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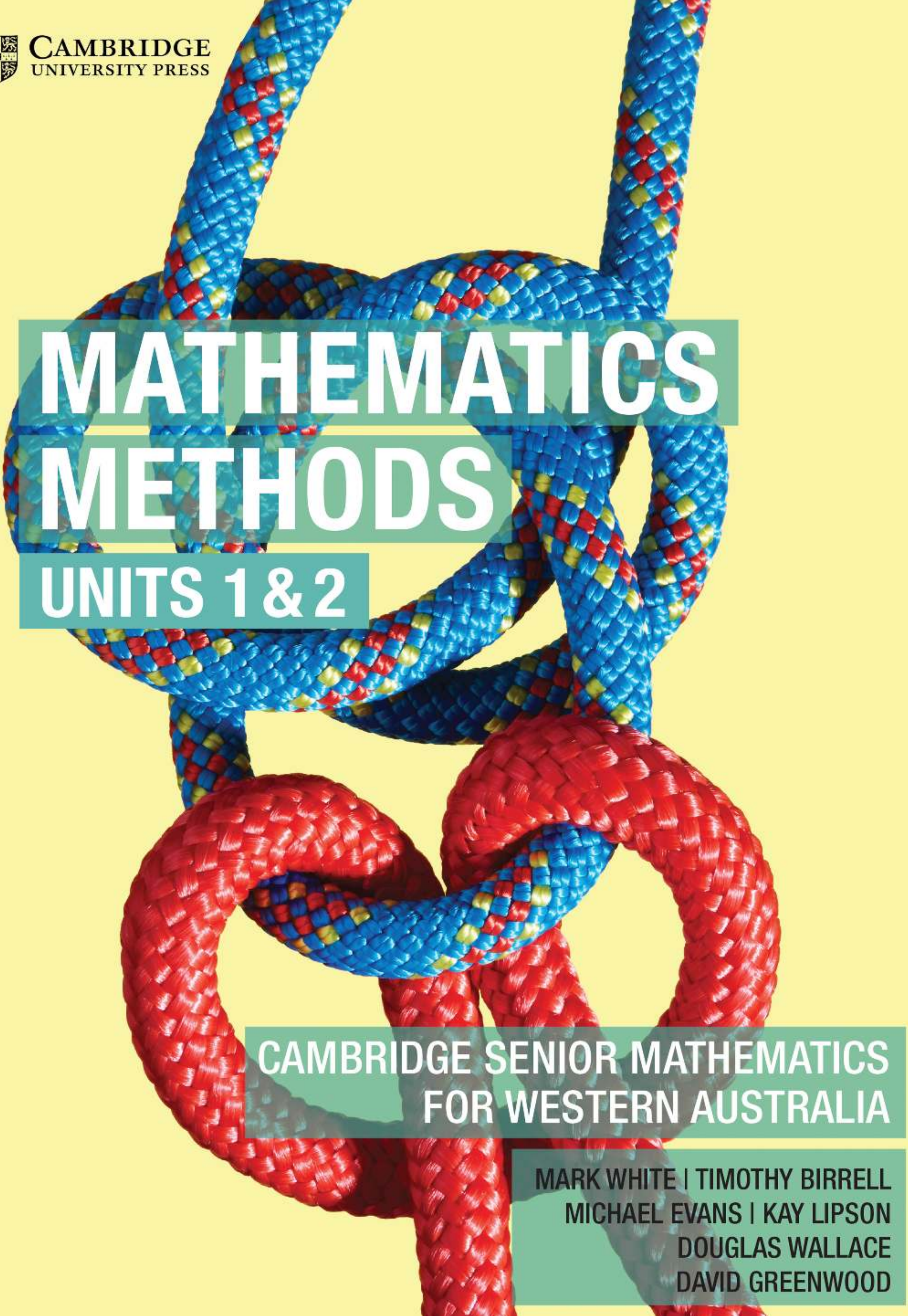

MATHEMATICS METHODS

UNITS 1 & 2

CAMBRIDGE SENIOR MATHEMATICS FOR WESTERN AUSTRALIA

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Introduction

Cambridge Mathematics Methods for Western Australia Units 1&2 is a new edition aligned specifically to the Western Australian Mathematics Methods Year 11 syllabus. Covering both Units 1 and 2 in one resource, it has been written with understanding as its chief aim and with ample practice offered through the worked examples and exercises.

The textbook starts by reviewing linear equations and provides an introductory study of simple elementary functions, algebra, calculus, probability and statistics and their applications in a variety of practical and theoretical contexts. The course is designed as preparation for the Western Australian Mathematics Methods Year 12 course (covered in the new Cambridge Units 3&4 textbook) and contains the prerequisite knowledge and skills for these units. Worked examples utilising CAS calculators are provided throughout, with screenshots and detailed user instructions for both ClassPad and TI-Nspire included for each CAS example.

Compared to the previous Australian Curriculum edition, this WA edition has undergone a number of revisions. Careful adjustments to notation and language have been made throughout to match that used in the WA syllabus and the Year 12 exam and in WA classrooms more generally. Sections on logarithms, part variation, piece-wise defined functions, and trigonometric problems in three dimensions have all been removed. All multiple-choice questions that were formerly located in the chapter reviews and revision chapters have also been removed.

The book contains five revision chapters. These provide technology-free and extended-response questions and are intended to help prepare students for examinations and other assessments, and the grading of the questions and the inclusion of challenging problems ensure that WA students have the opportunity to achieve at the highest standards.

The TI-Nspire calculator examples and instructions have been completed by Russell Brown and those for the Casio ClassPad have been completed by Maria Schaffner.

The integration of the features of the textbook and the new digital components of the package, powered by Cambridge HOTmaths, are illustrated on pages ix to xiii.

About Cambridge HOTmaths

Cambridge HOTmaths is a comprehensive, award-winning mathematics learning system – an interactive online maths learning, teaching and assessment resource for students and teachers, for individuals or whole classes, for school and at home. Its digital engine or platform is used to host and power the Interactive Textbook and the Online Teaching Suite, and selected topics from HOTmaths' own Years 9 and 10 courses area are available for revision of prior knowledge. All this is included in the price of the textbook.

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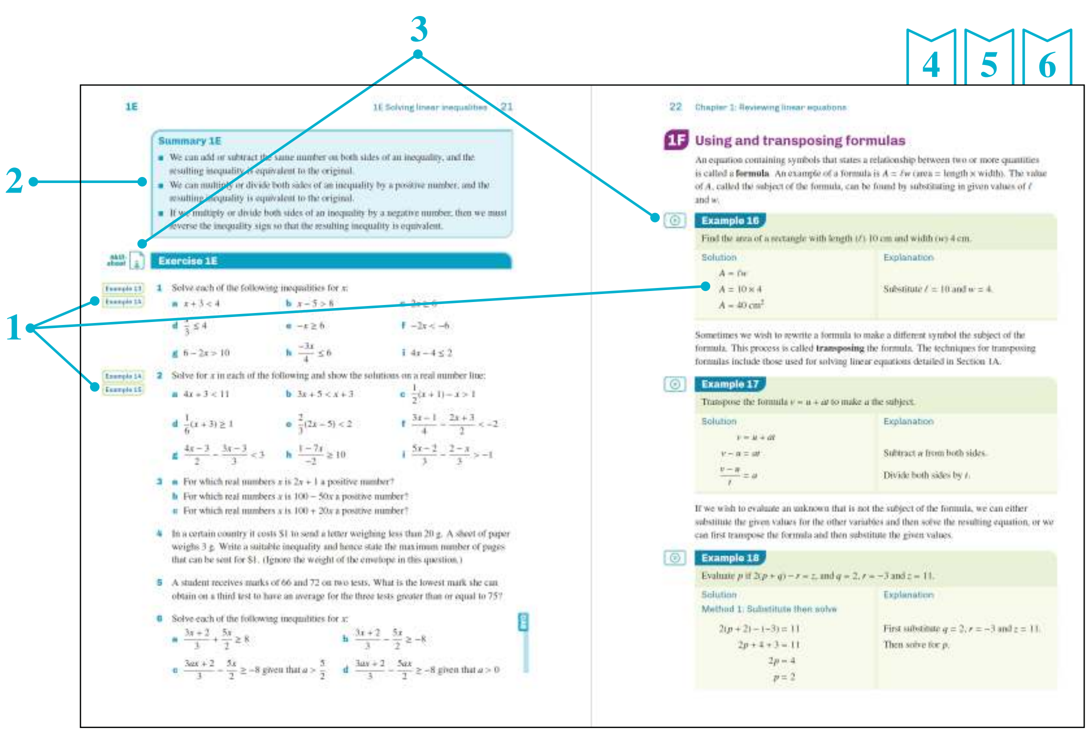
Overview

Overview of the print book

1. Graded step-by-step worked examples with precise explanations (and video versions) encourage independent learning, and are linked to exercise questions.
2. Section summaries provide important concepts in boxes for easy reference.
3. Additional linked resources in the Interactive Textbook are indicated by icons, such as skillsheets and video versions of examples.
4. Chapter reviews contain a chapter summary and short-answer and extended-response questions.
5. Revision chapters provide comprehensive revision and preparation for assessment.
6. The glossary includes page numbers of the main explanation of each term.

Numbers refer to descriptions above.



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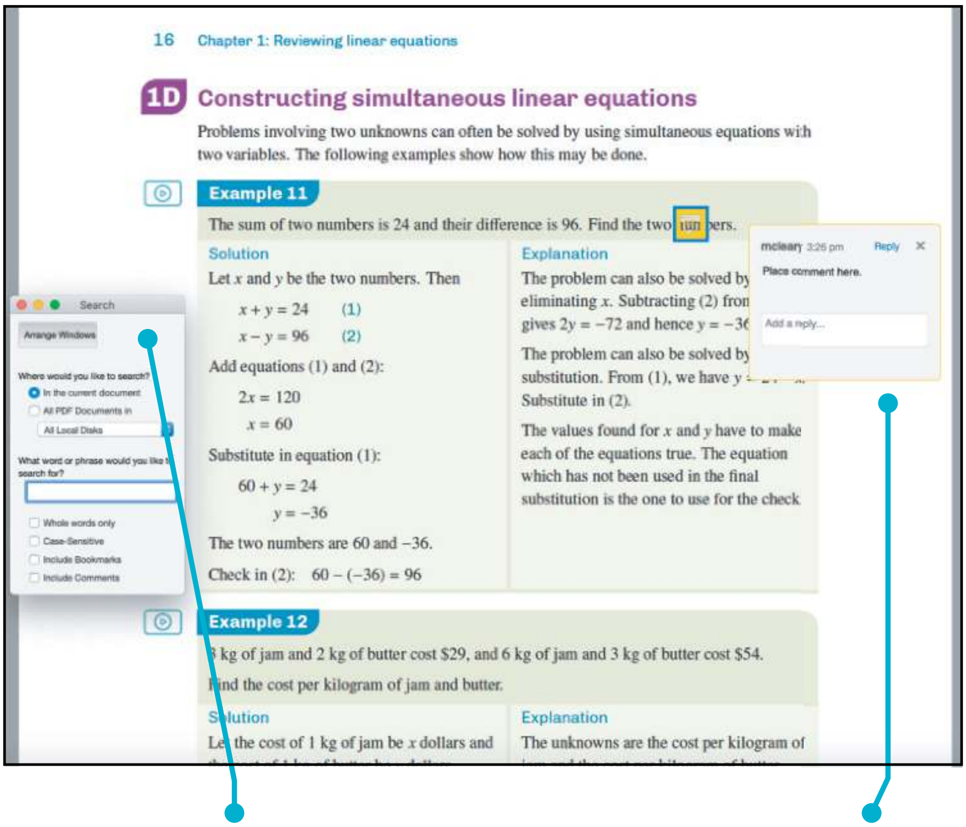
Overview of the downloadable PDF textbook

7 The convenience of a downloadable PDF textbook has been retained for times when users cannot go online.

8 PDF annotation and search features are enabled.



8 8

Overview of the Interactive Textbook

The **Interactive Textbook (ITB)** is an online HTML version of the print textbook powered by the HOTmaths platform, included with the print book or available as a separate purchase.

9 The material is formatted for on screen use with a convenient and easy-to-use navigation system and links to all resources.

10 **Workspaces** for all questions, which can be enabled or disabled by the teacher, allow students to enter working and answers online and to save them. Input is by typing, with the help of a symbol palette, handwriting and drawing on tablets, or by uploading images of writing or drawing done on paper.

11 **Self-assessment tools** enable students to check answers, mark their own work, and rate their confidence level in their work. This helps develop responsibility for learning and communicates progress and performance to the teacher. Student accounts can be linked to the learning management system used by the teacher in the Online Teaching Suite, so that teachers can review student self-assessment and provide feedback or adjust marks.

12 All worked examples have **video versions** to encourage independent learning.

13 **Worked solutions** are included and can be enabled or disabled in the student ITB accounts by the teacher.

14 An expanded and revised set of **Desmos interactives** and activities based on embedded graphics calculator and geometry tool windows demonstrate key concepts and enable students to visualise the mathematics.

15 The **Desmos graphics calculator, scientific calculator**, and **geometry tool** are also embedded for students to use for their own calculations and exploration.

16 **Revision of prior knowledge** is provided with links to diagnostic tests and Year 10 HOTmaths lessons.

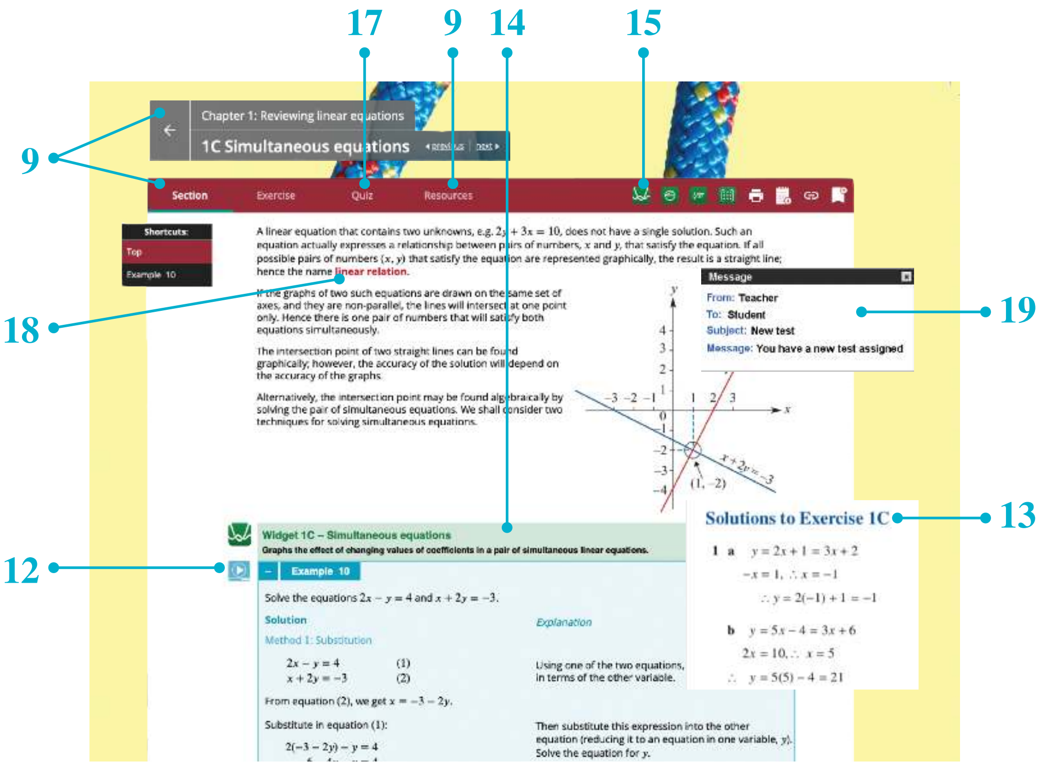
17 **Quick quizzes** containing automarked multiple-choice questions have been thoroughly expanded and revised, enabling students to check their understanding.

18 **Definitions** pop up for key terms in the text, and are also provided in a dictionary.

19 Messages from the teacher assign tasks and tests.

INTERACTIVE TEXTBOOK POWERED BY THE HOTmaths PLATFORM

A selection of features is shown. Numbers refer to the descriptions on pages xi–xii. HOTmaths platform features are updated regularly



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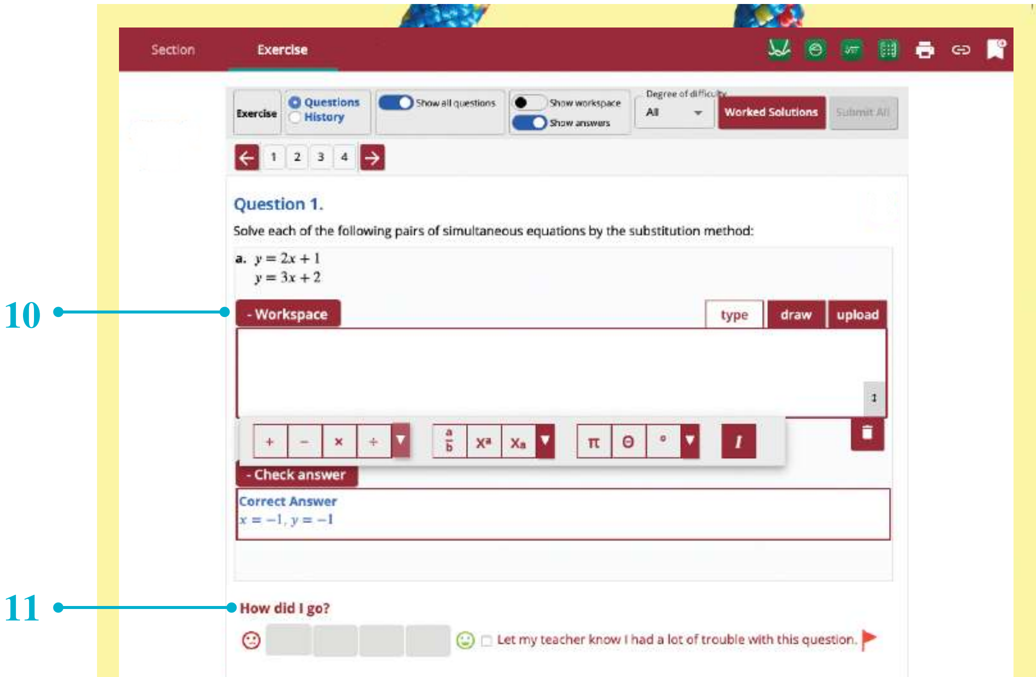
WORKSPACES AND SELF-ASSESSMENT



Overview of the Online Teaching Suite powered by the HOTmaths platform

The Online Teaching Suite is automatically enabled with a teacher account and is integrated with the teacher's copy of the Interactive Textbook. All the teacher resources are in one place for easy access. The features include:

20 The HOTmaths learning management system with class and student analytics and reports, and communication tools.

21 Teacher's view of a student's working and self-assessment which enables them to modify the student's self-assessed marks, and respond where students flag that they had diffculty.

22 A HOTmaths-style test generator.

23 A suite of chapter tests and assignments.

24 Editable curriculum grids and teaching programs.

25 A brand-new **Exam Generator**, allowing the creation of customised printable and online trial exams (see below for more).

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More about the Exam Generator

The Online Teaching Suite includes a comprehensive bank of SCSA exam questions, augmented by exam-style questions written by experts, to allow teachers to create custom trial exams.

Custom exams can model end-of-year exams, or target specific topics or types of questions that students may be having difficulty with.

Features include:

- Filtering by question-type, topic and degree of difficulty
- Searchable by key words
- Answers provided to teachers
- Worked solutions for all questions
- SCSA marking scheme
- All custom exams can be printed and completed under exam-like conditions or used as revision.

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1 Reviewing linear equations

In this chapter

Syllabus references

Topic: Functions and graphs

Subtopic: 1.1.6

Many problems may be solved by first translating them into mathematical equations and then solving the equations using algebraic techniques. An equation is solved by finding the value or values of the variables that would make the statement true.

Consider the equation $2x + 11 = 3 - 2x$. If $x = -2$, then

$$\text{LHS} = 2(-2) + 11 = 7 \quad \text{and} \quad \text{RHS} = 3 - 2(-2) = 7$$

The statement is true when $x = -2$. The solution to the equation is therefore $x = -2$. In this case there is no other value of x that would give a true statement.

The equations that we deal with in this chapter are called linear equations since they are related to the equation of a straight line.

Linear equations have either one solution (as shown in the example above), no solutions or infinitely many solutions. The equation $2x + 3 = 2x + 4$ has no solutions, since no value of x makes the statement true. The equation $2(x + 3) = 2x + 6$ has infinitely many solutions, since it is true for all values of x.

We note that the equation $x^2 = 16$ has exactly two solutions, $x = 4$ or $x = -4$, but this equation is not linear.

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1A Linear equations

A **linear equation** (in one unknown) is a particular type of polynomial equation in which the variable is to the first power. The following are examples of linear equations:

$$3x - 5 = 11, \qquad 7 - 2t = 8t - 11, \qquad \frac{z-3}{4} + \frac{2z-5}{3} = 11$$

In each of these equations, the variable is to the first power.

The following are examples of non-linear polynomial equations:

$$x^2 - x - 12 = 0 \quad \text{(quadratic)}, \qquad 2x^3 - x = 0 \quad \text{(cubic)}, \qquad x^4 = 16 \quad \text{(quartic)}$$

In each of these equations, the highest power of the variable is not the first power. You have met some of these in your previous study, and they are also in later chapters of this book.

Solving linear equations

It is important when setting out the solution to an equation that each step is written under the previous one with the equals signs aligned. This careful setting out makes the algebra easy to check. Unsystematic methods, such as guess and check, will generally be of limited use for more complicated equations.

It is often helpful to look at how the equation has been constructed so that the steps necessary to 'undo' the equation can be identified. It is most important that the steps taken to solve the equation are done in the correct order.

Linear equations of the form $ax + b = c$

Many linear equations that arise in applications are of the form $ax + b = c$.

Example 1

Solve the equation $3x + 4 = 16$ for x.

Solution	Explanation
$3x + 4 = 16$	
$3x = 12$	Subtract 4 from both sides.
$x = 4$	Divide both sides by 3.
Check:	Once a solution has been found it may be checked by substituting the value back into both sides of the original equation to ensure that the left-hand side (LHS) equals the right-hand side (RHS).
LHS $= 3(4) + 4 = 16$	
RHS $= 16$	
$\therefore$ the solution is correct.	

The first three equations in the above example are equivalent equations. Obtaining the second two equations enables us to solve the first equation.

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Given an equation, an equivalent equation can be formed by:

- adding or subtracting the same number on both sides of the equation
- multiplying or dividing both sides of the equation by the same non-zero number.

Importantly, two equivalent equations have the same solution. By forming suitable equivalent equations, we solve linear equations.

Equations with the unknown on both sides

Group all the terms containing the variable on one side of the equation and the remaining terms on the other side.

Example 2

Solve $4x + 3 = 3x - 5$.

Solution

$$\begin{aligned} 4x + 3 &= 3x - 5 \\ x + 3 &= -5 \\ x &= -8 \end{aligned}$$

Check:

$$\begin{aligned} \text{LHS} &= 4(-8) + 3 = -29 \\ \text{RHS} &= 3(-8) - 5 = -29 \end{aligned}$$

$\therefore$ the solution is correct.

Explanation

Subtract $3x$ from both sides and then subtract 3 from both sides.

The solution can be checked as previously shown.

Equations containing brackets

A frequently used first step is to remove brackets and then to follow the procedure for solving an equation without brackets.

Example 3

Solve $3(2x + 5) = 27$.

Solution

$$\begin{aligned} 3(2x + 5) &= 27 \\ 6x + 15 &= 27 \\ 6x &= 12 \\ x &= 2 \end{aligned}$$

Check:

$$\begin{aligned} \text{LHS} &= 3(2 \times 2 + 5) = 27 \\ \text{RHS} &= 27 \end{aligned}$$

$\therefore$ the solution is correct.

Explanation

We note that since 27 is divisible by 3, the following method is also possible:

$$\begin{aligned} 3(2x + 5) &= 27 \\ 2x + 5 &= 9 \\ 2x &= 4 \\ x &= 2 \end{aligned}$$

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Equations containing fractions

A frequently used first step is to multiply both sides of the equation by the lowest common multiple of the denominators of the fractions.

Example 4

Solve $\frac{x}{5} - 2 = \frac{x}{3}$.

Solution

$$\frac{x}{5} - 2 = \frac{x}{3}$$

$$\frac{x}{5} \times 15 - 2 \times 15 = \frac{x}{3} \times 15$$

$$3x - 30 = 5x$$

$$-2x = 30$$

$$x = -15$$

Check: $\text{LHS} = \frac{-15}{5} - 2 = -3 - 2 = -5$

$\text{RHS} = \frac{-15}{3} = -5$

∴ the solution is correct.

Explanation

The denominators of the fractions are 3 and 5. The lowest common multiple of 3 and 5 is 15.

Multiply both sides of the equation by 15. This means that each term of the LHS and the RHS of the equation is multiplied by 15.

Example 5

Solve $\frac{x-3}{2} - \frac{2x-4}{3} = 5$.

Solution

$$\frac{x-3}{2} \times 6 - \frac{2x-4}{3} \times 6 = 5 \times 6$$

$$3(x-3) - 2(2x-4) = 30$$

$$3x - 9 - 4x + 8 = 30$$

$$-x = 31$$

$$x = -31$$

Check:

$\text{LHS} = \frac{-31-3}{2} - \frac{2 \times (-31) - 4}{3}$

$= \frac{-34}{2} - \frac{-66}{3} = -17 + 22 = 5$

$\text{RHS} = 5$

∴ the solution is correct.

Explanation

Remember that the line separating the numerator and the denominator (the vinculum) acts as brackets.

Multiply both sides of the equation by 6, the lowest common multiple of 2 and 3.

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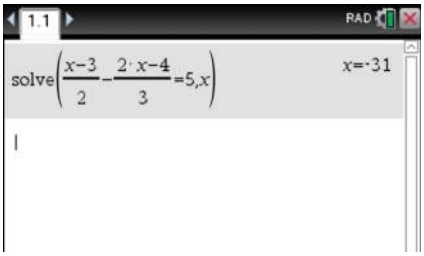
Using the TI-Nspire



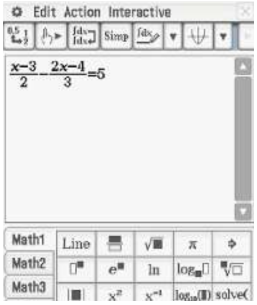
- To find the solution to the linear equation, use a **Calculator** application.
- Select (menu) > **Algebra** > **Solve**.
- Enter the equation

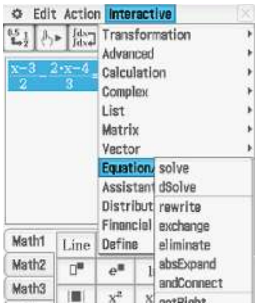
$$\frac{x-3}{2} - \frac{2x-4}{3} = 5$$

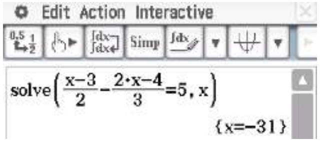
- Press (enter) to obtain the solution.

Note: A template for fractions may be obtained by pressing (ctrl) (÷).
For more details on the use of the calculator refer to the TI-Nspire appendix in the Interactive Textbook.

Using the Casio ClassPad







- Go to the $\sqrt[\text{Main}]{\alpha}$ screen and turn on the keyboard.
- Select the fraction icon [fraction icon] found in the (Math1) keyboard.
- Enter the equation

$$\frac{x-3}{2} - \frac{2x-4}{3} = 5$$

- Highlight the equation using the stylus and select **Interactive** > **Equation/Inequality** > **solve**.
- Tap on OK to obtain the solution. (Note that the default variable is x.)

Note: For more details on the use of the calculator refer to the Casio ClassPad appendix in the Interactive Textbook.

Literal equations

An equation for the variable x in which all the coefficients of x, including the constants, are pronumerals is known as a **literal equation**.

Example 6

Solve $ax + b = cx + d$ for x.

Solution	Explanation
$ax + b = cx + d$	
$ax - cx = d - b$	Collect terms in x on the left-hand side and constants on the right-hand side.
$(a - c)x = d - b$	Factorise the left-hand side.
$x = \dfrac{d-b}{a-c}$	

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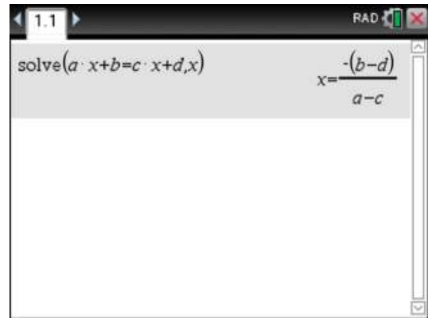
Using the TI-Nspire

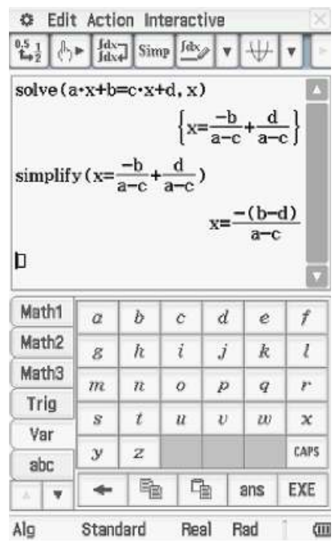
- To solve the literal equation $ax + b = cx + d$, use a **Calculator** application.
- Select (menu) > **Algebra** > **Solve**.
- Enter $ax + b = cx + d$ as shown.
- Press (enter) to obtain the solution.

Note: Ensure a multiplication sign is placed between the letters of the expression, otherwise the calculator will read them as a single variable rather than a product. That is, enter $a \times x$ and not ax.



Using the Casio ClassPad

- To solve the literal equation $ax + b = cx + d$, first go to the Main $\sqrt{\alpha}$ screen and turn on the keyboard.
- Select the [Var] keyboard. This will bring up the variables.
- Enter the equation $ax + b = cx + d$ and highlight it.
- Select **Interactive** > **Equation/Inequality** > **solve** and ensure that the variable selected is x.
- Tap on OK to obtain the solution.
- If necessary, the answer may be simplified further by copying the answer into the next line and then selecting **Interactive** > **Transformation** > **simplify**.



Summary 1A

- An equation is solved by finding the value or values of the variables that would make the statement true.
- A linear equation is one in which the variable is to the first power.
- There are often several different ways to solve a linear equation. The following steps provide some suggestions:
  1. Expand brackets and, if the equation involves fractions, multiply through by the lowest common denominator of the terms.
  2. Group all of the terms containing a variable on one side of the equation and the terms without the variable on the other side.

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Exercise 1A

1 Solve each of the following equations for x:

a $x + 3 = 6$ **b** $x - 3 = 6$ **c** $3 - x = 2$ **d** $x + 6 = -2$

e $2 - x = -3$ **f** $2x = 4$ **g** $3x = 5$ **h** $-2x = 7$

i $-3x = -7$ **j** $\frac{3x}{4} = 5$ **k** $\frac{-3x}{5} = 2$ **l** $\frac{-5x}{7} = -2$

2 Solve each of the following literal equations for x:

a $x - b = a$ **b** $x + b = a$ **c** $ax = b$ **d** $\frac{x}{a} = b$ **e** $\frac{ax}{b} = c$

Example 1

3 Solve the following linear equations:

a $2y - 4 = 6$ **b** $3t + 2 = 17$ **c** $2y + 5 = 2$ **d** $7x - 9 = 5$

e $2a - 4 = 7$ **f** $3a + 6 = 14$ **g** $\frac{y}{8} - 11 = 6$ **h** $\frac{t}{3} + \frac{1}{6} = \frac{1}{2}$

i $\frac{x}{3} + 5 = 9$ **j** $3 - 5y = 12$ **k** $-3x - 7 = 14$ **l** $14 - 3y = 8$

Example 2

4 Solve the following linear equations:

a $6x - 4 = 3x$ **b** $x - 5 = 4x + 10$ **c** $3x - 2 = 8 - 2x$

5 Solve the following linear equations:

Example 3

a $2(y + 6) = 10$ **b** $2y + 6 = 3(y - 4)$ **c** $2(x + 4) = 7x + 2$

d $5(y - 3) = 2(2y + 4)$ **e** $x - 6 = 2(x - 3)$ **f** $\frac{y + 2}{3} = 4$

Example 4

g $\frac{x}{2} + \frac{x}{3} = 10$ **h** $x + 4 = \frac{3}{2}x$ **i** $\frac{7x + 3}{2} = \frac{9x - 8}{4}$

Example 5

j $\frac{2(1 - 2x)}{3} - 2x = -\frac{2}{5} + \frac{4(2 - 3x)}{3}$ **k** $\frac{4y - 5}{2} - \frac{2y - 1}{6} = y$

Example 6

6 Solve the following literal equations for x:

a $ax + b = 0$ **b** $cx + d = e$ **c** $a(x + b) = c$ **d** $ax + b = cx$

e $\frac{x}{a} + \frac{x}{b} = 1$ **f** $\frac{a}{x} + \frac{b}{x} = 1$ **g** $ax - b = cx - d$ **h** $\frac{ax + c}{b} = d$

7 Solve each of the following for x:

a $0.2x + 6 = 2.4$ **b** $0.6(2.8 - x) = 48.6$ **c** $\frac{2x + 12}{7} = 6.5$

d $0.5x - 4 = 10$ **e** $\frac{1}{4}(x - 10) = 6$ **f** $6.4x + 2 = 3.2 - 4x$

8 Solve $\frac{b - cx}{a} + \frac{a - cx}{b} + 2 = 0$ for x.

9 Solve $\frac{a}{x + a} + \frac{b}{x - b} = \frac{a + b}{x + c}$ for x.

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1B Constructing linear equations

As stated earlier, many problems can be solved by translating them into mathematical language and using an appropriate mathematical technique to find the solution. By representing the unknown quantity in a problem with a symbol and constructing an equation from the information, the value of the unknown can be found by solving the equation.

Before constructing the equation, each symbol and what it stands for (including the units) should be stated. It is essential to remember that all the elements of the equation must be in units of the same system.

Example 7

A chef uses the following rule for cooking a turkey:
'Allow 30 minutes for each kilogram weight of turkey and then add an extra 15 minutes.'

If the chef forgot to weigh a turkey before cooking it, but knew that it had taken 3 hours to cook, calculate how much it weighed.

Solution

Let the weight of the turkey be x kilograms.
Then the time taken is $(30x + 15)$ minutes.

$$\begin{aligned} \therefore \quad 30x + 15 &= 180 \\ 30x &= 165 \\ x &= 5.5 \end{aligned}$$

The turkey weighed 5.5 kilograms.

Explanation

Assign a variable to the quantity that is to be found. In this example, the weight of the turkey is x kilograms.

Find, in terms of x, the time to cook the turkey. Then form the equation. Note that 3 hours is 180 minutes.

State the solution to the problem in words.

Example 8

Find the area of a rectangle whose perimeter is 1.08 m, if it is 8 cm longer than it is wide.

Solution

Let length $= \ell$ cm.
Then width $= (\ell - 8)$ cm.

$$\begin{aligned} \text{Perimeter} &= 2 \times \text{length} + 2 \times \text{width} \\ &= 2\ell + 2(\ell - 8) \\ &= (4\ell - 16)\text{ cm} \end{aligned}$$

$$\text{Perimeter} = 108\text{ cm}$$

$$\begin{aligned} \therefore \quad 4\ell - 16 &= 108 \\ 4\ell &= 124 \\ \ell &= 31\text{ cm} \end{aligned}$$

The length is 31 cm and the width is 23 cm.

Therefore the area is $31 \times 23 = 713\text{ cm}^2$.

Explanation

We know that

$$\text{Perimeter} = 2 \times \text{length} + 2 \times \text{width}$$

and that the width is 8 cm less than the length. Let ℓ cm be the length. Then the width is $(\ell - 8)$ cm.

Find the perimeter in terms of ℓ. Find the length and width, and hence find the area.

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Example 9

Adam normally takes 5 hours to travel between Higett and Logett. One day he increases his speed by 4 km/h and finds the journey from Higett to Logett takes half an hour less than the normal time. Find his normal speed.

Solution

Let x km/h be his normal speed.

The distance from Higett to Logett is $x \times 5 = 5x$ kilometres.

Adam's new speed is $(x + 4)$ km/h.

Hence
$$(x + 4) \times \frac{9}{2} = 5x$$
$$9(x + 4) = 10x$$
$$9x + 36 = 10x$$
$$36 = x$$

His normal speed is 36 km/h.

Explanation

In problems such as this, the speed is the average speed.

We note that

$$\text{distance} = \text{speed} \times \text{time}$$

Summary 1B

Steps for solving a word problem with a linear equation:

- Read the question carefully and write down the known information clearly.
- Identify the unknown quantity that is to be found.
- Assign a variable to this quantity.
- Form an expression in terms of x (or the variable being used) and use the other relevant information to form the equation.
- Solve the equation.
- Write a sentence answering the initial question.

Exercise 1B

1 For each of the following, write an equation using the variable x, then solve the equation for x:

a A number plus two is equal to six.

b A number multiplied by three is equal to ten.

c Six is added to a number multiplied by three and the result is twenty-two.

d Five is subtracted from a number multiplied by three and the result is fifteen.

e Three is added to a number. If the result of this is multiplied by six, then fifty-six is obtained.

f Five is added to a number and the result divided by four gives twenty-three.

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2 \$48 is divided among three students, A, B and C. If B receives three times as much as A, and C receives twice as much as A, how much does each receive?

3 The sum of two numbers is 42, and one number is twice the other. Find the two numbers.

Example 7

4 A chef uses the following rule for cooking food on a spit: 'Allow 20 minutes for each kilogram weight and then add an extra 20 minutes.' If the chef forgot to weigh the food before cooking it but knew that it had taken 3 hours to cook, calculate how much it weighed.

Example 8

5 Find the area of a rectangle whose perimeter is 4.8 m, if it is 0.5 m longer than it is wide.

6 Find three consecutive whole numbers with a sum of 150.

7 Find four consecutive odd numbers with a sum of 80.

8 Two tanks contain equal amounts of water. They are connected by a pipe and 3000 litres of water is pumped from one tank to the other. One tank then contains 6 times as much water as the other. How many litres of water did each tank contain originally?

9 A 120-page book has p lines to a page. If the number of lines were reduced by three on each page, the number of pages would need to be increased by 20 to give the same amount of writing space. How many lines were there on each page originally?

Example 9

10 A rower travels upstream at 6 km/h and back to the starting place at 10 km/h. The total journey takes 48 minutes. How far upstream did the rower go?

11 A shopkeeper buys a crate of eggs at \$1.50 per dozen. He buys another crate, containing 3 dozen more than the first crate, at \$2.00 per dozen. He sells them all for \$2.50 a dozen and makes \$15 profit. How many dozens were there in each of the crates?

Example 9

12 Jess walked for 45 minutes at 3 km/h and then ran for half an hour at x km/h. At the end of that time she was 6 km from the starting point. Find the value of x.

13 A man travels from A to B at 4 km/h and from B to A at 6 km/h. The total journey takes 45 minutes. Find the distance travelled.

14 A boy is 24 years younger than his father. In two years' time the sum of their ages will be 40. Find the present ages of father and son.

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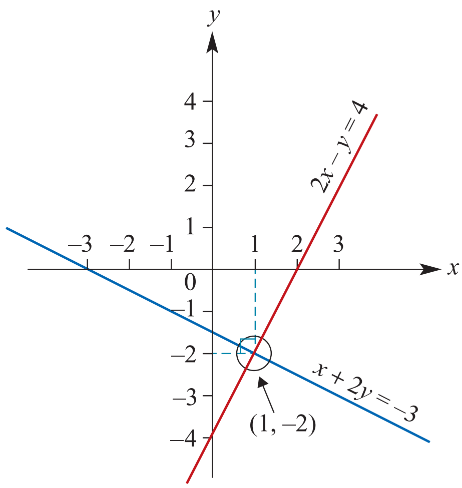
1C Simultaneous equations

A linear equation that contains two unknowns, e.g. $2y + 3x = 10$, does not have a single solution. Such an equation actually expresses a relationship between pairs of numbers, x and y, that satisfy the equation. If all possible pairs of numbers (x, y) that satisfy the equation are represented graphically, the result is a straight line; hence the name **linear relation**.

If the graphs of two such equations are drawn on the same set of axes, and they are non-parallel, the lines will intersect at one point only. Hence there is one pair of numbers that will satisfy both equations simultaneously.



The intersection point of two straight lines can be found graphically; however, the accuracy of the solution will depend on the accuracy of the graphs.

Alternatively, the intersection point may be found algebraically by solving the pair of simultaneous equations. We shall consider two techniques for solving simultaneous equations.

Example 10

Solve the equations $2x - y = 4$ and $x + 2y = -3$.

Solution

Method 1: Substitution

$$2x - y = 4 \quad (1)$$
$$x + 2y = -3 \quad (2)$$

Using one of the two equations, express one variable in terms of the other variable.

From equation (2), we get $x = -3 - 2y$.

Substitute in equation (1):

$$2(-3 - 2y) - y = 4$$
$$-6 - 4y - y = 4$$
$$-5y = 10$$
$$y = -2$$

Then substitute this expression into the other equation (reducing it to an equation in one variable, y). Solve the equation for y.

Substitute the value of y into (2):

$$x + 2(-2) = -3$$
$$x = 1$$

Substitute this value for y in one of the equations to find the other variable, x.

Check in (1): LHS $= 2(1) - (-2) = 4$

RHS $= 4$

A check can be carried out with the other equation.

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Method 2: Elimination

$$2x - y = 4 \qquad (1)$$
$$x + 2y = -3 \qquad (2)$$

To eliminate x, multiply equation (2) by 2 and subtract the result from equation (1).

When we multiply equation (2) by 2, the pair of equations becomes:

$$2x - y = 4 \qquad (1)$$
$$2x + 4y = -6 \qquad (2')$$

Subtract (2′) from (1):

$$-5y = 10$$
$$y = -2$$

Now substitute for y in equation (2) to find x, and check as in the substitution method.

If one of the variables has the same coefficient in the two equations, we can eliminate that variable by subtracting one equation from the other.

It may be necessary to multiply one of the equations by a constant to make the coefficients of x or y the same in the two equations.

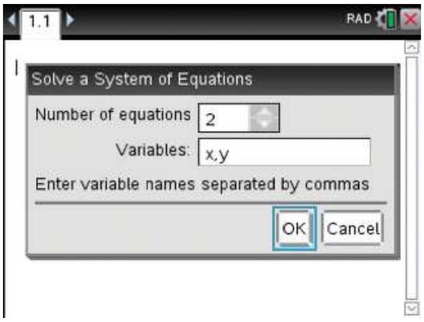
Note: This example shows that the point $(1, -2)$ is the point of intersection of the graphs of the two linear relations.

Using the TI-Nspire

Calculator application

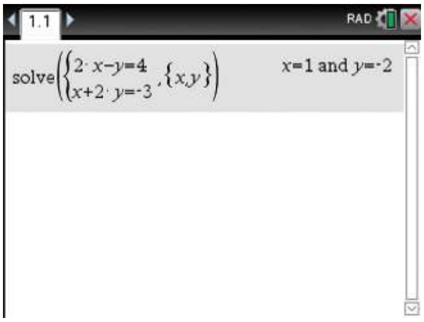
Simultaneous equations can be solved in a **Calculator** application.

- Use (menu) > **Algebra** > **Solve System of Equations** > **Solve System of Equations**.
- Complete the pop-up screen.



- Enter the equations as shown to give the solution to the simultaneous equations $2x - y = 4$ and $x + 2y = -3$.

Note: The solution can also be found with solve$(2x - y = 4$ and $x + 2y = -3, x, y)$.



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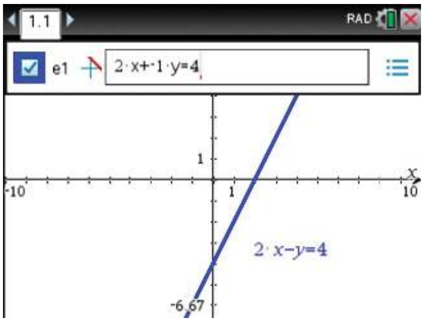
Graphs application

The simultaneous equations can also be solved graphically in a **Graphs** application.

Entering the equations:

- The equations can be entered directly in the form $a \cdot x + b \cdot y = c$ using (menu) > **Graph Entry/Edit** > **Equation** > **Line** > **a·x + b·y = c**.
- Enter the equations as shown.

Hint: Use ▼ to enter the second equation.



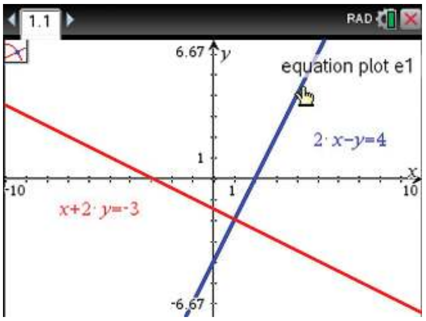
Alternatively:

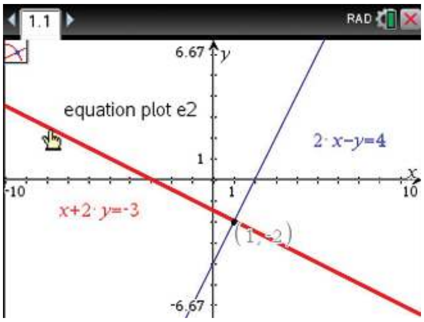
- The equations can be rearranged to make y the subject. The equations in this form are $f1(x) = 2x - 4$ and $f2(x) = \dfrac{-3 - x}{2}$.
- Enter these in the default function entry line.

Note: If the entry line is not visible, press (tab) or double click in an open area. Pressing (enter) will hide the entry line.

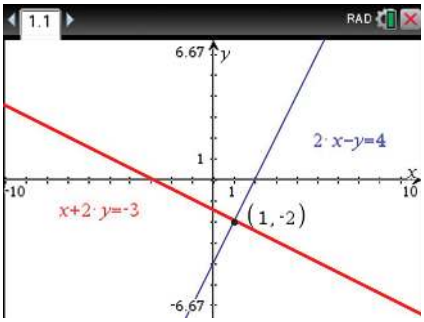
Finding the intersection point:

- Use (menu) > **Geometry** > **Points & Lines** > **Intersection Point(s)**.
- Use the touchpad to move the cursor to select each of the two graphs.





- The intersection point's coordinates will appear on the screen. Press (esc) to exit the **Intersection Point(s)** tool.



Note: You can also find the intersection point using (menu) > **Analyze Graph** > **Intersection**.

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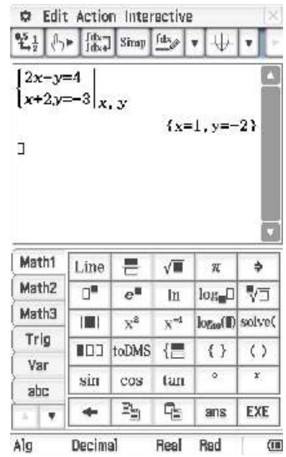
Using the Casio ClassPad

To solve the simultaneous equations algebraically:

- Go to the $\sqrt{\alpha}$ (Main) screen and turn on the keyboard.
- Open the Math1 keyboard and select the simultaneous equations icon {■.
- Enter the two equations

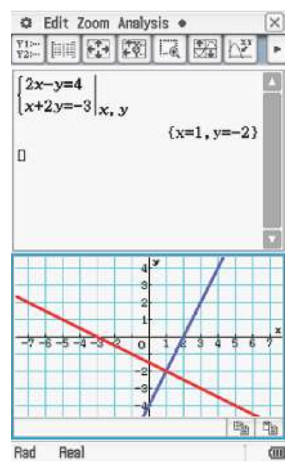
$$2x - y = 4$$
$$x + 2y = -3$$

  into the two lines.
- Type x, y in the bottom-right square to indicate the variables.
- Select EXE.

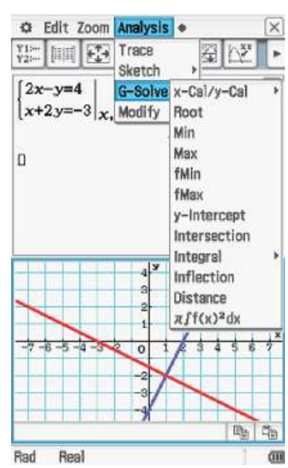


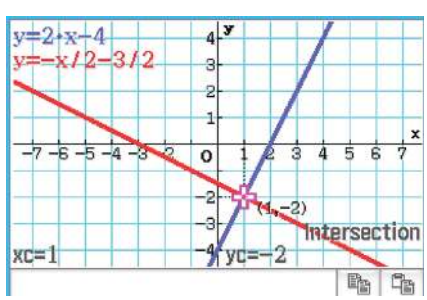
The simultaneous equations can also be solved graphically:

- Tap on the graph icon to display the graph screen.
- Using your stylus, highlight the first equation $2x - y = 4$ and drag it down into the graph screen. Lift the stylus off the screen for the graph to appear.
- Repeat by highlighting the second equation $x + 2y = -3$ and dragging it down into the graph screen. Lift the stylus off the screen for the second graph to appear.



- To find the solution, tap into the graph screen to select it, and then select **Analysis** > **G-Solve** > **Intersection**.





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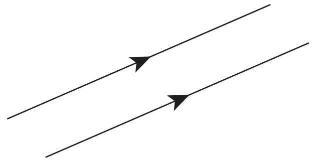
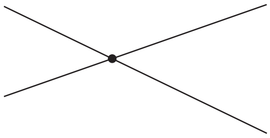
The geometry of simultaneous equations

Two distinct straight lines are either parallel or meet at a point.





There are three cases for a system of two linear equations with two variables.

	Example	Solutions	Geometry
Case 1	$2x + y = 5$ $x - y = 4$	Unique solution: $x = 3,\ y = -1$	Two lines meeting at a point
Case 2	$2x + y = 5$ $2x + y = 7$	No solutions	Distinct parallel lines
Case 3	$2x + y = 5$ $4x + 2y = 10$	Infinitely many solutions	Two copies of the same line

This is further discussed in Chapter 2.

Summary 1C

We have two methods for solving simultaneous linear equations in two variables by hand.

1 Substitution

Make one of the variables the subject in one of the equations and substitute for that variable in the other equation.

For example, for the equations:

$$3x + y = 6 \qquad (1)$$
$$5x + 3y = -2 \qquad (2)$$

- From (1), we have $y = 6 - 3x$.
- Substitute this into (2), and solve the resulting equation $5x + 3(6 - 3x) = -2$ for x.
- Substitute this value for x back into one of the original equations to find the value of the other variable, y.
- Check that your solution satisfies the original equations.

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2 Elimination

In this method, one of the variables is eliminated. For example, with the same equations considered above:

$$3x + y = 6 \qquad (1)$$
$$5x + 3y = -2 \qquad (2)$$

- Decide which variable to eliminate. We will eliminate y.
- If necessary, multiply both sides of one or both equations by a number to obtain the same or opposite coefficient for the variable to be eliminated. With this example, multiply (1) by 3:

$$9x + 3y = 18 \qquad (1')$$

- Add or subtract the equations to eliminate the chosen variable. In this example, subtract (2) from (1′).
- Solve the resulting equation, and then proceed as with the substitution method.

Exercise 1C

1 Solve each of the following pairs of simultaneous equations by the substitution method:

a $y = 2x + 1$, $y = 3x + 2$
b $y = 5x - 4$, $y = 3x + 6$
c $y = 2 - 3x$, $y = 5x + 10$
d $y - 4 = 3x$, $y - 5x + 6 = 0$
e $y - 4x = 3$, $2y - 5x + 6 = 0$
f $y - 4x = 6$, $2y - 3x = 4$

2 Solve each of the following pairs of simultaneous equations by the elimination method:

a $x + y = 6$, $x - y = 10$
b $y - x = 5$, $x + y = 3$
c $x - 2y = 6$, $x + 6y = 10$

Example 10

3 Solve each of the following pairs of simultaneous linear equations by either the substitution or the elimination method:

a $2x - 3y = 7$, $y = 5 - 3x$
b $2x - 5y = 10$, $4x + 3y = 7$
c $2m - 1 = n$, $2n + m = 8$
d $7x - 6y = 20$, $3x + 4y = 2$
e $3s - 1 = t$, $5s + 2t = 20$
f $4x - 3y = 1$, $4y - 5x = 2$
g $15x - 4y = 6$, $9x - 2y = 5$
h $2p + 5q = -3$, $7p - 2q = 9$
i $2x - 4y = -12$, $2y + 3x - 2 = 0$

4 For each of the following pairs of simultaneous linear equations, state whether there is one, none or infinitely many solutions:

a $3x + y = 6$, $6x + 2y = 7$
b $3x + y = 6$, $6x + 2y = 12$
c $3x + y = 6$, $6x - 2y = 7$
d $3x - y = 6$, $6x + 2y = 7$

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1D Constructing simultaneous linear equations

Problems involving two unknowns can often be solved by using simultaneous equations with two variables. The following examples show how this may be done.

Example 11

The sum of two numbers is 24 and their difference is 96. Find the two numbers.

Solution

Let x and y be the two numbers. Then

$x + y = 24 \quad (1)$

$x - y = 96 \quad (2)$

Add equations (1) and (2):

$2x = 120$

$x = 60$

Substitute in equation (1):

$60 + y = 24$

$y = -36$

The two numbers are 60 and −36.

Check in (2): $60 - (-36) = 96$

Explanation

The problem can also be solved by eliminating x. Subtracting (2) from (1) gives $2y = -72$ and hence $y = -36$.

The problem can also be solved by substitution. From (1), we have $y = 24 - x$. Substitute in (2).

The values found for x and y have to make each of the equations true. The equation which has not been used in the final substitution is the one to use for the check.

Example 12

3 kg of jam and 2 kg of butter cost \$29, and 6 kg of jam and 3 kg of butter cost \$54.

Find the cost per kilogram of jam and butter.

Solution

Let the cost of 1 kg of jam be x dollars and the cost of 1 kg of butter be y dollars.

Then $3x + 2y = 29 \quad (1)$

and $6x + 3y = 54 \quad (2)$

Multiply (1) by 2: $6x + 4y = 58 \quad (1')$

Subtract (1′) from (2): $-y = -4$

$y = 4$

Substitute in (2): $6x + 3(4) = 54$

$6x = 42$

$x = 7$

Jam costs \$7 per kg and butter \$4 per kg.

Explanation

The unknowns are the cost per kilogram of jam and the cost per kilogram of butter.

Three kilograms of jam and two kilograms of butter cost \$29.

Six kilograms of jam and three kilograms of butter cost \$54.

Check in the original problem:

3 kg of jam = \$21 and 2 kg of butter = \$8
Total = \$29

6 kg of jam = \$42 and 3 kg of butter = \$12
Total = \$54

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Summary 1D

Steps for solving a word problem with simultaneous linear equations:

- Read the question carefully and write down the known information clearly.
- Identify the two unknown quantities that are to be found.
- Assign variables to these two quantities.
- Form expressions in terms of x and y (or other suitable variables) and use the other relevant information to form the two equations.
- Solve the system of equations.
- Write a sentence answering the initial question.

Skill-sheet

Exercise 1D

1 Find two numbers whose sum is 138 and whose difference is 88.

2 Find two numbers whose sum is 36 and whose difference is 9.

3 Six stools and four chairs cost \$58, while five stools and two chairs cost \$35.

a How much do ten stools and four chairs cost?

b How much do four stools cost?

c How much does one stool cost?

4 A belt and a wallet cost \$42, while seven belts and four wallets cost \$213.

a How much do four belts and four wallets cost?

b How much do three belts cost?

c How much does one belt cost?

Use simultaneous equations to solve the following.

Example 11

5 Find a pair of numbers whose sum is 45 and whose difference is 11.

6 In four years' time a mother will be three times as old as her son. Four years ago she was five times as old as her son. Find their present ages.

7 A party was organised for thirty people at which they could have either a hamburger or a pizza. If there were five times as many hamburgers as pizzas, calculate the number of each.

8 Two children had 110 marbles between them. After one child had lost half her marbles and the other had lost 20 they had an equal number. How many marbles did each child start with and how many did they finish with?

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9 One hundred and fifty tickets were sold for a basketball match and \$560 was the total amount collected. Adult tickets were sold at \$4.00 each and child tickets were sold at \$1.50 each. How many adult tickets and how many child tickets were sold?

10 The sum of the numerator and denominator of a fraction expressed in simplest form is 17. If 3 is added to the numerator, the value of the fraction will be 1. What is the fraction?

11 Linda thinks of a two-digit number. The sum of the digits is 8. If she reverses the digits, the new number is 36 greater than her original number. What was Linda's original number?

12 Tickets to a musical cost \$30 for adults and \$12 for children. At one particular performance 960 people attended and \$19 080 was collected in ticket sales. Find the number of adults and the number of children who attended the performance.

13 An investor received \$1400 interest per annum from a sum of money, with part of it invested at 10% and the remainder at 7% simple interest. This investor found that if she interchanged the amounts she had invested she could increase her return by \$90 per annum. Calculate the total amount invested.

Example 12 **14** A shopkeeper sold his entire stock of shirts and ties in a sale for \$10 000. The shirts were priced at 3 for \$100 and the ties \$20 each. If he had sold only half the shirts and two-thirds of the ties he would have received \$6000. How many of each did he sell in the sale?

15 A tent manufacturer produces two models, the Outback and the Bush Walker. From earlier sales records it is known that 20 per cent more of the Outback model is sold than the Bush Walker. A profit of \$200 is made on each Outback sold, but \$350 is made on each Bush Walker. If during the next year a profit of \$177 000 is planned, how many of each model must be sold?

16 Oz Jeans has factories in Mydney and Selbourne. At the Mydney factory, fixed costs are \$28 000 per month and the cost of producing each pair of jeans is \$30. At the Selbourne factory, fixed costs are \$35 200 per month and the cost of producing each pair of jeans is \$24. During the next month Oz Jeans must manufacture 6000 pairs of jeans. Calculate the production order for each factory, if the total manufacturing costs for each factory are to be the same.

17 A tea wholesaler blends together three types of tea that normally sell for \$10, \$11 and \$12 per kilogram so as to obtain 100 kilograms of tea worth \$11.20 per kilogram. If the same amounts of the two higher priced teas are used, calculate how much of each type must be used in the blend.

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1E Solving linear inequalities

An **inequality** is a mathematical statement that contains an inequality symbol rather than an equals sign: for example, $2x + 1 < 4$. When you solve the inequality $2x + 1 < 4$, you answer the question:

'Which numbers for x satisfy the property that $2x + 1$ is less than 4?'

You will find that your answers can be described using a number line. This is a good way to represent the solution, as there are infinitely many numbers that satisfy an inequality such as $2x + 1 < 4$. For example:

$$2(1) + 1 = 3 < 4, \quad 2(0) + 1 = 1 < 4, \quad 2\left(\frac{1}{2}\right) + 1 = 2 < 4, \quad 2(-1) + 1 = -1 < 4$$

To solve linear inequalities, proceed exactly as for equations with the following exception:

- When multiplying or dividing both sides by a negative number, the 'direction' of the inequality symbol is reversed.

Example 13

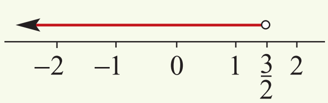
Solve the inequality $2x + 1 < 4$.

Solution

$2x + 1 < 4$

$2x < 3$ — Subtract 1 from both sides.

$x < \frac{3}{2}$ — Divide both sides by 2.



The solution can be represented on a real number line.

Note: In a number-line diagram, the 'endpoint' of an interval is indicated with a closed circle if the point is included and with an open circle if it is not.

Example 14

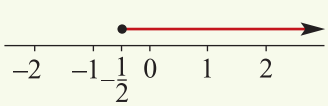
Solve the inequality $3 - 2x \le 4$.

Solution

$3 - 2x \le 4$

$-2x \le 1$ — Subtract 3 from both sides.

$x \ge -\frac{1}{2}$ — Divide both sides by -2. Note that the inequality symbol is reversed.



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Example 15

Solve the inequality $\dfrac{2x+3}{5} > \dfrac{3-4x}{3} + 2$.

Solution

$$\frac{2x+3}{5} > \frac{3-4x}{3} + 2$$

$$3(2x+3) > 5(3-4x) + 30$$

$$3(2x+3) - 5(3-4x) > 30$$

$$6x + 9 - 15 + 20x > 30$$

$$26x - 6 > 30$$

$$x > \frac{36}{26}$$

$$\therefore \quad x > \frac{18}{13}$$

Explanation

Multiply both sides by 15, the lowest common denominator of 5 and 3.

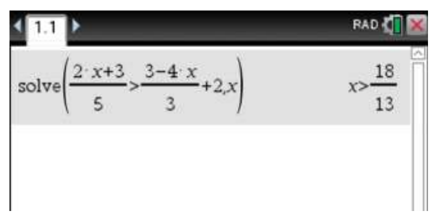
Collect the terms containing x on the left-hand side of the inequality.

Using the TI-Nspire

The inequality can be solved in a **Calculator** application.

- Choose **solve( )** from the **Algebra** menu to give the solution to

$$\frac{2x+3}{5} > \frac{3-4x}{3} + 2$$



Note: For the inequality signs template, press ctrl =.

Using the Casio ClassPad

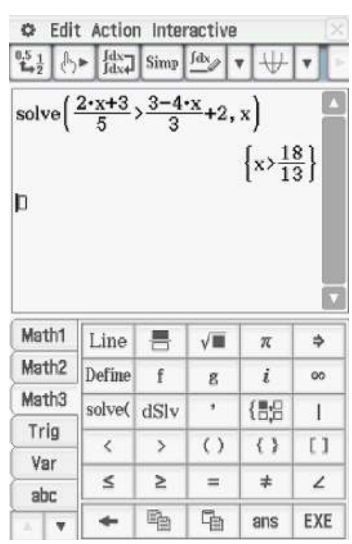
To solve the inequality:

- Go to the Main $\sqrt{\alpha}$ screen and turn on the keyboard.
- Select the fraction icon found in Math1.
- Enter the inequality

$$\frac{2x+3}{5} > \frac{3-4x}{3} + 2$$

Note: The inequality sign can be found in Math3.

- Highlight the inequality using the stylus.
- Select **Interactive** > **Equation/Inequality** > **solve** and ensure that the variable selected is x.
- Tap on OK to obtain the solution.



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Summary 1E

- We can add or subtract the same number on both sides of an inequality, and the resulting inequality is equivalent to the original.
- We can multiply or divide both sides of an inequality by a positive number, and the resulting inequality is equivalent to the original.
- If we multiply or divide both sides of an inequality by a negative number, then we must reverse the inequality sign so that the resulting inequality is equivalent.

Skill-sheet

Exercise 1E

Example 13
Example 14

1 Solve each of the following inequalities for x:

a $x + 3 < 4$ **b** $x - 5 > 8$ **c** $2x \geq 6$

d $\frac{x}{3} \leq 4$ **e** $-x \geq 6$ **f** $-2x < -6$

g $6 - 2x > 10$ **h** $\frac{-3x}{4} \leq 6$ **i** $4x - 4 \leq 2$

Example 14
Example 15

2 Solve for x in each of the following and show the solutions on a real number line:

a $4x + 3 < 11$ **b** $3x + 5 < x + 3$ **c** $\frac{1}{2}(x + 1) - x > 1$

d $\frac{1}{6}(x + 3) \geq 1$ **e** $\frac{2}{3}(2x - 5) < 2$ **f** $\frac{3x - 1}{4} - \frac{2x + 3}{2} < -2$

g $\frac{4x - 3}{2} - \frac{3x - 3}{3} < 3$ **h** $\frac{1 - 7x}{-2} \geq 10$ **i** $\frac{5x - 2}{3} - \frac{2 - x}{3} > -1$

3 **a** For which real numbers x is $2x + 1$ a positive number?

b For which real numbers x is $100 - 50x$ a positive number?

c For which real numbers x is $100 + 20x$ a positive number?

4 In a certain country it costs \$1 to send a letter weighing less than 20 g. A sheet of paper weighs 3 g. Write a suitable inequality and hence state the maximum number of pages that can be sent for \$1. (Ignore the weight of the envelope in this question.)

5 A student receives marks of 66 and 72 on two tests. What is the lowest mark she can obtain on a third test to have an average for the three tests greater than or equal to 75?

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1F Using and transposing formulas

An equation containing symbols that states a relationship between two or more quantities is called a **formula**. An example of a formula is $A = \ell w$ (area = length × width). The value of A, called the subject of the formula, can be found by substituting in given values of ℓ and w.

Example 16

Find the area of a rectangle with length (ℓ) 10 cm and width (w) 4 cm.

Solution	Explanation
$A = \ell w$	
$A = 10 \times 4$	Substitute $\ell = 10$ and $w = 4$.
$A = 40\text{ cm}^2$	

Sometimes we wish to rewrite a formula to make a different symbol the subject of the formula. This process is called **transposing** the formula. The techniques for transposing formulas include those used for solving linear equations detailed in Section 1A.

Example 17

Transpose the formula $v = u + at$ to make a the subject.

Solution	Explanation
$v = u + at$	
$v - u = at$	Subtract u from both sides.
$\dfrac{v-u}{t} = a$	Divide both sides by t.

If we wish to evaluate an unknown that is not the subject of the formula, we can either substitute the given values for the other variables and then solve the resulting equation, or we can first transpose the formula and then substitute the given values.

Example 18

Evaluate p if $2(p + q) - r = z$, and $q = 2$, $r = -3$ and $z = 11$.

Solution	Explanation
Method 1: Substituting then solving	
$2(p + 2) - (-3) = 11$	First substitute $q = 2$, $r = -3$ and $z = 11$.
$2p + 4 + 3 = 11$	Then solve for p.
$2p = 4$	
$p = 2$	

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Method 2: Transposing then substituting

$$2(p+q) - r = z$$
$$2(p+q) = z + r$$
$$p + q = \frac{z+r}{2}$$
$$p = \frac{z+r}{2} - q$$

First solve for p.

$$\therefore \quad p = \frac{11 + (-3)}{2} - 2$$
$$p = 2$$

Substitute $q = 2$, $r = -3$ and $z = 11$.

Example 19

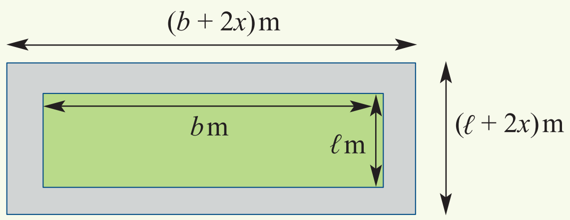
A path x metres wide surrounds a rectangular lawn. The lawn is ℓ metres long and b metres wide. The total area of the path is $A\ \text{m}^2$.

a Find A in terms of ℓ, b and x.

b Find b in terms of ℓ, A and x.

Solution

a



The area of the path is

$$A = (b+2x)(\ell+2x) - b\ell$$
$$= b\ell + 2x\ell + 2xb + 4x^2 - b\ell$$
$$\therefore \quad A = 2x\ell + 2xb + 4x^2$$

b $A - (2x\ell + 4x^2) = 2xb$

Therefore

$$b = \frac{A - (2x\ell + 4x^2)}{2x}$$

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Example 20

For each of the following, make c the subject of the formula:

a $e = \sqrt{3c - 7a}$

b $\dfrac{1}{a} - \dfrac{1}{b} = \dfrac{1}{c-2}$

Solution

a $$e = \sqrt{3c - 7a}$$

Square both sides of the equation:

$$e^2 = 3c - 7a$$

Therefore

$$3c = e^2 + 7a$$

$$c = \frac{e^2 + 7a}{3}$$

b $$\frac{1}{a} - \frac{1}{b} = \frac{1}{c-2}$$

Establish common denominator on the left-hand side of the equation:

$$\frac{b-a}{ab} = \frac{1}{c-2}$$

Take the reciprocal of both sides:

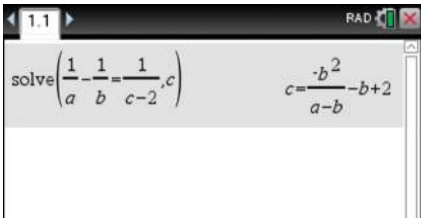
$$\frac{ab}{b-a} = c - 2$$

Therefore $c = \dfrac{ab}{b-a} + 2$

Using the TI-Nspire

Literal equations can be solved for a given variable in a **Calculator** application.

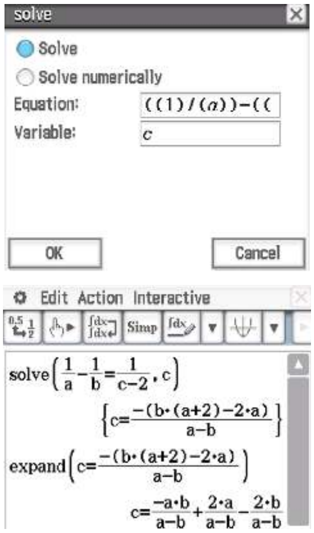
- Use **solve( )** from the **Algebra** menu to make c the subject of the formula
$$\frac{1}{a} - \frac{1}{b} = \frac{1}{c-2}$$



Using the Casio ClassPad

To solve a literal equation for a given variable:

- Go to the Main $\sqrt{\alpha}$ screen and turn on the keyboard.
- Select the fraction icon found in Math1.
- Select the Var keyboard.
- Enter and highlight the equation
$$\frac{1}{a} - \frac{1}{b} = \frac{1}{c-2}$$
- Select **Interactive** > **Equation/Inequality** > **solve** and ensure that you change the selected variable to c.
- Tap on OK to obtain the solution.
- If necessary, copy the solution and use **Interactive** > **Transformation** > **expand** to produce a neater answer.



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Summary 1F

- A formula relates different quantities: for example, the formula $A = \pi r^2$ relates the radius r with the area A of the circle.
- The variable on the left is called the subject of the formula: for example, in the formula $A = \pi r^2$, the subject is A.
- To calculate the value of a variable which is not the subject of a formula:

  Method 1 Substitute the values for the known variables, then solve the resulting equation for the unknown variable.

  Method 2 Rearrange to make the required variable the subject, then substitute values.

Skill-sheet

Exercise 1F

1 For each of the following, find the value of the letter in parentheses:

a $c = ab,\ a = 6,\ b = 3\quad (c)$
b $r = p + q,\ p = 12,\ q = -3\quad (r)$
c $c = ab,\ a = 6,\ c = 18\quad (b)$
d $r = p + q,\ p = 15,\ r = -3\quad (q)$
e $c = \sqrt{a},\ a = 9\quad (c)$
f $c = \sqrt{a},\ c = 9\quad (a)$
g $p = \dfrac{u}{v},\ u = 10,\ v = 2\quad (p)$
h $p = \dfrac{u}{v},\ p = 10,\ v = 2\quad (u)$

2 For each of the following, construct a formula using the given symbols:

a S, the sum of three numbers a, b and c
b P, the product of two numbers x and y
c the cost, \$$C$, of five CDs which each cost \$$p$
d the total cost, \$$T$, of d chairs which cost \$$p$ each and c tables which cost \$$q$ each
e the time, T, in minutes, of a train journey that takes a hours and b minutes

Example 16

3 Find the values of the following:

a $E = IR$, when $I = 5$ and $R = 3$
b $C = pd$, when $p = 3.14$ and $d = 10$
c $P = \dfrac{RT}{V}$, when $R = 60$, $T = 150$ and $V = 9$
d $I = \dfrac{E}{R}$, when $E = 240$ and $R = 20$
e $A = \pi r\ell$, when $\pi = 3.14$, $r = 5$ and $\ell = 20$
f $S = 90(2n - 4)$, when $n = 6$

Example 17

4 For each of the following, make the symbol indicated the subject of the formula:

a $PV = c;\ V$
b $F = ma;\ a$
c $I = Prt;\ P$
d $w = H + Cr;\ r$
e $S = P(1 + rt);\ t$
f $V = \dfrac{2R}{R - r};\ r$

Example 18

5 Find the value of the unknown symbol in each of the following:

a $D = \dfrac{T + 2}{P}$, when $D = 10$, $P = 5$
b $A = \dfrac{1}{2}bh$, when $A = 40$, $h = 10$
c $V = \dfrac{1}{3}\pi r^2 h$, when $\pi = 3.14$, $V = 100$, $r = 5$
d $A = \dfrac{1}{2}h(a + b)$, when $A = 50$, $h = 5$, $a = 10$

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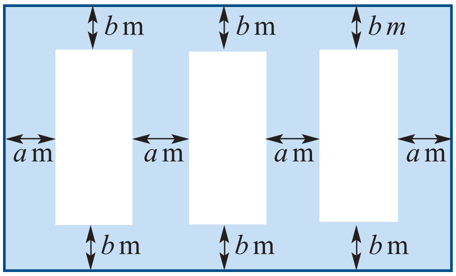
6 The diagram represents the brick wall of a dwelling with three windows. Each of the windows is h m high and w m wide. Other dimensions are as shown.



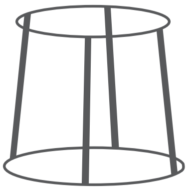
a Find the length of the wall.

b Find the height of the wall.

c Find the total area of the three windows.

d Find the total area of brickwork.

7 A lampshade has a metal frame consisting of two circular hoops of radii p cm and q cm joined by four straight struts of length h cm. The total length of metal is T cm.



a **i** Find an expression for T in terms of p, q and h.

ii Find T when $p = 20$, $q = 24$ and $h = 28$.

b The area of the material covering the frame is A cm^2, where $A = \pi h(p + q)$. Find an expression for p in terms of A, h, q and π.

8 Find the value of the unknown symbol in each of the following:

a $P = \dfrac{T - M}{D}$, $P = 6$, $T = 8$ and $M = 4$

b $H = \dfrac{a}{3} + \dfrac{a}{b}$, $H = 5$ and $a = 6$

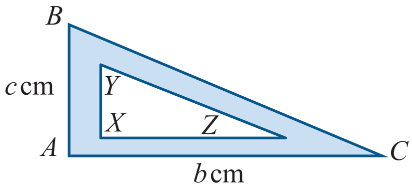
c $a = \dfrac{90(2n - 4)}{n}$, $a = 6$

d $R = \dfrac{r}{a} + \dfrac{r}{3}$, $a = 2$ and $R = 4$

Example 19

9 Right-angled triangles XYZ and ABC are similar.

$$\frac{XY}{AB} = \frac{YZ}{BC} = \frac{ZX}{CA} = k$$



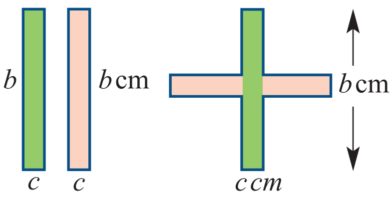
If $AB = c$ cm and $AC = b$ cm, find:

a the area, D cm^2, of the shaded region in terms of c, b and k

b k in terms of D, b and c

c the value of k if $D = 2$, $b = 3$ and $c = 4$.

10 Two rectangles each with dimensions c cm $\times$ b cm are used to form a cross as shown. The arms of the cross are all of equal length.



a Find the perimeter, P cm, of the cross in terms of b and c.

b Find the area, A cm^2, of the cross in terms of b and c.

c Find b in terms of A and c.

Example 20

11 For each of the following, make the symbol in brackets the subject of the formula:

a $a = \sqrt{a + 2b}$ (b)

b $\dfrac{a + x}{a - x} = \dfrac{b - y}{b + y}$ (x)

c $px = \sqrt{3q - r^2}$ (r)

d $\dfrac{x}{y} = \sqrt{1 - \dfrac{v^2}{u^2}}$ (v)

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Chapter summary

Assignment

Nrich

- A **linear equation** is one in which the variable is to the first power.
- It is often helpful to look at how the equation has been constructed so that the steps necessary to 'undo' the equation can be identified. It is most important that the steps taken to solve the equation are done in the correct order.
- An equation for the variable x in which all the coefficients of x, including the constants, are pronumerals is known as a **literal equation**: for example, $ax + b = c$.
- The two methods for solving simultaneous linear equations are **substitution** and **elimination**.
- An **inequality** is a mathematical statement that contains an inequality symbol rather than an equals sign: for example, $2x + 1 < 4$.
- To solve linear inequalities, proceed exactly as for equations except that, when multiplying or dividing both sides by a negative number, the 'direction' of the inequality symbol is reversed.
- An equation containing symbols that states a relationship between two or more quantities is called a **formula**. An example of a formula is $A = \ell w$ (area = length × width). The subject of this formula is A.
- If we wish to evaluate an unknown that is not the subject of the formula, we can either substitute the given values for the other variables and then solve the resulting equation, or we can first transpose the formula and then substitute the given values.

Short-answer questions

1 Solve each of the following equations for x:

a $2x + 6 = 8$ **b** $3 - 2x = 6$ **c** $2x + 5 = 3 - x$

d $\dfrac{3 - x}{5} = 6$ **e** $\dfrac{x}{3} = 4$ **f** $\dfrac{13x}{4} - 1 = 10$

g $3(2x + 1) = 5(1 - 2x)$ **h** $\dfrac{3x + 2}{5} + \dfrac{3 - x}{2} = 5$

2 Solve each of the following for t:

a $a - t = b$ **b** $\dfrac{at + b}{c} = d$ **c** $a(t - c) = d$

d $\dfrac{a - t}{b - t} = c$ **e** $\dfrac{at + b}{ct - b} = 1$ **f** $\dfrac{1}{at + c} = d$

3 Solve each of the following inequalities for x:

a $2 - 3x > 0$ **b** $\dfrac{3 - 2x}{5} \geq 60$

c $3(58x - 24) + 10 < 70$ **d** $\dfrac{3 - 2x}{5} - \dfrac{x - 7}{6} \leq 2$

4 **a** Make x the subject of the formula $z = \dfrac{1}{2}x - 3t$.

b Find x when $z = 4$ and $t = -3$.

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5 A number d is equal to the square of a number e plus twice a number f.

a Find a formula for d in terms of e and f.

b Make f the subject of the formula.

c Find f when $d = 10$ and $e = 3$.

6 The surface area of a sphere of radius r is given by the formula $A = 4\pi r^2$. Calculate the surface area of a sphere of radius 10 cm. Give your answer in terms of π.

7 The volume of metal in a tube is given by the formula $V = \pi\ell[r^2 - (r - t)^2]$, where ℓ is the length of the tube, r is the radius of the outside surface and t is the thickness of the material. Find V when:

a $\ell = 100$, $r = 5$ and $t = 0.2$

b $\ell = 50$, $r = 10$ and $t = 0.5$

8 For each of the following, make the variable in brackets the subject of the formula:

a $A = \pi rs \quad (r)$

b $T = P(1 + rw) \quad (w)$

c $v = \sqrt{\dfrac{n - p}{r}} \quad (r)$

d $ac = b^2 + bx \quad (x)$

9 Let $s = \left(\dfrac{u + v}{2}\right)t$.

a Find the value of s if $u = 10$, $v = 20$ and $t = 5$.

b Find the value of t if $u = 10$, $v = 20$ and $s = 120$.

10 The volume, V cm^3, of a cylinder is given by $V = \pi r^2 h$, where r cm is the radius and h cm is the height. Find the radius of the cylinder if the volume of the cylinder is 500π cm^3 and the height is 10 cm.

11 A rope of length 205 m is cut into 10 pieces of one length and 5 pieces of another length. The total length of three of the first 10 lengths exceeds that of two of the second length by 2 m. Find the lengths of the pieces.

12 If I add one to the numerator of a fraction $\dfrac{m}{n}$ it simplifies to $\dfrac{1}{5}$. If I subtract one from the denominator it simplifies to $\dfrac{1}{7}$. Find the fraction $\dfrac{m}{n}$.

13 Mr Adonis earns \$7200 more than Mr Apollo, and Ms Aphrodite earns \$4000 less than Mr Apollo. If the total of the three incomes is \$303 200, find the income of each person.

14 Solve each of the following pairs of simultaneous equations for a and b:

a $4a - b = 11$
$3a + 2b = 6$

b $a = 2b + 11$
$4a - 3b = 11$

15 A motorist travelled a total distance of 424 km, and had an average speed of 80 km/h on highways and 24 km/h while passing through towns. If the journey took six hours, find how long the motorist spent travelling on highways.

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Review

Extended-response questions

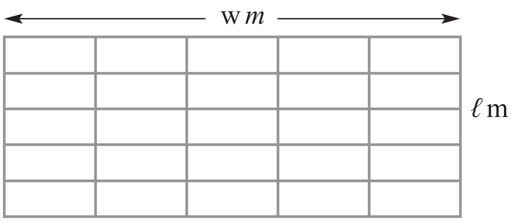
1 The formula for converting degrees Celsius, C, to degrees Fahrenheit, F, is $F = \frac{9}{5}C + 32$.

a Convert 30°F to Celsius.
b Convert 30°C to Fahrenheit.
c If x°C = x°F find x.
d If $(x + 10)$°C = x°F find x.
e If $2x$°C = x°F find the value of x.
f If k°F = $(-3k)$°C find k.

2 For a spherical mirror of radius r cm, $\frac{1}{v} + \frac{1}{u} = \frac{2}{r}$, where u cm is the distance from the mirror to the object and v cm is the distance from the mirror to the image. The magnification is given by $m = \frac{v - r}{r - u}$.

a Find r in terms of v and u from the first formula.
b Find m in terms of v and u only.

3 The diagram shows a section of wire mesh w metres in width and ℓ metres in length.



a Find an expression in terms of w and ℓ for the total length of wire required for the mesh.

b **i** If $w = 3\ell$, find an expression in terms of w for the total length of wire required.
ii If the length of wire used is 100 m, find the value of w and the value of ℓ.

c The total length of wire, L m, required for another type of rectangular mesh of dimensions x m by y m, is given by the formula $L = 6x + 8y$.
i Find y in terms of x and L.
ii Find y if $L = 200$ and $x = 4$.

d A third type of mesh can also be used to cover a rectangular region of dimensions x m by y m. In this case, the type of mesh introduced in part c requires 100 m of wire and so $6x + 8y = 100$. This third type of mesh requires 80 m and this gives the equation $3x + 2y = 40$. Find the values of x and y.

4 Tom leaves town A and travels towards town B at a constant speed of u km/h. At the same time, Julie leaves town B and travels towards town A at a constant speed of v km/h. Town B is d km from town A.

a How far has each travelled after t hours?
b By considering that the sum of their distances travelled must be d km when they meet, find:
i the time it takes for them to meet
ii their distance from town A when they meet.
c If $u = 30$, $v = 50$ and $d = 100$, find the time it takes for them to meet and their distance from town A.

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5 Xiu travels from town A to town B at u km/h and then returns at v km/h. Town A is d km from town B.

a Find the average speed at which Xiu travels for the complete journey, in terms of u and v. Remember that

$$\text{average speed} = \frac{\text{total distance travelled}}{\text{total time taken}}$$

b If it takes T hours to travel from A to B, find the time taken:

i for the return trip from B to A, in terms of T, u and v

ii for the entire trip, in terms of T, u and v.

6 A man on a bicycle rides one-third of the way from town A to town B at a speed a km/h and the remainder of the way at $2b$ km/h.

a If the distance between the two towns is 9 km, find the time taken to ride from A to B.

If the man had travelled at a uniform rate of $3c$ km/h, he could have ridden from A to B and back again in the same time.

b Show that $\frac{2}{c} = \frac{1}{a} + \frac{1}{b}$.

c **i** Make c the subject of this formula.

ii Find c, when $a = 10$ and $b = 20$.

7 A man walks 70 km. He walks x km at 8 km/h and y km at 10 km/h.

a Find the length of time he was walking at 8 km/h in terms of x, and the length of time he was walking at 10 km/h in terms of y.

b Find his average speed in terms of x and y.

c If the man walks at 10 km/h for the time he was walking at 8 km/h and at 8 km/h for the time he was walking at 10 km/h, he walks 72 km. Find x and y.

8 Prove that the lines with equations $2y - x = 2$, $y + x = 7$ and $y - 2x = -5$ meet at the one point.

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2 Coordinate geometry and linear relations

In this chapter

Syllabus references

Topic: Functions and graphs

Subtopics: 1.1.1 – 1.1.5

The number plane (Cartesian plane) is divided into four quadrants by two perpendicular axes. These axes intersect at a point called the origin. The position of any point in the plane can be represented by an **ordered pair** of numbers (x, y), called the **coordinates** of the point. Given the coordinates of two points, we can find the equation of the straight line through the two points, the distance between the two points and the midpoint of the line segment joining the points. These are the beginning ideas of coordinate geometry. The topic of calculus, which is introduced later in this book, builds on these ideas.

A **relation** is defined as a set of ordered pairs in the form (x, y). Sometimes we can give a rule relating the x-value to the y-value of each ordered pair, such as $y = 2x + 1$, and this is a more convenient way of describing the relation. A relation may also be represented graphically on a set of axes. If the resulting graph is a straight line, then the relation is called a **linear relation**.

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2A Distance and midpoints

In this first section we look at the method to find the coordinates of the midpoint of a line segment and we apply Pythagoras' theorem to find the distance between two points.

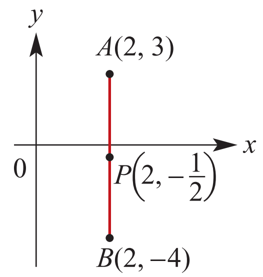
Midpoint of a line segment

Finding the **midpoint of a line segment** parallel to an axis is a simple special case, and it is useful in obtaining the more general result.

A line segment parallel to an axis

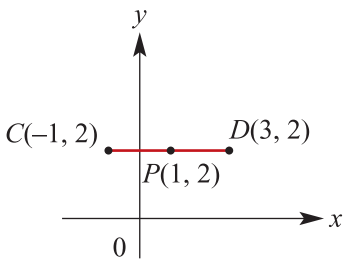
The midpoint of the line segment AB with endpoints $A(2, 3)$ and $B(2, -4)$ is the point P with coordinates $\left(2, -\frac{1}{2}\right)$.

Note that $-\frac{1}{2}$ is the average of 3 and −4. The line through A and B is parallel to the y-axis.



Similarly for the line segment CD with endpoints $C(-1, 2)$ and $D(3, 2)$, the midpoint is the point P with coordinates $(1, 2)$.

Note that 1 is the average of −1 and 3. The line through C and D is parallel to the x-axis.

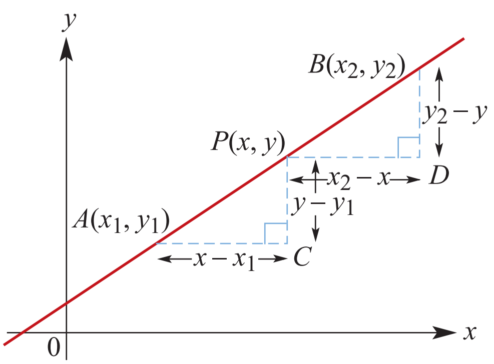


A line segment not parallel to one of the axes

Let $P(x, y)$ be the midpoint of the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$, where the line through A and B is not parallel to either axis.

Let points C and D be chosen so that AC and PD are parallel to the x-axis, and PC and BD are parallel to the y-axis.

The triangles APC and PBD are congruent (AAS). Hence



$$AC = PD \qquad \text{and} \qquad PC = BD$$

$$\therefore \quad x - x_1 = x_2 - x \qquad y - y_1 = y_2 - y$$

$$2x = x_1 + x_2 \qquad 2y = y_1 + y_2$$

$$x = \frac{x_1 + x_2}{2} \qquad y = \frac{y_1 + y_2}{2}$$

We have proved the following result.

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The coordinates of the midpoint P of the line segment AB joining $A(x_1, y_1)$ and $B(x_2, y_2)$ are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

That is, we take the average of the x-coordinates and the average of the y-coordinates.

Example 1

Find the midpoint of the line segment joining $A(2, 6)$ with $B(-3, -4)$.

Solution

The midpoint of line segment AB has coordinates

$$\left(\frac{2 + (-3)}{2}, \frac{6 + (-4)}{2}\right) = \left(-\frac{1}{2}, 1\right)$$

Explanation

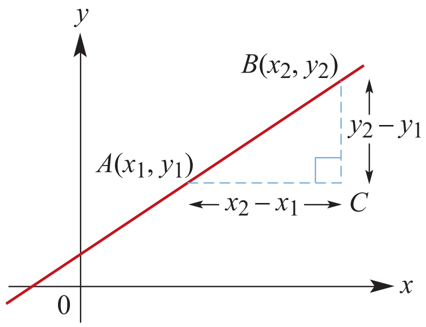
The coordinates of the midpoint of the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

The distance between two points

The **distance between given points** $A(x_1, y_1)$ and $B(x_2, y_2)$ can be found by applying Pythagoras' theorem to the triangle ABC:

$$\begin{aligned} AB^2 &= AC^2 + BC^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \end{aligned}$$



Therefore, the distance between the two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

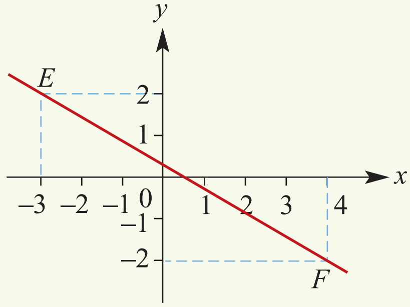
Example 2

Calculate the distance EF if E is $(-3, 2)$ and F is $(4, -2)$.

Solution

$$\begin{aligned} EF &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(4 - (-3))^2 + (-2 - 2)^2} \\ &= \sqrt{7^2 + (-4)^2} \\ &= \sqrt{65} \\ &= 8.06 \quad \text{(to two decimal places)} \end{aligned}$$

Explanation



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Summary 2A

- The coordinates of the midpoint of the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

- The distance between points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Skill-sheet

Exercise 2A

Example 1

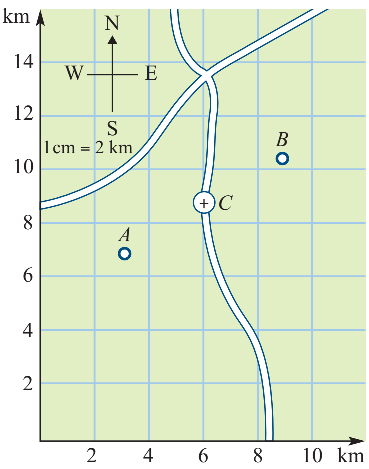
1 Find the coordinates of M, the midpoint of AB, where A and B have the following coordinates:

a $A(2, 12)$, $B(8, 4)$ **b** $A(-3, 5)$, $B(4, -4)$

c $A(-1.6, 3.4)$, $B(4.8, -2)$ **d** $A(3.6, -2.8)$, $B(-5, 4.5)$

2 Find the midpoints of each of the sides of a triangle ABC, where A is $(1, 1)$, B is $(5, 5)$ and C is $(11, 2)$.

3 The secretary of a motocross club wants to organise two meetings on the same weekend. One is a hill climb starting from point $A(3.1, 7.1)$ and the other is a circuit event with the start at $B(8.9, 10.5)$, as shown on the map. Only one ambulance can be provided. The ambulance can be called up by radio, so it is decided to keep it at C, halfway between A and B.
What are the coordinates of C?



4 If M is the midpoint of XY, find the coordinates of Y when X and M have the following coordinates:

a $X(-4, 2)$, $M(0, 3)$ **b** $X(-1, -3)$, $M(0.5, -1.6)$

c $X(6, -3)$, $M(2, 1)$ **d** $X(4, -3)$, $M(0, -3)$

5 **a** Find the coordinates of the midpoint of the line segment joining $(1, 4)$ and (a, b), in terms of a and b.

b If $(5, -1)$ is the midpoint, find the values of a and b.

Example 2

6 Find the distance between each of the following (correct to two decimal places):

a $(3, 6)$ and $(-4, 5)$ **b** $(4, 1)$ and $(5, -3)$

c $(-2, -3)$ and $(-5, -8)$ **d** $(6, 4)$ and $(-7, 4)$

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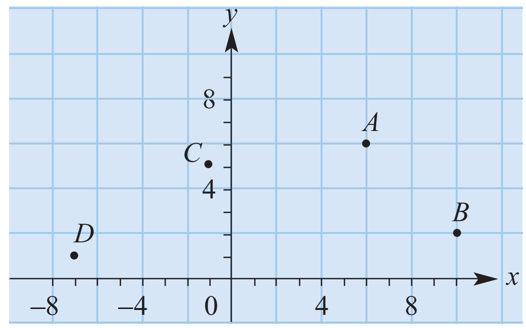
7 Calculate the perimeter of a triangle with vertices $(-3, -4)$, $(1, 5)$ and $(7, -2)$.

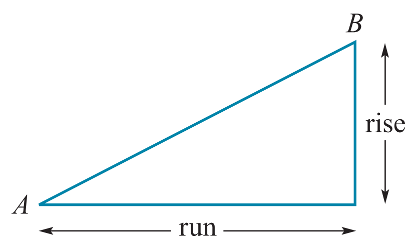
8 The diagram shows the four points $A(6, 6)$, $B(10, 2)$, $C(-1, 5)$ and $D(-7, 1)$. If the midpoint of AB is P and the midpoint of CD is M, calculate the distance PM.



9 There is an off-shore oil drilling platform in Bass Strait situated at $D(0, 6)$, where 1 unit = 5 km. Pipes for this oil drill come ashore at $M(-6, 1)$ and $N(3, -1)$. Assuming the pipelines are straight, which is the shorter DM or DN?

2B The gradient of a straight line

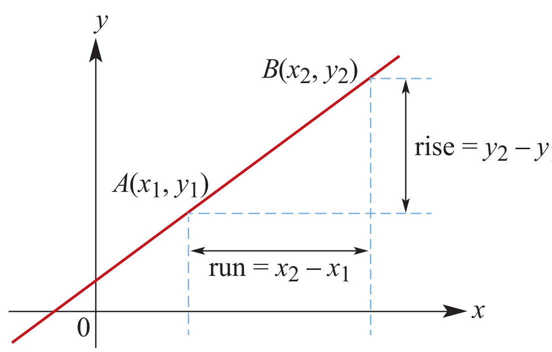
Through any two points it is only possible to draw a single straight line. Therefore a straight line is defined by any two points on the line.



In coordinate geometry the standard way to define the **gradient of a line segment** AB is $\frac{\text{rise}}{\text{run}}$ where:

- rise is the change in the y-values as you move from A to B
- run is the change in the x-values as you move from A to B.

The **gradient of a line** is defined to be the gradient of any segment within the line. This definition depends on the fact that any two segments of a line have the same gradient. Hence given any two points on the line, $A(x_1, y_1)$ and $B(x_2, y_2)$, the gradient of the line can be found. The symbol used for gradient is m.



$$\text{Gradient } m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

Note that since

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

it does not matter which point we take as the first and which point we take as the second.

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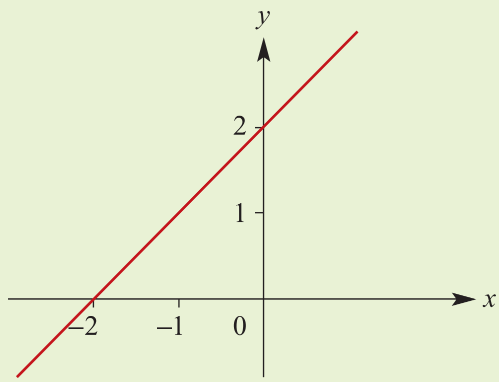
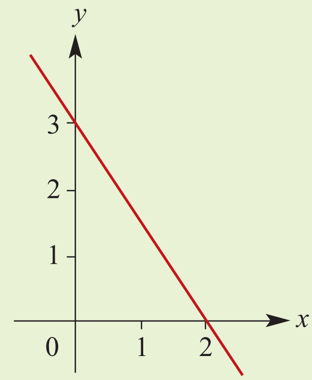
Example 3

Find the gradient of each line:

a



b



Solution

a Gradient $m = \dfrac{2-0}{0-(-2)}$

$= \dfrac{2}{2}$

$= 1$

b Gradient $m = \dfrac{0-3}{2-0}$

$= -\dfrac{3}{2}$

Explanation

Let $(x_1, y_1) = (-2, 0)$ and $(x_2, y_2) = (0, 2)$.

Gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Let $(x_1, y_1) = (0, 3)$ and $(x_2, y_2) = (2, 0)$.

Gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Notes:

- The gradient of a line that slopes upwards from left to right is **positive**, as illustrated in Example 3a.
- The gradient of a line that slopes downwards from left to right is **negative**, as illustrated in Example 3b.
- The gradient of a **horizontal line** (parallel to the x-axis) is zero, since $y_2 - y_1 = 0$.
- The gradient of a **vertical line** (parallel to the y-axis) is undefined, since $x_2 - x_1 = 0$.

Example 4

Find the gradient of the line that passes through the points $(1, 6)$ and $(-3, 7)$.

Solution

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{7-6}{-3-1}$$

$$= -\frac{1}{4}$$

Explanation

The gradient can also be found using

$$m = \frac{y_1 - y_2}{x_1 - x_2}$$

$$= \frac{6-7}{1-(-3)} = -\frac{1}{4}$$

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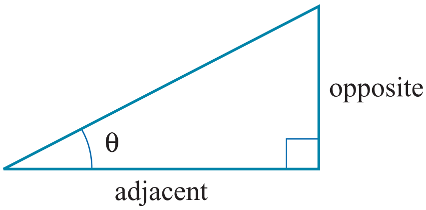
The tangent of the angle of slope

We will look first at the case when the gradient is positive and then when the gradient is negative.

Positive gradient

From Year 10 you will be familiar with the trigonometric ratio

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$



Consider a straight line with positive gradient. The line forms an acute angle, θ, with the positive direction of the x-axis.

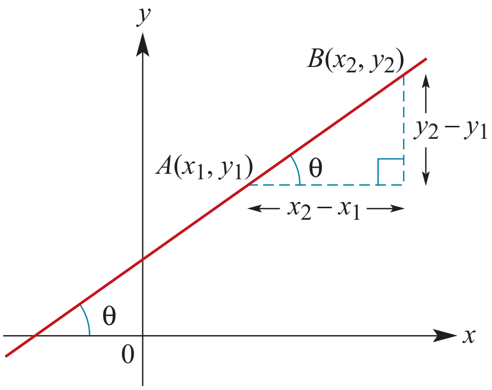
The gradient, m, of the line is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad (x_1 \neq x_2)$$

From the diagram, it follows that

$$m = \tan\theta$$

where θ is the angle that the line makes with the positive direction of the x-axis.



Example 5

Determine the gradient of the line passing through the points (3, 2) and (5, 7) and the angle θ that the line makes with the positive direction of the x-axis.

Solution

$$m = \frac{7-2}{5-3} = \frac{5}{2}$$

$$\tan\theta = \frac{5}{2}$$

$$\therefore \quad \theta = 68.1986\ldots^\circ = 68.20^\circ$$

correct to two decimal places.

Explanation

The gradient is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The gradient is positive and so the angle θ is acute.

The angle can be found with a calculator using inverse tan.

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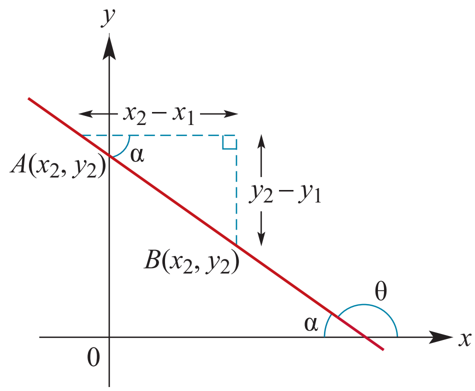
Negative gradient

Now consider a line with negative gradient. The line forms an acute angle α with the negative direction of the x-axis, and an obtuse angle θ with the positive direction of the x-axis. We have $\theta = 180° - \alpha$.



From the diagram, we see that the gradient satisfies

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-(y_1 - y_2)}{x_2 - x_1} = -\tan\alpha$$

From your work on trigonometric functions in Year 10 you may recall that

$$\tan\theta = \tan(180° - \alpha) = -\tan\alpha$$

Thus the gradient satisfies

$$m = -\tan\alpha = \tan\theta$$

Example 6

Determine the gradient of the line passing through the points $(5, -3)$ and $(-1, 5)$ and the angle θ that the line makes with the positive direction of the x-axis.

Solution

$$m = \frac{5 - (-3)}{-1 - 5}$$

$$= -\frac{4}{3}$$

$$\tan\theta = -\frac{4}{3}$$

$$\therefore \quad \theta = 180° - (53.130\ldots°)$$

$$= 126.87°$$

correct to two decimal places.

Explanation

The gradient is negative and so the angle θ between the line and the positive direction of the x-axis is obtuse.

You first use your calculator to find the adjacent supplementary angle α, which is acute. You do this by finding the inverse tangent of $\frac{4}{3}$. The magnitude of this angle is $53.130\ldots°$.

You subtract this from 180° to obtain θ.

Summary 2B

- The gradient of a line segment AB joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$\frac{y_2 - y_1}{x_2 - x_1} \qquad \text{provided } x_1 \neq x_2$$

- The gradient of a line is defined as the gradient of any line segment of that line.
- A horizontal line has gradient zero.
- A vertical line does not have a gradient.

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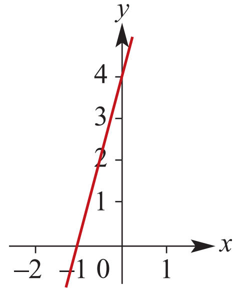
- Let θ be the angle that a line makes with the positive direction of the x-axis.
  - The gradient m is positive if and only if θ is acute.
  - The gradient m is negative if and only if θ is obtuse.
  - If θ is acute or obtuse, then $m = \tan\theta$.
  - If $\theta = 0°$, then $m = \tan 0° = 0$.
  - If $\theta = 90°$, then the gradient is not defined.

Exercise 2B

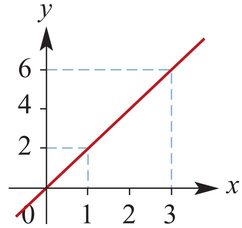
Example 3

1 Calculate the gradient of each of the following lines:

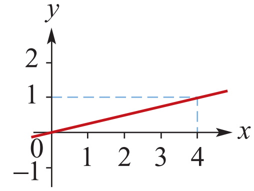
a



b



c



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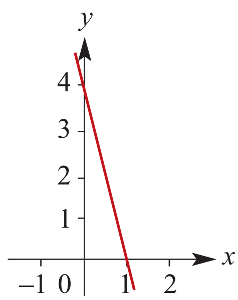

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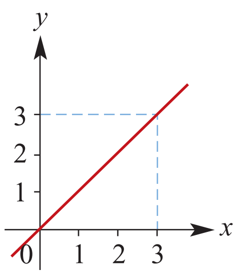

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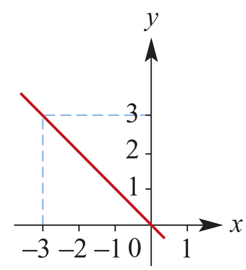

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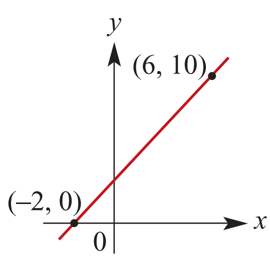

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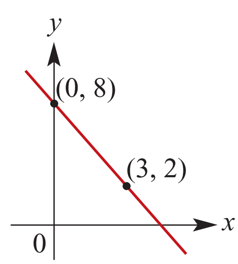

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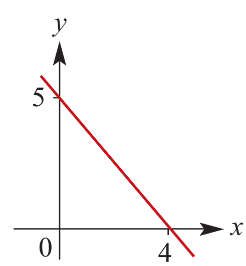

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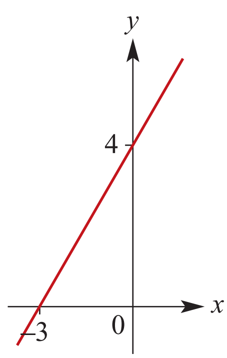

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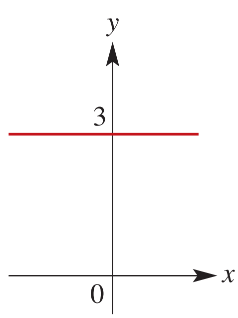

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2 Sketch a graph of a line with gradient 1.

3 Sketch a graph of a line with gradient 0 which passes through the point $(1, 6)$.

Example 4

4 For each of the following, find the gradient of the line that passes through the two points with the given coordinates:

a $(6, 3),\ (2, 4)$
b $(-3, 4),\ (1, -6)$
c $(6, 7),\ (11, -3)$
d $(5, 8),\ (6, 0)$
e $(6, 0),\ (-6, 0)$
f $(0, -6),\ (-6, 0)$
g $(3, 9),\ (4, 16)$
h $(5, 25),\ (6, 36)$
i $(-5, 25),\ (-8, 64)$
j $(1, 1),\ (10, 100)$
k $(1, 1),\ (10, 1000)$
l $(5, 125),\ (4, 64)$

5 **a** Find the gradient of the straight line that passes through the points with coordinates $(5a, 2a)$ and $(3a, 6a)$.
b Find the gradient of the straight line that passes through the points with coordinates $(5a, 2a)$ and $(5b, 2b)$.

6 **a** A line has gradient 6 and passes through the points with coordinates $(-1, 6)$ and $(7, a)$. Find the value of a.
b A line has gradient -6 and passes through the points with coordinates $(1, 6)$ and $(b, 7)$. Find the value of b.

7 Find the angle, correct to two decimal places, that the lines joining the given points make with the positive direction of the x-axis:

a $(0, 3),\ (-3, 0)$
b $(0, -4),\ (4, 0)$
c $(0, 2),\ (-4, 0)$
d $(0, -5),\ (-5, 0)$

Example 5
Example 6

8 Find the angle, correct to two decimal places, that the lines joining the given points make with the positive direction of the x-axis:

a $(-4, -2),\ (6, 8)$
b $(2, 6),\ (-2, 4)$
c $(-3, 4),\ (6, 1)$
d $(-4, -3),\ (2, 4)$
e $(3b, a),\ (3a, b)$
f $(c, b),\ (b, c)$

9 Find the gradient of a straight line which is:

a inclined at an angle of $45°$ to the positive direction of the x-axis
b inclined at an angle of $135°$ to the positive direction of the x-axis
c inclined at an angle of $60°$ to the positive direction of the x-axis
d inclined at an angle of $120°$ to the positive direction of the x-axis.

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2C The equation of a straight line

In this section we discuss different ways of determining the equation of a straight line. In general two 'independent pieces of information' are required. The following given information is considered:

- gradient and y-axis intercept
- gradient and a point
- two points.

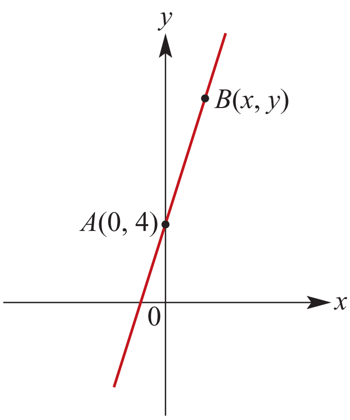
Sketching straight lines given the equation is discussed in Section 2D.

Gradient–intercept form of the equation of a straight line

We first consider an example before stating the general result. The argument in the general case is exactly the same.

The line $y = 2x + 4$

Consider the line with gradient 2 and y-axis intercept 4. This line passes through the point $A(0, 4)$. Let $B(x, y)$ be any other point on the line.



Gradient of line segment $AB = \dfrac{y-4}{x-0} = \dfrac{y-4}{x}$

We know that the gradient of the line is 2. Therefore

$$\frac{y-4}{x} = 2$$

$$y - 4 = 2x$$

$$y = 2x + 4$$

So the coordinates (x, y) satisfy the equation $y = 2x + 4$.

Conversely, if a point $B(x, y)$ in the plane satisfies $y = 2x + 4$, then

$$\frac{y-4}{x} = 2$$

Thus we know that the gradient of the line segment joining point B to the point $A(0, 4)$ is 2. Therefore the line through $A(0, 4)$ and $B(x, y)$ has gradient 2 and y-axis intercept 4.

The line $y = mx + c$

In the same way as for the line $y = 2x + 4$, we can show that:

- The line with gradient m and y-axis intercept c has equation $y = mx + c$.
- Conversely, the line with equation $y = mx + c$ has gradient m and y-axis intercept c.

This is called the **gradient–intercept form**.

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Example 7

Find the gradient and y-axis intercept of the line $y = 3x - 4$.

Solution

The gradient is 3 and the y-axis intercept is -4.

Explanation

$m = 3$ and $c = -4$

Example 8

Find the equation of the line with gradient -3 and y-axis intercept 5.

Solution

$y = -3x + 5$

Explanation

$y = mx + c$

Example 9

State the gradient and y-axis intercept of the line $3y + 6x = 9$.

Solution

$$3y + 6x = 9$$
$$3y = 9 - 6x$$
$$y = \frac{9 - 6x}{3}$$
$$y = 3 - 2x$$
$$\text{i.e.} \quad y = -2x + 3$$

Therefore $m = -2$ and $c = 3$.

Explanation

Rearrange the equation $3y + 6x = 9$ into gradient–intercept form.

Now the gradient and y-axis intercept can be read directly from the equation.

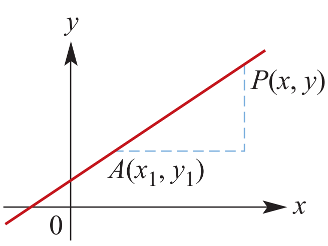
Point–gradient form of the equation of a straight line

If $A(x_1, y_1)$ is a point on a line with gradient m and $P(x, y)$ is any other point on the line, then

$$\frac{y - y_1}{x - x_1} = m$$

and so we have

$$y - y_1 = m(x - x_1)$$



The **point–gradient form** of the equation of a straight line is

$$y - y_1 = m(x - x_1)$$

where (x_1, y_1) is a point on the line and m is the gradient.

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Example 10

Find the equation of the line which passes through the point $(-1, 3)$ and has gradient 4.

Solution

Method 1

$(x_1, y_1) = (-1, 3)$ and $m = 4$.

The equation is

$$y - 3 = 4(x - (-1))$$
$$y = 4(x + 1) + 3$$
$$= 4x + 4 + 3$$
$$= 4x + 7$$

Explanation

We use the equation $y - y_1 = m(x - x_1)$. (It is not necessary to work from first principles every time.)

Rearrange to make y the subject and write the equation in the form $y = mx + c$.

Method 2

Since $m = 4$, the equation is of the form $y = 4x + c$.

When $x = -1$, $y = 3$.

Therefore

$$3 = 4 \times (-1) + c$$
$$7 = c$$

The equation is $y = 4x + 7$.

Explanation

We can also use the equation $y = mx + c$ and find the value of c. The gradient is 4.

The point $(-1, 3)$ lies on the line.

Solve for c.

Example 11

Find the equation of the line that passes through the point $(3, 2)$ and has a gradient of -2.

Solution

$$y - 2 = -2(x - 3)$$
$$y - 2 = -2x + 6$$
$$y = -2x + 8$$

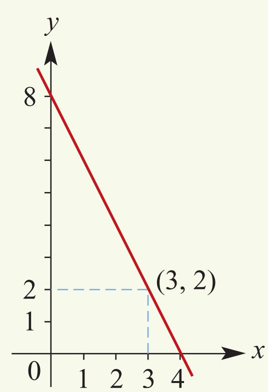
The equation is

$$y = -2x + 8$$

which could also be expressed as

$$2x + y - 8 = 0$$

Explanation



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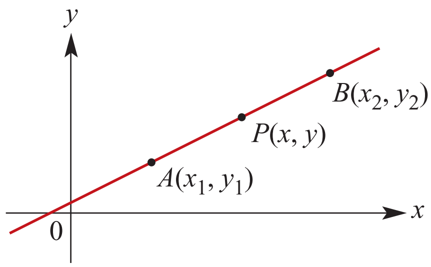
A line through two points

To find the equation of the line through two given points (x_1, y_1) and (x_2, y_2), first find the gradient

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

and then use the point–gradient form

$$y - y_1 = m(x - x_1)$$



We can also find the equation directly by taking the point $P(x, y)$ and noting that

$$\frac{y - y_1}{x - x_1} = m$$

Example 12

Find the equation of the straight line passing through the points $(1, -2)$ and $(3, 2)$.

Solution

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - (-2)}{3 - 1} = \frac{4}{2} = 2$$

$$\therefore \quad 2 = \frac{y - (-2)}{x - 1}$$

$$2x - 2 = y + 2$$

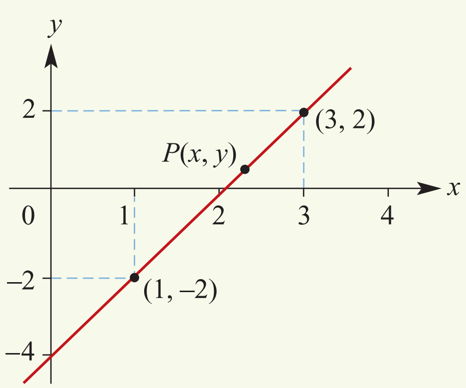
$$\therefore \quad y = 2x - 4$$

Explanation

First find the gradient m and then use

$$\frac{y - y_1}{x - x_1} = m$$

Choose $(x_1, y_1) = (1, -2)$.



Example 13

Find the equation of the straight line with y-axis intercept -3 which passes through the point with coordinates $(1, 10)$.

Solution

The gradient is

$$m = \frac{10 - (-3)}{1 - 0} = 13$$

Therefore the equation is $y = 13x - 3$.

Explanation

Find the gradient using $(x_1, y_1) = (0, -3)$ and $(x_2, y_2) = (1, 10)$.

The general equation of a line with y-axis intercept -3 is $y = mx - 3$.

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Two intercepts

A special case of finding the equation of a line given the coordinates of two points is when the intercept with each axis is known, that is, finding the line through $(a, 0)$ and $(0, b)$, where $a, b \neq 0$.

In this case, the gradient is

$$m = \frac{b - 0}{0 - a} = -\frac{b}{a}$$

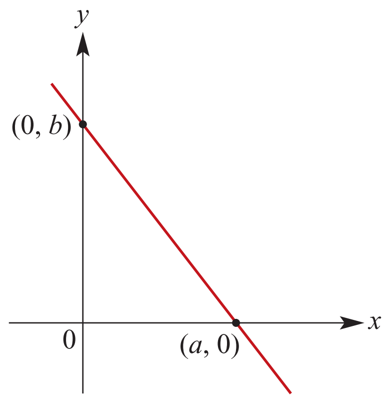
Thus the equation of the line is

$$y - 0 = -\frac{b}{a}(x - a)$$

Multiplying both sides of the equation by a gives

$$ay + bx = ab$$

Dividing both sides of the equation by ab gives the following:



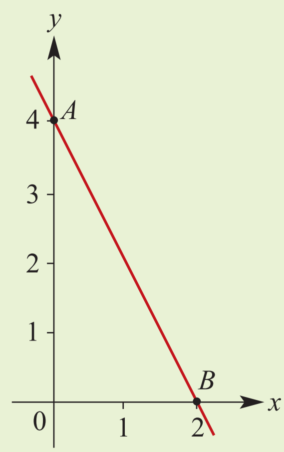
The **intercept form** of the equation of a straight line is

$$\frac{x}{a} + \frac{y}{b} = 1$$

where a and b are the x-axis intercept and y-axis intercept respectively.

Example 14

Find the equation of the line shown in the graph.



Solution

The intercept form of the equation is

$$\frac{x}{2} + \frac{y}{4} = 1$$

Multiply both sides by 4:

$$2x + y = 4$$

The equation of the line is $y = -2x + 4$.

Explanation

The coordinates of A and B are $(0, 4)$ and $(2, 0)$.

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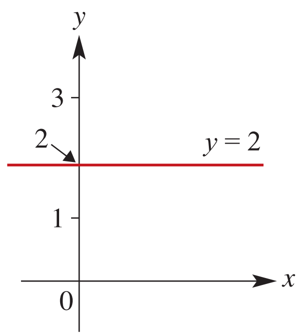
Vertical and horizontal lines

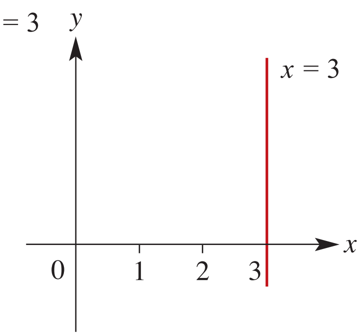
If a line is **horizontal**, then its gradient $m = 0$ and its equation is simply $\boldsymbol{y = c}$, where c is the y-axis intercept.

If a line is **vertical**, then its gradient is undefined and its equation is $\boldsymbol{x = a}$, where a is the x-axis intercept.

Equation $y = 2$



Equation $x = 3$



Note that the equation of a vertical line is not of the form $y = mx + c$.

General form of the equation of a straight line

We have seen that all points on the line through two given points satisfy an equation of the form $mx + ny + p = 0$, with m and n not both 0. Conversely, any 'linear equation' $mx + ny + p = 0$ is the equation of a (straight) line. This is called the **general form** of the equation of a line.

Summary 2C

- Gradient–intercept form: The line with gradient m and y-axis intercept c has equation $y = mx + c$. Conversely, the line with equation $y = mx + c$ has gradient m and y-axis intercept c.
- Point–gradient form: If (x_1, y_1) is a point on a line with gradient m and (x, y) is any other point on the line, then

$$\frac{y - y_1}{x - x_1} = m$$

which can be written as

$$y - y_1 = m(x - x_1)$$

- Two points: To find the equation of the line through two given points (x_1, y_1) and (x_2, y_2), first find the gradient

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

and then use the point–gradient form

$$y - y_1 = m(x - x_1)$$

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- **Intercept form:** If a line has x-axis intercept a and y-axis intercept b, the equation of the line is
$$\frac{x}{a} + \frac{y}{b} = 1$$
- **Horizontal line:** A line parallel to the x-axis through the point (a, c) has equation $y = c$.
- **Vertical line:** A line parallel to the y-axis through the point (a, c) has equation $x = a$.
- **General form:** Every straight line satisfies an equation of the form $mx + ny + p = 0$, with m and n not both 0. Conversely, any 'linear equation' $mx + ny + p = 0$ is the equation of a straight line.

Skill-sheet

Exercise 2C

Example 7

1 State the gradient and y-axis intercept of the graph of each equation:

a $y = 3x + 6$ **b** $y = -6x + 7$ **c** $y = 3x - 6$ **d** $y = -x - 4$

Example 8

2 **a** Find the equation of the straight line with gradient 3 and y-axis intercept 5.

b Find the equation of the straight line with gradient -4 and y-axis intercept 6.

c Find the equation of the straight line with gradient 3 and y-axis intercept -4.

Example 9

3 State the gradient and y-axis intercept of the graph of each equation:

a $3x - y = 6$ **b** $4x - 2y = 8$ **c** $5x - 10y = 20$ **d** $2x - 6y = 10$

4 Express in gradient–intercept form and hence state the gradient and y-axis intercept of each of the following linear relations:

a $2x - y = 9$ **b** $3x + 4y = 10$ **c** $-x - 3y = 6$ **d** $5x - 2y = 4$

Example 10

5 **a** Find the equation of the straight line that has gradient 3 and passes through the point with coordinates $(6, 7)$.

Example 11

b Find the equation of the straight line that has gradient -2 and passes through the point with coordinates $(1, 7)$.

Example 12

6 Find the equations of the straight lines passing through the following pairs of points. (Express your answer in gradient–intercept form.)

a $(-1, 4)$, $(2, 3)$ **b** $(0, 4)$, $(5, -3)$ **c** $(3, -2)$, $(4, -4)$ **d** $(5, -2)$, $(8, 9)$

Example 13

7 For the straight line that has y-axis intercept 6 and passes through the point with coordinates $(1, 8)$, find:

a the gradient **b** the equation.

8 Find the equation of the straight line that passes through the point $(1, 6)$ and has gradient:

a 2 **b** -2

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9 Write, in the form $y = mx + c$, the equations of the lines which have the given gradient and pass through the given point:

a $m = 2;\quad (-1, 4)$ **b** $m = -2;\quad (0, 4)$ **c** $m = -5;\quad (3, 0)$

Example 14 **10** Find equations defining the lines which pass through the following pairs of points:

a $(0, 4),\ (6, 0)$ **b** $(-3, 0),\ (0, -6)$

c $(0, 4),\ (4, 0)$ **d** $(2, 0),\ (0, 3)$

11 Find the equations, in the form $y = mx + c$, of the lines which pass through the following pairs of points:

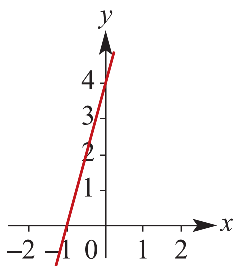
a $(0, 4),\ (3, 6)$ **b** $(1, 0),\ (4, 2)$

c $(-3, 0),\ (3, 3)$ **d** $(-2, 3),\ (4, 0)$

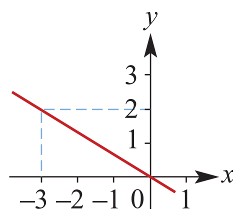
e $(-1.5, 2),\ (4.5, 8)$ **f** $(-3, 1.75),\ (4.5, -2)$

12 Find the equation of each of the following lines:

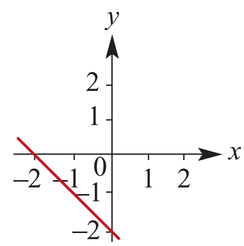
a



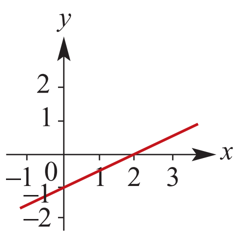
b



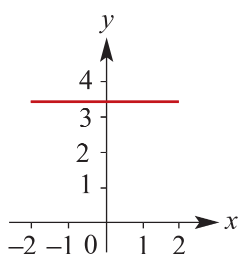
c



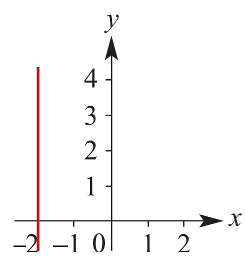
d



e



f



13 Do the points $P(1, -3)$, $Q(2, 1)$ and $R\left(2\frac{1}{2}, 3\right)$ lie on the same straight line?

14 For which of the following does the line pass through the origin?

A $y + x = 1$ **B** $y + 2x = 2(x + 1)$ **C** $x + y = 0$ **D** $x - y = 1$

15 **a** Find the equation of the line that is parallel to the y-axis and passes through the point with coordinates $(4, 7)$.

b Find the equation of the line that is parallel to the x-axis and passes through the point with coordinates $(-4, 11)$.

c Find the equation of the line that is parallel to the y-axis and passes through the point with coordinates $(11, -7)$.

d Find the equation of the line that is parallel to the x-axis and passes through the point with coordinates $(5, -1)$.

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2D Graphing straight lines

In the previous section we discussed methods of finding the equation of a straight line given suitable information. In this section we look at sketching a straight line from an equation. To sketch the graph we need to derive the coordinates of two points on the line. A convenient way to sketch graphs of straight lines is to plot the two axis intercepts.

Example 15

Sketch the graph of $2x + 4y = 10$.

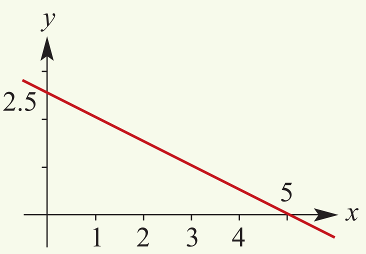
Solution

x-axis intercept ($y = 0$): $2x + 4(0) = 10$

$x = 5$

y-axis intercept ($x = 0$): $2(0) + 4y = 10$

$y = 2.5$



Example 16

Sketch the graph of $y = 2x - 6$ by first finding the intercepts.

Solution

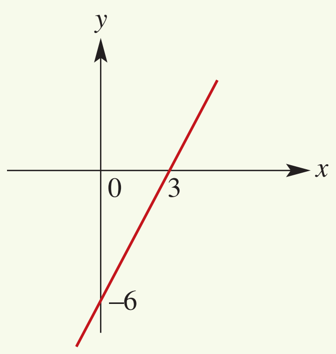
x-axis intercept ($y = 0$): $0 = 2x - 6$

$x = 3$

y-axis intercept ($x = 0$): $y = 2(0) - 6$

$y = -6$

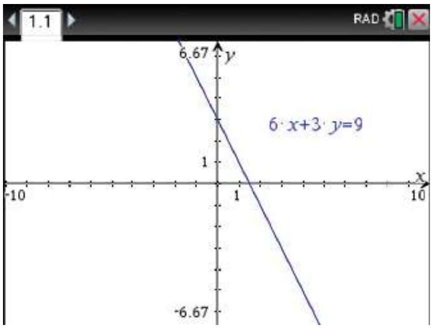
Note: You can also obtain the y-axis intercept directly from the equation.



Using the TI-Nspire

To sketch the graph of $6x + 3y = 9$:

- Open a **Graphs** application: press on and select the **Graphs** icon, or use ctrl I and select **Add Graphs**.
- Equations of the form $ax + by = c$ can be entered directly using menu > **Graph Entry/Edit** > **Equation** > **Line**. Enter as $6x + 3y = 9$.



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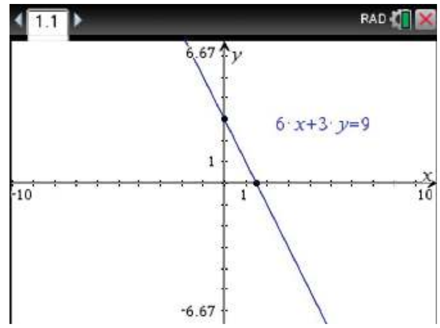
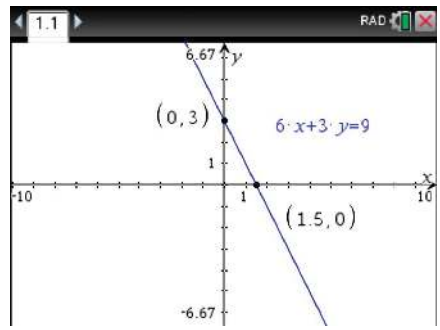
Note: The window settings (menu > **Window/Zoom** > **Window Settings**) will have to be changed if the axis intercepts do not appear on the screen.

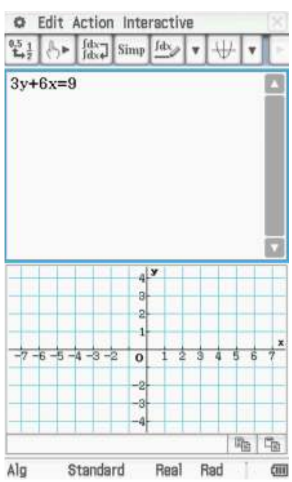


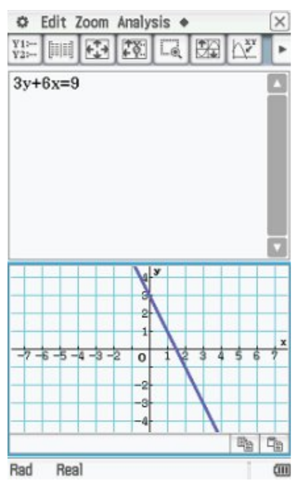
- The axis intercepts can be found using menu > **Geometry** > **Points & Lines** > **Intersection Point(s)**. Select the x-axis and the graph to display the x-axis intercept. Select the y-axis and the graph to display the y-axis intercept.
- To show the coordinates of these points, use menu > **Actions** > **Coordinates and Equations** and double click on each of the points.
- Press esc to exit the **Coordinates and Equations** tool.



Using the Casio ClassPad

- Type the equation $3y + 6x = 9$ in the main screen Main $\sqrt{\alpha}$. Tap on the graph icon to display the graph screen.
- Using your stylus, highlight the equation and drag it down into the graph screen. Lift the stylus off the screen for the graph to appear.





- The graph window setting can be adjusted using the window setting icon .
- Ensure the graph window is selected and the intercepts are visible on the graph.

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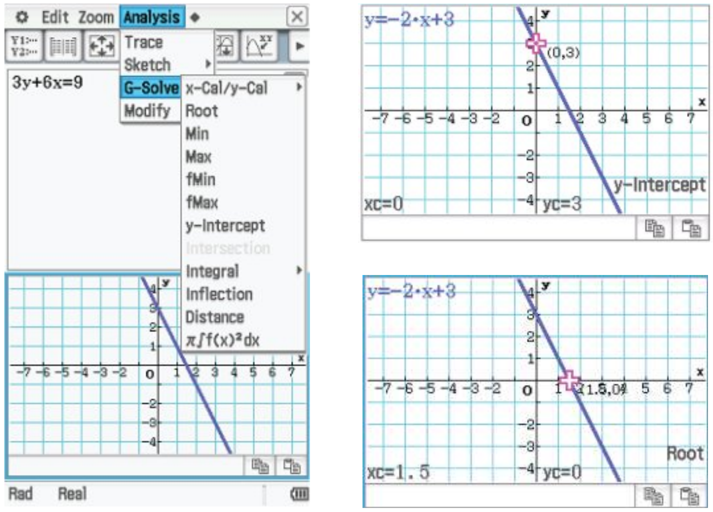
- To find the intercepts, go to **Analysis** > **G-Solve**. Select **y-Intercept** for the y-axis intercept and select **Root** for the x-axis intercept.



Note: The equation is displayed in gradient–intercept form in the top-left corner of the graph screen.

From Section 2B we know that the gradient of a line is the tangent of the angle of slope (that is, the angle formed by the line with the positive direction of the x-axis).

Example 17

For each of the following lines, find the magnitude of the angle θ (correct to two decimal places) that the line makes with the positive direction of the x-axis:

a $y = 2x + 3$

b $3y = 3x - 6$

c $y = -0.3x + 1.5$

Solution

a
$$y = 2x + 3$$
Gradient $= 2$

Hence $\tan\theta = 2$

Therefore $\theta = 63.43°$

correct to two decimal places

b
$$3y = 3x - 6$$
$$y = x - 2$$
Gradient $= 1$

Hence $\tan\theta = 1$

Therefore $\theta = 45°$

c
$$y = -0.3x + 1.5$$
Gradient $= -0.3$

Hence $\tan\theta = -0.3$

Therefore $\theta = (180 - 16.699\ldots)°$

$= 163.30°$ correct to two decimal places

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Summary 2D

- The most practical way to sketch a straight line is to plot two points known to be on the required line and draw the line through them.
  - Two important points are the intercept with the x-axis and the intercept with the y-axis. These are the best two points to use in order to sketch a line that does not pass through the origin and is not parallel to one of the axes.
  - To sketch a line with equation of the form $y = mx$, plot one other point on the line besides the origin.
- The gradient m of a line $y = mx + c$ is equal to $\tan\theta$, where θ is the angle of slope measured between the line and the positive direction of the x-axis.

Exercise 2D

1 For each of the following, give the coordinates of the axis intercepts:

a $x + y = 4$ **b** $x - y = 4$ **c** $-x - y = 6$ **d** $y - x = 8$

Example 15

2 Sketch the graphs of each of the following linear relations:

a $2x - 3y = 12$ **b** $x - 4y = 8$ **c** $-3x + 4y = 24$
d $-5x + 2y = 20$ **e** $4x - 3y = 15$ **f** $7x - 2y = 15$

Example 16

3 For each of the following, sketch the graph by first finding the axis intercepts:

a $y = x - 1$ **b** $y = x + 2$ **c** $y = 2x - 4$

4 Sketch the graphs of each of the following by first determining the axis intercepts:

a $y = 2x - 10$ **b** $y = 3x - 9$ **c** $y = 5x + 10$ **d** $y = -2x + 10$

5 Sketch the graphs of each of the following:

a $y = x + 2$ **b** $y = -x + 2$ **c** $y = 2x + 1$ **d** $y = -2x + 1$

6 Sketch the graphs of each of the following:

a $x + y = 1$ **b** $x - y = 1$ **c** $y - x = 1$ **d** $-x - y = 1$

7 Sketch the graphs of each of the following:

a $y = x + 3$ **b** $y = 3x + 1$ **c** $y = 4 - \frac{1}{2}x$ **d** $y = 3x - 2$
e $4y + 2x = 12$ **f** $3x + 6y = 12$ **g** $4y - 6x = 24$ **h** $8x - 3y = 24$

8 Sketch the graphs of each of the following:

a $y = 3$ **b** $x = -2$ **c** $y = -2$ **d** $x = 5$

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Example 17

9 Find the magnitude of the angle, correct to two decimal places where necessary, made by each of the following with the positive direction of the x-axis:

a $y = x$ **b** $y = -x$ **c** $y = x + 1$
d $x + y = 1$ **e** $y = 2x$ **f** $y = -2x$

10 Find the magnitude of the angle, correct to two decimal places, made by each of the following with the positive direction of the x-axis:

a $y = 3x + 2$ **b** $2y = -2x + 1$ **c** $2y - 2x = 6$ **d** $3y + x = 7$

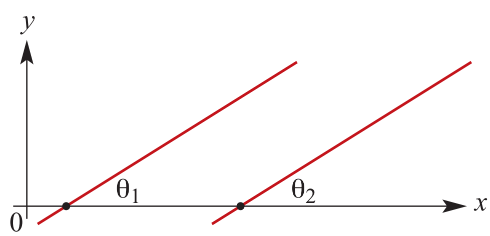
11 A straight line has equation $y = 3x - 4$. The points with coordinates $(0, a)$, $(b, 0)$, $(1, d)$ and $(e, 10)$ lie on the line. Find the values of a, b, d and e.

2E Parallel and perpendicular lines

Parallel lines

- Two non-vertical lines are **parallel** if they have the same gradient.
- Conversely, if two non-vertical lines are parallel, then they have the same gradient.

This is easily proved through considering the angles of inclination of such lines to the positive direction of the x-axis and using the following two results:



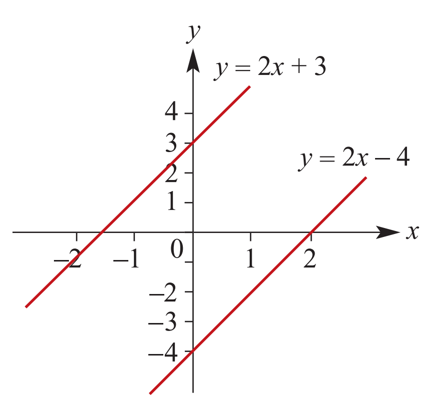
1 Two non-vertical lines are parallel if and only if the corresponding angles θ_1 and θ_2 formed by the lines and the positive direction of the x-axis are equal.

2 If θ_1 and θ_2 are two angles which are acute, obtuse or zero, then $\tan\theta_1 = \tan\theta_2$ implies $\theta_1 = \theta_2$.

For example, consider the lines

$$y = 2x + 3$$
$$y = 2x - 4$$

Both lines have gradient 2, and so they are parallel.



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Perpendicular lines

We prove that two lines are **perpendicular** if and only if the product of their gradients is -1 (or if one is horizontal and the other vertical).

Two lines with gradients m_1 and m_2 (both non-zero) are perpendicular if and only if $m_1 m_2 = -1$.

Proof Initially we consider the case where the two lines intersect at the origin.

Step 1 Draw two lines passing through the origin with one of the lines having positive gradient, m_1, and the other negative gradient, m_2. Form right-angled triangles OPQ and OAB with $OQ = OB$.

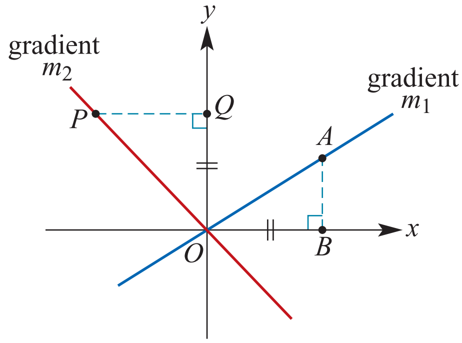
$$\text{Gradient } m_1 = \frac{AB}{BO}$$

$$\text{Gradient } m_2 = -\frac{OQ}{PQ}$$

$$\text{Product } m_1 m_2 = -\frac{OQ}{PQ} \times \frac{AB}{BO}$$

$$= -\frac{OQ}{PQ} \times \frac{AB}{OQ}$$

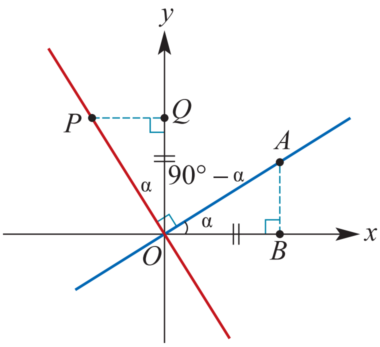
$$= -\frac{AB}{PQ}$$



Step 2 We now prove: *If two lines passing through the origin are perpendicular, then the product of their gradients is* -1.

If the lines are perpendicular, then $\angle POQ = \angle AOB$. Therefore triangles OPQ and OAB are congruent. So $PQ = AB$ and therefore the product of the gradients is

$$m_1 m_2 = -\frac{AB}{PQ} = -\frac{AB}{AB} = -1$$



Step 3 We next prove the converse: *If for two lines passing through the origin the product of their gradients is* -1, *then the lines are perpendicular.*

If the product $m_1 m_2 = -1$, then $AB = PQ$, which implies that the triangles OAB and OPQ are congruent. Therefore $\angle POQ = \angle AOB$ and so $\angle AOP = 90°$.

Step 4 If we are given two lines anywhere in the plane, we can draw lines through the origin parallel to the original two lines. The slopes of the new lines are the same. So the result holds for lines that do not necessarily pass through the origin.

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Example 18

Find the equation of the straight line which passes through $(1, 2)$ and is:

a parallel to the line with equation $2x - y = 4$

b perpendicular to the line with equation $2x - y = 4$.

Solution

The equation $2x - y = 4$ is equivalent to $y = 2x - 4$. The line $y = 2x - 4$ has gradient 2.

a The required line passes through $(1, 2)$ and has gradient 2. Hence

$$y - 2 = 2(x - 1)$$

Therefore the line has equation $y = 2x$.

b The required line passes through $(1, 2)$ and has gradient $-\frac{1}{2}$. Hence

$$y - 2 = -\frac{1}{2}(x - 1)$$

Therefore $2y - 4 = -x + 1$ and equivalently $2y + x = 5$.

The line has equation $2y + x = 5$.

Explanation

A line parallel to $y = 2x - 4$ has gradient 2.

We use $y - y_1 = m(x - x_1)$ where $(x_1, y_1) = (1, 2)$ and $m = 2$.

A line perpendicular to a line with gradient m has gradient $-\frac{1}{m}$ (provided $m \neq 0$).

We use $y - y_1 = m(x - x_1)$ where $(x_1, y_1) = (1, 2)$ and $m = -\frac{1}{2}$.

Example 19

The coordinates of the vertices of a triangle ABC are $A(0, -1)$, $B(2, 3)$ and $C\left(3, -2\frac{1}{2}\right)$.

Show that the side AB is perpendicular to the side AC.

Solution

Let m_1 be the gradient of the line AB and let m_2 be the gradient of the line AC.

$$m_1 = \frac{3 - (-1)}{2 - 0} = 2$$

$$m_2 = \frac{-2\frac{1}{2} - (-1)}{3 - 0} = \frac{-1\frac{1}{2}}{3} = -\frac{1}{2}$$

Since $m_1 \times m_2 = 2 \times \left(-\frac{1}{2}\right) = -1$, the lines AB and AC are perpendicular to each other.

Explanation

We use the fact that two lines with gradients m_1 and m_2 (both non-zero) are perpendicular if and only if $m_1 m_2 = -1$.

We show the product of the gradients is -1. Hence the lines are perpendicular.

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Summary 2E

- Two non-vertical lines are **parallel** if they have the same gradient. Conversely, if two non-vertical lines are parallel, then they have the same gradient.
- Two lines are **perpendicular** if the product of their gradients is -1 (or if one is horizontal and the other vertical). Conversely, if two lines are perpendicular, then the product of their gradients is -1 (or one is horizontal and the other vertical).

Skill-sheet

Exercise 2E

Example 18

1 Find the equation of the straight line which passes through $(4, -2)$ and is:

a parallel to the line with equation $y = 2x + 1$
b perpendicular to the line with equation $y = 2x + 1$
c parallel to the line with equation $y = -2x + 1$
d perpendicular to the line with equation $y = -2x + 1$
e parallel to the line with equation $2x - 3y = 4$
f perpendicular to the line with equation $2x - 3y = 4$
g parallel to the line with equation $x + 3y = 5$
h perpendicular to the line with equation $x + 3y = -4$.

2 For which of the following pairs of equations are the corresponding lines parallel to each other?

a $2y = 6x + 4; \quad y = 3x + 4$
b $x = 4 - y; \quad 2x + 2y = 6$
c $3y - 2x = 12; \quad y + \frac{1}{3} = \frac{2}{3}x$
d $4y - 3x = 4; \quad 3y = 4x - 3$

3 Find the equation of the line:

a perpendicular to the line $x = 3$ and which passes through the point $(3, 4)$
b perpendicular to the line $y = 3$ and which passes through the point $(2, 3)$
c perpendicular to the line $x = -2$ and which passes through the point $(-2, 4)$
d perpendicular to the line $y = -4$ and which passes through the point $(3, -4)$.

4 Find the equation of the straight line which passes through the point $(1, 4)$ and is perpendicular to the line with equation $y = -\frac{1}{2}x + 6$.

5 Points A and B have coordinates $(1, 5)$ and $(-3, 7)$ respectively. Find the coordinates of the midpoint M of the line segment AB and find the equation of the line which passes through the point M and is perpendicular to the line AB.

Example 19

6 If the points A, B and C have the coordinates $A(5, 2)$, $B(2, -3)$ and $C(-8, 3)$, show that the triangle ABC is a right-angled triangle.

7 Given the points $A(3, 7)$, $B(6, 1)$ and $C(20, 8)$, prove that AB is perpendicular to BC.

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8 Show that $RSTU$ is a rectangle if the coordinates of the vertices are respectively $R(2, 6)$, $S(6, 4)$, $T(2, -4)$ and $U(-2, -2)$.

9 Given that the lines $4x - 3y = 10$ and $4x - \ell y = m$ are perpendicular and intersect at the point $(4, 2)$, find the values of ℓ and m.

10 The line $y = 2x + 3$ intersects the y-axis at A. The points B and C on this line are such that $AB = BC$. The line through B perpendicular to AC passes through the point $D(-1, 6)$. Find:

a the equation of BD

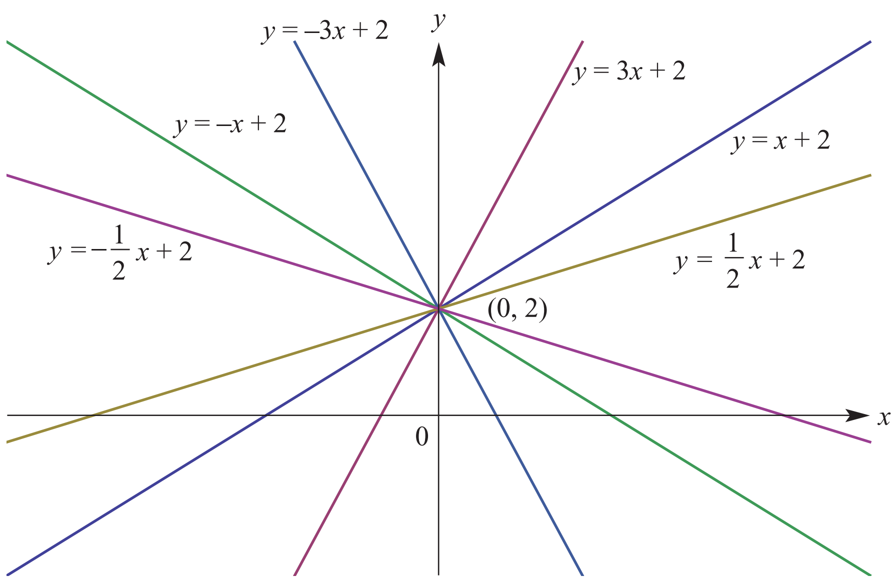
b the coordinates of B

c the coordinates of C.

2F Families of straight lines

Here are three families of straight lines:

- $y = mx$, where the gradient m of the lines varies – the graphs are the straight lines through the origin.
- $y = 3x + c$, where the y-axis intercept c of the lines varies – the graphs are the straight lines with gradient 3.
- $y = mx + 2$, where the gradient m of the lines varies – the graphs are the straight lines with y-axis intercept 2. Some graphs in this family are illustrated below.



The variable m is called a **parameter**. We will consider other families of graphs in later chapters of this book.

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Example 20

Find the value of m if the line $y = mx + 2$ passes through the point $(3, 11)$.

Solution

We can write

$$11 = 3m + 2$$

Therefore $3m = 9$ and hence $m = 3$. It is the line $y = 3x + 2$.

Example 21

A family of lines have equations of the form $y = mx + 2$, where m is a negative number.

a Find the x-axis intercept of a line in this family in terms of m.

b For which values of m is the x-axis intercept greater than 3?

c Find the equation of the line perpendicular to the line $y = mx + 2$ at the point $(0, 2)$.

Solution

a When $y = 0$:
$$\begin{aligned} mx + 2 &= 0 \\ mx &= -2 \\ x &= -\frac{2}{m} \end{aligned}$$

The x-axis intercept is $-\frac{2}{m}$.

b
$$\begin{aligned} -\frac{2}{m} &> 3 \\ -2 &< 3m \\ -\frac{2}{3} &< m \end{aligned}$$

Therefore the x-axis intercept is greater than 3 for $-\frac{2}{3} < m < 0$.

c The equation is $y - 2 = -\frac{1}{m}x$ and the gradient–intercept form is $y = -\frac{1}{m}x + 2$.

Explanation

a To find the x-axis intercept put $y = 0$.

b Multiply both sides of the inequality by m. Since m is negative, the inequality sign is reversed. Then divide both sides by 3.

c The perpendicular line has gradient $-\frac{1}{m}$ and passes through $(0, 2)$.

Summary 2F

Families of straight lines can be described through the use of a parameter. For example:

- All the non-vertical lines passing through $(0, 2)$ have equation of the form $y = mx + 2$, where m is a real number. (If $m = 0$, the line is horizontal.)
- All the lines with gradient 3 have equation of the form $y = 3x + c$, where c is a real number.
- All the lines with x-axis intercept 4 have equation of the form $\frac{x}{4} + \frac{y}{b} = 1$, where b is a non-zero real number.

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Skill-sheet

Exercise 2F

Example 20

1 The line with equation $y = mx - 3$, where m is a positive number, passes through the point $(2, 7)$. Find the value of m.

2 The line with equation $y = 2x + c$, where c is a number, passes through the point $(3, 11)$. Find the value of c.

3 A line has equation $y = mx + 3$, where $m \neq 0$. A second line is perpendicular to this line and passes through the point $(0, 3)$.

a Find the equation of the second line in terms of m.

b Find the value of m if the second line passes through the point $(1, -4)$.

4 The line with equation $y = mx + 2$, where m is a positive number, passes through the point $(3, 8)$. Find the value of m.

Example 21

5 A family of lines have equations of the form $y = mx - 3$, where m is a positive number.

a Find the x-axis intercept of a line in this family in terms of m.

b Find the value of m if the line passes through the point $(5, 6)$.

c For which values of m is the x-axis intercept less than or equal to 1?

d Find the equation of the line perpendicular to the line $y = mx - 3$ at the point $(0, -3)$.

6 A family of lines have equations of the form $y = 2x + c$, where c is a number.

a Find the x-axis intercept of a line in this family in terms of c.

b Find the value of c if the line passes through the point $(5, 6)$.

c For which values of c is the x-axis intercept less than or equal to 1?

d Find the equation of the line perpendicular to the line $y = 2x + c$ at the point $(0, c)$.

7 A family of lines have equations of the form $\frac{x}{a} - \frac{y}{12} = 4$, where a is a non-zero number.

a Find the x-axis intercept of a line in this family in terms of a.

b Find the gradient of the line in terms of a.

c Find the value of a if the gradient is:

i 2 **ii** -2

8 A family of lines have equations of the form $y = -2x + c$, where c is a positive number.

a Find the x-axis intercept of a line in this family in terms of c.

b Find the value of c if the line passes through the point $(1, 7)$.

c For which values of c is the x-axis intercept less than or equal to 1?

d Find the equation of the line perpendicular to $y = -2x + c$ at the point $(0, c)$.

e Let O denote the origin, and let A and B be the x-axis and y-axis intercepts of $y = -2x + c$.

i If the midpoint M of the line segment AB is $(3, 6)$, find the value of c.

ii If the triangle AOB has area 4, find the value of c.

iii If the midpoint M of AB is such that $OM = 2\sqrt{5}$, find the value of c.

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9 A family of lines have equations of the form $3x + by = 12$, where b is a non-zero number.

a Find the y-axis intercept of a line in this family in terms of b.

b Find the gradient of the line in terms of b.

c Find the value of b if the gradient is:

i 1

ii −2

d Find the equation of the line perpendicular to the line $3x + by = 12$ at the point $(4, 0)$.

2G Linear models

There are many practical situations where a linear relation can be used.

Example 22

Austcom's rates for local calls from private telephones consist of a quarterly rental fee of \$50 plus 25c for every call. Construct a cost function that describes the quarterly telephone bill and sketch the linear graph for this.

Solution

Let

C = cost (\$) of quarterly telephone bill

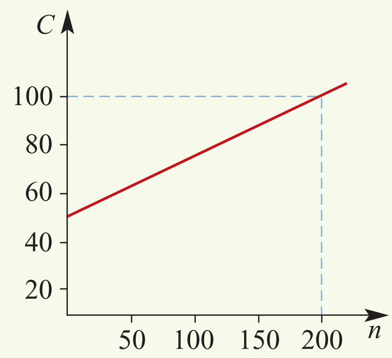
n = number of calls

Then

$$C = 0.25n + 50$$

The number of calls is counted in whole numbers including zero.

Draw the graph of the function $C = 0.25n + 50$ where n is a whole number including zero.



Note: The graph should be a series of discrete points rather than a continuous line, because n is a whole number. With the scale used it is not practical to show it correctly.

An important linear relation is the relation between distance travelled and time taken when an object is travelling with constant speed. If a car travels at 40 km/h, the relationship between distance travelled (s kilometres) and time taken (t hours) is

$$s = 40t \qquad (\text{for } t \geq 0)$$

The graph of s against t is a straight-line graph through the origin. The gradient of this graph is 40.

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Example 23

A car starts from point A on a highway 10 kilometres past the Corumbilla post office. The car travels at a constant speed of 90 km/h towards picnic stop B, which is 120 kilometres further on from A. Let t hours be the time after the car leaves point A.

a Find an expression for the distance d_1 of the car from the post office at time t hours.

b Find an expression for the distance d_2 of the car from point B at time t hours.

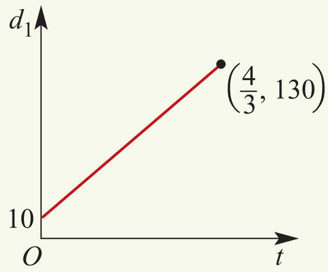
c On separate sets of axes, sketch the graphs of d_1 against t and d_2 against t and state the gradient of each graph.

Solution

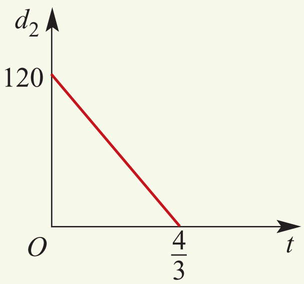
a At time t the distance of the car from the post office is $(10 + 90t)$ kilometres.

b At time t the distance of the car from B is $(120 - 90t)$ kilometres.

c



Gradient = 90



Gradient = −90

Exercise 2G

Skill-sheet

Example 22

1 The weekly wage, $\$w$, of a vacuum cleaner salesperson consists of a fixed sum of \$350 plus \$20 for each cleaner sold. If n cleaners are sold per week, construct a rule that describes the weekly wage of the salesperson.

Example 23

2 **a** A train moves at 50 km/h in a straight line away from a town. Give a rule for the distance, d_1 km, from the town at time t hours after leaving the town.

b A train has stopped at a siding 80 km from the town and then moves at 40 km/h in a straight line away from the siding towards the town. Give a rule for the distance, d_2 km, from the town at time t hours after leaving the siding.

c On separate sets of axes, sketch the graphs of d_1 against t $(0 \le t \le 4)$ and d_2 against t and state the gradient of each graph.

3 **a** An initially empty container is being filled with water at a rate of 5 litres per minute. Give a rule for the volume, V litres, of water in the container at time t minutes after the filling of the container starts.

b A container contains 10 litres of water. Water is then poured in at a rate of 5 litres per minute. Give a rule for the volume, V litres, of water in the container at time t minutes after the pouring starts.

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4 The reservoir feeding an intravenous drip contains 500 mL of a saline solution. The drip releases the solution into a patient at the rate of 2.5 mL/minute.

a Construct a rule which relates the amount of solution left in the reservoir, v mL, to time, t minutes.

b State the possible values of t and v.

c Sketch the graph of the relation.

5 The cost ($$C$) of hiring a taxi consists of two elements, a fixed flagfall and an amount that varies with the number (n) of kilometres travelled. If the flagfall is $2.60 and the cost per kilometre is $1.50, determine a rule which gives C in terms of n.

6 A car rental company charges $85, plus an additional amount of 24c per kilometre.

a Write a rule to determine the total charge $$C$ for hiring a car and travelling x kilometres.

b What would be the cost to travel 250 kilometres?

7 Two towns A and B are 200 km apart. A man leaves town A and walks at a speed of 5 km/h towards town B. Find the distance (d km) of the man from town B at time t hours after leaving town A.

8 The following table shows the extension of a spring when weights are attached to it.

x, extension (cm)	0	1	2	3	4	5	6
w, weight (g)	50	50.2	50.4	50.6	50.8	51.0	51.2

a Sketch a graph to show the relationship between x and w.

b Write a rule that describes the graph.

c What will be the extension if $w = 52.5$ g?

9 A printing firm charges $35 for printing 600 sheets of headed notepaper and $47 for printing 800 sheets.

a Find a formula, assuming the relationship is linear, for the charge, $$C$, in terms of number of sheets printed, n.

b How much would they charge for printing 1000 sheets?

10 An electronic bank teller registered $775 after it had counted 120 notes and $975 after it had counted 160 notes.

a Find a formula for the sum registered ($$C$) in terms of the number of notes (n) counted.

b Was there a sum already on the register when counting began?

c If so, how much?

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2H Simultaneous linear equations

In this section we revisit the geometry of simultaneous equations, first introduced in Section 1C, and also make use of parameters to explore these properties. Finally we consider some of the many applications of simultaneous equations.

The geometry of simultaneous equations

There are three possible outcomes when considering a system of two simultaneous linear equations in two unknowns:

- There is a unique solution. (Lines intersect at a point.)
- There are infinitely many solutions. (Lines coincide.)
- There is no solution. (Lines are parallel.)

Example 24

Explain why the simultaneous equations $2x + 3y = 6$ and $4x + 6y = 24$ have no solution.

Solution

First write the equations in the form $y = mx + c$. They become

$$y = -\frac{2}{3}x + 2 \qquad \text{and} \qquad y = -\frac{2}{3}x + 4$$

Each of the lines has gradient $-\frac{2}{3}$. The y-axis intercepts are 2 and 4 respectively. The equations have no solution as they correspond to parallel lines and they are different lines.

Example 25

The simultaneous equations $2x + 3y = 6$ and $4x + 6y = 12$ have infinitely many solutions. Describe these solutions through the use of a parameter.

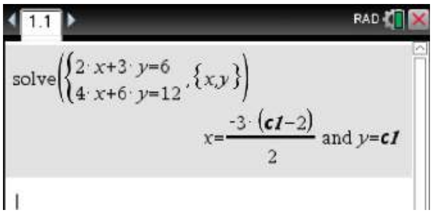
Solution

The two lines coincide, and so the solutions are all points on this line. We make use of a third variable λ as the parameter. If $y = \lambda$, then $x = \frac{6 - 3\lambda}{2}$. The points on the line are all points of the form $\left(\frac{6 - 3\lambda}{2}, \lambda\right)$.

Using the TI-Nspire

Simultaneous equations can be solved in a **Calculator** application.

- Use menu > **Algebra** > **Solve System of Equations** > **Solve System of Equations**.
- Complete the pop-up screen.



The solution to this system of equations is given by the calculator as shown. The variable $c1$ takes the place of λ.

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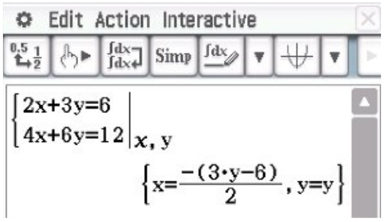
Using the Casio ClassPad

To solve the simultaneous equations $2x + 3y = 6$ and $4x + 6y = 12$:

- Open the Math1 keyboard.
- Select the simultaneous equations icon.
- Enter the two equations into the two lines and type x, y in the bottom-right square to indicate the variables.
- Select EXE.



Choose $y = \lambda$ to obtain the solution $x = \dfrac{6 - 3\lambda}{2}$, $y = \lambda$ where λ is any real number.

Example 26

The family of lines $y = mx + 2$ with varying gradient m all pass through the point $(0, 2)$.

a For what values of m does the line $y = mx + 2$ not intersect the line $y = 5x - 3$?

b For what values of m does the line $y = mx + 2$ intersect the line $y = 5x - 3$?

c If the line $y = mx + 2$ intersects the line $y = 5x - 3$ at the point $(5, 22)$, find the value of m.

Solution

a The y-axis intercept of $y = mx + 2$ is 2 and the y-axis intercept of $y = 5x - 3$ is -3. The lines will not intersect if they are parallel, that is, if they have the same gradient. So $m = 5$.

b The lines intersect when $m \neq 5$.

c If $(5, 22)$ lies on the line $y = mx + 2$, then

$$22 = 5m + 2$$
$$20 = 5m$$
$$m = 4$$

Thus the lines intersect at $(5, 22)$ when $m = 4$.

Example 27

The lines $y = x + k$ and $y = mx + 4$ intersect at $(1, 3)$. Find the values of m and k.

Solution

When $x = 1$, $y = 3$.

So $3 = 1 + k$ and $3 = m + 4$.

Hence $k = 2$ and $m = -1$.

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Example 28

The lines $(m-2)x + y = 2$ and $mx + 2y = k$ intersect at $(2, 8)$. Find the values of m and k.

Solution

$$(m-2)x + y = 2 \qquad (1)$$
$$mx + 2y = k \qquad (2)$$

When $x = 2$, $y = 8$. Substituting these values into equations (1) and (2), we have the equations

$$2(m-2) + 8 = 2 \qquad (3)$$
$$2m + 16 = k \qquad (4)$$

From (3), we have $2m - 4 + 8 = 2$. Therefore $m = -1$.

From (4), we have $2(-1) + 16 = k$. Therefore $k = 14$.

Example 29

Consider the simultaneous linear equations $(m-2)x + y = 2$ and $mx + 2y = k$. Find the values of m and k such that the system of equations has:

a no solution **b** infinitely many solutions **c** a unique solution.

Solution

$$(m-2)x + y = 2 \qquad (1)$$
$$mx + 2y = k \qquad (2)$$

We know that for *no solution* or *infinitely many solutions*, the corresponding lines are parallel or coincide. If the corresponding lines are parallel or coincide, the gradients are the same.

Gradient of line (1) $= 2 - m$

Gradient of line (2) $= -\dfrac{m}{2}$

Hence
$$2 - m = -\frac{m}{2}$$
$$4 - 2m = -m$$
$$m = 4$$

Substitute $m = 4$ in equations (1) and (2). We have

$$2x + y = 2$$
$$4x + 2y = k$$

a There is no solution if $m = 4$ and $k \neq 4$.

b If $m = 4$ and $k = 4$, there are infinitely many solutions as the equations are equivalent.

c The solution is unique if $m \neq 4$ and k is any real number.

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Applications of simultaneous equations

Example 30

There are two possible methods for paying gas bills:

Method A A fixed charge of \$25 per quarter + 50c per unit of gas used

Method B A fixed charge of \$50 per quarter + 25c per unit of gas used

Determine the number of units which must be used before method B becomes cheaper than method A.

Solution

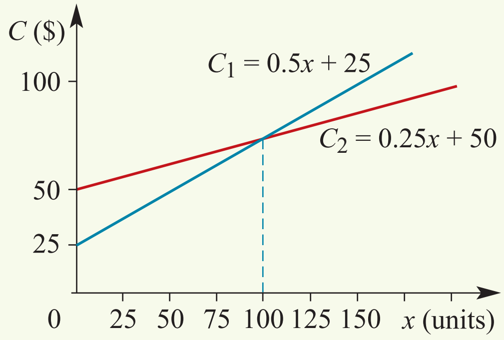
Let C_1 = charge (\$) using method A

C_2 = charge (\$) using method B

x = number of units of gas used

Then $C_1 = 25 + 0.5x$

$C_2 = 50 + 0.25x$



From the graph we see that method B is cheaper if the number of units exceeds 100.

The solution can be obtained by solving simultaneous linear equations:

$$C_1 = C_2$$
$$25 + 0.5x = 50 + 0.25x$$
$$0.25x = 25$$
$$x = 100$$

Example 31

Robyn and Cheryl race over 100 metres. Robyn runs so that it takes a seconds to run 1 metre, and Cheryl runs so that it takes b seconds to run 1 metre. Cheryl wins the race by 1 second. The next day they again race over 100 metres but Cheryl gives Robyn a 5-metre start so that Robyn runs 95 metres. Cheryl wins this race by 0.4 seconds. Find the values of a and b and the speed at which Robyn runs.

Solution

For the first race: Time for Robyn − time for Cheryl = 1 s.

$$100a - 100b = 1 \qquad (1)$$

For the second race: Time for Robyn − time for Cheryl = 0.4 s.

$$95a - 100b = 0.4 \qquad (2)$$

Subtract (2) from (1). This gives $5a = 0.6$ and therefore $a = 0.12$.

Substitute in (1) to find $b = 0.11$.

Robyn's speed $= \frac{1}{0.12} = \frac{25}{3}$ m/s.

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Summary 2H

- There are three cases for a system of two linear equations with two variables:
  - unique solution (lines intersect at a point), e.g. $y = 2x + 3$ and $y = 3x + 3$
  - infinitely many solutions (lines coincide), e.g. $y = 2x + 3$ and $2y = 4x + 6$
  - no solution (lines are parallel), e.g. $y = 2x + 3$ and $y = 2x + 4$.
- There are many applications of simultaneous linear equations with two variables. The problems often arise by working with two quantities both changing at a constant but often different rate.

Skill-sheet

Exercise 2H

Example 24

1 Explain why the following simultaneous equations have no solution:

$$x + y = 6$$
$$2x + 2y = 13$$

Example 25

2 The simultaneous equations $x + y = 6$ and $2x + 2y = 12$ have infinitely many solutions. Describe these solutions through the use of a parameter.

Example 26

3 The family of lines $y = mx + 6$ with varying gradient m all pass through the point $(0, 6)$.

a For what values of m does the line $y = mx + 6$ not intersect the line $y = 4x - 5$?

b For what values of m does the line $y = mx + 6$ intersect the line $y = 4x - 5$?

c If the line $y = mx + 6$ intersects the line $y = 4x - 5$ at the point $(5, 15)$, find the value of m.

Example 27

4 The lines $y = 2x + k$ and $y = mx - 4$ intersect at $(2, 6)$. Find the values of m and k.

Example 28

5 The lines $(m - 2)x + y = 4$ and $mx + 3y = k$ intersect at $(2, 8)$. Find the values of m and k.

Example 29

6 Find the value of m for which the simultaneous equations $mx - y = 5$ and $3x + y = 6$ have no solution.

7 Find the value of m for which the pair of simultaneous equations $3x + my = 5$ and $(m + 2)x + 5y = m$ have:

a infinitely many solutions

b no solutions.

Example 30

8 Two bicycle hire companies have different charges. Company A charges \$$C$, according to the rule $C = 10t + 20$, where t is the time in hours for which a bicycle is hired. Company B charges \$$C$, according to the rule $C = 8t + 30$.

a Sketch each of the graphs on the same set of axes.

b Find the time, t, for which the charge of both companies is the same.

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Example 31

9 John and Michael race over 50 metres. John runs so that it takes at a seconds to run 1 metre and Michael runs so that it takes b seconds to run 1 metre. Michael wins the race by 1 second. The next day they again race over 50 metres but Michael gives John a 3-metre start so that John runs 47 metres. Michael wins this race by 0.1 seconds. Find the values of a and b and the speed at which Michael runs.

10 The distances, d_A km and d_B km, of cyclists A and B travelling along a straight road from a town hall step are given respectively by $d_A = 10t + 15$ and $d_B = 20t + 5$, where t is the time in hours after 1 p.m.

a Sketch each of the graphs on the one set of axes.

b Find the time in hours at which the two cyclists are at the same distance from the town hall step.

11 A helicopter can be hired for \$210 per day plus a distance charge of \$1.60 per km or, alternatively, at a fixed charge of \$330 per day for an unlimited distance.

a For each of the methods of hiring, find an expression for cost, \$$C$, in terms of the distance travelled, x km.

b On one set of axes, draw the graph of cost versus distance travelled for each of the methods.

c Determine for what distances the fixed-charge method is cheaper.

12 Three power boats in a 500 km handicap race leave at 5 hourly intervals. Boat A leaves first and has a speed for the race of 20 km/h. Boat B leaves 5 hours later and travels at an average speed of 25 km/h. Boat C leaves last, 5 hours after B, and completes the race at a speed of 40 km/h.

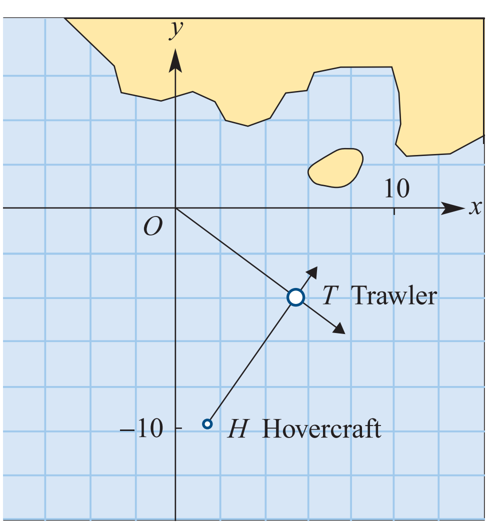
a Draw a graph of each boat's journey on the same set of axes.

b Use your graphs to find the winner of the race.

c Check your answer algebraically.

d Write a short description of what happened to each boat in the race.

13 If the line OT has the equation $y = -\frac{3}{4}x$ and the line HT has the equation $y = \frac{3}{2}x - 12$, determine the point over which both craft would pass.



Scale: 1 unit = 1 nautical mile

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14 A school wishes to take some of its students on an excursion. If they travel by public transport, it will cost the school \$2.80 per student. Alternatively, the school can hire a bus at a cost of \$54 for the day plus a charge of \$1 per student.

- **a** For each mode of transport, write an expression for the cost (\$$C$) of transport in terms of the number of students (x).
- **b** On one set of axes, draw the graph of cost, \$$C$, versus number of students, x, for each mode of transport.
- **c** Determine for how many students it will be more economical to hire the bus.

15 Anne and Maureen live in towns that are 57 km apart. Anne sets out at 9 a.m. one day to ride her bike to Maureen's town at a constant speed of 20 km/h. At the same time Maureen sets out to ride to Anne's town at a constant speed of 18 km/h.

- **a** Write down a rule for the distance, d km, that each of them is from Anne's place at a time t minutes after 9 a.m.
- **b** On the same set of axes, draw graphs of the distance, d km, versus time, t minutes after 9 a.m., for each cyclist.
- **c** Find the time at which they will meet.
- **d** How far has each of them travelled when they meet?

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Chapter summary

Assign-ment

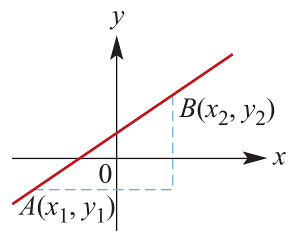
Nrich

- The **midpoint** of the line segment joining (x_1, y_1) and (x_2, y_2) is the point with coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

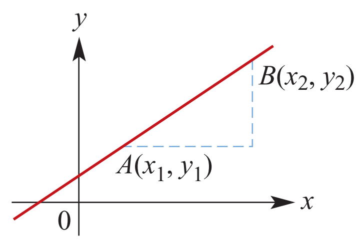
- The **distance** between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



- The **gradient** of a straight line joining two points:

$$\text{Gradient } m = \frac{y_2 - y_1}{x_2 - x_1}$$



- For a line with gradient m, the **angle of slope** (θ) can be found using

$$m = \tan\theta$$

where θ is the angle the line makes with the positive direction of the x-axis.

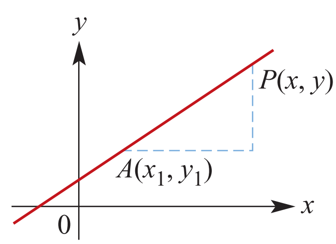
- The gradient–intercept form of the equation of a straight line is

$$y = mx + c$$

where m is the gradient and c is the y-axis intercept.

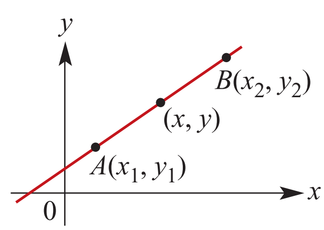
- The equation of a line passing through a given point (x_1, y_1) and having gradient m is

$$y - y_1 = m(x - x_1)$$



- The equation of a line passing through two given points (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = m(x - x_1) \quad \text{where} \quad m = \frac{y_2 - y_1}{x_2 - x_1}$$



- Two straight lines are **perpendicular** to each other if and only if the product of their gradients is -1 (or if one is horizontal and the other vertical):

$$m_1 m_2 = -1$$

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Short-answer questions

1 Find the length and the coordinates of the midpoint of the line segment joining each of the following pairs of points:

a $A(1, 2)$ and $B(5, 2)$ **b** $A(-4, -2)$ and $B(3, -7)$ **c** $A(3, 4)$ and $B(7, 1)$

2 Find the gradients of the lines joining each of the following pairs of points:

a $(4, 3)$ and $(8, 12)$ **b** $(-3, 4)$ and $(8, -6)$ **c** $(2, 1)$ and $(2, 9)$

d $(0, a)$ and $(a, 0)$ **e** $(0, 0)$ and (a, b) **f** $(0, b)$ and $(a, 0)$

3 Find the equation of the straight line of gradient 4 which passes through the point with coordinates:

a $(0, 0)$ **b** $(0, 5)$ **c** $(1, 6)$ **d** $(3, 7)$

4 **a** The point $(1, a)$ lies on the line with equation $y = 3x - 5$. Find the value of a.

b The point $(b, 15)$ lies on the line with equation $y = 3x - 5$. Find the value of b.

5 Find the equation of the straight line joining the points $(-5, 2)$ and $(3, -4)$.

6 Find the equation of the straight line of gradient $-\frac{2}{3}$ which passes through $(-4, 1)$.

7 Write down the equation of the straight line that:

a passes through $(5, 11)$ and is parallel to the x-axis

b passes through $(0, -10)$ and is parallel to the line with equation $y = 6x + 3$

c passes through the point $(0, -1)$ and is perpendicular to the line with equation $3x - 2y + 5 = 0$.

8 Find the equation of the straight line which passes through the point $(2, 3)$ and is inclined at $45°$ to the positive direction of the x-axis.

9 Find the equation of the straight line which passes through the point $(-2, 3)$ and makes an angle of $135°$ with the positive direction of the x-axis.

10 Find the equation of the straight line passing through the point $(4, 8)$ and which is perpendicular to the line with equation $y = -3x + 2$.

11 A straight line has equation $y = 2x + 1$. The points with coordinates $(0, a)$, $(b, 0)$, $(2, d)$ and $(e, 7)$ lie on this line. Find the values of a, b, d and e.

12 Sketch the graph of each of the following by first determining axis intercepts. Clearly label each axis intercept.

a $y = 2x - 8$ **b** $3x + y = 6$ **c** $3x + y + 6 = 0$

d $y - 2x - 8 = 0$ **e** $y = -6x + 6$ **f** $2x + 5y + 10 = 0$

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13 **a** Find the equation of the line parallel to the x-axis passing through the point $(8, -9)$. Sketch the graph.

b Find the equation of the line parallel to the y-axis passing through the point $(8, -9)$. Sketch the graph.

c Find the equation of the straight line that passes through the origin and has gradient:

i 3 **ii** -3

Sketch the graphs of each of these.

14 A car travels at a constant 60 km/h.

a Write down the formula for the distance d km travelled by the car in t hours.

b What is the gradient of the graph of d against t?

15 The weekly wage $\$S$ of a car salesperson consists of a fixed sum of \$800 plus \$500 for each car sold. If n cars are sold in a week, construct a formula that describes the weekly wage of the salesperson in terms of n.

16 A family of straight lines satisfy the rule $y = ax + 2$.

a Find the equation of the straight line in this family for which $y = 6$ when $x = 2$.

b **i** Find the x-axis intercept of the line with equation $y = ax + 2$.

ii If $a < 0$, find the values of a for which the x-axis intercept is greater than 1.

c Find the coordinates of the point of intersection of the line with equation $y = x + 3$ and the line with equation $y = ax + 2$, given that $a \neq 1$.

Extended-response questions

1 The cost of hiring a motor cruiser consists of a down payment of \$500 and a running charge of \$100 per day, or part of a day. The cost of fuel is \$27.50 per day. There is also a charge of \$50 for filling the freshwater tanks and charging the batteries. Food for a cruise of n days costs \$62.50 per day.

a Give a formula for C, the total cost in dollars of hiring the cruiser for n days (all costs to be included).

b For how many days can a cruiser be hired if the cost of a cruise is to be no more than \$3000?

c A rival company has a fixed rate of \$300 per day. For how many days would it be cheaper to hire from this company?

2 The cost of fitting a new plug and cable for an electric drill is $\$C$, when the length of the cable is x metres and $C = 4.5 + 1.8x$.

a What meaning could be given for the constant term 4.5?

b What could be the meaning of the coefficient 1.8?

c What would be the gradient of the graph of C against x?

d What length of cable would give a total cost of \$24.50?

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Review

3 The profit made on a single journey of an Easyride bus tour is $\$P$, when there are x empty seats and $P = 1020 - 24x$.

a What do you think is the meaning of the constant term 1020?

b What is the least number of empty seats which would result in a loss on a single journey?

c Suggest a meaning for the coefficient 24.

4 A quarterly electricity bill shows the following charges:

- For the first 50 kWh (kilowatt hours): 9.10c per kWh
- For the next 150 kWh: 5.80c per kWh
- Thereafter: 3.56c per kWh

a Write down a formula relating cost, $\$C$, to n, the number of kWh of electricity used:

i for the first 50 kWh

ii for the next 150 kWh

iii for more than 200 kWh.

b Draw a graph of C against n. Use the graph, or otherwise, to determine the charges for:

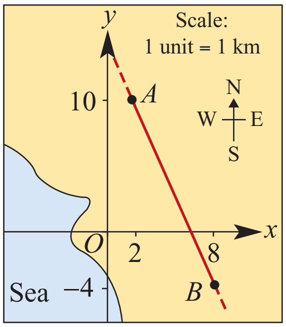
i 30 kWh **ii** 90 kWh **iii** 300 kWh

c How much electricity could be used for a cost of $20?

5 O is the position of the air traffic control tower at an airport. An aircraft travelling in a straight line is identified at $A(2, 10)$ and again at $B(8, -4)$.

a What is the equation that describes the flight path of the aircraft?

b How far south of O is the aircraft when $x = 15$ km?



6 A construction company estimates that for every 1% of air left in concrete as it is being laid the strength of the hardened concrete decreases by 7%. Let x represent the percentage of air in the concrete (by volume), and the strength of the concrete be s units, where $s = 100$ when $x = 0$.

a Write a formula for s in terms of x.

b Sketch a graph of s against x.

c Calculate how much air can be allowed to remain in the concrete for a strength of at least 95%.

d Estimate how much air the concrete will contain at 0% strength.

e Is the model sensible at 0% strength?

f State the possible values of x.

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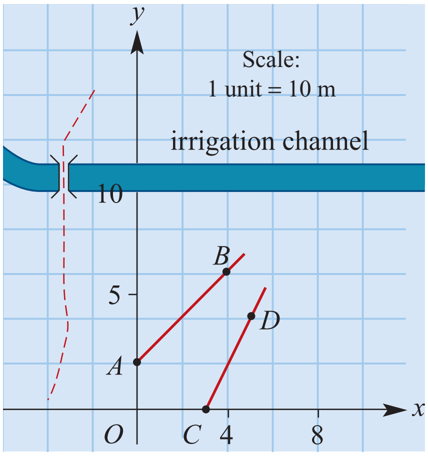
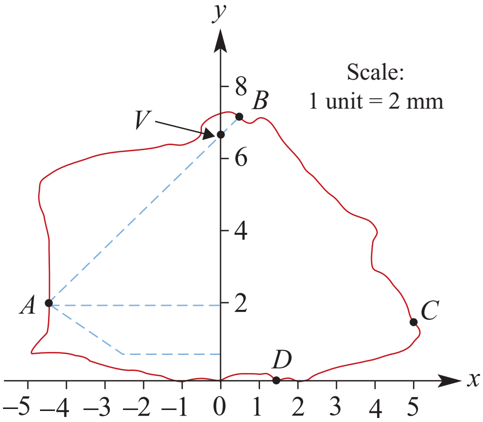
7 The diagram shows a plan view of a paddock over which a Cartesian framework has been superimposed. From an observation point O, a rabbit has been spotted first at $A(0, 2)$ and then at $B(4, 6)$. A fox is seen at $C(3, 0)$ and later at $D(5, 4)$.



a Find the equations of the lines AB and CD.

b Assuming that both the rabbit and the fox were running along straight lines, calculate whether the fox's path would cross the rabbit's track before the irrigation channel.

8 The diagram shows the side view of a rough, uncut diamond fixed in position on a computer-controlled cutting machine. The diamond is held at the points $A(-4.5, 2)$, $B(0.25, 7)$, $C(5, 1.5)$ and $D(1.5, 0)$.

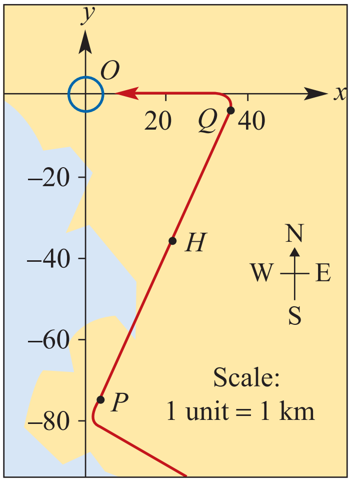


a If a straight cut is made joining A and B, find the y-coordinate of the point V at which the cut will cross the vertical axis.

b Find the equation of the line joining V and C.

c Would the cuts AB and VC be equally inclined to the vertical axis? Explain your answer.

9 A new light beacon is proposed at $P(4, -75)$ for air traffic flying into an airport located at $O(0, 0)$. It is intended that the aircraft should follow a course over beacons at P and $Q(36, -4)$, turning at Q towards the runway at O.



a Would a direct line from P to Q pass directly over a hospital located at $H(20, -36)$?

b If not, state how far east or west of H the aircraft would be when the y-coordinate of an aircraft's flight path is -36.

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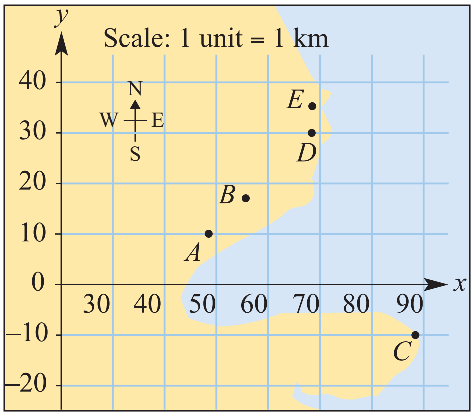
10 The map shows an area where it is proposed to construct a new airport. It is thought that the main runway of the airport will have one end of its centre line at $A(48, 10)$, but the position of the other end of this line, B, has not been decided. There is a light aircraft airport at $E(68, 35)$ and a radio beacon at $C(88, -10)$.



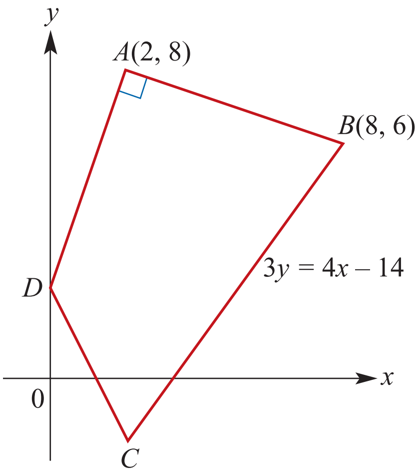
a What is the equation that will define the new runway if aircraft coming in to land from the east must be on the extended central line of the new runway when they are 5 km due south of E?

b If B is to be 8 km to the east of A, what will be its coordinates?

c A marker beacon is to be built at $D(68, 30)$ and it is proposed that several auxiliary beacons should be placed on the line CD. What is the equation of the line CD?

d If one of the auxiliary beacons is to be placed due east of A, what are the coordinates of its position?

11 The diagram shows a quadrilateral. Angle BAD is a right angle and C lies on the perpendicular bisector of AB. The equation of the line through points B and C is $3y = 4x - 14$. Find:



a the equation of the line AD

b the coordinates of D

c the equation of the perpendicular bisector of AB

d the coordinates of C

e the area of triangle ADC

f the area of the quadrilateral $ABCD$.

12 Wheelrite, a small company that manufactures garden wheelbarrows, has overhead expenses of \$30 000 per year. In addition, it costs \$40 to manufacture each wheelbarrow.

a Write a rule which determines the total cost, $\$C$, of manufacturing x wheelbarrows per year.

b If the annual production is 6000 wheelbarrows, what is the overall cost per wheelbarrow?

c How many wheelbarrows must be made so that the overall cost is \$46 per wheelbarrow?

d Wheelrite sells wheelbarrows to retailers for \$80 each. Write a rule which determines the revenue, $\$R$, from the sale of x wheelbarrows to retailers.

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e Sketch the graphs for C and R against x on the same axes.

f What is the minimum number of wheelbarrows that must be produced for Wheelrite to make a profit each year?

g Write a rule which determines the profit, $\$P$, from the manufacture and sale of x number of wheelbarrows.

13 An electricity supply authority is offering customers a choice of two methods of paying electricity bills. Method 1 involves payment annually and method 2 involves payment each quarter (that is, every three months). The charges for each method are as follows:

Method 1 – per year		Method 2 – per quarter	
Fixed charge	\$100	Fixed charge	\$27.50
Price per unit	\$0.08125	Price per unit	\$0.075

a Suppose a customer used 1560 units of electricity in a year. Calculate which is the cheaper method of payment.

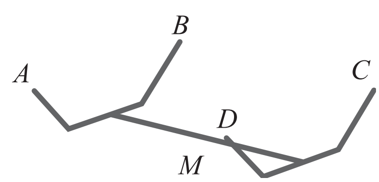
b Copy and then complete the following table:

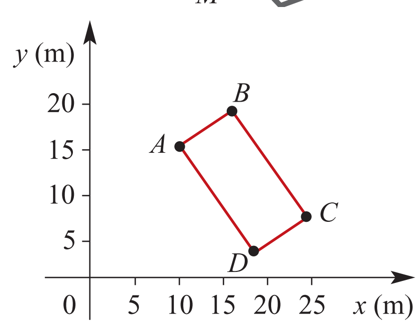
	Number of units of electricity			
	0	1000	2000	3000
Cost (\$) calculated by method 1				
Cost (\$) calculated by method 2				

c Use these values to plot graphs of the costs for each method of paying for electricity. Clearly indicate the approximate number of units of electricity for which the cost is the same for both methods of payment.

d If C_1 is the cost by method 1, C_2 is the cost by method 2, and x is the number of units of electricity used in a year, write down the two formulas which show the cost of x units calculated by each method. Use these formulas to calculate the exact number of units for which the cost is the same for both methods.

14 In a metal fabricating yard which has been flooded by overflow from a local river, a large steel frame has been partly submerged. The ends A, B, C and D are the only parts visible above the level of the flood water. The coordinates of the ends relative to an overhead crane are $A(10, 16)$, $B(16, 20)$, $C(24, 8)$ and $D(18, 4)$. The overhead crane moves east–west along its rail, and the distance east from a point $O(0, 0)$ is denoted by x. The crane's hook moves north–south across the frame and the distance to the north of the south rail is denoted by y. Units are in metres. The steel frame is to be raised out of the water by lifting it at the midpoint, M, of its middle section.





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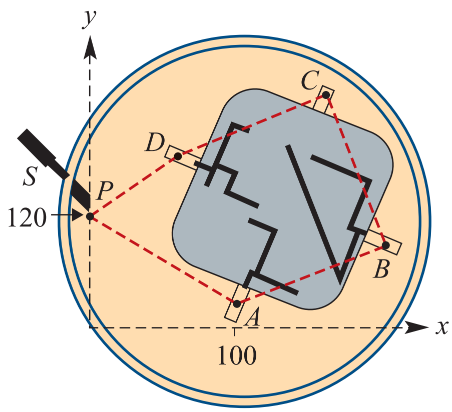
a Find the coordinates, x and y, of the point to which the hook must be moved so that it will be directly above the midpoint, M, of the steel frame.

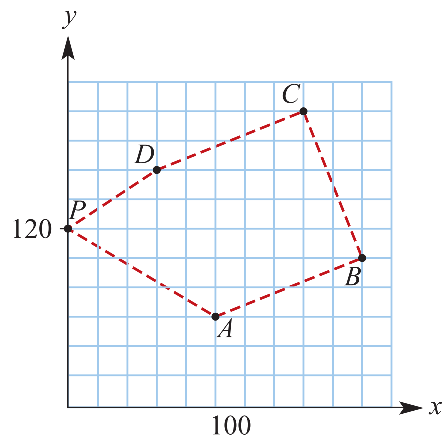
b In order to minimise the risk of the hook slipping, the hook will be moved slowly along a line parallel to AB. Find the equation of the line along which the hook will be moved.

15 The diagram below shows part of a micro-electronics circuit, as seen through a magnifying glass; the circuit has been etched onto a chip of plated silica.
The four points A, B, C and D stand away from the chip itself. A is $(100, 60)$, B is $(200, 100)$, C is $(160, 200)$ and D is $(60, 160)$. Units are in $\frac{1}{25}$ mm.
The unit S is a moveable micro-soldering unit, its tip being at $P(0, 120)$. It is desired to program the tip of the soldering iron, P, to solder wires to the points A, B, C and D, moving along the dashed lines as shown in the graph.

a Find equations for the lines defining each section of the path along which P must be programmed to move.

b Will any of the turns be through right angles? Explain.





Scale: 1 unit = $\frac{1}{25}$ mm

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3 Quadratics

In this chapter

Syllabus references

Topic: Quadratic relationships

Subtopics: 1.1.7 – 1.1.12

A **polynomial function** has a rule of the type

$$y = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is a natural number or zero, and $a_0, a_1, \dots, a_n$ are numbers called **coefficients**.

The **degree** of a polynomial is given by the value of n, the highest power of x with a non-zero coefficient. For example:

- $y = 2x + 3$ is a polynomial function of degree 1
- $y = 2x^2 + 3x - 2$ is a polynomial function of degree 2.

This chapter deals with polynomials of degree 2. These are called **quadratic polynomials**.

The graph of a linear polynomial function, $y = mx + c$, is a straight line and the graph of a quadratic polynomial function, $y = ax^2 + bx + c$, $a \neq 0$, is a **parabola**. Polynomials of higher degree will be studied in Chapter 7.

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3A Expanding and collecting like terms

In order to sketch graphs of quadratics, we need to find the x-axis intercepts (if they exist), and to do this we need to solve quadratic equations. As an introduction to the methods of solving quadratic equations, the first two sections of this chapter review the basic algebraic processes of expansion and factorisation.

An algebraic expression is the sum of its **terms**. For example:

- The terms of the linear expression $3x - 1$ are $3x$ and -1.
- The terms of the quadratic expression $-2x^2 + 3x - 4$ are $-2x^2$, $3x$ and -4.

Example 1

Simplify $2(x - 5) - 3(x + 5)$ by first expanding.

Solution

$$\begin{aligned} 2(x - 5) - 3(x + 5) &= 2x - 10 - 3x - 15 \\ &= 2x - 3x - 10 - 15 \\ &= -x - 25 \end{aligned}$$

Explanation

Expand each bracket.

Collect like terms.

Example 2

Expand $2x(3x - 2) + 3x(x - 2)$.

Solution

$$\begin{aligned} 2x(3x - 2) + 3x(x - 2) &= 6x^2 - 4x + 3x^2 - 6x \\ &= 9x^2 - 10x \end{aligned}$$

For expansions of the type $(a + b)(c + d)$, proceed as follows:

$$\begin{aligned} (a + b)(c + d) &= a(c + d) + b(c + d) \\ &= ac + ad + bc + bd \end{aligned}$$

Example 3

Expand the following:

a $(x + 3)(2x - 3)$ **b** $(x - 3)(2x - 2\sqrt{2})$

Solution

a $(x + 3)(2x - 3)$

$= x(2x - 3) + 3(2x - 3)$

$= 2x^2 - 3x + 6x - 9$

$= 2x^2 + 3x - 9$

Explanation

Each term in the second pair of brackets is multiplied by each term in the first.

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b $(x - 3)(2x - 2\sqrt{2})$

$= x(2x - 2\sqrt{2}) - 3(2x - 2\sqrt{2})$

$= 2x^2 - 2\sqrt{2}x - 6x + 6\sqrt{2}$

$= 2x^2 - (2\sqrt{2} + 6)x + 6\sqrt{2}$

Be careful with negative signs.

You can also complete binomial expansions with a table; this emphasises the terms.

	x	-3
$2x$	$2x^2$	$-6x$
$-2\sqrt{2}$	$-2\sqrt{2}x$	$6\sqrt{2}$

You add the terms to complete the expansion.

Example 4

Expand $(2x - 1)(3x^2 + 2x + 4)$.

Solution

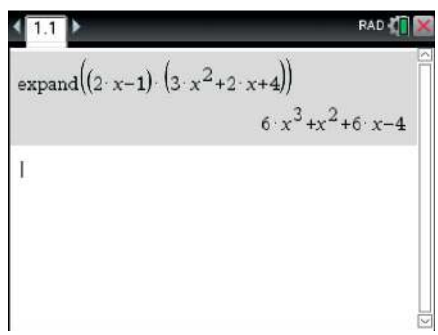
$$\begin{aligned}(2x - 1)(3x^2 + 2x + 4) &= 2x(3x^2 + 2x + 4) - 1(3x^2 + 2x + 4)\\ &= 6x^3 + 4x^2 + 8x - 3x^2 - 2x - 4\\ &= 6x^3 + x^2 + 6x - 4\end{aligned}$$

Using the TI-Nspire

To expand the expression

$(2x - 1)(3x^2 + 2x + 4)$

use menu > **Algebra** > **Expand**.

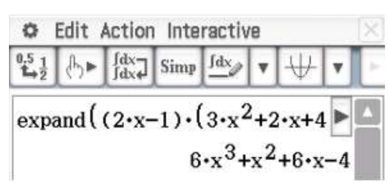


Using the Casio ClassPad

Enter the expression

$(2x - 1)(3x^2 + 2x + 4)$

into Main $\sqrt{\alpha}$. Highlight the expression and select **Interactive** > **Transformation** > **expand**.



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Perfect squares

Consider the expansion of a perfect square, $(a + b)^2$:

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a(a+b) + b(a+b)\\ &= a^2 + ab + ab + b^2\\ &= a^2 + 2ab + b^2\end{aligned}$$

Thus the general result can be stated as:

$$(a+b)^2 = a^2 + 2ab + b^2$$

That is, to expand $(a + b)^2$ take the sum of the squares of the terms and add twice the product of the terms.

Example 5

Expand $(3x - 2)^2$.

Solution

$$\begin{aligned}(3x-2)^2 &= (3x)^2 + 2(3x)(-2) + (-2)^2\\ &= 9x^2 - 12x + 4\end{aligned}$$

Explanation

Use the expansion $(a + b)^2 = a^2 + 2ab + b^2$.
Here $a = 3x$ and $b = -2$.

Difference of two squares

Consider the expansion of $(a + b)(a - b)$:

$$\begin{aligned}(a+b)(a-b) &= a(a-b) + b(a-b)\\ &= a^2 - ab + ab - b^2\\ &= a^2 - b^2\end{aligned}$$

Thus the expansion of the difference of two squares has been obtained:

$$(a+b)(a-b) = a^2 - b^2$$

Example 6

Expand:

a $(2x - 4)(2x + 4)$
b $(x - 2\sqrt{7})(x + 2\sqrt{7})$

Solution

a $$\begin{aligned}(2x-4)(2x+4) &= (2x)^2 - (4)^2\\ &= 4x^2 - 16\end{aligned}$$

b $$\begin{aligned}(x-2\sqrt{7})(x+2\sqrt{7}) &= x^2 - (2\sqrt{7})^2\\ &= x^2 - 28\end{aligned}$$

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Example 7

Expand $(2a - b + c)(2a - b - c)$.

Solution

$$\begin{aligned}(2a - b + c)(2a - b - c) &= ((2a - b) + c)((2a - b) - c)\\ &= (2a - b)^2 - c^2\\ &= 4a^2 - 4ab + b^2 - c^2\end{aligned}$$

Summary 3A

- A **polynomial function** has a rule of the type

  $$y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

  where n is a natural number or zero, and $a_0, a_1, \ldots, a_n$ are numbers called **coefficients**.
- The **degree** of a polynomial is given by the value of n, the highest power of x with a non-zero coefficient.
- A polynomial function of degree 2 is called a **quadratic function**. The general rule is of the form $y = ax^2 + bx + c$, where $a \neq 0$. The graph of a quadratic function is called a **parabola**.
- General binomial expansion:

  $$\begin{aligned}(a + b)(c + d) &= a(c + d) + b(c + d)\\ &= ac + ad + bc + bd\end{aligned}$$
- Perfect square expansion:

  $$(a + b)^2 = a^2 + 2ab + b^2$$
- Difference of two squares expansion:

  $$(a + b)(a - b) = a^2 - b^2$$

Exercise 3A

1 Expand each of the following:

a $2(x - 4)$ **b** $-2(x - 4)$ **c** $3(2x - 4)$
d $-3(4 - 2x)$ **e** $x(x - 1)$ **f** $2x(x - 5)$

2 Collect like terms in each of the following:

a $2x + 4x + 1$ **b** $2x - 6 + x$ **c** $3x + 1 - 2x$ **d** $-x + 2x - 3 + 4x$

Example 1
3 Simplify each of the following by expanding and collecting like terms:

a $8(2x - 3) - 2(x + 4)$ **b** $2x(x - 4) - 3x$
c $4(2 - 3x) + 4(6 - x)$ **d** $4 - 3(5 - 2x)$

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Example 2

4 Simplify each of the following by expanding and collecting like terms:

a $2x(x - 4) - 3x$ **b** $2x(x - 5) + x(x - 5)$ **c** $2x(-10 - 3x)$
d $3x(2 - 3x + 2x^2)$ **e** $3x - 2x(2 - x)$ **f** $3(4x - 2) - 6x$

Example 3

5 Simplify each of the following by expanding and collecting like terms:

a $(3x - 7)(2x + 4)$ **b** $(x - 10)(x - 12)$ **c** $(3x - 1)(12x + 4)$
d $(4x - 5)(2x - 3)$ **e** $(x - \sqrt{3})(x - 2)$ **f** $(2x - \sqrt{5})(x + \sqrt{5})$
g $(3x - 2\sqrt{7})(x + \sqrt{7})$ **h** $(5x - 3)(x + 2\sqrt{2})$ **i** $(\sqrt{5}x - 3)(\sqrt{5}x - 32\sqrt{2})$

Example 4

6 Simplify each of the following by expanding the brackets:

a $(2x - 3)(3x^2 + 2x - 4)$ **b** $(x - 1)(x^2 + x + 1)$
c $(6 - 2x - 3x^2)(4 - 2x)$ **d** $(5x - 3)(x + 2) - (2x - 3)(x + 3)$
e $(2x + 3)(3x - 2) - (4x + 2)(4x - 2)$

Example 5

7 Simplify each of the following by expanding and collecting like terms:

a $(x - 4)^2$ **b** $(2x - 3)^2$ **c** $(6 - 2x)^2$
d $\left(x - \frac{1}{2}\right)^2$ **e** $(x - \sqrt{5})^2$ **f** $(x - 2\sqrt{3})^2$

Example 6

8 Simplify each of the following by expanding and collecting like terms:

a $(x - 3)(x + 3)$ **b** $(2x - 4)(2x + 4)$
c $(9x - 11)(9x + 11)$ **d** $(2x - 3)(2x + 3)$
e $(2x + 5)(2x - 5)$ **f** $(x - \sqrt{5})(x + \sqrt{5})$
g $(2x + 3\sqrt{3})(2x - 3\sqrt{3})$ **h** $(\sqrt{3}x - \sqrt{7})(\sqrt{3}x + \sqrt{7})$

Example 7

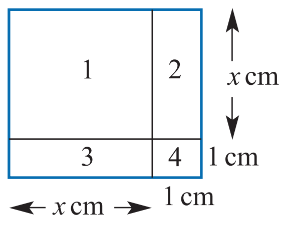
9 Simplify each of the following by expanding and collecting like terms:

a $(x - y + z)(x - y - z)$ **b** $(2a - b + c)(2a - b - c)$
c $(3w - 4z + u)(3w + 4z - u)$ **d** $(2a - \sqrt{5}b + c)(2a + \sqrt{5}b + c)$

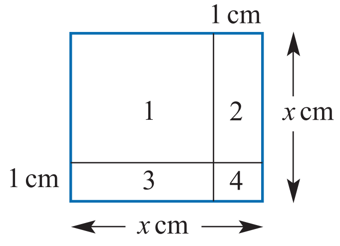
10 Find the area of each of the following by:

i adding the areas of the four 'non-overlapping' rectangles (two of which are squares)
ii multiplying length by width of the undivided square (boundary in blue).

a



b



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3B Factorising

Four different types of factorisation will be considered.

Factorisation using common factors

If each term in an algebraic expression to be factorised contains a **common factor**, then this common factor is a factor of the entire expression. To find the other factor, divide each term by the common factor. The common factor is placed outside the brackets. This process is known as 'taking the common factor outside the brackets'.

Example 8

a Factorise $9x^2 + 81x$.

b Factorise $2a^2 - 8ax^2$.

Solution	Explanation
a $9x^2 + 81x = 9x \times x + 9x \times 9$ $= 9x(x + 9)$	The common factor $9x$ is 'taken out' of the brackets.
b $2a^2 - 8ax^2 = 2a \times a - 2a \times 4x^2$ $= 2a(a - 4x^2)$	The common factor $2a$ is 'taken out' of the brackets.

Note: The answers can be checked by expanding.

In general, take out as many common factors as possible.

Example 9

Factorise $7x^2y - 35xy^2$.

Solution	Explanation
$7x^2y - 35xy^2 = 7xy(x - 5y)$	The common factor $7xy$ is 'taken out' of the brackets.

Grouping of terms

This method can be used for expressions containing four terms.

Example 10

Factorise $x^3 + 4x^2 - 3x - 12$.

Solution	Explanation
$x^3 + 4x^2 - 3x - 12 = (x^3 + 4x^2) - (3x + 12)$	The terms in this expression can be grouped as shown.
$= x^2(x + 4) - 3(x + 4)$ $= (x^2 - 3)(x + 4)$	The common factor $(x + 4)$ is 'taken out' of the brackets.

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Difference of two squares

You will recall the following identity from the previous section:

$$(a + b)(a - b) = a^2 - b^2$$

We can now use the result the other way in order to factorise:

$$a^2 - b^2 = (a + b)(a - b)$$

Example 11

a Factorise $3x^2 - 75$.

b Factorise $9x^2 - 36$.

Solution	Explanation
a $3x^2 - 75 = 3(x^2 - 25)$	First 'take out' the common factor 3.
$= 3(x + 5)(x - 5)$	Use the difference of squares identity.
b $9x^2 - 36 = 9(x^2 - 4)$	First 'take out' the common factor 9.
$= 9(x + 2)(x - 2)$	Use the difference of squares identity.

Example 12

Factorise $(x - y)^2 - 16y^2$.

Solution

$$(x - y)^2 - 16y^2 = (x - y)^2 - (4y)^2$$
$$= (x - y + 4y)(x - y - 4y)$$
$$= (x + 3y)(x - 5y)$$

Explanation

Use the difference of squares identity $a^2 - b^2 = (a + b)(a - b)$ with $a = (x - y)$ and $b = 4y$.

Factorising quadratic polynomials

A quadratic polynomial is an expression of the form $ax^2 + bx + c$ with $a \neq 0$. We have seen in the previous section that we can expand a product of two binomial factors to obtain a quadratic expression. For example:

$$(x + 2)(x - 4) = x(x - 4) + 2(x - 4)$$
$$= x^2 - 4x + 2x - 8$$
$$= x^2 - 2x - 8$$

We want to be able to reverse this process. That is, we want to start from the expanded expression and obtain the factorised form. We have already done this for expressions that are differences of squares. We now turn our attention to the general case.

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Example 13

Factorise $x^2 - 2x - 8$.

Solution

Using the method described in the explanation opposite, we can factorise without any further setting out:

$$x^2 - 2x - 8 = (x - 4)(x + 2)$$

Alternatively, we can reverse the process we used for expanding:

$$\begin{aligned} x^2 - 2x - 8 &= x^2 - 4x + 2x - 8 \\ &= x(x - 4) + 2(x - 4) \\ &= (x - 4)(x + 2) \end{aligned}$$

Explanation

We want

$$\begin{aligned} x^2 - 2x - 8 &= (x + a)(x + b) \\ &= x^2 + (a + b)x + ab \end{aligned}$$

The values of a and b are such that $ab = -8$ and $a + b = -2$.

Values of a and b which satisfy these two conditions are $a = -4$ and $b = 2$.

A quadratic polynomial is called a **monic quadratic polynomial** if the coefficient of x^2 is 1. The quadratic polynomial $x^2 - 2x - 8$ factorised in the previous example is monic.

Factorising non-monic quadratic polynomials involves a slightly different approach. We need to consider all possible combinations of factors of the x^2 term and the constant term. The next example and the following discussion give two methods.

Example 14

Factorise $6x^2 - 13x - 15$.

Solution

There are several combinations of factors of $6x^2$ and -15 to consider. Only one combination is correct.

$$6x^2 - 13x - 15 = (6x + 5)(x - 3)$$

Factors of $6x^2$	Factors of -15	'Cross-products' add to give $-13x$
$6x$	$+5$	$+5x$
x	-3	$-18x$
		$-13x$

Here is a second method for factorising $6x^2 - 13x - 15$ which still requires some trial and error but is more systematic. It is the reverse process of expanding $(x - 3)(6x + 5)$.

We let

$$ax^2 + bx + c = (\alpha x + \gamma)(\beta x + \delta)$$

Expanding the right-hand side gives

$$ax^2 + bx + c = \alpha\beta x^2 + (\gamma\beta + \alpha\delta)x + \gamma\delta$$

Note that $ac = \alpha\beta\gamma\delta$ and $b = \gamma\beta + \alpha\delta$.

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We now apply this to factorising $6x^2 - 13x - 15$.

First we look for two numbers that multiply together to give ac and add to give b. That is, we look for two numbers whose product is $6 \times (-15) = -90$ and whose sum is -13.

The two numbers are -18 and 5. We write:

$$\begin{aligned} 6x^2 - 13x - 15 &= 6x^2 - 18x + 5x - 15 \\ &= 6x(x - 3) + 5(x - 3) \\ &= (x - 3)(6x + 5) \end{aligned}$$

Example 15

Factorise $8x^2 + 2x - 15$.

Solution

$$\begin{aligned} 8x^2 + 2x - 15 &= 8x^2 + 12x - 10x - 15 \\ &= 4x(2x + 3) - 5(2x + 3) \\ &= (4x - 5)(2x + 3) \end{aligned}$$

Explanation

$ac = 8 \times (-15) = -120$ and $b = 2$.

The two numbers are 12 and -10. So we write $2x = 12x - 10x$.

It is sometimes possible to take out a common factor first to simplify the factorisation.

Example 16

Factorise $2x^2 + 6x - 20$.

Solution

$$\begin{aligned} 2x^2 + 6x - 20 &= 2(x^2 + 3x - 10) \\ &= 2(x + 5)(x - 2) \end{aligned}$$

Explanation

The common factor 2 is ‘taken out’ first.

Example 17

Factorise $(x + 1)^2 - 2(x + 1) - 3$.

Solution

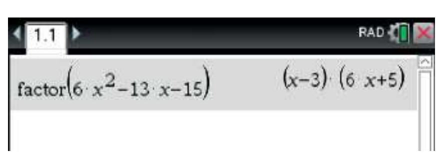
$$\begin{aligned} (x + 1)^2 - 2(x + 1) - 3 &= a^2 - 2a - 3 \\ &= (a - 3)(a + 1) \\ &= (x + 1 - 3)(x + 1 + 1) \\ &= (x - 2)(x + 2) \end{aligned}$$

Explanation

The substitution $a = x + 1$ makes it easier to recognise the required factorisation.

Using the TI-Nspire

To factorise the expression $6x^2 - 13x - 15$, use menu > **Algebra** > **Factor**.



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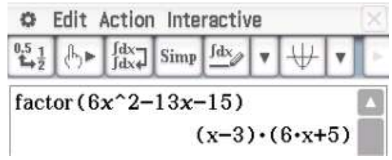
Using the Casio ClassPad

Enter the expression

$$6x^2 - 13x - 15$$

into Main $\sqrt{\alpha}$. Highlight the expression and select
Interactive > Transformation > factor.



Summary 3B

- Difference of two squares identity: $a^2 - b^2 = (a + b)(a - b)$.
- Factorisation of monic quadratics: To factorise a quadratic of the form $x^2 + bx + c$, find two numbers whose sum is the coefficient of x and whose product is the constant term.
- Factorisation of general quadratics: To factorise a quadratic of the form $ax^2 + bx + c$, find two numbers e and f whose product is ac and whose sum is b. Split the middle term bx as $ex + fx$ and then factorise by grouping.

Exercise 3B

1 Factorise each of the following:

a $2x + 4$ **b** $4a - 8$ **c** $6 - 3x$
d $2x - 10$ **e** $18x + 12$ **f** $24 - 16x$

Example 8

2 Factorise:

a $4x^2 - 2xy$ **b** $8ax + 32xy$ **c** $6ab - 12b$
d $6xy + 14x^2y$ **e** $x^2 + 2x$ **f** $5x^2 - 15x$
g $-4x^2 - 16x$ **h** $7x + 49x^2$ **i** $2x - x^2$

Example 9

3 Factorise:

a $6x^3y^2 + 12y^2x^2$ **b** $7x^2y - 6y^2x$ **c** $8x^2y^2 + 6y^2x$

Example 10

4 Factorise:

a $x^3 + 5x^2 + x + 5$ **b** $xy + 2x + 3y + 6$ **c** $x^2y^2 - x^2 - y^2 + 1$
d $ax + ay + bx + by$ **e** $a^3 - 3a^2 + a - 3$ **f** $2ab - 12a - 5b + 30$
g $2x^2 - 2x + 5x - 5$ **h** $x^3 - 4x + 2x^2 - 8$ **i** $x^3 - bx^2 - a^2x + a^2b$

Example 11

5 Factorise:

a $x^2 - 36$ **b** $x^2 - 81$ **c** $x^2 - a^2$
d $4x^2 - 81$ **e** $9x^2 - 16$ **f** $25x^2 - y^2$
g $3x^2 - 48$ **h** $2x^2 - 98$ **i** $3ax^2 - 27a$
j $a^2 - 7$ **k** $2a^2 - 5$ **l** $x^2 - 12$

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Example 12

6 Factorise:

a $(x-2)^2-16$ **b** $25-(2+x)^2$ **c** $3(x+1)^2-12$

d $(x-2)^2-(x+3)^2$ **e** $(2x-3)^2-(2x+3)^2$ **f** $(2x-1)^2-(3x+6)^2$

Example 13

7 Factorise:

a $x^2-7x-18$ **b** $y^2-19y+48$ **c** $a^2-14a+24$

d $a^2+18a+81$ **e** $x^2-5x-24$ **f** $x^2-2x-120$

Example 14
Example 15

8 Factorise:

a $3x^2-7x+2$ **b** $6x^2+7x+2$ **c** $5x^2+23x+12$

d $2x^2+9x+4$ **e** $6x^2-19x+10$ **f** $6x^2-7x-3$

g $12x^2-17x+6$ **h** $5x^2-4x-12$ **i** $5x^3-16x^2+12x$

Example 16

9 Factorise:

a $3y^2-12y-36$ **b** $2x^2-18x+28$ **c** $4x^2-36x+72$

d $3x^2+15x+18$ **e** $ax^2+7ax+12a$ **f** $48x-24x^2+3x^3$

Example 17

10 Factorise:

a $(x-1)^2+4(x-1)+3$

b $2(x-1)^2+5(x-1)-3$

c $(2x+1)^2+7(2x+1)+12$

3C Quadratic equations

This section looks at the solution of quadratic equations by simple factorisation. There are three steps to solving a quadratic equation by factorisation:

Step 1 Write the equation in the form $ax^2+bx+c=0$.

Step 2 Factorise the quadratic expression.

Step 3 Use the result that $mn=0$ implies $m=0$ or $n=0$ (or both); this is known as the **null factor theorem**.

For example, to solve the equation $x^2-x=12$:

$$\begin{aligned} x^2-x&=12 \\ x^2-x-12&=0 && \text{(Step 1)} \\ (x-4)(x+3)&=0 && \text{(Step 2)} \\ \therefore\quad x-4=0 \quad &\text{or} \quad x+3=0 && \text{(Step 3)} \\ x=4 \quad &\text{or} \quad x=-3 \end{aligned}$$

In the simplest cases, the first two steps may have been done already.

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Example 18

Solve $x^2 + 11x + 24 = 0$.

Solution

x^2	$+24$	$+11x$
x	$+3$	$+3x$
x	$+8$	$+8x$
		$+11x$

Factorising gives

$$x^2 + 11x + 24 = 0$$
$$(x + 3)(x + 8) = 0$$
$$\therefore \quad x + 3 = 0 \quad \text{or} \quad x + 8 = 0$$
$$x = -3 \quad \text{or} \quad x = -8$$

Explanation

The quadratic can also be factorised in the following way:

$$\begin{aligned} x^2 + 11x + 24 &= x^2 + 8x + 3x + 24 \\ &= x(x + 8) + 3(x + 8) \\ &= (x + 8)(x + 3) \end{aligned}$$

Note: We can check the answer for this example by substituting into the equation:

$$(-3)^2 + 11(-3) + 24 = 0$$
$$(-8)^2 + 11(-8) + 24 = 0$$

Example 19

Solve $2x^2 + 5x - 12 = 0$.

Solution

$2x^2$	-12	$+5x$
$2x$	-3	$-3x$
x	$+4$	$+8x$
		$+5x$

Factorising gives

$$2x^2 + 5x - 12 = 0$$
$$(2x - 3)(x + 4) = 0$$
$$\therefore \quad 2x - 3 = 0 \quad \text{or} \quad x + 4 = 0$$
$$x = \frac{3}{2} \quad \text{or} \quad x = -4$$

Explanation

The quadratic can also be factorised in the following way:

$$\begin{aligned} 2x^2 + 5x - 12 &= 2x^2 + 8x - 3x - 12 \\ &= 2x(x + 4) - 3(x + 4) \\ &= (2x - 3)(x + 4) \end{aligned}$$

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Applications of quadratic equations

Problems involving the solution of quadratic equations arise in many situations. We will meet more such problems in Section 3L.

Example 20

The perimeter of a rectangle is 20 cm and its area is 24 cm^2. Calculate the length and width of the rectangle.

Solution

Let x cm be the length of the rectangle and y cm the width.

Then $2(x + y) = 20$ and thus $y = 10 - x$.

The area is 24 cm^2 and therefore $xy = x(10 - x) = 24$.

i.e. $10x - x^2 = 24$

$x^2 - 10x + 24 = 0$

$(x - 6)(x - 4) = 0$

Thus the length is 6 cm or 4 cm. The width is 4 cm or 6 cm.

Summary 3C

To solve a quadratic equation by factorisation:

- **Step 1** Write the equation in the form $ax^2 + bx + c = 0$.
- **Step 2** Factorise the quadratic polynomial.
- **Step 3** Use the result that $mn = 0$ implies $m = 0$ or $n = 0$ (or both).

Skill-sheet

Exercise 3C

1 Solve each of the following for x:

a $(x - 2)(x - 3) = 0$ **b** $x(2x - 4) = 0$ **c** $(x - 4)(2x - 6) = 0$

d $(3 - x)(x - 4) = 0$ **e** $(2x - 6)(x + 4) = 0$ **f** $2x(x - 1) = 0$

g $(5 - 2x)(6 - x) = 0$ **h** $x^2 = 16$

2 Use a CAS calculator to solve each of the following equations. Give your answer correct to two decimal places.

a $x^2 - 4x - 3 = 0$ **b** $2x^2 - 4x - 3 = 0$ **c** $-2x^2 - 4x + 3 = 0$

Example 18

3 Solve for x in each of the following:

a $x^2 - x - 72 = 0$ **b** $x^2 - 6x + 8 = 0$ **c** $x^2 - 8x - 33 = 0$

d $x(x + 12) = 64$ **e** $x^2 + 5x - 14 = 0$ **f** $x^2 = 5x + 24$

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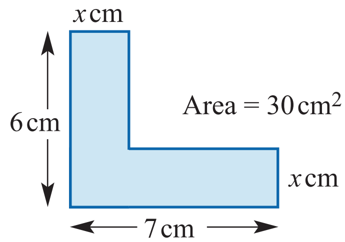
Example 19

4 Solve for x in each of the following:

a $2x^2 + 5x + 3 = 0$ **b** $4x^2 - 8x + 3 = 0$ **c** $6x^2 + 13x + 6 = 0$
d $2x^2 - x = 6$ **e** $6x^2 + 15 = 23x$ **f** $2x^2 - 3x - 9 = 0$
g $10x^2 - 11x + 3 = 0$ **h** $12x^2 + x = 6$ **i** $4x^2 + 1 = 4x$
j $x(x + 4) = 5$ **k** $\frac{1}{7}x^2 = \frac{3}{7}x$ **l** $x^2 + 8x = -15$
m $5x^2 = 11x - 2$

5 Calculate the value of x.



6 The bending moment, M, of a simple beam used in bridge construction is given by the formula

$$M = \frac{w\ell}{2}x - \frac{w}{2}x^2$$

If $\ell = 13$ m, $w = 16$ kg/m and $M = 288$ kg m, calculate the value of x.

7 The height, h metres, reached by a projectile after t seconds travelling vertically upwards is given by the formula $h = 70t - 16t^2$. Calculate t if h is 76 metres.

8 A polygon with n sides has $\frac{n(n-3)}{2}$ diagonals. How many sides has a polygon with 65 diagonals?

9 For a particular electric train, the tractive 'resistance' R at speed v km/h is given by $R = 1.6 + 0.03v + 0.003v^2$. Find v when the tractive resistance is 10.6.

Example 20

10 The perimeter of a rectangle is 16 cm and its area is 12 cm^2. Calculate the length and width of the rectangle.

11 The altitude of a triangle is 1 cm shorter than the base. If the area of the triangle is 15 cm^2, calculate the altitude.

12 Tickets for a concert are available at two prices. The more expensive ticket is \$30 more than the cheaper one. Find the cost of each type of ticket if a group can buy 10 more of the cheaper tickets than the expensive ones for \$1800.

13 The members of a club hire a bus for \$2100. Seven members withdraw from the club and the remaining members have to pay \$10 more each to cover the cost. How many members originally agreed to go on the bus?

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3D Graphing quadratics

A quadratic polynomial function is defined by the general rule

$$y = ax^2 + bx + c$$

where a, b and c are constants and $a \neq 0$. This is called **polynomial form**.

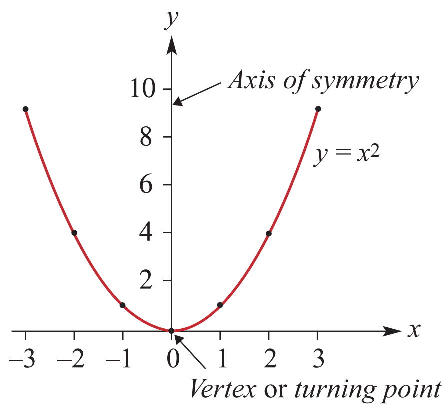
The parabola $y = x^2$

The simplest quadratic function is $y = x^2$. If a table of values is constructed for $y = x^2$ for $-3 \le x \le 3$, these points can be plotted and then connected to produce a continuous curve.

x	−3	−2	−1	0	1	2	3
y	9	4	1	0	1	4	9

Features of the graph of $y = x^2$:

- The graph is called a **parabola**.
- The possible y-values are all positive real numbers and 0. (This is called the **range** of the quadratic and is discussed in a more general context in Chapter 6.)
- The graph is symmetrical about the y-axis. The line about which the graph is symmetrical is called the **axis of symmetry**.
- The graph has a **vertex** or **turning point** at the origin $(0, 0)$.
- The minimum value of y is 0 and it occurs at the turning point.



Transformations of $y = x^2$

By a process called **completing the square** (to be discussed in Section 3E), all quadratics in polynomial form $y = ax^2 + bx + c$ may be transposed into what will be called the **turning point form**:

$$y = a(x - h)^2 + k$$

We first consider the effect of changing the value of a for our basic graph of $y = x^2$.

We then consider the effect of changing h and k for graphs of the form $y = ax^2$. Graphs of the form $y = a(x - h)^2 + k$ are formed by **translating** the graph of $y = ax^2$. The graph of $y = a(x - h)^2 + k$ is exactly the same shape as $y = ax^2$. All of these graphs are indeed congruent to $y = ax^2$ and each other.

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Graphs of $y = ax^2$

We first consider graphs of the form $y = ax^2$. In this case both $h = 0$ and $k = 0$. In the basic graph of $y = x^2$, the value of a is 1.

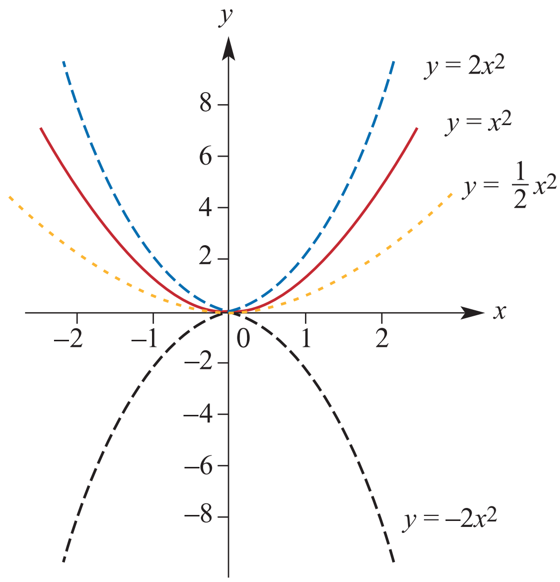
The following graphs are shown on the same set of axes:

$y = x^2$

$y = 2x^2 \quad (a = 2)$

$y = \frac{1}{2}x^2 \quad \left(a = \frac{1}{2}\right)$

$y = -2x^2 \quad (a = -2)$



If $a > 1$, the graph is 'narrower'. If $0 < a < 1$, the graph is 'broader'. The transformation which produces the graph of $y = 2x^2$ from the graph of $y = x^2$ is called a **dilation of factor 2 parallel to the *y*-axis**.

When a is negative, the graph is reflected in the x-axis. The transformation which produces the graph of $y = -x^2$ from the graph of $y = x^2$ is called a **reflection in the *x*-axis**.

Graphs of $y = x^2 + k$

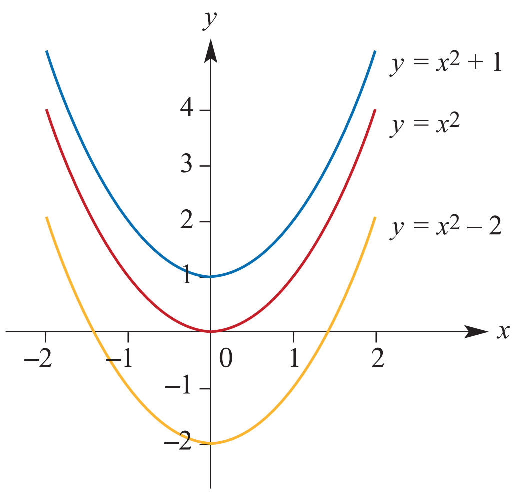
On this set of axes are the graphs of

$y = x^2$

$y = x^2 - 2 \quad (k = -2)$

$y = x^2 + 1 \quad (k = 1)$

As can be seen, changing k moves the basic graph of $y = x^2$ in a vertical direction.



- When $k = -2$ the graph is **translated** 2 units in the negative direction of the y-axis. The vertex is now $(0, -2)$ and the range is now all real numbers greater than or equal to -2.
- When $k = 1$ the graph is **translated** 1 unit in the positive direction of the y-axis. The vertex is now $(0, 1)$ and the range is now all real numbers greater than or equal to 1.

All other features of the graph are unchanged. The axis of symmetry is still the y-axis.

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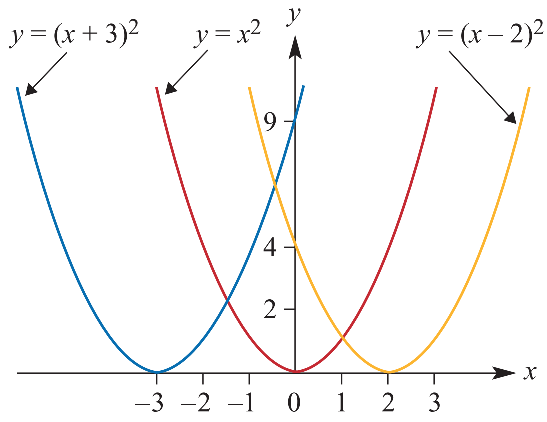
Graphs of $y = (x - h)^2$

On this set of axes are the graphs of

$y = x^2$

$y = (x-2)^2$ $(h = 2)$

$y = (x+3)^2$ $(h = -3)$



As can be seen, changing h moves the graph in a horizontal direction.

- When $h = 2$ the graph is **translated** 2 units in the positive direction of the x-axis. The vertex is now $(2, 0)$ and the axis of symmetry is now the line $x = 2$.
- When $h = -3$ the graph is **translated** 3 units in the negative direction of the x-axis. The vertex is now $(-3, 0)$ and the axis of symmetry is now the line $x = -3$.

In both cases, the range is unchanged and is still all non-negative real numbers.

Examples of transformations

By combining dilations, reflections and translations, we can sketch the graph of any quadratic expressed in the form $y = a(x-h)^2 + k$:

- The vertex is the point (h, k).
- The axis of symmetry is $x = h$.
- If h and k are positive, then the graph of $y = a(x-h)^2 + k$ is obtained from the graph of $y = ax^2$ by translating h units in the positive direction of the x-axis and k units in the positive direction of the y-axis.
- Similar results hold for different combinations of h and k positive and negative.

Example 21

Sketch the graph of $y = x^2 - 3$.

Solution

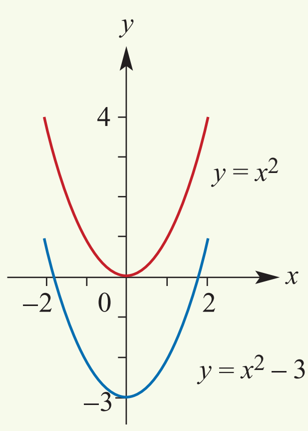
The graph of $y = x^2 - 3$ is obtained from the graph of $y = x^2$ by translating 3 units in the negative direction of the y-axis.

The vertex is now at $(0, -3)$. The axis of symmetry is the line with equation $x = 0$.

To find the x-axis intercepts, let $y = 0$:

$$0 = x^2 - 3$$
$$x^2 = 3$$
$$\therefore \quad x = \pm\sqrt{3}$$

Hence the x-axis intercepts are $\pm\sqrt{3}$.



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Example 22

Sketch the graph of $y = -(x + 1)^2$.

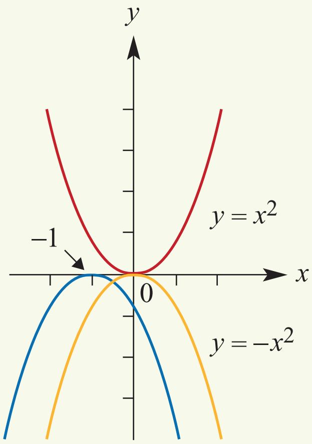
Solution

The graph of $y = -(x + 1)^2$ is obtained from the graph of $y = x^2$ by a reflection in the x-axis followed by a translation of 1 unit in the negative direction of the x-axis.

The vertex is now at $(-1, 0)$.

The axis of symmetry is the line with equation $x = -1$.

The x-axis intercept is -1.



Example 23

Sketch the graph of $y = 2(x - 1)^2 + 3$.

Solution

The graph of $y = 2x^2$ is translated 1 unit in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

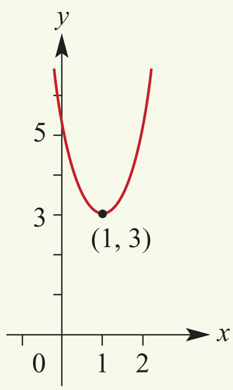
The vertex has coordinates $(1, 3)$.

The axis of symmetry is the line $x = 1$.

The graph will be narrower than $y = x^2$.

The range will be $y \geq 3$.

To add further detail to our graph, we can find the axis intercepts:



y-axis intercept

When $x = 0$, $y = 2(0 - 1)^2 + 3 = 5$.

x-axis intercepts

In this example, the minimum value of y is 3, and so y cannot be 0. Therefore this graph has no x-axis intercepts.

Note: Another way to see this is to let $y = 0$ and try to solve for x:

$$0 = 2(x - 1)^2 + 3$$
$$-3 = 2(x - 1)^2$$
$$-\frac{3}{2} = (x - 1)^2$$

As the square root of a negative number is not a real number, this equation has no real solutions.

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Example 24

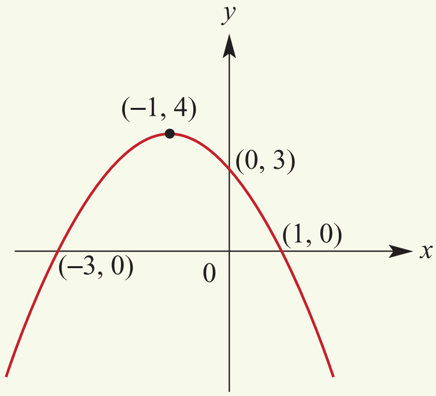
Sketch the graph of $y = -(x + 1)^2 + 4$.

Solution

The vertex has coordinates $(-1, 4)$ and so the axis of symmetry is the line $x = -1$.

