calculate the actual length of power lines.

f Compare your answer to part **e** with your results from part **b**. How close was the position you chose to the Fermat point?

A proof for the position of the Fermat point is included in the students and teacher ebooks.

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What happens if there are more than three vertices?

g The photograph below shows the soap film which formed when there were four vertices instead of three. Measure the angles formed by the soap film. How many nodes are there?

h Predict what might happen if there were five or six vertices. How many modes do you think there might be in each case? If possible test your predictions.

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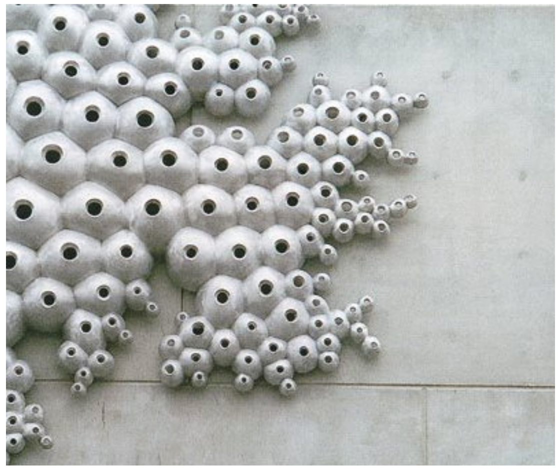
14.11 Honeycombs and bubbles

This photograph shows a sculpture called *Colony*, by sculptor Simon Perry.



Although Simon Perry's inspiration for the sculpture came from the idea of growth, the design also resembles the formation of bubbles in a foam or cells in a honeycomb. Notice that many of the shapes are hexagonal.

The hexagon shape that occurs in honeycomb has aroused interest for at least 2000 years. The mathematician Pappus of Alexandria compared the three regular polygons that tessellate: the equilateral triangle, the square and the hexagon. He concluded that if the same quantity of material is used to construct cells based on these shapes, it is the hexagonal cells that will hold more honey. This became known as the **honeycomb conjecture**.

According to the honeycomb conjecture, for a given perimeter, the regular hexagon should have a larger area than other shapes with the same perimeter.

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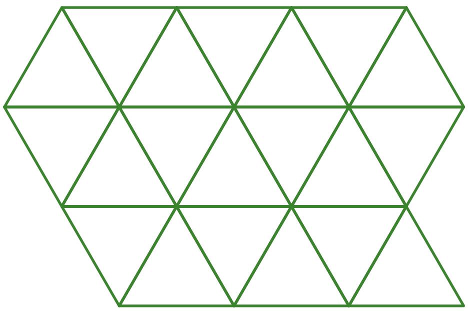
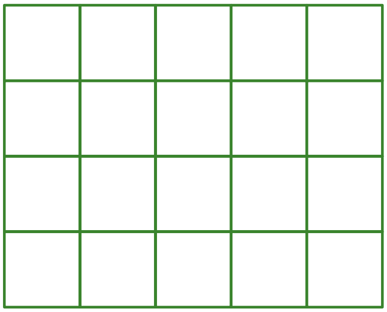
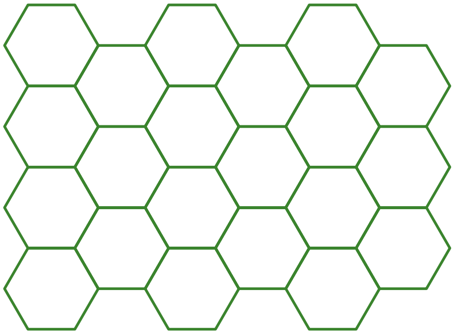
In this investigation, you will compare the area for equilateral triangles, squares and regular hexagons that each have a perimeter of 12 cm.







a What is the side length of a square with perimeter 12 cm?

b Calculate the area of the square.

c What is the side length of an equilateral triangle with perimeter 12 cm?

d Using any appropriate method, calculate the area of the equilateral triangle, correct to one decimal place.

e What is the side length of a regular hexagon with perimeter 12 cm?

f Using any appropriate method, calculate the area of the regular hexagon, correct to one decimal place.

g Copy and complete the following table. Do the results support the honeycomb conjecture?

Areas of polygons with perimeters of exactly 12 cm	
Equilateral triangle	
Square	
Regular hexagon	

We have considered only regular polygons. In order to prove the honeycomb conjecture, shapes other than regular polygons, including shapes with curved edges, must be considered. It was not until 1998 that Thomas C. Hales gave a proof of the honeycomb conjecture.

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14.12 Braking distance

The rule for calculating the braking distance d m of a car travelling at S km/h is

$$d = \frac{S^2}{254f}$$

where f is the friction value for the friction between the tyres and the road surface.

The total stopping distance D m is the sum of the braking distance and the distance travelled by the car before braking, dependent on the driver's reaction time. This time is normally taken to be 2.5 seconds. A speed of S km/h is equivalent to a speed of $\frac{S \times 1000}{3600}$ m/sec, so the distance travelled in 2.5 seconds would be $\frac{S \times 1000 \times 2.5}{3600}$ m, that is, $\frac{25S}{36}$ m. Adding these two distances gives the rule

$$D = \frac{S^2}{254f} + \frac{25S}{36}.$$

a Set up a spreadsheet with the headings shown here.

◇	A	B	C	D
1	Speed S (km/h)	Braking distance d (m)	Reaction distance (m)	Total stopping distance D (m)
2	0	(m)		
3	5			
4	10			
5	15			
6	20			
7	25			
8	30			

Using $f = 0.75$ for the friction value, enter a formula

- in cell B2 to calculate the braking distance d
- in cell C2 to calculate the distance travelled during the reaction time
- in cell D2 to calculate the total stopping distance D m.

Fill down the columns and extend the spreadsheet to find the total stopping distance for speeds up to 120 km/h. Round columns B, C and D to one decimal place.

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b Which is the dependent variable?

c Which is the independent variable?

d Select all four columns of data and construct an XY scatter graph. You should have three graphs on the one set of axes.

e Which of the graphs is linear?

f What is the total stopping distance at a speed of 100 km/h? How does this compare with the total stopping distance at 70 km/h?

g Which part of the total distance—the braking distance or the reaction distance—contributes most to the total stopping time?

h What difference do you think a wet road surface would make to the value of f?

i What effect would this have on total stopping time?

j What other factors apart from a wet road surface would affect the value of f?

ISBN 978 1 4202 2963 9

14.13 Standard drinks

There are many different kinds of alcohol, most of them used for industrial purposes. The alcohol in alcoholic drinks is called ethanol. It is formed by the fermentation of various plant products, such as hops, barley and grapes.

As ethanol is highly dangerous in large quantities, alcoholic drinks contain only small percentages of ethanol. The remaining liquid is mainly water, with various substances from the fermentation and production processes giving each drink its characteristic taste.

The amount of ethanol in a drink depends on

- the size of the drink
- the concentration of ethanol in the drink.

The alcohol content of various drinks must conform with government standards. In Australia, a **standard drink** is defined as a drink that contains 10 g of alcohol. The density of alcohol is approximately 0.8 g/mL, so 10 g of alcohol is equivalent to approximately 12 mL of pure alcohol.

The chart on the facing page shows the number of standard drinks that various alcoholic drinks are equivalent to.

a Which of the drinks shown contains the lowest percentage of alcohol?

b Which of the drinks shown contains the highest percentage of alcohol?

c Full strength beer contains 4.9% alcohol. What volume of alcohol is in a 375 mL can of beer?

d Explain why this is equivalent to approximately 1.5 standard drinks.

e If the label on a particular 750 mL bottle of wine states that it contains 13% alcohol, explain why the label also states that it contains 7.5 standard drinks.

f If a 750 mL bottle of wine contains approximately 8 standard drinks, what volume of wine would be equivalent to one standard drink? Estimate where this volume would come to in a 200 mL cup.

g If a 700 mL bottle of spirits contains approximately 23 standard drinks, what volume of spirits would be equivalent to one standard drink? Estimate where this volume would come to in a 200 mL cup.

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14.13

Note. Labels on alcoholic drink containers state the number of Standard Drinks in the container.

Check the label to find out how many Standard Drinks are in the bottle or can.

The Standard Drinks shown are calculated to one decimal place. To make counting your drinks easier, you can round the numbers up or down. For example count 0.9 as 1.0 and 1.6 as 1.5.

ISBN 978 1 4202 2963 9

Answers

Chapter 1

exercise 1.1

1 a 4^5 b 10^8
c $3^7 \times 7^3$ d $2^3 \times 3^5 \times 5^2$

2 a a^4 b k^5 c x^3y^4
d a^2b^5 e $3^2m^3n^2$ f $2^2c^4d^3$
g $200a^4b^4$ h $288d^3k^5$

3 a $2\times2\times2\times2\times5\times5\times5$
b $3\times3\times3\times3\times3\times7\times7\times7\times7\times11\times11$
c $3\times a\times a\times a\times b\times b$
d $5\times m\times m\times m\times m\times n\times n\times n$
e $2\times d\times d\times d\times d\times d\times y$
f $7\times a\times a\times m\times m\times m\times m\times m$
g $5\times a\times b\times b\times b\times b\times b\times b\times b$
h $7\times m\times m\times m\times p$
i $13\times x\times x\times x\times x\times y\times y\times y\times y\times y$

4 a 4^5 b 11^{10} c 2^{21} d 5^9
e 10^{18} f 5^4 g 10^7 h 7^4
i 3^8 j 10^7 k 6^6 l 2^8

5 a $3^5\times2^{12}$ b $5^6\times2^6$ c $2^{11}\times7^8$ d $3^5\times10^5$
e $2^7\times5^3$ f $2^8\times7^3$ g $3^5\times5^2$ h $2^5\times3^5\times5^3$
i $3^2\times7^9$

6 a a^7 b m^7 c x^7 d b^5 e r^7s^{12}
f v^8w^{10} g m^4n^5 h $x^{12}y^8$ i g^6h^8 j $a^{11}b^6$
k x^7y^8 l a^5bc^7

7 a 12^4 b 8^2 c 10^{11} d 3^4 e 2^4 f 10^7
g 5^4 h 4 i 3^5 j 5^2 k 2^8 l 3^6

8 a $2^2\times3^3$ b $5^2\times7^4$ c $2^4\times7^4$ d $5^6\times11^3$

9 a a^3 b k^5 c m^8 d $4y^4$
e $3x^4$ f $\frac{7b^6}{2}$ g 4 h $\frac{3a^6}{2}$
i m^3n j a^3b^5 k xy^2 l h^4k^4

10 a 3^6 b 10^6 c $4^2\times9^2$ d 5^2
e 8^2 f 6^2 g 7^4 h 3
i 6^5 j $2^3\times7^3$ k $3^4\times11^2$ l 3^3

11 a 27 b 25 c 64 d 36

12 a e^3 b t^5 c d^6 d y^2 e u^4 f g^5
g e^{22} h n i a^2 j t k w^6 l c^4

13 a $4ab$ b $3ab^2$ c $\frac{4x^2y}{3}$ d $\frac{a^2b^2}{3}$
e $\frac{3x^3}{2}$ f bx^3 g $\frac{3a^4b^2}{2}$ h $\frac{3xy}{4}$

14 a 1 b 1 c 2 d 1 e 9 f 4
g 1 h 26 i 9 j 1 k a l 5
m $3x$ n 2 o -3 p 0

15 a i 3 ii 4 iii 5 iv 6
b an index of 0.5 means the square root, e.g.
$x^{0.5}\times x^{0.5} = x^{0.5+0.5} = x^1 = x$

exercise 1.2

1 a 3^6 b 3^6 c 10^{10} d 4^9 e 2^{20}
f 3^0 g 3^0 h 10^{12} i 5^8 j 7^0
k 2^{15} l 6^0

2 a a^8 b a^8 c x^{12} d x^{12} e b^0
f b^0 g m^{15} h y^{10} i x^{15} j d^{24}
k k^9 l t^{16}

3 D

4 a $4b^2$ b $81y^4$ c $32a^5$
d $125x^3$ e 1 f $64h^3$
g $243x^5$ h $1\,000\,000k^6$ i $216a^3$
j $256x^8$ k $256n^4$ l 1

5 a a^3b^3 b x^4y^4 c a^6b^3
d x^2y^4 e $m^{12}n^8$ f c^6d^8
g $16x^8y^4$ h $27k^{12}t^9$ i $81a^8b^{12}$
j $32m^5p^{15}$ k $121x^4y^6$ l $125c^{12}d^3$

6 a $\frac{1}{8}$ b $\frac{1}{9}$ c $\frac{8}{125}$ d $\frac{81}{625}$
e $\frac{27}{64}$ f $\frac{343}{1000}$ g $\frac{256}{2401}$ h $\frac{64}{125}$
i $\frac{1}{64}$ j $\frac{81}{16}$ k $\frac{32}{243}$ l $\frac{25}{64}$

7 a $\frac{a^3}{b^3}$ b $\frac{x^4}{y^4}$ c $\frac{a^4b^4}{c^4}$ d $\frac{c^3d^3}{e^6}$
e $\frac{x^2}{4}$ f $\frac{a^3}{27}$ g $\frac{4y^2}{25}$ h $\frac{9a^2}{16}$

8 a $\frac{p^4q^8}{r^4}$ b $\frac{8x^6}{y^9}$ c $\frac{4m^2}{9n^2}$ d $\frac{4a^2}{9b^2}$
e $\frac{81x^8}{16y^4}$ f $\frac{c^6d^9}{8}$ g $\frac{625e^{12}h^4}{16}$ h $\frac{27x^3y^6}{64}$

9 a 54 b 288 c 16
d 216 e 576 f 65 610 000
g 1.8 h 15

10 a i 4 ii 9 iii 16 iv 25
b $a^{\frac{1}{2}\times2} = a^1 = a$ c cube root of a
d $8^{\frac{1}{3}} = 2$, $27^{\frac{1}{3}} = 3$ so confirms conjecture in part c

exercise 1.3

1 a $\frac{1}{6^3}$ b $\frac{1}{2^8}$ c $\frac{1}{4^7}$ d $\frac{1}{3}$
e 3^2 f 2^5 g 5^2 h 6^3

2 a $\frac{1}{a^2}$ b $\frac{1}{b^6}$ c $\frac{1}{x^3}$ d $\frac{1}{m^5}$
e y^4 f n^2 g x^6 h k^3

3 a $\frac{1}{16}$ b $\frac{1}{25}$ c $\frac{1}{1000}$
d $\frac{1}{1\,000\,000}$ e 3 f 8
g 81 h 1 000 000 i 8

j $1\frac{7}{9}$ **k** $6\frac{1}{4}$ **l** 32

4 **a** 2^{-1} **b** 3^{-5} **c** 5^{-2} **d** 10^{-3}
e 10^{-9} **f** 10^{-7} **g** 7^{-5} **h** 2^{6}
i 1 **j** 3^{3} **k** 2^{3} **l** $3^{5}\times 5^{-7}$

5 **a** 2^{12} **b** 3^{13} **c** 5^{8} **d** 10^{12}
e 10^{-9} **f** 10^{19} **g** 2^{-11} **h** 10^{-21}
i 3^{-4} **j** 2^{3} **k** 4^{3} **l** 5^{6}

6 **a** p **b** $\frac{1}{a^{15}}$ **c** $\frac{1}{m^{3}}$ **d** b^{2}
e $\frac{1}{a^{6}}$ **f** x **g** $\frac{1}{y^{6}}$ **h** x^{-6}

7 **a** x^{3} **b** y^{13} **c** $\frac{1}{m^{3}}$ **d** m
e b **f** $\frac{1}{m^{4}}$ **g** $\frac{1}{c^{10}}$ **h** $\frac{1}{p^{6}}$

8 **a** 2 **b** 5^{2} **c** 10^{-2} **d** 10^{30}
e 3^{2} **f** 1 **g** 6^{9} **h** 3^{-6}
i 2^{-7} **j** 2^{-13} **k** 3^{8} **l** 5^{9}

9 **a** p^{-9} **b** m^{2} **c** p^{-10} **d** 1
e f **f** b^{-7} **g** d^{-5} **h** n^{-3}

10 **a** **i**
$$\begin{aligned}\frac{p^3}{p^6}\div\frac{p^2}{p^8} &= \frac{p^3}{p^6}\times\frac{p^8}{p^2}\\ &= \frac{p^3\times p^8}{p^6\times p^2}\\ &= \frac{p^{11}}{p^8}\\ &= p^3\end{aligned}$$

ii
$$\begin{aligned}\frac{p^3}{p^6}\div\frac{p^2}{p^8} &= p^{3-6}\div p^{2-8}\\ &= p^{-3}\div p^{-6}\\ &= p^{-3-(-6)}\\ &= p^3\end{aligned}$$

b **i**
$$\begin{aligned}\frac{x}{x^9}\div\frac{x^3}{x^{15}} &= \frac{x}{x^9}\times\frac{x^{15}}{x^3}\\ &= \frac{x\times x^{15}}{x^9\times x^3}\\ &= \frac{x^{16}}{x^{12}}\\ &= x^4\end{aligned}$$

ii
$$\begin{aligned}\frac{x}{x^9}\div\frac{x^3}{x^{15}} &= x^{1-9}\div x^{3-15}\\ &= x^{-8}\div x^{-12}\\ &= x^{-8-(-12)}\\ &= x^4\end{aligned}$$

11 **a** $\frac{12}{x}$ **b** $\frac{14}{m^3}$ **c** $\frac{30}{a^4}$ **d** $-12x$
e $\frac{3a^2}{2}$ **f** $\frac{6x^9}{5}$ **g** $\frac{3b^3}{2}$ **h** $\frac{3x^6}{2}$

12 $\frac{1}{4374}$

exercise 1.4

1 **a** 1.2×10^{3} **b** 4.25×10^{5} **c** 3.6×10^{13}
d 1.75×10^{21} **e** 8.6×10^{7} **f** 3.864×10^{7}
g 1.86478×10^{5} **h** 8.7216×10^{3} **i** 4.784×10^{2}
j 1.845×10^{1} **k** 3.9267×10^{3} **l** 2.6732×10^{1}
m 4.682×10^{2} **n** 3.8×10^{6} **o** 1.75894×10^{4}
p 3.94768×10^{3} **q** 6.743×10^{9} **r** 9.5×10^{12}

2 **a** 5×10^{3} **b** 6×10^{6}
c 2.3×10^{7} **d** 5.96×10^{5}
e 1.94×10^{5} **f** 2.57×10^{8}
g 5.8×10^{10} **h** 1.7×10^{5}
i 1.86×10^{7} **j** 2.7×10^{6}
k 1.23×10^{10} **l** 5.8×10^{9}

3

Planet	Average distance from the Sun (km)
Mercury	5.8×10^{7}
Venus	1.08×10^{8}
Earth	1.5×10^{8}
Mars	2.28×10^{8}
Jupiter	7.78×10^{8}
Saturn	1.429×10^{9}
Uranus	2.871×10^{9}
Neptune	4.504×10^{9}

4 **a** 60 000 **b** 2000
c 600 000 **d** 90 000 000
e 7 143 000 **f** 80 900 000 000
g 40 160 000 **h** 10 800

5 **a** 8.1774×10^{11} **b** 1.326×10^{10}
c 1.6×10^{3} **d** 2.25×10^{4}

6 **a** 6.5×10^{-3} **b** 7.2×10^{-5}
c 1.0×10^{-5} **d** 9.0×10^{-9}
e 5.03×10^{-3} **f** 7.3×10^{-10}
g 9.12×10^{-6} **h** 8.34×10^{-16}
i 7.3×10^{-12} **j** 7.02×10^{-5}
k 8.4×10^{-8} **l** 9.55×10^{-11}
m 1.02×10^{-8} **n** 7.5×10^{-15}
o 3.2×10^{-9}

7 **a** 0.00376 **b** 0.0049
c 0.0000068 **d** 0.0002185
e 0.0000132 **f** 0.000000051
g 0.00000040 **h** 0.09925

8 **a** 7.44×10^{-8} **b** 2.135×10^{-6}
c 1.2×10^{-10} **d** 1.344×10^{-6}
e 3×10^{-8} **f** 4.152×10^{-7}
g 3.2×10^{-10} **h** 1.472×10^{-6}
i 2.8×10^{-6} **j** 3.7×10^{-9}
k 3.7×10^{-3} **l** 3.465×10^{-8}

9 **a** 8.1774×10^{11} **b** 1.326×10^{10}
c 1.6×10^{3} **d** 2.25×10^{4}

10 5.344×10^{-23} g

11 3.2×10^{11}

12 9.4608×10^{12} km

13 $1.270846648 \times 10^{12}$

14 2×10^{-6} cm to 4×10^{-5} cm

15 **a** 5×10^{9}
b 5×10^{12}
c 3×10^{13}
d 8.1×10^{21}
e 0.0079 mm

exercise 1.5

1 one decimal place

2 **a** 3 **b** 2 **c** 5 **d** 2
e 4 **f** 3 **g** 3 **h** 2
i 4 **j** 3 **k** 6 **l** 4

3 **a** **i** 47 **ii** 47.34
b **i** 170 **ii** 170.88
c **i** 5.1 **ii** 5.07
d **i** 0.000088 **ii** 0.00
e **i** 110 **ii** 107.65
f **i** 3.0 **ii** 3.01
g **i** 800 **ii** 800.40
h **i** 200 **ii** 200.19
i **i** 1.0 **ii** 1.01
j **i** 130 **ii** 130.42
k **i** 5.0 **ii** 5.00
l **i** 0.0075 **ii** 0.01

4 **a** 1870 **b** 2.4 **c** 160
d 1 000 000 **e** 0.0008 **f** 20 100
g 4.008 **h** 20.08 **i** 10
j 6.0 **k** 0.00059200 **l** 4.01

5 **a** 0.000604 **b** 18 000 **c** 196 000
d 60.1 **e** 8.48 **f** 20.1
g 480 000 000 **h** 0.0000607 **i** 72.1
j 10.0 **k** 380 000 **l** 1.00

6 **a** 1959 **b** 46.9 **c** 0.129 **d** 0.078

7 **a** 6.94 cm^2 **b** 25 m^2 **c** 190 cm^3 **d** 2.96 m

8 **a** 8.60 **b** 4.16 **c** 0.018 **d** 0.0001

Revision

1 B

2 C

3 D

4 B

5 C

6 **a** 3^8 **b** 3^{-4} **c** 7^8 **d** 5^5

7 **a** $\frac{1}{2}$ **b** 64 **c** 0.0001 **d** 25

8 **a** 17 **b** 16 **c** 0.0001 **d** 81

9 **a** m **b** $2pq^3$ **c** m^3n^2 **d** $2a^6b^4$
e s^6 **f** ab^3

10 **a** 3^{11} **b** a^{12} **c** x^4 **d** 8^5
e 3^6 **f** m^{20} **g** $15a^2b^5$ **h** $60x^7y^7$
i $4a^2$ **j** $3b^3$ **k** $8y^3$ **l** $24d^5e^6$

11 **a** 1 **b** 5 **c** $2p$ **d** 1 **e** 1 **f** 4

12 **a** k^{20} **b** $25x^2y^8$ **c** 1
d $\dfrac{8a^9}{27b^6}$ **e** $\dfrac{m^6n^8}{100}$ **f** $40a^{14}b^7$

13 **a** x^{13} **b** $\dfrac{1}{m^3}$ **c** $\dfrac{1}{a^4b^2}$
d $\dfrac{15b^5}{a^3}$ **e** p^7 **f** $\dfrac{12}{x^7y}$

14 **a** x^5y^8 **b** x^5 **c** 5

15 **a** 1.089×10^2 **b** 7.6×10^{-10}
c 5.94×10^8 **d** 4.2×10^{-5}

16 **a** 15 600 000 **b** 0.00045
c 86 700 000 000 **d** 0.000000060

17 **a** 8×10^{-4} **b** 4.736×10^7
c 1.05×10^{-4} **d** 2.24×10^{-2}

18 **a** 28 800 000 **b** 184
c 4.01 **d** 0.00730

19 **a** 4.5×10^{12} m **b** 9.4608×10^{15} m
c 4.76×10^{-4} light years **d** 1.5×10^4 s

Chapter 2

exercise 2.1

1 **a** m **b** y **c** g **d** j

2 a and c are Pythagorean triples.
a $8^2 + 15^2 = 64 + 225 = 289$
$17^2 = 289$
c $7^2 + 24^2 = 49 + 576 = 625$
$25^2 = 625$

3 C

4 **a** **i** 6, 8, 10 **ii** 9, 12, 15
iii 12, 16, 20 **iv** 15, 20, 25
b **i** 10, 24, 26 **ii** 15, 36, 39
iii 20, 48, 52 **iv** 25, 60, 65
c **i** 16, 30, 34 **ii** 24, 45, 51
iii 32, 60, 68 **iv** 40, 75, 85
d **i** 14, 48, 50 **ii** 21, 72, 75
iii 28, 96, 100 **iv** 35, 120, 125

5 **a** 45 **b** 45 **c** 48 **d** 21

6 **a** no **b** no **c** yes **d** yes
e yes **f** yes

7 **a** odd numbers, increasing by two each time
b value of c is always one greater than b
c 9, 25, 49 so $b + c = a^2$
d 9, 40, 41 and 11, 60, 61 and they do form Pythagorean triples

exercise 2.2

1 **a** 30 cm **b** 91 cm **c** 82 cm **d** 75 cm
e 117 cm **f** 85 cm **g** 35.5 cm **h** 148.2 cm
i 49 cm **j** 94.3 cm **k** 195.5 cm **l** 102.7 cm

2 **a** $\sqrt{85}$ cm, 9.2 cm **b** $\sqrt{20}$ m, 4.5 m
c $\sqrt{34}$ m, 5.8 m **d** $\sqrt{41}$ m, 6.4 m
e $\sqrt{73}$ m, 8.5 m **f** $\sqrt{61}$ m, 7.8 m
g $\sqrt{113}$ m, 10.6 m **h** $\sqrt{170}$ m, 13.0 m
i $\sqrt{45}$ cm, 6.7 cm **j** $\sqrt{117}$ m, 10.8 m
k $\sqrt{250}$ cm, 15.8 cm **l** $\sqrt{180}$ m, 13.4 m

3 **a** 5.40 m **b** 48.3 cm **c** 55.5 m
d 51.8 cm **e** 6.34 m **f** 54.1 m
g 613 mm **h** 170 cm **i** 47.9 cm
j 990 mm **k** 102.7 cm **l** 1037 mm

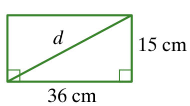
4 **a** 9, 12, 15 hypotenuse is 15 cm
b 10, 24, 26 hypotenuse is 26 cm
c 14, 48, 50 hypotenuse is 50 cm
d 15, 36, 39 hypotenuse is 39 cm
e 1.5, 2, 2.5 hypotenuse is 2.5 m
f 16, 30, 34 hypotenuse is 34 cm

5 17 cm

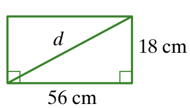
6 6.02 m

7 3.16 m

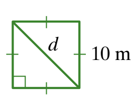
8 32.1 m

9 **a** **i**


ii 39 cm

b **i**


ii 58.8 cm

c **i**


ii 14.1 m

10 no, the guy rope would have to be a minimum length of 1.52 m to allow it to be pegged 0.6 m from the base of the tent pole

11 There are many possible answers, e.g. $\sqrt{17}$ m, $\sqrt{8}$ m, 2 m, $\sqrt{21}$ m, $\sqrt{10}$ m, $\sqrt{15}$ m

exercise 2.3

1 **a** $a = 60$ **b** $b = 48$ **c** $b = 63$ **d** $a = 45$
e $a = 68$ **f** $b = 44.4$ **g** $a = 100$ **h** $b = 9.85$
i $a = 13.2$ **j** $a = 21.7$ **k** $b = 103.2$ **l** $b = 50$

2 **a** $a = 5$ m **b** $b = 25$ cm **c** $c = 24$ mm
d $d = 72$ cm **e** $e = 8$ cm **f** $f = 14$ mm
g $g = 33$ m **h** $h = 50$ mm

3 **a** **i** $\sqrt{40}$ cm **ii** 6.3 cm
b **i** $\sqrt{85}$ cm **ii** 9.2 cm
c **i** $\sqrt{80}$ cm **ii** 8.9 cm
d **i** $\sqrt{128}$ cm **ii** 11.3 cm
e **i** $\sqrt{176}$ cm **ii** 13.3 cm
f **i** $\sqrt{32}$ cm **ii** 5.7 cm
g **i** $\sqrt{65}$ cm **ii** 8.1 cm
h **i** $\sqrt{115}$ cm **ii** 10.7 cm
i **i** $\sqrt{40}$ cm **ii** 6.3 cm
j **i** $\sqrt{153}$ cm **ii** 12.4 cm
k **i** $\sqrt{51}$ cm **ii** 7.1 cm
l **i** $\sqrt{800}$ cm **ii** 28.3 cm

4 B

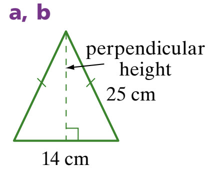
5 2.3 m

6 7.5 cm

7 1.26 m

8 52.4 cm

9 **a** $\sqrt{288}$ cm
b 17.0 cm

10 **a, b**


c 24 cm

11 5 cm

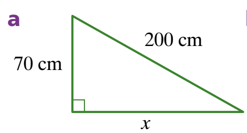
exercise 2.4

1 **a** 26.6 m **b** 328 mm **c** 408 cm
d 64.2 cm **e** 487 mm **f** 1204 cm
g 52.9 cm **h** 49.1 m **i** 529 mm
j 85.0 m **k** 50.2 m **l** 327 mm

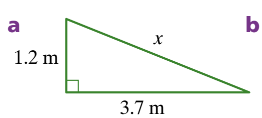
2 C

3 D

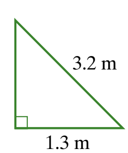
4 2.82 m

5 **a**


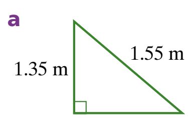
b 187 cm

6 **a**


b 3.9 m

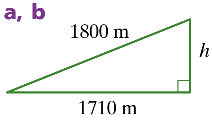
7 **a**


b 2.9 m

8 **a**


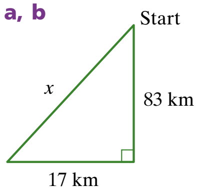
b 0.76 m

9 **a, b**



c 560 m

10 **a, b**



c 85 km

11 6.2 km

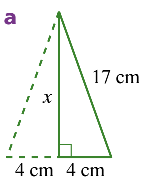
12 81 km

13 $a = 15$, $b = 17$

14 75 cm

15 7.1 cm

16 **a**



b 16.5 cm

exercise 2.5

1 **a** $a = 84.5$ **b** $b = 111.0$
c $c = 8.6$ **d** $d = 51.0$
e $e = 8.8$ **f** $f = 32.3$
g $g = 136.7$ **h** $h = 1.2$
i $i = 90.5$ **j** $j = 0.8$

2 **a** $a = 12.8, b = 14.6$ **b** $x = 52.5, y = 61.5$
c $a = 14.9, b = 15.6$ **d** $a = 45.0, b = 51.0$
e $a = 93.3, b = 116.6$ **f** $c = 31.6, d = 70.0$
g $m = 39.9, n = 55.0$ **h** $x = 22.4, y = 68.2$
i $a = 26.5, b = 71.3$ **j** $h = 8.4, b = 10.0$