When $x = 0$, $y = -(0 + 1)^2 + 4 = 3$.
$\therefore$ the y-axis intercept is 3.

When $y = 0$,
$$-(x + 1)^2 + 4 = 0$$
$$(x + 1)^2 = 4$$
$$x + 1 = \pm 2$$
$$x = \pm 2 - 1$$
$\therefore$ the x-axis intercepts are 1 and -3.



Summary 3D

- The graph of $y = x^2$ is called a parabola. The vertex (or turning point) is the point $(0, 0)$ and the axis of symmetry is the y-axis.
- The graph of $y = -x^2$ is the reflection of the graph of $y = x^2$ in the x-axis.
- For $y = ax^2$ and $a > 1$, the graph is 'narrower' than the graph of $y = x^2$.
- For $y = ax^2$ and $0 < a < 1$, the graph is 'broader' than the graph of $y = x^2$.
- All quadratic functions in polynomial form $y = ax^2 + bx + c$ may be transposed into the turning point form $y = a(x - h)^2 + k$.
- The graph of $y = a(x - h)^2 + k$ is a parabola congruent to the graph of $y = ax^2$.
  - The vertex (or turning point) is the point (h, k).
  - The axis of symmetry is $x = h$.
  - If h and k are positive numbers, then the graph of $y = a(x - h)^2 + k$ is obtained from the graph of $y = ax^2$ by translating h units in the positive direction of the x-axis and k units in the positive direction of the y-axis.
  - Similar results hold for different combinations of h and k positive and negative.

Exercise 3D

For each of the following, find:

i the coordinates of the turning point
ii the axis of symmetry
iii the x-axis intercepts (if any)

and use this information to help sketch the graph.

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Example 21

1 **a** $y = x^2 - 4$ **b** $y = x^2 + 2$ **c** $y = -x^2 + 3$

d $y = -2x^2 + 5$ **e** $y = -x^2 + 4$ **f** $y = 3x^2 - 9$

Example 22

2 **a** $y = (x - 2)^2$ **b** $y = (x + 3)^2$ **c** $y = -(x + 1)^2$ **d** $y = -\frac{1}{2}(x - 4)^2$

Example 23

Example 24

3 **a** $y = (x - 2)^2 + 1$ **b** $y = (x - 2)^2 - 1$ **c** $y = (x - 1)^2 + 2$

d $y = (x + 1)^2 - 1$ **e** $y = -(x - 3)^2 + 1$ **f** $y = (x + 2)^2 - 4$

g $y = 2(x + 2)^2 - 18$ **h** $y = -3(x - 4)^2 + 3$ **i** $y = -\frac{1}{2}(x + 5)^2 - 2$

j $y = 3(x + 2)^2 - 12$ **k** $y = -4(x - 2)^2 + 8$ **l** $y = \frac{1}{3}(x - 1)^2 - 3$

3E Completing the square and turning points

In order to use the techniques from the previous section for sketching quadratics, it is necessary for the quadratic to be expressed in **turning point form**. This can be done by two different but related methods: by completing the square and by using the equation of the axis of symmetry.

Completing the square

To transpose a quadratic in polynomial form we can **complete the square**.

Consider the expansion of a perfect square:

$$(x + a)^2 = x^2 + 2ax + a^2$$

The last term of the expansion is the square of half the coefficient of the middle term.

Now consider the quadratic polynomial:

$$x^2 + 2x - 3$$

This is not a perfect square. However, by adding and subtracting a new term, we can form a perfect square as part of a new expression for the same polynomial.

We have that

$$x^2 + 2x + 1 = (x + 1)^2$$

which is a perfect square. In order to keep our original quadratic 'intact', we both add and subtract the 'correct' new term. For example:

$$x^2 + 2x - 3 = (x^2 + 2x + 1) - 1 - 3$$

This can now be simplified to

$$(x + 1)^2 - 4$$

Hence the quadratic $y = x^2 + 2x - 3$ is expressed in turning point form as $y = (x + 1)^2 - 4$, and so the **vertex** (turning point) of its graph is the point with coordinates $(-1, -4)$.

In the above example, the coefficient of x^2 was 1. If the coefficient is not 1, this coefficient must first be 'factored out' before proceeding to complete the square.

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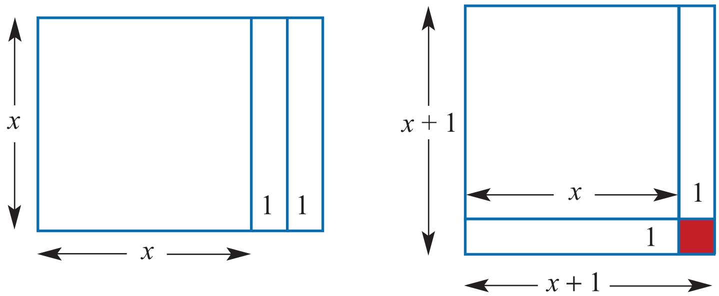
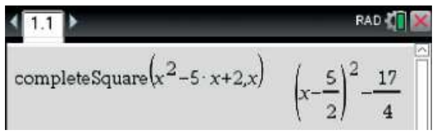
A geometric representation of completing the square

Completing the square for $x^2 + 2x$ is represented in the following diagrams. The diagram on the left shows $x^2 + 2x$. The small rectangle to the right is moved to the 'base' of the x by x square. The red square of area 1 unit is added. Thus $x^2 + 2x + 1 = (x + 1)^2$.



Using the TI-Nspire

Use (menu) > **Algebra** > **Complete the Square** to rearrange the expression $x^2 - 5x + 2$.



Solving equations by completing the square

The process of completing the square can also be used for the solution of equations.

Example 25

Solve each of the following equations for x by first completing the square:

a $x^2 - 3x + 1 = 0$ **b** $2x^2 - 3x - 1 = 0$

Solution

a Completing the square:

$$x^2 - 3x + 1 = 0$$

$$x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 1 = 0$$

$$\left(x - \frac{3}{2}\right)^2 - \frac{5}{4} = 0$$

$$\left(x - \frac{3}{2}\right)^2 = \frac{5}{4}$$

Therefore $x - \frac{3}{2} = \pm\frac{\sqrt{5}}{2}$

and so $x = \frac{3}{2} \pm \frac{\sqrt{5}}{2} = \frac{3 \pm \sqrt{5}}{2}$

Explanation

$\frac{1}{2} \times (-3) = -\frac{3}{2}$

We add and subtract $\left(-\frac{3}{2}\right)^2 = \frac{9}{4}$ on the left-hand side of the equation.

This gives an equivalent expression to the expression on the left-hand side.

Solve the equation as shown.

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b Completing the square:

$$2x^2 - 3x - 1 = 0$$

$$2\left(x^2 - \frac{3}{2}x - \frac{1}{2}\right) = 0$$

$$x^2 - \frac{3}{2}x + \left(-\frac{3}{4}\right)^2 - \left(-\frac{3}{4}\right)^2 - \frac{1}{2} = 0$$

$$\left(x - \frac{3}{4}\right)^2 = \frac{17}{16}$$

Therefore $x - \frac{3}{4} = \pm\frac{\sqrt{17}}{4}$

and so $x = \frac{3}{4} \pm \frac{\sqrt{17}}{4} = \frac{3 \pm \sqrt{17}}{4}$

Divide both sides by 2 before completing the square.

$\frac{1}{2} \times \left(-\frac{3}{2}\right) = -\frac{3}{4}$

We add and subtract $\left(-\frac{3}{4}\right)^2 = \frac{9}{16}$ on the left-hand side of the equation.

Sketching the graph of a quadratic polynomial after completing the square

Completing the square enables the quadratic rule to be written in turning point form. We have seen that this can be used to sketch the graphs of quadratic polynomials.

Example 26

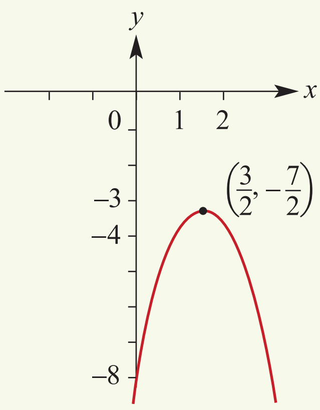
Find the coordinates of the vertex by completing the square and hence sketch the graph of $y = -2x^2 + 6x - 8$.

Solution

Take out -2 as a common factor and then complete the square:

$$\begin{aligned} y &= -2x^2 + 6x - 8 \\ &= -2(x^2 - 3x + 4) \\ &= -2\left(x^2 - 3x + \left(-\frac{3}{2}\right)^2 - \left(-\frac{3}{2}\right)^2 + 4\right) \\ &= -2\left(\left(x - \frac{3}{2}\right)^2 - \frac{9}{4} + 4\right) \\ &= -2\left(\left(x - \frac{3}{2}\right)^2 + \frac{7}{4}\right) \end{aligned}$$

$$\therefore \quad y = -2\left(x - \frac{3}{2}\right)^2 - \frac{7}{2}$$



Therefore the vertex is $\left(\frac{3}{2}, -\frac{7}{2}\right)$ and the axis of symmetry is $x = \frac{3}{2}$.

The y-axis intercept is -8.

The graph has maximum value of $-\frac{7}{2}$, and so there are no x-axis intercepts.

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The equation for the axis of symmetry of a parabola

We first complete the square for $y = ax^2 + bx + c$:

$$\begin{aligned} y &= ax^2 + bx + c \\ &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right) \qquad \text{completing the square} \\ &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c \end{aligned}$$

For a quadratic function written in polynomial form $y = ax^2 + bx + c$, the axis of symmetry of its graph has the equation $x = -\dfrac{b}{2a}$.

Therefore the x-coordinate of the turning point is $-\dfrac{b}{2a}$. Substitute this value into the quadratic polynomial to find the y-coordinate of the turning point.

Example 27

Use the axis of symmetry to find the turning point of the graph and hence express in turning point form:

a $y = x^2 - 4x + 3$ **b** $y = -2x^2 + 12x - 7$

Solution

a The x-coordinate of the turning point is 2.

When $x = 2$, $y = 4 - 8 + 3 = -1$.

The coordinates of the turning point are $(2, -1)$.

Hence the equation is $y = (x - 2)^2 - 1$.

Explanation

Here $a = 1$ and $b = -4$, so the axis of symmetry is $x = -\left(\dfrac{-4}{2}\right) = 2$.

For the turning point form $y = a(x - h)^2 + k$, we have found that $a = 1$, $h = 2$ and $k = -1$.

Solution

b The x-coordinate of the turning point is 3.

When $x = 3$, $y = -2 \times (3)^2 + 12 \times 3 - 7 = 11$.

The coordinates of the turning point are $(3, 11)$.

Hence the equation is $y = -2(x - 3)^2 + 11$.

Explanation

Here $a = -2$ and $b = 12$, so the axis of symmetry is $x = -\left(\dfrac{12}{-4}\right) = 3$.

For the turning point form $y = a(x - h)^2 + k$, we have found that $a = -2$, $h = 3$ and $k = 11$.

Summary 3E

- Quadratic equations can be solved by completing the square. The method of completing the square allows us to deal with all quadratic equations, even though there may be no solution for some quadratic equations.

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- To complete the square of $x^2 + bx + c$:
  - Take half the coefficient of x (that is, $\frac{b}{2}$) and add and subtract its square $\frac{b^2}{4}$.
- To complete the square of $ax^2 + bx + c$:
  - First take out a as a factor and then complete the square inside the bracket.
- The axis of symmetry of the graph of the quadratic function $y = ax^2 + bx + c$ is $x = -\frac{b}{2a}$.
- To convert the quadratic function $y = ax^2 + bx + c$ into turning point form using the axis of symmetry:
  1. The x-coordinate h of the vertex of the parabola is $-\frac{b}{2a}$.
  2. Find the y-coordinate k of the vertex by substituting in $y = ax^2 + bx + c$.
  3. Substitute these values for h and k in $y = a(x - h)^2 + k$.

Skill-sheet

Exercise 3E

1 Expand each of the following:

a $(x - 1)^2$ **b** $(x + 2)^2$ **c** $(x - 3)^2$ **d** $(-x + 3)^2$

e $(-x - 2)^2$ **f** $(x - 5)^2$ **g** $\left(x - \frac{1}{2}\right)^2$ **h** $\left(x - \frac{3}{2}\right)^2$

2 Factorise each of the following:

a $x^2 - 4x + 4$ **b** $x^2 - 12x + 36$ **c** $-x^2 + 4x - 4$ **d** $2x^2 - 8x + 8$

e $-2x^2 + 12x - 18$ **f** $x^2 - x + \frac{1}{4}$ **g** $x^2 - 3x + \frac{9}{4}$ **h** $x^2 + 5x + \frac{25}{4}$

Example 25

3 Solve each of the following equations for x by first completing the square:

a $x^2 - 2x - 1 = 0$ **b** $x^2 - 4x - 2 = 0$ **c** $x^2 - 6x + 2 = 0$

d $x^2 - 5x + 2 = 0$ **e** $2x^2 - 4x + 1 = 0$ **f** $3x^2 - 5x - 2 = 0$

g $x^2 + 2x + k = 0$ **h** $kx^2 + 2x + k = 0$ **i** $x^2 - 3kx + 1 = 0$

Example 26

4 Express each of the following in the form $y = a(x - h)^2 + k$ by completing the square. Hence state the coordinates of the turning point and sketch the graph in each case.

a $y = x^2 - 2x + 3$ **b** $y = x^2 + 4x + 1$ **c** $y = x^2 - 3x + 1$

5 Express each of the following in the form $y = a(x - h)^2 + k$ by completing the square. Hence state the coordinates of the turning point and sketch the graph in each case.

a $y = 2x^2 - 2x - 5$ **b** $y = 4x^2 + 8x + 8$ **c** $y = 3x^2 - 6x - 4$

Example 27

6 Express each of the following in the form $y = a(x - h)^2 + k$ using the axis of symmetry. Hence state the coordinates of the turning point and sketch the graph in each case.

a $y = x^2 - 8x + 12$ **b** $y = x^2 - x - 2$ **c** $y = 2x^2 + 4x - 2$

d $y = -x^2 + 4x + 1$ **e** $y = -2x^2 - 12x - 12$ **f** $y = 3x^2 - 6x + 12$

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3F Graphing quadratics in polynomial form

It is not always essential to convert a quadratic to turning point form in order to sketch its graph. We can sometimes find the x- and y-axis intercepts and the axis of symmetry from polynomial form by other methods and use these details to sketch the graph.

Step 1 Find the y-axis intercept

Let $x = 0$. For the general quadratic $y = ax^2 + bx + c$, this gives

$$y = a(0)^2 + b(0) + c$$
$$y = c$$

Hence the y-axis intercept is always equal to c.

Step 2 Find the x-axis intercepts

Let $y = 0$. In general, this gives

$$0 = ax^2 + bx + c$$

In order to solve such an equation it is necessary to factorise the right-hand side and then use the **null factor theorem**.

Step 3 Find the equation of the axis of symmetry

Once the x-axis intercepts have been found, the equation of the axis of symmetry can be found by using the symmetry properties of the parabola. The axis of symmetry is the perpendicular bisector of the line segment joining the x-axis intercepts.

Step 4 Find the coordinates of the turning point

The axis of symmetry gives the x-coordinate of the turning point. Substitute this into the quadratic polynomial to obtain the y-coordinate.

Example 28

Find the x- and y-axis intercepts and the turning point, and hence sketch the graph of $y = x^2 - 4x$.

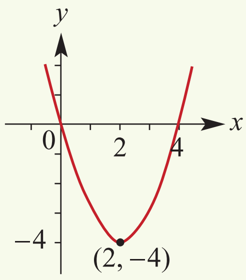
Solution

Step 1 $c = 0$. Therefore the y-axis intercept is 0.

Step 2 Let $y = 0$. Then

$$0 = x^2 - 4x$$
$$0 = x(x - 4)$$
$$\therefore \quad x = 0 \quad \text{or} \quad x = 4$$

The x-axis intercepts are 0 and 4.



Step 3 The axis of symmetry is the line with equation $x = \dfrac{0 + 4}{2}$, that is, $x = 2$.

Step 4 When $x = 2$, $y = (2)^2 - 4(2) = -4$. The turning point has coordinates $(2, -4)$.

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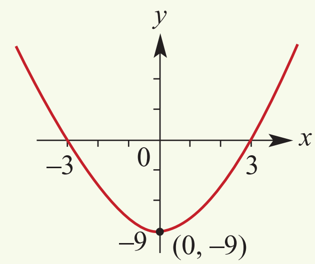
Example 29

Find the x- and y-axis intercepts and the turning point, and hence sketch the graph of $y = x^2 - 9$.

Solution

Step 1 $c = -9$. Therefore the y-axis intercept is -9.



Step 2 Let $y = 0$. Then

$$0 = x^2 - 9$$
$$0 = (x+3)(x-3)$$
$$\therefore \quad x = -3 \text{ or } x = 3$$

The x-axis intercepts are -3 and 3.

Step 3 The axis of symmetry is the line with equation $x = \dfrac{-3+3}{2}$, that is, $x = 0$.

Step 4 When $x = 0$, $y = (0)^2 - 9 = -9$

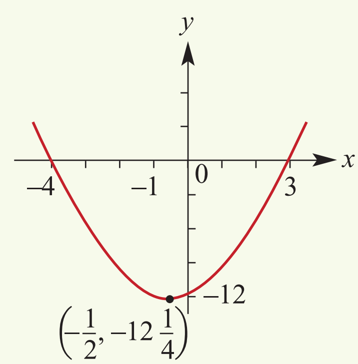
The turning point has coordinates $(0, -9)$.

Example 30

Find the x- and y-axis intercepts and the turning point, and hence sketch the graph of $y = x^2 + x - 12$.

Solution

Step 1 $c = -12$. Therefore the y-axis intercept is -12.



Step 2 Let $y = 0$. Then

$$0 = x^2 + x - 12$$
$$0 = (x+4)(x-3)$$
$$\therefore \quad x = -4 \text{ or } x = 3$$

The x-axis intercepts are -4 and 3.

Step 3 The axis of symmetry is the line with equation $x = \dfrac{-4+3}{2} = -\dfrac{1}{2}$

Step 4 When $x = -\dfrac{1}{2}$, $y = \left(-\dfrac{1}{2}\right)^2 + \left(-\dfrac{1}{2}\right) - 12$

$$= -12\frac{1}{4}$$

The turning point has coordinates $\left(-\dfrac{1}{2}, -12\dfrac{1}{4}\right)$.

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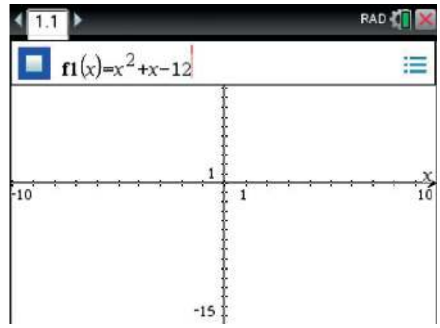
Using the TI-Nspire

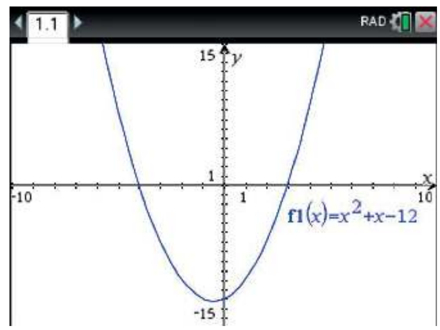
To graph the quadratic function with rule $y = x^2 + x - 12$:

- Enter the rule in the entry line of a **Graphs** application as shown, and press enter.



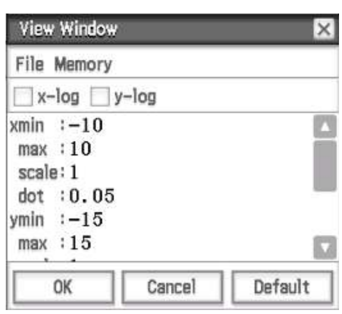
- Using menu > **Window/Zoom** > **Window Settings**, select the window settings $-10 \le x \le 10$ and $-15 \le y \le 15$ to obtain the graph shown.

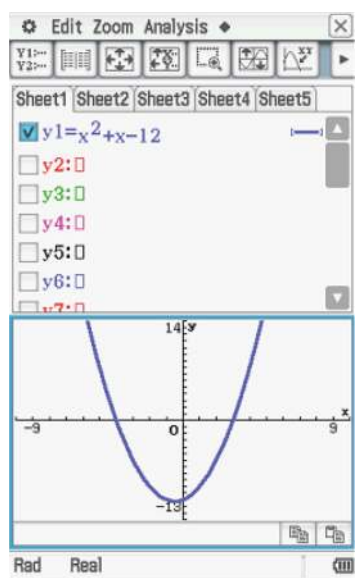
Note: You can also double click on the end values to change the window settings.



Using the Casio ClassPad

- Open the menu Menu; select **Graph & Table**.
- Type the expression $x^2 + x - 12$ in $y1$.
- Tick the box and tap the graph icon.
- It may be necessary to change the view window by using and the settings shown below.





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Summary 3F

Steps for sketching the graph of a quadratic function given in polynomial form:

Step 1 Find the y-axis intercept.

Step 2 Find the x-axis intercepts.

Step 3 Find the equation of the axis of symmetry.

Step 4 Find the coordinates of the turning point.

Exercise 3F

1 **a** A parabola has x-axis intercepts 4 and 10. State the x-coordinate of the vertex.

b A parabola has x-axis intercepts 6 and 8. State the x-coordinate of the vertex.

c A parabola has x-axis intercepts -6 and 8. State the x-coordinate of the vertex.

2 **a** A parabola has vertex $(2, -6)$ and one of the x-axis intercepts is at 6. Find the other x-axis intercept.

b A parabola has vertex $(2, -6)$ and one of the x-axis intercepts is at -4. Find the other x-axis intercept.

c A parabola has vertex $(2, -6)$ and one of the x-axis intercepts is at the origin. Find the other x-axis intercept.

Example 28
Example 29

3 Sketch each of the following parabolas, clearly showing the axis intercepts and the turning point:

a $y = x^2 - 1$
b $y = x^2 + 6x$
c $y = 25 - x^2$
d $y = x^2 - 4$
e $y = 2x^2 + 3x$
f $y = 2x^2 - 4x$
g $y = -2x^2 - 3x$
h $y = x^2 + 1$

Example 30

4 Sketch each of the following parabolas, clearly showing the axis intercepts and the turning point:

a $y = x^2 + 3x - 10$
b $y = x^2 - 5x + 4$
c $y = x^2 + 2x - 3$
d $y = x^2 + 4x + 3$
e $y = 2x^2 - x - 1$
f $y = 6 - x - x^2$
g $y = -x^2 - 5x - 6$
h $y = x^2 - 5x - 24$

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3G Solving quadratic inequalities

In Chapter 1 we looked at solving linear inequalities. The situation is a little more complex for quadratic inequalities. We suggest one possible approach.

To solve a quadratic inequality (for example, $x^2 + x - 12 > 0$):

Step 1 Solve the corresponding equation (for example, $x^2 + x - 12 = 0$).

Step 2 Sketch the graph of the quadratic polynomial (for example, $y = x^2 + x - 12$).

Step 3 Use the graph to determine the set of x-values which satisfy the inequality.

Example 31

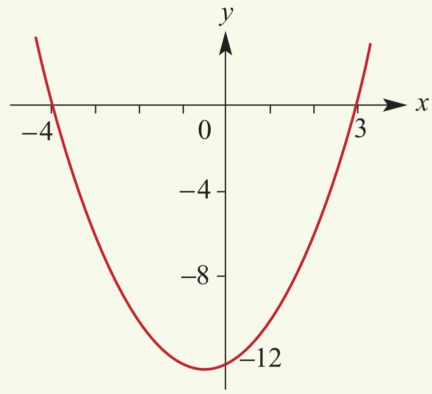
Solve $x^2 + x - 12 > 0$.

Solution

Step 1 Solve the equation

$$x^2 + x - 12 = 0$$
$$(x + 4)(x - 3) = 0$$
$$\therefore \quad x = -4 \text{ or } x = 3$$

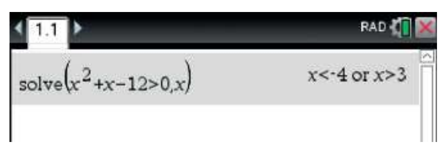
Step 2 Sketch the graph of the quadratic $y = x^2 + x - 12$.



Step 3 From the graph it can be seen that $x^2 + x - 12 > 0$ when $x < -4$ or $x > 3$.

Using the TI-Nspire

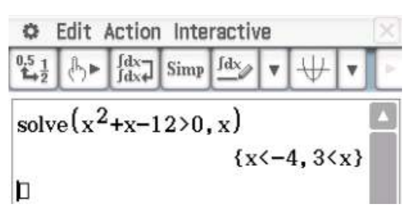
The calculator may be used to solve quadratic inequalities.



Using the Casio ClassPad

To solve the inequality $x^2 + x - 12 > 0$:

- Enter the inequality in the main screen Main $\sqrt{\alpha}$. (The inequality symbol can be found in the Math3 keyboard.)
- Highlight the inequality using the stylus.
- Select **Interactive** > **Equation/Inequality** > **solve** and ensure the variable is x.
- Tap on OK to obtain the solution.



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Summary 3G

When solving quadratic inequalities of the form $ax^2 + bx + c \le 0$ (or with $\ge$, $>$ or $<$), it is best to sketch the graph of $y = ax^2 + bx + c$.

Skill-sheet

Exercise 3G

Example 31

1 **a** Solve the equation $x^2 - 2x - 8 = 0$.
b Sketch the graph of $y = x^2 - 2x - 8$.
c Solve the inequality $x^2 - 2x - 8 \le 0$.
d Solve the inequality $x^2 - 2x - 8 > 0$.

2 Solve each of the following inequalities:

a $(x - 3)(x + 2) \ge 0$
b $(x + 4)(x + 3) < 0$
c $(2x - 1)(x + 4) \le 0$
d $(x - 6)(2x - 4) > 0$
e $(2x - 6)(2x - 4) < 0$
f $(7 - 2x)(2x - 3) \ge 0$
g $(2x + 7)(2x - 4) < 0$
h $(3x + 6)(2x - 5) \le 0$
i $(5 - 2x)(5 + x) < 0$
j $(7 - 2x)(x + 2) \ge 0$
k $(7 - 2x)(5x - 2) < 0$
l $(11 - 2x)(5 - 2x) \ge 0$

3 Solve each of the following inequalities:

a $(5 - x)(5 + x) < 0$
b $4 - 9y^2 \ge 0$
c $16 - y^2 < 0$
d $36 - 25x^2 \ge 0$
e $1 - 16y^2 \le 0$
f $25 - 36y^2 < 0$

4 Solve each of the following inequalities:

a $x^2 + 2x - 8 \ge 0$
b $x^2 - 5x - 24 < 0$
c $x^2 - 4x - 12 \le 0$
d $2x^2 - 3x - 9 > 0$
e $6x^2 + 13x < -6$
f $-x^2 - 5x - 6 \ge 0$
g $12x^2 + x > 6$
h $10x^2 - 11x \le -3$
i $x(x - 1) \le 20$
j $4 + 5p - p^2 \ge 0$
k $3 + 2y - y^2 < 0$
l $x^2 + 3x \ge -2$

5 Solve each of the following inequalities:

a $x^2 + 3x - 5 \ge 0$
b $x^2 - 5x + 2 < 0$
c $2x^2 - 3x - 1 \le 0$
d $8 - 3x - x^2 > 0$
e $2x^2 + 7x + 1 < 0$
f $2x^2 - 8x + 5 \ge 0$

6 Explain why $(x - 3)^2 \ge 0$ for all x.

7 Explain why $-(x - 1)^2 \le 0$ for all x.

8 Complete the square for $x^2 + 2x + 7$ and hence show that $x^2 + 2x + 7 \ge 6$ for all x.

9 Complete the square for $-x^2 - 2x - 7$ and hence show that $-x^2 - 2x - 7 \le -6$ for all x.

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3H The general quadratic formula

Not all quadratics can be factorised by inspection, and it is often difficult to find the x-axis intercepts this way. There is a general formula for finding the solutions of a quadratic equation in polynomial form. This formula comes from 'completing the square' for the general quadratic.

In Section 3E we showed that

$$\begin{aligned} y &= ax^2 + bx + c \\ &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c \end{aligned}$$

We can use this to solve the general quadratic equation:

$$\begin{aligned} ax^2 + bx + c &= 0 \\ a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c &= 0 \\ a\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a} - c \end{aligned}$$

Now divide both sides by a:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\therefore \quad x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note: The quadratic formula provides an alternative method for solving quadratic equations to 'completing the square', but it is probably not as useful for curve sketching as 'completing the square', which gives the turning point coordinates directly.

It should be noted that the equation of the axis of symmetry can be derived from this general formula: the axis of symmetry is the line with equation

$$x = -\frac{b}{2a}$$

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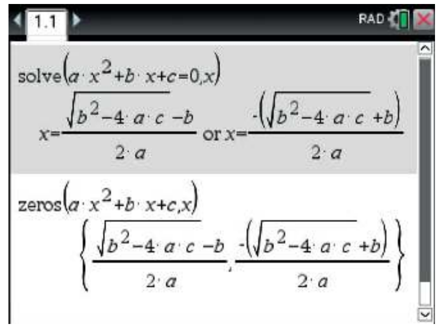
Also, from the formula it can be seen that:

- If $b^2 - 4ac > 0$, there are two solutions.
- If $b^2 - 4ac = 0$, there is one solution.
- If $b^2 - 4ac < 0$, there are no real solutions.

This will be further explored in the next section.

A CAS calculator gives the result shown opposite.



Example 32

Solve each of the following equations for x by using the quadratic formula:

a $x^2 - x - 1 = 0$ **b** $x^2 - 2kx - 3 = 0$

Solution

a $x^2 - x - 1 = 0$

Here $a = 1$, $b = -1$ and $c = -1$.

The formula gives

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times (-1)}}{2 \times 1}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

b $x^2 - 2kx - 3 = 0$

Here $a = 1$, $b = -2k$ and $c = -3$.

The formula gives

$$x = \frac{-(-2k) \pm \sqrt{(-2k)^2 - 4 \times 1 \times (-3)}}{2 \times 1}$$

$$= \frac{2k \pm \sqrt{4k^2 + 12}}{2}$$

$$= k \pm \sqrt{k^2 + 3}$$

Explanation

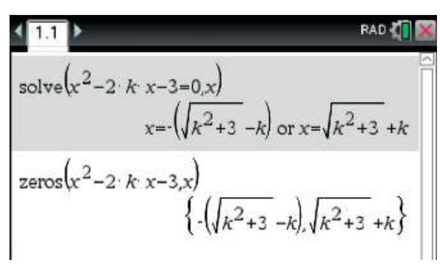
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note that $k^2 + 3 \geq 3$ for all values of k, since $k^2 \geq 0$.

Using the TI-Nspire

- Use (menu) > **Algebra** > **Solve** to solve the equation $x^2 - 2kx - 3 = 0$ for x.
- Alternatively, use (menu) > **Algebra** > **Zeros**.

Note: You must use a multiplication sign between the k and x.



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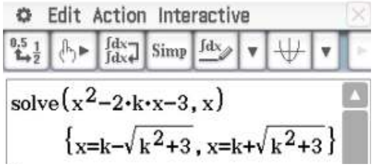
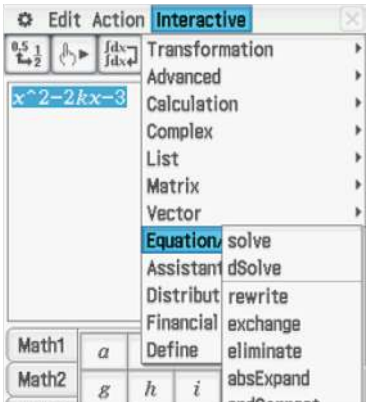
Using the Casio ClassPad

To solve the equation $x^2 - 2kx - 3 = 0$ for x:

- Enter and highlight the equation. (Use the Var keyboard to enter the variables.)
- Select **Interactive > Equation/Inequality > solve** and set the variable to x.





Example 33

Sketch the graph of $y = -3x^2 - 12x - 7$. Use the quadratic formula to calculate the x-axis intercepts.

Solution

Since $c = -7$, the y-axis intercept is -7.

Axis of symmetry
$$x = -\frac{b}{2a} = -\left(\frac{-12}{2 \times (-3)}\right) = -2$$

Turning point

When $x = -2$, $y = -3(-2)^2 - 12(-2) - 7 = 5$. The turning point coordinates are $(-2, 5)$.

x-axis intercepts

$$-3x^2 - 12x - 7 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

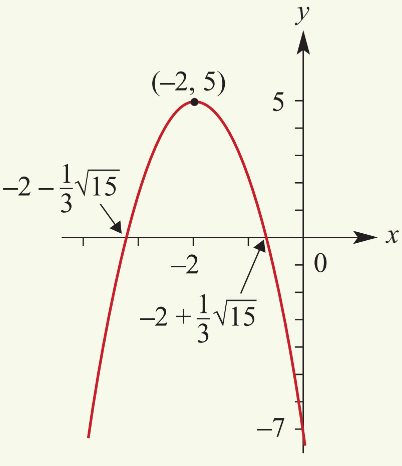
$$= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(-3)(-7)}}{2(-3)}$$

$$= \frac{12 \pm \sqrt{60}}{-6}$$

$$= \frac{12 \pm 2\sqrt{15}}{-6}$$

$$= -2 \pm \frac{1}{3}\sqrt{15}$$

$$\approx -3.29 \text{ or } -0.71 \quad \text{(to two decimal places)}$$



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Summary 3H

The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

From the formula it can be seen that:

- If $b^2 - 4ac > 0$, there are two solutions.
- If $b^2 - 4ac = 0$, there is one solution.
- If $b^2 - 4ac < 0$, there are no real solutions.

Exercise 3H

1 For each of the following, the coefficients a, b and c of a quadratic $y = ax^2 + bx + c$ are given. Find:

i $b^2 - 4ac$ **ii** $\sqrt{b^2 - 4ac}$ in simplest surd form

a $a = 2$, $b = 4$ and $c = -3$
b $a = 1$, $b = 10$ and $c = 18$
c $a = 1$, $b = 10$ and $c = -18$
d $a = -1$, $b = 6$ and $c = 15$
e $a = 1$, $b = 9$ and $c = -27$

2 Simplify each of the following:

a $\dfrac{2 + 2\sqrt{5}}{2}$ **b** $\dfrac{9 - 3\sqrt{5}}{6}$ **c** $\dfrac{5 + 5\sqrt{5}}{10}$ **d** $\dfrac{6 + 12\sqrt{2}}{6}$

Example 32

3 Solve each of the following for x. Give exact answers.

a $x^2 + 6x = 4$
b $x^2 - 7x - 3 = 0$
c $2x^2 - 5x + 2 = 0$
d $2x^2 + 4x - 7 = 0$
e $2x^2 + 8x = 1$
f $5x^2 - 10x = 1$
g $-2x^2 + 4x - 1 = 0$
h $2x^2 + x = 3$
i $2.5x^2 + 3x + 0.3 = 0$
j $-0.6x^2 - 1.3x = 0.1$
k $2kx^2 - 4x + k = 0$
l $2(1 - k)x^2 - 4kx + k = 0$

Example 33

4 Sketch the graphs of the following parabolas. Use the quadratic formula to find the x-axis intercepts (if they exist) and the axis of symmetry and, hence, the turning point.

a $y = x^2 + 5x - 1$
b $y = 2x^2 - 3x - 1$
c $y = -x^2 - 3x + 1$
d $y + 4 = x^2 + 2x$
e $y = 4x^2 + 5x + 1$
f $y = -3x^2 + 4x - 2$
g $y = -x^2 + 5x + 6$
h $y = 4x^2 - 3x + 2$
i $y = 3x^2 - x - 4$

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3I The discriminant

In the previous section we found that the solutions to the quadratic equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression under the square root sign is called the **discriminant**. We write

$$\Delta = b^2 - 4ac$$

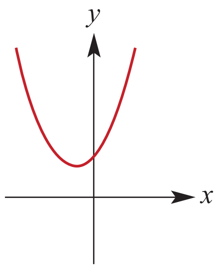
The number of x-axis intercepts

There are three different possibilities for the number of x-axis intercepts of a parabola:

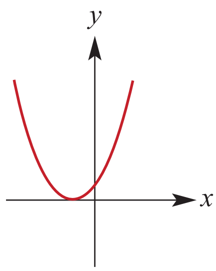
- zero – the graph is either all above or all below the x-axis
- one – the graph touches the x-axis and the turning point is the x-axis intercept
- two – the graph crosses the x-axis.

For a parabola $y = ax^2 + bx + c$, we can use the discriminant $\Delta = b^2 - 4ac$ to determine when each of these three situations occur.

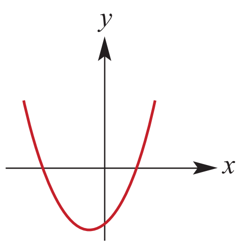
- If the discriminant $b^2 - 4ac < 0$, then the equation $ax^2 + bx + c = 0$ has no solutions and the corresponding parabola will have no x-axis intercepts.



- If the discriminant $b^2 - 4ac = 0$, then the equation $ax^2 + bx + c = 0$ has one solution and the corresponding parabola will have one x-axis intercept. (We sometimes say the equation has two coincident solutions.)



- If the discriminant $b^2 - 4ac > 0$, then the equation $ax^2 + bx + c = 0$ has two solutions and the corresponding parabola will have two x-axis intercepts.



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Example 34

Find the discriminant of each of the following quadratics and state whether the graph of each crosses the x-axis, touches the x-axis or does not intersect the x-axis.

a $y = x^2 - 6x + 8$ **b** $y = x^2 - 8x + 16$ **c** $y = 2x^2 - 3x + 4$

Solution

a Discriminant $\Delta = b^2 - 4ac$

$= (-6)^2 - (4 \times 1 \times 8)$

$= 4$

As $\Delta > 0$, the graph intersects the x-axis at two distinct points, i.e. there are two distinct solutions of the equation $x^2 - 6x + 8 = 0$.

b $\Delta = b^2 - 4ac$

$= (-8)^2 - (4 \times 1 \times 16)$

$= 0$

As $\Delta = 0$, the graph touches the x-axis, i.e. there is one solution of the equation $x^2 - 8x + 16 = 0$.

c $\Delta = b^2 - 4ac$

$= (-3)^2 - (4 \times 2 \times 4)$

$= -23$

As $\Delta < 0$, the graph does not intersect the x-axis, i.e. there are no real solutions for the equation $2x^2 - 3x + 4 = 0$.

Example 35

Find the values of m for which the equation $3x^2 - 2mx + 3 = 0$ has:

a one solution **b** no solution **c** two distinct solutions.

Solution

For the quadratic $3x^2 - 2mx + 3$, the discriminant is $\Delta = 4m^2 - 36$.

a For one solution:

$\Delta = 0$

i.e. $4m^2 - 36 = 0$

$m^2 = 9$

$\therefore$ $m = \pm 3$

b For no solution:

$\Delta < 0$

i.e. $4m^2 - 36 < 0$

From the graph, this is equivalent to

$-3 < m < 3$

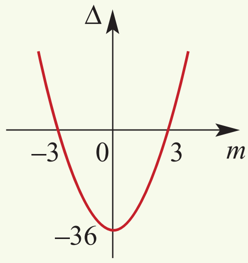
c For two distinct solutions:

$\Delta > 0$

i.e. $4m^2 - 36 > 0$

From the graph it can be seen that

$m > 3$ or $m < -3$



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The nature of the solutions of a quadratic equation

The discriminant can be used to assist in the identification of the particular type of solution for a quadratic equation $ax^2 + bx + c = 0$.

For a, b and c rational numbers:

- If $\Delta = b^2 - 4ac$ is a perfect square and $\Delta \neq 0$, then the quadratic equation has two rational solutions.
- If $\Delta = b^2 - 4ac = 0$, then the quadratic equation has one rational solution.
- If $\Delta = b^2 - 4ac$ is not a perfect square and $\Delta > 0$, then the quadratic equation has two irrational solutions.

Example 36

Show that the solutions of the equation $3x^2 + (m - 3)x - m = 0$ are rational for all rational values of m.

Solution

$$\begin{aligned}\Delta &= (m-3)^2 - 4 \times 3 \times (-m)\\ &= m^2 - 6m + 9 + 12m\\ &= m^2 + 6m + 9\\ &= (m+3)^2 \geq 0 \quad \text{for all } m\end{aligned}$$

Furthermore, Δ is a perfect square for all m.

Summary 3I

The **discriminant** Δ of a quadratic polynomial $ax^2 + bx + c$ is

$$\Delta = b^2 - 4ac$$

For the equation $ax^2 + bx + c = 0$:

- If $\Delta > 0$, there are two solutions.
- If $\Delta = 0$, there is one solution.
- If $\Delta < 0$, there are no real solutions.

For the equation $ax^2 + bx + c = 0$ where a, b and c are rational numbers:

- If Δ is a perfect square and $\Delta \neq 0$, then the equation has two rational solutions.
- If $\Delta = 0$, then the equation has one rational solution.
- If Δ is not a perfect square and $\Delta > 0$, then the equation has two irrational solutions.

Skill-sheet

Exercise 3I

1 Determine the discriminant of each of the following quadratics:

a $x^2 + 2x - 4$ **b** $x^2 + 2x + 4$ **c** $x^2 + 3x - 4$

d $2x^2 + 3x - 4$ **e** $-2x^2 + 3x + 4$

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Example 34

2 Without sketching the graphs of the following quadratics, determine whether they cross or touch the x-axis:

a $y = x^2 - 5x + 2$ **b** $y = -4x^2 + 2x - 1$ **c** $y = x^2 - 6x + 9$

d $y = 8 - 3x - 2x^2$ **e** $y = 3x^2 + 2x + 5$ **f** $y = -x^2 - x - 1$

3 By examining the discriminant, find the number of distinct solutions of:

a $x^2 + 8x + 7 = 0$ **b** $3x^2 + 8x + 7 = 0$ **c** $10x^2 - x - 3 = 0$

d $2x^2 + 8x - 7 = 0$ **e** $3x^2 - 8x - 7 = 0$ **f** $10x^2 - x + 3 = 0$

4 By examining the discriminant, state the nature and number of distinct solutions for each of the following:

a $9x^2 - 24x + 16 = 0$ **b** $-x^2 - 5x - 6 = 0$ **c** $x^2 - x - 4 = 0$

d $25x^2 - 20x + 4 = 0$ **e** $6x^2 - 3x - 2 = 0$ **f** $x^2 + 3x + 2 = 0$

Example 35

5 Find the values of m for which each of the following equations:

i has no solutions **ii** has one solution **iii** has two distinct solutions.

a $x^2 - 4mx + 20 = 0$ **b** $mx^2 - 3mx + 3 = 0$

c $5x^2 - 5mx - m = 0$ **d** $x^2 + 4mx - 4(m - 2) = 0$

Example 36

6 For m and n rational numbers show that $mx^2 + (2m + n)x + 2n = 0$ has rational solutions.

7 Find the values of p for which the equation $px^2 + 2(p + 2)x + p + 7 = 0$ has no real solution.

8 Find the values of p for which the equation $(1 - 2p)x^2 + 8px - (2 + 8p) = 0$ has one solution.

9 Find the value(s) of p for which:

a $px^2 - 6x + p = 0$ has one solution **b** $2x^2 - 4x + 3 = p$ has two solutions

c $3x^2 = 2x + p - 1$ has two solutions **d** $x^2 - 2x + 2 = p$ has two solutions.

10 Find the values of p for which the graph of $y = px^2 + 8x + p - 6$ crosses the x-axis.

11 Show that the equation $(p^2 + 1)x^2 + 2pqx + q^2 = 0$ has no real solution for any values of p and q $(q \neq 0)$.

12 **a** Find the discriminant of $x^2 + 4mx + 24m - 44$.

b Show the equation $x^2 + 4mx + 24m - 44 = 0$ has a solution for all values of m.

13 **a** Find the discriminant of $4mx^2 + 4(m - 1)x + m - 2$.

b Show the equation $4mx^2 + 4(m - 1)x + m - 2 = 0$ has a solution for all values of m.

14 Find the discriminant of the equation $4x^2 + (m - 4)x - m = 0$, where m is a rational number, and hence show that the equation has rational solution(s).

15 Find the discriminant of the equation $x^2 - (m + 2n)x + 2mn = 0$, where m and n are rational numbers, and hence show that the equation has rational solution(s).

16 If both a and c are positive, what can be said about the graph of $y = ax^2 + bx - c$?

17 If a is negative and c is positive, what can be said about the graph of $y = ax^2 + bx + c$?

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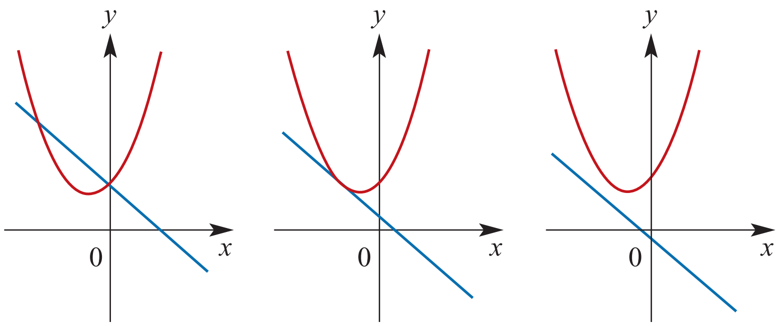
3J Solving simultaneous linear and quadratic equations

As discussed in Section 2H, when solving simultaneous linear equations we are actually finding the point of intersection of the two corresponding linear graphs.

If we wish to find the point or points of intersection between a straight line and a parabola, we can solve the equations simultaneously.

It should be noted that depending on whether the straight line intersects, touches or does not intersect the parabola we may get two, one or zero points of intersection.



Two points of intersection One point of intersection No point of intersection

If there is one point of intersection between the parabola and the straight line, then the line is a **tangent** to the parabola.

As we usually have the quadratic equation written with y as the subject, it is necessary to have the linear equation written with y as the subject. Then the linear expression for y can be substituted into the quadratic equation.

Example 37

Find the points of intersection of the line with equation $y = -2x + 4$ and the parabola with equation $y = x^2 - 8x + 12$.

Solution

At the point of intersection:

$$x^2 - 8x + 12 = -2x + 4$$
$$x^2 - 6x + 8 = 0$$
$$(x - 2)(x - 4) = 0$$

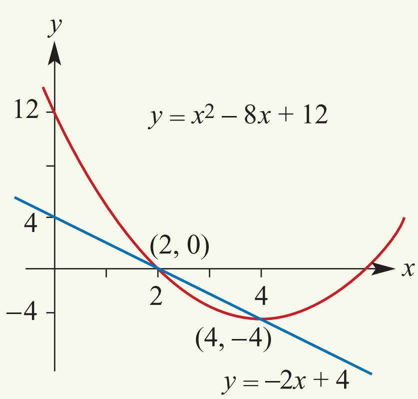
Hence $x = 2$ or $x = 4$.

When $x = 2$, $y = -2(2) + 4 = 0$.

When $x = 4$, $y = -2(4) + 4 = -4$.

Therefore the points of intersection are $(2, 0)$ and $(4, -4)$.

The result can be shown graphically.



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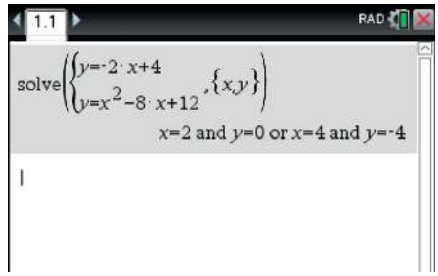
Using the TI-Nspire

To solve the simultaneous equations
$y = -2x + 4$ and $y = x^2 - 8x + 12$:

- Use (menu) > **Algebra** > **Solve System of Equations** > **Solve System of Equations**.
- Press (enter) to accept the default settings of two equations with variables x and y, and then complete the template as shown.

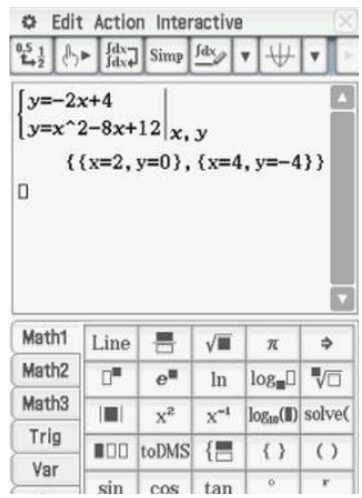


Using the Casio ClassPad

- In the Main $\sqrt{\alpha}$ screen, turn on the keyboard and select the simultaneous equations icon from (Math1).
- Enter the simultaneous equations

$$y = -2x + 4$$
$$y = x^2 - 8x + 12$$

into the two lines, and enter x, y as the variables.
- Tap (EXE).



Example 38

Prove that the straight line with the equation $y = 1 - x$ meets the parabola with the equation $y = x^2 - 3x + 2$ once only.

Solution

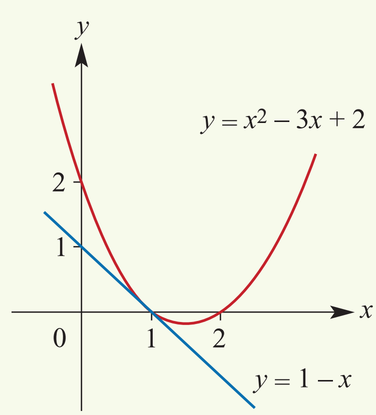
At the point of intersection:

$$x^2 - 3x + 2 = 1 - x$$
$$x^2 - 2x + 1 = 0$$
$$(x - 1)^2 = 0$$

Therefore $x = 1$ and $y = 1 - 1 = 0$.

The straight line just touches the parabola at $(1, 0)$.

This can be illustrated graphically.



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Summary 3J

To find the points of intersection of a straight line $y = mx + c_2$ and a parabola $y = ax^2 + bx + c_1$:

- Form the quadratic equation

$$ax^2 + bx + c_1 = mx + c_2$$

- Rearrange the equation so that the right-hand side is zero:

$$ax^2 + (b - m)x + (c_1 - c_2) = 0$$

- Solve the equation for x and substitute these x-values into the equation of the line to find the corresponding y-values.

The discriminant applied to the second equation, $ax^2 + (b - m)x + (c_1 - c_2) = 0$, can be used to determine the number of intersection points:

- If $\Delta > 0$, there are two intersection points.
- If $\Delta = 0$, there is one intersection point.
- If $\Delta < 0$, there are no intersection points.

Exercise 3J

Example 37

1 **a** Find the points of intersection of the line with equation $y = x - 2$ and the parabola with equation $y = x^2 - x - 6$.

b Find the points of intersection of the line with equation $x + y = 6$ and the parabola with equation $y = x^2$.

c Find the points of intersection of the line with equation $5x + 4y = 21$ and the parabola with equation $y = x^2$.

d Find the points of intersection of the line with equation $y = 2x + 1$ and the parabola with equation $y = x^2 - x + 3$.

2 Solve each of the following pairs of equations:

a $y = x^2 + 2x - 8$, $y = 2 - x$

b $y = x^2 - x - 3$, $y = 4x - 7$

c $y = x^2 + x - 5$, $y = -x - 2$

d $y = x^2 + 6x + 6$, $y = 2x + 3$

e $y = 6 - x - x^2$, $y = -2x - 2$

f $y = x^2 + x + 6$, $y = 6x + 8$

Example 38

3 Prove that, for each of the following pairs of equations, the straight line meets the parabola only once:

a $y = x^2 - 6x + 8$, $y = -2x + 4$

b $y = x^2 - 2x + 6$, $y = 4x - 3$

c $y = 2x^2 + 11x + 10$, $y = 3x + 2$

d $y = x^2 + 7x + 4$, $y = -x - 12$

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4 Solve each of the following pairs of equations:

a $y = x^2 - 6x$
$y = 8 + x$

b $y = 3x^2 + 9x$
$y = 32 - x$

c $y = 5x^2 + 9x$
$y = 12 - 2x$

d $y = -3x^2 + 32x$
$y = 32 - 3x$

e $y = 2x^2 - 12$
$y = 3(x - 4)$

f $y = 11x^2$
$y = 21 - 6x$

5 **a** Find the value of c such that $y = x + c$ is a tangent to the parabola $y = x^2 - x - 12$.
Hint: Consider the discriminant of the resulting quadratic.

b **i** Sketch the parabola with equation $y = -2x^2 - 6x + 2$.

ii Find the values of m for which the straight line $y = mx + 6$ is tangent to the parabola. Hint: Use the discriminant of the resulting quadratic.

6 **a** Find the value of c such that the line with equation $y = 2x + c$ is tangent to the parabola with equation $y = x^2 + 3x$.

b Find the possible values of c such that the line with equation $y = 2x + c$ twice intersects the parabola with equation $y = x^2 + 3x$.

7 Find the value(s) of a such that the line with equation $y = x$ is tangent to the parabola with equation $y = x^2 + ax + 1$.

8 Find the value of b such that the line with equation $y = -x$ is tangent to the parabola with equation $y = x^2 + x + b$.

9 Find the equation of the straight line(s) which pass through the point $(1, -2)$ and is (are) tangent to the parabola with equation $y = x^2$.

3K Families of quadratic polynomial functions

In Chapter 2 we considered the information that is necessary to determine the equation of a straight line and we also studied families of straight lines. In this section these two ideas are extended for our study of quadratic polynomials.

Families of quadratics

Here are some examples of families of quadratic polynomial functions:

$y = ax^2,\ a \neq 0$	The parabolas with their vertices at the origin.
$y = a(x - 2)^2 + 3,\ a \neq 0$	The parabolas with turning point at $(2, 3)$.
$y = a(x - 2)(x + 5),\ a \neq 0$	The parabolas with x-axis intercepts 2 and -5.
$y = a(x - h)(x - 2),\ a \neq 0$	The parabolas with x-axis intercept 2.
$y = ax^2 + bx,\ a \neq 0$ and $b \neq 0$	The parabolas with two x-axis intercepts, one of which is the origin.

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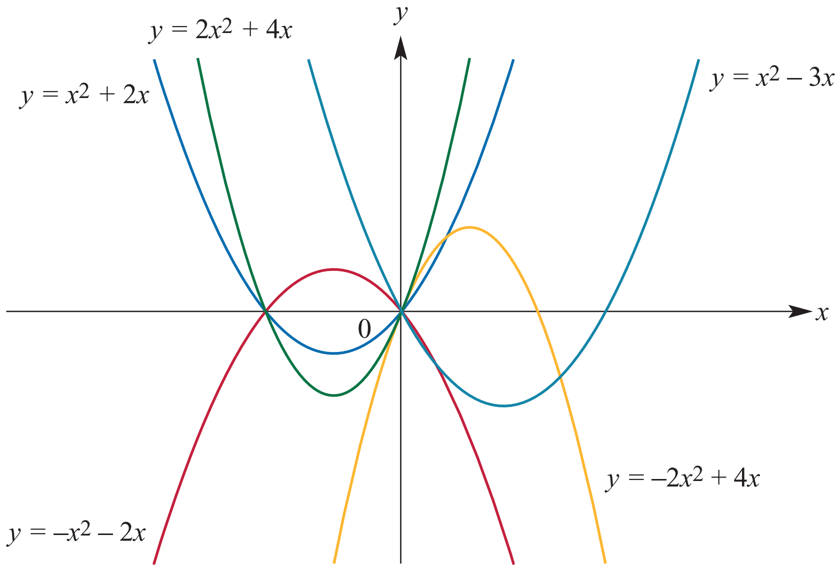
We recall from Chapter 2 that the letters a, b and h are called parameters. Varying the parameter produces different parabolas. For example, for $y = ax^2 + bx$ some possible curves are shown below.



Example 39

A family of parabolas have rules of the form $y = ax^2 + c$. For the parabola in this family that passes through the points $(1, 7)$ and $(2, 10)$, find the values of a and c.

Solution

When $x = 1$, $y = 7$ and when $x = 2$, $y = 10$.

$$7 = a + c \qquad (1)$$
$$10 = 4a + c \qquad (2)$$

Subtract (1) from (2):
$3 = 3a$ and hence $a = 1$.

Substitute in (1):
$7 = 1 + c$ and therefore $c = 6$.

The equation is $y = x^2 + 6$.

Explanation

Substitute $x = 1$, $y = 7$ in the equation $y = ax^2 + c$ to obtain (1).

Substitute $x = 2$, $y = 10$ in the equation $y = ax^2 + c$ to obtain (2).

Example 40

A family of parabolas have rules of the form $y = ax^2 + bx + 2$, where $a \neq 0$.

a For a parabola in this family with its turning point on the x-axis, find a in terms of b.

b If the turning point is at $(4, 0)$, find the values of a and b.

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Solution

a The discriminant $\Delta = b^2 - 8a$.

We have $\Delta = 0$ and therefore $a = \dfrac{b^2}{8}$.

b We have $-\dfrac{b}{2a} = 4$, which implies $b = -8a$.

From part a, we have $a = \dfrac{b^2}{8}$.

Hence $a = \dfrac{64a^2}{8} = 8a^2$.

Thus $a(1 - 8a) = 0$ and, since $a \neq 0$, $a = \dfrac{1}{8}$.

Substituting for a in $b = -8a$ gives $b = -1$.

Explanation

The discriminant of $ax^2 + bx + c$ is $\Delta = b^2 - 4ac$. In this case $c = 2$.

The discriminant $\Delta = 0$ since the parabola touches the x-axis at its turning point.

The axis of symmetry has equation $x = -\dfrac{b}{2a}$.

Determining quadratic rules

At the beginning of this section we looked at different families of quadratic polynomial functions. We now consider three important such families which can be used as a basis for finding a quadratic rule from given information. These are certainly not the only useful forms. You will see others in the worked examples.

1. $y = a(x - e)(x - f)$ This can be used if two x-axis intercepts and the coordinates of one other point are known.
2. $y = a(x - h)^2 + k$ This can be used if the coordinates of the turning point and one other point are known.
3. $y = ax^2 + bx + c$ This can be used if the coordinates of three points on the parabola are known.

Example 41

A parabola has x-axis intercepts -3 and 4 and it passes through the point $(1, 24)$. Find the rule for this parabola.

Solution

$y = a(x + 3)(x - 4)$

When $x = 1$, $y = 24$.

Therefore
$$\begin{aligned} 24 &= a(1 + 3)(1 - 4) \\ 24 &= -12a \\ a &= -2 \end{aligned}$$

The rule is $y = -2(x + 3)(x - 4)$.

Explanation

Two x-axis intercepts are given. Therefore use the form $y = a(x - e)(x - f)$.

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Example 42

The coordinates of the turning point of a parabola are $(2, 6)$ and the parabola passes through the point $(3, 3)$. Find the rule for this parabola.

Solution

$y = a(x-2)^2 + 6$

When $x = 3$, $y = 3$.

Therefore

$$3 = a(3-2)^2 + 6$$
$$3 = a + 6$$
$$a = -3$$

The rule is $y = -3(x-2)^2 + 6$.

Explanation

The coordinates of the turning point and one other point on the parabola are given. Therefore use $y = a(x-h)^2 + k$.

Example 43

A parabola passes through the points $(1, 4)$, $(0, 5)$ and $(-1, 10)$. Find the rule for this parabola.

Solution

$y = ax^2 + bx + c$

When $x = 1$, $y = 4$.
When $x = 0$, $y = 5$.
When $x = -1$, $y = 10$.

$$4 = a + b + c \qquad (1)$$
$$5 = c \qquad (2)$$
$$10 = a - b + c \qquad (3)$$

Substitute from equation (2) into equations (1) and (3):

$$-1 = a + b \qquad (1')$$
$$5 = a - b \qquad (3')$$

Add (1′) and (3′):

$$4 = 2a$$
$$a = 2$$

Using equation (1′), we obtain $b = -3$.

The rule is $y = 2x^2 - 3x + 5$.

Explanation

The coordinates of three points on the parabola are given. Therefore substitute values into the polynomial form $y = ax^2 + bx + c$ to obtain three equations in three unknowns.

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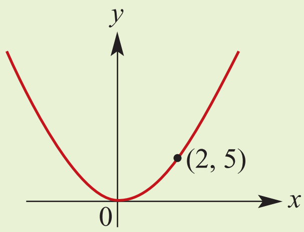
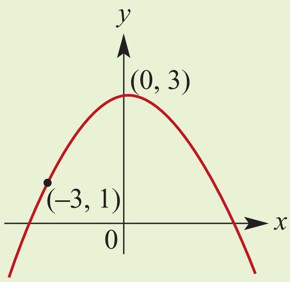
Example 44

Determine the quadratic rule for each of the following parabolas:

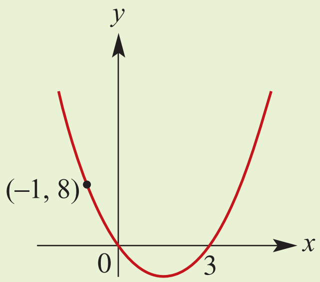
a



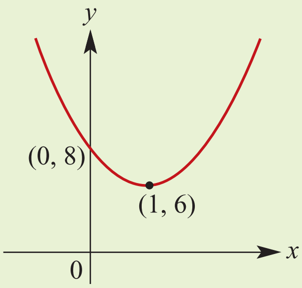
b



c



d



Solution

a This is of the form $y = ax^2$

For (2, 5): $5 = 4a$

$\therefore \quad a = \frac{5}{4}$

Hence the rule is $y = \frac{5}{4}x^2$

b This is of the form $y = ax^2 + c$

For (0, 3): $3 = a(0) + c$

$\therefore \quad c = 3$

For (−3, 1): $1 = a(-3)^2 + 3$

$1 = 9a + 3$

$\therefore \quad a = -\frac{2}{9}$

Hence the rule is $y = -\frac{2}{9}x^2 + 3$

c This is of the form $y = ax(x - 3)$

For (−1, 8): $8 = -a(-1 - 3)$

$8 = 4a$

$\therefore \quad a = 2$

Hence the rule is $y = 2x(x - 3)$

$y = 2x^2 - 6x$

d This is of the form $y = a(x - 1)^2 + 6$

For (0, 8): $8 = a + 6$

$\therefore \quad a = 2$

Hence the rule is $y = 2(x - 1)^2 + 6$

$y = 2(x^2 - 2x + 1) + 6$

$y = 2x^2 - 4x + 8$

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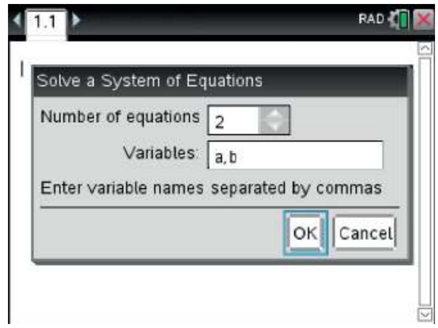
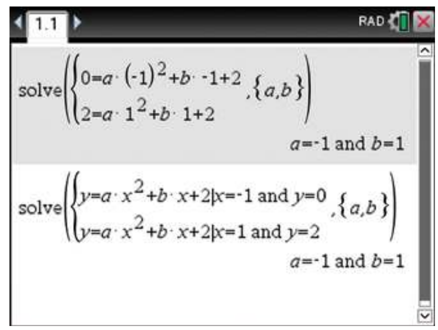
Using the TI-Nspire

The equation $y = ax^2 + bx + 2$ and the two points $(-1, 0)$ and $(1, 2)$ are used to generate equations in a and b. These equations are then solved simultaneously to find a and b.



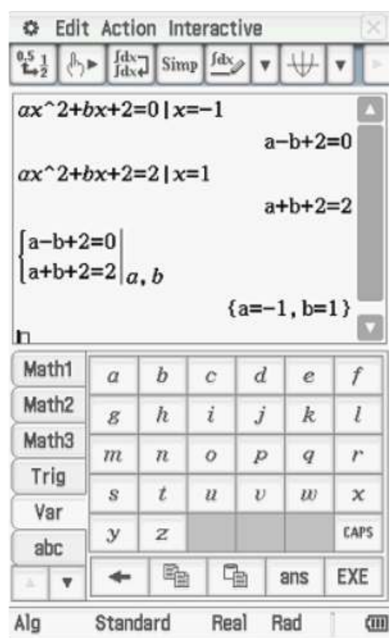
You can either substitute the values for x, y prior to entering or substitute in the command line as shown.



Using the Casio ClassPad

Substituting the points $(-1, 0)$ and $(1, 2)$ in the equation $y = ax^2 + bx + 2$ generates two equations in a and b.

These equations are then solved simultaneously to find a and b.



Note: To substitute $x = -1$ into $ax^2 + bx + 2 = 0$, use the symbol | found in Math2. Remember to use Var to enter the variables a and b.

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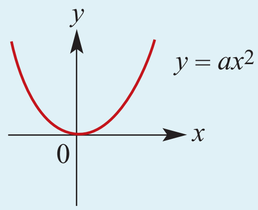
Summary 3K

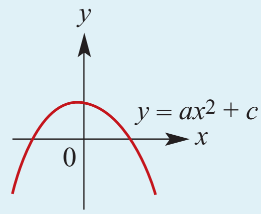
To find a quadratic rule to fit given points, first choose the best form of quadratic expression to work with. Then substitute in the coordinates of the known points to determine the unknown parameters. Some possible forms are given here:

i



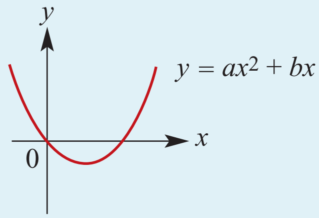
One point is needed to determine a.

ii



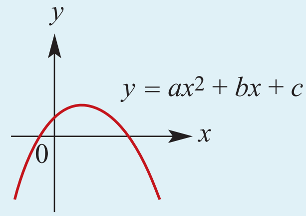
Two points are needed to determine a and c.

iii



Two points are needed to determine a and b.

iv



Three points are needed to determine a, b and c.

Skill-sheet

Exercise 3K

Example 39

1 A family of parabolas have rules of the form $y = ax^2 + c$. For the parabola in this family that passes through the points $(-1, 2)$ and $(0, 6)$, find the values of a and c.

Example 40

2 A family of parabolas have rules of the form $y = ax^2 + bx + 4$, where $a \neq 0$.

a Find the discriminant of the quadratic polynomial $ax^2 + bx + 4$.

b For a parabola in this family with its turning point on the x-axis, find a in terms of b.

c If the turning point is at $(-4, 0)$, find the values of a and b.

Example 41

3 **a** A parabola has x-axis intercepts -2 and 6 and it passes through the point $(1, -30)$. Find the rule for this parabola.

Example 42

b The coordinates of the turning point of a parabola are $(-2, 4)$ and the parabola passes through the point $(3, -46)$. Find the rule for this parabola.

Example 43

c A parabola passes through the points $(1, -2)$, $(0, -3)$ and $(-1, -6)$. Find the rule for this parabola.

4 A quadratic rule for a particular parabola is of the form $y = ax^2$. The parabola passes through the point with coordinates $(2, 8)$. Find the value of a.

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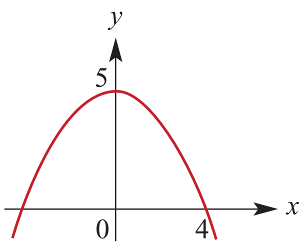
5 A quadratic rule for a particular parabola is of the form $y = ax^2 + bx$. The parabola passes through the point with coordinates $(-1, 4)$ and one of its x-axis intercepts is 6. Find the values of a and b.

6 A quadratic rule for a particular parabola is of the form $y = a(x - b)^2 + c$. The parabola has vertex $(1, 6)$ and passes through the point with coordinates $(2, 4)$. Find the values of a, b and c.

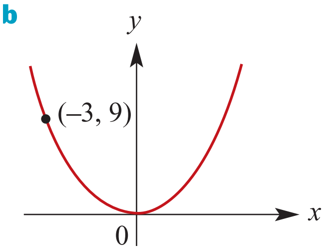
Example 44

7 Determine the equation of each of the following parabolas:

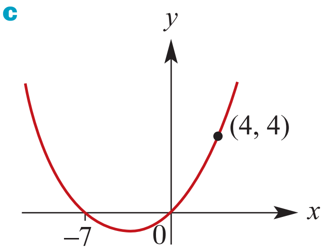
a



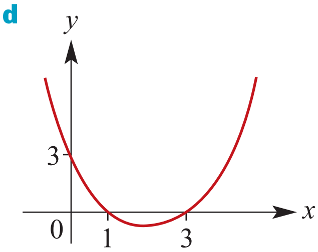
b



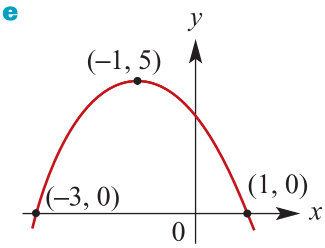
c



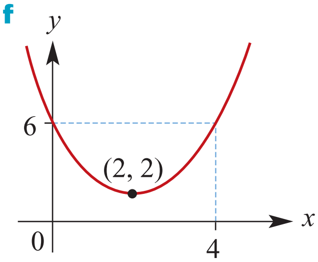
d



e



f

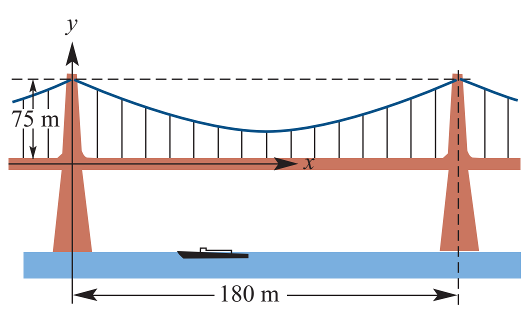


8 A parabola has vertex with coordinates $(-1, 3)$ and passes through the point with coordinates $(3, 8)$. Find the equation for the parabola.

9 A parabola has x-axis intercepts 6 and -3 and passes through the point $(1, 10)$. Find the equation of the parabola.

10 A parabola has vertex with coordinates $(-1, 3)$ and y-axis intercept 4. Find the equation for the parabola.

11 Assuming that the suspension cable shown in the diagram forms a parabola, find the rule which describes its shape. The minimum height of the cable above the roadway is 30 m.



12 A parabola has the same shape as $y = 2x^2$, but its turning point is $(1, -2)$. Write its equation.

13 A parabola has its vertex at $(1, -2)$ and passes through the point $(3, 2)$. Write its equation.

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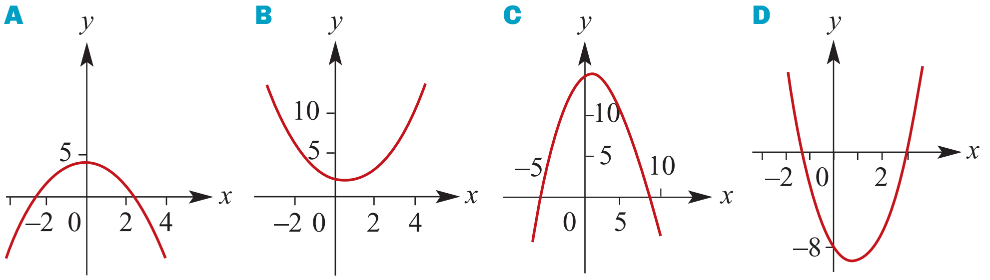
14 Which of the curves could be most nearly defined by each of the following?

a $y = \frac{1}{3}(x + 4)(8 - x)$ **b** $y = x^2 - x + 2$

c $y = -10 + 2(x - 1)^2$ **d** $y = \frac{1}{2}(9 - x^2)$



15 A family of parabolas satisfies the rule $y = ax^2 + 2x + a$.

a Express $ax^2 + 2x + a$ in the form $a(x + b)^2 + c$ for real numbers b and c.

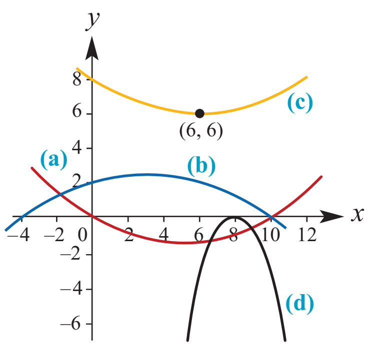
b Give the coordinates of the turning point of the graph of $y = ax^2 + 2x + a$ in terms of a.

c For which values of a is $ax^2 + 2x + a$ a perfect square?

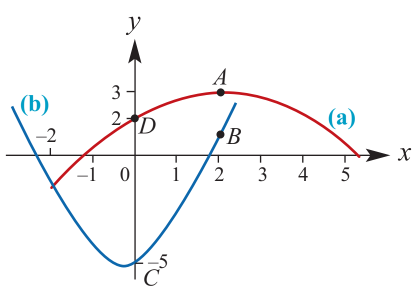
d For which values of a does the graph of $y = ax^2 + 2x + a$ have two x-axis intercepts?

16 A parabola has its vertex at $(2, 2)$ and passes through $(4, -6)$. Write its equation.

17 Write down four quadratic rules that have graphs similar to those in the diagram.



18 Find quadratic expressions which could represent the two curves in this diagram, given that the coefficient of x is 1 in each case. The labelled points are $A(2, 3)$, $B(2, 1)$, $C(0, -5)$ and $D(0, 2)$.



19 The rate of rainfall during a storm t hours after it began was 3 mm per hour when $t = 5$, 6 mm per hour when $t = 9$ and 5 mm per hour when $t = 13$. Assuming that a quadratic model applies, find an expression for the rate of rainfall, r mm per hour, in terms of t.

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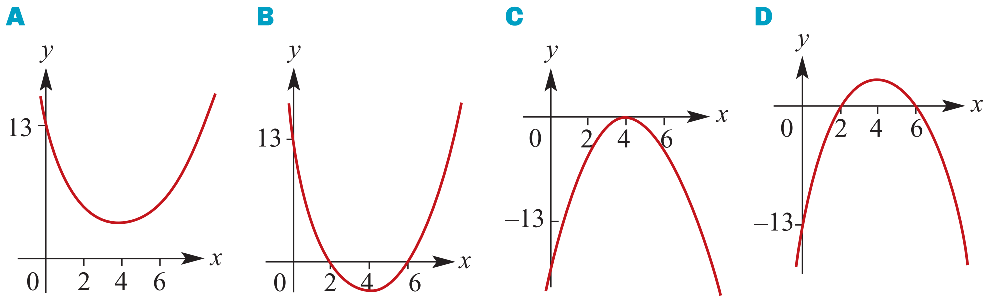
20 **a** Which of the graphs shown below could represent the equation $y = (x - 4)^2 - 3$?

b Which graph could represent $y = 3 - (x - 4)^2$?



21 Find the equation of the quadratic which passes through the points with coordinates:

a $(-2, -1)$, $(1, 2)$, $(3, -16)$ **b** $(-1, -2)$, $(1, -4)$, $(3, 10)$

c $(-3, 5)$, $(3, 20)$, $(5, 57)$

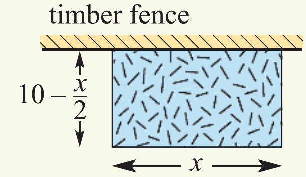
3L Quadratic models

In this section it is shown how quadratics can be used to solve worded problems, including problems which involve finding the maximum or minimum value of a quadratic polynomial that has been used to model a 'practical' situation.

Example 45

Jenny wishes to fence off a rectangular vegetable garden in her backyard. She has 20 m of fencing wire which she will use to fence three sides of the garden, with the existing timber fence forming the fourth side. Calculate the maximum area she can enclose.

Solution



Let A = area of the rectangular garden

x = length of the garden

Then width $= \dfrac{20 - x}{2} = 10 - \dfrac{x}{2}$

Therefore

$$A = x\left(10 - \frac{x}{2}\right)$$

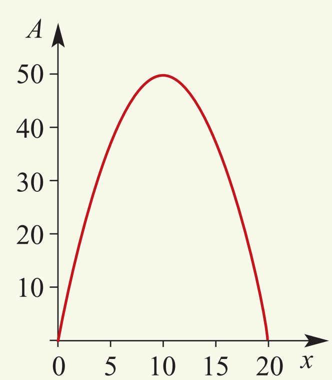
$$= 10x - \frac{x^2}{2}$$

$$= -\frac{1}{2}(x^2 - 20x + 100 - 100)$$

(completing the square)

$$= -\frac{1}{2}(x^2 - 20x + 100) + 50$$

$$= -\frac{1}{2}(x - 10)^2 + 50$$



Hence the maximum area is $50\,\text{m}^2$ when $x = 10$.

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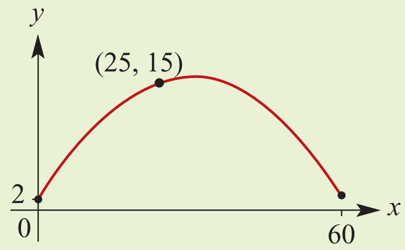
Example 46

A cricket ball is thrown by a fielder. It leaves his hand at a height of 2 metres above the ground and the wicketkeeper takes the ball 60 metres away again at a height of 2 metres. It is known that after the ball has gone 25 metres it is 15 metres above the ground. The path of the cricket ball is a parabola with equation $y = ax^2 + bx + c$.



a Find the values of a, b and c.

b Find the maximum height of the ball above the ground.

c Find the height of the ball when it is 5 metres horizontally before it hits the wicketkeeper's gloves.

Solution

a The data can be used to obtain three equations:

$$2 = c \qquad (1)$$

$$15 = (25)^2a + 25b + c \qquad (2)$$

$$2 = (60)^2a + 60b + c \qquad (3)$$

Substitute equation (1) in equations (2) and (3):

$$13 = 625a + 25b \qquad (2')$$

$$0 = 3600a + 60b \qquad (3')$$

Simplify (3′) by dividing both sides by 60:

$$0 = 60a + b \qquad (3')$$

Multiply this by 25 and subtract from equation (2′):

$$13 = -875a$$

$$\therefore \quad a = -\frac{13}{875} \quad \text{and} \quad b = \frac{156}{175}$$

The path of the ball has equation

$$y = -\frac{13}{875}x^2 + \frac{156}{175}x + 2$$

b The maximum height occurs when $x = 30$ and $y = \frac{538}{35}$.

$\therefore$ maximum height is $\frac{538}{35}$ m.

c When $x = 55$, $y = \frac{213}{35}$.

$\therefore$ height of the ball is $\frac{213}{35}$ m.

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Skill-sheet

Exercise 3L

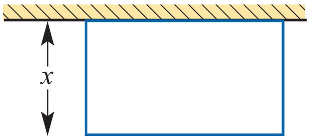
Example 45

1 A farmer has 60 m of fencing with which to construct three sides of a rectangular yard connected to an existing fence.

a If the width of the yard is x m and the area inside the yard is A m^2, write down the rule connecting A and x.

b Sketch the graph of A against x.

c Determine the maximum area that can be formed for the yard.



2 A rectangle has a perimeter of 20 m. Let x m be the length of one side. Find a formula for the area A of the rectangle in terms of x. Hence find the maximum area A.

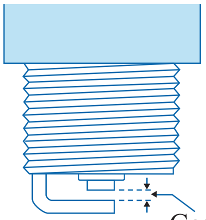
3 The efficiency rating, E, of a particular spark plug when the gap is set at x mm is said to be $400(x - x^2)$.

a Sketch the graph of E against x for $0 \le x \le 1$.

b What values of x give a zero efficiency rating?

c What value of x gives the maximum efficiency rating?

d Use the graph, or otherwise, to determine the values of x between which the efficiency rating is 70 or more.



4 A piece of wire 68 cm in length is bent into the shape of a rectangle.

a If x cm is the length of the rectangle and A cm^2 is the area enclosed by the rectangular shape, write down a formula which connects A and x.

b Sketch the graph of A against x for suitable x-values.

c Use your graph to determine the maximum area formed.

5 A piece of wire 80 cm long is to be cut into two pieces. One piece is to be bent into a square and the other into a rectangle four times as long as it is wide.

a Let x cm be the length of a side of the square and y cm be the width of the rectangle. Write a formula connecting y and x.

b Let A cm^2 be the sum of the areas of the square and the rectangle.

i Find a formula for A in terms of x.

ii Find the length of both pieces of wire if A is to be a minimum.

6 A construction firm has won a contract to build cable-car pylons at various positions on the side of a mountain. Because of difficulties associated with construction in alpine areas, the construction firm will be paid an extra amount \$$C$ for each pylon, given by the formula $C = 240h + 100h^2$, where h is the height in km above sea level.

a Sketch the graph of C as a function of h. Comment on the possible values of h.

b Does C have a maximum value?

c What is the value of C for a pylon built at an altitude of 2500 m?

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7 A tug-o-war team produces a tension in a rope described by the rule

$$T = 290(8t - 0.5t^2 - 1.4) \text{ units}$$

where t is the number of seconds after commencing the pull.

a Sketch a graph of T against t, stating the practical domain.

b What is the greatest tension produced during a 'heave'?

Example 46

8 A cricketer struck a cricket ball such that its height, d metres, after it had travelled x metres horizontally was given by the rule $d = 1 + \frac{3}{5}x - \frac{1}{50}x^2$, $x \geq 0$.

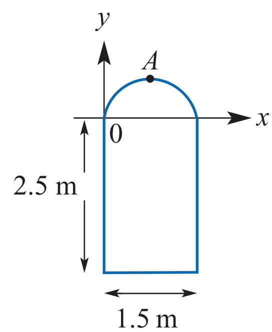
a Use a CAS calculator to graph d against x for values of x ranging from 0 to 30.

b **i** What was the maximum height reached by the ball?

ii If a fielder caught the ball when it was 2 m above the ground, how far was the ball from where it was hit?

iii At what height was the ball when it was struck?

9 An arch on the top of a door is parabolic in shape. The point A is 3.1 m above the bottom of the door. The equation $y = ax^2 + bx + c$ can be used to describe the arch. Find the values of a, b and c.



10 It is known that the daily spending of a government department follows a quadratic model. Let t be the number of days after 1 January and s be the spending in hundreds of thousands of dollars on a particular day, where $s = at^2 + bt + c$.

t	30	150	300
s	7.2	12.5	6

a Find the values of a, b and c.

b Sketch the graph for $0 \leq t \leq 360$. (Use a CAS calculator.)

c Find an estimate for the spending when:

i $t = 180$

ii $t = 350$

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Review

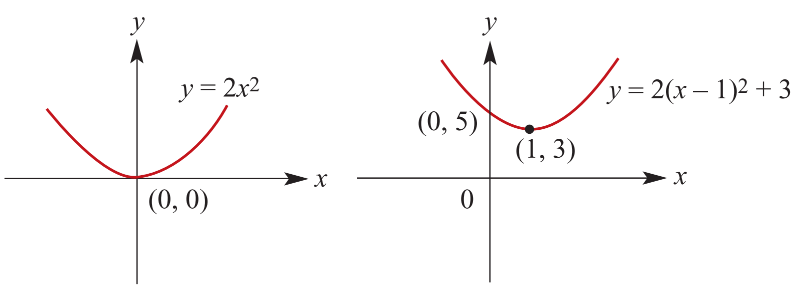
Chapter summary

Assign-ment

Nrich

- The general expression for a quadratic function is $y = ax^2 + bx + c$.
- Methods for **factorising**:
  - Taking out a common factor
    e.g. $9x^3 + 27x^2 = 9x^2(x + 3)$
  - Grouping of terms
    e.g. $$\begin{aligned} x^3 + 4x^2 - 3x - 12 &= (x^3 + 4x^2) - (3x + 12) \\ &= x^2(x + 4) - 3(x + 4) \\ &= (x^2 - 3)(x + 4) \end{aligned}$$
  - Difference of two squares: $x^2 - a^2 = (x + a)(x - a)$
    e.g. $16x^2 - 49 = (4x - 7)(4x + 7)$
  - Factorising quadratic expressions
    e.g. $x^2 + 2x - 8 = (x + 4)(x - 2)$
    $6x^2 - 13x - 15 = (6x + 5)(x - 3)$
- The graph of a quadratic may be sketched by first expressing the rule in **turning point** form, $y = a(x - h)^2 + k$. The graph can then be obtained from the graph of $y = ax^2$ by translating h units in the positive direction of the x-axis and k units in the positive direction of the y-axis (for h, k positive).

  e.g. for $y = 2(x - 1)^2 + 3$



- A quadratic equation $ax^2 + bx + c = 0$ may be solved by:
  - Factorising
  - Completing the square
  - Using the general **quadratic formula** $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- The following steps can be used to sketch the graph of a quadratic function in **polynomial** form, $y = ax^2 + bx + c$:
  - If $a > 0$, the function has a minimum value.
  - If $a < 0$, the function has a maximum value.
  - The value of c gives the y-axis intercept.
  - The equation of the axis of symmetry is $x = -\dfrac{b}{2a}$.
  - The x-axis intercepts are determined by solving the equation $ax^2 + bx + c = 0$.

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- The number of solutions of a quadratic equation $ax^2 + bx + c = 0$ can be found from the **discriminant** $\Delta = b^2 - 4ac$:
  - If $\Delta > 0$, the quadratic equation has two distinct solutions.
  - If $\Delta = 0$, the quadratic equation has one solution.
  - If $\Delta < 0$, the quadratic equation has no real solutions.
- To find a quadratic rule to fit given points, choose an appropriate form. For example:

$y = a(x - e)(x - f)$	This can be used if two x-axis intercepts and the coordinates of one other point are known.
$y = a(x - h)^2 + k$	This can be used if the coordinates of the turning point and one other point are known.
$y = ax^2 + bx + c$	This can be used if the coordinates of three points on the parabola are known.

Short-answer questions

1 Express each of the following in the form $(ax + b)^2$:

a $x^2 + 9x + \frac{81}{4}$ **b** $x^2 + 18x + 81$ **c** $x^2 - \frac{4}{5}x + \frac{4}{25}$

d $x^2 + 2bx + b^2$ **e** $9x^2 - 6x + 1$ **f** $25x^2 + 20x + 4$

2 Expand each of the following products:

a $-3(x - 2)$ **b** $-a(x - a)$

c $(7a - b)(7a + b)$ **d** $(x + 3)(x - 4)$

e $(2x + 3)(x - 4)$ **f** $(x + y)(x - y)$

g $(a - b)(a^2 + ab + b^2)$ **h** $(2x + 2y)(3x + y)$

i $(3a + 1)(a - 2)$ **j** $(x + y)^2 - (x - y)^2$

k $u(v + 2) + 2v(1 - u)$ **l** $(3x + 2)(x - 4) + (4 - x)(6x - 1)$

3 Express each of the following as a product of factors:

a $4x - 8$ **b** $3x^2 + 8x$ **c** $24ax - 3x$

d $4 - x^2$ **e** $au + 2av + 3aw$ **f** $4a^2b^2 - 9a^4$

g $1 - 36x^2a^2$ **h** $x^2 + x - 12$ **i** $x^2 + x - 2$

j $2x^2 + 3x - 2$ **k** $6x^2 + 7x + 2$ **l** $3x^2 - 8x - 3$

m $3x^2 + x - 2$ **n** $6a^2 - a - 2$ **o** $6x^2 - 7x + 2$

4 Solve each of the following equations for x by first factorising:

a $x^2 - 2x - 15 = 0$ **b** $x^2 - 9x = 0$ **c** $2x^2 - 10x + 12 = 0$

d $x^2 - 24x - 25 = 0$ **e** $3x^2 + 15x + 18 = 0$ **f** $x^2 - 12x + 36 = 0$

g $2x^2 - 5x - 3 = 0$ **h** $12x^2 - 8x - 15 = 0$ **i** $5x^2 + 7x - 12 = 0$

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Review

5 Sketch the graphs of each of the following:

a $y = 2x^2 + 3$ **b** $y = -2x^2 + 3$ **c** $y = 2(x - 2)^2 + 3$
d $y = 2(x + 2)^2 + 3$ **e** $y = 2(x - 4)^2 - 3$ **f** $y = 9 - 4x^2$
g $y = 3(x - 2)^2$ **h** $y = 2(2 - x)^2 + 3$

6 Express in the form $y = a(x - h)^2 + k$ and hence sketch the graphs of the following:

a $y = x^2 - 4x - 5$ **b** $y = x^2 - 6x$ **c** $y = x^2 - 8x + 4$
d $y = 2x^2 + 8x - 4$ **e** $y = -3x^2 - 12x + 9$ **f** $y = -x^2 + 4x + 5$

7 For each of the following, find

i the axis intercepts
ii the axis of symmetry
iii the turning point

and hence sketch the graph:

a $y = x^2 - 7x + 6$ **b** $y = -x^2 - x + 12$ **c** $y = -x^2 + 5x + 14$
d $y = x^2 - 10x + 16$ **e** $y = 2x^2 + x - 15$ **f** $y = 6x^2 - 13x - 5$
g $y = 9x^2 - 16$ **h** $y = 4x^2 - 25$

8 Find the value(s) of p that will make the quadratic $(5p - 1)x^2 - 4x + (2p - 1)$ a perfect square.

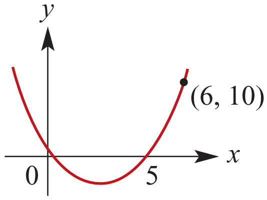
9 Solve the following quadratic inequalities:

a $x^2 > x$ **b** $(x + 2)^2 \le 34$ **c** $3x^2 + 5x - 2 \le 0$
d $-2x^2 + 13x \ge 15$

10 Use the quadratic formula to solve each of the following:

a $x^2 + 6x + 3 = 0$ **b** $x^2 + 9x + 12 = 0$ **c** $x^2 - 4x + 2 = 0$
d $2x^2 + 7x + 2 = 0$ **e** $2x^2 + 7x + 4 = 0$ **f** $3x^2 + 9x - 1 = 0$

11 Find the equation of the quadratic, the graph of which is shown.



12 A parabola has the same shape as $y = 3x^2$ but its vertex is at $(5, 2)$. Find the equation corresponding to this parabola.

13 Find the values of m if $(2m - 3)x^2 + (5m - 1)x + (3m - 2) = 0$ has two solutions.

14 Two numbers have a sum of 30. Find the maximum value of the product of such numbers.

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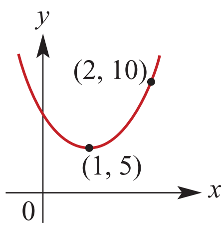
15 Find the rule of the quadratic function which describes the graph.



16 Find the coordinates of the points of intersection of the graphs with equations:

a $y = 2x + 3$ and $y = x^2$

b $y = 8x + 11$ and $y = 2x^2$

c $y = 3x^2 + 7x$ and $y = 2$

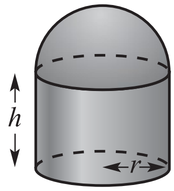
d $y = 2x^2$ and $y = 2 - 3x$

17 **a** A parabola has x-axis intercepts -4 and 1 and it passes through the point $(-1, -12)$. Find the rule for this parabola.

b The coordinates of the turning point of a parabola are $(-1, 3)$ and the parabola passes through the point $(1, -5)$. Find the rule for this parabola.

c A parabola passes through the points $(1, -3)$, $(0, -3)$ and $(-1, 1)$. Find the rule for this parabola.

18 The surface area, S, of a cylindrical tank with a hemispherical top is given by the formula $S = ar^2 + brh$, where $a = 9.42$ and $b = 6.28$. What is the radius of a tank of height 6 m which has a surface area of 125.6 m^2?



19 **a** For what value(s) of m does the equation $2x^2 + mx + 1 = 0$ have exactly one solution?

b For what values of m does the equation $x^2 - 4mx + 20 = 0$ have real solutions?

20 Consider the family of quadratics with rules of the form $y = x^2 + bx$, where b is a non-zero real number.

a Find the x-axis intercepts.

b Find the coordinates of the vertex of the parabola.

c **i** Find the coordinates of the points of intersection of the graph of $y = x^2 + bx$ with the line $y = x$, in terms of b.

ii For what value(s) of b is there one point of intersection?

iii For what value(s) of b are there two points of intersection?

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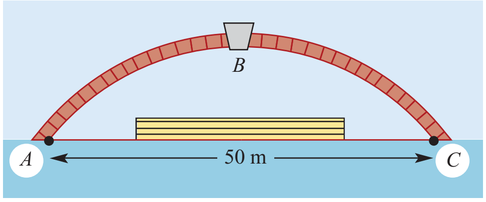
Extended-response questions

1 The diagram shows a masonry arch bridge of span 50 m. The shape of the curve, ABC, is a parabola. The line AC is the water level and B is the highest point of the bridge.



a Taking A as the origin and the maximum height of the arch above the water level as 4.5 m, write down a formula for the curve of the arch where y is the height of the arch above AC and x is the horizontal distance from A.

b Calculate a table of values and accurately plot the graph of the curve.

c At what horizontal distance from A is the height of the arch above the water level equal to 3 m?

d What is the height of the arch at a horizontal distance from A of 12 m?

e A floating platform 20 m wide is towed under the bridge. What is the greatest height of the deck above water level if the platform is to be towed under the bridge with at least 30 cm horizontal clearance on either side?

2 A piece of wire 12 cm long is cut into two pieces. One piece is used to form a square shape and the other a rectangular shape in which the length is twice its width.

a If x cm is the side length of the square, write down the dimensions of the rectangle in terms of x.

b Formulate a rule for A, the combined area of the square and rectangle in cm^2, in terms of x.

c Determine the lengths of the two pieces if the sum of the areas is to be a minimum.

3 Water is pumped into an empty metal tank at a steady rate of 0.2 litres/min. After 1 hour the depth of water in the tank is 5 cm; after 5 hours the depth is 10 cm.

a If the volume of water in the tank is V litres when the depth is x cm and there is a quadratic relationship between V and x, write down a rule which connects V and x.

b It is known that the maximum possible depth of water in the tank is 20 cm. For how long, from the beginning, can water be pumped into the tank at the same rate without overflowing?

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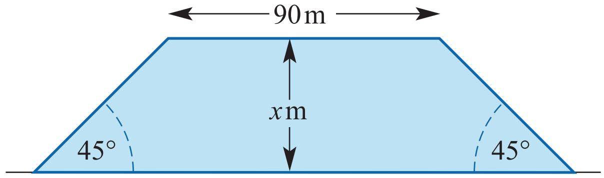
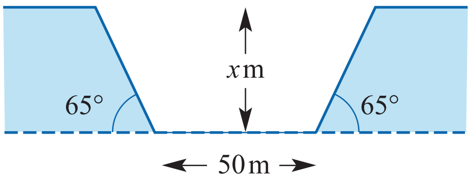
4 The figure shows a section view of a freeway embankment to be built across a flood-prone river flat. The height of the embankment is x m and the width at the top is 90 m.



a Find a formula, in terms of x, for V, the volume of earth in m^3 required to build a 120 m length of freeway embankment.

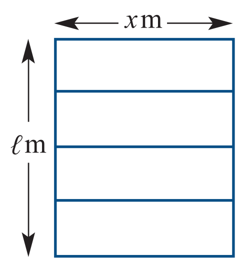
This figure shows another section of the freeway which is to be constructed by cutting through a hillside. The depth of the cutting is x m and the width of the cutting at the base is 50 m.



b Find a formula for the volume of earth, in m^3, which would have to be excavated to form a straight 100 m section of the cutting.

c If $x = 4$ m, what length of embankment could be constructed from earth taken from the cutting?

5 100 m of angle steel is used to make a rectangular frame with three crossbars as shown in the figure.



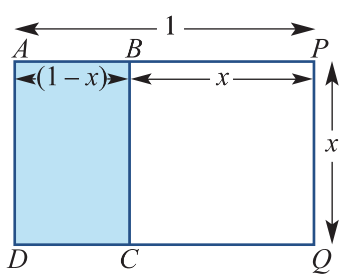
a If the width of the frame is x m, determine an expression for ℓ, the length of the frame in metres, in terms of x.

b The frame is to be covered by light aluminium sheeting. If the area of this sheeting is A m^2, formulate a rule connecting A and x.

c Sketch a graph of A against x, stating the axis intercepts and the turning point.

d What is the maximum area and the value of x which gives this area?

6 A shape which has been of interest to architects and artists over the centuries is the 'golden rectangle'. Many have thought that it has the perfect proportions for buildings. The rectangle is such that, if a square is drawn on one of the longer sides, then the new rectangle is similar to the original. Let the length of $AP = 1$ unit; then $AB = (1 - x)$ units and $\dfrac{AP}{AD} = \dfrac{AD}{AB}$.



Find the value of x. (This value is the reciprocal of the 'golden ratio'.)

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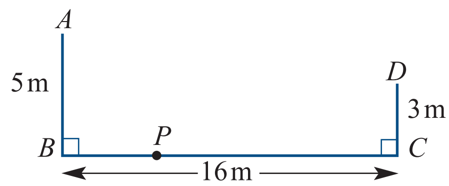
7 The point P is x m from B along the line BC.



a Find distance PA in terms of x.

b **i** Find distance PC in terms of x.

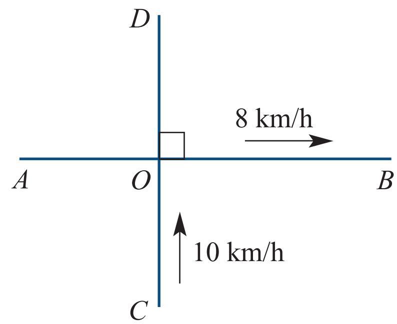
ii Find distance PD in terms of x.

c Find x if $PA = PD$.

d Find x if $PA = 2PD$. (Answer correct to three decimal places.)

e Find x if $PA = 3PD$. (Answer correct to three decimal places.)

8 AB and CD are crossroads. A jogger runs along road AB at a speed of 8 km/h and passes O at 1 p.m. Another runner is moving along road CD. The second runner is moving at 10 km/h and passes O at 1:30 p.m.



a Let y km be their distance apart t hours after 1 p.m.

i Find an expression for y in terms of t.

ii Plot the graph of y against t on a CAS calculator.

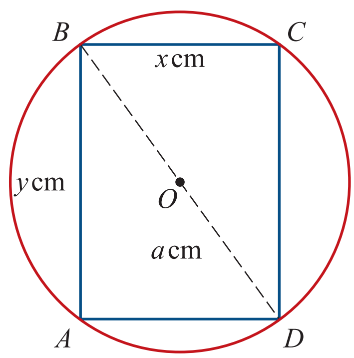
iii Find the time(s) when the runners are 4 km apart. (Use a CAS calculator.)

iv Find the time at which the runners are closest and their distance apart at this time.

b Find the exact value(s) of t for which:

i $y = 5$ **ii** $y = 6$

9 A rectangle of perimeter b cm is inscribed in a circle of radius a cm. The rectangle has width x cm and length y cm.



a Apply Pythagoras' theorem in triangle BCD to show that $x^2 + y^2 = 4a^2$.

b Form a second equation involving x, y and b.

c Eliminate y from these equations to form a quadratic equation in terms of x, a and b.

d As x, y and $2a$ are the sides of a triangle, $x + y > 2a$. Use this result and apply the discriminant to the quadratic equation formed in part c to show that the rectangle can be inscribed in the circle only if $4a < b \leq 4\sqrt{2}a$.

e **i** If $a = 5$ and $b = 24$, find the values of x and y.

ii If $b = 4\sqrt{2}a$, find the values of x and y in terms of a.

f If $\dfrac{b}{a} = 5$, find the values of x and y in terms of a.

g Write a program to solve the quadratic equation found in part c for suitable choices of a and b and state the values of x and y. (Answers correct to two decimal places.)

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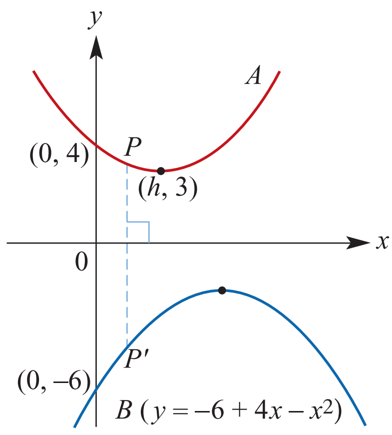
10 The equation of curve B is $y = -6 + 4x - x^2$.



a $(h, 3)$ is the vertex of a parabola A, with equation $y = x^2 + bx + c$. Find the values of b, c and h for $h > 0$.

b Let P be a point on curve A, and P' be a point on curve B such that PP' is perpendicular to the x-axis.

i The coordinates of P are $(x, x^2 + bx + c)$. State the coordinates of P' in terms of x.

ii Find the coordinates of M, the midpoint of PP', in terms of x.

iii Find the coordinates of M for $x = 0, 1, 2, 3, 4$.

iv Give the equation of the straight line on which all of these points lie. (This is called the locus of the midpoints.)

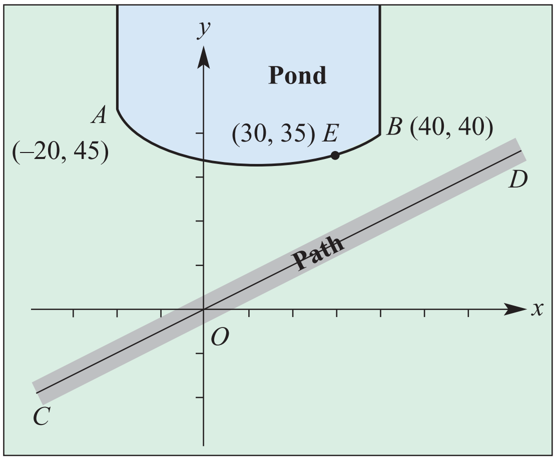
c Let d be the distance PP'.

i Express d in terms of x.

ii Sketch the graph of d against x.

iii Find the minimum value of d and the value of x for which this occurs.

11 A path cuts across a park. Its centreline can be described by the equation $y = \frac{x}{2}$, where the origin is at a point O in the park. The path starts at a point $C(-30, -15)$ and finishes at a point $D(60, 30)$.



a How long is the path?

One boundary of the pond in the park is parabolic in shape. The boundary passes through the points $A(-20, 45)$, $B(40, 40)$ and $E(30, 35)$. The equation of the parabola is of the form $y = ax^2 + bx + c$.

b **i** Find the equation of the parabola.

ii Find the coordinates of the vertex of the parabola.

c On the one set of axes sketch the graphs of $y = \frac{x}{2}$ and the parabola. (Use a CAS calculator to help.)

d Consider the rule $y = (ax^2 + bx + c) - \frac{1}{2}x$, where a, b and c have been determined in part b i.

i What does this expression determine?

ii Find the minimum value of this expression and the value of x for which this occurs.

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4 A gallery of graphs

In this chapter

Syllabus references

Topics: Inverse proportion; Powers and polynomials; Graphs of relations

Subtopics: 1.1.14, 1.1.15, 1.1.21, 1.1.22

In Chapter 2, we looked at linear graphs, sketching them and determining their rules given sufficient information. All linear graphs can be considered as transformations of $y = x$. The features we concentrated on for linear graphs were the x-axis intercept, the y-axis intercept and the gradient.

In Chapter 3, we considered quadratics written in 'turning point form' and sketched their graphs by using transformations of the graph of the basic quadratic $y = x^2$. The features we concentrated on for graphs of quadratic polynomials were the x-axis intercepts, the y-axis intercept and the coordinates of the turning point (vertex).

In this chapter, we study some other common algebraic relations, and develop methods similar to those used in Chapter 3 to sketch the graphs of these relations. The relations in this chapter have different types of key features. For example, we introduce asymptotes for graphs of rectangular hyperbolas, and the coordinates of the centre and the length of the radius are key features in the study of circles.

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4A Rectangular hyperbolas

Consider the rule

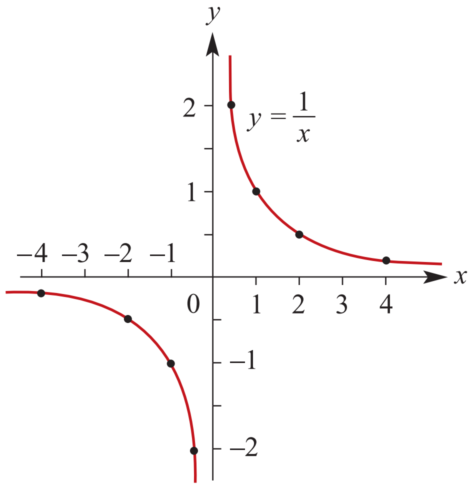
$$y = \frac{1}{x} = x^{-1} \qquad \text{for } x \neq 0$$

We can construct a table of values for $y = \frac{1}{x}$ for values of x between −4 and 4 as follows:

x	−4	−3	−2	−1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2	3	4
y	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{2}$	−1	−2	2	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$

We can plot these points and then connect the dots to produce a continuous curve. A graph of this type is an example of a **rectangular hyperbola**.

Note that y is undefined when $x = 0$, and that there is no x-value that will produce the value $y = 0$.



Asymptotes

There are two lines associated with this graph that help to describe its shape.

Horizontal asymptote

From the graph we see that, as x approaches infinity in either direction, the value of y approaches zero. The following notation will be used to state this:

- As $x \to \infty$, $y \to 0^+$. This is read: 'As x approaches infinity, y approaches 0 from the positive side.'
- As $x \to -\infty$, $y \to 0^-$. This is read: 'As x approaches negative infinity, y approaches 0 from the negative side.'

The graph approaches the x-axis (the line $y = 0$) but does not cross this line. The line $y = 0$ is a **horizontal asymptote**.

Vertical asymptote

As x approaches zero from either direction, the magnitude of y becomes very large. The following notation will be used to state this:

- As $x \to 0^+$, $y \to \infty$. This is read: 'As x approaches zero from the positive side, y approaches infinity.'
- As $x \to 0^-$, $y \to -\infty$. This is read: 'As x approaches zero from the negative side, y approaches negative infinity.'

The graph approaches the y-axis (the line $x = 0$) but does not cross this line. The line $x = 0$ is a **vertical asymptote**.

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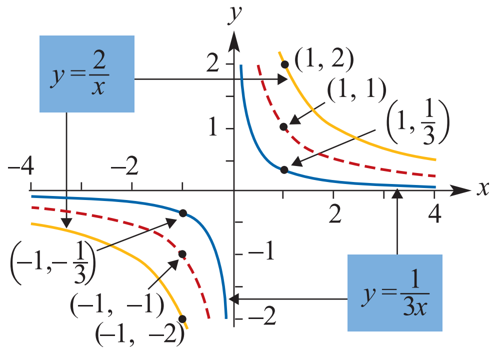
Dilations parallel to an axis

The diagram on the right shows the graphs of

$$y = \frac{1}{x}, \quad y = \frac{2}{x} \quad \text{and} \quad y = \frac{1}{3x}$$

The asymptotes are the x-axis and the y-axis, and they have equations $y = 0$ and $x = 0$ respectively.



As can be seen from the diagram, the graphs of $y = \frac{2}{x}$ and $y = \frac{1}{3x}$ have the same 'shape' and asymptotes as the graph of $y = \frac{1}{x}$, but they have been 'stretched'.

The transformation that takes the graph of $y = \frac{1}{x}$ to the graph of $y = \frac{2}{x}$ is called the **dilation** of factor 2 parallel to the y-axis. For example, the point $(1, 1)$ on the graph of $y = \frac{1}{x}$ is taken to the point $(1, 2)$ on the graph of $y = \frac{2}{x}$. Dilations will be considered formally in Chapter 6.

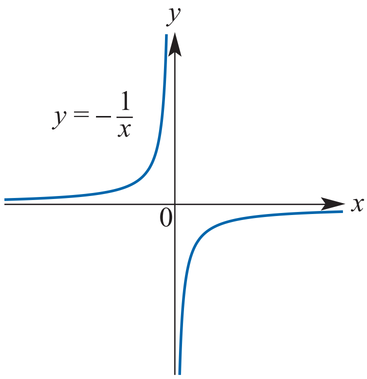
Reflection in the x-axis

When the graph of $y = \frac{1}{x}$ is reflected in the x-axis, the result is the graph of $y = -\frac{1}{x}$.

The asymptotes are still the two axes, that is, the lines $x = 0$ and $y = 0$.

Similarly, $y = -\frac{2}{x}$ is the reflection of $y = \frac{2}{x}$ in the x-axis.

Reflecting in the y-axis gives the same result for these two graphs.



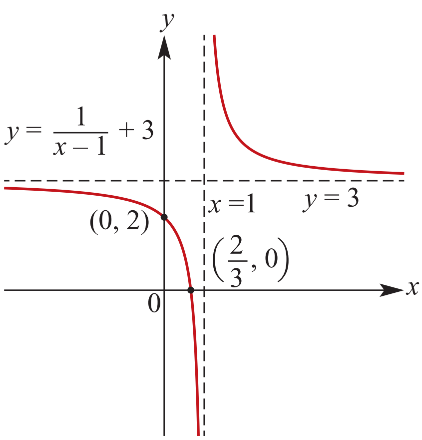
Translations

Now let us consider the graph of $y = \frac{1}{x-1} + 3$.

The basic graph of $y = \frac{1}{x}$ has been translated 1 unit to the right and 3 units up.

Asymptotes The equation of the vertical asymptote is now $x = 1$, and the equation of the horizontal asymptote is now $y = 3$.

Intercepts with the axes The graph now has x-axis and y-axis intercepts. These can be calculated in the usual way to add further detail to the graph.



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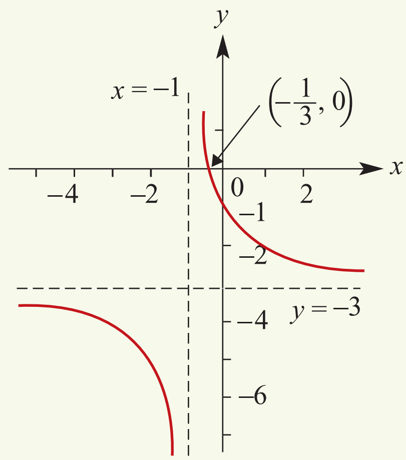
Sketching rectangular hyperbolas

Using dilations, reflections and translations, we are now able to sketch the graphs of all rectangular hyperbolas of the form $y = \frac{a}{x-h} + k$.

Example 1

Sketch the graph of $y = \frac{2}{x+1} - 3$.

Solution



Explanation

The graph of $y = \frac{2}{x}$ has been translated 1 unit to the left and 3 units down. The asymptotes have equations $x = -1$ and $y = -3$.

When $x = 0$, $y = \frac{2}{0+1} - 3 = -1$.

$\therefore$ the y-axis intercept is -1.

When $y = 0$,

$$0 = \frac{2}{x+1} - 3$$

$$3 = \frac{2}{x+1}$$

$$3(x+1) = 2$$

$$x = -\frac{1}{3}$$

$\therefore$ the x-axis intercept is $-\frac{1}{3}$.

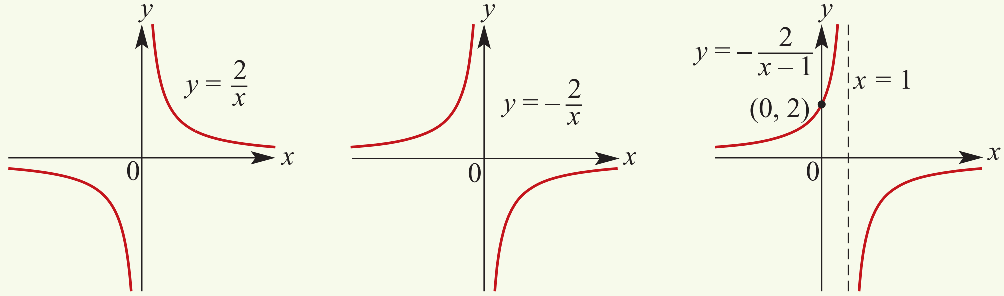
Example 2

Sketch the graph of $y = \frac{-2}{x-1}$.

Solution

The graph of $y = -\frac{2}{x}$ is obtained from the graph of $y = \frac{2}{x}$ by reflection in the x-axis.

This graph is then translated 1 unit to the right to obtain the graph of $y = \frac{-2}{x-1}$.



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Summary 4A

- For $a > 0$, a dilation of factor a parallel to the y-axis transforms the graph of $y = \frac{1}{x}$ to the graph of $y = \frac{a}{x}$.
- A reflection in the x-axis transforms the graph of $y = \frac{a}{x}$ to the graph of $y = -\frac{a}{x}$.
- For $h, k \geq 0$, a translation of h to the right and k upwards transforms the graph of $y = \frac{a}{x}$ to the graph of $y = \frac{a}{x-h} + k$.
- A rectangular hyperbola with rule of the form $y = \frac{a}{x-h} + k$ has:
  - vertical asymptote $x = h$
  - horizontal asymptote $y = k$.

Exercise 4A

Example 1
Example 2

1 Sketch the graphs of the following, showing all important features of the graphs:

a $y = \frac{1}{x}$ **b** $y = \frac{2}{x}$ **c** $y = \frac{1}{2x}$ **d** $y = \frac{-3}{x}$

e $y = \frac{1}{x} + 2$ **f** $y = \frac{1}{x} - 3$ **g** $y = \frac{2}{x} - 4$ **h** $y = \frac{-1}{2x} + 5$

i $y = \frac{1}{x-1}$ **j** $y = \frac{-1}{x+2}$ **k** $y = \frac{1}{x+1} + 3$ **l** $y = \frac{-2}{x-3} - 4$

2 Write down the equations of the asymptotes for each of the graphs in Question 1.

3 **a** We can write $y = \frac{1}{3x+6}$ as $y = \frac{1}{3(x+2)}$.
Sketch the graph of $y = \frac{1}{3x}$ and hence the graph of $y = \frac{1}{3x+6}$.

b We can write $y = \frac{3}{2x+4}$ as $y = \frac{3}{2(x+2)}$.
Sketch the graph of $y = \frac{3}{2x}$ and hence the graph of $y = \frac{3}{2x+4}$.

c We can write $y = \frac{-1}{2x+4}$ as $y = -\frac{1}{2(x+2)}$.
Sketch the graph of $y = -\frac{1}{2x}$ and hence the graph of $y = \frac{-1}{2x+4}$.

d We can write $y = \frac{1}{2x+1}$ as $y = \frac{1}{2(x+\frac{1}{2})}$.
Sketch the graph of $y = \frac{1}{2x}$ and hence the graph of $y = \frac{1}{2x+1}$.

4 Sketch the graphs of the following, showing all important features of the graphs:

a $y = \frac{1}{3x+1}$ **b** $y = \frac{1}{3x+1} - 1$ **c** $y = \frac{-1}{3x+1} - 1$ **d** $y = \frac{-2}{3x+1}$

e $y = \frac{-2}{3x+1} - 4$ **f** $y = \frac{-2}{3x+1} + 3$ **g** $y = \frac{2}{3x+2} - 1$ **h** $y = \frac{3}{3x+4} - 1$

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5 Show that $\dfrac{x+3}{x-1} = \dfrac{4}{x-1} + 1$ and hence sketch the graph of $y = \dfrac{x+3}{x-1}$.

6 Show that $\dfrac{2x+3}{x+1} = \dfrac{1}{x+1} + 2$ and hence sketch the graph of $y = \dfrac{2x+3}{x+1}$.

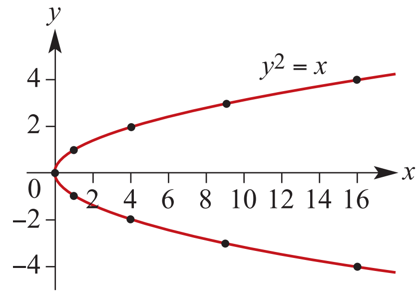
7 Show that $\dfrac{3-2x}{x-2} = -\dfrac{1}{x-2} - 2$ and hence sketch the graph of $y = \dfrac{3-2x}{x-2}$.

4B The graph of $y^2 = x$

Now consider the rule $y^2 = x$. We can construct a table of values for y between −4 and 4:

y	−4	−3	−2	−1	0	1	2	3	4
x	16	9	4	1	0	1	4	9	16

We plot these points with y against x and then connect the dots to produce a continuous curve.



The graph of $y^2 = x$ is a parabola. It can be obtained from the graph of $y = x^2$ by a reflection in the line $y = x$.

The vertex of the parabola is at $(0, 0)$, and the axis of symmetry is the x-axis.

The transformations considered in the previous section can be applied to the graph of $y^2 = x$. All graphs of the form

$$(y-k)^2 = a^2(x-h)$$

will have the same basic parabola shape. The vertex of the parabola will be at the point (h, k), and the axis of symmetry will be the line $y = k$.

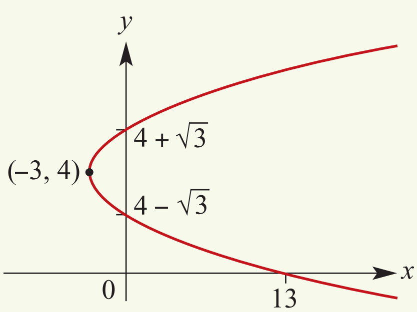
Example 3

Sketch the graph of:

a $(y-4)^2 = x+3$ **b** $y^2 + 2y = 2x + 3$

Solution

a



Explanation

The graph of $y^2 = x$ is translated 3 units to the left and 4 units up. The vertex is $(-3, 4)$.

To find the y-axis intercepts, let $x = 0$:

$$(y-4)^2 = 3 \Rightarrow y = 4 \pm \sqrt{3}$$

To find the x-axis intercept, let $y = 0$:

$$16 = x + 3 \Rightarrow x = 13$$

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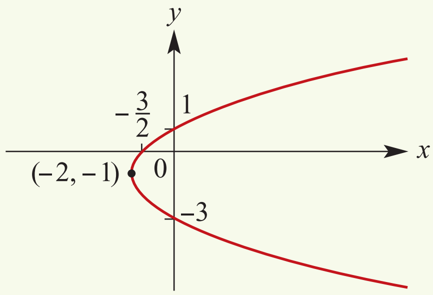
b Complete the square:

$$y^2 + 2y = 2x + 3$$
$$y^2 + 2y + 1 = 2x + 4$$
$$(y + 1)^2 = 2(x + 2)$$



The graph of $(y + 1)^2 = 2(x + 2)$ is obtained from the graph of $y^2 = x$ by a dilation of factor $\sqrt{2}$ parallel to the y-axis and then a translation 2 units to the left and 1 unit down.

The vertex has coordinates $(-2, -1)$.

To find the y-axis intercepts, let $x = 0$:

$$(y + 1)^2 = 4 \Rightarrow y = -1 \pm 2$$
$$\Rightarrow y = 1 \text{ or } y = -3$$

To find the x-axis intercept, let $y = 0$:

$$1 = 2x + 4 \Rightarrow x = -\frac{3}{2}$$

Exercise 4B

Example 3

1 Sketch the graph of each of the following relations, showing all important features:

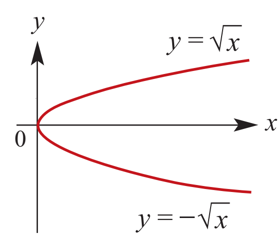
a $(y - 2)^2 = x - 3$ **b** $(y + 2)^2 = x + 4$ **c** $y^2 = 2x$
d $y^2 = 2(x + 5)$ **e** $(y - 4)^2 = 2(x + 3)$ **f** $(y + 4)^2 = 2x$
g $(y + 3)^2 = 2x - 4$ **h** $y^2 = \frac{x}{2}$ **i** $y^2 + 4y = 2x + 4$
j $y^2 + 6y - 2x + 3 = 0$ **k** $y^2 + y - x = 0$ **l** $y^2 + 7y - 5x + 3 = 0$
m $y^2 = -x$ **n** $y^2 + 2y - x = 0$

4C The graph of $y = \sqrt{x}$

The rule

$$y = \sqrt{x} = x^{\frac{1}{2}} \qquad \text{for } x \geq 0$$

corresponds to the upper part of the graph shown opposite. It is one arm of the parabola $y^2 = x$.

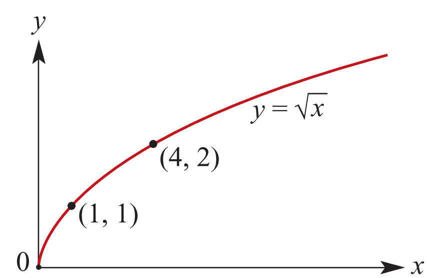


Coordinates of points on the graph of $y = \sqrt{x}$ include $(0, 0)$, $(1, 1)$, $(4, 2)$ and $(9, 3)$.

All graphs of the form

$$y = a\sqrt{x - h} + k$$

will have the same basic shape as the graph of $y = \sqrt{x}$.



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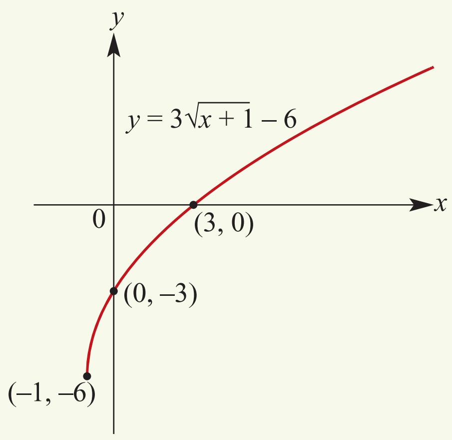
Example 4

Sketch the graph of $y = 3\sqrt{x+1} - 6$.

Solution



When $x = 0$, $y = -3$

When $y = 0$, $3\sqrt{x+1} - 6 = 0$

$$3\sqrt{x+1} = 6$$
$$\sqrt{x+1} = 2$$
$$x + 1 = 4$$
$$x = 3$$

Explanation

The graph is formed by dilating the graph of $y = \sqrt{x}$ parallel to the y-axis by factor 3 and then translating 1 unit to the left and 6 units down.

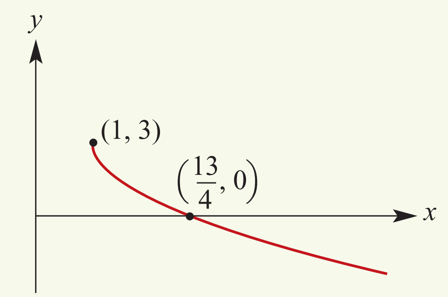
The rule is defined for $x \geq -1$.

The set of values the rule can take (the range) is all numbers greater than or equal to -6, i.e. $y \geq -6$.

Example 5

Sketch the graph of $y = -2\sqrt{x-1} + 3$.

Solution



When $y = 0$: $\quad -2\sqrt{x-1} + 3 = 0$

$$2\sqrt{x-1} = 3$$

Square both sides: $\quad 4(x-1) = 9$

Therefore $x = \frac{9}{4} + 1 = \frac{13}{4}$

Explanation

The graph is formed by dilating the graph of $y = \sqrt{x}$ parallel to the y-axis by factor 2, reflecting this in the x-axis and then translating it 1 unit to the right and 3 units up.

The rule is defined for $x \geq 1$.

The set of values the rule can take (the range) is all numbers less than or equal to 3, i.e. $y \leq 3$.

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The graph of $y = \sqrt{-x}$

The rule

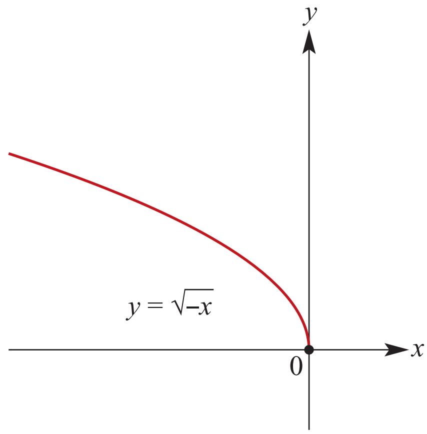
$$y = \sqrt{-x} \qquad \text{for } x \le 0$$

yields a graph which is the reflection of the graph of $y = \sqrt{x}$ in the y-axis.

All graphs of the form

$$y = a\sqrt{-(x-h)} + k$$

will have the same basic shape as the graph of $y = \sqrt{-x}$.

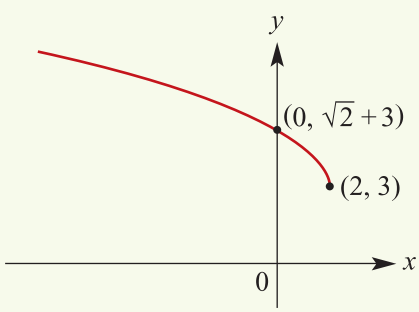


Example 6

Sketch the graph of $y = \sqrt{2-x} + 3$.

Note: $\sqrt{2-x} = \sqrt{-(x-2)}$

Solution



When $x = 0$, $y = \sqrt{2} + 3$.

Explanation

We can write the rule as

$$y = \sqrt{-(x-2)} + 3$$

The rule is defined for $x \le 2$. The set of values the rule can take (the range) is all numbers greater than or equal to 3, i.e. $y \ge 3$.

Summary 4C

- All graphs of the form $y = a\sqrt{x-h} + k$ will have the same basic shape as the graph of $y = \sqrt{x}$. The graph will have endpoint (h, k).
- The graph of $y = \sqrt{-x}$ is the reflection in the y-axis of the graph of $y = \sqrt{x}$.

Exercise 4C

Example 4

1 For each of the following rules, sketch the corresponding graph, giving the axis intercepts when they exist, the set of x-values for which the rule is defined and the set of y-values which the rule takes:

a $y = 2\sqrt{x} + 3$ **b** $y = \sqrt{x-2} + 3$ **c** $y = \sqrt{x-2} - 3$

d $y = \sqrt{x+2} + 1$ **e** $y = -\sqrt{x+2} + 3$ **f** $y = 2\sqrt{x+2} - 3$

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Example 5
Example 6

2 For each of the following rules, sketch the corresponding graph, giving the axis intercepts when they exist, the set of x-values for which the rule is defined and the set of y-values which the rule takes:

a $y = -\sqrt{x-2} + 3$

b $y = \sqrt{-(x-4)} - 2$

c $y = -2\sqrt{-(x+4)} - 1$

d $y = 2\sqrt{3-x}$

e $y = -2\sqrt{3-x}$

f $y = 4\sqrt{3-x} - 4$

3 For each of the following rules, sketch the corresponding graph, giving the axis intercepts when they exist, the set of x-values for which the rule is defined and the set of y-values which the rule takes:

a $y = \sqrt{3x}$

b $y = \sqrt{3(x-1)}$

c $y = -\sqrt{2x}$

d $y = \sqrt{2(3-x)}$

e $y = -2\sqrt{4(2-x)}$

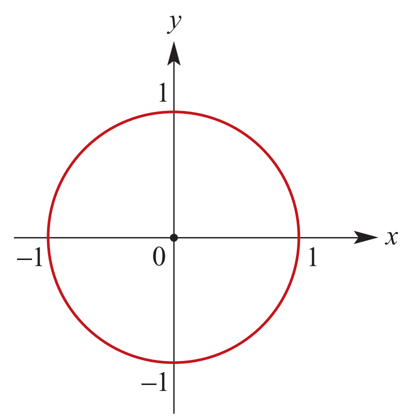
f $y = 4\sqrt{2(3-x)} - 4$

4D Circles

Consider a circle in the coordinate plane with centre the origin and radius r. If $P(x, y)$ is a point on the circle, its distance from the origin is r and so by Pythagoras' theorem $x^2 + y^2 = r^2$.

Conversely, if a point $P(x, y)$ in the plane satisfies the equation $x^2 + y^2 = r^2$, its distance from the origin is r, so it lies on a circle with centre the origin and radius r.

To the right is the graph of the circle with equation $x^2 + y^2 = 1$.



All circles can be considered as being transformations of this basic graph.

As has been seen with other graphs, the basic graph may be translated horizontally and vertically.

The equation for a circle is

$$(x-h)^2 + (y-k)^2 = r^2$$

where the **centre** of the circle is the point (h, k) and the **radius** is r.

If the radius and the coordinates of the centre of the circle are given, the equation of the circle can be determined.

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Example 7

Write down the equation of the circle with centre $(-3, 5)$ and radius 2.

Solution

If the radius is 2 and the centre is the point $(-3, 5)$, then the equation will be

$$(x - (-3))^2 + (y - 5)^2 = 4$$
$$(x + 3)^2 + (y - 5)^2 = 4$$

If the equation of the circle is given, the radius and the centre of the circle can be determined and the graph sketched.

Example 8

Find the centre and radius of the circle $(x - 1)^2 + (y - 2)^2 = 4$.

Solution

The equation $(x - 1)^2 + (y - 2)^2 = 4$ defines a circle of radius 2 with centre at $(1, 2)$.

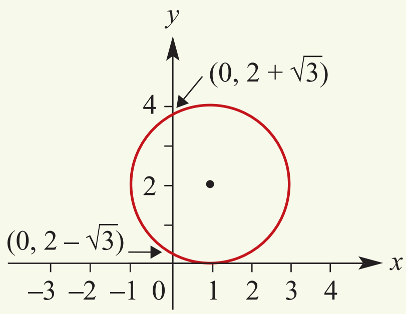
Explanation

We can sketch the circle with a little extra work.

When $x = 0$,

$$1 + (y - 2)^2 = 4$$
$$(y - 2)^2 = 3$$

Hence $y = 2 \pm \sqrt{3}$



Example 9

Sketch the graph of the circle $(x + 1)^2 + (y + 4)^2 = 9$.

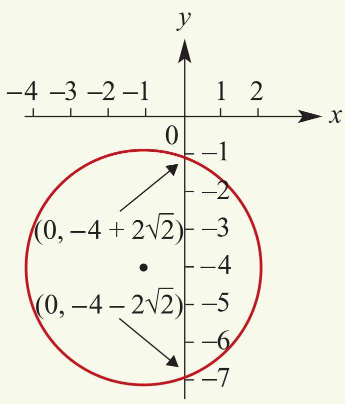
Solution

When $x = 0$,

$$1 + (y + 4)^2 = 9$$
$$(y + 4)^2 = 8$$

Hence $y = -4 \pm \sqrt{8}$

$= -4 \pm 2\sqrt{2}$



Explanation

The circle has radius 3 and centre $(-1, -4)$.

The y-axis intercepts can be found in the usual way.

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The equation of a circle may not always be written in the form $(x-h)^2+(y-k)^2=r^2$.

Expanding the general equation of a circle gives

$$\begin{aligned}(x-h)^2+(y-k)^2&=r^2\\x^2-2hx+h^2+y^2-2ky+k^2&=r^2\\x^2+y^2-2hx-2ky+h^2+k^2-r^2&=0\end{aligned}$$

Let $c=h^2+k^2-r^2$. Then we obtain an alternative form for the equation of a circle:

The **general form** for the equation of a circle is

$$x^2+y^2-2hx-2ky+c=0$$

You will note that there is some similarity with the general form of a straight line, $ax+by+c=0$.

Notice that in the general form of the circle equation, the coefficients of x^2 and y^2 are both 1 and there is no xy term.

In order to sketch a circle with equation expressed in this form, the equation can be converted to the 'centre–radius' form by completing the square for both x and y.

Example 10

Find the radius and the coordinates of the centre of the circle with equation

$$x^2+y^2-6x+4y-12=0$$

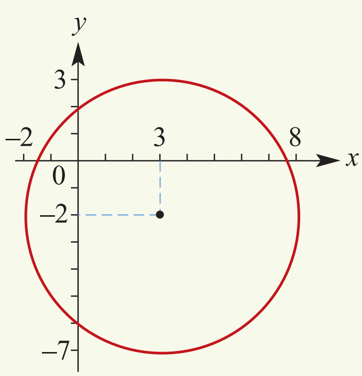
and hence sketch the graph.

Solution

By completing the square for both x and y we have

$$\begin{aligned}x^2+y^2-6x+4y-12&=0\\(x^2-6x+9)-9+(y^2+4y+4)-4-12&=0\\(x^2-6x+9)+(y^2+4y+4)&=25\\(x-3)^2+(y+2)^2&=5^2\end{aligned}$$

The radius is 5 and the centre is at (3, −2).



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Semicircles

Transposing the general equation of the circle $x^2 + y^2 = r^2$ to make y the subject, we have

$$y^2 = r^2 - x^2$$
$$y = \pm\sqrt{r^2 - x^2}$$

We can now consider two separate rules

$$y = +\sqrt{r^2 - x^2} \quad \text{and} \quad y = -\sqrt{r^2 - x^2}$$

which correspond to the top half and bottom half of the circle respectively.

Similarly, solving for x will give you the semicircles to the left and right of the y-axis:

$$x = \pm\sqrt{r^2 - y^2}$$

Example 11

Sketch the graphs of:

a $y = +\sqrt{4 - x^2}$

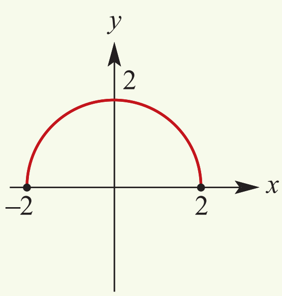
b $y = -\sqrt{4 - x^2}$

c $x = -\sqrt{4 - y^2}$

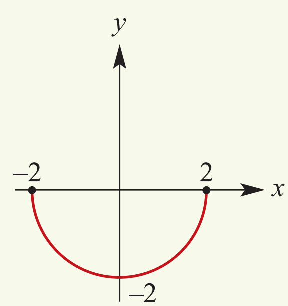
d $x = +\sqrt{4 - y^2}$

Solution

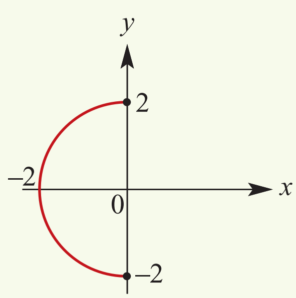
a $y = +\sqrt{4 - x^2}$



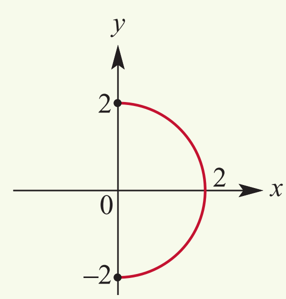
b $y = -\sqrt{4 - x^2}$



c $x = -\sqrt{4 - y^2}$



d $x = +\sqrt{4 - y^2}$



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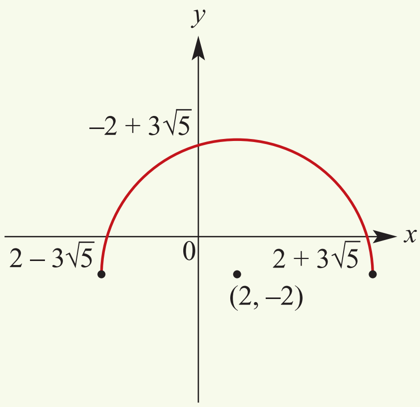
Example 12

Sketch the graph of $y = -2 + \sqrt{49 - (x - 2)^2}$.

Solution



When $x = 0$,

$$\begin{aligned} y &= -2 + \sqrt{45} \\ &= -2 + 3\sqrt{5} \end{aligned}$$

When $y = 0$,

$$\begin{aligned} -2 + \sqrt{49 - (x-2)^2} &= 0 \\ \sqrt{49 - (x-2)^2} &= 2 \\ 49 - (x-2)^2 &= 4 \\ (x-2)^2 &= 45 \\ x &= 2 \pm 3\sqrt{5} \end{aligned}$$

Explanation

It is a semicircle of the circle

$$(x - 2)^2 + (y + 2)^2 = 49$$

The centre is at the point $(2, -2)$.

It is the semicircle $y = \sqrt{49 - x^2}$ translated 2 units the right and 2 units down.

In the usual way, we find the x-axis intercepts and the y-axis intercept.

Summary 4D

- The equation of a circle with centre (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2$$

- The **general form** for the equation of a circle is

$$x^2 + y^2 - 2hx - 2ky + c = 0$$

- The two separate rules for semicircles with their base on the x-axis are

$$y = +\sqrt{r^2 - x^2} \quad \text{and} \quad y = -\sqrt{r^2 - x^2}$$

They correspond to the top half and bottom half of the circle respectively.

- The two separate rules for semicircles with their base on the y-axis are

$$x = +\sqrt{r^2 - y^2} \quad \text{and} \quad x = -\sqrt{r^2 - y^2}$$

They correspond to the right half and left half of the circle respectively.

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Skill-sheet

Exercise 4D

Example 7

1 Write down the equation of each of the following circles, with centre at $C(h, k)$ and radius r:

a $C(0, 0),\ r = 3$ **b** $C(0, 0),\ r = 4$ **c** $C(1, 3),\ r = 5$

d $C(2, -4),\ r = 3$ **e** $C(-3, 4),\ r = \frac{5}{2}$ **f** $C(-5, -6),\ r = 4.6$

Example 8

2 Find the centre, C, and the radius, r, of the following circles:

a $(x-1)^2 + (y-3)^2 = 4$ **b** $(x-2)^2 + (y+4)^2 = 5$

c $(x+3)^2 + (y-2)^2 = 9$ **d** $(x+5)^2 + (y-4)^2 = 8$

Example 9

3 Sketch the graphs of each of the following:

a $x^2 + y^2 = 64$ **b** $x^2 + (y-4)^2 = 9$

c $(x+2)^2 + y^2 = 25$ **d** $(x+1)^2 + (y-4)^2 - 169 = 0$

e $(2x-3)^2 + (2y-5)^2 = 36$ **f** $(x+5)^2 + (y-5)^2 = 36$

Example 10

4 Find the centre, C, and the radius, r, of the following circles:

a $x^2 + y^2 - 6y - 16 = 0$ **b** $x^2 + y^2 - 8x + 12y + 10 = 0$

c $x^2 + y^2 - 6x + 4y + 9 = 0$ **d** $x^2 + y^2 + 4x - 6y - 12 = 0$

e $x^2 + y^2 - 8x + 4y + 1 = 0$ **f** $x^2 + y^2 - x + 4y + 2 = 0$

Example 10

5 Sketch the graphs of each of the following:

a $x^2 + y^2 - 6y - 16 = 0$

b $x^2 + y^2 + 4x - 6y - 3 = 0$

c $x^2 + y^2 - 8x + 22y + 27 = 0$

Example 11

6 Sketch the graphs of each of the following:

a $y = +\sqrt{9 - x^2}$ **b** $x = +\sqrt{9 - y^2}$ **c** $y = -\sqrt{16 - x^2}$

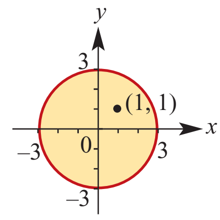
d $y = -\sqrt{25 - x^2}$ **e** $x = -\sqrt{49 - y^2}$ **f** $x = \sqrt{\frac{25}{4} - y^2}$

Example 12

7 Sketch the graphs of each of the following:

a $y = \sqrt{36 - (x-2)^2}$ **b** $y - 2 = \sqrt{4 - (x+2)^2}$

8 The graph of $x^2 + y^2 \le 9$ is as shown. Note that $(1, 1)$ satisfies $1^2 + 1^2 \le 9$. The coordinates of every point in the shaded region satisfy the inequality.



Sketch the graphs of each of the following. Use a dotted line to indicate that the boundary is not included.

a $x^2 + y^2 \le 4$ **b** $x^2 + y^2 > 1$ **c** $x^2 + y^2 \le 5$

d $x^2 + y^2 > 9$ **e** $x^2 + y^2 \ge 6$ **f** $x^2 + y^2 < 8$

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4E Determining rules

In Chapters 2 and 3 we looked at some sufficient conditions for determining the rules for straight lines and parabolas. For straight lines these included:

- the coordinates of two points
- the gradient and a point.

For parabolas these included:

- the coordinates of three points
- the coordinates of the vertex and the coordinates of one other point.

In this section we are looking at some sufficient conditions for determining the rules for the graphs of this chapter.

Example 13

a The rectangular hyperbola $y = \frac{a}{x} + 8$ passes through the point $(-2, 6)$. Find the value of a.

b The rectangular hyperbola $y = \frac{a}{x} + k$ passes through the points $(2, 7)$ and $(-1, 1)$. Find the values of a and k.

Solution

a When $x = -2$, $y = 6$. Hence

$$6 = \frac{a}{-2} + 8$$

$$-2 = \frac{a}{-2}$$

$$a = 4$$

The equation is $y = \frac{4}{x} + 8$.

Explanation

The general technique is to substitute the given values into the general equation

$$y = \frac{a}{x - h} + k$$

In this case $h = 0$ and $k = 8$.

Solution

b When $x = 2$, $y = 7$. When $x = -1$, $y = 1$.
So we have the equations

$$7 = \frac{a}{2} + k \qquad (1)$$

$$1 = -a + k \qquad (2)$$

Subtract (2) from (1):

$$6 = \frac{a}{2} + a \qquad (3)$$

Multiply both sides of equation (3) by 2:

$$12 = a + 2a$$

$$a = 4$$

From equation (2): $k = 5$.

The equation is $y = \frac{4}{x} + 5$.

Explanation

The general technique is to substitute the given values into the general equation

$$y = \frac{a}{x - h} + k$$

In this case $h = 0$ and the values of a and k are unknown.

Simultaneous equations need to be formed and then solved.

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Example 14

A graph which has rule $y = a\sqrt{x-h}$ passes through the points $(4, 2)$ and $(7, 4)$. Find the values of a and h.

Solution

When $x = 4$, $y = 2$. When $x = 7$, $y = 4$.

We have the equations

$$2 = a\sqrt{4-h} \qquad (1)$$

$$4 = a\sqrt{7-h} \qquad (2)$$

Divide (2) by (1):

$$2 = \frac{\sqrt{7-h}}{\sqrt{4-h}} \qquad (3)$$

Multiply both sides of equation (3) by $\sqrt{4-h}$:

$$2\sqrt{4-h} = \sqrt{7-h}$$

Square both sides of the equation:

$$\begin{aligned} 4(4-h) &= 7-h \\ 16-4h &= 7-h \\ 3h &= 9 \\ h &= 3 \end{aligned}$$

Substitute in (1) to find $a = 2$.

The required equation is $y = 2\sqrt{x-3}$.

Explanation

The general technique is to substitute the given values into the general equation

$$y = a\sqrt{x-h} + k$$

In this case $k = 0$ and the values of a and h are unknown.

Simultaneous equations need to be formed and then solved. Note that $h \neq 4$ from equation (1).

Example 15

Find the equation of the circle whose centre is at the point $(1, -1)$ and which passes through the point $(4, 3)$.

Solution

Let r be the length of the radius. Then

$$\begin{aligned} r &= \sqrt{(4-1)^2 + (3-(-1))^2} \\ &= \sqrt{3^2 + 4^2} \\ &= 5 \end{aligned}$$

Hence the equation of the circle is

$$(x-1)^2 + (y+1)^2 = 25$$

Explanation

We use the centre–radius form for the equation of a circle:

$$(x-h)^2 + (y-k)^2 = r^2$$

The centre is $(1, -1)$. We need to find the radius.

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Skill-sheet

Exercise 4E

Example 13a

1 The rectangular hyperbola $y = \frac{a}{x} + 3$ passes through the point $(1, 8)$. Find the value of a.

2 A rectangular hyperbola with rule of the form

$$y = \frac{a}{x - h} + k$$

has vertical asymptote $x = 3$, horizontal asymptote $y = 4$ and passes through the point $(0, 6)$. Find the values of a, h and k.

Example 13b

3 The rectangular hyperbola $y = \frac{a}{x} + k$ passes through the points $(1, 8)$ and $(-1, 7)$. Find the values of a and k.

4 A rectangular hyperbola with rule of the form

$$y = \frac{a}{x - h} + k$$

has vertical asymptote $x = 2$, horizontal asymptote $y = -4$ and passes through the point $(0, 4)$. Find the values of a, h and k.

5 A graph which has rule $y = a\sqrt{x}$ passes through the point $(2, 8)$. Find the value of a.

Example 14

6 A graph which has rule $y = a\sqrt{x - h}$ passes through the points $(1, 2)$ and $(10, 4)$. Find the values of a and h.

Example 15

7 Find the equation of the circle whose centre is at the point $(2, 1)$ and which passes through the point $(4, -3)$.

8 Find the equation of the circle whose centre is at the point $(-2, 3)$ and which passes through the point $(-3, 3)$.

9 Find the equation of the circle whose centre is at the point $(-2, 3)$ and which passes through the point $(2, 3)$.

10 Find the equation of the circle with centre $(2, -3)$ which touches the x-axis.

11 Find the equation of the circle whose centre lies on the line $y = 4$ and which passes through the points $(2, 0)$ and $(6, 0)$.

12 Find the equations of the circles which touch the x-axis, have radius 5 and pass through the point $(0, 8)$.

13 Find the equation of a circle which passes through the points $A(0, 2)$, $B(2, 0)$ and $C(-4, 0)$.

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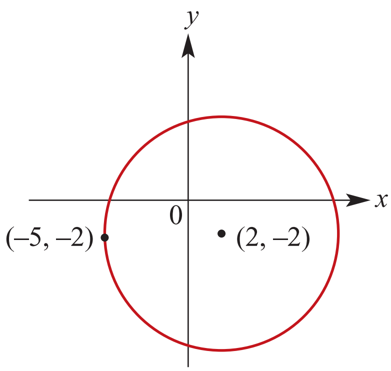
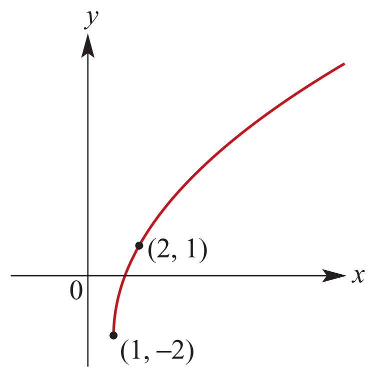
14 Find the rule for each of the following graphs. The general form of the rule is given for each graph.

a $(x - h)^2 + (y - k)^2 = r^2$

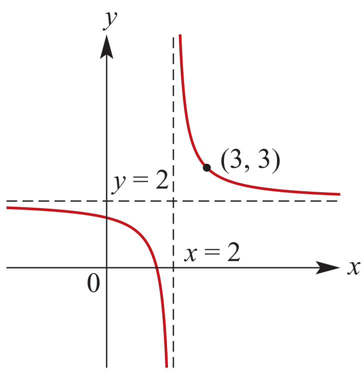


Centre at $(2, -2)$

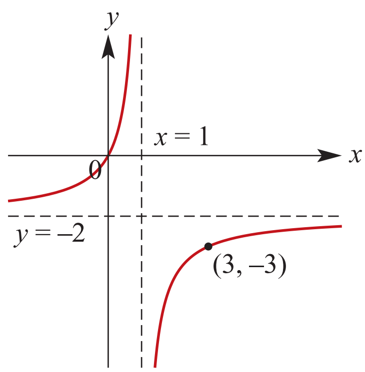
b $y = a\sqrt{x - h} + k$



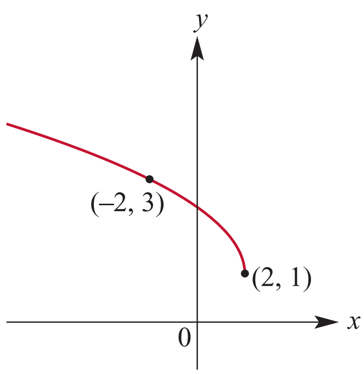
c $y = \dfrac{a}{x - h} + k$



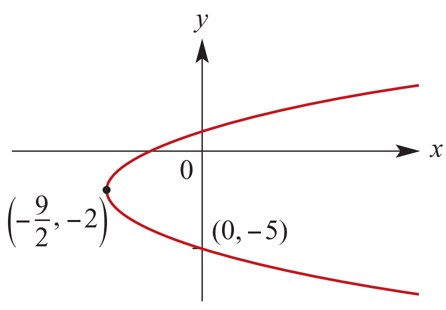
d $y = \dfrac{a}{x - h} + k$



e $y = a\sqrt{h - x} + k$



f $(y - k)^2 = b(x - h)$



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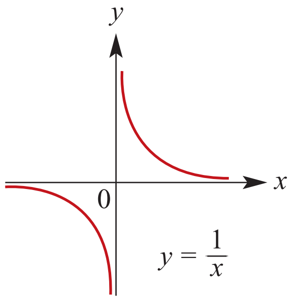
Chapter summary

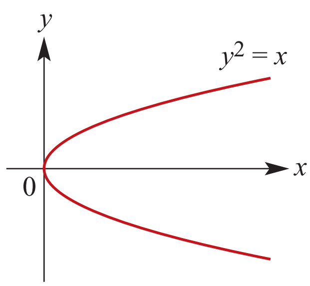
Assign-ment

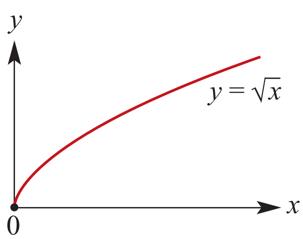
Nrich

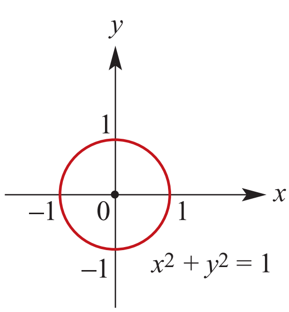
- The standard graphs:



Rectangular hyperbola

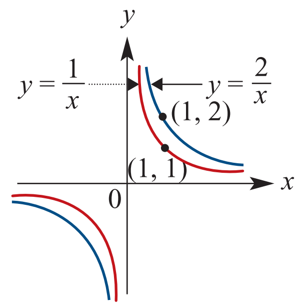


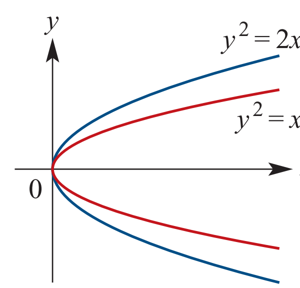


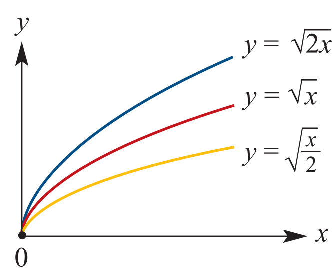


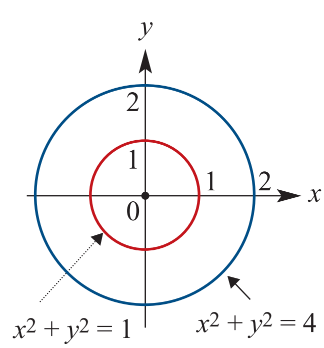
Circle

- Dilations of these graphs:









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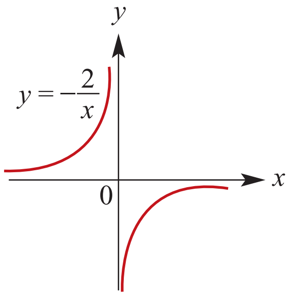
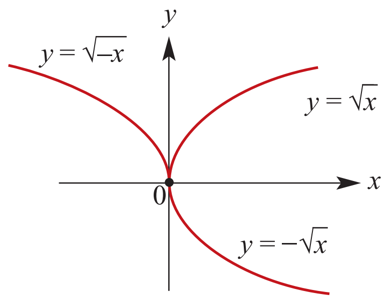
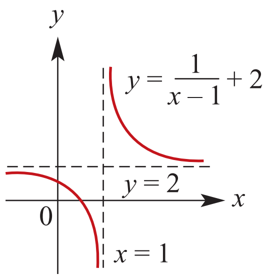
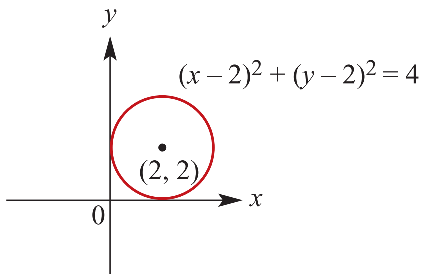
- Reflections in the axes:





- Translations of graphs:





- Equation for a circle with centre at (h, k) and radius r:

$$(x-h)^2 + (y-k)^2 = r^2$$

Alternative form:

$$x^2 + y^2 - 2hx - 2ky + c = 0$$

where $c = h^2 + k^2 - r^2$.

Short-answer questions

1 Sketch the graphs of each of the following:

a $y = \frac{-3}{x}$ **b** $y = \frac{1}{x-1}$ **c** $y = \frac{2}{x+1} + 1$ **d** $y = \frac{-1}{x-1}$

e $y = \frac{4}{2-x} + 3$ **f** $y^2 = x - 2$ **g** $(y-1)^2 = x + 1$ **h** $y = 2\sqrt{x} + 2$

i $y = 2\sqrt{x-3} + 2$ **j** $y = -2\sqrt{x+2} + 2$

2 By completing the square, write each of the following equations in the form $(x-a)^2 + (y-b)^2 = r^2$:

a $x^2 + y^2 - 6x + 4y - 12 = 0$ **b** $x^2 + y^2 - 3x + 5y - 4 = 0$

c $2x^2 + 2y^2 - x + y - 4 = 0$ **d** $x^2 + y^2 + 4x - 6y = 0$

e $x^2 + y^2 = 6(x + y)$ **f** $x^2 + y^2 = 4x - 6y$

3 For the circle $x^2 + y^2 - 4x + 6y = 14$, find the equation of the diameter which passes through the origin.

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4 For the circle $x^2 + y^2 - 3x + 2y = 26$, find the equation of the diameter which cuts the x-axis at an angle of 45°.

5 Find the equation of the circle with centre C and radius r for each of the following and sketch the graph:

a $C(3, 4),\ r = 5$
b $C(-1, 0),\ r = 1$
c $C(4, 4),\ r = 2$
d $C\left(\frac{1}{2}, -\frac{1}{3}\right),\ r = \frac{1}{6}$

6 The equation of a circle is $x^2 + y^2 + 4x - 6y = 23$. Find the centre and radius.

7 Find the length cut off on the x-axis and y-axis by the circle $x^2 + y^2 - 2x - 4y = 20$.

8 Sketch the graphs of the following semicircles:

a $y = \sqrt{9 - x^2}$
b $y = -\sqrt{16 - (x + 1)^2}$
c $y - 2 = -\sqrt{1 - x^2}$
d $y + 3 = \sqrt{4 - (x + 2)^2}$

Extended-response questions

The following questions also involve techniques developed in Chapters 2 and 3.

1 The line with equation $y = mx$ is tangent to the circle with centre $(10, 0)$ and radius 5 at the point $P(x, y)$.

a Find the equation of the circle.
b Show that the x-coordinate of the point P satisfies the equation $(1 + m^2)x^2 - 20x + 75 = 0$.
c Use the discriminant for this equation to find the exact value of m.
d Find the coordinates of P. (There are two such points.)
e Find the distance of P from the origin.

2 A circle has its centre at the origin and radius 4.

a Find the equation of this circle.
b Two lines which pass through the point $(8, 0)$ are tangents to this circle.
  i Show that the equations of these tangents are of the form $y = mx - 8m$.
  ii Use techniques similar to those used in Question 1 to find the value of m and, hence, the equations of the tangents.

3 A circle has centre at the origin and radius 5. The point $P(3, 4)$ lies on the circle.

a Find the gradient of the line segment OP, where O is the origin.
b Find the gradient of the tangent to the circle at P.
c Find the equation of the tangent at P.
d If the tangent crosses the x-axis at A and the y-axis at B, find the length of line segment AB.

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4 Let $P(x_1, y_1)$ be a point on the circle with equation $x^2 + y^2 = a^2$.

a **i** Give the gradient of the line segment OP, where O is the origin.

ii Give the gradient of the tangent to the circle at P.

b Show that the equation of the tangent at $P(x_1, y_1)$ is $x_1 x + y_1 y = a^2$.

c If $x_1 = y_1$ and $a = 4$, find the equations of the possible tangents.

5 An equilateral triangle ABC circumscribes the circle with equation $x^2 + y^2 = a^2$. The side BC of the triangle has equation $x = -a$.

a Find the equations of AB and AC.

b Find the equation of the circle circumscribing triangle ABC.

6 Consider the curve with equation $y = \sqrt{x - b} + c$.

a Show that if the curve meets the line with equation $y = x$ at the point (a, a), then a satisfies the equation $a^2 - (2c + 1)a + c^2 + b = 0$.

b **i** If the line with equation $y = x$ is a tangent to the curve, show that $c = \dfrac{4b - 1}{4}$.

ii Sketch the graph of $y = \sqrt{x} - \dfrac{1}{4}$ and find the coordinates of the point on the graph at which the line with equation $y = x$ is a tangent.

c Find the values of k for which the line with equation $y = x + k$:

i meets the curve with equation $y = \sqrt{x} - \dfrac{1}{4}$ twice

ii meets the curve with equation $y = \sqrt{x} - \dfrac{1}{4}$ once

iii does not meet the curve with equation $y = \sqrt{x} - \dfrac{1}{4}$.

7 For the curve with equation $y = \sqrt{x} - 1$ and the straight line with equation $y = kx$, find the values of k such that:

a the line meets the curve twice

b the line meets the curve once.

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5 Variation

In this chapter

Syllabus references

Topic: Inverse proportion

Subtopic: 1.1.13

People working in science, finance and many other areas look for relationships between two quantities of interest. These relationships often turn out to be either linear or hyperbolic. That is, the graph relating these quantities is either a straight line or a hyperbola. In the first case we have **direct variation** and in the second case **inverse variation**.

For example, consider the formula $d = vt$ connecting distance travelled (d), time taken (t) and speed (v).

- For a fixed speed, the distance travelled varies directly as the time taken.
- For a fixed distance, the time taken varies inversely as the speed: $t = \frac{d}{v}$.

Another familiar example is when an object is moving with constant acceleration a. The distance travelled varies directly as the square of the time taken: $d = \frac{1}{2}at^2$.

Throughout this chapter we explore many more applications of variation.

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5A Direct variation

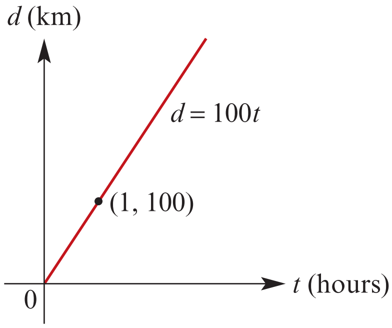
Emily drives from her home in Appleton to visit her friend Kim, who lives 600 km away in Brownsville. She drives at a constant speed, and each hour notes how far she has travelled.

Time (t hours)	1	2	3	4	5	6
Distance (d km)	100	200	300	400	500	600

It can be seen that, as t increases, d also increases. The rule relating time and distance is $d = 100t$.

This is an example of **direct variation**, and 100 is the **constant of variation**.

We can say that the distance travelled **varies directly** as the time taken, or that d is **proportional** to t. The graph of d against t is a straight line passing through the origin.



- The variable y is said to **vary directly** as x if $y = kx$, for some positive constant k.
- The constant k is called the **constant of variation**.
- The statement 'y varies directly as x' is written symbolically as $y \propto x$.

Note: For the above example, we can write $d \propto t$, and the constant of variation is the speed. Rates are one of the most common examples of variation.

Variation involving powers

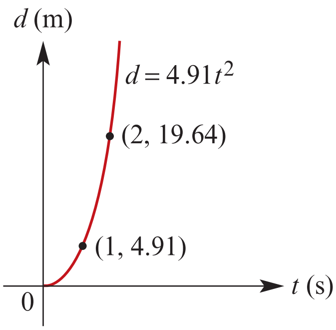
A metal ball is dropped from the top of a tall building and the distance it has fallen is recorded each second.

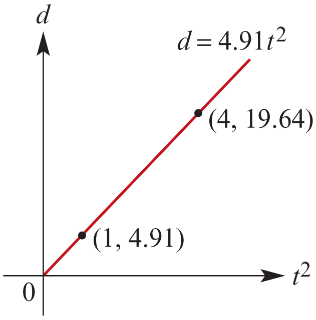
Time (t s)	0	1	2	3	4	5
Distance (d m)	0	4.91	19.64	44.19	78.56	122.75

As t increases, d also increases. The rule relating time and distance is $d = 4.91t^2$.

This is another example of direct variation. In this case, we say that the distance fallen varies directly as the square of the time taken, or that d is proportional to t^2. We write $d \propto t^2$.

The graph of d against t^2 is a straight line passing through the origin.





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- If $y \propto x^n$, then $y = kx^n$, where k is a **constant of variation**.
- If $y \propto x^n$, then the graph of y against x^n is a straight line passing through the origin.

For all examples of direct variation (where k is positive), as one variable increases the other will also increase. The graph of y against x will show an upward trend. It should be noted, however, that not all increasing trends are examples of direct variation.

Determining the constant of variation

If $y \propto x^n$, then the constant of variation can be found provided just one value of x and the corresponding value of y are given.

Example 1

In each case, use the table of values to determine the constant of variation k, and hence complete the table:

a $y \propto x^2$

x	2	4	6	
y	12		108	192

b $y \propto \sqrt{x}$ (i.e. $y \propto x^{\frac{1}{2}}$)

x	2	4	6	
y		1	1.225	1.414

Solution

a If $y \propto x^2$, then

$$y = kx^2$$

When $x = 2$, $y = 12$, and so

$$12 = k(2^2)$$
$$k = 3$$
$$\therefore \quad y = 3x^2$$

Check:

When $x = 6$, $y = 3(6^2) = 108$

To complete the table, consider the following:

When $x = 4$, $\quad y = 3(4^2) = 48$

When $y = 192$, $\quad 192 = 3x^2$
$$64 = x^2$$
$$x = 8$$

x	2	4	6	8
y	12	48	108	192

b If $y \propto \sqrt{x}$, then

$$y = k\sqrt{x}$$

When $x = 4$, $y = 1$, and so

$$1 = k\sqrt{4}$$
$$k = 0.5$$
$$\therefore \quad y = 0.5\sqrt{x}$$

Check:

When $x = 6$, $y = 0.5\sqrt{6} \approx 1.225$

To complete the table, consider the following:

When $x = 2$, $\quad y = 0.5\sqrt{2} \approx 0.707$

When $y = 1.414$, $\quad 1.414 = 0.5\sqrt{x}$
$$2.828 = \sqrt{x}$$
$$x \approx 8$$

x	2	4	6	8
y	0.707	1	1.225	1.414

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Example 2

In an electrical wire, the resistance (R ohms) varies directly as the length (L m) of the wire.

a If a 6 m wire has a resistance of 5 ohms, what is the resistance of a 4.5 m wire?

b How long is a wire for which the resistance is 3.8 ohms?

Solution

First determine the constant of variation.

$$R \propto L$$
$$\therefore \quad R = kL$$

When $L = 6$, $R = 5$, and so

$$5 = k(6)$$
$$\therefore \quad k = \frac{5}{6}$$

Hence $R = \dfrac{5L}{6}$

a When $L = 4.5$, $R = \dfrac{5 \times 4.5}{6}$

$= 3.75$

A wire of length 4.5 m has a resistance of 3.75 ohms.

b When $R = 3.8$, $3.8 = \dfrac{5L}{6}$

$L = 4.56$

A wire of resistance 3.8 ohms has a length of 4.56 m.

Increasing and decreasing

If one quantity varies directly as another, we can work out what happens to one of the quantities when the other is changed.

Example 3

The volume of a sphere varies directly as the cube of its radius. By what percentage will the volume increase if the radius is:

a doubled **b** increased by 20%?

Solution

$V \propto r^3$, i.e. $V = kr^3$

Initially, set the radius equal to 1. Then $V = k(1^3) = k$.

a If r is doubled, set $r = 2$.
Then $V = k(2^3) = 8k$.
The volume has increased from k to $8k$, an increase of $7k$.

$$\therefore \quad \% \text{ increase in volume} = \frac{7k}{k} \times \frac{100}{1} = 700\%$$

b If r is increased by 20%, set $r = 1.2$.
Then $V = k(1.2^3) = 1.728k$.

$$\therefore \quad \% \text{ increase in volume} = 72.8\%$$

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Summary 5A

Direct variation

- The variable y **varies directly** as x if $y = kx$, for some positive constant k.
  We can also say that y is **proportional** to x, and we can write $y \propto x$.
- The constant k is called the **constant of variation** or **constant of proportionality**.
- If y is proportional to x, then the graph of y against x is a straight line through the origin. The gradient of the line is the constant of proportionality.
- If $y \propto x$, then

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} = k$$

for any two non-zero values x_1 and x_2 and the corresponding values y_1 and y_2.

Exercise 5A

Example 1

1 For each of the following, determine the constant of variation k and hence complete the table of values:

a $y \propto x^2$

x	2	4	6	
y	8	32		128

b $y \propto x$

x	$\frac{1}{2}$	1	$\frac{3}{2}$	
y	$\frac{1}{6}$		$\frac{1}{2}$	$\frac{2}{3}$

c $y \propto \sqrt{x}$

x	4	9	49	
y	6	9		90

d $y \propto x^{\frac{1}{5}}$

x	$\frac{1}{32}$	1	32	
y	$\frac{1}{5}$	$\frac{2}{5}$		$\frac{8}{5}$

2 If $V \propto r^3$ and $V = 125$ when $r = 2.5$, find:

a V when $r = 3.2$

b r when $V = 200$

3 If $a \propto b^{\frac{2}{3}}$ and $a = \frac{2}{3}$ when $b = 1$, find:

a a when $b = 2$

b b when $a = 2$

Example 2

4 The area (A) of a triangle of fixed base length varies directly as its perpendicular height (h). If the area of the triangle is 60 cm^2 when its height is 10 cm, find:

a the area when its height is 12 cm

b the height when its area is 120 cm^2.

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5 The extension in a spring (E) varies directly with the weight (w) suspended from it. If a weight of 452 g produces an extension of 3.2 cm, find:

a the extension produced by a weight of 810 g

b the weight that would produce an extension of 10 cm.

6 The weight (W) of a square sheet of lead varies directly with the square of its side length (L). If a sheet of side length 20 cm weighs 18 kg, find the weight of a sheet that has an area of 225 cm^2.

7 The volume (V) of a sphere varies directly with the cube of its radius (r). A sphere of radius 10 cm has a volume of 4188.8 cm^3. Find the radius of a sphere that has a volume of 1 cubic metre.

Example 3

8 The surface area of a sphere varies directly as the square of its radius. By what percentage will the surface area increase if the radius is:

a doubled **b** tripled **c** increased by 10%?

9 The energy generated by a wind turbine is proportional to the cube of the wind speed. By what percentage will the energy increase if the wind speed increases by 15%?

10 The time taken for one complete oscillation of a pendulum is called its period. The period (T) of a pendulum varies directly with the square root of the length (L) of the pendulum. A pendulum of length 60 cm has a period of 1.55 seconds. Find the period of a pendulum that is one and a half times as long.

11 The distance (d) to the visible horizon varies directly with the square root of the height (h) of the observer above sea level. An observer 1.8 m tall can see 4.8 km out to sea when standing on the shoreline.

a How far could the person see if they climbed a 4 m tower?

b If the top of a 10 m mast on a yacht is just visible to the observer in the tower, how far out to sea is the yacht?

12 For each of the following, calculate the percentage change in y when x is:

a doubled **b** halved **c** reduced by 20% **d** increased by 40%

i $y \propto x^2$ **ii** $y \propto \sqrt{x}$ **iii** $y \propto x^3$

5B Inverse variation

A builder employs a number of bricklayers to build a brick wall. Three bricklayers will complete the wall in 8 hours. But if he employs six bricklayers, the wall will be complete in half the time. The more bricklayers he employs, the shorter the time taken to complete the wall. The time taken (t) decreases as the number of bricklayers (b) increases.

This is an example of **inverse variation**: the time taken to complete the wall **varies inversely** as the number of bricklayers. We can also say that t is **inversely proportional** to b.

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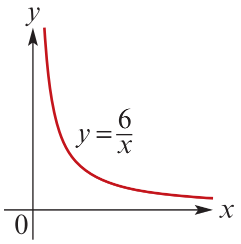
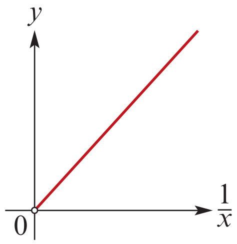
- The variable y is said to **vary inversely** as x if $y = \frac{k}{x}$, for some positive constant k called the **constant of variation**.
- If y varies inversely as x, then y varies directly as $\frac{1}{x}$ and so we can write $y \propto \frac{1}{x}$.

Note: $y \propto \frac{1}{x}$ is equivalent to $xy = k$, for a positive constant k. That is, the product is constant. This is often a useful way to check for inverse variation when given data in table form.

For example, if $y = \frac{6}{x}$, then $y \propto \frac{1}{x}$ and the constant of variation is 6.

x	$\frac{1}{3}$	$\frac{1}{2}$	1	2
$\frac{1}{x}$	3	2	1	$\frac{1}{2}$
y	18	12	6	3





The points given by the first and third rows of the table lie on the left-hand graph, and the points given by the second and third rows lie on the right-hand graph.

In general, if $y \propto \frac{1}{x^n}$, then the graph of y against $\frac{1}{x^n}$ will be a straight line. However, since $\frac{1}{x^n}$ never equals 0, the line will not be defined at the origin.

For all examples of inverse variation, as one variable increases the other will decrease, and vice versa. The graph of y against x will show a downward trend. It should be noted, however, that not all decreasing trends are examples of inverse variation.

Example 4

In each case, determine the constant of variation k and hence complete the table:

a $y \propto \frac{1}{x^2}$

x	2	5	10	
y	0.1	0.016		0.001

b $y \propto \frac{1}{\sqrt{x}}$

x	1	4		100
y	10		2	1

Solution

a $y = \frac{k}{x^2}$

When $x = 2$, $y = 0.1$, and so

$$0.1 = \frac{k}{2^2}$$

$$k = 0.4$$

$$\therefore \quad y = \frac{0.4}{x^2}$$

b $y = \frac{k}{\sqrt{x}}$

When $x = 1$, $y = 10$, and so

$$10 = \frac{k}{\sqrt{1}}$$

$$k = 10$$

$$\therefore \quad y = \frac{10}{\sqrt{x}}$$

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a continued

Check:

When $x = 5$, $y = \frac{0.4}{5^2} = 0.016$

To complete the table, consider the following:

When $x = 10$, $y = \frac{0.4}{10^2} = 0.004$

When $y = 0.001$, $0.001 = \frac{0.4}{x^2}$

$x^2 = \frac{0.4}{0.001}$

$x = 20$

x	2	5	10	20
y	0.1	0.016	0.004	0.001

b continued

Check:

When $x = 100$, $y = \frac{10}{\sqrt{100}} = 1$

To complete the table, consider the following:

When $x = 4$, $y = \frac{10}{\sqrt{4}} = 5$

When $y = 2$, $2 = \frac{10}{\sqrt{x}}$

$\sqrt{x} = 5$

$x = 25$

x	1	4	25	100
y	10	5	2	1

Example 5

For a cylinder of fixed volume, the height (h cm) is inversely proportional to the square of the radius (r cm).

a What percentage change in the height would result if its radius were reduced by 25%?

b If a cylinder of height 15 cm has a base radius of 4.2 cm, how high would a cylinder of equivalent volume be if its radius were 3.5 cm?

Solution

a $h \propto \frac{1}{r^2}$, i.e. $h = \frac{k}{r^2}$

If $r = 1$, then $h = \frac{k}{1^2} = k$.

If r is reduced by 25%, set $r = 0.75$.

Then $h = \frac{k}{(0.75)^2}$

$= \frac{k}{0.5625}$

$\approx 1.778k$ (to 3 d.p.)

Hence h is increased by 77.8%.

b $h = \frac{k}{r^2}$

When $h = 15$, $r = 4.2$, and so

$15 = \frac{k}{(4.2)^2}$

$k = 15(4.2)^2 = 264.6$

$\therefore\ h = \frac{264.6}{r^2}$

If $r = 3.5$, then

$h = \frac{264.6}{(3.5)^2} = 21.6$

A cylinder of radius 3.5 cm has a height of 21.6 cm.

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Summary 5B

Inverse variation

- The variable y **varies inversely** as x if $y = \frac{k}{x}$, for some positive constant k.
  We can also say that y is **inversely proportional** to x, and we can write $y \propto \frac{1}{x}$.
- If y varies inversely as x, then the graph of y against $\frac{1}{x}$ is a straight line (not defined at the origin) and the gradient is the constant of variation.
- If $y \propto \frac{1}{x}$, then $x_1y_1 = x_2y_2 = k$, for any two values x_1 and x_2 and the corresponding values y_1 and y_2.

Skill-sheet

Exercise 5B

Example 4

1 For each of the following, determine the constant of variation k and hence complete the table of values:

a $y \propto \frac{1}{x}$

x	2	4	6	
y	1	$\frac{1}{2}$		$\frac{1}{16}$

b $y \propto \frac{1}{\sqrt{x}}$

x	$\frac{1}{4}$	1		9
y	1	$\frac{1}{2}$	$\frac{1}{4}$	

c $y \propto \frac{1}{x^2}$

x	1	2	3	
y	3	$\frac{3}{4}$		$\frac{1}{12}$

d $y \propto \frac{1}{x^{\frac{1}{3}}}$

x	$\frac{1}{8}$	1		125
y	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{9}$	

2 If $a \propto \frac{1}{b^3}$ and $a = 4$ when $b = \sqrt{2}$, find:

a a when $b = 2\sqrt{2}$ **b** b when $a = \frac{1}{16}$

3 If $a \propto \frac{1}{b^4}$ and $a = 5$ when $b = 2$, find:

a a when $b = 4$ **b** b when $a = 20$

4 The gas in a cylindrical canister occupies a volume of 22.5 cm^3 and exerts a pressure of 1.9 kg/cm^2. If the volume (V) varies inversely as the pressure (P), find the pressure if the volume is reduced to 15 cm^3.

Example 5

5 The current (I amperes) that flows in an electrical appliance varies inversely as the resistance (R ohms). If the current is 3 amperes when the resistance is 80 ohms, find:

a the current when the resistance is 100 ohms

b the increase in resistance required to reduce the current to 80% of its original value.

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6 The intensity of illumination (I) of a light is inversely proportional to the square of the distance (d) from the light. At a distance of 20 m, a light has an intensity of 100 candelas. Find the intensity of the light at a distance of 25 m.

7 The radius (r) of a cylinder of fixed volume varies inversely as the square root of its height (h). If the height is 10 cm when the radius is 5.64 cm, find the radius if the height is 12 cm.

8 For each of the following, calculate the percentage change in y when x is:

a doubled **b** halved **c** reduced by 20% **d** increased by 40%

i $y \propto \frac{1}{x^2}$ **ii** $y \propto \frac{1}{\sqrt{x}}$ **iii** $y \propto \frac{1}{x^3}$

5C Fitting data

Sometimes the type of relationship that exists between two variables a and b is not known. By inspection of a table of values, it may be possible to ascertain whether the relationship between the variables is direct or inverse proportion. Analysis is required to establish the rule that best fits the given data. This may involve graphing the data.

Example 6

Establish the relationship between the two variables for the given table of values.

b	0	2	4	6	8
a	0	12	48	108	192

Solution

From the table, we see that as b increases, a also increases, and that when $b = 0$, $a = 0$. Therefore we can conjecture that some type of direct variation exists.

Assume that $a \propto b^n$ for some positive number n.

i.e. $a = kb^n \quad \therefore \quad k = \frac{a}{b^n}$

Choose a value for n (it must be a positive number) and test each pair of values in the table, except $(0, 0)$. If the value of k is the same for each pair, the choice of n is correct.

- Let $n = 1$. Then $k = \frac{a}{b}$.

  Consider $\frac{a}{b}$ for the values in the table.

  Testing: $\frac{12}{2} = 6, \quad \frac{48}{4} = 12,$

  $\frac{108}{6} = 18, \quad \frac{192}{8} = 24$

  The quotients are not all the same, and so $n \neq 1$.

- Let $n = 2$. Then $k = \frac{a}{b^2}$.

  Consider $\frac{a}{b^2}$ for the values in the table.

  Testing: $\frac{12}{4} = 3, \quad \frac{48}{16} = 3,$

  $\frac{108}{36} = 3, \quad \frac{192}{64} = 3$

  The quotients are all equal to 3. Thus $k = 3$ and $n = 2$. Hence $a = 3b^2$.

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Example 7

Establish the relationship between the two variables for the given table of values.

x	1	3	6	12	15
y	30	10	5	2.5	2

Solution

From the table, we see that as x increases, y decreases. Therefore we conjecture that some type of inverse variation exists.

Assume that $y \propto \frac{1}{x^n}$ for some positive number n.

i.e. $y = \frac{k}{x^n}$ $\quad \therefore \quad k = yx^n$

- Let $n = 1$. Then $k = yx$.
  Consider the product yx for the values given in the table.

  Testing: $30 \times 1 = 30$, $\quad 10 \times 3 = 30$, $\quad 5 \times 6 = 30$, $\quad 2.5 \times 12 = 30$, $\quad 2 \times 15 = 30$

  Thus $k = 30$ and $n = 1$. Hence $y = \frac{30}{x}$.

The type of variation can also be investigated by graphical analysis. By plotting the graph of a against b, an upward trend *may* indicate direct variation or a downward trend *may* indicate inverse variation.

To find the specific type of variation that exists, the following can be used as a guide:

- **Direct variation** If $a \propto b^n$, then the graph of a against b^n will be a straight line through the origin. The gradient of this line will be the constant of variation k.
- **Inverse variation** If $a \propto \frac{1}{b^n}$, then the graph of a against $\frac{1}{b^n}$ will be a straight line not defined at the origin. The gradient of this line will be the constant of variation k.

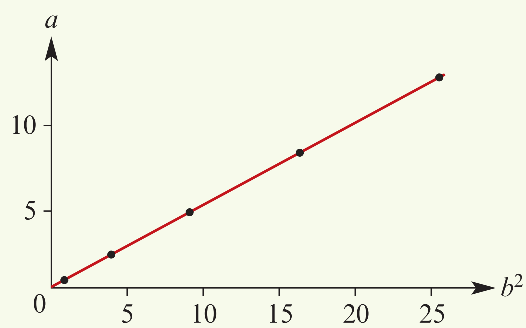
Example 8

For the given table of values, plot the graph of a against b^2 and hence establish the rule relating a to b.

b	1	2	3	4	5
a	0.5	2	4.5	8	12.5

Solution

b^2	1	4	9	16	25
a	0.5	2	4.5	8	12.5



The graph is a straight line, and so it can be conjectured that the relationship is $a = kb^2$, where k is the gradient of the line.

From the graph, it can be seen that $a = \frac{1}{2}b^2$.

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If the relationship between two variables x and y is of the form $y = kx^n$, where $k > 0$ and $n \neq 0$, then a CAS calculator can be used to find n and k given sufficient information.

Example 9

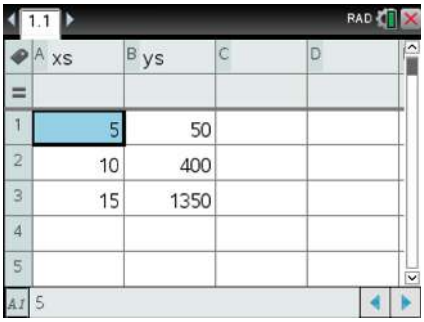
The following data was collected, recording the number of calls (N) made to a company D days after the start of an advertising campaign.

Days (D)	5	10	15
Number of calls (N)	50	400	1350

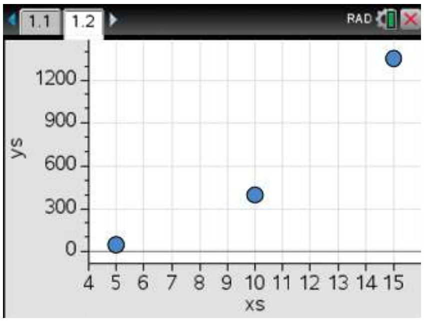
Find a relationship between N and D using a CAS calculator.

Using the TI-Nspire

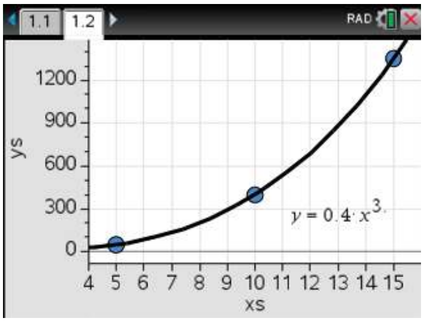
- In a **Lists & Spreadsheet** page, enter the data in lists *xs* and *ys* as shown.



- Insert a **Data & Statistics** page (ctrl I > **Add Data & Statistics**).
- Click on the 'Click to add variable' box and select *xs* from the dropdown menu.
- Repeat for the y-axis and select *ys*.



- Use menu > **Analyze** > **Regression** > **Show Power**.
- This gives $y = 0.4x^3$ and so the required relationship is $N = 0.4D^3$.



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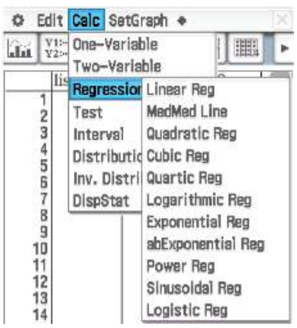
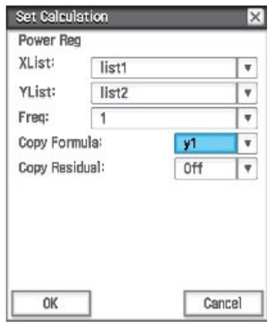
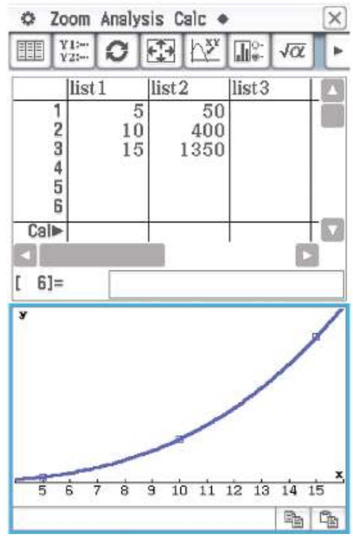
Using the Casio ClassPad

- Go to Menu and open **Statistics** Statistics.
- Enter the data values as shown.
- Select **Calc** > **Regression** > **Power Reg**.
- Ensure that the **Set Calculation** settings are as shown below.

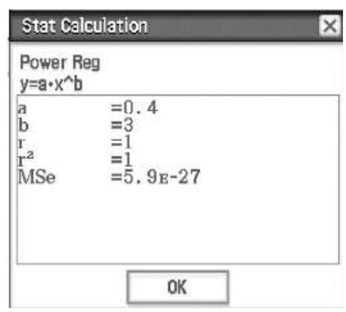






Note: Setting 'Copy Formula' to $y1$ will store the regression equation in the **Graph & Table** application.

- Tap OK to obtain the equation $y = 0.4x^3$. The required relationship is $N = 0.4D^3$.
- Tap OK a second time to view the graph of the regression curve.



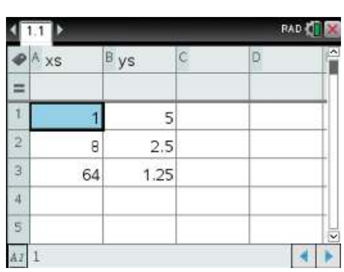
Example 10

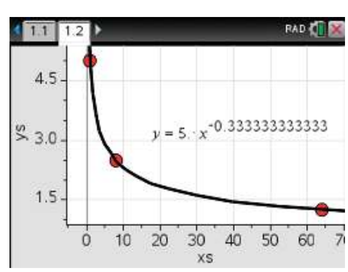
Using the data in the table, establish a rule connecting y and x.

x	1	8	64
y	5	2.5	1.25

Using the TI-Nspire

The rule is $y = 5x^{-\frac{1}{3}} = \dfrac{5}{\sqrt[3]{x}}$





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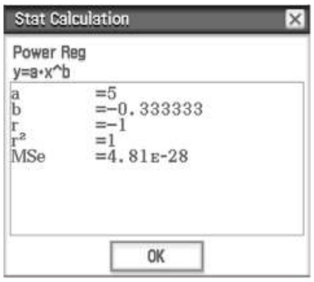
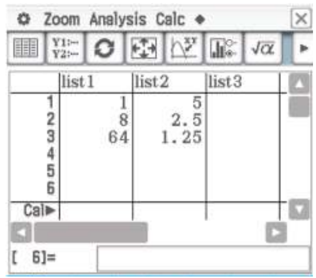
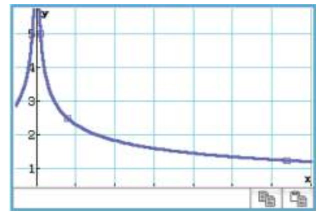
Using the Casio ClassPad

The rule is $y = 5x^{-\frac{1}{3}} = \dfrac{5}{\sqrt[3]{x}}$







Summary 5C

- **Direct variation** If $a \propto b^n$, then the graph of a against b^n will be a straight line through the origin. The gradient of this line will be the constant of variation k.
- **Inverse variation** If $a \propto \dfrac{1}{b^n}$, then the graph of a against $\dfrac{1}{b^n}$ will be a straight line not defined at the origin. The gradient of this line will be the constant of variation k.

Exercise 5C

Example 6
Example 7

1 Each of the tables in parts a to e fits one of the following five types of variation:

- direct $y \propto x$
- inverse $y \propto \dfrac{1}{x}$
- direct square $y \propto x^2$
- inverse square $y \propto \dfrac{1}{x^2}$
- direct square root $y \propto \sqrt{x}$

Establish the relationship between x and y in each case.

a

x	0	3	6	9	12
y	0	2	4	6	8

b

x	1	2	3	4	5
y	4	16	36	64	100

c

x	20	15	10	5	1
y	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1	5

d

x	1	2	3	4	5
y	2	2.828	3.464	4	4.472

e

x	1	1.5	2	2.5	3
y	4	1.78	1	0.64	0.444

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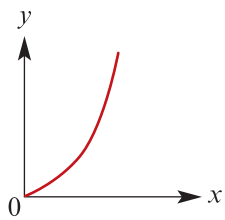
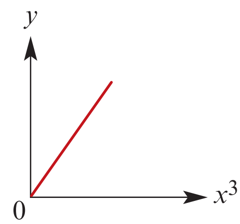
2 Which of the following graphs could represent examples of direct variation?

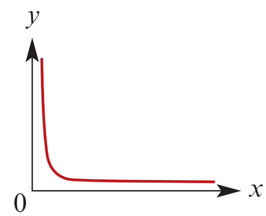
a



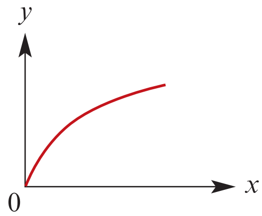
b



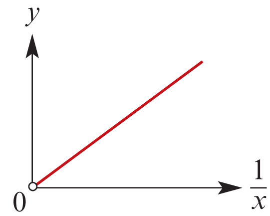
c



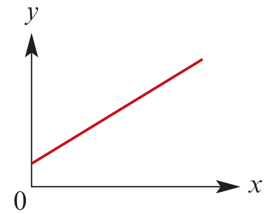
d



e

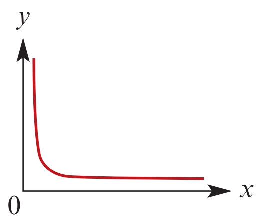


f

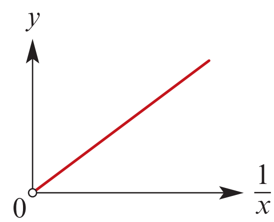


3 Which of the following graphs could represent examples of inverse variation?

a



b



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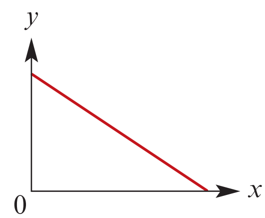
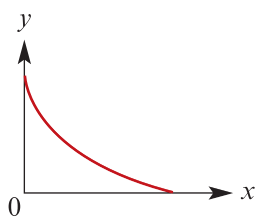
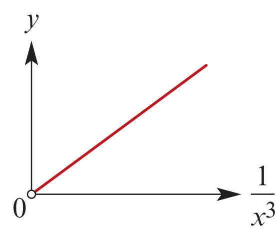


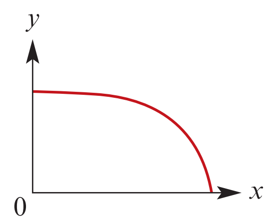
d



e

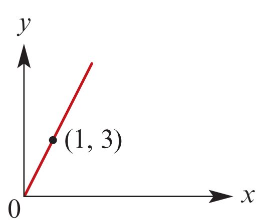


f

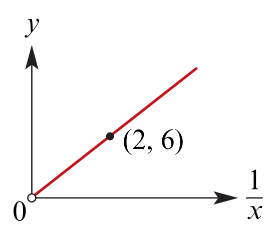


4 Give the rule connecting y and x for each of the following:

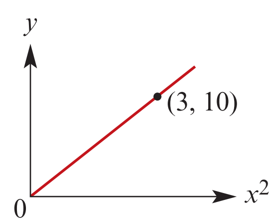
a



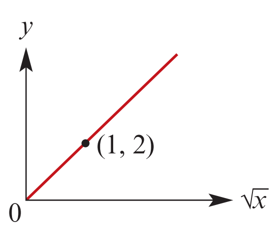
b



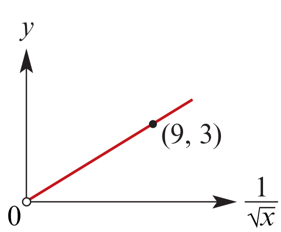
c



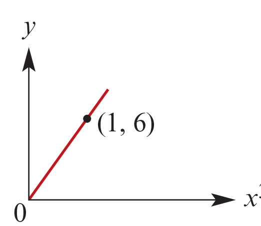
d



e



f



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Example 8

5 Plot the graph of y against x^2 and hence establish the relationship between x and y.

x	2	2.5	3	3.5	4
y	9.6	15	21.6	29.4	38.4

6 Plot the graph of y against $\sqrt{x}$ and hence establish the relationship between x and y.

x	1	4	9	16	25
y	1.5	3	4.5	6	7.5

7 Plot the graph of y against $\frac{1}{x^2}$ and hence establish the relationship between x and y.

x	0.2	0.3	0.4	0.5	1
y	50	22.2	12.5	8	2

Example 9
Example 10

8 Given that $y = ax^b$ for each of the following, use your CAS calculator's power regression function to establish the values of a and b:

a

x	4.00	8.00	12.00	16.00
y	0.50	0.71	0.87	1.00

b

x	1	5	10	15
y	2.00	14.95	35.57	59.04

c

x	1	10	100	1000
y	3.50	8.79	22.08	55.47

d

x	10	20	30	40
y	46.42	73.68	96.55	116.96

e

x	1	2	3	4
y	2.00	0.35	0.13	0.06

f

x	1	3	5	7
y	3.20	2.06	1.68	1.47

9 The concentration of antibodies (C) in an animal's bloodstream depends on the time (t hours) after which the animal is injected with an antigen. The relationship can be modelled by a rule of the form $C = at^b$. The following data is collected.

t	1	2	3	4
C	100	114.87	124.57	131.95

a Find values for a and b. **b** Find the concentration after 10 hours.

10 The level of infestation (I) of a pest in a crop depends on the time (t days) after which the crop is sprayed with an insecticide. The relationship can be modelled by a rule of the form $I = at^b$, for $t \geq 1$. The following data is collected.

t	1	2	3	4
I	1500	1061	866	750

a Find values for a and b. **b** Find the level of infestation after 10 days.

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5D Joint variation

There are many situations where one variable depends on more than one other variable. The variable is said to **vary jointly** as the other variables. For example, the volume of a cylinder varies jointly as the square of the radius and the height:

$$V \propto r^2 h$$

i.e. $V = kr^2 h$

(In this example, the value of k is known to be π.)

Example 11

Given that $y \propto \dfrac{x^2}{z}$, use the following table of values to find the constant of variation k, and hence complete the table.

x	2	4		10
z	10	8	50	
y	2		2.5	4

Solution

$$y = \frac{kx^2}{z}$$

When $x = 2$ and $z = 10$, $y = 2$:

$$2 = \frac{k(2^2)}{10}$$

$$k = 5$$

$$\therefore \quad y = \frac{5x^2}{z}$$

To complete the table, consider the following:

- When $x = 4$ and $z = 8$,

$$y = \frac{5(4^2)}{8} = 10$$

- When $z = 50$ and $y = 2.5$,

$$2.5 = \frac{5(x^2)}{50}$$

$$25 = x^2$$

$$x = 5$$

- When $x = 10$ and $y = 4$,

$$4 = \frac{5(10^2)}{z}$$

$$4z = 500$$

$$z = 125$$

x	2	4	5	10
z	10	8	50	125
y	2	10	2.5	4

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Example 12

The speed (s) of a conveyor belt varies jointly as the diameter (d) of the cog around which it passes and the number of revolutions per second (n) the cog makes. The speed of a belt that passes around a cog of diameter 0.3 m revolving 20 times per second is 18.85 m/s. Find the value of:

a the constant of variation

b the speed of a belt passing around a cog half as big revolving 30 times per second.

Solution

a $s \propto dn$

i.e. $s = kdn$

We know that $s = 18.85$ when $d = 0.3$ and $n = 20$. Therefore

$$18.85 = k(0.3)(20)$$

$$\therefore \quad k = 3.142 \qquad \text{(to 3 d.p.)}$$

Hence $s = 3.142dn$.

b When $d = 0.15$ and $n = 30$,

$$s = 3.142(0.15)(30)$$

$$= 14.14 \text{ m/s} \qquad \text{(to 2 d.p.)}$$

Summary 5D

Joint variation

- If z varies directly as xy, then z is said to **vary jointly** as x and y. We write $z \propto xy$.
- For example, the area (A) of a triangle varies jointly as the base (b) and the height (h). We write $A \propto bh$.

Skill-sheet

Exercise 5D

Example 11

1 Given that $y \propto \dfrac{x}{z}$, use this table of values to determine the constant of variation k, and hence complete the table.

x	2	4		10
z	10	2	60	
y	1	10	0.5	4

2 Given that $y \propto xz$, use this table of values to determine the constant of variation k, and hence complete the table.

x	2	4		10
z	10	8	50	
y	10		25	15

3 Given that $y \propto \dfrac{z}{x^2}$, use this table of values to determine the constant of variation k, and hence complete the table.

x	2	3		10
z	10	4	50	
y	$\frac{15}{2}$	$\frac{4}{3}$	6	4

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4 Assume that a varies directly as b^2 and inversely as c. If $a = 0.54$ when $b = 1.2$ and $c = 2$, find a when $b = 2.6$ and $c = 3.5$.

5 Assume that z varies as the square root of x and inversely as the cube of y. If $z = 1.46$ when $x = 5$ and $y = 1.5$, find z when $x = 4.8$ and $y = 2.3$.

Example 12

6 The potential energy (PE) of an object varies jointly as the mass (m kg) of the object and the height (h m) it is held above the Earth's surface. The potential energy of an 80 kg mass held 10 m above the Earth's surface is 7840 J. Find the value of:

a the constant of variation

b the potential energy of an object with half the mass at a height of 15 m.

7 The simple interest (I) earned on an investment varies jointly as the interest rate (r) and the time (t) for which it is invested. If a sum of money invested at 6.5% per annum for two years earns \$130, how much interest would the same amount of money earn if it were invested at 5.8% for three years?

8 The kinetic energy (E) of an object varies directly as its mass (m) and the square of its velocity (v). If the kinetic energy of an object with a mass of 2.5 kg moving at 15 m/s is 281.25 J, find the energy of an object with a mass of 1.8 kg moving at 20 m/s.

9 The resistance (R) in an electrical wire varies directly as its length (ℓ) and inversely as the square of its diameter (d). Find the percentage change in R if:

a ℓ is increased by 50% and d is decreased by 50%

b ℓ is decreased by 50% and d is increased by 50%.

10 The weight (W) that can be supported by a wooden beam varies directly as the square of its diameter (d) and inversely as its length (ℓ).

a What percentage increase in the diameter would be necessary for a beam twice as long to support an equivalent weight?

b What percentage change in the weight would be capable of being supported by a beam three times as long with twice the diameter?

11 If p varies as the square of q and inversely as the square root of r, what is the effect on p if:

a both q and r are doubled

b q is doubled and r is halved?

12 **a** The tension in a spring (T) varies directly with the extension (x) and inversely with the natural length (ℓ) of the spring. Compare the tension in a spring with a natural length of 3 m that is extended by 1 m with the tension in a second spring with a natural length of 2.7 m that is extended by 0.9 m.

b The work done (W) in stretching a spring varies directly with the square of the extension (x) and inversely with the natural length of the spring (ℓ). Compare the work done on the two springs in part a.

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Review

Assignment

Nrich

Chapter summary

- **Direct variation**

  If $y \propto x^n$, for some $n > 0$:
  - We say that y varies directly as x^n.
  - This means that $y = kx^n$, where k is the constant of variation ($k > 0$).
  - As x increases, y will also increase.
  - The graph of y against x^n is a straight line through the origin.

- **Inverse variation**

  If $y \propto \dfrac{1}{x^n}$, for some $n > 0$:
  - We say that y varies inversely as x^n.
  - This means that $y = \dfrac{k}{x^n}$, where k is the constant of variation ($k > 0$).
  - As x increases, y will decrease.
  - The graph of y against $\dfrac{1}{x^n}$ is a straight line not defined at the origin.

- **Joint variation**

  One variable varies with more than one other variable; this may be a combination of direct and inverse variation. For example:
  - $V \propto r^2 h$ implies $V = kr^2 h$, for some constant k
  - $a \propto \dfrac{c}{\sqrt{b}}$ implies $a = \dfrac{kc}{\sqrt{b}}$, for some constant k

Short-answer questions

1 **a** If $a \propto b^2$ and $a = \dfrac{3}{2}$ when $b = 2$, find a when $b = 4$ and find b when $a = 8$.

b If $y \propto x^{\frac{1}{3}}$ and $y = 10$ when $x = 2$, find y when $x = 27$ and find x when $y = \dfrac{1}{8}$.

c If $y \propto \dfrac{1}{x^2}$ and $y = \dfrac{1}{3}$ when $x = 2$, find y when $x = \dfrac{1}{2}$ and find x when $y = \dfrac{4}{27}$.

d Assume a varies directly as b and inversely as $\sqrt{c}$. If $a = \dfrac{1}{4}$ when $b = 1$ and $c = 4$, find a when $b = \dfrac{4}{9}$ and $c = \dfrac{16}{9}$.

2 The distance, d metres, which an object falls varies directly as the square of the time, t seconds, for which it has been falling. If an object falls 78.56 m in 4 s, find:

a the formula connecting d and t

b the distance fallen in 10 s

c the time taken to fall 19.64 m.

3 The velocity of a falling object (v m/s) varies directly as the square root of the distance (s m) through which it has fallen. An object has a velocity of 7 m/s after falling 2.5 m.

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a Find its velocity after falling 10 m.

b Find the distance through which it falls to attain a velocity of 28 m/s.

c What variables would be plotted on the axes to obtain a straight-line graph?

4 The time taken for a journey is inversely proportional to the average speed of travel. If it takes 4 hours travelling at 30 km/h, how long will it take travelling at 50 km/h?

5 If y varies inversely as x, what is the effect on:

a y if x is doubled

b x if y is doubled

c y if x is halved

d x if y is halved?

6 The cost of running an electrical appliance varies jointly as the time it is run, the electrical resistance and the square of the current. It costs 9 cents to use an appliance of resistance 60 ohms that draws 4 amps of current for 2.5 hours. How much will it cost to use an appliance of resistance 80 ohms that draws 3 amps of current for 1.5 hours?

7 For a constant resistance, the voltage (v volts) of an electrical circuit varies directly as the current (I amps). If the voltage is 24 volts when the current is 6 amps, find the current when the voltage is 72 volts.

8 The intensity of sound varies inversely as the square of the distance of the observer from the source. If the observer moves to twice the distance from the source, compare the second intensity I_2 with the first intensity I_1.

9 If y varies directly as x^2 and inversely as z, find the percentage change in y when x is increased by 10% and z is decreased by 10%.

Extended-response questions

1 A certain type of hollow sphere is designed in such a way that the mass varies as the square of the diameter. Three spheres of this type are made. The first has mass 0.10 kg and diameter 9 cm, the second has diameter 14 cm and the third has mass 0.15 kg. Find:

a the mass of the second sphere

b the diameter of the third sphere.

2 The height (h m) to which a centrifugal pump raises water is proportional to the square of its speed of rotation (n rpm). If the pump raises water to a height of 13.5 m when it is rotating at 200 rpm, find:

a the formula connecting h and n

b the height to which the water can be raised when the pump is rotating at 225 rpm

c the speed required to raise the water to a height of 16 m.

3 The maximum speed of yachts of normal dimensions varies as the square root of their length. If a yacht 20 m long can maintain a maximum speed of 15 knots, find the maximum speed of a yacht 15 m long.

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4 **a** The air in a tube occupies 43.5 cm^3 and the pressure is 2.8 kg/cm^2. If the volume (V) varies inversely as the pressure (P), find the formula connecting V and P.

b Calculate the pressure when the volume is decreased to 12.7 cm^3.

5 The weight (w kg) which a beam supported at each end will carry without breaking varies inversely as the distance (d m) between supports. A beam which measures 6 m between supports will just carry a load of 500 kg.

a Find the formula connecting w and d.

b What weight could the beam carry if the distance between the supports were 5 m?

c What weight could the beam carry if the distance between the supports were 9 m?

6 The table shows the relationship between the pressure and the volume of a fixed mass of gas when the temperature is constant.

Pressure (p)	12	16	18
Volume (v)	12	9	8

a What is a possible equation relating p and v?

b Using this equation, find:

i the volume when the pressure is 72 units

ii the pressure when the volume is 3 units.

c Sketch the graph relating v and $\frac{1}{p}$.

7 Data about the number of pies sold at football matches and the size of the crowds attending has been recorded as follows.

Attendance ($N \times 1000$)	20	30	60
Number of pies sold (P)	15 650	19 170	27 110

a Use a CAS calculator to find an approximate relationship between N and P of the form $P = aN^b$.

b The crowd predicted for a forthcoming match is 55 000. Assuming the model found in part a applies, how many pies would the caterers anticipate selling on that day?

c The caterers have only 25 000 pies available for sale. Again assuming the model found in part a applies, what is the maximum crowd the caterers might be hoping for if they are able to satisfy all customers wanting pies?

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Review

8 The effectiveness of an anaesthetic drug is being tested by varying the dose (d mL) given to patients and recording both the time (t min) for the patient to lose consciousness and the time (T min) for the patient to regain consciousness. The following data has been recorded.

Dosage (d mL)	10	30	60
Time to lose consciousness (t min)	36	4	1
Time to regain consciousness (T min)	14	126	504

a Establish the relationship between d and t. (Assume that t is proportional to a power of d.)

b Establish the relationship between d and T. (Assume that T is proportional to a power of d.)

c If it is desirable to have a particular patient unconscious for no longer than 80 minutes, what is the maximum dose of the drug that should be given?

d How long would it take that patient to lose consciousness?

e Another patient is given a dose of 20 mL. How long will it take for the patient to lose consciousness and how long will they remain unconscious?

9 The German astronomer Johannes Kepler collected data on the mean distance from the Sun to the planets ($R \times 10^6$ km) and the period of the orbit (T years). He was able to establish a relationship between R and T.

a **i** Using the data in the table (approximations only), establish the relationship between R and T. (Assume that T is proportional to a power of R.)

ii Complete the table by finding the period of orbit of the remaining planets.

Planet	Approximate radius of orbit ($R \times 10^6$ km)	Period of orbit (T years)
Mercury	58	0.24
Venus	108	0.61
Earth	150	1
Mars	228	
Jupiter	779	
Saturn	1427	
Uranus	2870	
Neptune	4497	

b A comet orbits the Sun every 70 years. What is its radius of orbit?

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10 To test the effectiveness of an advertising campaign for cheap flights to Hawaii, a travel agent keeps a record of the number of enquiries she receives. It is estimated that the number of enquiries, E, depends on the number of times, n, that the advertisement is shown on television. The following data is collected.

Number of advertisements (n)	10	20	30
Number of enquiries (E)	30	40	47

a Assuming that the relationship between the number of enquiries and the number of advertisements is modelled by the rule $E = an^b$, use your CAS calculator to find values for a and b.

b Predict the number of enquiries received if the advertisement is shown 100 times.

After two weeks, the advertisement has been shown 50 times and the advertising campaign is stopped. However, the travel agent continues to receive enquiries. It is now estimated that the number of enquiries, E, depends on the number of days, d, since the advertising campaign stopped. The following data is collected.

Number of days (d)	3	5	7	10
Number of enquiries (E)	45	25	17	11

c Assuming that the relationship between the number of enquiries and the number of days is modelled by $E = kd^p$, use your CAS calculator to find values for k and p.

d Predict the number of enquiries received on the 14th day after the advertising campaign finished.

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6 Functions, relations and transformations

In this chapter

Syllabus references

Topics: Functions; Language of events and sets

Subtopics: 1.1.23 – 1.1.28; 1.3.6 – 1.3.8

In this chapter we introduce the notation that will be used throughout the rest of the book. You will have met much of it before and this will serve as revision. The language introduced in this chapter helps to express important mathematical ideas precisely. Initially they may seem unnecessarily abstract, but later in the book you will find them used more and more in practical situations.

In Chapters 2 and 3 we looked at linear polynomials and quadratic polynomials, and in Chapter 4 we studied rectangular hyperbolas, square-root graphs, circles and semicircles. These are all examples of relations. You will meet them all again in this chapter, but using a new notation which will be carried through into the following chapters of this book.

We have studied transformations in the earlier chapters of this book without any systematic consideration of transformations of the points of the plane in general. In this chapter we offer the opportunity to develop techniques for both applying and identifying transformations.

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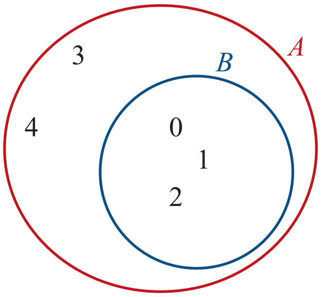
6A Set notation and sets of numbers

Set notation

Set notation is used widely in mathematics and in this book where appropriate. This section summarises all of the set notation you will need.

- A **set** is a collection of objects.
- The objects that are in the set are known as **elements** or members of the set.
- If x is an element of a set A, we write $x \in A$. This can also be read as 'x is a member of the set A' or 'x belongs to A' or 'x is in A'. For example: $2 \in$ set of even numbers.
- If x is **not an element** of A, we write $x \notin A$. For example: $2 \notin$ set of odd numbers.
- Set B is called a **subset** of set A if every element of B is also an element of A. We write $B \subseteq A$. This expression can also be read as 'B is contained in A' or 'A contains B'.

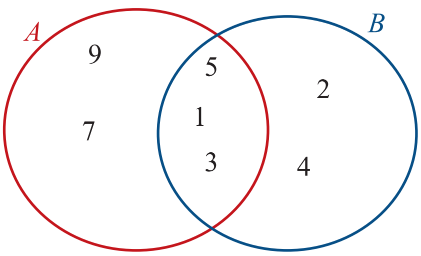
For example, let $B = \{0, 1, 2\}$ and $A = \{0, 1, 2, 3, 4\}$. Then B is a subset of A, as illustrated in the diagram opposite.



This diagram is called a **Venn diagram**. Venn diagrams are revisited in Chapter 9.

- The set of elements common to two sets A and B is called the **intersection** of A and B, and is denoted by $A \cap B$. Thus $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.
- If the sets A and B have no elements in common, we say A and B are **disjoint**, and write $A \cap B = \emptyset$. The set $\emptyset$ is called the **empty set**.
- The set of elements that are in A or in B (or in both) is called the **union** of sets A and B, and is denoted by $A \cup B$.

For example, let $A = \{1, 3, 5, 7, 9\}$ and $B = \{1, 2, 3, 4, 5\}$. The intersection and union are illustrated by the Venn diagram shown opposite:



$$A \cap B = \{1, 3, 5\}$$

$$A \cup B = \{1, 2, 3, 4, 5, 7, 9\}$$

Example 1

For $A = \{1, 2, 3, 7\}$ and $B = \{3, 4, 5, 6, 7\}$, find:

a $A \cap B$ **b** $A \cup B$

Solution	Explanation
a $A \cap B = \{3, 7\}$	The elements 3 and 7 are common to sets A and B.
b $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$	The set $A \cup B$ contains all elements that belong to A or B (or both).

Note: In Example 1, we have $3 \in A$ and $5 \notin A$ and $\{2, 3\} \subseteq A$.

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Sets of numbers

We begin by recalling that the elements of $\{1, 2, 3, 4, \ldots\}$ are called the **natural numbers**, and the elements of $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ are called **integers**.

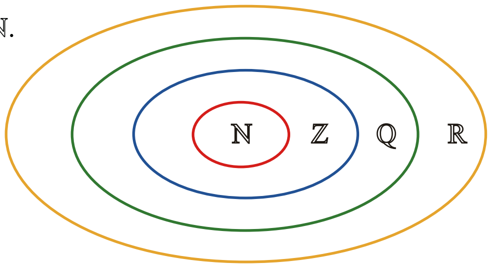
The numbers of the form $\frac{p}{q}$, with p and q integers, $q \neq 0$, are called **rational numbers**.

The real numbers which are not rational are called **irrational** (e.g. π and $\sqrt{2}$).

The rationals may be characterised as being those real numbers that can be written as a terminating or recurring decimal.

- The set of real numbers will be denoted by $\mathbb{R}$.
- The set of rational numbers will be denoted by $\mathbb{Q}$.
- The set of integers will be denoted by $\mathbb{Z}$.
- The set of natural numbers will be denoted by $\mathbb{N}$.

It is clear that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, and this may be represented by the diagram on the right.



Describing a set

It is not always possible to list the elements of a set. There is an alternative way of describing sets that is especially useful for infinite sets.

'The set of all x such that ___ ' is denoted by $\{x : \_\_\_\}$. Thus, for example:

- $\{x \in \mathbb{R} : 0 < x < 1\}$ is the set of all real numbers strictly between 0 and 1
- $\{x \in \mathbb{R} : x \geq 3\}$ is the set of all real numbers greater than or equal to 3
- $\{x \in \mathbb{R} : x \neq 0\}$ is the set of all real numbers excluding 0
- $\{x \in \mathbb{Q} : x > 0\}$ is the set of all positive rational numbers
- $\{2n \in \mathbb{Z} : n = 0, 1, 2, \ldots\}$ is the set of all non-negative even numbers
- $\{2n + 1 \in \mathbb{N} : n = 0, 1, 2, \ldots\}$ is the set of all non-negative odd numbers.

Interval notation

Among the most important subsets of $\mathbb{R}$ are the **intervals**. The following is an exhaustive list of the various types of intervals and the standard notation for them. We suppose that a and b are real numbers with $a < b$.

$$(a, b) = \{x \in \mathbb{R} : a < x < b\} \qquad [a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\} \qquad [a, b) = \{x \in \mathbb{R} : a \leq x < b\}$$

$$(a, \infty) = \{x \in \mathbb{R} : a < x\} \qquad [a, \infty) = \{x \in \mathbb{R} : a \leq x\}$$

$$(-\infty, b) = \{x \in \mathbb{R} : x < b\} \qquad (-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$$

Intervals may be represented by diagrams as shown in Example 2.

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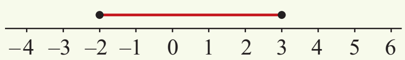
Example 2

Illustrate each of the following intervals of real numbers:

a $[-2, 3]$ **b** $(-3, 4]$ **c** $(-\infty, 5]$ **d** $(-2, 4)$ **e** $(-3, \infty)$

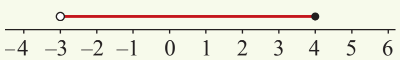
Solution / **Explanation**

a $[-2, 3]$



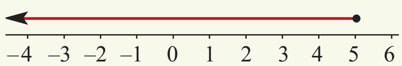
The square brackets indicate that the endpoints are included; this is shown with closed circles.

b $(-3, 4]$



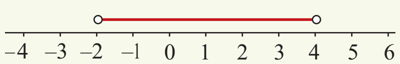
The round bracket indicates that the left endpoint is not included; this is shown with an open circle. The right endpoint is included.

c $(-\infty, 5]$



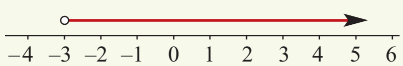
The symbol $-\infty$ indicates that the interval continues indefinitely (i.e. forever) to the left; it is read as 'negative infinity'. The right endpoint is included.

d $(-2, 4)$



Both brackets are round; the endpoints are not included.

e $(-3, \infty)$



The symbol ∞ indicates that the interval continues indefinitely (i.e. forever) to the right; it is read as 'infinity'. The left endpoint is not included.

Notes:

- The 'closed' circle (●) indicates that the number is included.
- The 'open' circle (○) indicates that the number is not included.

Summary 6A

- If x is an element of a set A, we write $x \in A$.
- If x is not an element of a set A, we write $x \notin A$.
- If every element of B is an element of A, we say B is a **subset** of A and write $B \subseteq A$.
- The set $A \cap B$ is the **intersection** of A and B, where $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.
- The set $A \cup B$ is the **union** of A and B, where $x \in A \cup B$ if and only if $x \in A$ or $x \in B$.
- If the sets A and B have no elements in common, we say A and B are **disjoint** and write $A \cap B = \emptyset$. The set $\emptyset$ is called the **empty set**.
- Sets of numbers:
  - Real numbers: $\mathbb{R}$
  - Rational numbers: $\mathbb{Q}$
  - Integers: $\mathbb{Z}$
  - Natural numbers: $\mathbb{N}$

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- For real numbers a and b with $a < b$, we can consider the following **intervals**:

$$(a, b) = \{ x \in \mathbb{R} : a < x < b \} \qquad [a, b] = \{ x \in \mathbb{R} : a \le x \le b \}$$
$$(a, b] = \{ x \in \mathbb{R} : a < x \le b \} \qquad [a, b) = \{ x \in \mathbb{R} : a \le x < b \}$$
$$(a, \infty) = \{ x \in \mathbb{R} : a < x \} \qquad [a, \infty) = \{ x \in \mathbb{R} : a \le x \}$$
$$(-\infty, b) = \{ x \in \mathbb{R} : x < b \} \qquad (-\infty, b] = \{ x \in \mathbb{R} : x \le b \}$$

Skill-sheet

Exercise 6A

Example 1

1 For $A = \{1, 2, 3, 5, 7, 11, 15\}$, $B = \{7, 11, 25, 30, 32\}$ and $C = \{1, 7, 11, 25, 30\}$, find:

a $A \cap B$ **b** $A \cap B \cap C$ **c** $A \cup C$
d $A \cup B$ **e** $A \cup B \cup C$ **f** $(A \cap B) \cup C$

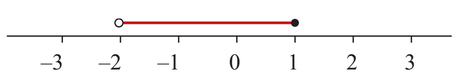
Example 2

2 Illustrate each of the following intervals on a number line:

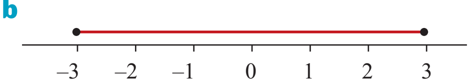
a $[-3, 4)$ **b** $(-\infty, 3]$ **c** $[-2, -1]$
d $(-2, \infty)$ **e** $(-2, 3)$ **f** $(-2, 4]$

3 Describe each of the following subsets of the real number line using the interval notation $[a, b)$, (a, b), etc.

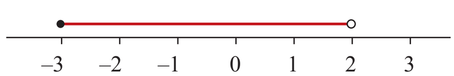
a



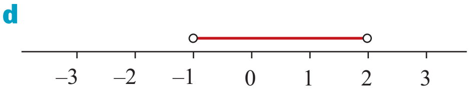
b



c



d



4 Use the appropriate interval notation (i.e. $[a, b]$, (a, b), etc.) to describe each of the following sets:

a $\{ x \in \mathbb{R} : -1 \le x \le 2 \}$ **b** $\{ x \in \mathbb{R} : -4 < x \le 2 \}$

c $\{ y \in \mathbb{R} : 0 < y < \sqrt{2} \}$ **d** $\left\{ y \in \mathbb{R} : -\frac{\sqrt{3}}{2} < y \le \frac{1}{\sqrt{2}} \right\}$

e $\{ x \in \mathbb{R} : x > -1 \}$ **f** $\{ x \in \mathbb{R} : x \le -2 \}$

g $\{ x \in \mathbb{R} \}$

5 For $B = \{7, 11, 25, 30, 32\}$, find:

a $(-2, 10] \cap B$ **b** $(3, \infty) \cap B$
c $(2, \infty) \cup B$ **d** $(25, \infty) \cap B$

6 For each of the following, use one number line on which to represent the sets:

a $[-2, 5]$, $[3, 4]$, $[-2, 5] \cap [3, 4]$
b $[3, \infty)$, $(-\infty, 7]$, $[3, \infty) \cap (-\infty, 7]$

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7 Write each of the following sets as a union of two intervals:

a $\{ x \in \mathbb{R} : x \neq -2 \}$ **b** $\{ x \in \mathbb{R} : x \neq 3 \}$ **c** $\{ x \in \mathbb{R} : x \neq 4 \}$

8 Illustrate each of these sets on a number line:

a $[-3, 2] \cup [4, 8]$ **b** $(-\infty, 2] \cup [4, \infty)$ **c** $(-\infty, -3) \cup (0, \infty)$

d $(-5, -2] \cup (2, 6]$ **e** $(-\infty, 2) \cup (2, \infty)$ **f** $(-\infty, -3) \cup (-3, \infty)$

9 Describe each of the following intersections of intervals as simply as possible:

a $(-\infty, -3) \cap (-6, \infty)$ **b** $(-\infty, 1) \cap (4, \infty)$ **c** $(-\infty, 0] \cap [-6, \infty)$

d $[-3, 2] \cap [-1, 8]$ **e** $[-3, 1] \cap [1, 8]$ **f** $(-\infty, -1] \cap (-10, \infty)$

6B Relations, domain and range

In previous chapters we have looked at how to sketch the graphs of various mathematical relations. We will now look at this aspect of representing relations in a more formal way.

- An **ordered pair**, denoted (x, y), is a pair of elements x and y in which x is considered to be the first coordinate and y the second coordinate.
- A **relation** is a set of ordered pairs. The following are examples of relations:
  a $S = \{(1, 1), (1, 2), (3, 4), (5, 6)\}$
  b $T = \{(-3, 5), (4, 12), (5, 12), (7, -6)\}$
- Every relation determines two sets:
  - The set of all the first coordinates of the ordered pairs is called the **domain**.
  - The set of all the second coordinates of the ordered pairs is called the **range**.

For the above examples:

a domain of $S = \{1, 3, 5\}$, range of $S = \{1, 2, 4, 6\}$

b domain of $T = \{-3, 4, 5, 7\}$, range of $T = \{5, 12, -6\}$

Some relations may be defined by a **rule** relating the elements in the domain to their corresponding elements in the range. In order to define the relation fully, we need to specify both the rule and the domain. For example, the set

$$\{ (x, y) : y = x + 1, \ x \in \{1, 2, 3, 4\} \}$$

is the relation

$$\{(1, 2), (2, 3), (3, 4), (4, 5)\}$$

The **domain** is the set $X = \{1, 2, 3, 4\}$ and the **range** is the set $Y = \{2, 3, 4, 5\}$.

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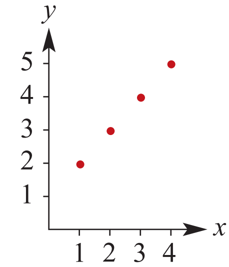
Representing relations

Graphing relations

We can represent a relation as a graph on a set of Cartesian axes.

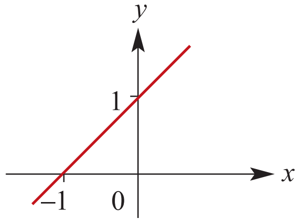


On the right is the graph of the relation

$$\{ (x, y) : y = x + 1, \ x \in \{1, 2, 3, 4\} \}$$

Note that we only graph the individual points of this relation.

If the domain of the relation is the set of real numbers, $\mathbb{R}$, then there are infinitely many points. For example, the graph of



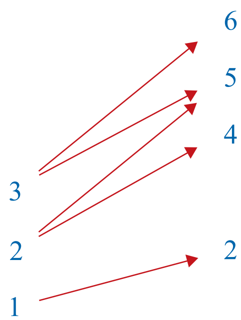
$$\{ (x, y) : y = x + 1, \ x \in \mathbb{R} \}$$

is a continuous straight line.

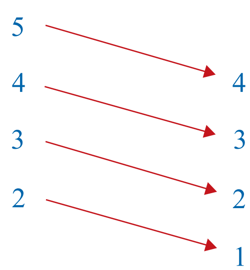
Arrow diagrams

A relation may also be represented by an arrow diagram.

This diagram represents the relation $\{(3, 6), (3, 5), (2, 5), (2, 4), (1, 2)\}$:



This diagram represents the relation $\{(5, 4), (4, 3), (3, 2), (2, 1)\}$:



- A relation may be written as:
  - a listed set of ordered pairs (not always convenient or possible)
  - a rule with a specified or implied domain.
- A relation may be represented by a graph or an arrow diagram.

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Example 3

Sketch a graph of each of the following relations and state its domain and range:

a $\{(-2, -1), (-1, -1), (-1, 1), (0, 1), (1, -1)\}$

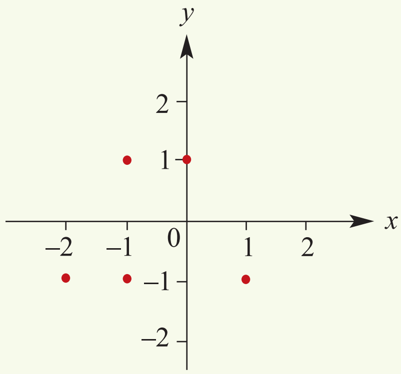
b $\{ (x, y) : x^2 + y^2 = 1,\ x \in [-1, 1] \}$

c $\{ (x, y) : 2x + 3y = 6,\ x \geq 0 \}$

d $\{ (x, y) : y = 2x - 1,\ x \in [-1, 2] \}$

Solution

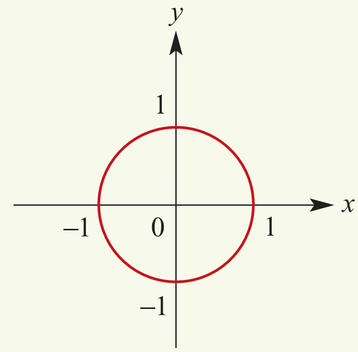
a



Domain = $\{-2, -1, 0, 1\}$

Range = $\{-1, 1\}$

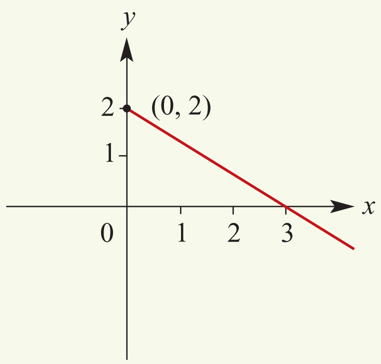
b



Domain = $\{ x \in \mathbb{R} : -1 \leq x \leq 1 \}$

Range = $\{ y \in \mathbb{R} : -1 \leq y \leq 1 \}$

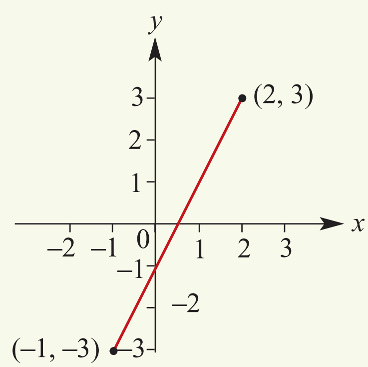
c



Domain = $[0, \infty)$

Range = $(-\infty, 2]$

d



Domain = $[-1, 2]$

Range = $[-3, 3]$

Often set notation is not used in the specification of a relation. For example:

- $\{ (x, y) : y = x^2 \}$ is written as $y = x^2$
- $\{ (x, y) : y = x + 1 \}$ is written as $y = x + 1$.

This has been the case in your previous considerations of relations.

Note: In order to determine the range of a relation it is necessary to consider the graph. This strategy is used in the following examples.

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Example 4

For each of the following, complete the square, sketch the graph and state the range. The domain is $\mathbb{R}$.

a $y = x^2 - 4x + 5$

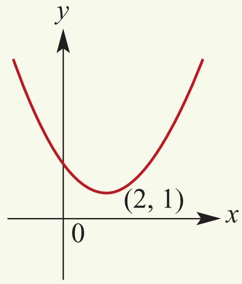
b $y = -x^2 + 4x - 5$

Solution

a $y = x^2 - 4x + 5$

$= (x - 2)^2 + 1$

The range is $[1, \infty)$.



Explanation

Complete the square:

$x^2 - 4x + 5 = x^2 - 4x + 4 - 4 + 5$

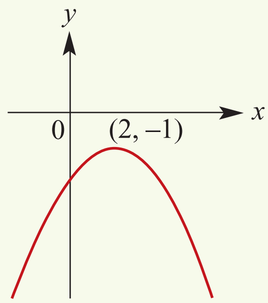
$= (x - 2)^2 + 1$

The vertex is at (2, 1). The minimum value of $y = x^2 - 4x + 5$ is 1.

b $y = -x^2 + 4x - 5$

$= -(x - 2)^2 - 1$

The range is $(-\infty, -1]$.



Complete the square:

$-x^2 + 4x - 5 = -(x^2 - 4x + 5)$

$= -[x^2 - 4x + 4 + 1]$

$= -[(x - 2)^2 + 1]$

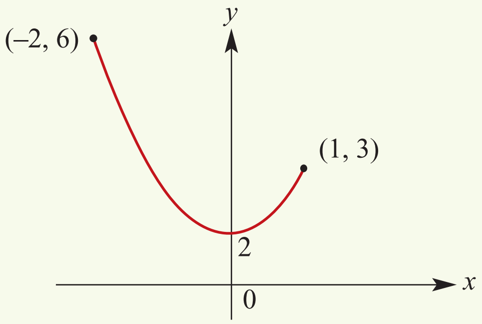
$= -(x - 2)^2 - 1$

The vertex is at (2, −1). The maximum value is −1.

Example 5

Sketch the graph of the relation $y = x^2 + 2$ for $x \in [-2, 1]$ and state the range.

Solution



The range is $[2, 6]$.

Explanation

Note that the range is not determined by considering the endpoints alone. The minimum value is 2, not 3.

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Natural (implied) domain

When the rule for a relation is written and no domain is stipulated, then it is understood that the domain taken is the largest for which the rule has meaning. This domain is called the **natural** or **implied domain**. (In this book we will generally use the term natural domain.)

For example, the natural domain of $y = x^2$ is $\mathbb{R}$, and the natural domain of $x^2 + y^2 = 1$ is $[-1, 1]$. This concept is considered again in Section 6C.

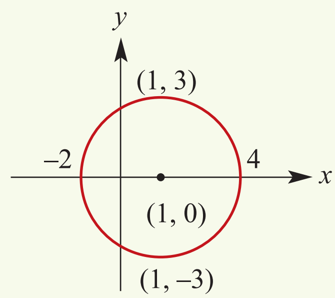
Example 6

For each of the following relations, state the natural domain and the range:

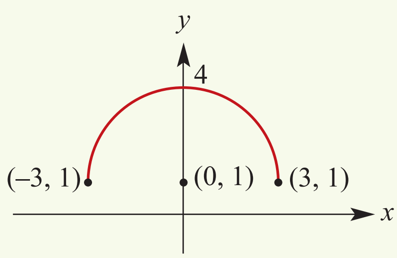
a $(x-1)^2 + y^2 = 9$ **b** $y = \sqrt{9 - x^2} + 1$

Solution

a This relation is a circle with centre $(1, 0)$ and radius 3. The natural domain is $[-2, 4]$ and the range is $[-3, 3]$.



b This relation is a semicircle with centre $(0, 1)$ and radius 3. The natural domain is $[-3, 3]$ and the range is $[1, 4]$.