3 D

4 **a** **i** $a = 5.66$ **ii** $40\,\text{cm}^2$
b **i** $b = 12.02$ **ii** $361\,\text{cm}^2$
c **i** $c = 48$ **ii** $1920\,\text{cm}^2$
d **i** $d = 9.9$ **ii** $75\,\text{m}^2$
e **i** $h = 8$ **ii** $60\,\text{m}^2$
f **i** $x = 7$ **ii** $126\,\text{m}^2$
g **i** $x = 7, y = 7$ **ii** $28\,\text{m}^2$
h **i** $a = 4.9, b = 5.9$ **ii** $18\,\text{m}^2$
i **i** $a = 94.5, b = 115.2$ **ii** $7981\,\text{m}^2$
j **i** $d = 54.3$ **ii** $1658\,\text{m}^2$

5 **a** $a = 31.68$, $b = 30.30$, $c = 28.29$
b block A = $941\,\text{m}^2$, block B = $1334\,\text{m}^2$, block C = $1564\,\text{m}^2$

Revision

1 D

2 B

3 C

4 D

5 E

6 **a** $a = 2.5$ **b** $b = 24$

7 **a** 36 cm **b** 72 cm
c 41.6 m **d** 10.92 m

8 **a** 9.4 cm **b** 6.7 cm
c 8.0 m **d** 59.4 m

9 **a** 11.5 cm **b** 35.4 cm

10 5.1 m

11 **a** $\sqrt{20}$ **b** $\sqrt{45}$
c **i** $15 - \sqrt{20} - \sqrt{45}$ km **ii** 3.8 km

12 **a** $d = 99$ **b** $7547\,\text{m}^2$

Chapter 3

exercise 3.1

1 **a** **i** 4 **ii** 7 **iii** -2 **iv** 8
b **i** 4 **ii** 1 **iii** 5 **iv** -8
c **i** 3 **ii** -1 **iii** 0 **iv** 10
d **i** 4 **ii** -1 **iii** -2 **iv** 18

2 A

3 D

4 **a** $3x, 6x, 8x$ **b** $3y, -7y$
c $2d, -3d$ **d** $4gh, 7gh, -3gh$
e $a^2, -5a^2$ **f** $4cd, 5dc, -8cd$
g $5ab, 10ba$ **h** $xy, 4xy, -5yx$
i $45zy, 2yz$ **j** $3rst, 5str, -3srt$
k none **l** $h^2m, 3mh^2$

5 D

6 **a** $7a + 4b$ **b** $16x + 12y$
c $3 + 8b$ **d** $-xy - 3x$
e $9m^2 - 14n$ **f** $2k - 10km$
g $15 + 2k - 4k^3$ **h** $x^2 - 8x + 2$
i $8a^2 - 5ab - 6b^2$ **j** $5x^4 - 11x^2 + 2$
k $15ab$ **l** $-7abc$
m $5x^2y + 3xy^2$ **n** $7k - 2kl$
o $3cd - 2d$ **p** $x^3 - 4x^2 - 5x - 3$

7 **a** $72a^3b^2$ **b** $60xy$
c $12abc$ **d** $12a^2b^2$
e $-35a^2b$ **f** $-15m^3n$
g $30a^2b^2$ **h** $6xy$
i $50pq$ **j** $40a^2b$
k $24a^2b^2c$ **l** $30m^3n^2$
m $-12a^2b^3c^2d^3$ **n** $40x^3yz$
o $-24x^3y^2$ **p** $54a^3b$

8 **a** $7x$ **b** x^2 **c** $3x$ **d** $7x$
e $3a$ **f** $\frac{3}{b}$ **g** $-3x$ **h** 6
i $-5a$ **j** $-6m$ **k** $6x^2$ **l** $9ab$

9 **a** $15a^2b^3$ **b** $27x$

exercise 3.2

1 **a** −27 **b** 26 **c** −46 **d** −78
e −1 **f** 80 **g** 40 **h** 16
i 16 **j** 23 **k** 4 **l** 72

2 **a** 35 **b** 12 **c** 81 **d** 95

3 C

4 **a**

x	−4	−3	−1	0	2
y	−5	−3	1	3	7

b

x	−3	−1	0	2	5
y	17	9	5	−3	−15

c

x	−8	−5	−1	0	4
y	29	20	8	5	−7

d

x	−3	−1	0	1	3
y	13	5	4	5	13

e

x	−5	−2	0	1	2
y	22	1	−3	−2	1

f

x	−6	−4	−2	0	2
y	34	16	6	4	10

g

x	−2	−1	0	1	2
y	19	10	5	4	7

h

x	0.1	0.2	0.5	2	10
y	10	5	2	0.5	0.1

i

x	−2	0	2	5	9
y	−3	3	1	0.5	0.3

j

x	−10	−1	0	1	3
y	0.5	2	3	6	−6

5 B

6 D

7 **a** 600 **b** 880

8 **a** 23.1 **b** 735.6 **c** 886.7 **d** 4.4

9 **a** 159.5 **b** 97.94

10 **a** 122.5 **b** 118.4

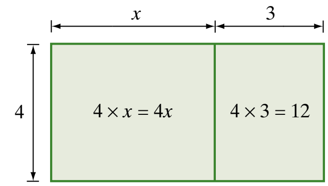
11 **a** 1 **b** 109 **c** 40 **d** 1
e 0 **f** 13 **g** 32 **h** −56

12 **a** 58 **b** 191 **c** 157 **d** 50

13 **a** 264 **b** 950 **c** 5050 **d** 40 200

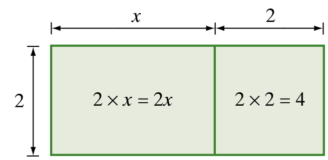
exercise 3.3

1 **a**



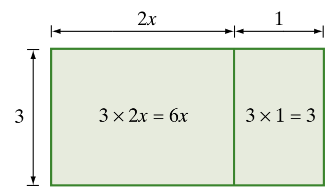
So, $4(x + 3) = 4x + 12$

b



So, $2(x + 2) = 2x + 4$

c



So, $3(2x + 1) = 6x + 3$

2 **a** $2x+8$ **b** $3a-15$ **c** $8b-4$
d $18m+12$ **e** $-3x-6$ **f** $-a^2-3a$
g $3x-6$ **h** $10a+5$ **i** p^2+3p
j $6x-8x^2$ **k** $-2a-12$ **l** $-2m^2-7m$
m $-h^2+7h$ **n** $-15+9x$ **o** x^2y+2xy^2

3 **a** $5a+13$ **b** $5b+1$ **c** $7a-50$
d $7c$ **e** $3x+1$ **f** $6y-20$
g $5d+27$ **h** $9x-29$ **i** $-2x+17$
j $2a-9$ **k** $-11k+19$ **l** $-5n+51$

4 **a** $5p^2-14p-4$ **b** k^2+9k+4
c $-2a^2-2a$ **d** $6p^2-33p+27$
e $d^2-8d-15$ **f** $2p^2+8p-12$
g $2x^2-2x+20$ **h** $2x^2-29x+6$
i $-2p^2+2pq+q^2$ **j** $9a^2-13a$
k $-2n^2+3n-20$ **l** $-12x^2+7xy+4y^2$

5 A

6 A

7 **a** **i** $P = 2(x + 2) + 2(2x + 4)$
ii $P = 6x + 12$
iii No

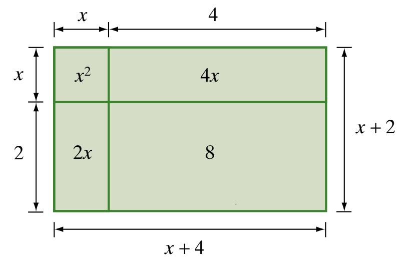
b **i** $P = 2(2x + 1) + 2(6)$
ii $P = 4x + 14$
iii No

c **i** $P = 2(3x + 2) + 2(3 + 1)$
ii $P = 6x + 12$
iii No

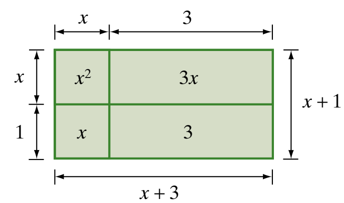
8 **a** $a = 3, b = 6$ **b** $a = -3, b = 2$
c $a = -14, b = 2$ **d** $a = -1, b = -13$

exercise 3.4

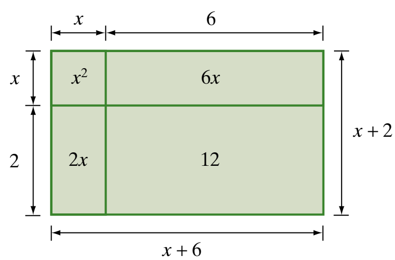
1 **a** $x^2 + 4x + 2x + 8 = x^2 + 6x + 8$



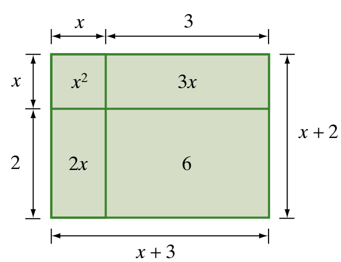
b $x^2 + 3x + x + 3 = x^2 + 4x + 3$



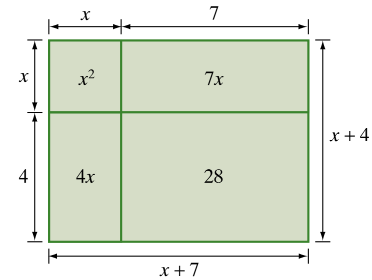
c $x^2 + 6x + 2x + 12 = x^2 + 8x + 12$



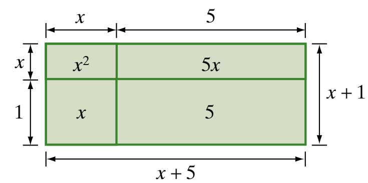
d $x^2 + 3x + 2x + 6 = x^2 + 5x + 6$



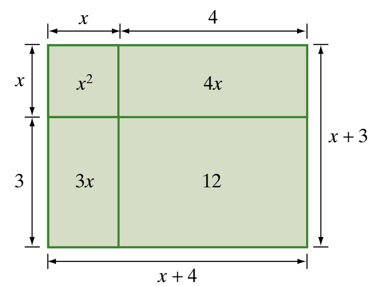
e $x^2 + 4x + 7x + 28 = x^2 + 11x + 28$



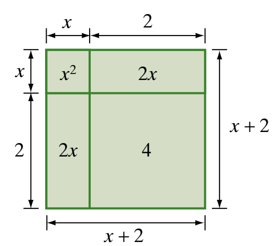
f $x^2 + 5x + x + 5 = x^2 + 6x + 5$



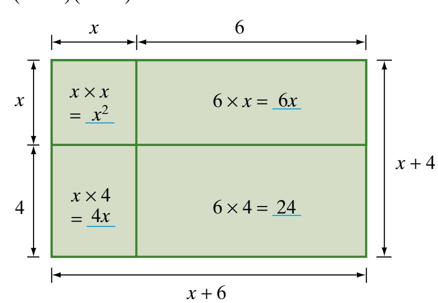
g $x^2 + 4x + 3x + 12 = x^2 + 7x + 12$



h $x^2 + 2x + 2x + 4 = x^2 + 4x + 4$



2 **a** $(x + 4)(x + 6) = x^2 + 10x + 24$



b $d^2 + 7d + 12$

3 **a** $x^2 + 3x + 2$ **b** $y^2 + 9y + 18$
c $z^2 + 12z + 32$ **d** $a^2 + 11a + 30$
e $b^2 - 3b - 10$ **f** $c^2 - 6c - 7$
g $d^2 + 2d - 15$ **h** $e^2 - 2e - 15$
i $f^2 + 3f - 4$ **j** $g^2 - 2g - 63$
k $h^2 - 7h + 10$ **l** $k^2 - 8k + 7$
m $l^2 - 11l + 28$ **n** $m^2 - 9m + 18$
o $x^2 + x - 20$ **p** $x^2 - 9x + 14$

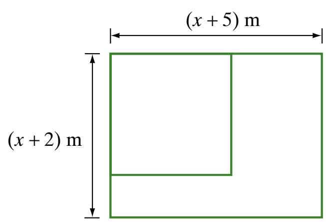
4 **a** $ab + a + 2b + 2$ **b** $st + 4s + 2t + 8$
c $mn + 2m - 3n - 6$ **d** $xy - 2x + 3y - 6$
e $ax + ay + bx + by$ **f** $mp + 2mq - np - 2nq$
g $cg - ch + dg - dh$ **h** $2am - 2bm + an - bn$
i $6kl - 3k - 2l + 1$ **j** $8p - 2pq - 12 + 3q$
k $ab^2 + 2abc - 3b - 6c$
l $3ac - 2ad + 6bc - 4bd$

5 **a** $2a^2 + 7a + 3$ **b** $6b^2 + 19b + 10$
c $3c^2 + 2c - 8$ **d** $4d^2 + 19d - 5$
e $6e^2 - 23e + 21$ **f** $4f^2 - 15f + 9$
g $-4g^2 - g + 14$ **h** $5g^2 - 8g + 3$
i $4x^2 + 4x - 3$ **j** $3a^2 + 11a - 20$
k $2b^2 - b - 28$ **l** $10x^2 + x - 21$

6 E

7 **a** $(x - 4)$ and $(x - 6)$ **b** $x^2 - 10x + 24$
c 624 cm^2

8 **a**



b Area = $(x^2 + 7x + 10)$ m^2
c The area has increased by $(7x + 10)$ m^2

9 **a** $x^2 - 5x + 6$ **b** $x^2 + x - 6$
c $x^2 - x - 6$ **d** $x^2 + 5x + 6$

10 **a** $3x^2 + 5x + 2$ **b** $8x^2 + 26x + 15$

11 **a** Length: $(10 + 2x)$ cm **b** Width: $(15 + 2x)$ cm
c $(150 + 50x + 4x^2)$ cm^2

12 **a** no, if she had chosen another value it would have shown that her expansion was wrong
b $x^2 - 5x + 6$

13 **a** $x^3 + 5x^2 + 7x + 3$ **b** $a^3 - b^3$
c $2p^3 + 3p^2 - p - 1$ **d** $a^4 - 2a^2 + 1$
e $a^4 - 2a^3b + 2ab^3 - b^4$
f $y^4 - 2y^3 - 5y + 2$

exercise 3.5

1

Term	Factors
9	1, 3, 9
12	1, 2, 3, 4, 6, 12
x	1, x
$7b$	1, 7, b, $7b$
$8p$	1, 2, 4, 8, p, $2p$, $4p$, $8p$
$4b$	1, 2, 4, b, $2b$, $4b$
$6x$	1, 2, 3, 6, x, $2x$, $3x$, $6x$
xy	x, y, xy
$7gh$	1, 7, g, h, gh, $7g$, $7h$, $7gh$
x^2	1, x, x^2
$3a^2$	1, 3, a, $3a$, a^2, $3a^2$
$2x^2y$	1, 2, x, y, $2x$, $2y$, x^2, $2x^2$, xy, $2xy$, x^2y, $2x^2y$

2 **a** $2a$ **b** a **c** 5
d $5d$ **e** -2 **f** $-3xy$
g $5ab$ **h** $2x$ **i** $2a$
j $-5ab$ **k** $6x$ **l** 3

3 **a** $3(x + 4) = 3x + 12$ **b** $x(x + 3y) = x^2 + 3xy$
c $b(4a + c) = 4ab + bc$ **d** $a(a + 2) = a^2 + 2a$

4 **a** $3(x + 2)$ **b** $a(5a - 8b)$
c $3(2a + 7b)$ **d** $-3pq(5p + 6q)$
e $-7(x + 2)$ **f** $-3(x + 4)$
g $-5(x + 4)$ **h** $-2x(3x + 5)$
i $-4x(3y - 5)$ **j** $-7(4 + 3x)$
k $-3x(5 - 4x)$ **l** $-8a(3a - 2)$

5 **a** $3(a + 3)$ **b** $2(y - 2)$
c $x(x + y)$ **d** $b(b + 3)$
e $5(1 - 2x)$ **f** $2(2x - 7)$
g $4(8a - 5b)$ **h** $3(2m - 9)$
i $x(3 + x)$ **j** $3x(1 + x)$
k $5x(1 - 3x)$ **l** $x(3 - 8x)$
m $2xy(2x - 3y)$ **n** $ab(a - 1)$
o $-2(x + 2)$ **p** $-6p(1 + 3p)$
q $-3(1 + 2x)$ **r** $-4x(3y + 2x)$
s $-4ab(5a + 4b)$ **t** $-6x(6y + 7x)$
u $6mn(3n + 2m)$ **v** $5pq(3 - 2p + q)$
w $-2(3x^2 + 2x - 5)$ **x** $-a(12a + 20b + 15)$

6 **a** $2(2x + 5) = 4x + 10$ **b** $x(x + 5) = x^2 + 5x$

7 **a** $6 - 3x = 3(2 - x)$
b yes, the calculator places the x term first so it took out a negative to make the x term positive. This changed the signs inside the bracket, but the expressions are equivalent

8 **a** $2\pi r(r + h)$ **b** $\sqrt{2}(p - q)$

c $\frac{ab}{7}\left(a-\frac{3}{2}\right)$ d $5\left(\frac{1}{x}-\frac{1}{y}\right)$

exercise 3.6

1 a $(a+2)(a+6)$ b $(x-5)(3+x)$
c $(2y+1)(y-2)$ d $(x-3)(x-1)$
e $(x+1)(a+3)$ f $(x-7)(4+x)$
g $(2p+3)(p-2)$ h $(a+3)(a+1)$
i $(2w-5)(3w+4)$ j $(x+3)(x+2)$
k $(k-7)(k-2)$ l $(m-5)(m+3)$
m $(4-p)(7-p)$ n $(a-5)(3a-4)$
o $(n^2+1)(4n+3)$ p $(2m+3)^2$
q $(p-1)^2$ r $(x+5)^2$

2 a $bx+by+7x+7y$
$=b(x+y)+7(x+y)$
$=(x+y)(b+7)$
b $k^2-k+pk-p$
$=k(k-1)+p(k-1)$
$=(k-1)(k+p)$
c $6z+36+az+6a$
$=6(z+6)+a(z+6)$
$=(z+6)(6+a)$
d $ab+7b-2a-14$
$=b(a+7)-2(a+7)$
$=(a+7)(b-2)$
e $2ab+4b+3a+6$
$=2b(a+2)+3(a+2)$
$=(a+2)(2b+3)$
f $4c-12+bc-3b$
$=4(c-3)+b(c-3)$
$=(c-3)(4+b)$
g $3b^2-6b+4b-8$
$=3b(b-2)+4(b-2)$
$=(b-2)(3b+4)$
h $2x^2-10x-3x+15$
$=2x(x-5)-3(x-5)$
$=(x-5)(2x-3)$
i $x^2y+6x+xy^2+6y$
$=x(xy+6)+y(xy+6)$
$=(xy+6)(x+y)$
j $3m^2-m^2n+3k-kn$
$=m^2(3-n)+k(3-n)$
$=(3-n)(m^2+k)$

3 a $(x+5)(x+3)$ b $(m+2)(n+9)$
c $(a+6)(4+b)$ d $(b+2)(a+c)$
e $(x-5)(y+2)$ f $(a+4)(x+6)$
g $(x+2)(x-8)$ h $(k+4)(k-6)$
i $(x-8)(x-3)$ j $(2-m)(7+n)$
k $(a-6)(a-5)$ l $(c+b)(d-1)$
m $(b+1)(a-1)$ n $(b+2)(b-a)$
o $(x+11)(x-3)$

4 a i $x(x+y)+3(x+y)$ ii $(x+y)(x+3)$
b i $3(3x-4)-(3x-4)$ ii $2(3x-4)$

5 a $(x+y)(x-4)$ b $(a-3)(b+4)$

exercise 3.7

1 c, d, and g

2 D

3 a x^2+5x+6 b $(x+2)(x+3)$

4 a $(x+2)(x+1)$ b $(x+4)(x+2)$
c $(x+10)(x+1)$ d $(x+6)(x+3)$
e $(x+8)(x+2)$ f $(x+8)(x+3)$
g $(x+15)(x+2)$ h $(x+5)(x+1)$
i $(x+7)(x+8)$ j $(x+14)(x+2)$
k $(x+6)(x+4)$ l $(x+7)(x+3)$

5 a $(x-13)(x+1)$ b $(x-4)(x-2)$
c $(x-5)(x-2)$ d $(x-11)(x+2)$
e $(x-7)(x+1)$ f $(x-4)(x-7)$
g $(x+11)(x+2)$ h $(x+6)(x+1)$
i $(x+9)(x+2)$ j $(x+7)(x+5)$
k $(x-7)(x-1)$ l $(x-7)(x-2)$

6 a $x^2+10x+16=(x+8)(x+2)$
b $x^2+10x+24=(x+6)(x+4)$
c $x^2+7x+12=(x+4)(x+3)$
d $x^2-12x-13=(x-13)(x+1)$
e $x^2-11x+24=(x-8)(x-3)$
f $x^2-6x+8=(x-4)(x-2)$

7 a $(x+16)(x-2)$ b $(x+5)(x-4)$
c $(x-14)(x+5)$ d $(x+12)(x-4)$
e $(x-13)(x-2)$ f $(x-5)^2$
g $(x+7)(x+9)$ h $(x+9)(x+4)$
i $(x-5)(x-3)$ j $(x-11)(x-3)$
k $(x-4)(x-3)$ l $(x-6)(x-2)$
m $(x-12)(x-1)$ n $(x-12)(x-2)$
o $(x+11)(x-1)$ p $(x-11)(x+1)$
q $(x+6)(x-4)$ r $(x-6)(x+5)$
s $(x+8)(x-6)$ t $(x+14)(x-4)$
u $(x-7)(x+6)$ v $(x+16)(x-3)$
w $(x-8)(x-5)$ x $(x-15)(x+2)$

8 a $(x+3)(x+2)$ b $(x-3)(x-2)$
c $(x-6)(x-1)$ d $(x+6)(x+1)$
e $(x-6)(x+1)$ f $(x+6)(x-1)$
g $(x-3)(x+2)$ h $(x+3)(x-2)$

9 a $(x-6)(x+4)$, so one sister is 6 years younger and the other sister is 4 years older than Jack.
b 144

10 a Area $=(x+8)(x-2)$, so length $=x+8$, width $=x-2$
b Length = 12, width = 2
c $x-2$ must be greater than zero, so x must be greater than 2.

11 a $2(x+1)(x+3)$ b $3(x+2)(x+5)$
c $2(x+1)(x+7)$ d $5(x-6)(x+1)$
e $4(x-3)(x+4)$ f $2(x-10)(x+2)$
g $2(x-1)(x+5)$ h $-(x+3)(x+4)$
i $2(x-13)(x-2)$

12 $a=3, b=8, c=1$ (or $a=3, b=1, c=8$)

13 a $(x+a)(x+2a)$ b $(x+a)(x-b)$
c $(x-p)(x-q)$
d $(a+2+3)(a+2+2)=(a+5)(a+4)$

exercise 3.8

1 a $\frac{2x}{5}$ b $\frac{5x}{8}$ c $\frac{x}{7}$ d $\frac{x}{11}$
e $\frac{2b}{9}$ f $\frac{3a}{5}$ g $\frac{6x}{7}$ h $\frac{7b}{8}$
i $\frac{4a}{7}$ j $\frac{6x}{5}$ k $\frac{2a}{3}$ l $\frac{5x}{11}$

2 a $\frac{7x}{12}$ b $\frac{7a}{6}$ c $\frac{16a}{15}$ d $\frac{29b}{24}$
e $\frac{11x}{16}$ f $\frac{7b}{15}$ g $\frac{x}{8}$ h $-\frac{x}{6}$

i $\frac{7x}{12}$ j $\frac{13a}{15}$ k $\frac{29x}{24}$ l $\frac{7x}{12}$

3 a $\frac{x}{3}$ b $\frac{2x}{3}$ c $\frac{a}{2}$ d a
e $\frac{x}{3}$ f $\frac{b}{2}$ g $\frac{2x}{3}$ h $\frac{5a}{4}$
i $2x$ j $\frac{33x}{20}$ k $\frac{4x}{3}$ l $\frac{3x}{8}$

4 $\frac{31x-7}{24}$

Revision

1 C
2 D
3 D
4 C
5 B
6 a $7a - 3b - 9ab$ b $-x^2y + xy + 3xy^2$
c $-60xy$ d $\frac{9m}{8}$
7 a $y = 11$ b $y = 1\frac{1}{2}$
8 $v = 4$
9 a $6x - 15$ b $2y - 10$
c $-4x + 7$
10 a $x(x+5)$ b $2ab(a-2b)$
c $3xy(x-3y+5)$
11 a $(x+3)(x-2)$ b $(a+2)(a+b)$
c $(m+5)(m-n)$ d $(x+2)(y-9)$
12 a $x^2, 3x, 2x, 6$
b i $(x+3)(x+2)$ ii x^2+5x+6
13 a $x^2+7x+10$ b a^2-a-6
c $p^2-13p+40$ d $2y^2-5y-3$
e $8e^2+6e-9$ f $2x^2 - 7x - 15$
14 a $(x+5)(x+2)$ b $(y-6)(y-2)$
c $(m-4)(m+3)$ d $(k+5)(k-3)$
15 a $\frac{3a}{11}$ b $\frac{5x}{6}$ c $\frac{19x}{12}$ d $2x$
16 a i $(x - 2)(x + 4)$ ii $x^2 + 2x - 8$
b For areas to be equal,

$$\begin{aligned} x^2 &= x^2 + 2x - 8 \\ x^2 - x^2 &= x^2 - x^2 + 2x - 8 \\ 0 &= 2x - 8 \\ 8 &= 2x \\ x &= 4 \end{aligned}$$

Chapter 4

exercise 4.1

1 a $x = 14$ b $x = 32$ c $x = 7$ d $x = 22$
e $x = -9$ f $x = -42$ g $x = 7$ h $x = -8$
i $x = 6$ j $x = 6$ k $x = 300$ l $x = -4$

2 b, d, e and f
3 a $x = 7$ b $x = 3$ c $x = 5$
d $x = 9$ e $x = -4\frac{1}{2}$ f $x = -4\frac{1}{5}$
g $x = -3$ h $x = -9$ i $x = -8$
j $x = -9$ k $x = -3\frac{5}{9}$ l $x = -1.4$
m $x = 5$ n $x = -2$ o $x = -2$
4 a yes
b subtract 8 from both sides first and then divide by 4.
5 D
6 a $a = 56$ b $x = 15$ c $x = -20$
d $x = -2\frac{2}{5}$ e $a = 22$ f $m = 20$
g $x = 21$ h $a = 9\frac{4}{5}$ i $a = -4\frac{1}{2}$
j $x = -16$ k $x = 3\frac{1}{2}$ l $x = -2\frac{4}{5}$
m $x = 66$ n $x = 64$ o $x = 40$
p $x = -6$ q $x = 15$ r $x = 7\frac{1}{2}$
7 a $a = -6$ b $x = \frac{3}{4}$ c $x = 6$
d $x = -39\frac{1}{2}$ e $x = 3\frac{1}{2}$ f $x = \frac{1}{12}$
g $n = 6\frac{1}{2}$ h $x = 11$ i $b = 23$
j $x = -2$ k $t = 2$ l $x = 5$
8 a $x = 5$ b $a = 2$ c $x = \frac{1}{2}$
d $m = 6$ e $x = 1\frac{1}{9}$ f $b = 1$
g $x = 1\frac{3}{4}$ h $a = 4\frac{1}{6}$ i $d = 2\frac{2}{5}$
j $x = 5\frac{4}{5}$ k $k = -26$ l $x = -18\frac{1}{5}$
9 a $x = 30$ b $x = 1\frac{1}{5}$ c $x = 40$
d $x = 45$ e $x = \frac{1}{2}$ f $x = 52\frac{1}{2}$
g $x = \frac{1}{5}$ h $x = 24$ i $x = 105$
j $x = 80$ k $x = 7\frac{1}{2}$ l $x = 120$
10 a $a = 3$ b $x = 9$ c $x = 7$
d $k = 10$ e $x = 13$ f $x = 3$
11 Answers will vary. Check with your teacher.

exercise 4.2

1 a $a = 4$ b $e = \frac{3}{5}$ c $d = 1$
d $x = -\frac{1}{2}$ e $m = -\frac{1}{5}$ f $k = -2\frac{1}{2}$

g $x=-1\frac{1}{3}$ **h** $x=-\frac{2}{5}$ **i** $a=\frac{3}{5}$

j $y=1$ **k** $j=\frac{4}{9}$ **l** $e=\frac{4}{5}$

m $a=8$ **n** $b=-2\frac{1}{3}$ **o** $a=\frac{5}{6}$

2 D

3 **a** $x=-\frac{1}{5}$ **b** $a=-1\frac{3}{5}$ **c** $x=-6\frac{1}{2}$

d $x=-26$ **e** $x=\frac{2}{3}$ **f** $m=3$

g $x=-10$ **h** $x=27$ **i** $x=-5.8$

j $x=1\frac{1}{7}$ **k** $x=8$ **l** $x=0$

m $x=1$ **n** $x=-6.1$

4 **a** $x=1\frac{3}{4}$ **b** $x=4$ **c** $a=31$

d $x=\frac{1}{4}$ **e** $m=\frac{11}{13}$ **f** $x=-1\frac{4}{7}$

g $p=-22$ **h** $x=8$ **i** $y=4\frac{1}{4}$

j $x=8$ **k** $m=5\frac{1}{2}$ **l** $p=2$

m $b=4\frac{1}{3}$ **n** $k=3\frac{5}{7}$

5 **a** yes
b answers will vary
c expand the brackets first, then move pronumeral to one side to solve.

6 **a** $x=-4\frac{2}{5}$ **b** $x=2\frac{13}{16}$

7 when x is subtracted from both sides there is no x left to solve for

8 There is no solution. When both sides are multiplied by 2 there is a $6x$ term on both sides. So when $6x$ is subtracted from both sides there is no x left to solve for.

exercise 4.3

1 **a** $a-9=12, a=21$ **b** $2b+5=19, b=7$
c $3c-7=20, c=9$ **d** $\frac{d}{2}+6=10, d=8$
e $4e+8=40, e=8$ **f** $5(f+3)=25, f=2$
g $2(g+8)=-6, g=-11$ **h** $3(h-4)=12, h=8$

2 $2x+5=51$
$x=23$
$\therefore$ 23 yellow lollies

3 $2x+40=136$
$x=48$
$\therefore$ lowest grade was 48.

4 $2x+6=40$
$x=17$
$\therefore$ two pieces were 17 cm and 23 cm.

5 $\frac{n+3}{2}+2=7$
$n=7$
$\therefore$ she had 7 plants.

6 $\frac{x}{2}+\frac{2x}{5}=18$
$x=30$
$\therefore$ 30 students in the class

7 $\frac{x}{4}+\frac{x}{5}=45$
$x=100$
$\therefore$ 100 cars altogether

8 $x-\left(\frac{x}{3}+\frac{x}{4}\right)=280$
$x=672$
$\therefore$ she was paid \$672.

9 $2x+1=43$
$x=21$
$\therefore$ numbers are 21, 22.

10 $2x+2=70$
$x=34$
$\therefore$ numbers are 34, 36.

11 $3x+6=255$
$x=83$
$\therefore$ numbers are 83, 85, 87.

12 $2(2x+15)=70$
$x=10$
$\therefore$ width = 10 m

13 $2(12+3x)=144$
$x=20$
$\therefore$ length is 52 m and width is 20 m.