Summary 6B

- An **ordered pair**, denoted (x, y), is a pair of elements x and y in which x is considered to be the first coordinate and y the second coordinate.
- A **relation** is a set of ordered pairs.
  - The set of all the first coordinates of the ordered pairs is called the **domain**.
  - The set of all the second coordinates of the ordered pairs is called the **range**.
- Some relations may be defined by a rule relating the elements in the domain to their corresponding elements in the range. In order to define the relation fully, we need to specify both the rule and the domain. For example:

$$\{(x, y) : y = x + 1,\ x \geq 0\}$$

- For a relation described by a rule with y in terms of x, the domain is the x-values and the range is the y-values.
- The **natural** or **implied domain** is the largest domain for which the rule of the relation has meaning.

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Skill-sheet

Exercise 6B

Example 3

1 Sketch a graph of each of the following relations and state its domain and range:

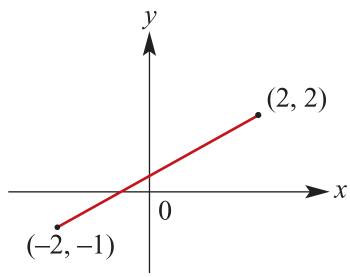
a $\{(-3, -4), (-1, -1), (-6, 7), (1, 5)\}$ **b** $\{(-4, 1), (-4, -1), (-6, 7), (-6, 8)\}$

c $\{ (x, y) : x^2 + y^2 = 4,\ x \in [-2, 2] \}$ **d** $\{ (x, y) : 3x + 2y = 12,\ x \geq 0 \}$

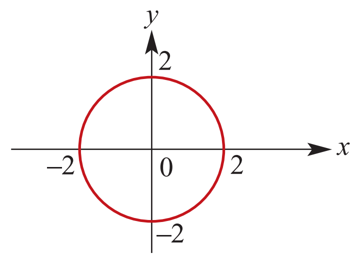
e $\{ (x, y) : x - y = 4,\ x \in [-1, 2] \}$ **f** $\{ (x, y) : y = 2x + 3,\ x \in [-4, 1] \}$

2 State the domain and range for the relations represented by each of the following graphs:

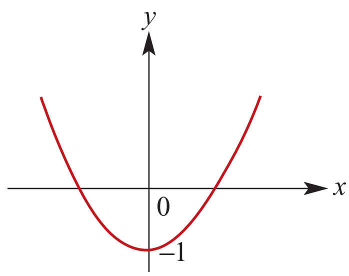
a



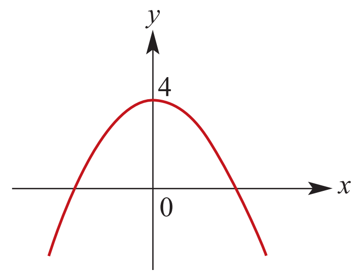
b



c



d



Example 4

3 For each of the following, complete the square, sketch the graph and state the range:

a $y = x^2 + 6x + 10$ **b** $y = -x^2 - 4x - 6$ **c** $y = 2x^2 - 4x + 6$

4 Sketch the graphs of each of the following and state the range of each:

a $y = x + 1,\ x \in [2, \infty)$ **b** $y = -x + 1,\ x \in [2, \infty)$

c $y = 2x + 1,\ x \in [-4, \infty)$ **d** $y = 3x + 2,\ x \in (-\infty, 3)$

e $y = x + 1,\ x \in (-\infty, 3]$ **f** $y = -3x - 1,\ x \in [-2, 6]$

g $y = -3x - 1,\ x \in [-5, -1]$ **h** $y = 5x - 1,\ x \in (-2, 4)$

Example 5

5 Sketch the graphs of each of the following and state the range of each:

a $y = x^2 + 3,\ x \in [-1, 1]$ **b** $y = x^2 + 4,\ x \in [-2, 1]$

c $y = x^2 - 4,\ x \in [-1, 2]$ **d** $y = 2x^2 + 1,\ x \in [-2, 3]$

6 Sketch the graphs of each of the following relations, stating the range of each:

a $\{ (x, y) : y = x^2 + 1 \}$ **b** $\{ (x, y) : y = x^2 + 2x + 1 \}$

c $\{ (x, y) : y = 4 - x^2,\ x \in [-2, 2] \}$ **d** $\{ (x, y) : y = x^2 + 2x + 3 \}$

e $\{ (x, y) : y = -x^2 + 2x + 3 \}$ **f** $\{ (x, y) : y = x^2 - 2,\ x \in [-1, 2] \}$

g $\{ (x, y) : y = 2x^2 - 3x + 6 \}$ **h** $\{ (x, y) : y = 6 - 3x + x^2 \}$

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Example 6

7 Sketch the graphs of each of the following relations, stating the natural domain and range of each:

a $\{(x, y) : x^2 + y^2 = 9\}$
b $(x-2)^2 + (y-3)^2 = 16$
c $(2x-1)^2 + (2y-4)^2 = 1$
d $y = \sqrt{25 - x^2}$
e $y = -\sqrt{25 - x^2}$
f $\{(x, y) : y = -\sqrt{25 - (x-2)^2}\}$

8 Sketch the graphs of each of the following relations, stating the natural domain and range of each:

a $y = \dfrac{2}{2x-5} + 3$
b $y = \sqrt{2x-5}$
c $y = \sqrt{5-2x}$
d $y = \sqrt{4 - (x-5)^2}$
e $y = (4-2x)^{\frac{1}{2}}$

9 Sketch the graphs of each of the following relations, stating the natural domain and range of each:

a $x = y^2$
b $x = y^2 + 1$
c $x = y^2 - 1$
d $x = (y-3)^2 + 1$

6C Functions

A **function** is a relation such that for each x-value there is only one corresponding y-value. This means that, if (a, b) and (a, c) are ordered pairs of a function, then $b = c$. In other words, a function cannot contain two different ordered pairs with the same first coordinate.

Example 7

Which of the following sets of ordered pairs defines a function?

a $\{(-3,-4), (-1,-1), (-6,7), (1,5)\}$
b $\{(-4,1), (-4,-1), (-6,7), (-6,8)\}$

Solution

a $\{(-3,-4), (-1,-1), (-6,7), (1,5)\}$ is a function, because for each x-value there is only one y-value.

b $\{(-4,1), (-4,-1), (-6,7), (-6,8)\}$ is *not* a function, because there is an x-value with two different y-values. The relation contains two ordered pairs, $(-4, 1)$ and $(-4, -1)$, with the same first coordinate.

One way to identify whether a relation is a function is to draw a graph of the relation and then apply the following test.

Vertical-line test

If a vertical line can be drawn anywhere on the graph and it only ever intersects the graph a maximum of once, then the relation is a **function**.

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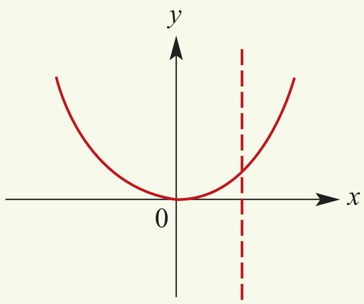
Example 8

a Is $y = x^2$ a function? State the natural domain and range of $y = x^2$.

b Is $x^2 + y^2 = 4$ a function? State the natural domain and range of $x^2 + y^2 = 4$.

Solution

a



The vertical-line test shows that $y = x^2$ is a function. The natural domain is $\mathbb{R}$ and the range is $[0, \infty)$.

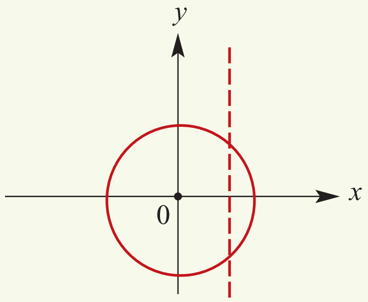
Explanation

For each x-value there is only one y-value.

The ordered pairs of the relation are all of the form (a, a^2).

Solution

b



The vertical-line test shows that $x^2 + y^2 = 4$ is *not* a function. The natural domain is $[-2, 2]$ and the range is $[-2, 2]$.

Explanation

Note that $(\sqrt{2}, \sqrt{2})$ and $(\sqrt{2}, -\sqrt{2})$ are ordered pairs of the relation.

There is an x-value with more than one y-value.

Function notation

Functions are usually denoted with lowercase letters such as f, g, h.

If f is a function, then for each x in the domain of f there is a unique element y in the range such that $(x, y) \in f$. The element y is called 'the **image** of x under f' or 'the **value** of f at x', and the element x is called 'a **pre-image** of y'.

Since the y-value obtained is a *function* of the x-value, we use the notation $f(x)$, read as 'f of x', in place of y.

For example, instead of $y = 2x + 1$ we can write $f(x) = 2x + 1$. Then $f(2)$ means the y-value obtained when $x = 2$.

e.g. $f(2) = 2(2) + 1 = 5$

$f(-4) = 2(-4) + 1 = -7$

$f(a) = 2a + 1$

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By incorporating this notation, we have an alternative way of writing functions:

- For the function $\{(x, y) : y = x^2\}$ with domain $\mathbb{R}$, we write $f(x) = x^2, \{x \in \mathbb{R}\}$.
- For the function $\{(x, y) : y = 2x - 1,\ x \in [0, 4]\}$, we write $f(x) = 2x - 1$, $\{x \in \mathbb{R} : 0 \le x \le 4\}$.
- For the function $\left\{(x, y) : y = \frac{1}{x}\right\}$ with domain $x \in (-\infty, 0) \cup (0, \infty)$, we write $f(x) = \frac{1}{x}$, $\{x \in \mathbb{R} : x \neq 0\}$.

If the domain is $\mathbb{R}$, we often just write the rule. For example: $f(x) = x^2$.

A function $f(x) = a$ is called a **constant function**. For such a function f, we have domain $\mathbb{R}$ and range $\{a\}$. For example, let $f(x) = 7$. This has domain $\mathbb{R}$ and range $\{7\}$.

A function $f(x) = mx + c$ is called a **linear function**. For example, let $f(x) = 3x + 1$. Then we have domain $\mathbb{R}$ and range $\mathbb{R}$. Note that if the domain of a linear function is $\mathbb{R}$ and $m \neq 0$, then the range is $\mathbb{R}$.

Example 9

If $f(x) = 2x^2 + x$, find:

a $f(3)$ **b** $f(-2)$ **c** $f(x - 1)$ **d** $f\left(\frac{1}{a}\right)$, $a \neq 0$

Solution

a $f(3) = 2(3)^2 + 3 = 21$

b $f(-2) = 2(-2)^2 - 2 = 6$

c
$$\begin{aligned} f(x-1) &= 2(x-1)^2 + x - 1 \\ &= 2(x^2 - 2x + 1) + x - 1 \\ &= 2x^2 - 3x + 1 \end{aligned}$$

d
$$\begin{aligned} f\left(\frac{1}{a}\right) &= 2\left(\frac{1}{a}\right)^2 + \frac{1}{a} \\ &= \frac{2}{a^2} + \frac{1}{a} \\ &= \frac{2 + a}{a^2} \end{aligned}$$

Example 10

Consider the function defined by $f(x) = 2x - 4$ for all $x \in \mathbb{R}$.

a Find the value of $f(2)$ and $f(t)$.

b Find the value of x for which $f(x) = 6$.

c Find the value of x for which $f(x) = 0$.

d For what values of t is $f(t) = t$?

e For what values of x is $f(x) \ge x$?

f For what values of x is $f(x) \le 3x$?

Solution

a $f(2) = 2(2) - 4 = 0$

$f(t) = 2t - 4$

b
$$\begin{aligned} f(x) &= 6 \\ 2x - 4 &= 6 \\ 2x &= 10 \\ \therefore\quad x &= 5 \end{aligned}$$

c
$$\begin{aligned} f(x) &= 0 \\ 2x - 4 &= 0 \\ 2x &= 4 \\ \therefore\quad x &= 2 \end{aligned}$$

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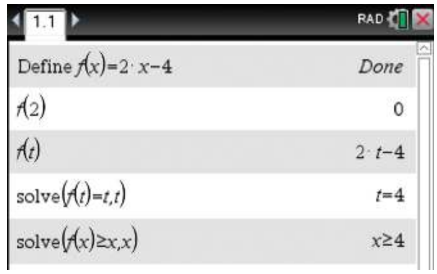
d	**e**	**f**
$f(t) = t$	$f(x) \ge x$	$f(x) \le 3x$
$2t - 4 = t$	$2x - 4 \ge x$	$2x - 4 \le 3x$
$t - 4 = 0$	$x - 4 \ge 0$	$-4 \le x$
$\therefore \quad t = 4$	$\therefore \quad x \ge 4$	$\therefore \quad x \ge -4$

Using the TI-Nspire

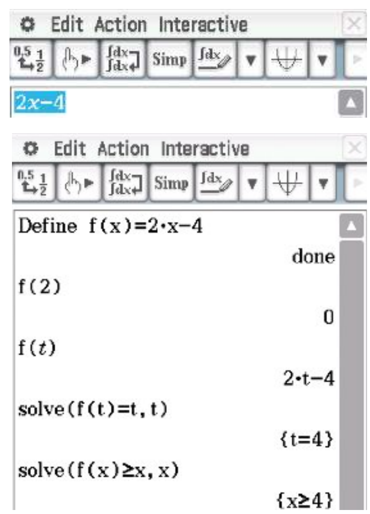
- Use (menu) > **Actions** > **Define** to define the function $f(x) = 2x - 4$. Find $f(2)$ and $f(t)$.
- Use (menu) > **Algebra** > **Solve** to solve the equation $f(t) = t$ and the inequality $f(x) \ge x$.

Note: The symbol $\ge$ can be accessed from the symbols palette (ctrl)(menu) or by using (ctrl)(=) and selecting $\ge$.



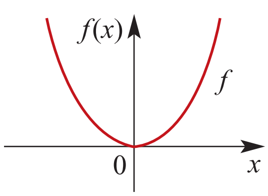
Using the Casio ClassPad

- In the main screen $\sqrt{\alpha}$ (Main), type $2x - 4$.
- Highlight the expression and go to **Interactive** > **Define**. Tap OK.
- Type $f(2)$ by using the (abc) keyboard. Alternatively, copy $f(x)$ and replace x with 2. Tap (EXE) for the answer.
- Type $f(t)$ and tap (EXE).
- Type $f(t) = t$, highlight it and go to **Interactive** > **Equation/Inequality** > **solve**. Remember to change the variable to t.
- The inequality $f(x) \ge x$ can be solved similarly.

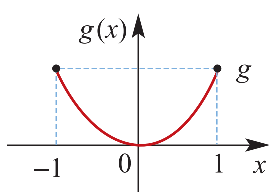


Restriction of a function

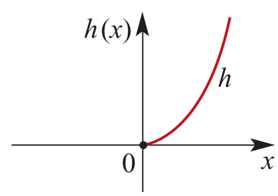
Consider the following functions:



$f(x) = x^2, \ x \in \mathbb{R}$



$g(x) = x^2, \ -1 \le x \le 1$



$h(x) = x^2, \ x \in [0, \infty)$

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The different letters, f, g and h, used to name the functions emphasise the fact that there are three different functions, even though they all have the same rule. They are different because they are defined for different domains. We say that g and h are **restrictions** of f, since their domains are subsets of the domain of f.

Example 11

Sketch the graph of each of the following functions and state its range:

a $f(x) = x, \{ x \in \mathbb{R} : -1 \leq x \leq 2 \}$

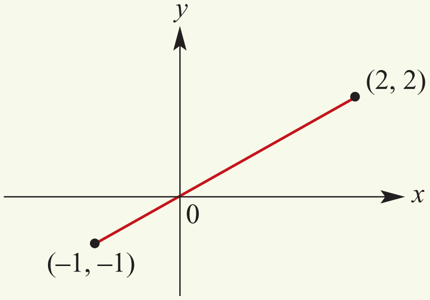
b $f(x) = x^2 + x, \{ x \in \mathbb{R} : -1 \leq x \leq 1 \}$

c $f(x) = \dfrac{1}{x}, \{ x \in \mathbb{R} : 0 < x \leq 2 \}$

d $f(x) = x^2 - 2x + 8$

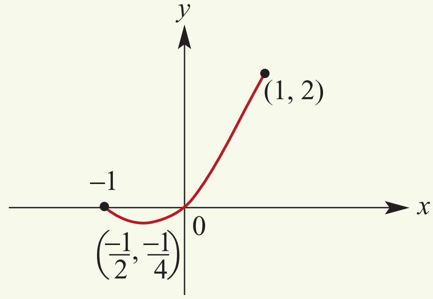
Solution

a



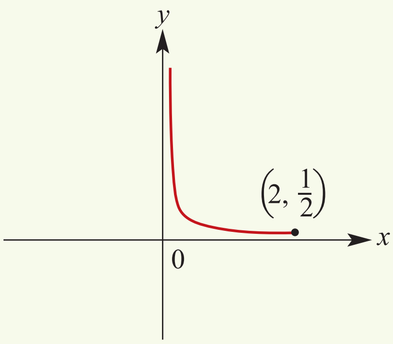
Range is $[-1, 2]$

b



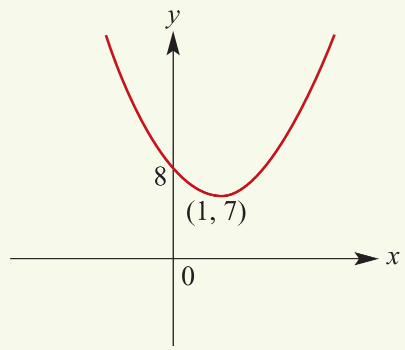
Range is $[-\frac{1}{4}, 2]$

c



Range is $[\frac{1}{2}, \infty)$

d



$f(x) = x^2 - 2x + 8 = (x - 1)^2 + 7$

Range is $[7, \infty)$

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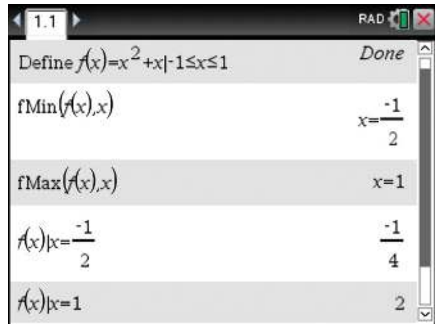
Using the TI-Nspire

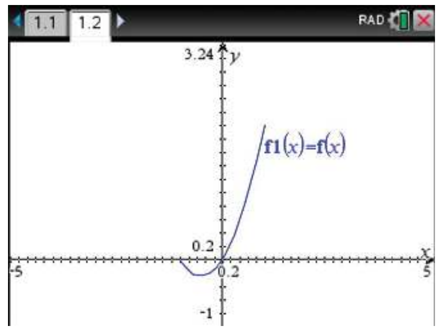
- In a **Calculator** application, use menu > **Actions** > **Define** to define the function $f: [-1, 1] \to \mathbb{R}, f(x) = x^2 + x$.

Note: The 'with' symbol | and the inequality signs can be accessed using ctrl =.

- Use menu > **Calculus** > **Function Minimum** and menu > **Calculus** > **Function Maximum** to help determine the range of this restricted function. The range is $[-\frac{1}{4}, 2]$.
- The graph of $y = f(x)$ is plotted by entering $f_1(x) = f(x)$ in a **Graphs** application.
- Use menu > **Analyze Graph** > **Minimum or Maximum** to show the key points.

Note: You can also enter the restricted function directly in the function entry line in the **Graphs** application if preferred.

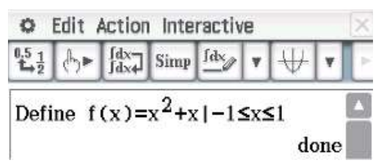


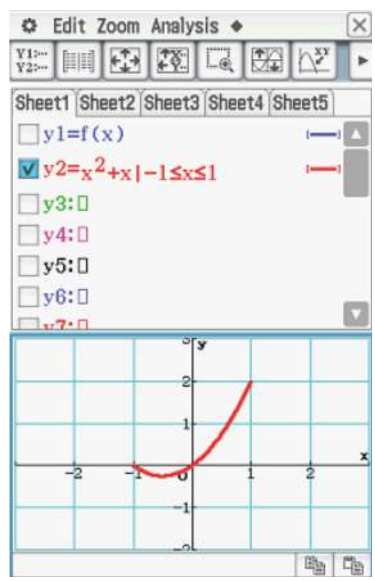


Using the Casio ClassPad

- In the Main $\sqrt{\alpha}$ screen, type $x^2 + x \mid -1 \le x \le 1$. (The symbol | is found in Math2.)
- Highlight the expression together with the restricted domain and go to **Interactive** > **Define**. Tap OK.
- Graphing in the main screen: Select the graph icon. Highlight the definition of $f(x)$ and drag into the graph window.
- Alternatively, go to the menu and select **Graph & Table**. Either enter $f(x)$ in $y1$ or enter $x^2 + x \mid -1 \le x \le 1$ directly.

Note: The window can be adjusted using the icon. Alternatively, use Zoom Box to manually select an appropriate window.





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Natural (implied) domain

We considered natural domains for relations in Section 6B. We recall our definition but this time we do so for functions in particular.

The **natural (implied) domain** of a function is the set of all real numbers for which the rule of the function has meaning.

For example:

$f(x) = 3x^2 - 2x$ has natural domain $\mathbb{R}$

$g(x) = \sqrt{x}$ has natural domain $[0, \infty)$

Example 12

State the natural domain, sketch the graph and find the corresponding range of each of the following:

a $f(x) = \sqrt{2x - 5}$ **b** $g(x) = \dfrac{1}{2x - 5}$ **c** $h(x) = 4 - x^2$

Solution

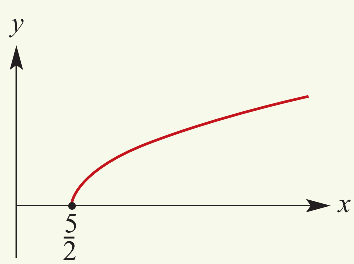
a For $f(x)$ to be defined, we need

$$2x - 5 \geq 0$$

$$\therefore \quad x \geq \frac{5}{2}$$

Hence the natural domain is $[\frac{5}{2}, \infty)$.

The range of the function is $[0, \infty)$.



b For $g(x)$ to be defined, we need

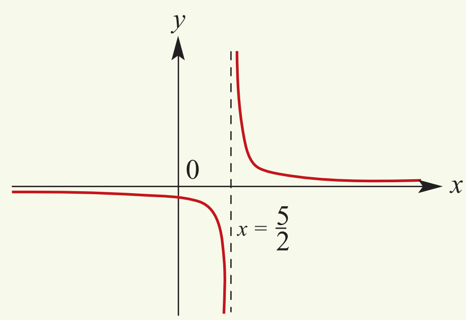
$$2x - 5 \neq 0$$

$$\therefore \quad x \neq \frac{5}{2}$$

Hence the natural domain is

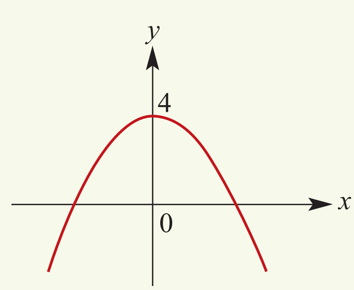
$\{\, x \in \mathbb{R} : x \neq \frac{5}{2} \,\}$.

The range is $\{\, y \in \mathbb{R} : y \neq 0 \,\}$.



c Since $h(x) = 4 - x^2$ is defined for all x, the natural domain is $\mathbb{R}$.

The range is $(-\infty, 4]$.



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Summary 6C

- A **function** is a relation such that for each x-value there is only one corresponding y-value.
- **Vertical-line test**: If a vertical line can be drawn anywhere on the graph and it only ever intersects the graph a maximum of once, then the relation is a function.
- Functions are usually denoted with lowercase letters such as f, g, h.
- For an ordered pair (x, y) of a function f, we say that y is the **image** of x under f or that y is the value of f at x, and we say that x is a **pre-image** of y.
- Since the y-value obtained is a function of the x-value, we use the notation $f(x)$, read as 'f of x', in place of y.
- Notation for defining functions: For example, we write $f(x) = 2x - 1$, $\{ x \in \mathbb{R} : 0 \le x \le 4 \}$ to define a function f with domain $[0, 4]$ and rule $f(x) = 2x - 1$.
- A **restriction** of a function has the same rule but a 'smaller' domain.
- When the domain of a function is not explicitly stated, it is assumed to consist of all real numbers for which the rule has meaning. We refer to the **natural (implied) domain** of a function, because the domain is implied by the rule.

Skill-sheet

Exercise 6C

Example 7

1 Which of the following relations are functions? State the domain and range for each:

a $\{(0, 1), (0, 2), (1, 2), (2, 3), (3, 4)\}$

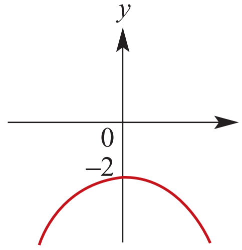
b $\{(-2, -1), (-1, -2), (0, 2), (1, 4), (2, -5)\}$

c $\{(0, 1), (0, 2), (-1, 2), (3, 4), (5, 6)\}$

d $\{(1, 3), (2, 3), (4, 3), (5, 3), (6, 3)\}$

2 Each of the following is the graph of a relation. State which are the graph of a function.

a



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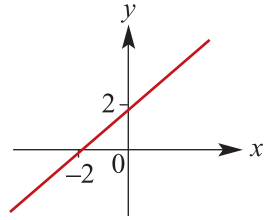

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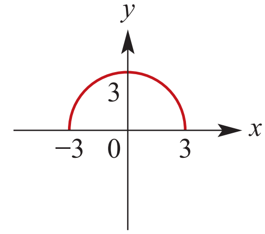

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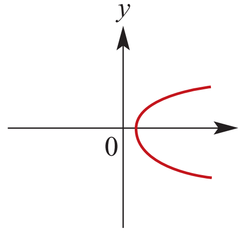

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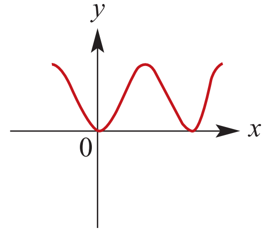

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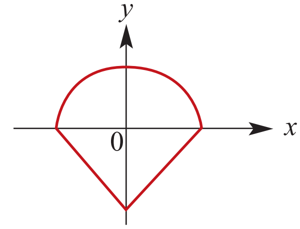

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Example 8

3 Sketch the graph of each of the following relations, then state the range of each and specify whether the relation is a function or not:

a $y = x^2,\ x \in [0, 4]$

b $\{ (x, y) : x^2 + y^2 = 4,\ x \in [0, 2] \}$

c $\{ (x, y) : 2x + 8y = 16,\ x \in [0, \infty) \}$

d $y = \sqrt{x},\ x \in (0, \infty)$

e $\left\{ (x, y) : y = \dfrac{1}{x},\ x > 0 \right\}$

f $y = x^2,\ x \in [-1, 4]$

g $\{ (x, y) : y^2 = x,\ x \geq 0 \}$

4 Which of the following relations are functions? State the domain and range for each:

a $\{ (x, -2) : x \in \mathbb{R} \}$

b $\{ (3, y) : y \in \mathbb{Z} \}$

c $y = -x + 3$

d $y = x^2 + 5$

e $\{ (x, y) : x^2 + y^2 = 9 \}$

Example 9

5 **a** Given that $f(x) = 2x - 3$, find:

i $f(0)$ **ii** $f(4)$ **iii** $f(-1)$ **iv** $f(6)$

v $f(x - 1)$ **vi** $f\left(\dfrac{1}{a}\right)$

b Given that $g(x) = \dfrac{4}{x}$, find:

i $g(1)$ **ii** $g(-1)$ **iii** $g(3)$ **iv** $g(2)$

c Given that $g(x) = (x - 2)^2$, find:

i $g(4)$ **ii** $g(-4)$ **iii** $g(8)$ **iv** $g(a)$

d Given that $f(x) = 1 - \dfrac{1}{x}$, find:

i $f(1)$ **ii** $f(1 + a)$ **iii** $f(1 - a)$ **iv** $f\left(\dfrac{1}{a}\right)$

Example 10

6 Consider the function defined by $f(x) = 2x + 1$ for all $x \in \mathbb{R}$.

a Find the value of $f(2)$ and $f(t)$.

b Find the value of x for which $f(x) = 6$.

c Find the value of x for which $f(x) = 0$.

d For what values of t is $f(t) = t$?

e For what values of x is $f(x) \geq x$?

f For what values of x is $f(x) \leq 3x$?

7 Find the value(s) of x for which the function has the given value:

a $f(x) = 5x - 2,\ f(x) = 3$

b $f(x) = \dfrac{1}{x},\ f(x) = 6$

c $f(x) = x^2,\ f(x) = 9$

d $f(x) = (x + 1)(x - 4),\ f(x) = 0$

e $f(x) = x^2 - 2x,\ f(x) = 3$

f $f(x) = x^2 - x - 6,\ f(x) = 0$

8 Let $g(x) = x^2 + 2x$ and $h(x) = 2x^3 - x^2 + 6$.

a Evaluate $g(-1)$, $g(2)$ and $g(-2)$.

b Evaluate $h(-1)$, $h(2)$ and $h(-2)$.

c Express the following in terms of x:

i $g(-3x)$ **ii** $g(x - 5)$ **iii** $h(-2x)$ **iv** $g(x + 2)$ **v** $h(x^2)$

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9 Consider the function $f(x) = 2x^2 - 3$. Find:

a $f(2), f(-4)$ **b** the range of f

10 Consider the function $f(x) = 3x + 1$. Find:

a the image of 2 **b** the pre-image of 7 **c** $\{ x : f(x) = 2x \}$

11 Consider the function $f(x) = 3x^2 + 2$. Find:

a the image of 0 **b** the pre-image(s) of 5 **c** $\{ x : f(x) = 11 \}$

12 Consider the functions $f(x) = 7x + 6$ and $g(x) = 2x + 1$. Find:

a $\{ x : f(x) = g(x) \}$ **b** $\{ x : f(x) > g(x) \}$ **c** $\{ x : f(x) = 0 \}$

Example 11

13 Sketch the graphs of each of the following functions and state the range of each:

a $f(x) = x^2, \{ x \in \mathbb{R} : -1 \leq x \leq 2 \}$

b $f(x) = x^2 + 2x, \{ x \in \mathbb{R} : -2 \leq x \leq 2 \}$

c $f(x) = \frac{1}{x}, \{ x \in \mathbb{R} : 0 < x \leq 3 \}$

d $f(x) = x^2 - 2x + 3$

e $f(x) = x^2 - 4x + 6, \{ x \in \mathbb{R} : 1 < x < 6 \}$

f $f(x) = x^2 - 2x + 1, \{ x \in \mathbb{R} : -3 \leq x \leq 6 \}$

Example 12

14 For each of the following, find the natural domain and the corresponding range for the function defined by the rule:

a $y = 7 - x$ **b** $y = 2\sqrt{x}$ **c** $y = x^2 + 1$

d $y = -\sqrt{9 - x^2}$ **e** $y = \frac{1}{\sqrt{x}}$ **f** $y = 3 - 2x^2$

g $y = \sqrt{x - 2}$ **h** $y = \sqrt{2x - 1}$ **i** $y = \sqrt{3 - 2x}$

j $y = \frac{1}{2x - 1}$ **k** $y = \frac{1}{(2x - 1)^2}$ **l** $y = \frac{1}{2x - 1} + 2$

15 For each of the following, state the natural domain and range:

a $f(x) = \sqrt{x - 4}$ **b** $f(x) = \sqrt{4 - x}$ **c** $f(x) = 2\sqrt{x - 2} + 3$

d $f(x) = \frac{1}{x - 4}$ **e** $f(x) = \frac{1}{x - 4} + 3$ **f** $f(x) = \frac{3}{x + 2} - 3$

16 Each of the following is the rule of a function. In each case write down the natural domain and the range:

a $f(x) = 3x + 4$ **b** $g(x) = x^2 + 2$

c $y = -\sqrt{16 - x^2}$ **d** $y = \frac{1}{x + 2}$

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6D Applying function notation

The first five chapters of this book involve functions without using function notation. This section presents further questions which arise from the first five chapters of this book but where function notation can now be used.

Example 13

The volume of a sphere of radius r is determined by the function with rule $V(r) = \frac{4}{3}\pi r^3$.

State the practical domain of the function V and find $V(10)$.

Solution

The practical domain is $(0, \infty)$.

$V(10) = \frac{4}{3} \times \pi \times 10^3 = \frac{4000\pi}{3}$

(The volume of a sphere of radius 10 is $\frac{4000\pi}{3}$ cubic units.)

Example 14

If $f(x) = ax + b$ such that $f(1) = 7$ and $f(5) = 19$, find a and b and sketch the graph of $y = f(x)$.

Solution

Since $f(1) = 7$ and $f(5) = 19$,

$7 = a + b \quad (1)$

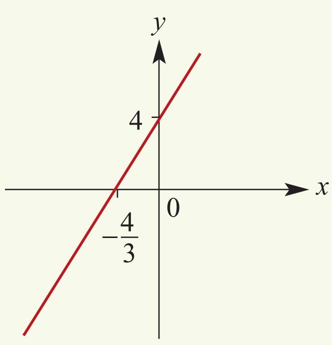
and $\quad 19 = 5a + b \quad (2)$

Subtract (1) from (2):

$12 = 4a$

Thus $a = 3$ and substituting in (1) gives $b = 4$.

Hence $f(x) = 3x + 4$.



Example 15

Find the quadratic function f such that $f(4) = f(-2) = 0$ and $f(0) = 16$.

Solution

Since 4 and -2 are solutions to the quadratic equation $f(x) = 0$, we have

$f(x) = k(x - 4)(x + 2)$

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Since $f(0) = 16$, we obtain

$$16 = k(-4)(2)$$

$$\therefore \quad k = -2$$

Hence $f(x) = -2(x-4)(x+2)$

$$= -2(x^2 - 2x - 8)$$

$$= -2x^2 + 4x + 16$$

Skill-sheet

Exercise 6D

Example 13

1 A metal bar is L cm long when its temperature is T°C. The quantities L and T are approximately related by the formula $L = 0.002T + 25$.

a L is a function of T and the rule can be written $L(T) = 0.002T + 25$. State a possible practical domain for the function.

b Find:

i $L(30)$ **ii** $L(16)$ **iii** $L(100)$ **iv** $L(500)$

Example 14

2 If $f(x) = a + bx$ with $f(4) = -1$ and $f(8) = 1$:

a find a and b

b solve the equation $f(x) = 0$.

3 Find a linear function f such that $f(0) = 7$ and whose graph is parallel to that of the function with rule $g(x) = 2 - 5x$.

4 f is a linear function such that $f(-5) = -12$ and $f(7) = 6$.

a Find:

i $f(0)$ **ii** $f(1)$

b Solve the equation $f(x) = 0$.

Example 15

5 Find the quadratic function f such that $f(2) = f(4) = 0$ and 7 is the greatest value of $f(x)$.

6 Write $f(x) = x^2 - 6x + 16$ in the form $f(x) = (x-3)^2 + p$ and hence state the range of f.

7 $f(x) = ax^2 + bx + c$. Find a, b and c if $f(0) = 2$, $f(4) = 0$ and $f(5) = 0$.

8 Find two quadratic functions f and g such that $f(1) = 0$, $g(1) = 0$ and $f(0) = 10$, $g(0) = 10$ and both have a maximum value of 18.

9 **a** Find the set of values of k for which $f(x) = 3x^2 - 5x - k$ is greater than 1 for all real x.

b Show that, for all k, the minimum value of $f(x)$ occurs when $x = \frac{5}{6}$. Find k if this minimum value is zero.

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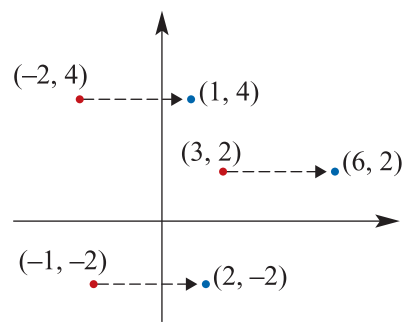
6E Translations of functions

The **Cartesian plane** is represented by the set $\mathbb{R}^2$ of all ordered pairs of real numbers. That is, $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$. The transformations considered in this book associate each ordered pair of $\mathbb{R}^2$ with a unique ordered pair. We can refer to them as examples of **transformations of the plane**.

For example, the translation 3 units in the positive direction of the x-axis (to the right) associates with each ordered pair (x, y) a new ordered pair $(x + 3, y)$. This translation is a transformation of the plane. Each point in the plane is mapped to a unique second point. Furthermore, every point in the plane is an image of another point under this translation.

Notation

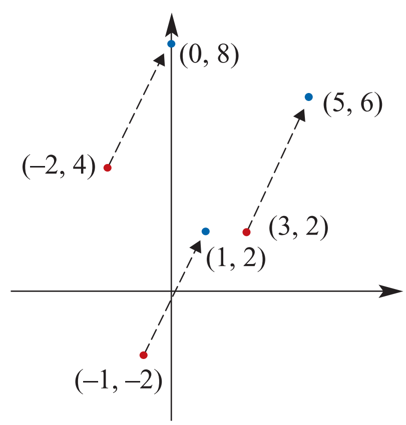
The translation 3 units to the right can be written $(x, y) \to (x + 3, y)$. This reads as '(x, y) maps to $(x + 3, y)$'.



For example, $(-1, -2) \to (-1 + 3, -2)$.

In applying this translation, it is useful to think of every point (x, y) in the plane as being mapped to a new point (x', y'). This point (x, y) is the only point which maps to (x', y'). The following can be written for this translation:

$$x' = x + 3 \quad \text{and} \quad y' = y$$



As another example, consider the translation 2 units in the positive direction of the x-axis (to the right) and 4 units in the positive direction of the y-axis (up). This can be described by the rule $(x, y) \to (x + 2, y + 4)$.

For example, $(3, 2) \to (3 + 2, 2 + 4)$.

The following can be written for this translation:

$$x' = x + 2 \quad \text{and} \quad y' = y + 4$$

A translation of h units in the positive direction of the x-axis and k units in the positive direction of the y-axis is described by the rule

$$(x, y) \to (x + h, y + k)$$

or $\quad x' = x + h \quad \text{and} \quad y' = y + k$

where h and k are positive numbers.

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A translation of h units in the negative direction of the x-axis and k units in the negative direction of the y-axis is described by the rule

$$(x, y) \to (x - h, y - k)$$

or $\quad x' = x - h \quad$ and $\quad y' = y - k$

where h and k are positive numbers.

Notes:

- Under a translation, if $(a', b') = (c', d')$, then $(a, b) = (c, d)$.
- For a particular translation $(x, y) \to (x + h, y + k)$, for each point $(a, b) \in \mathbb{R}^2$ there is a point (p, q) such that $(p, q) \to (a, b)$. (It is clear that $(a - h, b - k) \to (a, b)$ under this translation.)

Applying translations to sketch graphs

We look at a particular example.

Translate the set of points defined by the function

$$\{(x, y): y = x^2\}$$

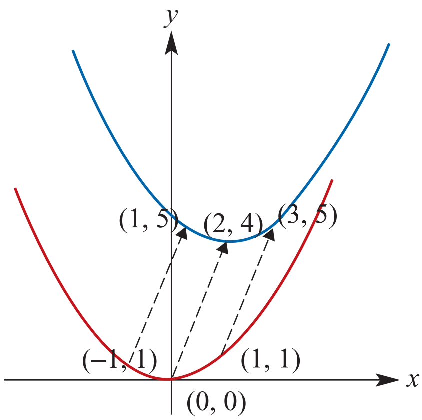
by the translation defined by the rule

$$(x, y) \to (x + 2, y + 4)$$

$$x' = x + 2 \quad \text{and} \quad y' = y + 4$$

For each point (x, y) there is a unique point (x', y') and vice versa.

We have $x = x' - 2$ and $y = y' - 4$.



This means the points on the curve with equation $y = x^2$ are mapped to the curve with equation $y' - 4 = (x' - 2)^2$.

Hence $\{(x, y): y = x^2\}$ maps to $\{(x', y'): y' - 4 = (x' - 2)^2\}$.

For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the translation $(x, y) \to (x + h, y + k)$ to the graph of $y = f(x)$.
- Replacing x with $x - h$ and y with $y - k$ in the equation to obtain $y - k = f(x - h)$ and graphing the result.

Proof A point (a, b) is on the graph of $y = f(x)$

$\Leftrightarrow \quad f(a) = b$

$\Leftrightarrow \quad f(a + h - h) = b$

$\Leftrightarrow \quad f(a + h - h) = b + k - k$

$\Leftrightarrow \quad (a + h, b + k)$ is a point on the graph of $y - k = f(x - h)$

Note: The double arrows indicate that the steps are reversible.

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Example 16

Find the equation for the image of the curve with equation $y = f(x)$, where $f(x) = \frac{1}{x}$, under a translation 3 units in the positive direction of the x-axis and 2 units in the negative direction of the y-axis.

Solution

Let (x', y') be the image of the point (x, y), where (x, y) is a point on the graph of $y = f(x)$.

Then $x' = x + 3$ and $y' = y - 2$.

Hence $x = x' - 3$ and $y = y' + 2$.

The graph of $y = f(x)$ is mapped to the graph of $y' + 2 = f(x' - 3)$

i.e. $y = \frac{1}{x}$ is mapped to

$$y' + 2 = \frac{1}{x' - 3}$$

Explanation

The rule is $(x, y) \to (x + 3, y - 2)$.

Substitute $x = x' - 3$ and $y = y' + 2$ into $y = f(x)$.

Recognising that a transformation has been applied makes it easy to sketch many graphs.

For example, in order to sketch the graph of

$$y = \frac{1}{x - 2}$$

note that it is of the form $y = f(x - 2)$ where $f(x) = \frac{1}{x}$. That is, the graph of $y = \frac{1}{x}$ is translated 2 units in the positive direction of the x-axis.

Examples of two other functions to which this translation is applied are:

$$f(x) = x^2 \qquad f(x - 2) = (x - 2)^2$$

$$f(x) = \sqrt{x} \qquad f(x - 2) = \sqrt{x - 2}$$

Summary 6E

For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the translation $(x, y) \to (x + h, y + k)$ to the graph of $y = f(x)$.
- Replacing x with $x - h$ and y with $y - k$ in the equation to obtain $y - k = f(x - h)$ and graphing the result.

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Exercise 6E

1 Find the image of the point $(-3, 4)$ after a mapping of a translation:

a of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis

b of 2 units in the negative direction of the x-axis and 4 units in the positive direction of the y-axis

c of 3 units in the negative direction of the x-axis and 2 units in the negative direction of the y-axis

d defined by the rule $(x, y) \to (x - 4, y + 5)$

e defined by the rule $(x, y) \to (x - 2, y - 1)$.

Example 16

2 In each of the following, find the rule for the image of the graph of $y = f(x)$ under the given translation:

a $f(x) = \frac{1}{x}$ under a translation 2 units in the positive direction of the x-axis and 1 unit in the negative direction of the y-axis

b $f(x) = \frac{1}{x}$ under a translation 4 units in the positive direction of the x-axis and 3 units in the positive direction of the y-axis

c $f(x) = x^2$ under a translation 2 units in the negative direction of the x-axis and 3 units in the negative direction of the y-axis

d $f(x) = x^2$ under a translation 4 units in the positive direction of the x-axis and 2 units in the negative direction of the y-axis

e $f(x) = \sqrt{x}$ under a translation 2 units in the positive direction of the x-axis and 1 unit in the negative direction of the y-axis.

3 For $y = f(x) = \frac{1}{x}$, sketch the graph of each of the following, labelling asymptotes and axis intercepts:

a $y = f(x - 1)$ **b** $y = f(x) + 1$ **c** $y = f(x + 3)$
d $y = f(x) - 3$ **e** $y = f(x + 1)$ **f** $y = f(x) - 1$

4 For $y = f(x) = x^2$, sketch the graph of each of the following, labelling axis intercepts:

a $y = f(x - 1)$ **b** $y = f(x) + 1$ **c** $y = f(x + 3)$
d $y = f(x) - 3$ **e** $y = f(x + 1)$ **f** $y = f(x) - 1$

5 For $y = f(x) = x^2$, sketch the graph of each of the following, labelling axis intercepts:

a $y = f(x - 1) + 2$ **b** $y = f(x - 3) + 1$ **c** $y = f(x + 3) - 5$
d $y = f(x + 1) - 3$ **e** $y + 2 = f(x + 1)$ **f** $y = f(x - 5) - 1$

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6F Dilations and reflections

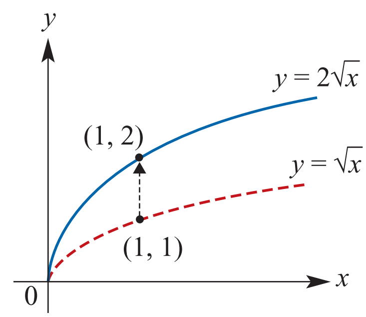
The same techniques can be applied to dilations from an axis and reflections.

Dilation parallel to the y-axis

A dilation of factor 2 parallel to the y-axis can be defined by the rule $(x, y) \to (x, 2y)$.

Hence the point with coordinates $(1, 1) \to (1, 2)$.

Consider the curve with equation $y = \sqrt{x}$ and the dilation of factor 2 parallel to the y-axis.



- Let (x', y') be the image of the point with coordinates (x, y) on the curve.
- Hence $x' = x$ and $y' = 2y$, and thus $x = x'$ and $y = \dfrac{y'}{2}$.
- Substituting for x and y, we see that the curve with equation $y = \sqrt{x}$ maps to the curve with equation $\dfrac{y'}{2} = \sqrt{x'}$, i.e. the curve with equation $y = 2\sqrt{x}$.

For b a positive constant, a dilation of factor b parallel to the y-axis is described by the rule

$$(x, y) \to (x, by)$$

or $\quad x' = x \quad$ and $\quad y' = by$

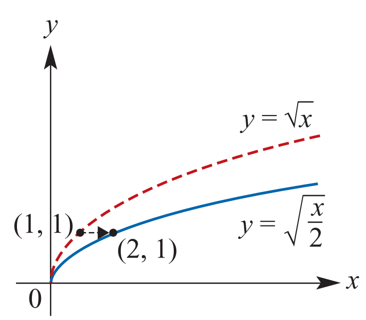
For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the dilation parallel to the y-axis $(x, y) \to (x, by)$ to the graph of $y = f(x)$.
- Replacing y with $\dfrac{y}{b}$ in the equation to obtain $y = bf(x)$ and graphing the result.

Dilation parallel to the x-axis

A dilation of factor 2 parallel to the x-axis can be defined by the rule $(x, y) \to (2x, y)$. Hence the point with coordinates $(1, 1) \to (2, 1)$.

Again, consider the curve with equation $y = \sqrt{x}$.



- Let (x', y') be the image of the point with coordinates (x, y) on the curve.
- Hence $x' = 2x$ and $y' = y$, and thus $x = \dfrac{x'}{2}$ and $y = y'$.
- The curve with equation $y = \sqrt{x}$ maps to the curve with equation $y' = \sqrt{\dfrac{x'}{2}}$.

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For a a positive constant, a dilation of factor a parallel to the x-axis is described by the rule

$$(x, y) \to (ax, y)$$

or $\quad x' = ax \quad$ and $\quad y' = y$

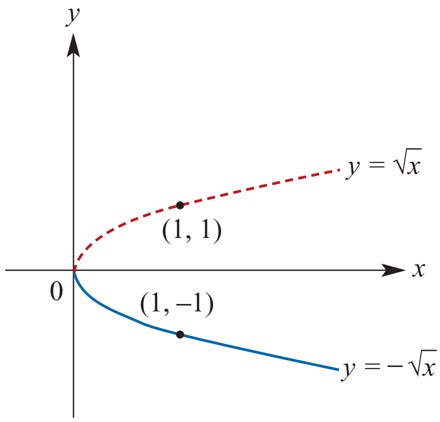
For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the dilation parallel to the x-axis $(x, y) \to (ax, y)$ to the graph of $y = f(x)$.
- Replacing x with $\frac{x}{a}$ in the equation to obtain $y = f\left(\frac{x}{a}\right)$ and graphing the result.

Reflection in the *x*-axis

A reflection in the x-axis can be defined by the rule $(x, y) \to (x, -y)$. Hence the point with coordinates $(1, 1) \to (1, -1)$.

- Let (x', y') be the image of the point (x, y).
- Hence $x' = x$ and $y' = -y$, which gives $x = x'$ and $y = -y'$.
- The curve with equation $y = \sqrt{x}$ maps to the curve with equation $-y' = \sqrt{x'}$, i.e. the curve with equation $y = -\sqrt{x}$.



A reflection in the x-axis is described by the rule

$$(x, y) \to (x, -y)$$

or $\quad x' = x \quad$ and $\quad y' = -y$

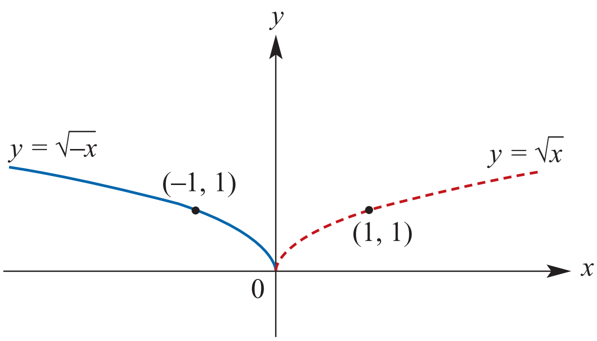
For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the reflection in the x-axis $(x, y) \to (x, -y)$ to the graph of $y = f(x)$.
- Replacing y with $-y$ in the equation to obtain $y = -f(x)$ and graphing the result.

Reflection in the *y*-axis

A reflection in the y-axis can be defined by the rule $(x, y) \to (-x, y)$. Hence the point with coordinates $(1, 1) \to (-1, 1)$.

- Let (x', y') be the image of the point (x, y).
- Hence $x' = -x$ and $y' = y$, which gives $x = -x'$ and $y = y'$.
- The curve with equation $y = \sqrt{x}$ maps to the curve with equation $y' = \sqrt{-x'}$, i.e. the curve with equation $y = \sqrt{-x}$.



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A reflection in the y-axis is described by the rule

$$(x, y) \to (-x, y)$$

or $\quad x' = -x \quad$ and $\quad y' = y$

For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the reflection in the y-axis $(x, y) \to (-x, y)$ to the graph of $y = f(x)$.
- Replacing x with $-x$ in the equation to obtain $y = f(-x)$ and graphing the result.

Example 17

Determine the rule of the image when the graph of $y = x^2$ is dilated by a factor of 3:

a parallel to the x-axis **b** parallel to the y-axis.

Solution

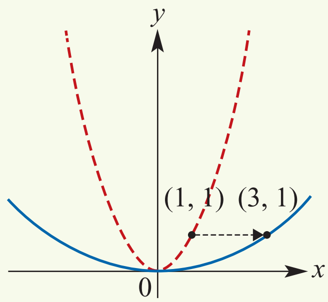
a $(x, y) \to (3x, y)$

Let (x', y') be the coordinates of the image of (x, y), so $x' = 3x$, $y' = y$.

Rearranging gives $x = \dfrac{x'}{3}$, $y = y'$.

Therefore $y = x^2$ becomes $y' = \left(\dfrac{x'}{3}\right)^2$.

The rule of the transformed function is $y = \dfrac{x^2}{9}$.



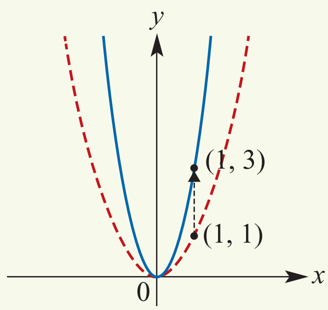
b $(x, y) \to (x, 3y)$

Let (x', y') be the coordinates of the image of (x, y), so $x' = x$, $y' = 3y$.

Rearranging gives $x = x'$, $y = \dfrac{y'}{3}$.

Therefore $y = x^2$ becomes $\dfrac{y'}{3} = (x')^2$.

The rule of the transformed function is $y = 3x^2$.



Applying dilations and reflections to sketch graphs

In order to sketch the graph of $y = \sqrt{\dfrac{x}{2}}$, note that it is of the form $y = f\left(\dfrac{x}{2}\right)$ where $f(x) = \sqrt{x}$. This is the graph of $y = \sqrt{x}$ dilated by factor 2 parallel to the x-axis.

Examples of other functions under this dilation are:

$$f(x) = x^2 \qquad f\left(\frac{x}{2}\right) = \left(\frac{x}{2}\right)^2 = \frac{x^2}{4}$$

$$f(x) = \frac{1}{x} \qquad f\left(\frac{x}{2}\right) = \frac{1}{\frac{x}{2}} = \frac{2}{x}$$

It should be noted that each of these functions formed by a dilation of factor 2 parallel to the x-axis can also be formed by a dilation parallel to the y-axis. This result is not true in general, as will be seen when new functions are introduced in Chapters 12 and 14.

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- For the graph of $y = \sqrt{\frac{x}{2}}$, we can write $y = \frac{1}{\sqrt{2}}\sqrt{x} = \frac{1}{\sqrt{2}}f(x)$, where $f(x) = \sqrt{x}$. That is, it is formed by a dilation of factor $\frac{1}{\sqrt{2}}$ parallel to the y-axis.
- For the graph of $y = \frac{x^2}{4}$, we can write $y = \frac{1}{4}x^2 = \frac{1}{4}f(x)$, where $f(x) = x^2$. That is, it is formed by a dilation of factor $\frac{1}{4}$ parallel to the y-axis.
- For the graph of $y = \frac{2}{x}$, we can write $y = 2f(x)$, where $f(x) = \frac{1}{x}$. That is, it is formed by a dilation of factor 2 parallel to the y-axis.

Summary 6F

For the graph of $y = f(x)$, we have the following four pairs of equivalent processes:

1. - Applying the **dilation parallel to the y-axis** $(x, y) \to (x, by)$ to the graph of $y = f(x)$.
   - Replacing y with $\frac{y}{b}$ in the equation to obtain $y = bf(x)$ and graphing the result.
2. - Applying the **dilation parallel to the x-axis** $(x, y) \to (ax, y)$ to the graph of $y = f(x)$.
   - Replacing x with $\frac{x}{a}$ in the equation to obtain $y = f\left(\frac{x}{a}\right)$ and graphing the result.
3. - Applying the **reflection in the x-axis** $(x, y) \to (x, -y)$ to the graph of $y = f(x)$.
   - Replacing y with $-y$ in the equation to obtain $y = -f(x)$ and graphing the result.
4. - Applying the **reflection in the y-axis** $(x, y) \to (-x, y)$ to the graph of $y = f(x)$.
   - Replacing x with $-x$ in the equation to obtain $y = f(-x)$ and graphing the result.

Exercise 6F

1 Find the image of the point $(-2, -3)$ after:

a a reflection in the x-axis
b a reflection in the y-axis
c a dilation of factor 4 parallel to the y-axis
d a dilation of factor 4 parallel to the x-axis.

Example 17

2 Write down the equation of the image obtained when the graph of each of the functions below is transformed by:

i a dilation of factor $\frac{1}{2}$ parallel to the x-axis
ii a dilation of factor 5 parallel to the x-axis
iii a dilation of factor $\frac{2}{3}$ parallel to the y-axis
iv a dilation of factor 4 parallel to the y-axis
v a reflection in the x-axis
vi a reflection in the y-axis.

a $y = x^2$ **b** $y = \frac{1}{x}$ **c** $y = \sqrt{x}$

3 Sketch the graph of each of the following:

a $y = 3\sqrt{x}$ **b** $y = -\frac{1}{x}$ **c** $y = \frac{3}{x}$ **d** $y = \frac{1}{2x}$ **e** $y = \sqrt{3x}$ **f** $y = \frac{3}{2x}$

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6G Combinations of transformations

In this section, we look at sequences of transformations. For example, first consider:

- a dilation of factor 2 parallel to the y-axis
- followed by a reflection in the x-axis.

The rule becomes

$$(x, y) \to (x, 2y) \to (x, -2y)$$

First the dilation is applied and then the reflection. For example, $(1, 1) \to (1, 2) \to (1, -2)$.

Another example is:

- a dilation of factor 2 parallel to the y-axis
- followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis.

The rule becomes

$$(x, y) \to (x, 2y) \to (x + 2, 2y - 3)$$

First the dilation is applied and then the translation. For example, $(1, 1) \to (1, 2) \to (3, -1)$.

Example 18

Find the equation of the image of $y = \sqrt{x}$ under:

a a dilation of factor 2 parallel to the y-axis followed by a reflection in the x-axis

b a dilation of factor 2 parallel to the y-axis followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis.

Solution

a From the discussion above, the rule is $(x, y) \to (x, 2y) \to (x, -2y)$.

If (x, y) maps to (x', y'), then $x' = x$ and $y' = -2y$. Thus $x = x'$ and $y = \dfrac{y'}{-2}$.

The image has equation $\dfrac{y'}{-2} = \sqrt{x'}$ and hence $y' = -2\sqrt{x'}$.

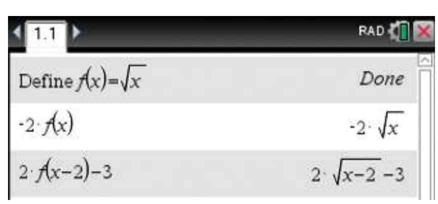
b From the discussion above, the rule is $(x, y) \to (x, 2y) \to (x + 2, 2y - 3)$.

If (x, y) maps to (x', y'), then $x' = x + 2$ and $y' = 2y - 3$. Thus $x = x' - 2$ and $y = \dfrac{y' + 3}{2}$.

The image has equation $\dfrac{y' + 3}{2} = \sqrt{x' - 2}$ and hence $y' = 2\sqrt{x' - 2} - 3$.

Using the TI-Nspire

Use (menu) > **Actions** > **Define** to define the function $f(x) = \sqrt{x}$. Complete as shown.



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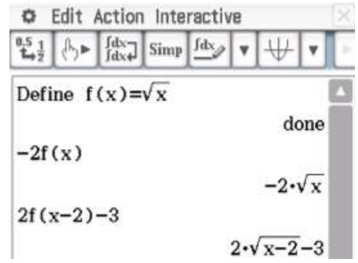
Using the Casio ClassPad

Use **Interactive** > **Define** to define the function $f(x) = \sqrt{x}$, and then complete as shown.

Note: The symbol $\sqrt{}$ is found in Math1.



Summary 6G

A sequence of transformations can be applied, and the rule for transforming points of the plane can be described. For example, the sequence

- a dilation of factor 3 parallel to the y-axis
- followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis
- followed by a reflection in the x-axis

can be described by the rule $(x, y) \to (x, 3y) \to (x + 2, 3y - 3) \to (x + 2, 3 - 3y)$.

Let $x' = x + 2$ and $y' = 3 - 3y$. Then $x = x' - 2$ and $y = \dfrac{3 - y'}{3}$.

The graph of $y = f(x)$ maps to $\dfrac{3 - y'}{3} = f(x' - 2)$. That is, the graph of $y = f(x)$ maps to the graph of $y = 3 - 3f(x - 2)$.

Skill-sheet

Exercise 6G

1 Find the equation of the image of the graph $y = \sqrt{x}$ when each of the following sequences of transformations have been applied:

Example 18

a a translation of 2 units in the positive direction of the x-axis followed by a dilation of factor 3 parallel to the y-axis

b a translation of 3 units in the negative direction of the x-axis followed by a reflection in the x-axis

c a reflection in the x-axis followed by a dilation of factor 3 parallel to the y-axis

d a reflection in the x-axis followed by a dilation of factor 2 parallel to the x-axis

e a dilation of factor 2 parallel to the y-axis followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis

f a dilation of factor 2 parallel to the x-axis followed by a translation of 2 units in the negative direction of the x-axis and 3 units in the negative direction of the y-axis.

2 Repeat Question 1 for $y = \dfrac{1}{x}$.

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6H Determining transformations

The method that has been used to find the effect of transformations can be reversed to determine the sequence of transformations used to take a graph to its image.

For example, in order to find the sequence of transformations which maps $y = \sqrt{x}$ to $y' = -2\sqrt{x'}$, work backwards through the steps in the solution of Example 18a:

- $y = \sqrt{x}$ maps to $\dfrac{y'}{-2} = \sqrt{x'}$.
- Hence $x = x'$ and $y = \dfrac{y'}{-2}$, and therefore $x' = x$ and $y' = -2y$.
- The transformation is a dilation of factor 2 parallel to the y-axis followed by a reflection in the x-axis.

This can also be done by inspection, of course, if you recognise the form of the image. For the combinations of transformations in this course, it is often simpler to do this.

Example 19

a Find a sequence of transformations which takes the graph of $y = x^2$ to the graph of $y = 2(x - 2)^2 + 3$.

b Find a sequence of transformations which takes the graph of $y = \sqrt{x}$ to the graph of $y = \sqrt{5x - 2}$.

Solution

a By inspection

By inspection, it is a dilation of factor 2 parallel to the y-axis followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

By the method

$y = x^2$ maps to $y' = 2(x' - 2)^2 + 3$. Rearranging this equation gives

$$\frac{y' - 3}{2} = (x' - 2)^2$$

We choose to write $y = \dfrac{y' - 3}{2}$ and $x = x' - 2$.

Solving for x' and y' gives

$$x' = x + 2 \quad \text{and} \quad y' = 2y + 3$$

The transformation is a dilation of factor 2 parallel to the y-axis followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

b We have $y' = \sqrt{5x' - 2}$ and $y = \sqrt{x}$. We choose to write $y = y'$ and $x = 5x' - 2$. Hence

$$x' = \frac{x + 2}{5} = \frac{x}{5} + \frac{2}{5} \quad \text{and} \quad y' = y$$

The transformation is a dilation of factor $\dfrac{1}{5}$ parallel to the x-axis followed by a translation of $\dfrac{2}{5}$ units in the positive direction of the x-axis.

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Example 20

a Find a sequence of transformations which takes the graph of $y = \frac{3}{x-1} + 6$ to the graph of $y = \frac{1}{x}$.

b Find a sequence of transformations which takes the graph of $y = (5x-1)^2 + 6$ to the graph of $y = x^2$.

Solution

a Write $\frac{y-6}{3} = \frac{1}{x-1}$ and $y' = \frac{1}{x'}$. The points (x, y) satisfying $\frac{y-6}{3} = \frac{1}{x-1}$ are mapped to the points (x', y') satisfying $y' = \frac{1}{x'}$.

Hence we choose to write

$$y' = \frac{y-6}{3} \quad \text{and} \quad x' = x - 1$$

One transformation is a translation of 6 units in the negative direction of the y-axis and 1 unit in the negative direction of the x-axis followed by a dilation of factor $\frac{1}{3}$ parallel to the y-axis.

b Write $y - 6 = (5x-1)^2$ and $y' = (x')^2$. The points (x, y) satisfying $y - 6 = (5x-1)^2$ are mapped to the points (x', y') satisfying $y' = (x')^2$.

Hence we choose to write

$$y' = y - 6 \quad \text{and} \quad x' = 5x - 1$$

One transformation is a dilation of factor 5 parallel to the x-axis followed by a translation of 6 units in the negative direction of the y-axis and 1 unit in the negative direction of the x-axis.

We note that the transformations we found are far from being the only possible answers. In fact there are infinitely many choices.

Summary 6H

The notation developed in this chapter can be used to help find the transformation that takes the graph of a function to its image.

For example, if the graph of $y = f(x)$ is mapped to the graph of $y' = 2f(x' - 3)$, we can see that the transformation

$$x' = x + 3 \quad \text{and} \quad y' = 2y$$

is a suitable choice. This is a translation of 3 units to the right followed by a dilation of factor 2 parallel to the y-axis.

There are infinitely many transformations that take the graph of $y = f(x)$ to the graph of $y' = 2f(x' - 3)$. The one we chose is conventional.

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Skill-sheet

Exercise 6H

Example 19

1 For each of the following, find a sequence of transformations that takes:

a the graph of $y = x^2$ to the graph of

i $y = 2(x - 1)^2 + 3$ **ii** $y = -(x + 1)^2 + 2$ **iii** $y = (2x + 1)^2 - 2$

b the graph of $y = \frac{1}{x}$ to the graph of

i $y = \frac{2}{x + 3}$ **ii** $y = \frac{1}{x + 3} + 2$ **iii** $y = \frac{1}{x - 3} - 2$

c the graph of $y = \sqrt{x}$ to the graph of

i $y = \sqrt{x + 3} + 2$ **ii** $y = 2\sqrt{3x}$ **iii** $y = -\sqrt{x} + 2$

Example 20

2 **a** Find a sequence of transformations that takes the graph of $y = (3x + 2)^2 + 5$ to the graph of $y = x^2$.

b Find a sequence of transformations that takes the graph of $y = -3(3x + 1)^2 + 7$ to the graph of $y = x^2$.

c Find a sequence of transformations that takes the graph of $y = 2\sqrt{4 - x}$ to the graph of $y = \sqrt{x}$.

d Find a sequence of transformations that takes the graph of $y = 2\sqrt{4 - x} + 3$ to the graph of $y = -\sqrt{x} + 6$.

6I Functions and modelling exercises

In the following examples we see how function notation can be used when applying mathematics in 'real' situations.

Example 21

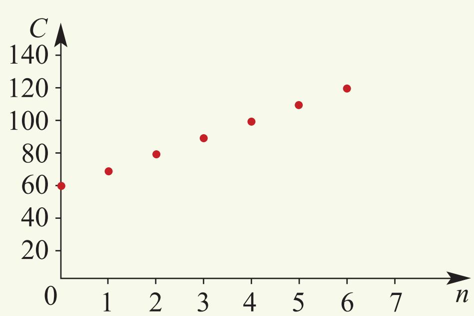
A book club has a membership fee of \$60.00 and each book purchased is \$10.00. Construct a cost function that can be used to determine the cost of different numbers of books, then sketch its graph.

Solution

Let $C(n)$ denote the cost (in dollars) when n books are purchased. Then

$$C(n) = 60 + 10n$$

The domain of this function is $\mathbb{N} \cup \{0\}$, the set of non-negative integers, and its graph will be as shown.



The range of this function is $\{10x\colon x \in \mathbb{N} \text{ and } x \geq 6\}$. Sometimes to simplify the situation we represent such functions by a continuous line. Strictly, this is not mathematically correct, but it may aid our understanding of the situation.

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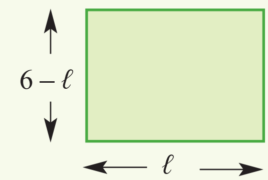
Example 22

A householder has six laying hens and wishes to construct a rectangular enclosure to provide a maximum area for the hens, using a 12 m length of fencing wire. Construct a function that will give the area of the enclosure, A, in terms of its length, ℓ. By sketching a graph, find the maximum area that can be fenced.

Solution



Let ℓ = length of the enclosure

Then width $= \dfrac{12 - 2\ell}{2} = 6 - \ell$

The area is

$$\begin{aligned} A(\ell) &= \ell(6 - \ell) \\ &= 6\ell - \ell^2 \end{aligned}$$

The domain of A is the interval $[0, 6]$.

The maximum area is $9\ \text{m}^2$ and occurs when $\ell = 3$ m, i.e. when the enclosure is a square.

Exercise 6I

Example 21

1 Vicom's rates for local calls from private telephones consist of a quarterly rental fee of \$45 plus 15c for every call. Construct a cost function that describes the quarterly telephone bill.

2 Self-Travel, a car rental firm, has two methods of charging for car rental:

Method 1 $64 per day + 25 cents per kilometre

Method 2 $89 per day with unlimited travel

a Write a rule for each method if C_1 is the cost, in dollars, using method 1 for x kilometres travelled, and C_2 is the cost using method 2.

b Draw a graph of each rule on the same axes.

c Determine, from the graph, the distance which must be travelled per day if method 2 is cheaper than method 1.

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Example 22

3 A piece of wire 100 cm long is bent to form a rectangle. Let x cm be the width of the rectangle.

a Find the length of the rectangle in terms of x.

b Find the rule $A(x)$ for the function A that gives the area of the rectangle in cm^2.

c Find the allowable values for x.

d Find the maximum possible area of the rectangle and the value of x for which this maximum occurs.

4 Consider the diagram shown. Assume that angles that look like right angles are right angles.

a **i** Find an expression for the area A in terms of x and y.

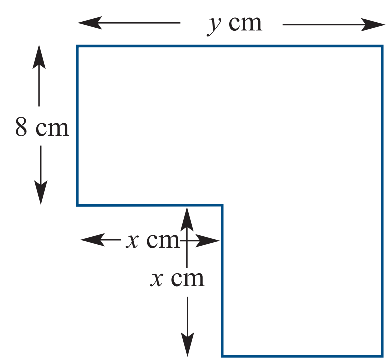
ii Find an expression for the perimeter P in terms of x and y.

b **i** If $P = 64$ cm, find A in terms of x.

ii Find the allowable values for x.

iii Sketch the graph of A against x for these values.

iv What is the maximum area?



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Chapter summary

Assign-ment

Nrich

■ **Set notation**

$x \in A$	x is an element of A
$x \notin A$	x is not an element of A
$A \subseteq B$	A is a subset of B
$A \cap B$	$x \in A \cap B$ if and only if $x \in A$ and $x \in B$
$A \cup B$	$x \in A \cup B$ if and only if $x \in A$ or $x \in B$

■ **Sets of numbers**

$\mathbb{N}$	Natural numbers	$\mathbb{Z}$	Integers
$\mathbb{Q}$	Rational numbers	$\mathbb{R}$	Real numbers

■ **Interval notation**

$$(a, b) = \{ x \in \mathbb{R} : a < x < b \} \qquad [a, b] = \{ x \in \mathbb{R} : a \le x \le b \}$$

$$(a, b] = \{ x \in \mathbb{R} : a < x \le b \} \qquad [a, b) = \{ x \in \mathbb{R} : a \le x < b \}$$

$$(a, \infty) = \{ x \in \mathbb{R} : a < x \} \qquad [a, \infty) = \{ x \in \mathbb{R} : a \le x \}$$

$$(-\infty, b) = \{ x \in \mathbb{R} : x < b \} \qquad (-\infty, b] = \{ x \in \mathbb{R} : x \le b \}$$

■ **Relations**

- A **relation** is a set of ordered pairs.
- The **domain** is the set of all the first coordinates of the ordered pairs in the relation.
- The **range** is the set of all the second coordinates of the ordered pairs in the relation.

■ **Functions**

- For a function f and an element x of the domain of f, there is a unique element y in the range such that $(x, y) \in f$. The element y is called the **image** of x under f, and the element x is called a **pre-image** of y.
- The **natural domain** (or **implied domain**) of a function is the largest subset of $\mathbb{R}$ for which the rule is defined.
- For a function f with domain D, a new function g may be defined with domain $A \subseteq D$ and rule given by $g(x) = f(x)$ for all $x \in A$. The function g is called a **restriction** of f.

■ **Transformations of the graphs of functions**

Mapping	Rule	Image of $y = f(x)$
Reflection in the x-axis	$x' = x,\ y' = -y$	$y = -f(x)$
Reflection in the y-axis	$x' = -x,\ y' = y$	$y = f(-x)$
Dilation of factor a parallel to the x-axis	$x' = ax,\ y' = y$	$y = f\left(\frac{x}{a}\right)$
Dilation of factor b parallel to the y-axis	$x' = x,\ y' = by$	$y = bf(x)$
Translation	$x' = x + h,\ y' = y + k$	$y - k = f(x - h)$

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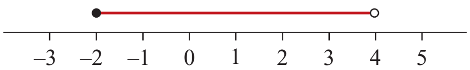
Short-answer questions

1 Describe each of the following using interval notation:

a


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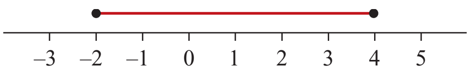

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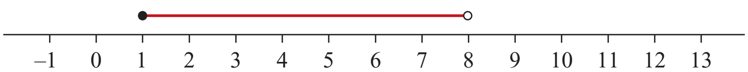
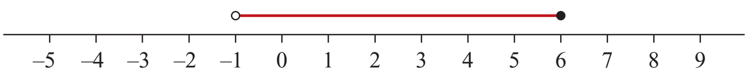


d


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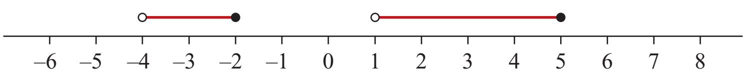

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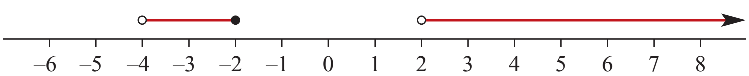
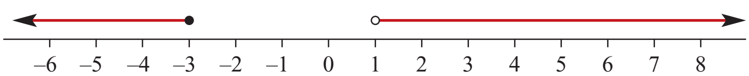


g


2 If f is the function with rule $f(x) = 2 - 6x$, find:

a $f(3)$ **b** $f(-4)$ **c** the value of x for which f maps x to 6.

3 For $f(x) = 6 - x$, $\{ x \in \mathbb{R} : -1 \le x \le 6 \}$:

a sketch the graph of f **b** state the range of f.

4 Sketch the graphs of each of the following, stating the range of each:

a $\{ (x, y) : 3x + y = 6 \}$

b $\{ (x, y) : y = 3x - 2,\ x \in [-1, 2] \}$

c $\{ (x, y) : y = x^2,\ x \in [-2, 2] \}$

d $\{ (x, y) : y = 9 - x^2 \}$

e $\{ (x, y) : y = x^2 + 4x + 6 \}$

f $\{(1, 2), (3, 4), (2, -6)\}$

g $f(x) = (x - 2)^2$

h $f(x) = \dfrac{1}{x} + 2$, $\{ x \in \mathbb{R} : x \ne 0 \}$

i $(x - \dfrac{1}{2})^2 + (y + 2)^2 = 9$

j $f(x) = x$, $\{ x \in \mathbb{R} : -1 \le x \le 3 \}$,

5 The function f has rule $f(x) = \dfrac{a}{x} + b$ such that $f(1) = \dfrac{3}{2}$ and $f(2) = 9$.

a Find the values of a and b. **b** State the natural domain of f.

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6 Given that $f(x) = 2x - x^2$, $\{x \in \mathbb{R} : 0 \le x \le 2\}$:

a sketch the graph **b** state the range.

7 Given that $f(x) = ax + b$, $f(5) = 10$ and $f(1) = -2$, find the values of a and b.

8 Given that $f(x) = ax^2 + bx + c$, $f(0) = 0$, $f(4) = 0$ and $f(-2) = -6$, find the values of a, b and c.

9 State the natural (implied) domain for each of the following:

a $y = \dfrac{1}{x-2}$ **b** $f(x) = \sqrt{x-2}$ **c** $y = \sqrt{25 - x^2}$

d $f(x) = \dfrac{1}{2x-1}$ **e** $g(x) = \sqrt{100 - x^2}$ **f** $h(x) = \sqrt{4 - x}$

10 Sketch the graph of each of the following and state the range:

a $f(x) = (x-1)^2$, $\{x \in \mathbb{R} : 0 \le x \le 3\}$

b $f(x) = (x+2)^2 + 1$, $\{x \in \mathbb{R} : -4 \le x \le 1\}$

c $f(x) = \dfrac{1}{2x-1}$, $\{x \in \mathbb{R} : 1 \le x \le 5\}$

d $f(x) = -x^2 + 3$, $\{x \in \mathbb{R} : -2 \le x \le 3\}$

11 State the natural domain and range of each of the following:

a $f(x) = \sqrt{x-1}$ **b** $f(x) = \sqrt{1-x}$ **c** $f(x) = 1 - \sqrt{x}$

12 State the natural domain and range of each of the following:

a $f(x) = \dfrac{2}{x-1}$ **b** $f(x) = \dfrac{2}{x+1}$ **c** $f(x) = \dfrac{2}{x-1} + 3$

13 State the natural domain and range of each of the following:

a $f(x) = \sqrt{1-x^2}$ **b** $f(x) = \sqrt{9-x^2}$ **c** $f(x) = \sqrt{1-x^2} + 3$

14 If $f(x) = 2x + 5$, find:

a $f(p)$ **b** $f(p+h)$ **c** $f(p+h) - f(p)$ **d** $f(p+1) - f(p)$

15 If $f(x) = 3 - 2x$, find $f(p+1) - f(p)$.

16 State the range of each of the following:

a $f(x) = -2x^2 + x - 2$ **b** $f(x) = 2x^2 - x + 4$

c $f(x) = -x^2 + 6x + 11$ **d** $g(x) = -2x^2 + 8x - 5$

17 $f(x) = 5 - 3x$, $\{x \in \mathbb{R} : -1 \le x \le 6\}$

a Sketch the graph of f. **b** State the range of f.

18 $f(x) = (x-2)^2$, $\{x \in \mathbb{R} : -1 \le x \le 8\}$

a Sketch the graph of f. **b** State the range of f.

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19 State the natural domain and range of each of the following relations:

a $x^2 + y^2 = 9$

b $(x - 2)^2 + y^2 = 1$

c $(2x - 1)^2 + (2y - 1)^2 = 1$

d $(x - 4)^2 + y^2 = 25$

e $(y - 2)^2 + x^2 = 16$

20 The domain of the function f is $\{1, 2, 3, 4\}$. Find the range of f if:

a $f(x) = 2x$ **b** $f(x) = 5 - x$ **c** $f(x) = x^2 - 4$ **d** $f(x) = \sqrt{x}$

21 Sketch the graph of each of the following, labelling asymptotes and axis intercepts:

a $y = \frac{1}{x} + 3$

b $y = \sqrt{x - 2}$

c $y = \frac{1}{x - 1}$

d $y = \frac{1}{x} - 4$

e $y = \frac{1}{x + 2}$

f $y = \frac{1}{x - 3}$

g $f(x) = \frac{1}{x - 1} + 1$

h $f(x) = \frac{1}{x - 2} + 2$

22 For each of the following, find a sequence of transformations that takes:

a the graph of $y = x^2$ to the graph of

i $y = 3(x + 1)^2 + 2$ **ii** $y = -2(x + 2)^2 + 3$ **iii** $y = (3x + 1)^2 - 1$

b the graph of $y = \frac{1}{x}$ to the graph of

i $y = \frac{4}{x + 2}$ **ii** $y = \frac{1}{x + 6} - 12$ **iii** $y = \frac{4}{x - 3} - 5$

c the graph of $y = \sqrt{x}$ to the graph of

i $y = \sqrt{x - 4} + 2$ **ii** $y = 2\sqrt{2x}$ **iii** $y = -2\sqrt{x} + 3$

Extended-response questions

1 A parking meter is designed to accept 200 twenty-cent coins.

a Write a rule which gives the number of hours parking, P, in terms of the number n of twenty-cent coins inserted, when the cost of parking is 20c for each half hour.

b Sketch the function, stating the domain and range.

2 The Exhibition Centre hires a graphics company to produce a poster for an exhibit. The graphics company charges \$1000 and an additional \$5 for each poster produced.

a **i** Write the rule for a function, $C(n)$, which describes the cost to the Exhibition Centre of obtaining n posters.

ii Sketch the graph of C against n (use a continuous model).

b The Exhibition Centre is going to sell the posters for \$15 each.

i Write down the rule for the function $P(n)$ which gives the profit when the Exhibition Centre sells n posters.

ii Sketch the graph of this function (use a continuous model).

3 An article depreciates by 5% of its original cost each year. If the original cost was \$8000, find an expression for the value, V, of the item n years after purchase.

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Review

4 The organisers of a sporting event know that, on average, 50 000 people will visit the venue each day. They are presently charging \$15.00 for an admission ticket. Each time in the past when they have raised the admission price, an average of 2500 fewer people have come to the venue for each \$1.00 increase in ticket price. Let x represent the number of \$1.00 increases.

a Write the rule for a function which gives the revenue, R, in terms of x.

b Sketch the graph of R against x.

c Find the price which will maximise the revenue.

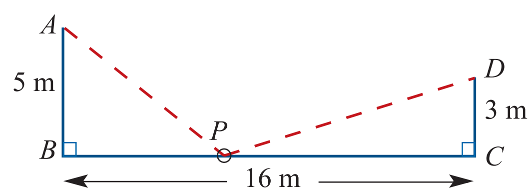
5 A thin wire of length a cm is bent to form the perimeter of a pentagon $ABCDE$ in which $BCDE$ is a rectangle and ABE is an equilateral triangle. Let x cm be the length of CD and let $A(x)$ be the area of the pentagon.

a Find $A(x)$ in terms of x.

b State the allowable values for x.

c Show that the maximum area is $\dfrac{a^2}{4(6-\sqrt{3})}$ cm^2.

6 Let P be a point between B and C on the line BC.
Let $d(x)$ be the distance $(PA + PD)$ m, where x is the distance of P from B.



a **i** Find an expression for $d(x)$.

ii Find the allowable values of x.

b **i** Use a calculator to plot the graph of $y = d(x)$ for a suitable window setting.

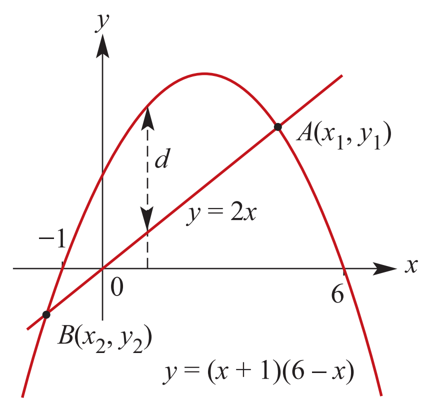
ii Find the value of x if $d(x) = 20$ (correct to two decimal places).

iii Find the values of x for which $d(x) = 19$ (correct to two decimal places).

c **i** Find the minimum value of $d(x)$ and the value of x for which this occurs.

ii State the range of the function.

7 **a** Find the coordinates of the points $A(x_1, y_1)$ and $B(x_2, y_2)$.



b Let $d(x)$ be the 'vertical' distance between the graphs for $x \in [x_2, x_1]$.

i Find $d(x)$ in terms of x.

ii Use a calculator to plot the graph of $d(x)$ against x for $x \in [x_2, x_1]$, and on the same screen plot the graphs of $y = 2x$ and $y = (x + 1)(6 - x)$.

c **i** State the maximum value of the function defined by $d(x)$ for $x \in [x_2, x_1]$.

ii State the range of this function.

d Repeat with the graphs $y = 5x$ and $y = (x + 1)(6 - x)$.

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8 Let $f(x) = x^2$.

a Find the value of k such that the line $y = x$ is tangent to the graph of $y = f(x) + k$.

b Find the value of h such that the line $y = x$ is tangent to the graph of $y = f(x - h)$.

9 **a** The graph of $f(x) = x^2$ is translated to the graph of $y = f(x + h)$. Find the possible values of h if $f(1 + h) = 8$.

b The graph of $f(x) = x^2$ is transformed to the graph of $y = f(ax)$. Find the possible values of a if the graph of $y = f(ax)$ passes through the point with coordinates $(1, 8)$.

c The quadratic with equation $y = ax^2 + bx$ has vertex with coordinates $(1, 8)$. Find the values of a and b.

10 A quadratic function g has rule $g(x) = x^2 + 4x - 6$.

a Find the value of k for which the equation $g(x) + k = 0$ has one solution.

b Find the values of h for which the equation $g(x - h) = 0$ has:

i two positive solutions

ii two negative solutions

iii one positive and one negative solution.

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7 Polynomials

In this chapter

Syllabus references

Topic: Powers and polynomials

Subtopics: 1.1.15 – 1.1.20

In Chapter 3 we looked at polynomial functions of degree 2, or quadratics.

A polynomial function of degree 3 is called a **cubic function**. The general rule for such a function is

$$f(x) = ax^3 + bx^2 + cx + d, \quad a \neq 0$$

A polynomial function of degree 4 is called a **quartic function**. The general rule for such a function is

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e, \quad a \neq 0$$

In Chapter 3 it was shown that all quadratic functions can be written in 'turning point form' and that the graph of a quadratic has one basic form, the parabola.

This is not true of cubic or quartic functions. There is a range of different graph 'shapes' for cubic and quartic functions, depending on the values of the coefficients (a, b, c, d and e).

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7A The language of polynomials

- A **polynomial function** is a function that can be written in the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where n is a natural number or zero, and the coefficients $a_0, \ldots, a_n$ are real numbers with $a_n \neq 0$.
- The number 0 is called the **zero polynomial**.
- The **leading term**, $a_n x^n$, of a polynomial is the term of highest index among those terms with a non-zero coefficient.
- The **degree of a polynomial** is the index n of the leading term.
- A **monic polynomial** is a polynomial whose leading term has coefficient 1.
- The **constant term** is the term of index 0. (This is the term not involving x.)

Example 1

Let $P(x) = x^4 - 3x^3 - 2$. Find:

a $P(1)$ **b** $P(-1)$ **c** $P(2)$ **d** $P(-2)$

Solution

a $P(1) = 1^4 - 3 \times 1^3 - 2$
$= 1 - 3 - 2$
$= -4$

b $P(-1) = (-1)^4 - 3 \times (-1)^3 - 2$
$= 1 + 3 - 2$
$= 2$

c $P(2) = 2^4 - 3 \times 2^3 - 2$
$= 16 - 24 - 2$
$= -10$

d $P(-2) = (-2)^4 - 3 \times (-2)^3 - 2$
$= 16 + 24 - 2$
$= 38$

Example 2

a Let $P(x) = 2x^4 - x^3 + 2cx + 6$. If $P(1) = 21$, find the value of c.

b Let $Q(x) = 2x^6 - x^3 + ax^2 + bx + 20$. If $Q(-1) = Q(2) = 0$, find the values of a and b.

Solution

a $P(x) = 2x^4 - x^3 + 2cx + 6$ and $P(1) = 21$.

$P(1) = 2(1)^4 - (1)^3 + 2c + 6$
$= 2 - 1 + 2c + 6$
$= 7 + 2c$

Since $P(1) = 21$,

$7 + 2c = 21$
$c = 7$

Explanation

We will substitute $x = 1$ into $P(x)$ to form an equation and solve.

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b $Q(x) = 2x^6 - x^3 + ax^2 + bx + 20$ and $Q(-1) = Q(2) = 0$.

First find $Q(-1)$ and $Q(2)$ in terms of a and b.

$$\begin{aligned} Q(-1) &= 2(-1)^6 - (-1)^3 + a(-1)^2 - b + 20 \\ &= 2 + 1 + a - b + 20 \\ &= 23 + a - b \end{aligned}$$

$$\begin{aligned} Q(2) &= 2(2)^6 - (2)^3 + a(2)^2 + 2b + 20 \\ &= 128 - 8 + 4a + 2b + 20 \\ &= 140 + 4a + 2b \end{aligned}$$

Since $Q(-1) = Q(2) = 0$, this gives

Form simultaneous equations in a and b by putting $Q(-1) = 0$ and $Q(2) = 0$.

$$23 + a - b = 0 \qquad (1)$$

$$140 + 4a + 2b = 0 \qquad (2)$$

Divide (2) by 2:

$$70 + 2a + b = 0 \qquad (3)$$

Add (1) and (3):

$$\begin{aligned} 93 + 3a &= 0 \\ a &= -31 \end{aligned}$$

Substitute in (1):

$$b = -8$$

Hence $a = -31$ and $b = -8$.

The arithmetic of polynomials

The operations of addition, subtraction and multiplication for polynomials are naturally defined, as shown in the following examples.

Let $P(x) = x^3 + 3x^2 + 2$ and $Q(x) = 2x^2 + 4$. Then

$$\begin{aligned} P(x) + Q(x) &= (x^3 + 3x^2 + 2) + (2x^2 + 4) \\ &= x^3 + 5x^2 + 6 \end{aligned}$$

$$\begin{aligned} P(x) - Q(x) &= (x^3 + 3x^2 + 2) - (2x^2 + 4) \\ &= x^3 + x^2 - 2 \end{aligned}$$

$$\begin{aligned} P(x)Q(x) &= (x^3 + 3x^2 + 2)(2x^2 + 4) \\ &= (x^3 + 3x^2 + 2) \times 2x^2 + (x^3 + 3x^2 + 2) \times 4 \\ &= 2x^5 + 6x^4 + 4x^2 + 4x^3 + 12x^2 + 8 \\ &= 2x^5 + 6x^4 + 4x^3 + 16x^2 + 8 \end{aligned}$$

The sum, difference and product of two polynomials is a polynomial.

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Example 3

Let $P(x) = x^3 - 6x + 3$ and $Q(x) = x^2 - 3x + 1$. Find:

a $P(x) + Q(x)$ **b** $P(x) - Q(x)$ **c** $P(x)Q(x)$

Solution

a $P(x) + Q(x)$

$= x^3 - 6x + 3 + x^2 - 3x + 1$

$= x^3 + x^2 - 6x - 3x + 3 + 1$

$= x^3 + x^2 - 9x + 4$

b $P(x) - Q(x)$

$= x^3 - 6x + 3 - (x^2 - 3x + 1)$

$= x^3 - 6x + 3 - x^2 + 3x - 1$

$= x^3 - x^2 - 6x + 3x + 3 - 1$

$= x^3 - x^2 - 3x + 2$

c $P(x)Q(x) = (x^3 - 6x + 3)(x^2 - 3x + 1)$

$= x^3(x^2 - 3x + 1) - 6x(x^2 - 3x + 1) + 3(x^2 - 3x + 1)$

$= x^5 - 3x^4 + x^3 - 6x^3 + 18x^2 - 6x + 3x^2 - 9x + 3$

$= x^5 - 3x^4 + (x^3 - 6x^3) + (18x^2 + 3x^2) - (6x + 9x) + 3$

$= x^5 - 3x^4 - 5x^3 + 21x^2 - 15x + 3$

We use the notation $\deg(f)$ to denote the degree of a polynomial f. For $f, g \neq 0$, we have

$$\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$$

$$\deg(f \times g) = \deg(f) + \deg(g)$$

Equating coefficients

Two polynomials P and Q are equal only if their corresponding coefficients are equal. For two cubic polynomials, $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and $Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$, they are equal if and only if $a_3 = b_3$, $a_2 = b_2$, $a_1 = b_1$ and $a_0 = b_0$.

For example, if

$$P(x) = 4x^3 + 5x^2 - x + 3 \quad \text{and} \quad Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$$

then $P(x) = Q(x)$ if and only if $b_3 = 4$, $b_2 = 5$, $b_1 = -1$ and $b_0 = 3$.

Example 4

The polynomial $P(x) = x^3 + 3x^2 + 2x + 1$ can be written in the form $(x - 2)(x^2 + bx + c) + r$ where b, c and r are real numbers. Find the values of b, c and r.

Solution

$(x - 2)(x^2 + bx + c) + r$

$= x(x^2 + bx + c) - 2(x^2 + bx + c) + r$

$= x^3 + bx^2 + cx - 2x^2 - 2bx - 2c + r$

$= x^3 + (b - 2)x^2 + (c - 2b)x - 2c + r$

Explanation

We first expand the brackets of

$$(x - 2)(x^2 + bx + c) + r$$

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We have $x^3 + 3x^2 + 2x + 1 = x^3 + (b - 2)x^2 + (c - 2b)x - 2c + r$	
for all real numbers x. This implies $b - 2 = 3 \quad \therefore\ b = 5$ $c - 2b = 2 \quad \therefore\ c = 2b + 2 = 12$ $-2c + r = 1 \quad \therefore\ r = 2c + 1 = 25$	We now equate coefficients: first the coefficients of x^2, then the coefficients of x, and finally the constants.
Hence $b = 5$, $c = 12$ and $r = 25$. This means that $P(x) = (x - 2)(x^2 + 5x + 12) + 25$	Substitute the values for b, c and r into $(x - 2)(x^2 + bx + c) + r$.

The expansion of $(a + b)^n$

We know that

$$(a + b)^2 = a^2 + 2ab + b^2$$

This is called an **identity**; it is true for all a and b. If we multiply both sides of this identity by $(a + b)$, then we obtain

$$\begin{aligned}(a + b)^3 = (a + b)(a + b)^2 &= (a + b)(a^2 + 2ab + b^2)\\ &= a(a^2 + 2ab + b^2) + b(a^2 + 2ab + b^2)\\ &= a^3 + 2a^2b + ab^2 + ba^2 + 2ab^2 + b^3\\ &= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

So we have a new identity:

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

We can continue to build identities in this way:

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

We will give the general expansion of $(a + b)^n$ in Chapter 10.

Summary 7A

- A **polynomial function** is a function that can be written in the form

  $$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

  where n is a natural number or zero, and the coefficients $a_0, \ldots, a_n$ are real numbers with $a_n \neq 0$. The **leading term** is $a_n x^n$ (the term of highest index) and the **constant term** is a_0 (the term not involving x).
- The **degree of a polynomial** is the index n of the leading term.
- The sum, difference and product of two polynomials is a polynomial. Division does not always lead to another polynomial.
- Two polynomials P and Q are equal only if their corresponding coefficients are equal. Two cubic polynomials, $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and $Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$, are equal if and only if $a_3 = b_3$, $a_2 = b_2$, $a_1 = b_1$ and $a_0 = b_0$.

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Skill-sheet

Exercise 7A

Example 1

1 Let $P(x) = x^3 - 3x^2 - 2x + 1$. Find:

a $P(1)$ **b** $P(-1)$ **c** $P(2)$ **d** $P(-2)$

2 Let $P(x) = 8x^3 - 4x^2 - 2x + 1$. Find:

a $P\left(\frac{1}{2}\right)$ **b** $P\left(-\frac{1}{2}\right)$

3 Let $P(x) = x^3 + 4x^2 - 2x + 6$. Find:

a $P(0)$ **b** $P(1)$ **c** $P(2)$

d $P(-1)$ **e** $P(a)$ **f** $P(2a)$

Example 2

4 **a** Let $P(x) = x^3 + 5x^2 - ax - 20$. If $P(2) = 0$, find the value of a.

b Let $P(x) = 2x^3 + ax^2 - 5x - 7$. If $P(3) = 68$, find the value of a.

c Let $P(x) = x^4 + x^3 - 2x + c$. If $P(1) = 6$, find the value of c.

d Let $P(x) = 3x^6 - 5x^3 + ax^2 + bx + 10$. If $P(-1) = P(2) = 0$, find the values of a and b.

e Let $P(x) = x^5 - 3x^4 + ax^3 + bx^2 + 24x - 36$. If $P(3) = P(1) = 0$, find the values of a and b.

Example 3

5 Let $f(x) = x^3 - 2x^2 + x$, $g(x) = 2 - 3x$ and $h(x) = x^2 + x$. Simplify each of the following:

a $f(x) + g(x)$ **b** $f(x) + h(x)$ **c** $f(x) - g(x)$

d $3f(x)$ **e** $f(x)g(x)$ **f** $g(x)h(x)$

g $f(x) + g(x) + h(x)$ **h** $f(x)h(x)$

6 Expand each of the following products and collect like terms:

a $(x - 2)(x^2 - 2x + 3)$ **b** $(x - 4)(x^2 - 2x + 3)$ **c** $(x - 1)(2x^2 - 3x - 4)$

d $(x - 2)(x^2 + bx + c)$ **e** $(2x + 1)(x^2 - 4x - 3)$

Example 4

7 It is known that $x^3 - 7x^2 + 4x + 12 = (x + 1)(x^2 + bx + c)$ for all values of x, for suitable values of b and c.

a Expand $(x + 1)(x^2 + bx + c)$ and collect like terms.

b Find b and c by equating coefficients.

c Hence write $x^3 - 7x^2 + 4x + 12$ as a product of three linear factors.

8 Let $x^2 + 6x - 2 = (x - b)^2 + c$. Find the values of b and c so that this is true for all x.

9 **a** Expand $(a + b)^5$.

b Expand $(a + b)^6$.

10 Use the identity $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ to expand:

a $(x - y)^4$ (Let $a = x$ and $b = -y$.)

b $(2x + y)^4$ (Let $a = 2x$ and $b = y$.)

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7B Division of polynomials

In order to sketch the graphs of many cubic and quartic functions (as well as higher degree polynomials) it is often necessary to find the x-axis intercepts. As with quadratics, finding x-axis intercepts can be done by factorising and then solving the resulting equation using the null factor theorem.

All cubics will have at least one x-axis intercept. Some will have two and others three.

We shall first look at the techniques for dividing one polynomial by another. One process for division of polynomials is exactly the same as the long division process for numbers.

Long division with positive integers

We show the process for $274 \div 13$.

$$\begin{array}{r@{\,}l} & 21 \\ 13 \,\big)\!\! & \overline{274} \\ & \underline{26} \\ & 14 \\ & \underline{13} \\ & 1 \end{array}$$

We have

$$274 = 13 \times 21 + 1$$

Here 274 is the **dividend**, 13 the **divisor**, 21 the **quotient** and 1 the **remainder**.

When we divide the number p by d we obtain two integers, q the quotient and r the remainder, such that

$$p = dq + r \quad \text{and} \quad 0 \leq r < d$$

For example,

$$27 = 4 \times 6 + 3$$

If $r = 0$, then d is a **factor** of p. For example, $24 = 4 \times 6$.

Long division with polynomials

The process for dividing a polynomial by a linear polynomial follows very similar steps. For example, $(x^2 + 7x + 11) \div (x - 2)$ gives

$x + 9$	
$x - 2\,\big)\,\overline{x^2 + 7x + 11}$	Divide x^2 by x. This gives x.
$\underline{x^2 - 2x}$	Multiply $x - 2$ by x and subtract from $x^2 + 7x + 11$.
$9x + 11$	This leaves $9x + 11$. Now x into $9x$ goes 9 times.
$\underline{9x - 18}$	Multiply $x - 2$ by 9 and subtract from $9x + 11$.
29	This leaves 29 remainder.

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Thus $(x^2 + 7x + 11) \div (x - 2) = x + 9$ with remainder 29. We write

$$x^2 + 7x + 11 = (x - 2)(x + 9) + 29$$

We can see in this example that $x - 2$ is *not* a factor of $x^2 + 7x + 11$. We can also write the result as

$$\frac{x^2 + 7x + 11}{x - 2} = x + 9 + \frac{29}{x - 2}$$

In this example:

- $x^2 + 7x + 11$ is the dividend
- $x - 2$ is the divisor
- 29 is the remainder.

When we divide the polynomial $P(x)$ by the polynomial $D(x)$ we obtain two polynomials, $Q(x)$ the **quotient** and $R(x)$ the **remainder**, such that

$$P(x) = D(x)Q(x) + R(x)$$

and either $R(x) = 0$ or $R(x)$ has degree less than $D(x)$.

Here $P(x)$ is the **dividend** and $D(x)$ is the **divisor**.

Note: If $R(x) = 0$, then $D(x)$ is a **factor** of $P(x)$. For example, let $P(x) = x^2 + 6x + 8$ and $D(x) = x + 2$. Then $P(x) = (x + 2)(x + 4) = D(x)(x + 4) + 0$.

Example 5

Divide $x^3 + x^2 - 14x - 24$ by $x + 2$.

Solution

$$\begin{array}{r|l} & x^2 - x - 12 \\ \hline x + 2 & x^3 + x^2 - 14x - 24 \\ & x^3 + 2x^2 \\ \hline & -x^2 - 14x - 24 \\ & -x^2 - 2x \\ \hline & -12x - 24 \\ & -12x - 24 \\ \hline & 0 \end{array}$$

Explanation

- Divide x, from $x + 2$, into the leading term x^3 to get x^2.
- Multiply x^2 by $x + 2$ to give $x^3 + 2x^2$.
- Subtract from $x^3 + x^2 - 14x - 24$, leaving $-x^2 - 14x - 24$.
- Now divide x, from $x + 2$, into $-x^2$ to get $-x$.
- Multiply $-x$ by $x + 2$ to give $-x^2 - 2x$.
- Subtract from $-x^2 - 14x - 24$, leaving $-12x - 24$.
- Divide x into $-12x$ to get -12.
- Multiply -12 by $x + 2$ to give $-12x - 24$.
- Subtract from $-12x - 24$, leaving remainder of 0.

In this example we see that $x + 2$ is a factor of $x^3 + x^2 - 14x - 24$, as the remainder is zero. Thus $(x^3 + x^2 - 14x - 24) \div (x + 2) = x^2 - x - 12$ with zero remainder.

$$\therefore \quad \frac{x^3 + x^2 - 14x - 24}{x + 2} = x^2 - x - 12$$

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Example 6

Divide $3x^3 + x - 3$ by $x - 2$.

Solution

$$\begin{array}{r}
3x^2 + 6x + 13 \\
x - 2 \overline{\smash{\big)}\, 3x^3 + 0x^2 + x - 3} \\
3x^3 - 6x^2 \\
\hline
6x^2 + x - 3 \\
6x^2 - 12x \\
\hline
13x - 3 \\
13x - 26 \\
\hline
23
\end{array}$$

Explanation

Here there is no term in x^2, however we can rewrite the polynomial as $3x^3 + 0x^2 + x - 3$.

- Divide x, from $x - 2$, into $3x^3$ to get $3x^2$.
- Multiply $3x^2$ by $x - 2$ to give $3x^3 - 6x^2$.
- Subtract from $3x^3 + 0x^2 + x - 3$, leaving $6x^2 + x - 3$.
- Now divide x, from $x - 2$, into $6x^2$ to get $6x$.
- Multiply $6x$ by $x - 2$ to give $6x^2 - 12x$.
- Subtract from $6x^2 + x - 3$, leaving $13x - 3$.
- Divide x into $13x$ to get 13.
- Multiply 13 by $x - 2$ to give $13x - 26$.
- Subtract from $13x - 3$, leaving remainder of 23.

From this example, we have

$$3x^3 + x - 3 = (x - 2)(3x^2 + 6x + 13) + 23$$

Alternatively, we can write

$$\frac{3x^3 + x - 3}{x - 2} = 3x^2 + 6x + 13 + \frac{23}{x - 2}$$

Equating coefficients to divide

We will briefly outline how to carry out divisions by equating coefficients as shown in the first section of this chapter.

To divide $x^3 - 7x^2 + 5x - 4$ by $x - 3$, first write the identity

$$x^3 - 7x^2 + 5x - 4 = (x - 3)(x^2 + bx + c) + r$$

We first find b, then c and finally r by equating coefficients of the left-hand side and right-hand side of this identity.

x^2 **term** Left-hand side: $-7x^2$. Right-hand side: $-3x^2 + bx^2 = (-3 + b)x^2$.
Therefore $-3 + b = -7$. Hence $b = -4$.

x **term** Left-hand side: $5x$. Right-hand side: $12x + cx = (12 + c)x$.
Therefore $12 + c = 5$. Hence $c = -7$.

constant term Left-hand side: -4. Right-hand side: $21 + r$.
Therefore $21 + r = -4$. Hence $r = -25$.

So we can write $x^3 - 7x^2 + 5x - 4 = (x - 3)(x^2 - 4x - 7) - 25$.

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We do the following example using this method. You can see how the long division has the same arithmetic steps.

Example 7

Divide $3x^3 + 2x^2 - x - 2$ by $2x + 1$.

Solution

$$
\begin{array}{r l}
 & \frac{3}{2}x^2 + \frac{1}{4}x - \frac{5}{8} \\
2x + 1 \,\big) & 3x^3 + 2x^2 - x - 2 \\
 & 3x^3 + \frac{3}{2}x^2 \\ \hline
 & \qquad \frac{1}{2}x^2 - x - 2 \\
 & \qquad \frac{1}{2}x^2 + \frac{1}{4}x \\ \hline
 & \qquad\qquad -\frac{5}{4}x - 2 \\
 & \qquad\qquad -\frac{5}{4}x - \frac{5}{8} \\ \hline
 & \qquad\qquad\qquad -1\frac{3}{8}
\end{array}
$$

Explanation

We show the alternative method here.

First write the identity

$3x^3 + 2x^2 - x - 2 = (2x + 1)(ax^2 + bx + c) + r$

Equate coefficients of x^3:

$3 = 2a$. Therefore $a = \frac{3}{2}$.

Equate coefficients of x^2:

$2 = a + 2b$. Therefore $b = \frac{1}{2}\left(2 - \frac{3}{2}\right) = \frac{1}{4}$.

Equate coefficients of x:

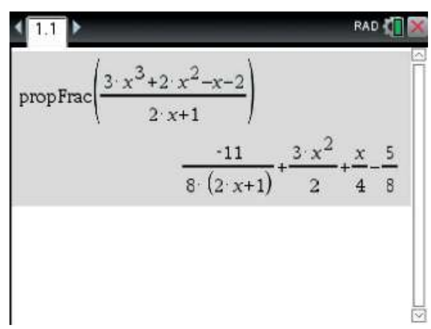
$-1 = 2c + b$. Therefore $c = \frac{1}{2}\left(-1 - \frac{1}{4}\right) = -\frac{5}{8}$.

Equate constant terms:

$-2 = c + r$. Therefore $r = -2 + \frac{5}{8} = -\frac{11}{8}$.

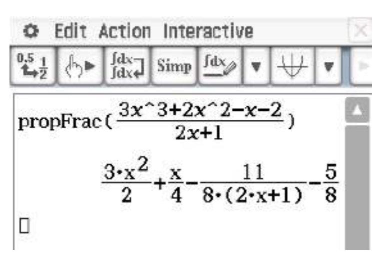
Using the TI-Nspire

Use **propFrac( )** from (menu) > **Algebra** > **Fraction Tools** > **Proper Fraction** as shown.



Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, select [fraction template] from the (Math1) keyboard.
- Enter the expression $\dfrac{3x^3 + 2x^2 - x - 2}{2x + 1}$.
- Highlight the expression and go to **Interactive** > **Transformation** > **Fraction** > **propFrac**.



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Dividing by a non-linear polynomial

We give one example of dividing by a non-linear polynomial. The technique is exactly the same as when dividing by a linear polynomial.

Example 8

Divide $3x^3 - 2x^2 + 3x - 4$ by $x^2 - 1$.

Solution

$$\begin{array}{r|l} & 3x - 2 \\ x^2 + 0x - 1 & 3x^3 - 2x^2 + 3x - 4 \\ & 3x^3 + 0x^2 - 3x \\ \hline & \quad -2x^2 + 6x - 4 \\ & \quad -2x^2 + 0x + 2 \\ \hline & \qquad\quad 6x - 6 \end{array}$$

$\therefore 3x^3 - 2x^2 + 3x - 4 = (x^2 - 1)(3x - 2) + 6x - 6$

Explanation

We write $x^2 - 1$ as $x^2 + 0x - 1$.

Summary 7B

- When we divide the polynomial $P(x)$ by the polynomial $D(x)$ we obtain two polynomials, $Q(x)$ the **quotient** and $R(x)$ the **remainder**, such that

  $$P(x) = D(x)Q(x) + R(x)$$

  and either $R(x) = 0$ or $R(x)$ has degree less than $D(x)$. The polynomial $P(x)$ is the **dividend**.
- Two methods for dividing polynomials are long division and equating coefficients.

Exercise 7B

Example 5

1 For each of the following, divide the polynomial by the accompanying linear expression:

a $x^3 + x^2 - 2x + 3, \ x - 1$ **b** $2x^3 + x^2 - 4x + 3, \ x + 1$

c $3x^3 - 4x^2 + 2x + 1, \ x + 2$ **d** $2x^3 - 3x^2 + x - 2, \ x - 3$

Example 6

2 For each of the following, divide the polynomial by the accompanying linear expression:

a $x^3 + 3x - 4, \ x + 1$ **b** $2x^3 + 17x + 15, \ x + 4$

c $x^3 + 4x^2 + 2, \ x + 3$ **d** $x^3 - 3x^2 + 6x, \ x - 2$

3 For each of the following, divide the polynomial by the accompanying linear expression and hence show that the linear expression is a factor of the polynomial:

a $x^3 - x^2 + 3x + 5, \ x + 1$ **b** $2x^3 + 6x^2 - 14x - 24, \ x + 4$

c $x^3 - 5x^2 + 18, \ x - 3$ **d** $3x^3 - 7x^2 - 4x + 12, \ x - 2$

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4 Find the quotient and remainder when the first polynomial is divided by the second:

a $x^3 + 2x^2 - 3x + 1,\ x + 2$
b $x^3 - 3x^2 + 5x - 4,\ x - 5$
c $2x^3 - x^2 - 3x - 7,\ x + 1$
d $5x^3 - 3x + 7,\ x - 4$

Example 7

5 For each of the following, divide the polynomial by the accompanying linear expression:

a $x^3 + 6x^2 + 8x + 11,\ 2x + 5$
b $2x^3 + 5x^2 - 4x - 5,\ 2x + 1$
c $2x^3 + 3x^2 - 32x + 15,\ 2x - 1$
d $x^3 - 3x^2 + 1,\ 3x - 1$

6 **a** Write $\dfrac{x^3 + 2x^2 + 5x + 1}{x - 1}$ in the form $P(x) + \dfrac{a}{x - 1}$, where $P(x)$ is a quadratic expression and a is a real number.

b Write $\dfrac{2x^3 - 2x^2 + 5x + 3}{2x - 1}$ in the form $P(x) + \dfrac{a}{2x - 1}$, where $P(x)$ is a quadratic expression and a is a real number.

Example 8

7 For each of the following, divide the polynomial $P(x)$ by the polynomial $D(x)$:

a $P(x) = 2x^3 - 6x^2 - 4x + 12,\ D(x) = x^2 - 2$
b $P(x) = x^3 - 6x^2 + x - 8,\ D(x) = x^2 + 1$
c $P(x) = 2x^3 - 6x^2 - 4x + 54,\ D(x) = x^2 - 2$
d $P(x) = x^4 - 2x^3 - 7x^2 + 7x + 5,\ D(x) = x^2 + 2x - 1$
e $P(x) = x^4 - x^3 + 7x + 2,\ D(x) = x^2 + 2x - 1$
f $P(x) = 2x^4 + x^3 + 13x + 10,\ D(x) = 2x^2 - x + 4$

7C Factorisation of polynomials

Remainder theorem

Since the aim of factorising a cubic is usually to solve an equation or to find the x-axis intercepts of a graph, the first step is to establish whether a particular linear expression is a factor of the given cubic or not. It is possible to do this without actually doing the division process.

Let $P(x) = x^3 + 3x^2 + 2x + 1$.

Divide $P(x)$ by $x - 2$:

$$\begin{array}{r|l} & x^2 + 5x + 12 \\ \hline x - 2 & x^3 + 3x^2 + 2x + 1 \\ & x^3 - 2x^2 \\ \hline & \quad 5x^2 + 2x + 1 \\ & \quad 5x^2 - 10x \\ \hline & \qquad 12x + 1 \\ & \qquad 12x - 24 \\ \hline & \qquad\quad 25 \end{array}$$

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The remainder is 25.

Now $P(2) = (2)^3 + 3(2)^2 + 2(2) + 1$

$= 8 + 12 + 4 + 1$

$= 25$

The example suggests that, when $P(x)$ is divided by $x - \alpha$, the remainder is equal to $P(\alpha)$. This is in fact true, and the result is called the **remainder theorem**.

It is proved as follows. Suppose that, when the polynomial $P(x)$ is divided by $x - \alpha$, the quotient is $Q(x)$ and the remainder is R. Then

$$P(x) = (x - \alpha)Q(x) + R$$

Now, as the two expressions are equal for all values of x, they are equal for $x = \alpha$.

$$\therefore \quad P(\alpha) = (\alpha - \alpha)Q(\alpha) + R \qquad \therefore \quad R = P(\alpha)$$

i.e. the remainder when $P(x)$ is divided by $x - \alpha$ is equal to $P(\alpha)$. We therefore have

$$P(x) = (x - \alpha)Q(x) + P(\alpha)$$

More generally:

Remainder theorem

When $P(x)$ is divided by $\beta x + \alpha$, the remainder is $P\left(-\frac{\alpha}{\beta}\right)$.

Example 9

Use the remainder theorem to find the value of the remainder when:

a $P(x) = x^3 - 3x^2 + 2x + 6$ is divided by $x - 2$

b $P(x) = x^3 - 2x + 4$ is divided by $2x + 1$.

Solution	Explanation
a $P(2) = (2)^3 - 3(2)^2 + 2(2) + 6$ $= 8 - 12 + 4 + 6$ $= 6$ The remainder is 6.	We apply the remainder theorem by evaluating $P(2)$.
b $P\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^3 - 2\left(-\frac{1}{2}\right) + 4$ $= -\frac{1}{8} + 1 + 4$ $= \frac{39}{8}$ The remainder is $\frac{39}{8}$.	We apply the remainder theorem by evaluating $P\left(-\frac{1}{2}\right)$.

Note: It is not necessary to perform polynomial division to find the remainder.

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Example 10

When $P(x) = x^3 + 2x + a$ is divided by $x - 2$, the remainder is 4. Find the value of a.

Solution

$P(2) = 8 + 4 + a = 4$

Therefore $a = -8$.

Explanation

We apply the remainder theorem to form a linear equation in a.

Factor theorem

Now, in order for $x - \alpha$ to be a factor of the polynomial $P(x)$, the remainder must be zero. We state this result as the **factor theorem**.

Factor theorem

For a polynomial $P(x)$:

- If $P(\alpha) = 0$, then $x - \alpha$ is a factor of $P(x)$.
- Conversely, if $x - \alpha$ is a factor of $P(x)$, then $P(\alpha) = 0$.

More generally:

- If $\beta x + \alpha$ is a factor of $P(x)$, then $P\left(-\frac{\alpha}{\beta}\right) = 0$.
- Conversely, if $P\left(-\frac{\alpha}{\beta}\right) = 0$, then $\beta x + \alpha$ is a factor of $P(x)$.

Example 11

Show that $x + 1$ is a factor of $x^3 - 4x^2 + x + 6$ and hence find the other linear factors.

Solution

Let $\quad P(x) = x^3 - 4x^2 + x + 6$

Then $\quad P(-1) = (-1)^3 - 4(-1)^2 + (-1) + 6$

$= 0$

Thus $x + 1$ is a factor (by the factor theorem).

Divide by $x + 1$ to find the other factor:

$$
\begin{array}{r|l}
 & x^2 - 5x + 6 \\
\hline
x + 1 & x^3 - 4x^2 + x + 6 \\
 & x^3 + x^2 \\
\hline
 & -5x^2 + x + 6 \\
 & -5x^2 - 5x \\
\hline
 & 6x + 6 \\
 & 6x + 6 \\
\hline
 & 0
\end{array}
$$

Explanation

We can use the factor theorem to find one factor, and then divide to find the other two linear factors.

Here is an alternative method:

Once we have found that $x + 1$ is a factor, we know that we can write

$$x^3 - 4x^2 + x + 6 = (x + 1)(x^2 + bx + c)$$

By equating constant terms, we have $6 = 1 \times c$. Hence $c = 6$.

By equating coefficients of x^2, we have $-4 = 1 + b$. Hence $b = -5$.

$$\therefore \quad x^3 - 4x^2 + x + 6 = (x + 1)(x^2 - 5x + 6)$$

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$$\therefore x^3 - 4x^2 + x + 6 = (x+1)(x^2 - 5x + 6)$$
$$= (x+1)(x-3)(x-2)$$

The linear factors of $x^3 - 4x^2 + x + 6$ are $(x+1)$, $(x-3)$ and $(x-2)$.

Thinking about the numbers involved in the process of factorisation gives us a way of searching for factors. For example, consider the polynomial $x^3 - 2x^2 - 5x + 6$. Assume this polynomial has a linear factor $x - \alpha$, where α is an integer.

Then we can write

$$x^3 - 2x^2 - 5x + 6 = (x - \alpha)(x^2 + bx + c)$$
$$= x^3 - (\alpha - b)x^2 - (\alpha b - c)x - \alpha c$$

By considering the constant term, it can be seen that $\alpha c = -6$. Therefore α divides 6. (Since α is an integer, it follows that b and c are too.)

Thus only the factors of 6 need be considered (i.e. $\pm 1, \pm 2, \pm 3, \pm 6$).

Try these in turn until a value for α makes $P(\alpha) = 0$. This process is completed in the following example.

Example 12

Factorise $x^3 - 2x^2 - 5x + 6$.

Solution

$P(1) = 1 - 2 - 5 + 6 = 0$

$\therefore x - 1$ is a factor.

Now divide to find the other factors:

$$\begin{array}{r|l} & x^2 - x - 6 \\ \hline x - 1 & x^3 - 2x^2 - 5x + 6 \\ & x^3 - x^2 \\ \hline & \quad -x^2 - 5x + 6 \\ & \quad -x^2 + x \\ \hline & \qquad -6x + 6 \\ & \qquad -6x + 6 \\ \hline & \qquad\qquad 0 \end{array}$$

$$\therefore x^3 - 2x^2 - 5x + 6 = (x-1)(x^2 - x - 6)$$
$$= (x-1)(x-3)(x+2)$$

Explanation

The factors of 6 are $\pm 1, \pm 2, \pm 3, \pm 6$.

We evaluate the first option, $P(1)$, which in fact equals 0. If $P(1)$ did not equal 0, we would try the other factors of 6 in turn until a zero result is found.

Note that, for some cubics, the quadratic factor may not be able to be factorised. For such cubics there will only be one linear factor. The implications of this will be discussed later in the chapter when considering the graphs of cubic functions.

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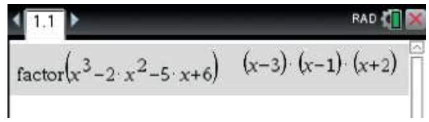
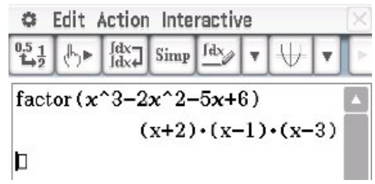
Using the TI-Nspire

Use **factor( )** from (menu) > **Algebra** > **Factor** to factorise the expression $x^3 - 2x^2 - 5x + 6$.



Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter the expression $x^3 - 2x^2 - 5x + 6$.
- Highlight the expression and go to **Interactive** > **Transformation** > **factor**.



Rational-root theorem

Consider the cubic polynomial

$$P(x) = 2x^3 - x^2 - x - 3$$

We can easily show that $P(1) \neq 0$, $P(-1) \neq 0$, $P(3) \neq 0$ and $P(-3) \neq 0$. Hence the equation $P(x) = 0$ has no solution that is an integer.

Does it have a rational solution, that is, a fraction for a solution?

The **rational-root theorem** helps us with this. It says that if α and β have highest common factor 1 (i.e. α and β are relatively prime) and $\beta x + \alpha$ is a factor of $2x^3 - x^2 - x - 3$, then β divides 2 and α divides −3.

Therefore, if $-\frac{\alpha}{\beta}$ is a solution of the equation $P(x) = 0$ (where α and β are relatively prime), then β must divide 2 and α must divide −3. So the only value of β that needs to be considered is 2, and $\alpha = \pm 3$ or $\alpha = \pm 1$.

We can test these through the factor theorem. That is, check $P\left(\pm\frac{1}{2}\right)$ and $P\left(\pm\frac{3}{2}\right)$. We find

$$\begin{aligned} P\left(\frac{3}{2}\right) &= 2\left(\frac{3}{2}\right)^3 - \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right) - 3 \\ &= 2 \times \frac{27}{8} - \frac{9}{4} - \frac{3}{2} - 3 \\ &= 0 \end{aligned}$$

We have found that $2x - 3$ is a factor.

Dividing through we find that

$$2x^3 - x^2 - x - 3 = (2x - 3)(x^2 + x + 1)$$

We can show that $x^2 + x + 1$ has no linear factors by showing that the discriminant of this quadratic is negative.

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Example 13

Factorise $3x^3 + 8x^2 + 2x - 5$.

Solution

Let $P(x) = 3x^3 + 8x^2 + 2x - 5$.

Then

$$P(1) = 8 \neq 0,$$
$$P(-1) = -2 \neq 0,$$
$$P(5) = 580 \neq 0,$$
$$P(-5) = -190 \neq 0,$$
$$P\left(-\frac{5}{3}\right) = 0$$

Therefore $3x + 5$ is a factor.

Dividing gives

$$3x^3 + 8x^2 + 2x - 5 = (3x + 5)(x^2 + x - 1)$$

We complete the square for $x^2 + x - 1$ to factorise:

$$\begin{aligned} x^2 + x - 1 &= x^2 + x + \frac{1}{4} - \frac{1}{4} - 1 \\ &= \left(x + \frac{1}{2}\right)^2 - \frac{5}{4} \\ &= \left(x + \frac{1}{2} + \frac{\sqrt{5}}{2}\right)\left(x + \frac{1}{2} - \frac{\sqrt{5}}{2}\right) \end{aligned}$$

Hence

$$P(x) = (3x + 5)\left(x + \frac{1}{2} + \frac{\sqrt{5}}{2}\right)\left(x + \frac{1}{2} - \frac{\sqrt{5}}{2}\right)$$

Explanation

The only possible integer solutions are ± 5 or ± 1. So there are no integer solutions. We now use the rational-root theorem.

If $-\frac{\alpha}{\beta}$ is a solution, the only value of β that needs to be considered is 3 and $\alpha = \pm 5$ or $\alpha = \pm 1$.

Here is the complete statement of the theorem:

Rational-root theorem

Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree n with all the coefficients a_i integers. Let α and β be integers such that the highest common factor of α and β is 1 (i.e. α and β are relatively prime).

If $\beta x + \alpha$ is a factor of $P(x)$, then β divides a_n and α divides a_0.

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Special cases: sums and differences of cubes

Example 14

Factorise $x^3 - 27$.

Solution

Let $P(x) = x^3 - 27$

Then $P(3) = 27 - 27 = 0$

$\therefore x - 3$ is a factor.

Divide to find the other factor:

$$\begin{array}{r} x^2 + 3x + 9 \\ x - 3 \,\overline{)\, x^3 + 0x^2 + 0x - 27} \\ \underline{x^3 - 3x^2 } \\ 3x^2 + 0x - 27 \\ \underline{3x^2 - 9x } \\ 9x - 27 \\ \underline{9x - 27} \\ 0 \end{array}$$

$\therefore x^3 - 27 = (x - 3)(x^2 + 3x + 9)$

Explanation

The division can also be performed using the method of equating coefficients:

Let $x^3 - 27 = (x - 3)(x^2 + bx + c)$.

Equating constant terms gives $c = 9$.

Equating coefficients of x^2 gives $-3 + b = 0$, and so $b = 3$.

In general, if $P(x) = x^3 - a^3$, then $x - a$ is a factor and so by division:

$$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$$

If a is replaced by $-a$, then

$$x^3 - (-a)^3 = (x - (-a))(x^2 + (-a)x + (-a)^2)$$

This gives:

$$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$$

Example 15

Factorise $8x^3 + 64$.

Solution

$$\begin{aligned} 8x^3 + 64 &= (2x)^3 + (4)^3 \\ &= (2x + 4)(4x^2 - 8x + 16) \end{aligned}$$

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Summary 7C

- **Remainder theorem** When $P(x)$ is divided by $\beta x + \alpha$, the remainder is $P\left(-\frac{\alpha}{\beta}\right)$.
- **Factor theorem**
  - If $\beta x + \alpha$ is a factor of $P(x)$, then $P\left(-\frac{\alpha}{\beta}\right) = 0$.
  - Conversely, if $P\left(-\frac{\alpha}{\beta}\right) = 0$, then $\beta x + \alpha$ is a factor of $P(x)$.
- A cubic function can be factorised using the factor theorem to find the first linear factor and then using polynomial division or the method of equating coefficients to complete the process.
- **Rational-root theorem** Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree n with all the coefficients a_i integers. Let α and β be integers such that the highest common factor of α and β is 1 (i.e. α and β are relatively prime). If $\beta x + \alpha$ is a factor of $P(x)$, then β divides a_n and α divides a_0.
- Difference of two cubes: $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$
- Sum of two cubes: $x^3 + a^3 = (x + a)(x^2 - ax + a^2)$

Skill-sheet

Exercise 7C

Example 9

1 Without dividing, find the remainder when the first polynomial is divided by the second:

a $x^3 - x^2 - 3x + 1,\ x - 1$
b $x^3 - 3x^2 + 4x - 1,\ x + 2$
c $2x^3 - 2x^2 + 3x + 1,\ x - 2$
d $x^3 - 2x + 3,\ x + 1$
e $x^3 + 2x - 5,\ x - 2$
f $2x^3 + 3x^2 + 3x - 2,\ x + 2$
g $6 - 5x + 9x^2 + 10x^3,\ 2x + 3$
h $10x^3 - 3x^2 + 4x - 1,\ 2x + 1$
i $108x^3 - 27x^2 - 1,\ 3x + 1$

Example 10

2 Find the value of a for each of the following:

a $x^3 + ax^2 + 3x - 5$ has remainder -3 when divided by $x - 2$
b $x^3 + x^2 - 2ax + a^2$ has remainder 8 when divided by $x - 2$
c $x^3 - 3x^2 + ax + 5$ has remainder 17 when divided by $x - 3$
d $x^3 + x^2 + ax + 8$ has remainder 0 when divided by $x - 1$

Example 11

3 Without dividing, show that the first polynomial is exactly divisible by the second polynomial:

a $x^3 - x^2 + x - 1,\ x - 1$
b $x^3 + 3x^2 - x - 3,\ x - 1$
c $2x^3 - 3x^2 - 11x + 6,\ x + 2$
d $2x^3 - 13x^2 + 27x - 18,\ 2x - 3$

4 Find the value of m if the first polynomial is exactly divisible by the second:

a $x^3 - 4x^2 + x + m,\ x - 3$
b $2x^3 - 3x^2 - (m + 1)x - 30,\ x - 5$
c $x^3 - (m + 1)x^2 - x + 30,\ x + 3$

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Example 12

5 Factorise each of the following:

a $2x^3 + x^2 - 2x - 1$
b $x^3 + 3x^2 + 3x + 1$
c $6x^3 - 13x^2 + 13x - 6$
d $x^3 - 21x + 20$
e $2x^3 + 3x^2 - 1$
f $x^3 - x^2 - x + 1$
g $4x^3 + 3x - 38$
h $4x^3 + 4x^2 - 11x - 6$

6 Find the remainder when $(1 + x)^4$ is divided by $x + 2$.

Example 13

7 Use the rational-root theorem to help factorise each of the following cubic polynomials:

a $2x^3 - 7x^2 + 16x - 15$
b $2x^3 - 3x^2 + 8x + 5$
c $2x^3 - 3x^2 - 12x - 5$
d $2x^3 - x^2 - 8x - 3$

Example 14
Example 15

8 Factorise each of the following:

a $x^3 - 1$
b $x^3 + 64$
c $27x^3 - 1$
d $64x^3 - 125$
e $1 - 125x^3$
f $8 + 27x^3$
g $64m^3 - 27n^3$
h $27b^3 + 8a^3$

9 Factorise each of the following:

a $x^3 + x^2 - x + 2$
b $3x^3 - 7x^2 + 4$
c $x^3 - 4x^2 + x + 6$
d $6x^3 + 17x^2 - 4x - 3$

10 Find the values of a and b and factorise the polynomial $P(x) = x^3 + ax^2 - x + b$, given that $P(x)$ is divisible by $x - 1$ and $x + 3$.

11 **a** Show that, for any constant a and any natural number n, $x - a$ is a factor of $x^n - a^n$.

b Find conditions (if any) on n that are required in order that:

i $x + a$ is a factor of $x^n + a^n$

ii $x + a$ is a factor of $x^n - a^n$.

12 The polynomial $P(x)$ has a remainder of 2 when divided by $x - 1$ and a remainder of 3 when divided by $x - 2$. The remainder when $P(x)$ is divided by $(x - 1)(x - 2)$ is $ax + b$, i.e. $P(x)$ can be written as $P(x) = (x - 1)(x - 2)Q(x) + ax + b$.

a Find the values of a and b.

b **i** Given that $P(x)$ is a cubic polynomial with coefficient of x^3 being 1, and -1 is a solution of the equation $P(x) = 0$, find $P(x)$.

ii Show that the equation $P(x) = 0$ has no other real solutions.

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7D Solving cubic equations

In order to solve a cubic equation, the first step is often to factorise. We start with an example of a cubic already written in factorised form.

Example 16

Solve $(x - 2)(x + 1)(x + 3) = 0$.

Solution

Using the null factor theorem,
$(x - 2)(x + 1)(x + 3) = 0$ implies

$$x - 2 = 0 \quad \text{or} \quad x + 1 = 0 \quad \text{or} \quad x + 3 = 0$$

Thus the solutions are $x = 2, -1$ and -3.

Explanation

In this example, the cubic has already been factorised.

In the following example, a common factor of x is first taken out.

Example 17

Solve each of the following equations for x:

a $2x^3 - x^2 - x = 0$

b $x^3 + 2x^2 - 10x = 0$

Solution

a
$$2x^3 - x^2 - x = 0$$
$$x(2x^2 - x - 1) = 0$$
$$x(2x + 1)(x - 1) = 0$$
$$\therefore \ x = 0 \text{ or } x = -\frac{1}{2} \text{ or } x = 1$$

b
$$x^3 + 2x^2 - 10x = 0$$
$$x(x^2 + 2x - 10) = 0$$
$$x(x^2 + 2x + 1 - 11) = 0$$
$$x(x + 1 - \sqrt{11})(x + 1 + \sqrt{11}) = 0$$
$$\therefore \ x = 0 \text{ or } x = -1 + \sqrt{11} \text{ or } x = -1 - \sqrt{11}$$

In the following example, grouping is used to factorise.

Example 18

Solve each of the following equations for x:

a $x^3 - 4x^2 - 11x + 44 = 0$

b $x^3 - ax^2 - 11x + 11a = 0$

Solution

a $x^3 - 4x^2 - 11x + 44 = 0$

$x^2(x - 4) - 11(x - 4) = 0$

Therefore $(x - 4)(x^2 - 11) = 0$

Hence $x = 4$ or $x = \pm\sqrt{11}$

b $x^3 - ax^2 - 11x + 11a = 0$

$x^2(x - a) - 11(x - a) = 0$

Therefore $(x - a)(x^2 - 11) = 0$

Hence $x = a$ or $x = \pm\sqrt{11}$

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In the following two examples, the factor theorem is used to find a linear factor.

Example 19

Solve $x^3 - 4x^2 - 11x + 30 = 0$.

Solution

Let $P(x) = x^3 - 4x^2 - 11x + 30$

Then $P(1) = 1 - 4 - 11 + 30 \neq 0$

$P(-1) = -1 - 4 + 11 + 30 \neq 0$

$P(2) = 8 - 16 - 22 + 30 = 0$

$\therefore x - 2$ is a factor.

By division or inspection,

$$x^3 - 4x^2 - 11x + 30 = (x-2)(x^2 - 2x - 15)$$
$$= (x-2)(x-5)(x+3)$$

$\therefore (x-2)(x-5)(x+3) = 0$

$\therefore x - 2 = 0$ or $x - 5 = 0$ or $x + 3 = 0$

$\therefore x = 2, 5$ or -3

Explanation

In this example we first identify a linear factor using the factor theorem.

The factorisation is completed using one of the methods given in the previous section.

Example 20

Solve $2x^3 - 5x^2 + 5x - 2 = 0$.

Solution

Let $P(x) = 2x^3 - 5x^2 + 5x - 2$

Then $P(1) = 2 - 5 + 5 - 2 = 0$

$\therefore x - 1$ is a factor.

By division or inspection,

$$2x^3 - 5x^2 + 5x - 2 = (x-1)(2x^2 - 3x + 2)$$

$\therefore (x-1)(2x^2 - 3x + 2) = 0$

$\therefore x = 1$

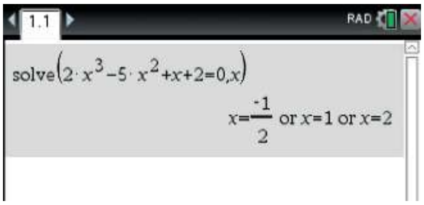
Explanation

First find a linear factor using the factor theorem. Then find the quadratic factor by division.

The discriminant of this quadratic is a negative number, so this quadratic cannot be factorised further. Hence there is only one linear factor and therefore only one solution.

Using the TI-Nspire

Use **solve( )** from (menu) > **Algebra** > **Solve** to solve the equation $2x^3 - 5x^2 + x + 2 = 0$.



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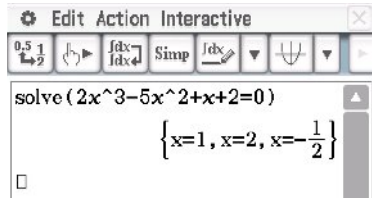
Using the Casio ClassPad

- Go to the main screen Main $\sqrt{\alpha}$.
- Select **solve(** from the Math1 keyboard.
- Enter the equation $2x^3 - 5x^2 + x + 2 = 0$.
- Close the bracket and tap EXE.



Summary 7D

Cubic polynomial equations can be solved by first using an appropriate factorisation technique. Factorisation may involve:

- extracting a simple common factor
- using the factor theorem
- polynomial division or equating coefficients
- sum or difference of two cubes
- using the quadratic formula to complete the factorisation.

Exercise 7D

Example 16

1 Solve each of the following:

a $(x - 1)(x + 2)(x - 4) = 0$ **b** $(x - 4)^2(x - 6) = 0$

c $(2x - 1)(x - 3)(3x + 2) = 0$ **d** $x(x + 3)(2x - 5) = 0$

Example 17

2 Solve each of the following:

a $x^3 - 2x^2 - 8x = 0$ **b** $x^3 + 2x^2 - 11x = 0$

c $x^3 - 3x^2 - 40x = 0$ **d** $x^3 + 2x^2 - 16x = 0$

Example 18

3 Use grouping to solve each of the following:

a $x^3 - x^2 + x - 1 = 0$ **b** $x^3 + x^2 + x + 1 = 0$

c $x^3 - 5x^2 - 10x + 50 = 0$ **d** $x^3 - ax^2 - 16x + 16a = 0$

Example 19

4 Solve each of the following:

a $x^3 - 19x + 30 = 0$ **b** $3x^3 - 4x^2 - 13x - 6 = 0$

c $x^3 - x^2 - 2x + 2 = 0$ **d** $5x^3 + 12x^2 - 36x - 16 = 0$

e $6x^3 - 5x^2 - 2x + 1 = 0$ **f** $2x^3 - 3x^2 - 29x - 30 = 0$

Example 20

5 Solve each of the following for x:

a $x^3 + x^2 - 24x + 36 = 0$ **b** $6x^3 + 13x^2 - 4 = 0$

c $x^3 - x^2 - 2x - 12 = 0$ **d** $2x^3 + 3x^2 + 7x + 6 = 0$

e $x^3 - x^2 - 5x - 3 = 0$ **f** $x^3 + x^2 - 11x - 3 = 0$

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6 Solve each of the following equations for x:

a $2x^3 = 16x$ **b** $2(x-1)^3 = 32$ **c** $x^3 + 8 = 0$

d $2x^3 + 250 = 0$ **e** $1000 = \frac{1}{x^3}$

7 Factorise each of the following cubic expressions, using a calculator to help find at least one linear factor:

a $2x^3 - 22x^2 - 250x + 2574$ **b** $2x^3 + 27x^2 + 52x - 33$

c $2x^3 - 9x^2 - 242x + 1089$ **d** $2x^3 + 51x^2 + 304x - 165$

7E Cubic functions of the form $f(x) = a(x-h)^3 + k$

In Chapter 3 we saw that all quadratic functions can be written in 'turning point form' and that the graphs of all quadratics have one basic form, the parabola. This is not true of cubic functions. Let us first consider those cubics that are of the form

$$f(x) = a(x-h)^3 + k$$

The graphs of these functions can be formed by simple transformations of the graph of $f(x) = x^3$.

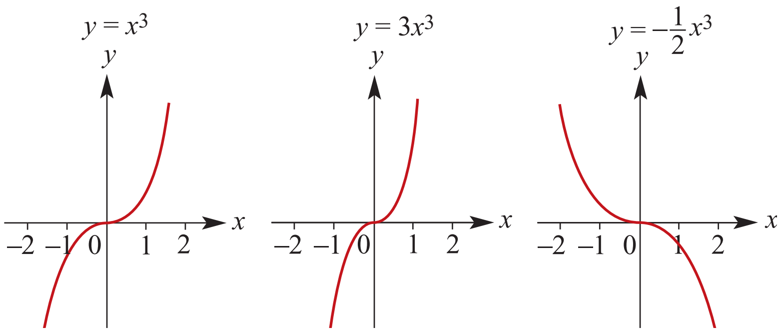
For example, the graph of $f(x) = (x-1)^3 + 3$ is obtained from the graph of $f(x) = x^3$ by a translation of 1 unit in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

Transformations of the graph of $f(x) = x^3$

Dilations parallel to an axis and reflections in an axis

As with other graphs it has been seen that changing the value of a simply narrows or broadens the graph without changing its fundamental shape. Again, if $a < 0$, the graph is reflected in an axis. Note that reflecting in the x-axis and reflecting in the y-axis result in the same graph. This is because $(-x)^3 = -x^3$.

For example:



It should be noted that the natural **domain** of all cubics is $\mathbb{R}$ and the **range** is also $\mathbb{R}$.

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Point of inflection

The significant feature of the graph of a cubic of this form is the **point of inflection** (a point of zero gradient). This will be discussed fully in Chapter 18, but for the moment we note that it is the 'flat point' of the graph.

The point of inflection of $y = x^3$ is at the origin $(0, 0)$.

Vertical translations

By adding or subtracting a constant term to $y = x^3$, the graph moves either 'up' or 'down'.

The graph of $y = x^3 + k$ is the basic graph moved k units up (for $k > 0$). The point of inflection becomes $(0, k)$. In this case, the graph of $y = x^3$ is translated k units in the positive direction of the y-axis.

Horizontal translations

The graph of $y = (x - h)^3$ is simply the basic graph moved h units to the 'right' (for $h > 0$). The point of inflection is at $(h, 0)$. In this case, the graph of $y = x^3$ is translated h units in the positive direction of the x-axis.

General form

For the graph of a cubic function of the form

$$y = a(x - h)^3 + k$$

the point of inflection is at (h, k).

When sketching cubic graphs of the form $y = a(x - h)^3 + k$, first identify the point of inflection. To add further detail to the graph, find the x-axis and y-axis intercepts.

Example 21

Sketch the graph of the function $y = (x - 2)^3 + 4$.

Solution

The graph of $y = x^3$ is translated 2 units to the right and 4 units up.

Point of inflection is $(2, 4)$.

x-axis intercept:

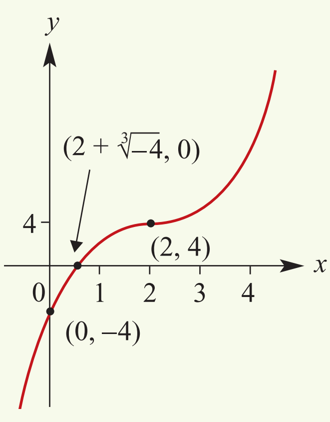
Let $y = 0$

$$0 = (x - 2)^3 + 4$$
$$-4 = (x - 2)^3$$
$$\sqrt[3]{-4} = x - 2$$
$$x = 2 + \sqrt[3]{-4}$$
$$\approx 0.413$$

y-axis intercept:

Let $x = 0$

$$y = (0 - 2)^3 + 4$$
$$y = -8 + 4$$
$$y = -4$$



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Summary 7E

- The graph of $y = a(x - h)^3 + k$ has the same shape as $y = ax^3$ but is translated h units in the positive x-axis direction and k units in the positive y-axis direction (where h and k are positive constants).
- The natural domain of all cubic functions is $\mathbb{R}$.

Exercise 7E

Example 21

1 Using the method of horizontal and vertical translations, sketch the graph of each of the following:

a $y = (x + 2)^3 - 1$
b $y = (x - 1)^3 - 1$
c $y = (x + 3)^3 + 2$
d $y = (x - 2)^3 + 5$
e $y = (x + 2)^3 - 5$

2 Sketch the graphs of the following functions:

a $y = 2x^3 + 3$
b $y = 2(x - 3)^3 + 2$
c $3y = x^3 - 5$
d $y = 3 - x^3$
e $y = (3 - x)^3$
f $y = -2(x + 1)^3 + 1$
g $y = \frac{1}{2}(x - 3)^3 + 2$

7F Graphs of factorised cubic functions

The general cubic function written in **polynomial form** is

$$y = ax^3 + bx^2 + cx + d$$

There is a variety of graph shapes for cubic functions, depending on the values of the coefficients. The graph of a cubic function is not necessarily a simple transformation (dilations, translations, reflections) of the graph of $y = x^3$.

All cubics have at least one x-axis intercept. We have seen that cubic functions of the form $f(x) = a(x - h)^3 + k$ have only one x-axis intercept, but these are not the only cubic functions with one x-axis intercept. Some cubic functions have two and others have three.

The y-axis intercept is easily found by letting $x = 0$, and it is the point $(0, d)$.

When sketching the graphs of cubics which are not of the form $f(x) = a(x - h)^3 + k$, begin by finding the x-axis intercepts.

In the following example, the cubic is in factored form already.

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Example 22

Sketch the graph of $y = (x-1)(x+2)(x+1)$. Do not give coordinates of turning points.

Solution

To find the x-axis intercepts, let $y = 0$.

Then $0 = (x-1)(x+2)(x+1)$

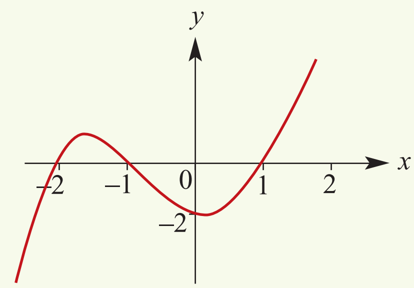
$\therefore x - 1 = 0$ or $x + 2 = 0$ or $x + 1 = 0$

$\therefore$ x-axis intercepts are 1, −1 and −2.

To find the y-axis intercept, let $x = 0$.

Then $y = (0-1)(0+2)(0+1) = -2$.

$\therefore$ y-axis intercept is −2.



Check the following by substituting values:

- When $x > 1$, $y > 0$.
- When $-1 < x < 1$, $y < 0$.
- When $-2 < x < -1$, $y > 0$.
- When $x < -2$, $y < 0$.

(Notice how the sign of the y-value changes from one side of an x-axis intercept to the other.) Finally, consider what happens to the graph 'beyond' the x-axis intercepts:

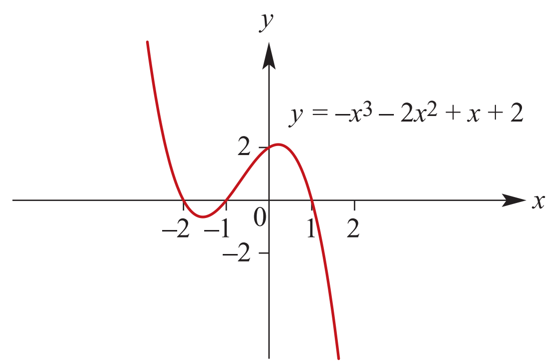
- For $x > 1$, $y > 0$ and as x increases y increases.
- For $x < -2$, $y < 0$ and as x decreases y decreases.

The polynomial form of the cubic in this example is $y = x^3 + 2x^2 - x - 2$. The coefficient of x^3 is positive. We now see what happens when the coefficient of x^3 is negative.

The graph of the cubic function

$$y = -x^3 - 2x^2 + x + 2$$

is the reflection in the x-axis of the graph of the cubic function considered in Example 22.



- When $x > 1$, $y < 0$.
- When $-1 < x < 1$, $y > 0$.
- When $-2 < x < -1$, $y < 0$.
- When $x < -2$, $y > 0$.

- For $x > 1$, $y < 0$ and as x increases y decreases.
- For $x < -2$, $y > 0$ and as x decreases y increases.

At this stage the location of the turning points is unspecified. However, it is important to note that, unlike quadratic graphs, the turning points are not symmetrically located between x-axis intercepts. How to determine the exact values of the coordinates of the turning points will be shown later in this book.

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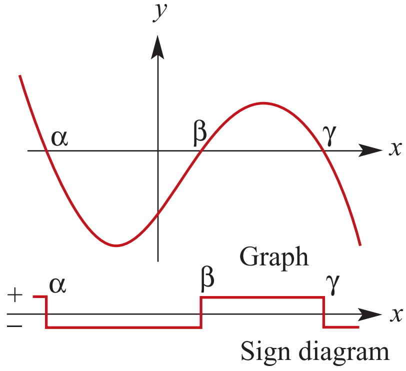
Sign diagrams

A sign diagram is a number-line diagram which shows when an expression is positive or negative.

The following is a sign diagram for a cubic function, the graph of which is also shown.



Using a sign diagram requires that the factors, and the x-axis intercepts, be found. The y-axis intercept and sign diagram can then be used to complete the graph.

This procedure is shown in Example 23.

Example 23

Sketch the graph of $y = x^3 + 2x^2 - 5x - 6$.

Solution

Let $\quad P(x) = x^3 + 2x^2 - 5x - 6$

Then $\quad P(1) = 1 + 2 - 5 - 6 \neq 0$

$\quad P(-1) = -1 + 2 + 5 - 6 = 0$

$\therefore$ $x + 1$ is a factor.

By division, $y = (x + 1)(x - 2)(x + 3)$.

$\therefore$ x-axis intercepts are -1, 2 and -3.

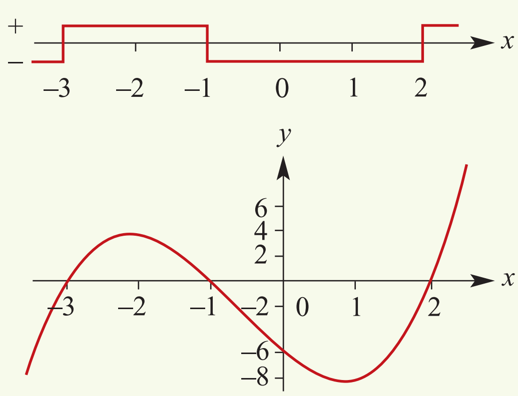
When $x < -3$, y is negative.

When $-3 < x < -1$, y is positive.

When $-1 < x < 2$, y is negative.

When $x > 2$, y is positive.

This gives the sign diagram.



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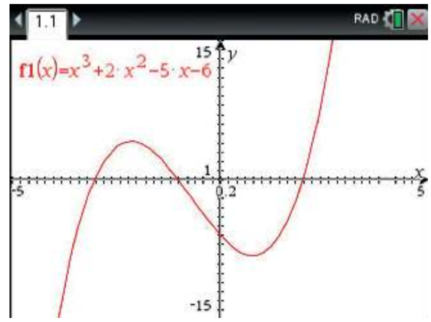
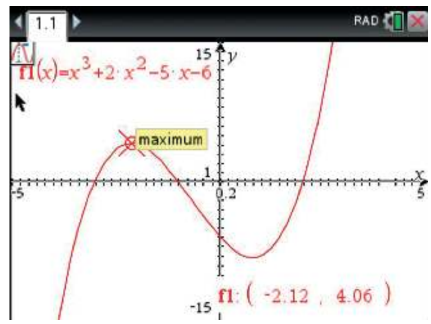
Using the TI-Nspire

In order to provide more detail, the coordinates of the turning points can be found with a CAS calculator.



- Enter $f_1(x) = x^3 + 2x^2 - 5x - 6$ in a **Graphs** application.
- Choose a suitable window (menu > **Window/Zoom** > **Window Settings**).

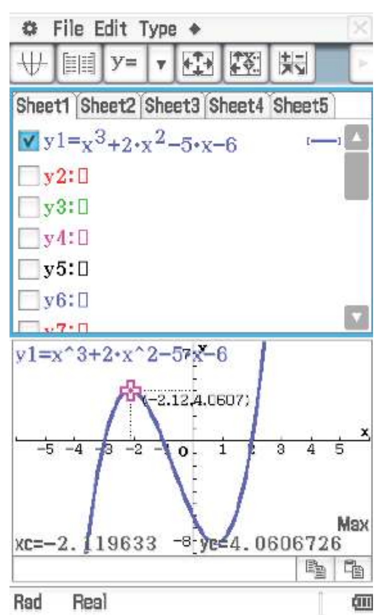


- Use menu > **Analyze Graph** > **Maximum**.
- Move the cursor to the left of point (lower bound), click, move to the right of point (upper bound) and click to display the coordinates.
- Repeat for other points of interest.

Note: Alternatively, use menu > **Trace** > **Graph Trace** to find the coordinates of the two turning points. A label will appear near a turning point to indicate that the calculator has found a local maximum or a local minimum.

Using the Casio ClassPad

In order to provide more detail, the coordinates of the turning points can be found with a CAS calculator.



- Go to the menu, select **Graph & Table** and tap the cursor next to $y1$.
- Enter $x^3 + 2x^2 - 5x - 6$.
- Tick the box and select to produce the graph.
- Choose a suitable window using or a combination of **Zoom Out** and **Zoom Box**.
- Tap in the graph screen to select it, then use **Analysis** > **G-Solve** > **Max** to find the local maximum and **Min** to find the local minimum.

Note: The maximum and minimum points must be visible on the screen before carrying out the analysis step.

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Repeated factors

The polynomial function $f(x) = (x-1)^2(x+3)$ has a **repeated factor**. In this case $(x-1)$ is repeated. Since the repeated factor is squared, it is easy to see that the sign of the y-value is the same 'close in' on either side of the corresponding x-axis intercept.

If the factorised cubic has a repeated factor and another linear factor, there are only two x-axis intercepts and the repeated factor corresponds to one of the turning points.

Example 24

Sketch the graph of $y = x^2(x-1)$.

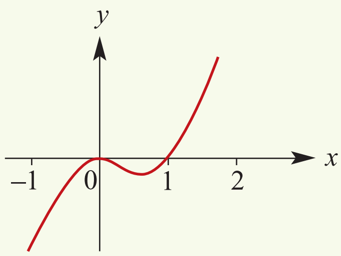
Solution

To find the x-axis intercepts, let $y = 0$.

Then $x^2(x-1) = 0$.

Thus the x-axis intercepts are at $x = 0$ and $x = 1$.
Because the repeated factor is x^2, there is also a turning point at $x = 0$.

The y-axis intercept (letting $x = 0$) is at $y = 0$.



Cubics with one x-axis intercept

Cubics of the form $y = (x-a)^3$ have only one x-axis intercept. Some other cubics also have only one x-axis intercept because, when they are factorised, they are found to have only one linear factor, with the remaining quadratic factor unable to be factorised further.

Example 25

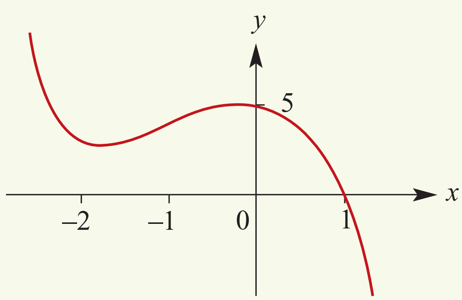
Sketch the graph of $y = -(x-1)(x^2+4x+5)$.

Solution

To find the x-axis intercept, let $y = 0$.

First, we note that the factor $x^2 + 4x + 5$ cannot be factorised further:

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= 4^2 - 4(1)(5) \\ &= -4\end{aligned}$$



$\therefore$ there are no further linear factors.

Hence, when solving the equation $-(x-1)(x^2+4x+5) = 0$, there is only one solution.

$\therefore$ x-axis intercept is $x = 1$.

To find the y-axis intercept, let $x = 0$. Then $y = -(0-1)(0^2+4(0)+5) = 5$.

On a CAS calculator it is found that the turning points are at $(0, 5)$ and $(-1.82, 2.91)$, where the values for the coordinates of the second point are given to two decimal places.

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Summary 7F

- The graph of a cubic function can have one, two or three x-axis intercepts.
- If a cubic can be written as the product of three linear factors, $y = a(x - \alpha)(x - \beta)(x - \gamma)$, then its graph can be sketched by following these steps:
  - Find the y-axis intercept.
  - Find the x-axis intercepts.
  - Prepare a sign diagram.
  - Consider the y-values as x increases to the right of all x-axis intercepts.
  - Consider the y-values as x decreases to the left of all x-axis intercepts.
- If there is a repeated factor to the power 2, the y-values have the same sign immediately to the left and right of the corresponding x-axis intercept.

Exercise 7F

Example 22

1 Sketch the graph for each of the following and draw a sign diagram. Label your sketch graph showing the axis intercepts. (Do not determine coordinates of turning points.)

a $y = x(x - 1)(x - 3)$ **b** $y = (x - 1)(x + 1)(x + 2)$
c $y = (2x - 1)(x - 2)(x + 3)$ **d** $y = (x - 1)(x - 2)(x - 3)$

Example 23

2 Sketch the graph for each of the following and draw a sign diagram. Label your sketch graph showing the axis intercepts. (Do not determine coordinates of turning points.)

a $y = x^3 - 9x$ **b** $y = x^3 - 4x^2 - 3x + 18$ **c** $y = -x^3 + x^2 + 3x - 3$
d $y = 3x^3 - 4x^2 - 13x - 6$ **e** $y = 6x^3 - 5x^2 - 2x + 1$ **f** $y = 2x^3 - 9x^2 + 7x + 6$

Example 24

3 Sketch the graph for each of the following and draw a sign diagram. Label your sketch graph showing the axis intercepts. (Do not determine coordinates of turning points.)

a $y = (x - 1)(x - 2)^2$ **b** $y = x^2(x - 4)$ **c** $y = 2(x + 1)^2(x - 3)$
d $y = x^3 + x^2$ **e** $y = 4x^3 - 8x^2 + 5x - 1$ **f** $y = x^3 - 5x^2 + 7x - 3$

Example 25

4 Sketch the graph for each of the following and draw a sign diagram. Label your sketch graph showing the axis intercepts. (Do not determine coordinates of turning points.) Use your calculator to help sketch each of them.

a $y = (x - 1)(x^2 + 1)$ (Note: There is no turning point or ‘flat point’ of this cubic.)
b $y = (x^2 + 2)(x - 4)$ (Note: There are two turning points.)

5 Sketch the graph for each of the following, using a CAS calculator to find the coordinates of axis intercepts and local maximum and local minimum values:

a $y = -4x^3 - 12x^2 + 37x - 15$ **b** $y = -4x^3 + 19x - 15$
c $y = -4x^3 + 0.8x^2 + 19.8x - 18$ **d** $y = 2x^3 + 11x^2 + 15x$
e $y = 2x^3 + 6x^2$ **f** $y = 2x^3 + 6x^2 + 6$

6 Show that the graph of f, where $f(x) = x^3 - x^2 - 5x - 3$, cuts the x-axis at one point and touches it at another. Find the values of x at these points.

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7G Families of cubic polynomial functions

In Chapter 2 we considered the information that is necessary to determine the equation of a straight line. In Chapter 3 this was considered for quadratic functions, and in Chapter 4 for rectangular hyperbolas, circles and other types of functions.

Here are some examples of families of cubic polynomial functions:

$y = ax^3,\ a > 0$	The cubic graphs that are dilations from the x-axis of $y = x^3$.
$y = a(x-h)^3 + k,\ a \neq 0$	The cubic graphs that are translations of $y = ax^3$.
$y = a(x-2)(x+5)(x-4),\ a \neq 0$	The cubic graphs with x-axis intercepts 2, −5 and 4.
$y = ax^3 + bx^2 + cx,\ a \neq 0$	The cubic graphs that pass through the origin.

Recall that in this context we call a, b, c, h and k parameters.

Finding rules for cubic polynomial functions

The method used for finding the equation from the graph of a cubic will depend on what information is given in the graph.

If the cubic function has rule of the form $f(x) = a(x-h)^3 + k$ and the point of inflection (h, k) is given, then only one other point needs to be known in order to find the value of a.

For those that are not of this form, the information given may be some or all of the x-axis intercepts as well as the coordinates of other points including possibly the y-axis intercept.

Example 26

a A cubic function has rule of the form $y = a(x-2)^3 + 2$. The point $(3, 10)$ is on the graph of the function. Find the value of a.

b A cubic function has rule of the form $y = a(x-1)(x+2)(x-4)$. The point $(5, 16)$ is on the graph of the function. Find the value of a.

Solution

a $y = a(x-2)^3 + 2$

When $x = 3$, $y = 10$. Solve for a:

$$10 = a(3-2)^3 + 2$$
$$8 = a \times 1^3$$
$$a = 8$$

b $y = a(x-1)(x+2)(x-4)$

When $x = 5$, $y = 16$.

$$16 = a(5-1)(5+2)(5-4)$$
$$16 = 28a$$
$$a = \frac{4}{7}$$

Explanation

In each of these problems we substitute in the given values to find the unknown.

The coordinates of the point of inflection of a graph which is a translation of $y = ax^3$ are known and the coordinates of one further point are known.

Three x-axis intercepts are known and the coordinates of a fourth point are known.

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Example 27

A cubic function has rule of the form $f(x) = ax^3 + bx$. The points $(1, 16)$ and $(2, 30)$ are on the graph of the function. Find the values of a and b.

Solution

Since $f(1) = 16$ and $f(2) = 30$, we obtain the simultaneous equations

$$16 = a + b \qquad (1)$$
$$30 = a(2)^3 + 2b \qquad (2)$$

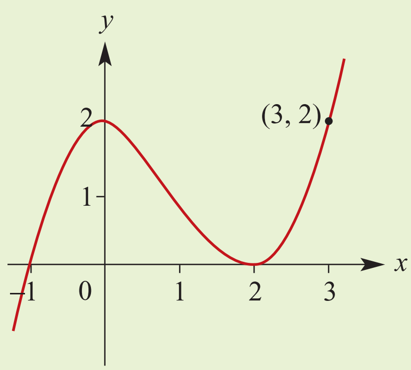
Multiply (1) by 2 and subtract from (2). This gives $-2 = 6a$ and hence $a = -\frac{1}{3}$.

Substitute in (1) to find $b = \frac{49}{3}$.

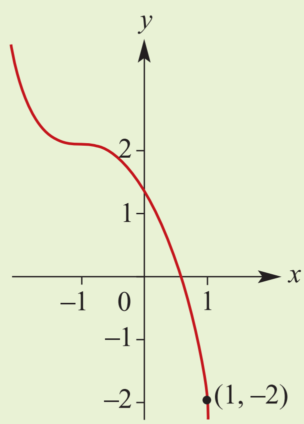
Example 28

Determine the rule for the cubic function shown in each of the following graphs:

a



b



Solution

a $y = a(x + 1)(x - 2)^2$

Put $(3, 2)$ in the equation:

$$2 = a(4)(1)$$
$$\frac{1}{2} = a$$

The rule is $y = \frac{1}{2}(x + 1)(x - 2)^2$.

b $y - 2 = a(x + 1)^3$

To determine a, put the known point $(1, -2)$ into the equation:

$$-2 - 2 = a(2)^3$$
$$-4 = 8a$$
$$-\frac{1}{2} = a$$

The rule is $y - 2 = -\frac{1}{2}(x + 1)^3$.

Explanation

a The x-axis intercepts are -1 and 2, and the graph touches the x-axis at 2. So the cubic has a repeated factor $(x - 2)$.

Therefore the form of the rule appears to be $y = a(x + 1)(x - 2)^2$.

b This graph appears to be of the form $y = a(x - h)^3 + k$. As can be seen from the graph, $k = 2$ and $h = -1$.

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Example 29

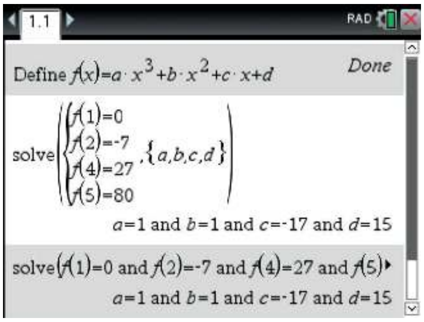
A cubic function f has rule $f(x) = ax^3 + bx^2 + cx + d$. If

$$f(1) = 0, \quad f(2) = -7, \quad f(4) = 27, \quad f(5) = 80$$

find the values of a, b, c and d.

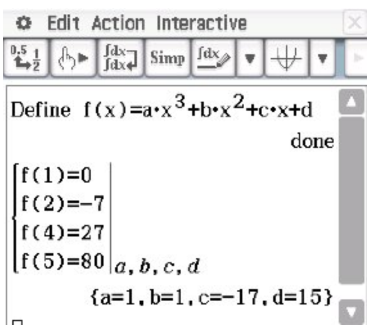
Using the TI-Nspire

- Define the function $f(x) = ax^3 + bx^2 + cx + d$.
- Use menu > **Algebra** > **Solve System of Equations** > **Solve System of Equations**. Complete the pop-up screen and enter the equations as shown to give the solution to the simultaneous equations.
- An alternative method is also shown.



Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter $f(x) = ax^3 + bx^2 + cx + d$. (Remember to use Var when entering the variables a, b, c, d and abc when entering the function name f.)
- Highlight the expression and go to **Interactive** > **Define**. Use the default function name f and variable x.
- In Math1, select the simultaneous equations icon. Tap it twice more to expand for four simultaneous equations.
- Enter the known values $f(1) = 0$, $f(2) = -7$, $f(4) = 27$ and $f(5) = 80$ into the four lines and enter the variables a, b, c, d in the bottom right separated by commas.



Summary 7G

The rule of a cubic function can be determined if:

- the coordinates of four points on the graph are known
- the form of the function is known to be $f(x) = a(x - \alpha)^2(x - \beta)$, and α and β and the coordinates of one other point are known
- the form of the function is known to be $f(x) = a(x - h)^3 + k$, and the coordinates of the inflection point (h, k) and one other point are known.

There are other sets of information which can be used to determine the rule of a cubic function and more of these will be given in Chapter 18.

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Skill-sheet

Exercise 7G

Example 26

1 **a** A cubic function has rule of the form $y = a(x - 3)^3 + 1$. The point $(4, 12)$ is on the graph of the function. Find the value of a.

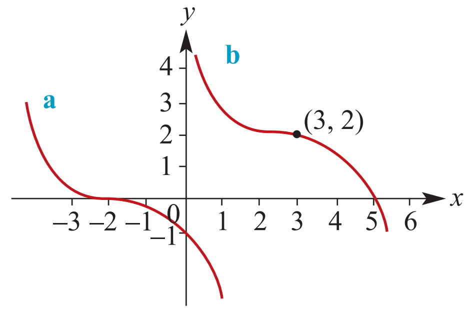
b A cubic function has rule of the form $y = a(x - 2)(x + 3)(x - 1)$. The point $(3, 24)$ is on the graph of the function. Find the value of a.

Example 27

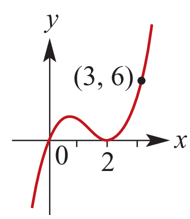
c A cubic function has rule of the form $y = ax^3 + bx$. The points $(1, 16)$ and $(2, 40)$ are on the graph of the function. Find the values of a and b.

Example 28

2 The graphs shown are similar to the basic curve $y = -x^3$. Find possible cubic functions which define each of the curves.



3 Find the equation of the cubic function for which the graph is shown.



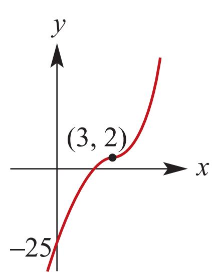
4 Find a cubic function whose graph touches the x-axis at $x = -4$, cuts it at the origin, and has a value 6 when $x = -3$.

5 The graph of a cubic function has x-axis intercepts 1, 3 and -1 and y-axis intercept -6. Find the rule for this cubic function.

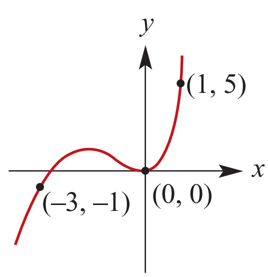
6 A cubic function f has rule $f(x) = (x - 3)(x^2 + a)$ and $f(6) = 216$. Find the value of a.

7 The graphs below have equations of the form shown. In each case, determine the equation.

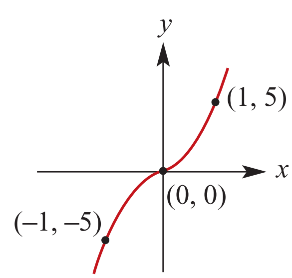
a $y = a(x - h)^3 + k$



b $y = ax^3 + bx^2$



c $y = ax^3$



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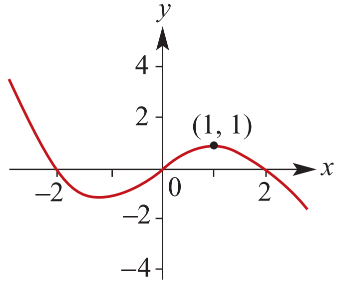
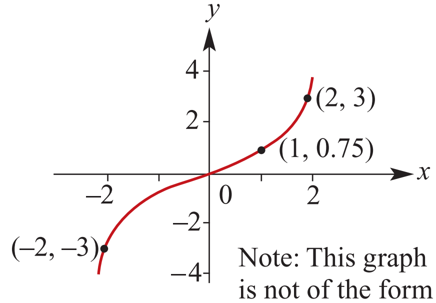
8 Find the expressions which define the following cubic curves:

a



b



Note: This graph is not of the form $y = a(x - h)^3 + k$.

Example 29

9 For each of the following, use a CAS calculator to find the values of a, b, c, d in the cubic equation $y = ax^3 + bx^2 + cx + d$, given that the following points lie on its graph:

a $(0, 270), (1, 312), (2, 230), (3, 0)$

b $(-2, -406), (0, 26), (1, 50), (2, -22)$

c $(-2, -32), (2, 8), (3, 23), (8, 428)$

d $(1, -1), (2, 10), (3, 45), (4, 116)$

e $(-3, -74), (-2, -23), (-1, -2), (1, -2)$

f $(-3, -47), (-2, -15), (1, -3), (2, -7)$

g $(-4, 25), (-3, 7), (-2, 1), (1, -5)$

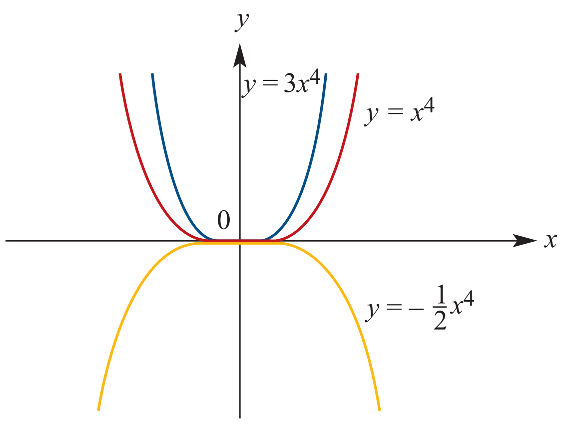
7H Quartic and other polynomial functions

In this section we look at polynomial functions of degree 4 and greater.

Quartic functions of the form $f(x) = a(x - h)^4 + k$

The graph of $f(x) = (x - 1)^4 + 3$ is obtained from the graph of $y = x^4$ by a translation of 1 unit in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

As with other graphs it has been seen that changing the value of a simply narrows or broadens the graph without changing its fundamental shape. Again, if $a < 0$, the graph is inverted.



The significant feature of the graph of a quartic of this form is the **turning point** (a point of zero gradient). The turning point of $y = x^4$ is at the origin $(0, 0)$.

For the graph of a quartic function of the form

$$y = a(x - h)^4 + k$$

the turning point is at (h, k).

When sketching quartic graphs of the form $y = a(x - h)^4 + k$, first identify the turning point. To add further detail to the graph, find the x-axis and y-axis intercepts.

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Example 30

Sketch the graph of the function $y = (x - 2)^4 - 1$.

Solution

Turning point is $(2, -1)$.

x-axis intercepts:

Let $y = 0$

$0 = (x - 2)^4 - 1$

$1 = (x - 2)^4$

$\pm\sqrt[4]{1} = x - 2$

$x = 2 + 1$ or $x = 2 - 1$

$x = 3$ or $x = 1$

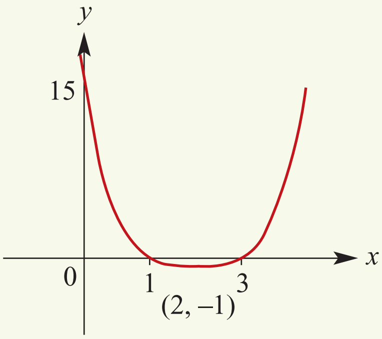
y-axis intercept:

Let $x = 0$

$y = (0 - 2)^4 - 1$

$= 16 - 1$

$= 15$

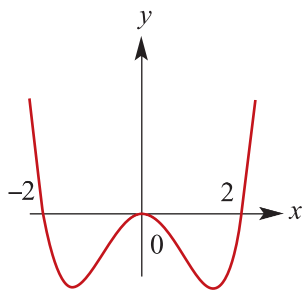


The natural **domain** of all quartics is $\mathbb{R}$, but unlike cubics the range is not $\mathbb{R}$.

Other quartic functions

The techniques for graphing quartic functions in general are very similar to those employed for cubic functions. A CAS calculator is to be used in the graphing of these functions. Great care needs to be taken in this process as it is easy to miss key points on the graph using these techniques.

The graph of $y = 2x^4 - 8x^2$ is shown.



Example 31

Solve each of the following equations for x:

a $x^4 - 8x = 0$ **b** $2x^4 - 8x^2 = 0$ **c** $x^4 - 2x^3 - 24x^2 = 0$

Solution

a $x^4 - 8x = 0$

Factorise to obtain

$x(x^3 - 8) = 0$

$\therefore x = 0$ or $x^3 - 8 = 0$

Thus $x = 0$ or $x = 2$.

b $2x^4 - 8x^2 = 0$

Factorise to obtain

$2x^2(x^2 - 4) = 0$

$\therefore 2x^2 = 0$ or $x^2 - 4 = 0$

Thus $x = 0$ or $x = 2$ or $x = -2$.

c $x^4 - 2x^3 - 24x^2 = 0$

Factorise to obtain $x^2(x^2 - 2x - 24) = 0$

$\therefore x^2 = 0$ or $x^2 - 2x - 24 = 0$

i.e. $x = 0$ or $(x - 6)(x + 4) = 0$

Thus $x = 0$ or $x = 6$ or $x = -4$.

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Odd and even polynomials

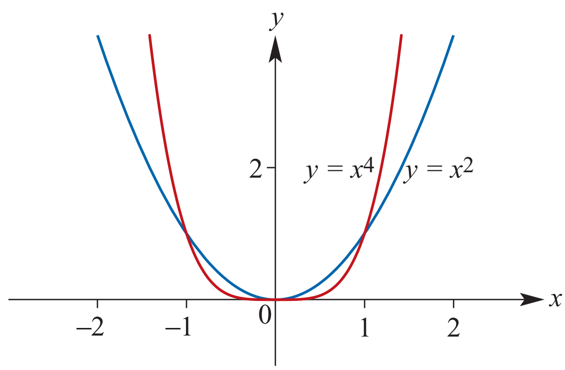
In this subsection we look briefly at odd and even polynomial functions. Knowing that a function is even or that it is odd is very helpful when sketching its graph.

- A function f is **even** if $f(-x) = f(x)$. This means that the graph is symmetric about the y-axis. That is, the graph appears the same after reflection in the y-axis.
- A function f is **odd** if $f(-x) = -f(x)$. The graph of an odd function has rotational symmetry with respect to the origin: the graph remains unchanged after rotation of 180° about the origin.

A **power function** is a function f with rule $f(x) = x^r$ where r is a non-zero real number. In this book we focus on the cases where r is a positive integer or $r \in \{-1, \frac{1}{2}\}$.

Even-degree power functions

The functions with rules $f(x) = x^2$ and $f(x) = x^4$ are examples of even-degree power functions.



The following are properties of all even-degree power functions:

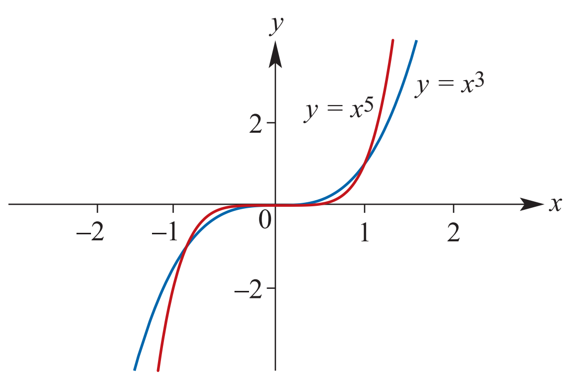
- $f(-x) = (x)$ for all x
- $f(0) = 0$
- As $x \to \pm\infty$, $y \to \infty$.

Note that, if m and n are positive even integers with $m > n$, then:

- $x^m > x^n$ for $x > 1$ or $x < -1$
- $x^m < x^n$ for $-1 < x < 1$ with $x \neq 0$
- $x^m = x^n$ for $x = 1$ or $x = -1$ or $x = 0$.

Odd-degree power functions

The functions with rules $f(x) = x^3$ and $f(x) = x^5$ are examples of odd-degree power functions.



The following are properties of all odd-degree power functions:

- $f(-x) = -f(x)$ for all x
- $f(0) = 0$
- As $x \to \infty$, $y \to \infty$ and as $x \to -\infty$, $y \to -\infty$.

Note that, if m and n are positive odd integers with $m > n$, then:

- $x^m > x^n$ for $x > 1$ or $-1 < x < 0$
- $x^m < x^n$ for $x < -1$ or $0 < x < 1$
- $x^m = x^n$ for $x = 1$ or $x = -1$ or $x = 0$.

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Example 32

State whether each of the following polynomials is even or odd:

a $f(x) = 6x^4 - 3x^2$

b $g(x) = 3x^5 - x^3 + x$

c $h(x) = x^6 - 3x^2 + 2$

d $m(x) = x^7 - 4x$

Solution

a $f(-x) = 6(-x)^4 - 3(-x)^2$
$= 6x^4 - 3x^2$
$= f(x)$

The function is even.

b $g(-x) = 3(-x)^5 - (-x)^3 + (-x)$
$= -3x^5 + x^3 - x$
$= -g(x)$

The function is odd.

c $h(-x) = (-x)^6 - 3(-x)^2 + 2$
$= x^6 - 3x^2 + 2$
$= h(x)$

The function is even.

d $m(-x) = (-x)^7 - 4(-x)$
$= -x^7 + 4x$
$= -m(x)$

The function is odd.

The results of the example are not surprising since:

- The sum of two even functions is even, and any constant multiple of an even function is even.
- The sum of two odd functions is odd, and any constant multiple of an odd function is odd.

Not every polynomial is even or odd. For example, the polynomial $f(x) = x^2 + x$ is neither.

Example 33

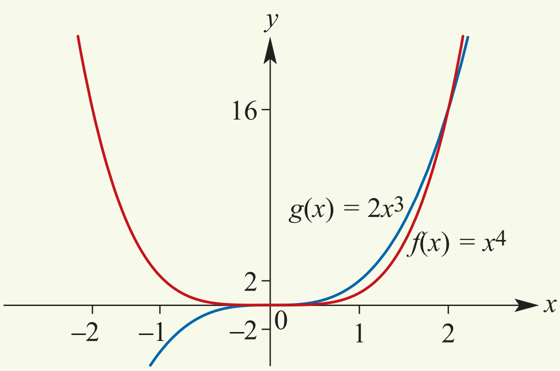
a On the one set of axes sketch the graphs of $f(x) = x^4$ and $g(x) = 2x^3$.

b Solve the equation $f(x) = g(x)$.

c Solve the inequality $f(x) \le g(x)$.

Solution

a



b
$f(x) = g(x)$
$x^4 = 2x^3$
$x^4 - 2x^3 = 0$
$x^3(x - 2) = 0$
$x = 0$ or $x = 2$

c $f(x) \le g(x)$
$x^4 \le 2x^3$

From the graphs and part b, $f(x) \le g(x)$ for $x \in [0, 2]$.

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Summary 7H

- The graph of $y = a(x - h)^4 + k$ has the same shape as $y = ax^4$ but is translated h units in the positive x-axis direction and k units in the positive y-axis direction (for h and k positive constants).
- The natural domain of all quartic functions is $\mathbb{R}$.
- A function f is **even** if $f(-x) = f(x)$. This means that the graph is symmetric about the y-axis.
- A function f is **odd** if $f(-x) = -f(x)$. The graph of an odd function has rotational symmetry with respect to the origin.
- A **power function** is a function f with rule $f(x) = x^r$ where r is a non-zero real number.

Exercise 7H

Example 30

1 Using the method of horizontal and vertical translations, sketch the graph of each of the following:

a $y = (x + 2)^4 - 1$ **b** $y = (x - 1)^4 - 1$ **c** $y = (x + 3)^4 + 2$
d $y = (x - 2)^4 + 5$ **e** $y = (x + 2)^4 - 5$

2 Sketch the graphs of the following functions:

a $y = 2x^4 + 3$ **b** $y = 2(x - 3)^4 + 2$ **c** $y = x^4 - 16$
d $y = 16 - x^4$ **e** $y = (3 - x)^4$ **f** $y = -2(x + 1)^4 + 1$

Example 31

3 Solve each of the following equations for x:

a $x^4 - 27x = 0$ **b** $(x^2 - x - 2)(x^2 - 2x - 15) = 0$
c $x^4 + 8x = 0$ **d** $x^4 - 6x^3 = 0$
e $x^4 - 9x^2 = 0$ **f** $81 - x^4 = 0$
g $x^4 - 16x^2 = 0$ **h** $x^4 - 7x^3 + 12x^2 = 0$
i $x^4 - 9x^3 + 20x^2 = 0$ **j** $(x^2 - 4)(x^2 - 9) = 0$
k $(x - 4)(x^2 + 2x + 8) = 0$ **l** $(x + 4)(x^2 + 2x - 8) = 0$

4 Use a CAS calculator to help draw the graph of each of the following. Give x-axis intercepts and coordinates of turning points. (Values of coordinates of turning points to be given correct to two decimal places.)

a $y = x^4 - 125x$ **b** $y = (x^2 - x - 20)(x^2 - 2x - 24)$
c $y = x^4 + 27x$ **d** $y = x^4 - 4x^3$
e $y = x^4 - 25x^2$ **f** $y = 16 - x^4$
g $y = x^4 - 81x^2$ **h** $y = x^4 - 7x^3 + 12x^2$
i $y = x^4 - 9x^3 + 20x^2$ **j** $y = (x^2 - 16)(x^2 - 25)$
k $y = (x - 2)(x^2 + 2x + 10)$ **l** $y = (x + 4)(x^2 + 2x - 35)$

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Example 32

5 State whether each of the following polynomials is even or odd:

a $f(x) = 5x^6 - 3x^2$
b $g(x) = 7x^{11} - x^3 + 2x$
c $h(x) = x^4 - 3x^2 + 2$
d $m(x) = x^5 - 4x^3$

Example 33

6 **a** On the one set of axes sketch the graphs of $f(x) = x^3$ and $g(x) = 2x^2$.
b Solve the equation $f(x) = g(x)$.
c Solve the inequality $f(x) \le g(x)$.

7 **a** On the one set of axes sketch the graphs of $f(x) = x^4$ and $g(x) = 9x^2$.
b Solve the equation $f(x) = g(x)$.
c Solve the inequality $f(x) \le g(x)$.

8 **a** On the one set of axes sketch the graphs of $f(x) = x^3$ and $g(x) = 4x$.
b Solve the equation $f(x) = g(x)$.
c Solve the inequality $f(x) \le g(x)$.

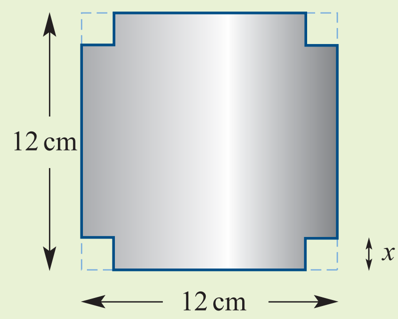
7I Applications of polynomial functions

Example 34

A square sheet of tin measures 12 cm × 12 cm.

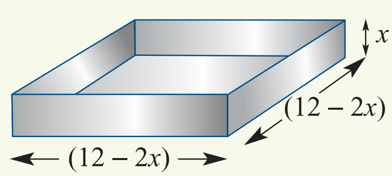
Four equal squares of edge x cm are cut out of the corners and the sides are turned up to form an open rectangular box. Find:

a the values of x for which the volume is 100 cm^3
b the maximum volume.



Solution

The figure shows how it is possible to form many open rectangular boxes with dimensions $12 - 2x$, $12 - 2x$ and x.



The volume of the box is

$$V = x(12 - 2x)^2, \quad 0 \le x \le 6$$

which is a cubic model.

We complete the solution using a CAS calculator as follows.

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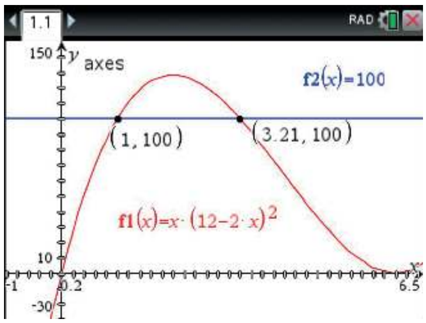
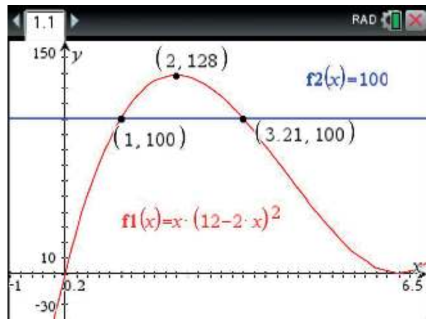
Using the TI-Nspire

Plot the graph of $V = x(12 - 2x)^2$.

a To find the values of x for which $V = 100$, plot the graph of $V = 100$ on the same screen and find the intersection points using (menu) > **Geometry** > **Points & Lines** > **Intersection Point(s)**.

b To find the maximum volume, use (menu) > **Trace** > **Graph Trace** or (menu) > **Analyze Graph** > **Maximum**.



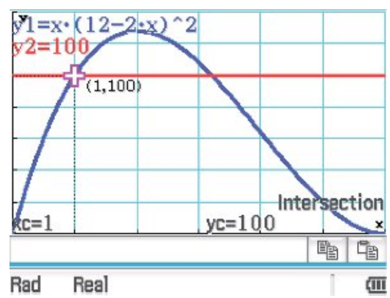


Using the Casio ClassPad

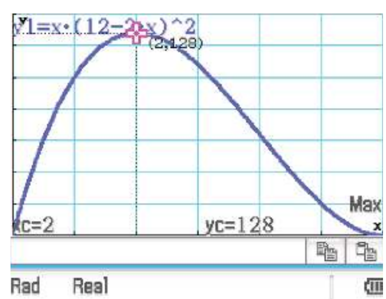
Plot the graph of $V = x(12 - 2x)^2$.

a To find the values of x for which $V = 100$:

- Plot the graph of $V = 100$ on the same screen.
- To adjust your window screen use [window icon]. Set $x_{min} = 0$ and $x_{max} = 6$. Then go to **Zoom Auto**. This will automatically adjust the window to the given domain. The grid appears when the scale is adjusted to 20 units.
- Select the graph window [graph icon] and go to **Analysis** > **G-Solve** > **Intersection**.
- Press the right arrow on the hard keyboard to find the other point of intersection.



b The maximum volume of the box may be found using **Analysis** > **G-Solve** > **Max**. (You must first remove the tick for $y2$ and redraw the graph.)



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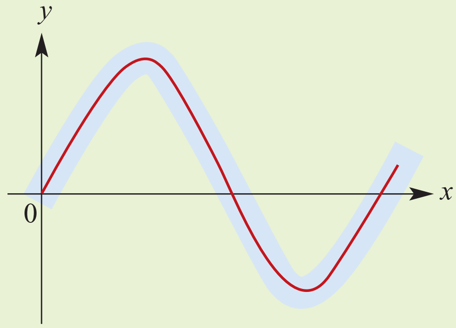
Example 35

It is found that 250 metres of the path of a stream can be modelled by a cubic function. The cubic passes through the points $(0, 0)$, $(100, 22)$, $(150, -10)$, $(200, -20)$.

a Find the equation of the cubic function.

b Find the maximum deviation of the graph from the x-axis for $x \in [0, 250]$.

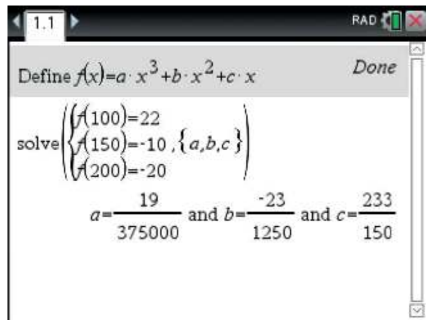


Using the TI-Nspire

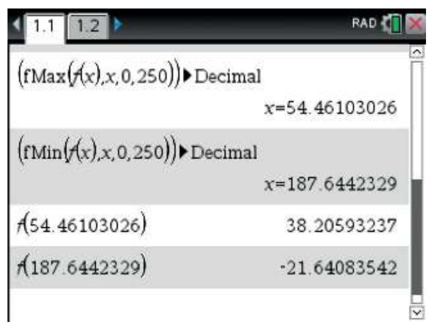
- **Define** $f(x) = ax^3 + bx^2 + cx$.
- Solve using the **Solve System of Equations** command. Enter using the following function notation:

  $$f(100) = 22, \quad f(150) = -10,$$
  $$f(200) = -20$$

  Proceed as shown in the first screen.



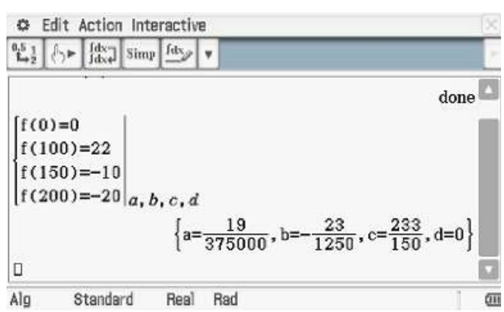
- **Store** these values as a, b and c respectively.
- Use **fMax( )** from (menu) > **Calculus** > **Function Maximum** to find where f obtains its maximum value.
- Use **fMin( )** from (menu) > **Calculus** > **Function Minimum** to find where f obtains its minimum value.



The maximum deviation is 38.21 metres.

Using the Casio ClassPad

- Define $f(x) = ax^3 + bx^2 + cx + d$.
- Enter the four equations shown as simultaneous equations with variables set as a, b, c, d.



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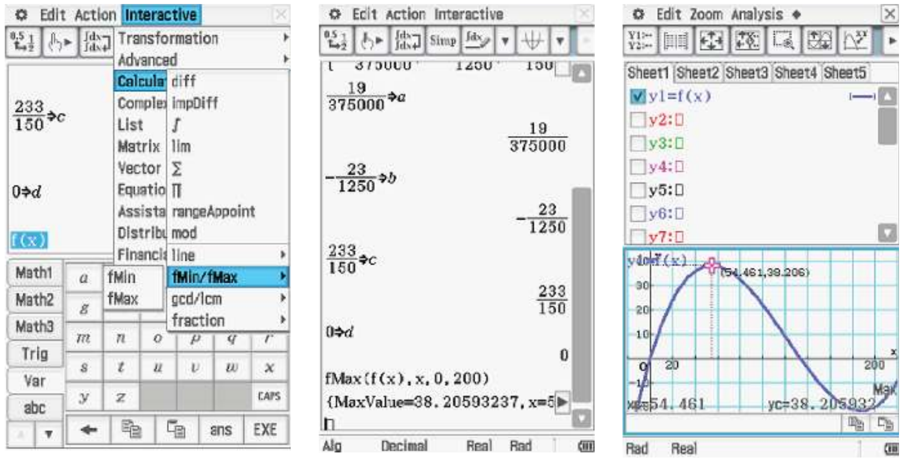
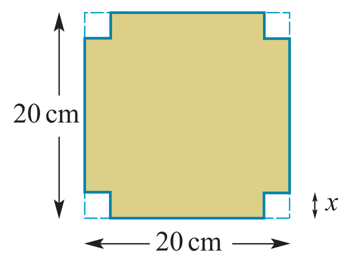
- Store the values found for a, b, c, d as shown below. (**Edit** > **Copy** and **Edit** > **Paste** are useful here. The symbol ⇒ is found in Math1.)
- The maximum value can be found in the main screen. Type $f(x)$, highlight it and go to **Interactive** > **Calculation** > **fMin/fMax**.
- Alternatively, find the maximum value in Graph& Table. Enter and graph $y1 = f(x)$ and then use **Analysis** > **G-Solve** > **Max**.



Exercise 7I

Example 34

1 A square sheet of cardboard has edges of length 20 cm. Four equal squares of edge length x cm are cut out of the corners and the sides are turned up to form an open rectangular box.



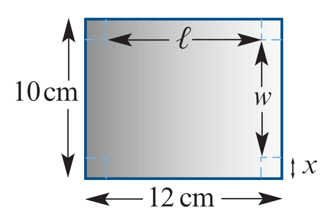
a Find the length of each edge of the base of the box in terms of x.

b Find the volume, V cm^3, of the box in terms of x.

c Find the volume of the box when $x = 5$.

d Find the values of x for which the volume is 500 cm^3.

2 A rectangular sheet of metal measuring 10 cm × 12 cm is to be used to construct an open rectangular tray. The tray will be constructed by cutting out four equal squares from each corner of the sheet as shown in the diagram.



a If the edge of each cut-out square is x cm, express ℓ and w in terms of x.

b Write down a rule for the volume, V cm^3, of the open tray in terms of x.

c Use a CAS calculator to help draw the graph of V against x for suitable values of x.

d Find the value of V when $x = 1$.

e Find the values of x for which $V = 50$.

f Find the maximum volume of the box and the value of x for which this occurs.

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3 The outside surface area of an open box is 75 cm^2. The base is a square with each edge x cm. Let h cm be the height of the box.

a Find the surface area of the box in terms of x and h.

b Hence, find h in terms of x.

c Find V in terms of x if $V\text{ cm}^3$ is the volume of the box.

d Find V when:

i $x = 2$ **ii** $x = 5$ **iii** $x = 8$

e Given that $V = 59$ when $x = 4$, find the other value of x for which $V = 59$.

4 In an upright triangular prism, the triangular base has sides of length $5x$ cm, $12x$ cm and $13x$ cm. The height of the prism is h cm. The sum of the lengths of all of its edges is 180 cm.

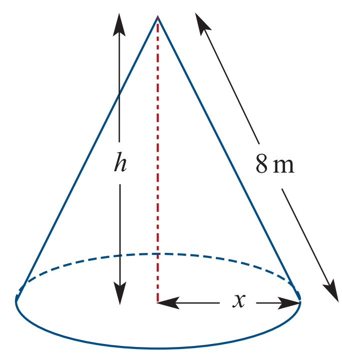
a Find h in terms of x.

b Find V in terms of x where $V\text{ cm}^3$ is the volume of the prism.

c Find V when $x = 3$.

d Find the values of x for which $V = 1200$.

5 The diagram shows a conical heap of gravel. The slant height of the heap is 8 m, the radius of the base x m, and the height h m.



a Express x in terms of h.

b Construct a function which expresses V, the volume of the heap in m^3, in terms of h.

c Use a CAS calculator to help draw the graph of V against h.

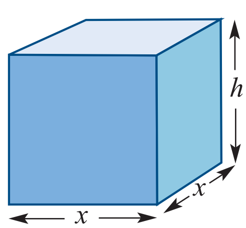
d State the domain for the function.

e Find the value of V when $h = 4$.

f Find the values of h for which $V = 150$.

g Find the maximum volume of the cone and the corresponding value of h.

6 The figure shows a rectangular prism with a square cross-section.



a If the sum of the dimensions, length plus width plus height, is 160 cm, express the height, h, in terms of x.

b Write down an expression for the volume, $V\text{ cm}^3$, of the prism in terms of x.

c State the domain.

d Use a CAS calculator to help draw the graph of V against x.

e Find the value(s) of x for which $V = 50\,000$.

f Find the maximum volume of the prism.

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Review

Chapter summary

Assign-ment

Nrich

- The sum, difference and product of two polynomials is a polynomial.
- Division of one polynomial by another does not always result in a polynomial.
- **Remainder theorem** When $P(x)$ is divided by $x - \alpha$, the remainder is equal to $P(\alpha)$.

  e.g. When $P(x) = x^3 + 3x^2 + 2x + 1$ is divided by $x - 2$, the remainder is

  $$P(2) = (2)^3 + 3(2)^2 + 2(2) + 1 = 25$$
- **Factor theorem** For a polynomial $P(x)$, if $P(\alpha) = 0$, then $x - \alpha$ is a factor. Conversely, if $x - \alpha$ is a factor of $P(x)$, then $P(\alpha) = 0$.

  e.g. For $P(x) = x^3 - 4x^2 + x + 6$,

  $$P(-1) = (-1)^3 - 4(-1)^2 + (-1) + 6 = 0$$

  and so $x + 1$ is a factor of $P(x)$.
- Sums and differences of cubes:

  $$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$$

  e.g. $$x^3 - 27 = (x - 3)(x^2 + 3x + 9)$$

  $$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$$

  e.g. $$\begin{aligned} 8x^3 + 64 &= (2x)^3 + 4^3 \\ &= (2x + 4)(4x^2 - 8x + 16) \end{aligned}$$
- **Rational-root theorem** Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree n with all the coefficients a_i integers. Let α and β be integers such that the highest common factor of α and β is 1 (i.e. α and β are relatively prime). If $\beta x + \alpha$ is a factor of $P(x)$, then β divides a_n and α divides a_0.
- The following are the steps in the process of solving cubic equations:

  i Determine factors by using the factor theorem and dividing.

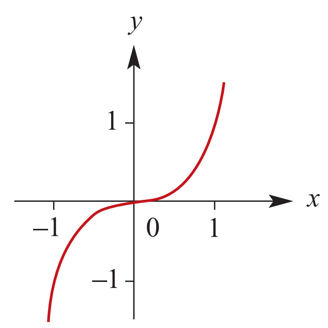
  ii Use the null factor theorem to determine solutions.

  e.g. Solve $x^3 - 4x^2 - 11x + 30 = 0$.
  Since $P(2) = 8 - 16 - 22 + 30 = 0$, we know that $x - 2$ is a factor.
  Dividing $x - 2$ into $x^3 - 4x^2 - 11x + 30$ gives

  $$\begin{aligned} x^3 - 4x^2 - 11x + 30 &= (x - 2)(x^2 - 2x - 15) \\ &= (x - 2)(x - 5)(x + 3) \end{aligned}$$

  $\therefore \quad (x - 2)(x - 5)(x + 3) = 0$

  $\therefore \quad x = 2, \ 5 \text{ or } -3$
- The basic shape of the curve defined by $y = x^3$ is shown in the graph.



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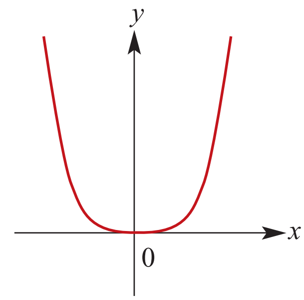
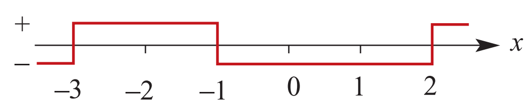
- The natural domain of all polynomial functions is $\mathbb{R}$.
- The graph of $y = a(x - h)^3 + k$ has the same shape as $y = ax^3$ but is translated h units in the positive x-axis direction and k units in the positive y-axis direction (where h and k are positive constants).
- The basic shape of the curve defined by $y = x^4$ is shown in the graph.



- The graph of $y = a(x - h)^4 + k$ has the same shape as $y = ax^4$ but is translated h units in the positive x-axis direction and k units in the positive y-axis direction (where h and k are positive constants). The turning point is at (h, k).
- Sign diagrams assist in sketching graphs of cubic functions.

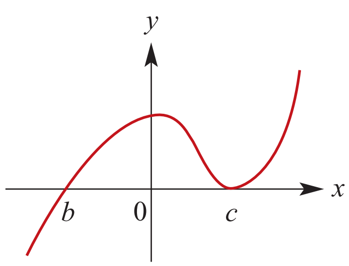
  e.g. $$\begin{aligned} y &= x^3 + 2x^2 - 5x - 6 \\ &= (x + 1)(x - 2)(x + 3) \end{aligned}$$

  When $x < -3$, y is negative.
  When $-3 < x < -1$, y is positive.
  When $-1 < x < 2$, y is negative.
  When $x > 2$, y is positive.



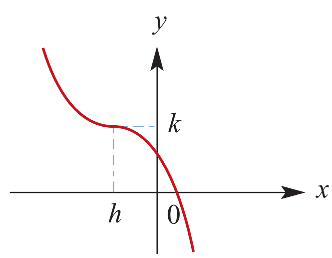
- To sketch the graph of a cubic function $y = ax^3 + bx^2 + cx + d$, the steps are:
  - **i** Use the factor theorem and division to determine the x-axis intercepts.
  - **ii** The y-axis intercept is d.
  - **iii** Draw a sign diagram.
- Finding equations for given cubic graphs. The following may assist:

i



Form: $y = a(x - b)(x - c)^2$
Assume b and c are known, substitute another known point to calculate a.

ii



Form: $y = a(x - h)^3 + k$
Substitute known values to determine a.

Alternatively, use the general form $y = ax^3 + bx^2 + cx + d$ and the known points to determine a, b, c and d.

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- A function f is **even** if $f(-x) = f(x)$. This means that the graph is symmetric about the y-axis.
- A function f is **odd** if $f(-x) = -f(x)$. The graph of an odd function has rotational symmetry with respect to the origin.
- A **power function** is a function f with rule $f(x) = x^r$ where r is a non-zero real number.

Short-answer questions

1 Sketch the graph of each of the following:

a $y = (x - 1)^3 - 2$ **b** $y = (2x - 1)^3 + 1$ **c** $y = 3(x - 1)^3 - 1$
d $y = -3x^3$ **e** $y = -3x^3 + 1$ **f** $y = -3(x - 2)^3 + 1$
g $y = 4(x + 2)^3 - 3$ **h** $y = 1 - 3(x + 2)^3$

2 Sketch the graph of each of the following:

a $y = (x - 1)^4$ **b** $y = (2x - 1)^4 + 1$ **c** $y = (x - 1)^4 - 1$
d $y = -2x^4$ **e** $y = -3x^4 + 1$ **f** $y = -(x - 2)^4 + 1$
g $y = 2(x + 1)^4 - 3$ **h** $y = 1 - 2(x + 2)^4$

3 Solve each of the following equations for x:

a $2x^3 + 3x^2 = 11x + 6$ **b** $x^2(5 - 2x) = 4$ **c** $x^3 + 4x + 12 = 7x^2$

4 **a** Use the factor theorem to show that $2x - 3$ and $x + 2$ are factors of $6x^3 + 5x^2 - 17x - 6$. Find the other factor.

b Solve the equation $2x^3 - 3x^2 - 11x + 6 = 0$.

c Solve the equation $x^3 + x^2 - 11x - 3 = 8$.

d **i** Show that $3x - 1$ is a factor of $3x^3 + 2x^2 - 19x + 6$.

ii Find the factors of $3x^3 + 2x^2 - 19x + 6$.

5 Let $f(x) = x^3 - kx^2 + 2kx - k - 1$.

a Show that $f(x)$ is divisible by $x - 1$.

b Factorise $f(x)$.

6 Find the values of a and b for which $x^3 + ax^2 - 10x + b$ is divisible by $x^2 + x - 12$.

7 Draw a sign diagram for each of the following and hence sketch the graph:

a $y = (x + 2)(3 - x)(x + 4)$ **b** $y = (x - 2)(x + 3)(x - 4)$
c $y = 6x^3 + 13x^2 - 4$ **d** $y = x^3 + x^2 - 24x + 36$

8 Without actually dividing, find the remainder when the first polynomial is divided by the second:

a $x^3 + 4x^2 - 5x + 1,\ x + 6$ **b** $2x^3 - 3x^2 + 2x + 4,\ x - 2$
c $3x^3 + 2x + 4,\ 3x - 1$

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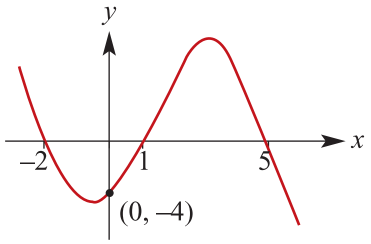
9 Find the rule of the cubic for which the graph is shown.



10 Find a cubic function whose graph touches the x-axis at $x = -4$, passes through the origin and has a value of 10 when $x = 5$.

11 Let $f(x) = 2x^3 + ax^2 - bx + 3$. When $f(x)$ is divided by $x - 2$ the remainder is 15 and $f(1) = 0$.

a Calculate the values of a and b.

b Find the other two factors of $f(x)$.

12 For each of the following, find a sequence of transformations that takes the graph of $y = x^3$ to the graph of:

a $y = 2(x - 1)^3 + 3$

b $y = -(x + 1)^3 + 2$

c $y = (2x + 1)^3 - 2$

Extended-response questions

1 The volume of a cylinder is given by $V = \pi r^2 h$. It is given that $h + r = 6$.

a Write $V(r)$ in terms of r.

b State the values that r can have.

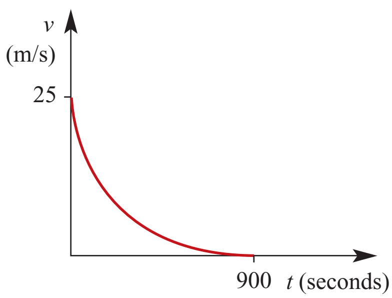
c Find $V(3)$.

d Find the values of r for which $V(r) = 27\pi$.

e Use your CAS calculator to find the maximum possible volume of the cylinder.

2 There is a proposal to provide a quicker, more efficient and more environmentally friendly system of inner-city public transport by using electric taxis. The proposal necessitates the installation of power sources at various locations as the taxis can only be driven for a limited time before requiring recharging.

The graph shows the speed v m/s that the taxi will maintain if it is driven at constant speed in such a way that it uses all its energy up in t seconds. The curve is a section of a parabola which touches the t-axis at $t = 900$. When $t = 0$, $v = 25$.



a Construct a rule for v in terms of t.

b If s metres is the distance that a taxi can travel before running out of electrical energy, write down a rule connecting s and t.

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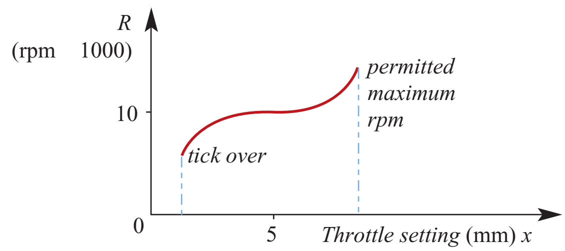
c Use a CAS calculator to help draw the graph of s against t.

d Originally the power sources were to be located at 2 km intervals.
However there is a further proposal to place them at 3.5 km intervals.
Is this new distance feasible?

e With the power sources at 2 km intervals, use your graph to determine approximately both the maximum and minimum speeds recommended for drivers. Explain your answer.

3 The figure shows part of a cubic graph that represents the relationship between the engine speed, R rpm, and the throttle setting, x mm from the closed position, for a new engine.

It can be seen from the graph that the engine has a 'flat spot' where an increase in x has very little effect on R.

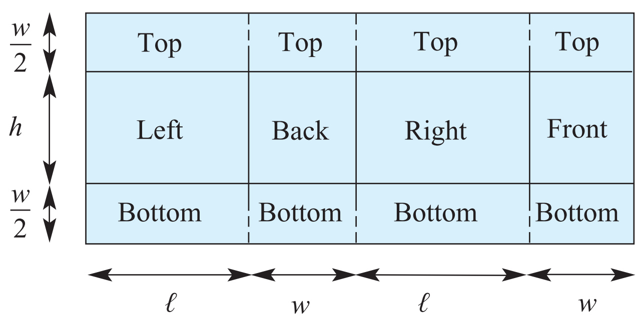


a Develop a cubic expression for R in terms of x of the form $R = a(x - h)^3 + k$.

b Find a if when the graph is extended it passes through the origin.

c In a proposed modification to the design, the 'flat spot' will occur when $x = 7$ mm. The speed of the engine in this case will be 12 000 rpm when $x = 7$ mm. Assuming that a cubic model still applies and that $R = 0$ when $x = 0$, write down an expression for R as a function of x.

4 A net for making a cardboard box with overlapping flaps is shown in the figure. The dotted lines represent cuts and the solid lines represent lines along which the cardboard is folded.



a If $\ell = 35$ cm, $w = 20$ cm and $h = 23$ cm, calculate the area of the net.

b If the area of the net is to remain constant at the value calculated in part a and $\ell = h$, write down an expression for V, the volume of the box in cm^3, as a function of ℓ. (The maximum volume of the box will occur when $\ell = h$).

c Use a CAS calculator to help draw the graph of V against ℓ.

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d Find the value of ℓ when the volume of the box is:

i $14\,000\text{ cm}^3$

ii $10\text{ litres} = 10\,000\text{ cm}^3$

e Find the maximum volume of the box and the value of ℓ for which this occurs.

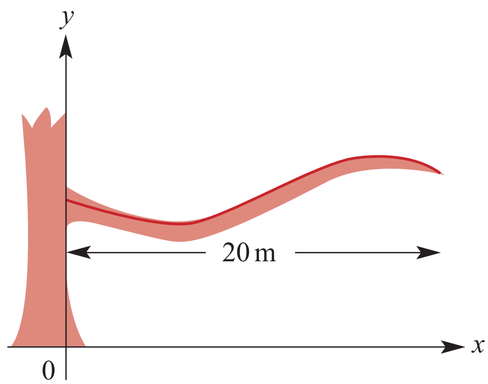
5 It is found that the shape of a branch of a eucalyptus tree can be modelled by a cubic function. The coordinates of several points on the branch are (0, 15.8), (10, 14.5), (15, 15.6), (20, 15).

a The rule for the function is of the form $y = ax^3 + bx^2 + cx + d$. Find the values of a, b, c and d.

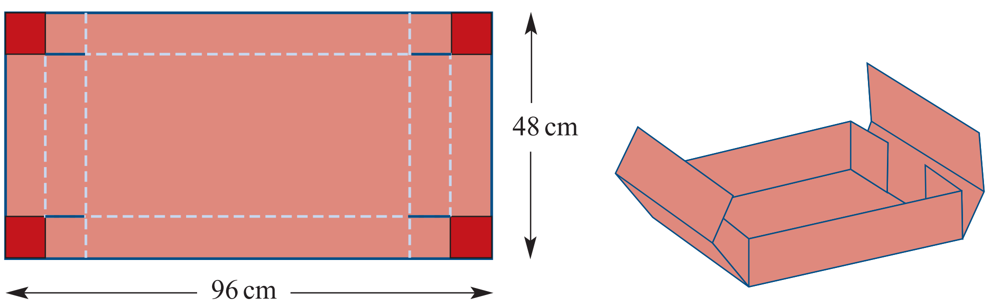
b Find the coordinates of the point on the branch that is:

i closest to the ground

ii furthest from the ground.



6 A reinforced box is made by cutting congruent squares of side length x cm from the four corners of a rectangular piece of cardboard that measures 48 cm by 96 cm. The flaps are folded up.



a Find an expression for V, the volume of the box formed.

b Plot a graph of V against x on your CAS calculator.

i What is the domain of the function V?

ii Using your CAS calculator, find the maximum volume of the box and the value of x for which this occurs (approximate values required).

c Find the volume of the box when $x = 10$.

d It is decided that $0 \le x \le 5$. Find the maximum volume possible.

e If $5 \le x \le 15$, what is the minimum volume of the box?

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8 Revision of Chapters 2–7

8A Short-answer questions

1 The points $A(-2, 1)$, $B(3, -4)$ and $C(5, 7)$ are the vertices of triangle ABC.

a Let M and N be the midpoints of line segments AB and AC respectively. Find the coordinates of M and N.

b Show that MN is parallel to BC.

2 Let $P(x) = 8x^3 + 4x - 3$.

a Find the remainder when $P(x)$ is divided by $2x + 1$.

b Find the remainder when $P(x)$ is divided by $2 - x$.

c Find the remainder when $P(x + 1)$ is divided by $x + 2$.

3 If $g(x) = 3x^2 - 4$, then find:

a $g(2a)$ **b** $g(a - 1)$ **c** $g(a + 1) - g(a - 1)$

4 Let $f(x) = 4 - 5x$ and $g(x) = 7 + 2x$.

a Is it true that $f(2) + f(3) = f(2 + 3)$?

b Solve the equation $f(x) = g(x)$ for x.

c Solve the inequality $f(x) \geq g(x)$.

d Find the value of k for which $f(2k) = g(3k)$.

5 Solve the simultaneous equations:

$$x + y = 5$$
$$(x + 1)^2 + (y + 1)^2 = 25$$

6 Show that the points $A(0, -5)$, $B(-1, 2)$, $C(4, 7)$ and $D(5, 0)$ form a rhombus $ABCD$.

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7 Write each of the following in turning point form:

a $y = x^2 + 4x - 9$ **b** $y = x^2 - 3x - 11$ **c** $y = 2x^2 - 3x + 11$

8 Find the coordinates of the points of intersection of the graphs for each of the following pairs of functions:

a $y = 4x + 1$ and $y = x^2 + 3x - 9$ **b** $y = 2x + 2$ and $y = x^2 - 2x + 6$

c $y = -3x + 2$ and $y = x^2 + 5x + 18$

9 Solve each of the following inequalities:

a $x^2 + 3x - 5 > 0$ **b** $2x^2 - 5x - 5 \geq 0$

10 State the natural (implied) domain of each of the following functions:

a $f(x) = \dfrac{3}{2x - 5} - 7$ **b** $g(x) = \sqrt{5 - x} - 4$ **c** $f(x) = x^2 - 4x - 7$

11 Given that $(x + 4)$ is a factor of $3x^3 + x^2 + px + 24$, find the value of p and hence factorise the expression completely.

12 If $5x^3 - 3x^2 + ax + 7 = (x + 2)Q_1(x) + R$ and $4x^3 + ax^2 + 7x - 4 = (x + 2)Q_2(x) + 2R$, find the values of a and R.

13 Find the range of each of the following functions:

a $f(x) = x^2, \{ x \in \mathbb{R} : 1 \leq x \leq 2 \}$ **b** $h(x) = 2 - x, \{ x \in \mathbb{R} : -1 \leq x \leq 2 \}$

c $g(x) = x^2 - 4, \{ x \in \mathbb{R} : x < 0 \}$ **d** $f(x) = \sqrt{2 - x} + 3, \{ x \in \mathbb{R} : x \leq 2 \}$

e $f(x) = (x - 2)^3 + 8$

14 At Big Bob's Burger Bar, a Big Bob burger costs $\$b$ and a regular fries costs $\$f$. Three Big Bob burgers and two regular fries cost \$18.20.

a Write an equation to show this.

b If one Big Bob burger costs \$4.20, what is the cost of one regular fries?

15 $4x + ky = 7$ and $y = 3 - 4x$ are the equations of two lines. Find k if:

a the lines are parallel **b** the lines are perpendicular.

16 Line ℓ_1 has an x-axis intercept at $(5, 0)$ and a y-axis intercept at $(0, -2)$. Line ℓ_2 is perpendicular to ℓ_1 and passes through the point $(1, 6)$.

a Find the gradient of line ℓ_1.

b Find the equation of line ℓ_2 expressing your answer in the form $ax + by + c = 0$, where a, b and c are integers and $a > 0$.

17 In the first few days of the life of an earthworm, its length (ℓ cm) is thought to be proportional to the square root of the time (n hours) since its birth. If an earthworm is 2 cm long after 1 hour, how long will it be after 4 hours? How long will it take to grow to a length of 14 cm?

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18 A family of parabolas have rules of the form $y = ax^2 + 2x + a$, for $a \neq 0$.

- **a** Express $ax^2 + 2x + a$ in the form $a(x + b)^2 + c$ for real numbers b and c.
- **b** Give the coordinates of the turning point of the graph of $y = ax^2 + 2x + a$ in terms of a.
- **c** For which values of a is $ax^2 + 2x + a$ a perfect square?
- **d** For which values of a does the graph of $y = ax^2 + 2x + a$ have two x-axis intercepts?

19
- **a** Sketch the graph of $y = 1 + \dfrac{1}{2 + x}$, where $x \neq -2$.
- **b** The graph crosses the y-axis at A and the x-axis at B. Give the coordinates of A and B.
- **c** Find the equation of line AB.
- **d** Find the coordinates of the midpoint M of AB.
- **e** Find the equation of the straight line passing through M perpendicular to AB.

20 The force of attraction (F newtons) between two magnets varies inversely as the square of the distance (d cm) between them. When the magnets are 2 cm apart, the attractive force is 1 newton. How far are they apart if the attractive force is 2 newtons?

21 The mass, m, of a particular type of object varies jointly as its density, d, and the cube of its height, h. When the height is 1 cm and the density is 8 g/cm^3, the mass is 32 g. Find the mass of a similar object of height 2 cm and density 6 g/cm^3.

8B Extended-response questions

1 The general equation of the circle can be written as $x^2 + y^2 + bx + cy + d = 0$. A circle passes through the points with coordinates $(-4, 5)$, $(-2, 7)$ and $(4, -3)$.

- **a** Write three simultaneous equations in b, c and d.
- **b** Determine the equation of the circle.

2 A circle passes through the origin. It has equation $x^2 + y^2 + bx + cy = 0$. The circle also passes through the point $(4, 4)$.

- **a** Find c in terms of b.
- **b** Find the x-axis intercepts in terms of b.
- **c** Find the y-axis intercepts in terms of b.
- **d** For what value of b does the circle touch the y-axis?

3 A family of functions have rules of the form $f(x) = \sqrt{a - x}$, where a is a positive real number.

- **a** State the natural domain of f.
- **b** Find the coordinates of the point of intersection of the graph of $y = f(x)$ with the graph of $y = x$.

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c For what value of a does the line with equation $y = x$ intersect the graph of $y = f(x)$ at the point with coordinates $(1, 1)$?

d For what value of a does the line with equation $y = x$ intersect the graph of $y = f(x)$ at the point with coordinates $(2, 2)$?

e For what value of a does the line with equation $y = x$ intersect the graph of $y = f(x)$ at the point with coordinates (c, c), where c is a positive real number?

4 A particular plastic plate manufactured at a factory sells at \$11.50. The cost of production consists of an initial cost of \$3500 and then \$10.50 a plate. Let x be the number of plates produced.

a Let \$$C$ be the cost of production of x plates. Write an expression for C in terms of x.

b Let \$$I$ be the income from selling x plates. Write an expression for I in terms of x.

c On the one set of axes sketch the graphs of I against x and C against x.

d How many plates must be sold for the income to equal the cost of production?

e Let $P = I - C$. Sketch the graph of P against x. What does P represent?

f How many plates must be sold for a profit of \$2000 to be made?

5 A swimming pool initially contains 45 000 litres of water. At 12 p.m., an inlet valve is opened letting 40 litres of water per minute into the pool. Assume no water evaporates.

a Find an expression for the volume, V litres, in the pool m minutes after 12 p.m.

b When will the pool reach its maximum capacity of 55 000 litres?

c Sketch the graph of V against m (use the appropriate domain).

6 From a piece of wire 42 cm long, a length $10x$ cm is cut off and bent into a rectangle whose length is one and a half times its width. The remainder is bent to form a square.

a State the area of the rectangle in terms of x.

b State the area of the square in terms of x.

c State the possible values for x.

d Find an expression for the total area, A cm^2, of the square and the rectangle.

e Sketch the graph of A against x.

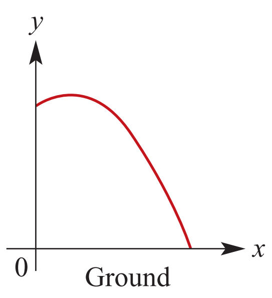
f What is the maximum possible total area?

g If the combined area of the rectangle and the square is 63 cm^2, find their dimensions.

7 A stone is projected from a balcony as shown. The path is described by the equation

$$y = -\frac{1}{10}(x + 10)(x - 20), \quad x \geq 0$$

where y metres is the height above the ground when the stone has travelled x metres horizontally.



a What is the height of the stone at the point of the projection?

b What is the horizontal distance travelled before the stone hits the ground?

c What is the maximum height reached by the stone?

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8 A rectangular block is 2 cm wider than it is high and twice as long as it is wide. Let x cm be the height of the block.

a Find an expression for the total surface area, A cm^2, in terms of x.

b Find A if:

i $x = 1$ **ii** $x = 2$

c Find the value of x if $A = 190$.

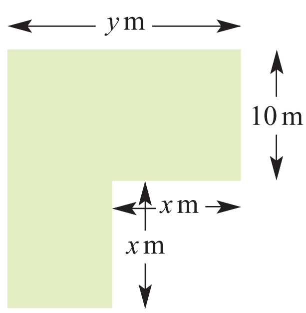
d Sketch the graph of A against x for suitable values of x.

e Find an expression for the volume, V cm^3, of the block in terms of x.

f If the volume of the block is 150 cm^3, find the value of x.

g If the volume of the block is 1000 cm^3, find the value of x.

9 A region is enclosed as shown.



a Find expressions for:

i A (m^2), the area of the region in terms of x and y

ii P (m), the perimeter of the region in terms of x and y.

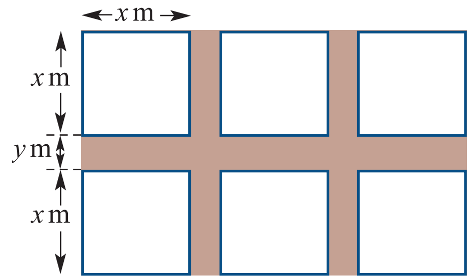
b **i** If the perimeter is 100 m, find A in terms of x.

ii What is the maximum area possible?

iii State the possible values for x.

iv Sketch the graph of A against x for these values.

10 The diagram shows a window with six equal square panes divided by equally thick timber framing.



a Write down the total area of the whole window in terms of x and y.

b Show that the total area of the dividing wood is $7xy + 2y^2$.

c The total area of glass is 1.5 m^2 and the total area of the dividing wood is 1 m^2.

i Find x. **ii** Find y.

11 A fountain is 5 metres high. Water is coming vertically out of the fountain. The function which describes the height of a drop of water above the spout t seconds after leaving the spout of the fountain is given by $h(t) = -4.9t^2 + 30t + 5$.

a How high is the drop of water after 3 seconds?

b When will the drop be back at the height of the spout?

c Sketch the graph of the height of the drop above the fountain against time.

d If there is a little wind, the drop will not travel along a vertical line but its time in the air will not be affected and h will still describe its height above the ground. How long will it take for a drop of water to hit the ground?

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12 An open rectangular cardboard box is 7 cm high and its length is 5 cm greater than its breadth, which is x cm.

a Find the length of the box in terms of x.

b Find the volume (V cm^3) of the box in terms of x.

c Find the surface area (S cm^2) in terms of x.

d Sketch the graphs of S against x and V against x for a suitable domain on the one set of axes.

e For what value of x is $V = S$?

f If the total surface area is 500 cm^2, find x.

13 Two points A and C have coordinates $(1, 3)$ and $(7, 7)$.

a Find the equation of the perpendicular bisector of AC.

b B is a point on the y-axis which is equidistant from A and C, and $ABCD$ is a rhombus. Find:

i the coordinates of B **ii** the coordinates of D

c Find the area of the rhombus $ABCD$.

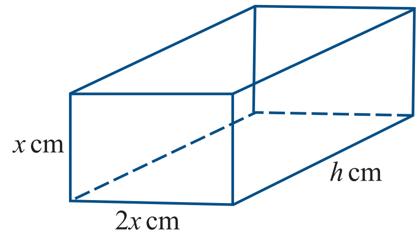
d Calculate the perpendicular distance of A from BC.

14 **a** A train travels 300 km at a constant speed of V km/h. If the train had travelled 5 km/h faster, the journey would have taken two hours less. Find the speed of the train travelling at the slower speed.

b A tank can be filled by two taps A and B in $33\frac{1}{3}$ minutes (33 minutes 20 seconds) when they are running together. Tap A running by itself fills the tank in 15 minutes less than tap B. Find the time taken for each tap running by itself to fill the tank.

c A hall can be paved with 200 square tiles of a certain size. If each tile were 1 cm longer and wider it would take 128 tiles. Find the length of each tile.

15 A piece of wire 400 cm long is used to make the 12 edges of a cuboid with dimensions as shown.



a Find h in terms of x.

b Find the volume, V cm^3, in terms of x.

c State the possible values for x.

d Plot the graph of V against x on a CAS calculator for the domain determined in part c.

e State the values of x (correct to three decimal places) which will result in a volume of:

i 30 000 cm^3 **ii** 20 000 cm^3.

f State the maximum volume (correct to three decimal places) and the corresponding value of x.

g The cuboid is covered in paper.

i Find the surface area, S cm^2, of the cuboid in terms of x.

ii Find the maximum value of S and the value of x for which this occurs.

h Find the values of x for which $S = V$.

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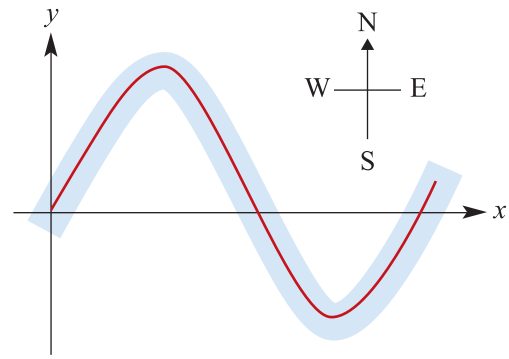
16 A section of a path along the edge of a river can be modelled by a cubic function for $x \in [0, 250]$. Measurements are in metres. The following are the coordinates of points on the path: $(0, 0)$, $(100, 33)$, $(50, 57)$ and $(150, -15)$.

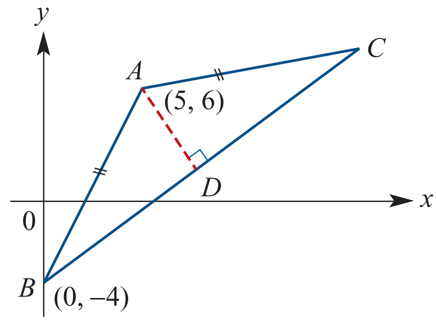


a Find the equation of the graph of the cubic function which passes through these points.

b The bank of the river is 5 metres to the north of the centre of the path. Find the equation of the cubic function which models the river bank.

c Find the largest deviation of the path from the x-axis.

17 ABC is an isosceles triangle. The coordinates of A and B are $(5, 6)$ and $(0, -4)$ respectively. Given that the gradient of BC is $\frac{3}{4}$ and D is the midpoint of BC, find:



a the equation of BC

b the equation of AD

c the coordinates of D

d the length of the perpendicular AD

e the area of triangle ABC.

18 The potential energy, P joules, of an object varies jointly as the mass, m kg, of the object and the height, h m, of the object above the ground.

a For an object of mass 5 kg:

i Find P in terms of h if $P = 980$ when $h = 20$.

ii Sketch the graph of P against h.

iii Find P if $h = 23.2$.

b **i** Find P in terms of h and m if $P = 980$ when $h = 20$ and $m = 5$.

ii Find the percentage change in potential energy if the height (h m) is doubled and the mass remains constant.

iii Find the percentage change in potential energy if the object has one-quarter of the original height (h m) and double the original mass (m kg).

c If an object is dropped from a height (h m) above ground level, its speed (V m/s) when it reaches the ground is given by $V = \sqrt{19.6h}$.

i Find V when $h = 10$. **ii** Find V when $h = 90$.

d In order to double the speed that a given object has when it hits the ground, by what factor must the height from which it is dropped be increased?

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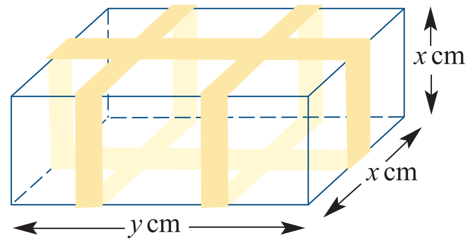
19 A parcel in the form of a rectangular block is held together by three pieces of tape as shown.
The parcel has square ends with side length x cm. The length of the parcel is y cm.
The total length of tape is 500 cm.



a **i** Find y in terms of x.
ii Find V, the volume of the parcel in cm^3, in terms of x.

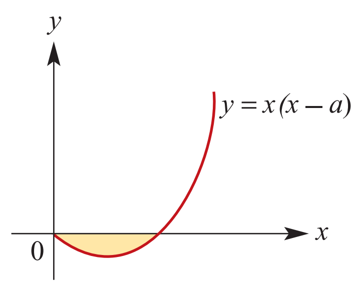
b Draw the graph of V against x for suitable values of x.

c State the domain of this function.

d Find the values of x for which $V = 25\,000$.

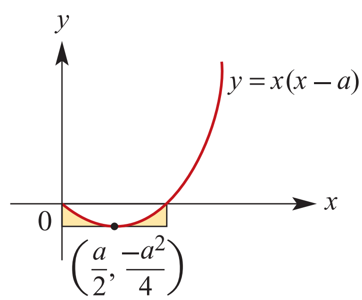
e Find the maximum volume of the parcel and the corresponding values of x and y.

20 **a** The area, A, of the shaded region varies directly as the cube of a.



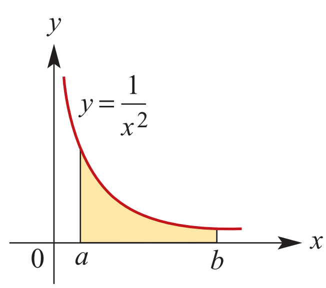
i If $A = \frac{4}{3}$ when $a = 2$, find an expression for A in terms of a.
ii Find A when $a = 3$.
iii If $A = 4500$, find a.

b The area, A_1, of the shaded region varies directly as the cube of a.



i If $A_1 = 1152$ when $a = 24$, find an expression for A_1 in terms of a.
ii Find A_1 when $a = 18$.
iii Find a when $A_1 = 3888$.

c The area, A_2, of the shaded region varies partly as the reciprocal of a and partly as the reciprocal of b.



i Given that $A_2 = \frac{1}{2}$ when $a = 1$ and $b = 2$, and that $A_2 = \frac{1}{12}$ when $a = 3$ and $b = 4$, find A_2 in terms of a and b.
ii Find A_2 when $a = 1$ and $b = 6$.
iii Find A_2 when $a = \frac{1}{4}$ and $b = 3$.
iv Find A_2 when $a = \frac{1}{100}$ and $b = 100$.
v Find A_2 when $a = \frac{1}{1000}$ and $b = 1000$.

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9 Probability

In this chapter

Syllabus references

Topics: Language of events and sets; Review of the fundamentals of probability; Conditional probability and independence

Subtopics: 1.3.6 – 1.3.17

Uncertainty is involved in much of the reasoning we undertake every day of our lives. We are often required to make decisions based on the chance of a particular occurrence. Some events can be predicted from our present store of knowledge, such as the time of the next high tide. Others, such as whether a head or tail will show when a coin is tossed, are not predictable.

However, whether through our prior observations or through a theoretical understanding of the circumstances, we are often able to assign a numerical value, or **probability**, to each possible outcome of an experiment. This probability, which will take a value between 0 and 1, gives us an indication as to how likely we are to observe the outcome. A probability of 0 means that the outcome is impossible and a probability of 1 means that it is certain. Generally, the probability will be somewhere in between, with a higher value meaning that the outcome is more likely.