14 **a** $j+12$ **b** $j+12=3(j-28)$
c Justine is 48 now. **d** 22 years

15 $3(x+2)-2+x=32$
$x=7$
$\therefore$ Anna is 7 and Bella is 25.

16 $150+2.4k = 120+2.7k$
$k=100$
$\therefore$ He can travel 100 km.

exercise 4.4

1 **a** $b=a-x$ **b** $a=\frac{F}{m}$ **c** $T=\frac{D}{S}$

d $T=\frac{I}{PR}$ **e** $r=\frac{C}{2\pi}$ **f** $m=\frac{E}{c^2}$

g $w=\frac{3V}{lh}$ **h** $b=2M-a$ **i** $h=\frac{P}{2}-1$

j $r=\sqrt{\frac{A}{\pi}}$ **k** $t=\frac{v-u}{a}$ **l** $V=\frac{k}{P}$

m $b=\frac{2A}{h}-a$ **n** $h=\frac{S}{b}-2b$

o $C=\frac{5(F-32)}{9}$ **p** $a=\frac{2(s-ut)}{t^2}$

q $h = \frac{3V}{\pi r^2}$ r $r = \sqrt{\frac{3V}{\pi h}}$

2 a $x = c - a - b$ b $x = \frac{bc}{a}$

c $x = \frac{cd - b}{a}$ d $x = \frac{\frac{de}{a} - c}{b}$ or $\frac{de - ac}{ab}$

e $x = \frac{ac}{b - a}$ or $-\frac{ac}{a - b}$ f $x = \frac{bde}{ad + bc}$

Revision

1 A
2 E
3 D
4 A
5 B
6 a $x = -2$ b $x = 6$ c $x = 20$
d $x = 12$ e $x = 3$ f $x = 2$
g $x = -5$ h $x = -\frac{1}{2}$ i $x = 31$
j $x = 4$ k $x = 36$ l $x = 40$
m $x = -2\frac{3}{8}$ n $x = -10$ o $x = 3\frac{1}{4}$
p $x = 9$
7 a $a = \frac{v - u}{t}$ b $r = \sqrt{\frac{S}{4\pi}}$ c $P = \frac{3W}{5T^2}$
8 a 77°F b $37\frac{7}{9}$°C c −40
9 a $48x + 7$
b $48x + 7 = 535$
c $x = 11$, so Fin purchased 11 tickets

Chapter 5

exercise 5.1

1 $\frac{2}{7}$
2 a $\frac{1}{4}$ b 1 c $\frac{3}{4}$ d $\frac{3}{13}$
3 a $\frac{3}{8}$ b $\frac{1}{2}$ c $\frac{1}{8}$ d $\frac{3}{4}$
4 a $\frac{2}{11}$ b $\frac{1}{11}$ c $\frac{7}{11}$ d $\frac{5}{11}$
5 a $\frac{1}{7}$ b $\frac{1}{2}$ c $\frac{1}{7}$ d $\frac{3}{14}$
e $\frac{3}{28}$ f $\frac{7}{28}$
g $\frac{1}{28}$ if born 1st to 28th and 0 if born 29th to 31st.
6 a $\frac{3}{5}$ b $\frac{7}{10}$
7 a $\frac{2}{11}$ b $\frac{4}{11}$ c $\frac{4}{11}$ d $\frac{3}{11}$
8 100
9 200
10 $\frac{3}{22}$

exercise 5.2

1 $\frac{5}{8}$
2 a i 0.33 ii 0.67
b no, a fair coin would have relative frequencies for heads and tails of approximately 0.5. This shows you are twice as likely to get a tail.
3 a

Car colour	Tally	Frequency	Relative frequency
White	卌 卌 卌 卌 卌 IIII	29	0.29
Silver	卌 卌 卌 III	18	0.18
Red	卌 卌 卌 I	16	0.16
Black	卌 卌 卌 卌 I	21	0.21
Blue	卌 I	6	0.06
Green	III	3	0.03
Other	卌 II	7	0.07

b 100 c white d 0.21 e 0.84
f By increasing the number of cars in the survey.

4 a

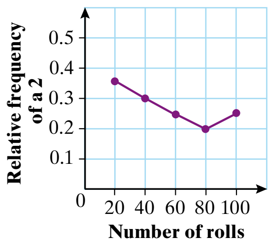
Number on die	Tally	Frequency
1	卌 III	8
2	卌 IIII	9
3	卌 II	7
4	卌 卌 II	12
5	卌 卌 III	13
6	卌 卌 I	11

b

Relative frequency
$\frac{2}{15}$
$\frac{3}{20}$
$\frac{7}{60}$
$\frac{1}{5}$
$\frac{13}{60}$
$\frac{11}{60}$

c i 24 ii 120
d i 20 ii 100

5 **a**



b no, because the relative frequency of rolling a 2 on an unbiased die should be approximately 0.167

6 **a** **i** 8 **ii** 4 **iii** 15 **iv** 16

b according to the graph there are eight 3s in the first 10 spins therefore there cannot be less than eight 3s in the first 20 spins. So the point for 20 spins must be wrong.

7 **a**

Coin Toss	Frequency
Head	15
Tail	5
Total	20

b 38 **c** 12 **d** 0.752 **e** 0.248

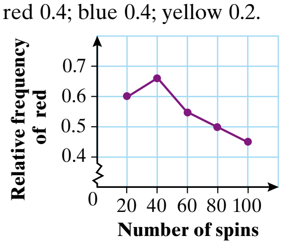
f no, because the relative frequency of obtaining a head should be 0.5

8 **a**

Number of spins	Number of reds	Relative frequency
20	12	0.6
40	26	0.65
60	33	0.55
80	40	0.5
100	45	0.45

b red 0.4; blue 0.4; yellow 0.2.

c



d yes

9 102

10 6090

11 **a** 960

b better

12 **a** $\frac{4}{5}$ **b** 400

13 **a** 0.2 **b** 0.8 **c** 80

14 $\frac{5}{6}$

15 **a** $\frac{2}{3}$ **b** 40

16 **a** **i** $\frac{14}{25}$ **ii** $\frac{8}{25}$ **iii** $\frac{3}{25}$

b 126

17 **a** $\frac{1}{2}$ **b** 9

18 **a** $\frac{2}{25}$ **b** 64

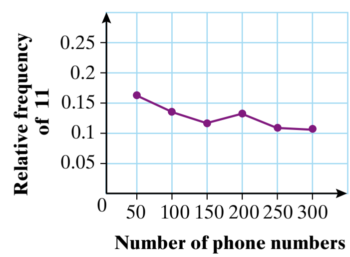
19 **a** 0.44 **b** 0.56 **c** 0.18 **d** 0.82

20 3 chewy, 2 nut, 6 fruit and 1 mint

21 **a**

Number of phone numbers	50	100	150	200	250	300
Number of 1s	8	13	18	25	28	33
Relative frequency of 1s	0.16	0.13	0.12	0.125	0.112	0.11

b



c 0.11 **d** 0.1

e **i** yes

ii check more phone numbers

exercise 5.3

1 **a** $S=\{(\mathrm{H,H}),(\mathrm{H,T}),(\mathrm{T,H}),(\mathrm{T,T})\}$

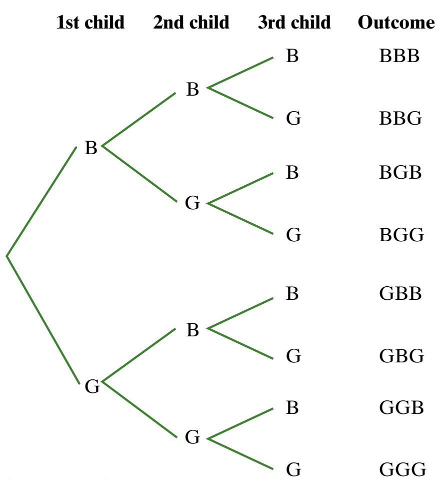
b $S=\{(1,\mathrm{H}),(2,\mathrm{H}),(3,\mathrm{H}),(4,\mathrm{H}),(5,\mathrm{H}),(6,\mathrm{H}),(1,\mathrm{T}),(2,\mathrm{T}),(3,\mathrm{T}),(4,\mathrm{T}),(5,\mathrm{T}),(6,\mathrm{T})\}$

c $S=$ (red,red),(red,purple),(red,yellow),(red,blue),
(purple,purple),(purple,red),(purple,yellow),(purple,blue),
(yellow,yellow),(yellow,red),(yellow,purple),(yellow,blue),
(blue,blue),(blue,red),(blue,purple),(blue,yellow)

d $S=\left\{\begin{array}{l}(\text{clubs,clubs}),(\text{clubs,diamonds}),(\text{clubs,hearts}),(\text{clubs,spades}),\\(\text{diamonds,clubs}),(\text{diamonds,diamonds}),(\text{diamonds,hearts}),(\text{diamonds,spades}),\\(\text{hearts,clubs}),(\text{hearts,diamonds}),(\text{hearts,hearts}),(\text{hearts,spades}),\\(\text{spades,clubs}),(\text{spades,diamonds}),(\text{spades,hearts}),(\text{spades,spades})\end{array}\right\}$

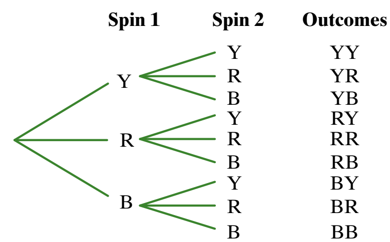
2 **a** $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** $\frac{3}{4}$

d

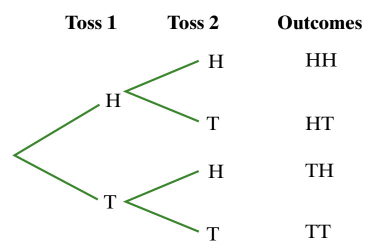


e $\frac{1}{4}$ **f** $\frac{1}{2}$

3 **a**



b

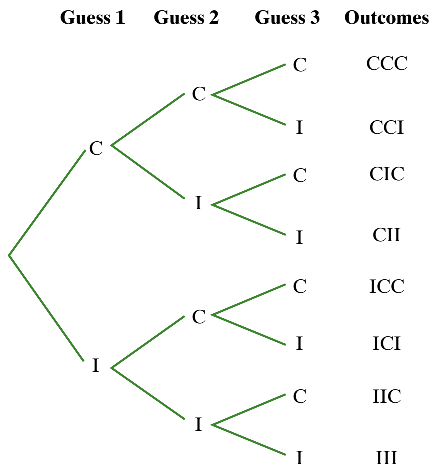


3 **c** see table below

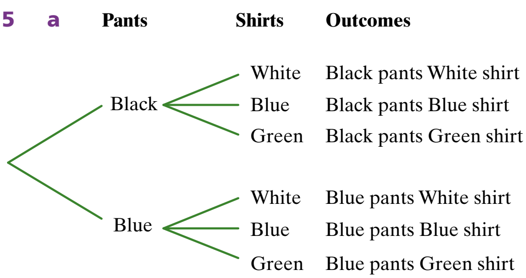
3 **d** $\frac{1}{6}$

3 **c**

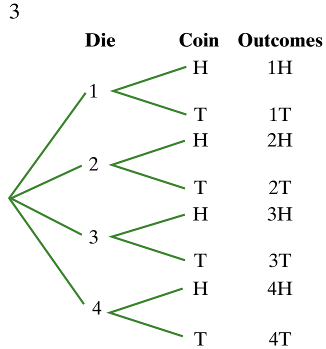
	YY	YR	YB	RY	RR	RB	BY	BR	BB
HH	YYHH	YRHH	YBHH	RYHH	RRHH	RBHH	BYHH	BRHH	BBHH
HT	YYHT	YRHT	YBHT	RYHT	RRHT	RBHT	BYHT	BRHT	BBHT
TH	YYTH	YRTH	YBTH	RYTH	RRTH	RBTH	BYTH	BRTH	BBTH
TT	YYTT	YRTT	YBTT	RYTT	RRTT	RBTT	BYTT	BRTT	BBTT

4 a 

b $\frac{1}{8}$ **c** $\frac{1}{2}$

5 a 

b $\frac{1}{3}$

6 a 

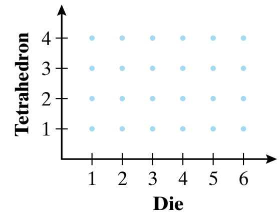
b i $\frac{1}{8}$ **ii** $\frac{1}{4}$

7 a $\frac{1}{16}$ **b** $\frac{1}{3}$

8 a i $\frac{1}{18}$ **ii** $\frac{1}{9}$ **iii** $\frac{1}{36}$ **iv** $\frac{11}{36}$

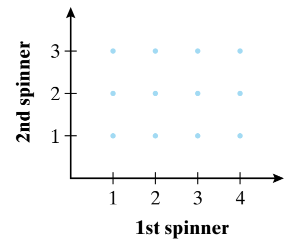
v $\frac{1}{6}$ **vi** $\frac{5}{12}$

b 7

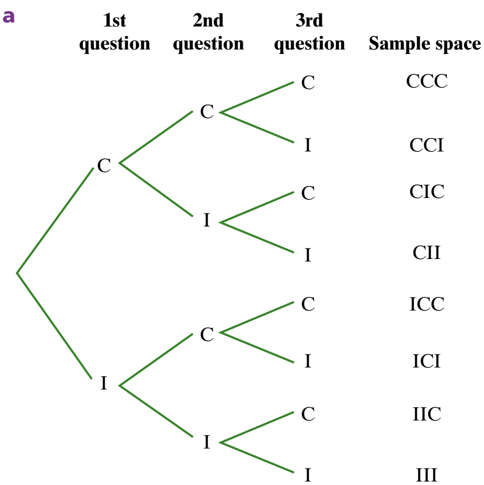
9 a 

b i 0 **ii** $\frac{3}{8}$ **iii** $\frac{1}{6}$ **iv** $\frac{1}{4}$

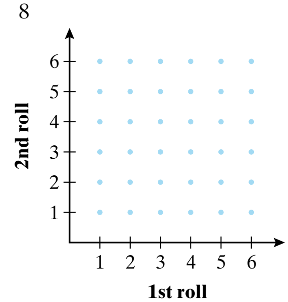
v $\frac{3}{4}$ **vi** $\frac{1}{4}$

10 a 

b i $\frac{1}{6}$ **ii** $\frac{1}{4}$ **iii** $\frac{3}{4}$

11 a 

b $\frac{3}{8}$

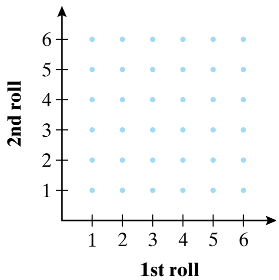
12 a 

b i $\frac{1}{2}$ **ii** $\frac{1}{3}$ **iii** $\frac{1}{9}$

iv $\frac{5}{12}$ v $\frac{1}{3}$ vi $\frac{1}{2}$

c a sum of 7

13 a



b 36

c i $\frac{1}{36}$ ii $\frac{1}{6}$ iii $\frac{1}{12}$ iv $\frac{5}{9}$

14 a i $\frac{1}{36}$ ii $\frac{1}{12}$ iii $\frac{1}{12}$ iv $\frac{7}{12}$

v $\frac{5}{12}$ vi $\frac{1}{6}$ vii $\frac{1}{6}$

b 7, probability is $\frac{1}{6}$

15 a $\frac{1}{32}$ b $\frac{1}{4}$ c $\frac{1}{16}$

16 $\frac{6}{35}$

17 a $\frac{1}{2}$ b $\frac{5}{6}$ c $\frac{2}{3}$ d $\frac{5}{6}$

18 a i $\frac{3}{8}$ ii $\frac{3}{8}$

b $\frac{1}{2}$

19 a $\frac{1}{25}$ b $\frac{6}{25}$ c $\frac{6}{25}$

20 a 0.0045 b 0.8245

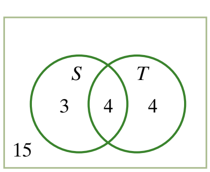
exercise 5.4

1 a $\frac{2}{15}$ b $\frac{17}{30}$ c $\frac{1}{3}$

d $\frac{3}{10}$ e $\frac{13}{15}$ f $\frac{7}{30}$

2 a $\frac{7}{17}$ b $\frac{7}{16}$

3 a



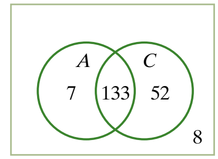
$n = 26$

b i $\frac{4}{13}$ ii $\frac{7}{26}$ iii $\frac{2}{13}$ iv $\frac{11}{26}$

v $\frac{15}{26}$

c $\frac{3}{7}$

4 a



$n = 200$

b

	C	C'	Total
A	133	7	140
A'	52	8	60
Total	185	15	200

c

	C	C'	Total
A	0.665	0.035	0.7
A'	0.26	0.04	0.3
Total	0.925	0.075	1

d 0.665 e 0.035

5 a 70 b 89

c i $\frac{89}{200}$ ii $\frac{13}{20}$ iii $\frac{43}{100}$

d $\frac{22}{65}$ e $\frac{9}{14}$

6 a

	S	S'	Total
F	0.42	0.18	0.6
F'	0.28	0.12	0.4
Total	0.7	0.3	1

b 0.12 c 0.6

7 a

	A	A'	Total
B	0.28	0.08	0.36
B'	0.17	0.47	0.64
Total	0.45	0.55	1

b 0.47 c 0.08

8 a $\frac{17}{30}$ b $\frac{1}{6}$ c $\frac{7}{30}$ d $\frac{2}{5}$

9 a

		Toss 1			
		1	**2**	**3**	**4**
Toss 2	**1**	2	3	4	5
	2	3	4	5	6
	3	4	5	6	7
	4	5	6	7	8

b **i** $\frac{1}{8}$ **ii** $\frac{5}{16}$ **iii** $\frac{3}{8}$

iv $\frac{3}{16}$ **v** $\frac{13}{16}$ **vi** $\frac{3}{8}$

10 a

		Toss 1			
		1	**2**	**3**	**4**
Toss 2	**1**	1, 1	2, 1	3, 1	4, 1
	2	1, 2	2, 2	3, 2	4, 2
	3	1, 3	2, 3	3, 3	4, 3
	4	1, 4	2, 4	3, 4	4, 4
	5	1, 5	2, 5	3, 5	4, 5
	6	1, 6	2, 6	3, 6	4, 6
	7	1, 7	2, 7	3, 7	4, 7
	8	1, 8	2, 8	3, 8	4, 8

b **i** 32

ii the number of sides on each die multiplied together

c $\frac{1}{2}$ **d** $\frac{1}{8}$ **e** $\frac{1}{8}$

11 a $\frac{1}{36}$ **b** $\frac{1}{6}$

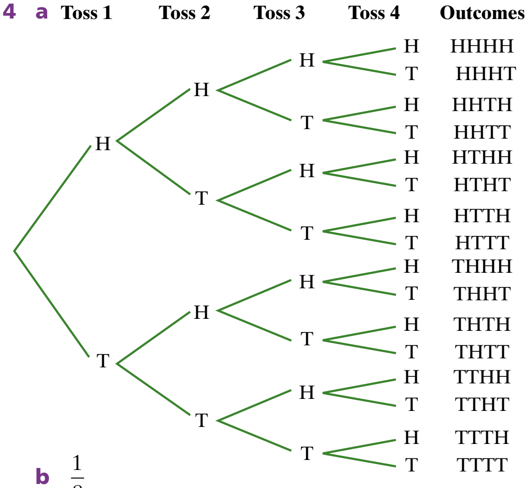
12 a

	Bag 1			
Bag 2	**W**	**W**	**R**	**Y**
R	W, R	W, R	R, R	Y, R
R	W, R	W, R	R, R	Y, R
W	W, W	W, W	R, W	Y, W
Y	W, Y	W, Y	R, Y	Y, Y

b $\frac{5}{16}$ **c** $\frac{1}{2}$ **d** $\frac{7}{16}$

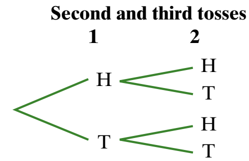
13 a $\frac{2}{5}$ **b** $\frac{2}{3}$

14 a



b $\frac{1}{8}$

c **i**



ii $\frac{1}{4}$

Revision

1 D
2 E
3 B
4 A
5 A
6 **a** 9 **b** 21
7 **a** $\frac{1}{2}$ **b** $\frac{3}{4}$ **c** $\frac{3}{13}$ **d** $\frac{2}{13}$ **e** $\frac{3}{52}$
8 **a** blue, red, green, yellow
b no
c **i** $\frac{1}{4}$ **ii** $\frac{1}{4}$ **iii** $\frac{3}{8}$ **iv** $\frac{7}{8}$
d 75

9 a

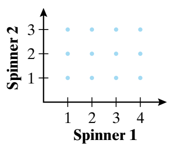
	Brown	Not Brown	
Adult	15	15	30
Child	12	8	20
	27	23	50

b **i** $\frac{3}{5}$ **ii** $\frac{27}{50}$ **iii** $\frac{3}{10}$

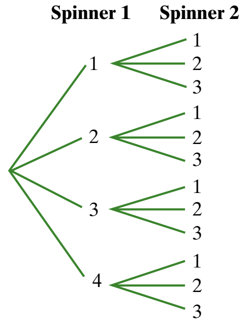
10 a $\frac{1}{4}$

b **i** $\frac{5}{19}$ **ii** $\frac{4}{19}$ **iii** $\frac{14}{19}$

11 **a**

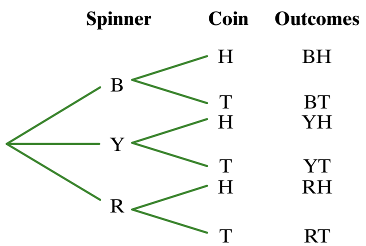


b



c $\frac{1}{3}$

12 **a**



b **i** $\frac{1}{6}$ **ii** $\frac{1}{6}$ **iii** $\frac{2}{3}$ **iv** 0

13 **a** See below

b **i** $\frac{1}{12}$ **ii** $\frac{1}{4}$ **iii** $\frac{1}{3}$

Chapter 6

exercise 6.1

1 **a** $(2, 1)$ **b** $(-1, 2)$ **c** $(2, 4)$
d $\left(1, \frac{1}{2}\right)$ **e** $(-1, 1)$ **f** $(1, -3)$

2 **a** $(6, 6)$ **b** $(8, 3)$ **c** $(5, 5)$
d $(-6, 1)$ **e** $(-1, 0)$ **f** $(-5, -1)$
g $(0, 1)$ **h** $(-2, 2)$ **i** $\left(3, 6\frac{1}{2}\right)$
j $\left(\frac{1}{2}, -3\right)$ **k** $\left(7\frac{1}{2}, 3\right)$ **l** $(-3, -8)$

3 **a** $(-3, -1)$ **b** $(-1, 4)$ **c** $(1, 0)$ **d** $(1, -1)$
e $(1, -3)$ **f** $(1, -2)$ **g** $(-3, -1)$ **h** $(-1, 1)$
i $\left(-1\frac{1}{2}, 1\right)$ **j** $(2, 3)$

4 **a** $(7, 7)$ **b** $(5, 9)$ **c** $(2, 3)$
d $(-6, 6)$ **e** $(2, 3)$ **f** $(0, -1)$
g $(-4, 0)$ **h** $\left(5\frac{1}{2}, 7\right)$ **i** $(1, 1)$
j $\left(\frac{1}{2}, 6\frac{1}{2}\right)$ **k** $\left(-3, 6\frac{1}{2}\right)$ **l** $\left(-3, -3\frac{1}{2}\right)$

5 $(-11, 23)$

6 $a = -23$, $b = -3$

exercise 6.2

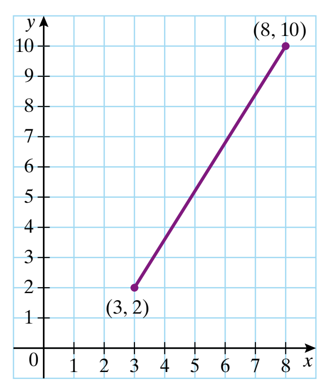
1 **a** 8 **b** 6 **c** 4 **d** 7
e 8 **f** 6

2 **a** 6 **b** 8 **c** 6 **d** 4
e 10 **f** 6 **g** 12 **h** 10
i 5 **j** 9 **k** 7 **l** 4

3 **a** 5 **b** 5 **c** 13 **d** 17
e 10 **f** 13

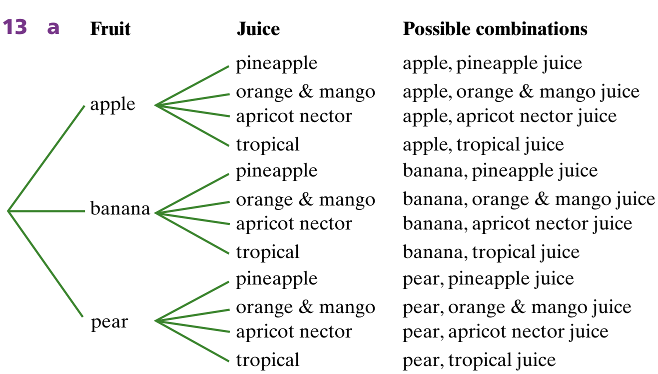
4 **a** 5.1 **b** 5.8 **c** 5.4 **d** 7.2
e 6.4 **f** 8.6 **g** 7.1 **h** 14.1
i 18.4

5 **a** **i**

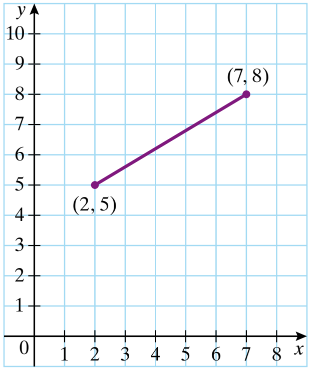


ii 9.4

13 **a**

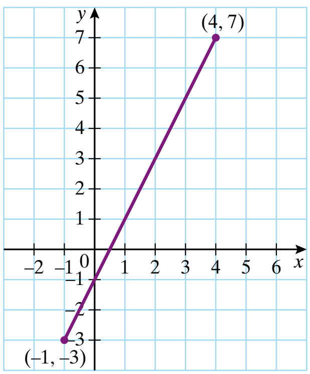


b i



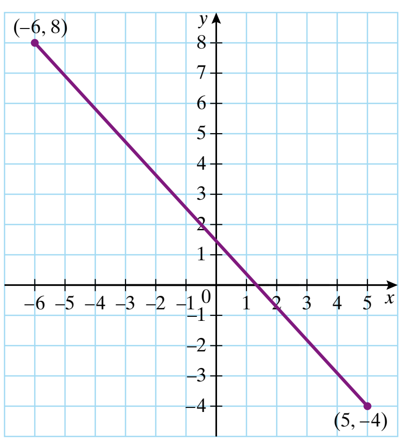
ii 5.8

c i



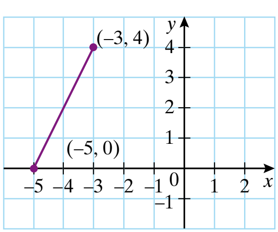
ii 11.2

d i



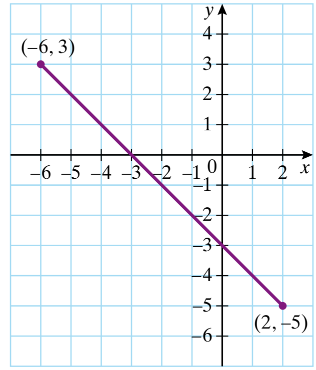
ii 16.3

e i



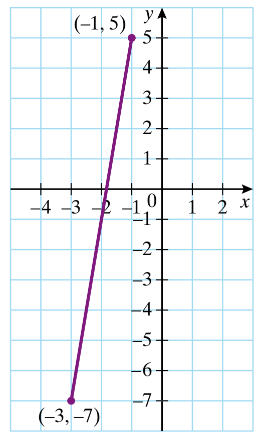
ii 4.5

f i



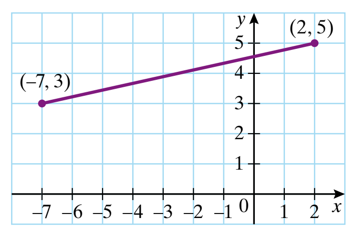
ii 11.3

g i



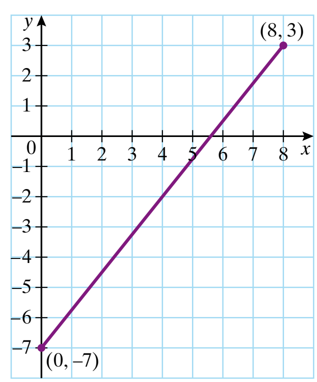
ii 12.2

h i



ii 9.2

i i

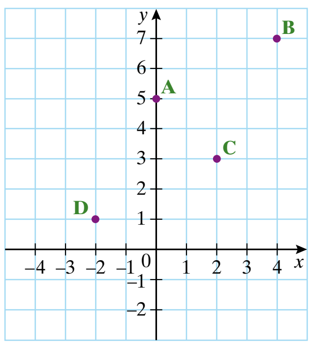


ii 12.8

6 $AB = 5$ and $AC = 5$ therefore ABC is an isosceles triangle because at least two sides are equal.

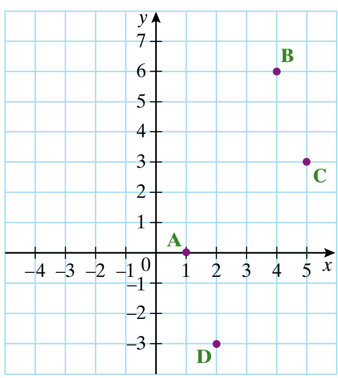
7 $AD = 5$, $AB = 5$, $DC \approx 8.5$, $BC \approx 8.5$ therefore $ABCD$ is a kite because it has two pairs of adjacent equal sides..

8 **a**



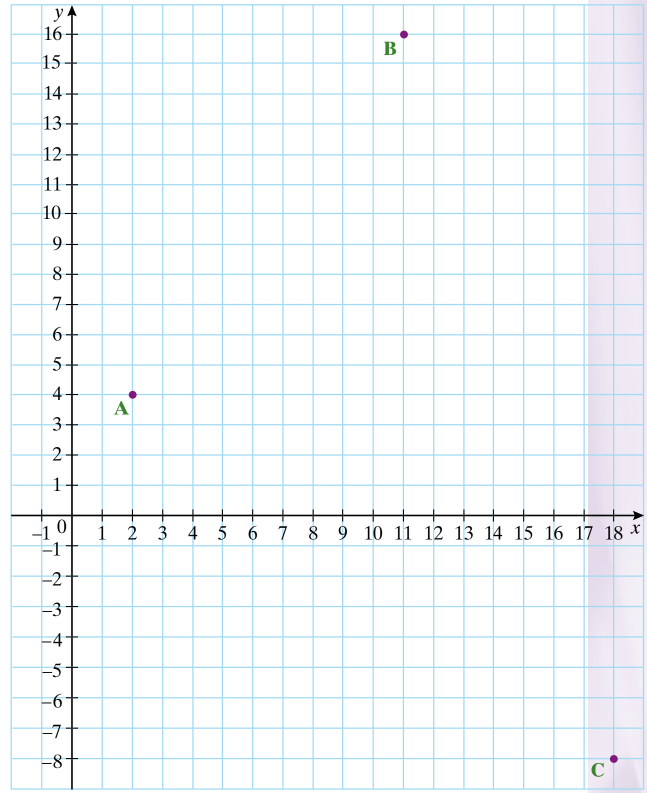
b $AD \approx 4.5$, $AB \approx 4.5$, $DC \approx 4.5$, $BC \approx 4.5$ therefore $ABCD$ is a rhombus because it has four equal sides.