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9A Sample spaces and probability

In this section we look at two essential components of probability: a sample space, which is the set of all possible outcomes of an experiment, and a set of probabilities, one for each outcome.

Sample spaces

Suppose we toss a coin to see whether a head (H) or a tail (T) appears uppermost. The toss of the coin can be termed a single **trial** of a **random experiment**. The word 'random' is used here because, while the outcome observed must be either a head or a tail, on a particular toss we don't know which will be observed. However, we do know that the outcome observed will be one of a known set of possible outcomes, and the set of all possible outcomes is called the **sample space** for the experiment.

Set notation can be used in listing all the elements in the sample space. For example, the sample space for the tossing of a coin would be written as

$$\{H, T\}$$

where H indicates head and T indicates tail. Throughout this chapter, the Greek letter ε (epsilon) will be used to denote the sample space.

For example, the following table lists the sample spaces for each of the random experiments described.

Random experiment	Sample space
The number observed when a die is rolled	$\varepsilon = \{1, 2, 3, 4, 5, 6\}$
The number of brown eggs in a carton of 12 eggs	$\varepsilon = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
The result when two coins are tossed	$\varepsilon = \{HH, HT, TH, TT\}$
The number of calls to your phone in the next two hours	$\varepsilon = \{0, 1, 2, 3, 4, \dots\}$
The time, in hours, it takes to complete your homework	$\varepsilon = \{\, t : t \ge 0 \,\}$

Events

An **event** is a subset of the sample space. It may consist of a single outcome, or it may consist of several outcomes. For example, when rolling a die, the event of interest may be 'getting a six', which consists of just one outcome and is described by the set $\{6\}$. However, the event 'getting an odd number' can be achieved by rolling 1, 3 or 5 and is described by the set $\{1, 3, 5\}$.

It is convenient to use set notation to list the elements of the event. In general we use capital letters, $A, B, C, \dots$, to denote events.

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The following table lists the experiments described earlier and gives the sample space and an example of an event for each one.

Sample space	An event
The number observed when a die is rolled $\varepsilon = \{1, 2, 3, 4, 5, 6\}$	'An even number' $= \{2, 4, 6\}$
The number of brown eggs in a carton of 12 eggs $\varepsilon = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$	'More than half brown' $= \{7, 8, 9, 10, 11, 12\}$
The result when two coins are tossed $\varepsilon = \{HH, HT, TH, TT\}$	'Two heads' $= \{HH\}$
The number of calls to your phone in the next two hours $\varepsilon = \{0, 1, 2, 3, 4, \dots\}$	'Fewer than two phone calls' $= \{0, 1\}$
The time, in hours, it takes to complete your homework $\varepsilon = \{ t : t \geq 0 \}$	'More than two hours' $= \{ t : t > 2 \}$

Note: Both a sample space and an event can be discrete or continuous, finite or infinite.

Example 1

A bag contains seven marbles numbered from 1 to 7 and a marble is withdrawn.

a Give the sample space for this experiment.

b List the outcomes (elements) of the event 'a marble with an odd number is withdrawn'.

Solution	Explanation
a $\{1, 2, 3, 4, 5, 6, 7\}$	Any number from 1 to 7 could be observed.
b $\{1, 3, 5, 7\}$	This set contains the odd numbers in the sample space.

Determining probabilities for equally likely outcomes

There are many situations for which we can develop a simple model that can be used to assign a probability to an event. The most obvious of these is when it is reasonable to assume that all of the outcomes are equally likely, such as when a die is rolled.

We require that the probabilities of all the outcomes in the sample space sum to 1, and that the probability of each outcome is a non-negative number. This means that the probability of each outcome must lie in the interval $[0, 1]$. Since six outcomes are possible when rolling a die, we can assign the probability of each outcome to be $\frac{1}{6}$. That is,

$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

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When the sample space is finite, the **probability of an event** is equal to the sum of the probabilities of the outcomes in that event.

For example, let A be the event that an even number is rolled on the die. Then $A = \{2, 4, 6\}$ and $\text{P}(A) = \text{P}(2) + \text{P}(4) + \text{P}(6) = \frac{1}{2}$. Since the outcomes are equally likely, we can calculate this more easily as

$$\text{P}(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}} = \frac{3}{6} = \frac{1}{2}$$

Equally likely outcomes

In general, if the sample space ε for an experiment contains n outcomes, all of which are equally likely to occur, we assign a probability of $\frac{1}{n}$ to each of these outcomes.

Then the probability of any event A which contains m of these outcomes is the ratio of the number of elements in A to the number of elements in ε. That is,

$$\text{P}(A) = \frac{n(A)}{n(\varepsilon)} = \frac{m}{n}$$

where the notation $n(A)$ is used to represent the number of elements in set A.

Of course, there are many situations where the outcomes are not equally likely. For example, it has been established worldwide that the probability of a male birth is in fact 0.51, not 0.5. However, in many situations the assumption of equally likely is justified, and allows us to assign probabilities reasonably.

Example 2

Suppose a number is drawn at random from the numbers 7, 8, 9, 10, 11, 12, 13, 14. What is the probability of choosing a prime number?

Solution

Let A be the event the chosen number is prime. Then

$$\text{P}(A) = \frac{3}{8}$$

Explanation

Since the number is drawn at random, we can assume each number is equally likely to be drawn.

$A = \{7, 11, 13\}, \; n(A) = 3, \; n(\varepsilon) = 8$

Example 3

Suppose that a card is drawn from a pack of 52 playing cards, and that each card has equal likelihood of being drawn. Find:

a the probability that the card is black

b the probability that the card is a king

c the probability that the card is a black king.

Solution

a $\text{P(black card)} = \frac{26}{52} = \frac{1}{2}$

Explanation

There are 52 cards in a pack and 26 are black.

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b P(king) $= \frac{4}{52} = \frac{1}{13}$

There are 52 cards in a pack and 4 are kings.

c P(black king) $= \frac{2}{52} = \frac{1}{26}$

There are 52 cards in a pack and 2 are black kings.

The following rules of probability hold for finite sample spaces:

- $P(A) \geq 0$, for any event A.
- The sum of the probabilities of all the outcomes of a random experiment must equal 1.

The second of these two rules can be used to determine probabilities as follows.

Example 4

A random experiment may result in 1, 2, 3 or 4. If $P(1) = \frac{1}{13}$, $P(2) = \frac{2}{13}$ and $P(3) = \frac{3}{13}$, find the probability of obtaining a 4.

Solution

$$P(4) = 1 - \left(\frac{1}{13} + \frac{2}{13} + \frac{3}{13}\right)$$
$$= 1 - \frac{6}{13} = \frac{7}{13}$$

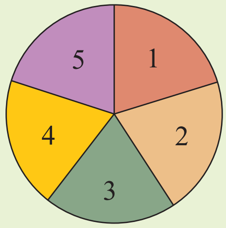
Explanation

The sum of the probabilities is 1.

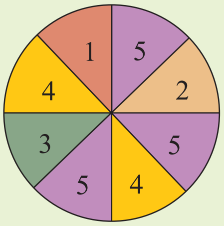
Example 5

Find the probability that each of the possible outcomes is observed for the following spinners:

a



b



Solution

a $P(1) = P(2) = P(3) = P(4) = P(5) = \frac{1}{5}$

b $P(1) = P(2) = P(3) = \frac{1}{8} = 0.125$

$P(4) = \frac{2}{8} = \frac{1}{4} = 0.25$

$P(5) = \frac{3}{8} = 0.375$

Note that in both these cases
$P(1) + P(2) + P(3) + P(4) + P(5) = 1$

Explanation

On spinner a, there are five equally likely outcomes.

Since there are 8 equal segments, we assume each has a probability of $\frac{1}{8}$.

The results 1, 2 and 3 appear once.
The result 4 appears twice.
The result 5 appears three times.

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Complementary events

When two events have no elements in common and together they make up the entire sample space, they are said to be **complementary events**. The complement of event A is the event A', which consists of all the outcomes in ε that are not in A. Since the sum of the probabilities is 1, we can write

$$P(A') = 1 - P(A)$$

Example 6

A card is drawn at random from a pack of 52 cards. What is the probability that the card is:

a not a heart

b not an ace?

Solution

a Let H be the event a heart is drawn.

Then $P(H') = 1 - P(H)$

$$= 1 - \frac{13}{52}$$

$$= 1 - \frac{1}{4}$$

$$= \frac{3}{4}$$

b Let A be the event an ace is drawn.

Then $P(A') = 1 - P(A)$

$$= 1 - \frac{4}{52}$$

$$= 1 - \frac{1}{13}$$

$$= \frac{12}{13}$$

Combining our knowledge of the rules of probability enables us to solve more complex problems.

Example 7

A random experiment may result in outcomes A, B, C, D or E, where A, B, C, D are equally likely and E is twice as likely as A. Find:

a $P(E)$

b $P(B')$

Solution	Explanation
a Let $P(A) = P(B) = P(C) = P(D) = x$. Then $P(E) = 2x$.	Summarise the information in the question in terms of one unknown.
$x + x + x + x + 2x = 1$ $6x = 1$ $x = \frac{1}{6}$	The sum of the probabilities is 1.
Thus $P(E) = 2x = \frac{1}{3}$	
b $P(B') = 1 - P(B) = 1 - \frac{1}{6} = \frac{5}{6}$	Since B' is the complement of B, the probabilities will add to 1.

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Summary 9A

- The **sample space**, ε, for a random experiment is the set of all possible outcomes.
- An **event** is a subset of the sample space. The probability of an event A occurring is denoted by $\mathrm{P}(A)$.
- Rules of probability for finite sample spaces:
  - $\mathrm{P}(A) \geq 0$, for each event A.
  - The sum of the probabilities of all the outcomes of a random experiment must be equal to 1.
- Equally likely outcomes If the sample space ε for an experiment contains n outcomes, all of which are equally likely to occur, we assign a probability of $\frac{1}{n}$ to each outcome. Then the probability of an event A is given by

$$\mathrm{P}(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}} = \frac{n(A)}{n(\varepsilon)}$$

- If two events have no elements in common and together they make up the entire sample space, they are said to be **complementary events**. The complement of any event A is denoted A' and we can write

$$\mathrm{P}(A') = 1 - \mathrm{P}(A)$$

Exercise 9A

Example 1

1 List the sample space for the toss of a coin.

2 List the sample space for the outcomes when a die is rolled.

3 Answer the following for a normal deck of playing cards:

a How many cards are there?
b How many suits are there?
c What are the suits called?
d Which suits are red and which suits are black?
e How many cards are there in each suit?
f Which cards are known as the 'picture cards'?
g How many aces are there in the deck?
h How many 'picture cards' are there in the deck?

4 List the sample spaces for the following experiments:

a the number of picture cards in a hand of five cards
b the number of female children in a family with six children
c the number of female students on a committee of three students chosen from a class of 10 male and 10 female students

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5 List the sample spaces for the following experiments:

a the number of cars which pass through a particular intersection in a day

b the number of people on board a bus licensed to carry 40 passengers

c the number of times a die is rolled before a six is observed

6 List the outcomes associated with the following events:

a 'an even number' when a die is rolled

b 'more than two female students' when three students are chosen for a committee from a class of 10 male and 10 female students

c 'more than four aces' when five cards are dealt from a standard pack of 52 cards

Example 2

7 A number is drawn at random from the set $\{1, 2, 3, \dots, 20\}$. What is the probability that the number is:

a divisible by 2

b divisible by 3

c divisible by both 2 and 3?

8 A bag has 15 marbles numbered $1, 2, 3, \dots, 15$. If one marble is drawn at random from the bag, what is the probability that the number on the marble is:

a less than 5

b greater than or equal to 6

c a number from 5 to 8 inclusive?

Example 3

9 A card is drawn at random from a well-shuffled pack of 52 cards. Find the probability that the card is:

a a club

b red

c a picture card (ace, king, queen, jack)

d a red picture card.

10 A card is drawn at random from a well-shuffled pack of 52 cards. Find the probability that the card is:

a less than 10

b less than or equal to 10

c an even number

d an ace.

11 Suppose that in a certain city the same number of people were born on each of the 365 days of the year, and that nobody was born on 29 February. Find the probability that the birthday of a person selected at random:

a is 29 November

b is in November

c falls between 15 January and 15 February, not including either day

d is in the first three months of the year.

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12 One letter is drawn at random from the letters in the word AUSTRALIA. Find the probability that the letter is:

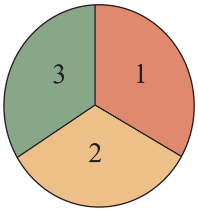
a a T **b** an A **c** a vowel **d** a consonant.

Example 4 **13** A random experiment results in 1, 2, 3, 4, 5 or 6. If $P(1) = \frac{1}{12}$, $P(2) = \frac{1}{6}$, $P(3) = \frac{1}{8}$, $P(5) = \frac{1}{6}$ and $P(6) = \frac{1}{8}$, find the probability of obtaining a 4.

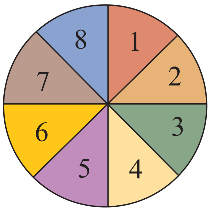
14 A random experiment results in 1, 2, 3 or 4. If $P(1) = 0.2$, $P(3) = 0.1$ and $P(4) = 0.3$, find $P(2)$.

Example 5 **15** Consider the following spinners. In each case, what is the chance of the pointer stopping in region 1?

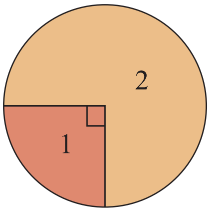
a



b



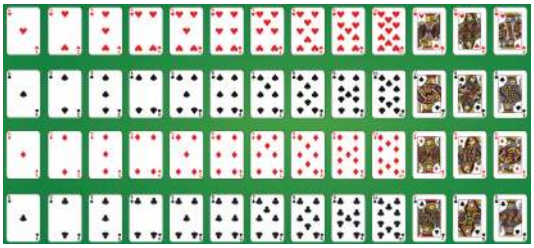
c



Example 6 **16** Assume that the probability of a baby being born on a certain day is the same for each day of the week. Find the probability that a randomly chosen person was born:

a on a Wednesday **b** not on the weekend.

17 A card is drawn at random from a well-shuffled pack of 52 cards. Find the probability that the card is:



a not a club
b not red
c not a picture card
d not a red picture card.

Example 7 **18** A random experiment results in 1, 2, 3 or 4. If 1, 2 and 3 are equally likely to occur, and 4 is twice as likely to occur as 3, find the probability of each of the possible outcomes.

19 For a particular biased six-sided die it is known that the numbers 2, 3, 4 and 5 are equally likely to occur, that the number 6 occurs twice as often as the number 2, and that the number 1 occurs half as often as the number 2.

a Find the probability of each of the possible outcomes.
b Find the probability that the number observed is not a 6.

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9B Estimating probabilities

When we are dealing with a random experiment which does not have equally likely outcomes, other methods of estimating probability are required.

Subjective probabilities

Sometimes, the probability is assigned a value just on the basis of experience. For example, a sports journalist may suggest that Australia has a 60% chance of winning the next Ashes series relying on his or her own judgement. Another journalist might well assign this probability an entirely different value. Such probabilities are called subjective probabilities, and whether or not they are accurate estimates of the true probability would be open to dispute.

Probabilities from data

A better way to estimate an unknown probability is by experimentation: by performing the random experiment leading to the event of interest many times and recording the results. This information can then be used to estimate the chances of it happening again in the future. The proportion of trials that resulted in this event is called the **relative frequency** of the event. (For most purposes we can consider proportion and relative frequency as interchangeable.) That is,

$$\text{Relative frequency of event } A = \frac{\text{number of times event } A \text{ occurs}}{\text{number of trials}}$$

Suppose, for example, that we are interested in determining the probability that a drawing pin will land 'point up' when it is tossed. Since a drawing pin is not symmetrical, the assumption of equally likely outcomes cannot be used to determine probabilities.

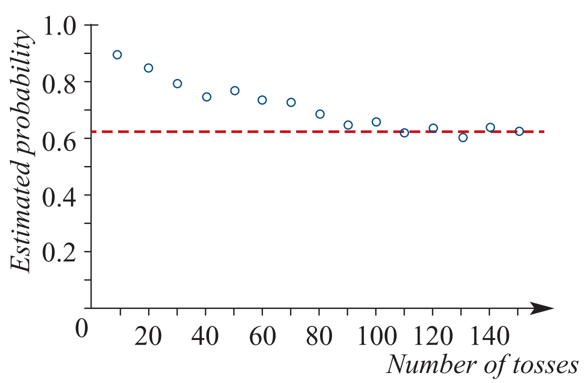
Our strategy to estimate this probability is to toss the drawing pin many times and count the number of times it lands point up. We can then calculate the relative frequency:

$$\text{Relative frequency of 'point up'} = \frac{\text{number of times drawing pin landed 'point up'}}{\text{number of trials}}$$

This proportion, or relative frequency, is an estimate of the probability of a drawing pin landing with the point up.

The graph opposite shows the results of one experiment where a drawing pin is tossed 150 times, with the probability of the drawing pin landing point up estimated every 10 throws.

From the graph it may be seen that, as the number of trials (repetitions of the experiment) increases, the estimated probability converges to a value and then stays fairly stable.



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In general, if the same experiment is repeated many, many times, the relative frequency of any particular event will stabilise to a constant value. This limiting value of the relative frequency is then considered to be the probability of the event.

When the number of trials is sufficiently large, the observed relative frequency of an event A becomes close to the probability $\mathrm{P}(A)$. That is,

$$\mathrm{P}(A) \approx \frac{\text{number of times event } A \text{ occurs}}{\text{number of trials}} \qquad \text{for a large number of trials}$$

If the experiment was repeated, it would generally be found that the results were slightly different. One might conclude that relative frequency is not a very good way of estimating probability. In many situations, however, experiments are the only way to get at an unknown probability. One of the most valuable lessons to be learnt is that such estimates are not exact, and will in fact vary from sample to sample.

Understanding the variation between estimates is extremely important in the study of statistics, and will be the topic of your later studies in mathematics. At this stage it is valuable to realise that the variation does exist, and that the best estimates of the probabilities will result from using as many trials as possible.

Example 8

In order to investigate the probability that a drawing pin lands point up, Katia decides to toss it 50 times and to count the number of favourable outcomes, which turns out to be 33. Mikki repeats the experiment, but she tosses the same drawing pin 100 times and counts 62 favourable outcomes.

a What is Katia's estimate of the probability of the drawing pin landing point up?

b What is Mikki's estimate?

c Which of these is the preferred estimate of the probability from these experiments?

d Based on the information available, what would be the preferred estimate of the probability?

Solution

a From Katia's information: $\mathrm{P}(\text{point up}) \approx \dfrac{33}{50} = 0.66$

b From Mikki's information: $\mathrm{P}(\text{point up}) \approx \dfrac{62}{100} = 0.62$

c Since Mikki has estimated the probability from a larger number of trials, her estimate would be preferred to Katia's.

d Based on the information available, the preferred estimate of the probability would be found by combining the data from both experiments, and so maximising the number of trials. In total, 95 favourable outcomes were observed in 150 tosses, and this gives a 'best' estimate of the probability of $\dfrac{95}{150} = 0.63$.

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Thus, probability can be considered as the proportion of times that an event will occur in the long run. This interpretation also defines the minimum and maximum values of probability as 0 (the event never occurs) and 1 (the event always occurs), and confirms that the sum of the probabilities for all possible outcomes will equal 1.

Simulation

The word simulate means to pretend or to imitate. In statistics, simulation is a way to model a random experiment, such that simulated outcomes closely match real-world outcomes. Simulation does not involve repeating the actual experiment. Instead, more complex probabilities can be estimated via multiple trials of an experiment which approximates the actual experiment, but can be carried out quickly and easily. A more detailed discussion of simulation is found in Section 9H.

Probabilities from area

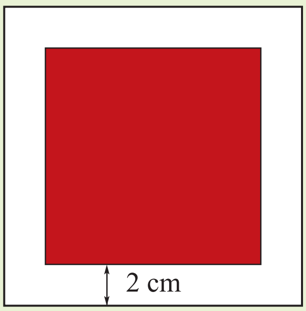
In the previous section we used the model of equally likely outcomes to determine probabilities. We counted both the outcomes in the event and the outcomes in the sample space, and used the ratio to determine the probability of the event.

This idea can be extended to calculate probabilities when areas are involved, by assuming that the probabilities of all points in the region (which can be considered to be the sample space) are equally likely.

Example 9

Suppose that a square dartboard consists of a red square drawn inside a larger white square of side length 12 cm, as shown.

If a dart thrown at the board has equal chance of landing anywhere on the board, what is the probability it lands in the red area? (Ignore the possibility that it might land on the line or miss the board altogether!)



Solution

$$P(\text{landing in the red area}) = \frac{\text{area of red square}}{\text{area of dartboard}}$$

$$= \frac{64}{144}$$

$$= \frac{4}{9}$$

Explanation

There are really only two outcomes for this experiment: landing in the red area or landing in the white area.

Assume that the probability of landing in an area is proportional to the size of the area.

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Summary 9B

- When a probability is unknown, it can be estimated by the relative frequency obtained through repeated trials of the random experiment under consideration. In this case,

$$\mathrm{P}(A) \approx \frac{\text{number of times event } A \text{ occurs}}{\text{number of trials}} \qquad \text{for a large number of trials}$$

- Whichever method of estimating probability is used, the rules of probability hold:
  - $\mathrm{P}(A) \geq 0$, for each event A
  - $\mathrm{P}(\varepsilon) = 1$
  - The sum of the probabilities of all the outcomes of a random experiment equals 1.

Skill-sheet

Exercise 9B

Example 8

1 Estimate the probability of the event specified occurring, using the data given:

a P(head) if a coin is tossed 100 times and 34 heads observed

b P(ten) if a spinner is spun 200 times and lands on the 'ten' 20 times

c P(two heads) if two coins are tossed 150 times and two heads are observed on 40 occasions

d P(three sixes) if three dice are rolled 200 times and three sixes observed only once

2 A student decides to toss two coins and notes the results.

a Do you think relative frequencies obtained from 20 trials would make for a good estimate of the probabilities?

b Perform the experiment 20 times and estimate P(two heads), P(one head) and P(no heads).

c Combine your results with those of your friends, so that you have results from at least 100 trials. Use these results to again estimate the probabilities.

d Do you think the data from 100 trials give better estimates of the probabilities?

e How many trials would you need to find the probabilities exactly?

3 Two misshapen six-sided dice were used for the following experiment. The first die was thrown 500 times and 78 sixes were observed. The second die was thrown 700 times and 102 sixes were observed. If you wished to throw a six, which die would you choose to throw, and why?

4 A bowl contains 340 red and 60 black balls.

a State the proportion of red balls in the bowl.

b A random sample of 60 balls is taken from the bowl and is found to have 48 red balls. Find the proportion of red balls in the sample.

c Another random sample of 60 balls is taken from the bowl and is found to have 54 red balls. Find the proportion of red balls in the sample.

d What is the expected number of red balls in a sample of 60?

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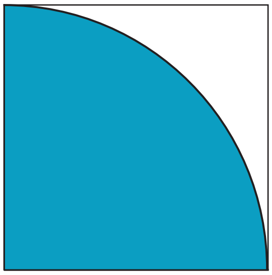
5 In a survey of 2000 people, 890 indicated that they regularly use social media to keep in touch with friends. What is an estimate for the probability that the next person surveyed also uses social media?

Example 9

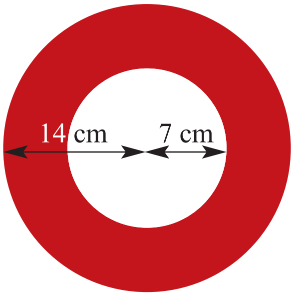
6 A square of side length 1 metre contains a blue one-quarter of a circular disc centred at the bottom-left vertex of the square, as shown.



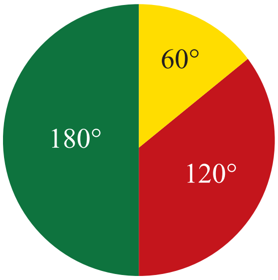
a What proportion of the square is blue?

b If a dart thrown at the square is equally likely to hit any part of the square, and it hits the square every time, find the probability of it hitting the blue region.

7 A dart is thrown at random onto a board that has the shape of a circle as shown. Calculate the probability that the dart will hit the shaded region.



8 A spinner is as shown in the diagram.
Find the probability that when spun the pointer will land on:

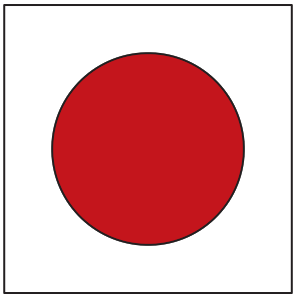


a the red section

b the yellow section

c any section except the yellow section.

9 In a sideshow at a fete, a dart is thrown at a square with side length 1 metre. The circle shown has a radius of 0.4 metres. The dart is equally likely to hit any point on the square. Find the probability that the dart will hit:



a the shaded part of the square

b the unshaded part of the square.

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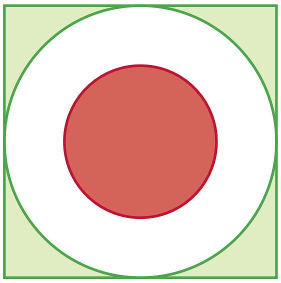
10 A dart is thrown at random onto the board shown. The board is a square of side length x, the larger circle is of radius $\frac{x}{2}$ and the smaller circle is of radius $\frac{x}{4}$.



a Find, in terms of x:

i the area of the square

ii the area of the larger circle

iii the area of the smaller circle.

b Hence find the probability that the dart will land:

i inside the smaller circle

ii in the white region

iii in the outer shaded region.

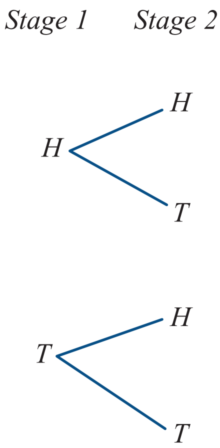
9C Multi-stage experiments

Often we are interested in probabilities which arise from more complex **multi-stage experiments**. That is, they are concerned with experiments which could be considered to take place in more than one stage.

For example, when considering the outcomes from tossing two coins (or tossing one coin twice) we should consider the possible outcomes in two stages:

- the outcome from coin 1
- followed by the outcome from coin 2.

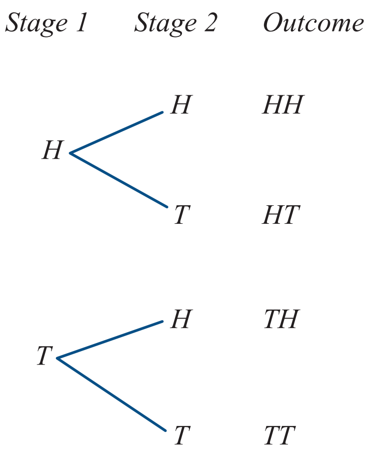
In such cases, it is helpful to list the elements of the sample space systematically by means of a **tree diagram** as shown.



Each path along the branches of the tree gives an outcome, which we determine by reading along the branches, making sure we maintain the order of the outcome at each stage carefully.

Collecting together all the outcomes, we see that the sample space is

$$\varepsilon = \{HH, HT, TH, TT\}$$



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When the outcomes of the multi-stage experiment are equally likely, we can still determine the probability of an event occurring by dividing the number of outcomes in the event by the number of outcomes in the sample space.

Example 10

Find the probability that when a fair coin is tossed twice:

a one head is observed

b at least one head is observed

c both heads or both tails are observed.

Solution

a P(one head) $= \frac{2}{4} = \frac{1}{2}$

Explanation

List the outcomes of the event:
'one head' $= \{HT, TH\}$.
There are 2 outcomes in the event and 4 in the sample space (see tree diagram).

b P(at least one head) $= \frac{3}{4}$

List the outcomes of the event:
'at least one head' $= \{HH, HT, TH\}$.
There are 3 outcomes in the event and 4 in the sample space.

c P(both heads or both tails) $= \frac{2}{4} = \frac{1}{2}$

List the outcomes of the event:
'both heads or both tails' $= \{HH, TT\}$.
There are 2 outcomes in the event and 4 in the sample space.

When listing the outcomes for a two-stage experiment, it can also be convenient to display the sample space in a table. For example, when rolling two dice (or a single die twice) there is the possibility of $\{1, 2, 3, 4, 5, 6\}$ on die 1 (or the first roll), and $\{1, 2, 3, 4, 5, 6\}$ on die 2 (or the second roll). So the sample space for this experiment can be written as:

		Die 2					
		1	2	3	4	5	6
Die 1	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

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Example 11

Find the probability that when two fair dice are rolled:

a the same number shows on both dice (a double)

b the sum of the two numbers shown is greater than 10.

Solution

a $P(\text{double}) = \frac{6}{36} = \frac{1}{6}$

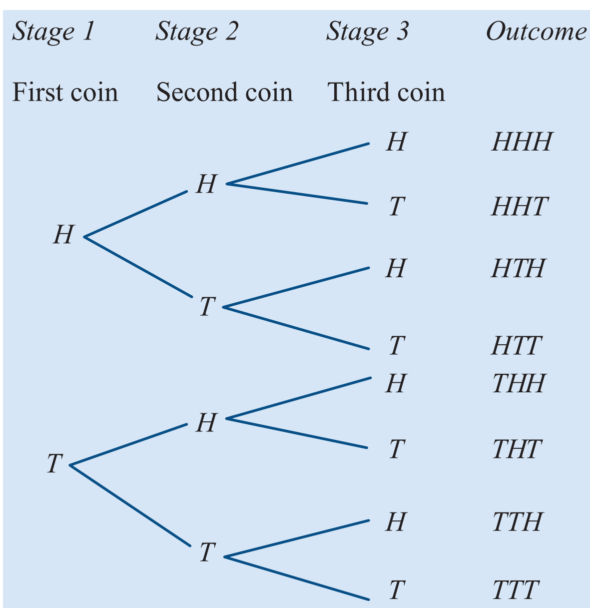
Explanation

'double' = {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)}.
There are 6 outcomes in the event and 36 in the sample space.

b $P(\text{sum} > 10) = \frac{3}{36} = \frac{1}{12}$

'sum is greater than 10' = {(5, 6), (6, 5), (6, 6)}.
There are 3 outcomes in the event and 36 in the sample space.

When the experiment involves more than two stages, it is best to use a **tree diagram** to determine all of the possible outcomes. Suppose, for example, that three coins are tossed and the outcomes noted. The three-stage tree diagram for listing the sample space for this experiment is as follows:



Thus the required sample space is

$$\varepsilon = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

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Example 12

Find the probability that when a coin is tossed three times:

a one head is observed
b at least one head is observed
c the second toss results in a head
d all heads or all tails are observed.

Solution	Explanation
a P(one head) $= \frac{3}{8}$	'one head' $= \{HTT, THT, TTH\}$. There are 3 outcomes in the event and 8 in the sample space.
b P(at least one head) $= \frac{7}{8}$	'at least one head' $= \{HHH, HHT, HTH, THH, HTT, THT, TTH\}$. There are 7 outcomes in the event and 8 in the sample space.
c P(second toss is a head) $= \frac{4}{8} = \frac{1}{2}$	'second toss is a head' $= \{HHH, HHT, THH, THT\}$. There are 4 outcomes in the event and 8 in the sample space.
d P(all heads or all tails) $= \frac{2}{8} = \frac{1}{4}$	'all heads or all tails' $= \{HHH, TTT\}$. There are 2 outcomes in the event and 8 in the sample space.

Summary 9C

The sample space for a two-stage experiment can be displayed using a tree diagram or a table. If an experiment involves more than two stages, then a tree diagram should be used.

Exercise 9C

Example 10

1 Two fair coins are tossed. Use the sample space for this experiment to find the probability of observing:

a no heads **b** more than one tail.

2 A fair coin is tossed twice. Find the probability that:

a the first toss is a head **b** the second toss is a head **c** both tosses are heads.

Example 11

3 Two regular dice are rolled. Use the sample space for this experiment to find the probability that the sum of the numbers showing is:

a even **b** 3 **c** less than 6.

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4 Two regular dice are rolled. Use the sample space for this experiment to find the probability that the sum of the numbers showing is:

a equal to 10 **b** odd **c** less than or equal to 7.

Example 12

5 A fair coin is tossed three times. Use the sample space for this experiment to find the probability that:

a exactly one tail is observed **b** exactly two tails are observed

c exactly three tails are observed **d** no tails are observed.

6 A fair coin is tossed three times. Use the sample space for this experiment to find the probability that:

a the third toss is a head

b the second and third tosses are heads

c at least one head and one tail are observed.

7 An experiment consists of rolling a die and tossing a coin. Use a tree diagram to list the sample space for the experiment. Find the probability of obtaining a head and an even number.

8 Two coins are tossed and a die is rolled.

a Draw a tree diagram to show all the possible outcomes.

b Find the probability of observing:

i two heads and a 6

ii one head, one tail and an even number

iii two tails and an odd number

iv an odd number on the die.

9 Madison has a choice of two entrees (soup or salad), three main courses (fish, chicken or steak) and three desserts (ice-cream, lemon tart or cheese).

a Draw a tree diagram to show all her possible dinner combinations.

b If Madison chooses all three courses, and is equally likely to choose any of the options at each course, find the probability that:

i she chooses soup, fish and lemon tart

ii she chooses fish

iii she chooses salad and chicken

iv she doesn't have the lemon tart.

c Suppose Madison has the choice to omit the entree and/or the dessert course altogether. Find the probability that:

i she chooses soup, fish and lemon tart

ii she chooses all three courses

iii she chooses only two courses

iv she has only the main course.

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10 A bag contains five balls, numbered 1 to 5. A ball is chosen at random, the number noted and the ball replaced. A second ball is then chosen at random and its number noted.

a Draw up a table of ordered pairs to show the sample space for the experiment.

b Find the probability that:

i the sum of the two numbers is 5

ii the two numbers are different

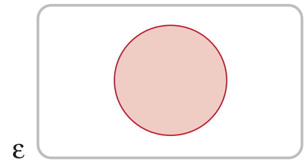
iii the second number is two more than the first.

9D Combining events

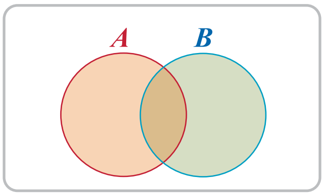
Before proceeding with the discussion of probability, a review of sets and set notation is necessary.

The **empty set**, denoted by $\varnothing$, is the set consisting of no elements. This is different from $\{0\}$, which is a set containing one element, 0.

Sets, and the relationships between sets, can be illustrated clearly by using **Venn diagrams**. The universal set ε is usually shown as a rectangle, and a subset of ε as a circle.



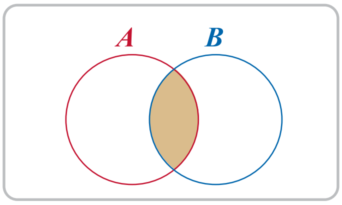
If A and B are any two sets, then the **union** of A and B, denoted $A \cup B$, is the set of all elements in A or B (or both). This is shown on a Venn diagram by shading both sets A and B.



Union

For example, if A is the set of students in a school who play hockey, and B the set of students who play tennis, then the union of A and B is the set of students who play either hockey or tennis or both.

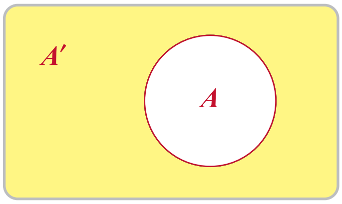
The **intersection** of A and B, denoted $A \cap B$, is the set of elements that are in both A and B. This is shown on a Venn diagram by shading only the area contained in both A and B.



Intersection

For example, the intersection of the two sets previously described is the set of students who play both hockey and tennis.

As previously, note that the **complement** of A, denoted A', is the set of all elements of ε that are not in A. This is shown on a Venn diagram by shading only the area outside A.



Complement

The complement of the set of students who play hockey in a school is the set of students who do not play hockey.

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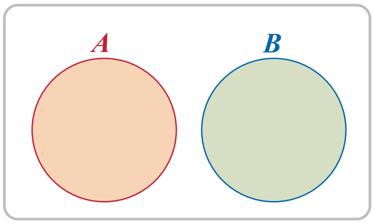
Two sets A and B are said to be **disjoint** or **mutually exclusive** if they have no elements in common, that is, if $A \cap B = \varnothing$. The Venn diagram opposite shows two sets that are mutually exclusive.

If A is the set of girls who play hockey in a school and B is the set of boys who play hockey, then A and B are mutually exclusive, as no student can belong to both sets.



Disjoint sets

Finally, the number of elements in a set A is usually denoted $n(A)$. For example, if $A = \{2, 4, 6\}$, then $n(A) = 3$.

Venn diagrams can be used to help us solve practical problems involving sets.

Example 13

Fifty students were asked what they did on the weekends. A total of 35 said they went to football matches, the movies or both. Of the 22 who went to football matches, 12 said they also went to the movies. Show this information on a Venn diagram.

a How many students went to the movies but not to football matches?

b How many went neither to football matches nor to the movies?

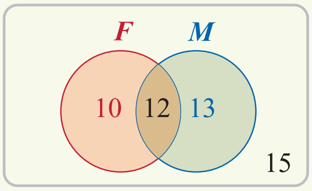
Solution

Let F denote the set of students who attend football matches and M denote the set of students who attend movies.

Hence, from the information given, $n(F \cup M) = 35$, $n(F) = 22$ and $n(F \cap M) = 12$.

a Students who go to the movies but not to football matches are found in the region $F' \cap M$, and from the diagram $n(F' \cap M) = 13$.

b Those who attend neither are found in the region $F' \cap M'$, and from the diagram $n(F' \cap M') = 15$.



$n = 50$

Example 14

Consider Example 13. What is the probability that a student chosen at random from this group of 50:

a went to the movies but not to football matches

b went neither to football matches nor to the movies?

Solution

a $\mathrm{P}(F' \cap M) = \dfrac{n(F' \cap M)}{n(\varepsilon)} = \dfrac{13}{50}$

b $\mathrm{P}(F' \cap M') = \dfrac{n(F' \cap M')}{n(\varepsilon)} = \dfrac{15}{50} = \dfrac{3}{10}$

Explanation

To determine the probability of these events, divide by the size of the sample space in each case.

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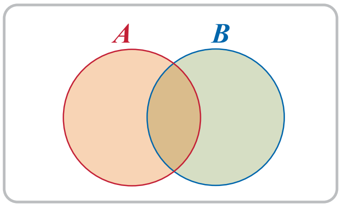
The addition rule

Venn diagrams can be used to illustrate a very important rule that will enable us to calculate probabilities for more complex events. If A and B are two events in a sample space ε and $A \cap B \neq \emptyset$, then the relationship between them can be represented by a Venn diagram, as shown.



From the Venn diagram we can see that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

(As the intersection has been counted twice, in both $n(A)$ and $n(B)$, we must subtract it.)

Dividing through by $n(\varepsilon)$ gives

$$\frac{n(A \cup B)}{n(\varepsilon)} = \frac{n(A)}{n(\varepsilon)} + \frac{n(B)}{n(\varepsilon)} - \frac{n(A \cap B)}{n(\varepsilon)}$$

Now, if each of the outcomes in ε is equally likely to occur, then each term in this expression is equal to the probability of that event occurring. This can be rewritten as:

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

So the probability of A or B or both occurring can be calculated using

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

This is called the **addition rule** for combining probabilities. This rule also applies in situations where the outcomes are not equally likely; it is always true.

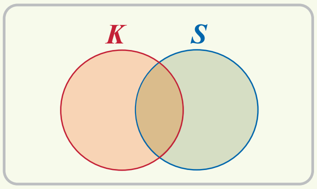
This rule can be used to help solve more complex problems in probability.

Example 15

If one card is chosen at random from a well-shuffled deck, what is the probability that the card is a king or a spade?

Solution

Let event K be 'a king'. Then K = {king of spades, king of hearts, king of diamonds, king of clubs} and $n(K) = 4$.



Let event S be 'a spade'. Then S = {ace of spades, king of spades, queen of spades, . . . } and $n(S) = 13$.

The event 'a king or a spade' corresponds to the union of sets K and S. We have

$$\mathrm{P}(K) = \frac{4}{52}, \qquad \mathrm{P}(S) = \frac{13}{52}, \qquad \mathrm{P}(K \cap S) = \frac{1}{52}$$

and so, using the addition rule, we find

$$\mathrm{P}(K \cup S) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = 0.3077 \quad \text{(correct to four decimal places)}$$

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Summary 9D

- Venn diagrams are often useful for solving problems involving sets.
- For any two events A and B, the **addition rule** can be applied:

  $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- If the two events A and B are mutually exclusive, then $P(A \cap B) = 0$ and therefore $P(A \cup B) = P(A) + P(B)$.

Exercise 9D

1 $\varepsilon = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6\}$.
Show these sets on a Venn diagram and use your diagram to find:

a $A \cup B$ **b** $A \cap B$ **c** A'
d $A \cap B'$ **e** $(A \cap B)'$ **f** $(A \cup B)'$

2 $\varepsilon = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, A = {multiples of four}, B = {even numbers}.
Show these sets on a Venn diagram and use your diagram to find:

a A' **b** B' **c** $A \cup B$
d $(A \cup B)'$ **e** $A' \cap B'$

3 ε = {different letters of the word MATHEMATICS}
A = {different letters of the word ATTIC}
B = {different letters of the word TASTE}
Show ε, A and B on a Venn diagram, entering all the elements. Hence list the sets:

a A' **b** B' **c** $A \cup B$
d $(A \cup B)'$ **e** $A' \cup B'$ **f** $A' \cap B'$

Example 13

4 In a survey of 100 university students, a market research company found that 70 students owned smartphones, 45 owned cars and 35 owned smartphones and cars. Use a Venn diagram to help you answer the following questions:

a How many students owned neither a car nor a smartphone?
b How many students owned either a car or a smartphone, but not both?

Example 14

5 Let $\varepsilon = \{1, 2, 3, 4, 5, 6\}$, where the outcomes are equally likely. If $A = \{2, 4, 6\}$ and $B = \{3\}$, find:

a $P(A \cup B)$ **b** $P(A \cap B)$ **c** $P(A')$ **d** $P(B')$

Example 15

6 Let $\varepsilon = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, where the outcomes are equally likely. If A is the event 'an even number' and B is the event 'a multiple of three', find:

a $P(A)$ **b** $P(B)$ **c** $P(A \cap B)$ and hence $P(A \cup B)$

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7 In Sam's class at school many of the students are on the swimming team. There are 18 students in the class. Seven of the students swim freestyle and four swim backstroke. Two students swim both freestyle and backstroke. Draw a Venn diagram and use it to find the probability that a student randomly chosen from Sam's class:

a swims freestyle **b** swims backstroke

c swims freestyle and backstroke **d** is on the swimming team.

8 Suppose that A is the set of factors of 12, and B is the set of prime numbers less than 10. If a number is chosen at random from the integers from 1 to 20, draw a Venn diagram and use it to find:

a $\text{P}(A)$ **b** $\text{P}(B)$ **c** $\text{P}(A \cap B)$ **d** $\text{P}(A \cup B)$

9 Suppose $\text{P}(A) = 0.5$, $\text{P}(B) = 0.4$ and $\text{P}(A \cap B) = 0.2$. Find $\text{P}(A \cup B)$.

10 Suppose $\text{P}(A) = 0.35$, $\text{P}(B) = 0.24$ and $\text{P}(A \cap B) = 0.12$. Find $\text{P}(A \cup B)$.

11 Suppose $\text{P}(A) = 0.28$, $\text{P}(B) = 0.45$ and $A \subseteq B$. Find:

a $\text{P}(A \cap B)$ **b** $\text{P}(A \cup B)$

12 Suppose $\text{P}(A) = 0.58$, $\text{P}(B) = 0.45$ and $B \subseteq A$. Find:

a $\text{P}(A \cap B)$ **b** $\text{P}(A \cup B)$

13 Suppose $\text{P}(A) = 0.3$, $\text{P}(B) = 0.4$ and $A \cap B = \varnothing$. Find:

a $\text{P}(A \cap B)$ **b** $\text{P}(A \cup B)$

14 Suppose $\text{P}(A) = 0.08$, $\text{P}(B) = 0.15$, and A and B are disjoint. Find:

a $\text{P}(A \cap B)$ **b** $\text{P}(A \cup B)$

15 Suppose $\text{P}(A) = 0.3$, $\text{P}(B) = 0.4$ and $\text{P}(A \cup B) = 0.5$. Find $\text{P}(A \cap B)$.

16 Suppose $\text{P}(A) = 0.24$, $\text{P}(B) = 0.44$ and $\text{P}(A \cup B) = 0.63$. Find $\text{P}(A \cap B)$.

17 Suppose $\text{P}(A) = 0.3$, $\text{P}(B) = 0.4$ and $\text{P}(A \cap B') = 0.2$. Find $\text{P}(A \cup B')$.

18 Suppose that in a certain school the probability that a student plays soccer is 0.18, the probability that a student plays tennis is 0.25, and probability that a student plays both soccer and tennis is 0.11. Find the probability that a student plays either or both of these sports.

19 Suppose that in a certain school the probability that a student studies Chinese is 0.22, the probability that a student studies French is 0.35, and the probability that a student studies both languages is 0.14.

a Find the probability that a student in that school studies at least one of these languages.

b Find the probability that a student in that school studies exactly one of these languages.

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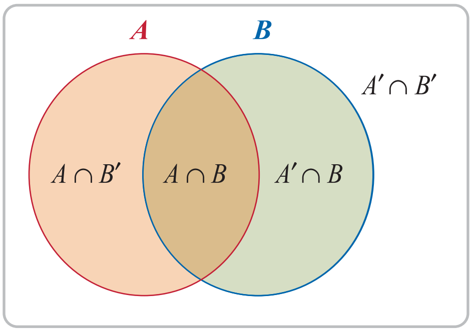
9E Probability tables

A **probability table** is an alternative to a Venn diagram when illustrating a probability problem diagrammatically. Consider the Venn diagram which illustrates two intersecting sets A and B.



From the Venn diagram it can be seen that the sample space is divided by the sets into four disjoint regions: $A \cap B$, $A \cap B'$, $A' \cap B$ and $A' \cap B'$. These regions may be represented in a table as follows. Such a table is sometimes referred to as a **Karnaugh map**.

	B	B'
A	$A \cap B$	$A \cap B'$
A'	$A' \cap B$	$A' \cap B'$

In a probability table, the entries give the probabilities of each of these events occurring.

		Column 1	Column 2
		B	B'
Row 1	A	$\Pr(A \cap B)$	$\Pr(A \cap B')$
Row 2	A'	$\Pr(A' \cap B)$	$\Pr(A' \cap B')$

Further, from the Venn diagram we can see that set A is the union of the part of set A that intersects with set B and the part of set A that does not intersect with set B. That is,

$$A = (A \cap B) \cup (A \cap B')$$

The sets $A \cap B$ and $A \cap B'$ are mutually exclusive, so

$$\Pr(A \cap B) + \Pr(A \cap B') = \Pr(A) \qquad \text{(row 1)}$$

and thus summing the probabilities in row 1 gives $\Pr(A)$. Similarly:

$$\Pr(A' \cap B) + \Pr(A' \cap B') = \Pr(A') \qquad \text{(row 2)}$$

$$\Pr(A \cap B) + \Pr(A' \cap B) = \Pr(B) \qquad \text{(column 1)}$$

$$\Pr(A \cap B') + \Pr(A' \cap B') = \Pr(B') \qquad \text{(column 2)}$$

Finally, since $\Pr(A) + \Pr(A') = 1$ and $\Pr(B) + \Pr(B') = 1$, the totals for both column 3 and row 3 are equal to 1. Thus, the completed table becomes:

		Column 1	Column 2	Column 3
		B	B'	
Row 1	A	$\Pr(A \cap B)$	$\Pr(A \cap B')$	$\Pr(A)$
Row 2	A'	$\Pr(A' \cap B)$	$\Pr(A' \cap B')$	$\Pr(A')$
Row 3		$\Pr(B)$	$\Pr(B')$	1

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These tables can be useful when solving problems involving probability, as shown in the next two examples.

Example 16

If A and B are events such that $P(A) = 0.7$, $P(A \cap B) = 0.4$ and $P(A' \cap B) = 0.2$, find:

a $P(A \cap B')$ **b** $P(B)$ **c** $P(A' \cap B')$ **d** $P(A \cup B)$

Solution

		Column 1	Column 2	Column 3
		B	B'	
Row 1	A	$P(A \cap B) = 0.4$	$P(A \cap B')$	$P(A) = 0.7$
Row 2	A'	$P(A' \cap B) = 0.2$	$P(A' \cap B')$	$P(A')$
Row 3		$P(B)$	$P(B')$	1

The given information has been entered in the table in red.

a From row 1: $P(A \cap B') = P(A) - P(A \cap B) = 0.7 - 0.4 = 0.3$

b From column 1: $P(B) = P(A \cap B) + P(A' \cap B) = 0.4 + 0.2 = 0.6$

c From column 3: $P(A') = 1 - P(A) = 1 - 0.7 = 0.3$
From row 2: $P(A' \cap B') = 0.3 - 0.2 = 0.1$

d Using the addition rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= 0.7 + 0.6 - 0.4$$

$$= 0.9$$

The completed table is shown below.

		Column 1	Column 2	Column 3
		B	B'	
Row 1	A	$P(A \cap B) = 0.4$	$P(A \cap B') = 0.3$	$P(A) = 0.7$
Row 2	A'	$P(A' \cap B) = 0.2$	$P(A' \cap B') = 0.1$	$P(A') = 0.3$
Row 3		$P(B) = 0.6$	$P(B') = 0.4$	1

Example 17

Records indicate that, in Australia, 65% of secondary students participate in sport, and 71% of secondary students are Australian by birth. They also show that 53% of students are Australian by birth and participate in sport. Use this information to find:

a the probability that a student selected at random is not Australian by birth

b the probability that a student selected at random is not Australian by birth and does not participate in sport.

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Solution

The information in the question may be entered into a table as shown. We use A to represent 'Australian by birth' and S to represent 'participates in sport'.

	S	S'	
A	0.53		0.71
A'			
	0.65		1

All the empty cells in the table may now be filled in by subtraction.

In column 1: $P(A' \cap S) = 0.65 - 0.53 = 0.12$

In column 3: $P(A') = 1 - 0.71 = 0.29$

In row 1: $P(A \cap S') = 0.71 - 0.53 = 0.18$

In row 3: $P(S') = 1 - 0.65 = 0.35$

In row 2: $P(A' \cap S') = 0.29 - 0.12 = 0.17$

	S	S'	
A	0.53	0.18	0.71
A'	0.12	0.17	0.29
	0.65	0.35	1

a The probability that a student selected at random is not Australian by birth is given by $P(A') = 0.29$.

b The probability that a student selected at random is not Australian by birth and does not participate in sport is given by $P(A' \cap S') = 0.17$.

Skill-sheet

Exercise 9E

Example 16

1 If A and B are events such that $P(A) = 0.6$, $P(A \cap B) = 0.4$ and $P(A' \cap B) = 0.1$, find:

a $P(A \cap B')$ **b** $P(B)$ **c** $P(A' \cap B')$ **d** $P(A \cup B)$

2 If A and B are events such that $P(A') = 0.25$, $P(A' \cap B) = 0.12$ and $P(B) = 0.52$, find:

a $P(A)$ **b** $P(A \cap B)$ **c** $P(A \cup B)$ **d** $P(B')$

3 If C and D are events such that $P(C \cup D) = 0.85$, $P(C) = 0.45$ and $P(D') = 0.37$, find:

a $P(D)$ **b** $P(C \cap D)$ **c** $P(C \cap D')$ **d** $P(C' \cup D')$

4 If E and F are events such that $P(E \cup F) = 0.7$, $P(E \cap F) = 0.15$ and $P(E') = 0.55$, find:

a $P(E)$ **b** $P(F)$ **c** $P(E' \cap F)$ **d** $P(E' \cup F)$

5 If A and B are events such that $P(A) = 0.8$, $P(B) = 0.7$ and $P(A' \cap B') = 0.1$, find:

a $P(A \cup B)$ **b** $P(A \cap B)$ **c** $P(A' \cap B)$ **d** $P(A \cup B')$

Example 17

6 In a recent survey of senior citizens, it was found that 85% favoured giving greater powers of arrest to police, 60% favoured longer sentences for convicted persons, and 50% favoured both propositions.

a What percentage favoured at least one of the two propositions?

b What percentage favoured neither proposition?

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7 Suppose a card is selected at random from an ordinary deck of 52 playing cards.
Let A = event a picture card is selected (i.e. jack, queen, king or ace)
C = event a heart is selected

a List the outcomes corresponding to events A and C.

b Determine the following probabilities and express your results in words:

i $\Pr(A)$ **ii** $\Pr(C)$ **iii** $\Pr(A \cap C)$ **iv** $\Pr(A \cup C)$ **v** $\Pr(A \cup C')$

8 The following information applies to a particular class:

- The probability that a student's name begins with M and the student studies French is $\frac{1}{6}$.
- The probability that a student's name begins with M is $\frac{3}{10}$.
- The probability that a student does not study French is $\frac{7}{15}$.

Find the probability that a student chosen at random from this class:

a studies French

b has a name which does not begin with M

c has a name which does begin with M, but does not study French

d has a name which does not begin with M and does not study French.

9 A frame is chosen at random from a shop where picture frames are sold. It is known that in this shop:

- the probability that the frame is made of wood is 0.72
- the probability that the frame is freestanding is 0.65
- the probability that the frame is not made of wood and is not freestanding is 0.2.

Find the probability that the randomly chosen frame:

a is made of wood or is freestanding

b is made of wood and is freestanding

c is not made of wood

d is not made of wood but is freestanding.

10 A book is chosen at random from a bookshop. It is known that in this bookshop:

- the probability that the book is a hardback but not a novel is 0.05
- the probability that the book is not hardback but is a novel is 0.12
- the probability that the book is not a novel is 0.19.

Find the probability that the randomly chosen book is:

a a novel

b a hardback novel

c a hardback

d a novel or a hardback.

11 At a school camp consisting of 60 students, sailing was offered as an activity one morning, and bushwalking in the afternoon. Every student attended at least one activity. If 32 students went sailing and 40 students went bushwalking, find the probability that a student chosen at random:

a undertook neither of these activities

b has sailed or bushwalked

c has sailed and bushwalked

d has sailed but not bushwalked.

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12 At a barbecue attended by 50 people, hamburgers and sausages were available. It was found that 35 hamburgers and 38 sausages were eaten, and six people were noted to have eaten neither a hamburger nor a sausage. If no person ate more than one hamburger or one sausage, find the probability that a person chosen at random ate:

a a hamburger or a sausage
b a hamburger and a sausage
c only one serve of food
d only a hamburger.

9F Conditional probability

We are often interested in calculating the probability of one event in the light of whether another event has or has not already occurred. For example, consider tossing a coin twice. What is the probability that the second toss shows a head, if we know that the first toss shows a head? Is the probability the same as if the first toss was a tail?

Suppose that we define event A as 'the second toss is a head', and event B as 'the first toss is a head'. Then the probability that the second toss shows a head, given that the first toss shows a head, is written $\mathrm{P}(A \mid B)$ and is an example of conditional probability.

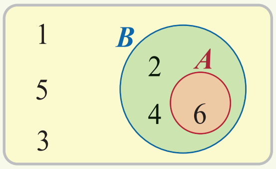
The probability of an event A occurring when it is known that some event B has occurred is called **conditional probability** and is written $\mathrm{P}(A \mid B)$. This is usually read as 'the probability of A given B', and can be thought of as a means of adjusting probability in the light of new information.

Example 18

Suppose we roll a fair die and define event A as 'rolling a six' and event B as 'rolling an even number'. What is the probability of rolling a six given the information that an even number was rolled?

Solution

The events A and B can be shown on a Venn diagram.



We know that event B has already occurred so we know that the outcome was 2, 4 or 6. Thus

P(six is rolled given an even number is rolled)

$$= \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$$

$$= \frac{n(A)}{n(B)}$$

$$= \frac{1}{3}$$

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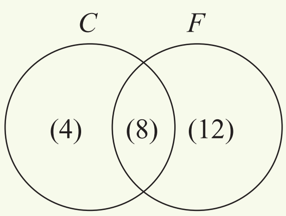
Example 19

In Stephen's class 12 students study Chinese, 20 study French, and 8 study both Chinese and French.

a Given that a student in his class studies Chinese (C), what is the probability that they also study French (F)?

b Given that a student in his class studies French, what is the probability that they also study Chinese?

Solution



a $\mathrm{P}(F \mid C) = \dfrac{8}{12} = \dfrac{2}{3}$

b $\mathrm{P}(C \mid F) = \dfrac{8}{20} = \dfrac{2}{5}$

Explanation

Display the information in the question in a Venn diagram. The numbers in brackets indicate the number of elements in each region.

If we know that the student studies Chinese, the sample space is restricted to those 12 students. From the Venn diagram we can see that 8 of these students also study French.

If we know that the student studies French, the sample space is restricted to those 20 students. From the Venn diagram we can see that 8 of these students also study Chinese.

This example clearly demonstrates that, in general, $\mathrm{P}(A \mid B) \neq \mathrm{P}(B \mid A)$. So care needs to be taken when determining conditional probabilities.

Conditional probabilities can also be calculated from a table, as shown in Example 20.

Example 20

500 people were questioned and classified according to age and whether or not they regularly use social media. The results are shown in the table.

Do you regularly use social media?

	Age < 25	Age ≥ 25	Total
Yes	200	100	300
No	40	160	200
Total	240	260	500

One person is selected at random from these 500. Given that the selected person is less than 25 years of age, what is the probability that they regularly use social media?

Solution

$\mathrm{P}(\text{Yes} \mid \text{Age} < 25) = \dfrac{200}{240} = \dfrac{5}{6}$

Explanation

If we know the person is less than 25 years old, then the sample space is restricted to those 240 people. Of these, 200 regularly use social media.

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Note that, in Example 20,

$$\Pr(\text{Yes} \cap \text{Age} < 25) = \frac{200}{500} \qquad \text{and} \qquad \Pr(\text{Age} < 25) = \frac{240}{500}$$

Hence we have

$$\frac{\Pr(\text{Yes} \cap \text{Age} < 25)}{\Pr(\text{Age} < 25)} = \frac{\frac{200}{500}}{\frac{240}{500}} = \frac{200}{240} = \frac{5}{6}$$

which is equal to the conditional probability $\Pr(\text{Yes} \mid \text{Age} < 25)$.

This illustrates a general principle which is always true.

The conditional probability of an event A, given that event B has already occurred, is given by

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} \qquad \text{if } \Pr(B) \neq 0$$

This formula may be rearranged to give the **multiplication rule of probability**:

$$\Pr(A \cap B) = \Pr(A \mid B) \times \Pr(B)$$

Example 21

Given that for two events A and B, $\Pr(A) = 0.7$, $\Pr(B) = 0.3$ and $\Pr(B \mid A) = 0.4$, find:

a $\Pr(A \cap B)$

b $\Pr(A \mid B)$

Solution

a $\Pr(A \cap B) = \Pr(B \mid A) \times \Pr(A)$

$= 0.4 \times 0.7 = 0.28$

b $\Pr(A \mid B) = \dfrac{\Pr(A \cap B)}{\Pr(B)} = \dfrac{0.28}{0.3} = \dfrac{14}{15}$

Example 22

In a particular school 55% of the students are male and 45% are female. Of the male students 13% say mathematics is their favourite subject, while of the female students 18% prefer mathematics. Find the probability that:

a a student chosen at random prefers mathematics and is female

b a student chosen at random prefers mathematics and is male.

Solution

Let us use M to represent male, F for female, and P for prefers mathematics. Then

$$\Pr(M) = 0.55, \quad \Pr(F) = 0.45, \quad \Pr(P \mid M) = 0.13, \quad \Pr(P \mid F) = 0.18$$

We can use the multiplication rule to find the required probabilities:

a The event 'prefers mathematics and is female' is represented by $P \cap F$, with

$$\Pr(P \cap F) = \Pr(P \mid F) \times \Pr(F) = 0.18 \times 0.45 = 0.081$$

b The event 'prefers mathematics and is male' is represented by $P \cap M$, with

$$\Pr(P \cap M) = \Pr(P \mid M) \times \Pr(M) = 0.13 \times 0.55 = 0.0715$$

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The law of total probability

As has already been seen, the tree diagram is an efficient way of listing a multi-stage sample space. If the probabilities associated with each stage are also added to the tree diagram, it becomes a very useful way of calculating the probability for each outcome. The probabilities at each stage are conditional probabilities that the particular path will be followed and the multiplication rule says that the probability of reaching the end of a given branch is the product of the probabilities associated with each segment of that branch.

Example 23

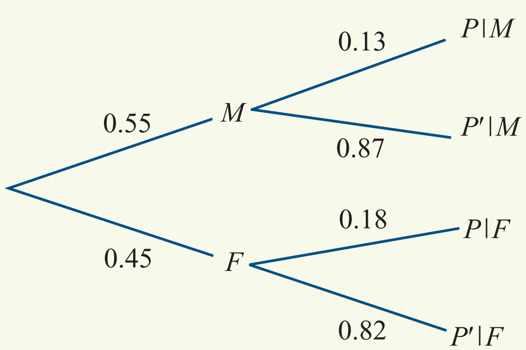
Using the information from Example 22, construct a tree diagram and use it to determine:

a the probability that a student selected is female and does not prefer mathematics

b the overall percentage of students who prefer mathematics.

Solution

The situation described can be represented by a tree diagram as follows:



$P(P \cap M) = 0.55 \times 0.13 = 0.0715$

$P(P' \cap M) = 0.55 \times 0.87 = 0.4785$

$P(P \cap F) = 0.45 \times 0.18 = 0.081$

$P(P' \cap F) = 0.45 \times 0.82 = 0.369$

a To find the probability that a student is female and does not prefer mathematics we multiply along the appropriate branches thus:

$$P(F \cap P') = P(F) \times P(P' \mid F) = 0.45 \times 0.82 = 0.369$$

b Now, to find the overall percentage of students who prefer mathematics we recall that:

$$P = (P \cap F) \cup (P \cap M)$$

Since $P \cap F$ and $P \cap M$ are mutually exclusive,

$$P(P) = P(P \cap F) + P(P \cap M) = 0.081 + 0.0715 = 0.1525$$

Thus 15.25% of all students prefer mathematics.

The solution to part b of Example 23 is an application of a rule known as the law of total probability. This can be expressed in general terms as follows:

The **law of total probability** states that, in the case of two events A and B,

$$P(A) = P(A \mid B)P(B) + P(A \mid B')P(B')$$

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A further example of the use of the law of total probability is given in the following example.

Example 24

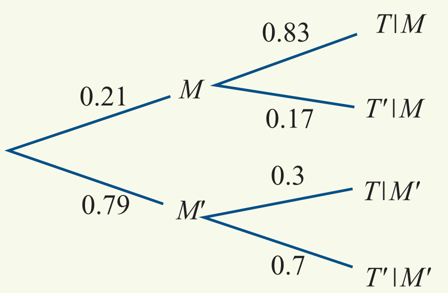
In a certain town, the probability that it rains on any Monday is 0.21. If it rains on Monday, then the probability that it rains on Tuesday is 0.83. If it does not rain on Monday, then the probability of rain on Tuesday is 0.3. For a given week, find the probability that it rains:

a on both Monday and Tuesday **b** on Tuesday.

Solution

Let M represent the event 'rain on Monday' and T represent the event 'rain on Tuesday'.

The situation described in the question can be represented by a tree diagram. You can check that the probabilities are correct by seeing if they add to 1.



$\mathrm{P}(T \cap M) = 0.21 \times 0.83 = 0.1743$

$\mathrm{P}(T' \cap M) = 0.21 \times 0.17 = 0.0357$

$\mathrm{P}(T \cap M') = 0.79 \times 0.3 = 0.237$

$\mathrm{P}(T' \cap M') = 0.79 \times 0.7 = 0.553$

a The probability that it rains on both Monday and Tuesday is given by

$$\mathrm{P}(T \cap M) = 0.21 \times 0.83 = 0.1743$$

b The probability that it rains on Tuesday is given by

$$\mathrm{P}(T) = \mathrm{P}(T \cap M) + \mathrm{P}(T \cap M') = 0.1743 + 0.237 = 0.4113$$

Summary 9F

- The probability of an event A occurring when it is known that some event B has already occurred is called conditional probability and is written $\mathrm{P}(A \mid B)$.
- In general, the **conditional probability** of an event A, given that event B has already occurred, is given by

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} \qquad \text{if } \mathrm{P}(B) \neq 0$$

  This formula may be rearranged to give the **multiplication rule of probability**:

$$\mathrm{P}(A \cap B) = \mathrm{P}(A \mid B) \times \mathrm{P}(B)$$

- The **law of total probability** states that, in the case of two events A and B,

$$\mathrm{P}(A) = \mathrm{P}(A \mid B)\mathrm{P}(B) + \mathrm{P}(A \mid B')\mathrm{P}(B')$$

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Skill-sheet

Exercise 9F

Example 18

1 Suppose that a fair die is rolled, and event A is defined as 'rolling a six' and event B as 'rolling a number greater than 2'. Find $\mathrm{P}(A \mid B)$.

2 Suppose that a fair die is rolled, and event A is defined as 'rolling a three' and event B as 'rolling an odd number'. Draw a Venn diagram and use it to find $\mathrm{P}(A \mid B)$.

Example 19

3 Suppose that a card is drawn from a pack of 52 cards, and event A is defined as 'selecting an ace' and event B as 'selecting a club'. Draw a Venn diagram and use it to find the probability that the card drawn is an ace, given that it is a club.

4 In Sam's class 12 students play violin, 12 play piano and 7 play both. Draw a Venn diagram and use it to find the probability that a randomly chosen student plays the violin given that they play the piano.

5 Two dice are rolled and the outcomes observed. Given that the dice both show the same number, what is the probability that it's a 'double six'?

6 In Annabelle's class 17 students own an iPhone, 7 own an iPad, and 4 own both.

a What is the probability that a student owns an iPad, given that they own an iPhone?

b What is the probability that a student owns an iPhone, given that they own an iPad?

Example 20

7 100 people were questioned and classified according to sex and whether or not they think private individuals should be allowed to carry guns. The results are shown in the table.

Do you think private individuals should be allowed to carry guns?

	Male	Female	Total
Yes	35	30	65
No	25	10	35
Total	60	40	100

One person is selected at random from these 100. Given that the selected person is male, what is the probability that they think private individuals should be allowed to carry guns?

8 A group of 500 students were asked whether they would rather spend their recreational time playing sport or listening to music. The results, as well as the sex of the student, are given in the table.

	Male	Female	Total
Sport	225	150	375
Music	75	50	125
Total	300	200	500

One student is selected at random from these 500. Find:

a the probability that the student prefers sport

b the probability that the student prefers sport, given that they are male.

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9 The following data was derived from accident records on a highway noted for its above-average accident rate.

Type of accident	Probable cause: Speed	Alcohol	Reckless driving	Other	Total
Fatal	42	61	22	12	137
Non-fatal	88	185	98	60	431
Total	130	246	120	72	568

Use the table to estimate:

a the probability that speed is the cause of the accident

b the probability that the accident is fatal

c the probability that the accident is fatal, given that speed is the cause

d the probability that the accident is fatal, given that alcohol is the cause.

Example 21 **10** Given that for two events A and B, $\mathrm{P}(A) = 0.6$, $\mathrm{P}(B) = 0.3$ and $\mathrm{P}(B \mid A) = 0.1$, find:

a $\mathrm{P}(A \cap B)$ **b** $\mathrm{P}(A \mid B)$

11 For events A and B:

a if $\mathrm{P}(A) = 0.7$ and $\mathrm{P}(A \cap B) = 0.4$, find $\mathrm{P}(B \mid A)$

b if $\mathrm{P}(A \mid B) = 0.6$ and $\mathrm{P}(B) = 0.5$, find $\mathrm{P}(A \cap B)$

c if $\mathrm{P}(A \mid B) = 0.44$ and $\mathrm{P}(A \cap B) = 0.3$, find $\mathrm{P}(B)$.

12 In a random experiment $\mathrm{P}(A) = 0.5$, $\mathrm{P}(B) = 0.4$ and $\mathrm{P}(A \cup B) = 0.7$. Find:

a $\mathrm{P}(A \cap B)$ **b** $\mathrm{P}(A \mid B)$ **c** $\mathrm{P}(B \mid A)$

13 In a random experiment $\mathrm{P}(A) = 0.6$, $\mathrm{P}(B) = 0.54$ and $\mathrm{P}(A \cap B') = 0.4$. Find:

a $\mathrm{P}(A \cap B)$ **b** $\mathrm{P}(A \mid B)$ **c** $\mathrm{P}(B \mid A)$

14 In a random experiment $\mathrm{P}(A) = 0.4$, $\mathrm{P}(A \mid B) = 0.6$ and $\mathrm{P}(B) = 0.5$. Find:

a $\mathrm{P}(A \cap B)$ **b** $\mathrm{P}(B \mid A)$

Example 22 **15** The current football fixture has the local team playing at home for 60% of its matches. When it plays at home, the team wins 80% of the time. When it plays away, the team wins only 40% of the time. What percentage of its games does the team play away and win?

16 The probability that a car will need an oil change is 0.15, the probability that it needs a new oil filter is 0.08, and the probability that both the oil and the filter need changing is 0.03. Given that the oil has to be changed, what is the probability that a new oil filter is also needed?

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17 A card is selected from a pack of 52 playing cards. The card is replaced and a second card chosen. Find the probability that:

a both cards are hearts
b both cards are aces
c the first card is red and the second is black
d both cards are picture cards.

18 A card is selected from a pack of 52 playing cards and **not** replaced. Then a second card is chosen. Find the probability that:

a both cards are hearts
b both cards are aces
c the first card is red and the second is black
d both cards are picture cards.

19 A person is chosen at random from the employees of a large company. Let W be the event that the person chosen is a woman, and let A be the event that the person chosen is 25 years or older. Suppose the probability of selecting a woman is $\text{P}(W) = 0.652$ and the probability of a woman being 25 years or older is $\text{P}(A \mid W) = 0.354$. Find the probability that a randomly chosen employee is a woman aged 25 years or older.

20 In a class of 28 students there are 15 girls. Of the students in the class, six girls and eight boys play basketball. A student is chosen at random from the class. If G represents the event that a girl student is chosen and B represents the event that the student chosen plays basketball, find:

a $\text{P}(G)$ **b** $\text{P}(B)$ **c** $\text{P}(B')$ **d** $\text{P}(B \mid G)$
e $\text{P}(G \mid B)$ **f** $\text{P}(B \mid G')$ **g** $\text{P}(B' \cap G')$ **h** $\text{P}(B \cap G)$

21 In a recent survey it was found that 85% of the population eats red meat. Of those who eat red meat, 60% preferred lamb. A person is chosen at random from the population. If R represents the event that the person eats red meat and L represents the event that the person prefers lamb, find:

a $\text{P}(R)$ **b** $\text{P}(L \mid R)$ **c** $\text{P}(L \cap R)$ **d** $\text{P}(L)$

Example 23 **22** In a senior college, 25% of the Year 11 students and 40% of the Year 12 students would prefer not to wear school uniform. This particular college has 320 Year 11 students and 280 Year 12 students. Find the probability that a randomly chosen student is in Year 11 and is in favour of wearing school uniform. What is the overall percentage of students who are in favour of wearing school uniform?

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23 At a certain school it was found that 35% of the 500 boys and 40% of the 400 girls enjoyed bushwalking. One student from the school is chosen at random. Let G represent the event that the student is a girl, and B represent the event that the student enjoys bushwalking.

a Find, correct to two decimal places:

i $\mathrm{P}(G)$ **ii** $\mathrm{P}(B \mid G)$ **iii** $\mathrm{P}(B \mid G')$

iv $\mathrm{P}(B \cap G)$ **v** $\mathrm{P}(B \cap G')$

b Find $\mathrm{P}(B)$.

c Hence find:

i $\mathrm{P}(G \mid B)$ **ii** $\mathrm{P}(G \mid B')$

24 In a factory two machines produce a particular circuit board. The older machine produces 480 boards each day, of which an average of 12% are defective. The newer machine produces 620 boards each day, of which an average of 5% are defective.

A board is chosen at random and checked. Let N represent the event that the board comes from the newer machine, and D represent the event that the board is defective.

a Find, correct to two decimal places:

i $\mathrm{P}(N)$ **ii** $\mathrm{P}(D \mid N)$ **iii** $\mathrm{P}(D \mid N')$

iv $\mathrm{P}(D \cap N)$ **v** $\mathrm{P}(D \cap N')$

b Find $\mathrm{P}(D)$.

c Hence find $\mathrm{P}(N \mid D)$, correct to two decimal places.

25 Jane has three bags of lollies. In bag 1 there are three mints and three toffees, in bag 2 there are three mints and two toffees, and in bag 3 there are two mints and one toffee. Jane selects a bag at random, and then selects a lolly at random. Find:

a the probability she chooses a mint from bag 1

b the probability she chooses a mint

c the probability that Jane chose bag 1, given that she selects a mint.

26 Assuming a finite sample space, describe the relationship between events A and B if:

a $\mathrm{P}(A \mid B) = 1$ **b** $\mathrm{P}(A \mid B) = 0$ **c** $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A)}{\mathrm{P}(B)}$

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9G Independent events

Let us again consider the question of the probability that the second toss shows a head, given that the first toss shows a head, when tossing a coin twice. If we define A as the event 'the second toss is a head' and B as the event 'the first toss is a head', then what is $\mathrm{P}(A \mid B)$?

Using the definition of conditional probability:

$$\begin{aligned}\mathrm{P}(A \mid B) &= \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} \\ &= \frac{\mathrm{P}(\text{both tosses show heads})}{\mathrm{P}(\text{first toss shows a head})} \\ &= \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}\end{aligned}$$

That is, the probability of the second toss showing a head does not seem to be affected by the outcome of the first toss. This is an example of independent events.

Two events A and B are **independent** if the occurrence of one event has no effect on the probability of the occurrence of the other, that is, if

$$\mathrm{P}(A \mid B) = \mathrm{P}(A)$$

If $\mathrm{P}(B) \neq 0$, then the multiplication rule of probability gives

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$$

Thus, when events A and B are independent, we can equate the two expressions for $\mathrm{P}(A \mid B)$ to obtain

$$\mathrm{P}(A) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$$

and therefore

$$\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$$

In fact, we can also use this final equation as a test for independence:

Events A and B are independent if and only if

$$\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$$

Notes:

- For events A and B with $\mathrm{P}(A) \neq 0$ and $\mathrm{P}(B) \neq 0$, the following three conditions are all equivalent conditions for the independence of A and B:
  - $\mathrm{P}(A \mid B) = \mathrm{P}(A)$
  - $\mathrm{P}(B \mid A) = \mathrm{P}(B)$
  - $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$
- In the special case that $\mathrm{P}(A) = 0$ or $\mathrm{P}(B) = 0$, the condition $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$ holds since both sides are zero, and so we say that A and B are independent.
- Sometimes this definition of independence is referred to as **pairwise independence**.

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Example 25

500 people were questioned and classified according to age and whether or not they regularly use social media. The results are shown in the table.

Is the regular use of social media independent of the respondent's age?

Do you regularly use social media?

	Age < 25	Age ≥ 25	Total
Yes	200	100	300
No	40	160	200
Total	240	260	500

Solution

From the table:

$$\text{P(Age} < 25 \cap \text{Yes)} = \frac{200}{500} = 0.4$$

$$\text{P(Age} < 25) \times \text{P(Yes)} = \frac{240}{500} \times \frac{300}{500} = 0.48 \times 0.6 = 0.288$$

Hence

$$\text{P(Age} < 25 \cap \text{Yes)} \neq \text{P(Age} < 25) \times \text{P(Yes)}$$

and therefore these events are not independent.

Example 26

An experiment consists of drawing a number at random from $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Let $A = \{1, 2, 3, 4\}$, $B = \{1, 3, 5, 7\}$ and $C = \{4, 6, 8\}$.

a Are A and B independent?

b Are A and C independent?

c Are B and C independent?

Solution

$\text{P}(A) = \frac{1}{2}, \ \text{P}(B) = \frac{1}{2}, \ \text{P}(C) = \frac{3}{8}$

a $\text{P}(A \cap B) = \frac{1}{4}$ — since $A \cap B = \{1, 3\}$

$\text{P}(A) \times \text{P}(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$\therefore \ \text{P}(A) \times \text{P}(B) = \text{P}(A \cap B)$

Thus A and B are independent — since these two probabilities are equal.

b $\text{P}(A \cap C) = \frac{1}{8}$ — since $A \cap C = \{4\}$

$\text{P}(A) \times \text{P}(C) = \frac{1}{2} \times \frac{3}{8} = \frac{3}{16}$

$\therefore \ \text{P}(A) \times \text{P}(C) \neq \text{P}(A \cap C)$

Thus A and C are not independent — since these two probabilities are not equal.

c $\text{P}(B \cap C) = 0$ — since $B \cap C = \varnothing$

$\text{P}(B) \times \text{P}(C) = \frac{1}{2} \times \frac{3}{8} = \frac{3}{16}$

$\therefore \ \text{P}(B) \times \text{P}(C) \neq \text{P}(B \cap C)$

Thus B and C are not independent — since these two probabilities are not equal.

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The concept of mathematical independence is sometimes confused with that of physical independence. If two events are physically independent, then they are also mathematically independent, but the converse is not necessarily true. The following example illustrates this.

Example 27

Suppose we roll a die twice and define the following events:

A = the first roll shows a 4

B = the second roll shows a 4

C = the sum of the numbers showing is at least 10

a Are A and B independent events?

b What about A and C?

Solution

a Since A and B are physically independent, they must also be mathematically independent, but we can also check this directly.

We have

$$P(A) \times P(B) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

If we write the sample space as ordered pairs, in which the first entry is the result of the first throw and the second is the result of the second throw, then

$$\varepsilon = \{(1,1), (1,2), (1,3), \dots, (6,5), (6,6)\}$$

and $\quad n(\varepsilon) = 36$

The event $A \cap B$ corresponds to the outcome $(4, 4)$, and so $n(A \cap B) = 1$.

Thus

$$P(A \cap B) = \frac{1}{36} = P(A) \times P(B)$$

and so A and B are independent.

b We have

$$C = \{(4,6), (5,5), (5,6), (6,4), (6,5), (6,6)\}$$

so $\quad n(C) = 6$

Thus

$$P(A) \times P(C) = \frac{1}{6} \times \frac{6}{36} = \frac{1}{36}$$

The event $A \cap C$ corresponds to the outcome $(4, 6)$, and so $n(A \cap C) = 1$.

Thus

$$P(A \cap C) = \frac{1}{36} = P(A) \times P(C)$$

This means that A and C are also independent events.

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Knowing that events are independent means that we can determine the probability of their intersection by multiplying together their individual probabilities. This is illustrated in the following example.

Example 28

Suppose that the probability that a family in a certain town owns a television set (T) is 0.75, and the probability that a family owns a station wagon (S) is 0.25. If these events are independent, find the following probabilities:

a A family chosen at random owns both a television set and a station wagon.

b A family chosen at random owns at least one of these items.

Solution

a The event 'owns both a television set and a station wagon' is represented by $T \cap S$, with

$$\begin{aligned} P(T \cap S) &= P(T) \times P(S) && \text{(as } T \text{ and } S \text{ are independent)} \\ &= 0.75 \times 0.25 = 0.1875 \end{aligned}$$

b The event 'owns at least one of these items' is represented by $T \cup S$, with

$$\begin{aligned} P(T \cup S) &= P(T) + P(S) - P(T \cap S) && \text{(from the addition rule)} \\ &= 0.75 + 0.25 - 0.75 \times 0.25 && \text{(as } T \text{ and } S \text{ are independent)} \\ &= 0.8125 \end{aligned}$$

Confusion often arises between independent and mutually exclusive events. That two events A and B are mutually exclusive means that $A \cap B = \varnothing$ and hence that $P(A \cap B) = 0$. Thus, if two events are independent, they cannot also be mutually exclusive, unless the probability of at least one of the events is zero.

Summary 9G

- The probability of an event A occurring when it is known that some event B has already occurred is called **conditional probability** and is written $P(A \mid B)$, where

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \qquad \text{if } P(B) \neq 0$$

- Two events A and B are **independent** if the occurrence of one event has no effect on the probability of the occurrence of the other, that is, if

$$P(A \mid B) = P(A)$$

- Events A and B are independent if and only if

$$P(A \cap B) = P(A) \times P(B)$$

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Exercise 9G

Example 25

1 100 people were questioned and classified according to sex and whether or not they think private individuals should be allowed to carry guns. The results are shown in the table.

Do you think private individuals should be allowed to carry guns?

	Male	Female	Total
Yes	35	30	65
No	25	10	35
Total	60	40	100

Is support for private individuals carrying guns independent of sex?

2 A group of 500 students were asked whether they would rather spend their recreational time playing sport or listening to music. The results, as well as the sex of the student, are given in the following table.

	Male	Female	Total
Sport	225	150	375
Music	75	50	125
Total	300	200	500

Is preference for playing sport or listening to music independent of sex?

3 An analysis of traffic accidents in a certain city classified the accident as serious or minor, as well as whether the driver was speeding or not.

Type of accident	Speeding		
	Yes	No	Total
Serious	42	61	103
Minor	88	185	273
Total	130	246	376

Is the seriousness of the accident independent of whether the driver was speeding or not?

Example 26

4 An experiment consists of drawing a number at random from $\{1, 2, 3, \dots, 12\}$. Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{1, 3, 5, 7, 9, 11\}$ and $C = \{4, 6, 8, 9\}$.

a Are A and B independent?

b Are A and C independent?

c Are B and C independent?

Example 27

5 A die is thrown and the number uppermost is recorded. Events A and B are defined as 'an even number' and 'a square number' respectively. Show that A and B are independent.

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6 Two events A and B are such that $P(A) = 0.3$, $P(B) = 0.1$ and $P(A \cap B) = 0.1$. Are A and B independent?

7 If A and B are independent events with $P(A) = 0.6$ and $P(B) = 0.7$, find:

a $P(A \mid B)$ **b** $P(A \cap B)$ **c** $P(A \cup B)$

8 If A and B are independent events with $P(A) = 0.5$ and $P(B) = 0.2$, find $P(A \cup B)$.

Example 28

9 A man and a woman decide to marry. Assume that the probability that each will have a specific blood group is as follows:

Blood group	O	A	B	AB
Probability	0.5	0.35	0.1	0.05

If the blood group of the husband is independent of that of his wife, find the probability that:

a the husband is group A **b** the husband is group A and his wife is group B

c both are group A **d** the wife is group AB and her husband is group O.

10 The 165 subjects volunteering for a medical study are classified by sex and blood pressure (high (H), normal (N) and low (L)).

	H	N	L
M	88	22	10
F	11	22	12
Total	99	44	22

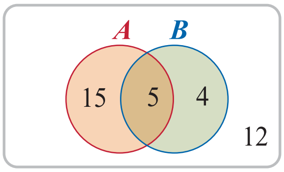
If a subject is selected at random, find:

a $P(N)$ **b** $P(F \cap H)$ **c** $P(F \cup H)$

d $P(F \mid L)$ **e** $P(L \mid F)$

Are F and L independent? Explain.

11 Events A and B are as shown in the Venn diagram. Show that A and B are independent.



12 The probability that a married woman watches a certain television show is 0.4, and the probability that her husband watches the show is 0.5. The television viewing habits of a husband and wife are clearly not independent. In fact, the probability that a married woman watches the show, given that her husband does, is 0.7. Find the probability that:

a both the husband and wife watch the show

b the husband watches the show given that his wife watches it.

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13 The 65 middle managers in a company are classified by age and income as follows:

	Age		
Income	30–39 (T)	40–49 (F)	50–69 (S)
Low (L)	13	4	1
Moderate (M)	8	10	3
High (H)	2	16	8
Total	23	30	12

A middle manager is selected at random from the company. Find:

a $\text{P}(L)$ **b** $\text{P}(S)$ **c** $\text{P}(T)$ **d** $\text{P}(M)$
e $\text{P}(L \cap F)$ **f** $\text{P}(T \cap M)$ **g** $\text{P}(L \mid F)$ **h** $\text{P}(T \mid M)$

Is income independent of age? Explain your answer.

14 A consumer research organisation has studied the services provided by the 150 TV repair persons in a certain city and their findings are summarised in the following table.

	Good service (G)	Poor service (G')
Factory trained (F)	48	16
Not factory trained (F')	24	62

a One of the TV repairers is randomly selected. Calculate the following probabilities:
  i $\text{P}(G \mid F)$, the probability that a factory-trained repairer is one who gives good service
  ii $\text{P}(G \cap F)$, the probability that the repairer is giving good service and is factory trained
  iii $\text{P}(G \cup F)$, the probability that the repairer is giving good service or is factory trained or both

b Are events G and F independent?

c Are the events G and F mutually exclusive?

9H Solving probability problems using simulation

Simulation is a very powerful and widely used procedure which enables us to find approximate answers to difficult probability questions. It is a technique which imitates the operation of the real-world system being investigated. Some problems are not able to be solved directly and simulation allows a solution to be obtained where otherwise none would be possible. In this section some specific probability problems are looked at which may be solved by using simulation, a valuable and legitimate tool for the statistician.

Suppose, for example, we would like to know the probability that a family of five children will include at least four girls. There are ways of determining this probability exactly (which will be covered in Year 12), but we don't know how to do this yet. What we can do, however, is estimate the probability by simulation.

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Firstly, we need to make some assumptions so we can decide on a suitable **model** for the simulation. We will assume:

- There is a probability of 0.5 of each child being female.
- The sex of each child is independent of the sex of the other children. That is, the probability of a female child is always 0.5.

Since the probability of a female child is 0.5, a suitable simulation model would be tossing a fair coin. Let a head represent a female child and a tail a male child. A trial consists of tossing the coin five times to represent one complete family of five children, and the result of the trial is the number of heads obtained in the trial.

To estimate the required probability, several trials need to be conducted. How many trials are needed to estimate the probability? As we have already noted in Section 9B, the more repetitions of an experiment the better the estimate of the probability. Initially about 50 trials could be considered.

An example of the results that might be obtained from 10 trials is:

Trial number	Simulation results	Number of heads
1	$T\,H\,H\,T\,T$	2
2	$H\,H\,H\,T\,H$	4
3	$H\,H\,H\,T\,H$	4
4	$H\,T\,T\,T\,H$	2
5	$H\,T\,H\,H\,H$	4
6	$H\,T\,T\,T\,H$	2
7	$T\,T\,H\,H\,H$	3
8	$H\,T\,H\,H\,T$	3
9	$T\,T\,T\,H\,H$	2
10	$H\,H\,T\,T\,T$	2

Continuing in this way, the following results were obtained for 50 trials:

Number of heads	0	1	2	3	4	5
Number of times obtained	1	8	17	13	10	1

The results in the table can be used to estimate the required probability. Since at least four heads were obtained in 11 trials, estimate the probability of at least four female children as $\frac{11}{50}$ or 0.22. Of course, since this probability has been estimated experimentally, repeating the simulations would give a slightly different result, but we would expect to obtain approximately this value most of the time.

Simulation is also widely used to estimate the values of other quantities which are of interest in a probability problem. We may wish to know the average result, the largest result, the number of trials required to achieve a certain result, and so on. An example of this type of problem is given in Example 29.

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Example 29

A pizza shop is giving away football cards with each pizza bought. There are six different cards available, and a fan decides to continue buying the pizzas until all six are obtained. How many pizzas will need to be bought, on average, to obtain the complete set of cards?

Solution

As there are more than two outcomes of interest, a coin is not a suitable simulation model, but a fair six-sided die could be used. Each of the six different cards is represented by one of the six sides of the die. Rolling the die and observing the outcome is equivalent to buying a pizza and noting which card was obtained. This simulation model is based on the following assumptions:

- The six cards all occur with equal frequency.
- The card obtained with one pizza is independent of the cards obtained with the other pizzas.

A trial would consist of rolling the die until all of the six numbers 1, 2, 3, 4, 5 and 6 have been observed, and the result of the trial is the number of rolls necessary to do this. The results of one trial are shown:

5 2 5 2 2 2 3 3 1 2 6 3 5 4

In this instance, 14 pizzas were bought before the whole set was obtained. Of course, we would not expect to buy 14 pizzas every time – this is just the result from one trial. To obtain an appropriate estimate, we would need to conduct several trials.

The following is an example of the results that might be obtained from 50 trials. In each case the number listed represents the number of pizzas that were bought to obtain a complete set of football cards:

14	8	12	11	16	8	8	11	15	26	14	20	11	13	35
23	19	14	10	10	20	9	10	14	29	13	7	15	15	22
9	10	14	16	14	17	12	10	24	13	19	27	31	11	9
16	21	22	8	9										

To estimate the number of pizzas that need to be bought, the average of the numbers obtained in these simulations is calculated. Thus we estimate that, in order to collect the complete set of cards, it would be necessary to buy approximately

$$\frac{14 + 8 + 12 + 11 + 16 + \cdots + 16 + 21 + 22 + 8 + 9}{50} \approx 15 \text{ pizzas}$$

In practice there are situations where coins and dice may not be useful. Other methods of simulation need to be adopted to deal with a wide range of situations. Suppose we wished to determine how many pizzas would need to be bought, on average, to obtain a complete set of eight souvenirs. This time we need to generate random numbers from 1 to 8 and a six-sided die would no longer be appropriate, but there are other methods that could be used.

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We could construct a spinner with eight equal sections marked from 1 to 8, or we could mark eight balls from 1 to 8 and draw them (with replacement) from a bowl, or one of a number of other methods. Generally, when we wish to simulate we use random number generators on a calculator or computer.

Summary 9H

- Simulation is a simple and legitimate method for finding solutions to problems when an exact solution is difficult, or impossible, to find.
- In order to use simulation to solve a problem, a clear statement of the problem and the underlying assumptions must be made.
- A model must be selected to generate outcomes for a simulation. Possible choices for physical simulation models are coins, dice and spinners. Random number tables, calculators and computers may also be used.
- Each trial should be defined and repeated several times (at least 50).
- The results from all the trials should be recorded and summarised appropriately to provide an answer to a problem.

Exercise 9H

1 Use simulation to estimate the probability that a family with three children have all boys.

2 A teacher gives her class a test consisting of five 'true or false' questions. Use simulation to estimate the probability that a student who guesses the answer to every question gets at least three correct.

3 A teacher gives the class a test consisting of 10 multiple-choice questions, each with five alternatives. Use simulation to estimate the probability that a student who guesses the answer to every question gets at least five correct.

4 Use simulation to estimate the number of pizzas we would need to buy if the number of football cards described in Example 29 was extended to 10.

5 Eight players are entered into a tennis tournament.
In round one, every player plays (four matches).
In round two, the four winners from round one play (two matches).
In round three, the two winners from round two play (one match).

a Suppose Shaun has a 50% chance of winning each match he plays. Use simulation to determine how many matches he will play, on average, in the tournament.

b Suppose he has a 70% chance of winning each match he plays. Use simulation to determine how many matches he will play, on average, in the tournament.

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Chapter summary

Assign-ment

Nrich

- Probability is a numerical measure of the chance of a particular event occurring and may be determined experimentally or by symmetry.
- Whatever method is used to determine the probability, the following rules will hold:
  - $0 \le \Pr(A) \le 1$ for all events $A \subseteq \varepsilon$
  - $\Pr(\varepsilon) = 1$
  - $\Pr(\varnothing) = 0$
  - $\Pr(A') = 1 - \Pr(A)$, where A' is the complement of A
  - $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$, the **addition rule**.
- Probabilities associated with combined events are sometimes able to be calculated more easily from a probability table.
- Two events A and B are **mutually exclusive** if $A \cap B = \varnothing$.
- If events A and B are mutually exclusive, then $\Pr(A \cap B) = 0$ and $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.
- The **conditional probability** of event A occurring, given that event B has already occurred, is

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} \qquad \text{if } \Pr(B) \neq 0$$

giving $\Pr(A \cap B) = \Pr(A \mid B) \times \Pr(B)$ (the **multiplication rule**)

- The probabilities associated with multi-stage experiments can be calculated by constructing an appropriate tree diagram and multiplying along the relevant branches (from the multiplication rule).
- The **law of total probability** states that, in the case of two events A and B,

$$\Pr(A) = \Pr(A \mid B)\Pr(B) + \Pr(A \mid B')\Pr(B')$$

- Two events A and B are **independent** if

$$\Pr(A \mid B) = \Pr(A)$$

so whether or not B has occurred has no effect on the probability of A occurring.

- Events A and B are independent if and only if $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$.

Short-answer questions

1 Two six-sided dice are tossed. Find the probability that:

a the sum of the values of the uppermost faces is 7 **b** the sum is not 7.

2 The probability that a computer chip is operational is 0.993. What is the probability that it is not operational?

3 A whole number between 1 and 300 (inclusive) is chosen at random. Find the probability that the number is:

a divisible by 3 **b** divisible by 4 **c** divisible by 3 or by 4.

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4 A drawer contains 30 red socks and 20 blue socks.

a If a sock is chosen at random, its colour noted, the sock replaced and a second sock withdrawn, what is the probability that both socks are red?

b If replacement doesn't take place, what is the probability that both socks are red?

5 Box A contains five pieces of paper numbered 1, 3, 5, 7, 9.
Box B contains three pieces of paper numbered 1, 4, 9.
One piece of paper is removed at random from each box. Find the probability that the two numbers obtained have a sum that is divisible by 3.

6 A three-digit number is formed by arranging the digits 1, 5 and 6 in a random order.

a List the sample space.

b Find the probability of getting a number larger than 400.

c What is the probability that an even number is obtained?

7 A letter is chosen at random from the word STATISTICIAN.

a What is the probability that it is a vowel?

b What is the probability that it is a T?

8 Ivan and Joe are chess players. In any game the probabilities of Ivan beating Joe, Joe beating Ivan or the game resulting in a draw are 0.6, 0.1 or 0.3 respectively. They play a series of three games. Calculate the probability that:

a they win alternate games, with Ivan winning the first game

b the three games are drawn

c exactly two of the games are drawn

d Joe does not win a game.

9 A die with two red faces and four blue faces is thrown three times. Each face is equally likely to face upward. Find the probability of obtaining the following:

a three red faces

b a blue on the first, a red on the second and a blue on the third

c exactly one red face

d at least two blue faces.

10 If $\mathrm{P}(A) = 0.6$ and $\mathrm{P}(B) = 0.5$, can A and B be mutually exclusive? Why or why not?

11 Events A and B are such that $\mathrm{P}(A) = 0.6$, $\mathrm{P}(B) = 0.5$ and $\mathrm{P}(A' \cap B) = 0.4$. Construct a probability table and use it to find:

a $\mathrm{P}(A \cap B')$

b $\mathrm{P}(A' \cap B')$

c $\mathrm{P}(A \cup B)$

12 In Minh's class 18 students study mathematics, 14 study music, and 7 study both mathematics and music.

a Given that a student in his class studies mathematics, what is the probability that they also study music?

b Given that a student in his class studies music, what is the probability that they also study mathematics?

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13 Given $P(B) = \frac{1}{3}$, $P(A \mid B) = \frac{2}{3}$ and $P(A \mid B') = \frac{3}{7}$, determine:

a $P(A \cap B')$ **b** $P(A)$ **c** $P(B' \mid A)$

14 A group of executives is classified according to body weight and incidence of hypertension. The proportion of the various categories is as shown.

	Overweight	Normal weight	Underweight
Hypertensive	0.10	0.08	0.02
Not hypertensive	0.15	0.45	0.20

a What is the probability that a person selected at random from this group will have hypertension?

b A person, selected at random from this group, is found to be overweight. What is the probability that this person is also hypertensive?

15 Given an experiment such that $P(A) = 0.3$, $P(B) = 0.6$ and $P(A \cap B) = 0.2$, find:

a $P(A \cup B)$ **b** $P(A' \cap B')$ **c** $P(A \mid B)$ **d** $P(B \mid A)$

16 For a finite sample space, explain the implication of each of the following in terms of the relationship between events A and B:

a $P(A \mid B) = 1$ **b** $P(A \mid B) = 0$ **c** $P(A \mid B) = P(A)$

Extended-response questions

1 To have a stage production ready for opening night there are three tasks which must be done and, as the same people are involved in each task, these must be done in sequence. The following probabilities are estimated for the duration of the activities:

Task	6 days	7 days	8 days
Build scenery	0.3	0.3	0.4
Paint scenery	0.6	0.3	0.1
Print programs	0.4	0.4	0.2

a What is the probability that the building and painting of the scenery will together take exactly 15 days?

b What is the probability that all three tasks will together take exactly 22 days?

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Review

2 Two bowls each contain eight pieces of fruit. In bowl A there are five oranges and three apples; in bowl B there is one orange and seven apples.

a For each bowl, find the probability that two pieces of fruit chosen at random will both be apples, if the first piece of fruit is not replaced before the second piece of fruit is chosen.

b For each bowl, find the probability that two pieces of fruit chosen at random will both be apples, when the first piece of fruit is replaced before the second is chosen.

c One bowl is chosen at random and from it two pieces of fruit are chosen at random without replacement. If both pieces of fruit are apples, find the probability that bowl A was chosen.

d One bowl is chosen at random and from it two pieces of fruit are chosen at random, the first piece of fruit being replaced before the second is chosen. If both pieces of fruit are apples, find the probability that bowl A was chosen.

3 Rachel is a keen runner. She is supposed to attend running training five days per week. Rachel finds that if she runs one day, the probability that she will run again the next day is $\frac{4}{5}$, and if she does not run one day, the probability that she will not run the next day is $\frac{3}{4}$. Suppose that Rachel runs one day:

a What is the probability that she runs the next day?

b What is the probability that she runs the day after that?

c What is the probability that she runs exactly twice in the next three days?

4 Sixteen players are entered in a tennis tournament.

- In round one, every player plays (eight matches).
- In round two, the eight winners from round one play (four matches).
- In round three, the four winners from round two play (two matches).
- In round four, the two winners from round three play (one match).

Use simulation to estimate how many matches a player will play, on average:

a if the player has a 50% chance of winning each match

b if the player has a 70% chance of winning each match.

5 Consider a finals series of games in which the top four teams play off as follows:

Game 1 Team A vs Team B

Game 2 Team C vs Team D

Game 3 Winner of game 2 plays loser of game 1

Game 4 Winner of game 3 plays winner of game 1

The winner of game 4 is then the winner of the series.

a Assuming all four teams are equally likely to win any game, use simulation to model the series.

b Use the results of the simulation to estimate the probability that each of the four teams wins the series.

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10 Counting methods

In this chapter

Syllabus references

Topic: Combinations

Subtopics: 1.3.1 – 1.3.5

When determining the probability of an event occurring, we often need to know the number of outcomes contained in the event and in the sample space. To do this in Chapter 9, we listed the sample space using a tree diagram or a table, and counted the number of outcomes.

When dealing with more complicated probability problems, listing the sample space and the event becomes too difficult. There may be hundreds of outcomes for a particular experiment, and even if they were comparatively easy to list we would soon tire of the task. In this chapter we will look at ways of counting the number of outcomes for various experiments and this will enable us to deal with more complicated probability problems.

For example, suppose there are 25 students in a class. The number of different ways of choosing a group of 5 students from the class is 53 130, if it doesn't matter about the order of choice. The number of ways of choosing 5 students in order from the class is 6 375 600. If it takes you 1 minute to choose one ordered group of 5 students, then it would take you over 12 years to obtain every possible ordered group. It is evident that we need to develop some methods to be able to 'count' in these situations.

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10A Addition and multiplication principles (optional content)

Before we start to consider probabilities, we need to formalise some simple rules for determining the number of possible outcomes.

The addition rule

Some people find the decision about what to wear when they get up in the morning to be very difficult, and the more clothes they own, the more complex the decision becomes! Let us consider the number of choices they might have by looking at some examples.

Example 1

Sandi can't decide whether to wear a windcheater or a jacket. She has four windcheaters and two jackets. How many choices does she have?

Solution

As Sandi is going to wear a windcheater *or* a jacket, she has a total of six choices from among these two items.

Example 2

Sandi's next choice is whether to wear jeans or a skirt. She has three pairs of jeans and four skirts. How many choices does she have?

Solution

Once again, as Sandi will wear jeans *or* a skirt, she has a total of seven choices from these two items.

Addition rule

In general, to choose between alternatives simply add up the number of choices available for each alternative.

This rule is generally associated with the use of the word 'or' in the question.

Example 3

At the library Alan is having trouble deciding which book to borrow. He has a choice of three mystery novels, three biographies or two science fiction books. How many choices of book does he have?

Solution

As he is choosing between alternatives (mystery novels *or* biographies *or* science fiction), he has a total of $3 + 3 + 2 = 8$ choices.

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The multiplication rule

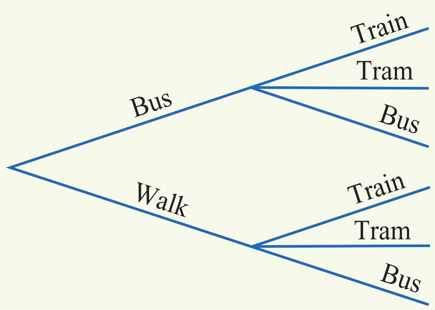
Sometimes the question arises of determining the number of possibilities when making successive choices, as in multi-stage experiments.

Example 4

When travelling from home to school James first takes a bus or walks to the main road, where he can then catch a train or a tram or another bus to his destination. How many ways does James have for travelling to school?

Solution

A tree diagram may help to answer this question.



By counting the branches of the tree diagram, it is found that there are six choices.

This answer could also be found by noting that there are two choices for the first part of the journey and three choices for the second, and $2 \times 3 = 6$.

Multiplication rule

When sequential choices are involved, the total number of possibilities is found by multiplying the number of options at each successive stage.

For example, for two stages, if there are m choices for the first stage and n choices for the second stage, then there are $m \times n$ choices altogether.

This rule is generally associated with the word 'and'.

Example 5

Consider Sandi's situation again. She has six choices of windcheaters or jackets, and seven choices of jeans or skirts. How many choices does she have for a complete outfit?

Solution

As Sandi will wear either a windcheater or a jacket *and* jeans or a skirt, we cannot consider these to be alternative choices. We could draw a tree diagram to list the possibilities, but this would be arduous. Using the multiplication rule, however, we can quickly determine the number of choices to be $6 \times 7 = 42$.

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Summary 10A

There are two simple rules which greatly assist in determining the number of possibilities in a sample space.

- **Addition rule** When choosing between alternatives, the total number of possibilities is found by adding the number of options available for each alternative. This rule is generally associated with the use of the word 'or' in the question.
- **Multiplication rule** When sequential choices are involved, the total number of possibilities is found by multiplying the number of options at each successive stage. For example, for two stages, if there are m choices for the first stage and n choices for the second stage, then there are $m \times n$ choices altogether.

Exercise 10A

Example 3

1 Find how many choices of book are possible if one book is to be selected from the following:

a eight novels, three dictionaries
b three mysteries, two dramas, seven science fiction
c twenty-two romances, fourteen mysteries, one autobiography
d ten novels, three biographies, twelve encyclopedias, four atlases

Example 4
Example 5

2 Find how many different meals are possible if three courses (one entree, one main course and one dessert) are to be selected from a menu that lists:

a three entrees, four main courses, five desserts
b ten entrees, ten main courses, five desserts
c five entrees, seven main courses, ten desserts
d eight entrees, eight main courses, eight desserts.

3 The menu in a restaurant lists four choices of entree, eight of main course and four of dessert. Find the number of choices of meal possible:

a if one of each of the three courses must be chosen
b if you can choose to omit the entree.

4 John cannot decide how to spend his evening. He can read a book, watch a video or go to the movies. If he can choose between three books, seven videos and ten movies, how many different choices does he have?

5 A student has to select a two-unit study for her course. She has to choose one unit in each semester. In semester one she has to choose one of two mathematics units, three language units and four science units. In semester two she has a choice of two history units, three geography units and two art units. How many choices does she have for the complete study?

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6 Dominic is travelling from Adelaide to Brisbane. He could fly directly from Adelaide to Brisbane on one of three airlines, or he could fly from Adelaide to Sydney on one of four airlines and then travel from Sydney to Brisbane with one of five bus lines, or he could go on one of three bus lines directly from Adelaide to Brisbane. In how many ways could Dominic travel from Adelaide to Brisbane?

7 A particular new model of car is available in five choices of colour, three choices of transmission, four types of interior and two types of engine. Air conditioning is optional. How many different types of car are possible?

8 A company uses one letter followed by four digits for product codes. If any of the letters A–Z is allowed in the first position, and any of the digits 0–9 in the next four positions, how many different product codes are possible? (The letters and digits may be used more than once.)

9 Suppose that a licence plate must consist of three letters followed by three numbers. If any of the letters A–Z is allowed in the first three positions, and any of the digits 0–9 in the second three positions, how many different licence plates are possible? (The letters and digits may be used more than once.)

10 Morse code consists of a succession of dots and dashes. The symbols formed by the code may consist of one, two, three or four dots or dashes. How many different symbols may be represented by this code?

10B Arrangements (optional content)

The number of **arrangements** of a set of objects is the number of different ways these objects can be arranged in a distinct order. Consider, for example, arrangements of the letters A, B, C. The possibilities are ABC, ACB, BAC, BCA, CAB, CBA. Each of these is considered a different arrangement (even though they all use the same letters), and counting them we can see that there are six arrangements in total.

While it is quite simple to determine the number of arrangements by listing the possibilities when there are only three objects, this is not a good strategy when there are more than three. However, we can work out the number of arrangements by the following logic.

Consider that we have three boxes to be filled:

- We have 3 choices of letter for the first box (A, B or C).
- We only have 2 choices for the second box (because we have already used one letter).
- We only have 1 choice for the third box (because we have already used two letters).

3	2	1

Using the multiplication rule, we see that the total number of arrangements (or choices) is

$$3 \times 2 \times 1 = 6$$

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Example 6

How many ways are there of arranging four different books on a shelf?

Solution

Consider the bookshelf as having four possible positions in which books can be placed:

As we have four books:

- There are 4 choices of book to place in position 1.
- There are 3 choices of book to place in position 2.
- There are 2 choices of book to place in position 3.
- There is 1 choice of book to place in position 4.

4	3	2	1

Using the multiplication rule, we know that the total number of choices will be the product of these individual choices:

The number of arrangements of four books in a row is

$$4 \times 3 \times 2 \times 1 = 24$$

In general, if n objects are arranged in a row, then there are n choices for the first position, $n - 1$ choices for the second position, $n - 2$ choices for the third position, and so on, until 1 is reached.

Thus, the number of ways of arranging n objects in a row is

$$n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 2 \times 1$$

Example 7

A photo is to be taken of a group of 12 students. How many ways are there of arranging the group if they all sit in a row?

Solution

As there are 12 students, the number of arrangements is

$$12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 479\,001\,600$$

which is rather a large number of choices!

Continuing to write out the expression $n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$ can be rather cumbersome. So for convenience this is written as $n!$, which is read as 'n factorial'.

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The notation $n!$ (read as 'n factorial') is an abbreviation for the product of all the integers from n down to 1:

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 2 \times 1$$

A CAS calculator can be used to evaluate $n!$. See the appropriate appendix in the Interactive Textbook for directions.

Sometimes we wish to arrange objects, but not use all of them every time. Suppose, for example, we wanted to determine how many arrangements there are of the letters A, B, C when taken in groups of two. The possibilities are AB, BA, AC, CA, BC, CB; six in total.

Once again, we can use our 'box' approach to work out the number of arrangements without listing them. This time we have two boxes to be filled:

- We have 3 choices of letter for the first box (A, B or C).
- We have 2 choices for the second box (because we have already used one letter).

3	2

Using the multiplication rule again, we see that the total number of arrangements (or choices) is $3 \times 2 = 6$.

Example 8

A painter is to paint the five circles of the Olympic flag. He cannot remember the colours to use for any of the circles, but he knows they should all be different. He has eight colours of paint available. In how many ways can he paint the circles on the flag?

Solution

We represent the painter's choices with five boxes:

- For the first box (paint colour) there are 8 choices.
- For the second box (paint colour) there are 7 choices.
- For the third box (paint colour) there are 6 choices.
- For the fourth box (paint colour) there are 5 choices.
- For the fifth box (paint colour) there are 4 choices.

8	7	6	5	4

Thus the total number of arrangements possible is $8 \times 7 \times 6 \times 5 \times 4 = 6720$.

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Could the factorial notation be used to express the answer to Example 8? In that example, the number of arrangements of eight objects in groups of five was to determined to be

$$8 \times 7 \times 6 \times 5 \times 4$$

Multiplying this answer by 1, we can write

$$8 \times 7 \times 6 \times 5 \times 4 = (8 \times 7 \times 6 \times 5 \times 4) \times \frac{3 \times 2 \times 1}{3 \times 2 \times 1} = \frac{8!}{3!}$$

Thus, the number of ways of choosing and arranging five objects from eight objects can be written as

$$\frac{8!}{3!} \quad \text{or} \quad \frac{8!}{(8-5)!}$$

In general, the number of arrangements of n objects in groups of size r is given by

$$\frac{n!}{(n-r)!} = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)$$

Arrangements are also called **permutations** in mathematics, and so this expression for the number of arrangements of n objects in groups of size r is often denoted by the symbol ${}^nP_r$.

Example 9

Find the number of different four-digit numbers that can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, if each digit:

a can only be used once

b can be used more than once

Solution

a As we are arranging nine objects (n) in groups of four (r):

$$\frac{9!}{(9-4)!} = \frac{9!}{5!} = 9 \times 8 \times 7 \times 6 = 3024$$

b There are nine choices for each of the four positions, so the total number of choices is $9 \times 9 \times 9 \times 9 = 9^4 = 6561$.

Definition of 0!

For us to be able to use our formula for the number of arrangements in all situations, we need to define a value for 0!. Consider the number of arrangements of n objects in groups of size n. From first principles, we have found that this is equal to $n!$. Using our formula for the number of arrangements of n objects in groups of size n gives us the answer

$${}^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!}$$

For this to make sense, we need to define

$$0! = 1$$

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Arrangements with restrictions

If a more complicated arrangement is required, then it is usually best to deal with the restriction first, as shown in the following example.

Example 10

How many different even four-digit numbers can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, if each digit may be used only once?

Solution

We have four boxes to be filled. Dealing with the restriction first, there are 4 choices for the final (fourth) box:

			4

Having now selected the last digit, we have seven digits remaining:

- 7 choices for the first box
- 6 choices for the second box
- 5 choices for the third box.

7	6	5	4

Multiplying gives $7 \times 6 \times 5 \times 4 = 840$.

Explanation

The restriction is that the number must be even, so it ends in 2, 4, 6 or 8.

The number of choices is reduced by one each time we use a digit.

Summary 10B

- An **arrangement** is a grouping of objects, such that the order of the objects is important. That is, the arrangements ABC and CBA are different.
- The number of ways of arranging n objects in a row is

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 2 \times 1$$

- The number of ways of arranging n objects in groups of size r is

$$^nP_r = \frac{n!}{(n-r)!} = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)$$

Skill-sheet

Exercise 10B

1 Evaluate:

a $3!$ **b** $5!$ **c** $7!$ **d** $2!$ **e** $0!$ **f** $1!$

2 Evaluate:

a $\frac{5!}{3!}$ **b** $\frac{9!}{7!}$ **c** $\frac{3!}{0!}$ **d** $\frac{8!}{6!}$ **e** $\frac{5!}{0!}$ **f** $\frac{10!}{7!}$

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Example 6 **3** In how many ways can five books be arranged in a row on a shelf?

Example 7 **4** In how many ways can seven students be seated on a row of seven chairs?

5 In how many ways can four pictures be hung in a row on a wall?

6 In how many ways can six cups be hung on six hooks?

Example 8 **7** In how many ways can three pictures be hung along a wall if there are ten pictures available?

8 If there are eight swimmers in the final of the 1500 m freestyle event, in how many ways can the first three places be filled?

9 Find the number of ways in which the letters of the word TROUBLE can be arranged:

a if they are all used **b** if they are used three at a time.

10 Find the number of ways in which the letters of the word PANIC can be arranged:

a if they are all used **b** if they are used four at a time.

Example 9 **11** Find the number of four-letter code words that can be made from the letters of the word COMPLEX:

a if no letter can be used more than once **b** if the letters can be re-used.

12 Find how many code words of three or four letters can be made from the letters of the word NUMBER:

a if no letter can be used more than once **b** if the letters can be re-used.

Example 10 **13** If no digit can be used more than once, find how many numbers that can be formed from the digits 3, 4, 5, 6, 7 are:

a three-digit numbers **b** three-digit numbers and even

c greater than 700.

14 If no digit can be used more than once, find how many numbers that can be formed from the digits 3, 4, 5, 6, 7, 8 are:

a two- or three-digit numbers **b** six-digit numbers and even

c greater than 7000.

15 Four boys and two girls sit in a line on stools in front of a counter. Find the number of ways in which they can arrange themselves:

a if there are no restrictions **b** if the two girls wish to be at the ends.

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10C Selections

In the previous section, methods for counting the number of ways in which objects can be chosen and then arranged were discussed. Sometimes the interest is solely in the number of different groups of objects that can be selected. That is, the order is unimportant.

Consider again the letters A, B, C when taken in groups of two. We saw that there are six arrangements: AB, BA, AC, CA, BC, CB. However, there are only three distinct selections: $\{A, B\}$, $\{A, C\}$, $\{B, C\}$.

Example 11

Four flavours of ice-cream – vanilla, chocolate, strawberry and caramel – are available at the school canteen. How many different double-scoop selections are possible if two different flavours must be used?

Solution

The possibilities are:

- vanilla and chocolate
- vanilla and strawberry
- vanilla and caramel
- chocolate and strawberry
- chocolate and caramel
- strawberry and caramel

giving a total of six different selections.

In this example, the selection 'vanilla and chocolate' is considered to be the same as 'chocolate and vanilla', and so is counted only once. Such choices without regard to order are called **selections** or **combinations**. The notation ${}^nC_r$ is used to represent the number of different ways in which groups of size r can be chosen from a total of n objects when order is unimportant.

When the total group size n is not large, the combinations can be listed. But obviously a more efficient method is preferable. Consider again Example 8 concerning the colours on the Olympic flag. Suppose that first the five colours to be used are chosen, and then they are arranged on the flag. This is shown as:

Choose the colours	Arrange them	Possible arrangements
${}^8C_5$	$\times\, 5!$	$= \dfrac{8!}{3!}$

So, since ${}^8C_5 \times 5! = \dfrac{8!}{3!}$, we can find an expression for ${}^8C_5$ by dividing both sides by 5!.

$${}^8C_5 = \frac{8!}{3!5!}$$

Note that the two figures on the bottom line (3 and 5) add to 8.

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In general, the number of combinations of n objects in groups of size r is

$$^nC_r = \frac{^nP_r}{r!} = \frac{n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)}{r!} = \frac{n!}{r!\,(n-r)!}$$

A commonly used alternative notation for ^nC_r is $\binom{n}{r}$.

A CAS calculator can be used to determine values of ^nC_r. See the appropriate appendix in the Interactive Textbook for directions.

Example 12

Consider the situation from Example 11 again: If four flavours of ice-cream are available, how many double-scoop selections are possible if two different flavours must be used?

Solution

The number of combinations of four flavours in groups of size two is

$$^4C_2 = \frac{4!}{2!2!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1} = 6$$

This is the same as the answer we found before by listing the combinations.

Once again, not all combination problems are so straightforward, as shown by the following example.

Example 13

A team of three boys and three girls is to be chosen from a group of eight boys and five girls. How many different teams are possible?

Solution

Three boys can be chosen from eight in $^8C_3$ ways, and three girls from five in $^5C_3$ ways. Since we are choosing boys and girls, the total number of possible teams is

$$^8C_3 \times {}^5C_3 = 56 \times 10 = 560$$

Selections of any size

The following result is useful when asked to count all the combinations of any size from a group of n objects.

For n objects,

$$^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_{n-1} + {}^nC_n = 2^n$$

We can see why it is true using the following argument for three objects.

There are three different books A, B and C, and David can choose whichever ones he likes (including choosing none of them). How many ways can this choice be made?

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Let $S = \{A, B, C\}$. The subsets are

$$\emptyset, \quad \{A\}, \quad \{B\}, \quad \{C\}, \quad \{A, B\}, \quad \{A, C\}, \quad \{B, C\}, \quad \{A, B, C\}$$

So there are eight choices.

We can use two different methods to count these choices. First we focus on how many books David chooses:

0 He can choose no books in ${}^3C_0 = 1$ way. (This corresponds to $\emptyset$.)

1 He can choose one book in ${}^3C_1 = 3$ ways. (This corresponds to $\{A\}, \{B\}, \{C\}$.)

2 He can choose two books in ${}^3C_2 = 3$ ways. (This corresponds to $\{A, B\}, \{A, C\}, \{B, C\}$.)

3 He can choose three books in ${}^3C_3 = 1$ way. (This corresponds to $\{A, B, C\}$.)

By the addition rule, the total number of choices is

$$ {}^3C_0 + {}^3C_1 + {}^3C_2 + {}^3C_3 $$

Now, instead, we focus on each book one at a time:

A Book A can be chosen or not (2 ways).

B Book B can be chosen or not (2 ways).

C Book C can be chosen or not (2 ways).

By the multiplication rule, there are a total of $2 \times 2 \times 2 = 2^3$ choices. This gives

$$ {}^3C_0 + {}^3C_1 + {}^3C_2 + {}^3C_3 = 2^3 $$

This argument can easily be extended to prove the result for n objects.

Example 14

Nick is making an invitation list for his party, and has seven friends to choose from. If he may choose to invite any number of friends (from one to all seven), how many possible party lists does he have? (Assume he will invite at least one person to his party.)

Solution

Nick may invite one person to the party, and he has ${}^7C_1 = 7$ ways of doing this. If he invites two people to the party, he has ${}^7C_2 = 21$ ways of doing this.

Continuing in this way, we can see that Nick's total number of choices is

$$ {}^7C_1 + {}^7C_2 + {}^7C_3 + {}^7C_4 + {}^7C_5 + {}^7C_6 + {}^7C_7 $$

Since we know that

$$ {}^7C_0 + {}^7C_1 + {}^7C_2 + {}^7C_3 + {}^7C_4 + {}^7C_5 + {}^7C_6 + {}^7C_7 = 2^7 $$

the required answer is

$$ \begin{aligned} 2^7 - {}^7C_0 &= 2^7 - 1 \\ &= 128 - 1 \\ &= 127 \end{aligned} $$

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Summary 10C

- A **selection** (or **combination**) is a grouping of objects, such that the order of the objects is not important. That is, the selections ABC and CBA are the same.
- Knowing the number of distinct selection of objects, when taking groups of a particular size, may enable us to determine the number of elements in the sample space without having to list them.
- In general, the number of combinations of n objects in groups of size r is

$$^nC_r = \frac{^nP_r}{r!} = \frac{n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)}{r!} = \frac{n!}{r!\,(n-r)!}$$

- A commonly used alternative notation for ^nC_r is $\binom{n}{r}$.
- For n objects, $^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_{n-1} + {}^nC_n = 2^n$.

Skill-sheet

Exercise 10C

Example 11

1 For each of the following examples, determine the number of selections possible by listing the possibilities:

a An ice-cream with two different scoops is selected from a choice of vanilla, chocolate and strawberry.

b Two students from the group of Jaime, Georgia and Wey are chosen to represent the class in debating.

c Two students from the group of Thomas, William, Jackson and Phillip are chosen for the tennis team.

d Three scarves are chosen from a blue scarf, a green scarf, a red scarf and a white scarf.

Example 12

2 Evaluate:

a $^5C_3$ **b** $^5C_2$ **c** $^7C_4$ **d** $^7C_3$

Compare your answers for parts a and b, and for parts c and d.

3 Evaluate:

a $^{20}C_{18}$ **b** $^{100}C_{99}$ **c** $^{100}C_2$ **d** $^{250}C_{248}$

4 Evaluate:

a $\binom{6}{3}$ **b** $\binom{7}{1}$ **c** $\binom{8}{2}$ **d** $\binom{50}{48}$

5 How many netball teams of seven can be chosen from 13 players?

6 An ice-cream parlour has 25 different flavours of ice-cream available. How many different three-scoop ice-cream sundaes are available if three different flavours are to be used and the order of the scoops does not matter?

7 How many different hands of seven cards can be dealt from a normal pack of 52 cards?

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8 In Tattslotto six numbers are selected from 45. How many different possible selections are there? (Do not attempt to consider supplementary numbers.)

Example 13 9 A student has the choice of three mathematics subjects and four science subjects. In how many ways can they choose to study one mathematics and two science subjects?

10 A survey is to be conducted, and eight people are to be chosen from a group of 30.

- a In how many different ways could the eight be chosen?
- b If the group contains 10 men and 20 women, how many groups containing exactly two men are possible?

11 From a standard 52-card deck, how many 7-card hands have 5 spades and 2 hearts?

12 In how many ways can a committee of five be selected from eight women and four men:

- a without restriction
- b if there must be exactly three women on the committee?

13 Six females and five males are interviewed for five positions. If all are acceptable for any position, in how many ways could the following combinations be selected?

- a three males and two females
- b four females and one male
- c five females
- d five people regardless of sex
- e at least four females

14 The selectors of a sporting team need to choose 10 athletes from the 15 track and 12 field athletes who have qualified to compete.

- a How many groups are possible?
- b How many groups would contain track athletes only?
- c How many groups would contain field athletes only?
- d How many groups would contain half track and half field athletes?

15 A student representative committee of five is to be chosen from four male and six female students. How many committees could be selected which contain more female than male students?

Example 14 16 Joanne is offered a selection of five different sweets. She can choose to pass or to select any number of them. In total how many choices does she have?

17 Eight people have auditioned for roles in a play. The director can choose none, or any number of them for the production. In how many ways can selections be made from these eight people, any number at a time?

18 How many colours can be obtained by mixing a set volume of red, blue, green, orange and white paints if any number of paints can be used at a time?

19 How many varieties of fruit salad, using at least two fruits, can be obtained from apples, oranges, pears, passionfruit, kiwifruit and nectarines, taken any number at a time?

20 In how many ways can a group of six people be divided into:

- a two equal groups
- b two unequal groups, if there must be at least one person in each group?

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10D Applications to probability (optional content)

As discussed in Chapter 9, for a sample space with equally likely outcomes, the probability of an event occurring may be determined by dividing the number of outcomes in the event by the total number of possible outcomes. Establishing the number of outcomes in the event and the total number of outcomes is often achieved by using permutations and combinations.

Example 15

Four-letter 'words' are to be made by arranging letters of the word SPECIAL. What is the probability that the 'word' will start with a vowel?

Solution

There are 7 letters to be arranged in groups of 4. So the total number of possible arrangements is $7 \times 6 \times 5 \times 4 = 840$.

Now consider words which start with a vowel. Since there are three vowels, we have 3 choices for the first letter. Having done this, we have six letters remaining which are to be placed in the three remaining positions, and this can be done in $6 \times 5 \times 4 = 120$ ways.

Thus the number of arrangements which start with a vowel is

$$3 \times 6 \times 5 \times 4 = 360$$

Hence, the probability of the word starting with a vowel is

$$\frac{\text{number of outcomes in the event}}{\text{total number of outcomes}} = \frac{360}{840} = \frac{3}{7}$$

Example 16

Three students are to be chosen to represent the class in a debate. If the class consists of six boys and four girls, what is the probability that the team will contain:

a exactly one girl

b at least two girls?

Solution

Since there is a total of 10 students, the number of possible teams is $^{10}C_3 = 120$.

a One girl can be chosen for the team in $^{4}C_1 = 4$ different ways. Having placed one girl, the other two places must be filled by boys, and this can be done in $^{6}C_2 = 15$ different ways. Thus the total number of teams containing one girl and two boys is $4 \times 15 = 60$, and the probability that the team contains exactly one girl is $\frac{60}{120} = 0.5$.

b If the team is to contain at least two girls, then it may contain two *or* three girls. The number of teams containing:

- exactly two girls is $^{6}C_1 \times {}^{4}C_2 = 36$
- exactly three girls is $^{6}C_0 \times {}^{4}C_3 = 4$

Thus the total number of teams containing at least two girls is 40, and the probability of this is $\frac{40}{120} = \frac{1}{3}$.

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Summary 10D

Using our knowledge of arrangements and selections, we can calculate probabilities for sample spaces with equally like outcomes:

- First determine the total number of possible outcomes.
- Then determine the number of outcomes in the event of interest.
- The required probability is equal to

$$\frac{\text{number of outcomes in the event}}{\text{total number of outcomes}}$$

Skill-sheet

Exercise 10D

1 A four-digit number (with no repetitions) is to be formed from the set of digits $\{1, 2, 3, 4, 5, 6\}$. Find the probability that the number:

a is even **b** is odd.

Example 15

2 Three-letter ‘words’ are to be made by arranging the letters of the word COMPUTER. What is the probability that the ‘word’ will start with a vowel?

3 Three letters are chosen at random from the word HEART and arranged in a row. Find the probability that:

a the letter H is first **b** the letter H is chosen **c** both vowels are chosen.

4 Three men and three women are to be randomly seated in a row. Find the probability that both the end places will be filled by women.

Example 16

5 A netball team of seven players is to be chosen from six men and seven women. Find the probability that the selected team contains more men than women.

6 Bill is making a sandwich. He may choose any combination of the following: lettuce, tomato, carrot, cheese, cucumber, beetroot, onion, ham. Find the probability that:

a the sandwich contains ham

b the sandwich contains three ingredients

c the sandwich contains at least three ingredients.

7 A bag contains five white, six red and seven blue balls. If three balls are selected at random, without replacement, find the probability they are:

a all red **b** all different colours.

8 Susie chooses four pieces of bubble gum from a jar containing five red, two blue and three green pieces of bubble gum. Calculate the probability that Susie selects:

a no green bubble gum

b at least one green bubble gum

c at least one green bubble gum and at least one red bubble gum

d at least one red bubble gum, given that there is at least one green bubble gum.

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9 A four-digit number (with no repetitions) is to be formed from the set of digits $\{0, 1, 2, 3, 4, 5, 6, 7\}$. Find the probability that the number:

a is even **b** is odd **c** is less than 4000
d is less than 4000, given that it is greater than 3000.

10 A hand of five cards is dealt from a normal pack of 52 cards. Find the probability that the hand will contain:

a no aces **b** at least one ace **c** the ace of spades
d the ace of spades, given that there is at least one ace.

11 A committee of three is to be chosen from a group of four men and five women. Find the probability that the committee contains:

a all women **b** at least one woman
c exactly two men, given that at least one man is chosen.

10E Pascal's triangle and the binomial theorem

There is a connection between the selection problems we have encountered and the expansion of expressions of the form $(a + b)^n$. In this section we explore this connection.

Pascal's triangle

Consider the possibilities when choosing from a small group of objects. In the first instance, consider when the group has only one object, A. We can choose either no objects or one object from this group, each in only one way:

$$^1C_0 = 1 \qquad ^1C_1 = 1$$

Next, consider when the group has two objects, A and B. We can choose either no objects, one object or two objects from this group:

$$^2C_0 = 1 \qquad ^2C_1 = 2 \qquad ^2C_2 = 1$$

Choosing from three objects we have:

$$^3C_0 = 1 \qquad ^3C_1 = 3 \qquad ^3C_2 = 3 \qquad ^3C_3 = 1$$

We can continue in this way to create the following pattern of numbers. We also add a top line for a group of 0 objects.

$n = 0$:	$^0C_0$	1
$n = 1$:	$^1C_0 \quad ^1C_1$	1 1
$n = 2$:	$^2C_0 \quad ^2C_1 \quad ^2C_2$	1 2 1
$n = 3$:	$^3C_0 \quad ^3C_1 \quad ^3C_2 \quad ^3C_3$	1 3 3 1
$n = 4$:	$^4C_0 \quad ^4C_1 \quad ^4C_2 \quad ^4C_3 \quad ^4C_4$	1 4 6 4 1
$n = 5$:	$^5C_0 \quad ^5C_1 \quad ^5C_2 \quad ^5C_3 \quad ^5C_4 \quad ^5C_5$	1 5 10 10 5 1

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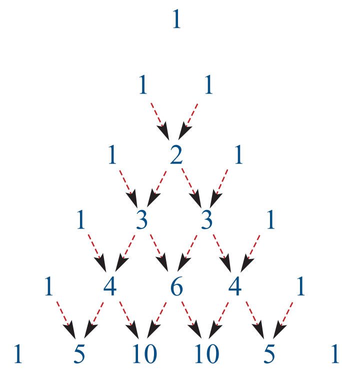
This triangle of numbers is usually called **Pascal's triangle** after the French mathematician Blaise Pascal who made ingenious use of it in his studies of probability.

The pattern can be continued indefinitely by noting:

- The first and last numbers in every row are always 1.
- The number in any position in a row is the sum of the two numbers in the row above which are to the left and right of it, as shown.

This is proved by the following result.



Pascal's identity

$^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r \quad \text{for } 0 < r < n$

Proof Using the formula for ^nC_r, we find

$$\begin{aligned} {}^{n-1}C_{r-1} + {}^{n-1}C_r &= \frac{(n-1)!}{(n-r)!\,(r-1)!} + \frac{(n-1)!}{(n-1-r)!\,r!} \\ &= \frac{(n-1)!}{(n-r-1)!\,(r-1)!}\left(\frac{1}{n-r} + \frac{1}{r}\right) \\ &= \frac{(n-1)!}{(n-r-1)!\,(r-1)!}\left(\frac{n}{(n-r)r}\right) \\ &= \frac{n!}{(n-r)!\,r!} \\ &= {}^nC_r \end{aligned}$$

The binomial theorem

In Chapter 7, we briefly looked at expansions of $(a+b)^n$ and showed how you could build up the identities. We are now able to apply the techniques of this chapter to obtain the general result that

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n-1}ab^{n-1} + b^n$$

This may look very complicated, but by substituting in $n = 3$ and $n = 4$ we obtain the results we know:

$$\begin{aligned} (a+b)^3 &= a^3 + \binom{3}{1}a^{3-1}b + \binom{3}{2}a^{3-2}b^2 + \binom{3}{3}a^{3-3}b^3 \\ &= a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3 \end{aligned}$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

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Before proceeding further, we note that the coefficients can be found from Pascal's triangle. We present the triangle in this form so the coefficient of the b^i term can easily be read off.

n	b^0	b^1	b^2	b^3	b^4	b^5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

From this we could read off the coefficients of $(a+b)^5$, for example:

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Using combinations, we can now see why this identity holds without having to multiply out the left-hand side.

If we expand $(a+b)^5$, we know that the terms will be of the form

$$c_0a^5 \qquad c_1a^4b \qquad c_2a^3b^2 \qquad c_3a^2b^3 \qquad c_4ab^4 \qquad c_5b^5$$

where the c_i are the coefficients. Writing

$$(a+b)^5 = (a+b)(a+b)(a+b)(a+b)(a+b)$$

we can use our knowledge of combinations to find these coefficients.

For finding the coefficient c_1 of a^4b, the relevant terms when multiplying out are

$$aaaab \qquad aaaba \qquad aabaa \qquad abaaa \qquad baaaa$$

Each of these terms comes from choosing either a or b from each of the five brackets. There are $\binom{5}{1}$ ways of choosing exactly one b from the five brackets. Therefore the coefficient of a^4b is $c_1 = \binom{5}{1} = 5$.

Similarly, there are $\binom{5}{2}$ ways of choosing exactly two bs from the five brackets. Therefore the coefficient of a^3b^2 is $c_2 = \binom{5}{2} = 10$.

Continuing in this way, we can see that the remaining coefficients are $c_3 = \binom{5}{3} = 10$, $c_4 = \binom{5}{4} = 5$ and $c_5 = \binom{5}{5} = 1$.

This argument in the special case of $(a+b)^5$ can be extended to $(a+b)^n$ to give the following general result.

Binomial theorem

For each positive integer n,

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n-1}ab^{n-1} + b^n$$

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Example 17

Use the binomial theorem to expand $(2 - 3x)^4$.

Solution

We know $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

Let $a = 2$ and $b = -3x$. Then

$$\begin{aligned}(2 + (-3x))^4 &= 2^4 + 4 \times 2^3(-3x) + 6 \times 2^2(-3x)^2 + 4 \times 2(-3x)^3 + (-3x)^4 \\ &= 16 + 4 \times 8 \times (-3)x + 6 \times 4 \times (-3)^2x^2 + 4 \times 2 \times (-3)^3x^3 + (-3)^4x^4 \\ &= 16 - 96x + 216x^2 - 216x^3 + 81x^4\end{aligned}$$

Summary 10E

- Pascal's triangle

n	b^0	b^1	b^2	b^3	b^4	b^5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

- Binomial theorem

For each positive integer n,

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n-1}ab^{n-1} + b^n$$

Exercise 10E

Example 17

1 Expand each of the following using the binomial theorem:

a $(x + 2)^4$ **b** $(2x + 1)^4$ **c** $(2x - 3)^4$ **d** $(3x - 1)^3$

e $(1 - 2x)^4$ **f** $(1 - 2x)^5$ **g** $(1 - 3x)^5$ **h** $(3 - 2x)^4$

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Review

Chapter summary

Assign-ment

Nrich

- $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$ and $0! = 1$, where the notation $n!$ is read as 'n factorial'.
- An **arrangement** (or **permutation**) is a grouping of objects, such that the order of the objects is important. That is, the arrangements ABC and CBA are different.
- The number of ways of arranging n objects in a row is

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 2 \times 1$$

- The number of ways of arranging n objects in groups of size r is

$$^nP_r = \frac{n!}{(n-r)!} = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)$$

- A **selection** (or **combination**) is a grouping of objects, such that the order of the objects is not important. That is, the selections ABC and CBA are the same.
- The number of combinations of n objects in groups of size r is

$$^nC_r = \frac{^nP_r}{r!} = \frac{n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)}{r!} = \frac{n!}{r!\,(n-r)!}$$

- A commonly used alternative notation for ^nC_r is $\binom{n}{r}$.
- For n objects, $^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_{n-1} + {}^nC_n = 2^n$.
- When the combination problem involves restrictions, deal with these first.
- Combinations may be used when determining probabilities. In the appropriate cases, the probability is given by dividing the number of outcomes in an event by the total number of outcomes.
- Binomial theorem:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n-1}ab^{n-1} + b^n$$

Short-answer questions

1 Evaluate:

a $^{1000}C_{998}$ **b** $^{1\,000\,000}C_{999\,999}$ **c** $^{1\,000\,000}C_1$

2 How many integers from 100 to 999, inclusive, have three different digits?

3 How many different three-digit house numbers can be constructed from six brass numerals 1, 2, 3, 4, 5, 6?

4 A supermarket sells n different brands of dishwashing liquid. Each brand offers four different sized bottles (small, medium, large and economy), and each is available as either lemon-scented or pine-scented. How many different types of dishwashing liquid bottles are available at this supermarket?

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5 Of the integers from 1000 to 9999, how many have at least one digit a 5 or 7?

6 A bushwalking club has 80 members: 50 men and 30 women. A committee consisting of two men and one woman is to be selected. How many different committees are possible?

7 There are five vowels and 21 consonants in the English alphabet. How many different four-letter 'words' can be formed that contain two different vowels and two different consonants?

8 A pizza restaurant offers the following toppings: onions, green peppers, mushrooms, anchovies and pepperoni.

a How many different kinds of pizza with three different toppings can be ordered?

b How many different kinds with any number of toppings (between none and all five) can be ordered?

9 Seven people are to be seated in a row. Calculate the number of ways in which this can be done so that two particular persons, A and B, always have exactly one of the others between them.

10 Three letters are chosen at random from the word OLYMPICS and arranged in a row. What is the probability that:

a the letter O is first

b the letter Y is chosen

c both vowels are chosen?

11 Using Pascal's triangle and the binomial theorem, expand $(x - 1)^6$.

Extended-response questions

1 Permutations are formed using all of the digits 1, 2, 3, 4, . . . , 9 without repetition. Determine the number of possible permutations in each of the following cases:

a Even and odd digits alternate.

b The digits 1 and 2 are together but not necessarily in that order.

2 There are 10 chairs in a row.

a In how many ways can three people be seated?

b In how many of these will the two end chairs be occupied?

c In how many of these will the two end chairs be empty?

3 All possible three-digit numbers are formed from the odd digits $\{1, 3, 5, 7, 9\}$.

a How many such numbers are possible if each digit is used only once?

b How many of the numbers from part a are larger than 350?

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4 In how many ways can a committee of four be chosen from five married couples if:

a all individuals are eligible for selection

b the committee must consist of two women and two men

c a husband and wife cannot both be selected?

5 Geoff has five flat batteries and ten charged batteries. Unfortunately his little brother mixes them up, and he can't tell them apart. He selects four batteries at random for his calculator.

a How many different combinations of the 15 batteries could Geoff select?

b In how many of these are all four batteries charged?

c In how many of these is at least one battery flat?

6 There are seven mints and 11 jubes in the lolly jar. Steve puts his hand in the jar and selects four lollies at random.

a How many different combinations of the lollies are there?

b In how many of these are there no mints?

c In how many of these are there two mints and two jubes?

7 In Tattslotto, a player picks a selection of six numbers from the numbers 1 to 45. To determine the winners, eight numbers are chosen at random – the first six are designated as the winning numbers, and the other two as the supplementary numbers. Prizes are determined as follows.

Division 1: 6 winning numbers

Division 2: 5 winning numbers and 1 supplementary

Division 3: 5 winning numbers

Division 4: 4 winning numbers

Division 5: 3 winning numbers and 1 supplementary

Find the number of combinations which satisfy each of the divisions, and hence the probabilities of winning each of the five divisions.

8 In Keno, a player selects between three and ten numbers from 1 to 80. Each selection is called a 'spot'. If you select five numbers, you are playing a 'Spot 5' game.
To determine the winners, 20 numbers are drawn randomly from the 80 numbers. If all your selected numbers are among the 20, you win. The amount you win depends on the 'spot' you are playing.
Find the probability of winning:

a a 'Spot 6' game

b a 'Spot 5' game.

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11 Revision of Chapters 9–10

11A Short-answer questions

1 A six-sided die is rolled twice. Calculate the probability that:

a the sum of the numbers showing is 5

b the sum of the numbers showing is not 5.

2 A number is formed by arranging the digits 3, 4 and 8 in random order.

a List the sample space.

b Calculate the probability that a number less than 500 is obtained.

c What is the probability that the number obtained is even?

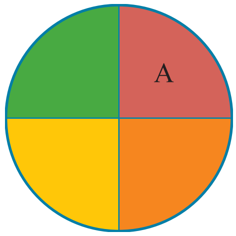
3 A card is drawn at random from a pack of 52 cards. What is the probability that the card is:

a not red

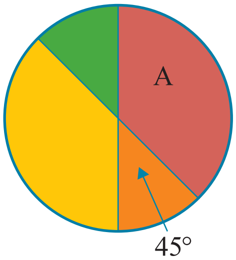
b not an ace?

4 Consider the following spinners. In each case, what is the chance of the pointer stopping in region A?

a



b



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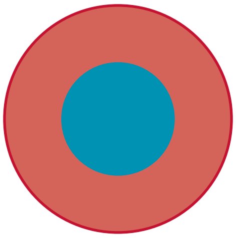
5 For a particular biased six-sided die, it is known that the numbers 1, 2, 3 and 5 are equally likely to occur, that the number 4 occurs four times as often as the number 2, and that the number 6 occurs half as often as the number 2.

a Find the probability of each of the possible outcomes.

b Find the probability that the number observed is not a 4.

6 The dartboard shown consists of two concentric circular regions. The radius of the inner circle is 10 cm, and the radius of the outer circle is 20 cm. If Mike throws a dart, it is equally likely to hit any part of the dartboard, and it hits the dartboard every time. Find the probability of the dart hitting the blue region.



7 Suppose that in a certain school the probability that a student plays basketball is 0.3, the probability that a student plays hockey is 0.4, and probability that a student plays both basketball and hockey is 0.1.

a Find the probability that a student plays either or both of these sports.

b Find the probability that a student plays hockey, given that they play basketball.

8 At a school holiday program for 60 students, painting was offered as an additional activity one morning, and music as an additional activity in the afternoon. Every student went to at least one additional activity, with 45 students attending painting and 30 students attending music. Construct a probability table, and use it to find the probability that a student chosen at random:

a went to music

b went to painting

c went to painting but not music

d went to music but not painting.

9 In a certain town, the probability that it will be sunny on any Saturday is 0.6. If it is sunny on Saturday, then the probability that it is sunny on Sunday is 0.8. If it is not sunny on Saturday, then the probability that it is sunny on Sunday is 0.2. Find the probability that:

a it is sunny all weekend

b it is sunny on Sunday.

10 For two events A and B such that $\mathrm{P}(A) = 0.5$, $\mathrm{P}(B) = 0.2$ and $\mathrm{P}(B \mid A) = 0.1$, find:

a $\mathrm{P}(A \cap B)$

b $\mathrm{P}(A \mid B)$

11 If A and B are independent events with $\mathrm{P}(A) = 0.4$ and $\mathrm{P}(B) = 0.5$, find:

a $\mathrm{P}(A \mid B)$

b $\mathrm{P}(A \cap B)$

c $\mathrm{P}(A \cup B)$

12 If there are ten runners in the final of the 5000 metres, in how many ways can the first three places be filled?

13 How many different hands of seven cards can be dealt from a normal pack of 52 cards? (Give your answer using factorial notation.)

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14 A committee of three is to be chosen to represent a class at the School Council. If everyone in the class is equally likely to be chosen, and the class consists of five boys and seven girls, what is the probability that the committee will contain:

a one girl **b** two girls?

11B Extended-response questions

1 The heights (measured to the nearest centimetre) of 56 plants grown under experimental conditions are given in the following table.

Note: The table gives the number of plants for which the measured height lies in a given interval.

Interval	(0, 10]	(10, 20]	(20, 30]	(30, 40]	(40, 50]	(50, 60]	(60, 70]
Number of plants	1	2	4	6	13	22	8

a If one plant is selected at random, find the probability that its height is:

i greater than 50 cm

ii either greater than 50 cm or less than or equal to 30 cm

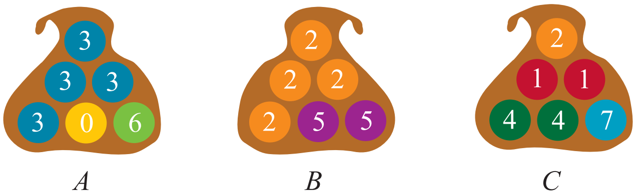
iii greater than 40 cm, given that it is greater than 30 cm.

b The probability that a randomly selected plant produces a flower is $\frac{6}{7}$ and the probability that it will suffer from a disease is $\frac{1}{4}$. Assume independence.

i Find the probability that a single plant selected at random produces a flower but does not suffer from disease.

ii Find the probability that a single plant selected at random produces a flower, does not suffer from disease and has a height greater than 50 cm.

2 Three people, A, B and C, are each given a bag containing six numbered balls as shown in the diagram below, and each of them draws a ball from the bag that they were given.



a Calculate the probability that B draws a higher number than C.

b Calculate the probability that A draws a higher number than both B and C.

3 Eight cards bearing the letters P, A, R, A, L, L, E, L are placed in a box. Three cards are drawn out at random without replacement. Calculate the probability that:

a the second card bears the letter L

b the three cards bear the letters A, L, E in that order

c the three cards bear the letters A, L, E in any order

d the first two cards bear different letters.

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Revision

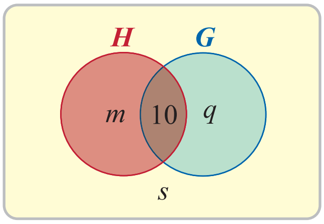
4 Of the employees in a large factory, $\frac{1}{8}$ travel to work by bus, $\frac{3}{8}$ by train, and the remainder by car. Those travelling by bus have a probability of $\frac{1}{3}$ of being late, those by train will be late with probability $\frac{1}{5}$, and those by car will be late with probability $\frac{3}{4}$.

a Draw and complete a tree diagram and calculate the probability that an employee chosen at random will be late.

b If an employee is late, calculate the probability that he travelled by car.

5 Of the 100 students at a particular level in a school, 40 take history, 45 take geography and 10 take both subjects.

a A Venn diagram interpreting this situation is as shown:



m is the number of students who take history only.

q is the number of students who take geography only.

s is the number of students who take neither subject.

i Find the values of m, q and s.

ii Find the number of students who take either history or geography but not both.

b If a student is chosen at random, find the probability that he or she takes history but not geography.

c If a student is chosen randomly from the students who do not take history, find the probability that he or she takes geography.

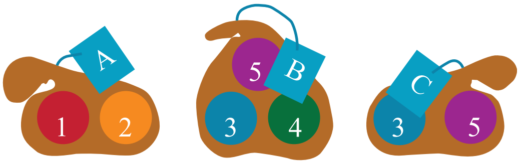
6 In order to choose a team for a quiz, a class is split into three groups. Group A contains two boys and three girls, group B contains one boy and three girls, and group C contains two boys and one girl. An unbiased die is thrown. If a 1, 2 or 3 appears, a random choice will be made from group A. If a 4 or 5 appears, a random choice will be made from group C. If a 6 appears, a random choice will be made from group B.

a Calculate the probability that a boy will be chosen when the choice is made.

b Given that a girl is chosen when the choice is made, calculate the probability that she will be chosen from:

i group A **ii** group B.

7 Suppose that in a game of chance bag A contains balls numbered 1 and 2, bag B contains balls numbered 3, 4 and 5, and bag C contains balls numbered 3 and 5.



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a David chooses one bag at random and then draws one ball randomly from the bag.

i Find the probability that the ball drawn is a 4.

ii Find the probability that the ball drawn is a 3.

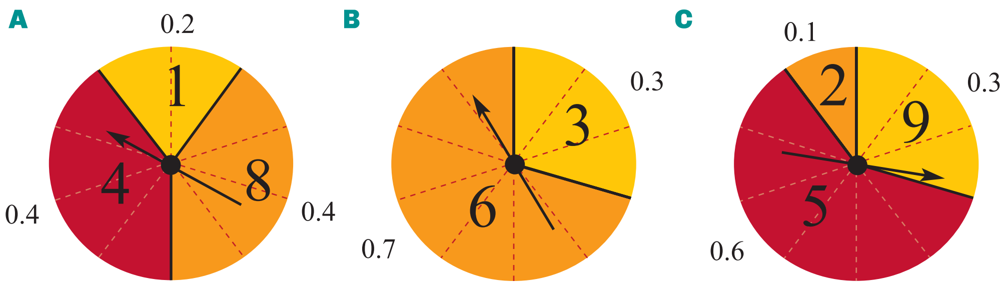
b After David has drawn a ball in the way described in part a, he puts it back into the original bag. Next, Sally chooses one bag at random and then draws one ball randomly from the bag.

Find the probability that:

i the balls drawn by David and Sally are both numbered 4

ii the balls drawn by David and Sally are both numbered 3 and from the same bag.

8 Players A, B and C each have a disc and a spinning arrow which when spun comes to rest with its head pointing towards one sector. The probability of the arrowhead stopping at any sector and the score recorded by the player is shown in the diagram. For example, player A scores 1, 4 or 8 with probabilities 0.2, 0.4 or 0.4 respectively.



If A, B and C all spin their arrows simultaneously, find the probability (assuming independence) that:

a the total of the scores is 23

b B scores more than C

c C scores more than A.

9 The journey from town A to town D passes through towns B and C. There are three roads from A to B, four from B to C and five from C to D.

a Find the number of different routes that can be taken from A to D.

There are road works on one of the roads between A and B, on two roads between B and C and on three roads between C and D.

b Find the number of different routes that can be taken from A to D without encountering any road works.

c Calculate the probability that a traveller from A to D, picking a route at random, encounters road works at each stage of their journey.

10 A, B and C fire one shot each at a target. The probability that A will hit the target is $\frac{1}{5}$, the probability that B will hit the target is $\frac{1}{4}$, and the probability that C will hit the target is $\frac{1}{3}$. If they fire together, calculate the probability that:

a all three shots hit the target

b only C's shot hits the target

c at least one shot hits the target

d given that only one shot hits the target, it is the shot by C.

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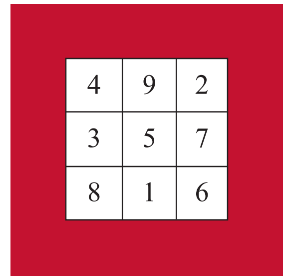
11 In a game at a fair, darts are thrown at a board. The outer square is of side length 100 cm and each of the nine inner squares is of side length 20 cm.



a Calculate, in cm^2, the area of:

i the larger outer square

ii one of the inner squares

iii the shaded region.

The number shown in each region is the score obtained by a dart hitting that region. A dart hitting the shaded region scores 0. Assume that all darts hit the board and that each point on the board is equally likely to be hit.

b Find the probability that with one dart the score will be:

i 7 **ii** at least 7 **iii** 0

c Each turn consists of throwing three darts. Calculate the probability of scoring:

i a total of 18 with the first two darts **ii** a total of 24 with the three darts.

12 Eighteen numbers are arranged in three groups of six as follows:

Group A: 0, 6, 6, 6, 6, 10

Group B: 1, 1, 1, 7, 7, 11

Group C: 3, 3, 3, 3, 8, 8

One number is drawn at random from each group. Given that a, b and c denote the numbers drawn from groups A, B and C respectively, calculate the probability that:

a $c < b$ **b** a is greater than both b and c **c** $c > a + b$

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12 Trigonometric functions

In this chapter

Syllabus references

Topics: Circular measure and radian measure; Trigonometric functions

Subtopics: 1.2.5, 1.2.7 – 1.2.16

In this chapter, the work on functions in this course is continued. Following on from our study of polynomial functions, we meet a further three important functions. Again we use the notation developed in Chapter 6 for describing functions and their properties.

You have studied trigonometry in earlier years, mainly for angles between 0° and 90°. In this chapter we see how the trigonometry you have studied may be extended to form three new functions: sine, cosine and tangent. We will see that the first two of these functions have the real numbers as their domain, and the third the real numbers without the odd multiples of $\frac{\pi}{2}$.

An important property of these three functions is that they are periodic. That is, they each repeat their values in regular intervals or periods. In general, a function f is **periodic** if there is a positive constant a such that $f(x + a) = f(x)$. The sine and cosine functions each have period 2π, while the tangent function has period π.

The sine and cosine functions are used to model wave motion, and are therefore central to the application of mathematics to any problem in which periodic motion is involved – from the motion of the tides and ocean waves to sound waves and modern telecommunications.

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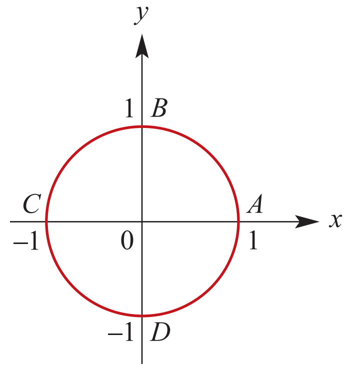
12A Measuring angles in degrees and radians

The diagram shows a **unit circle**, i.e. a circle of radius 1 unit.



The circumference of the unit circle $= 2\pi \times 1$

$= 2\pi$ units

Thus, the distance in an anticlockwise direction around the circle from

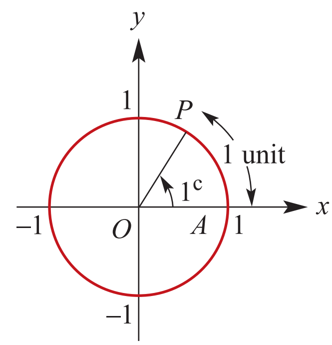
$$A \text{ to } B = \frac{\pi}{2} \text{ units}$$

$$A \text{ to } C = \pi \text{ units}$$

$$A \text{ to } D = \frac{3\pi}{2} \text{ units}$$

Definition of a radian

In moving around the circle a distance of 1 unit from A to P, the angle POA is defined. The measure of this angle is 1 radian.



One **radian** (written 1^c) is the angle subtended at the centre of the unit circle by an arc of length 1 unit.

Note: Angles formed by moving **anticlockwise** around the unit circle are defined as **positive**; those formed by moving **clockwise** are defined as **negative**.

Degrees and radians

The angle, in radians, swept out in one revolution of a circle is $2\pi^c$.

$$2\pi^c = 360^\circ$$

$$\therefore \quad \pi^c = 180^\circ$$

$$\therefore \quad 1^c = \frac{180^\circ}{\pi} \quad \text{or} \quad 1^\circ = \frac{\pi^c}{180}$$

Example 1

Convert 30° to radians.

Solution	Explanation
$1^\circ = \dfrac{\pi^c}{180}$	
$\therefore \quad 30^\circ = \dfrac{30 \times \pi}{180} = \dfrac{\pi^c}{6}$	Multiply by $\dfrac{\pi}{180}$ and simplify by cancelling.

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Example 2

Convert $\frac{\pi^c}{4}$ to degrees.

Solution	Explanation
$1^c = \frac{180^\circ}{\pi}$	
$\therefore \quad \frac{\pi^c}{4} = \frac{\pi \times 180}{4 \times \pi} = 45^\circ$	Multiply by $\frac{180}{\pi}$ and simplify by cancelling.

Note: Often the symbol for radians, ^{c}, is omitted.
For example, the angle 45° is written as $\frac{\pi}{4}$ rather than $\frac{\pi^c}{4}$.

Using the TI-Nspire

To convert 32 degrees to radians, type 32° ► **Rad** as shown.

- The degree symbol ° is found in the symbols palette (ctrl 📖) or the catalog (📖 4).
- The ► **Rad** command can be found in the catalog (📖 1 R).

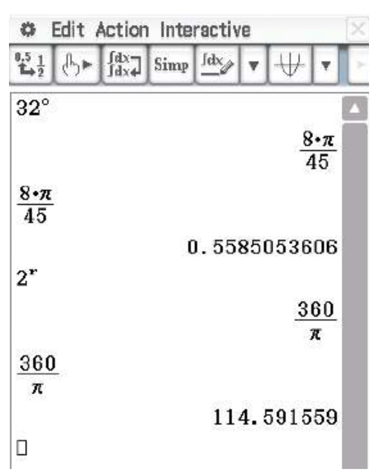
To convert 2 radians to degrees, type 2^r ► **DD** as shown.

- The radian symbol ^{r} is found in the symbols palette (ctrl 📖) or the catalog (📖 4).
- The ► **DD** command can be found in the catalog (📖 1 D).

Note: If the calculator is in radian mode, you can convert 32° to radians by simply typing 32° then enter. If the calculator is in degree mode, type 2^r then enter.

Using the Casio ClassPad

To convert 32 degrees to radians:



- Ensure your calculator is in radian mode (with **Rad** in the status bar at the bottom of the main screen).
- Enter 32° and tap EXE. The degree symbol ° is found in the Math1 keyboard.
- The answer can be displayed exactly, as shown, or highlight the answer and tap [½ ⇄ 0.5] to convert to decimal.

To convert 2 radians to degrees:

- Ensure your calculator is in degree mode (**Deg**).
- Enter 2^r and tap EXE. The radian symbol ^{r} is found in the Math1 keyboard.

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Summary 12A

- One **radian** (written 1^c) is the angle subtended at the centre of the unit circle by an arc of length 1 unit.
- To convert:
  - degrees to radians, multiply by $\frac{\pi}{180}$
  - radians to degrees, multiply by $\frac{180}{\pi}$.

Exercise 12A

Example 1

1 Express the following angles in radian measure in terms of π:

a $60°$ **b** $144°$ **c** $240°$ **d** $330°$ **e** $420°$ **f** $480°$

Example 2

2 Express, in degrees, the angles with the following radian measures:

a $\frac{2\pi}{3}$ **b** $\frac{5\pi}{6}$ **c** $\frac{7\pi}{6}$ **d** 0.9π

e $\frac{5\pi}{9}$ **f** $\frac{9\pi}{5}$ **g** $\frac{11\pi}{9}$ **h** 1.8π

3 Use a calculator to convert the following angles from radians to degrees:

a 0.6 **b** 1.89 **c** 2.9 **d** 4.31

e 3.72 **f** 5.18 **g** 4.73 **h** 6.00

4 Use a calculator to express the following in radian measure:

a $38°$ **b** $73°$ **c** $107°$ **d** $161°$

e $84.1°$ **f** $228°$ **g** $136.4°$ **h** $329°$

5 Express, in degrees, the angles with the following radian measures:

a $-\frac{\pi}{3}$ **b** -4π **c** -3π **d** $-\pi$

e $\frac{5\pi}{3}$ **f** $\frac{-11\pi}{6}$ **g** $\frac{23\pi}{6}$ **h** $\frac{-23\pi}{6}$

6 Express each of the following in radian measure in terms of π:

a $-360°$ **b** $-540°$ **c** $-240°$ **d** $-720°$ **e** $-330°$ **f** $-210°$

7 **a** On a set of axes, draw a unit circle centred at the origin and indicate the position on the unit circle corresponding to each of the following:

i $\frac{\pi}{4}$ **ii** $\frac{3\pi}{4}$ **iii** $\frac{5\pi}{4}$ **iv** $\frac{7\pi}{4}$

b On a set of axes, draw a unit circle centred at the origin and indicate the position on the unit circle corresponding to each of the following:

i $\frac{\pi}{3}$ **ii** $-\frac{\pi}{3}$ **iii** $\frac{2\pi}{3}$ **iv** $\frac{4\pi}{3}$

c On a set of axes, draw a unit circle centred at the origin and indicate the position on the unit circle corresponding to each of the following:

i $\frac{\pi}{6}$ **ii** $-\frac{7\pi}{6}$ **iii** $\frac{13\pi}{6}$ **iv** $\frac{17\pi}{6}$

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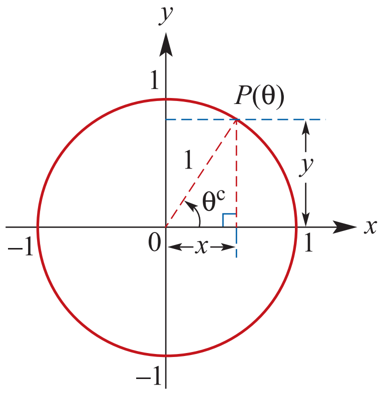
12B Defining trigonometric functions: sine and cosine

Consider the unit circle.



The position of point P on the circle can be described by relating the angle θ to the Cartesian coordinates x and y.

The point P on the circle corresponding to an angle θ is written $P(\theta)$.

The x-coordinate of $P(\theta)$ is determined by the angle θ. Similarly, the y-coordinate of $P(\theta)$ is determined by the angle θ. So we can define two functions, called sine and cosine, as follows:

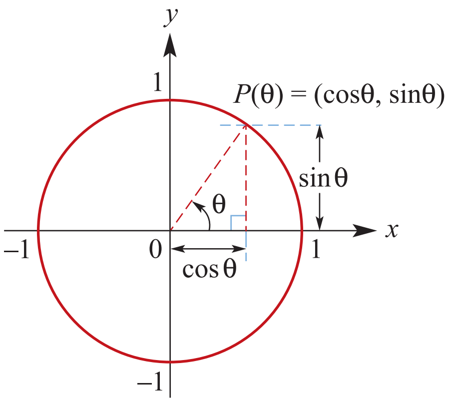
The x-coordinate of $P(\theta)$ is given by $x = \text{cosine } \theta$, for $\theta \in \mathbb{R}$.

The y-coordinate of $P(\theta)$ is given by $y = \text{sine } \theta$, for $\theta \in \mathbb{R}$.

These functions are usually written in an abbreviated form as follows:

$$x = \cos\theta$$
$$y = \sin\theta$$

Hence the coordinates of $P(\theta)$ are $(\cos\theta, \sin\theta)$.



Note: Adding 2π to the angle results in a return to the same point on the unit circle. Thus $\cos(2\pi + \theta) = \cos\theta$ and $\sin(2\pi + \theta) = \sin\theta$.

Example 3

Evaluate $\sin\pi$ and $\cos\pi$.

Solution

After moving through an angle of π, the position is $P(\pi) = (-1, 0)$.

$\therefore \quad \cos\pi = -1 \quad$ and $\quad \sin\pi = 0$

Example 4

Evaluate $\sin\left(-\frac{3\pi}{2}\right)$ and $\cos\left(-\frac{\pi}{2}\right)$.

Solution	Explanation
$\sin\left(-\frac{3\pi}{2}\right) = 1$	The point $P\left(-\frac{3\pi}{2}\right)$ has coordinates (0, 1).
$\cos\left(-\frac{\pi}{2}\right) = 0$	The point $P\left(-\frac{\pi}{2}\right)$ has coordinates (0, −1).

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Example 5

Evaluate $\sin\left(\frac{5\pi}{2}\right)$ and $\sin\left(\frac{7\pi}{2}\right)$.

Solution

$$\sin\left(\frac{5\pi}{2}\right) = \sin\left(2\frac{1}{2}\pi\right) = \sin\left(2\pi + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

$$\sin\left(\frac{7\pi}{2}\right) = \sin\left(3\frac{1}{2}\pi\right) = \sin\left(2\pi + \frac{3\pi}{2}\right) = \sin\left(\frac{3\pi}{2}\right) = -1$$

Example 6

Evaluate $\sin\left(\frac{9\pi}{2}\right)$ and $\cos(27\pi)$.

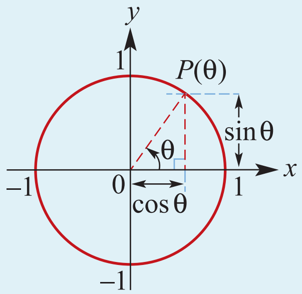
Solution

$$\sin\left(\frac{9\pi}{2}\right) = \sin\left(4\pi + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

$$\cos(27\pi) = \cos(26\pi + \pi) = \cos\pi = -1$$

Summary 12B

- $P(\theta) = (\cos\theta, \sin\theta)$
- $\cos(2\pi + \theta) = \cos\theta$ and $\sin(2\pi + \theta) = \sin\theta$
- If an angle is an odd multiple of $\frac{\pi}{2}$, then its sine is either 1 or -1, and its cosine is zero.
- If an angle is an even multiple of $\frac{\pi}{2}$, then its sine is zero, and its cosine is either 1 or -1.



Exercise 12B

Example 4, Example 5

1 For each of the following angles, t, determine the values of $\sin t$ and $\cos t$:

a $t = 0$ **b** $t = \frac{3\pi}{2}$ **c** $t = -\frac{3\pi}{2}$ **d** $t = \frac{5\pi}{2}$

Example 5, Example 6

e $t = -3\pi$ **f** $t = \frac{9\pi}{2}$ **g** $t = \frac{7\pi}{2}$ **h** $t = 4\pi$

2 Evaluate using your calculator. (Check that your calculator is in radian mode.)

a $\sin 1.9$ **b** $\sin 2.3$ **c** $\sin 4.1$ **d** $\cos 0.3$

e $\cos 2.1$ **f** $\cos(-1.6)$ **g** $\sin(-2.1)$ **h** $\sin(-3.8)$

Example 6

3 For each of the following angles, θ, determine the values of $\sin\theta$ and $\cos\theta$:

a $\theta = 27\pi$ **b** $\theta = -\frac{5\pi}{2}$ **c** $\theta = \frac{27\pi}{2}$ **d** $\theta = -\frac{9\pi}{2}$

e $\theta = \frac{11\pi}{2}$ **f** $\theta = 57\pi$ **g** $\theta = 211\pi$ **h** $\theta = -53\pi$

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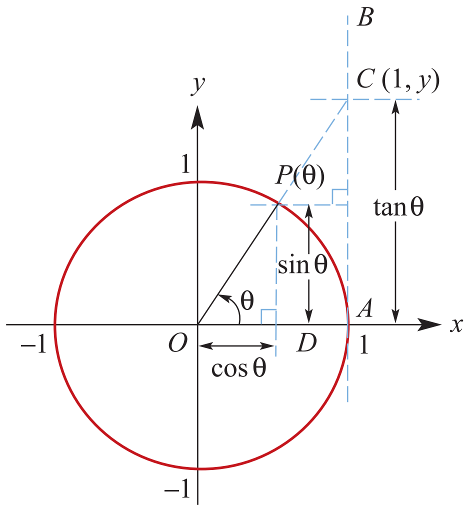
12C Another trigonometric function: tangent

Again consider the unit circle.

If we draw a tangent to the unit circle at A, then the y-coordinate of C, the point of intersection of the extension of OP and the tangent, is called **tangent** θ (abbreviated to $\tan\theta$).



By considering the similar triangles OPD and OCA:

$$\frac{\tan\theta}{1} = \frac{\sin\theta}{\cos\theta}$$

$$\therefore \quad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

Note that $\tan\theta$ is undefined when $\cos\theta = 0$.

Hence $\tan\theta$ is undefined when $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots$

The domain of tan is $\mathbb{R} \setminus \{\, \theta : \cos\theta = 0 \,\}$.

Example 7

Evaluate using a calculator. (Give answers to two decimal places.)

a $\tan 1.3$ **b** $\tan 1.9$ **c** $\tan(-2.8)$ **d** $\tan 59^\circ$ **e** $\tan 138^\circ$

Solution	Explanation
a $\tan 1.3 = 3.60$	Don't forget that your calculator must be in radian mode.
b $\tan 1.9 = -2.93$	$\cos 1.9$ is negative.
c $\tan(-2.8) = 0.36$	Both $\cos(-2.8)$ and $\sin(-2.8)$ are negative, so tan is positive.
d $\tan 59^\circ = 1.66$	Calculate in degree mode.
e $\tan 138^\circ = -0.90$	

Exercise 12C

1 Evaluate:

a $\tan\pi$ **b** $\tan(-\pi)$ **c** $\tan\left(\frac{7\pi}{2}\right)$ **d** $\tan(-2\pi)$ **e** $\tan\left(\frac{5\pi}{2}\right)$ **f** $\tan\left(-\frac{\pi}{2}\right)$

Example 7

2 Use a calculator to find correct to two decimal places:

a $\tan 1.6$ **b** $\tan(-1.2)$ **c** $\tan 136^\circ$ **d** $\tan(-54^\circ)$
e $\tan 3.9$ **f** $\tan(-2.5)$ **g** $\tan 239^\circ$

3 For each of the following values of θ, find $\tan\theta$:

a $\theta = 180^\circ$ **b** $\theta = 360^\circ$ **c** $\theta = 0$
d $\theta = -180^\circ$ **e** $\theta = -540^\circ$ **f** $\theta = 720^\circ$

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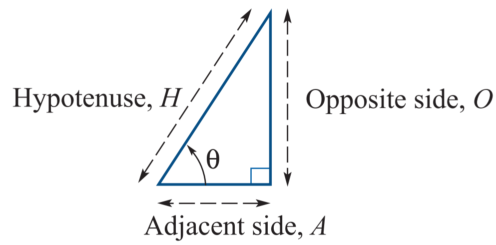
12D Reviewing trigonometric ratios

For right-angled triangles:

$$\sin\theta = \frac{O}{H}$$

$$\cos\theta = \frac{A}{H}$$

$$\tan\theta = \frac{O}{A}$$

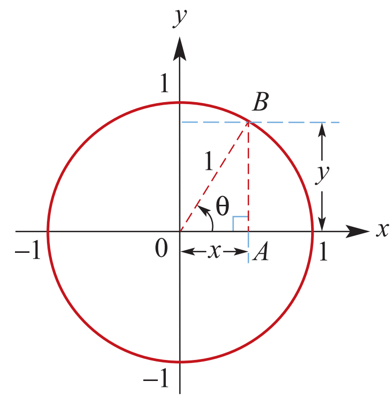


Applying these trigonometric ratios to the right-angled triangle, OAB, in the unit circle:

$$\sin\theta = \frac{O}{H} = \frac{y}{1} = y$$

$$\cos\theta = \frac{A}{H} = \frac{x}{1} = x$$

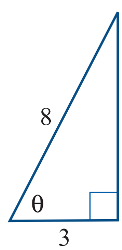
$$\tan\theta = \frac{O}{A} = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$

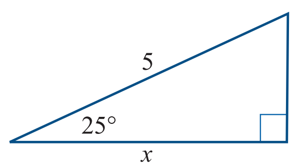


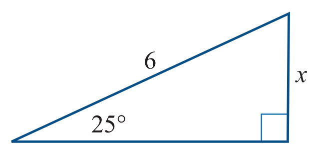
For $0 < \theta < \frac{\pi}{2}$, the functions sin, cos and tan defined by the trigonometric ratios agree with the trigonometric functions introduced in this chapter.

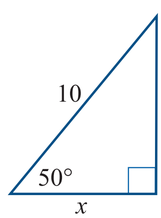
Exercise 12D

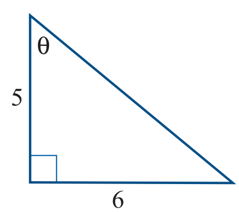
1 Find the value of the pronumeral for each of the following:

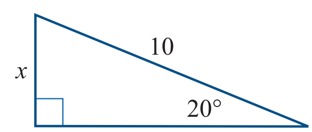
a


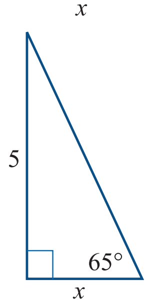
b


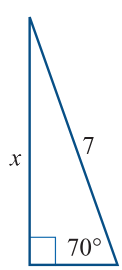
c


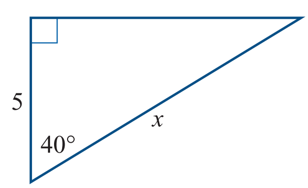
d


e


f


g


h


i


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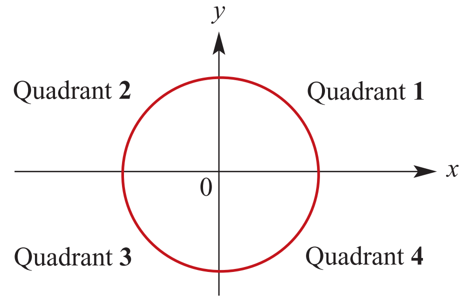
12E Symmetry properties of trigonometric functions

The coordinate axes divide the unit circle into four quadrants. The quadrants can be numbered, anticlockwise from the positive direction of the x-axis, as shown.



Using symmetry, we can determine relationships between the trigonometric functions for angles in different quadrants.

Quadrant 2

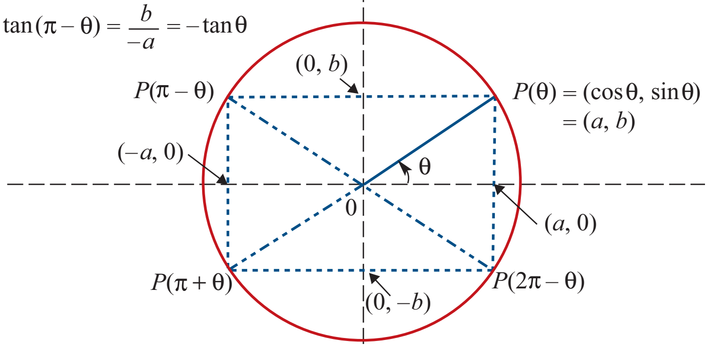
By symmetry:

$\sin(\pi - \theta) = b = \sin\theta$

$\cos(\pi - \theta) = -a = -\cos\theta$

$\tan(\pi - \theta) = \dfrac{b}{-a} = -\tan\theta$

Quadrant 1



Quadrant 3

$\sin(\pi + \theta) = -b = -\sin\theta$

$\cos(\pi + \theta) = -a = -\cos\theta$

$\tan(\pi + \theta) = \dfrac{-b}{-a} = \tan\theta$

Quadrant 4

$\sin(2\pi - \theta) = -b = -\sin\theta$

$\cos(2\pi - \theta) = a = \cos\theta$

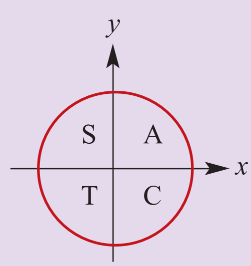
$\tan(2\pi - \theta) = \dfrac{-b}{a} = -\tan\theta$

Note: These relationships are true for all values of θ.

Signs of trigonometric functions

Using these symmetry properties, the signs of sin, cos and tan for the four quadrants can be summarised as follows:

1st quadrant	all are positive	(A)
2nd quadrant	sin is positive	(S)
3rd quadrant	tan is positive	(T)
4th quadrant	cos is positive	(C)



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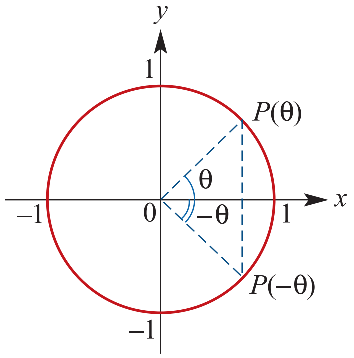
Negative of angles

By symmetry:

$$\cos(-\theta) = \cos\theta$$

$$\sin(-\theta) = -\sin\theta$$

$$\tan(-\theta) = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$



Example 8

If $\sin x = 0.6$, find the value of:

a $\sin(\pi - x)$ **b** $\sin(\pi + x)$ **c** $\sin(2\pi - x)$ **d** $\sin(-x)$

Solution

a $\sin(\pi - x) = \sin x = 0.6$

b $\sin(\pi + x) = -\sin x = -0.6$

c $\sin(2\pi - x) = -\sin x = -0.6$

d $\sin(-x) = -\sin x = -0.6$

Example 9

If $\cos x^\circ = 0.8$, find the value of:

a $\cos(180 - x)^\circ$ **b** $\cos(180 + x)^\circ$ **c** $\cos(360 - x)^\circ$ **d** $\cos(-x)^\circ$

Solution

a $\cos(180 - x)^\circ = -\cos x^\circ = -0.8$

b $\cos(180 + x)^\circ = -\cos x^\circ = -0.8$

c $\cos(360 - x)^\circ = \cos x^\circ = 0.8$

d $\cos(-x)^\circ = \cos x^\circ = 0.8$

Exercise 12E

Example 8

1 If $\sin\theta = 0.42$, $\cos x = 0.7$ and $\tan\alpha = 0.38$, write down the values of:

a $\sin(\pi + \theta)$ **b** $\cos(\pi - x)$ **c** $\sin(2\pi - \theta)$ **d** $\tan(\pi - \alpha)$

e $\sin(\pi - \theta)$ **f** $\tan(2\pi - \alpha)$ **g** $\cos(\pi + x)$ **h** $\cos(2\pi - x)$

2 **a** If $\cos x = -\cos\left(\frac{\pi}{6}\right)$ and $\frac{\pi}{2} < x < \pi$, find the value of x.

b If $\cos x = -\cos\left(\frac{\pi}{6}\right)$ and $\pi < x < \frac{3\pi}{2}$, find the value of x.

c If $\cos x = \cos\left(\frac{\pi}{6}\right)$ and $\frac{3\pi}{2} < x < 2\pi$, find the value of x.

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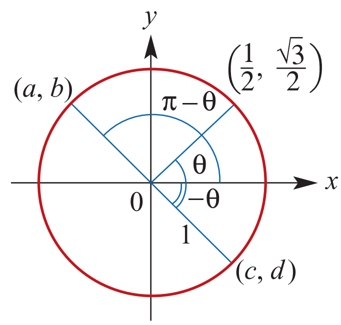
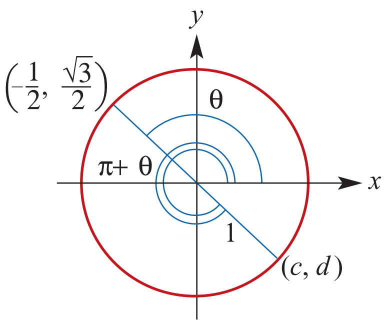
3 For the diagram shown, write down the values of:

a $a = \cos(\pi - \theta)$
b $b = \sin(\pi - \theta)$
c $c = \cos(-\theta)$
d $d = \sin(-\theta)$
e $\tan(\pi - \theta)$
f $\tan(-\theta)$

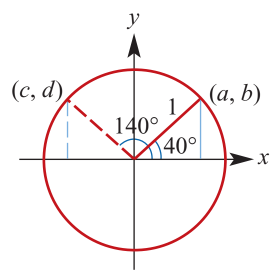


4 For the diagram shown, write down the values of:

a $d = \sin(\pi + \theta)$
b $c = \cos(\pi + \theta)$
c $\tan(\pi + \theta)$
d $\sin(2\pi - \theta)$
e $\cos(2\pi - \theta)$



5 **a** For the diagram shown, use your calculator to find a and b correct to four decimal places.
b Hence find the values of c and d.
c **i** Use your calculator to find $\cos 140°$ and $\sin 140°$.
ii Write $\cos 140°$ in terms of $\cos 40°$.



Example 9

6 If $\sin x° = 0.7$, $\cos \theta° = 0.6$ and $\tan \alpha° = 0.4$, write down the values of:

a $\sin(180 + x)°$
b $\cos(180 + \theta)°$
c $\tan(360 - \alpha)°$
d $\cos(180 - \theta)°$
e $\sin(360 - x)°$
f $\sin(-x)°$
g $\tan(360 + \alpha)°$
h $\cos(-\theta)°$

7 **a** If $\sin x° = \sin 60°$ and $90° < x° < 180°$, find the value of x.
b If $\sin x° = -\sin 60°$ and $180° < x° < 270°$, find the value of x.
c If $\sin x° = -\sin 60°$ and $-90° < x° < 0°$, find the value of x.
d If $\cos x° = -\cos 60°$ and $90° < x° < 180°$, find the value of x.
e If $\cos x° = -\cos 60°$ and $180° < x° < 270°$, find the value of x.
f If $\cos x° = \cos 60°$ and $270° < x° < 360°$, find the value of x.

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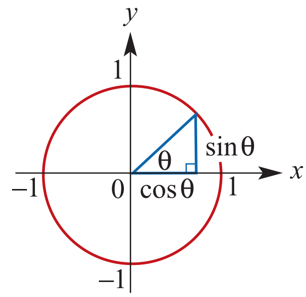
12F Exact values of trigonometric functions

A calculator can be used to find the values of the trigonometric functions for different values of θ. For many values of θ, the calculator gives an approximation. We now consider some values of θ such that sin, cos and tan can be calculated exactly.

Exact values for 0 (0°) and $\frac{\pi}{2}$ (90°)



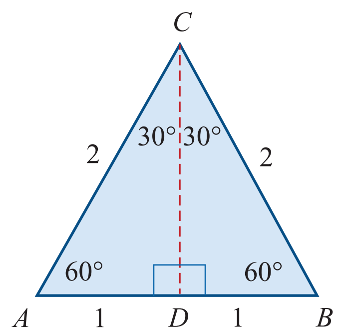
From the unit circle:

$\sin 0° = 0$ $\quad$ $\sin 90° = 1$

$\cos 0° = 1$ $\quad$ $\cos 90° = 0$

$\tan 0° = 0$ $\quad$ $\tan 90°$ is undefined

Exact values for $\frac{\pi}{6}$ (30°) and $\frac{\pi}{3}$ (60°)



Consider an equilateral triangle ABC of side length 2 units.
In $\triangle ACD$, by Pythagoras' theorem, $CD = \sqrt{AC^2 - AD^2} = \sqrt{3}$.

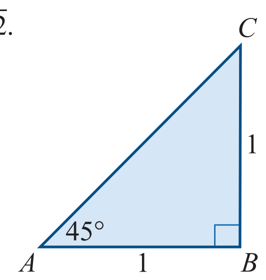
$$\sin 30° = \frac{AD}{AC} = \frac{1}{2} \qquad \sin 60° = \frac{CD}{AC} = \frac{\sqrt{3}}{2}$$

$$\cos 30° = \frac{CD}{AC} = \frac{\sqrt{3}}{2} \qquad \cos 60° = \frac{AD}{AC} = \frac{1}{2}$$

$$\tan 30° = \frac{AD}{CD} = \frac{1}{\sqrt{3}} \qquad \tan 60° = \frac{CD}{AD} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

Exact values for $\frac{\pi}{4}$ (45°)

For the triangle ABC shown on the right, we have $AC = \sqrt{1^2 + 1^2} = \sqrt{2}$.



$$\sin 45° = \frac{BC}{AC} = \frac{1}{\sqrt{2}}$$

$$\cos 45° = \frac{AB}{AC} = \frac{1}{\sqrt{2}}$$

$$\tan 45° = \frac{BC}{AB} = 1$$

Example 10

Evaluate:

a $\cos 150°$ $\quad$ **b** $\sin 690°$

Solution

a $\cos 150° = \cos(180 - 30)°$

$= -\cos 30° = -\frac{\sqrt{3}}{2}$

b $\sin 690° = \sin(2 \times 360 - 30)°$

$= \sin(-30°) = -\frac{1}{2}$

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Example 11

Evaluate:

a $\cos\left(\frac{5\pi}{4}\right)$ **b** $\sin\left(\frac{11\pi}{6}\right)$

Solution

a $\cos\left(\frac{5\pi}{4}\right) = \cos\left(\pi + \frac{\pi}{4}\right)$

$= -\cos\left(\frac{\pi}{4}\right)$ (by symmetry)

$= -\frac{1}{\sqrt{2}}$

b $\sin\left(\frac{11\pi}{6}\right) = \sin\left(2\pi - \frac{\pi}{6}\right)$

$= -\sin\left(\frac{\pi}{6}\right)$ (by symmetry)

$= -\frac{1}{2}$

Summary 12F

As an aid to memory, the exact values for trigonometric functions can be tabulated.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$ (30°)	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$ (45°)	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$ (60°)	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$ (90°)	1	0	undefined

Exercise 12F

Example 10

1 Without using a calculator, evaluate the sin, cos and tan of each of the following:

a 120° **b** 135° **c** 210° **d** 240° **e** 315°
f 390° **g** 420° **h** −135° **i** −300° **j** −60°

Example 11

2 Write down the exact values of:

a $\sin\left(\frac{2\pi}{3}\right)$ **b** $\cos\left(\frac{3\pi}{4}\right)$ **c** $\tan\left(\frac{5\pi}{6}\right)$

d $\sin\left(\frac{7\pi}{6}\right)$ **e** $\cos\left(\frac{5\pi}{4}\right)$ **f** $\tan\left(\frac{4\pi}{3}\right)$

g $\sin\left(\frac{5\pi}{3}\right)$ **h** $\cos\left(\frac{7\pi}{4}\right)$ **i** $\tan\left(\frac{11\pi}{6}\right)$

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3 Write down the exact values of:

a $\sin\left(-\frac{2\pi}{3}\right)$ **b** $\cos\left(\frac{11\pi}{4}\right)$ **c** $\tan\left(\frac{13\pi}{6}\right)$ **d** $\tan\left(\frac{15\pi}{6}\right)$

e $\cos\left(\frac{14\pi}{4}\right)$ **f** $\cos\left(-\frac{3\pi}{4}\right)$ **g** $\sin\left(\frac{11\pi}{4}\right)$ **h** $\cos\left(-\frac{21\pi}{3}\right)$

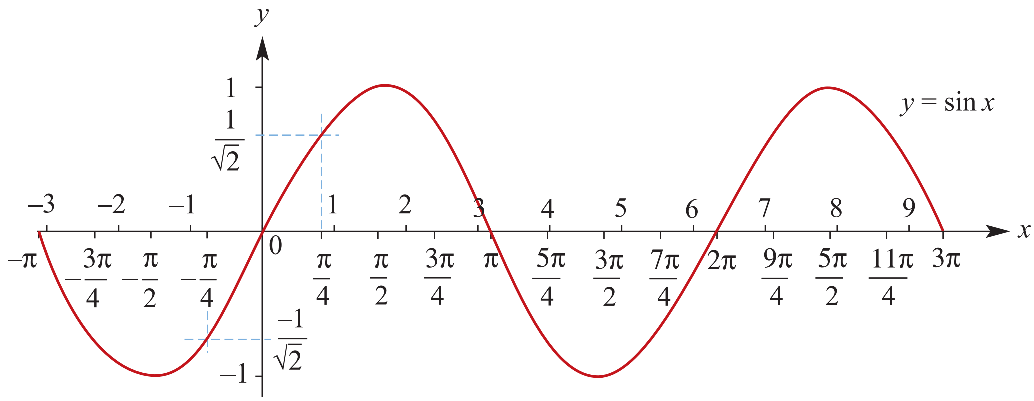
12G Graphs of sine and cosine

Graph of the sine function

A table of exact values for $y = \sin x$ is given below.

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π
y	0	$\frac{-1}{\sqrt{2}}$	-1	$\frac{-1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$\frac{-1}{\sqrt{2}}$	-1	$\frac{-1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0

A calculator can be used to plot the graph of $y = \sin x$ (for $-\pi \le x \le 3\pi$). Note that radian mode must be selected.



Observations from the graph of $y = \sin x$

- The graph repeats itself after an interval of 2π units.
  A function which repeats itself regularly is called a **periodic** function, and the interval between the repetitions is called the **period** of the function (also called the wavelength). Thus $\sin x$ has a period of 2π units.
- The maximum and minimum values of $\sin x$ are 1 and -1 respectively.
  The distance between the mean position and the maximum position is called the **amplitude**. The graph of $y = \sin x$ has an amplitude of 1.

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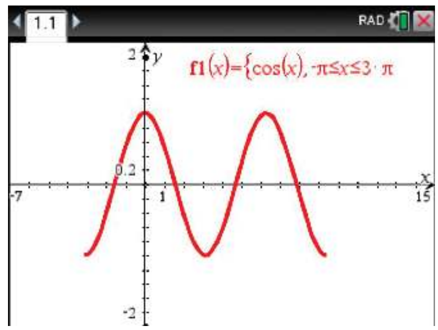
Graph of the cosine function

A table of exact values for $y = \cos x$ is given below.

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π
y	-1	$\frac{-1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$\frac{-1}{\sqrt{2}}$	-1	$\frac{-1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$\frac{-1}{\sqrt{2}}$	-1

Using the TI-Nspire

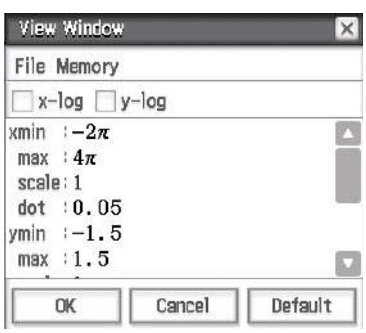
- A graph of $y = \cos x$ for $-\pi \le x \le 3\pi$ can be plotted in a **Graphs** application by entering $f_1(x) = \cos(x) \mid -\pi \le x \le 3\pi$.
- Change the window to suit (menu > **Window/Zoom** > **Window Settings**).

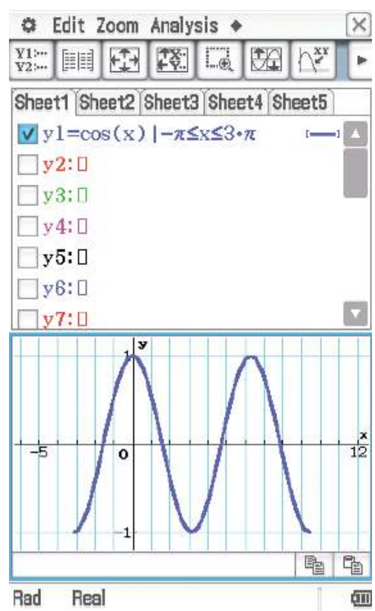


Using the Casio ClassPad

To plot the graph of $y = \cos x$ for $-\pi \le x \le 3\pi$:

- In **Graph & Table**, enter the equation in $y1$ as shown, then tick to select and tap to produce the graph.
- Select the icon to adjust the window settings as shown below.





Observations from the graph of $y = \cos x$

- The period is 2π and the amplitude is 1.
- The graph of $y = \cos x$ is the graph of $y = \sin x$ translated $\frac{\pi}{2}$ units in the negative direction of the x-axis.

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Sketch graphs of $y = a\sin(nt)$ and $y = a\cos(nt)$

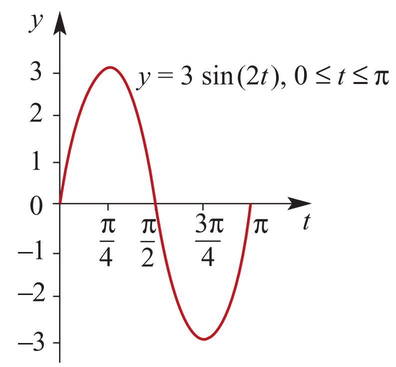
The graphs of functions of the forms $y = a\sin(nt)$ and $y = a\cos(nt)$ are transformations of the graphs of $y = \sin t$ and $y = \cos t$ respectively. We first consider the case where a and n are positive numbers.

Transformations: dilations

Graph of $y = 3\sin(2t)$ The image of the graph of $y = \sin t$ under a dilation of factor 3 parallel to the y-axis and a dilation of factor $\frac{1}{2}$ parallel to the t-axis is $y = 3\sin(2t)$.

Note: Let $f(t) = \sin t$. Then the graph of $y = f(t)$ is transformed to the graph of $y = 3f(2t)$. The point with coordinates (t, y) is mapped to the point with coordinates $\left(\frac{t}{2}, 3y\right)$.

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$y = 3\sin(2t)$	0	3	0	-3	0

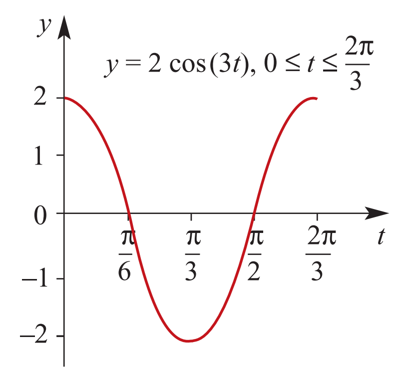


We make the following observations about the graph of $y = 3\sin(2t)$:

- amplitude is 3
- period is π

Graph of $y = 2\cos(3t)$ The image of the graph of $y = \cos t$ under a dilation of factor 2 parallel to the y-axis and a dilation of factor $\frac{1}{3}$ parallel to the t-axis is $y = 2\cos(3t)$.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
$y = 2\cos(3t)$	2	0	-2	0	2



We make the following observations about the graph of $y = 2\cos(3t)$:

- amplitude is 2
- period is $\frac{2\pi}{3}$

Amplitude and period Comparing these results with those for $y = \sin t$ and $y = \cos t$, the following general rules can be stated for a and n positive:

Function	Amplitude	Period
$y = a\sin(nt)$	a	$\frac{2\pi}{n}$
$y = a\cos(nt)$	a	$\frac{2\pi}{n}$

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Example 12

For each of the following functions with domain $\mathbb{R}$, state the amplitude and period:

a $f(t) = 2\sin(3t)$

b $f(t) = -\frac{1}{2}\sin\left(\frac{t}{2}\right)$

c $f(t) = 4\cos(3\pi t)$

Solution

a Amplitude is 2

Period is $\frac{2\pi}{3}$

b Amplitude is $\frac{1}{2}$

Period is $2\pi \div \frac{1}{2} = 4\pi$

c Amplitude is 4

Period is $\frac{2\pi}{3\pi} = \frac{2}{3}$

Graphs of $y = a\sin(nt)$ and $y = a\cos(nt)$

In general, for a and n positive numbers, the graph of $y = a\sin(nt)$ (or $y = a\cos(nt)$) is obtained from the graph of $y = \sin t$ (or $y = \cos t$) by a dilation of factor a parallel to the y-axis and a dilation of factor $\frac{1}{n}$ parallel to the t-axis.

The point with coordinates (t, y) is mapped to the point with coordinates $\left(\frac{t}{n}, ay\right)$.

The following are important properties of both of the functions $f(t) = a\sin(nt)$ and $g(t) = a\cos(nt)$:

- The period is $\frac{2\pi}{n}$.
- The amplitude is a.
- The natural domain is $\mathbb{R}$.
- The range is $[-a, a]$.

Example 13

For each of the following, give a sequence of transformations which takes the graph of $y = \sin x$ to the graph of $y = g(x)$, and state the amplitude and period of $g(x)$:

a $g(x) = 3\sin(2x)$

b $g(x) = 4\sin\left(\frac{x}{2}\right)$

Solution

a The graph of $y = 3\sin(2x)$ is obtained from the graph of $y = \sin x$ by a dilation of factor 3 parallel to the y-axis and a dilation of factor $\frac{1}{2}$ parallel to the x-axis.

The function $g(x) = 3\sin(2x)$ has amplitude 3 and period $\frac{2\pi}{2} = \pi$.

b The graph of $y = 4\sin\left(\frac{x}{2}\right)$ is obtained from the graph of $y = \sin x$ by a dilation of factor 4 parallel to the y-axis and a dilation of factor 2 parallel to the x-axis.

The function $g(x) = 4\sin\left(\frac{x}{2}\right)$ has amplitude 4 and period $2\pi \div \frac{1}{2} = 4\pi$.

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Example 14

Sketch the graph of each of the following functions:

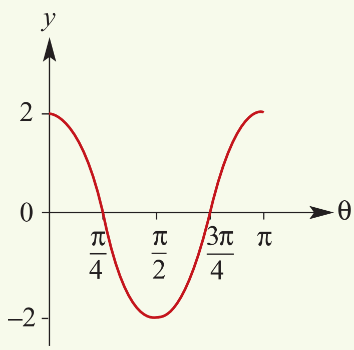
a $y = 2\cos(2\theta)$

b $y = \frac{1}{2}\sin\left(\frac{x}{2}\right)$

In each case, show one complete cycle.

Solution

a $y = 2\cos(2\theta)$



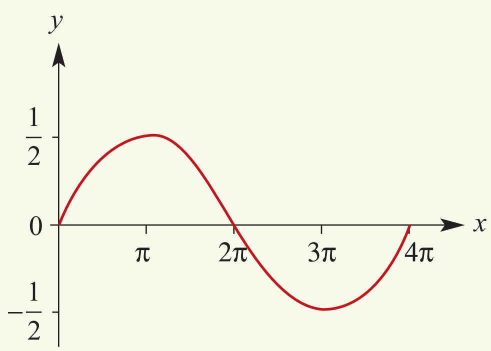
Explanation

The amplitude is 2.

The period is $\frac{2\pi}{2} = \pi$.

The graph of $y = 2\cos(2\theta)$ is obtained from the graph of $y = \cos\theta$ by a dilation of factor 2 parallel to the y-axis and a dilation of factor $\frac{1}{2}$ parallel to the θ-axis.

b $y = \frac{1}{2}\sin\left(\frac{x}{2}\right)$



The amplitude is $\frac{1}{2}$.

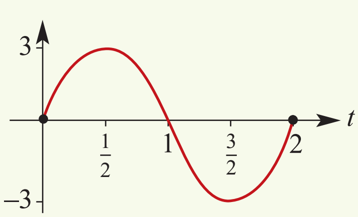
The period is $2\pi \div \frac{1}{2} = 4\pi$.

The graph of $y = \frac{1}{2}\sin\left(\frac{x}{2}\right)$ is obtained from the graph of $y = \sin x$ by a dilation of factor $\frac{1}{2}$ parallel to the y-axis and a dilation of factor 2 parallel to the x-axis.

Example 15

Sketch the graph of $f(t) = 3\sin(\pi t)$, $\{ t \in \mathbb{R} : 0 \le t \le 2 \}$.

Solution



Explanation

The amplitude is 3.

The period is $2\pi \div \pi = 2$.

The graph of $f(t) = 3\sin(\pi t)$ is obtained from the graph of $y = \sin t$ by a dilation of factor 3 parallel to the y-axis and a dilation of factor $\frac{1}{\pi}$ parallel to the t-axis.

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Transformations: reflection in the horizontal axis

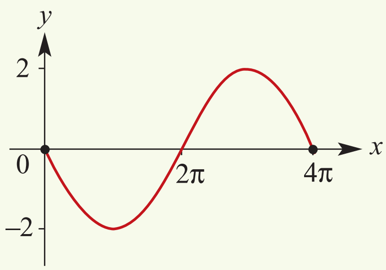
Example 16

Sketch the following graphs for $x \in [0, 4\pi]$:

a $f(x) = -2\sin\left(\frac{x}{2}\right)$ **b** $y = -\cos(2x)$

Solution

a

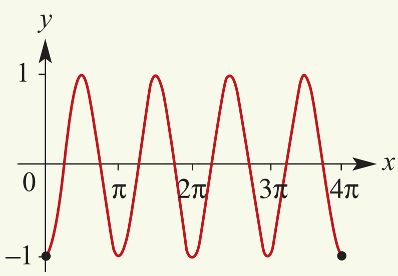


Explanation

The graph of $f(x) = -2\sin\left(\frac{x}{2}\right)$ is obtained from the graph of $y = 2\sin\left(\frac{x}{2}\right)$ by a reflection in the x-axis.

The amplitude is 2 and the period is 4π.

b



The graph of $y = -\cos(2x)$ is obtained from the graph of $y = \cos(2x)$ by a reflection in the x-axis.

The amplitude is 1 and the period is π.

Transformations: reflection in the vertical axis

Remember that $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$.

Hence, when reflected in the y-axis:

- The graph of $y = \cos x$ transforms onto itself. (The point (α, β) is mapped to the point $(-\alpha, \beta)$. This second point is on the curve $y = \cos x$ by the observation above.)
- The graph of $y = \sin x$ transforms onto the graph of $y = -\sin x$. (The point (α, β) is mapped to the point $(-\alpha, \beta)$. This second point is on the curve $y = -\sin x$ by the observation above.)

Summary 12G

For positive numbers a and n, the graphs of $y = a\sin(nt)$, $y = -a\sin(nt)$, $y = a\cos(nt)$ and $y = -a\cos(nt)$ all have the following properties:

- The period is $\frac{2\pi}{n}$.
- The amplitude is a.
- The natural domain is $\mathbb{R}$.
- The range is $[-a, a]$.

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Skill-sheet

Exercise 12G

Example 12

1 Write down **i** the period and **ii** the amplitude of each of the following:

a $2\sin\theta$ **b** $3\sin(2\theta)$ **c** $\frac{1}{2}\cos(3\theta)$

d $3\sin\left(\frac{1}{2}\theta\right)$ **e** $4\cos(3\theta)$ **f** $-\frac{1}{2}\sin(4\theta)$

g $-2\cos\left(\frac{1}{2}\theta\right)$ **h** $2\cos(\pi t)$ **i** $-3\sin\left(\frac{\pi t}{2}\right)$

Example 13

2 For each of the following, give a sequence of transformations which takes the graph of $y = \sin x$ to the graph of $y = g(x)$, and state the amplitude and period of $g(x)$:

a $g(x) = 3\sin x$ **b** $g(x) = \sin(5x)$

c $g(x) = \sin\left(\frac{x}{3}\right)$ **d** $g(x) = 2\sin(5x)$

Example 14

3 Sketch the graph of each of the following, showing one complete cycle. State the amplitude and period.

a $y = 3\sin(2x)$ **b** $y = 2\cos(3\theta)$ **c** $y = 4\sin\left(\frac{\theta}{2}\right)$

d $y = \frac{1}{2}\cos(3x)$ **e** $y = 4\sin(3x)$ **f** $y = 5\cos(2x)$

g $y = -3\cos\left(\frac{\theta}{2}\right)$ **h** $y = 2\cos(4\theta)$ **i** $y = -2\sin\left(\frac{\theta}{3}\right)$

Example 15

4 Sketch the graph of:

a $f(t) = 2\cos(\pi t), \{ t \in \mathbb{R} : 0 \le t \le 2 \}$ **b** $f(t) = 3\sin(2\pi t), \{ t \in \mathbb{R} : 0 \le t \le 2 \}$

Example 16

5 Sketch the graph of:

a $f(x) = \sin(2x)$ for $x \in [-2\pi, 2\pi]$ **b** $f(x) = 2\sin\left(\frac{x}{3}\right)$ for $x \in [-6\pi, 6\pi]$

c $f(x) = 2\cos(3x)$ for $x \in [0, 2\pi]$ **d** $f(x) = -2\sin(3x)$ for $x \in [0, 2\pi]$

6 Sketch the graph of $f(x) = \frac{5}{2}\cos\left(\frac{2x}{3}\right), \{ x \in \mathbb{R} : 0 \le x \le 2\pi \}$.
Hint: For the endpoints, find $f(0)$ and $f(2\pi)$.

7 For each of the following, give a sequence of transformations which takes the graph of $y = \sin x$ to the graph of $y = g(x)$, and state the amplitude and period of $g(x)$:

a $g(x) = -\sin(5x)$ **b** $g(x) = \sin(-x)$ **c** $g(x) = 2\sin\left(\frac{x}{3}\right)$

d $g(x) = -4\sin\left(\frac{x}{2}\right)$ **e** $g(x) = 2\sin\left(\frac{-x}{3}\right)$

8 **a** On the one set of axes, sketch the graphs of $f(x) = \sin x, \{ x \in \mathbb{R} : 0 \le x \le 2\pi \}$ and $g(x) = \cos x, \{ x \in \mathbb{R} : 0 \le x \le 2\pi \}$.

b By inspection from these graphs, state the values of x for which $\sin x = \cos x$.

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12H Solution of trigonometric equations

In this section we introduce methods for solving equations of the form $a\sin(nt) = b$ and $a\cos(nt) = b$.

Solving equations of the form $\sin t = b$ and $\cos t = b$

First we look at the techniques for solving equations of the form $\sin t = b$ and $\cos t = b$. These same techniques will be applied to solve more complicated trigonometric equations later in this section.

Example 17

Find all solutions to the equation $\sin\theta = \frac{1}{2}$ for $\theta \in [0, 4\pi]$.

Solution

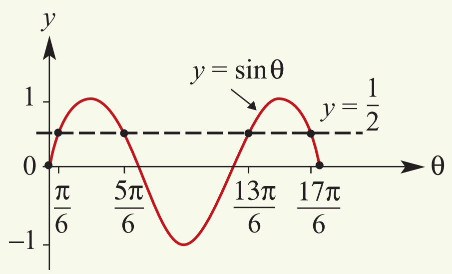
The solution for $\theta \in \left[0, \frac{\pi}{2}\right]$ is $\theta = \frac{\pi}{6}$.

The second solution is $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

The third solution is $\theta = 2\pi + \frac{\pi}{6} = \frac{13\pi}{6}$.

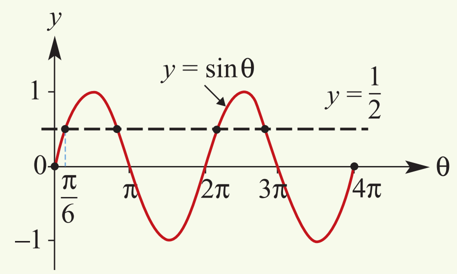
The fourth solution is $\theta = 2\pi + \frac{5\pi}{6} = \frac{17\pi}{6}$.

These four solutions are shown on the graph below.



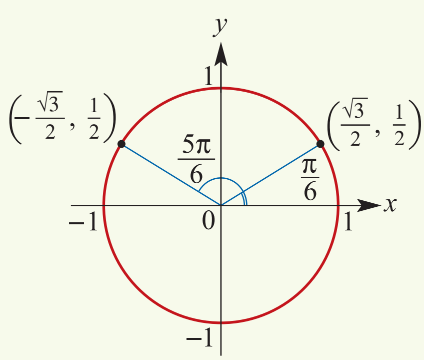
Explanation

By sketching a graph, we can see that there are four solutions in the interval $[0, 4\pi]$.



The first solution can be obtained from a knowledge of exact values or by using $\sin^{-1}$ on your calculator.

The second solution is obtained using symmetry. The sine function is positive in the 2nd quadrant and $\sin(\pi - \theta) = \sin\theta$.



Further solutions are found by adding 2π, since $\sin\theta = \sin(2\pi + \theta)$.

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Example 18

Find two values of x:

a $\sin x = -0.3$ with $0 \le x \le 2\pi$

b $\cos x^\circ = -0.7$ with $0^\circ \le x^\circ \le 360^\circ$

Solution

a First solve the equation $\sin \alpha = 0.3$ for $\alpha \in \left[0, \frac{\pi}{2}\right]$. Use your calculator to find the solution $\alpha = 0.30469\ldots$

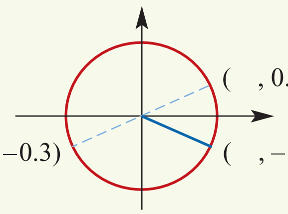
Now the value of $\sin x$ is negative for $P(x)$ in the 3rd and 4th quadrants. From the symmetry relationships (or from the graph of $y = \sin x$):

3rd quadrant: $x = \pi + 0.30469\ldots$

$= 3.446$ (to 3 d.p.)

4th quadrant: $x = 2\pi - 0.30469\ldots$

$= 5.978$ (to 3 d.p.)



∴ If $\sin x = -0.3$, then $x = 3.446$ or $x = 5.978$.

b First solve the equation $\cos \alpha^\circ = 0.7$ for $\alpha^\circ \in [0^\circ, 90^\circ]$. Use your calculator to find the solution $\alpha^\circ = 45.57^\circ$.

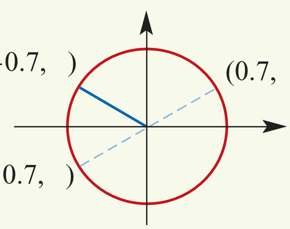
Now the value of $\cos x^\circ$ is negative for $P(x^\circ)$ in the 2nd and 3rd quadrants.

2nd quadrant: $x^\circ = 180^\circ - 45.57^\circ$

$= 134.43^\circ$

3rd quadrant: $x^\circ = 180^\circ + 45.57^\circ$

$= 225.57^\circ$



∴ If $\cos x^\circ = -0.7$, then $x^\circ = 134.43^\circ$ or $x^\circ = 225.57^\circ$.

Example 19

Find all the values of θ° between 0° and 360° for which:

a $\cos \theta^\circ = \frac{\sqrt{3}}{2}$ **b** $\sin \theta^\circ = -\frac{1}{2}$ **c** $\cos \theta^\circ - \frac{1}{\sqrt{2}} = 0$

Solution

a $\cos \theta^\circ = \frac{\sqrt{3}}{2}$

$\theta^\circ = 30^\circ$ or $\theta^\circ = 360^\circ - 30^\circ$

$\theta^\circ = 30^\circ$ or $\theta^\circ = 330^\circ$

Explanation

$\cos \theta^\circ$ is positive, and so $P(\theta^\circ)$ lies in the 1st or 4th quadrant.

$\cos(360^\circ - \theta^\circ) = \cos \theta^\circ$

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b $\sin\theta^\circ = -\dfrac{1}{2}$

$\theta^\circ = 180^\circ + 30^\circ$ or $\theta^\circ = 360^\circ - 30^\circ$

$\theta^\circ = 210^\circ$ or $\theta^\circ = 330^\circ$

$\sin\theta^\circ$ is negative, and so $P(\theta^\circ)$ lies in the 3rd or 4th quadrant.

$\sin(180^\circ + \theta^\circ) = -\sin\theta^\circ$

$\sin(360^\circ - \theta^\circ) = -\sin\theta^\circ$

c $\cos\theta^\circ - \dfrac{1}{\sqrt{2}} = 0$

$\therefore \quad \cos\theta^\circ = \dfrac{1}{\sqrt{2}}$

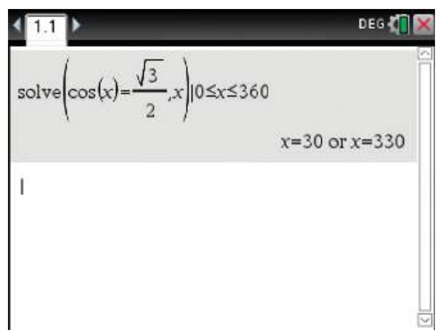
$\theta^\circ = 45^\circ$ or $\theta^\circ = 360^\circ - 45^\circ$

$\theta^\circ = 45^\circ$ or $\theta^\circ = 315^\circ$

$\cos\theta^\circ$ is positive, and so $P(\theta^\circ)$ lies in the 1st or 4th quadrant.

Using the TI-Nspire

For Example 19a, make sure the calculator is in degree mode and complete as shown.

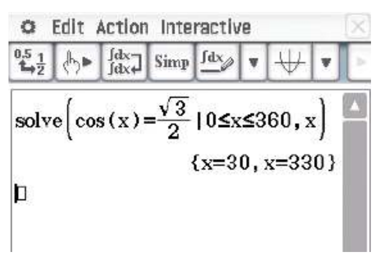


Using the Casio ClassPad

- Ensure your calculator is in degree mode (with **Deg** in the status bar at the bottom of the main screen).
- Use the [Math1] and [Math3] keyboards to enter the equation

$$\cos(x) = \frac{\sqrt{3}}{2} \mid 0 \le x \le 360$$

- Highlight the equation and domain. Then select **Interactive** > **Equation/Inequality** > **solve** and ensure the variable is set to x.



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Solving equations of the form $a\sin(nt) = b$ and $a\cos(nt) = b$

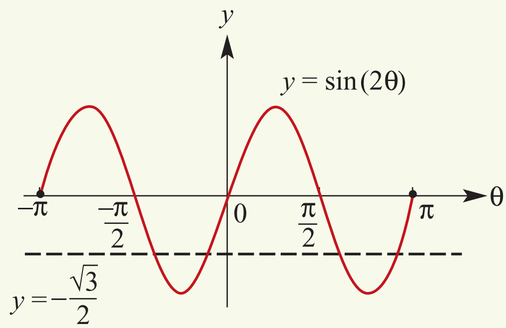
The techniques introduced above can be applied in a more general situation. This is achieved by a simple substitution, as shown in the following example.

Example 20

Solve the equation $\sin(2\theta) = -\frac{\sqrt{3}}{2}$ for $\theta \in [-\pi, \pi]$.

Solution

It is clear from the graph that there are four solutions.



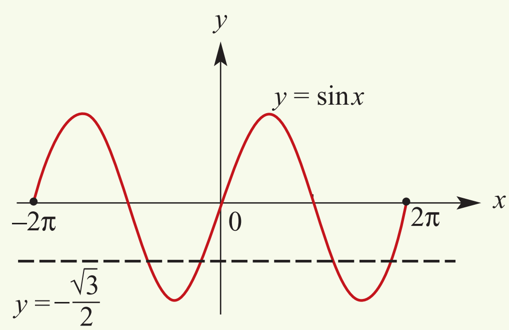
To solve the equation, let $x = 2\theta$.

Note: If $\theta \in [-\pi, \pi]$, then we have
$x = 2\theta \in [-2\pi, 2\pi]$.

Now consider the equation

$$\sin x = -\frac{\sqrt{3}}{2} \quad \text{for } x \in [-2\pi, 2\pi]$$

The 1st quadrant solution to the equation $\sin\alpha = \frac{\sqrt{3}}{2}$ is $\alpha = \frac{\pi}{3}$.



Using symmetry, the solutions to $\sin x = -\frac{\sqrt{3}}{2}$ for $x \in [0, 2\pi]$ are

$x = \pi + \frac{\pi}{3}$ and $x = 2\pi - \frac{\pi}{3}$

i.e. $x = \frac{4\pi}{3}$ and $x = \frac{5\pi}{3}$

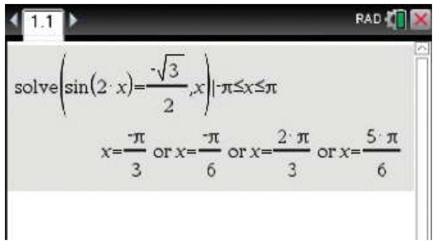
The other two solutions (obtained by subtracting 2π) are $x = \frac{4\pi}{3} - 2\pi$ and $x = \frac{5\pi}{3} - 2\pi$.

$\therefore$ The required solutions for x are $-\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{4\pi}{3}$ and $\frac{5\pi}{3}$.

$\therefore$ The required solutions for θ are $-\frac{\pi}{3}, -\frac{\pi}{6}, \frac{2\pi}{3}$ and $\frac{5\pi}{6}$.

Using the TI-Nspire

Ensure that the calculator is in radian mode and complete as shown.



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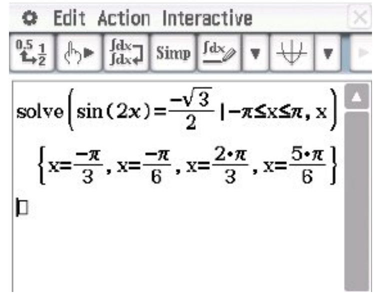
Using the Casio ClassPad

- Ensure your calculator is in radian mode (with **Rad** in the status bar at bottom of the main screen).
- Enter the equation

$$\sin(2x) = \frac{-\sqrt{3}}{2} \mid -\pi \le x \le \pi$$

- Highlight the equation and domain. Then select **Interactive** > **Equation/Inequality** > **solve** and ensure the variable is set to x.



Summary 12H

- For solving equations of the form $\sin t = b$ and $\cos t = b$:
  - First find the solutions in the interval $[0, 2\pi]$. This can be done using your knowledge of exact values and symmetry properties, or with the aid of a calculator.
  - Further solutions can be found by adding and subtracting multiples of 2π.
- For solving equations of the form $a\sin(nt) = b$ and $a\cos(nt) = b$:
  - First substitute $x = nt$. Work out the interval in which solutions for x are required. Then proceed as in the case above to solve for x.
  - Once the solutions for x are found, the solutions for t can be found.

  For example: To solve $\sin(3t) = \frac{1}{2}$ for $t \in [0, 2\pi]$, first let $x = 3t$. The equation becomes $\sin x = \frac{1}{2}$ and the required solutions for x are in the interval $[0, 6\pi]$.

Skill-sheet

Exercise 12H

Example 17

1 Find all the values of x between 0 and 4π for which:

a $\cos x = \frac{1}{2}$ **b** $\sin x = \frac{1}{\sqrt{2}}$ **c** $\sin x = \frac{\sqrt{3}}{2}$

Example 18

2 Find, correct to two decimal places, all the values of x between 0 and 2π for which:

a $\sin x = 0.8$ **b** $\cos x = -0.4$ **c** $\sin x = -0.35$

d $\sin x = 0.4$ **e** $\cos x = -0.7$ **f** $\cos x = -0.2$

Example 19

3 Without using a calculator, find all the values of $\theta°$ between 0° and 360° for each of the following:

a $\cos\theta° = -\frac{\sqrt{3}}{2}$ **b** $\sin\theta° = \frac{1}{2}$ **c** $\cos\theta° = -\frac{1}{2}$

d $2\cos(\theta°) + 1 = 0$ **e** $2\sin\theta° = \sqrt{3}$ **f** $\sqrt{2}\sin(\theta°) - 1 = 0$

4 Without using a calculator, find all the values of x between 0 and 2π for each of the following:

a $2\cos x = \sqrt{3}$ **b** $\sqrt{2}\sin(x) + 1 = 0$ **c** $\sqrt{2}\cos(x) - 1 = 0$

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5 Find all the values of x between $-\pi$ and π for which:

a $\cos x = -\frac{1}{\sqrt{2}}$ **b** $\sin x = \frac{\sqrt{3}}{2}$ **c** $\cos x = -\frac{1}{2}$

6 **a** Sketch the graph of $f(x) = \cos x$, $\{ x \in \mathbb{R} : -2\pi \le x \le 2\pi \}$.

b On the graph, mark the points with y-coordinate $\frac{1}{2}$ and give the associated x-values.

c On the graph, mark the points with y-coordinate $-\frac{1}{2}$ and give the associated x-values.

Example 20

7 Solve the following equations for $\theta \in [0, 2\pi]$:

a $\sin(2\theta) = -\frac{1}{2}$ **b** $\cos(2\theta) = \frac{\sqrt{3}}{2}$ **c** $\sin(2\theta) = \frac{1}{2}$

d $\sin(3\theta) = -\frac{1}{\sqrt{2}}$ **e** $\cos(2\theta) = -\frac{\sqrt{3}}{2}$ **f** $\sin(2\theta) = -\frac{1}{\sqrt{2}}$

8 Solve the following equations for $\theta \in [0, 2\pi]$:

a $\sin(2\theta) = -0.8$ **b** $\sin(2\theta) = -0.6$ **c** $\cos(2\theta) = 0.4$ **d** $\cos(3\theta) = 0.6$

12I Sketch graphs of $y = a \sin n(t \pm \varepsilon)$ and $y = a \cos n(t \pm \varepsilon)$

In this section, we consider translations of graphs of functions of the form $f(t) = a\sin(nt)$ and $g(t) = a\cos(nt)$ in the direction of the t-axis.

When a translation of $\frac{\pi}{4}$ units in the positive direction of the t-axis is applied to the graph of $y = f(t)$, the resulting image has equation

$$y = f\left(t - \frac{\pi}{4}\right)$$

For example, the graph of $f(t) = 3\sin(2t)$ is mapped to the graph of $y = 3\sin 2\left(t - \frac{\pi}{4}\right)$.

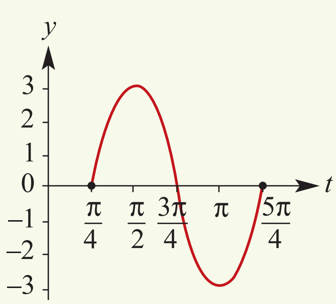
Example 21

On separate axes, draw the graphs of the following functions. Use a calculator to help establish the shape. Set the window appropriately by noting the range and period.

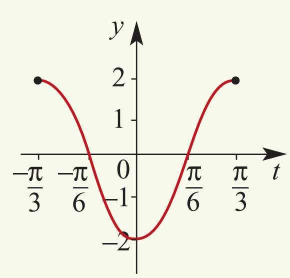
a $y = 3\sin 2\left(t - \frac{\pi}{4}\right)$, $\frac{\pi}{4} \le t \le \frac{5\pi}{4}$ **b** $y = 2\cos 3\left(t + \frac{\pi}{3}\right)$, $-\frac{\pi}{3} \le t \le \frac{\pi}{3}$

Solution

a The range is $[-3, 3]$ and the period is π.



b The range is $[-2, 2]$ and the period is $\frac{2\pi}{3}$.



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Observations from the example

a The graph of $y = 3 \sin 2\left(t - \frac{\pi}{4}\right)$ is the same shape as $y = 3 \sin(2t)$, but is translated $\frac{\pi}{4}$ units in the positive direction of the t-axis.

b The graph of $y = 2 \cos 3\left(t + \frac{\pi}{3}\right)$ is the same shape as $y = 2 \cos(3t)$, but is translated $\frac{\pi}{3}$ units in the negative direction of the t-axis.

The effect of $\pm\varepsilon$ is to translate the graph parallel to the t-axis. (Here $\pm\varepsilon$ is called the phase.)

Note: To determine the sequence of transformations, the techniques of Chapter 6 can also be used. The graph of $y = \sin t$ is transformed to the graph of $y = 3 \sin 2\left(t - \frac{\pi}{4}\right)$.

Write the second equation as $\frac{y'}{3} = \sin 2\left(t' - \frac{\pi}{4}\right)$.

From this it can be seen that $y = \frac{y'}{3}$ and $t = 2\left(t' - \frac{\pi}{4}\right)$. Thus $y' = 3y$ and $t' = \frac{t}{2} + \frac{\pi}{4}$.

Hence the sequence of transformations is:

- dilation of factor 3 parallel to the y-axis
- dilation of factor $\frac{1}{2}$ parallel to the t-axis
- translation of $\frac{\pi}{4}$ units in the positive direction of the t-axis.

Alternatively, we can find this sequence by observing that graph of $y = f(t)$ is transformed to the graph of $y = 3f\left(2\left(t - \frac{\pi}{4}\right)\right)$, where $f(t) = \sin t$.

Summary 12I

The graphs of $y = a \sin n(t \pm \varepsilon)$ and $y = a \cos n(t \pm \varepsilon)$ are translations of the graphs of $y = a \sin(nt)$ and $y = a \cos(nt)$ respectively.

The graphs are translated $\mp\varepsilon$ units parallel to the t-axis, where $\pm\varepsilon$ is called the phase.

Exercise 12I

Example 21

1 Sketch the graph of each of the following, showing one complete cycle. State the period and amplitude, and the greatest and least values of y.

a $y = 3 \sin\left(\theta - \frac{\pi}{2}\right)$ **b** $y = \sin 2(\theta + \pi)$ **c** $y = 2 \sin 3\left(\theta + \frac{\pi}{4}\right)$

d $y = \sqrt{3} \sin 2\left(\theta - \frac{\pi}{2}\right)$ **e** $y = 3 \sin(2x)$ **f** $y = 2 \cos 3\left(\theta + \frac{\pi}{4}\right)$

g $y = \sqrt{2} \sin 2\left(\theta - \frac{\pi}{3}\right)$ **h** $y = -3 \sin(2x)$ **i** $y = -3 \cos 2\left(\theta + \frac{\pi}{2}\right)$

2 For the function $f(x) = \cos\left(x - \frac{\pi}{3}\right)$, $\{x \in \mathbb{R} : 0 \le x \le 2\pi\}$:

a find $f(0)$, $f(2\pi)$ **b** sketch the graph of f.

3 For the function $f(x) = \sin 2\left(x - \frac{\pi}{3}\right)$, $\{x \in \mathbb{R} : 0 \le x \le 2\pi\}$:

a find $f(0)$, $f(2\pi)$ **b** sketch the graph of f.

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4 For the function $f(x) = \sin 3\left(x + \frac{\pi}{4}\right)$, $\{x \in \mathbb{R} : -\pi \le x \le \pi\}$:

a find $f(-\pi)$, $f(\pi)$ **b** sketch the graph of f.

5 Find the equation of the image of $y = \sin x$ for each of the following transformations:

a dilation of factor 2 from the y-axis followed by dilation of factor 3 from the x-axis

b dilation of factor $\frac{1}{2}$ from the y-axis followed by dilation of factor 3 from the x-axis

c dilation of factor 3 from the y-axis followed by dilation of factor 2 from the x-axis

d dilation of factor $\frac{1}{2}$ from the y-axis followed by translation of $\frac{\pi}{3}$ units in the positive direction of the x-axis

e dilation of factor 2 from the y-axis followed by translation of $\frac{\pi}{3}$ units in the negative direction of the x-axis

12J Sketch graphs of $y = a \sin n(t \pm \varepsilon) \pm b$ and $y = a \cos n(t \pm \varepsilon) \pm b$

We now consider translations parallel to the y-axis.

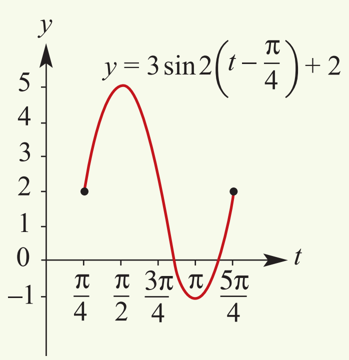
Example 22

Sketch each of the following graphs. Use a calculator to help establish the shape.

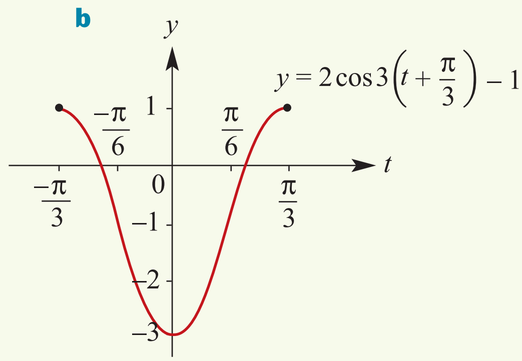
a $y = 3\sin 2\left(t - \frac{\pi}{4}\right) + 2$, $\frac{\pi}{4} \le t \le \frac{5\pi}{4}$ **b** $y = 2\cos 3\left(t + \frac{\pi}{3}\right) - 1$, $-\frac{\pi}{3} \le t \le \frac{\pi}{3}$

Solution

a



b



Observations from the example

a The graph of $y = 3\sin 2\left(t - \frac{\pi}{4}\right) + 2$ is the same shape as the graph of $y = 3\sin 2\left(t - \frac{\pi}{4}\right)$, but is translated 2 units in the positive direction of the y-axis.

b The graph of $y = 2\cos 3\left(t + \frac{\pi}{3}\right) - 1$ is the same shape as the graph of $y = 2\cos 3\left(t + \frac{\pi}{3}\right)$, but is translated 1 unit in the negative direction of the y-axis.

In general, the effect of $\pm b$ is to translate the graph $\pm b$ units parallel to the y-axis.

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Finding axis intercepts

Example 23

Sketch the graph of each of the following for $x \in [0, 2\pi]$. Clearly indicate axis intercepts.

a $y = \sqrt{2}\sin(x) + 1$ **b** $y = 2\cos(2x) - 1$ **c** $y = 2\sin 2\left(x - \frac{\pi}{3}\right) - \sqrt{3}$

Solution

a To determine the x-axis intercepts, the equation $\sqrt{2}\sin(x) + 1 = 0$ must be solved.

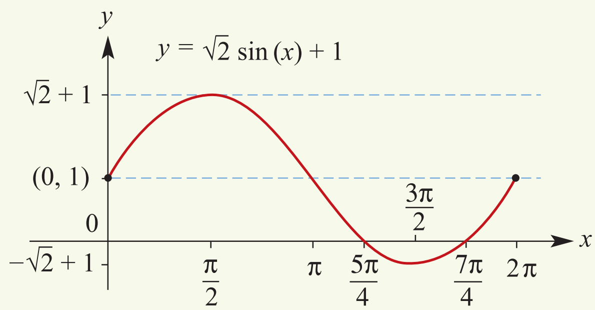
$$\sqrt{2}\sin(x) + 1 = 0$$

$$\therefore \quad \sin x = -\frac{1}{\sqrt{2}}$$

$$\therefore x = \pi + \frac{\pi}{4} \text{ or } 2\pi - \frac{\pi}{4}$$

$$\therefore x = \frac{5\pi}{4} \text{ or } \frac{7\pi}{4}$$

The x-axis intercepts are $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$.



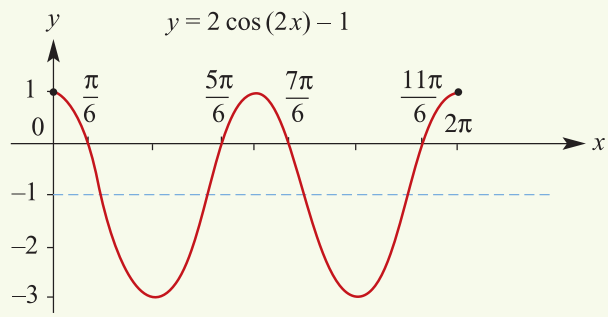
b $2\cos(2x) - 1 = 0$

$$\therefore \quad \cos(2x) = \frac{1}{2}$$

$$\therefore 2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3} \text{ or } \frac{11\pi}{3}$$

$$\therefore \quad x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6} \text{ or } \frac{11\pi}{6}$$

The x-axis intercepts are $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$ and $\frac{11\pi}{6}$.



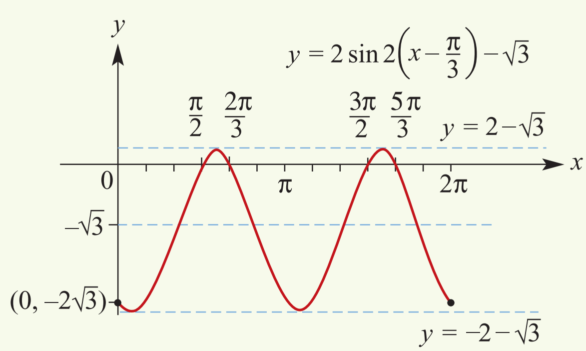
c $\sin 2\left(x - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$$\therefore 2\left(x - \frac{\pi}{3}\right) = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3} \text{ or } \frac{8\pi}{3}$$

$$\therefore \quad x - \frac{\pi}{3} = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6} \text{ or } \frac{4\pi}{3}$$

$$\therefore \quad x = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{2} \text{ or } \frac{5\pi}{3}$$

The x-axis intercepts are $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{2}$ and $\frac{5\pi}{3}$.