9 **a**



b $AD \approx 3.2$, $AB \approx 6.7$, $DC \approx 6.7$, $BC \approx 3.2$ therefore $ABCD$ is a parallelogram because opposite sides are equal.

10 **a**



b $AB = 15$, $AC = 20$, $BC = 25$

$$\begin{aligned} AB^2 + AC^2 &= 15^2 + 20^2 \\ &= 625 \\ &= BC^2 \end{aligned}$$

So, satisfies Pythagoras' Theorem. Therefore ABC is a right-angled triangle.

exercise 6.3

1 **a** **i** $6\frac{2}{3}$ m **ii** $\frac{1}{15}$ m $= 0.0\dot{6}$ m

b **i** 12 m **ii** 0.12 m

2 **a** $\frac{1}{4}$ **b** $\frac{3}{5}$ **c** 1.4 **d** 0.5

3 **a** **i** 6 **ii** 6 **iii** 1

b **i** 10 **ii** 5 **iii** 2

c **i** 9 **ii** 6 **iii** $1\frac{1}{2}$

d **i** 5 **ii** 2 **iii** $2\frac{1}{2}$

e **i** 12 **ii** 8 **iii** $1\frac{1}{2}$

f **i** 12 **ii** 3 **iii** 4

g **i** 5 **ii** 6 **iii** $\frac{5}{6}$

h **i** 8 **ii** 6 **iii** $1\frac{1}{3}$

4 **a** Positive. Graph slopes upward to the right.
b Negative. The graph slopes downward to the right.
c Positive. Graph slopes upward to the right.
d Positive. Graph slopes upward to the right.
e Positive. Graph slopes upward to the right.
f Negative. Graph slopes downward to the right.

5 **a** **i** –6 **ii** 3 **iii** –2
b **i** –5 **ii** 5 **iii** –1
c **i** –6 **ii** 9 **iii** $-\frac{2}{3}$
d **i** 6 **ii** 9 **iii** $\frac{2}{3}$
e **i** –7 **ii** 4 **iii** $-1\frac{3}{4}$
f **i** –12 **ii** 8 **iii** $-1\frac{1}{4}$
g **i** –5 **ii** 7 **iii** $-\frac{5}{7}$
h **i** –5 **ii** 10 **iii** $-\frac{1}{2}$

6 **a** **i** 8 **ii** 10 **iii** $\frac{4}{5}$
b **i** 10 **ii** 5 **iii** 2
c **i** 10 **ii** 5 **iii** 2
d **i** 4 **ii** 12 **iii** $\frac{1}{3}$
e **i** 5 **ii** 15 **iii** $\frac{1}{3}$
f **i** 4 **ii** 8 **iii** $\frac{1}{2}$
g **i** 11 **ii** 6 **iii** $1\frac{5}{6}$
h **i** 5 **ii** 12 **iii** $\frac{5}{12}$

7 **a** **i** –8 **ii** 2 **iii** –4
b **i** –5 **ii** 10 **iii** $-\frac{1}{2}$
c **i** –6 **ii** 12 **iii** $-\frac{1}{2}$
d **i** –12 **ii** 10 **iii** $-1\frac{5}{6}$
e **i** –12 **ii** 4 **iii** –3
f **i** –4 **ii** 12 **iii** $-\frac{1}{3}$
g **i** –9 **ii** 6 **iii** $-1\frac{1}{2}$
h **i** –12 **ii** 4 **iii** –3

8 C

9 E

10 **a** **i** negative **ii** –1
b **i** positive **ii** 3
c **i** positive **ii** $\frac{1}{3}$
d **i** neither **ii** 0
e **i** positive **ii** 4
f **i** negative **ii** $-\frac{7}{9}$

11 A

12 **a** 2 **b** 1 **c** –4 **d** 3
e –3 **f** $-\frac{1}{2}$ **g** 1 **h** –3
i $1\frac{2}{3}$ **j** $-1\frac{4}{5}$ **k** $1\frac{1}{4}$ **l** $-1\frac{1}{2}$
m $\frac{4}{5}$ **n** $-\frac{1}{2}$ **o** $\frac{7}{12}$ **p** $-\frac{2}{5}$
q –1 **r** $2\frac{1}{2}$

13 **a** positive **b** zero **c** negative
d zero **e** positive **f** undefined
g undefined **h** negative

14 10

Revision

1 D

2 C

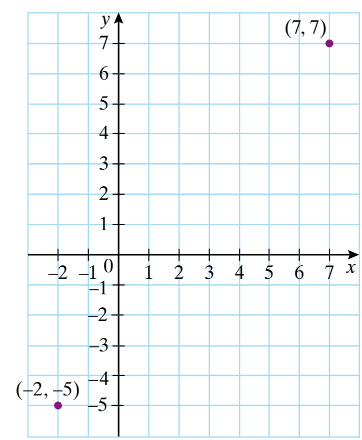
3 E

4 C

5 A

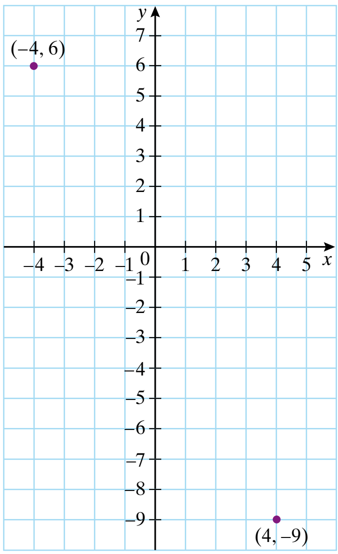
6 **a** (6, 10)
b (–1, –1)

7 **a** **i**



ii 15

b **i**



ii 17

8 19.2

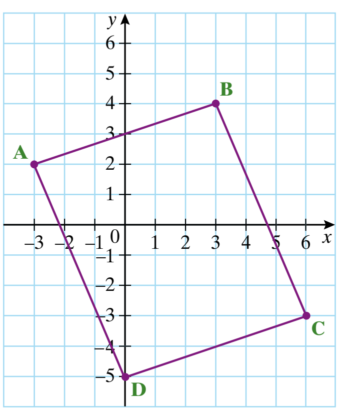
9 **a** 3

b $-\frac{1}{2}$

10 –8

11 $AB \approx 8.1$, $BC \approx 4.2$ and $AC \approx 8.1$ therefore ABC is an isosceles triangle because at least two sides are equal.

12 **a**



b $AD \approx 7.6$, $AB \approx 6.3$, $DC \approx 6.3$, $BC \approx 7.6$ therefore $ABCD$ is a parallelogram because opposite sides are equal.

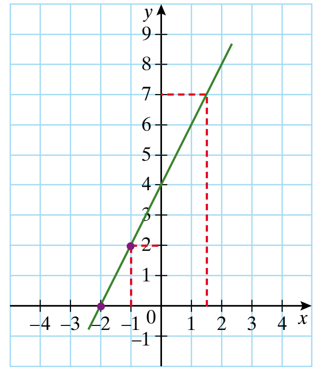
Chapter 7

exercise 7.1

1 **a**

x	–2	–1	0	1	2
y	0	2	4	6	8

b



c **i** $y = 7$ **ii** $x = -1$

d **i** $y = 2 \times 1.5 + 4 = 7$

ii $2 = 2x + 4$

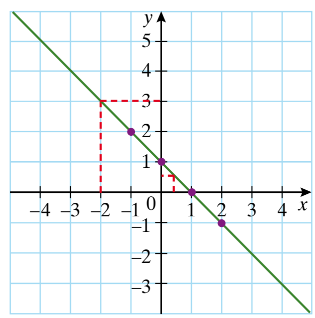
$-2 = 2x$

$x = -1$

2 **a**

x	–2	–1	0	1	2
y	3	2	1	0	–1

b



c **i** $y = 0.5$ **ii** $x = -2$

d **i** $y = -1 \times 0.5 + 1 = 0.5$

ii $3 = -x + 1$

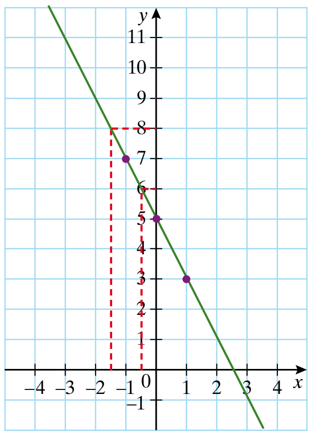
$-2 = -x$

$x = -2$

3 **a**

x	–3	–2	–1	0	1
y	11	9	7	5	3

b



c **i** $y = 8$ **ii** $x = -0.5$

d **i** $y = -2 \times -1.5 + 5 = 8$

ii $6 = -2x + 5$

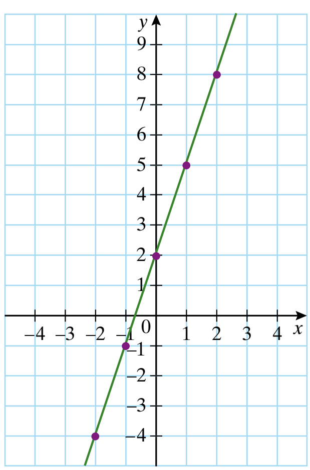
$1 = -2x$

$x = -0.5$

4 **a**

x	–2	–1	0	1	2
y	–4	–1	2	5	8

b



c **i** $y = 2$ **ii** $x \approx 0.7$

d **i** $y = 3 \times 0 + 2 = 2$

ii $0 = 3x + 2$

$-2 = 3x$

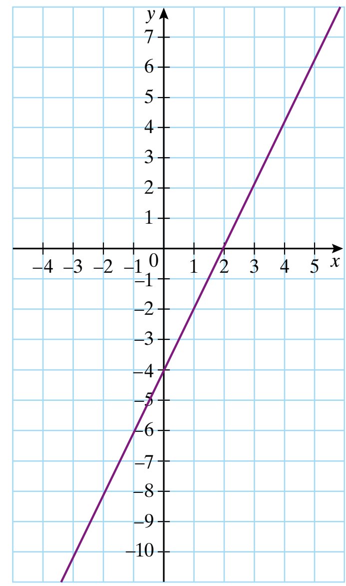
$x = \dfrac{-2}{3}$

5 **a** $2x - 4 = y$

b

x	–3	–2	–1	0	1	2	3	4	5
y	–10	–8	–6	–4	–2	0	2	4	6

c



d 2

e same as doubling

f –4

g it's the amount subtracted.

exercise 7.2

1 **a** linear **b** non-linear
c linear **d** linear
e non-linear **f** linear
g linear **h** non-linear
i linear **j** linear

2

	i	ii	iii
a	1	(0, –1)	$y = x - 1$
b	2	(0, 0)	$y = 2x$
c	1	(0, 5)	$y = x + 5$
d	4	(0, 0)	$y = 4x$
e	2	(0, 5)	$y = 2x + 5$
f	–1	(0, 0)	$y = -x$
g	–1	(0, 2)	$y = -x + 2$
h	6	(0, 1)	$y = 6x + 1$
i	–5	(0, 0)	$y = -5x$
j	0.5	(0, 0)	$y = 0.5x$

3 C

4

	i	ii	iii
a	$y = 2x + 3$	2	(0, 3)
b	$y = \frac{1}{2}x - 1$	$\frac{1}{2}$	(0, –1)
c	$x + y = 3$	–1	(0, 3)

exercise 7.3

1

Equation	Gradient	y-intercept
$y = 3x - 5$	3	–5
$y = -5x + 4$	–5	4
$y = x + 3$	1	3
$y = -2x + 1$	–2	1
$y = x$	1	0
$y = 3x + 2$	3	2
$y = -4x - 1$	–4	–1
$y = -x - 5$	–1	–5
$y = 2x + 7$	2	7
$y = 0.5x + 3.1$	0.5	3.1

2 **a** $y = 2x - 6$ **b** $y = -3x + 4$
c $y = \frac{2}{3}x - 1$ **d** $y = -\frac{4}{5}x + 9$
e $y = \frac{5}{7}x - 3$ **f** $y = -3x + \frac{1}{2}$

3 **a** **i** 1 **ii** (0, 2) **iii** $y = x + 2$
b **i** 1 **ii** (0, 4) **iii** $y = x + 4$
c **i** –1 **ii** (0, 4) **iii** $y = -x + 4$
d **i** $-3\frac{1}{3}$ **ii** $\frac{-6}{7}$ **iii** $y = \frac{-6}{7} - 3\frac{1}{3}$
e **i** 4 **ii** (0, –3) **iii** $y = 4x - 3$
f **i** 3 **ii** (0, 4) **iii** $y = 3x + 4$
g **i** 2 **ii** (0, 3) **iii** $y = 2x + 3$
h **i** –2 **ii** (0, –1) **iii** $y = -2x - 1$

4 D

5 C

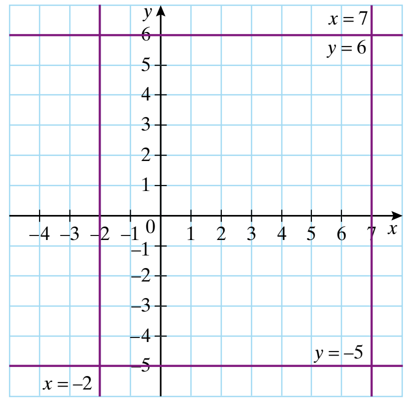
6 **a** $y = 2x + 7$ **b** $y = -5x + 38$
c $y = -3x - 2$ **d** $y = 4x - 13$
e $y = -\frac{5}{4}x + 11\frac{3}{4}$ **f** $y = \frac{2}{3}x - 2\frac{1}{3}$
g $y = \frac{3}{4}x - 2\frac{3}{4}$ **h** $y = -\frac{1}{2}x + 2\frac{1}{2}$

7 D

8 **a** If y-intercept is zero (i.e. the constant is zero).
b If gradient is positive it is upwards to the right, if negative then downward to the right.

9 **a** $y = x$ **b** $x = -3$

10



11 $y = -3x + 2$, $y = -3x$, $y = -3x + 7$, $y = -3x + 2$ $\}$ are parallel

12 **a** $7x - 3y - 24 = 0$ **b** $2x - 5y - 35 = 0$
c $2x + 11y + 11 = 0$ **d** $3x - 2y + 5 = 0$
e $4x + 5y + 9 = 0$ **f** $3x + 4y + 10 = 0$

13 **a** $y = -\frac{1}{2}x + 1\frac{1}{4}$

$m = -\frac{1}{2}, b = 1\frac{1}{4}$

b $y = 2\frac{1}{2}x - 5$

$m = 2\frac{1}{2}, b = -5$

c $y = 1\frac{1}{3}x + 5$

$m = 1\frac{1}{3}, b = 5$

d $y = \frac{5}{8}x - 2\frac{1}{2}$

$m = \frac{5}{8}, b = -2\frac{1}{2}$

e $y = 1\frac{1}{2}x - 6\frac{1}{4}$

$m = 1\frac{1}{2}, b = -6\frac{1}{4}$

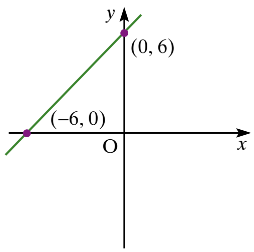
f $y = -1\frac{3}{4}x + 2\frac{1}{2}$

$m = -1\frac{3}{4}, b = 2\frac{1}{2}$

exercise 7.4

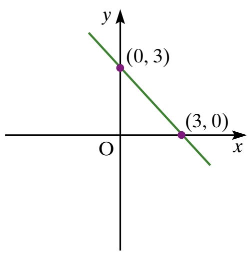
1 **a** **i** (0, 6)(−6, 0)

ii



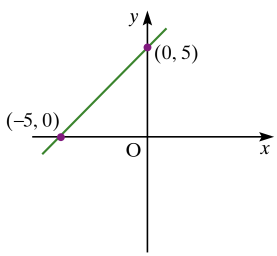
b **i** (0, 3)(3, 0)

ii



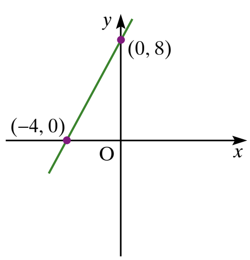
c **i** (0, 5)(−5, 0)

ii



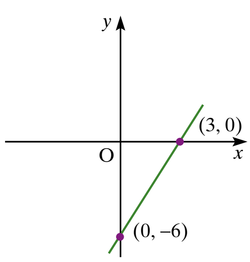
d **i** (0, 8)(−4, 0)

ii



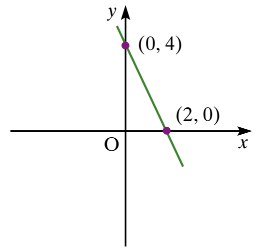
e **i** (0, −6)(3, 0)

ii



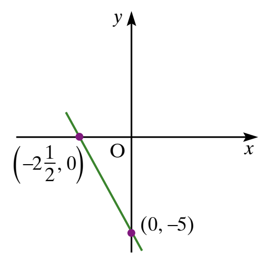
f **i** (0, 4)(2, 0)

ii



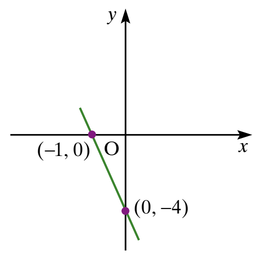
g **i** $(0, -5)(-2\frac{1}{2}, 0)$

ii



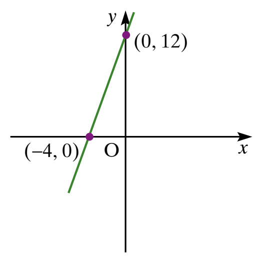
h **i** (0, −4)(−1, 0)

ii



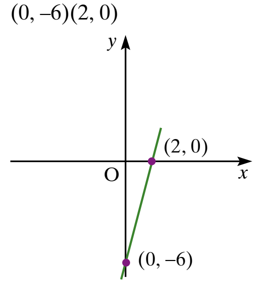
i **i** (0, 12)(−4, 0)

ii



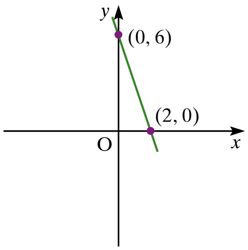
j **i** (0, −6)(2, 0)

ii



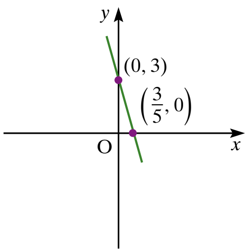
k **i** (0, 6)(2, 0)

ii



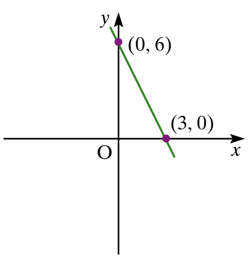
l **i** $(0, 3)(\frac{3}{5}, 0)$

ii



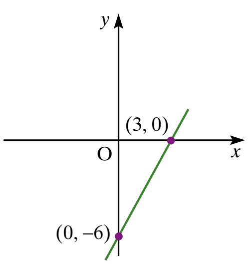
2 **a** **i** (0, 6)(3, 0)

ii



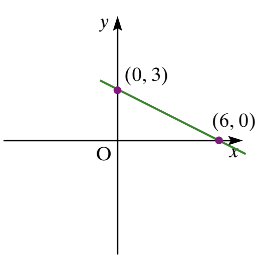
b **i** (0, –6)(3, 0)

ii



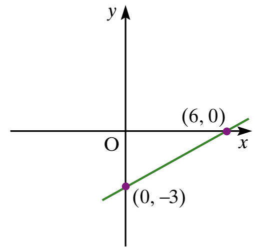
c **i** (0, 3)(6, 0)

ii



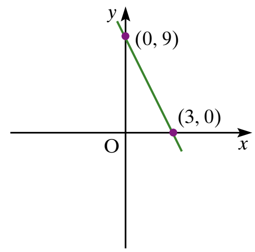
d **i** (0, –3)(6, 0)

ii



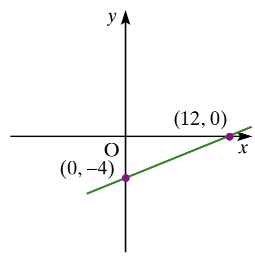
e **i** (0, 9)(3, 0)

ii



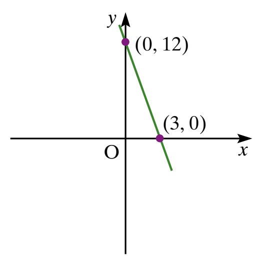
f **i** (0, –4)(12, 0)

ii



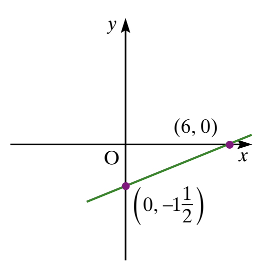
g **i** (0, 12)(3, 0)

ii



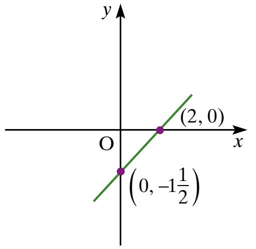
h **i** $(0, -1\frac{1}{2})(6, 0)$

ii



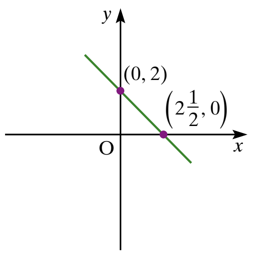
i **i** $(0, -1\frac{1}{2})(2, 0)$

ii



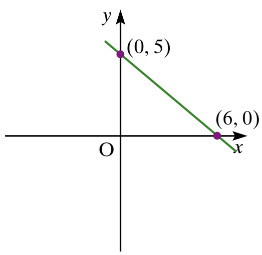
j **i** $(0, 2)(2\frac{1}{2}, 0)$

ii



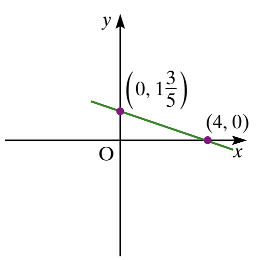
k **i** $(0, 5)(6, 0)$

ii

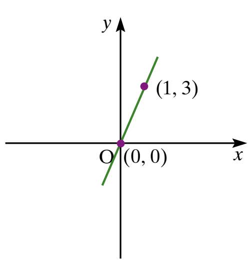


l **i** $(0, 1\frac{3}{5})(4, 0)$

ii



3 **a**



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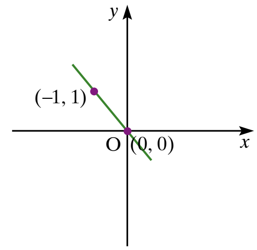

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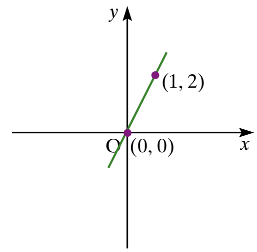

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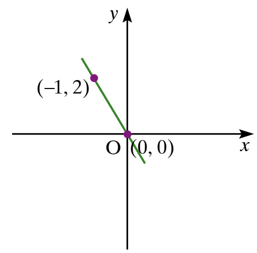

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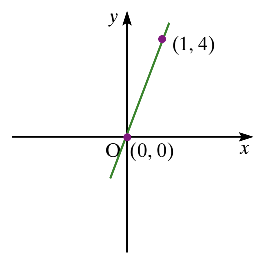

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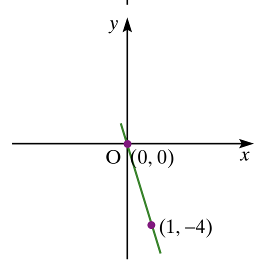
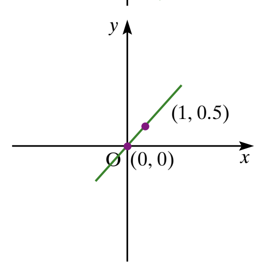
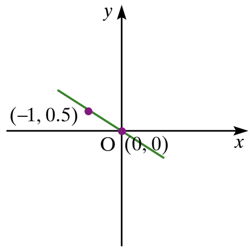


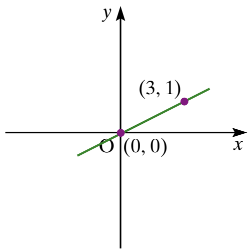
g



h



i



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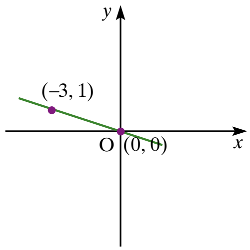

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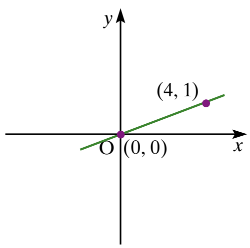

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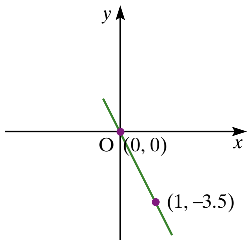
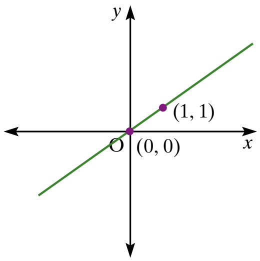


4 E

5 C

6 a



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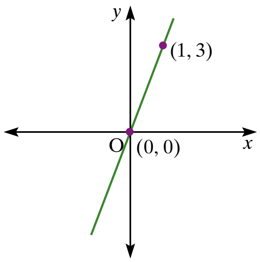

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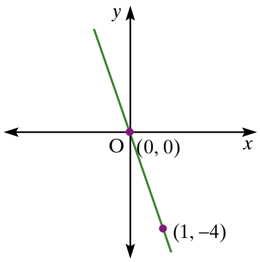
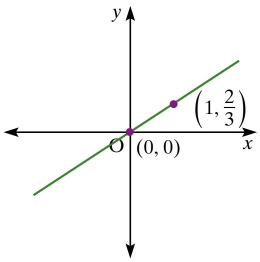


d



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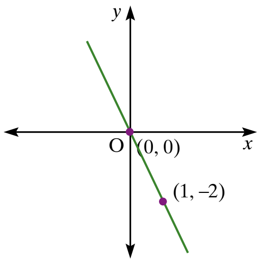

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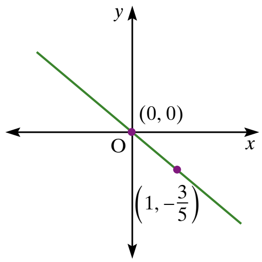


7 C

8 E

9 **a** B **b** C **c** A **d** D

exercise 7.5

1 **a** $T = 50 + 40h$ **b** $C = 10 + 5k$
c $A = 40 + 6d$ **d** $d = 30 - 5t$
e $F = 55 + 5n$

2 **a**

t	0	1	2	3	4	5	6	7
d	700	600	500	400	300	200	100	0

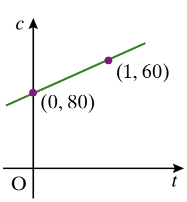
b When the values of t increase by 1, the values of d decrease by a constant amount of 100. This shows that the relationship is linear.
c –100. It is has the value of the speed. It is negative because positive is away from home and the travel is towards home.
d (0, 700). At the start the distance is 700 km away from home.
e $d = 700 - 100t$

3 **a**

t	0	1	2	3	4
d	60	115	170	225	280

b When the values of t increase by 1, the values of d increase by a constant amount of 55. This shows that the relationship is linear.
c 55. This is the hourly rate
d (0, 60). This is the fixed fee at the start
e $d = 60 + 55t$

4



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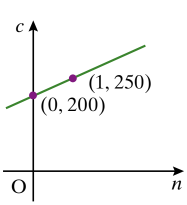

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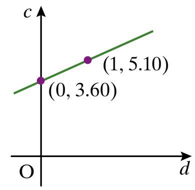
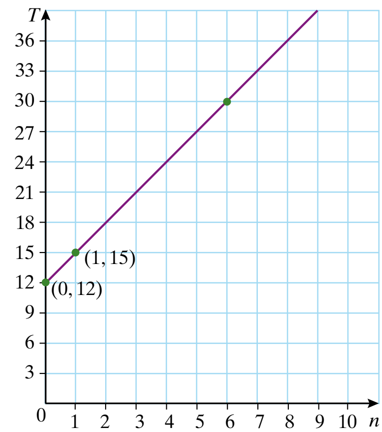


7 **a** 8000 litres **b** 150 litres per hour

8 **a** \$60 **b** \$20 per week

9 **a** 5.5 metres **b** 1.5 metres

10 **a** $T = 12 + 3n$
b



c **i** \$30 **ii** \$39
d **i** $T = 12 + 3 \times 6$
$= \$30$
ii $T = 12 + 3 \times 9$
$= \$39$
e 11 rides
f $45 = 12 + 3n$
$33 = 3n$
$11 = n$
$\therefore$ 11 rides

11 **a**

t	0	1	2	3	4	5
d	80	170	260	350	440	530

b When the values of t increase by 1, the values of d increase by a constant amount of 90. This shows that the relationship is linear.
c gradient = 90, y-intercept = (0, 80)
d $d = 80 + 90t$
e 395 km
f 8 hours

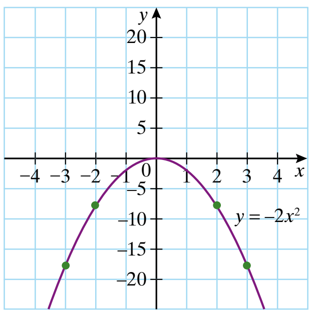
12 **a** distance travelled **b** travel expenses
c $T = 155 + 0.21k$ **d** \$276.80

exercise 7.6

1 a i

x	–3	–2	–1	0	1	2	3
y	–18	–8	–2	0	–2	–8	–18

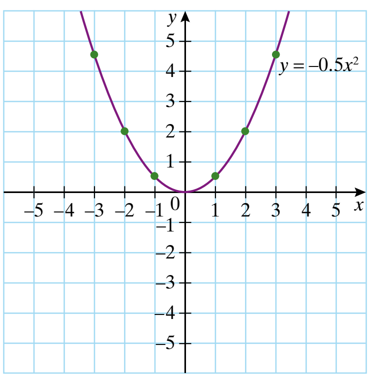
ii



b i

x	–3	–2	–1	0	1	2	3
y	4.5	2	0.5	0	0.5	2	4.5

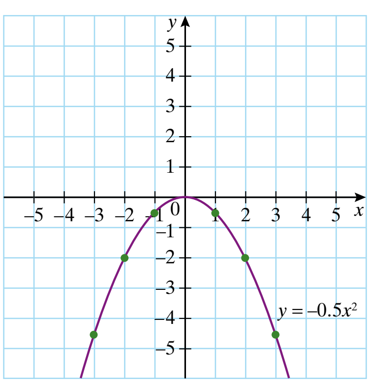
ii



c i

x	–3	–2	–1	0	1	2	3
y	–4.5	–2	–0.5	0	–0.5	–2	–4.5

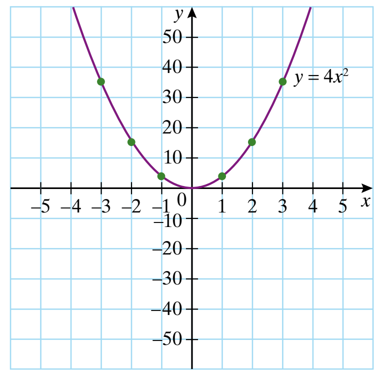
ii



d i

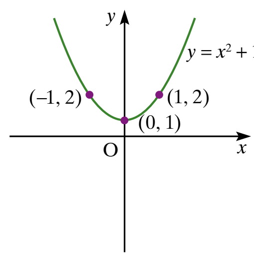
x	–3	–2	–1	0	1	2	3
y	36	16	4	0	4	16	36

ii

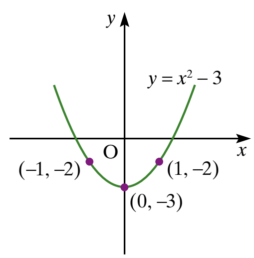


2 All have an axis of symmetry: $x = 0$

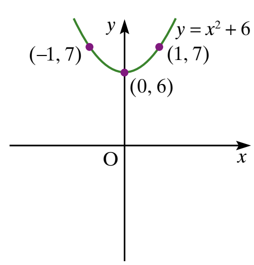
a



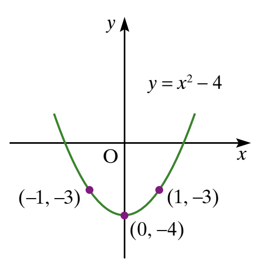
b



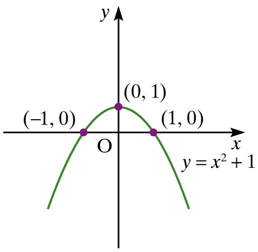
c



d



e



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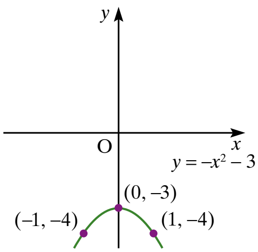