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Skill-sheet

Exercise 12J

Example 23

1 Sketch the graph of each of the following for $x \in [0, 2\pi]$. List the x-axis intercepts of each graph for this interval.

a $y = 2\sin(x) + 1$ **b** $y = 2\sin(2x) - \sqrt{3}$ **c** $y = \sqrt{2}\cos(x) + 1$

d $y = 2\sin(2x) - 2$ **e** $y = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right) + 1$

2 Sketch the graph of each of the following for $x \in [-\pi, 2\pi]$:

a $y = 2\sin(3x) - 2$ **b** $y = 2\cos 3\left(x - \frac{\pi}{4}\right)$ **c** $y = 2\sin(2x) - 3$

d $y = 2\cos(2x) + 1$ **e** $y = 2\cos 2\left(x - \frac{\pi}{3}\right) - 1$ **f** $y = 2\sin 2\left(x + \frac{\pi}{6}\right) + 1$

3 Sketch the graph of each of the following for $x \in [-\pi, \pi]$:

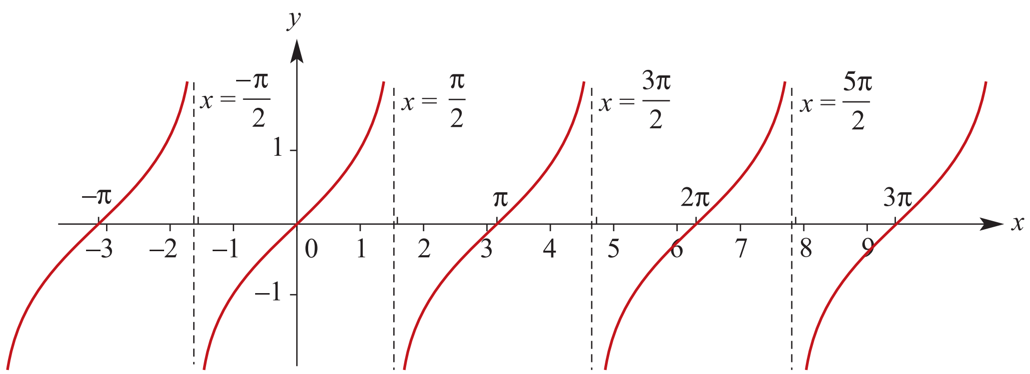
a $y = 2\sin 2\left(x + \frac{\pi}{3}\right) + 1$ **b** $y = -2\sin 2\left(x + \frac{\pi}{6}\right) + 1$ **c** $y = 2\cos 2\left(x + \frac{\pi}{4}\right) + \sqrt{3}$

12K The tangent function

A table of values for $y = \tan x$ is given below. Use a calculator to check these values and plot the graph of $y = \tan x$.

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π
y	0	1	ud	−1	0	1	ud	−1	0	1	ud	−1	0	1	ud	−1	0

Note: 'ud' is undefined



Note: The lines $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$ and $x = \frac{5\pi}{2}$ are asymptotes.

Observations from the graph of $y = \tan x$

- The graph repeats itself every π units, i.e. the period of tan is π.
- The range of tan is $\mathbb{R}$.
- The equations of the asymptotes are of the form $x = \frac{(2k+1)\pi}{2}$, where k is an integer.
- The x-axis intercepts occur when $\sin x = 0$, which is for $x = k\pi$, where k is an integer.

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Transformations of $y = \tan x$

Consider a dilation of factor $\frac{1}{2}$ parallel to the x-axis and a dilation of factor 3 parallel to the y-axis:

$$(x, y) \to \left(\frac{1}{2}x, 3y\right)$$

If the image of (x, y) under the transformation is (x', y'), then $x' = \frac{1}{2}x$ and $y' = 3y$. Hence $x = 2x'$ and $y = \frac{y'}{3}$.

Thus the graph of $y = \tan x$ is transformed to the graph of $\frac{y'}{3} = \tan(2x')$. That is, it is transformed to the graph of $y = 3\tan(2x)$. The period of the graph will be $\frac{\pi}{2}$.

Graph of $y = a\tan(nt)$

In general, for a and n positive numbers, the graph of $y = a\tan(nt)$ is obtained from the graph of $y = \tan t$ by a dilation of factor a parallel to the y-axis and a dilation of factor $\frac{1}{n}$ parallel to the t-axis.

The following are important properties of the function $f(t) = a\tan(nt)$:

- The period is $\frac{\pi}{n}$.
- The range is $\mathbb{R}$.
- The asymptotes have equations $t = \frac{(2k+1)\pi}{2n}$, where k is an integer.
- The t-axis intercepts are $t = \frac{k\pi}{n}$, where k is an integer.

Example 24

Sketch the graph of each of the following for $x \in [-\pi, \pi]$:

a $y = 3\tan(2x)$

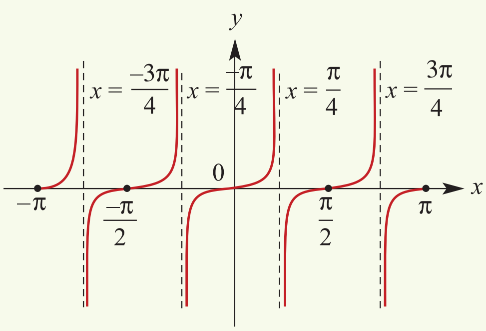
b $y = -2\tan(3x)$

Solution

a Period $= \frac{\pi}{n} = \frac{\pi}{2}$

Asymptotes: $x = \frac{(2k+1)\pi}{4}$, $k \in \mathbb{Z}$

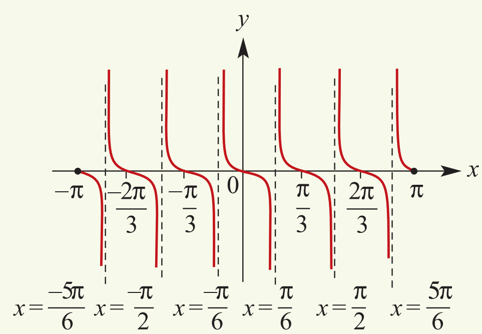
Axis intercepts: $x = \frac{k\pi}{2}$, $k \in \mathbb{Z}$



b Period $= \frac{\pi}{n} = \frac{\pi}{3}$

Asymptotes: $x = \frac{(2k+1)\pi}{6}$, $k \in \mathbb{Z}$

Axis intercepts: $x = \frac{k\pi}{3}$, $k \in \mathbb{Z}$



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Solution of equations

The techniques for solving equations of the form

$$a\tan(nt) = b$$

are similar to those for solving equations of the form $a\sin(nt) = b$ and $a\cos(nt) = b$ discussed in Section 12H. An important difference is that the period of tan is π. The method for obtaining further solutions is to add and subtract multiples of π.

Example 25

Solve each of the following equations for $x \in [-\pi, \pi]$:

a $\tan x = -1$

b $\tan(2x) = \sqrt{3}$

c $2\tan(3x) = 0$

Solution

a $\tan x = -1$

$\therefore \quad x = \dfrac{3\pi}{4}$ or $\dfrac{-\pi}{4}$

Explanation: Since $\tan x$ is negative, the point $P(x)$ lies in the 2nd or 4th quadrant. Solutions are required for $x \in [-\pi, \pi]$.

Use $\tan(\pi - x) = -\tan x$ and $\tan(-x) = -\tan x$.

b $\tan(2x) = \sqrt{3}$

Let $a = 2x$. The equation becomes

$\tan a = \sqrt{3}$

$\therefore \quad a = \dfrac{\pi}{3}$ or $\dfrac{4\pi}{3}$ or $\dfrac{-2\pi}{3}$ or $\dfrac{-5\pi}{3}$

$\therefore \quad x = \dfrac{\pi}{6}$ or $\dfrac{4\pi}{6}$ or $\dfrac{-2\pi}{6}$ or $\dfrac{-5\pi}{6}$

$= \dfrac{\pi}{6}$ or $\dfrac{2\pi}{3}$ or $\dfrac{-\pi}{3}$ or $\dfrac{-5\pi}{6}$

Explanation: Consider solutions for $a \in [-2\pi, 2\pi]$.

Since $\tan a$ is positive, the point $P(a)$ lies in the 1st or 3rd quadrant.

Use $\tan(\pi + x) = \tan x$.

Subtract π from each of the first two solutions to obtain the second two.

c $2\tan(3x) = 0$

$\tan(3x) = 0$

$3x = -3\pi, \ -2\pi, \ -\pi, \ 0, \ \pi, \ 2\pi$ or 3π

$\therefore \ x = -\pi, \ \dfrac{-2\pi}{3}, \ \dfrac{-\pi}{3}, \ 0, \ \dfrac{\pi}{3}, \ \dfrac{2\pi}{3}$ or π

Explanation: The solutions for $3x$ are to be in the interval $[-3\pi, 3\pi]$.

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Example 26

Sketch the graph of $y = \tan(2x) + 1$ for $x \in [-\pi, \pi]$.

Solution

The graph of $y = \tan(2x) + 1$ is obtained from the graph of $y = \tan(2x)$ by a translation of 1 unit in the positive direction of the y-axis.

For the y-axis intercept, let $x = 0$. Then $y = \tan 0 + 1 = 1$.

For the x-axis intercepts, consider

$$\tan(2x) + 1 = 0$$

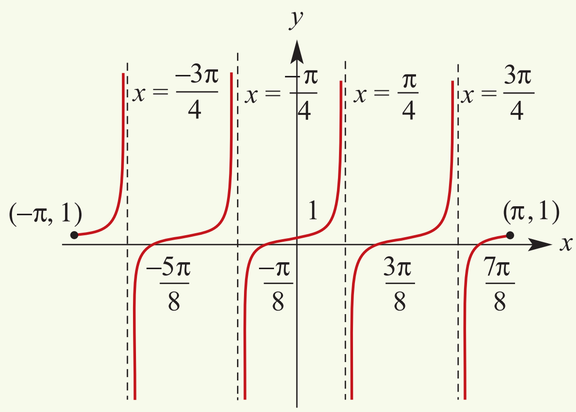
$$\tan(2x) = -1$$

Hence $2x = \frac{3\pi}{4}, \frac{-\pi}{4}, \frac{7\pi}{4}$ or $\frac{-5\pi}{4}$

and so $x = \frac{3\pi}{8}, \frac{-\pi}{8}, \frac{7\pi}{8}$ or $\frac{-5\pi}{8}$

The asymptotes are the same as those for $y = \tan(2x)$,

that is, $x = \frac{(2k + 1)\pi}{4}, k \in \mathbb{Z}$



Exercise 12K

1 For each of the following, state the period:

a $y = \tan(4x)$ **b** $y = \tan\left(\frac{2x}{3}\right)$ **c** $y = -3\tan(2x)$

Example 24

2 Sketch the graph of each of the following for $x \in [-\pi, \pi]$:

a $y = \tan(2x)$ **b** $y = 2\tan(3x)$ **c** $y = -\tan x$

Example 25

3 Solve each of the following equations for $x \in [-\pi, \pi]$:

a $2\tan(2x) = 2$

b $3\tan(3x) = \sqrt{3}$

c $2\tan(2x) = 2\sqrt{3}$

d $3\tan(3x) = -\sqrt{3}$

Example 26

4 Sketch the graph of each of the following for $x \in [-\pi, \pi]$:

a $y = 3\tan(x) + \sqrt{3}$

b $y = \tan(x) + 2$

c $y = 3\tan(x) - 3$

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12L Further symmetry properties and the Pythagorean identity

Complementary relationships

From the diagram to the right,

$$\sin\left(\frac{\pi}{2} - \theta\right) = a$$

and, since $a = \cos\theta$,

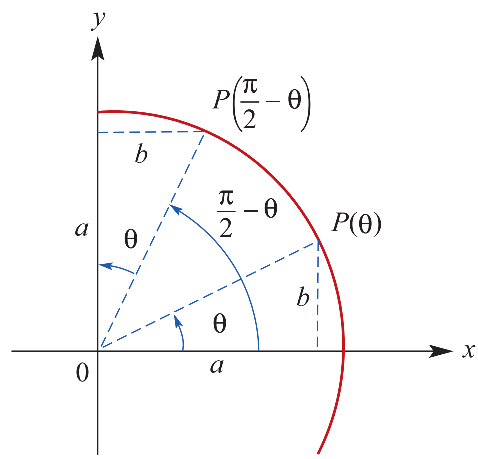
$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

From the same diagram,

$$\cos\left(\frac{\pi}{2} - \theta\right) = b$$

and, since $b = \sin\theta$,

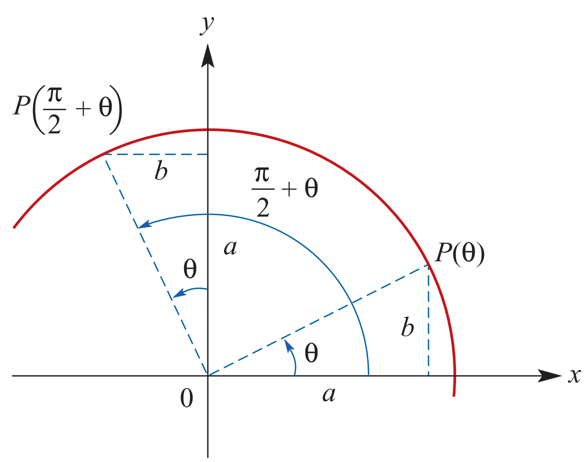
$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$



From the diagram to the right:

$$\sin\left(\frac{\pi}{2} + \theta\right) = a = \cos\theta$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = -b = -\sin\theta$$



Example 27

If $\sin\theta = 0.3$ and $\cos\alpha = 0.8$, find the values of:

a $\sin\left(\frac{\pi}{2} - \alpha\right)$ **b** $\cos\left(\frac{\pi}{2} + \theta\right)$ **c** $\sin(-\theta)$

Solution

a $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha$
$= 0.8$

b $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$
$= -0.3$

c $\sin(-\theta) = -\sin\theta$
$= -0.3$

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The Pythagorean identity

Consider a point, $P(\theta)$, on the unit circle.

By Pythagoras' theorem,

$$OP^2 = OM^2 + MP^2$$

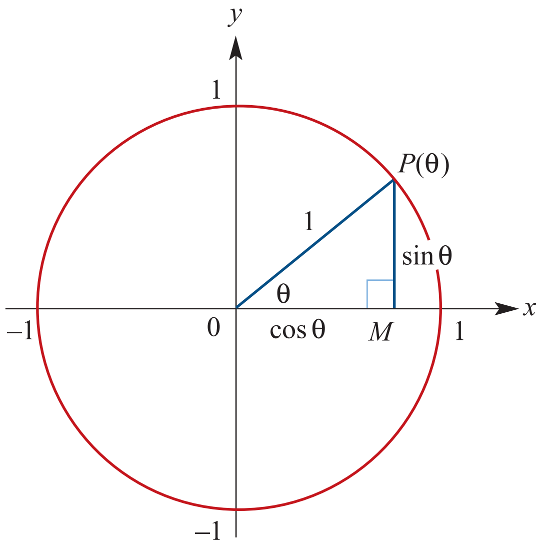
$$\therefore \quad 1 = (\cos\theta)^2 + (\sin\theta)^2$$

Now $(\cos\theta)^2$ and $(\sin\theta)^2$ may be written as $\cos^2\theta$ and $\sin^2\theta$.

$$\therefore \quad 1 = \cos^2\theta + \sin^2\theta$$

As this is true for all values of θ, it is called an identity. In particular, this is called the **Pythagorean identity**:

$$\cos^2\theta + \sin^2\theta = 1$$



Example 28

Given that $\sin x = \frac{3}{5}$ and $\frac{\pi}{2} < x < \pi$, find:

a $\cos x$

b $\tan x$

Solution

a Substitute $\sin x = \frac{3}{5}$ into the Pythagorean identity:

$$\cos^2 x + \sin^2 x = 1$$

$$\cos^2 x + \frac{9}{25} = 1$$

$$\cos^2 x = 1 - \frac{9}{25}$$

$$= \frac{16}{25}$$

Therefore $\cos x = \pm\frac{4}{5}$. But x is in the 2nd quadrant, and so $\cos x = -\frac{4}{5}$.

b Using part a, we have

$$\tan x = \frac{\sin x}{\cos x}$$

$$= \frac{3}{5} \div \left(-\frac{4}{5}\right)$$

$$= \frac{3}{5} \times \left(-\frac{5}{4}\right)$$

$$= -\frac{3}{4}$$

Summary 12L

- **Complementary relationships**

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

- **Pythagorean identity**

$$\cos^2\theta + \sin^2\theta = 1$$

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Exercise 12L

Example 27

1 If $\sin x = 0.3$, $\cos \alpha = 0.6$ and $\tan \theta = 0.7$, find the values of:

a $\cos(-\alpha)$ **b** $\sin\left(\frac{\pi}{2} + \alpha\right)$ **c** $\tan(-\theta)$ **d** $\cos\left(\frac{\pi}{2} - x\right)$

e $\sin(-x)$ **f** $\tan\left(\frac{\pi}{2} - \theta\right)$ **g** $\cos\left(\frac{\pi}{2} + x\right)$ **h** $\sin\left(\frac{\pi}{2} - \alpha\right)$

i $\sin\left(\frac{3\pi}{2} + \alpha\right)$ **j** $\cos\left(\frac{3\pi}{2} - x\right)$

2 **a** Given that $0 < \theta < \frac{\pi}{2}$ and $\cos\theta = \sin\left(\frac{\pi}{6}\right)$, find the value of θ.

b Given that $0 < \theta < \frac{\pi}{2}$ and $\sin\theta = \cos\left(\frac{\pi}{6}\right)$, find the value of θ.

c Given that $0 < \theta < \frac{\pi}{2}$ and $\cos\theta = \sin\left(\frac{\pi}{12}\right)$, find the value of θ.

d Given that $0 < \theta < \frac{\pi}{2}$ and $\sin\theta = \cos\left(\frac{3\pi}{7}\right)$, find the value of θ.

Example 28

3 Given that $\cos x = \frac{3}{5}$ and $\frac{3\pi}{2} < x < 2\pi$, find $\sin x$ and $\tan x$.

4 Given that $\sin x = \frac{5}{13}$ and $\frac{\pi}{2} < x < \pi$, find $\cos x$ and $\tan x$.

5 Given that $\cos x = \frac{1}{5}$ and $\frac{3\pi}{2} < x < 2\pi$, find $\sin x$ and $\tan x$.

12M Addition formulas and double angle formulas

There are many interesting and useful relationships between the trigonometric functions. The most fundamental is the Pythagorean identity:

$$\sin^2 A + \cos^2 A = 1$$

Some of these identities were discovered a very long time ago. For example, the following two results were discovered by the Indian mathematician Bhāskara II in the twelfth century:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

In this section, we will establish and use these two formulas. They are of great importance in many areas of mathematics, including calculus.

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Addition formulas

Addition formulas for cosine

1 $\cos(u+v) = \cos u \cos v - \sin u \sin v$

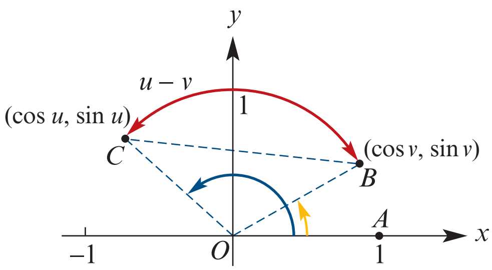
2 $\cos(u-v) = \cos u \cos v + \sin u \sin v$

Proof Consider a unit circle as shown:

arc length $AB = v$ units

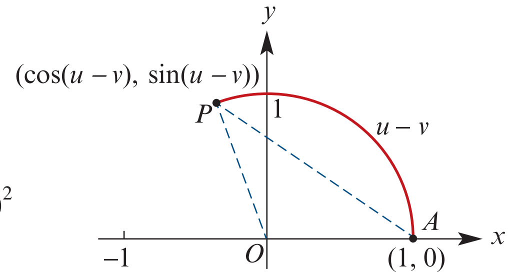
arc length $AC = u$ units

arc length $BC = u - v$ units



Rotate ΔOCB so that B is coincident with A. Then C is moved to

$P(\cos(u-v), \sin(u-v))$



Since the triangles CBO and PAO are congruent, we have $CB = PA$.

Using the coordinate distance formula:

$$\begin{aligned} CB^2 &= (\cos u - \cos v)^2 + (\sin u - \sin v)^2 \\ &= 2 - 2(\cos u \cos v + \sin u \sin v) \end{aligned}$$

$$\begin{aligned} PA^2 &= (\cos(u-v) - 1)^2 + (\sin(u-v) - 0)^2 \\ &= 2 - 2\cos(u-v) \end{aligned}$$

Since $CB = PA$, this gives

$$2 - 2\cos(u-v) = 2 - 2(\cos u \cos v + \sin u \sin v)$$

$$\therefore \quad \cos(u-v) = \cos u \cos v + \sin u \sin v$$

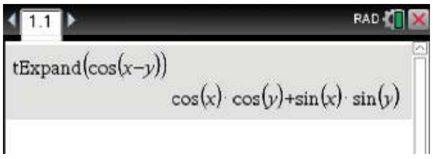
We can now obtain the first formula from the second by replacing v with $-v$:

$$\begin{aligned} \cos(u+v) &= \cos(u-(-v)) \\ &= \cos u \cos(-v) + \sin u \sin(-v) \\ &= \cos u \cos v - \sin u \sin v \end{aligned}$$

Note: Here we used $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.

Using the TI-Nspire

Access the **tExpand( )** command from menu > **Algebra** > **Trigonometry** > **Expand** and complete as shown.



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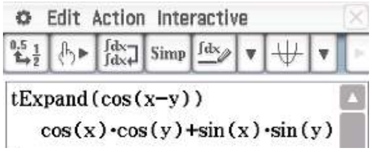
Using the Casio ClassPad

- In $\sqrt[\text{Main}]{\alpha}$, enter and highlight $\cos(x - y)$.
- Go to **Interactive** > **Transformation** > **tExpand** and tap OK.



Example 29

Evaluate $\cos 75°$.

Solution

$$\begin{aligned}\cos 75° &= \cos(45° + 30°)\\ &= \cos 45° \cos 30° - \sin 45° \sin 30°\\ &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Explanation

Write $75°$ as $45° + 30°$.

Use the addition formula for cosine.

We can express the answer in a neater form by avoiding surds in the denominator.

Addition formulas for sine

1 $\sin(u + v) = \sin u \cos v + \cos u \sin v$

2 $\sin(u - v) = \sin u \cos v - \cos u \sin v$

Proof We use the symmetry properties $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ and $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$:

$$\begin{aligned}\sin(u + v) &= \cos\left(\frac{\pi}{2} - (u + v)\right)\\ &= \cos\left(\left(\frac{\pi}{2} - u\right) - v\right)\\ &= \cos\left(\frac{\pi}{2} - u\right)\cos v + \sin\left(\frac{\pi}{2} - u\right)\sin v\\ &= \sin u \cos v + \cos u \sin v\end{aligned}$$

We can now obtain the second formula from the first by replacing v with $-v$:

$$\begin{aligned}\sin(u - v) &= \sin u \cos(-v) + \cos u \sin(-v)\\ &= \sin u \cos v - \cos u \sin v\end{aligned}$$

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Example 30

Evaluate:

a $\sin 75^\circ$

b $\sin 15^\circ$

Solution

a
$$\begin{aligned}\sin 75^\circ &= \sin(30^\circ + 45^\circ)\\ &= \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ\\ &= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}\\ &= \frac{1+\sqrt{3}}{2\sqrt{2}}\\ &= \frac{1+\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{2}+\sqrt{6}}{4}\end{aligned}$$

b
$$\begin{aligned}\sin 15^\circ &= \sin(45^\circ - 30^\circ)\\ &= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ\\ &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Addition formulas for tangent

1 $\tan(u+v) = \dfrac{\tan u + \tan v}{1 - \tan u \tan v}$

2 $\tan(u-v) = \dfrac{\tan u - \tan v}{1 + \tan u \tan v}$

Proof To obtain the first formula, we write

$$\tan(u+v) = \frac{\sin(u+v)}{\cos(u+v)} = \frac{\sin u \cos v + \cos u \sin v}{\cos u \cos v - \sin u \sin v}$$

Now divide the numerator and denominator by $\cos u \cos v$. The second formula can be obtained from the first by using $\tan(-\theta) = -\tan\theta$.

Example 31

If u and v are acute angles such that $\tan u = 4$ and $\tan v = \frac{3}{5}$, show that $u - v = \frac{\pi}{4}$.

Solution

$$\begin{aligned}\tan(u-v) &= \frac{\tan u - \tan v}{1 + \tan u \tan v}\\ &= \frac{4 - \frac{3}{5}}{1 + 4 \times \frac{3}{5}}\\ &= \frac{20-3}{5+4\times 3}\\ &= 1\end{aligned}$$

$$\therefore \quad u - v = \frac{\pi}{4}$$

Note: Since u and v are acute angles with $u > v$, the angle $u - v$ is in the interval $\left(0, \frac{\pi}{2}\right)$.

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Double angle formulas (optional content)

Using the addition formulas, we can easily derive useful expressions for $\sin(2u)$, $\cos(2u)$ and $\tan(2u)$.

Double angle formulas for cosine

$$\begin{aligned}\cos(2u) &= \cos^2 u - \sin^2 u \\ &= 2\cos^2 u - 1 && (\text{since } \sin^2 u = 1 - \cos^2 u) \\ &= 1 - 2\sin^2 u && (\text{since } \cos^2 u = 1 - \sin^2 u)\end{aligned}$$

Proof $\cos(u + u) = \cos u \cos u - \sin u \sin u$

$$= \cos^2 u - \sin^2 u$$

Double angle formula for sine

$$\sin(2u) = 2\sin u \cos u$$

Proof $\sin(u + u) = \sin u \cos u + \cos u \sin u$

$$= 2\sin u \cos u$$

Double angle formula for tangent

$$\tan(2u) = \frac{2\tan u}{1 - \tan^2 u}$$

Proof $\tan(u + u) = \dfrac{\tan u + \tan u}{1 - \tan u \tan u}$

$$= \frac{2\tan u}{1 - \tan^2 u}$$

Example 32

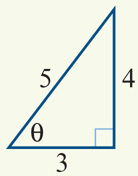
If $\tan\theta = \frac{4}{3}$ and $0 < \theta < \frac{\pi}{2}$, evaluate:

a $\sin(2\theta)$ **b** $\tan(2\theta)$

Solution

a $\sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5}$

$$\begin{aligned}\therefore\ \sin(2\theta) &= 2\sin\theta\cos\theta \\ &= 2 \times \frac{4}{5} \times \frac{3}{5} \\ &= \frac{24}{25}\end{aligned}$$



b
$$\begin{aligned}\tan(2\theta) &= \frac{2\tan\theta}{1 - \tan^2\theta} \\ &= \frac{2 \times \frac{4}{3}}{1 - \frac{16}{9}} \\ &= \frac{2 \times 4 \times 3}{9 - 16} \\ &= -\frac{24}{7}\end{aligned}$$

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Summary 12M

- **Addition formulas**

$$\cos(u + v) = \cos u \cos v - \sin u \sin v$$
$$\cos(u - v) = \cos u \cos v + \sin u \sin v$$
$$\sin(u + v) = \sin u \cos v + \cos u \sin v$$
$$\sin(u - v) = \sin u \cos v - \cos u \sin v$$
$$\tan(u + v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$
$$\tan(u - v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

- **Double angle formulas**

$$\cos(2u) = \cos^2 u - \sin^2 u = 2\cos^2 u - 1 = 1 - 2\sin^2 u$$
$$\sin(2u) = 2\sin u \cos u$$
$$\tan(2u) = \frac{2\tan u}{1 - \tan^2 u}$$

Exercise 12M

Example 29

1 By using the appropriate addition formulas, find exact values for the following:

a $\cos 15°$ **b** $\cos 105°$

Example 30

2 By using the appropriate addition formulas, find exact values for the following:

a $\sin 165°$ **b** $\tan 75°$

3 Find the exact value of:

a $\cos\left(\frac{5\pi}{12}\right)$ **b** $\sin\left(\frac{11\pi}{12}\right)$ **c** $\tan\left(\frac{-\pi}{12}\right)$

Example 31

4 For acute angles u and v such that $\sin u = \frac{12}{13}$ and $\sin v = \frac{3}{5}$, evaluate $\sin(u + v)$.

5 Simplify the following:

a $\sin\left(\theta + \frac{\pi}{6}\right)$ **b** $\cos\left(\varphi - \frac{\pi}{4}\right)$ **c** $\tan\left(\theta + \frac{\pi}{3}\right)$ **d** $\sin\left(\theta - \frac{\pi}{4}\right)$

6 Simplify:

a $\cos(u - v)\sin v + \sin(u - v)\cos v$ **b** $\sin(u + v)\sin v + \cos(u + v)\cos v$

Example 32

7 If $\sin\theta = \frac{-3}{5}$, with θ in the 3rd quadrant, and $\cos\varphi = \frac{-5}{13}$, with φ in the 2nd quadrant, evaluate each of the following without using a calculator:

a $\cos(2\varphi)$ **b** $\sin(2\theta)$ **c** $\tan(2\theta)$

d $\sin(\theta + \varphi)$ **e** $\cos(\theta - \varphi)$

8 For acute angles u and v such that $\tan u = \frac{4}{3}$ and $\tan v = \frac{5}{12}$, evaluate:

a $\tan(u + v)$ **b** $\tan(2u)$ **c** $\cos(u - v)$ **d** $\sin(2u)$

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9 If $\sin\alpha = \frac{3}{5}$ and $\sin\beta = \frac{24}{25}$, with $\frac{\pi}{2} < \beta < \alpha < \pi$, evaluate:

a $\cos(2\alpha)$ **b** $\sin(\alpha - \beta)$ **c** $\tan(\alpha + \beta)$ **d** $\sin(2\beta)$

10 If $\sin\theta = -\frac{\sqrt{3}}{2}$ and $\cos\theta = \frac{1}{2}$, evaluate:

a $\sin(2\theta)$ **b** $\cos(2\theta)$

11 Simplify each of the following expressions:

a $(\sin\theta - \cos\theta)^2$ **b** $\cos^4\theta - \sin^4\theta$

12N Applications of trigonometric functions

Example 33

It is suggested that the height, $h(t)$ metres, of the tide above mean sea level on 1 January at Warnung is given approximately by the rule $h(t) = 4\sin\left(\frac{\pi}{6}t\right)$, where t is the number of hours after midnight.

a Draw the graph of $y = h(t)$ for $0 \le t \le 24$.

b When was high tide?

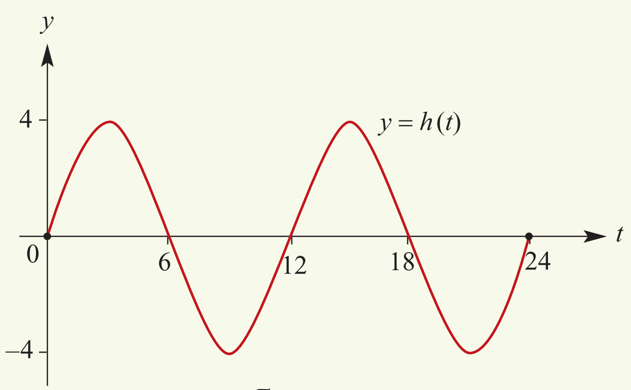
c What was the height of the high tide?

d What was the height of the tide at 8 a.m.?

e A boat can only cross the harbour bar when the tide is at least 1 metre above mean sea level. When could the boat cross the harbour bar on 1 January?

Solution

a



Note: Period $= 2\pi \div \frac{\pi}{6} = 12$

b High tide occurs when $h(t) = 4$:

$$4\sin\left(\frac{\pi}{6}t\right) = 4$$

$$\sin\left(\frac{\pi}{6}t\right) = 1$$

$$\frac{\pi}{6}t = \frac{\pi}{2}, \frac{5\pi}{2}$$

$$\therefore \quad t = 3, \ 15$$

i.e. high tide occurs at 03:00 and 15:00 (3 p.m.).

c The high tide has height 4 metres above the mean height.

d $h(8) = 4\sin\left(\frac{8\pi}{6}\right) = 4\sin\left(\frac{4\pi}{3}\right) = 4 \times \frac{-\sqrt{3}}{2} = -2\sqrt{3}$

The water is $2\sqrt{3}$ metres below the mean height at 8 a.m.

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e We first consider $4\sin\left(\frac{\pi}{6}t\right) = 1$.

Thus $\sin\left(\frac{\pi}{6}t\right) = \frac{1}{4}$

$\therefore \quad \frac{\pi}{6}t = 0.2526,\ 2.889,\ 6.5358,\ 9.172$

$\therefore \quad t = 0.4824,\ 5.5176,\ 12.4824,\ 17.5173$

i.e. the water is at height 1 metre at 00:29, 05:31, 12:29, 17:31.
Thus the boat can pass across the harbour bar between 00:29 and 05:31, and between 12:29 and 17:31.

Skill-sheet

Exercise 12N

Example 33

1 It is suggested that the height, $h(t)$ metres, of the tide above mean sea level during a particular day at Seabreak is given approximately by the rule

$$h(t) = 5\sin\left(\frac{\pi}{6}t\right)$$

where t is the number of hours after midnight.

a Draw the graph of $y = h(t)$ for $0 \le t \le 24$.
b When was high tide?
c What was the height of the high tide?
d What was the height of the tide at 2 a.m.?
e What was the height of the tide at 2 p.m.?
f A boat can only cross the harbour bar when the tide is at least 2.5 metres above mean sea level. When could the boat cross the harbour bar on this day?

2 A particle moves on a straight line, OX, and its distance x metres from O at time t seconds is given by $x = 3 + 2\sin(3t)$.

a Find its greatest distance from O.
b Find its least distance from O.
c Find the times at which it is 5 metres from O for $0 \le t \le 5$.
d Find the times at which it is 3 metres from O for $0 \le t \le 3$.
e Describe the motion of the particle.

3 A particle moves on a straight line, OX, and its distance x metres from O at time t seconds is given by $x = 5 + 2\sin(2\pi t)$.

a Find its greatest distance from O.
b Find its least distance from O.
c Find the times at which it is 7 metres from O for $0 \le t \le 5$.
d Find the times at which it is 6 metres from O for $0 \le t \le 3$.
e Describe the motion of the particle.

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4 A particle moves in a vertical circle of radius 10 m. The height, $h(t)$ m, of the particle above the ground at time t seconds is given by the function

$$h(t) = 10\sin\left(\frac{\pi t}{3}\right) + 10 \quad \text{where } t \geq 0$$

a Find the height of the particle above the ground for:

i $t = 0$ **ii** $t = 1$ **iii** $t = 2$ **iv** $t = 4$ **v** $t = 5$

b Find the period of the motion of the particle.

c Find the greatest height of the particle above the ground.

d Find the first four times that the particle is at a height 15 m above the ground.

e Find the first four times that the particle is at a height 5 m above the ground.

5 The temperature, T°C, in a town over a day is modelled by the function with rule

$$T = 17 - 8\cos\left(\frac{\pi t}{12}\right)$$

where t is the time in hours after midnight, $0 \leq t \leq 24$.

a What is the temperature at midnight?

b What are the maximum and minimum temperatures reached?

c At what times of the day, to the nearest minute, are temperatures warmer than 20°C?

d Sketch the graph for the temperatures over a day.

6 The depth, $D(t)$ metres, of water at the entrance to a harbour at t hours after midnight on a particular day is given by $D(t) = 10 + 3\sin\left(\frac{\pi t}{6}\right)$, $0 \leq t \leq 24$.

a Sketch the graph of $D(t)$ for $0 \leq t \leq 24$.

b Find the values of t for which $D(t) \geq 8.5$.

c Boats which need a depth of w metres are permitted to enter the harbour only if the depth of the water at the entrance is at least w metres for a continuous period of 1 hour. Find, correct to one decimal place, the largest value of w which satisfies this condition.

7 The depth of water at the entrance to a harbour t hours after high tide is D metres, where $D = p + q\cos(rt)°$ for suitable constants p, q, r. At high tide the depth is 7 m; at low tide, 6 hours later, the depth is 3 m.

a Show that $r = 30$ and find the values of p and q.

b Sketch the graph of D against t for $0 \leq t \leq 12$.

c Find how soon after low tide a ship that requires a depth of at least 4 m of water will be able to enter the harbour.

8 For each of the following, construct a formula involving a trigonometric function which could be used to model the situation described:

a Water depths in a canal vary between a minimum of 3 metres and a maximum of 6 metres over a 24-hour period.

b At a certain town just south of the Arctic circle, the number of hours of daylight varies between 2 and 22 hours during a 365-day year.

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Chapter summary

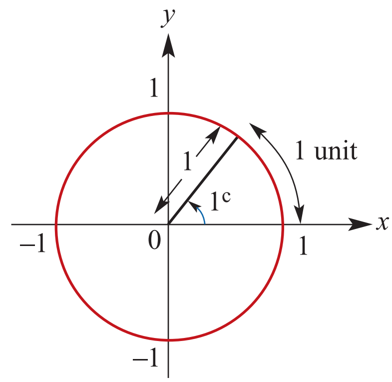
Assign-ment

Nrich

- **Definition of a radian**

One radian (written 1^c) is the angle formed at the centre of the unit circle by an arc of length 1 unit.

$$1^c = \frac{180°}{\pi} \qquad 1° = \frac{\pi^c}{180}$$



- **Sine and cosine**

x-coordinate of $P(\theta)$ on unit circle:

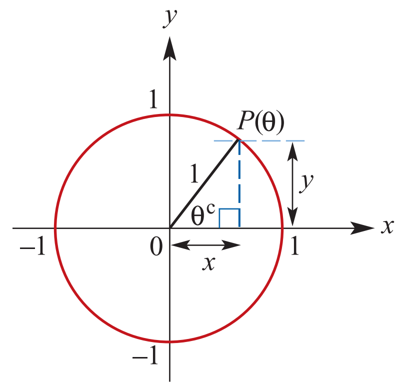
$$x = \text{cosine } \theta, \quad \theta \in \mathbb{R}$$

y-coordinate of $P(\theta)$ on unit circle:

$$y = \text{sine } \theta, \quad \theta \in \mathbb{R}$$

Abbreviated to

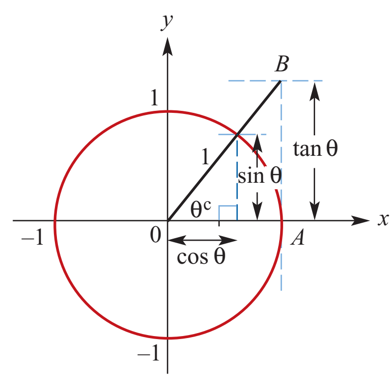
$$x = \cos\theta$$
$$y = \sin\theta$$



- **Tangent**

If the tangent to the unit circle at A is drawn, then the y-coordinate of B is called tangent θ (abbreviated to $\tan\theta$). By using similar triangles:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

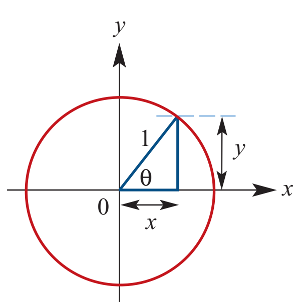


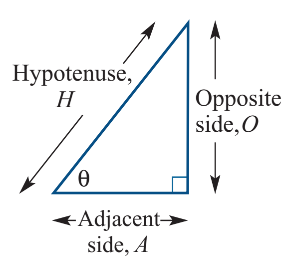
- **Trigonometric functions and trigonometric ratios**

$$\sin\theta = \frac{O}{H} = \frac{y}{1} = y$$

$$\cos\theta = \frac{A}{H} = \frac{x}{1} = x$$

$$\tan\theta = \frac{O}{A} = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$





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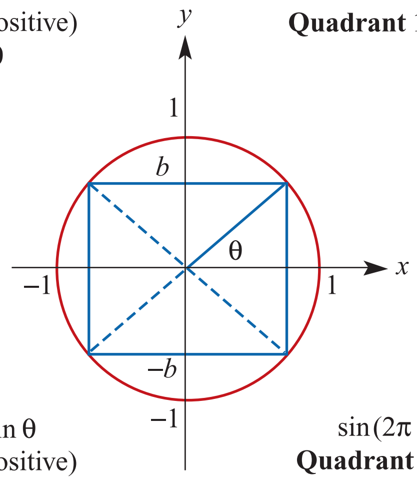
Symmetry properties of trigonometric functions

Quadrant 2 (sin is positive)
$\sin(\pi - \theta) = b = \sin\theta$

Quadrant 1 (all functions are positive)
e.g. $\sin\theta = b$



$\sin(\pi + \theta) = -b = -\sin\theta$
Quadrant 3 (tan is positive)

$\sin(2\pi - \theta) = -b = -\sin\theta$
Quadrant 4 (cos is positive)

Exact values of trigonometric functions

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	undefined

Further symmetry properties

Negative angles:

$$\cos(-\theta) = \cos\theta$$
$$\sin(-\theta) = -\sin\theta$$
$$\tan(-\theta) = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$

Complementary angles:

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$
$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$
$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$$
$$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$

Addition formulas

$$\cos(u + v) = \cos u\cos v - \sin u\sin v$$
$$\cos(u - v) = \cos u\cos v + \sin u\sin v$$
$$\sin(u + v) = \sin u\cos v + \cos u\sin v$$
$$\sin(u - v) = \sin u\cos v - \cos u\sin v$$
$$\tan(u + v) = \frac{\tan u + \tan v}{1 - \tan u\tan v}$$
$$\tan(u - v) = \frac{\tan u - \tan v}{1 + \tan u\tan v}$$

Double angle formulas

$$\cos(2u) = \cos^2 u - \sin^2 u$$
$$= 2\cos^2 u - 1$$
$$= 1 - 2\sin^2 u$$
$$\sin(2u) = 2\sin u\cos u$$
$$\tan(2u) = \frac{2\tan u}{1 - \tan^2 u}$$

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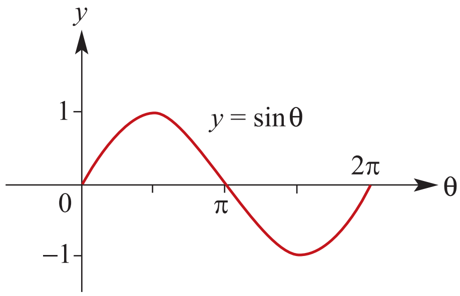
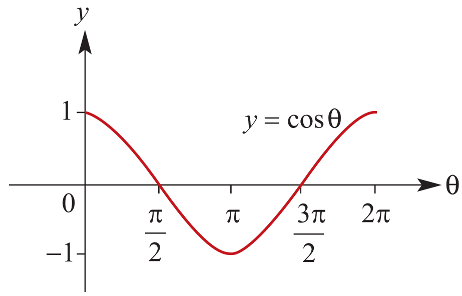
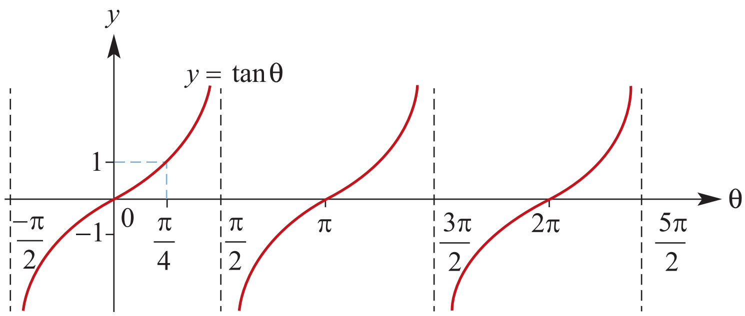
- **Graphs of trigonometric functions**



Amplitude = 1
Period = 2π



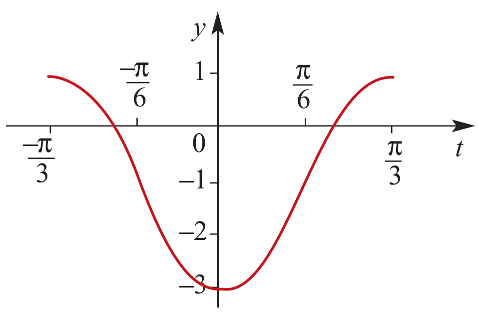
Amplitude = 1
Period = 2π



Amplitude is undefined
Period = π

- Graphs of trigonometric functions of the type $y = a \sin n(t \pm \varepsilon) \pm b$ and $y = a \cos n(t \pm \varepsilon) \pm b$

  e.g. $y = 2\cos 3\left(t + \frac{\pi}{3}\right) - 1$

  - Amplitude, $a = 2$
  - Period $= \frac{2\pi}{n} = \frac{2\pi}{3}$
  - The graph is the same shape as $y = 2\cos(3t)$ but is translated $\frac{\pi}{3}$ units in the negative direction of the t-axis and 1 unit in the negative direction of the y-axis.

- Solutions of trigonometric equations of the type $\sin x^\circ = a$ and $\cos x^\circ = a$

  e.g. Solve $\cos x^\circ = -0.7$ for $x \in [0, 360]$.
  First look at the 1st quadrant: If $\cos \alpha^\circ = 0.7$, then $\alpha = 45.6$.
  Since $\cos x^\circ$ is negative for $P(x^\circ)$ in the 2nd and 3rd quadrants, the solutions are

  $$x = 180 - 45.6 = 134.4 \quad \text{and} \quad x = 180 + 45.6 = 225.6$$

Short-answer questions

1 Convert each of the following to radian measure in terms of π:

a 330° **b** 810° **c** 1080° **d** 1035° **e** 135°
f 405° **g** 390° **h** 420° **i** 80°

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2 Convert each of the following to degree measure:

a $\frac{5\pi^c}{6}$ **b** $\frac{7\pi^c}{4}$ **c** $\frac{11\pi^c}{4}$ **d** $\frac{3\pi^c}{12}$ **e** $\frac{15\pi^c}{2}$

f $-\frac{3\pi^c}{4}$ **g** $-\frac{\pi^c}{4}$ **h** $-\frac{11\pi^c}{4}$ **i** $-\frac{23\pi^c}{4}$

3 Give exact values of each of the following:

a $\sin\left(\frac{11\pi}{4}\right)$ **b** $\cos\left(-\frac{7\pi}{4}\right)$ **c** $\sin\left(\frac{11\pi}{6}\right)$ **d** $\cos\left(-\frac{7\pi}{6}\right)$

e $\cos\left(\frac{13\pi}{6}\right)$ **f** $\sin\left(\frac{23\pi}{6}\right)$ **g** $\cos\left(-\frac{23}{3}\pi\right)$ **h** $\sin\left(-\frac{17}{4}\pi\right)$

4 State the amplitude and period of each of the following:

a $2\sin\left(\frac{\theta}{2}\right)$ **b** $-3\sin(4\theta)$ **c** $\frac{1}{2}\sin(3\theta)$

d $-3\cos(2x)$ **e** $-4\sin\left(\frac{x}{3}\right)$ **f** $\frac{2}{3}\sin\left(\frac{2x}{3}\right)$

5 Sketch the graph of each of the following (showing one cycle):

a $y = 2\sin(2x)$ **b** $y = -3\cos\left(\frac{x}{3}\right)$ **c** $y = -2\sin(3x)$

d $y = 2\sin\left(\frac{x}{3}\right)$ **e** $y = \sin\left(x - \frac{\pi}{4}\right)$ **f** $y = \sin\left(x + \frac{2\pi}{3}\right)$

g $y = 2\cos\left(x - \frac{5\pi}{6}\right)$ **h** $y = -3\cos\left(x + \frac{\pi}{6}\right)$

6 Solve each of the following equations for θ:

a $\sin\theta = -\frac{\sqrt{3}}{2},\ \theta \in [-\pi, \pi]$ **b** $\sin(2\theta) = -\frac{\sqrt{3}}{2},\ \theta \in [-\pi, \pi]$

c $\sin\left(\theta - \frac{\pi}{3}\right) = -\frac{1}{2},\ \theta \in [0, 2\pi]$ **d** $\sin\left(\theta + \frac{\pi}{3}\right) = -1,\ \theta \in [0, 2\pi]$

e $\sin\left(\frac{\pi}{3} - \theta\right) = -\frac{1}{2},\ \theta \in [0, 2\pi]$

7 If $\sin A = \frac{5}{13}$ and $\sin B = \frac{8}{17}$, where A and B are acute, find:

a $\cos(A + B)$ **b** $\sin(A - B)$ **c** $\tan(A + B)$

8 If $\tan A = 2$ and $\tan(\theta + A) = 4$, find the exact value of $\tan\theta$.

Extended-response questions

1 The number of hours of daylight at a point on the Antarctic Circle is given approximately by $d = 12 + 12\cos\left(\frac{1}{6}\pi\left(t + \frac{1}{3}\right)\right)$, where t is the number of months which have elapsed since 1 January.

a **i** Find d on 21 June ($t \approx 5.7$). **ii** Find d on 21 March ($t \approx 2.7$).

b When will there be 5 hours of daylight?

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2 The temperature, A°C, inside a house at t hours after 4 a.m. is given by the rule $A = 21 - 3\cos\left(\frac{\pi t}{12}\right)$, for $0 \le t \le 24$. The temperature, B°C, outside the house at the same time is given by $B = 22 - 5\cos\left(\frac{\pi t}{12}\right)$, for $0 \le t \le 24$.

- **a** Find the temperature inside the house at 8 a.m.
- **b** Write down an expression for $D = A - B$, the difference between the inside and outside temperatures.
- **c** Sketch the graph of D for $0 \le t \le 24$.
- **d** Determine when the inside temperature is less than the outside temperature.

3 At a certain time of the year the depth of water, d metres, in the harbour at Bunk Island is given by the rule $d = 3 + 1.8\cos\left(\frac{\pi}{6}t\right)$, where t is the time in hours after 3 a.m.

- **a** Sketch the graph of d against t for a 24-hour period from 3 a.m. to 3 a.m.
- **b** At what time(s) does high tide occur for $t \in [0, 24]$?
- **c** At what time(s) does low tide occur for $t \in [0, 24]$?

A ferry operates between Main Beach and Bunk Island. It takes 50 minutes to go from Main Beach to Bunk Island. The ferry only runs between 8 a.m. and 8 p.m., and is only able to enter the harbour at Bunk Island if the depth of water is at least 2 metres.

- **d** What is the earliest time the ferry should leave Main Beach so that it arrives at Bunk Island and can immediately enter the harbour?
- **e** The time to go from Bunk Island to Main Beach is also 50 minutes. The minimum time the ferry takes at Bunk Island harbour is 5 minutes. The minimum time at Main Beach is also 5 minutes.
  - **i** What is the latest time the ferry can leave Main Beach to complete a round trip in 105 minutes?
  - **ii** How many complete round trips can the ferry make in a day?

4 The depth of water, D metres, at the end of a pier t hours after low tide is given by the rule $D = p - 2\cos(rt)$, where p and r are suitable constants. At low tide ($t = 0$) the depth is 2 metres; at high tide, which occurs 8 hours later, the depth is 6 metres.

- **a** Show that $r = \frac{\pi}{8}$ and $p = 4$.
- **b** Sketch the graph of $D = 4 - 2\cos\left(\frac{\pi}{8}t\right)$ for $0 \le t \le 16$.
- **c** If the first low tide occurs at 4 a.m., when will the next low tide occur?
- **d** At what times will the depth be equal to 4 metres?

The poles that support the pier stand 7.5 metres above the sea bed.

- **e** How much of a particular pole is exposed at:
  - **i** high tide **ii** 2 p.m.?

Over the years mussels have attached themselves to the pole. A particular mussel is attached 4 metres from the top of the pole so that some of the time it is exposed and some of the time it is covered by water.

- **f** For how long will the mussel be covered by water during the time from one low tide to the next?

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13 Trigonometric ratios and applications

In this chapter

Syllabus references

Topics: Cosine and sine rules; Circular measure and radian measure

Subtopics: 1.2.1 – 1.2.4, 1.2.6

Trigonometry deals with the side lengths and angles of a triangle: the word *trigonometry* comes from the Greek words for triangle and measurement.

We start this chapter by revising the four standard congruence tests for triangles. If you have the information about a triangle given in one of the congruence tests, then the triangle is uniquely determined (up to congruence). You can find the unknown side lengths and angles of the triangle using the **sine rule** or the **cosine rule**. In this chapter, we establish these rules, and apply them in two- and three-dimensional problems.

We also apply trigonometry to measurement problems involving circles. We will determine lengths and angles associated with chords and arcs in circles, and find the areas of sectors and segments of circles. We will see that such problems are greatly simplified by measuring angles in radians rather than degrees.

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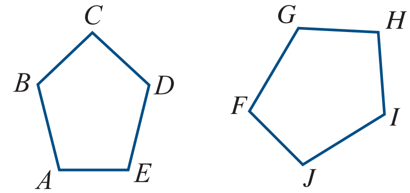
13A Reviewing trigonometry

Congruent triangles

Two plane figures are called **congruent** if one figure can be moved on top of the other figure, by a sequence of translations, rotations and reflections, so that they coincide exactly.

Congruent figures have exactly the same shape and size. For example, the two figures shown are congruent. We can write:

pentagon $ABCDE \equiv$ pentagon $FGHIJ$

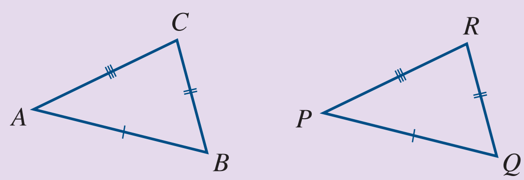


When two figures are congruent, we can find a transformation that pairs up every part of one figure with the corresponding part of the other, so that:

- paired angles have the same size
- paired line segments have the same length
- paired regions have the same area.

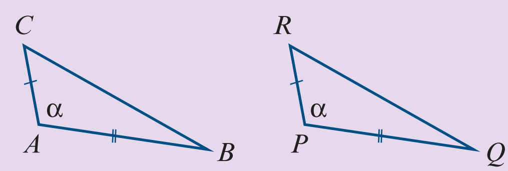
There are four standard tests for two triangles to be congruent.

- **The SSS congruence test**
  If the three sides of one triangle are respectively equal to the three sides of another triangle, then the two triangles are congruent.



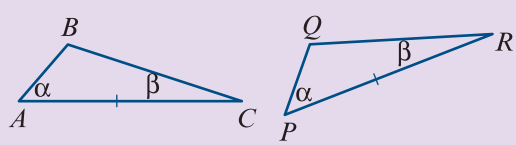
$\triangle ABC \equiv \triangle PQR$

- **The SAS congruence test**
  If two sides and the included angle of one triangle are respectively equal to two sides and the included angle of another triangle, then the two triangles are congruent.



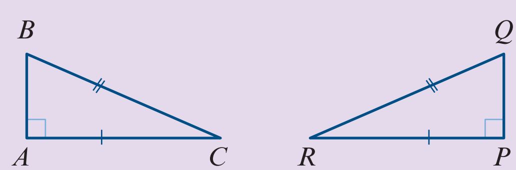
$\triangle ABC \equiv \triangle PQR$

- **The AAS congruence test**
  If two angles and one side of one triangle are respectively equal to two angles and the matching side of another triangle, then the two triangles are congruent.



$\triangle ABC \equiv \triangle PQR$

- **The RHS congruence test**
  If the hypotenuse and one side of one right-angled triangle are respectively equal to the hypotenuse and one side of another right-angled triangle, then the two triangles are congruent.



$\triangle ABC \equiv \triangle PQR$

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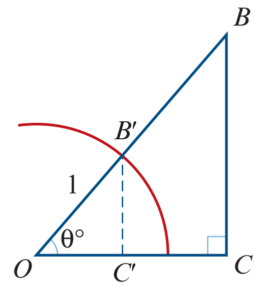
The trigonometric ratios

For acute angles, the unit-circle definition of sine and cosine given in Section 12B is equivalent to the ratio definition.

For a right-angled triangle OBC, we can construct a similar triangle $OB'C'$ that lies in the unit circle. From the diagram:



$$B'C' = \sin(\theta^\circ) \quad \text{and} \quad OC' = \cos(\theta^\circ)$$

As triangles OBC and $OB'C'$ are similar, we have

$$\frac{BC}{OB} = \frac{B'C'}{1} \quad \text{and} \quad \frac{OC}{OB} = \frac{OC'}{1}$$

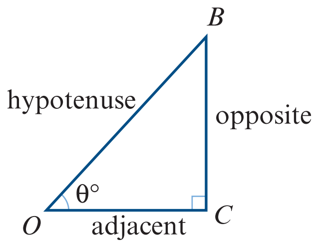
$$\therefore \quad \frac{BC}{OB} = \sin(\theta^\circ) \quad \text{and} \quad \frac{OC}{OB} = \cos(\theta^\circ)$$

This gives the ratio definition of sine and cosine for a right-angled triangle. The naming of sides with respect to an angle θ° is as shown.

$$\sin(\theta^\circ) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(\theta^\circ) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan(\theta^\circ) = \frac{\text{opposite}}{\text{adjacent}}$$

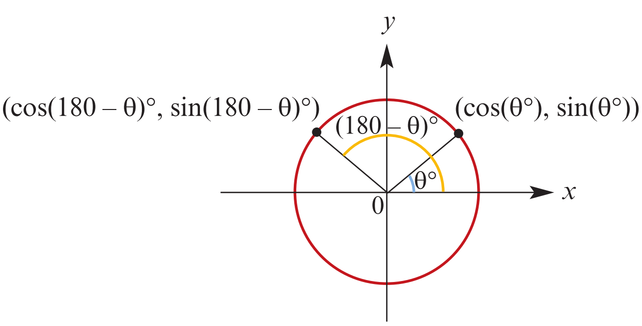


Obtuse angles

From the unit circle, we see that

$$\sin(180 - \theta)^\circ = \sin(\theta^\circ)$$

$$\cos(180 - \theta)^\circ = -\cos(\theta^\circ)$$



For example:

$$\sin 135^\circ = \sin 45^\circ$$

$$\cos 135^\circ = -\cos 45^\circ$$

In this chapter, we will generally use the ratio definition of tangent for acute angles. But we can also find the tangent of an obtuse angle by defining

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

We will not consider angles greater than 180° or less than 0° in this chapter, since we are dealing with triangles.

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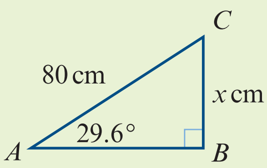
Solving right-angled triangles

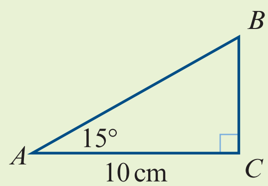
Here we provide some examples of using the trigonometric ratios.

Example 1

a Find the value of x correct to two decimal places.



b Find the length of the hypotenuse correct to two decimal places.



Solution

a

$$\frac{x}{80} = \sin 29.6°$$

$$\therefore\ x = 80 \sin 29.6°$$

$$= 39.5153\ldots$$

Hence $x = 39.52$, correct to two decimal places.

b

$$\frac{10}{AB} = \cos 15°$$

$$10 = AB \cos 15°$$

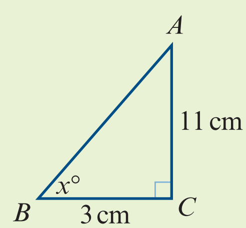
$$\therefore\ AB = \frac{10}{\cos 15°}$$

$$= 10.3527\ldots$$

The length of the hypotenuse is 10.35 cm, correct to two decimal places.

Example 2

Find the magnitude of $\angle ABC$.



Solution

$$\tan x = \frac{11}{3}$$

$$\therefore\ x = \tan^{-1}\left(\frac{11}{3}\right)$$

$$= (74.7448\ldots)°$$

Hence $x = 74.74°$, correct to two decimal places.

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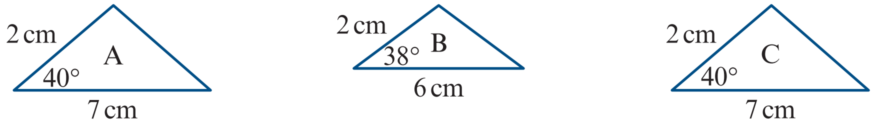
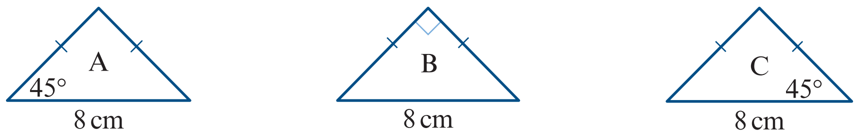
Exercise 13A

1 In each part, find pairs of congruent triangles. State the congruence tests used.

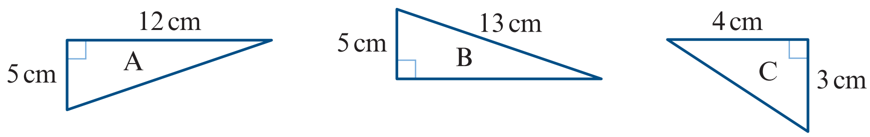
a



b



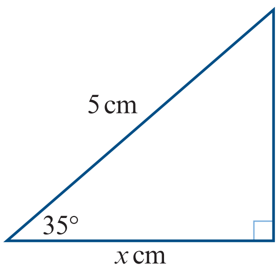
c



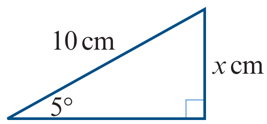
Example 1
Example 2

2 Find the value of x in each of the following:

a



b



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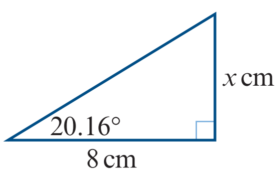
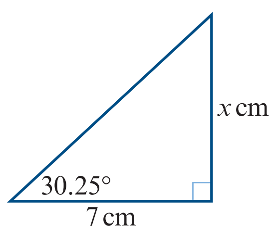
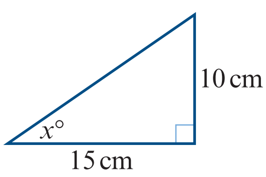


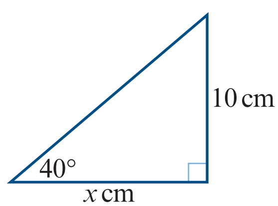
d



e



f



3 An equilateral triangle has altitudes of length 20 cm. Find the length of one side.

4 The base of an isosceles triangle is 12 cm long and the equal sides are 15 cm long. Find the magnitude of each of the three angles of the triangle.

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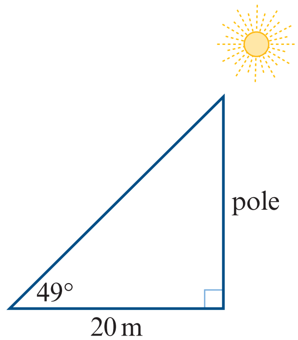
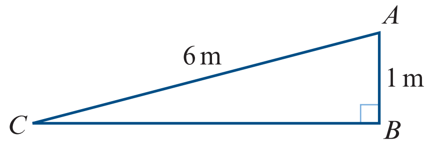
5 A pole casts a shadow 20 m long when the altitude of the sun is 49°. Calculate the height of the pole.



6 This figure represents a ramp.

a Find the magnitude of angle ACB.

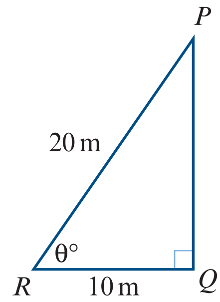
b Find the distance BC.



7 This figure shows a vertical mast PQ, which stands on horizontal ground. A straight wire 20 m long runs from P at the top of the mast to a point R on the ground, which is 10 m from the foot of the mast.

a Calculate the angle of inclination, $\theta°$, of the wire to the ground.

b Calculate the height of the mast.



8 A ladder leaning against a vertical wall makes an angle of 26° with the wall. If the foot of the ladder is 3 m from the wall, calculate:

a the length of the ladder

b the height it reaches above the ground.

9 An engineer is designing a straight concrete entry ramp, 60 m long, for a car park that is 13 m above street level. Calculate the angle of the ramp to the horizontal.

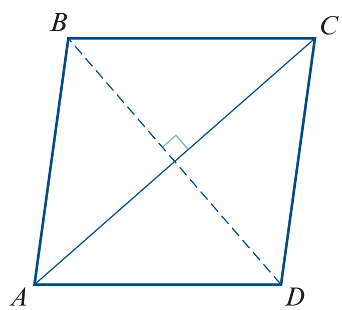
10 A vertical mast is secured from its top by straight cables 200 m long fixed at the ground. The cables make angles of 66° with the ground. What is the height of the mast?

11 A mountain railway rises 400 m at a uniform slope of 16° with the horizontal. What is the distance travelled by a train for this rise?

12 The diagonals of a rhombus bisect each other at right angles. If $BD = AC = 10$ cm, find:

a the length of the sides of the rhombus

b the magnitude of angle ABC.



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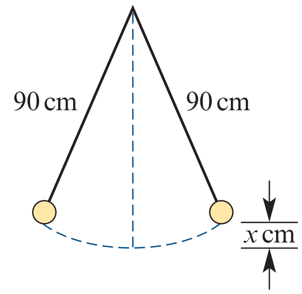
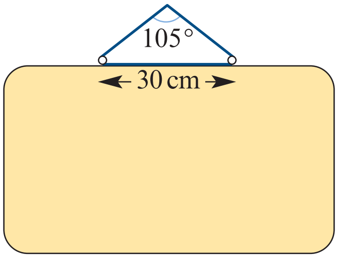
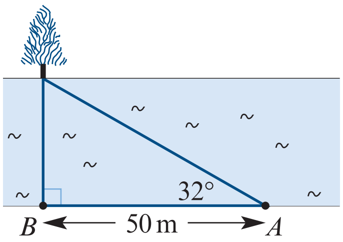
13 A pendulum swings from the vertical through an angle of 15° on each side of the vertical. If the pendulum is 90 cm long, what is the distance, x cm, between its highest and lowest points?



14 A picture is hung symmetrically by means of a string passing over a nail, with the ends of the string attached to two rings on the upper edge of the picture. The distance between the rings is 30 cm, and the string makes an angle of 105° at the nail. Find the length of the string.

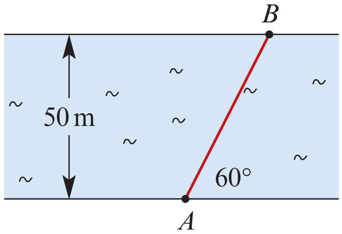


15 The distance AB is 50 m. If the line of sight to the tree of a person standing at A makes an angle of 32° with the bank, how wide is the river?



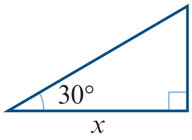
16 A ladder 4.7 m long is placed against a wall. The foot of the ladder must not be placed in a flower bed, which extends a distance of 1.7 m out from the base of the wall. How high up the wall can the ladder reach?

17 A river is known to be 50 m wide. A swimmer sets off from A to cross the river, and the path of the swimmer AB is as shown. How far does the person swim?



18 A rope is tied to the top of a flagpole. When it hangs straight down, it is 2 m longer than the pole. When the rope is pulled tight with the lower end on the ground, it makes an angle of 60° to the horizontal. How tall is the flagpole?

19 The triangle shown has perimeter 10. Find the value of x.



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13B The sine rule

In the previous section, we focused on right-angled triangles. In this section and the next, we consider non-right-angled triangles.

The **sine rule** is used to find unknown side lengths or angles of a triangle in the following two situations:

1 one side and two angles are given

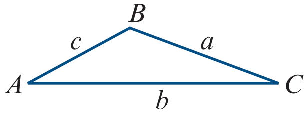
2 two sides and a non-included angle are given (that is, the given angle is not 'between' the two given sides).

In the first case, the triangle is uniquely defined up to congruence. In the second case, there may be two triangles.

Labelling triangles

The following convention is used in the remainder of this chapter:

- Interior angles are denoted by uppercase letters.
- The length of the side opposite an angle is denoted by the corresponding lowercase letter.

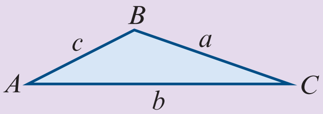


For example, the magnitude of angle BAC is denoted by A, and the length of side BC is denoted by a.

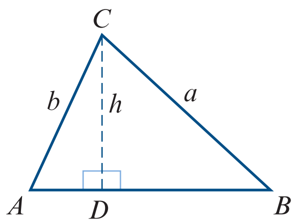
Sine rule

For triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Proof We will give a proof for acute-angled triangles. The proof for obtuse-angled triangles is similar.



In triangle ACD:

$$\sin A = \frac{h}{b}$$

$$\therefore \quad h = b \sin A$$

In triangle BCD:

$$\sin B = \frac{h}{a}$$

$$\therefore \quad a \sin B = b \sin A$$

$$\text{i.e.} \quad \frac{a}{\sin A} = \frac{b}{\sin B}$$

Similarly, starting with a perpendicular from A to BC would give

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

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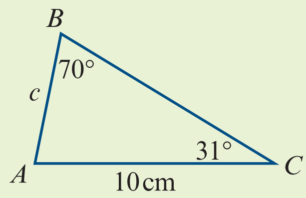
One side and two angles

When one side and two angles are given, this corresponds to the AAS congruence test. The triangle is uniquely defined up to congruence.

Example 3

Use the sine rule to find the length of AB.



Solution

$$\frac{c}{\sin 31^\circ} = \frac{10}{\sin 70^\circ}$$

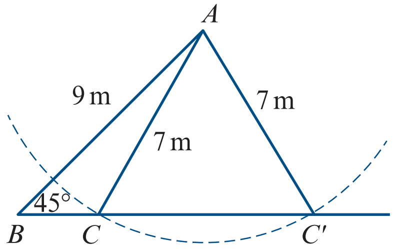
$$\therefore \quad c = \frac{10 \sin 31^\circ}{\sin 70^\circ}$$

$$= 5.4809\ldots$$

The length of AB is 5.48 cm, correct to two decimal places.

Two sides and a non-included angle

Suppose that we are given the two side lengths 7 m and 9 m and a non-included angle of 45°. There are two triangles that satisfy these conditions, as shown in the diagram.



Warning

- When you are given two sides and a non-included angle, you must consider the possibility that there are two such triangles.
- An angle found using the sine rule is possible if the sum of the given angle and the found angle is less than 180°.

Note: If the given angle is obtuse or a right angle, then there is only one such triangle.

The following example illustrates the case where there are two possible triangles.

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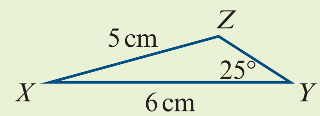
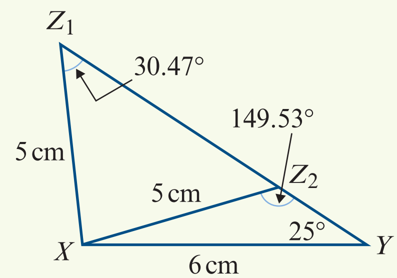
Example 4

Use the sine rule to find the magnitude of angle XZY in the triangle, given that $Y = 25°$, $y = 5$ cm and $z = 6$ cm.



Solution



$$\frac{5}{\sin 25°} = \frac{6}{\sin Z}$$

$$\frac{\sin Z}{6} = \frac{\sin 25°}{5}$$

$$\sin Z = \frac{6 \sin 25°}{5}$$

$$= 0.5071\ldots$$

$\therefore \quad Z = (30.473\ldots)° \quad \text{or} \quad Z = (180 - 30.473\ldots)°$

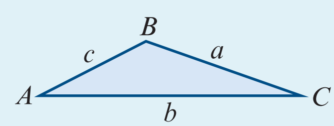
Hence $Z = 30.47°$ or $Z = 149.53°$, correct to two decimal places.

Note: Remember that $\sin(180 - \theta)° = \sin(\theta°)$.

Summary 13B

- **Sine rule** For triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



- When to use the sine rule:
  - one side and two angles are given (AAS)
  - two sides and a non-included angle are given.

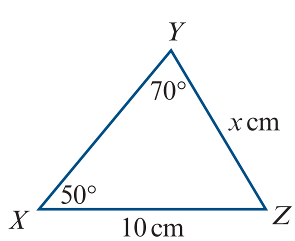
Skill-sheet

Exercise 13B

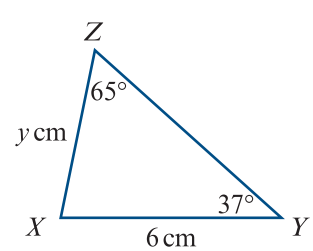
Example 3

1 Find the value of the pronumeral for each of the following triangles:

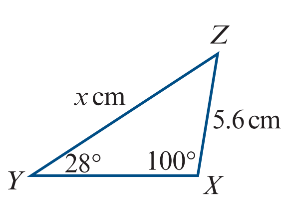
a



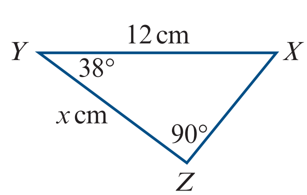
b



c



d



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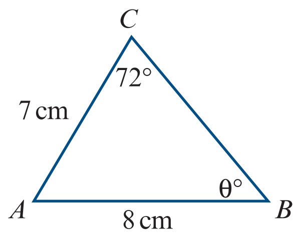
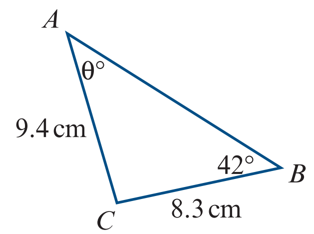
Example 4

2 Find the value of θ for each of the following triangles:

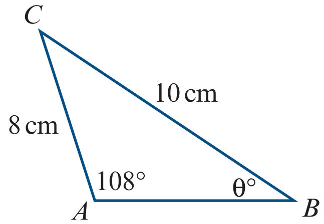
a



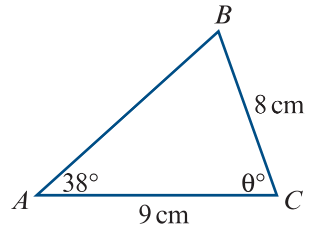
b



c



d



3 Solve the following triangles (i.e. find all sides and angles):

a $a = 12,\ B = 59°,\ C = 73°$

b $A = 75.3°,\ b = 5.6,\ B = 48.25°$

c $A = 123.2°,\ a = 11.5,\ C = 37°$

d $A = 23°,\ a = 15,\ B = 40°$

e $B = 140°,\ b = 20,\ A = 10°$

4 Solve the following triangles (i.e. find all sides and angles):

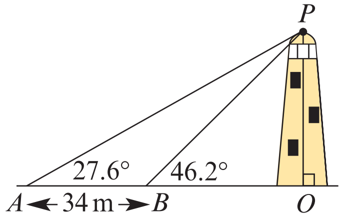
a $b = 17.6,\ C = 48.25°,\ c = 15.3$

b $B = 129°,\ b = 7.89,\ c = 4.56$

c $A = 28.35°,\ a = 8.5,\ b = 14.8$

5 A landmark A is observed from two points B and C, which are 400 m apart. The magnitude of angle ABC is measured as 68° and the magnitude of angle ACB as 70°. Find the distance of A from C.

6 P is a point at the top of a lighthouse. Measurements of the length AB and angles PBO and PAO are as shown in the diagram. Find the height of the lighthouse.

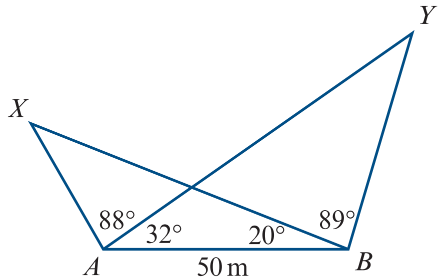


7 A and B are two points on a coastline, and C is a point at sea. The points A and B are 1070 m apart. The angles CAB and CBA have magnitudes of 74° and 69° respectively. Find the distance of C from A.

8 Find:

a AX

b AY



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13C The cosine rule

The **cosine rule** is used to find unknown side lengths or angles of a triangle in the following two situations:

1 two sides and the included angle are given
2 three sides are given.

In each case, the triangle is uniquely defined up to congruence.

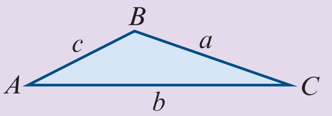
Cosine rule

For triangle ABC:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

or equivalently

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$



Proof We will give a proof for acute-angled triangles. The proof for obtuse-angled triangles is similar.

In triangle ACD:

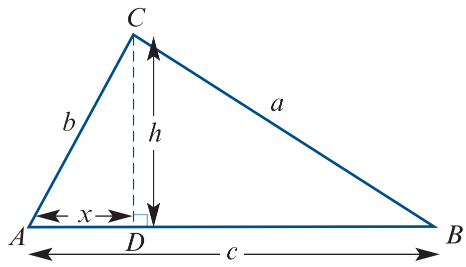
$$\cos A = \frac{x}{b}$$

$$\therefore \quad x = b \cos A$$

Using Pythagoras' theorem in triangles ACD and BCD:

$$b^2 = x^2 + h^2$$

$$a^2 = (c - x)^2 + h^2$$



Expanding gives

$$a^2 = c^2 - 2cx + x^2 + h^2$$

$$= c^2 - 2cx + b^2 \qquad (\text{as } b^2 = x^2 + h^2)$$

$$\therefore \quad a^2 = b^2 + c^2 - 2bc \cos A \qquad (\text{as } x = b \cos A)$$

Note: By symmetry, the following results also hold:

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Two sides and the included angle

When two sides and the included angle are given, this corresponds to the SAS congruence test. The triangle is uniquely defined up to congruence.

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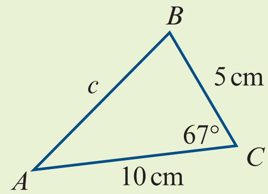
Example 5

For triangle ABC, find the length of AB in centimetres correct to two decimal places.



Solution

$$c^2 = 5^2 + 10^2 - 2 \times 5 \times 10 \cos 67^\circ$$
$$= 85.9268\ldots$$
$$\therefore \quad c = 9.2696\ldots$$

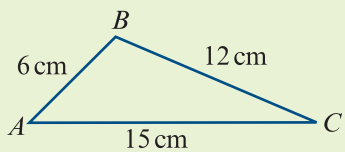
The length of AB is 9.27 cm, correct to two decimal places.

Three sides

When three sides are given, this corresponds to the SSS congruence test. The triangle is uniquely defined up to congruence.

Example 6

Find the magnitude of angle ABC.



Solution

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$= \frac{12^2 + 6^2 - 15^2}{2 \times 12 \times 6}$$
$$= -0.3125$$
$$\therefore \quad B = (108.2099\ldots)^\circ$$

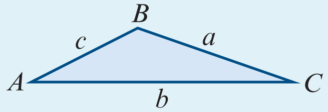
The magnitude of angle ABC is 108.21°, correct to two decimal places.

Summary 13C

- **Cosine rule** For triangle ABC:

$$a^2 = b^2 + c^2 - 2bc \cos A \quad \text{or} \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

- When to use the cosine rule:
  - two sides and the included angle are given (SAS)
  - three sides are given (SSS).



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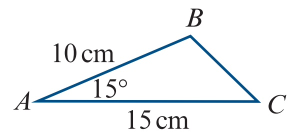
Skill-sheet

Exercise 13C

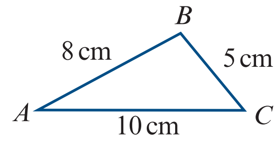
Example 5

1 Find the length of BC.



Example 6

2 Find the magnitudes of angles ABC and ACB.



3 For triangle ABC with:

a $A = 60°$ $b = 16$ $c = 30$, find a

b $a = 14$ $B = 53°$ $c = 12$, find b

c $a = 27$ $b = 35$ $c = 46$, find the magnitude of angle ABC

d $a = 17$ $B = 120°$ $c = 63$, find b

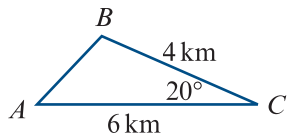
e $a = 31$ $b = 42$ $C = 140°$, find c

f $a = 10$ $b = 12$ $c = 9$, find the magnitude of angle BCA

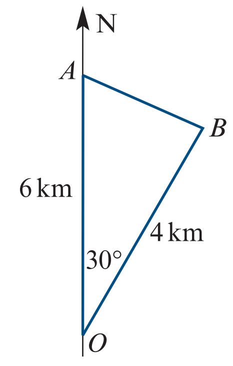
g $a = 11$ $b = 9$ $C = 43.2°$, find c

h $a = 8$ $b = 10$ $c = 15$, find the magnitude of angle CBA.

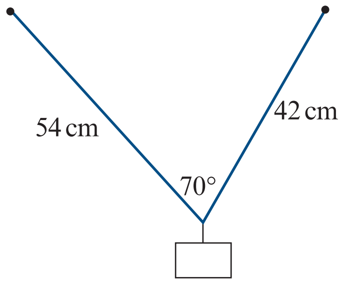
4 A section of an orienteering course is as shown. Find the length of leg AB.



5 Two ships sail in different directions from a point O. At a particular time, their positions A and B are as shown. Find the distance between the ships at this time.



6 A weight is hung from two hooks in a ceiling by strings of length 54 cm and 42 cm, which are inclined at 70° to each other. Find the distance between the hooks.



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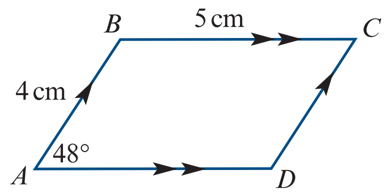
7 *ABCD* is a parallelogram. Find the lengths of the diagonals:

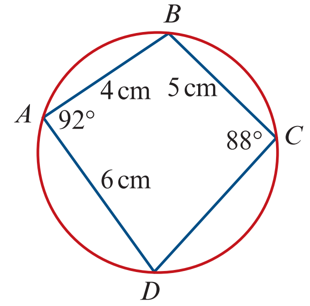
a *AC*

b *BD*



8 **a** Find the length of diagonal *BD*.

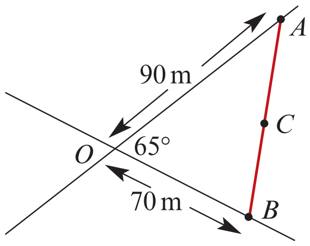
b Use the sine rule to find the length of *CD*.



9 Two straight roads intersect at an angle of 65°. A point *A* on one road is 90 m from the intersection and a point *B* on the other road is 70 m from the intersection, as shown.

a Find the distance of *A* from *B*.

b If *C* is the midpoint of *AB*, find the distance of *C* from the intersection.

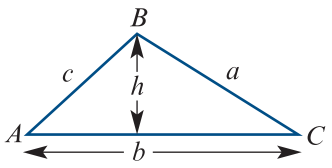


13D The area of a triangle

The area of a triangle is given by

$$\text{Area} = \frac{1}{2} \times \text{base length} \times \text{height}$$

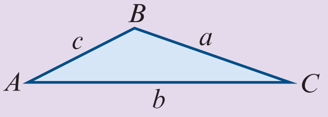
$$= \frac{1}{2}bh$$



By observing that $h = c \sin A$, we obtain the following useful formula.

For triangle *ABC*:

$$\text{Area} = \frac{1}{2}bc \sin A$$



That is, the area is half the product of the lengths of two sides and the sine of the angle included between them.

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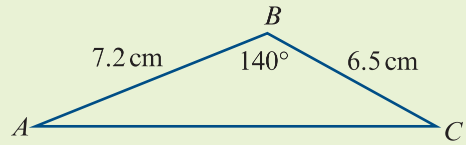
Example 7

Find the area of triangle ABC shown in the diagram.



Solution

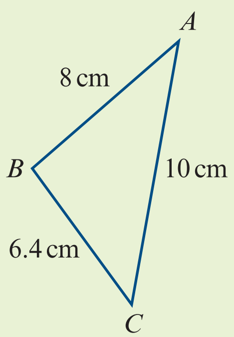
$$\text{Area} = \frac{1}{2} \times 7.2 \times 6.5 \sin 140^\circ$$

$$= 15.04 \text{ cm}^2 \qquad \text{(correct to two decimal places)}$$

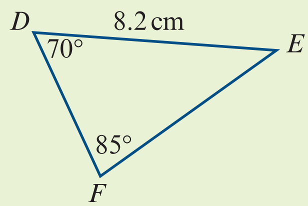
Example 8

Find the area of each of the following triangles, correct to three decimal places, assuming that triangle GHI is an obtuse-angled triangle:

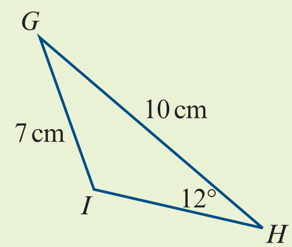
a



b



c



Solution

a Using the cosine rule:

$$8^2 = 6.4^2 + 10^2 - 2 \times 6.4 \times 10 \cos C$$

$$64 = 140.96 - 128 \cos C$$

$$\cos C = 0.60125$$

$$\therefore \quad C^\circ = (53.0405\ldots)^\circ \qquad \text{(store exact value on your calculator)}$$

$$\text{Area } \triangle ABC = \frac{1}{2} \times 6.4 \times 10 \times \sin C$$

$$= 25.570 \text{ cm}^2 \qquad \text{(correct to three decimal places)}$$

b Note that $E^\circ = (180 - (70 + 85))^\circ = 25^\circ$.

Using the sine rule:

$$DF = \sin 25^\circ \times \frac{8.2}{\sin 85^\circ}$$

$$= 3.4787\ldots \qquad \text{(store exact value on your calculator)}$$

$$\text{Area } \triangle DEF = \frac{1}{2} \times 8.2 \times DF \times \sin 70^\circ$$

$$= 13.403 \text{ cm}^2 \qquad \text{(correct to three decimal places)}$$

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c Using the sine rule:

$$\sin I = 10 \times \frac{\sin 12^\circ}{7}$$
$$= 0.2970\ldots$$

$$\therefore \quad I^\circ = (180 - 17.27\ldots)^\circ \qquad \text{(since } I \text{ is an obtuse angle)}$$
$$= (162.72\ldots)^\circ \qquad \text{(store exact value on your calculator)}$$

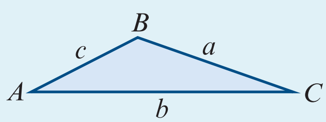
$$\therefore \quad G^\circ = (180 - (12 + I))^\circ$$
$$= (5.27\ldots)^\circ \qquad \text{(store exact value on your calculator)}$$

$$\text{Area } \triangle GHI = \frac{1}{2} \times 10 \times 7 \times \sin G$$
$$= 3.220\,\text{cm}^2 \qquad \text{(correct to three decimal places)}$$

Summary 13D

For triangle ABC:

$$\text{Area} = \frac{1}{2} bc \sin A$$



That is, the area is half the product of the lengths of two sides and the sine of the angle included between them.

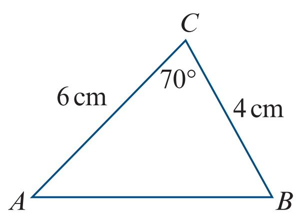
Skill-sheet

Exercise 13D

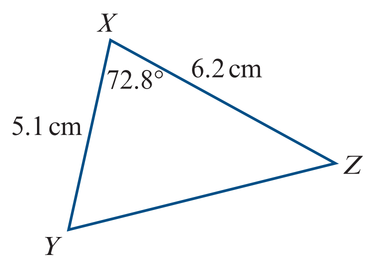
Example 7

1 Find the area of each of the following triangles:

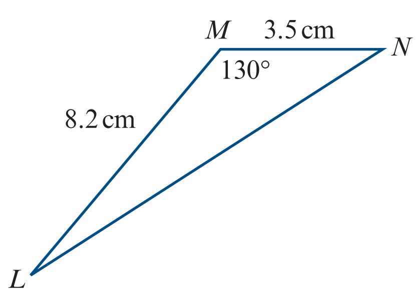
a



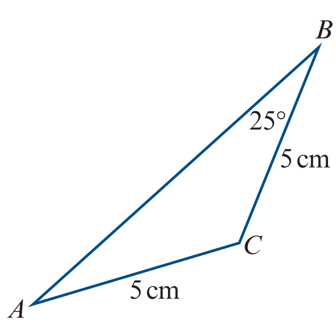
b



c



d



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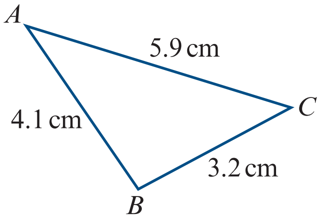
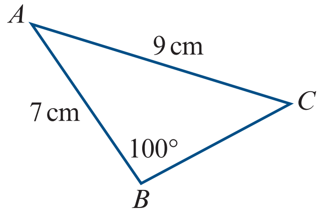
Example 8

2 Find the area of each of the following triangles, correct to three decimal places:

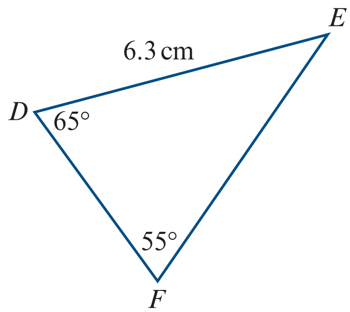
a



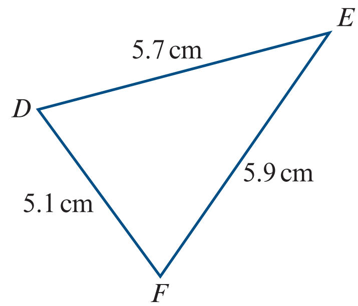
b



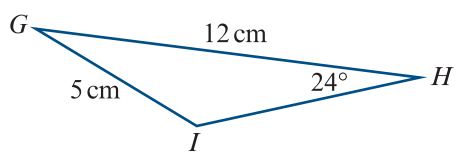
c



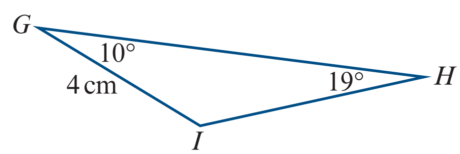
d



e



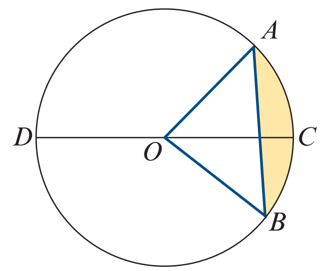
f



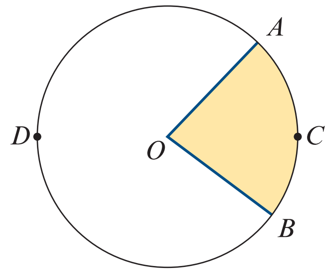
13E Circle mensuration

Terminology

In the diagram, the circle has centre O.



- Chords A **chord** of a circle is a line segment with endpoints on the circle; e.g. line segment AB in the diagram. A chord passing through the centre of the circle is called a **diameter**; e.g. line segment CD in the diagram.
- Arcs Any two points on a circle divide the circle into arcs. The shorter arc is called the **minor arc** and the longer is the **major arc**. In the diagram, arc ACB is a minor arc and arc ADB is a major arc. The arcs DAC and DBC are called **semicircular arcs**.
- Segments Every chord divides the interior of a circle into two regions called segments. The smaller is called the **minor segment** and the larger is the **major segment**. In the above diagram, the minor segment has been shaded.
- Sectors Two radii and an arc define a region called a sector. In this diagram, with circle centre O, the shaded region is a **minor sector** and the unshaded region is a **major sector**.



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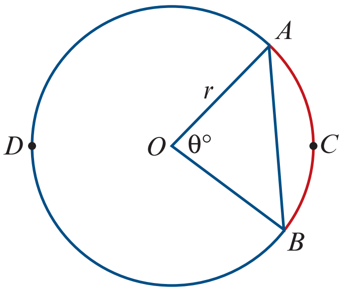
Arc length

The circle in the diagram has centre O and radius r. The arc ACB and the corresponding chord AB are said to **subtend** the angle $\angle AOB$ at the centre of the circle.



The magnitude $\theta°$ of angle $\angle AOB$ is a fraction of $360°$. The length ℓ of arc ACB will be the same fraction of the circumference of the circle, $2\pi r$.

Length of an arc using degrees

$$\ell = \frac{\theta}{360} \times 2\pi r$$

$$= \frac{\pi r \theta}{180} \qquad \text{(where } \theta \text{ is measured in degrees)}$$

Radian measure of angles was introduced in Section 12A.

We recall that, in the unit circle, an arc of length θ units subtends an angle of θ radians at the centre. A circle of radius r is similar to the unit circle, with similarity factor r, and therefore an arc of length $r\theta$ units subtends an angle of θ radians at the centre.

Length of an arc using radians

$$\ell = r\theta \qquad \text{(where } \theta \text{ is measured in radians)}$$

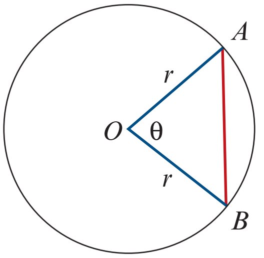
Note: As there are 2π radians in a circle, the arc length is $\ell = \frac{\theta}{2\pi} \times 2\pi r = r\theta$.

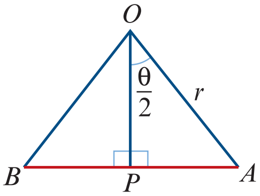
Chord length

In triangle OAP:

$$AP = r\sin\left(\frac{\theta}{2}\right)$$

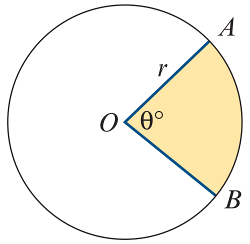
$$\therefore \quad AB = 2r\sin\left(\frac{\theta}{2}\right)$$





Area of a sector

The magnitude $\theta°$ of angle $\angle AOB$ is a fraction of $360°$. The area of the sector will be the same fraction of the area of the circle, πr^2.



Using degrees: Area of sector $= \dfrac{\pi r^2 \theta}{360}$

Using radians: Area of sector $= \dfrac{1}{2} r^2 \theta$

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Example 9

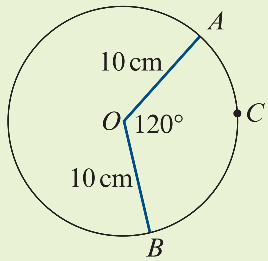
The circle shown has centre O and radius length 10 cm. The angle subtended at O by arc ACB has magnitude $120°$. Find:

a **i** the exact length of the chord AB

ii the exact length of the arc ACB

b the exact area of the minor sector AOB

c the magnitude of angle AOC, in degrees, if the minor arc AC has length 4 cm.



Solution

a **i** Chord length $= 2r\sin\left(\frac{\theta}{2}\right)$

$= 20\sin 60°$ since $r = 10$ and $\theta = 120°$

$= 20 \times \frac{\sqrt{3}}{2}$

$= 10\sqrt{3}$

Length of chord is $10\sqrt{3}$ cm.

ii Arc length $\ell = r\theta$ using radians

$= 10 \times \frac{2\pi}{3}$ since $r = 10$ and $\theta = \frac{2\pi}{3}$

$= \frac{20\pi}{3}$

Length of arc is $\frac{20\pi}{3}$ cm.

Check: Verify that length of arc is greater than length of chord.

b Area of sector $= \frac{1}{2}r^2\theta$ using radians

$= \frac{1}{2} \times 10^2 \times \frac{2\pi}{3}$ since $r = 10$ and $\theta = \frac{2\pi}{3}$

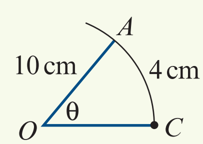
$= \frac{100\pi}{3}$

Area of minor sector AOB is $\frac{100\pi}{3}$ cm^2.

c Using radians: $\ell = r\theta$

$4 = 10\theta$

$\therefore \quad \theta = \frac{4}{10}$



Convert to degrees: $\angle AOC = 0.4 \times \frac{180}{\pi}$

$= (22.9183\ldots)°$

$= 22.92°$ (correct to two decimal places)

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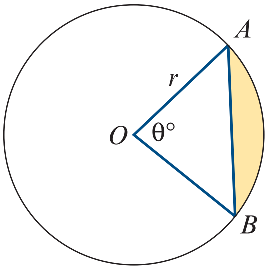
Area of a segment

The area of the shaded segment is found by subtracting the area of $\triangle AOB$ from the area of the minor sector OAB.



Using degrees: Area of segment $= \dfrac{\pi r^2\theta}{360} - \dfrac{1}{2}r^2 \sin\theta$

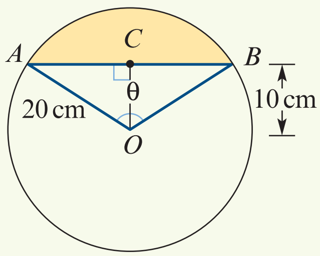
Using radians: Area of segment $= \dfrac{1}{2}r^2\theta - \dfrac{1}{2}r^2 \sin\theta$

Example 10

A circle, with centre O and radius length 20 cm, has a chord AB that is 10 cm from the centre of the circle. Calculate the area of the minor segment formed by this chord.

Solution

The area of the segment is $\frac{1}{2}r^2(\theta - \sin\theta)$. We know $r = 20$, but we need to find θ.



In $\triangle OCB$: $\cos\left(\dfrac{\theta}{2}\right) = \dfrac{10}{20}$

$$\frac{\theta}{2} = \frac{\pi}{3}$$

$$\therefore \quad \theta = \frac{2\pi}{3}$$

$$\text{Area of segment} = \frac{1}{2} \times 20^2\left(\frac{2\pi}{3} - \sin\left(\frac{2\pi}{3}\right)\right)$$

$$= 200\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

$$= 200\left(\frac{4\pi - 3\sqrt{3}}{6}\right)$$

$$= \frac{100(4\pi - 3\sqrt{3})}{3}\ \text{cm}^2$$

Summary 13E

- Circle mensuration formulas with θ in radians
  - Arc length $= r\theta$
  - Chord length $= 2r\sin\left(\dfrac{\theta}{2}\right)$
  - Area of sector $= \dfrac{1}{2}r^2\theta$
  - Area of segment $= \dfrac{1}{2}r^2(\theta - \sin\theta)$
- Circle mensuration formulas with θ in degrees
  - Arc length $= \dfrac{\pi r\theta}{180}$
  - Chord length $= 2r\sin\left(\dfrac{\theta}{2}\right)$
  - Area of sector $= \dfrac{\pi r^2\theta}{360}$
  - Area of segment $= \dfrac{\pi r^2\theta}{360} - \dfrac{1}{2}r^2\sin\theta$

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Skill-sheet

Exercise 13E

Example 9

1 Find the length of an arc which subtends an angle of magnitude 105° at the centre of a circle of radius length 25 cm.

2 Find the magnitude, in degrees, of the angle subtended at the centre of a circle of radius length 30 cm by:

a an arc of length 50 cm

b a chord of length 50 cm.

Example 10

3 A chord of length 6 cm is drawn in a circle of radius 7 cm. Find:

a the length of the minor arc cut off by the chord

b the area of the smaller region inside the circle cut off by the chord.

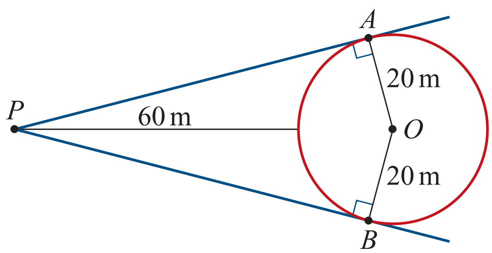
4 Sketch, on the same set of axes, the graphs of $A = \{ (x, y) : x^2 + y^2 \leq 16 \}$ and $B = \{ (x, y) : y \geq 2 \}$. Find the area measure of the region $A \cap B$.

5 Find the area of the region between an equilateral triangle of side length 10 cm and the circumcircle of the triangle (the circle that passes through the three vertices of the triangle).

6 A person stands on level ground 60 m from the nearest point of a cylindrical tank of radius length 20 m. Calculate:

a the circumference of the tank

b the percentage of the circumference that is visible to the person.



Note: In the diagram, the point O is the centre of the circle and $\angle PAO = \angle PBO = 90°$.

7 The minute hand of a large clock is 4 m long.

a How far does the tip of the minute hand move between 12:10 p.m. and 12:35 p.m.?

b What is the area covered by the minute hand between 12:10 p.m. and 12:35 p.m.?

8 Two circles of radii 3 cm and 4 cm have their centres 5 cm apart. Calculate the area of the region common to both circles.

9 A sector of a circle has perimeter 32 cm and area 63 cm^2. Find the radius length and the magnitude of the angle subtended at the centre of the two possible sectors.

10 Two wheels (pulleys) have radii of length 15 cm and 25 cm and have their centres 60 cm apart. What is the length of the belt required to pass tightly around the pulleys without crossing?

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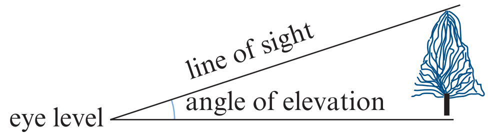
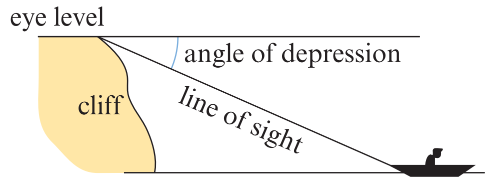
13F Angles of elevation, angles of depression and bearings

The **angle of elevation** is the angle between the horizontal and a direction above the horizontal.



The **angle of depression** is the angle between the horizontal and a direction below the horizontal.



Example 11

The pilot of a helicopter flying at 400 m observes a small boat at an angle of depression of 1.2°. Calculate the horizontal distance of the boat to the helicopter.

Solution

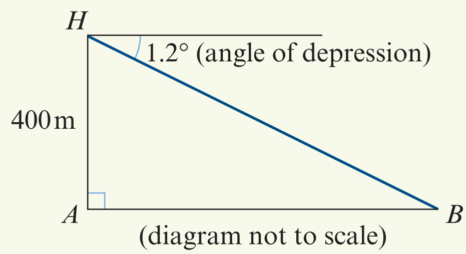
Note that $\angle ABH = 1.2°$, using alternate angles.

Thus

$$\frac{AH}{AB} = \tan 1.2°$$

$$\frac{400}{AB} = \tan 1.2°$$

$$\therefore \quad AB = \frac{400}{\tan 1.2°} = 19\,095.800\ldots$$



The horizontal distance is 19 100 m, correct to the nearest 10 m.

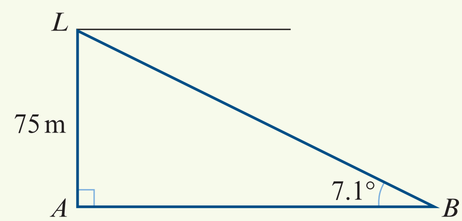
Example 12

The light on a cliff-top lighthouse, known to be 75 m above sea level, is observed from a boat at an angle of elevation of 7.1°. Calculate the distance of the boat from the lighthouse.

Solution

$$\frac{75}{AB} = \tan 7.1°$$

$$\therefore \quad AB = \frac{75}{\tan 7.1°} = 602.135\ldots$$



The distance of the boat from the lighthouse is 602 m, correct to the nearest metre.

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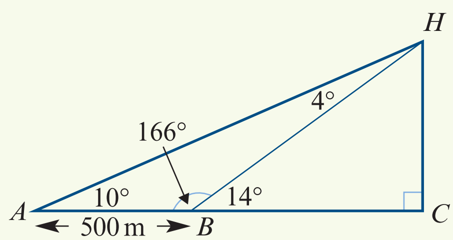
Example 13

From the point A, a man observes that the angle of elevation of the summit of a hill is 10°. He then walks towards the hill for 500 m along flat ground. The summit of the hill is now at an angle of elevation of 14°. Find the height of the hill above the level of A.

Solution

Magnitude of $\angle HBA = (180 - 14)^\circ = 166^\circ$

Magnitude of $\angle AHB = (180 - (166 + 10))^\circ = 4^\circ$



Using the sine rule in triangle ABH:

$$\frac{500}{\sin 4^\circ} = \frac{HB}{\sin 10^\circ}$$

$$\therefore \quad HB = \frac{500 \sin 10^\circ}{\sin 4^\circ}$$

$$= 1244.67\ldots$$

In triangle BCH:

$$\frac{HC}{HB} = \sin 14^\circ$$

$$\therefore \quad HC = HB \sin 14^\circ$$

$$= 301.11\ldots$$

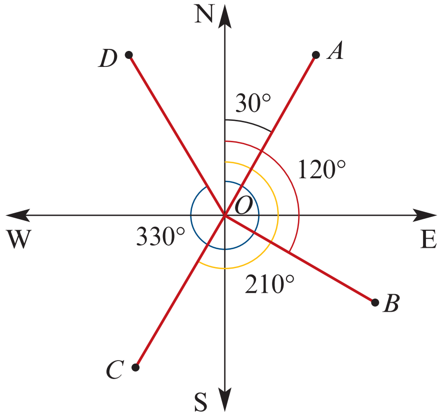
The height of the hill is 301 m, correct to the nearest metre.

Bearings

The **bearing** (or compass bearing) is the direction measured from north clockwise.

For example:

- The bearing of A from O is 030°.
- The bearing of B from O is 120°.
- The bearing of C from O is 210°.
- The bearing of D from O is 330°.



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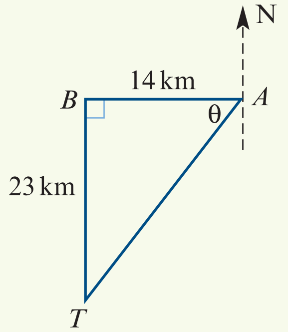
Example 14

The road from town A runs due west for 14 km to town B. A television mast is located due south of B at a distance of 23 km. Calculate the distance and bearing of the mast from the centre of town A.

Solution



$$\tan\theta = \frac{23}{14}$$

$\therefore \quad \theta = 58.67°$ (to two decimal places)

Thus the bearing is

$$180° + (90 - 58.67)° = 211.33°$$

To find the distance, use Pythagoras' theorem:

$$\begin{aligned} AT^2 &= AB^2 + BT^2 \\ &= 14^2 + 23^2 \\ &= 725 \end{aligned}$$

$\therefore \quad AT = 26.925\ldots$

The mast is 27 km from the centre of town A (to the nearest kilometre) and on a bearing of 211.33°.

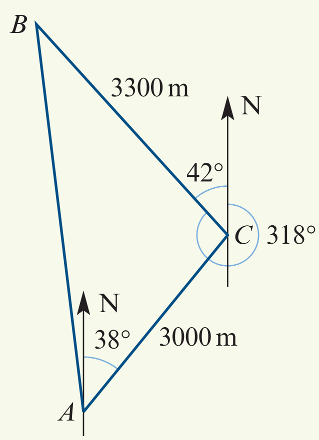
Example 15

A yacht starts from a point A and sails on a bearing of 038° for 3000 m. It then alters its course to a bearing of 318° and after sailing for a further 3300 m reaches a point B. Find:

a the distance AB

b the bearing of B from A.

Solution



a The magnitude of angle ACB needs to be found so that the cosine rule can be applied in triangle ABC:

$$\angle ACB = (180 - (38 + 42))° = 100°$$

In triangle ABC:

$$\begin{aligned} AB^2 &= 3000^2 + 3300^2 - 2 \times 3000 \times 3300 \cos 100° \\ &= 23\,328\,233.917\ldots \end{aligned}$$

$\therefore \quad AB = 4829.931\ldots$

The distance of B from A is 4830 m (to the nearest metre).

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b To find the bearing of B from A, the magnitude of angle BAC must first be found. Using the sine rule:

$$\frac{3300}{\sin A} = \frac{AB}{\sin 100°}$$

$$\therefore \quad \sin A = \frac{3300 \sin 100°}{AB}$$

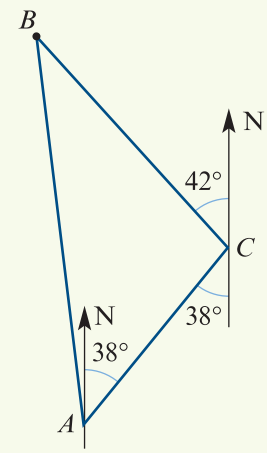
$$= 0.6728\ldots$$

$$\therefore \quad A = (42.288\ldots)°$$

The bearing of B from $A = 360° - (42.29° - 38°)$

$= 355.71°$

The bearing of B from A is 356° to the nearest degree.



Exercise 13F

Example 11

1 From the top of a vertical cliff 130 m high, the angle of depression of a buoy at sea is 18°. What is the distance of the buoy from the foot of the cliff?

Example 12

2 The angle of elevation of the top of an old chimney stack at a point 40 m from its base is 41°. Find the height of the chimney.

3 A hiker standing on top of a mountain observes that the angle of depression to the base of a building is 41°. If the height of the hiker above the base of the building is 500 m, find the horizontal distance from the hiker to the building.

4 A person lying down on top of a cliff 40 m high observes the angle of depression to a buoy in the sea below to be 20°. If the person is in line with the buoy, find the distance between the buoy and the base of the cliff, which may be assumed to be vertical.

Example 13

5 A person standing on top of a cliff 50 m high is in line with two buoys whose angles of depression are 18° and 20°. Calculate the distance between the buoys.

Example 14

6 A ship sails 10 km north and then sails 15 km east. What is its bearing from the starting point?

7 A ship leaves port A and travels 15 km due east. It then turns and travels 22 km due north.

a What is the bearing of the ship from port A?

b What is the bearing of port A from the ship?

Example 15

8 A yacht sails from point A on a bearing of 035° for 2000 m. It then alters course to a direction with a bearing of 320° and after sailing for 2500 m it reaches point B.

a Find the distance AB.

b Find the bearing of B from A.

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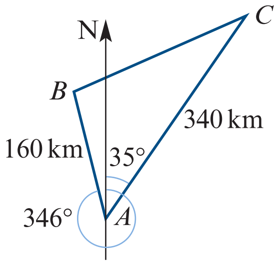
9 The bearing of a point A from a point B is 207°. What is the bearing of B from A?

10 The bearing of a ship S from a lighthouse A is 055°. A second lighthouse B is due east of A. The bearing of S from B is 302°. Find the magnitude of angle ASB.

11 A yacht starts from L and sails 12 km due east to M. It then sails 9 km on a bearing of 142° to K. Find the magnitude of angle MLK.

12 The bearing of C from A is 035°. The bearing of B from A is 346°. The distance of C from A is 340 km. The distance of B from A is 160 km.



a Find the magnitude of angle BAC.

b Use the cosine rule to find the distance from B to C.

13 From a ship S, two other ships P and Q are on bearings 320° and 075° respectively. The distance PS is 7.5 km and the distance QS is 5 km. Find the distance PQ.

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Chapter summary

Assign-
ment

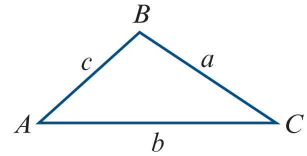
Nrich

Triangles

■ **Labelling triangles**

- Interior angles are denoted by uppercase letters.
- The length of the side opposite an angle is denoted by the corresponding lowercase letter.

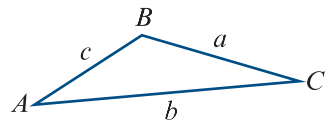
For example, the magnitude of angle BAC is denoted by A, and the length of side BC by a.



■ **Sine rule**

For triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



The sine rule is used to find unknown quantities in a triangle in the following cases:

- one side and two angles are given
- two sides and a non-included angle are given.

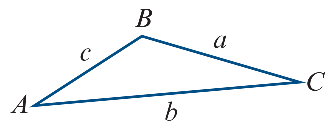
In the first case, the triangle is uniquely defined. But in the second case, there may be two triangles.

■ **Cosine rule**

For triangle ABC:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$



The symmetrical results also hold:

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

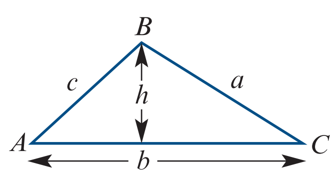
The cosine rule is used to find unknown quantities in a triangle in the following cases:

- two sides and the included angle are given
- three sides are given.

■ **Area of a triangle**

$$\text{Area} = \frac{1}{2}bh$$

$$\text{Area} = \frac{1}{2}bc\sin A$$



That is, the area of a triangle is half the product of the lengths of two sides and the sine of the angle included between them.

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Review

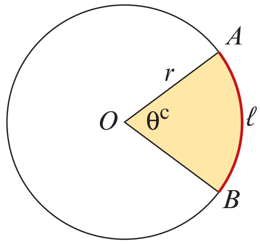
Circles

- Length of minor arc AB (red curve) is given by

$$\ell = r\theta$$

- Area of sector AOB (shaded) is given by

$$\text{Area} = \frac{1}{2}r^2\theta$$

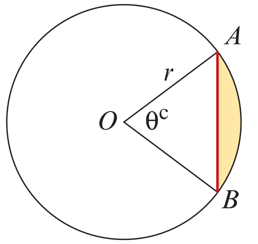


- Length of chord AB (red line) is given by

$$\ell = 2r\sin\left(\frac{\theta}{2}\right)$$

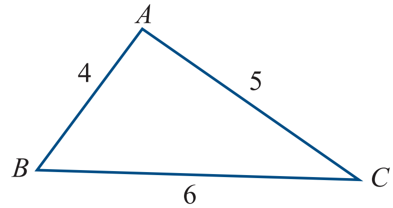
- Area of segment (shaded) is given by

$$\text{Area} = \frac{1}{2}r^2(\theta - \sin\theta)$$

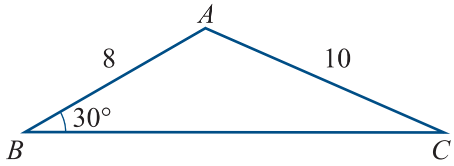


Short-answer questions

1 The triangle shown has sides $AB = 4$, $BC = 6$ and $CA = 5$. Find $\cos(\angle BAC)$.



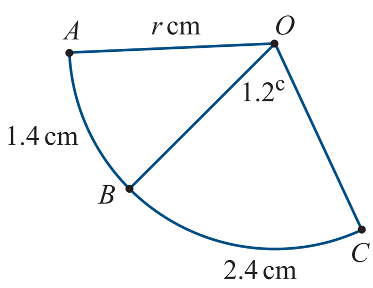
2 In triangle ABC, $AB = 8$, $AC = 10$ and $\angle ABC = 30°$. Find $\sin(\angle ACB)$.



3 The diagram shows part of a circle with centre O and radius r cm. The length of arc AB is 1.4 cm and the length of arc BC is 2.4 cm. The magnitude of $\angle BOC$ is 1.2 radians. Calculate:

a the value of r

b the magnitude of $\angle AOB$ in radians.



4 Triangle ABC has $AB = BC = 10$ cm and $\angle ABC = 120°$. Find AC.

5 In a quadrilateral $ABCD$, $AB = 5$ cm, $BC = 5$ cm, $CD = 7$ cm, $B = 120°$ and $C = 90°$. Find:

a the length of the diagonal AC

b the area of triangle ABC

c the area of triangle ADC

d the area of the quadrilateral.

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6 If $\sin x = \sin 37°$ and x is obtuse, find x.

7 A point T is 10 km due north of a point S. A point R, which is east of the straight line joining T and S, is 8 km from T and 7 km from S. Calculate the cosine of the bearing of R from S.

8 In $\triangle ABC$, $AB = 5$ cm, $\angle BAC = 60°$ and $AC = 6$ cm. Calculate the sine of $\angle ABC$.

9 The area of a sector of a circle with radius 6 cm is 33 cm^2. Calculate the angle of the sector.

10 A boat sails 11 km from a harbour on a bearing of 220°. It then sails 15 km on a bearing of 340°. How far is the boat from the harbour?

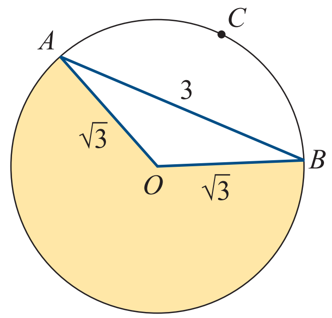
11 A helicopter leaves a heliport A and flies 2.4 km on a bearing of 150° to a checkpoint B. It then flies due east to its base C. If the bearing of C from A is 120°, find the distances AC and BC.

12 A sector of a circle has an arc length of 30 cm. If the radius of the circle is 12 cm, find the area of the sector.

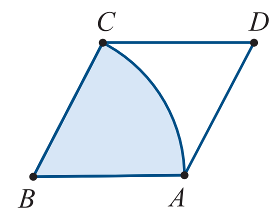
13 The diagram shows a circle of radius length $\sqrt{3}$ cm and a chord AB of length 3 cm. Calculate:

a the length of arc ACB

b the area of the shaded region.



14 The diagram shows a rhombus $ABCD$ with each side of length 4 cm. An arc AC, centre B, has length 5 cm. Calculate the area of the white region ACD.



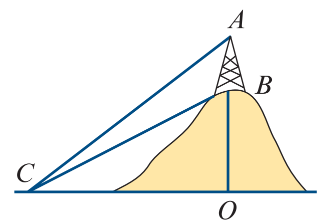
Extended-response questions

1 AB is a tower 60 m high on top of a hill. The magnitude of $\angle ACO$ is 49° and the magnitude of $\angle BCO$ is 37°.

a Find the magnitudes of $\angle ACB$, $\angle CBO$ and $\angle CBA$.

b Find the length of BC.

c Find the height of the hill, i.e. the length of OB.



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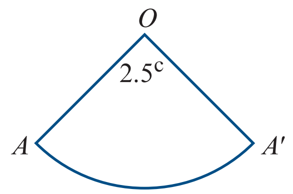
Review

2 The angle of a sector of a circle, centre O and radius length 12 cm, has magnitude 2.5 radians. The sector is folded so that OA and OA' are joined to form a cone. Calculate:

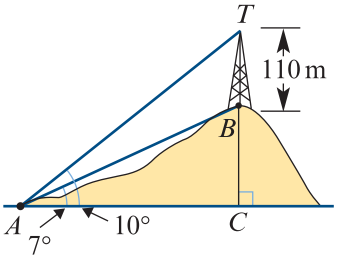


a the base radius length of the cone

b the curved surface area of the cone

c the shortest distance between two points diametrically opposite on the edge of the base.

3 A tower 110 m high stands on the top of a hill. From a point A at the foot of the hill, the angle of elevation of the bottom of the tower is 7° and that of the top is 10°.

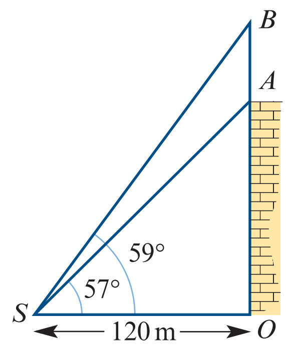


a Find the magnitudes of angles TAB, ABT and ATB.

b Use the sine rule to find the length of AB.

c Find CB, the height of the hill.

4 Point S is a distance of 120 m from the base of a building. On the building is an aerial, AB. The angle of elevation from S to A is 57°. The angle of elevation from S to B is 59°. Find:

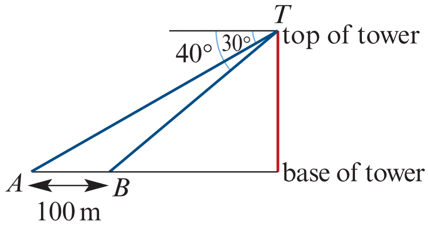


a the distance OA

b the distance OB

c the distance AB.

5 From the top of a communications tower, T, the angles of depression of two points A and B on a horizontal line through the base of the tower are 30° and 40°. The distance between the points is 100 m. Find:



a the distance AT

b the distance BT

c the height of the tower.

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14 Exponential functions

In this chapter

Syllabus references

Topics: Indices and the index laws; Exponential functions

Subtopics: 2.1.1 – 2.1.7

The function $f(x) = ka^x$, where k is a non-zero constant and the base a is a positive real number other than 1, is called an **exponential function**.

Consider the following example of an exponential function. Assume that a particular biological organism reproduces by dividing every minute. The following table shows the population, P, after n one-minute intervals (assuming that all organisms are still alive).

n	0	1	2	3	4	5	6	n
P	1	2	4	8	16	32	64	2^n

Thus P defines a function which has the rule $P = 2^n$, an exponential function.

In this chapter, many of the concepts introduced in Chapter 6 – such as domain, range and transformations – are used in the context of exponential functions.

There are many applications of exponential functions, and some of these are investigated in the final section of this chapter.

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14A The index laws

The expression a^n is called a **power**, where a is a non-zero number called the **base** and n is a number called the **exponent** or **index**. In this section we concentrate on indices that are integers.

We note that, if n is positive, then $0^n = 0$. But if n is negative or zero, then 0^n is undefined.

Index law 1: Multiplying powers

Index law 1

To **multiply** two powers with the same base, **add** the indices.

$$a^m \times a^n = a^{m+n}$$

If m and n are positive integers,

then $$a^m = \underbrace{a \times a \times \cdots \times a}_{m \text{ terms}}$$

and $$a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ terms}}$$

$$\therefore \quad a^m \times a^n = (\underbrace{a \times a \times \cdots \times a}_{m \text{ terms}}) \times (\underbrace{a \times a \times \cdots \times a}_{n \text{ terms}})$$

$$= \underbrace{a \times a \times \cdots \times a}_{(m+n) \text{ terms}}$$

$$= a^{m+n}$$

Example 1

Simplify each of the following:

a $2^3 \times 2^{12}$ **b** $x^2y^3 \times x^4y$ **c** $2^x \times 2^{x+2}$ **d** $3a^2b^3 \times 4a^3b^3$

Solution	Explanation
a $2^3 \times 2^{12} = 2^{3+12} = 2^{15}$	When multiplying powers with the same base, add the indices.
b $x^2y^3 \times x^4y = x^2 \times x^4 \times y^3 \times y = x^6y^4$	In part b, the indices of the base-x powers are added, and the indices of the base-y powers are added. Note that $y = y^1$.
c $2^x \times 2^{x+2} = 2^{x+x+2} = 2^{2x+2}$	In part c, we use $x + x + 2 = 2x + 2$.
d $3a^2b^3 \times 4a^3b^3 = 3 \times 4 \times a^2 \times a^3 \times b^3 \times b^3 = 12a^5b^6$	In part d, the indices of the base-a powers are added, the indices of the base-b powers are added, and the factors that are numerals are multiplied together.

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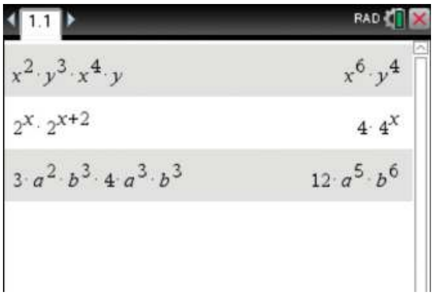
Using the TI-Nspire

Parts b, c and d can be simplified as shown: the TI-Nspire will simplify automatically.

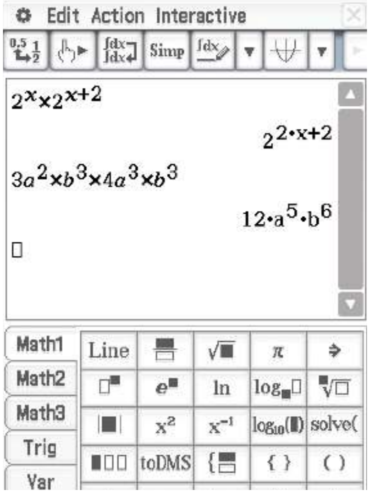
Note: When using [∧] to enter indices, you need to use either ► or ▼ to return to the baseline.



Using the Casio ClassPad

- To enter expressions involving powers, use the power button [□■] found in the [Math1] keyboard.
- Tap [EXE] to obtain the simplified expression.

Note: Tap the stylus under the index to return the cursor to the baseline.
The pronumerals *a* and *b* must be selected from the [Var] keyboard.



Index law 2: Dividing powers

Index law 2

To **divide** two powers with the same base, **subtract** the indices.

$$a^m \div a^n = a^{m-n}$$

If m and n are positive integers with $m > n$, then

$$
\begin{aligned}
a^m \div a^n &= \frac{\overbrace{a \times a \times \cdots \times a}^{m \text{ terms}}}{\underbrace{a \times a \times \cdots \times a}_{n \text{ terms}}} \\
&= \underbrace{a \times a \times \cdots \times a}_{(m-n) \text{ terms}} \qquad \text{(by cancelling)} \\
&= a^{m-n}
\end{aligned}
$$

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Example 2

Simplify each of the following:

a $\dfrac{x^4y^3}{x^2y^2}$ **b** $\dfrac{b^{4x} \times b^{x+1}}{b^{2x}}$ **c** $\dfrac{16a^5b \times 4a^4b^3}{8ab}$

Solution

a $\dfrac{x^4y^3}{x^2y^2} = x^{4-2}y^{3-2}$
$= x^2y$

b $\dfrac{b^{4x} \times b^{x+1}}{b^{2x}} = b^{4x+x+1-2x}$
$= b^{3x+1}$

c $\dfrac{16a^5b \times 4a^4b^3}{8ab} = \dfrac{16 \times 4}{8} \times a^{5+4-1} \times b^{1+3-1}$
$= 8a^8b^3$

Explanation

When dividing powers with the same base, subtract the indices.

In part a, the indices of the base-x powers are subtracted, and the indices of the base-y powers are subtracted.

In parts b and c, both index law 1 and index law 2 are used. In part c, the factors that are numerals are grouped together and simplified.

The zero index and negative integer indices

Define $a^0 = 1$ for $a \neq 0$, and define $a^{-n} = \dfrac{1}{a^n}$ for $a \neq 0$.

Index laws 1 and 2 also hold for negative indices m, n. For example:

$$2^4 \times 2^{-2} = \frac{2^4}{2^2} = \frac{2 \times 2 \times 2 \times 2}{2 \times 2} = 2^2 \qquad (\text{i.e. } 2^{4+(-2)})$$

$$2^{-4} \div 2^2 = \frac{1}{2^4} \times \frac{1}{2^2} = \frac{1}{2^4 \times 2^2} = 2^{-6} \qquad (\text{i.e. } 2^{-4-2})$$

$$2^3 \div 2^3 = 2^3 \times \frac{1}{2^3} = 1 = 2^0 \qquad (\text{i.e. } 2^{3-3})$$

The reciprocal of a fraction such as $\dfrac{2}{3}$ is $\dfrac{3}{2}$. For fractions, the index -1 means 'the reciprocal of'. For example:

$$\left(\frac{2}{3}\right)^{-1} = \frac{3}{2}$$

When raising a fraction to other negative indices, take the reciprocal first. For example:

$$\left(\frac{5}{6}\right)^{-2} = \left(\frac{6}{5}\right)^2 = \frac{36}{25}$$

Example 3

Evaluate each of the following:

a 8^{-2} **b** $\left(\dfrac{1}{2}\right)^{-4}$ **c** $\left(\dfrac{3}{4}\right)^{-3}$

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Solution

a $8^{-2} = \left(\frac{1}{8}\right)^2 = \frac{1}{64}$

b $\left(\frac{1}{2}\right)^{-4} = 2^4 = 16$

c $\left(\frac{3}{4}\right)^{-3} = \left(\frac{4}{3}\right)^3 = \frac{64}{27}$

Index law 3: Raising the power

Consider the following:

$$(2^3)^2 = 2^3 \times 2^3 = 2^{3+3} = 2^6 = 2^{3\times 2}$$

$$(4^3)^4 = 4^3 \times 4^3 \times 4^3 \times 4^3 = 4^{3+3+3+3} = 4^{12} = 4^{3\times 4}$$

$$(a^2)^5 = a^2 \times a^2 \times a^2 \times a^2 \times a^2 = a^{2+2+2+2+2} = a^{10} = a^{2\times 5}$$

In general, $(a^m)^n = a^{m\times n}$.

Index law 3

To raise a power to another power, **multiply** the indices.

$$(a^m)^n = a^{m\times n}$$

This rule holds for all integers m and n.

Example 4

Simplify each of the following:

a $(a^5)^3$

b $\left(\left(\frac{1}{2}\right)^{-3}\right)^2$

c $(b^3)^2 \times (b^2)^{-1}$

Solution

a $(a^5)^3 = a^{15}$

b $\left(\left(\frac{1}{2}\right)^{-3}\right)^2 = \left(\frac{1}{2}\right)^{-6} = 2^6 = 64$

c $(b^3)^2 \times (b^2)^{-1} = b^6 \times b^{-2} = b^4$

Explanation

Index law 3 is used.

For part b, the following calculation is probably preferable:

$$\left(\left(\frac{1}{2}\right)^{-3}\right)^2 = (2^3)^2 = 8^2 = 64$$

In part c, index law 1 is also used.

Index laws 4 and 5: Products and quotients

Index law 4

$$(ab)^n = a^n b^n$$

If n is a positive integer, then

$$(ab)^n = \underbrace{(ab) \times (ab) \times \cdots \times (ab)}_{n \text{ terms}}$$

$$= \underbrace{(a \times a \times \cdots \times a)}_{n \text{ terms}} \times \underbrace{(b \times b \times \cdots \times b)}_{n \text{ terms}}$$

$$= a^n b^n$$

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Index law 5

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

If n is a positive integer, then

$$\begin{aligned}\left(\frac{a}{b}\right)^n &= \frac{a}{b} \times \frac{a}{b} \times \cdots \times \frac{a}{b} \\ &= \frac{a^n}{b^n}\end{aligned}$$

Example 5

Simplify each of the following:

a $(2a^2b^3)^3 \times (3ab^4)^{-2}$ **b** $\left(\frac{2a^3b^2}{abc^2}\right)^3 \div (ab^{-1}c)^3$

Solution

a $$\begin{aligned}(2a^2b^3)^3 \times (3ab^4)^{-2} &= 8a^6b^9 \times 3^{-2}a^{-2}b^{-8} \\ &= \frac{8a^4b}{9}\end{aligned}$$

b $$\begin{aligned}\left(\frac{2a^3b^2}{abc^2}\right)^3 \div (ab^{-1}c)^3 &= \frac{8a^9b^6}{a^3b^3c^6} \times \frac{1}{a^3b^{-3}c^3} \\ &= \frac{8a^3b^6}{c^9}\end{aligned}$$

Explanation

In part a, index law 4 is used, and then laws 3 and 1 and the fact that $3^{-2} = \frac{1}{9}$.

In part b, index law 5 is used. Dividing by a fraction always means multiply by the reciprocal of that fraction.

Working with a negative base

The power $(-a)^n$ can be written as $(-1 \times a)^n = (-1)^n(a)^n$. We note that:

- If n is even, then $(-1)^n = 1$.
- If n is odd, then $(-1)^n = -1$.

Hence, if a is a positive number, then the number $(-a)^n$ is positive when n is even and negative when n is odd.

Example 6

Simplify each of the following:

a $(-3)^4$ **b** $(-5a)^3$ **c** $(-2a)^3 \times 3a^2$

Solution

a $(-3)^4 = 81$

b $(-5a)^3 = -125a^3$

c $$\begin{aligned}(-2a)^3 \times 3a^2 &= -8a^3 \times 3a^2 \\ &= -24a^5\end{aligned}$$

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Using prime decomposition

Bases that are composite numbers are often best factored into primes before further calculations are undertaken.

Example 7

Simplify the following, expressing the answers in positive-index form:

a $12^n \times 18^{-2n}$ **b** $\dfrac{3^{-3} \times 6^4 \times 12^{-3}}{9^{-4} \times 2^{-2}}$ **c** $\dfrac{3^{2n} \times 6^n}{8^n \times 3^n}$

Solution

a
$$\begin{aligned} 12^n \times 18^{-2n} &= (3 \times 2^2)^n \times (3^2 \times 2)^{-2n} \\ &= 3^n \times 2^{2n} \times 3^{-4n} \times 2^{-2n} \\ &= 3^{-3n} \times 2^0 \\ &= \frac{1}{3^{3n}} \end{aligned}$$

b
$$\begin{aligned} \frac{3^{-3} \times 6^4 \times 12^{-3}}{9^{-4} \times 2^{-2}} &= \frac{3^{-3} \times 2^4 \times 3^4 \times 2^{-6} \times 3^{-3}}{3^{-4} \times 3^{-4} \times 2^{-2}} \\ &= \frac{3^{-2} \times 2^{-2}}{3^{-8} \times 2^{-2}} \\ &= 3^6 \end{aligned}$$

c
$$\begin{aligned} \frac{3^{2n} \times 6^n}{8^n \times 3^n} &= \frac{(3^n \times 3^n) \times (3^n \times 2^n)}{2^{3n} \times 3^n} \\ &= \frac{3^n \times 3^n}{2^{2n}} \\ &= \left(\frac{3}{2}\right)^{2n} \end{aligned}$$

Explanation

The prime decomposition of 12 is

$$12 = 2^2 \times 3$$

The prime decomposition of 18 is

$$18 = 2 \times 3^2$$

Each number in this question can be expressed using powers of 2 and 3.

Index law 4 is used in each of the parts. For example:

$$\begin{aligned} 12^{-3} &= (2^2 \times 3)^{-3} \\ &= (2^2)^{-3} \times 3^{-3} \\ &= 2^{-6} \times 3^{-3} \end{aligned}$$

Summary 14A

- The expression a^n is called a **power**, where a is a non-zero number called the **base** and n is a number called the **exponent** or **index**.
- Index laws The following results hold for all non-zero numbers a and b and all integers m and n:

  1 $a^m \times a^n = a^{m+n}$ **2** $a^m \div a^n = a^{m-n}$ **3** $(a^m)^n = a^{mn}$

  4 $(ab)^n = a^n b^n$ **5** $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$
- For every non-zero number a and positive integer n:
  - $a^0 = 1$
  - $a^{-n} = \dfrac{1}{a^n}$ and $\dfrac{1}{a^{-n}} = a^n$
- $0^n = 0$, if n is a positive integer, and 0^0 is undefined.

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Skill-sheet

Exercise 14A

1 For each of the following, use the stated rule to give an equivalent expression in simplest form:

a $x^2 \times x^3$	**b** $2 \times x^3 \times x^4 \times 4$	Index law 1
c $\dfrac{x^5}{x^3}$	**d** $\dfrac{4x^6}{2x^3}$	Index law 2
e $(a^3)^2$	**f** $(2^3)^2$	Index law 3
g $(xy)^2$	**h** $(x^2y^3)^2$	Index law 4 (also use law 3 for h)
i $\left(\dfrac{x}{y}\right)^3$	**j** $\left(\dfrac{x^3}{y^2}\right)^2$	Index law 5 (also use law 3 for j)

Example 1

2 Simplify each of the following:

a $3^5 \times 3^{12}$ **b** $x^3y^2 \times x^4y^3$ **c** $3^{x+1} \times 3^{3x+2}$ **d** $5a^3b^2 \times 6a^2b^4$

Example 2

3 Simplify each of the following:

a $\dfrac{x^5y^2}{x^3y}$ **b** $\dfrac{b^{5x} \times b^{2x+1}}{b^{3x}}$ **c** $\dfrac{8a^2b \times 3a^5b^2}{6a^2b^2}$

Example 3

4 Evaluate each of the following:

a 7^{-2} **b** $\left(\dfrac{1}{4}\right)^{-3}$ **c** $\left(\dfrac{5}{2}\right)^{-3}$

Example 4

5 Simplify each of the following:

a $(b^5)^2$ **b** $\left(\left(\dfrac{1}{3}\right)^{-2}\right)^3$ **c** $(b^5)^2 \times (b^2)^{-3}$

Example 5

6 Simplify each of the following:

a $(3a^4b^3)^3 \times (4a^2b^4)^{-2}$ **b** $\left(\dfrac{5a^3b^3}{ab^2c^2}\right)^3 \div (a^2b^{-1}c)^3$

Example 6

7 Simplify each of the following:

a $(-2)^6$ **b** $(-3a)^3$ **c** $(-2a)^5 \times 3a^{-2}$

Example 7

8 Simplify the following:

a $36^n \times 12^{-2n}$ **b** $\dfrac{2^{-3} \times 8^4 \times 32^{-3}}{4^{-4} \times 2^{-2}}$ **c** $\dfrac{5^{2n} \times 10^n}{8^n \times 5^n}$

9 Simplify the following:

a $x^3 \times x^4 \times x^2$ **b** $2^4 \times 4^3 \times 8^2$

c $3^4 \times 9^2 \times 27^3$ **d** $(q^2p)^3 \times (qp^3)^2$

e $a^2b^{-3} \times (a^3b^2)^3$ **f** $(2x^3)^2 \times (4x^4)^3$

g $m^3p^2 \times (m^2n^3)^4 \times (p^{-2})^2$ **h** $2^3a^3b^2 \times (2a^{-1}b^2)^{-2}$

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10 Simplify the following:

a $\dfrac{x^3y^5}{xy^2}$ **b** $\dfrac{16a^5b \times 4a^4b^3}{8ab}$ **c** $\dfrac{(-2xy)^2 \times 2(x^2y)^3}{8(xy)^3}$ **d** $\dfrac{(-3x^2y^3)^2}{(2xy)^3} \times \dfrac{4x^4y^3}{(xy)^3}$

11 Simplify each of the following, expressing your answer in positive-index form:

a $m^3n^2p^{-2} \times (mn^2p)^{-3}$ **b** $\dfrac{x^3yz^{-2} \times 2(x^3y^{-2}z)^2}{xyz^{-1}}$ **c** $\dfrac{a^2b \times (ab^{-2})^{-3}}{(a^{-2}b^{-1})^{-2}}$

d $\dfrac{a^2b^3c^{-4}}{a^{-1}b^2c^{-3}}$ **e** $\dfrac{a^{2n-1} \times b^3 \times c^{1-n}}{a^{n-3} \times b^{2-n} \times c^{2-2n}}$

12 Simplify each of the following:

a $3^{4n} \times 9^{2n} \times 27^{3n}$ **b** $\dfrac{2^n \times 8^{n+1}}{32^n}$ **c** $\dfrac{3^{n-1} \times 9^{2n-3}}{6^2 \times 3^{n+2}}$

d $\dfrac{2^{2n} \times 9^{2n-1}}{6^{n-1}}$ **e** $\dfrac{25^{2n} \times 5^{n-1}}{5^{2n+1}}$ **f** $\dfrac{6^{x-3} \times 4^x}{3^{x+1}}$

g $\dfrac{6^{2n} \times 9^3}{27^n \times 8^n \times 16^n}$ **h** $\dfrac{3^{n-2} \times 9^{n+1}}{27^{n-1}}$ **i** $\dfrac{8 \times 2^5 \times 3^7}{9 \times 2^7 \times 81}$

13 Simplify and evaluate:

a $\dfrac{(8^3)^4}{(2^{12})^2}$ **b** $\dfrac{125^3}{25^2}$ **c** $\dfrac{81^4 \div 27^3}{9^2}$

14B Rational indices

Let a be a positive real number and let $n \in \mathbb{N}$. Then $a^{\frac{1}{n}}$ is defined to be the nth root of a. That is, $a^{\frac{1}{n}}$ is the positive number whose nth power is a. We can also write this as $a^{\frac{1}{n}} = \sqrt[n]{a}$. For example: $9^{\frac{1}{2}} = 3$, since $3^2 = 9$.

We define $0^{\frac{1}{n}} = 0$, for each natural number n, since $0^n = 0$.

If n is odd, then we can also define $a^{\frac{1}{n}}$ when a is negative. If a is negative and n is odd, define $a^{\frac{1}{n}}$ to be the number whose nth power is a. For example: $(-8)^{\frac{1}{3}} = -2$, as $(-2)^3 = -8$.

In all three cases we can write:

$$a^{\frac{1}{n}} = \sqrt[n]{a} \quad \text{with} \quad \left(a^{\frac{1}{n}}\right)^n = a$$

Using this notation for square roots:

$$\sqrt{a} = \sqrt[2]{a} = a^{\frac{1}{2}}$$

Further, the expression a^x can be defined for rational indices, i.e. when $x = \dfrac{m}{n}$, where m and n are integers, by defining

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$$

To employ this definition we will always first write the fractional power in simplest form.

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Example 8

Evaluate:

a $(-64)^{\frac{1}{3}}$ **b** $9^{-\frac{1}{2}}$ **c** $16^{\frac{5}{2}}$ **d** $64^{-\frac{2}{3}}$

Solution

a $(-64)^{\frac{1}{3}} = -4$

b $9^{-\frac{1}{2}} = \dfrac{1}{9^{\frac{1}{2}}} = \dfrac{1}{\sqrt{9}} = \dfrac{1}{3}$

c $16^{\frac{5}{2}} = \left(16^{\frac{1}{2}}\right)^5 = \left(\sqrt[2]{16}\right)^5 = 4^5 = 1024$

d $64^{-\frac{2}{3}} = \dfrac{1}{64^{\frac{2}{3}}} = \dfrac{1}{\left(64^{\frac{1}{3}}\right)^2} = \dfrac{1}{\left(\sqrt[3]{64}\right)^2} = \dfrac{1}{4^2} = \dfrac{1}{16}$

Note: In the previous section, we stated the index laws for m and n integers:

1 $a^m \times a^n = a^{m+n}$ **2** $a^m \div a^n = a^{m-n}$ **3** $(a^m)^n = a^{m \times n}$

These laws are applicable for all rational indices:

1 $a^{\frac{m}{q}} \times a^{\frac{n}{p}} = a^{\frac{m}{q}+\frac{n}{p}}$ **2** $a^{\frac{m}{q}} \div a^{\frac{n}{p}} = a^{\frac{m}{q}-\frac{n}{p}}$ **3** $\left(a^{\frac{m}{q}}\right)^{\frac{n}{p}} = a^{\frac{m}{q}\times\frac{n}{p}}$

Example 9

Simplify:

a $\dfrac{3^{\frac{1}{4}} \times \sqrt{6} \times \sqrt[4]{2}}{16^{\frac{3}{4}}}$ **b** $(x^{-2}y)^{\frac{1}{2}} \times \left(\dfrac{x}{y^{-3}}\right)^4$

Solution

a

$$\frac{3^{\frac{1}{4}} \times \sqrt{6} \times \sqrt[4]{2}}{16^{\frac{3}{4}}} = \frac{3^{\frac{1}{4}} \times 3^{\frac{1}{2}} \times 2^{\frac{1}{2}} \times 2^{\frac{1}{4}}}{\left(16^{\frac{1}{4}}\right)^3}$$

$$= \frac{3^{\frac{1}{4}} \times 3^{\frac{1}{2}} \times 2^{\frac{1}{2}} \times 2^{\frac{1}{4}}}{2^3}$$

$$= \frac{3^{\frac{3}{4}} \times 2^{\frac{3}{4}}}{2^3}$$

$$= \frac{3^{\frac{3}{4}}}{2^{\frac{12}{4}-\frac{3}{4}}} = \frac{3^{\frac{3}{4}}}{2^{\frac{9}{4}}}$$

b

$$(x^{-2}y)^{\frac{1}{2}} \times \left(\frac{x}{y^{-3}}\right)^4 = x^{-1}y^{\frac{1}{2}} \times \frac{x^4}{y^{-12}}$$

$$= x^3 \times y^{\frac{25}{2}}$$

Explanation

$\sqrt{6} = \sqrt{3} \times \sqrt{2} = 3^{\frac{1}{2}} \times 2^{\frac{1}{2}}$ and $\sqrt[4]{2} = 2^{\frac{1}{4}}$

$2^3 = 2^{\frac{12}{4}}$

$\dfrac{x^4}{y^{-12}} = x^4 \times y^{12}$

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Summary 14B

- Let a be a positive real number and let $n \in \mathbb{N}$. Then $a^{\frac{1}{n}}$ is defined to be the nth root of a. That is, $a^{\frac{1}{n}}$ is the positive number whose nth power is a.
- Define $0^{\frac{1}{n}} = 0$, for each $n \in \mathbb{N}$.
- If n is odd, then we can define $a^{\frac{1}{n}}$ when a is negative. If a is negative and n is odd, define $a^{\frac{1}{n}}$ to be the number whose nth power is a.
- In all three cases we can write:

$$a^{\frac{1}{n}} = \sqrt[n]{a} \quad \text{with} \quad \left(a^{\frac{1}{n}}\right)^n = a$$

- The index laws can be extended to rational indices:

1 $a^{\frac{m}{q}} \times a^{\frac{n}{p}} = a^{\frac{m}{q}+\frac{n}{p}}$ **2** $a^{\frac{m}{q}} \div a^{\frac{n}{p}} = a^{\frac{m}{q}-\frac{n}{p}}$ **3** $\left(a^{\frac{m}{q}}\right)^{\frac{n}{p}} = a^{\frac{m}{q}\times\frac{n}{p}}$

Skill-sheet

Exercise 14B

Example 8

1 Evaluate each of the following:

a $125^{\frac{2}{3}}$ **b** $243^{\frac{3}{5}}$ **c** $81^{-\frac{1}{2}}$ **d** $64^{\frac{2}{3}}$

e $\left(\frac{1}{8}\right)^{\frac{1}{3}}$ **f** $32^{-\frac{2}{5}}$ **g** $125^{-\frac{2}{3}}$ **h** $32^{\frac{4}{5}}$

i $1000^{-\frac{4}{3}}$ **j** $10\,000^{\frac{3}{4}}$ **k** $81^{\frac{3}{4}}$ **l** $\left(\frac{27}{125}\right)^{\frac{1}{3}}$

m $(-8)^{\frac{1}{3}}$ **n** $125^{-\frac{4}{3}}$ **o** $(-32)^{\frac{4}{5}}$ **p** $\left(\frac{1}{49}\right)^{-\frac{3}{2}}$

Example 9

2 Simplify:

a $\sqrt[3]{a^2b} \div \sqrt{ab^3}$ **b** $(a^{-2}b)^3 \times \left(\frac{1}{b^{-3}}\right)^{\frac{1}{2}}$

c $\dfrac{45^{\frac{1}{3}}}{9^{\frac{3}{4}} \times 15^{\frac{3}{2}}}$ **d** $2^{\frac{3}{2}} \times 4^{-\frac{1}{4}} \times 16^{-\frac{3}{4}}$

e $\left(\dfrac{x^3y^{-2}}{3^{-3}y^{-3}}\right)^{-2} \div \left(\dfrac{3^{-3}x^{-2}y}{x^4y^{-2}}\right)^2$ **f** $\left(\sqrt[5]{a^2}\right)^{\frac{3}{2}} \times \left(\sqrt[3]{a^5}\right)^{\frac{1}{5}}$

3 Simplify each of the following:

a $(2x-1)\sqrt{2x-1}$ **b** $(x-1)^2\sqrt{x-1}$

c $(x^2+1)\sqrt{x^2+1}$ **d** $(x-1)^3\sqrt{x-1}$

e $\dfrac{1}{\sqrt{x-1}} + \sqrt{x-1}$ **f** $(5x^2+1)\sqrt[3]{5x^2+1}$

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14C Standard form

Often when dealing with real-world problems, the numbers involved may be very small or very large. For example:

- The distance from Earth to the Sun is approximately 150 000 000 kilometres.
- The mass of an oxygen atom is approximately 0.000 000 000 000 000 000 000 026 grams.

To help deal with such numbers, we can use a more convenient way to express them. This involves expressing the number as a product of a number between 1 and 10 and a power of 10 and is called **standard form** or **scientific notation**.

These examples written in standard form are:

- 1.5×10^8 kilometres
- 2.6×10^{-23} grams

Multiplication and division with very small or very large numbers can often be simplified by first converting the numbers into standard form. When simplifying algebraic expressions or manipulating numbers in standard form, a sound knowledge of the index laws is essential.

Example 10

Write each of the following in standard form:

a 3 453 000 **b** 0.006 75

Solution

a $3\,453\,000 = 3.453 \times 10^6$ **b** $0.006\,75 = 6.75 \times 10^{-3}$

Example 11

Find the value of $\dfrac{32\,000\,000 \times 0.000\,004}{16\,000}$.

Solution

$$\begin{aligned}\frac{32\,000\,000 \times 0.000\,004}{16\,000} &= \frac{3.2 \times 10^7 \times 4 \times 10^{-6}}{1.6 \times 10^4}\\ &= \frac{12.8 \times 10^1}{1.6 \times 10^4}\\ &= 8 \times 10^{-3}\\ &= 0.008\end{aligned}$$

Significant figures

When measurements are made, the result is recorded to a certain number of significant figures. For example, if we say that the length of a piece of ribbon is 156 cm to the nearest centimetre, this means that the length is between 155.5 cm and 156.5 cm. The number 156 is said to be correct to three significant figures. Similarly, we may record π as being 3.1416, correct to five significant figures.

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When rounding off to a given number of significant figures, first identify the last significant digit and then:

- if the next digit is 0, 1, 2, 3 or 4, round down
- if the next digit is 5, 6, 7, 8 or 9, round up.

It can help with rounding off if the original number is first written in scientific notation.

So $\pi = 3.141\,592\,653\ldots$ is rounded off to $3,\ 3.1,\ 3.14,\ 3.142,\ 3.1416,\ 3.14159,$ etc. depending on the number of significant figures required.

Writing a number in scientific notation makes it clear how many significant figures have been recorded. For example, it is unclear whether 600 is recorded to one, two or three significant figures. However, when written in scientific notation as $6.00 \times 10^2,\ 6.0 \times 10^2$ or 6×10^2, it is clear how many significant figures are recorded.

Example 12

Evaluate $\dfrac{\sqrt[5]{a}}{b^2}$ if $a = 1.34 \times 10^{-10}$ and $b = 2.7 \times 10^{-8}$.

Solution

$$\frac{\sqrt[5]{a}}{b^2} = \frac{\sqrt[5]{1.34 \times 10^{-10}}}{(2.7 \times 10^{-8})^2}$$

$$= \frac{(1.34 \times 10^{-10})^{\frac{1}{5}}}{2.7^2 \times (10^{-8})^2}$$

$$= 1.45443\ldots \times 10^{13}$$

$$= 1.45 \times 10^{13} \quad \text{to three significant figures}$$

Many calculators can display numbers in scientific notation. The format will vary from calculator to calculator. For example, the number $3\,245\,000 = 3.245 \times 10^6$ may appear as 3.245E6 or 3.245^{06}.

Using the TI-Nspire

Insert a **Calculator** page, then use [on] > **Settings** > **Document Settings** and change the **Exponential Format** field to **Scientific**. If you want this change to apply only to the current page, select OK to accept the change. Select **Current** to return to the current page.

Using the Casio ClassPad

The ClassPad calculator can be set to express decimal answers in various forms. To select a fixed number of decimal places, including specifying scientific notation with a fixed decimal accuracy, go to **Settings** ⚙ and in **Basic format** tap the arrow to select from the various Number formats available.

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Summary 14C

- A number is said to be in **scientific notation** (or **standard form**) when it is written as a product of a number between 1 and 10 and an integer power of 10.
  For example: $6547 = 6.457 \times 10^3$ and $0.789 = 7.89 \times 10^{-1}$
- Writing a number in scientific notation makes it clear how many **significant figures** have been recorded.
- When rounding off to a given number of significant figures, first identify the last significant digit and then:
  - if the next digit is 0, 1, 2, 3 or 4, round down
  - if the next digit is 5, 6, 7, 8 or 9, round up.

Exercise 14C

Example 10

1 Express each of the following numbers in standard form:

a 47.8 **b** 6728 **c** 79.23 **d** 43 580
e 0.0023 **f** 0.000 000 56 **g** 12.000 34 **h** 50 million
i 23 000 000 000 **j** 0.000 000 0013 **k** 165 thousand **l** 0.000 014 567

2 Express each of the following in scientific notation:

a X-rays have a wavelength of 0.000 000 01 cm.
b The mass of a hydrogen atom is 0.000 000 000 000 000 000 000 001 67 g.
c Visible light has wavelength 0.000 05 cm.
d One nautical mile is 1853.18 m.
e A light year is 9 461 000 000 000 km.
f The speed of light is 29 980 000 000 cm/s.

3 Express each of the following as an ordinary number:

a The star Sirius is approximately 8.128×10^{13} km from Earth.
b A single red blood cell contains 2.7×10^8 molecules of haemoglobin.
c The radius of an electron is 2.8×10^{-13} cm.

4 Write each of the following in scientific notation, correct to the number of significant figures indicated in the brackets:

a 456.89 (4) **b** 34 567.23 (2) **c** 5679.087 (5)
d 0.045 36 (2) **e** 0.090 45 (2) **f** 4568.234 (5)

Example 11

5 Find the value of:

a $\dfrac{324\,000 \times 0.000\,000\,7}{4000}$ **b** $\dfrac{5\,240\,000 \times 0.8}{42\,000\,000}$

Example 12

6 Evaluate the following correct to three significant figures:

a $\dfrac{\sqrt[3]{a}}{b^4}$ if $a = 2 \times 10^9$ and $b = 3.215$ **b** $\dfrac{\sqrt[4]{a}}{4b^4}$ if $a = 2 \times 10^{12}$ and $b = 0.05$

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14D Graphs of exponential functions

Two types of graphs of exponential functions will be examined.

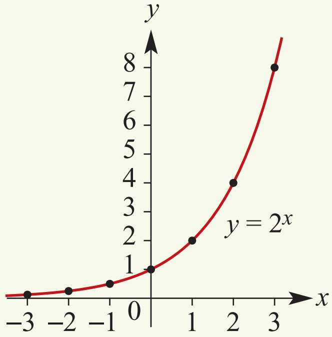
Graph of $y = a^x$ when $a > 1$

Example 13

Plot the graph of $y = 2^x$ and examine the table of values for $-3 \leq x \leq 3$. A calculator can be used.

Solution

x	-3	-2	-1	0	1	2	3
$y = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



We can make the following observations about graphs of the form $y = a^x$ where $a > 1$:

- As the magnitude of the negative x-values becomes larger and larger, the y-values get closer and closer to zero, but never reach zero. That is, the graph of $y = a^x$ approaches the x-axis from above. The x-axis is said to be an **asymptote**.

  We can write: As $x \to -\infty$, $y \to 0^+$.

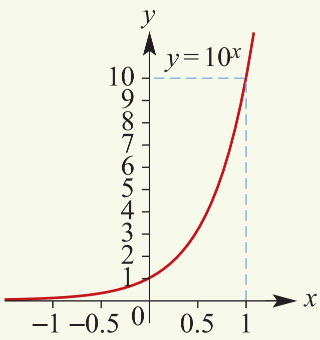
  This is read: As x approaches negative infinity, y approaches 0 from the positive side.
- As the x-values increase, the y-values increase.
- The y-axis intercept is at (0, 1).
- The range of the function is $y \in (0, \infty)$.

Example 14

Plot the graph of $y = 10^x$ and examine the table of values for $-1 \leq x \leq 1$. A calculator can be used to obtain approximate values.

Solution

x	-1	-0.5	0	0.5	1
$y = 10^x$	0.1	≈ 0.316	1	≈ 3.16	10



- The x-axis is an asymptote.
- The y-axis intercept is at (0, 1).
- As the x-values increase, the y-values increase.
- For a given value of x, this rate of increase for $y = 10^x$ is greater than that for $y = 2^x$.

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Note that, for any numbers a and b greater than 1, there is a positive number k with $a^k = b$. This can be seen from the graphs of $y = 2^x$ and $y = 10^x$. Using a calculator to solve $2^k = 10$ gives $k = 3.3219\ldots$. Hence $10^x = (2^{3.3219\ldots})^x$ and therefore the graph of $y = 10^x$ can be obtained from the graph of $y = 2^x$ by a dilation of factor $\frac{1}{k} = \frac{1}{3.3219\ldots}$ parallel to the x-axis.

All graphs of the form $y = a^x$, where $a > 1$, are related to each other by dilations parallel to the x-axis. (This will be discussed again later in the chapter.)

Graph of $y = a^x$ when $0 < a < 1$

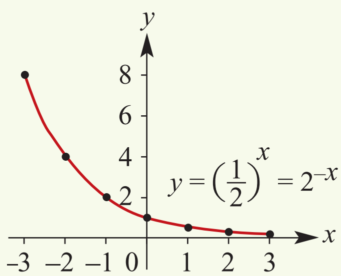
Example 15

Plot the graph of $y = \left(\frac{1}{2}\right)^x$ and examine the table of values for $-3 \le x \le 3$. A calculator can be used.

Solution

Note that $y = \left(\frac{1}{2}\right)^x = 2^{-x}$.

x	-3	-2	-1	0	1	2	3
$y = (\frac{1}{2})^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

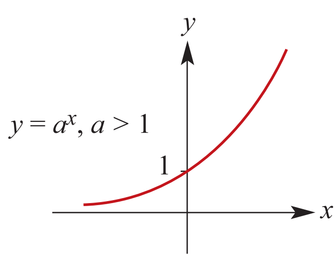


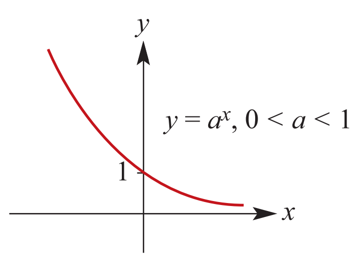
We can make the following observations about graphs of the form $y = a^x$ where $0 < a < 1$:

- The x-axis is an asymptote. As the x-values increase, the graph approaches the x-axis from above. This is written: As $x \to \infty$, $y \to 0^+$.
- The y-axis intercept is at $(0, 1)$.
- The range of the function is $y \in (0, \infty)$.

Graphs of $y = a^x$ in general

In general:





In both cases $a > 1$ and $0 < a < 1$, we can write $y = a^x$ as $y = b^{-x}$, where $b = \frac{1}{a}$. The graph of $y = b^{-x}$ is obtained from the graph of $y = b^x$ by a reflection in the y-axis.

Thus, for example, the graph of $y = \left(\frac{1}{2}\right)^x$ is obtained from the graph of $y = 2^x$ by a reflection in the y-axis, and vice versa. Using function notation: Let $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$. Then

$$g(x) = \left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x} = f(-x)$$

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Example 16

Plot the graph of $y = 2^x$ on a CAS calculator and hence find (correct to three decimal places):

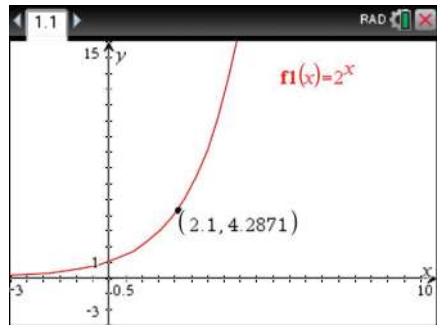
a the value of y when $x = 2.1$ **b** the value of x when $y = 9$.

Using the TI-Nspire

Plot the graph of $y = 2^x$.

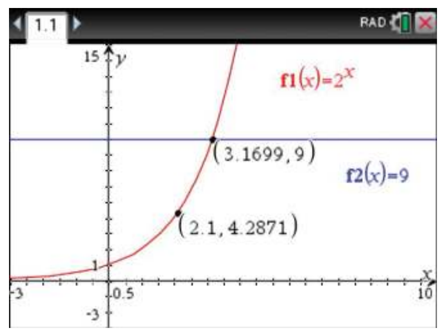
a
- To go to the point with x-coordinate 2.1, use (menu) > **Trace** > **Graph Trace** and type 2.1 (enter).
- Press (enter) to paste the coordinates to the point.
- Press (esc) to exit the **Graph Trace** tool.

When $x = 2.1$, $y = 4.287$ (correct to three decimal places).



b
- To find the value of x for which $y = 9$, plot the graph of $y = 9$ on the same screen and use (menu) > **Geometry** > **Points & Lines** > **Intersection Point(s)**.
- Press (esc) to exit the **Intersection Point(s)** tool.

When $y = 9$, $x = 3.170$ (correct to three decimal places).

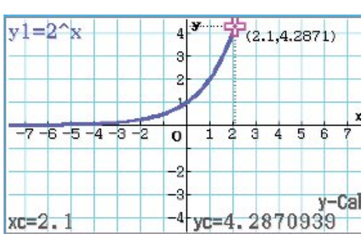


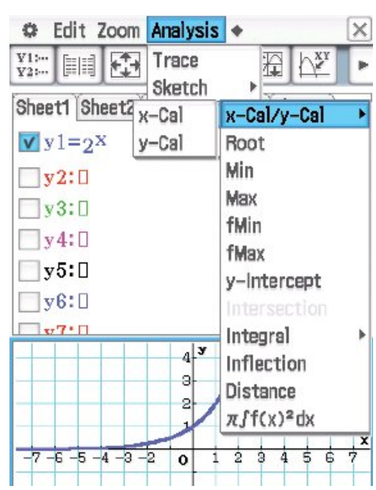
Note: Alternatively, find the intersection point using (menu) > **Analyze Graph** > **Intersection**.

Using the Casio ClassPad

In Graph & Table, enter 2^x in y1. Tick the box and select the graph icon.

a Tap in the graph window, select **Analysis** > **G-Solve** > **x-Cal/y-Cal** > **y-Cal** and enter $x = 2.1$.





When $x = 2.1$, $y = 4.287$ (correct to three decimal places).

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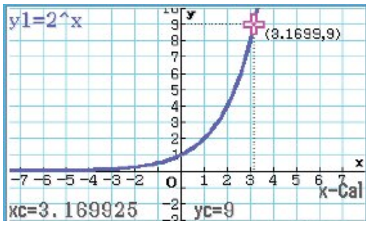
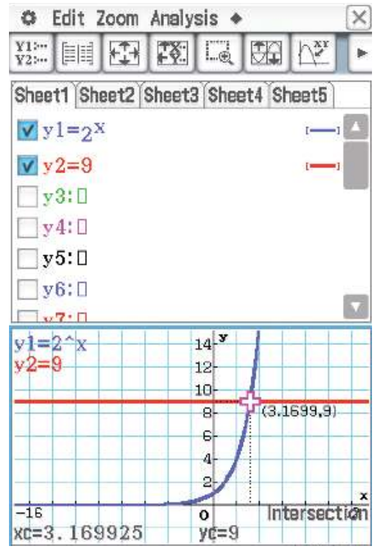
b Select **Analysis** > **G-Solve** > **x-Cal/y-Cal** > **x-Cal** and enter $y = 9$.



When $y = 9$, $x = 3.170$ (correct to three decimal places).

Note: An alternative method for part b is to enter $y2 = 9$ and use **Analysis** > **G-Solve** > **Intersection**, as shown on the right.



Transformations of exponential graphs

The techniques for transformations that were introduced in earlier chapters are now applied to the graphs of exponential functions.

Example 17

Sketch the graphs of each of the following pairs of functions. For the second function in each pair, state the equation of the asymptote, the y-axis intercept and the range. (The x-axis intercepts need not be given.)

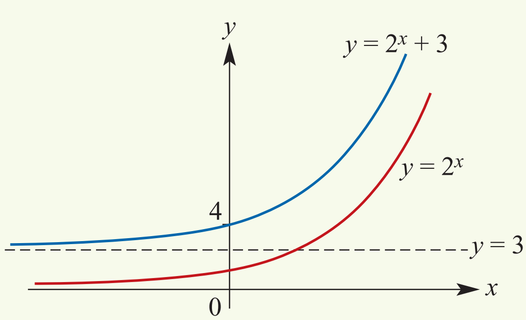
a $f(x) = 2^x$ and $g(x) = 2^x + 3$

b $f(x) = 3^x$ and $g(x) = 2 \times 3^x + 1$

c $f(x) = 3^x$ and $g(x) = -3^x + 2$

Solution

a



Graph of $g(x) = 2^x + 3$:

- The asymptote has equation $y = 3$.
- The y-axis intercept is $g(0) = 2^0 + 3 = 4$.
- The range of the function g is $(3, \infty)$.

Explanation

The graph of $y = 2^x + 3$ is obtained by transforming the graph of $y = 2^x$ by a translation of 3 units in the positive direction of the y-axis.

The asymptote of $y = 2^x$ is the line with equation $y = 0$, which is transformed to the line with equation $y = 3$.

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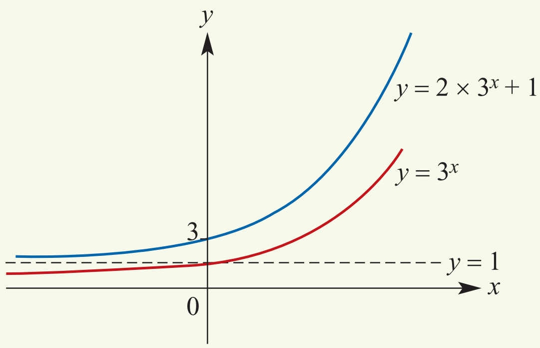
b



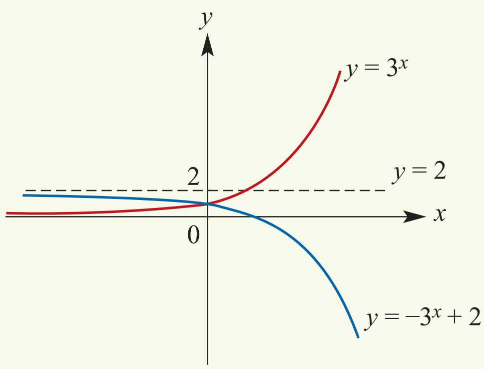
Graph of $g(x) = 2 \times 3^x + 1$:

- The asymptote has equation $y = 1$.
- The y-axis intercept is $g(0) = 2 \times 3^0 + 1 = 3$.
- The range of the function g is $(1, \infty)$.

The graph of $y = 2 \times 3^x + 1$ is obtained by transforming the graph of $y = 3^x$ by a dilation of factor 2 parallel to the y-axis, followed by a translation of 1 unit in the positive direction of the y-axis.

The asymptote of $y = 3^x$ is the line $y = 0$, which is transformed to the line $y = 1$.

c



Graph of $g(x) = -3^x + 2$:

- The asymptote has equation $y = 2$.
- The y-axis intercept is $g(0) = -3^0 + 2 = 1$.
- The range of the function g is $(-\infty, 2)$.

The graph of $y = -3^x + 2$ is obtained by transforming the graph of $y = 3^x$ by a reflection in the x-axis, followed by a translation of 2 units in the positive direction of the y-axis.

The asymptote of $y = 3^x$ is the line $y = 0$, which is transformed to the line $y = 2$.

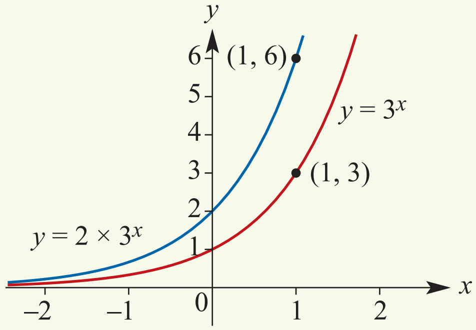
Example 18

Sketch the graph of each of the following:

a $y = 2 \times 3^x$ **b** $y = 3^{2x}$ **c** $y = -3^{2x} + 4$

Solution

a



Explanation

The graph of $y = 2 \times 3^x$ is obtained from the graph of $y = 3^x$ by a dilation of factor 2 parallel to the y-axis.

Both graphs have a horizontal asymptote with equation $y = 0$.

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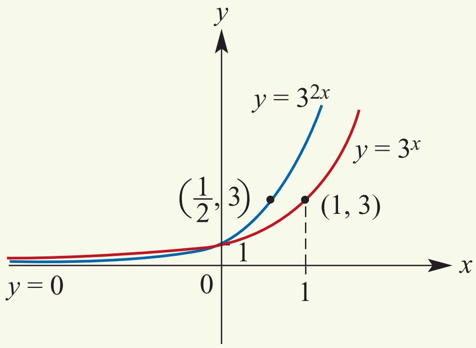
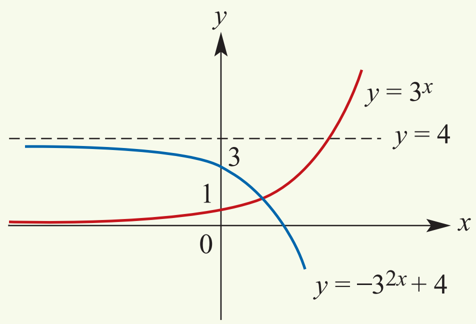
b



The graph of $y = 3^{2x}$ is obtained from the graph of $y = 3^x$ by a dilation of factor $\frac{1}{2}$ parallel to the x-axis. (See the note below.)

Both graphs have a horizontal asymptote with equation $y = 0$.

c



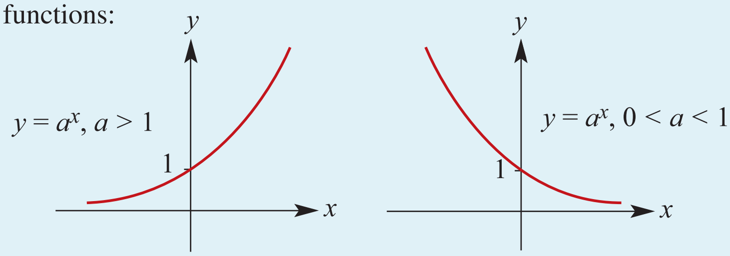
The graph of $y = -3^{2x} + 4$ is obtained from the graph of $y = 3^x$ by a dilation of factor $\frac{1}{2}$ parallel to the x-axis, followed by a reflection in the x-axis and then a translation of 4 units in the positive direction of the y-axis.

The graph of $y = -3^{2x} + 4$ has a horizontal asymptote with equation $y = 4$.

Note: In the notation introduced in Chapter 6, write the transformation for part b as $(x, y) \to \left(\frac{1}{2}x, y\right)$. Then describe the transformation as $x' = \frac{1}{2}x$ and $y' = y$, and hence $x = 2x'$ and $y = y'$. The graph of $y = 3^x$ is mapped to the graph of $y' = 3^{2x'}$.

Summary 14D

- Graphs of exponential functions:



- In both cases $a > 1$ and $0 < a < 1$, the graph of $y = a^x$ has the following properties:
  - The x-axis is an asymptote.
  - The y-axis intercept is 1.
  - The y-values are always positive.
  - There is no x-axis intercept.
- All graphs of the form $y = a^x$, where $a > 1$, are related to each other by dilations parallel to the x-axis. Similarly, all graphs of the form $y = a^x$, where $0 < a < 1$, are related to each other by dilations parallel to the x-axis.
- Let $a > 1$. If $f(x) = a^x$ and $g(x) = \left(\frac{1}{a}\right)^x$, then $g(x) = f(-x)$ and so the graph of $y = g(x)$ is the reflection in the y-axis of the graph of $y = f(x)$.

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Exercise 14D

1 Using a calculator, plot the graphs of the following and comment on the similarities and differences between them:

a $y = 1.8^x$ **b** $y = 2.4^x$ **c** $y = 0.9^x$ **d** $y = 0.5^x$

2 Using a calculator, plot the graphs of the following and comment on the similarities and differences between them:

a $y = 2 \times 3^x$ **b** $y = 5 \times 3^x$ **c** $y = -2 \times 3^x$ **d** $y = -5 \times 3^x$

Example 16

3 Plot the graph of $y = 2^x$ on a CAS calculator and hence find the solution of the equation $2^x = 14$.

4 Plot the graph of $y = 10^x$ on a CAS calculator and hence find the solution of the equation $10^x = 6$.

Example 17
Example 18

5 Sketch the graphs of the following functions. Give equations of asymptotes and y-axis intercepts, and state the range of each function. (The x-axis intercepts need not be given.)

a $f(x) = 3 \times 2^x + 2$ **b** $f(x) = 3 \times 2^x - 3$

c $f(x) = -3^x - 2$ **d** $f(x) = -2 \times 3^x + 2$

e $f(x) = \left(\frac{1}{2}\right)^x + 2$ **f** $f(x) = -2 \times 3^x - 2$

6 Sketch the graph of each of the following:

a $y = 2 \times 5^x$ **b** $y = 3^{3x}$ **c** $y = 5^{\frac{x}{2}}$ **d** $y = -3^{2x} + 2$

14E Solving exponential equations

Solution of equations

One method without using a calculator is to express both sides of the equation as powers with the same base and then equate the indices (since $a^x = a^y$ implies $x = y$, for any $a \in (0, 1) \cup (1, \infty)$).

Example 19

Find the value of x for which:

a $4^x = 256$ **b** $3^{x-1} = 81$ **c** $5^{2x-4} = 25^{-x+2}$

Solution

a
$$4^x = 256$$
$$4^x = 4^4$$
$$\therefore\ x = 4$$

b
$$3^{x-1} = 81$$
$$3^{x-1} = 3^4$$
$$\therefore\ x - 1 = 4$$
$$x = 5$$

c
$$5^{2x-4} = 25^{-x+2}$$
$$= (5^2)^{-x+2}$$
$$= 5^{-2x+4}$$
$$\therefore\ 2x - 4 = -2x + 4$$
$$4x = 8$$
$$x = 2$$

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Sometimes solving an exponential equation involves solving a polynomial equation first. In the following example, the solution of a quadratic equation is necessary.

Example 20

Solve:

a $9^x = 12 \times 3^x - 27$

b $3^{2x} = 27 - 6 \times 3^x$

Solution

a We have $(3^x)^2 = 12 \times 3^x - 27$.

Let $a = 3^x$. The equation becomes

$$a^2 = 12a - 27$$
$$a^2 - 12a + 27 = 0$$
$$(a - 3)(a - 9) = 0$$

Therefore

$$a - 3 = 0 \quad \text{or} \quad a - 9 = 0$$
$$a = 3 \quad \text{or} \quad a = 9$$

Hence $3^x = 3^1$ or $3^x = 3^2$

and so $x = 1$ or $x = 2$

b We have $(3^x)^2 = 27 - 6 \times 3^x$.

Let $a = 3^x$. The equation becomes

$$a^2 = 27 - 6a$$
$$a^2 + 6a - 27 = 0$$
$$(a + 9)(a - 3) = 0$$

Therefore

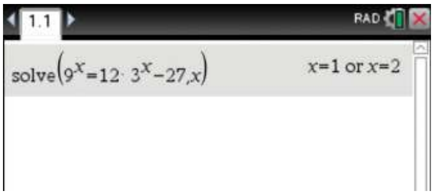
$$a = -9 \quad \text{or} \quad a = 3$$

Hence $3^x = -9$ or $3^x = 3^1$

There is only one solution, $x = 1$, since $3^x > 0$ for all values of x.

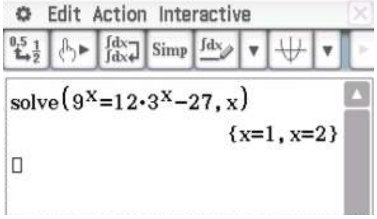
Using the TI-Nspire

Use (menu) > **Algebra** > **Solve** to solve the equation.



Using the Casio ClassPad

- Go to the $\sqrt[\text{Main}]{\alpha}$ screen and turn on the keyboard.
- Enter the equation $9^x = 12 \times 3^x - 27$.
- Highlight the equation using the stylus and select **Interactive** > **Equation/Inequality** > **solve**.
- Tap on OK to obtain the solution. (Note that the default variable is x.)



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The calculator can be used to obtain approximate answers as shown in the following example. For the equation $5^x = 10$ we can find an exact solution, but logarithms are involved in the final answer. Logarithms are discussed in the following section.

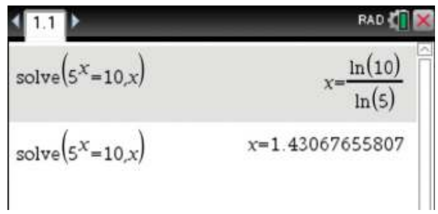
Example 21

Solve $5^x = 10$ correct to two decimal places.

Using the TI-Nspire

Press ctrl enter to obtain the answer as a decimal number.

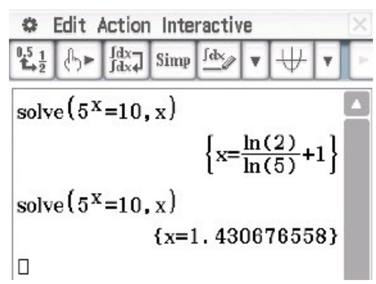
The solution is $x = 1.43$ (correct to two decimal places).



Using the Casio ClassPad

To answer the question as required, you may need to highlight the answer and tap $\frac{0.5}{\frac{1}{2}}$ to convert from the exact solution to a decimal approximation.

The solution is $x = 1.43$ (correct to two decimal places).



Summary 14E

- One method for solving an exponential equation, without using a calculator, is first to express both sides of the equation as powers with the same base and then to equate the indices (since $a^x = a^y$ implies $x = y$, for any base $a \in (0, 1) \cup (1, \infty)$).
  For example: $2^{x+1} = 8 \Leftrightarrow 2^{x+1} = 2^3 \Leftrightarrow x + 1 = 3 \Leftrightarrow x = 2$

Skill-sheet

Exercise 14E

Example 19

1 Solve for x in each of the following:

a $3^x = 27$ **b** $4^x = 64$ **c** $49^x = 7$

d $16^x = 8$ **e** $125^x = 5$ **f** $5^x = 625$

g $16^x = 256$ **h** $4^{-x} = \frac{1}{64}$ **i** $5^{-x} = \frac{1}{125}$

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2 Solve for n in each of the following:

a $5^n \times 25^{2n-1} = 125$ **b** $3^{2n-4} = 1$ **c** $3^{2n-1} = \frac{1}{81}$

d $\frac{3^{n-2}}{9^{1-n}} = 1$ **e** $3^{3n} \times 9^{-2n+1} = 27$ **f** $2^{-3n} \times 4^{2n-2} = 16$

g $2^{n-6} = 8^{2-n}$ **h** $9^{3n+3} = 27^{n-2}$ **i** $4^{n+1} = 8^{n-2}$

j $32^{2n+1} = 8^{4n-1}$ **k** $25^{n+1} = 5 \times 390\,625$ **l** $125^{4-n} = 5^{6-2n}$

m $4^{2-n} = \frac{1}{2048}$

3 Solve the following exponential equations:

a $2^{x-1} \times 4^{2x+1} = 32$ **b** $3^{2x-1} \times 9^x = 243$ **c** $(27 \times 3^x)^2 = 27^x \times 3^{\frac{1}{2}}$

Example 20

4 Solve for x:

a $4(2^{2x}) = 8(2^x) - 4$ **b** $8(2^{2x}) - 10(2^x) + 2 = 0$

c $3 \times 2^{2x} - 18(2^x) + 24 = 0$ **d** $9^x - 4(3^x) + 3 = 0$

Example 21

5 Use a calculator to solve each of the following, correct to two decimal places:

a $2^x = 5$ **b** $4^x = 6$ **c** $10^x = 18$ **d** $10^x = 56$

14F Exponential models and applications

In this section we begin by looking at several situations that can be modelled by exponential functions, and then discuss some examples where data is fitted by an exponential function.

Exponential growth and decay

In the following, we consider a variable A that is subject to exponential change. Let A be the quantity at time t. Then $A = A_0 b^t$, where A_0 is the initial quantity and b is a positive constant.

If $b > 1$, the model represents **growth**:

- growth of cells
- population growth
- continuously compounded interest

If $b < 1$, the model represents **decay**:

- radioactive decay
- cooling of materials

Cell growth

Suppose a particular type of bacteria cell divides into two new cells every T_D minutes. Let N_0 be the initial number of cells of this type. Then after t minutes the number of cells, N, is given by the formula

$$N = N_0 2^{\frac{t}{T_D}}$$

where T_D is called the **generation time**. Here we are only dealing with the type of reproduction where the cell divides in two. For most known bacteria that can be cultured, generation times range from about 15 minutes to 1 hour.

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Example 22

What is the generation time of a bacterial population that increases from 5000 cells to 100 000 cells in four hours of growth?

Solution

In this example, $N_0 = 5000$ and $N = 100\,000$ when $t = 240$.

Hence $100\,000 = 5000 \times 2^{\frac{240}{T_D}}$

$$20 = 2^{\frac{240}{T_D}}$$

Using a CAS calculator, $T_D \approx 55.53$ (correct to two decimal places).

The generation time is approximately 55.53 minutes.

Radioactive decay

Radioactive materials decay so that the amount of radioactive material, A, present at time t (in years) is given by the formula

$$A = A_0 2^{-kt}$$

where A_0 is the initial amount and k is a positive constant that depends on the type of material. A radioactive substance is often described in terms of its **half-life**, which is the time required for half the material to decay.

Example 23

After 1000 years, a sample of radium-226 has decayed to 64.7% of its original mass. Find the half-life of radium-226.

Solution

We use the formula $A = A_0 2^{-kt}$. When $t = 1000$, $A = 0.647A_0$. Thus

$$0.647A_0 = A_0 2^{-1000k}$$

$$0.647 = 2^{-1000k}$$

Using a CAS calculator to solve for k:

$$k \approx 0.000\,628$$

To find the half-life, we consider when $A = \frac{1}{2}A_0$:

$$A_0 2^{-kt} = \frac{1}{2}A_0$$

$$2^{-kt} = \frac{1}{2}$$

$$2^{-kt} = 2^{-1}$$

$$-kt = -1$$

$$t = \frac{1}{k} \approx 1591.95$$

The half-life of radium-226 is approximately 1592 years.

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Population growth

It is sometimes possible to model population growth through exponential models.

Example 24

There are approximately ten times as many red kangaroos as grey kangaroos in a certain area. If the population of grey kangaroos increases at a rate of 11% per annum while that of the red kangaroos decreases at 5% per annum, find how many years must elapse before the proportions are reversed, assuming the same rates continue to apply.

Solution

Let G_0 be the population of grey kangaroos at the start.

Then the number of grey kangaroos after n years is $G = G_0(1.11)^n$, and the number of red kangaroos after n years is $R = 10G_0(0.95)^n$.

When the proportions are reversed:

$$\begin{aligned} G &= 10R \\ G_0(1.11)^n &= 10 \times 10G_0(0.95)^n \\ (1.11)^n &= 100(0.95)^n \\ \frac{(1.11)^n}{(0.95)^n} &= 100 \\ \left(\frac{1.11}{0.95}\right)^n &= 100 \end{aligned}$$

Using a CAS calculator to solve for n:

$$n \approx 29.6$$

i.e. the proportions of the kangaroo populations will be reversed after 30 years.

Determining exponential rules

We have looked at determining rules for functions in Chapters 2 to 7. We look at one very useful case for exponential functions.

Example 25

The points $(1, 6)$ and $(5, 96)$ are known to lie on the curve $y = a \times b^x$, where $a > 0$ and $b > 0$. Find the values of a and b.

Solution

We can write

$$a \times b^1 = 6 \qquad (1)$$

$$a \times b^5 = 96 \qquad (2)$$

Dividing equation (2) by equation (1) gives $b^4 = 16$. Thus $b = 16^{\frac{1}{4}} = 2$, and substituting into equation (1) gives $a = 3$.

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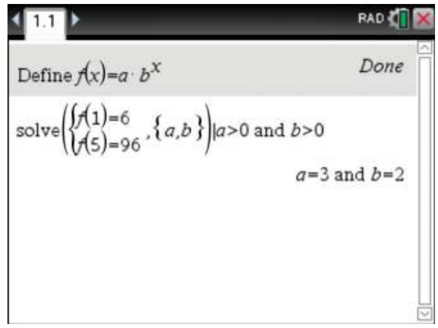
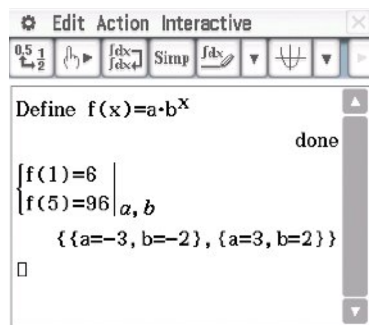
Using the TI-Nspire

- Define $f(x) = a \times b^x$.
- Solve for $a > 0$ and $b > 0$ using the simultaneous equations template with $f(1) = 6$ and $f(5) = 96$ as shown.



Using the Casio ClassPad

- Define $f(x) = a \times b^x$.
- Solve the simultaneous equations $f(1) = 6$ and $f(5) = 96$ for $a > 0$ and $b > 0$ as shown.



In many practical situations, the relationship between variables is exponential.

Example 26

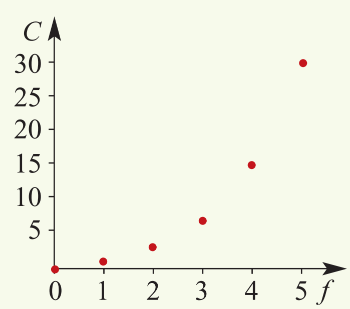
Take a rectangular piece of paper approximately 30 cm × 6 cm. Fold the paper in half, successively, until you have folded it five times. Tabulate the times folded, f, and the number of creases in the paper, C.

Solution

Times folded, f	0	1	2	3	4	5
Creases, C	0	1	3	7	15	31

The rule connecting C and f is

$$C = 2^f - 1, \quad f \in \mathbb{N} \cup \{0\}$$



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Fitting data

Example 27

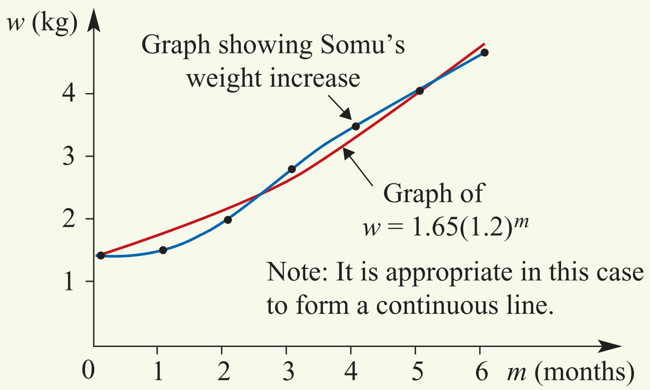
The table below shows the increase in weight of Somu, an orang-utan born at the Eastern Plains Zoo. Draw a graph to show Somu's weight increase for the first six months.

Data values

Months, m	0	1	2	3	4	5	6
Weight (kg), w	1.65	1.7	2.2	3.0	3.7	4.2	4.8

Solution

Plotting these data values:



On the same set of axes is the graph of the exponential function $w = 1.65(1.2)^m$, $0 \le m \le 6$. A table of values for this function is shown below.

Values from model

m	0	1	2	3	4	5	6
w	1.65	1.98	2.38	2.85	3.42	4.1	4.93

It can be seen from the graphs in Example 27 that the exponential model $w = 1.65(1.2)^m$ approximates the actual weight gain and would be a useful model to predict weight gains for any future orang-utan births at the zoo. This model describes a growth rate of 20% per month for the first 6 months.

This problem can also be attempted with a CAS calculator.

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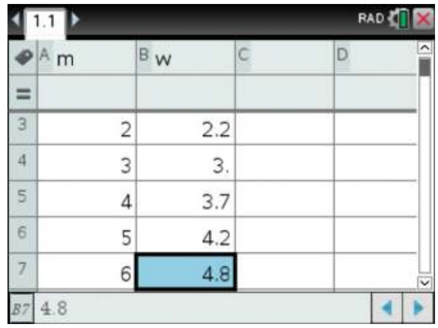
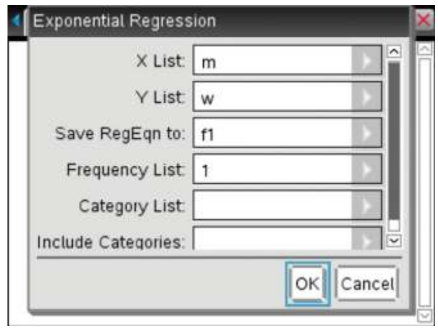
Using the TI-Nspire

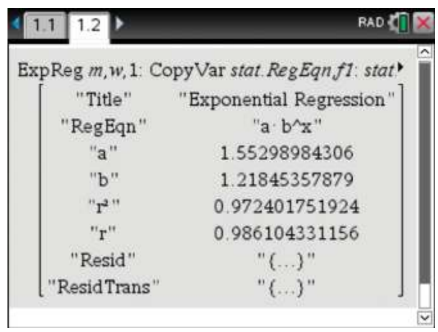
- Enter the data either in a **Calculator** application as lists or in a **Lists & Spreadsheet** application as shown.



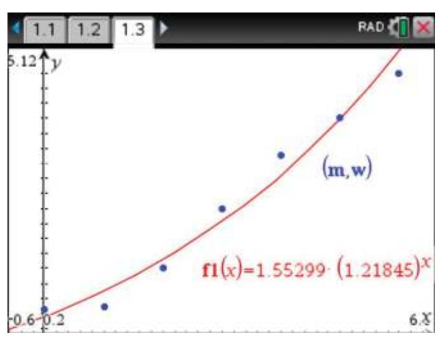
- Insert a **Calculator** page and use (menu) > **Statistics** > **Stat Calculations** > **Exponential Regression**. Complete as shown:
  - Use (tab) to move between fields.
  - Use the selection tool to open a field. Then use the arrows ▲ and ▼ to choose the correct entry, and select this entry using the selection tool.



- This now gives the values of a and b, and the equation has been entered in $f_1(x)$.



- The curve can be shown in a **Graphs** application together with the scatter plot ((menu) > **Graph Type** > **Scatter Plot**) using an appropriate window ((menu) > **Window/Zoom**).



Note: Alternatively, the scatter plot and regression curve can be obtained using the **Data & Statistics** application.

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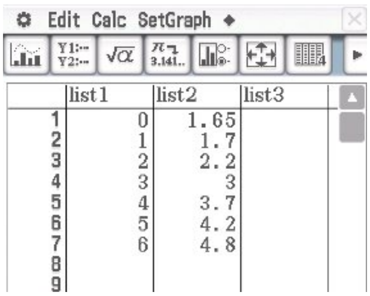
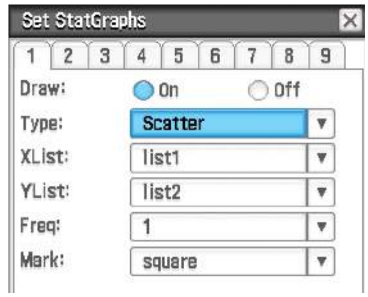
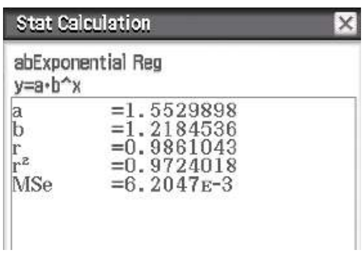
Using the Casio ClassPad

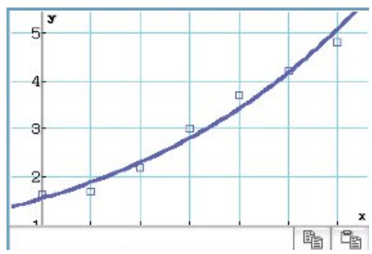
- In **Statistics**, enter the data in lists 1 and 2 as shown below on the left.
- Select the graph icon and ensure that you set the graph type to Scatter and the lists to list 1 and list 2.





- Now select **Calc** > **Regression** > **abExponential Reg**. Confirm the settings by selecting OK.





Summary 14F

There are many situations in which a varying quantity can be modelled by an exponential function. Let A be the quantity at time t. Then $A = A_0 b^t$, where A_0 is the initial quantity and b is a positive constant.

If $b > 1$, the model represents **growth**:

- growth of cells
- population growth
- continuously compounded interest

If $b < 1$, the model represents **decay**:

- radioactive decay
- cooling of materials

Skill-sheet

Exercise 14F

Example 22

1 A population of 1000 E. coli bacteria doubles every 15 minutes.

a Determine the formula for the number of bacteria at time t minutes.

b How long will it take for the population to reach 10 000? (Give your answer to the nearest minute.)

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Example 23

2 The half-life of plutonium-239 is 24 000 years. If 10 grams are present now, how long will it take until only 10% of the original sample remains? (Give your answer to the nearest year.)

3 Carbon-14 is a radioactive substance with a half-life of 5730 years. It is used to determine the age of ancient objects. A Babylonian cloth fragment now has 40% of the carbon-14 that it contained originally. How old is the fragment of cloth?

4 The pressure, P, in the Earth's atmosphere decreases exponentially as you rise above the surface. The pressure in millibars at a height of h kilometres is given approximately by the function $P(h) = 1000 \times 10^{-0.05428h}$.

a Find the pressure at a height of 5 km. (Give your answer to the nearest millibar.)

b Find the height at which the pressure is 400 millibars. (Give your answer to the nearest metre.)

5 A biological culture contains 500 000 bacteria at 12 p.m. on Sunday. The culture increases by 10% every hour. At what time will the culture exceed 4 million bacteria?

6 When a liquid is placed into a refrigerator, its temperature T°C at time t minutes is given by the formula $T = T_0 10^{-kt}$. The temperature is initially 100°C and drops to 40°C in 5 minutes. Find the temperature of the liquid after 15 minutes.

7 Iodine-131 is a radioactive isotope used in the treatment of the thyroid gland. It decays so that, after t days, 1 unit of the isotope is reduced to 0.9174^t units. How many days does it take for the amount to fall to less than 0.2 units?

Example 24

8 The populations (in millions), p and q, of two neighbouring American states, P and Q, over a period of 50 years from 1950 are modelled by functions $p = 1.2 \times 2^{0.08t}$ and $q = 1.7 \times 2^{0.04t}$, where t is the number of years since 1950.

a Plot the graphs of both functions using a calculator.

b Find when the population of state P is:

i equal to the population of state Q

ii twice the population of state Q.

Example 25

9 Each of the following pairs of points is known to lie on a curve $y = a \times b^x$, where $a > 0$ and $b > 0$. Find the values of a and b in each case.

a (1, 15) and (4, 1875) **b** (2, 1) and $\left(5, \frac{1}{8}\right)$ **c** $\left(1, \frac{15}{2}\right)$ and $\left(\frac{1}{2}, \frac{5\sqrt{6}}{2}\right)$

10 Five kilograms of sugar is gradually dissolved in a vat of water. After t hours, the amount S kg of undissolved sugar remaining is given by $S = 5 \times 10^{-kt}$.

a Calculate k given that $S = 3.2$ when $t = 2$.

b At what time will there be 1 kg of sugar remaining?

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11 The number of bacteria, N, in a culture increases exponentially with time according to the rule $N = a \times b^t$, where time t is measured in hours. When observation started, there were 1000 bacteria, and five hours later there were 10 000 bacteria.

a Find the values of a and b.

b Find, to the nearest minute, when there were 5000 bacteria.

c Find, to the nearest minute, when the number of bacteria first exceeds 1 000 000.

d How many bacteria would there be 12 hours after the first observation?

12 Find a and k such that the graph of $y = a10^{kx}$ passes through the points $(2, 6)$ and $(5, 20)$.

Example 27 **13** Find an exponential model of the form $y = ab^x$ to fit the following data:

x	0	2	4	5	10
y	1.5	0.5	0.17	0.09	0.006

14 Find an exponential model of the form $p = ab^t$ to fit the following data:

t	0	2	4	6	8
p	2.5	4.56	8.3	15.12	27.56

15 A sheet of paper 0.2 mm thick is cut in half, and one piece is stacked on top of the other.

a If this process is repeated, complete the following table:

Cuts, n	Sheets	Total thickness, T (mm)
0	1	0.2
1	2	0.4
2	4	0.8
3	8	
⋮	⋮	⋮
10		

b Write down a formula which shows the relationship between T and n.

c Draw a graph of T against n for $n \leq 10$.

d What would be the total thickness, T, after 30 cuts?

16 In the initial period of its life a particular species of tree grows in the manner described by the rule $d = d_0 10^{mt}$, where d is the diameter (in cm) of the tree t years after the beginning of this period. The diameter is 52 cm after 1 year, and 80 cm after 3 years. Calculate the values of the constants d_0 and m.

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Chapter summary

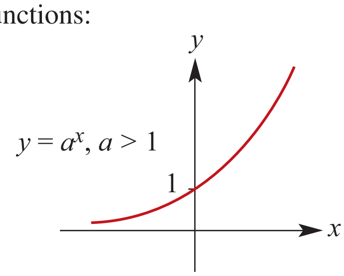
Assignment

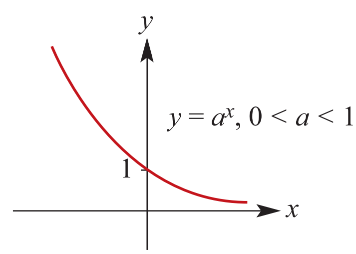
Nrich

- Index laws
  - To **multiply** two powers with the same base, **add** the indices: $a^m \times a^n = a^{m+n}$
  - To **divide** two powers with the same base, **subtract** the indices: $a^m \div a^n = a^{m-n}$
  - To raise a power to another power, **multiply** the indices: $(a^m)^n = a^{m \times n}$
- Rational indices:

$$a^{\frac{1}{n}} = \sqrt[n]{a} \quad \text{and} \quad a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$$

- A number is expressed in **standard form** or **scientific notation** when written as a product of a number between 1 and 10 and an integer power of 10; e.g. 1.5×10^8
- Graphs of exponential functions:





- For $a \in (0, 1) \cup (1, \infty)$:

$$\text{if } a^x = a^y, \text{ then } x = y$$

Short-answer questions

1 Simplify each of the following, expressing your answer with positive index:

a $\dfrac{a^6}{a^2}$ **b** $\dfrac{b^8}{b^{10}}$ **c** $\dfrac{m^3n^4}{m^5n^6}$ **d** $\dfrac{a^3b^2}{(ab^2)^4}$

e $\dfrac{6a^8}{4a^2}$ **f** $\dfrac{10a^7}{6a^9}$ **g** $\dfrac{8(a^3)^2}{(2a)^3}$ **h** $\dfrac{m^{-1}n^2}{(mn^{-2})^3}$

i $(p^{-1}q^{-2})^2$ **j** $\dfrac{(2a^{-4})^3}{5a^{-1}}$ **k** $\dfrac{6a^{-1}}{3a^{-2}}$ **l** $\dfrac{a^4 + a^8}{a^2}$

2 Express the product $32 \times 10^{11} \times 12 \times 10^{-5}$ in standard form.

3 The human body can produce 2.5 million red blood cells per second. If a person donates 500 mL of blood, how long will it take to replace the red blood cells if a litre of blood contains 5×10^{12} red blood cells?

4 The Sun is approximately 1.5×10^8 km from Earth and a comet is approximately 3×10^6 km from Earth. How many times further from Earth than the comet is the Sun?

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5 Solve each of the following for x:

a $3^x(3^x - 27) = 0$ **b** $(2^x - 8)(2^x - 1) = 0$

c $2^{2x} - 2^{x+1} = 0$ **d** $2^{2x} - 12 \times 2^x + 32 = 0$

6 Sketch the graph of:

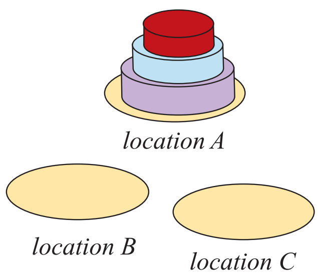
a $y = 2 \times 2^x$ **b** $y = -3 \times 2^x$ **c** $y = 5 \times 2^{-x}$

d $y = 2^{-x} + 1$ **e** $y = 2^x - 1$ **f** $y = 2^x + 2$

7 Solve $2 \times 4^{a+1} = 16^{2a}$ for a.

Extended-response questions

1 This problem is based on the so-called 'Tower of Hanoi' puzzle. Given a number of different sized discs, the problem is to move a pile of discs to a second location (if starting at A, then either to B or C) according to the following rules:



- Only one disc can be moved at a time.
- A total of only three locations (A, B and C) can be used to 'rest' the discs.
- A larger sized disc cannot be placed on top of a smaller disc.
- The task must be completed in the smallest possible number of moves.

a First complete the puzzle using two coins. Repeat with three coins and then with four coins, and thus complete the following table:

Number of discs, n	1	2	3	4
Minimum number of moves, M	1			

b Work out the formula which shows the relationship between M and n. Use your formula to extend the table of values for $n = 5, 6, 7$.

c Plot the graph of M against n.

d Investigate, for both $n = 3$ and $n = 4$, to find whether there is a pattern for the number of times each particular disc is moved.

2 To control an advanced electronic machine, a total of 2187 different switch positions are required. There are two kinds of switches available:

Switch 1 These can be set in 9 different positions.

Switch 2 These can be set in 3 different positions.

If n of switch type 1 and $n + 1$ of switch type 2 are used, calculate the value of n that gives the required total number of switch positions.

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3 The gravitational force between two objects, F N, is given by the formula

$$F = \frac{6.67 \times 10^{-11} m_1 m_2}{r^2}$$

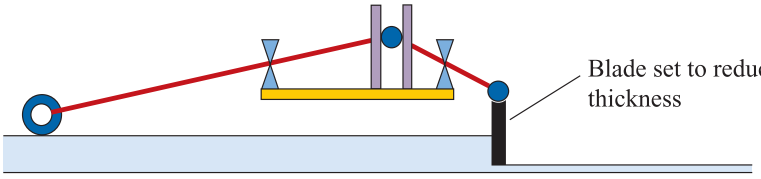
where m_1 and m_2 are the masses (in kilograms) of the two objects and r is the distance (in metres) between them.

a What is the gravitational force between two objects each weighing 200 kg if they are 12 m apart? Express the answer in standard form (to two significant figures).

b Transpose the above formula to make m_1 the subject.

c The gravitational force between a planet and an object 6.4×10^6 m away from the centre of the planet is found to be 2.4×10^4 N. If the object has a mass of 1500 kg, calculate the approximate mass of the planet, giving the answer in standard form (to two significant figures).

4 Research is being carried out to investigate the durability of paints of different thicknesses. The automatic machine shown in the diagram is proposed for producing a coat of paint of a particular thickness.



The paint is spread over a plate and a blade sweeps over the plate reducing the thickness of the paint. The process involves the blade moving at three different speeds.

a Operating at the initial setting, the blade reduces the paint thickness to one-eighth of the original thickness. This happens n times. What fraction of the paint thickness remains? Express this as a power of $\frac{1}{2}$.

b The blade is then reset so that it removes three-quarters of the remaining paint. This happens $(n - 1)$ times. At the end of this second stage, express the remaining thickness as a power of $\frac{1}{2}$.

c The third phase of the process involves the blade being reset to remove half of the remaining paint. This happens $(n - 3)$ times. At what value of n would the machine have to be set to reduce a film of paint 8192 units thick to 1 unit thick?

5 A hermit has little opportunity to replenish supplies of tea and so, to eke out supplies for as long as possible, he dries out the tea leaves after use and then stores the dried tea in an airtight box. He estimates that after each re-use of the leaves the amount of tannin in the used tea will be half the previous amount. He also estimates that the amount of caffeine in the used tea will be one-quarter of the previous amount.
The information on the label of the tea packet states that the tea contains 729 mg of caffeine and 128 mg of tannin.

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a Write down expressions for the amount of caffeine when the tea leaves are re-used for the first, second, third and nth times.

b Do the same for the amount of tannin remaining.

c Find the number of times he can re-use the tea leaves if a 'tea' containing more than three times as much tannin as caffeine is undrinkable.

6 A new type of red synthetic carpet was produced in two batches. The first batch had a brightness of 15 units and the second batch 20 units. After a period of time it was discovered that the first batch was losing its brightness at the rate of 5% per year while the second was losing brightness at the rate of 6% per year.

a Write down expressions for the brightness of each batch after n years.

b A person bought some carpet from the first batch when it was a year old and some new carpet from the second batch. How long would it be before the brightness of the two carpets was the same?

7 In the year 2000, in an African game park it was estimated that there were approximately 700 wildebeest and that their population was increasing at 3% per year. At the same time, in the park there were approximately 1850 zebras and their population was decreasing at the rate of 4% per year. Use a calculator to plot the graphs of both functions.

a After how many years was the number of wildebeest greater than the number of zebras?

b It is also estimated that there were 1000 antelope and their numbers were increasing by 50 per year. After how many years was the number of antelope greater than the number of zebras?

8 Students conducting a science experiment on cooling rates measure the temperature of a beaker of liquid over a period of time. The following measurements were taken.

Time (minutes)	3	6	9	12	15	18	21
Temperature (°C)	71.5	59	49	45.5	34	28	23.5

a Find an exponential model to fit the data collected.

b Use this model to estimate:

i the initial temperature of the liquid

ii the temperature of the liquid after 25 minutes.

It is suspected that one of the temperature readings was incorrect.

c Re-calculate the model to fit the data, omitting the incorrect reading.

d Use the new model to estimate:

i the initial temperature of the liquid

ii the temperature of the liquid at $t = 12$.

e If the room temperature is 15°C, find the approximate time at which the cooling of the liquid ceased.

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15 Sequences and series

In this chapter

Syllabus references

Topics: Arithmetic sequences; Geometric sequences

Subtopics: 2.2.1 – 2.2.9

The following are examples of **sequences** of numbers:

a 1, 3, 5, 7, 9, . . .

b 0.1, 0.11, 0.111, 0.1111, 0.111 11, . . .

c $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$

d 10, 7, 4, 1, −2, . . .

e 0.6, 1.7, 2.8, . . . , 9.4

Each sequence is a list of numbers, with order being important. Sequence **e** is an example of a finite sequence, and the others are infinite sequences.

For some sequences of numbers, we can give a rule for getting from one number to the next:

- a rule for sequence **a** is: add 2
- a rule for sequence **c** is: multiply by $\frac{1}{3}$
- a rule for sequence **d** is: subtract 3
- a rule for sequence **e** is: add 1.1

In this chapter we study the theory required to solve problems involving sequences, and we look at some applications of sequences.

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15A Introduction to sequences

The numbers of a sequence are called its **terms**. The nth term of a sequence is denoted by the symbol t_n. So the first term is t_1, the 12th term is t_{12}, and so on.

Recurrence relations

A sequence may be defined by a rule which enables each subsequent term to be found from the previous term. This type of rule is called a **recurrence relation**, a **recursive formula** or an **iterative rule**. For example:

- The sequence $1, 3, 5, 7, 9, \dots$ may be defined by $t_1 = 1$ and $t_n = t_{n-1} + 2$.
- The sequence $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \dots$ may be defined by $t_1 = \frac{1}{3}$ and $t_n = \frac{1}{3}t_{n-1}$.

Example 1

Use the recurrence relation to find the first four terms of the sequence

$$t_1 = 3, \quad t_n = t_{n-1} + 5$$

Solution

$$t_1 = 3$$
$$t_2 = t_1 + 5 = 8$$
$$t_3 = t_2 + 5 = 13$$
$$t_4 = t_3 + 5 = 18$$

The first four terms are 3, 8, 13, 18.

Example 2

Find the recurrence relation for the following sequence:

$$9, -3, 1, -\frac{1}{3}, \dots$$

Solution

$$-3 = -\frac{1}{3} \times 9 \qquad \text{i.e. } t_2 = -\frac{1}{3}t_1$$

$$1 = -\frac{1}{3} \times -3 \qquad \text{i.e. } t_3 = -\frac{1}{3}t_2$$

The sequence is defined by $t_1 = 9$ and $t_n = -\frac{1}{3}t_{n-1}$.

A sequence may also be defined explicitly by a rule that is stated in terms of n. For example:

- The rule $t_n = 2n$ defines the sequence $t_1 = 2, t_2 = 4, t_3 = 6, t_4 = 8, \dots$
- The rule $t_n = 2^{n-1}$ defines the sequence $t_1 = 1, t_2 = 2, t_3 = 4, t_4 = 8, \dots$
- The sequence $1, 3, 5, 7, 9, \dots$ can be defined by $t_n = 2n - 1$.
- The sequence $t_1 = \frac{1}{3}, t_n = \frac{1}{3}t_{n-1}$ can be defined by $t_n = \frac{1}{3^n}$.

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For an infinite sequence, there is a term t_n of the sequence for each natural number n. Therefore we can consider an infinite sequence to be a function whose domain is the natural numbers. For example, we can write $t_n = 2n + 3$, $\{ n \in \mathbb{N} \}$.

Example 3

Find the first four terms of the sequence defined by the rule $t_n = 2n + 3$.

Solution

$$t_1 = 2(1) + 3 = 5$$
$$t_2 = 2(2) + 3 = 7$$
$$t_3 = 2(3) + 3 = 9$$
$$t_4 = 2(4) + 3 = 11$$

The first four terms are 5, 7, 9, 11.

Example 4

Find a rule for the nth term of the sequence 1, 4, 9, 16 in terms of n.

Solution

$$t_1 = 1 = 1^2$$
$$t_2 = 4 = 2^2$$
$$t_3 = 9 = 3^2$$
$$t_4 = 16 = 4^2$$
$$\therefore \quad t_n = n^2$$

Example 5

At a particular school, the number of students studying Mathematical Methods increases each year. There are presently 40 students studying Mathematical Methods.

a Set up the recurrence relation if the number is increasing by five students each year.

b Write down an expression for t_n in terms of n for the recurrence relation found in part a.

c Find the number of students expected to be studying Mathematical Methods at the school in five years' time.

Solution

a $t_n = t_{n-1} + 5$

b $t_1 = 40$

$$t_2 = t_1 + 5 = 45 = 40 + 1 \times 5$$
$$t_3 = t_2 + 5 = 50 = 40 + 2 \times 5$$

Therefore $t_n = 40 + (n - 1) \times 5$

$$= 35 + 5n$$

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c Five years from now implies $n = 6$:

$$t_6 = 40 + 5 \times 5 = 65$$

Sixty-five students will be studying Mathematical Methods in five years.

Example 6

The height of a sand dune is increasing by 10% each year. It is currently 4 m high.

a Set up the recurrence relation that describes the height of the sand dune.

b Write down an expression for t_n in terms of n for the recurrence relation found in part a.

c Find the height of the sand dune seven years from now.

Solution

a $t_n = t_{n-1} \times 1.1$

b $t_1 = 4$

$t_2 = 4 \times 1.1 = 4.4$

$t_3 = 4 \times (1.1)^2 = 4.84$

Therefore $t_n = 4 \times (1.1)^{n-1}$

c Seven years from now implies $n = 8$:

$$t_8 = 4 \times (1.1)^7 \approx 7.795$$

The sand dune will be 7.795 m high in seven years.

Using a calculator with explicitly defined sequences

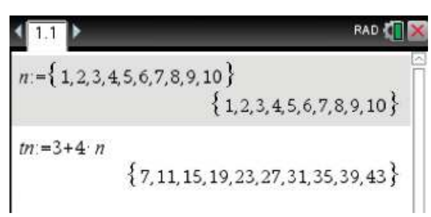
Example 7

Use a calculator to generate the first 10 terms of the sequence of numbers defined by the rule $t_n = 3 + 4n$.

Using the TI-Nspire

Sequences defined in terms of n can be investigated in a **Calculator** application.

- To generate the first 10 terms of the sequence defined by the rule $t_n = 3 + 4n$, complete as shown.



Note: Assigning (storing) the resulting list as *tn* enables the sequence to be graphed. If preferred, the variable *tn* can be entered as t_n using the subscript template $\square_{\square}$, which is accessed via [template key].

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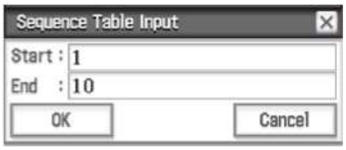
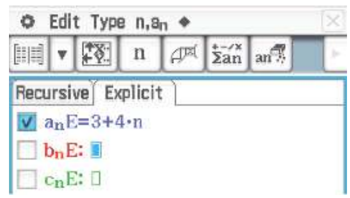
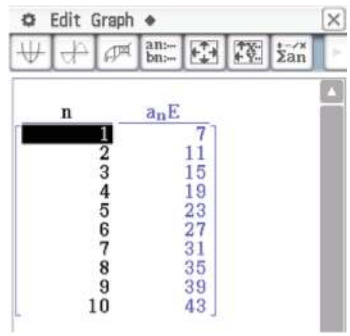
Using the Casio ClassPad

- Open the menu Menu; select **Sequence** Sequence.
- Ensure that the **Explicit** window is activated.
- Select the setting n.
- Tap the cursor next to a_nE and enter $3 + 4n$.
- Tick the box or tap EXE.
- Tap to view the sequence values.
- Tap to open the Sequence Table Input window and complete as shown below; tap OK.







- Tap Resize to see the sequence of numbers.

Using a calculator with recursively defined sequences

Example 8

Use a CAS calculator to generate the sequence defined by the recurrence relation

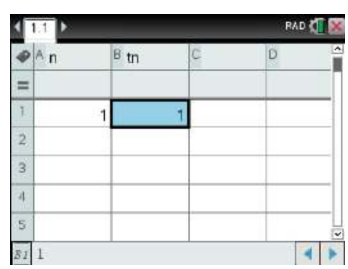
$$t_n = t_{n-1} + 3, \quad t_1 = 1$$

and plot the graph of the sequence against n.

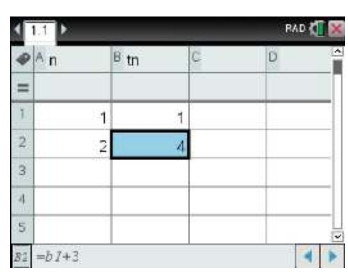
Using the TI-Nspire

- In a **Lists & Spreadsheet** page, name the first two lists n and tn respectively.
- Enter 1 in cell A1 and enter 1 in cell B1.

Note: If preferred, the variable tn can be entered as t_n using the subscript template, which is accessed via.



- Enter $= a1 + 1$ in cell A2 and enter $= b1 + 3$ in cell B2.



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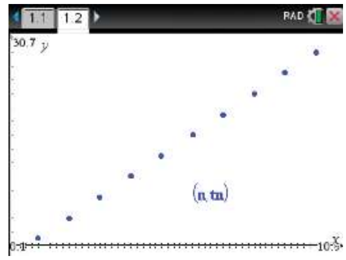
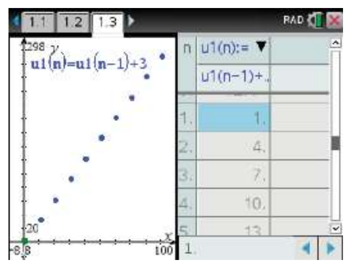
- Highlight the cells A2 and B2 using (shift) and the arrows.
- Use (menu) > **Data** > **Fill** and arrow down to row 10.
- Press (enter) to populate the lists.



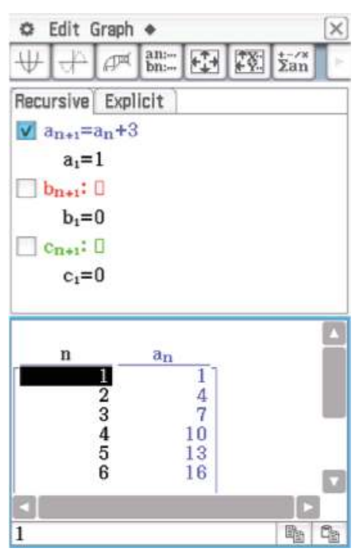
- To graph the sequence, open a **Graphs** application ((ctrl) (I) > **Add Graphs**).
- Graph the sequence as a scatter plot using (menu) > **Graph Entry/Edit** > **Scatter Plot**. Enter the list variables as n and tn in their respective fields.
- Set an appropriate window using (menu) > **Window/Zoom** > **Zoom – Data**.

Note: It is possible to see the coordinates of the points: (menu) > **Trace** > **Graph Trace**. The scatter plot can also be graphed in a **Data & Statistics** page.



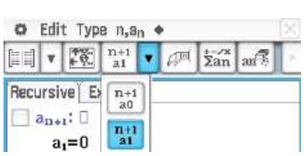
- Alternatively, the sequence can be graphed directly in the sequence plotter ((menu) > **Graph Entry/Edit** > **Sequence** > **Sequence**).
- Enter the rule $u1(n) = u1(n - 1) + 3$ and the initial value 1. Change **nStep** to 10.
- Set an appropriate window using (menu) > **Window/Zoom** > **Zoom – Fit**.
- Use (ctrl) (T) to show a table of values.

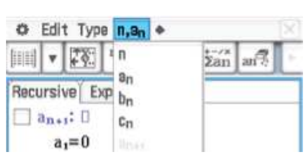
Using the Casio ClassPad



- Open the menu Menu; select **Sequence** Sequence.
- Ensure that the **Recursive** window is activated.
- Select the setting $\frac{n+1}{a1}$ as shown below.
- Tap the cursor next to a_{n+1} and enter $a_n + 3$.

Note: The symbol a_n can be found in the dropdown menu in the toolbar as shown below.





- Enter 1 for the value of the first term, a_1.
- Tick the box. Tap the table icon to view the sequence values.

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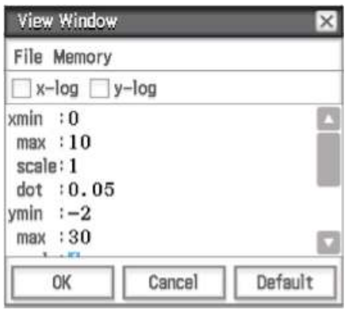
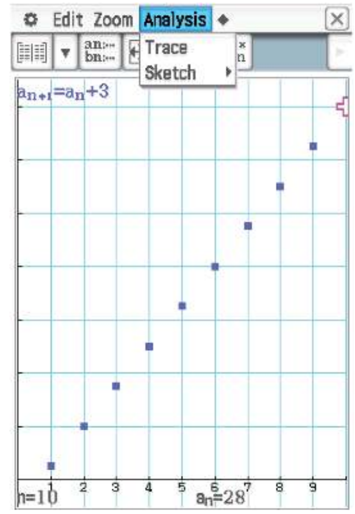
- Tap [graph icon] to view the graph.
- Tap Resize [icon] and then [icon]. Set the View Window as shown below.





- Select **Analysis** > **Trace** and use the cursor ► to view each value in the sequence.

Summary 15A

A sequence may be defined by a rule which enables each subsequent term to be found from the previous term. This type of rule is called a **recurrence relation** and we say that the sequence has been defined **recursively**. For example:

- The sequence $1, 3, 5, 7, 9, \ldots$ is defined by $t_1 = 1$ and $t_n = t_{n-1} + 2$.
- The sequence $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$ is defined by $t_1 = \frac{1}{3}$ and $t_n = \frac{1}{3}t_{n-1}$.

Exercise 15A

Example 1

1 In each of the following, a recursive definition for a sequence is given. List the first five terms.

a $t_1 = 3,\ t_n = t_{n-1} + 4$ **b** $t_1 = 5,\ t_n = 3t_{n-1} + 4$ **c** $t_1 = 1,\ t_n = 5t_{n-1}$

d $t_1 = -1,\ t_n = t_{n-1} + 2$ **e** $t_{n+1} = 2t_n + t_{n-1},\ t_1 = 1,\ t_2 = 3$

Example 2

2 For each of the following sequences, find the recurrence relation:

a $3, 6, 9, 12, \ldots$ **b** $1, 2, 4, 8, \ldots$ **c** $3, -6, 12, -24, \ldots$

d $4, 7, 10, 13, \ldots$ **e** $4, 9, 14, 19, \ldots$

Example 3

3 Each of the following is a rule for a sequence. In each case, find t_1, t_2, t_3, t_4.

a $t_n = \frac{1}{n}$ **b** $t_n = n^2 + 1$ **c** $t_n = 2n$

d $t_n = 2^n$ **e** $t_n = 3n + 2$ **f** $t_n = (-1)^n n^3$

g $t_n = 2n + 1$ **h** $t_n = 2 \times 3^{n-1}$

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Example 4 **4** For each of the following sequences, find a possible rule for t_n in terms of n:

a $3, 6, 9, 12, \dots$ **b** $1, 2, 4, 8, \dots$ **c** $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \dots$

d $3, -6, 12, -24, \dots$ **e** $4, 7, 10, 13, \dots$ **f** $4, 9, 14, 19, \dots$

5 Consider a sequence for which $t_n = 3n + 1$. Find t_{n+1} and t_{2n}.

Example 5 **6** Hamish collects football cards. He currently has 15 and he adds three to his collection every week.

a Set up the recurrence relation that will generate the number of cards Hamish has in any given week.

b Write down an expression for t_n in terms of n for the recurrence relation from part a.

c Find the number of cards Hamish should have after another 12 weeks.

Example 6 **7** Isobel can swim 100 m in 94.3 s. She aims to reduce her time by 4% each week.

a Set up the recurrence relation that generates Isobel's time for the 100 m in any given week.

b Write down an expression for t_n in terms of n for the recurrence relation from part a.

c Find the time in which Isobel expects to be able to complete the 100 m after another 8 weeks.

8 Stephen is a sheep farmer with a flock of 100 sheep. He wishes to increase the size of his flock by both breeding and buying new stock. He estimates that 80% of his sheep will produce one lamb each year and he intends to buy 20 sheep to add to the flock each year. Assuming that no sheep die:

a Write the recurrence relation for the expected number of sheep at the end of each year. (Let $t_0 = 100$.)

b Calculate the number of sheep at the end of each of the first five years.

9 Alison invests \$2000 at the beginning of the year. At the beginning of each of the following years, she puts a further \$400 into the account. Compound interest of 6% p.a. is paid on the investment at the end of each year.

a Write down the amount of money in the account at the end of each of the first three years.

b Set up a recurrence relation to generate the sequence for the investment. (Let t_1 be the amount in the account at the end of the first year.)

c With a calculator or spreadsheet, use the recurrence relation to find the amount in the account after 10 years.

Example 7 **10** For each of the following, use a CAS calculator to find the first six terms of the sequence and plot the graph of these terms against n:

a $t_n = 3n - 2$ **b** $t_n = 5 - 2n$ **c** $t_n = 2^{n-2}$ **d** $t_n = 2^{6-n}$

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Example 8

11 For each of the following, use a CAS calculator to find the first six terms of the sequence and plot the graph of these terms against n:

a $t_n = (t_{n-1})^2, \; t_1 = 1.1$

b $t_n = \frac{2}{3}t_{n-1}, \; t_1 = 27$

c $t_n = 2t_{n-1} + 5, \; t_1 = -1$

d $t_n = 4 - t_{n-1}, \; t_1 = -3$

12 **a** For a sequence for which $t_n = 2^{n-1}$, find t_1, t_2 and t_3.

b For a sequence for which $u_n = \frac{1}{2}(n^2 - n) + 1$, find u_1, u_2 and u_3.

c What do you notice?

d Find t_4 and u_4.

13 If $S_n = an^2 + bn$, for constants $a, b \in \mathbb{R}$, find S_1, S_2, S_3 and $S_{n+1} - S_n$.

14 For the sequence defined by $t_1 = 1$ and $t_{n+1} = \frac{1}{2}\left(t_n + \frac{2}{t_n}\right)$, find t_2, t_3 and t_4.

The terms of this sequence are successive rational approximations of a real number. Can you recognise the number?

15 The Fibonacci sequence is defined by $t_1 = 1$, $t_2 = 1$ and $t_{n+2} = t_{n+1} + t_n$ for $n \in \mathbb{N}$. Use the rule to find t_3, t_4 and t_5. Show that $t_{n+2} = 2t_n + t_{n-1}$ for all $\{ n \in \mathbb{N} : n \neq 1 \}$.

15B Arithmetic sequences

A sequence in which each successive term is found by adding a fixed amount to the previous term is called an **arithmetic sequence**. That is, an arithmetic sequence has a recurrence relation of the form $t_n = t_{n-1} + d$, where d is a constant.

For example: $2, 5, 8, 11, 14, 17, \ldots$ is an arithmetic sequence.

The nth term of an arithmetic sequence is given by

$$t_n = a + (n-1)d$$

where a is the first term and d is the **common difference** between successive terms, that is, $d = t_k - t_{k-1}$, for all $k > 1$.

Note: In an arithmetic sequence, the nth term t_n is a linear function of n.

Example 9

Find the 10th term of the arithmetic sequence $-4, -1, 2, 5, \ldots$.

Solution

$a = -4$, $d = 3$

$$t_n = a + (n-1)d$$

$$\therefore \quad t_{10} = -4 + (10-1) \times 3$$

$$= 23$$

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Example 10

If 41 is the nth term in the arithmetic sequence $-4, -1, 2, 5, \dots$, find the value of n.

Solution

$a = -4$, $d = 3$

$$\begin{aligned} t_n &= a + (n-1)d = 41 \\ \therefore \quad -4 + (n-1) \times 3 &= 41 \\ 3(n-1) &= 45 \\ n - 1 &= 15 \\ n &= 16 \end{aligned}$$

Hence 41 is the 16th term of the sequence.

Example 11

The 12th term of an arithmetic sequence is 9 and the 25th term is 100. Find a and d, and hence find the 8th term.

Solution

An arithmetic sequence has rule

$$t_n = a + (n-1)d$$

Since the 12th term is 9, we have

$$9 = a + 11d \qquad (1)$$

Since the 25th term is 100, we have

$$100 = a + 24d \qquad (2)$$

To find a and d, we solve the two equations simultaneously.

Subtract (1) from (2):

$$\begin{aligned} 91 &= 13d \\ \therefore \quad d &= 7 \end{aligned}$$

From (1), we have

$$\begin{aligned} 9 &= a + 11(7) \\ \therefore \quad a &= -68 \end{aligned}$$

Therefore

$$\begin{aligned} t_8 &= a + 7d \\ &= -68 + 7 \times 7 \\ &= -19 \end{aligned}$$

The 8th term of the sequence is -19.

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Example 12

A national park has a series of huts along one of its mountain trails. The first hut is 5 km from the start of the trail, the second is 8 km from the start, the third 11 km and so on.

a How far from the start of the trail is the sixth hut?

b How far is it from the sixth hut to the twelfth hut?

Solution

The distances of the huts from the start of the trail form an arithmetic sequence with $a = 5$ and $d = 3$.

a For the sixth hut:

$$t_6 = a + 5d$$
$$= 5 + 5 \times 3 = 20$$

The sixth hut is 20 km from the start of the trail.

b For the twelfth hut:

$$t_{12} = a + 11d$$
$$= 5 + 11 \times 3 = 38$$

The distance from the sixth hut to the twelfth hut is $t_{12} - t_6 = 38 - 20 = 18$ km.

Arithmetic mean

The **arithmetic mean** of two numbers a and b is defined as $\dfrac{a+b}{2}$.

If the numbers a, c, b are consecutive terms of an arithmetic sequence, then

$$c - a = b - c$$
$$2c = a + b$$
$$\therefore \quad c = \frac{a+b}{2}$$

That is, the middle term c is the arithmetic mean of a and b.

Summary 15B

- An **arithmetic sequence** has a recurrence relation of the form $t_n = t_{n-1} + d$, where d is a constant. Each successive term is found by adding a fixed amount to the previous term. For example: $2, 5, 8, 11, \dots$
- The nth term of an arithmetic sequence is given by

  $$t_n = a + (n-1)d$$

  where a is the first term and d is the **common difference** between successive terms, that is, $d = t_k - t_{k-1}$, for all $k > 1$.

Exercise 15B

1 For the arithmetic sequence where $t_n = a + (n-1)d$, find the first four terms given that:

a $a = 0,\ d = 2$

b $a = -3,\ d = 5$

c $a = d = -\sqrt{5}$

d $a = 11,\ d = -2$

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Example 9

2 **a** If an arithmetic sequence has a first term of 5 and a common difference of −3, find the 13th term.

b If an arithmetic sequence has a first term of −12 and a common difference of 4, find the 10th term.

c For the arithmetic sequence with $a = 25$ and $d = -2.5$, find the ninth term.

d For the arithmetic sequence with $a = 2\sqrt{3}$ and $d = \sqrt{3}$, find the fifth term.

3 Find the rule of the arithmetic sequence whose first few terms are:

a $3, 7, 11$ **b** $3, -1, -5$ **c** $-\frac{1}{2}, \frac{3}{2}, \frac{7}{2}, \frac{11}{2}$ **d** $5 - \sqrt{5},\ 5,\ 5 + \sqrt{5}$

Example 10

4 In each of the following, t_n is the nth term of an arithmetic sequence:

a If 54 is the nth term in the sequence $6, 10, 14, 18, \dots$, find the value of n.

b If −16 is the nth term in the sequence $5, 2, -1, -4, \dots$, find the value of n.

c Find n if $t_1 = 16$, $t_2 = 13$ and $t_n = -41$.

d Find n if $t_1 = 7$, $t_2 = 11$ and $t_n = 227$.

Example 11

5 For an arithmetic sequence with fourth term 7 and thirtieth term 85, find the values of a and d, and hence find the seventh term.

6 If an arithmetic sequence has $t_3 = 18$ and $t_6 = 486$, find the rule for the sequence, i.e. find t_n.

7 For the arithmetic sequence with $t_7 = 0.6$ and $t_{12} = -0.4$, find t_{20}.

8 The number of laps that a swimmer swims each week follows an arithmetic sequence. In the 5th week she swims 24 laps and in the 10th week she swims 39 laps. How many laps does she swim in the 15th week?

9 For an arithmetic sequence, find t_6 if $t_{10} = 31$ and $t_{20} = 61$.

Example 12

10 A small company producing wallets plans an increase in output. In the first week it produces 280 wallets. The number of wallets produced each week is to be increased by 8 per week until the weekly number produced reaches 1000.

a How many wallets are produced in the 50th week?

b In which week does the production reach 1000?

11 An amphitheatre has 25 seats in row A, 28 seats in row B, 31 seats in row C, and so on.

a How many seats are there in row P?

b How many seats are there in row X?

c Which row has 40 seats?

12 The number of people who go to see a movie over a period of a week follows an arithmetic sequence. On the first day only three people go to the movie, but on the sixth day 98 people go. Find the rule for the sequence and hence determine how many attend on the seventh day.

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13 An arithmetic sequence contains 10 terms. If the first is 4 and the tenth is 30, what is the eighth term?

14 The number of goals kicked by a team in the first six games of a season follows an arithmetic sequence. If the team kicked 5 goals in the first game and 15 in the sixth, how many did they kick in each of the other four games?

15 The first term of an arithmetic sequence is a and the mth term is 0. Find the rule for t_n for this sequence.

16 Find the arithmetic mean of:

a 8 and 15 **b** 21 and 79

17 Find x if $3x - 2$ is the arithmetic mean of $5x + 1$ and 11.

18 If a, $4a - 4$ and $8a - 13$ are successive terms of an arithmetic sequence, find a.

19 If $t_m = n$ and $t_n = m$, prove that $t_{m+n} = 0$. (Here t_m and t_n are the mth and nth terms of an arithmetic sequence).

20 If a, $2a$ and a^2 are consecutive terms of an arithmetic sequence, find a ($a \neq 0$).

21 Show that there is no infinite arithmetic sequence whose terms are all prime numbers.

15C Arithmetic series

The sum of the terms in a sequence is called a **series**. If the sequence is arithmetic, then the series is called an **arithmetic series**.

The symbol S_n is used to denote the sum of the first n terms of a sequence. That is,

$$S_n = a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d)$$

Writing this sum in reverse order, we have

$$S_n = (a + (n - 1)d) + (a + (n - 2)d) + \cdots + (a + d) + a$$

Adding these two expressions together gives

$$2S_n = n(2a + (n - 1)d)$$

Therefore

$$S_n = \frac{n}{2}\big(2a + (n - 1)d\big)$$

Since the last term $\ell = t_n = a + (n - 1)d$, we can also write

$$S_n = \frac{n}{2}(a + \ell)$$

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Example 13

For the arithmetic sequence 2, 5, 8, 11, . . . , calculate the sum of the first 14 terms.

Solution

$a = 2$, $d = 3$, $n = 14$

$$S_n = \frac{n}{2}\big(2a + (n-1)d\big)$$

$$\therefore \quad S_{14} = \frac{14}{2}(2 \times 2 + 13 \times 3) = 301$$

Example 14

For the arithmetic sequence 27, 23, 19, 15, . . . , −33, find:

a the number of terms

b the sum of the terms.

Solution

a $a = 27$, $d = -4$, $\ell = t_n = -33$

$$\begin{aligned} t_n &= a + (n-1)d \\ -33 &= 27 + (n-1)(-4) \\ -60 &= (n-1)(-4) \\ 15 &= n - 1 \\ n &= 16 \end{aligned}$$

There are 16 terms in the sequence.

b $a = 27$, $\ell = t_n = -33$, $n = 16$

$$S_n = \frac{n}{2}(a + \ell)$$

$$\begin{aligned} \therefore \quad S_{16} &= \frac{16}{2}(27 - 33) \\ &= -48 \end{aligned}$$

The sum of the terms is −48.

Example 15

For the arithmetic sequence 3, 6, 9, 12, . . . , calculate:

a the sum of the first 25 terms

b the number of terms in the series if $S_n = 1395$.

Solution

a $a = 3$, $d = 3$, $n = 25$

$$S_n = \frac{n}{2}\big(2a + (n-1)d\big)$$

$$\begin{aligned} \therefore \quad S_{25} &= \frac{25}{2}\big(2(3) + (24)(3)\big) \\ &= 975 \end{aligned}$$

b $a = 3$, $d = 3$, $S_n = 1395$

$$\begin{aligned} S_n = \frac{n}{2}\big(2a + (n-1)d\big) &= 1395 \\ \frac{n}{2}\big(2(3) + (n-1)(3)\big) &= 1395 \\ n(6 + 3n - 3) &= 2790 \\ 3n + 3n^2 &= 2790 \\ 3n^2 + 3n - 2790 &= 0 \\ n^2 + n - 930 &= 0 \\ (n-30)(n+31) &= 0 \end{aligned}$$

Therefore $n = 30$, since $n > 0$.
Hence there are 30 terms in the series.

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Example 16

A hardware store sells nails in a range of packet sizes. Packet A contains 50 nails, packet B contains 75 nails, packet C contains 100 nails, and so on.

a Find the number of nails in packet J.

b Lachlan buys one each of packets A to J. How many nails in total does Lachlan have?

c Assuming he buys one of each packet starting at A, how many packets does he need to buy to have a total of 1100 nails?

Solution

a $a = 50,\ d = 25$

$$t_n = a + (n-1)d$$

For packet J, we take $n = 10$:

$$\begin{aligned} t_{10} &= 50 + 9 \times 25 \\ &= 275 \end{aligned}$$

Packet J contains 275 nails.

b $a = 50,\ d = 25$

$$S_n = \frac{n}{2}\big(2a + (n-1)d\big)$$

$$\begin{aligned} \therefore \quad S_{10} &= \frac{10}{2}\big(2 \times 50 + 9 \times 25\big) \\ &= 1625 \end{aligned}$$

Packets A to J contain 1625 nails.

c $a = 50,\ d = 25,\ S_n = 1100$

$$\begin{aligned} S_n = \frac{n}{2}\big(2a + (n-1)d\big) &= 1100 \\ \frac{n}{2}\big(2(50) + (n-1)(25)\big) &= 1100 \\ n(100 + 25n - 25) &= 2200 \\ 25n^2 + 75n - 2200 &= 0 \\ n^2 + 3n - 88 &= 0 \\ (n+11)(n-8) &= 0 \end{aligned}$$

Thus $n = 8$, since $n > 0$. If Lachlan buys one each of the first eight packets (A to H), he will have exactly 1100 nails.

Example 17

The sum of the first 10 terms of an arithmetic sequence is $48\frac{3}{4}$. If the fourth term is $3\frac{3}{4}$, find the first term and the common difference.

Solution

$$t_4 = a + 3d = 3\frac{3}{4}$$

$$\therefore \quad a + 3d = \frac{15}{4} \qquad (1)$$

$$S_{10} = \frac{10}{2}(2a + 9d) = 48\frac{3}{4}$$

$$\therefore \quad 10a + 45d = \frac{195}{4} \qquad (2)$$

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Solve equations (1) and (2) simultaneously:

$(1) \times 40: \quad 40a + 120d = 150$

$(2) \times 4: \quad 40a + 180d = 195$

$$60d = 45$$

$$\therefore \quad d = \frac{3}{4}$$

Substitute in (1) to obtain $a + 3\left(\frac{3}{4}\right) = \frac{15}{4}$ and therefore $a = \frac{3}{2}$.

The first term is $1\frac{1}{2}$ and the common difference is $\frac{3}{4}$.

Summary 15C

The sum of the first n terms of an arithmetic sequence

$$S_n = a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d)$$

is given by

$$S_n = \frac{n}{2}\big(2a + (n - 1)d\big) \quad \text{or} \quad S_n = \frac{n}{2}(a + \ell), \text{ where } \ell = t_n$$

Skill-sheet

Exercise 15C

Example 13

1 For each arithmetic sequence, find the specified sum:

a $8, 13, 18, \ldots$ find S_{12}

b $-3.5, -1.5, 0.5, \ldots$ find S_{10}

c $\frac{1}{\sqrt{2}}, \sqrt{2}, \frac{3}{\sqrt{2}}, \ldots$ find S_{15}

d $-4, 1, 6, \ldots$ find S_8

2 Greg goes fishing every day for a week. On the first day he catches seven fish and each day he catches three more than the previous day. How many fish did he catch in total?

3 Find the sum of the first 16 multiples of 5.

4 Find the sum of all the even numbers between 1 and 99.

Example 14

5 For the arithmetic sequence $-3, 1, 5, 9, \ldots, 49$, find:

a the number of terms

b the sum of the terms.

6 For the arithmetic sequence $24, 20, 16, 12, \ldots, -52$, find:

a the number of terms

b the sum of the terms.

7 For the arithmetic sequence $\frac{1}{2}, 2, \frac{7}{2}, 5, \ldots, 17$, find:

a the number of terms

b the sum of the terms.

Example 15

8 For the sequence $4, 8, 12, \ldots$, find:

a the sum of the first 9 terms

b $\{\, n : S_n = 180 \,\}$.

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9 There are 110 logs to be put in a pile, with 15 logs in the bottom layer, 14 in the next, 13 in the next, and so on. How many layers will there be?

10 The sum of the first m terms of an arithmetic sequence with first term -5 and common difference 4 is 660. Find m.

11 Evaluate $54 + 48 + 42 + \cdots + (-54)$.

Example 16

12 Dora's walking club plans 15 walks for the summer. The first walk is a distance of 6 km, the last walk is a distance of 27 km, and the distances of the walks form an arithmetic sequence.

a How far is the 8th walk?

b How far does the club plan to walk in the first five walks?

c Dora's husband, Alan, can only complete the first n walks. If he walks a total of 73.5 km, how many walks does he complete?

d Dora goes away on holiday and misses the 9th, 10th and 11th walks, but completes all other walks. How far does Dora walk in total?

13 Liz has to proofread 500 pages of a new novel. She plans to read 30 pages on the first day and to increase the number of pages she reads by five each day.

a How many days will it take her to complete the proofreading?

She has only five days to complete the task. She therefore decides to read 50 pages on the first day and to increase the number she reads by a constant amount each day.

b By how many should she increase the number of pages she reads each day if she is to meet her deadline?

14 An assembly hall has 50 seats in row A, 54 seats in row B, 58 seats in row C, and so on. That is, there are four more seats in each row.

a How many seats are there in row J?

b How many seats are there altogether if the back row is row Z?

On a particular day, the front four rows are reserved for parents (and there is no other seating for parents).

c How many parents can be seated?

d How many students can be seated?

The hall is extended by adding more rows following the same pattern.

e If the final capacity of the hall is 3410, how many rows were added?

15 A new golf club is formed with 40 members in its first year. Each following year, the number of new members exceeds the number of retirements by 15. Each member pays \$120 p.a. in membership fees. Calculate the amount received from fees in the first 12 years of the club's existence.

Example 17

16 For the arithmetic sequence with $t_2 = -12$ and $S_{12} = 18$, find a, d, t_6 and S_6.

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17 The sum of the first 10 terms of an arithmetic sequence is 120, and the sum of the first 20 terms is 840. Find the sum of the first 30 terms.

18 If $t_6 = 16$ and $t_{12} = 28$, find S_{14}.

19 For an arithmetic sequence, find t_n if:

a $t_3 = 6.5$ and $S_8 = 67$

b $t_6 = 17$ and $S_{10} = 155$

20 For the sequence with $t_n = bn$, where $b \in \mathbb{R}$, find:

a $t_{n+1} - t_n$

b $t_1 + t_2 + \cdots + t_n$

21 For a sequence where $t_n = 15 - 5n$, find t_5 and find the sum of the first 25 terms.

22 An arithmetic sequence has a common difference of d and the sum of the first 20 terms is 25 times the first term. Find the sum of the first 30 terms in terms of d.

23 The sum of the first n terms of a particular sequence is given by $S_n = 17n - 3n^2$.

a Find an expression for the sum of the first $(n - 1)$ terms.

b Find an expression for the nth term of the sequence.

c Show that the sequence is arithmetic and find a and d.

24 Three consecutive terms of an arithmetic sequence have a sum of 36 and a product of 1428. Find the three terms.

25 Show that the sum of the first $2n$ terms of an arithmetic sequence is n times the sum of the two middle terms.

15D Geometric sequences

A sequence in which each successive term is found by multiplying the previous term by a fixed amount is called a **geometric sequence**. That is, a geometric sequence has a recurrence relation of the form $t_n = rt_{n-1}$, where r is a constant.

For example: $2, 6, 18, 54, \dots$ is a geometric sequence.

The nth term of a geometric sequence is given by

$$t_n = ar^{n-1}$$

where a is the first term and r is the **common ratio** of successive terms, that is, $r = \dfrac{t_k}{t_{k-1}}$, for all $k > 1$.

Note: In a geometric sequence, the nth term t_n is an exponential function of n.

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Example 18

Find the 10th term of the sequence 2, 6, 18, ...

Solution

$a = 2$, $r = 3$

$$t_n = ar^{n-1}$$
$$\therefore \quad t_{10} = 2 \times 3^{10-1} = 39\,366$$

Example 19

For a geometric sequence, the first term is 18 and the fourth term is 144. Find the common ratio.

Solution

$a = 18$, $t_4 = 144$

$$t_4 = 18 \times r^{4-1} = 144$$
$$18r^3 = 144$$
$$r^3 = 8$$
$$\therefore \quad r = 2$$

The common ratio is 2.

Example 20

For a geometric sequence 36, 18, 9, . . . , the nth term is $\frac{9}{16}$. Find the value of n.

Solution

$a = 36$, $r = \frac{1}{2}$

$$t_n = 36 \times \left(\frac{1}{2}\right)^{n-1} = \frac{9}{16}$$
$$\left(\frac{1}{2}\right)^{n-1} = \frac{9}{576}$$
$$\left(\frac{1}{2}\right)^{n-1} = \frac{1}{64}$$
$$\left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^{6}$$
$$n - 1 = 6$$
$$\therefore \quad n = 7$$

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Example 21

The third term of a geometric sequence is 10 and the sixth term is 80. Find the common ratio and the first term.

Solution

$$t_3 = ar^2 = 10 \qquad (1)$$
$$t_6 = ar^5 = 80 \qquad (2)$$

Divide (2) by (1):

$$\frac{ar^5}{ar^2} = \frac{80}{10}$$
$$r^3 = 8$$
$$\therefore \quad r = 2$$

Substitute in (1):

$$a \times 4 = 10$$
$$\therefore \quad a = \frac{5}{2}$$

The common ratio is 2 and the first term is $\frac{5}{2}$.

Example 22

Georgina draws a pattern consisting of a number of equilateral triangles. The first triangle has sides of length 4 cm and the side length of each successive triangle is one and a half times the side length of the previous one.

a What is the side length of the fifth triangle?

b Which triangle has a side length of $45\frac{9}{16}$ cm?

Solution

a $a = 4, r = \frac{3}{2}$

$$t_n = ar^{n-1}$$
$$\therefore \quad t_5 = ar^4 = 4 \times \left(\frac{3}{2}\right)^4$$
$$= 20\frac{1}{4}$$

The fifth triangle has a side length of $20\frac{1}{4}$ cm.

b $a = 4, r = \frac{3}{2}, t_n = 45\frac{9}{16}$

$$t_n = ar^{n-1} = 45\frac{9}{16}$$
$$4 \times \left(\frac{3}{2}\right)^{n-1} = \frac{729}{16}$$
$$\left(\frac{3}{2}\right)^{n-1} = \frac{729}{64} = \left(\frac{3}{2}\right)^6$$

Therefore $n - 1 = 6$ and so $n = 7$.

The seventh triangle has a side length of $45\frac{9}{16}$ cm.

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Compound interest

One application of geometric sequences is **compound interest**. Compound interest is interest calculated at regular intervals on the total of the amount originally invested and the amount accumulated in the previous years.

Assume that \$1000 is invested at 10% per annum. At the end of the first year, the amount will have grown to

$$1000 + 10\%(1000) = \$1100$$

At the end of the second year, it will have grown to

$$(1000 + 10\%(1000)) + 10\%(1000 + 10\%(1000)) = \$1210$$

The value of the investment at the end of each year forms a geometric sequence. For this example, we have

$$a = 1000 \text{ and } r = 1.1, \quad \text{i.e. } r = 100\% + 10\%$$

Example 23

Marta invests \$2500 at 7% p.a. compounded annually. Find:

a the value of her investment after 5 years

b how long it takes until her investment is worth \$10 000.

Solution

Let t_n be the value at the end of the $(n-1)$st year. Then $a = 2500$ and $r = 1.07$.

a
$$\begin{aligned} t_6 &= ar^5 \\ &= 2500(1.07)^5 \\ &= 3506.38 \end{aligned}$$

The value of the investment after 5 years is \$3506.38.

b
$$\begin{aligned} t_n = ar^{n-1} &= 10\,000 \\ 2500(1.07)^{n-1} &= 10\,000 \\ 1.07^{n-1} &= 4 \\ \frac{1.07^n}{1.07^1} &= 4 \\ 1.07^n &= 4.28 \end{aligned}$$

Using a CAS calculator to solve for n:

$$n \approx 21.49$$

By the end of the 21st year, the investment will be worth over \$10 000.

Note: For part b, the number of years can also be found by trial and error.

Geometric mean

The **geometric mean** of two numbers a and b is defined as $\sqrt{ab}$.

If positive numbers a, c, b are consecutive terms of a geometric sequence, then

$$\frac{c}{a} = \frac{b}{c} \qquad \therefore \quad c = \sqrt{ab}$$

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Summary 15D

- A **geometric sequence** has a recurrence relation of the form $t_n = rt_{n-1}$, where r is a constant. Each successive term is found by multiplying the previous term by a fixed amount. For example: $2, 6, 18, 54, \ldots$
- The nth term of a geometric sequence is given by

  $$t_n = ar^{n-1}$$

  where a is the first term and r is the **common ratio** of successive terms, that is, $r = \dfrac{t_k}{t_{k-1}}$, for all $k > 1$.

Exercise 15D

1 For a geometric sequence $t_n = ar^{n-1}$, find the first four terms given that:

a $a = 3,\ r = 2$ **b** $a = 3,\ r = -2$

c $a = 10\,000,\ r = 0.1$ **d** $a = r = 3$

Example 18

2 Find the specified term in each of the following geometric sequences:

a $\frac{15}{7}, \frac{5}{7}, \frac{5}{21}, \ldots$ find t_6 **b** $1, -\frac{1}{4}, \frac{1}{16}, \ldots$ find t_5

c $\sqrt{2}, 2, 2\sqrt{2}, \ldots$ find t_{10} **d** $a^x, a^{x+1}, a^{x+2}, \ldots$ find t_6

3 Find the rule for the geometric sequence whose first few terms are:

a $3, 2, \frac{4}{3}$ **b** $2, -4, 8, -16$ **c** $2, 2\sqrt{5}, 10$

Example 19

4 Find the common ratio for the following geometric sequences:

a the first term is 2 and the sixth term is 486

b the first term is 25 and the fifth term is $\frac{16}{25}$

Example 20

5 A geometric sequence has first term $\frac{1}{4}$ and common ratio 2. Which term of the sequence is 64?

6 If t_n is the nth term of the following geometric sequences, find n in each case:

a $2, 6, 18, \ldots$ $t_n = 486$ **b** $5, 10, 20, \ldots$ $t_n = 1280$

c $768, 384, 192, \ldots$ $t_n = 3$ **d** $\frac{8}{9}, \frac{4}{3}, 2, \ldots$ $t_n = \frac{27}{4}$

e $-\frac{4}{3}, \frac{2}{3}, -\frac{1}{3}, \ldots$ $t_n = \frac{1}{96}$

Example 21

7 The 12th term of a geometric sequence is 2 and the 15th term is 54. Find the 7th term.

8 A geometric sequence has $t_2 = \dfrac{1}{2\sqrt{2}}$ and $t_4 = \sqrt{2}$. Find t_8.

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9 The number of fish in the breeding tanks of a fish farm follow a geometric sequence. The third tank contains 96 fish and the sixth tank contains 768 fish.

a How many fish are in the first tank? **b** How many fish are in the 10th tank?

Example 22 **10** An algal bloom is growing in a lake. The area it covers triples each day.

a If it initially covers an area of $10\,\text{m}^2$, how many square metres will it cover after one week?

b If the lake has a total area of $200\,000\,\text{m}^2$, how long before the entire lake is covered?

11 A ball is dropped from a height of 2 m and continues to bounce so that it rebounds to three-quarters of the height from which it previously falls. Find the height it rises to on the fifth bounce.

12 An art collector has a painting that is increasing in value by 8% each year. If the painting is currently valued at \$2500:

a How much will it be worth in 10 years?

b How many years before its value exceeds \$100 000?

13 The Tour de Moravia is a cycling event which lasts for 15 days. On the first day the cyclists must ride 120 km, and each successive day they ride 90% of the distance of the previous day.

a How far do they ride on the 8th day?

b On which day do they ride 30.5 km?

14 A child negotiates a new pocket-money deal with her unsuspecting father in which she receives 1 cent on the first day of the month, 2 cents on the second, 4 cents on the third, 8 cents on the fourth, and so on, until the end of the month. How much would the child receive on the 30th day of the month? (Give your answer to the nearest thousand dollars.)

Example 23 **15** \$5000 is invested at 6% p.a. compounded annually.

a Find the value of the investment after 6 years.

b Find how long it will take for the original investment to double in value.

16 How much would need to be invested at 8.5% p.a. compounded annually to yield a return of \$8000 after 12 years?

17 What annual compound interest rate would be required to triple the value of an investment of \$200 in 10 years?

18 The first three terms of a geometric sequence are 4, 8, 16. Find the first term in the sequence which exceeds 2000.

19 The first three terms of a geometric sequence are 3, 9, 27. Find the first term in the sequence which exceeds 500.

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20 The number of 'type A' apple bugs present in an orchard is estimated to be 40 960, and the number is reducing by 50% each week. At the same time it is estimated that there are 40 'type B' apple bugs, whose number is doubling each week. After how many weeks will there be the same number of each type of bug?

21 Find the geometric mean of:

a 5 and 720 **b** 1 and 6.25 **c** $\frac{1}{\sqrt{3}}$ and $\sqrt{3}$ **d** x^2y^3 and x^6y^{11}

22 The fourth, seventh and sixteenth terms of an arithmetic sequence also form consecutive terms of a geometric sequence. Find the common ratio of the geometric sequence.

23 Consider the geometric sequence $1, a, a^2, a^3, \dots$. Suppose that the sum of two consecutive terms in the sequence gives the next term in the sequence. Find a.

24 A bottle contains 1000 mL of pure ethanol. Then 300 mL is removed and the bottle is topped up with pure water. The mixture is stirred.

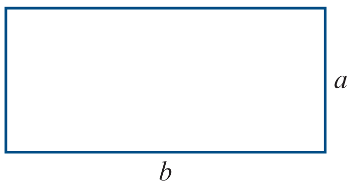
a What is the volume of ethanol in the bottle if this process is repeated five times in total?

b How many times should the process be repeated for there to be less than 1 mL of ethanol in the bottle?

25 The rectangle shown has side lengths a and b.

a Find the side length of a square with the same perimeter. Comment.

b Find the side length of a square with the same area Comment.



15E Geometric series

The sum of the terms in a geometric sequence is called a **geometric series**. An expression for S_n, the sum of the first n terms of a geometric sequence, can be found using a similar method to that used for arithmetic series.

Let $\quad S_n = a + ar + ar^2 + \cdots + ar^{n-1} \quad (1)$

Then $\quad rS_n = ar + ar^2 + ar^3 + \cdots + ar^n \quad (2)$

Subtract (1) from (2):

$$rS_n - S_n = ar^n - a$$

$$S_n(r-1) = a(r^n - 1)$$

Therefore

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

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For values of r such that $-1 < r < 1$, it is often more convenient to use the equivalent formula

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

which is obtained by multiplying both the numerator and the denominator by -1.

Example 24

Find the sum of the first nine terms of the sequence $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \dots$.

Solution

$a = \frac{1}{3}$, $r = \frac{1}{3}$, $n = 9$

$$\therefore \quad S_9 = \frac{\frac{1}{3}\left(1 - (\frac{1}{3})^9\right)}{1 - \frac{1}{3}}$$

$$= \frac{1}{2}\left(1 - \left(\frac{1}{3}\right)^9\right)$$

$$\approx 0.499975$$

Example 25

For the geometric sequence $1, 3, 9, \dots$, find how many terms must be added together to obtain a sum of 1093.

Solution

$a = 1$, $r = 3$, $S_n = 1093$

$$S_n = \frac{a(r^n - 1)}{r - 1} = 1093$$

$$\frac{1(3^n - 1)}{3 - 1} = 1093$$

$$3^n - 1 = 1093 \times 2$$

$$\therefore \quad 3^n = 2187$$

A CAS calculator can be used to find $n = 7$.

Seven terms are required to give a sum of 1093.

Example 26

In the 15-day Tour de Moravia, the cyclists must ride 120 km on the first day, and each successive day they ride 90% of the distance of the previous day.

a How far do they ride in total to the nearest kilometre?

b After how many days will they have ridden half that distance?

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Solution

a $a = 120$, $r = 0.9$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$\therefore \quad S_{15} = \frac{120(1 - (0.9)^{15})}{1 - 0.9}$$

$$= 952.93$$

They ride 953 km.

b $a = 120$, $r = 0.9$, $S_n = 476.5$

$$S_n = \frac{a(1 - r^n)}{1 - r} = 476.5$$

$$\frac{120(1 - (0.9)^n)}{1 - 0.9} = 476.5$$

$$1 - 0.9^n = \frac{476.5 \times 0.1}{120}$$

$$1 - 0.9^n = 0.3971$$

$$0.9^n = 1 - 0.3971$$

$$\therefore \quad 0.9^n = 0.6029$$

A CAS calculator can be used to find $n \approx 4.8$. Thus they pass the halfway mark on the fifth day.

Summary 15E

The sum of the first n terms of a geometric sequence

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

is given by

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{or} \quad S_n = \frac{a(1 - r^n)}{1 - r}$$

Exercise 15E

Example 24

1 Find the specified sum for each of the following geometric series:

a $5 + 10 + 20 + \cdots$ find S_{10}

b $1 - 3 + 9 - \cdots$ find S_6

c $-\frac{4}{3} + \frac{2}{3} - \frac{1}{3} + \cdots$ find S_9

2 Find:

a $2 - 6 + 18 - \cdots + 1458$

b $-4 + 8 - 16 + \cdots - 1024$

c $6250 + 1250 + 250 + \cdots + 2$

Example 25

3 For the geometric sequence $3, 6, 12, \ldots$, find how many terms must be added together to obtain a sum of 3069.

4 For the geometric sequence $24, -12, 6, \ldots$, find how many terms must be added together to obtain a sum of $16\frac{1}{8}$.

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Example 26

5 Gerry owns a milking cow. On the first day he milks the cow, it produces 600 mL of milk. On each successive day, the amount of milk increases by 10%.

- **a** How much milk does the cow produce on the seventh day?
- **b** How much milk does it produce in the first week?
- **c** After how many days will it have produced a total in excess of 10 000 mL?

6 On Monday, William spends 20 minutes playing the piano. On Tuesday, he spends 25 minutes playing, and on each successive day he increases the time he spends playing in the same ratio.

- **a** For how many minutes does he play on Friday?
- **b** How many minutes in total does he play from Monday to Friday?
- **c** On which day of the following week will his total time playing pass 15 hours?

7 A ball dropped from a height of 15 m rebounds from the ground to a height of 10 m. With each successive rebound, it rises to two-thirds of the height of the previous rebound. What total distance will it have travelled when it strikes the ground for the 10th time?

8 An insurance broker makes $15 000 commission on sales in her first year. Each year, she increases her sales by 5%.

- **a** How much commission would she make in her fifth year?
- **b** How much commission would she make in total over 5 years?

9 Andrew invests $1000 at 20% simple interest for 10 years. Bianca invests her $1000 at 12.5% compound interest for 10 years. At the end of 10 years, whose investment is worth more?

10 For a geometric sequence with nth term t_n:

- **a** if $t_3 = 20$ and $t_6 = 160$, find S_5
- **b** if $t_3 = \sqrt{2}$ and $t_8 = 8$, find S_8.

11
- **a** How many terms of the geometric sequence where $t_1 = 1$, $t_2 = 2$, $t_3 = 4, \ldots$ must be taken for $S_n = 255$?
- **b** Let $S_n = 1 + 2 + 4 + \cdots + 2^{n-1}$. Find $\{ n : S_n > 1\,000\,000 \}$.

12 Find $1 - x^2 + x^4 - x^6 + x^8 - \cdots + x^{2m}$, where m is even.

13 A sheet of A4 paper is about 0.05 mm thick. The paper is torn in half, and each half is again torn in half, and this process is repeated for a total of 40 times.

- **a** How high will the stack of paper be if the pieces are placed one on top of the other?
- **b** How many times would the process have to be repeated for the stack to first reach the moon, 384 400 km away?

14 Which would you prefer: $1 million given to you every week for a year, or 1c in the first week, 2c in the second, 4c in the third, and so on, until the end of the year?

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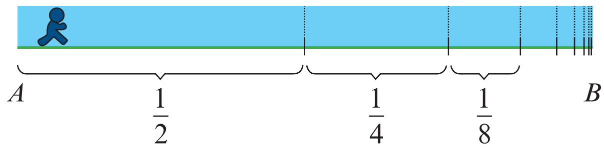
15F Zeno's paradox and infinite geometric series

A runner wants to go from point A to point B. To do this, he would first have to run half the distance, then half the remaining distance, then half the remaining distance, and so on.



The Greek philosopher Zeno of Elea, who lived about 450 BC, argued that since the runner has to complete an infinite number of stages to get from A to B, he cannot do this in a finite amount of time, and so he cannot reach B. In this section we see how to resolve this paradox.

Infinite geometric series

If a geometric sequence has a common ratio with magnitude less than 1, that is, if $-1 < r < 1$, then each successive term is closer to zero. For example, consider the sequence

$$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \dots$$

In Example 24 we found that the sum of the first 9 terms is $S_9 \approx 0.499\,975$. The sum of the first 20 terms is $S_{20} \approx 0.499\,999\,999\,86$. We might conjecture that, as we add more and more terms of the sequence, the sum will get closer and closer to 0.5, that is, $S_n \to 0.5$ as $n \to \infty$.

An infinite series $t_1 + t_2 + t_3 + \cdots$ is said to be **convergent** if the sum of the first n terms, S_n, approaches a limiting value as $n \to \infty$. This limit is called the **sum to infinity** of the series.

If $-1 < r < 1$, then the infinite geometric series $a + ar + ar^2 + \cdots$ is convergent and the sum to infinity is given by

$$S_\infty = \frac{a}{1-r}$$

Proof We know that

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

As $n \to \infty$, we have $r^n \to 0$ and so $\dfrac{ar^n}{1-r} \to 0$. Hence $S_n \to \dfrac{a}{1-r}$ as $n \to \infty$.

Resolution of Zeno's paradox Assume that the runner is travelling at a constant speed and that he takes 1 minute to run half the distance from A to B. Then he takes $\frac{1}{2}$ minute to run half the remaining distance, and so on. The total time taken is

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

This is an infinite geometric series, and the formula gives $S_\infty = \dfrac{a}{1-r} = \dfrac{1}{1-\frac{1}{2}} = 2$.

This fits with our common sense: If the runner takes 1 minute to cover half the distance, then he will take 2 minutes to cover the whole distance.

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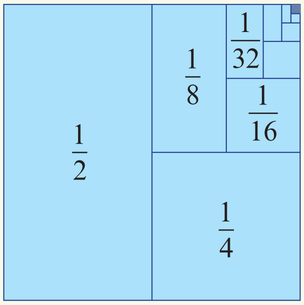
Example 27

Find the sum to infinity of the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$.

Solution

$a = \frac{1}{2}$, $r = \frac{1}{2}$ and so $S_\infty = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$

Note: This result is illustrated by the unit square shown. Divide the square in two, then divide one of the resulting rectangles in two, and so on. The sum of the areas of the rectangles equals the area of the square.



Example 28

A square has a side length of 40 cm. A copy of the square is made so that the area of the copy is 80% of the original. The process is repeated so that each time the area of the new square is 80% of the previous one. If this process is repeated indefinitely, find the total area of all the squares.

Solution

The area of the first square is $40^2 = 1600\text{ cm}^2$.

We have $a = 1600$ and $r = 0.8$, giving

$$S_\infty = \frac{1600}{1 - 0.8} = 8000\text{ cm}^2$$

Example 29

Express the recurring decimal $0.\dot{3}\dot{2}$ as the ratio of two integers.

Solution

$$0.\dot{3}\dot{2} = 0.32 + 0.0032 + 0.000\,032 + \cdots$$

We have $a = 0.32$ and $r = 0.01$, giving

$$S_\infty = \frac{0.32}{0.99} = \frac{32}{99}$$

i.e. $0.\dot{3}\dot{2} = \frac{32}{99}$

Skill-sheet

Exercise 15F

Example 27

1 Find:

a $1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots$

b $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \cdots$

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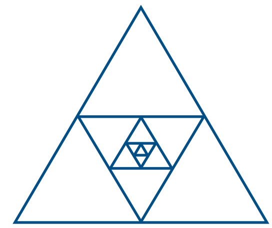
Example 28

2 An equilateral triangle has perimeter p cm. The midpoints of the sides are joined to form another triangle, and this process is repeated. Find the perimeter and area of the nth triangle, and find the limits as $n \to \infty$ of the sum of the perimeters and the sum of the areas of the first n triangles.



3 A rocket is launched into the air so that it reaches a height of 200 m in the first second. Each subsequent second it gains 6% less height. Find how high the rocket will climb.

4 A patient has an infection that, if it exceeds a certain level, will kill him. He is given a drug to inhibit the spread of the infection. The drug acts such that each day the level of infection only increases by 65% of the previous day's level. The level of infection on the first day is 450, and the critical level is 1280. Will the infection kill him?

5 A man can walk 3 km in the first hour of a journey, but in each succeeding hour walks half the distance covered in the preceding hour. Can he complete a journey of 6 km? Where does this problem cease to be realistic?

6 A frog standing 10 m from the edge of a pond sets out to jump towards it. Its first jump is 2 m, its second jump is $1\frac{1}{2}$ m, its third jump is $1\frac{1}{8}$ m, and so on. Show that the frog will never reach the edge of the pond.

7 A computer virus acts in such a way that on the first day it blocks out one-third of the area of the screen of an infected computer. Each successive day it blocks out more of the screen: an area one-third of that it blocked the previous day. If this continues indefinitely, what percentage of the screen will eventually be blocked out?

8 A stone is thrown so that it skips across the surface of a lake. If each skip is 30% less than the previous skip, how long should the first skip be so that the total distance travelled by the stone is 40 m?

9 A ball dropped from a height of 15 m rebounds from the ground to a height of 10 m. With each successive rebound it rises two-thirds of the height of the previous rebound. If it continues to bounce indefinitely, what is the total distance it will travel?

Example 29

10 Express each of the following periodic decimals as the ratio of a pair of integers:

a $0.\dot{4}$ **b** $0.0\dot{3}$ **c** $10.\dot{3}$ **d** $0.0\dot{3}\dot{5}$ **e** $0.\dot{9}$ **f** $4.\dot{1}$

11 The sum of the first four terms of a geometric series is 30 and the sum to infinity is 32. Find the first two terms.

12 Find the third term of a geometric sequence that has a common ratio of $-\frac{1}{4}$ and a sum to infinity of 8.

13 Find the common ratio of a geometric sequence with first term 5 and sum to infinity 15.

14 For any number $x > 2$, show that there is an infinite geometric series such that $a = 2$ and the sum to infinity is x.

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Review

Chapter summary

Assign-ment

Nrich

- The nth term of a sequence is denoted by t_n.
- A **recurrence relation** enables each subsequent term to be found from previous terms. A sequence specified in this way is said to be defined **recursively**.

  e.g. $t_1 = 1, \quad t_n = t_{n-1} + 2$
- A sequence may also be defined by a rule that is stated in terms of n.

  e.g. $t_n = 2n$

Arithmetic sequences and series

- An **arithmetic sequence** has a rule of the form

$$t_n = a + (n-1)d$$

  where a is the first term and d is the **common difference** (i.e. $d = t_k - t_{k-1}$ for all $k > 1$).
- The **arithmetic mean** of two numbers a and b is $\dfrac{a+b}{2}$.
- The sum of the terms in an arithmetic sequence is called an **arithmetic series**.
- The sum of the first n terms of an arithmetic sequence is given by

$$S_n = \frac{n}{2}\big(2a + (n-1)d\big) \quad \text{or} \quad S_n = \frac{n}{2}(a + \ell), \text{ where } \ell = t_n$$

Geometric sequences and series

- A **geometric sequence** has a rule of the form

$$t_n = ar^{n-1}$$

  where a is the first term and r is the **common ratio** (i.e. $r = \dfrac{t_k}{t_{k-1}}$ for all $k > 1$).
- The **geometric mean** of two numbers a and b is $\sqrt{ab}$.
- The sum of the terms in a geometric sequence is called a **geometric series**.
- For $r \neq 1$, the sum of the first n terms of a geometric sequence is given by

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{or} \quad S_n = \frac{a(1 - r^n)}{1 - r}$$

- For $-1 < r < 1$, the sum S_n approaches a limiting value as $n \to \infty$, and the series is said to be **convergent**. This limit is called the **sum to infinity** and is given by $S_\infty = \dfrac{a}{1-r}$.

Short-answer questions

1 Find the first six terms of the following sequences:

a $t_1 = 3, \ t_n = t_{n-1} - 4$

b $t_1 = 5, \ t_n = 2t_{n-1} + 2$

2 Find the first six terms of the following sequences:

a $t_n = 2n$

b $t_n = -3n + 2$

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3 Nick invests \$5000 at 5% p.a. compound interest at the beginning of the year. At the beginning of each of the following years, he puts a further \$500 into the account.

a Write down the amount of money in the account at the end of each of the first two years.

b Set up a recurrence relation to generate the sequence for the investment.

4 The 4th term of an arithmetic sequence is 19 and the 7th term is 43. Find the 20th term.

5 For an arithmetic sequence with $t_5 = 0.35$ and $t_9 = 0.15$, find t_{14}.

6 For an arithmetic sequence with $t_6 = -24$ and $t_{14} = 6$, find S_{10}.

7 For the arithmetic sequence $-5, 2, 9, \dots$, find $\{ n : S_n = 402 \}$.

8 The 6th term of a geometric sequence is 9 and the 10th term is 729. Find the 4th term.

9 One thousand dollars is invested at 3.5% p.a. compounded annually. Find the value of the investment after n years.

10 The first term of a geometric sequence is 9 and the third term is 4. Find the possible values for the second and fourth terms.

11 The sum of three consecutive terms of a geometric sequence is 24 and the sum of the next three terms is also 24. Find the sum of the first 12 terms.

12 Find the sum of the first eight terms of the geometric sequence with first term 6 and common ratio -3.

13 Find the sum to infinity of $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$.

14 The numbers x, $x + 4$, $2x + 2$ are three successive terms of a geometric sequence. Find the value of x.

Extended-response questions

1 A do-it-yourself picture-framing kit is available in various sizes. Size 1 contains 0.8 m of moulding, size 2 contains 1.5 m, size 3 contains 2.2 m, and so on.

a Form the sequence of lengths of moulding.

b Is the sequence of lengths of moulding an arithmetic sequence?

c Find the length of moulding contained in the largest kit, size 12.

2 A firm proposes to sell coated seeds in packs containing the following number of seeds: $50, 75, 100, 125, \dots$.

a Is this an arithmetic sequence?

b Find a formula for the nth term.

c Find the number of seeds in the 25th size packet.

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3 A number of telegraph poles are to be placed in a straight line between two towns, A and B, which are 32 km apart. The first is placed 5 km from town A, and the last is placed 3 km from town B. The poles are placed so that the intervals starting from town A and finishing at town B are

$$5,\ 5-d,\ 5-2d,\ 5-3d,\ \ldots,\ 5-6d,\ 3$$

There are seven poles. How far is the fifth pole from town A, and how far is it from town B?

4 A new electronic desktop telephone exchange, for use in large organisations, is available in various sizes.

Size 1 can handle 20 internal lines.
Size 2 can handle 36 internal lines.
Size 3 can handle 52 internal lines.
Size 4 can handle 68 internal lines, and so on.
Size n can handle T_n internal lines.

a Continue the sequence up to T_8.
b Write down a formula for T_n in terms of n.
c A customer needs an exchange to handle 196 lines. Is there a version of the desktop exchange which will just do this? If so, which size is it? If not, which is the next largest size?

5 A firm makes nylon thread in the following deniers (thicknesses):

$$2, 9, 16, 23, 30, \ldots$$

a Find the denier number, D_n, of the firm's nth thread in order of increasing thickness.

A request came in for some very heavy 191 denier thread, but this turned out to be one stage beyond the thickest thread made by the firm.

b How many different thicknesses does the firm make?

6 A new house appears to be slipping down a hillside. The first year it slipped 4 mm, the second year 16 mm, and the third year 28 mm. If it goes on like this, how far will it slip during the 40th year?

7 Anna sends 16 Christmas cards the first year, 24 the second year, 32 the next year, and so on. How many Christmas cards will she have sent altogether after 10 years if she keeps increasing the number sent each year in the same way?

8 Each time Lee rinses her hair after washing it, the result is to remove a quantity of shampoo from her hair. With each rinse, the quantity of shampoo removed is one-tenth of that removed by the previous rinse.

a If Lee washes out 90 mg of shampoo with the first rinse, how much will she have washed out altogether after six rinses?
b How much shampoo do you think was present in her hair at the beginning?

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9 A prisoner is trapped in an underground cell, which is inundated by a sudden rush of water. The water comes up to a height of 1 m, which is one-third of the height of the ceiling (3 m). After an hour another inundation occurs, and the water level in the cell rises by $\frac{1}{3}$ m. After a second hour another inundation raises the water level by $\frac{1}{9}$ m. If this process continues for 6 hours, write down:

a the amount the water level will rise at the end of the sixth hour

b the total height of the water level then.

If this process continues, do you think the prisoner, who cannot swim, will drown? Why?

10 After an undetected leak in a storage tank, the staff at an experimental station were subjected to 500 curie hours of radiation the first day, 400 the second day, 320 the third day, and so on. Find the number of curie hours they were subjected to:

a on the 14th day

b during the first five days of the leak.

11 A rubber ball is dropped from a height of 81 m. Each time it strikes the ground, it rebounds two-thirds of the distance through which it has fallen.

a Find the height that the ball reaches after the sixth bounce.

b Assuming that the ball continues to bounce indefinitely, find the total distance travelled by the ball.

12 In payment for loyal service to the king, a wise peasant asked to be given one grain of rice for the first square of a chessboard, two grains for the second square, four for the third square, and so on for all 64 squares of the board. The king thought that this seemed fair and readily agreed, but was horrified when the court mathematician informed him how many grains of rice he would have to pay the peasant. How many grains of rice did the king have to pay? (Leave your answer in index form.)

13 **a** In its first month of operation, a cement factory, A, produces 4000 tonnes of cement. In each successive month, production rises by 250 tonnes per month. This growth in production is illustrated for the first five months in the table shown.

Month number (n)	1	2	3	4	5
Cement produced (tonnes)	4000	4250	4500	4750	5000

i Find an expression, in terms of n, for the amount of cement produced in the nth month.

ii Find an expression, in terms of n, for the total amount of cement produced in the first n months.

iii In which month is the amount of cement produced 9250 tonnes?

iv In month m, the amount of cement produced is T tonnes. Find m in terms of T.

v The total amount of cement produced in the first p months is 522 750 tonnes. Find the value of p.

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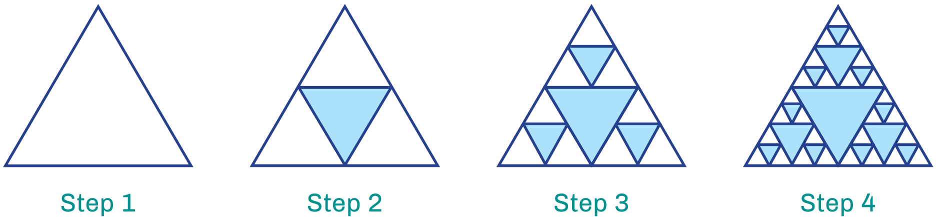
Review

b A second factory, B, commences production at exactly the same time as the first. In its first month it produces 3000 tonnes of cement. In each successive month, production increases by 8%.

i Find an expression for the total amount of cement produced by this factory after n months.

ii Let Q_A be the total amount of cement produced by factory A in the first n months and Q_B the total amount of cement produced by factory B in the first n months. Find an expression in terms of n for $Q_B - Q_A$ and find the smallest value of n for which $Q_B - Q_A \geq 0$.

14 The following diagrams show the first four steps in forming the Sierpinski triangle.



The diagrams are produced in the following way:

Step 1 Start with an equilateral triangle of side length 1 unit.

Step 2 Subdivide it into four smaller congruent equilateral triangles and colour the central one blue.

Step 3 Repeat Step 2 with each of the smaller white triangles.

Step 4 Repeat again.

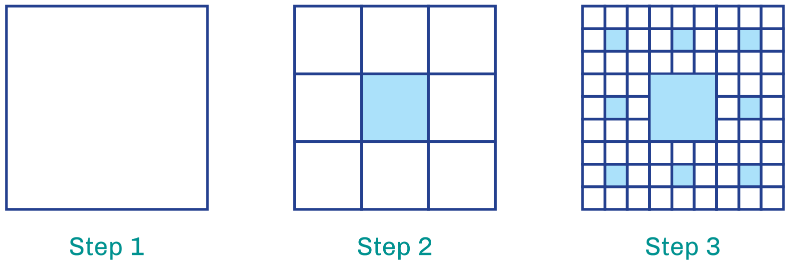
a How many white triangles are there in the nth diagram (that is, after Step n)?

b What is the side length of a white triangle in the nth diagram?

c What fraction of the area of the original triangle is still white in the nth diagram?

d Consider what happens as n approaches infinity.

15 The Sierpinski carpet is formed from a unit square in a way similar to the Sierpinski triangle. The following diagrams show the first three steps.



a How many white squares are there in the nth diagram (that is, after Step n)?

b What is the length of a side of a white square in the nth diagram?

c What is the fraction of the area of square which is white in the nth diagram?

d Consider what happens as n approaches infinity.

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16 Revision of Chapters 12–15

16A Short-answer questions

1 Convert to radians:

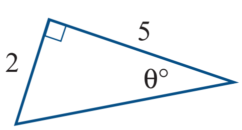
a 60° **b** 270° **c** 140°

2 Evaluate:

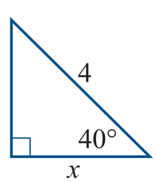
a $\sin\left(-\frac{\pi}{2}\right)$ **b** $\cos\left(\frac{3\pi}{2}\right)$ **c** $\tan(3\pi)$ **d** $\tan\left(-\frac{\pi}{2}\right)$

3 Find the value of the pronumeral:

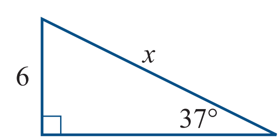
a



b



c



4 Find:

a $\sin(2\pi - \theta)$ if $\sin\theta = 0.3$

b $\cos(-\theta)$ if $\cos\theta = -0.5$

c $\tan(\pi + \theta)$ if $\tan\theta = 1.6$

d $\sin(\pi + \theta)$ if $\sin\theta = 0.6$

e $\sin\left(\frac{\pi}{2} - \theta\right)$ if $\cos\theta = 0.1$

f $\cos\theta$ if $\sin\theta = \frac{3}{5}$ and $0 < \theta < \frac{\pi}{2}$

5 Write down the exact values of:

a $\sin\left(\frac{2\pi}{3}\right)$ **b** $\cos\left(\frac{5\pi}{6}\right)$ **c** $\tan\left(\frac{-\pi}{4}\right)$

d $\sin\left(\frac{-7\pi}{6}\right)$ **e** $\cos\left(\frac{-7\pi}{4}\right)$ **f** $\tan\left(\frac{5\pi}{3}\right)$

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6 Consider the function $f(x) = 2\cos\left(\frac{x}{2}\right)$.

a State the period and amplitude.

b Sketch a graph for one complete cycle.

c Give the sequence of transformations which takes the graph of $y = \cos x$ to the graph of $y = f(x)$.

7 Sketch $f(x) = -3\sin(2\pi x)$ for $x \in [0, 2]$.

8 Solve these equations for $\theta \in [-2\pi, 2\pi]$:

a $\cos\theta = -\frac{\sqrt{3}}{2}$ **b** $\sqrt{2}\sin\theta = 1$ **c** $\sin(2\theta) = -\frac{1}{2}$ **d** $\tan\theta = -\sqrt{3}$

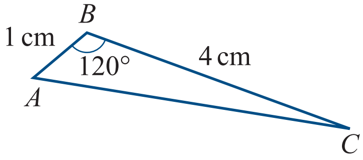
9 Sketch $y = 2\sin 2\left(x - \frac{\pi}{4}\right) + 1$ for $x \in [0, 2\pi]$ and determine the x-axis intercepts.

10 Sketch the graph of $y = \frac{1}{2}\tan(2x)$ for $x \in [-\pi, \pi]$.

11 For acute angles A and B such that $\sin A = \frac{4}{5}$ and $\sin B = \frac{1}{\sqrt{2}}$, evaluate $\sin(A - B)$.

12 Prove that $\frac{1 - \cos(2A)}{1 + \cos(2A)} = \tan^2 A$.

13 The triangle ABC has side lengths $AB = 1$ cm and $BC = 4$ cm. The magnitude of angle ABC is 120°.



a Find the length of AC.

b Find the sine of angle BAC.

c Find the area of the triangle.

14 A sector cut from a circle of radius 5 cm has a perimeter of 16 cm. Find the area of this sector.

15 A triangle ABC satisfies $\sin A : \sin B = 1 : 2$. If $BC = 6$ cm, find AC.

16 Simplify the following, expressing the answers in positive-index form:

a $(-2a^2)^3 \times 3a^4$ **b** $\frac{5a^4 \times 2ab^2}{20a^2b^4}$ **c** $\frac{(xy^{-2})^{-1}}{y} \times \frac{3x^{-1}y^2}{4(xy)^3}$

d $\left(\frac{4a^2}{ab}\right)^3 \div (2ab^{-1})^3$ **e** $\sqrt{x^{-1}y^2} \times \left(\frac{y}{x}\right)^{-\frac{1}{3}}$ **f** $\sqrt{2x-1} \times (2x-1)^{-1}$

17 Simplify and evaluate:

a $\left(\frac{3}{5}\right)^{-2}$ **b** $\left(\frac{4^2}{2^6}\right)^{-2}$ **c** $\frac{27^2 \times 9^3}{81^2}$ **d** $(-27)^{-\frac{1}{3}}$

18 Simplify $\frac{9^{2n} \times 8^n \times 16^n}{6^n}$.

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19 Sketch graphs the following functions. Give equations of asymptotes and y-axis intercepts, and state the range of each.

a $f(x) = 2^x - 3$ **b** $f(x) = -3 \times 2^x + 1$

20 Solve these equations for x:

a $4^x = 8^{x-1}$ **b** $4^x = 5 \times 2^x - 4$

21 Find the common difference for the arithmetic sequence with general term $t_n = 6n - 2$.

22 Find the sum of the first 12 terms of the sequence $\frac{-3}{4}, \frac{-11}{4}, \frac{-19}{4}, \ldots$.

23 For a certain arithmetic sequence, the sum of the first n terms is given by $S_n = 2n^2 + 3n$.

a Obtain and simplify an expression which gives S_{n-1} in terms of n.

b Find an expression for the nth term of the sequence in terms of n.

c Find the value of the first term.

d Find the value of the common difference.

24 How many terms are in the sequence $3, 6, 12, 24, \ldots$ if the sum of the terms is 189?

25 A frog's first jump is 4 m, the second is 2 m, the third is 1 m, and so on. If the frog continues to jump indefinitely, how far will it get?

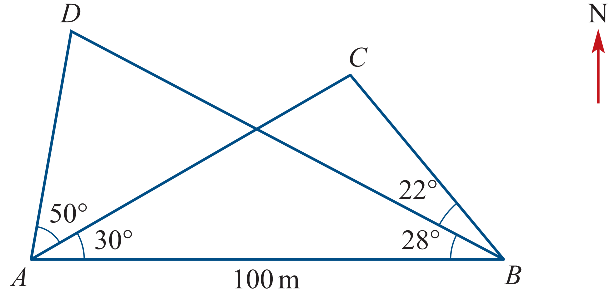
26 A triangle is such that the lengths of its sides form the first three terms in a geometric sequence. Given that the length of the longest side is 36 cm and the perimeter is 76 cm, find the length of the shortest side.

27 Of the first 100 natural numbers, find the sum of those that:

a are even **b** are not divisible by 3.

16B Extended-response questions

1 A triangular survey method was used to calculate the distance from a point A, on land, to two platforms C and D, out to sea. Sightings were made from two accessible points A and B, which are 100 m apart.



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a Find the distance from A to each of the platforms (to the nearest metre).

b Find the distance between the two platforms (to the nearest metre).

c Find the bearing of the platform D from the platform C (to the nearest degree).

2 The height of the tide, h metres, at a harbour at any time during a 24-hour period is given by the rule $h(t) = 10 + 4\sin(15t)^\circ$, where t is measured in hours.

a Sketch the graph of h against t for $0 \leq t \leq 24$.

b Find the times at which $h = 13$ during the 24-hour period.

c A boat can leave the harbour when the height of the tide is at 11 metres or more. State the times during the 24 hours when the boat can leave the harbour.

3 Medical researchers studying the growth of a strain of bacteria observe that the number of bacteria present after t hours is given by the formula $N(t) = 40 \times 2^{1.5t}$.

a State the number of bacteria present at the start of the experiment.

b State the number of bacteria present after:

i 2 hours

ii 4 hours

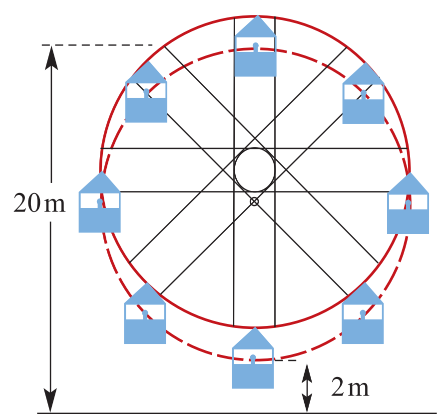
iii 12 hours

c Sketch the graph of N against t.

d How many minutes does it take for the number of bacteria to double?

4 For a ride on a Ferris wheel, the height above the ground, h metres, of a person at time t seconds is given by

$$h(t) = 11 + 9\cos\left(\frac{\pi}{30}(t - 10)\right)$$



a How long does it take for the Ferris wheel to make one complete revolution?

b Sketch the graph of h against t for one revolution.

c State the range of the function.

d At what time(s) is the person at a height of 2 metres above the ground?

e At what time(s) is the person at a height of 15.5 metres above the ground?

5 The voltage, V, in a circuit after t seconds is given by $V = 120\cos(60\pi t)$.

a Sketch the graph of V against t for one cycle.

b Find the first time the voltage is 60.

c Find all times at which the voltage is maximised.

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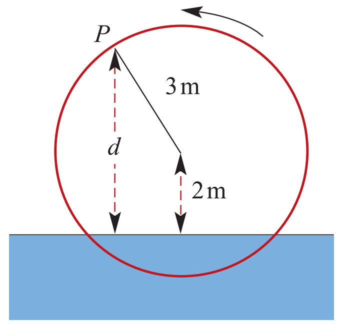
6 The figure shows a waterwheel rotating at 4 revolutions per minute. The distance, d in metres, of a point P from the surface of the water as a function of time, t in seconds, can be modelled by a rule of the form

$$d = a + b\sin c(t - h)$$



a Find:

i the period

ii the amplitude

iii c

b If $d = 0$ when $t = 0$, find h.

c Sketch the graph of d against t for $0 \le t \le 30$.

7 A forest fire has burned out 30 hectares by 11 a.m. It then spreads according to the formula

$$h(t) = 30(1.65)^t$$

where $h(t)$ is the area burned (in hectares) at time t (in hours after 11 a.m.).

a Find h when:

i $t = 0$

ii $t = 1$

iii $t = 2$

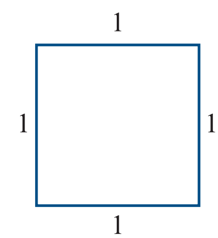
b Find k such that $h(t + 1) = k\,h(t)$.

c How long does it take for 900 hectares to be burned?

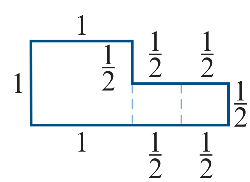
d Sketch the graph of h against t.

8 The square shown has each side of length 1 unit.

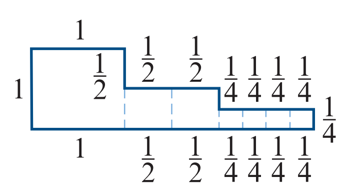
a The perimeter of the square is denoted by P_1. What is the value of P_1?



b A new figure is formed by joining two squares of side length $\frac{1}{2}$ to this square, as shown. The new perimeter is denoted by P_2. What is the value of P_2?



c What is the perimeter, P_3, of this figure?



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d It is known that $P_1, P_2, P_3, \ldots$ are the terms of an arithmetic sequence with first term P_1. What is the common difference?

e **i** Find P_4.

ii Find P_n in terms of P_{n-1}.

iii Find P_n in terms of n.

iv Draw the diagram of the figure corresponding to P_4.

9 A bowl of water is initially at boiling point (i.e. 100°C). The temperature, θ°C, of the water t minutes after beginning to cool is $\theta = 80(2^{-t}) + 20$.

a Copy and complete this table:

t	0	1	2	3	4	5
θ						

b Draw a graph of θ against t.

c At what time is the temperature 60°C?

d Find θ when $t = 3.5$.

10 A, B and C are three countries. Each of them now has a population of 70 million.

- Country A's population grows uniformly by 30 million in every period of 10 years.
- Country B's population grows uniformly by 50 million in every period of 10 years.
- Country C's population is multiplied by 1.3 every 10 years.

a Give an equation for the population of each country at time t.

b On the same set of axes, carefully draw the graphs of the three equations.

c From your graph find out when the population of C overtakes:

i the population of A

ii the population of B.

11 An estimate for the population of the Earth, P in billions, is

$$P = 4 \times 2^{\frac{(t-1975)}{35}}$$

where t is the year.

a Evaluate P for:

i $t = 1975$ **ii** $t = 1995$ **iii** $t = 2005$

b When will the population of the Earth be twice that in 1997?

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Revision

12 Two tanks simultaneously start to leak. Tank A contains $V_1(t)$ litres of water and tank B contains $V_2(t)$ litres of water, where

$$V_1(t) = 1000 \times 3^{\frac{-t}{10}} \qquad t \geq 0$$
$$V_2(t) = 1000 - 40t \qquad 0 \leq t \leq 25$$

and t is the time after the tanks start to leak.

a Find $V_1(0)$ and $V_2(0)$.

b Sketch the graphs of $y = V_1(t)$ and $y = V_2(t)$ for $0 \leq t \leq 25$.

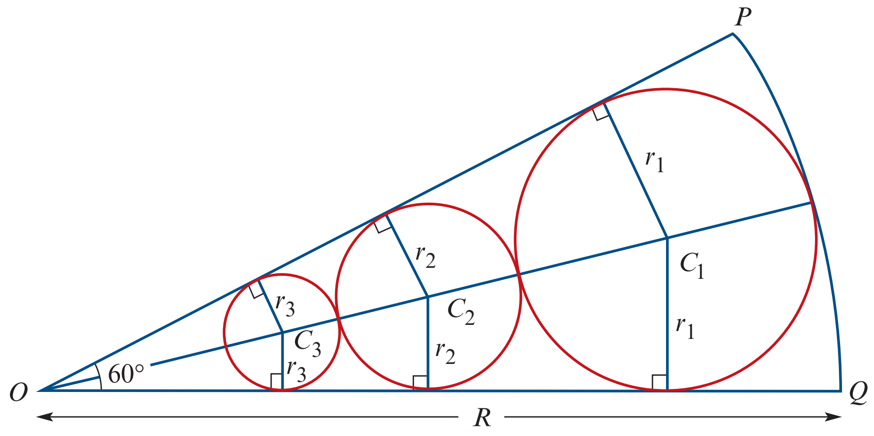
c How much water is in tank A when tank B is empty?

d Find the times at which the two tanks have equal amounts of water in them.

13 **a** In the diagram, OPQ is a sector of radius R. A circle with centre C_1 and radius r_1 is inscribed in this sector.

i Express OC_1 in terms of R and r_1.

ii Show that $\dfrac{r_1}{OC_1} = \dfrac{1}{2}$ and hence express r_1 in terms of R.



b Another circle, centre C_2, is inscribed in the sector as shown.

i Express OC_2 in terms of r_2 and R.

ii Express r_2 in terms of R.

c Circles with centres at $C_3, C_4, C_5, \ldots$ are constructed in a similar way. Their radii are $r_3, r_4, r_5, \ldots$ respectively. It is known that $r_1, r_2, r_3, \ldots$ is a geometric sequence.

i Find the common ratio.

ii Find r_n.

iii Find the sum to infinity of the sequence, and interpret the result geometrically.

iv Find, in terms of R and π, the sum to infinity of the areas of the circles with radii $r_1, r_2, r_3, \ldots$.

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14 Two companies produce the same chemical. For Company A, the number of tonnes produced increases by 80 tonnes per month. For Company B, production increases by 4% per month. Each company produced 1000 tonnes in January 2013.
(Let n be the number of months of production. Use $n = 1$ for January 2013.)

a Find, to the nearest tonne where appropriate:

i the production of Company A in the nth month

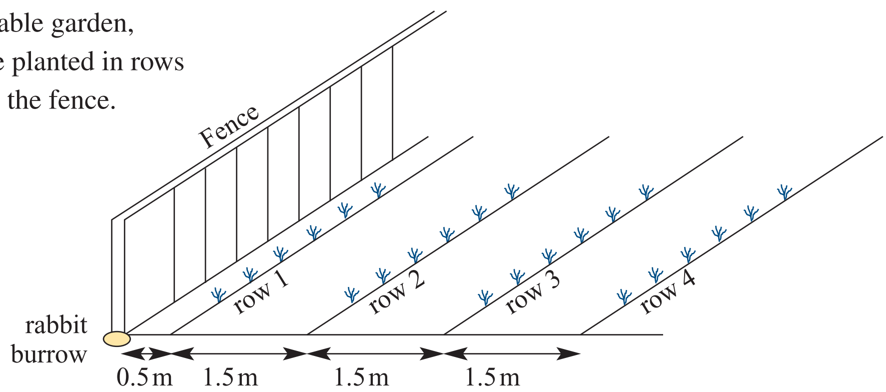
ii the production of each company in December 2014 (i.e. for $n = 24$)

iii the total production of Company A over n months (starting with $n = 1$ for January 2013)

iv the total production of each company for the period from January 2013 to December 2014 inclusive.

b Find in which month of which year the total production of Company A passed 100 000 tonnes.

15 In a vegetable garden, carrots are planted in rows parallel to the fence.

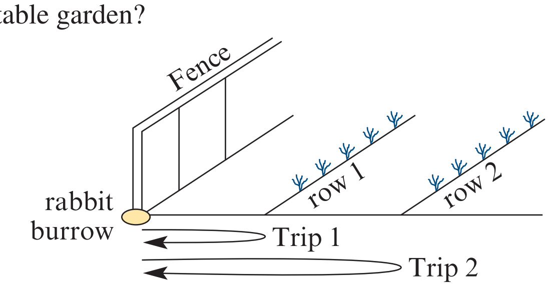


a Calculate the distance between the fence and the 10th row of carrots.

b If t_n represents the distance between the fence and the nth row, find a formula for t_n in terms of n.

c Given that the last row of carrots is less than 80 m from the fence, what is the largest number of rows possible in this vegetable garden?

d A systematic rabbit has its burrow under the fence as shown in the diagram. The rabbit runs to the first row, takes a carrot and returns it to the burrow. The rabbit then runs to the second row, takes a carrot and returns it to the burrow.



The rabbit continues in this way until it has 15 carrots. Calculate the shortest distance the rabbit has to run to accomplish this.

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Revision

16 In a certain country, the total population and grain production are recorded each year:

- In December 1996, the population of the country was 12.5 million.
- In 1996, the grain production was 10 million tonnes.

It was found that, since then, the population has grown by 5% each year and grain production has increased by 0.9 million tonnes each year.

Let P_1 denote the population in December 1996, let P_2 denote the population in December 1997 and, in general, let P_n denote the population $n - 1$ years after December 1996.

Let t_1 denote the grain production in 1996, let t_2 denote the grain production in 1997 and, in general, let t_n denote the grain production in the $(n - 1)$st year after 1996.

a Find, in millions of tonnes, the grain production in:

i 2002 **ii** 2009

b Find an expression for t_n.

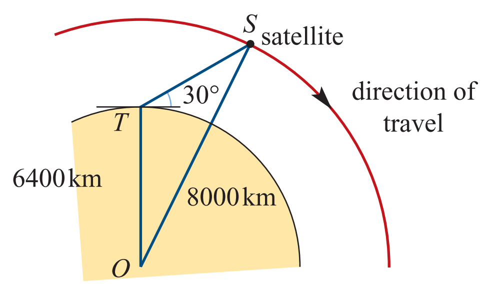
c Find the total grain production for the 20 years starting 1996.

d How many years does it take for the grain production to double?

e Find an expression for P_n.

f How many years does it take for the population to double?

17 A satellite travelling in a circular orbit 1600 km above the Earth is due to pass directly over a tracking station at 12 p.m. Assume that the satellite takes two hours to make an orbit and that the radius of the Earth is 6400 km.

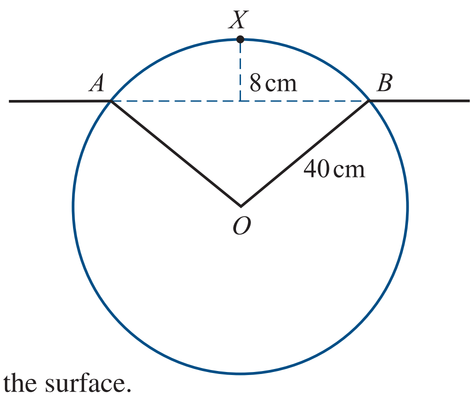


a If the tracking station antenna is aimed at 30° above the horizon, at what time will the satellite pass through the beam of the antenna?

b Find the distance between the satellite and the tracking station at 12:06 p.m.

c At what angle above the horizon should the antenna be aimed so that its beam will intercept the satellite at 12:06 p.m.?

18 The figure shows the circular cross-section of a uniform log of radius 40 cm floating in water. The points A and B are on the surface of the water and the highest point X is 8 cm above the surface.



a Show that the magnitude of $\angle AOB$ is approximately 1.29 radians.

b **i** Find the length of arc AXB.

ii Find the area of the cross-section below the surface.

iii Find the percentage of the volume of the log below the surface.

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19 A river gate is used to control the height of water in a river.

- On one side of the gate, the height of the water is subject to tides. The height of the water (in metres) on this side is given by

$$h_1(t) = 18 + 10\sin\left(\frac{\pi}{6}t\right)$$

where t is the time in hours past midnight.

- On the other side of the gate, the height of the water (in metres) is rising according to the rule

$$h_2(t) = 8 + 6t$$

where t is the time in hours past midnight.

a Sketch the graphs of $y = h_1(t)$ and $y = h_2(t)$, for $0 \le t \le 6$, on the one set of axes.

b Find the time at which $h_1(t) = h_2(t)$.

c When the water levels on both sides of the gate are the same, the gate is opened and the height of the water on both sides is given by $y = h_1(t)$.

i The gate is closed again when $h_1(t)$ reaches its minimum value. At what time does this happen?

ii Water flows in on the non-tidal side so that the height increases by 6 metres every hour. What will be the height of the river t hours after the gates have been closed?

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17 Differentiation and antidifferentiation of polynomials

In this chapter

Syllabus references

Topics: Rates of change; The concept of the derivative; Computation of derivatives; Properties of derivatives; Anti-derivatives

Subtopics: 2.3.1 – 2.3.15, 2.3.22

Throughout this book, we have been looking at situations where there is a relationship between two variables. We have developed and applied the idea that one variable, say y, is a function of another variable, say x. We have also represented such relationships graphically.

This graphical representation can be used to see how the relationship is changing.

In applications, how the relationship is changing is of critical importance in establishing how accurately a given rule models the relationship between the variables in question. For example, if x increases, does y also increase, or does it decrease, or remain unaltered? And, if it does change, does it do so consistently, quickly, slowly, indefinitely, etc.?

In this chapter, we begin our study of calculus, which will enable us to talk about these ideas more precisely. The discovery of calculus is attributed to two great mathematicians – Isaac Newton and Gottfried Leibniz – in the late seventeenth century. Calculus is now one of the most fundamental areas of mathematics, with applications in a wide range of fields including economics, meteorology and computer graphics.

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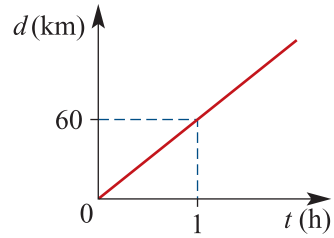
17A Average rate of change

If a car is moving at a constant speed of 60 km/h, then the graph of distance travelled, d km, against time taken, t hours, is a straight line with gradient 60. The gradient is the rate of change of distance with respect to time.



However, many moving objects do not travel with constant speed. For example, the speedometer of a car being driven in city traffic rarely stays still for long.

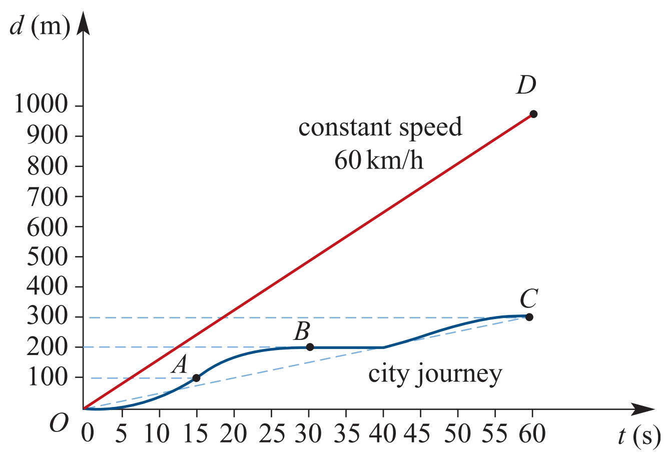
Similarly, not all functions are linear, and so not all functions have a constant rate of change. For a function that is non-linear, the rate of change of the function varies, and may in fact be different for every different point on the graph of the function.

Average speed

We will use a distance–time graph to illustrate the idea of average speed.

The graph below shows the motion of two cars both travelling in a straight line away from a fixed point O, where d is the distance travelled (in metres) at time t (in seconds).

The straight-line graph through D shows a constant speed of 60 km/h. By comparison, the graph through points A, B and C shows a motorist travelling at varying speeds. The motorist accelerates to reach 60 km/h at A before slowing for the lights at B, where there is a 10-second standstill. There is then another short burst of speed before another standstill at C.



Although we do not know the actual speed of the car travelling in the city at any particular time (other than when it is stationary), we can work out the average speed of the car over the full 60 seconds.

The average speed is given by

$$\frac{\text{distance travelled}}{\text{time taken}} = \frac{300}{60}$$

which gives an average speed of 5 metres per second. The average speed is the gradient of the line OC.

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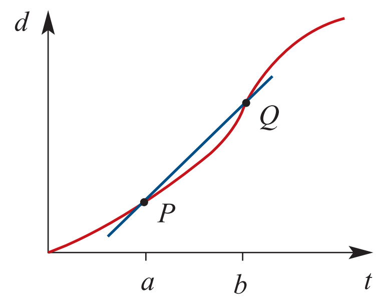
The average speed may also be calculated for any given time interval. For example, for the time interval from $t = 15$ to $t = 30$, the average speed is given by the gradient of the line joining points A and B. This is $\frac{100}{15} = 6\frac{2}{3}$ metres per second.

In general:

$$\text{average speed} = \frac{\text{total distance travelled}}{\text{total time taken}}$$

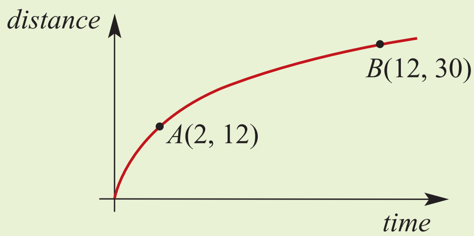
So the average speed of an object for $a \le t \le b$ is given by the gradient of the line passing through points P and Q.



Example 1

The graph of distance travelled (metres) against time (seconds) for the motion of an object is shown.

Find the average speed of the object in m/s over the interval from $t = 2$ to $t = 12$.



Solution

$$\text{Average speed} = \frac{\text{total distance travelled}}{\text{total time taken}}$$

$$= \frac{30 - 12}{12 - 2}$$

$$= \frac{18}{10}$$

$$= 1.8 \text{ m/s}$$

Average rate of change for a function

The line which passes through two points on a curve is called a **secant**.

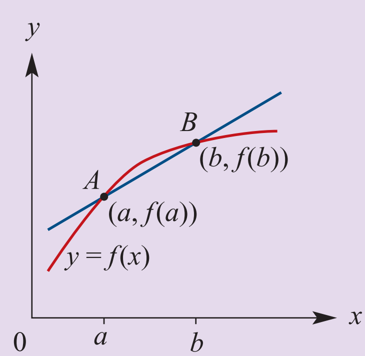
The line segment joining two points on a curve is called a **chord**.

Average rate of change

For any function $y = f(x)$, the **average rate of change** of y with respect to x over the interval $[a, b]$ is the gradient of the line through $A(a, f(a))$ and $B(b, f(b))$ (secant AB).

That is,

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a}$$



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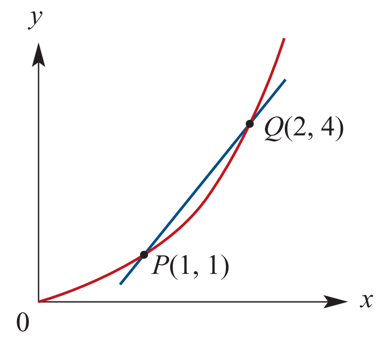
For example, for the function with the graph shown, the average rate of change of y with respect to x over the interval $[1, 2]$ is given by the gradient of the secant PQ:

$$\text{gradient} = \frac{4-1}{2-1} = 3$$



Example 2

Find the average rate of change of the function with rule $f(x) = x^2 - 2x + 5$ as x changes from 1 to 5.

Solution

$$\text{Average rate of change} = \frac{\text{change in } y}{\text{change in } x}$$

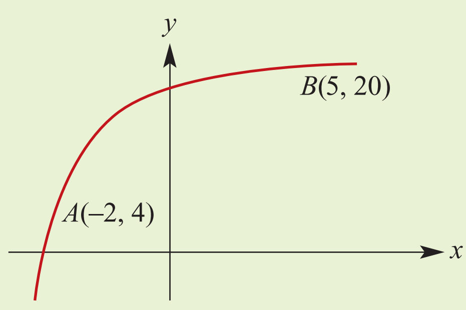
$f(1) = (1)^2 - 2(1) + 5 = 4$

$f(5) = (5)^2 - 2(5) + 5 = 20$

$$\text{Average rate of change} = \frac{20-4}{5-1} = 4$$

Example 3

Find the average rate of change of the function depicted in the graph for the interval $[-2, 5]$.



Solution

Average rate of change for the interval $[-2, 5]$

$$= \frac{20-4}{5-(-2)}$$

$$= \frac{16}{7}$$

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Example 4

The air temperature, T°C, at a weather station on a particular evening is modelled by the equation $T = \dfrac{600}{t^2 + 2t + 30}$, where t is the time in hours after 6 p.m.

a Find the temperature at 6 p.m.

b Find the temperature at midnight.

c Find the average rate of change of the air temperature from 6 p.m. until midnight.

Solution

a At 6 p.m., $t = 0$. Hence

$$T = \frac{600}{(0)^2 + 2(0) + 30} = 20°\text{C}$$

b At midnight, $t = 6$. Hence

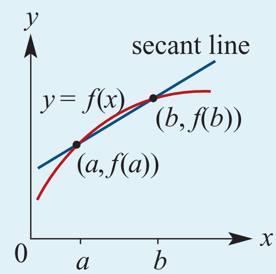
$$T = \frac{600}{(6)^2 + 2(6) + 30} = \frac{100}{13} = 7.69°\text{C} \quad \text{(correct to two decimal places)}$$

c Average rate of change of temperature $= \dfrac{\frac{100}{13} - 20}{6 - 0} = -\dfrac{80}{39} = -2.05$°C per hour

Summary 17A

- The line which passes through two points on a curve is called a **secant**.
- The line segment joining two points on a curve is called a **chord**.
- For a function $y = f(x)$, the **average rate of change** of y with respect to x over the interval $[a, b]$ is the gradient of the secant line through $(a, f(a))$ and $(b, f(b))$.
  That is,

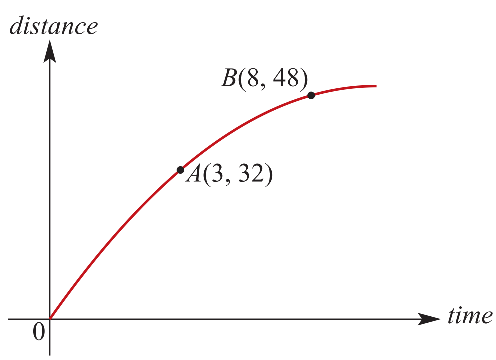
$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a}$$



Exercise 17A

Example 1

1 The graph of distance travelled (metres) against time (seconds) for the motion of an object is shown. Find the average speed of the object in m/s over the interval from $t = 3$ to $t = 8$.



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Example 2

2 For each function, find the average rate of change over the stated interval:

a $f(x) = 2x + 5, \quad x \in [0, 3]$

b $f(x) = 3x^2 + 4x - 2, \quad x \in [-1, 2]$

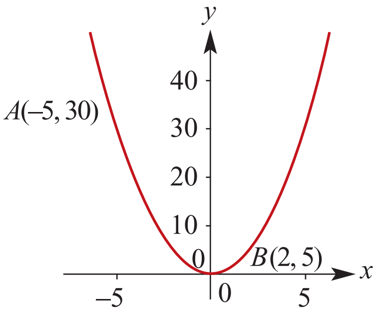
c $f(x) = \dfrac{2}{x-3} + 4, \quad x \in [4, 7]$

d $f(x) = \sqrt{5 - x}, \quad x \in [0, 4]$

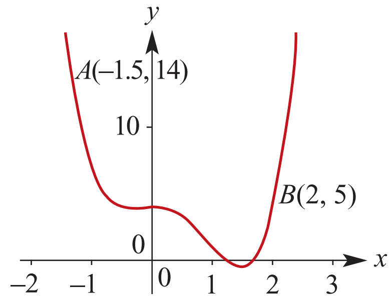
Example 3

3 Find the average rate of change of y with respect to x from point A to point B for each of the following graphs:

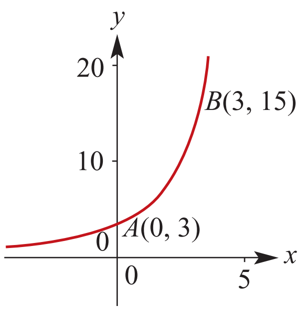
a



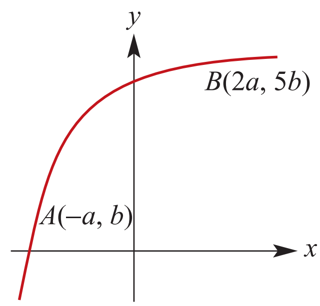
b



c



d



4 The distance (in metres) from a point O of an object t seconds after it starts to move in a straight line is given by the function $S(t) = t^3 + t^2 - 2t + 2, t \geq 0$. Find the average rate of change with respect to time of the distance of the object from O:

a in the first 2 seconds

b in the next 2 seconds.

5 A person invests \$2000 dollars, which increases in value by 7% per year for three years.

a Calculate the value of the investment after three years.

b Calculate the average rate of change in the value of the investment over that time.

Example 4

6 The depth, d cm, of water in a bath tub t minutes after the tap is turned on is modelled by the function $d(t) = \dfrac{-300}{t+6} + 50, t \geq 0$. Find the average rate of change of the depth of the water with respect to time over the first 10 minutes after the tap is turned on.

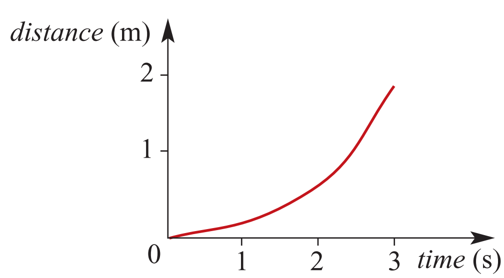
7 Using the information in the graph on the right, the average speed from $t = 0$ to $t = 3$ is

A 2 m/s

B 1 m/s

C $\dfrac{2}{3}$ m/s

D $1\dfrac{1}{2}$ m/s



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17B Instantaneous rate of change

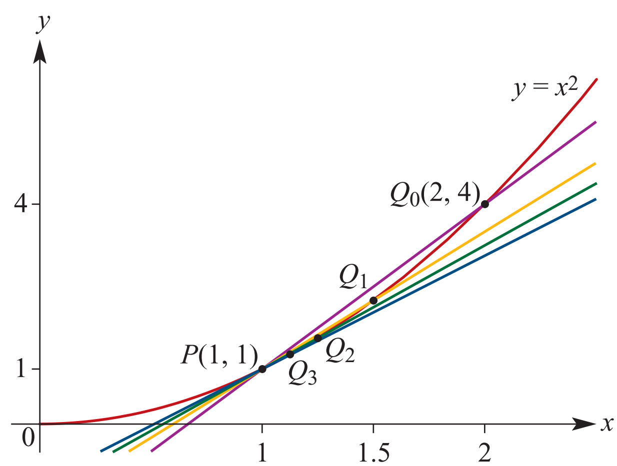
In the previous section, we investigated the average rate of change of a function over a stated interval. We saw that, in general (except for linear functions), the average rate of change of a function over different intervals in the domain of the function is not constant.

In this section, we investigate the idea of instantaneous rate of change.

Tangent line at a point

We talk about the idea of the tangent to a curve at a point informally here. It is a line which has 'the same slope' as the graph at this point. Of course, we don't know exactly what this means, but if we think of a very, very, ..., very small section of the curve around the point, we can consider it to be a line segment which can be extended out to a straight line. This straight line is what we call the 'tangent line' to the curve at the point.

We can illustrate this idea with a specific example. Part of the graph of $y = x^2$ is shown below. We will find the tangent line at the point $P(1, 1)$.



We start with the secant PQ_0 passing through $P(1, 1)$ and $Q_0(2, 4)$.

The gradient of PQ_0 is $\dfrac{4-1}{2-1} = 3$, and so the equation of the secant PQ_0 is $y = 3x - 2$.

The points $Q_1, Q_2, Q_3, \dots, Q_n, \dots$ on the curve $y = x^2$ are chosen so that they get closer and closer to P in the following way:

- The x-coordinate of Q_1 is $\frac{1}{2}(1 + 2) = \frac{3}{2}$.
- The x-coordinate of Q_2 is $\frac{1}{2}\left(1 + \frac{3}{2}\right) = \frac{5}{4}$.
- The x-coordinate of Q_3 is $\frac{1}{2}\left(1 + \frac{5}{4}\right) = \frac{9}{8}$.

We now look at the sequence of secants $PQ_0, PQ_1, PQ_2, PQ_3, \dots, PQ_n, \dots$. The following table shows the gradient and the equation for each secant.

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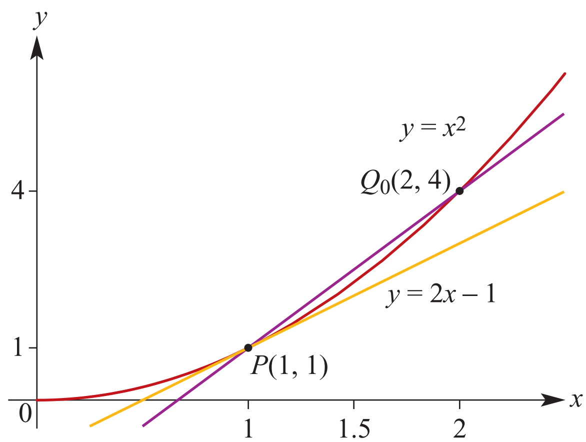
Secants of the curve $y = x^2$ through the point $P(1, 1)$

Step	Endpoint	Secant	Gradient	Equation of secant
0	$Q_0(2, 4)$	PQ_0	3	$y = 3x - 2$
1	$Q_1\left(\frac{3}{2}, \frac{9}{4}\right)$	PQ_1	$\frac{5}{2}$	$y = \frac{5}{2}x - \frac{3}{2}$
2	$Q_2\left(\frac{5}{4}, \frac{25}{16}\right)$	PQ_2	$\frac{9}{4}$	$y = \frac{9}{4}x - \frac{5}{4}$
3	$Q_3\left(\frac{9}{8}, \frac{81}{64}\right)$	PQ_3	$\frac{17}{8}$	$y = \frac{17}{8}x - \frac{9}{8}$
n	$Q_n(1 + 2^{-n}, (1 + 2^{-n})^2)$	PQ_n	$2 + 2^{-n}$	$y = (2 + 2^{-n})x - (1 + 2^{-n})$

The sequence of gradients is $3, \frac{5}{2}, \frac{9}{4}, \frac{17}{8}, \dots, 2 + \frac{1}{2^n}, \dots$

We can see that the gradients get closer and closer to 2. This is particularly evident from the general gradient, $2 + \frac{1}{2^n}$, since as $n \to \infty$, $\frac{1}{2^n} \to 0$.

We can also see that the secants get closer and closer to the line with equation $y = 2x - 1$. This line is the **tangent line** at the point P, and the gradient of the tangent line is the **instantaneous rate of change** of y with respect to x at the point P.



We will give the definition of a tangent line in the next section. In this section, we only consider approximations to the instantaneous rate of change.

Example 5

Estimate the instantaneous rate of change of y with respect to x at the point $P(2, 9)$ on the curve $y = x^3 + 1$ by considering the secant PQ, where $Q = (2.01, (2.01)^3 + 1)$.

Solution

Gradient of $PQ = \dfrac{(2.01)^3 + 1 - 9}{2.01 - 2} = 12.0601$

Note: An even better approximation can be made by choosing the points $P(2, 9)$ and $Q(2.001, (2.001)^3 + 1)$. Using the approach taken for $y = x^2$ in the discussion above, we would find that the instantaneous rate of change for this example is 12.

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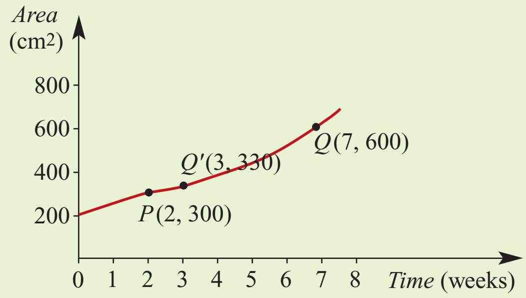
Example 6

The graph represents the area covered by a spreading plant. Area is measured in square centimetres and time in weeks.



a Find the gradient of the secant PQ.

b The point Q' has coordinates $(3, 330)$. Find the average rate of change of area with respect to time for the interval $[2, 3]$, and hence estimate the instantaneous rate of change of the area of the plant at $t = 2$.

Solution

a Gradient of $PQ = \dfrac{600 - 300}{7 - 2}$

$$= \frac{300}{5}$$

$$= 60$$

The average rate of change of area from $t = 2$ to $t = 7$ is $60\,\text{cm}^2$ per week.

b Gradient of $PQ' = \dfrac{330 - 300}{1} = 30$

$\therefore$ Gradient at P is approximately 30.

The instantaneous rate of change of the area of the plant with respect to time when $t = 2$ is approximately $30\,\text{cm}^2$ per week.

Example 7

Consider the curve $y = 2^x$.

a Using the secant through the points where $x = 3$ and $x = 3.1$, estimate the instantaneous rate of change of y with respect to x at the point where $x = 3$.

b Repeat for the points where $x = 3$ and $x = 3.001$.

Solution

a When $x = 3$, $y = 8$ and when $x = 3.1$, $y = 8.5742$ (correct to four decimal places).
The gradient of the line through $(3, 8)$ and $(3.1, 8.5742)$ is 5.7419.
Thus an estimate for the instantaneous rate of change of $y = 2^x$ at $x = 3$ is 5.742.

b When $x = 3.001$, $y = 8.005547$.
The gradient of the line through $(3, 8)$ and $(3.001, 8.005547)$ is 5.547.

Note: The true instantaneous rate of change of y with respect to x at $x = 3$ is 5.5452 (correct to four decimal places).

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Using the graph window of your calculator

The graph of $y = 0.5x^3 - 2x + 1$ is shown. We will investigate the gradient at the point (0, 1).

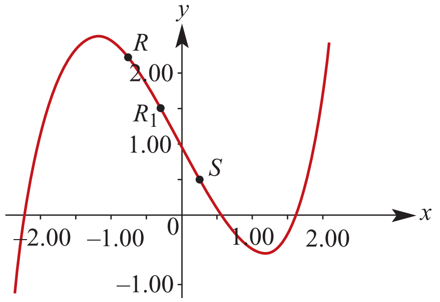
First find the gradient of the secant RS, where $R = (-0.75, 2.2891)$ and $S = (0.25, 0.5078)$:

gradient of $RS = -1.7813$

(The coordinates of R and S are given to four decimal places.)

Now consider another secant R_1S, where $R_1 = (-0.25, 1.4922)$ and $S = (0.25, 0.5078)$:

gradient of $R_1S = -1.9688$

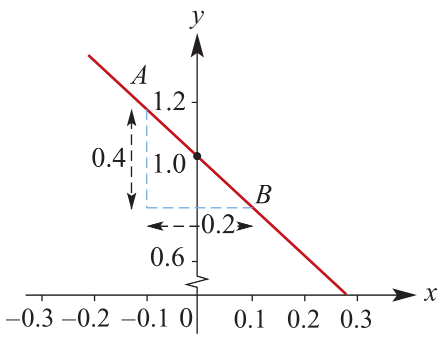


Using a calculator, zoom in on the graph near $x = 0$. As you zoom further in, this section of the curve appears increasingly linear. By assuming that this section of the curve is in fact linear and finding its gradient, we can approximate the gradient of the curve at $x = 0$.

This diagram shows a 'zoomed in' section of the graph around the point (0, 1).

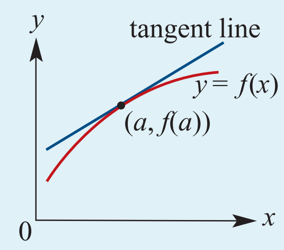
Consider the secant AB where $A = (-0.1, 1.1995)$ and $B = (0.1, 0.8005)$. The gradient of this line is approximately -2.

Therefore we make the approximation that the gradient of the curve $y = 0.5x^3 - 2x + 1$ at the point (0, 1) is -2.



Summary 17B

For a function $y = f(x)$, the **instantaneous rate of change** of y with respect to x at the point $(a, f(a))$ is the gradient of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$.



Exercise 17B

Example 5

1 By considering the secant through the points where $x = 1.2$ and $x = 1.3$, estimate the instantaneous rate of change of y with respect to x at the point on the curve $y = x^3 + x^2$ where $x = 1.3$.

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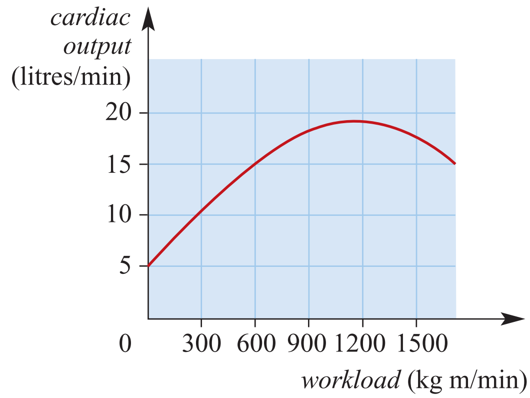
Example 6

2 Cardiac output is an important factor in athletic endurance. The graph shows a stress-test graph of cardiac output (measured in litres/min of blood) versus workload (measured in kg m/min).

a Estimate the average rate of change of cardiac output with respect to workload as the workload increases from 0 to 1200 kg m/min.

b Estimate the instantaneous rate of change of cardiac output with respect to workload at the point where the workload is 450 kg m/min.



Example 7

3 Let $y = 10^x$.

a Find the average rate at which y changes with respect to x over each of the following intervals:

i $[0, 1]$ **ii** $[0, 0.5]$ **iii** $[0, 0.1]$

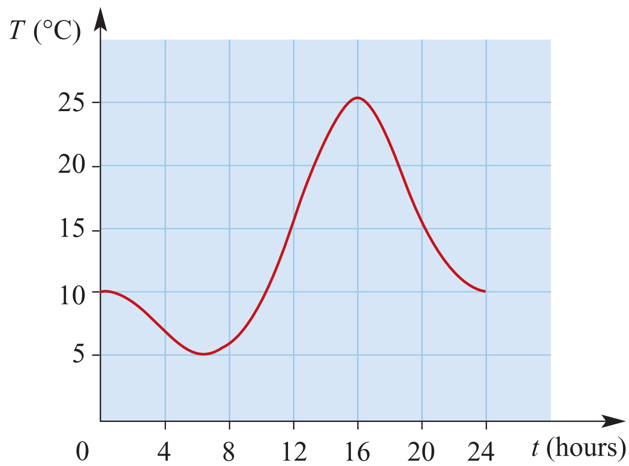
b Estimate the instantaneous rate of change of y with respect to x when $x = 0$.

4 Temperature (T°C) varies with time (t hours) over a 24-hour period, as illustrated in the graph.

a Estimate the maximum temperature and the time at which this occurs.

b The temperature rise between 10:00 and 14:00 is approximately linear. Estimate the rate at which the temperature is increasing in this period.

c Estimate the instantaneous rate of change of temperature at $t = 20$.



5 By considering the secant through the points at which $x = 1.2$ and $x = 1.4$, estimate the instantaneous rate of change of y with respect to x of the curve $y = \dfrac{1}{x}$ at $x = 1.2$.

6 Draw the graph of $y = \sqrt{16 - x^2}$, $-4 \le x \le 4$. Use an appropriate technique to find an estimate of the instantaneous rate of change of y with respect to x at the points:

a $x = 0$ **b** $x = 2$ **c** $x = 3$

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7 It is known that the straight line $y = 4x - 4$ touches the curve $y = x^2$ at the point $(2, 4)$. Sketch the graphs of both of these functions on the one set of axes.
Find the instantaneous rate of change of y with respect to x at the point at $(2, 4)$ on the curve $y = x^2$.

8 Water is being collected in a water tank. The volume, V cubic metres, of water in the tank after t minutes is given by $V = 3t^2 + 4t + 2$.

a Find the average rate of change of volume with respect to time between times $t = 1$ and $t = 3$.

b Find an estimate for the instantaneous rate of change of volume with respect to time at $t = 1$.

9 A population of bacteria is growing. The population, P million, after time t minutes is given by $P = 3 \times 2^t$.

a Find the average rate of change of population between times $t = 2$ and $t = 4$.

b Find an estimate for the instantaneous rate of change of population with respect to time at $t = 2$.

10 Water is flowing out of a water tank. The volume, V cubic metres, of water in the tank after t minutes is given by $V = 5 \times 10^5 - 10^2 \times 2^t$, $0 \le t \le 12$.

a Find the average rate of change of volume with respect to time between times $t = 0$ and $t = 5$.

b Find an estimate for the instantaneous rate of change of volume with respect to time when $t = 6$.

c Find an estimate for the rate of change of volume when $t = 12$.

11 Use the technique of Examples 5 and 7 to estimate the instantaneous rate of change of y with respect to x for each of the following at the stated point:

a $y = x^3 + 2x^2$, $(1, 3)$

b $y = 2x^3 + 3x$, $(1, 5)$

c $y = -x^3 + 3x^2 + 2x$, $(2, 8)$

d $y = 2x^3 - 3x^2 - x + 2$, $(3, 26)$

12 The volume, V, of a cube with edge length x is given by $V = x^3$.

a Find the average rate at which the volume of the cube changes with respect to x, as x increases from $x = 2$ to $x = 4$.

b Find an estimate for the instantaneous rate at which V changes with respect to x when $x = 2$.

13 Let $y = \sin x$.

a Find the average rate at which y changes with respect to x over each of the following intervals:

i $\left[0, \frac{\pi}{2}\right]$ **ii** $\left[0, \frac{\pi}{4}\right]$ **iii** $[0, 0.5]$ **iv** $[0, 0.1]$

b Estimate the instantaneous rate of change of y with respect to x when $x = 0$.

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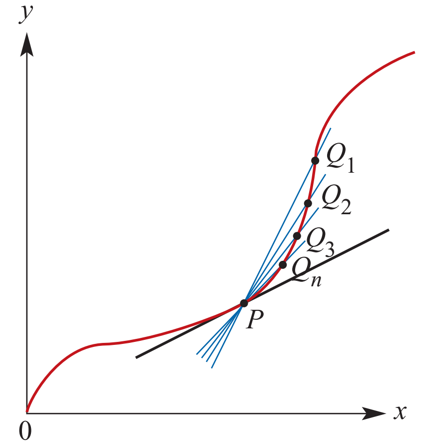
17C The derivative

In the previous section we considered what happened when we looked at a sequence of secant lines $PQ_1, PQ_2, \ldots, PQ_n, \ldots$ of a curve, where the points Q_i get closer and closer to P. The idea of the instantaneous rate of change at P was introduced.

In this section we focus our attention on the gradient of the tangent line at P.



The tangent to a curve at a point

Consider the function $f(x) = x^2$. We will find the gradient of the tangent to the curve $y = f(x)$ at the point $P(a, a^2)$.

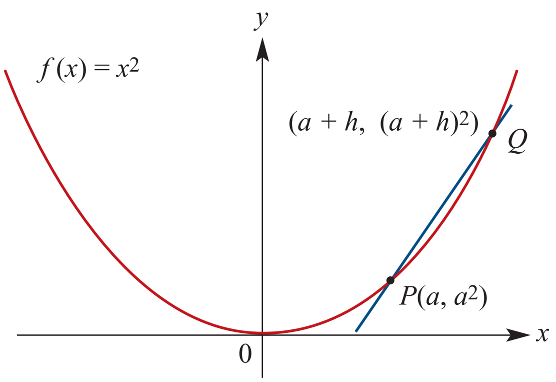
Choose a point Q on the curve close to P. The coordinates of such a point can be expressed in the form $Q(a + h, (a + h)^2)$, where h is a non-zero real number with small magnitude.

The gradient of the secant line PQ is given by

$$\text{gradient of } PQ = \frac{(a+h)^2 - a^2}{a + h - a}$$

$$= \frac{a^2 + 2ah + h^2 - a^2}{h}$$

$$= 2a + h$$



Now consider what happens if we take values of h with smaller and smaller magnitude, so that the point Q gets closer and closer to P. The gradient of the secant PQ gets closer and closer to $2a$.

The limit of $2a + h$ as h approaches 0 is $2a$, and so we say that the gradient of the tangent at P is $2a$.

The straight line that passes through the point P and has gradient $2a$ is called the **tangent** to the curve at P.

It can be seen that there is nothing special about a here. The same calculation works for any real number x. The gradient of the tangent to the graph of $y = x^2$ at any point x is $2x$.

We say that the **derivative of x^2 with respect to x is $2x$**, or more briefly, we can say that the **derivative of x^2 is $2x$**.

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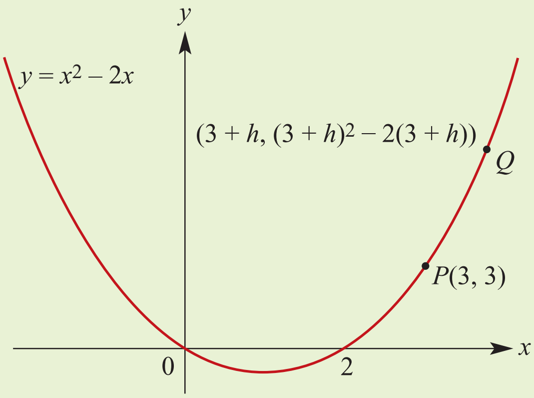
Example 8

By first considering the gradient of the secant PQ, find the gradient of the tangent line to $y = x^2 - 2x$ at the point P with coordinates $(3, 3)$.



Solution

$$\text{Gradient of } PQ = \frac{(3+h)^2 - 2(3+h) - 3}{3 + h - 3}$$

$$= \frac{9 + 6h + h^2 - 6 - 2h - 3}{h}$$

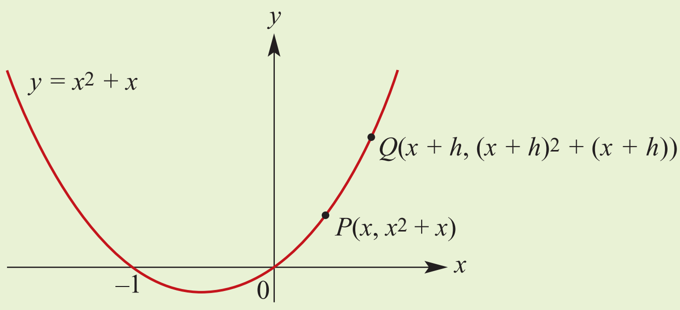
$$= \frac{4h + h^2}{h}$$

$$= 4 + h$$

Now consider the gradient of PQ as h approaches 0. The gradient of the tangent line at the point $P(3, 3)$ is 4.

Example 9

Find the gradient of the secant PQ and hence find the derivative of $x^2 + x$.



Solution

$$\text{Gradient of } PQ = \frac{(x+h)^2 + (x+h) - (x^2 + x)}{x + h - x}$$

$$= \frac{x^2 + 2xh + h^2 + x + h - x^2 - x}{h}$$

$$= \frac{2xh + h^2 + h}{h}$$

$$= 2x + h + 1$$

From this it is seen that the derivative of $x^2 + x$ is $2x + 1$.

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Limit notation

The notation for the limit of $2x + h + 1$ as h approaches 0 is

$$\lim_{h \to 0} (2x + h + 1)$$

The derivative of a function with rule $f(x)$ may be found by:

1. finding an expression for the gradient of the line through $P(x, f(x))$ and $Q(x + h, f(x + h))$
2. finding the limit of this expression as h approaches 0.

Example 10

Consider the function $f(x) = x^3$. By first finding the gradient of the secant through $P(2, 8)$ and $Q(2 + h, (2 + h)^3)$, find the gradient of the tangent to the curve at the point $(2, 8)$.

Solution

$$\begin{aligned} \text{Gradient of } PQ &= \frac{(2+h)^3 - 8}{2 + h - 2} \\ &= \frac{8 + 12h + 6h^2 + h^3 - 8}{h} \\ &= \frac{12h + 6h^2 + h^3}{h} \\ &= 12 + 6h + h^2 \end{aligned}$$

The gradient of the tangent line at $(2, 8)$ is $\lim_{h \to 0} (12 + 6h + h^2) = 12$.

The following example provides practice in determining limits.

Example 11

Find:

a $\lim_{h \to 0} (22x^2 + 20xh + h)$

b $\lim_{h \to 0} \dfrac{3x^2h + 2h^2}{h}$

c $\lim_{h \to 0} 3x$

d $\lim_{h \to 0} 4$

Solution

a $\lim_{h \to 0} (22x^2 + 20xh + h) = 22x^2$

b $$\begin{aligned} \lim_{h \to 0} \frac{3x^2h + 2h^2}{h} &= \lim_{h \to 0} (3x^2 + 2h) \\ &= 3x^2 \end{aligned}$$

c $\lim_{h \to 0} 3x = 3x$

d $\lim_{h \to 0} 4 = 4$

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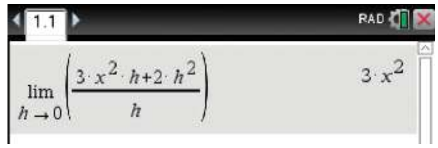
Using the TI-Nspire

To calculate a limit, use menu > **Calculus** > **Limit** and complete as shown.



Note: The limit template can also be accessed from the 2D-template palette.
When you insert the limit template, you will notice a superscript field (small box) on the template – generally this will be left empty.

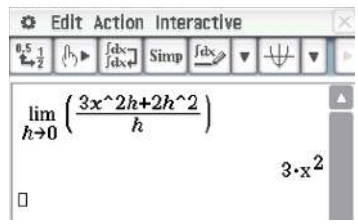
Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight the expression

$$\frac{3x^2h + 2h^2}{h}$$

Note: Use h from the Var keyboard.

- Select lim from the Math2 keyboard and tap EXE.
- Enter h and 0 in the spaces provided as shown.

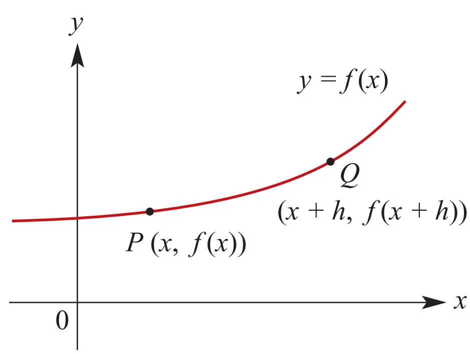


Definition of the derivative

In general, consider the graph $y = f(x)$ of a function f.

$$\text{Gradient of secant } PQ = \frac{f(x+h) - f(x)}{x + h - x}$$
$$= \frac{f(x+h) - f(x)}{h}$$

The gradient of the tangent to the graph of $y = f(x)$ at the point $P(x, f(x))$ is the limit of this expression as h approaches 0.



Derivative of a function

The **derivative** of the function f is denoted f' and is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The **tangent line** to the graph of the function f at the point $(a, f(a))$ is defined to be the line through $(a, f(a))$ with gradient $f'(a)$.

Warning: This definition of the derivative assumes that the limit exists. For polynomial functions, such limits always exist. But it is not true that for every function you can find the derivative at every point of its domain. This is discussed further in Sections 17H and 17I.

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Differentiation by first principles

Determining the derivative of a function by evaluating the limit is called **differentiation by first principles**.

Example 12

For $f(x) = x^2 + 2x$, find $f'(x)$ by first principles.

Solution

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 + 2(x+h) - (x^2 + 2x)}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + 2x + 2h - x^2 - 2x}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2 + 2h}{h} \\
&= \lim_{h\to 0} (2x + h + 2) \\
&= 2x + 2
\end{aligned}$$

$\therefore \quad f'(x) = 2x + 2$

Example 13

For $f(x) = 2 - x^3$, find $f'(x)$ by first principles.

Solution

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{2 - (x+h)^3 - (2 - x^3)}{h} \\
&= \lim_{h\to 0} \frac{2 - (x^3 + 3x^2h + 3xh^2 + h^3) - (2 - x^3)}{h} \\
&= \lim_{h\to 0} \frac{-3x^2h - 3xh^2 - h^3}{h} \\
&= \lim_{h\to 0} (-3x^2 - 3xh - h^2) \\
&= -3x^2
\end{aligned}$$

$\therefore \quad f'(x) = -3x^2$

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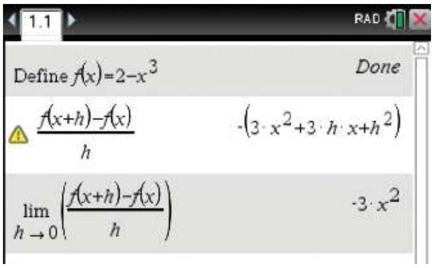
Using the TI-Nspire

- Define $f(x) = 2 - x^3$.
- Use (menu) > **Calculus** > **Limit** or the 2D-template palette, and complete as shown.



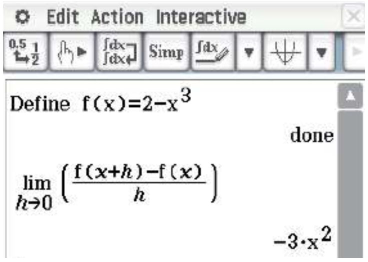
Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight the expression $2 - x^3$. Select **Interactive** > **Define** and tap OK.
- Now enter and highlight the expression

$$\frac{f(x+h) - f(x)}{h}$$

Note: Select f from the (abc) keyboard and x, h from the (Var) keyboard.

- Select (lim) from the (Math2) keyboard and tap (EXE).
- Enter h and 0 in the spaces provided as shown.



Summary 17C

- The **derivative** of the function f is denoted f' and is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- The **tangent line** to the graph of the function f at the point $(a, f(a))$ is defined to be the line through $(a, f(a))$ with gradient $f'(a)$.

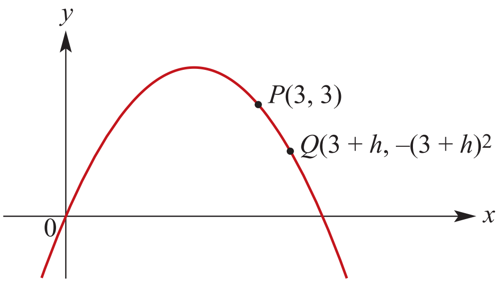
Exercise 17C

Example 8

1 Let $f(x) = -x^2 + 4x$. The graph of $y = f(x)$ is shown opposite.

a Find the gradient of PQ.

b Find the gradient of the curve at the point P by considering what happens as h approaches 0.



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2 Let $f(x) = x^2 - 3x$. Then the points $P(4, 4)$ and $Q(4 + h, (4 + h)^2 - 3(4 + h))$ are on the curve $y = f(x)$.

a Find the gradient of the secant PQ.

b Find the gradient of the tangent line to the curve at the point P by considering what happens as h approaches 0.

Example 9

3 The points $P(x, x^2 - 2x)$ and $Q(x + h, (x + h)^2 - 2(x + h))$ are on the curve $y = x^2 - 2x$. Find the gradient of PQ and hence find the derivative of $x^2 - 2x$.

Example 10

4 By first considering the gradient of the secant through $P(2, 16)$ and $Q(2 + h, (2 + h)^4)$ for the curve $y = x^4$, find the gradient of the tangent to the curve at the point $(2, 16)$.

Hint: $(x + h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$

5 Water is being collected in a water tank. The volume, V cubic metres, of water in the tank after t minutes is given by $V = 3t^2 + 4t + 2$. By finding the gradient of the secant through the points where $t = 1$ and $t = 1 + h$, calculate the rate of change of volume with respect to time at $t = 1$.

6 A population of insects grows so that the size of the population, P, at time t (days) is given by $P = 1000 + t^2 + t$. By finding the gradient of the secant through the points where $t = 3$ and $t = 3 + h$, calculate the rate of growth of the insect population at time $t = 3$.

Example 11

7 Find:

a $\lim\limits_{h \to 0} \dfrac{2x^2h^3 + xh^2 + h}{h}$ **b** $\lim\limits_{h \to 0} \dfrac{3x^2h - 2xh^2 + h}{h}$ **c** $\lim\limits_{h \to 0} (20 - 10h)$

d $\lim\limits_{h \to 0} \dfrac{30hx^2 + 2h^2 + h}{h}$ **e** $\lim\limits_{h \to 0} 5$ **f** $\lim\limits_{h \to 0} \dfrac{30hx^3 + 2h^2 + 4h}{h}$

8 Find:

a $\lim\limits_{h \to 0} \dfrac{(x + h)^2 + 2(x + h) - (x^2 + 2x)}{h}$ i.e. the derivative of $y = x^2 + 2x$

b $\lim\limits_{h \to 0} \dfrac{(5 + h)^2 + 3(5 + h) - 40}{h}$ i.e. the gradient of $y = x^2 + 3x$ at $x = 5$

c $\lim\limits_{h \to 0} \dfrac{(x + h)^3 + 2(x + h)^2 - (x^3 + 2x^2)}{h}$ i.e. the derivative of $y = x^3 + 2x^2$

9 For the curve with equation $y = 3x^2 - x$:

a Find the gradient of the secant PQ, where P is the point $(1, 2)$ and Q is the point $(1 + h,\ 3(1 + h)^2 - (1 + h))$.

b Find the gradient of PQ when $h = 0.1$.

c Find the gradient of the tangent to the curve at P.

10 For the curve with equation $y = \dfrac{2}{x}$:

a Find the gradient of the chord AB, where $A = (2, 1)$ and $B = \left(2 + h,\ \dfrac{2}{2 + h}\right)$.

b Find the gradient of AB when $h = 0.1$.

c Find the gradient of the tangent to the curve at A.

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11 For the curve with equation $y = x^2 + 2x - 3$:

a Find the gradient of the secant PQ, where P is the point $(2, 5)$ and Q is the point $(2 + h,\ (2 + h)^2 + 2(2 + h) - 3)$.

b Find the gradient of PQ when $h = 0.1$.

c Find the gradient of the tangent to the curve at P.

Example 12
Example 13

12 For each of the following, find $f'(x)$ by finding $\lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$:

a $f(x) = 3x^2$ **b** $f(x) = 4x$ **c** $f(x) = 3$

d $f(x) = 3x^2 + 4x + 3$ **e** $f(x) = 2x^3 - 4$ **f** $f(x) = 4x^2 - 5x$

g $f(x) = 3 - 2x + x^2$

13 By first considering the gradient of the secant through $P(x, f(x))$ and $Q(x + h, f(x + h))$ for the curve $f(x) = x^4$, find the derivative of x^4.

Hint: $f(x + h) = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$

17D Rules for differentiation

The derivative of x^n where n is a positive integer

From your work in the previous section, you may have noticed that differentiating from first principles gives the following:

- For $f(x) = x$, $f'(x) = 1$.
- For $f(x) = x^2$, $f'(x) = 2x$.
- For $f(x) = x^3$, $f'(x) = 3x^2$.

This suggests the following general result.

> For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where $n = 1, 2, 3, \dots$

Proof We can prove this result using the binomial theorem, which was introduced in Section 10E. Let $f(x) = x^n$, where $n \in \mathbb{N}$ with $n \geq 2$.

Then
$$\begin{aligned} f(x + h) - f(x) &= (x + h)^n - x^n \\ &= x^n + {}^nC_1 x^{n-1}h + {}^nC_2 x^{n-2}h^2 + \cdots + {}^nC_{n-1} xh^{n-1} + h^n - x^n \\ &= {}^nC_1 x^{n-1}h + {}^nC_2 x^{n-2}h^2 + \cdots + {}^nC_{n-1} xh^{n-1} + h^n \\ &= nx^{n-1}h + {}^nC_2 x^{n-2}h^2 + \cdots + {}^nC_{n-1} xh^{n-1} + h^n \end{aligned}$$

and so
$$\begin{aligned} \frac{f(x + h) - f(x)}{h} &= \frac{1}{h}\left(nx^{n-1}h + {}^nC_2 x^{n-2}h^2 + \cdots + {}^nC_{n-1} xh^{n-1} + h^n\right) \\ &= nx^{n-1} + {}^nC_2 x^{n-2}h + \cdots + {}^nC_{n-1} xh^{n-2} + h^{n-1} \end{aligned}$$

$$\therefore \quad \begin{aligned} \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} &= \lim_{h \to 0}\left(nx^{n-1} + {}^nC_2 x^{n-2}h + \cdots + {}^nC_{n-1} xh^{n-2} + h^{n-1}\right) \\ &= nx^{n-1} \end{aligned}$$

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The derivative of a polynomial function

The following results are very useful when finding the derivative of a polynomial function. We will prove two of these results in the final section of this chapter.

- **Constant function:** If $f(x) = c$, then $f'(x) = 0$.
- **Linear function:** If $f(x) = mx + c$, then $f'(x) = m$.
- **Multiple:** If $f(x) = k\,g(x)$, where k is a constant, then $f'(x) = k\,g'(x)$.
  That is, the derivative of a number multiple is the multiple of the derivative.
  For example: if $f(x) = 5x^2$, then $f'(x) = 5(2x) = 10x$.
- **Sum:** If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.
  That is, the derivative of the sum is the sum of the derivatives.
  For example: if $f(x) = x^2 + 2x$, then $f'(x) = 2x + 2$.
- **Difference:** If $f(x) = g(x) - h(x)$, then $f'(x) = g'(x) - h'(x)$.
  That is, the derivative of the difference is the difference of the derivatives.
  For example: if $f(x) = x^2 - 2x$, then $f'(x) = 2x - 2$.

You will meet rules for the derivative of products and quotients in Year 12.

The process of finding the derivative function is called **differentiation**.

Example 14

Find the derivative of $x^5 - 2x^3 + 2$, i.e. differentiate $x^5 - 2x^3 + 2$ with respect to x.

Solution

Let $\quad f(x) = x^5 - 2x^3 + 2$

Then $\quad f'(x) = 5x^4 - 2(3x^2) + 0$

$\qquad\qquad = 5x^4 - 6x^2$

Explanation

We use the following results:

- the derivative of x^n is nx^{n-1}
- the derivative of a number is 0
- the multiple, sum and difference rules.

Example 15

Find the derivative of $f(x) = 3x^3 - 6x^2 + 1$ and thus find $f'(1)$.

Solution

Let $\quad f(x) = 3x^3 - 6x^2 + 1$

Then $\quad f'(x) = 3(3x^2) - 6(2x) + 0$

$\qquad\qquad = 9x^2 - 12x$

$\therefore \quad f'(1) = 9 - 12$

$\qquad\qquad = -3$

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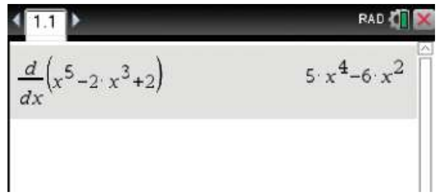
Using the TI-Nspire

For Example 14:

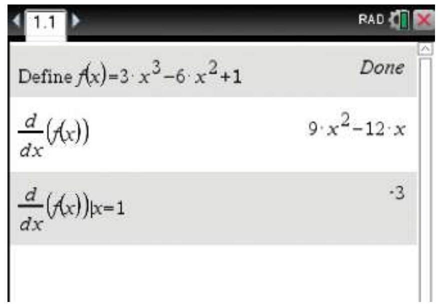
- Use menu > **Calculus** > **Derivative** and complete as shown.



Note: The derivative template can also be accessed from the 2D-template palette. Alternatively, using shift – will paste the derivative template to the screen.

For Example 15:

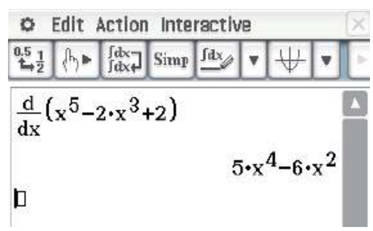
- Define $f(x) = 3x^3 - 6x^2 + 1$.
- Use menu > **Calculus** > **Derivative** to differentiate as shown.
- To find the value of the derivative at $x = 1$, use menu > **Calculus** > **Derivative at a Point**.



Using the Casio ClassPad

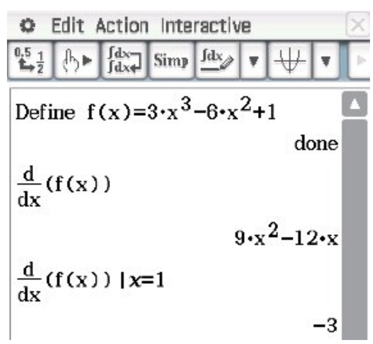
For Example 14:

- In Main, enter and highlight the expression $x^5 - 2x^3 + 2$.
- Go to **Interactive** > **Calculation** > **diff** and tap OK.

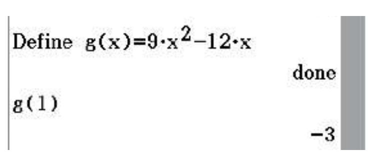


For Example 15:

- In Main, enter and highlight the expression $3x^3 - 6x^2 + 1$.
- Go to **Interactive** > **Calculation** > **diff** and tap OK; this will give the derivative only.
- To find the value of the derivative at $x = 1$, tap the stylus at the end of the entry line. Select | from the Math3 keyboard and type $x = 1$. Then tap EXE.



- Alternatively, define the derivative as $g(x)$ and find $g(1)$.



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Finding the gradient of a tangent line

We discussed the tangent line at a point on a graph in Section 17C. We recall the following:

> The **tangent line** to the graph of the function f at the point $(a, f(a))$ is defined to be the line through $(a, f(a))$ with gradient $f'(a)$.

Example 16

For the curve determined by the rule $f(x) = 3x^3 - 6x^2 + 1$, find the gradient of the tangent line to the curve at the point $(1, -2)$.

Solution

Now $f'(x) = 9x^2 - 12x$ and so $f'(1) = 9 - 12 = -3$.

The gradient of the tangent line at the point $(1, -2)$ is -3.

Alternative notations

It was mentioned in the introduction to this chapter that the German mathematician Gottfried Leibniz was one of the two people to whom the discovery of calculus is attributed. A form of the notation he introduced is still in use today.

Leibniz notation

An alternative notation for the derivative is the following:

> If $y = x^3$, then the derivative can be denoted by $\dfrac{dy}{dx}$, and so we write $\dfrac{dy}{dx} = 3x^2$.

In general, if y is a function of x, then the derivative of y with respect to x is denoted by $\dfrac{dy}{dx}$.

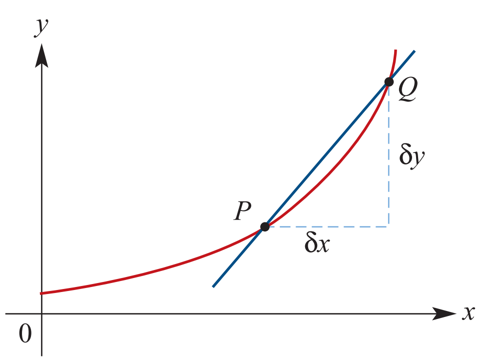
Similarly, if z is a function of t, then the derivative of z with respect to t is denoted $\dfrac{dz}{dt}$.

Warning: In this notation, the symbol d is not a factor and cannot be cancelled.

This notation came about because, in the eighteenth century, the standard diagram for finding the limiting gradient was labelled as shown:

- δx means a small difference in x
- δy means a small difference in y

where δ (delta) is the lowercase Greek letter d.



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Example 17

a If $y = t^2$, find $\frac{dy}{dt}$.

b If $x = t^3 + t$, find $\frac{dx}{dt}$.

c If $z = \frac{1}{3}x^3 + x^2$, find $\frac{dz}{dx}$.

Solution

a $y = t^2$

$\frac{dy}{dt} = 2t$

b $x = t^3 + t$

$\frac{dx}{dt} = 3t^2 + 1$

c $z = \frac{1}{3}x^3 + x^2$

$\frac{dz}{dx} = x^2 + 2x$

Example 18

a For $y = (x + 3)^2$, find $\frac{dy}{dx}$.

b For $z = (2t - 1)^2(t + 2)$, find $\frac{dz}{dt}$.

c For $y = \frac{x^2 + 3x}{x}$, find $\frac{dy}{dx}$.

d Differentiate $y = 2x^3 - 1$ with respect to x.

Solution

a It is first necessary to write $y = (x + 3)^2$ in expanded form:

$$y = x^2 + 6x + 9$$

$$\therefore \quad \frac{dy}{dx} = 2x + 6$$

b Expanding:

$$\begin{aligned} z &= (4t^2 - 4t + 1)(t + 2) \\ &= 4t^3 - 4t^2 + t + 8t^2 - 8t + 2 \\ &= 4t^3 + 4t^2 - 7t + 2 \end{aligned}$$

$$\therefore \quad \frac{dz}{dt} = 12t^2 + 8t - 7$$

c First simplify:

$$y = x + 3 \quad (\text{for } x \neq 0)$$

$$\therefore \quad \frac{dy}{dx} = 1 \quad (\text{for } x \neq 0)$$

d $y = 2x^3 - 1$

$$\therefore \quad \frac{dy}{dx} = 6x^2$$

Operator notation

'Find the derivative of $2x^2 - 4x$ with respect to x' can also be written as 'find $\frac{d}{dx}(2x^2 - 4x)$'.

In general: $\frac{d}{dx}(f(x)) = f'(x)$.

Example 19

Find:

a $\frac{d}{dx}(5x - 4x^3)$

b $\frac{d}{dz}(5z^2 - 4z)$

c $\frac{d}{dz}(6z^3 - 4z^2)$

Solution

a $\frac{d}{dx}(5x - 4x^3)$

$= 5 - 12x^2$

b $\frac{d}{dz}(5z^2 - 4z)$

$= 10z - 4$

c $\frac{d}{dz}(6z^3 - 4z^2)$

$= 18z^2 - 8z$

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Example 20

For each of the following curves, find the coordinates of the points on the curve at which the gradient of the tangent line at that point has the given value:

a $y = x^3$, gradient = 8

b $y = x^2 - 4x + 2$, gradient = 0

c $y = 4 - x^3$, gradient = −6

Solution

a $y = x^3$ implies $\dfrac{dy}{dx} = 3x^2$

$$\therefore \quad 3x^2 = 8$$

$$\therefore \quad x = \pm\sqrt{\frac{8}{3}} = \frac{\pm 2\sqrt{6}}{3}$$

The points are $\left(\dfrac{2\sqrt{6}}{3}, \dfrac{16\sqrt{6}}{9}\right)$ and $\left(\dfrac{-2\sqrt{6}}{3}, \dfrac{-16\sqrt{6}}{9}\right)$.

b $y = x^2 - 4x + 2$ implies $\dfrac{dy}{dx} = 2x - 4$

$$\therefore \quad 2x - 4 = 0$$

$$\therefore \quad x = 2$$

The only point is $(2, -2)$.

c $y = 4 - x^3$ implies $\dfrac{dy}{dx} = -3x^2$

$$\therefore \quad -3x^2 = -6$$

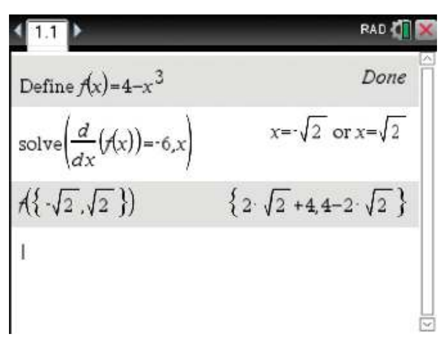
$$\therefore \quad x^2 = 2$$

$$\therefore \quad x = \pm\sqrt{2}$$

The points are $\left(2^{\frac{1}{2}}, 4 - 2^{\frac{3}{2}}\right)$ and $\left(-2^{\frac{1}{2}}, 4 + 2^{\frac{3}{2}}\right)$.

Using the TI-Nspire

- Define $f(x) = 4 - x^3$.
- Solve the equation $\dfrac{d}{dx}(f(x)) = -6$.
- Substitute in $f(x)$ to find the y-coordinates.



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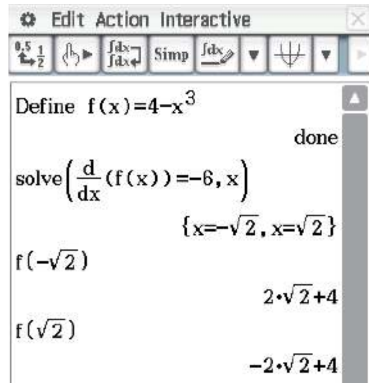
Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight the expression $4 - x^3$.
- Go to **Interactive** > **Define** and tap OK.
- In the next entry line, type and highlight $f(x)$.
- Go to **Interactive** > **Calculation** > **diff** and tap OK.
- Type $= -6$ after $\dfrac{d}{dx}(f(x))$. Highlight the equation and use **Interactive** > **Equation/Inequality** > **solve**.
- Enter $f(-\sqrt{2})$ and $f(\sqrt{2})$ to find the required y-values.



Summary 17D

- For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where $n = 1, 2, 3, \ldots$
- Constant function: If $f(x) = c$, then $f'(x) = 0$.
- Linear function: If $f(x) = mx + c$, then $f'(x) = m$.
- Multiple: If $f(x) = k\,g(x)$, where k is a constant, then $f'(x) = k\,g'(x)$.
  That is, the derivative of a number multiple is the multiple of the derivative.
- Sum: If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.
  That is, the derivative of the sum is the sum of the derivatives.
- Difference: If $f(x) = g(x) - h(x)$, then $f'(x) = g'(x) - h'(x)$.
  That is, the derivative of the difference is the difference of the derivatives.

For example, if $f(x) = 5x^3 - 10x^2 + 7$, then $f'(x) = 5(3x^2) - 10(2x) + 0 = 15x^2 - 20x$.

Exercise 17D

Example 14

1 Find the derivative of each of the following with respect to x:

a $x^2 + 4x$ **b** $2x + 1$ **c** $x^3 - x$

d $\frac{1}{2}x^2 - 3x + 4$ **e** $5x^3 + 3x^2$ **f** $-x^3 + 2x^2$

2 For each of the following, find $f'(x)$:

a $f(x) = x^{12}$ **b** $f(x) = 3x^7$ **c** $f(x) = 5x$

d $f(x) = 5x + 3$ **e** $f(x) = 3$ **f** $f(x) = 5x^2 - 3x$

g $f(x) = 10x^5 + 3x^4$ **h** $f(x) = 2x^4 - \frac{1}{3}x^3 - \frac{1}{4}x^2 + 2$

Example 15

3 For each of the following, find $f'(1)$:

a $f(x) = x^6$ **b** $f(x) = 4x^5$ **c** $f(x) = 5x$

d $f(x) = 5x^2 + 3$ **e** $f(x) = 3$ **f** $f(x) = 5x^2 - 3x$

g $f(x) = 10x^4 - 3x^3$ **h** $f(x) = 2x^4 - \frac{1}{3}x^3$ **i** $f(x) = -10x^3 - 2x^2 + 2$

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4 For each of the following, find $f'(-2)$:

a $f(x) = 5x^3$ **b** $f(x) = 4x^2$

c $f(x) = 5x^3 - 3x$ **d** $f(x) = -5x^4 - 2x^2$

Example 16

5 Find the gradient of the tangent line to the graph of f at the given point:

a $f(x) = x^2 + 3x, \quad (2, 10)$ **b** $f(x) = 3x^2 - 4x, \quad (1, -1)$

c $f(x) = -2x^2 - 4x, \quad (3, -30)$ **d** $f(x) = x^3 - x, \quad (2, 6)$

Example 17
Example 18

6 For each of the following, find $\frac{dy}{dx}$:

a $y = -x$ **b** $y = 10$ **c** $y = 4x^3 - 3x + 2$

d $y = \frac{1}{3}(x^3 - 3x + 6)$ **e** $y = (x + 1)(x + 2)$ **f** $y = 2x(3x^2 - 4)$

g $y = \frac{10x^5 + 3x^4}{2x^2}, \ x \neq 0$

7 **a** For $y = (x + 4)^2$, find $\frac{dy}{dx}$.

b For $z = (4t - 1)^2(t + 1)$, find $\frac{dz}{dt}$.

c For $y = \frac{x^3 + 3x}{x}$, find $\frac{dy}{dx}$.

8 **a** For the curve with equation $y = x^3 + 1$, find the gradient of the tangent line at points:

i $(1, 2)$ **ii** $(a, a^3 + 1)$

b Find the derivative of $x^3 + 1$ with respect to x.

9 **a** Given that $y = x^3 - 3x^2 + 3x$, find $\frac{dy}{dx}$. Hence show that $\frac{dy}{dx} \geq 0$ for all x, and interpret this in terms of the graph of $y = x^3 - 3x^2 + 3x$.

b Given that $y = \frac{x^2 + 2x}{x}$, for $x \neq 0$, find $\frac{dy}{dx}$.

c Differentiate $y = (3x + 1)^2$ with respect to x.

10 For each of the following curves, find the y-coordinate of the point on the curve with the given x-coordinate, and find the gradient of the tangent line at that point:

a $y = x^2 - 2x + 1, \quad x = 2$ **b** $y = x^2 + x + 1, \quad x = 0$

c $y = x^2 - 2x, \quad x = -1$ **d** $y = (x + 2)(x - 4), \quad x = 3$

e $y = 3x^2 - 2x^3, \quad x = -2$ **f** $y = (4x - 5)^2, \quad x = \frac{1}{2}$

11 **a** For each of the following, first find $f'(x)$ and $f'(1)$. Then, for $y = f(x)$, find the set $\{(x, y) : f'(x) = 1\}$. That is, find the coordinates of the points where the gradient of the tangent line is 1.

i $f(x) = 2x^2 - x$ **ii** $f(x) = 1 + \frac{1}{2}x + \frac{1}{3}x^2$

iii $f(x) = x^3 + x$ **iv** $f(x) = x^4 - 31x$

b What is the interpretation of $\{(x, y) : f'(x) = 1\}$ in terms of the graphs?

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Example 19 **12** Find:

a $\frac{d}{dt}(3t^2 - 4t)$ **b** $\frac{d}{dx}(4 - x^2 + x^3)$ **c** $\frac{d}{dz}(5 - 2z^2 - z^4)$

d $\frac{d}{dy}(3y^2 - y^3)$ **e** $\frac{d}{dx}(2x^3 - 4x^2)$ **f** $\frac{d}{dt}(9.8t^2 - 2t)$

Example 20 **13** For each of the following curves, find the coordinates of the points on the curve at which the gradient of the tangent line has the given value:

a $y = x^2$, gradient = 8 **b** $y = x^3$, gradient = 12

c $y = x(2 - x)$, gradient = 2 **d** $y = x^2 - 3x + 1$, gradient = 0

e $y = x^3 - 6x^2 + 4$, gradient = −12 **f** $y = x^2 - x^3$, gradient = −1

17E Differentiating x^n where n is a negative integer (optional content)

In the previous section we have seen how to differentiate polynomial functions. In this section we add to the family of functions that we can differentiate. In particular, we will consider functions which involve linear combinations of powers of x, where the indices may be negative integers.

e.g. $f(x) = x^{-1}, \{x \in \mathbb{R} : x \neq 0\}$

$f(x) = 2x + x^{-1}, \{x \in \mathbb{R} : x \neq 0\}$

$f(x) = x + 3 + x^{-2}, \{x \in \mathbb{R} : x \neq 0\}$

Note: We have reintroduced function notation to emphasise the need to consider domains.

Example 21

Let $f(x) = \frac{1}{x}, \{x \in \mathbb{R} : x \neq 0\}$. Find $f'(x)$ by first principles.

Solution

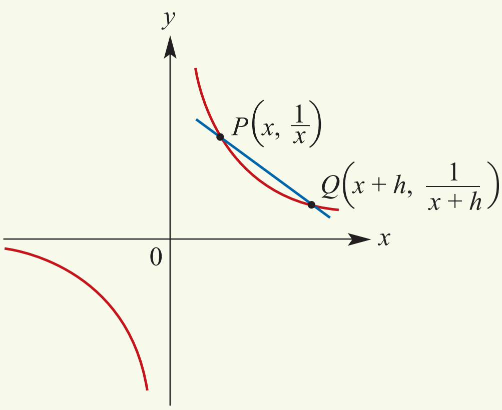
The gradient of secant PQ is given by

$$\frac{f(x+h) - f(x)}{x + h - x} = \left(\frac{1}{x+h} - \frac{1}{x}\right) \times \frac{1}{h}$$

$$= \frac{x - (x+h)}{(x+h)x} \times \frac{1}{h}$$

$$= \frac{-h}{(x+h)x} \times \frac{1}{h}$$

$$= \frac{-1}{(x+h)x}$$



So the gradient of the curve at P is

$$\lim_{h \to 0} \frac{-1}{(x+h)x} = \frac{-1}{x^2} = -x^{-2}$$

Hence $f'(x) = -x^{-2}$.

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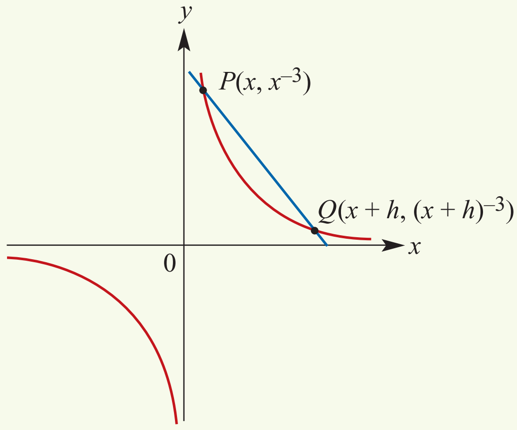
Example 22

Let $f(x) = x^{-3}, \{ x \in \mathbb{R} : x \neq 0 \}$. Find $f'(x)$ by first principles.

Solution

The gradient of secant PQ is given by

$$\frac{(x+h)^{-3} - x^{-3}}{h}$$
$$= \frac{x^3 - (x+h)^3}{(x+h)^3 x^3} \times \frac{1}{h}$$
$$= \frac{x^3 - (x^3 + 3x^2h + 3xh^2 + h^3)}{(x+h)^3 x^3} \times \frac{1}{h}$$
$$= \frac{-3x^2h - 3xh^2 - h^3}{(x+h)^3 x^3} \times \frac{1}{h}$$
$$= \frac{-3x^2 - 3xh - h^2}{(x+h)^3 x^3}$$



So the gradient of the curve at P is given by

$$\lim_{h \to 0} \frac{-3x^2 - 3xh - h^2}{(x+h)^3 x^3} = \frac{-3x^2}{x^6} = -3x^{-4}$$

Hence $f'(x) = -3x^{-4}$.

We are now in a position to state the generalisation of the result we found in Section 17D.

For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where n is a non-zero integer.

For $f(x) = c$, $f'(x) = 0$, where c is a constant.

When n is positive, we take the domain of f to be $\mathbb{R}$, and when n is negative, we take the domain of f to be $\{ x \in \mathbb{R} : x \neq 0 \}$.

Note: This result can be proved by again using the binomial theorem – the previous example gives the idea of the proof.

Example 23

Find the derivative of $x^4 - 2x^{-3} + x^{-1} + 2$, $x \neq 0$.

Solution

If $f(x) = x^4 - 2x^{-3} + x^{-1} + 2$ (for $x \neq 0$)

then $f'(x) = 4x^3 - 2(-3x^{-4}) + (-x^{-2}) + 0$

$= 4x^3 + 6x^{-4} - x^{-2}$ (for $x \neq 0$)

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Example 24

Find the gradient of the tangent to the curve determined by the function $f(x) = x^2 + \frac{1}{x}$, $\{ x \in \mathbb{R} : x \neq 0 \}$ at the point $(1, 2)$.

Solution

Let f' have domain $\{ x \in \mathbb{R} : x \neq 0 \}$

$$f'(x) = 2x + (-x^{-2}) = 2x - x^{-2}$$

Therefore $f'(1) = 2 - 1 = 1$. The gradient of the curve is 1 at the point $(1, 2)$.

Example 25

Show that the derivative of the function $f(x) = x^{-3}$, $\{ x \in \mathbb{R} : x \neq 0 \}$ is always negative.

Solution

Let f' have domain $\{ x \in \mathbb{R} : x \neq 0 \}$

$$f'(x) = -3x^{-4} = \frac{-3}{x^4}$$

Since x^4 is positive for all $x \neq 0$, we have $f'(x) < 0$ for all $x \neq 0$.

Summary 17E

For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where n is a non-zero integer.

For $f(x) = c$, $f'(x) = 0$, where c is a constant.

Skill-sheet

Exercise 17E

Example 21

1 **a** $f(x) = \frac{1}{x-3}$, $\{ x \in \mathbb{R} : x \neq 3 \}$. Find $f'(x)$ by first principles.

b $f(x) = \frac{1}{x+2}$, $\{ x \in \mathbb{R} : x \neq -2 \}$. Find $f'(x)$ by first principles.

Example 22

2 **a** $f(x) = x^{-2}$, $\{ x \in \mathbb{R} : x \neq 0 \}$. Find $f'(x)$ by first principles.

b $f(x) = x^{-4}$, $\{ x \in \mathbb{R} : x \neq 0 \}$. Find $f'(x)$ by first principles.

Hint: Remember that $(x + h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$.

Example 23

3 Differentiate each of the following with respect to x:

a $3x^{-2} + 5x^{-1} + 6$ **b** $\frac{3}{x^2} + 5x^2$ **c** $\frac{5}{x^3} + \frac{4}{x^2} + 1$

d $3x^2 + \frac{5}{3}x^{-4} + 2$ **e** $6x^{-2} + 3x$ **f** $\frac{3x^2 + 2}{x}$

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4 Find the derivative of each of the following:

a $\frac{3z^2 + 2z + 4}{z^2}$, $z \neq 0$ **b** $\frac{3 + z}{z^3}$, $z \neq 0$ **c** $\frac{2z^2 + 3z}{4z}$, $z \neq 0$

d $9z^2 + 4z + 6z^{-3}$, $z \neq 0$ **e** $9 - z^{-2}$, $z \neq 0$ **f** $\frac{5z - 3z^2}{5z}$, $z \neq 0$

5 **a** Find the derivative f' of $f(x) = 3x^4 - 6x^{-3} + x^{-1}, \{ x \in \mathbb{R} : x \neq 0 \}$.

b Find the derivative f' of $f(x) = 5x^4 + 4x^{-2} + x^{-1}, \{ x \in \mathbb{R} : x \neq 0 \}$.

6 With the help of your calculator, sketch the graph of $f(x) = \frac{1}{x^2}$, $x \neq 0$.

a Let P be the point $(1, f(1))$ and Q the point $(1 + h, f(1 + h))$. Find the gradient of the secant PQ.

b Hence find the gradient of the tangent line to the curve $y = \frac{1}{x^2}$ at $x = 1$.

Example 24

7 For each of the following curves, find the gradient of the tangent line to the curve at the given point:

a $y = x^{-2} + x^3$, $x \neq 0$, at $\left(2, 8\frac{1}{4}\right)$ **b** $y = \frac{x - 2}{x}$, $x \neq 0$, at $\left(4, \frac{1}{2}\right)$

c $y = x^{-2} - \frac{1}{x}$, $x \neq 0$, at $(1, 0)$ **d** $y = x(x^{-1} + x^2 - x^{-3})$, $x \neq 0$, at $(1, 1)$

8 For the curve with equation $f(x) = x^{-2}$, find the x-coordinate of the point on the curve at which the gradient of the tangent line is:

a 16 **b** −16

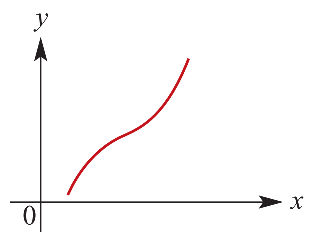
Example 25

9 Show that the derivative of the function $f(x) = x^{-1}, \{ x \in \mathbb{R} : x \neq 0 \}$ is always negative.

17F Graphs of the derivative function

Increasing and decreasing functions

We say a function f is **strictly increasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.



For example:

- The graph opposite shows a strictly increasing function.
- A straight line with positive gradient is strictly increasing.
- The function $f(x) = x^2, \{ x \in \mathbb{R} : x > 0 \}$ is strictly increasing.

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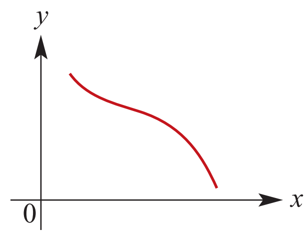
We say a function f is **strictly decreasing** on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.



For example:

- The graph opposite shows a strictly decreasing function.
- A straight line with negative gradient is strictly decreasing.
- The function $f(x) = x^2, \{ x \in \mathbb{R} : x < 0 \}$ is strictly decreasing.

Note: The word *strictly* refers to the use of the strict inequality signs $<, >$ rather than $\leq, \geq$.

If $f'(x) > 0$, for all x in the interval, then the function is strictly increasing.
(Think of the tangents at each point – they each have positive gradient.)

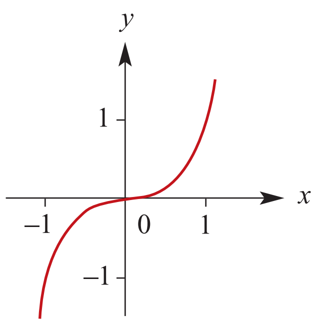
If $f'(x) < 0$, for all x in the interval, then the function is strictly decreasing.
(Think of the tangents at each point – they each have negative gradient.)

Warning: The function $f(x) = x^3$ is strictly increasing, but $f'(0) = 0$. This means that *strictly increasing does not imply $f'(x) > 0$.*

We can see that $f(x) = x^3$ is strictly increasing from its graph. Alternatively, consider

$$\begin{aligned} a^3 - b^3 &= (a - b)(a^2 + ab + b^2) \\ &= (a - b)\left(a^2 + ab + \left(\frac{1}{2}b\right)^2 + b^2 - \left(\frac{1}{2}b\right)^2\right) \\ &= (a - b)\left(\left(a + \frac{1}{2}b\right)^2 + \frac{3}{4}b^2\right) \end{aligned}$$

Hence $a > b$ implies $a^3 > b^3$.

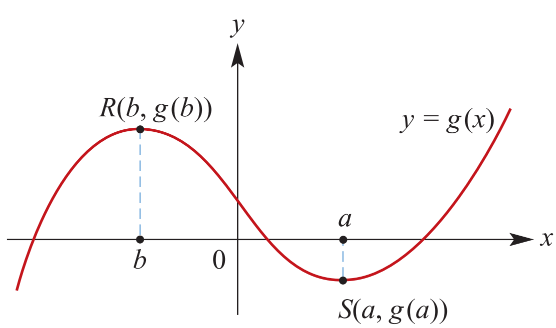


Sign of the derivative

Consider the graph of $y = g(x)$ shown opposite.

At a point $(a, g(a))$ on the graph, the gradient is $g'(a)$.

By noting whether the curve is sloping upwards or downwards at a particular point, we can tell the sign of the derivative at that point:



- For $x < b$, $g'(x) > 0$. This implies that g is strictly increasing on the interval $(-\infty, b)$.
- For $x = b$, $g'(b) = 0$.
- For $b < x < a$, $g'(x) < 0$. This implies that g is strictly decreasing on the interval (b, a).
- For $x = a$, $g'(a) = 0$.
- For $x > a$, $g'(x) > 0$. This implies that g is strictly increasing on the interval (a, ∞).

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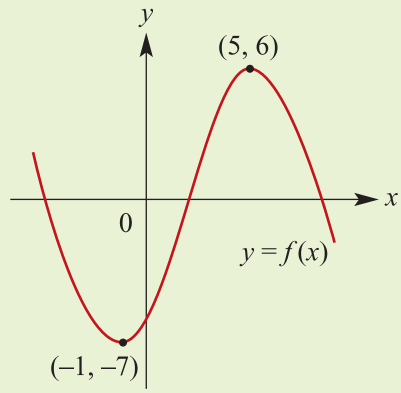
Example 26

For the graph of $f : \mathbb{R} \to \mathbb{R}$, find:

a $\{x : f'(x) > 0\}$

b $\{x : f'(x) < 0\}$

c $\{x : f'(x) = 0\}$



Solution

a $\{x : f'(x) > 0\} = \{x : -1 < x < 5\} = (-1, 5)$

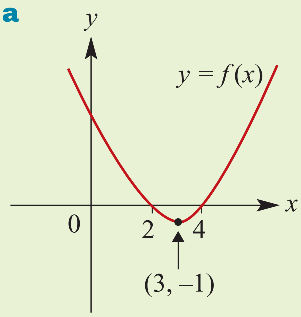
b $\{x : f'(x) < 0\} = \{x : x < -1\} \cup \{x : x > 5\} = (-\infty, -1) \cup (5, \infty)$

c $\{x : f'(x) = 0\} = \{-1, 5\}$

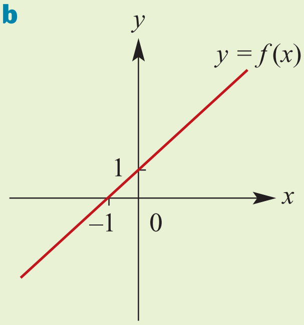
Example 27

Sketch the graph of $y = f'(x)$ for each of the following. (It is impossible to determine all features.)

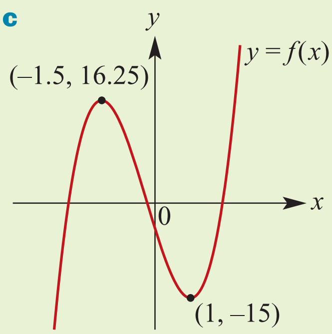
a



b



c

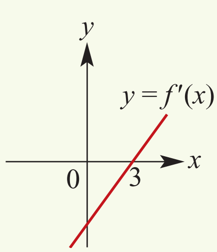


Solution

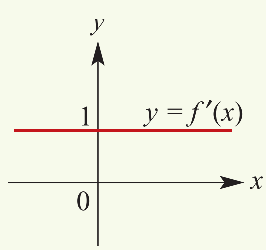
a $f'(x) > 0$ for $x > 3$

$f'(x) < 0$ for $x < 3$

$f'(x) = 0$ for $x = 3$



b $f'(x) = 1$ for all x

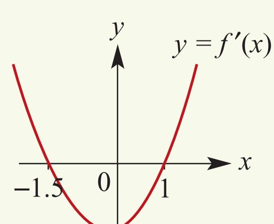


c $f'(x) > 0$ for $x > 1$

$f'(x) < 0$ for $-1.5 < x < 1$

$f'(x) > 0$ for $x < -1.5$

$f'(-1.5) = 0$ and $f'(1) = 0$



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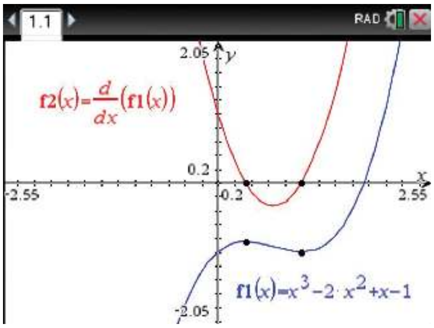
If the rule for the function is given, then a CAS calculator can be used to plot the graph of its derivative function.

Using the TI-Nspire

Plot the graphs of

$$f_1(x) = x^3 - 2x^2 + x - 1$$

$$f_2(x) = \frac{d}{dx}(f_1(x))$$



An angle associated with the gradient of a curve at a point

The gradient of a curve at a point is the gradient of the tangent at that point. A straight line, the tangent, is associated with each point on the curve.

If α is the angle a straight line makes with the positive direction of the x-axis, then the gradient, m, of the straight line is equal to $\tan\alpha$. That is, $m = \tan\alpha$.

For example:

- If $\alpha = 45°$, then $\tan\alpha = 1$ and the gradient is 1.
- If $\alpha = 20°$, then the gradient of the straight line is $\tan 20°$.
- If $\alpha = 135°$, then $\tan\alpha = -1$ and the gradient is -1.

Example 28

Find the coordinates of the points on the curve with equation $y = x^2 - 7x + 8$ at which the tangent line:

a makes an angle of 45° with the positive direction of the x-axis

b is parallel to the line $y = -2x + 6$.

Solution

a $\dfrac{dy}{dx} = 2x - 7$

$2x - 7 = 1 \qquad (\text{as } \tan 45° = 1)$

$2x = 8$

$\therefore \quad x = 4$

$y = 4^2 - 7 \times 4 + 8 = -4$

The coordinates are $(4, -4)$.

b The line $y = -2x + 6$ has gradient -2.

$2x - 7 = -2$

$2x = 5$

$\therefore \quad x = \dfrac{5}{2}$

The coordinates are $\left(\dfrac{5}{2}, -\dfrac{13}{4}\right)$.

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Example 29

The planned path for a flying saucer leaving a planet is defined by the equation

$$y = \frac{1}{4}x^4 + \frac{2}{3}x^3 \quad \text{for } x > 0$$

The units are kilometres. (The x-axis is horizontal and the y-axis vertical.)

a Find the direction of motion when the x-value is:

i 2 **ii** 3

b Find a point on the flying saucer's path where the path is inclined at 45° to the positive x-axis (i.e. where the gradient of the path is 1).

c Are there any other points on the path which satisfy the situation described in part b?

Solution

a $\dfrac{dy}{dx} = x^3 + 2x^2$

i When $x = 2$, $\dfrac{dy}{dx} = 8 + 8 = 16$

$\tan^{-1} 16 = 86.42°$ (to the x-axis)

ii When $x = 3$, $\dfrac{dy}{dx} = 27 + 18 = 45$

$\tan^{-1} 45 = 88.73°$ (to the x-axis)

b, c When the flying saucer is flying at 45° to the positive direction of the x-axis, the gradient of the curve of its path is given by $\tan 45°$. Thus to find the point at which this happens we consider the equation

$$\frac{dy}{dx} = \tan 45°$$

$$x^3 + 2x^2 = 1$$

$$x^3 + 2x^2 - 1 = 0$$

$$(x+1)(x^2 + x - 1) = 0$$

$$\therefore \quad x = -1 \text{ or } x = \frac{-1 \pm \sqrt{5}}{2}$$

The only acceptable solution is $x = \dfrac{-1 + \sqrt{5}}{2} \approx 0.62$, as the other two possibilities give negative values for x and we are only considering positive values for x.

Summary 17F

- A function f is **strictly increasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.
- A function f is **strictly decreasing** on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.
- If $f'(x) > 0$ for all x in the interval, then the function is strictly increasing.
- If $f'(x) < 0$ for all x in the interval, then the function is strictly decreasing.

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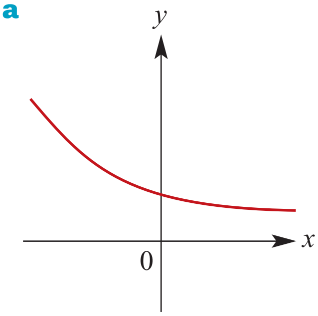
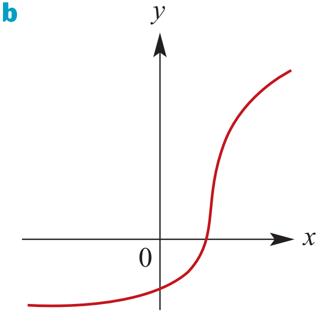
Skill-sheet Exercise 17F

1 For which of the following curves is $\dfrac{dy}{dx}$ positive for all values of x?

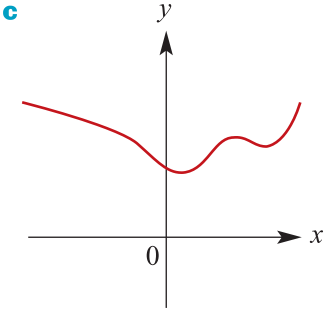
a



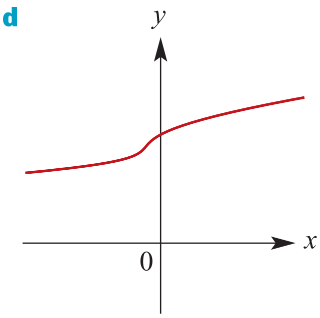
b



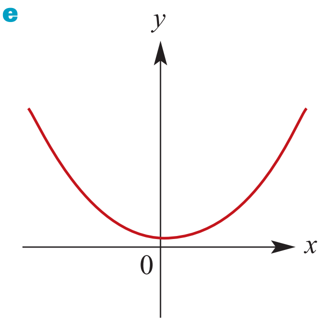
c



d

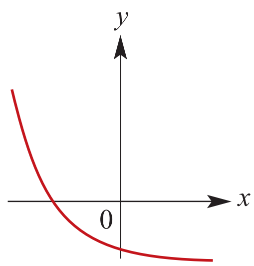


e

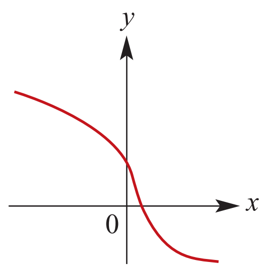


2 For which of the following curves is $\dfrac{dy}{dx}$ negative for all values of x?

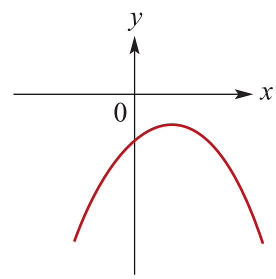
a



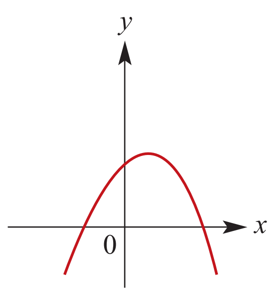
b



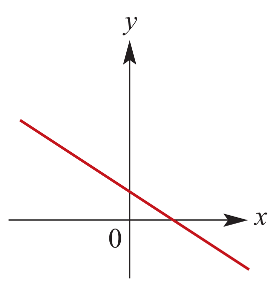
c



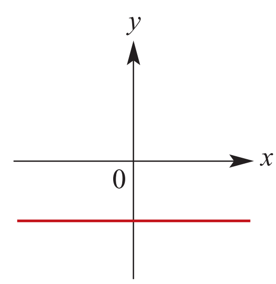
d



e



f



3 For the function $f(x) = 2(x - 1)^2$, find the values of x for which:

a $f(x) = 0$ **b** $f'(x) = 0$ **c** $f'(x) > 0$ **d** $f'(x) < 0$ **e** $f'(x) = -2$

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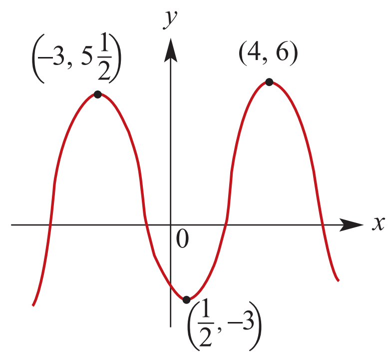
Example 26

4 For the graph of $y = h(x)$ shown here, find:

a $\{ x : h'(x) > 0 \}$

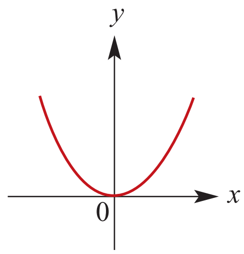
b $\{ x : h'(x) < 0 \}$

c $\{ x : h'(x) = 0 \}$

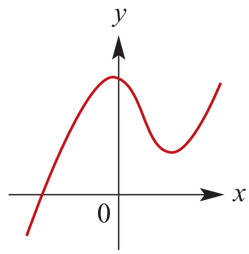


5 Which of the graphs labelled **A–F** correspond to each of the graphs labelled **a–f**?

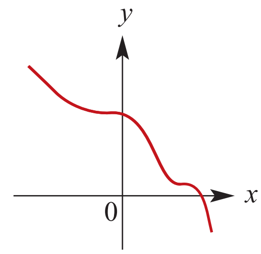
a



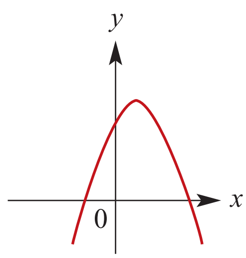
b



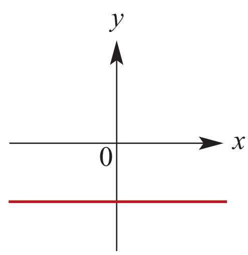
c



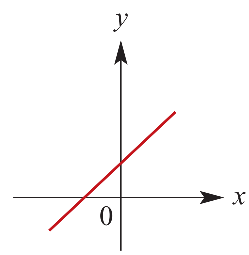
d



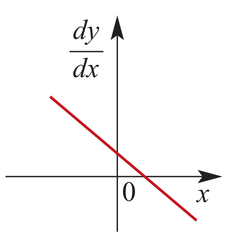
e



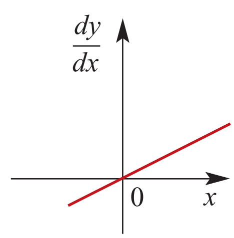
f



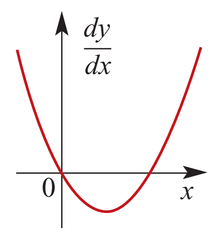
A



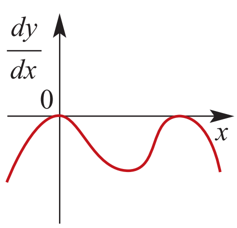
B



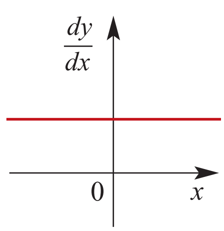
C



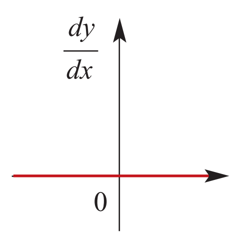
D



E



F



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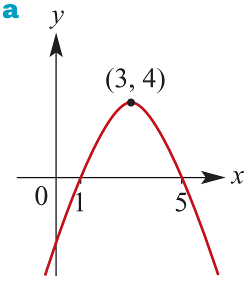
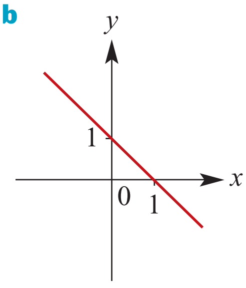
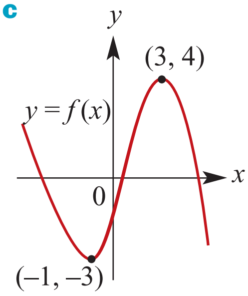
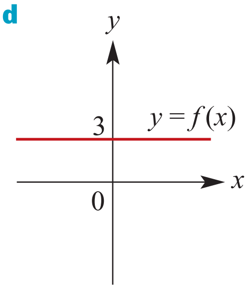
Example 27 **6** Sketch the graph of $y = f'(x)$ for each of the following:

a


b


c


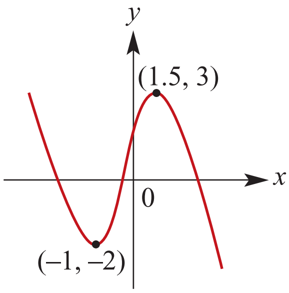
d


7 For the graph of $y = f(x)$ shown on the right, find:

a $\{ x : f'(x) > 0 \}$

b $\{ x : f'(x) < 0 \}$

c $\{ x : f'(x) = 0 \}$



Example 28 **8** Find the coordinates of the points on the curve $y = x^2 - 5x + 6$ at which the tangent:

a makes an angle of 45° with the positive direction of the x-axis

b is parallel to the line $y = 3x + 4$.

9 Find the coordinates of the points on the parabola $y = x^2 - x - 6$ at which:

a the gradient of the tangent is zero

b the tangent is parallel to the line $x + y = 6$.

10 Use a calculator to plot the graph of $y = f'(x)$ where:

a $f(x) = \sin x$ **b** $f(x) = \cos x$ **c** $f(x) = 2^x$

Example 29 **11** The path of a particle is defined by the equation $y = \frac{1}{3}x^3 + \frac{2}{3}x^2$, for $x > 0$. The units are metres. (The x-axis is horizontal and the y-axis vertical.)

a Find the direction of motion when the x-value is:

i 1 **ii** 0.5

b Find a point on the particle's path where the path is inclined at 45° to the positive direction of the x-axis.

c Are there any other points on the path which satisfy the situation described in part b?

12 The curve with equation $y = ax^2 + bx$ has a gradient of 3 at the point $(2, -2)$.

a Find the values of a and b.

b Find the coordinates of the point where the gradient is 0.

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17G Antidifferentiation of polynomial functions

The derivative of x^2 with respect to x is $2x$. Conversely, given that an unknown expression has derivative $2x$, it is clear that the unknown expression could be x^2. The process of finding a function from its derivative is called **antidifferentiation**.

Now consider the functions $f(x) = x^2 + 1$ and $g(x) = x^2 - 7$.

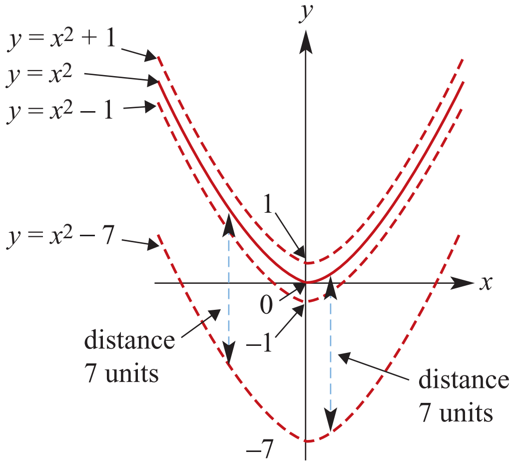
We have $f'(x) = 2x$ and $g'(x) = 2x$. So the two different functions have the same derivative function.

Both $x^2 + 1$ and $x^2 - 7$ are said to be **antiderivatives** of $2x$.

If two functions have the same derivative function, then they differ by a constant. So the graphs of the two functions can be obtained from each other by translation parallel to the y-axis.

The diagram shows several antiderivatives of $2x$.

Each of the graphs is a translation of $y = x^2$ parallel to the y-axis.



Notation

The general antiderivative of $2x$ is $x^2 + c$, where c is an arbitrary real number. We use the notation of Leibniz to state this with symbols:

$$\int 2x\,dx = x^2 + c$$

This is read as 'the **general antiderivative** of $2x$ with respect to x is equal to $x^2 + c$' or as 'the **indefinite integral** of $2x$ with respect to x is $x^2 + c$'.

To be more precise, the indefinite integral is the set of all antiderivatives and to emphasise this we could write:

$$\begin{aligned}\int 2x\,dx &= \{\, f(x) : f'(x) = 2x \,\}\\ &= \{\, x^2 + c : c \in \mathbb{R} \,\}\end{aligned}$$

This set notation is not commonly used, but it should be clearly understood that there is not a unique antiderivative for a given function. We will not use this set notation, but it is advisable to keep it in mind when considering further results.

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In general:

If $F'(x) = f(x)$, then $\int f(x)\,dx = F(x) + c$, where c is an arbitrary real number.

Rules for antidifferentiation

We know that:

$$f(x) = x^3 \quad \text{implies} \quad f'(x) = 3x^2$$
$$f(x) = x^8 \quad \text{implies} \quad f'(x) = 8x^7$$
$$f(x) = x \quad \text{implies} \quad f'(x) = 1$$
$$f(x) = x^n \quad \text{implies} \quad f'(x) = nx^{n-1}$$

Reversing this process we have:

$$\int 3x^2\,dx = x^3 + c \qquad \text{where } c \text{ is an arbitrary constant}$$
$$\int 8x^7\,dx = x^8 + c \qquad \text{where } c \text{ is an arbitrary constant}$$
$$\int 1\,dx = x + c \qquad \text{where } c \text{ is an arbitrary constant}$$
$$\int nx^{n-1}\,dx = x^n + c \qquad \text{where } c \text{ is an arbitrary constant}$$

We also have:

$$\int x^2\,dx = \frac{1}{3}x^3 + c$$
$$\int x^7\,dx = \frac{1}{8}x^8 + c$$
$$\int 1\,dx = x + c$$
$$\int x^{n-1}\,dx = \frac{1}{n}x^n + c$$

From this we see that:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c, \qquad n \in \mathbb{N} \cup \{0\}$$

Note: This result can be extended to include x^n where n is a negative integer other than -1. You will see the antiderivative of x^{-1} in Year 12.

We also record the following results, which follow immediately from the corresponding results for differentiation:

Sum $\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$

Difference $\int f(x) - g(x)\,dx = \int f(x)\,dx - \int g(x)\,dx$

Multiple $\int kf(x)\,dx = k\int f(x)\,dx$, where k is a real number

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Example 30

Find the general antiderivative (indefinite integral) of each of the following:

a $3x^5$ **b** $3x^2 + 4x^3 + 3$

Solution

a $\int 3x^5\,dx = 3\int x^5\,dx$

$= 3 \times \frac{x^6}{6} + c$

$= \frac{x^6}{2} + c$

b $\int 3x^2 + 4x^3 + 3\,dx$

$= 3\int x^2\,dx + 4\int x^3\,dx + 3\int 1\,dx$

$= \frac{3x^3}{3} + \frac{4x^4}{4} + \frac{3x}{1} + c$

$= x^3 + x^4 + 3x + c$

Given extra information, we can find a unique antiderivative.

Example 31

It is known that $f'(x) = x^3 + 4x^2$ and $f(0) = 0$. Find $f(x)$.

Solution

$\int x^3 + 4x^2\,dx = \frac{x^4}{4} + \frac{4x^3}{3} + c$

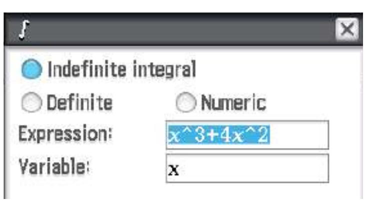
Thus $f(x) = \frac{x^4}{4} + \frac{4x^3}{3} + c$ for some real number c.

Since $f(0) = 0$, we have $c = 0$.

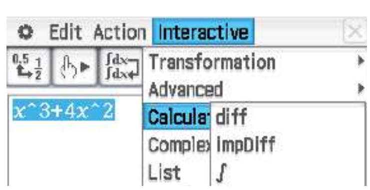
$\therefore \quad f(x) = \frac{x^4}{4} + \frac{4x^3}{3}$

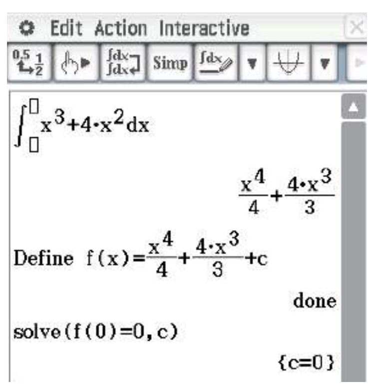
Using the Casio ClassPad

- To find the general antiderivative of $x^3 + 4x^2$, enter and highlight the expression in Main $\sqrt{\alpha}$.
- Select **Interactive** > **Calculation** > $\int$.
- Ensure that ‘Indefinite integral’ is selected, as shown below.



- Remember to add a constant c to the answer.
- To find the specific antiderivative, define the family of functions $f(x)$.
- Solve $f(0) = 0$ for c.





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Example 32

If the gradient of the tangent at a point (x, y) on a curve is given by $5x$ and the curve passes through $(0, 6)$, find the equation of the curve.

Solution

Let the curve have equation $y = f(x)$. Then $f'(x) = 5x$.

$$\int 5x\,dx = \frac{5x^2}{2} + c$$

$$\therefore \quad f(x) = \frac{5x^2}{2} + c$$

This describes the family of curves for which $f'(x) = 5x$. Here we are given the additional information that the curve passes through $(0, 6)$, i.e. $f(0) = 6$.

Hence $c = 6$ and so $f(x) = \dfrac{5x^2}{2} + 6$.

Example 33

Find y in terms of x if:

a $\dfrac{dy}{dx} = x^2 + 2x$, and $y = 1$ when $x = 1$

b $\dfrac{dy}{dx} = 3 - x$, and $y = 2$ when $x = 4$

Solution

a $\displaystyle\int x^2 + 2x\,dx = \frac{x^3}{3} + x^2 + c$

$$\therefore \quad y = \frac{x^3}{3} + x^2 + c$$

As $y = 1$ when $x = 1$,

$$1 = \frac{1}{3} + 1 + c$$

$$c = -\frac{1}{3}$$

Hence $y = \dfrac{x^3}{3} + x^2 - \dfrac{1}{3}$

b $\displaystyle\int 3 - x\,dx = 3x - \frac{x^2}{2} + c$

$$\therefore \quad y = 3x - \frac{x^2}{2} + c$$

As $y = 2$ when $x = 4$,

$$2 = 3 \times 4 - \frac{4^2}{2} + c$$

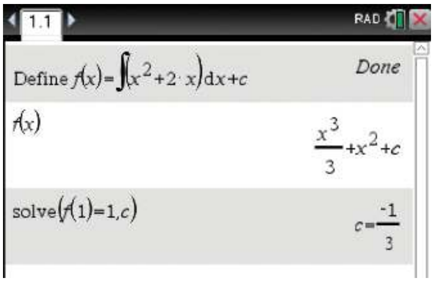
$$c = -2$$

Hence $y = 3x - \dfrac{x^2}{2} - 2$

Using the TI-Nspire

For Example 33a:

- To find the general antiderivative, define the function $f(x)$ using menu > **Calculus** > **Integral** as shown.
- Check that c has not been assigned a value.
- For the specific antiderivative, find the value of c by solving $f(1) = 1$.



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Summary 17G

- Antiderivative of x^n, for $n \in \mathbb{N} \cup \{0\}$:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$$

- Rules of antidifferentiation:
  - $\int f(x) + g(x) \, dx = \int f(x) \, dx + \int g(x) \, dx$
  - $\int f(x) - g(x) \, dx = \int f(x) \, dx - \int g(x) \, dx$
  - $\int kf(x) \, dx = k \int f(x) \, dx$, where k is a real number

Exercise 17G

Example 30

1 Find:

a $\int \frac{1}{2}x^3 \, dx$ **b** $\int 3x^2 - 2 \, dx$ **c** $\int 5x^3 - 2x \, dx$

d $\int \frac{4}{5}x^3 - 2x^2 \, dx$ **e** $\int (x-1)^2 \, dx$ **f** $\int x\left(x + \frac{1}{x}\right) dx, \; x \neq 0$

g $\int 2z^2(z-1) \, dz$ **h** $\int (2t-3)^2 \, dt$ **i** $\int (t-1)^3 \, dt$

Example 31

2 It is known that $f'(x) = 4x^3 + 6x^2 + 2$ and $f(0) = 0$. Find $f(x)$.

Example 32

3 If the gradient at a point (x, y) on a curve is given by $6x^2$ and the curve passes through $(0, 12)$, find the equation of the curve.

Example 33

4 Find y in terms of x in each of the following:

a $\frac{dy}{dx} = 2x - 1$, and $y = 0$ when $x = 1$

b $\frac{dy}{dx} = 3 - x$, and $y = 1$ when $x = 0$

c $\frac{dy}{dx} = x^2 + 2x$, and $y = 2$ when $x = 0$

d $\frac{dy}{dx} = 3 - x^2$, and $y = 2$ when $x = 3$

e $\frac{dy}{dx} = 2x^4 + x$, and $y = 0$ when $x = 0$

5 Assume that $\frac{dV}{dt} = t^2 - t$ for $t > 1$, and that $V = 9$ when $t = 3$.

a Find V in terms of t.

b Calculate the value of V when $t = 10$.

6 The gradient of the tangent at any point $(x, f(x))$ on the curve with equation $y = f(x)$ is given by $3x^2 - 1$. Find $f(x)$ if the curve passes through the point $(1, 2)$, i.e. $f(1) = 2$.

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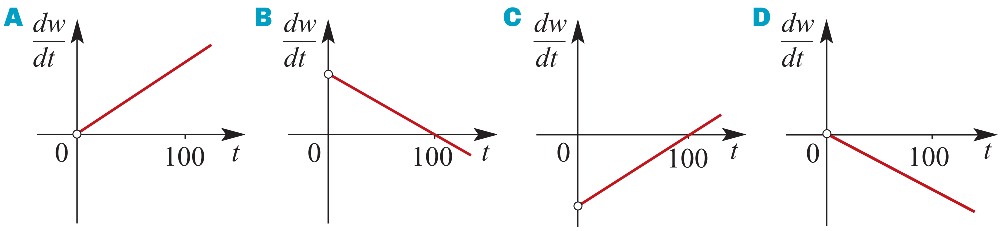
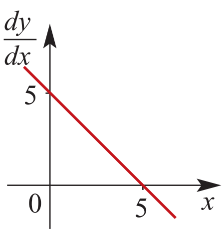
7 **a** Which one of the following graphs represents $\frac{dw}{dt} = 2000 - 20t$, $t > 0$?



b Find w in terms of t if when $t = 0$, $w = 100\,000$.

8 The graph shows $\frac{dy}{dx}$ against x for a certain curve with equation $y = f(x)$.

Find $f(x)$, given that the point $(0, 4)$ lies on the curve.



9 Find the equation of the curve $y = f(x)$ which passes through the point $(2, -6)$ and for which $f'(x) = x^2(x - 3)$.

10 The curve $y = f(x)$ for which $f'(x) = 4x + k$, where k is a constant, has a turning point at $(-2, -1)$.

a Find the value of k.

b Find the coordinates of the point at which the curve meets the y-axis.

11 Given that $\frac{dy}{dx} = ax^2 + 1$ and that when $x = 1$, $\frac{dy}{dx} = 3$ and $y = 3$, find the value of y when $x = 2$.

12 The curve for which $\frac{dy}{dx} = 2x + k$, where k is a constant, is such that the tangent at $(3, 6)$ passes through the origin. Find the gradient of this tangent and hence determine:

a the value of k

b the equation of the curve.

13 The curve $y = f(x)$ for which $f'(x) = 16x + k$, where k is a constant, has a turning point at $(2, 1)$.

a Find the value of k.

b Find the value of $f(x)$ when $x = 7$.

14 Suppose that a point moves along some unknown curve $y = f(x)$ in such a way that, at each point (x, y) on the curve, the tangent line has slope x^2. Find an equation for the curve, given that it passes through $(2, 1)$.

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Review

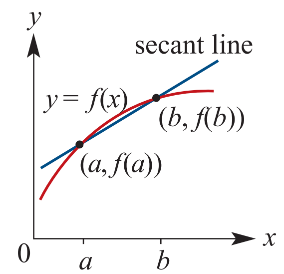
Chapter summary

Assign-ment

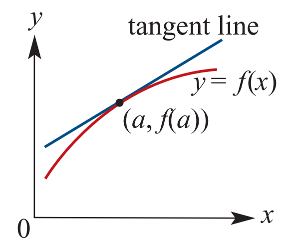
Nrich

Rates of change

- For a function $y = f(x)$, the **average rate of change** of y with respect to x over the interval $[a, b]$ is the gradient of the secant line through $(a, f(a))$ and $(b, f(b))$.
  That is,
  $$\text{average rate of change} = \frac{f(b) - f(a)}{b - a}$$

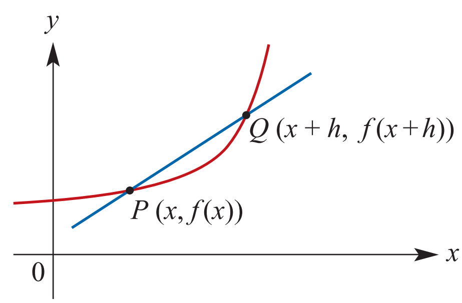


- For a function $y = f(x)$, the **instantaneous rate of change** of y with respect to x at the point $(a, f(a))$ is the gradient of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$.

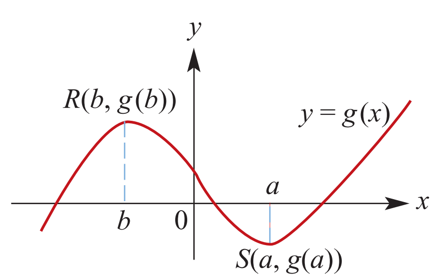


The derivative

- The notation for the limit as h approaches 0 is $\lim_{h \to 0}$.
- For the graph of $y = f(x)$:
  - The gradient of the secant PQ is given by
    $$\frac{f(x + h) - f(x)}{h}$$
  - The gradient of the tangent to the graph at the point P is given by
    $$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$



- The **derivative** of the function f is denoted f' and is defined by
  $$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$
- At a point $(a, g(a))$ on the curve $y = g(x)$, the gradient is $g'(a)$.
  For the graph shown:
  - $g'(x) > 0$ for $x < b$ and for $x > a$
  - $g'(x) < 0$ for $b < x < a$
  - $g'(x) = 0$ for $x = b$ and for $x = a$.



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Rules for differentiation

- For $f(x) = c$, $f'(x) = 0$. That is, the derivative of a constant function is zero.
  For example:
  - $f(x) = 1$, $f'(x) = 0$
  - $f(x) = 27.3$, $f'(x) = 0$
- For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where n is a non-zero integer.
  For example:
  - $f(x) = x^2$, $f'(x) = 2x$
  - $f(x) = x^4$, $f'(x) = 4x^3$
  - $f(x) = x^{-1}$, $f'(x) = -x^{-2}$
  - $f(x) = x^{-3}$, $f'(x) = -3x^{-4}$
- For $f(x) = k\,g(x)$, where k is a constant, $f'(x) = k\,g'(x)$.
  That is, the derivative of a number multiple is the multiple of the derivative.
  For example:
  - $f(x) = 3x^2$, $f'(x) = 3(2x) = 6x$
  - $f(x) = 4x^3$, $f'(x) = 4(3x^2) = 12x^2$
- For $f(x) = g(x) + h(x)$, $f'(x) = g'(x) + h'(x)$.
  That is, the derivative of a sum is the sum of the derivatives.
  For example:
  - $f(x) = x^2 + x^3$, $f'(x) = 2x + 3x^2$
  - $f(x) = 3x^2 + 4x^3$, $f'(x) = 6x + 12x^2$

Antiderivatives

- If $F'(x) = f(x)$, then $\int f(x)\,dx = F(x) + c$, where c is an arbitrary real number.
- $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c$, where $n \in \mathbb{N} \cup \{0\}$
- $\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$
- $\int kf(x)\,dx = k\int f(x)\,dx$, where k is a real number.

Short-answer questions

1 Let $y = x^3$. Find the average rate at which y changes with respect to x over each of the following intervals:

a $[0, 1]$ **b** $[1, 3]$

2 Find the derivative of each of the following by first principles:

a $y = 3x + 1$ **b** $y = 4 - x^2$ **c** $y = x^2 + 5x$
d $y = x^3 + x$ **e** $y = x^2 + 2x + 1$ **f** $y = 3x^2 - x$

3 Find $\dfrac{dy}{dx}$ when:

a $y = 3x^2 - 2x + 6$ **b** $y = 5$ **c** $y = 2x(2 - x)$
d $y = 4(2x - 1)(5x + 2)$ **e** $y = (x + 1)(3x - 2)$ **f** $y = (x + 1)(2 - 3x)$

4 Find $\dfrac{dy}{dx}$ when:

a $y = -x$ **b** $y = 10$ **c** $y = \dfrac{(x + 3)(2x + 1)}{4}$
d $y = \dfrac{2x^3 - x^2}{3x}$ **e** $y = \dfrac{x^4 + 3x^2}{2x^2}$

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Review

5 For each of the following functions, find the y-coordinate and the gradient of the tangent at the point on the curve for the given value of x:

a $y = x^2 - 2x + 1, \quad x = 2$
b $y = x^2 - 2x, \quad x = -1$
c $y = (x + 2)(x - 4), \quad x = 3$
d $y = 3x^2 - 2x^3, \quad x = -2$

6 Find the coordinates of the points on the curves given by the following equations at which the gradient of the tangent at that point has the given value:

a $y = x^2 - 3x + 1, \quad \dfrac{dy}{dx} = 0$
b $y = x^3 - 6x^2 + 4, \quad \dfrac{dy}{dx} = -12$
c $y = x^4 - 2x^3 + 1, \quad \dfrac{dy}{dx} = 0$
d $y = x(x - 3)^2, \quad \dfrac{dy}{dx} = 0$

7 For the function with rule $f(x) = 3(2x - 1)^2$, find the values of x for which:

a $f(x) = 0$
b $f'(x) = 0$
c $f'(x) > 0$
d $f'(x) < 0$
e $f(x) > 0$
f $f'(x) = 3$

8 The graph of $y = ax^2 + bx$ has a tangent with gradient 3 at the point $(1, 1)$. Find:

a the values of a and b
b the coordinates of the points where the gradient is 0.

9 Find:

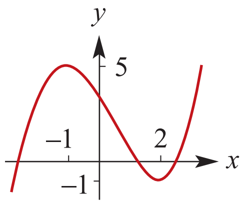
a $\int \frac{1}{2}\, dx$
b $\int \frac{1}{2}x^2\, dx$
c $\int (x^2 + 3x)\, dx$
d $\int (2x + 3)^2\, dx$
e $\int at\, dt$
f $\int \frac{1}{3}t^3\, dt$
g $\int (t + 1)(t - 2)\, dt$
h $\int (2 - t)(t + 1)\, dt$

10 The curve $y = f(x)$ passes through the point $(3, -1)$ and $f'(x) = 2x + 5$. Find $f(x)$.

11 The curve with equation $y = f(x)$ passes through the origin and $f'(x) = 3x^2 - 8x + 3$.

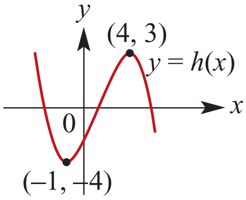
a Find $f(x)$.
b Find the intercepts of the curve with the x-axis.

12 The graph of $y = f(x)$ is shown. Sketch the graph of $y = f'(x)$. (Not all details can be determined, but the x-axis intercepts and the shape of graph can be determined.)



13 For the graph of $y = h(x)$, find:

a $\{x : h'(x) > 0\}$
b $\{x : h'(x) < 0\}$
c $\{x : h'(x) = 0\}$



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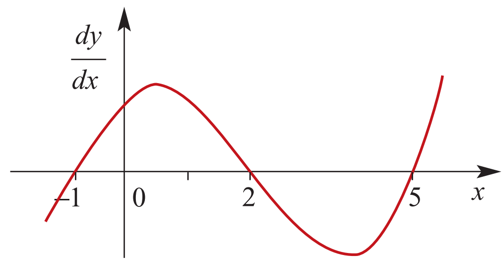
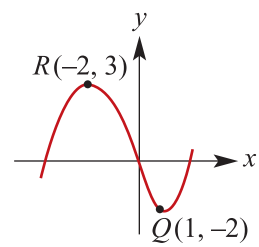
Extended-response questions

1 The diagram to the right shows part of the graph of $\frac{dy}{dx}$ against x.
Sketch a possible shape for the graph of y against x over the same interval if:
- $y = -1$ when $x = -1$
- $y = 0$ when $x = 0$
- $y = 1$ when $x = 2$.



2 The graph shown is that of a polynomial function of the form $P(x) = ax^3 + bx^2 + cx + d$. Find the values of a, b, c and d.
Note: $Q(1, -2)$ is not a turning point.



3 A body moves in a path described by the equation $y = \frac{1}{5}x^5 + \frac{1}{2}x^4$, for $x \geq 0$.
Units are in kilometres, and x and y are the horizontal and vertical axes respectively.
- **a** What will be the direction of motion (give your answer as the angle between the direction of motion and the x-axis) when the x-value is:
  - **i** 1 km **ii** 3 km?
- **b** Find a value of x for which the gradient of the path is 32.

4 A trail over a mountain pass can be modelled by the curve $y = 2 + 0.12x - 0.01x^3$, where x and y are the horizontal and vertical distances respectively, measured in kilometres, and $0 \leq x \leq 3$.
- **a** Find the gradients at the beginning and the end of the trail.
- **b** Calculate the point where the gradient is zero, and calculate also the height of the pass.

5
- **a** Show that the gradients of the curve $y = x(x - 2)$ at the points $(0, 0)$ and $(2, 0)$ only differ in sign. What is the geometrical interpretation for this?
- **b** If the gradients of the curve $y = x(x - 2)(x - 5)$ at the points $(0, 0)$, $(2, 0)$ and $(5, 0)$ are ℓ, m and n respectively, show that $\frac{1}{\ell} + \frac{1}{m} + \frac{1}{n} = 0$.

6 In the following, $f(x)$ is the rule for a well-behaved function f.
Assume that, for $y = f(x)$, the average rate of change of y with respect to x is m, over the interval $[a, b]$. Find the average rate of change of y with respect to x over the same interval $[a, b]$ for:

a $y = f(x) + c$ **b** $y = cf(x)$ **c** $y = -f(x)$

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18 Applications of differentiation of polynomials

In this chapter

Syllabus references

Topic: Applications of derivatives

Subtopics: 2.3.16 – 2.3.21

In this chapter we continue our study of calculus. There are two main aspects of this chapter. One is to apply our knowledge of the derivative to sketching graphs and solving maximum and minimum problems. The other is to see that the derivative can be used to define instantaneous rate of change.

The new techniques for sketching graphs of polynomial functions are a useful addition to the skills that were introduced in Chapter 7. At that stage, rather frustratingly, we were only able to determine the coordinates of turning points of cubic and quartic functions using technology. The new techniques are also used for determining maximum or minimum values for problems set in a 'real world' context.

The use of the derivative to determine instantaneous rates of change is a very important application of calculus. One of the first areas of applied mathematics to be studied in the seventeenth century was motion in a straight line. The problems of kinematics were the motivation for Newton's work on calculus.

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18A Tangents and normals

The derivative of a function is a new function which gives the measure of the gradient of the tangent at each point on the curve. If the gradient is known, it is possible to find the equation of the tangent for a given point on the curve.

Suppose (x_1, y_1) is a point on the curve $y = f(x)$. Then, if f is differentiable at $x = x_1$, the equation of the tangent at (x_1, y_1) is given by $y - y_1 = f'(x_1)(x - x_1)$.

Example 1

Find the equation of the tangent to the curve $y = x^3 + \frac{1}{2}x^2$ at the point $x = 1$.

Solution

When $x = 1$, $y = \frac{3}{2}$, and so $\left(1, \frac{3}{2}\right)$ is a point on the tangent.

Since $\frac{dy}{dx} = 3x^2 + x$, the gradient of the tangent to the curve at $x = 1$ is 4.

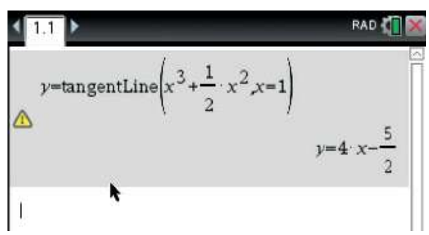
Hence the equation of the tangent is

$$y - \frac{3}{2} = 4(x - 1)$$

which becomes $y = 4x - \frac{5}{2}$.

Using the TI-Nspire

Use (menu) > **Calculus** > **Tangent Line** to calculate the tangent to the curve at $x = 1$.

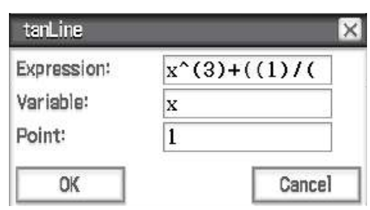


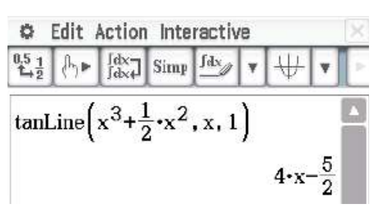
Using the Casio ClassPad

The tangent to the graph of $y = x^3 + \frac{1}{2}x^2$ at the point $x = 1$ can be found in two ways.

Method 1

- In Main $\sqrt{\alpha}$, enter and highlight the expression $x^3 + \frac{1}{2}x^2$.
- Select **Interactive** > **Calculation** > **line** > **tanLine**.
- The pop-up window shown will appear. Enter the value 1 for the point and tap OK.





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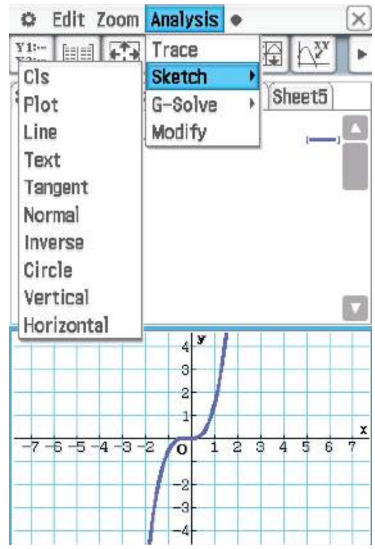
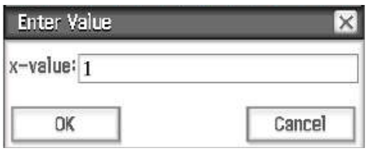
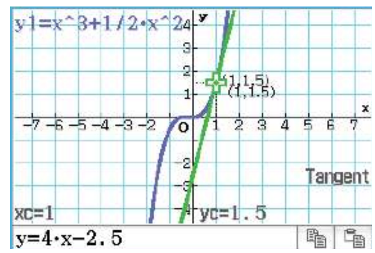
Method 2

- In [Graph & Table icon], enter the expression $x^3 + \frac{1}{2}x^2$ in $y1$.
- Tick the box for $y1$ and select the graph icon [graph icon].
- Select **Analysis** > **Sketch** > **Tangent**.
- When the graph appears, press the x-value of interest, in this case $x = 1$, and the window shown below will appear. Tap OK.





- The tangent at $x = 1$ is drawn on the graph.
- To view the equation of the tangent, tap EXE.
- The tangent equation is shown at the bottom of the screen.



The **normal** to a curve at a point on the curve is the line which passes through the point and is perpendicular to the tangent at that point.

Recall from Chapter 2 that two lines with gradients m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$.

Thus, if a tangent has gradient m, the normal has gradient $-\frac{1}{m}$.

Example 2

Find the equation of the normal to the curve with equation $y = x^3 - 2x^2$ at the point $(1, -1)$.

Solution

The point $(1, -1)$ lies on the normal.

Since $\frac{dy}{dx} = 3x^2 - 4x$, the gradient of the tangent at $x = 1$ is -1.

Thus the gradient of the normal at $x = 1$ is $\frac{-1}{-1} = 1$.

Hence the equation of the normal is

$$y - (-1) = 1(x - 1)$$

i.e. the equation of the normal is $y = x - 2$.

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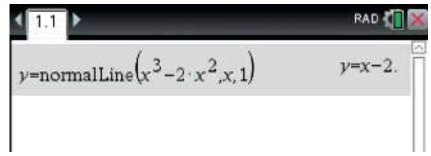
Using the TI-Nspire

Use (menu) > **Calculus** > **Normal Line** to calculate the normal to the curve at the point $(1, -1)$, i.e. when $x = 1$.

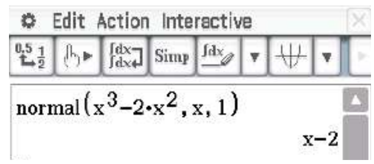


Using the Casio ClassPad

The normal to the graph of $y = x^3 - 2x^2$ at the point $(1, -1)$ can be found in two ways.

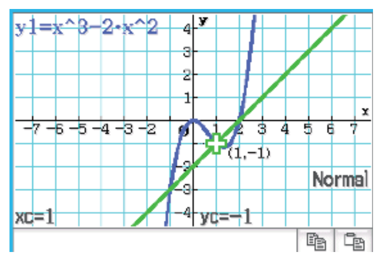
Method 1

- In Main $\sqrt{\alpha}$, enter and highlight $x^3 - 2x^2$.
- Select **Interactive** > **Calculation** > **line** > **normal**.
- In the pop-up window that appears, enter the value 1 for the point and tap OK.



Method 2

- In Graph&Table, enter the expression $x^3 - 2x^2$ in $y1$.
- Tick the box for $y1$ and select the graph icon.
- Select **Analysis** > **Sketch** > **Normal**.
- When the graph appears, press the x-value of interest, in this case $x = 1$. Tap OK.
- The normal at $x = 1$ is drawn on the graph.
- To view the equation of the normal, tap (EXE).



Summary 18A

- Equation of a tangent Suppose (x_1, y_1) is a point on the curve $y = f(x)$. Then, if f is differentiable at $x = x_1$, the equation of the tangent to the curve at (x_1, y_1) is given by $y - y_1 = f'(x_1)(x - x_1)$.
- Gradient of normal If a tangent has gradient m, the normal has gradient $-\frac{1}{m}$.

Exercise 18A

Example 1
Example 2

1 Find the equation of the tangent and the normal at the given point for:

a $f(x) = x^2$, $(2, 4)$
b $f(x) = (2x - 1)^2$, $(2, 9)$
c $f(x) = 3x - x^2$, $(2, 2)$
d $f(x) = 9x - x^3$, $(1, 8)$

2 Find the equation of the tangent to the curve with equation $y = 3x^3 - 4x^2 + 2x - 10$ at the point of intersection with the y-axis.

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3 Find the equation of the tangent to $y = x^2$ at the point $(1, 1)$ and the equation of the tangent to $y = \frac{1}{6}x^3$ at the point $\left(2, \frac{4}{3}\right)$.
Show that these tangents are parallel and find the perpendicular distance between them.

4 Find the equations of the tangents to the curve $y = x^3 - 6x^2 + 12x + 2$ which are parallel to the line $y = 3x$.

5 The curve with the equation $y = (x - 2)(x - 3)(x - 4)$ cuts the x-axis at the points $P = (2, 0)$, $Q = (3, 0)$ and $R = (4, 0)$.

a Prove that the tangents at P and R are parallel.

b At what point does the normal to the curve at Q cut the y-axis?

6 For the curve with equation $y = x^2 + 3$, show that $y = 2ax - a^2 + 3$ is the equation of the tangent at the point $(a, a^2 + 3)$.
Hence find the coordinates of the two points on the curve, the tangents of which pass through the point $(2, 6)$.

7 **a** Find the equation of the tangent at the point $(2, 4)$ to the curve $y = x^3 - 2x$.

b Find the coordinates of the point where the tangent meets the curve again.

8 **a** Find the equation of the tangent to the curve $y = x^3 - 9x^2 + 20x - 8$ at the point $(1, 4)$.

b At what points on the curve is the tangent parallel to the line $4x + y - 3 = 0$?

18B Rates of change

The derivative of a function was defined geometrically in Section 17C. But the process of differentiation may be used to tackle many kinds of problems involving rates of change.

For the function with rule $f(x)$:

- The **average rate of change** for $x \in [a, b]$ is given by $\dfrac{f(b) - f(a)}{b - a}$.
- The **instantaneous rate of change** of f with respect to x when $x = a$ is defined to be $f'(a)$.

Average rate of change has been discussed in Section 17A.

The instantaneous rate of change of y with respect to x is given by $\dfrac{dy}{dx}$, that is, by the derivative of y with respect to x.

- If $\dfrac{dy}{dx} > 0$, the change is an increase in the value of y corresponding to an increase in x.
- If $\dfrac{dy}{dx} < 0$, the change is a decrease in the value of y corresponding to an increase in x.

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Example 3

For the function with rule $f(x) = x^2 + 2x$, find:

a the average rate of change for $x \in [2, 3]$

b the average rate of change for the interval $[2, 2 + h]$

c the instantaneous rate of change of f with respect to x when $x = 2$.

Solution

a Average rate of change $= \dfrac{f(3) - f(2)}{3 - 2} = 15 - 8 = 7$

b Average rate of change $= \dfrac{f(2 + h) - f(2)}{2 + h - 2}$

$$= \frac{(2 + h)^2 + 2(2 + h) - 8}{h}$$

$$= \frac{4 + 4h + h^2 + 4 + 2h - 8}{h}$$

$$= \frac{6h + h^2}{h}$$

$$= 6 + h$$

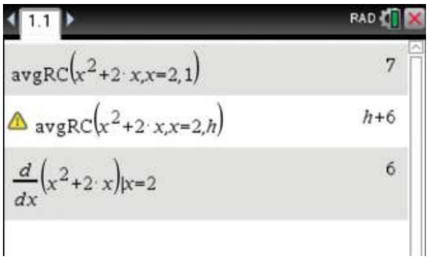
c The derivative is $f'(x) = 2x + 2$. When $x = 2$, the instantaneous rate of change is $f'(2) = 6$. This can also be seen from the result of part b.

Using the TI-Nspire

- For parts a and b, use the catalog to access the **Average Rate of Change** command (📖 1 A) and enter as:

  avgRC(*expression*, $x =$ *initial value*, *step size*)

- For part c, use menu > **Calculus** > **Derivative at a Point** and complete as shown.



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Example 4

A balloon develops a microscopic leak and gradually decreases in volume.

Its volume, V (cm^3), at time t (seconds) is $V = 600 - 10t - \frac{1}{100}t^2$, $t > 0$.

a Find the rate of change of volume after:

  i 10 seconds **ii** 20 seconds.

b For how long could the model be valid?

Solution

a $V = 600 - 10t - \frac{1}{100}t^2$

$$\frac{dV}{dt} = -10 - \frac{t}{50}$$

i When $t = 10$, $\frac{dV}{dt} = -10 - \frac{1}{5}$

$$= -10\frac{1}{5}$$

i.e. the volume is decreasing at a rate of $10\frac{1}{5}$ cm^3 per second.

ii When $t = 20$, $\frac{dV}{dt} = -10 - \frac{2}{5}$

$$= -10\frac{2}{5}$$

i.e. the volume is decreasing at a rate of $10\frac{2}{5}$ cm^3 per second.

b The model will not be meaningful when $V < 0$.

Consider $V = 0$:

$$600 - 10t - \frac{1}{100}t^2 = 0$$

$$\therefore \quad t = \frac{10 \pm \sqrt{100 + 4 \times 0.01 \times 600}}{-0.02}$$

$\therefore \quad t = -1056.78$ or $t = 56.78$ (to two decimal places)

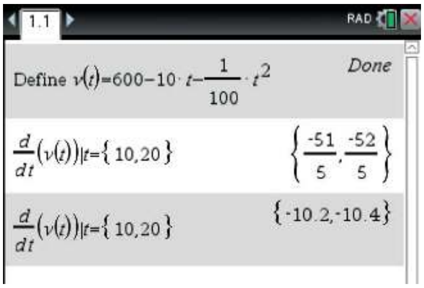
Hence the model may be suitable for $0 < t < 56.78$.

Using the TI-Nspire

Define $v(t) = 600 - 10t - (1/100)t^2$.

a Use menu > **Calculus** > **Derivative** and enter the required t-values using the | symbol (ctrl =) to evaluate the derivative of $v(t)$ at $t = 10$ and $t = 20$.

Press ctrl enter to obtain the answer as a decimal number.



Note: If you used menu > **Calculus** > **Derivative at a Point** instead, then each t-value would need to be evaluated separately.

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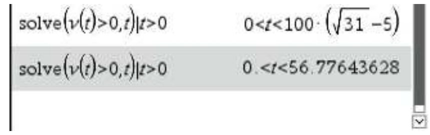
b To find the domain, use:

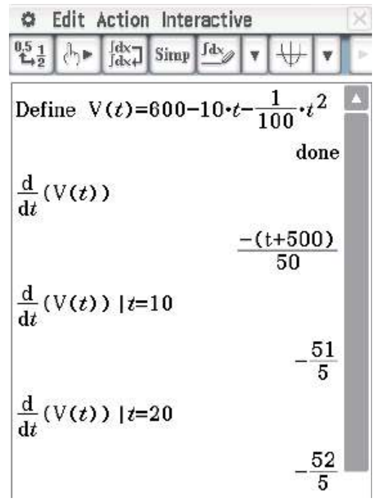
$$\text{solve}(v(t) > 0, t) \mid t > 0$$

Press ctrl enter to obtain the answer as a decimal number.

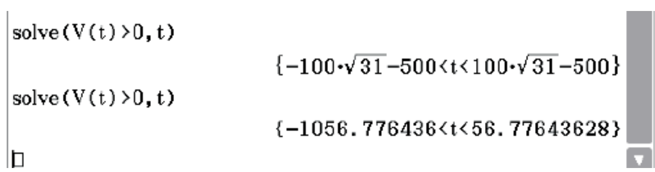


Using the Casio ClassPad

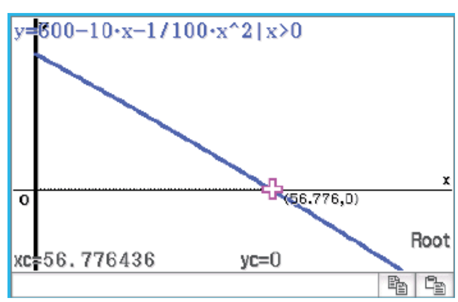
- In Main $\sqrt{\alpha}$, enter and highlight $600 - 10t - \frac{1}{100}t^2$.
- Select **Interactive** > **Define**: enter the function name V using the abc keyboard, set the variable to t using the Var keyboard, and then tap OK.
- In the next entry line, enter and highlight $V(t)$.
- Go to **Interactive** > **Calculation** > **diff**. Set the variable to t and tap OK.
- To substitute values of t, insert | from the Math3 keyboard and type $t = 10$ or $t = 20$ after the derivative, as shown.



- To find the domain, enter and highlight $V(t) > 0$.
- Go to **Interactive** > **Equation/Inequality** > **solve**. Set the variable to t and tap OK.



- Disregard negative values, since $t > 0$. So the answer is $0 < t < 56.78$.
- The domain can also be obtained graphically by finding where $V = 0$.



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Summary 18B

For the function with rule $f(x)$:

- The average rate of change for $x \in [a, b]$ is given by $\dfrac{f(b) - f(a)}{b - a}$.
- The instantaneous rate of change of f with respect to x when $x = a$ is $f'(a)$.

Exercise 18B

Example 3

1 Let $y = 35 + 12x^2$.

- **a** Find the change in y as x changes from 1 to 2. What is the average rate of change of y with respect to x in this interval?
- **b** Find the change in y as x changes from $2 - h$ to 2. What is the average rate of change of y with respect to x in this interval?
- **c** Find the rate of change of y with respect to x when $x = 2$.

Example 4

2 According to a business magazine, the expected assets, $\$M$, of a proposed new company will be given by $M = 200\,000 + 600t^2 - \dfrac{200}{3}t^3$, where t is the number of months after the business is set up.

- **a** Find the rate of growth of assets at time t months.
- **b** Find the rate of growth of assets at time $t = 3$ months.
- **c** When will the rate of growth of assets be zero?

3 As a result of a survey, the marketing director of a company found that the revenue, $\$R$, from pricing 100 produced items at $\$P$ each is given by the rule $R = 30P - 2P^2$.

- **a** Find $\dfrac{dR}{dP}$ and explain what it means.
- **b** Calculate $\dfrac{dR}{dP}$ when $P = 5$ and $P = 10$.
- **c** For what selling prices is revenue rising?

4 The population, P, of a new housing estate t years after 30 January 2012 is given by the rule $P = 100(5 + t - 0.25t^2)$. Find the rate of change of the population after:

- **a** 1 year
- **b** 2 years
- **c** 3 years.

5 Water is being poured into a flask. The volume, V mL, of water in the flask at time t seconds is given by $V(t) = \dfrac{5}{8}\left(10t^2 - \dfrac{t^3}{3}\right)$, $0 \le t \le 20$.

- **a** Find the volume of water in the flask at time:
  - **i** $t = 0$
  - **ii** $t = 20$.
- **b** Find the rate of flow of water into the flask at time t.
- **c** Sketch the graph of $V'(t)$ against t for $0 \le t \le 20$.

6 The area, A km^2, of an oil slick is growing according to the rule $A = \dfrac{t}{2} + \dfrac{t^2}{10}$, where t is the time in hours since the leak started.

- **a** Find the area covered at the end of 1 hour.
- **b** Find the rate of increase of the area after 1 hour.

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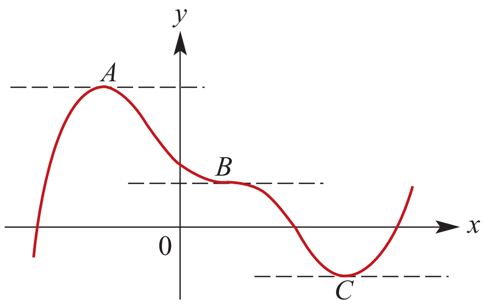
18C Stationary points

In the previous chapter, we have seen that the gradient of the tangent at a point $(a, f(a))$ on the curve with rule $y = f(x)$ is given by $f'(a)$.

A point $(a, f(a))$ on a curve $y = f(x)$ is said to be a **stationary point** if $f'(a) = 0$.

Equivalently, for $y = f(x)$, if $\frac{dy}{dx} = 0$ when $x = a$, then $(a, f(a))$ is a stationary point.

In the graph shown, there are stationary points at A, B and C. At such points the tangents are parallel to the x-axis (illustrated as dashed lines).



The reason for the name *stationary points* becomes clear later in this chapter when we look at the application to the motion of a particle.

Example 5

Find the stationary points of the following functions:

a $y = 9 + 12x - 2x^2$ **b** $p = 2t^3 - 5t^2 - 4t + 13$ for $t > 0$ **c** $y = 4 + 3x - x^3$

Solution

a $y = 9 + 12x - 2x^2$

$\frac{dy}{dx} = 12 - 4x$

The stationary points occur when $\frac{dy}{dx} = 0$, i.e. when $12 - 4x = 0$, i.e. at $x = 3$.

When $x = 3$, $y = 9 + 12 \times 3 - 2 \times 3^2 = 27$. Thus the stationary point is at $(3, 27)$.

b $p = 2t^3 - 5t^2 - 4t + 13 \quad (t > 0)$

$\frac{dp}{dt} = 6t^2 - 10t - 4 \quad (t > 0)$

Thus, $\frac{dp}{dt} = 0$ implies $2(3t^2 - 5t - 2) = 0$

$(3t + 1)(t - 2) = 0$

$\therefore t = -\frac{1}{3}$ or $t = 2$

But $t > 0$, therefore the only acceptable solution is $t = 2$.

When $t = 2$, $p = 16 - 20 - 8 + 13 = 1$.

So the corresponding stationary point is $(2, 1)$.

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c $y = 4 + 3x - x^3$

$$\frac{dy}{dx} = 3 - 3x^2$$

Thus, $\frac{dy}{dx} = 0$ implies $3(1 - x^2) = 0$

$\therefore \quad x = \pm 1$

The stationary points occur at $(1, 6)$ and $(-1, 2)$.

Example 6

The curve with equation $y = x^3 + ax^2 + bx + c$ passes through $(0, 5)$ and has a stationary point at $(2, 7)$. Find a, b and c.

Solution

When $x = 0$, $y = 5$. Thus $c = 5$.

We have $\frac{dy}{dx} = 3x^2 + 2ax + b$ and at $x = 2$, $\frac{dy}{dx} = 0$.

Therefore

$$0 = 12 + 4a + b \qquad (1)$$

The point $(2, 7)$ is on the curve and so

$$7 = 2^3 + 2^2a + 2b + 5$$

$$2 = 8 + 4a + 2b$$

$$4a + 2b + 6 = 0 \qquad (2)$$

Subtract (2) from (1):

$$-b + 6 = 0$$

$$\therefore \quad b = 6$$

Substitute in (1):

$$0 = 12 + 4a + 6$$

$$-18 = 4a$$

$$\therefore \quad -\frac{9}{2} = a$$

Hence $a = -\frac{9}{2}$, $b = 6$ and $c = 5$.

Summary 18C

- A point $(a, f(a))$ on a curve $y = f(x)$ is said to be a **stationary point** if $f'(a) = 0$.
- Equivalently, for $y = f(x)$, if $\frac{dy}{dx} = 0$ when $x = a$, then $(a, f(a))$ is a stationary point.

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Skill-sheet

Exercise 18C

Example 5

1 Find the coordinates of the stationary points of each of the following functions:

a $f(x) = x^2 - 6x + 3$

b $y = x^3 - 4x^2 - 3x + 20$ for $x > 0$

c $z = x^4 - 32x + 50$

d $q = 8t + 5t^2 - t^3$ for $t > 0$

e $y = 2x^2(x - 3)$

f $y = 3x^4 - 16x^3 + 24x^2 - 10$

Example 6

2 The curve with equation $y = ax^2 + bx + c$ passes through $(0, -1)$ and has a stationary point at $(2, -9)$. Find a, b and c.

3 The curve with equation $y = ax^2 + bx + c$ has a stationary point at $(1, 2)$. When $x = 0$, the slope of the curve is 45°. Find a, b and c.

4 The curve with equation $y = ax^2 + bx$ has a gradient of 3 at the point $(2, -2)$.

a Find the values of a and b.

b Find the coordinates of the turning point.

5 The curve with equation $y = x^2 + ax + 3$ has a stationary point when $x = 4$. Find a.

6 The curve with equation $y = x^2 - ax + 4$ has a stationary point when $x = 3$. Find a.

7 Find the coordinates of the stationary points of each of the following:

a $y = x^2 - 5x - 6$

b $y = (3x - 2)(8x + 3)$

c $y = 2x^3 - 9x^2 + 27$

d $y = x^3 - 3x^2 - 24x + 20$

e $y = (x + 1)^2(x + 4)$

f $y = (x + 1)^2 + (x + 2)^2$

8 The curve with equation $y = ax^2 + bx + 12$ has a stationary point at $(1, 13)$. Find a and b.

9 The curve with equation $y = ax^3 + bx^2 + cx + d$ has a gradient of -3 at $\left(0, 7\frac{1}{2}\right)$ and a turning point at $(3, 6)$. Find a, b, c and d.

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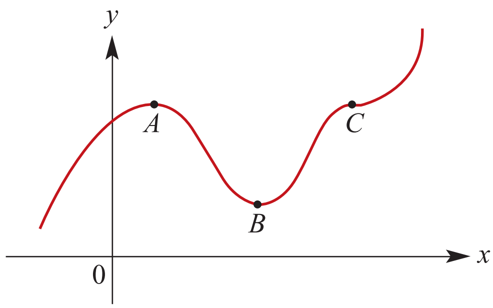
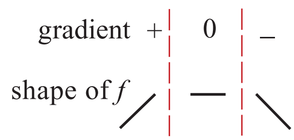
18D Types of stationary points

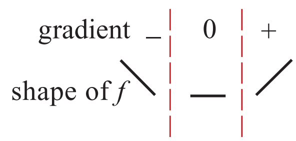
The graph of $y = f(x)$ below has three stationary points $A(a, f(a))$, $B(b, f(b))$, $C(c, f(c))$.



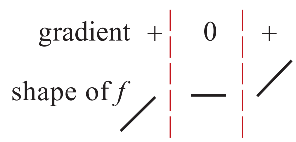
A Point A is called a **local maximum** point.
Notice that $f'(x) > 0$ immediately to the left of A, and that $f'(x) < 0$ immediately to the right of A. This means that f is strictly increasing immediately to the left of A, and strictly decreasing immediately to the right of A.



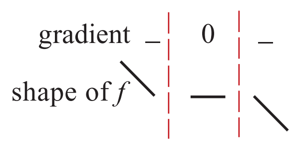
B Point B is called a **local minimum** point.
Notice that $f'(x) < 0$ immediately to the left of B, and that $f'(x) > 0$ immediately to the right of B. This means that f is strictly decreasing immediately to the left of B, and strictly increasing immediately to the right of B.



C The point C is called a **stationary point of inflection**.
Notice that $f'(x) > 0$ immediately to the left and right of C.

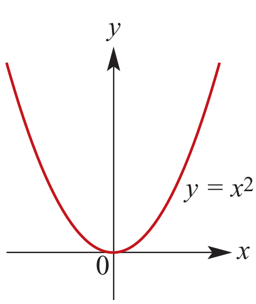


Clearly it is also possible to have stationary points of inflection with $f'(x) < 0$ immediately to the left and right.

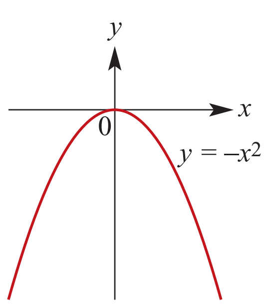


Stationary points of types A and B are referred to as **turning points**.

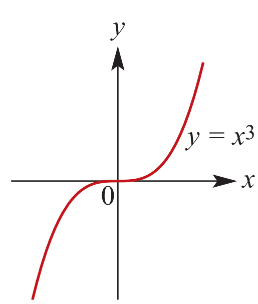
Before proceeding with some more complicated functions, it is worth referring back to some of the functions we met earlier in this book.



$f(x) = x^2$
Local minimum at $(0, 0)$.



$f(x) = -x^2$
Local maximum at $(0, 0)$.



$f(x) = x^3$
Stationary point of inflection at $(0, 0)$.

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Example 7

For the function $f(x) = 3x^3 - 4x + 1$:

a Find the stationary points and state their nature.

b Sketch the graph.

Solution

a The derivative is $f'(x) = 9x^2 - 4$.

The stationary points occur where $f'(x) = 0$:

$$9x^2 - 4 = 0$$

$$\therefore \quad x = \pm\frac{2}{3}$$

There are stationary points at $\left(-\frac{2}{3}, f\left(-\frac{2}{3}\right)\right)$ and $\left(\frac{2}{3}, f\left(\frac{2}{3}\right)\right)$, that is, at $\left(-\frac{2}{3}, 2\frac{7}{9}\right)$ and $\left(\frac{2}{3}, -\frac{7}{9}\right)$.

So $f'(x)$ is of constant sign for each of

$$\left\{x : x < -\frac{2}{3}\right\}, \quad \left\{x : -\frac{2}{3} < x < \frac{2}{3}\right\} \quad \text{and} \quad \left\{x : x > \frac{2}{3}\right\}$$

To calculate the sign of $f'(x)$ for each of these sets, simply choose a representative number in the set.

Thus $f'(-1) = 9 - 4 = 5 > 0$

$f'(0) = 0 - 4 = -4 < 0$

$f'(1) = 9 - 4 = 5 > 0$

We can now put together the table shown on the right.

x		$-\frac{2}{3}$		$\frac{2}{3}$	
$f'(x)$	+	0	−	0	+
shape of f	/	—	\	—	/

There is a local maximum at $\left(-\frac{2}{3}, 2\frac{7}{9}\right)$ and a local minimum at $\left(\frac{2}{3}, -\frac{7}{9}\right)$.

b To sketch the graph of this function we need to find the axis intercepts and investigate the behaviour of the graph for $x > \frac{2}{3}$ and $x < -\frac{2}{3}$.

The y-axis intercept is $f(0) = 1$.

To find the x-axis intercepts, consider $f(x) = 0$, which implies $3x^3 - 4x + 1 = 0$.

Using the factor theorem, we find that $x - 1$ is a factor of $3x^3 - 4x + 1$.

By division:

$$3x^3 - 4x + 1 = (x - 1)(3x^2 + 3x - 1)$$

Now $(x - 1)(3x^2 + 3x - 1) = 0$ implies that $x = 1$ or $3x^2 + 3x - 1 = 0$.

We have

$$3x^2 + 3x - 1 = 3\left[\left(x + \frac{1}{2}\right)^2 - \frac{1}{4} - \frac{1}{3}\right] = 3\left[\left(x + \frac{1}{2}\right)^2 - \frac{21}{36}\right]$$

$$= 3\left(x + \frac{1}{2} - \frac{\sqrt{21}}{6}\right)\left(x + \frac{1}{2} + \frac{\sqrt{21}}{6}\right)$$

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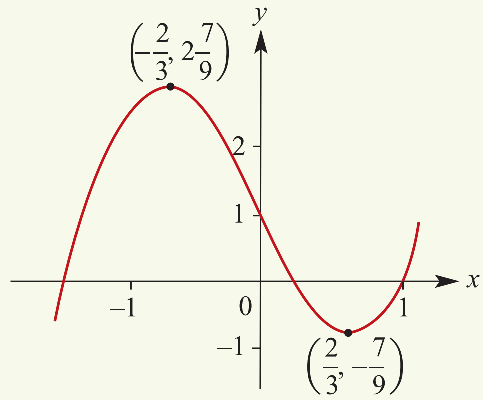
Thus the x-axis intercepts are at

$$x = -\frac{1}{2} + \frac{\sqrt{21}}{6}, \quad x = -\frac{1}{2} - \frac{\sqrt{21}}{6}, \quad x = 1$$

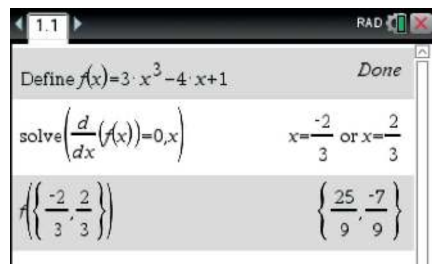
For $x > \frac{2}{3}$, $f(x)$ becomes larger.

For $x < \frac{2}{3}$, $f(x)$ becomes smaller.

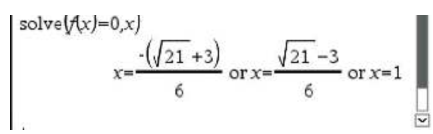


Using the TI-Nspire

- Define the function $f(x) = 3x^3 - 4x + 1$.
- Use (menu) > **Algebra** > **Solve** and (menu) > **Calculus** > **Derivative** to solve the equation $\frac{d}{dx}(f(x)) = 0$ and determine the coordinates of the stationary points.



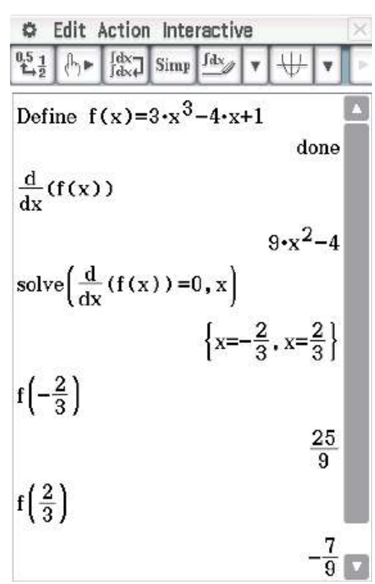
- Find the x-axis intercepts by solving the equation $f(x) = 0$.



Using the Casio ClassPad

To determine the exact coordinates of the stationary points:

- In Main $\sqrt{\alpha}$, define the function $f(x) = 3x^3 - 4x + 1$.
- Solve the equation $\frac{d}{dx}(f(x)) = 0$ for x.
- Evaluate the function f at each x-value to find the corresponding y-value.



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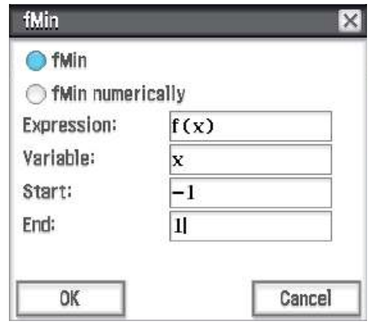
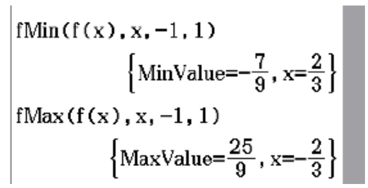
To determine the nature of these stationary points:

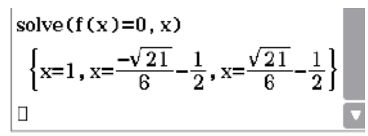
- In Main $\sqrt{\alpha}$, enter and highlight $f(x)$.
- Select **Interactive** > **Calculation** > **fMin**.
- Enter a known interval in which the stationary points are located, e.g. start at −1 and end at 1.
- This gives the coordinates of the local minimum.
- The coordinates of the local maximum can be found similarly using **fMax**.





To find the x-axis intercepts:

- Solve the equation $f(x) = 0$.



Summary 18D

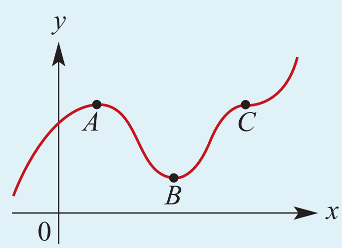
A point $(a, f(a))$ on a curve $y = f(x)$ is said to be a **stationary point** if $f'(a) = 0$.

Types of stationary points

A Point A is a **local maximum**:
- $f'(x) > 0$ immediately to the left of A
- $f'(x) < 0$ immediately to the right of A.

B Point B is a **local minimum**:
- $f'(x) < 0$ immediately to the left of B
- $f'(x) > 0$ immediately to the right of B.

C Point C is a **stationary point of inflection**.



Stationary points of types A and B are called **turning points**.

Skill-sheet

Exercise 18D

Example 7

1 For each of the following, find all stationary points and state their nature. Sketch the graph of each function.

a $y = 9x^2 - x^3$ **b** $y = x^3 - 3x^2 - 9x$ **c** $y = x^4 - 4x^3$

2 Find the stationary points (and state their type) for each of the following functions:

a $y = x^2(x - 4)$ **b** $y = x^2(3 - x)$ **c** $y = x^4$

d $y = x^5(x - 4)$ **e** $y = x^3 - 5x^2 + 3x + 2$ **f** $y = x(x - 8)(x - 3)$

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3 Sketch the graph of each of the following functions:

a $y = 2 + 3x - x^3$ **b** $y = 2x^2(x - 3)$ **c** $y = x^3 - 3x^2 - 9x + 11$

4 The graph corresponding to each of the following equations has a stationary point at $(-2, 10)$. For each graph, find the nature of the stationary point at $(-2, 10)$.

a $y = 2x^3 + 3x^2 - 12x - 10$

b $y = 3x^4 + 16x^3 + 24x^2 - 6$

5 For the function $y = x^3 - 6x^2 + 9x + 10$:

a Find the values of x for which $\frac{dy}{dx} > 0$, i.e. find $\left\{ x : \frac{dy}{dx} > 0 \right\}$.

b Find the stationary points on the curve corresponding to $y = x^3 - 6x^2 + 9x + 10$.

c Sketch the curve carefully between $x = 0$ and $x = 4$.

6 For the function $f(x) = 1 + 12x - x^3$, determine the values of x for which $f'(x) > 0$.

7 Let $f(x) = 3 + 6x - 2x^3$.

a Find the values of x such that $f'(x) > 0$.

b Find the values of x such that $f'(x) < 0$.

8 Let $f(x) = x(x + 3)(x - 5)$.

a Find the values of x for which $f'(x) = 0$.

b Sketch the graph of $y = f(x)$ for $-5 \le x \le 6$, giving the coordinates of the intersections with the axes and the coordinates of the turning points.

9 Sketch the graph of $y = x^3 - 6x^2 + 9x - 4$. State the coordinates of the axis intercepts and the turning points.

10 Find the coordinates of the points on the curve $y = x^3 - 3x^2 - 45x + 2$ where the tangent is parallel to the x-axis.

11 Let $f(x) = x^3 - 3x^2$.

a Find:

i $\{ x : f'(x) < 0 \}$

ii $\{ x : f'(x) > 0 \}$

iii $\{ x : f'(x) = 0 \}$

b Sketch the graph of $y = f(x)$.

12 Sketch the graph of $y = x^3 - 9x^2 + 27x - 19$ and state the coordinates of the stationary points.

13 Sketch the graph of $y = x^4 - 8x^2 + 7$. All axis intercepts and all turning points should be identified and their coordinates given.