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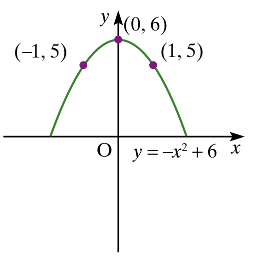

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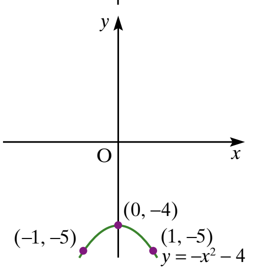
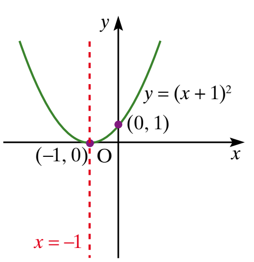


3 **a**



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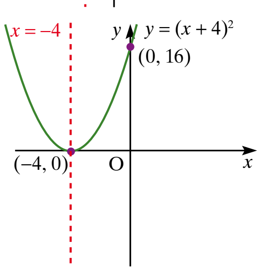

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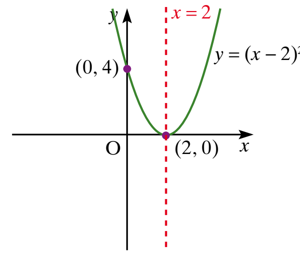

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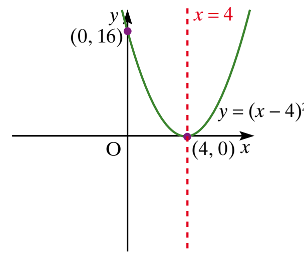

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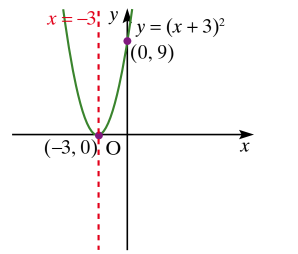

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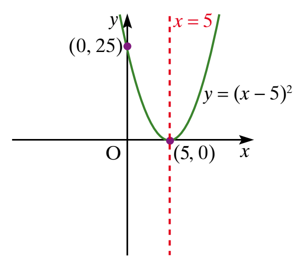

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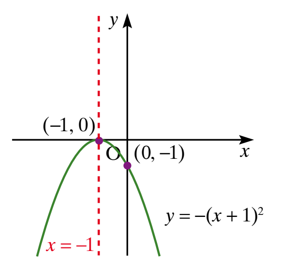

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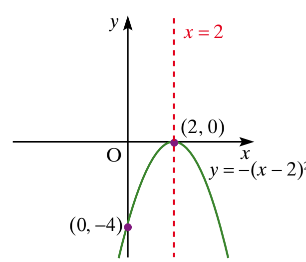
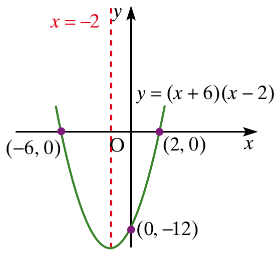


4 **a**



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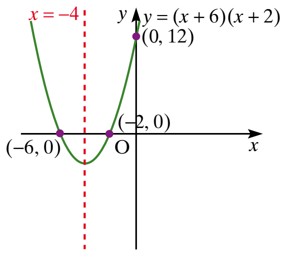

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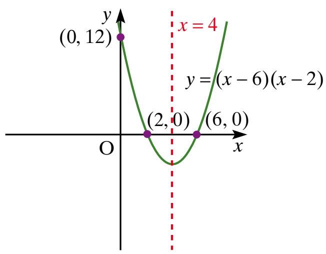

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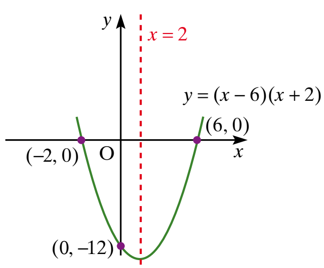

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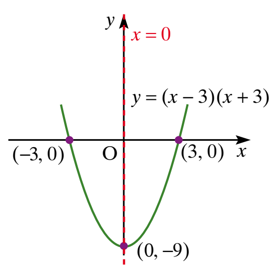

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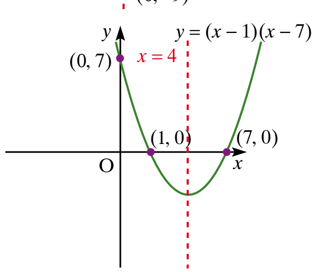

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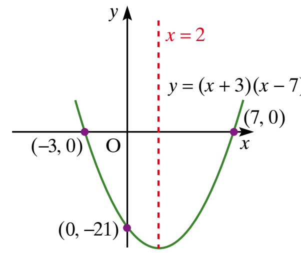

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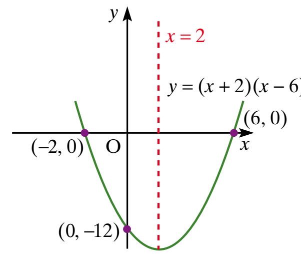

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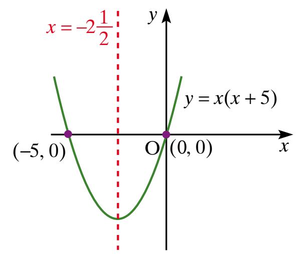

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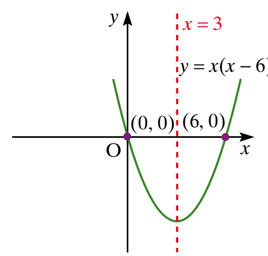

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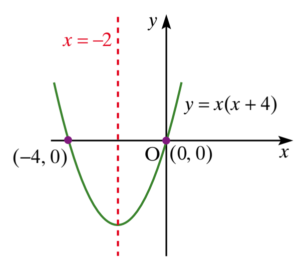

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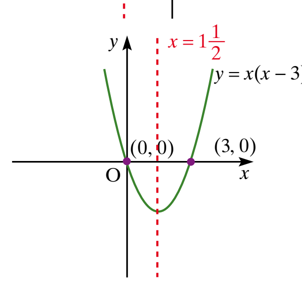
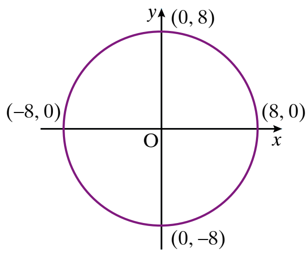
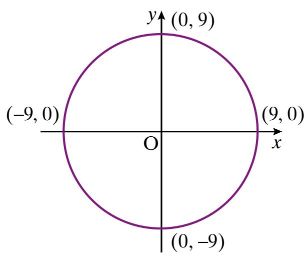


5 **a** $x^2 + y^2 = 9$ **b** $x^2 + y^2 = 49$
c $x^2 + y^2 = 36$ **d** $x^2 + y^2 = 121$
e $x^2 + y^2 = 225$ **f** $x^2 + y^2 = 12$
g $x^2 + y^2 = 50$ **h** $x^2 + y^2 = 18$

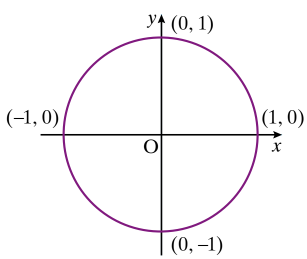
6 **a** **i** 8
ii



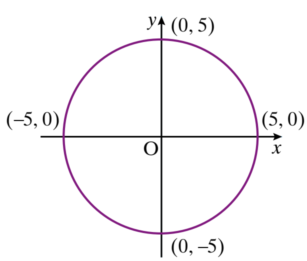
b **i** 9
ii



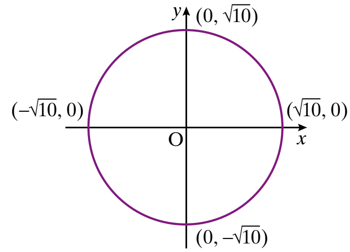
c **i** 1
ii



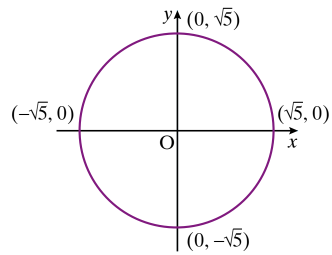
d **i** 5
ii



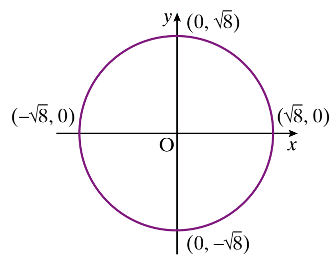
e **i** $\sqrt{10}$
ii



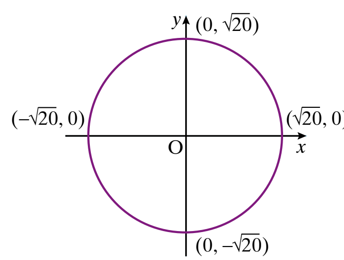
f **i** $\sqrt{5}$
ii



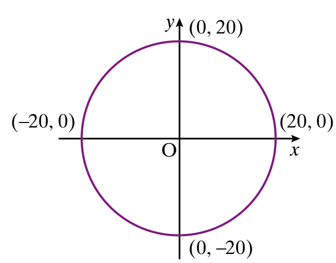
g **i** $\sqrt{8}$
ii



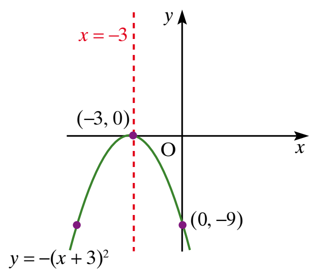
h **i** $\sqrt{20}$
ii



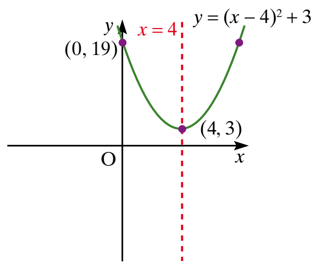
i **i** 20
ii



7 **a**



b



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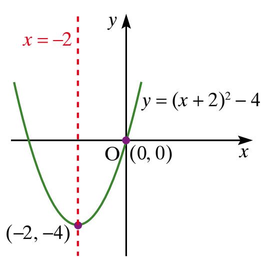

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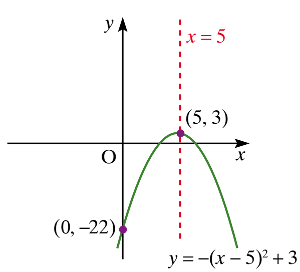


Revision

1 C

2 D

3 A

4 A

5 E

6 **a** $y = -3x + 5$ **b** $y = \frac{3}{4}x - 7$

7 **a** gradient is 2, y-intercept is –7
b gradient is –1.5, y-intercept is –3.5

8 **a** $y = 4x + 6$ **b** $y = -\frac{1}{2}x + 6\frac{1}{2}$
c $y = \frac{1}{2}x + 1\frac{1}{2}$ **d** $y = -\frac{3}{4}x + 6$

9 **a** $y = -5$ **b** $x = -1$

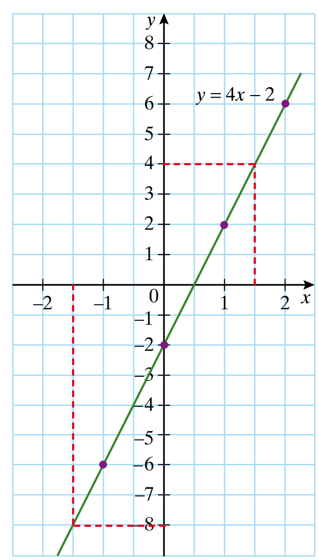
10 **a** $y = \frac{3}{4}x - 4\frac{1}{2}$, so gradient is $\frac{3}{4}$ and y-intercept is $\left(0, -4\frac{1}{2}\right)$
b $y = \frac{5}{6}x - 4$, so gradient is $\frac{5}{6}$ and y-intercept is (0, –4)

11 **a** $2x + 5y - 30 = 0$ **b** $3x - 8y - 40 = 0$

12 **a**

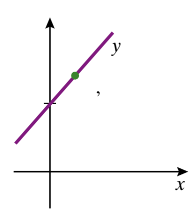
x	–2	–1	0	1	2
y	–10	–6	–2	2	6

b

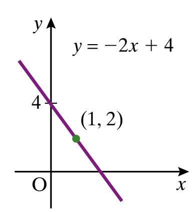


c **i** –8 **ii** 1.5

13 **a**



b



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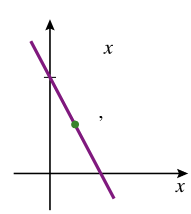

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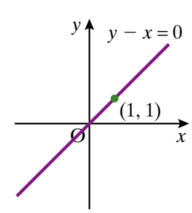

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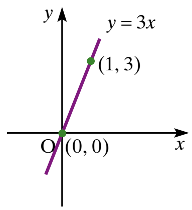

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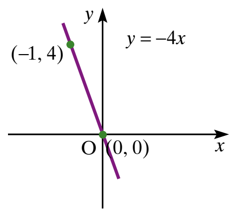

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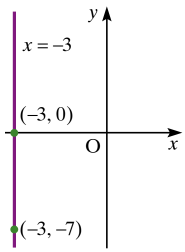

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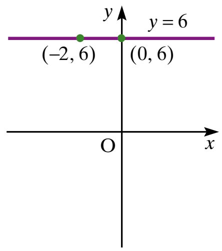

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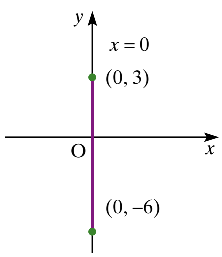
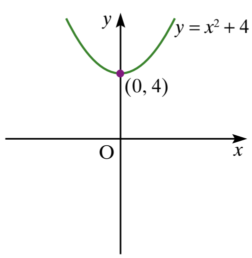
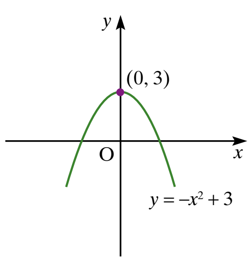
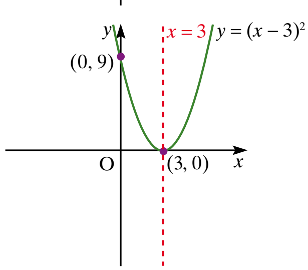
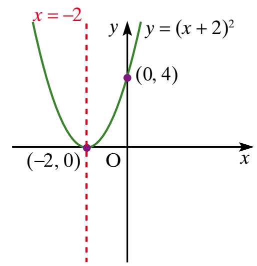
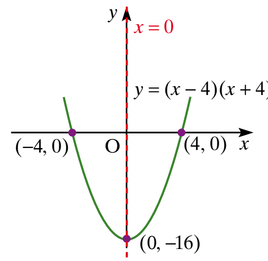


14 **a**


b


c


d


15 **a**


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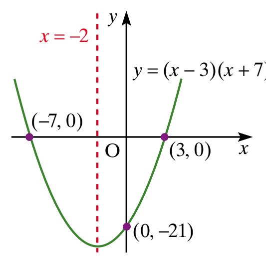
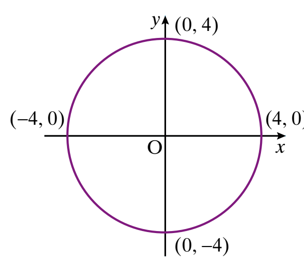


16


17 $x^2 + y^2 = 25$

18 **a** \$140 **b** \$180

c number of weeks

d

Week	1	2	3	4
Savings	\$140	\$180	\$220	\$260

e $y = 40x + 100$ **f** 5 weeks

Chapter 8

exercise 8.1

1 **a** **i** $W \propto m$ **ii** $W = km$

b **i** $C \propto n$ **ii** $C = kn$

c **i** $p \propto t$ **ii** $p = kt$

d i $C \propto r$ ii $C = kr$

e i $V \propto h$ ii $V = kh$

f i $I \propto t$ ii $I = kt$

2 a

A	0	1	2	3	4	5
n	0	38	76	114	152	190

b

A	0	1	2	3	4	5
n	0	19	38	57	76	95

c

A	0	1	2	3	4	5
n	0	5	10	15	20	25

d

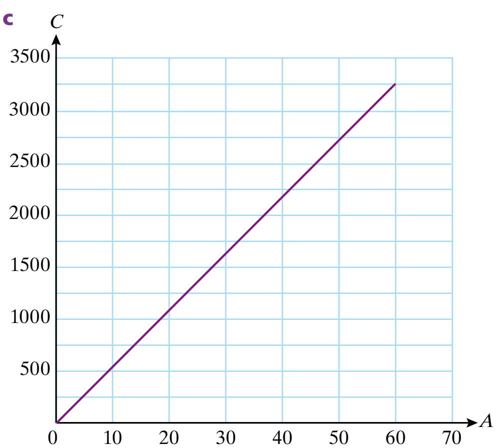
A	0	1	2	3	4	5
n	0	6.25	12.5	18.75	25	31.25

3 a $C = 55A$

b

A	0	10	20	30	40	50	60
C	0	550	1100	1650	2200	2750	3300

c



d The graph is a straight line passing through zero.

e 17 f \$1430

4 a $C \propto L, C = kL$ b 80

c $C = 80L$

d

Length of fence (L m)	0	10	20	30	40	50	56
Cost (\$C)	0	800	1600	2400	3200	4000	4480

e \$5440

5 a

Area of wall (A m^2)	0	1	2	3	4	5	10
Number of blocks (n)	0	27	54	81	108	135	270

b $n \propto A, n = kA$ c $n = 27A$

d approximately 18.5 m^2

6 a $d \propto t, d = kt$ b 96

c 96 km/h d $d = 96t$

e 240 km

f 3.25 hours = 3 hours 15 minutes

7 a $d \propto t, d = kt$ b 95

c 95 km/h d $d = 95t$

e 213.75 km

f 3.2 hours = 3 hrs 12 minutes

8 750 mL

9 a $W \propto I, W = kI$ b 240

c $W = 240I$ d 0.25

10 a $C \propto d, C = kd$ b π

11 200 g

12 21.85 L

13 a $d \propto m,\ d = km$ b 1.5

c $d = 1.5m$ d 63 mg

14 a $w \propto m, w = km$ b 9.8

c $w = 9.8$ m d 666.4 Newtons

15

ethane	60	120	200	500
oxygen	224	448	$746\frac{2}{3}$	$1866\frac{2}{3}$
carbon dioxide	176	352	$586\frac{2}{3}$	$1466\frac{2}{3}$
water	108	216	360	900

exercise 8.2

1 a i

× 2 × 2

x	1	2	3	4	5
y	30	15	10	7.5	6

÷ 2 ÷ 2

ii 30 iii $y = \frac{30}{x}$

b i

× 2 × 2

x	1	2	4	5	10
y	40	20	10	8	4

÷ 2 ÷ 2

ii 40 iii $y = \frac{40}{x}$

c **i**

x	0.5	1	2	4	8
y	32	16	8	4	2

ii 16 **iii** $y = \dfrac{16}{x}$

d **i**

x	2	3	4	8	15
y	60	40	30	15	8

ii 120 **iii** $y = \dfrac{120}{x}$

2 **a** $W \propto \dfrac{1}{A}$, $W = \dfrac{k}{A}$ **b** $T \propto \dfrac{1}{S}$, $T = \dfrac{k}{S}$

c $I \propto \dfrac{1}{R}$, $I = \dfrac{k}{R}$ **d** $n \propto \dfrac{1}{C}$, $n = \dfrac{k}{C}$

e $n \propto \dfrac{1}{A}$, $n = \dfrac{k}{A}$ **f** $V \propto \dfrac{1}{P}$, $V = \dfrac{k}{P}$

3 **a** **i**

x	1	2	5	8	10	20
y	120	60	24	15	12	6

ii 120 **iii** $y = \dfrac{120}{x}$

b **i**

x	1	2	3	4	5	6
y	36	18	12	9	7.2	6

ii 36 **iii** $y = \dfrac{36}{x}$

c **i**

x	1	2	3	5	6	10
y	150	75	50	30	25	15

ii 150 **iii** $y = \dfrac{150}{x}$

d **i**

A	1	2	3	4	5	6
n	144	72	48	36	28.8	24

ii 144 **iii** $n = \dfrac{144}{A}$

4 **a**

n	1	2	3	4	6
c	24	12	8	6	4

b As the number of people increases, the number of chocolates that each person receives decreases.

c $c \propto \dfrac{1}{n}$, $c = \dfrac{k}{n}$ **d** 24 **e** $c = \dfrac{24}{n}$

5 **a**

Size (s mL)	30	60	90	120	144	180
Number (n)	24	12	8	6	5	4

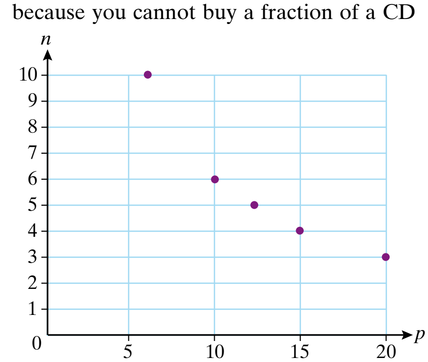
b $n \propto \dfrac{1}{S}$, $n = \dfrac{k}{S}$ **c** 720 **d** $n = \dfrac{720}{S}$

6 **a** 4 **b** 5

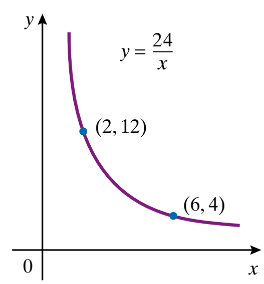
c

p	6	10	12	15	29
n	10	6	5	4	3

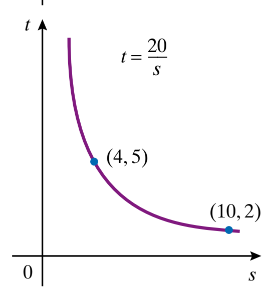
d no, it is not appropriate to join the points because you cannot buy a fraction of a CD



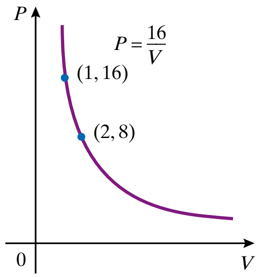
7 **a**



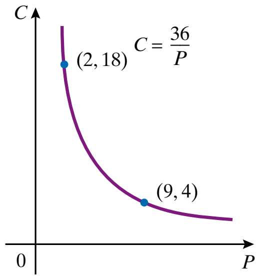
b



c



d



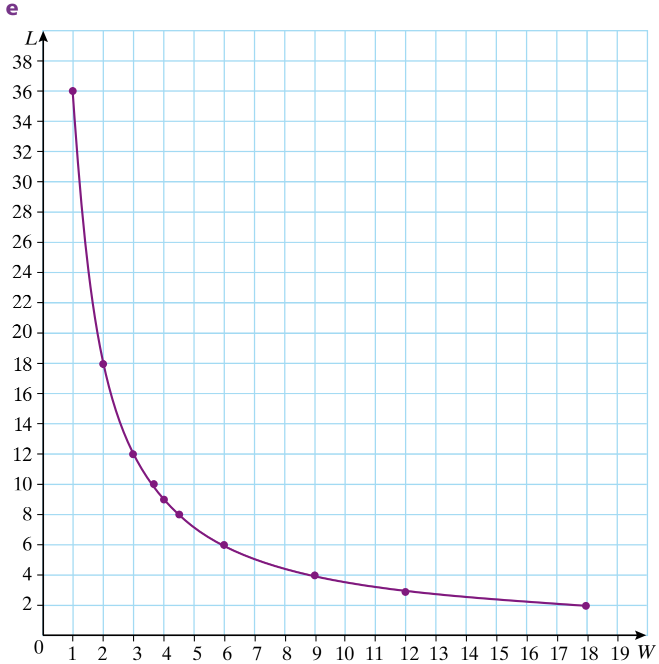
8 **a** answers will vary, but an example of the table is shown below

L	36	18	12	10	9	8	6	4	3	2
W	1	2	3	3.6	4	4.5	6	9	12	18

b $W \propto \frac{1}{L}$, $W = \frac{k}{L}$ **c** 36

d $W = \frac{36}{L}$

e



f yes, because they are rectangle length and width and do not have to be whole numbers

9 **a** $e \propto \frac{1}{n}$, $e = \frac{k}{n}$

b 360

c $e = \frac{360}{n}$

d

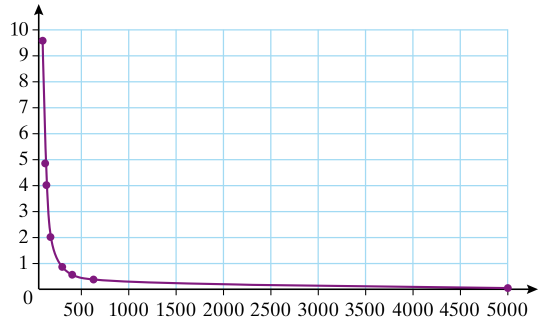
n	3	4	5	6	8	10
e	120	90	72	60	45	36

e 30° **f** 24

10 **a**

Appliance	Resistance *R* ohm	Current *I* amp
Fan heater	25	9.6
Toaster	50	4.8
Refrigerator	60	4.0
Heated towel rail	120	2.0
TV	300	0.8
Desktop computer	400	0.6
Incandescent light globe	600	0.4
Compact fluorescent light	5000	0.048

b



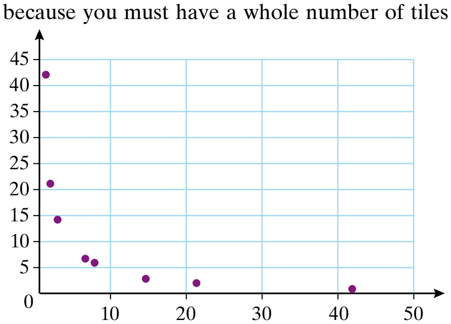
c 1200 ohm

11 **a**

n	1	2	3	6	7	14	21	42
S	42	21	14	7	6	3	2	1

b $S = \dfrac{42}{n}$

c no, it is not appropriate to join the points because you must have a whole number of tiles



d no, because zero multiplied by any number is zero

Revision

1 B

2 E

3 C

4 B

5 C

6 **a** **i** $C \propto n$ **ii** $C = kn$

b **i** $F \propto m$ **ii** $F = km$

c **i** $A \propto w$ **ii** $A = kw$

d **i** $P \propto T$ **ii** $P = kT$

7 **a**

n	0	1	2	3	4	5
A	0	27	54	81	108	135

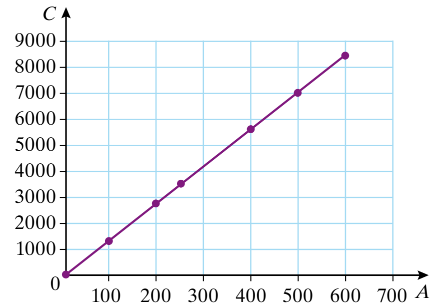
b 27 **c** $A = 27n$

8 **a** $C = 14A$

b

A	0	100	200	250	400	500	600
C	0	1400	2800	3500	5600	7000	8400

c



d the graph is a straight line passing through zero. As the area painted increases, the cost increases in the same proportion.

e 357 m^2

f \$8960

9 **a** $C \propto L$, $C = kL$ **b** 38 **c** $C = 38L$

d

Length of guttering (L m)	0	50	108	120	150	200
Cost (\$$C$)	0	1900	4104	4560	5700	7600

10 **a** 1.609

b

d km	1000	2000	3600	8000	12 000
n	1609	3218	5792.4	12 872	19 308

c $n = 1.609d$ **d** 24 778.6 **e** 18 646 km

11 **a** **i** 36 668 **ii** 7334

b \$22.82

12 **a** $L \propto \dfrac{1}{W}$, $L = \dfrac{k}{W}$ **b** $R \propto \dfrac{1}{I}$, $R = \dfrac{k}{I}$

c $T \propto \dfrac{1}{n}$, $T = \dfrac{k}{n}$ **d** $n \propto \dfrac{1}{R}$, $n = \dfrac{k}{R}$

13 **a**

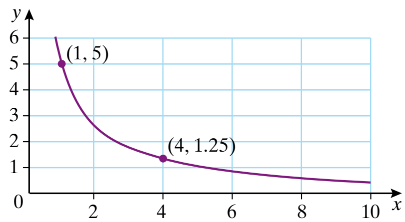
× 2 × 2

x	1	2	4	5	8	10
y	5	2.5	1.25	1	0.625	0.5

÷ 2 ÷ 2

b 5 **c** $y = \dfrac{5}{x}$

d



e 0.2

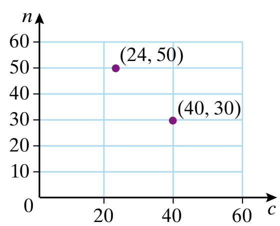
14 **a**

C	24	25	30	40	48
n	50	48	40	30	25

b $n \propto \frac{1}{C}$, $n = \frac{k}{C}$ **c** 1200

d $n = \frac{1200}{C}$

e



Chapter 9

exercise 9.1

1 **a** not congruent because although they have two matching sides the angle in between them is not equal.
b congruent (AAS)
c not congruent because although they have two matching sides the angle in between them is not equal.
d congruent (SAS) using isosceles triangles
e not congruent because although they have two matching sides the angle in between them is not equal.
f congruent (SSS) using Pythagoras's theorem
g not congruent because although the angles are equal in both triangles, matching sides are not equal
h congruent (AAS) using isosceles triangles.