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18E Applications to maximum and minimum problems

Many practical problems involve finding a maximum or minimum value of a function. We have solved some of these in Chapters 3 and 7. In the case of quadratic functions, we wrote the quadratic in turning point form and hence determined the maximum or minimum value. In the case of cubic functions, we used a CAS calculator to find the maximum or minimum values.

In this section we use calculus to solve problems which involve finding a local maximum or local minimum.

Example 8

A loop of string of length 100 cm is to be formed into a rectangle. Find the maximum area of this rectangle.

Solution

Let the length of the rectangle be x cm and the width y cm.

Then $2x + 2y = 100$. Thus $x + y = 50$ and hence

$$y = 50 - x \qquad (1)$$

It is clear that, for this problem, we must have $0 \le x \le 50$.

The area, A cm^2, is given by the formula $A = xy$.

Substituting from (1) gives

$$\begin{aligned} A &= x(50 - x) \\ &= 50x - x^2 \end{aligned}$$

Differentiating with respect to x:

$$\frac{dA}{dx} = 50 - 2x$$

Thus $\frac{dA}{dx} = 0$ implies $x = 25$.

Since the coefficient of x^2 is negative, this stationary point is a local maximum. (Alternatively, we could check the sign of $A'(x)$ immediately to the left and the right of $x = 25$.)

The maximum area is formed when the rectangle is a square with side length 25 cm, and so the maximum area is 625 cm^2.

Note: It is clear that we could have completed this question without calculus by using our knowledge of quadratic functions.

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Example 9

Given that $x + 2y = 4$, calculate the minimum value of $x^2 + xy - y^2$.

Solution

Rearranging $x + 2y = 4$, we have $x = 4 - 2y$.

Let $P = x^2 + xy - y^2$. Substituting for x gives

$$\begin{aligned} P &= (4 - 2y)^2 + (4 - 2y)y - y^2 \\ &= 16 - 16y + 4y^2 + 4y - 2y^2 - y^2 \\ &= 16 - 12y + y^2 \end{aligned}$$

$$\therefore \quad \frac{dP}{dy} = -12 + 2y$$

Stationary values occur when $\frac{dP}{dy} = 0$:

$$\begin{aligned} -12 + 2y &= 0 \\ y &= 6 \end{aligned}$$

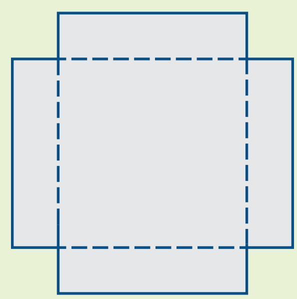
y		6	
$\frac{dP}{dy}$	−	0	+
shape of P	\	—	/

From the diagram, there is a minimum when $y = 6$.

When $y = 6$, $x = -8$. Thus the minimum value of $x^2 + xy - y^2$ is -20.

Example 10

From a square piece of metal of side length 2 m, four squares are removed as shown in the diagram. The metal is then folded along the dashed lines to form an open box with height x m.



a Show that the volume of the box, V m^3, is given by $V = 4x^3 - 8x^2 + 4x$.

b Find the value of x that gives the box its maximum volume and show that the volume is a maximum for this value.

c Sketch the graph of V against x for a suitable domain.

d Find the value(s) of x for which $V = 0.5$ m^3.

Solution

a The box has length and width $2 - 2x$ metres, and has height x metres. Thus

$$\begin{aligned} V &= (2 - 2x)^2 x \\ &= (4 - 8x + 4x^2)x \\ &= 4x^3 - 8x^2 + 4x \end{aligned}$$

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b Let $V = 4x^3 - 8x^2 + 4x$. The maximum volume will occur when $\frac{dV}{dx} = 0$.

We have $\frac{dV}{dx} = 12x^2 - 16x + 4$, and so $\frac{dV}{dx} = 0$ implies that

$$12x^2 - 16x + 4 = 0$$
$$3x^2 - 4x + 1 = 0$$
$$(3x - 1)(x - 1) = 0$$
$$\therefore \quad x = \frac{1}{3} \text{ or } x = 1$$

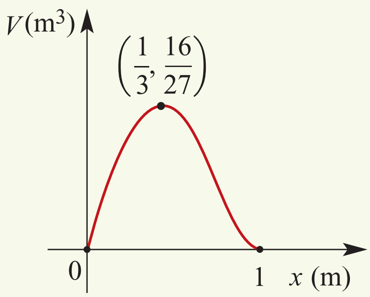
But, when $x = 1$, the length of the box is $2 - 2x = 0$. Therefore the only value to be considered is $x = \frac{1}{3}$. We show the entire chart for completeness.

A maximum occurs when $x = \frac{1}{3}$.

$$\therefore \text{Maximum volume} = \left(2 - 2 \times \frac{1}{3}\right)^2 \times \frac{1}{3}$$
$$= \frac{16}{27} \text{ m}^3$$

x		$\frac{1}{3}$		1
$\frac{dV}{dx}$	+	0	−	0
shape	/	—	\	—

c



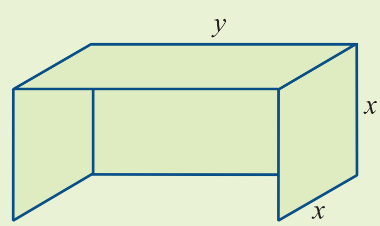
d To find the value(s) of x for which $V = 0.5\text{ m}^3$, we need to solve the equation $V = 0.5$, i.e. $4x^3 - 8x^2 + 4x = 0.5$.

Using a CAS calculator gives $x = \frac{1}{2}$ or $x = \frac{3 \pm \sqrt{5}}{4}$.

But the domain of V is $[0, 1]$. Hence $x = \frac{1}{2}$ or $x = \frac{3 - \sqrt{5}}{4}$.

Example 11

A canvas shelter is made up with a back, two square sides and a top. The area of canvas available is 24 m^2. Let $V\text{ m}^3$ be the volume enclosed by the shelter.



a Find the dimensions of the shelter that will create the largest possible enclosed volume.

b Sketch the graph of V against x for a suitable domain.

c Find the values of x and y for which $V = 10\text{ m}^3$.

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Solution

a The volume $V = x^2y$. One of the variables must be eliminated.
We know that the area is $24\ \text{m}^2$.

$$\therefore \quad 2x^2 + 2xy = 24$$

Rearranging gives $y = \dfrac{24 - 2x^2}{2x}$, i.e. $y = \dfrac{12}{x} - x$.
Substituting in the formula for volume gives

$$V = 12x - x^3$$

Differentiation now gives

$$\frac{dV}{dx} = 12 - 3x^2$$

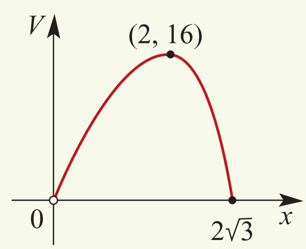
Stationary points occur when $\dfrac{dV}{dx} = 0$, which implies $12 - 3x^2 = 0$.

So stationary points occur when $x^2 = 4$, i.e. when $x = \pm 2$. But negative values have no meaning in this problem, so the only solution is $x = 2$.

Hence the maximum is at $x = 2$. The dimensions are 2 m, 2 m, 4 m.

x		2	
$\dfrac{dV}{dx}$	+	0	−
shape of V	/	—	\

b Note that $x > 0$ and $y \geq 0$.
This implies $x > 0$ and $12 - x^2 \geq 0$, i.e. $0 < x \leq 2\sqrt{3}$.



c Using a CAS calculator, solve the equation $12x - x^3 = 10$ numerically.
The solutions are $x = 2.9304\ldots$ and $x = 0.8925\ldots$
Possible dimensions to the nearest centimetre are 2.93 m, 2.93 m, 1.16 m and 0.89 m, 0.89 m, 12.55 m.

Maximum or minimum at an endpoint

Calculus can be used to find a local maximum or local minimum, but these are often not the actual maximum or minimum values of the function.

For a function defined on an interval:

- the actual maximum value of the function is called the **absolute maximum**
- the actual minimum value of the function is called the **absolute minimum**.

The corresponding points on the graph of the function are not necessarily stationary points.

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Example 12

Let $f(x) = x^2 + 2$, $\{ x \in \mathbb{R} : -2 \le x \le 4 \}$. Find the absolute maximum value and the absolute minimum value of the function.

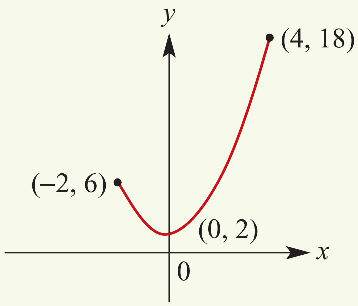
Solution

The maximum value is 18 and occurs when $x = 4$.

The minimum value is 2 and occurs when $x = 0$.

The minimum value occurs at a stationary point of the graph, but the endpoint $(4, 18)$ is not a stationary point.

The absolute maximum value is 18 and the absolute minimum value is 2.



Example 13

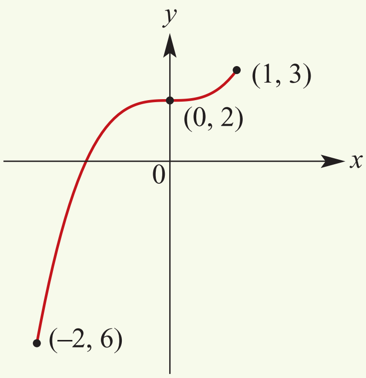
Let $f(x) = x^3 + 2$, $\{ x \in \mathbb{R} : -2 \le x \le 1 \}$. Find the maximum and minimum values of the function.

Solution

The maximum value is 3 and occurs when $x = 1$.

The minimum value is -6 and occurs when $x = -2$.

The absolute maximum and the absolute minimum do not occur at stationary points.



Example 14

In Example 10, the maximum volume of a box was found. The maximum value corresponded to a local maximum of the graph of $V = 4x^3 - 8x^2 + 4x$. This was also the absolute maximum value.

If the height of the box must be at most 0.3 m (i.e. $x \le 0.3$), what will be the maximum volume of the box?

Solution

The local maximum of $V(x)$ for $x \in [0, 1]$ was at $x = \frac{1}{3}$. But $\frac{1}{3}$ is greater than 0.3.

For the new problem, we have $V'(x) > 0$ for all $x \in [0, 0.3]$, and so $V(x)$ is strictly increasing on the interval $[0, 0.3]$.

Therefore the maximum volume occurs when $x = 0.3$ and is 0.588 m^3.

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Summary 18E

Here are some steps for solving maximum and minimum problems:

- Where possible, draw a diagram to illustrate the problem. Label the diagram and designate your variables and constants. Note the values that the variables can take.
- Write an expression for the quantity that is going to be maximised or minimised. Form an equation for this quantity in terms of a single independent variable. This may require some algebraic manipulation.
- If $y = f(x)$ is the quantity to be maximised or minimised, find the values of x for which $f'(x) = 0$.
- Test each point for which $f'(x) = 0$ to determine whether it is a local maximum, a local minimum or neither.
- If the function $y = f(x)$ is defined on an interval, such as $[a, b]$ or $[0, \infty)$, check the values of the function at the endpoints.

Skill-sheet

Exercise 18E

Example 8

1 A loop of string of length 200 cm is to be formed into a rectangle. Find the maximum area of this rectangle.

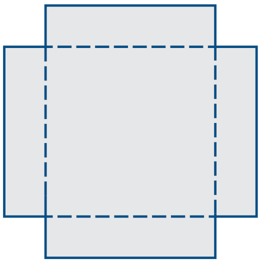
2 Find the maximum value of the product of two numbers x and $10 - x$.

Example 9

3 Given that $x + y = 2$, calculate the minimum value of $x^2 + y^2$.

Example 10

4 From a square piece of metal of side length 6 m, four squares are removed as shown in the diagram. The metal is folded along the dashed lines to form an open box with height x m.

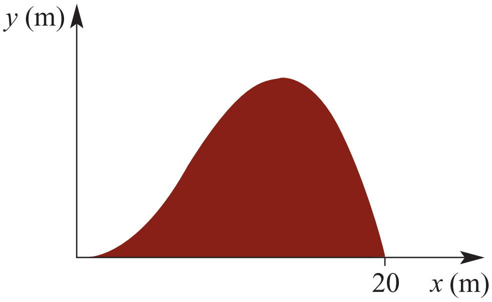


a Show that the volume of the box, V m^3, is given by $V = 4x^3 - 24x^2 + 36x$.

b Find the value of x that gives the box its maximum volume and find the maximum volume.

5 A bank of earth has cross-section as shown in the diagram. The curve defining the bank has equation

$$y = \frac{x^2}{400}(20 - x) \quad \text{for } x \in [0, 20]$$



a Find the height of the bank where:

i $x = 5$ **ii** $x = 10$ **iii** $x = 15$

b Find the value of x for which the height is a maximum and state the maximum height of the bank.

c Find the values of x for which:

i $\frac{dy}{dx} = \frac{1}{8}$ **ii** $\frac{dy}{dx} = -\frac{1}{8}$

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Example 11

6 A cuboid has a total surface area of 150 cm^2 and a square base of side length x cm.

a Show that the height, h cm, of the cuboid is given by $h = \dfrac{75 - x^2}{2x}$.

b Express the volume of the cuboid in terms of x.

c Hence determine its maximum volume as x varies.

d If the maximum side length of the square base of the cuboid is 4 cm, what is the maximum volume possible?

7 The volume of a cylinder is given by the formula $V = \pi r^2 h$. Find the maximum value of V if $r + h = 12$.

8 A rectangular sheet of metal measures 50 cm by 40 cm. Congruent squares of side length x cm are cut from each of the corners and not used further. The sheet is then folded up to make a tray of depth x cm. Find the value of x for which the volume of the tray is a maximum.

Example 12

9 Let $f(x) = 2 - 8x^2$, $\{ x \in \mathbb{R} : -2 \le x \le 2 \}$. Find the absolute maximum value and the absolute minimum value of the function.

Example 13

10 Let $f(x) = x^3 + 2x + 3$, $\{ x \in \mathbb{R} : -2 \le x \le 1 \}$. Find the absolute maximum value and the absolute minimum value of the function for its domain.

11 Let $f(x) = 2x^3 - 6x^2$, $\{ x \in \mathbb{R} : 0 \le x \le 4 \}$. Find the absolute maximum and the absolute minimum values of the function.

12 Let $f(x) = 2x^4 - 8x^2$, $\{ x \in \mathbb{R} : -2 \le x \le 5 \}$. Find the absolute maximum and the absolute minimum values of the function.

Example 14

13 A rectangular block is such that the sides of its base are of length x cm and $3x$ cm. The sum of the lengths of all its edges is 20 cm.

a Show that the volume, V cm^3, is given by $V = 15x^2 - 12x^3$.

b Find the derivative $\dfrac{dV}{dx}$.

c Find the local maximum for the graph of V against x for $x \in [0, 1.25]$.

d If $x \in [0, 0.8]$, find the absolute maximum value of V and the value of x for which this occurs.

e If $x \in [0, 1]$, find the absolute maximum value of V and the value of x for which this occurs.

14 For the variables x, y and z, it is known that $x + y = 20$ and $z = xy$.

a If $x \in [2, 5]$, find the possible values of y.

b Find the maximum and minimum values of z.

15 For the variables x, y and z, it is known that $z = x^2 y$ and $2x + y = 50$. Find the maximum value of z if:

a $x \in [0, 25]$ **b** $x \in [0, 10]$ **c** $x \in [5, 20]$.

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16 A piece of string 10 metres long is cut into two pieces to form two squares.

a If one piece of string has length x metres, show that the combined area of the two squares is given by $A = \frac{1}{8}(x^2 - 10x + 50)$.

b Find $\frac{dA}{dx}$.

c Find the value of x that makes A a minimum.

d What is the minimum total area of the two squares?

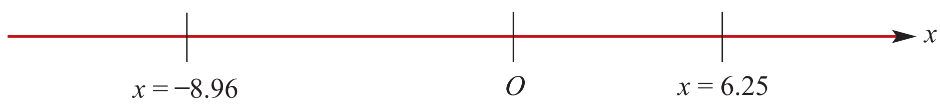
18F Applications of differentiation to kinematics

One of the key applications of rates of change is in the study of the motion of a particle. In this section, we consider motion in a straight line.

Position

The **position** of a particle is a specification of its location relative to a reference point, which is called the **origin**.

Consider motion on a straight line with reference point O.



We say that position to the right of O is positive and to the left of O is negative.

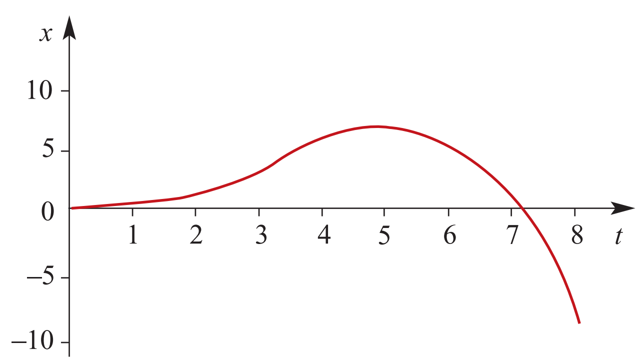
A particle is moving along the straight line. Let x metres denote its position relative to O at time t (where time is measured in seconds).

- At time $t = 0$, $x = 0$.
- At time $t = 5$, $x = 6.25$.
- At time $t = 8$, $x = -8.96$.

At $t = 0$, the particle starts from rest and moves to the right. At $t = 5$, the particle stops and moves back in the opposite direction. Its position–time graph is shown below.

Note that for $0 < t < 7.1$, the position is positive, i.e. the particle is to the right of O.

For $t > 7.1$, the position is negative, i.e. the particle is to the left of O.



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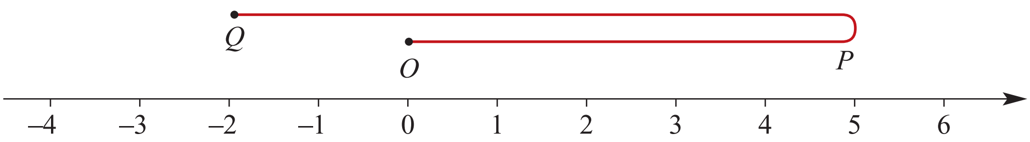
Distance and displacement

The **displacement** of a particle is defined as the change in position of the particle.

It is important to distinguish between the scalar quantity **distance** and the vector quantity displacement (which has a direction).

For example, consider a particle that starts at O and moves first 5 units to the right to point P, and then 7 units to the left to point Q.



The difference between its final position and its initial position is -2. So the displacement of the particle is -2 units. However, the distance it has travelled is 12 units.

Velocity

In this section, we focus on the instantaneous rates of change which arise when studying the motion of a particle travelling in a straight line. In particular, we define the velocity and acceleration of a particle.

Average velocity

The average rate of change of position with respect to time is **average velocity**.

A particle's average velocity for a time interval $[t_1, t_2]$ is given by

$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}} = \frac{x_2 - x_1}{t_2 - t_1}$$

where x_1 is the position at time t_1 and x_2 is the position at time t_2.

Instantaneous velocity

The instantaneous rate of change of position with respect to time is **instantaneous velocity**. We will refer to the instantaneous velocity as simply the **velocity**.

If a particle's position, x, at time t is given as a function of t, then the velocity of the particle at time t is determined by differentiating the rule for position with respect to time.

If x is the position of a particle at time t, then

$$\text{velocity } v = \frac{dx}{dt}$$

Velocity may be positive, negative or zero. If the velocity is positive, the particle is moving to the right, and if it is negative, the particle is moving to the left. A velocity of zero means the particle is instantaneously at rest.

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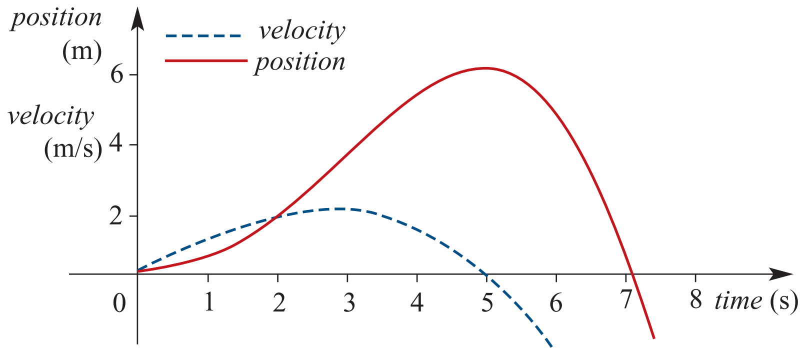
In the graph shown below, the position–time graph is the same as the one from the start of this section. This graph illustrates the relationship between position and velocity. The vertical axis is in both metres per second for velocity and metres for position.



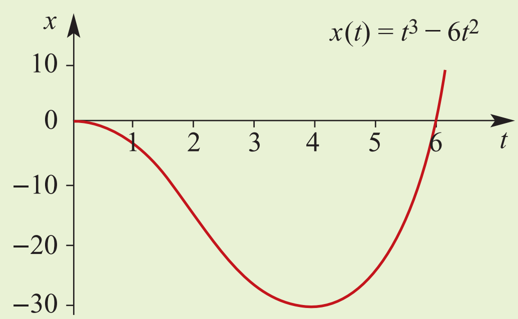
We make the following observations from the graph:

- For $0 < t < 5$, the velocity is positive: the particle is travelling from left to right.
- For $t > 5$, the velocity is negative: the particle is travelling from right to left.
- For $t = 5$, the velocity is zero: the particle is instantaneously at rest.

Note: The velocity at time t is given by the gradient of the tangent to the position–time graph at time t.

Example 15

A particle is moving in a straight line so that its position, x m, at time t seconds is given by the function $x(t) = t^3 - 6t^2$, $t \geq 0$. The graph of x against t is shown.



a For what values of t is the particle's velocity positive?

b For what values of t is the particle's velocity negative?

c Find the average velocity of the particle from $t = 0$ to $t = 4$.

d Find the velocity of the particle at $t = 2$.

Solution

a From the position–time graph it can be seen that the velocity is positive for $t > 4$.

b From the position–time graph it can be seen that the velocity is negative for $0 < t < 4$.

c $$\text{Average velocity} = \frac{x(4) - x(0)}{4 - 0} = \frac{-32 - 0}{4} = -8\text{ m/s}$$

d $$x(t) = t^3 - 6t^2$$

$$v(t) = \frac{dx}{dt} = 3t^2 - 12t$$

$$\therefore \quad v(2) = 3(2)^2 - 12(2) = -12$$

At time $t = 2$, the velocity is -12 m/s.

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Speed and average speed

- **Speed** is the magnitude of the velocity.
- **Average speed** for a time interval $[t_1, t_2]$ is given by $\dfrac{\text{distance travelled}}{t_2 - t_1}$

Units of measurement

Common units for velocity (and speed) are:

$$\begin{aligned} &1 \text{ metre per second} &&= 1 \text{ m/s} &&= 1 \text{ m s}^{-1} \\ &1 \text{ centimetre per second} &&= 1 \text{ cm/s} &&= 1 \text{ cm s}^{-1} \\ &1 \text{ kilometre per hour} &&= 1 \text{ km/h} &&= 1 \text{ km h}^{-1} \end{aligned}$$

The first and third units are connected in the following way:

$$\begin{aligned} 1 \text{ km/h} &= 1000 \text{ m/h} \\ &= \frac{1000}{60 \times 60} \text{ m/s} \\ &= \frac{5}{18} \text{ m/s} \\ \therefore \quad 1 \text{ m/s} &= \frac{18}{5} \text{ km/h} \end{aligned}$$

Example 16

A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = t^2 - 7t + 6$, $t \geq 0$.

a Find its initial velocity.

b When does its velocity equal zero, and what is its position at this time?

c What is its average velocity for the first 4 seconds?

d Determine its average speed for the first 4 seconds.

Solution

a $x = t^2 - 7t + 6$

$$v = \frac{dx}{dt} = 2t - 7$$

At $t = 0$, $v = -7$. The particle is initially moving to the left at 7 cm/s.

b $\dfrac{dx}{dt} = 0$ implies $2t - 7 = 0$, i.e. $t = 3.5$

When $t = 3.5$, $x = (3.5)^2 - 7(3.5) + 6$

$= -6.25$

So, at $t = 3.5$ seconds, the particle is at rest 6.25 cm to the left of O.

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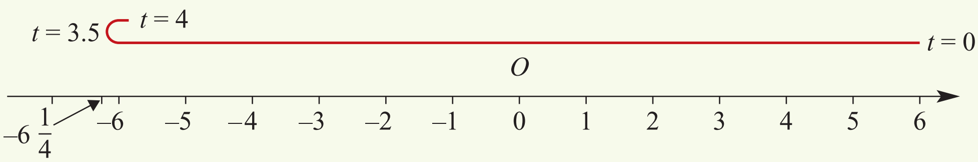
c Average velocity = $\dfrac{\text{change in position}}{\text{change in time}}$

Position is given by $x = t^2 - 7t + 6$. So at $t = 4$, $x = -6$, and at $t = 0$, $x = 6$.

$\therefore$ Average velocity = $\dfrac{-6 - 6}{4} = -3$ cm/s

d Average speed = $\dfrac{\text{distance travelled}}{\text{change in time}}$



The particle stopped at $t = 3.5$ and began to move in the opposite direction. So we must consider the distance travelled in the first 3.5 seconds (from $x = 6$ to $x = -6.25$) and then the distance travelled in the final 0.5 seconds (from $x = -6.25$ to $x = -6$).

Total distance travelled = 12.25 + 0.25 = 12.5

$\therefore$ Average speed = $\dfrac{12.5}{4} = 3.125$ cm/s

Note: Remember that speed is the magnitude of the velocity. However, we can see from this example that average speed is *not* the magnitude of the average velocity.

Acceleration

The acceleration of a particle is the rate of change of its velocity with respect to time.

- **Average acceleration** for the time interval $[t_1, t_2]$ is given by $\dfrac{v_2 - v_1}{t_2 - t_1}$, where v_2 is the velocity at time t_2 and v_1 is the velocity at time t_1.
- **Instantaneous acceleration** $a = \dfrac{dv}{dt} = \dfrac{d}{dt}\left(\dfrac{dx}{dt}\right) = \dfrac{d^2x}{dt^2}$

Note: The second derivative $\dfrac{d^2x}{dt^2}$ is just the derivative of the derivative. You will study the second derivative in more detail in Year 12.

Acceleration may be positive, negative or zero. Zero acceleration means the particle is moving at a constant velocity.

The direction of motion and the acceleration need not coincide. For example, a particle may have a positive velocity, indicating it is moving to the right, but a negative acceleration, indicating it is slowing down.

Also, although a particle may be instantaneously at rest, its acceleration at that instant need not be zero. If acceleration has the same sign as velocity, then the particle is ‘speeding up’. If the sign is opposite, the particle is ‘slowing down’.

The most commonly used units for acceleration are cm/s^2 and m/s^2.

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Example 17

A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = t^3 - 6t^2 + 5$, $t \geq 0$.

a Find its initial position, velocity and acceleration, and hence describe its motion.

b Find the times when it is instantaneously at rest and determine its position and acceleration at those times.

Solution

a $x = t^3 - 6t^2 + 5$

$$v = \frac{dx}{dt} = 3t^2 - 12t$$

$$a = \frac{dv}{dt} = 6t - 12$$

So when $t = 0$, we have $x = 5$, $v = 0$ and $a = -12$.

Initially, the particle is instantaneously at rest 5 cm to the right of O, with an acceleration of -12 cm/s^2.

b $v = 0$ implies $3t^2 - 12t = 0$

$$3t(t - 4) = 0$$

$$\therefore \quad t = 0 \text{ or } t = 4$$

The particle is initially at rest and stops again after 4 seconds.

At $t = 0$, $x = 5$ and $a = -12$.

At $t = 4$, $x = (4)^3 - 6(4)^2 + 5 = -27$ and $a = 6(4) - 12 = 12$.

After 4 seconds, the particle's position is 27 cm to the left of O, and its acceleration is 12 cm/s^2.

Example 18

A car starts from rest and moves a distance s metres in t seconds, where $s = \frac{1}{6}t^3 + \frac{1}{4}t^2$.

What is the initial acceleration and the acceleration when $t = 2$?

Solution

We are given

$$s = \frac{1}{6}t^3 + \frac{1}{4}t^2$$

The car's velocity is given by

$$v = \frac{ds}{dt} = \frac{1}{2}t^2 + \frac{1}{2}t$$

The car's acceleration is given by

$$a = \frac{dv}{dt} = t + \frac{1}{2}$$

When $t = 0$, $a = \frac{1}{2}$, and when $t = 2$, $a = 2\frac{1}{2}$.

Hence the required accelerations are $\frac{1}{2}$ m/s^2 and $2\frac{1}{2}$ m/s^2.

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Summary 18F

- The **position** of a particle moving in a straight line is determined by its distance from a fixed point O on the line, called the **origin**, and whether it is to the right or left of O. By convention, the direction to the right of the origin is positive.
- **Average velocity** for a time interval $[t_1, t_2]$ is given by

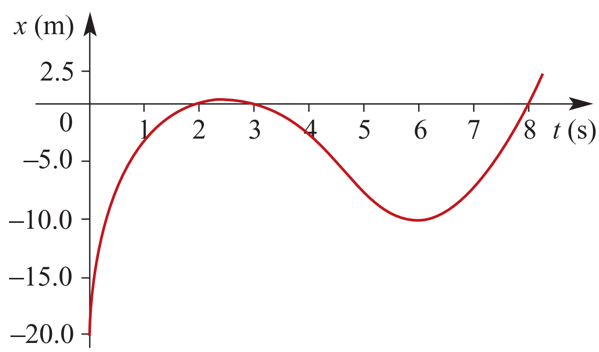
$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}} = \frac{x_2 - x_1}{t_2 - t_1}$$

  where x_2 is the position at time t_2 and x_1 is the position at time t_1.
- The instantaneous rate of change of position with respect to time is called the **instantaneous velocity**, or simply the **velocity**.
  If x is the position of the particle at time t, then its velocity is $v = \dfrac{dx}{dt}$
- **Speed** is the magnitude of the velocity.
- **Average speed** for a time interval $[t_1, t_2]$ is $\dfrac{\text{distance travelled}}{t_2 - t_1}$
- **Average acceleration** for a time interval $[t_1, t_2]$ is given by $\dfrac{v_2 - v_1}{t_2 - t_1}$, where v_2 is the velocity at time t_2 and v_1 is the velocity at time t_1.
- **Instantaneous acceleration** $a = \dfrac{dv}{dt} = \dfrac{d}{dt}\left(\dfrac{dx}{dt}\right) = \dfrac{d^2x}{dt^2}$

Exercise 18F

Example 15

1 A particle is moving in a horizontal straight line. Position is measured from a point O. The particle starts at a point 20 m to the left of O. The position–time graph for the motion of the particle is as shown.



a At which times is the particle at O?

b For what values of t is the velocity positive?

c For what values of t is the velocity equal to zero?

Example 16

2 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = t^2 - 12t + 11$, $t \geq 0$.

a Find its initial velocity.

b When does its velocity equal zero, and what is its position at this time?

c What is its average velocity for the first 3 seconds?

d Determine its average speed for the first 3 seconds.

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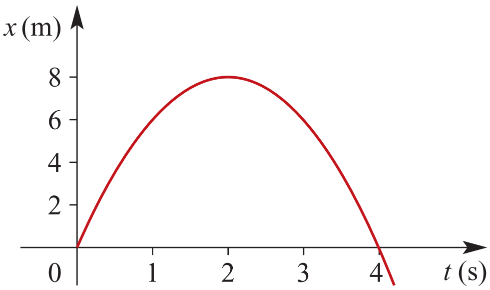
3 A particle moves along a horizontal straight line, starting at a point O. The position–time graph for the motion of the particle is as shown.



a For which values of t is the particle stationary?

b For which values of t is the particle moving to the right?

c How far from O does the particle go to the right?

d How long does it take to return to O?

e The position of the particle, x m, at time t seconds is given by a rule of the form $x = pt^2 + qt + r$ for $t \geq 0$. From the graph, determine the values of p, q and r.

f Find the velocity of the particle at time $t = 3$.

4 The position of a body moving in a straight line, x cm from the origin, at time t seconds ($t \geq 0$) is given by $x = \frac{1}{3}t^3 - 12t + 6$.

a Find the rate of change of position with respect to time at $t = 3$.

b Find the time at which the velocity is zero.

Example 17

5 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = 4t^3 - 6t^2 + 5$, $t \geq 0$.

a Find its initial position, velocity and acceleration, and hence describe its motion.

b Find the times when it is instantaneously at rest and determine its position and acceleration at those times.

Example 18

6 A car starts from rest and moves a distance s metres in t seconds, where $s = t^4 + t^2$.

a What is the acceleration when $t = 0$? **b** What is the acceleration when $t = 2$?

7 Let $s = 10 + 15t - 4.9t^2$ be the height (in metres) of an object at time t (in seconds).

a Find the velocity at time t. **b** Find the acceleration at time t.

8 The position, x metres, at time t seconds ($t \geq 0$) of a particle moving in a straight line is given by $x = t^2 - 7t + 10$.

a When does its velocity equal zero?

b Find its acceleration at this time.

c Find the distance travelled in the first 5 seconds.

d When does its velocity equal −2 m/s, and what is its position at this time?

9 A particle moves along a straight line so that after t seconds its position, s m, relative to a fixed point O on the line is given by $s = t^3 - 3t^2 + 2t$.

a When is the particle at O?

b What is its velocity and acceleration at these times?

c What is the average velocity during the first second?

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10 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds ($t \geq 0$) is given by $x = t^2 - 7t + 12$.

a Find its initial position.

b What is its position at $t = 5$?

c Find its initial velocity.

d When does its velocity equal zero, and what is its position at this time?

e What is its average velocity in the first 5 seconds?

f What is its average speed in the first 5 seconds?

11 A particle moves in a straight line so that after t seconds its position, s metres, is given by $s = t^4 + 3t^2$.

a Find the acceleration when $t = 1$, $t = 2$, $t = 3$.

b Find the average acceleration between $t = 1$ and $t = 3$.

12 A particle moving in a straight line has position x cm relative to the point O at time t seconds ($t \geq 0$), where $x = t^3 - 11t^2 + 24t - 3$.

a Find its initial position and velocity.

b Find its velocity at any time t.

c At what times is the particle stationary?

d What is the position of the particle when it is stationary?

e For how long is the particle's velocity negative?

f Find its acceleration at any time t.

g When is the particle's acceleration zero? What is its velocity and its position at that time?

13 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds ($t \geq 0$) is given by $x = 2t^3 - 5t^2 + 4t - 5$.

a When is its velocity zero, and what is its acceleration at that time?

b When is its acceleration zero, and what is its velocity at that time?

14 A particle is moving in a straight line in such a way that its position, x cm, relative to the point O at time t seconds ($t \geq 0$) satisfies $x = t^3 - 13t^2 + 46t - 48$. When does the particle pass through O, and what is its velocity and acceleration at those times?

15 Two particles are moving along a straight path so that their positions, x_1 cm and x_2 cm, relative to a fixed point P at any time t seconds are given by $x_1 = t + 2$ and $x_2 = t^2 - 2t - 2$.

a Find the time when the particles are at the same position.

b Find the time when the particles are moving with the same velocity.

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18G Families of functions and transformations

In the earlier chapters of this book we looked at families of functions. We can now use calculus to explore such families further. It is assumed that a CAS calculator will be used throughout this section.

Example 19

Consider the family of functions with rules of the form $f(x) = (x - a)^2(x - b)$, where a and b are positive constants with $b > a$.

a Find the derivative of $f(x)$ with respect to x.

b Find the coordinates of the stationary points of the graph of $y = f(x)$.

c Show that the stationary point at $(a, 0)$ is always a local maximum.

d Find the values of a and b if the stationary points occur where $x = 3$ and $x = 4$.

Solution

a Use a CAS calculator to find that $f'(x) = (x - a)(3x - a - 2b)$.

b The coordinates of the stationary points are $(a, 0)$ and $\left(\dfrac{a + 2b}{3}, \dfrac{4(a - b)^3}{27}\right)$.

c If $x < a$, then $f'(x) > 0$, and if $a < x < \dfrac{a + 2b}{3}$, then $f'(x) < 0$.

Therefore the stationary point at $(a, 0)$ is a local maximum.

d Since $a < b$, we must have $a = 3$ and $\dfrac{a + 2b}{3} = 4$. Therefore $b = \dfrac{9}{2}$.

Example 20

The graph of the function $y = x^3 - 3x^2$ is translated by a units in the positive direction of the x-axis and b units in the positive direction of the y-axis (where a and b are positive constants).

a Find the coordinates of the turning points of the graph of $y = x^3 - 3x^2$.

b Find the coordinates of the turning points of its image.

Solution

a The turning points have coordinates $(0, 0)$ and $(2, -4)$.

b The turning points of the image are (a, b) and $(2 + a, -4 + b)$.

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Skill-sheet

Exercise 18G

Example 19

1 Consider the family of functions with rules of the form $f(x) = (x-2)^2(x-b)$, where b is a positive constant with $b > 2$.

a Find the derivative of $f(x)$ with respect to x.

b Find the coordinates of the stationary points of the graph of $y = f(x)$.

c Show that the stationary point at $(2, 0)$ is always a local maximum.

d Find the value of b if the stationary points occur where $x = 2$ and $x = 4$.

Example 20

2 The graph of the function $y = x^4 - 12x^3$ is translated by a units in the positive direction of the x-axis and b units in the positive direction of the y-axis (where a and b are positive constants).

a Find the coordinates of the stationary points of the graph of $y = x^4 - 12x^3$.

b Find the coordinates of the stationary points of its image.

3 Consider the function defined by $f(x) = x - ax^2$, where a is a real number with $a > 0$.

a Determine the intervals for which $f'(x)$ is:

i positive **ii** negative.

b Find the equation of the tangent to the graph of f at the point $\left(\frac{1}{a}, 0\right)$.

c Find the equation of the normal to the graph of f at the point $\left(\frac{1}{a}, 0\right)$.

d What is the range of f?

4 Consider the cubic function with rule $f(x) = (x-a)^2(x-1)$, where $a > 1$.

a Find the coordinates of the turning points of the graph of $y = f(x)$.

b State the nature of each of the turning points.

c Find the equation of the tangent at which:

i $x = 1$ **ii** $x = a$ **iii** $x = \frac{a+1}{2}$

5 A line with equation $y = mx + c$ is a tangent to the curve $y = (x-2)^2$ at a point P where $x = a$ such that $0 < a < 2$.

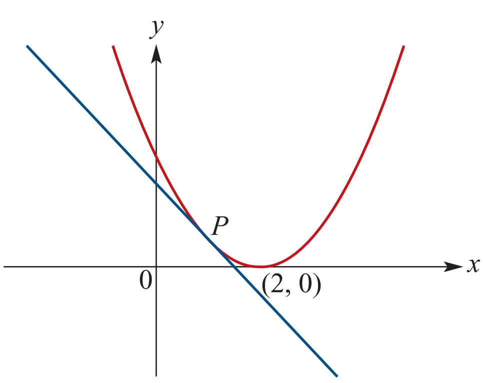
a **i** Find the gradient of the curve where $x = a$, for $0 < a < 2$.

ii Hence express m in terms of a.

b State the coordinates of the point P, expressing your answer in terms of a.

c Find the equation of the tangent where $x = a$.

d Find the x-axis intercept of the tangent.



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6 **a** The graph of $f(x) = x^3$ is translated to the graph of $y = f(x + h)$. Find the value of h if $f(1 + h) = 27$.

b The graph of $f(x) = x^3$ is transformed to the graph of $y = f(ax)$. Find the value of a if the graph of $y = f(ax)$ passes through the point $(1, 27)$.

c The cubic with equation $y = ax^3 - bx^2$ has a turning point with coordinates $(1, 8)$. Find the values of a and b.

7 The graph of the function $y = x^4 + 4x^2$ is translated by a units in the positive direction of the x-axis and b units in the positive direction of the y-axis (where a and b are positive constants).

a Find the coordinates of the turning points of the graph of $y = x^4 + 4x^2$.

b Find the coordinates of the turning points of its image.

8 Consider the quartic function with rule $f(x) = (x - 1)^2(x - b)^2$, where $b > 1$.

a Find the derivative of f.

b Find the coordinates of the turning points of f.

c Find the value of b such that the graph of $y = f(x)$ has a turning point at $(2, 1)$.

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Review

Chapter summary

Assign-ment

Nrich

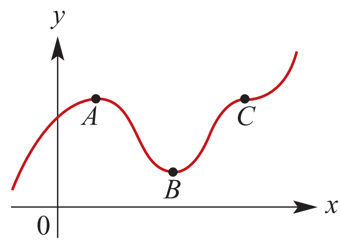
- **Tangents and normals**

  Let (x_1, y_1) be a point on the curve $y = f(x)$. If f is differentiable at $x = x_1$, then
  - the equation of the **tangent** to the curve at (x_1, y_1) is given by $y - y_1 = f'(x_1)(x - x_1)$
  - the equation of the **normal** to the curve at (x_1, y_1) is given by $y - y_1 = \dfrac{-1}{f'(x_1)}(x - x_1)$.

- **Stationary points**

  A point with coordinates $(a, f(a))$ on a curve $y = f(x)$ is a **stationary point** if $f'(a) = 0$.

  The graph shown has three stationary points: A, B and C.



  A Point A is a **local maximum** point. Notice that immediately to the left of A the gradient is positive, and immediately to the right the gradient is negative.

  B Point B is a **local minimum** point. Notice that immediately to the left of B the gradient is negative, and immediately to the right the gradient is positive.

  C Point C is a **stationary point of inflection**.

  Stationary points of types A and B are referred to as **turning points**.

- **Maximum and minimum values**

  For a continuous function f defined on an interval $[a, b]$:
  - if M is a value of the function such that $f(x) \le M$ for all $x \in [a, b]$, then M is the **absolute maximum** value of the function
  - if N is a value of the function such that $f(x) \ge N$ for all $x \in [a, b]$, then N is the **absolute minimum** value of the function.

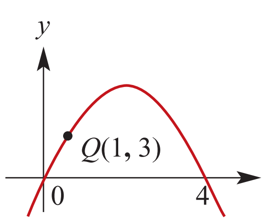
- **Motion in a straight line**

  For an object moving in a straight line with position x at time t:

  $$\text{velocity } v = \frac{dx}{dt} \qquad \text{acceleration } a = \frac{dv}{dt}$$

Short-answer questions

1 The graph of $y = 4x - x^2$ is shown.



a Find $\dfrac{dy}{dx}$.

b Find the gradient of the tangent to the curve at $Q(1, 3)$.

c Find the equation of the tangent at Q.

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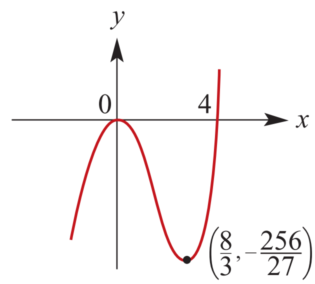
2 The graph of $y = x^3 - 4x^2$ is shown.



a Find $\frac{dy}{dx}$.

b Find the gradient of the tangent to the curve at the point $(2, -8)$.

c Find the equation of the tangent at the point $(2, -8)$.

d Find the coordinates of the point Q where the tangent crosses the curve again.

3 Let $y = x^3 - 12x + 2$.

a Find $\frac{dy}{dx}$ and the value(s) of x for which $\frac{dy}{dx} = 0$.

b State the nature of each of these stationary points.

c Find the corresponding y-value for each of these.

4 Write down the values of x for which each of the following derivative functions are zero. For each of the corresponding stationary points, determine whether it is a local maximum, local minimum or stationary point of inflection.

a $\frac{dy}{dx} = 3x^2$ **b** $\frac{dy}{dx} = -3x^3$

c $f'(x) = (x - 2)(x - 3)$ **d** $f'(x) = (x - 2)(x + 2)$

e $f'(x) = (2 - x)(x + 2)$ **f** $f'(x) = -(x - 1)(x - 3)$

g $\frac{dy}{dx} = -x^2 + x + 12$ **h** $\frac{dy}{dx} = 15 - 2x - x^2$

5 For each of the following, find all stationary points and state the nature of each:

a $y = 4x - 3x^3$ **b** $y = 2x^3 - 3x^2 - 12x - 7$ **c** $y = x(2x - 3)(x - 4)$

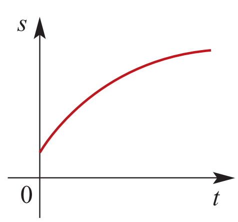
6 Sketch the graph of each of the following. Give the coordinates of the stationary points and the axis intercepts.

a $y = 3x^2 - x^3$ **b** $y = x^3 - 6x^2$ **c** $y = (x + 1)^2(2 - x)$

d $y = 4x^3 - 3x$ **e** $y = x^3 - 12x^2$

7 Match each position–time graph shown with its velocity–time graph:

a



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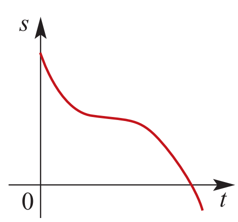
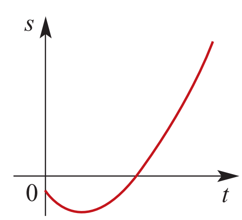
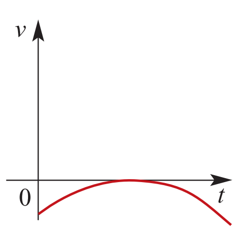


c



A



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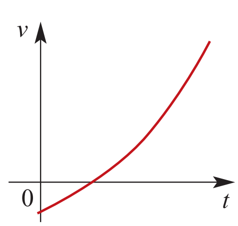
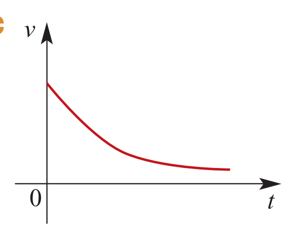


C



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8 A boy stands on the edge of a cliff of height 60 m. He throws a stone vertically upwards so that its distance, h m, above the cliff top is given by $h = 20t - 5t^2$.

a Calculate the maximum height reached by the stone above the cliff top.

b Calculate the time which elapses before the stone hits the beach (vertically below).

c Calculate the speed with which the stone hits the beach.

9 Find the least possible value of $x^2 + y^2$ given that $x + y = 12$.

Extended-response questions

Rate of change problems

1 The height, in metres, of a stone thrown vertically upwards from the surface of a planet is $2 + 10t - 4t^2$ after t seconds.

a Calculate the velocity of the stone after 3 seconds.

b Find the acceleration due to gravity.

2 A dam is being emptied. The quantity of water, V litres, remaining in the dam at any time t minutes after it starts to empty is given by $V(t) = 1000(30 - t)^3$, for $t \geq 0$.

a Sketch the graph of V against t.

b Find the time at which there are:

i 2 000 000 litres of water in the dam **ii** 20 000 000 litres of water in the dam.

c At what rate is the dam being emptied at any time t?

d How long does it take to empty the dam?

e At what time is the water flowing out at 8000 litres per minute?

f Sketch the graphs of $y = V(t)$ and $y = V'(t)$ on the one set of axes.

3 In a certain area of Australia the quantity of blackberries, W tonnes, ready for picking x days after 1 September is given by

$$W = \frac{x}{4000}\left(48\,000 - 2600x + 60x^2 - \frac{x^3}{2}\right) \qquad \text{for } 0 \leq x \leq 60$$

a Sketch the graph of W against x for $0 \leq x \leq 60$.

b After how many days will there be 50 tonnes of blackberries ready for picking?

c Find the rate of increase of W, in tonnes per day, when $x = 20$, 40 and 60.

d Find the value of W when $x = 30$.

4 A newly installed central heating system has a thermometer which shows the water temperature as it leaves the boiler (y°C). It also has a thermostat which switches off the system when $y = 65$.

The relationship between y and t, the time in minutes, is given by $y = 15 + \frac{1}{80}t^2(30 - t)$.

a Find the temperature at $t = 0$.

b Find the rate of increase of y with respect to t, when $t = 0$, 5, 10, 15 and 20.

c Sketch the graph of y against t for $0 \leq t \leq 20$.

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5 The sweetness, S, of a pineapple t days after it begins to ripen is found to be given by $S = 4000 + (t - 16)^3$ units.

a At what rate is S increasing when $t = 0$? **b** Find $\frac{dS}{dt}$ when $t = 4, 8, 12$ and 16.

c The pineapple is said to be unsatisfactory when our model indicates that the rate of increase of sweetness is zero. When does this happen?

d Sketch the graph of S against t up to the moment when the pineapple is unsatisfactory.

6 A slow train which stops at every station passes a certain signal box at 12 p.m. The motion of the train between the two stations on either side of the signal box is such that it is s km past the signal box at t minutes past 12 p.m., where $s = \frac{1}{3}t + \frac{1}{9}t^2 - \frac{1}{27}t^3$. (Note that, before the train reaches the signal box, both s and t will be negative.)

a Use a calculator to help sketch the graphs of s against t and $\frac{ds}{dt}$ against t on the one set of axes. Sketch for $t \in [-2, 5]$.

b Find the time of departure from the first station and the time of arrival at the second.

c Find the distance of each station from the signal box.

d Find the average velocity between the stations.

e Find the velocity with which the train passes the signal box.

7 Water is draining from a tank. The volume, V L, of water at time t (hours) is given by $V(t) = 1000 + (2 - t)^3$, for $t \geq 0$ and $V(t) \geq 0$.

a What are the possible values of t?

b Find the rate of draining when:

i $t = 5$ **ii** $t = 10$

8 A mountain path can be approximately described by the following rule, where y is the elevation, in metres above sea level, and x is the horizontal distance travelled in kilometres:

$$y = \frac{1}{5}(4x^3 - 8x^2 + 192x + 144) \qquad \text{for } 0 \leq x \leq 7$$

a How high above sea level is the start of the track, i.e. $x = 0$?

b When $x = 6$, what is the value of y?

c Use a calculator to draw a graph of the path. Sketch this graph.

d Does this model for the path make sense for $x > 7$?

e Find the gradient of the graph for the following distances (be careful of units):

i $x = 0$ **ii** $x = 3$ **iii** $x = 7$

Maximum and minimum problems

9 **a** On the one set of axes sketch the graphs of $y = x^3$ and $y = 2 + x - x^2$.

b Note that $2 + x - x^2 \geq x^3$ for $x \leq 0$. Find the value of x, with $x \leq 0$, for which the vertical distance between the two curves is a minimum and find the minimum distance.

Hint: Consider the function with rule $y = 2 + x - x^2 - x^3$ for $x \leq 0$.

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10 The number of mosquitos, $M(x)$ in millions, in a certain area depends on the average daily rainfall, x mm, during September and is approximated by

$$M(x) = \frac{1}{30}(50 - 32x + 14x^2 - x^3) \qquad \text{for } 0 \le x \le 10$$

Find the rainfall that will produce the maximum and the minimum number of mosquitos. (First plot the graph of $y = M(x)$ using a calculator.)

11 Given that $x + y = 5$ and $P = xy$, find:

a y in terms of x

b P in terms of x

c the maximum value of P and the corresponding values of x and y.

12 Given that $2x + y = 10$ and $A = x^2y$, where $0 \le x \le 5$, find:

a y in terms of x

b A in terms of x

c the maximum value of A and the corresponding values of x and y.

13 Given that $xy = 10$ and $T = 3x^2y - x^3$, find the maximum value of T for $0 < x < \sqrt{30}$.

14 The sum of two numbers x and y is 8.

a Write down an expression for y in terms of x.

b Write down an expression for s, the sum of the squares of these two numbers, in terms of x.

c Find the least value of the sum of their squares.

15 Find two positive numbers whose sum is 4, such that the sum of the cube of the first and the square of the second is as small as possible.

16 A rectangular patch of ground is to be enclosed with 100 metres of fencing wire. Find the dimensions of the rectangle so that the area enclosed will be a maximum.

17 The sum of two numbers is 24. If one number is x, find the value of x such that the product of the two numbers is a maximum.

18 A factory which produces n items per hour is found to have overhead costs of $\$\left(400 - 16n + \frac{1}{4}n^2\right)$ per hour. How many items should be produced every hour to keep the overhead costs to a minimum?

19 For $x + y = 100$, prove that the product $P = xy$ is a maximum when $x = y$, and find the maximum value of P.

20 A farmer has 4 km of fencing wire and wishes to fence in a rectangular piece of land through which a straight river flows. The river is to form one side of the enclosure. How can this be done to enclose as much land as possible?

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21 Two positive quantities p and q vary in such a way that $p^3q = 9$. Another quantity z is defined by $z = 16p + 3q$. Find values of p and q that make z a minimum.

22 A beam has a rectangular cross-section of depth x cm and width y cm. The perimeter of the cross-section of the beam is 120 cm. The strength, S, of the beam is given by $S = 5x^2y$.

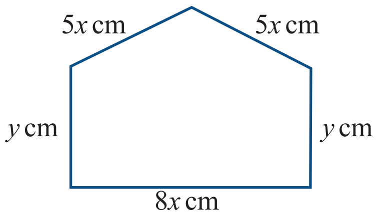
- **a** Find y in terms of x.
- **b** Express S in terms of x.
- **c** What are the possible values for x?
- **d** Sketch the graph of S against x.
- **e** Find the values of x and y which give the strongest beam.
- **f** If the cross-sectional depth of the beam must be less than or equal to 19 cm, find the maximum strength of the beam.

23 The number of salmon swimming upstream in a river to spawn is approximated by $s(x) = -x^3 + 3x^2 + 360x + 5000$, with x representing the temperature of the water in degrees (°C). (This model is valid only if $6 \le x \le 20$.) Find the water temperature that results in the maximum number of salmon swimming upstream.

24 A piece of wire 360 cm long is used to make the twelve edges of a rectangular box for which the length is twice the breadth.

- **a** Denoting the breadth of the box by x cm, show that the volume of the box, V cm^3, is given by $V = 180x^2 - 6x^3$.
- **b** Find the domain, S, of the function $V(x) = 180x^2 - 6x^3$ which describes the situation.
- **c** Sketch the graph of the function with rule $y = V(x)$.
- **d** Find the dimensions of the box that has the greatest volume.
- **e** Find the values of x (correct to two decimal places) for which $V = 20\,000$.

25 A piece of wire of length 90 cm is bent into the shape shown in the diagram.



- **a** Show that the area, A cm^2, enclosed by the wire is given by $A = 360x - 60x^2$.
- **b** Find the values of x and y for which A is a maximum.

26 A piece of wire 100 cm in length is to be cut into two pieces, one piece of which is to be shaped into a circle and the other into a square.

- **a** How should the wire be cut if the sum of the enclosed areas is to be a minimum? (Give your answer to the nearest centimetre.)
- **b** How should the wire be used to obtain a maximum area?

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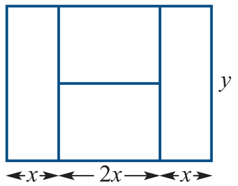
Review

27 A roll of tape 36 metres long is to be used to mark out the edges and internal lines of a rectangular court of length $4x$ metres and width y metres, as shown in the diagram. Find the length and width of the court for which the area is a maximum.



28 A rectangular chicken run is to be built on flat ground. A 16-metre length of chicken wire will be used to form three of the sides; the fourth side, of length x metres, will be part of a straight wooden fence.

a Let y be the width of the rectangle. Find an expression for A, the area of the chicken run, in terms of x and y.

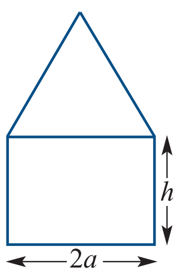
b Find an expression for A in terms of x.

c Find the possible values of x.

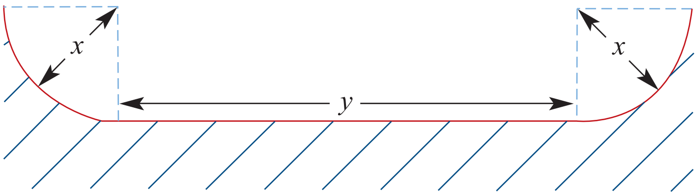
d Sketch the graph of A against x for these values of x.

e What is the largest area of ground the chicken run can cover?

29 The diagram illustrates a window that consists of an equilateral triangle and a rectangle. The amount of light that comes through the window is directly proportional to the area of the window.
If the perimeter of such a window must be 8000 mm, find the values of h and a (correct to the nearest mm) which allow the maximum amount of light to pass.



30 The diagram shows a cross-section of an open drainage channel. The flat bottom of the channel is y metres across and the sides are quarter circles of radius x metres. The total length of the bottom plus the two curved sides is 10 metres.



a Express y in terms of x.

b State the possible values that x can take.

c Find an expression for A, the area of the cross-section, in terms of x.

d Sketch the graph of A against x, for possible values of x.

e Find the value of x which maximises A.

f Comment on the cross-sectional shape of the drain.

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31 A cylinder closed at both ends has a total surface area of 1000 cm^2. The radius of the cylinder is x cm and the height h cm. Let V cm^3 be the volume of the cylinder.

a Find h in terms of x.

b Find V in terms of x.

c Find $\dfrac{dV}{dx}$.

d Find $\left\{ x : \dfrac{dV}{dx} = 0 \right\}$.

e Sketch the graph of V against x for a suitable domain.

f Find the maximum volume of the cylinder.

g Find the value(s) of x and h for which $V = 1000$, correct to two decimal places.

32 A cylindrical aluminium can able to contain half a litre of drink is to be manufactured. The volume of the can must therefore be 500 cm^3.

a Find the radius and height of the can which will use the least aluminium and therefore be the cheapest to manufacture.

b If the radius of the can must be no greater than 5 cm, find the radius and height of the can that will use the least aluminium.

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19 Revision of Chapters 17–18

19A Short-answer questions

1 The surface area, s cm^2, of a cube is given by the formula $s = 6x^2$, where x cm is the length of each edge of the cube. Find the average rate at which the surface area changes with respect to x, as x increases from $x = 2$ to $x = 4$.

2 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x(t) = 2t^3 - t$, $t \geq 0$.

a Find the average velocity in the first second.

b Find the average velocity in the next three seconds.

3 A curve has rule $y = 9 - x^2$.

a Find the average rate of change over the interval:

i $[1, 3]$ **ii** $[1, 2]$

b Find an expression for the average rate of change over the interval $[1, 1 + h]$.

c Use your result from part b to find the instantaneous rate of change at $x = 1$.

4 By first considering the gradient of the secant through the points $P(x, f(x))$ and $Q(x + h, f(x + h))$ on the curve $f(x) = \frac{1}{2}x^2 - x$, find the derivative of $\frac{1}{2}x^2 - x$.

5 Find the derivative of each of the following:

a $2x^3 - x + 1$ **b** $(x - 1)(x + 2)$ **c** $\dfrac{x^2 + 5x}{x}$

6 Find the gradient of the tangent to each of the following curves at the given point:

a $y = 3x^4 + x$, $(1, 4)$

b $y = 2x(1 - x)$, $(-2, -12)$

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7 For the function $f(x) = x - 2x^2$, find the values of x for which:

a $f(x) = 0$ **b** $f'(x) = 0$ **c** $f'(x) > 0$ **d** $f'(x) < 0$ **e** $f'(x) = -10$

8 Find the equations of the tangent and the normal to the curve $y = x^2 - 5x$ at $x = 1$.

9 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = \frac{1}{6}t^3 - \frac{1}{2}t^2 + 1$.

a At what times does the particle have zero velocity?

b Find its acceleration at those instants.

c Find the velocity when the acceleration is zero.

10 For $y = 2(x^3 - 4x)$, find all stationary points and state their nature.

19B Extended-response questions

1 An object follows a path (in a vertical plane) described by the equation $y = x - 0.01x^2$, where x is the horizontal distance travelled and y is the height above ground level of the object when it has travelled a horizontal distance x. The object travels from $(0, 0)$, a point at ground level.

a What is the horizontal distance travelled by the object before it returns to ground level?

b Find $\frac{dy}{dx}$.

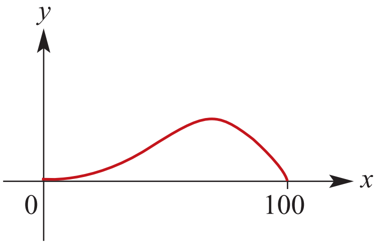
c Find the value of x for which $\frac{dy}{dx} = 0$ and the corresponding y-value.

d Sketch the graph of y against x.

e State the coordinates of the point on the path for which the gradient is:

i $\frac{1}{2}$ **ii** $-\frac{1}{2}$

2 A designer of roller coasters decides to use a polynomial model for a new construction, and tries the polynomial $y = -0.0001(x^3 - 100x^2)$, where $x \in [0, 100]$. The graph is as shown.



a What are the coordinates of the highest point reached?

b Find the gradient of the curve at:

i $x = 20$

ii $x = 80$

iii $x = 100$

c Describe the ride the roller coaster would provide.

d What alteration would you suggest?

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3 A rectangular block is such that the sides of its base are of length x cm and $3x$ cm. The sum of the length of all its edges is 20 cm.

a Find an expression in terms of x for the height of the block.

b Show that the volume, V cm^3, is given by $V = 15x^2 - 12x^3$.

c What values of x should be considered? (That is, find the domain of the function V.)

d Find $\frac{dV}{dx}$.

e Find $\left\{x : \frac{dV}{dx} = 0\right\}$ and hence find the maximum volume possible.

f Sketch the graph of V against x for the values of x determined in part c.

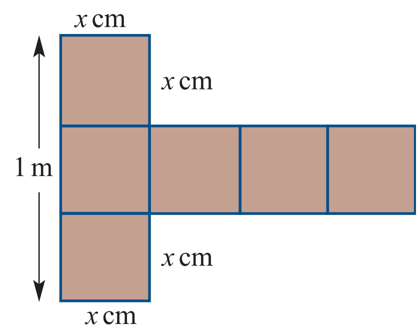
4 A stone is projected vertically upwards with a speed of 30 m/s. Its height, h m, above the ground after t seconds ($t < 6$) is given by $h = 30t - 5t^2$.

a Find $\frac{dh}{dt}$.

b Find the maximum height reached.

c Sketch the graph of h against t.

5 A box is made from the net shown.



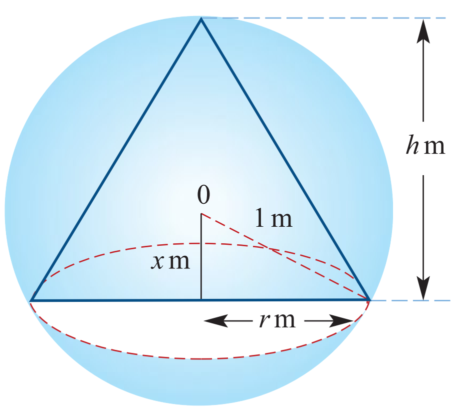
a Write down an expression for the surface area of the net in terms of x.

b Write down an expression for the volume, V cm^3, of the made-up box in terms of x.

c Sketch the graph of V against x.

d Find the dimensions and the volume of a box with maximum volume.

6 A right circular cone lies inside a sphere of radius 1 m as shown. The centre of the sphere, 0, lies x m from the base of the cone. The volume of a cone is given by the formula $V = \frac{1}{3}\pi r^2 h$.



a **i** Find r in terms of x.

ii Find h in terms of x.

b Show that $V = \frac{\pi}{3}(1 + x - x^2 - x^3)$.

c State a suitable domain for the function with rule $V = \frac{\pi}{3}(1 + x - x^2 - x^3)$.

d **i** Find $\frac{dV}{dx}$.

ii Find $\left\{x : \frac{dV}{dx} = 0\right\}$.

iii State the maximum possible volume of the cone.

e Sketch the graph of V against x.

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7 The number of insects in a colony at time t days after 1 January 2015 is approximated by the function with rule $P(t) = 1000 \times 2^{\frac{t}{20}}$, where $t = 0$ corresponds to 1 January 2015. This rule for the population is valid for the entire year.

a Find the approximate number of insects in the colony on 1 January.

b Find the approximate number of insects on 10 January (i.e. when $t = 9$).

c For what values of t is $P(t)$ equal to:

i 4000 **ii** 6000? (Give answer correct to two decimal places.)

d Find $P(20)$ and $P(15)$ and hence calculate the average rate of change of P with respect to time for the interval of time $[15, 20]$, giving your answer correct to two decimal places.

e **i** Find the average rate of change of P with respect to t for the interval $[15, 15 + h]$, in terms of h.

ii Explain how the instantaneous rate of change of P with respect to t, for $t = 15$, could be found by numerical methods.

8 A rectangular block, the length of whose base is twice its width, has total surface area of 300 m^2. The width of the base is x m and the height of the block is h m.

a Find h in terms of x, with h the subject of the formula.

b Find V, the volume of the block, in terms of x.

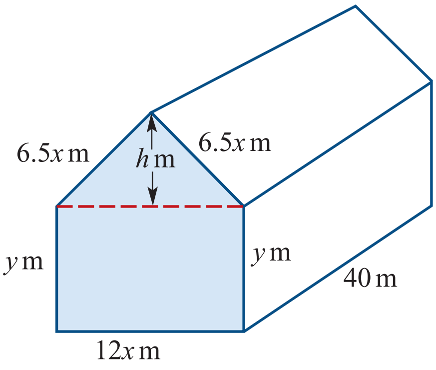
c Find $\dfrac{dV}{dx}$.

d State the positive values of x for which $V > 0$.

e Find the maximum value of V and the value of x for which this occurs.

f For the values of x established in part d, sketch the graph of V against x.

9 The metal frame of the front face of a building is as shown. Each length represents a steel girder. The total length of girder used for the front face is 70 metres. The building is 40 metres long.



a If $x = 2$, find the value of y.

b Show that $y = \dfrac{70 - 25x}{2}$.

c **i** Find the height h (marked on the diagram) in terms of x.

ii Show that the area of the front face of the building (shaded in the diagram) in terms of x and y is $A = 15x^2 + 12xy$.

d Find the volume, $V\text{ m}^3$, of the building in terms of x.

e **i** For what value of x is the volume of the building a maximum? What is the corresponding value of y?

ii State the corresponding maximum volume of the building.

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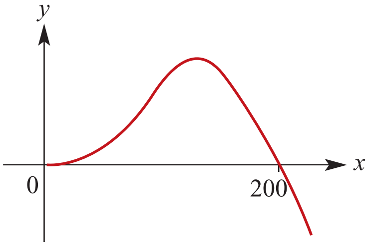
10 The equation of the curve is of the form $y = kx^2(a - x)$.



a Find the values of a and k given that when $x = 170$, $y = 8.67$, and when $x = 200$, $y = 0$.

b **i** Find the value of x for which there is a local maximum.

ii State the value of y which corresponds to this value of x.

c When $x = 105$, find:

i the y-value **ii** the gradient of the curve at this point.

d **i** Find the equation of the tangent at the point where $x = 105$.

ii State the y-axis intercept of the tangent.

e Find the average rate of change of y with respect to x for the interval $[0, 105]$.

f Carefully sketch the graph of $y = kx^2(a - x)$, for $x \in [0, 200]$, and on the same set of axes sketch the graph of the tangent at $x = 105$.

11 The population density (number of residents per unit area) of many cities depends on the distance from the city centre. For a particular city, the population density P (in thousands of people per square kilometre) at a distance of r kilometres from the centre is given approximately by $P = 10 + 40r - 20r^2$.

a What is the population density in the centre of the city?

b What are the possible values for r?

c Sketch a graph of P against r.

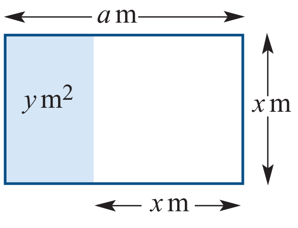
d **i** Find $\frac{dP}{dr}$.

ii Evaluate $\frac{dP}{dr}$ when $r = 0.5$, 1 and 2.

iii Sketch a graph of $\frac{dP}{dr}$ against r.

e Where is the population density greatest?

12 The diagram shows a rectangle with sides a m and x m. The area of the shaded region is y m^2. In the following, a is a constant.



a Find an expression for y in terms of x.

b Find the set of possible values for x in terms of a.

c Find the maximum value of y and the corresponding value of x (in terms of a).

d Explain briefly why this value of y is a maximum.

e **i** Sketch the graph of y against x when $a = 9$.

ii State the set of possible values for y.

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13 Water is being poured into a flask. The volume, V mL, of water in the flask at time t seconds is given by $V(t) = 0.6\left(20t^2 - \dfrac{2t^3}{3}\right)$, $0 \leq t \leq 20$.

a Find the volume of water in the flask when:

i $t = 0$ **ii** $t = 20$

b Find the rate of flow of water into the flask, i.e. find $\dfrac{dV}{dt}$.

c Sketch the graph of V against t for $t \in [0, 20]$.

d Sketch the graph of $\dfrac{dV}{dt}$ against t for $t \in [0, 20]$.

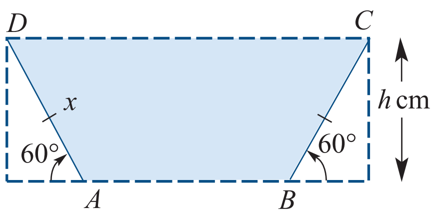
14 The function $y = ax^3 + bx^2$, where a and b are constants, has a stationary point at $(1, -1)$.

a Using the fact that $(1, -1)$ lies on the curve, form an equation involving a and b.

b Using the fact that $(1, -1)$ is also a stationary point, form a second equation and solve the two equations simultaneously to find a and b.

c Sketch the graph of y against x.

15 A trough is to be made by bending a long sheet of metal 80 cm wide to give a trapezoid cross-section with sides of equal length x cm inclined at 60° to the horizontal as shown.



a Find:

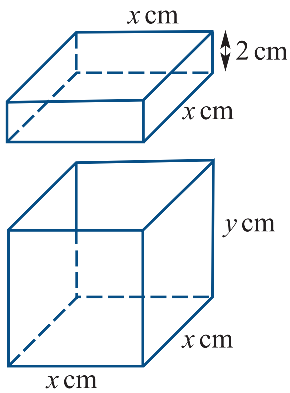
i length AB in terms of x

ii h in terms of x

b Find the cross-sectional area (shaded) in terms of x.

c Find the value of x for which the cross-sectional area is a maximum.

16 The diagrams opposite show an open cardboard box of dimensions x cm, x cm, y cm and the lid for the box of dimensions x cm, x cm, 2 cm. The total amount of cardboard used for the box and the lid is 1400 cm^2. Let V cm^3 be the volume of the box.



a Find y in terms of x.

b Find V in terms of x.

c Find $\dfrac{dV}{dx}$.

d Find $\left\{ x : \dfrac{dV}{dx} = 0 \right\}$.

e Sketch the graph of V against x.

f Find the maximum volume of the box.

g Find the values of x and y such that $V = 1000$.

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Revision

20 Revision of Chapters 1–19

20A Short-answer questions

1 Solve the linear equation $2x + 3(4 - x) = 8$ for x.

2 Solve the equation $\dfrac{at + b}{ct + d} = 2$ for t.

3 Solve the inequality $\dfrac{4x}{3} - 4 \le 2x - 3$.

4 Given that $-4 \le x \le 6$ and $2 \le y \le 8$, calculate:

a the smallest possible value of $x - y$

b the largest possible value of $\dfrac{x}{y}$

c the largest possible value of $x^2 + y^2$.

5 A man bought 20 books. Some of the books cost \$72 each and the other books cost \$24 each. If the man spent a total of \$720 on these books, how many of the \$24 books did he buy?

6 Solve the inequality $\dfrac{1 - 5x}{3} \ge -12$.

7 If $a = \dfrac{y^2 - xz}{10}$, find a when $x = -5$, $y = 7$ and $z = 6$.

8 **a** Find the coordinates of the midpoint of the line segment joining $(8, 14)$ and (a, b) in terms of a and b.

b If $(5, 10)$ is the midpoint, find the values of a and b.

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9 The line passing through the points $A(-2, 6)$ and $B(10, 15)$ meets the axes at P and Q.

a Find the equation of AB.

b Find the length of PQ.

10 If $A = (-7, 6)$ and $B = (11, -5)$, find:

a the coordinates of the midpoint of AB

b the distance between A and B

c the equation of AB

d the equation of the perpendicular bisector of AB.

11 Sketch the graph of $y = -x^2 + 4x + 2$, clearly labelling the x-axis and y-axis intercepts and the turning point.

12 The graph of the function $y = f(x)$ is a parabola that has turning point $(2, -6)$ and passes through the point $(6, 12)$. Find the rule for the function.

13 Find the value of a in the polynomial $ax^3 + 4x^2 + 3$ if the remainder is 3 when the polynomial is divided by $x - 2$.

14 A piece of wire 6000 cm long is used to make the edges of a cuboid. The length is $5x$ cm, the width is $4x$ cm and the height is w cm.

a Find w in terms of x.

b Find the volume, V cm^3, of the cuboid in terms of x.

c State the possible values of x.

d Find the volume if $x = 100$.

15 The number of square lino tiles needed to surface the floor of a hall varies inversely as the square of the side length of the tile used. If 900 tiles of side length 0.5 m would be needed to surface the floor of a certain hall, how many tiles of side length 0.75 m would be required?

16 A box contains 8 red cups and 10 blue cups.

a If a cup is chosen at random, the colour observed, the cup replaced and a second cup withdrawn, what is the probability that both cups drawn are red?

b If the first cup is not replaced, what is the probability that both cups drawn are red?

17 Box A contains three pieces of paper numbered 1, 3, 5.
Box B contains three pieces of paper numbered 2, 4, 6.
One piece of paper is removed at random from each box.
Find the probability that the two numbers obtained have a sum that is divisible by 3.

18 A letter is chosen at random from the word GOOGLE.

a What is the probability that the letter is a vowel?

b What is the probability that the letter is an O?

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Revision

19 David has either a sandwich or fruit salad for lunch. If he has a sandwich for lunch one day, the probability he has a sandwich for lunch the next day is 0.4. If he has fruit salad for lunch one day, the probability he has fruit salad for lunch the next day is 0.3. Suppose he has a sandwich for lunch on a Monday. What is the probability that he has fruit salad for lunch on the following Wednesday?

20 Solve the equation $\cos(3x) = \frac{1}{2}$ for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

21 The graph of $y = ax^3 + bx + c$ has axis intercepts at $(0, 6)$ and $(-2, 0)$ and has a stationary point where $x = 1$.

a Find the value of c.

b Write down two simultaneous equations in a and b from the given information.

c Hence find the values of a and b.

22 A tangent to the graph of $y = x^4$ has equation $y = -32x + a$. Find the value(s) of a.

23 For the function $f(x) = 4\cos(2x)$, $\{ x \in \mathbb{R} : -\pi \le x \le \pi \}$.

a Write down the amplitude and period of the function.

b Sketch the graph of the function f on a set of axes. Label the axis intercepts and the endpoints with their coordinates.

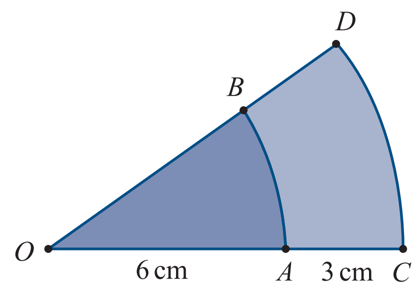
24 **a** Triangle ABC has $\angle ABC = 30°$, $AC = 10$ cm and $BC = 6$ cm. Find $\sin(\angle BAC)$.

b Triangle ABC has $AB = BC = 20$ cm and $\angle ABC = 45°$. Find AC^2.

25 In the figure, the arcs AB and CD are arcs of concentric circles with centre O. Given that $OA = 6$ cm, $AC = 3$ cm and the area of sector AOB is 12 cm^2, calculate:

a $\angle AOB$ in radians

b the area and perimeter of the region $ABDC$.



26 Four identical balls are numbered 1, 3, 5 and 7 and put into a box. A ball is randomly drawn from the box, and not returned to the box. A second ball is then randomly drawn from the box.

a What is the probability that the second ball drawn is numbered 1?

b What is the probability that the sum of the numbers on the two balls is 8?

c Given that the sum of the numbers on the two balls is 8, what is the probability that the second ball drawn is numbered 1?

27 The line $y = x + 1$ cuts the circle $x^2 + y^2 + 2x - 4y + 1 = 0$ at the points A and B.

a Find the midpoint of line segment AB.

b Sketch the graphs of $y = x + 1$ and $x^2 + y^2 + 2x - 4y + 1 = 0$ on the same axes.

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28 From a port P, a ship Q is 20 km away on a bearing of 112°, and a ship R is 12 km away on a bearing of 052°. Find the distance between the two ships.

29 Solve the following equations:

a $4^x - 5 \times 2^x - 24 = 0$ **b** $2^{5-3x} - 4^{x^2} = 0$

30 Find the value of $1 + 6 + 11 + \cdots + 61$.

31 If the sum to infinity of $x^2 + x^3 + x^4 + \cdots$ is equal to $\frac{1}{6}$, find the possible value(s) of x.

32 The curve for which $\frac{dy}{dx} = -4x + k$, where k is a constant, has a stationary point at $(1, 5)$. Find the value of k and the equation of the curve.

33 The curve $y = ax^3 - 2x^2 - x + 7$ has a gradient of 4 when $x = -1$. Find the value of a.

34 The polynomial $3x^2 + x + 10$ leaves the same remainder when divided by $x + b$ and by $x - 2b$, where $b \neq 0$. Calculate the value of b.

35 Show that the curves $y = x^3$ and $y = x^3 + x^2 + 6x + 9$ intersect at one point and find the gradient of each of these curves at that point.

36 **a** Find the coordinates of the stationary points on the graph of $y = x^3 - 75x - 10$ and sketch the graph.

b Find the values of p for which the equation $x^3 - 75x - 10 = p$ has more than one real solution.

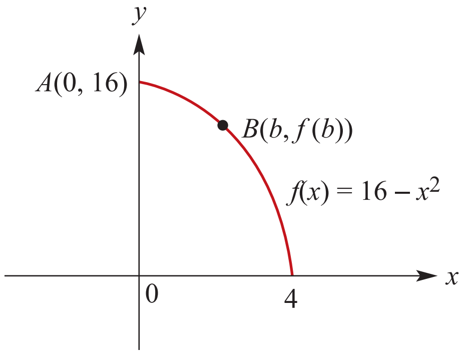
37 Find the maximal domains of the functions with the following rules:

a $f(x) = \frac{1}{x-3}$ **b** $f(x) = \frac{1}{(x-2)^2}$ **c** $f(x) = \sqrt{2-x}$

d $f(x) = \sqrt{x-4}$ **e** $f(x) = \frac{1}{\sqrt{5-x}}$

20B Extended-response questions

1 The diagram shows the graph of the function $f(x) = 16 - x^2$, $\{ x \in \mathbb{R} : 0 \leq x \leq 4 \}$. The line segment AB is drawn from the point $A(0, 16)$ to the point $B(b, f(b))$, where $0 < b < 4$.



a **i** What is the gradient of AB?

ii At what value of x between 0 and b does the tangent to the graph of f have the same gradient as AB?

b **i** What is the area, $S(b)$, of the trapezium bounded by line segment AB, the x-axis, the line $x = 0$ and the line $x = b$?

ii For what value of b does this area equal 28?

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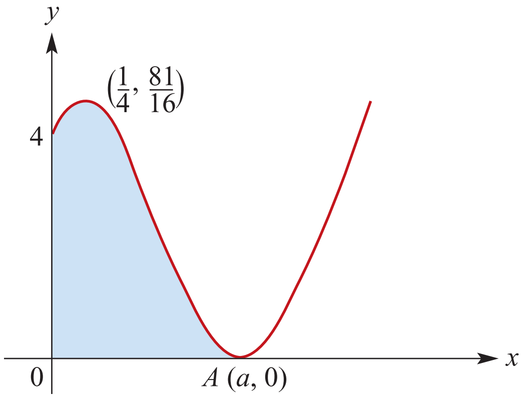
2 Consider the function given by $f(x) = (\sqrt{x} - 2)^2(\sqrt{x} + 1)^2$, $\{ x \in \mathbb{R} : x \geq 0 \}$.

The graph of $y = f(x)$ is shown below.



a Point $A(a, 0)$ is the x-axis intercept of $y = f(x)$. Find the value of a.

b There is a stationary point at $x = \frac{1}{4}$. State the interval for which the graph of f has negative gradient.

c The area of the shaded region of the graph is $\frac{136}{15}$.

Let O be the origin, C a point on the positive y-axis and B the point such that $OABC$ is a rectangle. Find the length OC such that the area of the rectangle $OABC$ is the same as the area of the shaded region.

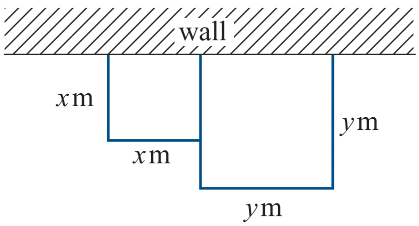
3 A piece of wire of length 52 cm is bent to form a trapezium in which the lengths of the two parallel sides are $2y$ cm and $(16x + 2y)$ cm. The other two sides both have length $10x$ cm.

a Find y in terms of x.

b Find the area, A cm^2, in terms of x.

c Find the values of x and y for which A is a maximum.

4 Kerrie has 5200 m of fencing to make two square enclosures using an existing wall as one side of the enclosure. The squares have side lengths x m and y m with $x \leq y$, as shown.



a Show that the total area, A m^2, of the two enclosures is given by

$$A = x^2 + \frac{(5200 - 2x)^2}{9}$$

b Find the minimum area and the values of x and y for which this occurs.

c Sketch the graph of A against x.

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5 For the function $f(x) = -x^3 + ax^2$, where a is a positive real number:

a Determine the intervals on which f has:

i negative gradient

ii positive gradient.

b Find the equation of the tangent to the graph of f at the point $(a, 0)$.

c Find the equation of the normal to the graph of f at the point $(a, 0)$.

6 A die has six faces numbered 0, 2, 4, 6, 8 and 10. A second die has faces numbered 1, 3, 5, 7, 9 and 11. Both dice are tossed and the results recorded.

a List the sample space of the experiment as a set of ordered pairs.

b Find the probability that the sum of the values from the two dice is:

i 1 **ii** 13 **iii** 9

c Find the probability of obtaining a sum of 15 given that the sum is greater than 7.

7 In its first month of operation, a soft-drink manufacturer produces 50 000 litres of soft drink. In each successive month, the production rises by 5000 litres per month.

a **i** The quantity of soft drink, t_n litres, produced in the nth month can be determined by a rule of the form

$$t_n = a + (n - 1)d$$

Find the values of a and d.

ii In which month will the factory double its original production?

iii How many litres in total will be produced in the first 36 months of operation?

b Another soft-drink manufacturer sets up a factory at the same time as the first. In the first month, the production is 12 000 litres. The production of this factory increases by 10% every month.

i Find a rule for q_n, the quantity of soft drink produced in the nth month.

ii Find the total amount of soft drink produced in the first 12 months.

c The two factories start production in the same month. In which month will the production of the second factory become faster than that of the first factory?

8 Consider the family of functions with rules of the form $f(x) = \sqrt{x - 2a}$, where a is a positive number.

a Find the maximal domain of such a function.

b Solve the equation $f(x) = x$ for x.

c For what value of a does the equation $f(x) = x$ have one solution?

d Sketch the graphs of $f(x) = \sqrt{x - \frac{1}{4}}$ and $y = x$ on the one set of axes and label the point at which they touch with its coordinates.

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9 Each night Frederick goes to the library or the study hall. If he goes to the library one night, the probability he goes to the library the next night is 0.7, and if he goes to the study hall one night, the probability he goes to the study hall the next night is 0.4. Suppose he goes to the library one Wednesday night.

a What is the probability that he goes to the library on each of the next three nights?

b What is the probability that he goes to the library on exactly two of the next three nights?

10 Platypus Sports sells hockey sticks directly to hockey clubs on a yearly basis. If a hockey club buys its sticks from Platypus one year, there is a 75% chance it will buy sticks from Platypus the next year. If a hockey club does not buy its sticks from Platypus one year, there is a 20% chance it will buy its sticks from Platypus the next year.
Suppose the Cross-Sticks club buys its hockey sticks from Platypus this year ($n = 0$).

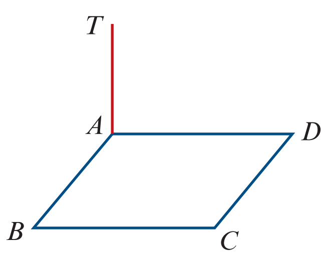
a What is the probability, correct to four decimal places, that Cross-Sticks buys its sticks from Platypus for the next three years?

b What is the probability, correct to four decimal places, that it buys its sticks from Platypus for exactly two of the next three years?

c What is the probability, correct to four decimal places, that it buys its sticks from Platypus in the third year ($n = 2$)?

11 **a** Adam notices a distinctive tree while orienteering on a flat horizontal plane. From where he is standing, the tree is 200 m away on a bearing of 050°. Two other people, Brian and Colin, who are both standing due east of Adam, each claim that the tree is 150 m away from them. Given that their claims are true and that Brian and Colin are not standing in the same place, how far apart are they? Give your answer to the nearest metre.

b A vertical tower of height 10 m stands in one corner of a rectangular courtyard. From the top of the tower, T, the angles of depression to the nearest corners B and D are 32° and 19° respectively. Find:



i AB, correct to two decimal places

ii AD, correct to two decimal places

iii the angle of depression from T to the corner C diagonally opposite the tower, correct to the nearest degree.

c Two circles, each of radius length 10 cm, have their centres 16 cm apart. Calculate the area common to both circles, correct to one decimal place.

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12 Consider the family of quadratics with rules of the form $f(x) = (p - 1)x^2 + 4x + (p - 4)$, where p is an arbitrary constant.

a Sketch the graph of f when:

i $p = 0$ **ii** $p = 2$

b Find the coordinates of the turning point of the graph of $y = f(x)$ in terms of p.

c For which values of p does the turning point of the graph of $y = f(x)$ lie on the x-axis?

d For which values of p are there two distinct solutions to the equation $f(x) = 0$?

e Let $f(x) = 2x^2 + 4x - 1$. (This is when $p = 3$.) Now let $y = g(x)$ be the image of the graph of $y = f(x)$ under a reflection in the y-axis. Find the equation for $y = g(x)$ and sketch the graphs of both $y = f(x)$ and $y = g(x)$ on the same set of axes.

13 A road near the shoreline is sometimes covered by water, and then the road is closed until it is uncovered again. The height, h metres, of the tide above mean sea level is modelled by the equation $h(t) = 2.3\cos(kt)$, where t hours is the time measured from 1 p.m. on a particular day. It is assumed that high tides occur every 12 hours.

a Find the value of k.

b If on this day the road was closed for 1 hour 30 minutes after 1 p.m., find the height of the road above mean sea level. Give your answer in centimetres.

c A raised footpath along the road is only under water for 1 hour after 1 p.m. Find the height of the footpath above mean sea level. Give your answer in centimetres.

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Glossary

A

Acceleration [p. 610] The acceleration of a particle is defined as the rate of change of its velocity with respect to time.

Acceleration, average [p. 610] The average acceleration of a particle for the time interval $[t_1, t_2]$ is given by $\frac{v_2 - v_1}{t_2 - t_1}$, where v_2 is the velocity at time t_2 and v_1 is the velocity at time t_1.

Acceleration, instantaneous [p. 610] $a = \frac{dv}{dt}$

Addition formulas [p. 409]

- $\cos(u + v) = \cos u \cos v - \sin u \sin v$
- $\cos(u - v) = \cos u \cos v + \sin u \sin v$
- $\sin(u + v) = \sin u \cos v + \cos u \sin v$
- $\sin(u - v) = \sin u \cos v - \cos u \sin v$
- $\tan(u + v) = \dfrac{\tan u + \tan v}{1 - \tan u \tan v}$
- $\tan(u - v) = \dfrac{\tan u - \tan v}{1 + \tan u \tan v}$

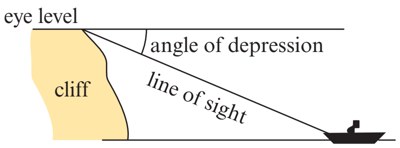
Addition rule for choices [p. 344] To determine the total number of choices from disjoint alternatives, simply add up the number of choices available for each alternative.

Addition rule for probability [p. 313] The probability of A or B or both occurring is given by

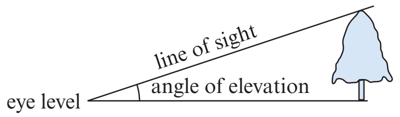
$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

Amplitude of trigonometric functions [p. 386] The distance between the mean position and the maximum position is called the amplitude. The graph of $y = \sin x$ has an amplitude of 1.

Angle of depression [p. 444] the angle between the horizontal and a direction below the horizontal



Angle of elevation [p. 444] the angle between the horizontal and a direction above the horizontal



Antiderivative [p. 572] To find the general antiderivative of $f(x)$: If $F'(x) = f(x)$, then

$\int f(x)\,dx = F(x) + c$

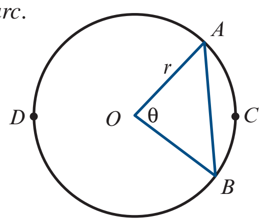
where c is an arbitrary real number.

Antidifferentiation rules [p. 573]

- $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c, \quad \{n \in \mathbb{Z} : n \neq -1\}$
- $\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$
- $\int kf(x)\,dx = k \int f(x)\,dx$

Arc [p. 439] Two points on a circle divide the circle into arcs; the shorter is the *minor arc*, and the longer is the *major arc*.

Arc length, ℓ [p. 440] The length of arc ACB is given by $\ell = r\theta$, where $\theta^c = \angle AOB$.



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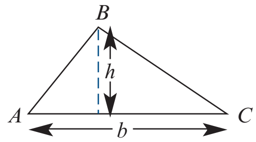
Area of a triangle [p. 436] given by half the product of the lengths of two sides and the sine of the angle included between them

$$\text{Area} = \frac{1}{2}bh$$

$$\text{Area} = \frac{1}{2}bc\sin A$$



Arithmetic sequence [p. 497] a sequence in which each successive term is found by adding a fixed amount to the previous term; e.g. $2, 5, 8, 11, \ldots$. An arithmetic sequence has a recurrence relation of the form $t_n = t_{n-1} + d$, where d is the common difference. The nth term can be found using $t_n = a + (n-1)d$, where $a = t_1$.

Arithmetic series [p. 501] the sum of the terms in an arithmetic sequence. The sum of the first n terms is given by the formula

$$S_n = \frac{n}{2}\big(2a + (n-1)d\big)$$

where $a = t_1$ and d is the common difference.

Arrangements [p. 347] counted when order is important. The number of ways of selecting and arranging r objects from a total of n objects is

$$\frac{n!}{(n-r)!} = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)$$

Asymptote [p. 143] A straight line is an asymptote of the graph of a function $y = f(x)$ if the graph of $y = f(x)$ gets arbitrarily close to the straight line. An asymptote can be horizontal, vertical or oblique.

B

Bearing [p. 445] the compass bearing; the direction measured from north clockwise

Binomial expansions [p. 361] In the expansion of $(a+b)^n$, each term is of the form ${}^nC_r\, a^{n-r}b^r$. For example:

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

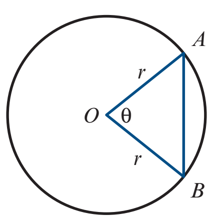
C

Chord [p. 439] a line segment with endpoints on a circle

Chord length [p. 440]

$$AB = 2r\sin\left(\frac{\theta}{2}\right)$$

where $\theta^c = \angle AOB$



Circle, general equation [p. 151] The general equation for a circle is

$$(x-h)^2 + (y-k)^2 = r^2$$

where the centre of the circle is the point (h, k) and the radius is r.

Coefficient [p. 234] the number that multiplies a power of x in a polynomial. E.g. for $2x^5 - 7x^2 + 4$, the coefficient of x^2 is -7.

Combinations [p. 353] *see* selections

Common difference, *d* [p. 497] the difference between two consecutive terms of an arithmetic sequence, i.e. $d = t_n - t_{n-1}$

Common ratio, *r* [p. 506] the quotient of two consecutive terms of a geometric sequence, i.e.

$$r = \frac{t_n}{t_{n-1}}$$

Compass bearing [p. 445] the direction measured from north clockwise

Complement, *A′* [pp. 297, 311] the set of outcomes that are in the sample space, ε, but not in A. The probability of the event A' is

$$\mathrm{P}(A') = 1 - \mathrm{P}(A)$$

Complementary relationships [p. 406]

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \qquad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

Compound angle formulas [p. 409] *see* addition formulas

Conditional probability [p. 320] the probability of an event A occurring when it is known that some event B has occurred, given by

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$$

Congruence tests [p. 423] Two triangles are congruent if one of the following conditions holds:

- **SSS** the three sides of one triangle are equal to the three sides of the other triangle
- **SAS** two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle
- **AAS** two angles and one side of one triangle are equal to two angles and the matching side of the other triangle
- **RHS** the hypotenuse and one side of a right-angled triangle are equal to the hypotenuse and one side of another right-angled triangle.

Congruent figures [p. 423] have exactly the same shape and size

Constant function [p. 202] a function $f(x) = a$

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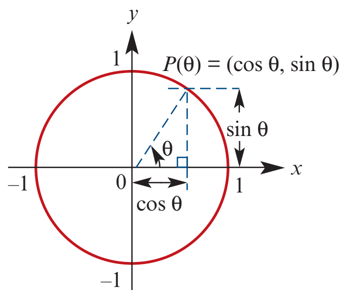
Convergent series [p. 516] An infinite series $t_1 + t_2 + t_3 + \cdots$ is convergent if the sum of the first n terms, S_n, approaches a limiting value as $n \to \infty$. An infinite geometric series is convergent if $-1 < r < 1$, where r is the common ratio.

Coordinates [p. 32] an ordered pair of numbers that identifies a point in the Cartesian plane; the first number identifies the position with respect to the x-axis, and the second number identifies the position with respect to the y-axis

Cosine function [p. 377] cosine θ is defined as the x-coordinate of the point P on the unit circle where OP forms an angle of θ radians with the positive direction of the x-axis.

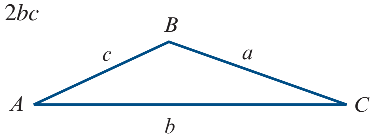


Cosine rule [p. 433] For triangle ABC:

$a^2 = b^2 + c^2 - 2bc\cos A$

or equivalently

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$



The cosine rule is used to find unknown quantities in a triangle given two sides and the included angle, or given three sides.

Cubic function [p. 233] A polynomial of degree 3 is called a cubic, and is a function f with rule $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$.

D

Degree of a polynomial [p. 79] given by the highest power of x with a non-zero coefficient. For example, the polynomial $2x^5 - 7x^2 + 4$ has degree 5.

Derivative function [p. 549] also called the gradient function. The derivative f' of a function f is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Diameter [p. 439] a chord of a circle that passes through the centre

Difference of two cubes [p. 250]
$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

Difference of two squares [pp. 82, 86]
$x^2 - y^2 = (x - y)(x + y)$

Differentiation rules [p. 553]

- $f(x) = x^n$, $f'(x) = nx^{n-1}$, for $\{ n \in \mathbb{Z} : n \neq 0 \}$
- Constant: $f(x) = c$, $f'(x) = 0$
- Sum: $f(x) = g(x) + h(x)$, $f'(x) = g'(x) + h'(x)$
- Multiple: $f(x) = k\,g(x)$, $f'(x) = k\,g'(x)$

Dilation parallel to the x-axis [p. 216] A dilation of factor a parallel to the x-axis is described by the rule $(x, y) \to (ax, y)$. The curve with equation $y = f(x)$ is mapped to the curve with equation $y = f\left(\frac{x}{a}\right)$.

Dilation parallel to the y-axis [p. 216] A dilation of factor b parallel to the y-axis is described by the rule $(x, y) \to (x, by)$. The curve with equation $y = f(x)$ is mapped to the curve with equation $y = bf(x)$.

Direct variation [p. 166] occurs when $y \propto x^n$, for some $n \in (0, \infty)$. We say that y varies directly as x^n, or that y is *directly proportional* to x^n. This means that $y = kx^n$, where k is the constant of variation ($k \in (0, \infty)$).

Discriminant, Δ, of a quadratic [p. 114] the expression $b^2 - 4ac$, which is part of the quadratic formula. For the quadratic equation $ax^2 + bx + c = 0$:

- If $b^2 - 4ac > 0$, there are two solutions.
- If $b^2 - 4ac = 0$, there is one solution.
- If $b^2 - 4ac < 0$, there are no real solutions.

Disjoint sets [pp. 190, 312] If sets A and B have no elements in common, we say A and B are disjoint and write $A \cap B = \emptyset$.

Displacement [p. 607] The displacement of a particle moving in a straight line is defined as the change in position of the particle.

Distance between two points [p. 34] The distance between the given points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Division of polynomials [p. 239] When we divide the polynomial $P(x)$ by the polynomial $D(x)$ we obtain two polynomials, $Q(x)$ the quotient and $R(x)$ the remainder, such that

$P(x) = D(x)Q(x) + R(x)$

and either $R(x) = 0$ or $R(x)$ has degree less than $D(x)$. The polynomial $P(x)$ is the dividend.

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Domain [p. 194] the set of all the first coordinates of the ordered pairs in a relation

Double angle formulas [p. 412]

- $\cos(2u) = \cos^2 u - \sin^2 u$
  $= 2\cos^2 u - 1$
  $= 1 - 2\sin^2 u$
- $\sin(2u) = 2\sin u \cos u$
- $\tan(2u) = \dfrac{2\tan u}{1 - \tan^2 u}$

E

Element [p. 190] a member of a set.

- If x is an element of a set A, we write $x \in A$.
- If x is *not* an element of a set A, we write $x \notin A$.

Empty set, ∅ [pp. 190, 311] the set that has no elements

Even function [p. 270] A function f is even if $f(-x) = f(x)$. This means that the graph is symmetric about the y-axis.

Event [p. 293] a subset of the sample space. It may consist of a single outcome, or it may consist of several outcomes.

Exponential function [p. 453] a function $f(x) = ka^x$, where k is a non-zero constant and the base a is a positive real number other than 1

F

Factor [pp. 239, 240] a number or expression that divides another number or expression without remainder

Factor theorem [p. 246] If $\beta x + \alpha$ is a factor of $P(x)$, then $P\left(-\dfrac{\alpha}{\beta}\right) = 0$. Conversely, if $P\left(-\dfrac{\alpha}{\beta}\right) = 0$, then $\beta x + \alpha$ is a factor of $P(x)$.

Factorise [p. 85] express as a product of factors

Formula [p. 23] an equation containing symbols that states a relationship between two or more quantities; e.g. $A = \ell w$ (area = length × width). The value of A, the subject of the formula, can be found by substituting given values of ℓ and w.

Function [p. 200] a relation such that for each x-value there is only one corresponding y-value. This means that, if (a, b) and (a, c) are ordered pairs of a function, then $b = c$.

Function, vertical-line test [p. 200] used to identify whether a relation is a function or not. If a vertical line can be drawn anywhere on the graph and it only ever intersects the graph a maximum of once, then the relation is a *function*.

G

Geometric sequence [p. 506] a sequence in which each successive term is found by multiplying the previous term by a fixed amount; e.g. 2, 6, 18, 54, ... A geometric sequence has a recurrence relation of the form $t_n = rt_{n-1}$, where r is the common ratio. The nth term can be found using $t_n = ar^{n-1}$, where $a = t_1$.

Geometric series [p. 512] the sum of the terms in a geometric sequence. The sum of the first n terms is given by the formula

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r}$$

where $a = t_1$ and r is the common ratio.

Gradient function *see* derivative function

Gradient of a line [p. 36] The gradient is

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

where (x_1, y_1) and (x_2, y_2) are the coordinates of two points on the line.

Gradient of a vertical line [p. 37] The gradient of a vertical line (parallel to the y-axis) is undefined.

I

Implied domain *see* natural domain

Indefinite integral *see* antiderivative

Independence [p. 329] Two events A and B are independent if $P(A \cap B) = P(A) \times P(B)$ or $P(A \mid B) = P(A)$ or $P(B \mid A) = P(B)$

Index laws [p. 454]

- To multiply two powers with the same base, add the indices: $a^m \times a^n = a^{m+n}$
- To divide two powers with the same base, subtract the indices: $a^m \div a^n = a^{m-n}$
- To raise a power to another power, multiply the indices: $(a^m)^n = a^{m \times n}$
- Rational indices: $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$
- For base $a \in (0, 1) \cup (1, \infty)$, if $a^x = a^y$, then $x = y$.

Inequality [p. 20] a mathematical statement that contains an inequality symbol rather than an equals sign; e.g. $2x + 1 < 4$

Infinite geometric series [p. 516] If $-1 < r < 1$, then the sum to infinity is given by

$$S_\infty = \frac{a}{1 - r}$$

where $a = t_1$ and r is the common ratio.

Integers [p. 191] the elements of $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

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Intersection of sets [pp. 190, 311] The intersection of two sets A and B, written $A \cap B$, is the set of all elements common to A and B.

Interval [p. 191] a subset of the real numbers of the form $[a, b]$, $[a, b)$, (a, ∞), etc.

Inverse variation [p. 170] occurs when $y \propto \frac{1}{x^n}$, for some $n \in (0, \infty)$. We say that y varies inversely as x^n, or that y is *inversely proportional* to x^n. This means that $y = \frac{k}{x^n}$, where k is the constant of variation ($k \in (0, \infty)$).

Irrational number [p. 191] a real number that is not rational; e.g. π and $\sqrt{2}$

Iterative rule [p. 490] *see* recurrence relation

J

Joint variation [p. 181] occurs when one variable varies with more than one other variable; this may be a combination of direct and inverse variation. E.g. the area (A) of a triangle varies jointly as the base (b) and the height (h); we write $A \propto bh$.

K

Karnaugh map [p. 316] a probability table

L

Law of total probability [p. 323] In the case of two events, A and B:

$\mathrm{P}(A) = \mathrm{P}(A \mid B)\mathrm{P}(B) + \mathrm{P}(A \mid B')\mathrm{P}(B')$

Leading term [p. 234] The leading term, $a_n x^n$, of a polynomial is the term of highest index among those terms with a non-zero coefficient. E.g. the leading term of $2x^5 - 7x^2 + 4$ is $2x^5$.

Linear equation [p. 2] a polynomial equation of degree 1; e.g. $2x + 1 = 0$

Linear function [p. 202] a function $f(x) = mx + c$; e.g. $f(x) = 3x + 1$

Literal equation [p. 5] an equation for the variable x in which the coefficients of x, including the constants, are pronumerals; e.g. $ax + b = c$

M

Maximum and minimum value [p. 602] For a continuous function f defined on an interval $[a, b]$:

- if M is a value of the function such that $f(x) \leq M$ for all $x \in [a, b]$, then M is the *absolute maximum* value of the function
- if N is a value of the function such that $f(x) \geq N$ for all $x \in [a, b]$, then N is the *absolute minimum* value of the function.

Midpoint of a line segment [p. 33] If $P(x, y)$ is the midpoint of the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$, then

$$x = \frac{x_1 + x_2}{2} \quad \text{and} \quad y = \frac{y_1 + y_2}{2}$$

Monic polynomial [p. 234] a polynomial whose leading term has coefficient 1; e.g. $x^3 + 4x^2 + 5$

Multiplication rule for choices [p. 345] When sequential choices are involved, the total number of possibilities is found by multiplying the number of options at each successive stage.

Multiplication rule for probability [p. 322] the probability of events A and B both occurring is

$\mathrm{P}(A \cap B) = \mathrm{P}(A \mid B) \times \mathrm{P}(B)$

Multi-stage experiment [p. 306] an experiment that could be considered to take place in more than one stage; e.g. tossing two coins

Mutually exclusive [p. 312] Two sets are said to be mutually exclusive if they have no elements in common.

N

$n!$ [p. 349] The notation $n!$ (read as 'n factorial') is an abbreviation for the product of all the integers from n down to 1:

$n! = n \times (n - 1) \times (n - 2) \times (n - 3) \times \cdots \times 2 \times 1$

Natural domain [pp. 198, 206] When the rule for a relation is given and no domain is specified, then the domain taken is the largest for which the rule has meaning.

Natural numbers [p. 191] the elements of $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

^nC_r [p. 354] the number of combinations of n objects in groups of size r:

$$^nC_r = \frac{n!}{r!\,(n-r)!}$$

An alternative notation for ^nC_r is $\binom{n}{r}$.

Normal, equation of [p. 584] Let (x_1, y_1) be a point on the curve $y = f(x)$. If f is differentiable at $x = x_1$, the equation of the normal at (x_1, y_1) is

$$y - y_1 = \frac{-1}{f'(x_1)}(x - x_1)$$

Null factor theorem [p. 90] If $ab = 0$, then $a = 0$ or $b = 0$.

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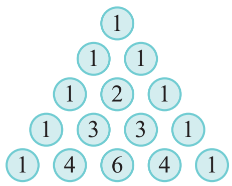
O

Odd function [p. 270] A function f is odd if $f(-x) = -f(x)$. The graph of an odd function has rotational symmetry with respect to the origin.

Ordered pair [p. 194] An ordered pair, denoted (x, y), is a pair of elements x and y in which x is considered to be the first coordinate and y the second coordinate.

P

Pascal's triangle [p. 360] a triangular pattern of numbers formed by the binomial coefficients ${}^nC_r$



Period of a function [p. 386] A function f with domain $\mathbb{R}$ is periodic if there is a positive constant a such that $f(x + a) = f(x)$ for all x. The smallest such a is called the period of f. For example, the period of the sine function is 2π, as $\sin(x + 2\pi) = \sin x$.

Permutations [p. 347] *see* arrangements

Polynomial function [p. 234] A polynomial has a rule of the type

$$y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad n \in \mathbb{N} \cup \{0\}$$

where $a_0, a_1, \ldots, a_n$ are numbers called coefficients.

Position [p. 606] For a particle moving in a straight line, the position of the particle relative to a point O on the line is determined by its distance from O and whether it is to the right or left of O. The direction to the right of O is positive.

Power function [p. 270] a function of the form $f(x) = x^r$, where r is a non-zero real number

Probability [p. 292] a numerical value assigned to the likelihood of an event occurring. If the event A is impossible, then $\mathrm{P}(A) = 0$; if the event A is certain, then $\mathrm{P}(A) = 1$; otherwise $0 < \mathrm{P}(A) < 1$.

Probability table [p. 316] a table used for illustrating a probability problem diagrammatically

Pythagorean identity [p. 407]
$\cos^2\theta + \sin^2\theta = 1$

Q

$\mathbb{Q}$ [p. 191] the set of all rational numbers

Quadratic formula [p. 110]
$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is the solution of the quadratic equation $ax^2 + bx + c = 0$

Quadratic function [p. 79] A quadratic has a rule of the form $y = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$.

Quadratic, turning point form [p. 94] The turning point form of a quadratic function is $y = a(x - h)^2 + k$, where (h, k) is the turning point.

Quartic function [p. 233] A polynomial of degree 4 is called a quartic, and is a function f with rule $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, where $a \neq 0$.

R

$\mathbb{R}^2$ [p. 212] $\{(x, y) : x, y \in \mathbb{R}\}$; i.e. $\mathbb{R}^2$ is the set of all ordered pairs of real numbers

Radian [p. 374] One radian (written 1^c) is the angle subtended at the centre of the unit circle by an arc of length 1 unit:

$$1^c = \frac{180^\circ}{\pi} \quad \text{and} \quad 1^\circ = \frac{\pi^c}{180}$$

Random experiment [p. 293] an experiment, such as the rolling of a die, in which the outcome of a single trial is uncertain but observable

Range [p. 194] the set of all the second coordinates of the ordered pairs in a relation

Rational number [p. 191] a number that can be written as $\dfrac{p}{q}$, for some integers p and q with $q \neq 0$

Rational-root theorem [p. 248]
Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree n with all coefficients a_i integers. Let α and β be integers such that the highest common factor of α and β is 1. If $\beta x + \alpha$ is a factor of $P(x)$, then β divides a_n and α divides a_0.

Rectangular hyperbola [p. 143] The basic rectangular hyperbola has equation $y = \dfrac{1}{x}$.

Recurrence relation [p. 490] a rule which enables each subsequent term of a sequence to be found from previous terms; e.g. $t_1 = 1$, $t_n = t_{n-1} + 2$

Reflection in the x-axis [p. 217] A reflection in the x-axis is described by the rule $(x, y) \to (x, -y)$. The curve with equation $y = f(x)$ is mapped to the curve with equation $y = -f(x)$.

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Reflection in the y-axis [p. 217] A reflection in the y-axis is described by the rule $(x, y) \to (-x, y)$. The curve with equation $y = f(x)$ is mapped to the curve with equation $y = f(-x)$.

Relation [p. 194] a set of ordered pairs; e.g. $\{ (x, y) : y = x^2 \}$. A relation may be written as a listed set of ordered pairs or as a rule with a specified or implied domain. A relation may be represented by a graph or by an arrow diagram.

Remainder theorem [p. 244] When a polynomial $P(x)$ is divided by $\beta x + \alpha$, the remainder is $P\left(-\frac{\alpha}{\beta}\right)$.

Repeated factor of a polynomial [p. 262] a factor that occurs more than once in the factorised form of a polynomial. For example, $x - a$ is a repeated factor of $P(x) = (x - a)^3(x - b)$.

S

Sample space [p. 293] the set of all possible outcomes for a random experiment, sometimes denoted ε

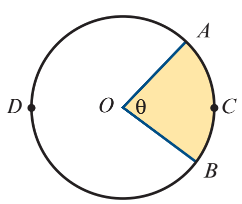
Scientific notation [p. 464] A number is in standard form when written as a product of a number between 1 and 10 and an integer power of 10; e.g. 6.626×10^{-34}.

Secant [p. 536] a straight line that passes through two points $(a, f(a))$ and $(b, f(b))$ on the graph of $y = f(x)$

Sector [pp. 439, 440] Two radii and an arc define a region called a sector. In this diagram, the shaded region is a *minor sector* and the unshaded region is a *major sector*.

Area of sector $= \frac{1}{2}r^2\theta$

where $\theta^c = \angle AOB$

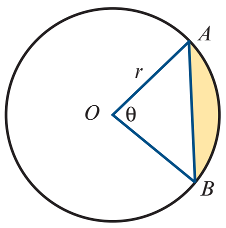


Segment [pp. 439, 442] Every chord divides the interior of a circle into two regions called segments; the smaller is the *minor segment* (shaded), and the larger is the *major segment*.

Area of segment

$= \frac{1}{2}r^2(\theta - \sin\theta)$

where $\theta^c = \angle AOB$



Selections [p. 353] counted when order is not important. The number of ways of selecting r objects from a total of n objects is

$${}^nC_r = \frac{n!}{r!\,(n-r)!}$$

An alternative notation for ${}^nC_r$ is $\binom{n}{r}$.

Sequence [p. 489] a list of numbers, with the order being important; e.g. $1, 1, 2, 3, 5, 8, 13, \ldots$ The numbers of a sequence are called its *terms*, and the nth term is often denoted by t_n.

Series [p. 501] the sum of the terms in a sequence

Set notation [p. 190]

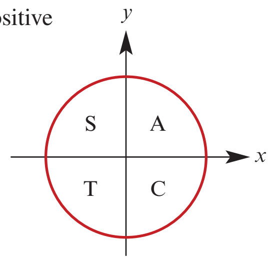
- $\in$ means 'is an element of'
- $\notin$ means 'is not an element of'
- $\subseteq$ means 'is a subset of'
- $\cap$ means 'intersection'
- $\cup$ means 'union'
- $\emptyset$ is the empty set, containing no elements

Sets of numbers [p. 191]

- $\mathbb{N}$ is the set of natural numbers
- $\mathbb{Z}$ is the set of integers
- $\mathbb{Q}$ is the set of rational numbers
- $\mathbb{R}$ is the set of real numbers

Signs of circular functions [p. 381]

1st quadrant	all are positive
2nd quadrant	sin is positive
3rd quadrant	tan is positive
4th quadrant	cos is positive



Simulation [p. 335] the process of finding an approximate solution to a probability problem by repeated trials using a simulation model

Simulation model [p. 335] a simple model which is analogous to a real-world situation. For example, the outcomes from a toss of a coin (head, tail) could be used as a simulation model for the sex of a child (male, female) under the assumption that in both situations the probabilities are 0.5 for each outcome.

Simultaneous equations [p. 11] equations of two or more lines or curves in the Cartesian plane, the solutions of which are the points of intersection of the lines or curves

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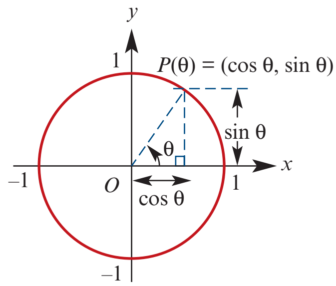
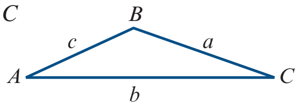
Sine function [p. 377] sine θ is defined as the y-coordinate of the point P on the unit circle where OP forms an angle of θ radians with the positive direction of the x-axis.



Sine rule [p. 429] For triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



The sine rule is used to find unknown quantities in a triangle given one side and two angles, or given two sides and a non-included angle.

Speed [p. 609] the magnitude of velocity

Speed, average [pp. 535, 609]

$$\text{average speed} = \frac{\text{total distance travelled}}{\text{total time taken}}$$

Standard form [p. 464] *see* scientific notation

Stationary point [pp. 591, 594] A point with coordinates $(a, f(a))$ on a curve $y = f(x)$ is said to be a stationary point if $f'(a) = 0$.

Straight line, equation given two points [p. 45] $y - y_1 = m(x - x_1)$, where $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Straight line, gradient–intercept form [p. 42] The gradient–intercept form of the equation of a straight line is $y = mx + c$, where m is the gradient and c is the y-axis intercept.

Straight lines, parallel [p. 54] Two non-vertical straight lines are parallel to each other if and only if they have the same gradient.

Straight lines, perpendicular [p. 55] Two straight lines are perpendicular to each other if and only if the product of their gradients is −1 (or if one is horizontal and the other vertical).

Strictly decreasing [p. 565] A function f is strictly decreasing on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.

Strictly increasing [p. 564] A function f is strictly increasing on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.

Subset [p. 190] A set B is called a subset of set A if every element of B is also an element of A. We write $B \subseteq A$.

Sum of two cubes [p. 250]
$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

Sum to infinity [p. 516] The sum to infinity of an infinite geometric series exists provided $-1 < r < 1$ and is given by

$$S_\infty = \frac{a}{1 - r}$$

where $a = t_1$ and r is the common ratio.

T

Tangent, equation of [p. 583] Let (x_1, y_1) be a point on the curve $y = f(x)$. Then, if f is differentiable at $x = x_1$, the equation of the tangent at (x_1, y_1) is given by
$y - y_1 = f'(x_1)(x - x_1)$

Tangent function [p. 379] The tangent function is given by

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Translation [p. 212] A translation of h units in the positive direction of the x-axis and k units in the positive direction of the y-axis is described by the rule $(x, y) \to (x + h, y + k)$, where h and k are positive numbers. The curve with equation $y = f(x)$ is mapped to the curve with equation $y - k = f(x - h)$.

Tree diagram [pp. 308, 323] a diagram representing the outcomes of a multi-stage experiment

Trigonometric functions [pp. 377, 379] the sine, cosine and tangent functions

Trigonometric functions, exact values [p. 384]

θ^c	$\theta°$	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0°	0	1	0
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	90°	1	0	undefined

Glossary T

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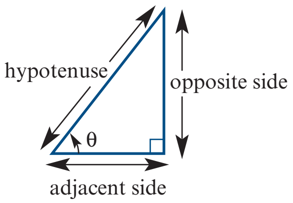
Trigonometric ratios [p. 424]

$\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$

$\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

$\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$



U

Union of sets [pp. 190, 311] The union of two sets A and B, written $A \cup B$, is the set of all elements which are in A or B or both.

V

Velocity [p. 607] The velocity of a particle is defined as the rate of change of its position with respect to time.

Velocity, average [p. 607]

$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}}$$

Velocity, instantaneous [p. 607] $v = \dfrac{dx}{dt}$

Venn diagram [pp. 190, 311] a diagram showing sets and the relationships between sets

Vertical-line test [p. 200] *see* function

Z

$\mathbb{Z}$ [p. 191] the set of all integers

Zero polynomial [p. 234] The number 0 is called the zero polynomial.

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Answers

Chapter 1

Exercise 1A

1 **a** 3 **b** 9 **c** 1 **d** −8
e 5 **f** 2 **g** $\frac{5}{3}$ **h** $\frac{-7}{2}$
i $\frac{7}{3}$ **j** $\frac{20}{3}$ **k** $\frac{-10}{3}$ **l** $\frac{14}{5}$

2 **a** $a+b$ **b** $a-b$ **c** $\frac{b}{a}$ **d** ab
e $\frac{bc}{a}$

3 **a** $y=5$ **b** $t=5$ **c** $y=-\frac{3}{2}$ **d** $x=2$
e $a=\frac{11}{2}$ **f** $a=\frac{8}{3}$ **g** $y=136$ **h** $t=1$
i $x=12$ **j** $y=-\frac{9}{5}$ **k** $x=-7$ **l** $y=2$

4 **a** $\frac{4}{3}$ **b** −5 **c** 2

5 **a** −1 **b** 18 **c** $\frac{6}{5}$ **d** 23
e 0 **f** 10 **g** 12 **h** 8
i $-\frac{14}{5}$ **j** $\frac{12}{5}$ **k** $\frac{7}{2}$

6 **a** $\frac{-b}{a}$ **b** $\frac{e-d}{c}$ **c** $\frac{c}{a}-b$ **d** $\frac{b}{c-a}$
e $\frac{ab}{b+a}$ **f** $a+b$ **g** $\frac{b-d}{a-c}$ **h** $\frac{bd-c}{a}$

7 **a** −18 **b** −78.2 **c** 16.75 **d** 28
e 34 **f** $\frac{3}{26}$

8 $x=\frac{a^2+b^2+2ab}{ac+bc}=\frac{a+b}{c}$

9 $x=\frac{ab}{a-b-c}$

Exercise 1B

1 **a** $x+2=6,\ 4$ **b** $3x=10,\ \frac{10}{3}$
c $3x+6=22,\ \frac{16}{3}$ **d** $3x-5=15,\ \frac{20}{3}$
e $6(x+3)=56,\ \frac{19}{3}$ **f** $\frac{x+5}{4}=23,\ 87$

2 $A=\$8$, $B=\$24$, $C=\$16$ **3** 14 and 28
4 8 kg **5** 1.3775 m^2 **6** 49, 50, 51
7 17, 19, 21, 23 **8** 4200 L **9** 21
10 3 km **11** 9 and 12 dozen
12 7.5 km/h **13** 3.6 km
14 Dad is 30 and son is 6.

Exercise 1C

1 **a** $x=-1, y=-1$ **b** $x=5, y=21$
c $x=-1, y=5$ **d** $x=5, y=19$
e $x=-4, y=-13$ **f** $x=-\frac{8}{5}, y=-\frac{2}{5}$

2 **a** $x=8, y=-2$ **b** $x=-1, y=4$
c $x=7, y=\frac{1}{2}$

3 **a** $x=2, y=-1$ **b** $x=2.5, y=-1$
c $m=2, n=3$ **d** $x=2, y=-1$
e $s=2, t=5$ **f** $x=10, y=13$
g $x=\frac{4}{3}, y=\frac{7}{2}$ **h** $p=1, q=-1$
i $x=-1, y=\frac{5}{2}$

4 **a** No solutions **b** Infinitely many solutions
c One solution **d** One solution

Exercise 1D

1 25, 113 **2** 22.5, 13.5
3 **a** \$70 **b** \$12 **c** \$3
4 **a** \$168 **b** \$45 **c** \$15

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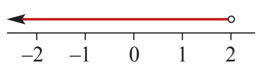
5 17 and 28 **6** 44 and 12
7 5 pizzas, 25 hamburgers
8 Started with 60 and 50; finished with 30 each
9 134 adults, 16 children **10** $\frac{7}{10}$
11 26 **12** 420 adults, 540 children
13 \$17 000 **14** 120 shirts, 300 ties
15 360 Outbacks, 300 Bush Walkers
16 2800 in Mydney, 3200 in Selbourne
17 20 kg at \$10, 40 kg at \$11 and 40 kg at \$12

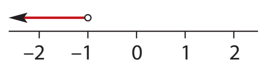
Exercise 1E

1 a $x < 1$ **b** $x > 13$ **c** $x \geq 3$ **d** $x \leq 12$
e $x \leq -6$ **f** $x > 3$ **g** $x < -2$ **h** $x \geq -8$
i $x \leq \frac{3}{2}$

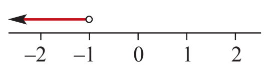
2 a $x < 2$



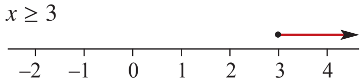
b $x < -1$



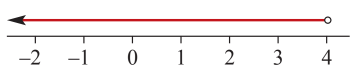
c $x < -1$



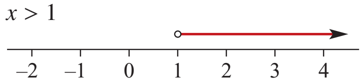
d $x \geq 3$



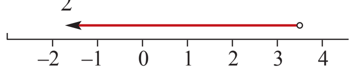
e $x < 4$



f $x > 1$

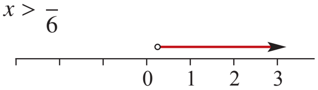


g $x < 3\frac{1}{2}$



h $x \geq 3$

i $x > \frac{1}{6}$



3 a $x > \frac{-1}{2}$ **b** $x < 2$ **c** $x > -5$

4 $3x < 20$, $x < \frac{20}{3}$, 6 pages **5** 87

Exercise 1F

1 a 18 **b** 9 **c** 3 **d** −18
e 3 **f** 81 **g** 5 **h** 20
2 a $S = a + b + c$ **b** $P = xy$ **c** $C = 5p$
d $T = dp + cq$ **e** $T = 60a + b$
3 a 15 **b** 31.4 **c** 1000 **d** 12
e 314 **f** 720

4 a $V = \frac{c}{p}$ **b** $a = \frac{F}{m}$ **c** $P = \frac{I}{rt}$
d $r = \frac{w - H}{C}$ **e** $t = \frac{S - P}{Pr}$ **f** $r = \frac{R(V-2)}{V}$
5 a $T = 48$ **b** $b = 8$ **c** $h = 3.82$ **d** $b = 10$
6 a $(4a + 3w)$ m **b** $(h + 2b)$ m **c** $3wh\,\text{m}^2$
d $(4ah + 8ab + 6wb)\,\text{m}^2$
7 a i $T = 2\pi(p + q) + 4h$ **ii** $88\pi + 112$
b $p = \frac{A}{\pi h} - q$
8 a $D = \frac{2}{3}$ **b** $b = 2$ **c** $n = \frac{60}{29}$ **d** $r = 4.8$
9 a $D = \frac{1}{2}bc(1 - k^2)$ **b** $k = \sqrt{1 - \frac{2D}{bc}}$
c $k = \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}$
10 a $P = 4b$ **b** $A = 2bc - c^2$ **c** $b = \frac{A + c^2}{2c}$
11 a $b = \frac{a^2 - a}{2}$ **b** $x = \frac{-ay}{b}$
c $r = \pm\sqrt{3q - p^2x^2}$ **d** $v = \pm\sqrt{u^2\left(1 - \frac{x^2}{y^2}\right)}$

Chapter 1 review

Short-answer questions

1 a 1 **b** $\frac{-3}{2}$ **c** $\frac{-2}{3}$ **d** −27
e 12 **f** $\frac{44}{13}$ **g** $\frac{1}{8}$ **h** 31
2 a $t = a - b$ **b** $\frac{cd - b}{a}$ **c** $\frac{d}{a} + c$
d $\frac{cb - a}{c - 1}$ **e** $\frac{2b}{c - a}$ **f** $\frac{1 - cd}{ad}$
3 a $x < \frac{2}{3}$ **b** $x \leq -148\frac{1}{2}$ **c** $x < \frac{22}{29}$ **d** $x \geq \frac{-7}{17}$
4 a $x = 2(z + 3t)$ **b** −10
5 a $d = e^2 + 2f$ **b** $f = \frac{d - e^2}{2}$ **c** $f = \frac{1}{2}$
6 $400\pi\,\text{cm}^2$
7 a 196π **b** $\frac{975\pi}{2}$
8 a $r = \frac{A}{\pi s}$ **b** $w = \frac{T - P}{Pr}$
c $r = \frac{n - p}{v^2}$ **d** $x = \frac{ac - b^2}{b}$
9 a $s = 75$ **b** $t = 8$
10 $5\sqrt{2}$ cm
11 12 m and 17 m
12 $m = 2$ and $n = 15$
13 Mr Apollo earns \$100 000, Mr Adonis earns \$107 200 and Ms Aphrodite earns \$96 000
14 a $a = \frac{28}{11}, b = -\frac{9}{11}$ **b** $a = -\frac{11}{5}, b = -\frac{33}{5}$
15 5 hours travelling on highways

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Extended-response questions

1 a $C = \frac{-10}{9}$ **b** $F = 86$ **c** $x = -40$
d $x = -62.5$ **e** $x = \frac{-160}{13}$ **f** $k = 5$

2 a $r = \frac{2uv}{u+v}$ **b** $m = \frac{v}{u}$

3 a $T = 6w + 6\ell$
b i $T = 8w$ **ii** $w = 12\frac{1}{2}$, $\ell = \frac{25}{6}$
c i $y = \frac{L - 6x}{8}$ **ii** $y = 22$
d $x = 10, y = 5$

4 a Distance Tom travelled = ut km
Distance Julie travelled = vt km
b i $t = \frac{d}{u+v}$ h
ii Distance from $A = \frac{ud}{u+v}$ km
c $t = 1.25$ h, distance from $A = 37.5$ km

5 a Average speed $= \frac{2uv}{u+v}$
b i $\frac{uT}{v}$ **ii** $\frac{vT + uT}{v}$

6 a $\frac{3}{a} + \frac{3}{b}$ **c i** $c = \frac{2ab}{a+b}$ **ii** $\frac{40}{3}$

7 a $\frac{x}{8}, \frac{y}{10}$ **b** $\frac{80(x+y)}{10x+8y}$
c $x = \frac{320}{9}, y = \frac{310}{9}$

8 The three lines intersect at the point (4, 3)

Chapter 2

Exercise 2A

1 a (5, 8) **b** $\left(\frac{1}{2}, \frac{1}{2}\right)$ **c** (1.6, 0.7)
d (−0.7, 0.85)

2 $M_{AB}(3, 3)$, $M_{BC}\left(8, 3\frac{1}{2}\right)$, $M_{AC}\left(6, 1\frac{1}{2}\right)$

3 Coordinates of C are (6, 8.8)

4 a (4, 4) **b** (2, −0.2) **c** (−2, 5) **d** (−4, −3)

5 a $\left(\frac{1+a}{2}, \frac{4+b}{2}\right)$ **b** $a = 9$, $b = -6$

6 a $5\sqrt{2} \approx 7.07$ **b** $\sqrt{17} \approx 4.12$
c $\sqrt{34} \approx 5.83$ **d** 13

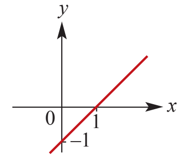
7 $\sqrt{97} + \sqrt{85} + \sqrt{104} \approx 29.27$

8 $PM = \sqrt{145} \approx 12.04$ **9** DN

Exercise 2B

1 a 4 **b** 2 **c** $\frac{1}{4}$ **d** −4 **e** 1 **f** −1
g $\frac{5}{4}$ **h** −2 **i** $\frac{-5}{4}$ **j** $\frac{4}{3}$ **k** 0

2 Any line parallel to the one shown $y = x - 1$



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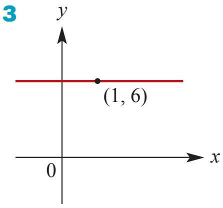

4 a $-\frac{1}{4}$ **b** $-\frac{5}{2}$ **c** −2 **d** −8 **e** 0 **f** −1
g 7 **h** 11 **i** −13 **j** 11 **k** 111 **l** 61

5 a −2 **b** $\frac{2}{5}$

6 a 54 **b** $\frac{5}{6}$

7 a 45° **b** 45° **c** 26.57° **d** 135°

8 a 45° **b** 26.57° **c** 161.57° **d** 49.4°
e 161.57° **f** 135°

9 a 1 **b** −1 **c** $\sqrt{3}$ **d** $-\sqrt{3}$

Exercise 2C

1 a $m = 3, c = 6$ **b** $m = -6, c = 7$
c $m = 3, c = -6$ **d** $m = -1, c = -4$

2 a $y = 3x + 5$ **b** $y = -4x + 6$ **c** $y = 3x - 4$

3 a $m = 3, c = -6$ **b** $m = 2, c = -4$
c $m = \frac{1}{2}, c = -2$ **d** $m = \frac{1}{3}, c = -\frac{5}{3}$

4 a $m = 2, c = -9$ **b** $m = -\frac{3}{4}, c = \frac{5}{2}$
c $m = -\frac{1}{3}, c = -2$ **d** $m = \frac{5}{2}, c = -2$

5 a $y = 3x - 11$ **b** $y = -2x + 9$

6 a $y = -\frac{1}{3}x + \frac{11}{3}$ **b** $y = -\frac{7}{5}x + 4$
c $y = -2x + 4$ **d** $y = \frac{11}{3}x - \frac{61}{3}$

7 a 2 **b** $y = 2x + 6$

8 a $y = 2x + 4$ **b** $y = -2x + 8$

9 a $y = 2x + 6$ **b** $y = -2x + 4$
c $y = -5x + 15$

10 a $y = -\frac{2}{3}x + 4$ **b** $y = -2x - 6$
c $y = -x + 4$ **d** $y = -\frac{3}{2}x + 3$

11 a $y = \frac{2}{3}x + 4$ **b** $y = \frac{2}{3}x - \frac{2}{3}$
c $y = \frac{1}{2}x + 1\frac{1}{2}$ **d** $y = -\frac{1}{2}x + 2$
e $y = x + 3.5$ **f** $y = -0.5x + 0.25$

12 a $y = 4x + 4$ **b** $y = -\frac{2}{3}x$ **c** $y = -x - 2$
d $y = \frac{1}{2}x - 1$ **e** $y = 3\frac{1}{2}$ **f** $x = -2$

13 Yes **14** Only C

15 a $x = 4$ **b** $y = 11$ **c** $x = 11$ **d** $y = -1$

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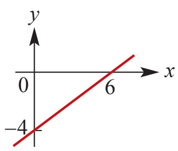
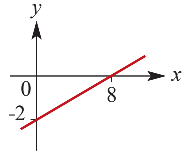
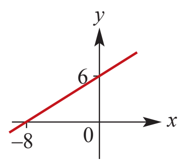
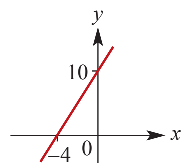
Exercise 2D

1 a (0, 4), (4, 0) **b** (0, −4), (4, 0)
c (0, −6), (−6, 0) **d** (0, 8), (−8, 0)

2 a


b


c


d


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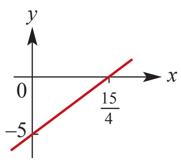

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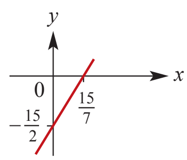

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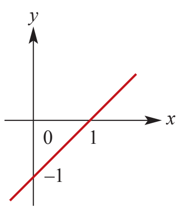

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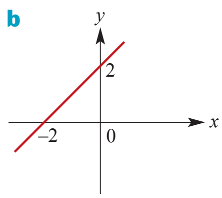

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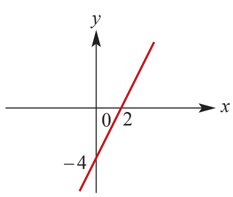

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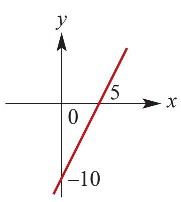

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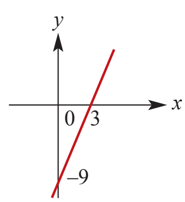

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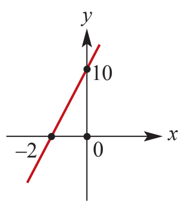

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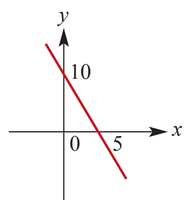

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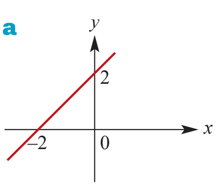

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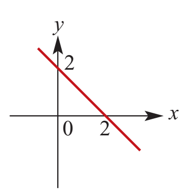

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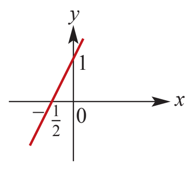

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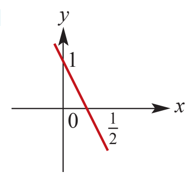

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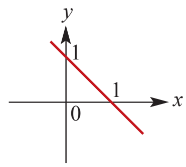

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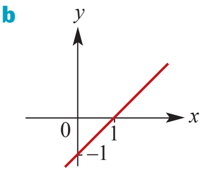

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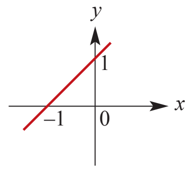

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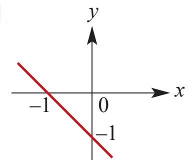

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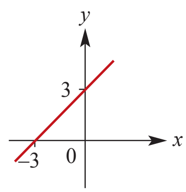

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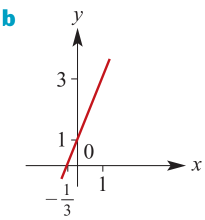

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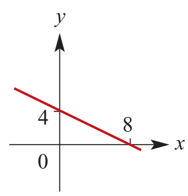

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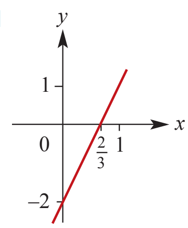

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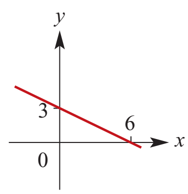

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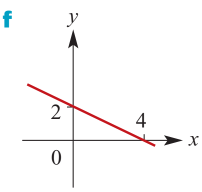

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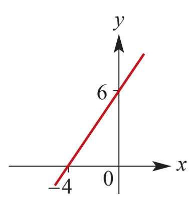

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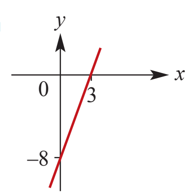

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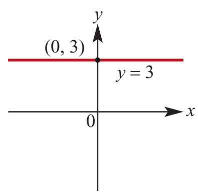

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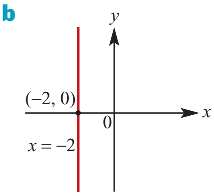

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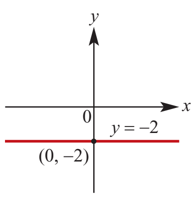
c



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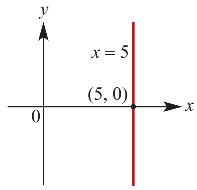


9 a 45° **b** 135° **c** 45° **d** 135°
e 63.43° (to 2 d.p.) **f** 116.57° (to 2 d.p.)

10 a 71.57° **b** 135° **c** 45° **d** 161.57°

11 $a = -4,\ b = \frac{4}{3},\ d = -1,\ e = \frac{14}{3}$

Exercise 2E

1 a $y = 2x - 10$ **b** $y = -\frac{1}{2}x$
c $y = -2x + 6$ **d** $y = \frac{1}{2}x - 4$
e $y = \frac{2}{3}x - \frac{14}{3}$ **f** $y = -\frac{3}{2}x + 4$
g $y = -\frac{1}{3}x - \frac{2}{3}$ **h** $y = 3x - 14$

2 Parallel lines: a, b, c; non-parallel lines: d

3 a $y = 4$ **b** $x = 2$ **c** $y = 4$ **d** $x = 3$

4 $y = 2x + 2$

5 Midpoint of AB is $(-1, 6)$; $y = 2x + 8$

6 $m_{BC} = -\frac{3}{5},\ m_{AB} = \frac{5}{3}$
$\therefore m_{BC} \times m_{AB} = -\frac{3}{5} \times \frac{5}{3} = -1$
$\therefore \triangle ABC$ is a right-angled triangle

7 $m_{AB} = -2,\ m_{BC} = \frac{1}{2}$ $AB \perp BC$

8 $m_{RS} = -\frac{1}{2},\ m_{ST} = 2 \therefore RS \perp ST$
$m_{UT} = -\frac{1}{2},\ m_{ST} = 2 \therefore UT \perp ST$
(Also need to show $SR = UT$.)
$\therefore RSTU$ is a rectangle

9 $\ell = -\frac{16}{3},\ m = \frac{80}{3}$

10 a $y = -\frac{1}{2}x + \frac{11}{2}$ **b** $B(1, 5)$ **c** $C(2, 7)$

Exercise 2F

1 $m = 5$

2 $c = 5$

3 a $y = -\frac{1}{m}x + 3$ **b** $m = \frac{1}{7}$

4 $m = 2$

5 a $x = \frac{3}{m}$ **b** $m = \frac{9}{5}$ **c** $m \geq 3$
d $y = -\frac{x}{m} - 3$

6 a $x = -\frac{c}{2}$ **b** $c = -4$ **c** $c \geq -2$
d $y = -\frac{1}{2}x + c$

7 a $x = 4a$ **b** $m = \frac{12}{a}$ **c i** $a = 6$ **ii** $a = -6$

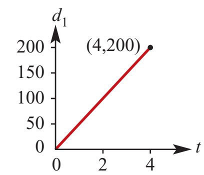
8 a $x = \frac{c}{2}$ **b** $c = 9$ **c** $c \leq 2$ **d** $y = \frac{1}{2}x + c$
e i $c = 12$ **ii** $c = 4$ **iii** $c = 8$

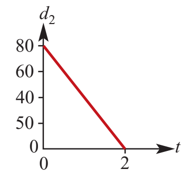
9 a $\frac{12}{b}$ **b** $-\frac{3}{b}$ **c i** $b = -3$ **ii** $b = \frac{3}{2}$
d $y = \frac{b}{3}x - \frac{4b}{3}$

Exercise 2G

1 $w = 20n + 350$ for $n \in \mathbb{N} \cup \{0\}$

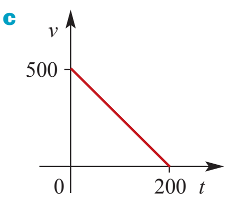
2 a $d_1 = 50t$ **b** $d_2 = 80 - 40t$
c Gradient = 50 Gradient = −40





3 a $V = 5t$ **b** $V = 10 + 5t$

4 a $v = 500 - 2.5t$
b Domain $0 \leq t \leq 200$; Range $0 \leq v \leq 500$
c

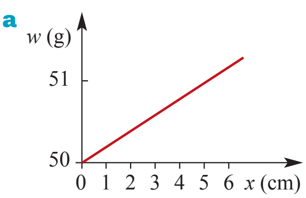


5 $C = 1.5n + 2.6$

6 a $C = 0.24x + 85$ **b** $145

7 $d = 200 - 5t$

8 a



b $w = 0.2x + 50$ **c** $x = 12.5$ cm

9 a $C = 0.06n - 1$ **b** $59

10 a $C = 5n + 175$ **b** Yes **c** $175

Exercise 2H

1 Both lines have gradient −1, but their y-axis intercepts are 6 and $\frac{13}{2}$

2 $(t, 6 - t)$ where t is a real number

3 a $m = 4$ **b** $m \neq 4$ **c** $m = \frac{9}{5}$

4 $k = 2,\ m = 5$

5 $k = 24,\ m = 0$

6 $m = -3$

7 a $m = -5$ **b** $m = 3$

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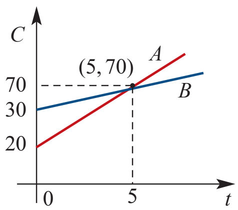
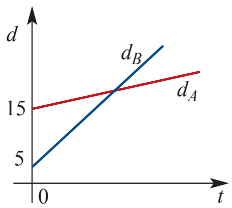
8 a



b $t = 5$

9 $b = 0.28$ and $a = 0.3$, $\frac{25}{7}$ m/s

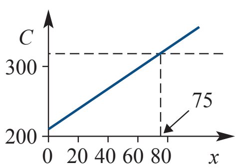
10 a



b 2 p.m.

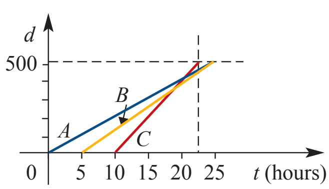
11 a $C_1 = 210 + 1.6x$, $C_2 = 330$

b



c Fixed-charge method is cheaper for $x > 75$

12 a



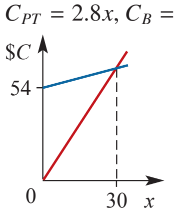
b C wins the race

d C leaves 5 hours after B, and overtakes B $13\frac{1}{3}$ hours after B had started, then overtakes A 20 hours after A had started. C wins with a total time of $22\frac{1}{2}$ hours ($12\frac{1}{2}$ hours for journey + 10 hours handicap), with A and B deadheating for second place, each with a total time of 25 hours.

13 Both craft will pass over the point $\left(5\frac{1}{3}, -4\right)$

14 a $C_{PT} = 2.8x$, $C_B = 54 + x$

b

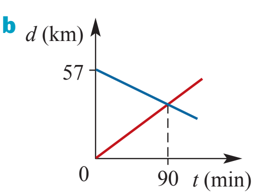


c More than 30

15 a $d_A = \frac{1}{3}t$

$d_M = 57 - \frac{3}{10}t$

b



c 10:30 a.m. **d** Anne 30 km, Maureen 27 km

Chapter 2 review

Short-answer questions

1 a Midpoint = (3, 2); Length = 4

b Midpoint = $\left(-\frac{1}{2}, -\frac{9}{2}\right)$; Length = $\sqrt{74}$

c Midpoint = $\left(5, \frac{5}{2}\right)$; Length = 5

2 a $\frac{9}{4}$ **b** $-\frac{10}{11}$ **c** Undefined

d -1 **e** $\frac{b}{a}$ **f** $\frac{-b}{a}$

3 a $y = 4x$ **b** $y = 4x + 5$

c $y = 4x + 2$ **d** $y = 4x - 5$

4 a $a = -2$ **b** $b = \frac{20}{3}$

5 $4y + 3x = -7$

6 $3y + 2x = -5$

7 a $y = 11$ **b** $y = 6x - 10$ **c** $3y + 2x = -3$

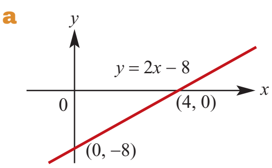
8 $y = x + 1$

9 $y + x = 1$

10 $y = \frac{1}{3}x + \frac{20}{3}$

11 $a = 1$, $b = -\frac{1}{2}$, $d = 5$, $e = 3$

12 a



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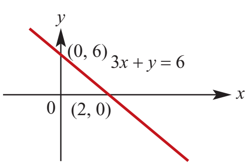

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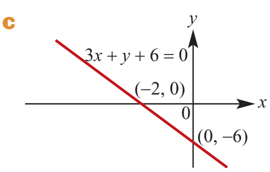

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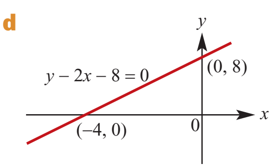

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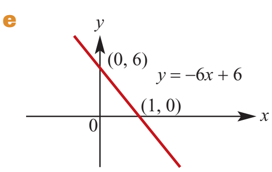

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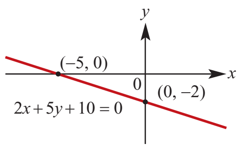
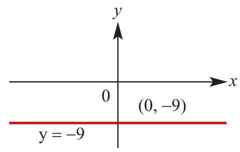
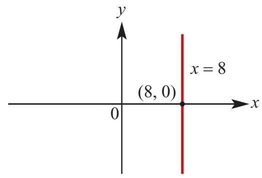


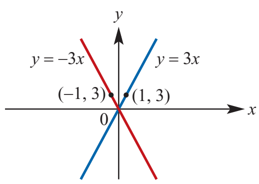
13 a $y = -9$



b $x = 8$



c **i** $y = 3x$ **ii** $y = -3x$



14 a $d = 60t$ **b** $m = 60$

15 $S = 800 + 500n$

16 a $y = 2x + 2$ **b i** $\frac{-2}{a}$ **ii** $-2 < a < 0$

c $\left(\frac{1}{a-1}, \frac{1}{a-1} + 3\right)$

Extended-response questions

1 a $C = 550 + 190n$ **b** 12 days

c Fewer than 5 days

2 a Cost of the plug

b Cost per metre of cable **c** 1.8 **d** $11\frac{1}{9}$ m

3 a The maximum profit (when $x = 0$)

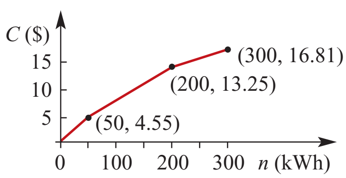
b 43 seats

c Profit reduces by \$24 for each empty seat

4 a i $C = 0.091n$ **ii** $C = 1.65 + 0.058n$

iii $C = 6.13 + 0.0356n$

b



i For 30 kWh, $C = 2.73$

ii For 90 kWh, $C = 6.87$

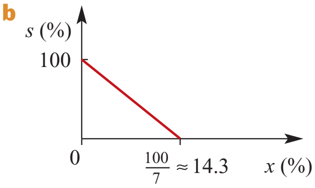
iii For 300 kWh, $C = 16.81$

c 389.61 kWh

5 a $y = -\frac{7}{3}x + 14\frac{2}{3}$ **b** $20\frac{1}{3}$ km south

6 a $s = 100 - 7x$

b



c $\frac{5}{7}\%$

d $14\frac{2}{7}\%$

e Probably not a realistic model when $s = 0$

f $0 \le x \le 14\frac{2}{7}$

7 a AB, $y = x + 2$; CD, $y = 2x - 6$

b Intersection is at (8, 10), on the near bank

8 a $\frac{128}{19}$

b $y = -\frac{199}{190}x + \frac{128}{19}$

c No, since gradient of AB is $\frac{20}{19} \approx 1.053$, whereas the gradient of VC is -1.047

9 a No **b** $1\frac{41}{71}$ km to the east of H

10 a $y = x - 38$ **b** $B(56, 18)$

c $y = -2x + 166$ **d** (78, 10)

11 a $y = 3x + 2$ **b** (0, 2)

c $y = 3x - 8$ **d** (2, −2)

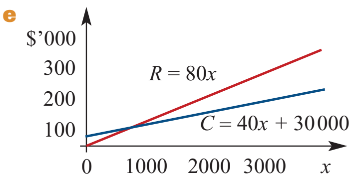
e Area = 10 square units

f Area = 40 square units

12 a $C = 40x + 30\,000$ **b** \$45 **c** 5000

d $R = 80x$

e



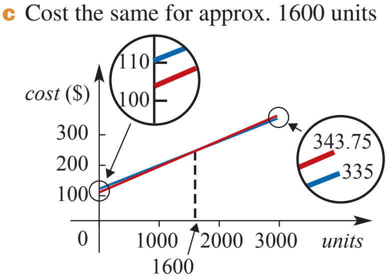
f 751 **g** $P = 40x - 30\,000$

13 a Cost with method 1 is \$226.75; cost with method 2 is \$227; so method 1 is cheaper

b

	0	1000	2000	3000
Method 1	100	181.25	262.50	343.75
Method 2	110	185	260	335

c Cost the same for approx. 1600 units



d $C_1 = 0.08125x + 100$, $C_2 = 0.075x + 110$, $x = 1600$

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14 a $(17, 12)$ **b** $3y = 2x + 2$

15 a PD: $y = \frac{2}{3}x + 120$; DC: $y = \frac{2}{5}x + 136$;

CB: $y = -\frac{5}{2}x + 600$; BA: $y = \frac{2}{5}x + 20$;

AP: $y = -\frac{3}{5}x + 120$

b At B and C, since product of gradients is -1

E.g. $m_{DC} = \frac{2}{5}$, $m_{CB} = -\frac{5}{2}$, $m_{DC} \times m_{CB} = -1$

Chapter 3

Exercise 3A

1 a $2x - 8$ **b** $-2x + 8$ **c** $6x - 12$
d $-12 + 6x$ **e** $x^2 - x$ **f** $2x^2 - 10x$

2 a $6x + 1$ **b** $3x - 6$ **c** $x + 1$ **d** $5x - 3$

3 a $14x - 32$ **b** $2x^2 - 11x$
c $32 - 16x$ **d** $6x - 11$

4 a $2x^2 - 11x$ **b** $3x^2 - 15x$
c $-20x - 6x^2$ **d** $6x - 9x^2 + 6x^3$
e $2x^2 - x$ **f** $6x - 6$

5 a $6x^2 - 2x - 28$ **b** $x^2 - 22x + 120$
c $36x^2 - 4$ **d** $8x^2 - 22x + 15$
e $x^2 - (\sqrt{3} + 2)x + 2\sqrt{3}$
f $2x^2 + \sqrt{5}x - 5$ **g** $3x^2 + \sqrt{7}x - 14$
h $5x^2 + (10\sqrt{2} - 3)x - 6\sqrt{2}$
i $5x^2 - (3\sqrt{5} + 32\sqrt{10})x + 96\sqrt{2}$

6 a $6x^3 - 5x^2 - 14x + 12$ **b** $x^3 - 1$
c $24 - 20x - 8x^2 + 6x^3$ **d** $3x^2 + 4x + 3$
e $-10x^2 + 5x - 2$

7 a $x^2 - 8x + 16$ **b** $4x^2 - 12x + 9$
c $36 - 24x + 4x^2$ **d** $x^2 - x + \frac{1}{4}$
e $x^2 - 2\sqrt{5}x + 5$ **f** $x^2 - 4\sqrt{3}x + 12$

8 a $x^2 - 9$ **b** $4x^2 - 16$ **c** $81x^2 - 121$
d $4x^2 - 9$ **e** $4x^2 - 25$ **f** $x^2 - 5$
g $4x^2 - 27$ **h** $3x^2 - 7$

9 a $x^2 + y^2 - z^2 - 2xy$ **b** $4a^2 - 4ab + b^2 - c^2$
c $9w^2 + 8uz - 16z^2 - u^2$
d $4a^2 - 5b^2 + 4ac + c^2$

10 a i $x^2 + 2x + 1$ **ii** $(x + 1)^2$
b i $(x - 1)^2 + 2(x - 1) + 1$ **ii** x^2
Note: The answers to **10ai** and **10aii** are equivalent and that the answers to **10bi** and **10bii** are equivalent.

Exercise 3B

1 a $2(x + 2)$ **b** $4(a - 2)$ **c** $3(2 - x)$
d $2(x - 5)$ **e** $6(3x + 2)$ **f** $8(3 - 2x)$

2 a $2x(2x - y)$ **b** $8x(a + 4y)$ **c** $6b(a - 2)$
d $2xy(3 + 7x)$ **e** $x(x + 2)$ **f** $5x(x - 3)$
g $-4x(x + 4)$ **h** $7x(1 + 7x)$ **i** $x(2 - x)$

3 a $6x^2y^2(x + 2)$ **b** $xy(7x - 6y)$
c $2xy^2(4x + 3)$

4 a $(x^2 + 1)(x + 5)$ **b** $(x + 3)(y + 2)$
c $(x - 1)(x + 1)(y - 1)(y + 1)$
d $(a + b)(x + y)$ **e** $(a^2 + 1)(a - 3)$
f $(2a - 5)(b - 6)$ **g** $(2x + 5)(x - 1)$
h $(x + 2)(x - 2)(x + 2)$
i $(x - a)(x + a)(x - b)$

5 a $(x - 6)(x + 6)$ **b** $(x - 9)(x + 9)$
c $(x - a)(x + a)$ **d** $(2x - 9)(2x + 9)$
e $(3x - 4)(3x + 4)$ **f** $(5x - y)(5x + y)$
g $3(x - 4)(x + 4)$ **h** $2(x - 7)(x + 7)$
i $3a(x - 3)(x + 3)$ **j** $(a - \sqrt{7})(a + \sqrt{7})$
k $(\sqrt{2}a - \sqrt{5})(\sqrt{2}a + \sqrt{5})$
l $(x - 2\sqrt{3})(x + 2\sqrt{3})$

6 a $(x - 6)(x + 2)$ **b** $(7 + x)(3 - x)$
c $3(x - 1)(x + 3)$ **d** $-5(2x + 1)$
e $-24x$ **f** $-5(x + 7)(x + 1)$

7 a $(x - 9)(x + 2)$ **b** $(y - 16)(y - 3)$
c $(a - 2)(a - 12)$ **d** $(a + 9)^2$
e $(x - 8)(x + 3)$ **f** $(x - 12)(x + 10)$

8 a $(3x - 1)(x - 2)$ **b** $(2x + 1)(3x + 2)$
c $(5x + 3)(x + 4)$ **d** $(2x + 1)(x + 4)$
e $(3x - 2)(2x - 5)$ **f** $(3x + 1)(2x - 3)$
g $(3x - 2)(4x - 3)$ **h** $(x - 2)(5x + 6)$
i $x(5x - 6)(x - 2)$

9 a $3(y - 6)(y + 2)$ **b** $2(x - 7)(x - 2)$
c $4(x - 3)(x - 6)$ **d** $3(x + 2)(x + 3)$
e $a(x + 3)(x + 4)$ **f** $3x(4 - x)^2$

10 a $x(x + 2)$ **b** $(2x - 3)(x + 2)$
c $2(2x + 5)(x + 2)$

Exercise 3C

1 a 2 or 3 **b** 0 or 2 **c** 4 or 3 **d** 4 or 3
e 3 or -4 **f** 0 or 1 **g** $\frac{5}{2}$ or 6 **h** -4 or 4

2 a $-0.65, 4.65$ **b** $-0.58, 2.58$ **c** $-2.58, 0.58$

3 a $9, -8$ **b** $4, 2$ **c** $11, -3$ **d** $4, -16$
e $2, -7$ **f** $-3, 8$

4 a $-\frac{3}{2}, -1$ **b** $\frac{1}{2}, \frac{3}{2}$ **c** $-\frac{2}{3}, -\frac{3}{2}$ **d** $-\frac{3}{2}, 2$
e $\frac{5}{6}, 3$ **f** $-\frac{3}{2}, 3$ **g** $\frac{1}{2}, \frac{3}{5}$ **h** $-\frac{3}{4}, \frac{2}{3}$
i $\frac{1}{2}$ **j** $-5, 1$ **k** $0, 3$ **l** $-5, -3$
m $\frac{1}{5}, 2$

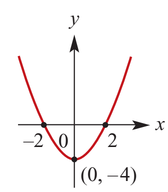
5 3 **6** 4 or 9 **7** $2, 2\frac{3}{8}$

8 13 **9** 50 **10** 6 cm, 2 cm

11 5 cm **12** \$90, \$60 **13** 42

Exercise 3D

1 a i $(0, -4)$
ii $x = 0$
iii $(-2, 0), (2, 0)$



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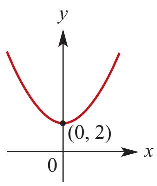
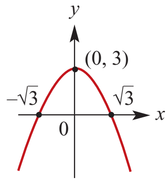
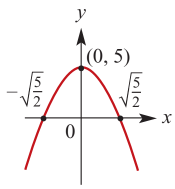
b **i** (0, 2)
ii $x = 0$
iii None



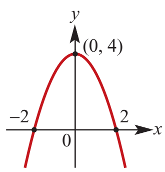
c **i** (0, 3)
ii $x = 0$
iii $(-\sqrt{3}, 0)$, $(\sqrt{3}, 0)$



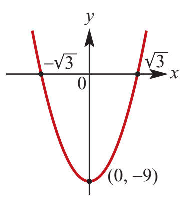
d **i** (0, 5)
ii $x = 0$
iii $\left(-\sqrt{\frac{5}{2}}, 0\right)$, $\left(\sqrt{\frac{5}{2}}, 0\right)$



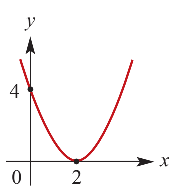
e **i** (0, 4)
ii $x = 0$
iii $(-2, 0)$, $(2, 0)$



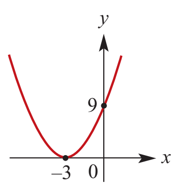
f **i** (0, –9)
ii $x = 0$
iii $(-\sqrt{3}, 0)$, $(\sqrt{3}, 0)$



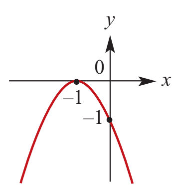
2 a **i** (2, 0)
ii $x = 2$
iii (2, 0)



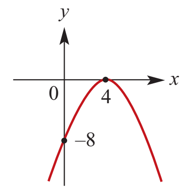
b **i** (–3, 0)
ii $x = -3$
iii (–3, 0)



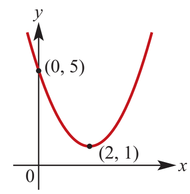
c **i** (–1, 0)
ii $x = -1$
iii (–1, 0)



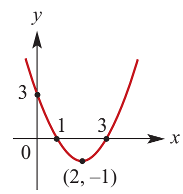
d **i** (4, 0)
ii $x = 4$
iii (4, 0)



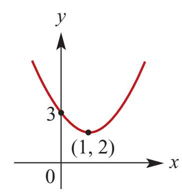
3 a **i** (2, 1)
ii $x = 2$
iii None



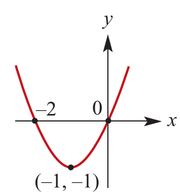
b **i** (2, –1)
ii $x = 2$
iii (1, 0), (3, 0)



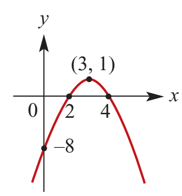
c **i** (1, 2)
ii $x = 1$
iii None



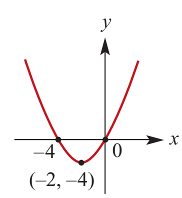
d **i** (–1, –1)
ii $x = -1$
iii (–2, 0), (0, 0)



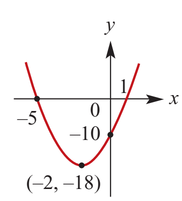
e **i** (3, 1)
ii $x = 3$
iii (2, 0), (4, 0)



f **i** (–2, –4)
ii $x = -2$
iii (–4, 0), (0, 0)



g **i** (–2, –18)
ii $x = -2$
iii (–5, 0), (1, 0)



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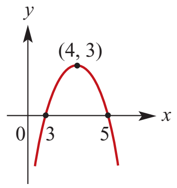
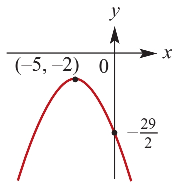
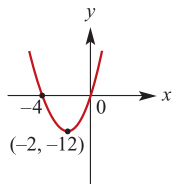
h **i** (4, 3)
ii $x = 4$
iii (3, 0), (5, 0)



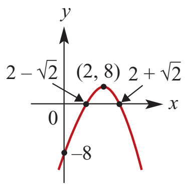
i **i** (−5, −2)
ii $x = -5$
iii None



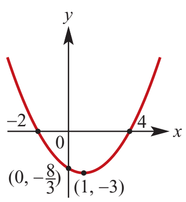
j **i** (−2, −12)
ii $x = -2$
iii (0, 0), (−4, 0)



k **i** (2, 8)
ii $x = 2$
iii $(2 - \sqrt{2}, 0)$, $(2 + \sqrt{2}, 0)$



l **i** (1, −3)
ii $x = 1$
iii (−2, 0), (4, 0)



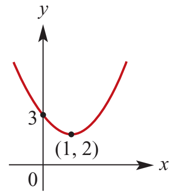
Exercise 3E

1 **a** $x^2 - 2x + 1$ **b** $x^2 + 4x + 4$
c $x^2 - 6x + 9$ **d** $x^2 - 6x + 9$
e $x^2 + 4x + 4$ **f** $x^2 - 10x + 25$
g $x^2 - x + \frac{1}{4}$ **h** $x^2 - 3x + \frac{9}{4}$

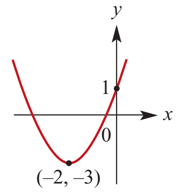
2 **a** $(x - 2)^2$ **b** $(x - 6)^2$ **c** $-(x - 2)^2$
d $2(x - 2)^2$ **e** $-2(x - 3)^2$ **f** $\left(x - \frac{1}{2}\right)^2$
g $\left(x - \frac{3}{2}\right)^2$ **h** $\left(x + \frac{5}{2}\right)^2$

3 **a** $1 \pm \sqrt{2}$ **b** $2 \pm \sqrt{6}$ **c** $3 \pm \sqrt{7}$
d $\frac{5 \pm \sqrt{17}}{2}$ **e** $1 \pm \frac{1}{\sqrt{2}}$ **f** $-\frac{1}{3}, 2$
g $-1 \pm \sqrt{1 - k}$ **h** $\frac{-1 \pm \sqrt{1 - k^2}}{k}$
i $\frac{3k \pm \sqrt{9k^2 - 4}}{2}$

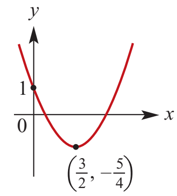
4 **a** $y = (x - 1)^2 + 2$
t. pt (1, 2)



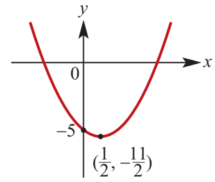
b $y = (x + 2)^2 - 3$
t. pt (−2, −3)



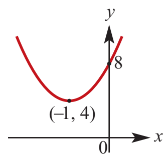
c $y = \left(x - \frac{3}{2}\right)^2 - \frac{5}{4}$
t. pt $\left(\frac{3}{2}, -\frac{5}{4}\right)$



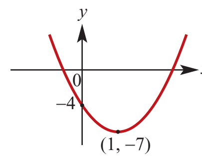
5 **a** $y = 2\left(x - \frac{1}{2}\right)^2 - \frac{11}{2}$
t. pt $\left(\frac{1}{2}, -\frac{11}{2}\right)$



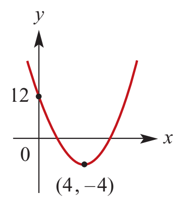
b $y = 4(x + 1)^2 + 4$
t. pt (−1, 4)



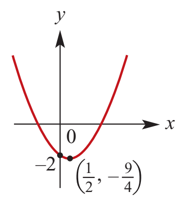
c $y = 3(x - 1)^2 - 7$
t. pt (1, −7)



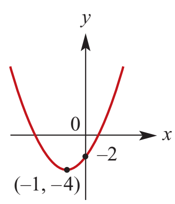
6 **a** $y = (x - 4)^2 - 4$
t. pt (4, −4)



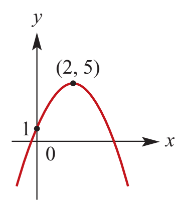
b $y = \left(x - \frac{1}{2}\right)^2 - \frac{9}{4}$
t. pt $\left(\frac{1}{2}, -\frac{9}{4}\right)$



c $y = 2(x + 1)^2 - 4$
t. pt (−1, −4)



d $y = -(x - 2)^2 + 5$
t. pt (2, 5)



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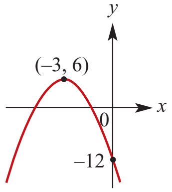
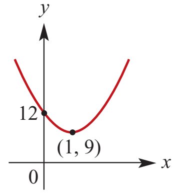
e $y = -2(x+3)^2 + 6$
t. pt $(-3, 6)$



f $y = 3(x-1)^2 + 9$
t. pt $(1, 9)$

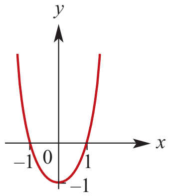


Exercise 3F

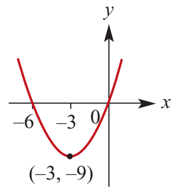
1 a 7 **b** 7 **c** 1

2 a −2 **b** 8 **c** 4

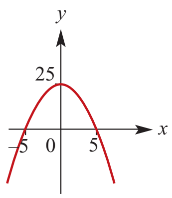
3 a



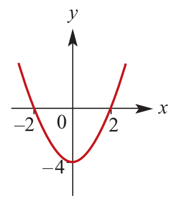
b



c



d



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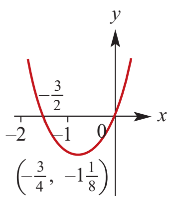

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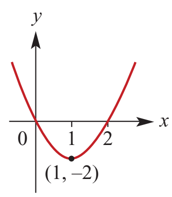

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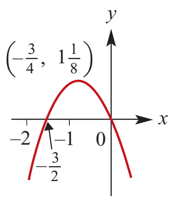

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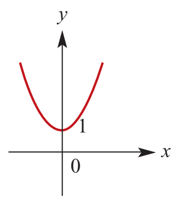
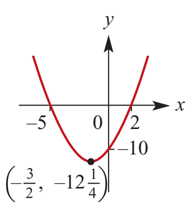


4 a



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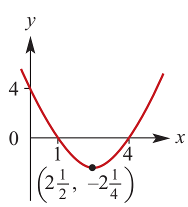

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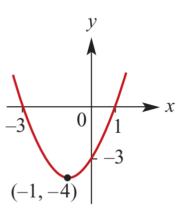

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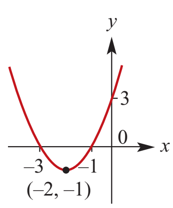

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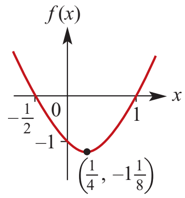

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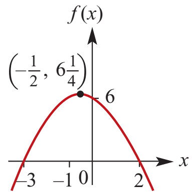

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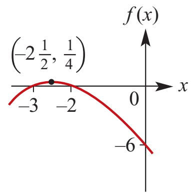

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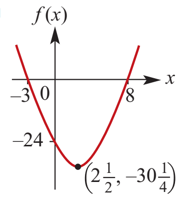
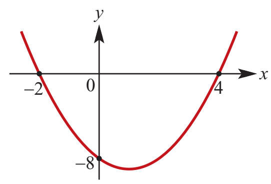


Exercise 3G

1 a −2, 4 **b**



c $-2 \le x \le 4$ **d** $x < -2$ or $x > 4$

2 a $x \le -2$ or $x \ge 3$ **b** $-4 < x < -3$

c $-4 \le x \le \frac{1}{2}$ **d** $x < 2$ or $x > 6$

e $2 < x < 3$ **f** $\frac{3}{2} \le x \le \frac{7}{2}$

g $-\frac{7}{2} < x < 2$ **h** $-2 \le x \le \frac{5}{2}$

i $x < -5$ or $x > \frac{5}{2}$ **j** $-2 \le x \le \frac{7}{2}$

k $x < \frac{2}{5}$ or $x > \frac{7}{2}$ **l** $x \le \frac{5}{2}$ or $x \ge \frac{11}{2}$

3 a $x < -5$ or $x > 5$ **b** $-\frac{2}{3} \le y \le \frac{2}{3}$

c $y < -4$ or $y > 4$ **d** $-\frac{6}{5} \le x \le \frac{6}{5}$

e $y \le -\frac{1}{4}$ or $y \ge \frac{1}{4}$ **f** $y < -\frac{5}{6}$ or $y > \frac{5}{6}$

4 a $x \ge 2$ or $x \le -4$ **b** $-3 < x < 8$

c $-2 \le x \le 6$ **d** $x > 3$ or $x < -\frac{3}{2}$

e $-\frac{3}{2} < x < -\frac{2}{3}$ **f** $-3 \le x \le -2$

g $x > \frac{2}{3}$ or $x < -\frac{3}{4}$ **h** $\frac{1}{2} \le x \le \frac{3}{5}$

i $-4 \le x \le 5$

j $\frac{1}{2}(5 - \sqrt{41}) \le p \le \frac{1}{2}(5 + \sqrt{41})$

k $y < -1$ or $y > 3$ **l** $x \le -2$ or $x \ge -1$

5 a $x \le \frac{-3 - \sqrt{29}}{2}$ or $x \ge \frac{-3 + \sqrt{29}}{2}$

b $\frac{5 - \sqrt{17}}{2} < x < \frac{5 + \sqrt{17}}{2}$

c $\frac{3 - \sqrt{17}}{4} \le x \le \frac{3 + \sqrt{17}}{4}$

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d $\frac{-3-\sqrt{41}}{2} < x < \frac{-3+\sqrt{41}}{2}$

e $\frac{-7-\sqrt{41}}{4} < x < \frac{-7+\sqrt{41}}{4}$

f $x \le \frac{4-\sqrt{6}}{2}$ or $x \ge \frac{4+\sqrt{6}}{2}$

6 The square of any number is greater than or equal to zero.

7 The negative of the square of any number is less than or equal to zero.

8 $x^2 + 2x + 7 = (x+1)^2 + 6$. For all x, we have $(x+1)^2 \ge 0$ and so $(x+1)^2 + 6 \ge 6$

9 $-x^2 - 2x - 7 = -(x+1)^2 - 6$. For all x, we have $-(x+1)^2 \le 0$ and so $-(x+1)^2 - 6 \le -6$

Exercise 3H

1 a i 40 ii $2\sqrt{10}$
b i 28 ii $2\sqrt{7}$
c i 172 ii $2\sqrt{43}$
d i 96 ii $4\sqrt{6}$
e i 189 ii $3\sqrt{21}$

2 a $1+\sqrt{5}$ **b** $\frac{3-\sqrt{5}}{2}$ **c** $\frac{1+\sqrt{5}}{2}$ **d** $1+2\sqrt{2}$

3 a $-3 \pm \sqrt{13}$ **b** $\frac{7 \pm \sqrt{61}}{2}$ **c** $\frac{1}{2}, 2$

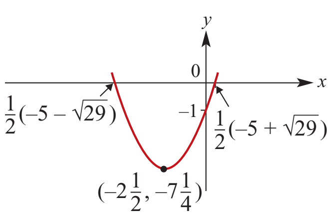
d $-1 \pm \frac{3}{2}\sqrt{2}$ **e** $-2 \pm \frac{3}{2}\sqrt{2}$ **f** $1 \pm \frac{\sqrt{30}}{5}$

g $1 \pm \frac{\sqrt{2}}{2}$ **h** $1, \frac{-3}{2}$ **i** $\frac{-3 \pm \sqrt{6}}{5}$

j $\frac{-13 \pm \sqrt{145}}{12}$ **k** $\frac{2 \pm \sqrt{4-2k^2}}{2k}$

l $\frac{2k \pm \sqrt{6k^2-2k}}{2(1-k)}$

4 a



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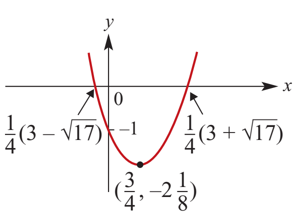

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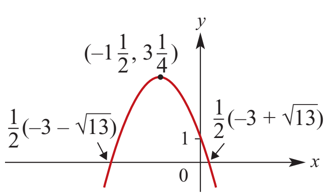

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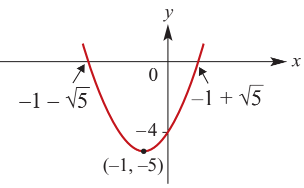

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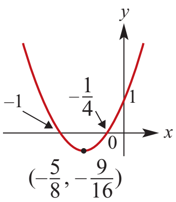
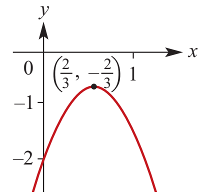


f



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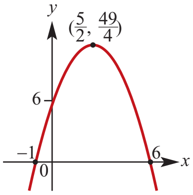

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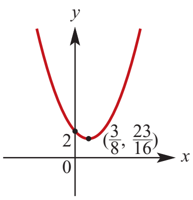
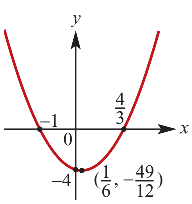


i



Exercise 3I

1 a 20 **b** −12 **c** 25 **d** 41 **e** 41

2 a Crosses the x-axis **b** Does not cross
c Just touches the x-axis
d Crosses the x-axis **e** Does not cross
f Does not cross

3 a Two real solutions **b** No real solutions
c Two real solutions **d** Two real solutions
e Two real solutions **f** No real solutions

4 a $\Delta = 0$, one rational solution
b $\Delta = 1$, two rational solutions
c $\Delta = 17$, two irrational solutions
d $\Delta = 0$, one rational solution
e $\Delta = 57$, two irrational solutions
f $\Delta = 1$, two rational solutions

5 a i $-\sqrt{5} < m < \sqrt{5}$ ii $m = \pm\sqrt{5}$
iii $m > \sqrt{5}$ or $m < -\sqrt{5}$

b i $0 < m < \frac{4}{3}$ ii $m = 0$ or $m = \frac{4}{3}$
iii $m > \frac{4}{3}$ or $m < 0$

c i $-\frac{4}{5} < m < 0$ ii $m = 0$ or $m = -\frac{4}{5}$
iii $m < -\frac{4}{5}$ or $m > 0$

d i $-2 < m < 1$ ii $m = -2$ or $m = 1$
iii $m > 1$ or $m < -2$

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6 $\Delta = (2m-n)^2$, a perfect square **7** $p > \frac{4}{3}$

8 $p = \frac{-1}{2}$

9 a $p = \pm 3$ **b** $p > 1$ **c** $p > \frac{2}{3}$ **d** $p > 1$

10 $-2 < p < 8$

11 $\Delta = -4q^2 < 0$ for all values of q

12 a $\Delta = 16m^2 - 96m + 176 = 16(m-3)^2 + 32$
b $\Delta \geq 32$; therefore two solutions

13 a $\Delta = 16$
b $\Delta > 0$; therefore two solutions

14 $\Delta = (m+4)^2$, a perfect square; therefore rational solutions

15 $\Delta = (m-2n)^2$, a perfect square; therefore rational solutions

16 The graph will cross the x-axis twice

17 The graph will cross the x-axis twice

Exercise 3J

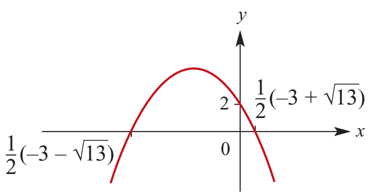
1 a $(1-\sqrt{5}, -1-\sqrt{5}), (1+\sqrt{5}, -1+\sqrt{5})$
b $(-3, 9), (2, 4)$ **c** $(-3, 9), \left(\frac{7}{4}, \frac{49}{16}\right)$
d $(1, 3), (2, 5)$

2 a $(2, 0), (-5, 7)$ **b** $(1, -3), (4, 9)$
c $(1, -3), (-3, 1)$ **d** $(-1, 1), (-3, -3)$
e $\left(\frac{1+\sqrt{33}}{2}, -3-\sqrt{33}\right), \left(\frac{1-\sqrt{33}}{2}, -3+\sqrt{33}\right)$
f $\left(\frac{5+\sqrt{33}}{2}, 23+3\sqrt{33}\right), \left(\frac{5-\sqrt{33}}{2}, 23-3\sqrt{33}\right)$

3 a Touch at $(2, 0)$ **b** Touch at $(3, 9)$
c Touch at $(-2, -4)$ **d** Touch at $(-4, -8)$

4 a $x = 8, y = 16$ and $x = -1, y = 7$
b $x = -\frac{16}{3}, y = 37\frac{1}{3}$ and $x = 2, y = 30$
c $x = \frac{4}{5}, y = 10\frac{2}{5}$ and $x = -3, y = 18$
d $x = 10\frac{2}{3}, y = 0$ and $x = 1, y = 29$
e $x = 0, y = -12$ and $x = \frac{3}{2}, y = -7\frac{1}{2}$
f $x = 1.14, y = 14.19$ and $x = -1.68, y = 31.09$

5 a -13
b i



ii $m = -6 \pm \sqrt{32} = -6 \pm 4\sqrt{2}$

6 a $c = \frac{-1}{4}$ **b** $c > \frac{-1}{4}$

7 $a = 3$ or $a = -1$

8 $b = 1$

9 $y = (2+2\sqrt{3})x - 4 - 2\sqrt{3}$ and $y = (2-2\sqrt{3})x - 4 + 2\sqrt{3}$

Exercise 3K

1 $a = -4, c = 6$

2 a $\Delta = b^2 - 16a$ **b** $a = \frac{b^2}{16}$ **c** $a = \frac{1}{4}, b = 2$

3 a $y = 2(x+2)(x-6)$ **b** $y = -2(x+2)^2 + 4$
c $y = -x^2 + 2x - 3$

4 2 **5** $a = \frac{4}{7}, b = \frac{-24}{7}$

6 $a = -2, b = 1, c = 6$

7 a $y = -\frac{5}{16}x^2 + 5$ **b** $y = x^2$
c $y = \frac{1}{11}x^2 + \frac{7}{11}x$ **d** $y = x^2 - 4x + 3$
e $y = -\frac{5}{4}x^2 - \frac{5}{2}x + \frac{15}{4}$ **f** $y = x^2 - 4x + 6$

8 $y = \frac{5}{16}(x+1)^2 + 3$ **9** $y = -\frac{1}{2}(x^2 - 3x - 18)$

10 $y = (x+1)^2 + 3$ **11** $y = \frac{1}{180}x^2 - x + 75$

12 $y = 2x^2 - 4x$ **13** $y = x^2 - 2x - 1$

14 a C **b** B **c** D **d** A

15 a $y = a\left(x + \frac{1}{a}\right)^2 + a - \frac{1}{a}$ **b** $\left(-\frac{1}{a}, a - \frac{1}{a}\right)$
c $a = \pm 1$ **d** $-1 < a < 1$

16 $y = -2x^2 + 8x - 6$

17 a $y = ax(x-10), a > 0$
b $y = a(x+4)(x-10), a < 0$
c $y = \frac{1}{18}(x-6)^2 + 6$ **d** $y = a(x-8)^2, a < 0$

18 a $y = -\frac{1}{4}x^2 + x + 2$ **b** $y = x^2 + x - 5$

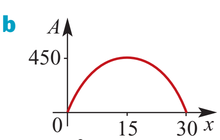
19 $r = -\frac{1}{8}t^2 + 2\frac{1}{2}t - 6\frac{3}{8}$

20 a B **b** D

21 a $y = -2x^2 - x + 5$ **b** $y = 2x^2 - x - 5$
c $y = 2x^2 + \frac{5}{2}x - \frac{11}{2}$

Exercise 3L

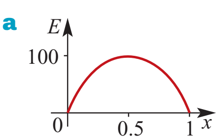
1 a $A = 60x - 2x^2$ **b**



c Maximum area = 450 m^2

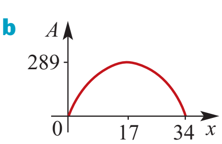
2 $A = x(10-x)$; Maximum area = 25 m^2

3 a



b 0 and 1 **c** 0.5
d 0.23 and 0.77

4 a $A = 34x - x^2$ **b**



c 289 cm^2

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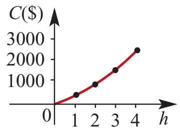
5 a $4x + 10y = 80$

b **i** $A = 1.64x^2 - 25.6x + 256$

ii 31.22 and 48.78

6 a

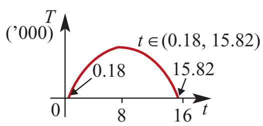


The domain depends on the height of the alpine area. In Australia, the highest mountain is approx. 2 km high and the minimum alpine height would be approx. 1 km. Thus, for Australia, domain = [1, 2].

b Theoretically, no. But of course there is a practical maximum.

c $1225

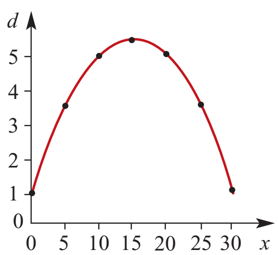
7 a



b 8874 units

8 a

x	0	5	10	15	20	25	30
d	1	3.5	5	5.5	5	3.5	1



b **i** 5.5 m

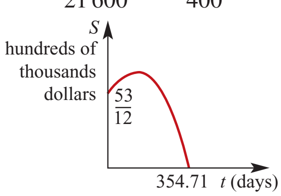
ii $\{15 - 5\sqrt{7}\}$ m or $\{15 + 5\sqrt{7}\}$ m from the bat

iii 1 m above the ground

9 $a = -\dfrac{16}{15}, b = \dfrac{8}{5}, c = 0$

10 a $a = -\dfrac{7}{21\,600}, b = \dfrac{41}{400}, c = \dfrac{53}{12}$

b



c **i** S = $1 236 666 **ii** S = $59 259

Chapter 3 review

Short-answer questions

1 a $\left(x + \dfrac{9}{2}\right)^2$ **b** $(x + 9)^2$ **c** $\left(x - \dfrac{2}{5}\right)^2$

d $(x + b)^2$ **e** $(3x - 1)^2$ **f** $(5x + 2)^2$

2 a $-3x + 6$ **b** $-ax + a^2$

c $49a^2 - b^2$ **d** $x^2 - x - 12$

e $2x^2 - 5x - 12$ **f** $x^2 - y^2$

g $a^3 - b^3$ **h** $6x^2 + 8xy + 2y^2$

i $3a^2 - 5a - 2$ **j** $4xy$

k $2u + 2v - uv$ **l** $-3x^2 + 15x - 12$

3 a $4(x - 2)$ **b** $x(3x + 8)$

c $3x(8a - 1)$ **d** $(2 - x)(2 + x)$

e $a(u + 2v + 3w)$ **f** $a^2(2b - 3a)(2b + 3a)$

g $(1 - 6ax)(1 + 6ax)$ **h** $(x + 4)(x - 3)$

i $(x + 2)(x - 1)$ **j** $(2x - 1)(x + 2)$

k $(3x + 2)(2x + 1)$ **l** $(3x + 1)(x - 3)$

m $(3x - 2)(x + 1)$ **n** $(3a - 2)(2a + 1)$

o $(3x - 2)(2x - 1)$

4 a $x = 5$ or $x = -3$ **b** $x = 9$ or $x = 0$

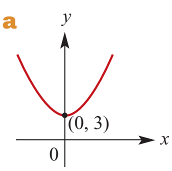
c $x = 2$ or $x = 3$ **d** $x = -1$ or $x = 25$

e $x = -3$ or $x = -2$ **f** $x = 6$

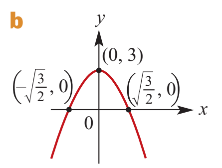
g $x = -\dfrac{1}{2}$ or $x = 3$ **h** $x = -\dfrac{5}{6}$ or $x = \dfrac{3}{2}$

i $x = -\dfrac{12}{5}$ or $x = 1$

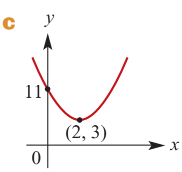
5 a



b



c



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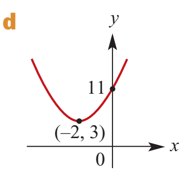

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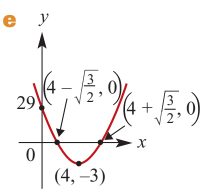
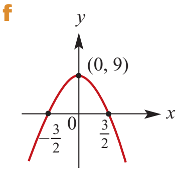


f



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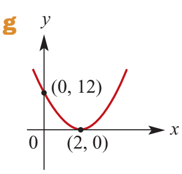

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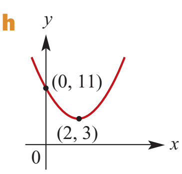

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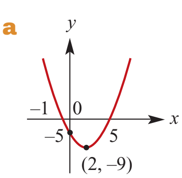

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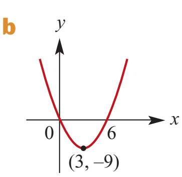

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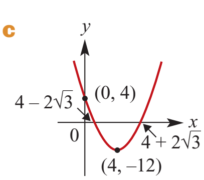

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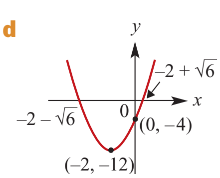

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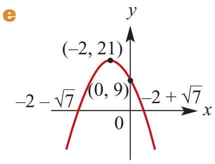
e



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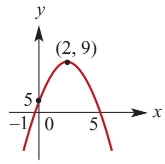
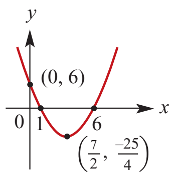
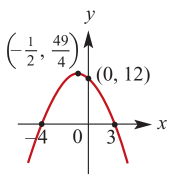


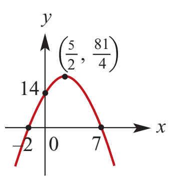
7 a $x = \frac{7}{2}$



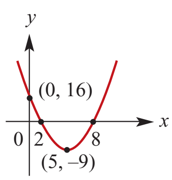
b $x = -\frac{1}{2}$



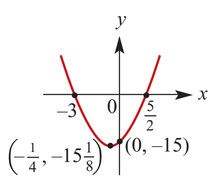
c $x = \frac{5}{2}$



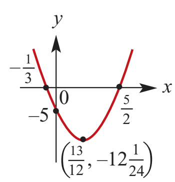
d $x = 5$



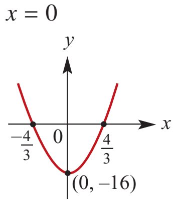
e $x = -\frac{1}{4}$



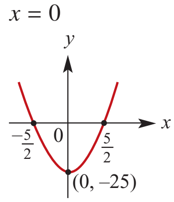
f $x = \frac{13}{12}$



g $x = 0$



h $x = 0$



8 $p = 1$ or $p = -\frac{3}{10}$

9 a $x < 0$ or $x > 1$

b $-2 - \sqrt{34} \le x \le -2 + \sqrt{34}$

c $-2 \le x \le \frac{1}{3}$ **d** $\frac{3}{2} \le x \le 5$

10 a $-3 \pm \sqrt{6}$ **b** $\frac{-9 \pm \sqrt{33}}{2}$

c $2 \pm \sqrt{2}$ **d** $\frac{-7 \pm \sqrt{33}}{4}$

e $\frac{-7 \pm \sqrt{17}}{4}$ **f** $\frac{-9 \pm \sqrt{93}}{6}$

11 $y = \frac{5}{3}x(x - 5)$

12 $y = 3(x - 5)^2 + 2$

13 $m < -21 - 4\sqrt{29}$ or $m > -21 + 4\sqrt{29}$

14 225

15 $y = 5(x - 1)^2 + 5$

16 a $(3, 9), (-1, 1)$

b $\left(\frac{4 - \sqrt{38}}{2}, 27 - 4\sqrt{38}\right)$, $\left(\frac{4 + \sqrt{38}}{2}, 27 + 4\sqrt{38}\right)$

c $\left(\frac{-7 - \sqrt{73}}{6}, 2\right), \left(\frac{-7 + \sqrt{73}}{6}, 2\right)$

d $(\frac{1}{2}, \frac{1}{2}), (-2, 8)$

17 a $y = 2(x + 4)(x - 1)$ **b** $y = -2(x + 1)^2 + 3$

c $y = 2x^2 - 2x - 3$

18 2.16 m

19 a $m = \pm\sqrt{8} = \pm 2\sqrt{2}$ **b** $m \le -\sqrt{5}$ or $m \ge \sqrt{5}$

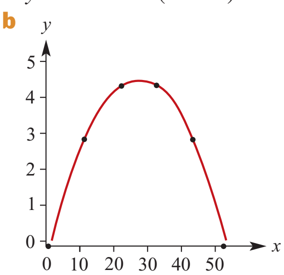
20 a $x = 0$ and $x = -b$ **b** $\left(-\frac{b}{2}, -\frac{b^2}{4}\right)$

c i $(0, 0), (1 - b, 1 - b)$ **ii** $b = 1$ **iii** $b \ne 1$

Extended-response questions

1 a $y = -0.0072x(x - 50)$

b



c 10.57 m and 39.43 m $\left(25 \pm \frac{25\sqrt{3}}{3} \text{ m}\right)$

d 3.2832 m **e** 3.736 m (correct to 3 d.p.)

2 a Width $= \frac{12 - 4x}{6}$ cm; length $= \frac{12 - 4x}{3}$ cm

b $A = \frac{17}{9}x^2 - \frac{16}{3}x + 8$

c Length for square $= \frac{96}{17} \approx 5.65$ cm and length for rectangle $= \frac{108}{17} \approx 6.35$ cm

3 a $V = 0.72x^2 - 1.2x$ **b** 22 hours

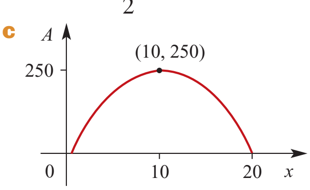
4 a $V = 10\,800x + 120x^2$

b $V = 46.6x^2 + 5000x$

c $\ell = 55.18$ m

5 a $\ell = 50 - \frac{5x}{2}$ **b** $A = 50x - \frac{5}{2}x^2$

c



d Maximum area $= 250\text{ m}^2$ when $x = 10$ m

6 $x = \frac{-1 + \sqrt{5}}{2}$

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7 a $\sqrt{25 + x^2}$

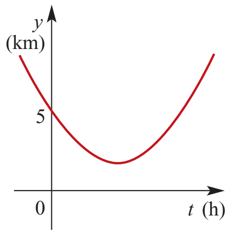
b i $16 - x$ **ii** $\sqrt{x^2 - 32x + 265}$

c 7.5 **d** 10.840 **e** 12.615

8 a i $y = \sqrt{64t^2 + 100(t - 0.5)^2}$

$= \sqrt{164t^2 - 100t + 25}$

ii



iii $t = \frac{1}{2}$, 1:30 p.m.; $t = \frac{9}{82}$, 1:07 p.m.

iv 0.305; 1:18 p.m.; distance 3.123 km

b i $0, \frac{25}{41}$ **ii** $\frac{25 \pm 2\sqrt{269}}{82}$

9 b $2x + 2y = b$

c $8x^2 - 4bx + b^2 - 16a^2 = 0$

e i $x = 6 \pm \sqrt{14}$, $y = 6 \mp \sqrt{14}$

ii $x = y = \sqrt{2}a$

f $x = \frac{(5 \pm \sqrt{7})a}{4}$, $y = \frac{(5 \mp \sqrt{7})a}{4}$

10 a $b = -2$, $c = 4$, $h = 1$

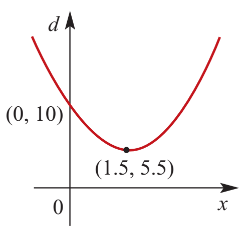
b i $(x, -6 + 4x - x^2)$ **ii** $(x, x - 1)$

iii $(0, -1)$, $(1, 0)$, $(2, 1)$, $(3, 2)$, $(4, 3)$

iv $y = x - 1$

c i $d = 2x^2 - 6x + 10$

ii



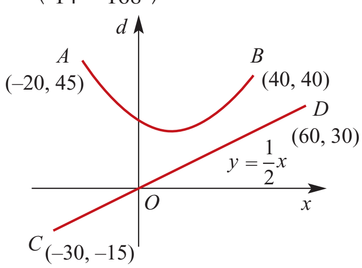
iii Min value of $d = 5.5$ when $x = 1.5$

11 a $45\sqrt{5}$

b i $y = \frac{1}{600}(7x^2 - 190x + 20\,400)$

ii $\left(\frac{190}{14}, \frac{5351}{168}\right)$

c



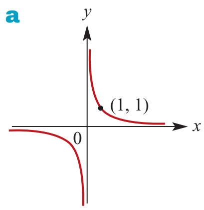
d i The distance (measured parallel to the y-axis) between path and pond

ii Minimum value $= \frac{473}{24}$ when $x = 35$

Chapter 4

Exercise 4A

1 a



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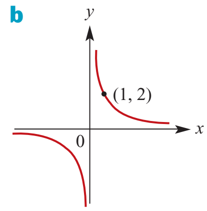

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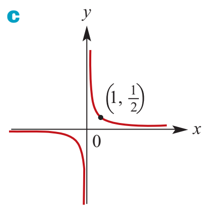

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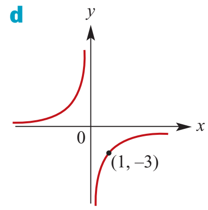

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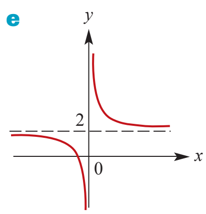

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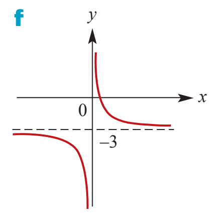

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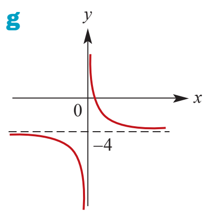

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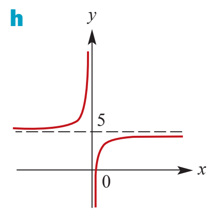

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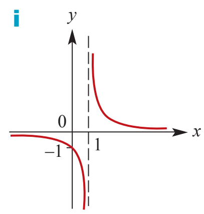

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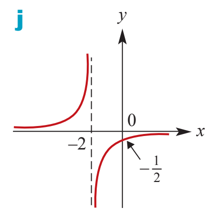

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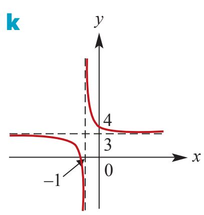

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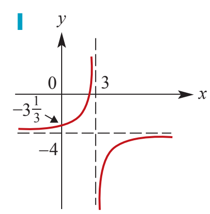

2 a $y = 0, x = 0$ **b** $y = 0, x = 0$

c $y = 0, x = 0$ **d** $y = 0, x = 0$

e $y = 2, x = 0$ **f** $y = -3, x = 0$

g $y = -4, x = 0$ **h** $y = 5, x = 0$

i $y = 0, x = 1$ **j** $y = 0, x = -2$

k $y = 3, x = -1$ **l** $y = -4, x = 3$

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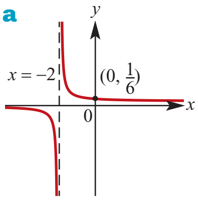
3 a



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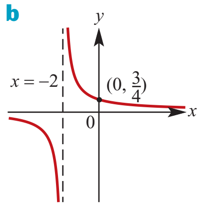

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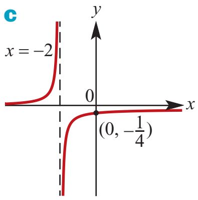

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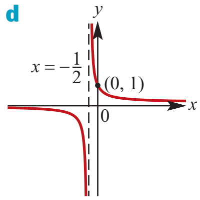
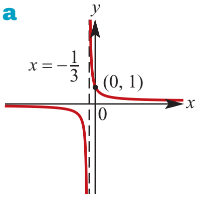


4 a



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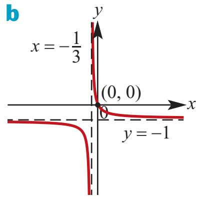

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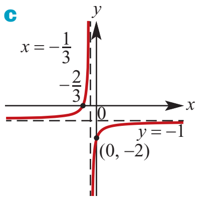

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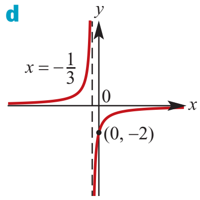
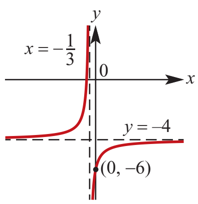


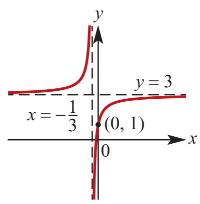
e x-axis intercept $-\frac{1}{2}$

y-axis intercept -6



f x-axis intercept $-\frac{1}{9}$

y-axis intercept 1



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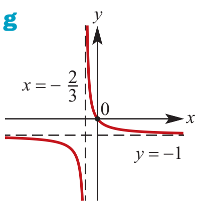
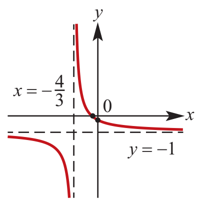


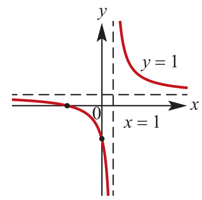
h x-axis intercept $-\frac{1}{3}$

y-axis intercept $-\frac{1}{4}$



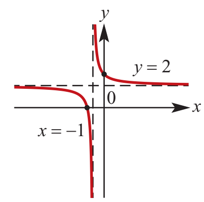
5 x-axis intercept -3

y-axis intercept -3



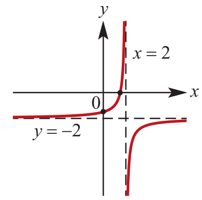
6 x-axis intercept $-\frac{3}{2}$

y-axis intercept 3



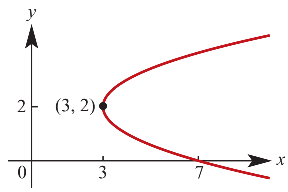
7 x-axis intercept $\frac{3}{2}$

y-axis intercept $-\frac{3}{2}$



Exercise 4B

1 a



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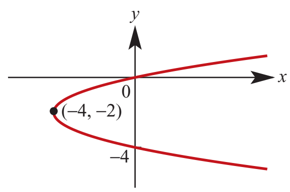

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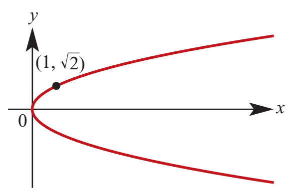

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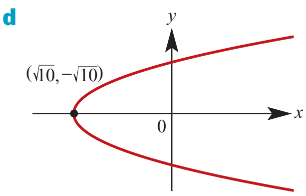

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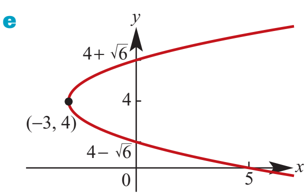

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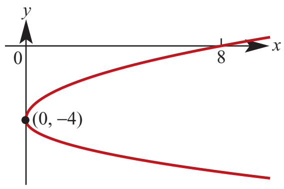

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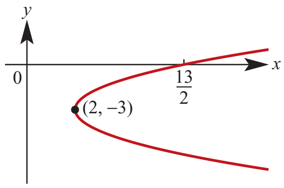

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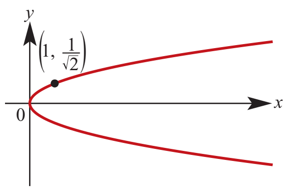

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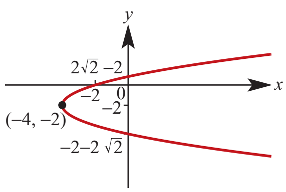

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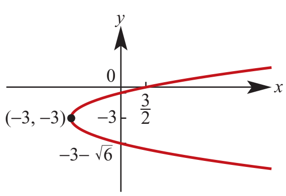

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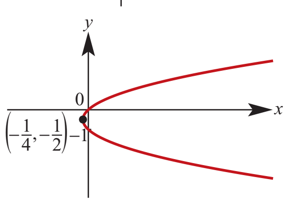

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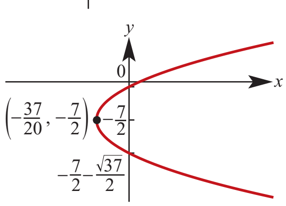

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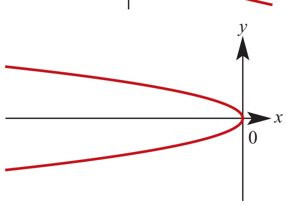

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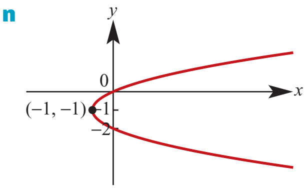
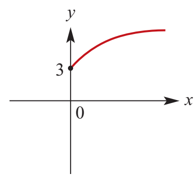
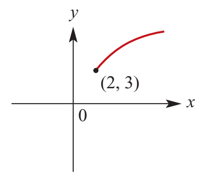


Exercise 4C

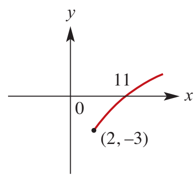
1 a $x \geq 0$ and $y \geq 3$



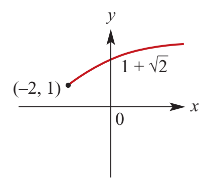
b $x \geq 2$ and $y \geq 3$



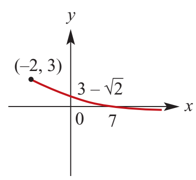
c $x \geq 2$ and $y \geq -3$



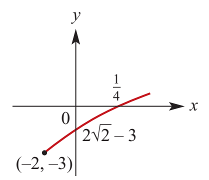
d $x \geq -2$ and $y \geq 1$



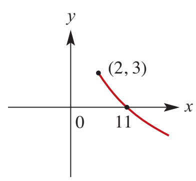
e $x \geq -2$ and $y \leq 3$



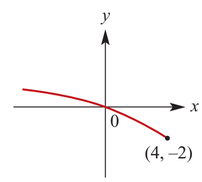
f $x \geq -2$ and $y \geq -3$



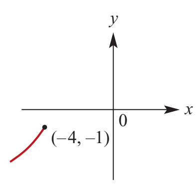
2 a $x \geq 2$ and $y \leq 3$



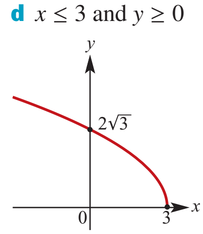
b $x \leq 4$ and $y \geq -2$



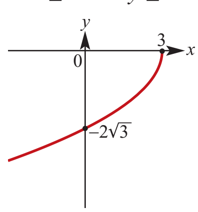
c $x \leq -4$ and $y \leq -1$



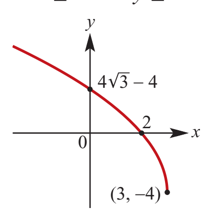
d $x \leq 3$ and $y \geq 0$



e $x \leq 3$ and $y \leq 0$



f $x \leq 3$ and $y \geq -4$



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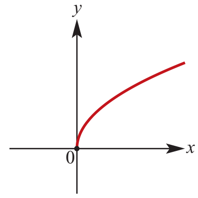
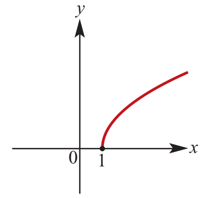
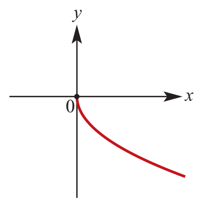
3 a $x \geq 0$ and $y \geq 0$



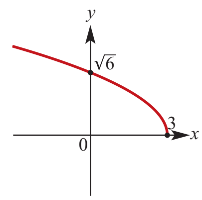
b $x \geq 1$ and $y \geq 0$



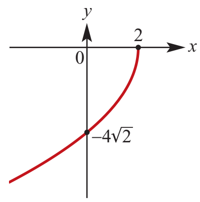
c $x \geq 0$ and $y \leq 0$



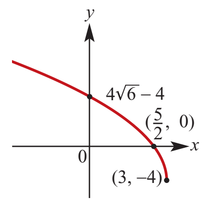
d $x \leq 3$ and $y \geq 0$



e $x \leq 2$ and $y \leq 0$



f $x \leq 3$ and $y \geq -4$



Exercise 4D

1 a $x^2 + y^2 = 9$ **b** $x^2 + y^2 = 16$

c $(x-1)^2 + (y-3)^2 = 25$

d $(x-2)^2 + (y+4)^2 = 9$

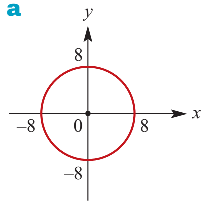
e $(x+3)^2 + (y-4)^2 = \dfrac{25}{4}$

f $(x+5)^2 + (y+6)^2 = (4.6)^2$

2 a $C(1, 3)$, $r = 2$ **b** $C(2, -4)$, $r = \sqrt{5}$

c $C(-3, 2)$, $r = 3$ **d** $C(-5, 4)$, $r = \sqrt{8}$

3 a



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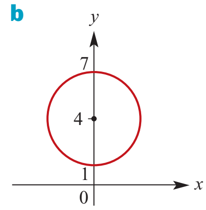

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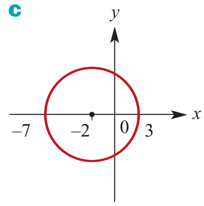

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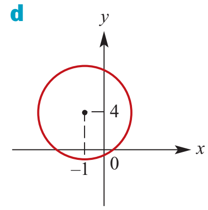

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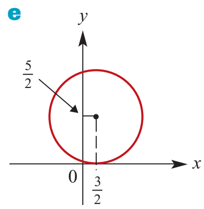

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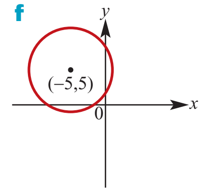
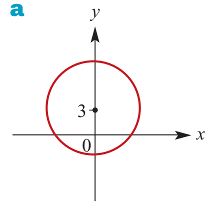


4 a $C(0, 3)$, $r = 5$ **b** $C(4, -6)$, $r = \sqrt{42}$

c $C(3, -2)$, $r = 2$ **d** $C(-2, 3)$, $r = 5$

e $C(4, -2)$, $r = \sqrt{19}$ **f** $C\left(\dfrac{1}{2}, -2\right)$, $r = \dfrac{3}{2}$

5 a



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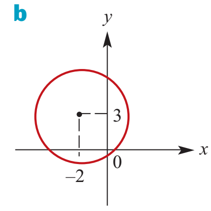

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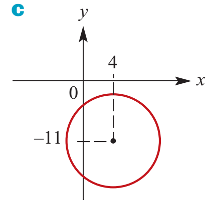
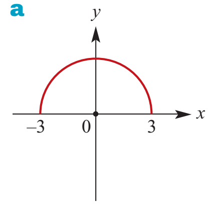


6 a



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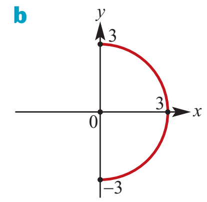

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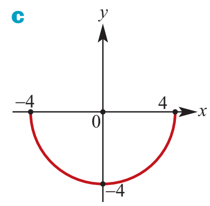

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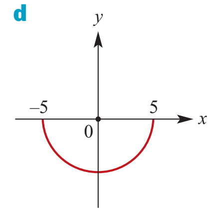

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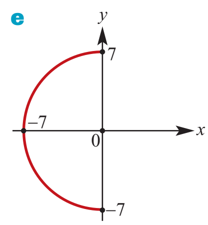

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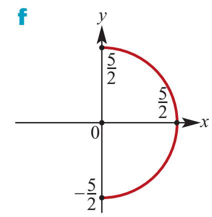

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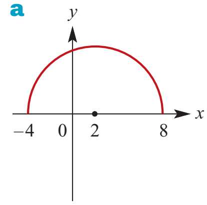

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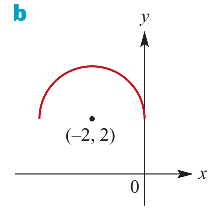
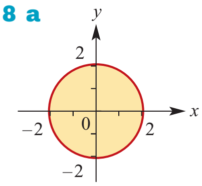


8 a



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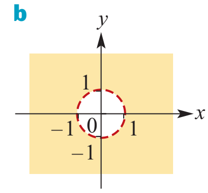

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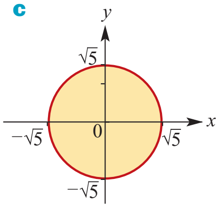

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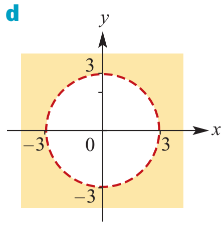

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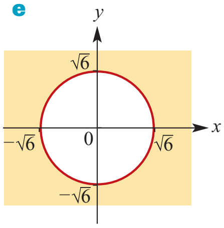

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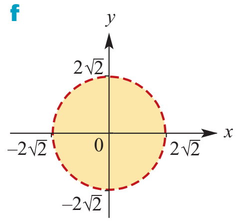


Exercise 4E

1 $a = 5$ **2** $a = -6, h = 3, k = 4$

3 $a = \frac{1}{2}, k = \frac{15}{2}$ **4** $a = -16, h = 2, k = -4$

5 $a = 4\sqrt{2}$ **6** $a = \frac{2\sqrt{3}}{3}, h = -2$

7 $(x-2)^2 + (y-1)^2 = 20$

8 $(x+2)^2 + (y-3)^2 = 1$

9 $(x+2)^2 + (y-3)^2 = 16$

10 $(x-2)^2 + (y+3)^2 = 9$

11 $(x-4)^2 + (y-4)^2 = 20$

12 $(x-4)^2 + (y-5)^2 = 25$ and $(x+4)^2 + (y-5)^2 = 25$

13 $(x+1)^2 + (y+1)^2 = 10$

14 a $(x-2)^2 + (y+2)^2 = 49$

b $y = 3\sqrt{x-1} - 2$ **c** $y = \frac{1}{x-2} + 2$

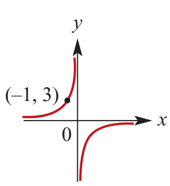
d $y = -\frac{2}{x-1} - 2$ **e** $y = \sqrt{2-x} + 1$

f $(y+2)^2 = 2x + 9$

Chapter 4 review

Short-answer questions

1 a



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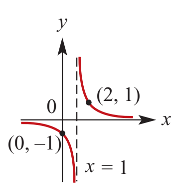

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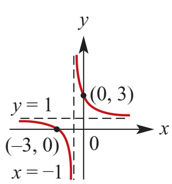

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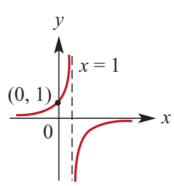

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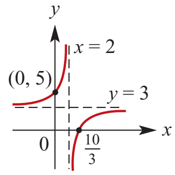

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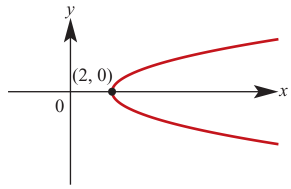

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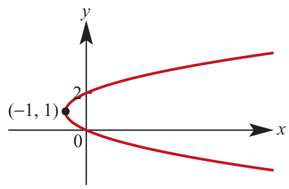

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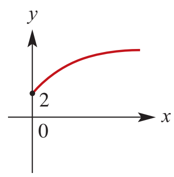

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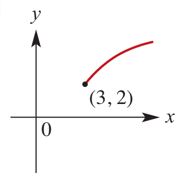

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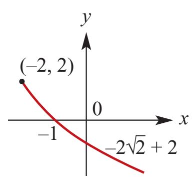


2 a $(x-3)^2 + (y+2)^2 = 25$

b $\left(x - \frac{3}{2}\right)^2 + \left(y + \frac{5}{2}\right)^2 = \frac{25}{2}$

c $\left(x - \frac{1}{4}\right)^2 + \left(y + \frac{1}{4}\right)^2 = \frac{17}{8}$

d $(x+2)^2 + (y-3)^2 = 13$

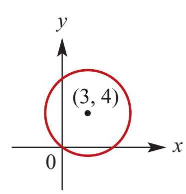
e $(x-3)^2 + (y-3)^2 = 18$

f $(x-2)^2 + (y+3)^2 = 13$

3 $2y + 3x = 0$

4 $2x + 2y = 1$ or $y = x - \frac{5}{2}$

5 a $(x-3)^2 + (y-4)^2 = 25$



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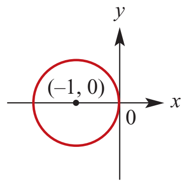
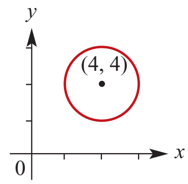
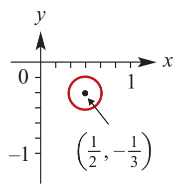
b $(x+1)^2 + y^2 = 1$



c $(x-4)^2 + (y-4)^2 = 4$



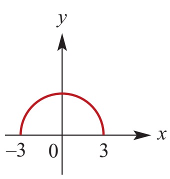
d $\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{1}{3}\right)^2 = \frac{1}{36}$



6 $C(-2, 3)$, $r = 6$

7 y-axis: $4\sqrt{6}$; x-axis: $2\sqrt{21}$

8 a



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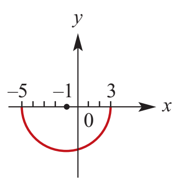

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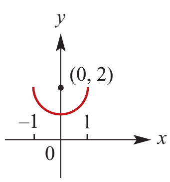
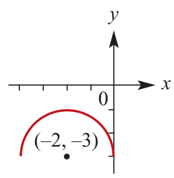


d



Extended-response questions

1 a $(x-10)^2 + y^2 = 25$ **c** $m = \pm\frac{\sqrt{3}}{3}$

d $P\left(\frac{15}{2}, \frac{\pm 5\sqrt{3}}{2}\right)$ **e** $5\sqrt{3}$

2 a $x^2 + y^2 = 16$

b ii $m = \pm\frac{\sqrt{3}}{3}$; $y = \frac{\sqrt{3}}{3}x - \frac{8\sqrt{3}}{3}$

$y = -\frac{\sqrt{3}}{3}x + \frac{8\sqrt{3}}{3}$

3 a $\frac{4}{3}$ **b** $\frac{-3}{4}$ **c** $4y + 3x = 25$ **d** $\frac{125}{12}$

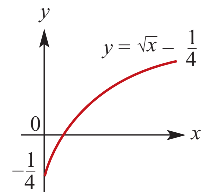
4 a i $\frac{y_1}{x_1}$ **ii** $\frac{-x_1}{y_1}$

c $\sqrt{2}x + \sqrt{2}y = 8$ or $\sqrt{2}x + \sqrt{2}y = -8$

5 a $y = \frac{-\sqrt{3}}{3}x + \frac{2\sqrt{3}}{3}a$, $y = \frac{\sqrt{3}}{3}x - \frac{2\sqrt{3}}{3}a$

b $x^2 + y^2 = 4a^2$

6 b ii $\left(\frac{1}{4}, \frac{1}{4}\right)$



c i $\frac{-1}{4} < k < 0$ **ii** $k = 0$ or $k < \frac{-1}{4}$

iii $k > 0$

7 a $0 < k < \frac{1}{4}$ **b** $k = \frac{1}{4}$ or $k \le 0$

Chapter 5

Exercise 5A

1 a $k = 2$

x	2	4	6	8
y	8	32	72	128

b $k = \frac{1}{3}$

x	$\frac{1}{2}$	1	$\frac{3}{2}$	2
y	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$

c $k = 3$

x	4	9	49	900
y	6	9	21	90

d $k = \frac{2}{5}$

x	$\frac{1}{32}$	1	32	1024
y	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{8}{5}$

2 a $V = 262.144$ **b** $r \approx 2.924$

3 a $a \approx 1.058$ **b** $b \approx 5.196$

4 a $72\,\text{cm}^2$ **b** 20 cm

5 a $\frac{648}{113}$ cm **b** 1412.5 g

6 10.125 kg

7 62.035 cm

8 a 300% **b** 800% **c** 21%

9 52%

10 1.898 s

11 a 8.616 km **b** 19.93 km

12 a i 300% increase **ii** 41% increase

iii 700% increase

b i 75% decrease **ii** 29% decrease

iii 87.5% decrease

c i 36% decrease **ii** 11% decrease

iii 48.8% decrease

d i 96% increase **ii** 18% increase

iii 174.4% increase

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Exercise 5B

1 a $k = 2$

x	2	4	6	32
y	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{16}$

b $k = \frac{1}{2}$

x	$\frac{1}{4}$	1	4	9
y	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{6}$

c $k = 3$

x	1	2	3	6
y	3	$\frac{3}{4}$	$\frac{1}{3}$	$\frac{1}{12}$

d $k = \frac{1}{3}$

x	$\frac{1}{8}$	1	27	125
y	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{15}$

2 a $a = \frac{1}{2}$ **b** $b \approx 5.657$

3 a $a = 0.3125$ **b** $b = \sqrt{2}$

4 2.85 kg/cm^2

5 a 2.4 amperes **b** 25%

6 64 candelas

7 5.15 cm

8 a i 75% decrease **ii** 29% decrease **iii** 87.5% decrease
b i 300% increase **ii** 41% increase **iii** 700% increase
c i 56.25% increase **ii** 12% increase **iii** 95% increase
d i 49% decrease **ii** 15% decrease **iii** 64% decrease

Exercise 5C

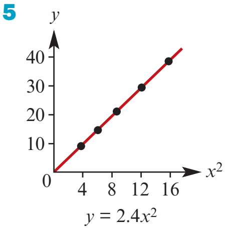
1 a Direct
b Direct square
c Inverse
d Direct square root
e Inverse square

2 a $y \propto x^2$ (possibly) **b** $y \propto x^3$
d $y \propto \sqrt{x}$ (possibly)

3 a, b, e

4 a $y = 3x$ **b** $y = \frac{3}{x}$ **c** $y = \frac{10}{3}x^2$
d $y = 2\sqrt{x}$ **e** $y = \frac{1}{3\sqrt{x}}$ **f** $y = 6x^3$

5



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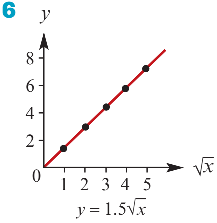

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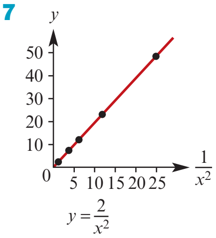

8 a $y = \frac{1}{4}\sqrt{x}$ **b** $y = 2x^{\frac{5}{4}}$ **c** $y = 3.5x^{0.4}$
d $y = 10x^{\frac{2}{3}}$ **e** $y = 2x^{-\frac{5}{2}}$ **f** $y = 3.2x^{-0.4}$

9 a $a = 100,\ b = 0.2$ **b** 158.49

10 a $a = 1500,\ b = -0.5$ **b** 474.34

Exercise 5D

1 $k = 5$

x	2	4	6	10
z	10	2	60	12.5
y	1	10	0.5	4

2 $k = \frac{1}{2}$

x	2	4	1	10
z	10	8	50	3
y	10	16	25	15

3 $k = 3$

x	2	3	5	10
z	10	4	50	$\frac{400}{3}$
y	$\frac{15}{2}$	$\frac{4}{3}$	6	4

4 $a \approx 1.449$

5 $z \approx 0.397$

6 a 9.8 J/kg.m **b** 5880 J

7 $174

8 360 J

9 a 500% increase **b** 78% decrease

10 a 41% increase **b** 33% increase

11 a 183% increase **b** 466% increase

12 a Tensions are the same
b Work done by the second spring is 90% of the work done by the first

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Chapter 5 review

Short-answer questions

1 a When $b = 4$, $a = 6$; when $a = 8$, $b = \pm\frac{8}{\sqrt{3}}$

b When $x = 27$, $y = \frac{30}{2^{\frac{1}{3}}}$;
when $y = \frac{1}{8}$, $x = \frac{1}{256\,000}$

c When $x = \frac{1}{2}$, $y = \frac{16}{3}$; when $y = \frac{4}{27}$, $x = \pm 3$

d $a = \frac{1}{6}$

2 a $d = 4.91t^2$ **b** 491 m
c 2 s, correct to one decimal place

3 a 14 m/s **b** 40 m **c** v and $\sqrt{s}$

4 2.4 hours

5 a y is halved
b x is halved
c y is doubled
d x is doubled

6 4.05 cents

7 18 amps

8 $I_2 = \frac{1}{4}I_1$

9 34% increase

Extended-response questions

1 a 0.24 kg **b** 11 cm

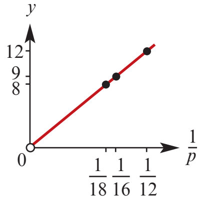
2 a $h = 0.000\,3375n^2$
b 17.1 m
c 218 rpm

3 13 knots

4 a $V = \frac{121.8}{P}$ **b** 9.6 kg/cm^2

5 a $w = \frac{3000}{d}$ **b** 600 kg **c** 333 kg

6 a $v = \frac{144}{p}$
b i $v = 2$ **ii** $p = 48$
c



7 a $P = 3498.544 \times N^{0.5}$ **b** 25 956
c 51 023

8 a $t = \frac{3600}{d^2}$ **b** $T = 0.14d^2$ **c** 23.9 mL
d 6.3 min **e** 9 min; 56 min

9 a i $T = 0.000\,539 \times R^{1.501}$
ii Mars 1.87; Jupiter 11.86; Saturn 29.45; Uranus 84.09; Neptune 165.05
b 2.540×10^9 km

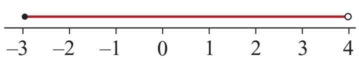
10 a $a = 11.7$, $b = 0.41$ **b** 77
c $k = 163$, $p = -1.167$ **d** 7

Chapter 6

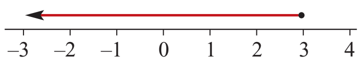
Exercise 6A

1 a {7, 11} **b** {7, 11}
c {1, 2, 3, 5, 7, 11, 15, 25, 30}
d {1, 2, 3, 5, 7, 11, 15, 25, 30, 32}
e {1, 2, 3, 5, 7, 11, 15, 25, 30, 32}
f {1, 7, 11, 25, 30}

2 a



b



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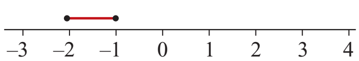

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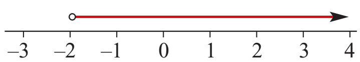

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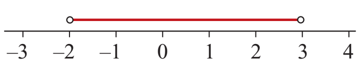
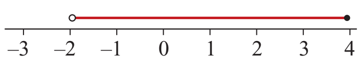


f

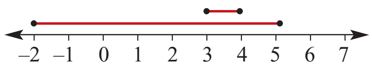


3 a $(-2, 1]$ **b** $[-3, 3]$ **c** $[-3, 2)$ **d** $(-1, 2)$

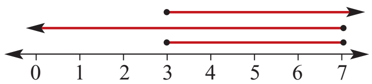
4 a $[-1, 2]$
b $(-4, 2]$
c $(0, \sqrt{2})$
d $\left(-\frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}\right]$
e $(-1, \infty)$
f $(-\infty, -2]$
g $(-\infty, \infty)$

5 a {7} **b** B, i.e. {7, 11, 25, 30, 32}
c $(2, \infty)$ **d** {30, 32}

6 a



b



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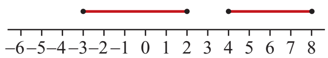
7 a $(-\infty, -2) \cup (-2, \infty)$ **b** $(-\infty, 3) \cup (3, \infty)$
c $(-\infty, 4) \cup (4, \infty)$

8 a


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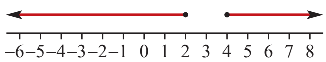

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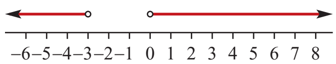

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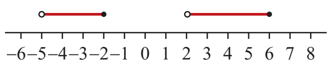
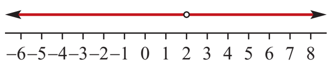
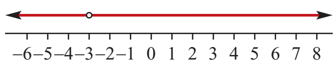


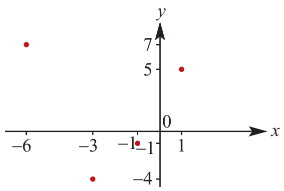
e


f


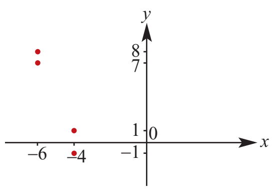
9 a $(-6, -3)$ **b** $\varnothing$ **c** $[-6, 0]$
d $[-1, 2]$ **e** $\{1\}$ **f** $(-10, -1]$

Exercise 6B

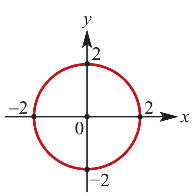
1 a Domain = $\{-3, -1, -6, 1\}$
Range = $\{-4, -1, 7, 5\}$



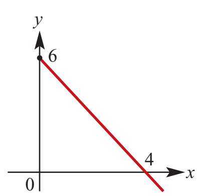
b Domain = $\{-4, -6\}$; Range = $\{-1, 1, 7, 8\}$



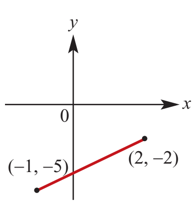
c Domain = $[-2, 2]$
Range = $[-2, 2]$



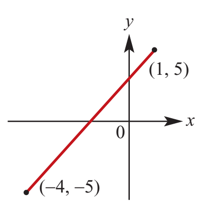
d Domain = $[0, \infty)$
Range = $(-\infty, 6]$



e Domain = $[-1, 2]$
Range = $[-5, -2]$

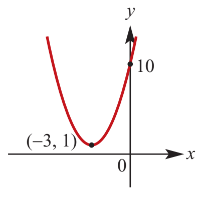


f Domain = $[-4, 1]$
Range = $[-5, 5]$

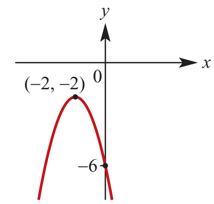


2 a Domain = $[-2, 2]$; Range = $[-1, 2]$
b Domain = $[-2, 2]$; Range = $[-2, 2]$
c Domain = $\mathbb{R}$; Range = $[-1, \infty)$
d Domain = $\mathbb{R}$; Range = $(-\infty, 4]$

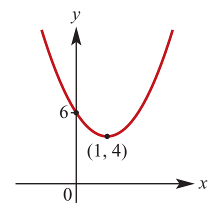
3 a $y = (x + 3)^2 + 1$
Range = $[1, \infty)$



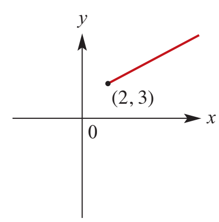
b $y = -(x + 2)^2 - 2$
Range = $(-\infty, -2]$



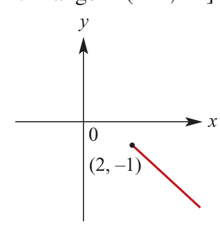
c $y = 2(x - 1)^2 + 4$
Range = $[4, \infty)$



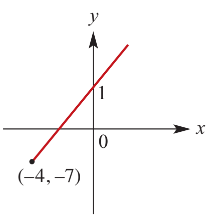
4 a Range = $[3, \infty)$



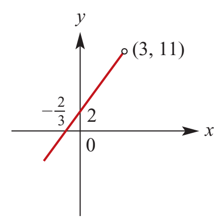
b Range = $(-\infty, -1]$



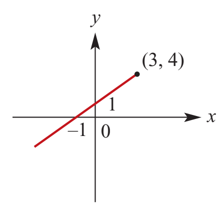
c Range = $[-7, \infty)$



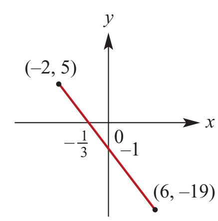
d Range = $(-\infty, 11)$



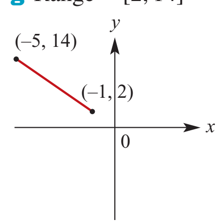
e Range = $(-\infty, 4]$



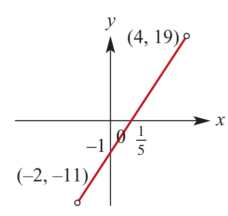
f Range = $[-19, 5]$



g Range = $[2, 14]$



h Range = $(-11, 19)$



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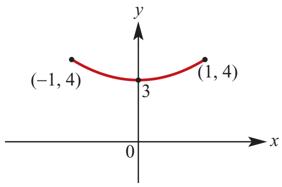
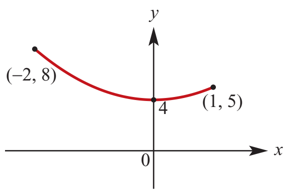
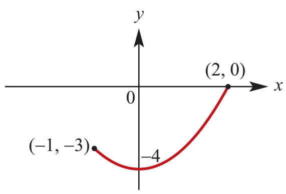
5 **a** Range = $[3, 4]$



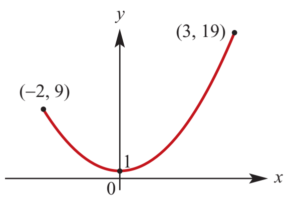
b Range = $[4, 8]$



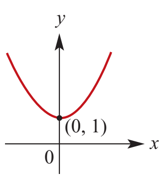
c Range = $[-4, 0]$



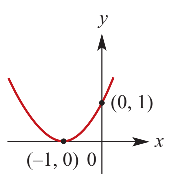
d Range = $[1, 19]$



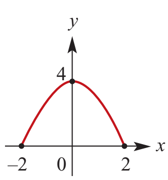
6 **a** Range = $[1, \infty)$



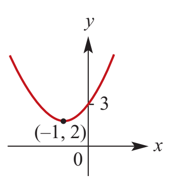
b Range = $[0, \infty)$



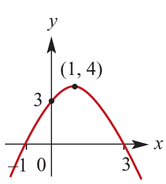
c Range = $[0, 4]$



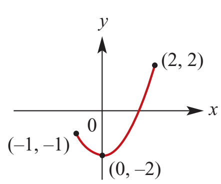
d Range = $[2, \infty)$



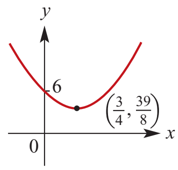
e Range = $(-\infty, 4]$



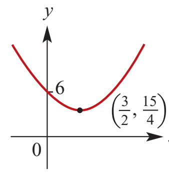
f Range = $[-2, 2]$



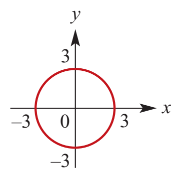
g Range = $\left[\frac{39}{8}, \infty\right)$



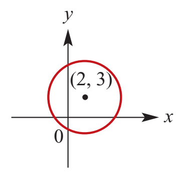
h Range = $\left[\frac{15}{4}, \infty\right)$



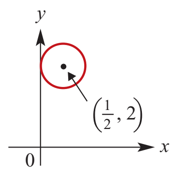
7 **a** Domain $-3 \le x \le 3$
Range $-3 \le y \le 3$



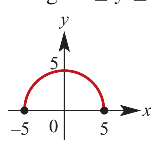
b Domain $-2 \le x \le 6$
Range $-1 \le y \le 7$



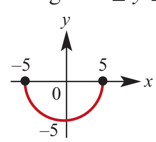
c Domain $0 \le x \le 1$
Range $1\frac{1}{2} \le y \le 2\frac{1}{2}$



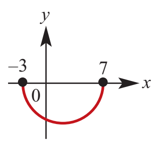
d Domain $-5 \le x \le 5$
Range $0 \le y \le 5$



e Domain $-5 \le x \le 5$
Range $-5 \le y \le 0$

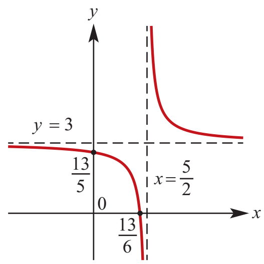


f Domain $-3 \le x \le 7$
Range $-5 \le y \le 0$



8

a Domain = $\left\{x \in \mathbb{R} : x \neq \frac{5}{2}\right\}$;
Range = $\{ y \in \mathbb{R} : y \neq 3 \}$



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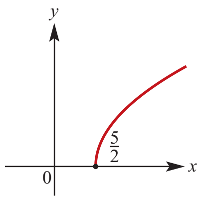
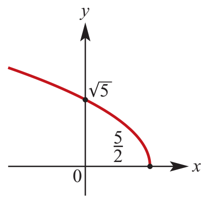
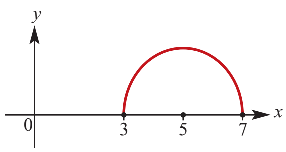
b Domain = $\left[\frac{5}{2}, \infty\right)$
Range = $[0, \infty)$



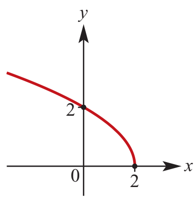
c Domain = $\left(-\infty, \frac{5}{2}\right]$
Range = $[0, \infty)$



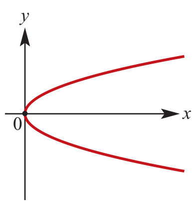
d Domain = $[3, 7]$; Range = $[0, 2]$



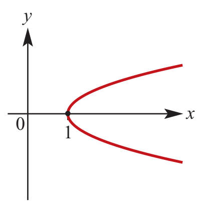
e Domain = $(-\infty, 2]$
Range = $[0, \infty)$



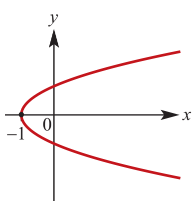
9 a Domain = $[0, \infty)$
Range = $\mathbb{R}$



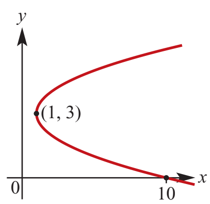
b Domain = $[1, \infty)$
Range = $\mathbb{R}$



c Domain = $[-1, \infty)$
Range = $\mathbb{R}$



d Domain = $[1, \infty)$
Range = $\mathbb{R}$

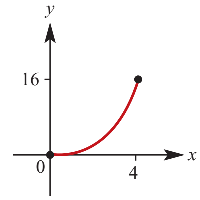


Exercise 6C

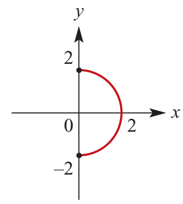
1 a Not a function; Domain = $\{0, 1, 2, 3\}$;
Range = $\{1, 2, 3, 4\}$
b A function; Domain = $\{-2, -1, 0, 1, 2\}$;
Range = $\{-5, -2, -1, 2, 4\}$
c Not a function; Domain = $\{-1, 0, 3, 5\}$;
Range = $\{1, 2, 4, 6\}$
d A function; Domain = $\{1, 2, 4, 5, 6\}$;
Range = $\{3\}$

2 Functions: a, b, c, e

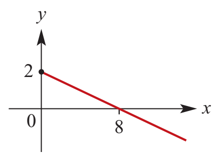
3 a A function
Domain = $[0, 4]$
Range = $[0, 16]$



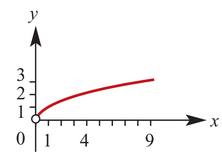
b Not a function
Domain = $[0, 2]$
Range = $[-2, 2]$



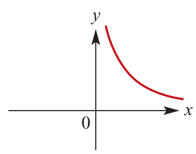
c A function
Domain = $[0, \infty)$
Range = $(-\infty, 2]$



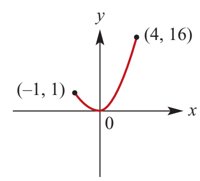
d A function
Domain = $(0, \infty)$
Range = $(0, \infty)$



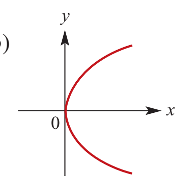
e A function
Domain = $(0, \infty)$
Range = $(0, \infty)$



f A function
Domain = $[-1, 4]$
Range = $[0, 16]$



g Not a function
Domain = $[0, \infty)$
Range = $\mathbb{R}$



4 a A function; Domain = $\mathbb{R}$; Range = $\{-2\}$
b Not a function; Domain = $\{3\}$; Range = $\mathbb{Z}$
c A function; Domain = $\mathbb{R}$; Range = $\mathbb{R}$
d A function; Domain = $\mathbb{R}$; Range = $[5, \infty)$
e Not a function; Domain = $[-3, 3]$;
Range = $[-3, 3]$

5 a **i** -3 **ii** 5 **iii** -5 **iv** 9
v $2x - 5$ **vi** $\frac{2}{a} - 3$
b **i** 4 **ii** -4 **iii** $\frac{4}{3}$ **iv** 2
c **i** 4 **ii** 36 **iii** 36 **iv** $(a-2)^2$
d **i** 0 **ii** $\frac{a}{1+a}$ **iii** $\frac{-a}{1-a}$ **iv** $1 - a$

6 a $5, 2t + 1$ **b** $x = \frac{5}{2}$ **c** $x = -\frac{1}{2}$
d $t = -1$ **e** $x \geq -1$ **f** $x \geq 1$

7 a 1 **b** $\frac{1}{6}$ **c** ± 3 **d** $-1, 4$ **e** $-1, 3$ **f** $-2, 3$

8 a $g(-1) = -1, g(2) = 8, g(-2) = 0$
b $h(-1) = 3, h(2) = 18, h(-2) = -14$

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c **i** $g(-3x) = 9x^2 - 6x$
ii $g(x - 5) = x^2 - 8x + 15$
iii $h(-2x) = -16x^3 - 4x^2 + 6$
iv $g(x + 2) = x^2 + 6x + 8$
v $h(x^2) = 2x^6 - x^4 + 6$

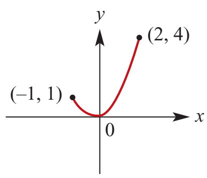
9 **a** $f(2) = 5, f(-4) = 29$
b Range $= [-3, \infty)$

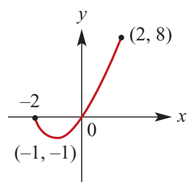
10 **a** $f(2) = 7$ **b** $x = 2$ **c** $x = -1$

11 **a** 2 **b** ± 1 **c** $x = \pm\sqrt{3}$

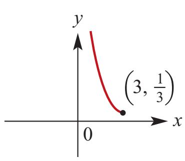
12 **a** $x = -1$ **b** $x > -1$ **c** $x = -\frac{6}{7}$

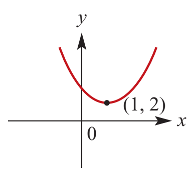
13 **a** Range $= [0, 4]$ **b** Range $= [-1, 8]$



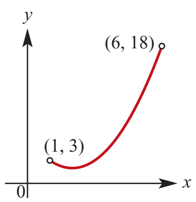


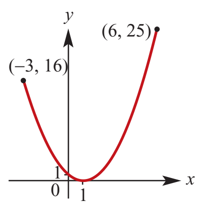
c Range $= \left[\frac{1}{3}, \infty\right)$ **d** Range $= [2, \infty)$





e Range $= [2, 18)$ **f** Range $= [0, 25]$





14 **a** Domain $= \mathbb{R}$; Range $= \mathbb{R}$
b Domain $= [0, \infty)$; Range $= [0, \infty)$
c Domain $= \mathbb{R}$; Range $= [1, \infty)$
d Domain $= [-3, 3]$; Range $= [-3, 0]$
e Domain $= (0, \infty)$; Range $= (0, \infty)$
f Domain $= \mathbb{R}$; Range $= (-\infty, 3]$
g Domain $= [2, \infty)$; Range $= [0, \infty)$
h Domain $= \left[\frac{1}{2}, \infty\right)$; Range $= [0, \infty)$
i Domain $= \left(-\infty, \frac{3}{2}\right]$; Range $= [0, \infty)$
j Domain $= \left\{x \in \mathbb{R} : x \neq \frac{1}{2}\right\}$;
Range $= \{ y \in \mathbb{R} : y \neq 0 \}$
k Domain $= \left\{x \in \mathbb{R} : x \neq \frac{1}{2}\right\}$;
Range $= (0, \infty)$
l Domain $= \left\{x \in \mathbb{R} : x \neq \frac{1}{2}\right\}$;
Range $= \{ y \in \mathbb{R} : y \neq 2 \}$

15 **a** Domain $= [4, \infty)$; Range $= [0, \infty)$
b Domain $= (-\infty, 4]$; Range $= [0, \infty)$
c Domain $= [2, \infty)$; Range $= [3, \infty)$
d Domain $= \{ x \in \mathbb{R} : y \neq 4 \}$;
Range $= \{ y \in \mathbb{R} : y \neq 0 \}$
e Domain $= \{ x \in \mathbb{R} : y \neq 4 \}$;
Range $= \{ y \in \mathbb{R} : y \neq 3 \}$
f Domain $= \{ x \in \mathbb{R} : x \neq -2 \}$;
Range $= \{ y \in \mathbb{R} : y \neq -3 \}$

16 **a** Domain $= \mathbb{R}$; Range $= \mathbb{R}$
b Domain $= \mathbb{R}$; Range $= [2, \infty)$
c Domain $= [-4, 4]$; Range $= [-4, 0]$
d Domain $= \{ x \in \mathbb{R} : x \neq -2 \}$;
Range $= \{ y \in \mathbb{R} : y \neq 0 \}$

Exercise 6D

1 **b** **i** 25.06 **ii** 25.032 **iii** 25.2 **iv** 26

2 **a** $a = -3, b = \frac{1}{2}$ **b** 6

3 $f(x) = 7 - 5x$

4 **a** **i** $f(0) = -\frac{9}{2}$ **ii** $f(1) = -3$
b 3

5 $f(x) = -7(x - 2)(x - 4)$

6 $f(x) = (x - 3)^2 + 7$, Range $= [7, \infty)$

7 $a = \frac{1}{10}, b = -\frac{9}{10}, c = 2$

8 $f(x) = -2(x - 1)(x + 5)$
$g(x) = -50(x - 1)\left(x + \frac{1}{5}\right)$

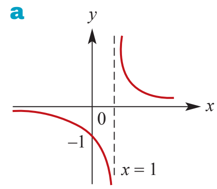
9 **a** $k < \frac{-37}{12}$ **b** $k = -\frac{25}{12}$

Exercise 6E

1 **a** $(-1, 1)$ **b** $(-5, 8)$
c $(-6, 2)$ **d** $(-7, 9)$
e $(-5, 3)$

2 **a** $g(x) = \frac{1}{x - 2} - 1$ **b** $g(x) = \frac{1}{x - 4} + 3$
c $g(x) = (x + 2)^2 - 3$ **d** $g(x) = (x - 4)^2 - 2$
e $g(x) = \sqrt{x - 2} - 1$

3 **a**



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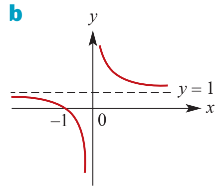

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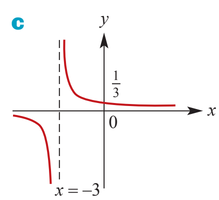
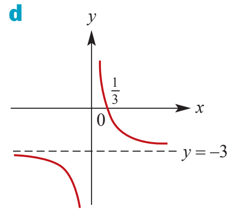


d



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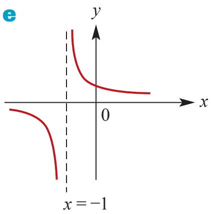

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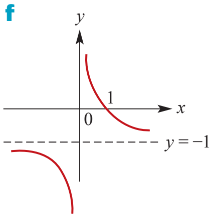

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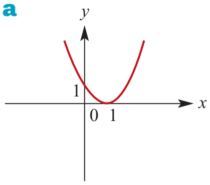

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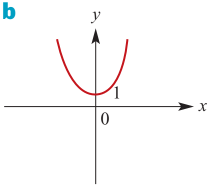

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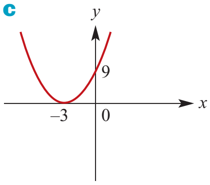

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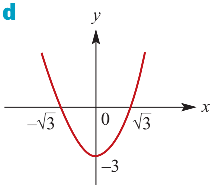

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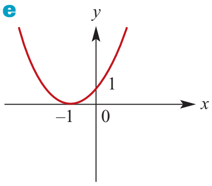

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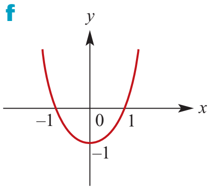
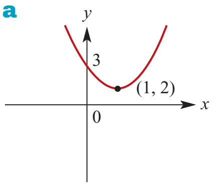


5 a


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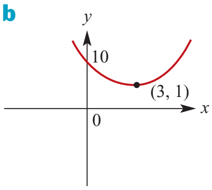

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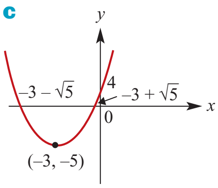

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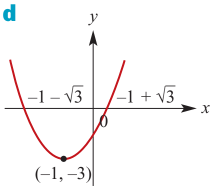

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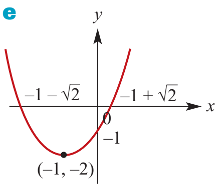

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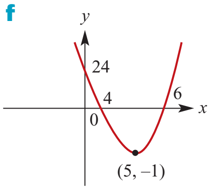


Exercise 6F

1 a $(-2, 3)$ **b** $(2, -3)$ **c** $(-2, -12)$ **d** $(-8, -3)$

2 a i $y = 4x^2$ **ii** $y = \dfrac{x^2}{25}$ **iii** $y = \dfrac{2x^2}{3}$

iv $y = 4x^2$ **v** $y = -x^2$ **vi** $y = x^2$

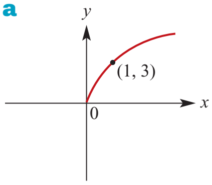
b i $y = \dfrac{1}{2x}$ **ii** $y = \dfrac{5}{x}$ **iii** $y = \dfrac{2}{3x}$

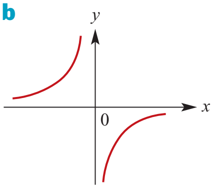
iv $y = \dfrac{4}{x}$ **v** $y = \dfrac{-1}{x}$ **vi** $y = \dfrac{-1}{x}$

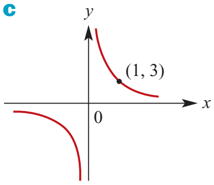
c i $y = \sqrt{2x}$ **ii** $y = \sqrt{\dfrac{x}{5}}$

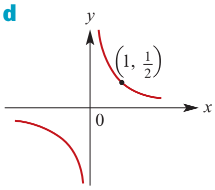
iii $y = \dfrac{2\sqrt{x}}{3}$ **iv** $y = 4\sqrt{x}$

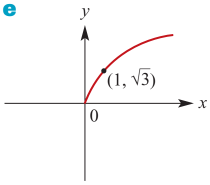
v $y = -\sqrt{x}$ **vi** $y = \sqrt{-x}, x \le 0$

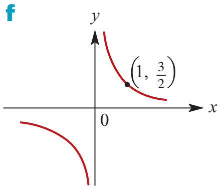
3 a


b


c


d


e


f


Exercise 6G

1 a $y = 3\sqrt{x-2}$ **b** $y = -\sqrt{x+3}$

c $y = -3\sqrt{x}$ **d** $y = -\sqrt{\dfrac{x}{2}}$

e $y = 2\sqrt{x-2} - 3$ **f** $y = \sqrt{\dfrac{x+2}{2}} - 3$

2 a $y = \dfrac{3}{x-2}$ **b** $y = \dfrac{-1}{x+3}$

c $y = -\dfrac{3}{x}$ **d** $y = -\dfrac{2}{x}$

e $y = \dfrac{2}{x-2} - 3$ **f** $y = \dfrac{2}{x+2} - 3$

Exercise 6H

1 a i A dilation of factor 2 parallel to the y-axis, then a translation of 1 unit in the positive direction of the x-axis and 3 units in the positive direction of the y-axis

ii A reflection in the x-axis, then a translation of 1 unit in the negative direction of the x-axis and 2 units in the positive direction of the y-axis

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iii A dilation of factor $\frac{1}{2}$ parallel to the x-axis, then a translation of $\frac{1}{2}$ unit in the negative direction of the x-axis and 2 units in the negative direction of the y-axis

b **i** A dilation of factor 2 parallel to the y-axis, then a translation of 3 units in the negative direction of the x-axis

ii A translation of 3 units in the negative direction of the x-axis and 2 units in the positive direction of the y-axis

iii A translation of 3 units in the positive direction of the x-axis and 2 units in the negative direction of the y-axis

c **i** A translation of 3 units in the negative direction of the x-axis and 2 units in the positive direction of the y-axis

ii A dilation of factor $\frac{1}{3}$ parallel to the x-axis, then a dilation of factor 2 parallel to the y-axis

iii A reflection in the x-axis, then a translation of 2 units in the positive direction of the y-axis

2 **a** $(x, y) \to (3x + 2, y - 5)$

b $(x, y) \to \left(3x + 1, -\frac{y-7}{3}\right)$

c $(x, y) \to \left(-(x - 4), \frac{y}{2}\right)$

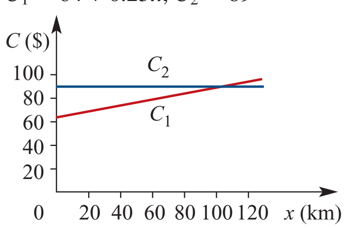
d $(x, y) \to \left(-(x - 4), \frac{15 - y}{2}\right)$

Exercise 6I

1 $C = 0.15n + 45$ where n is the number of calls

2 **a** $C_1 = 64 + 0.25x$, $C_2 = 89$

b



c $x > 100$ km

3 **a** Length $= (50 - x)$ cm

b $A(x) = x(50 - x)$ **c** $0 \le x \le 50$

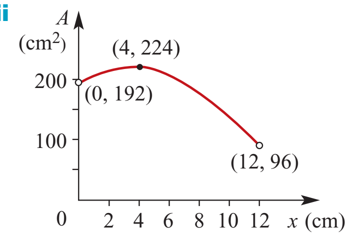
d Maximum area $= 625\text{ cm}^2$ when $x = 25$

4 **a** **i** $A = (8 + x)y - x^2$

ii $P = 2x + 2y + 16$

b **i** $A = 192 + 16x - 2x^2$ **ii** $0 < x < 12$

iii



iv 224 cm^2

Chapter 6 review

Short-answer questions

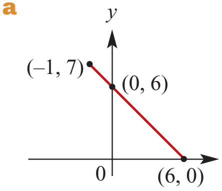
1 **a** $[-2, 4)$ **b** $[-2, 4]$ **c** $[1, 8)$ **d** $(-1, 6]$

e $(-4, -2] \cup (1, 5]$ **f** $(-4, -2] \cup (2, \infty)$

g $(-\infty, -3] \cup (1, \infty)$

2 **a** -16 **b** 26 **c** $-\frac{2}{3}$

3 **a**



b Range $= [0, 7]$

4 **a** Range $= \mathbb{R}$ **b** Range $= [-5, 4]$

c Range $= [0, 4]$ **d** Range $= (-\infty, 9]$

e Range $= [2, \infty)$ **f** Range $= \{-6, 2, 4\}$

g Range $= [0, \infty)$

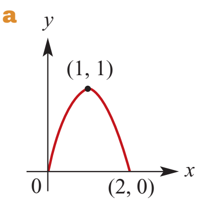
h Range $\{ y \in \mathbb{R} : y \ne 2 \}$

i Range $= [-5, 1]$ **j** Range $= [-1, 3]$

5 **a** $a = -15, b = \frac{33}{2}$

b Domain $= \{ x \in \mathbb{R} : x \ne 0 \}$

6 **a**



b Range $= [0, 1]$

7 $a = 3, b = -5$

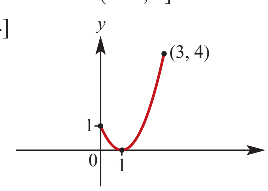
8 $a = -\frac{1}{2}, b = 2, c = 0$

9 **a** $\{ x \in \mathbb{R} : x \ne 2 \}$ **b** $[2, \infty)$

c $[-5, 5]$ **d** $\left\{ x \in \mathbb{R} : x \ne \frac{1}{2} \right\}$

e $[-10, 10]$ **f** $(-\infty, 4]$

10 **a** Range $= [0, 4]$



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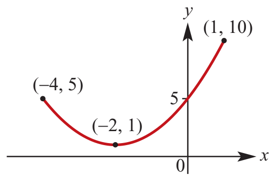
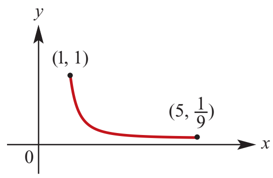
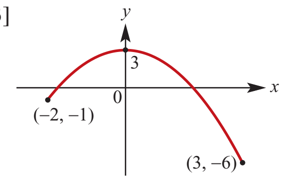
b Range = [1, 10]



c Range = $\left[\frac{1}{9}, 1\right]$



d Range = [−6, 3]



11 a Domain = $[1, \infty)$; Range = $[0, \infty)$
b Domain = $(-\infty, 1]$; Range = $[0, \infty)$
c Domain = $[0, \infty)$; Range = $(-\infty, 1]$

12 a Domain = $\{ x \in \mathbb{R} : x \neq 1 \}$;
Range = $\{ y \in \mathbb{R} : y \neq 0 \}$
b Domain = $\{ x \in \mathbb{R} : x \neq -1 \}$;
Range = $\{ y \in \mathbb{R} : y \neq 0 \}$
c Domain = $\{ x \in \mathbb{R} : x \neq 1 \}$;
Range = $\{ y \in \mathbb{R} : y \neq 3 \}$

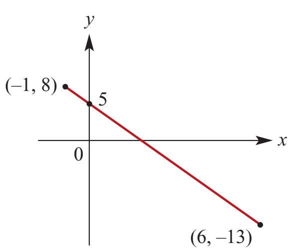
13 a Domain = [−1, 1]; Range = [0, 1]
b Domain = [−3, 3]; Range = [0, 3]
c Domain = [−1, 1]; Range = [3, 4]

14 a $2p + 5$ **b** $2(p + h) + 5$
c $2h$ **d** 2

15 −2

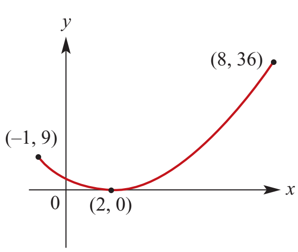
16 a $\left(-\infty, -\frac{15}{8}\right]$ **b** $\left[3\frac{7}{8}, \infty\right)$
c $(-\infty, 20]$ **d** $(-\infty, 3]$

17 a



b Range = [−13, 8]

18 a

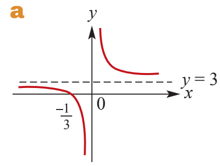


b Range = [0, 36]

19 a Domain $-3 \leq x \leq 3$; Range $-3 \leq y \leq 3$
b Domain $1 \leq x \leq 3$; Range $-1 \leq y \leq 1$
c Domain $0 \leq x \leq 1$; Range $0 \leq y \leq 1$
d Domain $-1 \leq x \leq 9$; Range $-5 \leq y \leq 5$
e Domain $-4 \leq x \leq 4$; Range $-2 \leq y \leq 6$

20 a {2, 4, 6, 8} **b** {4, 3, 2, 1}
c {−3, 0, 5, 12} **d** $\{1, \sqrt{2}, \sqrt{3}, 2\}$

21 a



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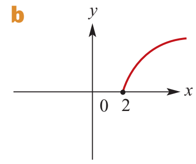
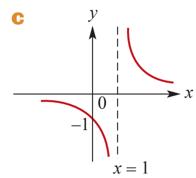


c



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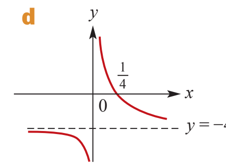
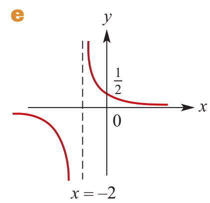
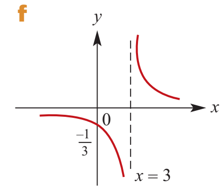


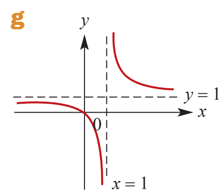
e



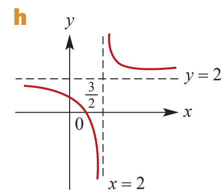
f



g



h

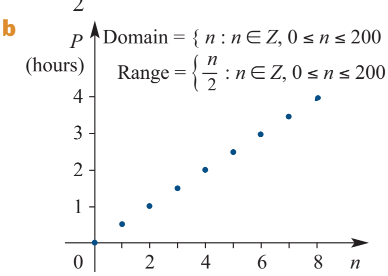


22 a i $(x, y) \to (x - 1, 3y + 2)$
ii $(x, y) \to (x - 2, -2y + 3)$
iii $(x, y) \to \left(\frac{x - 1}{3}, y - 1\right)$
b i $(x, y) \to (x - 2, 4y)$
ii $(x, y) \to (x - 6, y - 12)$
iii $(x, y) \to (x + 3, 4y - 5)$
c i $(x, y) \to (x + 4, y + 2)$
ii $(x, y) \to \left(\frac{x}{2}, 2y\right)$
iii $(x, y) \to (x, -2y + 3)$

Extended-response questions

1 a $P = \frac{1}{2}n$

b



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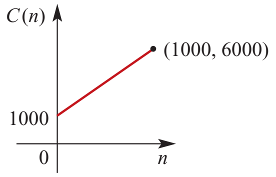
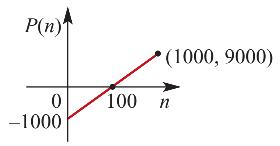
2 a **i** $C(n) = 1000 + 5n, n > 0$

ii



b **i** $P(n) = 15n - (1000 + 5n)$
$= 10n - 1000$

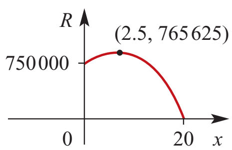
ii



3 $V = 8000(1 - 0.05n) = 8000 - 400n$

4 a $R = (50\,000 - 2500x)(15 + x)$
$= 2500(x + 15)(20 - x)$

b



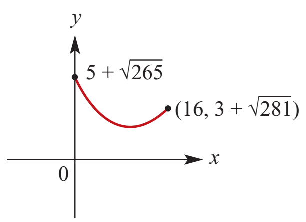
c Price for max revenue = \$17.50

5 a $A(x) = \frac{x}{4}(2a - (6 - \sqrt{3})x)$ **b** $0 < x < \frac{a}{3}$

6 a **i** $d(x) = \sqrt{x^2 + 25} + \sqrt{(16 - x)^2 + 9}$

ii $0 \le x \le 16$

b **i**



ii 1.54

iii 3.40 or 15.04

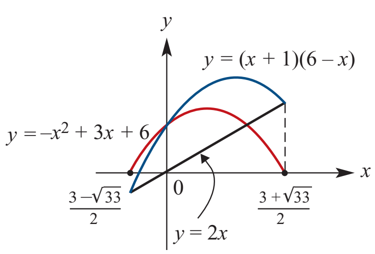
c **i** Minimum is $8\sqrt{5}$, occurs when $x = 10$

ii Range = $[8\sqrt{5}, 5 + \sqrt{265}]$

7 a $A\left(\frac{3 + \sqrt{33}}{2}, 3 + \sqrt{33}\right), B\left(\frac{3 - \sqrt{33}}{2}, 3 - \sqrt{33}\right)$

b **i** $d(x) = -x^2 + 3x + 6$

ii



c **i** Maximum value of $d(x)$ is 8.25

ii Range = $[0, 8.25]$

d $A(2.45, 12.25), B(-2.45, -12.25)$
$d(x) = -x^2 + 6$
Maximum value of $d(x)$ is 6; Range = $[0, 6]$

8 a $k = \frac{1}{4}$ **b** $h = -\frac{1}{4}$

9 a $h = -1 \pm 2\sqrt{2}$ **b** $a = \pm 2\sqrt{2}$

c $a = -8, b = 16$

10 a $k = 10$

b **i** $h > 2 + \sqrt{10}$ **ii** $h < 2 - \sqrt{10}$

iii $2 - \sqrt{10} < h < 2 + \sqrt{10}$

Chapter 7

Exercise 7A

1 a -3 **b** -1 **c** -7 **d** -15

2 a 0 **b** 0

3 a 6 **b** 9 **c** 26 **d** 11

e $a^3 + 4a^2 - 2a + 6$ **f** $8a^3 + 16a^2 - 4a + 6$

4 a $a = 4$ **b** $a = 4$ **c** $c = 6$

d $a = -33, b = -15$ **e** $a = -9, b = 23$

5 a $x^3 - 2x^2 - 2x + 2$ **b** $x^3 - x^2 + 2x$

c $x^3 - 2x^2 + 4x - 2$ **d** $3x^3 - 6x^2 + 3x$

e $-3x^4 + 8x^3 - 7x^2 + 2x$

f $-3x^3 - x^2 + 2x$ **g** $x^3 - x^2 - x + 2$

h $x^5 - x^4 - x^3 + x^2$

6 a $x^3 - 4x^2 + 7x - 6$ **b** $x^3 - 6x^2 + 11x - 12$

c $2x^3 - 5x^2 - x + 4$

d $x^3 + (b - 2)x^2 + (c - 2b)x - 2c$

e $2x^3 - 7x^2 - 10x - 3$

7 a $x^3 + (b + 1)x^2 + (c + b)x + c$

b $b = -8, c = 12$ **c** $(x + 1)(x - 6)(x - 2)$

8 $b = -3, c = -11$

9 a $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

b $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

10 a $(x - y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

b $(2x + y)^4 = 16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4$

Exercise 7B

1 a $x^2 + 2x + \frac{3}{x - 1}$ **b** $2x^2 - x - 3 + \frac{6}{x + 1}$

c $3x^2 - 10x + 22 - \frac{43}{x + 2}$

d $2x^2 + 3x + 10 + \frac{28}{x - 3}$

2 a $x^2 - x + 4 - \frac{8}{x + 1}$

b $2x^2 - 8x + 49 - \frac{181}{x + 4}$

c $x^2 + x - 3 + \frac{11}{x + 3}$

d $x^2 - x + 4 + \frac{8}{x - 2}$

3 a $x^2 - 2x + 5$ **b** $2x^2 - 2x - 6$

c $x^2 - 2x - 6$ **d** $3x^2 - x - 6$

4 a Quotient $x^2 - 3$; Remainder 7

b Quotient $x^2 + 2x + 15$; Remainder 71

c Quotient $2x^2 - 3x$; Remainder -7

d Quotient $5x^2 + 20x + 77$; Remainder 315

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5 a $\frac{1}{2}x^2 + \frac{7}{4}x - \frac{3}{8} + \frac{103}{8(2x+5)}$

b $x^2 + 2x - 3 - \frac{2}{2x+1}$ **c** $x^2 + 2x - 15$

d $\frac{1}{3}x^2 - \frac{8}{9}x - \frac{8}{27} + \frac{19}{27(3x-1)}$

6 a $x^2 + 3x + 8 + \frac{9}{x-1}$

b $x^2 - \frac{x}{2} + \frac{9}{4} + \frac{21}{4(2x-1)}$

7 a Quotient $2x - 6$; Remainder 0

b Quotient $x - 6$; Remainder -2

c Quotient $2x - 6$; Remainder 42

d Quotient $x^2 - 4x + 2$; Remainder $-x + 7$

e Quotient $x^2 - 3x + 7$; Remainder $-10x + 9$

f Quotient $x^2 + x - \frac{3}{2}$; Remainder $\frac{15}{2}x + 16$

Exercise 7C

1 a -2 **b** -29 **c** 15 **d** 4 **e** 7
f -12 **g** 0 **h** -5 **i** -8

2 a $a = -3$ **b** $a = 2$ **c** $a = 4$ **d** $a = -10$

3 a $P(1) = 0$ **b** $P(1) = 0$

c $P(-2) = 0$ **d** $P\left(\frac{3}{2}\right) = 0$

4 a 6 **b** 28 **c** $-\frac{1}{3}$

5 a $(x-1)(x+1)(2x+1)$ **b** $(x+1)^3$

c $(x-1)(6x^2 - 7x + 6)$

d $(x-1)(x+5)(x-4)$ **e** $(x+1)^2(2x-1)$

f $(x+1)(x-1)^2$

g $(x-2)(4x^2 + 8x + 19)$

h $(x+2)(2x+1)(2x-3)$

6 1

7 a $(2x-3)(x^2 - 2x + 5)$

b $(2x+1)(x^2 - 2x + 5)$

c $(2x+1)(x - 1 - \sqrt{6})(x - 1 + \sqrt{6})$

d $(2x+3)(x - 1 - \sqrt{2})(x - 1 + \sqrt{2})$

8 a $(x-1)(x^2 + x + 1)$

b $(x+4)(x^2 - 4x + 16)$

c $(3x-1)(9x^2 + 3x + 1)$

d $(4x-5)(16x^2 + 20x + 25)$

e $(1-5x)(1 + 5x + 25x^2)$

f $(3x+2)(9x^2 - 6x + 4)$

g $(4m - 3n)(16m^2 + 12mn + 9n^2)$

h $(3b + 2a)(9b^2 - 6ab + 4a^2)$

9 a $(x+2)(x^2 - x + 1)$

b $(3x+2)(x-1)(x-2)$

c $(x-3)(x+1)(x-2)$

d $(3x+1)(x+3)(2x-1)$

10 $a = 3, b = -3, P(x) = (x-1)(x+3)(x+1)$

11 b i n odd **ii** n even

12 a $a = 1, b = 1$

b i $P(x) = x^3 - 2x^2 + 3$

Exercise 7D

1 a $1, -2, 4$ **b** $4, 6$ **c** $\frac{1}{2}, 3, -\frac{2}{3}$ **d** $0, -3, \frac{5}{2}$

2 a $-2, 0, 4$ **b** $0, -1 \pm 2\sqrt{3}$ **c** $-5, 0, 8$

d $0, -1 \pm \sqrt{17}$

3 a 1 **b** -1 **c** $5, \pm\sqrt{10}$ **d** $\pm 4, a$

4 a $2, 3, -5$ **b** $-1, -\frac{2}{3}, 3$ **c** $1, -\sqrt{2}, \sqrt{2}$

d $-\frac{2}{5}, -4, 2$ **e** $-\frac{1}{2}, \frac{1}{3}, 1$ **f** $-2, -\frac{3}{2}, 5$

5 a $-6, 2, 3$ **b** $-2, -\frac{2}{3}, \frac{1}{2}$ **c** 3

d -1 **e** $-1, 3$ **f** $3, -2 \pm \sqrt{3}$

6 a $0, \pm 2\sqrt{2}$ **b** $1 + 2\sqrt[3]{2}$ **c** -2

d -5 **e** $\frac{1}{10}$

7 a $2(x-9)(x-13)(x+11)$

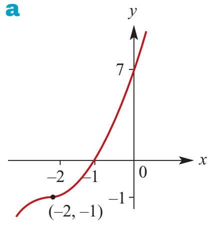
b $(x+11)(x+3)(2x-1)$

c $(x+11)(2x-9)(x-11)$

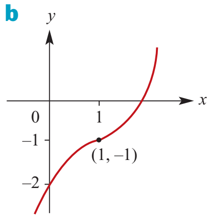
d $(2x-1)(x+11)(x+15)$

Exercise 7E

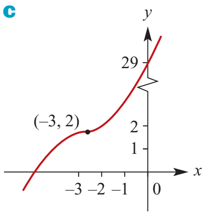
1 a



b



c



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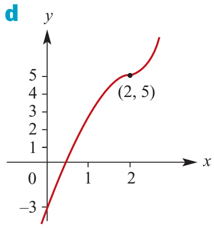
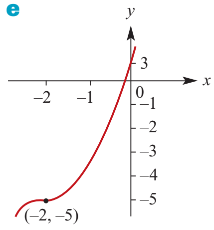
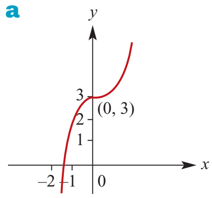


e



2 a



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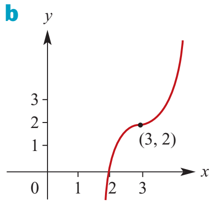

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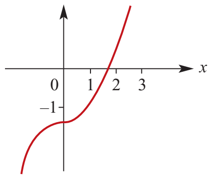
c



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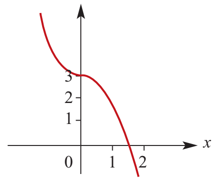

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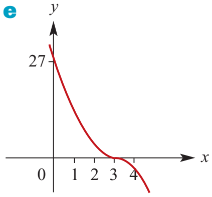

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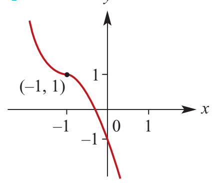

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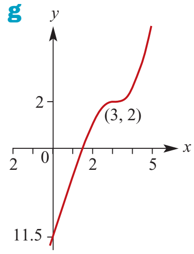
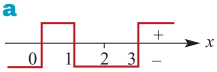
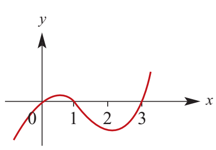


Exercise 7F

1 a





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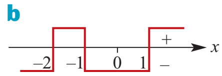

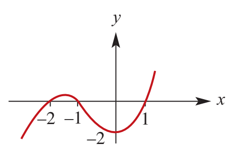

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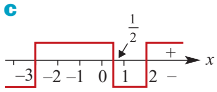

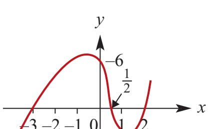

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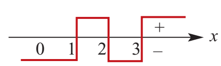

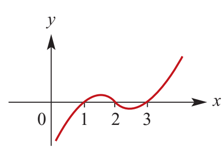

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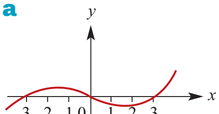

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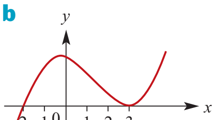

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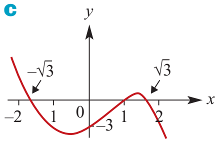

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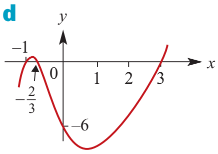

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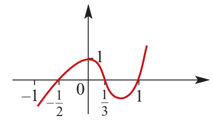

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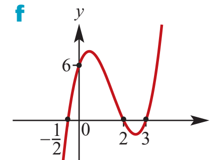

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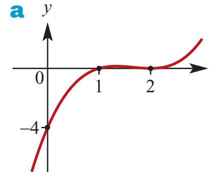

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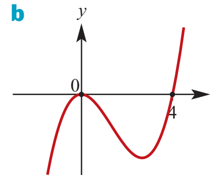

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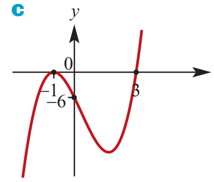

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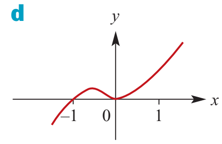

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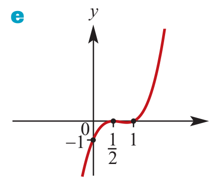

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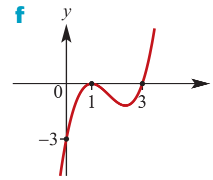
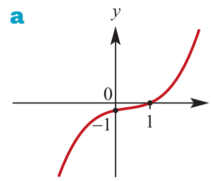


4 a



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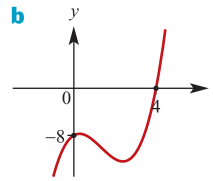

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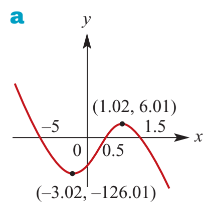

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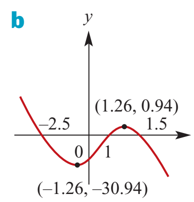

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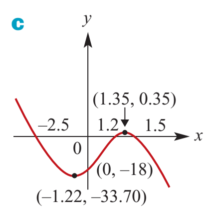

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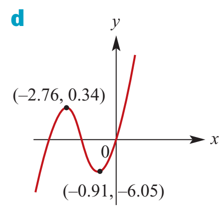

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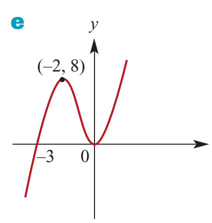

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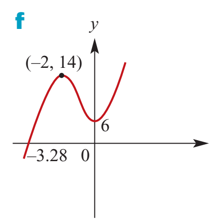

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6 $f(x) = (x+1)^2(x-3)$, so graph just touches the x-axis at $x = -1$ and cuts it at $x = 3$

Exercise 7G

1 a $a = 11$ **b** $a = 2$ **c** $a = \frac{4}{3}, b = \frac{44}{3}$

2 a $y = -\frac{1}{8}(x+2)^3$ **b** $y - 2 = -\frac{1}{4}(x-3)^3$

3 $y = 2x(x-2)^2$ **4** $y = -2x(x+4)^2$

5 $y = -2(x-1)(x-3)(x+1)$ **6** $a = 36$

7 a $y = (x-3)^3 + 2$ **b** $y = \frac{23}{18}x^3 + \frac{67}{18}x^2$

c $y = 5x^3$

8 a $y = -\frac{1}{3}x^3 + \frac{4}{3}x$ **b** $y = \frac{1}{4}x(x^2+2)$

9 a $y = -4x^3 - 50x^2 + 96x + 270$

b $y = 4x^3 - 60x^2 + 80x + 26$

c $y = x^3 - 2x^2 + 6x - 4$

d $y = 2x^3 - 3x$

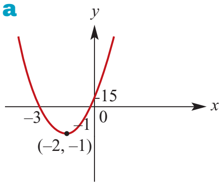
e $y = 2x^3 - 3x^2 - 2x + 1$

f $y = x^3 - 3x^2 - 2x + 1$

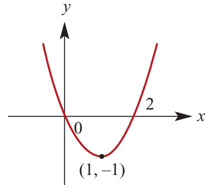
g $y = -x^3 - 3x^2 - 2x + 1$

Exercise 7H

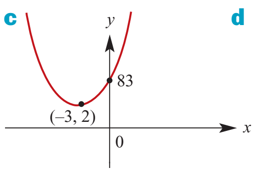
1 a



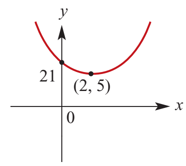
b



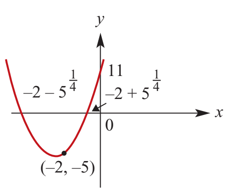
c



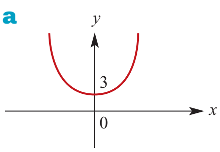
d



e



2 a



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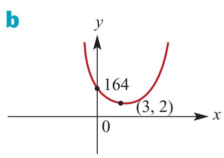

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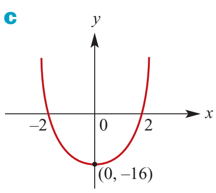

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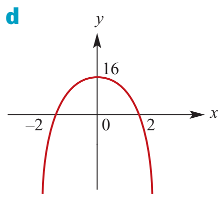

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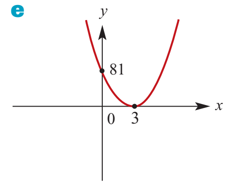

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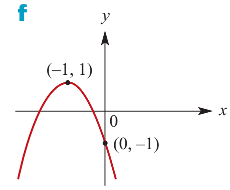


3 a $x = 0$ or $x = 3$

b $x = 2$ or $x = -1$ or $x = 5$ or $x = -3$

c $x = 0$ or $x = -2$ **d** $x = 0$ or $x = 6$

e $x = 0$ or $x = 3$ or $x = -3$

f $x = 3$ or $x = -3$

g $x = 0$ or $x = 4$ or $x = -4$

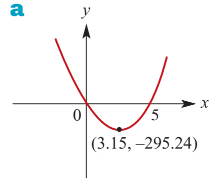
h $x = 0$ or $x = 4$ or $x = 3$

i $x = 0$ or $x = 4$ or $x = 5$

j $x = 2$ or $x = -2$ or $x = 3$ or $x = -3$

k $x = 4$ **l** $x = -4$ or $x = 2$

4 a



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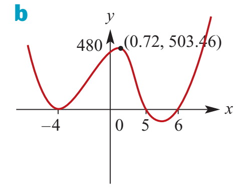

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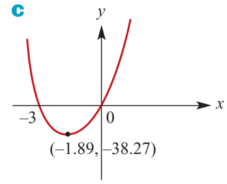

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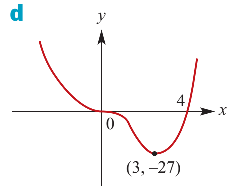

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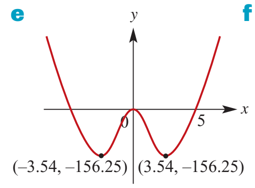

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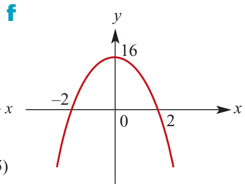

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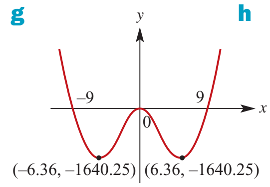

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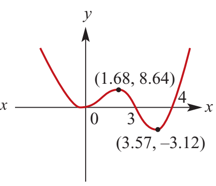

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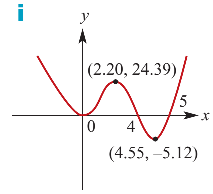

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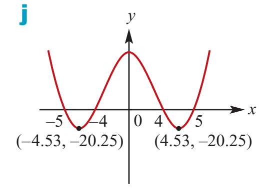

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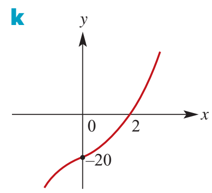

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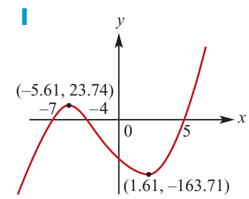

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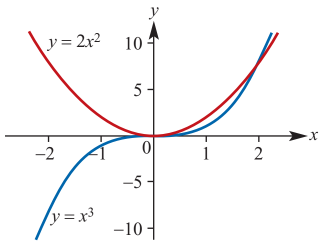
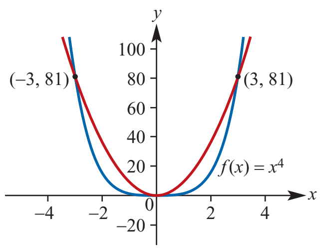
5 a Even **b** Odd **c** Even **d** Odd

6 a



b $x = 0, x = 2$ **c** $\{ x : x \le 2 \}$

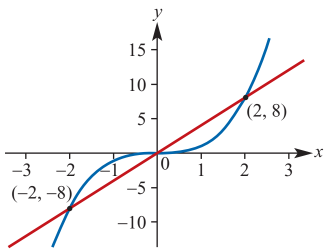
7 a



b $x = 0, x = 3, x = -3$

c $\{ x : -3 \le x \le 3 \}$

8 a



b $x = 0, x = 2, x = -2$

c $\{ x : x \le -2 \} \cup \{ x : 0 \le x \le 2 \}$

Exercise 7I

1 a Length of each edge $= 20 - 2x$

b $V(x) = 4x(10 - x)^2$

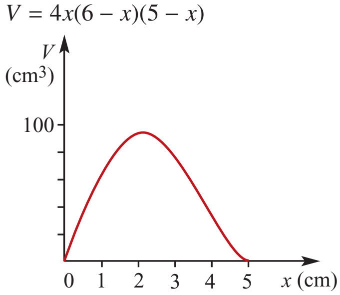
c $V(5) = 500$; Volume of box $= 500\,\text{cm}^3$

d $x = 5$ or $x = \frac{5}{2}(3 - \sqrt{5})$

2 a $\ell = 12 - 2x, w = 10 - 2x$

b $V = 4x(6 - x)(5 - x)$

c



d $V = 80$ **e** $x = 3.56$ or $x = 0.51$

f $V_{\max} = 96.8\,\text{cm}^3$ when $x = 1.81$

3 a Surface area $= x^2 + 4xh$

b $h = \dfrac{75 - x^2}{4x}$ **c** $V = \frac{1}{4}(75x - x^3)$

d i $\frac{71}{2}$ **ii** $\frac{125}{2}$ **iii** 22

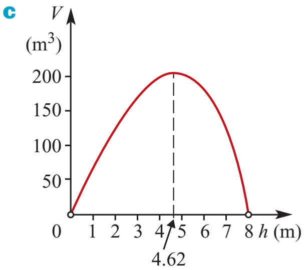
e $x = -2 + 3\sqrt{7}$

4 a $h = 60 - 20x$ **b** $V = 600x^2(3 - x)$

c 0 **d** $x = 1$ or $x = 1 + \sqrt{3}$

5 a $x = \sqrt{64 - h^2}$ **b** $V = \dfrac{\pi h}{3}(64 - h^2)$

c



d Domain $= \{ h : 0 < h < 8 \}$ **e** 64π

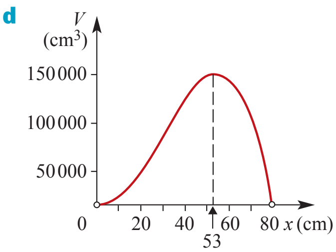
f $h = 2.48$ or $h = 6.47$

g $V_{\max} \approx 206.37\,\text{m}^3$ when $h = 4.62$

6 a $h = 160 - 2x$ **b** $V = x^2(160 - 2x)$

c Domain $= (0, 80)$

d



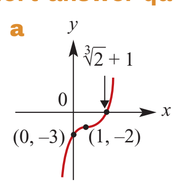
e $x = 20.498$ or $x = 75.63$

f $V_{\max} \approx 151\,703.7\,\text{cm}^3$ when $x \approx 53$

Chapter 7 review

Short-answer questions

1 a



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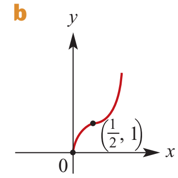

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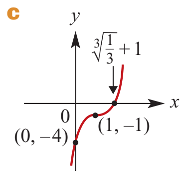

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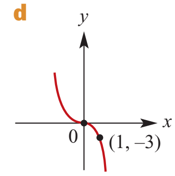

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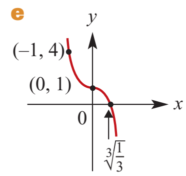
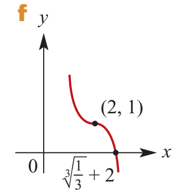


f



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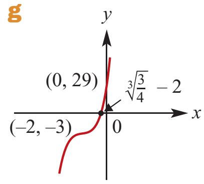
g



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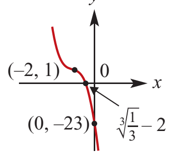
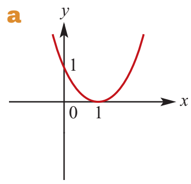


2 a



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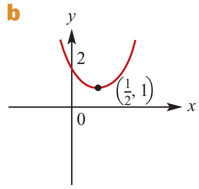

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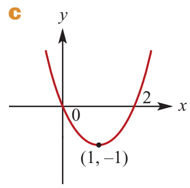

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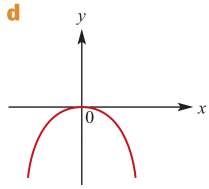

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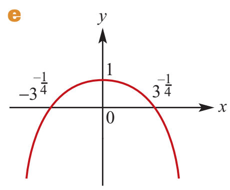

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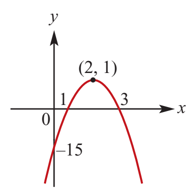

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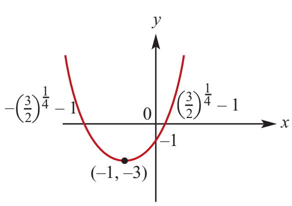

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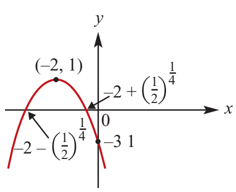


3 a $x = 2, x = -\dfrac{1}{2}, x = -3$

b $x = 2, x = \dfrac{\sqrt{17}+1}{4}, x = \dfrac{1-\sqrt{17}}{4}$

c $x = -1, x = 2, x = 6$

4 a $P\left(\dfrac{3}{2}\right) = 0$ and $P(-2) = 0$; $(3x + 1)$

b $x = -2, \dfrac{1}{2}, 3$

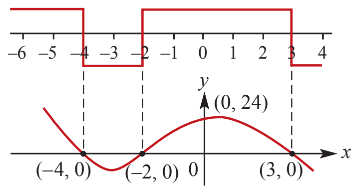
c $x = -1, -\sqrt{11}, +\sqrt{11}$

d i $P\left(\dfrac{1}{3}\right) = 0$ **ii** $(3x - 1)(x + 3)(x - 2)$

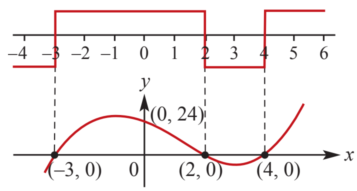
5 a $f(1) = 0$ **b** $(x - 1)(x^2 + (1 - k)x + k + 1)$

6 $a = 3, b = -24$

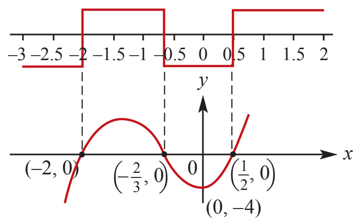
7 a



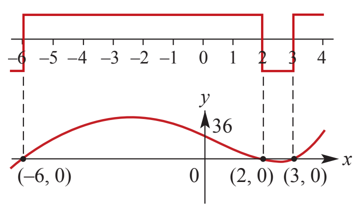
b



c



d



8 a −41 **b** 12 **c** $\dfrac{43}{9}$

9 $y = -\dfrac{2}{5}(x + 2)(x - 1)(x - 5)$

10 $y = \dfrac{2}{81}x(x + 4)^2$

11 a $a = 3, b = 8$ **b** $(x + 3)(2x - 1)(x - 1)$

12 a Dilation of factor 2 parallel to the y-axis, then translation of 1 unit in the positive direction of the x-axis and 3 units in the positive direction of the y-axis

b Reflection in the x-axis, then translation of 1 unit in the negative direction of the x-axis and 2 units in the positive direction of the y-axis

c Dilation of factor $\dfrac{1}{2}$ parallel to the x-axis then translation of $\dfrac{1}{2}$ unit in the negative direction of the x-axis and 2 units in the negative direction of the y-axis

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Extended-response questions

1 a $V = \pi r^2(6 - r)$ **b** $0 \le r \le 6$

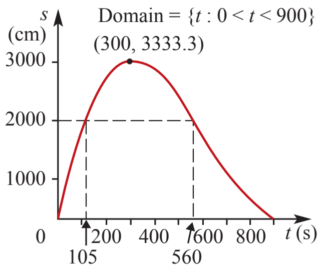
c $V(3) = 27\pi$ **d** $r = 3$ or $r = \frac{3}{2}(1 + \sqrt{5})$

e Maximum ≈ 100.53 (correct to 2 d.p.)

2 a $v = \frac{1}{32\,400}(t - 900)^2$

b $s = \frac{t}{32\,400}(t - 900)^2$

c



d No, it is not feasible as the maximum range of the taxi is less than 3.5 km (≈ 3.33 km)

e Maximum speed ≈ $\frac{2000}{105}$ = 19 m/s

Minimum speed ≈ $\frac{2000}{560}$ = 3.6 m/s

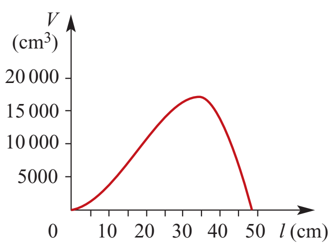
3 a $R = a(x - 5)^3 + 10$ **b** $a = \frac{2}{25}$

c $R = \frac{12}{343}(x - 7)^3 + 12$

4 a 4730 cm^2

b $V = \ell^2(\sqrt{2365} - \ell)$

c



d **i** $\ell = 23.69$ or $\ell = 39.79$

ii $\ell = 18.1$ or $\ell = 43.3$

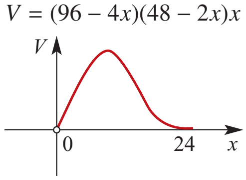
e $V_{max} \approx 17\,039$ cm^3 when $\ell \approx 32.42$ cm

5 a $a = \frac{-43}{15\,000}$, $b = 0.095$, $c = \frac{-119}{150}$, $d = 15.8$

b **i** (5.59, 13.83) **ii** (0, 15.8)

6 a $V = (96 - 4x)(48 - 2x)x = 8x(24 - x)^2$

b



i $0 < x < 24$

ii $V_{max} = 16\,384$ cm^3 when $x = 8.00$

c 15 680 cm^3 **d** 14 440 cm^3 **e** 9720 cm^3

Chapter 8

Short-answer questions

1 a $M\left(\frac{1}{2}, -\frac{3}{2}\right)$, $N\left(\frac{3}{2}, 4\right)$ **b** $m_{BC} = m_{MN} = \frac{11}{2}$

2 a −6 **b** 69 **c** −15

3 a $12a^2 - 4$ **b** $3a^2 - 6a - 1$

c $12a$

4 a No **b** $x = -\frac{3}{7}$ **c** $x \le -\frac{3}{7}$ **d** $k = -\frac{3}{16}$

5 $x = 2$ and $y = 3$, or $x = 3$ and $y = 2$

6 $AB = BC = CD = DA = 5\sqrt{2}$,
$m_{BC} = m_{AD} = 1$ and $m_{AB} = m_{CD} = -7$

7 a $y = (x + 2)^2 - 13$ **b** $y = \left(x - \frac{3}{2}\right)^2 - \frac{53}{4}$

c $y = 2\left(x - \frac{3}{4}\right)^2 + \frac{79}{8}$

8 a $\left(\frac{1 - \sqrt{41}}{2}, 3 - 2\sqrt{41}\right), \left(\frac{\sqrt{41} + 1}{2}, 2\sqrt{41} + 3\right)$

b (2, 6) **c** (−4, 14)

9 a $x < \frac{-3 - \sqrt{29}}{2}$ or $x > \frac{-3 + \sqrt{29}}{2}$

b $x \le \frac{5 - \sqrt{65}}{4}$ or $x \ge \frac{5 + \sqrt{65}}{4}$

10 a $\left\{x \in \mathbb{R} : x \ne \frac{5}{2}\right\}$ **b** $(-\infty, 5]$ **c** $\mathbb{R}$

11 $p = -38$, $(x - 3)(x + 4)(3x - 2)$

12 $a = -5$, $R = -35$

13 a [1, 4] **b** [0, 3] **c** $(-4, \infty)$

d $[3, \infty)$ **e** $\mathbb{R}$

14 a $3b + 2f = 18.20$ **b** \$2.80

15 a $k = 1$ **b** $k = -16$

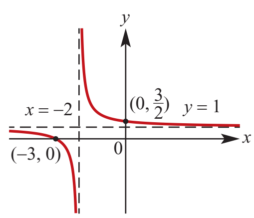
16 a $\frac{2}{5}$ **b** $2y + 5x - 17 = 0$

17 4 cm; 49 hours

18 a $a\left(x + \frac{1}{a}\right)^2 + \frac{a^2 - 1}{a}$ **b** $\left(-\frac{1}{a}, \frac{a^2 - 1}{a}\right)$

c $a = \pm 1$ **d** $a \in (-1, 1)$

19 a



b $A\left(0, \frac{3}{2}\right)$, $B(-3, 0)$ **c** $y = \frac{1}{2}x + \frac{3}{2}$

d $\left(-\frac{3}{2}, \frac{3}{4}\right)$ **e** $y = -2x - \frac{9}{4}$

20 $\sqrt{2}$ cm

21 192 g

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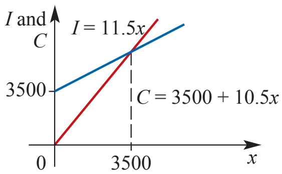
Extended-response questions

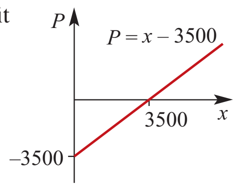
1 a $4b - 5c - d = 41,\ 2b - 7c - d = 53,$
$-4b + 3c - d = 25$
b $x^2 + y^2 - 2x - 4y - 29 = 0$

2 a $c = -b - 8$ **b** $x = 0$ or $x = -b$
c $y = 0$ or $y = b + 8$ **d** $b = -8$

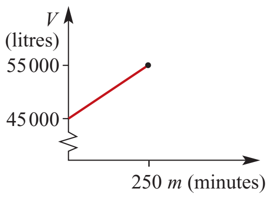
3 a $x \le a$ **b** $\left(\dfrac{\sqrt{4a+1}-1}{2}, \dfrac{\sqrt{4a+1}-1}{2}\right)$
c $a = 2$ **d** $a = 6$ **e** $a = c^2 + c$

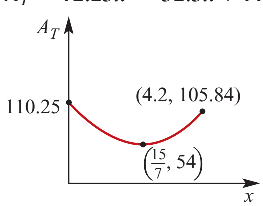
4 a $C = 3500 + 10.5x$ **b** $I = 11.5x$
c


d 3500

e Profit


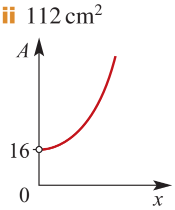
f 5500

5 a $V = 45\,000 + 40m$ **b** 4 hours 10 minutes
c


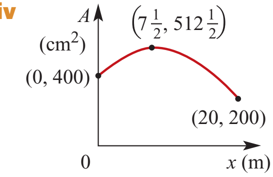
6 a $A_R = 6x^2$
b $A_S = (10.5 - 2.5x)^2$
c $0 \le x \le 4.2$
d $A_T = 12.25x^2 - 52.5x + 110.25$
e


f $110.25\ \text{cm}^2$ (when area of rectangle = 0)
g Rectangle 9×6 and square 3×3,
or rectangle $\dfrac{27}{7} \times \dfrac{18}{7}$ and square $\dfrac{51}{7} \times \dfrac{51}{7}$

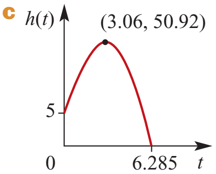
7 a 20 m **b** 20 m **c** 22.5 m

8 a $A = 10x^2 + 28x + 16$
b i $54\ \text{cm}^2$ **ii** $112\ \text{cm}^2$
c 3 cm **d**


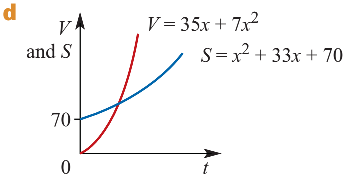
e $V = 2x^3 + 8x^2 + 8x$ **f** $x = 3$ **g** $x = 6.66$

9 a i $A = (10 + x)y - x^2$ **ii** $P = 2(y + x + 10)$
b i $A = 400 + 30x - 2x^2$
ii $512\frac{1}{2}\ \text{m}^2$ **iii** $0 \le x \le 20$
iv


10 a $A = 6x^2 + 7xy + 2y^2$
c i $x = 0.5$ m **ii** $y = 0.25$ m

11 a 50.9 m **b** $t = 6.12$ seconds
c


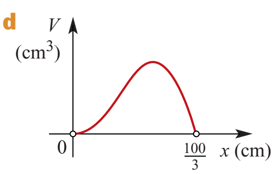
d 6.285 seconds

12 a $x + 5$ **b** $V = 35x + 7x^2$
c $S = x^2 + 33x + 70$
d


e 3.25 cm **f** 10 cm

13 a $2y + 3x = 22$
b i $B(0, 11)$ **ii** $D(8, -1)$
c $52\ \text{units}^2$
d 6.45 units

14 a 25 km/h **b** Tap A 60 min; Tap B 75 min
c 4 cm

15 a $h = 100 - 3x$ **b** $V = 2x^2(100 - 3x)$
c $0 < x < \dfrac{100}{3}$ **d**


e i $x = 18.142$ or $x = 25.852$
ii $x = 12.715$ or $x = 29.504$
f $V_{max} = 32\,921.811\ \text{cm}^3$ when $x = 22.222$
g i $S = 600x - 14x^2$
ii $S_{max} = \dfrac{45\,000}{7}\ \text{cm}^2$ when $x = \dfrac{150}{7}$
h $x = 3.068$ or $x = 32.599$

16 a $y = (7.6 \times 10^{-5})x^3 - 0.0276x^2 + 2.33x$
b $y = (7.6 \times 10^{-5})x^3 - 0.0276x^2 + 2.33x + 5$
c 57.31 m

17 a $y = \dfrac{3}{4}x - 4$ **b** $y = -\dfrac{4}{3}x + \dfrac{38}{3}$
c $D(8, 2)$ **d** 5 units **e** $50\ \text{units}^2$

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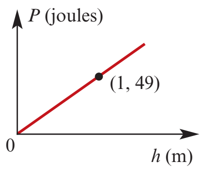
18 a i $P = 49h$ **ii**



iii 1136.8

b i $P = 9.8mh$ **ii** 100% increase

iii 50% decrease

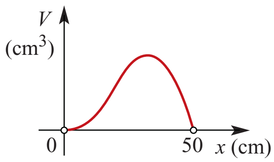
c i 14 **ii** 42

d 4

19 a i $y = 250 - 5x$

ii $V = x^2(250 - 5x) = 5x^2(50 - x)$

b



c (0, 50)

d $x = 11.378$ or $x = 47.813$

e $V_{max} = 92\,592.59$ cm^3 when $x = 33.33$ and $y = 83.33$

20 a i $A = \frac{a^3}{6}$ **ii** 4.5 **iii** 30

b i $A_1 = \frac{a^3}{12}$ **ii** 486 **iii** 36

c i $A_2 = \frac{1}{a} - \frac{1}{b}$ **ii** $\frac{5}{6}$ **iii** $\frac{11}{3}$

iv 99.99 **v** 999.999

Chapter 9

Exercise 9A

1 $\{H, T\}$

2 $\{1, 2, 3, 4, 5, 6\}$

3 a 52 **b** 4

c Clubs ♣, hearts ♥, spades ♠, diamonds ♦

d Clubs and spades are black; diamonds and hearts are red **e** 13

f Ace, king, queen, jack **g** 4

h 16

4 a $\{0, 1, 2, 3, 4, 5\}$ **b** $\{0, 1, 2, 3, 4, 5, 6\}$

c $\{0, 1, 2, 3\}$

5 a $\{0, 1, 2, 3, \dots\}$ **b** $\{0, 1, 2, 3, \dots, 41\}$

c $\{1, 2, 3, \dots\}$

6 a $\{2, 4, 6\}$ **b** $\{FFF\}$ **c** $\varnothing$

7 a $\frac{1}{2}$ **b** $\frac{3}{10}$ **c** $\frac{3}{20}$

8 a $\frac{4}{15}$ **b** $\frac{2}{3}$ **c** $\frac{4}{15}$

9 a $\frac{1}{4}$ **b** $\frac{1}{2}$ **c** $\frac{4}{13}$ **d** $\frac{2}{13}$

10 a $\frac{9}{13}$ **b** $\frac{10}{13}$ **c** $\frac{5}{13}$ **d** $\frac{1}{13}$

11 a $\frac{1}{365}$ **b** $\frac{30}{365}$ **c** $\frac{30}{365}$ **d** $\frac{90}{365}$

12 a $\frac{1}{9}$ **b** $\frac{1}{3}$ **c** $\frac{5}{9}$ **d** $\frac{4}{9}$

13 $\frac{1}{3}$

14 0.4

15 a $\frac{1}{3}$ **b** $\frac{1}{8}$ **c** $\frac{1}{4}$

16 a $\frac{1}{7}$ **b** $\frac{5}{7}$

17 a $\frac{3}{4}$ **b** $\frac{1}{2}$ **c** $\frac{10}{13}$ **d** $\frac{23}{26}$

18 $\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}$

19 a $\frac{1}{13}, \frac{2}{13}, \frac{2}{13}, \frac{2}{13}, \frac{2}{13}, \frac{4}{13}$ **b** $\frac{9}{13}$

Exercise 9B

1 a $\frac{17}{50}$ **b** $\frac{1}{10}$ **c** $\frac{4}{15}$ **d** $\frac{1}{200}$

2 a No **b** Answers will vary

c Answers will vary **d** Yes

e As the number of trials approaches infinity, the relative frequency approaches the value of the probability

3 P(a 6 from first die) $\approx \frac{78}{500} = 0.156$

P(a 6 from second die) $\approx \frac{102}{700} \approx 0.146$

Choose first die, as higher probability of a 6

4 a $\frac{17}{20}$ **b** $\frac{4}{5}$ **c** $\frac{9}{10}$ **d** 51

5 0.445

6 a $\frac{\pi}{4}$ **b** $\frac{\pi}{4} \approx 0.7855$

7 $\frac{3}{4}$

8 a $\frac{1}{3}$ **b** $\frac{1}{6}$ **c** $\frac{5}{6}$

9 a $\frac{4\pi}{25}$ **b** $1 - \frac{4\pi}{25} \approx 0.4973$

10 a i x^2 **ii** $\frac{1}{4}\pi x^2$ **iii** $\frac{1}{16}\pi x^2$

b i $\frac{\pi}{16}$ **ii** $\frac{3\pi}{16}$ **iii** $1 - \frac{\pi}{4}$

Exercise 9C

1 a $\frac{1}{4}$ **b** $\frac{1}{4}$

2 a $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** $\frac{1}{4}$

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3 a $\frac{1}{2}$ **b** $\frac{1}{18}$ **c** $\frac{5}{18}$

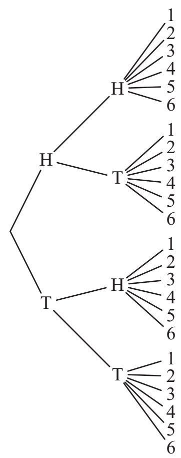
4 a $\frac{1}{12}$ **b** $\frac{1}{2}$ **c** $\frac{7}{12}$

5 a $\frac{3}{8}$ **b** $\frac{3}{8}$ **c** $\frac{1}{8}$ **d** $\frac{1}{8}$

6 a $\frac{1}{2}$ **b** $\frac{1}{4}$ **c** $\frac{3}{4}$

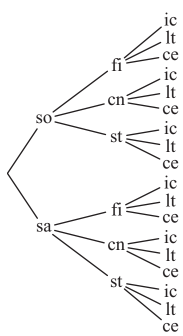
7 $\frac{1}{4}$

8 a



b i $\frac{1}{24}$ **ii** $\frac{1}{4}$ **iii** $\frac{1}{8}$ **iv** $\frac{1}{2}$

9 a



b i $\frac{1}{18}$ **ii** $\frac{1}{3}$ **iii** $\frac{1}{6}$ **iv** $\frac{2}{3}$

c i $\frac{1}{36}$ **ii** $\frac{1}{2}$ **iii** $\frac{5}{12}$ **iv** $\frac{1}{12}$

10 a

		2nd ball				
		1	2	3	4	5
1st ball	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)

b i $\frac{4}{25}$ **ii** $\frac{4}{5}$ **iii** $\frac{3}{25}$

Exercise 9D

1 a $\{1, 2, 3, 4, 6\}$ **b** $\{2, 4\}$
c $\{5, 6, 7, 8, 9, 10\}$ **d** $\{1, 3\}$
e $\{1, 3, 5, 6, 7, 8, 9, 10\}$ **f** $\{5, 7, 8, 9, 10\}$

2 a $\{1, 2, 3, 5, 6, 7, 9, 10, 11\}$
b $\{1, 3, 5, 7, 9, 11\}$ **c** $\{2, 4, 6, 8, 10, 12\}$
d $\{1, 3, 5, 7, 9, 11\}$ **e** $\{1, 3, 5, 7, 9, 11\}$

3 a {E, H, M, S} **b** {C, H, I, M}
c {A, C, E, I, S, T} **d** {H, M}
e {C, E, H, I, M, S} **f** {H, M}

4 a 20 **b** 45

5 a $\frac{2}{3}$ **b** 0 **c** $\frac{1}{2}$ **d** $\frac{5}{6}$

6 a $\frac{1}{2}$ **b** $\frac{1}{3}$ **c** $\frac{1}{6}, \frac{2}{3}$

7 a $\frac{7}{18}$ **b** $\frac{4}{18} = \frac{2}{9}$ **c** $\frac{2}{18} = \frac{1}{9}$ **d** $\frac{1}{2}$

8 a $\frac{3}{10}$ **b** $\frac{1}{5}$ **c** $\frac{1}{10}$ **d** $\frac{2}{5}$

9 $\mathrm{P}(A \cup B) = 0.7$

10 $\mathrm{P}(A \cup B) = 0.47$

11 a $\mathrm{P}(A \cap B) = 0.28$ **b** $\mathrm{P}(A \cup B) = 0.45$

12 a $\mathrm{P}(A \cap B) = 0.45$ **b** $\mathrm{P}(A \cup B) = 0.58$

13 a $\mathrm{P}(A \cap B) = 0$ **b** $\mathrm{P}(A \cup B) = 0.7$

14 a $\mathrm{P}(A \cap B) = 0$ **b** $\mathrm{P}(A \cup B) = 0.23$

15 $\mathrm{P}(A \cap B) = 0.2$

16 $\mathrm{P}(A \cap B) = 0.05$

17 $\mathrm{P}(A \cup B') = 0.7$

18 0.32

19 a 0.43 **b** 0.29

Exercise 9E

1 a 0.2 **b** 0.5 **c** 0.3 **d** 0.7

2 a 0.75 **b** 0.4 **c** 0.87 **d** 0.48

3 a 0.63 **b** 0.23 **c** 0.22 **d** 0.77

4 a 0.45 **b** 0.40 **c** 0.25 **d** 0.70

5 a 0.9 **b** 0.6 **c** 0.1 **d** 0.9

6 a 95% **b** 5%

7 a A = {J♥, Q♥, K♥, A♥, J♠, Q♠, K♠, A♠, J♦, Q♦, K♦, A♦, J♣, Q♣, K♣, A♣}

C = {2♥, 3♥, 4♥, 5♥, 6♥, 7♥, 8♥, 9♥, 10♥, J♥, Q♥, K♥, A♥}

b i P(a picture card) $= \frac{4}{13}$

ii P(a heart) $= \frac{1}{4}$

iii P(a heart picture card) $= \frac{1}{13}$

iv P(a picture card or a heart) $= \frac{25}{52}$

v P(a picture card or a club, diamond or spade) $= \frac{43}{52}$

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8 a $\frac{8}{15}$ **b** $\frac{7}{10}$ **c** $\frac{2}{15}$ **d** $\frac{1}{3}$

9 a 0.8 **b** 0.57 **c** 0.28 **d** 0.08

10 a 0.81 **b** 0.69 **c** 0.74 **d** 0.86

11 a 0 **b** 1 **c** $\frac{1}{5}$ **d** $\frac{1}{3}$

12 a 0.88 **b** 0.58 **c** 0.30 **d** 0.12

Exercise 9F

1 $\frac{1}{4}$ **2** $\frac{1}{3}$ **3** $\frac{1}{13}$ **4** $\frac{7}{19}$ **5** $\frac{1}{6}$

6 a $\frac{4}{17}$ **b** $\frac{4}{7}$

7 $\frac{7}{12}$

8 a $\frac{375}{500} = \frac{3}{4}$ **b** $\frac{225}{300} = \frac{3}{4}$

9 a $\frac{65}{284}$ **b** $\frac{137}{568}$ **c** $\frac{21}{65}$ **d** $\frac{61}{246}$

10 a 0.06 **b** 0.2

11 a $\frac{4}{7}$ **b** 0.3 **c** $\frac{15}{22}$

12 a 0.2 **b** 0.5 **c** 0.4

13 a 0.2 **b** $\frac{10}{27}$ **c** $\frac{1}{3}$

14 a 0.3 **b** 0.75

15 16% **16** $\frac{1}{5}$

17 a $\frac{1}{16}$ **b** $\frac{1}{169}$ **c** $\frac{1}{4}$ **d** $\frac{16}{169}$

18 a $\frac{1}{17}$ **b** $\frac{1}{221}$ **c** $\frac{13}{51}$ **d** $\frac{20}{221}$

19 $0.230\,808 \approx 0.231$

20 a $\frac{15}{28}$ **b** $\frac{1}{2}$ **c** $\frac{1}{2}$ **d** $\frac{2}{5}$
e $\frac{3}{7}$ **f** $\frac{8}{13}$ **g** $\frac{5}{28}$ **h** $\frac{3}{14}$

21 a 0.85 **b** 0.6 **c** 0.51 **d** 0.51

22 0.4, 68%

23 a i 0.444 **ii** 0.4 **iii** 0.35 **iv** 0.178 **v** 0.194
b 0.372
c i 0.478 **ii** 0.425

24 a i 0.564 **ii** 0.05 **iii** 0.12 **iv** 0.0282 **v** 0.052
b 0.081
c 0.35

25 a $\frac{1}{6}$ **b** $\frac{53}{90}$ **c** $\frac{15}{53}$

26 a $B \subseteq A$ **b** $A \cap B = \varnothing$ **c** $A \subseteq B$

Exercise 9G

1 P(male and support guns) = 0.35;
P(male) × P(support guns) = 0.6 × 0.65 = 0.39 ≠ 0.35; therefore not independent

2 P(male and prefer sport) = 0.45;
P(male) × P(prefer sport) = 0.45;
therefore independent

3 P(speeding and serious) = $\frac{42}{376} \approx 0.112$;
P(speeding) × P(serious) = $\frac{130}{376} \times \frac{103}{376} \approx$ 0.095 ≠ 0.112; therefore not independent

4 a Yes **b** Yes **c** No

5 $P(A) \times P(B) = \frac{3}{6} \times \frac{2}{6} = \frac{1}{6} = P(A \cap B)$

6 No

7 a 0.6 **b** 0.42 **c** 0.88

8 0.6

9 a 0.35 **b** 0.035 **c** 0.1225 **d** 0.025

10 a $\frac{4}{15}$ **b** $\frac{1}{15}$ **c** $\frac{133}{165}$ **d** $\frac{6}{11}$ **e** $\frac{4}{15}$
No, as $P(L \mid F) \neq P(L)$

11 $P(A) \times P(B) = \frac{20}{36} \times \frac{9}{36} = \frac{5}{36} = P(A \cap B)$

12 a 0.35 **b** 0.875

13 a $\frac{18}{65}$ **b** $\frac{12}{65}$ **c** $\frac{23}{65}$ **d** $\frac{21}{65}$ **e** $\frac{4}{65}$
f $\frac{8}{65}$ **g** $\frac{2}{15}$ **h** $\frac{8}{21}$; No

14 a i 0.75 **ii** 0.32 **iii** 0.59 **b** No **c** No

Exercise 9H

1 Approx. 0.125

2 Approx. 0.5

3 Approx. 0.033

4 Approx. 29.29

5 a Approx. 1.75 **b** Approx. 2.19

Chapter 9 review

Short-answer questions

1 a $\frac{1}{6}$ **b** $\frac{5}{6}$

2 0.007

3 a $\frac{1}{3}$ **b** $\frac{1}{4}$ **c** $\frac{1}{2}$

4 a 0.36 **b** $\frac{87}{245}$

5 $\frac{4}{15}$

6 a $\{156, 165, 516, 561, 615, 651\}$ **b** $\frac{2}{3}$ **c** $\frac{1}{3}$

7 a $\frac{5}{12}$ **b** $\frac{1}{4}$

8 a 0.036 **b** 0.027 **c** 0.189 **d** 0.729

9 a $\frac{1}{27}$ **b** $\frac{4}{27}$ **c** $\frac{4}{9}$ **d** $\frac{20}{27}$

10 No

11 a 0.5 **b** 0 **c** 1

12 a $\frac{7}{18}$ **b** $\frac{1}{2}$

13 a $\frac{2}{7}$ **b** $\frac{32}{63}$ **c** $\frac{9}{16}$

14 a 0.2 **b** 0.4

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15 a 0.7 **b** 0.3 **c** $\frac{1}{3}$ **d** $\frac{2}{3}$

16 a $B \subseteq A$ **b** $A \cap B = \varnothing$
c A and B are independent

Extended-response questions

1 a 0.15 **b** 0.148

2 a A: $\frac{3}{28}$; B: $\frac{3}{4}$ **b** A: $\frac{9}{64}$; B: $\frac{49}{64}$
c $\frac{1}{8}$ **d** $\frac{9}{58}$

3 a $\frac{4}{5}$ **b** 0.69 **c** 0.208

4 a 1.6 **b** 2.9

5 A and B: $\frac{3}{8}$; C and D: $\frac{1}{8}$

Chapter 10

Exercise 10A

1 a 11 **b** 12 **c** 37 **d** 29
2 a 60 **b** 500 **c** 350 **d** 512
3 a 128 **b** 160
4 20 **5** 63 **6** 26
7 240 **8** 260 000
9 17 576 000 **10** 30

Exercise 10B

1 a 6 **b** 120 **c** 5040 **d** 2 **e** 1 **f** 1
2 a 20 **b** 72 **c** 6 **d** 56 **e** 120 **f** 720
3 120 **4** 5040 **5** 24
6 720 **7** 720 **8** 336
9 a 5040 **b** 210
10 a 120 **b** 120
11 a 840 **b** 2401
12 a 480 **b** 1512
13 a 60 **b** 24 **c** 252
14 a 150 **b** 360 **c** 1560
15 a 720 **b** 48

Exercise 10C

1 a 3 **b** 3 **c** 6 **d** 4
2 a 10 **b** 10 **c** 35 **d** 35
3 a 190 **b** 100 **c** 4950 **d** 31 125
4 a 20 **b** 7 **c** 28 **d** 1225
5 1716 **6** 2300
7 133 784 560 **8** 8 145 060 **9** 18
10 a 5 852 925 **b** 1 744 200
11 100 386
12 a 792 **b** 336
13 a 150 **b** 75 **c** 6 **d** 462 **e** 81
14 a 8 436 285 **b** 3003 **c** 66 **d** 2 378 376

15 186 **16** 32 **17** 256 **18** 31 **19** 57
20 a 10 **b** 21

Exercise 10D

1 a 0.5 **b** 0.5
2 0.375
3 a 0.2 **b** 0.6 **c** 0.3
4 0.2 **5** $\frac{329}{858}$
6 a $\frac{2^7}{2^8 - 1} \approx 0.502$ **b** $\frac{56}{255}$ **c** $\frac{73}{85}$
7 a $\frac{5}{204}$ **b** $\frac{35}{136}$
8 a $\frac{1}{6}$ **b** $\frac{5}{6}$ **c** $\frac{17}{21}$ **d** $\frac{34}{35}$
9 a $\frac{25}{49}$ **b** $\frac{24}{49}$ **c** $\frac{3}{7}$ **d** $\frac{1}{5}$
10 a 0.659 **b** 0.341 **c** 0.096 **d** 0.282
11 a $\frac{5}{42}$ **b** $\frac{20}{21}$ **c** $\frac{15}{37}$

Exercise 10E

1 a $x^4 + 8x^3 + 24x^2 + 32x + 16$
b $16x^4 + 32x^3 + 24x^2 + 8x + 1$
c $16x^4 - 96x^3 + 216x^2 - 216x + 81$
d $27x^3 - 27x^2 + 9x - 1$
e $16x^4 - 32x^3 + 24x^2 - 8x + 1$
f $-32x^5 + 80x^4 - 80x^3 + 40x^2 - 10x + 1$
g $-243x^5 + 405x^4 - 270x^3 + 90x^2 - 15x + 1$
h $16x^4 - 96x^3 + 216x^2 - 216x + 81$

Chapter 10 review

Short-answer questions

1 a 499 500 **b** 1 000 000 **c** 1 000 000
2 648 **3** 120 **4** $8n$
5 5416 **6** 36 750 **7** 50 400
8 a 10 **b** 32
9 1200
10 a $\frac{1}{8}$ **b** $\frac{3}{8}$ **c** $\frac{3}{28}$
11 $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$

Extended-response questions

1 a 2880 **b** 80 640
2 a 720 **b** 48 **c** 336
3 a 60 **b** 45
4 a 210 **b** 100 **c** 80
5 a 1365 **b** 210 **c** 1155
6 a 3060 **b** 330 **c** 1155
7 Div. 1: 1.228×10^{-7} Div. 2: 1.473×10^{-6}
Div. 3: 2.726×10^{-5} Div. 4: 1.365×10^{-3}
Div. 5: 3.362×10^{-3}
8 a 1.290×10^{-4} **b** 6.449×10^{-4}

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Chapter 11

Short-answer questions

1 a $\frac{1}{9}$ **b** $\frac{8}{9}$

2 a $\{348, 384, 438, 483, 834, 843\}$, $n(\varepsilon) = 6$
b $\frac{2}{3}$ **c** $\frac{2}{3}$

3 a $\frac{1}{2}$ **b** $\frac{12}{13}$

4 a $\frac{1}{4}$ **b** $\frac{3}{8}$

5 a $P(1) = P(2) = P(3) = P(5) = \frac{2}{17}$, $P(4) = \frac{8}{17}$, $P(6) = \frac{1}{17}$
b $\frac{9}{17}$

6 $\frac{1}{4}$

7 a 0.6 **b** $\frac{1}{3}$

8 a $\frac{1}{2}$ **b** $\frac{3}{4}$ **c** $\frac{1}{2}$ **d** $\frac{1}{4}$

9 a 0.48 **b** 0.56

10 a $P(A \cap B) = 0.05$ **b** $P(A \mid B) = 0.25$

11 a 0.4 **b** 0.2 **c** 0.7

12 720

13 $\frac{52!}{7!45!}$

14 a $\frac{7}{22}$ **b** $\frac{21}{44}$

Extended-response questions

1 a i $\frac{15}{28}$ **ii** $\frac{37}{56}$ **iii** $\frac{43}{49}$
b i $\frac{9}{14}$ **ii** $\frac{135}{392}$

2 a $\frac{1}{2}$ **b** $\frac{13}{36}$

3 a $\frac{3}{8}$ **b** $\frac{1}{56}$ **c** $\frac{3}{28}$ **d** $\frac{6}{7}$

4 a $\frac{59}{120}$ **b** $\frac{45}{59}$

5 a i $m = 30, q = 35, s = 25$
ii $m + q = 65$
b $\frac{3}{10}$
c $\frac{7}{12}$

6 a $\frac{167}{360}$
b i $\frac{108}{193}$ **ii** $\frac{45}{193}$

7 a i $\frac{1}{9}$ **ii** $\frac{5}{18}$
b i $\frac{1}{81}$ **ii** $\frac{13}{324}$

8 a 0.084 **b** 0.52 **c** 0.68

9 a 60 **b** 8 **c** 0.1

10 a $\frac{1}{60}$ **b** $\frac{1}{5}$ **c** $\frac{3}{5}$ **d** $\frac{6}{13}$

11 a i $10\,000\ \text{cm}^2$ **ii** $400\ \text{cm}^2$ **iii** $6400\ \text{cm}^2$
b i 0.04 **ii** 0.12 **iii** 0.64
c i 0.0016 **ii** 0.000 64

12 a $\frac{7}{18}$ **b** $\frac{13}{36}$ **c** $\frac{23}{108}$

Chapter 12

Exercise 12A

1 a $\frac{\pi}{3}$ **b** $\frac{4\pi}{5}$ **c** $\frac{4\pi}{3}$ **d** $\frac{11\pi}{6}$ **e** $\frac{7\pi}{3}$ **f** $\frac{8\pi}{3}$

2 a 120° **b** 150° **c** 210° **d** 162°
e 100° **f** 324° **g** 220° **h** 324°

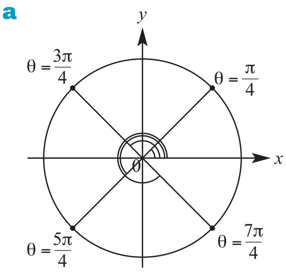
3 a 34.38° **b** 108.29° **c** 166.16° **d** 246.94°
e 213.14° **f** 296.79° **g** 271.01° **h** 343.78°

4 a 0.66 **b** 1.27 **c** 1.87 **d** 2.81
e 1.47 **f** 3.98 **g** 2.38 **h** 5.74

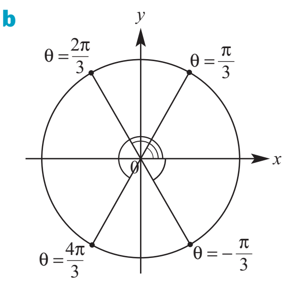
5 a −60° **b** −720° **c** −540° **d** −180°
e 300° **f** −330° **g** 690° **h** −690°

6 a -2π **b** -3π **c** $-\frac{4\pi}{3}$ **d** -4π
e $-\frac{11\pi}{6}$ **f** $-\frac{7\pi}{6}$

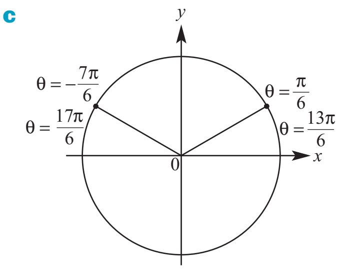
7 a



b



c



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Exercise 12B

1 a 0, 1 **b** −1, 0 **c** 1, 0 **d** 1, 0
e 0, −1 **f** 1, 0 **g** −1, 0 **h** 0, 1

2 a 0.95 **b** 0.75 **c** −0.82 **d** 0.96
e −0.5 **f** −0.03 **g** −0.86 **h** 0.61

3 a 0, −1 **b** −1, 0 **c** −1, 0 **d** −1, 0
e −1, 0 **f** 0, −1 **g** 0, −1 **h** 0, −1

Exercise 12C

1 a 0 **b** 0 **c** Undefined **d** 0
e Undefined **f** Undefined

2 a −34.23 **b** −2.57 **c** −0.97 **d** −1.38
e 0.95 **f** 0.75 **g** 1.66

3 a 0 **b** 0 **c** 0 **d** 0 **e** 0 **f** 0

Exercise 12D

1 a 67.98° **b** 4.5315 **c** 2.5357
d 6.4279 **e** 50.19° **f** 3.4202
g 2.3315 **h** 6.5778 **i** 6.5270

Exercise 12E

1 a −0.42 **b** −0.7 **c** −0.42 **d** −0.38
e 0.42 **f** −0.38 **g** −0.7 **h** 0.7

2 a $\frac{5\pi}{6}$ **b** $\frac{7\pi}{6}$ **c** $\frac{11\pi}{6}$

3 a $-\frac{1}{2}$ **b** $\frac{\sqrt{3}}{2}$ **c** $\frac{1}{2}$ **d** $\frac{-\sqrt{3}}{2}$
e $-\sqrt{3}$ **f** $-\sqrt{3}$

4 a $-\frac{\sqrt{3}}{2}$ **b** $\frac{1}{2}$ **c** $-\sqrt{3}$ **d** $-\frac{\sqrt{3}}{2}$ **e** $-\frac{1}{2}$

5 a $a = 0.7660$, $b = 0.6428$
b $c = -0.7660$, $d = 0.6428$
c i $\cos 140° = -0.7660$, $\sin 140° = 0.6428$
ii $\cos 140° = -\cos 40°$

6 a −0.7 **b** −0.6 **c** −0.4 **d** −0.6
e −0.7 **f** −0.7 **g** 0.4 **h** 0.6

7 a 120° **b** 240° **c** −60° **d** 120°
e 240° **f** 300°

Exercise 12F

1 a $\sin = \frac{\sqrt{3}}{2}$, $\cos = -\frac{1}{2}$, $\tan = -\sqrt{3}$

b $\sin = \frac{1}{\sqrt{2}}$, $\cos = -\frac{1}{\sqrt{2}}$, $\tan = -1$

c $\sin = -\frac{1}{2}$, $\cos = -\frac{\sqrt{3}}{2}$, $\tan = \frac{1}{\sqrt{3}}$

d $\sin = -\frac{\sqrt{3}}{2}$, $\cos = -\frac{1}{2}$, $\tan = \sqrt{3}$

e $\sin = -\frac{1}{\sqrt{2}}$, $\cos = \frac{1}{\sqrt{2}}$, $\tan = -1$

f $\sin = \frac{1}{2}$, $\cos = \frac{\sqrt{3}}{2}$, $\tan = \frac{1}{\sqrt{3}}$

g $\sin = \frac{\sqrt{3}}{2}$, $\cos = \frac{1}{2}$, $\tan = \sqrt{3}$

h $\sin = -\frac{1}{\sqrt{2}}$, $\cos = -\frac{1}{\sqrt{2}}$, $\tan = 1$

i $\sin = \frac{\sqrt{3}}{2}$, $\cos = \frac{1}{2}$, $\tan = \sqrt{3}$

j $\sin = -\frac{\sqrt{3}}{2}$, $\cos = \frac{1}{2}$, $\tan = -\sqrt{3}$

2 a $\frac{\sqrt{3}}{2}$ **b** $-\frac{1}{\sqrt{2}}$ **c** $-\frac{1}{\sqrt{3}}$
d $-\frac{1}{2}$ **e** $-\frac{1}{\sqrt{2}}$ **f** $\sqrt{3}$
g $-\frac{\sqrt{3}}{2}$ **h** $\frac{1}{\sqrt{2}}$ **i** $-\frac{1}{\sqrt{3}}$

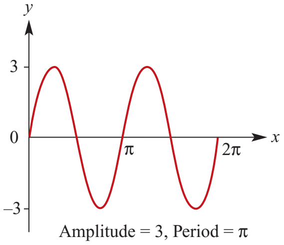
3 a $-\frac{\sqrt{3}}{2}$ **b** $-\frac{1}{\sqrt{2}}$ **c** $\frac{1}{\sqrt{3}}$
d Undefined **e** 0 **f** $-\frac{1}{\sqrt{2}}$
g $\frac{1}{\sqrt{2}}$ **h** −1

Exercise 12G

1 a i 2π **ii** 2 **b i** π **ii** 3 **c i** $\frac{2\pi}{3}$ **ii** $\frac{1}{2}$
d i 4π **ii** 3 **e i** $\frac{2\pi}{3}$ **ii** 4 **f i** $\frac{\pi}{2}$ **ii** $\frac{1}{2}$
g i 4π **ii** 2 **h i** 2 **ii** 2 **i i** 4 **ii** 3

2 a Dilation of factor 3 parallel to the y-axis; Amplitude = 3; Period = 2π
b Dilation of factor $\frac{1}{5}$ parallel to the x-axis; Amplitude = 1; Period $\frac{2\pi}{5}$
c Dilation of factor 3 parallel to the x-axis; Amplitude = 1; Period = 6π
d Dilation of factor 2 parallel to the y-axis and dilation of factor $\frac{1}{5}$ parallel to the x-axis; Amplitude = 2; Period = $\frac{2\pi}{5}$

3 a



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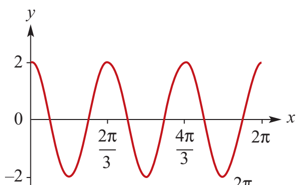
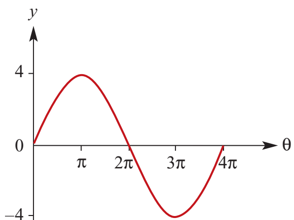
b



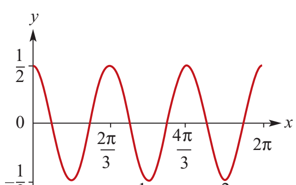
Amplitude = 2, Period = $\frac{2\pi}{3}$

c



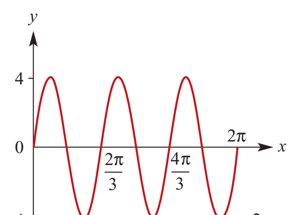
Amplitude = 4, Period = 4π

d



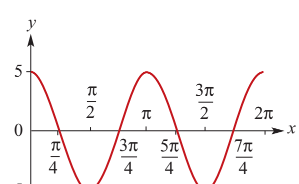
Amplitude = $\frac{1}{2}$, Period = $\frac{2\pi}{3}$

e



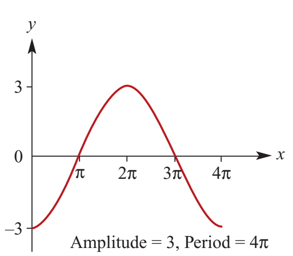
Amplitude = 4, Period = $\frac{2\pi}{3}$

f



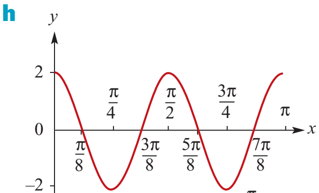
Amplitude = 5, Period = π

g



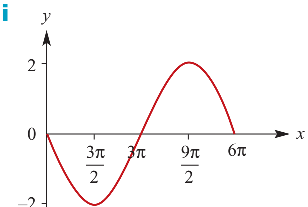
Amplitude = 3, Period = 4π

h



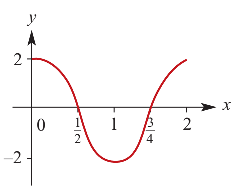
Amplitude = 2, Period = $\frac{\pi}{2}$

i

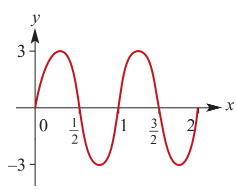


Amplitude = 2, Period = 6π

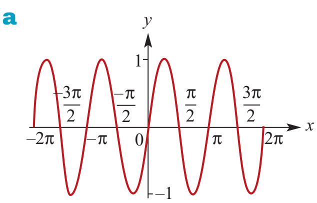
4 a



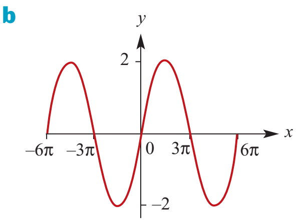
b



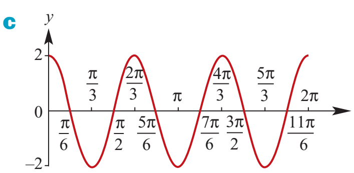
5 a



b



c



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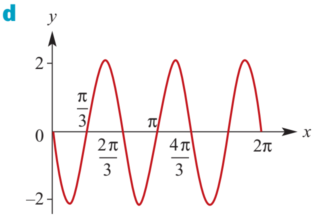
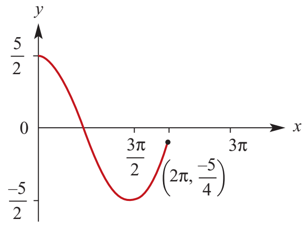
d



6



7 a Dilation of factor $\frac{1}{5}$ parallel to the x-axis and reflection in the x-axis; Amplitude = 1; Period = $\frac{2\pi}{5}$

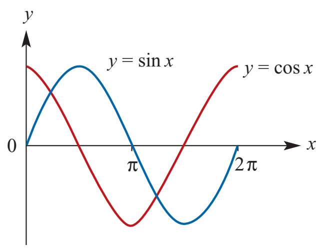
b Reflection in the y-axis; Amplitude = 1; Period = 2π

c Dilation of factor 3 parallel to the x-axis and dilation of factor 2 parallel to the y-axis; Amplitude = 2; Period = 6π

d Dilation of factor 2 parallel to the x-axis, dilation of factor 4 parallel to the y-axis and reflection in the x-axis; Amplitude = 4; Period = 4π

e Dilation of factor 3 from the y-axis, dilation of factor 2 parallel to the y-axis and reflection in the y-axis; Amplitude = 2; Period = 6π

8 a



b $\frac{\pi}{4}, \frac{5\pi}{4}$

Exercise 12H

1 a $\frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$ **b** $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$

c $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$

2 a 0.93, 2.21 **b** 4.30, 1.98 **c** 3.50, 5.93

d 0.41, 2.73 **e** 2.35, 3.94 **f** 1.77, 4.51

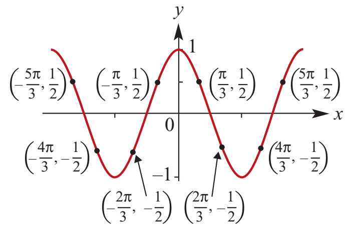
3 a 150, 210 **b** 30, 150 **c** 120, 240

d 120, 240 **e** 60, 120 **f** 45, 135

4 a $\frac{\pi}{6}, \frac{11\pi}{6}$ **b** $\frac{5\pi}{4}, \frac{7\pi}{4}$ **c** $\frac{\pi}{4}, \frac{7\pi}{4}$

5 a $\frac{3\pi}{4}, -\frac{3\pi}{4}$ **b** $\frac{\pi}{3}, \frac{2\pi}{3}$ **c** $\frac{2\pi}{3}, -\frac{2\pi}{3}$

6



7 a $\frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$

b $\frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$

c $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

d $\frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{15\pi}{12}, \frac{21\pi}{12}, \frac{23\pi}{12}$

e $\frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}$

f $\frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$

8 a 2.034, 2.678, 5.176, 5.820

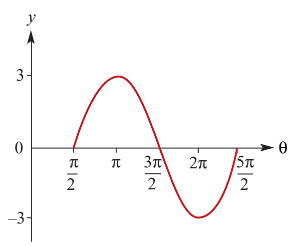
b 1.892, 2.820, 5.034, 5.961

c 0.580, 2.562, 3.721, 5.704

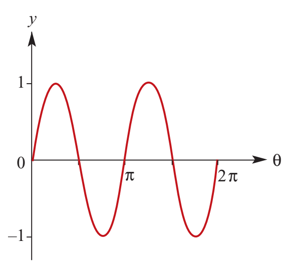
d 0.309, 1.785, 2.403, 3.880, 4.498, 5.974

Exercise 12I

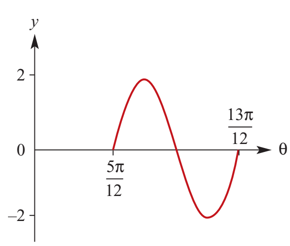
1 a Period = 2π, Amplitude = 3, $y = \pm 3$



b Period = π, Amplitude = 1, $y = \pm 1$



c Period = $\frac{2\pi}{3}$, Amplitude = 2, $y = \pm 2$



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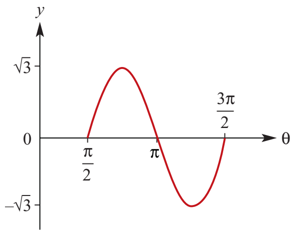
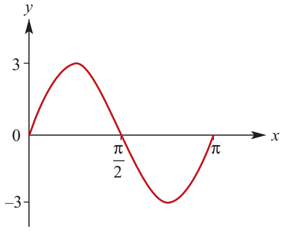
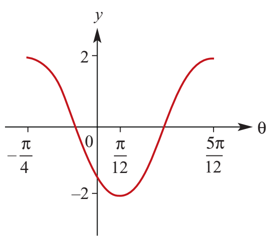
d Period = π, Amplitude = $\sqrt{3}$, $y = \pm\sqrt{3}$



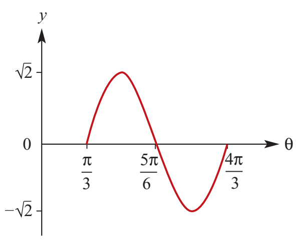
e Period = π, Amplitude = 3, $y = \pm 3$



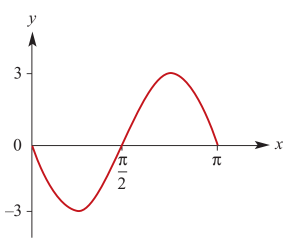
f Period = $\frac{2\pi}{3}$, Amplitude = 2, $y = \pm 2$



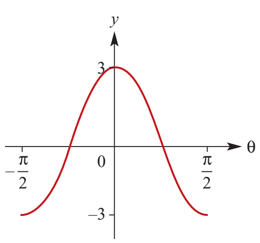
g Period = π, Amplitude = $\sqrt{2}$, $y = \pm\sqrt{2}$



h Period = π, Amplitude = 3, $y = \pm 3$

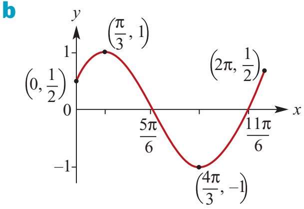


i Period = π, Amplitude = 3, $y = \pm 3$



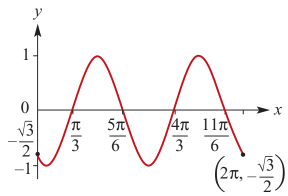
2 a $f(0) = \frac{1}{2}$, $f(2\pi) = \frac{1}{2}$

b



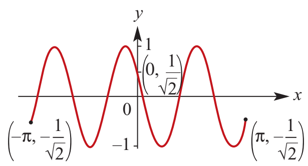
3 a $f(0) = -\frac{\sqrt{3}}{2}$, $f(2\pi) = -\frac{\sqrt{3}}{2}$

b



4 a $f(-\pi) = -\frac{1}{\sqrt{2}}$, $f(\pi) = -\frac{1}{\sqrt{2}}$

b



5 a $y = 3\sin\left(\frac{x}{2}\right)$

b $y = 3\sin(2x)$

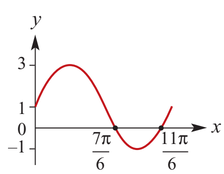
c $y = 2\sin\left(\frac{x}{3}\right)$

d $y = \sin 2\left(x - \frac{\pi}{3}\right)$

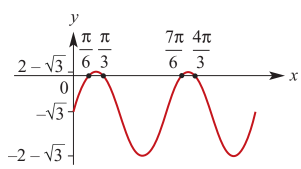
e $y = \sin \frac{1}{2}\left(x + \frac{\pi}{3}\right)$

Exercise 12J

1 a



b



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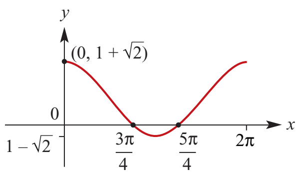
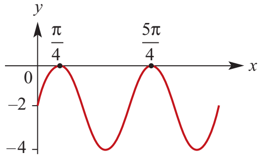
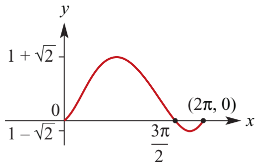
c



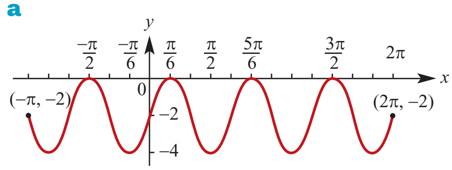
d



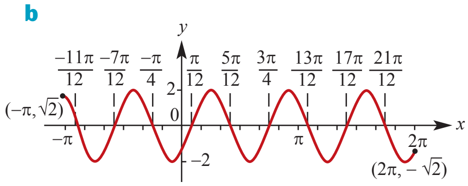
e



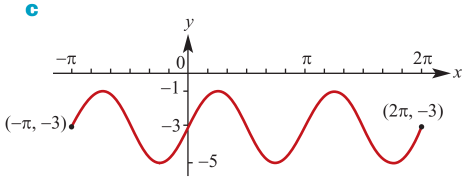
2 a



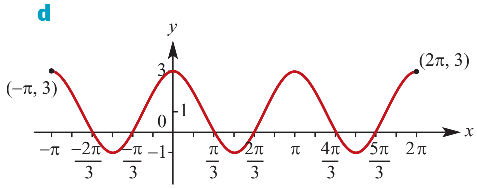
b



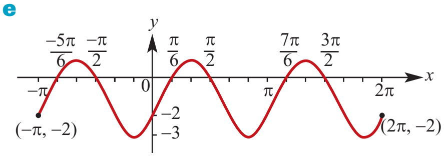
c



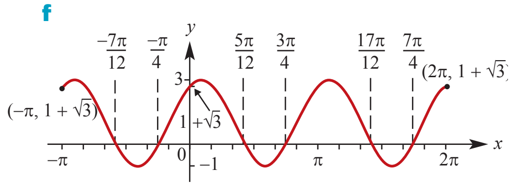
d



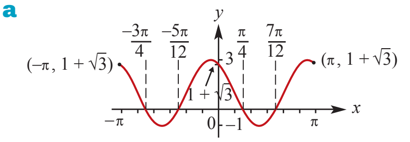
e



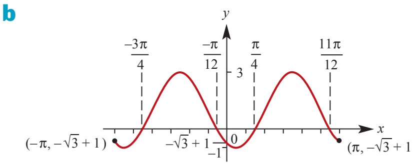
f



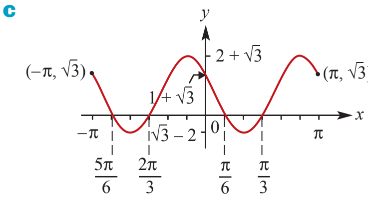
3 a



b



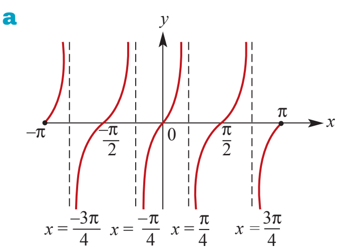
c



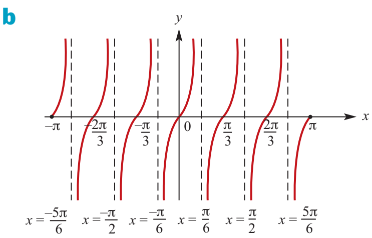
Exercise 12K

1 a $\frac{\pi}{4}$ **b** $\frac{3\pi}{2}$ **c** $\frac{\pi}{2}$

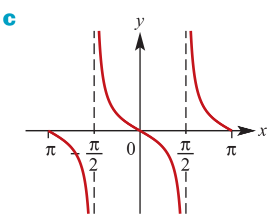
2 a



b



c



3 a $\frac{-7\pi}{8}, \frac{-3\pi}{8}, \frac{\pi}{8}, \frac{5\pi}{8}$

b $\frac{-17\pi}{18}, \frac{-11\pi}{18}, \frac{-5\pi}{18}, \frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}$

c $\frac{-5\pi}{6}, \frac{-\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}$

d $\frac{-13\pi}{18}, \frac{-7\pi}{18}, \frac{-\pi}{18}, \frac{5\pi}{18}, \frac{11\pi}{18}, \frac{17\pi}{18}$

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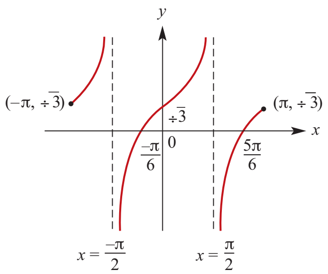
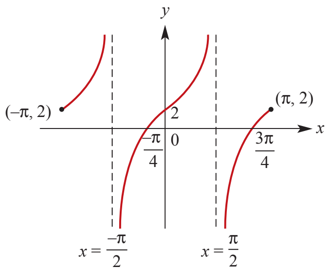
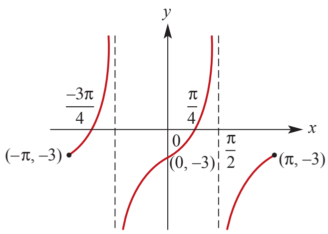
4 a



b



c



Exercise 12L

1 a 0.6 **b** 0.6 **c** −0.7 **d** 0.3
e −0.3 **f** $\frac{10}{7}$ **g** −0.3 **h** 0.6
i −0.6 **j** −0.3

2 a $\frac{\pi}{3}$ **b** $\frac{\pi}{3}$ **c** $\frac{5\pi}{12}$ **d** $\frac{\pi}{14}$

3 $\sin x = \frac{-4}{5}$, $\tan x = \frac{-4}{3}$

4 $\cos x = \frac{-12}{13}$, $\tan x = \frac{-5}{12}$

5 $\sin x = \frac{-2\sqrt{6}}{5}$, $\tan x = -2\sqrt{6}$

Exercise 12M

1 a $\frac{1+\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}+\sqrt{6}}{4}$ **b** $\frac{1-\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}-\sqrt{6}}{4}$

2 a $\frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$ **b** $\frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3}$

3 a $\frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$

b $\frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$

c $\frac{1-\sqrt{3}}{1+\sqrt{3}} = -2+\sqrt{3}$

4 $\sin(u+v) = \frac{63}{65}$

5 a $\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta$ **b** $\frac{1}{\sqrt{2}}(\cos\varphi + \sin\varphi)$

c $\frac{\tan\theta + \sqrt{3}}{1-\sqrt{3}\tan\theta}$ **d** $\frac{1}{\sqrt{2}}(\sin\theta - \cos\theta)$

6 a $\sin u$ **b** $\cos u$

7 a $-\frac{119}{169}$ **b** $\frac{24}{25}$ **c** $\frac{24}{7}$

d $-\frac{33}{65}$ **e** $-\frac{16}{65}$

8 a $\frac{63}{16}$ **b** $-\frac{24}{7}$ **c** $\frac{56}{65}$ **d** $\frac{24}{25}$

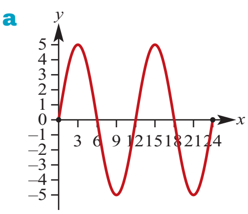
9 a $\frac{7}{25}$ **b** $\frac{3}{5}$ **c** $\frac{117}{44}$ **d** $\frac{-336}{625}$

10 a $-\frac{\sqrt{3}}{2}$ **b** $-\frac{1}{2}$

11 a $1-\sin(2\theta)$ **b** $\cos(2\theta)$

Exercise 12N

1 a



b $t = 3$ and $t = 15$

c 5 m above mean sea level

d $\frac{5\sqrt{3}}{2}$ m above mean sea level

e $\frac{5\sqrt{3}}{2}$ m above mean sea level

f $t \in [1, 5] \cup [13, 17]$

2 a 5 metres **b** 1 metres

c $t = 0.524, 2.618$ or 4.712 seconds

d $t = 0, 1.047$ or 2.094 seconds

e Particle oscillates between $x = 1$ and $x = 5$

3 a 7 m **b** 3 m

c $t = \frac{1}{4}, \frac{5}{4}, \frac{9}{4}, \frac{13}{4}$ or $\frac{17}{4}$

d $t = \frac{1}{12}, \frac{5}{12}, \frac{13}{12}, \frac{17}{12}, \frac{25}{12}$ or $\frac{29}{12}$

e Particle oscillates between $x = 3$ and $x = 7$

4 a i 10 **ii** $10 + 5\sqrt{3}$ **iii** $10 + 5\sqrt{3}$

iv $10 - 5\sqrt{3}$ **v** $10 - 5\sqrt{3}$

b 6 seconds

c 20 metres

d $\frac{1}{2}, \frac{5}{2}, \frac{13}{2}, \frac{17}{2}$ seconds

e $\frac{7}{2}, \frac{11}{2}, \frac{19}{2}, \frac{23}{2}$ seconds

Answers 12L → 12N

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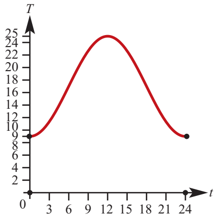
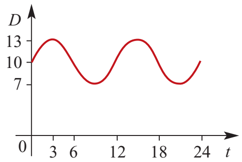
5 a 9°C **b** Maximum = 25°C; Minimum = 9°C
c Between 7:28 and 16:32
d

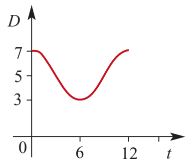


6 a



b $\{ t : D(t) \geq 8.5 \} = [0, 7] \cup [11, 19] \cup [23, 24]$
c 12.9 m

7 a $p = 5, q = 2$ **b**



c A ship can enter 2 hours after low tide

Chapter 12 review

Short-answer questions

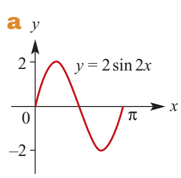
1 a $\frac{11\pi}{6}$ **b** $\frac{9\pi}{2}$ **c** 6π **d** $\frac{23\pi}{4}$ **e** $\frac{3\pi}{4}$
f $\frac{9\pi}{4}$ **g** $\frac{13\pi}{6}$ **h** $\frac{7\pi}{3}$ **i** $\frac{4\pi}{9}$

2 a 150° **b** 315° **c** 495° **d** 45°
e 1350° **f** −135° **g** −45° **h** −495°
i −1035°

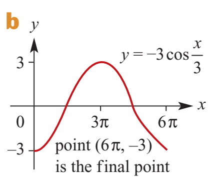
3 a $\frac{1}{\sqrt{2}}$ **b** $\frac{1}{\sqrt{2}}$ **c** $-\frac{1}{2}$ **d** $-\frac{\sqrt{3}}{2}$
e $\frac{\sqrt{3}}{2}$ **f** $-\frac{1}{2}$ **g** $\frac{1}{2}$ **h** $-\frac{1}{\sqrt{2}}$

4 a $2, 4\pi$ **b** $3, \frac{\pi}{2}$ **c** $\frac{1}{2}, \frac{2\pi}{3}$ **d** $3, \pi$
e $4, 6\pi$ **f** $\frac{2}{3}, 3\pi$

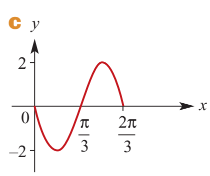
5 a



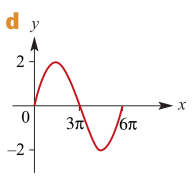
b



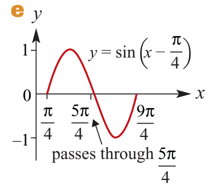
c



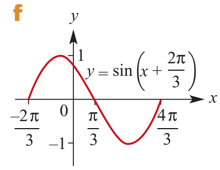
d



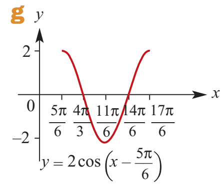
e



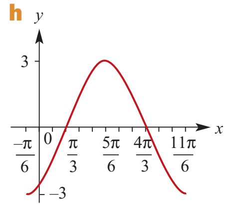
f



g



h



6 a $-\frac{2\pi}{3}, -\frac{\pi}{3}$ **b** $-\frac{\pi}{3}, -\frac{\pi}{6}, \frac{2\pi}{3}, \frac{5\pi}{6}$
c $\frac{\pi}{6}, \frac{3\pi}{2}$ **d** $\frac{7\pi}{6}$ **e** $\frac{\pi}{2}, \frac{7\pi}{6}$

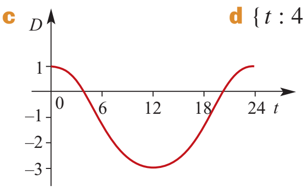
7 a $\frac{140}{221}$ **b** $\frac{-21}{221}$ **c** $\frac{171}{140}$

8 $\frac{2}{9}$

Extended-response questions

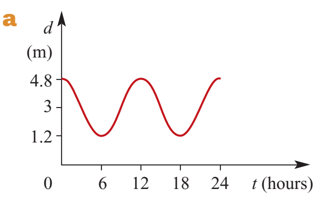
1 a i 1.83×10^{-3} hours **ii** 11.79 hours
b 26 April ($t = 3.86$), 14 August ($t = 7.48$)

2 a 19.5°C **b** $D = -1 + 2\cos\left(\frac{\pi t}{12}\right)$
c



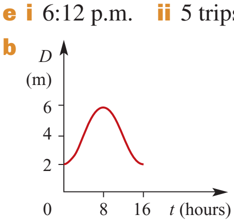
d $\{ t : 4 < t < 20 \}$

3 a



b 3 a.m., 3 p.m., 3 a.m.
c 9 a.m., 9 p.m. **d** 10:03 a.m.
e i 6:12 p.m. **ii** 5 trips

4 b



c $t = 16$ (8 p.m.)
d $t = 4$ and $t = 12$ (8 a.m. and 4 p.m.)
e i 1.5 m **ii** 2.086 m
f 9 hours 17 minutes

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Chapter 13

Exercise 13A

1 **a** A and C (SAS) **b** All of them (AAS) **c** A and B (SSS)

2 **a** 4.10 **b** 0.87 **c** 2.94 **d** 4.08 **e** 33.69° **f** 11.92

3 $\frac{40}{\sqrt{3}} = \frac{40\sqrt{3}}{3}$ cm

4 66.42°, 66.42°, 47.16°

5 23 m

6 **a** 9.59° **b** $\sqrt{35}$ m

7 **a** 60° **b** $10\sqrt{3}$ m

8 **a** 6.84 m **b** 6.15 m

9 12.51° **10** 182.7 m **11** 1451 m

12 **a** $5\sqrt{2}$ cm **b** 90°

13 3.07 cm **14** 37.8 cm **15** 31.24 m

16 4.38 m **17** 57.74 m

18 $\frac{2\sqrt{3}}{2 - \sqrt{3}} \approx 12.93$ m **19** $\frac{10}{1 + \sqrt{3}} \approx 3.66$

Exercise 13B

1 **a** 8.15 **b** 3.98 **c** 11.75 **d** 9.46

2 **a** 56.32° **b** 36.22° **c** 49.54° **d** 98.16° or 5.84°

3 **a** $A = 48°$, $b = 13.84$, $c = 15.44$
b $a = 7.26$, $C = 56.45°$, $c = 6.26$
c $B = 19.8°$, $b = 4.66$, $c = 8.27$
d $C = 117°$, $b = 24.68$, $c = 34.21$
e $C = 30°$, $a = 5.40$, $c = 15.56$

4 **a** $B = 59.12°$, $A = 72.63°$, $a = 19.57$ or $B = 120.88°$, $A = 10.87°$, $a = 3.87$
b $C = 26.69°$, $A = 24.31°$, $a = 4.18$
c $B = 55.77°$, $C = 95.88°$, $c = 17.81$ or $B = 124.23°$, $C = 27.42°$, $c = 8.24$

5 554.26 m

6 35.64 m

7 1659.86 m

8 **a** 26.60 m **b** 75.12 m

Exercise 13C

1 5.93 cm

2 $\angle ABC = 97.90°$, $\angle ACB = 52.41°$

3 **a** 26 **b** 11.74 **c** 49.29° **d** 73 **e** 68.70 **f** 47.22° **g** 7.59 **h** 38.05°

4 2.626 km **5** 3.23 km **6** 55.93 cm

7 **a** 8.23 cm **b** 3.77 cm

8 **a** 7.326 cm **b** 5.53 cm

9 **a** 87.61 m **b** 67.7 m

Exercise 13D

1 **a** 11.28 cm^2 **b** 15.10 cm^2 **c** 10.99 cm^2 **d** 9.58 cm^2

2 **a** 6.267 cm^2 **b** 15.754 cm^2 **c** 19.015 cm^2 **d** 13.274 cm^2 **e** 24.105 cm^2 or 29.401 cm^2 **f** 2.069 cm^2

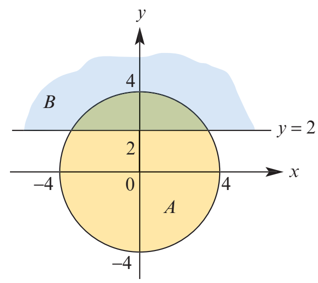
Exercise 13E

1 45.81 cm

2 **a** 95.5° **b** 112.88°

3 **a** 6.20 cm **b** 2.73 cm^2

4



Area of $A \cap B = 9.83$ square units

5 61.42 cm^2

6 **a** 125.66 m **b** 41.96%

7 **a** 10.47 m **b** 20.94 m^2

8 6.64 cm^2

9 $r = 7$ cm, $\theta = \left(\frac{18}{7}\right)^c$ or $r = 9$ cm, $\theta = \left(\frac{14}{9}\right)^c$

10 247.33 cm

Exercise 13F

1 400.10 m **2** 34.77 m **3** 575.18 m **4** 109.90 m **5** 16.51 m **6** 056°

7 **a** 034° **b** 214°

8 **a** 3583.04 m **b** 353°

9 027° **10** 113° **11** 22.01°

12 **a** $\angle BAC = 49°$ **b** 264.24 km

13 10.63 km

Chapter 13 review

Short-answer questions

1 $\frac{1}{8}$

2 $\frac{2}{5}$

3 **a** 2 **b** 0.7^c

4 $10\sqrt{3}$ cm

5 **a** $5\sqrt{3}$ cm **b** $\frac{25\sqrt{3}}{4}$ cm^2 **c** $\frac{105}{4}$ cm^2 **d** $\frac{5(21 + 5\sqrt{3})}{4}$ cm^2

6 143° **7** $\frac{17}{28}$ **8** $\frac{3\sqrt{93}}{31}$ **9** $\left(\frac{11}{6}\right)^c$

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10 $\sqrt{181}$ km

11 $AC = \dfrac{12\sqrt{3}}{5}$ km, $BC = 2.4$ km

12 180 cm^2

13 a $\dfrac{2\sqrt{3}\pi}{3}$ cm **b** $2\pi\text{ cm}^2$

14 $16\sin\left(\dfrac{5}{4}\right) - 10 \approx 5.18\text{ cm}^2$

Extended-response questions

1 a $\angle ACB = 12°,\ \angle CBO = 53°,\ \angle CBA = 127°$
b 189.33 m **c** 113.94 m

2 a 4.77 cm **b** 180 cm^2 **c** 9.55 cm

3 a $\angle TAB = 3°,\ \angle ABT = 97°,\ \angle ATB = 80°$
b 2069.87 m **c** 252.25 m

4 a 184.78 m **b** 199.71 m **c** 14.93 m

5 a 370.17 m **b** 287.94 m **c** 185.08 m

Chapter 14

Exercise 14A

1 a x^5 **b** $8x^7$ **c** x^2 **d** $2x^3$ **e** a^6
f 2^6 **g** x^2y^2 **h** x^4y^6 **i** $\dfrac{x^3}{y^3}$ **j** $\dfrac{x^6}{y^4}$

2 a 3^{17} **b** x^7y^5 **c** 3^{4x+3} **d** $30a^5b^6$

3 a x^2y **b** b^{4x+1} **c** $4a^5b$

4 a $\dfrac{1}{49}$ **b** 64 **c** $\dfrac{8}{125}$

5 a b^{10} **b** 729 **c** b^4

6 a $\dfrac{27a^8b}{16}$ **b** $\dfrac{125b^6}{c^9}$

7 a 64 **b** $-27a^3$ **c** $-96a^3$

8 a 2^{-2n} **b** 2^4 **c** $\dfrac{5^{2n}}{2^{2n}}$

9 a x^9 **b** 2^{16} **c** 3^{17} **d** q^8p^9 **e** $a^{11}b^3$
f 2^8x^{18} **g** $m^{11}n^{12}p^{-2}$ **h** $2a^5b^{-2}$

10 a x^2y^3 **b** $8a^8b^3$ **c** x^5y^2 **d** $\dfrac{9}{2}x^2y^3$

11 a $\dfrac{1}{n^4p^5}$ **b** $\dfrac{2x^8z}{y^4}$ **c** $\dfrac{b^5}{a^5}$ **d** $\dfrac{a^3b}{c}$
e $a^{n+2}b^{n+1}c^{n-1}$

12 a 3^{17n} **b** 2^{3-n} **c** $\dfrac{3^{4n-11}}{2^2}$
d $2^{n+1}3^{3n-1}$ **e** 5^{3n-2} **f** $2^{3x-3} \times 3^{-4}$
g $3^{6-n} \times 2^{-5n}$ **h** $3^3 = 27$ **i** 6

13 a $2^{12} = 4096$ **b** $5^5 = 3125$ **c** $3^3 = 27$

Exercise 14B

1 a 25 **b** 27 **c** $\dfrac{1}{9}$ **d** 16
e $\dfrac{1}{2}$ **f** $\dfrac{1}{4}$ **g** $\dfrac{1}{25}$ **h** 16
i $\dfrac{1}{10\,000}$ **j** 1000 **k** 27 **l** $\dfrac{3}{5}$
m -2 **n** $\dfrac{1}{625}$ **o** 16 **p** 343

2 a $a^{\frac{1}{6}}b^{-\frac{7}{6}}$ **b** $a^{-6}b^{\frac{9}{2}}$ **c** $3^{-\frac{7}{3}} \times 5^{-\frac{7}{6}}$
d $\dfrac{1}{4}$ **e** x^6y^{-8} **f** $a^{\frac{14}{15}}$

3 a $(2x-1)^{\frac{3}{2}}$ **b** $(x-1)^{\frac{5}{2}}$ **c** $(x^2+1)^{\frac{3}{2}}$
d $(x-1)^{\frac{7}{2}}$ **e** $x(x-1)^{-\frac{1}{2}}$ **f** $(5x^2+1)^{\frac{4}{3}}$

Exercise 14C

1 a 4.78×10 **b** 6.728×10^3
c 7.923×10 **d** 4.358×10^4
e 2.3×10^{-3} **f** 5.6×10^{-7}
g $1.200\,034 \times 10$ **h** 5.0×10^7
i 2.3×10^{10} **j** 1.3×10^{-9}
k 1.65×10^5 **l** 1.4567×10^{-5}

2 a 1.0×10^{-8} **b** 1.67×10^{-24}
c 5×10^{-5} **d** $1.853\,18 \times 10^3$
e 9.461×10^{12} **f** 2.998×10^{10}

3 a 81 280 000 000 000 **b** 270 000 000
c 0.000 000 000 000 28

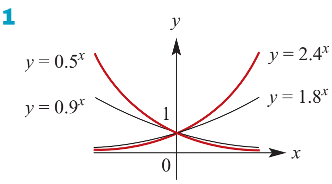
4 a 4.569×10^2 **b** 3.5×10^4
c 5.6791×10^3 **d** 4.5×10^{-2}
e 9.0×10^{-2} **f** 4.5682×10^3

5 a 0.000 0567 **b** $\dfrac{262}{2625}$

6 a 11.8 **b** 4.76×10^7

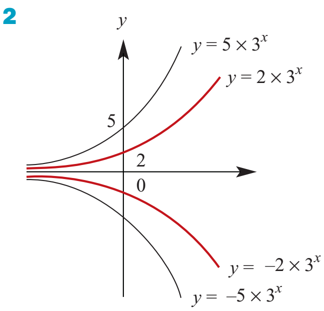
Exercise 14D

1



- All pass through (0, 1)
- Horizontal asymptote $y = 0$
- Increasing for base > 1
- Decreasing for base < 1

2



- $y = a \times b^x$ has y-axis intercept at $(0, a)$
- Horizontal asymptote $y = 0$
- Graphs c and d are reflections in the x-axis of graphs a and b

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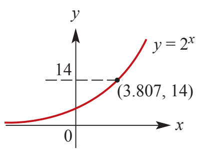
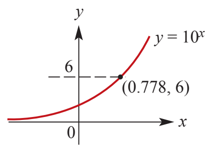
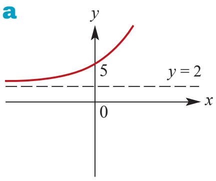
3 $x = 3.807$



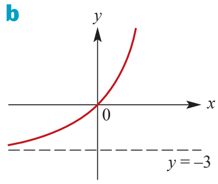
4 $x = 0.778$



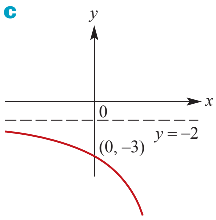
5 a



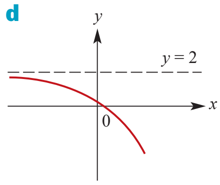
b



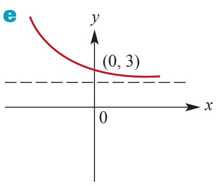
c



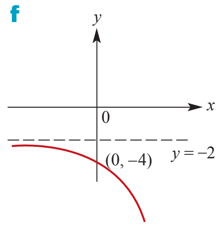
d



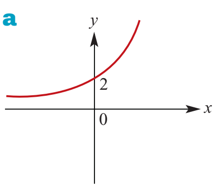
e



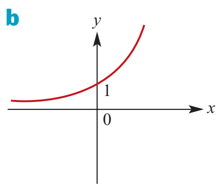
f



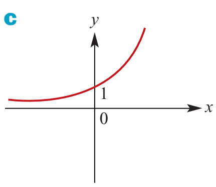
6 a



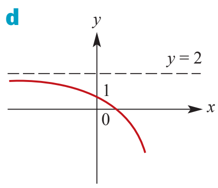
b



c



d



Exercise 14E

1 a 3 **b** 3 **c** $\frac{1}{2}$ **d** $\frac{3}{4}$ **e** $\frac{1}{3}$
f 4 **g** 2 **h** 3 **i** 3

2 a 1 **b** 2 **c** $-\frac{3}{2}$ **d** $\frac{4}{3}$ **e** −1
f 8 **g** 3 **h** −4 **i** 8 **j** 4
k $3\frac{1}{2}$ **l** 6 **m** $7\frac{1}{2}$

3 a $\frac{4}{5}$ **b** $\frac{3}{2}$ **c** $5\frac{1}{2}$

4 a 0 **b** 0, −2 **c** 1, 2 **d** 0, 1

5 a 2.32 **b** 1.29 **c** 1.26 **d** 1.75

Exercise 14F

1 a $N = 1000 \times 2^{\frac{t}{15}}$ **b** 50 minutes

2 79 726 years

3 7575 years

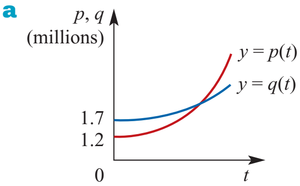
4 a 535 millibars **b** 7331 metres

5 22 hours

6 6.4°C

7 $t > 18.668\ldots$

8 a



b i $t = 12.56\ldots$ (mid 1962)
ii $t = 37.56\ldots$ (mid 1987)

9 a $y = 3 \times 5^x$ **b** $y = 4 \times \left(\frac{1}{2}\right)^x$
c $y = 5 \times \left(\frac{3}{2}\right)^x$

10 a $k = 0.0969\ldots$ **b** 7.212 hours

11 a $N = 1000 \times 10^{\frac{t}{5}}$ **b** 210 minutes
c 15 hours **d** 251 189 bacteria

12 $a = 6 \times \left(\frac{10}{3}\right)^{-\frac{2}{3}}$ and $k = 0.1742\ldots$

13 $y = 1.5 \times 0.575^x$

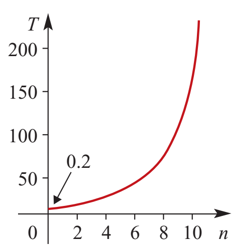
14 $p = 2.5 \times 1.35^t$

15 a

Cuts, n	Sheets	Thickness, T (mm)
0	1	0.2
1	2	0.4
2	4	0.8
3	8	1.6
4	16	3.2
5	32	6.4
6	64	12.8
7	128	25.6
8	256	51.2
9	512	102.4
10	1024	204.8

b $T = 0.2 \times 2^n$

c



d 214 748.4 m

16 $d_0 = 41.92$, $m = 0.094$

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Chapter 14 review

Short-answer questions

1 a a^4 **b** $\frac{1}{b^2}$ **c** $\frac{1}{m^2n^2}$ **d** $\frac{1}{ab^6}$

e $\frac{3a^6}{2}$ **f** $\frac{5}{3a^2}$ **g** a^3 **h** $\frac{n^8}{m^4}$

i $\frac{1}{p^2q^4}$ **j** $\frac{8}{5a^{11}}$ **k** $2a$ **l** $a^2 + a^6$

2 3.84×10^8

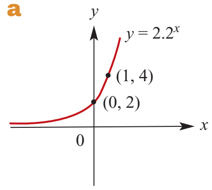
3 10^6 seconds or $11\frac{31}{54}$ days

4 50

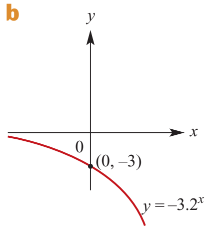
5 a $x = 3$ **b** $x = 3$ or $x = 0$

c $x = 1$ **d** $x = 2$ or $x = 3$

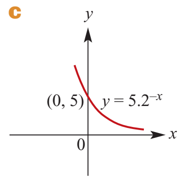
6 a



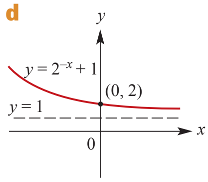
b



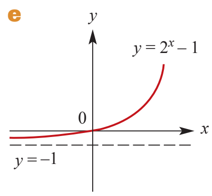
c



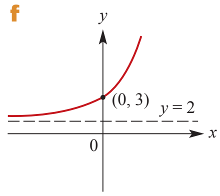
d



e



f



7 $a = \frac{1}{2}$

Extended-response questions

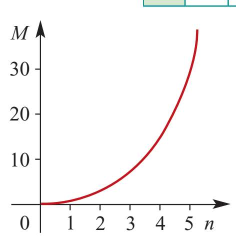
1 a

n	1	2	3	4
M	1	3	7	15

b $M = 2^n - 1$

n	5	6	7
M	31	63	127

c



d

Three discs	1	2	3
Times moved	4	2	1

Four discs	1	2	3	4
Times moved	8	4	2	1

2 $n = 2$

3 a 1.9×10^{-8} N **b** $m_1 = \frac{Fr^2 10^{11}}{6.67m_2}$

c 9.8×10^{24} kg

4 a $\left(\frac{1}{2}\right)^{3n}$ **b** $\left(\frac{1}{2}\right)^{5n-2}$ **c** $n = 3$

5 a $729\left(\frac{1}{4}\right)^n$ **b** $128\left(\frac{1}{2}\right)^n$

c 4 times

6 a Batch 1: $15(0.95)^n$; Batch 2: $20(0.94)^n$

b 32 years

7 a 13.81 years **b** 7.38 years

8 a Temperature $= 87.065 \times 0.94^t$

b i 87.1°C **ii** 18.56°C

c Temperature $= 85.724 \times 0.94^t$

d i 85.72°C **ii** 40.82°C

e 28.19 minutes

Chapter 15

Exercise 15A

1 a 3, 7, 11, 15, 19 **b** 5, 19, 61, 187, 565

c 1, 5, 25, 125, 625 **d** −1, 1, 3, 5, 7

e 1, 3, 7, 17, 41

2 a $t_n = t_{n-1} + 3,\ t_1 = 3$ **b** $t_n = 2t_{n-1},\ t_1 = 1$

c $t_n = -2t_{n-1},\ t_1 = 3$ **d** $t_n = t_{n-1} + 3,\ t_1 = 4$

e $t_n = t_{n-1} + 5,\ t_1 = 4$

3 a $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ **b** 2, 5, 10, 17

c 2, 4, 6, 8 **d** 2, 4, 8, 16

e 5, 8, 11, 14 **f** −1, 8, −27, 64

g 3, 5, 7, 9 **h** 2, 6, 18, 54

4 a $t_n = 3n$ **b** $t_n = 2^{n-1}$

c $t_n = \frac{1}{n^2}$ **d** $t_n = 3(-2)^{n-1}$

e $t_n = 3n + 1$ **f** $t_n = 5n - 1$

5 $t_{n+1} = 3n + 4,\ t_{2n} = 6n + 1$

6 a $t_1 = 15,\ t_n = t_{n-1} + 3$

b $t_n = 12 + 3n$ **c** $t_{13} = 51$

7 a $t_1 = 94.3,\ t_n = 0.96t_{n-1}$

b $t_n = 94.3(0.96)^{n-1}$ **c** $t_9 = 68.03$

8 a $t_0 = 100,\ t_n = 1.8t_{n-1} + 20$

b $t_1 = 200,\ t_2 = 380,\ t_3 = 704,\ t_4 = 1287,$ $t_5 = 2336$

9 a 1st year \$2120; 2nd year \$2671.20; 3rd year \$3255.47

b $t_n = 1.06(t_{n-1} + 400),\ t_1 = 2120$

c \$8454.02

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10 a 1, 4, 7, 10, 13, 16 **b** 3, 1, −1, −3, −5, −7
c $\frac{1}{2}$, 1, 2, 4, 8, 16 **d** 32, 16, 8, 4, 2, 1

11 a 1.1, 1.21, 1.4641, 2.144, 4.595, 21.114
b 27, 18, 12, 8, $\frac{16}{3}$, $\frac{32}{9}$
c −1, 3, 11, 27, 59, 123
d −3, 7, −3, 7, −3, 7

12 a $t_1 = 1, t_2 = 2, t_3 = 4$
b $u_1 = 1, u_2 = 2, u_3 = 4$
c $t_1 = u_1, t_2 = u_2, t_3 = u_3$
d $t_4 = 8, u_4 = 7$

13 $S_1 = a + b,\ S_2 = 4a + 2b,\ S_3 = 9a + 3b,$
$S_{n+1} - S_n = 2an + a + b$

14 $t_2 = \frac{3}{2}, t_3 = \frac{17}{12}, t_4 = \frac{577}{408}$; the number is $\sqrt{2}$

15 $t_3 = 2,\ t_4 = 3,\ t_5 = 5$

Exercise 15B

1 a 0, 2, 4, 6 **b** −3, 2, 7, 12
c $-\sqrt{5}, -2\sqrt{5}, -3\sqrt{5}, -4\sqrt{5}$ **d** 11, 9, 7, 5

2 a −31 **b** 24 **c** 5 **d** $6\sqrt{3}$

3 a $a = 3,\ d = 4,\ t_n = 4n - 1$
b $a = 3,\ d = -4,\ t_n = 7 - 4n$
c $a = -\frac{1}{2},\ d = 2,\ t_n = 2n - \frac{5}{2}$
d $a = 5 - \sqrt{5},\ d = \sqrt{5},\ t_n = \sqrt{5}n + 5 - 2\sqrt{5}$

4 a 13 **b** 8 **c** 20 **d** 56

5 $a = -2,\ d = 3,\ t_7 = 16$

6 $t_n = 156n - 450$ **7** −2

8 54 **9** 19

10 a 672 **b** 91st week

11 a 70 **b** 94 **c** Row F

12 117

13 $\frac{218}{9}$

14 7, 9, 11, 13

15 $t_n = a - \frac{a(n-1)}{m-1}$

16 a 11.5 **b** 50

17 16

18 5

20 3

Exercise 15C

1 a 426 **b** 55 **c** $60\sqrt{2}$ **d** 108

2 112

3 680

4 2450

5 a 14 **b** 322

6 a 20 **b** −280

7 a 12 **b** 105

8 a 180 **b** {9}

9 11

10 20

11 0

12 a 16.5 km **b** 45 km **c** 7 walks **d** 189 km

13 a 10 days **b** 25 per day

14 a 86 **b** 2600 **c** 224 **d** 2376
e 5 extra rows

15 \$176 400

16 $a = -15,\ d = 3,\ t_6 = 0,\ S_6 = -45$

17 2160

18 266

19 a $t_n = \frac{5}{4}n + \frac{11}{4}$ **b** $t_n = 3n - 1$

20 a b **b** $\frac{n}{2}(b + bn)$

21 $t_5 = -10,\ S_{25} = -1250$

22 $1575d$

23 a $S_{n-1} = 23n - 3n^2 - 20$
b $t_n = 20 - 6n$ **c** $a = 14,\ d = -6$

24 7, 12, 17

Exercise 15D

1 a 3, 6, 12, 24 **b** 3, −6, 12, −24
c 10 000, 1000, 100, 10
d 3, 9, 27, 81

2 a $\frac{5}{567}$ **b** $\frac{1}{256}$ **c** 32 **d** a^{x+5}

3 a $t_n = 3\left(\frac{2}{3}\right)^{n-1}$ **b** $t_n = 2(-2)^{n-1}$
c $t_n = 2(\sqrt{5})^{n-1}$

4 a 3 **b** $\pm\frac{2}{5}$

5 t_9

6 a 6 **b** 9 **c** 9 **d** 6 **e** 8

7 $\frac{2}{3^5}$

8 $16\sqrt{2}$

9 a 24 **b** 12 288

10 a 21 870 m^2 **b** 9th day

11 47.46 cm

12 a \$5397.31 **b** 48th year

13 a 57.4 km **b** 14th day

14 \$5 369 000

15 a \$7092.60 **b** 12 years

16 \$3005.61 **17** 11.6% p.a.

18 $t_{10} = 2048$ **19** $t_6 = 729$

20 5 weeks

21 a 60 **b** 2.5 **c** 1 **d** x^4y^7

22 3 or 1

23 $a = \frac{1 \pm \sqrt{5}}{2}$

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24 a 168.07 mL **b** 20 times

25 a side lengths $\frac{a+b}{2}$ **b** side lengths $\sqrt{ab}$

Exercise 15E

1 a 5115 **b** −182 **c** $-\frac{57}{64}$

2 a 1094 **b** −684 **c** 7812

3 10

4 7

5 a 1062.9 mL **b** 5692.27 mL **c** 11 days

6 a 49 minutes (to nearest minute)
b 164 minutes **c** Friday

7 $\frac{481\,835}{6561} \approx 73.44$ m

8 a \$18 232.59 **b** \$82 884.47

9 Bianca \$3247.32; Andrew \$3000

10 a 155 **b** $\frac{15\sqrt{2}}{2} + 15$

11 a 8 **b** $\{ n : n > 19 \}$

12 $\frac{x^{2m+2}+1}{x^2+1}$

13 a 54 976 km **b** 43 times

14 Option 1: \$52 million;
Option 2: \$45 040 000 million

Exercise 15F

1 a $\frac{5}{4}$ **b** $\frac{3}{5}$

2 Perimeter $p\left(\frac{1}{2}\right)^{n-1}$; Area $\frac{p^2\sqrt{3}}{9 \times 4^n}$;
Sum of perimeters $2p$; Sum of areas $\frac{p^2\sqrt{3}}{27}$

3 $3333\frac{1}{3}$ **4** Yes

5 Yes, as the number of hours approaches infinity, but the problem becomes unrealistic after 4 to 5 hours

6 $S_\infty = 8$ **7** $\frac{1}{2}$

8 12 m **9** 75 m

10 a $\frac{4}{9}$ **b** $\frac{1}{30}$ **c** $\frac{31}{3}$ **d** $\frac{7}{198}$ **e** 1 **f** $\frac{37}{9}$

11 $r = \frac{1}{2}$, $t_1 = 16$, $t_2 = 8$;
$r = -\frac{1}{2}$, $t_1 = 48$, $t_2 = -24$

12 $\frac{5}{8}$ **13** $\frac{2}{3}$

Chapter 15 review

Short-answer questions

1 a 3, −1, −5, −9, −13, −17
b 5, 12, 26, 54, 110, 222

2 a 2, 4, 6, 8, 10, 12
b −1, −4, −7, −10, −13, −16

3 a \$5250, \$6037.50
b $t_1 = 5250$, $t_n = 1.05(t_{n-1} + 500)$

4 147 **5** −0.1

6 −258.75 **7** {12}

8 1 **9** 1000×1.035^n

10 $t_2 = 6$, $t_4 = \frac{8}{3}$ or $t_2 = -6$, $t_4 = -\frac{8}{3}$

11 96 **12** −9840

13 $\frac{3}{4}$ **14** $x = 8$ or $x = -2$

Extended-response questions

1 a 0.8, 1.5, 2.2, . . . **b** Yes **c** 8.5 m

2 a Yes **b** $t_n = 25n + 25$ **c** 650

3 $22\frac{1}{7}$ km from town A; $9\frac{6}{7}$ km from town B

4 a 20, 36, 52, 68, 84, 100, 116, 132
b $T_n = 16n + 4$ **c** Yes, size 12

5 a $D_n = 7n - 5$ **b** 27

6 472 mm **7** 520

8 a 99.9999 mg **b** 100 mg

9 a $\frac{1}{729}$ m **b** 1.499 m
No, maximum height of water is 1.5 m

10 a 27.49 **b** 1680.8

11 a $7\frac{1}{9}$ m **b** 405 m

12 $2^{64} - 1 = 1.845 \times 10^{19}$

13 a i $t_n = 3750 + 250n$
ii $S_n = 3875n + 125n^2$
iii $n = 22$
iv $m = \frac{T - 4000}{250} + 1$
v $p = 51$

b i $S_n = 37\,500(1.08^n - 1)$
ii $Q_B - Q_A = 37\,500(1.08^n - 1) - 3875n - 125n^2$; $n = 18$

14 a 3^{n-1} **b** $\left(\frac{1}{2}\right)^{n-1}$ **c** $\left(\frac{3}{4}\right)^{n-1}$
d Area of white region approaches zero

15 a 8^{n-1} **b** $\left(\frac{1}{3}\right)^{n-1}$ **c** $\left(\frac{8}{9}\right)^{n-1}$
d Area of white region approaches zero

Chapter 16

Short-answer questions

1 a $\frac{\pi}{3}$ **b** $\frac{3\pi}{2}$ **c** $\frac{7\pi}{9}$

2 a −1 **b** 0 **c** 0 **d** Undefined

3 a 21.80° **b** 3.06 **c** 9.97

4 a −0.3 **b** −0.5 **c** 1.6 **d** −0.6 **e** 0.1 **f** $\frac{4}{5}$

5 a $\frac{\sqrt{3}}{2}$ **b** $-\frac{\sqrt{3}}{2}$ **c** −1 **d** $\frac{1}{2}$ **e** $\frac{1}{\sqrt{2}}$ **f** $-\sqrt{3}$

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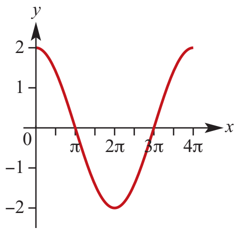
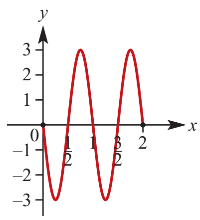
6 a Period = 4π; Amplitude = 2

b



c Dilation of factor 2 parallel to the y-axis and dilation of factor 2 parallel to the x-axis

7



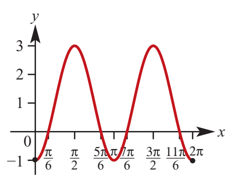
8 a $\frac{-7\pi}{6}, \frac{-5\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$

b $\frac{-7\pi}{4}, \frac{-5\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$

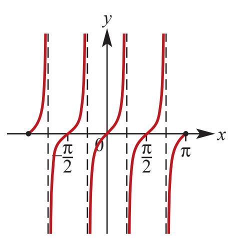
c $\frac{-17\pi}{12}, \frac{-13\pi}{12}, \frac{-5\pi}{12}, \frac{-\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$

d $\frac{-4\pi}{3}, \frac{-\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}$

9



10



11 $\sin(A - B) = \frac{1}{5\sqrt{2}} = \frac{\sqrt{2}}{10}$

12 $\frac{1 - \cos(2A)}{1 + \cos(2A)} = \frac{1 - (1 - 2\sin^2 A)}{1 + (2\cos^2 A - 1)}$

$= \frac{2\sin^2 A}{2\cos^2 A} = \tan^2 A$

13 a $\sqrt{21}$ cm **b** $\frac{2}{\sqrt{7}}$ **c** $\sqrt{3}$ cm^2

14 15 cm^2

15 12 cm

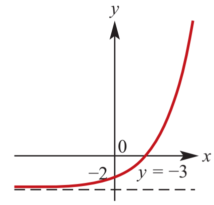
16 a $-24a^{10}$ **b** $\frac{a^3}{2b^2}$ **c** $\frac{3}{4x^5}$ **d** 8

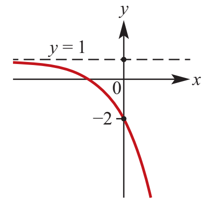
e $\frac{y^{\frac{2}{3}}}{x^{\frac{1}{6}}}$ **f** $\frac{1}{(2x-1)^{\frac{1}{2}}}$

17 a $\frac{25}{9}$ **b** 16 **c** 81 **d** $-\frac{1}{3}$

18 $2^{6n} \times 3^{3n}$

19 a Range $(-3, \infty)$ **b** Range $(-\infty, 1)$





20 a $x = 3$ **b** $x = 0, 2$

21 6

22 −141

23 a $S_{n-1} = 2n^2 - n - 1$ **b** $t_n = 4n + 1$

c $t_1 = 5$ **d** $d = 4$

24 6

25 8 m

26 16 cm

27 a 2550 **b** 3367

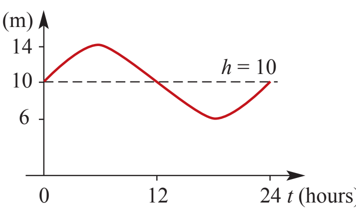
Extended-response questions

1 a 78 m to C, 49 m to D

b 60 m

c 279°

2 a



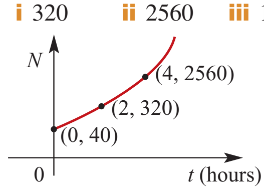
b $t = 3.2393$ and $t = 8.7606$

c $t \in [0.9652, 11.0348]$

3 a 40 bacteria

b i 320 **ii** 2560 **iii** 10 485 760

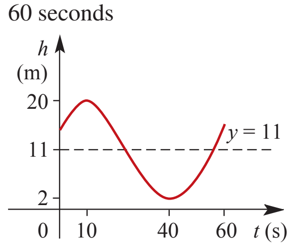
c



d 40 minutes $\left(= \frac{2}{3} \text{ hours}\right)$

4 a 60 seconds

b



c [2, 20]

d First at height 2 metres after 40 seconds; then every 60 seconds after this first time

e At $t = 0$, $t = 20$ and $t = 60$, for $t \in [0, 60]$

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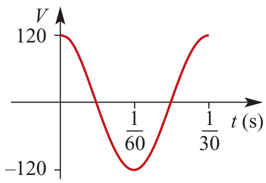
5 a



b $t = \dfrac{1}{180}$ s

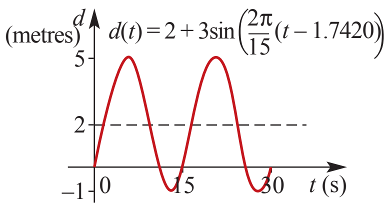
c $t = \dfrac{k}{30}$ s, for $k = 0, 1, 2, \ldots$

6 a i Period = 15 seconds

ii Amplitude = 3 **iii** $c = \dfrac{2\pi}{15}$

b $h = 1.74202$

c

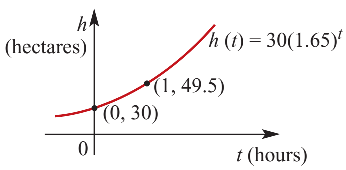


7 a i 30 **ii** 49.5 **iii** 81.675

b $k = 1.65$

c 6.792 hours

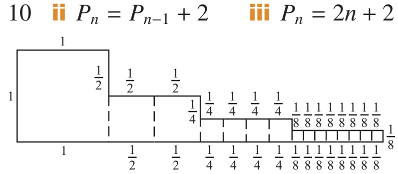
d



8 a 4 **b** 6 **c** 8 **d** 2

e i 10 **ii** $P_n = P_{n-1} + 2$ **iii** $P_n = 2n + 2$

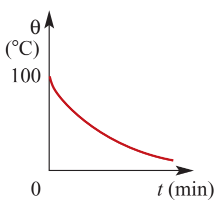
iv



9 a

t	0	1	2	3	4	5
θ	100	60	40	30	25	22.5

b



c 1 minute

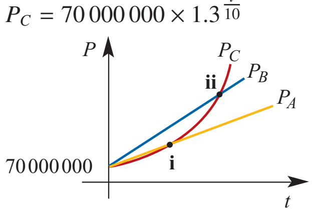
d 27.071

10 a $P_A = 70\,000\,000 + 3\,000\,000t$

$P_B = 70\,000\,000 + 5\,000\,000t$

$P_C = 70\,000\,000 \times 1.3^{\frac{t}{10}}$

b



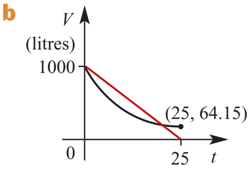
c i 35 years **ii** 67 years

11 a i 4 billion **ii** 5.944 billion **iii** 7.25 billion

b 2032

12 a $V_1(0) = V_2(0) = 1000$

b



c 64.15 litres **d** $t = 0$ and $t = 23.00$

13 a i $OC_1 = R - r_1$ **ii** $r_1 = \dfrac{R}{3}$

b i $OC_2 = \dfrac{R}{3} - r_2$ **ii** $r_2 = \dfrac{R}{9}$

c i $r = \dfrac{1}{3}$ **ii** $r_n = \dfrac{R}{3^n}$

iii $S_\infty = \dfrac{R}{2}$ **iv** $S_\infty = \dfrac{\pi R^2}{8}$

14 a i $80n + 920$

ii A: 2840 tonnes; B: 2465 tonnes

iii $40n(n + 24)$

iv A: 46 080 tonnes; B: 39 083 tonnes

b April 2016

15 a 14 m **b** $t_n = 1.5n - 1$ **c** 53 **d** 330 m

16 a i 15.4 million tonnes

ii 21.7 million tonnes

b $t_n = 0.9n + 9.1$ **c** 371 million tonnes

d 12 years **e** $P_n = 12.5(1.05)^{n-1}$

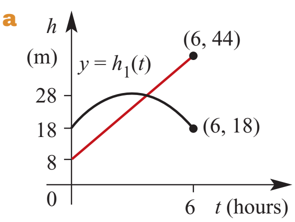
f 15 years

17 a 12:05 p.m. **b** 2752 km **c** 26.1°

18 b i 51.48 cm **ii** 4764.95 cm^2

iii 94.80%

19 a



b 3:19 a.m. to nearest minute ($t = 3.31$)

c i 9 a.m. **ii** $(8 + 6t)$ metres

Chapter 17

Exercise 17A

1 3.2 m/s

2 a 2 **b** 7 **c** $\dfrac{-1}{2}$ **d** $\dfrac{1-\sqrt{5}}{4}$

3 a $\dfrac{-25}{7}$ **b** $\dfrac{-18}{7}$ **c** 4 **d** $\dfrac{4b}{3a}$

4 a 4 m/s **b** 32 m/s

5 a $2450.09 **b** $150.03 per year

6 3.125 cm/min **7** C

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Exercise 17B

1 7.19

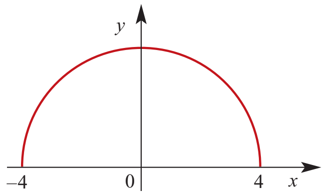
2 a 0.015 **b** $\frac{1}{60} \approx 0.0167$

3 a i 9 **ii** 4.3246 **iii** 2.5893
b 2.30

4 a 25°C at 16:00 **b** ≈ 3°C/h **c** −2.5°C/h

5 −0.5952

6



a 0 **b** −0.6 **c** −1.1

7 4

8 a 16 m^3/min **b** 10 m^3/min

9 a 18 million/min **b** 8.3 million/min

10 a 620 m^3/min flowing out
b 4440 m^3/min flowing out
c 284 000 m^3/min flowing out

11 a 7 **b** 9 **c** 2 **d** 35

12 a 28 **b** 12

13 a i $\frac{2}{\pi} \approx 0.637$ **ii** $\frac{2\sqrt{2}}{\pi} \approx 0.9003$
iii 0.959 **iv** 0.998
b 1

Exercise 17C

1 a $-2-h$ **b** −2

2 a $5+h$ **b** 5

3 $2x-2$ **4** 32 **5** 10 m^3/min

6 7 per day

7 a 1 **b** $3x^2+1$ **c** 20
d $30x^2+1$ **e** 5 **f** $30x^3+4$

8 a $2x+2$ **b** 13 **c** $3x^2+4x$

9 a $5+3h$ **b** 5.3 **c** 5

10 a $\frac{-1}{2+h}$ **b** −0.48 **c** $\frac{-1}{2}$

11 a $6+h$ **b** 6.1 **c** 6

12 a $6x$ **b** 4 **c** 0 **d** $6x+4$
e $6x^2$ **f** $8x-5$ **g** $-2+2x$

13 $4x^3$

Exercise 17D

1 a $2x+4$ **b** 2 **c** $3x^2-1$ **d** $x-3$
e $15x^2+6x$ **f** $-3x^2+4x$

2 a $12x^{11}$ **b** $21x^6$ **c** 5 **d** 5
e 0 **f** $10x-3$ **g** $50x^4+12x^3$
h $8x^3-x^2-\frac{1}{2}x$

3 a 6 **b** 20 **c** 5 **d** 10 **e** 0
f 7 **g** 31 **h** 7 **i** −34

4 a 60 **b** −16 **c** 57 **d** 168

5 a 7 **b** 2 **c** −16 **d** 11

6 a −1 **b** 0 **c** $12x^2-3$
d x^2-1 **e** $2x+3$ **f** $18x^2-8$
g $15x^2+3x$

7 a $2(x+4)$ **b** $48t^2+16t-7$ **c** $2x$

8 a i 3 **ii** $3a^2$ **b** $3x^2$

9 a $\frac{dy}{dx} = 3(x-1)^2 \geq 0$ for all x;
therefore gradient of graph ≥ 0 for all x
b $\frac{dy}{dx} = 1$ for $x \neq 0$
c $18x+6$

10 a 1, Gradient = 2 **b** 1, Gradient = 1
c 3, Gradient = −4 **d** −5, Gradient = 4
e 28, Gradient = −36 **f** 9, Gradient = −24

11 a i $4x-1$, 3, $\left(\frac{1}{2}, 0\right)$
ii $\frac{1}{2}+\frac{2}{3}x$, $\frac{7}{6}$, $\left(\frac{3}{4}, \frac{25}{16}\right)$
iii $3x^2+1$, 4, (0, 0)
iv $4x^3-31$, −27, (2, −46)
b Coordinates of the point where gradient is 1

12 a $6t-4$ **b** $-2x+3x^2$ **c** $-4z-4z^3$
d $6y-3y^2$ **e** $6x^2-8x$ **f** $19.6t-2$

13 a (4, 16) **b** (2, 8), (−2, −8) **c** (0, 0)
d $\left(\frac{3}{2}, -\frac{5}{4}\right)$ **e** (2, −12) **f** $\left(-\frac{1}{3}, \frac{4}{27}\right)$, (1, 0)

Exercise 17E

1 a $-\frac{1}{(x-3)^2}$ **b** $-\frac{1}{(x+2)^2}$

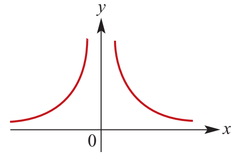
2 a $-\frac{2}{x^3}$ **b** $-\frac{4}{x^5}$

3 a $-6x^{-3}-5x^{-2}$ **b** $-6x^{-3}+10x$
c $-15x^{-4}-8x^{-3}$ **d** $6x-\frac{20}{3}x^{-5}$
e $-12x^{-3}+3$ **f** $3-2x^{-2}$

4 a $-2z^{-2}-8z^{-3}$, $z \neq 0$ **b** $-9z^{-4}-2z^{-3}$, $z \neq 0$
c $\frac{1}{2}$, $z \neq 0$ **d** $18z+4-18z^{-4}$, $z \neq 0$
e $2z^{-3}$, $z \neq 0$ **f** $-\frac{3}{5}$, $z \neq 0$

5 a $f'(x) = 12x^3+18x^{-4}-x^{-2}$
b $f'(x) = 20x^3-8x^{-3}-x^{-2}$

6



a Gradient of $PQ = \frac{-2-h}{(1+h)^2}$ **b** −2

7 a $11\frac{3}{4}$ **b** $\frac{1}{8}$ **c** −1 **d** 5

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8 a $-\frac{1}{2}$ **b** $\frac{1}{2}$

9 $f'(x) = -\frac{1}{x^2} < 0$ for all $x \neq 0$

Exercise 17F

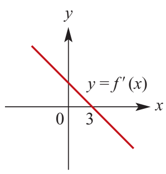
1 Graphs b and d **2** Graphs a, b and e

3 a $x = 1$ **b** $x = 1$ **c** $x > 1$ **d** $x < 1$
e $x = \frac{1}{2}$

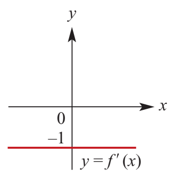
4 a $(-\infty, -3) \cup \left(\frac{1}{2}, 4\right)$ **b** $\left(-3, \frac{1}{2}\right) \cup (4, \infty)$
c $\left\{-3, \frac{1}{2}, 4\right\}$

5 a B **b** C **c** D **d** A **e** F **f** E

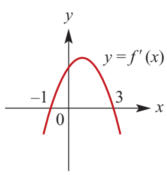
6 a



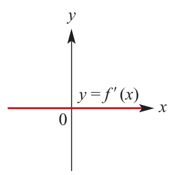
b



c



d

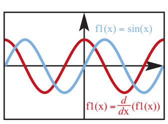


7 a $(-1, 1.5)$ **b** $(-\infty, -1) \cup (1.5, \infty)$
c $\{-1, 1.5\}$

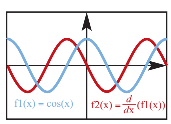
8 a $(3, 0)$ **b** $(4, 2)$

9 a $\left(\frac{1}{2}, -6\frac{1}{4}\right)$ **b** $(0, -6)$

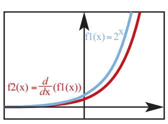
10 a



b



c



11 a i 66.80° **ii** 42.51°
b $(0.5352, 0.2420)$
c No

12 a $a = 2, b = -5$ **b** $\left(\frac{5}{4}, -\frac{25}{8}\right)$

Exercise 17G

1 a $\frac{x^4}{8} + c$ **b** $x^3 - 2x + c$ **c** $\frac{5x^4}{4} - x^2 + c$
d $\frac{x^4}{5} - \frac{2x^3}{3} + c$ **e** $\frac{x^3}{3} - x^2 + x + c$
f $\frac{x^3}{3} + x + c$ **g** $\frac{z^4}{2} - \frac{2z^3}{3} + c$
h $\frac{4t^3}{3} - 6t^2 + 9t + c$ **i** $\frac{t^4}{4} - t^3 + \frac{3t^2}{2} - t + c$

2 $f(x) = x^4 + 2x^3 + 2x$

3 $y = 2x^3 + 12$

4 a $y = x^2 - x$ **b** $y = 3x - \frac{x^2}{2} + 1$
c $y = \frac{x^3}{3} + x^2 + 2$ **d** $y = 3x - \frac{x^3}{3} + 2$
e $y = \frac{2x^5}{5} + \frac{x^2}{2}$

5 a $V = \frac{t^3}{3} - \frac{t^2}{2} + \frac{9}{2}$ **b** $\frac{1727}{6} \approx 287.83$

6 $f(x) = x^3 - x + 2$

7 a B **b** $w = 2000t - 10t^2 + 100\,000$

8 $f(x) = 5x - \frac{x^2}{2} + 4$

9 $f(x) = \frac{x^4}{4} - x^3 - 2$

10 a $k = 8$ **b** $(0, 7)$

11 $8\frac{2}{3}$

12 a $k = -4$ **b** $y = x^2 - 4x + 9$

13 a $k = -32$ **b** $f(7) = 201$

14 $y = \frac{1}{3}(x^3 - 5)$

Chapter 17 review

Short-answer questions

1 a 1 **b** 13

2 a 3 **b** $-2x$ **c** $2x + 5$
d $3x^2 + 1$ **e** $2x + 2$ **f** $6x - 1$

3 a $6x - 2$ **b** 0 **c** $4 - 4x$
d $4(20x - 1)$ **e** $6x + 1$ **f** $-6x - 1$

4 a -1 **b** 0 **c** $\frac{4x + 7}{4}$ **d** $\frac{4x - 1}{3}$
e x

5 a 1; 2 **b** 3; −4 **c** −5; 4 **d** 28; −36

6 a $\left(\frac{3}{2}, -\frac{5}{4}\right)$ **b** $(2, -12)$ **c** $(0, 1), \left(\frac{3}{2}, -\frac{11}{16}\right)$
d $(3, 0), (1, 4)$

7 a $x = \frac{1}{2}$ **b** $x = \frac{1}{2}$ **c** $x > \frac{1}{2}$ **d** $x < \frac{1}{2}$
e $x \neq \frac{1}{2}$ **f** $x = \frac{5}{8}$

8 a $a = 2, b = -1$ **b** $\left(\frac{1}{4}, -\frac{1}{8}\right)$

9 a $\frac{x}{2} + c$ **b** $\frac{x^3}{6} + c$ **c** $\frac{x^3}{3} + \frac{3x^2}{2} + c$
d $\frac{4x^3}{3} + 6x^2 + 9x + c$ **e** $\frac{at^2}{2} + c$ **f** $\frac{t^4}{12} + c$
g $\frac{t^3}{3} - \frac{t^2}{2} - 2t + c$ **h** $\frac{-t^3}{3} + \frac{t^2}{2} + 2t + c$

10 $f(x) = x^2 + 5x - 25$

11 a $f(x) = x^3 - 4x^2 + 3x$
b 0, 1, 3

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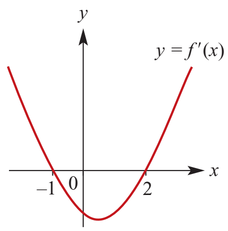
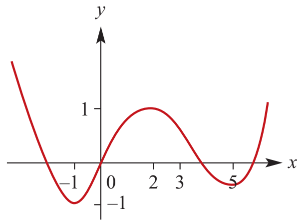
12



13 a $(-1, 4)$ **b** $(-\infty, -1) \cup (4, \infty)$ **c** $\{-1, 4\}$

Extended-response questions

1



2 $y = \frac{7}{36}x^3 + \frac{1}{36}x^2 - \frac{20}{9}x$

3 a i 71.57° **ii** 89.58°
b 2 km

4 a 0.12, −0.15
b $x = 2$, $y = 2.16$; Height 2.16 km

5 a At $x = 0$, gradient is −2; at $x = 2$, gradient is 2. Angles of inclination to the positive direction of the x-axis are supplementary.

6 a m **b** cm **c** $-m$

Chapter 18

Exercise 18A

1 a $y = 4x - 4$, $4y + x = 18$
b $y = 12x - 15$, $12y + x = 110$
c $y = -x + 4$, $y = x$
d $y = 6x + 2$, $6y + x = 49$

2 $y = 2x - 10$

3 $y = 2x - 1$, $y = 2x - \frac{8}{3}$;
Both have gradient 2; Distance apart $= \frac{\sqrt{5}}{3}$

4 $y = 3x + 2$, $y = 3x + 6$

5 a Tangents both have gradient 2 **b** $(0, -3)$

6 $(3, 12)$, $(1, 4)$

7 a $y = 10x - 16$ **b** $(-4, -56)$

8 a $y = 5x - 1$ **b** $(2, 4)$, $(4, -8)$

Exercise 18B

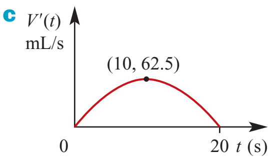
1 a 36; $\frac{36}{1} = 36$ **b** $48 - 12h$ **c** 48

2 a $1200t - 200t^2$ **b** $1800 per month
c At $t = 0$ and $t = 6$

3 a $30 - 4P$
b 10, −10
c For $P < 7.5$ revenue increases as P increases

4 a 50 people per year **b** 0 people per year
c Decreasing by 50 people per year

5 a i 0 mL **ii** $833\frac{1}{3}$ mL
b $V'(t) = \frac{5}{8}(20t - t^2)$
c



6 a 0.6 km^2 **b** 0.7 km^2/h

Exercise 18C

1 a $(3, -6)$ **b** $(3, 2)$ **c** $(2, 2)$ **d** $(4, 48)$
e $(0, 0)$, $(2, -8)$ **f** $(0, -10)$, $(2, 6)$

2 $a = 2$, $b = -8$, $c = -1$

3 $a = -\frac{1}{2}$, $b = 1$, $c = 1\frac{1}{2}$

4 a $a = 2$, $b = -5$ **b** $\left(\frac{5}{4}, -\frac{25}{8}\right)$

5 $a = -8$

6 $a = 6$

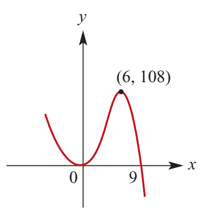
7 a $(2.5, -12.25)$ **b** $\left(\frac{7}{48}, -\frac{625}{96}\right)$
c $(0, 27)$, $(3, 0)$ **d** $(-2, 48)$, $(4, -60)$
e $(-3, 4)$, $(-1, 0)$ **f** $(-1.5, 0.5)$

8 $a = -1$, $b = 2$

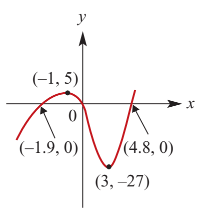
9 $a = -\frac{2}{9}$, $b = \frac{3}{2}$, $c = -3$, $d = 7\frac{1}{2}$

Exercise 18D

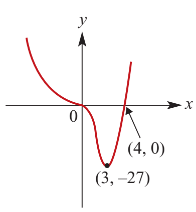
1 a min $(0, 0)$
max $(6, 108)$



b min $(3, -27)$
max $(-1, 5)$



c inflection $(0, 0)$
min $(3, -27)$



2 a $(0, 0)$ max; $\left(\frac{8}{3}, -\frac{256}{27}\right)$ min
b $(0, 0)$ min; $(2, 4)$ max **c** $(0, 0)$ min

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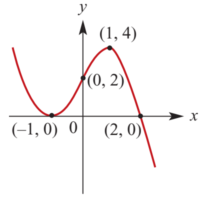
d $\left(\frac{10}{3}, \frac{-200\,000}{729}\right)$ min; $(0, 0)$ inflection

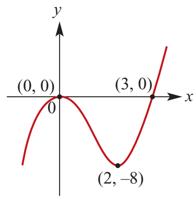
e $(3, -7)$ min; $\left(\frac{1}{3}, \frac{67}{27}\right)$ max

f $(6, -36)$ min; $\left(\frac{4}{3}, \frac{400}{27}\right)$ max

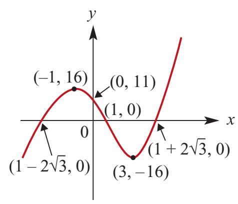
3 a max at $(1, 4)$
min at $(-1, 0)$
intercepts $(2, 0)$, $(-1, 0)$



b min at $(2, -8)$
max at $(0, 0)$
intercepts $(3, 0)$, $(0, 0)$

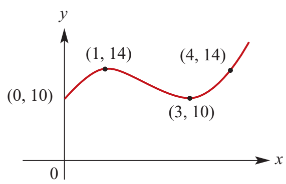


c min at $(3, -16)$
max at $(-1, 16)$
intercepts $(0, 11)$,
$(1 \pm 2\sqrt{3}, 0)$, $(1, 0)$



4 a Local maximum
b Stationary point of inflection

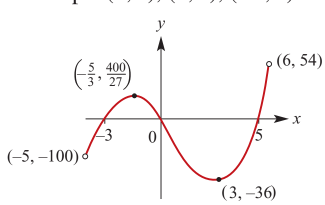
5 a $(-\infty, 1) \cup (3, \infty)$
b $(1, 14)$ max; $(3, 10)$ min
c



6 $\{x : -2 < x < 2\}$

7 a $x \in (-1, 1)$ **b** $x \in (-\infty, -1) \cup (1, \infty)$

8 a $x = -\frac{5}{3}$, $x = 3$
b max at $\left(-\frac{5}{3}, \frac{400}{27}\right)$, min at $(3, -36)$
intercepts $(5, 0)$, $(0, 0)$, $(-3, 0)$



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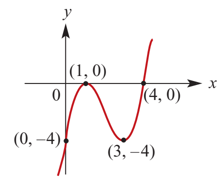
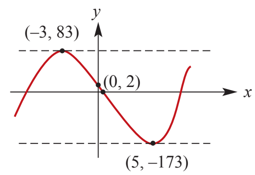
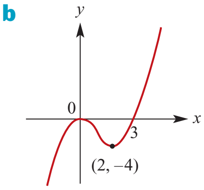


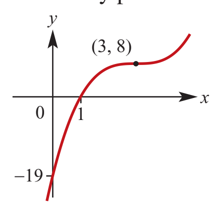
10



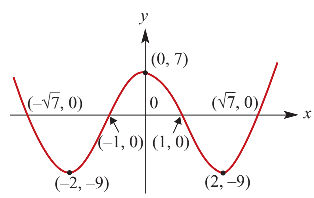
11 a i $(0, 2)$ **ii** $(-\infty, 0) \cup (2, \infty)$ **iii** $\{0, 2\}$
b



12 Stationary point of inflection at $(3, 8)$



13 min at $(-2, -9)$ and $(2, -9)$; max at $(0, 7)$
intercepts $(\pm\sqrt{7}, 0)$, $(\pm 1, 0)$, $(0, 7)$



Exercise 18E

1 2500 cm^2 **2** 25 **3** 2

4 a $V(x) = (6 - 2x)^2 x$
b $V_{max} = 16\text{ m}^3$ when $x = 1$

5 a i 0.9375 m **ii** 2.5 m **iii** 2.8125 m
b $x = \frac{40}{3}$, $y = \frac{80}{27}$
c i $x = 11.937$, $x = 1.396$ **ii** $x = 14.484$

6 b $V = \frac{75x - x^3}{2}$ **c** 125 cm^3 **d** 118 cm^3

7 256π

8 $x = \frac{5}{3}(9 - \sqrt{21})$

9 Absolute max = 2; Absolute min = −30

10 Absolute max = 6; Absolute min = −9

11 Absolute max = 32; Absolute min = −8

12 Absolute max = 1050; Absolute min = −8

13 b $\frac{dV}{dx} = 30x - 36x^2$ **c** $\frac{125}{36}$ **d** $\frac{432}{125}$
e $\frac{125}{36}$ when $x = \frac{5}{6}$

14 a $15 \le y \le 18$ **b** Max 75, min 36

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15 a $\dfrac{125\,000}{27}$ **b** 3000 **c** $\dfrac{125\,000}{27}$

16 b $\dfrac{dA}{dx} = \dfrac{1}{8}(2x - 10)$ **c** $x = 5$ **d** $\dfrac{25}{8}$ m^2

Exercise 18F

1 a $t = 2, t = 3$ and $t = 8$
b $0 \le t < 2.5$ and $t > 6$
c $t = 2.5$ and $t = 6$

2 a -12 cm/s **b** $t = 6, x = -25$
c -9 cm/s **d** 9 cm/s

3 a $t = 2$ **b** $0 \le t < 2$ **c** 8 m
d 4 s **e** $p = -2,\ q = 8,\ r = 0$
f -4 m/s

4 a -3 cm/s **b** $2\sqrt{3}$ s

5 a $x = 5$ cm, $v = 0$ cm/s, $a = -12$ cm/s^2
b $t = 0, x = 5, a = -12$; $\quad t = 1, x = 3, a = 12$

6 a 2 m/s^2 **b** 50 m/s^2

7 a $(15 - 9.8t)$ m/s **b** -9.8 m/s^2

8 a 3.5 s **b** 2 m/s^2 **c** 14.5 m
d When $t = 2.5$ s; the particle is 1.25 m to the left of O

9 a 0 s, 1 s, 2 s
b 2 m/s, -1 m/s, 2 m/s; -6 m/s^2, 0 m/s^2, 6 m/s^2
c 0 m/s

10 a 12 cm to the right of O
b 2 cm to the right of O
c Moving to the left at 7 cm/s
d $t = 3.5$ s; particle is 0.25 cm to the left of O
e -2 cm/s **f** 2.9 cm/s

11 a 18 m/s^2, 54 m/s^2, 114 m/s^2, **b** 58 m/s^2

12 a 3 cm to the left of O, moving to the right at 24 cm/s
b $v = 3t^2 - 22t + 24$
c At $\frac{4}{3}$ s and 6 s
d $11\frac{22}{27}$ cm to the right of O and 39 cm to the left of O
e $4\frac{2}{3}$ s
f $a = 6t - 22$
g When $t = \frac{11}{3}$ s and the particle is $13\frac{16}{27}$ cm left of O moving to the left at $16\frac{1}{3}$ cm/s

13 a When $t = \frac{2}{3}$ s and $a = -2$ cm/s^2, and when $t = 1$ and $a = 2$ cm/s^2
b When $t = \frac{5}{6}$ and the particle is moving to the left at $\frac{1}{6}$ cm/s

14 When $t = 2$ s, $v = 6$ cm/s, $a = -14$ cm/s^2
When $t = 3$ s, $v = -5$ cm/s, $a = -8$ cm/s^2
When $t = 8$ s, $v = 30$ cm/s, $a = 22$ cm/s^2

15 a $t = 4$ s and $t = -1$ s **b** $t = \frac{3}{2}$ s

Exercise 18G

1 a $(x - 2)(3x - 2(b + 1))$
b $(2, 0), \left(\dfrac{2(b + 1)}{3}, \dfrac{-4(b - 2)^3}{27}\right)$ **d** $b = 5$

2 a $(0, 0)$ and $(9, -2187)$
b (a, b) and $(9 + a, -2187 + b)$

3 a i $\left(-\infty, \dfrac{1}{2a}\right)$ **ii** $\left(\dfrac{1}{2a}, \infty\right)$
b $y = -x + \dfrac{1}{a}$ **c** $y = x - \dfrac{1}{a}$ **d** $\left(-\infty, \dfrac{1}{4a}\right]$

4 a $(a, 0)$ and $\left(\dfrac{a + 2}{3}, \dfrac{4(a - 1)^3}{27}\right)$
b Local minimum at $(a, 0)$
Local maximum at $\left(\dfrac{a + 2}{3}, \dfrac{4(a - 1)^3}{27}\right)$
c i $y = (a - 1)^2(x - 1)$ **ii** $y = 0$
iii $y = \dfrac{-(a - 1)^2}{4}(x - a)$

5 a i $2(a - 2)$ **ii** $m = 2(a - 2)$
b $(a, (a - 2)^2)$
c $y = 2(a - 2)x - a^2 + 4$ **d** $\dfrac{a + 2}{2}$

6 a $h = 2$ **b** $a = 3$ **c** $a = -16, b = -24$

7 a $(0, 0)$ **b** (a, b)

8 a $2(x - 1)(x - b)(2x - b - 1)$
b $(1, 0), (b, 0), \left(\dfrac{b + 1}{2}, \dfrac{(b - 1)^4}{16}\right)$ **c** $b = 3$

Chapter 18 review

Short-answer questions

1 a $\dfrac{dy}{dx} = 4 - 2x$ **b** 2 **c** $y = 2x + 1$

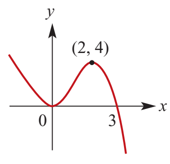
2 a $3x^2 - 8x$ **b** -4 **c** $y = -4x$ **d** $(0, 0)$

3 a $3x^2 - 12$; $x = \pm 2$
b Local minimum when $x = 2$
Local maximum when $x = -2$
c $x = 2, y = -14$; $\ x = -2, y = 18$

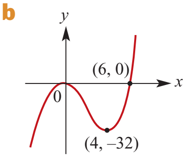
4 a Stationary point of inflection at $x = 0$
b Maximum at $x = 0$
c Min at $x = 3$, max at $x = 2$
d Min at $x = 2$, max at $x = -2$
e Max at $x = 2$, min at $x = -2$
f Max at $x = 3$, min at $x = 1$
g Max at $x = 4$, min at $x = -3$
h Max at $x = 3$, min at $x = -5$

5 a $\left(-\dfrac{2}{3}, -\dfrac{16}{9}\right)$ minimum, $\left(\dfrac{2}{3}, \dfrac{16}{9}\right)$ maximum
b $(-1, 0)$ maximum, $(2, -27)$ minimum
c $\left(\dfrac{2}{3}, \dfrac{100}{27}\right)$ maximum, $(3, -9)$ minimum

6 a



b



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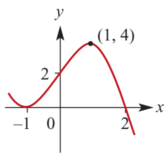
c



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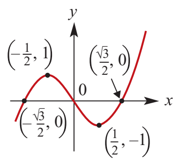

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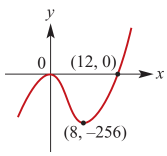


7 a C **b** A **c** B

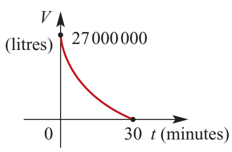
8 a 20 m **b** 6 s **c** 40 m/s

9 72

Extended-response questions

1 a −14 m/s **b** -8 m/s^2

2 a

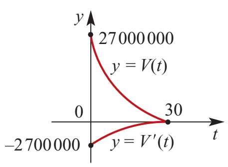


b i 17.4 minutes **ii** 2.9 minutes

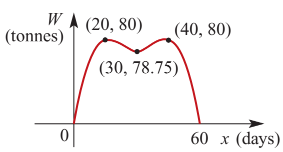
c $\frac{dV}{dt} = -3000(30 - t)^2$

d 30 minutes **e** 28.36 minutes

f



3 a



b From 5.71 days until 54.29 days

c When $x = 20$ and when $x = 40$, $\frac{dW}{dx} = 0$;

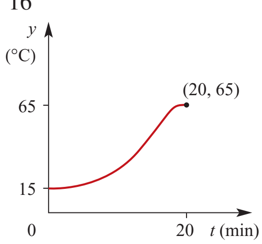
When $x = 60$, $\frac{dW}{dx} = -12$ tonnes per day

d When $x = 30$, $W = 78.75$

4 a 15°C

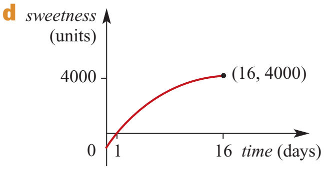
b 0°C/min, $\frac{45}{16}$ °C/min, $\frac{15}{4}$ °C/min, $\frac{45}{16}$ °C/min, 0°C/min

c

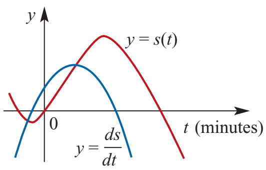


5 a 768 units/day **b** 432, 192, 48, 0

c $t = 16$ **d**



6 a



b 11:59 a.m., 12:03 p.m.

c $\frac{5}{27}$ km, 1 km

d $\frac{8}{27}$ km/min $= 17\frac{7}{9}$ km/h

e $\frac{1}{3}$ km/min $= 20$ km/h

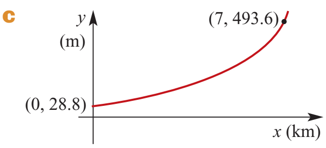
7 a $0 \le t \le 12$

b i 27 L/h **ii** 192 L/h

8 a 28.8 m

b 374.4

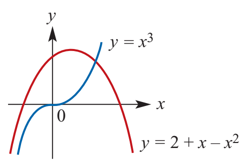
c



d Path gets too steep after 7 km

e i 0.0384 **ii** 0.0504 **iii** 0.1336

9 a



b For $x \le 0$, the minimum vertical distance occurs when $x = -1$; Min distance = 1 unit

10 8 mm for maximum and $\frac{4}{3}$ mm for minimum

11 a $y = 5 - x$ **b** $P = x(5 - x)$

c Max value is 6.25 when $x = 2.5$ and $y = 2.5$

12 a $y = 10 - 2x$ **b** $A = x^2(10 - 2x)$

c $A = \frac{1000}{27}$, $x = \frac{10}{3}$, $y = \frac{10}{3}$

13 $20\sqrt{10}$

14 a $y = 8 - x$ **b** $s = x^2 + (8 - x)^2$ **c** 32

15 $\frac{4}{3}, \frac{8}{3}$

16 Maximum area is 625 m^2 for 25 m × 25 m

17 $x = 12$ **18** 32

19 Maximum value of P is 2500

20 Maximum area is 2 km^2 for 2 km × 1 km

21 $p = \frac{3}{2}$, $q = \frac{8}{3}$

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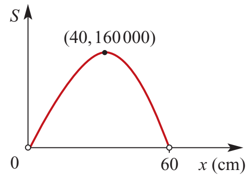
22 a $y = 60 - x$ **b** $S = 5x^2(60 - x)$

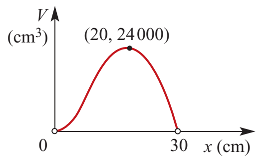
c $0 < x < 60$ **d**



e $x = 40, y = 20$ **f** 74 005

23 12°C

24 b $0 < x < 30$ **c**



d 20 cm, 40 cm, 30 cm

e $x = 14.82$ or $x = 24.4$

25 b Maximum when $x = 3$ and $y = 18$

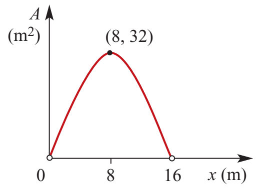
26 a Use 44 cm for circle and 56 cm for square

b Use all the wire for the circle

27 Length 7.2 metres, width 4.5 metres

28 a $A = xy$ **b** $A = \left(8 - \frac{x}{2}\right)x$

c $0 < x < 16$ **d**

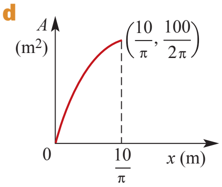


e 32 m^2

29 $h = 1188, a = 937$

30 a $y = 10 - \pi x$ **b** $0 \le x \le \frac{10}{\pi}$

c $A = \frac{x}{2}(20 - \pi x)$ **d**

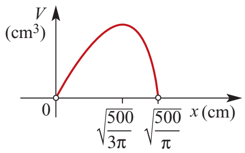


e Maximum at $x = \frac{10}{\pi}$ **f** A semicircle

31 a $h = \frac{500}{\pi x} - x$ **b** $V = 500x - \pi x^3$

c $\frac{dV}{dx} = 500 - 3\pi x^2$ **d** $x = 10\sqrt{\frac{5}{3\pi}} \approx 7.28$

e



f 2427.89 cm^3

g $x = 2.05, h = 75.41$ or $x = 11.46, h = 2.42$

32 a $r = 4.3$ cm, $h = 8.6$ cm

b $r = 4.3$ cm, $h = 8.6$ cm

Chapter 19

Short-answer questions

1 $36\text{ cm}^2/\text{cm}$

2 a 1 cm/s **b** 41 cm/s

3 a i -4 **ii** -3 **b** $-2 - h$ **c** -2

4 $x - 1$

5 a $6x^2 - 1$ **b** $2x + 1$ **c** 1

6 a 13 **b** 10

7 a $x = 0$ or $x = \frac{1}{2}$ **b** $x = \frac{1}{4}$ **c** $x < \frac{1}{4}$

d $x > \frac{1}{4}$ **e** $x = \frac{11}{4}$

8 Tangent $y = -3x - 1$; Normal $y = \frac{1}{3}x - \frac{13}{3}$

9 a $t = 0$ and $t = 2$

b $t = 0, a = -1\text{ cm/s}^2$; $t = 2, a = 1\text{ cm/s}^2$

c $-\frac{1}{2}$ cm/s

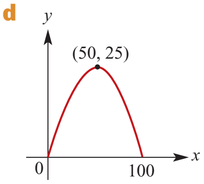
10 Local minimum $\left(\frac{2}{\sqrt{3}}, -\frac{32}{3\sqrt{3}}\right)$

Local maximum $\left(-\frac{2}{\sqrt{3}}, \frac{32}{3\sqrt{3}}\right)$

Extended-response questions

1 a 100 **b** $\frac{dy}{dx} = 1 - 0.02x$

c $x = 50, y = 25$ **d**



e i (25, 18.75) **ii** (75, 18.75)

2 a $\left(66\frac{2}{3}, 14\frac{22}{27}\right)$

b i 0.28 **ii** -0.32 **iii** -1

c A gradual rise to the turning point and a descent which becomes increasingly steep (in fact, alarmingly steep)

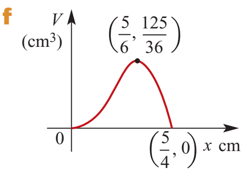
d Smooth out the end of the trip

3 a $h = 5 - 4x$ **c** $0 < x < \frac{5}{4}$

d $\frac{dV}{dx} = 30x - 36x^2$

e $\left\{0, \frac{5}{6}\right\}$; Maximum volume $= 3\frac{17}{36}\text{ cm}^3$

f



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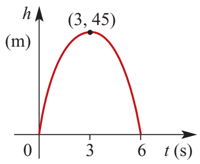
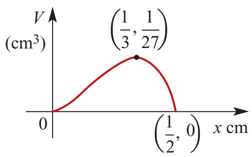
4 **a** $\frac{dh}{dt} = 30 - 10t$ **b** 45 m

c



5 **a** $A = 4x - 6x^2$ **b** $V = x^2 - 2x^3$

c



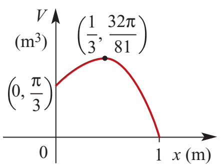
d $\frac{1}{3}$ cm $\times$ $\frac{1}{3}$ cm $\times$ $\frac{1}{3}$ cm; Volume = $\frac{1}{27}$ cm^3

6 **a** **i** $r = \sqrt{1 - x^2}$ **ii** $h = 1 + x$

c $0 < x < 1$

d **i** $\frac{dV}{dx} = \frac{\pi}{3}(1 - 2x - 3x^2)$ **ii** $\left\{\frac{1}{3}\right\}$ **iii** $\frac{32\pi}{81}$ m^3

e



7 **a** 1000 insects **b** 1366 insects

c **i** $t = 40$ **ii** $t = 51.70$

d 63.64

e **i** $\frac{1000 \times 2^{\frac{3}{4}}\left(2^{\frac{h}{20}} - 1\right)}{h}$

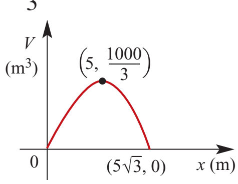
ii Consider h decreasing and approaching zero; instantaneous rate of change = 58.286 insects/day

8 **a** $h = \frac{150 - 2x^2}{3x}$ **b** $V = \frac{2}{3}(150x - 2x^3)$

c $\frac{dV}{dx} = 2(50 - 2x^2)$ **d** $0 < x < 5\sqrt{3}$

e $\frac{1000}{3}$ m^3 when $x = 5$

f



9 **a** 10

c **i** $h = 2.5x$

d $V = 40(420x - 135x^2)$

e **i** $x = \frac{14}{9}, y = \frac{140}{9}$ **ii** $13\,066\frac{2}{3}$ m^3

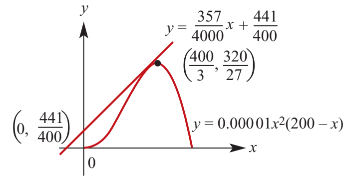
10 **a** $a = 200, k = 0.000\,01$

b **i** $\frac{400}{3}$ **ii** $\frac{320}{27}$ **c** **i** $\frac{8379}{800}$ **ii** $\frac{357}{4000}$

d **i** $y = \frac{357}{4000}x + \frac{441}{400}$ **ii** $\frac{441}{400}$

e 0.099 75

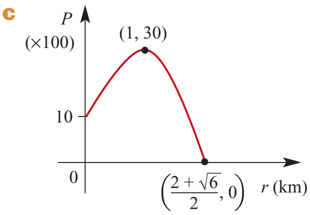
f



11 **a** 10 000 people/km^2

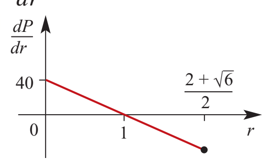
b $0 \le r \le \frac{2 + \sqrt{6}}{2}$

c



d **i** $\frac{dP}{dr} = 40 - 40r$ **ii** 20, 0, −40

iii

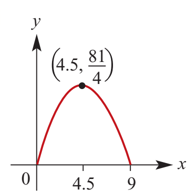


e At $r = 1$

12 **a** $y = ax - x^2$ **b** $0 < x < a$ **c** $\frac{a^2}{4}, \frac{a}{2}$

d Negative coefficient of x^2 for quadratic function

e **i**

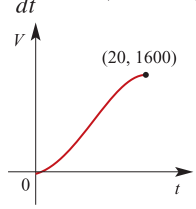


ii $\left(0, \frac{81}{4}\right]$

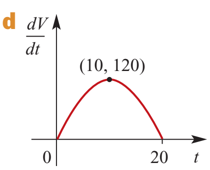
13 **a** **i** 0 **ii** 1600

b $\frac{dV}{dt} = 0.6(40t - 2t^2)$

c

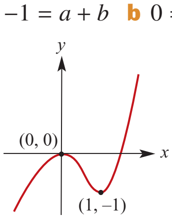


d



14 **a** $-1 = a + b$ **b** $0 = 3a + 2b, a = 2, b = -3$

c



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15 a i $80 - 2x$ **ii** $h = \frac{\sqrt{3}}{2}x$

b $A = \frac{\sqrt{3}}{4}x(160 - 3x)$ **c** $x = \frac{80}{3}$

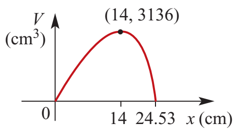
16 a $y = \frac{1400 - 2x^2 - 8x}{4x}$

b $V = -\frac{x^3}{2} - 2x^2 + 350x$

c $\frac{dV}{dx} = -\frac{3}{2}x^2 - 4x + 350$

d $x = 14$

e



f Maximum volume is $3136\,\text{cm}^3$

g $x = 22.83$ and $y = 1.92$, or $x = 2.94$ and $y = 115.45$

Chapter 20

Short-answer questions

1 $x = 4$

2 $t = \frac{2d - b}{a - 2c}$

3 $x \geq -\frac{3}{2}$

4 a -12 **b** 3 **c** 100

5 15

6 $x \leq \frac{37}{5}$

7 $a = 7.9$

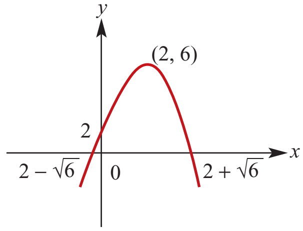
8 a $\left(\frac{a+8}{2}, \frac{b+14}{2}\right)$ **b** $a = 2, b = 6$

9 a $4y - 3x = 30$ **b** $\frac{25}{2}$

10 a $\left(2, \frac{1}{2}\right)$ **b** $\sqrt{445}$ **c** $11x + 18y = 31$

d $22y - 36x + 61 = 0$

11



12 $f(x) = \frac{9}{8}(x - 2)^2 - 6$

13 $a = -2$

14 a $w = 1500 - 9x$ **b** $V = 20x^2(1500 - 9x)$

c $0 \leq x \leq \frac{500}{3}$ **d** $120\,000\,000\,\text{cm}^3$

15 400 tiles

16 a $\frac{16}{81}$ **b** $\frac{28}{153}$

17 $\frac{1}{3}$

18 a $\frac{1}{2}$ **b** $\frac{1}{3}$

19 0.42

20 $-\frac{\pi}{9}, \frac{\pi}{9}$

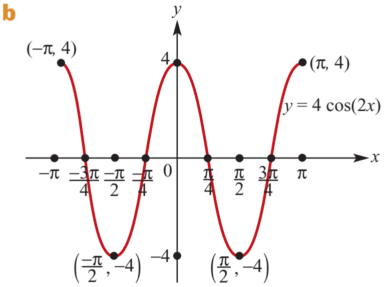
21 a $c = 6$ **b** $0 = -8a - 2b + 6, 0 = 3a + b$

c $a = 3, b = -9$

22 $a = -48$

23 a Amplitude = 4; Period = π

b



24 a 0.3 **b** $400(2 - \sqrt{2})$

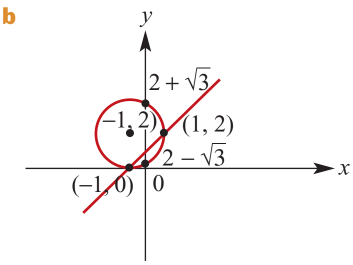
25 a $\left(\frac{2}{3}\right)^c$

b Area = $15\,\text{cm}^2$; Perimeter = $16\,\text{cm}$

26 a $\frac{1}{4}$ **b** $\frac{1}{3}$ **c** $\frac{1}{4}$

27 a $(0, 1)$

b



28 $4\sqrt{19}\,\text{km}$

29 a $x = 3$ **b** $x = -\frac{5}{2}$ or $x = 1$

30 403

31 $x = \frac{1}{3}$ or $x = -\frac{1}{2}$

32 $k = 4$ and $y = -2x^2 + 4x + 3$

33 $a = \frac{1}{3}$

34 $b = -\frac{1}{3}$

35 Intersect at $(-3, -27)$; both curves have gradient 27 at this point

Answers 20 revision

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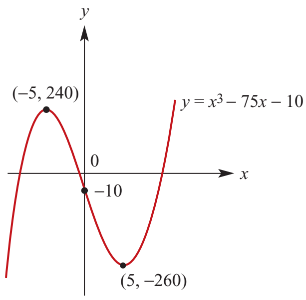
36 a (5, −260) and (−5, 240)



b $-260 \le p \le 240$

37 a $\{ x \in \mathbb{R} : x \neq 3 \}$ **b** $\{ x \in \mathbb{R} : x \neq 2 \}$
c $(-\infty, 2]$ **d** $[4, \infty)$
e $(-\infty, 5)$

Extended-response questions

1 a i $-b$ **ii** $x = \frac{b}{2}$

b i $S(b) = \frac{b}{2}(32 - b^2)$ **ii** $b = 2$

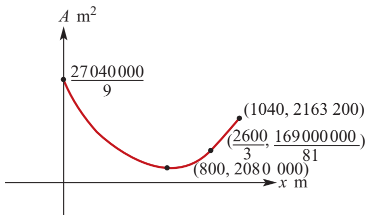
2 a $a = 4$ **b** $\left(\frac{1}{4}, 4\right)$ **c** $\frac{34}{15}$

3 a $y = 13 - 9x$ **b** $A = 156x - 60x^2$

c $x = \frac{13}{10}$ and $y = \frac{13}{10}$

4 b $2\,080\,000\ \text{m}^2$ when $x = 800$ and $y = 1200$

c



5 a i $(-\infty, 0) \cup \left(\frac{2a}{3}, \infty\right)$ **ii** $\left(0, \frac{2a}{3}\right)$

b $y = -a^2(x - a)$ **c** $y = \frac{x}{a^2} - \frac{1}{a}$

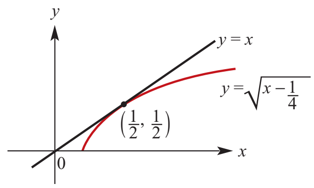
6 a {(0, 1), (0, 3), (0, 5), (0, 7), (0, 9), (0, 11),
(2, 1), (2, 3), (2, 5), (2, 7), (2, 9), (2, 11),
(4, 1), (4, 3), (4, 5), (4, 7), (4, 9), (4, 11),
(6, 1), (6, 3), (6, 5), (6, 7), (6, 9), (6, 11),
(8, 1), (8, 3), (8, 5), (8, 7), (8, 9), (8, 11),
(10, 1), (10, 3), (10, 5), (10, 7), (10, 9), (10, 11)}

b i $\frac{1}{36}$ **ii** $\frac{5}{36}$ **iii** $\frac{5}{36}$ **c** $\frac{2}{13}$

7 a i $a = 50\,000$, $d = 5000$
ii 11th month **iii** 4 950 000 litres
b i $q_n = 12\,000(1.1)^{n-1}$
ii 256 611 litres
c 31st month

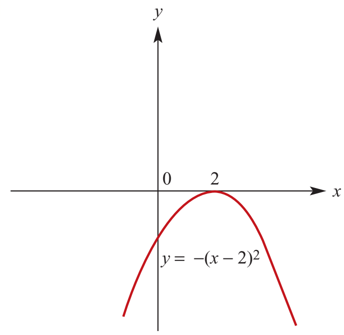
8 a $x \ge 2a$ **b** $x = \frac{1 \pm \sqrt{1 - 8a}}{2}$ **c** $a = \frac{1}{8}$

d

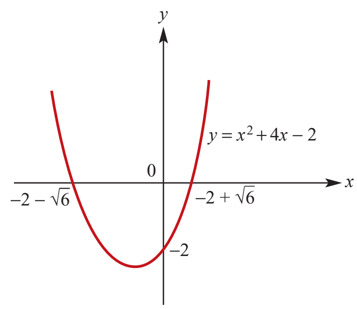


9 a 0.343 **b** 0.399
10 a 0.4219 **b** 0.2156 **c** 0.6125
11 a 155 m
b i 16.00 m **ii** 29.04 m **iii** 17°
c $32.7\ \text{cm}^2$

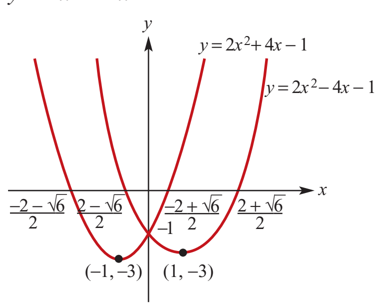
12 a i



ii



b $\left(\frac{2}{1 - p}, \frac{p^2 - 5p}{p - 1}\right)$ **c** $p = 0$ or $p = 5$
d $0 < p < 5$ and $p \neq 1$
e $y = 2x^2 - 4x - 1$



13 a $\frac{\pi}{6}$ **b** $115\sqrt{2}$ cm **c** $115\sqrt{3}$ cm

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Appendix A TI-Nspire CAS CX with OS4.0

Keystroke actions and short cuts for the TI-Nspire CAS CX

esc: removes menus and dialogue boxes

ctrl + esc: undo last move

⇧shift + esc: redo last move

tab: move to next entry box (field)

ctrl + tab: switch applications in split screen

Navpad (Touchpad)

ctrl: accesses secondary (blue) commands

ctrl + ▲: displays page sorter

ctrl + ◀: displays previous page

ctrl + ▶: displays next page

on: displays icon page to select applications, mode, My Documents and start a new document

menu: options for each application

ctrl + menu: contextual menus (same as right mouse click)

(mouse pointer icon): mouse pointer (cursor). Selects items.

ctrl + (mouse pointer icon): grab

del: backspace, deletes a character

(catalogue icon): catalogue

(template icon): 2D maths template

ctrl + ÷: adds fraction template

enter: completes commands and displays results

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Mode Settings

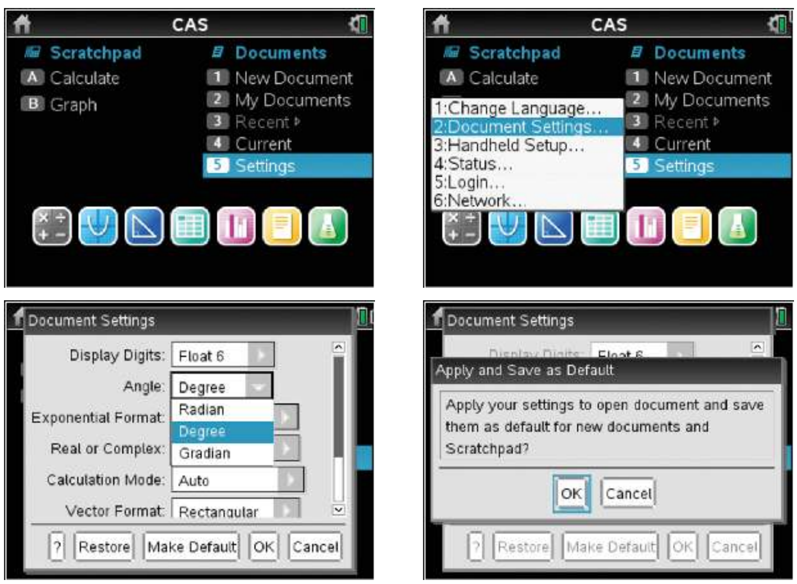
How to set in Degree mode

For this subject it is necessary to set the calculator to **Degree** mode right from the start. This is very important for the Trigonometry topic. The calculator will remain in this mode unless you change the setting again or update the operating system.

Press [⌂ on] and move to **Settings>Document Settings**, arrow down to the **Angle** field, press ▶ and select **Degree** from the list then arrow down to the **Make Default** tab. Select **OK** to accept the change.

Note that there is a separate settings menu for the **Graphs** and **Geometry** pages. These settings are accessed from the relevant pages. For Mathematics it is not necessary for you to change these settings.

Note: When you start your new document you will see **DEG** in the top status line.



How to set in Approximate (Decimal) mode

For this subject it is useful to set the calculator to **Approximate (Decimal)** mode right from the start. The calculator will remain in this mode unless you change the setting again or update the operating system.

Press [⌂ on] and move to **Settings>Document Settings**, arrow down to the **Calculation Mode** field, press ▶ and select **Approximate** from the list then arrow down to the **Make Default** tab. Select **OK** to accept the change.

Note: You can make both the **Degree** and **Approximate Mode** selections at the same time if desired.

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The home screen is divided into two main areas – **Scratchpad** and **Documents**.

All instructions given in the text, and in the Appendix, are based on the **Documents** platform.

Documents

Documents can be used to access all the functionality required for this subject including all calculations, graphing, statistics and geometry.

Starting a new document

1 To start a new document, press [on] and select **New Document**.

2 If prompted to save an existing document move the cursor to **No** and press [enter].

Note: Pressing [ctrl]+[N] will also start a new document.

A: Calculator page - this is a fully functional CAS calculation platform that can be used for calculations such as arithmetic, algebra, finance, trigonometry and matrices. When you open a new document select **Add Calculator** from the list.

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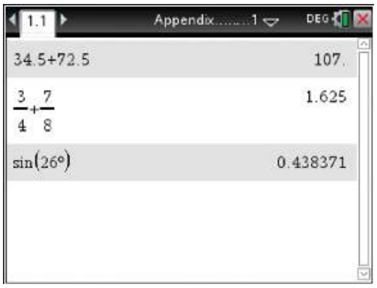
1 You can enter fractions using the fraction template if you prefer. Press ctrl ÷ to paste the fraction template and enter the values. Use the tab key or arrows to move between boxes. Press enter to display the answer. Note that all answers will be either whole numbers or decimals because the mode was set to approximate (decimal).



2 For problems that involve angles (e.g. evaluate sin(26°)) it is good practice to include the degree symbol even if the mode is set to degree (DEG) as recommended.

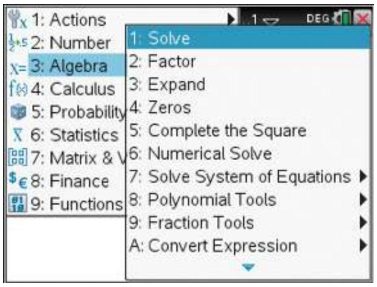
Note: If the calculator is accidentally left in radian (RAD) mode the degree symbol will override this and compute using degree values.

The degree symbol can be accessed using ?!▸. Alternatively select from the **Symbols** palette ctrl.To enter trigonometry functions such as *sin, cos*, press the trig key or just type them in with an opening parenthesis.

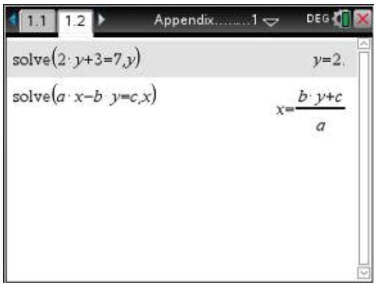
Solving equations

Using the **Solve command** Solve $2y + 3 = 7$ for y.

In a **Calculator** page press menu **>Algebra>Solve** and complete the **Solve** command as shown opposite. You must include the variable you are making the subject at the end of the command line.



Hint: You can also type in **solve(** directly from the keypad but make sure you include the opening bracket.



Literal equations such as $ax - by = c$ can be solved in a similar way.

Note that you must use a multiplication sign between two letters.

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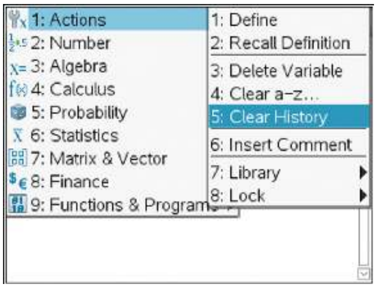
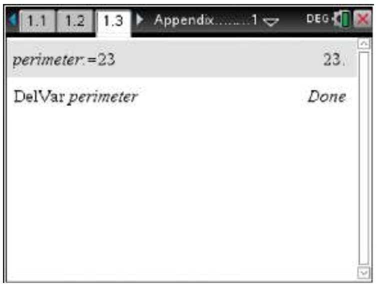
Clearing the history area

Once you have pressed [enter] the computation becomes part of the **History** area. To clear a line from the History area, press ▲ repeatedly until the expression is highlighted and press [enter]. To completely clear the History area, press [menu]**>Actions>Clear History** and press [enter] again.



Alternatively press [ctrl]+[menu] to access the contextual menu.

It is also useful occasionally to clear any previously stored values. Clearing **History** does not clear stored variables.
Pressing [menu]**>Actions>Clear a-z...** will clear any stored values for single letter variables that have been used.
Use [menu]**>Actions>Delete Variable** if the variable name is more than one letter. For example, to delete the variable perimeter, then use **DelVar *perimeter***.



Note: When you start a new document any previously stored variables are deleted.

How to construct parallel boxplots from two data lists

Construct parallel boxplots to display the pulse rates of 23 adult females and 23 adult males.

Pulse rate (beats per minute)	
Females	Males
65 73 74 81 59 64 76 83 95 70 73 79 64 77 80 82 77 87 66 89 68 78 74	80 73 73 78 75 65 69 70 70 78 58 77 64 76 67 69 72 71 68 72 67 77 73

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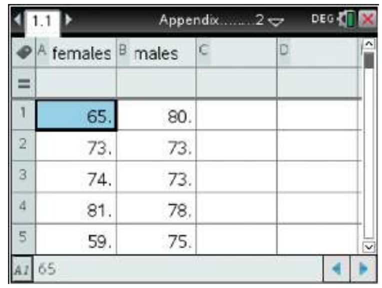
Steps

1 Start a new document: ctrl + N.

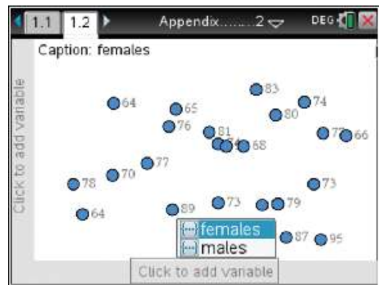
2 Select **Add Lists & Spreadsheet**. Enter the data into lists called ***females*** and ***males*** as shown.



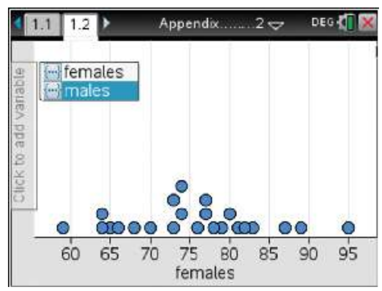
3 Statistical graphing is done through the **Data & Statistics** application.

Press ctrl + I and select **Add Data & Statistics** (or press on and arrow to and press enter).

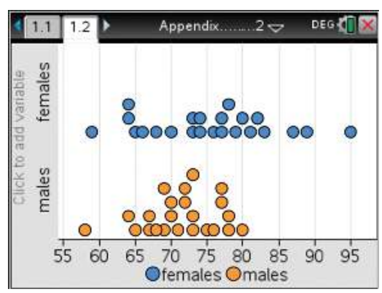
Note: A random display of dots will appear – this is to indicate list data is available for plotting. It is not a statistical plot.



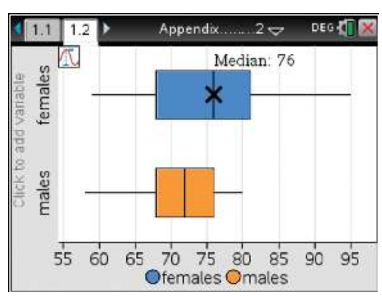
a Press tab, or navigate and click on the "Click to add variable" box to show the list of variables. Select the variable, ***females***. Press enter or to paste the variable to the x-axis. A dot plot is displayed by default as shown.



b To add another variable to the x-axis press menu **>Plot Properties>Add X Variable**, then enter. Select the variable ***males***. Parallel dot plots are displayed by default.



c To change the plots to box plots press menu **>Plot Type>Box Plot**, then press enter. Your screen should now look like that shown below.



Use ▼ to trace the other plot.

Press esc to exit the **Graph Trace** tool.

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4 Data analysis

Use menu **>Analyze>Graph Trace** and use the cursor arrows to navigate through the key points. Alternatively just move the cursor over the key points. Starting at the far left of the plots, we see that, for females, the

- minimum value is 59: **MinX** = **59**
- first quartile is 68: **Q1**= **68**
- median is 76: **Median** = **76**
- third quartile is 81: **Q3** = **81**
- maximum value is 95: **MaxX** = **95**

and for males, the

- minimum value is 58: **MinX** = **58**
- first quartile is 68: **Q1** = **68**
- median is 72: **Median** = **72**
- third quartile is 76: **Q3** = **76**
- maximum value is 80: **MaxX** = **80**

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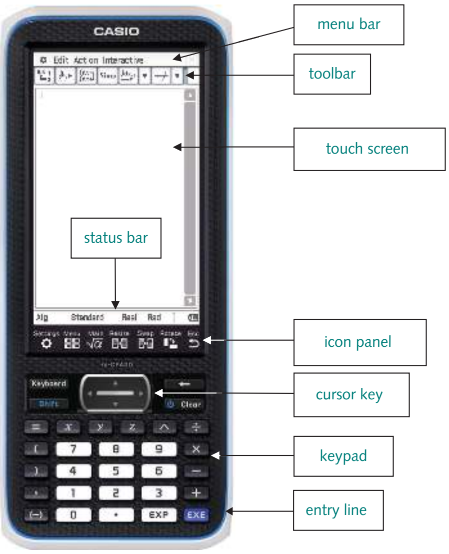
Appendix B Casio ClassPad II

Operating system

Written for operating system 2.0 or above.

Terminology

Some of the common terms used with the ClassPad are:



The menu bar

The toolbar

The touch screen contains the work area where the input is displayed on the left and the output is displayed on the right. Use your finger or stylus to tap and perform calculations.

The icon panel contains seven permanent icons that access settings, applications and different view settings. Press **escape** to cancel a calculation that causes the calculator to freeze.

The cursor key works in a similar way to a computer cursor keys.

The keypad refers to the hard keyboard.

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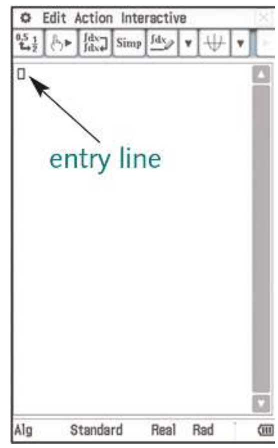
Calculating

Tap Menu from the **icon panel** to display the application menu if it is not already visible.

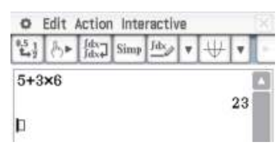
Tap Main to open the **Main** application.

Note: There are two application menus. Alternate between the two by tapping on the screen selector at the bottom of the screen.

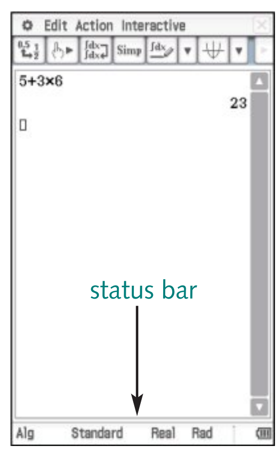
1 The main screen consists of an entry line which is recognised by a flashing vertical line (cursor) inside a small square. The history area, showing previous calculations, is above the entry line.



2 To calculate, enter the required expression in the entry line and press EXE. For example, if we wish to evaluate $5 + 3 \times 6$, type the expression in the entry line and press EXE.
You can move between the entry line and the history area by tapping or using the cursor keys (i.e. ▼, ◀, ▲, ▶).



3 The ClassPad gives answers in either exact form or as a decimal approximation. Tapping settings in the **status bar** will toggle between the available options.



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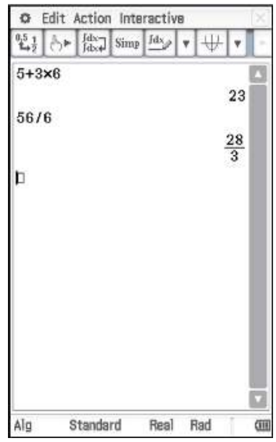
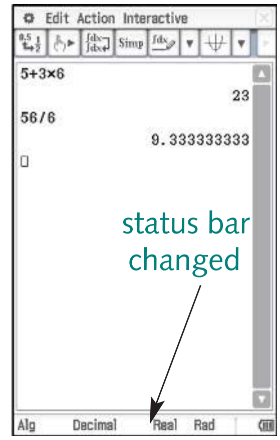
4 For example, if an exact answer is required for the calculation 56 ÷ 6, the **Standard** setting must be selected.

5 If a decimal approximation is required, change the **Standard** setting to **Decimal** by tapping it and press EXE.

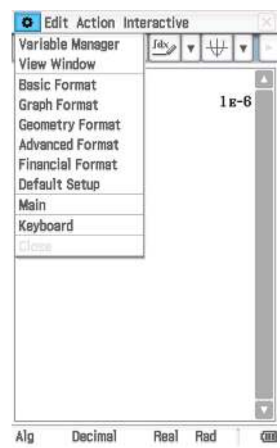




Extremely large and extremely small numbers

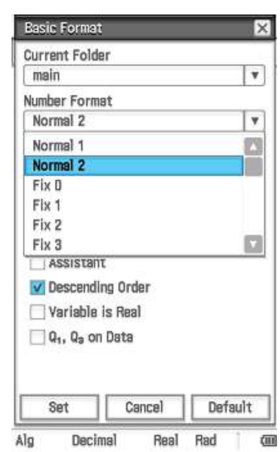
When solving problems that involve large or small numbers the calculator's default setting will give answers in scientific form.

For example, one millionth, or 1/1000000, in scientific form is written as 1×10^{-6} and the calculator will present this as 1E-6.



To change this setting, tap on the settings icon and select **Basic Format.**

Under the Number Format select **Normal 2** and tap SET.



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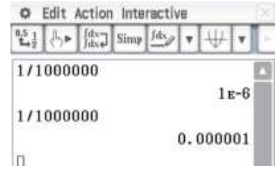
In the Main screen type 1/1000000 and press **EXE**.

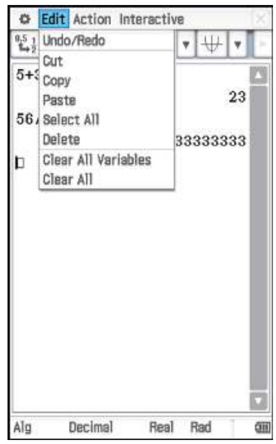
The answer will now be presented in decimal form 0.000001

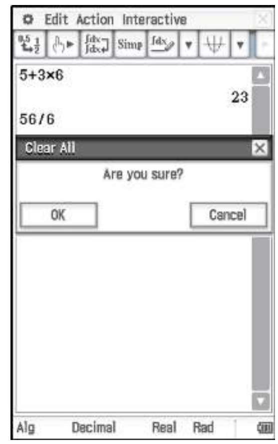
This setting will remain until the calculator is reset.



Clearing the history screen

To clear the **Main** application screen, select **Edit** from the menu bar and then tap **Clear All**. Confirm your selection by tapping **OK**. The entire screen is now cleared. To clear the entry line only, press **Clear** on the front calculator.

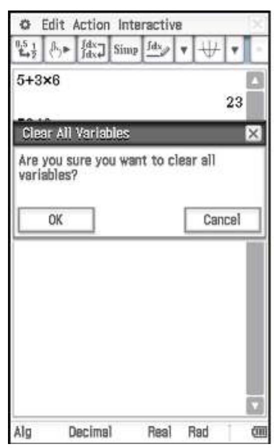


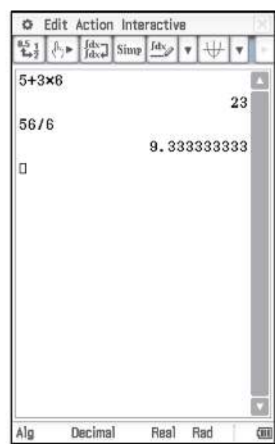


Clearing variables

To clear stored variable values, select **Edit** from the menu bar and then tap **Clear All Variables**. Confirm your selection by tapping OK.

The variables are cleared but the history created on the main screen is kept.





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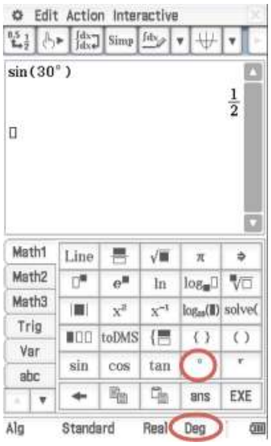
Degree mode

When solving problems in trigonometry, your calculator should be kept in **Degree** mode. In the main screen, the status bar displays the angle mode.

To change the angle mode, tap on the angle unit in the status bar until **Deg** is displayed.

In addition, it is recommended that you always insert the degree symbol after any angle. This overrides any mode changes and reminds you that you should be entering an angle, not a length.

The degree symbol is found in the Math1 keyboard.



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