2 **a** $a = 43$, $b = 109$, $c = 33$, $d = 24$
b $h = 76$, $m = 60$
c $x = 360$, $y = 39$
d $t = 38$, $u = 24$

3 In $\triangle ADB$ and $\triangle ADC$,
AB = AC ($\triangle ABC$ is isosceles)
AD is common
$\angle ADB = \angle ADC$ ($AD \perp BC$)
So $\triangle ADB \equiv \triangle ADC$ (RHS)
So BD = DC (matching sides in congruent triangles)
So AD bisects BC.

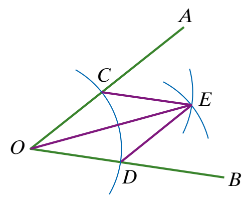
4 **a** In $\triangle ADC$ and $\triangle ABC$,
AD = AB (definition of a kite)
DC = BC (definition of a kite)
AC is common
So $\triangle ADC \equiv \triangle ABC$ (SSS)
Thus the long diagonal of a kite divides the kite into two congruent triangles.
b $\angle ADC = \angle ABC$ (matching angles in congruent triangles)
Thus the kite has one pair of opposite angles equal.

5 **a** In $\triangle AOB$ and $\triangle COD$, $\angle AOB = \angle COD$ (vertically opposite angles equal)
$\angle OBA = \angle ODC$ (alternate angles equal because $AB \parallel DC$, DB is transversal)
$AB = DC$ (opposite sides of parallelogram are equal)
So $\triangle AOB \equiv \triangle COD$ (AAS)
b So $AO = OC$ and $DO = OB$ (matching sides in congruent triangles)
So the diagonals of a parallelogram bisect each other.

6 **a** In $\triangle AMD$, $\triangle AMB$, $\triangle CMD$ and $\triangle CMB$
$\left.\begin{array}{l} AM = MC \\ MD = MB \end{array}\right\}$ diagonals of a parallelogram bisect each other
$AB = BC = CD = DA$ (sides of a rhombus are all equal)
So $\triangle AMD \equiv \triangle AMB \equiv \triangle CMD \equiv \triangle CMB$ (SSS)
b So $\angle AMD = \angle AMB = \angle CMD = \angle CMB$ (matching angles in congruent triangles)
So each angle must be 90°
So the diagonals of a rhombus intersect at right angles.

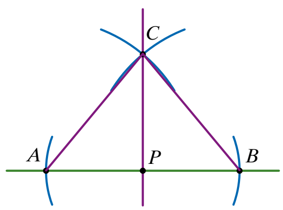
7 **a** If CE and DE are joined, two triangles are formed.
In $\triangle OCE$ and $\triangle ODE$
$OC = OD$ (arcs with equal radius)
$CE = DE$ (arcs with equal radius)
OE is common
So $\triangle OCE \equiv \triangle ODE$ (SSS)
So $\angle COE = \angle DOE$ (matching angles in congruent triangles) so, OE exactly bisects $\angle AOB$.

b If the arcs were not the same radius, CE would not equal DE so $\triangle OCE$ and $\triangle ODE$ would not be congruent and $\angle AOB$ would not be bisected.



8 **a** The radius of the arcs drawn from A and B needs to be greater than AP, otherwise the two arcs would either not meet or just meet at P instead of intersecting.

b Draw segments AC and BC



In $\triangle ACP$ and $\triangle BCP$
$AP = PB$ (arcs with equal radius)
$AC = BC$ (arcs with equal radius)
CP is common
So $\triangle ACP \equiv \triangle BCP$ (SSS)
$\angle CPA = \angle CPB$ (matching angles in congruent triangles)
But $\angle CPA + \angle CPB = 180°$
So $\angle CPA = \angle CPB = 90°$
So $CP \perp AB$.

c The method depends on $\triangle ACP$ and $\triangle BCP$ being congruent and if the arcs weren't equal then AC ≠ BC so they would not be congruent.

exercise 9.2

1 **a** **i** enlargement **ii** 2
b **i** reduction **ii** 0.75
c **i** enlargement **ii** 1.25
d **i** reduction **ii** 0.6

2 A and C

3 The four angles of quadrilateral $ABCD$ are equal to the matching angles of quadrilateral $JKLM$.

$$\frac{JK}{AB} = \frac{5.85}{3.9} = 1.5; \ \frac{KL}{BC} = \frac{6.0}{4.0} = 1.5; \ \frac{LM}{CD} = \frac{7.2}{4.8} = 1.5;$$
$$\frac{MJ}{DA} = \frac{5.25}{3.5} = 1.5$$

The two quadrilaterals are similar because their matching angles are equal and their matching sides are in the same proportion.

4 **a** 1.5 **b** $a = 9$

5 **a** The length and width are each enlarged by 125% so the proportions of the picture are still the same.
b 45 cm by 25 cm

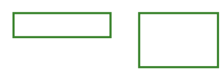
6 Quadrilaterals B, C and D have matching angles equal, but the sides of quadrilateral D are not in the same proportion. Each side of quadrilateral B is 1.5 times the matching side of quadrilateral C so quadrilaterals B and C are similar.

7 Five angles of polygon $ABCDEF$ are equal to the corresponding five angles of polygon $GHIJKL$. The sixth angles must also be equal because the angles of both hexagons must add to the same amount.

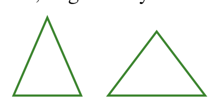
$$\frac{GH}{AB} = \frac{48}{32} = 1.5; \ \frac{HI}{BC} = \frac{60}{40} = 1.5; \ \frac{IJ}{CD} = \frac{12}{8} = 1.5;$$
$$\frac{JK}{DE} = \frac{21}{14} = 1.5; \ \frac{KL}{EF} = \frac{18}{12} = 1.5; \ \frac{LG}{FA} = \frac{45}{30} = 1.5$$

The two polygons are similar because their matching angles are equal and their matching sides are in the same proportion.

8 **a** Yes, all squares have four angles of 90°, and because the four sides are equal any square will have its four sides in proportion to all other squares.
b No, the length and width may be in different proportions. e.g.



c No, angles may be different, e.g.,



d Yes, all equilateral triangles have three angles of 60° and because the three sides are equal they will be in proportion to all other equilateral triangles.

exercise 9.3

1 **a** **i** 2
ii 200 cm^2; 800 cm^2
iii 4 times
b **i** 3
ii 12 cm^2; 108 cm^2
iii 9 times

2 **a** 1 cm^2; 4 cm^2; 9 cm^2; 16 cm^2
b 4; 9; 4; 9

3 **a** 2 **b** 16 cm
c **i** 32 cm **ii** 32 cm
d 12 cm^2
e **i** 48 cm^2 **ii** 48 cm^2

4 **a** 1.5 **b** 2.25 times

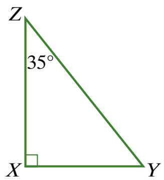
5 **a** 1.2 **b** 25:36 **c** 72 cm^2

6 **a** 8:13 **b** 32.5 cm

7 **a** $\frac{10\sqrt{8}}{\sqrt{5}}$ **b** 12.6

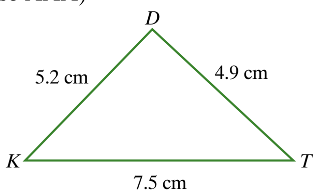
exercise 9.4

1 **a** **i**



ii $\triangle ZXY$ is similar to $\triangle ABC$ ($\angle BAC = 32°$ so AAA)

b **i**

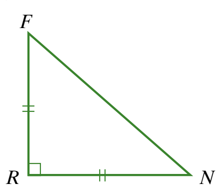


ii $\triangle DKT$ is not similar to $\triangle ABC$ ($\frac{5.2}{4} = 1.3$, $\frac{4.9}{3} \approx 1.6$, $\frac{7.5}{5} = 1.5$ so the sides are not in proportion)

c **i** already same

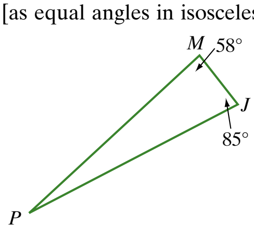
ii $\triangle PMG$ is similar to $\triangle ABC$ ($\angle ABC = \angle ACB = 73°$ so AAA)

d **i**



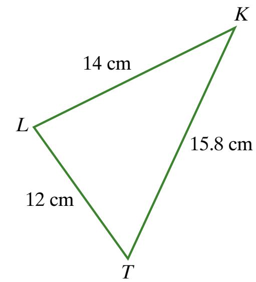
ii $\triangle FRN$ is similar to $\triangle ABC$ ($\angle BAC = \angle CAB = \angle RFN = \angle RNF = 45°$ [as equal angles in isosceles $\triangle$] so AAA)

e **i**



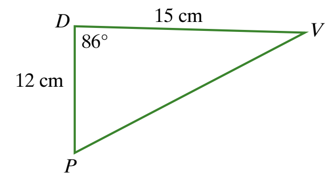
ii $\triangle MJP$ is similar to $\triangle ABC$ ($\angle CAB = 58°$ so AAA)

f **i**



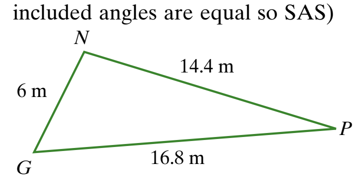
ii $\triangle KLT$ is not similar to $\triangle ABC$ ($\frac{14}{7} = 2$, $\frac{12}{6} = 2$ but $\frac{15.8}{7.8} \neq 2$ so the sides are not in proportion)

g **i**



ii $\triangle DVP$ is similar to $\triangle ABC$ ($\frac{20}{15} = \frac{4}{3}$, $\frac{16}{12} = \frac{4}{3}$, included angles are equal so SAS)

h **i**



ii $\triangle NPG$ is similar to $\triangle ABC$ ($\frac{14.4}{12} = 1.2$, $\frac{16.8}{14} = 1.2$, $\frac{6}{5} = 1.2$ so the sides are in proportion so SSS)

2 A

3 **a** **i** 1.5 **ii** a = 15

b **i** $\frac{2}{3}$ **ii** b = 8

c **i** 1.4 **ii** c = 12.6

d **i** 1.45 **ii** d = 12.18

e **i** $1\frac{1}{3}$ **ii** $e = 13\frac{1}{3}$

f **i** $\frac{15}{26}$ **ii** $f = 23\frac{1}{13}$
≈ 23.1

g **i** $1\frac{35}{107}$ **ii** $g = 23\frac{404}{535}$
≈ 23.8

h **i** $1\frac{19}{33}$ **ii** $h = 11\frac{31}{11.5}$
≈ 11.2

i $i = 5\frac{37}{55}$
≈ 5.7

4 **a** 8 cm; $DF = \frac{1}{2}AC$ so $ED = \frac{1}{2}BA$

b $\angle EDF = 60°$ because the triangles are similar and $LEDF$ corresponds to $\angle BAC$.

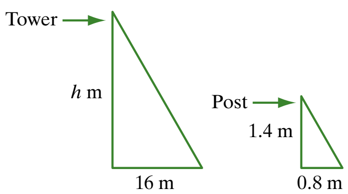
5 **a** **i** 1.25 **ii** $g = 22.5$

b **i** 1.625 **ii** $h = 11.7$, $i = 6.5$

6 **a** 3.2 **b** 3.2 m

7 8.4 m

8 **a**



b 28 m

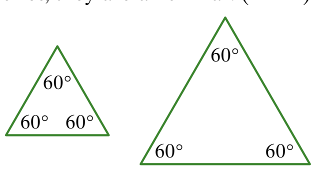
9 **a** In $\triangle ABX$ and $\triangle CDX$, $\angle ABX = \angle CDX$ and $\angle BAX = \angle DCX$ (alternate angles formed by transversals AC and BD cutting across parallel segments AB and DC)
$\angle AXB = \angle CXD$ (vertically opposite angles)
So $\triangle ABX$ is similar to $\triangle CDX$ (AAA)

b CX **c** $d = 28$, $e = 24$

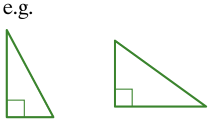
10 $x = 7.68$

11 In $\triangle ADE$ and $\triangle ABC$
$\angle DAE$ is common
$\angle ADE = \angle ABC$ (corresponding angles, $DE \parallel BC$)
$\angle AED = \angle ACB$ (corresponding angles, $DE \parallel BC$)
So, $\triangle ADE$ is similar to $\triangle ABC$ (AAA).

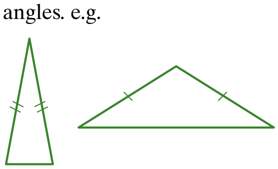
12 **a** True. All equilateral triangles have three angles of 60°. Because they all have the same angle sizes, they are all similar. (AAA) e.g.



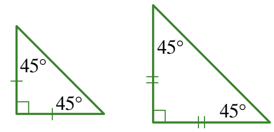
b False. Not all right-angled have the same angles. e.g.



c False. Not all isosceles triangles have the same angles. e.g.



d True. All right-angled isosceles triangles have angles of 45°, 45° and 90° (AAA) e.g.



13 **a** $\frac{40}{20} = 2$ and $\frac{24}{12} = 2$. The included angles are vertically opposite so are equal. Thus the triangles are similar (SAS).

b The triangles are similar so $\angle ABC$ and $\angle BCD$ are equal and $\angle BAD$ and $\angle ADC$ are equal. These are both sets of alternate angles. So the top AB and floor CD must be parallel, as alternate angles are equal.

14 **a** In $\triangle PBA$ and $\triangle ABC$
$\angle APB = \angle CAB = 90°$ (given)
$\angle PBA = \angle ABC$ (same angle)
So $\triangle PBA$ is similar to $\triangle ABC$ (AAA)
In $\triangle PAC$ and $\triangle ABC$,
$\angle APC = \angle CAB = 90°$ (given)
$\angle ACP = \angle BCA$ (same angle)
So $\triangle PAC$ is similar to $\triangle ABC$ (AAA)

b $AP = 4.8$ cm, $PB = 3.6$ cm

c $DQ = 6.72$ cm, $EQ = 23.04$ cm

15 $a = 6.4$, $b = 7.5$

16 $BE = 8$ cm, $CE = 12.5$ cm

Revision

1 D

2 C

3 D

4 C

5 E

6 **a** $\angle BAC = 42°$ so the triangles are not congruent

b $\triangle ABC \equiv \triangle DSY$ (RHS)

c $\triangle ABC = \triangle FHN$ (SAS)

d $\triangle ABC \equiv \triangle EGT$ (SSS)

7 In $\triangle ABD$ and $\triangle CDB$,
$AD = CB$ (opposite sides of rectangle $ABCD$)
$AB = CD$ (opposite sides of rectangle $ABCD$)
BD is common
$\therefore \triangle ABD \equiv \triangle CBD$ (SSS)
So, the diagonal BD divides the rectangle $ABCD$ into two congruent triangles.

8 **a** 1.4 **b** $a = 2.52$ **c** 11.6 m

d **i** 16.24 m **ii** 16.24 m

e 7.2 m^2

f **i** 14.112 m^2 **ii** 14.112 m^2

9 **a** $\frac{13.75}{11} = 1.25$; $\frac{5}{4} = 1.25$; $\frac{15}{12} = 1.25$

So $\triangle ABC$ and $\triangle MPH$ are similar (SSS)

b $\frac{50}{28} \approx 1.7857$; $\frac{28}{16} = 1.75$. Even though the included angles are equal, the two pairs of corresponding sides are not in the same proportion, so $\triangle ABC$ and $\triangle XPJ$ are not similar.

10 **a** 1.5 **b** $a = 39.6$; $b = 23.2$

11 **a** 3

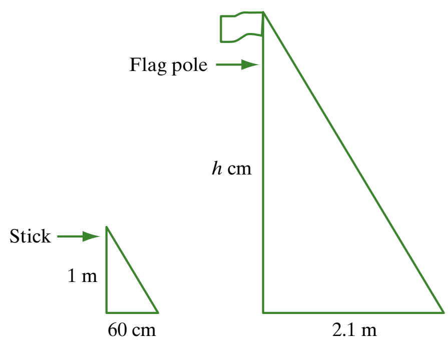
b triangle A:1 cm^2; triangle B:9 cm^2

c 9 times

12 13.5 m

13 $a = 7.5$, $b = 8.1$

14 **a**



b 3.5 m

Chapter 10

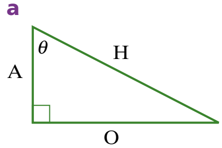
exercise 10.1

1 **a** adjacent **b** hypotenuse

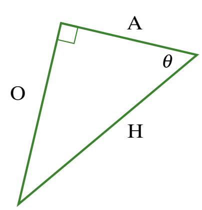
c opposite **d** adjacent

e opposite **f** hypotenuse

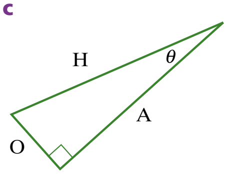
2 **a**



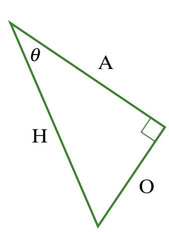
b



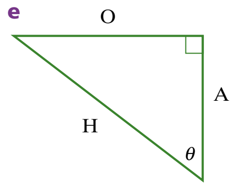
c



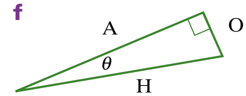
d



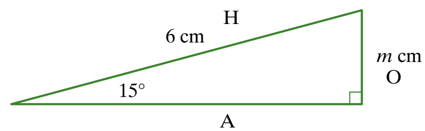
e



f



3 **a**

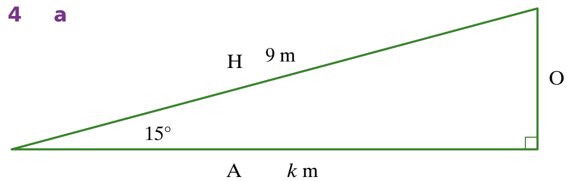


b $\frac{\text{Opp}}{\text{Hyp}} = 0.259$ **c** $\frac{m}{6} = 0.259$

$m = 1.6$

d 1.55 cm = 1.6 cm. So, yes, it is correct.

4 **a**



b $\frac{\text{Adj}}{\text{Hyp}} = 0.966$ **c** $\frac{k}{9} = 0.966$

$k = 8.7$

d 8.7 cm. So, yes, it is correct.

5 **a**

$\frac{\text{Opp}}{\text{Adj}} = \frac{3.5}{8.4}$ $= 0.42$	$\frac{\text{Opp}}{\text{Hyp}} = \frac{3.5}{9.0}$ $= 0.39$	$\frac{\text{Adj}}{\text{Hyp}} = \frac{8.4}{9.0}$ $= 0.93$

b $\triangle DEF$ and $\triangle KLM$ both have angles of 23° and 90° so the third angles of each must be 67°. So the triangles are similar (AAA)

c $\frac{\text{Opp}}{\text{Hyp}} = 0.39$ **d** $a \approx 2.7$

e $\frac{\text{Adj}}{\text{Hyp}} = 0.93$ **f** $b \approx 6.4$

g $KL \approx 2.7$ cm; $LM \approx 6.4$ cm. So answers are correct.

6

Calculated ratio from measurements for 15° Worked example 7	$\frac{\text{Opp}}{\text{Adj}}$ $= 0.268$	$\frac{\text{Opp}}{\text{Hyp}}$ $= 0.259$	$\frac{\text{Adj}}{\text{Hyp}}$ $= 0.966$
Calculator value	tan 15° = 0.268	sin 15° = 0.259	cos 15° = 0.966
Calculated ratio from measurements for 23° Question 5a	$\frac{\text{Opp}}{\text{Adj}}$ $= 0.42$	$\frac{\text{Opp}}{\text{Adj}}$ $= 0.39$	$\frac{\text{Opp}}{\text{Adj}}$ $= 0.92$
Calculator value	tan 23° = 0.424	sin 23° = 0.391	cos 23° = 0.921

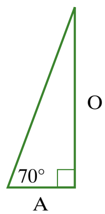
The values calculated from measurements are very close to the calculator values.

exercise 10.2

1

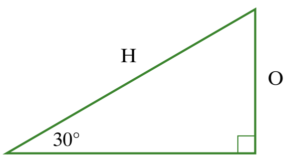
a 0.6018	**b** 0.8829	**c** 7.1154
d 0.8660	**e** 0.8387	**f** 0.9325
g 0.0872	**h** 0.9962	**i** 0.9994
j 0.2126	**k** 0.4540	**l** 5.6713
m 0.5000	**n** 0.9004	**o** 0.8151
p 0.9646	**q** 0.4390	**r** 0.8348
s 0.9070	**t** 3.9812	

2 **a** **i**



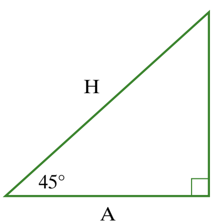
ii tan 70° = 2.747

b **i**



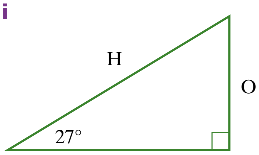
ii sin 30° = 0.500

c **i**



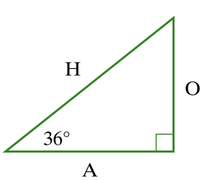
ii cos 45° = 0.707

d **i**



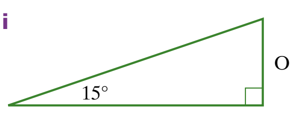
ii sin 27° = 0.454

e **i**



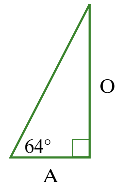
ii cos 36° = 0.809

f **i**



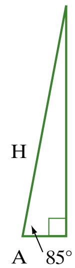
ii tan 15° = 0.268

g **i**



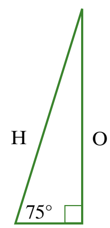
ii tan 64° = 2.050

h **i**



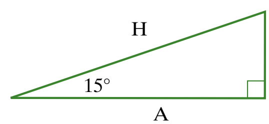
ii cos 85° = 0.087

i **i**



ii sin 75° = 0.966

j **i**



ii cos 15° = 0.966

3

a 1.1	**b** 11.4	**c** 22.8	**d** 5.4
e 42.8	**f** 26.6	**g** 30.2	**h** 10.4
i 1.4	**j** 3.4	**k** 4.0	**l** 13.7

4

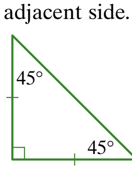
a 20.1	**b** 125.9	**c** 341.5	**d** 278.2
e 38.2	**f** 43.8	**g** 21.0	**h** 123.4
i 34.4	**j** 46.9	**k** 20.0	**l** 93.3

5

a 7.7	**b** 10.3	**c** 5.1	**d** 4.6
e 20.2	**f** 112.6	**g** 224.8	**h** 6.4
i 685.9	**j** 140.5	**k** 340.9	**l** 13.5

6 **a** 1

b A right-angled triangle with 45° angles is isosceles, so the opposite side will equal the adjacent side. Therefore tan 45° = 1.



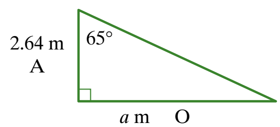
exercise 10.3

1	**a** $a = 2.5$	**b** $b = 4.2$	**c** $c = 36.9$
	d $d = 3.9$	**e** $e = 67.2$	**f** $f = 6.0$
	g $g = 79.6$	**h** $h = 3.6$	**i** $i = 38.0$
	j $j = 9.4$	**k** $k = 27.7$	**l** $l = 11.3$
2	**a** $a = 10.2$	**b** $b = 242.5$	**c** $c = 11.3$
	d $d = 214.7$	**e** $e = 104.0$	**f** $f = 21.4$
	g $g = 12.3$	**h** $h = 112.6$	**i** $i = 617.6$
	j $j = 16.1$	**k** $k = 31.0$	**l** $l = 114.7$
3	**a** $a = 125.0$	**b** $b = 13.8$	**c** $c = 273.6$
	d $d = 3.5$	**e** $e = 1.1$	**f** $f = 643.8$
	g $g = 45.0$	**h** $h = 64.3$	**i** $i = 4.4$
	j $j = 68.0$	**k** $k = 36.9$	**l** $l = 190.2$

4 D

5 C

6 **a** **i**

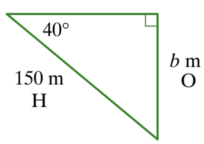


ii tangent

iii $\tan 65° = \dfrac{a}{2.64}$

$a = 5.7$

b **i**

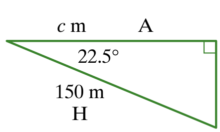


ii sine

iii $\sin 40° = \dfrac{b}{150}$

$b = 96.4$

c **i**

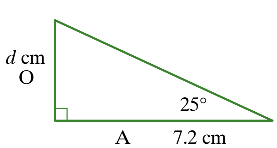


ii cosine

iii $\cos 22.5° = \dfrac{c}{150}$

$c = 138.6$

d **i**

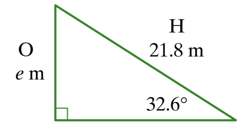


ii tangent

iii $\tan 25° = \dfrac{d}{7.2}$

$d = 3.4$

e **i**

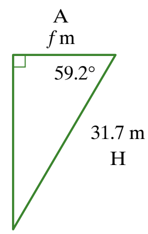


ii sine

iii $\sin 32.6° = \dfrac{e}{21.8}$

$e = 11.7$

f **i**

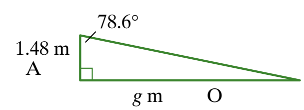


ii cosine

iii $\cos 59.2° = \dfrac{f}{31.7}$

$f = 16.2$

g **i**

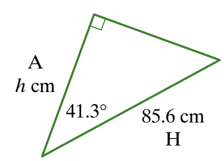


ii tangent

iii $\tan 78.6° = \dfrac{g}{1.48}$

$g = 7.3$

h **i**

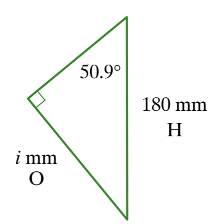


ii cosine

iii $\cos 41.3° = \dfrac{h}{85.6}$

$h = 64.3$

i **i**

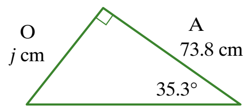


ii sine

iii $\sin 50.9° = \dfrac{i}{180}$

$i = 139.7$

j **i**

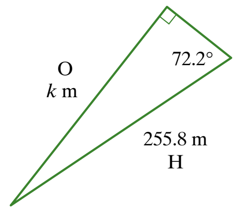


ii tangent

iii $\tan 35.3° = \frac{j}{73.8}$

$j = 52.3$

k **i**

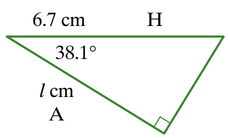


ii sine

iii $\sin 72.2° = \frac{k}{255.8}$

$k = 243.6$

l **i**



ii cosine

iii $\cos 38.1° = \frac{l}{6.7}$

$l = 5.3$

7 **a** **i** sine **ii** $\sin 28° = \frac{a}{38.2}$
iii $a = 17.9$

b **i** tangent **ii** $\tan 53° = \frac{b}{1.63}$
iii $b = 2.2$

c **i** sine **ii** $\sin 67° = \frac{c}{18.95}$
iii $c = 17.4$

d **i** cosine **ii** $\cos 49° = \frac{d}{1184}$
iii $d = 776.8$

e **i** sine **ii** $\sin 78° = \frac{e}{38.4}$
iii $e = 37.6$

f **i** tangent **ii** $\tan 60° = \frac{f}{2.85}$
iii $f = 4.9$

g **i** cosine **ii** $\cos 31° = \frac{g}{4.73}$
iii $g = 4.1$

h **i** tangent **ii** $\tan 49° = \frac{h}{214}$
iii $h = 246.2$

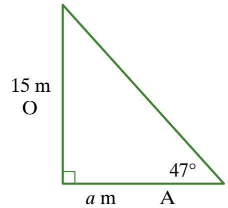
i **i** sine **ii** $\sin 63° = \frac{i}{37.5}$
iii $i = 33.4$

j **i** cosine **ii** $\cos 36° = \frac{j}{95.6}$
iii $j = 77.3$

k **i** tangent **ii** $\tan 50° = \frac{k}{232}$
iii $k = 276.5$

l **i** sine **ii** $\sin 59° = \frac{l}{384}$
iii $l = 329.2$

8 **a** **i**

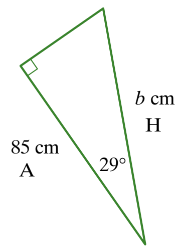


ii tangent

iii $\tan 47° = \frac{15}{a}$

$a = 14.0$

b **i**

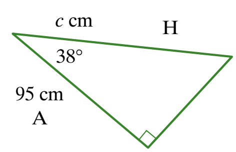


ii cosine

iii $\cos 29° = \frac{85}{b}$

$b = 97.2$

c **i**

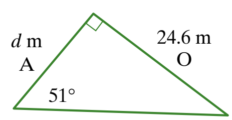


ii cosine

iii $\cos 38° = \frac{95}{c}$

$c = 120.6$

d **i**

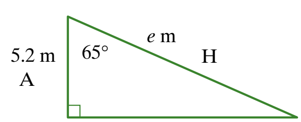


ii tangent

iii $\tan 51° = \frac{24.6}{d}$

$d = 19.9$

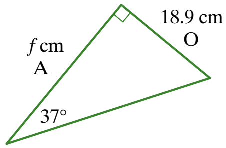
e **i**



ii cosine

iii $\cos 65° = \frac{5.2}{e}$

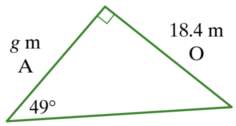
$e = 12.3$

f **i** 

ii tangent

iii $\tan 37° = \dfrac{18.9}{f}$

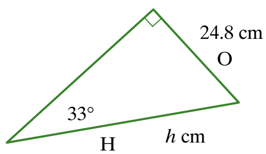
$f = 25.1$

g **i** 

ii tangent

iii $\tan 49° = \dfrac{18.4}{g}$

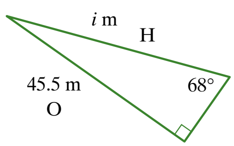
$g = 16.0$

h **i** 

ii sine

iii $\sin 33° = \dfrac{24.8}{h}$

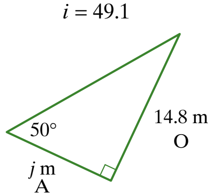
$h = 45.5$

i **i** 

ii sine

iii $\sin 68° = \dfrac{45.5}{i}$

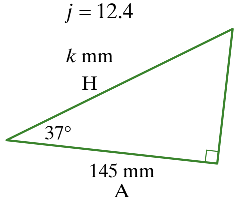
$i = 49.1$

j **i** 

ii tangent

iii $\tan 50° = \dfrac{14.8}{j}$

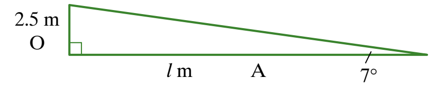
$j = 12.4$

k **i** 

ii cosine

iii $\cos 37° = \dfrac{145}{k}$

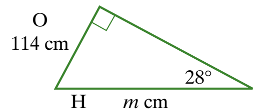
$k = 181.6$

l **i** 

ii tangent

iii $\tan 7° = \dfrac{2.5}{l}$

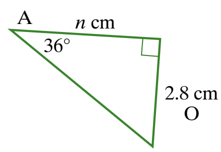
$l = 20.4$

m **i** 

ii sine

iii $\sin 28° = \dfrac{114}{m}$

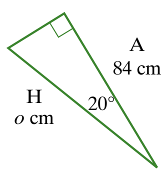
$m = 242.8$

n **i** 

ii tangent

iii $\tan 36° = \dfrac{2.8}{n}$

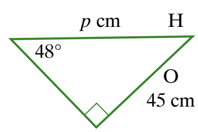
$n = 3.9$

o **i** 

ii cosine

iii $\cos 20° = \dfrac{84}{o}$

$o = 89.4$

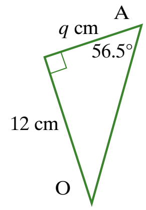
p **i** 

ii sine

iii $\sin 48° = \dfrac{45}{p}$

$p = 60.6$

q **i**

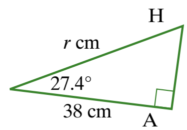


ii tangent

iii $\tan 56.5° = \frac{12}{q}$

$q = 7.9$

r **i**



ii cosine

iii $\cos 27.4° = \frac{38}{r}$

$r = 42.8$

9 B

10 E

11 **a** **i** $\cos 56° = \frac{25}{a}$ **ii** $a = 44.7$

b **i** $\sin 62° = \frac{b}{125}$ **ii** $b = 110.4$

c **i** $\cos 33° = \frac{9.5}{c}$ **ii** $c = 11.3$

d **i** $\tan 52° = \frac{d}{15}$ **ii** $d = 19.2$

e **i** $\sin 64.8° = \frac{e}{5.2}$ **ii** $e = 4.7$

f **i** $\tan 62° = \frac{18.9}{f}$ **ii** $f = 10.0$

g **i** $\tan 44° = \frac{g}{18.7}$ **ii** $g = 18.1$

h **i** $\sin 71.2° = \frac{h}{45}$ **ii** $h = 42.6$

i **i** $\cos 35° = \frac{18.5}{i}$ **ii** $i = 22.6$

j **i** $\sin 25° = \frac{125}{j}$ **ii** $j = 295.8$

k **i** $\cos 60° = \frac{8}{k}$ **ii** $k = 16.0$

l **i** $\tan 31.4° = \frac{l}{40}$ **ii** $l = 24.4$

12 **a** **i** $a = 16.9$ **ii** α $b = 67.9$
β $b = 67.9$

b **i** $a = 127.7$ **ii** α $b = 114.8$
β $b = 114.8$

c **i** $a = 13.1$ **ii** α $b = 7.4$
β $b = 7.4$

d **i** $a = 53.7$ **ii** α $b = 83.5$
β $b = 83.5$

e **i** $a = 400.6$ **ii** α $b = 453.7$
β $b = 453.7$

f **i** $a = 62.0$ **ii** α $b = 113.8$
β $b = 113.8$

g **i** $a = 111.1$ **ii** α $b = 93.2$
β $b = 93.2$

h **i** $a = 15.1$ **ii** α $b = 26.2$
β $b = 26.2$

i **i** $a = 7.5$ **ii** α $b = 7.4$
β $b = 7.4$

13 **a** The other angle is 50°. The other sides have lengths 62.2 cm and 47.7 cm (correct to one decimal place).

b The other angle is 59°. The other sides have lengths 46.3 cm and 27.8 cm (correct to one decimal place).

14 **a** **i** 18.2 **ii** 18.2

b they are the same

15 **a** **i** $\cos 20° \approx 0.9397$ **ii** $\sin 70° \approx 0.9397$

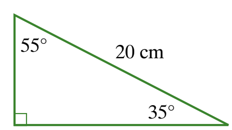
b $\cos 20° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{c}$;

$\sin 70° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}$. So $\cos 20° = \sin 70°$.

c The cosine of an angle is equal to the sine of the complementary angle.

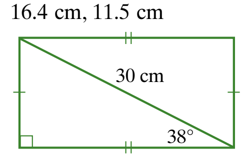
d **ii** $\cos 85° = \sin 5° = 0.087$
iii $\sin 0° = \cos 90° = 0.000$
iv $\cos 45° = \sin 45° = 0.707$
v $\sin 58° = \cos 32° = 0.848$

16 **a**



b 16.4 cm, 11.5 cm

17 **a**



b 23.6 cm; 18.5 cm

exercise 10.4

1 **a** 21.1° **b** 63.3° **c** 57.1° **d** 33.4°
e 86.9° **f** 86.6° **g** 84.3° **h** 89.0°
i 60.0° **j** 45.0° **k** 45.9° **l** 30.9°
m 53.1° **n** 48.6° **o** 84.9° **p** 76.5°
q 88.5° **r** 36.9° **s** 84.3° **t** 89.0°
u 31.8° **v** 40.4° **w** 39.8° **x** 31.6°

2 **a** 50° **b** 32° **c** 17° **d** 28°
e 63° **f** 84° **g** 75° **h** 56°
i 29° **j** 87° **k** 26° **l** 9°
m 24° **n** 63° **o** 57° **p** 48°

3 a i $\tan\theta = \frac{12}{15}$ ii $\theta = 39°$

b i $\tan\theta = \frac{7}{17}$ ii $\theta = 22°$

c i $\tan\theta = \frac{59}{43}$ ii $\theta = 54°$

d i $\tan\theta = \frac{40}{75}$ ii $\theta = 28°$

e i $\tan\theta = \frac{122}{125}$ ii $\theta = 44°$

f i $\tan\theta = \frac{8.9}{6.7}$ ii $\theta = 53°$

g i $\tan\theta = \frac{16}{16}$ ii $\theta = 45°$

h i $\tan\theta = \frac{53}{76}$ ii $\theta = 35°$

i i $\tan\theta = \frac{0.9}{0.2}$ ii $\theta = 77°$

4 D

5 a i $\sin\theta = \frac{13}{15}$ ii $\theta = 60°$

b i $\sin\theta = \frac{5.3}{6.8}$ ii $\theta = 51°$

c i $\sin\theta = \frac{44}{110}$ ii $\theta = 24°$

d i $\sin\theta = \frac{53}{56}$ ii $\theta = 71°$

e i $\sin\theta = \frac{78}{104}$ ii $\theta = 49°$

f i $\sin\theta = \frac{76}{90}$ ii $\theta = 58°$

g i $\sin\theta = \frac{8.5}{27.5}$ ii $\theta = 18°$

h i $\sin\theta = \frac{38.5}{47.0}$ ii $\theta = 55°$

i i $\sin\theta = \frac{64}{72}$ ii $\theta = 63°$

6 a i $\cos\theta = \frac{20}{25}$ ii $\theta = 37°$

b i $\cos\theta = \frac{5.0}{9.2}$ ii $\theta = 57°$

c i $\cos\theta = \frac{64}{65}$ ii $\theta = 10°$

d i $\cos\theta = \frac{38.5}{57.5}$ ii $\theta = 48°$

e i $\cos\theta = \frac{32}{52}$ ii $\theta = 52°$

f i $\cos\theta = \frac{52}{88}$ ii $\theta = 54°$

g i $\cos\theta = \frac{16}{17}$ ii $\theta = 20°$

h i $\cos\theta = \frac{9.5}{55.0}$ ii $\theta = 80°$

i i $\cos\theta = \frac{39}{47}$ ii $\theta = 34°$

7 E

8 A

9 C

10 a i $\sin\theta = \frac{9}{20}$ ii $\theta = 27°$ iii $\theta = 26.7°$

b i $\tan\theta = \frac{21}{6}$ ii $\theta = 74°$ iii $\theta = 74.1°$

c i $\cos\theta = \frac{57}{98}$ ii $\theta = 54°$ iii $\theta = 54.4°$

d i $\sin\theta = \frac{96}{113}$ ii $\theta = 58°$ iii $\theta = 58.2°$

e i $\tan\theta = \frac{128}{53}$ ii $\theta = 68°$ iii $\theta = 67.5°$

f i $\cos\theta = \frac{4.2}{5.4}$ ii $\theta = 39°$ iii $\theta = 38.9°$

g i $\sin\theta = \frac{39.7}{52.5}$ ii $\theta = 49°$ iii $\theta = 49.1°$

h i $\cos\theta = \frac{50}{70.71}$ ii $\theta = 45°$ iii $\theta = 45.0°$

i i $\tan\theta = \frac{3.4}{4.6}$ ii $\theta = 36°$ iii $\theta = 36.5°$

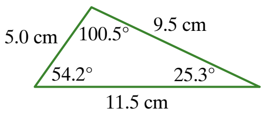
11 a i $\theta = 51.7°$ ii 115 m
b i $\theta = 54.4°$ ii 407 mm
c i $\theta = 73.3°$ ii 138 cm
d i $\theta = 39.0°$ ii 43.7 m
e i $\theta = 7.1°$ ii 8.06 m
f i $\theta = 64.3°$ ii 407 mm
g i $\theta = 60.0°$ ii 48.0 m
h i $\theta = 23.1°$ ii 57.5 cm
i i $\theta = 54.6°$ ii 76.7 m
j i $\theta = 63.1°$ ii 852 mm
k i $\theta = 9.5°$ ii 75.4 m
l i $\theta = 53.7°$ ii 583 mm
m i $\theta = 23.1°$ ii 115 cm
n i $\theta = 58.2°$ ii 43.7 m
o i $\theta = 55.1°$ ii 43.9 m
p i $\theta = 61.9°$ ii 193 mm
q i $\theta = 58.4°$ ii 44.8 m
r i $\theta = 76.3°$ ii 191 cm

12 a $a = 46.4$

using $\sin^{-1}\left(\frac{21}{29}\right)$

$\cos^{-1}\left(\frac{20}{29}\right)$

$\tan^{-1}\left(\frac{21}{20}\right)$

13


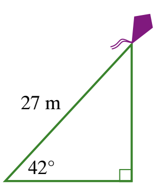
14 $\angle ABC = \angle ADC = 49°$, $\angle DAB = \angle BCD = 131°$

15 **a** $BP \approx 7.4$ cm **b** $\angle BCA \approx 32°$
c $AC \approx 20.1$ cm

16 **a** tangent
b 63°
c **i** 45° **ii** 45°
iii 72° **iv** 63°

exercise 10.5

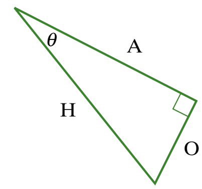
1 3.3 m
2 1.7 m
3 2.7 m
4 180 cm; 156 cm
5 18°
6 20°
7 6.9 m
8 **a** 5.77 m
b **i** 11.55 m **ii** 11.55 m
9 35 m
10 **a**

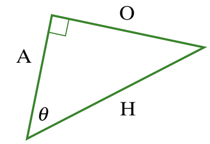

b 5.72 m
c 5.74 m
11 14.0 m
12 **a**


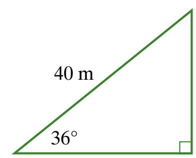
b 19.3 m

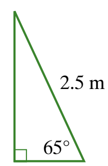
Revision

1 A
2 E
3 B
4 A
5 B
6 **a** 1.1106 **b** 0.2924 **c** 0.7826 **d** 0.8660
7 **a** 10.9 **b** 18.0 **c** 26.4 **d** 75.1
8 **a** 10.9 **b** 106.0 **c** 16.1 **d** 29.7
9 **a** 85° **b** 41° **c** 81° **d** 7°
10 **a** 65° **b** 40° **c** 18° **d** 34°

11 **a**


b


12 **a** $a = 2.1$ **b** $b = 4.3$ **c** $c = 3.6$ **d** $d = 61.4$
13 **a** 19° **b** 49°
14 3.0 m
15 **a** 53°
b **i** 5.6 m **ii** 5.6 m
16 **a**


b 24 m
17 **a**


b 1.1 m

Chapter 11

exercise 11.1

1 **a** $171.1\,\text{m}^2$ **b** $636.3\,\text{cm}^2$ **c** $2388.8\,\text{cm}^2$
d $272.4\,\text{mm}^2$ **e** $122.2\,\text{cm}^2$ **f** $173.3\,\text{m}^2$
g $4704.4\,\text{m}^2$ **h** $1631.6\,\text{cm}^2$ **i** $2783\,\text{m}^2$

2 **a** $359.0\,\text{cm}^2$ **b** $1040\,\text{mm}^2$ **c** $6.55\,\text{m}^2$
d $150.7\,\text{m}^2$ **e** $1206\,\text{mm}^2$ **f** $13.0\,\text{m}^2$
g $878.0\,\text{m}^2$ **h** $2688\,\text{cm}^2$ **i** $18.1\,\text{m}^2$
j $637.9\,\text{mm}^2$ **k** $12\,272\,\text{mm}^2$ **l** $1086.9\,\text{cm}^2$

3 **a** $a = 4.90$ **b** $b = 1.96$ **c** $c = 2.90$
d $d = 175$ **e** $e = 15$ **f** $f = 4.2$
g $g = 48$ **h** $h = 76$ **i** $i = 9$

4 **a** $226\,\text{cm}^2$ **b** $2.7\,\text{m}^2$ **c** $424\,\text{mm}^2$
d $6.2\,\text{cm}^2$ **e** $547\,\text{cm}^2$ **f** $34.4\,\text{m}^2$

5 **a** $1357.2\,\text{cm}^2$ **b** $51.5\,\text{m}^2$
c $23.4\,\text{m}^2$ **d** $478.3\,\text{cm}^2$

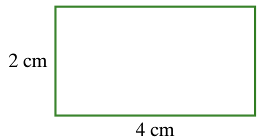
6 **a** $24.5\,\text{m}^2$ **b** $50.0\,\text{m}^2$ **c** $69.5\,\text{cm}^2$
d $50.5\,\text{cm}^2$ **e** $10.4\,\text{m}^2$ **f** $0.4\,\text{m}^2$
g $15.9\,\text{m}^2$ **h** $5.9\,\text{m}^2$ **i** $9.2\,\text{m}^2$
j $10.8\,\text{m}^2$ **k** $1633.6\,\text{cm}^2$ **l** $13.0\,\text{m}^2$

7 **a** $60\,\text{cm}^2$ **b** $9.72\,\text{m}^2$

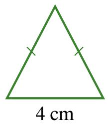
8 **a** $1433.1\,\text{cm}^2$ **b** 36 cm

exercise 11.2

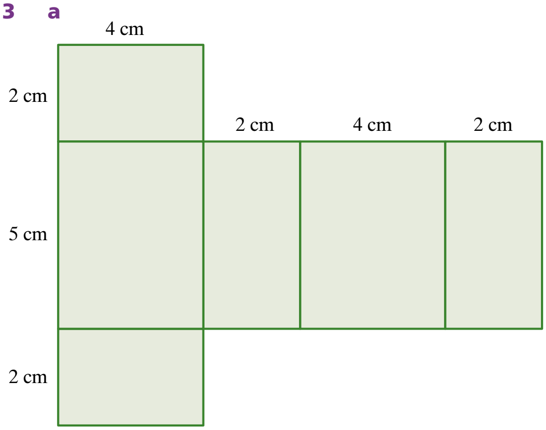
1



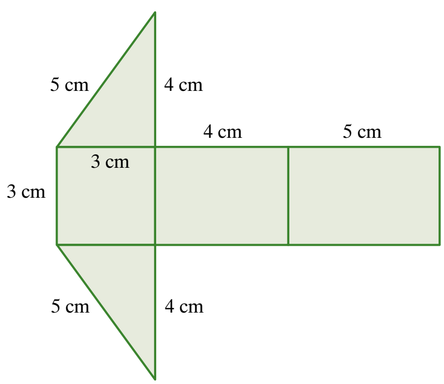
2



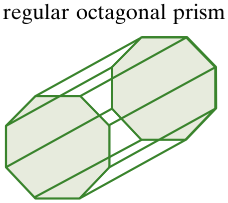
3 **a**



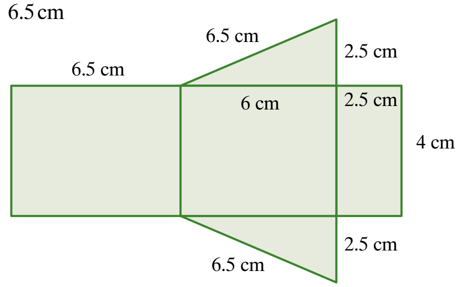
b



4 **a** regular octagonal prism
b



5 **a** check with your teacher
b 6.5 cm
c



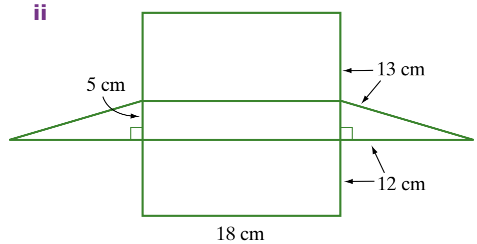
exercise 11.3

1 **a** 486 cm^2 **b** 3456 cm^2 **c** 8.64 cm^2
d 2354 cm^2 **e** 28 cm^2 **f** 142 cm^2
g 2642 cm^2 **h** 47.36 cm^2 **i** 3330 cm^2
j 711.6 cm^2 **k** 4.08 m^2 **l** 1296 cm^2
m 6908 mm^2 **n** 2280 cm^2 **o** 5.46 m^2

2 D

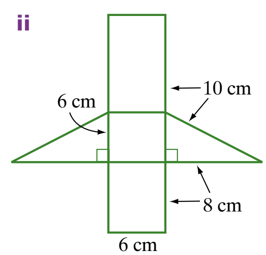
3 **a** 6 m^2; 6.1 m^2 **b** Shape 1

4 **a** **i** 13 cm
ii



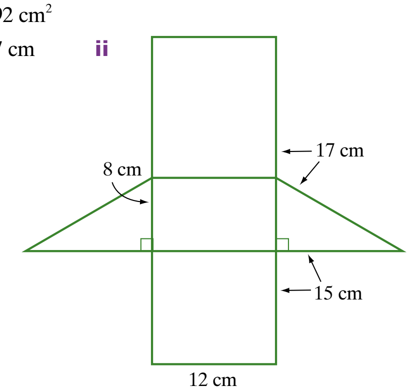
iii 600 cm^2

b **i** 10 cm **ii**



iii 192 cm^2

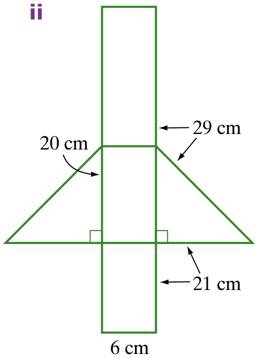
c **i** 17 cm **ii**



iii 600 cm^2

d **i** 21 cm

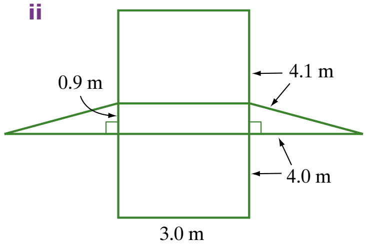
ii



iii 840 cm^2

e **i** 4.1 m

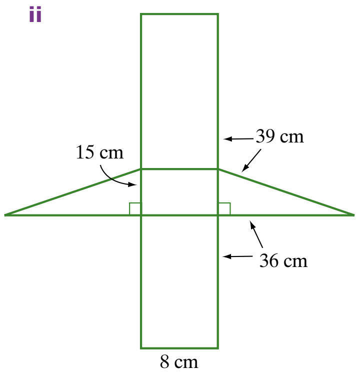
ii



iii 30.6 m^2

f **i** 36 cm

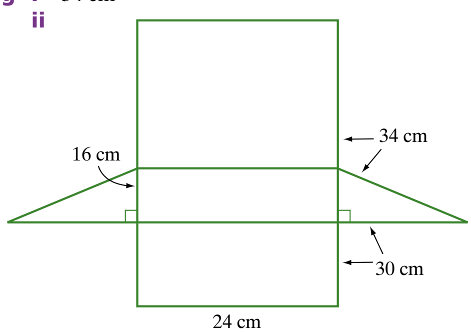
ii



iii 1260 cm^2

g **i** 34 cm

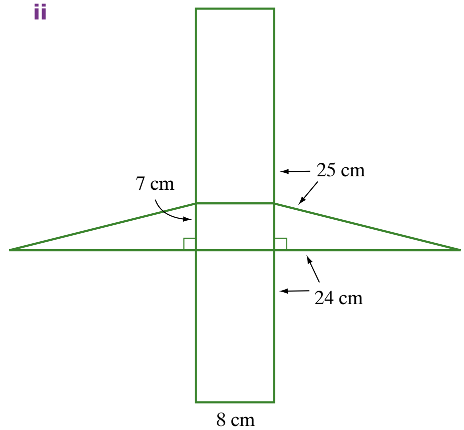
ii



iii 2400 cm^2

h **i** 24 cm

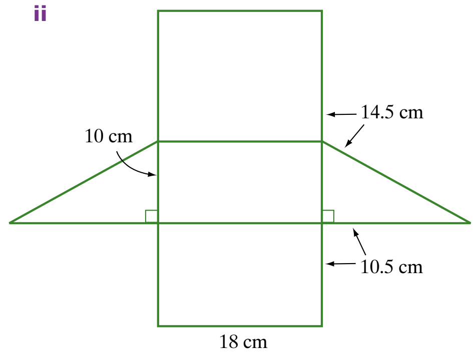
ii



iii 616 cm^2

i **i** 14.5 cm

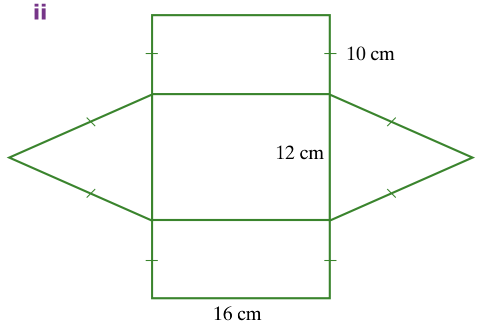
ii



iii 735 cm^2

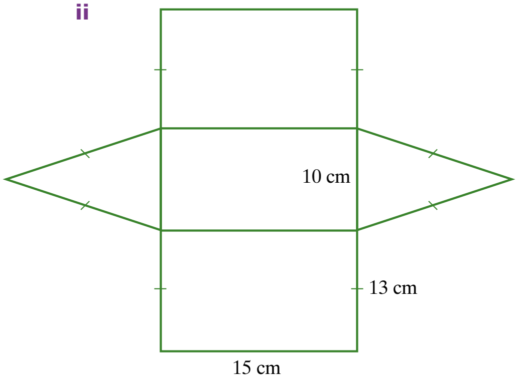
5 **a** **i** 8 cm

ii



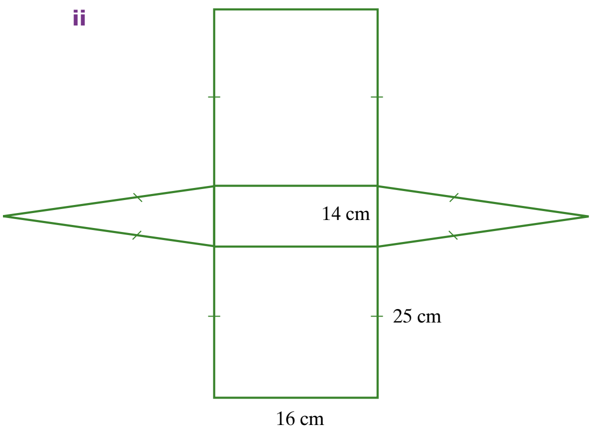
iii 608 cm^2

b **i** 12 cm
ii



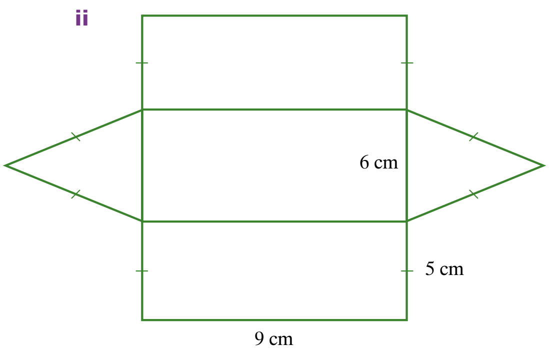
iii 660 cm^2

c **i** 24 cm
ii



iii 1360 cm^2

d **i** 4 cm
ii



iii 168 cm^2

6 B

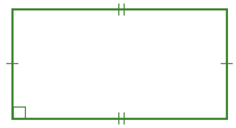
7 **a** 600 cm^2; 670 cm^2; 610 cm^2
b shape 1

8 **a** **i** 47.85 cm^2 **ii** 287.1 cm^2 **iii** 3214.2 cm^2
b **i** 60.5 cm^2 **ii** 484 cm^2 **iii** 2968 cm^2
c **i** 34.5 cm^2 **ii** 172.5 cm^2 **iii** 1845 cm^2
d **i** 62.4 cm^2 **ii** 374.4 cm^2 **iii** 2908.8 cm^2

9 **a** 57.6 cm^2
b 59.4 cm^2

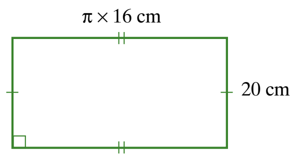
exercise 11.4

1 **a**



b 23.6 cm long and 10.6 cm wide
c 250 cm^2

2 **a**



b 1005 cm^2 **c** 1407 cm^2

3 D

4 **a** **i** 3619 cm^2 **ii** 5228 cm^2
b **i** 339 cm^2 **ii** 848 cm^2
c **i** 2513 cm^2 **ii** 5027 cm^2
d **i** 1206 cm^2 **ii** 2111 cm^2
e **i** 785 cm^2 **ii** 942 cm^2
f **i** 30 m^2 **ii** 55 m^2
g **i** 12 m^2 **ii** 21 m^2
h **i** 151 cm^2 **ii** 207 cm^2
i **i** 18 850 cm^2 **ii** 28 903 cm^2
j **i** 3 m^2 **ii** 12 m^2
k **i** 21 cm^2 **ii** 32 cm^2
l **i** 39 cm^2 **ii** 79 cm^2

5 D

6 B

7 **a** 6.4 m^2 **b** 19.2 m^2 **c** 578.1 cm^2
d 15.9 m^2 **e** 2721.4 cm^2 **f** 6.8 m^2

8 297 cm^2

9 67 cm

10 1838 cm^2

11 **a** 5027 cm^2 **b** 25 133 cm^2 **c** 30 160 cm^2

12 **a** cylinder: 49 359 mm^2; triangle prism: 65 785 mm^2
b the cylinder is more economical

exercise 11.5

1 **a** 189 m^3 **b** 552 cm^3 **c** 456 cm^3
d 1920 cm^3 **e** 2212 cm^3 **f** 1178 cm^3

2 **a** 729 cm^3 **b** 13 824 cm^3 **c** 1.728 cm^3
d 4680 cm^3 **e** 8 cm^3 **f** 105 cm^3
g 3360 cm^3 **h** 47 250 cm^3 **i** 8100 cm^3
j 1044 cm^3 **k** 0.36 m^3 **l** 3024 cm^3
m 27.36 cm^3 **n** 7200 cm^3 **o** 0.676 m^3

3 **a** 7.65 cm^3 **b** 144 cm^3 **c** 720 cm^3
d 1260 cm^3 **e** 5.4 m^3 **f** 2160 cm^3
g 5760 cm^3 **h** 672 cm^3 **i** 945 cm^3

4 **a** 768 cm^3 **b** 900 cm^3
c 2688 cm^3 **d** 108 cm^3

5 96.75 cm^3

6 a 11 484 cm^3 b 12 100 cm^3
c 5175 cm^3 d 11 232 cm^3

7 4.05 m^3

8 a 1.496 m^3 b 1496 L

9 34 560 cm^3

10 40.32 m^3

11 12.5 m^2

exercise 11.6

1 a 28 953 cm^3 b 1527 cm^3
c 25 133 cm^3 d 7238 cm^3
e 1963 cm^3 f 30 m^3
g 7 m^3 h 226 cm^3
i 376 991 cm^3 j 2 m^3
k 14 cm^3 l 49 cm^3

2 D

3 C

4 a i 375 cm^3 ii 375 mL
b it is the same

5 449 cm^3

6 a 1.5 m^3 b 1.5 kL

7 a 8445 mm^3 b 8 cm^3

8 5.7 m^3

9 30.16 cm^3

10 282 743.34 cm^3

11 a 19 cm^3 b 54 cm^3
c 777 cm^3 d 735 cm^3

12 17.4 cm

13 a 500 m^3 b 500 kL

14 a 13.2 m^3 b 1.05 m

15 a round: 1816 cm^3; rectangular: 1470 cm^3
b 1429 cm^3 c rectangular tin

16 a 5.08 cm^3 b 40 g

17 a 16.6 m^3 b 11 cm c 12.5 m^3
d 12.5 kL e 4.1 m^3 f 10 tonnes
g 20 tonnes

Revision

1 D

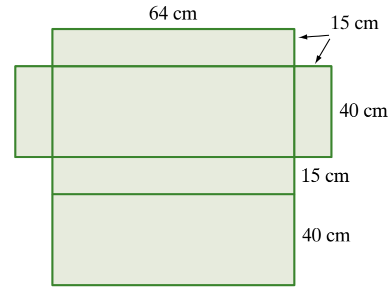
2 E

3 C

4 E

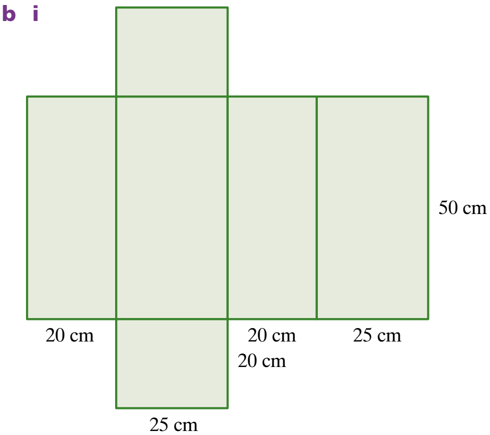
5 C

6 a i



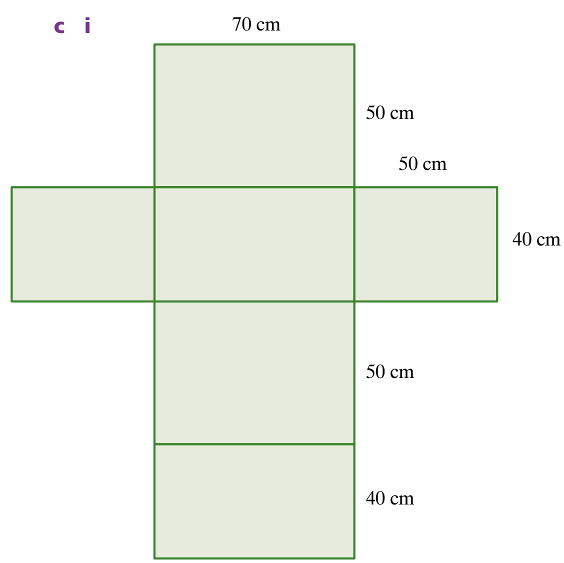
ii 8240 cm^2

b i



iii 5500 cm^2

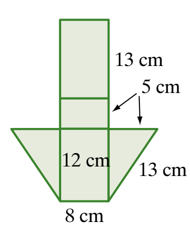
c i



iii 16 600 cm^2

7 a 38 400 cm^3 b 25 000 cm^3 c 140 000 cm^3

8 a



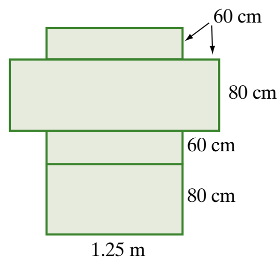
b 300 cm^2 **c** 240 cm^3

9 **a** 1140 cm^2 **b** 2160 cm^3

10 **a** **i** 132 m^2 **ii** 358 m^2
b **i** 12 252 cm^2 **ii** 21 809 cm^2
c **i** 2195 cm^2 **ii** 3374 cm^2

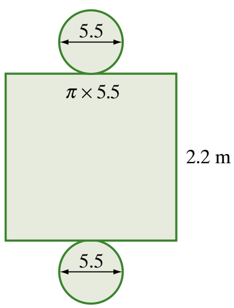
11 **a** 396 m^3 **b** 238 918 cm^3 **c** 15 036 cm^3

12 **a** **i**



ii 0.6 m^3 **iii** 0.6 kL

b **i**



ii 52.3 m^3 **iii** 52.3 kL

13 **a** 8640 cm^3 **b** 8.64 L

14 **a** 4131 cm **b** 4 L
c 972 cm^2 **d** 1426 cm^3

15 **a** 1.08 kL **b** 30 kL
c 99 kL **d** 165 L

Chapter 12

exercise 12.1

1 **a** $\frac{7}{25}$ **b** 0.28

2 **a** 45.7 gigalitres **b** 90 gigalitres

3 7936

4 84%

5 675

6 3.1%

7 **a** 2430 **b** 556.8 **c** 11 875 **d** 399

8 **a** 525 **b** 84 **c** 6232 **d** 578.2

9 \$5.53

10 14 809

11 \$64 375

12 \$23 300

13 \$2 322 600

14 44

15 5426 million tonnes

16 6250

exercise 12.2

1 \$638

2 \$107.42

3 \$249

4 31.5%

5 **a** \$16.80 **b** \$18.45

6 **a** \$765 **b** \$841.50

7 \$20.50

8 \$47.10

9 **a** \$900 **b** \$180

10 **a** \$38.40 **b** $83\frac{1}{3}\%$ **c** $16\frac{2}{3}\%$

d Instead of doing percentage discount as $\frac{\text{discount}}{\text{normal price}} \times 100$ they did $\frac{\text{discount}}{\text{sale price}} \times 100$.

exercise 12.3

1 **a** \$98.60 **b** \$52.20 **c** \$1600 **d** \$6000

2 **a** \$200; \$2200 **b** \$135; \$735
c \$42; \$792 **d** \$7200; \$22 200
e \$956.25; \$9456.25 **f** \$7500; \$31 500

3 **a** \$29 **b** \$87

4 \$12.83

5 \$864

6 \$52 627.80

7 \$4428

8 **a** \$1.50 **b** \$37.50
c \$102.08 **d** \$3328.77
e \$129.38 **f** \$165

9 **a** \$78.90 **b** \$9.47

10 **a** \$69 **b** \$4069

11 **a** \$39.06 **b** \$2539.06

12 **a** \$177.53 **b** \$30 177.53

13 **a** 4 **b** 5 **c** 8 **d** 5

14 10 years

15 **a** \$71.25 **b** 5 years

16 5 years

17 20 years

18 25 years

19 D

20 6 years

21 **a** \$675 **b** \$500 **c** \$5000 **d** \$10 000

22 **a** \$400 **b** \$1000 **c** \$7500 **d** \$12 000

23 \$2500

24 a 3% b 4% c 5% d 3.125%
25 6.9%
26 a $5512.50 b $5788.13

exercise 12.4

1 a $1080 b $7080
2 a $6528
3 a $168.30 b $1018.30 c $9.79
4 a $2880 b $10 880 c $302.22
5 $789.33
6 $195
7 $600
8 a $6792 b $37 356 c $3396 d 5%
9 a $98 b $686 c $31.57

exercise 12.5

1 a i 1.4583% ii 0.47945%
b i 1.25% ii 0.041096%
c i 1.5416% ii 0.050685%
d i 1.7875% ii 0.058767%
e i 1.9% ii 0.062466%
f i 1.66583% ii 0.054767%
2 a i $25 ii $690
b i $46 ii $2254
c i $25 ii $190
d i $37 ii $1813
3 $13.49
4 $23.24
5 $16.60
6 $48.85
7 $9.21
8 $53.50
9 a 0.065479% b $3.28 c $1.66
d No, there is a rounding error: the amounts are the same if one calculation is used for the total interest.
e $9.44
10 a $40 b $76
c $36 d $2964
e 19 years f $3064
g 18 months h $370

Revision

1 D
2 B
3 A
4 C
5 C
6 a 37.5% b $150 c 708
d 2700 e 2.8%
7 a $242.25 b $32.00
8 $669.90
9 a 23.2% b −10.96%
c −11.68% d −10.75%
10 a 32% b 23.5%
11 a $1040 b $4032
12 $172.60
13 12 years
14 a $108 b $376.03 c $91.23
15 $1848
16 a $48 b $54
17 6 years
18 $2400
19 7.5%
20 $322.83
21 a $5400 b $44 640
c $8640 d 6%

Chapter 13

exercise 13.1

1 a N b N c C
d C e C f N
2 a How many text messages do you send each week?
b How many hours each week would you spend doing a strenuous physical activity?
c What is the length of your arm span (in cm)? What is your height (in cm)?
3 a How often do you drink water each day? (never/rarely/sometimes/often)
b Do you use internet chat rooms?
c Would you say that your foot size is smaller, similar to, or larger than the average foot size for your age?
4 a DN (e.g. 3 females)
b NC (e.g. green)
c CN (e.g. approximately 12 m^2)
d NC (e.g. green, turquoise, aqua)
e OC (e.g. never)
f DN (e.g. 4 colours)
g OC (e.g. 1970)
h DN (e.g. 3)
i CN (e.g. 7.1 km)
j DN (e.g. 17)
5 check with your teacher
6 a Dwelling → what built of
→ complete or not
inhabited or not

person: civil conditions, religion, occupation

b 8

c 5 adults, 3 children

d 2

e The 3 children

f The adult female

g Answers will vary. Check with your teacher.

7 a

Stem	*Leaf*
0	7 9
1	6
2	2 4 5 6 7 8 8
3	2 5 5 7 8 8
4	2 8
5	7
6	2

(Key: 6|2 means 62)

b 50%

8 a

Stem	*Leaf*
2	5 6 8 9
3	0 0 2 2 3 3 3 4 4
3	5 6 7 7 7 7 8 9 9 9
4	0 2
4	0
5	
5	5
6	
6	5 6
7	
7	
8	1

(Key: 2|5 means 0.25)

The majority of times were between 0.25 and 0.40 seconds, with an outlier at 2.96 seconds (not plotted).

b The outlier at 2.96 has not been plotted, as it would make an extremely long stem plot (too many blank rows before value of 29|6). This value may have occurred because of a student badly mis-timing the reaction, or it could be a recording error.

c It is sensible to eliminate outliers from the plot (but list them individually) so that the plot focuses on the data reasonably close to the middle.

exercise 13.2

1

	Mean	Median	Mode
a	19	19	20
b	30.4	22	14, 15, 42
c	34.2	33.1	27.5
d	11.0	5	5

2 a 4 **b** 4.35

3 a 2, 10, 12, 16, or any four numbers that add to 40.

b Any 8 numbers, as long as the average of the two middle numbers (when they are in order) is 8.

c –5, –3, –1, 2, 2, 5, or any set of numbers whose sum is 0 and the difference between the first and last number is 10.

4 23

5 a mean = 7.08; median = 7; modes = 4, 5, 8, 9, 10; range = 6

b The number of Es in a line of text doesn't vary a great deal.

6 8

7 2.32

8 a 34 **b** 51 **c** 23.1 **d** 28

9 a $83 000

b The suburb could have a few homes in the very high price range.

c The median is not affected by a few very low or very high prices.

10 a 45 **b** 47.35 **c** 46 **d** 32

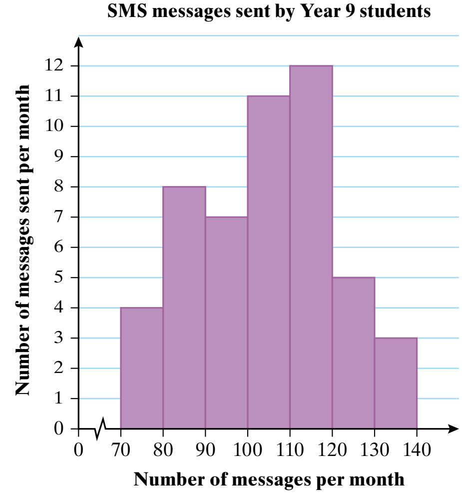
exercise 13.3

1 a

Number of messages sent per month	Number of students (Frequency)
70–<80	4
80–<90	8
90–<100	7
100–<110	11
110–<120	12
120–<130	5
130–<140	3

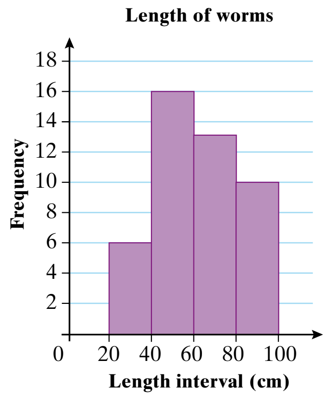
b

SMS messages sent by Year 9 students

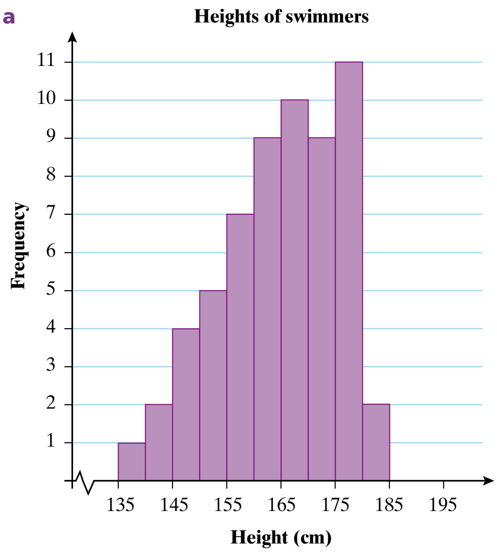


c 105 messages

2

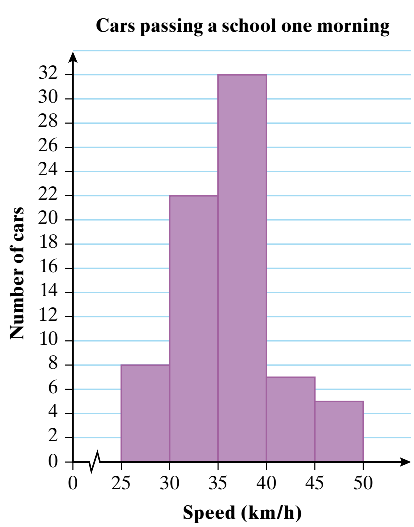


3 **a**



b 175–<180 **c** 19

4 **a**



b 35–<40 **c** 12

5

Mass (kg)	Number of watermelons
1.0–<1.5	8
1.5–<2.0	7
2.0–<2.5	14
2.5–<3.0	8
3.0–<3.5	4

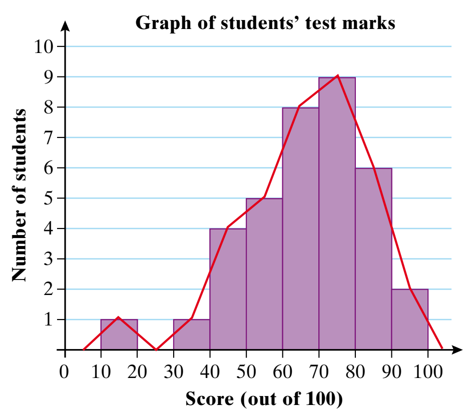
6 **a**

Height	Frequency
125–<130	1
130–<135	1
135–<140	2
140–<145	4
145–<150	5
150–<155	6
155–<160	5
160–<165	3
165–<170	2
170–<175	1
TOTAL	30

b 30

7 **a** The data appears to be centred around a mark of 70, with most students scoring between 50 and 90.

b

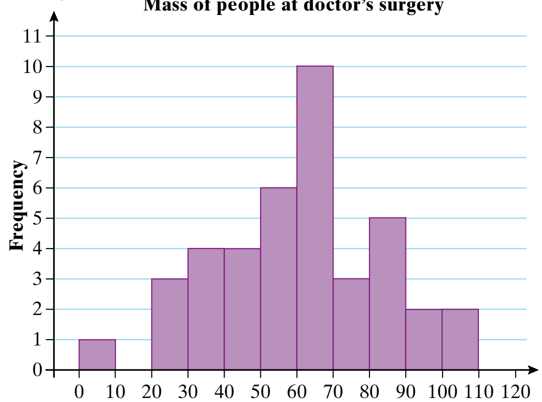


c slightly negatively skewed.

8 **a**

Mass (kg)	Frequency
0–<10	1
10–<20	0
20–<30	3
30–<40	4
40–<50	4
50–<60	6
60–<70	10
70–<80	3
80–<90	5
90–<100	2
100–<110	2

b **Mass of people at doctor's surgery**



c

Stem	*Leaf*
0	8
1	
2	3 4 9
3	2 4 5 7
4	3 4 8 8
5	2 5 6 6 9 9
6	0 2 3 3 3 4 4 5 5 8
7	1 3 8
8	1 1 2 2 8
9	1 7
10	1 6

(Key: 3|4 means 34 kg)

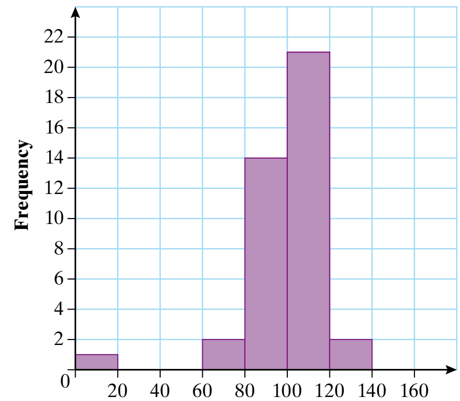
d The steam and leaf plot gives the same visual shape information as the histogram but it retains the individual data values.

9 **a** Because distances can take any value within a reasonable range, even though values are usually rounded (e.g. to nearest cm).

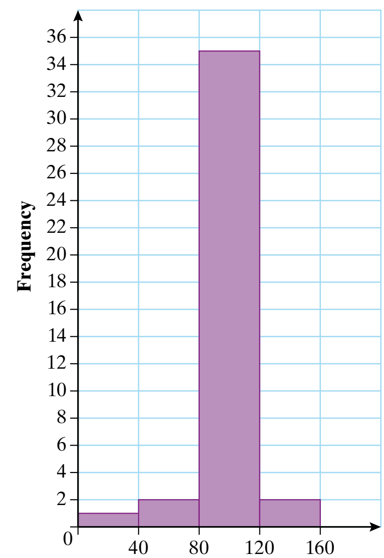
b 23 out of 40 (approximately 58%)

c Perhaps the student misinterpreted the question and made the wrong measurement.

d **i** **Distance from ground to belly button**



ii **Distance from ground to belly button**



e The detail is less visible, but a broad picture emerges: for the vast majority of students, the distance from the ground to their belly button was between 80 and 120 cm.

10 **a** positively skewed **b** none of these
c symmetrical **d** negatively skewed

11 **a** negatively skewed

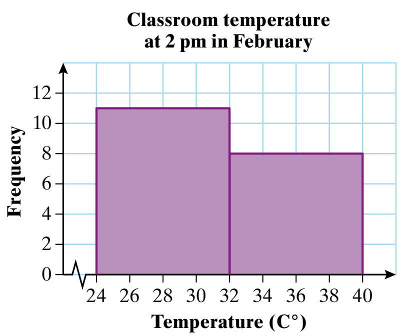
b

Score	Frequency
10–<20	1
20–<30	0
30–<40	1
40–<50	4
50–<60	5
60–<70	8
70–<80	9
80–<90	6
90–<100	2

12 **a** 2 degrees **b** 19 days

c 8 out of 19 days (approximately 42%)

d



13 **a** approximately symmetrical

b **i** 4 **ii** 61

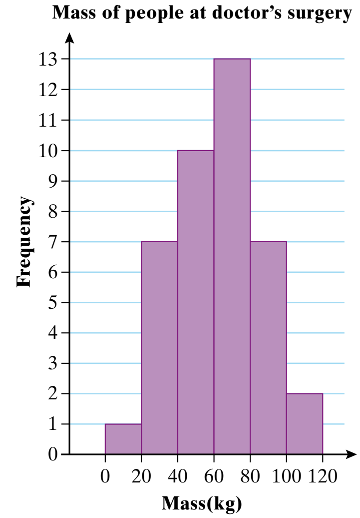
c 105 to 110 **d** 51

e continuous

14 **a**

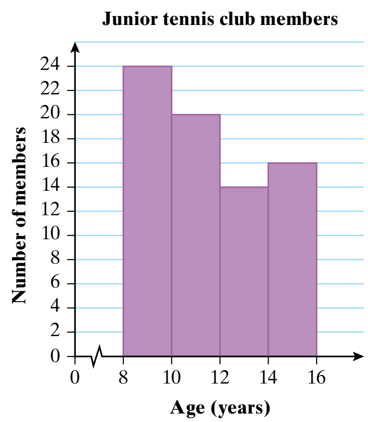
Mass (kg)	Frequency
0–<20	1
20–<40	7
40–<60	10
60–<80	13
80–<100	7
100–<120	2

b



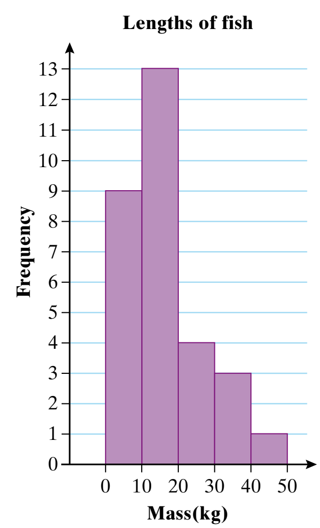
c Answers will vary. The one with class intervals of 10 gives you a clearer idea and is more effective.

15



16 **a** Answers will vary.

b



c $16\frac{1}{3}$

17 a

Length interval (cm)	Frequency	Cumulative frequency
0–<10	9	9
10–<20	13	22
20–<30	4	26
30–<40	3	29
40–<50	1	30
TOTAL	30	

b 26

18 a, b

Height	Frequency	Cumulative frequency	Cumulative frequency %
125–<130	1	1	$3.\dot{3}$
130–<135	1	2	$6.\dot{6}$
135–<140	2	4	$13.\dot{3}$
140–<145	4	8	$26.\dot{6}$
145–<150	5	13	$43.\dot{3}$
150–<155	6	19	$63.\dot{3}$
155–<160	5	24	80
160–<165	3	27	90
165–<170	2	29	$96.\dot{6}$
170–<175	1	30	100
TOTAL	30		

b $43.\dot{3}\%$

19 a approximately 28%
b 40%
c approximately 4.3 cm

20 a 65
b

Time(s)	Frequency	fx
20–<25	2	45
25–<30	4	110
30–<35	6	195
35–<40	3	112.5
40–<45	2	85
45–<50	3	142.5
50–<55	3	157.5
55–<60	2	115
60–<65	2	125
65–<70	1	67.5
70–<75	4	290
75–<80	2	155
80–<85	1	82.5
85–<90	2	175
90–<95	5	462.5
95–<100	3	292.5
100–<105	2	205
105–<110	0	0
110–<115	0	0
115–<120	0	0
120–<125	0	0
125–<130	1	127.5
130–<135	0	0
135–<140	0	0
140–<145	0	0
145–<150	1	147.5
150–<155	0	0
155–<160	0	0
160–<165	0	0
165–<170	0	0
170–<175	0	0
175–<180	1	177.5
	$\Sigma f = 50$	$\Sigma fx = 3270$

mean ≈ 65.4

c

Time(s)	Frequency	fx
20–<30	6	150
30–<40	9	315
40–<50	5	225
50–<60	5	275
60–<70	3	195
70–<80	6	450
80–<90	3	255
90–<100	8	760
100–<110	2	210
110–<120	0	0
120–<130	1	125
130–<140	0	0
140–<150	1	145
150–<160	0	0
160–<170	0	0
170–<180	1	175
	$\Sigma f = 50$	$\Sigma fx = 3280$

mean ≈ 65.6

d

Time(s)	Frequency	fx
20–<40	15	450
40–<60	10	500
60–<80	9	630
80–<100	11	990
100–<120	2	220
120–<140	1	130
140–<160	1	150
160–<180	1	170
	$\Sigma f = 50$	$\Sigma fx = 3240$

mean ≈ 64.8

e i They are all very close to the mean value of 65 seconds.

ii The grouping of the data has not had much effect on the accuracy of the estimate for the mean.

exercise 13.4

1 a 50% of men in their fifties and 90% of men in their thirties

b Men in their fifties tend to have larger waists. There is a greater range of waist measurements for men in their fifties.

2 a girls : 36 boys : 35
The girls' median time is slightly higher.

b girls : 18 boys : 49
The girls' data was grouped close together whilst the boys' was quite spread out.

c The girls' data is approximately symmetrical around the mid 30s whilst the boys' data is positively skewed.

3 a Set A: median = 19.5, range = 16
Set B: median = 23, range = 30

b **Number of vowels**

Set A		Set B
Leaf	*Stem*	*Leaf*
	0	7 9
4 3 1 1	1	4
9 9 8 5 5 5	1	5 6 7
4 2 2 2 1 0 0	2	0 1 1 3 3 3 4
7 7 6	2	8
	3	0 0 0
	3	5 5 7

(Key: 4|1 = 14; 1|6 = 16)

c Set A is grouped quite close and is approximately symmetrical whilst Set B is quite spread out and has a much greater range.

d Italian words tend to have more vowels than English words, and the number of vowels in Italian is more variable.

4 a Set A: median = 169, range = 27
Set B: median = 156.5, range = 13

b Set B would be the girls, as 14-year-old boys tend to be taller than girls. Also, the greater spread of set A reflects the variation in age when boys reach their adolescent growth spurt.

5 a

Class 1		Class 2
Leaf	*Stem*	*Leaf*
	1	0
0	2	
0	3	0 0
0	4	0
0 0 0	5	0 0 0 0
0 0 0 0	6	0 0 0 0 0 0
0 0 0 0 0 0	7	0 0 0 0 0
0 0 0 0 0 0	8	0 0 0
0 0 0 0	9	0 0 0
0	10	0 0

(Key: 0|4 = 40; 5|0 = 50)

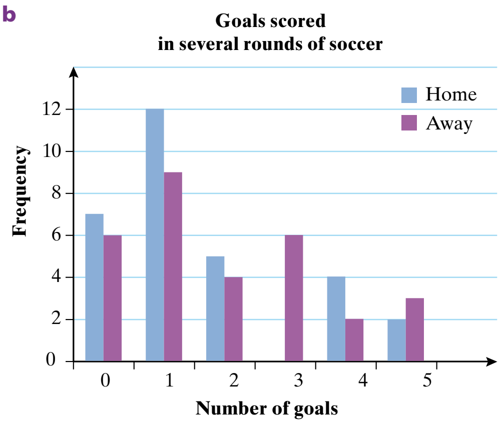
b class 2 c They both had the same

d class 1 performed better because their mean score was higher. It is easy to see that class 1 is grouped around 70–80 whilst class 2 is grouped around 60–70.

6 **a**

Number of goals	Home goals	Away goals
0	7	6
1	12	9
2	5	4
3	0	6
4	4	2
5	2	3

b



7 **a** aluminium bat data is negatively skewed, wooden bat data is approximately symmetrical
b aluminium bats are better at hitting the ball at high speed than wooden bats

exercise 13.5

1 **a** sample **b** sample
c census **d** sample

2 If all batteries are tested then there will be none left to sell.

3 It would be time-consuming and expensive to survey all 1300 students.

4 **a** The sample of hockey students does not accurately represent the population as these students are known to exercise.
b The sample of Year 9 students is not representative of all secondary students as they are all in the same year level.
c The number of cars using the street between 9 am and 10 am each day is not necessarily representative of the number of cars using the street in a 24 hour period.
d The sample of people at a dog club does not accurately represent the population as these people are known to own pets.

5 answers will vary

6 answers will vary

7 answers will vary

8 **a** 17 **b** 23

9 **a** convenience sampling
b random sampling
c systematic sampling
d convenience sampling
e stratified sampling

10

Age (years)	0–19	20–39	40–59	60–79	80–
Number of members in sample	13	17	12	7	1

11 **a** There would be 250 sets of data to analyse if a census was used. Use of a sample saves time and money.
b The sample needs to be representative of the three offices and representative of men and women.
c Milton 24; Newmarket 12; Albion 24
d 19
e Using a random or systematic sampling technique

exercise 13.6

Answers will vary for whole exercise. Example answers are given.

1 **a** Not a large sample, and it's concerning that such a large proportion didn't respond.
b 68.7%
c You would expect that the results of average number of friends would be higher as the trend is that the younger people have more friends.
d It doesn't say how the sample was obtained, how many people didn't respond, what age the respondants were, whether representative of New Zealand population.
e You would expect if it was an unbiased sample in both cases that they would be similar results.

2 **a** A good size sample. As long as all done in an unbiased and representative way the results should be a good estimate for the population.
b Whether the proportion of students in the sample in different categories is representative of the population (i.e. regional v city, boys v girls, 13-year-olds, 14-year-olds, 15-year-olds, etc.)
c With such as large sample you would expect it to be representative.

3 **a** That there were 4000 people in total asked. People from each capital city were included in the sample.
b Did the proportion of people from each capital city in the sample match the proportion of people from each in the population? How were people selected? What percentage of the total population does the 4000 represent?

4 **a** People who read the Sydney Morning Herald, have access to the internet and probably have strong opinions on the topic.
b It isn't a representative, unbiased sample.

5 **a** Was the proportion of city/country residents in the sample the same as the proportion in the population? How was the sample chosen?
b Whether it was a representive, unbiased survey.
c Check with your teacher.
d Yes. With the breaking of the drought and the 2011 floods there may be a different perception.

Revision

1 D

2 B

3 E

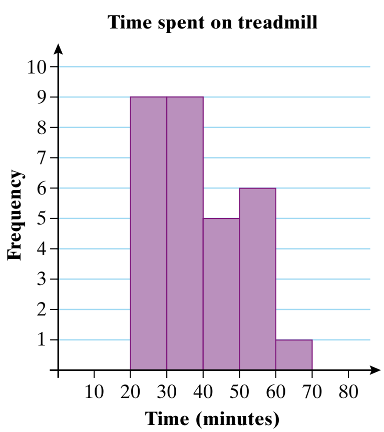
4 C

5 B

6 **a** mean = 38.1 minutes, median = 37.5 minutes
b 39
c

Time (minutes)	Tally	Frequency
20–29	卌 ||||	9
30–39	卌 ||||	9
40–49	卌	5
50–59	卌		6
60–69			1

d



e positively skewed

7 **a** convenience sampling
b stratified sampling
c systematic sampling
d random sampling

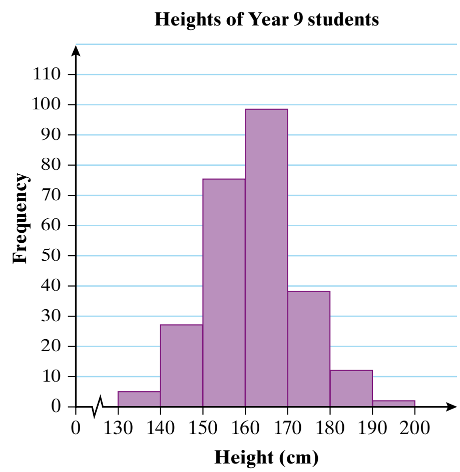
8 **a**

French Leaf	Stem	Mathematics Leaf
5	5	
9 2	6	
6 6 5 4	7	5 6 7 8 8
9 7 6 2 1	8	2 6 7 8
	9	2 7
	10	0

(Key: 10|0 means 100)

b French
c Median: Mathematics 84, French 76
d Students did better in Mathematics than French.

9 **a** 257 students
b 162 cm
c



d approximately symmetrical.

10 **a**

Stem	Leaf
1	
2	3 5 5 6 6 8 9 9 9 9 9 9
3	0 1 1 1 1 1 2 2 2 2 2 2 2 2 3 4 4 5 5 5 6 6 6 7 9 9 9 9 9
4	0 1 1 2 2
5	0 1 4
6	
7	
8	
9	7 8

(Key: 2|3 means 0.23)

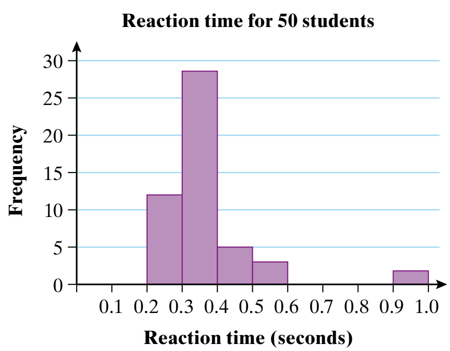
b 0.325 seconds
c 0.32 seconds
d 0.37 seconds

e

Reaction time (x seconds)	Tally	Frequency (f)	xf
0.00–<0.10		0	0.00
0.10–<0.20		0	0.00
0.20–<0.30	𝍸 𝍸 ||	12	3.00
0.30–<0.40	𝍸 𝍸 𝍸 𝍸 𝍸 |||	28	9.80
0.40–<0.50	𝍸	5	2.25
0.50–<0.60	|||	3	1.65
0.60–<0.70		0	0.00
0.70–<0.80		0	0.00
0.80–<0.90		0	0.00
0.90–<1.00	||	2	1.90
Totals		$\Sigma f = 50$	$\Sigma xf = 18.60$

f 0.37, which is the same as the value calculated in part d by adding all 50 values and dividing by 50.

g



h Yes: 0.97 and 0.98 seconds

Chapter 14

Answers to all the extending and investigating problems are contained in the Teacher book.

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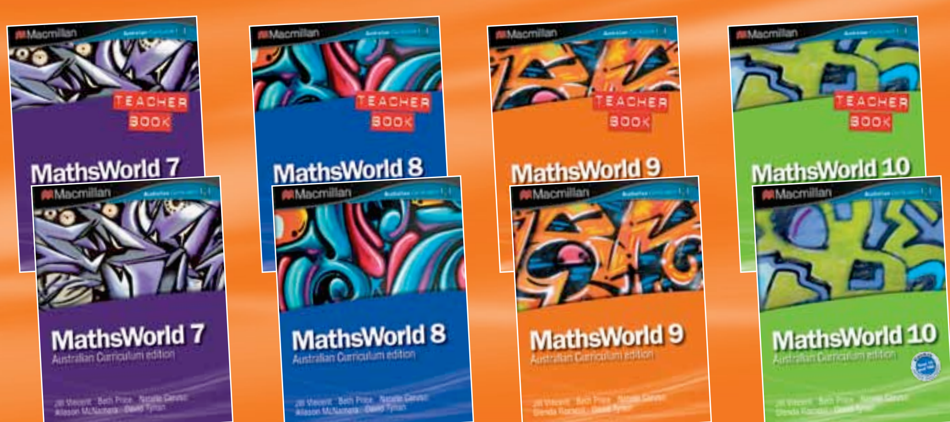

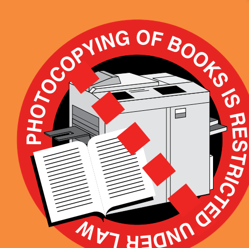

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